Abstract

Using \((2+\epsilon)\)-dimensional quantum gravity recently formulated by Kawai, Kitazawa and Ninomiya, we calculate the scaling dimensions of manifestly generally covariant operators in two-dimensional quantum gravity coupled to \((p,q)\) minimal conformal matter. Although the spectrum includes all the scaling dimensions of the scaling operators in the matrix model except the boundary operators, there are also many others which do not appear in the matrix model. We argue that the partial agreement of the scaling dimensions should be considered as accidental and that the operators considered give a new series of operators in two-dimensional quantum gravity.
1 Introduction

Two-dimensional quantum gravity [1, 2], which is important not only as a toy model of quantum gravity but also as a noncritical string theory, has been studied intensively these several years mainly by the matrix model [3, 4, 5] and Liouville theory [6, 7]. Although the equivalence of the two approaches is almost confirmed based on the agreement of the correlation functions of the operators [8, 9, 10, 11, 12], the notion of operators comes out in each approach in quite a different way. In the matrix model, the scaling operators appear when a macroscopic loop on the surface is shrunk. They form a complete set in the sense that their correlators satisfy closed recursive relations [13, 14]. We must say, however, that they come out in such a geometrical way that it is not clear how they can be written in terms of the metric and the matter fields. In Liouville theory, on the other hand, one can carry out the BRST cohomological analysis [15, 16] to obtain the physical operators, whose scaling dimensions have the same spectrum as that appearing in the matrix model except for those operators in the matrix model known as the boundary operators [17] or the redundant operators [17, 18]. Here the operators with zero ghost number can be understood as primary fields with gravitational dressing, while the operators with nonzero ghost number do not allow such a clear interpretation. Alternatively, without taking Felder’s resolution [19], one can construct the gravitationally dressed primary fields inside and outside the minimal Kac table, which have a one-to-one correspondence to the scaling operators in the matrix model up to the correlation function level [9, 12]. The inside ones are nothing but the operators with zero ghost number in the BRST analysis, while the outside ones include the operators with nonzero ghost number in the BRST analysis and the boundary operators. Here the physical meaning of the dressed primary fields outside the minimal Kac table is quite obscure.

In these circumstances, we think it is worth while studying manifestly generally covariant operators, whose physical meaning is clear. Specifically, we consider in this paper manifestly generally covariant operators written as a volume integral of a local scalar density composed of the metric and the matter fields. For example, in the case of pure gravity, the operators we consider are \( \int \sqrt{g} R^n d^2x \), where \( n = 0, 1, 2, \cdots \). In spite of the clarity of their physical meaning, such operators are difficult to study in the conventional approaches. In Liouville theory, there is no unambiguous way to define such composite operators, while in the matrix model, or in dynamical triangulation in general, one may consider their formal counterparts by identifying the scalar curvature with the deficit angle per volume, but it is not clear
whether they really correspond to the desired operators in the continuum limit.

There is a formalism, however, which seems most suitable for our purpose, namely \((2+\epsilon)\)-dimensional quantum gravity recently developed by Kawai, Kitazawa and Ninomiya \([20]\). Although their primary motivation was to explore the nature of higher-dimensional quantum gravity as in refs. \([21, 22, 23, 27]\), they succeeded in showing how they can take the \(\epsilon \to 0\) limit in their formalism to reproduce the results in two dimensions. Here we would like to generalize their calculation to the scaling dimensions of the manifestly generally covariant operators explained above, which was not accessible by the conventional approaches.

The paper is organized as follows. In the next section, we review the formalism of \((2+\epsilon)\)-dimensional quantum gravity. In Section 3, we explain how to calculate the scaling dimensions of the manifestly generally covariant operators using the formalism. In Section 4, we compare the obtained spectrum with that appearing in the matrix model or in Liouville theory. Section 5 is devoted to the summary and the discussion. The appendices contain some details of the calculation.

## 2 Formalism of \((2+\epsilon)\)-dimensional quantum gravity

We first review briefly the formalism of \((2+\epsilon)\)-dimensional quantum gravity which was developed by the authors of ref. \([20]\). Considering that the conformal mode plays the central role in two-dimensional quantum gravity as is learned through Liouville theory, they parametrize the metric in such a way that the conformal mode is explicitly separated as

\[
g_{\mu\nu} = \hat{g}_{\mu\rho}(e^{h})_{\rho}^{\nu}e^{-\phi},
\]

where \(h_{\mu\nu}\) is a traceless hermite tensor. Starting from the Einstein action and the action for \(c\) species of scalar field in \(D = 2 + \epsilon\) dimensions they calculate the one-loop divergence and obtain the counterterm as

\[
S_{\text{c.t.}} = -\frac{25 - c}{24\pi} \frac{\mu^{\epsilon}}{\epsilon} \int d^{D}x \sqrt{g}R.
\]

This causes, however, an oversubtraction problem, for the conformal mode, since the kinetic term of the conformal mode in the counterterm is \(O(1)\), whereas the divergent one-loop diagram for the conformal mode two-point function gives \(O(\epsilon)\) quantity. In this sense, the ordinary renormalization procedure breaks down unless we give up the general covariance of the procedure.
They argue, however, in the two-dimension limit, this oversubtraction problem can be taken care of in the following way. On the grounds that the oversubtraction problem is nothing but the counterterm dominance for the kinetic term of the conformal mode, they redefine the conformal mode propagator by summing up conformal mode propagators with arbitrary times of insertion of the counterterm $\frac{25-c}{24\pi} \partial_{\mu} \phi \partial_{\mu} \phi$. The conformal mode propagator after this resummation becomes

$$
\left( -\frac{2G}{\epsilon} \frac{1}{p^2} \right) \sum_{n=0}^{\infty} \left[ \left( -\frac{25-c}{24\pi} \frac{1}{2p^2} \right) \left( -\frac{2G}{\epsilon} \frac{1}{p^2} \right) \right]^n
= \left( -\frac{2G}{\epsilon} \frac{1}{p^2} \right) \frac{1}{1 - \frac{25-c}{24\pi} \frac{2G}{\epsilon}}
= -\frac{2G_0 \mu^{\epsilon}}{\epsilon} \frac{1}{p^2},
$$

