Correlations in one dimensional quantum impurity problems with an external field or a temperature.

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We discuss in more details the theory of low energy excitations in quantum impurity problems with an external field at vanishing temperature, giving further support to results of the previous paper. We then extend these results to the next order at low frequency, obtaining in particular the exact expression, as a function of the bias, of the first two derivatives of the response function $\chi''(\omega)$ at $\omega = 0$ in the double well problem of dissipative quantum mechanics.

We also extend our approach to the case of non vanishing temperature and no external field. Fendley et al. had obtained in that case an expression for the Hall conductance with a single impurity using a Landauer-Büttiker type approach. We recover their result in the framework of linear response theory using renormalized form-factors. We also obtain, as a function of the temperature, the first derivative of the response function $\chi''(\omega)$ at $\omega = 0$ in the double well problem of dissipative quantum mechanics.
1. Introduction

A lot of progress has been accomplished recently in the non perturbative computation of correlation functions for integrable quantum field theories in 1+1 dimensions \[1\], \[2\], \[3\]. Interesting physical applications have been found in particular in one dimensional quantum impurity problems \[4\], \[5\], \[6\]. Unfortunately, many of these computations have been restricted to the case of vanishing temperature and external field, preventing in most cases comparison with experimental data; a notable exception concerns the DC properties for tunneling between edge states in fractional quantum Hall devices \[3\].

The introduction of a temperature or an external field, while it does not break integrability, renders the computation of correlators using form-factors more difficult (putting a temperature seems somewhat more natural in the approach of \[2\], which however produces only rather implicit results). Few works have addressed this question so far, except in the free fermion case, where however the problem is already quite non trivial for spin operators \[7\]. We note that some interesting results have also been obtained in the case of interacting massive bulk theories \[3\] in the limit of very low temperatures. In \[3\] (henceforth refered to as I) we have considered quantum impurity problems at \(T = 0\) but with an external field. We were able to determine two important low energy properties, the limit \(\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega}\) of the dynamical susceptibility in the double well problem of dissipative quantum mechanics, and the \(|\omega|\) component of the noises in a four terminal geometry for tunneling between edges in fractional quantum Hall devices. Some global properties, like the existence of potential singularities, were also examined.

The first purpose of this sequel is to place the low energy analysis of I on firmer grounds, and to compute the next order at low frequency, eg the \(\omega^3\) term in \(\chi''(\omega)\). This is the subject of sections 2 and 3.

Our second purpose is to carry out a similar analysis for the case of a non vanishing temperature (section 4). We discuss the formula for the conductance through a point contact that had been obtained previously by Fendley et al. using a Landauer-Büttiker \[10\] type approach, and recover it, at the price of some reasonable hypothesis, in the framework of linear response theory. Building on this analysis, we obtain the limit \(\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega}\) of the dynamical susceptibility in the double well problem of dissipative quantum mechanics at finite temperature.
2. A closer look at low energy excitations with $T = 0, V \neq 0$

This paper is a sequel to I and the notation introduced there will be used here. The hamiltonian we study,

$$H = \frac{1}{2} \int_{-\infty}^{0} dx \left[ 8 \pi g \Pi^2 + \frac{1}{8 \pi g} (\partial_x \phi)^2 \right] + H_B,$$

consists of a free bulk theory with a boundary interaction. The detailed form of this interaction, $H_B$, is different for the dissipative quantum mechanics and the tunneling problems.

The principle of our approach using massless scattering is described in full details in I. In brief, the bulk degrees of freedom are described by a massless sine-Gordon theory which has solitons/anti-solitons and bound states as fundamental “bare” excitations. In the presence of a voltage, the ground state consists of a Fermi sea filled with solitons, in a way easily controlled using the Bethe ansatz. Physical properties of interest are determined by the structure of excitations over this voltage dependent ground state. The boundary interaction is finally taken into account by the introduction of reflection matrices.

In this section we discuss the voltage dependent ground state and its excitations more thoroughly. The main problem we are addressing is the scattering of these excitations, which is “dressed” by the presence of the Fermi sea, and the resulting dressed form-factors. Most of the discussion is centered on the scattering at coincident rapidities, where, for non-neutral excitations, a non trivial phase seems to appear. This phase, and its possible effects on the low energy properties, are discussed in details.

2.1. The Fermi sea

Recall that at zero temperature and in the presence of an external field, the ground state contains only solitons. The model is massless and a convenient way to use the Bethe ansatz in this problem is to “unfold” the half-line problem to obtain a problem on the full line. Then we can, for instance, consider a system with only right movers. The system is therefore described by the Bethe ansatz equations (setting $\hbar = 1$)

$$I_\alpha = \frac{\mu L}{2\pi} e^{\theta_\alpha} + \sum_\beta \delta(\theta_\alpha - \theta_\beta),$$

(2.2)
where we have reinstated an arbitrary energy scale \( \mu \) (set equal to one in \( I \)) and \( \delta \) is the shift coming from the soliton-soliton S matrix. In the ground state, solitons fill the interval \( \theta \in [-\infty, A] \) with a density \( \rho \) obeying

\[
\rho(\theta) = \frac{\mu e^\theta}{2\pi} + \int_{-\infty}^{A} \Phi(\theta - \theta') \rho(\theta')d\theta',
\]

(2.3)

where \( S_{++}^+ = S = e^{2i\pi \delta} \), \( \Phi(\theta) = \frac{1}{2\pi} \frac{d\ln S}{d\theta} \). It should be pointed out that the quantization condition (2.2) should also involve the reflection matrix but its contribution to the density is of order \( 1/L \), and this is negligible for our purposes. Equations of the type (2.3) for a general function \( f \)

\[
f(\theta) = g(\theta) + \int_{-\infty}^{A} \Phi(\theta - \theta') f(\theta')d\theta'
\]

will occur repeatedly in the subsequent analysis. Let us introduce the functional \( \hat{K} \) such that

\[
\hat{K} \circ f(\theta) = \int_{-\infty}^{A} \Phi(\theta - \theta') f(\theta')d\theta'
\]

and similarly

\[
\hat{I} \circ f(\theta) = f(\theta)
\]

Then the previous equation reads \( (\hat{I} - \hat{K}) \circ f = g \), and the solution follows by introducing an operator \( L \) such that

\[
(\hat{I} + \hat{L})(\hat{I} - \hat{K}) = \hat{I},
\]

that is

\[
f(\theta) = g(\theta) + \int_{-\infty}^{A} L(\theta, \theta') g(\theta')d\theta'.
\]

The functions \( L \) is symmetric of its two arguments \( L(\theta, \theta') = L(\theta', \theta) \) and can be written formally as

\[
L(\theta, \theta') = \Phi(\theta - \theta') + \int_{-\infty}^{A} \Phi(\theta - \theta'') \Phi(\theta'' - \theta')d\theta'' + \ldots
\]

(2.4)

It is important to stress that \( L \) is not a function of the difference of its arguments, as the previous equation clearly shows. Observe that one has \( (\hat{I} + \hat{L}) \hat{K} = \hat{L} \), or more explicitly

\[
\int_{-\infty}^{A} [\delta(\theta - \theta') + L(\theta, \theta')] \Phi(\theta' - \theta'')d\theta' = L(\theta, \theta'').
\]

\(^1\) As in I we use rapidities to parametrize the energy and momentum of a particle.
In the particular case of the density we have \( \rho(\theta) = \frac{\mu}{2\pi} \left[ e^{\theta} + \int_{-\infty}^{A} L(\theta, \theta') e^{\theta'} d\theta' \right] \). For \( \theta < A \), \( \rho \) is the density of solitons in the sea. For \( \theta > A \), \( \rho \) is the density of holes of solitons above the sea. The function \( \rho \) can be written maybe more explicitly as an infinite series of exponentials using Wiener Hopf integration techniques, see I. We recall the value \( \rho(A) = \frac{V}{4\pi} \sqrt{2g} \). The Fermi rapidity \( A \) is determined self consistently by introducing the quantity

\[
\epsilon(\theta) = \mu e^{\theta} - \frac{V}{2} + \int_{-\infty}^{A} \Phi(\theta - \theta') \epsilon(\theta') d\theta',
\]

and requiring \( \epsilon(A) = 0 \). Observe the identity

\[
\frac{d\epsilon(\theta)}{d\theta} = 2\pi \rho(\theta).
\]

2.2. Neutral excitations

Let us now discuss excitations over the ground state. First, consider a particle hole excitation. In order not to make confusion with the other types of particles (eg antisolitons) which also appear in this problem at high energy, we call a soliton above the sea a volton, and a hole in the sea an antivolton. Suppose the volton has rapidity \( \theta_p \) and the antivolton rapidity \( \theta_h \). The particles are interacting and this induces a shift of the rapidities in the sea: a rapidity equal to \( \theta_\alpha \) initially becomes \( \theta_\alpha + \delta^{(2)} \theta_\alpha \) with the conditions

\[
I_\alpha = \frac{\mu L}{2\pi} e^{\theta_\alpha} + \sum_{\beta} \delta(\theta_\alpha - \theta_\beta)
\]

\[
I_\alpha = \frac{\mu L}{2\pi} e^{\theta_\alpha + \delta^{(2)} \theta_\alpha} + \sum_{\beta} \delta(\theta_\alpha - \theta_\beta + \delta^{(2)} \theta_\alpha - \delta^{(2)} \theta_\beta)
\]  

\[
+ \delta(\theta_\alpha - \theta_p) - \delta(\theta_\alpha - \theta_h).
\]

