Casimir energy for spherical boundaries

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Calculations of the Casimir energy for spherical geometries which are based on integrations of the stress tensor are critically examined. It is shown that despite their apparent agreement with numerical results obtained from mode summation methods, they contain a number of serious errors. Specifically, these include (1) an improper application of the stress tensor to spherical boundaries, (2) the neglect of pole terms in contour integrations, and (3) the imposition of inappropriate boundary conditions upon the relevant propagators. A calculation which is based on the stress tensor and which avoids such problems is shown to be possible. It is, however, equivalent to the mode summation method and does not therefore constitute an independent calculation of the Casimir energy.

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In 1948 Casimir [1] first predicted that two infinite parallel plates in vacuum would attract each other. This remarkable result has its origin in the zero point energy of the electromagnetic field. While the latter is highly divergent, the change associated with this quantity for specific plate configurations is found to be finite and thus in principle observable. Early work to detect this small effect [2] was characterized by relatively large experimental uncertainties which left the issue in some doubt. More recent efforts [3] have provided quite remarkable data, but are based on a different geometry from that of Casimir. Since a rigorous theoretical calculation has never been carried out for the latter configuration, there remains room for skepticism as to whether the Casimir effect is as well established as is frequently asserted.

The extension of Casimir’s result to problems with nonplanar boundaries has been of considerable interest and fraught with difficulties. It was achieved for the case of the conducting spherical shell by Boyer [4] in a remarkable but intricate calculation. His result was subsequently verified by various methods [5-7], including in particular that of direct mode summation [8-9]. One of the methods developed in ref. 7 consisted of integrating the radial component of the stress tensor over the bounding surface, a technique which was subsequently applied to the case of the cylinder [10], the circle [11], and the Dirichlet problem of a D-dimensional sphere [12]. In view of the increasingly wide application of this technique, it is clearly of interest to ascertain its validity.

Since the electromagnetic sphere is the only known case in which a finite Casimir energy can be obtained for a nonplanar geometry using conventional (e.g., exponential) regularization techniques, it is convenient to use this case as a specific framework for the present work. One begins by defining the usual stress tensor for the uncoupled electromagnetic field

\[ T^{\mu \nu} = F^\mu \alpha F^\nu \alpha - \frac{1}{4} g^{\mu \nu} F^{\alpha \beta} F_{\alpha \beta} \]  

where \( g^{\mu \nu} = (1, 1, 1, -1) \) and

\[ F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \]

with \( A^\mu \) being the usual vector potential [13]. The latter is taken most conveniently to be in the radiation gauge \( \partial_k A^k = 0, A^0 = 0 \). The vacuum expectation value of \( T^{\mu \nu} \) can clearly be obtained from appropriate derivatives of the time ordered product

\[ G^{ij}(x, t; x', t') = i \langle 0 | (A^i(x, t) A^j(x', t'))_+ | 0 \rangle \]

where the latter is to be evaluated in the presence of spherical conducting boundaries at \( r = a \) and \( r = R \) where \( r = |x| \). The limit \( R \to \infty \) is to be taken as the final step of the calculation.

The differential equation for \( G^{ij} \)

\[ \left( -\nabla^2 + \frac{\partial^2}{\partial t^2} \right) G^{ij}(x, t; x', t') = \delta^{ij}(x - x') \delta(t - t') \]

with \( \delta^{ij}(x) \) the usual transverse delta function is to be solved for \( 0 \leq r, r' \leq a \) and for \( a \leq r, r' \leq R \) subject to the usual causal boundary conditions, namely positive (negative) frequencies for \( t - t' > 0 \) (\( t - t' < 0 \)). This is an issue
of some importance since all previous stress tensor calculations of the Casimir energy for non-planar boundaries have imposed outgoing wave conditions in the region exterior to the surface of interest. While causal boundary conditions follow directly from the existence of the vacuum as the state of lowest energy, there is no way to infer outgoing wave conditions [14] for the propagator (2).

