UPPER BOUNDING FOR PACKING DIMENSION IN VECTORIAL MULTIFRACTAL FORMALISM

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Abstract. We establish an other upper bounding for packing dimension in the framework of the vectorial multifractal formalism that is in some cases finer than that established by J. Peyrière.

1. Introduction

The multifractal analysis was developed around 1980, following the work of B. Mandelbrot [5, 6], when he studied the multiplicative cascades for energy dissipation in a context of turbulence. In 1992, G. Brown, G. Michon and J. Peyrière [2] have established the first general and rigorous theorems of the multifractal formalism. Their work prompted the three past decades, several mathematicians [8, 3, 7, 4, 1, 9, ...], to develop their research in various contexts by generalizing or improving the multifractal formalism.

In this paper, we take place in the framework of the vectorial multifractal formalism introduced by J. Peyrière [9] in 2004. We recall at the end of this paragraph the results of this formalism that we are going to use later. In the second section, we give an other upper bounding for packing dimension [10] of the set

\[ X_\chi (\alpha, E) = \left\{ x \in X; \limsup_{r \to 0} \frac{\langle q, \chi(x, r) \rangle}{\log r} \leq \langle q, \alpha \rangle, \forall q \in E \right\}, \]

where \( X \) is a metric space verifying the Besicovitch covering property, \( E \) is a subset of a separable real Banach space \( E \), \( \chi \) is a function from \( X \times [0, 1] \) to the dual \( E' \) and \( \alpha \in E' \).

In the third section, we present some situations where our inequality is finer than that made by J. Peyrière in [9].

In what follows, we recall the vectorial multifractal formalism introduced by J. Peyrière in [9].

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For $A \subset X$, $q \in \mathbb{E}$, $t \in \mathbb{R}$ and $\varepsilon \in [0,1]$, we set
\[
\mathcal{P}^{q,t}_{\chi,\varepsilon}(A) = \sup \left\{ \sum_i r_i^t e(q, \chi(x_i, r_i)) \right\},
\]
where the supremum is taken over all the centered $\varepsilon-$packing $(B(x_i, r_i))_{i \in I}$ of $A$.

Then, we set
\[
\overline{\mathcal{P}}^{q,t}_{\chi}(A) = \lim_{\varepsilon \to 0} \mathcal{P}^{q,t}_{\chi,\varepsilon}(A)
\]
and
\[
\underline{\mathcal{P}}^{q,t}_{\chi}(A) = \inf \left\{ \sum_i \mathcal{P}^{q,t}_{\chi}(A_i); \ A \subset \bigcup_i A_i \right\}.
\]

It is clear that
\[
\mathcal{P}^{q,t}_{\chi}(A) = \inf \left\{ \sum_i \mathcal{P}^{q,t}_{\chi}(A_i); \ A = \bigcup_i A_i \right\}
\]
and
\[
\mathcal{P}^{q,t}_{\chi}(A) = \inf \left\{ \sum_i \mathcal{P}^{q,t}_{\chi}(A_i); \ \text{\textit{\bigcup}_i A_i \text{ is a partition of } A} \right\}.
\]

We denote by $\Delta^q_{\chi}(A)$ and $\Dim^q_{\chi}(A)$ the dimensions of $A$ characterized by
\[
\mathcal{P}^{q,t}_{\chi}(A) = \begin{cases} +\infty, & \text{if } t < \Delta^q_{\chi}(A), \\ 0, & \text{if } t > \Delta^q_{\chi}(A), \end{cases}
\]
and
\[
\mathcal{P}^{q,t}_{\chi}(A) = \begin{cases} +\infty, & \text{if } t < \Dim^q_{\chi}(A), \\ 0, & \text{if } t > \Dim^q_{\chi}(A). \end{cases}
\]

For $X = \mathbb{R}^d$, $\mathbb{E} = \mathbb{R}$, $\mu$ a Borel probability measure on $\mathbb{R}^d$, and considering the function $\chi$ defined by
\[
\langle q, \chi(x_i, r_i) \rangle = q \log \mu(B(x_i, r_i)).
\]
for all centered $\varepsilon-$packing $(B(x_i, r_i))_{i \in I}$ of $A$, we found the formalism introduced by L. Olsen [7], in particular we get
\[
\Delta^q_{\chi}(A) = \Delta^q_{\mu}(A) \quad \text{and} \quad \Dim^q_{\chi}(A) = \Dim^q_{\mu}(A).
\]

Furthermore, note also that for the trivial case $\chi = 0$, we obtain the prepacking dimension and the packing dimension of $A$, ie
\[
\Delta^q_{\chi}(A) = \Delta(A) \quad \text{and} \quad \Dim^q_{\chi}(A) = \Dim(A).
\]

The following proposition and theorem are established in [9].
Proposition 1. Write $\Lambda_{\chi}(q) = \Delta_{\chi}(X)$ and $B_{\chi}(q) = \text{Dim}_{\chi}(X)$. Then

i. $B_{\chi} \leq \Lambda_{\chi}$.

ii. The functions $\Lambda_{\chi} : q \mapsto \Lambda_{\chi}(q)$ and $B_{\chi} : q \mapsto B_{\chi}(q)$ are convex.

Theorem 1. For $\alpha \in E'$ and $E \subset E$ we set

$$X_{\chi}(\alpha, E) = \left\{ x \in X; \limsup_{r \to 0} \frac{\langle q, \chi(x, r) \rangle}{\log r} \leq \langle q, \alpha \rangle, \ \forall q \in E \right\},$$

then

$$\text{Dim}(X_{\chi}(\alpha, E)) \leq \inf_{q \in E} (\langle q, \alpha \rangle + B_{\chi}(q)).$$

2. An other upper bounding for $\text{Dim}(X_{\chi}(\alpha, E))$

Let $\varepsilon > 0$ be a real number and $k \geq 1$ be an integer. A family $(B(x_i, r_i))_{i \in I}$ is called a centered $\varepsilon - k$—Besicovitch packing of a set $A$ when $I = I_1 \cup \ldots \cup I_s$ with $1 \leq s \leq k$ and $(B(x_i, r_i))_{i \in I_j}$ a centered $\varepsilon$—packing of $A$ for all $1 \leq j \leq s$.

