Dynamics on Leibniz manifolds

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Abstract

This paper shows that various relevant dynamical systems can be described as vector fields associated to smooth functions via a bracket that defines what we call a Leibniz structure. We show that gradient flows, some dissipative systems, and nonholonomically constrained simple mechanical systems, among other dynamical behaviors, can be described using this mathematical construction that generalizes the standard Poisson bracket currently used in Hamiltonian mechanics. The symmetries of these systems and the associated reduction procedures are described in detail. A number of examples illustrate the theoretical developments in the paper.

1 Introduction

It is well known that most of classical mechanics can be formulated using a Poisson structure (see for instance [AM78, LM87], and references therein). A Poisson structure on a manifold \( P \) is a bilinear map \( \{ \cdot, \cdot \} : C^\infty(P) \times C^\infty(P) \rightarrow C^\infty(P) \) that defines a Lie algebra structure on the algebra \( C^\infty(P) \) of smooth functions on that manifold and that is a derivation on each entry. This property allows the association of a vector field \( X_h \) to any smooth function \( h \in C^\infty(P) \), usually referred to as the Hamiltonian vector field of the Hamiltonian function \( h \). The properties of the bracket \( \{ \cdot, \cdot \} \) have important consequences on the dynamical features of the vector field \( X_h \). For instance, its antisymmetry implies that the Hamiltonian function \( h \) is a constant of the motion for \( X_h \). Additionally, the fact that \( \{ \cdot, \cdot \} \) satisfies the Jacobi identity implies that the flow of \( X_h \) is a Poisson map, that is, it respects the bracket.

It has been noticed in recent times that a weakening of the defining conditions of a Poisson system is sometimes necessary in order to accommodate the description of more general dynamical systems. A well known example is the use of brackets that do not satisfy the Jacobi identity, known as almost Poisson brackets (see Section 3), in the context of nonholonomically constrained mechanical systems. In this paper we go all the way in this direction and we work with a bracket, first introduced in [GraUrb], that is just required to be bilinear and a derivation on each of its entries. The derivation property, also known as the Leibniz rule, justifies why we refer to this structure as \textit{Leibniz bracket}. The properties exhibited by the dynamical systems defined in this way are in general very different from those presented by standard Hamiltonian systems since most of the features of those systems are based on the Lie algebraic properties of the bracket that we have chosen to drop. This construction should not be mistaken with the Leibniz structures (also called Loday algebras) introduced by Loday [Lod93] in the algebraic context.

The introduction of the notion of Leibniz system, whose elementary properties are presented in Section 2, is justified by the great variety of relevant systems whose natural underlying mathematical structure seems to be based in this kind of brackets. A number of these systems are described in Sections 3 and 4. To be more specific, in Section 3 we have identified the Leibniz structure inherent to a number of systems that can be found in the literature and Section 4 is devoted to the Leibniz formulation of simple mechanical systems subjected to nonholonomic constraints. One of the main differences between our treatment of this problem and other bracket formulations for nonholonomic systems is the fact that our bracket is defined in the entire phase space of the unconstrained system, unlike other approaches that provide a bracket only on the constraint submanifold. The Leibniz bracket that we construct in that section associates to the Hamiltonian of the unconstrained system a vector

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field that, when restricted to the constraint submanifold, coincides with the evolution vector field of the constrained system. As we point out in Section 4, the construction of this bracket involves the use of certain extensions that make it not to be uniquely determined by the dynamics that we want to describe. This leeway can be used in specific examples (see Section 4) to encode in a single bracket entire families of constraints, which may be very useful in the study of bifurcation problems in the nonholonomic context.

Section 5 contains a first approach to the study of the symmetries and the reduction of a Leibniz system. We introduce a notion of momentum map associated to symmetries that do not necessarily preserve the Leibniz bracket but that nevertheless produce non trivial conservation laws as long as the Hamiltonian function is invariant. We also formulate a theorem that spells out, under certain regularity assumptions, how the orbit space of a symmetry that does respect the Leibniz structure (in a weak sense that is introduced in the text) is again a Leibniz manifold. This result reproduces in the Leibniz context the well known result for Poisson manifolds. The analog of symplectic or Marsden–Weinstein reduction (see [MW74]) in this context is the subject of ongoing research and will be treated elsewhere.

Finally, Section 5 contains the generalization to the Leibniz context of the Poisson reduction results in [ACC91, MR86, OR98] that characterize the situations in which a new Leibniz structure can be obtained by restriction to a subset and projection to the orbit space of a (pseudo)group of symmetries or to the leaf space of a distribution.

## 2 Leibniz systems

**Definition 2.1** Let $P$ be a smooth manifold and let $C^\infty(P)$ be the ring of smooth functions on it. A **Leibniz bracket** on $P$ is a bilinear map $[\cdot,\cdot] : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ that is a derivation on each entry, that is,

$$[fg,h] = [f,h]g + f[gh], \quad [f,gh] = g[f,h] + h[f,g],$$

for any $f,g,h \in C^\infty(P)$. We will say that the pair $(P,[\cdot,\cdot])$ is a **Leibniz manifold**. If the bracket $[\cdot,\cdot]$ is antisymmetric, that is, it satisfies

$$[f,g] = -[g,f]$$

for every pair of functions $f,g \in C^\infty(P)$ then we say that $(P,[\cdot,\cdot])$ is an **almost Poisson manifold**. We will usually denote the almost Poisson brackets with the symbol $\{\cdot,\cdot\}$.

A function $f \in C^\infty(P)$ such that $[f,g] = 0$ (respectively, $[g,f] = 0$) for any $g \in C^\infty(P)$ is called a **left** (respectively, **right**) **Casimir** of the Leibniz manifold $(P,[\cdot,\cdot])$.

**Definition 2.2** Let $(P,\{\cdot,\cdot\})$ be an almost Poisson manifold. We define the **Jacobiator** of the bracket $\{\cdot,\cdot\}$ as the map $\mathfrak{J} : C^\infty(P) \times C^\infty(P) \times C^\infty(P) \to C^\infty(P)$ given by

$$\mathfrak{J}(f,g,h) = \{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}.$$  \hspace{1cm} (2.1)

A **Poisson structure** on $P$ is an almost Poisson structure on $P$ for which the Jacobiator is the zero map.

The following lemmas are a direct consequence of the fact that the Leibniz structure is a derivation.

Let $(P,[\cdot,\cdot])$ be a Leibniz manifold and let $h$ be a smooth function on $P$. There exist two vector fields $X^R_h$ and $X^L_h$ on $P$ uniquely characterized by the relations

$$X^R_h[f] = [f,h] \quad \text{and} \quad X^L_h[f](m) = -[h,f], \quad \text{for any } f \in C^\infty(P).$$

Given two smooth functions $g,h \in C^\infty(P)$ there exists a unique vector field $X_{g,h}$ on $P$ such that

$$X_{g,h}[f](m) = \mathfrak{J}(f,g,h)(m), \quad \text{for any } f \in C^\infty(P).$$

We will call $X^R_h$ the **Leibniz vector field** associated to the **Hamiltonian function** $h \in C^\infty(P)$. In this paper, the abbreviation $X_h$ will always denote $X^R_h$. The flow $F_t$ of the vector field $X_h$ satisfies

$$\left.\frac{d}{dt}\right|_{t=0} g(F_t(m)) = [g,h](F_t(m)), \quad \text{for any } g \in C^\infty(P).$$  \hspace{1cm} (2.2)
A straightforward corollary of (2.2) is that in the context of almost Poisson manifolds the Hamiltonian function is a constant of motion, that is, if \( F_t \) is flow of \( X_h \) then \( h \circ F_t = h \), for any \( h \in C^\infty(P) \). Note that since \([\cdot,\cdot]\) and \( \mathfrak{z} \) are a derivation on each of their arguments they only depend on the first derivatives of the functions and thus, we can define two tensor maps \( B : T^*P \times T^*P \to \mathbb{R} \) and \( B_J : T^*P \times T^*P \times T^*P \to \mathbb{R} \) by
\[
B(df, dg) = [f, g] \quad \text{and} \quad B_J(df, dg, dh) = \mathfrak{z}(f, g, h),
\]
for any \( f, g, h \in C^\infty(P) \). We can associate to the tensor \( B \) two vector bundle maps \( B_L^t : T^*P \to TP \) and \( B_R^t : T^*P \to TP \) defined by the relations
\[
B(\alpha, \beta) = -\langle \beta, B_L^t(\alpha) \rangle \quad \text{and} \quad B(\alpha, \beta) = \langle \alpha, B_R^t(\beta) \rangle
\]
for any \( \alpha, \beta \in T^*P \). Notice that when the bracket \([\cdot,\cdot]\) is symmetric (respectively, antisymmetric) we have that \( B_R^t = -B_L^t \) (respectively, \( B_R^t = B_L^t \)) and \( X_h^R = -X_h^L \) (respectively, \( X_h^R = X_h^L \)), for any \( h \in C^\infty(P) \). We say that the Leibniz manifold \((P, [\cdot,\cdot])\) is non degenerate whenever the maps \( B_L^t \) and \( B_R^t \) are vector bundle isomorphisms.

**Definition 2.3** Let \((P, [\cdot,\cdot])\) be a Leibniz manifold. We define the left and right characteristic distributions \( \mathcal{E}_L \) and \( \mathcal{E}_R \), respectively, by
\[
\mathcal{E}_L := \text{span}\{X_h^L | h \in C^\infty(P)\} = B_L^t(T^*P) \quad \text{and} \quad \mathcal{E}_R := \text{span}\{X_h^R | h \in C^\infty(P)\} = B_R^t(T^*P).
\]
Notice that if the Leibniz bracket \([\cdot,\cdot]\) is either symmetric or antisymmetric then both distributions coincide. If additionally the Leibniz manifold \((P, [\cdot,\cdot])\) is non degenerate then \( \mathcal{E}_L = \mathcal{E}_R = TP \) and we can define a tensor field \( \omega : TP \times TP \to \mathbb{R} \) of type \((0,2)\) on \( P \) by
\[
\omega(X_f, X_g) = [f, g],
\]
for any \( f, g \in C^\infty(P) \). Given any point \( m \in P \) and any vector subspace \( V \subset T_mP \) we denote
\[
V^\omega := \{w \in TP | \omega(m)(v, w) = 0, \ for \ any \ v \in V\}.
\]
If the tensor \( \omega \) is antisymmetric (respectively, symmetric) then it is a two–form (respectively, a pseudo–metric) on \( P \). If additionally the form \( \omega \) is closed we say that \( \omega \) is a symplectic form on \( P \) and that the pair \((P, \omega)\) is a symplectic manifold.

