Kinetic phase transition with global coupling in the resonantly driven atomic trap

Geol Moon$^1$, Yonghee Kim$^{1,3}$, Myoung-Sun Heo$^{1,4}$, Daegun Ahn$^1$, Jina Park$^1$, Soyoung Shin$^1$, Heung-Ryoul Noh$^2$ and Wonho Jhe$^{1,5}$

$^1$ Department of Physics and Astronomy, Seoul National University, Seoul 151-747, Korea
$^2$ Department of Physics, Chonnam National University, Gwangju 500-757, Korea
E-mail: whjhe@snu.ac.kr

New Journal of Physics 15 (2013) 103030 (13pp)
Received 12 July 2013
Published 28 October 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/10/103030

Abstract. We present an experimental observation and comprehensive theory that address the effect of interaction on the phase transition in a many-body system, which is far from equilibrium and does not have detailed balance. In the resonantly driven cold atomic trap, we observe kinetic phase transition (KPT) that is manifested by substantial enhancement of fluctuations, similar to the first-order phase transition in thermal equilibrium. Moreover, we can control the attractive interaction between atoms trapped in the coexisting states by adjusting the total number of atoms, which induces shift of the phase boundaries of KPT. The demonstrated effect is the many-body analogue of KPT that was previously predicted and observed in a single-driven oscillator. Our work provides a unique model platform for quantitative study of nonequilibrium nonlinear dynamics in many-body systems.

---

3 Current address: Quantum Optics Division, Korea Atomic Energy Research Institute, Daejeon 305-353, Korea.
4 Current address: Korea Research Institute of Standards and Science, Daejeon 305-340, Korea.
5 Author to whom any correspondence should be addressed.
1. Introduction

The last few decades have witnessed an immense amount of research activity, both in experimental and theoretical work, focused on understanding the detailed dynamics of quantum [1] and classical systems [2–5] that are under time-dependent external perturbations. In general, such systems exhibit an interplay of three characteristic components: (i) nonlinearity, (ii) nonequilibrium behavior and (iii) quantum tunneling (for quantum systems) or large rare fluctuation (for classical systems). In particular, the large rare fluctuations provide a variety of phase transitions far from equilibrium in classical nonlinear dynamical systems [6–10].

The study of large rare fluctuations has mainly been focused on prominent qualitative changes of an equilibrium system, such as the nucleation at phase transitions, chemical reactions, mutations in DNA sequences and protein transport in biological cells [11–14]. On the other hand, the fluctuating systems of interest are far from thermal equilibrium; for example, lasers, pattern-forming systems, radio-frequency-driven Josephson junctions [15], micromechanical and nanomechanical oscillators [16–18] and periodically driven trapped electrons [19–22]. In particular, in the periodically driven nonlinear dynamic system out of equilibrium, which exhibits intrinsic bistable or multistable states, the fluctuation triggers noise-induced switching between stable states [5, 8, 9], where the fluctuation-induced switching dynamics shows characteristics similar to the equilibrium phase transition [5, 8].

It is well known that, in thermodynamic systems, the increase of fluctuations at the critical point is a universal characteristic of phase transition [23]. In particular, even in the single-particle-driven nonlinear systems [5], such as the Duffing oscillator, significant enhancement of fluctuations that arise from noise-induced switching between coexisting states was also observed in the measured spectral density of fluctuations (SDFs) near the specific driving frequency, which is indicative of a phase transition in nonequilibrium systems—the so-called kinetic phase transition (KPT) [8]. Nonetheless, in such a driven mechanical system, it is experimentally difficult to go beyond the single-particle behavior and investigate many-body effects associated with the inter-particle interactions between bistable states, although the coupled oscillator arrays have been explored theoretically [24–27]. In the parametrically driven cold atom trap [9, 10], on the other hand, the globally coupled atom–atom attractive interaction between attractors was shown to produce numerous qualitative changes in the nonequilibrium phase transition, while exhibiting the ideal mean-field transition. Therefore, it is essential to make an experimental
Figure 1. Weakly interacting atomic Duffing oscillators under fluctuation-induced switching. (a) The fluctuations due to spontaneous emission of photons triggers noise-induced switching between two attractors. The images of oscillatory atomic motion are captured at regular intervals such that the LS atomic cloud is fully apart from the trap center (dashed horizontal line) whereas the SS cloud is not in the maximum vibrational amplitude because of the phase difference between LS and SS. The number of atoms is about $2.27 \times 10^7$ and the measured $\gamma$ and $\omega_0$ are $50.44 \text{ s}^{-1}$ and $2\pi \times 28.26 \text{ Hz}$, respectively. (b) Schematic representation of experiment setup and the typical intensity profile of atoms in LS and SS. (c) The change of atomic population in each state versus the driving frequency $\nu_F = \omega_F / 2\pi$. KPT occurs when the two populations are equal in the CS. (d) The measured intensity of SDF is maximized at the KPT point, leading to the onset of large fluctuations linked to transitions between two states (semi-log scale). The error bars show the standard deviation.

study of KPT in a weakly interacting ensemble system [28], which still lacks despite many KPT works performed so far.

