A MODIFIED LIU-STOREY-CONJUGATE DESCENT HYBRID PROJECTION METHOD FOR CONVEX CONSTRAINED NONLINEAR EQUATIONS AND IMAGE RESTORATION

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ABSTRACT. We present an iterative method for solving the convex constraint nonlinear equation problem. The method incorporates the projection strategy by Solodov and Svaiter with the hybrid Liu-Storey and Conjugate descent method by Yang et al. for solving the unconstrained optimization problem. The proposed method does not require the Jacobian information, nor does it require to store any matrix at each iteration. Thus, it has the potential to solve large-scale non-smooth problems. Under some standard assumptions, the convergence analysis of the method is established. Finally, to show the applicability of the proposed method, the proposed method is used to solve the $\ell_1$-norm regularized problems to restore blurred and noisy images. The numerical experiment indicates that our result is a significant improvement compared with the related methods for solving the convex constraint nonlinear equation problem.

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1. Introduction. Let $\Omega$ be a nonempty closed and convex subset of $\mathbb{R}^n$ and $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonlinear monotone and continuous mapping. By monotonicity, we mean that $M$ satisfies the following inequality:

$$(b - y)^T (M(b) - M(y)) \geq 0, \forall b, y \in \mathbb{R}^n.$$ 

The concept of the monotone operator was first introduced by Minty [35]. The concept has aided several studies such as the abstract study of electrical networks. Interest in the study of monotone operators stems mainly from their firm connection with optimization problems. In general, a problem of interest in the study of nonlinear monotone operators in $\mathbb{R}^n$ is the following:

Find $b \in \Omega$ such that $M(b) = 0$. \hspace{1cm} (1)

Our interest in this paper is to propose an efficient derivative-free iterative method for approximating solutions of large scale nonlinear monotone equations.

Several problems in Mathematics, for example, split monotone inclusion problems, monotone variational inequality problems, fixed point problems of pseudo-contractive mappings and so on, can be transformed into the nonlinear monotone equation (1) (see, e.g., [14]). In applications, concrete problems arising from chemical equilibrium systems, economic equilibrium problems, image restoration and signal processing can be modeled as (1) (see, e.g., [34, 17, 40]). These interesting applications of the nonlinear monotone equation (1) have attracted the attention of many researchers. Thus, numerous iterative methods have been proposed by many authors to approximate solutions of (1). Among the early methods introduced and studied in the literature are Newton method, quasi-Newton method, Gauss-Newton method, Levenberg-Marquardt method and their modifications (see, e.g., [15, 32, 45, 46] and the references therein). However, the aforementioned methods are not efficient for solving large scale nonlinear monotone equations, because they involve the computation of the Jacobian matrix or its approximation per iteration, which is well-known to require large amount of storage.

Over the years, several Jacobian-free alternatives of these early methods have been proposed. Amongst these methods are conjugate gradient methods, spectral conjugate gradient methods and spectral gradient methods, which were employed in solving large-scale unconstrained optimization problems. Extensions of the conjugate gradient method and its variant to solve large scale nonlinear equations have been obtained by several authors. The literature on these extensions abound (see [19, 2, 22, 21, 23, 3, 6, 25, 24, 10, 27, 9, 8, 26, 20, 36, 5, 4, 7, 37] and references therein). The PRP conjugate gradient method [38] was extended to solve unconstrained monotone equations by Cheng [13]. La Cruz and Raydan [30] introduced a spectral algorithm (SANE) using the spectral gradient method, which solves a large scale unconstrained monotone equations. Later, a derivative-free SANE algorithm for a class of nonlinear unconstrained monotone equations was introduced by La Cruz et al. [29]. A similar approach was considered by Zhang and Zhou [44]. They generalized the spectral gradient algorithm and proposed a derivative-free spectral gradient projection method (SP) for unconstrained monotone equations. Yu et al. [43] considered the SP method for the case of constrained nonlinear monotone equations. Results obtained from numerical implementations of these derivative-free methods are promising. Thus, extending conjugate gradient methods for unconstrained optimization problems to solve (1) is meaningful.

Motivated by the idea of Liu-Storey- Conjugate descent hybrid method [42] and the projection method [39], our goal is to introduce a modified hybrid Liu-Storey
and Conjugate descent method to solve the convex constraint nonlinear equation problem. The proposed method can be viewed as an extension of the hybrid Liu-Storey and Conjugate descent method by Yang et al. [42] for solving unconstrained optimization problem. The global convergence result is established under some standard assumptions. Numerical experiments are presented to illustrate the efficiency of the proposed method. The proposed method is proved to be effective and suitable as an iterative method to solve large-scaled nonlinear equations with convex constraints. Moreover, we also successfully apply the proposed method to restore blurred and noisy images in compressive sensing.