We then define the shift function, describing the change in the Fermi sea, by \( L \rho(\theta) \delta^{(2)} \theta \equiv F(\theta|\theta_p, \theta_h) \). By standard manipulations [2], one finds the equation obeyed by the shift

\[
F(\theta|\theta_p, \theta_h) - \int_{-\infty}^{A} \Phi(\theta - \theta') F(\theta'|\theta_p, \theta_h) d\theta' = \delta(\theta - \theta_h) - \delta(\theta - \theta_p).
\]

A formal solution of this equation follows as

\[
F(\theta|\theta_p, \theta_h) = \int_{\theta_h}^{\theta_p} L(\theta, \theta') d\theta'.
\]
The change of energy of the system with this excitation is given by the bare energies of the particle and the hole and the contribution from the shift of the Fermi sea

\[ \delta^{(2)} E = \mu e^{\theta_p} - \mu e^{\theta_h} + \int_{-\infty}^{A} F(\theta|\theta_p, \theta_h) \mu e^{\theta} d\theta, \]  

and this can be shown [2] to coincide with

\[ \delta^{(2)} E = \epsilon(\theta_p) - \epsilon(\theta_h), \]  

with \( \epsilon \) defined in (2.5). One has \( \epsilon(\theta) < 0 \) for \( \theta < A \), so the energy of hole excitations is positive as it should. We note the value \( \epsilon(-\infty) = -\mu gV \). Similar computations can be carried out for the momentum, with the result that, for these neutral excitations, \( \delta^{(2)} P = \delta^{(2)} E \); hence the excitations are massless, as physically expected.

2.3. Non neutral excitations

Although physical excitations will always be made of volton-antivolton pairs, it is necessary, to understand the scattering properties, to consider excitations with only one volton or one antivolton. Such excitations will involve a “Fermi” momentum, that is \( \delta^{(1)} P = \delta^{(1)} E \pm p_f \). Assuming for the moment that the excitation energy of a volton (resp. an antivolton) is still given by \( \epsilon \) (resp. \(-\epsilon\)), let us now determine the Fermi momentum. Recall that the proper definition of \( p \) is via the phase shift collected when a particle goes around the world. For a volton excitation this phase reads

\[ L_p = L \mu e^{\theta} + 2\pi \sum_{\alpha} \delta(\theta - \theta_{\alpha}^{(1)}) + 2\pi \sum_{\alpha} \delta(\infty). \]

Here, we have taken the usual phase shift \( \delta = \frac{1}{2i\pi} \ln S \) where \( S \) is the soliton soliton sine-Gordon S-matrix. However, when taking the massless limit of the sine-Gordon model, left and right movers do not become totally independent. There remains a RL constant scattering phase, \( \delta(\infty) \). This phase appears very rarely in computations, and its meaning is not totally clear. But when we pass a particle through the system, this RL shift contributes to the momentum, and this is the meaning of the last term in the foregoing equation. At leading order, we can neglect the shift of rapidities in the sea and reexpress this as

\[ L_p = L \mu e^{\theta} + 2\pi \int_{-\infty}^{A} [\delta(\theta - \theta') + \delta(\infty)] \rho(\theta') d\theta'. \]
Then, using the relation $\rho = \frac{1}{2\pi} \frac{d\rho}{d\theta}$, we can write

$$L_\rho = L\epsilon(\theta) + L\frac{V}{2} + 2\mu L g V \delta(\infty).$$

One checks that $\delta(\infty) = -\delta(-\infty) = \frac{1}{2} - \frac{1}{4g}$, and it follows that

$$p_f = \mu g V.$$

Consider now the operator destroying a right moving volton and creating a left moving one. Its correlation function, from this extra Fermi momentum, will be alternating with a $\cos 2p_F x$ part. On the other hand, the alternating part can be computed by observing that the potential $V$ can be absorbed by a redefinition of the field $\phi_{L,R} \rightarrow \phi_{L,R} \pm g V x$ in the original hamiltonian. This leads to the identification of this operator with $\exp i(\phi_{L} - \phi_{R})$, of conformal weights $h = \bar{h} = g$. More generally, the operator $\exp i\alpha(\phi_{L} - \phi_{R})$ has conformal weight $h = g\alpha^2$ and its correlation function involves a part $\cos 2\alpha g V x$. This corresponds to a charge $Q = 2g\alpha$, in conventions where the soliton has unit charge.

Notice that the LR phase shift should appear in (2.2). However, as long as the number of L particles is a constant, its presence would simply shift all the $I_\alpha$ by a constant, without changing the results for densities or excitation energies. Things are different when the number of particles is changed. For instance, adding a R soliton adds up a phase $\delta(-\infty)$ to the rhs of Bethe equations for L movers. Without any correction term, this phase in turn will move the Fermi sea, resulting in an induced L charge, an effect which is not expected for a purely R excitation. This means that, to observe physical non neutral excitations, one has to complement the addition or removal of a particle by an additional phase shift, corresponding presumably to changing boundary conditions. Determining the value of this phase is not obvious and, in the rest of this section, we embark in a long discussion of the properties of the system with this phase to see its effect. Consider for instance a volton excitation, associated with an additional phase shift $\exp 2i\pi \delta_b$. The following holds

$$I_\alpha = \frac{\mu L}{2\pi} e^{\theta_\alpha} + \sum \delta(\theta_\alpha - \theta_\beta)$$

$$I_\alpha + \delta_b = \frac{\mu L}{2\pi} e^{\theta_\alpha + \delta(1)\theta_\alpha} + \sum \delta(\theta_\alpha - \theta_\beta + \delta(1)\theta_\alpha - \delta(1)\theta_\beta) + \delta(\theta_\alpha - \theta_\rho),$$

(2.13)
with $\theta_p$ the volton rapidity. By the same logic as before, defining $F_b(\theta|\theta_p) = L\rho(\theta)\delta^{(1)}(\theta)$, one has
\[ F_b(\theta|\theta_p) - \int_{-\infty}^{\infty} \Phi(\theta - \theta')F_b(\theta'|\theta_p) = \delta_b - \delta(\theta - \theta_p). \tag{2.14} \]
If we were to add an antivolton, the additional phase shift would be $\exp(-2i\pi\delta_b)$, so of course for pairs voltons antivoltons the total shift would be as before, as one has
\[ F(\theta|\theta_p,\theta_h) = F_b(\theta|\theta_p) - F_b(\theta|\theta_h) \]
for any $\delta_b$, so (2.10),(2.11) hold independently of this phase. Now, for non “neutral” excitations we have to require more, namely that the excitation energy also is given by $\delta^{(1)}E = \epsilon(\theta_p)$. Let us see whether this can be satisfied: On the one hand, the solution of (2.14) reads
\[ \epsilon(\theta_p) = \mu e^{\theta_p} - V + \int_{-\infty}^{\infty} \mu e^\theta L(\theta_p, \theta)d\theta - \frac{V}{2} \int_{-\infty}^{\infty} L(\theta_p, \theta)d\theta. \tag{2.15} \]
On the other hand, the excitation energy due to the volton can be written
\[ \varepsilon(\theta_p) = \mu e^{\theta_p} - V + \int_{-\infty}^{\infty} \mu e^{\theta'} F_b(\theta'|\theta_p)d\theta' - \frac{V}{2} F_b(-\infty|\theta_p). \tag{2.16} \]
The last term was not present for neutral excitations. First, using the formal solution (2.4), one checks that $F_b(-\infty, \theta_p)$ is in fact independent of $\theta_p$. For neutral excitations, this term would be cancelled by its counterpart due to the antivolton. The meaning of $F_b(-\infty, \theta_p)$ is simple: it is (minus) the number of solitons that are pushed “out” of the Fermi sea in the presence of an added soliton above the sea. While these particles have a vanishing kinetic energy, they have a non vanishing potential energy, which must be put by hand.

With the shift $\delta_b$, the solution of (2.14) reads
\[ F_b(\theta|\theta_p) = [\delta_b - \delta(\infty)] \left[ 1 + \int_{-\infty}^{\infty} L(\theta, \theta')d\theta' \right] + \int_{-\infty}^{\theta_p} L(\theta, \theta')d\theta'. \tag{2.17} \]
From the defining equation for $F_b$ (2.14), we can also deduce
\[ \frac{d}{d\theta} F_b(\theta|\theta_p) = - L(\theta, \theta_p) - L(\theta, A)F_b(A|\theta_p) \tag{2.18} \]
and
\[ \frac{d}{d\theta_p} F_b(\theta|\theta_p) = L(\theta, \theta_p). \]
Integration by parts in (2.16) gives rise to the alternate expression

\[ \varepsilon(\theta_p) = \mu e^{\theta_p} - \frac{V}{2} + \mu e^A F_b(A|\theta_p) \]

\[ + \int_{-\infty}^{A} \mu e^\theta L(\theta, \theta_p) d\theta + F_b(A|\theta_p) \int_{-\infty}^{A} \mu e^\theta L(\theta, A) d\theta - \frac{V}{2} F_b(-\infty|\theta_p). \]  

(2.19)

Using \( \epsilon(A) = 0 \), the equality \( \epsilon = \varepsilon \) will hold iff

\[ F_b(A|\theta_p) \left[ 1 + \int_{-\infty}^{A} L(A, \theta) d\theta \right] - F_b(-\infty|\theta_p) = -\int_{-\infty}^{A} L(\theta_p, \theta) d\theta. \]  