The Fourier transform of the propagator is thus inferred from the boundary conditions to have the form

$$G^{ij}(\mathbf{x}, \mathbf{x}'; \omega) = \sum_n \frac{A_n^i(\mathbf{x})A_n^j(\mathbf{x}')}{\omega_n^2 - \omega^2 - i\epsilon}$$

(3)

where the sum is to be taken over all normalized eigenfunctions $A_n^i$ which satisfy the wave equation

$$(\nabla^2 + \omega_n^2)A_n(\mathbf{r}) = 0$$

subject to appropriate boundary conditions at $r = 0, a, R$ together with the transversality condition $\partial_k A^k = 0$. From $\mathbf{r} \cdot \mathbf{E} = 0$ for $r = a, R$ where $E_k = F^{0k}$ and $B_k = \frac{1}{2}\epsilon_{klm} F^m$, one readily obtains the form of these functions. Using $f_n(\omega_n^{(1)} r)$ and $g_n(\omega_n^{(2)} r)$ respectively to denote the TE and TM modes the eigenfunctions are seen to be of the form

$$A^{(1)}_{lmn}(\mathbf{x}) = f_n(\omega_n^{(1)} r) X_{lm}(\theta, \phi)$$

and

$$A^{(2)}_{lmn}(\mathbf{x}) = \frac{1}{\omega_n^{(2)}} \nabla \times g_n(\omega_n^{(2)} r) X_{lm}(\theta, \phi)$$

where $X_{lm}(\theta, \phi)$ with $l \geq 1$ denotes the vector spherical harmonics [15], and $f_n$ and $g_n$ are those real linear combinations of the spherical Bessel functions $j_l$ and $n_l$ which satisfy the boundary conditions.

Denoting the interior and exterior radial functions by $(f_n, g_n)$ and $(f_n, g_n)$ respectively, it follows that for $0 \leq r \leq a$, $f_n$ and $g_n$ are proportional to $j_l$ and required to satisfy the conditions

$$j_l(\omega_n^{(1)} a) = 0$$

$$\frac{d}{dr}[rj_l(\omega_n^{(2)} r)]_{r=a} = 0$$

from which the interior eigenfrequencies $\omega_n^{(\lambda)}$ are determined. For $a \leq r \leq R$ the relevant radial functions are

$$f_n^\gamma(\omega_n^{(1)} r) = a_l j_l(\omega_n^{(1)} r) + b_l n_l(\omega_n^{(1)} r)$$

and

$$g_n^\gamma(\omega_n^{(2)} r) = c_l j_l(\omega_n^{(2)} r) + d_l n_l(\omega_n^{(2)} r)$$

where $a_l$, $b_l$, $c_l$, and $d_l$ are constants. The ratios $b_l/a_l$ and $d_l/c_l$ together with the exterior eigenfrequencies $\omega_n^{(\lambda)} (\lambda = 1, 2)$ are fixed by the vanishing of $f_n^\gamma$ and $\frac{d}{dr}g_n^\gamma$ at $r = a$ and $r = R$. This enables (3) to be written more explicitly as

$$G^{ij}(\mathbf{x}, \mathbf{x}' ; \omega) = \sum_{lmn, \lambda} \frac{A^{(\lambda)}_{lmn}(\mathbf{x})A^{(\lambda)*}_{m'n'}(\mathbf{x}')}{\omega_n^{(\lambda)2} - \omega^2 - i\epsilon}$$

(4)

with $A_{lmn}^{(\lambda)}(\mathbf{x})$ normalized according to the prescription

$$\int d\mathbf{x} A_{lmn}^{(\lambda)}(\mathbf{x}) \cdot A^{(\lambda)*}_{m'n'}(\mathbf{x}) = \delta_{\lambda'} \delta_{l'} \delta_{m,m'} \delta_{n,n'}$$

(5)

which in turn is seen to imply the conditions

$$\int_0^a r^2 dr f_n^\gamma(\omega_n^{(1)} r)^2 = \int_a^R r^2 dr f_n^\gamma(\omega_n^{(2)} r)^2 = 1$$
and correspondingly for $q_{ln}^r$ and $q_{ln}^g$. It is to be noted that (4) applies both to the interior and exterior domains provided only that the appropriate eigenfrequencies $\omega_{ln}^{(\lambda)}$ and eigenfunctions $A_{lnm}^{(\lambda)}$ are applied in each case.

