The stochastic interpolation method:
A simple scheme to prove replica formulas
in Bayesian inference

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Abstract
In recent years important progress has been achieved towards proving the validity of the replica predictions for the (asymptotic) mutual information (or “free energy”) in Bayesian inference problems. The proof techniques that have emerged appear to be quite general, despite they have been worked out on a case-by-case basis. Unfortunately, a common point between all these schemes is their relatively high level of technicality. We present a new proof scheme that is quite straightforward with respect to the previous ones. We call it the stochastic interpolation method because it can be seen as an extension of the interpolation method developed by Guerra and Toninelli in the context of spin glasses, with an interpolation constructed out of a stochastic process. In order to illustrate our method we show how to prove the replica formula for three non-trivial inference problems. The first one is symmetric rank-one matrix estimation (or factorisation), which is the simplest problem considered here and the one for which the method is presented in full details. Then we generalize to symmetric tensor estimation and random linear estimation. We believe that the present method has a much wider range of applicability and also sheds new insights on the reasons for the validity of replica formulas in Bayesian inference.

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I. INTRODUCTION

A very interesting development in probability theory in recent years has been the progress on a coherent mathematical theory of the predictions of the replica and cavity methods in statistical physics of spin glasses. In this respect one of the most important tools is the invention of the interpolation method by Guerra and Toninelli which eventually led to a remarkable proof of the Parisi formula for the free energy of the Sherrington-Kirkpatrick model.

In more recent years the interpolation method has been fruitfully extended and adapted to problems of interest in a wide range of applications such as in coding theory, communications, signal processing and theoretical computer science, well beyond the realm of traditional statistical mechanics. Among these we highlight applications of the interpolation method to error correcting codes, random linear estimation and compressive sensing, and constraint satisfaction problems. Most of these problems are inference problems and when a Bayesian framework is adopted, they can be solved with a replica symmetric scheme (constraint satisfaction is not, as such at least, an inference problem and does not fall in this category). The replica symmetric formulas for the free energies, mutual informations and error performance measures typically predict interesting first order phase transitions, with associated metastable states with infinite lifetime, which pose interesting algorithmic challenges of great importance in practical applications as well as challenges from the analysis point of view. It has turned out that one can learn a great deal about the fundamental limitations for important classes of (message-passing) algorithms by studying these replica solutions (we refer to [27] for a general reference and come back to this point in the conclusion).

In spite of their complexity, for all the inference problems cited above, complete proofs of the replica symmetric formulas have been found. These proofs usually combine Guerra-Toninelli interpolation bounds with some other non-trivial idea or method, namely algorithmic approaches involving so-called spatially coupled models, information theoretic methods or rigorous versions of the cavity method using the Aizenman-Sims-Starr principle. While each of these methods has its own merit and sheds interesting light, they all lead to quite long and technically involved proofs. Besides, although each method can probably be tailored for each problem, it would clearly be more satisfactory to have a more or less unified approach.

In this paper we develop a new unified and self-contained interpolation method. We illustrate how it works for three different problems, namely rank-one symmetric matrix and tensor factorization, as well as random linear estimation and compressive sensing. Our method allows to prove at the same time matching lower and upper bounds on the free energy with much less effort than all known current proofs. All these problems are “spin systems” defined for “dense graphs” (complete graphs or hypergraphs). The ideas of this paper can also be adapted to error correcting codes that are akin to spin systems on “sparse” random graphs and we plan to come back to this aspect elsewhere.

Roughly speaking, our new scheme interpolates between the original problem and the mean-field replica solution in small steps, each step involving its own set of trials parameters and Gaussian mean-fields in the spirit of Guerra and Toninelli. We are then able to choose the set of trial parameters in various ways so that we get both upper and lower bounds that eventually match. One can interpret the succession of Gaussian mean-fields in each step as a Wiener process. For this reason we call this new approach “the stochastic interpolation method.” The interpretation in terms of a Wiener process is in fact not really needed, and here we choose a more pedestrian path, but we believe this is an aspect of the method that may be of further interest and briefly discuss it at the end of the paper.

An important aspect of our method is the need for concentration properties of the suitable overlap parameters. It was already proven long ago in that a concentration hypothesis for overlaps implies that the replica symmetric solution is exact (an implication that was known to physicists). However for typical spin glass systems (e.g. the Sherrington-Kirkpatrick or p-spin glass) this hypothesis can only hold in some high temperature phase, and it is also difficult to prove. We refer to and [1] for pioneering works on such proofs with the help of cavity-like methods. In the framework of Bayesian inference the situation is more favourable. The Bayes rule immediately implies a special set of identities obeyed by suitable “correlation functions” often known as Nishimori identities. These identities then allow to deduce the concentration of overlaps from the concentration of the free energy in the whole phase diagram. This is also the reason why Bayesian inference problems generally lead to replica symmetric solutions.

The paper is organized as follows. Section II gives a pedagogic introduction to the stochastic interpolation method for one of the simplest, yet non-trivial problems, namely random linear estimation. In particular, we provide a much simpler and transparent proof than all other existing proofs of the replica formula (see Theorem 4.1). As explained in the previous paragraph, for all these problems our analysis also rests on concentration properties of the overlap parameters in the whole phase diagram. The proof of such results for random linear estimation can be found in and boils down to the important identity (95) in the present paper. Here we apply the same proof ideas and show all details for the pedagogic case of rank-one symmetric matrix estimation to obtain the relevant concentration Lemma 2.6. This analysis is the object of sections V, VI and appendix IX. The same concentration...
results with identical proofs (at the expense of heavier notations) also apply to the tensor case. Finally in section VII we detail the interpretation of the method in terms of a Wiener process and in the conclusion we briefly point out algorithmic consequences of our results and a few open issues.

II. THE STOCHASTIC INTERPOLATION METHOD: A “DIVIDE AND CONQUER” APPROACH

A. Symmetric rank-one matrix estimation: Setting

Consider the following probabilistic rank-one matrix estimation problem: One has access to noisy observations \( w = [w_{ij}]_{i,j=1}^n \) of the pair-wise product of the components of a vector \( s = [s_1, \ldots, s_n]^T \in \mathbb{R}^n \) with i.i.d components distributed as \( S_i \sim P_0, \ i = 1, \ldots, n \). We often abuse notation and simply denote \( S \sim P_0 \). A standard and natural setting is the case of additive white Gaussian noise of known variance \( \Delta \),

\[
    w_{ij} = \frac{s_is_j}{\sqrt{n}} + z_{ij} \sqrt{\Delta} \text{ for } 1 \leq i \leq j \leq n, \tag{1}
\]

where \( z = [z_{ij}]_{i,j=1}^n \) is a symmetric matrix with i.i.d entries \( Z_{ij} \sim N(0,1) \) for \( 1 \leq i \leq j \leq n \). This is denoted \( Z \sim N(0,1) \) for simplicity. The goal is to estimate \( s \) from \( w \) assuming that both \( P_0 \) and \( \Delta \) are known and independent of \( n \) (the noise is symmetric so that \( w_{ij} = w_{ji} \)).

We consider a Bayesian setting and associate to the model (1) its posterior distribution. The likelihood of the (component-wise independent) observation matrix \( w \) is

\[
    P(w|x) = \frac{e^{-\frac{1}{n} \sum_{i \leq j} (w_{ij} - \frac{s_is_j}{\sqrt{n}})^2}}{(2\pi\Delta)^{n(n+1)/2}}, \tag{2}
\]

and from the Bayes formula we get the posterior distribution

\[
    P(x|w = \frac{ss^T}{\sqrt{n}} + z\sqrt{\Delta}) = \frac{\prod_{i=1}^n P_0(x_i)P(w|x)}{\int dx \prod_{i=1}^n P_0(x_i)P(w|x)} = \frac{\prod_{i=1}^n P_0(x_i)e^{-\frac{1}{n}H(x,s,z)}}{\int dx \prod_{i=1}^n P_0(x_i)e^{-\frac{1}{n}H(x,s,z)}} \tag{3}
\]

where we call

\[
    H(x,s,z) := \frac{1}{\Delta} \sum_{i \leq j} \left( \frac{x_i^2x_j^2}{2n} - \frac{x_ix_j s_i s_j}{n} - \frac{x_i x_j x_i x_j s_i s_j \sqrt{\Delta}}{\sqrt{n}} \right) \tag{4}
\]

the Hamiltonian of the model. In order to obtain the last form of the posterior distribution we replaced \( w_{ij} \) using (1), developed the square in \( P(w|x) \), and simplified the x-independent terms in the numerator and denominator. The normalization factor is by definition the partition function

\[
    Z(s,z) := \int dx \prod_{i=1}^n P_0(x_i)e^{-\frac{1}{n}H(x,s,z)} = \mathbb{E}_x[e^{-\frac{1}{n}H(x,s,z)}]. \tag{5}
\]

Our principal quantity of interest is the average free energy per component defined by \( f := -\frac{1}{n} \mathbb{E}_s,z[\ln Z(S,Z)] \). Its explicit expression is

\[
    f = -\frac{1}{n} \mathbb{E}_{s,z} \left[ \ln \mathbb{E}_x \left[ e^{-\frac{1}{n} \sum_{i \leq j} \left( \frac{x_i^2x_j^2}{2n} - \frac{x_ix_j s_i s_j}{n} - \frac{x_i x_j x_i x_j s_i s_j \sqrt{\Delta}}{\sqrt{n}} \right) } \right] \right], \tag{6}
\]

where \( S, X \sim P_0 \) and \( Z \sim N(0,1) \).

Define the replica symmetric (RS) potential \( f_{RS}(m;\Delta) \) as

\[
    f_{RS}(m;\Delta) := \frac{m^2}{4\Delta} + f_{den}(\Sigma(m;\Delta)), \tag{7}
\]

with \( \Sigma(m;\Delta) := \Delta/m \). Here \( f_{den}(\Sigma) \) is the free energy associated with a scalar Gaussian denoising model: \( y = s + \tilde{z} \Sigma \) where \( S \sim P_0, \tilde{Z} \sim N(0,1) \). The free energy \( f_{den}(\Sigma) \) is minus the average logarithm of the normalization of the posterior distribution \( P(x|s + \tilde{z} \Sigma) \propto \exp(-\Sigma^{-2}(x^2/2 - xs - x\tilde{z} \Sigma)) P_0(x) \). Let \( X \sim P_0 \). Then

\[
    f_{den}(\Sigma) := -\mathbb{E}_{s,\tilde{z}} \left[ \ln \mathbb{E}_x \left[ e^{-\frac{1}{2\Sigma} \left( \frac{x^2}{2} - xs - x\tilde{z} \Sigma \right) } \right] \right]. \tag{8}
\]

Our first theorem illustrating the stochastic interpolation method is

**Theorem 2.1 (RS formula for symmetric rank-one matrix estimation):** Fix \( \Delta > 0 \). For any \( P_0 \) with bounded support, the asymptotic free energy of the symmetric rank-one matrix estimation model (1) verifies

\[
    \lim_{n \to \infty} f = \min_{m \geq 0} f_{RS}(m;\Delta). \tag{9}
\]

\(^1\)For all other models considered in this paper we directly write the explicit expression of the free energy, but the derivation is always similar.
The bounded support property hypothesis for $P_0$ is not really a requisite of the stochastic interpolation method, but simply makes the necessary concentration proofs for the free energy simpler. There is no condition on the size of the support, and it is presumably possible to take a support equal to the whole real line by a limiting process as long as the first four moments of $P_0$ are finite.

This theorem has already been obtained recently in [22, 31] (with varying hypothesis on $P_0$) by the more elaborate methods mentioned in the introduction. In the next paragraphs we introduce the stochastic interpolation method through a pedagogical and new proof of this theorem.

**Remark 2.2 (Free energy, mutual information and algorithms):** In Bayesian inference the average free energy is related to the mutual information $I(S; W)$ between the observation and the unknown vector (which is formally expressed as a difference of Shannon entropies: $I(S; W) = H(W) - H(W|S)$). For model (1), a straightforward computation shows that when $P_0$ has bounded first four moments

$$
\frac{I(S; W)}{n} = f + \frac{\mathbb{E}[S^2]^2}{4\Delta} + \mathcal{O}(n^{-1}),
$$

where $S \sim P_0$. The $n \to \infty$ limit of the mutual information (or equivalently of the average free energy) is an interesting object to compute because it allows to locate the phase transition(s) occurring in the inference problem, which corresponds to its non-analyticity point(s) as a function of $\Delta$. This phase transition threshold usually separates a low-noise regime where inference is information theoretically possible from a high-noise regime where inference is impossible. In this high-noise regime the observation simply does not carry enough information for reconstructing the signal. Furthermore, remarkably, the replica formula for the mutual information (or average free energy) also allows to determine an algorithmic noise threshold, below the phase transition threshold, which separates the information theoretic possible phase in two regions: An “easy” phase where there exist low complexity message-passing algorithms for optimal inference and a “hard” phase where message-passing algorithms yield suboptimal inference. For further information and rigorous results on these issues for model (1) we refer to [22]. A few more pointers to the literature are given in the conclusion.

**Remark 2.3 (Channel universality):** The Gaussian noise setting (1) is actually sufficient to completely characterize the generic model where the entries of $w$ are observed through a noisy element-wise (possibly non-linear) output probabilistic channel $P_{\text{out}}(w|s_i s_j/\sqrt{n})$. This is made possible by a theorem of channel universality [21] (conjectured in [41] and already proven for community detection in [42]). This theorem states that given an output channel $P_{\text{out}}(w|y)$, such that $\ln P_{\text{out}}(w|y = 0)$ is three times differentiable with bounded second and third derivatives, then the mutual information satisfies

$$
I(S; W) = I(S; S^T/\sqrt{n} + Z\sqrt{\Delta}) + \mathcal{O}(\sqrt{n}),
$$

where $\Delta$ is the inverse Fisher information (at $y = 0$) of the output channel: $\Delta^{-1} := \int dw P_{\text{out}}(w|0)(\partial_y \log P_{\text{out}}(w|y)|_{y=0})^2$.

Informally, this means that we only have to compute the mutual information for a Gaussian channel to take care of a wide range of problems, which can be expressed in terms of their Fisher information.

