Path Integral for non-relativistic Generalized Uncertainty Principle corrected Hamiltonian

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Abstract

Generalized Uncertainty Principle (GUP) has brought the idea of existence of minimum measurable length in Quantum physics. Depending on this GUP, non-relativistic Hamiltonian at the Planck scale is modified. In this article, we construct the kernel for this GUP corrected Hamiltonian for free particle by applying the Hamiltonian path integral approach and check the validity conditions for this kernel thoroughly. Interestingly, the probabilistic interpretation of this kernel induces a momentum upper bound in the theory which is comparable with GUP induced maximum momentum uncertainty.

1 Introduction:

There are many indications that in quantum gravity there might exist a minimal observable distance of the order of the Planck length, \(l_{pl} \approx 10^{-33} \text{ cm}\). Generalized Uncertainty Principle (GUP) \([1]\) naturally encodes the idea of existence of a minimum measurable length through modifications in the Poisson brackets of position \(x\) and momentum \(p\). The Heisenberg Uncertainty Principle (HUP) says that uncertainty in position decreases with increasing energy (\(\Delta x \sim \frac{\hbar}{\Delta p}\)). But HUP breaks down for energies close to Planck scale, at which point the Schwarzschild radius becomes comparable to Compton wavelength. Higher energies result in a further increase of the Schwarzschild radius, inducing the following relation: \(\Delta x \approx l_{pl}^{2} \frac{\Delta p}{\hbar}\). Consistent with the above, the following form of GUP has been proposed, postulated to hold in all scales \([3]\)

\[
\Delta x_i \Delta p_i \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta p^2 + \langle p \rangle^2) + 2\beta (\Delta p_i^2 + \langle p_i \rangle^2) \right] \quad i = 1, 2, 3
\]

(1)

where \([\beta] = (\text{momentum})^{-2}\) and we will assume that \(\beta = \beta_0 / (M_{pl}c)^2 = l_{pl}^2 / 2 \hbar^2\), \(M_{pl}\) = Planck mass, and \(M_{pl}c^2\) = Planck energy \(\approx 10^{19} \text{ GeV}\). It is evident that the parameter \(\beta_0\) is dimensionless and normally assumed to be \(\beta_0 \approx 1\). In one dimension the above inequality takes the form:

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + 3\beta (\Delta p^2 + \langle p \rangle^2) \right]
\]

(2)
from which we have
\[ \Delta p \leq \frac{\Delta x}{3\hbar} + \sqrt{\left(\frac{\Delta x}{3\hbar}\right)^2 - \frac{1 + 3\beta(p)^2}{3\beta}}. \] (3)

Since \( \Delta p \) is a real quantity, we have
\[ \left(\frac{\Delta x}{3\hbar}\right)^2 \geq \frac{1 + 3\beta(p)^2}{3\beta} \Rightarrow \Delta x \geq \hbar \sqrt{3\beta \sqrt{1 + 3\beta(p)^2}} \] (4)

which gives the minimum bound for \( \Delta x \) as:
\[ \Delta x_{\text{min}} = \hbar \sqrt{3\beta}. \] (5)

Here one should notice that the condition \( \langle p \rangle = 0 \) gives this minimum bound (5), which is also consistent with the above inequality (3) and GUP relation (2). Now using relation (5) along with the condition \( \langle p \rangle = 0 \) we get the maximum bound of \( \Delta p \) as
\[ \Delta p_{\text{max}} = \frac{1}{\sqrt{3\beta}}. \] (6)

It can be shown \[3\] that the inequality (11) follows from the modified Heisenberg algebra
\[ [x_i, p_j] = i\hbar (\delta_{ij} + \beta \delta_{ij} p^2 + 2\beta p_i p_j). \] (7)

To satisfy the Jacobi identity, the above bracket (7) gives \( [x_i, x_j] = [p_i, p_j] = 0 \), to first order in \( O(\beta) \) \[4\]. Now defining
\[ x_i = x_{0i}, \quad p_i = p_{0i}(1 + \beta p_0^2) \] (8)

where \( p_0^2 = \sum_{j=1}^{3} p_{0j} p_{0j} \) and \( x_{0i}, p_{0j} \) satisfying the canonical commutation relations \( [x_{0i}, p_{0j}] = i\hbar \delta_{ij} \), it is easy to show that the above commutation relation (7) is satisfied, to first order of \( \beta \). Henceforth, we neglect terms of order \( \beta^2 \) and higher. The effects of this GUP \[11\] in Lamb shift and Landau levels have been studied in \[5\]. Also, formulation of coherent states for this GUP has been described in \[6\]. In this work, we successfully derive the kernel for this GUP model by Hamiltonian path integral formulation \[7\] and show that this GUP corrected kernel induces a maximum momentum bound in the theory.

