On the exact linearisation and control of flat discrete-time systems

Bernd Kolar, Johannes Diwold, Conrad Gstöttner and Markus Schöberl

Magna Powertrain Engineering Center Steyr GmbH & Co KG, St. Valentin, Austria; Institute of Automatic Control and Control Systems Technology, Johannes Kepler University, Linz, Austria

ABSTRACT
The paper addresses the exact linearisation of flat nonlinear discrete-time systems by generalised static or dynamic feedbacks which may also depend on forward-shifts of the new input. We first investigate the question which forward-shifts of a given flat output can be chosen in principle as a new input, and subsequently how to actually introduce the new input by a suitable feedback. With respect to the choice of a feasible input, easily verifiable conditions are derived. Introducing such a new input requires a feedback which may in general depend not only on this new input itself but also on its forward-shifts. This is similar to the continuous-time case, where feedbacks which depend on time derivatives of the closed-loop input – and in particular quasi-static ones – have already been used successfully for the exact linearisation of flat systems since the nineties of the last century. For systems with a flat output that does not depend on forward-shifts of the input, it is shown how to systematically construct a new input such that the total number of the corresponding forward-shifts of the flat output is minimal. Furthermore, it is shown that in this case the calculation of a linearising feedback is particularly simple, and the subsequent design of a discrete-time flatness-based tracking control is discussed. The presented theory is illustrated by the discretised models of a wheeled mobile robot and a 3DOF helicopter.

1. Introduction

The concept of flatness has been introduced in the 1990s by Fliess, Lévine, Martin and Rouchon for nonlinear continuous-time systems (see e.g. Fliess et al. (1995, 1999)). Roughly speaking, a continuous-time system is flat if all system variables can be parameterised by a flat output and its time derivatives, which in turn depends on the system variables and their time derivatives. In other words, there exists a one-to-one correspondence between the trajectories of a flat system and the trajectories of a trivial system. Since these properties allow an elegant solution to motion planning problems and a systematic design of tracking controllers, flatness belongs doubtlessly to the most popular nonlinear control concepts.

For the practical implementation of a flatness-based control, it is important to evaluate the continuous-time control law at a sufficiently high sampling rate. If this is not possible, it can be advantageous to design the controller directly for a suitable discretisation of the continuous-time system, see e.g. Diwold et al. (2022a). Thus, transferring the flatness concept to discrete-time systems is both interesting from a theoretical and an application point of view. In fact, there are two equally obvious possibilities: The first one is to replace the time derivatives of the continuous-time definition by forward-shifts. This point of view has been adopted e.g. in Kaldmäe and Kotta (2013), Sira-Ramirez and Agrawal (2004), or Kolar et al. (2016), and is consistent with the linearity of a discrete-time endogenous dynamic feedback as it is defined in Aranda-Bricaire and Moog (2008). The second approach is based on the one-to-one correspondence of the system trajectories to the trajectories of a trivial system. In contrast to the first approach, in this case the flat output may also depend on backward-shifts of the input – and in particular quasi-static ones – have already been used successfully for the exact linearisation of flat systems since the nineties of the last century. For systems with a flat output that does not depend on forward-shifts of the input, it is shown how to systematically construct a new input such that the total number of the corresponding forward-shifts of the flat output is minimal. Furthermore, it is shown that in this case the calculation of a linearising feedback is particularly simple, and the subsequent design of a discrete-time flatness-based tracking control is discussed. The presented theory is illustrated by the discretised models of a wheeled mobile robot and a 3DOF helicopter.

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demonstrated in Diwold et al. (2022a) for the practical example of a gantry crane that also an exact linearisation which introduces lower-order forward-shifts of the flat output as a new input is in principle applicable. This allows to achieve a lower-order tracking error dynamics, but requires a feedback which depends also on forward-shifts of the new input. Since in the particular case of the gantry crane the required feedback does not have a dynamic part, it is a discrete-time quasi-static feedback – see e.g. Aranda-Bricaire and Kotta (2001), where such a type of feedback is used in a different context. However, according to the authors’ best knowledge, a thorough theoretical analysis of the problem of exact linearisation and tracking control design for flat discrete-time systems does not yet exist in the literature. Thus, the aim of the present paper is to close this gap: We investigate which (lower-order) forward-shifts of a flat output can be used in principle as a new input, and how to actually introduce such a new input by a suitable feedback. After discussing the general case, we consider systems with a flat output that does not depend on future values of the control input. We show how to systematically construct a new input such that the sum of the orders of the corresponding forward-shifts of the components of the flat output is minimal, and that deriving a linearising feedback is particularly simple in this case. Furthermore, we show how such an exact linearisation can be used as a basis for the systematic design of a flatness-based tracking control, and demonstrate our results by two examples.

The employed mathematical methods are similar as in Gstöttner et al. (2021a), where we have proven that every flat continuous-time system with a flat output that is independent of time derivatives of the input can be exactly linearised by a quasi-static feedback of its classical state. Hence, it is particularly important to emphasise that like in Gstöttner et al. (2021a) we restrict ourselves to feedbacks of the classical state system, and do not consider feedbacks of generalised states as e.g. in Delaleau and Rudolph (1998) or Rudolph (2021). However, it should be noted that the considered feedbacks are more general than a usual static or dynamic feedback in the sense that they may also depend on forward-shifts of the closed-loop input.

The paper is organised as follows: In Sections 2 and 3 we introduce some notation and recapitulate the concept of flatness for discrete-time systems. In Section 4 we investigate the exact linearisation of flat discrete-time systems in general, and derive further results for systems with a flat output that does not depend on future values of the control input. The design of a flatness-based tracking control is discussed in Section 5, and in Section 6 the developed theory is finally applied to the discretised models of a wheeled mobile robot and a 3DOF helicopter.

2. Notation

Throughout the paper we make use of some basic differential-geometric concepts. Let \( \mathcal{X} \) be an \( n \)-dimensional smooth manifold equipped with local coordinates \( x^i, i = 1, \ldots, n \), and \( h: \mathcal{X} \rightarrow \mathbb{R}^m \) some smooth function. Then we denote by \( \partial_{x^i} h \) the \( m \times n \) Jacobian matrix of \( h = (h^1, \ldots, h^m) \) with respect to \( x = (x^1, \ldots, x^n) \). The partial derivative of a single component \( h^i \) with respect to a coordinate \( x^j \) is denoted by \( \partial_{x^j} h^i \). Furthermore, \( dh^i = \partial_{x^j} h^i dx^j + \cdots + \partial_{x^n} h^i dx^n \) denotes the differential (exterior derivative) of the function \( h^i \), where \( dx^i, i = 1, \ldots, n \) are the differentials corresponding to the local coordinates. We frequently use \( dh \) as an abbreviation for the set \( \{dh^1, \ldots, dh^m\} \), and with e.g. \( \text{span}(dh^1, \ldots, dh^m) \) we mean the span over the ring \( \mathbb{C}^\infty(\mathcal{X}) \) of smooth functions. The symbols \( \subset \) and \( \supset \) are used in the sense that they also include equality.

To denote forward- and backward-shifts of the system variables, we use subscripts in brackets. For instance, the \( \alpha \) forward- or backward-shift of a component \( y^i \) of a flat output \( y \) with \( \alpha \in \mathbb{Z} \) is denoted by \( y^i_{[\alpha]} \). We define \( y^i_{[\alpha]} = (y^i_{[\alpha]1}, \ldots, y^i_{[\alpha]m}) \). To keep expressions which depend on different numbers of shifts of different components of a flat output readable, we use multi-indices. If \( A = (a^1, \ldots, a^m) \) and \( B = (b^1, \ldots, b^m) \) are two multi-indices with \( A \leq B \), i.e. \( a^j \leq b^j \) for \( j = 1, \ldots, m \), then

\[
y^i_{[A]} = (y^i_{[a^1]}, \ldots, y^i_{[a^m]})
\]

and

\[
y^i_{[A,B]} = (y^i_{[a^1,b^1]}, \ldots, y^i_{[a^m,b^m]})
\]

with \( y^i_{[a^1,b^1]} = (y^i_{[a^1]}, \ldots, y^i_{[b^1]}) \). In the case \( a^j > b^j \) we define \( y^i_{[a^1,b^1]} \) as empty. The addition and subtraction of multi-indices is performed componentwise, and for an integer \( c \) we define \( A \pm c = (a^1 \pm c, \ldots, a^m \pm c) \). Furthermore, \( \#A = \sum_{j=1}^m a^j \) denotes the sum over all components of a multi-index. As an example consider the tuple \( y = (y^1, y^2) \), an integer \( c = 2 \), and multi-indices \( A = (0,2), B = (1,2) \). Then we have \( y_{[c]} = (y^i_{[2,1]}, y^i_{[2,2]}) \), \( y_{[A]} = (y^1, y^2) \), \( y_{[A,B]} = (y^1, y^1_{[1,1]}) \), and \( y_{[A+c]} = (y^i_{[2,1]}, y^i_{[2,2]}) \) as well as \( \#A = 2 \) and \( \#B = 3 \).

Frequently, it is also convenient to decompose the components of a flat output \( y \) or the input \( u \) into several blocks like e.g.

\[
y = (y^1, \ldots, y^m, y^{m+1}, \ldots, y^{m+3}).
\]

Since such blocks are also denoted by a subscript, in this case the first subscript always refers to the block, and shifts are denoted by a second subscript in brackets. For instance,

\[
y_{[\alpha]} = (y^i_{[\alpha]1}, \ldots, y^i_{[\alpha]m}, y^{m+1}_{[\alpha]1}, \ldots, y^{m+3}_{[\alpha]1})
\]

with some integer \( \alpha \in \mathbb{Z} \), or \( y_{[1,A_1]} = (y^i_{[1,\alpha_1]1}, \ldots, y^{m+3}_{[1,\alpha_1]1}) \) with some multi-index \( A_1 = (a^1_1, \ldots, a^m_1) \).