**B. The $(k, t)$–interpolating model**

Let $z^{(k)} = [z^{(k)}_{ij}]_{i,j=1}^n$, $Z^{(k)} = [Z^{(k)}_{ij}]_{i,j=1}^n$, $z^{(k)}_{ij} = Z^{(k)}_{ij} \sim \mathcal{N}(0, 1)$, $\tilde{Z}^{(k)}_i \sim \mathcal{N}(0, 1)$ for $k=1, \ldots, K$ be Gaussian noise symmetric matrices and vectors. It is important to keep in mind that these are indexed both by the vertex indices $i, j$ and the discrete
global interpolation parameter \( k \). Define also \( \theta := \{s, \{\tilde{z}^{(k)}\}_{k=1}^{K}\}_{k=1}^{K} \) the collection of all quenched random variables \( \tilde{z} \) will appear next.

The \((k, t)\)-interpolating Hamiltonian is

\[
H_{k,t}(x; \theta) := \sum_{k'=k+1}^{K} h\left( x, s, z^{(k')}, K\Delta \right) + \sum_{k'=1}^{k-1} h_{mf}\left( x, s, z^{(k')}, K\Delta \right) + h\left( x, s, z^{(k)}, K\Delta \right) + h_{mf}\left( x, s, \tilde{z}^{(k)}, \frac{K\Delta}{1-t} \right) + h_{mf}\left( x, s, \tilde{z}^{(k)}, \frac{K\Delta}{t m_k} \right),
\]

(12)

where the parameters \( \{m_k\}_{k=1}^{K} \) are to be fixed later (these will be chosen \( O(1) \) with respect to (w.r.t) \( n \) and can be interpreted as signal-to-noise ratios), \( t \in [0, 1] \) the continuous local interpolation parameter, and

\[
h(x, s, z, \sigma^2) := \frac{1}{\sigma^2} \sum_{i,j=1}^{n} \left( \frac{x_i^2 x_j^2}{2n} - \frac{x_i x_j s_i s_j}{n} - \frac{\sigma x_i x_j \tilde{z}_i \tilde{z}_j}{\sqrt{n}} \right),
\]

(13)

\[
h_{mf}(x, s, z, \sigma^2) := \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_i s_i - \sigma x_i \tilde{z}_i \right).
\]

(14)

A possible interpretation of the scheme is the following. The \((k, t)\)-interpolating model corresponds to the following inference model: One has access to the sets of noisy observations about the signal \( s \) where each noise realization is independent:

\[
\left\{ w^{(k')} = \frac{ss^T}{\sqrt{n}} + z^{(k')} \sqrt{K\Delta} \right\}_{k'=k+1}^{K},
\]

(15)

\[
\left\{ y^{(k')} = s + z^{(k')} \sqrt{K\Delta} \right\}_{k'=1}^{k-1},
\]

(16)

\[
w^{(k)} = \frac{ss^T}{\sqrt{n}} + z^{(k)} \sqrt{\frac{K\Delta}{1-t}},
\]

(17)

\[
y^{(k)} = s + z^{(k)} \sqrt{\frac{K\Delta}{t m_k}}.
\]

(18)

The first and third sets of observations correspond to similar inference channels as the original model (1) but with a much higher noise variance proportional to \( K \). These correspond to the first and third terms, respectively, of the \((k, t)\)-interpolating Hamiltonian (12). The second and fourth sets instead correspond to decoupled Gaussian denoising models, with associated “mean field” second and fourth terms in (12). The noise variances are proportional to \( K \) because the total number of observations is \( K \) and we want the total signal-to-noise ratio to be \( O(1) \). At fixed \( k \), letting \( t \) going from 0 to 1 increases by one unit the number of decoupled observations (16) by continuously adding the observation (18) in the sense that its signal-to-noise ratio that vanishes at \( t = 0 \) (which is equivalent to not having access to this observation) becomes finite and equal to the signal-to-noise ratio of the individual observations in the set (16) at \( t = 1 \). Simultaneously it reduces by one the number of observations of the form (15) by “removing” the observation (17) since its signal-to-noise ratio, which is finite at \( t = 0 \), vanishes at \( t = 1 \). From (15)–(18) it is clear that the \((k, t = 1) \) and \((k + 1, t = 0)\)-interpolating models are statistically equivalent. A complementary and more graphical illustration of the interpolation scheme is found on Figure 1.

In order to use an important concentration lemma later on, we will need a slightly more general Hamiltonian, and consider the following perturbed version of (12):

\[
H_{k,t,\epsilon}(x; \theta) := H_{k,t}(x; \theta) + \epsilon \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_i s_i - \frac{x_i \tilde{z}_i}{\sqrt{\epsilon}} \right),
\]

(19)

with i.i.d \( \tilde{Z}_i \sim N(0, 1) \). It should be kept in mind that the signal-to-noise ratio \( \epsilon \) of this additional Gaussian “side-channel” \( y = s \sqrt{\epsilon} + \tilde{z} \) will tend to 0 at the end of the proofs. Therefore we always consider \( \epsilon \in [0, 1] \).

The \((k, t)\)-interpolating model has an associated Gibbs expectation \( \langle \cdot \rangle_{k,t,\epsilon} \) and \((k, t)\)-interpolating free energy \( f_{k,t,\epsilon} \):

\[
P_{k,t,\epsilon}(x|\theta) := \frac{\prod_{i=1}^{n} P_{0}(x_i) e^{-H_{k,t,\epsilon}(x; \theta)}}{E_{\Theta} [e^{-H_{k,t,\epsilon}(X; \theta)}]},
\]

(20)

\[
\langle A \rangle_{k,t,\epsilon} := \int dx A(x) P_{k,t,\epsilon}(x|\theta),
\]

(21)

\[
f_{k,t,\epsilon} := \frac{1}{n} E_{\Theta} \left[ \ln E_{X} [e^{-H_{k,t,\epsilon}(X; \Theta)}] \right].
\]

(22)

where \( X \sim P_{0} \). In the following, we simply denote \( E_{\Theta} \) by \( E \).

Remark 2.4 (Thermodynamic limit): The interpolation methods used in [22] imply super-additivity of \( f_{k,t,\epsilon} \) and thus (by Fekete’s lemma) the existence of the thermodynamic limit \( \lim_{n \to \infty} f_{k,t,\epsilon} \). Moreover it is easy to show from (19) that \( \lim_{n \to \infty} f_{k,t,\epsilon} \) is concave and thus continuous in \( \epsilon \) on any compact set containing \( \epsilon = 0 \), which implies \( \lim_{\epsilon \to 0} \lim_{n \to \infty} f_{k,t,\epsilon} = \lim_{n \to \infty} f_{k,t,\epsilon=0} \) (note that the free energy is bounded for any \( \epsilon \), so \( \epsilon = 0 \) can be included in the compact subset).
C. The initial and final models

Let us compute the \((k, t)\)-interpolating free energy \(f_{1,0;0}\) associated with the initial \((k = 1, t = 0)\) model. Using (12) and (13),

\[
\mathcal{H}_{1,0;0}(\mathbf{x}; \mathbf{\theta}) = \sum_{k=1}^{K} h\left(\mathbf{x}, \mathbf{z}^{(k)}; K\Delta\right) = \frac{1}{\Delta} \sum_{i,j=1}^{n} \left( \frac{x_i^2 - x_i s_j^2}{2n} - \frac{x_i x_j}{\sqrt{n}} \sum_{k=1}^{K} \frac{z_{ij}^{(k)}}{\sqrt{K}} \right) \tag{23}
\]

As the \(z_{ij}^{(k)}\), \(1 \leq i \leq j \leq n\), are i.i.d. \(\mathcal{N}(0,1)\) random variables, they possess the stability property, namely \(z_{ij} := \sum_{k=1}^{K} z_{ij}^{(k)}/\sqrt{K}\) are i.i.d. \(\mathcal{N}(0,1)\) random variables as well (and symmetric). Let \(z = [z_{ij}]_{i,j=1}^{n}\). Using this we obtain

\[
f_{1,0;0} = \frac{1}{n} \mathbb{E}_{\mathbf{s}, \mathbf{z}} \left[ \ln \mathbb{E}_{\mathbf{x}} \left[ e^{-\mathcal{H}_{1,0;0}(\mathbf{x}; \mathbf{\theta})} \right] \right] \tag{24}
\]

which is actually the free energy (6) of the original model. We thus have:

\[
f_{1,0;0} = f. \tag{25}
\]

Let us now consider the free energy \(f_{K,1;0}\) of the final model. Using (12) and (14) we get

\[
\mathcal{H}_{K,1;0}(\mathbf{x}; \mathbf{\theta}) = \sum_{k=1}^{K} h_{mf}(\mathbf{x}, \mathbf{\tilde{z}}^{(k)}; K\Delta/m_k) = \sum_{k=1}^{K} \frac{m_k}{K} \sum_{i=1}^{n} \left( \frac{x_i^2}{K} - x_i s_i - x_i \sum_{k=1}^{K} \frac{z_{i}^{(k)}}{\sqrt{m_k}} \right) \tag{26}
\]

Define \(m_{mf} := K^{-1} \sum_{k=1}^{K} m_k\). Simple algebra leads to

\[
\mathcal{H}_{K,1;0}(\mathbf{x}; \mathbf{\theta}) = \frac{m_{mf}}{\Delta} \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_i s_i - x_i \sqrt{\frac{m_{mf}}{K}} \sum_{k=1}^{K} \frac{z_{i}^{(k)}}{\sqrt{m_k}} \right). \tag{27}
\]

We now proceed as previously using again the stability property of the Gaussian noise variables. Since \(\tilde{z}_{i}^{(k)}\) are i.i.d. \(\mathcal{N}(0,1)\), then \(\tilde{z}_{i} := \sum_{k=1}^{K} \tilde{z}_{i}^{(k)}/\sqrt{m_k/K} \sim \mathcal{N}(0,1)\) and are i.i.d. Let \(\tilde{z} = [\tilde{z}_{i}]_{i=1}^{n}\). Using (22) we find that \(f_{K,1;0}\) can also be expressed as

\[
f_{K,1;0} = \frac{1}{n} \mathbb{E}_{\mathbf{s}, \mathbf{\tilde{z}}} \left[ \ln \mathbb{E}_{\mathbf{x}} \left[ e^{-\mathcal{H}_{K,1;0}(\mathbf{x}; \mathbf{\theta})} \right] \right] = -\mathbb{E}_{\mathbf{s}, \mathbf{\tilde{z}}} \left[ \ln \mathbb{E}_{\mathbf{x}} \left[ e^{-\frac{m_{mf}}{\Delta} \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_i s_i - x_i \tilde{z}_i \sqrt{\frac{m_{mf}}{m_k}} \right)} \right] \right]. \tag{28}
\]

Expression (28) is nothing else than the free energy (8) associated with the following scalar denoising model: \(y = s + \tilde{z} \Sigma(m_{mf}; \Delta)\), which leads to

\[
f_{K,1;0} = f_{den}(\Sigma(m_{mf}; \Delta)). \tag{29}
\]

D. Evaluating the free energy change along the stochastic interpolation path

By construction of (12) we have the following coherence property (see Figure 1): The \((k, t = 1)\) and \((k+1, t = 0)\) models are equivalent (the Hamiltonian (12) is invariant under this change) and thus \(f_{k,1;\epsilon} = f_{k+1,0;\epsilon}\) for any \(k\). This implies that the \((k, t)\)-interpolating free energy (22) verifies

\[
f_{1,0;\epsilon} = f_{K,1;\epsilon} + \sum_{k=1}^{K} (f_{k,0;\epsilon} - f_{k,1;\epsilon}) = f_{K,1;\epsilon} - \sum_{k=1}^{K} \int_{0}^{1} dt \frac{df_{k,t;\epsilon}}{dt}. \tag{30}
\]

Let us evaluate \(df_{k,t;\epsilon}/dt\). Define the overlap \(q_{\mathbf{x};\mathbf{s}} := n^{-1} \sum_{i=1}^{n} x_i s_i\). Starting from (22), lengthy but simple algebra (see sec. II-G1 for the details) leads that as long as \(P_1\) has bounded first four moments,

\[
\frac{df_{k,t;\epsilon}}{dt} = \frac{1}{4\Delta K} \mathbb{E}_{\mathbf{x}}[(q_{\mathbf{x};\mathbf{s}}^2 - 2m_k q_{\mathbf{x};\mathbf{s}})_{k,t;\epsilon}] + O((nK)^{-1}). \tag{31}
\]

This, with (30) and (29) yields

\[
f_{1,0;\epsilon} = f_{K,1;\epsilon} - \frac{1}{4\Delta K} \sum_{k=1}^{K} \int_{0}^{1} dt \mathbb{E}_{\mathbf{x}}[(q_{\mathbf{x};\mathbf{s}}^2 - 2m_k q_{\mathbf{x};\mathbf{s}})_{k,t;\epsilon}] + O(n^{-1})
\]

\[
= (f_{K,1;\epsilon} - f_{k,1;0}) + f_{den}(\Sigma(m_{mf}; \Delta)) - \frac{1}{4\Delta} \left\{ - \frac{1}{K} \sum_{k=1}^{K} m_k^2 + \frac{1}{K} \sum_{k=1}^{K} \int_{0}^{1} dt \mathbb{E}_{\mathbf{x}}[(q_{\mathbf{x};\mathbf{s}} - m_k)^2] \right\} + O(n^{-1})
\]

\[
= (f_{K,1;\epsilon} - f_{K,1;0}) + m_{mf}(m_{mf}; \Delta) + \frac{V(m_{mf})}{4\Delta} - \frac{1}{4\Delta K} \sum_{k=1}^{K} \int_{0}^{1} dt \mathbb{E}_{\mathbf{x}}[(q_{\mathbf{x};\mathbf{s}} - m_k)^2] + O(n^{-1}), \tag{32}
\]

\[
\]
where in the last equality we used (7) and introduced the non-negative “variance”
\[
V(\{m_k\}) := \frac{1}{K} \sum_{k=1}^{K} m_k^2 - m_m^2 = \frac{1}{K} \sum_{k=1}^{K} m_k^2 - \left( \frac{1}{K} \sum_{k=1}^{K} m_k \right)^2.
\]  
(33)

The fundamental identity (32) can now be used to prove the replica symmetric formula.