Having determined the relevant propagators the problem of computing the Casimir energy $E_c$ can now be addressed. Using the energy density given in (1), the propagators (4), and the normalization conditions (5), the appropriate derivatives can be taken together with the limits $x' \to x$ and $t' \to t$. This readily yields the result

$$E_c = \int_0^R r^2 dr \int d\Omega |0\rangle \frac{1}{2}(E^2 + B^2) |0\rangle$$

$$= \sum_{ln\lambda} \left( l + \frac{1}{2} \right) \omega_{ln}^{(\lambda)}$$

where the summation is to be taken over both the interior and exterior eigenmodes of the system. This sum has been carefully evaluated in ref.9 using an exponential cutoff with results entirely consistent with previous calculations. Thus it only remains to be determined whether similar conclusions follow from the stress tensor method.

To pursue this issue it is to be noted that the interpretation of $T^{kl}(x)$ as the $k$ component of the force per unit area normal to a surface $\sigma_l$ comes from the $k$ component of the conservation law

$$\partial_v T^{\mu\nu} = 0.$$ 

Thus

$$\frac{\partial}{\partial t} \int_V d\mathbf{x} T^{k0} = - \int_{\sigma_l} d\sigma_l T^{kl}$$

where the integration on the rhs is to be taken over the surface $\Sigma$ which bounds the volume $V$. Since the lhs is the time rate of change of the $k$ component of the momentum, it can then be claimed that $T^{kl}(x)$ is the correct force per unit area so long as the calculation is performed in cartesian coordinates. However, the claim has been made [7,10-12] that it is also valid in spherical coordinates, i.e., that the force per unit area $F/A$ on the spherical surface is simply given by

$$F/A = \langle 0 | [T_{rr}(r = a - \epsilon) - T_{rr}(r = a + \epsilon)] | 0 \rangle. \quad (6)$$

This, of course, is an assertion which can be examined by direct calculation, a task to which attention is now directed.

At $r = a$ the vacuum expectation value of $T_{rr}$ can be written as $\langle 0 | \frac{1}{2}(B^2 - E^2) | 0 \rangle$. Application of the boundary conditions at $r = a$ together with the identity

$$|Y_{lm}(\theta, \phi)|^2 = \frac{2l + 1}{4\pi}$$

yields the vacuum expectation value of $T_{rr}(r = a - \epsilon)$ as the angular independent form

$$\langle 0 | T_{rr}(r = a - \epsilon) | 0 \rangle = \sum_{ln} \frac{l + \frac{1}{2}}{8\pi} \left\{ \frac{1}{\omega_{ln}^{(1)}} \right\}$$

$$\left[ \frac{d}{dr} f_{ln}^c(\omega_{ln}^{(1)} a) \right]^2 + \left[ \omega_{ln}^{(2)} - \frac{l(l+1)}{\omega_{ln}^{(2)} a^2} \right] g_{ln}^c(\omega_{ln}^{(2)} a)^2 \right\} \quad (7)$$

where $\omega_{ln}^{(\lambda)}$ refers to the internal eigenfrequencies. The corresponding result for $\langle 0 | T_{rr}(r = a + \epsilon) | 0 \rangle$ is simply obtained from (6) by using the corresponding external eigenmodes together with the replacements $f^c \to f^g$ and $g^c \to g^g$.