3. Discrete-time systems and flatness

In this contribution, we consider time-invariant discrete-time nonlinear systems

\[
x^{i,+} = f^i(x, u), \quad i = 1, \ldots, n
\]

with \( \dim(x) = n, \dim(u) = m \) and smooth functions \( f^i(x, u) \). In addition, we assume that system (1) meets the submersivity condition

\[
\text{rank}(\partial_{(x,u)} f) = n,
\]

which is common in the discrete-time literature and necessary for accessibility.
Like in Diwold et al. (2022b), we call a discrete-time system (1) flat if there exists a one-to-one correspondence between its solutions \((x(k), u(k))\) and solutions \(y(k)\) of a trivial system (arbitrary trajectories that need not satisfy any difference equation) with the same number of inputs. Before we state a more rigorous definition of discrete-time flatness, let us consider the coupling of the trajectories \(x(k)\) and \((k)\) by the system equations (1). By a repeated application of (1), all forward-shifts \(x(k + \alpha), \alpha \geq 1\) of the state variables are obviously determined by \(x(k)\) and the input trajectory \(u(k + \alpha)\) for \(\alpha \geq 0\). A similar argument holds for the backward-direction: Because of the submersivity condition (2), there always exist \(m\) functions \(g(x, u)\) such that the \((n+m)\times(n+m)\) Jacobian matrix

\[
\begin{bmatrix}
\frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial x} & \frac{\partial y}{\partial y}
\end{bmatrix}
\]

is regular and the map

\[
x^+ = f(x, u) \\
\zeta = g(x, u)
\]

hence locally invertible. By a repeated application of its inverse

\[
(x, u) = \psi(x^+, \zeta),
\]

all backward-shifts \(x(k - \beta), u(k - \beta)\) of the state- and input variables for \(\beta \geq 1\) are determined by \(x(k)\) and backward-shifts \(\zeta(k - \beta)\), \(\beta \geq 1\) of the system variables \(\zeta\) defined by (3). Hence, if only a finite time-interval is considered, the system trajectories \((x(k), u(k))\) can be identified with points of a manifold \(\zeta_{[-l, \cdots, 1]} \times X \times U[0,1]\) with coordinates \((\zeta_{[-l, \cdots, 1]}, x, u, u[1], \cdots, u[1])\) and suitably chosen integers \(l, l\). If \(h \in C^\infty(\zeta_{[-l, \cdots, 1]} \times X \times U[0,1])\) is a function defined on this manifold, then its future values can be determined by a repeated application \(\delta^\beta\) of the forward-shift operator

\[
\delta(h(\cdots, \zeta_{[-l, \cdots, 1]}, x, u, u[1], \cdots)) = h(\cdots, \zeta_{[-l, \cdots, 1]}, g(x, u), f(x, u), u[1], u[2], \cdots).
\]

Likewise, its past values can be determined by a repeated application \(\delta^{-\beta}\) of the backward-shift operator

\[
\delta^{-1}(h(\cdots, \zeta_{[-l, \cdots, 1]}, x, u, u[1], u[2], \cdots)) = h(\cdots, \zeta_{[-l, \cdots, 1]}, g(x, u), f(x, u), u[1], u[2], \cdots)
\]

where \(\psi_x\) and \(\psi_u\) are the corresponding components of (4). Since we work in a finite-dimensional framework, it is important to emphasise that \(\delta\) only yields the correct forward-shift if the integer \(l\) is chosen large enough such that the considered function \(h\) does not already depend on \(u[1]\). Likewise, \(\delta^{-1}\) only yields the correct backward-shift if \(l\) is chosen large enough such that the function \(h\) does not already depend on \(\zeta_{[-l]}\). Thus, throughout this contribution we assume that \(l\) and \(l\) are chosen large enough such that (5) and (6) act as correct forward- and backward-shifts on all considered functions.

Like the discrete-time static feedback linearisation problem, discrete-time flatness is considered in a suitable neighbourhood of an equilibrium \((x_0, u_0)\) of the system (1). On the manifold \(\zeta_{[-l, \cdots, 1]} \times X \times U[0,1]\), an equilibrium corresponds to a point with coordinates \((\zeta_0, \cdots, \zeta_0, x_0, u_0, \cdots, u_0)\) with \(\zeta_0 = g(x_0, u_0)\) according to (3). Hence, evaluated at an equilibrium point (or, in other words, at an equilibrium trajectory), the functions \(\delta(h)\) and \(\delta^{-1}(h)\) have the same value as the function \(h\) itself.

**Definition 3.1:** (Diwold et al. (2022b)) The system (1) is said to be flat around an equilibrium \((x_0, u_0)\), if the \(n + m\) coordinate functions \(x\) and \(u\) can be expressed locally by an \(m\)-tuple of functions

\[
y^j = \psi^j(\zeta_{[-l_1]}, \cdots, \zeta_{[-l_1]}, x, u, \cdots, u[1]), \quad j = 1, \cdots, m
\]

and their forward-shifts

\[
y^1 = \delta(\psi(\zeta_{[-l_1]}, \cdots, \zeta_{[-l_1]}, x, u, \cdots, u[1])) \\
y^2 = \delta^2(\psi(\zeta_{[-l_1]}, \cdots, \zeta_{[-l_1]}, x, u, \cdots, u[1])) \\
\vdots
\]

up to some finite order. The \(m\)-tuple (7) is called a flat output.

The representation of \(x\) and \(u\) by a flat output (7) is unique and has the form

\[
x^i = F^i(y, \cdots, y[\infty - 1]), \quad i = 1, \cdots, n
\]

\[
u^j = F^m(y, \cdots, y[\infty]), \quad j = 1, \cdots, m.
\]

The multi-index \(R = (r^1, \cdots, r^n)\) consists of the number of forward-shifts of each component of the flat output (7) that are needed to express \(x\) and \(u\). After substituting (8) into (9), the equations are satisfied identically. Because of Lemma A.1 (see the Appendix), this is equivalent to the condition

\[
\text{dx} \in \text{span} \{\text{d} \psi^1, \cdots, \text{d} \delta^R(\psi)\}
\]

\[
\text{du} \in \text{span} \{\text{d} \psi^1, \cdots, \text{d} \delta^R(\psi)\}
\]

which we shall use later. The uniqueness of the map (9) is a consequence of the fact that all forward- and backward-shifts of a flat output are functionally independent, see Diwold et al. (2022b). Furthermore, the rows of the Jacobian matrix of the right-hand side of (9) with respect to \(y[0,\infty]\) are linearly independent, i.e. the map (9) is a submersion. With a restriction to flat outputs that are independent of backward-shifts of the system variables, Definition 3.1 leads to the concept of forward-flatness considered e.g. in Sira-Ramirez and Agrawal (2004), Kaldmæ and Kotta (2013), or Kolar et al. (2016).

**Definition 3.2:** (Diwold et al. (2022b)) The system (1) is said to be forward-flat, if it meets the conditions of Definition 3.1 with a flat output of the form \(y^j = \psi^j(x, u, \cdots, u[1])\).

For continuous-time systems, the computation of flat outputs is known to be a challenging problem. Recent research in this field can be found e.g. in Nicolau and Respondek (2017), Nicolau and Respondek (2019), or Göttnet et al. (2021b). For discrete-time systems, in contrast, we have shown in Kolar et al. (2021) that every forward-flat system can be decomposed by coordinate transformations into a smaller-dimensional
forward-flat subsystem and an endogenous dynamic feedback. Because of this property, it is possible to check the forward-flatness of a system (1) similar to the well-known static feedback linearisation test by computing a certain sequence of distributions, see Kolar et al. (2022). However, even though ideas for an extension of this approach to the general case of Definition 3.1 can be found in Kaldmæ (2021), a computationally efficient test does not yet exist. Hence, within the present paper, we assume that a flat output is given and do not address its computation.

4. Exact linearisation

In Diwold et al. (2022b), it has been shown that every flat discrete-time system (1) can be exactly linearised by a dynamic feedback which leads to an input-output behaviour of the form

\[ y'_j = \nu, \quad j = 1, \ldots, m \]

the order of the tracking error dynamics is given by \#A = \sum_{j=1}^{m} a^j instead of \#R = \sum_{j=1}^{m} r^j.

**Example 4.1:** Consider the system

\[
\begin{align*}
x_1' &= x_1 + u^1 \\
x_2' &= x_2 + \frac{x_3}{u_1 + 1} \\
x_3' &= u^2
\end{align*}
\]

with the flat output

\[
\begin{align*}
y_1 &= \varphi^1(x) = x_1 \\
y_2 &= \varphi^2(x) = x_2.
\end{align*}
\]

The corresponding parameterisation (9) of the system variables by the flat output is given by

\[
\begin{align*}
x_1 &= y^1 \\
x_2 &= y^2 \\
x_3 &= y_1^1 \left(1 - y^1 + y_1^1\right) \\
u^1 &= y_1^1 - y^1 \\
u^2 &= y_2^1 \left(1 - y_1^1 + y_2^1\right)
\end{align*}
\]

i.e. there occur forward-shifts of the flat output up to the order \(R = (2, 2)\). Thus, as shown in Diwold et al. (2022b), by applying a dynamic feedback it is definitely possible to introduce a new input \(v = (\delta^2(\varphi^1), \delta^2(\varphi^2))\) and hence achieve an input-output behaviour

\[ y_2^1 = v^1 \]

However, it is actually also possible to introduce lower-order forward-shifts of the flat output (11) as a new input: Let us define

\[
\begin{align*}
v^1 &= \delta(\varphi^1) = x^1 + u^1 \\
v^2 &= \delta^2(\varphi^2) = \frac{u^2}{u_1^1 + 1}
\end{align*}
\]

(i.e. \(A = (1, 2)\)) and complement these equations by the forward-shift

\[ v_1^1 = \delta^1(\varphi^1) = x^1 + u^1 + u_1^1. \]

If we solve the resulting set of equations for \(u^1\) and \(u^2\) as well as \(u_1^1\), the original inputs are given by

\[
\begin{align*}
u^1 &= v^1 - x^1 \\
u^2 &= \left(1 - v^1 + v_1^1\right) v^2.
\end{align*}
\]

With the feedback (13), it is possible to introduce the input (12) and achieve an input-output behaviour

\[
\begin{align*}
y_1^1 &= v^1 \\
y_2^2 &= v^2.
\end{align*}
\]

However, as already mentioned in the introduction, this requires a feedback which also depends on forward-shifts of the new input.

In this introductory example, we have chosen the new input \(v = (\delta(\varphi^1), \delta^2(\varphi^2))\) without a prior theoretical justification. The criterion for the feasibility of an \(m\)-tuple of forward-shifts \(\delta^A(\varphi)\) of a flat output (7) as a new input \(v\) is the possibility to realise arbitrary trajectories \(v(k + \alpha), \alpha \geq 0\) independently of the previous trajectory of the system. More precisely, like for the original input \(u\), at every time step \(k\) the system (1) must permit arbitrary trajectories \(v(k + \alpha), \alpha \geq 0\) independently of its current state \(x(k)\) and past values \(\zeta(k - \beta), \beta \geq 1\). That is, for every possible state \(x(k)\) and past values \(\zeta(k - \beta), \beta \geq 1\) there must exist a trajectory \(u(k + \alpha), \alpha \geq 0\) of the original control input such that the desired trajectory \(v(k + \alpha), \alpha \geq 0\) can be realised. The practical importance of this criterion is obvious, since otherwise there would be no guarantee that a trajectory \(v(k + \alpha), \alpha \geq 0\) requested e.g. by a controller for the exactly linearised system can actually be achieved. Similar considerations for flat continuous-time systems can be found in Gstöttner et al. (2021a), where it has to be ensured that arbitrary trajectories \(v(t)\) can be realised.

