LARGE SUBSETS OF DISCRETE HYPERSURFACES IN $\mathbb{Z}^d$
CONTAIN ARBITRARILY MANY COLLINAR POINTS

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Abstract. In 1977 L.T. Ramsey showed that any sequence in $\mathbb{Z}^2$ with bounded gaps contains arbitrarily many collinear points. Thereafter, in 1980, C. Pomerance provided a density version of this result, relaxing the condition on the sequence from having bounded gaps to having gaps bounded on average.

We give a higher dimensional generalization of these results. Our main theorem is the following.

Theorem. Let $d \in \mathbb{N}$, let $f : \mathbb{Z}^d \to \mathbb{Z}^{d+1}$ be a Lipschitz map and let $A \subset \mathbb{Z}^d$ have positive upper Banach density. Then $f(A)$ contains arbitrarily many collinear points.

Note that Pomerance’s theorem corresponds to the special case $d = 1$. In our proof, we transfer the problem from a discrete to a continuous setting, allowing us to take advantage of analytic and measure theoretic tools such as Rademacher’s theorem.

1. Introduction

Ramsey theory deals with the problem of finding structured configurations in suitably large but possibly disordered sets. The nature of the desired configurations can range from complete subgraphs of a graph to arithmetic progressions in $\mathbb{Z}$ to solutions of equations, such as $x + y = z$, in a countable commutative semigroup. In this paper we deal with configurations consisting of finitely many collinear points in $\mathbb{Z}^d$. The following theorem, which deals with this type of configurations, was obtained by L. T. Ramsey in 1977:

Theorem 1.1 ([6, Lemma 1]). Let $M \in \mathbb{N}$ and suppose $\vec{u}_1, \vec{u}_2, \ldots \in \mathbb{Z}^2$ satisfies

$$\|\vec{u}_{i+1} - \vec{u}_i\|_2 \leq M \quad \forall i \in \mathbb{N}. \quad (1)$$

Then the sequence $\vec{u}_1, \vec{u}_2, \ldots$ contains arbitrarily many collinear points. More precisely, for each $k \in \mathbb{N}$ there exists a set $X \subset \mathbb{N}$ with cardinality $|X| = k$ such that the set $\{\vec{u}_i : i \in X\}$ is contained in a single line.

A sequence that satisfies (1) is said to have bounded gaps. The above theorem can be interpreted as an analogue of van der Waerden’s theorem on arithmetic progressions [8], which, in one of its many forms, states that any sequence $u_1, u_2, \ldots \in \mathbb{Z}$ with bounded gaps contains arbitrarily long arithmetic progressions. The fact that a sequence in $\mathbb{Z}^2$ with bounded gaps may not contain arbitrarily long arithmetic progressions is a non-trivial result first obtained by J. Justin [4]. When properly interpreted, Justin’s construction gives a sequence with bounded gaps in $\mathbb{Z}^2$ without a five term arithmetic progression. This construction was later improved by F.
M. Dekking, who built a sequence \( \vec{u}_1, \vec{u}_2, \ldots \in \mathbb{Z}^2 \) with \( \|\vec{u}_{i+1} - \vec{u}_i\|_2 \leq 1 \) that does not contain a four term arithmetic progression [1].

It is natural to ask whether a result similar to Theorem 1.1 holds in higher dimensions. It follows, as an easy corollary, that any sequence in \( \mathbb{Z}^d \) with bounded gaps will contain arbitrarily many points in the same \((d - 1)\)-dimensional hyperplane. To see this, simply project any given sequence in \( \mathbb{Z}^d \) onto \( \mathbb{Z}^2 \) and take the preimage under this projection of the set of collinear points guaranteed by Theorem 1.1.

One could naively attempt to extend Theorem 1.1 by asking whether a sequence with bounded gaps in higher dimensional lattices contains arbitrarily many collinear points. However, J. L. Gerver and L. T. Ramsey constructed a sequence in \( \mathbb{Z}^3 \) with bounded gaps (actually with gaps bounded by 1) with no more than 511 points contained in a single line [2]. This example shows that one needs to change the framework to obtain non-trivial generalizations of Theorem 1.1 to higher dimensions.

A sequence in \( \mathbb{Z}^2 \) with bounded gaps can be viewed as a Lipschitz function \( f: \mathbb{Z} \to \mathbb{Z}^2 \). Using this language, Theorem 1.1 asserts that the image of any such Lipschitz function contains arbitrarily many collinear points. In order to increase the dimension of the range from \( \mathbb{Z}^2 \) to a higher dimensional space \( \mathbb{Z}^{d+1} \) one must also increase the dimension of the domain from \( \mathbb{Z} \) to \( \mathbb{Z}^d \) in order to get similar qualitative results. We will prove the following:

**Theorem A.** Let \( d \in \mathbb{N} \) and let \( f: \mathbb{Z}^d \to \mathbb{Z}^{d+1} \) be a Lipschitz map. Then there are arbitrarily many collinear points in the image of \( f \). More precisely for any \( k \in \mathbb{N} \) there exists a set \( X \subset \mathbb{Z}^d \) with \( |X| = k \) such that \( f(X) \) is contained in a single line.

Observe that Theorem 1.1 can be derived from Theorem A by setting \( d = 1 \). One can intuitively interpret Theorem A as stating that any discrete hypersurface in \( \mathbb{Z}^{d+1} \) contains arbitrarily many collinear points; this interpretation becomes rigorous if one defines a discrete hypersurface as a set quasi-isometric\(^1\) to \( \mathbb{Z}^d \).

A density version of van der Waerden’s theorem, known as Szemerédi’s theorem, was obtained in [7].

**Szemerédi’s theorem.** Let \( A \subset \mathbb{Z} \) have positive upper Banach density, i.e.,

\[
d^*(A) := \limsup_{L \to \infty} \sup \left\{ \frac{|A \cap [N, N + L]|}{L} \mid N \in \mathbb{Z} \right\} > 0.
\]

Then \( A \) contains arbitrarily long arithmetic progressions.

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\(^1\)A map \( f: X \to Y \) between metric spaces is a quasi-isometry if there exist \( C, M \geq 1 \) such that \( 1/Md(x, y) - C < d(f(x), f(y)) < Md(x, y) + C \) and for every \( y \in Y \) there exists \( x \in X \) such that \( d(f(x), y) < C \).
for infinitely many $m \in \mathbb{N}$, contains arbitrarily long arithmetic progressions. In 1978, in analogy with Szemerédi’s theorem, C. Pomerance presented a proof of the following density version of Theorem 1.1:

**Theorem 1.2 ([5]).** Let $M \in \mathbb{N}$ and suppose the sequence $\vec{u}_1, \vec{u}_2, \ldots \in \mathbb{Z}^2$ satisfies

$$\frac{1}{m} \sum_{i=1}^{m} \|\vec{u}_{i+1} - \vec{u}_i\|_2 \leq M$$

for infinitely many $m \in \mathbb{N}$. Then the sequence $\vec{u}_1, \vec{u}_2, \ldots$ contains arbitrarily many collinear points.

It is a corollary of Pomerance’s theorem that if a sequence $u_1, u_2, \ldots \in \mathbb{Z}$ has gaps bounded on average, then the sequence defined by $\vec{u}_i = (i, u_i) \in \mathbb{Z}^2, i \in \mathbb{N}$, contains arbitrarily many collinear points.

It turns out that an extension of Pomerance’s theorem to higher dimensions along the lines of Theorem A holds as well. Our main theorem is then the common generalization of Theorems A and 1.2.

**Theorem B.** Suppose $f : \mathbb{Z}^d \rightarrow \mathbb{Z}^{d+1}$ is a Lipschitz map and $A \subset \mathbb{Z}^d$ has positive upper Banach density (defined in (3)). Then given any positive integer $k$, there exists $X \subset A$ with $|X| = k$ such that $f(X)$ is contained in a line.