Using (8), we start with the corresponding Hamiltonian of the form
\[ H = \frac{p^2}{2m} + V(\vec{r}) \] (9)

which can be written as
\[ H = H_0 + H_1 + O(\beta^2), \] (10)

where
\[ H_0 = \frac{p_0^2}{2m} + V(\vec{r}), \quad H_1 = \frac{\beta}{m} p_0^4. \] (11)
Thus, we see that any system with an well defined quantum (or even classical) Hamiltonian \( H_0 \), is perturbed by \( H_1 \), near the Planck scale. In other words, Quantum Gravity effects are in some sense universal. Now the modified Schrodinger equation corresponding to the above Hamiltonian (10) is given by

\[
- \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + \frac{\beta \hbar^4}{m} \frac{\partial^4}{\partial x^4} \psi(x,t) + V(x)\psi = i\hbar \frac{\partial}{\partial t} \psi(x,t).
\] (12)

In this paper we will see that path integral method [8] is applicable to this higher energy cases and we will evaluate the free particle kernel for GUP corrected Hamiltonian (10). For this purpose, we shall briefly recall the notion of basic properties of kernel in Hamiltonian path integral formalism. The kernel in Hamiltonian path integral is given by [7]

\[
K(x'', x') = \int \left[ e^{i \int (\vec{p} - \vec{H}) dt} \right] \frac{dp_1}{2\pi \hbar} \frac{dp_2}{2\pi \hbar} ... \frac{dp_N}{2\pi \hbar} dx_1 ... dx_{N-1}.
\] (13)

It has been shown in [7] that the above kernel (13) can be written in the form

\[
K(x'', x', \Delta t) = \delta(\vec{x}'' - \vec{x}') - \frac{i\Delta t}{\hbar} \left[ - \frac{\hbar^2 \nabla^2}{2m} \times \delta(\vec{x}'' - \vec{x}') + V(x) \delta(\vec{x}'' - \vec{x}') \right]
\] (14)

from which one can easily obtain Schrodinger equation using the relation

\[
\psi(x'', t'') = \int K(x'', t''; x', t') \psi(x', t') \, dx'.
\] (15)

Now taking the complex conjugate of the above equation, we have

\[
\psi^*(x'', t'') = \int K^*(x'', t''; x', t') \psi^*(x', t') \, dx'.
\] (16)

Since

\[
\int \psi^*(x'', t'') \psi(x'', t'') \, dx'' = \int \psi^*(x', t') \psi(x', t') \, dx',
\]

using the relations (15), (16) we have

\[
\int \int K^*(x'', t''; x_1', t') K(x'', t''; x', t') \psi^*(x_1', t') \psi(x', t') \, dx'' \, dx_1' \, dx' = \int \psi^*(x', t') \psi(x', t') \, dx',
\] (17)

which immediately implies the following relation

\[
\int \int K^*(x'', t''; x_1', t') K(x'', t''; x', t') \psi^*(x_1', t') \, dx'' \, dx_1' = \psi^*(x', t').
\] (18)

Also, if \( \psi(x, t) \) is the solution of the Schrodinger equation

\[
- \frac{\hbar^2}{2m} \nabla^2 \psi(x, t) + V(x)\psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t),
\] (19)

then the kernel also satisfy Schrodinger equation at the end point \( x = x'' \), i.e

\[
- \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x''^2} K(x'', t''; x', t') + V(x'') K(x'', t''; x', t') = i\hbar \frac{\partial}{\partial x''} K(x'', t''; x', t').
\] (20)

Equation (15), (18) and (20) are the basic properties of the kernel \( K(x, t) \).
2 Kernel for GUP corrected Hamiltonian:

Path integral method \[8\] is applicable in all cases where the change of action, corresponding to the variation of path, is large enough compared to \(\hbar\). As the above Hamiltonian \[10\] is associated with higher energy, so a small variation on paths other than the path of least action make enormous change in phase for which cosine or sine will oscillate exceedingly rapidly between plus and minus value and cancel out their total contribution. So only the least action path will contribute in kernel. This is similar to the idea of path integral in quantum mechanics. We now therefore consider the Hamiltonian \[10\] in one dimension

\[
H = \frac{p^2}{2m} + \frac{\beta}{m}p^4 + V(x).
\]  

If we consider that the particle goes from \(x'\) to \(x''\) during the short time interval \(\Delta t\), then the kernel is of the form

\[
K(x'', t' + \Delta t ; x', t') = \int e^{i\int_{t'}^{t' + \Delta t} \left( p_0 \dot{x} - \frac{m}{2} \dot{x}^2 - \frac{\beta}{m} p_0^4 - V(x) \right) dt} \frac{dp_0}{2\pi\hbar} = \int e^{\frac{i}{\hbar} \int_{t'}^{t' + \Delta t} \left( \frac{p_0^2}{2m} - \frac{\beta}{m} p_0^4 - \Delta t V(x) \right) dt} \frac{dp_0}{2\pi\hbar}
\]

\[
= \int e^{\frac{i}{\hbar} \int_{t'}^{t' + \Delta t} \left[ \frac{p_0^2}{2m} + \frac{\beta}{m} p_0^4 + \Delta t \bar{V}(x) \right] dt} \frac{dp_0}{2\pi\hbar},
\]  

(22)

where \(\bar{V}(x)\) is the average of \(V(x)\) over the straight line connecting \(x''\) and \(x'\). Expanding the second exponential function in (22) and neglecting the second and higher order terms of \(\Delta t\), we have

\[
K(x'', t' + \Delta t ; x', t') = \int e^{\frac{i}{\hbar} \int_{t'}^{t' + \Delta t} \left( p_0 \dot{x} - \frac{m}{2} \dot{x}^2 - \frac{\beta}{m} p_0^4 + \bar{V}(x) \right) dt} \frac{dp_0}{2\pi\hbar}
\]

\[
= \delta(x'' - x') - \frac{i\Delta t}{\hbar} \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x'^2} \delta(x'' - x') + \frac{\beta \hbar^4}{m} \frac{\partial^4}{\partial x'^4} \delta(x'' - x') + \bar{V}(x) \delta(x'' - x') \right].
\]  

(23)

It is interesting to note that the kernel (23) boils down to the same form as in [7] in the limit \(\beta \to 0\). But it is very difficult to deal with above form of kernel (23) as it contains derivative of delta function. Therefore we are going to derive the delta-independent equivalent form of kernel. For this we consider the kernel for free particle,

\[
K(x'', t' + \Delta t ; x', t') = \int e^{\frac{i}{\hbar} \int_{t'}^{t' + \Delta t} \left( p_0 \dot{x} - \frac{m}{2} \dot{x}^2 \right) dt} \frac{dp_0}{2\pi\hbar},
\]  

(24)

for short time interval \(\Delta t\). Now expanding the exponential series of the last term in (24) and neglecting the terms containing higher order of \(\beta\) we have,

\[
K(x'', t' + \Delta t ; x', t') = \int e^{-\frac{i\Delta t}{\hbar} \left( p_0 \dot{x} - \frac{m}{2} \dot{x}^2 \right) \Delta t} \left( 1 - \frac{i}{\hbar} \frac{\beta \Delta t}{m} p_0^4 \right) \frac{dp_0}{2\pi\hbar}.
\]  

(25)

After some calculation we get the kernel as,

\[
K(x'', t' + \Delta t ; x', t') = \sqrt{\frac{m}{2\pi i \hbar \Delta t}} \left[ 1 + \frac{3\beta i \hbar m}{\Delta t} - \frac{6\beta m^2 (x'' - x')^2}{\Delta t^2} - \frac{i\beta m^3 (x'' - x')^4}{h \Delta t^3} \right] e^{\frac{i\beta m^3 (x'' - x')^4}{2m \Delta t^3}}.
\]  