Let $(u_\varepsilon)_{\varepsilon > 0}$ be a decreasing family of numbers such that $\varepsilon \leq u_\varepsilon$ and $\lim_{\varepsilon \to 0} u_\varepsilon = 0$.

For $q \in E$, $A \subset X$ and $(B(x_i, r_i))_{i \in I}$ a centered $\varepsilon$—packing of $A$, we consider all the families $(B(y_i, \delta_i))_{i \in I}$ that are centered $u_\varepsilon - k$—Besicovitch packing of $A$ and we set

$$L_{q,k}^\varepsilon, (B(x_i, r_i))_{i \in I} (A) = \inf \left( \sup_{i \in I} \left( \frac{\langle q, \chi(y_i, \delta_i) \rangle}{\log r_i} \right) \right),$$

where the infimum is taken over all the centered $u_\varepsilon - k$—Besicovitch packing $(B(y_i, \delta_i))_{i \in I}$ of $A$.

It is clear that

$$(2.1) \quad L_{q,k}^\varepsilon, (B(x_i, r_i))_{i \in I} (A) \leq \sup_{i \in I} \left( \frac{\langle q, \chi(x_i, r_i) \rangle}{\log r_i} \right).$$

Write

$$L_{q,k}^\varepsilon (A) = \sup \left\{ L_{q,k}^\varepsilon, (B(x_i, r_i))_{i \in I} (A) \right\},$$

where the supremum is taken over all the centered $\varepsilon$—packing $(B(x_i, r_i))_{i \in I}$ of $A$.

We remark that for $\varepsilon < \varepsilon'$, $L_{q,k}^{\varepsilon'}(A) > L_{q,k}^{\varepsilon}(A)$, then we define

$$L_{q,k}^\varepsilon (A) = \lim_{\varepsilon \to 0} L_{q,k}^{\varepsilon} (A).$$

As the sequence $(L_{q,k}^{q,k}(A))_k$ is decreasing, write

$$L_{q}^\varepsilon (A) = \lim_{k \to +\infty} L_{q,k}^{q,k} (A).$$
Before giving our new inequality involving $\text{Dim}(X_\chi(\alpha, E))$, we first illustrate our main idea on the set $A^{(q,\alpha)}$ defined for $\alpha \in \mathbb{E}'$, $q \in E$ and $r_0 > 0$ by

$$A^{(q,\alpha)} = \{ x \in \mathbb{X}; r^{(q,\alpha)} \leq e^{(q,\chi(x,r))}, \text{ for } r < r_0 \}. \hspace{2cm} (2.2)$$

**Proposition 2.**

$$L^q(A^{(q,\alpha)}) \leq \langle q, \alpha \rangle. \hspace{2cm} \Box$$

**Proof.** Let $\varepsilon < r_0$ and $(B(x_i,r_i))_{i \in I}$ a centered $\varepsilon-$packing of $A^{(q,\alpha)}$. Thanks to the characteristic property of $A^{(q,\alpha)}$ (2.2), it comes for all $i \in I$,

$$\frac{\langle q, \chi(x_i,r_i) \rangle}{\log r_i} \leq \langle q, \alpha \rangle,$$

hence

$$\sup_{i \in I} \frac{\langle q, \chi(x_i,r_i) \rangle}{\log r_i} \leq \langle q, \alpha \rangle,$$

from the inequality (2.1), we deduce that

$$L^{q,k}_{\varepsilon,(B(x_i,r_i))_{i \in I}}(A^{(q,\alpha)}) \leq \langle q, \alpha \rangle.$$ \hspace{2cm} (2.3)

while considering the supremum over all centered $\varepsilon-$packing, it results that

$$L^{q,k}_{\varepsilon}(A^{(q,\alpha)}) \leq \langle q, \alpha \rangle.$$ \hspace{2cm} (2.4)

Letting $\varepsilon \to 0$, we obtain that

$$L^q(A^{(q,\alpha)}) \leq \langle q, \alpha \rangle,$$

then letting $k \to +\infty$, it comes that

$$L^q(A^{(q,\alpha)}) \leq \langle q, \alpha \rangle.$$

\hspace{2cm} $\Box$

**Theorem 2.** Let $\alpha \in \mathbb{E}'$ and $q \in E$.

For $t < 0$ we set $\Phi_q(t) = \inf \{ \gamma > 0; t \langle q, \alpha \rangle > B_\chi((\gamma - t)q) \}$. Then,

$$\text{Dim}(A^{(q,\alpha)}) \leq \Phi_q(t)L^q(A^{(q,\alpha)}).$$

**Proof.** For $t < 0$ and $\gamma > 0$ such that $t \langle q, \alpha \rangle > B_\chi((\gamma - t)q$, it is clear that $P^{(\gamma-t)q,t(q,\alpha)}_\chi(\mathbb{X}) = 0$. Then $P^{(\gamma-t)q,t(q,\alpha)}_\chi(A^{(q,\alpha)}) = 0$. From the equality (1.1), we write

$$A^{(q,\alpha)} = \bigcup_{m \in M} A_m \hspace{2cm} \text{(2.3)}$$

such that for all $m \in M$,

$$P^{(\gamma-t)q,t(q,\alpha)}_\chi(A_m) < +\infty. \hspace{2cm} \text{(2.4)}$$
Let \( \lambda > L^q(A^{(q,\alpha)}) \), let us prove first that for all \( m \in M \),
\[
\Delta(A_m) \leq \gamma \lambda.
\]

As \( A_m \subset A^{(q,\alpha)} \), and \( \lambda > L^q(A_m) \), then there exist an integer \( k \geq 1 \) and a real number \( \varepsilon_0 < r_0 \) such that for all \( \varepsilon < \varepsilon_0 \),
\[
L_{\varepsilon}^{q,k}(A_m) < \lambda.
\]