Although not apparent at first, Theorem 1.2 and the special case $d = 1$ of Theorem B are equivalent. We give a proof of this fact in Section 2. An intuitive interpretation of Theorem B is that large subsets of discrete hypersurfaces contain arbitrarily many collinear points.

The paper is organized as follows: In Section 2 we explore some equivalent and related statements to our main theorem. In Section 3 we outline the proof and state our main technical result, which is Lemma 3.4. In Section 4 we prove Lemma 3.4 by reducing it to a statement about Lipschitz functions on $\mathbb{R}^n$. Finally, Section 5 finishes the proof of Theorem B.

**Acknowledgements.** The authors wish to thank Vitaly Bergelson for helpful comments and remarks, as well as the anonymous referees for their many pertinent suggestions.

2. Equivalent Formulations and Corollaries of the Main Theorems

For the remainder of this paper we fix a dimension $d \in \mathbb{N}$. For $p \in \{1, 2\}$ we define

$$\|\vec{x} - \vec{y}\|_p = \sum_{i=1}^{d} |x_i - y_i|^p.$$  

The upper Banach density $d^*$ of a set $A \subset \mathbb{Z}^d$ is defined as:

$$d^*(A) := \lim_{L \to \infty} \sup L \sup \left\{ \frac{|A \cap \prod [N_i, N_i + L]|}{L^d} \middle| (N_1, \ldots, N_d) \in \mathbb{Z}^d \right\}. \quad (3)$$

Whenever $f$ is a function and $X$ is a subset of its domain, we denote by $f(X)$ the set $\{f(x) : x \in X\}$. For $a \in \mathbb{N}$ we denote by $[a]$ the set $\{1, \ldots, a\}$. For a finite set $X$ we let $|X|$ be its cardinality. For $x \in \mathbb{R}$ let $|x|$ be defined as the largest integer no bigger than $x$ and for $\vec{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let $|\vec{x}|$ be defined
as the vector \((|x_1|, \ldots, |x_d|)\) \in \mathbb{Z}^d. We denote by \(S^d \subset \mathbb{R}^{d+1}\) the unit sphere, \(S^d = \{\bar{x} \in \mathbb{R}^{d+1}: ||x||_2 = 1\}\).

Given \(d, h \in \mathbb{N}\), \(M > 0\) and a set \(Z \subset \mathbb{R}^d\), a function \(f : Z \to \mathbb{R}^h\) is Lipschitz with Lipschitz constant \(M\) if \(\|f(\bar{x}) - f(\bar{y})\|_2 \leq M||\bar{x} - \bar{y}||_2\) for all \(\bar{x}, \bar{y} \in Z\).

First let us assume a seemingly more general, but in fact, equivalent version of Theorem B.

**Theorem 2.1.** Let \(d, h \in \mathbb{N}\). Suppose \(f : \mathbb{Z}^d \to \mathbb{Z}^{d+h}\) is a Lipschitz map and \(A \subset \mathbb{Z}^d\) has positive upper Banach density. Then given any positive integer \(k\), there exists \(X \subset A\) with cardinality \(|X| = k\) such that \(f(X)\) is contained in a \(h\)-dimensional hyperplane of \(\mathbb{Z}^{d+h}\).

When \(h = 1\), Theorem 2.1 reduces to Theorem B. To deduce Theorem 2.1 from Theorem B, compose \(f\) with the projection \(\pi : \mathbb{Z}^{d+h} \to \mathbb{Z}^{d+1}\), find a line in \(\mathbb{Z}^{d+1}\) which contains \(\pi(f(X))\) and notice that the pre-image of a line under \(\pi\) is an \(h\)-dimensional affine subspace in \(\mathbb{Z}^{d+h}\).

As is usual with Ramsey theory results, there is an equivalent formulation of Theorem B in finitistic terms:

**Theorem 2.2.** Let \(d, k \in \mathbb{N}\) and let \(\delta, M > 0\). There exists \(L = L(d, k, \delta, M) \in \mathbb{N}\) such that for any Lipschitz function \(f : \mathbb{Z}^d \to \mathbb{Z}^{d+1}\) with Lipschitz constant \(M\) and any \(A \subset \mathbb{Z}^d\) with cardinality \(|A| > \delta L^d\) one can find a subset \(X \subset A\) with \(|X| = k\) such that \(f(X)\) is contained in a line.

**Proposition 2.3.** Theorem 2.2 and Theorem B are equivalent.

Proof. It is trivial to see that Theorem 2.2 implies Theorem B. To prove the converse suppose, for the sake of a contradiction, that Theorem 2.2 is false. Thus there are \(d, k, \delta, M\) such that for every \(L \in \mathbb{N}\) one can find a set \(A_L \subset [L]^d\) with \(|A_L| > \delta L^d\) and a Lipschitz function \(f_L : [L]^d \to \mathbb{Z}^{d+1}\) with Lipschitz constant \(M\) such that for any \(X \subset A_L\) with \(|X| = k\), the image \(f_L(X)\) is not contained in a single line.

Let the sequence \((\tilde{N}_L)_{L=1}^\infty\) in \(\mathbb{Z}^{d+1}\) be defined recursively by letting \(\tilde{N}_1 = \vec{0}\) and, for each \(L > 1\), by choosing \(\tilde{N}_L \in \mathbb{Z}^{d+1}\) such that \(\tilde{N}_L + f_L([L]^d)\) is disjoint from all the lines which contain at least two points of the union \(\bigcup_{j=1}^{L-1} \tilde{N}_j + f_L([j]^d)\).

Next let \((M_L)_{L=1}^\infty\) be a sequence in \(\mathbb{Z}^d\) such that the \(1\)-distance between \(M_L + [L]^d\) and the union \(\bigcup_{j=1}^{L-1} M_j + [j]^d\) is at least \(||\tilde{N}_L||_1\). Let \(A\) be the union of \(M_L + A_L\) over all \(L \in \mathbb{N}\); it is clear that \(d^*(A) \geq \delta\).

Finally, define \(g : \mathbb{Z}^d \to \mathbb{Z}^{d+1}\) to be a Lipschitz function with Lipschitz constant \(M\) such that for every \(L \in \mathbb{N}\) and \(\vec{x} \in \tilde{M}_L + [L]^d\) we have \(g(\vec{x}) = \tilde{N}_L + f_L(\vec{x} - \tilde{M}_L)\).

According to Theorem B one can find \(X \subset A\) with \(|X| = k + 1\) such that \(g(X)\) is contained in a line. Find the maximal \(L \in \mathbb{N}\) for which there exists some \(\vec{x} \in X\) with \(\vec{x} \in \tilde{M}_L + [L]^d\). Observe that \(g(\vec{x}) \in \tilde{N}_L + f_L([L]^d)\), so the line which contains \(g(X)\) cannot contain more than one point from \(\bigcup_{j=1}^{L-1} \tilde{N}_j + f_L([j]^d)\). Therefore, there is a subset \(Y \subset X\) with \(|Y| = k\) such that \(Y \subset \tilde{M}_L + [L]^d\). Since \(g(Y)\) is still contained in a single line and \(g(Y) = \tilde{N}_L + f_L(Y - \tilde{M}_L)\), the set \(Y = Y - \tilde{M}_L\) is a subset of \(A_L\), with cardinality \(k\) such that \(f(Y)\) is contained in a line, thus contradicting the construction. This contradiction finishes the proof. \(\square\)
Pomerance’s original formulation of Theorem 1.2 in [5] was in finitistic terms. More precisely, he showed that for every \( k, M \in \mathbb{N} \) there exists \( n = n(k, M) \in \mathbb{N} \) such that whenever \( \vec{u}_0, \ldots, \vec{u}_n \in \mathbb{Z}^2 \) satisfy \( \sum_{i=1}^{n} \|\vec{u}_i - \vec{u}_{i-1}\|_2 \leq nM \), there are \( k \) collinear points among \( \vec{u}_1, \ldots, \vec{u}_n \). This statement clearly implies Theorem 1.2; the reverse implication can be deduced similarly to the proof of Proposition 2.3.

