Heavy-light Bootstrap from Lorentzian Inversion Formula

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ABSTRACT

We study heavy-light four-point function by employing Lorentzian inversion formula, where the conformal dimension of heavy operator is as large as central charge \( C_T \rightarrow \infty \). Implementing Lorentzian inversion formula back and forth reveals the universality of lowest-twist multi-stress-tensor \( T^k \) as well as large spin double-twist operators \([O_H O_L]_{n',J'}\). In this way, an algorithm is proposed to bootstrap heavy-light four-point function with extracting relevant OPE coefficients and anomalous dimensions. Following the algorithm, examples of \( d = 4 \) are exhibited up to triple-stress-tensor, and moreover, general dimensional heavy-light bootstrap up to double-stress-tensor is discussed with ending up presenting an infinite series representation of lowest-twist double-stress-tensor OPE coefficient. Exact expressions of lowest-twist double-stress-tensor OPE coefficients in \( d = 6, 8, 10 \) are also obtained as further examples.

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1 Introduction

AdS/CFT correspondence (holography) serves as a bridge connecting gravity theories in anti-de Sitter (AdS) spacetime and strong-coupled CFT living in the AdS boundary \[1-3\], enabling us to exploit conformal field theories (CFT) with sparse spectrum \[4\] at strong coupling without counting on any specific CFT theories. From AdS side, investigations of Witten diagrams could enlighten us the organization and universality associated with CFT correlation functions. On the other hand, although directly studying strongly-coupled CFT is a hard task, recent development of conformal bootstrap makes it achievable. Without referring to any Lagrangian, conformal bootstrap focuses on conformal symmetry itself \[5\] combined with crossing symmetry \[6-10\], unitarity \[11,12\] and other physical consistency conditions, from which properties of conformal dimensions and operator product expansion (OPE) coefficients can be efficiently explored. In turn, the progress of strongly-coupled CFT can be expected to shed a light on some essential aspects of quantum gravity.

In parallel to numerical bootstrap which aims to precisely constrain and even determine those CFT data for numerous specific models like Ising model (see \[13\] for a recent review), analytic bootstrap has been developed to probe universality of CFT data provided with some parametric limit. By analyzing the singularities from crossing symmetry near light-cone limit, the universal spectrum and OPE coefficients of large spin operators were understood \[8,9\]. This progress boosted the large spin perturbation theory \[10,14,16\], those universal spectrum can be expanded as inverse of spin \(1/J\), which surprisingly remains intact down to finite spin \[17,18\]. This incredible validity can be explained by analyticity in spin which was made manifest by Caron-Huot Lorentzian inversion formula \[19-21\]. The Lorentzian inversion formula encapsulates the large spin systematics and allows us to compute OPE coefficients and anomalous dimensions more efficiently even with finite spin \[22,23\].

Naturally, Lorentzian inversion formula was applied to investigate quantum gravity and AdS/CFT, for example, it allows us to study correlators up to loop level in supergravity \[24,25\] and to understand the growth of extra dimension in AdS/CFT \[26\]. However, these explorations only involve pure AdS and do not include any heavy states. Undoubtedly, four-point functions with heavy states, i.e. heavy-light four-point functions \(O_H O_H O_L O_L\) are interesting and important aspects both in pure CFT and AdS/CFT. In fact, the knowledge of heavy-light four-point functions is essential for understanding of various topics, e.g. information loss and black hole collapse \[27,30\], entanglement entropy \[31,34\] and chaos \[35\], which are well-understood in AdS\(_3\)/CFT\(_2\) thanks to Virasoro symmetry in CFT\(_2\). Roughly speaking, at large central charge limit \(C_T \to \infty\), the heavy-light four-point func-
tion (conformal dimension of heavy operator is heavy as $\Delta_H \sim C_T$ while $\Delta_L \ll C_T$) is captured by Virasoro block in the channel $O_H O_H O_{\Delta,J} \times O_{\Delta,J} O_L O_L$ where identity 1 and all exchanged multi-stress-tensors $T^m$ are packaged universally \[28\] \[31\] \[36\] \[37\]. However, the Virasoro symmetry is not available in $d \geq 3$ CFT. Studying heavy-light four-point functions in $d \geq 3$ CFT is thus necessary.

By crossing symmetry, it would be probably simpler to investigate $O_H O_L O_L O_H$ with exchanging double-twist operators $[O_H O_L]_{n,J}$ in channel $O_H O_L O_{\Delta,J} \times O_{\Delta,J} O_L O_H$ firstly. Holographically, the underlying exchanged operators in this channel were studied considerably recently by either computing the bulk phase shift \[35\] \[41\] or adopting Hamiltonian perturbation theory \[16\] \[28\] in \[42\] \[43\]. In parallel, to search for universality of multi-stress-tensor $T^m$ OPE coefficients in high dimensions similar to CFT$_2$, \[44\] initiated holographic studies as hints, and it is evidently that the lowest-twist multi-stress-tensor OPE coefficients exhibit universality by only depending on $\Delta_H, \Delta_L$ and $C_T$. However, a CFT origin of this universality is not clear. By studying stress-tensor commutation relation without calling holography, \[45\] shows that Virasoso-like structure indeed exists at light-cone limit. Before long, lowest-twist double-stress-tensor OPE coefficients were conjectured in \[46\] by equating two channels provided with borrowing some of holographic anomalous dimensions in $O_H O_L O_{\Delta,J} \times O_{\Delta,J} O_L O_H$ channel \[42\] \[43\]. It turns out that lowest-twist double-stress-tensor OPE proposed in \[46\] is exactly same as one found from holography \[47\]. A recent progress was made in \[48\] where in $d = 4$ lowest-twist double-stress-tensor OPE coefficients, triple-stress-tensor OPE coefficients exchanged in $O_H O_H O_{\Delta,J} \times O_{\Delta,J} O_L O_L$, and those double-twist OPE coefficients and anomalous dimensions exchanged in $O_H O_L O_{\Delta,J} \times O_{\Delta,J} O_L O_H$ at same order can all be extracted by using crossing symmetry back and forth, without any holography coming in. Even some results in $d = 6$ were also achieved up to $T^2$ \[48\]. Remarkably, it can be verified that the data exchanged in $O_H O_L O_{\Delta,J} \times O_{\Delta,J} O_L O_H$ is consistent with predictions from holography \[42\] \[43\].

These results are exciting, but can still be improved. At the first glimpse, the groundwork of \[48\] is exponentiated heavy-light four-point function ansatz near light-cone limit resembling CFT$_2$. In addition, there are some other mysteries are raised up. Presumably, the holographic computations for exchanged operators in $O_H O_L O_{\Delta,J} \times O_{\Delta,J} O_L O_H$ should be only valid at eikonal limit (Regge limit \[49\]), however, it is equal to those extracted near light-cone limit \[48\]. Such a connection between universality in eikonal region and lowest-twist region was also discussed in \[47\]. In the end, the framework of \[48\] could not exhibit

\[\text{Recently, the exponentiated Virasoro block is proved in \[37\].}\]
lowest-twist multi-stress-tensor universality in general. In this paper, we apply Lorentzian inversion formula to heavy-light four-point functions back and forth, and we surprisingly find these questions are answered by Lorentzian inversion formula partly, with agreement of those results existed in the literature.

The paper is organized as follows. In section 2 we briefly introduce some basics of conformal blocks and Lorentzian inversion formula. Notations and more backgrounds of heavy-light four-point functions are attached in section 2.3 with summarizing our main conclusions. In section 3 we show that heavy-light four-point functions can indeed be bootstrapped by implementing Lorentzian inversion formula back and forth. In this sense the resulting CFT data is shown to be universal. After commenting on $\Delta_L$ poles, we propose an algorithm to manipulate heavy-light bootstrap with extracting all universal data. In section 4 we adopt our algorithm to work on $d = 4$ examples up to triple-stress-tensor $T^3$. In section 5 we have an attempt at heavy-light bootstrap in general dimension up to double-stress-tensor $T^2$. In particular, an infinite series representation of lowest-twist $T^2$ OPE coefficients is presented. In section 6 the paper is summarized and some future directions are discussed. In Appendix A we collect some missing steps of the main text. In Appendix B more examples of lowest-twist $T^2$ OPE coefficients are worked out, includes $d = 6, 8, 10$ and a generic pattern.

2 Generalities

In this section, we assemble some preliminaries that will be used throughout this paper, includes conformal blocks, Lorentzian inversion formula and some background knowledge of heavy-light four-point function.

2.1 Conformal blocks

A four-point function $\langle O_1 O_2 O_3 O_4 \rangle$ can be expanded by conformal blocks

$$\langle O_1(0)O_2(z, \bar{z})O_3(1)O_4(\infty) \rangle = \frac{G(z, \bar{z})}{(z\bar{z})^{\frac{\Delta_1 + \Delta_2}{2}}}, \quad G(z, \bar{z}) = \sum c_{\Delta, J}G^{a,b}_{\Delta, J}(z, \bar{z}), \quad (2.1)$$

where $a = (\Delta_2 - \Delta_1)/2$, $b = (\Delta_3 - \Delta_4)/2$ and $c_{\Delta, J}$ is the OPE coefficient. The conformal block is the solution of the quadratic Casimir equation

$$C_2 G^{a,b}_{\Delta, J}(z, \bar{z}) = (\Delta(\Delta - d) + J(J + d - 2))G^{a,b}_{\Delta, J}(z, \bar{z}), \quad (2.2)$$
where

\[ C_2 = D_z + D_{\bar{z}} + 2(d - 2) \frac{z \bar{z}}{z - \bar{z}} ((1 - z) \partial_z - (1 - \bar{z}) \partial_{\bar{z}}), \]

\[ D_z = 2(z^2(1 - z) \partial_z^2 - (1 + a + b)z^2 \partial_z - abz). \] (2.3)

In \( d = 4 \), the closed form of conformal block for scalar four-point function \( \langle O_1 O_2 O_3 O_4 \rangle \) is known

\[ G_{\Delta,J}^{a,b}(z,\bar{z}) = \frac{z \bar{z}}{z - \bar{z}} (k_{\Delta+J}^{a,b}(z) k_{\Delta-J-2}^{a,b}(\bar{z}) - k_{\Delta+J}^{a,b}(\bar{z}) k_{\Delta-J-2}^{a,b}(z)), \] (2.4)

where \( k_{\beta}^{a,b}(x) \) is SL(2,R) block and is given by

\[ k_{\beta}^{a,b}(x) = x^{\beta} F_1 \left( a + \frac{\beta}{2}, b + \frac{\beta}{2}, \beta, x \right). \] (2.5)

The conformal block (2.4) is symmetric under \( (z \rightarrow \bar{z}, \bar{z} \rightarrow z) \). However, in general dimensions, the exact solutions are hard to come by.

Fortunately, we can probe some useful information hidden in conformal blocks by series expansion without knowing exact conformal block. The colinear expansion around \( z \rightarrow 0 \) is very useful for our purpose in this paper. The leading term is

\[ G_{\Delta,J}^{a,b} |_{z \rightarrow 0} = z^{\Delta - J / 2} k_{\Delta+J}^{a,b}(\bar{z}). \] (2.6)

Specializing (2.6) in \( d = 4 \) and comparing with exact block in \( d = 4 \) (2.4), it is obvious that this expansion has lost control of another part of \( z \) power, therefore sometimes (2.6) is called power law [19] in the sense that it only captures the essential power \( z^{(\Delta - J)/2} \). Group theoretically, the full colinear expansion is expected to take the form as

\[ G_{\Delta,J}^{a,b} = \sum_n \sum_{m=-n} B_{n,m}^{a,b} z^{\frac{\Delta - J}{2} + n} k_{\beta + 2m}^{a,b}(\bar{z}), \] (2.7)

where for simplicity we write \( \Delta - J = \tau \) and \( \Delta + J = \beta \). The coefficients \( B_{n,m}^{a,b} \) can be solved by quadratic Casimir equation, see, e.g. [19] and Appendix [A.1].
2.2 Lorentzian inversion formula

Lorentzian inversion formula is a powerful formula to extract the OPE data associated with the $s$-channel of four-point function $\langle O_1 O_2 O_3 O_4 \rangle$ [19][21]. The formula is given as

$$c(\Delta, J) = \frac{1 + (-1)^J}{4} \kappa_{\Delta + J}^{a,b} \int dzd\bar{z} \mu^{a,b}(z, \bar{z}) G_{J+d-1,\Delta-d+1}^{a,b}(z, \bar{z}) d\text{Disc}[G(z, \bar{z})], \quad (2.8)$$

where $\mu^{a,b}(z, \bar{z})$ is given by

$$\mu^{a,b}(z, \bar{z}) = \frac{z \bar{z}}{z - \bar{z}} \left( \frac{(1 - z)(1 - \bar{z})}{(z \bar{z})^2} \right)^{a+b}, \quad (2.9)$$

and $\kappa_{\Delta + J}^{a,b}$ is

$$\kappa_{\beta}^{a,b} = \Gamma\left(\frac{\beta}{2} - a\right) \Gamma\left(\frac{\beta}{2} + a\right) \Gamma\left(\frac{\beta}{2} - b\right) \Gamma\left(\frac{\beta}{2} + b\right) \frac{2\pi^2 \Gamma(\beta - 1) \Gamma(\beta)}{\Gamma\left(\frac{\beta}{2}\right)}. \quad (2.10)$$

Moreover, $d\text{Disc}$ represents the double-discontinuity, which is defined by expectation value of multiplication of 14 and 23 commutators and can be computed by

$$d\text{Disc}[G(z, \bar{z})] = \cos(\pi(a + b)) G(z, \bar{z}) - \frac{e^{-i(a+b)}}{2} G^\circ(z, \bar{z}) - \frac{e^{i(a+b)}}{2} G^\circ(z, \bar{z}), \quad (2.11)$$

where $G^\circ$ and $G^\circ$ are two different analytic continuations for $\bar{z}$ around 1. Notice that in Lorentzian inversion formula (2.8), there is a conformal block with spin and conformal dimension interchanged $G_{J+d-1,\Delta-d+1}^{a,b}$ which is called funny block and is related to light-transform [21]. Notably, the formula is analytic in spin except for $(-1)^J$, which could be set to 1 in this paper since exchanged operators can only have even spin. Practically, we should expand $G(z, \bar{z})$ by cross-channel conformal blocks, for a certain block with $(\Delta, J)$ it should be

$$G(z, \bar{z}) = \frac{(zz)^{\Delta_3 + \Delta_4}}{(1-z)(1-\bar{z})} G_{\Delta, J}^{\tilde{a}, \tilde{b}}(1-\bar{z}, 1-z), \quad (2.12)$$

where $\tilde{a} = (\Delta_3 - \Delta_2)/2$ and $\tilde{b} = (\Delta_4 - \Delta_1)/2$. Then we could integrate $z$ and $\bar{z}$ to obtain $c(\Delta, J)$.