In order to relate this result to the mode summation method, however, an explicit construction of the relevant eigenfunctions is required. For the interior region $r \leq a$ the normalization of the functions $f_{ln}^c$ and $g_{ln}^c$ leads to

$$f_{ln}^c(\omega_{ln}^{(1)} r) = \left\{ \frac{a^3}{2\omega_{ln}^{(1)}} \left[ \frac{d}{dr} j_l(\omega_{ln}^{(1)} a) \right]^2 \right\}^{-\frac{1}{2}} j_l(\omega_{ln}^{(1)} r)$$
and
\[
g_{\ln}^{(2)}(\omega_{\ln} r) = \left\{ \frac{a}{2} \left[ a^2 - l(l+1) \right] j_l(\omega_{\ln} a^2)^2 \right\}^{-\frac{1}{2}} j_l(\omega_{\ln} r).
\]
Upon using these results it is found that
\[
F/A|_{r=a-\epsilon} = \frac{1}{a} \left( \frac{1}{4\pi a^2} \right) \sum_{\ln\lambda} (l + \frac{1}{2}) \omega_{\ln}^{(\lambda)}
\]
with the sum to be performed over the interior eigenfrequencies. If a corresponding result were to obtain for \( T_{rr} \) over the exterior of the sphere, it would then follow that
\[
F/A|_{a-\epsilon} + F/A|_{a+\epsilon} = \frac{1}{a} \left( \frac{1}{4\pi a^2} \right) E_c
\]
\[
= -(\frac{1}{4\pi a^2}) \partial/\partial a E_c
\]
in agreement with the claim of [7]. However, for the exterior of the sphere there is, of course, a contribution only from \( r = a \) while the normalization depends upon both \( a \) and \( R \), a circumstance which in fact prevents one from obtaining a result analogous to (8). To see this it is convenient to go to manifestly normalized forms by the replacements
\[
f_{\ln}^{(1)}(\omega_{\ln} r) \rightarrow \left\{ \frac{r^3}{2\omega_{\ln}^{(1)2}} \left[ \frac{d}{dr} f_{\ln}^{(1)}(\omega_{\ln} r) \right]^2 \right\}^{1/2} f_{\ln}^{(1)}(\omega_{\ln} r)
\]
and
\[
g_{\ln}^{(2)}(\omega_{\ln} r) \rightarrow \left\{ \frac{r}{2} \left[ r^2 - \frac{l(l+1)}{\omega_{\ln}^{(2)2}} \right] g_{\ln}^{(2)}(\omega_{\ln} r)^2 \left[ \frac{d}{dr} g_{\ln}^{(2)}(\omega_{\ln} r) \right]^2 \right\}^{1/2} g_{\ln}^{(2)}(\omega_{\ln} r).
\]
Upon using these expressions in the evaluation of \( |0| T_{rr} |r = a + \epsilon \rangle \langle 0| \) there results
\[
F/A|_{r=a+\epsilon} = \frac{1}{a} \left( \frac{1}{4\pi a^2} \right) \sum_{\ln} (l + \frac{1}{2}) \left[ \omega_{\ln}^{(1)} \alpha_{\ln}^{(1)} + \omega_{\ln}^{(2)} \alpha_{\ln}^{(2)} \right]
\]
where
\[
\alpha_{\ln}^{(1)} = - \frac{r^3}{r^3} \left[ \frac{d}{dr} f_{\ln}^{(1)}(\omega_{\ln} r) \right]^2 \bigg|_{r=a} \frac{R^2}{R_a}
\]
and
\[
\alpha_{\ln}^{(2)} = - \frac{r}{r^2} \left[ \frac{d}{dr} g_{\ln}^{(2)}(\omega_{\ln} r) \right]^2 \bigg|_{r=a} \frac{R}{R_a}
\]
Since the coefficients \( \alpha_{\ln}^{(1,2)} \) vanish in the large \( R \) limit, it follows that this implies a suppression of the contribution of the external modes in the calculation of the Casimir energy, thereby contradicting Eq.(8). Thus the stress tensor approach does not constitute a valid approach to this problem for the case of spherical boundaries. Similar conclusions are readily obtained for Dirichlet and Neumann boundary conditions.

