On homogeneous Besov spaces for 1D Hamiltonians without zero resonance

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Abstract
We consider 1-D Laplace operator with short range potential $V(x)$, such that

$$(1 + |x|)\gamma V(x) \in L^1(\mathbb{R}), \quad \gamma > 1.$$  

We study the equivalence of classical homogeneous Besov type spaces $\dot{B}^s_p(\mathbb{R})$, $p \in (1, \infty)$ and the corresponding perturbed homogeneous Besov spaces associated with the perturbed Hamiltonian $H = -\partial_x^2 + V(x)$ on the real line. It is shown that the assumptions $1/p < \gamma - 1$ and zero is not a resonance guarantee that the perturbed and unperturbed homogeneous Besov norms of order $s \in [0, 1/p)$ are equivalent. As a corollary, the corresponding wave operators leave classical homogeneous Besov spaces of order $s \in [0, 1/p)$ invariant.

1 Introduction.
The wave operator methods have been used frequently in the study of the evolution flow generated by Hamiltonians, typically considered as perturbations of free Hamiltonians. The wave operators are defined by the relation

$$W_\pm = s - \lim_{t \to \pm \infty} e^{itH}e^{-itH_0},$$

where $H_0$ is the free Hamiltonian (self-adjoint non-negative operator), $H$ is the perturbed one and $s - \lim$ means strong limit. The existence and completeness of the wave operators in standard Hilbert space (typically Lebesgue space $L^2$) in case of short range perturbations is well known (see [8], [9], [7] and the references therein).

The functional calculus for the perturbed non-negative operator $H$ can be introduced with a relation involving $W_\pm$

$$g(H) = W_+ g(H_0) W_+^* = W_- g(H_0) W_-^*, \quad (1.1)$$

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for any function \( g \in L^\infty_{\text{loc}}(0, \infty) \). Moreover, the wave operators map unperturbed Sobolev spaces in the perturbed ones,

\[
W_{\pm} : D(\mathcal{H}_0^{s/2}) \to D(\mathcal{H}^{s/2}),
\]

and we have the equivalence of the Sobolev norms (see [12] for more general Sobolev norms)

\[
\|\mathcal{H}^{s/2}f\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})} \sim \|\mathcal{H}_0^{s/2}f\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}.
\] (1.2)

The study of the dispersive properties of the evolution flow in some cases of short range perturbations shows (see [2]) that we have stronger equivalence between homogeneous Sobolev norms

\[
\|\mathcal{H}^{s/2}f\|_{L^2(\mathbb{R}^n)} \sim \|\mathcal{H}_0^{s/2}f\|_{L^2(\mathbb{R}^n)},
\] (1.3)

provided \( s < n/2 \).

Our first goal in this work is to show that the requirement \( s < n/2 \) is optimal at least for \( n = 1, 2 \) and \( \mathcal{H}_0 = -\Delta = -\partial^2_x \), i.e. we shall prove the following result:

**Theorem 1.** If \( n = 1, 2 \), and \( V(x) \) is a positive potential such that

\[
\int_{\mathbb{R}^n} V^{n/2}(x)dx \leq C < \infty,
\] (1.4)

then (1.3) with \( s = n/2 \) is not true.

The mapping properties for the case of Sobolev spaces \( \dot{B}^s_p(\mathbb{R}^n) \) are studied in ([12], [10]) and they show examples of spaces invariant under the action of the wave operators.

Our unperturbed Hamiltonian \( \mathcal{H}_0 \) is the self-adjoint realization of \(-\partial^2_x \) on the real line \( \mathbb{R} \). The perturbed Hamiltonian is \( \mathcal{H} = -\partial^2_x + V(x) \). The results in [10] deal with short range assumptions that guarantee \( \dot{B}^s_p(\mathbb{R}) \) boundedness of \( W_{\pm} \). The \( L^p(\mathbb{R}) \) boundedness is studied in [3].

Our key goal in this work is to study how classical homogeneous Besov spaces \( \dot{B}^s_p(\mathbb{R}) \) are transformed under the action of the wave operators.

The splitting property (1.11) implies that

\[
W_{\pm} : \dot{B}^s_p(\mathbb{R}) \Longrightarrow \dot{B}^s_{p,\mathcal{H}}(\mathbb{R}), \quad \forall s \geq 0, \ 1 < p < \infty,
\]

where \( \dot{B}^s_{p,\mathcal{H}}(\mathbb{R}) \) is the perturbed Besov space generated by the Hamiltonian \( \mathcal{H} \). More precisely, \( \dot{B}^s_{p,\mathcal{H}}(\mathbb{R}) \) is the homogeneous Besov spaces associated with the perturbed Hamiltonian \( \mathcal{H} = -\partial^2_x + V \) as the closure of \( S(\mathbb{R}) \) functions \( f \) with respect to the norm

\[
\| f \|_{\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})} = \left( \sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}^2 \right)^{1/2}.
\] (1.5)

Here and below \( \varphi(\tau) \in C_{0}^\infty(\mathbb{R} \setminus 0) \) is a non-negative even function, such that

\[
\sum_{j \in \mathbb{Z}} \varphi \left( \frac{s}{2^j} \right) = 1, \quad \forall s \in \mathbb{R} \setminus 0.
\]

Some basic properties of these Besov spaces and the independence of the Besov space of the choice of the Paley-Littlewood function \( \varphi \) can be found in [13].

The equivalence of the homogeneous Besov norm

\[
\sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}^2 \sim \sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}^2,
\] (1.6)

imply that the homogeneous Besov space \( \dot{B}^s_{p,\mathcal{H}}(\mathbb{R}) \) is also invariant under the action of the wave operators \( W_{\pm} \). The natural restriction \( 0 \leq s < 1/p \) can be justified by Theorem 1.

Our approach to establish (1.6) is based on establishing estimates of this kind

\[
\left\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{2^j} \right) \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^k} \right) f \right\|_{L^p(\mathbb{R})} \leq C \frac{1}{2^{j-k+s}} \| f \|_{L^p(\mathbb{R})}, \quad \forall s > 0, \ s < \frac{1}{p},
\] (1.7)

where \( j, k \in \mathbb{Z} \) will satisfy certain relations.
2 Assumptions and main results

We shall assume that the potential $V : \mathbb{R} \rightarrow \mathbb{R}$ is a real-valued potential, $V \in L^1(\mathbb{R})$ and $V$ is decaying sufficiently rapidly at infinity, namely following [11] we require

$$\|(x)\gamma V\|_{L^1(\mathbb{R})} < \infty, \quad \gamma > 1 + 1/p, \quad 1 < p < \infty,$$

or equivalently we assume $V \in L^1_\gamma(\mathbb{R})$, where

$$L^1_\gamma(\mathbb{R}) = \{ f \in L^1_{\text{loc}}(\mathbb{R}); \langle x \rangle^\gamma f(x) \in L^1(\mathbb{R}) \}, \quad \langle x \rangle^2 = 1 + x^2.$$

We shall impose for simplicity in this work the assumption that the point spectrum of $\mathcal{H} = -\partial_x^2 + V(x)$ is empty, i.e.

$$\mathcal{H}f - zf = 0, \quad f \in L^2(\mathbb{R}), \quad z \in \mathbb{C} \Rightarrow f = 0. \quad (2.2)$$

Moreover, we are looking for appropriate decomposition of the kernel of the Paley-Littlewood localization operator

$$\varphi \left( \frac{\sqrt{\mathcal{H}}}{2^j} \right), \quad (2.3)$$

where $\varphi(\tau) \in C_0^\infty(\mathbb{R} \setminus 0)$ is an even function and $j \in \mathbb{Z}$. We plan to decompose the kernel of the operator (2.2) into a leading term, involving similar localization operators for the unperturbed Hamiltonian $\mathcal{H}_0$

$$\left| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^j} \right)(x, y) \right| \leq \frac{C 2^j}{(2^j(x - y))^2} \quad \forall j \in \mathbb{Z}, \quad (2.4)$$

and a remainder satisfying better kernel estimates.

The existence of the wave operators $W_\pm$ is well known according to the results in [10], [1], [3], so $W_\pm$ are well defined operators in $L^p(\mathbb{R})$, $1 < p < \infty$.

The splitting property

$$\mathcal{H}W_\pm = W_\pm \mathcal{H}_0$$

implies that

$$W_\pm : \hat{B}_p^s(\mathbb{R}) \rightarrow \hat{B}_p^s(\mathbb{R}), \quad \forall s \geq 0, \quad 1 < p < \infty.$$

The functional calculus for the perturbed operator $\mathcal{H}$ can be defined as follows

$$g(\mathcal{H}) = W_+ g(\mathcal{H}_0)W_+^* = W_- g(\mathcal{H}_0)W_-^*$$

for any function $g \in L_{\text{loc}}^\infty(\mathbb{R})$.

The functional calculus enables one to introduce a Paley-Littlewood partition of unity

$$1 = \sum_{j \in \mathbb{Z}} \varphi \left( \frac{t}{2^j} \right), \quad t > 0$$

for an appropriate non-negative cutoff $\varphi \in C_0^\infty(\mathbb{R}_+)$, such that $\text{supp}\varphi \subseteq [1/2, 2]$.

The homogeneous Besov spaces $\hat{B}_p^s(\mathbb{R})$ for $p, 1 \leq p \leq \infty$ and $s \geq 0$ can be defined as the closure of $S(\mathbb{R})$ functions $f$ with respect to the norm

$$\|f\|_{\hat{B}_p^s(\mathbb{R})} = \left( \sum_{j=-\infty}^{\infty} 2^{js} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}^2 \right)^{1/2}. \quad (2.5)$$

The perturbed Besov spaces $\hat{B}_p^s(\mathbb{R})$ have been already defined in [1],[3].

Using the classical result due to Weder [10], one can derive the following $L^p$ estimate.

$$\left\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{M} \right) f \right\|_{L^p(\mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R})} \quad (2.6)$$

for $M > 0$, $f \in S(\mathbb{R})$, $1 < p < \infty$.

First we prove the following high energy kernel estimate needed in the proof of the equivalence of homogeneous Besov norms.
Lemma 2.1. Suppose the condition (2.1) is fulfilled with $\gamma > 1 + 1/p$ and the operator $\mathcal{H}$ has no point spectrum. If $\varphi$ is an even non-negative function, such that $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, then for any $M \in [1, \infty)$, $\sigma \in (0,1) \cap (0,\gamma - 1)$ we have

$$\left| \varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y) - \varphi \left( \sqrt{\mathcal{H}_0/M} \right)(x,y) \right| \leq C \left( \sum_{\pm} \frac{1}{\langle M(x \pm y) \rangle^\sigma} \right) \left( \frac{1}{\langle x \rangle^{\gamma-\sigma}} + \frac{1}{\langle y \rangle^{\gamma-\sigma}} \right).$$

(2.7)

The proof is based on careful evaluation of the kernel of the operator $\varphi \left( \sqrt{\mathcal{H}/M} \right)$, having the representation

$$\varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y) = -\frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\tau/M) T(\tau)f_+(y,\tau)f_-(x,\tau) d\tau,$$

when $x < y$, (2.8)

and

$$\varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y) = -\frac{1}{2\pi} \int_{\mathbb{R}} \varphi(\tau/M) T(\tau)f_-(y,\tau)f_+(x,\tau) d\tau,$$

otherwise. (2.9)

Here and below $T(\tau)$ is the transmission coefficient (see (6.22) for its definition). Moreover,

$$f_{\pm}(x,\tau) = e^{\pm i\tau x} m_{\pm}(x,\tau),$$

and $f_{\pm}(x,\tau)$ are the Jost functions (see Section 2 in [4]) satisfying the integral equations (Marchenko type equations)

$$m_+(x,\tau) = 1 + \int_{x}^{\infty} D(t-x,\tau)V(t)m_+(t,\tau)dt,$$

$$m_-(x,\tau) = 1 + \int_{-\infty}^{x} D(x-t,\tau)V(t)m_-(t,\tau)dt,$$

with

$$D(t,\tau) = \frac{e^{2i\tau\tau} - 1}{2i\tau} = \int_{0}^{t} e^{2i\tau y} dy.$$  

(2.11)

The kernel $\varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y)$ is symmetric, that it

$$\varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y) = \varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y).$$

(2.12)

For the low energy domain $M \in (0,1]$, we have the following estimate.

