Time-Varying Second-Order Sliding Mode Control for Systems Subject to External Disturbance

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ABSTRACT In this article, the design problem of a class of novel second-order sliding mode controller for systems subject to external disturbance is investigated, where the sliding variable is affected by a time-varying regulation. On the one hand, for the unmatched disturbance bounded by positive functions, a second-order sliding mode controller is proposed to prove the finite-time stability of aforementioned system through the backstepping-like method and virtual control strategy. On the other hand, for the non-vanishing bounded disturbance, a non-singular terminal sliding mode surface containing disturbance observations is designed via a finite-time disturbance observation method and the finite-time stability of the system controlled output is realized. The claimed performances of the proposed methods are validated by two simulation examples.

INDEX TERMS Finite-time stability, second-order sliding mode, time-varying systems, external disturbance, virtual control, backstepping-like method.

I. INTRODUCTION

Over the past decades, finite-time control method has received much research attention owing to its advantages in fast transient and high-precision performances [1]–[3]. Many inherently nonlinear systems, which cannot be stabilized by any smooth feedback control, can be stabilized by using finite-time control methods [1]. Roughly speaking, finite-time stability (FTS) characterizes a phenomenon in which system reaches an equilibrium state in finite time. Compared with asymptotically stable systems, FTS systems have better convergence performance around the equilibrium point and better disturbance rejection performance. Lots of interesting results on FTS have been raised from theoretical and practical points of view. [4] proposed Lyapunov theorem on FTS of continuous autonomous systems, which provided a basic tool for analysis of FTS of nonlinear control systems. Moreover, FTS properties have been studied for various systems, such as impulsive systems [5], non-autonomous systems [6], time-varying systems [7], [8]. To avoid confusion, it should be pointed out that the FTS considered in this article is different from another FTS concept adopted in [9], which dealt with the finite-time boundedness.

Sliding mode control has long been recognized as a powerful control method to counteract external disturbance and unmodeled dynamics. This method is based on deliberately introducing sliding motion in the control system. In addition, since the motion along the sliding manifold is proven to be free from matched disturbance, the closed-loop system guarantees strong robustness to significant disturbance and model uncertainty [10]. Due to these advantages and the simplicity of implementation, sliding mode controllers have widely been used in various applications [10]–[12]. Among various sliding mode control, a terminal sliding mode (TSM) controller has developed in recent decades [13], [14]. Compared with sliding mode based on linear hyperplanes, TSM have some superior properties, such as fast, efficient, and finite-time convergence. This controller is particularly useful for high precision control as it speeds up the rate of convergence near an equilibrium point. However, the limitation of sliding variable and the chattering problem restrict the widespread applications of sliding mode control. Therefore, to overcome the above drawbacks, the second-order sliding mode (SOSM) control methodology has been developed in [15], [16]. On one hand, SOSM brings more flexibility on the choice of sliding variable. On the other hand, SOSM provides effective tools for the reduction or even practical elimination of the chattering, without compromising the
benefits of the standard sliding mode [17]. Based on aforementioned two advantages, the SOSM control problems have been widely studied. Through TSM technology and homogeneous system theory, [18] studied the global FTS of the double integrator. [1] showed that globally FTS of uncertain nonlinear systems that are dominated by a lower-triangular system can be achieved by Hölder continuous state feedback. The results of [1] in the second-order case were somewhat similar to the controller design in [19]. By comparison, a new feature of FTS controller in [1] was its capability of achieving globally FTS for a family of uncertain nonlinear systems that go beyond planar systems. For the singularity of TSM controller, [19] presented a global non-singular terminal sliding mode (NTSM) controller for a class of nonlinear dynamical systems with parameter uncertainty and external disturbance. A new NTSM manifold was proposed to overcome the singularity problem. Furthermore, while taking the issue of singularity into account, [20] proposed an improved version of TSM, resulting in a new continuous TSM control for robotic manipulators with global FTS.