(2.3)

where $G_0$ is the bare coupling constant which is related to the renormalized coupling constant $G$ through

$$
\frac{1}{G_0} = \mu^{\epsilon} \left( \frac{1}{G} - \frac{25-c}{24\pi} \frac{1}{\epsilon} \right),
$$

(2.4)

as can be read off from eq. (2.2). This leads them to use $G_0 \mu^{\epsilon}$ as an expansion parameter instead of $G$. Although it might cause trouble in the dynamics of the $h$-field, they claim that the expansion can be performed in a consistent way, expecting that the conformal mode determines the dynamics in the two-dimension limit. From (2.4), one can calculate the $\beta$-function as

$$
\beta(G) = \mu \frac{\partial G}{\partial \mu} = \epsilon G - \frac{25-c}{24\pi} G^2,
$$

(2.5)

which means there is an ultraviolet fixed point

$$
G^* = \frac{24\pi}{25-c} \epsilon,
$$

(2.6)

as long as $c < 25$. Finally they claim that the $\epsilon \to 0$ limit should be taken in the strong coupling regime ($G \gg G^* = O(\epsilon)$), which means

$$
G_0 \mu^{\epsilon} \longrightarrow -\frac{24\pi}{25-c} \epsilon.
$$

(2.7)

In this way, a perturbative expansion in terms of $G_0 \mu^{\epsilon}$ is turned into an expansion in terms of $\frac{24\pi}{25-c}$ in the $\epsilon \to 0$ limit, providing a well-defined formalism of two-dimensional quantum gravity.
A technically important point in their formalism is that actually the dynamics is completely determined by the conformal mode in the sense that the other fields can be dropped from the beginning, as they have checked explicitly up to the two-loop level in the case of the renormalization of $\int \sqrt{g}^{1-\Delta_0} \Phi_{\Delta_0} d^2x$ type operators, where $\Phi_{\Delta_0}$ is a spinless primary field with conformal dimension $\Delta_0$. After this simplification the theory can be reduced to a free field theory, which enables them to perform a full order calculation of the scaling dimensions of $\int \sqrt{g}^{1-\Delta_0} \Phi_{\Delta_0} d^2x$ type operators. The calculation reproduces the exact result of refs. [6, 7], which seems rather surprising considering the subtlety in their procedure described above.

3 Calculation of the scaling dimensions of manifestly generally covariant operators

Using the formalism described in the previous section, we first calculate the scaling dimensions of $\int \sqrt{g} R^n d^2x$ type operators in pure gravity. Dropping the $h$-field, the Einstein action can be written in terms of the conformal mode as

$$\int \sqrt{g} R d^Dx = \int d^Dx \left[ \sqrt{\hat{g}} \hat{R} e^{-\frac{4}{\epsilon} \phi} - \frac{\epsilon(D-1)}{4} \sqrt{\hat{g}} e^{-\frac{4}{\epsilon} \phi} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \text{(total derivative term)} \right]. \quad (3.1)$$

By introducing a new variable $\psi$ through

$$e^{-\frac{4}{\epsilon} \phi} = 1 + \frac{\epsilon}{4} \psi \quad (3.2)$$

the action can be written in terms of $\psi$ as

$$\int \sqrt{g} R d^Dx = \int d^Dx \left[ \sqrt{\hat{g}} \hat{R} \left( 1 + \frac{\epsilon}{4} \psi \right) - \frac{\epsilon(D-1)}{4} \sqrt{\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right]. \quad (3.3)$$

Following the usual prescription of the background field method, we drop the linear term and arrive at the following bare action

$$S = \frac{1}{G_0} \int d^Dx \left[ \frac{\epsilon^2}{16} \sqrt{\hat{g}} \hat{R} \psi^2 - \frac{D-1}{4} \epsilon \sqrt{\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right]. \quad (3.4)$$

We make use of the general covariance of the theory to proceed further; namely, instead of keeping the full background field dependence, we expand the background field around the flat metric as

$$\hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu} \quad (3.5)$$
and, after calculating the one-point function of an operator up to sufficient order in $\hat{h}_{\mu\nu}$, we read off the corresponding generally covariant form to reproduce the full result. Defining $H$ and $G_{\mu\nu}$ through

$$\sqrt{\hat{g}} = 1 + H$$  
(3.6)  
$$\sqrt{\hat{g}}\hat{g}^{\mu\nu} = \delta_{\mu\nu} + G_{\mu\nu},$$  
(3.7)

the bare action reads

$$S = \frac{1}{G_0} \int d^Dx \left[ \frac{\epsilon^2}{16} \hat{R}(1 + H)^2 - \frac{D-1}{4} \epsilon (\partial_{\mu}\psi \partial_{\mu}\psi + G_{\mu\nu} \partial_{\mu}\psi \partial_{\nu}\psi) \right].$$  
(3.8)