Lemma 2.2. Suppose the condition (2.1) is fulfilled with $\gamma > 1 + 1/p$, the operator $\mathcal{H}$ has no point spectrum and 0 is not a resonance point for $\mathcal{H}$. If $\varphi$ is an even non-negative function, such that $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, then for any $M \in (0,1]$ and $\sigma \in (0,1) \cap (0,\gamma - 1)$ we have

$$\left| \varphi \left( \sqrt{\mathcal{H}/M} \right)(x,y) - K_M(x,y) \right| \leq C M \left( \sum_{\pm} \frac{1}{\langle M(x \pm y) \rangle^\sigma} \right) \left( \frac{1}{\langle x \rangle^{\gamma-\sigma}} + \frac{1}{\langle y \rangle^{\gamma-\sigma}} \right),$$

(2.13)

where

$$K_M(x,y) = c \int_{\mathbb{R}} e^{-i\tau(x-y)} \varphi \left( \frac{\tau}{M} \right) b(x,y,\tau) d\tau.$$  

(2.14)
with symmetric kernel \( b(x, y, \tau) = b(y, x, \tau) \) and

\[
b(x, y, \tau) = \begin{cases} 
T(\tau) & x < 0 < y, \\
(R_+ (\tau) + 1) e^{2i\tau x} - e^{2i\tau x} + 1 & 0 < x < y, \\
(R_- (\tau) + 1) e^{-2i\tau y} - e^{-2i\tau y} + 1 & x < y < 0.
\end{cases}
\]

**Remark 2.3.** The precise definition of the notion of resonance point at the origin is given in Definition 6.5 by the aid of the relation

\( T(0) = 0 \).

Our next result treats the equivalence of the homogeneous Besov norms for the free and perturbed Hamiltonians. Here we meet the natural obstruction to cover all positive values of \( s \) so we impose a condition

\[
s < \frac{1}{p},
\]

similar to the Hardy inequality restrictions.

**Theorem 2.** Suppose

\( V \in L^1_\gamma (\mathbb{R}), \; \gamma > 1 + 1/p, \; 0 \leq s < 1/p, \; p \in (1, \infty), \)

the operator \( H \) has no point spectrum and 0 is not a resonance for \( H \). Then we have

\[
\| f \|_{\dot{B}^s_{p, \infty} (\mathbb{R})} \sim \| f \|_{\dot{B}^s_{p, 1} (\mathbb{R})}.
\]

As immediate consequence we have the following.

**Corollary 1.** Suppose the assumptions of Theorem 2 are fulfilled. Then for any \( p \in (1, \infty) \), any \( s \in [0, 1/p) \), we have

\[
W_{\pm} : \dot{B}^s_p (\mathbb{R}) \to \dot{B}^s_p (\mathbb{R}).
\]

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We shall present the plan of the work.

### 3 Counterexample for equivalence of homogeneous Besov spaces

In this section we consider the simplest case \( p = 2 \) and we shall prove Theorem 1 therefore we shall show that the equivalence property

\[
\|(H_0 + V)^{n/4} u\|_{L^2 (\mathbb{R}^n)} \sim \|(H_0)^{n/4} u\|_{L^2 (\mathbb{R}^n)}
\]

is not true for \( n = 1, 2 \).

**Proof of Theorem 1.** Let us suppose that the relation (3.1) holds. First, we use an interpolation argument and show that

\[
\| \mathcal{H}_0^a u \|^2_{L^2 (\mathbb{R}^n)} \geq \| V^a u \|^2_{L^2 (\mathbb{R}^n)}, \tag{3.2}
\]

provided \( 0 \leq \text{Re} a \leq 1/2 \). Indeed, we have the property

\[
\| \mathcal{H}_0^a u \|^2_{L^2 (\mathbb{R}^n)} = \| u \|^2_{L^2 (\mathbb{R}^n)}, \forall b \in \mathbb{R},
\]

and

\[
\| V^{ib} u \|^2_{L^2 (\mathbb{R}^n)} = \| u \|^2_{L^2 (\mathbb{R}^n)}, \forall b \in \mathbb{R},
\]

so we have to check (3.2) only for \( a = 1/2 \). The equivalence of the norms (3.3) implies that

\[
\| \mathcal{H}_0^{1/2} u \|_{L^2 (\mathbb{R}^n)} \approx \| (-\Delta + V)^{1/2} u \|^2_{L^2 (\mathbb{R}^n)} = \langle (-\Delta + V) u, u \rangle_{L^2 (\mathbb{R}^n)} \geq \langle V u, u \rangle_{L^2 (\mathbb{R}^n)} = \| V^{1/2} u \|^2_{L^2 (\mathbb{R}^n)},
\]

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and we conclude that \((3.2)\) is true. Then, assuming \((3.1)\) is fulfilled and applying the proved inequality with 
\(a = n/4 \leq 1/2\), we get
\[
\int_{\mathbb{R}^n} (V(x))^{n/2} |u(x)|^2 \, dx \leq C \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)}, \quad D = (-\Delta)^{1/2}.
\] (3.3)

Taking \(u\) in the Schwartz class \(S(\mathbb{R}^n)\) of rapidly decreasing function, we can apply a rescaling argument. Indeed, considering the dilation
\[ u_\lambda(x) = u(x\lambda), \]
we find
\[
\|D^{n/2}u_\lambda\|^2_{L^2(\mathbb{R}^n)} = \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)} \text{ constant in } \lambda
\]
and
\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^n} V^{n/2}(x)|u_\lambda(x)|^2 \, dx = \left( \int_{\mathbb{R}^n} V^{n/2}(x) \, dx \right) |u(0)|^2.
\]
In this way we deduce
\[
|u(0)|^2 \left( \int_{\mathbb{R}^n} V^{n/2}(x) \, dx \right) \leq C \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)}.
\] (3.4)

The homogeneous norm
\[
\|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)}
\]
is also invariant under translations, i.e. setting
\[ u(\tau)(x) = u(x + \tau), \]
we have
\[ \widehat{u(\tau)}(\xi) = e^{-i\tau \xi} \widehat{u}(\xi) \]
and
\[
\|D^{n/2}u(\tau)\|^2_{L^2(\mathbb{R}^n)} = \|\xi|^{n/2} \widehat{u(\tau)}\|^2_{L^2(\mathbb{R}^n)} = \|\xi|^{n/2} \widehat{u}\|^2_{L^2(\mathbb{R}^n)} = \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)},
\]
so applying \((3.4)\) with \(u(\tau)\) in the place of \(u\), we find
\[
|u(\tau)|^2 \int_{\mathbb{R}^n} V^{n/2}(x) \, dx \leq C \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)},
\]
or equivalently
\[
\|u\|^2_{L^\infty(\mathbb{R}^n)} \leq C_1 \|D^{n/2}u\|^2_{L^2(\mathbb{R}^n)}, \quad (3.5)
\]
where
\[
C_1 = \frac{C}{\|V^{n/2}\|_{L^1(\mathbb{R}^n)}}.
\]

The substitution \(\phi = D^{n/2}u\) enables us to rewrite \((3.5)\) as
\[
\|I_{n/2}(\phi)\|^2_{L^\infty(\mathbb{R}^n)} \leq C_1 \|\phi\|^2_{L^2(\mathbb{R}^n)}, \quad (3.6)
\]
where
\[
I_\alpha(\phi)(x) = D^{-\alpha}(\phi)(x) = c \int_{\mathbb{R}^n} |x - y|^{-n+\alpha} \phi(y) \, dy, \quad \alpha \in (0, n)
\]
are the Riesz operators.

It is easy to show that \((3.6)\) leads to a contradiction. Indeed, taking
\[
\phi_N(x) = \sum_{j=0}^{N} |x|^{-n/2} \mathbf{1}_{2^j \leq |x| \leq 2^{j+1}}(x),
\]
with $N \geq 2$ sufficiently large and being $1_A(x)$ the characteristic function of the set $A$, we can use the estimates
\[
I_{n/2}(\phi_N)(0) \geq \left( \sum_{j=0}^{N/2} \int_{2j}^{2j+1} \frac{r^{n-1}dr}{r^n} \right) \geq CN
\]
and
\[
\|\phi_N\|_{L^2(\mathbb{R}^n)}^2 = \sum_{j=0}^{N/2} \int_{2j}^{2j+1} \frac{r^{n-1}dr}{r^n} \leq C'N.
\]
Hence, from (3.6) we deduce
\[
CN^2 \leq \|I_{n/2}(\phi)\|_{L^\infty(\mathbb{R}^n)} \leq C_1 \|\phi\|_{L^2(\mathbb{R}^n)}^2 \leq C_2 N,
\]
for any $N$ sufficiently big and this is impossible. This completes the proof of the Theorem.

4 Functional calculus kernels and their asymptotic expansions

The functional calculus for the perturbed Hamiltonian $H$ is based on the relations (2.8) and (2.9). In the low energy domain we have the kernel expansion proposed in Lemma 2.2. We shall prove this kernel estimate below.

**Proof of Lemma 2.2.** We assume $x < y$ for determinacy and consider three cases.

- **Case A:** $x < 0 < y$
- **Case B:** $0 \leq x < y$
- **Case C:** $x < y \leq 0$

In the **Case A**, we can use the representation
\[
T(\tau)m_+(y, \tau)m_-(x, \tau) = T(\tau) + T(\tau) \underbrace{m_{0\text{rem},+}(y, \tau) + m_{0\text{rem},-}(x, \tau)}_{= a_1(x,\tau)}
\]
\[
+ T(\tau) \underbrace{m_{0\text{rem},-}(x, \tau)}_{= a_2(x,\tau)} + T(\tau) \underbrace{m_{0\text{rem},+}(y, \tau)m_{0\text{rem},-}(x, \tau)}_{= a_3(x,y,\tau)},
\]
where
\[
m_{0\text{rem},\pm}(x, \tau) = m_{\pm}(x, \tau) - 1.
\]
In this way, from (2.8), we have the representation
\[
\varphi \left( \sqrt{\frac{\mathcal{H}}{M}} \right)(x, y) = c \varphi_M(x - y) + c \sum_{j=1}^{3} I_M(a_j)(x, y),
\]
where
\[
I_M(a)(x, y) = M \int_{\mathbb{R}} \varphi(\tau) T(M\tau)a(x, y, M\tau)e^{-iM\tau(x-y)}d\tau
\]
and
\[
\varphi_M(\tau) = T(\tau)\varphi \left( \frac{\tau}{M} \right).
\]
We can put the term $\widehat{\varphi_M}(x - y)$ in the leading term $K_M(x, y)$ defined in (2.14), indeed, we have

$$\varphi_M(x - y) = \mathbb{1}_{x < 0}\mathbb{1}_{y > 0} \int_\mathbb{R} e^{-ir(x,y)} \frac{\tau}{M} T(\tau) \, d\tau.$$ 