Although the conclusions reached here have been based on a very specific calculation, they can be placed in a more general context by using techniques of covariant differentiation. To this end one takes the conservation law for \( T^{\mu\nu} \) in the form
\[
T^{\mu\nu}_{\mu\nu} = 0
\]
\[ \partial_{\mu} T^{\mu \nu} + \left\{ \frac{\mu}{\alpha} \nu \right\} T^{\alpha \nu} + \left\{ \frac{\nu}{\alpha} \mu \right\} T^{\mu \alpha} = 0. \]  

(10)

For spherical coordinates the nonvanishing independent Christoffel symbols are

\[ \left\{ \frac{\theta}{r} \theta \right\} = \left\{ \frac{\phi}{r} \phi \right\} = 1/r, \]

\[ \left\{ \frac{r}{\theta} \theta \right\} = -r, \quad \left\{ \frac{r}{\phi} \phi \right\} = -r^2 \sin^2 \theta, \]

\[ \left\{ \frac{\theta}{\phi} \phi \right\} = -\sin \theta \cos \theta, \quad \text{and} \quad \left\{ \frac{\phi}{\theta} \theta \right\} = \cot \theta. \]

This yields for the \( r \) component of (10) that

\[ \partial_0 T^{r0} + \frac{1}{r^3} \partial_r r^3 T^{rr} + \frac{1}{\sin \theta} \partial_\theta \sin \theta T^{r \theta} + \partial_\phi T^{r \phi} = \frac{1}{r} T^{00} \]

where use has been made of the tracelessness of \( T^{\mu \nu} \). Since \( \langle 0 | T^{r0} | 0 \rangle \) is time independent and \( \langle 0 | T^{r \theta} | 0 \rangle = \langle 0 | T^{r \phi} | 0 \rangle = 0 \), it follows that

\[ \int dr d\Omega \partial_r (r^3 \langle 0 | T^{rr} | 0 \rangle) = \int r^2 dr d\Omega \langle 0 | T^{00} | 0 \rangle, \]

and consequently

\[ r^3 \langle 0 | T_{rr} (r) | 0 \rangle \bigg|_{r_1}^{r_2} = \int_{r_1}^{r_2} r^2 dr \langle 0 | T^{00} (r) | 0 \rangle \]  

(11)

for arbitrary \( r_1 \) and \( r_2 \). This clearly shows that the correct inference to be drawn from (10) is not the force equation (6) but rather the fact that the difference between the quantities \( r^3 \langle 0 | T_{rr} (r) | 0 \rangle \) when evaluated on the two bounding spherical surfaces is proportional to the total energy in that region.

It should also be noted that this result can also be inferred directly from the cartesian tensor result

\[ \partial_l \langle 0 | T^{kl} | 0 \rangle = 0. \]

which implies that

\[ x_k \partial_l \langle 0 | T^{kl} | 0 \rangle = \partial_l x_k \langle 0 | T_{kl} | 0 \rangle - \langle 0 | T^{kk} | 0 \rangle = 0. \]

Upon using once again the tracelessness of the energy momentum tensor Eq.(11) clearly follows.