In this paper, we experimentally realize the globally coupled Duffing oscillators in an optically modulated magneto-optical trap of $^{85}$Rb in the presence of long-range interatomic interactions associated with the shadow-induced light force [10] and advance a theory. This paper is organized as follows. In section 2, the experimental results are presented and compared to the theoretical results. In section 3, we demonstrate the theoretical approach of global coupling long-range interaction between dynamical bistable states consisting of many particles. Section 4 summarizes the results.

2. Experimental results and discussions

The resonantly driven nonlinear oscillator, Duffing oscillator, exhibits two coexisting dynamical states (CS) as well as the large-amplitude state (LS) and small-amplitude state (SS), which have differing vibrational amplitude and phase at a specific range of driving frequency (figure 1(a)). In the presence of noise, such as thermal fluctuations, there occurs noise-induced switching
between CS, and the transition probability as well as the occupied population between them are determined by the activation energy and noise intensity, given by

$$P_n = \frac{W_{mn}}{W_{nn} + W_{mm}}, \quad W_{nm} = C \exp(-R_n/D),$$

where $W_{nm}$ and $P_n$ are the transition probabilities from the $n$th state to the $m$th state ($n, m = 1, 2, n \neq m$) and the occupied population in the $n$th state, respectively. Here $P_{1(2)}$ is the normalized atomic populations of LS (SS), so that $P_1 + P_2 = 1$, and $C$ is a constant. The subscript 1 and 2 denote LS and SS, respectively. $R_n$ stands for the activation energy of the $n$th state and $D$ is the noise intensity. Notice that $W_{nm}$ and $P_n$ have been calculated previously for a single Duffing oscillator, which is expected to exhibit differing behavior for an interacting many-oscillator system.

The many-body system of the Duffing oscillators is experimentally realized in the magneto-optical trap of $^{85}$Rb atoms by modulating the intensity of trap lasers counterpropagating along the anti-Helmholtz coil axis with the phase difference between the two lasers maintained at $\pi$ (figure 1(b)). We measure the atomic populations in each state to investigate the noise-induced switching between two attractors, and compare experimental results with simulation. Near the KPT point where the populations of two attractors are comparable, we also observe the enhanced intensity of the SDFs. Notice that, in our ensemble system, $10^6$–$10^7$ atoms are distributed over dynamical states, and thus one can readily measure the steady-state populations occupied in each state without any statistical analysis. Figure 1(a) shows the atomic cloud ($T \approx 0.4$ mK) in the LS, CS and SS within a specific frequency range. Notice that because the distance between the two attractors is not far enough, there exists a region where the two atomic clouds overlap, manifested by the long cloud shape.

Figure 1(c) shows the normalized atomic populations versus $\omega_F/2\pi$ near the KPT point at 35 Hz. As shown, the two attractors are equally occupied at the KPT point because the activation energies of the two attractors are equal. Here the important feature of a phase transition is to observe large fluctuations associated with transitions between the two states.

To confirm this feature, we have obtained the fluctuations near the KPT point. In our system consisting of many particles, it is hard to obtain the time trace of single-particle trajectory. Instead of analyzing the time correlation of single-particle trajectory [2, 5], therefore, in our ensemble system of many atoms, we obtain the intensity $I$ of the SDF on the distance from KPT point, which represents the integrated spectral density, given by [8]

$$I \equiv \int_{-\infty}^{\infty} d\omega Q_\omega(\omega) = \frac{(z_{\text{max}}^1 - z_{\text{max}}^2)^2}{4} P_1 P_2,$$

where $z_{\text{max}}^{(i)}$ is the maximum amplitude of LS (SS) and $Q_\omega(\omega)$ is the fluctuational noise-induced spectral peak that arises due to noise-induced transition between two dynamical states, which $Q_\omega(\omega) \approx \frac{2m|\delta\omega|}{\hbar}(W_{12} + W_{21}) \exp\left(\frac{(z_{\text{max}}^1 - z_{\text{max}}^2)^2}{4} P_1 P_2\right)$ where $\delta\omega = \omega_f - \omega_0$. Figure 1(d) clearly shows the maximum SDF intensity near the driving frequency 35 Hz, as expected.