The most existing SOSM control methods, such as [19]–[21], obtained their sliding mode dynamics by directly taking two times derivative on the sliding variable, which can only be used to deal with the matched disturbance. However, in practical applications, some useful information may be included in the first derivative of the sliding variable and the direct derivative may cause the terms in the controller much large. Hence, it is meaningful to consider the SOSM controller design subject to unmatched disturbance and there exist some (but very little) results focus on this topic. For instance, [22] derived a novel SOSM control method to handle sliding mode dynamics with unmatched term through a so called backstepping-like method. [23] concerned the synchronization problem for a class of hyperchaotic chaotic systems and proposed a robust control scheme to make most of the synchronization errors of the systems to zero for matched and unmatched uncertainty. [24] introduced a continuous NTSM control method for the finite-time control of system subject to unmatched bounded disturbance via a finite-time disturbance observer. To avoid “explosion of complexity” caused by repeated calculations of virtual controllers in basic backstepping approaches [1], [25] studied a discontinuous finite-time exact tracking control of lower-triangular nonlinear systems with unmatched time-varying disturbance. However, it should be mentioned that the sliding variable involved in above theoretical results is linearly regulated. When the sliding variable in above TSM control method is affected by a time-varying regulation, there is no corresponding research on the design problem of TSM controller. Hence, in this article, we will investigate the time-varying SOSM controller design for systems subject to external disturbance, and strive to achieve the FTS of system under two situations of disturbance: disturbance bounded by positive functions and non-vanishing bounded disturbance.

The rest of the paper is organized as follows. In Section 2, we firstly address some essential stability definitions and lemmas. Main contribution to design the sliding mode controller is given in Section 3. By some examples, we verify the main results in Section 4. Finally, Section 5 collects some concluding remarks.

**Notations:** Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of non-negative numbers, $\mathbb{R}^n$ the n-dimensional real spaces equipped with the Euclidean norm $| \cdot |$, $\mathbb{N}$, $\mathbb{D} \subset \mathbb{R}^n$ the subspaces of $\mathbb{R}^n$, $a \lor b$ and $a \land b$ are the maximum and minimum of $a$ and $b$, respectively. The notation $\mathbb{A}^T$ denotes the transpose of $A$. For integers $j$ and $k$ satisfying $0 \leq j \leq k$, let $\mathbb{N}_{j,k} = \{j, j+1, \ldots, k\}$ be a set of non-negative integers. The symbol $\mathbb{C}^i$ denotes the set of all differentiable functions whose first ith time derivatives are continuous.

**II. PRELIMINARIES**

Consider the nonlinear system given by

$$\begin{align*}
\dot{x}(t) &= f(t, x(t)), \\
x(0) &= x_0,
\end{align*}$$

(1)

where $t \geq 0$, $x \in \mathbb{R}^n$ denotes the state vector, $f(\cdot): \mathbb{R}_+ \times \mathbb{D} \rightarrow \mathbb{R}^n$ is a continuous function with $f(t, 0) = 0$ for all $t \geq 0$. We assume that $f(t, x)$ satisfies suitable conditions so the solution $x(t) = x(t, x_0)$ with initial state $x_0 \in \mathbb{D}$ uniquely exists in forward time for all initial conditions except possibly the origin, see [4], [6].

*Definition 1* [4]: System (1) is said to be FTS, if there exist a function $T: \mathbb{R}^n \mapsto \mathbb{R}_+$ and an open neighborhood $\mathbb{N} \subseteq \mathbb{D}$ such that the following statements hold:

(i) **Finite-time convergence (FTC):** For every $x_0 \in \mathbb{N}/\{0\}$, $x(t) \in \mathbb{N}/\{0\}$ holds for all $t \in [0, T(x_0))$ and $x(t) \rightarrow 0$ as $t \rightarrow T(x_0)^-$. 

(ii) **Lyapunov stability (LS):** For every $\epsilon$ ball $B_\epsilon$ around the origin, there exist a $\delta$ ball $B_\delta$ around the origin such that, for every $x_0 \in B_\delta \setminus \{0\}, x(t) \in B_\epsilon$ for all $t \in [0, T(x_0))$.

Give the following definition

$$\dot{V}(t, x(t)) \triangleq \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x(t)),$$

for a continuously differentiable function $V(t, x): \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$, $V(t, 0) = 0$, and $V(t, \cdot)$ is a positive definite and radially unbounded if there exists a positive definite and radially unbounded continuous function $W(x): \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $V(t, x) \geq W(x)$, for $t \geq 0, x \in \mathbb{R}^n$.