The rest of this paper is organized as follows. The algorithm and its global convergence are presented in Section 2. In Section 3, we present the numerical experiments on nonlinear equations with convex constraints. Section 4, we provide numerical experiments to show its practical performance, and apply it to deal with the restoration of blurred and noisy images in compressive sensing. Finally, we finish this paper with a conclusion.

2. The Algorithm and Its Convergence Analysis. We begin this section with the definition of the projection map, followed by the assumptions that will be used in establishing the global convergence of the proposed method.

Definition 2.1. Let $B \subseteq \mathbb{R}^n$ be a nonempty closed convex set. Then for any $t \in \mathbb{R}^n$, its projection onto $B$, denoted by $P_B[t]$, is defined by

$$P_B[t] := \arg \min \{ \| t - y \| : y \in B \}. \quad (2)$$

The projection operator $P_B$ has a well-known property, that is, for any $t, y \in \mathbb{R}^n$, the following nonexpansive property hold

$$\| P_B[t] - P_B[y] \| \leq \| t - y \|. \quad (3)$$

Assumption 2.2.

1. The solution set of (1), denoted by $B^\star$, is nonempty.
2. The mapping $M$ is Lipschitz continuous on $\mathbb{R}^n$. That is, there exists a constant $L > 0$ such that

$$\| M(\alpha) - M(\beta) \| \leq L \| \alpha - \beta \| \quad \forall \alpha, \beta \in \mathbb{R}^n \quad (4)$$

3. For any $t \in B^\star$ and $u \in \mathbb{R}^n$, it holds that

$$M(u)^T (u - t) \geq 0. \quad (5)$$

We will now formally present the iterative procedures/steps of our method. The following Lemma shows that, the direction $p_k$ generated by Algorithm 1 satisfies the descent condition.

Lemma 2.3. Let $\{M(b_k)\}$ and $\{p_k\}$ be generated by Algorithm 1, then for all $k \geq 0$,

$$M(b_k)^T d_k \leq -c \| M(b_k) \|^2, \quad c > 0. \quad (10)$$

Proof. For $k = 0$, multiplying both sides of (6) by $M(b_0)^T$, we have

$$M(b_0)^T p_0 = -\|M(b_0)\|^2.$$

Also for $k \geq 1$, we have two cases. Suppose $\beta_k^{MLSCD} = 0$. Then we can easily see that (10) holds. For the case when $\beta_k^{MLSCD} = \min \{ \beta_k^{MLS}, \beta_k^{MCD} \}$, we have the following two cases.
Algorithm 1

**Input.** Set an initial point \( b_0 \in \mathcal{B} \), the positive constants: \( Tol > 0, r \in (0, 1), n \in (0, 2), \ a > 0, \mu > 0, \ d > 1 \). Set \( k = 0 \).

**Step 0.** Compute \( M(b_k) \). If \( \| M(b_k) \| \leq Tol \), stop. Otherwise, generate the search direction \( p_k \) by

\[
p_k := \begin{cases} -M(b_k) & \text{if } k = 0, \\ -M(b_k) + \beta_{k, MLSCD} p_{k-1} & \text{if } k > 0, \end{cases}
\]

where,

\[
\beta_{k, MLSCD} := \max \left\{ 0, \min \left\{ \beta_{k, MLS}, \beta_{k, MCD} \right\} \right\},
\]

\[
\beta_{k, MLS} := \frac{M(b_k)^T y_{k-1}}{\max\{d\|p_{k-1}\|\|y_{k-1}\|, \|M(b_{k-1})\|^2\}}, \quad y_{k-1} = M(b_k) - M(b_{k-1}),
\]

\[
\beta_{k, MCD} := \frac{\|M_k\|^2}{\max\{d\|p_{k-1}\|\|M(b_k)\|, dM(b_k)^T d_{k-1}\}}.
\]

**Step 1.** Determine the step-size \( \varepsilon_k = \max\{ar^i | i \geq 0\} \) such that \( M(b_k + \varepsilon_k p_k)^T p_k \geq \mu \varepsilon_k p_k \|p_k\|^2 \).

