The coloring problem for \( \{P_5, \overline{P_5}\}\)-free graphs and \( \{P_5, K_p - e\}\)-free graphs is polynomial

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Abstract

We show that determining the chromatic number of a \( \{P_5, \overline{P_5}\}\)-free graph or a \( \{P_5, K_p - e\}\)-free graph can be done in polynomial time.

Keywords: computational complexity, coloring problem, hereditary class, efficient algorithm

1 Introduction

A coloring is an arbitrary mapping from the set of vertices or edges of a graph into a set of colors of the graph such that any adjacent vertices (or edges) are colored with different colors. The minimal number of colors sufficient for coloring a graph \( G \) is said to be the chromatic number of \( G \) denoted by \( \chi(G) \). The coloring problem is to decide whether \( \chi(G) \leq k \) or not for given graph \( G \) and a number \( k \). A similar \( k \)-colorability problem is to check whether a given graph can be colored with at most \( k \) colors. Both problems can be naturally defined in another way via partition into independent sets. An independent set of graph is an arbitrary set of pairwise nonadjacent vertices. A coloring is partitioning of vertex set of a graph into independent subsets called color classes.

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There is a natural lower bound for the chromatic number of a graph. A *clique* in a graph is a subset of pairwise adjacent vertices. The size of a maximum clique in a graph \( G \) is called the *clique number* of \( G \) denoted by \( \omega(G) \). Clearly, \( \chi(G) \geq \omega(G) \). Sometimes, computing \( \omega(G) \) helps to determine \( \chi(G) \) \([8, 14]\).

A class of graphs is called *hereditary* if it is closed under isomorphism and deletion of vertices. It is well known that any hereditary (and only hereditary) graph class \( \mathcal{X} \) can be defined by a set of its forbidden induced subgraphs \( \mathcal{S} \). We write \( \mathcal{X} = \text{Free}(\mathcal{S}) \) in this case, and graphs in \( \mathcal{X} \) are said to be \( \mathcal{S} \)-free. If \( \mathcal{S} = \{G\} \), then we write ”\( G \)-free” instead of ”\( \{G\} \)-free”.

We say that \( \mathcal{X} \) is *easy* for the coloring problem if \( \mathcal{X} \) is hereditary and the problem can be polynomially solved for it.

The computational complexity of the coloring problem was completely determined for all classes of the form \( \text{Free}(\{G\}) \) \([11]\). A study of forbidden pairs was also initiated in \([11]\). A complete complexity dichotomy appeared hard to obtain even in the cases of two four-vertex and connected five-vertex forbidden induced subgraphs \([12]\) \([13]\). For all but three cases either NP-completeness or polynomial-time solvability was shown in the family of hereditary classes defined by four-vertex forbidden induced structures \([12]\). The remaining three classes are stubborn. A similar result was obtained in \([13]\) for two connected five-vertex forbidden induced fragments, where the number of open cases was 13. Recently, it was reduced to 11 \([14]\). We reduce the number to nine by showing that the coloring problem can be solved for \( \{P_5, P_5\} \)-free and \( \{P_5, K_p - e\} \)-free graphs in polynomial time.

**2 Notation**

As usual, \( P_n, C_n, O_n, K_n \) stand respectively for the simple path, the chordless cycle, the empty graph, the complete graph with \( n \) vertices respectively. A graph \( K_p - e \) is obtained from \( K_p \) by deleting an arbitrary edge. A formula \( N(x) \) means the neighborhood of a vertex \( x \) of some graph. For a graph \( G \) and a set \( V' \subseteq V(G) \), \( G(V') \) denotes the subgraph of \( G \) induced by \( V' \).