(26)
For a finite interval, we divide the time interval into \( N \) subintervals of equal length \( \Delta t \) and then we calculate the kernel as:

\[
K(x''; t''; x', t') = \left( \frac{m}{2\pi i\hbar \Delta t} \right)^{N} \int dx_{1} dx_{2} \ldots dx_{N-1} \ e^{i\frac{\hbar}{\Delta t}(x_{1}-x_{0})^{2}+(x_{2}-x_{1})^{2}+\ldots+(x_{N}-x_{N-1})^{2}} \\
\times \{ 1 + \frac{3\beta \hbar m}{\Delta t} - \frac{6\beta m^{2}(x_{1}-x_{0})^{2}}{\hbar \Delta t} \} \{ 1 + \frac{3\beta \hbar m}{\Delta t} - \frac{6\beta m^{2}(x_{2}-x_{1})^{2}}{\hbar \Delta t} \} } \\
\ldots \{ 1 + \frac{3\beta \hbar m}{\Delta t} - \frac{6\beta m^{2}(x_{N}-x_{N-1})^{2}}{\hbar \Delta t} \} \} \\
\right]
\]

(27)

After a bit of lengthy algebra (see appendix: 1), the final form of the kernel becomes

\[
K(x''; t''; x', t') = \sqrt{\frac{m}{2\pi i\hbar (t'' - t')}} \ e^{i\frac{\hbar}{2\hbar}(x'' - x')^{2}} \left[ 1 + \frac{3\beta \hbar m}{(t'' - t')} - \frac{6\beta m^{2}(x'' - x')^{2}}{(t'' - t')^{2}} - \frac{i\beta m^{3}(x'' - x')^{4}}{h(t'' - t')^{3}} \right],
\]

(28)

where \( t'' - t' = N\Delta t \). This kernel (28) is exactly of the same form as the kernel for the infinitesimal interval (26). It can be shown that the above kernel (28) satisfies the modified Schrodinger equation

\[
-\frac{\hbar^{2}}{2m} \frac{\partial^{2}}{\partial x^{2}} \psi(x, t) + \frac{\beta \hbar^{4}}{m} \frac{\partial^{4}}{\partial x^{4}} \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t),
\]

(29)

at the point \( x = x'' \), \( t = t'' \) (see appendix: 2). Now the solution of this Schrodinger equation (29) is given by [5]

\[
\psi(x, t) = \left( A e^{i(k(1-\beta^{2}k^{2})x - \frac{\beta E}{\hbar})} + B e^{-i(k(1-\beta^{2}k^{2})x - \frac{\beta E}{\hbar})} + C e^{\sqrt{2\beta}x - \frac{\beta E}{\hbar}} + D e^{-\sqrt{2\beta}x - \frac{\beta E}{\hbar}} \right).
\]

(30)

With this solution (30) in hand, we can show that the kernels (2) and (26) indeed propagates the wave function \( \psi(x, t) \) from a point \( (x', t') \) to the point \( (x'', t'') \), for a chosen time interval \( \Delta t = t'' - t' \), such that \( \frac{\Delta t}{4\beta \hbar m} = D \) a dimensionless quantity \( o(\beta) \). Thus the following relation holds:

\[
\psi(x'', t'') = \int K(x'', t''; x', t') \psi(x', t') \, dx',
\]

(31)

where we use the fact that

\[
e^{-\frac{iEt'}{\hbar}} = e^{-\frac{iEt'}{\hbar}} + \frac{iE \Delta t}{\hbar} = e^{-\frac{iEt'}{\hbar}} + 4i E m (\beta D) = e^{-\frac{iEt'}{\hbar}} + 4i E m O(\beta^{2}) = e^{-\frac{iEt'}{\hbar}}
\]

(32)

Now for finite interval, we divide the time interval \( (t'' - t') \) into \( N \) parts of equal length \( \Delta t \) in such a manner that \( \frac{\Delta t}{4\beta \hbar m} = D = o(\beta) \). Then applying equation (31) for each interval, we have

\[
\int K(x'', t''; x', t') \psi(x', t') \, dx' \\
= \int \int \ldots K(x'', t''; x_{N-1}, t_{N-1}) K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \ldots K(x_{1}, t_{1}; x', t') \, dx' \, dx_{1} \ldots dx_{N-1} \\
= \int \int \ldots K(x'', t''; x_{N-1}, t_{N-1}) K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) \ldots K(x_{2}, t_{2}; x_{1}, t_{1}) \psi(x_{1}, t_{1}) \, dx_{1} \, dx_{2} \ldots dx_{N-1} \\
= \ldots = \psi(x'', t'')
\]