**Step 2.** Compute \( o_k = b_k + \varepsilon_k p_k \), where \( o_k \) is a trial point.

**Step 3.** If \( o_k \in \mathcal{B} \) and \( \| M(o_k) \| = 0 \), stop. Otherwise, compute the next iterate by

\[
b_{k+1} = P_\mathcal{B} \left[ b_k - n \frac{M(o_k)^T (b_k - o_k)}{\|M(o_k)\|^2} M(o_k) \right],
\]

where \( n = \frac{1}{\mu} \).

**Step 4.** Finally, set \( k = k + 1 \) and return to step 0.

1. When \( \min \{ \beta_{k, MLS}, \beta_{k, MCD} \} = \beta_{k, MLS} \), then by pre-multiplying (6) with \( M(b_k)^T \), we have the following

\[
M(b_k)^T p_k = -\|M(b_k)\|^2 + \beta_{k, MLS} M(b_k)^T p_{k-1}
\]

\[
= -\|M(b_k)\|^2 + \frac{M(b_k)^T y_{k-1}}{\max\{d\|p_{k-1}\|\|y_{k-1}\|, \|M(b_{k-1})\|^2\}} M(b_k)^T p_{k-1}
\]

\[
\leq -\|M(b_k)\|^2 + \frac{\|M(b_k)\|\|y_{k-1}\|}{d\|p_{k-1}\|\|y_{k-1}\|} \|M(b_k)\| \|p_{k-1}\|
\]

\[
= -\left( 1 - \frac{1}{d} \right) \|M(b_k)\|^2.
\]

2. min \( \{ \beta_{k, MLS}, \beta_{k, MCD} \} = \beta_{k, MCD} \), we can deduce that,

\[
M(b_k)^T p_k = -\|M(b_k)\|^2 + \beta_{k, MCD} M(b_k)^T p_{k-1}
\]

\[
= -\|M(b_k)\|^2 + \frac{\|M_k\|^2}{\max\{d\|p_{k-1}\|\|M(b_k)\|, dM(b_k)^T d_{k-1}\}} M(b_k)^T p_{k-1}
\]

\[
\leq -\|M(b_k)\|^2 + \frac{\|M(b_k)\|^2}{dM(b_k)^T p_{k-1}} M(b_k)^T p_{k-1}
\]

\[
= -\left( 1 - \frac{1}{d} \right) \|M(b_k)\|^2.
\]

Therefore, the result (10) is true.
Lemma 2.4. Let \( \{p_k\} \) and \( \{b_k\} \) be two sequences generated by Algorithm 1. Then, there exists a step size \( \varepsilon_k \) satisfying the line search (8) for all \( k \geq 0 \).

Proof. For any \( i \geq 0 \), suppose (8) does not hold for the iterate \( t_0 \)-th, then we have

\[-M(b_{k_0} + ar^i p_{k_0})^T p_{k_0} < \mu ar^i \|p_{k_0}\|^2.\]

Thus, by the continuity of \( M \) and with \( 0 < r < 1 \), it follows that by letting \( i \to \infty \), we have

\[-M(b_{k_0})^T p_{k_0} \leq 0,\]

which contradicts (10).

Lemma 2.5. Let the sequences \( \{b_k\} \) and \( \{o_k\} \) be generated by the Algorithm 1 method under Assumption 2.2, then

\[\varepsilon_k \geq \max \left\{ a, \frac{rc\|M(b_k)\|^2}{(L + \mu)\|p_k\|^2} \right\},\] (11)

Proof. Let \( \varepsilon_k = \varepsilon_k r^{-1} \). Assume \( \varepsilon_k \neq a \), from (8), \( \varepsilon_k \) does not satisfy (8). That is,

\[-M(b_k + \varepsilon_k p_k)^T p_k < \mu \varepsilon_k \|p_k\|^2.\]

From (4) and (10), it can be obviously seen that

\[c\|M(b_k)\|^2 \leq -M(b_k)^T p_k = (M(b_k + \varepsilon_k p_k) - M(b_k))^T p_k - M(b_k + \varepsilon_k p_k)^T p_k\]

\[\leq L \varepsilon_k \|p_k\|^2 + \mu \varepsilon_k \|p_k\|^2\]

\[\leq \varepsilon_k (L + \mu) \|p_k\|^2.\]

This gives the desired inequality (11).