We refer to textbooks in graph theory for any graph terminology undefined here.
3 Auxilliary results

3.1 Decomposition by clique separators and its applications to the coloring problem

A clique separator in a graph is a clique whose removal increases the number of connected components. For example, the graph \( K_p - e \) has a clique separator with \( p - 2 \) vertices. If a graph \( G \) has a clique separator \( Q \), then \( V(G) \setminus Q \) can be arbitrarily partitioned into nonempty subsets \( A \) and \( B \) such that no vertex of \( A \) is adjacent to a vertex of \( B \). Let \( G_1 \triangleq G(A \cup Q) \) and \( G_2 \triangleq G(B \cup Q) \). We repeat a similar decomposition until no further decomposition is possible. The whole process can be represented by a binary decomposition tree whose leaves correspond to some induced subgraphs of \( G \) without clique separators. There exists an \( O(mn) \)-time algorithm for constructing some binary decomposition tree for any graph with \( n \) vertices and \( m \) edges [15].

**Lemma 1** For each graph \( G \), \( \chi(G) = \max(\chi(G_1), \chi(G_2)) \).

**Proof.** Without loss of generality, \( \chi(G_1) \leq \chi(G_2) \). Let us consider a partial coloring of \( G \) induced by an optimal coloring of \( G_2 \) and color classes of \( G_1 \) in its optimal coloring containing all vertices of \( Q \). These color classes can be colored with colors assigned to elements of \( Q \) in the partial coloring. To color the remaining part of \( A \), it is enough \( \chi(G_1) - |Q| \) colors distinct to the colors of \( Q \). The set \( B \) has \( \chi(G_2) - |Q| \geq \chi(G_1) - |Q| \) color classes with colors of this type. Hence, \( G \) can be colored with \( \chi(G_2) \) colors. So, \( \chi(G) = \chi(G_2) \).

A maximal induced subgraph of a given graph without proper clique separators will be called a C-block of the graph. Leaves of a decomposition tree of any graph correspond to its C-blocks. Let \( \mathcal{X} \) be a class of graphs. The set of all graphs whose every C-block belongs to \( \mathcal{X} \) will be called the C-closure of \( \mathcal{X} \) denoted by \( [\mathcal{X}]_C \).

**Theorem 1** If \( \mathcal{X} \) is easy for the coloring problem, then it is so for \( [\mathcal{X}]_C \).

**Proof.** Clearly, \( [\mathcal{X}]_C \) is hereditary. All C-blocks of a graph \( G \in [\mathcal{X}]_C \) belong to \( \mathcal{X} \), and the coloring problem can be solved in polynomial time for them. A decomposition tree for \( G \) can be constructed in polynomial time. Hence, by the previous lemma, \( [\mathcal{X}]_C \) is easy for the coloring problem. ■
3.2 Modular decomposition and its applications to the weighted coloring problem

A set \( M \subseteq V(G) \) is a module in a graph \( G \) if either \( x \) is adjacent to all elements of \( M \) or none of them for each \( x \in V(G) \setminus M \). Each vertex of \( G \) and the set \( V(G) \) constitute a module called trivial. A module \( M \) is a nontrivial module in \( G \) if \( |M| > 1 \) and \( M \neq V(G) \). A graph containing no nontrivial modules is said to be prime. For instance, \( P_4 \) is prime and \( C_4 \) does not.

Modular decomposition of graphs is an algorithmic technique based on the following decomposition theorem due to T. Gallai.

**Theorem 2** Let \( G \) be a graph with at least two vertices. Then exactly one of the following conditions holds:

1. \( G \) is not connected
2. \( \overline{G} \) is not connected
3. \( G \) and \( \overline{G} \) are connected, and there is a set \( V' \) with at least four elements and an unique partition \( P(G) \) of \( V(G) \) such that
   
   (a) \( G(V') \) is a maximal prime induced subgraph of \( G \)
   
   (b) for each \( V'' \in P(G) \), \( V'' \) is a module (perhaps, trivial) in \( G \) and \( |V'' \cap V'| = 1 \).