Two functions smooth functions \( h_1, h_2 \in C^\infty(P) \) on the Leibniz manifold \((P, [\cdot,\cdot])\) are said to be equivalent if and only if \([f, h_1 - h_2] = 0\), for any \( f \in C^\infty(P) \) or equivalently, whenever \( X_{h_1} = X_{h_2} \). Notice that this definition establishes an equivalence relation on the set \( C^\infty(P) \).

**Definition 2.4** A Leibniz map between two Leibniz manifolds \((P_1, [\cdot,\cdot]_1)\) and \((P_2, [\cdot,\cdot]_2)\) is a smooth map \( \phi : P_1 \to P_2 \) that satisfies
\[
\phi^*[f, g]_2 = [\phi^*f, \phi^*g]_1, \quad \text{for any} \ f, g \in C^\infty(P_2).
\]

**Lemma 2.5** Let \( \phi : (P_1, [\cdot,\cdot]_1) \to (P_2, [\cdot,\cdot]_2) \) be a Leibniz map. Let \( h \in C^\infty(P_2) \), \( F_t^2 \) be the flow of the Leibniz vector field \( X_h, F_t^1 \) the flow of \( X_{h_{\phi^*}} \), and \( \text{Dom}(F_t^1) \) and \( \text{Dom}(F_t^2) \) the domains of definition of \( F_t^1 \) and \( F_t^2 \), respectively. Then \( \text{Dom}(F_t^1) \subset \phi^{-1}(\text{Dom}(F_t^2)) \) and
\[
F_t^2 \circ \phi(z) = \phi \circ F_t^1(z), \quad \text{for any} \ z \ in \ the \ domain \ \text{Dom}(F_t^1) \ \text{of} \ F_t^1.
\]
Additionally, \( X_{h_{\phi^*}} \) and \( X_h \) are \( \phi \)-related, that is, \( T\phi \circ X_{h_{\phi^*}} = X_h \circ \phi \).

**Proof.** Let \( z \in \text{Dom}(F_t^1) \) and \( g \in C^\infty(P_2) \) arbitrary. Using the Leibniz condition on the map \( \phi \) we can write
\[
geq dg((\phi \circ F_t^1)(z)) = \frac{d}{dt}g((\phi \circ F_t^1)(z)) = \frac{d}{dt}(g \circ \phi)(F_t^1(z)) = g \circ \phi \circ h \circ \phi)(F_t^1(z)) = [g, h](\phi \circ F_t^1)(z) = \mathbf{d}g((\phi \circ F_t^1)(z)) \cdot X_h((\phi \circ F_t^1)(z)).
\]
Since the function $g$ is arbitrary, this equality implies that
\[
\frac{d}{dt}(\phi \circ F^1_t)(z) = X_h((\phi \circ F^1_t)(z)),
\]
which allows us to conclude that $(\phi \circ F^1_t)(z)$ is an integral curve of $X_h$ through the point $\phi(z)$. Since $F^2_t$ is the flow of $X_h$ this automatically implies that $\phi(z) \in \text{Dom}(F^2_t)$. As $z \in \text{Dom}(F^1_t)$ is arbitrary we get that $\phi(\text{Dom}(F^1_t)) \subset \text{Dom}(F^2_t)$ which implies that $\text{Dom}(F^1_t) \subset \phi^{-1}(\text{Dom}(F^2_t))$. Additionally, the uniqueness property of the flow of a smooth vector field allows us to write that $(\phi \circ F^1_1)(z) = (F^2_1 \circ \phi)(z)$.

The $\phi$–relatedness of $X_{h \circ \phi}$ and $X_h$ follows from taking the time derivative of \((2.4)\) at $t = 0$, recalling that $\text{Dom}(F^1_t)$ becomes the entire manifold $P_1$ when $t$ goes to zero. $\blacksquare$

**Proposition 2.6** Let $\phi : (P, \{\cdot, \cdot\}_P) \rightarrow (Q, \{\cdot, \cdot\}_Q)$ be a surjective Leibniz map. If $(P, \{\cdot, \cdot\}_P)$ is a Poisson manifold then so is $(Q, \{\cdot, \cdot\}_Q)$.

**Proof.** Let $f, g \in C^\infty(Q)$ arbitrary. The surjectivity of $\phi$ implies that any element in $Q$ can be written as $\phi(z)$, for some $z \in P$. Hence,
\[
[f, g]_Q(\phi(z)) = \{f \circ \phi, g \circ \phi\}_P(z) = -\{g \circ \phi, f \circ \phi\}_P(z) = -[g, f]_Q(\phi(z)),
\]
which proves the antisymmetry of $\{\cdot, \cdot\}_Q$. Analogously, in order to prove that $\{\cdot, \cdot\}_Q$ satisfies the Jacobi identity consider $f, g, h \in C^\infty(Q)$ and $z \in P$. Since $\phi$ is a Leibniz map we can write
\[
[f, [g, h]]_Q\phi(z) + [g, [h, f]]_Q\phi(z) + [h, [f, g]]_Q\phi(z)
= \{f \circ \phi, \{g \circ \phi, h \circ \phi\}_P\}_P + \{g \circ \phi, \{h \circ \phi, f \circ \phi\}_P\}_P + \{h \circ \phi, \{f \circ \phi, g \circ \phi\}_P\}_P = 0. \blacksquare
\]

## 3 Examples

As we already said symplectic and Poisson manifolds are particular cases of Leibniz manifolds. We now briefly introduce other non trivial examples.

(i) **Pseudometric brackets and gradient dynamical systems.** Let $g : TP \times TP \rightarrow \mathbb{R}$ be a pseudometric on the smooth manifold $P$, that is, a symmetric non degenerate tensor field of type $(0, 2)$ on $P$. Let $g^2 : T^*P \rightarrow TP$ and $g^3 : T^*P \rightarrow T^*P$ be the associated vector bundle maps. Given any smooth function $h \in C^\infty(P)$ we define its gradient $\nabla h : P \rightarrow TP$ as the vector field on $P$ given by $\nabla h := g^2 dh$. Let $\{\cdot, \cdot\} : C^\infty(P) \times C^\infty(P) \rightarrow \mathbb{R}$ be the Leibniz bracket defined by
\[
[f, h] := g(\nabla f, \nabla h),
\]
for any $f, h \in C^\infty(P)$. We will refer to this bracket as the **pseudometric bracket** associated to $g$. This bracket is clearly symmetric and non degenerate and the Leibniz vector field $X_h$ associated to any function $h \in C^\infty(P)$ is such that $X_h = \nabla h$. These brackets are also called **Beltrami brackets**, see \cite{Cro91, Vul84}.

(ii) **The three–wave interaction.** A very relevant problem in dynamics is the study of the interaction between non linear oscillators and the energy exchange between them. This problem can be viewed as an interaction between waves of different frequencies with different resonance conditions. A particular case that has deserved special attention is the so called three–wave or triad interaction\cite{Ala98}. Following \cite{BOO} this problem can be formulated as a dynamical system in $\mathbb{R}^3$ that satisfies the differential equations given by
\[
\frac{dx}{dt} = s_1 \gamma_1 yz, \quad \frac{dy}{dt} = s_2 \gamma_2 xz, \quad \frac{dz}{dt} = s_3 \gamma_3 xy,
\]
where the parameters $s_1, s_2, s_3 \in \{-1, 1\}$ and $\gamma_1, \gamma_2, \gamma_3$ are real numbers that satisfy $\gamma_1 + \gamma_2 + \gamma_3 = 0$. This system happens to be a particular case of point (i) by taking the Leibniz bracket induced by the constant pseudometric
\[
g = \begin{pmatrix}
\frac{1}{s_1 \gamma_1} & 0 & 0 \\
0 & \frac{1}{s_2 \gamma_2} & 0 \\
0 & 0 & \frac{1}{s_3 \gamma_3}
\end{pmatrix}
\]
and the Hamiltonian function \( H(x, y, z) = xyz \).

(iii) **Double bracket dissipation.** As we already said the Leibniz dynamical systems induced by an almost Poisson bracket are energy preserving. Brockett [Br88, Br93] has proposed the modelling of certain dissipative phenomena by adding a symmetric bracket to a known antisymmetric one, that is,

\[ [\cdot, \cdot]_{\text{Leibniz}} = [\cdot, \cdot]_{\text{skew}} + [\cdot, \cdot]_{\text{sym}} \]

where the bracket \([\cdot, \cdot]_{\text{skew}}\) is skewsymmetric, \([\cdot, \cdot]_{\text{sym}}\) is symmetric, and hence the sum is a Leibniz bracket. This scheme allows the modeling of a surprising number of physical examples. The reader is encouraged to check with [Mars92, Blal96] for an account of applications and references in this direction.

A particularly simple example that fits into this framework is the equation arising from the Landau–Lifschitz model for the magnetization vector \( M \) in an external vector field \( B \),

\[ \dot{M} = \gamma M \times B + \frac{\lambda}{\|M\|^2} (M \times (M \times B)) \]  

(3.1)

where \( \gamma \) and \( \lambda \) are physical parameters. This equation is Leibniz in our sense if we take the Leibniz bracket on \( \mathbb{R}^3 \) given by the sum of the two brackets

\[ \{f, g\}_{\text{skew}}(M) := M \cdot (\nabla f(M) \times \nabla g(M)) \quad \text{and} \quad \{f, g\}_{\text{sym}}(M) := \frac{\lambda (M \times \nabla f(M)) (M \times \nabla g(M))}{\gamma \|M\|^2}, \]

where the symbol \( \times \) denotes the standard cross product on \( \mathbb{R}^3 \) and \( \nabla \) is the Euclidean gradient. With this bracket the differential equation (3.1) corresponds to the expression of the Leibniz vector field determined by the function

\[ h(M) = \gamma B \cdot M. \]

Another related example is the differential equation satisfied by a rigid body subjected to certain dissipation (see [Mars92])

\[ \dot{M} = M \times \Omega + \alpha (M \times (M \times \Omega)). \]  

(3.2)

In this expression \( M \) is the momentum vector of the solid and \( \Omega \) its angular velocity, both in body coordinates. Recall that

\[ \Omega := \left( \frac{M_1}{I_1}, \frac{M_2}{I_2}, \frac{M_3}{I_3} \right), \]

where \( (I_1, I_2, I_3) \) are the components of the inertia tensor of the body with respect to a basis in which this tensor is diagonal. If we take the same bracket as before with \( \alpha = \frac{\lambda}{\gamma \|M\|^2} \), the equations (3.2) coincide with the Hamilton equations corresponding to the function

\[ h(M) = \frac{1}{2} \left( \frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} \right). \]