E. Upper bound

From (32) we recover the upper bound usually obtained by the classical method of Guerra and Toninelli [43] and applied in [21] to symmetric rank-one matrix estimation (but see also [20] which already fully proved the replica formula in the binary case). Choose \( m_k = \arg\min_{m \geq 0} f_{\text{RS}}(m; \Delta) \) for all \( k = 1, \ldots, K \). This implies \( m_m = \arg\min_{m \geq 0} f_{\text{RS}}(m; \Delta) \) as well as \( V(\{m_k\}) = 0 \). Thus since the integrand in (32) is non-negative we get the bound
\[
\lim_{n \to \infty} f_{1,0;\epsilon} \leq (f_{K,1;\epsilon} - f_{K,1;0}) + \min_{m \geq 0} f_{\text{RS}}(m; \Delta).
\]  
(34)

From Remark 2.4 we can just set \( \epsilon = 0 \) is this inequality and using (25) we obtain the desired upper bound.

Proposition 2.5 (Upper bound): Fix \( \Delta > 0 \). For any \( P_0 \) with bounded first four moments,
\[
\lim_{n \to \infty} f \leq \min_{m \geq 0} f_{\text{RS}}(m; \Delta).
\]  
(35)

F. Lower bound

The converse bound is generally the one requiring extra technical tools, such as the use of spatial coupling [17, 18, 22, 28, 44] or the Aizenman-Sims-Starr scheme, see [31–33, 35]. Thanks to the stochastic interpolation method the proof is quite straightforward. As in all of the existing methods, we need a concentration lemma which takes the following form in the present context (see sec. V for the proof).

Lemma 2.6 (Overlap concentration): For any \( P_0 \) with bounded support, any \( k \in \{1, \ldots, K\} \) and \( t \in [0,1] \) and for almost every (a.e) \( \epsilon \),
\[
\mathbb{E} \left[ \left( \langle q_{X,S} - \mathbb{E}[q_{X,S}] \rangle_{k,t;\epsilon} \right)^2 \right] = o_n(1),
\]  
(36)

where \( \lim_{n \to \infty} o_n(1) = 0 \) uniformly in \( K \). Note that \( o_n(1) \) is however not uniform in \( \epsilon \).

Using this lemma, (32) becomes for a.e \( \epsilon \)
\[
f_{1,0;\epsilon} = (f_{K,1;\epsilon} - f_{K,1;0}) + f_{\text{RS}}(m_m; \Delta) + \frac{V(\{m_k\})}{4\Delta} - \frac{1}{4\Delta K} \sum_{k=1}^{K} \int_{0}^{1} dt \left( \mathbb{E}[q_{X,S}]_{k,t;\epsilon} - m_k \right)^2 + o_n(1).
\]  
(37)

Remark 2.7 (No need to control the concentration rate): A powerful feature of the stochastic interpolation method is that as long as the overlap can be shown to concentrate, its concentration rate with \( n \) does not need to be controlled. Indeed, when employing Lemma 2.6 in (32), the overlap fluctuation \( o_n(1) \) is averaged over \( k = 1, \ldots, K \), not summed up.

At this point we need another crucial lemma (see sec. II-G2 for the proof) which is made possible by construction of the stochastic interpolation method.

Lemma 2.8 (Weak t-dependence at fixed k): For \( P_0 \) with bounded first four moments and any \( k \in \{1, \ldots, K\} \) and \( t \in [0,1] \),
\[
\left| \mathbb{E}[q_{X,S}]_{k,t;\epsilon} - \mathbb{E}[q_{X,S}]_{k,0;\epsilon} \right| = O\left( \frac{n}{K} \right).
\]  
(38)

Using this lemma and considering \( K \gg n \), e.g. \( K = \Omega(n^a) \) with \( a > 1 \), (37) takes the following convenient form for a.e \( \epsilon \):
\[
f_{1,0;\epsilon} = (f_{K,1;\epsilon} - f_{K,1;0}) + f_{\text{RS}}(m_m; \Delta) + \frac{V(\{m_k\})}{4\Delta} - \frac{1}{4\Delta K} \sum_{k=1}^{K} (\mathbb{E}[q_{X,S}]_{k,0;\epsilon} - m_k)^2 + o_n(1).
\]  
(39)

We now use the last crucial lemma which is a fundamental property of the stochastic interpolation.

Lemma 2.9 (Freedom of choice for the mean-field parameters): For a given \( \epsilon \), one can freely select
\[
m_k = \mathbb{E}[q_{X,S}]_{k,0;\epsilon}.
\]  
(40)

Proof: This is authorized by construction of the stochastic interpolation method. Indeed, the \((k = 1, t = 0)\)-interpolating model (see the Hamiltonian \( H_{1,0;\epsilon}(x; \theta) \) in (12)) is independent of \( \{m_k\}_{k=1}^{K} \). Thus we can freely set \( m_1 = \mathbb{E}[q_{X,S}]_{1,0;\epsilon} \). Once \( m_1 \) is fixed for a given \( \epsilon \), we go to the next step and set \( m_2 = \mathbb{E}[q_{X,S}]_{2,0;\epsilon} \), which again is possible due to the fact that the Hamiltonian \( H_{2,0;\epsilon}(x; \theta) \) and the Gibbs average \( \langle \cdot \rangle_{2,0;\epsilon} \) as well depend only on \( m_1 \) which has already been fixed. And so forth: As seen from Fig. 1, the Gibbs average \( \langle \cdot \rangle_{k,0;\epsilon} \) depends only on \( \{m_{k'}\}_{k'=1}^{k-1} \) which were already fixed in the previous steps so that the choice (40) is valid. Note that \( \mathbb{E}[q_{X,S}]_{k,0;\epsilon} \geq 0 \) which is important as the \( m_k \)'s play the role of signal-to-noise ratios, and thus must be positive.
With this particular choice of mean-field parameters \( \{m_k\}_{k=1}^K \), the sum over \( k = 1, \ldots, K \) in (39) is set to zero. Since \( V \) is non-negative, (39) directly implies the following lower bound for a.e. \( \epsilon \):

\[
\begin{align*}
 f_{1,0;\epsilon} &= (f_{K,1;\epsilon} - f_{K,1;0}) + \mathbb{E}\left[\sum_{k=1}^K \mathbb{E}[\langle q_{S_k}, S_k, \epsilon, \cdot \rangle; \Delta]\right] + \frac{V(\mathbb{E}[\langle q_{S_k}, S_k, \epsilon, \cdot \rangle; \Delta])}{4\Delta} + \sigma_n(1) \\
 &\geq (f_{K,1;\epsilon} - f_{K,1;0}) + \min_{m \geq 0} f_{\text{RS}}(m; \Delta) + \sigma_n(1),
\end{align*}
\]

(41)

which implies when letting \( n \to \infty \) and \( \epsilon \to 0 \) (using also Remark 2.4 and (25)):

**Proposition 2.10 (Lower bound):** Fix \( \Delta > 0 \). For any \( P_0 \) with bounded support,

\[
\lim_{n \to \infty} f \geq \min_{m \geq 0} f_{\text{RS}}(m; \Delta).
\]

(43)

Propositions 2.5 and 2.10 end the proof of Theorem 2.1.

**Remark 2.11 (The overlap must concentrate):** Note that it is not at all obvious that one can find \( \{m_k\} \) which directly cancel the integrals in the fundamental identity (32) without using the overlap concentration of Lemma 2.6. Overlap concentration is a fundamental requirement of the above proof. This agrees with the statistical physics assumption that a necessary condition for the validity of the replica symmetric method is precisely the overlap concentration [5].

We present an alternative way to obtain the lower bound that is not directly based on the positivity of \( V \) (this positivity can be traced back to the convexity of \( m^2 \) in (7)). This alternative route is a little bit more complicated but very handy for the more complicated models in the following sections. Note that defining

\[
\tilde{f}_{\text{RS}}(\{m_k\}; \Delta) := \frac{1}{4\Delta K} \sum_{k=1}^K m_k^2 + f_{\text{den}}(\Sigma(m_{\text{mf}}; \Delta)) = f_{\text{RS}}(m_{\text{mf}}; \Delta) + \frac{V(\{m_k\})}{4\Delta},
\]

(44)

the identity (41) is equivalent to

\[
\begin{align*}
 f_{1,0;\epsilon} &= (f_{K,1;\epsilon} - f_{K,1;0}) + \mathbb{E}\left[\sum_{k=1}^K \mathbb{E}[\langle q_{S_k}, S_k, \epsilon, \cdot \rangle; \Delta]\right] + \frac{V(\mathbb{E}[\langle q_{S_k}, S_k, \epsilon, \cdot \rangle; \Delta])}{4\Delta} + \sigma_n(1) \\
 &\geq (f_{K,1;\epsilon} - f_{K,1;0}) + \min_{\{m_k\} \geq 0} \tilde{f}_{\text{RS}}(\{m_k\}; \Delta) + \sigma_n(1),
\end{align*}
\]

(45)

and thus, when taking the limits \( n \to \infty \) and \( \epsilon \to 0 \) (recall Remark 2.4) and using (25),

\[
\lim_{n \to \infty} f \geq \min_{\{m_k\} \geq 0} \tilde{f}_{\text{RS}}(\{m_k\}; \Delta).
\]

(46)

Simple algebra starting from \( \partial_{m_k} \tilde{f}_{\text{RS}}(\{m_k\}; \Delta) = 0 \) implies that the minimum of \( \tilde{f}_{\text{RS}}(\{m_k\}; \Delta) \) is attained for \( m_k = m_* = -\frac{2}{4\Delta^2} f_{\text{den}}(\Sigma(m_{\text{mf}}; \Delta)) \geq 0 \) for all \( k = 1, \ldots, K \). This is also the argmin of \( f_{\text{RS}}(m; \Delta) \) given by (7). Using \( \tilde{f}_{\text{RS}}(\{m_k = m_*\}; \Delta) = \min_{m \geq 0} f_{\text{RS}}(m; \Delta) \), identity (46) leads Proposition 2.10.

**G. Proofs**

1) **Proof of the identity (31):** We will need a fundamental identity\(^2\) which is straightforward consequence of the Bayes law. Let \( X, X' \) be two i.i.d replicas drawn according to the product distribution \( P_{k,t,\epsilon}(x|\Theta)P_{k,t,\epsilon}(x'|\Theta) \). Then for any function \( g \),

\[
\mathbb{E}[\langle g(X,S)\rangle_{k,t,\epsilon}] = \mathbb{E}[\langle g(X',S')\rangle_{k,t,\epsilon}].
\]

(47)

Let us now compute \( df_{k,t,\epsilon}/dt \). Starting from (22), (12) (19), one obtains

\[
\frac{df_{k,t,\epsilon}}{dt} = \frac{1}{n} \mathbb{E}\left[\frac{dH_{k,t,\epsilon}(X, \Theta)}{dt}\right] = \frac{1}{n} \mathbb{E}\left[\frac{d}{dt} h_{\text{mf}}(X, S, Z^{(k)}, \frac{K\Delta}{t,m_k}) + \frac{d}{dt} h(X, S, Z^{(k)}, \frac{K\Delta}{1-t})\right]_{k,t,\epsilon}
\]

\[
= \frac{1}{nK\Delta} \mathbb{E}\left[\sum_{i=1}^n \left(\frac{X_i^2}{2} - X_iS_i - X_iZ_i^{(k)} \frac{K\Delta}{t,m_k}\right) - \sum_{i \leq j} \left(\frac{X_iX_j}{2n} - \frac{X_iX_jZ^{(k)}_i}{2\sqrt{n}} \frac{K\Delta}{1-t}\right)\right]_{k,t,\epsilon}.
\]

Now we integrate by part the Gaussian noise using the elementary formula \( \mathbb{E}_Z[Z\{f(Z)\}] = \mathbb{E}_Z[f'(Z)] \). This leads to

\[
\frac{df_{k,t,\epsilon}}{dt} = \frac{1}{nK\Delta} \mathbb{E}\left[\sum_{i=1}^n \left(\frac{X_i^2}{2} - X_iS_i\right) - \sum_{i \leq j} \left(\frac{X_iX_j}{2n} \frac{X_iX_j}{2n} - \frac{X_iX_jS_iS_j}{n}\right)\right]_{k,t,\epsilon}.
\]

(48)

\(^2\)This identity has been abusively called “Nishimori identity” in the statistical physics literature. One should however note that it is a simple consequence of Bayes formula (see e.g. appendix B of [18]). The “true” Nishimori identity [45] concerns models with one extra feature, namely a gauge symmetry which allows to eliminate the input signal, and the expectation over \( S \) in (47) can therefore be dropped (see e.g. [20]).
where $\mathbf{X}$, $\mathbf{X}'$ are the two i.i.d replicas drawn according to (20). An application of identity (47) then leads to

$$
\frac{d\langle q_{k,t}\rangle}{dt} = \frac{1}{2K} \mathbb{E}
\left[
\frac{1}{n^2} \sum_{i \leq j=1}^{n} X_i X_j S_i S_j - \frac{m}{n} \sum_{i=1}^{n} X_i S_i
\right]_{k,t,e}
$$

$$
= \frac{1}{2K} \mathbb{E}
\left[
\frac{1}{2n^2} \sum_{i,j=1}^{n} X_i X_j S_i S_j + \frac{1}{2n^2} \sum_{i=1}^{n} X_i^2 S_i^2 - \frac{m}{n} \sum_{i=1}^{n} X_i S_i
\right]_{k,t,e}.
$$

(49)

The Cauchy-Schwarz inequality and (47) imply that $\mathbb{E}[\langle n^{-2} \sum_{i=1}^{n} X_i^2 S_i^2 \rangle_{k,t,e}] = \mathcal{O}(n^{-1})$ as long as $P_0$ has bounded fourth moment. Indeed, by Cauchy-Schwarz

$$
\mathbb{E}
\left[
\langle n^{-1} \sum_{i=1}^{n} X_i^2 S_i^2 \rangle_{k,t,e}
\right] \leq \left( \mathbb{E}\left[\langle n^{-1} \sum_{i=1}^{n} X_i^4 \rangle_{k,t,e}\right]\right)^{1/2}\left( \mathbb{E}\left[\langle n^{-1} \sum_{i=1}^{n} S_i^4 \rangle\right]\right)^{1/2}
$$

(50)

and by (47) we have $\mathbb{E}[\langle X_i^4 \rangle_{k,t,e}] = \mathbb{E}[S_i^4]$ for $i = 1, \ldots, n$, thus we get

$$
\mathbb{E}
\left[
\langle n^{-1} \sum_{i=1}^{n} X_i^2 S_i^2 \rangle_{k,t,e}
\right] \leq \mathbb{E}[S_i^4].
$$

(51)

Finally, expressing the two other terms in (49) with the help of the overlap $q_{k,s} = n^{-1} \sum_{i=1}^{n} x_i s_i$ we find (31).

