A direct numerical method for solving inverse heat source problems

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Abstract. Inverse source problems is a classical ill-posed problem. There exist several literatures on inverse heat source problems. In this study we use some stable methods to deal with the inverse problems of determining space-wise and time-wise dependent heat source in the heat diffusion equation. The problems of inverse heat source sources are transferred to the problems of numerical differentiation. By simple and effective methods which are used to regularize the problems of numerical differentiation before, we can obtain stable solutions. The numerical results show that our proposed procedure yields stable and accurate approximation.

1. Introduction
In this paper, we consider the problems of identifying the heat source from measured data for heat equation. This model has many important applications in practice, e.g., in finding a pollution source intensity and also for designing the final state in melting and freezing processes. The problem is ill-posed and any small change in the input data may result in a dramatic change in the solution. To obtain a stable numerical solution for these kinds of ill-posed problems, some regularization strategies should be applied.

The problems of identification of heat source have been discussed by many authors. For theoretical papers, we can refer to the references [3, 4, 5, 13, 17, 18, 19, 22]. For computational papers, we can refer to the references [2, 8, 9, 10, 14, 15, 23] and the references therein.

Recently, in the case of space-wise dependent heat source, Johansson and Lesnic proposed an iterative method [11] and a variational method [12] for solving the problem of this kind numerically. However, these methods need to solve a direct problem at each iterative step. The computational cost is very large. In the case of time-wise dependent heat source, Yan et al in [23] provided a fundamental solution method for solving the problem. The method has one drawback: it needs source points outside the domain which are not easy to place properly.
In this paper, we present an alternative method for solving these problems. The method is simple and effective. The main aim of this paper is to provide a new method for solving the problems of unknown heat source. Furthermore, our method can be extended to the case of higher dimensions easily.

2. The inverse problem of space-wise dependent heat source
We consider the following inverse problem: find the temperature $u$ and the heat source $f$ which satisfy the heat conduction equation, namely

$$u_t(x,t) = u_{xx}(x,t) + f(x), \quad 0 < x < 1, \quad 0 < t < T,$$

with initial data and boundary conditions

$$u(x,0) = u_0(x), \quad 0 < x < 1$$

$$u(0,t) = s(t), \quad u(1,t) = l(t), \quad 0 < t < T,$$

where we need to reconstruct $f(x)$ from the over-specified data $u(x,T) = g(x)$. Assume that the given functions satisfy the compatibility conditions

$$u_0(0) = s(0), \quad u_0(1) = l(0),$$

$$g(0) = s(T), \quad g(1) = l(T).$$

One may ask a question: How much is the degree of ill-posedness of this problem? In order to answer this question, for convenience of analysis, let $s(t) = l(t) = 0$. Here we use finite difference method to give the answer in Subsection 2.1.

2.1. Analysis on degree of ill-posedness
First we divide the domain $(x,t) \in [0,1] \times [0,T]$ into several sub-domains, i.e. $x_1 = 0 < x_2 < \cdots < x_i = (i-1)h < \cdots < x_{n+1} = 1$, $t_1 = 0 < \cdots < t_i = (i-1)\Delta t < \cdots < t_{m+1} = T$, where $h = 1/n$, $\Delta t = T/m$. Let $u_t(x_i,t_j) \approx \frac{1}{\Delta t} (u_{i+1}^{j+1} - u_i^j)$, $u_{xx}(x_i,t_j) \approx \frac{1}{h^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j)$, where $u_i^j = u(x_i,t_j)$, let $f(x_i) = f_i$, $g(x_i) = g_i$, the equation (1)-(3) can be discretized as

$$\frac{1}{\Delta t} (u_{i+1}^{j+1} - u_i^j) - 1/h^2 (u_{i+1}^j - 2u_i^j + u_{i-1}^j) = f_i,$$

Let $r = \Delta t/h^2$, then

$$u_{i+1}^{j+1} = ru_{i+1}^j + (1-2r)u_i^j + ru_{i-1}^j + \Delta tf_i,$$

we can rewrite the above equation as

$$U_{i+1}^j = AU_i^j + \Delta t F,$$

where $U_i^j = [u(x_2,t_j), \cdots, u(x_n,t_j)]^tr$, $G = [g_2, \cdots, g_n]^tr$, the superscript $tr$ denotes the transposition of a matrix,

$$A = \begin{pmatrix}
  1 - 2r & r & r & \cdots & r \\
  r & 1 - 2r & r & \cdots & r \\
  r & r & 1 - 2r & \cdots & r \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  r & r & r & \cdots & 1 - 2r
\end{pmatrix}_{(n-1)\times(n-1)}.$$
By induction, we have

\[ U^2 = AU^1 + \Delta t G \]
\[ U^3 = AU^2 + \Delta t G = A[AU^1 + \Delta t G] + \Delta t G \]
\[ = A^2 U^1 + \Delta t AG + \Delta t G \]
\[ \vdots \]
\[ U^{m+1} = A^m U^1 + (A^{m-1} + A^{m-2} + \cdots + A + I)\Delta t F. \]