To estimate the terms $I_M(a_j)(x,y)$ we are going to use the following fractional integration by parts estimate:

$$\left|\int_\mathbb{R} e^{irM\xi} g(\tau) d\tau\right| \leq \frac{C}{(M\xi)^\sigma} \|g\|_{C^{0,\sigma}(\mathbb{R})}, \quad \forall \sigma \in (0, 1). \tag{4.3}$$

Hence we have

$$|I_M(a_j)(x,y)| \leq C \frac{M}{(M(x-y))^\sigma} \left\|\varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_j(x,y,M\tau)\right\|_{C^{0,\sigma}(\mathbb{R})}. \tag{4.4}$$

Then, using the estimates proved in Lemma (6.2) combined with the following estimates for $T(\tau)$

$$\left\|\frac{T(\tau)}{\tau}\right\|_{C^{0,\sigma}(\mathbb{R})} + \left|\frac{T(\tau)}{\tau}\right| \leq C, \quad \sigma \in (0, 1) \cap (0, \gamma - 1)$$

and with the fact that $\varphi \in C_0^\infty(\mathbb{R}^+)$, such that supp$\varphi \subseteq [1/2, 2]$, we get

$$\left\|\varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_1(x,y,M\tau)\right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle y \rangle^{\gamma}} + \frac{M^\sigma}{\langle y \rangle^{\gamma - \sigma}}\right),$$

$$\left\|\varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_2(x,y,M\tau)\right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle x \rangle^{\gamma}} + \frac{M^\sigma}{\langle x \rangle^{\gamma - \sigma}}\right),$$

$$\left\|\varphi(\tau) \frac{T(M\tau)}{M\tau} M\tau a_3(x,y,M\tau)\right\|_{C^{0,\sigma}(\mathbb{R})} \leq C \left(\frac{1}{\langle y \rangle^{\gamma - 1\gamma}} + \frac{1}{\langle y \rangle^{\gamma - \sigma}}\right)\left(\frac{1}{\langle x \rangle^{\gamma - 1\gamma}} + \frac{M^\sigma}{\langle y \rangle^{\gamma - \sigma - 1}}\right).$$

Turning back to (4.2) and using the estimates (4.4) together with Holder estimates above, we obtain

$$\left|\varphi\left(\frac{\sqrt{M}}{M}\right) - c\varphi_M(x - y)\right| \leq C \frac{M}{(M(x-y))^\sigma} \left(\frac{1}{\langle y \rangle^{\gamma - \sigma}} + \frac{1}{\langle x \rangle^{\gamma - \sigma}}\right)$$

with $\sigma \in (0, 1) \cap (0, \gamma - 1)$, and $x < 0 < y$, i.e. we get (2.13) in the Case A.

In the Case B since we have $x \geq 0$, we want to write $m_-(x, \tau)$ in term of $m_+(x, \pm \tau)$. In order to do this, we can use the relation

$$T(\tau)m_-(x,\tau) = R_+(\tau) e^{2irx} m_+(x,\tau) + m_+(x,-\tau). \tag{4.5}$$

Then we can write

$$T(\tau)m_+(y,\tau)m_-(x,\tau) = \left(R_+(\tau) + 1\right) e^{2irx} m_+(y,\tau)m_+(x,\tau) - e^{2irx} m_+(y,\tau)m_+(x,\tau) + m_+(y,\tau)m_+(x,-\tau).$$

Using the remainders introduced in (4.1) we can represent the kernel $\varphi\left(\frac{\sqrt{M}}{M}\right)(x,y)$ as a sum of kernels of three types:

$$I_M(x,y) = \int_\mathbb{R} \varphi\left(\frac{\tau}{M}\right) \left((R_+(\tau) + 1) e^{2irx} m_+(y,\tau)m_+(x,\tau)\right) e^{-ir(x-y)} d\tau,$$

$$II_M(x,y) = M\mathfrak{f}(M(x-y)) - M\mathfrak{f}(M(x+y))$$

$$III_M(x,y) = \sum_{j=1}^2 K_j(x,y;M),$$

with $g$ is a compactly supported function in $C_0^\infty(\mathbb{R})$ such that $0 \notin \text{supp} g$. \footnote{Here $g$ is a compactly supported function in $C_0^\infty(\mathbb{R})$ such that $0 \notin \text{supp} g$.}
\[ K_1(x, y; M) = M \int_{\mathbb{R}} e^{-iM\tau(x-y)}\varphi(\tau)b_1(x, y, M\tau) d\tau, \quad (4.6) \]

\[ K_2(x, y; M) = -M \int_{\mathbb{R}} e^{iM\tau(x+y)}\varphi(\tau)b_2(x, y, M\tau) d\tau, \quad (4.7) \]

where

\[
\begin{align*}
K_1(x, y; M) &= M \int_{\mathbb{R}} e^{-iM\tau(x-y)}\varphi(\tau)b_1(x, y, M\tau) d\tau, \\
K_2(x, y; M) &= -M \int_{\mathbb{R}} e^{iM\tau(x+y)}\varphi(\tau)b_2(x, y, M\tau) d\tau,
\end{align*}
\]

and

\[
\begin{align*}
b_1(x, y, M\tau) &= m_0^{\text{rem.}+}(y, M\tau) + m_0^{\text{rem.}+}(x, -M\tau) + m_0^{\text{rem.}+}(y, M\tau)m_0^{\text{rem.}+}(x, -M\tau), \\
b_2(x, y, M\tau) &= m_0^{\text{rem.}+}(y, M\tau) + m_0^{\text{rem.}+}(x, M\tau) + m_0^{\text{rem.}+}(y, M\tau)m_0^{\text{rem.}+}(x, M\tau).
\end{align*}
\]

As before, we firstly estimate the terms \( I_{M}^{(x,y)} \) and \( III_{M}^{(x,y)} \) with the fractional integration by parts estimate (4.3) and then we use Lemma 6.2 combined with the estimates

\[
\left\| \frac{R_{\pm}(\tau) + 1}{\tau} \right\|_{C^0, \sigma(C_{\pm})} + \left\| \frac{R_{\pm}(\tau) + 1}{\tau} \right\|_{C^0, \sigma(C_{\pm})} \leq C, \quad \sigma \in (0, 1) \cap (0, \gamma - 1),
\]

and the properties of the function \( \varphi \) to prove (2.13) in the (Case B). Here

\[
b(x, y, \tau) = (R_{\pm}(\tau) + 1)e^{2i\tau x} - e^{2i\tau x} + 1
\]

and \( 0 \leq x < y \).

In the (Case C) we follow the argument used in the (Case B) but this time we replace (4.5) by

\[
T(\tau) = R_{\pm}(\tau) = R_{\pm}(y, \tau) + m(y, -\tau), \quad (4.8)
\]

and we derive (2.13) using the argument of case (Case B).

This completes the proof of (2.13). \( \square \)

**Proof of Lemma 2.7** In the high energy domain \( M > 1 \) we can follow the proof of Lemma 2.2. Using the estimates

\[
T(\tau) = 1 + O(\tau^{-1}), \quad R(\tau) = O(\tau^{-1})
\]

near \( \tau \rightarrow \infty \), we can absorb the factor \( M > 1 \) that appears in

\[
\varphi \left( \frac{\sqrt{H}}{M} \right) (x, y) - \varphi \left( \frac{\sqrt{H_0}}{M} \right) (x, y) =
\]

\[
= M \int_{\mathbb{R}} \varphi(\tau) [T(\tau M)m_{\pm}(y, \tau M)m_{\pm}(x, \tau M) - 1] e^{-i\tau M(x-y)} d\tau.
\]

Then, proceeding as in the proof of Lemma 2.2, we obtain the following estimate

\[
\left| \varphi \left( \frac{\sqrt{H}}{M} \right) (x, y) - \varphi \left( \frac{\sqrt{H_0}}{M} \right) (x, y) \right| \leq
\]

\[
\leq C \left\{ \sum_{\pm} \frac{1}{\langle M(x \pm y) \rangle^\sigma} + \frac{1}{\langle x \rangle^{1-\sigma}} + \frac{1}{\langle y \rangle^{1-\sigma}} \right\},
\]

i.e. the inequality (2.7). \( \square \)
5 Equivalence of homogeneous Besov norms

The comparison of homogeneous Besov spaces $\dot{B}^s_p(R)$ and $\dot{B}^s_{p,H}(R)$ is closely connected with the definition and properties of fractional power of the Hamiltonians $H$ and $H_0$.

For sectorial operators $A$ in $L^p(R)$ with spectrum $\sigma(A)$ satisfying

$$z \in \sigma(A) \setminus \{0\} \implies \text{Re} z \geq c |\text{Im} z|, \ c > 0$$

we can define for any $\sigma \in (0,1)$ the fractional negative powers of $A$ as follows (see Theorem 1.4.2 in [3])

$$A^{-\sigma} = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty \lambda^{-\sigma} (\lambda + A)^{-1} d\lambda. \quad (5.1)$$

The above relation suggests to introduce the fractional powers $H^{s/2}$ for $s \in [0,1)$ by the aid of the relation

$$H^{s/2} = H^{-s} H = \frac{\sin(\pi \sigma)}{\pi} \int_0^\infty \lambda^{-s} H(\lambda + H)^{-1} d\lambda \quad (5.2)$$

with $\sigma = 1 - s/2$.

Sectorial properties of $H$ are studied in [3] under the assumption that 0 is not resonance for $H$. The convergence of the integral in (5.2) near $\theta = 0$ needs justification based on limiting absorption type estimates, obtained in [3] in the case 0 is not a resonance point for $H$.

The existence of the wave operators $W_{\pm}$ is well known according to the results in [10], [1], [3], so $W_{\pm}$ are well defined operators in $L^p(R)$, $1 < p < \infty$.

The splitting property

$$H W_{\pm} = W_{\pm} H_0$$

implies that

$$W_{\pm} : \dot{B}^s_{p,H}(R) \to \dot{B}^s_{p,H}(R), \ \forall s \geq 0, \ 1 < p < \infty.$$

The functional calculus for the perturbed operator $H$ can be defined as follows

$$g(H) = W_+ g(H_0) W_+^* = W_- g(H_0) W_-^*$$

for any function $g \in L^\infty_{loc}(R)$.

Using the existence of the wave operators, its boundness in $L^p(R)$ and the splitting property, one can derive the following $L^p$ estimate (partial of Bernstein inequality)

$$\left\| \varphi \left( \frac{\sqrt{H}}{M} \right) f \right\|_{L^p(R)} \leq C \| f \|_{L^p(R)} \quad (5.3)$$

for $M > 0$ and $f \in S(R)$.

For completeness we can also mention that from Lemma 2.1 combined with Young convolution inequality we get the Bernstein inequality for $M > 1$

$$\left\| \varphi \left( \frac{\sqrt{H}}{M} \right) f - \varphi \left( \frac{\sqrt{H_0}}{M} \right) f \right\|_{L^p(R)} \leq C M^{1/p - 1/q - \delta} \| f \|_{L^p(R)}, \quad (5.4)$$

where $1 \leq p \leq q \leq \infty$ and $\delta > 0$, $\delta = s - \sigma$ according with the notations used in Lemma 2.1.

For the low energy domain, $0 < M \leq 1$, we need the following estimate.