The terms with $H$ and $G_{\mu\nu}$ will be treated perturbatively. When we calculate the one-point function of $\int \sqrt{g} R^g d^2x$, we have to keep terms up to $O(\hat{h}^n)$, since $\hat{R} = \partial^2 \hat{h}_{\mu\nu} - \partial_{\mu} \partial_{\nu} \hat{h}_{\mu\nu}$. Special care should be taken for the $n = 1$ case, which will be treated later. $\int \sqrt{g} R^g d^2x$ can be expressed in terms of $\psi$ as

$$\int \sqrt{g} R^n d^Dx = \int d^Dx \sqrt{\hat{g}} e^{\frac{1}{4} (-\frac{D}{2} + n + 1)} \left\{ \hat{R} - (D-1) \epsilon \hat{g}^{\mu\nu} \nabla_{\mu} \partial_{\nu} \phi + \frac{1}{4} \epsilon (D-1) \hat{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi \right\}.$$  
(3.9)

In the following, we set $H = 0$ and $G_{\mu\nu} = 0$ in the action (3.8) and replace $\hat{g}^{\mu\nu} \nabla_{\mu} \partial_{\nu} \phi$ in the expression (3.9) with $\partial^2$. That this does not affect the result is shown in Appendix A. The expectation value of the expression (3.9) can be written down for $n = 2$, for example, as

$$\langle \int \sqrt{g} R^2 d^Dx \rangle = \int d^Dx \sqrt{\hat{g}} (e^{\frac{1}{4} (-\frac{D}{2} + 2 + \log(1 + \psi))} \hat{R}^2 + 2(D-1) \int d^Dx \sqrt{\hat{g}} (e^{\frac{1}{4} (-\frac{D}{2} + 2 + \log(1 + \psi))} \partial^2 \phi) \hat{R} + (D-1)^2 \int d^Dx \sqrt{\hat{g}} (e^{\frac{1}{4} (-\frac{D}{2} + 2 + \log(1 + \psi))} (\partial^2 \phi)^2).$$  
(3.10)

Since we are dealing with free field theory, the expectation value within each term can be calculated to full order. For the details of the calculation, we refer the reader to Appendix B, where we show that the relevant $\frac{1}{\epsilon}$ divergence comes from the $e^{\frac{1}{4} (-\frac{D}{2} + 2 + \log(1 + \psi))}$ in each term. The same argument holds for arbitrary $n$, and considering that

$$\sqrt{\hat{g}}^{1-\Delta_0} \sim \sqrt{\hat{g}}^{1-\Delta_0} e^{-\frac{D}{2}(1-\Delta_0)} \phi + \ldots$$  
(3.11)  
$$\sqrt{\hat{g}} R^n \sim \sqrt{\hat{g}} \hat{R}^n e^{(-\frac{D}{2}+n)\phi} + \ldots,$$  
(3.12)
we can obtain the scaling dimension of \( \int \sqrt{g} R^n \, d^2 x \) by substituting \( \Delta_0 \) with \( n \) in the expression for the scaling dimension of \( \int \sqrt{g}^{1-\Delta_0} \Phi_{\Delta_0} \, d^2 x \) \[\text{(20)}\]

\[
\Delta \left( \int \sqrt{g}^{1-\Delta_0} \Phi_{\Delta_0} \, q^2 \, d^2 x \right) = 1 - \frac{2(\Delta_0 - 1) - \frac{25-c}{12} (\sqrt{1 + \frac{24}{25-c}} (\Delta_0 - 1) - 1)^2}{2(\Delta_0^{(0)} - 1) - \frac{25-c}{12} (\sqrt{1 + \frac{24}{25-c}} (\Delta_0^{(0)} - 1) - 1)^2}
\]

\[= \frac{\sqrt{1 - c + 24\Delta_0} - \sqrt{1 - c + 24\Delta_0^{(0)}}}{\sqrt{25 - c} - \sqrt{1 - c + 24\Delta_0^{(0)}}}, \quad (3.13)\]

where \( c = 0 \) and \( \Delta_0^{(0)} = 0 \) for pure gravity. Thus, we obtain

\[
\Delta \left( \int \sqrt{g} R^n \, d^2 x \right) = \frac{\sqrt{1 + 24n - 1}}{4}. \quad (3.14)
\]

For \( n = 1 \), since the \( O(\hat{h}) \) contribution to \( \int \sqrt{g} R \, d^2 x \) is a total derivative, we have to look at the \( O(\hat{h}^2) \) contributions instead of the \( O(\hat{h}) \) contributions. In this case, however, the exponent of \( e^{-\frac{4}{\epsilon} (-\frac{2}{2} + n) \log(1+\frac{4}{\psi})} \) in \( (3.9) \) gets an extra \( O(\epsilon) \) factor and therefore we do not have any \( \frac{1}{\epsilon} \) divergence, which means the scaling dimension is unity and the expression \( (3.14) \) holds for \( n = 1 \) as well.

Let us extend the above result to two-dimensional quantum gravity coupled to \((p, q)\) minimal conformal matter. Recall that \( p \) and \( q \) are coprime integers and satisfy \( p < q \). The central charge of the \((p, q)\) minimal model is

\[
c = 1 - \frac{6(p - q)^2}{pq}, \quad (3.15)
\]

and the conformal weight of the \((r, s)\) primary field \( \Phi_{r,s} \) is given by the Kac table as

\[
h_{r,s} = \left(\frac{qr - ps}{4pq}\right)^2 - \left(\frac{p - q}{pq}\right)^2, \quad (3.16)
\]

where \( r \) and \( s \) are positive integers which satisfy

\[
ps < qr, \quad r < p, \quad \text{and} \quad s < q. \quad (3.17)
\]

\( \Phi_{1,1} \) corresponds to the identity operator, whose conformal weight is 0.