The atom–atom interaction induces change of the activation energy, which is measurable because the cumulative effect of the atomic interactions between clouds becomes significant [10, 29]. Figure 2 presents the simulation results on the effects of atomic interactions on the activation energy and the associated atomic populations. To control the cumulative atomic interaction, one can change $N_{\text{int}}$; the long-range attractive shadow force increases with $N_{\text{int}}$ because the atomic clouds in each state, while absorbing the incoming laser beams, attract each other due to the resulting imbalance of laser intensity.
Figure 2. Simulation results on the activation energy and the atomic population versus the driving frequency. (a) The activation energy $R_n$ ($R_n^{(0)}$) with (without) interaction; $R_{1(2)}$ of LS (SS). (b) The activation energy difference $\Delta R$ between LS and SS decreases with the interaction or $N_{\text{tot}}$. (c) The change of normalized populations in LS and SS in the weakly interacting regime. Here $N_0 = 10^6$, $\gamma$ and $\omega_0$ are 40.58 s$^{-1}$ and $2\pi \times 32.68$ Hz.

As shown in figure 2(a), as $N_{\text{tot}}$ increases, the atom–atom interaction induces the change of $R_1$ and $R_2$; the activation energy of LS (SS) increases (decreases) with $N_{\text{tot}}$. Figure 2(b) shows that the change of $R_{1(2)}$ then shifts the KPT point, where $\Delta R = R_2 - R_1 = 0$, toward the higher $\omega_F$ at the higher $N_{\text{tot}}$. In addition, at a fixed modulation frequency $\omega_F$, $\Delta R$ decreases with the increase of $N_{\text{tot}}$. Figure 2(c) presents the resulting population change; $P_1$ of LS increases with $N_{\text{tot}}$ (solid line) where $P_2$ of SS decreases (dot-dashed line), shifting the KPT point toward the higher $\omega_F$.

We have experimentally observed the effects of atom–atom interactions on the single-particle behavior of KPT (i.e. KPT without interaction). As shown in figure 3(a), as $N_{\text{tot}}$ increases, atoms in SS transfer to LS in a unidirectional way, so that the atom–atom interaction plays the role as the one-way bias field (notice that the effect of the interaction in the parametric resonance system is to induce the time-translational symmetry, breaking which results in the ideal mean-field transition [10]). To detail the characteristics of the interaction as a one-way bias field, we have measured the difference of atomic populations of each state, $P_1 - P_2$. Figure 3(b) shows that the population difference grows larger at the higher $N_{\text{tot}}$. The experimental results are in qualitatively good agreement with the theoretical results (figure 3(c)).

The characteristic interaction of the one-way bias field originates from the difference of the optimal path and equilibrium position of each stable state because the interatomic force depends...
on the position of all individual atoms. The coefficient $\alpha_{nm}$ in equation (17) is related to the interatomic force, where the terms with $n = m$ and $n \neq m$ indicate the force from the atoms in the same and in the different cloud, respectively. In our system, we find $\alpha_{11} \neq \alpha_{22}$ and $\alpha_{12} \neq \alpha_{21}$, which means the interatomic force that an atom in the cloud $m$ experiences by atoms in the same as well as in different clouds is not equal to that for an atom in the cloud $n$. In particular, notice that the effective switching activation energy $R_n^{(1)}$ (equation (16)) linearly depends on the number of atoms in the clouds, and also numerical calculation shows that $\alpha_{11} > \alpha_{22}, \alpha_{12}, \alpha_{21}$. For example, when $\alpha_{11} > 0$ while $\alpha_{22}, \alpha_{12}$ and $\alpha_{21}$ are negative, we observe that the effective switching activation energy of LS (SS) becomes positive (negative), so that the total activation energy of LS (SS) increases (decreases) with the total number of atoms, as shown in figure 2(a).

Figure 4 presents the phase-transition map in the parameter space of driving frequency $\omega_F$ and modulation amplitude $\epsilon$ ($\propto F_0$). In figure 4(a), the calculated KPT lines (black and green dashed lines) are shown, where the changes of the nonequilibrium first-order phase transition boundary are expected to depend on the total number of trapped atoms. The experimental results in figure 4(b) are in qualitatively good agreement (the differing absolute values of $N_{tot}$ may...
Figure 4. (a) Theoretical results on the shift of the KPT phase boundary in the $\nu_F$-$\epsilon$ parameter space with respect to the cumulative effect of atom–atom interactions (or $N_{\text{tot}}$). $\epsilon$ is proportional to $F_0$. (b) Experimental results on the shift of the KPT boundary. Measured $\gamma$ and $\omega_0$ are 40.56 s$^{-1}$ and $2\pi \times 28.07$ Hz, respectively. Red and blue dashed lines represent the boundary where the transition between a monostable state (LS or SS) and a coexisting bistable state (CS) occurs. Note that these boundary lines remain unaffected under the weak atom–atom interaction. As shown in the CS region, the typical KPT lines shift from black to green line as $N_{\text{tot}}$ is increased. Each circle denotes the experimental data, which show qualitatively good agreement with the numerical results except the absolute values of $N_{\text{tot}}$.

be attributed to the fact that the experiment is three-dimensional whereas the theory is one-dimensional).