The OPE coefficients are encoded in $c(\Delta, J)$ by [19]

$$c_{\Delta, J} = -\text{Res}_{\Delta = \Delta'} c(\Delta', J). \quad (2.13)$$
This implies that \( c(\Delta', J) \) has poles around physical operators

\[
c(\Delta', J) \sim \frac{c_{\Delta,J}}{\Delta - \Delta'},
\]

(2.14)

In fact, \( z \) integral in Lorentzian inversion formula is responsible for creating these poles, while \( \bar{z} \) integral provides other factors. To end this subsection, we would like to mention that an integral formula against \( \bar{z} \) from [19] would be useful throughout our calculation

\[
I_{a,b}^{\hat{\tau}}(\beta) = \int_0^1 \frac{d\bar{z}}{\bar{z}^2}(1 - \bar{z})^{a+b} \kappa^{a,b}_{\beta} \kappa^{\bar{a},\bar{b}} \hat{G}(\bar{z}) \text{dDisc}[\left(\frac{1 - \bar{z}}{\bar{z}}\right)^{-b} - \frac{1}{\bar{z}}],
\]

(2.15)

2.3 Heavy-light four-point function

Our interest is the heavy-light four-point function \( \langle O_H O_H O_L O_L \rangle \) in both \( s \)-channel and \( t \)-channel of large central charge \( C_T \) CFT \( (C_T \sim N^2) \) in higher dimension \( d > 2 \), where the conformal dimension of heavy operator is comparable to large \( C_T \), i.e. \( \Delta_H \sim O(C_T) \), and the conformal dimension of light operator is of course \( \Delta_L \ll C_T \). To study such a four-point function, we would like to choose a convenient conformal frame in \( s \)-channel

\[
\langle O_H(\infty) O_H(1) O_L(z, \bar{z}) O_L(0) \rangle,
\]

(2.16)

where \( z, \bar{z} \) are cross ratios, and expand it in terms of conformal blocks. Typically, in most cases, we still prefer similar \( (z, \bar{z}) \) rather than \( (1 - \bar{z}, 1 - z) \) in \( t \)-channel. Thus for clarity, we should clarify the notation used throughout this paper

Notations

1.1. HHLL \( s \)-channel: \( O_H O_H O_{\Delta,J} \times O_{\Delta,J} O_L O_L \)

\[
\langle O_H(\infty) O_H(1) O_L(z, \bar{z}) O_L(0) \rangle = \frac{G(z, \bar{z})}{(z \bar{z})^{\Delta_L}}, \quad G(z, \bar{z}) = \sum_{\Delta,J} c_{\Delta,J} G_{\Delta,J}^{0,0}(z, \bar{z}),
\]

(2.17)

where \( c_{\Delta,J} \) is the OPE coefficient.
1.2. HHLL $t$-channel: the cross-channel of HHLL $s$-channel $O_H O_L O_{\Delta',J'} \times O_{\Delta',J'} O_L O_H$

\[
\langle O_H(\infty) O_H(1) O_L(z, \bar{z}) O_L(0) \rangle = \frac{\tilde{G}(1 - \bar{z}, 1 - z)}{(1 - z)(1 - \bar{z})} \delta_{\Delta H + \Delta L},
\]

\[
\tilde{G}(1 - \bar{z}, 1 - z) = \sum_{\Delta',J'} \tilde{c}_{\Delta',J'} G^{a,b}_{\Delta',J'} (1 - \bar{z}, 1 - z),
\]

(2.18)

where the exchanged operators in cross-channel are denoted with prime, the cross-channel OPE coefficients are denoted with tilde and $a = b = (\Delta_L - \Delta_H)/2$.

2.1. HLLH $s$-channel: $O_H O_L O_{\Delta',J'} \times O_{\Delta',J'} O_L O_H$

\[
\langle O_H(0) O_L(z, \bar{z}) O_L(1) O_H(\infty) \rangle = \frac{G(z, \bar{z})}{(z\bar{z})} \delta_{\Delta H + \Delta L}, \quad G(z, \bar{z}) = \sum_{\Delta',J'} \tilde{c}_{\Delta',J'} G^{a,b}_{\Delta',J'} (z, \bar{z}),
\]

(2.19)

which is actually equivalent to HHLL $t$-channel but with different conformal frame for latter convenience.

2.2. HLLH $t$-channel: cross-channel of HLLH $s$-channel above $O_H O_H O_{\Delta,J} \times O_{\Delta,J} O_L O_L$

\[
\langle O_H(0) O_L(z, \bar{z}) O_L(1) O_H(\infty) \rangle = \frac{\tilde{G}(1 - \bar{z}, 1 - z)}{(1 - z)(1 - \bar{z})} \delta_{\Delta L},
\]

\[
\tilde{G}(1 - \bar{z}, 1 - z) = \sum_{\Delta,J} c_{\Delta,J} G^{0,0}_{\Delta,J} (1 - \bar{z}, 1 - z),
\]

(2.20)

which is actually equivalent to HHLL $s$-channel but with different conformal frame for latter convenience.

3. We always use $s$-channel, i.e. HHLL $s$-channel and HLLH $s$-channel to indicate what is the underlying OPE expansion. $t$-channel terminology, as the cross-channel of $s$-channel, will only be used when we are implementing Lorentzian inversion formula.

Usually, in large $C_T$ CFT, the OPE coefficients should be expanded in powers of $1/C_T$. If we are only interested in $O(1)$ OPE, then target theory is called generalized free field theory. In generalized free field theory, operators that can be exchanged in HLLH $s$-channel (let us assume $\Delta_H \sim \Delta_L \ll C_T$ for the moment) are double-twist operators

\[
[O_H O_L]_{\mu',J'} = O_H \Box^{\mu'} \partial_{\mu_1} \cdots \partial_{\mu_J} O_L, \quad \Delta' - J' = \Delta_H + \Delta_L + 2n',
\]

(2.21)
where \( n' \) is integer. Thus it would be more convenient to denote the OPE coefficients with twists \( \tilde{c}_{n',J'} \). There are an infinite number of double-twist operators, and they are contributing to identity exchange in HHLL s-channel. The exact free OPE coefficients can be computed by Euclidean inversion formula elegantly and in fact they are well-known \[50\]

\[
\tilde{c}_{n',J'}^{\text{free}} = \frac{(\Delta_H + 1 - \frac{d}{2})_{n'}(\Delta_L + 1 - \frac{d}{2})_{n'}(\Delta_H)_{n'+J'}(\Delta_L)_{n'+J'}}{n'!J'!(\Delta_H + \Delta_L + n' + 1 - d)_{n'}(\Delta_H + \Delta_L + 2n' + J' - 1)_{J'}}
\times \frac{1}{(\Delta_H + \Delta_L + n' + J' - \frac{d}{2})_{n'}(J' + \frac{d}{2})_{J'}.}
\] (2.22)

It behaves like \( J'\Delta_L^{-1} \) at heavy-limit and large \( J' \) limit \[42, 43, 46, 48\]. Typically, as one goes to next order and even higher of large \( C_T \) expansion, not only OPE coefficients will be corrected by \( 1/C_T^n \) with \( n \geq 1 \), but also double-twist operators will acquire anomalous dimensions suppressed by \( 1/C_T^n \) with \( n \geq 1 \). From holographic viewpoint, these corrections and anomalous dimensions come from tree-level exchange (\( n = 1 \)) and loop effects of Witten diagrams (\( n > 1 \)). When an additional parametrically large conformal dimension \( \Delta_H \sim C_T \) is available in the spectrum, higher order \( 1/C_T \) suppressions have their chance to be compensated by \( \Delta_H \), consequently, OPE corrections and anomalous dimensions may have \( \mathcal{O}(1) \) and can not be neglected. Instead, OPE coefficients and anomalous dimensions for double-twist operators exchanged in HHLL s-channel could be expanded by \( \Delta_H/C_T \). Follow the convention from \[42, 43, 46, 48\] and for latter convenience, we introduce a parameter \( \mu \)

\[
\mu = \frac{4\Gamma(d + 2)}{(d - 1)^2 \Gamma(\frac{d}{2})^2} \frac{\Delta_H}{C_T}.
\] (2.23)

Naturally, we can organize the double-twist OPE coefficients and anomalous dimensions by

\[
\tilde{c}_{n',J'}(\mu) = \tilde{c}_{n',J'}^{\text{free}} \sum_k \mu^k \tilde{c}_{n',J'}^{(k)}, \quad \tilde{\gamma}_{n',J'}(\mu) = \tilde{\gamma}_{n',J'}^{\text{free}} \sum_k \mu^k \tilde{\gamma}_{n',J'}^{(k)}.
\] (2.24)

It is worth commenting that the expansion (2.24) is an natural organization: presumably, we can start from full \( 1/C_T \) expansion and collect those terms having enough power of \( \Delta_H \) to reorganize an expansion by arranging \( \mu \) order. For \( \mathcal{O}(\mu) \), \( \tilde{c}_{n',J'}^{(1)} \) and \( \tilde{\gamma}_{n',J'}^{(1)} \) are contributed by single-stress-tensor exchange in HHLL s-channel which is shaped by Ward identity and is proportional to \( \mu \)

\[
c_{\Delta=d,J=2} = \frac{d^2 \Delta_L \Delta_H}{4(d - 1)^2 C_T} = \mu \frac{\Delta_L \Gamma(\frac{d}{2} + 1)^2}{4\Gamma(d + 2)}.
\] (2.25)
Then $\tilde{c}^{(1)}_{n',J'}$ and $\tilde{\gamma}^{(1)}_{n',J'}$ could be extracted \[42,43\] by using the impact parameter representation at Regge limit \[38–41\]. By dimensional analysis, $O(\mu^k)$ corrections of HLLH $s$-channel OPE coefficients and anomalous dimensions are contributed by multi-stress-tensor exchange $T^k$ in HHLL $s$-channel, however, we almost know nothing about their OPE coefficients beyond single-stress-tensor. Hence beyond $O(\mu)$, the expansion (2.24) can only be calculated by holographic studies either by bulk phase shift \[42,43\] or Hamiltonian perturbation theory \[42\]. Those holographic investigations are limited to Regge limit where OPE coefficients and anomalous dimensions are restricted to large spin limit $\Delta_H \gg J' \gg 1$, in which the holographic investigations also suggest \[42,43\]

$$
\tilde{c}^{(k)}_{n',J'}, \tilde{\gamma}^{(k)}_{n',J'} \sim \frac{1}{J'^{d-2}}.
$$

(2.26)

These data calculated from bulk is universal in the sense that any higher-derivative gravity corrections will be suppressed by $1/J'$. Typically, in this paper, we will show the large spin behavior \[22,24\] for HLLH $s$-channel data is indeed valid from CFT point of view by using Lorentzian inversion formula.