As mentioned in the Introduction, the case \( d = 1 \) of Theorem B is equivalent to Theorem 1.2. To see how Pomerance’s theorem implies the case \( d = 1 \) of Theorem B, we will use the finitistic versions of both theorems. Let \( k, \delta, M \) be as in Theorem 2.2 and let \( f : \mathbb{Z} \to \mathbb{Z}^2 \) be a Lipschitz function with constant \( M \). Let \( L = M/\delta \) and let \( n = n(k, L) \) be given by Pomerance’s theorem. Finally let \( N \geq n/\delta \).

Take any \( A \subset [N] \) with \( |A| > \delta N \geq n \) and order it, \( A = \{a_1 < \cdots < a_n\} \). Let \( \vec{u}_i = f(a_i) \). It now suffices to show that the average gap of the sequence \( \vec{u}_1, \ldots, \vec{u}_n \) is at most \( L \) and the result will follow by Pomerance’s theorem. Indeed we have

\[
\sum_{i=1}^{n-1} \|\vec{u}_{i+1} - \vec{u}_i\|_2 = \sum_{i=1}^{n-1} \|f(a_{i+1}) - f(a_i)\|_2 \leq M \sum_{i=1}^{n-1} a_{i+1} - a_i = M (a_n - a_1) \leq nL.
\]

To prove the converse direction (i.e., that Theorem B with \( d = 1 \) implies Theorem 1.2), let \( \vec{u}_1, \vec{u}_2, \ldots \) be a sequence in \( \mathbb{Z}^2 \) with gaps bounded on average by \( M \). For each consecutive pair \( \vec{u}_i, \vec{u}_{i+1} \), consider a path of minimal \( \| \cdot \|_2 \) length connecting \( \vec{u}_i \) with \( \vec{u}_{i+1} \). Each such path will have length \( \|\vec{u}_{i+1} - \vec{u}_i\|_2 \) and stringing them together defines a Lipschitz function \( f : \mathbb{Z} \to \mathbb{Z}^2 \). Next construct the set \( A = \{a_i\}_{i \in \mathbb{N}} \) recursively by setting \( a_1 = 1 \) and \( a_{i+1} = a_i + \|\vec{u}_{i+1} - \vec{u}_i\|_2 \). It is then easy to check that \( A \) has density bounded from below by \( 1/M \) and that \( f(a_i) = \vec{u}_i \). Thus by applying Theorem B we can find \( X \subset A \) with \( |X| = k \) such that \( f(X) \subset \{\vec{u}_1, \vec{u}_2, \ldots\} \) is collinear.

As a Corollary of Theorem 2.1 we immediately obtain the following “coloring” version of our main theorem:

**Corollary 2.4.** Let \( n, h, M \in \mathbb{N} \), let \( f : \mathbb{Z}^n \to \mathbb{Z}^{n+h} \) be a Lipschitz map and suppose \( \mathbb{Z}^{n+h} \) has been colored with finitely many colors. Then given any positive integer \( k \), there exists a subset \( X \subset \mathbb{Z}^n \) of size \( k \) with \( f(X) \) monochromatic and contained in a \( n \)-dimensional subspace of \( \mathbb{Z}^{n+h} \).

Similarly to Proposition 2.3, the case \( h = 1 \) of this corollary is equivalent to Theorem A.

### 3. Outlining the proof of Theorem B

Throughout the rest of this paper, let \( d, k \in \mathbb{N}, \ M \in \mathbb{R}^+ \) and \( A \subset \mathbb{Z}^d \) with \( d(A) > 0 \) be arbitrary but fixed. In the following, these four parameters will be invisible in the notation to reduce the amount of subscripts.

Let \( L_\mathbb{Z} \) denote the set of all Lipschitz functions \( f : \mathbb{Z}^d \to \mathbb{Z}^{d+1} \) with Lipschitz constant \( M \), and with the property that there exists no set \( X \subset A \) with \( |X| \geq k \) such that \( f(X) \) is collinear. Thus Theorem B is proven if we can show that \( L_\mathbb{Z} \) is in fact the empty set for all \( d, k, M, A \).

**Definition 3.1.** A generalized line segment is a function \( \ell : [0, 1] \to \mathbb{Z}^d \) of the form

\[
\ell(t) = [(1-t)\vec{x} + t\vec{y}]
\]
for some $\vec{x}, \vec{y} \in \mathbb{R}^d$. Given a generalized line segment $\ell$ we denote by $m_\ell$ the distance $m_\ell = \|\ell(1) - \ell(0)\|_2$.

The underlying argument of the proof goes back to Ramsey’s paper [6], and was adapted by Pomerance in [5]. The basic idea is to find a long, narrow cylinder in $\mathbb{Z}^{d+1}$ which contains “many” points from $f(A)$. We can then cover this cylinder with not too many lines that are almost parallel to the axis of the cylinder, which allows us to find some line containing at least $k$ points. However, our methods to find such a cylinder differ significantly from both Ramsey and Pomerance, mainly due to our appeal to the classical Rademacher’s theorem:

**Theorem 3.2** (Rademacher’s Theorem, cf. [3, Theorem 3.1]). Let $d, h \in \mathbb{N}$ be arbitrary dimensions, let $U \subset \mathbb{R}^d$ and let $f : U \to \mathbb{R}^h$ be Lipschitz. Then $f$ is differentiable at (Lebesgue) almost every point $\vec{x} \in U$.

Rademacher’s theorem tells us that a Lipschitz function is almost everywhere locally ‘flat’ in a certain sense. We will use this property to find the cylinder with the desired properties.

**Definition 3.3.** Assume $L_Z$ is non-empty. Given $f \in L_Z$, $\varepsilon, \delta > 0$ and $\vec{w} = (w_1, \ldots, w_{d+1}) \in S^d \subset \mathbb{R}^{d+1}$ we define $X_Z(f, \varepsilon, \delta, \vec{w})$ to be the collection of all generalized line segments $\ell : [0, 1] \to \mathbb{Z}^d$ with $\varepsilon m_\ell > 14\sqrt{d}$ and satisfying the following properties:

(z-i) $\|\vec{v}_\ell - \vec{w}\|_2 < \varepsilon$, where $\vec{v}_\ell$ denotes the ‘mean slope’ of $(f \circ \ell)$,

$$\vec{v}_\ell = \frac{(f \circ \ell)(1) - (f \circ \ell)(0)}{\| (f \circ \ell)(1) - (f \circ \ell)(0) \|_2}.$$  

(z-ii) For every $t \in [0, 1]$,

$$\| (f \circ \ell)(t) - [(1 - t)(f \circ \ell)(0) + t(f \circ \ell)(1)] \|_2 < \varepsilon M m_\ell.$$  

Roughly speaking, this condition states that the image of the generalized line segment $\ell$ under $f$ remains relatively close to a line.

(z-iii) If we let $K_Z = K_Z(\varepsilon, \ell)$ be the cylinder defined by

$$K_Z = \{ \vec{z} \in \mathbb{Z}^d : \min_{t \in [0, 1]} \| \vec{z} - \ell(t) \|_2 \leq \varepsilon m_\ell \}$$

then $|A \cap K_Z| > \delta |K_Z|$.

**Lemma 3.4.** Suppose $L_Z$ is non-empty. Then for every $f \in L_Z$ there exists $\delta > 0$ and $\vec{w} \in S^d$ such that the set $X_Z(f, \varepsilon, \delta, \vec{w})$ is non-empty for all sufficiently small $\varepsilon > 0$.