It has been noted here that many of the existing calculations of the Casimir effect for a sphere have not used correct boundary conditions on the underlying propagators. More serious is the fact that those which have used the stress tensor method have applied an approach which fails in its application to nonplanar boundaries. In concluding this work it needs to be pointed out that there exists yet a third failure in many of these calculations. To illustrate this point it is convenient to refer to Eq.(3.14) of ref.[7] which gives the Casimir energy as

\[ E_c = \frac{i}{2a} \sum_l (2l + 1) \int_{-\infty}^{\infty} \frac{d(\omega a)}{2\pi} e^{-i\omega \tau z} \left\{ \frac{(z j_l)'^2}{z j_l} + \frac{(z j_l)''^2}{z j_l} + \frac{(z h_l^{(1)})'^2}{z h_l^{(1)}} + \frac{(z h_l^{(1)})''^2}{z h_l^{(1)}} \right\}. \]

where the \( j_l \) terms are associated with the interior of the sphere and the \( h_l^{(1)} \) terms with the exterior. In order to bring this result to the usual integral over Bessel functions of imaginary argument it is necessary to perform a ninety
degree rotation of the contour. However, for the case of the exterior mode part of $E_c$ such a rotation fails since it does not take into account the existence of poles of the Hankel functions in the lower half plane. This is a noteworthy reminder of the remarks made earlier concerning the fact that the outgoing spherical wave condition for $r > a$ cannot be derived for Casimir propagators.

Having displayed some of the problems associated with the calculation of the Casimir energy for the case of spherical boundaries it is of interest to note that calculations which employ the stress tensor method claim to obtain the same result as that found by direct mode summation. This issue is dealt with in an appendix which shows that when appropriate attention is paid to the issue of contour rotation one does not in fact obtain the usual result.

**APPENDIX**

It has been stated in this work that the discontinuity in the stress tensor across a boundary cannot be used to calculate the Casimir pressure on that surface. Since, however, such calculations invariably claim to obtain the usual result, it is of considerable interest to display explicitly the flaw in such calculations. In order to avoid inessential complications one can choose to deal only with $T_{rr}(r = a + \epsilon)$ for the case of the TE modes with corresponding results for $T_{rr}(r = a - \epsilon)$ and for the TM modes then following immediately. This is, of course, equivalent to considering the scalar field case with Dirichlet boundary conditions. For this case one finds in analogy to [7] and [12] that the relevant Green’s function is obtained by solving the equation

$$(-\nabla^2 - \omega^2) G(x, x'; \omega) = \delta(x - x')$$

subject to Dirichlet boundary conditions at $r = a, R$. Upon writing

$$G(x, x'; \omega) = \sum_{l=0}^{\infty} G_l(r, r'; \omega) Y_l^m(\theta, \phi) Y_l^{m*}(\theta', \phi')$$

with $Y_l^m$ denoting the usual spherical harmonics, it follows that

$$G_l(r, r'; \omega) = -\frac{1}{r^2 W[f_1^<, f_1^>]} f_1^<(\omega r_<) f_1^>(\omega r_>)$$

where $W[f_1^<, f_1^>]$ denotes the Wronskian and $r_>$ ($r_<$) denotes the greater (lesser) of $r$ and $r'$. In terms of the spherical Bessel functions $j_l$ and $n_l$ one readily finds that $f_1^<(\omega r_<)$ and $f_1^>(\omega r_>)$ have the form

$$f_1^<(\omega r) = j_l(\omega r)n_l(\omega a) - j_l(\omega a)n_l(\omega r)$$

and

$$f_1^>(\omega r) = j_l(\omega r)n_l(\omega R) - j_l(\omega R)n_l(\omega r).$$

This allows the partial wave Green’s function to be written in the form

$$G_l(r, r'; \omega) = \frac{\omega}{n_l(\omega a)j_l(\omega R) - n_l(\omega R)j_l(\omega a)} f_1^<(\omega r_<) f_1^>(\omega r_>).$$

(A1)

Using the fact that the zeros of the Wronskian determine the eigenmodes of the system $G_l(r, r'; t - t')$ is found to be given by

$$G_l(r, r'; t - t') = i \sum_n \omega_l n f_1^<(\omega_l n r) f_1^>(\omega_l n r') e^{-i\omega_l |t - t'|}$$

(A2)

where the sum is taken over all eigenmodes of the system corresponding to eigenfrequencies $\omega_l n$. This form of the propagator is recognizable as the Fourier transform of the scalar version of (3) with explicitly normalized eigenfunctions. An immediate consequence is that the calculation of Casimir energy by direct mode summation can now be carried out using the approach of [9] to obtain the usual result. In addition the analysis following Eq.(9) can be repeated, again finding that there is no contribution to $\langle 0 | T_{rr}(r = a + \epsilon)|0 \rangle$ in the large $R$ limit.