(33)
Therefore for a finite interval \((t'' - t')\), we have

\[
\psi(x'', t'') = \int K(x'', t''; x', t') \psi(x', t') dx'.
\] (34)

The alternative way to check the above relation is the condition

\[
\int \int K^*(x'', t''; x_1', t') K(x'', t''; x', t') \psi^*(x_1', t') dx'' dx_1' = \psi^*(x', t').
\] (35)

which is proved in appendix: 3. Therefore the free particle kernel satisfies the basic properties of kernel, which we have stated in earlier section.

For usual free particle case, the probability that a particle arrives at the point \(x''\) is proportional to the absolute square of the kernel \(K(x'', x', t'' - t')\), i.e. for usual free-particle kernel the probability is given by

\[
P(x'')dx = \frac{m}{2\pi \hbar (t'' - t')} dx.
\] (36)

Now, for the GUP corrected kernel \([25]\), the corresponding probability is given by

\[
P(x'')dx = K^*(x'', x', t'' - t') K(x'', x', t'' - t') dx = \left(1 - \frac{12\beta m^2 (x'' - x')^2}{(t'' - t')^2}\right) \frac{m}{2\pi \hbar (t'' - t')} dx.
\] (37)

It is clearly observable that the term \(1 - \frac{12\beta m^2 (x'' - x')^2}{(t'' - t')^2}\) in (37) is less than 1 as \(\beta > 0\). Then we conclude that the probability value in this case is less than the corresponding value in the free particle case. Also, since probability is non-negative, this term \(1 - \frac{12\beta m^2 (x'' - x')^2}{(t'' - t')^2}\) should also be non-negative. Thus we have the following relation:

\[
1 - \frac{12\beta m^2 (x'' - x')^2}{(t'' - t')^2} \geq 0,
\] (38)

which immediately implies the bound for momentum as

\[
p \leq p_{\text{max}} = \frac{1}{2\sqrt{3}\beta},
\] (39)

where \(p = \frac{m x}{\epsilon}\). Thus, from (39) we see that GUP induces a momentum upper bound in the theory which is comparable to maximum momentum uncertainty \([1]\) induced by GUP.

If we consider higher order terms of \(\beta\), i.e. terms up to \(o(\beta^2)\), the Hamiltonian \([1]\) becomes

\[
H = \frac{p_0^2}{2m} + \frac{\beta p_0^4}{m} + \frac{\beta^2 p_0^6}{2m},
\] (40)

and the corresponding path integral becomes

\[
K(x'', t' + \Delta t; x', t') = \int \exp \left[ i \frac{p_0}{m} \left( x - \frac{x'^2}{2m} - \frac{\beta p_0^4}{m} - \frac{\beta^2 p_0^6}{2m} \right) \right] dp_0 \frac{2\pi \hbar}{2\pi \hbar}.
\] (41)

Proceeding in the same way as before we get the final form of kernel as

\[
K(x'', t' + \Delta t; x', t') = \sqrt{\frac{m}{2\pi \hbar \Delta t}} \left(1 - \frac{6\beta m^2 (x'' - x')^2}{\Delta t^2} - \frac{105\beta^2 \hbar^2 m^2}{2\Delta t^2}\right),
\]
Now, as we know that the final form of kernel is same for infinitesimal and finite interval, so we can deal with the above form of kernel for infinitesimal interval as well. In this case, the probability of finding the particle at the region $dx$ enclosing the point $x''$ is

$$K^*(x'', t''; x', t') K(x'', t''; x', t') \, dx = \frac{m}{2\pi \hbar \Delta t} \left( 1 - \frac{12\beta m^2(x'' - x')^2}{\Delta t^2} - \frac{81\beta^2 \hbar^2 m^2}{\Delta t^2} + \frac{225\beta^2 m^4(x'' - x')^4}{\Delta t^4} \right) \, dx$$