If \( r < 1/2 \), then \( \|A\|_2 < 1, \ A^{m-1} + A^{m-2} + \cdots + A + I = (I - A)^{-1}(I - A^m) \), Noting that \( U^{m+1} = G \), we have

\[ G = A^m U^1 + (I - A)^{-1}(I - A^m)\Delta t F. \] (8)

Denote \( K = (I - A)^{-1}(I - A^m)\Delta t, \) we can get

\[ KF = G - A^m U^1. \] (9)

As \( A \) is a symmetric matrix, we can easily obtain its eigenvalues

\[ \lambda_j = 1 - 4r \sin^2\left(\frac{\pi j}{2}h\right), \ 1 \leq j \leq n - 1, \] (10)

then the eigenvalues of \( K \) are

\[ \mu_j = \left[ (1 - \lambda_j^m)/(1 - \lambda_j) \right] \Delta t, \ 1 \leq j \leq n - 1. \]

By a simple calculation,

\[ \mu_j = \frac{1 - [1 - 4r \sin^2(\frac{\pi j}{2}h)]^m}{4r \sin^2(\frac{\pi j}{2}h)} \Delta t, \ 1 \leq j \leq n - 1. \]

Recall that \( r = \Delta t/h^2 = n^2/m \), and \( r \) is fixed and satisfies \( r \leq 1/2 \). If \( c > 0, \) \( n \) is large sufficiently, we have \( (1 - c/n)^n \approx \exp(-c) \), \( \sin(\frac{\pi j}{2}h) \approx j \frac{\pi}{2}h \), then

\[ \mu_j \approx \frac{1 - [1 - 4r j^2 \frac{\pi^2}{4m^2}]^{n^2/r}}{4r j^2 \frac{\pi^2}{4m^2}} \frac{r}{n^2}, \ 1 \leq j \leq n - 1. \]

We get

\[ \mu_j \approx \frac{1 - e^{-n^2 j^2}}{j^2 \pi^2}, \ 1 \leq j \leq n - 1. \] (11)

If \( j \) is large enough, then

\[ \mu_j \approx \frac{1}{j^2 \pi^2}, \ 1 \leq j \leq n - 1. \] (12)

From above formula, we can conclude that the degree of the problem (1)-(3) is equivalent to that of second-order numerical differentiation.
2.2. A simple method for the inverse problem

In this subsection, we assume that \( s(t) \) and \( l(t) \) in (1)-(3) belong to \( C^1(0, T) \), and \( g(x) \), \( u_0(x) \) belong to \( C^2(0, 1) \).

Differentiating Equation (1) with respect to \( t \), it yields

\[
  u_{tt} = u_{xxt},
\]

let \( w = u_t \), and

\[
  u(x, t) = \int_0^t w(x, \tau)d\tau + u_0(x).
\]

Thus, (13) becomes

\[
  w_t = w_{xx},
\]

with boundary conditions

\[
  w(0, t) = s'(t), \quad w(1, t) = l'(t).
\]

Integrating Equation (13) with respect to \( t \) in the interval \([0, T]\), it is obtained

\[
  \int_0^T u_{tt} dt = \int_0^T u_{xxt} dt,
\]

i.e.

\[
  w(x, T) - w(x, 0) = g''(x) - u''_0(x).
\]

Integrating Equation (13) with respect to \( t \) in the interval \([t, T]\), we obtain

\[
  u_t(x, t) = u_{xx}(x, t) + u_t(x, T) - g''(x).
\]

Therefore, we can get

\[
  f(x) = w(x, T) - g''(x).
\]