Since \( \sqrt{g}^{1-h_{r,s}} \Phi_{r,s} \) is a scalar density, we can define a set of manifestly generally covariant operators by

\[
\int \sqrt{g}^{1-h_{r,s}} \Phi_{r,s} R^n \, d^2 x \quad (n = 0, 1, 2, \cdots). \quad (3.18)
\]
The cosmological term, which we take as a standard scale to define the scaling dimensions, is identified, as in Liouville theory, with the operator
\[ \int \sqrt{g}^{1-h_{\text{min}}} \Phi_{\text{min}} \, d^2x, \] (3.19)
where \( \Phi_{\text{min}} \) is the primary field with the least conformal weight \( h_{\text{min}} \) given by
\[ h_{\text{min}} = \frac{1 - (p - q)^2}{4pq}. \] (3.20)
For unitary models \((q = p + 1)\), \( h_{\text{min}} = 0 \) and \( \Phi_{\text{min}} = \Phi_{1,1} \) (the identity operator), and therefore (3.19) reduces to the naive cosmological term \( \int \sqrt{g} \, d^2x \). The scaling dimension \( \Delta^{\text{MGC}}_{r,s;n} \) of the operator \( \int \sqrt{g}^{1-h_{r,s}} \Phi_{r,s} R^n d^2x \) can be obtained by setting \( \Delta_0 = n + h_{r,s} \) and \( \Delta_0^{(0)} = h_{\text{min}} \) in the expression (3.13), which gives
\[ \Delta^{\text{MGC}}_{r,s;n} = \frac{\sqrt{1 - c + 24(h_{r,s} + n)} - \sqrt{1 - c + 24h_{\text{min}}}}{\sqrt{25 - c} \sqrt{1 - c + 24h_{\text{min}}}} = \frac{\sqrt{(qr - ps)^2 + 4pqn - 1}}{p + q - 1}, \] (3.21)
where (3.15), (3.16) and (3.20) are used in the last equality. One can see that the scaling dimension of \( \int \sqrt{g} R \, d^2x \) is 1, which is to be expected since \( \int \sqrt{g} R \, d^2x \) is topological in the sense that it is a constant for a fixed topology.

We comment here that there are also such generally covariant operators as \( \int \sqrt{g} R \Delta R d^2x \), which we do not consider in this paper. The only difficulty in dealing with such operators is that the argument made in Appendix A does not work in this case. Consequently even the renormalizability of such operators is not obvious. We can say, however, that if they are renormalizable at all, they have the same scaling dimension as a \( \int \sqrt{g} R^n d^2x \) type operator with the same canonical dimension.

4 Comparison of the spectrum with that appearing in the matrix model

We compare the spectrum of the scaling dimensions obtained in the previous section with that appearing in the matrix model. Let us begin with the case of pure gravity. In the matrix model, we have a set of scaling operators \( \mathcal{O}_k \) \((k = 1, 3, 5 \cdots)\) whose scaling dimension
Our result (3.14) agrees with this scaling dimension when
\[ n = \frac{k^2 - 1}{24} = \frac{(k + 1)(k - 1)}{24}. \]  
(4.1)

Since \( k \) is a positive odd integer, the righthand side of the above expression becomes integer except when \( k = 0 \mod 3 \). Thus we have confirmed that in the case of pure gravity our spectrum includes all the scaling dimensions of the scaling operators in the matrix model except \( \mathcal{O}_k (k = 0 \mod 3) \), which are called the boundary operators due to the fact that \( \mathcal{O}_3 \) can be interpreted as a ‘cosmological term’ for the boundary of the surface [17].

Let us next examine the case in which \((p,q)\) minimal conformal matter is coupled. In the matrix model, we have a set of scaling operators \( \mathcal{O}_k (k > 0, k \not\equiv 0 \mod p) \) whose scaling dimension is given by
\[ \Delta_{\text{MM}}^k = \frac{k - 1}{p + q - 1}. \]  
(4.2)

We can check explicitly that when
\[ n = \begin{cases} 
  pqt^2 + (qr + ps)t + rs & \text{with } t \in \mathbb{Z}, \\
  pqt^2 + (qr - ps)t & \text{else}
\end{cases} \]  
(4.3)

with \( t \in \mathbb{Z} \), our result (3.21) reduces to
\[ \frac{|2pq + qr \pm ps| - 1}{p + q - 1}, \]  
(4.4)

which agrees with the spectrum obtained in the BRST analysis of the Liouville theory [15, 16]. Note that the righthand side of (4.3) is a non-negative integer for any \( t \in \mathbb{Z} \). This means that, just as in pure gravity, our spectrum includes all the scaling dimensions of the scaling operators in the matrix model except the boundary operators \( \mathcal{O}_k (k = 0 \mod q) \).

One should note, however, that in our spectrum there are also many generically irrational scaling dimensions which do not appear in the matrix model. This may be a clue that the operators considered in this paper, except for the ones with \( n = 0 \), are completely different from those appearing in the matrix model. Indeed we argue in the next section that the partial agreement of the scaling dimensions should be considered as accidental.

To illustrate our result, we show, in Tables 1, 2 and 3, our spectrum as well as that appearing in the matrix model for three typical cases: pure gravity \((p = 2, q = 3)\), the \( k = 3 \) case of Kazakov’s \( k \)-series \((p = 2, q = 5)\), and quantum gravity coupled to the critical Ising model \((p = 3, q = 4)\).
5 Summary and Discussion

In this paper, we have calculated the scaling dimensions of manifestly generally covariant operators using \((2 + \epsilon)\)-dimensional quantum gravity. The spectrum we obtained includes all the scaling dimensions of scaling operators in the matrix model except the boundary operators. Yet there are also many others which do not appear in the matrix model or in any other formalism considered so far.

