CONFORMAL CLASSES REALIZING THE YAMABE INVARIANT

HEATHER MACBETH

ABSTRACT. We give a characterization of conformal classes realizing a compact manifold’s Yamabe invariant. This characterization is the analogue of an observation of Nadirashvili for metrics realizing the maximal first eigenvalue, and of Fraser and Schoen for metrics realizing the maximal first Steklov eigenvalue.

1. INTRODUCTION

1.1. The result. Three decades ago, Schoen’s ground-breaking solution [Sch84] of the Yamabe problem established that, within any conformal equivalence class $c$ of Riemannian metrics on a compact smooth $n$-manifold $M$, the total scalar curvature functional

$$g \mapsto \frac{\int_M R(g) d\text{Vol}_g}{\left(\int_M d\text{Vol}_g\right)^{1-2/n}}$$

attains its infimum. The quantity

$$I(c) := \min_{g \in c} \frac{\int_M R(g) d\text{Vol}_g}{\left(\int_M d\text{Vol}_g\right)^{1-2/n}},$$

which is therefore well-defined, is known as the the Yamabe constant of $c$, and metrics $g$ attaining this minimum are referred to as Yamabe metrics. Computing the Euler-Lagrange equation of the restriction of the total scalar curvature functional to $c$ shows that Yamabe metrics have constant scalar curvature.

One can study the properties of $I(c)$ as it varies over all conformal classes $c$ on a smooth manifold. In particular the Yamabe invariant of $M$ is defined to be the minimax expression

$$Y(M) := \sup_c I(c) = \sup_c \inf_{g \in c} \frac{\int_M R(g) d\text{Vol}_g}{\left(\int_M d\text{Vol}_g\right)^{1-2/n}};$$

This quantity is finite, as follows from the observation of Aubin [Aub76a] that $I(c) \leq n(n-1)\omega_n^{2/n}$ for all $c$ (it is a corollary of the solution of the Yamabe problem that equality holds if and only if $c$ is the conformal class of the round sphere).
In this paper we study manifolds $M$ whose Yamabe invariant $Y(M)$ is attained. Our result is the following algebraic constraint on the set of Yamabe metrics in a conformal class attaining the Yamabe invariant:

**Theorem 1.1.** Let $M$ be a compact smooth $n$-manifold, and $c$ a conformal class attaining $Y(M)$. There exists a finite set $(g_i)$ of Yamabe metrics in $c$, such that

$$0 = \sum_i \left( \text{Ric}(g_i) - \frac{1}{n} R(g_i)g_i \right) |d\text{Vol}_{g_i}|^{1 - 2/n}. $$

**Remarks.**

1. Here $|d\text{Vol}_{g_i}|$ is the density associated to the volume form $d\text{Vol}_{g_i}$; this density is a positive section of the density bundle $|\Lambda^n M|$, an oriented line bundle, and so $|d\text{Vol}_{g_i}|^{1 - 2/n}$ is a well-defined section of $|\Lambda^n M|^{1 - 2/n}$.

2. More explicitly: let $g$ be a representative of $c$; then there exists a finite set $(u^2_i g)$ of Yamabe metrics in $c$, such that

$$0 = \sum_i u_i^{n-2} \left[ \text{Ric}(u^2_i g) - \frac{1}{n} R(u^2_i g)u^2_i g \right].$$

The interest of Theorem 1.1 lies in its connection to a subtle and very appealing open problem:

**Question 1.2.** Do all conformal classes attaining the Yamabe invariant contain an Einstein metric?

This question is motivated by the examples of manifolds for which the Yamabe invariant has to date been computed, which fall into two classes:

- Einstein manifolds $(M, g)$ for which it has been proved that $[g]$, the conformal class of the Einstein metric, attains $Y(M)$; these include
  - $S^n$ ([Aub76b]; see [LP87] Section 3 for an exposition);
  - manifolds admitting flat metrics ([SY79], [GL83]; see [Sch89] Proposition 1.3 for an exposition);
  - general-type and Calabi-Yau complex surfaces ([LeB96], [LeB99];
  - $\mathbb{CP}^2$ ([LeB97]);
  - $\mathbb{RP}^3$ ([BN04]);