It is our goal to use Rademacher’s theorem to deduce Lemma 3.4. In order to do this, we need first to convert Lemma 3.4 into a continuous version; this is done by Theorem 4.2 in Section 4. In Section 5 we use Lemma 3.4 together with the methods developed by Ramsey and Pomerance to finish the proof.

4. **Deducing Lemma 3.4 from a Continuous Version**

We use $\lambda$ to represent the Lebesgue measure on $\mathbb{R}^d$ and define the ball $B_{\mathbb{R}}(\vec{x}, r) = \{ \vec{y} \in \mathbb{R}^d : \| \vec{x} - \vec{y} \|_2 \leq r \}$ for any $\vec{x} \in \mathbb{R}^d$ and $r > 0$. 
**Definition 4.1.** Let $T : [-1,1]^d \to [-1,1]^{d+1}$ be a Lipschitz function with Lipschitz constant $1$, let $\phi : [-1,1]^d \to [0,1]$ be Lebesgue measurable, let $\bar{x} \in [-1,1]^d$ and let $\bar{w} \in S^d \subset \mathbb{R}^{d+1}$. For each $\varepsilon, \delta > 0$ we define the set $X_R(T, \varepsilon, \delta, \bar{w}, \bar{x})$ as the set of all $\bar{y} \in [-1,1]^d$ with the following properties:

(r-i) \[
\left\| \frac{T(\bar{y}) - T(\bar{x})}{\left\| T(\bar{y}) - T(\bar{x}) \right\|_2} - \bar{w} \right\|_2 < \varepsilon.
\]

This asserts that the direction of the line segment connecting $T(\bar{x})$ and $T(\bar{y})$ is approximately equal to $\bar{w}$.

(r-ii) For every $t \in [0,1]$, \[
\left\| T((1-t)\bar{x} + t\bar{y}) - [(1-t)T(\bar{x}) + tT(\bar{y})] \right\|_2 < \varepsilon \|y - \bar{x}\|_2.
\]

Similar to condition (z-ii), this condition states that the image under $T$ of the line segment connecting $\bar{x}$ and $\bar{y}$ remains relatively close to a line.

(r-iii) If we let \[
K_R = K_R(\varepsilon, \bar{x}, \bar{y}) = \{(1-t)\bar{x} + t\bar{y} : t \in [0,1]\} + B_R(\bar{0}, \varepsilon \|\bar{y} - \bar{x}\|_2)
\]
then \[
\frac{1}{\lambda(K_R)} \int_{K_R} \phi \ d\lambda > \delta.
\]

In other words, $\phi$ gives enough mass to a thin cylinder around the segment connecting $\bar{x}$ and $\bar{y}$.

We denote by $L_R$ the set of Lipschitz functions $T : [-1,1]^d \to [-1,1]^{d+1}$ with Lipschitz constant $1$ and with the property that for any point $\bar{x} \in (-1,1)^d$ where $T$ is differentiable, the derivative is nonzero (i.e., some partial derivative is nonzero).

**Theorem 4.2.** Let $T \in L_R$ and let $\phi \in L^\infty([-1,1]^d)$ be non-negative with $\int \phi d\lambda > 0$. Then there exist $\delta > 0$, $\bar{x} \in [-1,1]^d$ and $\bar{w} \in S^d$ such that for every sufficiently small $\varepsilon > 0$ the set $X_R(T, \varepsilon, \delta, \bar{w}, \bar{x})$ is non-empty.

In order to prove Theorem 4.2 we will need the following Lemma.

**Lemma 4.3.** Let $R = \{r_n : n \in \mathbb{N}\}$ be an infinite subset of $\mathbb{R}^+$, let $\phi \in L^\infty([-1,1]^d)$ be a non-negative function and let $\eta > 0$. Assume that

(4) \[
\frac{1}{\lambda(B_R(\bar{0}, r))} \int_{B_R(\bar{0}, r)} \phi \ d\lambda \geq \eta \quad \forall r \in R.
\]

Let $\mu$ denote the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. Then there exists some constant $c_1 > 0$ that only depends on the dimension $d$ and a set $P \subset S^{d-1}$ with $\mu(P) > 0$ and such that for any $\bar{\varepsilon} \in P$ there exists an infinite subset $R'(\bar{\varepsilon}) \subset R$ such that

\[
\liminf_{\varepsilon \to 0} \frac{1}{\lambda(r K_R)} \int_{r K_R} \phi \ d\lambda \geq c_1 \eta, \quad \forall r \in R'(\bar{\varepsilon})
\]
where $K_R = K_R(\varepsilon, 0, \bar{\varepsilon})$ is as in Definition 4.1.
Proof. Without loss of generality we assume that \(|\|\phi\|_\infty \leq 1\). For each \(r > 0\), the measure space \((B_\mathbb{R}(\vec{0}, r), \lambda)\) can be decomposed as the product of the measure spaces \((S^{d-1}, \mu)\) and \(([0, r], t^{d-1}dt)\). Letting

\[
\psi_r(\vec{z}) = \int_{[0, r]} \phi(t \vec{z}) t^{d-1} dt,
\]

we deduce from (4) that for every \(r \in \mathbb{R}\) we have

\[
\eta \lambda(B_\mathbb{R}(\vec{0}, r)) \leq \int_{S^{d-1}} \psi_r(\vec{z}) d\mu(\vec{z}).
\]

Let \(A_r\) be the set of those \(\vec{z} \in S^{d-1}\) for which

\[
\psi_r(\vec{z}) > \frac{\eta \lambda(B_\mathbb{R}(\vec{0}, r))}{2\mu(S^{d-1})}.
\]

Observe that \(|\psi_r(\vec{z})| \leq r^d/d\), and hence

\[
\eta \lambda(B_\mathbb{R}(\vec{0}, r)) \leq \frac{r^d}{d} \mu(A_r) + \frac{\eta \lambda(B_\mathbb{R}(\vec{0}, r))}{2\mu(S^{d-1})} \mu(S^{d-1}).
\]

It follows that \(\mu(A_r) \geq c_0 \eta/2\) for some constant \(c_0\) which only depends on the dimension \(d\).

Next we apply Lebesgue’s differentiation theorem to find a set \(B_r \subset A_r\) with \(\mu(A_r) = \mu(B_r)\) and such that for every \(\vec{z} \in B_r\)

\[
\psi_r(\vec{z}) = \lim_{\varepsilon \to 0} \frac{1}{\mu(D_\varepsilon(\vec{z}))} \int_{D_\varepsilon(\vec{z})} \psi_r d\mu
\]

where \(D_\varepsilon(\vec{z}) = B_\mathbb{R}(\vec{z}, \varepsilon) \cap S^{d-1}\). It follows from (reverse) Fatou’s lemma that the set

\[
P = \limsup_{r \to \infty} B_r = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_{r_m}
\]

has measure \(\mu(P) \geq c_0 \eta/2 > 0\). By construction, for every \(\vec{z} \in P\) there exists an infinite subset \(R' \subset R\) such that

\[
\vec{z} \in \bigcap_{r \in R'} B_r.
\]

This implies that for every \(r \in R'\) and sufficiently small \(\varepsilon\) we can assume that

\[
\int_{D_\varepsilon(\vec{z})} \psi_r d\mu \geq \frac{\eta \lambda(B_\mathbb{R}(\vec{0}, r))}{3\mu(S^{d-1})} \mu(D_\varepsilon(\vec{z})).
\]

Let \(C_r = \{t\vec{u} : t \in [0, r], \ \vec{u} \in D_\varepsilon(\vec{z})\}\). Then for \(r \in R'\)

\[
\int_{C_r} \phi d\lambda = \int_{D_\varepsilon(\vec{z})} \int_{[0, r]} \phi(t\vec{u}) t^{d-1} dt \ d\mu(\vec{u})
\]

\[
= \int_{D_\varepsilon(\vec{z})} \psi_r(\vec{u}) d\mu(\vec{u})
\]

\[
\geq \frac{\eta \lambda(B_\mathbb{R}(\vec{0}, r))}{3\mu(S^{d-1})} \mu(D_\varepsilon(\vec{z})).
\]
Finally we note that the cylinder \( rK_R = K_R(\varepsilon, \vec{0}, r\vec{z}) \) contains the cone \( C_r \) and that for fixed \( d \) the quotient
\[
\frac{\lambda(B_R(\vec{0}, r))\mu(D_\varepsilon(\vec{z}))}{3\lambda(rK_R)\mu(S^{d-1})}
\]
is constant. From this the lemma follows. \( \Box \)

**Remark 4.4.** Observe that condition (4) in Lemma 4.3 can be replaced with

\[
\frac{1}{\lambda(B_R(\varepsilon, r))}\int_{B_R(\varepsilon, r)} \phi \, d\lambda \geq \eta
\]

for an arbitrary point \( \vec{x} \in [-1, 1]^d \). In this case, the cylinder \( K_R \) in the conclusion becomes \( K_R = K_R(\varepsilon, \vec{x}, \vec{z}) \).