**Lemma 5.1.** Assume that $V \in L^1_{c}(R)$ with

$$\gamma > 1 + 1/p, \ 0 < s < \frac{1}{p}, \ 1 < p < \infty.$$

Then for any even function $\varphi(\tau) \in C_0^\infty(R \setminus 0)$ there exists a constant $C = C(\|V\|_{L^1_{c}(R)})$ so that for any pair of real positive numbers $\Lambda, M$ such that $0 < \Lambda \leq M$, $M \leq 1$ and for any $f \in S(R)$, the following inequality holds:

$$\left\| \varphi \left( \frac{\sqrt{H}}{M} \right) \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right) f \right\|_{L^p(R)} \leq C \frac{\Lambda^s}{M^s} \| f \|_{L^p(R)}. \quad (5.5)$$
Proof. We can assume that the support of \( \varphi \) is in \([1/2,2]\).
We note that if \( \Lambda/4 \leq M \leq 4\Lambda \), or \( M/4 \leq \Lambda \leq 4M \) then we can use the fact that \( \varphi(\sqrt{H}/M) \) and \( \varphi(\sqrt{H}/\Lambda) \) are \( L^p \) bounded operators, so the estimates (5.5) is obvious in this case.

Since if \( V \in L^1_c(R) \) with \( \gamma > 1 + 1/p \) then \( V \in L^{1+s}_c(R) \) for any \( s \in [0,1/p) \).

Our first step is the proof of (5.5), assuming

\[
\Lambda < M/4. 
\] (5.6)

Our goal is to check the inequality

\[
\left\| \int_R \varphi \left( \frac{\sqrt{H}}{M} \right)(x,y) f_\Lambda(y) dy \right\|_{L^p(R)} \leq C \left( \frac{\Lambda}{M} \right)^s \| f \|_{L^p(R)} \] (5.7)

where \( f_\Lambda = \varphi(\sqrt{H}/\Lambda)f \).

We can apply the kernel estimate (2.13) from Lemma 2.2 so we get

\[
CM \left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |x|^{1+s-\sigma}} \right\|_{L^p(R)} + CM \left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |y|^{1+s-\sigma}} \right\|_{L^p(R)},
\] (5.8)

for any \( \sigma \in (0,s] \).

The terms in the right side of the inequality above can be evaluated using Hardy-Sobolev estimates. To be more precise, the equivalence between the Lebesgue spaces \( L^p(R) \) and the Lorentz ones \( L^{p,p}(R) \) in the case \( 1 < p < \infty \) allows us to use the sharp inequalities in Lorentz spaces. Indeed, we recall that \( |x|^{-1/\beta} \in L^{\beta,\infty}(R) \), for any \( \beta \geq 1 \). Hence, using the relation

\[
1 + \frac{1}{p} = \sigma + (1+s-\sigma) + \left( \frac{1}{p} - s \right),
\] (5.9)

we are in position to apply Young and Hölder inequalities in Lorentz spaces to get

\[
\left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |x|^{1+s-\sigma}} \right\|_{L^{p,p}(R)} \leq C \frac{1}{M^\sigma} \left\| \frac{1}{|x|^{\sigma}} \right\|_{L^{1,\infty}(R)} \left\| \frac{1}{|y|^{1+s-\sigma}} \right\|_{L^{1+s-\sigma,\infty}(R)} \| f \|_{L^{p,p}(R)},
\]

where

\[
\frac{1}{q} = \frac{1}{p} - s.
\]

This estimate can be combined with the Sobolev embedding in Lorentz spaces

\[
\| f \|_{L^{q,p}(R)} \leq C \| D^s f \|_{L^{p,p}(R)}, \quad \frac{1}{q} = \frac{1}{p} - s, \quad 0 < s < 1/p,
\] (5.10)

so that we obtain that the second term in the right side in (5.8) is bounded from

\[
CM^{1-\sigma} \Lambda^s \| f \|_{L^p(R)} \leq CA^s \| f \|_{L^p(R)}.
\]

One can proceed similarly to find

\[
\left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |x|^{1+s-\sigma}} \right\|_{L^p(R)} \leq C \frac{1}{M^\sigma} \Lambda^s \| f \|_{L^p(R)}.
\]

Hence we have proved that in the case \( 0 < \Lambda < M \leq 1 \)

\[
M \left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |x|^{1+s-\sigma}} \right\|_{L^p(R)} + M \left\| \int_R \frac{|f_\Lambda(y)| dy}{\langle M(x-y) \rangle^\sigma |y|^{1+s-\sigma}} \right\|_{L^p(R)} \leq \left( \frac{\Lambda}{M} \right)^s \| f \|_{L^p(R)} \leq C \frac{M}{M^\sigma} \Lambda^s \| f \|_{L^p(R)},
\] (5.11)
with $0 < \sigma \leq s$. So we have established the (5.5).

Now we need to estimate the leading terms

$$ A_{M,\Lambda}(f)(x) = \int_{\mathbb{R}} K_M(x, y) f(y) dy, \quad (5.12) $$

caracterezied in (2.14). We start with the study of the kernel

$$ K_M(x, y) = \mathbf{1}_{x > 0} \mathbf{1}_{y > 0} \int e^{-i\tau (x-y)} \varphi \left( \frac{\tau}{M} \right) d\tau. \quad (5.13) $$

At first we look for the kernel $\tilde{K}_{M,\Lambda}(x, y)$, such that

$$ A_{M,\Lambda}(f)(x) = \int \tilde{K}_{M,\Lambda}(x, y) f(y) dy $$

and then we will find suitable bounds for $|\tilde{K}_{M,\Lambda}(x, y)|$ in order to estimate $\|A_{M,\Lambda}(f)\|_{L^p(\mathbb{R})}$. We can neglect the characteristic function $\mathbf{1}_{x > 0}$. On the other side, the presence of $\mathbf{1}_{y > 0}$ and the integration in $dy$ imply that

$$ A_{M,\Lambda}(f)(x) = \int \int e^{-i\tau x} e^{-i\xi \varphi \left( \frac{\tau}{M} \right) \left( \frac{\xi}{\Lambda} \right)} \tilde{f}(\xi) d\xi d\tau. $$

We note that $\tau - \xi \neq 0$ since we are considering the case $\Lambda < M/4$.

By the definition of Fourier transform $\hat{f}(\xi) = \int e^{-iy\xi} f(y) dy$, we get the expression of the kernel

$$ \hat{K}_{M,\Lambda}(x, y) = \int \int e^{-i\tau x} e^{-i\xi \varphi \left( \frac{\tau}{M} \right) \left( \frac{\xi}{\Lambda} \right)} \tilde{f}(\xi) d\xi d\tau. \quad (5.14) $$

Operating the change of variables $\tau \mapsto M\tau$ and $\xi \mapsto \Lambda \xi$ we obtain

$$ \tilde{K}_{M,\Lambda}(x, y) = MA \int \int e^{-i\tau x} e^{-i\Lambda \xi \varphi \left( \tau \right) \varphi \left( \xi \right) \left( \frac{1}{M\tau - \Lambda \xi} \right)} d\xi d\tau. $$

Integrating two times by parts in $\tau$ and then in $\xi$ we find the following estimate

$$ |\tilde{K}_{M,\Lambda}(x, y)| \leq \frac{MA}{(M\tau)^2 (\Lambda \xi)^2} \int \left| \partial^2_{\xi} \partial^2_{\tau} \varphi \left( \tau \right) \varphi \left( \xi \right) \left( \frac{1}{M\tau - \Lambda \xi} \right) \right| d\xi d\tau \leq C \frac{AM}{(M\tau)^2 (\Lambda \xi)^2} \max (M, \Lambda). \quad (5.15) $$

Now we can apply Hölder inequality to get

$$ \|A_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \leq \frac{\Lambda}{M^{1/p} \Lambda^{1-1/p}} \|f\|_{L^p(\mathbb{R})}. \quad (5.16) $$

We can proceed similarly for the kernels

$$ K_M(x, y) = \mathbf{1}_{x < 0} \mathbf{1}_{y > 0} \int e^{-i\tau (x-y)} \varphi \left( \frac{\tau}{M} \right) T(\tau) d\tau $$

$$ K_M(x, y) = \mathbf{1}_{x > 0} \mathbf{1}_{y < 0} \int e^{-i\tau (|x|+|y|)} \varphi \left( \frac{\tau}{M} \right) (R_{x} (\tau) + 1) d\tau. $$

using the assumption $T(\tau) \sim \tau$, $(R_{x} (\tau) + 1) \sim \tau$ near $\tau = 0$ and fractional integration by parts.
Indeed, from the Theorem 2.3 in [11] we have that $T(\tau)$ is $C^1(\mathbb{R})$ and $R_\pm(\tau) \in C^{0,\alpha}(\mathbb{R})$ with $\alpha < \gamma - 1$. Applying $\alpha$ integration by parts we have that

$$|\tilde{K}_{M,\Lambda}(x, y)| \leq C \frac{\Lambda M}{(Mx)^\alpha (My)^\alpha} \frac{1}{\max(M, \Lambda)}$$

and

$$\|A_{M,\Lambda}(f)\|_{L^p(\mathbb{R})} \leq C \frac{\Lambda}{M^{1/p} \Lambda^{1-1/p}} \|f\|_{L^p(\mathbb{R})}$$

where we have chosen $\alpha > 1/p$ thanks to the hypothesis $\gamma > 1 + 1/p$.

In conclusion the estimate (5.5) is checked and it holds whenever $0 < \Lambda \leq M \leq 1$. \qed

Our proof of the equivalence of the high energy part of the homogeneous Besov norms (1.6) for the perturbed Hamiltonian and the corresponding unperturbed homogeneous Besov norms is based also on the estimate of the operator

$$\varphi \left( \frac{\sqrt{H}}{M} \right) \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right).$$

More precisely, we have the following estimates.

**Lemma 5.2.** Assume that $V \in L^1_\gamma(\mathbb{R})$, $\gamma > 1 + 1/p$, the operator $\mathcal{H}$ has no point spectrum and resonance at zero. Then for any even function $\varphi(\tau) \in C^\infty(\mathbb{R} \setminus 0)$ there exists a constant $C = C(\|V\|_{L^1_\gamma(\mathbb{R})})$ so that for any pair of real positive numbers $\Lambda, M$ and for any $f \in S(\mathbb{R})$, the following inequalities hold:

$$\left\| \varphi \left( \frac{\sqrt{H}}{M} \right) \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right) f \right\|_{L^p_\gamma(\mathbb{R})} \leq C \left( \frac{\Lambda}{M} \right)^{1/p} \|f\|_{L^p_\gamma(\mathbb{R})}, \forall \ 0 < \Lambda \leq M, \ M \geq 1$$

(5.19)

and

$$\left\| \varphi \left( \frac{\sqrt{H}}{M} \right) \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right) f \right\|_{L^p_\gamma(\mathbb{R})} \leq C \left( \frac{M}{\Lambda} \right)^{1/p} \|f\|_{L^p_\gamma(\mathbb{R})}, \forall \ \Lambda \geq M, \ M \geq 1,$$

(5.20)

with $1 < p < \infty$.