(2.20)

The left hand side is linear in \( \delta_b \) with the coefficient

\[ \left[ 1 + \int_{-\infty}^{A} L(A, \theta) d\theta \right]^2 - 1 - \int_{-\infty}^{A} L(-\infty, \theta) d\theta \]

(for the second term, the upper integration bound has no importance since the region where the integrand is finite is concentrated around \( -\infty \)). By using the equations for \( \rho \) and \( \epsilon \) together with the values of \( \rho(A) \) and \( \epsilon(-\infty) \) one finds the results

\[ \int_{-\infty}^{A} L(A, \theta) d\theta = \sqrt{2g} - 1 \]  

(2.21)

\[ \int_{-\infty}^{A} L(-\infty, \theta) d\theta = 2g - 1. \]

Hence the left hand side of (2.20) actually is independent of \( \delta_b \). Let us chose the particular value \( b^* \) of \( \delta_b \) such that \( F_{b^*}(A, A) = 0 \). Using (2.18) it follows that in that case \( F_{b^*}(\theta, A) = -F_{b^*}(A, \theta) \). One has then

\[ F_{b^*}(\theta_p|A) = \frac{\int_{-\infty}^{A} [L(\theta_p, \theta) - L(A, \theta)] d\theta}{1 + \int_{-\infty}^{A} L(A, \theta) d\theta}, \]  

(2.22)

from which (2.20) follows together with \( F_{b^*}(-\infty, \theta) = \int_{-\infty}^{A} L(A, \theta) d\theta = \sqrt{2g} - 1 \). Hence, indeed, the function \( \epsilon \) gives the excitation energy of non neutral excitations too, and this for any value of \( \delta_b \).

Another quantity of interest is the charge of excitations. For a volton for instance, it is equal to +1, the bare charge, minus the number of solitons ejected from the sea at \( -\infty \), which reads \(-F_b(-\infty|\theta_p)\). So we have

\[ q_{\pm}(\theta) = \pm [1 + F_b(-\infty|\theta)] \]  

(2.23)
where from now on, we designate voltons and antivoltons by a label $\pm$. The modulus of this charge is a constant, so neutral excitations still consist of pairs voltons-antivoltons, as physically expected. For a generic choice of shift $\delta_b$, one finds that $F_b(-\infty|\theta) = 2g[\delta_b - \delta(\infty)] + 2g - 1$. For this value, one has $F_b(A|A) = \sqrt{2g}\delta_b - \delta(\infty)] + \sqrt{2g} - 1$.

At this stage, it is useful to consider the effect of shifting the edge of the Fermi sea. We would like to compute the corresponding change in energy to order $1/L$ and see if that will furnish more constraints on $\delta_b$. So far, we did all computations without worrying about such terms. It is for instance not possible to use straightforwardly the formula for excitation energies $\epsilon$ when one wants such corrections. The safest is to go back to original definitions. The energy reads as a discrete sum. When replacing the sum by an integral in the ground state, the Euler-Mac Laurin formula can be applied. It shows that there is a term proportional to $L$, no term of order one (as physically expected when there is no boundary nor impurity), and a term of order $1/L$ that determines the central charge. If we shift the edge by a small amount $\delta A$, this term of order $1/L$ will have a variation of order $1/L^2$ which we do not look for. Only the variation of the extensive term is therefore of interest. We have

$$\frac{\tilde{E}}{L} = \int_{-\infty}^{A+\delta A} \left( \mu e^{\theta} - \frac{V}{2} \right) \tilde{\rho}(\theta) d\theta$$

where

$$\tilde{\rho}(\theta) = \frac{\mu e^{\theta}}{2\pi} + \int_{-\infty}^{A+\delta A} \Phi(\theta - \theta') \tilde{\rho}(\theta') d\theta'.$$

As in [2] we can easily rewrite

$$\frac{\tilde{E}}{L} = \frac{\mu}{2\pi} \int_{-\infty}^{A+\delta A} \tilde{\epsilon}(\theta) e^{\theta} d\theta$$

and a similar equation for non tilde quantities, where

$$\tilde{\epsilon}(\theta) = \mu e^{\theta} - \frac{V}{2} + \int_{-\infty}^{A+\delta A} \Phi(\theta - \theta') \tilde{\epsilon}(\theta') d\theta'.$$

To get a corection of order $1/L$ to the energy, we expand $\tilde{E}$ and $\tilde{\epsilon}$ as functions of $\delta A$. One finds

$$\left. \frac{\partial \tilde{\epsilon}(\theta)}{\partial \delta A} \right|_{\delta A=0} = 0$$

$$\left. \frac{\partial^2 \tilde{\epsilon}(\theta)}{\partial^2 \delta A} \right|_{\delta A=0} = \dot{\epsilon}(A)L(\theta, A),$$
from which it follows that the first derivative of $E$ at $\delta A = 0$ vanishes, and for the second derivative one has
\[
\frac{1}{L} \frac{\partial^2 \tilde{E}}{\partial^2 \delta A} \bigg|_{\delta A = 0} = \frac{\mu}{2\pi} \dot{\epsilon}(A) \left[ \frac{V}{2} + \frac{V}{2} \int_{-\infty}^{A} L(\theta, A) d\theta \right].
\]

It follows finally that
\[
\tilde{E} - E = \mu L \frac{gV^2}{8\pi} (\delta A)^2.
\] (2.24)

Now suppose we want to create an excitation of charge $Q$. For each volton added, a certain number of voltons leave the Fermi sea, so for $dn$ voltons added, we have a charge $Q = \left[ 1 + F_b(-\infty | A) \right] dn$. By which amount does the Fermi edge shift if we add one volton with minimum possible energy? First, the rapidity of the soliton immediately below the added volton shifts, by an amount $\delta_1 = \frac{F(A|A)}{L\rho(A)}$. Second, the added volton must lie immediately above, at a rapidity differing from this one by $\delta_2 = \frac{1}{L\rho(A)}$. This leads to $\delta A = \frac{F(A|A)}{L\rho(A)} + \frac{1}{L\rho(A)}$ and thus a change of energy, using (2.24)
\[
\tilde{E} - E = \frac{1}{L} \frac{gV^2}{8\pi} \left( \frac{F(A|A) + 1}{\rho(A)} \right)^2
\]

If we add $dn$ voltons, the change of energy will thus read
\[
\tilde{E} - E = \frac{\pi}{L} [1 + F_b(A|A)]^2 (dn)^2.
\]

Using the formulas for the charge, together with the usual formula between the gaps and the conformal weights [11] leads to
\[
h = \frac{1}{2} [1 + F_b(A|A)]^2 = \frac{1}{4g} [1 + F_b(-\infty | A)]^2 (dn)^2 = \frac{Q^2}{4g}.
\] (2.25)

This agrees with what is expected from the Lagrangian. Notice that, once again, the result holds for any value of $\delta_b$: so far, we have obtained no constraint for this unknown parameter. Its value is however crucial for the scattering theory, as we now discuss.

2.4. Renormalized S matrix

To proceed, we consider the S matrix of excitations. Take for instance two voltons at rapidities $\theta_1, \theta_2$, with $\theta_1 > \theta_2$. With the first volton only, the rapidities in the sea are shifted to $\theta^{(1)}_{\alpha}$, with both particles they are shifted to $\theta^{(12)}_{\alpha}$. Passing the first volton through the system in the presence of the (shifted) Fermi sea only, one gets a phase $\phi_1$,
while passing it through the system in the presence of the (shifted) Fermi sea and the second particle, one gets a phase $\phi_{12}$. These two phases read respectively,

$$\phi_1 = Le^{\theta_1} + 2\pi \sum_\alpha \delta(\theta_1 - \theta_1^{(1)})$$

$$\phi_{12} = Le^{\theta_1} + 2\pi \sum_\alpha \delta(\theta_1^p - \theta_1^{(12)}) + \delta(\theta_1 - \theta_2).$$

(2.26)

Straightforward computations give

$$\phi_{12} - \phi_1 = -2\pi F_b(\theta_1|\theta_2),$$

(2.27)

with $F$ given in (2.17). Here, we have assumed that with $n$ voltons, the boundary conditions will be changed so that a term $n\delta_b$ will be added to the left hand side of quantization equations (similar to (2.13)) for solitons in the sea. We then define the S-matrix of voltons by

$$S_{++}(\theta_1, \theta_2) = -\exp\left[-2i\pi F_b(\theta_1|\theta_2)\right], \theta_1 > \theta_2.$$  \hspace{1cm} (2.28)

The previous computation does not require $\theta_1 > \theta_2$ to be algebraically correct. What does require this inequality however is the identification of the S matrix with the physical process of passing the particle 1 through the system.