It comes that for all tout centered \( \varepsilon \)-packing \( (B(x_i, r_i)) \) of \( A_m \), there exists a centered \( u_\varepsilon - k \)-Besicovitch packing \( (B(y_i, \delta_i))_{i \in I} \) of \( A_m \) such that for all \( i \in I \),
\[
\frac{\langle q, \chi(y_i, \delta_i) \rangle}{\log r_i} < \lambda,
\]
so that
\[
(2.5) \quad r_i^\lambda < e^{\langle q, \chi(y_i, \delta_i) \rangle}.
\]

Thanks to the characteristic property of \( A^{(q,\alpha)} \) \( (2.2) \), it comes that
\[
(2.6) \quad \delta_i^{(q,\alpha)} < e^{\langle q, \chi(y_i, \delta_i) \rangle}.
\]

Thus from the inequalities \( (2.5) \) and \( (2.6) \), we obtain that for all \( \gamma > 0 \) and \( t < 0 \),
\[
r_i^{\gamma \lambda} < e^{(\gamma-t)\langle q, \chi(y_i, \delta_i) \rangle} \delta_i^{t(q,\alpha)}.
\]

Using the equality \( I = I_1 \cup \ldots \cup I_s \) with \( 1 \leq s \leq k \) and \( (B(x_i, r_i))_{i \in I_j} \) a centered \( \varepsilon \)-packing of \( A_m \) for all \( 1 \leq j \leq s \), it follows that
\[
\sum_{i \in I} r_i^{\gamma \lambda} \leq \sum_{i \in I} e^{(\gamma-t)\langle q, \chi(y_i, \delta_i) \rangle} \delta_i^{t(q,\alpha)} = \sum_{j=1}^{s} \sum_{i \in I_j} e^{(\gamma-t)\langle q, \chi(y_i, \delta_i) \rangle} \delta_i^{t(q,\alpha)}.
\]

It results that
\[
(2.7) \quad \sum_{i \in I} r_i^{\gamma \lambda} \leq k P_{\chi, u_\varepsilon}^{(\gamma-t)q,t(q,\alpha)}(A_m).
\]

We note that from the inequality \( (2.4) \), there exists \( \varepsilon_1 > 0 \) such that for \( u_\varepsilon < \varepsilon_1 \),
\[
P_{\chi, u_\varepsilon}^{(\gamma-t)q,t(q,\alpha)}(A_m) < +\infty.
\]

Then from the inequality \( (2.7) \), it comes that for all \( m \in M \),
\[
\Delta(A_m) \leq \gamma \lambda.
\]

Therefore
\[
\text{Dim}(A_m) \leq \gamma \lambda, \ m \in M.
\]

From the equality \( (2.3) \), we write
\[
\text{Dim}(A^{(q,\alpha)}) \leq \gamma \lambda, \ m \in M.
\]
Finally, for all $t < 0$, we obtain
\[
\text{Dim}(A^{(q,\alpha)}) \leq \Phi_q(t)L^q(A^{(q,\alpha)}).
\]
\[\square\]

Thereafter, let $\alpha \in E'$ and $q \in E$. We set
\[
\Phi_q = \inf_{t < 0} (\Phi_q(t)).
\]
and
\[
X^q(\alpha) = \left\{ x \in X; \limsup_{r \to 0} \frac{\langle q, \chi(x, r) \rangle}{\log r} \leq \langle q, \alpha \rangle \right\}.
\]
For all real number $\eta > 0$ and $p \geq 1$ an integer, we set
\[
X_{(q,\alpha)}(\eta, p) = \left\{ x \in X(\alpha); r^{(q,\alpha)+\eta} \leq e^{\langle q, \chi(x, r) \rangle} \text{ for } r < \frac{1}{p} \right\}
\]
and
\[
T^q(\alpha, \eta, p) = \sup_{A \subset X_{(q,\alpha)}(\eta, p)} L^q(A),
\]
\[
T^q(\alpha, \eta) = \lim_{p \to +\infty} T^q(\alpha, \eta, p),
\]
\[
T^q(\alpha) = \lim_{\eta \to 0^+} T^q(\alpha, \eta).
\]

**Theorem 3.**
\[
\text{Dim}(X(\alpha, E)) \leq \inf_{q \in E} \left\{ \Phi_q T^q(\alpha) \right\}.
\]

**Proof.** Let $q \in E$ and suppose that $X^q(\alpha) \neq \emptyset$. From the theorem it comes that for all $\eta > 0$,
\[
\text{Dim}(X_{(q,\alpha)}(\eta, p)) \leq \Phi_q(t)L^q(X_{(q,\alpha)}(\eta, p)).
\]
Thus
\[
\text{Dim}(X_{(q,\alpha)}(\eta, p)) \leq \Phi_q(t)T^q(\alpha, \eta, p).
\]
Then for all $\eta > 0$,
\[
\text{Dim}(\bigcup_p X_{(q,\alpha)}(\eta, p)) \leq \Phi_q(t)T^q(\alpha, \eta).
\]
We remark that for all $\eta > 0$,
\[
X^q(\alpha) \subset \bigcup_p X_{(q,\alpha)}(\eta, p).
\]
It results that for all $\eta > 0$,
\[
\text{Dim}(X^q(\alpha)) \leq \Phi_q(t)T^q(\alpha, \eta).
\]
Letting $\eta \to 0$, we obtain that
\[
\text{Dim}(X^q(\alpha)) \leq \Phi_q(t)T^q(\alpha).
\]
So it comes that
\[ \dim(X_\chi^q(\alpha)) \leq \inf_{t<0} (\Phi_q(t)) T_\chi^q(\alpha). \]
It is clear that \( X_\chi^q(\alpha, E) = \bigcap_{q \in E} X_\chi^q(\alpha) \). Then for all \( q \in E \)
\[ \dim(X_\chi(\alpha, E)) \leq \Phi_q T_\chi^q(\alpha). \]
Finally it follows that
\[ \dim(X_\chi(\alpha, E)) \leq \inf_{q \in E} \{ \Phi_q T_\chi^q(\alpha) \}. \]
\[ \square \]

We have just established an other upper bounding for \( \dim(X_\chi(\alpha, E)) \) which is in some cases thinner than that established by J. Peyrière in \cite{9} as shown in the example below.