**Theorem 4.2:** Let \(A = (a^1, \ldots, a^m)\) denote a multi-index with \(a^j \geq 0, j = 1, \ldots, m\). The system (1) permits arbitrary trajectories \(v(k + \alpha), \alpha \geq 0\) for the forward-shifts \(v = \delta^A(\varphi)\) of a flat output (7) regardless of its current state \(x(k)\) and past values \(\zeta(k - \beta), \beta \geq 1\) if and only if the differentials

\[
d\zeta[-q_1, +A], \ldots, d\zeta[-1], dx, \delta^A(\varphi), \ldots, \delta^{R-1}(\varphi)
\]

are linearly independent.
**Proof:** The system permits arbitrary trajectories $v(k + \alpha)$, $\alpha \geq 0$ if and only if there does not exist any non-trivial relation of the form

$$\psi(\ldots, \zeta(k - 2), \zeta(k - 1), x(k), v(k), v(k + 1), \ldots) = 0. \tag{15}$$

Otherwise, (15) could be solved by the implicit function theorem for at least one component of some $v(k + \alpha)$, $\alpha \geq 0$, which would thus be uniquely determined by the other quantities appearing in (15). In our differential-geometric framework with the manifold $\xi_{[-k,-1]} \times \mathcal{X} \times U_{[0,k]}$, this corresponds to the non-existence of any non-trivial relation

$$\psi(\ldots, \zeta[-2], \zeta[-1], x, \delta^A(\psi), \delta^{A+1}(\psi), \ldots) = 0.$$  

Because of Lemma A.1, this condition is equivalent to the linear independence of the differentials

$$\ldots, d\zeta[-2], d\zeta[-1], dx, d\delta^A(\psi), d\delta^{A+1}(\psi), \ldots \tag{16}$$

However, it is not necessary to check the linear independence of all these differentials. Since the flat output (7) is independent of the variables $\ldots, \zeta_{[-q-1],-2}], \zeta_{[-q-1]},$ its forward-shift $\delta^A(\psi)$ is independent of $\ldots, \zeta_{[-q+A-1],-2}], \zeta_{[-q+A-1]}$. Thus, we do not need to consider the corresponding differentials. Furthermore, since the differentials of a flat output and all its forward- and backward-shifts are linearly independent and the fact that

$$dx \in \text{span}\{d\psi, \ldots, d\delta^{R-1}(\psi)\}$$

as well as

$$d\zeta_{[-1]} \in \text{span}\{d\delta^{-1}(\psi), d\psi, \ldots, d\delta^{R-1}(\psi)\}$$

$$d\zeta_{[-2]} \in \text{span}\{d\delta^{-2}(\psi), d\delta^{-1}(\psi), \ldots, d\delta^{R-2}(\psi)\},$$

there is also no need to consider the differentials $d\delta^R(\psi), d\delta^{R+1}(\psi), \ldots$. Consequently, the linear independence of the differentials (14) implies the linear independence of the differentials (16), which completes the proof. \hfill \blacksquare

An immediate consequence of Theorem 4.2 is that the choice of an input $v = \delta^A(\psi)$ with $A \geq R$ is always possible.

**Example 4.3:** Consider again the system (10) with the flat output (11) of Example 4.1. For the chosen new input (12) with $A = (1, 2)$ and $R = (2, 2)$, the differentials (14) of Theorem 4.2 are given by

$$dx^1$$

$$dx^2$$

$$dx^3$$

$$d\delta(\psi^1) = dx^1 + du^1$$

and obviously linearly independent. Thus, the system permits indeed arbitrary trajectories $v(k + \alpha)$, $\alpha \geq 0$ for the new input (12) independently of its current state $x(k)$ and past values of the system trajectory. The latter play no role in this case, since the flat output (11) is a forward-flat output. If, however, we would try to use the flat output (11) itself as a new input

$$v^1 = \varphi^1 = x^1$$

$$v^2 = \varphi^2 = x^2,$$

then it is obvious that the possible trajectories $v(k + \alpha), \alpha \geq 0$ are restricted by the current state $x(k)$. Accordingly, it can be observed that the differentials

$$dx^1$$

$$dx^2$$

$$dx^3$$

$$d\varphi^1 = dx^1$$

$$d\varphi^2 = dx^2$$

$$d\delta(\psi^1) = dx^1 + du^1$$

$$\frac{1}{u^1+1}dx^3 - \frac{x^3}{(u^1+1)^2}du^1$$

of condition (14) with $A = (0, 0)$ are not linearly independent.

### 4.1 Construction of the linearising feedback in the general case

Theorem 4.2 ensures that every trajectory $v(k + \alpha), \alpha \geq 0$ of the new input $v = \delta^A(\psi)$ can be realised independently of the previous trajectory of the system up to the time instant $k$ by applying a suitable trajectory of the control input $u(k + \alpha), \alpha \geq 0$. In the following, we show how the required trajectory $u(k + \alpha)$ can be determined by a suitable state feedback. In other words, we derive a feedback which actually introduces $v = \delta^A(\psi)$ as new input. For simplicity we assume $A \leq R$, since the choice $A = R$ is possible anyway. The basic idea for the construction of the linearising feedback is similar as in the continuous-time case in Gstöttner et al. (2021a). However, due to the different transformation laws of continuous-time and discrete-time systems, the proof is adapted accordingly.

Because of the linear independence of the differentials (14) and $dx \in \text{span}\{d\psi_{[0,R-1]}\}$, there exists a selection $d\psi_c$ of $\#A - n$ differentials from the set $d\psi_{[0,A-1]}$ such that

$$\text{span}\{d\psi_{[0,R]}\} = \text{span}\{d\psi_{[0,A-1]}, d\psi_{[A,R]}\}$$

$$= \text{span}\{dx, d\psi_c, d\psi_{[A,R]}\}.$$ 

As a consequence of Lemma A.1, there exists a diffeomorphism $\Psi : \mathbb{R}^{\#R+m} \rightarrow \mathbb{R}^{\#R+m}$ such that locally

$$\varphi_{[0,R]} = \Psi(x, \psi_c, \varphi_{[A,R]}) \tag{17}$$

holds identically. Its inverse is given by

$$x = F_\epsilon(\varphi_{[0,R-1]}$$

$$\psi_c = \varphi_c$$

$$\varphi_{[A,R]} = \varphi_{[A,R]}),$$

where $F_\epsilon$ is the parameterisation of the state variables according to (9). Based on the diffeomorphism (17), the quantities...
$v = \delta A(\varphi)$ can be introduced as new input of the system (1) by a dynamic feedback with the controller state $z = \varphi_c$. Since the functions $\varphi_c$ belong to the set $\varphi_{[A-1]}$, their forward-shifts $\varphi_{c,[1]} = \delta(\varphi_c)$ belong to the set $\varphi_{[1,A]} \subseteq \varphi_{[0,R]}$ and can hence be expressed as functions of $x$, $\varphi_c$, and $\varphi_{[A,R]}$. The corresponding components of (17) are denoted in the following as $\varphi_{c,[1]} = \psi_{c,[1]}(x, \varphi_c, \varphi_{[A,R]})$.

**Theorem 4.4:** Consider a system (1) with a flat output (7) and a multi-index $A \leq R$ which satisfies the condition of Theorem 4.2. With a feedback

$$z^+ = \psi_{c,[1]}(x, z, v_{[0,R-A-1]})$$

$$u = F_u \circ \Psi(x, z, v_{[0,R-A-1]})$$

with $\dim(z) = \#A - n$, the closed-loop system

$$x^+ = f(x, F_u \circ \Psi(x, z, v_{[0,R-A-1]}))$$

$$z^+ = \psi_{c,[1]}(x, z, v_{[0,R-A-1]})$$

has the input-output behaviour $y_{[A]} = v$.

**Proof:** First, let us extend the feedback (18) by the trivial equations $v_{[0,R-A-1]}^+ = v_{[1,R-A]}$, such that the extended closed-loop system

$$x^+ = f(x, F_u \circ \Psi(x, z, v_{[0,R-A-1]}))$$

$$z^+ = \psi_{c,[1]}(x, z, v_{[0,R-A-1]})$$

has the form of a classical state representation with the input $v_{[R-A]}$. In the following, we show that with the transformation

$$y_{[0,R]} = \Psi(x, z, v_{[0,R-A-1]})$$

derived from (17) the system is equivalent to the discrete-time Brunovsky normal form

$$y_{[1]}^A \quad \ldots \quad y_{[m]}^A = y_{[1]}^m$$

$$y_{[1]}^A \quad \ldots \quad y_{[m]}^A = y_{[1]}^m$$

From the inverse

$$x = F_x(y_{[0,R-1]})$$

$$z = \gamma_c$$

$$v_{[0,R-A-1]}^+ = y_{[A,R-1]}$$

$$v_{[R-A]}^+ = y_{[R]}$$

of (21) and $\gamma_c \subset y_{[0,A-1]} \subset y_{[0,R-1]}$, it is clear that the transformation is actually a state transformation for the extended closed-loop system (20). The input is only renamed according to $v_{[R-A]} = y_{[R]}$. Because of the transformation law for discrete-time systems, applying the transformation (23) to the Brunovsky normal form (22) yields

$$x^+ = F_x(y_{[1,R]} \circ \Psi(x, z, v_{[0,R-A-1]})$$

$$v_{[0,R-A-1]}^+ = y_{[A+1,R]} \circ \Psi(x, z, v_{[0,R-A-1]}).$$

Using the identity

$$F_x(y_{[1,R]} = f(F_x(y_{[0,R-1]}), F_u(y_{[0,R]}))$$

as well as $F_u \circ \Psi(x, z, v_{[0,R-A-1]} = x$ and $y_{[A+1,R]} \circ \Psi(x, z, v_{[0,R-A-1]} = v_{[1,R-A-1]}$, the system representation (20) follows. Thus, the extended closed-loop system (20) is equivalent to the Brunovsky normal form (22) via a state transformation and a renaming of the input. Consequently, it has the linear input-output behaviour $y_{[R]} = v_{[R-A]}$. Since the closed-loop system (19) has the input $v$ instead of $v_{[R-A]}$, it has the input-output behaviour $y_{[A]} = v$.