We now build a Faddeev Zamolodchikov [12] algebra for our excitations, setting

$$|\theta_1, \theta_2 >_{++} = S_{++}(\theta_1, \theta_2)|\theta_2, \theta_1 >_{++}, \theta_1 > \theta_2.$$  \hspace{1cm} (2.29)

From this, we see that

$$|\theta_1, \theta_2 >_{++} = S_{++}^{-1}(\theta_2, \theta_1)|\theta_2, \theta_1 >_{++}, \theta_1 < \theta_2.$$  \hspace{1cm} (2.30)

We will henceforth set

$$S_{++}(\theta_1, \theta_2) = -\exp\left[-2i\pi F_b(\theta_1|\theta_2)\right], \theta_1 > \theta_2$$

$$= -\exp\left[2i\pi F_b(\theta_2|\theta_1)\right], \theta_2 > \theta_1.$$  \hspace{1cm} (2.31)

The S-matrix satisfies

$$S_{++}(\theta_1, \theta_2)S_{++}(\theta_2, \theta_1) = 1.$$  \hspace{1cm} (2.32)

It is singular at $\theta_1 = \theta_2$, and is not a function of $\theta_1 - \theta_2$ only. The same analysis for antivoltons results in

$$S_{--}(\theta_1, \theta_2) = -\exp\left[-2i\pi F_b(\theta_1|\theta_2)\right], \theta_1 < \theta_2$$

$$-\exp\left[2i\pi F_b(\theta_1|\theta_2)\right], \theta_1 > \theta_2.$$  \hspace{1cm} (2.33)
Observe that the phases are opposite to the ones in (2.31). This is because the momentum of an antivolton, \( p = p_f + \epsilon(\theta) \), is a decreasing function of \( \theta \) for \( \theta < A \). Finally, one can also scatter a volton and an antivolton. One has

\[
|\theta_1, \theta_2 >_{+-} = S_{+-}(\theta_1, \theta_2)|\theta_2, \theta_1 >_{-+}, \tag{2.34}
\]

with

\[
S_{+-}(\theta_1, \theta_2) = -\exp[2i\pi F_b(\theta_1|\theta_2)], \tag{2.35}
\]

together with

\[
S_{-+}(\theta_1, \theta_2) = -\exp[-2i\pi F_b(\theta_2|\theta_1)]. \tag{2.36}
\]

In the foregoing equations, there is an additional minus sign compared with the phase shifts like (2.27) because our definition of the S-matrix is through the Faddeev Zamolodchikov algebra [13]. Indeed, consider for instance a pair of particles in some scattering theory. If \( \phi_{12} \) is the phase one gets passing the first through the second (this \( \phi_{12} \) should not be confused with the one in equation (2.27)), the coordinate wave function reads, before symmetrization or antisymmetrization

\[
e^{-i\phi_{12}/2}e^{i(p_1 x_1 + p_2 x_2)}, \quad x_1 < x_2
\]

\[
e^{i\phi_{12}/2}e^{i(p_1 x_1 + p_2 x_2)}, \quad x_1 > x_2
\]

from which

\[
|p_1 p_2 > = \int_{x_1 < x_2} e^{-i\phi_{12}/2}e^{i(p_1 x_1 + p_2 x_2)}\chi^+(x_1)\chi^+(x_2)|0 > + \int_{x_1 > x_2} e^{i\phi_{12}/2}e^{i(p_1 x_1 + p_2 x_2)}\chi^+(x_1)\chi^+(x_2)|0 > = -e^{-i\phi_{12}}|p_2 p_1 >, \]

where we used that the \( \chi \) operators are fermionic, and \( \phi_{12} + \phi_{21} = 0 \).

2.5. Form-factors

Still assuming a generic \( \delta_b \), let us consider the physics at low energy. Right moving, low energy excitations are made of volton-antivolton pairs near the Fermi rapidity \( \theta = A \). Because there is an energy scale \( V \), the theory is not relativistically invariant, even though neutral excitations have dispersion relation \( \epsilon = p \). This means, as observed above, that the S-matrix does not depend only on the ratio of energies. However, very close to \( A \), the ratio \( \epsilon/V \) goes to zero while the S-matrix elements go to constants. In that limit, we deal
again with a relativistically invariant theory, now depending on $\delta_b$. The key parameter now is

$$\kappa \equiv F_b(A|A) = \sqrt{2g}[\delta_b - \delta(\infty)] + \sqrt{2g} - 1. \quad (2.37)$$

Since the theory is relativistic, it is convenient to parametrise the energy in terms of a renormalised rapidity $\beta$ (see I). For voltons, we set $\epsilon(\theta) = \mu e^\beta$, $\theta > A$, and for antivoltons, we set $\epsilon(\theta) = -\mu e^\beta$, $\theta < A$. We normalize asymptotic states to $2\pi \delta(\beta_1 - \beta_2)$. Note that due to the minus sign, for antivoltons, the function $\beta(\theta)$ is decreasing so increasing momentum means increasing renormalized rapidities.

All form factors are now expected to depend on differences of $\beta$’s, which are renormalized rapidities. The choice $b = b^*$ or $\kappa = 0$ is the only one for which the S-matrix is analytic at coincident rapidities. This by itself is probably enough to dictate that choice. However, we wish to accumulate more evidence, and keep studying the generic case first.

We discuss the form factor $<0|j(z)|\beta_2, \beta_1 >_{-+}$, where the right hand side physically describes a hole particle pair and $z = x - t$. The $z$ dependence of this correlator follows trivially from kinematic considerations, $\exp \mu iz[e^{\beta_1} + e^{\beta_2}]$, so we restrict to the case $z = 0$. We call this form factor $f_{+-}(\beta_1, \beta_2)$\footnote{The notation is similar to [I], $<0|j|\beta_2, \beta_1 > = f(\beta_1, \beta_2)$.}. The first axiom to be satisfied is

$$f_{+-}(\beta_1, \beta_2)S_{+-}(\beta_1, \beta_2) = f_{-+}(\beta_2, \beta_1) \quad (2.38)$$

with the $S$ matrix given by its low energy limit described by (2.37). An obvious solution to (2.38) is simply to set

$$f_{+-}(\beta_1, \beta_2) \propto \exp[-i\pi \kappa]$$

$$f_{-+}(\beta_2, \beta_1) \propto -\exp[i\pi \kappa]. \quad (2.39)$$

The second axiom is

$$f_{+-}(\beta_1, \beta_2 + 2i\pi) = S_{+-}(\beta_1, \beta_2)f_{+-}(\beta_1, \beta_2)$$

$$f_{+-}(\beta_1 + 2i\pi, \beta_2) = S_{+-}^{-1}(\beta_1, \beta_2)f_{+-}(\beta_1, \beta_2). \quad (2.40)$$

And a solution to (2.40) is of the form

$$f_{+-}[\beta_1, \beta_2] \propto \mu e^{\beta_1/2}e^{\beta_2/2} (e^{\beta_2-\beta_1})^\kappa, \quad (2.41)$$

where we have put the correct dimension $1/\text{length}$ of the form factor. Apart from the dimensional factor, the form factor is expected to depend on the massless relativistic
invariant $s \equiv e^{\beta_2 - \beta_1}$. We see from (2.41) that there is a cut in the $s$ plane along the negative real axis. With $\kappa \neq 0$, such cut is unavoidable since the S matrix is singular at coincident rapidities.

The last axiom in the relativistic case stems from crossing:

$$\langle \beta_1 | j(0) | \beta_2 \rangle_{-} = f_{+-}(\beta_1 - i\pi, \beta_2). \quad (2.42)$$

Leading to the minimal conjecture for the two particle form-factor

$$f_{+-}(\beta_1, \beta_2) = i\mu q_- e^{-i\pi\kappa} e^{\beta_1/2} e^{\beta_2/2} (e^{\beta_2 - \beta_1})^\kappa. \quad (2.43)$$

The normalisation of the form factor is related to the charge term which originates from crossing, as we now demonstrate. Indeed, we must satisfy the relation

$$\int_{-\infty}^{\infty} \langle \beta_1 | j(z) | \beta_2 \rangle_{-} dz = 2\pi \delta(\beta_1 - \beta_2) q_- \quad (2.44)$$

After crossing, performing the integral over $z$, constrains $\beta_1 = \beta_2$ and brings a jacobian. The ratio of $\epsilon$’s becomes equal to unity and disappears. The only remaining part, therefore is the charge $q_-$, as desired. In the case $\kappa = 0$, $\delta_b = \delta_{b^*} = -\frac{1}{2} + \frac{1}{\sqrt{2g}} - \frac{1}{4g}$, $q_- = -\sqrt{2g}$, so one has then

$$f_{+-}(\beta_1, \beta_2) = -i\mu \sqrt{2g} e^{\beta_1/2} e^{\beta_2/2}. \quad (2.45)$$

Note that instead of (2.39), (2.38) could be solved also by setting,

$$f_{+-}(\beta_1, \beta_2) \propto (e^{\beta_1} - e^{\beta_2})^\kappa \quad \text{and} \quad f_{--}(\beta_2, \beta_1) \propto - (e^{\beta_2} - e^{\beta_1})^\kappa. \quad (2.46)$$

However, in the computation of the charge, this would result in an additional factor $e^{\kappa\beta}$ in the right hand side of (2.44), which is impossible.

The same logic can be applied to form-factors with higher number of particles. A reasonable guess for the four particle form-factor is

$$\langle 0 | j(z) | \beta_4, \beta_3, \beta_2, \beta_1 \rangle_{-} = q_- \mu (e^{2i\pi\kappa} - 1) (e^{\beta_3 + \beta_4 - \beta_1 - \beta_2})^\kappa (e^{\beta_1} + e^{\beta_2} + e^{\beta_3} + e^{\beta_4})$$

$$\frac{\sinh \frac{\beta_1 - \beta_2}{2} \sinh \frac{\beta_3 - \beta_4}{2}}{\cosh \frac{\beta_1 - \beta_2}{2} \cosh \frac{\beta_3 - \beta_4}{2}} \frac{\sinh \frac{\beta_2 - \beta_3}{2}}{\cosh \frac{\beta_2 - \beta_3}{2} \cosh \frac{\beta_2 - \beta_3}{2}} \frac{\sinh \frac{\beta_2 - \beta_3}{2}}{\cosh \frac{\beta_2 - \beta_3}{2} \cosh \frac{\beta_2 - \beta_3}{2}} \quad \beta_1 > \beta_2, \beta_3 > \beta_4. \quad (2.47)$$

Form factors where the $\beta$’s have been exchanged will take similar forms, up to phases arising from the S-matrix as in (2.39) - reproducing these phases by terms depending on
the differences of the energies would make the pole axioms and the charge computation impossible as in (2.40). Note that these phases are singular at coincident rapidities $\beta_1 = \beta_2$ or $\beta_3 = \beta_4$, since the $S$ matrix is.