**Proof.** We shall prove first (5.19). Take $p \in (1, \infty)$ and $f, g \in S(\mathbb{R})$. Set

$$f_\Lambda(x) = \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right) f.$$

We use the relation

$$\varphi \left( \frac{\sqrt{H}}{M} \right) f_\Lambda = M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \mathcal{H}_0^{s/2} f_\Lambda + M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \left( \mathcal{H}^{s/2} - \mathcal{H}_0^{s/2} \right) f_\Lambda,$$

where $s > 0$ will be chosen later on and

$$\varphi_1(\tau) = \varphi(\tau) \tau^{-s}.$$

Hence we have the representation formula

$$\varphi \left( \frac{\sqrt{H}}{M} \right) f_\Lambda = M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \left( \mathcal{H}_0^{s/2} f_\Lambda + G_{M,\Lambda}(f) \right),$$

(5.22)

where

$$G_{M,\Lambda} = M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \left( \mathcal{H}^{s/2} - \mathcal{H}_0^{s/2} \right) \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right).$$

(5.23)

By (5.3), we can write

$$\left\| M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \mathcal{H}_0^{s/2} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq \frac{C}{M^s} \left\| \mathcal{H}_0^{s/2} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq \frac{C\Lambda^s}{M^s} \|f\|_{L^p(\mathbb{R})},$$

(5.23)
so we have the estimate
\[
\left\| M^{-s} \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \mathcal{H}_0^{s/2} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq \frac{CA^s}{M^s} \| f \|_{L^p(\mathbb{R})}.
\] (5.24)

The operator \( \mathcal{H}^{s/2} - \mathcal{H}_0^{s/2} \), entering in the right side of (5.23) can be substituted by
\[
C \int_0^\infty \lambda^{-1+s/2} \left[ \mathcal{H}(\lambda + \mathcal{H})^{-1} - \mathcal{H}_0(\lambda + \mathcal{H}_0)^{-1} \right] d\lambda =
\]
\[
= C \int_0^\infty \lambda^{s/2}(\lambda + \mathcal{H})^{-1}V(\lambda + \mathcal{H}_0)^{-1} d\lambda
\]
due to (5.2). Hence
\[
\| G_{M, \Lambda}(f) \|_{L^p(\mathbb{R})} \leq \frac{C}{M^s} \int_0^\infty \lambda^{s/2} h(\lambda, \Lambda, M) d\lambda,
\] (5.25)
where
\[
h(\lambda, \Lambda, M) = \left\| \varphi_1 \left( \frac{\sqrt{H}}{M} \right) (\lambda + \mathcal{H})^{-1}V(\lambda + \mathcal{H}_0)^{-1} f_\Lambda \right\|_{L^p(\mathbb{R})}.
\]

By (5.4) and the standard estimate
\[
\left\| \varphi \left( \frac{\sqrt{H_0}}{\Lambda} \right) (\lambda + \mathcal{H}_0)^{-1} f \right\|_{L^p(\mathbb{R})} \leq \frac{CA^{1/p-1/2}}{\lambda + \Lambda^2} \| f \|_{L^p(\mathbb{R})},
\] (5.26)
we can write
\[
h(\lambda, \Lambda, M) = \left\| \varphi_1 \left( \frac{\sqrt{H}}{M} \right) (\lambda + \mathcal{H})^{-1}V(\lambda + \mathcal{H}_0)^{-1} f_\Lambda \right\|_{L^p(\mathbb{R})} \leq
\]
\[
\leq \frac{CM^{1-1/p}}{\lambda + M^2} \| V(\lambda + \mathcal{H}_0)^{-1} f_\Lambda \|_{L^1(\mathbb{R})} \leq
\]
\[
\leq \frac{CM^{1-1/p}}{\lambda + M^2} \| (\lambda + \mathcal{H}_0)^{-1} f_\Lambda \|_{L^\infty(\mathbb{R})} \leq \frac{CM^{1-1/p} \Lambda^{1/p}}{(\lambda + M^2)(\lambda + \Lambda^2)} \| f \|_{L^p(\mathbb{R})}
\]
so we derive from (5.26) the inequality
\[
\| G_{M, \Lambda}(f) \|_{L^p(\mathbb{R})} \leq CM^{1-1/p-s} \Lambda^{1/p} \int_0^\infty \frac{\lambda^{s/2} d\lambda}{(\lambda + M^2)(\lambda + \Lambda^2)} \| f \|_{L^p(\mathbb{R})}.
\] (5.27)

Now we can use the inequalities
\[
\int_0^\infty \frac{\lambda^{s/2} d\lambda}{(\lambda + M^2)(\lambda + \Lambda^2)} \leq \int_0^\infty \frac{\lambda^{s/2-1} d\lambda}{(\lambda + M^2)} = CM^{s-2}
\] (5.28)
and via (5.27) we find
\[
\| G_{M, \Lambda}(f) \|_{L^p(\mathbb{R})} \leq CM^{1-1/p-2s} \Lambda^{1/p} \| f \|_{L^p(\mathbb{R})}.
\] (5.29)

From this estimate, \( M \geq 1 \), the identity (5.22) and the inequality (5.24), we see that taking \( s = 1/p \), we obtain
\[
\| \varphi \left( \frac{\sqrt{H}}{M} \right) f_\Lambda \|_{L^p(\mathbb{R})} \leq \frac{CA^{1/p}}{M^{1/p}} \| f_\Lambda \|_{L^p(\mathbb{R})} + \frac{CA^{1/p}}{M^{1+1/p}} \| f \|_{L^p(\mathbb{R})} \leq \frac{CA^{1/p}}{M^{1/p}} \| f \|_{L^p(\mathbb{R})}.
\]

Similarly we can prove the case \( \Lambda \geq M \), \( M \geq 1 \). Indeed we can use the relation
\[
\varphi \left( \frac{\sqrt{H}}{M} \right) f_\Lambda = M^s \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \mathcal{H}_0^{-s/2} f_\Lambda + M^s \varphi_1 \left( \frac{\sqrt{H}}{M} \right) \left( \mathcal{H}^{-s/2} - \mathcal{H}_0^{-s/2} \right) f_\Lambda,
\]
where \( \varphi_1(\tau) = \varphi(\tau) \tau^s \).

We can write \( \mathcal{H}^{-s/2} - \mathcal{H}_{0}^{-s/2} \) and \( \mathcal{H}_{0}^{-s/2} \) via (5.1). Then operating computations similar to the previous case and using \( M \geq 1 \), we get

\[
\| \varphi \left( \frac{\sqrt{\mathcal{H}}}{M} \right) f \|_{L^p(\mathbb{R})} \leq C \frac{M^s}{\Lambda^s} \| f \|_{L^p(\mathbb{R})},
\]

for any \( s \in (0, 1) \). In particular it holds for \( s = 1/p \).

**Corollary 5.3.** Assume that \( V \in L^1(\mathbb{R}) \) with

\[ \gamma > 1 + 1/p, \quad 0 < s < \frac{1}{p}, \quad 1 < p < \infty, \]

and assume that the operator \( \mathcal{H} \) has no point spectrum and resonance at zero. Then for any even function \( \varphi(\tau) \in C^\infty(\mathbb{R} \setminus 0) \) there exists a constant \( C = C(\| V \|_{L^1(\mathbb{R})}) \) so that for any pair of real positive numbers \( \Lambda, M \) such that \( 0 < \Lambda \leq M, \quad M \leq 1 \) and for any \( f \in S(\mathbb{R}) \), the following inequality holds:

\[
\| \varphi \left( \frac{\sqrt{\mathcal{H}_{0}}}{M} \right) \varphi \left( \frac{\sqrt{\mathcal{H}}}{\Lambda} \right) f \|_{L^p(\mathbb{R})} \leq C \frac{\Lambda^s}{M^s} \| f \|_{L^p(\mathbb{R})},
\]

(5.30)

**Proof.** By Lemma 2.2 we have that

\[
\varphi \left( \frac{\sqrt{\mathcal{H}}}{\Lambda} \right) = K_\Lambda + \text{Rem}_\Lambda,
\]

where the kernel \( K_\Lambda(x, y) \) is defined in (2.14) and the kernel of the remainder \( \text{Rem}_\Lambda(x, y) \) satisfies the estimate (2.13).

We first estimate the remainder term. By (2.13) we have

\[
\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) \text{Rem}_\Lambda f \|_{L^p(\mathbb{R})} \leq C A \int_\mathbb{R} \left( \sum_{\pm} \frac{1}{\left( \Lambda(x \pm y) \right)^\sigma} \left( \frac{1}{\langle x \rangle^{\gamma-\sigma}} + \frac{1}{\langle y \rangle^{\gamma-\sigma}} \right) \right) |f(y)| dy \|_{L^p(\mathbb{R})},
\]

where \( \sigma \in (0, 1) \cap (0, \gamma - 1) \) will be choose small enough. Applying Hölder and Young inequalities in Lorentz spaces with the following index relation

\[
\frac{1}{p} + 1 = \sigma + (1 - \sigma) + \frac{1}{p},
\]

we get

\[
\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) \text{Rem}_\Lambda f \|_{L^p(\mathbb{R})} \leq C \frac{\Lambda}{\Lambda^s} \| f \|_{L^p(\mathbb{R})} \leq C \frac{\Lambda^s}{M^s} \| f \|_{L^p(\mathbb{R})}.
\]

Now we turn to estimate the leading term. We consider

\[
K_\Lambda(x, y) = c_1 \mathbb{1}_{x > 0} \mathbb{1}_{y > 0} \int_\mathbb{R} e^{-i\tau(x-y)} \varphi \left( \frac{\tau}{\Lambda} \right) d\tau
\]

since we can proceed similarly for the other terms defined in (2.14).

We look for the kernel \( K_{M, \Lambda}(x, y) \) such that

\[
\varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) \left( \int_\mathbb{R} K_\Lambda(\cdot, y) f(y) dy \right)(x) = \int_\mathbb{R} K_{M, \Lambda}(x, y) f(y) dy,
\]
We put
\[ h(x) = \int dy \int d\tau \, e^{-i(x-y)\tau} \varphi \left( \frac{\tau}{\Lambda} \right) f(y) 1_{y>0} \]
and
\[ g(x) = c 1_{x>0} h(x). \]
Using the notation above we have that
\[ \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) g(x) = c \int e^{i\xi \varphi \left( \frac{\xi}{M} \right)} \hat{g}(\xi) \, d\xi \]
and
\[ \tilde{h}(\eta) = c \varphi \left( \frac{\eta}{\Lambda} \right) \int dy e^{i\eta y} f(y) 1_{y>0}. \]
Hence we deduce
\[ \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) g(x) = \int dy \int d\xi \int d\eta e^{i\xi \varphi \left( \frac{\xi}{M} \right)} \varphi \left( \frac{\eta}{\Lambda} \right) \frac{1}{\xi - \eta} f(y) 1_{y>0}. \]

Then, as in (5.11), integrating by parts we get
\[ \left| K_{M,\Lambda}(x, y) \right| \leq C \frac{MA}{(M\gamma)^2} \frac{1}{\max(M, \Lambda)}. \]
We note that we are considering the \( 0 < \Lambda < M \leq 1 \). Then, using Hölder inequality combined with a scaling argument we get
\[ \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) g \right\|_{L^p(\mathbb{R})} \leq C \frac{1}{M^{1/p} \Lambda^{1-1/p}} \| f \|_{L^p(\mathbb{R})}. \]

**Corollary 5.4.** Assume that \( V \in L^1_0(\mathbb{R}), \) \( \gamma > 1 + 1/p, \) the operator \( \mathcal{H} \) has no point spectrum and resonance at zero. Then for any even function \( \varphi(\tau) \in C^\infty_0(\mathbb{R}) \setminus 0 \) there exists a constant \( C = C(\| V \|_{L^1_0(\mathbb{R})}) \) so that for any pair of real positive numbers \( \Lambda, M \) and for any \( f \in \mathcal{S}(\mathbb{R}) \), the following inequalities hold:
\[ \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) \varphi \left( \frac{\sqrt{\mathcal{H}}}{\Lambda} \right) f \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{\Lambda}{M} \right)^{1/p} \| f \|_{L^p(\mathbb{R})}, \quad \forall \ 0 < \Lambda \leq M, \ M \geq 1 \]  
(5.31)
and
\[ \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{M} \right) \varphi \left( \frac{\sqrt{\mathcal{H}}}{\Lambda} \right) f \right\|_{L^p(\mathbb{R})} \leq C \left( \frac{\Lambda}{M} \right)^{1/p} \| f \|_{L^p(\mathbb{R})}, \quad \forall \ \Lambda \geq M, \ M \geq 1, \]  
(5.32)
with \( 1 < p < \infty \).