As was mentioned in the Introduction, although the scaling operators in the matrix model form a complete set, their physical picture is not clear except for the ones which can be understood as primary fields with gravitational dressing. Our result might suggest the interesting possibility that the rest of the scaling operators correspond to \(\int \sqrt{g}^{1-\Delta_{r,s}} \Phi_{r,s} R^n d^2x\) \((n = 1, 2, 3, \ldots)\) except for the boundary operators. Moreover, one might expect that the indices \(r\) and \(s\) in the spectrum (4.4) obtained in the BRST analysis are nothing but those of the \((r, s)\) primary field \(\Phi_{r,s}\) and that the ghost number \(-(2t + 1)\) or \(2t\) respectively for the plus/minus sign in the expression (4.4) is related to the \(n\) of \(R^n\) through the expression (4.3), though the correspondence at the correlation function level between the physical operators in the BRST analysis and the scaling operators in the matrix model has not been proved yet for the operators with nonzero ghost number.

We argue, however, that this seems not the case. Firstly the operators \(\int \sqrt{g}^{1-\Delta_{r,s}} \Phi_{r,s} R^n d^2x\), which seem naively to be written in terms of the Liouville field only and without ghosts for the gravity sector, cannot be identified with the operators with nonzero ghost number in the BRST analysis. This argument is not strict, though, since we might pick up the ghost contribution when we regularize such composite operators as \(\int \sqrt{g}^{1-\Delta_{r,s}} \Phi_{r,s} R^n d^2x\) in Liouville theory. One can also argue as follows [26]. Take, for example, the operators in pure gravity, \(\mathcal{O}_7\) and \(\int \sqrt{g} R^2 d^2x\), which have been shown to have the same scaling dimension \(3/2\). Recently the theory with \(\int \sqrt{g} R^2 d^2x\) in the action has been investigated and the partition function is shown to behave as a function of the area as [27]

\[
f(A) \sim A^{\text{str} - 3} e^{-\frac{\text{const.}}{m^2A}},
\]

(5.1)

for \(m^2A \ll 1\), where \(1/m^2\) is the coefficient of the \(R^2\) term in the action. On the other hand, the theory with the action \(S = t\mathcal{O}_1 + \mathcal{O}_5 + x_7\mathcal{O}_7\) in the matrix model gives the string equation

\[
t + f^2 + x_7f^3 = 0,
\]

(5.2)
which means that the area dependence of the partition function for this case gives a power behavior, which is obviously different from that in the $R^2$ gravity. We conclude, therefore, that $\mathcal{O}_7$ and $\int \sqrt{g} R^2 d^2 x$ cannot be identified, in spite of the agreement of the scaling dimensions.

Our result together with the arguments presented above suggests that the operators $\int \sqrt{g}^{1-\Delta_{r,s}} \Phi_{r,s} R^n d^2 x \ (n = 1, 2, 3, \cdots)$ give a new series of operators in two-dimensional quantum gravity.

We would like to thank Prof. H. Kawai for stimulating discussions. We are also grateful to Prof. Y. Kitazawa, Dr. K.-J. Hamada, Dr. K. Hori, Dr. M. Oshikawa and Dr. Y. Watabiki for fruitful conversations and to Dr. N. McDougall for carefully reading the manuscript.
Appendix A

In this appendix, we check that the result is not affected by the simplification we have made concerning the background field dependence. We expand the background field around the flat metric as

\[ \hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu}. \]  

(A.3)

Then the action and the operator considered are written as

\[ S = \frac{1}{G_0} \int d^Dx \left[ \frac{\epsilon^2}{16} \hat{R}(1 + H)\psi^2 - \frac{D - 1}{4} \epsilon(\partial_\mu \psi \partial_\mu \psi + G_{\mu\nu} \partial_\mu \psi \partial_\nu \psi) \right], \]  

(A.4)

\[ \int d^Dx \sqrt{\hat{g}} R^n = \int d^Dx \sqrt{\hat{g}} e^{-\frac{4}{\epsilon}(-\frac{D}{2} + n) \log(1 + \frac{1}{\epsilon} \psi)} \cdot \left\{ \hat{R} + \partial^2 \psi - \hat{\Gamma}_{\mu\nu}^\nu \partial_\nu \psi + I_{\mu\nu}(\partial_\mu \partial_\nu \psi - \hat{\Gamma}_{\lambda\mu\nu}^\lambda \partial_\lambda \psi) \right\}^n, \]  

(A.5)

respectively, where \( H, G_{\mu\nu} \) and \( I_{\mu\nu} \) are \( O(\hat{h}) \) quantities defined through

\[ \sqrt{\hat{g}} = 1 + H \]  

(A.6)

\[ \sqrt{\hat{g}} \hat{g}_{\mu\nu} = \delta_{\mu\nu} + G_{\mu\nu} \]  

(A.7)

\[ \hat{g}_{\mu\nu} = \delta_{\mu\nu} + I_{\mu\nu}. \]  

(A.8)

We have set \( H = 0, G_{\mu\nu} = 0, I_{\mu\nu} = 0 \) and \( \hat{\Gamma}_{\mu\nu}^\nu = 0 \) at the beginning of our calculation. In order to justify this simplification, we have to check that there is no extra \( \frac{1}{\epsilon} \) divergence coming from the terms with the above \( O(\hat{h}) \) coefficients.