Lemma 2.6. Suppose that Assumption 2.2 holds. Let \( \{b_k\} \) and \( \{o_k\} \) be sequences generated by the Algorithm 1, then for any solution \( b^* \) contained in the solution set \( \mathcal{B}^* \), the inequality

\[\|b_{k+1} - b^*\|^2 \leq \|b_k - b^*\|^2 - \mu^2 \|b_k - o_k\|^4.\] (12)

holds. In addition, \( \{b_k\} \) is bounded and

\[\sum_{k=0}^{\infty} \|b_k - o_k\|^4 < +\infty.\] (13)

Proof. First, we begin by using the weakly monotonicity assumption (Assumption 2.2 (iii)) on the mapping \( M \). Thus, for any solution \( b^* \in \mathcal{B}^* \),

\[M(o_k)^T (b_k - b^*) \geq M(o_k)^T (b_k - o_k).\]

The above inequality together with (8) gives

\[M(b_k + \varepsilon_k p_k)^T (b_k - o_k) \geq \mu \varepsilon_k^2 \|p_k\|^2 \geq 0.\] (14)

From (3) and (14), we have the following

\[\|b_{k+1} - b^*\|^2 = \|P_{\mathcal{B}} \left[ b_k - n \frac{M(o_k)^T (b_k - o_k)}{\|M(o_k)\|^2} M(o_k) \right] - b^* \|^2\]
To show this, we have the following cases:

that

Having in view that from Lemma 2.6 that the sequences \( \{ s_k \} \) generated by Algorithm 1, then

Thus, the sequence \( \{ \| b_k - b^* \| \} \) has a nonincreasing and convergent property. Therefore, this makes \( \{ b_k \} \) to be bounded and therefore the following holds.

\[
\mu^2 \sum_{k=0}^{\infty} \| b_k - o_k \|^4 < \| b_0 - b^* \|^2 < +\infty.
\]

Remark 1. Taking into account of the definition of \( o_k \) and also by (13), it can be deduced that

\[
\lim_{k \to \infty} \varepsilon_k \| p_k \| = 0.
\]

Theorem 2.7. Suppose Assumption 2.2 holds. Let \( \{ b_k \} \) and \( \{ o_k \} \) be sequences generated by Algorithm 1, then

\[
\liminf_{k \to \infty} \| M(b_k) \| = 0.
\]

Proof. Suppose (16) is not valid, that is, there exist a constant say \( s > 0 \) such that

\[
s \leq \| M(b_k) \|, \quad k \geq 0.
\]

Then this along with (10) implies that

\[
\| p_k \| \geq cs, \quad \forall k \geq 0.
\]

Having in view that from Lemma 2.6 that the sequences \( \{ b_k \} \) is bounded by a positive constant say \( k_0 \). In addition with the continuity of \( M \), it further implies that \( \{ \| M(b_k) \| \} \) is bounded by a constant say \( u \). Thus, we show that \( p_k \) is bounded. To show this, we have the following cases:

- if \( \beta_k^{MLSCD} = 0 \), it is easy to see that

\[
\| p_k \| = \| M(b_k) \| \leq u.
\]

- if \( \beta_k^{MLSCD} = \beta_k^{MLS} \),

\[
\| p_k \| = \| M(b_k) + \beta_k^{MLS} p_{k-1} \|
\leq \| M(b_k) \| + \frac{\| M(b_k) \| \| y_{k-1} \|}{d \| p_{k-1} \|} \| y_{k-1} \| \| p_{k-1} \|
\leq \| M(b_k) \| + \frac{\| M(b_k) \|}{d}
\]
= \left(1 + \frac{1}{d}\right)\|M(b_k)\|^2
= \left(1 + \frac{1}{d}\right)u^2 \triangleq Q.

- if \beta_k^{MLSCD} = \beta_k^{MCD},

\|p_k\| = \| - M(b_k) + \beta_k^{MCD}p_{k-1}\|
\leq \|M(b_k)\| + \frac{\|M(b_k)\|^2}{d\|p_{k-1}\|\|M(b_k)\|\|p_{k-1}\|}
= \|M(b_k)\| + \frac{\|M(b_k)\|}{d}
= \left(1 + \frac{1}{d}\right)\|M(b_k)\|^2
= \left(1 + \frac{1}{d}\right)u^2 \triangleq Q.

Therefore, we can deduce that for all \(k\),

\|p_k\| \leq Q.

From (11), we have

\varepsilon_k\|p_k\| \geq \max \left\{ a, \frac{r_c\|M(b_k)\|^2}{(L + \mu)\|p_k\|^2}\right\}\|p_k\|
\geq \max \left\{ acs, \frac{r_c s^2}{(L + \mu)Q}\right\} > 0,

which contradicts (15). Hence (16) is valid.