3. Example

To build the example, we first define the metric space \( \mathbb{X} \) and then choose the function \( \chi \).

We denote by \( \mathcal{A} \) the set \( \{0, 1\} \) and by \( \mathcal{A}^n \) all words of length \( n \) constructed with \( \mathcal{A} \) as alphabet. The empty word is denoted by \( \epsilon \). For all \( j \in \mathcal{A}^n \), we set \( N_0(j) \) the number of occurrence of the letter 0 in the word \( j \).

Let \( j \) and \( j' \) two words, we denote by \( jj' \) the concatenation of \( j \) and \( j' \).

We denote by \( \mathbb{X} \) the symbolic space \( \{0, 1\}^\mathbb{N} \), ie the set of sequences \( (x_i)_{i \geq 0} \) of elements of \( \{0, 1\} \). Is defined in the same way the concatenation of a finite word and an infinite word.

If \( x = (x_i)_{i \geq 0}, y = (y_i)_{i \geq 0} \in \mathbb{X}, \) we set
\[ d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 2^{-n}, & \text{if } x_n \neq y_n \text{ and } x_i = y_i \text{ for all } 0 \leq i < n. \end{cases} \]

If \( j = j_0 j_1 \ldots j_{n-1} \in \mathcal{A}^n \), we set the cylinder
\[ [j] = [j_0 j_1 \ldots j_{n-1}] = \{ jx, \ x \in \mathbb{X} \}. \]

It is clear that if \( x = (x_i)_{i \geq 0} \in \mathbb{X} \) and \( 2^{-n-1} \leq r < 2^{-n} \), then
\[ B(x, r) = [x_0 x_1 \ldots x_n]. \]

Let \( \mathcal{L} \) be a family of cylinders, any element \([j]\) of \( \mathcal{L} \) is called selected cylinder.

Let \( 0 < p_0 \leq p_1 \) such that \( p_0 + p_1 = 1 \).
We associate the measure $\mu$ on $X$ such that for any cylinder $[j]$ and $l \in \{0, 1\}$,

$$\mu([jl]) = \begin{cases} \frac{p_l \mu([j])}{2}, & \text{if } [j] \text{ contains a selected cylinder,} \\
\mu([j]) & \text{otherwise.}
\end{cases}$$

For the construction of the example we choose the part $\mathcal{L}$ as follows. Let $\beta_1, \beta_2, \gamma_1$ and $\gamma_2$ be real numbers such that

$$\frac{1}{2} < \beta_1 < \gamma_1 < \beta_2 < \gamma_2 < \frac{1}{3}.$$  

We say that the cylinder $[j]$ such that $j \in \mathcal{A}^n$ is of

- type $T_1$, if $\beta_1 < \frac{N_0(j)}{n} < \gamma_1$,

- type $T_2$, if $\beta_2 < \frac{N_0(j)}{n} < \gamma_2$.

Let $j \in \mathcal{A}^n$ such that $[j]$ is of type 1 (respectively of type 2), put $\widetilde{[j]}$ the set of the cylinders $[j'], j' \in \mathcal{A}^{n+6}$, contained in $[j]$ and of the same type than $[j]$.

Let $n_0 \in \mathbb{N}$ be a multiple of 6 and $(n_p)$ the sequence of integers defined by

$$n_0, n_{3i+1} = 2n_3 n_0, n_{3i+2} = 2n_{3i+1} \text{ and } n_{3i+3} = 2n_{3i+2}.$$  

For $k \in \mathbb{N}$ we construct the family $\mathcal{G}_k$ of disjoint cylinders $[j], j \in \mathcal{A}^{n_0+6k}$ such that:

- any element $[j]$ of $\mathcal{G}_k$ such that $j \in \mathcal{A}^n$ satisfies the relation

$$\beta_1 < \frac{N_0(j)}{n} < \gamma_2;$$

- $\mathcal{G}_0$ contains two cylinders $[j^1]$ and $[j^2]$ respectively of type $T_1$ and $T_2$,

- any element of $\mathcal{G}_{k+1}$ is contained in an element of $\mathcal{G}_k$ called his father,

- all elements of $\mathcal{G}_k$ beget the same number of son in $\mathcal{G}_{k+1}$, and from the generation $\mathcal{G}_k$ to generation $\mathcal{G}_{k+1}$ we distinguish the following three cases:

1st case: If $n_{3i} \leq n_0 + 6k < n_{3i+1}$, then for all $[j] \in \mathcal{G}_k$ we select two cylinders in $\widetilde{[j]}$. Then $\mathcal{G}_{k+1}$ is the union of all these selected cylinders.

2nd case: If $n_{3i+1} \leq n_0 + 6k < n_{3i+2}$, then for all $[j] \in \mathcal{G}_k$ of type $T_1$, we select a cylinder in $\widetilde{[j]}$ and for all $[j] \in \mathcal{G}_k$ of type $T_2$ we select a cylinder $[j']$, $j' \in \mathcal{A}^{n_0+6(k+1)}$ containing a cylinder selected in $\mathcal{G}_{n_{3i+2}}$ of type $T_1$. Then $\mathcal{G}_{k+1}$ is the union of all these selected cylinders. Note that all cylinders in $\mathcal{G}_{n_{3i+2}}$ are of type $T_1$.

3rd case: If $n_{3i+2} \leq n_0 + 6k < n_{3i+3}$, then for all $[j] \in \mathcal{G}_k$ having an
ancestor in $\mathcal{G}_{n_{3i+1}}$ of type $T_1$, we select a cylinder $[j']$, $j' \in A^{n_0+6(k+1)}$, containing a selected cylinder in $\mathcal{G}_{n_{3i+1}}$ of type $T_2$, and for all $[j] \in \mathcal{G}_k$ of type $T_1$ we select a cylinder in $[j]$. Then $\mathcal{G}_{k+1}$ is the union of all these selected cylinders.