To see this one can apply Lemma 4.3 to the function \( \tilde{\phi}(\vec{y}) = \phi(\vec{y} - \vec{x}) \).

**Proof of Theorem 4.2.** First let us invoke Lebesgue’s differentiation theorem as well as Rademacher’s Theorem to find a set \( X \subset [-1, 1]^d \) with full Lebesgue measure such that for every \( \vec{x} \in X \) the map \( T \) is differentiable at \( \vec{x} \) and

\[
\phi(\vec{x}) = \lim_{r \to 0} \frac{1}{\lambda(B(\varepsilon, r))} \int_{B(\varepsilon, r)} \phi \, d\lambda.
\]

Pick any point \( \vec{x} \in X \) such that \( \phi(\vec{x}) > 0 \). Then, since \( T \) is differentiable at \( \vec{x} \), there exists a linear map \( J : \mathbb{R}^d \to \mathbb{R}^{d+1} \), the Jacobian of \( T \) at \( \vec{x} \), such that \( T(\vec{y}) \) can be written as

\[
T(\vec{y}) = T(\vec{x}) + J(\vec{y} - \vec{x}) + e(\vec{y} - \vec{x})\|\vec{y} - \vec{x}\|_2,
\]

where the error term \( e(\vec{z}) \) is continuous and satisfies \( e(\vec{0}) = 0 \). Since \( T \in L_{R^*}, J \neq 0 \).

Next take \( R = \{\frac{1}{n}\}_{n \geq n_0} \). For \( n_0 \) large enough we can apply Lemma 4.3 to \( R, \phi \) and \( \vec{x} \). Let \( P \subset S^{d-1} \) be the set obtained this way. Since \( P \) has positive measure, it spans \( \mathbb{R}^d \), and because \( J \) is a non-zero linear map, there exists some \( \vec{z} \in P \) for which \( J : \vec{z} \neq 0 \). Since \( \vec{z} \in P \) we can find an infinite set \( R' \subset R \) such that

\[
\liminf_{\varepsilon \to 0} \frac{1}{\lambda(rK_R)} \int_{rK_R} \phi \, d\lambda \geq c_1 \phi(\vec{x}), \quad \forall r \in R'.
\]

where \( K_R = K_R(\varepsilon, \vec{x}, \vec{z}) \) is as in Definition 4.1.

Set \( \delta = \frac{c_1 \phi(\vec{x})}{2} \) and set \( \vec{w} = \frac{\vec{z}}{\|\vec{z}\|_2} \in S^{d-1} \). We claim that with this choice of \( \delta \) and \( \vec{w} \) the set \( X_R(T, \varepsilon, \delta, \vec{w}, \vec{x}) \) is non-empty for all sufficiently small \( \varepsilon > 0 \). To show this, take \( r \in R' \) sufficiently small such that \( e(\vec{u}) < \min(\varepsilon/2, \|J \cdot \vec{z}\|_2 \varepsilon/3) \) for all \( \vec{u} \) with \( \|\vec{u}\|_2 \leq r \). Thereafter set \( \vec{y} = \vec{x} + r\vec{z} \). It follows from (6) that

\[
\left\| \frac{T(\vec{y}) - T(\vec{x})}{\|T(\vec{y}) - T(\vec{x})\|_2} - \vec{w} \right\|_2 < \varepsilon.
\]

Also, provided that \( \varepsilon \) was chosen sufficiently small, we have

\[
\int_{K_R(\varepsilon, \vec{x}, \vec{y})} \phi \, d\lambda \geq \delta \lambda(K_R(\varepsilon, \vec{x}, \vec{y})).
\]

At last, note that the distance between \( T(t\vec{x} + (1-t)\vec{y}) \) and \( tT(\vec{x}) + (1-t)T(\vec{y}) \) is equal to \( (1-t)r \) times the distance between \( e((1-t)r(\vec{z})) \) and \( e(r\vec{z}) \), which indeed is smaller than \( \varepsilon \|\vec{y} - \vec{x}\|_2 = \varepsilon r \). \( \Box \)
The rest of this section is dedicated to deriving Lemma 3.4 from Theorem 4.2. Assume \( L_Z \) is non-empty and let \( f \in L_Z \). Recall that every \( f \) in \( L_Z \) has Lipschitz constant \( M \). By definition (see (3)) one can find a sequence \((\ddot{z}_r)_{r \in \mathbb{N}}\) in \( \mathbb{Z}^d \) such that

\[
\limsup_{r \to \infty} \frac{|A \cap ([-r, r]^d + \ddot{z}_r)|}{(2r)^d} = d^*(A).
\] (7)

One can rarefy the sequence \((r)_{r \in \mathbb{N}}\), to say \((r(i))_{i \in \mathbb{N}}\), so that the lim sup in (7) is replaced by \( \lim \). For each \( i \) let \( V_i : [-1, 1]^d \to [-1,1]^{d+1} \) be the map

\[
V_i(\ddot{x}) = \frac{1}{Mr(i)} \left[ f\left( r(i)\ddot{x}\right) + \dddot{z}_{r(i)}\right] - f(\dddot{z}_{r(i)}).
\] (8)

One can further rarefy the sequence \((r(i))_{i \in \mathbb{N}}\), so that

\[
T(\ddot{x}) := \lim_{i \to \infty} V_i(\ddot{x})
\]

exists for every \( \ddot{x} \in [-1, 1]^d \cap Q^d \). One can easily deduce that for any \( \ddot{x}, \ddot{y} \in [-1, 1]^d \cap Q^d \) we have

\[
\|T(\ddot{x}) - T(\ddot{y})\|_2 \leq \|\ddot{x} - \ddot{y}\|_2.
\]

This implies that \( T \) can be extended to \([-1, 1]^d \) as a Lipschitz function with Lipschitz constant \( 1 \). Since \( Q^d \) is dense in \( \mathbb{R}^d \) (and Lipschitz functions are continuous), this extension is unique.

**Lemma 4.5.** \( V_i \to T \) uniformly on \([-1, 1]^d \).

**Proof.** Fix \( \varepsilon > 0 \). One can find a finite set \( F \subset [-1, 1]^d \) such that any \( \ddot{x} \in [-1, 1]^d \) satisfies \( \|\ddot{x} - \ddot{y}\|_2 < \varepsilon/4 \) for some \( \ddot{y} = \ddot{y}(\ddot{x}) \in F \). Let \( i \in \mathbb{N} \) be large enough so that \( \|V_j(\ddot{y}) - T(\ddot{y})\|_2 < \varepsilon/4 \) for all \( j \geq i \) and \( \ddot{y} \in F \), and such that \( d/r(i) < \varepsilon/4 \).