3. Numerical Experiments. This section focuses on the numerical experiments for testing the efficiency of the proposed method based on the Dolan and More performance profile [18]. Nine test problems were considered for the experiments with all codes written and implemented in Matlab. The proposed method is compared with methods that share similar characteristics. For example, the derivative-free iterative method for nonlinear monotone equations with convex constraints (PDY) [33] and the conjugate gradient method to solve convex constrained monotone equations (CGD) [41]. All algorithms are terminated when

\|M(b_k)\| \leq 10^{-6}.

Our proposed algorithm (denoted as MLSCD) was implemented using the following parameters: \(a = 1\), \(r = 0.8\), \(n = 1.2\), \(\mu = 10^{-4}\), \(d = 1.2\); while PDY and CGD were implemented using the parameters reported in their respective papers. More so, our experiment made use of various dimensions of 1000, 5000, 10,000, 50,000, 100,000 and initial points \(b_1 = (0.1, 0.1, \cdots , 0.1)^T\), \(b_2 = (0.2, 0.2, \cdots , 0.2)^T\), \(b_3 = (0.5, 0.5, \cdots , 0.5)^T\), \(b_4 = (1.2, 1.2, \cdots , 1.2)^T\), \(b_5 = (1.5, 1.5, \cdots , 1.5)^T\), \(b_6 = (2, 2, \cdots , 2)^T\), \(b_7 = \text{rand}(0, 1)\).

The test problems with \(M = (M_1, M_2, \cdots , M_n)\) are given below:
Problem 1 [29] Exponential Function.

\[ M_i(b) = e^{b_i} - 1, \]
\[ M_i(b) = e^{b_i} + b_i - 1, \quad \text{for } i = 2, 3, ..., n, \]
and \( B = \mathbb{R}^n_{+}. \)

Problem 2 [29] Modified Logarithmic Function.

\[ M_i(b) = \ln(b_i + 1) - \frac{b_i}{n}, \quad \text{for } i = 1, 2, 3, ..., n, \]
and \( B = \left\{ b \in \mathbb{R}^n : \sum_{i=1}^{n} b_i \leq n, b_i > -1, i = 1, 2, \cdots, n \right\}. \)

Problem 3 [28]

\[ M_i(b) = \min \left( \min(|b_i|, b_i^2), \max(|b_i|, b_i^2) \right), \quad \text{for } i = 2, 3, ..., n, \]
and \( B = \mathbb{R}^n_{+}. \)

Problem 4 [29] Strictly Convex Function I.

\[ M_i(b) = e^{b_i} - 1, \quad \text{for } i = 1, 2, ..., n, \]
and \( B = \mathbb{R}^n_{+}. \)

Problem 5 [29] Strictly Convex Function II.

\[ M_i(b) = \frac{i}{n} e^{b_i} - 1, \quad \text{for } i = 1, 2, ..., n, \]
and \( B = \mathbb{R}^n_{+}. \)

Problem 6 [11] Tridiagonal Exponential Function.

\[ M_1(b) = b_1 - e^{\cos(h(b_1 + b_2))}, \]
\[ M_i(b) = b_i - e^{\cos(h(b_{i-1} + b_i + b_{i+1}))}, \quad \text{for } i = 2, ..., n - 1, \]
\[ M_n(b) = b_n - e^{\cos(h(b_{n-1} + b_n))}, \]
\[ h = \frac{1}{n + 1} \]

Problem 7 [43] Nonsmooth Function.

\[ M_i(b) = b_i - \sin |b_i - 1|, \quad i = 1, 2, 3, ..., n, \]
and \( B = \left\{ q \in \mathbb{R}^n : \sum_{i=1}^{n} b_i \leq n, b_i \geq -1, i = 1, 2, \cdots, n \right\}. \)

Problem 8 [29] The Trig exp function

\[ M_1(b) = 3b_1^3 + 2b_2 - 5 + \sin(b_1 - b_2) \sin(b_1 + b_2) \]
\[ M_i(b) = 3b_i^3 + 2b_{i+1} - 5 + \sin(b_i - b_{i+1}) \sin(b_i + b_{i+1}) + 4b_i - b_{i-1} e^{b_i - 1} - b_i - 3 \]
\[ \text{for } i = 2, 3, ..., n - 1 \]
\[ M_n(b) = b_{n-1} e^{b_{n-1} - b_n} - 4b_n - 3, \quad \text{where } h = \frac{1}{m + 1} \text{ and } B = \mathbb{R}^n_{+}. \]
Problem 9 \cite{16}

\[ t_i = \sum_{i=1}^{n} b_i^2, \quad d = 10^{-5} \]

\[ M_i(b) = 2d(b_i - 1) + 4(t_i - 0.25)b_i, \quad i = 1, 2, 3, ..., n, \]

and \( \mathcal{B} = \mathbb{R}_n^+ \).