By the theorem, there are decomposition operations of three types. First, if \( G \) is not connected, then disconnect it into connected components \( G_1, \ldots, G_p \). Second, if \( \overline{G} \) has connected components \( \overline{G}_1, \ldots, \overline{G}_q \), then decompose \( G \) into \( G_1, \ldots, G_q \). At length, if \( G \) and \( \overline{G} \) are connected, then its maximal modules are pairwise disjoint, and they form the partition \( P(G) \). The graph \( G \) is decomposed into subgraphs in \( \{ G(V'') \mid V'' \in P(G) \} \). Additionally, each class of \( P(G) \) is contracted to obtain a graph which is isomorphic to \( G(V') \). In other words, \( G(V') \) is an induced subgraph of \( G \) producing by taking one element in each class of \( P(G) \).

The decomposition process above can be represented by an uniquely determined tree called the modular decomposition tree of \( G \). Its vertices are induced subgraphs of \( G \). A vertex \( G \) has the connected components of \( G \) or \( \overline{G} \) as the children in the first two cases; the children are subgraphs of the form \( G(V''), V'' \in P(G) \) in the third one. Moreover, we associate the graph \( G(V'') \) with the vertex \( G \). The modular decomposition tree can be determined in \( O(n + m) \)-time for any graph with \( n \) vertices and \( m \) edges.

The weighted coloring problem is to find, for given \( G \) and a function \( w : V(G) \to \mathbb{N} \), the smallest number \( k \) such that there is a function \( c : \)
\[ V(G) \to 2^{\{1,2,\ldots,k\}} \] such that \(|c(v)| = w(v)\) for any \(v\) and \(c(v_1) \cap c(v_2) = \emptyset\) for any adjacent \(v_1\) and \(v_2\). The elements of \(c(v)\) are called the \textit{colors of} \(v\). This \(k\) is denoted by \(\chi_w(G)\) and called the \textit{weighted chromatic number} of \(G\). For every graph \(G\), \(\chi_w'(G) = \chi(G)\), where \(w'\) maps every vertex to 1.

Clearly, for each function \(w\), we have \(\chi_w(G) = \max_i (\chi_w(G_i))\), where \(G_1, \ldots, G_p\) are connected components of \(G\). Similarly, if \(\overline{G}_1, \ldots, \overline{G}_q\) are connected components of \(\overline{G}\), then \(\chi_w(G) = \sum_{i=1}^{q} \chi_w(G_i)\).

**Lemma 2** Let \(G\) be a graph, \(P(G)\) be its modular decomposition, \(w : V(G) \to \mathbb{N}\) be an arbitrary function. Then \(\chi_w(G) = \chi_w(G(V'))\), where \(w'(v) = \chi_w(G(V''))\) for each \(v \in V', V'' \in P(G), \{v\} = V' \cap V''\).

**Proof.** Contraction of \(V''\) to \(v\) and assignment \(w(v) = \chi_w(G(V''))\) produces a subgraph whose weighted chromatic number is at most \(\chi_w(G)\). On the other hand, each element of \(N(v)\) cannot have some \(\chi_w(G(V''))\) colors of \(v\). Hence, the weighted chromatic number of the subgraph is equal to \(\chi_w(G)\). Therefore, \(\chi_w(G) = \chi_w'(G(V'))\).

Let \([\mathcal{X}]_P\) be the set of graphs whose every prime induced subgraph belongs to \(\mathcal{X}'\). Clearly, \([\mathcal{X}]_P\) is hereditary whenever \(\mathcal{X}\) is hereditary. The theorem below follows from the previous lemma and \([3]\).

**Theorem 3** If \(\mathcal{X}\) is an easy class for the coloring problem, then it is so for \([\mathcal{X}]_P\).

### 3.3 Bipartite Ramsey theorem

A famous Ramsey theorem claims that any graph has a sufficiently large independent set or a sufficiently large clique. There are numerous its analogues for different classes of graphs, e.g. for bipartite graphs. Recall that a graph is \textit{bipartite} if its vertex set can be partitioned into at most two independent sets. These independent sets are called \textit{parts}. A \textit{matching} in a graph is a subset of pairwise nonadjacent edges. The following result is a corollary of theorem 2 from \([5]\) for \(H = K_{s,s}\).