(iv) **Almost Poisson manifolds and non holonomically constrained mechanical systems.**

The equations of motion of a simple mechanical system subjected to a constraint can be written using D’Alembert’s Principle. When the constraints can be expressed as a linear function on the velocities these equations admit a Leibniz formulation that corresponds to the almost Poisson bracket introduced in Definition 2.1. See [vdSMa94, Mar95, Cual95, Bal96, Caa99, Snia01, B03] and references therein. If the constraints do not satisfy the linearity condition the almost Poisson formulation ceases to be valid in general. Nevertheless, if the constraints are affine on the velocities the problem still admits a formulation in the context of Leibniz manifolds using a bracket that in general is not antisymmetric. We discuss this point in detail in the following section.
4 Example: nonholonomic constraints

Let us consider a simple mechanical system characterized by a hyperregular Lagrangian \( L : TQ \rightarrow \mathbb{R} \) on the tangent bundle \( TQ \) of a configuration space \( Q \). One way to impose kinematic constraints on that system consists of fixing an affine subbundle \( C \subset TQ \), usually referred to as the set of admissible kinematical states. The hyperregularity of \( L \) implies that the associated Legendre transform \( FL : TQ \rightarrow T^*Q \) is a diffeomorphism that can be used to define an associated Hamiltonian dynamical system on \( T^*Q \) with Hamiltonian function \( H \), as well as the Hamiltonian constraint submanifold \( D := FL(C) \) on \( T^*Q \). D’Alembert’s Principle defines (see [Mar95]) a vector subbundle \( W \subset TD(T^*Q) \) such that if \( TD \cap W = \{0\} \) and the Hamiltonian vector field \( X_H \mid_D \) is a section of \( TD \oplus W \) then the corresponding splitting

\[
X_H \mid_D = X_B^H + X_W^H
\]

is well defined and \( X_B^H \) is the vector field whose flow describes the motion of the constrained dynamical system. The vector field \( X_B^H \) is usually referred to as the evolution vector field and the complementary vector field \( X_W^H \) as the constraint force field.

Our goal in the following paragraphs consists of endowing \( T^*Q \) with a Leibniz structure \([\cdot, \cdot] \) such that the Leibniz vector field \( X_B^H \) associated to \( H \) is such that

\[
X_B^H(z) = X_B^H(z) \quad \forall z \in D
\]

**Theorem 4.1** Assume \( T^*Q \) paracompact. Let \( H \in C^\infty(T^*Q) \) be a smooth function with Hamiltonian vector field associated \( X_H \). Let \( D \subset T^*Q \) be a closed and embedded constraint submanifold and \( W \subset TD(T^*Q) \) a smooth vector subbundle such that \( TD(T^*Q) = TD \oplus W \). There exists a Leibniz structure \([\cdot, \cdot] \) on \( T^*Q \) such that

\[
X_B^H(z) = \pi X_H(z) = : X_B^H(z) \quad z \in D
\]

where \( \pi : TD \oplus W \rightarrow TD \) is the natural projection.

**Proof.** Consider the bilinear mapping

\[
[\cdot, \cdot]_D : C^\infty(T^*Q) \times C^\infty(T^*Q) \rightarrow C^\infty(D)
\]

defined by \([f, g]_D(z) := (df(z), \pi B^\sharp (z) (dg(z)))\), for any \( z \in D \), and where \( B^\sharp : T^* (T^*Q) \rightarrow T^* (T^*Q) \) is the vector bundle isomorphism induced by the canonical symplectic form of \( T^*Q \). This bracket has a smooth section \( B_D : D \rightarrow T^2_0(T^*Q) \) associated given by

\[
B_D(z) (\alpha_2, \beta_2) := \langle \alpha_2, \pi B^\sharp (z)(\beta_2) \rangle,
\]

for any \( \alpha_2, \beta_2 \in T_z^* (T^*Q) \). By the smooth Tietze extension theorem (see for instance [AMRSS, Theorem 5.5.9]) \( B_D \) can be extended to a smooth section \( B : T^*Q \rightarrow T^2_0(T^*Q) \). For any \( f, g \in C^\infty(T^*Q) \) we define

\[
[f, g]_L(m) = B(m)(df(m), dg(m))
\]

The bracket \([\cdot, \cdot]_L : C^\infty(T^*Q) \times C^\infty(T^*Q) \rightarrow C^\infty(T^*Q) \) endows \( T^*Q \) with a Leibniz structure. Finally, we show that (4.1) holds. Indeed, for any \( z \in D \),

\[
X_B^H(z) = B^\sharp (z)(dH(z)) = B_D(z)(dH(z)) = \pi B^\sharp (z)(dH(z)) = X_B^H(z).
\]

**Remark 4.2** The Leibniz structures on \( T^*Q \) for which (4.1) holds are not unique. This freedom in the construction of the bracket can be used in specific applications to study families of systems instead of just a particular one, which may be of relevance in bifurcation theoretical problems. We make this comment more specific with the following elementary example.

Consider a particle of mass \( m \) constrained to move in a rotating hoop of mass \( M \) whose axis of rotation is parallel to the gravity and contains a diameter of the hoop. Let \( \phi \) be the angle that parametrizes the position of the bead in the hoop and \( \psi \) the angle that characterizes the position of the hoop. This setup
can be seen as a simple mechanical system with configuration space the two torus $\mathbb{T}^2$ and subjected to the affine constraint $\dot{\psi} - \omega = 0$, where $\omega \in \mathbb{R}$ is the constant angular speed of the hoop.

From the point of view of the formalism that we introduced above, for each value $\omega$, there is a Hamiltonian constraint submanifold, $D_\omega$ and a vector subbundle $W = \text{span} \left\{ \frac{\partial}{\partial \psi} \right\}$ provided by D’Alembert’s principle (see [Mar95]) such that

$$T(T^*\mathbb{T}^2) = TD_\omega \oplus W.$$ 

As in the proof of the theorem, there exists for each $\omega$ a smooth section $\tilde{B}_{D_\omega} : T_0^2(T^*\mathbb{T}^2) \to T^2(T^*\mathbb{T}^2)$.

Then the level sets of the momentum map are $X^H_{\tilde{t}}(z) = X^H_{D_\omega}(z)$ for any $z \in D_\omega$ and any $\omega \in \mathbb{R}$.

## 5 Symmetries and reduction of Leibniz systems

The symmetries of a dynamical system are in general very useful to simplify its study. In the particular case of symplectic and Poisson manifolds this idea has been specifically implemented using a procedure that is generically known as reduction (see [MW74] [MR86] [OR98] [OR03], an references therein). In the next two sections we will adapt some aspects of the reduction theory of symplectic and Poisson systems to the context of Leibniz manifolds.

In this section we will consider symmetries of Leibniz systems that are encoded under the form of Lie group and Lie algebra actions. Let $\mathfrak{g}$ be a Lie algebra and $P$ a smooth manifold. We recall that a right (left) Lie algebra action of $\mathfrak{g}$ on $P$ is a Lie algebra (anti)homomorphism $\xi : \mathfrak{g} \to \xi_P \in \mathfrak{X}(P)$ such that the mapping $(m, \xi) \in P \times \mathfrak{g} \mapsto \xi_P(m) \in TP$ is smooth. The symbol $\mathfrak{X}(P)$ denotes the set of smooth vector fields on $P$. We will denote by $C^\infty(P)^g$ the set of $\mathfrak{g}$–invariant smooth functions on $P$, that is, $C^\infty(P)^g := \{ f \in C^\infty(P) \mid df \cdot \xi_P = 0 \text{ for any } \xi \in \mathfrak{g} \}$

**Definition 5.1** Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $B \in T_0^2(P)$ the associated Leibniz tensor. Let $G$ be a Lie group (respectively, a Lie algebra) acting on $P$. We say that this $G$–action (respectively, $\mathfrak{g}$–action) is a weak symmetry of $(P, [\cdot, \cdot])$ whenever the algebra $C^\infty(P)^G$ (respectively, $C^\infty(P)^\mathfrak{g}$) of $G$–invariant functions on $P$ is closed under the Leibniz bracket. We say that $G$ (respectively, $\mathfrak{g}$) is a strong symmetry if $G$ acts on $P$ by Leibniz maps (respectively, if $L_{\xi_P}B = 0$, for any $\xi \in \mathfrak{g}$). Such actions will be sometimes referred to as canonical.

**Definition 5.2** Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $\mathfrak{g}$ a Lie algebra acting on $P$. We say that the $\mathfrak{g}$–action on $P$ admits a momentum map $J : P \to \mathfrak{g}^*$ whenever, for any $\xi \in \mathfrak{g}$, there exists a smooth function $f_\xi \in C^\infty(P)$ such that the component $J^\xi := \langle J, \xi \rangle$ is also smooth and

$$X^\xi_J = f_\xi \xi_P.$$ 

We will call the function $f_\xi$ the $\xi$–integrating factor.

The definition that we just introduced allows the formulation of a Noether’s Theorem in the Leibniz context. Notice that, unlike the classical result, only the invariance of the Hamiltonian function is required in the following statement and that there are no compatibility assumptions between the action and the Leibniz bracket.

**Proposition 5.3** Let $(P, [\cdot, \cdot])$ be a Leibniz manifold and $\mathfrak{g}$ a Lie algebra acting on $P$. Assume that the $\mathfrak{g}$–action on $P$ admits a momentum map $J : P \to \mathfrak{g}^*$. Then the level sets of the momentum map are preserved by the flows of the Leibniz vector fields associated to any $\mathfrak{g}$–invariant function on $P$.

**Proof.** Let $h \in C^\infty(P)^\mathfrak{g}$ and $\xi \in \mathfrak{g}$ be arbitrary. Let $F_t$ be the flow of the Leibniz vector field $X^R_h$. Then

$$\frac{d}{dt} J^\xi \circ F_t = X^R_h[J^\xi] = [J^\xi, h] = X^L_{J^\xi}[h] = f_\xi \xi_P[h] = f_\xi dh \cdot \xi_P = 0. \quad \blacksquare$$
Remark 5.4 The introduction of the integrating factors in the definition of the momentum map is not motivated by particular needs of the Leibniz category. Indeed, as we show in the following example, even in the symplectic or in the Poisson category this seems to be the only way to associate non trivial conservation laws to non canonical symmetries.