For $n_0$ large enough, this construction is possible and we can impose the following separation condition:

For all $[j], [j'] \in \mathcal{G}_k$ such that $j, j' \in A^n$, for all $k \geq 0$, the distance between $[j]$ and $[j']$ is larger than $\frac{1}{2^{n-2}}$ and for all $k \geq 1$, the distance between $[j]$ and an element of his father is larger than $\frac{1}{2^{n-1}}$.

We choose $L = \bigcup_{k \geq 0} \mathcal{G}_k$ and we associate the following relation on $L$:

the two elements of $\mathcal{G}_0$ are related and two element of $\mathcal{G}_{k+1}$ are related if their fathers elements of $\mathcal{G}_k$, are related.

Now put

$$E = \{ q = (q_1, q_2) \in \mathbb{R}^2; \ q_1 + q_2 \geq 0 \}.$$ 

The function $\chi : \mathbb{X} \times [0,1] \to \mathbb{R}'$ is defined such that for all $q = (q_1, q_2) \in \mathbb{R}^2$ and for all $\lambda > 0$, there exists $r_0 > 0$ such that for $x \in \mathbb{X}$ and $r < r_0$

$$r^\lambda \mu(B(x, r))^{(q_1+q_2)} \leq e^{\langle q, \chi(x, r) \rangle} \leq r^{-\lambda} \mu(B(x, r))^{(q_1+q_2)}.$$ 

Let $a > 0$, for all $q = (q_1, q_2) \in \mathbb{R}^2$ we set

$$\langle q, \alpha \rangle = a(q_1 + q_2).$$

We denote

$$\overline{X}^a = \left\{ x \in \mathbb{X}, \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \leq a \right\}.$$ 

**Proposition 3.**

$$X_\chi(\alpha, E) = \overline{X}^a.$$ 

**Proof.** From the inequalities

$$r^\lambda \mu(B(x, r))^{(q_1+q_2)} \leq e^{\langle q, \chi(x, r) \rangle} \leq r^{-\lambda} \mu(B(x, r))^{(q_1+q_2)}$$

we deduce that

$$-\lambda + (q_1+q_2) \frac{\log(\mu(B(x, r)))}{\log r} \leq \langle q, \chi(x, r) \rangle \frac{1}{\log r} \leq \lambda + (q_1+q_2) \frac{\log(\mu(B(x, r)))}{\log r}.$$ 

It follows that

$$X_\chi(\alpha, E) = \left\{ x \in \mathbb{X}, \ (q_1 + q_2) \limsup_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r} \leq a(q_1 + q_2) \right\}. $$
Let us recall that for all \( q = (q_1, q_2) \in E; q_1 + q_2 \geq 0 \), then it is easy to obtain that
\[
X_\chi (\alpha, E) = \left\{ x \in \mathbb{X}, \limsup_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \leq a \right\}
\]
or
\[
X_\chi (\alpha, E) = \overline{X}^a.
\]
\[\square\]

We write for all real number \( \theta \),
\[
\Lambda_\mu(\theta) = \Delta^\theta(\text{supp } \mu) \quad \text{and} \quad B_\mu(\theta) = \text{Dim}^\theta(\text{supp } \mu),
\]

Let us recall that it is already established in \([7]\) that
\[
B_\mu \leq \Lambda_\mu
\]
\[
B_\mu(1) = \Lambda_\mu(1) = 0
\]
and that the functions \( \Lambda_\mu : \theta \mapsto \Lambda_\mu(\theta) \) and \( B_\mu : \theta \mapsto B_\mu(\theta) \) are convex and decreasing.

**Proposition 4.** For \( q = (q_1, q_2) \in \mathbb{R}^2 \),
\[
\Lambda_\chi(q) = \Lambda_\mu(q_1 + q_2)
\]
and
\[
B_\chi(q) = B_\mu(q_1 + q_2).
\]

**Proof.** It suffices to note that for \( t \in \mathbb{R}, q = (q_1, q_2) \in \mathbb{R}^2 \) and for \( \lambda > 0 \), there exists \( r_0 > 0 \) such that for \( r_i < r_0 \) and \( (B(x_i, r_i))_{i \in I} \) a centered \( \varepsilon \)-packing of \( \mathbb{X} \) with \( \varepsilon \leq r_0 \),
\[
\sum_i r_i^{t+\lambda} \mu(B(x_i, r_i))^{(q_1 + q_2)} \leq \sum_i r_i^{t} e(q_\chi(x_i, r_i)) \leq \sum_i r_i^{-\lambda} \mu(B(x_i, r_i))^{(q_1 + q_2)}.
\]

Then
\[
\mathcal{P}^{(q_1+q_2),t+\lambda}_{\mu,\varepsilon}(\mathbb{X}) \leq \mathcal{P}^{q_\chi,t}_{\mu,\varepsilon}(\mathbb{X}) \leq \mathcal{P}^{(q_1+q_2),t-\lambda}_{\mu,\varepsilon}(\mathbb{X}),
\]
letting \( \varepsilon \to 0 \), it comes that
\[
\mathcal{P}^{(q_1+q_2),t+\lambda}_{\mu}(\mathbb{X}) \leq \mathcal{P}^{q_\chi,t}(\mathbb{X}) \leq \mathcal{P}^{(q_1+q_2),t-\lambda}_{\mu}(\mathbb{X}),
\]
and letting \( \lambda \to 0 \), we obtain the equality
\[
\mathcal{P}^{q_\chi,t}(\mathbb{X}) = \mathcal{P}^{(q_1+q_2),t}_{\mu}(\mathbb{X}).
\]
Then it is clear that
\[
\Lambda_\chi(q) = \Lambda_\mu(q_1 + q_2)
\]
and
\[
B_\chi(q) = B_\mu(q_1 + q_2).
\]
\[\square\]
Proposition 5.
\[ \inf_{q \in E} (\langle q, \alpha \rangle + B_X(q)) = \inf_{\theta \geq 0} (a\theta + B_\mu(\theta)). \]

Proof. As \( B_X(q) = B_\mu(q_1 + q_2) \) it is clear that
\[ \inf_{q \in E} (\langle q, \alpha \rangle + B_X(q)) = \inf_{q_1 + q_2 \geq 0} (a(q_1 + q_2) + B_\mu(q_1 + q_2)) \]
or
\[ \inf_{q \in E} (\langle q, \alpha \rangle + B_X(q)) = \inf_{\theta \geq 0} (a\theta + B_\mu(\theta)). \]
\[ \square \]

We find in the following corollary a theorem obtained by L. Olsen in [7].