*Lemma 1* [26]: Consider system (1), if there exist a positive definite, continuously differentiable function $V: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$, an integrable function $c(t): \mathbb{R}_+ \mapsto \mathbb{R}_+$, and two real numbers $\alpha \in (0, 1)$, $\sigma > 0$, such that the derivative of $V$ along the solution $x(t) = x(t, x_0)$ of system (1) satisfies

$$\dot{V}(t, x(t)) \leq -c(t)V^\alpha(t, x(t)), \forall t \geq 0,$$

where $c(t)$ satisfies

$$\int_{0}^{+\infty} c(s)ds \triangleq \beta \geq \frac{\sigma^{1-\alpha}}{1-\alpha}.$$
Then system (1) is locally FTS with respect to $x_0$ satisfying $V_0 \leq \sigma$, where $V_0 \triangleq V(0, x_0)$. The settling-time function $T : \mathbb{R}^n \mapsto \mathbb{R}_+$, depending on the initial state $x_0$, is bounded by

$$T(x_0) \leq \inf \{ t > 0 : \int_0^t c(s)ds = \frac{V_{01} - \alpha}{1 - \alpha} \}.$$ 

Moreover, when $\beta = +\infty$ and $V$ is radially unbounded, system (1) is globally FTS.

To simplify the expression, we denote $[x]^a = \text{sign}(x)|x|^a$.

**Lemma 2** [27]: If $p_1 > 0$ and $0 < p_2 < 1$, then for any function $\gamma > 0$, $x, y \in \mathbb{R}$, we have

$$\|x\|^{p_1 p_2} - \|y\|^{p_1 p_2} \leq 2^{1-p_2} \|x\|^{p_1} - \|y\|^{p_1} \|p_2\|.$$

**Lemma 3** [28]: Let $c$ and $d$ be positive constants. Given any function $\gamma > 1$, the following inequality holds for all $x, y \in \mathbb{R}$

$$|x|^c |y|^d \leq \frac{c}{c + d} \gamma |x|^c + \frac{d}{c + d} \gamma^{-\frac{1}{p}} |y|^{c+d}.$$

**Lemma 4** [29]: Let $p$ be a real number with $0 < p < 1$. Then for all $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, we have

$$\left( |x_1|^1 + \cdots + |x_n|^p \right)^p \leq |x_1|^p + \cdots + |x_n|^p.$$

**III. MAIN RESULTS**

Consider the nonlinear system of the following form

$$\begin{align*}
\dot{x}_1(t) &= w_1(t)x_2 + d_1(t, x_1), \\
\dot{x}_2(t) &= u,
\end{align*}$$

(2)

where $x = (x_1, x_2)^T$ and $u$ are the states and control input, respectively. $w_1(t) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is an integrable function, $d_1(t, x_1) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ denotes the disturbance term.

**Assumption 1**: There exists a continuous function $\rho(t, s) : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+$ such that $|d_1(t, s)| \leq \rho(t, s)|s|^{\frac{1}{2}}$, for all $t \geq 0$, $s \in \mathbb{R}$.

**Theorem 1**: Assume that Assumption 1 holds, then the nonlinear system (2) is FTS under the feedback controller

$$u = -\left\{ \frac{1}{4} \beta_0(\|w_1(t)\| + [c_1(\beta_0) + c_2(x_1, \beta_0) + c_3(x_1, \beta_0)]) \right\} \times \text{sign} \left[ x_2 \frac{\pi}{2} + \beta_1 \frac{\pi}{2} (x_1 \frac{\pi}{2n}) \right],$$

(3)

where $w_1(t) \geq 1$ for all $t \geq 0$, $p > a > r_1 = 2r_2$ are positive constants, smooth function

$$\beta_1(s) = \sup_{\tau \geq 0} \frac{\rho(t, s)}{w_1(t)} + \beta_0, \ \beta_0 > 0,$$

and

$$\lambda(s) = \left( \frac{d\beta_1^{\frac{\pi}{2}}(s)}{ds} [s] + \frac{a\beta_1^{\frac{\pi}{2}}(s)}{r_1} \right), \ c_1(\beta_0) = \frac{2^{1-\frac{a}{p}} r_2}{2p}.$$
From lemma 2, we also have
\[ |x_2| = |x_2 - x_2^* + x_2^*| \leq 2^{1 - \frac{2}{p}} |\xi_2|^{\frac{2p}{p-2}} + \beta_1(x_1)|\xi_1|^{\frac{2p}{p-2}}, \]
then we can get that
\[ \frac{d[x_2^*]}{dx_1} \leq \frac{\lambda(x_1)|\xi_1|^{1 - \frac{2}{p}}}{a} \left\{ w_1(t)(2^{1 - \frac{2}{p}} |\xi_2|^{\frac{2p}{p-2}} + \beta_1(x_1)|\xi_1|^{\frac{2p}{p-2}} + \rho(t, x_1)|\xi_1|^{\frac{2p}{p-2}} + \rho(t, x_1)|\xi_1|^{\frac{2p}{p-2}}) \right\} \]
(9)