2) Proof of Lemma 2.8: The proof of this lemma uses another interpolation:

$$
\mathbb{E}[\langle q_{k,s} \rangle_{k,t,e}] - \mathbb{E}[\langle q_{k,s} \rangle_{k,0,e}] = \int_0^t ds \mathbb{E}\left[\langle q_{k,s} \rangle_{k,s,e} d/ds \left(\mathcal{H}_{k,s,e}(X'; \Theta) - \mathcal{H}_{k,s,e}(X; \Theta)\right)\right]_{k,s,e},
$$

(52)

where $X$, $X'$, $X''$ etc are i.i.d replicas distributed according to (20). The same computations as in sec. II-G1 lead to

$$
\mathbb{E}[\langle q_{k,s} \rangle_{k,t,e}] - \mathbb{E}[\langle q_{k,s} \rangle_{k,0,e}] = \frac{1}{K} \int_0^t ds \mathbb{E}[\langle q_{k,s} \rangle_{k,s,e} d/ds \left(g(X', X''; S) - g(X, X'; S)\right)\right]_{k,s,e}
$$

(53)

where we define

$$
g(X, X'; S) := \frac{m}{\Delta} \sum_{i=1}^{n} \left(\frac{x_i x'_i}{2} - x_i s_i\right) - \frac{1}{\Delta} \sum_{i,j=1}^{n} \left(\frac{x_i x_j x'_i x'_j}{4n} - \frac{x_i x_j s_i s_j}{n}\right).
$$

(54)

Finally from (53) and Cauchy-Schwarz, one obtains

$$
\mathbb{E}[\langle q_{k,s} \rangle_{k,t,e}] - \mathbb{E}[\langle q_{k,s} \rangle_{k,0,e}] = \mathcal{O}\left(\frac{1}{K} \sqrt{\mathbb{E}[\langle q_{k,s}^2 \rangle_{k,s,e}] \mathbb{E}[\langle g(X, X'; S)^2 \rangle_{k,s,e}]}ight) = \mathcal{O}\left(\frac{n}{K}\right).
$$

(55)

The last equality is true as long as the prior $P_0$ has bounded first four moments. We prove this claim now. Let us start by studying $\mathbb{E}[\langle q_{k,s}^2 \rangle_{k,s,e}]$. Using Cauchy-Schwarz for the inequality and (47) for the subsequent equality,

$$
\mathbb{E}[\langle q_{k,s}^2 \rangle_{k,s,e}] = \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i X_j S_i S_j \rangle_{k,s,e}] \leq \frac{1}{n^2} \sum_{i,j=1}^{n} \sqrt{\mathbb{E}[\langle X_i^2 \rangle_{k,s,e}] \mathbb{E}[\langle S_i^2 \rangle_{k,s,e}]^2} = \frac{n}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[S_i^2 S_j^2] = \mathcal{O}(1),
$$

(56)

where the last equality is valid for $P_0$ with bounded second and fourth moments. For $\mathbb{E}[\langle g(X, X'; S)^2 \rangle_{k,s,e}]$ we proceed similarly by decoupling the expectations using Cauchy-Schwarz and then using (47) to make appear only terms depending on the signal $s$. One finds that under the same conditions on the moments of $P_0$, $\mathbb{E}[\langle g(X, X'; S)^2 \rangle_{k,s,e}] = \mathcal{O}(n^2)$, which combined with (56) leads to the last equality of (55) and ends the proof.

III. APPLICATION TO RANK-ONE SYMMETRIC TENSOR ESTIMATION

The present method can be extended to cover rank-one symmetric tensor estimation, which amounts to treat a kind of $p$-spin model on the Nishimori line. For binary spins the Guerra-Toninelli bound was proven in [20] for any value of $p$, the replica symmetric formula was proved in the whole phase diagram for $p = 2$, and also in a restricted region away from the first order phase transition for $p \geq 3$. A complete proof for all $p \geq 2$ and general spins was finally achieved in [32] by a rigorous version of the cavity method and the Aizenman-Sims-Starr principle.
A. Symmetric rank-one tensor estimation: Setting

The symmetric tensor problem is very close to the matrix case presented in full details in sec. II so we only sketch the main steps. The observed symmetric tensor \( w \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) is obtained through the following estimation model:

\[
w_{i_1, i_2, \ldots, i_p} = \sqrt{\frac{(p-1)!}{n^{p-1}}} s_{i_1} s_{i_2} \ldots s_{i_p} + z_{i_1, i_2, \ldots, i_p} \sqrt{\Delta} \quad \text{for} \quad 1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq n, \quad (57)
\]

where \( s \in \mathbb{R}^n \) with i.i.d components distributed according to a known prior \( P_0 \), \( Z \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_p} \) is a symmetric Gaussian noise tensor with i.i.d (up to the symmetry constraint) \( \mathcal{N}(0, 1) \) entries.

We note that, like in the case of symmetric matrix estimation of sec. II-A, the channel universality property (see remark 2.3) is valid in the present setting. This means that by covering the additive white Gaussian noise \( (57) \), we actually treat a wide range of (component-wise) inference channels \( P_{\text{out}}(w_{i_1, i_2, \ldots, i_p} \sqrt{\frac{(p-1)!}{n^{p-1}}} s_{i_1} s_{i_2} \ldots s_{i_p}) \). We refer to [32, 41, 46] for more details on this point.

The free energy of the model is

\[
f := -\frac{1}{n} \mathbb{E}_{S, Z} \left[ \ln \mathbb{E}_X \left[ e^{-\frac{1}{\Delta} \sum_{i_1 \leq i_2 \leq \ldots \leq i_p} \frac{(p-1)!}{n^{p-1}} X_{i_1}^2 \ldots X_{i_p}^2 - \frac{(p-1)!}{n^{p-1}} X_{i_1} S_{i_1} \ldots X_{i_p} S_{i_p} - \sqrt{\frac{(p-1)!}{n^{p-1}}} Z_{i_1} z_{i_1} \ldots X_{i_p} x_{i_p}} \right] \right]. \quad (58)
\]

For a \( P_0 \) with bounded first four moments the free energy is related to the mutual information \( I(S; W) \) through

\[
\frac{I(S; W)}{n} = f + \frac{\mathbb{E}[S|^2]}{2p\Delta} + O(n^{-1}). \quad (59)
\]

Define the RS potential for symmetric tensor estimation as

\[
f_{\text{RS}}(m; \Delta) := \frac{(p-1)m^p}{2p\Delta} + f_{\text{den}}(\Sigma(m; \Delta)) \quad (60)
\]

where \( \Sigma(m; \Delta)^2 := \Delta/m^{p-1} \) and \( f_{\text{den}}(\Sigma) \) is given by (8). Next we prove the RS formula.

**Theorem 3.1 (RS formula for symmetric rank-one tensor estimation):** Fix \( \Delta > 0 \). For any \( P_0 \) with bounded support, the asymptotic free energy of the symmetric tensor estimation model \( (57) \) verifies

\[
\lim_{n \to \infty} f = \min_{m \geq 0} f_{\text{RS}}(m; \Delta). \quad (61)
\]

Again, we note that the bounded support property of \( P_0 \) is only needed for concentration proofs and does not impose any upper limit on the size of the support. We believe this can be removed as long as \( P_0 \) has bounded first four moments.

B. Proof of the RS formula

The (perturbed) \((k, t)\)-interpolating Hamiltonian is

\[
\mathcal{H}_{k,t}(x; \theta) := \sum_{k'=k+1}^K h\left( x, s, z^{(k')}, K \Delta \right) + \sum_{k'=1}^{k-1} h_{\text{mf}}\left( x, s, z^{(k')}, K \Sigma(m_k; \Delta)^2 \right) + h\left( x, s, z^{(k)}; K \frac{\Delta}{1-t} \right) + \sum_{k'=1}^{K-1} \frac{n}{t} \left( \frac{x_i^2}{2} - x_i s_i - x_i z_i \right), \quad (62)
\]

where the trial signal-to-noise ratios \( \{m_k\}_K \) are to be fixed later and

\[
h\left( x, s, z, \sigma^2 \right) := \frac{1}{\sigma^2} \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_p} \left( \frac{(p-1)!}{2np-1} x_{i_1}^2 \ldots x_{i_p}^2 - \frac{(p-1)!}{n^{p-1}} x_{i_1} s_{i_1} \ldots x_{i_p} s_{i_p} - \sigma \sqrt{\frac{(p-1)!}{n^{p-1}}} z_{i_1} x_{i_1} \ldots x_{i_p} \right), \quad (63)
\]

\[
h_{\text{mf}}\left( x, s, z, \sigma^2 \right) := \frac{1}{\sigma^2} \sum_{i=1}^n \left( \frac{x_i^2}{2} - x_i s_i - \sigma z_i x_i \right). \quad (64)
\]

The associated \((k, t)\)-interpolating model, Gibbs expectation and \((k, t)\)-interpolating free energy are defined respectively by (20), (21) and (22). Using the stability property of the Gaussian noise variables, one can check that the initial and final \((k, t)\)-interpolating models are such that

\[
\begin{align*}
&f_{1,0} = f, \quad (65) \\
&f_{K,1} = f_{\text{den}}(\Sigma_{\text{mf}}(\{m_k\}; \Delta)), \quad (66)
\end{align*}
\]

where

\[
\Sigma_{\text{mf}}(\{m_k\}; \Delta) := \left( \frac{1}{K} \sum_{k=1}^K \Sigma(m_k; \Delta)^{-2} \right)^{-1/2} = \frac{1}{\Delta K} \left( \sum_{k=1}^K m_k \right)^{-1/2}. \quad (67)
\]
By a trivial generalization of the calculations of sec. II-G1, we obtain the variation of the \((k, t)\)-interpolating free energy:

\[
\frac{df_{k,t,\epsilon}}{dt} = \frac{1}{2p\Delta K} \mathbb{E}[\langle q_{X,S}^p - p m_k^{p-1} q_{X,S} \rangle_{k,t,\epsilon}] + O((nK)^{-1}),
\]

where the overlap is \(q_{k,S} := n^{-1} \sum_i^n x_is_i\).

The analysis of sec. VI can be straightforwardly generalized to the present setting (at the expense of more heavy notations) and then the results of sec. V directly follow which implies that the overlap concentrates for a.e. \(\epsilon\). Using this concentration together with (30), (65), (66) and (68) yields

\[
f_{1,0,\epsilon} = (f_{K,1,\epsilon} - f_{K,1,0}) + \tilde{f}_{RS}(\{m_k\}; \Delta)
\]

\[
- \frac{1}{2p\Delta K} \sum_{k=1}^K \int_0^1 dt \left( \mathbb{E}[\langle q_{X,S}^p \rangle_{k,t,\epsilon} - p m_k^{p-1} \mathbb{E}[\langle q_{X,S} \rangle_{k,t,\epsilon}] + (p - 1)m_k^p \right) + o_n(1),
\]

\[
\tilde{f}_{RS}(\{m_k\}; \Delta) := \frac{1}{2p\Delta K} \sum_{k=1}^K (p - 1)m_k^p + f_\text{det}(\Sigma_{\text{det}}(\{m_k\}; \Delta)).
\]

One can in addition show the weak \(t\)-dependence of the average overlaps \(\mathbb{E}[\langle q_{X,S} \rangle_{k,t,\epsilon}]\) at fixed \(k\) by generalizing the proof of sec. II-G2. Then Lemma 2.8 also applies here which permits to replace \(\mathbb{E}[\langle q_{X,S} \rangle_{k,t,\epsilon}]\) by \(\mathbb{E}[\langle q_{X,S} \rangle_{k,0,\epsilon}]\) in (69) and thus to remove the \(t\) integral. Once this is done, we freely select \(m_k = m^*_k \in \mathbb{R}^m\) in order to cancel the sum in (69) term by term (the proof of the validity of this choice is the same as the proof of Lemma 2.9; this is authorized by construction of the stochastic interpolation method). This implies the same inequalities as (45) and (46) (after taking the limits \(n \to \infty\) and \(\epsilon \to 0\)). Finally, arguments similar to those exposed below (46) apply here and lead to a statement analogous to Proposition 2.10 for the present setting: The asymptotic free energy is lower bounded by the minimum of the RS potential (with the proper free energy (58) and RS potential (60)).

To obtain the converse bound, observe that by convexity of \(x^p\) for \(x \geq 0\) we have for all \(x, y \geq 0\) that the polynomial \(x^p - px^{p-1}x + (p - 1)y^p \geq 0\). Select \(m_k = m^*_k := \arg\min_{m \geq 0} f_{RS}(m; \Delta)\) for all \(k = 1, \ldots, K\) in (69). Since we have that both \(\mathbb{E}[\langle q_{X,S} \rangle_{k,t,\epsilon}]\) and \(m^*_k \geq 0\) are non-negative, the integral is also non-negative. Thus using \(f_{RS}(\{m_k = m^*_k\}; \Delta) = \min_{m \geq 0} f_{RS}(m; \Delta)\) in (69) implies that the inequality (34) holds. Taking the proper limits, we obtain the analog of Proposition 2.5 for the present setting.

Combining this with the lower bound ends the proof of Theorem 3.1.