Since \( g(x) \) is measured data and there exist measurement errors, i.e. only the noisy data \( g^\delta(x) \) is available. This involves an ill-posed problem of second-order numerical differentiation. Hence we should deal with the second-order numerical differentiation by a stable approximation method firstly. The problem of numerical differentiation is discussed by several authors, we can consult the references [1, 16, 20, 21]. In the sequent section, we use the Tikhonov regularization method to approximate the second-order derivative of noisy data \( g^\delta \). Minimize the following functional

\[
  H_\alpha(g^\delta_\alpha) = \int_0^1 (g^\delta_\alpha(x) - g^\delta(x))^2 dx + \alpha \int_0^1 \left( \frac{d^2 g^\delta_\alpha(x)}{dx^2} \right)^2 dx,
\]

and we can get the minimizer \( g^\delta_\alpha(x) \). Thus we have a stable approximation \((g^\delta_\alpha)''(x)\) to the second-order derivative \( g''(x) \).

Now let us summarize what we have:

- **Step 1.** Use the Tikhonov method (or other methods such as central difference method) to stably approximate the second-order derivative \( g'' \).
- **Step 2.** Numerically solve the following problem:

\[
  \begin{align*}
    w_t(x, t) &= w_{xx}(x, t), \\
    w(0, t) &= s'(t), \quad w(1, t) = l'(t),
  \end{align*}
\]

and the following condition

\[
  w(x, T) - w(x, 0) = g''(x) - u''_0(x).
\]

- **Step 3.** Compute the heat source \( f(x) \) according to (20).
2.3. Generalization to two-dimensional case

We consider the following two-dimensional problem

\[ u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) + f(x, y), \ 0 < x < 1, 0 < y < 1, 0 < t < T, \]  

(24)

with initial data and boundary conditions

\[ u(x, y, 0) = \varphi(x, y), \ 0 < x < 1, 0 < y < 1, \]  

(25)

\[ u(0, y, t) = f_1(y, t), \ u(x, 0, t) = f_2(x, t), \ 0 < t < T, \]  

(26)

\[ u(1, y, t) = f_3(y, t), \ u(x, 1, t) = f_4(x, t), \ 0 < t < T, \]  

(27)

where the symbol \( a' \) denotes differentiating the function \( a \) with respect to \( t \) and the following condition

\[ w(x, y, T) - w(x, y, 0) = g_{xx}(x, y) + g_{yy}(x, y) - \varphi_{xx}(x, y) - \varphi_{yy}(x, y). \]  

(29)

- Step 3. Compute the heat source \( f(x, y) \) according to

\[ f(x, y) = w(x, y, T) - g_{xx}(x, y) - g_{yy}(x, y). \]  

(30)

3. The inverse problem of time-wise dependent heat source

3.1. A direct method

Similar to Section 2, we consider the following inverse problem: find the temperature \( u \) and the heat source \( f \) which satisfy the heat conduction equation, namely

\[ u_t(x, t) = u_{xx}(x, t) + f(t), \ 0 < x < 1, 0 < t < T, \]  

(31)

with initial data and boundary conditions

\[ u(x, 0) = u_0(x), \ 0 < x < 1, \]  

(32)

\[ u_x(0, t) = s(t), \ u_x(1, t) = l(t), \ 0 < t < T. \]  

(33)

We want to reconstruct \( f(t) \) from the over-specified data \( u(x_f, t) = g(t) \) where \( x_f \) is a fixed point inside \([0, 1] \). Assume that the given functions satisfy the compatibility conditions

\[ (u_0)_x(0) = s(0), \ (u_0)_x(1) = l(0), \ g(0) = u_0(x_f). \]  

(34)

Furthermore, we assume that \( s(t) \) and \( l(t) \) in (33) belong to \( C(0, T) \) and \( g(t) \) belongs to \( C^1(0, T) \), and \( u_0(x) \) belongs to \( C^1(0, 1) \).
Differentiating Equation (31) with respect to $x$, it yields

$$u_{tx} = u_{xxx}, \quad (35)$$

let $w = u_x$, and

$$u(x, t) = \int_{x_f}^{x} w(y, t) \, dy + g(t). \quad (36)$$

Thus, (35) becomes

$$w_t = w_{xx}, \quad (37)$$

with initial and boundary conditions

$$w(x, 0) = (u_0)_x(x), \quad w(0, t) = s(t), \quad w(1, t) = l(t). \quad (38)$$

Integrating Equation (35) with respect to $x$ in the interval $[x_f, x]$, it is obtained

$$\int_{x_f}^{x} u_{tx} \, dx = \int_{x_f}^{x} u_{xxx} \, dx, \quad (39)$$

i.e.

$$u_t(x, t) = u_{xx}(x, t) + g'(t) - u_{xx}(x_f, t). \quad (40)$$

Therefore, we can get

$$f(t) = g'(t) - u_{xx}(x_f, t). \quad (41)$$

Since $g(t)$ is measured data and there exist measurement errors, i.e. only the noisy data $g^\delta(t)$ is available. This involves an ill-posed problem of first-order numerical differentiation. It is well-known that the step lengths in the standard numerical method such as finite difference method have regularization effect. Hence we can choose appropriate step lengths to compute the first-order derivative stably.