Combining (8) and (9), we have
\[ A.2 \leq \frac{2^{3 - \frac{2}{p}} p}{a} \lambda(x_1)w_1(t)|\xi_1|^{1 - \frac{2}{p}} |\xi_2|^{\frac{2p}{p-2} + \frac{2}{p}} \]
(10)

Applying lemma 3 to (10), it can be concluded that
\[ B.1 \leq w_1(t) \left\{ \frac{a-r_1}{2p} \gamma |\xi_1|^{\frac{2p}{p-2}} + \frac{2p+r_1}{2p} - \frac{a}{2p} \gamma^{-\frac{a-r_1}{2p-2}} \right\} \]
(11)

where \( \gamma = \frac{\rho_0}{2(\sigma-r_1)} \). Similar to the above simplification process, we can get the estimated regions of B.2 and B.3, that is
\[ B.2 \leq w_1(t) \left\{ \frac{1}{4} \beta_0|\xi_1|^{\frac{2p}{p-2}} + c_2(x_1, \beta_0)|\xi_2|^{\frac{2p}{p-2}} \right\} \]
(12)
\[ B.3 \leq \frac{1}{4} \beta_0|\xi_1|^{\frac{2p}{p-2}} + c_2(t, x_1, \beta_0)|\xi_2|^{\frac{2p}{p-2}} \]
(13)

For the third marked term of (6), we have
\[ A.3 = [\xi_2]^{\frac{2p}{p-2}} u \]
(14)

When the feedback control law \( u \) is designed as (3), we get that
\[ \dot{V}_2(x_1, x_2) \leq -\frac{1}{4} \beta_0(w_1(t) - 1)(|\xi_1|^{\frac{2p}{p-2}} + |\xi_2|^{\frac{2p}{p-2}}) \]
(15)

(16)

Therefore, together with (5), (16), and lemma 4, we have
\[ \dot{V}_2(x_1, x_2) + C(t)V_2^k(x_1, x_2) \leq 0, \]
(17)

where \( C(t) = 2 \frac{\rho_0}{\sigma^{\frac{2p}{p-2}}} \beta_0(w_1(t) - 1) \geq 0, k = \frac{2p}{2p+2} \in (0, 1) \). It follows from lemma 1 that the nonlinear system (2) can be finite-time stabilized by controller (3). This completes the proof.

Remark 1: It should be pointed out that the control strategy in Theorem 1 can be divided into two steps. In the first step, based on the constraints for the unmatched disturbance, a virtual controller is designed to stabilize the first state component of system (2). In the second step, we design the actual controller so that the second component of the system (2) tracks the desired trajectory represented by the virtual controller in a finite time.

Remark 2: In (3), the parameter \( \beta_1(s) \) is obtained by the supremacy of \( \rho(s)(w_1(t)) \) with respect to \( t \). However, such supremacy does not exist in some specific cases. Hence, we can relax the restrictions on \( \beta_1(s) \) and choose an appropriate positive function \( \tilde{\beta}_1(t, s) \) such that \( \tilde{\beta}_1(t, s) - \beta_0 \geq \rho(t, s) \), for all \( t \geq 0, s \in \mathbb{R} \).

Remark 3: In Theorem 1, the unmatched disturbance is bounded by a positive function, which indicates that we cannot find the global Lipschitz constant for the unmatched term. Therefore, the methods proposed in [24], [25] in which the Lipschitz constant is involved cannot be applied to system (2). Although similar disturbance has been considered in [22], the proposed method was not suitable to system (2) since it involves time-varying structure.

The disturbance term in Theorem 1 is bounded by the positive function \( \rho(t, x_1)|x_1|^\frac{2p}{p-2} \), which is related to the system state. Meanwhile, when the system state reaches an equilibrium point, the disturbance term then will disappear. However, when disturbance term always exists, the above result may be invalid. To study the system behaviour under such case, the following discussions are proposed.