Interestingly, a similar phenomenon is observed in the first-order gas–liquid transition in equilibrium states \[6, 23\]. The gas–liquid transition described by the van der Waals equation takes into account the nonzero radius of atoms as well as the attractive interactions between atoms. In the pressure–temperature ($P$–$T$) map, the phase transition can be plotted where there is a coexistence region and a critical point. The atomic radius and interaction depend on the species of atom, and thus the coexistence line in the $P$–$T$ plane changes with respect to the atomic species. Therefore, the attractive interaction between cold Duffing-oscillator atoms in the coexisting periodic attractors out of equilibrium exhibits close similarity with the gas–liquid transition in equilibrium. These results provide another piece of evidence for the similarity of phase transitions between equilibrium and nonequilibrium systems.

3. Theory of nonlinear vibrations in an interacting cold-atom system

3.1. Model

The vibrations in the periodically driven cold $^{85}\text{Rb}$ atom system are subject to the combined influence of a periodic field, a weak random force and a long-range attractive interaction. The equation of motion for this many-body system is derived from the atom–photon interaction theory for a $(1+3)$ atomic energy structure model \[30, 31\], which is given by

\[
\ddot{z}_i + \gamma \dot{z}_i + \omega_0^2 z_i + B_0(z_i + \eta \dot{z}_i)^3 = F_0 \cos \omega_F t + F_{\text{sh}} + f(t) \quad (i = 1, \ldots, N_{\text{tot}}),
\]
where $z_i$ and $\dot{z}_i$ are the position and velocity of the $i$th atom, and $N_{tot}$ is the total number of trapped atoms. We ignore the term $\eta \dot{z}_i$ as $\eta \dot{z}_i \ll 1$. $\omega_0$ is the natural frequency, $\gamma$ is the damping coefficient, $B_0$ is the nonlinear coefficient and $F_0$ is the magnitude of the force. They are given by

$$\omega_0^2 = C_t \omega_i^2, \quad \gamma = C_d \frac{\hbar k}{\mu_B} \omega_i^2,$$

$$B_0 = C_n \frac{8\mu_B^2 b^2 (4\delta^2/\Gamma^2 - 1)}{\hbar^2 \Gamma^2 (4\delta^2/\Gamma^2 + 1)^2} \omega_i^2,$$

$$F_0 = C_d \frac{\hbar k \Gamma s_0 \epsilon}{m_4 (4\delta^2/\Gamma^2 + 1)},$$

where $\omega_i^2 = 8k \mu_B b s_0 (-\delta/\Gamma)/[m_4 (1 + 4\delta^2/\Gamma^2)^2]$, and correction factors $C_t$, $C_d$ and $C_n$, considering an atomic multilevel energy structure, are 0.54, 0.41 and 0.19, respectively. $s_0$ is the on-resonance saturation parameter. $\epsilon$ stands for the modulation amplitude of the laser intensity. $k$ is the wave vector, $\mu_B$ is the Bohr magneton, $m_4$ is the mass of an atom and $\Gamma$ is the decay rate of the excited state ($= 2\pi \times 6.07$ MHz). $\delta$ and $b$ are the detuning of the laser frequency relative to the transition line $F_0 = 3 \rightarrow F_e = 4$ and the magnetic gradient field, respectively.

$f(t)$ is the white noise, which originates from the atomic random motion due to spontaneous emission, and $\langle f(t) f(t') \rangle = 2\gamma (k_B T/m_4) \delta(t - t')$. $k_B$ and $T$ are the Boltzmann constant and the temperature of the atom. The long-range shadow force $F_{sh}$ results from ‘shielding’ of atoms from the laser light by other atoms, which produces all-to-all attractive coupling between atoms in the two CS [10]

$$F_{sh} = -f_{sh} \sum_{j=1}^{N_{tot}} \text{sgn}(z_i - z_j),$$

where the interaction strength $f_{sh}$ is a very small constant value ($9.19 \times 10^{-32}$ N kg$^{-1}$). The $\text{sgn}(z)$ indicates the sign function and $z_j$ represents the position of the $j$th atom of a stable state.

The nonlinear oscillator in the periodically perturbed magneto-optical trap consists of a number of atoms, and there is a long-range attractive interaction between atoms, unlike in the single nonlinear mechanical oscillator [32]. The atom–atom interaction modifies the activation energy $R_n$ in equation (1), including the interaction effect, and thus the switching probability $W_{nn}$ is changed. The influence of the variation of the switching probability owing to the atom–atom interaction remarkably displays the qualitative and quantitative discrepancies between a single particle and many particles in nonlinear dynamic systems, as reported in [10]. For understanding the interaction effect in the Duffing oscillator clearly, it is useful to transform equation (3) into the rotating frame, $(z_i, \dot{z}_i) \rightarrow (Q_i, P_i)$ [33].