- manifolds $M$ for which the existence of a conformal class attaining $Y(M)$ is unclear, or for which it has been proved that no conformal class attains $Y(M)$; these include
  - $S^1 \times S^{n-1}$ and connect sums thereof ([Kob87]);
  - $M \# k\mathbb{CP}^2$, where $M$ is a complex surface of general type ([LeB96]);
  - $M \# [S^1 \times S^3]$, where $M$ is a 4-manifold with $Y(M) \leq 0$ ([Pet98]);
  - $\mathbb{RP}^3 \# k(S^1 \times \mathbb{RP}^2)$ ([BN04]);

as well as by the observation that the Yamabe invariant is a minimax quantity associated with the total scalar curvature functional

$$g \mapsto \frac{\int_M R(g) d\text{Vol}_g}{(\int_M d\text{Vol}_g)^{1 - 2/n}}.$$
of which Einstein metrics are the critical points. If Question 1.2 were answered in the affirmative, it could perhaps be possible to find Einstein metrics on new manifolds by direct variational methods, by maximizing the functional $I$. Theorem 1.1 generalizes two previous results on Question 1.2. In the case when the finite set of Theorem 1.1 consists simply of a single metric $g$, the following well-known result is obtained:

**Lemma 1.3** (see for example the discussion preceding Lemma 1.2 of [Sch89]). Suppose that the conformal class $c$ attains $Y(M)$, and suppose that there is, up to rescaling, exactly one Yamabe metric $g$ in $c$. Then $g$ is Einstein.

**Proof.** given Theorem 1.1. The tensor $\text{Ric}(g) - \frac{1}{n} R(g)g$ vanishes precisely when $g$ is Einstein. $\Box$

In the case when the finite set of Theorem 1.1 consists of two metrics, we may write them explicitly as $g$ and $u^2 g$, and obtain the following relationship, previously derived by Anderson ([And05], equation 2.37):

$$u^{-1}(1 + u^{n-2})[\text{Ric}(g) - \frac{1}{n} R(g)g] = -(n - 2)[\text{Hess}_g(u^{-1}) - \frac{1}{n} \Delta_g(u^{-1})].$$

Anderson then gives a Bianchi-identity argument deducing a contradiction from this relationship unless the two metrics coincide, thus answering Question 1.2 in the affirmative in this case:

**Theorem 1.4** ([And05], Theorem 1.2). Suppose that the conformal class $c$ attains $Y(M)$, and suppose that there are at most two (modulo rescaling) Yamabe metrics in $c$. Then in fact there is exactly one Yamabe metric, and that metric is Einstein.

We note a different point of view, from which the available evidence regarding Question 1.2 is more equivocal. There is a bidirectional version of Lemma 1.3 which rephrases Question 1.2 as a question of uniqueness:

**Proposition 1.5** ([Oba72], see also [Sch89] Proposition 1.4). Let $M$ be a compact manifold other than the sphere, and suppose the conformal class $c$ attains $Y(M)$. Then there is an Einstein metric in $c$ if and only there is a unique (modulo rescaling) Yamabe metric in $c$; the Einstein metric is the Yamabe metric if so.

In a general (that is, not necessarily maximizing) conformal class, the question of uniqueness (modulo rescaling) of Yamabe metrics is well-studied. Let $\Omega$ be the subset of conformal classes containing a unique Yamabe metric. $\Omega$ is large (for instance it contains all conformal classes $c$ such that $I(c) \leq 0$ [Aub70], and it is open and dense [And05]) but its complement need not actually be empty; the simplest counterexample is the round sphere, and many others have been found ([Sch89], [BP13]).

We therefore hope that future work may use the results of this paper to provide a resolution of Question 1.2 in one of two ways. On the one
hand, there may exist a Bianchi-type argument generalizing Anderson’s, which uses Theorem 1.1 to answer the question in the affirmative. On the other hand, it may be possible to construct a finite set of metrics satisfying the algebraic criterion of Theorem 1.1, thus answering the question in the negative.

1.2. The technique. This paper was inspired by two closely analogous results in other geometric minimax contexts. Nadirashvili [Nad96] studies Riemannian metrics $g$ on a compact smooth manifold $M$ which maximize the weighted first Dirichlet eigenvalue $\lambda_1(g) \Vol(M, g)^{2/n}$, where

$$\lambda_1(g) = \inf_{\{u \in C^\infty(M): \ 0 = \int_M u \} \Vol_g} \int_M \frac{\abs{\nabla u}^2 \Vol_g}{\int_M u^2 \Vol_g}.$$