On the other hand, in HHLL $s$-channel, we expect the dominate exchanged operators are multi-stress-tensors $T^k$, for example

$$
k = 1, \quad T_{\mu\nu},
$$

$$
k = 2, \quad T_{\mu\nu} \Box^n \partial_{\mu_1} \cdots \partial_{\mu_{J-4}} T_{\rho\sigma},
$$

$$
k = 3, \quad T_{\mu\nu} T_{\rho\sigma} \Box^n \partial_{\mu_1} \cdots \partial_{\mu_{J-6}} T_{\alpha\beta}, \
$$

$$
\cdots.
$$

(2.27)

Similar to organization of HLLH $s$-channel data, the OPE coefficient for $T^k$ could be organized by factorizing $\mu$ out

$$
c_{\Delta,J} = \mu^k c^{(k)}_{n,J}, \quad \Delta = k(d-2) + J + 2n.
$$

(2.28)

However, as we mentioned previously, the multi-stress-tensor OPE coefficients are beyond our knowledge, impeding our efforts on understanding $O(\mu^k)$ correction of double-twist operators from pure CFT point of view. The efforts were made recently toward understanding multi-stress-tensor OPE coefficients holographically in \[44\]. By treating heavy operator as a black hole, the heavy-light four-point function could be understood as a two-point function under this black hole, a technique was then developed in \[44\] to read off multi-stress-tensor
OPE coefficients. The main conclusion of [44] is that they found, by considering arbitrary higher-derivative gravity models, the lowest-twist multi-stress-tensor OPE coefficients are universal. Although by applying crossing symmetry at light-cone limit with help of holography [46] or exponentiated HHLL block ansatz [48], [46] and [48] successfully extracted lowest-twist double-stress-tensor OPE coefficients as well as some low-lying double-twist $[\mathcal{O}_H \mathcal{O}_L]_{n',J'}$ data for which a precise agreement with holographic results [42,43,47] was observed, an insightful CFT understanding of this universality is still lacking. In this paper, we would employ Lorentzian inversion formula to fill this gap up to some extents. Considering it can be observed in [46, 47, 47, 48, 51] that multi-stress-tensor OPE coefficients have integer $\Delta_L$ poles in even dimension, we will assume $\Delta_L$ is neither an integer nor half-integer (see section 3.2) throughout this paper except for section 3.2. The origin of such poles could be easily observed in our framework and we will leave the comments in section 3.2. For clear and as a guide for readers, we list the main conclusion of this paper below provided with two assumptions

**Assumption:**

a. $\mathcal{O}_L$ belongs to an non-even-integer multiplet: additional light operators with conformal dimension $\tilde{\Delta}_L = \Delta_L + 2q$ (where $q$ is an integer) are not available in the spectrum.

b. $\Delta_L$ is not integer and half-integer.

**Main conclusion:**

1. We can bootstrap heavy-light four-point function by implementing Lorentzian inversion formula back and forth.

2. The large spin limit of double-twist OPE coefficients exchanged in HLLH $s$-channel are universal.

3. The lowest-twist multi-stress-tensor OPE coefficients exchanged in HHLL $s$-channel are universal.

4. This universality is valid from light-cone limit to Regge limit with respect to HLLH $s$-channel. (i.e. refer $\bar{z} \to 1$).

### 3 Bootstrapping heavy-light: the algorithm

In this subsection, we present the generic algorithm for bootstrapping heavy-light four-point functions. By bootstrapping heavy-light, we mean, ambitiously, we would like to
have a machine that both details of HHLL s-channel and HLLH s-channel can come out by following the algorithm. The machine should be Lorentzian inversion formula. The idea is that we could implement Lorentzian inversion formula back and forth to extract all universal CFT data, i.e. \( \cdots \text{HHLL} \rightarrow \text{HLLH} \rightarrow \text{HHLL} \cdots \) Typically, Lorentzian inversion formula is powerful to probe the universality of double-twist operators at large spin limit \[17\]18, elegantly and systematically capturing the large spin perturbation systematics \[10\]14–16, in which finite spin makes sense at the end of the day \[17\]18,22,23. More surprisingly, in this section, we will show that for heavy-light four-point function where \( \Delta_H \) is comparable to \( C_T \) charge, the Lorentzian inversion formula tells us the multi-stress-tensor exchanged in HHLL s-channel is universal and allows us to have an algorithm computing multi-stress-tensor OPE coefficients alongwith computing HLLH s-channel double-twist data at large spin limit.

3.1 Lowest-twist multi-stress-tensor OPE

3.1.1 HLLH large spin behavior

To exhibit that Lorentzian inversion formula can encode the multi-stress-tensor data, we would like to start with showing that the HHLL t-channel twist \( \tau = \Delta - J \) conformal block contribution in the correction of HLLH s-channel double-twist \([\mathcal{O}_H \mathcal{O}_L]_{n',j'}\) OPE and anomalous dimension behaves as \( 1/J'\tau/2 \) at large \( J' \) limit. Since we are not restricting ourselves in the leading-twist \( n' = 0 \), we shall keep all \( z \) expansion of HHLL funny conformal block in the Lorentzian inversion formula, in other words, we should adopt \((2.7)\) with \( \Delta \rightarrow J' + d - 1, J \rightarrow \Delta' - d + 1 \). Nevertheless, it is not necessary to know everything there, for example, the recursion coefficients \( B^{a,b} \) in \((2.7)\) actually plays no essential role for our purpose, since the recursion coefficients turn out to contribute \( \mathcal{O}(1) \). Generally, in Lorentzian inversion, we should consider following terms

\[
\frac{\kappa^{a,b}(\beta')}{\kappa^{a,b}(\beta' + 2m)}(1 - z)^{a+b}(1 - \frac{z}{\bar{z}})^{d-2} G^{a,b}_{J'+d-1,\Delta'-d+1} \bigg|_{n,m} \sim \tilde{B}^{a,b}_{n,m} z^{\frac{J'-\Delta'}{2} + n + d - 1} k^{a,b}_{\beta'+2m}(\bar{z}),
\]

where \( m \) can be integers from \(-n\) to \( n \), and \( \tilde{B}^{a,b}_{n,m} \) is some linear combination of \( B^{a,b} \). It turns out the contribution of \( \tilde{B} \) is of order 1, i.e. \( \mathcal{O}(1) \) at heavy and large spin limit, hence it is of no importance for large \( J' \) power behavior and can be slipped off here for simplicity.
On the other hand, the HHLL $t$-channel twist $\tau = \Delta - J$ conformal block is given by

$$G_{HHLL} \sim \frac{(zz)^{\Delta_H + \Delta_L}}{((1-z)(1-\bar{z}))^{\Delta_L}} G^{0,0}_{\Delta,J}(1-\bar{z},1-z). \quad (3.2)$$

To extract large $J'$ limit data, we can take the light-cone limit $\bar{z} \to 1$ of HLLH $s$-channel, in which $z$ and $\bar{z}$ dependence is factorized, then using (2.15) to integrate against $\bar{z}$ yields following function to be integrated against $z$

$$C(z,\beta') = \frac{1}{2}(2(n-1)+\Delta_H+\Delta_L-\tau) \frac{k_{\beta'}^0(1-z)}{(1-z)^{\Delta_L}} I^{(a,a)}_{\tau-\Delta_H-\Delta_L} (\beta'+2m). \quad (3.3)$$

The $z$ dependence in (3.3) will not introduce additional $J'$ and $\Delta_H$ dependent factors, and it does nothing but tells us the underlying exchanged operators are double-twist $[O_H O_L]^{n',J'}$. Hence, the large $J'$ behavior is encoded in the remaining factor $I^{(a,a)}_{\tau-\Delta_H-\Delta_L} (\beta'+2m)$ lying in the double-twist operator trajectories. For our purpose, we are supposed to take both the heavy and large $J'$ limit. Taking the limit is a little bit subtle here. Precisely we should consider $\Delta_H \gg J' \gg 1$. To achieve such a limit, we parameterize $\Delta_H \sim J'/\xi$ and take $\xi \to 0$, it ends like

$$I^{(a,a)}_{\tau-\Delta_H-\Delta_L} (\beta'+2m) \sim \frac{\Gamma(\Delta_L + J' + m + n)}{\Gamma(-\frac{\tau}{2} + \Delta_L)\Gamma(-\frac{\tau}{2} + J' + m + n + 1)} \frac{J' - \frac{\tau}{2} - 1 + \Delta_L}{\Gamma(-\frac{\tau}{2} + \Delta_L)}. \quad (3.4)$$

Recall that the free OPE coefficients go like $J'^{\Delta_L-1}$, we immediately have

$$\tilde{c}_{n',J'} \text{ and } \tilde{\gamma}_{n',J'} \sim J'^{-\frac{\tau}{2}}. \quad (3.5)$$

for any twist $n'$, where the superscript $\tau$ denotes that it is contributed by twist $\tau$ conformal block in the cross-channel. However, there is a gap in this rough proof, which is the large $J'$ behavior of $\tilde{B}^{a,b}_{n,m}$. By solving quadratic Casimir as in Appendix A.2 we find that for double-twist operators the heavy and large $J'$ limit of $\tilde{B}^{a,b}_{n,m}$ is

$$\tilde{B}^{a,b}_{n,m} = (-1)^n \frac{d}{\Gamma(n+1)} \frac{n}{\Gamma(n+1)} a_n \tilde{B}^{a,b}_{n,m} = 0. \quad (3.6)$$

Thus it does nothing to do with final large $J'$ behavior of HLLH OPE and anomalous dimension.
3.1.2 Finding lowest-twist multi-stress-tensor

Next, we would like to show that knowing $J_{n',J'}^{(d-2)}$ and $\tilde{J}_{n',J'}^{(d-2)}$ with $1 \leq k' \leq k$ as HLLH $t$-channel data allows us to find lowest-twist multi-stress-tensor $T^{k+1}$ exchanged in HLLH $s$-channel in Lorentzian inversion formula. The ingredient is the HLLH $t$-channel heavy block. The HLLH $s$-channel heavy block with twist $n'$ can be deduced from (2.7), i.e.

$$C_{\Delta',J'}^{a,b}(z,\bar{z}) = \sum_{m=-n}^{n} \sum_{m=-n}^{n} B_{n,m}^{a,b} \frac{1}{(1-z)(1-\bar{z})} \Delta_L^{\Delta_H + \Delta_L + \Delta_H^{\tilde{c}_{n',J'}(\mu)}} \frac{\Delta_H + \Delta_L + \Delta_H^{\tilde{c}_{n',J'}(\mu)}}{2} + J' + m + n'. \quad (3.7)$$

where $\Delta' = \Delta_H + \Delta_L + 2n' + \bar{\gamma}_{n',J'}(\mu)$. Crossing (3.7) by $(z \rightarrow 1 - \bar{z}, \bar{z} \rightarrow 1 - z)$ leads to HLLH $t$-channel heavy block. Note we are restricted in large $J'$ limit where the summation of $J'$ can be replaced by integration, we thus have

$$G_{\text{HLLH}} = \frac{(z\bar{z})^{\Delta_L}}{(1-z)(1-\bar{z})} \sum_{n'} \int_{0}^{\infty} dJ' \tilde{c}_{n',J'}(\mu) G_{\Delta',J'}^{a,b}(1 - \bar{z}, 1 - z). \quad (3.8)$$

It is worth noting that (3.8) only makes sense for $z \rightarrow 0$, since HLLH $s$-channel four-point function evaluated at large $J'$ limit by integrating against $J'$ is only consistent with $\bar{z} \rightarrow 1$ limit, namely $z \rightarrow 0$ after crossing. In other words, the large $J'$ data of HLLH $s$-channel evaluated before forces that we can only probe the lowest-twist data in HLLH $s$-channel.

Then as soon as we know $J_{n',J'}^{(d-2)}$ and $\tilde{J}_{n',J'}^{(d-2)}$ we can know the $O(\mu^{(k+1)(d-2)})$ order of $G_{\text{HLLH}}$ by expanding with respect to anomalous dimension in (3.7). Practically, the expansion up to $O(\mu^{(k+1)(d-2)})$ is permitted, since dDisc only keeps power $m \geq 2$ of log$^{m}$, while the unknown information $c_{n',J'}^{(k+1)(d-2)}$ and $\gamma_{n',J'}^{(k+1)(d-2)}$ is attached to linear log which will always be killed by dDisc. This is analogous to one-loop investigation of supergravity correlator, in which the one-loop effect can be computed by squaring the tree-level data due to the same reason here [24][25]. At the order $O(\mu^{(k+1)(d-2)})$, from (3.5) it follows that $c_{n',J'}^{(d-2)}$ and $\gamma_{n',J'}^{(d-2)}$ contributes as $J^{-(k+1)(d-2)/2}$ via many possible combinations, for example,

$$\tilde{J}_{n',J'}^{(d-2)} c_{n',J'}^{(d-2)}, \quad \tilde{J}_{n',J'}^{(d-2)} \gamma_{n',J'}^{(d-2)}, \quad J^{-(k-1)(d-2)/2} c_{n',J'}^{(d-2)}, \quad \gamma_{n',J'}^{(d-2)} c_{n',J'}^{(d-2)}, \ldots. \quad (3.9)$$

Note as for $\tilde{B}_{a,b}$ in (3.1), $B_{a,b}$ is also of order $O(1)$ at heavy and large $J'$ limit and hence does not contribute any $J'$ dependence. Precisely, $B_{a,b}$ is given by

$$B_{n,-n}^{a,b} = \frac{(\frac{4}{3} - 1)_n}{\Gamma(n + 1)}, \quad B_{n,m,-n}^{a,b} = 0, \quad (3.10)$$

for which the detail is presented in Appendix A.1. Then after integration against $J'$, the
only relevant factor is the power of \( z \)

\[
G \sim z^{(k+1)(d-2)} \Gamma \left( \Delta_L - (k + 1)(d - 2)/2 \right).
\]

(3.11)

All other factors like \( \bar{z} \) dependence, summing \( n' \) and other \( \Delta_L \) dependent coefficients are not relevant for our purpose, since the pole that signals the exchanged operators is encoded in \( z \) dependence. We keep a Gamma function for later comments in section 3.2. Then Lorentzian inversion formula provided with (3.11) now is

\[
c(\Delta, J) = \int_0^1 dz \, z^\frac{1}{2}(-2 - \tau + (k+1)(d-2))F,
\]

(3.12)

where \( F \) is some unknown but regular factors (up to \( \Delta_L \) poles) independent of \( z \). It is obvious from (3.12) that it encodes the OPE coefficients for lowest-twist multi-stress-tensor \( \tau = (k + 1)(d - 2) \) and we are allowed to compute them by using Lorentzian inversion formula as soon as we know all \( \tilde{c}_{n',J'}^{k(d-2)} \) and \( \tilde{\gamma}_{n',J'}^{k(d-2)} \) with \( 1 \leq k' \leq k \) in HLLH.