In contrast to a classical static or dynamic feedback, the feedback (18) depends besides the new input $v$ also on its forward-shifts up to the order $R-A$. This is similar to the continuous-time case, where feedbacks which depend also on time derivatives of the new input have been used successfully for the exact linearisation of flat systems since the nineties of the last century, see e.g. Delaleau and Rudolph (1998) or Rudolph (2021). For a practical application, this means that at every time step not only the value of $v$ itself needs to be specified but also its future values. Hence, if a control law for the exactly linearised system $y_{[A]} = v$ is designed, also expressions for the forward-shifts of $v$ occurring in (18) have to be derived. If the system $y_{[A]} = v$ shall be controlled by a pure feedforward control, this is of course straightforward as long as the desired reference trajectory $y_d$ is known a sufficient number of time steps ahead (note that with such a feedforward control already a dead-beat behaviour can be achieved). In Section 5, it is shown how the required forward-shifts of $v$ can be determined also for a more general type of tracking control.

**Remark 4.1:** For $\#A = n$, the controller state $z$ is empty and the feedback (18) degenerates to a feedback of the form $u = F_u \circ \Psi(x, v_{[0,R-A-1]}$. Since there is no controller state but the feedback depends on forward-shifts of the closed-loop input $v$, such a feedback is a discrete-time quasi-static feedback as it is defined in Aranda-Bricaire and Kotta (2001). For continuous-time systems, a quasi-static feedback depends on time derivatives instead of forward-shifts of the closed-loop input, see Delaleau and Rudolph (1998) or Rudolph (2021).

**4.2 Flat outputs that are independent of future values of the input**

In the remainder of the paper, we consider flat outputs of the form

$$y^j = \varphi^j(\xi_{[1]}, \ldots, \xi_{[-1]}, x, u), \quad j = 1, \ldots, m$$

that are independent of future values of the input $u$. With this restriction, it is possible to derive further results in a similar way as in Gstöttner et al. (2021a) for flat outputs of continuous-time systems which are independent of time derivatives of $u$. In the following, we show how to systematically construct a 'minimal' multi-index $\kappa = (k^1, \ldots, k^m)$ such that with $A = \kappa$ the condition of Theorem 4.2 is met and $\#\kappa \leq \#A$ for all other
feasible multi-indices $A$. The basic idea is to replace the coordinates $u, u[1], u[2], \ldots$ of the manifold $\xi[-1, -1] \times X \times \mathcal{U}[0, \mathcal{U}]$ step by step by forward-shifts $v, v[1], v[2], \ldots$ of the flat output (24) with $v[\alpha] = \delta^{\alpha} \phi(\cdot)$, $\alpha \geq 0$, such that finally we have coordinates $(\zeta[-1, -1], x, v, v[1], v[2], \ldots)$. For this purpose, we forward-shift every component of the flat output (24) until it depends explicitly on the input $u$, and introduce as many of these functions as possible as new coordinates. Subsequently, the other components of the flat output are further shifted until they depend explicitly on the remaining components of $u$, and again as many of these functions as possible are introduced as new coordinates. Continuing this procedure until all $m$ components of the original input $u$ have been replaced by forward-shifts of the flat output (24) yields a minimal multi-index $k = (k^1, \ldots, k^m)$ such that with $v = \delta^k \phi$ the condition of Theorem 4.2 is satisfied. In the following, we explain the procedure in detail.

In the first step, determine the multi-index $K_1 = (k^{1,1}, \ldots, k^{1,m})$ such that

$$
\delta^{k^{1,1}}(\phi^1) = \phi^1 \mid_{\zeta[-1, -1]}(\zeta[-q, -1], x)
$$

and define $m_1 = \text{rank}(\partial u \phi[1,1])$. Then reorder the components of the flat output (24) and the input $u$ such that rank($\partial u \phi[1,1]$) = $m_1$, where $\phi^1 = (\phi^1, \ldots, \phi^{m_1})$, $u_1 = (u^{1,1}, \ldots, u^{m_1})$, and $\zeta^1 = (k^{1,1}, \ldots, k^{1,m_1})$ consist of the first $m_1$ components of $\phi$, $u$, and $K_1$, respectively. Now apply the coordinate transformation

$$
\begin{align*}
&v_1 = \phi[1,1](\xi[-q, -1], x, u) \\
&u_{r_1,1} = (u^{m_1+1,1}, \ldots, u^{m_1}) \\
&v_{1,1} = \phi[1,1](\xi[-q, -1], x, u, u_1) \\
&u_{r_1,1} = (u^{m_1+1,1}, \ldots, u^{m_1}) \\
&v_{1,2} = \phi[1,2](\xi[-q, -1], x, u, u_1, u_2) \\
&u_{r_1,2} = (u^{m_1+1,2}, \ldots, u^{m_1}) \\
&\vdots
\end{align*}
$$

which replaces the inputs $u_1$ and their forward-shifts by $v_1$ and its forward-shifts. The remaining inputs $u_{r_1,1} = (u^{m_1+1,1}, \ldots, u^{m_1})$ and their forward-shifts are left unchanged.

**Remark 4.2:** The coordinate transformation (25) is indeed regular, since in a sufficiently small neighborhood of an equilibrium point the condition rank($\partial u \phi[1,1]$) = $m_1$ implies rank($\partial u \phi[1,1]$) = $m_1$ for $\alpha \geq 1$. In the new coordinates, the forward-shift operator (5) has the form

$$
\delta(h(\ldots, \zeta[-2], \zeta[-1], x, v_1, u_{r_1,1}, u_{r_1,2}, \ldots))
$$

with $\hat{\phi}$ denoting the inverse of the transformation (25).

After the coordinate transformation (25) we have

$$
\begin{align*}
y_1[0, q_1-1] &= \phi[1,0, q_1-1](\xi[-q_1, -1], x) \\
y_1[1,1] &= v_1 \\
y_{r_1,1} &= \phi[1,0, q_1-1](\xi[-q_1, -1], x, v_1, v_{1,1}) \\
y_{r_1,2} &= \phi[1,0, q_1-1](\xi[-q_1, -1], x, v_1, v_{1,1}, v_{1,2}) \\
y_{r_1,3} &= \phi[1,0, q_1-1](\xi[-q_1, -1], x, v_1, v_{1,1}, v_{1,2}, v_{1,3}) \\
&\vdots
\end{align*}
$$

which replaces the inputs $u_2 = (u^{m_1+1,1}, \ldots, u^{m_1+m_2})$ and their forward-shifts by $y_2$ and its forward-shifts. The remaining inputs $u_{r_1,2} = (u^{m_1+m_2,1}, \ldots, u^{m_1})$ and their forward-shifts are left unchanged. After the coordinate transformation (26) we have

$$
\begin{align*}
y_1[0, q_1-1] &= \phi[1,0, q_1-1](\xi[-q_1, -1], x) \\
y_1[1,1] &= v_1 \\
y_2[0, q_2-1] &= \phi[2,0, q_2-1](\xi[-q_2, -1], x, y_1, v_{1,1}, \ldots) \\
y_2[1,1] &= v_2 \\
y_{r_2,1} &= \phi[2,0, q_2-1](\xi[-q_2, -1], x, y_1, v_{1,1}, v_{1,2}) \\
y_{r_2,2} &= \phi[2,0, q_2-1](\xi[-q_2, -1], x, y_1, v_{1,1}, v_{1,2}, v_{1,3}) \\
&\vdots
\end{align*}
$$

where $y_2 = (y^{m_1+1,1}, \ldots, y^{m_1+m_2})$, $y_{r_2,1} = (y^{m_1+m_2+1,1}, \ldots, y^{m_1})$, $K_{r_2} = (k^{m_2+1,1}, \ldots, k^{m_2+m_1})$, and $\varphi_{r_2,1} = (\varphi_{r_2,1}^{m_2+1,1}, \ldots, \varphi_{r_2,1}^{m_2+m_1})$ with the inverse $\hat{\phi}$ of the transformation (26).
The functions $\psi_{rest_2[K_{rest_1}]}$ are again independent of $u_{rest_1}$, since otherwise rank$(\partial_{u_{rest_1}} \psi_{rest_1[K_i]})$ would have been larger than $m_2$.

This procedure is now continued until in some step $s$ we obtain a multi-index $K_s = (k_1^s, \ldots, k_{\ell}^{m_1-1})$ with

$$
\delta^j_{k_s} \psi_{rest_1, [K_s]}
= \psi_{j, rest_1, [K_s]}(x, v_1, v_{1,1}, \ldots, v_{s-1}, v_{s-1,1}, \ldots)
$$

$$
\delta^j_{k_s} \psi_{rest_1, [K_s]}(x, v_1, v_{1,1}, \ldots, v_{s-1}, v_{s-1,1}, \ldots, u_{rest_1})
$$

such that rank$(\partial_{u_{rest_1}} \psi_{rest_1, [K_s]}) = \dim(u_{rest_1})$. Thus, with $\psi_s = \psi_{rest_1}$ and $\kappa_s = K_s$ we can apply the coordinate transformation

$$
v_s = \psi_{s}[K_s](x, v_1, v_{1,1}, \ldots, v_{s-1}, v_{s-1,1}, \ldots, u_{rest_1}),
$$

which replaces the remaining inputs $u_{rest_1}$ by $v_s$. With the constructed coordinates, the flat output and its forward-shifts up to the orders $\kappa_i$ are finally given by

$$
y_1[0, \kappa_i-1] = \psi_{1}[0, \kappa_i-1](\zeta[-q_i,-1], x)
$$

$$
y_1[1, \kappa_i] = v_1
$$

$$
y_2[0, s_2-1] = \psi_{2}[0, s_2-1](\zeta[-q_i,-1], x, v_1, v_{1,1}, \ldots)
$$

$$
y_2[2, \kappa_2] = v_2
$$

$$
\vdots
$$

$$
y_{s-1}[0, \kappa_{s-1}-1] = \psi_{s-1}[0, \kappa_{s-1}-1](\zeta[-q_i,-1], x, v_1, v_{1,1}, \ldots, v_{s-2}, v_{s-2,1}, \ldots)
$$

$$
y_{s-1}[s_{s-1}] = v_{s-1}
$$

$$
y_s[0, \kappa_i-1] = \psi_{s}[0, \kappa_i-1](\zeta[-q_i,-1], x, v_1, v_{1,1}, \ldots, v_{s-1}, v_{s-1,1}, \ldots)
$$

$$
y_s[k_i] = v_s
$$

with $\dim(y_i) = m_i$ and $\kappa_i = (k_1^i, \ldots, k_{\ell}^{m_i})$.

**Theorem 4.5:** For every flat output (24) of the system (1), the above procedure terminates after $s \leq m$ steps. The multi-index $\kappa = (\kappa_1, \ldots, \kappa_s)$ formed by the constructed multi-indices $\kappa_i = (k_1^i, \ldots, k_{\ell}^{m_i})$ has the following properties:

(i) $\kappa \leq R$

(ii) $\#\kappa \geq n$, and $\#\kappa = n$ if and only if the flat output (24) is independent of the variables $\xi[-q_i], \ldots, \zeta[-1]$. Moreover, it can also be seen that there exist exactly $\#\kappa$ independent linear combinations of the multi-index $R$ corresponding to the components $y_i = (y_1^i, \ldots, y_\ell^i)$ of the flat output.