**Remark.** Alternatively the first-order numerical differentiation can be solved by other stable approximation methods. For example, we can use the Tikhonov regularization method to approximate the first-order derivative of noisy data $g^\delta$. Minimize the following functional

$$H_\alpha(g^\delta_\alpha) = \int_{0}^{T} (g^\delta_\alpha(t) - g^\delta(t))^2 \, dt + \alpha \int_{0}^{T} (\frac{dg^\delta_\alpha}{dt}(t))^2 \, dt \quad (42)$$

and we can get the minimizer $g^\delta_\alpha(t)$. Thus we have a stable approximation $(g^\delta_\alpha)'(t)$ to the first-order derivative $g'(t)$.

Now let us summarize what we have:

- **Step 1.** Use the central difference formula to stably approximate the first-order derivative $g'$ with appropriate step lengths.
- **Step 2.** Numerically solve the following problem to get $w_x(x_f, t)$:

$$
\begin{align*}
w_t(x, t) &= w_{xx}(x, t), & 0 < x < 1, & 0 < t < T, \\
w(0, t) &= (u_0)_x(x), & 0 < x < 1, \\
w(0, t) &= s(t), & w(1, t) = l(t), & 0 < t < T.
\end{align*} \quad (43)
$$
- **Step 3.** Compute the heat source $f(t)$ according to (41) by noting that $w = u_x$. 

6
3.2. Generalization to two-dimensional case
Consider the following two-dimensional problem

\[ u_t(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) + f(t), \quad 0 < x < 1, 0 < y < 1, 0 < t < T, \quad (44) \]

with initial data and boundary conditions

\[ u(x, y, 0) = \varphi(x, y), \quad 0 < x < 1, 0 < y < 1, \quad (45) \]
\[ u_x(0, y, t) = f_1(y, t), \quad u_y(x, 0, t) = f_2(x, t), \quad 0 < t < T, \quad (46) \]
\[ u_x(1, y, t) = f_3(y, t), \quad u_y(x, 1, t) = f_4(x, t), \quad 0 < t < T, \quad (47) \]

here we need to reconstruct \( f(t) \) from the over-specified data \( u(\bar{x}, \bar{y}, t) = g(t) \) where \( (\bar{x}, \bar{y}) \in (0, 1) \times (0, 1) \). Similarly, from the transform \( v(x, y, t) = u(x, y, t) - \int_0^t f(\tau) d\tau \), we have the following algorithm:

- Step 1. Use the central difference formula to stably approximate the first-order derivative \( g' \) with appropriate step sizes.
- Step 2. Numerically solve the following problem to get \( v(\bar{x}, \bar{y}, t) \):

\[
\begin{align*}
v_t(x, y, t) &= v_{xx}(x, y, t) + v_{yy}(x, y, t), \quad 0 < x < 1, 0 < y < 1, 0 < t < T, \\
v(x, y, 0) &= \varphi(x, y), \quad 0 < x < 1, 0 < y < 1, \\
v_x(0, y, t) &= f_1(y, t), \quad v_x(1, y, t) = f_3(y, t), \quad 0 < y < 1, 0 < t < T, \\
v_y(x, 0, t) &= f_2(x, t), \quad v_y(x, 1, t) = f_4(x, t), \quad 0 < x < 1, 0 < t < T.
\end{align*}
\quad (48)
\]

- Step 3. Compute the heat source \( f(t) \) according to

\[ f(t) = g'(t) - v_t(\bar{x}, \bar{y}, t). \quad (49) \]

4. Numerical examples
In this section, we implement the numerical algorithm by using Matlab in IEEE double precision with unit round off \( 1.1 \cdot 10^{-16} \) and use uniform grids in finite difference methods. Due to the similarity, we only give two numerical examples for the case of time-wise dependent heat source.

In the following numerical experiment, the problem (43) in Step 2 is solved by the explicit forward Euler method which is the simplest method and requires that \( \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \) for the reason of stability, where \( \Delta t \) and \( \Delta x \) are the time step length and space step length, respectively.