It is worth noting that one has to be cautious of the procedure discussed in this subsection. Typically, the double-twist operators \([\mathcal{O}_H \mathcal{O}_L]_{n',J'} \) in HLLH s-channel are likely to mix with other operators. For example, \([\mathcal{O}_H \mathcal{O}_L]_{n',J'} \) would be mixing with \([\mathcal{O}_H \tilde{\mathcal{O}}_L]_{n'-1,J'} \) where the conformal dimension of \( \tilde{\mathcal{O}}_L \) is \( \tilde{\Delta}_L = \Delta_L + 2 \): they share same conformal dimension, twist and spin. In this way, the OPE coefficients \( \tilde{c}_{n',J'}^{(k)} \) and anomalous dimensions \( \tilde{\gamma}_{n',J'}^{(k)} \) should be interpreted as the weighted average over degenerate operators. Under average, it is apparent that, e.g. \( \langle \tilde{c}_{n',J'}^{(k)(d-2)} \rangle \) is not equal to \( \langle \tilde{\gamma}_{n',J'}^{(k)(d-2)} \rangle \). Hence the simple multiplications are not trustable any more. Similar mixing problem appears in the efforts toward understanding loop level of supergravity correlators, e.g. \([24, 25, 52, 53]\). Therefore, an assumption should be made throughout this paper: there are no other light operators having conformal dimension \( \tilde{\Delta}_L = \Delta_L + 2q \) where \( q \) is an integer. This is assumption \( a \) listed in section 2.3 we shall call this assumption non-even-integer multiplet assumption.

3.1.3 The universality

Now as assumption \( a \) in section 2.3 is made, we are ready to show the main conclusions of this paper listed in section 2.3. The assumption \( b \) restricting \( \Delta_L \) to non-integer and non-half-integer could actually be quickly observed in factor of (3.11), nevertheless, we leave this to section 3.2.

The input is OPE coefficients of single stress-tensor that is completely fixed by Ward

\footnote{We would like to thank Simon Caron-Huot for pointing this out to us.}
identity \( (2.25) \). For convenience, we present it here again

\[
c_{\Delta=d,J=2} = \frac{d^2 \Delta_L \Delta_H}{4(d-1)^2 C_T} = \frac{\Delta_L \Gamma \left( \frac{d}{2} + 1 \right)^2}{4 \Gamma(d+2)}.
\]

(3.13)

Remarks are necessary here. This coefficient is exact, it does not require heavy limit of \( \Delta_H \). On the other hand, this coefficient is universal in the sense that it only depends on \( \Delta_L, \Delta_H \) and \( C_T \). Immediately, one can use (3.13) to calculate the order \( \mathcal{O}(\mu) \) of double-twist OPE correction and anomalous dimension at large spin limit via Lorentzian inversion formula. Since (3.13) is universal, and Lorentzian inversion formula will not introduce additional parameters, thus it follows that \( \mathcal{O}(\mu) \) HLLH s-channel data at large spin limit are universal. Then as discussed previously, we can keep going, use \( \mathcal{O}(\mu) \) HLLH large spin data to extract lowest-twist double-stress-tensor \( T^2 \) OPE coefficients which are universal due to the universality of \( \mathcal{O}(\mu) \) HLLH s-channel large spin data. In turn we could input double-stress-tensor OPE and extract \( \mathcal{O}(\mu^2) \) HLLH s-channel large spin data. Furthermore, \( \mathcal{O}(\mu^2) \) HLLH s-channel large spin data could tell us triple-stress-tensor \( T^3 \) OPE. We can employ Lorentzian inversion formula back and forth to do this iteratively, in principle all lowest-twist multi-stress-tensor OPE and large spin double-twist data could be bootstrapped in the present game. Typically, since our input is nothing else but universal data (3.13), all the relevant coefficients extracted by this method, including lowest-twist multi-stress-tensor and large spin double-twist data, are universal.

It would also be essential to comment the range of this universality. Follow the previous framework, it should be emphasized that the universal double-twist data is valid only at \( \bar{z} \to 1 \) in HLLH s-channel, i.e. \( \langle O_H(0)O_L(z, \bar{z})O_L(1)O_H(\infty) \rangle \) which is where \( \bar{z} \) direction of light operators nearly coincide with each other. While for \( T^n \) exchanged in HHLH s-channel, the universality is holding for lowest-twist, i.e. \( z \to 0 \) in, specifically, \( \langle O_H(\infty)O_H(1)O_L(z, \bar{z})O_L(0) \rangle \). Notably, this can be viewed as the cross channel of HLLH s-channel, i.e. HLLH s-channel = HHLH t-channel, thus \( z \to 0 \) in HHLH s-channel is actually equivalent to \( \bar{z} \to 1 \) in HLLH s-channel. This equivalence is implicit in the process of using Lorentzian inversion formula back and forth. Let us put our foot on HLLH s-channel, thus the universality of heavy-light four-point function we present here is valid at \( \bar{z} \to 1 \) limit of HLLH s-channel. Such a limit does not have any constraints on \( z \) and thus is way beyond light-cone limit for which we also require \( z \to 0 \). It is worth noting that \( z \to 1 \) limit is not forbidden by our construction, in this way, we could say this universality holds at both light-cone limit and Regge limit. This explains why the results of double-twist data obtained
by bulk phase shift in eikonal limit is consistent with light-cone limit treatment \[47,48\].

### 3.2 Comments on $\Delta_L$ poles

Before we finally propose the algorithm for bootstrapping heavy-light four-point function, we would like to leave a subsection commenting on the $\Delta_L$ poles and explaining why the assumption $b$ in section 2.3 is necessary. The holographic calculations in even dimensions \[44,51\] implies that the multi-stress-tensor would be suffering from poles of $1/(\Delta_L - n)$ where $n$ is integer. This phenomenon can also be observed from recent CFT investigations \[46,48\]. Typically, it shows a pattern, for examples, for double-stress-tensor OPE has poles $1/(\Delta_L - 2)$ in $d = 4$ and $1/((\Delta_L - 3)(\Delta_L - 4))$ in $d = 6$. The origin of these poles is clear from our framework, precisely, it comes from \[3.11\] alongwith the lowest-twist multi-stress-tensor $T^{k+1}$ trajectory. Let us write down the relevant factor here again

$$P(\Delta_L) = \Gamma(\Delta_L - (k + 1)(d - 2)/2). \quad (3.14)$$

Now the pattern of such poles is clear:

1. In even dimensions, all multi-stress-tensor OPE coefficients suffer from integer $\Delta_L$ poles.
2. In general dimensions, for even number of stress-tensors, e.g. $T^2, T^4, \cdots$, the corresponding OPE coefficients have some integer poles.
3. In odd dimensions, for odd number of stress-tensors, e.g. $T^3, T^5, \cdots$, the corresponding OPE coefficients have some half-integer poles.

As discussed in \[44\], these poles exist because HHLL $s$-channel double-twist operators $[O_L O_L]_{n,J}$ are not distinguishable from some of multi-stress-tensor operators for certain $\Delta_L$. Two sets of operators themselves both have such poles and as poles are approached, operators and conformal blocks merge with divergence identically canceled \[44,47,51\]. Typically, the holographic technique developed in \[44\] could not reveal HHLL $s$-channel double-twist $[O_L O_L]_{n,J}$ OPE coefficients, which require us to relate near boundary expansion to near horizon data and impose the regularity \[44,51\]. Moreover, when these two sets of operators are somehow mixing, the holographic techniques developed can no longer determine mixed OPE coefficients \[44,51\].

We hope our framework could resolve this situation: we expect that we can distinguishably extract both multi-stress-tensor OPE and HHLL $s$-channel double-twist OPE with
poles attached, and clearly observe they merge to eliminate the relevant pole as $\Delta_L$ approaching that pole. Unfortunately, for now, this remains unclear: without heavy-limit, Lorentzian inversion formula should tell us HHLL $s$-channel double-twist OPE coefficients, however, by taking the limit as in previous subsection, HHLL $s$-channel double-twist signals are lost due to some unknown reasons.

Nevertheless, we have to overcome this obstacle for the purpose of going to specific CFT, for examples, $d = 4, \mathcal{N} = 4$ super-conformal Yang-Mills theory, in which half-BPS operators all have integer conformal dimensions. From holographic point of view, sphere reductions from type IIB string theory or M theory are more likely to give rise to integer $\Delta_L$ in even dimensions. It thus deserves future investigations. On the other hand, it turns out that when $\Delta_L$ approaches a certain pole, the relevant operators acquire anomalous dimension for which the ratio between this anomalous dimension and OPE coefficient could be determined by Residue around that pole of relevant multi-stress-tensor OPE coefficient. We can also understand, from viewpoint of Lorentzian inversion formula, that this anomalous dimension should emerge. Note the relevant term in dDisc is $z^{\Delta_L - p} \Gamma(\Delta_L - p)$ where $p$ is the upper bound of involved poles, by expanding around a certain pole $p - p'$, it becomes

$$z^{\Delta_L - p} \Gamma(\Delta_L - p) \sim \frac{(-1)^{1+p'} z^{p'}}{\Gamma(1+p')} \left( \frac{1}{(p-p' - \Delta_L)} + \log z + \cdots \right),$$

where $\cdots$ denotes other irrelevant terms and the divergence term should be expected to be canceled by another set of operators. $\log z$ implies that the corresponding multi-stress-tensor or HHLL $s$-channel double-twist (now they mix with each other) acquire anomalous dimension. We hope our framework could also inspire the understanding of this anomalous dimension and verify the Residue relation proposed in future.

### 3.3 The algorithm

In this subsection, with assumptions listed in section in hands, we would explicitly propose the algorithm to bootstrap heavy-light four-point function below.

1. Start with the single-stress-tensor conformal block of HHLL $s$-channel, Lorentzian-invert to extract $\mathcal{O}(\mu)$ HLLH $s$-channel data (OPE coefficients and anomalous dimension of double-twist operators $[\mathcal{O}_H \mathcal{O}_L]_{n,J}$) in the heavy and large spin limit.

2. Take advantage of $\mathcal{O}(\mu)$ HLLH $s$-channel data to evaluate $\mathcal{O}(\mu^2)$ colinear ($z \to 0$) four-point function by summing over twists $n$ and integrating over spin.
3. Lorentzian-invert $O(\mu^2)$ colinear four-point function to obtain $O(\mu^2)$ HHLL $s$-channel OPE data which encodes double-stress-tensor OPE coefficients, read off double-stress-tensor OPE coefficients.

4. Input double-stress-tensor conformal block of HHLL $s$-channel, Lorentzian-invert to extract $O(\mu^2)$ HHLL $s$-channel data in the heavy and large spin limit.

5. Recursively repeat 1 to 4 to extract more and more $O(\mu^{\text{order}})$ HLLH $s$-channel data and $T^{\text{order}}$ OPE coefficients of HHLL $s$-channel.

4 Examples in four dimension up to $T^3$

In this section, we follow the algorithm introduced in the previous section to solve the heavy-light four-point function in four dimension up to $T^3$ as an explicit example.

4.1 $O(\mu)$ double-twist

In $d = 4$, the closed form of conformal block is known as (2.4), which simplifies things a lot. Since the conformal block (2.4) is explicitly invariant under interchanging $z$ and $\bar{z}$, making it possible to just use a half of it, thus we only need to evaluate

$$c(\Delta', J') = \int_0^1 dz d\bar{z} \frac{(z - \bar{z})}{(z \bar{z})^3} ((1 - z)(1 - \bar{z}))^{a + b} k_{a,b}^{a,b}(z) k_{2 - \tau'}(z) d\text{Disc}[G_T(z, \bar{z})], \quad (4.1)$$

where $G_T(z, \bar{z})$ is single-stress-tensor conformal block, which in $d = 4$ is specifically given by (still evaluate a half of (2.4))

$$G_T(z, \bar{z}) = -\frac{\Delta_L(z - 1)^{-1-\Delta_L(z - 1)^{-1-\Delta_L(z \bar{z})}}} {40(z - \bar{z})} \frac{(3(1 - z)^2 + (z^2 + 4z + 1) \log z)} {40(z - \bar{z})}, \quad (4.2)$$

where we have served $\mu$ as expansion parameter and thus slipped it off here as the organization in section 2.3. (4.2) should be automatically separated into two parts, one is free of log and one contains log $z$. The former would be evaluated to contribute the $O(\mu)$ correction of HLLH $s$-channel double-twist OPE coefficients, and the latter reflects that the HLLH $s$-channel double-twist operators acquires anomalous dimension at order $O(\mu)$. Evaluating the part without log and taking both the heavy limit $\xi \to 0$ and large spin limit $J' \to \infty$
yields
\[ \tilde{c}^{(1)}(\Delta', J') = \frac{3\Delta_L(2\Gamma(1-n')\Gamma(1-\Delta_L) + \Gamma(-n')\Gamma(2-\Delta_L))}{4\Gamma(2-n'-\Delta_L)\Gamma(-1+\Delta_L)} J'^{-2+\Delta_L}, \] (4.3)
in which we set \( \tau' = \Delta_H + \Delta_L + 2n' \). Note the free OPE coefficients with heavy and large spin limit specializing in \( d = 4 \) are
\[ \tilde{c}_{\text{free}}^{n', J'} = \frac{\Gamma(\Delta_L + n' - 1)}{\Gamma(n' + 1)\Gamma(\Delta_L)\Gamma(\Delta_L - 1)} J'^{\Delta_L - 1}. \] (4.4)
Then taking the Residue of interested twists integer \( n' \) and dividing by free OPE coefficients (4.4) leads to
\[ \tilde{c}^{(1)}_{n', J'} = -\frac{3\Delta_L(\Delta_L + 2n' - 1)}{4J'}. \] (4.5)
This result exactly agrees with examples of low-lying \( n' \) obtained in [42, 48].