Let \( \ddot{x} \in [-1, 1]^d \) be arbitrary, let \( \ddot{y} \in F \) be such that \( \|\ddot{x} - \ddot{y}\|_2 < \varepsilon/4M \) and let \( j \geq i \). Then

\[
\|T(\ddot{x}) - V_j(\ddot{x})\|_2 \leq \|T(\ddot{x}) - T(\ddot{y})\|_2 + \|T(\ddot{y}) - V_j(\ddot{y})\|_2 + \|V_j(\ddot{y}) - V_j(\ddot{x})\|_2 \\
\leq \|\ddot{x} - \ddot{y}\|_2 + \frac{\varepsilon}{4} + \|\ddot{x} - \ddot{y}\|_2 + \frac{d}{r(j)} \\
\leq \varepsilon.
\]

\( \square \)

Next consider the sequence \((\phi_i)_{i \in \mathbb{N}}\) in \( L^2([-1, 1]^d) \) defined by

\[
\phi_i(\ddot{x}) = 1_A \left( [r(i)\ddot{x}] + \dddot{z}_{r(i)}\right).
\]

Observe that

\[
\int \phi_id\lambda = \frac{|A \cap \left( [r(i)] + \dddot{z}_{r(i)}\right)|}{r(i)^d}
\]

and hence, using (7),

\[
\lim_{i \to \infty} \int \phi_id\lambda = 2^dd^*(A).
\]
Rearranging, we get

$$\varepsilon \text{ for some other constant } C.$$

and let $\tilde{x}$ be large enough so that $\|\tilde{y} - \tilde{x}\|_2 < \varepsilon \|\tilde{y} - \tilde{x}\|_2 \leq \varepsilon \delta$. Choose $i$ large enough so that $\|V_i(\tilde{y}) - T(\tilde{y})\|_2 < \varepsilon \delta$ for any $\tilde{y} \in [-1, 1]^d$ and let

$$\tilde{u} := Mr(i)V_i(\tilde{x}) + f(\tilde{z}_{r(i)}) = f(r(i)\tilde{x} + \tilde{z}_{r(i)}) \in \mathbb{Z}^{d+1}.$$ \(\text{Now take } \tilde{u} \in \tilde{z}_{r(i)} + r(i)B_R(\tilde{x}, \delta) \cap \mathbb{Z}^d \text{ and let } \tilde{g} = (\tilde{u} - \tilde{z}_{r(i)})/r(i). \text{ We have}

$$\|f(\tilde{u}) - \tilde{u}\|_2 = Mr(i)\|V_i(\tilde{y}) - V_i(\tilde{x})\|_2 \leq Mr(i)\left(\|T(\tilde{g}) - T(\tilde{x})\|_2 + \|V_i(\tilde{y}) - T(\tilde{g})\|_2 + \|T(x) - V_i(\tilde{x})\|_2\right) \leq Mr(i)\varepsilon \delta + 2\varepsilon \delta = 3Mr(i)\varepsilon \delta.$$ \(\text{We just showed that}

$f \left( \tilde{z}_{r(i)} + r(i)B_R(\tilde{x}, \delta) \cap \mathbb{Z}^d \right) \supset B_Z(\tilde{u}, 3Mr(i)\varepsilon \delta)$. \(\text{On the one hand,}

$$\|\tilde{z}_{r(i)} + r(i)B_R(\tilde{x}, \delta) \cap \mathbb{Z}^d\| \geq C_1(\delta r(i))^d,$$

for some $C_1 > 0$ that only depends on the dimension $d$. \(\text{On the other hand, the ball}

$B_Z(\tilde{u}, 3Mr(i)\varepsilon \delta)$ can be covered with no more than $C_2(3Mr(i)\varepsilon \delta)^d$ vertical lines, for some other constant $C_2 > 0$ that only depends on the dimension $d$. \(\text{Since each line in } \mathbb{Z}^{d+1} \text{ contains the image (under } f) \text{ of at most } k \text{ points, we deduce that}

$$C_1(\delta r(i))^d \leq kC_2(3Mr(i)\varepsilon \delta)^d.$$ \(\text{Rearranging, we get}

$$\varepsilon \geq \sqrt[d]{C_1/(3d^MkC_2)},$$

so, by choosing $\varepsilon$ small enough (depending only on $k$, $M$ and $d$), we obtain the desired contradiction. \(\square

Now, we can apply Theorem 4.2 to $T = \lim V_i$ and $\phi = \lim \phi_i$ in order to find $\delta' > 0$, $\tilde{x} \in [-1, 1]^d$ and $\tilde{w} \in S^d$ such that for every sufficiently small $\varepsilon' > 0$ the set $X_R(T, \varepsilon', \delta', \tilde{w}, \tilde{x})$ is nonempty. \(\text{We will show that}

$X_Z(f, \varepsilon, \delta, \tilde{w})$ is non-empty for $\delta := \frac{\varepsilon'}{2}$ and $\varepsilon := \varepsilon'/2$. \(\text{To prove this claim, take any } \tilde{y} \in X_R(T, \varepsilon', \delta', \tilde{w}, \tilde{x}) \text{ and put } \eta := \varepsilon \|\tilde{x} - \tilde{y}\|_2/16. \text{ In view of Lemma 4.5 we can find } i \in \mathbb{N} \text{ large enough so that the following three conditions are satisfied:}

$$\forall \tilde{x} \in [-1, 1]^d$$

(i-1) $\left\|V_i(\tilde{x}) - V_i(\tilde{y})\right\|_2 < \frac{\varepsilon}{2};$

(i-2) $\left\|V_i(\tilde{z}) - T(\tilde{z})\right\|_2 < \eta, \forall \tilde{z} \in [-1, 1]^d;$

(i-3) $r(i) > \frac{\sqrt[d]{\eta}}{\eta}.$
Let \( \ell \) be the generalized line segment defined by \( \ell(0) = [r(i)\vec{x}] + \vec{z}_{r(i)} \) and \( \ell(1) = [r(i)\vec{y}] + \vec{z}_{r(i)} \). The proof of Lemma 3.4 will be completed with the following lemma.

**Lemma 4.7.** \( \ell \in X_{Z}(f, \varepsilon, \delta, \vec{w}) \).

**Proof.** First notice that \( m_{\ell} = \| [r(i)\vec{x}] - [r(i)\vec{y}] \|_{2} \geq r(i)\| \vec{x} - \vec{y} \|_{2} - 2\sqrt{d} \), and hence by condition (i-3) we have \( \varepsilon m_{\ell} > 16\sqrt{d} - 2\varepsilon\sqrt{d} \geq 14\sqrt{d} \). By equation (8) we have \((f \circ \ell)(0) = r(i)MV_{i}(\vec{x}) + f(\vec{z}_{r(i)}) \) and \((f \circ \ell)(1) = r(i)MV_{i}(\vec{y}) + f(\vec{z}_{r(i)}) \). Therefore,

\[
\frac{\| (f \circ \ell)(1) - (f \circ \ell)(0) \|_{2} - \vec{w}}{\varepsilon} = \frac{\| r(i)MV_{i}(\vec{x}) - r(i)MV_{i}(\vec{y}) \|_{2} - \vec{w}}{\varepsilon} \\
= \frac{\| V_{i}(\vec{x}) - V_{i}(\vec{y}) \|_{2} - \vec{w}}{\varepsilon} \leq \varepsilon,
\]

where the first inequality follows from (i-1). This proves (z-i). For the second condition, (z-ii), let \( t \in [0, 1] \) and observe that on the one hand

\[
\ell(t) = [(1-t)\ell(0) + t\ell(1) = [(1-t)[r(i)\vec{x}] + t[r(i)\vec{y}]] + \vec{z}_{r(i)}
\]

which implies that

\[
(f \circ \ell)(t) = r(i)MV_{i}\left(\frac{(1-t)[r(i)\vec{x}] + t[r(i)\vec{y}]}{r(i)}\right) + f(\vec{z}_{r(i)})
\]

for some \( \bar{e} \) with \( \| \bar{e} \|_{\infty} \leq 2M\sqrt{d} \). On the other hand,

\[
(1-t)(f \circ \ell)(0) + t(f \circ \ell)(1) = (1-t)r(i)MV_{i}(\vec{x}) + tr(i)MV_{i}(\vec{y}) + f(\vec{z}_{r(i)})
\]