**Proof:** In every step $i \geq 1$ of the procedure, it is possible to forward-shift the remaining components $\psi_{rest_1}$ of the flat output until every component depends explicitly on one of the remaining inputs $u_{rest_1}$ ($\psi_{rest_1} = \varphi$ and $u_{rest_1} = u$ for $i = 1$). Otherwise, the property that all forward-shifts of a flat output up to arbitrary order are functionally independent could not hold. Since in every step we have rank$(\partial_{u_{rest_1}} \psi_{rest_1, [K_i]}) \geq 1$, at least one of the original inputs $u$ can be eliminated, and the procedure terminates after at most $\dim(u) = m$ steps.

Now let us prove that $\kappa \leq R$. If the parameterisation of $x$ and $\zeta[-q_i,-1]$ by the flat output is substituted into (27) and $y_{[\kappa+i]}$, $\alpha \geq 0$ renamed according to $y_{[\kappa+i]} = y_{[\alpha]}$, then the equations

$$
y_{k_i}[0, \kappa_i-1] = \psi_{s}[0, \kappa_i-1](\zeta[1,1], x, v_1, v_{1,1}, \ldots, v_{s-1}, v_{s-1,1}, \ldots)
$$

$$
v_i, v_{i,1}, \ldots, v_{i-1}, v_{i-1,1}, \ldots, j_i = 1, \ldots, m_i, \quad i = 1, \ldots, s
$$

must be satisfied identically. Since both $x$ and $\zeta[-q_i,-1]$ depend only on forward-shifts of the flat output up to the order $R-1$, and the forward-shifts $y_{k_i}[0, \kappa_i-1]$ of $y_i$ on the left-hand side of (29) are not contained in the quantities $v_i, v_{i,1}, \ldots, v_{i-1}, v_{i-1,1}, \ldots$ on the right-hand side, this can only hold in the case $\kappa \leq R$. By the same argument, it is clear that (27) is actually of the form (28).

To prove $\#\kappa \geq n$, recall that there exist exactly $n$ independent linear combinations of the differentials of a flat output and its forward-shifts which are contained in $\text{span}(dx)$. In the coordinates constructed during the above procedure, the forward-shifts of the flat output up to the order $\kappa$ are given by the expressions in (27), and the higher forward-shifts are forward-shifts of $v$. Thus, there can exist at most $\#\kappa$ independent linear combinations which are contained in $\text{span}(dx)$, and hence $\#\kappa \geq n$. If the flat output (24) is independent of $\zeta[-q_i], \ldots, \zeta[-1]$, then all expressions in (27) are independent of these variables. Consequently, there exist exactly $\#\kappa$ independent linear combinations of the differentials of the flat output and its forward-shifts which are contained in $\text{span}(dx)$, and hence $\#\kappa = n$.

To prove the relation with Theorem 4.2, we use again the representation (27). In these coordinates, it can be immediately observed that with $A = \kappa$ the differentials (14) are linearly independent. Moreover, it can also be seen that there exist exactly $\#\kappa$ independent linear combinations of the differentials
of the flat output and its forward-shifts which are contained in \( \text{span}(d\zeta_{[-q_1]}, \ldots, d\zeta_{[-1]}, dx) \). If there exists a multi-index \( A \) such that the differentials (14) are linearly independent, then there can exist at most \( |A| \) independent linear combinations of the differentials of the flat output and its forward-shifts which are contained in \( \text{span}(d\zeta_{[-q_1]}, \ldots, d\zeta_{[-1]}, dx) \). Thus, the existence of such a multi-index with \( |A| < \#\kappa \) would be a contradiction.

Since the condition of Theorem 4.2 is met, the forward-shifts \( y_{[k]} \) of the flat output (24) can be introduced as a new input \( v \) by a (dynamic) feedback according to Theorem 4.4, where the controller state \( z \) corresponds to suitable forward-shifts of the flat output that are contained in \( y_{[0:k-1]} \). However, the representation (28) of \( y_{[0:k-1]} \) offers a convenient alternative. With the map  
\[
\phi(\zeta_{[-q_1]}, \ldots, \zeta_{[-1]}, x, v_{[0:R-k]})
\]
defined by
\[
y_{1,[0:k-1]} = \Phi_{1,[0:k-1]}(\zeta_{[-q_1]}, \ldots, \zeta_{[-1]}, x)
y_{1,[k_1]} = \Phi_{1,[k_1]}(v_{[0:k_1]})
\]
\[
\vdots
\]
\[
y_{k,[0:k_2]} = \Phi_{k,[0:k_2]}(\zeta_{[-q_1]}, \ldots, \zeta_{[-1]}, x, v_{[0:k_2]})
y_{k,[k_3]} = \Phi_{k,[k_3]}(v_{[0:k_3]})
\]
we can formulate the following corollary.

**Corollary 4.6:** A flat system (1) with a flat output of the form (24) can be exactly linearised with respect to this flat output by a feedback of the form
\[
u = F_u \circ \phi(\zeta_{[-q_1]}, \ldots, \zeta_{[-1]}, x, v_{[0:R-k]}),
\]
such that the input-output behaviour of the closed-loop system is given by \( y_{[k]} = v \). If the flat output (24) is independent of the variables \( \zeta_{[-q_1]}, \ldots, \zeta_{[-1]} \), then the feedback has the form
\[
u = F_u \circ \phi(x, v_{[0:R-k]}),
\]
and \( \#\kappa = n \).

In contrast to a dynamic feedback (18) with the controller state \( z \), the feedback (30) depends only on the state \( x \) and past values \( \zeta_{[-q_1]}, \ldots, \zeta_{[-1]} \) of the system trajectory. Since the values of \( \zeta_{[-q_1]}, \ldots, \zeta_{[-1]} \) are available anyway from past measurements and/or past control inputs (depending on the choice of \( \zeta \), cf. (3)), the implementation of a feedback (30) is straightforward. However, even though there is no dedicated controller dynamics as in (18), the required past values \( \zeta_{[-q_1]}, \ldots, \zeta_{[-1]} \) have to be stored. Thus, the feedback (30) can be considered either as a special case of a dynamic feedback or a generalisation of the class of discrete-time quasi-static feedbacks (as they are defined in Aranda-Bricaire and Kotta (2001)) to backward-shifts \( \ldots, \zeta_{[-2]}, \zeta_{[-1]} \) of the system variables. A feedback of the special form (31) has been used in Diwold et al. (2022a) for the exact linearisation of the discrete-time model of a gantry crane.

**Remark 4.3:** If the flat output (24) is of the form \( y = \varphi(x) \) – i.e. independent of the input \( u \) as well as past values \( \zeta_{[-q_1]}, \ldots, \zeta_{[-1]} \) of the system trajectory – then the proposed procedure for the construction of a minimal multi-index \( \kappa \) is from a technical point of view similar to the inversion algorithm stated e.g. in Kotta (1995), see also Kotta and Nijmeijer (1991) or Kotta (1990). In the inversion algorithm, which deals with the forward right-invertibility of a system (1) with a (not necessarily flat) output \( y = \varphi(x) \) and possibly also \( \dim(y) \neq \dim(u) \), the components of the output are also shifted until the Jacobian matrices with respect to the input variables meet certain rank conditions. However, it should be noted that in every step of the inversion algorithm only one-fold shifts of the components of the output are performed. Furthermore, it should also be noted that the so-called invertibility indices computed by the inversion algorithm are related to but not the same as the components of the constructed minimal multi-index \( \kappa \).

### 5. Tracking control design

Like in the continuous-time case, the exact linearisation can be used as a first step in the design of a flatness-based tracking control. For an exact linearisation according to Theorem 4.4 with the choice \( A = R \), which is always possible, the closed-loop system has the form of a classical state representation. Thus, the design of a tracking control is straightforward. For \( A < R \), however, the closed-loop system depends also on forward-shifts \( v_{[0:R-A]} \) of the new input \( v \). Thus, when designing a control law for \( v \), also the corresponding expressions for these forward-shifts have to be derived. For a discussion of this problem in the continuous-time case see e.g. Delaleau and Rudolph (1998), Rudolph (2021), or Gstöttner et al. (2021a).

In the following, we demonstrate the design of a tracking control for flat outputs of the form (24) and an exact linearisation by a feedback (30) according to Corollary 4.6. We assume that the multi-index \( \kappa \) which determines the new input \( v \) has been constructed in accordance with the procedure of Section 4.2, and make use of the corresponding notation. With the control law
\[
\nu^i_{\gamma} = y^i_{\gamma} - \sum_{\beta=0}^{\kappa^i_{\gamma} - 1} d^i_{\beta} \gamma_{\beta},
\]
(32)
for the exactly linearised system \( y_{[k]} = v \), the tracking error \( e^i_{\gamma} = y^i_{\gamma} - y^i_{\gamma} \) with respect to an arbitrary reference trajectory \( y^i_{\gamma}(k) \) is subject to the tracking error dynamics
\[
\dot{e}^i_{\gamma} + \sum_{\beta=0}^{\kappa^i_{\gamma} - 1} d^i_{\beta} e^i_{\beta} = 0,
\]
(33)
for \( i = 1, \ldots, m, j = 1, \ldots, s \).