Among other things, he observes (Theorem 5) that such metrics are $\lambda_1$-minimal; that is, that there exist a set of first Dirichlet eigenfunctions for $g$, such that the map of $M$ into Euclidean space defined by those eigenfunctions is an isometric embedding as a minimal submanifold of the unit sphere. The key point of the proof is to find a finite set $(u_i)$ of first eigenfunctions, such that

$$g = \sum_i \left[ (du_i) \otimes (du_i) + \frac{1}{4} \Delta_g(u_i^2) g \right].$$

Fraser and Schoen [FS12] study Riemannian metrics $g$ on a compact smooth manifold with boundary $M$ which maximize the weighted first Steklov eigenvalue $\sigma_1(g) \Vol(\partial M, g|_{\partial M})^{1/(n-1)}$, where

$$\sigma_1(g) = \inf_{\{u \in C^\infty(M) : \ u \text{ harmonic} \}} \frac{\int_M \abs{\nabla u}^2 \Vol_g}{\int_{\partial M} u^2 \Vol_g|_{\partial M}}.$$

Among other things, they observe (Proposition 5.2) that such a metric $g$ has a set of first Steklov eigenfunctions, such that the map of $M$ into Euclidean space defined by those eigenfunctions is a conformal embedding as a minimal submanifold of the unit ball, isometric on $\partial M$. The key point of the proof is to find a finite set $(u_i)$ of first eigenfunctions, such that

$$0 = \sum_i \left[ (du_i) \otimes (du_i) - \frac{1}{2} \abs{du_i}^2 g \right], \text{ on } M,$$

$$1 = \sum_i u_i^2, \text{ on } \partial M.$$

Our argument, particularly in Section 4, closely follows Fraser and Schoen’s.

1.3. Outline. The organization of this paper is as follows. Section 2 reviews some preliminaries. Section 3 is dedicated to the proof of Proposition 3.1; this proposition establishes control on the formal derivative of the Yamabe
functional at a conformal class which attains the Yamabe invariant. A Hahn-Banach theorem argument in Section 4 completes the proof of Theorem 1.1.

1.4. Acknowledgments. Discussions with Rod Gover, Richard Schoen and my advisor Gang Tian have helped shape this paper, and I am grateful to them for their interest.

2. Preliminaries

2.1. Bounds on constant-scalar-curvature metrics. We give a version of the standard bound on metrics with constant scalar curvature, in which the dependence of the bounds on the metric is made explicit.

By convention we define

$$\Lambda := 4n(n-1) \left( \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+1\right)\omega_{n-1}}{\Gamma(n+1)} \right)^{2/n} = n(n-1)\omega_n^2.$$ 

Here $\omega_n$ is the volume of the $n$-sphere.

In a fixed conformal class, the standard a priori bounds on metrics with constant scalar curvature are essentially equivalent to the solution to the Yamabe problem for $(M, [g])$ with $I([g]) < \Lambda$, and are due to Trudinger [Tru68] and Aubin [Aub76a, Aub76b] (see also the exposition in [LP87]):

**Theorem 2.1.** Suppose given $m \geq 2$, $p > n$, $\eta$, and a smooth metric $g$ on $M$. There exists $C > 0$ dependent only on $n$, $m$, $p$, $\eta$, $g$, such that for each smooth positive function $\varphi$ on $M$ such that $\varphi^{4/(n-2)}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$, we have the uniform bound

$$||\varphi||_{W^{m,p},g} \leq C.$$ 

This theorem holds more generally for a compact set of conformal classes:

**Theorem 2.2.** Suppose given $m \geq 2$, $p > n$, $i$, $D$, $B$, $\eta$. There exists $C > 0$ dependent only on $n$, $m$, $p$, $i$, $D$, $B$, $\eta$, such that for each

- smooth metric $g$ on $M$ such that $\text{diam}(M, g) \leq D$, $\operatorname{injrad}(M, g) \geq i$,
  $$||\text{Ric}(g)||_{C^{m-2},g} \leq B,$$
- smooth positive function $\varphi$ on $M$ such that $\varphi^{4/(n-2)}g$ has volume 1 and constant scalar curvature $\lambda \leq \Lambda - \eta$,

we have the uniform bound

$$||\varphi||_{W^{m,p},g} \leq C.$$ 

Such generalizations from a single conformal class to a compact set of conformal classes have generally been understood to be folklore, see for instance [And05] (discussion following Equation 2.1), [KMS09] Lemma 10.1. The point to be checked is the dependence of the constant $C$ in some standard elliptic estimates on the background Riemannian metric $g$: the $L^p$ estimates $||u||_{W^{m+2,p},g} \leq C(||\Delta_g u||_{W^{m-2,p},g} + ||u||_{p,g})$ (where $C$ can be controlled in terms of the injectivity radius and $||\text{Ric}\||_{C^{m-2},g}$) and the Harnack inequality $\sup_M u \leq C \inf_M u$ (where $C$ can be controlled in terms of the diameter,
the injectivity radius and a lower Ricci bound); in each case one works first locally in harmonic co-ordinates ([HII97], [Heb96]) and then globally.