For each use of \( G_{\mu\nu} \partial_\mu \psi \partial_\nu \psi \) in the action, we have to use \( \partial^2 \psi \) in the operator in order to keep \( O(\hat{h}^n) \). The diagrams we have to consider are listed in Figure 1, where the dot represents a derivative. The first one, for example, gives

\[ \int \frac{d^Dp}{(2\pi)^D} \frac{(p + k)^2(p + k)_{\mu}p_{\nu}}{(p + k)^2p^2}. \]  

(A.9)

In order to have a logarithmic divergence, we have to factor out \( k^2 \) from the integrand, which is not possible due to the fact that \( (p + k)^2 \) in the numerator coming from the \( \partial^2 \psi \) in the operator cancels the propagator \( \frac{1}{(p + k)^2} \). This occurs for each of the diagrams in Figure 1 and one can also check that the above situation is not altered even if one takes into account the terms \( \hat{\Gamma}_{\mu\nu}^\nu \partial_\nu \psi, I_{\mu\nu} \partial_\mu \partial_\nu \psi \) and \( I_{\mu\nu} \hat{\Gamma}_{\lambda\mu\nu}^\lambda \partial_\lambda \psi \) in the operator and the \( \frac{\epsilon^2}{16} \hat{R} H \psi^2 \) term in the action.
Appendix B

In this appendix, we explain how to evaluate the expectation values appearing in eq. (3.10), namely,

\[
\langle \int \sqrt{g} R^2 d^D x \rangle = \int d^D x \sqrt{\hat{g}} \langle e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2}) \partial^2 \psi} \rangle \hat{R}^2 \\
+ 2(D - 1) \int d^D x \sqrt{\hat{g}} \langle e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2}) \partial^2 \psi} \rangle \hat{R} \\
+ (D - 1)^2 \int d^D x \sqrt{\hat{g}} \langle e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2}) (\partial^2 \psi)^2} \rangle. \tag{B.1}
\]

The diagrams which appear in calculating the expectation value in each term can be drawn generally as (a),(b) and (c) respectively in Figure 2, where the dot represents a derivative and the cross represents a mass insertion. Note that the expectation value in the first term is just the one we encounter in the case of \( \int \sqrt{g} \Phi \Delta \Phi d^2 x \) type operators. Since each plain loop contributes a factor

\[
- \frac{2G_0}{\epsilon} \int \frac{d^D p}{(2\pi)^D} \frac{1}{p^2} = - \frac{2G_0}{\epsilon} \left( - \frac{\mu}{2\pi \epsilon} \right) = \frac{G_0 \mu \epsilon}{\pi \epsilon^2}, \tag{B.2}
\]

we can calculate, for example, the diagram (a) by introducing a zero-dimensional field theory whose action is \( S(X) = \frac{1}{2} \frac{\pi \epsilon^2}{G_0 \mu \epsilon} X^2 \), and considering the expectation value of

\[
e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2}) X}. \tag{B.3}
\]

Thus we have

\[
\langle e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2})} \rangle = \frac{1}{Z} \int_{-\infty}^{\infty} dX e^{-\frac{4}{\epsilon}(-\Delta + 2 + \frac{\Delta}{2}) \log(1 + \frac{\Delta}{2}) X} e^{-\frac{\pi \epsilon^2}{G_0 \mu \epsilon} X^2}, \tag{B.4}
\]

where

\[
Z = \int_{-\infty}^{\infty} dX e^{-\frac{\pi \epsilon^2}{G_0 \mu \epsilon} X^2} = \sqrt{\frac{2G_0 \mu \epsilon}{\epsilon}}. \tag{B.5}
\]

Introducing a new variable \( Y = \frac{1}{4} \epsilon X \), the integral becomes

\[
\frac{\epsilon}{\sqrt{2G_0 \mu \epsilon}} \frac{4}{\epsilon} \int_{-\infty}^{\infty} dY e^{-\frac{1}{4} \left( -\frac{\Delta}{2} + 2 + \frac{\Delta}{2} \log(1 + Y) + \frac{8\pi \epsilon}{G_0 \mu \epsilon} Y^2 \right)} , \tag{B.6}
\]

whose asymptotic behavior for \( \epsilon \to 0 \) can be readily evaluated by means of the saddle-point method. The saddle point \( Y = \rho \) is given through

\[
\frac{d}{dY} \left[ 4 \left( -\frac{\Delta}{2} + 2 \right) \log(1 + Y) + \frac{8\pi \epsilon}{G_0 \mu \epsilon} Y^2 \right]_{Y=\rho} = 0, \tag{B.7}
\]
namely,
\[
4 \left( -\frac{D}{2} + 2 \right) \frac{1}{1 + \rho} + \frac{16\pi\epsilon}{G_0\mu^3} \rho = 0,
\] (B.8)

from which we obtain
\[
\rho = \frac{1}{2} \left\{-1 \pm \sqrt{1 - \frac{G_0\mu^3}{\pi\epsilon}} \right\}.
\] (B.9)

Thus we obtain the asymptotic behavior of the expectation value up to a factor of \(O(1)\) as
\[
\sim \exp \left[ -\frac{4}{\epsilon} \log(1 + \rho) \right.
- \frac{8\pi}{G_0\mu^3} \rho^2 \left. \right].
\] (B.10)

We have to choose ‘+’ for the double sign in the expression (B.9) so that we may reproduce the correct perturbative expansion. Let us now turn to the second term in eq. (B.1). The expectation value in this term can be evaluated with the diagram (b) as,
\[
\frac{1}{Z} \int_{-\infty}^{\infty} dX \left\{ \frac{d}{dX} e^{-\frac{4}{\epsilon}(-\frac{D}{2}+\frac{\epsilon}{4}) \log(1+\frac{\epsilon}{4}X)} \right\} e^{-\frac{1}{2} \frac{\pi^2}{G_0\mu^3} X^2}
\cdot \left( -\frac{2G_0}{\epsilon} \right)^2 \cdot \left( -\frac{1}{G_0} \frac{\epsilon^2}{8} \right) \int \frac{d^Dp}{(2\pi)^D} \frac{(-p^2)}{(p^2)^2}.
\]