**Proof.** The proof of the inequalities (5.31) and (5.32) follows repeating the same arguments of Lemma 5.2 and replacing \( \mathcal{H} \) with \( \mathcal{H}_0 \) and vice versa. 

Now we can turn to the following.

**Proof of Theorem 2.** We have to prove the equivalence of the norms in (5.25) and (5.26), i.e.
\[ \sum_{k=-\infty}^{\infty} 2^{2ks} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^k} \right) f \right\|_{L^p(\mathbb{R})}^2 \sim \sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}^2. \]
(5.33)

We set
\[ a_k = \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^k} \right) f \right\|_{L^p(\mathbb{R})}, \quad b_j = \left\| \varphi \left( \frac{\sqrt{\mathcal{H}_0}}{2^j} \right) f \right\|_{L^p(\mathbb{R})}. \]
Using the Paley-Littlewood partition

\[ f = \sum_{j=-\infty}^{\infty} f_j = \sum_{j=-\infty}^{\infty} \varphi \left( \frac{\sqrt{H_0}}{2^j} \right) f, \]

we take \( \psi(\tau) \in C_0^\infty(\mathbb{R}_+) \) such that \( \psi(\tau) = 1 \) on the support of \( \varphi \). Then we can use the identity

\[ \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) f = \sum_{j=-\infty}^{\infty} \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) \psi \left( \frac{\sqrt{H_0}}{2^j} \right) f_j. \] (5.34)

We distinguish the two cases \( k \geq 0 \) and \( k < 0 \).

Let \( k \geq 0 \) be fixed. We can apply Lemma 5.2 and we obtain that

\[ a_k = \left\| \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) f \right\|_{L^p(\mathbb{R})} \leq C \sum_{j=-\infty}^{\infty} 2^{-|k-j|(1/p)} \| f_j \|_{L^p(\mathbb{R})} = C \sum_{j=-\infty}^{\infty} 2^{-|k-j|(1/p)} b_j. \]

From this we deduce that

\[ \| 2^{ks} a_k \|_{\ell^2_{k \geq 0}} \leq C \| 2^{js} b_j \|_{\ell^2_j(\mathbb{Z})}. \] (5.35)

Indeed we have

\[ \| 2^{ks} a_k \|_{\ell^2_{k \geq 0}(\mathbb{Z})} \leq C \left\| \sum_{j \in \mathbb{Z}} 2^{-|j-k|(1/p)} 2^{-(j-k)s} 2^{js} f_j \right\|_{L^p(\mathbb{R})} \left\| \ell^2_{k \geq 0} \right\|. \] (5.36)

Using the discrete Young inequality combined with

\[ \| 2^{-|n(1/p)-ns|} \|_{\ell^1_{n}(\mathbb{Z})} \leq C, \] (5.37)

with \( 0 < s < 1/p \), we get the inequality (5.39).

Let \( k < 0 \) be fixed. Then we write

\[ 2^{ks} a_k \leq 2^{ks} \sum_{j \leq k} \left\| \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) \psi \left( \frac{\sqrt{H_0}}{2^j} \right) f_j \right\|_{L^p(\mathbb{R})} + 2^{ks} \sum_{j \geq k} \left\| \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) \psi \left( \frac{\sqrt{H_0}}{2^j} \right) f_j \right\|_{L^p(\mathbb{R})}. \]

Now we estimate the \( \ell^2_{k \leq 0} \) norm of the two addends above.

We can estimate the first addend as in the case \( k > 0 \) using the inequality (5.5) and the index \( s' \) such that \( 0 < s < s' < 1/p \). Then we can proceed as in (5.39), (5.37) replacing \( 1/p \) with \( s' \).

For the second addend the estimate is simpler. Indeed, using (5.3) we have

\[ 2^{ks} \sum_{j \geq k} \left\| \varphi \left( \frac{\sqrt{H_0}}{2^k} \right) \psi \left( \frac{\sqrt{H_0}}{2^j} \right) f_j \right\|_{L^p(\mathbb{R})} \leq C \sum_{j \geq k} 2^{ks} 2^{-j s} 2^{js} \| f_j \|_{L^p(\mathbb{R})}. \] (5.38)

Since we are considering the case \( j \geq k \) and \( k < 0 \), we can estimate the right side above with the sum

\[ C \sum_{n \in \mathbb{Z}} 2^{-|k-j|s} 2^{js} \| f_j \|_{L^p(\mathbb{R})}. \]

Now, computing the \( \ell^2_k \) norm and applying the discrete Young inequality we complete the proof of the estimate

\[ \| 2^{ks} a_k \|_{\ell^2_k(\mathbb{Z})} \leq C \| 2^{js} b_j \|_{\ell^2_j(\mathbb{Z})}. \] (5.39)

To prove that

\[ \| 2^{js} b_j \|_{\ell^2_j(\mathbb{Z})} \leq C \| 2^{ks} a_k \|_{\ell^2_k(\mathbb{Z})}, \] (5.40)
we use Corollary 5.3 and Corollary 5.4. Indeed, if we write
\[ \varphi \left( \frac{\sqrt{H_0}}{2^j} \right) f = \sum_{k=-\infty}^{\infty} \varphi \left( \frac{\sqrt{H_0}}{2^j} \right) \varphi \left( \frac{\sqrt{H}}{2^k} \right) f_k, \]

where
\[ f_k = \varphi \left( \frac{\sqrt{H_V}}{2^k} \right) f, \]
as before we can distinguish the case \( j \geq 0 \) and \( j < 0 \).

Computations similar to the ones used to prove (5.39) conclude the proof.

6 Estimates for the modified Jost functions

In this section we recall some classical results concerning the spectral decomposition of the perturbed Hamiltonian. Recall that the Jost functions are solutions
\[ f_\pm(x, \tau) = e^{\pm i \tau x} m_\pm(x, \tau) \] of \( Hu = \tau^2 u \) with
\[ \lim_{x \to +\infty} m_+(x, \tau) = 1 = \lim_{x \to -\infty} m_-(x, \tau). \]

We set \( x_+ := \max\{0, x\}, \quad x_- := \max\{0, -x\} \).

The estimate and the asymptotic expansions of \( m_\pm(x, \tau) \) are based on the following integral equations
\[ m_\pm(x, \tau) = 1 + K^{(\tau)}_\pm(m_\pm(\cdot, \tau))(x), \quad (6.1) \]
where \( K^{(\tau)}_\pm \) is the integral operator defined as follows
\[ K^{(\tau)}_\pm(f)(x) = \pm \int_x^{\pm\infty} D(\pm(t - x), \tau)V(t)f(t)dt \]
and
\[ D(t, \tau) = \frac{e^{2it\tau} - 1}{2i\tau} = \int_0^t e^{2iy\tau} dy; \quad (6.2) \]

The following lemma is well known.

**Lemma 6.1.** (see Lemma 1 p. 130 [4]) Assume \( V \in L^1_\gamma(\mathbb{R}) \). Then we have the properties:

a) for any \( x \in \mathbb{R} \) the function
\[ \tau \in \mathbb{C}_\pm \mapsto m_\pm(x, \tau), \quad \mathbb{C}_\pm = \{ \tau \in \mathbb{C}; \text{Im} \tau \geq 0 \} \]
is analytic in \( \mathbb{C}_\pm \) and \( C^1(\mathbb{C}_\pm) \);

b) there exist constants \( C_1 \) and \( C_2 > 0 \) such that for any \( x, \tau \in \mathbb{R} \):
\[ |m_\pm(x, \tau) - 1| \leq C_1 \langle x_\pm \rangle^{-1} \langle \tau \rangle^{-1}; \quad \text{(6.4)} \]
\[ |\partial_\tau m_\pm(x, \tau)| \leq C_2 \langle x \rangle^2. \quad \text{(6.5)} \]

A slight improvement is given in the next Lemma.

**Lemma 6.2.** Suppose \( V \in L^1_\gamma(\mathbb{R}) \) with \( \gamma \geq 1 \). Then we have the following properties:

a) There exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R}, \tau \in \mathbb{C}_\pm \), we have
\[ |m_\pm(x, \tau) - 1| \leq C \frac{\langle x_\pm \rangle}{\langle x_\pm \rangle^\gamma - 1}; \quad \text{(6.6)} \]
\[ |m_\pm (x, \tau) - 1| \leq C \frac{\langle x_\pm \rangle - \langle x_\pm \rangle}{\langle x_\pm \rangle^{\gamma - 1}}; \quad (6.7) \]

c) Let \( \sigma \in [0, 1) \). Then there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) we have
\[ \|m_\pm (x, \tau) - 1\|_{C^0, \sigma(\mathbb{R}^+)} \leq C \frac{\langle x_\pm \rangle^{1+\sigma}}{\langle x_\pm \rangle^{\gamma - 1 - \sigma}}, \gamma > 1, 0 \leq \sigma \leq \gamma - 1; \quad (6.8) \]

d) Let \( \sigma \in [0, 1) \). Then there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) we have
\[ \|m_\pm (x, \tau) - 1\|_{C^0, \sigma(\mathbb{R}^+)} \leq C \frac{\langle x_\pm \rangle^{1+\sigma}}{\langle x_\pm \rangle^{\gamma - 1}}, \gamma > 1. \quad (6.9) \]

Proof. We can fix for determinacy the sign + in the left sides of the inequalities \( (6.6) - (6.9) \), since the arguments are similar for the term \( m_- \).

We start proving the \((6.6)\). The right side of \((6.6)\) suggests to consider the quantity
\[ v(x, \tau) = \frac{\langle x_+ \rangle^{\gamma - 1}}{\langle x_- \rangle} |m_+ (x, \tau) - 1|. \]

We plan to use the integral equation for \( m_+(x, \tau) \) and to check inequality of type
\[ v(x) \leq a(x) + \int_{\tau}^{\infty} b(t) v(t) dt, \quad (6.10) \]

where \( b \in L^1(\mathbb{R}) \). Applying for \( v(x) \) a Gronwall type inequality (see Lemma \[7.1\] for the precise statement), we can derive apriori bound \( v(x) \leq C(a(x), \|b\|_{L^1(\mathbb{R})}) \).