The computation for log part is similar but more involved. Notably, in previous work on computing anomalous dimension via Lorentzian inversion formula, there is no \( z \) integral needs to be done. In most cases, one could just evaluates the \( \bar{z} \) integral and the remaining \( z \)-dependence will be same as \( z \)-dependence of integral associated with OPE data up to an overall \( \log z \). Therefore, by definition, the anomalous dimension can be easily worked out by dividing the \( z \)-dependence and pushing everything onto double-twist trajectories. However, in our case, discrepancy emerges for \( z \)-dependent of log part and OPE part, which is manifest in (4.2). The trick here is simply ignoring the overall \( \log z \) and integrating the remaining factor against \( z \). This integration does a job to make the double-twist trajectories visible. Subsequently, we should take the Residue to specify the value on the double-twist trajectories and then divided it by free OPE coefficients to end up with anomalous dimension. The limits \( \xi \to 0, J' \to \infty \) should be taken, we thus find
\[ \tilde{c}^{(1)}_{\log}(\Delta', J') = \frac{1}{4\Gamma(2-n'-\Delta_L)\Gamma(\Delta_L - 1)\Gamma(\Delta_L)} (\Delta_L(\Delta_L + 6n' - 1)\Gamma(-n')\Gamma(1-\Delta_L)\Gamma(\Delta_L) - 6(-1)^n\Gamma(2-n')\Gamma(2-n'-\Delta_L)\Gamma(\Delta_L + n' - 1)). \] (4.6)
Thus we end up with the anomalous dimension as
\[ \tilde{\gamma}^{(1)}_{n', J'} = -\frac{\Delta_L^2 + (6n' - 1)\Delta_L + 6n'(n' - 1)}{2J'}. \] (4.7)
It is matching with those examples obtained in [48].
4.2 Lowest-twist double-stress-tensor

Now we are ready to bootstrap the lowest-twist double-stress-tensor with (4.5) and (4.7) in hands. From (2.4), the full HLLH $s$-channel block in $d = 4$ with bare double-twist operators at the heavy-limit is given by

$$g_{n',J'} = \frac{(z\bar{z})^{n'} + \Delta \mu + \Delta L}{z - \bar{z}} (-z^{J'-1} + \bar{z}^{J'+1}) . \quad (4.8)$$

As a warm-up exercise, we would present the HLLH $s$-channel four-point function at $O(\mu)$ order. We present contribution of the twist $n'$, then we should sum over $n'$. For a certain twist $n'$ and $J'$ we have

$$\mathcal{G}_{n',J'}^{HLLH,s,(1)}(z, \bar{z}) = \tilde{c}_{\text{free},J'}^{n'}(\tilde{c}_{\text{free},J'}^{n'} + \frac{\tilde{\gamma}_{n',J'}^{(1)}}{2}(\log z + \log \bar{z}))g_{n',J'} , \quad (4.9)$$

where the superscript denotes that it is $O(\mu)$ order of HLLH $s$-channel. Substituting (4.4), (4.5) and (4.7), integrating $J'$ from 0 to $\infty$ and summing over all twists $n'$ yields (We also need to take $\bar{z} \to 1$ limit in the end for the consistency with large $J'$ limit)

$$\mathcal{G}_{n',J'}^{HLLH,s,(1)}(z, \bar{z}) = -\frac{\Delta L}{4}(1 - z)^{-2\Delta L}(1 - \bar{z})^{1-\Delta L}(3(1 - z^2) + (z^2 + 4z + 1) \log z)(z\bar{z})^{\Delta \mu + \Delta L}, \quad (4.10)$$

which is obviously consistent with the HHLL $t$-channel single-stress-tensor block (4.2). This is the double-check of this approach.

Then we move to the HLLH $s$-channel four-point function at the order of $O(\mu^2)$, specifically, what we are looking at is

$$\mathcal{G}_{n',J'}^{HLLH,s,(2)}(z, \bar{z}) = \frac{\tilde{c}_{\text{free},J'}^{n'}(\tilde{c}_{\text{free},J'}^{n'} + \frac{\tilde{\gamma}_{n',J'}^{(1)}}{2}(\log z + \log \bar{z}))^2}{4}(\log z + \log \bar{z})g_{n',J'} , \quad (4.11)$$

where we have already shut down the contribution from $\tilde{c}_{n',J'}^{(2)}$ and $\tilde{\gamma}_{n',J'}^{(2)}$ since they will be killed by dDisc. In fact, even $\tilde{c}_{n',J'}^{(1)}$ is useless here in the sense that it gives us linear log. Integrating $J'$, summing $n'$ and turning to cross-channel, we thus have (for simplicity we only keep $\log^2(1 - z)$ that survives under dDisc)

$$\mathcal{G}_{HLLH}^{(2)} = \frac{\Delta L}{32(\Delta_L - 2)} \frac{z^2}{\bar{z}^4}(\Delta_L(\Delta_L - 1)\bar{z}^4 - 12\Delta_L(\Delta_L + 2)\bar{z}^3 + 12(4\Delta_L + 3)(\Delta_L + 2)\bar{z}^2 + 36(\Delta_L + 2)(\Delta_L + 1)(2\bar{z} - 1))(\log z + \log \bar{z}) \log^2(1 - z) . \quad (4.12)$$

The pole $\Delta_L - 2$ in $T^2$ OPE observed in [44] already appears here. Then we just need to
work out the Lorentzian inversion formula \((2.8)\) with leading \(z \to 0\) term

\[
c(\Delta, J) = - \int dz d\bar{z} z^{-\frac{\Delta + 2}{2}} k^{0,0}_\beta(\bar{z}) d\text{Disc}[G^{(2)}_{HLLH}].
\]

(4.13)

Nevertheless, it is worth noting that we should not apply \((2.15)\) anymore, since now no \(\bar{z} \to 1\) limit is assumed. In other words, what we are interested in is finite \(J\) result. Following formula would be useful

\[
\int_0^1 d\bar{z} \bar{z}^{\alpha} \, _2F_1(\beta, \beta, 2\beta, \bar{z}) = \frac{1}{\alpha + 1} \, _3F_2(\alpha + 1, \beta, \beta; \alpha + 2, 2\beta; 1).
\]

(4.14)

The trick to do the integral is to expand the hypergeometric function as a series which makes the integral doable, and then sum the infinite series back to a finite result. Meanwhile, the integral of \(z\) is not necessary, since we know it will give rise to the pole \(\Delta - J - 4\), we only need to slip off \(z\) and assign the value \(\Delta = J + 4\) to the rest. After some algebra, we have

\[
c^{(2)}_{0,J} = \frac{2^{-5-2J} \sqrt{\pi} \Delta L \Gamma(J + 1)}{(\Delta L - 2)(J - 1)(J - 3)(J + 3)(J + 4)(J + 2) \Gamma(J + \frac{3}{2})} (a^{(2)}_0 + a^{(2)}_1 \Delta L + a^{(2)}_2 \Delta^2 L),
\]

\[
a^{(2)}_0 = 288, \quad a^{(2)}_1 = -(J^4 + 6J^3 - 37J^2 - 138J + 72), \quad a^{(2)}_2 = (J - 2)J(J + 3)(J + 5).
\]

(4.15)

One can straightforwardly verify that \((4.15)\) is exactly same as the holographic result in \([47]\) and also as conjectured in \([46]\).

### 4.3 \(O(\mu^2)\) double-twist and lowest-twist \(T^3\)

To go further and work on \(O(\mu^2)\), a practical problem arises. Typically, there are infinite number of lowest-twist double-stress-tensors with different spin \(J\), and one has to sum all of them for the purpose of using Lorentzian inversion formula. This would be a hard-core task, and \([46, 48]\) have done this by taking advantage of a complicated hypergeometric identity. In fact, the summed block exhibits a nice pattern at limit \(\bar{z} \to 1\) with respect to HHLL \(t\)-channel. Based on this nice pattern, \([48]\) proposed an ansatz to write down all multi-stress-tensor blocks, from which the computation was carried out to obtain HLLH \(s\)-channel data and HHLL \(s\)-channel \(T^3\) OPE coefficients that are partly overlapped with this section \([48]\). The summed lowest-twist double-stress-tensor four-point function is given
by (after crossing) \[G_{T^2} = \frac{\Delta_L}{28800(\Delta_L - 2)}((\Delta_L - 4)(\Delta_L - 3)(k_{6,0}^0(1 - z))^2 + \frac{15}{7}(\Delta_L - 8)k_{4,0}^0(1 - z)k_{8,0}^0(1 - z) + \frac{40}{7}(\Delta_L + 1)k_{2,0}^0(1 - z)k_{10,0}^0(1 - z)) . \] (4.16)

Then exactly as in (4.11) and previous subsections, we work out the integral and take heavy and large spin limit followed by taking the corresponding Residue to have the correction of double-twist OPE coefficients

\[
\tilde{c}_{n', J'}^{(2)} = \frac{1}{96J'^2} (27\Delta_L^4 + 4(27n' - 43)\Delta_L^3 + 3(36n'^2 - 208n' + 39)\Delta_L^2 - 4(129n'^2 + 27n' - 7)\Delta_L - 624n'(n' - 1)) ,
\] (4.17)

and the correction of double-twist anomalous dimensions

\[
\tilde{\gamma}_{n', J'}^{(2)} = -4\Delta_L^2 + 3(14n' - 1)\Delta_L^2 + (102n'^2 - 66n' - 1)\Delta_L + 34(2n' - 1)n'(n' - 1) \quad \frac{8}{J'^2} ,
\] (4.18)

which agrees with results obtained from Hamiltonian perturbation theory [42], and the low-lying examples \(n' = 0, 1, 2, 3\) exactly match those obtained in [48].

Then we would like to have attempt at solving \(T^3\) OPE coefficients. Expanding the HLLH s-channel heavy block associated with twist \(n'\) and spin \(J'\) up to \(O(q^3)\) leads to (ignoring linear log term)

\[
G_{HLLH,s}^{(3)} = \frac{\tilde{c}_{n', J'}^{(1)}(\tilde{c}_{n', J'}^{(1)} + 2\tilde{\gamma}_{n', J'}^{(2)}) + \frac{1}{6}(\tilde{\gamma}_{n', J'}^{(1)})^3(\log z + \log \bar{z})}{\log z + \log \bar{z})^2} .
\] (4.19)

By substituting the known data (4.5), (4.7), (4.17) and (4.18), we are allowed to integrate \(J'\) and sum \(n'\) to obtain \(G_{HLLH}^{(3)}\). Although the expression of \(G_{HLLH}^{(3)}\) is too cumbersome and complicated to be presented here, it is for sure that \(\log^3(1 - \bar{z})\) is involved. After doing the double-discontinuity, we are still left with \(\log(1 - \bar{z})\). In this way, at the order \(T^3\), we have to face with following integral

\[
\int_0^1 d\bar{z} \bar{z}^\alpha {}_2F_1(\beta, \beta, 2\beta, \bar{z}) \log(1 - \bar{z}) .
\] (4.20)

Unfortunately, at least to our knowledge, this integral (4.20) does not have a closed form.

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\[ \int_0^1 d\bar{z} \bar{z}^\alpha \binom{2}{\beta, 2\beta, \bar{z}} \log(1 - \bar{z}) = -\sum_{k=0}^{\infty} \frac{2^{2\beta-1} \Gamma(\beta + \frac{1}{2})^2 \Gamma(k + \beta)^2 (\gamma + \psi(\alpha + k + 2))}{\sqrt{\pi} \Gamma(k + \beta) \Gamma(2\beta + k)}. \]

(4.21)

Thus we are hindered to have lowest-twist \( T^3 \) OPE coefficients with symbolic \( J \) dependence. Nevertheless, for specific \( J \), the integral is easy to evaluate and we could steadily have many low-lying examples for lowest-twist \( T^3 \) OPE coefficients. We present some examples with low-lying \( J = 6, 8, 10, 12, 14 \)

\[ c^{(3)}_{0,6} = \frac{\Delta_L(1001\Delta_L^4 + 3575\Delta_L^3 + 7310\Delta_L^2 + 7500\Delta_L + 3024)}{10378368000(\Delta_L - 3)(\Delta_L - 2)}, \]

\[ c^{(3)}_{0,8} = \frac{\Delta_L(3003\Delta_L^4 + 6032\Delta_L^3 + 9029\Delta_L^2 + 7148\Delta_L + 2688)}{613476864000(\Delta_L - 3)(\Delta_L - 2)}, \]

\[ c^{(3)}_{0,10} = \frac{\Delta_L(2431\Delta_L^4 + 3077\Delta_L^3 + 3742\Delta_L^2 + 2216\Delta_L + 888)}{9468531072000(\Delta_L - 3)(\Delta_L - 2)}, \]

\[ c^{(3)}_{0,12} = \frac{\Delta_L(46865039\Delta_L^4 + 38644366\Delta_L^3 + 41210477\Delta_L^2 + 15350374\Delta_L + 8351544)}{34001495079520000(\Delta_L - 3)(\Delta_L - 2)}, \]

\[ c^{(3)}_{0,14} = \frac{\Delta_L(4892481\Delta_L^4 + 2593025\Delta_L^3 + 2625560\Delta_L^2 + 245300\Delta_L + 477744)}{6497406470370816000(\Delta_L - 3)(\Delta_L - 2)}. \]

(4.22)

The first three examples \( J = 6, 8, 10 \) are verified to be the same as those in [48].