Corollary 1.
\[ \text{Dim}(\mathcal{X}^n) \leq \inf_{\theta \geq 0} (a\theta + B_\mu(\theta)). \]

Proof. By applying the theorem established by J. Peyrière we get that
\[ \text{Dim}(X_\chi(\alpha, E)) \leq \inf_{q \in E} (\langle q, \alpha \rangle + B_X(q)) \]
which gives using the propositions and that
\[ \text{Dim}(\mathcal{X}^n) \leq \inf_{\theta \geq 0} (a\theta + B_\mu(\theta)). \]
\[ \square \]

Proposition 6.
\[ \lim_{\theta \to +\infty} B_\mu(\theta) = -\infty. \]

Proof. We note that for all \([j]\) such that \(j \in \mathcal{A}^n\),
\[ p_0^n \leq \mu([j]) \leq p_1^n. \]
Let \((B(x_i, r_i))_{i \in I}\) be a centered \(\varepsilon\)-packing of \(X\).
For all \(i \in I\), we consider the cylinder \([j]_i = B(x_i, r_i)\) such that \(j \in \mathcal{A}^{n+1}\) and
\[ \frac{1}{2^{n+1}} \leq r_i < \frac{1}{2^n}. \]
Given (3.1), we get that
\[ p_0^n \leq \mu(B(x_i, r_i)) \leq p_1^n. \]
From (3.2), we deduce that for all \(t\), there exist \(c_1\) and \(c_2\) such that for all \(n \in \mathbb{N}\),
\[ \frac{c_1}{2^{nt}} \leq r_i^t \leq \frac{c_2}{2^{nt}}. \]
and from (3.3), it comes that for all $\theta > 0$,
\begin{equation}
 p_0^{n^\theta} \leq \mu(B(x_i, r_i))^\theta \leq p_1^{n^\theta}.
\end{equation}

Then given (3.4) and (3.5), there exists $c_3$ such that
\begin{equation}
 \mu(B(x_i, r_i))^\theta r_i^t \leq c_3 p_1^{n^\theta 2^{-nt}}.
\end{equation}

It follows thanks to (3.6), that there exists $C$ which depends only on $\theta$ and $t$ such that
\begin{equation}
 \sum_{\frac{1}{2} \leq r_i \leq \frac{1}{\log 2}} \mu(B(x_i, r_i))^\theta r_i^t \leq C(p_1^{\theta 2^{-t}})^n.
\end{equation}

Writing for $\varepsilon > 0$ small enough,
\begin{align*}
 \sum_{i \in I} \mu(B(x_i, r_i))^\theta r_i^t &= \sum_{n \geq 0} \sum_{\frac{1}{2} \leq r_i < \frac{1}{2^n}} \mu(B(x_i, r_i))^\theta r_i^t, \\
 \sum_{i \in I} \mu(B(x_i, r_i))^\theta r_i^t &< +\infty, \quad t > \frac{1}{\log p_1 / \log 2}.
\end{align*}

We deduce that
\begin{equation*}
 \Lambda_\mu(\theta) \leq \theta \frac{\log p_1}{\log 2}, \quad \theta > 0.
\end{equation*}

Then
\begin{equation*}
 B_\mu(\theta) \leq \theta \frac{\log p_1}{\log 2}, \quad \theta > 0.
\end{equation*}

Finally we obtain that
\begin{equation*}
 \lim_{\theta \to +\infty} B_\mu(\theta) = -\infty.
\end{equation*}

\begin{proposition}
 Put $B'_\mu(1)$ the left derivative number of $B_\mu$ at 1. Then \[ B'_\mu(1) \leq -1. \]
\end{proposition}

\begin{proof}
 As $B_\mu(1) = 0$ and $B_\mu$ is convex, it comes that to prove that $B'_\mu(1) \leq -1$, it is sufficient to establish that for all $\theta < 1$,
\begin{equation*}
 B_\mu(\theta) \geq 1 - \theta,
\end{equation*}

which amounts given (1.2), to show that if $\bigcup E_i$ is a partition of $X$, then $\sum_{i \in I} P_\mu^\theta(E_i) = +\infty$.
\end{proof}
Let us consider the case where for all $i \in I$, $\mathcal{P}_{\mu}^{\theta,t}(E_i) < +\infty$. Let $0 < \varepsilon < \frac{1}{2n_0}$. For all $i \in I$, we choose $\delta_i < \varepsilon$ such that

$$P_{\mu,\delta_i}^{\theta,t}(E_i) \leq P_{\mu}^{\theta,t}(E_i) + \frac{1}{2i}. \quad (3.8)$$

As the space $\mathbb{X}$ satisfies the Besicovitch covering property, there exists an integer $\zeta$ (which depends only on $\mathbb{X}$) such that each $E_i$ is covered by $\bigcup_{u=1}^{\zeta} \left( \bigcup_{j} B(x_{ij}, \delta_i) \right)$ such that for all $1 \leq u \leq \zeta$, $(B(x_{ij}, \delta_i))_j$ is a packing.