Expression (2.47) vanishes when two voltons or antivoltons coincide, as physically required in the Bethe ansatz. It also has a simple pole as a volton and antivolton annihilate, e.g $\beta_4 = \beta_2 + i\pi$, with a residue that is proportional to the two particle form-factor in agreement with the axioms in [1] (it is not completely clear how exactly to generalize the pole axiom, and therefore what this residue exactly should be).

Let us now use these form factors to discuss physical quantities, relying heavily on I for the definitions of the impurity problems and the quantities under study. The important point about (2.47) is that it transforms as $FF \rightarrow e^{i\sigma}FF$ when $\beta \rightarrow \beta + \sigma$. As a result it will, together with all the higher form factors, contribute to the noise computation at order $\omega$. While the normalisation of successive form-factors are dictated by LSZ reduction formulas [1], and the normalisation of the first form-factor by the charge, a very non trivial sum rule has to be satisfied to ensure that the current correlator reproduces

$$S(\omega) = \int e^{i\omega t} \langle \{j(t), j(0)\} \rangle = \frac{g|\omega|}{\pi},$$

(2.48)

(see I). We suspect this sum rule would not be satisfied away from $\kappa = 0$, but we have no proof of this.

Some restrictions are obvious however. The two particle form factor by itself gives to the noise a contribution (assuming $\omega > 0$)

$$S^{(2)} = \frac{1}{2\pi} (q_-)^2 \int_0^\omega dx \left( \frac{x}{\omega - x} \right)^{2\kappa} = \frac{2\kappa \pi}{\sin 2\kappa \pi} \left( 1/2 + g + 2g\delta_b \right)^2 \frac{\omega}{2\pi}.$$  

(2.49)

The ratio involving $\kappa$ is clearly greater than one. Hence one needs $(1/2 + g + 2g\delta_b)^2 \leq 2g$. For instance, for $g < 1/2$, this excludes the choice $\delta_b = 0$. The choice $\delta_b = 1/2$ is always excluded. In fact, any choice $\delta_b = \text{cst}$ is excluded for $g$ small enough, so $\delta_b$ must have some non trivial dependence on $g$.

2.6. $\chi''$ revisited.

Following I, let us now add a boundary interaction. Without a voltage, the effect of the boundary is taken into account by reflection matrices $R$ (solutions of the boundary Yang-Baxter equation) for the “bare” excitations. With the voltage, one obtains new, dressed reflection matrices $\mathcal{R}$, for the excitations over the Fermi sea.
Consider again the computation of the low frequency behaviour of \( \chi''(\omega) \) in dissipative quantum mechanics along the same lines as in I. For a generic value of \( \kappa \), the two particle term would be proportional to, instead of eq. (5.8) in I,
\[
\int_0^\omega [\mathcal{R}(x)\mathcal{R}^*(x - \omega) - 1] \left( \frac{x}{\omega - x} \right)^{2\kappa} dx.
\] (2.50)

Change variables \( x \to \omega x \) to get
\[
\omega \int_0^1 [\mathcal{R}(\omega x)\mathcal{R}^*(x\omega - \omega) - 1] \left( \frac{x}{1 - x} \right)^{2\kappa} dx.
\] (2.51)

We are interested in the first non trivial real term, which occurs from the expansion of the integrand to order \( \omega^2 \). Setting \( \mathcal{R} = e^{i\phi} \), the bracket gives a total of four terms
\[
-\frac{1}{2}(\phi')^2 \omega^2 [x^2 + (x - 1)^2 - 2x(x - 1)] = -\frac{1}{2}(\phi')^2 \omega^2
\]
so the first contribution is equal to
\[
-\frac{1}{2}(\phi')^2 \omega^2 \int_0^1 \left( \frac{x}{1 - x} \right)^{2\kappa} dx
\]
Observe that this integral is exactly the one that would appear in the computation of the current correlator without impurity (2.49): the impurity dependence (\( \phi' \)) and the energy dependence actually factor out. This result is true at every order. For instance the next order term would be proportional to
\[
\int_0^\omega \int_0^\omega \int_0^\omega |f(x_1, x_2, x_3, \omega - x_1 - x_2 - x_3)|^2
\]
\[
[\mathcal{R}(x_1)\mathcal{R}(x_2)\mathcal{R}^*(-x_3)\mathcal{R}^*(x_1 + x_2 + x_3 - \omega) - 1] \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3 (\omega - x_1 - x_2 - x_3)},
\]
where \( f \) is the four particle form-factor (2.47). Now the same integral without the term in brackets is the one appearing in the computation of the noise, the fourth order term of a whole series that sums up to \( g\omega \). Contributing to the linear term of \( \chi'' \) we have to extract the term of third order in \( \omega \). Make a change of variables \( x \to \omega x \). Using homogeneity of the current form-factors, we can rewrite the integral as
\[
\omega \int_0^1 \int_0^1 \int_0^1 |f(x_1, x_2, x_3, 1 - x_1 - x_2 - x_3)|^2
\]
\[
\{\mathcal{R}(\omega x_1)\mathcal{R}(\omega x_2)\mathcal{R}^*(-\omega x_3)\mathcal{R}^*[\omega(x_1 + x_2 + x_3 - 1) - 1] \frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3 (1 - x_1 - x_2 - x_3)}
\]

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To get $\chi''$, we expand the integrand to order $\omega^2$. Setting again $R = e^{i\phi}$, the corresponding terms are identified, setting $x_4 = 1 - x_1 - x_2 - x_3$, as

$$-\frac{1}{2} (\phi')^2 \omega^2 \left[ \sum x_i^2 - 2 \sum_{i<j} x_i x_j \right] = -\frac{1}{2} (\phi')^2 \omega^2$$

Hence the term $-\frac{1}{2} (\phi')^2 \omega^2$ factors out, and the remaining integral is again the same as the one arising in the computation of the correlator without impurity. Since the form-factors are normalized to sum up these integrals to the same term $g\omega/\pi$ for any $\delta_b$, we get the same formula for the limit (as $\omega \to 0$) of $\chi''(\omega)/\omega$ for any value of $\delta_b$! Observing that $\phi' = \alpha e^{-\theta} \frac{d}{d\theta} \ln R(\theta)|_{\theta=A}$, where $\alpha = \frac{de}{d\theta}$ is given in I, we recover

$$\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega} = -\frac{\alpha^2}{4g\pi^2} \left( e^{-\theta} \frac{d}{d\theta} \ln R(\theta) \bigg|_{\theta=A} \right)^2. \quad (2.52)$$

This is true for any value of $\delta_b$.

2.7. Summary

The analysis of the theory of excitations over the new ground state, when done carefully, turns out to involve an extra phase $\delta_b$, which has a key influence on the dressed scattering theory, and thus on the response functions in the presence of a voltage. In this long analysis we have looked at various physical quantities to check whether we could fix $\delta_b$. The computation of $\lim_{\omega \to 0} \frac{\chi''(\omega)}{\omega}$ and the corresponding Shiba relation of I are not enough to settle the value of $\delta_b$; in fact, any value would give the right result, although the value that leads to $\kappa = 0$ is the most reasonable physically, and actually the only one for which the sum rules for the charge and noise can be explicitly checked. In contrast with dissipative quantum mechanics, the value of $\kappa$ is crucial for tunneling between edges in quantum Hall devices, also discussed in I. Indeed, if $\kappa$ was non zero and the theory admitted multiple particle, low energy excitations, the noise in the presence of an impurity would involve terms of an arbitrarily high degree in the reflection matrix. The same feature would be observed in the conductance, as will be discussed later in this paper. This is in contradiction with formulas that have been derived using a variety of approaches (in particular a Boltzmann type equation [5], or a Landauer-Büttiker approach [10], [14]), and checked against duality, perturbative expansions, numerical and experimental data. To summarize: while we have no proof that $\kappa = 0$, there is a large body of practical and conceptual evidence that it is so.
3. Beyond the low energy behaviour

Let us thus assume from now on that $\delta_b = \delta_{b^*}$ has been chosen so that $\kappa = 0$, and that, at low energy, the theory is made of free fermions. What can we say for higher energies? Let us restrict to energies $\omega < gV$ so the only processes involved concern volton-antivolton pairs. Since excitations have finite energy, we turn back to the original variables $\theta$. We consider the form-factor $\langle 0 | j(z) | \theta_2, \theta_1 \rangle_{-+}$. The $z$ dependence is now $\exp iz[\epsilon(\theta_1) - \epsilon(\theta_2)]$.