Before ending this section, we would like to comment what we have learned about the heavy-light bootstrap algorithm from \( d = 4 \) examples. Even though the algorithm is clear and in principle it could reveal HLLH \( s \)-channel data and HHLL \( s \)-channel multi-stress-tensor OPE coefficients up to any high order, some technical issues are impeding our efforts toward higher order. The most important technical issue is that higher order cross-channel four-point functions \( G \) needed in Lorentzian inversion formula require us to sum over twists \( n' \) and spins \( J \) for manipulation. In general, higher order calculations come with higher power of \( \log(1 - \bar{z}) \) in the integral, making the symbolic \( J \) formula for \( T^n \) OPE coefficients impossible, not mention summing them. Fortunately, the ansatz of HHLL four-point function proposed in [48] can release our pressure on summing all possible \( J \) in lowest-twist multi-stress-tensor blocks to pick up required HHLL four-point function \( G_{T^n} \).

Typically, \( G_{T^n} \) takes the form of that ansatz, where the undetermined coefficients could be

\[^3\text{We thank Junyu Liu, Wei Li and Jian-Dong Zhang for discussions on this integral.}\]
fixed by drawing references from some low-lying $J$ OPE coefficients of $T^m$. Thus the HHLL ansatz proposed in [18] is undoubtedly important for improving our algorithm, which could largely promote the efficiency. When it comes to summing twists $n'$, no difficulty appears in examples $d = 4$. However, we will see that this issue is inevitable in the next section. Some other issues exist and for the moment we are not aware of the resolution. For examples, we will see in next section that in general dimension even $O(\mu)$ order double-twist OPE coefficients can not be solved!

5 $O(\mu^2)$ bootstrap in general dimension

In this section, we would employ our algorithm to push on $O(\mu^2)$ heavy-light bootstrap in general dimensions. The main results are as follows:

1. We find a series representation of $O(\mu)$ correction to HLLH $s$-channel double-twist OPE coefficients in general dimension. Nicely, $O(\mu)$ order of HLLH $s$-channel double-twist anomalous dimension is found with a closed form as $3F_2$ function.

2. For lowest-twist double-stress-tensor OPE coefficient in general dimension, an infinite series representation is given.

5.1 A warm-up: free double-twist OPE

As a warm-up, we would like to reproduce the double-twist free OPE coefficients in this subsection. The key ingredient is HLLH $s$-channel funny block in general dimension, which is an infinite series and each term is shown in (3.6). For each term, we could take advantage of the nice formula (2.15) to integrate it and take the interested limit $\xi \to 0$ followed by $J' \to \infty$, in which we would like to recap the fact that only $\tilde{B}_{n,n}$ survives at heavy-limit as in (3.6), and we find

$$c(\Delta', J')|_n = \tilde{B}_{n,n} \frac{\Gamma(n - n' + 1)\Gamma(1 - \Delta_L)}{(n - n')\Gamma(1 + n - n' - \Delta_L)\Gamma(\Delta_L)}J^{\Delta_L - 1},$$

(5.1)

where we assume $\Delta' - J' = \Delta_H + \Delta_L + 2n'$ and $\tilde{B}_{n,n}$ can be found in (3.6). We are happy that the summation over $n$ is not hard, we find

$$c(\Delta', J') = \sum_{n=0}^{\infty} c(\Delta', J')|_n = -\frac{\Gamma(1 - n')\Gamma(\frac{d}{2} - n' - \Delta_L)}{n'\Gamma(\frac{d}{2} - n' - \Delta_L)\Gamma(\Delta_L)}J^{\Delta_L - 1}.$$
By taking the Residue around integer $n'$, it is straightforward to find
\[
\tilde{c}_{n',J'}^{\text{free}} = \frac{(-1)^n \Delta_L (d + 1) + n'}{\Gamma(n' + 1) \Gamma(\Delta_L)} J^j \Delta_L - 1
\]
which can be verified to be consistent with heavy and large $J'$ limit of (2.22) and comes back to (4.4) as soon as $d = 4$ is specified.

5.2 $O(\mu)$ double-twist

Now we turn to compute $O(\mu)$ correction of HLLH $s$-channel data. The essential ingredient is the form of $G_T$. Since we are only interested in large $J'$ limit, we could adopt the colinear block (2.6) in the cross-channel, we thus have
\[
G_T = \frac{\left\{ (1 - z)(1 - \bar{z}) \right\}^{\Delta_L} (1 - \bar{z})^{\frac{d-2}{2}} z^{\Delta_L \Delta_L - 1}}{(1 - \bar{z})^{\Delta_L \Delta_L - 1}} k_{d+2}^{0,0}(1 - z).
\]

The next step is to address $k_{d+2}^{0,0}(1 - z)$. The strategy is to expand it as an infinite series around $z \rightarrow 0$, and in the end sum back. Notice that the involved hypergeometric function is of the type $\text{I}_2F_1(\beta, \beta, 2\beta, 1 - z)$, specifically, $\beta = (d + 2)/2$, we should take following series
\[
\text{I}_2F_1(\beta, \beta, 2\beta, 1 - z) = \sum_{k=0}^{\infty} \frac{\Gamma(2\beta) \Gamma(\beta) (2(\psi_{k+1} - \psi_{k+\beta}) - \log z)}{(k!)^2 \Gamma(\beta)^2}.
\]

As expected, we have log free part and log part responsible for OPE and anomalous dimension respectively. Then we would like to obtain anomalous dimension at first by following the strategy demonstrated in section 4. For each $n$ and $k$ in the heavy and large spin limit we find
\[
c^{(1)}_{\log}(\Delta', J')|_{n,k} = \frac{(-1)^n \Delta_L \Gamma(d + k + 1) \Gamma(k + n - n') \Gamma(d - \Delta_L + 2) J^j \Delta_L - \frac{d}{2}}{d^2 \Gamma(d) \Gamma(k + 1) \Gamma(\frac{d}{2} - n) \Gamma(n + 1) \Gamma(d + k + n - n' - \Delta_L) \Gamma(- \Delta_L - \frac{d}{2} + 1)}.
\]

Fortunately, it is not difficult to sum $n$ and $k$ in (5.6)
\[
c^{(1)}_{\log}(\Delta', J') = \sum_{n,k=0}^{\infty} c^{(1)}_{\log}(\Delta', J')|_{n,k}
\]
\[
= \frac{\Delta_L \Gamma(-n') \Gamma(d - \Delta_L + 1) J^j \Delta_L - \frac{d}{2}}{4 \Gamma(d - \Delta_L + n' + 1) \Gamma(\Delta_L - \frac{d}{2} + 1)} \text{I}_3F_2 \left( \frac{d}{2} + 1, \frac{d}{2} + 1, -n'; 1, 1, d - n' - \Delta_L + 1; 1 \right). \]
After taking the Residue and dividing by free OPE (5.3), it gives rise to

\[ \tilde{\gamma}_{n',J'}^{(1)} = \frac{(-1)^n' \Gamma(\Delta_L + 1) \Gamma(d - \Delta_L + 1) \, _3F_2\left(\frac{d}{2} + 1, \frac{d}{2} + 1, -n'; 1, d - n - \Delta_L + 1; 1\right)}{2J'^{d-2} \Gamma(d - n - \Delta_L + 1) \Gamma(-\frac{d}{2} + n + \Delta_L + 1)}, \]

which is precisely what [42] obtained by using holographic technique of Hamiltonian perturbation theory.

For log free part, follow similar analysis, we find

\[ \tilde{c}^{(1)}(\Delta', J')|_{n,k} = -2c^{(1)}_{\log}(\Delta', J')\left(\psi_{k+1} - \psi_{k+(d+2)/2}\right). \] (5.9)

The difficulty thus arises. To our knowledge, we can only do summation over \( n \) in (5.9). When it comes to \( k \), polygamma functions are involved and the summation is hard to carry out. Nevertheless, we could take limit and Residue for each \( k \), in which a truncation in \( k \) summation becomes manifest if \( k_{\text{max}} = n' \), and we end up with

\[ c^{(1)}_{n',J'}|_{\text{free}} = \sum_{k=0}^{n'} \frac{(-1)^{n'-k+1} \Delta_L \Gamma(\frac{d}{2} + k + 1)^2 \Gamma(d - \Delta_L + 1) \Gamma(d + k - n' - \Delta_L + 1) \Gamma(\Delta_L - \frac{d}{2} + 1)}{2\Gamma(\frac{d}{2} + 1)^2 \Gamma(k + 1)\Gamma(n' - k + 1)\Gamma(\frac{d}{2} - \frac{d}{2} + 1).} \] (5.10)

The simplest case would be the leading-twist \( n' = 0 \), in general dimension we have

\[ \tilde{c}^{(1)}_{0,J'} = -\frac{\Gamma(\Delta_L + 1)(\gamma + \psi_{(d+2)/2})}{2\Gamma(\frac{d}{2} + 1)J'^{d-2} \Gamma(\Delta_L - \frac{d}{2} + 1)}. \] (5.11)

When \( d \) is even, it is not hard to implement the summation. Particularly, specializing (5.10) to \( d = 4 \) gives back to (4.5). Some other low-lying examples which are simple enough to present here are \( d = 6,8 \)

\[
\begin{align*}
\text{for } d = 6, & \quad \tilde{c}^{(1)}_{n',J'} = -\frac{\Delta_L(60n'(n' + \Delta_L - 2)) + 11(\Delta_L - 1)(\Delta_L - 2)}{12J'^2}, \\
\text{for } d = 8, & \quad \tilde{c}^{(1)}_{n',J'} = -\frac{5\Delta_L(\Delta_L + 2n' - 3)(5\Delta_L^2 + 42n' - 15)\Delta_L + 2(21n'^2 - 63n' + 5)}{24J'^3}.
\end{align*}
\] (5.12)

5.3 An infinite series of lowest-twist \( T^2 \)

In this section, we would like to see whether we can have access to something on \( T^2 \) OPE in general dimension. Although we do not even have a closed form for \( O(\mu) \) double-twist
OPE coefficients, they are not necessary to come in as we discussed in section 4, they are suppressed by double-discontinuity. Now to implement Lorentzian inversion formula we need the full heavy-block \( (3.7) \) with summing \( n \).

Thanks to the heavy-limit where we have \((3.10)\), we thus find the HLLH s-channel four-point function with bare double-twist operators is

\[
g_{n',J'}(z, \bar{z}) = \frac{\Delta_{\mu' + \Delta_L + n'} \Delta_{\mu' + d + n' + J' - 1}}{(\bar{z} - z)^{d - 2}}, \quad (5.13)
\]

which gives us the relevant term in \((4.8)\) when specializing in \( d = 4 \). Subsequently we will have exactly \((4.11)\) with \( \tilde{c}_{n',J}^{(1)} \) being slipped off (since it is irrelevant). However, we immediately encounter the problem. Following the algorithm, we are required to sum over twists \( n' \). Unfortunately, considering the anomalous dimension in general dimension \((5.8)\) is a generalized hypergeometric function without any simple identity to simplify, we are not likely to accomplish the summation. Nevertheless, as before, we could keep \( n' \) and apply Lorentzian inversion formula to each term with \( n' \). Although the process is very complicated and it is not appropriate to write all of them down, we manage to have a final answer for lowest-twist \( T^2 \) OPE contribute from each twist \( n' \) by following the standard steps as shown before. Hence, we end up with an infinite series representation for lowest-twist double-stress OPE coefficients

\[
c_{0,J}^{(2)} = \sum_{n'} H(\Delta_L, J) \frac{3F2 \left( \frac{d}{2} + 1, \frac{d}{2} + 1, -n'; 1, d - \Delta_L - n' + 1; 1 \right)^2}{\Gamma(d - \Delta_L - n' - 1) \Gamma(\Delta_L - \frac{d}{2} + n' + 1) \Gamma(\Delta_L + J - \frac{d}{2} - n' - 1)} \times 3F2 \left( d + J - 2, d + J - 2, \Delta_L + J + \frac{d}{2} - 2; 2(J + d - 2), \Delta_L + J + \frac{d}{2} + n' - 1; 1 \right), \quad (5.14)
\]

where \( H(\Delta_L, J) \) is given by

\[
H(\Delta_L, J) = \frac{16^{d - 2 - J} \pi^2 \Delta_L(d - \Delta_L)(d - \Delta_L - 1) \Gamma(\Delta_L + 1) \Gamma(J + d - 2) \Gamma(\Delta_L + J + \frac{d}{2} - 2)}{\Gamma(d + J - \frac{d}{2}) \Gamma(J + d - 2) \Gamma(\Delta_L - \frac{d}{2} + 1) \sin(\pi \Delta_L)}. \quad (5.15)
\]

However, it is rather difficult to start with the infinite series \((5.15)\) trying to work out examples with specific dimensions due to the existence of generalized hypergeometric function.