IV. APPLICATION TO GAUSSIAN RANDOM LINEAR ESTIMATION

A. Gaussian random linear estimation: Setting

In Gaussian random linear estimation (RLE) one is interested in reconstructing a signal \(s = [s_i]_{i=1}^n \in \mathbb{R}^n\) from few noisy measurements \(y = [y_{\mu}]_{\mu=1}^m \in \mathbb{R}^m\) obtained from the projection of \(s\) by a random i.i.d Gaussian measurement matrix \(\phi = [\phi_{\mu,i}]_{\mu=1,i=1}^{m,n} \in \mathbb{R}^{m \times n}\) with entries \(\phi_{\mu,i} \sim \mathcal{N}(0, 1/n)\). The measurement rate is \(\alpha := m/n\). We consider i.i.d additive white Gaussian noise of known variance \(\Delta\). Let the standardized noise components be \(Z_\mu \sim \mathcal{N}(0, 1), \mu = 1, \ldots, m\). Then the measurement model is \(y = \phi s + z\sqrt{\Delta}\), or equivalently

\[
y_{\mu} = \sum_{i=1}^n \phi_{\mu,i}s_i + z_{\mu}\sqrt{\Delta} \quad \text{for} \quad 1 \leq \mu \leq m.
\]

The signal has i.i.d components distributed according to a discrete prior \(P_0(s_i) = \sum_{a=1}^B p_\delta(s_i - a)\) with a finite number \(B\) of terms and \(\max_{i=1}^n |a_0| \leq s_{\max}\). Note that the more general case where the signal has i.i.d vectorial components, as considered in [17, 18], can be tackled with our proof technique exactly in the same way but we consider the scalar case for the sake of notational simplicity.

The free energy of the RLE model (71) (which is also equal to the mutual information per component \(I(S; Y)/n\) between the noisy observation and the signal) is defined as

\[
f := -\frac{1}{n} \mathbb{E}_{S,Z,\Phi} \left[ \ln \mathbb{E}_X \left[ e^{-\frac{1}{\alpha} \sum_{\mu=1}^m \left( \frac{1}{2} \phi_{\mu,i}^2 - |\phi_{\mu,i}|z_{\mu}\sqrt{\Delta} \right) \right] \right],
\]

where \(X \sim P_0, x := x - s, [\phi_{\mu,i}]_{\mu=1}^m \in \mathbb{R}^{n \times m}\). Let

\[
\Sigma(E; \Delta) := \frac{\Delta + E}{\alpha},
\]

\[
\psi(E; \Delta) := \frac{\alpha}{2} \left( \ln \left( 1 + \frac{E}{\Delta} \right) - \frac{E}{\Delta + E} \right).
\]
Define the following RS potential:

\[ f_{\text{RS}}(E; \Delta) := \psi(E; \Delta) + i_{\text{den}}(\Sigma(E; \Delta)), \]  

(75)

where \( i_{\text{den}}(\Sigma) \) is the mutual information \( I(S; Y) \) of the scalar Gaussian denoising model \( y = s + \tilde{z} \Sigma \) with \( S \sim P_0, \tilde{z} \sim \mathcal{N}(0,1) \):

\[ i_{\text{den}}(\Sigma) := -\mathbb{E}_{S, \tilde{z}} \left[ \ln \mathbb{E}_{\tilde{X}} \left[ e^{-\frac{1}{\bar{t}} (\tilde{x}^2 - \bar{x} \tilde{z} \Sigma)} \right] \right], \]

(76)

where \( X \sim P_0 \) and recall \( \bar{x} := x - s \). We will prove the RS formula (already proven in [17, 18, 29, 30]):

**Theorem 4.1 (RS formula for Gaussian RLE):** Fix \( \Delta > 0 \). For any discrete \( P_0 \), the asymptotic free energy of the RLE model (71) verifies

\[ \lim_{n \to \infty} f = \min_{E \geq 0} f_{\text{RS}}(E; \Delta). \]

(77)

**B. Proof of the RS formula**

Let \( \tilde{z}^{(k)} = [\tilde{z}_\mu^{(k)}]_{\mu=1}^m, \tilde{z}^{(k)} = [\tilde{z}_i^{(k)}]_{i=1}^n \) and \( \tilde{z} = [\tilde{z}_i]_{i=1}^n \) all with i.i.d \( \mathcal{N}(0,1) \) entries for \( k = 1, \ldots, K \). Define \( \Sigma_k := \Sigma(E_k; \Delta) \) where the trial parameters \( \{E_k\}_{k=1}^K \), interpreted as mean-square-errors, are fixed later on. The (perturbed) \((k, t)\)-interpolating Hamiltonian for the present model is

\[ \mathcal{H}_{k, t} (x; \theta) := \sum_{k'=k+1}^{K} h \left( x, s, z^{(k')}, \phi, K \Delta \right) + \sum_{k'=1}^{k-1} h_{\text{mf}} \left( x, s, z^{(k')}, K, \Sigma_k^2 \right) \]

\[ + \ h \left( x, s, z^{(k)}, \phi, K \gamma_k(t) \right) + h_{\text{mf}} \left( x, s, \tilde{z}^{(k)}, K \lambda_k(t) \right) + \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_i s_i - \frac{x_i \tilde{z}_i}{\sqrt{\epsilon}} \right). \]

(78)

Again, the last term is a small perturbation needed to use an important concentration lemma. Here \( \theta := \{s, \{z^{(k)}, \tilde{z}^{(k)}\}_{k=1}^K, \tilde{z}, \phi\} \), \( k \in \{1, \ldots, K\} \), \( t \in [0, 1] \) and

\[ h(x, s, z, \phi, \sigma^2) := \frac{1}{\sigma^2} \sum_{\mu=1}^{m} \left( \frac{[\phi \tilde{x}]_{\mu}^2}{2} - \sigma [\phi \tilde{x}]_{\mu} z_{\mu} \right), \]

(79)

\[ h_{\text{mf}} (x, s, \tilde{z}, \sigma^2) := \frac{1}{\sigma^2} \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - \sigma x_i \tilde{z}_i \right). \]

(80)

Moreover, the “signal-to-noise functions” \( \{\gamma_k(t), \lambda_k(t)\}_{k=1}^K \) verify

\[ \gamma_k(0) = \Delta^{-1}, \quad \lambda_k(0) = 0, \]

(81)

\[ \gamma_k(1) = 0, \quad \lambda_k(1) = \Sigma_k^{-2}, \]

(82)

as well as the following constraint (see [18] for an interpretation of this formula)

\[ \frac{\alpha}{\gamma_k(t)^{-1} + E_k} + \lambda_k(t) = \Sigma_k^{-2} \Rightarrow \frac{d \lambda_k(t)}{dt} = -\frac{d \gamma_k(t)}{dt} \left( 1 + \gamma_k(t) E_k \right)^2. \]

(83)

We also require \( \gamma_k(t) \) to be strictly decreasing with \( t \). The associated \((k, t)\)-interpolating model, Gibbs expectation and \((k, t)\)-interpolating free energy are defined respectively by (20), (21) and (22) with the Hamiltonian (78). Note that Remark 2.4 remains valid for the present model, see [18].

Similarly as in sec. II-C, and using again the stability property of the Gaussian random noise variables, it is easy to verify that the initial and final \((k, t)\)-interpolating models correspond to the RLE and denoising models respectively, that is

\[ f_{1,0;0} = f, \]

(84)

\[ f_{K,1;0} = i_{\text{den}}(\Sigma_{\text{mf}}(\{E_k\}; \Delta)). \]

(85)

where

\[ \Sigma_{\text{mf}}(\{E_k\}; \Delta) := \left( \frac{1}{K} \sum_{k=1}^{K} \Sigma_k^{-2} \right)^{-1/2}. \]

(86)
As before we use the identity (30) and compute the free energy change along the stochastic interpolation. Straightforward differentiation leads to

$$\frac{df_{k,t}}{dt} = \frac{1}{\mathcal{R}} \left( A_{k,t,e} + B_{k,t,e} \right),$$  \hspace{1cm} (87)

$$A_{k,t,e} := \frac{d\gamma_k(t)}{dt} \frac{1}{2n} \sum_{\mu=1}^{m} \mathbb{E} \left[ \left\langle \left| \Phi \tilde{X}_{\mu} \right|^2 - \frac{K}{\gamma_k(t)} \Phi \tilde{X}_{\mu} Z_{\mu}^{(k)} \right\rangle_{k,t,e} \right],$$  \hspace{1cm} (88)

$$B_{k,t,e} := \frac{d\lambda_k(t)}{dt} \frac{1}{2n} \sum_{i=1}^{n} \mathbb{E} \left[ \left\langle \tilde{X}_i^2 - \frac{K}{\lambda_k(t)} \bar{X}_i \bar{Z}_i^{(k)} \right\rangle_{k,t,e} \right],$$  \hspace{1cm} (89)

where as before $\mathbb{E}$ denotes the average w.r.t all quenched random variables $\theta$. The two quantities (88) and (89) can be simplified using Gaussian integration by parts. For example, integrating by parts w.r.t $E_k^{(k)}$, we proceed similarly with an integration by parts w.r.t $k, t$; $\bar{Z}_i^{(k)}$, and find

$$\sqrt{\frac{K}{\gamma_k(t)}} \mathbb{E}[\left\langle \Phi \tilde{X}_{\mu} \right\rangle_{k,t,e} Z_{\mu}^{(k)}] = \mathbb{E}[\left\langle \Phi \tilde{X} \right\rangle_{k,t,e} - \left\langle \Phi \tilde{X}_{\mu} \right\rangle_{k,t,e}]$$  \hspace{1cm} (90)

which allows to simplify $A_{k,t,e}$ as follows,

$$A_{k,t,e} = \frac{d\gamma_k(t)}{dt} \frac{1}{2n} \sum_{\mu=1}^{m} \mathbb{E}[\left\langle \left| \Phi \tilde{X}_{\mu} \right|^2 \right\rangle_{k,t,e}] = \frac{d\gamma_k(t)}{dt} \frac{1}{2m} \mathbb{E}[\left\langle \left| \Phi (X)_{k,t,e} - S \right| \right]^2 = \frac{d\gamma_k(t)}{dt} \frac{1}{2} \text{ymmse}_{k,t,e},$$  \hspace{1cm} (91)

where we recognized the “measurement minimum mean-square-error”

$$\text{ymmse}_{k,t,e} := \frac{1}{m} \mathbb{E}[\left\langle \left| \Phi (X)_{k,t,e} - S \right| \right]^2].$$  \hspace{1cm} (92)

For $B_{k,t,e}$ we proceed similarly with an integration by parts w.r.t $\bar{Z}_i^{(k)}$, and find

$$B_{k,t,e} = \frac{d\lambda_k(t)}{dt} \frac{1}{2n} \sum_{i=1}^{n} \mathbb{E}[\left\langle \tilde{X}_i \right\rangle_{k,t,e}] = \frac{d\lambda_k(t)}{dt} \frac{1}{2n} \mathbb{E}[\left\langle \left| \tilde{X}_i \right| \right\rangle_{k,t,e} - \left\langle \left| \tilde{X}_i \right| \right\rangle_{k,t,e} = - \frac{d\lambda_k(t)}{dt} \frac{1}{(1 + \gamma_k(t)E_k)^2} \frac{1}{2} \text{mmse}_{k,t,e},$$  \hspace{1cm} (93)

using (83) for the last equality, and the minimum mean-square-error (MMSE) defined as

$$\text{mmse}_{k,t,e} := \frac{1}{n} \mathbb{E}[\left\langle \left| (X)_{k,t,e} - S \right| \right]^2].$$  \hspace{1cm} (94)

The free energy can be shown to concentrate by generalizing the computations of Appendix E in [18] in order take into account that the noise variables $\{Z_{\mu}^{(k)}, \bar{Z}_i^{(k)}\}$ are indexed by the discrete interpolation parameter (this is what we do in sec. VI for matrix estimation). Since the free energy at fixed quenched random variables realization concentrates, both sec. VIII of [18] or sec. V of the present paper apply here (these are perfectly equivalent analyses and only require the identity (47) and the free energy concentration to be valid). Thus the overlap $q_{k,s} := n^{-1} \sum_i x_i \bar{x}_i$ concentrates too. As a consequence, Lemma 4.6 in [18] is valid here, and says that the MMSE and measurement MMSE are linked through the following relation for a.e $\epsilon$:

$$\text{ymmse}_{k,t,e} = \frac{\text{mmse}_{k,t,e}}{1 + \gamma_k(t) \text{mmse}_{k,t,e}} + o_n(1),$$  \hspace{1cm} (95)

where $\lim_{n \to \infty} o_n(1) = 0$. Here $o_n(1)$ is uniform in $K$ but not in $\epsilon$. This relation is the reason of the introduction of the $\epsilon$-perturbation in (78). Now combining (30), (84), (85), (87) and (91), (93), together with (95), we obtain for a.e $\epsilon$

$$f_{1,0;\epsilon} = (f_{K,1;\epsilon} - f_{K,1;0}) + i_{\text{den}}(\Sigma_{\text{mf}}(\{E_k\}; \Delta))$$

$$- \frac{\alpha}{2K} \sum_{k=1}^{K} \int_0^1 dt \frac{d\gamma_k(t)}{dt} \left[ \frac{\text{mmse}_{k,t,e}}{1 + \gamma_k(t) \text{mmse}_{k,t,e}} - \frac{\text{mmse}_{k,t,e}}{(1 + \gamma_k(t)E_k)^2} \right] + o_n(1).$$  \hspace{1cm} (96)

Again, the fluctuation $o_n(1)$ in Lemma 95 has been averaged over the stochastic interpolation (recall Remark 2.7) and we have used (85).