To be more general, consider the following nonlinear system subject to matched and unmatched bounded disturbance,
\[ \begin{aligned}
\dot{x}_1(t) &= w_2(t)x_2 + d_1(t), \\
\dot{x}_2(t) &= u + d_2(t), \\
y &= x_1,
\end{aligned} \]
(18)

where \( x = (x_1, x_2)^T, u, \) and \( y \) are the state vector, the control input, and the controlled output, respectively. \( w_2(t) : \mathbb{R}_+ \mapsto \mathbb{R}_+/\{0\} \) is a continuously differentiable function, \( d_1(t), d_2(t) : \mathbb{R}_+ \mapsto \mathbb{R} \) denote the disturbance term.

Assumption 2: The disturbance in system (18) satisfies \( d_i(t) \in \mathbb{C}^{3-i} \). Meanwhile, \( d_i^{(i)}(t) \) are bounded by \( sup_{t \geq 0} |d_i^{(i)}(t)| \leq K_i \), where \( K_i \) is a known positive constant, \( i \in \{0, 1, 2\}, i = 1, 2 \).

Suppose that the disturbance terms \( d_1(t), d_2(t) \) in system (18) satisfy Assumption 2, then a series of disturbance observers ([30]) are utilized to estimate the disturbance and
their derivatives in system (18), that is
\[
\begin{align*}
\dot{x}_1(t) - \hat{x}_1(t) & = \dot{d}_1(t) - d_1(t) - L_0^1 [\hat{x}_1(t) - x_1(t)], \\
\dot{x}_2(t) - \hat{x}_2(t) & = \dot{d}_2(t) - d_2(t) - L_0^2 [\hat{x}_2(t) - x_2(t)], \\
\dot{d}_1(t) - \hat{d}_1(t) & = \dot{d}_1(t) - d_1(t) - L_1^1 [\hat{x}_1(t) - x_1(t)], \\
\dot{d}_2(t) - \hat{d}_2(t) & \in [-K_2, K_2] - L_1^2 [\hat{x}_2(t) - x_2(t)], \\
\dot{\hat{d}}_1(t) - \hat{d}_1(t) & \in [-K_1, K_1] - L_1^1 [\hat{x}_1(t) - x_1(t)],
\end{align*}
\]  
(19)

where \(a_0, a_0^2, a_1, a_2^2, a_2 \) are positive constants and belong to \((0, 1)\); \(L_0^1, L_0^2, L_1^1, L_1^2 \) are observer gains to be assigned. The estimate of state, disturbance, and their derivatives are denoted by \(\hat{x}_i(t), \hat{d}_i(t), \) and \(\dot{\hat{d}}_i(t)\). Besides, the estimation errors are defined as
\[
\begin{align*}
e_0^i(t) & = \hat{x}_i(t) - x_i(t), \quad i = 1, 2, \\
e_1^i(t) & = \hat{d}_i(t) - d_i(t), \quad i = 1, 2, \\
e_2^i(t) & = \dot{\hat{d}}_i(t) - \dot{d}_i(t).
\end{align*}
\]  
(22)

Lemma 5 [30]: If the observer gain \(L_j^i \) is selected such that \(L_j^i > K_i\), then the observer estimation errors (22) are \(FTS\), which means that there exists a time instant \(T_e\) such that \(e_0^i(t) = e_1^i(t) = e_2^i(t) = 0\) for \(t \geq T_e\), where \(j \in \mathbb{N}_{0.3-i}, \quad i = 1, 2\). Moreover, the observer estimation errors (22), disturbance, and their derivatives are globally bounded for all \(t \geq 0\).

Theorem 2: Consider system (18) with Assumption 2. If the nonlinear dynamic sliding mode surface for system (18) is defined by
\[
s = x_1 + (w_2(t)x_2 + \dot{d}_1(t))^{\frac{p}{q}},
\]  
(23)

and the \(NTSM\) control law \(u\) is designed as
\[
u = -\left[|s|^\alpha + \dot{d}_2(t) + \frac{w_2(t)}{w_2(t)} x_2 + \frac{1}{w_2(t)} \dot{d}_1(t)ight] + \frac{q}{p w_2(t)} \left(w_2(t)x_2 + \dot{d}_1(t)\right)^{2 - \frac{q}{p}},
\]  
(24)

where \(\alpha \in (0, 1), \quad p, q\) are positive odd integers, and \(1 < \frac{q}{p} < 2\), then the controlled output \(y = x_1\) will converge to zero in finite time.