The eigenvalues of the sum \( \sum_{j=1}^{s} m_j = m \) decoupled tracking error systems (33) can be placed arbitrarily by a suitable choice of the coefficients \( d^i_{\beta} \in \mathbb{R} \). The forward-shifts \( v_{[0:R-k]} \) that are needed in the linearising feedback (30) can be determined by shifting (32) and using
\[
y^i_{\gamma} = \nu^i_{\gamma}, \gamma \geq 0,
\]
which leads to
equations of the form
\[ v_{i}[\gamma] = y_{i}^{d}[\gamma] - \sum_{\alpha=0}^{k_{i}-1} a_{i}^{\alpha} (v_{i}^{d}[\alpha] - y_{i}^{d}[\alpha]) - \sum_{\beta=\gamma}^{k_{i}-1} a_{i}^{\beta-\gamma} (v_{i}^{d}[\beta] - y_{i}^{d}[\beta]). \] (34)

Since the future values \(v[0,\xi-1]\) of the flat output which appear in (32) and (34) are in general not available as measurements, we use again the expressions (28) and finally obtain the system of equations
\[ v_{1}^{j} = y_{1}^{d}[1] - \sum_{\alpha=0}^{k_{1}-1} a_{1}^{\alpha} (v_{1}^{d}[\alpha] - y_{1}^{d}[\alpha]) \]
\[ v_{1}[1] = y_{1}^{d}[1] - a_{1}^{\alpha} (v_{1}^{d}[\alpha] - y_{1}^{d}[\alpha]) - \sum_{\beta=1}^{k_{1}-1} a_{1}^{\beta-1} (v_{1}^{d}[\beta] - y_{1}^{d}[\beta]) \]
\[ v_{j}^{j}[1,\xi-1] = y_{j}^{d}[1,\xi-1] - \sum_{\alpha=0}^{k_{j}[\xi-1]} a_{j}^{\alpha} (v_{j}^{d}[\alpha] - y_{j}^{d}[\alpha]) - \sum_{\beta=\gamma}^{k_{j}[\xi-1]} a_{j}^{\beta-\gamma} (v_{j}^{d}[\beta] - y_{j}^{d}[\beta]). \] (35)

Because of the triangular dependence of the functions \(v_{i}[0,\xi-1], v_{2}[0,\xi-1], \ldots\) of (28) on the variables \(v_{1}[0,\xi-k_{1}-1], v_{2}[0,\xi-k_{2}-1], \ldots\), Equations (35) have a triangular structure and can be solved systematically from top to bottom for all elements of \(v[0,R-\xi]\) as a function of \(\xi[-1]\), \(x\), and the reference trajectory \(y_{0}[0,R]\), i.e.
\[ v[0,R-\xi] = \rho(\xi[-1],x,y_{0}[0,R]). \] (36)

Substituting (36) into the linearising feedback (30) yields a tracking control law of the form
\[ u = \eta(\xi[-1],x,y_{0}[0,R]). \]

Besides the known reference trajectory \(y_{0}[0,R]\), this tracking control law depends like the linearising feedback (30) only on the state \(x\) and past values \(\xi[-1], \ldots, \xi[-1]\) of the system trajectory. Thus, if the state \(x\) of system (1) can be measured, an implementation is again straightforward.

**Remark 5.1:** If all coefficients \(a_{i}^{\alpha}\) in the control law (32) are set to zero, all eigenvalues of the tracking error dynamics (33) are located at the origin of the complex plane and a dead-beat control is obtained. In this case, the system of equations (35) drastically simplifies. Because of \(v_{i}^{j} = y_{i}^{d}[\alpha]\), the forward-shifts of \(v\) required in the linearising feedback (30) are simply higher-order forward-shifts of the reference trajectory \(y_{0}[0,R]\).

### 6. Examples

As already mentioned in the introduction, an important application for the concept of discrete-time flatness is discretised continuous-time systems. In the following, we illustrate our results by the discretised models of a wheeled mobile robot and a 3DOF helicopter.

#### 6.1 Wheeled mobile robot

As first example let us consider a wheeled mobile robot, which has already been studied in the context of discrete-time dynamic feedback linearisation in Orosco-Guerrero et al. (2004) and Aranda-Bricaire and Moog (2008). The continuous-time system is given by
\[ \dot{x}^{1} = u^{1} \cos(x^{3}) \]
\[ \dot{x}^{2} = u^{1} \sin(x^{3}) \]
\[ \dot{x}^{3} = u^{2}, \]
and is also known as kinematic car model, see e.g. Nijmeijer and van der Schaft (1990). The state variables \(x^{1}\) and \(x^{2}\) describe the position and \(x^{3}\) the angle of the mobile robot. The control inputs \(u^{1}\) and \(u^{2}\) represent the translatory and the angular velocity. An exact discretisation of system (37) with the sampling time \(T > 0\) yields the discrete-time system
\[ x^{1,+} = x^{1} + u^{1} T \cos(x^{3}) \sin(\frac{u^{2}T}{2}) \]
\[ x^{2,+} = x^{2} + u^{1} T \sin(x^{3}) \sin(\frac{u^{2}T}{2}) \]
\[ x^{3,+} = x^{3} + u^{2} T, \] (38)

for the variables $\varphi$ according to (3), the flat output is given by

$$y = (\xi_{[-1]}^1, x^1 \sin \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right) - x^2 \cos \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right)).$$

(41)

The corresponding parameterisation (9) of the system variables has the form

$$x = F_x(y^1, y^2, \ldots, y^1_{[2]}, y^1_{[1]})$$

$$\ddot{u} = F_u(y^1, y^2, \ldots, y^1_{[3]}, y^1_{[2]}).$$

(42)

Thus, the orders of the highest forward-shifts of the flat output that appear in (9) are given by $R = (3, 2)$, and an exact linearisation by a dynamic feedback which leads to an input-output behaviour

$$y^1_{[3]} = v_1^1$$

$$y^1_{[2]} = v_2^1$$

(43)

is possible with the standard approach discussed in Diwold et al. (2022b).

In the following, we investigate whether also lower-order forward-shifts of the flat output (41) can be chosen as a new input. Since the flat output is of the form (24), we can apply the procedure of Section 4.2 and use the corresponding notation. In the first step, both components of the flat output have to be shifted until they depend explicitly on the input $\ddot{u}$. Because of

$$\varphi^1 = \xi_{[-1]}^1$$

$$\delta(\varphi^1) = x^3$$

$$\delta^2(\varphi^1) = 2\ddot{u} - x^3$$

and

$$\varphi^2 = x^1 \sin \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right) - x^2 \cos \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right)$$

$$\delta(\varphi^2) = x^1 \sin (\ddot{u}^2) - x^2 \cos (\ddot{u}^2),$$

this is the case for the second and the first forward-shift, respectively. Hence, we obtain $K_1 = (2, 1)$ and since $\varphi_{[K_1]}^1$ is independent of $\ddot{u}$ we clearly have $m_1 = \text{rank}(\partial_{\ddot{u}}\varphi_{[K_1]}^1) = 1$. At this point, we can choose whether we introduce $\delta^2(\varphi^1)$ or $\delta(\varphi^2)$ as new input $v_1 = v_1^1$. 7 In the following, we proceed with $\varphi_1 = \varphi^1$ and $\varphi_{\text{rest}_1} = \varphi^2$. Consequently, we get $\kappa_1 = k_1^1 = 2$ and $K_{\text{rest}_1} = k_1^2 = 1$. After the coordinate transformation

$$v_1^1 = \varphi_{[1],2}^1 = 2\ddot{u}^2 - x^3$$

$$v_1^1_{[1]} = \varphi_{[1],3}^1 = 2\ddot{u}^1_{[1]} - 2\ddot{u}^2 + x^3,$$

which replaces $\ddot{u}^2$ and its forward-shifts by $v_1^1$ and its forward-shifts, we have

$$y_{1,[0,k_1-1]} = \left[\begin{array}{c} \varphi_{1,[1]}^1 \\ \varphi_{1,[1]}^1 \end{array}\right] = \left[\begin{array}{c} \xi_{[-1]}^1 \\ x^3 \end{array}\right]$$

$$y_{1,[k_1]} = \varphi_{1,[2]}^1 = v_1^1$$

$$y_{\text{rest}_1,[0,K_{\text{rest}_1}-1]} = \varphi_{\text{rest}_1}^1 = x^1 \sin \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right) - x^2 \cos \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right)$$

$$y_{\text{rest}_1,[K_{\text{rest}_1}]} = \varphi_{\text{rest}_1,[1]} = x^1 \sin \left(\frac{x^1 + v_1}{2}\right) - x^2 \cos \left(\frac{x^1 + v_1}{2}\right).$$

(44)

Because of $\text{rank}(\partial_{\ddot{u}}\varphi_{\text{rest}_1,[2]}) = \text{dim}(\ddot{u}^2) = 1$, the procedure terminates with $\varphi_2 = \varphi_{\text{rest}_1}$ and $\kappa_2 = K_2 = 2$. After introducing (44) as new input $v_2 = v_2^1$, the forward-shifts of the flat output (41) up to the orders $\kappa = (k_1, k_2) = (2, 2)$ are given by

$$y_{1,[0,k_1-1]} = \left[\begin{array}{c} \varphi_{1,[1]}^1 \\ \varphi_{1,[1]}^1 \end{array}\right] = \left[\begin{array}{c} \xi_{[-1]}^1 \\ x^3 \end{array}\right]$$

$$y_{1,[k_1]} = \varphi_{1,[2]}^1 = v_1^1$$

$$y_{2,[0,k_2-1]} = \left[\begin{array}{c} \varphi_{2,[1]}^1 \\ \varphi_{2,[1]}^1 \end{array}\right] = \left[\begin{array}{c} x^1 \sin \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right) - x^2 \cos \left(\frac{\xi_{[-1]}^1 + x^1}{2}\right) \\ x^1 \sin \left(\frac{x^1 + v_1^1}{2}\right) - x^2 \cos \left(\frac{x^1 + v_1^1}{2}\right) \end{array}\right]$$

$$y_{2,[k_2]} = \varphi_{2,[2]}^1 = v_2^1.$$
such that the input-output behaviour of the closed-loop system is given by
\[
\begin{align*}
    y_{1,2}^i &= v_1^i \\
    y_{2,2}^i &= v_2^i,
\end{align*}
\]
(47)
cf. Corollary 4.6. Thus, in contrast to (43), we can actually use the second instead of the third forward-shift of the first component of the flat output as a new input.

For the exactly linearised system (47), the design of a tracking control is now straightforward. The control law
\[
\begin{align*}
    v_1^i &= y_{1,2}^{id} - a_{11}^i(y_{1,1}^i - y_{1,1}^{id}) - a_{10}^i(y_1^i - y_1^{id}) \\
    v_2^i &= y_{2,2}^{id} - a_{21}^i(y_{2,1}^i - y_{2,1}^{id}) - a_{20}^i(y_2^i - y_2^{id})
\end{align*}
\]
results in the linear tracking error dynamics
\[
\begin{align*}
    e_{1,2}^i + a_{11}^i e_{1,1}^i + a_{10}^i e_1^i &= 0 \\
    e_{2,2}^i + a_{21}^i e_{2,1}^i + a_{20}^i e_2^i &= 0
\end{align*}
\]
with a total order of \(\#k = 2 + 2 = 4\) instead of \(\#R = 3 + 2 = 5\) with the standard approach. Since the linearising feedback (46) depends also on \(v_{1,1}^i\), the system of equations (35) is given by
\[
\begin{align*}
    v_1^i &= y_{1,2}^{id} - a_{11}^i(\varphi_{1,1}^i - y_{1,1}^{id}) - a_{10}^i(\varphi_1^i - y_1^{id}) \\
    v_1^i &= y_{1,1}^{id} - a_{11}^i(v_1 - y_{1,2}^i) - a_{10}^i(\varphi_1^i - y_1^{id}) \\
    v_2^i &= y_{2,2}^{id} - a_{21}^i(\varphi_{2,1}^i - y_{2,1}^{id}) - a_{20}^i(\varphi_2^i - y_2^{id}) \\
    v_2^i &= y_{2,1}^{id} - a_{21}^i(v_2 - y_{2,2}^i) - a_{20}^i(\varphi_2^i - y_2^{id})
\end{align*}
\]
with the functions \(\varphi_{1,1}^i, \varphi_{1,2}^i, \varphi_{2,1}^i, \varphi_{2,2}^i\) according to (45). This system of equations can be solved from top to bottom for \(v_1^i, v_{1,1}^i, v_{1,2}^i\) as a function of \(\zeta_{1,-1}^i\), \(x\), and the reference trajectory \(y_{0,R}^d\). Substituting the solution into the linearising feedback (46) yields a control law of the form
\[
\begin{align*}
    \bar{u}^1 &= \eta^i(\zeta_{1,-1}^i, x^1, x^2, x^3, y_{1,1}^{id}, y_{2,2}^{id}) \\
    \bar{u}^2 &= \eta^i(\zeta_{1,-1}^i, x^1, x^2, x^3, y_{1,2}^{id}, y_{2,2}^{id})
\end{align*}
\]
With the inverse of the input transformation (39), the corresponding control law for the system (38) or (37) with the original inputs \(u^1\) and \(u^2\) follows. The presence of the variable \(\zeta_{1,-1}^i\) is no obstacle for a practical implementation, since it simply represents a past value of \(x^3\).