Since the S-matrix describes exchange properties, it is natural to expect that the analog of (2.38) will be satisfied (we denote $F_{b^*} \equiv F$),

$$f_{+-}(\theta_1, \theta_2)S_{+-}(\theta_1, \theta_2) = f_{-+}(\theta_2, \theta_1). \tag{3.1}$$

Generalizing (2.39), this has the simple solution

$$f_{+-}(\theta_1, \theta_2) \propto \exp[-i\pi F(\theta_1, \theta_2)]$$
$$f_{-+}(\theta_2, \theta_1) \propto -\exp[i\pi F(\theta_1, \theta_2)]. \tag{3.2}$$

The analog of crossing is also clear. While voltons are originally defined for $\theta > A$ and antivoltons for $\theta < A$, one can continue the associated amplitudes by analyticity through the threshold $A$. Creating an antivolton at $\theta > A$ will simply mean, as before, destroying a soliton (called a volton here) at this rapidity. In other words, we expect the analog of (2.42):

$$\langle \theta_1 | j(0) | \theta_2 \rangle_{-} = \langle 0 | j(0) | \theta_2, \theta_1 \rangle_{-+} = f_{+-}(\theta_1, \theta_2). \tag{3.3}$$

The shift of $i\pi$ was necessary in the $\beta$ parametrization of the relativistic case to describe both sides of the thresholds, something we simply accomplish here by continuation through the threshold.

Away from the relativistic limit, it is not clear what the equivalent of (2.39) should be. However, the charge normalization can still be used. Indeed, even for excitations of finite energy, since $F(-\infty | \theta_p)$ is independent of $\theta_p$, the renormalized charge is as before $\sqrt{2g}$. From (3.3) we conjecture

$$f_{+-}(\theta_1, \theta_2) = -i\sqrt{2g} \exp[-i\pi F(\theta_1 | \theta_2)][\dot{\epsilon}(\theta_1)]^{1/2}[\dot{\epsilon}(\theta_2)]^{1/2} \left( \frac{\epsilon(\theta_2)}{\epsilon(\theta_1)} \right)^{F(\theta_1 | \theta_2)}. \tag{3.4}$$

Here, $\dot{\epsilon} \equiv \frac{d\epsilon}{dg}$. This formula agrees with (2.43) in the relativistic limit once the change of normalization $\delta(\theta_1 - \theta_2) \rightarrow \delta(\beta_1 - \beta_2)$ is taken into account. The last term is fixed by
dimensional analysis, together with the charge normalization (it is necessary to off set the overall $\exp[-i\pi F(\theta, \theta)]$ at coincident rapidities).

It is interesting to discuss this form factor by getting back to the noise without impurity. The contribution of the two particle form factor is proportional to (assuming $\omega > 0$)

$$
\int_{\theta}^{\infty} \int_{\theta}^{\infty} |f_{+-}(\theta_1, \theta_2)|^2 \delta[\epsilon(\theta_1) - \epsilon(\theta_2) - \omega] d\theta_1 d\theta_2 = \int_{\theta}^{e^{-1}(\omega)} \frac{1}{\epsilon(\theta)} |f_{+-}(\theta, e^{-1}[\epsilon(\theta) - \omega])|^2 d\theta.
$$

At first order, we simply set $q = \sqrt{2g}$, $\theta = A$, to get

$$(\sqrt{2g})^2 \int_{\theta}^{e^{-1}(\omega)} \epsilon(\theta) d\theta = 2g\omega$$

as requested. At next order, we have to take the other terms into account. Setting $\theta_1 = \theta$, observe that

$$F(\theta_1|\theta_2) = L(A, A)(\theta_1 - \theta_2)
= L(A, A)(\theta_1 - A + A - \theta_2)
= \frac{L(A, A)}{\epsilon(A)}[\epsilon(\theta_1) - \epsilon(\theta_2)]
= \lambda \omega$$

Here we defined

$$\lambda \equiv \frac{L(A, A)}{\epsilon(A)} = \frac{L(A, A)}{2\pi \rho(A)}. \quad (3.6)$$

The contribution of this term to the noise reads

$$(\sqrt{2g})^2 \int_{\theta}^{e^{-1}(\omega)} \epsilon(\theta) d\theta \left( \frac{\epsilon(\theta)}{\omega - \epsilon(\theta)} \right)^{\lambda \omega}
= 2g\omega \int_{0}^{1} \exp \left[ \lambda \omega \ln \frac{x}{1-x} \right].$$

Expanding the exponential to first order gives no $\omega^2$ correction due to $\int_{0}^{1} \ln \frac{x}{1-x} = 0$.

Of course, the form factor contributes a non vanishing term at order $\omega^3$. This should be offset by the next form-factor involving a pair of voltons and a pair of antivoltons. Indeed, while in the relativistic limit with $\kappa \neq 0$, such a form factor gave a contribution of order $\omega$ to the noise, now the residue axiom, reasonably generalized to the non-relativistic case, will lead to a form factor similar to (2.47), but with the term $e^{2i\pi\kappa} - 1$ replaced by a term of order $\bar{\phi}$, measuring the difference between the S-matrix and $-1$ away from the Fermi rapidity. In the computation of the noise, this will give a term of order $\omega(\omega/V)^2$. More
generally, the process involving $n$ voltons and $n$ antivoltons will give a leading contribution of order $\omega (\omega/V)^{2n-2}$.

Let us now discuss in more details the term $\omega^3$. The integral (3.5) gives for the noise a contribution (assuming $\omega > 0$)

$$S(\omega)^{(2)} = \frac{g\omega}{\pi} + \zeta \omega^3,$$

(3.7)

where the value of $\zeta$ will not be needed in what follows. Since the total noise is $S(\omega) = \frac{\omega}{\pi}$, this means that the 4 particle form factor must contribute to the noise by a leading term

$$S(\omega)^{(4)} = -\zeta \omega^3;$$

(3.8)

and thus we obtain the sum rule for the four particle form factor

$$\int_{A}^{\infty} \int_{A}^{\infty} |f_{++--}(\theta_1, \theta_2, \theta_3, \theta_4)|^2 2\pi \delta[\epsilon(\theta_1) + \epsilon(\theta_2) - \epsilon(\theta_3) - \epsilon(\theta_4) - \omega] \prod_{i=1}^{4} \frac{d\theta_i}{2\pi} = -\zeta \omega^3. \quad (3.9)$$

The point is, that this sum rule is enough to determine the next term in $\chi''(\omega)$. Indeed, the two particle form factor contributes to $\chi''$ by a term proportional to

$$\int_{A}^{\infty} \int_{A}^{\infty} |f_{+-}(\theta_1, \theta_2)|^2 \delta[\epsilon(\theta_1) - \epsilon(\theta_2) - \omega] \text{Re}[\mathcal{R}(\theta_1)\mathcal{R}^*(\theta_2) - 1]d\theta_1d\theta_2. \quad (3.10)$$

We are now interested in getting the terms of order $\omega^3$ and $\omega^5$ in this integral. The term of order $\omega^3$ was already evaluated in I. For the term of order $\omega^5$, since the integral combined with the delta function contributes an overall $1/\omega$, there will be two contributions: we can either expand the $\mathcal{R}$ bracket to order $\omega^4$ and take the leading expression ($O(\omega^2)$) for the two particle form-factor, or expand the $\mathcal{R}$ bracket only to order $\omega^2$ and take the next to leading contribution for the two particle form factor ($O(\omega^4)$).

Similarly, the four particle form factor contributes by a term proportional to

$$\int_{A}^{\infty} |f_{++--}(\theta_1, \theta_2, \theta_3, \theta_4)|^2 \delta[\epsilon(\theta_1) + \epsilon(\theta_2) - \epsilon(\theta_3) - \epsilon(\theta_4) - \omega] \text{Re}[\mathcal{R}(\theta_1)\mathcal{R}(\theta_2)\mathcal{R}^*(\theta_3)\mathcal{R}^*(\theta_4) - 1] \prod_{i=1}^{4} d\theta_i. \quad (3.11)$$

Taking the term of order $\omega^6$ in the four particle form factor would necessitate taking the order 0 in the $\mathcal{R}$ bracket, which vanishes. Hence, the only term we need to consider is order $\omega^2$ in the $\mathcal{R}$ bracket, and the leading order ($O(\omega^4)$) in the four particle form factor. Now
the point is, that like in the previous section (paragraph 2.5), the term of order $\omega^2$ in the $\mathcal{R}$ bracket is actually independent of the rapidities, and simply factors out as a constant. The integral that is left is the same as the one appearing in the noise. The same remark holds for the two particle form-factor contribution (3.10), so these two terms cancel out! Hence, all what remains to determine the $\omega^5$ order is the $\mathcal{R}$ bracket to order $\omega^4$, combined with the leading expression ($O(\omega^2)$) for the two particle form-factor. In other words, the expression that was obtained in I is good to get the $\omega^5$ order:

$$\chi''(\omega) = -\frac{1}{2g\pi^2\omega^2} \text{Re} \int_{-\infty}^{\ln \omega} d\beta_2 d\beta'_2 [\mathcal{R}^*(\theta_2)\mathcal{R}(\theta'_2) - 1] e^{\beta_2} e^{\beta'_2} \delta(\omega - e^{\beta_2} - e^{\beta'_2}), \quad (3.12)$$

Here we have reparametrized $\epsilon \rightarrow e^\beta$, which defines the $\theta \rightarrow \beta$ correspondence. Redefining $\mathcal{R}(\epsilon) = \mathcal{R}(\theta)$ we rewrite (3.12) as

$$\chi''(\omega) = -\frac{1}{2g\pi^2\omega^2} \text{Re} \int_0^\omega dx [\mathcal{R}(x)\mathcal{R}^*(x - \omega) - 1] dx \quad (3.13)$$

Laborious but straightforward manipulations lead to the expression

$$\chi''(\omega) \approx \frac{(\phi')^2}{4g\pi^2} + \frac{1}{2g\pi^2} \left[ \frac{(\phi'')^2}{24} + \frac{\phi'\phi'''}{15} - \frac{(\phi')^4}{40} \right] \omega^2 + O(\omega^4), \quad \omega \rightarrow 0 \quad (3.14)$$

where we have set as usual $\mathcal{R} = e^{i\phi}$, and primes denote successive derivatives with respect to the variable $x = \epsilon(\theta)$. Equation (3.14) can be made completely explicit using the exact form of the dressed R matrix eq. (5.4) of I, together with the relation between $\epsilon$ and $\theta$ as given in eq. (2.13) of I. No simplification emerges, and at this stage there seems little point in giving a more explicit expression of the second term in (3.14).