Given (3.8), it comes that

$$\sum_{u=1}^{\zeta} \sum_{j} \mu(B(x_{ij}, \delta_i))^\theta \delta_i^t \leq \zeta \left( P_{\mu}^{\theta,t}(E_i) + \frac{1}{2i} \right).$$

Then,

$$\sum_{i} \left( \sum_{u=1}^{\zeta} \sum_{j} \mu(B(x_{ij}, \delta_i))^\theta \delta_i^t \right) \leq \zeta \sum_{i} P_{\mu}^{\theta,t}(E_i) + \zeta. \quad (3.9)$$

let us consider the sum

$$\sum_{i} \left( \sum_{u=1}^{\zeta} \sum_{l} \mu(B(x_{il}, \delta_i))^\theta \delta_i^t \right), \quad (3.10)$$

where $\sum_{l}'$ is taken over all $l$ such that the distance between $x_{il}$ and $[j^1]$ (respectively $[j^2]$) is larger than $\frac{1}{2n_0-1}$.

In this case, there exists $c$ which depends only on $n_0$ such that

$$\mu(B(x_{il}, \delta_i)) \leq c m(B(x_{il}, \delta_i)), \quad \text{where } m \text{ is the Lebesgue measure.}$$

Then there exists $C$ such that

$$C^{\theta-1} \delta_i^{\theta-1+t} \leq \mu(B(x_{il}, \delta_i))^{\theta-1} \delta_i^t. \quad (3.11)$$

Moreover, the union of the balls contained in the sum (3.10) covers $\mathbb{X}$ deprived of at most 6 cylinders of the generation $n_0$, therefore given (3.11), we obtain that

$$\left(1 - \frac{6}{2^{n_0-1}}\right) C^{\theta-1} \varepsilon^{\theta-1+t} \leq \sum_{i} \left( \sum_{u=1}^{\zeta} \sum_{l} \mu(B(x_{il}, \delta_i))^\theta \delta_i^t \right).$$
We deduce according to (3.9),
\[
\left(1 - \frac{6}{2^n}\right) C^{\theta-1} \varepsilon^{\theta-1+t} \leq \zeta \sum_{i \in I} \mathcal{T}_{\mu}^{\theta,t}(E_i) + \zeta.
\]
Letting $\varepsilon \to 0$, it results that $\sum_{i \in I} \mathcal{T}_{\mu}^{\theta,t}(E_i) = +\infty$ while $t < 1 - \theta$, so that
\[
B_{\mu}(\theta) \geq 1 - \theta, \quad \theta < 1.
\]

**Proposition 8.** We set
\[
\mathcal{C} = \bigcap_{k \geq 1} \left( \bigcup_{[j] \in G_k} [j] \right)
\]
and the function $g$ defined on $[0, 1]$ by
\[
g(x) = -\frac{x \log \left( \frac{p_0}{p_1} \right) + \log p_1}{\log 2}.
\]
i. If $x \notin \mathcal{C}$, then
\[
\lim_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} = 1.
\]
ii. If $x \in \mathcal{C}$, then
\[
g(\beta_1) \leq \liminf_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \leq \limsup_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} \leq g(\gamma_2).
\]

**Proof.** i. Let $x \notin \mathcal{C}$. Thanks to the separation condition, for $r > 0$ small enough, the ball $B(x,r) = [j]$ such that $j \in \mathcal{A}^{n+1}$, $\frac{1}{2^{n+1}} \leq r < \frac{1}{2^n}$ and $[j]$ do not meet $\mathcal{C}$. There exists $c$ such that
\[
\mu([j]) = \frac{c}{2^n}.
\]
We deduce that
\[
\lim_{n \to +\infty} \frac{\log(\mu([j]))}{\log \left( \frac{1}{2^{n+1}} \right)} = 1
\]
i.e.
\[
\lim_{r \to 0} \frac{\log(\mu(B(x,r)))}{\log r} = 1.
\]
ii. It is clear that if $[j] \in G_k$ such that $j \in \mathcal{A}^n$, then
\[
\mu([j]) = p_0^{N_0(j)} p_1^{n-N_0(j)},
\]
so that
\[ \mu([j]) = \left( \frac{1}{2^n} \right)^{g\left( \frac{N_0(j)}{n} \right)}. \]

Furthermore, we recall that,
\[ \beta_1 < \frac{N_0(j)}{n} < \gamma_2. \]

The function \( g \) is strictly increasing, it comes that
\[ g(\beta_1) < \frac{\log(\mu([j]))}{\log\left( \frac{1}{2^n} \right)} < g(\gamma_2). \]

Let \( x \in \mathcal{C} \) and \( r < \frac{1}{2n+6} \), then \( B(x, r) \) is contained in one of the selected cylinders \([j^1]\) or \([j^2]\).

We consider the selected cylinder \([j]\) such that \( j \in \mathcal{A}^{n+1}, \frac{1}{2n+1} \leq r < \frac{1}{2^n} \) and \([j] = B(x, r)\), we can write
\[ \mu(B(x, r)) = \mu([j]). \]

Given (3.12), it results that
\[ g(\beta_1) \leq \lim_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r} \leq \limsup_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r} \leq g(\gamma_2). \]

\[ \square \]

Subsequently, even if we choose \( p_0 > \gamma_2 \), we stand in the case where \( g(\gamma_2) < 1 \).

Thus under the proposition \( \mathfrak{N} \)
\[ \overline{X}^{g(\gamma_2)} = \mathcal{C}. \]

Let \( a > 0 \) such that \( g(\gamma_1) < a \leq g(\gamma_2) \) and \( \overline{X}^a \neq \emptyset \).