### 6.2 3DOF helicopter

As a second example, let us consider the three-degrees-of-freedom helicopter laboratory experiment of Kiefer et al. (2004).

The continuous-time system is given by
\[
\begin{align*}
    q^1 &= \omega^1 \\
    q^2 &= \omega^2 \\
    q^3 &= \omega^3 \\
    \omega^1 &= b_1 \cos(q^2) \sin(q^3) u^1 \\
    \omega^2 &= a_1 \sin(q^2) + a_2 \cos(q^2) + b_2 \cos(q^3) u^1 \\
    \omega^3 &= a_3 \cos(q^2) \sin(q^3) + b_3 u^2
\end{align*}
\]
(48)
with the travel angle \(q^1\), the elevation angle \(q^2\), and the pitch angle \(q^3\) as well as the corresponding angular velocities \(\omega^1, \omega^2,\) and \(\omega^3\). The control inputs \(u^1\) and \(u^2\) are the sum and the difference of the thrusts of the two propellers. The constant coefficients \(a_1, a_2, a_3, b_1, b_2\) depend on the masses and the geometric parameters. As shown in Kiefer et al. (2004), the system (48) is flat and a flat output is given by
\[
y = (q^2, q^1).
\]
(49)
Since the system equations are significantly more complex than those of the wheeled mobile robot of Section 6.1, instead of an exact discretisation we perform an approximate discretisation with the Euler-method. The resulting system
\[
\begin{align*}
    q^{1,+} &= q^1 + T\omega^1 \\
    q^{2,+} &= q^2 + T\omega^2 \\
    q^{3,+} &= q^3 + T\omega^3 \\
    \omega^{1,+} &= \omega^1 + Tb_1 \cos(q^2) \sin(q^3) u^1 \\
    \omega^{2,+} &= \omega^2 + T(a_1 \sin(q^2) + a_2 \cos(q^2) + b_2 \cos(q^3) u^1) \\
    \omega^{3,+} &= \omega^3 + T(a_3 \cos(q^2) \sin(q^3) + b_3 u^2)
\end{align*}
\]
(50)
is forward-flat, and the flat output (49) is preserved. This can be checked either with the systematic tests proposed in Kolar et al. (2021) and Kolar et al. (2022), or by simply verifying that all state- and input variables of (50) can indeed be expressed by (49) and its forward-shifts. The corresponding parameterisation (9) is of the form
\[
\begin{align*}
    x &= F_x(y^1, y^2, \ldots, y_{[3]}^3, y_{[3]}^3) \\
    u &= F_u(y^1, y^2, \ldots, y_{[4]}^4, y_{[4]}^4)
\end{align*}
\]
(51)
with the orders of the highest required forward-shifts given by \(R = (4,4)\). Thus, an exact linearisation with a dynamic feedback according to the standard approach discussed in Divold et al. (2022b) would lead to an input-output behaviour
\[
\begin{align*}
    y_{[4]}^1 &= v^1 \\
    y_{[4]}^2 &= v^2
\end{align*}
\]
(52)
Since the flat output (49) is of the form (24), we can again apply the procedure of Section 4.2 in order to determine whether also lower-order forward-shifts can be used as new input. In the first step, both components of the flat output have to be shifted until they depend explicitly on the input \(u\). Because of
\[
\begin{align*}
    \varphi^1 &= q^2 \\
    \delta(\varphi^1) &= q^2 + T\omega^2 \\
    \delta^2(\varphi^1) &= q^2 + 2T\omega^2 + T^2(a_1 \sin(q^2) + a_2 \cos(q^2) + b_2 \cos(q^3) u^1)
\end{align*}
\]
and
\[
    \varphi^2 = q^1
\]
\[ \delta(\varphi^2) = q^1 + T\omega^1 \]
\[ \delta^2(\varphi^2) = q^1 + 2T\omega^1 + T^2b_1 \cos(q^2) \sin(q^3)u_1, \]

this is the case for the second forward-shifts. Hence, we obtain \( K_1 = (2,2) \), and since \( \varphi_{K[1]} \) is independent of \( u^2 \) we obviously have \( m_1 = \text{rank}(\partial_u\varphi_{K[1]}) = 1 \). Again, we can choose whether we introduce \( \delta(\varphi^1) \) or \( \delta^2(\varphi^2) \) as new input \( v_1 = v_1^1 \). In the following we proceed with \( \varphi_1 = \varphi^1 \) and \( \varphi_{\text{rest}_1} = \varphi^2 \), and get \( \kappa_1 = k_1^1 = 2 \) as well as \( K_{\text{rest}_1} = k_1^2 = 2 \). The other choice would lead to a feedback with a singularity for \( q^3 = 0 \), which is not suitable for a practical application since the pitch angle \( q^3 \) is zero in an equilibrium position. After the coordinate transformation

\[
v_1^1 = \varphi_{1,[2]} = q^2 + 2T\omega^2 \\
v_1^1_{\text{rest}} = \varphi_{1,[3]}(q^2, q^3, \omega^2, \omega^3, u_1, u_{1[1]}),
\]

which replaces \( u^1 \) and its forward-shifts by \( v_1^1 \) and its forward-shifts, we have

\[
y_{1,[0,k_1-1]} = \begin{bmatrix} \varphi_{1,[2]} \\ \varphi_{1,[1]} \end{bmatrix} = \begin{bmatrix} q^2 \\ q^2 + T\omega^2 \end{bmatrix} \\
y_{1,[k_1]} = \varphi_{1,[2]} = v_1^1 \\
y_{\text{rest}_1,[0,K_{\text{rest}_1}-1]} = \begin{bmatrix} \varphi_{\text{rest}_1,[2]} \\ \varphi_{\text{rest}_1,[1]} \end{bmatrix} = \begin{bmatrix} q^1 \\ q^1 + T\omega^1 \end{bmatrix} \\
y_{\text{rest}_1,[K_{\text{rest}_1}]} = \varphi_{\text{rest}_1,[2]} = q^1 + 2T\omega^1 \\
\quad - b_1 \cos(q^2) \tan(q^3) \\
\quad \times (q^2 - v_1^1 + 2T\omega^2 \\
\quad + T^2 (a_1 \sin(q^2) + a_2 \cos(q^2)))).
\]

In the second step, we have to shift the remaining component \( \varphi_{\text{rest}_1} = \varphi^2 \) of the flat output until it depends explicitly on the remaining input \( u^2 \). This is the case for its fourth forward-shift

\[
\delta^4(\varphi_{\text{rest}_1}) = \varphi_{\text{rest}_1,[4]}(q^1, q^2, q^3, \omega^1, \omega^2, \omega^3, v_1^1, v_{1,[1]}, v_{1,[2]}, u_2^1).
\]

(53)

Because of \( \text{rank}(\partial_u\varphi_{\text{rest}_1,[4]}^1) = \dim(u^2) = 1 \), the procedure terminates with \( \varphi_2 \), \( \varphi_{\text{rest}_1} \), and \( k_2 = K_2 = 4 \). After introducing (53) as new input \( v_2 = v_2^1 \), the forward-shifts of the flat output (49) up to the orders \( \kappa = (k_1, k_2) = (2,4) \) are given by

\[
y_{1,[0,k_1-1]} = \begin{bmatrix} \varphi_{1,[2]} \\ \varphi_{1,[1]} \end{bmatrix} = \begin{bmatrix} q^2 \\ q^2 + T\omega^2 \end{bmatrix} \\
y_{1,[k_1]} = \varphi_{1,[2]} = v_1^1 \\
y_{2,[0,k_2-1]} = \begin{bmatrix} \varphi_{2,[1]} \\ \varphi_{2,[2]} \\ \varphi_{2,[3]} \end{bmatrix} = \begin{bmatrix} q^1 \\ q^1 + T\omega^1 \\ \varphi_{2,[2]}(q^1, q^2, q^3, \omega^1, \omega^2, v_1^1) \\ \varphi_{2,[3]}(q^1, q^2, q^3, \omega^1, v_1^1, v_{1,[1]}^1) \end{bmatrix} \\
y_{2,[k_2]} = \varphi_{2,[4]} = v_2^1.
\]

(54)

Substituting (54) as well as \( y_{1,[3]}^1 = v_{1,[1]}^1 \) and \( y_{1,[4]}^1 = v_{1,[2]}^1 \) into the parameterisation (51) of the control input \( u \) yields a feedback of the form

\[
\begin{bmatrix} u \\ v \end{bmatrix} = F_u \circ \phi(q^1, q^2, q^3, \omega^1, \omega^2, \omega^3, v_{1,[1]}^1, v_{1,[2]}^1, v_{1,[3]}^1) \\
\]

such that the input-output behaviour of the closed-loop system is given by

\[
y_{1,[2]}^1 = v_1^1 \\
y_{2,[4]}^1 = v_2^1.
\]

(56)

Thus, in contrast to (52), we can actually use the second instead of the fourth forward-shift of the first component of the flat output as a new input. Since the linearising feedback (55) does not depend on backward-shifts of the system variables, it is a discrete-time quasi-static feedback in the sense of Aranda-Bricaire and Kotta (2001) – see the discussion after Corollary 4.6.