2.2. A compactness theorem for Yamabe metrics.

**Lemma 2.3.** \( I \) is upper semi-continuous.

**Proof.** It is the infimum of a continuous functional. \( \square \)

In fact \( I \) is continuous ([BB83]), but we will not need that here.

**Proposition 2.4.** Let \( m \geq 3 \). Let \((M, c)\) be a conformal manifold with \( I(c) < \Lambda \). Let \((c_k)\) be a sequence of smooth conformal classes on \( M \) which \( C^m \)-converges to \( c \), and let \((g_k)\) be volume-1 Yamabe metrics for the classes \((c_k)\).

Then there exists a subsequence \((g_{k_i})\) which \( C^{m-1} \)-converges to a volume-1 Yamabe metric for \( c \).

**Proof.** Choose \( p > n \), and choose representatives \((\overline{\eta}_k), \overline{\eta} \) of the classes \((c_k)\), \( c \) with \( \overline{\eta}_k \to \overline{\eta} \) in the \( C^m \) topology. For sufficiently large \( k \),

1. \( I(c_k) \leq \frac{1}{2} I(c) + \Lambda \) (by Lemma 2.3);
2. there are uniform bound on the expressions
   \[ \text{diam}(M, \overline{\eta}_k), \text{injrad}(M, \overline{\eta}_k), \|\text{Ric}(\overline{\eta}_k)\|_{C^{m-2}} \];
3. there exists a uniform \( C \) such that for \( k \) sufficiently large,
   \[ C^{-1} \| \|W^{m+1,p}_\overline{\eta}_k\| \leq \| \|W^{m+1,p}_\overline{\eta}\| \leq C \| \|W^{m+1,p}_\overline{\eta}_k\|. \]

Therefore, by Theorem 2.2 if \( \varphi_k \) are smooth positive functions on \( M \) such that \( \varphi_k^{4/(n-2)} \overline{\eta}_k \) are volume-1 Yamabe metrics for \( c_k \), then we have the uniform bound \( \| \varphi_k \|_{W^{m,p}_\overline{\eta}} \leq C \), hence by Morrey’s inequality the uniform bound \( \| \varphi_k \|_{C^{m-1,1-\frac{m}{n}}_\overline{\eta}} \leq C \), and so there exists a subsequence \((k_i)\) such that \( \varphi_{k_i} \) \( C^{m-1} \)-converges.

It remains to be checked that the limit, \( \varphi \), makes \( \varphi^{4/(n-2)} \overline{\eta} \) a Yamabe metric for \( c \). Indeed, \( \varphi_k^{4/(n-2)} \overline{\eta}_k \) all have volume 1 and constant scalar curvature \( I(c_k) \). So their \( C^{m-1} \)-limit \( g := \varphi^{4/(n-2)} \overline{\eta} \) has volume 1, and constant scalar curvature \( \lim_{k \to \infty} I(c_k) \), which is \( \leq I(c) \) by Lemma 2.3; this is a contradiction unless equality holds and \( \overline{\eta} \) is a Yamabe metric. \( \square \)

2.3. **Notation.** As sketched in the introduction, we denote by \(|\Lambda^n M|\) the bundle of densities on \( M \); it is a line bundle, equipped with a natural positive orientation, and hence all real powers \(|\Lambda^n M|^\alpha\) are well-defined. We write \(|\Omega|\) for the density associated to an \( n \)-form \( \Omega \).