Let \( \mathbb{R}_u^2 \) be the upper half plane considered as a symplectic manifold with form \( \omega = dx \wedge dy \). Let \((\mathbb{R}, +)\) act on \( \mathbb{R}_u^2 \) by \( a \cdot (x, y) := (x, e^ay) \), for any \((x, y) \in \mathbb{R}_u^2\) and any \(a \in \mathbb{R}\). This action is clearly not canonical but it still admits a momentum map with a non trivial integrating factor that leads to a conservation law via Proposition 5.3. Indeed, for any \(\xi \in \mathbb{R}\), the infinitesimal generator \(\xi_{\mathbb{R}^2}\) is given by \(\xi_{\mathbb{R}^2}(x, y) = (0, e^ay)\), \((x, y) \in \mathbb{R}_u^2\). This vector field is not Hamiltonian and hence this action does not have a traditional momentum map associated. However, there is a momentum map available in the sense of Definition 5.4 since the maps \(J : \mathbb{R}_u^2 \to \mathbb{R}\) and \(f_\xi \in C^\infty(\mathbb{R}_u^2)\) given by \(J(x, y) := x\) and \(f_\xi = - (\xi/e^ay)\), \((x, y) \in \mathbb{R}_u^2, \xi \in \mathbb{R}\), are such that \(X_{J} = f_\xi_{\mathbb{R}^2}\). The conservation law associated by Proposition 5.3 to the existence of this momentum map can be phrased by saying that the Hamiltonian flows corresponding to Hamiltonian functions that depend only on the \(x\) variable preserve the vertical lines in \(\mathbb{R}_u^2\).

The following result shows that the orbit space corresponding to the weak symmetry \(G\) of a Leibniz manifold \((P, \cdot, \cdot)\) is also a Leibniz manifold provided that the group action satisfies enough regularity assumptions to guarantee that the quotient \(P/G\) is a regular quotient manifold, that is, \(P/G\) can be endowed with a (unique) smooth structure that makes the projection \(\pi : P \to P/G\) a surjective submersion. This observation is consistent with the result of Bates and Sniatycki [BS93] in the context \(P/G\), surjective submersion. This observation is consistent with the result of Bates and Sniatycki [BS93] in the context of nonholonomic mechanics that shows that symmetry reduction for these systems preserves the form of the equations of motion.

Theorem 5.5 Let \((P, \cdot, \cdot)\) be a Leibniz manifold and let \(G\) be a Lie group acting on \(P\) in such way that the orbit space \(P/G\) is a regular quotient manifold (this is the case when, for instance, the action is free and proper). Assume that \(G\) is a weak symmetry of \((P, \cdot, \cdot)\). Then

(i) \((P/G, \cdot, \cdot)_{P/G}\) is a Leibniz manifold with bracket \([\cdot, \cdot]_{P/G}\) uniquely determined by the expression
\[
[f, g]_{P/G} \circ \pi = [\pi^* f, \pi^* g],
\]
for any \(f, g \in C^\infty(P/G)\) and where \(\pi : P \to P/G\) is the projection.

(ii) The Leibniz structure induced by the bracket \([\cdot, \cdot]_{P/G}\) on \(P/G\) is the only one for which the projection \(\pi : P \to P/G\) is a Leibniz map.

(iii) Let \(h \in C^\infty(P/G)\) be a smooth \(G\)-invariant function on \(P\) and \(h_{P/G} \in C^\infty(P/G)\) the function on the quotient uniquely determined by the expression \(h_{P/G} \circ \pi = h\). Let \(X_h\) and \(X_{h_{P/G}}\) be the corresponding Leibniz vector fields on \((P, \cdot, \cdot)\) and \((P/G, \cdot, \cdot)_{P/G}\), respectively, and \(F_t\) and \(F^{P/G}_t\) the associated flows. Then \(\text{Dom}(F_t) \subset \pi^{-1}(\text{Dom}(F^{P/G}_t))\) and
\[
F^{P/G}_t \circ \pi(x) = \pi \circ F_t(x),
\]
for any \(x \in \text{Dom}(F_t)\). The vector fields \(X_h\) and \(X_{h_{P/G}}\) are \(\pi\)-related.

Proof. (i) We first check that [5.3] is a good definition for the bracket \([\cdot, \cdot]_{P/G}\). Let \(m, m'\) be two points in \(P\) such that \(\pi(m) = \pi(m')\). This equality implies that there exists an element \(g \in G\) such that \(m' = g \cdot m\). Let now \(f, g \in C^\infty(P/G)\) arbitrary. By definition \([f, g]_{P/G}(\pi(m')) = [f \circ \pi, g \circ \pi](m')\). Since by hypothesis \(C^\infty(P/G)\) is closed under the bracket and \(f \circ \pi\) and \(g \circ \pi\) are \(G\)-invariant then so is \([f \circ \pi, g \circ \pi]\) and hence
\[
[f, g]_{P/G}(\pi(m')) = [f \circ \pi, g \circ \pi](m') = [f \circ \pi, g \circ \pi](g \cdot m) = [f \circ \pi, g \circ \pi](m) = [f, g]_{P/G}(\pi(m)),
\]
as required. The bracket \([\cdot, \cdot]_{P/G}\) is clearly bilinear and is a derivation on its two arguments. Therefore, \((P/G, \cdot, \cdot)_{P/G}\) is a Leibniz manifold. (ii) It is a consequence of the fact that the projection \(\pi\) is a surjective submersion. (iii) is a consequence of (ii) and Lemma 5.4. ■
We emphasize that the weak symmetry condition on the Leibniz bracket \( (P, [,]) \) and the \( G \)-invariance of the Hamiltonian \( h \) do not suffice to ensure the equivariance of the associated Leibniz flow \( F_t \) of \( X_h \). In general this is only true whenever the symmetry is strong.

**Remark 5.6** The second part of the theorem shows that if the \( G \)-action is a weak symmetry of a Leibniz manifold \( (P, [,]) \) then the quotient \( P/G \) admits a unique Leibniz structure \([,]_{P/G}\) with respect to which the projection \( \pi : P \to P/G \) is a Leibniz map. The converse is also true. Indeed, let \( f, g \in C^{\infty}(P)^G \) and \( \bar{f}, \bar{g} \in C^{\infty}(P/G) \) be the unique smooth functions such that \( \bar{f} \circ \pi = f \) and \( \bar{g} \circ \pi = g \). Then the bracket \([f, g] \in C^{\infty}(P)\) is such that for any \( h \in G \) and any \( z \in P \):

\[
[f, g](h \cdot z) = [\bar{f} \circ \pi, \bar{g} \circ \pi](h \cdot z) = [\bar{f}, \bar{g}]_{P/G}(\pi(h \cdot z)) = [\bar{f} \circ \pi, \bar{g} \circ \pi](z) = [f, g](z),
\]

which proves that \([f, g] \in C^{\infty}(P)^G\) and hence that the \( G \)-action is a weak symmetry.

**Remark 5.7** Proposition 5.6 guarantees that if the Leibniz manifold \( (P, [,]) \) in the previous theorem is actually Poisson then so is the reduced manifold \( (P/G, [,]_{P/G}) \).

**Examples 5.8** (i) **Double bracket dissipation.** Consider the systems with double bracket dissipation that we studied in §3. This time we will restrict our discussion to the points in \( \mathbb{R}^3 \) that do not lie in the third axis. Assume that the magnetic vector field \( B \) is constant and equal to the vector \((0, 0, 1)\) or, in the case of the rigid body subjected to a dissipative interaction, assume that the moments of inertia \( I_1 \) and \( I_2 \) are equal. In both cases, the rotations around the third axis, which is a free group action on the restricted phase space, leave invariant the Hamiltonian functions and constitute a strong symmetry for the Leibniz system which allows us to apply Theorem 5.5. For a concrete realization of the quotient Leibniz structure we use the invariant polynomials \( \sigma_1 = \frac{1}{2}(M_1^2 + M_2^2) \) and \( \sigma_2 = M_3 \) of this action. In these coordinates the Leibniz tensor associated to the reduced Leibniz structure takes the form

\[
B = \frac{\gamma}{2\sigma_1 + \sigma_2^2} \begin{pmatrix}
-2\sigma_1 \sigma_2^2 & 2\sigma_1 \sigma_2 \\
2\sigma_1 \sigma_2 & -2\sigma_1
\end{pmatrix}.
\]

The reduced Hamiltonian functions are \( h(\sigma_1, \sigma_2) = \gamma \sigma_2 \) in the first case and \( h = \frac{\sigma_1}{2} + \frac{\sigma_2^2}{2} \) in the second.

(ii) **Reduction of a Poisson system with a non-canonical symmetry.** Consider the Poisson dynamical system \((\mathbb{R}^3, \{\cdot, \cdot\}, H)\) on \( \mathbb{R}^3 := \mathbb{R}^3 \setminus \{(x, y, 0) \in \mathbb{R}^3\} \), where the Poisson bracket \( \{\cdot, \cdot\} \) is determined by the Poisson tensor

\[
B(x, y, z) = \begin{pmatrix}
0 & x & y \\
-x & 0 & x \\
-y & -x & 0
\end{pmatrix},
\]

for any \((x, y, z) \in \mathbb{R}^3\) and \( H = \frac{1}{2}(x^2 + y^2) \). Let \( G := (\mathbb{R}, +) \) act on \( \mathbb{R}^3 \) by the map \( \phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \) given by

\[
a \cdot (x, y, z) = (x, y, e^a z), \quad \text{for any } a \in \mathbb{R} \text{ and any } (x, y, z) \in \mathbb{R}^3.
\]

This action is not canonical since, for example, \( \phi_a^* \{y, z\} = x \neq e^a x = \{\phi_a^* y, \phi_a^* z\} \). However, notice that since the algebra of \( G \)-invariant functions is made by functions depending on just the first two variables, that is, \( C^{\infty}(P)^G = \{ f \in C^{\infty}(P) \mid f = f(x, y) \} \) then \( C^{\infty}(P)^G \) is closed under the Poisson bracket. Consequently, by Theorem 5.5 \((\mathbb{R}^3, \{\cdot, \cdot\}, H)\) can be reduced by this action. The reduced Poisson space is \( \mathbb{R}^2 \) with the Poisson structure given by the reduced Poisson tensor

\[
B_{\mathbb{R}^2/G}(x, y) = \begin{pmatrix}
0 & x \\
-x & 0
\end{pmatrix}
\]

and the reduced Hamiltonian is \( h(x, y) = \frac{1}{2} (x^2 + y^2) \).