4. The problem with $V = 0, T > 0$

4.1. Generalities

When $T \neq 0$, it is well known that thermal properties can be computed by evaluating correlators in a “thermal ground state”, that is, any state that is characterized by the equilibrium densities [2], [7]. For the problem at hand, restricting once again to $g = 1/t$,
recall that the particles are soliton, antisoliton and the n-breathers \( n = 1, 2, \ldots, t - 2 \).
Particles will be designated by the generic label \( j \). Introduce the functions \( \epsilon_j \) solution of
\[
\epsilon_j(\theta) = \mu_j e^\theta - T \sum_k \int_{-\infty}^{\infty} \Phi_{jk}(\theta - \theta') \ln \left[ 1 + e^{-\epsilon_k(\theta')/T} \right] d\theta'.
\] (4.1)
Then, equilibrium densities are given by
\[
\rho_j(\theta) = \frac{1}{2\pi} \frac{d\epsilon_j}{d\theta},
\] (4.2)
and the filling fractions are
\[
\frac{\rho_j}{\rho_j^0} \equiv \xi_j = \frac{1}{1 + e^{\epsilon_j/T}}.
\] (4.3)
The \( \epsilon \)'s diverge at infinity as in a free theory: \( \epsilon_j(\theta) \approx \mu_j e^\theta, e^\theta >> T \). At minus infinity, an interesting consequence of the interaction is that the \( \epsilon \)'s go to a constant of order \( T \). Explicitly one has
\[
e^{\epsilon_n/T} \approx (n + 1)^2 - 1, e^{\epsilon_{s.a}/T} \approx t - 1, \theta \rightarrow -\infty.
\] (4.4)
It is well known [2] (see also below) that the \( \epsilon \)'s are excitation energies: destroying (creating) a particle of type \( j \) decreases (increases) the energy by \( \epsilon_j \). Hence, due to (4.4), there are no single particle excitations of arbitrarily low energy: the low energy excitations are, as in the voltage case, fully obtained by particle hole pairs (there is a gap of order \( T \) for any processes that do not conserve the number of particles). Contrarily to the foregoing voltage case, such pairs can however be created at any (neighbouring) rapidities since at \( T > 0 \), the filling fractions are less than one.

As in the voltage case, we would like to know the form factors of the current. Observe that, since only particle hole pairs are involved in the very low energy limit, breathers won’t appear (since the current operator changes the parity of the number of breathers) and we can restrict to soliton/antsoliton excitations. To address the form factors, we need as before the renormalized charge and S-matrix.

### 4.2. Renormalized S matrix, charge and form-factors

We start with the shift function. Suppose for instance we add a particle of type \( k \) at rapidity \( \theta_p \). The shift of particles of type \( j \) obeys, similar to (2.14)
\[
F_{jk}(\theta|\theta_p) - \sum_l \int_{-\infty}^{\infty} \Phi_{jl}(\theta - \theta')\xi_l(\theta')F_{lk}(\theta'|\theta_p) = \delta_{b/jk}(\theta) - \delta_{jk}(\theta - \theta_p).
\] (4.5)
Observe that, since we consider massless excitations around arbitrary rapidities, we have allowed the additional phase shift $\delta_b$ to depend upon $\theta$.

To proceed, let us assume there is only one type of excitations. Dealing with several types makes the notation more cumbersome without changing the argument. Then the equation for the shift reads

$$F(\theta|\theta_p) - \int_{-\infty}^{\infty} \Phi(\theta - \theta')\xi(\theta') F(\theta'|\theta_p) = \delta_b(\theta) - \delta(\theta - \theta_p). \quad (4.6)$$

The excitation energy for adding this particle is

$$\varepsilon(\theta_p) = \mu e^{\theta_p} + \int_{-\infty}^{\infty} \mu e^{\theta'} \xi(\theta') F(\theta'|\theta_p). \quad (4.7)$$

Note that, compared with (2.16), there is no term arising from $-\infty$ since without a voltage, the bare (kinetic) energy of particles vanishes in that limit. From (4.7) we deduce

$$\frac{d\varepsilon(\theta_p)}{d\theta_p} = \mu e^{\theta_p} + \int_{-\infty}^{\infty} \mu e^{\theta'} \xi(\theta') \frac{dF(\theta'|\theta_p)}{d\theta_p}. \quad (4.8)$$

Now, from (4.6) we have

$$\frac{dF(\theta|\theta_p)}{d\theta_p} = \int_{-\infty}^{\infty} \Phi(\theta - \theta')\xi(\theta') \frac{dF(\theta'|\theta_p)}{d\theta_p} = \Phi(\theta - \theta_p)$$

Introduce the kernel, similar to (2.4)

$$\frac{L(\theta|\theta')}{\xi(\theta')} = \Phi(\theta - \theta') + \int_{-\infty}^{\infty} \Phi(\theta - \theta'')\xi(\theta'') \Phi(\theta'' - \theta') + \ldots. \quad (4.9)$$

It follows that

$$\frac{dF(\theta|\theta_p)}{d\theta_p} = \frac{L(\theta|\theta_p)}{\xi(\theta_p)} = \frac{L(\theta_p|\theta)}{\xi(\theta)}$$

and thus

$$\frac{d\varepsilon(\theta_p)}{d\theta_p} = \mu e^{\theta_p} + \int_{-\infty}^{\infty} \mu e^{\theta'} L(\theta_p|\theta') d\theta'$$

On the other hand, from (4.10) in the case of a single particle, we deduce

$$\frac{d\varepsilon}{d\theta} = \mu e^{\theta} + \int_{-\infty}^{\infty} \Phi(\theta - \theta')\xi(\theta') \frac{d\varepsilon}{d\theta'} d\theta', \quad (4.10)$$

so we find

$$\frac{d\varepsilon}{d\theta_p} = \frac{d\varepsilon}{d\theta_p}$$
hence $\varepsilon(\theta_p) = \epsilon(\theta_p) + \text{cst}$, the constant depending on the function $\delta_b(\theta)$.

Consider now excitations around some rapidity $\theta_0$. Shift the energy and momentum such that these excitations have $e = p = \varepsilon(\theta) - \varepsilon(\theta_0) = \epsilon(\theta) - \epsilon(\theta_0)$. In the low energy limit, parametrize $e = e^\beta$, we are then in a situation similar to excitations around the Fermi rapidity in the previous case. Chose the function $\delta_b(\theta)$ such that $F(\theta_0|\theta_0) = 0$ for any $\theta_0$. Then the analog of $\kappa$ vanishes. To completely determine the form factor of the current, we need to determine the charge of excitations. This is simpler than in the case of finite voltage. Indeed, since solitons and antisolitons behave in the same way except for their charge, we chose the same function $\delta_b$ for both species, and $F_s(-\infty,\theta_p) = F_a(-\infty,\theta_p)$. So, as many solitons as antisolitons move across the rapidity $-\infty$, resulting in a vanishing contribution to the charge. In other words the renormalized charge equals the bare charge.

Going back to $\theta$ variables, at low energy $\theta_1 \approx \theta_2$,

$$< 0|j(z)|\theta_2,\theta_1 >_{-+} = -i[\dot{\epsilon}_s(\theta_1)]^{1/2}[\dot{\epsilon}_s(\theta_2)]^{1/2} e^{iz[\varepsilon_s(\theta_1) - \varepsilon_s(\theta_2)]},$$

(4.11)

where we also used the symmetry (in the absence of applied voltage) between solitons and antisolitons $\epsilon_s = \epsilon_a$.

4.3. DC Conductance in the fractional quantum Hall effect using Kubo formula

From (4.11), the $\omega \to 0$ limit of the noise without impurity follows

$$\int dt \langle \{j(t), j(0)\} \rangle = 2 \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{4\pi^2} \xi[\varepsilon_s(\theta_1)] (1 - \xi[\varepsilon_s(\theta_2)]) \dot{\varepsilon}_s(\theta_1) \dot{\varepsilon}_s(\theta_2) 2\pi \delta[\varepsilon_s(\theta_1) - \varepsilon_s(\theta_2) - \omega] + (\omega \to -\omega)$$

$$= 2 \int_{-\infty}^{\infty} \frac{dx}{2\pi} \frac{1}{1 + e^{-x/T}} \frac{1}{1 + e^{x+\omega/T}} + (\omega \to -\omega) = \frac{4Tg}{2\pi} + O(\omega^2).$$

(4.12)

as required.