For each \( \alpha \) there is a well-defined “determinant” map of bundle sections,

\[
\text{det} : C^\infty(M, \text{Sym}^2(T^* M) \otimes |\Lambda^n M|^\alpha) \to C^\infty(M, |\Lambda^n M|^{2+\alpha}).
\]

For \( \alpha = 0 \) this is the square of the volume form, \( g \mapsto (\text{Vol}_g)^{2/n} \). The case \( \alpha = -2/n \) (that is, the bundle \( \text{Sym}^2(T^* M) \otimes |\Lambda^n M|^{-2/n} \) has the special feature that its determinant has range \( C^\infty(M, \mathbb{R}) \).
Denote by $\mathcal{M}$ the space of smooth Riemannian metrics on $M$, and by $\mathcal{V}$ the space of smooth positive densities. The map from $\mathcal{M}$ to $\mathcal{V} \times \mathcal{C}^\infty(M, \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n})$ given by
\[ g \mapsto (|d\text{Vol}_g|, g \otimes |d\text{Vol}_g|^{-2/n}) \]
is injective with image $\mathcal{V} \times \mathcal{C}$, where $\mathcal{C}$ denotes the set of smooth positive-definite sections of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$ with determinant 1. The inverse, an isomorphism from $\mathcal{V} \times \mathcal{C}$ to $\mathcal{M}$, is given by $(\Omega, c) \mapsto \Omega^{2/n} c$.

For any element $c \in \mathcal{C}$, the set of Riemannian metrics in $\mathcal{M}$ corresponding to $\mathcal{V} \times \{c\}$ under this isomorphism is precisely a conformal equivalence class of Riemannian metrics. We therefore identify $c$ with that conformal equivalence class, and $\mathcal{C}$ with the space of smooth conformal classes. The tangent space $T_c \mathcal{C}$ to $\mathcal{C}$ at $c$ is the kernel of the trace
\[ \text{tr}_c : \mathcal{C}^\infty(M, \text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}) \to \mathcal{C}^\infty(M, \mathbb{R}). \]

### 2.4. Continuity properties of the total scalar curvature functional.

We introduce the notation $Q : \mathcal{M} \to \mathbb{R}$ for the total scalar curvature functional,
\[ Q(g) = \int_M R(g)|d\text{Vol}_g| \left( \int_M d\text{Vol}_g \right)^{1-2/n}, \]
and, according to the isomorphism described in the previous subsection, have an equivalent functional $Q : \mathcal{V} \times \mathcal{C} \to \mathbb{R},$
\[ Q_\Omega(c) = Q(\Omega^{2/n} c). \]

We have, for a conformal class $c$,
\[ I(c) = \inf_{g \in c} Q(g) = \inf_{\Omega \in \mathcal{V}} Q_\Omega(c). \]

**Lemma 2.5.** Let $\Omega$ be a positive density. The functional $Q_\Omega : \mathcal{C} \to \mathbb{R}$ is $C^1$, and its derivative $D_c(Q_\Omega) : T_c \mathcal{C} \to \mathbb{R}$ at $c \in \mathcal{C}$ is
\[ D_c(Q_\Omega)(w) = \text{Vol}(\Omega)^{2/n-1} \int_M \langle w, -\text{Ric}(\Omega^{2/n} c)\Omega^{1-2/n} \rangle_c. \]

**Proof.** It is well-known (e.g. [Bes87] Proposition 4.17) that for a Riemannian metric $g$ and symmetric 2-tensor $h$,
\[ D_g(Q)(h) = \text{Vol}(g)^{2/n-1} \int_M \left\langle h, -\text{Ric}(g) + \frac{1}{2} \left[ R(g) - \frac{\int_M R(g)|d\text{Vol}_g|}{\text{Vol}(g)} \right] g \right\rangle_g |d\text{Vol}_g|, \]
Recall from the previous subsection that $T_c \mathcal{C}$ is the set of $c$-tracefree sections of $\mathcal{C}^\infty(M, \text{Sym}^2(T^*M))$. If $w$ is such a section, the tangent vector $(0, w)$ to $\mathcal{V} \times \mathcal{C}$ at $(\Omega, c)$ corresponds to the tangent vector $\Omega^{2/n} w$ to $\mathcal{M}$ at $\Omega^{2/n} c$, and since $\text{tr}_c w = 0$, the term $\langle \Omega^{2/n} w, \Omega^{2/n} c \rangle_{\Omega^{2/n} c}$ vanishes. \qed
Proposition 2.6 (Modulus of continuity of $I$). Let $(M, c_0)$ be a conformal manifold. Let $v$ be a smooth section of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{2/n}$, sufficiently small that $\det(c_0 + tv)$ is nonvanishing for $t \in [0, 1]$. Write
\[ c_t := \frac{c_0 + tv}{\det(c_0 + tv)^{\frac{1}{n}}}, \]
so that $(c_t)$ is a 1-parameter family of conformal classes starting at $c_0$. Let $\Omega$ be a density such that $g_1 = \Omega^{2/n} c_1$ is a Yamabe metric for $c_1$, and write $g_t = \Omega^{2/n} c_t$. Then
\[ \text{Vol}(\Omega)^{1-\frac{2}{n}} [I(c_1) - I(c_0)] \geq \int_0^1 \int_M \langle v, (-\text{Ric}(g_t) + \frac{1}{n} R(g_t) g_t) \det(c_0 + tv)^{-1/n} \Omega^{1-2/n} \rangle_{c_t} dt. \]