### 3.2. Rotating-wave approximation

The atomic dynamics in equation (3) can be described by changing to the rotating frame using a standard transformation

$$z_i = C_{\text{RWA}} [P_i \cos(\omega_R t) - Q_i \sin(\omega_R t)],$$

$$\dot{z}_i = -\omega_R C_{\text{RWA}} [P_i \sin(\omega_R t) + Q_i \cos(\omega_R t)].$$
with $C_{\text{RWA}} = (2\omega_0|\delta\omega/3|B_0)^{1/2}$ and $\delta\omega = \omega_0 - \omega$. In the rotating-wave approximation (RWA) \cite{10, 33}, the equation of motions for slow variables $\mathbf{q}_i \equiv (Q_i, P_i)$ in slow time $\tau = |\delta\omega|t$ are

$$\frac{d\mathbf{q}_i}{d\tau} = \mathbf{K}_i(\mathbf{q}_i) + f'(\tau), \quad \mathbf{K}_i(\mathbf{q}_i) = \mathbf{K}_i^{(0)}(\mathbf{q}_i) + \hat{\epsilon} \partial_{\mathbf{q}} H_{\text{sh}}$$

and

$$\mathbf{K}_i^{(0)}(\mathbf{q}_i) = -\frac{\gamma}{2|\delta\omega|} \mathbf{q}_i + \hat{\epsilon} \partial_{\mathbf{q}} H(\mathbf{q}_i),$$

$$H(\mathbf{q}_i) = \frac{1}{16} \left( Q_i^2 + P_i^2 \right)^2 - \frac{1}{2} \left( Q_i^2 + P_i^2 \right) - \frac{F_0}{2C_{\text{RWA}}|\delta\omega|} P_i,$$

where $i = 1, \ldots, N_{\text{tot}}$, $\partial_{\mathbf{q}} \equiv (\partial_{Q_i}, \partial_{P_i})$ and $f'(\tau)$ is white Gaussian noise with two asymptotically independent components

$$\left\{ f'_k(\tau) f'_{k'}(\tau') \right\} = 2D_k k_B T \delta_{kk'} \delta(\tau - \tau'),$$

where $D_k = 2D/ k_B T$ and $D = 3B_0 k_B T/(4\pi m_\gamma^3 \gamma)$.

The generalized force $\mathbf{K}_i(\mathbf{q}_i)$ on the $i$th atom depends on the dynamical variables $\mathbf{q}_i$ of all particles. The term $\mathbf{K}_i^{(0)}(\mathbf{q}_i)$ in equation (9) is the force in the absence of interaction. The tensor $\hat{\epsilon}$ is the permutation tensor: $\epsilon_{Q_i Q_j} = \epsilon_{P_i P_j} = 0, \epsilon_{Q_i P_j} = \epsilon_{P_i Q_j} = 1$. The Hamiltonian $H_{\text{sh}}$ that describes the shadow-effect-induced interaction has the form

$$H_{\text{sh}} = \frac{2f_{\text{sh}}}{\pi \omega T |\delta\omega|m_\gamma C_{\text{RWA}}} \sum_{i,j=1}^{N_{\text{tot}}} \sqrt{Q_{ij}^2 + P_{ij}^2},$$

where $Q_{ij} = Q_i - Q_j$ and $P_{ij} = P_i - P_j$. Thus the interaction from the shadow effect is described by a simple Hamiltonian in the slow variables.

3.3. Switching rate modification by the atom–atom interaction

The interaction between atoms leads to an extra force $F_{\text{sh}}$ in equation (3). This force is weak in the sense that it weakly affects the intracloud atomic dynamics. As reported in \cite{10, 33}, the modification of activation energy by the interatomic force on an atom in cloud $n$, to first order in $H_{\text{sh}}$, can be written as

$$R_n \approx R_n^{(0)} - \frac{1}{2} \int_{-\infty}^{\infty} d\tau \left[ \mathbf{q}_{\text{opt}} - \mathbf{K}_{\text{opt}}^{(0)} \right] \hat{\epsilon} \partial_{\mathbf{q}} H_{\text{sh}},$$

where $\mathbf{q}_{\text{opt}}$ is the optimal path for the $i$th atom in the absence of the interatomic interaction, and $\mathbf{K}_{\text{opt}}^{(0)} = \mathbf{K}^{(0)}(\mathbf{q}_{\text{opt}})$. Here, $\mathbf{q}_{\text{opt}} - \mathbf{K}_{\text{opt}}^{(0)}$ are the solutions of the variational problem of minimizing the functional $\mathcal{R}^{(0)}$ \cite{34}

$$\mathcal{R}^{(0)} = (4D_\tau)^{-1} \int d\tau f'(\tau)^2 + \int d\tau \lambda(\tau) \left[ \frac{d\mathbf{q}}{d\tau} - \mathbf{K}^{(0)}(\mathbf{q}) - f'(\tau) \right],$$

where $\lambda(\tau)$ is the Lagrange multiplier that relates the dynamics of the atom and the force to each other. In fact, $\mathbf{q}_{\text{opt}}$ determines the most probable path followed in switching.