Although this would seem to establish the inapplicability of the stress tensor method, it is instructive to continue the analysis to determine exactly at what point the latter method fails. To this end one notes that the evaluation of $\langle 0 | T_{rr}(r = a + \epsilon)|0 \rangle$ requires that derivatives with respect to both $r$ and $r'$ be taken at $r = r' = a$ and an integration be performed over $\omega$. Since the eigenmodes occur at real values of $\omega$, the path of integration must be carefully specified.
As deduced from the boundary conditions (and also as stated clearly in [7]) the appropriate path is just above (below) the real axis for \( \omega > 0 \) (\( \omega < 0 \)). The usual expression for the Casimir force is then obtained if (a) the contour can be rotated ninety degrees counterclockwise and (b) if \( f_l^>(\omega a) \) becomes proportional to the Hankel function \( h_l^{(1)}(\omega a) \) (\( h_l^{(2)}(\omega a) \)) in the upper (lower) half plane in the large \( R \) limit. Although the second condition (b) is satisfied, giving a \( R \) independent result which is formally identical to the usual expression for the Casimir energy, it is easy to see that the required contour rotation cannot be performed. To display this result one writes

\[
(0|T_{rr}(r = a + \epsilon)|0) = \lim_{t \to t' + \epsilon} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \frac{i}{2\pi^2} \left. \left[ \frac{d}{d\epsilon} \frac{d}{d\epsilon} f_l^>(\omega r) f_l^<(\omega r) \right] \right|_{r=a}
\]

which can be reduced to

\[
(0|T_{rr}(r = a + \epsilon)|0) = -\frac{i}{2a^2} \lim_{t \to t' + \epsilon} \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \left. \left[ \frac{d}{d\epsilon} \frac{d}{d\epsilon} \frac{j_l(\omega R)}{j_l(\omega R)n_l(\omega a) - n_l(\omega R)j_l(\omega a)} \right] \right|_{r=a}.
\]

While it is clear that the segment of the contour from \( \omega = -\infty \) to \( \omega = 0 \) can be rotated counterclockwise to the negative imaginary axis, no such rotation is possible for the segment \( \omega = 0 \) to \( \omega = \infty \) [16]. Conversely, in the case that the stress tensor is defined by the limit \( t = t' - \epsilon \) rather than \( t = t' + \epsilon \), it becomes possible to rotate that part of the contour along the positive real axis to the positive imaginary axis. This choice, however, precludes the possibility of a legitimate rotation of the contour along the negative real axis. This completes the proof that the stress tensor does not in fact yield the usual expression for the Casimir stress on the sphere. The demonstration in the body of this paper was based on the method of expansion in eigenfunctions and in this appendix on the Wronskian formulation of the Green’s function, the latter being the one more commonly used in published calculations of the stress tensor. The results of the two approaches are identical (note the remark [16]) and show that stress tensor calculations in curvilinear coordinates cannot be expected to yield the correct Casimir energy.

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[13] Throughout this work Greek indices range over 0,1,2,3 while Latin ones take values 1,2,3.

[14] Since the usual view of the Casimir effect sees it as arising from the zero point energy of the electromagnetic field, it is inappropriate to impose such time asymmetric boundary conditions which have the further consequence of eliminating any possible correspondence to the case of planar boundaries.

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[16] It is of interest to note here that if the rotation of the contour along the negative real axis is continued to just below the positive real axis a simple application of Cauchy’s theorem yields (A2). The latter is known to give zero stress in the limit of large \( R \) by the analysis following Eq.(9).