The relations
\[ m_+ (x, \tau) - 1 = \int_{\tau}^{\infty} D(t - x, \tau) V(t) m_+(t, \tau) dt, \quad (6.11) \]

\[ |D(t - x, \tau)| \leq C |t - x| \leq C(t + \langle x_- \rangle), \]

imply the following estimate
\[ v(x, \tau) \leq C \int_{\tau}^{\infty} \frac{\langle x_+ \rangle^{\gamma - 1}}{\langle x_- \rangle^{\gamma}} \frac{(t - x)}{\langle t \rangle} \langle t \rangle^{\gamma} V(t) (|m_+(t, \tau) - 1| + 1) dt \tau. \]

We set\(^2\)
\[ c_1 = \sup_{t \geq x} \frac{\langle x_+ \rangle^{\gamma - 1} \langle t - x \rangle \langle t_- \rangle}{\langle x_- \rangle \langle t \rangle \langle t \rangle^{\gamma}} \in \mathbb{R}_+, \quad \gamma \geq 1, \]
\[ c_2 = \sup_{t \geq x} \frac{\langle x_+ \rangle^{\gamma - 1} \langle t - x \rangle}{\langle x_- \rangle \langle t \rangle^{\gamma}} \in \mathbb{R}_+, \quad \gamma \geq 1 \]

and we deduce that
\[ v(x, \tau) \leq c_1 \int_{\tau}^{\infty} \langle t \rangle^{\gamma} |V(t)| v(t, \tau) dt + c_2 \|V\|_{L^1(\mathbb{R})}. \]

Now applying the Gronwall argument of Lemma \[7.1\] mentioned above, we find the \((6.6)\).

We follow the same idea to prove the other inequalities.

Indeed, to get the \((6.7)\) we define
\[ u(x, \tau) = |\tau \frac{\langle x_+ \rangle^{\gamma}}{\langle x_- \rangle} |m_+ (x, \tau) - 1|, \quad (6.12) \]

\(^2\)To prove that the quantity above are finite, we consider three different cases: \( x < t < 0, 0 < x < t, x < 0 < t \) separately. In the last case, we distinguish the behaviour if \( x \approx t, |x| << |t| \) and \( |t| >> |x| \).
This time we quote the estimates
\[ |D(t - x, \tau)| \leq C \min \left( (t - x), \frac{1}{\tau} \right). \]  
(6.13)

Hence, by the integral equation (6.11) and the estimates above follows
\[
u(x, \tau) \leq \int_x^{+\infty} \frac{(x^+)^\gamma (t^-)}{(x^-)^\gamma (t^+)^\gamma} |D(t - x, \tau)||V(t)||u(t, \tau)\,d\tau \\
+ \int_x^{+\infty} \frac{(x^+)^\gamma}{(x^-)^\gamma} |\tau||D(t - x, \tau)||V(t)|\,d\tau.
\]

As before\(^3\) we can set
\[ c_1 = \sup_{t \geq x} \frac{(x^+)^\gamma (t^-)}{(x^-)^\gamma (t^+)^\gamma} \in \mathbb{R}_+, \gamma \geq 1, \]
\[ c_2 = \sup_{t \geq x} \frac{(x^+)^\gamma}{(x^-)^\gamma} \in \mathbb{R}_+, \gamma \geq 1 \]
and via Gronwall argument we get \[ u(x, \tau) \leq C\|V\|_{L^\infty(\mathbb{R})}, \] i.e. (6.7).

Similarly to get the (6.8) we put
\[ g^\sigma(x, \tau_1, \tau_2) = \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^1}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} \frac{|m^+(x, \tau_1) - m^+(x, \tau_2)|}{|\tau_1 - \tau_2|^\sigma} \]
and by the estimate
\[ \frac{|D(t, \tau_1) - D(t, \tau_2)|}{|\tau_1 - \tau_2|^\sigma} \leq C(t - x)^{1+\sigma} \leq C((t)^{1+\sigma} + (x^-)^{1+\sigma}), \sigma \in (0, 1) \]
we get
\[
g^\sigma(x, \tau_1, \tau_2) \leq \int_x^{+\infty} \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^{1+\sigma}}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} |V(t)||m^+(t, \tau_1)|\,dt + \\
+ \int_x^{+\infty} \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^{1+\sigma}}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} |V(t)|g^\sigma(t, \tau_1, \tau_2)\,dt.
\]

Moreover, we can estimate \[ |m^+(t, \tau_1)| \] with (6.6).
If we consider \[ 1 < \gamma < 2 \] and \[ \sigma \leq \gamma - 1 \] or \[ \gamma \geq 2 \] and \[ \sigma \in (0, 1) \], we have that the following quantities are finite\(^4\)
\[ c_1 = \sup_{t \geq x} \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^{1+\sigma}}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} \in \mathbb{R}_+, \]
\[ c_2 = \sup_{t \geq x} \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^{1+\sigma}}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} \in \mathbb{R}_+, \]
\[ c_3 = \sup_{t \geq x} \frac{(x^+)^{\gamma - 1 - \sigma} (t^-)^{1+\sigma} (t^+)^{\gamma - 1 - \sigma}}{(x^-)^{\gamma - 1 + \sigma} (t^+)^{\gamma - 1 - \sigma}} \in \mathbb{R}_+. \]

Then we have \[ g^\sigma(x, \tau_1, \tau_2) \leq C\|V\|_{L^1(\mathbb{R})}. \]
Finally we prove the inequality (6.9) for any \[ \sigma \in (0, 1) \]. We rewrite (6.11) as
\[ \tau (m^+(x, \tau) - 1) = \tau \int_x^{+\infty} D(t - x, \tau)V(t)\,dt + \\
+ \int_x^{+\infty} \tau D(t - x, \tau)V(t) (m^+(t, \tau) - 1)\,dt. \]

\(^3\)One can see footnote 3 \(^4\)One can see the footnote 3
Setting now
\[ h^\sigma(x, \tau) = \frac{\langle x_+ \rangle^{\gamma - \sigma}}{(x_-)^{\sigma + 1}} \| \tau(m_+(x, \tau) - 1) \|_{C^{0,\sigma}(\mathbb{C}_+)} , \]
we can use the inequality
\[ \| fg \|_{C^{0,\sigma}} \leq C (\| f \|_{C^{0,\sigma}} \| g \|_{C^{0}} + \| f \|_{C^{0}} \| g \|_{C^{0,\sigma}}) \]
and arrive at the estimate
\[
\begin{align*}
& h^\sigma(x, \tau) \leq \int_x^\infty \frac{\langle x_+ \rangle^{\gamma - \sigma}}{(x_-)^{1+\sigma}} \| \tau D(t - x, \tau) \|_{C^{0,\sigma}(\mathbb{C}_+)} |V(t)| dt + \\
& \quad + \int_x^\infty \frac{\langle x_+ \rangle^{\gamma - \sigma}}{(x_-)^{1+\sigma}} \| \tau D(t - x, \tau) \|_{C^{0,\sigma}(\mathbb{C}_+)} |V(t)| \| m_+(t, \tau) - 1 \|_{C^{0}(\mathbb{C}_+)} dt + \\
& \quad + \frac{\langle x_+ \rangle^{\gamma - \sigma}}{(x_-)^{1+\sigma}} \int_x^\infty \| D(t - x, \tau) \|_{C^{0}(\mathbb{C}_+)} \frac{|V(t)| \| t_- \|^{\gamma + 1} h^\sigma(t, \tau) dt}{\| t_+ \|^{\gamma - \sigma}} \\
& \quad + \int_x^\infty \| D(t - x, \tau) \|_{C^{0}(\mathbb{C}_+)} |V(t)| dt .
\end{align*}
\]
We quote the inequalities
\[
\| t^{1-k} D(t - x, \tau) \|_{C^{0,\sigma}(\mathbb{C}_+)} \leq C \| t - x \|^{k+\sigma}, \quad k = 0, 1, \sigma \in [0, 1). \tag{6.15}
\]
For the term \( I(x) \) we use the estimate (6.15) with \( k = 0 \) and note that
\[ c_1 = \sup_{t \geq x} \frac{\langle x_+ \rangle^{\gamma - \sigma} \langle t - x \rangle^{\sigma}}{(x_-)^{1+\sigma} \langle t \rangle^\gamma} < \infty, \]
for \( 0 \leq \sigma \leq \gamma, \gamma \geq 1 \). Hence,
\[ I(x) \leq c_1 \| V \|_{L^1_\gamma(\mathbb{R})} . \]
In a similar way, for \( II(x) \) we use the estimate (6.15) with \( k = 0 \) combined with (6.16) and using the estimate
\[ c_2 = \sup_{t \geq x} \frac{\langle x_+ \rangle^{\gamma - \sigma} \langle t - x \rangle^{\sigma} \langle t_- \rangle^{\gamma + 1}}{(x_-)^{1+\sigma} \langle t \rangle^{\gamma + 1} \langle t_+ \rangle^{\gamma - 1}} \in \mathbb{R}_+, \]
for \( 0 \leq \sigma < 1, \gamma \geq 1 \), we arrive at
\[ II(x) \leq c_2 \| V \|_{L^1_\gamma(\mathbb{R})} . \]
Finally, for \( III(x) \) we use \( \| D(t - x, \tau) \|_{C^{0}(\mathbb{C}_+)} \leq C \| t - x \| \) and from
\[ c_3 = \sup_{t \geq x} \frac{\langle x_+ \rangle^{\gamma - \sigma} \langle t - x \rangle \langle t_- \rangle^{1+\sigma}}{(x_-)^{1+\sigma} \langle t \rangle^{\gamma + 1} \langle t_+ \rangle^{\gamma - 1}} < \infty, \]
we deduce
\[ III(x) \leq c_3 \int_x^\infty \langle t \rangle^\gamma |V(t)| h^\sigma(t, \tau) dt . \]
So, the application of the Gronwall argument implies \( h^\sigma(x, \tau) \leq C \) and the estimate (6.9) is established. This complete the proof.

Similarly, if we require more decay for the potential, we can get estimates also for the quantity \( \partial_t^k (m_+(x, \tau) - 1) \). In particular we have the following Lemma.

**Lemma 6.3.** Suppose \( V \in L^1_\gamma(\mathbb{R}) \) with \( \gamma \geq 1 \). Then we have the following properties:

\(^5\)the only case, when \( \sigma \leq \gamma \) is necessary is the case \( x < 0 < t, |x| \ll |t| \)
a) If \( \gamma \geq 2 \), then for any integer \( k \), \( 1 \leq k \leq \gamma - 1 \) the function in (6.3) is \( C^{k}(\mathbb{C}_{\pm}) \) and there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) and \( \tau \in (\mathbb{C}_{\pm}) \) we have
\[
|\partial^{k}_{x}(m_{\pm}(x, \tau) - 1)| \leq C\frac{(x_{\pm})^{1+k}}{(x_{\pm})^{\gamma-1-k}};
\]
(6.16)

b) For any integer \( k \), \( 1 \leq k \leq \gamma \), then the function in (6.3) is \( C^{k}(\mathbb{C}_{\pm}) \) and there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) and \( \tau \in (\mathbb{C}_{\pm} \setminus \{0\}) \) we have
\[
|\partial^{k}_{x}(m_{\pm}(x, \tau) - 1)| \leq C\frac{(x_{\pm})^{1+k}}{(x_{\pm})^{\gamma-k}};
\]
(6.17)

c) If \( \gamma > 2 \), then for any integer \( k \), \( 1 \leq k \leq \gamma - 1 \) and for any \( \sigma \in (0, 1) \) such that \( 0 \leq \sigma \leq \gamma - 1 - k \), there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) we have
\[
\|m_{\pm}(x, \tau) - 1\|_{C^{k,\sigma}(\mathbb{C}_{\pm})} \leq C\frac{(x_{\pm})^{1+k+\sigma}}{(x_{\pm})^{\gamma-1-k-\sigma}};
\]
(6.18)

d) If \( \gamma > 1 \), then for any integer \( k \), \( 1 \leq k \leq \gamma \) and for any \( \sigma \in (0, 1) \) such that \( 0 \leq \sigma \leq \gamma - k \), there exists a constant \( C > 0 \) such that for any \( x \in \mathbb{R} \) we have
\[
\|	au(m_{\pm}(x, \tau) - 1)\|_{C^{k,\sigma}(\mathbb{C}_{\pm})} \leq C\frac{(x_{\pm})^{1+k+\sigma}}{(x_{\pm})^{\gamma-k-\sigma}}.
\]
(6.19)

**Proof.** The proof of this Lemma follows the same spirit of the proof of the previous one.

We prove the inequality (6.16) fixing the sign + in the left side. The arguments are similar for the others inequalities and also for the terms \( m_{-} \).