\footnote{The reason another part in \((4.8)\) is missing in \((5.13)\) is that \((5.13)\) is deduced from \((2.7)\) which is actually the pure power law block where another power series of \( z \) is not essential and is simply omitted. On the other hand, \((4.8)\) is deduced from the full \( d = 4 \) conformal block \((2.4)\). In other words, a half of block is enough for our purpose.
Instead, one should start with anomalous dimension (5.8). We find, for even dimension, (5.8) could be reduced to be a nice finite series for which summing $n'$ to obtain $g_{HLLH}^{(2)}$ is manageable. Thereafter, lowest-twist $T^2$ OPE coefficients with symbolic $J$ can be steadily extracted by following the standard integration technique. We present some low-lying examples $d = 6, 8, 10$ in Appendix [B]. It should be commented that it seems even dimension is special, while odd dimension is harder to handle. This is consistent with holographic treatment of multi-stress-tensor OPE in [44, 51] where only even dimension case could be truncated to finite series such that the framework is applicable.

6 Conclusion and future directions

In this paper, we studied heavy-light four-point functions by implementing Lorentzian inversion formula back and forth. Focusing on non-degenerate scalar fields and assuming $\Delta_L$ is not integer and half-integer, we generally show (but not a serious proof) that Lorentzian inversion formula can probe the universality of lowest-twist multi-stress-tensor exchanged in HHLL $s$-channel and large spin OPE coefficients and anomalous dimensions of double-twist operators exchanged in HLLH $s$-channel. This universality holds at the region $\bar{z} \to 1$ with respect to HLLH $s$-channel. Moreover, an algorithm for computing these data was proposed. In this way, we could state that we can bootstrap heavy-light four-point functions. Applying the algorithm, examples of $d = 4$ up to triple-stress-tensor $T^3$ were presented, consistent with results in previous literatures. In addition, we also bootstrapped heavy-light four-point function up to $O(\mu^2)$ ($T^2$) order in general dimensions: we obtain $O(\mu^2)$ double-twist anomalous dimension in HLLH $s$-channel, series representations of $O(\mu^2)$ double-twist OPE coefficients in HLLH $s$-channel and series representations of lowest-twist double-stress-tensor OPE coefficients in HHLL $s$-channel.

Although now we can claim that universality of lowest-twist multi-stress-tensor in heavy-light four-point function is understood by Lorentzian inversion formula up to some extent, many related valuable questions are still far from clear. We would like to point out some important future directions

- The efficiency of our algorithm is somehow limited. [48] suggests that first few twists $n'$ of double-twist HLLH $s$-channel data and some low-lying spin $J$ of lowest-twist multi-stress-tensor OPE are enough to maintain the cycle of crossing back and forth and extract more data. It is thus important to investigate the necessary minimum number of twists $n'$ and spin $J$ examples in order to maintain the algorithm, which
could, enhance the efficiency and allow us to go to higher orders.

- It is clear from Lorentzian inversion formula that lowest-twist multi-stress-tensor OPE coefficients are suffering from some $\Delta_L$ poles. These poles are expected to be canceled by relevant double-twist operators $[O_L O_L]_{n,J}$ in HHLL $s$-channel with emergence of anomalous dimensions when $\Delta_L$ approaches the poles. Further understanding of this cancelation and inherent anomalous dimensions, along with extracting OPE of $[O_L O_L]_{n,J}$ is worthy and necessary whenever specific CFTs or supergravities are considered. This understanding, in turn, should shed light on holographic technique of relating near boundary data to near horizon regularity [44, 51].

- In order to touch specific CFTs or supergravities, it is also necessary to get rid of non-even-integer multiplet assumption. It is thus very important and interesting to include other light operators, forming a class of light operator where double-twist operators are mixed. In this situation, there should be extra index such that the double-twist OPE coefficients and anomalous dimensions in HLLH $s$-channel are matrixes and an appropriate diagonal basis is required.

- Our results achieve a precise agreement with [48], verifying the exponential ansatz in some sense. We wish, similar to Virasoro block in $d = 2$ [37], we could somehow directly solve the universal heavy-light conformal block of HHLL $s$-channel which is supposed to be exponentiated. This might be possible by using $6j$ symbol [56].

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A Details of $B_{n,m}^{a,b}$ and $\tilde{B}_{n,m}^{a,b}$

This Appendix is devoted to collect the skipped details in the main text about $B_{n,m}^{a,b}$ and $\tilde{B}_{n,m}^{a,b}$ in the heavy and large spin limit [31,10] and [35].
\section{A.1 $B_{n,m}^{a,b}$}

At first, we would like to keep track of full $B_{n,m}^{a,b}$ without any limits taken. The logic is simple, we just throw (2.7) into quadratic Casimir equation (2.2) with (2.3) and organize the resulting equation as recursion equation. We will frequently use two derivative identities for $k_{\beta}^{a,b}(\bar{z})$. The first one is

$$
\partial_{\bar{z}}^2 k_{\beta}^{a,b} = \frac{(4ab\bar{z} + \beta(\beta - 2)) k_{\beta}^{a,b} + 4(a + b + 1)\bar{z}^2 \partial_{\bar{z}} k_{\beta}^{a,b}}{4\bar{z}(\bar{z} - 1)},
$$

(A.1)

which connects second derivative to first derivative without shifting $\beta$. The second identity relates first derivative of $k_{\beta}^{a,b}$ to $k_{\beta}^{a,b}$ with $\beta$ shifted by $-2,0,2$, namely

$$
\partial_{\bar{z}} k_{\beta}^{a,b} = \frac{\beta}{2(1 - \bar{z})} k_{\beta - 2}^{a,b} - \frac{\beta(\beta - 2)(a + b) - 4ab}{2(\bar{z} - 1)\beta(\beta - 2)} k_{\beta}^{a,b} + \frac{(\beta^2 - 4a^2)(\beta^2 - 4b^2)(\beta - 2)}{32(\bar{z} - 1)(\beta - 1)\beta^2(\beta + 1)} k_{\beta + 2}^{a,b}.
$$

(A.2)

Then after expanding $z$ series and taking advantage of (A.1) and (A.2) to remove all derivatives, the Casimir equation becomes

$$
\sum_{m=-n}^{n} (A_{nm} B_{n,m}^{a,b} k_{\beta + 2m}^{a,b} + B_{n-1,m}^{a,b} k_{\beta + 2m}^{a,b}) + \sum_{p=1}^{n} \sum_{m=-n+p}^{n} \frac{1}{z^{p-1}} \left( -1 \frac{c_{1,0}^{1,0} k_{\beta + 2m}^{a,b}}{z} \right) B_{n-p,m}^{a,b} = 0,
$$

(A.3)

where all $A, B, C$ are given by

\begin{align*}
A_{nm} &= 2\left(m^2 + m(\beta - 1) + n(n + \tau - d + 1)\right), \quad c_{n}^{1,0} = -2(d - 2)n, \\
B_{n} &= -\frac{1}{2}(2a + 2n + \tau)(2b + 2n + \tau), \quad c_{m}^{2,-1} = (d - 2)(2m + \beta - \tau), \\
C_{m}^{2,0} &= \frac{(d - 2)(2a(\beta + 2m - 2)(\beta + 2m) + 4ab(\tau - 2) + (\beta + 2m - 2)(\beta + 2m)(2b + \tau + 4n))}{2(\beta + 2m - 2)(\beta + 2m)}, \\
C_{m}^{2,1} &= -\frac{(d - 2)((2m + \beta)^2 - 4a^2)((2m + \beta)^2 - 4b^2)(\beta + \tau + 2m - 2)}{16(\beta + 2m + 1)(\beta + 2m)^2(\beta + 2m - 1)}. 
\end{align*}

(A.4)
In addition, another important identity is necessary \[10\]

\[
\frac{k_{\beta}^{a,b}}{\bar{z}} = k_{\beta-2}^{a,b} + \frac{1}{2} - \frac{2ab}{\beta(\beta-2)}k_{\beta}^{a,b} + \left(\frac{a^2 - \frac{1}{2}\beta^2}{\beta^2(\beta-1)}\right)k_{\beta+2}^{a,b}.
\]  

(A.5)

By using this identity (A.5) to remove all extra \(1/\bar{z}\), the equation (A.3) boils down to a recursion relation that could be solved for \(B_{n,m}^{a,b}\) with boundary condition \(B_{0,m}^{a,b} = \delta_{0m}\). Take examples for \(n = 1\), we find

\[
B_{1,-1}^{a,b} = \frac{(d-2)(\beta - \tau)}{2(\beta - \tau + d - 4)},
\]

\[
B_{1,0}^{a,b} = \frac{1}{2}(a + b + \frac{2ab(4 + 2\beta - \beta^2 + d(\tau - 2) - 2\tau)}{(\beta - 2)(d - 2 - \tau)}),
\]

\[
B_{1,1}^{a,b} = \frac{(d-2)(\beta^2 - 4a^2)(\beta^2 - 4b^2)(\beta + \tau - 2)}{32(\beta - 1)\beta^2(\beta + 1)(\beta + \tau - d + 2)}.
\]

(A.6)

By solving \(B_{n,m}^{a,b}\) order by order and taking the relevant limits, the formula (3.10) would come out. However, this approach is not convincing enough in the sense that we could not find a well-organized closed formula as a solution of the full recursion (A.3).

In fact, we can restrict onto bare double-twist trajectories and take the heavy-limit at the very beginning and surprisingly the infinite recursion equation would be self-consistently truncated to be finite and simple one. Taking the heavy-limit reduces (2.7) to (3.7) with vanishing \(\gamma(\mu)\), i.e.

\[
G_{\Delta',J'}(z, \bar{z}) = \sum_{n=m}^{m=n} \sum_{m=-n}^{m=n} B_{n,m}^{a,b} z^{\frac{1}{2}(2(n'+n)+\Delta_L+\Delta_H)} \bar{z}^{\Delta_H+\Delta_L} z^{\Delta_H+\Delta_L} + J'+m+n'.
\]

(A.7)

Subsequently, the quadratic Casimir equation becomes a simple recursion equation

\[
B_{n,m}^{a,b} = -\frac{1}{A_{nm}}(A_{n-1,m}^{0,0}B_{n,m-1}^{a,b} + A_{n-1,m}^{1,0}B_{n-1,m}^{a,b} + A_{n-1,m+1}^{1,1}B_{n-1,m+1}^{a,b} + A_{n-2,m+1}^{2,1}B_{n-1,m+1}^{a,b}),
\]

(A.8)

where \(A\)'s are

\[
A_{nm}^{0,0} = -2(m^2 + m(\beta - 1) + n(\tau - d + n + 1)), \quad A_{nm}^{0,-1} = \frac{1}{2}(2m + 2a + \beta),
\]

\[
A_{nm}^{1,0} = -\frac{1}{2}(2(m - n) + \beta - \tau)(2(m + n + 2a - d + 2) + \beta + \tau),
\]

33
\[ A_{nm}^{1,1} = 2(m^2 + n^2) + 2m(\beta - d + 1) - (d - 2)(\beta - \tau) + 2n(\tau - 1), \]
\[ A_{n}^{2,1} = -\frac{1}{2}(2n + 2a + \tau)^2. \] (A.9)

We should emphasize that in above recursion (A.8) we have already specify \( b = a = \frac{1}{2}(\Delta_L - \Delta_H) \) as before, and in particular \( \tau = \Delta_H + \Delta_L + 2n' \) where \( n' \) is arbitrary twist. Then we can continue, take heavy and large spin limit in above \( A \) in recursion equation (A.8). We find for \( n > m > -n \)
\[ \frac{A_{n-1,m+1}^{1,1}}{A_{n,m}^{1,0}} = -1, \quad \frac{A_{n-1,m}^{1,0}}{A_{n,m}^{0,1}} = \frac{A_{n-1,m}^{2,1}}{A_{n,m}^{0,0}} = 0, \quad \text{for} \quad n > m > -n. \] (A.10)

For \( m = n \), all allowed terms are zero, it is thus clear from (A.10) that all \( B_{n,m>-n}^{a,b} = 0 \). Then we just need to figure out the recursion equation provided with \( m = -n \). Typically, when \( m = -n \), only the third term in the right hand side of (A.8) makes sense, and it is evaluated to be \((-d + 4 - 2n)/(2n)\). Then the recursion equation is largely simplified to be
\[ B_{n,-n}^{a,b} = \frac{d - 4 + 2n}{2n} B_{n-1,1-n}^{a,b}, \] (A.11)
which is easily to be solved by
\[ B_{n,-n}^{a,b} = \frac{(\frac{d}{2} - 1)^{n}}{\Gamma(n + 1)}. \] (A.12)

However, this shall not be the end of story. The reduced block that needs to be solved (A.7) suffers from ambiguity of \( m \). To be precise, for example, relevant \( \bar{z}^{m-1} \) in (A.7) could either be \( k_{\beta+2m}/\bar{z} \) or \( k_{\beta+2(m-1)}^{a,b} \). Fortunately, this ambiguity is of no significance here, because we could always use (A.5) to state \( k_{\beta+2m}/\bar{z} \) and \( k_{\beta+2(m-1)}^{a,b} \) is equivalent provided with the coefficients in (A.5) is vanishing in the heavy-limit. Till now, the proof of (3.10) is completed.