In the presence of an impurity, we can also compute the noise in a way similar to paper I. The LL and LR noise take the same form. The RR noise is a bit different because in the (thermal) ground state we have both solitons and antisolitons this time. In fact the

\[3\] To compare with the usual form of Nyquist theorem, recall that we have set $\hbar = 1$, so the dimensionless conductance of the pure Luttinger liquid reads $G = \frac{g}{2\pi}$. 

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RR noise is easily seen to be the same as the LL noise here, as is easily proven by changing basis $|\theta >_L \pm P|\theta >_L + Q|\theta >_L$. Hence

$$\int dt \left\langle \{j(t), j(0)\} \right\rangle = \frac{2}{2\pi} \int_{-\infty}^{\infty} \xi(\theta)[1 - \xi(\theta)]\dot{\xi}(\theta)|P(\theta)|^2d\theta.$$ 

From this we deduce the formula for the DC conductance using the Kubo formula

$$G = \frac{1}{T} \int_{-\infty}^{\infty} \xi_s(\theta)[1 - \xi_s(\theta)]\dot{\xi}_s(\theta)|P(\theta)|^2d\theta, \quad (4.13)$$

in agreement with $[5]$. Observe that, should multiple particle excitations be allowed as discussed briefly in the voltage case, terms involving higher orders of $|P|^2$ would appear in $G$. This seems unlikely in view of the success of formula (4.13) compared with numerical simulations. Also, (4.13) was initially derived using Landauer-Büttiker scattering formula, and although it is not entirely clear whether one can apply it to the quasiparticles of integrable systems, this alternate derivation still increases our confidence in (4.13).

### 4.4. Dynamical susceptibility in the double well problem

As an application, we would like to consider the low frequency behaviour of the dynamical susceptibility in the dissipative quantum mechanics problem and $T \neq 0$. We need to introduce a dressed reflection matrix in that case, which reads here

$$\mathcal{R}_j(\theta) = R_j(\theta) \exp \left[ \sum_k \int_{-\infty}^{\infty} \xi_k(\theta')F_{kj}(\theta'|\theta) \frac{d}{d\theta'} \ln R_k(\theta') \right]. \quad (4.14)$$

We define at $T \neq 0$,

$$\chi''(\omega) = \frac{1}{2} \int \frac{dt}{2\pi} e^{i\omega t} \left\langle \{S_z(t), S_z(0)\} \right\rangle$$

$$= \frac{1}{2} \tanh \frac{\omega}{2T} \int \frac{dt}{2\pi} e^{i\omega t} \left\langle \{S_z(t), S_z(0)\} \right\rangle. \quad (4.15)$$

The correlator of the spin anticommutator can be related with the correlator of current operators. The argument works exactly like at $T = 0$ $[\text{3}]$ (the propagator with Neumann boundary conditions eq. (5.4) of $[\text{3}]$ becomes a rational function of trigonometric functions at finite $T$. When $x \to 0$, it vanishes everywhere except at $y = y'$, and its integral for $y$ running on the interval $[0, 1/T]$ is still equal to 1). Using $[\text{3}]$ and the previous form-factors,

---

4 Here, we defined $G$ as $2\pi$ times the usual conductance.
one finds at low frequency (using again the symmetry between the two species at vanishing bias)
\[
\frac{2T}{\omega} \chi''(\omega) \approx -\frac{1}{g^2 \pi^2 \omega^2} \text{Re} \int_{-\infty}^{\infty} \xi[e_s(\theta)](1 - \xi[e_s(\theta) + \omega]) \dot{e}_s(\theta) (\mathcal{R}[e(\theta)]\mathcal{R}^*[e(\theta) + \omega] - 1) \, d\theta.
\]

Note that (4.16) differs from (3.12) by factors of 2, due to the fact that excitations involve solitons or antisolitons, and then can have positive or negative energy. There is also a factor of \(2g\) that differs, due to the fact that the renormalized charge is 1 with a temperature and no voltage, while it is \(\sqrt{2g}\) with a voltage and no temperature. From (4.16) one finds thus
\[
\lim_{\omega \to 0} \frac{2T}{\omega} \chi''(\omega) \approx -\frac{1}{2g^2 \pi^2} \int_{-\infty}^{\infty} \xi_s(\theta) \left[1 - \xi_s(\theta)\right] \left(\frac{d}{d\theta} \ln \mathcal{R}\right)^2 \, d\theta.
\]

In the case \(g = 1/2\), there is no dressing, and one finds
\[
\lim_{\omega \to 0} \frac{2T}{\omega} \chi''(\omega) = \frac{8}{\pi^2} \int_0^\infty dx \frac{1}{1 + e^{x/T}} \frac{1}{1 + e^{-x/T}} \left(\frac{T_B}{x^2 + T_B^2}\right)^2,
\]
so, after using the transformation
\[
\frac{1}{x + 1} = \frac{1}{x - 1} - \frac{2}{x^2 - 1}
\]
one finds, using standard integral representations of the \(\psi\) function,
\[
\lim_{\omega \to 0} \frac{2T}{\omega} \chi''(\omega) = \frac{1}{\pi^4 T} \left[\frac{\pi T}{T_B} \psi'\left(\frac{T_B}{2\pi T} + \frac{1}{2}\right) - \frac{1}{2} \psi''\left(\frac{T_B}{2\pi T} + \frac{1}{2}\right)\right],
\]
in agreement with the result of [15].

As small temperature, the sums are dominated by particles whose rapidities approach \(-\infty\). In that limit, one checks that \(\frac{d}{d\theta} \ln \mathcal{R} \approx \frac{4\pi}{T_B} \rho\), from which we recover \(\chi''(\omega) \approx \frac{\omega}{\theta(\pi T_B)^2}\) at \(T = 0\), as shown in I.

Eq. (4.17) can be made as explicit as necessary. Using the relation
\[
\frac{d}{d\theta} F_{kj}(\theta'|\theta) = \frac{L_{jk}(\theta'|\theta)}{\xi_k(\theta')}
\]
one finds
\[
\frac{d}{d\theta} \ln \mathcal{R}_j(\theta) = \frac{d}{d\theta} \ln R_j(\theta) + \sum_k \int_{-\infty}^{\infty} L_{kj}(\theta'|\theta') \frac{d}{d\theta'} \ln R_k(\theta').
\]
Here, one has
\[
L_{jk}(\theta'|\theta') = \Phi_{jk}(\theta - \theta')\xi_k(\theta') + \int_{-\infty}^{\infty} \Phi_{jl}(\theta - \theta'')\xi_l(\theta'')\Phi_{lk}(\theta'' - \theta')\xi_k(\theta') + \ldots
\]
Since functions \(\xi\) are easy to obtain by numerical solution of the thermodynamic Bethe ansatz equations, it is only a technical matter to determine (4.17).
5. Conclusion

We hope that the computations in this paper and the previous one open a possible direction for the study of correlators at finite $T$ and $V$ in quantum impurity problems. Our results (3.14), (4.17), although completely explicit, are unfortunately quite cumbersome: pushing the method further and getting results at arbitrarily large frequency looks much more involved than in the $T = 0$, $V = 0$ case (the situation looks more favorable for problems which are massive in the bulk, especially at low temperature \[8\]). Nevertheless, interesting questions are already raised by this approach: in particular, the results of the present paper and those found in I do not agree with those of perturbative methods. It would be very useful to investigate the matter further in experiments or numerical studies such as \[16\].

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References

[1] F.A. Smirnov, “Form factors in completely integrable models of quantum field theory”, World Scientific, and references therein.

[2] V.E. Korepin, N.M. Bogoliubov, A.G. Izergin, “Quantum inverse scattering method and correlation functions”, Cambridge University Press, (1993).

[3] G. Delfino, G. Mussardo, P. Simonetti, Phys. Rev. D51, 6620 (1995).

[4] F. Guinea, V. Hakim, A. Muramatsu, Phys. Rev. B32, 4410 (1985); S.A. Bulgadaev, JEPT vol. 38, 264 (1984), Sov. Phys. JETP, vol. 39, 314 (1984).

[5] P. Fendley, A.W.W. Ludwig, H. Saleur, Phys. Rev. B52, 8934 (1995), cond-mat/9503172.

[6] F. Lesage, H. Saleur, S. Skorik, Phys. Rev. Lett. 76 (1996) 3388, cond-mat/9512087; Nucl. Phys. B474 (1996), 602, cond-mat/9603043.

[7] A. Leclair, F. Lesage, S. Sachdev, H. Saleur, to appear in Nucl. Phys. B., cond-mat/9606104.

[8] S. Sachdev, K. Damle, “Low temperature spin diffusion in the one-dimensional quantum O(3) nonlinear σ model”, cond-mat/9610115.

[9] F. Lesage, H. Saleur, “Correlations in one dimensional quantum impurity problems with an external field”, submitted to Nucl. Phys. B; cond-mat/9611025.

[10] R. Landauer, Physica D38 (1989) 594; G.B. Lesovik, JETP Lett. 49 (1989) 594; M. Büttiker, Phys. Rev. Lett. 65 (1990) 2901.

[11] J. Cardy, J. Phys. A17 (1984) L385

[12] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.

[13] T. Klassen, E. Melzer, Int. J. Mod. Phys. A8 (1993) 4131.

[14] P. Fendley, H. Saleur, to appear in Phys. Rev. B., cond-mat/9601117.

[15] M. Sassetti, U. Weiss, Phys. Rev. A41 (1990) 5383.

[16] T.A. Costi, C. Kieffer, Phys. Rev. Lett. 76, 1683 (1996).