By combining both (and by using (i-2) and (i-3)) we get

\[
\frac{\| (f \circ \ell)(t) - [(1-t)(f \circ \ell)(0) + t(f \circ \ell)(1)] \|_{2}}{\varepsilon} = \frac{\| r(i)M\left(\frac{V_{i}((1-t)\vec{x} + t\vec{y}) - [(1-t)V_{i}(\vec{x}) + tV_{i}(\vec{y})]}{r(i)}\right) + \vec{e}_{r(i)} \|_{2}}{\varepsilon} \leq \frac{\| r(i)M\left(\frac{T((1-t)\vec{x} + t\vec{y}) - [(1-t)T(\vec{x}) + tT(\vec{y})]}{r(i)}\right) + \vec{e} \|_{2}}{\varepsilon} \leq \frac{\| r(i)M\left(\frac{\vec{e}'}{r(i)})\|_{2} + 2\eta + 2\sqrt{d}}{r(i)} \leq \| r(i)\| \| \vec{x} - \vec{y} \|_{2}.
\]

Also

\[
m_{\ell} = \| \ell(1) - \ell(0) \|_{2} = \| [(r(i)\vec{y}) - [r(i)\vec{x}] \|_{2} \geq r(i)\| \vec{x} - \vec{y} \|_{2} - \sqrt{d} \geq \| 3/4 \| r(i)\| \vec{x} - \vec{y} \|_{2}
\]

and this finishes the proof of the second condition.
Finally, we prove the third condition, (z-iii). Let \( g_i : \vec{u} \mapsto |r(i)\vec{u}| + \vec{z}_{r(i)} \). For a set \( U \subset g_i([-1,1]^d) \) we have
\[
|U| = r(i)^d \lambda(g_i^{-1}(U)) \quad \text{and} \quad |A \cap U| = r(i)^d \int_{g_i^{-1}(U)} \phi_i d\lambda.
\]
We wish to apply these two facts to \( U = K_Z = K_Z(\varepsilon, \ell) \) (as in Definition 3.3). The idea is to approximate \( g_i^{-1}(K_Z) \) with \( K_R(\varepsilon, \vec{x}, \vec{y}) \). Now let \( K_Z = K_Z(\varepsilon, \ell) \) and \( K_R = K_R(\varepsilon', \vec{x}, \vec{y}) \) be as in and Definition 4.1. For any \( \vec{u} \in K_R \) there is some \( t \in [0,1] \) such that
\[
\|\vec{u} - (1-t)\vec{x} - t\vec{y}\|_2 \leq \varepsilon'\|\vec{x} - \vec{y}\|_2.
\]
It follows from (i-2) that
\[
\left\| g_i(\vec{u}) - [(1-t)g_i(\vec{x}) + t g_i(\vec{y})] \right\|_2 \leq r(i)\varepsilon'\|\vec{x} - \vec{y}\|_2 + 2\sqrt{d} \leq \varepsilon_m \ell.
\]

This implies that \( K_R \subset g_i^{-1}(K_Z) \). Similarly, one can show that \( g_i^{-1}(K_Z(\varepsilon, \ell)) \subset K_R(4\varepsilon', \vec{x}, \vec{y}) \). Therefore we conclude that
\[
\frac{|A \cap K_Z|}{|K_Z|} \geq \frac{r(i)^d \int_{K_R} \phi_i d\lambda}{\lambda(K_R(4\varepsilon', \vec{x}, \vec{y}))} r(i)^d > \frac{\delta'}{4^d} = \delta.
\]
This finishes the proof. \( \square \)

5. Proof of Theorem B using Lemma 3.4

Assume \( L_Z \neq \emptyset \) and let \( f \in L_Z \). Let \( \delta > 0 \) and \( \vec{w} \in S^d \) be given by Lemma 3.4. We will assume without loss of generality that the first coordinate \( w_1 \) of \( \vec{w} \) has the highest absolute value. Since \( \|\vec{w}\|_2 = 1 \), this implies that
\[
|w_1| \geq d^{-1/2}.
\]
(9)

We will need the following form of Dirichlet’s approximation theorem.

**Lemma 5.1.** Let \((u_2, u_3, \ldots, u_{d+1}) \in \mathbb{R}^d \) and let \( N \in \mathbb{N} \). Then there exists a positive integer \( b \leq N^d \) and \( a_2, a_3, \ldots, a_{d+1} \in \mathbb{Z} \) such that
\[
|u_l - a_l / b| \leq \frac{1}{bN} \quad \forall l \in \{2, 3, \ldots, d+1\}.
\]

For the remainder of this section let \( N \) be any positive integer satisfying \( N > kc_2/(\delta c_3) \), where \( c_2 \) is the constant appearing on Lemma 5.3 and \( c_3 \) is the constant appearing in Lemma 5.2, both depending only on the fixed parameters \( d \) and \( M \). Also, assume that \( N \) is large enough such that \( X_1(\varepsilon, \ell) \neq \emptyset \) for all \( \varepsilon \leq \frac{1}{N^d} \) as guaranteed by Lemma 3.4. Apply Lemma 5.1 to find \( b, a_2, \ldots, a_{d+1} \in \mathbb{Z} \) satisfying
\[
\left| \frac{a_l}{b} - \frac{u_l - a_l}{bN} \right| \leq \frac{1}{bN} \quad \forall l \in \{2, 3, \ldots, d+1\}.
\]
(10)

Finally let \( \varepsilon = \frac{1}{N^d} \) and take some generalized line segment \( \ell \in X_1(\varepsilon, \ell) \).

We need a lower bound on the cardinality of \( K_Z \).

**Lemma 5.2.** There exists a constant \( c_3 \) that only depends on the dimension \( d \) such that for any generalized line segment \( \ell \) and any \( \varepsilon > 0 \) satisfying \( \varepsilon m_\ell > 14\sqrt{d} \), the cylinder \( K_Z \) has cardinality \( |K_Z| \geq c_3 \varepsilon^{d-1} m_\ell^d \).
We have that tells us $B$ and the $5.3$ f image, under lines. It follows from the pigeonhole principle that some line in $E$ shows that $\tilde{Z}$ is as in Definition $3.3$. Due to (z-iii) we have that $K_Z \cap A/|K_Z| > \delta$, therefore this will imply that there exists $X \subset K_Z \cap A$ with $|X| = k$ and such that the image $f(X)$ is contained in a line.

We can assume without loss of generality that $(f \circ \ell)(0) = 0$, as otherwise we can instead cover the set $f(K_Z) - (f \circ \ell)(0)$ with less than $\delta|K_Z|/k$ lines and this would then yield a covering of $f(K_Z)$ with the same number of lines.

Define $\tilde{s} = (b, a_2, a_3, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1}$ and let $E$ be the set of all lines in $\mathbb{R}^{d+1}$ of the form $\{\tilde{x} - t\tilde{s}: t \in \mathbb{R}\}$ for some $\tilde{x} \in f(K_Z)$. Thus the set $E$ covers all points in $f(K_Z)$.