### A.2 \( \tilde{B}_{n,m}^{a,b} \)

Now we turn to draw (3.6) for \( \tilde{B}_{n,m}^{a,b} \). We have to remind that this subsection is not a serious proof, but should be served as a strong evidence that (3.6) is correct. In fact, as soon as we solve \( B_{n,m}^{a,b} \) in (2.7) from (A.3), we could multiply by the overall factor \( \kappa_{a,b}(\beta'/\gamma')/\kappa_{a,b}(\beta' + 2m)(1 - z)^{a+b}(1 - z/\bar{z})^{d-2} \), then we re-expand \( z \), organize resulting expansion as (3.1) by
using \((A.5)\) and turn \((\Delta \rightarrow J + d - 1, J \rightarrow \Delta - d + 1)\), the coefficients \(\tilde{B}_{n,m}^{a,b}\) could thus be read off \(19\). Take the heavy and large spin limit, we can observe \((3.6)\). As in previous subsection on \(B_{n,m}^{a,b}\), this approach is not satisfactory since we are not allowed to solve \((3.6)\) in an apparent way.

A better way is to take the heavy-limit at the first place. One should note we have a factor \(\kappa^{a,b}(\beta')/\kappa^{a,b}(\beta' + 2m)\) attached to each \(m\) which is a little bit annoying and unnatural. For now, we simply do not consider this factor and aim to solve an auxiliary coefficients \(\tilde{B}_{n,m}^{a,b}\) in

\[
G_{j+d-1,\Delta-d+1}^{a,b} = \sum_{n} \sum_{m=-n}^{n} \tilde{B}_{n,m}^{a,b} (1 - z)^{-a-b} (1 - \frac{z}{2})^{2-d} z^{-\frac{\tau}{2} + d + n - 1} \frac{\beta}{2 + m}. \tag{A.13}
\]

The resulting recursion equation is infinite but neat

\[
\tilde{B}_{n,m}^{a,b} = -\frac{1}{\mathcal{A}_{n,m}} \left( \tilde{A}_{m-1,n}^{0,0} \tilde{B}_{n,m-1}^{a,b} + \tilde{B}_{n-1,m}^{1,0} \tilde{D}_{n-1,m}^{a,b} + \tilde{B}_{n-1,m+1}^{1,-1} \tilde{D}_{n-1,m+1}^{a,b} \right.
\]

\[
+ \sum_{p=2}^{n} \left( \tilde{C}_{n-p,m+p-1}^{1,0} \tilde{B}_{n-p,m+p-1}^{a,b} + \tilde{C}_{n-p,m+p+1}^{2,0} \tilde{D}_{n-p,m+p}^{a,b} \right) \tag{A.14}
\]

where the coefficients are given by

\[
\tilde{A}_{n,m}^{0,0} = 2(m^2 + m(\beta - 1) + n(n - \tau + d - 1)), \quad \tilde{A}_{m}^{0,-1} = -\frac{1}{2}(2a + 2m + \beta)(2b + 2m + \beta),
\]

\[
\tilde{B}_{n,m}^{1,0} = \frac{1}{2} \left( 4d(m - n) - 4m^2 + 4b(d + n) + 2d\beta - 4(\beta + 2m + 2d + b - 3) + 2(d - b)\tau 
\right.
\]

\[
\left. + \tau(4n - \tau) - 4a(b + \tau - n - d + 1) \right),
\]

\[
\tilde{B}_{n,m}^{1,-1} = -(d - 2)(\beta + \tau + 2m - 2n - 6), \quad \tilde{C}_{n,m}^{1,C} = (d - 2)(\beta + \tau + 2(m - n + a + b - 2p)),
\]

\[
\tilde{C}_{n,m}^{2,p} = -(d - 2)\left( \beta + \tau + 2(m - n - 2p - 1) \right). \tag{A.15}
\]

Then we take the heavy and large spin limit for these coefficients within double-twist trajectories. For \(n > m > -n\), first three terms in the right hand side of \((A.14)\) tend to zero. Furthermore, for \(m = n\), only the first term in the right hand side of \((A.14)\) makes sense, although it is not zero and actually diverges as \(J\), it expresses \(\tilde{B}_{n,m}^{a,b}\) in terms of \(\tilde{B}_{n,n-1\leq n}^{a,b}\) which is zero, indicating that all \(\tilde{B}_{n,m>-n}^{a,b} = 0\). Again we are left with \(\tilde{B}_{n,-n}^{a,b}\), for which the
recursion equation reduces to
\[
\hat{B}_{n,n}^{a,b} + \frac{d - 2}{2n} \sum_{p=1}^{n} \hat{B}_{n-p,-n+p}^{a,b} = 0, \tag{A.16}
\]
which is easily solved by
\[
\hat{B}_{n,n}^{a,b} = (-1)^n \left( \frac{d}{2} - n \right)_n \frac{n}{\Gamma(n + 1)}. \tag{A.17}
\]

Then we would like to recover the factor \( \kappa_{a,b}(\beta')/\kappa_{a,b}(\beta' + 2m) \) and translate \( \hat{B} \) to \( \tilde{B} \). One may naively multiply \( \kappa_{a,b}(\beta')/\kappa_{a,b}(\beta' - 2n) \), which, however, identically vanishes at heavy and large spin limits. This subtlety arises because of the ambiguity of \( \bar{z}^m \) exactly as previous subsection. Now we are not lucky enough to make \( k^{a,b}_{\beta+2m}/\bar{z} \) and \( k^{a,b}_{\beta+2(m-1)} \) equivalent, since the factor \( \kappa_{a,b}(\beta')/\kappa_{a,b}(\beta' + 2m) \) is different for each of them. The only possibility such that we have nontrivial result is\(^5\)
\[
G_{\beta+d-1,\Delta-d+1}^{a,b}|_{n} = \hat{B}_{n,-n}^{a,b}(1 - z)^{-a-b}(1 - \frac{z}{\bar{z}})^{2-dz-\frac{d}{2}+d+n-1} \frac{\kappa_{a,b}}{\bar{z}^n}. \tag{A.18}
\]

We then should adopt (A.5) \( n \) times to remove all additional \( 1/\bar{z} \) factor, and multiplying each term with corresponding \( \kappa_{a,b}(\beta')/\kappa_{a,b}(\beta' + 2m) \) factor. Note the factor \( \kappa_{a,b}(\beta')/\kappa_{a,b}(\beta' + 2m) \) goes like \( \xi^{-2m} \), while coefficients for second and third term in the right hand side of (A.5) behave as \( \xi \) and \( \xi^2 \) respectively, we finally find the only surviving term is \( \hat{B}_{n,-n}^{a,b} k^{a,b}_{\beta+2m} \), thus
\[
\hat{B}_{n,n}^{a,b} = \hat{B}_{n,-n}^{a,b}, \quad \hat{B}_{n,m<n}^{a,b} = 0, \tag{A.19}
\]
which is precisely (3.6).

B More examples for double-stress-tensor

In this subsection, we present some low-lying examples \( d = 6, 8, 10 \) for lowest-twist double-stress-tensor OPE coefficients. Actually, from our algorithm of bootstrapping heavy-light four-point function, it is not difficult to work out more even dimensional examples. Typically, we find that lowest-twist double-stress-tensor OPE coefficients in even dimension

\(^5\)Actually, a more general possibility should be an arbitrary linear combination \( \sum_{q=0}^{n} c_q k^{a,b}_{\beta - 2(n-q)/\bar{z}^q} \) with \( \sum_{q} c_q = 1 \). However, only \( c_n \) will come into the final answer while all other \( c_i \)'s are redundancies. Thus it is natural to shut them down while keep \( c_n = 1 \).
follow the pattern as

\[ a^{(2)}_{0,J} = \frac{2^{2-3d-2}J \sqrt{\pi} \Delta_L \Gamma(\Delta_L - d + 2) \Gamma\left(\frac{J+2}{2}\right) \Gamma\left(\frac{J+d-2}{2}\right) \Gamma(J + d - 3)}{\Gamma(\Delta_L - \frac{d}{2} + 1) \Gamma\left(\frac{J+d-3}{2}\right) \Gamma\left(\frac{J+2d}{2}\right) \Gamma(J + d - \frac{9}{2})} \sum_{i=0}^{d} a^{(2)}_i \Delta_L, \]

\[ a^{(2)}_{d/2} = \frac{\Gamma\left(\frac{J+d-2}{2}\right) \Gamma\left(\frac{J+2d-1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right) \Gamma\left(\frac{J+d-1}{2}\right)}, \quad a^{(2)}_0 = \text{const}, \quad a^{(2)}_{i \neq 0 \wedge d/2} = \sum_{p=0}^{p=d} b^{(2)}_{ip} J^p. \quad \text{(B.1)} \]

However we do not find patterns governing the constant \( a^{(2)}_0 \) and other \( b^{(2)}_{ip} \). We then just list other \( a^{(2)}_i \) or \( b^{(2)}_{ip} \) in various dimension below.

\( d = 6 \)

\[ a^{(2)}_0 = 86400, \quad b^{(2)}_{10} = 51840, \quad b^{(2)}_{11} = 45864, \quad b^{(2)}_{12} = 1288, \quad b^{(2)}_{13} = -1554, \quad b^{(2)}_{14} = 134, \]
\[ b^{(2)}_{15} = 42, \quad b^{(2)}_{16} = 2, \quad b^{(2)}_{20} = -8640, \quad b^{(2)}_{21} = 5796, \quad b^{(2)}_{22} = 9060, \quad b^{(2)}_{23} = 1323, \quad b^{(2)}_{24} = -273, \]
\[ b^{(2)}_{25} = -63, \quad b^{(2)}_{26} = -3. \quad \text{(B.2)} \]

\( d = 8 \)

\[ a^{(2)}_0 = 67737600, \quad b^{(2)}_{10} = 82252800, \quad b^{(2)}_{11} = 24783264, \quad b^{(2)}_{12} = -2374984, \quad b^{(2)}_{13} = 63624, \]
\[ b^{(2)}_{14} = 120746, \quad b^{(2)}_{15} = -9504, \quad b^{(2)}_{16} = -3676, \quad b^{(2)}_{17} = -264, \quad b^{(2)}_{18} = -6, \quad b^{(2)}_{20} = 12700800, \]
\[ b^{(2)}_{21} = 21699216, \quad b^{(2)}_{22} = 4826804, \quad b^{(2)}_{23} = -785444, \quad b^{(2)}_{24} = -171101, \quad b^{(2)}_{25} = 26224, \]
\[ b^{(2)}_{26} = 7006, \quad b^{(2)}_{27} = 484, \quad b^{(2)}_{28} = 11, \quad b^{(2)}_{30} = -1814400, \quad b^{(2)}_{31} = 231264, \quad b^{(2)}_{32} = 1878616, \]
\[ b^{(2)}_{33} = 710424, \quad b^{(2)}_{34} = 29146, \quad b^{(2)}_{35} = -22704, \quad b^{(2)}_{36} = -4076, \quad b^{(2)}_{37} = -264, \quad b^{(2)}_{38} = -6. \quad \text{(B.3)} \]

\( d = 10 \)

\[ a^{(2)}_0 = 109734912000, \quad b^{(2)}_{10} = 176795136000, \quad b^{(2)}_{11} = 29162885760, \quad b^{(2)}_{12} = -1932683616, \]
\[ b^{(2)}_{13} = 245131200, \quad b^{(2)}_{14} = -28845960, \quad b^{(2)}_{15} = -15354360, \quad b^{(2)}_{16} = 1926792, \quad b^{(2)}_{17} = 615600, \]
\[ b^{(2)}_{18} = 50760, \quad b^{(2)}_{19} = 1800, \quad b^{(2)}_{20} = 24, \quad b^{(2)}_{21} = 7193731200, \quad b^{(2)}_{22} = 41655168000, \]
\[ b^{(2)}_{23} = 1983391200, \quad b^{(2)}_{24} = -723441000, \quad b^{(2)}_{25} = 146322800, \quad b^{(2)}_{26} = 26696250, \]
\[ b^{(2)}_{27} = -5549250, \quad b^{(2)}_{28} = -1363500, \quad b^{(2)}_{29} = -107100, \quad b^{(2)}_{30} = -3750, \quad b^{(2)}_{31} = -50, \]
\[ b^{(2)}_{32} = 4267468800, \quad b^{(2)}_{33} = 11649074400, \quad b^{(2)}_{34} = 4893789960, \quad b^{(2)}_{35} = 1465905000, \]
\[ b^{(2)}_{36} = -176081150, \quad b^{(2)}_{37} = -9161775, \quad b^{(2)}_{38} = 5702655, \quad b^{(2)}_{39} = 1046250, \quad b^{(2)}_{40} = 765000, \]
\[ b^{(2)}_{41} = 2625, \quad b^{(2)}_{310} = 35, \quad b^{(2)}_{42} = -609638400, \quad b^{(2)}_{43} = -134438400, \quad b^{(2)}_{44} = 568117440, \]
\[ b^{(2)}_{45} = 332499000, \quad b^{(2)}_{46} = 53675800, \quad b^{(2)}_{47} = -4186350, \quad b^{(2)}_{48} = -2455530, \quad b^{(2)}_{49} = -337500, \]
\[ b_{48}^{(2)} = -22500, b_{49}^{(2)} = -750, b_{10}^{(2)} = -10. \quad (B.4) \]

The case \( d = 6 \) was obtained recently in [48], which is exactly same as ours.

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