Proof. Since $g$ is a Yamabe metric for $c_1$,
\[ Q_{|d\text{Vol}|}(c_1) = Q(g) = I(c_1), \]
so
\[ I(c_1) - I(c_0) = Q_{|d\text{Vol}|}(c_1) - \left( \inf_{\Omega \in V} Q_{\Omega}(c_0) \right) \geq Q_{|d\text{Vol}|}(c_1) - Q_{|d\text{Vol}|}(c_0). \]

We calculate
\[ \frac{dc_t}{d\tau} \bigg|_{\tau=t} = \frac{d}{d\tau} \bigg|_{\tau=t} \left[ \frac{c_0 + \tau v}{\det(c_0 + \tau v)^{\frac{1}{n}}} \right] \]
\[ = v \cdot \det(c_0 + tv)^{\frac{1}{n}} - (c_0 + tv) \cdot \frac{1}{n} \det(c_0 + tv)^{-\frac{1}{n}} \cdot \text{tr}_{c_0+tv} v \cdot \det(c_0 + tv) \]
\[ = \det(c_0 + tv)^{-\frac{1}{n}} \left[ v - \frac{1}{n} (\text{tr}_{c_t} v) c_t \right], \]
so by Lemma 2.5
\[ \text{Vol}(\Omega)^{1-\frac{2}{n}} \frac{d}{d\tau} \bigg|_{\tau=t} \left[ Q_{|d\text{Vol}|}(c_{\tau}) \right] \]
\[ = \int_M \langle v - \frac{1}{n} (\text{tr}_{c_t} v) c_t, -\text{Ric}(g_t) \det(c_0 + tv)^{-\frac{1}{n}} \Omega^{1-2/n} \rangle_{c_t} \]
\[ = \int_M \langle v, (-\text{Ric}(g_t) + \frac{1}{n} R(g_t) g_t) \det(c_0 + tv)^{-\frac{1}{n}} \Omega^{1-2/n} \rangle_{c_t}. \]
The result follows by the Fundamental Theorem of Calculus. \qed

3. An “Euler-Lagrange inequality”

Let $M$ be a compact connected smooth manifold, and suppose the conformal class $c$ on $M$ attains the Yamabe invariant of the manifold $M$. 
Proposition 3.1. For each distributional section $v$ of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$, there exists a Yamabe metric $g$ for $c$, such that

$$\int_M \langle v, (\text{Ric}(g) - \frac{1}{n} R(g) g) |\text{dVol}_g|^{1-2/n} \rangle_c \geq 0.$$ 

Proof. If $(M, c)$ is the conformally round sphere, then $c$ contains an Einstein metric $g$, the round metric; for this $g$, the tensor $\text{Ric}(g) - \frac{1}{n} R(g) g$ vanishes. The result follows.

If $(M, c)$ is not the conformally round sphere, by the solution of the Yamabe problem [Aub76a] [Sch84], $I(c) < \Lambda$, so the compactness result Proposition 2.4 applies. We now give a proof in these cases using it.

Let $(v_k)$ be a sequence of smooth $c$-tracefree sections of $\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{-2/n}$ which converge distributionally to $v$, and let $(t_k)$ be a sequence of positive reals such that for all $m$ we have $t_k ||v_k||_{C^m, g} \to 0$. (For instance, one might choose $t_k := \frac{1}{k} ||v_k||_{C^m, g}^{-1}$. Thus the sequence $c_k := c + t_k v_k$ of smooth conformal classes $C^\infty$-converges to $c$. Let $(\Omega_k)$ be densities such that the metrics $\Omega_k^{2/n} c_k$ are volume-1 Yamabe metrics for the classes $(c_k)$.