Proof: For the above sliding mode surface (23), its derivative along system (18) is
\[
\dot{s} = x_1 + \frac{p}{q} (w_2(t)x_2 + \dot{d}_1(t))^{\frac{p}{q}-1} (w_2(t)x_2 + w_2(t)x_2 + \dot{\hat{d}}_1).
\]  
(25)

Substituting the control law \(u\) into (25) yields
\[
\dot{s} = -e_1^1 - \rho(t, x_2) w_2(t) [s]^{\alpha} + e_1^2, \quad e_1^2 \geq 0 \quad \text{for all} \quad t \geq 0, \quad x_2 \in \mathbb{R}.
\]  
(26)

From lemma 5, the disturbance estimation errors \(e_1^1\) and \(e_1^2\) in (26) will converge to zero in finite time. Therefore, when \(t \geq T_e\), (26) reduces to
\[
\dot{s} = -\rho(t, x_2) w_2(t) [s]^{\alpha}.
\]  
(27)

Next we will show that (27) is \(FTS\). The idea of the proof procedure is inspired by [19]. For the case of \(w_2(t)x_2 + \dot{d}_1(t) \neq 0\), namely, \(\rho(t, x_2) > 0\), it follows from lemma 1 that (27) is \(FTS\). When \(w_2(t)x_2 + \dot{d}_1(t) = 0\), we have
\[
(w_2(t)x_2 + \dot{d}_1(t))' = -w_2(t)s^{\alpha}.
\]  
(28)

For both \(s > 0\) and \(s < 0\), it is obtained that \((w_2(t)x_2 + \dot{d}_1(t))' < 0\) and \((w_2(t)x_2 + \dot{d}_1(t))' > 0\), which shows that the trajectory corresponding to \((w_2(t)x_2 + \dot{d}_1(t)) = 0\) is not an attractor. Therefore, the \(NTSM\) manifold (23) can be reached in finite time.

Once the sliding surface \(s = 0\) is reached, it is derived from the sliding surface (23) and the system dynamics (18) that
\[
s = x_1 + (w_2(t)x_2 + \dot{d}_1(t))^{\frac{p}{q}} = x_1 + (w_2(t)x_2 + \dot{d}_1(t))^{\frac{p}{q}} = 0.
\]  
(29)

Through a transformation for (29), we have
\[
\dot{x}_1 = -x_1^{\frac{q}{p}},
\]  
(30)

which means the controlled output \(y = x_1\) of system (18) is \(FTS\). This completes the proof.

Remark 4: Recently, some significant results on \(TSM\) strategy have been reported in [13], [14], [19], [20]. However, they all considered traditional dynamic sliding mode surface, which lead to that the above mentioned \(TSM\) control method cannot drive the sliding variable reach the sliding mode surface (23) in finite time. In Theorem 2, we take the nonlinear dynamic sliding mode surface that containing time-varying structure into account. It is efficient to deal with the problem of a class of novel second-order sliding mode controller design for systems subject to external disturbance and the \(FTS\) of the controlled output of system (18) is realized.

Remark 5: It should be noted that (24) can effectively avoid the singularity problem since the parameters \(p, q\) are chosen as \(1 < \frac{q}{p} < 2\). Besides, to verifying that \(s = 0\) is the only attractor of (27), we apply a proof method similar to that in [19]. As stated in set theory, there exists a vicinity of \(w_2(t)x_2 + \dot{d}_1(t) = 0\) such that for a small \(\delta > 0\), we have \(|w_2(t)x_2 + \dot{d}_1(t)| < \delta\). Since \((w_2(t)x_2 + \dot{d}_1(t))' < 0\) for \(s > 0\) and \((w_2(t)x_2 + \dot{d}_1(t))' > 0\) for \(s < 0\), the crossing of the trajectory from the boundary of the vicinity \(w_2(t)x_2 + \dot{d}_1(t) = \delta\) to \(w_2(t)x_2 + \dot{d}_1(t) = -\delta\) for \(s > 0\) and from \(w_2(t)x_2 + \dot{d}_1(t) = -\delta\) to \(w_2(t)x_2 + \dot{d}_1(t) = \delta\) for \(s < 0\) occurs in finite time. That is to say, when any trajectory of (27) approaches the trajectory corresponding to \(w_2(t)x_2 + \dot{d}_1(t) = 0\), it will run away and eventually approach to \(s = 0\).