In this experiment, in order to investigate the stability of the numerical solution, the data \( g(t) \) is perturbed as

\[ G^\delta = G + \epsilon \cdot \text{randn(size}(G)), \quad (50) \]

where the \text{randn} function generates arrays of random numbers whose elements are normally distributed with mean 0, variance \( \sigma^2 = 1 \), and standard deviation \( \sigma = 1 \), \( G \) is the column vector discretized by \( g(t) \) at grid points, \( \epsilon \) is the noise level.

To test the accuracy of the approximate solution, we use the root mean square error (RMSE) and the relative root mean square error (RSE), which are defined as follows:

\[ \text{RMSE}(f) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (f(x_i) - f_c(x_i))^2}; \quad (51) \]
\[ RSE(f) = \sqrt{\frac{\sum_{i=1}^{N} (f_c(t_i) - f_e(t_i))^2}{\sum_{i=1}^{N} (f_e(t_i))^2}}, \]  

where \( N \) is the total number of test points in \([0, T]\), \( f_c \) and \( f_e \) represent the computed heat source and the exact heat source, respectively.

**Example 1.** We try to reconstruct the heat source defined by
\[ f(t) = 2\pi \cos(2\pi t), \quad t \in (0, 1). \]  
If we consider boundary conditions
\[ u_x(0, t) = 0 = s(t) \text{ and } u_x(1, t) = 2 = l(t) \text{ for } t \in [0, 1]. \]  
The initial condition is given by
\[ u(x, 0) = x^2 \text{ for } x \in [0, 1]. \]  
Then in this case the forward problem given by (31)-(33) with \( f \) given by (53) has the exact solution
\[ u(x, t) = x^2 + 2t + \sin 2\pi t, \text{ for } (x, t) \in [0, 1] \times [0, 1]. \]  
Based on (56), we can obtain the data at \( x_f = \frac{1}{2} \) given by
\[ g(t) = \frac{1}{4} + 2t + \sin 2\pi t, \text{ for } t \in [0, 1]. \]  

Figure 1-a shows the reconstruction effect with \( M = 6, N = 80 \) and \( \epsilon = 0.01 \), where \( M \) is the total number of grid points at \( x \)-axis and \( N \) is the total number of grid points at \( t \)-axis. Figure 1-b shows the approximation effect of \( f(t) \) with \( M = 10, N = 300 \) and \( \epsilon = 0.001 \). In the experiment, we find that the approximation effect varies with the time step length \( 1/N \) if \( \epsilon \) is fixed. This shows that the time step length has the effect of regularization.

**Example 2.** In this example, we want to reconstruct the heat source given by
\[ f(t) = \begin{cases} 
0, & 0 \leq t < \frac{1}{4}, \\
\frac{1}{2}, & \frac{1}{4} \leq t < \frac{1}{2}, \\
1, & \frac{1}{2} \leq t \leq 1. 
\end{cases} \]
with the initial and boundary conditions

\[ u_0(x) = x^2, \quad u_x(1, t) = u_x(0, t) = 0, \quad 0 \leq x \leq 1, \quad 0 \leq t \leq 1. \]  

(59)

In this case, we don’t have the exact solution \( u(x, t) \). We solve the direct problem numerically to get \( u(x_f, t) \), then we add random noise to the data \( g(t) := u(x_f, t) \), finally we reconstruct the heat source from the noisy data. The parameters in this example are given as follows: \( M = 6, \quad N = 80 \) and \( \epsilon = 0.001, \quad x_f = 1/2 \). The numerical results are shown as Figure 2. The large oscillation near zero in Fig. 2-b is caused by the incompatibility of the data \( g(0) = u_0(x_f) \) in the Eq. (34).

5. Conclusions

In this paper we can see that degree of ill-posedness of the problems of unknown time-wise and space-wise dependent heat source are as the same as that of the problem of first-order and second-order numerical differentiation, respectively. For sideways heat equation, as Eldén pointed in the References [6, 7], the ill-posedness is caused by time derivative and the time step length has regularization effect. Therefore, by choosing appropriate step lengths, we can make the direct numerical method work well with moderate noise levels. Furthermore, we can conclude that our method can be generalized to higher dimensional case. Further work should be considered, for example, how to reconstruct the style of heat source such as \( f(x, t) \) in (1)-(3)? This problem will be discussed in the forthcoming paper.

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