Now we need the following useful identity that can easily be checked using (74), (81), (82), (83):

$$\psi(E_k; \Delta) = \frac{\alpha}{2} \int_0^1 dt \frac{d\gamma_k(t)}{dt} \left[ \frac{E_k}{(1 + \gamma_k(t)E_k)^2} - \frac{E_k}{1 + \gamma_k(t)E_k} \right].$$  \hspace{1cm} (97)

Let us define

$$\bar{f}_{\text{RS}}(\{E_k\}; \Delta) := i_{\text{den}}(\Sigma_{\text{mf}}(\{E_k\}; \Delta)) + \frac{1}{K} \sum_{k=1}^{K} \psi(E_k; \Delta).$$  \hspace{1cm} (98)
With the help of (97) and (98) the identity (96) becomes

\[
\begin{align*}
 f_{1,0;\epsilon} &= (f_{K,1;\epsilon} - f_{K,1;0}) + \tilde{f}_{RS}(\{E_k\}; \Delta) \\
 &\quad - \frac{\alpha}{2K} \sum_{k=1}^{K} \int_{0}^{1} dt \frac{d\gamma_k(t)}{dt} \left( \frac{\text{mmse}_{k,t;\epsilon}}{1 + \gamma_k(t)\text{mmse}_{k,t;\epsilon}} - \frac{\text{mmse}_{k,t;\epsilon}}{1 + \gamma_k(t)\text{mmse}_{k,t;\epsilon}} \right) + \frac{E_k}{(1 + \gamma_k(t)\text{mmse}_{k,t;\epsilon})^2} - \frac{E_k}{1 + \gamma_k(t)\text{mmse}_{k,t;\epsilon}} + o_n(1)
\end{align*}
\]

(99)

We can now prove Theorem 4.1. We start with the upper bound. As in sec. II-E we choose \( E_k = E_* := \arg\min_{E \geq 0} f_{RS}(E; \Delta) \) for all \( k = 1, \ldots, K \) which implies that \( \Sigma_{ml}(\{E_k = E_*\}; \Delta) = \Sigma(E_*; \Delta) \) and thus, as seen from (98), \( \tilde{f}_{RS}(\{E_k = E_*\}; \Delta) = \min_{E \geq 0} f_{RS}(E; \Delta) \). Thus since the integrand in (99) is non-positive (recall that \( d\gamma_k(t)/dt \leq 0 \) and using the same arguments as in sec. II-E in order to take the \( \epsilon \to 0 \) limit, we reach:

**Proposition 4.2** (Upper bound): Fix \( \Delta > 0 \). For any discrete \( P_0 \),

\[
\lim_{n \to \infty} \frac{f}{\min_{E \geq 0} f_{RS}(E; \Delta)} = 0
\]

Let us now prove the converse bound. This bound required the use of spatial coupling in [17, 18] or “conditional central limit theorems” in [29, 30]. Here we derive the bound in a direct manner following the same steps as in sec. II-F. We first need the following identity: For any discrete \( P_0 \), any \( k \in \{1, \ldots, K\} \) and \( t \in [0, 1] \) and for a.e \( \epsilon \),

\[
|m\text{mse}_{k,t;\epsilon} - m\text{mse}_{k,0;\epsilon}| = O\left(\frac{n}{K}\right).
\]

(101)

Its proof is very similar to the one given in sec. II-G2 and uses (47) and Cauchy-Schwarz. Using this identity with \( K = \Omega(n^a) \), \( a > 1 \), in (99) and freely choosing \( E_k = m\text{mse}_{k,0;\epsilon} \) for all \( k = 1, \ldots, K \) (by the same arguments than those in the proof of Lemma 2.9), we reach

\[
\begin{align*}
 f_{1,0;\epsilon} &= (f_{K,1;\epsilon} - f_{K,1;0}) + \tilde{f}_{RS}(\{E_k = m\text{mse}_{k,0;\epsilon}\}; \Delta) + o_n(1) \geq (f_{K,1;\epsilon} - f_{K,1;0}) + \min_{\{E_k \geq 0\}} \tilde{f}_{RS}(\{E_k\}; \Delta) + o_n(1).
\end{align*}
\]

(102)

Thus, taking the limits \( n \to \infty \) and \( \epsilon \to 0 \) (recall Remark 2.4) and using (84),

\[
\lim_{n \to \infty} \frac{f}{\min_{E \geq 0} \tilde{f}_{RS}(\{E_k\}; \Delta)} = 0.
\]

(103)

Simple algebra starting from \( \partial_{E_k} \tilde{f}_{RS}(\{E_k\}; \Delta) = 0 \) shows that the minimum of \( \tilde{f}_{RS}(\{E_k\}; \Delta) \) is attained for a constant trial profile \( E_k = E_* = 2 \delta_{\Sigma_{i=1}^{n}i\text{den}(\Sigma)}[\Sigma_{ml}(\{E_k\}; \Delta)] \geq 0 \) for all \( k = 1, \ldots, K \). Then, thanks to the identity \( \tilde{f}_{RS}(\{E_k = E_*\}; \Delta) = \min_{E \geq 0} \tilde{f}_{RS}(E_*; \Delta) \) we get from (102) the desired bound.

**Proposition 4.3** (Lower bound): Fix \( \Delta > 0 \). For any discrete \( P_0 \),

\[
\lim_{n \to \infty} \frac{f}{\min_{E \geq 0} \tilde{f}_{RS}(E; \Delta)} = 0.
\]

(104)

Combining the two previous propositions ends the proof of Theorem 4.1.

V. CONCENTRATION OF OVERLAPS

The main goal of this section is the proof of Lemma 2.6. The proof strategy outlined here is very general and it will appear to the reader that it applies to essentially any inference problem for which the identity (47) is valid and as long as the free energy can be shown to concentrate. The ideas of such proofs go back to [12, 16, 20] for binary signals (in coding, CDMA and the gauge symmetric p-spin model) and have been extended more recently in random linear estimation for arbitrary signal distributions [18]. The exposition given here is a simplified and streamlined version.

Let

\[
\mathcal{L}_\epsilon := \frac{1}{n} \sum_{i=1}^{n} \left( \frac{x_i^2}{2} - x_is_i - \frac{x_i z_i}{2\sqrt{\epsilon}} \right).
\]

(105)

Note that up to the prefactor \( n^{-1} \) this quantity is the derivative of the perturbation in (19). We will show that Lemma 2.6 is a direct consequence of the following:

**Proposition 5.1** (Concentration of \( \mathcal{L}_\epsilon \) on \( \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle] \)): Let \( P_0 \) with bounded support in \([-M, M] \). For any \( 0 < a < 1 \),

\[
\lim_{n \to \infty} \int_{a}^{1} dx \mathbb{E}[\langle (\mathcal{L}_\epsilon - \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle])^2 \rangle_{k,t;\epsilon}] = 0.
\]

(106)

The proof of this proposition is broken in two parts. Notice that

\[
\mathbb{E}[\langle (\mathcal{L}_\epsilon - \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle])^2 \rangle_{k,t;\epsilon}] = \mathbb{E}[\langle (\mathcal{L}_\epsilon - \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle])^2 \rangle_{k,t;\epsilon}] + \mathbb{E}[\langle (\mathcal{L}_\epsilon - \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle])^2 \rangle_{k,t;\epsilon}].
\]

(107)
Thus it suffices to prove the two following lemmas. The first lemma expresses concentration w.r.t the posterior distribution (or “thermal fluctuations”) and is an elementary consequence of concavity properties of the free energy.

**Lemma 5.2 (Concentration of $\mathcal{L}_e$ on $\langle \mathcal{L}_e \rangle$):** For any $0 < a < 1$ we have
\[
\int_a^1 de \mathbb{E} \left[ \left( \langle \mathcal{L}_e - \langle \mathcal{L}_e \rangle \rangle_{k,t,e} \right)^2 \right] \leq \frac{\mathbb{E}[S^2]}{2n} \left( 1 + \frac{\ln a}{2} \right). \tag{108}
\]

The second lemma expresses the concentration of the average overlap w.r.t the realizations of quenched disorder variables.

**Lemma 5.3 (Concentration of $\langle \mathcal{L}_e \rangle$ on $\mathbb{E}[\langle \mathcal{L}_e \rangle]$):** Let $P_0$ with bounded support in $[-M, M]$. For any $0 < a < 1$ and $0 < \eta < 1/2$ we have
\[
\int_a^1 de \mathbb{E} \left[ \left( \langle \mathcal{L}_e \rangle_{k,t,e} - \mathbb{E}[\langle \mathcal{L}_e \rangle_{k,t,e}] \right)^2 \right] \leq \frac{C}{n^{4/2}} \left( \mathbb{E}[S^2] + \frac{2M}{\sqrt{a}} \right)^2. \tag{109}
\]

where $C$ is a numerical constant.

Thanks to the identity (112) that we will show in the appendix, the statements of Proposition 5.1 and Lemmas 5.2, 5.3 hold if we replace $\mathcal{L}_e$ by the overlap $q_{k,s}$.

The proof of the last lemma is based on an important but generic result concerning the concentration of the $(k, t)$-interpolating free energy for a single realization of quenched variables. Let
\[
F_{k,t,e}(\theta) := \frac{1}{n} \ln \mathbb{E}[e^{-H_{k,t,e}(X, \theta)}] \tag{110}
\]
where $X \sim P_0$ and recall that $f_{k,t,e} = \mathbb{E}[F_{k,t,e}(\Theta)]$.

**Proposition 5.4 (Concentration of the $(k, t)$-interpolating free energy):** Let $P_0$ with bounded support in $[-M, M]$. One can find $c > 0$ which depends only on $M$ and $\Delta$ such that for all $k = 1, \ldots, K$, $t \in [0, 1]$ and $\epsilon \in [0, 1]$,
\[
\mathbb{P} \left[ \left| F_{k,t,e}(\theta) - f_{k,t,e} \right| > u \right] \leq e^{-cnu^2} \tag{111}
\]
where $u > 0$. Explicit expressions for $c$ can be derived from (157) in sec. VI.

This proposition is proved in sec. VI. In the rest of this section we prove Lemmas 2.6, 5.2 and 5.3. In order to simplify the notations we set $\langle - \rangle_{k,t,e} \rightarrow \langle - \rangle$. The parameters $k$ and $t$ stay fixed and do not play any role, but it is important to be careful about the $\epsilon$ dependence.

**Proof of Lemma 2.6:** The proof is based on the exact formula
\[
\mathbb{E}[\langle (\mathcal{L}_e - \mathbb{E}[\langle \mathcal{L}_e \rangle])^2 \rangle] = \frac{1}{4} \left( \mathbb{E}[\langle q_{k,S}^2 \rangle] - \mathbb{E}[\langle q_{X,S} \rangle^2] \right) + \frac{1}{2} \left( \mathbb{E}[\langle q_{X,S} \rangle^2] - \mathbb{E}[\langle q_{X,S} \rangle^2] \right) + \frac{1}{4n^2} \sum_{i=1}^{n^2} \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2]. \tag{112}
\]

Its derivation is found in Appendix IX and involves lengthy algebra using identity (47) and integrations by parts w.r.t the Gaussian noise. This formula implies
\[
\mathbb{E}[\langle (\mathcal{L}_e - \mathbb{E}[\langle \mathcal{L}_e \rangle])^2 \rangle] \geq \frac{1}{4} \left( \mathbb{E}[\langle q_{X,S}^2 \rangle - \mathbb{E}[\langle q_{X,S} \rangle^2] \right) = 0. \tag{113}
\]

Therefore Proposition 5.1 immediately implies
\[
\lim_{n \to \infty} \int_a^1 de \mathbb{E}[\langle (q_{X,S} - \mathbb{E}[\langle q_{X,S} \rangle])^2 \rangle] = 0. \tag{114}
\]

The integrand is bounded uniformly in $n$ so by Lebesgue dominated convergence one can interchange the limit and integral, thus the statement of the Lemma follows, namely
\[
\lim_{n \to \infty} \mathbb{E}[\langle (q_{X,S} - \mathbb{E}[\langle q_{X,S} \rangle])^2 \rangle] = 0 \tag{115}
\]
for a.e $\epsilon \in [a, 1]$. Note that uniform boundedness in $n$ of the integrand does not require boundedness of the support of $P_0$ (this is only required for Lemma 5.3 and thus Proposition 5.1). Indeed it suffices to remark the following. From (47) $\mathbb{E}[\langle X_iX_j \rangle^2] = \mathbb{E}[S_iS_j \langle X_iX_j \rangle]$ therefore $n^{-2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_iX_j \rangle^2] = n^{-2} \sum_{i,j=1}^{n} \mathbb{E}[S_iS_j \langle X_iX_j \rangle] = \mathbb{E}[\langle q_{X,S}^2 \rangle]$ which implies
\[
\mathbb{E}[\langle q_{X,S}^2 \rangle] \leq \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i^2X_j^2 \rangle] = \frac{n^2 - n\mathbb{E}[S^2]^2 + n\mathbb{E}[S^4]}{n^2} \leq \mathbb{E}[S^2]^2 + \mathbb{E}[S^4]. \tag{116}
\]

This ends the proof.
We now turn to the proof of Lemmas 5.2 and 5.3. The main ingredient is a set of formulas for the first two derivatives of the free energy w.r.t \( \epsilon \). For any given realisation of the quenched disorder we have

\[
\frac{dF_{k,t,c}(\theta)}{d\epsilon} = \langle \mathcal{L}_\epsilon \rangle, \\
\frac{1}{n} \frac{d^2F_{k,t,c}(\theta)}{d\epsilon^2} = -\langle (\mathcal{L}_\epsilon^2) - \langle \mathcal{L}_\epsilon \rangle^2 \rangle + \frac{1}{4n^2\epsilon^{3/2}} \sum_{i=1}^{n} \langle X_i \rangle z_i. 
\]

(A17)

(A18)

Averaging (A17) and (A18) over \( \theta \), using a Gaussian integration by parts w.r.t \( \hat{z}_i \) and the identity \( \mathbb{E}[\langle X_i \rangle S_i] = \mathbb{E}[\langle X_i \rangle^2] \) (again a special case of (47)), we find

\[
\frac{dF_{k,t,c}}{d\epsilon} = \mathbb{E}[\langle \mathcal{L}_\epsilon \rangle] = -\frac{1}{2n} \sum_{i=1}^{n} \mathbb{E}[\langle X_i \rangle^2], \\
\frac{1}{n} \frac{d^2F_{k,t,c}}{d\epsilon^2} = -\mathbb{E}[\langle \mathcal{L}_\epsilon^2 \rangle - \langle \mathcal{L}_\epsilon \rangle^2] + \frac{1}{4n^2\epsilon} \sum_{i=1}^{n} \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2].
\]

(A19)

(A20)

There is another useful formula for \( d^2f_{k,t,c}/d\epsilon^2 \) that can be worked out directly (see Appendix IX) by differentiating (A19)

\[
\frac{1}{n} \frac{d^2F_{k,t,c}}{d\epsilon^2} = \frac{1}{2n} \sum_{i=1}^{n} \mathbb{E}[2\langle X_i \rangle \langle X_i \mathcal{L}_\epsilon \rangle - 2\langle X_i \rangle^2 \langle \mathcal{L}_\epsilon \rangle] = -\frac{1}{2n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle^2].
\]

(A21)

This formula clearly shows that \( f_{k,t,c} \) is a concave function of \( \epsilon \). It also shows that the apparent divergence for \( \epsilon \to 0 \) in (A20) is canceled by the first term. This also shows that the \( |\ln a| \) term in 5.2 is unavoidable.