**Lemma 3** Any bipartite graph \(G\) having parts \(A\) and \(B\) with \(n > s^{s+1}\) vertices contains subsets \(A' \subseteq A, B' \subseteq B, |A'| = |B'| = \lfloor (\frac{n}{2})^{\frac{1}{s}} \rfloor\) such that \(G(A' \cup B')\) is empty or complete bipartite.
3.4 Connected \( \{P_5, K_p - e\} \)-free graphs without clique separators

Let \( G \) be a connected \( \{P_5, K_p - e\} \)-free graph \( (p \geq 3) \) without clique separators, and let \( Q \) be its maximum clique.

**Lemma 4** The graph \( G \) is \( O_3 \)-free or \( |Q| \leq (p + 1)^{p+2}(p - 2) \).

**Proof.** Assume that \( |Q| > (p + 1)^{p+2}(p - 2) \). Let \( N(Q) \triangleq \{y \notin Q \mid \exists x \in Q, (y, x) \in E(G)\} \). Any element of \( N(Q) \) cannot be adjacent to \( p - 2 \) or more vertices of \( Q \). Let us consider a bipartite graph \( G' \) induced by edges between \( Q \) and \( N(Q) \). As \( G \) has no clique separators, \( Q \) and \( N(Q) \) are parts of \( G' \). Clearly, the graph \( G' \) has a matching with \( \left\lceil \frac{|Q|}{p-2} \right\rceil \) edges, and it is \( K_{p-2,p-2} \)-free.

Let \( N_1 \triangleq \{u_1, u_2, \ldots, u_k\} \) be a maximum subset of \( Q \) such that \( N(Q) \) has vertices \( v_1, v_2, \ldots, v_k \) with \( v_i \in N(u_i) \setminus \bigcup_{j \neq i} N(u_j) \) for each \( i \). By the previous lemma for \( s = p + 1, k \geq \left\lceil \frac{|Q|}{p-2} \right\rceil \geq p + 1 \). As \( p \geq 3 \), \( N_2 \triangleq \{v_1, v_2, \ldots, v_k\} \) must be an independent set or a clique to avoid an induced \( P_5 \). If \( N_2 \) is independent, then there is no a vertex \( v_i \) having a neighbor \( w \notin Q \cup N(Q) \). Otherwise, \( w \) must be adjacent to all vertices of \( N_2 \), and \( G \) is not \( P_5 \)-free. Hence, a possible neighbor \( w \notin Q \) of an element \( v_i \in N_2 \) must belong to \( N(Q) \). To avoid an induced \( P_5 \), \( w \) must be adjacent to all elements of \( N_2 \) or to \( v_i \) only. The second case is realized if and only if \( N_1 \) has only one neighbor of \( w \) coinciding with \( u_i \). In the first case, there are some three non-neighbors \( u_{i1}, u_{i2}, u_{i3} \) of \( w \), as \( G' \) is \( K_{p-2,p-2} \)-free. But \( v_i, w, v_{i2}, u_{i2}, u_{i3} \) induce \( P_5 \). Hence, any possible neighbor \( w \) of \( v \) that lies outside \( Q \) must be adjacent to \( u_i \) and nonadjacent to \( u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_k \). Similarly, \( N(u_i) \subseteq N(u_i) \cup \{u_i\} \). Hence, \( Q \) is a clique separator. Thus, \( N_2 \) must be a clique.