(iii) **A symmetry of the three-wave interaction.** The Hamiltonian \( H(x, y, z) = x y z \) of the three-wave interaction that we presented in Section 3 is invariant with respect to the action of the product of the two multiplicative groups \( G := \mathbb{R}^+ \times \mathbb{R}^+ \) by \((\lambda_1, \lambda_2) \cdot (x, y, z) := (\lambda_1 x, \lambda_2 y, \lambda_3 z)\), where
Let \((x, y, z) \in \mathbb{R}^3\), \((\lambda_1, \lambda_2) \in \mathbb{R}^+ \times \mathbb{R}^+\), and \(\lambda_3 := (\lambda_1 \cdot \lambda_2)^{-1}\). The infinitesimal generator for this action is given by \(\xi_{\mathbb{R}^3}(x, y, z) = (ax, by, -(a + b)z)\), for any \(\xi := (a, b) \in \mathbb{R}^2\). Even though this action is not even a weak Leibniz symmetry we can associate to it a momentum map \(J: \mathbb{R}^3 \to \mathbb{R}^2\) given by

\[
J(x, y, z) = \left( \frac{1}{2} \left( \frac{x^2}{s_1} - \frac{z^2}{s_3} \right), -\frac{1}{2} \left( \frac{y^2}{s_2} + \frac{z^2}{s_3} \right) \right).
\]

By Proposition 5.3, the components of \(J\) are constants of the motion for the flow of the Leibniz vector field \(X_H\).

(iv) Nonholonomically constrained particle. Consider a free particle in \(\mathbb{R}^3\). We will encode this setup as a Hamiltonian dynamical system on the cotangent bundle \(T^*\mathbb{R}^3\) endowed with its canonical symplectic structure. We will denote by \(B \in \Lambda^2(T^*(T^*\mathbb{R}^3))\) the Poisson tensor associated to this symplectic form. The Hamiltonian function of this system is \(H(x, y, z, p_x, p_y, p_z) = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2)\).

Suppose that the particle is forced to satisfy the affine constraint \(\dot{x} + y \dot{z} - a = 0\), where \(a \in \mathbb{R}\). In this particular case, the Hamiltonian constrained submanifold is given by:

\[
D_a = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 | p_x + yp_z - a = 0\}.
\]

Consequently,

\[
T(x, y, z, p_x, p_y, p_z) D_a = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial p_x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial p_y}, -y \frac{\partial}{\partial p_x} + \frac{\partial}{\partial p_z} \right\}
\]

Using D’Alembert’s principle (see Mar93) we choose the subbundle \(W_a \subset T_{D_a}(T^*\mathbb{R}^3)\) given by

\[
W_a(x, y, z, p_x, p_y, p_z) := \text{span} \left\{ \frac{\partial}{\partial p_x} + y \frac{\partial}{\partial p_z} \right\}
\]

that satisfies the regularity condition \(T_{D_a}(T^*\mathbb{R}^3) = TD_a \oplus W_a\). We now follow the scheme introduced in Section 4. A straightforward computation shows that the projection \(\pi_a : T_{D_a} \oplus W_a \to TD_a\) and the composition \(\overline{B}_a := \pi_a \circ B^t\) are given, using canonical coordinates, by the matrices

\[
\pi(m) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{p_y}{1+y^2} & 0 & \frac{y^2}{1+y^2} & 0 & \frac{p_z}{1+y^2} \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \frac{p_y}{1+y^2} & 0 & \frac{y^2}{1+y^2} & 0 & \frac{1}{1+y^2}
\end{pmatrix}, \quad \overline{B}_a^t(m) = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & \frac{p_z}{1+y^2} & 0 & \frac{y^2}{1+y^2} & 0 & \frac{p_y}{1+y^2} \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{1+y^2} & 0 & \frac{1}{1+y^2} & 0 & \frac{1}{1+y^2}
\end{pmatrix},
\]

where \(m = (x, y, z, p_x, p_y, p_z)\). Following the ideas introduced in the Remark 4.2 and noticing that the expression of \(\overline{B}_a^t\) does not depend on the parameter \(a\), we can trivially extend \(\overline{B}_a\) to a Leibniz tensor \(B \in \mathcal{T}_{D_a}(T^*\mathbb{R}^3)\) whose restriction to any \(D_a\), \(a \in \mathbb{R}\), coincides with \(\overline{B}_D\). Using this extension and the expressions in (5.2) we can write the evolution vector field of the constrained system as

\[
X_H^D = \overline{B}^t dH,
\]

which in canonical coordinates reads

\[
\dot{x} = p_x, \quad \dot{y} = p_y, \quad \dot{z} = p_z, \quad \dot{p_x} = \frac{-p_z p_y}{1+y^2}, \quad \dot{p_y} = 0, \quad \dot{p_z} = \frac{-yp_z p_y}{1+y^2}.
\]

Note that the constraint does not need to be included in the set of equations since for a set of initial conditions satisfying the constraint, the dynamics will preserve it automatically. From the point of view of the Leibniz formulation of the problem this remark can be phrased by saying that the function defining the constraint is a left Casimir of the Leibniz system \((T^*\mathbb{R}^3, \overline{B})\).
A momentum map. The Hamiltonian $H$ is symmetric with respect to the lifted action of the translations on the configuration space $\mathbb{R}^3$. The infinitesimal generators associated to this action are given by $\xi_{T\mathbb{R}^3}(x, y, z, p_x, p_y, p_z) = (\xi, 0)$, for any $\xi \in \mathbb{R}^3$. The lifted translations along the $OY$–axis admit a momentum map $J : T^*\mathbb{R}^3 \to \mathbb{R}$ with respect to the Leibniz structure $(T^*\mathbb{R}^3, \bar{B})$, given by

$$J(x, y, z, p_x, p_y, p_z) = p_y + \phi(p_x + yp_z)$$

where $\phi$ is an arbitrary smooth real valued function. By Proposition $5.3$, $J$ is preserved by the integral curves of the evolution vector field $X_H^J$.

Reduction. Consider now the group $G := (\mathbb{R}^2, +)$ acting on $T^*\mathbb{R}^3$ by lifting the translations in the coordinates $x$ and $z$. This action is a weak symmetry of $(T^*\mathbb{R}^3, \bar{B})$ and thus we can apply Theorem $5.5$.

Let $(T^*\mathbb{R}^3/G, [\cdot, \cdot]_{T^*\mathbb{R}^3/G}, \bar{h})$ be the reduced system. $T^*\mathbb{R}^3/G$ can be identified with $\mathbb{R}^4$ since the points in this orbit space correspond to the elements of the form $(y, p_x, p_y, p_z)$. Using this identification the reduced Hamiltonian can be written as $\bar{h} = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2)$ and the reduced Leibniz tensor $\bar{B}_1$ is given by

$$\bar{B}_1^\xi(y, p_x, p_y, p_z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \frac{-y}{1+y^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-p_z}{1+y^2} & 0 & 0 & 0 \end{pmatrix}.$$ 

This Leibniz structure has two independent left Casimirs $C_1^\xi(y, p_x, p_y, p_z) = p_y \cdot C_1^\xi(y, p_x, p_y, p_z) = p_z + yp_z$ and two independent right Casimirs $C_1^\xi(y, p_x, p_y, p_z) = p_z \cdot C_1^\xi(y, p_x, p_y, p_z) = p_z$. Consequently the new Hamiltonian function $\bar{h} := \bar{h} - \frac{1}{2}(C_1^\xi)^2 + (C_2^\xi)^2$, that is, $\bar{h}(y, p_x, p_y, p_z) = \frac{1}{2}p_y^2$, has the same evolution vector field than that of $h$. This equivalent Hamiltonian admits the symmetry of translations in the coordinates $p_x$ and $p_z$ and hence we can further reduce the system onto a two dimensional Leibniz one with tensor

$$\bar{B}_2^\xi(y, p_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and Hamiltonian function $\bar{h}_2 = \frac{1}{2}p_y^2$.

6 The reduction of a presheaf of Leibniz algebras

The reduction theorem that we presented in the previous section contains extremely strong regularity hypotheses that allowed us to have a smooth orbit space onto which the Leibniz bracket and the corresponding equivariant dynamics can be dropped. When these hypotheses are not present, the orbit space is not smooth anymore but nevertheless, the Leibniz algebra, or more specifically, the presheaf of Leibniz algebras associated to the bracket admits, under certain circumstances, a projection to the quotient. The algebraic approach to reduction that we introduce in the following paragraphs has its origins in the works $[\text{ACC91, OR98}]$ carried out in the context of symmetric Poisson manifolds.

We recall that a sheaf $\mathcal{F}$ of functions on a topological space $P$ is a map that assigns to any open set $U$ a set of real valued functions $\mathcal{F}(U)$ which is an algebra under multiplication. In the definition it is also required that for every inclusion $V \subset U$ of open sets there is a given homomorphism $\text{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ called the restriction from $U$ to $V$ that satisfies the following conditions:

(SH1) $\mathcal{F}(\emptyset) = \{0\}$ and $\text{res}^U_U : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.

(SH2) If $W \subset V \subset U$ are open sets, then $\text{res}^V_U \circ \text{res}^W_V = \text{res}^W_U$.

(SH3) Let $U$ be an open set and $\{V_i\}_{i \in I}$ an open covering of $U$. If $f \in \mathcal{F}(U)$ is such that the restriction $\text{res}^V_{V_i}(f)$ of $f$ to each $V_i$ is $0$, then $f = 0$.

(SH4) Let $U$ be an open set, $\{V_i\}_{i \in I}$ an open covering of $U$, and let $f_i \in \mathcal{F}(V_i)$ be given for each $i \in I$. Suppose that the restrictions of $f_i$ and $f_j$ to $V_i \cap V_j$ are equal for all $i, j \in I$. Then there exists a unique $f \in \mathcal{F}(U)$ whose restriction to each $V_i$ is $f_i$ for all $i \in I$.\)
When the map $\mathcal{F}$ satisfies only properties (SH1) and (SH2) we say that $\mathcal{F}$ is a presheaf. The elements in $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over $U$. The elements in $\mathcal{F}(P)$ are called global sections.

**Definition 6.1** Let $M$ be a topological space with a presheaf $\mathcal{F}$ of smooth functions. A presheaf of Leibniz algebras on $(P,\mathcal{F})$ is a map $[\cdot,\cdot]_U$ that assigns to each open set $U \subset M$ a bilinear operation $[\cdot,\cdot]_U : \mathcal{F}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ such that the pair $(\mathcal{F}(U),[\cdot,\cdot]_U)$ is a Leibniz algebra. A presheaf of Leibniz algebras will be usually denoted as a triple $(P,\mathcal{F},[\cdot,\cdot])$.

We say that the presheaf of Leibniz algebras $(P,\mathcal{F},[\cdot,\cdot])$ is non degenerate when if $f \in \mathcal{F}(U)$, is such that $[f,g]_U \mid V = 0$, for any $g \in \mathcal{F}(V)$ and any open set of $V$, then $f$ is constant in the connected components of $U$.