Lemma 5.3. There exists a constant $c_2$, depending only on the dimension $d$ and on the Lipschitz constant $M$, such that

$$|E| \leq c_2 \frac{m_\ell^d}{b^{d-1}N^d}.$$ 

Before we embark on the proof of this lemma let us first show how it implies Theorem B: On the one hand, it follows from Lemma 5.2, condition (z-iii) and the choice of $\varepsilon$ that $|A \cap K_Z| > \delta c_3 m_\ell^d/(bN)^{d-1}$. On the other hand, Lemma 5.3 tells us that we can cover the image of $K_Z$ under $f$ with no more than $c_2(m_\ell)^d/(b^{d-1}N^d)$ lines. It follows from the pigeonhole principle that some line in $E$ contains the image, under $f$, of at least

$$\frac{|A \cap K_Z|}{|E|} \geq \delta \left( c_3 \frac{m_\ell^d}{b^{d-1}N^d} \right) / \left( c_2 \frac{m_\ell^d}{b^{d-1}N^d} \right) = \frac{\delta c_3}{c_2} N$$

points from $A$. By choosing $N$ sufficiently large, depending only on $d, k, M$ and $\delta$, we deduce that some line in $E$ must contain the image of at least $k$ points from $A$. This contradicts the fact that $f \in L_{\mathbb{Z}}$, and this contradiction finishes the proof of Theorem B.

Now, all that remains to show is Lemma 5.3. Since all lines in $E$ are parallel, in order to count them, we can simply look at their intersection with the hyperplane $H = \{0\} \times \mathbb{R}^d$.

With this in mind, for a vector $\tilde{u} = (u_1, \ldots, u_{d+1})$ with $u_1 \neq 0$, we define the projection $P_{\tilde{u}}: \mathbb{R}^{d+1} \to \mathbb{R}^d$ by

$$P_{\tilde{u}}(x_1, \ldots, x_{d+1}) = (x_2, \ldots, x_{d+1}) - \frac{x_1}{u_1}(u_2, \ldots, u_{d+1}).$$
Note that \((0, P_x(\bar{x})) \in \mathbb{R}^{d+1}\) is the intersection of the line \(\{\bar{x} - t\bar{u} : t \in \mathbb{R}\}\) with \(H\). Thus \(E_0 \equiv P_x(f(K_\mathcal{Z}))\) is the set of intersections of lines in \(E\) with \(H\), and hence \(|E_0| = |E|\).

A simple calculation shows that if \(\bar{x}, \bar{y} \in \mathbb{Z}^{d+1}\) are such that the first coordinate of \(\bar{x}\) and the first coordinate of \(\bar{y}\) differ by a multiple of \(b\), then \(P_x(\bar{x}) - P_x(\bar{y}) \in \mathbb{Z}^d\). This implies
\[
E_0 \subset \bigcup_{t \in [0,b^{-1}]} \left( \bar{P}_x(l, 0, \ldots, 0) + \mathbb{Z}^d \right). \tag{11}
\]

Next we want to enclose \(E_0 = P_x(f(K_\mathcal{Z}))\) inside a convex set \(D \subset \mathbb{R}^d\). It follows from (9) that the operator norm of \(P_x\) is smaller than a constant \(c_4\) which only depends on \(d\). Let \(c_5 = 2c_4M\), let \(\bar{u} = P_x((f \circ \ell)(1)) \in \mathbb{R}^d\) and define
\[
D = \{t\bar{u} : t \in [0,1]\} + B_{\mathbb{R}}(0, c_5\varepsilon m_\ell).
\]

To see that \(E_0 \subset D\), let \(\bar{x} \in E_0\) be arbitrary. From the above construction we have \(\bar{x} = P_x(f(\bar{z}))\) for some \(\bar{z} \in K_\mathcal{Z}\). Therefore there exists some \(t \in [0,1]\) such that \(\|\bar{z} - \ell(t)\|_2 \leq \varepsilon m_\ell\) and hence \(\|f(\bar{z}) - (f \circ \ell)(t)\|_2 \leq \varepsilon M m_\ell\). Using (z-ii) we deduce that \(\|f(\bar{z}) - (f \circ \ell)(t)\|_2 \leq \varepsilon M m_\ell\). Thus
\[
\|\bar{x} - t\bar{u}\|_2 = \left\|P_x(f(\bar{z})) - P_x(t(f \circ \ell)(1))\right\|_2 \leq 2c_4\varepsilon M m_\ell.
\]
This shows \(\bar{x} \in D\) as desired.

Putting together the inclusion \(E_0 \subset D\) with (11) we deduce that
\[
|E_0| \leq c_6 b\lambda(D) \tag{12}
\]
for some constant \(c_6\) that only depends on \(d\). Moreover \(\lambda(D)\) can be bounded by
\[
\lambda(D) \leq c_7 (c_5\varepsilon m_\ell)^{d-1} \cdot (\|u\|_2 + 2c_5\varepsilon m_\ell) \tag{13}
\]
where \(c_7\) is the volume of the unit ball in \(\mathbb{R}^{d-1}\). Finally we need to estimate \(\|i\bar{u}\|_2\).

Let
\[
\bar{v} = (v_1, \ldots, v_d) = \frac{(f \circ \ell)(1) - (f \circ \ell)(0)}{(f \circ \ell)(1) - (f \circ \ell)(0)} = (f \circ \ell)(1)
\]
and define \(\bar{s} = (s_2, \ldots, s_{d+1})\) and \(\bar{v} = (v_2, \ldots, v_{d+1})\). We claim that there exists a constant \(c_8\) which only depends on \(d\) such that for all \(\bar{x} \in \mathbb{R}^{d+1}\) we have
\[
\|P_x(\bar{x}) - P_x(\bar{v})(\bar{x})\|_2 \leq c_8\varepsilon\|\bar{x}\|_2. \tag{14}
\]
To prove this claim, first observe that for \(\bar{x} \in \mathbb{R}^{d+1}\) we have
\[
\|P_x(\bar{x}) - P_x(\bar{v})(\bar{x})\|_2 = \left\|\frac{x_1}{v_1} \bar{v} - \frac{x_1}{b} \bar{s}\right\|_2 \leq \|x\|_2 \left\|\frac{\bar{v}}{v_1} - \frac{\bar{s}}{b}\right\|_2. \tag{15}
\]
Next, take an arbitrary \(i \in \{2, 3, \ldots, d+1\}\); it follows from (10) that
\[
\left|\frac{v_i}{v_1} - \frac{s_i}{b}\right| \leq \varepsilon + \left|\frac{v_i}{v_1} - \frac{w_i}{w_1}\right|. \tag{16}
\]
From (z-i) we get that \(\|\bar{w} - \bar{v}\|_2 \leq \varepsilon\), and so, in particular, \(|w_j - v_j| < \varepsilon\) for each \(j \in \{1, \ldots, d+1\}\). Recall that \(\varepsilon < d^{-1/2}/2\), \(w_1 > d^{-1/2}\) and \(|w_1|, |w_i| \leq 1\). We deduce that
\[
\left|\frac{v_i}{v_1} - \frac{w_i}{w_1}\right| = \left|\frac{v_iw_1 - v_1w_i}{w_1v_1}\right| = \left|\frac{(v_i - w_i)w_1 - (v_1 - w_1)w_i}{w_1v_1}\right| \leq \frac{2\varepsilon}{1/2d} = 4d\varepsilon.
\]
Putting this together with (16) and (15) we get (14) and this proves the claim.
Using (14) with $\vec{x} = (f \circ \ell)(1)$ and observing that $P_\ell((f \circ \ell)(1)) = 0$ we deduce that
\[ \|\vec{u}\|^2 = \|P_\ell(f \circ \ell)(1)\|^2 \leq c_8 \varepsilon \| (f \circ \ell)(1) \|^2 \leq c_8 \varepsilon M m_\ell. \]
Putting this together with (13) and (12) we conclude that
\[ |E| = |E_0| \leq c_6 b \lambda(D) \leq c_2 \frac{m_\ell^d}{b^{d-1} N^d} \]
with $c_2 = c_6 c_7 c_5^{d-1} (c_8 M + 2c_5)$. This finishes the proof of Lemma 5.3 and hence the proof of Theorem B.

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