Since $c$ attains the Yamabe invariant, we have that for each $k$,

$$0 \geq I(c_k) - I(c).$$

Thus, applying Proposition 2.6 to $c$ and $c_k$,

$$0 \geq \frac{I(c_k) - I(c)}{t_k} \geq \int_0^1 \int_M \langle v_k, (-\text{Ric}(g_{k, \tau}) + \frac{1}{n} R(g_{k, \tau}) g_{k, \tau}) \det(c_0 + \tau t_k v_k)^{-1/n} \Omega_k^{1-2/n} \rangle_{c_{k, \tau}} \text{dVol}_g,$$

where we define the 1-parameter families of conformal classes $c_{k, \tau}$ by

$$c_{k, \tau} := \frac{c + \tau t_k v_k}{\det(c + \tau t_k v_k)^{1/n}},$$

and metrics $g_{k, \tau}$ by $\Omega_k^{2/n} c_{k, \tau}$.

By the compactness result Proposition 2.4 and a diagonal argument, we may choose a subsequence $(k_i)$ such that $g_{k_i}$ converges in $C^\infty$ to a volume-1 Yamabe metric, $g$ say, for $c$, with $\Omega_{k_i}$ converging in $C^\infty$ to $|\text{dVol}_g|$. Therefore also the sequence of fields on $[0, 1] \times M$,

$$(x, \tau) \mapsto \langle \cdot, (-\text{Ric}(g_{k, \tau}) + \frac{1}{n} R(g_{k, \tau}) g_{k, \tau}) \det(c_0 + \tau t_k v_k)^{-1/n} \Omega_k^{1-2/n} \rangle_{c_{k, \tau}},$$

$C^\infty$-converges to the constant-in-$\tau$ field

$$(x, \tau) \mapsto \langle \cdot, (\text{Ric}(g) + \frac{1}{n} R(g) g) |\text{dVol}_g|^{1-2/n} \rangle_c.$$
By construction, the sequence \( (v_k) \) of constant-in-\( \tau \) fields on \([0, 1] \times M\) converges distributionally to the constant-in-\( \tau \) field \( v \). Pairing, it follows that

\[
\int_M \left\langle v, \left( -\text{Ric}(g) + \frac{1}{n} R(g) g \right) \right| \text{dVol}_g \right|^{1-2/n}_c \leq 0.
\]

□

4. HAHN-BANACH ARGUMENT

In this section we prove Theorem 1.1.

**Lemma 4.1.** A conformal equivalence class \( c \) induces canonical pairings

\[
\langle \cdot, \cdot \rangle : \Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^s) \times \mathcal{C}^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n-s}) \to \mathbb{R},
\]

(\( \Gamma \) denoting distributional sections) such that the induced maps

\[
\Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^s) \to (\mathcal{C}^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n-s}))^*
\]

are isomorphisms of topological vector spaces.

**Proof.** Pick a metric \( g \in c \). The induced pairing

\[
\langle v, w \rangle = \int g(v, w) \cdot \text{dVol}_g \right|^{4/n}
\]

is easily checked to be independent of the choice of \( g \in c \). □

**Theorem 4.2** (Hahn-Banach separation theorem, [Rud91] Theorem 3.4). Let \( X \) be a locally convex topological vector space over \( \mathbb{R} \), and \( A \) and \( B \) nonempty closed convex subsets of \( X \), with \( A \) compact. If for all functionals \( \varphi \in X^* \) we have

\[
\sup_{a \in A} \varphi(a) \geq \inf_{b \in B} \varphi(b),
\]

then \( A \cap B \neq \emptyset \).

Let \( M \) be a compact smooth \( n \)-manifold, and \( c \) a conformal class attaining \( Y(M) \). Denote by

\[
\mathcal{F} \subseteq \mathcal{C}^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n})
\]

the set of volume-1 Yamabe metrics in the conformal class \( c \).

Define a function

\[
Q : c \to \mathcal{C}^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|-2/n)
\]

on the set of metrics in the conformal class \( c \), by

\[
Q(g) := \left( \text{Ric}(g) - \frac{1}{n} R(g) g \right) \left| \text{dVol}_g \right|^{1-2/n}.
\]

**Proposition 4.3.** The convex hull of the set \( Q(\mathcal{F}) \) contains 0.
Proof. Let $K$ be the convex hull of the set $Q(F)$.