**Example 6.2** Any Leibniz manifold $(P,\cdot,\cdot)$ has a natural presheaf of Leibniz algebras on its presheaf of smooth functions that associates to any open subset $U$ of $P$ the restriction $[\cdot,\cdot]|_U$ of $[\cdot,\cdot]$ to $C^\infty(U) \times C^\infty(U)$.

### 6.1 Leibniz reduction by pseudogroups

The main goal of this section is the presentation of a result that fully characterizes the situations in which the presheaf of Leibniz algebras in Example 6.2 behaves properly under restriction to subsets and projection to the orbit spaces of pseudogroups of local Leibniz diffeomorphisms of $(P,\cdot,\cdot)$.

We start by introducing our terminology. Let $P$ be a smooth manifold and $\text{Diff}_L(P)$ the pseudogroup of local diffeomorphisms of $P$. More explicitly, the elements of $\text{Diff}_L(P)$ are diffeomorphisms $F : \text{Dom}(F) \subset P \to F(\text{Dom}(F))$ of an open subset $\text{Dom}(F) \subset P$ onto its image $F(\text{Dom}(F)) \subset P$. We will denote the elements of $\text{Diff}_L(P)$ as pairs $(F,\text{Dom}(F))$. The local diffeomorphisms can be composed using the binary operation defined as

$$(G,\text{Dom}(G)) \cdot (F,\text{Dom}(F)) := (G \circ F, F^{-1}(\text{Dom}(G)) \cap \text{Dom}(F)),$$

for all $(G,\text{Dom}(G)),(F,\text{Dom}(F)) \in \text{Diff}_L(P)$. It is easy to see that this operation is associative and has $(\mathbb{1},P)$, the identity map of $P$, as (unique) two sided identity element, which makes $\text{Diff}_L(P)$ into a monoid (set with an associative operation which contains a two–sided identity element). Notice that only the elements in $\text{Diff}(P) \subset \text{Diff}_L(P)$ have an inverse since, in general, for any $(F,\text{Dom}(F)) \in \text{Diff}_L(P)$, we have that

$$(F^{-1},F(\text{Dom}(F))) \cdot (F,\text{Dom}(F)) = (\mathbb{1}|_{\text{Dom}(F)},\text{Dom}(F)),$$

$$(F,\text{Dom}(F)) \cdot (F^{-1},F(\text{Dom}(F))) = (\mathbb{1}|_{F(\text{Dom}(F))},F(\text{Dom}(F))).$$

Consequently, the only way to obtain the identity element $(\mathbb{1},P)$ out of the composition of $F$ with its inverse is having $\text{Dom}(F) = P$. It follows from this argument that $\text{Diff}(P)$ is the biggest subgroup contained in the monoid $\text{Diff}_L(P)$ with respect to the composition law $(6.1)$. In the sequel we will frequently encounter submonoids $A$ of $\text{Diff}_L(P)$ that satisfy the following property:

(PS) for any $F : \text{Dom}(F) \to F(\text{Dom}(F))$ in $A$ there exists another element $F^{-1} : F(\text{Dom}(F)) \to \text{Dom}(F)$ also in $A$ that satisfies the identities $(6.2)$ and $(6.3)$.

Such submonoids will be referred to as pseudogroups of $\text{Diff}_L(P)$. Recall that $A$ being a submonoid implies that it is closed under composition and $(\mathbb{1},P) \in A$. One of the important features of pseudogroups is that they have an associated orbit space. Indeed, if $A$ is a pseudogroup we define the orbit $A \cdot m$ under $A$ of any element $m \in P$ as the set $A \cdot m := \{F(m) | F \in A, \text{ such that } m \in \text{Dom}(F)\}$. A being a pseudogroup implies that the relation being in the same $A$–orbit is an equivalence relation and induces a partition of $P$ into $A$–orbits. The space of $A$–orbits will be denoted by $P/A$. If we endow the space of orbits $P/A$ with the quotient topology, the projection $\pi_A : P \to P/A$ is a continuous and open map.

Let $S \subset P$ be a subset of $P$ endowed with a topology $\mathcal{T}$ that in general does not coincide with the relative or subspace topology. The presheaf $C^\infty_P$ of smooth functions on $P$ induces a quotient presheaf $C^\infty_{P/A}$ on the orbit space $P/A$. Consider now the subset

$$A_S := \{a \in A | a(s) \in S \text{ for any } s \in S \cap \text{Dom}(a)\}.$$
All along this section we will assume that $A_S$ is a subpseudogroup of $A$. This hypothesis will allow us to construct the quotients $S/A_S$ and $P/A_S$. Given that the quotient $S/A_S$ can be seen as a subset of $P/A_S$, there is a well defined presheaf of Whitney smooth functions $W^\infty_{S/A_S}$ on $S/A_S$ induced by $C^\infty_{P/A_S}$.

We recall (see [OR03]) that for any open set $V \subset S/A_S$, the elements $f \in W^\infty_{S/A_S}(V)$ are characterized by the fact that if $\pi_S : S \to S/A_S$ is the projection onto orbit space then for any $m \in \pi_S^{-1}(V)$ there exists an open $A_S$–invariant neighborhood of $m$ in $P$ and $F \in C^\infty_P(U_m)^A$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V) \cap U_m} = F|_{\pi_S^{-1}(V) \cap U_m}.$$  

We will say that $F$ is a local extension of $f \circ \pi_S$ at the point $m$.

**Definition 6.3** Let $P$ be a smooth manifold, $A \subset \text{Diff}_L(P)$ a pseudogroup of local diffeomorphisms of $P$, and $S$ a subset of $P$ endowed with a topology $T$ that is stronger than the relative topology. We say that the presheaf $W^\infty_{S/A_S}$ has the $(A,A_S)$–local extension property when $A_S$ is a subpseudogroup of $A$ and for any $f \in W^\infty_{S/A_S}(V)$ and $m \in \pi_S^{-1}(V)$ there exists an open $A$–invariant neighborhood $U_m$ of $m$ in $M$ and $F \in C^\infty_P(U_m)^A$ such that

$$f \circ \pi_S|_{\pi_S^{-1}(V) \cap U_m} = F|_{\pi_S^{-1}(V) \cap U_m}.$$  

We will say that $F$ is a $A$–invariant local extension of $f \circ \pi_S$ at $m$.

**Definition 6.4** Let $(P,[\cdot,\cdot])$ be a smooth Leibniz manifold and $A \subset \mathcal{P}_L(P)$ a pseudogroup of local diffeomorphisms of $P$ such that the presheaf of $A$–invariant functions on $P$ is closed under the Leibniz bracket $[\cdot,\cdot]$. Let $S \subset P$ be a subset of $P$ such that $W^\infty_{S/A_S}$ has the $(A,A_S)$–local extension property. We say that $(P,[\cdot,\cdot],A,S)$ is Leibniz reducible when $(S/A_S,W^\infty_{S/A_S},[\cdot,\cdot]^S_{/A_S})$ is a well defined presheaf of Leibniz algebras where, for any open set $V \subset S/A_S$, the bracket $[\cdot,\cdot]^S_{/A_S} : W^\infty_{S/A_S}(V) \times W^\infty_{S/A_S}(V) \to W^\infty_{S/A_S}(V)$ is given by

$$(f,g)^{S_{/A_S}}_{/V}(\pi_S(m)) = [F,G](m)$$  

for any $m \in \pi_S^{-1}(V)$ and where $F,G$ are $A$–invariant local extensions at $m$ of $f \circ \pi_S$ and $g \circ \pi_S$, respectively.

The following theorem generalizes the main reduction result in [OR03] to the context of Leibniz manifolds with locally defined Leibniz weak symmetries.

**Theorem 6.5** Let $(P,[\cdot,\cdot])$ be a smooth Leibniz manifold and $A \subset \mathcal{P}_L(P)$ a pseudogroup of local diffeomorphisms of $P$ such that the presheaf of $A$–invariant functions on $P$ is closed under the Leibniz bracket $[\cdot,\cdot]$. Let $S \subset P$ be a subset of $P$ such that $W^\infty_{S/A_S}$ has the $(A,A_S)$–local extension property. Let $B^L_R : T^*M \to TM$ be the left and right bundle maps respectively associated to the Leibniz tensor of $(P,[\cdot,\cdot])$. Then $(P,[\cdot,\cdot],A,S)$ is Leibniz reducible if and only if for any $m \in S$ we have that

$$B^L_R(\Delta_m) + B^R_L(\Delta_m) \subset \Delta^S_m,$$  

where $\Delta_m := \{\mathbf{d}F(m) \mid F \in C^\infty_P(U_m)^A$, for any open $A$–invariant neighborhood $U_m$ of $m$ in $P\}$, and where $\Delta^S_m = \{\mathbf{d}F(m) \in \Delta_m \mid F|_{U_m \cap S} \text{ is constant}\}$, for an open $A$–invariant neighborhood $U_m$ of $m$ in $P$ and an open $A_S$–invariant neighborhood $V_m$ of $m$ in $S$.

**Remark 6.6** If $S$ has the relative topology then $\Delta^S_m = \{\mathbf{d}F(m) \in \Delta_m \mid F|_{U_m \cap S} \text{ is constant}\}$, for an open $A$–invariant neighborhood $U_m$ of $m$ in $P$.

**Remark 6.7** If $A$ consists of local Leibniz diffeomorphisms then the condition on the presheaf of $A$–invariant functions on $P$ being closed under the Leibniz bracket $[\cdot,\cdot]$ is automatically satisfied.
Lemma 6.8 Let $P$ be a smooth manifold, $A \subset \text{Diff}_L(P)$ a pseudogroup of local transformations of $P$, and $S \subset P$ a subset whose topology is stronger than the relative topology and such that $A_S$ is a subpseudogroup of $A$. If $\pi_S : S \to S/A_S$ is the projection, $U \subset P$ is an open $A$-invariant subset of $P$, $F \in C_P^\infty(U)^A$, and $V := \pi_S(U \cap S)$ then there exists a unique function $f \in W_{S/A_S}^\infty(V)$ such that
\[
f \circ \pi_S|_{U \cap S} = f \circ \pi_S|_{\pi_S^{-1}(V) \cap U} = F|_{\pi_S^{-1}(V) \cap U}.
\]

Proof. Since by hypothesis the topology of $S$ is stronger than the relative topology we have that for any open $A$-invariant subset $U$ of $P$, the intersection $U \cap S$ is an open $A_S$-invariant subset of $S$. As the projection $\pi_S$ is an open map, the set $V := \pi_S(U \cap S)$ is open in $S/A_S$. Also, the $A_S$-invariance of $U \cap S$ implies that $U \cap S = \pi_S^{-1}(V)$ and hence
\[
\pi_S^{-1}(V) \cap U = U \cap S \cap U = U \cap S,
\]
which proves the first equality in (6.7).