Proof of Lemma 5.2: From (120) we have

\[
\mathbb{E}[\langle (\mathcal{L}_\epsilon - \langle \mathcal{L}_\epsilon \rangle)^2 \rangle] = -\frac{1}{n} \frac{d^2F_{k,t,c}}{d\epsilon^2} + \frac{1}{4n^2\epsilon} \sum_{i=1}^{n} \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2] \leq -\frac{1}{n} \frac{d^2f_{k,t,c}}{d\epsilon^2} + \frac{\mathbb{E}[S^2]}{4n\epsilon},
\]

where we used \( \mathbb{E}[\langle X_i^2 \rangle] = \mathbb{E}[S^2] \) (an application of (47)). Integrating over \( \epsilon \in [a,1] \) we obtain

\[
\int_{a}^{1} d\epsilon \mathbb{E}[\langle (\mathcal{L}_\epsilon - \langle \mathcal{L}_\epsilon \rangle)^2 \rangle] \leq \frac{1}{n} \frac{dF_{k,t,c}}{d\epsilon} \bigg|_{\epsilon=a} - \frac{1}{n} \frac{dF_{k,t,c}}{d\epsilon} \bigg|_{\epsilon=1} + \frac{\mathbb{E}[S^2]}{4n} \ln a \leq \frac{1}{n} \frac{dF_{k,t,c}}{d\epsilon} \bigg|_{\epsilon=1} + \frac{\mathbb{E}[S^2]}{4n} |\ln a|
\]

(A22)

using (A19) to assert that the first term of the r.h.s of the first inequality is negative. For the second term

\[
-\frac{1}{n} \frac{d}{d\epsilon} f_{k,t,c} \bigg|_{\epsilon=1} = \frac{1}{2n^2} \sum_{i=1}^{n} \mathbb{E}[\langle X_i \rangle^2_{k,t}] \leq \frac{1}{2n^2} \sum_{i=1}^{n} \mathbb{E}[\langle X_i \rangle^2_{k,t}] = \frac{\mathbb{E}[S^2]}{2n}.
\]

(A23)

This combined with (A23) allows to conclude the proof of Lemma 5.2.

Proof of Lemma 5.3: Recall that \( P_0 \) has bounded support in \([-M,M]\). Consider the two functions

\[
\tilde{F}(\epsilon) := F_{k,t,c}(\theta) + \sqrt{\epsilon} \sum_{i=1}^{n} M|\hat{z}_i|, \quad \tilde{f}(\epsilon) := f_{k,t,c} + \sqrt{\epsilon} \sum_{i=1}^{n} M\mathbb{E}[|\hat{z}_i|].
\]

(A24)

Because of (A18) we see that the second derivative of \( \tilde{F}(\epsilon) \) is negative, so this is a concave function of \( \epsilon \) (without this extra term \( F_{k,t,c}(\epsilon) \) is not necessarily concave, although \( f_{k,t,c} \) is concave). Note also that \( \tilde{f}(\epsilon) \) is concave. Concavity implies for any \( \delta > 0 \)

\[
\frac{d\tilde{F}(\epsilon)}{d\epsilon} - \frac{d\tilde{f}(\epsilon)}{d\epsilon} \leq \frac{\tilde{F}(\epsilon - \delta) - \tilde{F}(\epsilon) - \tilde{f}(\epsilon - \delta) + \tilde{f}(\epsilon)}{\delta}, \quad \frac{d\tilde{F}(\epsilon) - \delta}{d\epsilon} - \frac{d\tilde{f}(\epsilon) - \delta}{d\epsilon} \leq \frac{\tilde{F}(\epsilon) - \tilde{f}(\epsilon) - \tilde{f}(\epsilon - \delta) + \tilde{f}(\epsilon - \delta)}{\delta}.
\]

(A25)

(A26)

The difference between the derivatives appearing on the r.h.s of these inequalities cannot be considered small because at a first order transition point the derivatives have jump discontinuities. Set

\[
C^-(\epsilon) := \frac{d\tilde{f}(\epsilon + \delta) - d\tilde{f}(\epsilon)}{d\epsilon}, \quad C^+(\epsilon) := \frac{d\tilde{f}(\epsilon - \delta) - d\tilde{f}(\epsilon)}{d\epsilon} \geq 0,
\]

(A27)

where the signs of these quantities follow from concavity of \( \tilde{f}(\epsilon) \). From (A26), (A27) and (A28) we get

\[
\frac{\tilde{F}(\epsilon + \delta) - \tilde{F}(\epsilon) - \tilde{f}(\epsilon + \delta)}{\delta} - \frac{\tilde{F}(\epsilon) - \tilde{f}(\epsilon)}{\delta} - C^- (\epsilon) \leq \frac{d\tilde{F}(\epsilon) - d\tilde{f}(\epsilon)}{d\epsilon} \leq \frac{\tilde{F}(\epsilon) - \tilde{f}(\epsilon) - \tilde{f}(\epsilon - \delta) + \tilde{f}(\epsilon - \delta)}{\delta} - \frac{\tilde{F}(\epsilon - \delta) - \tilde{f}(\epsilon - \delta) - \tilde{f}(\epsilon) + C^+(\epsilon)}{\delta}.
\]

(A28)

(A29)
Now we will cast this inequality in a more usable form. From (125)
\[
\bar{F}(\epsilon) - \bar{f}(\epsilon) = F_{k,t,e}(\theta) - f_{k,t,e} + \sqrt{TM}A
\]
with
\[
A = \frac{1}{n} \sum_{i=1}^{n} (|\tilde{z}_i| - E[|\tilde{z}_i|])
\]
and from (117), (119),
\[
\frac{d\bar{F}(\epsilon)}{d\epsilon} - \frac{d\bar{f}(\epsilon)}{d\epsilon} = \langle L_{\epsilon} \rangle - E[\langle L_{\epsilon} \rangle] + \frac{M}{2\sqrt{\epsilon}} A.
\]
From (130), (132) it is easy to show that (129) implies
\[
|\langle L_{\epsilon} \rangle - E[\langle L_{\epsilon} \rangle]| \leq \delta^{-1} \sum_{u \in \{-\delta, \epsilon, \epsilon+\delta\}} (|F_{k,t,u}(\theta) - f_{k,t,u}| + M|A|\sqrt{u} + C^+(\epsilon) + C^-(\epsilon) + \frac{M}{2\sqrt{\epsilon}}|A|.
\]
At this point we use Proposition 5.4. A standard argument given at the end of this proof shows that this proposition implies
\[
E[(F_{k,t,e}(\theta) - f_{k,t,e})^2] = O(n^{-1+\eta})
\]
for any $0 < \eta < 1$. Squaring, then taking the expectation of (133) and using $E[A^2] = O(n^{-1})$ by the central limit theorem, $C^\pm(\epsilon) = O(1)$ and $(\sum_{i=1}^{p} v_i)^2 \leq p \sum_{i=1}^{p} v_i^2$ by convexity,
\[
\frac{1}{9} E[(\langle L_{\epsilon} \rangle - E[\langle L_{\epsilon} \rangle])^2] \leq \delta^{-2} O(n^{-1+\eta}) + 3\delta^{-2} M^2 (\epsilon + \delta) O(n^{-1}) + C^+(\epsilon)^2 + C^-(\epsilon)^2 + \frac{M^2}{4\epsilon} O(n^{-1}).
\]
Now fix $0 < a < 1$ and take $\delta < a$. Using $|d\bar{f}(\epsilon)/d\epsilon| \leq (E[S^2] + M/\sqrt{\epsilon})/2$ from (119) and (125), $C^\pm(\epsilon) \geq 0$ from (128) and the mean value theorem
\[
\int_a^1 \frac{1}{\epsilon} (C^+(\epsilon)^2 + C^-(\epsilon)^2) \leq \left( E[S^2] + \frac{M}{\sqrt{\epsilon}} \right) \int_a^1 \frac{1}{\epsilon} (C^+(\epsilon) + C^-(\epsilon))
\]
\[
= \left( E[S^2] + \frac{M}{\sqrt{\epsilon}} \right) \left( \bar{f}(1-\delta) - \bar{f}(1+\delta) + (\bar{f}(a+\delta) - \bar{f}(a-\delta)) \right)
\]
\[
\leq 2\delta \left( E[S^2] + \frac{M}{\sqrt{\epsilon}} \right)^2.
\]
Thus, integrating (135) over $\epsilon \in [a, 1]$ yields
\[
\frac{1}{9} \int_a^1 \frac{1}{\epsilon} E[(\langle L_{\epsilon} \rangle - E[\langle L_{\epsilon} \rangle])^2] \leq \delta^{-2} O(n^{-1+\eta}) + 3\delta^{-2} M^2 (1 + \delta) O(n^{-1}) + \frac{M^2}{4} \ln a |O(n^{-1}) + 2\delta \left( E[S^2] + \frac{M}{\sqrt{\epsilon}} \right)^2.
\]
Finally we choose $\delta = O(n^{-1+\frac{1}{2}\eta})$, $0 < \eta < 1/2$, and obtain for $n$ large enough (and $a$ fixed positive small)
\[
\int_a^1 \frac{1}{\epsilon} E[(\langle L_{\epsilon} \rangle - E[\langle L_{\epsilon} \rangle])^2] \leq C n^{-1+\frac{1}{2}\eta} \left( E[S^2] + \frac{2M}{\sqrt{\epsilon}} \right)^2
\]
for some large enough numerical constant $C$.

It remains to justify (134). By the Cauchy-Schwarz inequality and Proposition 5.4 we have
\[
E[(F_{k,t,e}(\theta) - f_{k,t,e})^2] = E[(F_{k,t,e}(\theta) - f_{k,t,e})^2 1(|F_{k,t,e}(\theta) - f_{k,t,e}| \leq u)]
\]
\[
+ E[(F_{k,t,e}(\theta) - f_{k,t,e})^2 1(|F_{k,t,e}(\theta) - f_{k,t,e}| > u)]
\]
\[
\leq u^2 + \sqrt{E[(F_{k,t,e}(\theta) - f_{k,t,e})^2]} \sqrt{E[1(|F_{k,t,e}(\theta) - f_{k,t,e}| > u)]}
\]
\[
\leq u^2 + \sqrt{E[(F_{k,t,e}(\theta) - f_{k,t,e})^2]} e^{-cu^2/2}.
\]
If we can show that the moments of the (random) free energy $F_{k,t,e}(\theta)$ are bounded uniformly in $n$, then the choice $u = n^{-(1/2)+\lambda}$ for any $0 < \eta < 1/2$ allows to conclude the proof. Let us briefly show how the moments are estimated. By the Jensen inequality
\[
E[e^{-H_{k,t,e}(X,\theta)}] \geq e^{-E[H_{k,t,e}(X,\theta)]}
\]
so we have
\[
F_{k,t,e}(\theta) \leq \frac{1}{n} E[H_{k,t,e}(X,\theta)].
\]
Our essential task is now to prove an upper bound on interpolating between the two realizations of the Gaussian disorder, with new interpolating parameter \( \hat{s} \). Guerra and Toninelli [47]. We fix the input signal realisation and we find that

The expectation over \( X \) is computed from (12) and one finds a polynomial in \( \{s_i, \{z_{ij}^{(k)}, \tilde{z}_i^{(k)}\}_{k=1}^K, \tilde{z}_i\}_{i=1}^n \) which all have bounded moments. On the other hand from (13), (14) by completing the squares we have

\[
\begin{align*}
    h(x, s, z, \sigma^2) &\geq -\frac{1}{2\sigma^2} \sum_{i \leq j=1}^n \left( \frac{s_is_j}{\sqrt{n}} + z_{ij}\right)^2, \\
    h_{\text{mf}}(x, s, \tilde{z}, \sigma^2) &\geq -\frac{1}{2\sigma^2} \sum_{i=1}^n (s_i + \tilde{z}_i)^2,
\end{align*}
\]

and we find that \( H_{k,t,e}(x; \theta) \) is lower bounded by a polynomial in \( \{s_i, \{z_{ij}^{(k)}, \tilde{z}_i^{(k)}\}_{k=1}^K, \tilde{z}_i\}_{i=1}^n \). This is also the case for \( F_{k,t,e}(\theta) \). With these upper and lower bounds on \( F_{k,t,e}(\theta) \) it is not hard to show that for any integer \( p \)

\[E[|F_{k,t,e}(\theta)|^p] \leq C_p\]

where \( C_p \) is independent of \( n \) and depends only on \( \Delta \) and moments of \( P_0 \).

VI. CONCENTRATION OF THE FREE ENERGY

In this section we prove Proposition 5.4. We will call \( E_Z, P_Z \) the expectation and probability law over all Gaussian variables, \( E_s, P_s \) the ones over the input signal variables, and \( E, P \) the ones over the joint law. The proof is broken up in two lemmas. We first show a lemma which expresses concentration w.r.t all Gaussian sources of disorder uniformly in the input signal.

**Lemma 6.1 (Concentration w.r.t the Gaussian quenched disorder):** Take \( P_0 \) with bounded support in \([-M, M]\). For any signal realisation \( s \) and all \( k = 1, \ldots, K \), \( t \in [0, 1] \) and \( \epsilon > 0 \) we have

\[
    P_Z[|F_{k,t,e}(\Theta) - E_Z[F_{k,t,e}(\Theta)]| > u/2] \leq 2 \exp \left( -\frac{nu^2}{16\Delta^2 + \epsilon u^2} \right),
\]

where \( u > 0 \).