By assuming that all the other atoms stay close to either of the attractors, the total activation energy $R_n (n = 1, 2)$ considering the interaction between atoms can be approximately written as

$$R_n = R_n^{(0)} + R_n^{(1)}, \quad R_n^{(1)} = \sum_{m=1,2} \alpha_{nm} N_m,$$

http://www.njp.org/
where $N_n$ is the number of atoms in cloud $n$, and $\alpha_{nm}$ is given by the explicit expressions

$$
\alpha_{nn} = \frac{f_{sh}}{\pi \omega_0 |\delta\omega|m_2C_{RWA}} \int_{-\infty}^{\infty} dt \left[ (\dot{P}_{\text{opt}}^{(n)} - K^{(0)}(P)) \eta_{Q}^{(n)}(t) - (\dot{Q}_{\text{opt}}^{(n)} - K^{(0)}(Q)) \eta_{P}^{(n)}(t) \right],
$$

$$
\alpha_{n3-n} = \frac{f_{sh}}{\pi \omega_0 |\delta\omega|m_2C_{RWA}} \int_{-\infty}^{\infty} dt \left[ (\dot{P}_{\text{opt}}^{(n)} - K^{(0)}(P)) \tilde{\eta}_{Q}^{(n)}(t) - (\dot{Q}_{\text{opt}}^{(n)} - K^{(0)}(Q)) \tilde{\eta}_{P}^{(n)}(t) \right].
$$

Here

$$
\eta_{Q}^{(n)}(t) = \frac{Q_{\text{opt}}^{(n)}(t) - Q_{\text{eq}}^{(n)}}{\sqrt{\xi_{1}^{(n)}(t)}}, \quad \tilde{\eta}_{Q}^{(n)}(t) = \frac{Q_{\text{opt}}^{(n)}(t) - Q_{\text{eq}}^{(3-n)}}{\sqrt{\xi_{2}^{(n)}(t)}},
$$

where $\eta_{P}^{(n)}$ and $\tilde{\eta}_{P}^{(n)}$ are obtained by changing $P$ and $Q$ with each other in $\eta_{Q}^{(n)}$ and $\tilde{\eta}_{Q}^{(n)}$.

The denominators in equation (18) are given by

$$
\xi_{1}^{(n)}(t) = (P_{\text{opt}}^{(n)}(t) - P_{\text{eq}}^{(n)})^2 + (Q_{\text{opt}}^{(n)}(t) - Q_{\text{eq}}^{(n)})^2,
$$

$$
\xi_{2}^{(n)}(t) = (P_{\text{opt}}^{(n)}(t) - P_{\text{eq}}^{(3-n)})^2 + (Q_{\text{opt}}^{(n)}(t) - Q_{\text{eq}}^{(3-n)})^2.
$$

$R_{n}^{(1)}$ in equation (16) shows that the effective switching activation energy linearly depends on the number of atoms in the clouds and $\alpha_{nm}$, for weak interatomic coupling. $\alpha_{nm}$ depends on the systemic parameters, optimal path ($Q_{\text{opt}}$, $P_{\text{opt}}$) and equilibrium position ($Q_{\text{eq}}$, $P_{\text{eq}}$) as shown in equation (17), and according to equation (16), the activation energy of each stable state is changed by the growth of the total number of atoms. From equations (1), (16) and (17) we can obtain the switching probability as follows:

$$
W_{nm}(N_n; N_{\text{tot}}) = W_{nm}^{(0)} \exp[(\alpha_{nm}^{0} + \beta_{nm}^{0})N_{\text{tot}} - 2\alpha_{nm}^{0}N_n] \quad (19)
$$

and

$$
W_{nm}^{(0)} = C_{W} \exp(-R_{n}^{(0)}/D),
$$

$$
\alpha_{nm}^{0} = (\alpha_{nn} - \alpha_{nm})/2D, \quad \beta_{nm}^{0} = -(\alpha_{nn} + \alpha_{nm})/2D. \quad (20)
$$

### 3.4. Master equation

To describe noise-induced switching between the bistable states, consisting of many particles, we need to solve the master equation describing the discrete jump process. Through the steady-state solution of the master equation, we can calculate the occupied population of each state as follows. Figure 5 depicts a schematic representation of the discrete jump process describing the master equation [35] for the time-dependent probability $P_1(N_1, t)$ of state 1, where $N_n$ and $N_{\text{tot}}$ mean the number of particles in state $n$ ($=1, 2$) and the total number of particles, respectively. The master equation for the time-dependent probability $P_1(N_1, t) \equiv P_1(N_1)$ of state 1, which means the probability of having $N_1$ atoms in cloud (state) 1 at time $t$, is as follows:

$$
\partial_t P_1(N_1) = -[\mu(N_1) + v(N_1)] P_1(N_1) + \nu(N_1 - 1) P_1(N_1 - 1) + \nu(N_1 + 1) P_1(N_1 + 1), \quad (21)
$$

where $\mu(N_1) = N_1 W_{12}(N_1; N_{\text{tot}})$ and $\nu(N_1) = (N_{\text{tot}} - N_1) W_{21}(N_{\text{tot}} - N_1; N_{\text{tot}})$. $W_{nm}(N_n; N_{\text{tot}})$ is the transition probability per unit time for one atom in state $n$, as a result of an $n \rightarrow m$ transition, and then for $N_n$ atoms in state $n$, the total transition probability per unit time becomes
Figure 5. Schematic representation of the discrete jump process describing the master equation for the time-dependent probability $P_1(N_1, t) \equiv P_1(N_1)$ of state 1, which means the probability of having $N_1$ atoms in cloud (stable state) 1 at time $t$.

$N_n W_{nm}(N_n; N_{tot})$. Therefore, in the case in which atoms transit from state 1 to state 2, the total transition probability is expressed by $\mu(N_1) = N_1 W_{12}(N_1; N_{tot})$. Conversely, in the case in which atoms transit from state 2 to state 1, $\nu(N_1) = N_2 W_{21}(N_2; N_{tot})$. Because the total number of atoms is conserved, it becomes $\nu(N_1) = (N_{tot} - N_1) W_{21}(N_{tot} - N_1; N_{tot})$. Let us calculate the time derivative of the average of the population in state 1, $\partial_t \langle N_1 \rangle$. It is given by

$$\partial_t \langle N_1 \rangle = \partial_t \left[ \sum_{N_1=0}^{N_{tot}} N_1 P_1(N_1) \right],$$

(22)

where $\sum_{N_1=0}^{N_{tot}} P_1(N_1) = 1$, and then

$$\partial_t \langle N_1 \rangle = \sum_{N_1=0}^{N_{tot}} N_1 \partial_t P_1(N_1)$$

$$= \sum_{N_1=0}^{N_{tot}} N_1 \left[-\mu(N_1) + \nu(N_1)\right] P_1(N_1) + \nu(N_1 - 1) P_1(N_1 - 1)$$

$$+ \mu(N_1 + 1) P_1(N_1 + 1).$$

(23)

Because $N_{tot} \gg 1$, $\partial_t \langle N_1 \rangle$ is simplified as follows:

$$\partial_t \langle N_1 \rangle = \sum_{N_1=0}^{\infty} \left[ \nu(N_1) - \mu(N_1) \right] P_1(N_1).$$

(24)

In the absence of fluctuations, we can obtain the deterministic equation from equation (24), which becomes $\partial_t N_1 = \nu(N_1) - \mu(N_1)$. The stationary state occurs when $\nu(N_1) = \mu(N_1)$, which provides the following result:

$$N_2 W_{21}(N_2; N_{tot}) = N_1 W_{12}(N_1; N_{tot}).$$

(25)

From the above formula, we can obtain the ratio of $N_1$ to $N_2$ as follows:

$$\frac{N_1}{N_2} = \frac{W_{21}(N_2; N_{tot})}{W_{12}(N_1; N_{tot})}$$

$$= \frac{W_{21}^{(0)} \exp[(\alpha^{21} + \beta^{21}) N_{tot} - 2\alpha^{21} N_2]}{W_{12}^{(0)} \exp[(\alpha^{12} + \beta^{12}) N_{tot} - 2\alpha^{12} N_1]}$$

$$= \exp\{-(\Delta R^{(0)} + \Delta R^{(1)})/D\},$$

(26)
where
\[ \Delta R^{(0)} = R_2^{(0)} - R_1^{(0)}, \]  
\[ \Delta R^{(1)} = D\{2(\alpha^{21}N_2 - \alpha^{12}N_1) - (\alpha^{21} - \alpha^{12} + \beta^{21} - \beta^{12})N_{tot}\}. \]

\( \Delta R^{(0)} \) is the activation energy difference between two states in the non-interacting case, and \( \Delta R^{(1)} \) is the activation energy difference between them, induced by the atom–atom interaction, depending on the number of atoms. Equation (26) can be simplified further in terms of \( P_1 (\equiv \frac{N_1}{N_{tot}}) \) as follows:
\[ P_1 = \frac{1}{1 + \exp[\Delta R(P_1)/D]}, \]  
where \( \Delta R(P_1) \equiv \Delta R^{(0)} + \Delta R^{(1)} \) and \( P_1 + P_2 = 1 \). According to the above equation, we can numerically calculate the density of the population, \( P_{1(2)} \).