**Proposition 9.**
\[ \inf_{q \in \mathcal{E}} \left\{ \Phi_q T_q^\alpha(\alpha) \right\} < \inf_{q \in \mathcal{E}} (q, \alpha) + B_\alpha(q). \]

**Proof.** Put \( s = \inf_{\theta} B_\mu(\theta) < 0 \). It is clear that
\[ \inf_{q \in \mathcal{E}} \left\{ \Phi_q T_q^\alpha(\alpha) \right\} \leq \inf_{q \in \mathcal{E}} \left\{ T_q^\alpha(\alpha) \inf_{t \leq 0} (\Phi_q(t)) \right\}. \]

We put
\[ F = \{ q = (q_1, q_2) \in \mathcal{E}; \ q_1 + q_2 = 1 \}. \]
It follows that for all \( q \in F \), \( T^q_\chi(\alpha) \) keeps a constant value which we denote by \( T^F_\chi(\alpha) \), we can write
\[
\inf_{q \in E} \{ \Phi q T^q_\chi(\alpha) \} \leq T^F_\chi(\alpha) \inf_{s < t < 0} (\Phi_q(t)).
\]
Also the equality
\[
\Phi_q(t) = \inf \{ \gamma > 0; \; ta(q_1 + q_2) > B_\mu((\gamma - t)(q_1 + q_2)) \}
\]
gives
\[
\inf_{q \in E} \{ \Phi q T^q_\chi(\alpha) \} \leq T^F_\chi(\alpha) \inf_{s < t < 0} \{ \inf \{ \gamma > 0; \; ta > B_\mu((\gamma - t)) \} \}.
\]
According to proposition 6, we verify that
\[
\inf_{s < t < 0} \{ \inf \{ \gamma > 0; \; ta > B_\mu((\gamma - t)) \} \} = 1
\]
\[
a \inf_{\theta \geq 1} \{ \alpha \theta + B_\mu(\theta) \}.
\]
It comes that
\[
\inf_{q \in E} \{ \Phi q T^q_\chi(\alpha) \} \leq \frac{T^F_\chi(\alpha)}{a} \inf_{\theta \geq 1} \{ \alpha \theta + B_\mu(\theta) \}.
\]
On the other hand, as \( a < 1 \) and by proposition 7, it follows that
\[
B'(1) - (1) \leq -a,
\]
therefore
\[
\inf_{\theta \geq 1} \{ a \theta + B_\mu(\theta) \} = \inf_{\theta \geq 0} \{ a \theta + B_\mu(\theta) \}.
\]
Then we deduce according to proposition 5
\[
\inf_{q \in E} \{ \Phi q T^q_\chi(\alpha) \} \leq \frac{T^F_\chi(\alpha)}{a} \inf_{q \in E} \{ \langle q, \alpha \rangle + B_\chi(q) \}.
\]
Remains to show that
\[
\frac{T^F_\chi(\alpha)}{a} < 1.
\]
Let \( A \subset X(q,\alpha)(\eta,p) \) and \((B(x_i, r_i))\) a centered \( \varepsilon \)-packing of \( A \). It is clear that for all \( i \in I \), \( x_i \in C \). Then we consider the selected cylinder \([j]_i = B(x_i, r_i)\) such that \( j \in A^{n+1} \) and \( \frac{1}{2^{n+1}} \leq r_i < \frac{1}{2^n} \).
We consider the partition \( I_1 \cup I_2 \) of \( I \) such that
\[
I_1 = \{ i \in I : [j]_i \text{ is of type } T_1 \} \quad \text{and} \quad I_2 = I \setminus I_1.
\]
We recall that each cylinder \([j]_i, i \in I_2 \) is related to a single cylinder type \( T_1 \) centered \( x'_i \in A \), denoted \([j']_i\). With the condition of separation, \( \left( B \left( x'_i, \frac{1}{2^{n+1}} \right) \right)_{i \in I_2} \) is a centered \( \varepsilon \)-packing of \( A \). Then we
consider the family \((B(y_i, \delta_i))_{i \in I}\) defined by

\[
B(y_i, \delta_i) = \begin{cases} 
B(x_i, r_i), & i \in I_1 \\
B(x'_i, \frac{1}{2^{n+1}}), & i \in I_2.
\end{cases}
\]

We verify that

\[
\frac{\log \mu(B(y_i, \delta_i))}{\log \delta_i} \leq \frac{\log \mu([j])}{\log \left(\frac{1}{2^{n+1}}\right)}, \quad i \in I_1
\]

and

\[
\frac{\log \mu(B(y_i, \delta_i))}{\log \delta_i} \leq \frac{\log \mu([j'])}{\log \left(\frac{1}{2^{n+1}}\right)}, \quad i \in I_2.
\]

Given (??) and the fact that the function \(g\) is increasing, we deduce that for all \(i \in I\),

\[
\frac{\log \mu(B(y_i, \delta_i))}{\log \delta_i} \leq g(\gamma_1).
\]

Yet for \(q \in F\) and \(\lambda > 0\), there exists \(r_0 > 0\) such that \(\varepsilon \leq r_0\) and \(\delta_i < r_0\)

\[
\langle q, \chi(y_i, \delta_i) \rangle \leq \lambda + \frac{\log \mu(B(y_i, \delta_i))}{\log \delta_i},
\]

then

\[
L^{q,2}_{\varepsilon, (B(x_i, r_i))_{i \in I}}(A) \leq \lambda + g(\gamma_1).
\]

It follows that \(L^{q,2}(A) \leq \lambda + g(\gamma_1)\), letting \(\varepsilon \to 0\) and \(\lambda \to 0\), we deduce that

\[
L^{q,2}(A) \leq g(\gamma_1).
\]

The sequence \((L^{q,k}(A))_k\) is decreasing, it results that

\[
L^{q}(A) \leq g(\gamma_1),
\]

then,

\[
T^q_{\chi}(\alpha) \leq g(\gamma_1),
\]

as \(g(\gamma_1) < a\) and \(T^q_{\chi}(\alpha) = T^F_{\chi}(\alpha)\), it follows that

\[
T^F_{\chi}(\alpha) < a.
\]

Finally we obtain

\[
\inf_{q \in E} \{ \Phi_q T^q_{\chi}(\alpha) \} < \inf_{q \in E} (\langle q, \alpha \rangle + B_\chi(q)).
\]
Corollary 2.

\[ \text{Dim}(X_\chi(\alpha, E)) \leq \inf_{q \in E} \left\{ \Phi_q T^q_\chi(\alpha) \right\} < \inf_{q \in E} \left( q, \alpha + B_\chi(q) \right). \]

Proof. Follows from theorem 3 and proposition 9. □

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