$$= \frac{m}{2\pi \hbar \Delta t} \left( 1 - 12\beta p^2 + 225\beta^2 p^4 - \frac{81\beta^2 \hbar^2 m^2}{\Delta t^2} \right) \, dx,$$

where we have neglected the terms of higher order than $\beta^2$. Since probability is always non-negative, we have the following relation

$$1 - 12\beta p^2 + 225\beta^2 p^4 - \frac{81\beta^2 \hbar^2 m^2}{\Delta t^2} \geq 0$$

from which we obtain the bound

$$p_{\text{max}} = \frac{1}{2\sqrt{3\beta}} \left( 1 - \frac{1}{2} \left( \frac{9\beta m}{\Delta t} \right)^2 \right)$$

$$\approx \frac{1}{2\sqrt{3\beta}} e^{-\frac{9\beta m}{2\Delta t}}.$$ 

From (2) we clearly see that the maximum momentum bound for $o(\beta^2)$ case is less than that obtained in the previous $o(\beta)$ case, though by a very small amount with the correction terms being of the order of or higher than $o(\beta^2)$.

### 3 Conclusions and future prospects:

We know that GUP gives rise additional terms in quantum mechanical Hamiltonian like $\beta p^4$, where $\beta \sim \frac{1}{(M_p c)^2}$ is the GUP parameter. This term plays important role at Planck energy level. Considering this term as a perturbation, we have shown that path integral is applicable on this GUP corrected non-relativistic cases. Here we have constructed the two form of kernel by applying Hamiltonian path integral method. The consistency properties of this kernel is then thoroughly verified. We have shown that the probability for finding a particle at a given point in case of the GUP model is less than the corresponding probability in the usual free particle case. We have also shown that probabilistic interpretation of this kernel induces a momentum upper bound in the theory. And this upper bounds changes slightly with $e^{-o(\beta^2)}$, if we consider...
higher order term of $\beta$ in the Hamiltonian (9). Now following the Hamiltonian path integral approach one can construct kernels and study their properties for other systems like particle in a step potential, Hydrogen atom etc, which is our future goal.

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5 Appendix: 1

Neglecting the term containing higher order of $\beta$ equation (27), becomes

$$K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i\Delta t}} \int e^{\frac{\imath m}{\Delta t}[(x_1-x_0)^2 + (x_2-x_1)^2 + ... + (x_N-x_{N-1})^2]} \left[ 1 + \frac{3\imath \hbar m}{\Delta t}N - \frac{6 \hbar m^2}{\Delta t^2} \{(x_1-x_0)^2 + (x_2-x_1)^2 + ... + (x_N-x_{N-1})^2\} \right] \, dx_1 dx_2 ... dx_{N-1} \, .$$

(46)

With the substitution $x_i = \sqrt{\frac{2\Delta t}{m}} y_i$ the above equation becomes

$$K(x'', t''; x', t') = \sqrt{\frac{m}{2\pi i\Delta t}} \left( \frac{1}{i\pi} \right)^{\frac{N-1}{2}} \int e^{i(y_1-y_0)^2 + i(y_2-y_1)^2 + ... + i(y_N-y_{N-1})^2} \left[ 1 + \frac{3\imath \hbar m}{\Delta t}N - \frac{12 \hbar m}{\Delta t} \{(y_1-y_0)^2 + (y_2-y_1)^2 + ... + (y_N-y_{N-1})^2\} \right] \, dy_1 dy_2 ... dy_{N-1} \, .$$

(47)

Now

$$\int e^{i(y_1-y_0)^2 + ... + i(y_N-y_{N-1})^2} \left\{ (y_1-y_0)^2 + ... + (y_N-y_{N-1})^2 \right\} \, dy_1 dy_2 ... dy_{N-1}$$

$$= \left\{ \frac{i(N-1)}{2} + \frac{(y_N-y_0)^2}{N} \right\} \left( \frac{i\pi}{\sqrt{N}} \right)^{\frac{N-1}{2}} e^{\frac{i(y_N-y_0)^2}{N}} \, .$$

(48)

which is done by adding $N$ separate integral of the form

$$\int e^{i(y_1-y_0)^2 + ... + i(y_N-y_{N-1})^2} (y_i-y_{i-1})^2 \, dy_1 dy_2 ... dy_{N-1}$$

(49)