4. Conclusions

In conclusion, two CS have been realized in a resonantly driven trapped cold atoms, interacting weakly and globally with each other. In contrast to the single-particle system, we directly measure the change of atomic population with respect to the variation of activation energy, and thereby confirm the fluctuation enhancement near the KPT point, which is characteristic of KPT far from equilibrium. Moreover, we observe that the attractive atom–atom interaction shifts the phase-transition boundaries of KPT, similar to the effect of interaction on the system governed by the van der Waals equation, which provides evidence of similarity between equilibrium and nonequilibrium phase transition. Our system serves as a unique simple platform suitable for investigating the nonlinear dynamics of many-body cold atoms far from equilibrium and relating this dynamics to other domains of physics (dynamical systems, statistical physics).

Acknowledgments

This work was supported by National Research Foundation of Korea (NRF) grant no. 2012-047677 funded by the Korean government (MEST). GM was supported by Korea Student Aid Foundation (KOSAF) grant no. S2-2009-000-01627-1 funded by the Korean government (MEST).

References

[1] Grifoni M and Hanggi P 1998 Phys. Rep. 304 229
[2] Chan H B, Dykman M I and Stambaugh C 2008 Phys. Rev. Lett. 100 130602
[3] Chan H B and Stambaugh C 2007 Phys. Rev. Lett. 99 060601
[4] Dykman M I, Mannella R, McClintock P V E and Stocks N G 1990 Phys. Rev. Lett. 65 48
[5] Stambaugh C and Chan H B 2006 Phys. Rev. Lett. 97 110602
[6] Kunz R E 1995 Dynamics of First-Order Phase Transitions (Frankfurt: Verlag Harri Deutsch)
[7] Horsthemke W and Lefever R 1984 Noise-Induced Transitions (New York: Springer)
[8] Dykman M I, Luchinsky D G, Mannella R, McClintock P V E, Stein N D and Stocks N G 1994 Phys. Rev. E 49 1198
[9] Kim K, Heo M S, Lee K H, Jang K, Noh H R, Kim D and Jhe W 2006 Phys. Rev. Lett. 96 150601

New Journal of Physics 15 (2013) 103030 (http://www.njp.org/)
[10] Heo M S, Kim Y, Kim K, Moon G, Lee J, Noh H R, Dykman M I and Jhe W 2010 Phys. Rev. E 82 031134
[11] La Mer V K 1952 Indust. Eng. Chem. 44 1270
[12] Moss F and McClintock P V E 1989 Noise in Nonlinear Dynamical Systems vol 2 (Cambridge: Cambridge University Press)
[13] Foster P L 2006 Methods Enzymol. 406 195
[14] Kosztin I and Schulten K 2004 Phys. Rev. Lett. 93 238102
[15] Siddiqui I et al 2005 Phys. Rev. Lett. 94 027005
[16] Stambaugh C and Chan H B 2006 Phys. Rev. B 73 172302
[17] Aldridge J S and Cleland A N 2005 Phys. Rev. Lett. 94 156403
[18] Badzey R L, Zolfagharkhani G, Gaidarzhy A and Mohanty P 2004 Appl. Phys. Lett. 85 3587
[19] Lapidus L J, Enzer D and Gabrielse G 1999 Phys. Rev. Lett. 83 899
[20] Cross M C and Hohenberg P C 1993 Rev. Mod. Phys. 65 851
[21] Dehmelt H 1990 Rev. Mod. Phys. 62 525
[22] Gabrielse G, Dehmelt H and Kells W 1985 Phys. Rev. Lett. 54 537
[23] Goldenfeld N 1992 Lectures on Phase Transitions and the Renormalization Group (Reading, MA: Addison-Wesley)
[24] Peles S and Wiesenfeld K 2003 Phys. Rev. E 68 026220
[25] In V, Longhini P, Kho N, Naik S, Palacios A and Neff J D 2011 Physica D 240 701
[26] Palacios A, Carretero-González R, Longhini P and Renz N 2005 Phys. Rev. E 72 026211
[27] Josić K and Peleś S 2004 J. Phys. A: Math. Gen. 37 11801
[28] Kim K, Heo M S, Lee K H, Ha H J, Jang K, Noh H R and Jhe W 2005 Phys. Rev. A 72 053402
[29] Dykman M I and Khazan I S 1979 Zh. Eksp. Teor. Fiz. 77 1488
[30] Moon G, Heo M S, Kim Y, Noh H R, Kim D and Jhe W 2010 Phys. Rev. A 81 033425
[31] Moon G, Noh H R and Jhe W 2011 J. Korean Phys. Soc. 58 1105
[32] Akerman N, Kotler S, Glickman Y, Dallal Y, Keselman A and Ozeri R 2010 Phys. Rev. A 82 061402
[33] Ryvkine D and Dykman M I 2006 Phys. Rev. E 74 061118
[34] Dykman M I, Golding B, McCann L I, Smelyanskiy V N, Luchinsky D G, Mannella R and McClintock P V E 2001 Chaos 11 587
[35] Gardiner C W 1983 Handbook of Stochastic Methods (New York: Springer)