From the numerical comparison (Tables 1-9 in the appendix section), we observe that our proposed algorithm is competitive and performs better than the CGD \cite{41} and PDY \cite{33} in terms of the number of iteration (\#IT), the number of function evaluation (FV) and CPU running time (RT) in all the problems considered. Also, in the problems (Problems 1-9), we observe that all the algorithms are independent of the initial point and dimension chosen. Finally, we use the Dolan and More performance profile to compare the efficiency and robustness of the algorithms. It is evident from Figs. 1-3 that our proposed algorithms perform better than CGD and PDY.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Performance profiles based on number of iterations.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Performance profiles based on number of function evaluations.}
\end{figure}
3.1. Image Restoration Problem. In this subsection, we consider the image restoration problem. Specifically, image restoration can be formulated by the inversion of the following observation model:

\[ b = Gu + \varepsilon, \tag{18} \]

where \( G = [g_1, g_2, \ldots, g_m] \) is a blurring matrix, \( b \) is the observed image (\( b \in \mathbb{R}^m \)) and \( \varepsilon \in \mathbb{R}^m \) is the random additive noise. Since the matrix \( G \) is highly ill-conditioned, a classical and widely studied restoration model is

\[ \min_u \left\{ \frac{1}{2} \| Gu - b \|^2 + \zeta \| u \|_1 \right\}, \tag{19} \]

where \( \zeta \) is a positive regularization parameter. The equivalence relation between the convex unconstrained minimization problem (19) and the convex constraint nonlinear equation (1) has been established in the literature (see, [41] and references therein).

In the following, numerical results are presented. The methods are tested on three colored images (Lenna, Tiffany and Malamute) of size 512 × 512 which were degraded using Gaussian noise operator and gaussian blur of standard deviation \( 10^{-2} \). We compare the performance of the proposed method (MLSCD) with other related methods such as MFRM [1] and ELSFR [24]. The implementation of the methods were carried out using MATLAB R2020b run on a HP (2.3 GHz Intel Core i3 processor, 8 GB RAM). The quality of the restored image is determined using the signal-to-noise ratio (SNR), Peak signal-to-noise ratio [12] and the Structural Similarity index (SSIM) [31]. The original test images and the recovered images by the various methods are illustrated in Figure 4.

For the implementation of the MLSCD, we set parameters as follows: \( a = 1, \; r = 0.5, \; n = 1, \; d = 0.5, \; \mu = 0.0001 \). The parameters for the compared methods were set as reported in their respective articles. Table 1 shows that MLSCD has a higher SNR, PSNR and SSIM than the other algorithms, thus, in restoring the blurred images, images restored by MLSCD method is better than MFRM and ELSFR method.

Note. In our experiments, all iterative procedure are terminated when the relative change between successive iterates falls below \( 10^{-4} \).
Figure 4. Image restoration: The original images (First column), Blurred and noisy images (Second Column), restored images by MLSCD (Third column), MFRM (Fourth column) and restored images by ELSFR (Fifth column)

Table 1. Numerical result for the image restoration problem

| Images | SNR  | PSNR | SSIM | SNR  | PSNR | SSIM | SNR  | PSNR | SSIM |
|--------|------|------|------|------|------|------|------|------|------|
| Lena   | 20.43| 25.76| 0.943| 18.02| 23.35| 0.924| 19.84| 25.18| 0.941|
| Tiffany| 24.27| 26.11| 0.933| 22.13| 23.96| 0.905| 23.75| 25.59| 0.935|
| Barbara| 16.03| 22.45| 0.706| 14.22| 20.64| 0.645| 15.65| 22.07| 0.701|

Conclusion. In this paper, we proposed, analysed and tested a family of efficient sufficient descent algorithms to solve non-linear constrained monotone equations. The proposed method does not require the Jacobian information and does not store any matrix at each iteration. Our motivation was simple and natural and mainly came from the hybrid LS and CD method by Yang et al [42]. Under some mild conditions, we proved its global convergence. Preliminary numerical experiments illustrated that the proposed method worked well and was competitive with its competitors.

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Appendix. Please see https://documentcloud.adobe.com/link/review?uri=urn:aaid:scds:US:ad2bc2b8-c83a-428b-b936-e41351d68faa

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