Again

$$\int e^{i(y_1-y_0)^2 + ... + i(y_N-y_{N-1})^2} (y_1-y_0)^4 + ... + (y_N-y_{N-1})^4 \, dy_1 dy_2 ... dy_{N-1}$$

$$= \left\{ -\frac{3}{4} \left( \frac{1^2}{2^2} + \frac{2^2}{3^2} + ... + \frac{N-1^2}{N^2} \right) + \left( \frac{1}{1.2^2} + \frac{1}{2.3^2} + ... + \frac{1}{(N-1).N^2} \right) + 2 \left( \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{N} \right) - \frac{2(N-1)}{N} \right\}$$

$$+ \frac{3i(N-1)}{N} (1 + 1 + ... + 1)_{N \text{times}} \left( \frac{y_N-y_0)^2}{N^2} \right) + \left( \frac{y_N-y_0)^4}{N^4} \right) \left( \frac{i\pi}{\sqrt{N}} \right)^{\frac{N-1}{2}} e^{\frac{i(y_N-y_0)^2}{N}} \, .$$
Now multiply (53) by $-i\hbar$ times of (52), which generates the equation
\[ i\hbar \frac{\partial}{\partial t} K(x''', t''', x', t') = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial t''^2} K(x''', t''', x', t') + \frac{\beta}{m} \hbar^4 \frac{\partial^4}{\partial t''^4} K(x''', t''', x', t') \]
where $\Delta t = t'' - t'$. Which is modified schrodinger equation for free particle.

7 Appendix: 3

Taking the complex conjugate of the equation (28), we have
\[ K^*(x''', t''', x', t') = \sqrt{-\frac{m}{2\pi i\hbar}} e^{-\frac{i m (x''' - x')^2}{2\pi i \hbar}} \left[ 1 - \frac{3i\beta m^2}{2\hbar \Delta t^2} \frac{(x''' - x')^2}{(t''' - t')^2} + \frac{i\beta m^4 (x'' - x')^4}{\hbar^2 (t''' - t')^3} \right] \]
Therefore
\[ \int K^*(x''', t''', x', t') K(x''', t''', x', t') dx'' 

Now multiplying both side of (7) by $\psi$ where we have used the relation
\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial t} + i \beta \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial x} + \frac{m}{\hbar} \frac{\partial f}{\partial \psi} \frac{\partial \psi}{\partial t},
\]
we get from above
\[
\int e^{-u(x'-x_1')} u^n du = 2\pi i^n \frac{d^n \delta(x' - x_1')} {d(x' - x_1')^n}. \tag{58}
\]

Now multiplying both side of (7) by $\psi^*(x_1', t')$ and integrating w.r.to $x_1'$ we have
\[
\int\int K^*(x''', t''; x_1', t') K(x'', t''; x', t') \psi^*(x_1', t') \, dx'' \, dx_1'
\]
\[
= \int\left[ 1 - \frac{6\beta m^2(x' - x_1')^2}{(t'' - t')^2} - \frac{6i\beta m^3(x' - x_1')^4}{h(t'' - t')^3} \right] \delta(x' - x_1') \psi^*(x_1', t')
\]
\[
- \left( \frac{12i\beta \hbar(x' - x_1')}{(t'' - t')} + \frac{4\beta m^2(x' - x_1')^3}{h(t'' - t')^2} \right) \delta'(x' - x_1') \psi^*(x_1', t')
\]
\[
+ \left( \frac{12\beta \hbar^2 - \frac{6i\beta \hbar m(x' - x_1')^2}{(t'' - t')}}{(t'' - t')} \right) \delta''(x' - x_1') \psi^*(x_1', t') + 4\beta \hbar^2 (x' - x_1') \delta'''(x' - x_1') \psi^*(x_1', t') \right] e^{-\frac{im(x'-x_1')^2}{2\hbar(t''-t')}} \, dx_1'. \tag{59}
\]

Using the relation
\[
\int f(x) \delta^n(x)dx = -\int \frac{df(x)}{dx} \delta^{n-1}(x)dx \tag{60}
\]
we get from above
\[
\int\int K^*(x'', t''; x_1', t') K(x'', t''; x', t') \psi^*(x_1', t') \, dx'' \, dx_1' = \psi^*(x', t'). \tag{61}
\]

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