Since the expression in the curly bracket gives
\[
-\frac{4}{\epsilon} \left( -\frac{D}{2} + 2 + \frac{\epsilon}{4} \right) \frac{1}{1 + \frac{\epsilon}{4} X} \frac{\epsilon}{4} e^{-\frac{4}{\epsilon}(-\frac{D}{2}+\frac{\epsilon}{4}) \log(1+\frac{\epsilon}{4}X)} = - \left( -\frac{D}{2} + 2 + \frac{\epsilon}{4} \right) e^{-\frac{4}{\epsilon}(-\frac{D}{2}+\frac{\epsilon}{4}) \log(1+\frac{\epsilon}{4}X)},
\] (B.11)

the result for the asymptotic behavior is the same as (B.10) up to a factor of \(O(1)\). As for the third term, there are two diagrams we have to consider, as is shown in Figure (2-c). The left one can be evaluated as
\[
\frac{1}{Z} \int_{-\infty}^{\infty} dX \left\{ \frac{d^2}{dX^2} e^{-\frac{4}{\epsilon}(-\frac{D}{2}+\frac{\epsilon}{4}) \log(1+\frac{\epsilon}{4}X)} \right\} e^{-\frac{1}{2} \frac{\pi^2}{G_0\mu^3} X^2}
\cdot \left[ \left( -\frac{2G_0}{\epsilon} \right)^2 \cdot \left( -\frac{1}{G_0} \frac{\epsilon^2}{8} \right) \int \frac{d^Dp}{(2\pi)^D} \frac{(-p^2)}{(p^2)^2} \right]^2,
\]

whose asymptotic behavior is also the same as (B.10) up to a factor of \(O(1)\), while the right one can be evaluated as
\[
\frac{1}{Z} \int_{-\infty}^{\infty} dX e^{-\frac{4}{\epsilon}(-\frac{D}{2}+\frac{\epsilon}{4}) \log(1+\frac{\epsilon}{4}X)} e^{-\frac{1}{2} \frac{\pi^2}{G_0\mu^3} X^2}
\cdot \left( -\frac{2G_0}{\epsilon} \right)^3 \cdot \left( -\frac{1}{G_0} \frac{\epsilon^2}{8} \right) \int \frac{d^Dp}{(2\pi)^D} \frac{(-p^2)^2}{(p^2)^3}.
\]
which has the asymptotic behavior of \((B.10)\) multiplied by an \(O(\epsilon)\) factor. Altogether, we get
\[
\langle \int \sqrt{gR^2} d^Dx \rangle \sim \exp \left[ -\frac{4}{\epsilon} \log(1 + \rho) - \frac{8\pi G_0 \mu}{\epsilon\rho^2} \right] \int \sqrt{\hat{g}\hat{R}^2} d^Dx, \tag{B.13}
\]
which means that the relevant \(\frac{1}{\epsilon}\) divergence in calculating the scaling dimension comes from the \(e^{-\frac{4}{\epsilon}(-D^2 + 2\log(1 + \frac{\rho}{\epsilon^2})}\) in each term of \((B.1)\).
References

[1] A.M. Polyakov, Mod. Phys. Lett. A2 (1987) 893.

[2] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.

[3] E. Brezin and V. Kazakov, Phys. Lett. B236 (1990) 144.

[4] M. Douglas and S. Shenker, Nucl. Phys. B335 (1990) 635.

[5] D.J. Gross and A.A. Migdal, Phys. Rev. Lett. 64 (1990) 717; Nucl. Phys. B340 (1990) 333.

[6] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 504.

[7] F. David, Mod. Phys. Lett. A3 (1988) 1651.

[8] M. Goulian and M. Li, Phys. Rev. Lett. 66 (1991) 2051.

[9] Y. Kitazawa, Phys. Lett. B265 (1991) 262.

[10] P. Di Francesco and D. Kutasov, Phys. Lett. B261 (1991) 385.

[11] Vl. S. Dotsenko, Mod. Phys. Lett. A6 (1991) 3601.

[12] K.-J. Hamada, preprint YITP/U-93-28, to appear in Nucl. Phys. B; preprint YITP/U-93-34.

[13] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385.

[14] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348 (1991) 435.

[15] B. Lian and G. Zuckermann, Phys. Lett. B254 (1991) 417.

[16] P. Bouwknegt, J. McCarthy and K. Pilch, Comm. Math. Phys. 145 (1992) 541.

[17] E. Martinec, G. Moore and N. Seiberg, Phys. Lett. B263 (1991) 190.

[18] M. Fukuma, H. Kawai and R. Nakayama, Comm. Math. Phys. 148 (1992) 101.

[19] G. Felder, Nucl. Phys. B324 (1989) 548.
[20] H. Kawai, Y. Kitazawa and M. Ninomiya, Nucl. Phys. **B393** (1993) 280; Nucl. Phys. **B404** (1993) 684.

[21] S. Weinberg, in General Relativity, an Einstein Centenary Survey, eds. S.W. Hawking and W. Israel (Cambridge University Press, 1979).

[22] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. **B133** (1978) 417.

[23] S.M. Christensen and M.J. Duff, Phys. Lett. **B79** (1978) 213.

[24] H. Kawai and M. Ninomiya, Nucl. Phys. **B336** (1990) 115.

[25] P. Ginsparg, M. Goulian, M.R. Plesser and J. Zinn-Justin, Nucl. Phys. **B342** (1990) 539.