Let \( Q' \) be a maximal clique that includes \( N_2 \). Suppose that \( v \in N(Q) \setminus Q' \). Since \( N_1 \) is maximum, \( v \) has neighbors in \( N_1 \), say, \( u_1, \ldots, u_q \). Clearly, \( q \leq p - 3 \). To avoid an induced \( P_5 \), \( v \) must be adjacent to at least \( k - q - 1 \) vertices among \( v_{q+1}, \ldots, v_k \). Similarly, \( v \) must be adjacent to \( v_1, \ldots, v_q \). Hence, \( v \) is adjacent to at least \( k - 1 \) vertices of \( N_2 \). To avoid an induced \( K_{p-2, p-2} \), \( v \in Q' \). Thus, \( N(Q) \setminus Q' = \emptyset \). In fact, \( Q' = N(Q) \) and \( V(G) = Q \cup N(Q) \), since \( N(Q) \) is a clique separator otherwise. So, \( G \) is \( O_3 \)-free. 

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3.5 Connected prime \( \{P_5, \overline{P_5}\} \)-free graphs

A graph is said to be perfect if the clique number and the chromatic number are equal for every its induced subgraph (not necessarily proper). The class of perfect graphs coincides with \( \text{Free}(\{C_5, \overline{C_5}, C_7, \overline{C_7}, \ldots\}) \), by the strong perfect graph theorem \( [2] \).

**Lemma 5** Any connected prime \( \{P_5, \overline{P_5}\} \)-free graph is perfect or isomorphic to \( C_5 \).

**Proof.** Every \( \{P_5, \overline{P_5}, C_5\} \)-free graph is perfect, by the strong perfect graph theorem. Let \( G \) be a connected prime \( \{P_5, \overline{P_5}\} \)-graph containing an induced \( C_5 \). Every element of \( V(G) \setminus V(C_5) \) is either adjacent to all vertices of \( C_5 \) or to none of them or to two nonadjacent or to three consecutive \( [6] \). Let \( V_i \) be the set of vertices of \( G \) adjacent to the \((i - 1)\)-th and \((i + 1)\)-th vertices of \( C_5 \) counting modulo 5. Let \( V_0 \) be the set of vertices adjacent to all vertices of \( C_5 \). Any element of \( V_i \) is adjacent to each element of \( V_0 \cup V_{i-1} \cup V_{i+1} \), nonadjacent to any element of \( V_i \setminus V(C_5) \) cannot have neighbors outside \( \bigcup_{i=0}^{5} V_i \cup V(C_5) \) \( [6] \). Suppose that \( G \) is not isomorphic to \( C_5 \). Then \( V_i \) has at least two elements for some \( i \) or \( V_0 \neq \emptyset \) and \( |V_1| = |V_2| = |V_3| = |V_4| = |V_5| = 1 \). The set \( V_i \) is a nontrivial module in the first case, and \( V(C_5) \) is a nontrivial module in the second one. We have a contradiction with the assumption. ■

4 Main result

**Theorem 4** The class \( \text{Free}(\{P_5, \overline{P_5}\}) \) and all classes of the form \( \text{Free}(\{P_5, K_p - e\}) \) are easy for the coloring problem.

**Proof.** It is known that for any \( P_5 \)-free graph \( G \) the inequality \( \chi(G) \leq 4^{w(G)-1} \) holds \( [9] \). Moreover, for each fixed \( k \), the \( k \)-colorability problem can solved in polynomial time for \( P_5 \)-free graphs \( [10] \). Hence, by these results, Theorem 1 and Lemma 4, the coloring problem for \( \{P_3, K_p - e\} \)-free graphs can be polynomially reduced to the same problem for \( O_3 \)-graphs. The coloring problem for \( O_3 \)-free graphs is polynomially equivalent to determining the sizes of maximum matchings in the complement graphs. The last problem is known to be polynomial \( [4] \). Hence, \( \{P_5, K_p - e\} \)-free graphs constitute
an easy class for the coloring problem. The class of perfect graphs is easy for the weighted coloring problem \[8\]. Perfect graphs can be recognized in polynomial time \[1\]. Hence, by these facts, Theorem 3 and Lemma 5, \(\text{Free}(\{P_5, \overline{P_5}\})\) is easy for the coloring problem. ■

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