Now, the invariance properties of $F$ and $S$ imply the existence of a unique map $f$ defined on $V$ such that $f \circ \pi_S|_{U \cap S} = F|_{U \cap S}$ or equivalently, by (6.8), $f \circ \pi_S|_{\pi_S^{-1}(V) \cap U} = F|_{\pi_S^{-1}(V) \cap U}$. Given that by construction $\pi_S^{-1}(V) \subset U$ then for any $m \in \pi_S^{-1}(V)$, the map $f$ satisfies (6.8) by taking in that characterization $U$ and $F$, which implies that $f \in W_{S/A_S}^\infty(V)$. 

Proof of Theorem 6.5. We first show that if $(P, [\cdot, \cdot], A, S)$ is Leibniz reducible then $\Delta_m^S \subset \left[ B_L^T(D_m) + B_R^T(D_m) \right]^\circ$, for all $m \in S$. Let $\alpha_m \in \Delta_m^S$; by definition there exists an open $A$-invariant neighborhood $U_m$ of $m$ in $P$ and a function $K \in C_P^\infty(U_m)^A$ such that $\alpha_m = dK(m)$ and $K|_{W_m \cap U_m}$ is constant for an open $A_S$-invariant neighborhood of $m$ in $S$. Notice now that by definition any element in $B_L^T(D_m) + B_R^T(D_m)$ can be written as $X^L_f(m) + X^R_f(m)$ with $F, G \in C^\infty(W_m)^A$, $W_m$ an open $A$-invariant neighborhood of $m$ in $P$. By Lemma 6.8 there exist functions $f \in W_{S/A_S}^\infty(\pi_S(U_m \cap S))$ and $g \in W_{S/A_S}^\infty(\pi_S(W_m \cap S))$ such that
\[
k \circ \pi_S|_{U_m \cap S} = K|_{U_m \cap S}, \quad f \circ \pi_S|_{W_m \cap S} = F|_{W_m \cap S}.
\]

Hence, by the Leibniz reducibility of $(P, [\cdot, \cdot], A, S)$ we have that
\[
\langle \alpha_m, X^L_f(m) + X^R_f(m) \rangle = [K, G](m) - [F, K](m) = [k|_W, g|_W]|_{W_{S/A_S}}(\pi_S(m)) - [f|_W, k|_W]|_{W_{S/A_S}}(\pi_S(m)),
\]
where $W = \pi_S(U_m \cap S) \cap \pi_S(W_m \cap S)$. However, given that the function $C$ on $P$ that is constant and equal to $K(m)$ is also an $A$-invariant local extension of $k \circ \pi_S$ at $m$, we have that
\[
[k|_W, g|_W]|_{W_{S/A_S}}(\pi_S(m)) - [f|_W, k|_W]|_{W_{S/A_S}}(\pi_S(m)) = [C, G] - [F, C] = 0,
\]
which implies that $\langle \alpha_m, X^L_f(m) + X^R_f(m) \rangle = 0$. Since $X^L_f(m) + X^R_f(m) \in B_L^T(D_m) + B_R^T(D_m)$ is arbitrary we have that $\alpha_m \in \left[ B_L^T(D_m) + B_R^T(D_m) \right]^\circ$.

Suppose now that the inclusion (6.9) holds and then we will prove the reducibility of $(P, [\cdot, \cdot], A, S)$. Let $f, g \in W_{S/A_S}^\infty(V)$ and $F, G \in C_P^\infty(U_m)^A$ be local $A$-invariant extensions of $f \circ \pi_S$ and $g \circ \pi_S$, respectively, at a point $m \in \pi_S^{-1}(V)$. We now show that the equality
\[
[f, g]|_{W_{S/A_S}}(\pi_S(m)) = [F, G]|_{U_m}(m)
\]
provides a well defined presheaf of Leibniz algebras. The only point that requires a proof is that the expression (6.9) does not depend on the local extensions utilized in the definition. The fact that $\langle [\cdot, \cdot]|_{W_{S/A_S}}(\pi_S(m)) \rangle$ determines a presheaf of Leibniz algebras is inherited from the properties of the bracket $[\cdot, \cdot]_P$ on $P$. Let $G' \in C_P^\infty(U_m)^A$ be another local extension of $g \circ \pi_S$ at $m$. This implies that $G - G'|_{\pi_S^{-1}(V) \cap U_m} = 0$ and hence $d(G - G')(m) \in \Delta_m^S \subset \left[ B_L^T(D_m) + B_R^T(D_m) \right]^\circ$. Consequently,
\[
0 = \langle d(G - G')(m), X^L_f(m) \rangle = -[F, G - G']|_{U_m}(m),
\]
which implies that \([F,G]|_{U_m}(m) = [F,G']|_{U_m}(m)\) and hence guarantees the independence of \((6.9)\) with respect to the choice of local extension for \(g \circ \pi_S\). A similar argument guarantees that this definition is also independent of the choice of extension for \(f \circ \pi_S\). Therefore, the expression \((6.9)\) defines a function \([f,g]|_{S/D}\) on \(V\) that actually belongs to \(W^\infty_{S/D}(V)\) because if \(F\) and \(G\) are local \(A\)-invariant extensions of \(f \circ \pi_S\) and \(g \circ \pi_S\), respectively, at any point \(m \in \pi_S^{-1}(V)\) then so is the function \(\{F,G\}\) with respect to \(\{f,g\}|_{V/S} \circ \pi_S\) by the hypothesis on the presheaf of \(A\)-invariant functions on \(P\) being closed under the Leibniz bracket \([\cdot,\cdot]\).

\[ \text{6.2 Leibniz reduction by distributions} \]

The Leibniz reduction theorem that we presented in Section \(6.1\) requires the presence of a pseudogroup of transformations defined in the entire manifold. However, sometimes one may want to reduce with respect to an invariance property defined only on a subset of the manifold in question. The study of this situation is the main goal of this section. We start by introducing the setup.

**Definition 6.9** Let \(P\) be a differentiable manifold and \(S \subset P\) a decomposed subset of \(P\). Let \(\{S_i\}_{i \in I}\) be the pieces of this decomposition. The topology of \(S\) is not necessarily the relative topology as a subset of \(P\). We say that \(D \subset TP|_S\) is a smooth distribution on \(S\) adapted to the decomposition \(\{S_i\}_{i \in I}\), if \(D \cap TS_i\) is a smooth distribution on \(S_i\) for all \(i \in I\). The distribution \(D\) is said to be integrable if \(D \cap TS_i\) is integrable for each \(i \in I\).

In the situation described by the previous definition and if \(D\) is integrable, the integrability of the distributions \(D_{S_i} := D \cap TS_i\) on \(S_i\) allows us to partition each \(S_i\) into the corresponding maximal integral manifolds. Thus, there is an equivalence relation on \(S_i\) whose equivalence classes are precisely these maximal integral manifolds. Doing this on each \(S_i\), we obtain an equivalence relation \(D_S\) on the whole set \(S\) by taking the union of the different equivalence classes corresponding to all the \(D_{S_i}\). We define the quotient space \(S/D_S\) as

\[ S/D_S := \bigcup_{i \in I} S_i/D_{S_i}. \]

We will denote by \(\pi_{D_S} : S \to S/D_S\) the natural projection.

**Definition 6.10** Let \((P, \cdot, \cdot)\) be a Leibniz manifold and \(D \subset TP\) a smooth distribution on \(P\). The distribution \(D\) is called **Leibniz** or **canonical**, if the condition \(df|_D = dg|_D = 0\), for any \(f, g \in C^\infty_P(U)\) and any open subset \(U \subset P\), implies that \(df|_D = dg|_D = 0\).

**The presheaf of smooth functions on \(S/D_S\).** In this section we will be considering a presheaf of smooth functions on \(S/D_S\) that require less invariance properties in their definition than those that appeared in the context of quotients by pseudogroups of transformations. We define the presheaf of smooth functions \(C^\infty_{S/D_S}\) on \(S/D_S\) as the map that associates to any open subset \(V\) of \(S/D_S\) the set of functions \(C^\infty_{S/D_S}(V)\) characterized by the following property: \(f \in C^\infty_{S/D_S}(V)\) if and only if for any \(z \in V\) there exists \(m \in \pi_{D_S}^{-1}(V), U_m\) open neighborhood of \(m\) in \(P\), and \(F \in C^\infty_P(U_m)\) such that

\[ f \circ \pi_{D_S}|_{\pi_{D_S}^{-1}(V) \cap U_m} = F|_{\pi_{D_S}^{-1}(V) \cap U_m}. \]

We say that \(F\) is a **local extension** of \(f \circ \pi_{D_S}\) at the point \(m \in \pi_{D_S}^{-1}(V)\). It can be proved (see [OR03]) that if \(S\) is a smooth embedded submanifold of \(P\) and \(D_S\) is a smooth, integrable, and regular distribution on \(S\) then the presheaf \(C^\infty_{S/D_S}\) coincides with the presheaf of smooth functions on \(S/D_S\) when considered as a regular quotient manifold.

We say that the presheaf \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-**local extension property** when the topology of \(S\) is stronger than the relative topology and, at the same time, the local extensions of \(f \circ \pi_{D_S}\) defined in \((6.10)\) can always be chosen so that

\[ df(n)|_{D(n)} = 0, \text{ for any } n \in \pi_{D_S}^{-1}(V) \cap U_m. \]

We say that \(F\) is a **local \(D\)-invariant extension** of \(f \circ \pi_{D_S}\) at the point \(m \in \pi_{D_S}^{-1}(V)\).
Definition 6.11 Let \((P, [\cdot, \cdot])\) be a Leibniz manifold, \(S\) a decomposed subset of \(P\), and \(D \subset TP|_S\) a Leibniz integrable generalised distribution adapted to the decomposition of \(S\). Assume that \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. We say that \((P, [\cdot, \cdot], D, S)\) is \textbf{Leibniz reducible} when \((S/D_S, C^\infty_{S/D_S}, [\cdot, \cdot]|_{S/D_S})\) is a well defined presheaf of Leibniz algebras where, for any open set \(V \subset S/D_S\), the bracket \([\cdot, \cdot]|_{S/D_S}: C^\infty_{S/D_S}(V) \times C^\infty_{S/D_S}(V) \rightarrow C^\infty_{S/D_S}(V)\) is given by

\[
[f, g]_{S/D_S}(\pi_{D_S}(m)) = [F, G](m),
\]
for any \(m \in \pi_{D_S}^{-1}(V)\). The maps \(F, G\) are local \(D\)-invariant extensions at \(m\) of \(f \circ \pi_{D_S}\) and \(g \circ \pi_{D_S}\), respectively.