[26] H. Kawai, private communication.

[27] H. Kawai and R. Nakayama, Phys. Lett. **B306** (1993) 224.
| scaling operator | scaling dimension | generally covariant operator | scaling dimension |
|------------------|-------------------|-----------------------------|------------------|
| \( \mathcal{O}_1 \) | 0 | \( f \sqrt{g} d^2 x \) | 0 |
| \( \mathcal{O}_3 \) | 1/2 | \( f \sqrt{g} R d^2 x \) | 1 |
| \( \mathcal{O}_5 \) | 1 | \( f \sqrt{g} R^2 d^2 x \) | 3/2 |
| \( \mathcal{O}_7 \) | 3/2 | \( f \sqrt{g} R^3 d^2 x \) | (\( \sqrt{73} - 1 \)/4) |
| \( \mathcal{O}_9 \) | 2 | \( f \sqrt{g} R^4 d^2 x \) | (\( \sqrt{97} - 1 \)/4) |
| \( \mathcal{O}_{11} \) | 5/2 | \( f \sqrt{g} R^5 d^2 x \) | 5/2 |
| \( \mathcal{O}_{13} \) | 3 | \( f \sqrt{g} R^6 d^2 x \) | (\( \sqrt{145} - 1 \)/4) |
| \( \mathcal{O}_{15} \) | 7/2 | \( f \sqrt{g} R^7 d^2 x \) | 3 |
| \( \mathcal{O}_{17} \) | 4 | \( f \sqrt{g} R^8 d^2 x \) | (\( \sqrt{193} - 1 \)/4) |
| \( \mathcal{O}_{19} \) | 9/2 | \( f \sqrt{g} R^9 d^2 x \) | (\( \sqrt{217} - 1 \)/4) |
| \( \mathcal{O}_{21} \) | 11/2 | \( f \sqrt{g} R^{10} d^2 x \) | (\( \sqrt{241} - 1 \)/4) |
| \( \mathcal{O}_{23} \) | 13/2 | \( f \sqrt{g} R^{11} d^2 x \) | (\( \sqrt{265} - 1 \)/4) |
| \( \mathcal{O}_{25} \) | 15/2 | \( f \sqrt{g} R^{12} d^2 x \) | 4 |
| \( \mathcal{O}_{27} \) | 17/2 | \( f \sqrt{g} R^{13} d^2 x \) | (\( \sqrt{313} - 1 \)/4) |
| \( \mathcal{O}_{29} \) | 19/2 | \( f \sqrt{g} R^{14} d^2 x \) | (\( \sqrt{337} - 1 \)/4) |
| \( \mathcal{O}_{31} \) | \( \vdots \) | \( f \sqrt{g} R^{15} d^2 x \) | \( \vdots \) |

Table 1: Comparison of the scaling dimensions in pure gravity.
Table 2: Comparison of the scaling dimensions in the $k = 3$ case of Kazakov's $k$-series ($p = 2$, $q = 5$). Note that the (2,5) minimal model has two primary fields, namely the identity operator and $\Phi_{1,2}$ which has a negative conformal weight ($h_{1,2} = -\frac{1}{2}$).
| scaling operator | scaling dimension | generally covariant operator | scaling dimension |
|------------------|------------------|------------------------------|------------------|
| \(O_1\)          | 0                | \(\int \sqrt{g} d^2 x\)     | 0                |
| \(O_2\)          | 1/6              | \(\int \sqrt{g}^{-h_{2,2}} \Phi_{2,2} d^2 x\) | 1/6              |
| \(O_4\)          | 1/2              | \(\int \sqrt{g}^{-h_{2,1}} \Phi_{2,1} d^2 x\) | 1                |
| \(O_5\)          | 2/3              | \(\int \sqrt{g} R d^2 x\)    | \((2\sqrt{13} - 1)/6\) |
| \(O_7\)          | 7/6              | \(\int \sqrt{g}^{-h_{2,2}} \Phi_{2,2} R d^2 x\) | \((\sqrt{73} - 1)/6\) |
| \(O_{10}\)       | 3/2              | \(\int \sqrt{g}^{-h_{2,2}} \Phi_{2,2} R^2 d^2 x\) | 3/2              |
| \(O_{11}\)       | 5/3              | \(\int \sqrt{g}^{-h_{2,1}} \Phi_{2,1} R^2 d^2 x\) | 5/3              |
| \(O_{13}\)       | 2                | \(\int \sqrt{g} R^3 d^2 x\)  | \((\sqrt{145} - 1)/6\) |
| \(O_{14}\)       | 13/6             | \(\int \sqrt{g}^{-h_{2,2}} \Phi_{2,2} R^4 d^2 x\) | 13/6             |

Table 3: Comparison of the scaling dimensions in two-dimensional quantum gravity coupled to the critical Ising model \((p = 3, q = 4)\). Note that the \((3, 4)\) minimal model has three primary fields, namely the identity operator, the energy density operator \(\Phi_{2,1} (h_{2,1} = \frac{1}{2})\) and the local spin operator \(\Phi_{2,2} (h_{2,2} = \frac{1}{16})\).
Figure captions

Fig. 1 The diagrams we have to evaluate in order to justify the simplification \( G_{\mu\nu} = 0 \). The dot represents a derivative and the arc connecting two dots implies a contraction.

Fig. 2 The diagrams which appear in calculating the expectation value of each term in (B.1). (a),(b) and (c) correspond to the first, second and third terms respectively. The dot represents a derivative and the arc connecting two dots implies a contraction, as in Figure 1. The cross represents a mass insertion using the \( \frac{\alpha^2}{16} \hat{R} \psi^2 \) term in the action.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9402050v2