The right side in (6.16) suggests us to define
\[
v^{(k)}(x) = \frac{(x_{+})^{\gamma-1-k}}{(x_{-})^{k+1}}|\partial^{k}_{x}(m_{+}(x, \tau) - 1)|.
\]

We intend to prove the (6.16), i.e.
\[
v^{(k)}(x) \leq C(\|V\|_{L^{1}_{\mathbb{C}}(\mathbb{R})}), \quad 0 \leq k \leq \gamma - 1,
\]
(6.20)

by induction in \( k \). Since the inequalities above for \( k = 0 \) is already established in (6.16), we suppose that the inequality (6.20) holds for any \( 0 \leq k \leq \gamma - 1 \) and so our goal will be to prove that
\[
v^{(k+1)}(x) \leq C, \quad k + 1 \leq \gamma - 1.
\]

The key tools here will be to consider the following formula
\[
\partial^{k+1}_{x}m_{+}(x, \tau) = \sum_{\ell=0}^{k+1} c_{k,\ell} \int_{x}^{\infty} \partial^{k+1-\ell}_{x}D(t-x, \tau)V(t)\partial^{\ell}_{x}m_{+}(t, \tau)dt
\]
(6.21)

and to quote the following inequalities
\[
|\partial^{k}_{x}D(t, x)| \leq C \min \left\{ \langle t \rangle^{k+1}, \frac{(\langle t \rangle)^{k}}{|\tau|} \right\}, \quad k = 0, 1, 2, \ldots, \tau \in \mathbb{C}_{+}.
\]

Then, from the boundness of the following quantities:
\[
c_{1} = \sup_{t \geq x} \frac{(x_{+})^{\gamma-2-k}(t-x)^{k+2}}{(x_{-})^{2+k}(t\gamma)} \in \mathbb{R}_{+},
\]

\[\text{To prove that the quantity above are finite, we consider three different cases: } x < t < 0, 0 < x < t, x < 0 < t \text{ separately. In the last case, we distinguish the behaviour if } x \approx t, |x| << |t| \text{ and } |t| << |x|.\]
The proof is based on the relations
\[ c_2 = \max_{0 \leq \ell \leq k+1} \sup_{t \geq x} \frac{(x_+)^{\gamma - 2 - k} (t - x)^{k+2 - \ell} (t_+)^{1 + \ell}}{(x_-)^{2+k} (t) \gamma^{\gamma - 1 - \ell}} \in \mathbb{R}_+ , \]
combined with a Gronwall argument we get (6.16).

We do not prove the inequalities (6.17), (6.18) and (6.19) to avoid the repetition of the same arguments. We just note that for the proof of the inequalities (6.18) and (6.19) we use also the following estimate
\[ \| x^{1-k} D(t-x,\tau) \|_{C^0,\sigma(C^+)} \leq C(t-x)^{k+\sigma}, \ k = 0, 1, \ \sigma \in [0,1). \]

\[
\square
\]

6.1 Expansions for transmission and reflection coefficients

The transmission coefficient \( T(\tau) \) and the reflection coefficients \( R_{\pm}(\tau) \) are defined by the formula
\[ T(\tau)m_{\pm}(x,\tau) = R_{\pm}(\tau)e^{-2i\tau x}m_{\pm}(x,\tau) + m_{\pm}(x,-\tau). \] (6.22)

From [4] and from [11] we have the following Lemma.

Lemma 6.4. We have the following properties of the transmissions and reflection coefficients.

1. \( T, R_{\pm} \in C(\mathbb{R}) \).
2. There exists \( C_1, C_2 > 0 \) such that:
   \[ |T(\tau) - 1| + |R_{\pm}(\tau)| \leq C_1|\tau|^{-1} \] (6.23)
   \[ |T(\tau)|^2 + |R_{\pm}(\tau)|^2 = 1. \] (6.24)

3. If \( T(0) = 0 \), (i.e. zero is not a resonance point), then for some \( \alpha \in \mathbb{C} \setminus \{0\} \) and for some \( \alpha_+, \alpha_- \in \mathbb{C} \)
   \[ T(\tau) = \alpha\tau + o(\tau), \ 1 + R_{\pm}(\tau) = \alpha_{\pm}\tau + o(\tau), \] (6.25)
   for \( \tau \in \mathbb{R}, \ \tau \to 0. \)

In particular, (6.23), (6.24) follow from Sect.3 [4] and (6.25) follows from Theorem 2.3 [11].

The property c) in the last Theorem suggest the following.

Definition 6.5. The origin is a resonance point for the hamiltonian \( \mathcal{H} \) if and only if
\[ T(0) \neq 0. \]

We can use the assumption \( V \in L_1^1(\mathbb{R}), \ \gamma \geq 1 \), to get some more precise bounds.

Lemma 6.6. Suppose \( V \in L_1^1(\mathbb{R}) \) with \( \gamma \geq 1 \) and \( T(0) = 0 \). Then for any integer \( k, \ 0 \leq k \leq \gamma - 1 \) we have:

1. \( T, R_{\pm} \in C^k(\mathbb{R}) \);
2. There exists \( C > 0 \) such that for any \( \tau \in \mathbb{R} \) we have:
   \[ \left| \frac{d^k}{d\tau^k} T(\tau) \right| + \left| \frac{d^k}{d\tau^k} R_{\pm}(\tau) \right| \leq C, \] (6.26)
   \[ \left| \frac{d^k}{d\tau^k} \left[ \tau (T(\tau) - 1) \right] \right| + \left| \frac{d^k}{d\tau^k} \left[ \tau R_{\pm}(\tau) \right] \right| \leq C. \] (6.27)

Proof. The proof is based on the relations
\[ \frac{\tau}{T(\tau)} = \tau - \frac{1}{2i} \int_{\mathbb{R}} V(t)m_+(t,\tau)dt, \ \tau \in \mathbb{R} \setminus \{0\}, \] (6.28)
\[ R_{\pm}(\tau) = \frac{T(\tau)}{2i\tau} \int_{\mathbb{R}} e^{\pm 2i\tau} V(t)m_{\pm}(t,\tau)dt, \ \tau \in \mathbb{R} \setminus \{0\} \] (6.29)
and the properties of the functions \( m_\pm(t, \tau) \) from Lemma 6.3. Indeed we can write

\[
\tau = T(\tau) \left( \tau - \frac{1}{2 \tau} \int_{\mathbb{R}} V(t)m_+(t, \tau) dt \right),
\]

(6.30)

and defer the boundness of the left sides in (6.26) and (6.27) from the relation above combined with the inequalities (6.16) and (6.17).

Remark 6.7. It is easy to see that

\[
\left| \frac{T(\tau)}{\tau} + \frac{R_\pm(\tau) + 1}{\tau} \right| \leq C \tag{6.31}
\]

and

\[
\left\| \frac{T(\tau)}{\tau} \right\|_{C^0,\sigma(\mathbb{R})} + \left\| \frac{R_\pm(\tau) + 1}{\tau} \right\|_{C^0,\sigma(\mathbb{R})} \leq C. \tag{6.32}
\]

Indeed (6.31) follows from relations (6.30), (6.29), using the inequality (6.6) and the property (6.25) in Lemma 6.4. Similarly (6.32) follows from relations (6.30), (6.29), using the inequality (6.8) and the property (6.25) in Lemma 6.4.

In the spirit of the Lemma before, we can establish the corresponding Hölder norm estimates for the transmission and the reflection coefficients.

Lemma 6.8. Suppose \( V \in L^1_\gamma(\mathbb{R}) \) with \( \gamma > 1 \) and \( T(0) = 0 \). Then for any integer \( k, 0 \leq k \leq \gamma - 1 \) and any \( \sigma \in (0, 1) \cap (0, \gamma - 1 - k) \) we have:

a) \( T, R_\pm \in C^{k,\sigma}(\mathbb{R}) \);

b) There exists \( C > 0 \) such that for any \( \tau \in \mathbb{R} \) we have:

\[
\left\| \frac{d^k}{d\tau^k} T(\tau) \right\|_{C^0,\sigma(\mathbb{R})} + \left\| \frac{d^k}{d\tau^k} R_\pm(\tau) \right\|_{C^0,\sigma(\mathbb{R})} \leq C, \tag{6.33}
\]

\[
\left\| \frac{d^k}{d\tau^k} [\tau (T(\tau) - 1)] \right\|_{C^0,\sigma(\mathbb{R})} + \left\| \frac{d^k}{d\tau^k} [\tau R_\pm(\tau)] \right\|_{C^0,\sigma(\mathbb{R})} \leq C. \tag{6.34}
\]

7 Appendix I: Gronwall’s lemma on the real line.

In this section we shall recall first some of modifications of the classical Gronwall’s inequality on \( \mathbb{R} \).

Lemma 7.1. If \( v(x), a(x), b(x) \) are continuous non negative functions on \( \mathbb{R} \), and for any real \( r \) we have

\[
a(x), v(x) \in L^\infty((r, \infty)), b(x) \in L^1((r, \infty)) \tag{7.1}
\]

that satisfy the inequality

\[
v(x) \leq a(x) + \int_x^\infty b(t)v(t)dt \tag{7.2}
\]

then we have

\[
v(x) \leq a(x) + \int_x^\infty a(t)b(t)exp \left( \int_x^t b(s)ds \right) dt. \tag{7.3}
\]

Proof. We shall sketch the proof for completeness. Set

\[
\varphi(x) = \int_x^\infty b(t)v(t)dt.
\]

The function is well-defined and \( C^1 \) due to the assumption (7.1). Then

\[
\varphi'(x) = -b(x)v(x) \geq -b(x)(\varphi(x) + a(x))
\]
and
\[
\left( e^{-B(x)} \varphi(x) \right)' \geq -e^{-B(x)} b(x) a(x)
\]
with \( B(x) = \int_x^\infty b(t) dt \). Integrating this inequality in the interval \((x, R)\), we get
\[
\varphi(x) \leq e^{B(x)-B(R)} \varphi(R) + \int_x^R e^{B(x)-B(t)} a(t) b(t) dt.
\]
Using again the assumption \((7.1)\), we see that
\[
\lim_{R \to \infty} B(R) = 0, \quad \lim_{R \to \infty} \varphi(R) = 0
\]
so we get
\[
\varphi(x) \leq \int_x^\infty e^{B(x)-B(t)} a(t) b(t) dt.
\]
Then \(\text{(7.2)}\) implies \(v(x) \leq a(x) + \varphi(x)\) and we arrive at \(\text{(7.3)}\). This completes the proof.

**Corollary 7.2.** If \(a(x)\) is a continuous \(L^\infty(\mathbb{R})\) function, such that
\[
a(x) = \begin{cases} 
C > 0, & \text{for } x \leq 0; \\
\text{decreasing positive function}, & \text{for } x > 0,
\end{cases}
\]
and \(b \in L^1(\mathbb{R})\), then the inequality \(\text{(7.2)}\) implies
\[
v(x) \leq Ca(x_+), \quad x_+ = \max(0, x).
\]

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