The proof of the following theorem mimics the corresponding implication in Theorem 6.5.

Theorem 6.12 Let \((P, [\cdot, \cdot])\) be a Leibniz manifold with associated Leibniz tensor \(B\), \(S\) a decomposed space, and \(D \subset TP|_S\) a Leibniz integrable generalised distribution adapted to the decomposition of \(S\). Assume that \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. Then \((P, [\cdot, \cdot], D, S)\) is Leibniz reducible if for any \(m \in S\)

\[
B^L_R(\Delta_m) + B^L_R(\Delta_m) \subset \Delta^S_m \quad (6.11)
\]

where \(\Delta_m := \{dF(m) \mid F \in C^\infty_P(U_m), dF(z)|_{D(z)} = 0\), for all \(z \in U_m \cap S\), and for any open neighborhood \(U_m\) of \(m\) in \(P\} \) and \(\Delta^S_m := \{dF(m) \in \Delta_m \mid F|_{U_m \cap V_m}\) is constant for an open neighborhood \(U_m\) of \(m\) in \(P\) and an open neighborhood \(V_m\) of \(m\) in \(S\} \).

Remark 6.13 If \(S\) is endowed with the relative topology then

\[
\Delta^S_m := \{dF(m) \in \Delta_m \mid F|_{U_m \cap V_m}\) is constant for an open neighborhood \(U_m\) of \(m\) in \(P\} \).
\]

Remark 6.14 As opposed to the situation in Theorem 6.5, the condition \((6.11)\) is sufficient for Leibniz reducibility but in general not necessary. The reason behind this circumstance is that the functions that define the spaces \(\Delta_m\) and \(\Delta^S_m\) are not defined on saturated open sets which prevents the formulation of a result similar to Lemma 6.8. As we will see in Theorem 6.15 an alternative hypothesis that makes this condition necessary and sufficient is, roughly speaking, the regularity of the distribution \(D_S := D \cap TS\).

Reduction by regular canonical distributions. Let \((P, [\cdot, \cdot])\) be a Leibniz manifold and \(S\) an embedded submanifold of \(P\). Let \(D \subset TP|_S\) be a subbundle of the tangent bundle of \(P\) restricted to \(S\) such that \(D_S := D \cap TS\) is a smooth, integrable, and regular distribution on \(S\) and \(D\) is Leibniz. Our next theorem is a generalization of the main result of [MRS06] to the context of Leibniz manifolds.

Theorem 6.15 Let \((P, [\cdot, \cdot])\) be a Leibniz manifold with associated Leibniz tensor \(B\) and \(S\) an embedded smooth submanifold of \(P\). Let \(D \subset TP|_S\) be a canonical subbundle of the tangent bundle of \(P\) restricted to \(S\) such that \(D_S := D \cap TS\) is a smooth, integrable, and regular distribution on \(S\). Then \((P, [\cdot, \cdot], D, S)\) is Leibniz reducible if and only if

\[
B^L_R(D^\circ) + B^L_R(D^\circ) \subset TS + D. \quad (6.12)
\]

Proof. We first prove that the condition \((6.12)\) implies the Leibniz reducibility of \((P, [\cdot, \cdot], D, S)\). This implication can be obtained as a corollary of Theorem 6.12. Indeed, a result whose proof can be found in [OR03] guarantees that the hypotheses on \(D_S\) imply that the presheaf \(C^\infty_{S/D_S}\) has the \((D, D_S)\)-local extension property. Hence, it suffices to show that in this situation

\[
\Delta_m = D(m)^\circ, \quad (6.13)
\]

\[
[\Delta^S_m]^\circ = T_mS + D(m). \quad (6.14)
\]

In order to prove \((6.13)\) notice first that by definition \(\Delta_m \subset D(m)^\circ\). To prove the converse inclusion take \(\alpha_m \in D(m)^\circ\) arbitrary and let \(U_m\) be a submanifold chart of \(S\) around \(m\) that we can think of as
We prove this inclusion by using again the same adapted local submanifold coordinates around the point \( u \in U \). We have that \( dF(u,0) \cdot w = \langle w, u \rangle = 0 \), which implies that \( dF(z)_{|D(z)} = 0 \), for any \( z \in U_m \cap S \), as required.

We now prove (6.13). By definition \( T_mS + D(m) \subset [\Delta_m]^0 \). Conversely, the inclusion \( [\Delta_m]^0 \subset T_mS + D(m) \) holds if and only if \( D(m)^0 \cap T_mS^0 \subset \Delta_m^0 \), which, by (6.13), amounts to \( \Delta_m \cap T_mS^0 \subset \Delta_m^0 \).

We prove this inclusion by using again the same adapted local submanifold coordinates around the point \( m \). Let \( \alpha_m \in \Delta_m \cap T_mS^0 \) arbitrary. As we saw above, there exists \( \alpha \in E^2 \) such that \( \alpha_m = dF(0,0) \), with \( F : U \times V \to \mathbb{R} \) given by \( F(u,v) := \langle \alpha, (u,v) \rangle, (u,v) \in U \times V \). Since \( \alpha_m \in (T_mS)^0 \) we have that \( dF(0,0) \cdot (u,0) = \langle u, \alpha \rangle = 0 \), for any \( u \in F_1 \). This equality implies that \( F(u,v) = \langle \alpha, u \rangle + \langle \alpha, (u,0) \rangle \), for any \( (u,v) \in U \times V \), and hence \( F \) is constant on \( U = U_m \cap S \) which shows that \( \alpha_m \in \Delta_m^0 \), as required.

We now show that the reducibility of \((P, [\cdot, \cdot], D, S)\) implies (6.14) or, equivalently, that for any \( m \in S \)

\[
T_mS^0 \cap D(m)^0 \subset \left( (B^2_E(m) + B^2_S((m))(D(m)^0)) \right)^0.
\]

(6.15)

We proceed again by using the same local coordinates. In this occasion we consider a non degenerate inner product \( \langle \cdot, \cdot \rangle_{F_1 \oplus F_2} \) on \( F_1 \oplus F_2 \) defined by \( \langle (u_1, u_2), (v_1, v_2) \rangle_{F_1 \oplus F_2} = \langle u_1, v_1 \rangle_{F_1} + \langle u_2, v_2 \rangle_{F_2} \), with \( \langle \cdot, \cdot \rangle_{F_1} \) and \( \langle \cdot, \cdot \rangle_{F_2} \) non degenerate inner products in \( F_1 \) and \( F_2 \), respectively, and \( u_1, v_1 \in F_1, u_2, v_2 \in F_2 \). Given that \( U_m \cap S = U \subset F_1 \), any element \( \alpha_m \in T_mS^0 \) can be written as \( \alpha_m = \langle 0, (u_0,0) \rangle \), for some \( u_0 \in F_2 \) or, analogously, as \( \alpha_m = dK(m) \), with \( K \in C^\infty(U_m) \) defined by

\[
K(u,v) := \langle (0, u_0), (u,v) \rangle_{F_1 \oplus F_2} = \langle u_0, v \rangle_{F_2}.
\]

(6.16)

Moreover, if \( \alpha_m \in T_mS^0 \cap D(m)^0 \) then as \( D \) in these coordinates looks like \( D = U \times E \) for some vector subspace \( E \subset F_1 \oplus F_2 \), we have that the function \( K \) defined in (6.16) is such that

\[
K|_{U_m \cap S} = 0 \quad \text{and} \quad dK(z)_{|D(z)} = 0, \quad \text{for any} \ z \in U_m \cap S.
\]

We have thus proven that any \( \alpha_m \in T_mS^0 \cap D(m)^0 \) can be written as \( \alpha_m = dK(m) \) with \( K \) a local \( D \)-invariant extension of the zero function in \( S \) at the point \( m \in S \).

Let now \( \beta_m \in D(m)^0 \). Due to the non degeneracy of the inner product \( \langle \cdot, \cdot \rangle_{F_1 \oplus F_2} \) there exists \( w_0 \in F_1 \oplus F_2 \) such that \( \beta_m = dF(w) \) with \( F(u) := \langle w_0, u \rangle_{F_1 \oplus F_2}, u \in U_m \), and such that \( \langle w_0, u \rangle_{F_1 \oplus F_2} = 0 \) for any \( u \in E \).

The regularity of the distribution \( D_S \) implies via a result of Godement (see Lemma 3.5.26 in [AMR95]) that the neighborhood \( U_m \) can be shrunk so that there exists a smooth submanifold \( T \) of \( U_m \cap S \) (called a local slice of \( D_S \)) and a smooth map \( s : U_m \cap S \to T \) such that \( s|_T \) is the identity map on \( T \) and the integral leaf \( L_u \) of \( D_u \) on \( S \) that contains any arbitrary point \( u \in U_m \) is such that \( L_u \cap T = \{ s(u) \} \).

Notice now that since \( \text{d}F|_{U_m \cap S} \cdot D_S|_{U_m \cap S} = 0, \) we can use the slice and the map \( F|_{U_m \cap S} \) to define another map \( f \in C^\infty_{D_S}(\pi_D, U_m \cap S) \) as the unique map that satisfies

\[
f \circ \pi_D(z) = F(z) = F(s(z)), \quad \text{for any} \ z \in U_m \cap S.
\]

Using the constructions in the last two paragraphs we can now write for any \( \alpha_m \in T_mS^0 \cap D(m)^0 \) and any \( \beta_m, \beta_m^2 \in D(m)^0 \)

\[
\langle \alpha_m, (B^2_L(\beta_m^1) + B^2_R(\beta_m^2)) \rangle
\]

\[
= [K,F](m) - [G,K] = [0, f]|_{\pi_SD_S(U_m \cap S)}(\pi_D_S(m)) - [g,0]|_{\pi_SD_S(U_m \cap S)}(\pi_D_S(m)) = 0,
\]

(6.17)

where in the last equality we used the Leibniz reducibility of \((P, [\cdot, \cdot], D, S)\) to write

\[
[0, f]|_{\pi_SD_S(U_m \cap S)}(\pi_D_S(m)) = [0, F](m) \quad \text{and} \quad [g,0]|_{\pi_SD_S(U_m \cap S)}(\pi_D_S(m)) = [G,0](m),
\]
since the zero function on $M$ is also an extension of the zero function on $S$ that can be used instead of $K$ in the definition of the bracket. The expression (6.17) establishes (6.15).

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