By Proposition 2.4 the set $F$ is compact, so the set $K$ is closed. The topological vector space of sections

$$C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n})$$

is Fréchet and therefore locally convex. Thus it suffices to verify the hypothesis of Theorem 4.2. Indeed, let $v$ be in

$$\Gamma(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n}) \cong (C^\infty(\text{Sym}^2(T^*M) \otimes |\Lambda^n M|^{1-2/n}))^*$$

(the isomorphism is by Lemma 4.1). By Proposition 3.1 there exists (rescaling if necessary) a metric $g \in F$ such that

$$\langle v, Q(g) \rangle \geq 0.$$

Since $Q(g) \in K$, it follows that indeed

$$\sup_{a \in K} \langle v, a \rangle \geq 0 = \langle v, 0 \rangle.$$

\[\square\]

We can now complete the proof of Theorem 1.1. By Proposition 4.3 the convex hull of the set $Q(F)$ contains 0; that is, there exist finite sets $(a_i)$ of positive reals, and $(g_i)$ of volume-1 Yamabe metrics in $c$, such that $1 = \sum_i a_i$ and

$$0 = \sum_i a_i \left( \text{Ric}(g_i) - \frac{1}{n} R(g_i) g_i \right) |d\text{Vol}_{g_i}|^{1-2/n}.$$

Let $g_i' = a_i^{2/(n-2)} g_i$; then the $g_i'$ are (not necessarily volume-1) Yamabe metrics and

$$0 = \sum_i \left( \text{Ric}(g_i') - \frac{1}{n} R(g_i') g_i' \right) |d\text{Vol}_{g_i'}|^{1-2/n}.$$

References

[And05] Michael T. Anderson. On uniqueness and differentiability in the space of Yamabe metrics. Commun. Contemp. Math., 7(3):299–310, 2005.

[Aub70] Thierry Aubin. Métriques riemanniennes et courbure. J. Differential Geometry, 4:383–424, 1970.

[Aub76a] Thierry Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pure Appl. (9), 55(3):269–296, 1976.

[Aub76b] Thierry Aubin. Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geometry, 11(4):573–598, 1976.

[BB83] Lionel Bérdar Bergery. Scalar curvature and isometry group. In Marcel Berger, Shingo Murakami, and Takushiro Ochiai, editors, Spectra of Riemannian Manifolds, pages 9–28. Kaigai Publications, Ltd., 1983.

[Bes87] Arthur L. Besse. Einstein manifolds, volume 10 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1987.

[BN04] Hubert L. Bray and André Neves. Classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{R}P^3$. Ann. of Math. (2), 159(1):407–424, 2004.
Renato G. Bettiol and Paolo Piccione. Multiplicity of solutions to the Yamabe problem on collapsing Riemannian submersions. *Pacific J. Math.*, 266(1):1–21, 2013.

A. Fraser and R. Schoen. Sharp eigenvalue bounds and minimal surfaces in the ball. arXiv:1209.3789, September 2012.

Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.

Emmanuel Hebey. *Sobolev spaces on Riemannian manifolds*, volume 1635 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

E. Hebey and M. Herzlich. Harmonic coordinates, harmonic radius and convergence of Riemannian manifolds. *Rend. Mat. Appl. (7)*, 17(4):569–605 (1998), 1997.

M. A. Khuri, F. C. Marques, and R. M. Schoen. A compactness theorem for the Yamabe problem. *J. Differential Geom.*, 81(1):143–196, 2009.

Osamu Kobayashi. Scalar curvature of a metric with unit volume. *Math. Ann.*, 279(2):253–265, 1987.

Claude LeBrun. Four-manifolds without Einstein metrics. *Math. Res. Lett.*, 3(2):133–147, 1996.

Claude LeBrun. Yamabe constants and the perturbed Seiberg-Witten equations. *Comm. Anal. Geom.*, 5(3):535–553, 1997.

Claude LeBrun. Einstein metrics and the Yamabe problem. In *Trends in mathematical physics (Knoxville, TN, 1998)*, volume 13 of *AMS/IP Stud. Adv. Math.*, pages 353–376. Amer. Math. Soc., Providence, RI, 1999.

John M. Lee and Thomas H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

Morio Obata. The conjectures on conformal transformations of Riemannian manifolds. *J. Differential Geometry*, 6:247–258, 1971/72.

Jimmy Petean. Computations of the Yamabe invariant. *Math. Res. Lett.*, 5(6):703–709, 1998.

Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

Richard Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

Richard M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

R. Schoen and S. T. Yau. On the structure of manifolds with positive scalar curvature. *Manuscripta Math.*, 28(1-3):159–183, 1979.

Neil S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

Department of Mathematics, Princeton University; Fine Hall, Washington Rd, Princeton, NJ 08544

E-mail address: macbeth@math.princeton.edu