For the exactly linearised system (56), the design of a tracking control is again straightforward. The control law

\[
v_1^1 = y_{1,[2]}^1 - a_{1,1}^1(y_{1,[1]}^1 - y_{1,[1]}^d) - a_{1,0}^1(y_1^1 - y_{1,d}^1) \\
v_2^1 = y_{2,[4]}^1 - \sum_{\beta=0}^3 a_{2,\beta}^1(y_{2,[\beta]}^1 - y_{2,[\beta]}^d)
\]

results in the linear tracking error dynamics

\[
e_{1,[2]}^1 + a_{1,1}^1 e_{1,[1]}^1 + a_{1,0}^1 e_1^1 = 0 \\
e_{2,[4]}^1 + \sum_{\beta=0}^3 a_{2,\beta}^1 e_{2,[\beta]}^1 = 0.
\]

with a total order of \( \#\kappa = 2 + 4 = 6 \) instead of \( \#R = 4 + 4 = 8 \).

The system of equations (35) is given by

\[
v_1^1 = y_{1,[2]}^1 - a_{1,1}^1(y_{1,[1]}^1 - y_{1,[1]}^d) - a_{1,0}^1(y_1^1 - y_{1,d}^1) \\
v_{1,[1]}^1 = y_{1,[3]}^1 - a_{1,1}^1(y_1^1 - y_{1,[2]}^d) - a_{1,0}^1(y_{1,[1]}^1 - y_{1,[1]}^d) \\
v_{1,[2]}^1 = y_{1,[4]}^1 - a_{1,1}^1(y_1^1 - y_{1,[3]}^d) - a_{1,0}^1(y_{1,[1]}^1 - y_{1,[2]}^d) \\
v_2^1 = y_{2,[4]}^1 - \sum_{\beta=0}^3 a_{2,\beta}^1(y_{2,[\beta]}^1 - y_{2,[\beta]}^d)
\]

with the functions \( \varphi_{1,[1]}^1, \varphi_{1,[2]}^1, \varphi_{1,[3]}^1, \varphi_{2,[1]}^1, \varphi_{2,[2]}^1, \varphi_{2,[3]}^1 \) according to (54), and can be solved from top to bottom for \( v_1^1, v_{1,[1]}^1, v_{1,[2]}^1 \),...
and $\nu_2^1$ as a function of $q^1, q^2, q^3, o^1, o^2, o^3$ and the reference trajectory $y_{y_0}^1$. Substituting the solution into the linearising feedback (55) finally yields a control law of the form

$$u^1 = \eta^1(q^1, q^2, q^3, o^1, o^2, o^3, y_{1,[0,4]}, y_{2,[0,4]}),$$

$$u^2 = \eta^2(q^1, q^2, q^3, o^1, o^2, o^3, y_{1,[0,4]}, y_{2,[0,4]}).$$

7. Conclusion

In this contribution we have investigated the exact linearisation of flat discrete-time systems. Since an exact linearisation can always be achieved by choosing the highest forward-shifts of the flat output in (9) as new input $v = y_{[A]}$, the point of departure of our considerations was the question whether also lower-order forward-shifts $v = y_{[A]}$ with $A \leq R$ can be used. Similar to the continuous-time case, this allows e.g. to achieve a lower-order error dynamics for a subsequently designed tracking control. Concerning the choice of a feasible new input $v = y_{[A]}$, we have derived conditions which are formulated in terms of the linear independence of certain differentials and can be checked in a straightforward way. Furthermore, we have shown how the new input $v$ can be introduced by a suitable dynamic feedback. For the practically quite important case of flat outputs (24) which do not depend on future values of the control input, we have shown how to construct a minimal multi-index $\kappa$ such that $v = y_{[\kappa]}$ is a feasible input and $\#\kappa \leq \# A$ for all other feasible inputs $v = y_{[A]}$. Such an input $v = y_{[\kappa]}$ can be introduced by a feedback (30) which depends only on the state $x$ as well as past values of the system variables. This is particularly convenient for an implementation, since past values of the system variables are available anyway from past measurements or past inputs, which only need to be stored. Moreover, we have shown that such an exact linearisation can be used as a basis for the design of a tracking control law which again only depends on $x$ as well as past values of the system variables and the reference trajectory. To illustrate our results, we have computed tracking control laws for the discretised models of a wheeled mobile robot and a 3DOF helicopter.

Notes

1. The feedback (13) is actually a discrete-time quasi-static feedback, see e.g. Aranda-Bricaire and Kotta (2001).

2. Note that because of (3) the quantities $\xi$ are functions of $x$ and $u$. Hence, their parameterisation by the flat output (7) can be obtained immediately from (9).

3. If the parameterisation (9) of the system variables is substituted into (1), then the equations are satisfied identically.

4. Since the manifold $\eta_{x,[1]} \times X \times U_{[A]}$ is finite-dimensional, in fact some of the higher-order forward-shifts of $u$ cannot be replaced by forward-shifts of the flat output and must be kept as coordinates (unless the system (1) is static feedback linearisable and (24) a linearising output).

5. The functions $\varphi_{x,[1]}$ depend on forward-shifts of $v_1$, and since we work on a finite-dimensional manifold $\eta_{x,[1]} \times X \times U_{[A]}$ these forward-shifts are only available up to the order $l_0$. Thus, some higher-order forward-shifts of $w_2$ must be kept as coordinates on $\eta_{x,[1]} \times X \times U_{[A]}$ and cannot be replaced by forward-shifts of $v_2$. However, as long as $l_0$ is chosen sufficiently large, this does not affect our considerations.

6. In contrast to Section 4.2, for the sake of simplicity we do not renumber the components $\nu_1^1$ and $\nu_2^2$ of the input.

7. To emphasise that $v_1$ could in general consist of more than one component we write $\nu_1^1$.

8. The components of the flat output are already sorted in such a way that we do not need a renumbering during our calculations.

9. Neither an exact nor an approximate discretisation necessarily preserve the flatness or the static feedback linearisability of a continuous-time system, see e.g. Diwold et al. (2022b), Diwold et al. (2022a), or Grizzle (1986) for a further discussion.

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ORCID

Bernd Kolar http://orcid.org/0000-0001-9710-8445

Johannes Diwold http://orcid.org/0000-0001-7878-4819

Conrad Götßtner http://orcid.org/0000-0003-2107-3009

Markus Schöberl http://orcid.org/0000-0001-5539-7015

References

Aranda-Bricaire, E., & Kotta, U. (2001). Generalized controlled invariance for discrete-time nonlinear systems with an application to the dynamic disturbance decoupling problem. IEEE Transactions on Automatic Control, 46(1), 165–171. https://doi.org/10.1109/9.898712

Aranda-Bricaire, E., & Moog, C. (2008). Linearization of discrete-time systems by exogenous dynamic feedback. Automatica, 44(7), 1707–1717. https://doi.org/10.1016/j.automatica.2007.10.030

Delaleau, E., & Rudolph, J. (1998). Control of flat systems by quasi-static feedback of generalized states. International Journal of Control, 71(5), 745–765. https://doi.org/10.1080/00207179.798221551

Diwold, J., Kolar, B., & Schöberl, M. (2022a). Discrete-time flatness-based control of a gantry crane. Control Engineering Practice, 119, Article ID 104980. https://doi.org/10.1016/j.conengprac.2021.104980

Diwold, J., Kolar, B., & Schöberl, M. (2022b). A trajectory-based approach to discrete-time flatness. IEEE Control Systems Letters, 6, 289–294. https://doi.org/10.1109/LCSYS.2021.3071177

Fliess, M., Lévine, J., Martin, P., & Rouchon, P. (1995). Flatness and defect of non-linear systems: Introductory theory and examples. International Journal of Control, 61(6), 1327–1361. https://doi.org/10.1080/00207179.1995.950892159

Fliess, M., Lévine, J., Martin, P., & Rouchon, P. (1999). A Lie–Bäcklund approach to equivalence and flatness of nonlinear systems. IEEE Transactions on Automatic Control, 44(5), 922–937. https://doi.org/10.1109/9.763209

Grizzle, J. (1986). Feedback linearization of discrete-time systems. In A. Bensoussan & J. Lions (Eds.), Analysis and optimization of systems (Vol. 83, pp. 273–281). Springer.

Götßtner, C., Kolar, B., & Schöberl, M. (2021). Control of $(x,u)$-flat systems by quasi-static feedback of classical states. arXiv e-prints. arXiv:2110.12995 [math.OC].

Götßtner, C., Kolar, B., & Schöberl, M. (2021b). Necessary and sufficient conditions for the linearity of two-input systems by a two-dimensional endogenous dynamic feedback. International Journal of Control. https://doi.org/10.1080/00207179.2021.2015542

Guillot, P., & Millérioux, G. (2020). Flatness and submersivity of discrete-time dynamical systems. IEEE Control Systems Letters, 4(2), 337–342. https://doi.org/10.1109/LCSYS.2020.3047863

Kaldmäe, A. (2021). Algebraic necessary and sufficient condition for difference flatness. International Journal of Control. https://doi.org/10.1080/00207179.2021.1908598

Kaldmäe, A., & Kotta, U. (2013). On flatness of discrete-time nonlinear systems. In Proceedings 9th IFAC symposium on nonlinear control systems (pp. 588–593).
Appendix. Supplements

The following lemma addresses the relation between the functional independence of functions and the linear independence of their differentials.

Lemma A.1: Consider a set of smooth functions \(g^1, \ldots, g^k\) as well as another smooth function \(h\) which are all defined on the same manifold. Then

\[ dh \in \text{span}\{dg^1, \ldots, dg^k\} \quad (A1) \]

is equivalent to the existence of a smooth function \(\psi: \mathbb{R}^k \mapsto \mathbb{R}\) such that locally

\[ h = \psi(g^1, \ldots, g^k) \quad (A2) \]

holds identically. If the differentials \(dg^1, \ldots, dg^k\) are linearly independent, then the function \(\psi\) is unique.

Proof: Let \(l \leq k\) denote the maximal number of linearly independent differentials from the set \(\{dg^1, \ldots, dg^k\}\), and assume that these differentials are given by \(dg^1, \ldots, dg^l\) (which is always possible by a renumbering). Then the functions \(g^1, \ldots, g^l\) can be introduced as (a part of the) coordinates

\[ z^i = g^i, \quad i = 1, \ldots, l \]

on the considered manifold. Moreover, by construction, also the functions \(g^{l+1}, \ldots, g^k\) can depend only on the coordinates \(z\). With such coordinates, (A1) is equivalent to

\[ dh \in \text{span}\{dz^1, \ldots, dz^l\}. \quad (A3) \]

Thus, the function \(h\) can only depend on \(z^1, \ldots, z^l\), and hence be written in original coordinates as

\[ h = \psi(g^1, \ldots, g^l). \]

Conversely, it is clear that

\[ h = \psi(z^1, \ldots, z^l, g^{l+1}(z), \ldots, g^k(z)) \]

implies (A3) and hence (A1). In the case \(l = k\), all functions \(g^1, \ldots, g^k\) can be introduced as new coordinates, and no choice of \(l\) independent ones is necessary. Thus, the representation (A2) is then unique.