IV. EXAMPLES

In this section, we will demonstrate the effectiveness of the proposed results through the following simulation examples.

Example 1: Consider system (2) with \(d_1(t) = \sin(tx_1^2)x_1^2\), \(w_1(t) = 1 + t\), \(\rho(t, x_1) = tx_1^2\), and \(\beta_0 = 1\),
It is obvious that \(|d_1(t, x_1)| \leq \rho(t, x_1)|x_1|^2\), for all \(t \geq 0, x_1 \in \mathbb{R}\). Hence, we can calculate that \(\beta_2(x_1) = \sup_{t \geq 0} \rho(t, x_1) + \beta_0 = x_1^2 + 1\).

It follows from Theorem 1 that system (2) is FTS under the control input

\[
u = -\left\{ \frac{t}{4} + (t + 1)\left[ \frac{49}{16} + 7 \cdot 3^{-\frac{8}{7}} \cdot 2^{\frac{38}{21}} \cdot (6x_1^2 + 1)^2 x_1 \right.ight.
\]
\[
+ \frac{3}{2} x_1^3 + 2^{\frac{26}{21}} \cdot 3^{-\frac{1}{7}} \left[ \left( (6x_1^2 + 1)^2 x_1^2 + \frac{3}{2} (x_1^2 + 1)^3 x_1^2 + 1 \right) \frac{8}{7} \right]
\]
\[
\left. + 2^{\frac{38}{21}} \cdot 3^{-\frac{4}{7}} \left[ (6x_1^2 + 1)^2 x_1^2 + \frac{3}{2} (x_1^2 + 1)^3 x_1^2 \right] \right\} \times \text{sign} \left\{ \lfloor x_2 \rfloor^3 + (x_1^2 + 1)^3 |x_1|^3 \right\}. \tag{31}
\]

Then the initial state is given by \(x_0 = (2.5, -2)\) and the numerical simulations for system (2) are as follows.

**FIGURE 1.** Simulation results of states for system (2) under the control input (31).

In Figure 1, it can be clearly seen that both state components of system (2) converge to zero in finite time. Meanwhile, the control input (31) changes over time and turns into 0 when \(x(t)\) converges to zero. Thus, all of these simulation results illustrate the validity of Theorem 1.

**Example 2:** Consider system (18) with \(d_1(t) = \sin(t), d_2(t) = \cos(t), w_2(t) = \sin(t) + 2, k_1 = k_2 = 1.1, \alpha = \frac{1}{2}\), and \(\frac{q}{\rho} = \frac{3}{7}\). For convenience, when designing the NTSM control method \(u\), we directly apply the given disturbance information to replace the disturbance observations. It is easy to check that the disturbance terms \(d_1(t), d_2(t)\) are bounded, and \(w_2(t) > 0\) for all \(t \geq 0\). According to Theorem 2, the nonlinear dynamic sliding mode surface for system (18) can be defined by

\[
s = x_1 + \left( (\sin(t) + 2)x_2 + \sin(t) \right) \frac{5}{7},
\]

and the NTSM control law \(u\) can be designed as

\[
u = -\left\{ x_1^\alpha + \cos(t) + \frac{\cos(t)x_2}{\sin(t) + 2} + \frac{\cos(t)}{\sin(t) + 2}
\right.
\]
\[
+ \frac{3}{5(\sin(t) + 2)} \left( (\sin(t) + 2)x_2 + \sin(t) \right) \frac{5}{7}. \tag{32}
\]

It then follows from Theorem 2 that the controlled output of system (18) is FTS under the control input (32).

For simulations, choose initial state \(x_0 = (-2, 2)\), then the state trajectory of system (18) is shown in Figure 2. One can observe from Figure 2 that the controlled output of system (18) is FTS under a bounded control input (32), which illustrates the validity of Theorem 2.

**V. CONCLUSION**

The problem of time-varying second-order sliding mode controller design for systems subject to external disturbance is investigated in this article, where the sliding variable is affected by a time-varying regulation. For the external disturbance bounded by positive functions and conventional constant upper bounds, we propose novel SOSM control methods, respectively. Further research topics would be considered to extend the main results of this article to other problems in the context of practical applications, such as the hybrid stabilization of a wheeled mobile robot, the FTS theorem for time-varying nonlinear systems in the sense of Lyapunov.

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