**Proof:** The proof method is again based on an interpolation (of a different kind) that goes back to a beautiful work of Guerra and Toninelli [47]. We fix the input signal realisation \( s \) and consider two i.i.d copies for the Gaussian quenched variables \( z^{(k,1)}_i = [z_{ij}^{(k,1)}]_{i,j=1}^n \) and \( z^{(k,2)}_i = [z_{ij}^{(k,2)}]_{i,j=1}^n \). We also need two copies of the extra Gaussian noise introduced in the perturbation term (19), namely \( \tilde{z}_{i}^{(1)} = [\tilde{z}_i^{(1)}]_{i=1}^n \) and \( \tilde{z}_{i}^{(2)} = [\tilde{z}_i^{(2)}]_{i=1}^n \). We define a Hamiltonian interpolating between the two realizations of the Gaussian disorder, with new interpolating parameter \( \tau \in [0, 1] \):

\[
    H_{k,t,e}(x; \tau) := \sum_{k' > k} h\left(x, s, \sqrt{\tau} z^{(k',1)} + \sqrt{1 - \tau} z^{(k',2)}, K \Delta \right) + \sum_{k' < k} h_{\text{mf}}\left(x, s, \sqrt{\tau} z^{(k,1)} + \sqrt{1 - \tau} z^{(k,2)}, K \Delta \right)
\]

\[
    + h\left(x, s, \sqrt{\tau} z^{(k,1)} + \sqrt{1 - \tau} z^{(k,2)}, \frac{K \Delta}{1 - \tau} \right) + h_{\text{mf}}\left(x, s, \sqrt{\tau} z^{(k,1)} + \sqrt{1 - \tau} z^{(k,2)}, \frac{K \Delta}{1 - \tau} \right)
\]

\[
    + \epsilon \sum_{i=1}^n \frac{x_i^2}{2} - \epsilon_1 s_i - \frac{1}{\epsilon} \epsilon_1 (\sqrt{\tau} z_i^{(1)} + \sqrt{1 - \tau} z_i^{(2)})^2.
\]

Let \( Z_{k,t,e}(\tau) := E_X[\exp(-H_{k,t,e}(x, \tau))], X \sim P_0 \), the partition function associated to \( H_{k,t,e}(x, \tau) \). Let \( s > 0 \) be a trial parameter to be fixed later on and let

\[
    \varphi_{k,t,e}(\tau) := \ln E_1 \left[ \exp \left( s E_2 \left[ \ln Z_{k,t,e}(\tau) \right] \right) \right],
\]

where \( E_1 \) and \( E_2 \) are the expectations w.r.t the two sets of Gaussian variables (note that \( \varphi_{k,t,e}(\tau) \) depends on the fixed signal instance \( s \)). Using the union bound for the first inequality and Markov’s inequality together with \( \exp(\varphi_{k,t,e}(1)) = E_Z[\exp(-snF_{k,t,e}(\Theta))] \) and \( \exp(\varphi_{k,t,e}(0)) = E_Z[\exp(-snF_{k,t,e}(\Theta))] \) for the second one, one deduces that

\[
    P_Z[|F_{k,t,e}(\Theta) - E_Z[F_{k,t,e}(\Theta)]| > u/2] \leq P_Z[e^{ns(\varphi_{k,t,e}(\Theta) - E_Z[F_{k,t,e}(\Theta)]) - u/2}] > 1 + P_Z[e^{ns(\varphi_{k,t,e}(\Theta) - E_Z[F_{k,t,e}(\Theta)]) - u/2}] > 1
\]

\[
    \leq \exp \left( \varphi_{k,t,e}(0) - \varphi_{k,t,e}(1) - sn/2 \right) + \exp \left( \varphi_{k,t,e}(1) - \varphi_{k,t,e}(0) - sn/2 \right)
\]

\[
    \leq 2 \exp \left( \left| \varphi_{k,t,e}(1) - \varphi_{k,t,e}(0) \right| - sn/2 \right)
\]

\[
    \leq 2 \exp \left( \int_0^1 d\tau |\varphi'_{k,t,e}(\tau)| - sn/2 \right).
\]

Our essential task is now to prove an upper bound on \( |\varphi'_{k,t,e}(\tau)| \). We have

\[
    \varphi'_{k,t,e}(\tau) = \frac{E_1 \left[ s E_2 \left[ \frac{Z_{k,t,e}(\tau)}{Z_{k,t,e}(1)} \right] \exp(s E_2[\ln Z_{k,t,e}(\tau)]) \right]}{E_1 \left[ \exp(s E_2[\ln Z_{k,t,e}(\tau)]) \right]}
\]
where

\[
\begin{align*}
\mathbb{E}_2 \left[ Z_{k,t,e}^{(1)}(\tau) \right] &= \frac{1}{2\sqrt{K\Delta t}} \sum_{k' > k} \sum_{i,j} z_{ij}^{(k',1)} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] - \frac{1}{2\sqrt{K\Delta (1-\tau) n}} \sum_{k' > k} \sum_{i,j} \mathbb{E}_2 [Z_{ij}^{(k',2)} (X_i X_j)_{k,t,e}] \\
&\quad + \frac{\sqrt{1-t}}{2\sqrt{K\Delta t}} \sum_{i,j} z_{ij}^{(k,1)} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] - \frac{\sqrt{1-t}}{2\sqrt{K\Delta (1-\tau) n}} \sum_i \mathbb{E}_2 [Z_{i}^{(k,2)} (X_i X_j)_{k,t,e}] \\
&\quad + \frac{\sqrt{t}m_k}{2\sqrt{K\Delta t}} \sum_i z_i^{(1)} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] - \frac{\sqrt{t}m_k}{2\sqrt{K\Delta (1-\tau) n}} \sum_i \mathbb{E}_2 [Z_{i}^{(1)} (X_i X_j)_{k,t,e}] \\
&\quad + \frac{\sqrt{t}}{2\sqrt{t}} \sum_i z_i^{(1)} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] - \frac{\sqrt{t}}{2\sqrt{1-\tau}} \sum_i \mathbb{E}_2 [Z_{i}^{(2)} (X_i X_j)_{k,t,e}].
\end{align*}
\]

We then replace this expression in the numerator of (147) and integrate by parts over all standard Gaussian variables of type \(z^{(1)}\) and \(z^{(2)}\). Doing so generates partial derivatives of the form \(\mathbb{E}_2[\frac{\partial}{\partial z^{(1)}} (-)]\) and \(\mathbb{E}_2[\frac{\partial}{\partial z^{(2)}} (-)]\) as well as derivatives of the form \(\frac{\partial}{\partial z^{(1)}} \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)])\). A lengthy but straightforward calculation shows that only the later survive. The numerator of (147) becomes

\[
\begin{align*}
\mathbb{E}_1 \left[ \frac{s}{2\sqrt{K\Delta t}} \sum_{k' > k} \sum_{i,j} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] \frac{\partial}{\partial z_{ij}^{(k',1)}} \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \mathbb{E}_1 \left[ \frac{\sqrt{1-t}}{2\sqrt{K\Delta t}} \sum_{i,j} \frac{\partial}{\partial z_{ij}^{(1)}} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \mathbb{E}_1 \left[ \frac{\sqrt{t}m_k}{2\sqrt{K\Delta t}} \sum_i \frac{\partial}{\partial z_{i}^{(1)}} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \mathbb{E}_1 \left[ \frac{\sqrt{t}}{2\sqrt{t}} \sum_i \frac{\partial}{\partial z_{i}^{(2)}} \mathbb{E}_2 [(X_i X_j)_{k,t,e}] \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right].
\end{align*}
\]

Working out the partial derivatives yields

\[
\begin{align*}
\frac{s^2}{2K\Delta} \sum_{k' > k} \frac{1}{n} \sum_{i,j} \mathbb{E}_1 \left[ \mathbb{E}_2 [(X_i X_j)_{k,t,e}]^2 \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \frac{s^2}{2K\Delta} \sum_{k' < k} m_{k'} \sum_i \mathbb{E}_1 \left[ \mathbb{E}_2 [(X_i X_j)_{k,t,e}]^2 \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \frac{s^2(1-t)}{2K\Delta} \frac{1}{n} \sum_{i,j} \mathbb{E}_1 \left[ \mathbb{E}_2 [(X_i X_j)_{k,t,e}]^2 \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \frac{s^2(t m_k)}{2K\Delta} \sum_i \mathbb{E}_1 \left[ \mathbb{E}_2 [(X_i X_j)_{k,t,e}]^2 \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right] \\
+ \frac{s^2}{2} \sum_i \mathbb{E}_1 \left[ \mathbb{E}_2 [(X_i X_j)_{k,t,e}]^2 \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right].
\end{align*}
\]

For bounded signals we have \(|x_i| < M\) as well as \(m_k \leq M^2\). Therefore the sum of these four terms is bounded by

\[
\frac{s^2 n}{2K\Delta} \left( \frac{2M^4}{\Delta} + \epsilon M^2 \right) \mathbb{E}_1 \left[ \exp(s \mathbb{E}_2[\ln Z_{k,t,e}(\tau)]) \right]
\]

for all \(k = 1, \ldots, K\). This is an upper bound for the numerator of (147), which implies \(|\varphi_{k,t,e}^{(1)}(\tau)| \leq s^2 n (2M^4/\Delta + \epsilon M^2/2)\). From (146)

\[
\mathbb{E}_Z \left[ |F_{k,t,e}(\Theta) - \mathbb{E}_Z[F_{k,t,e}(\Theta)]| \right] > u/2 \leq 2 \exp \left( \frac{s^2 n}{2K\Delta} \left( \frac{2M^4}{\Delta} + \epsilon M^2 \right) - s n u/2 \right)
\]

and the best possible value \(s = u(M^4/\Delta + \epsilon M^2/2)^{-1}\) yields (144) and ends the proof.

The second lemma expresses concentration w.r.t the input signal of the free energy averaged over the Gaussian disorder. Recall that \(\mathbb{P}_s\) is the probability law w.r.t the signal realisation.
Lemma 6.2 (Concentration w.r.t the signal realisation): Take $P_0$ with bounded support in $[-M,M]$. For all $k=1,\ldots,K$, $t\in[0,1]$, and $\epsilon\in[0,1]$ we have
\[
\mathbb{P}_s[|\mathbb{E}_z[F_{k,t,x}(\Theta)] - \mathbb{E}[F_{k,t,x}(\Theta)]| > u/2] \leq \exp\left(-\frac{n\epsilon^2}{32(M^4 + \epsilon M^2)^2}\right),
\] (150)
where $u > 0$.

Proof: We first prove a bounded difference property on $\mathbb{E}_z[F_{k,t,x}(\Theta)]$ and then apply the McDiarmid inequality [48,49]. Let $s$ and $s'$ two signal realisations that differ at the component $i$ only, i.e. $s_j = s'_j$ for $j \neq i$. We first consider the difference of Hamiltonians corresponding to these two realisations. From (12)–(19) we have
\[
\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s) - \mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s') = -\frac{1}{K\Delta n} \sum_{k' > k} \sum_{j=1,j\neq i}^n x_i x_j (s_i - s'_i) s_j - \frac{1}{K\Delta n} \sum_{k' < k} x_i^2 (s_i^2 - s'_i^2)
- \frac{1-t}{K\Delta n} \sum_{j=1,j\neq i}^n x_i x_j (s_i - s'_i) s_j - \frac{1-t}{K\Delta n} x_i^2 (s_i^2 - s'_i^2) - \frac{1}{K\Delta} \sum_{k' < k} m_k x_j (s_j - s'_j) - \frac{t}{K\Delta} x_i (s_i - s'_i) - \epsilon x_i (s_i - s'_i). 
\] (151)
For a signal distribution with bounded support $[-M,M]$ we get (recall $|m_k| \leq M^2$)
\[
|\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s) - \mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s')| \leq 2\left(\frac{M^4}{\Delta} + \epsilon M^2\right).
\] (152)
Now set $g(s_1,\ldots,s_n) := \mathbb{E}_z[F_{k,t,x}(\Theta)]$. We have
\[
g(s_1,\ldots,s_i,\ldots,s_n) - g(s_1,\ldots,s'_i,\ldots,s_n) = \frac{1}{n} \mathbb{E}_z\left[\ln \frac{\mathbb{E}_x[e^{-\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s)}]}{\mathbb{E}_x[e^{-\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s')]}]\right] = \frac{1}{n} \mathbb{E}_z\left[\ln \frac{\mathbb{E}_x[e^{-\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s)}] e^{\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s') - \mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s)\epsilon}}{\mathbb{E}_x[e^{-\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s)}]}\right] 
\] (153)
and since from (152)
\[
e^{-2\frac{M^4}{\Delta} + \epsilon M^2} \leq e^{-\mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s) - \mathcal{H}_{k,t,x}(\mathbf{x};\mathbf{z},\mathbf{z},\mathbf{z},s') \epsilon} \leq e^{2\frac{M^4}{\Delta} + \epsilon M^2} 
\] (154)
we readily obtain
\[
|g(s_1,\ldots,s_i,\ldots,s_n) - g(s_1,\ldots,s'_i,\ldots,s_n)| \leq c_i
\] (155)
with $c_i = 2(M^4/\Delta + \epsilon M^2)/n$, $i = 1,\ldots,n$. McDiarmid’s inequality states that
\[
\mathbb{P}_s[|g(S) - \mathbb{E}_s[g(S)]| \geq u/2] \leq \exp\left(-\frac{u^2}{8\sum_{i=1}^n c_i^2}\right)
\] (156)
which here reads (150) and ends the proof of the lemma.

Proof of Proposition 5.4: From the triangle inequality and the union bound
\[
\mathbb{P}[|F_{k,t,x}(\Theta) - F_{k,t,t}| > u] = \mathbb{P}[|F_{k,t,x}(\Theta) - \mathbb{E}[F_{k,t,x}(\Theta)]| + \mathbb{E}[F_{k,t,x}(\Theta)] - \mathbb{E}[F_{k,t,x}(\Theta)]| > u]
\leq \mathbb{P}[|F_{k,t,x}(\Theta) - \mathbb{E}[F_{k,t,x}(\Theta)]| + |\mathbb{E}[F_{k,t,x}(\Theta)] - \mathbb{E}[F_{k,t,x}(\Theta)]| > u]
\leq \mathbb{P}[|F_{k,t,x}(\Theta) - \mathbb{E}[F_{k,t,x}(\Theta)]| > u/2] + \mathbb{P}[|\mathbb{E}[F_{k,t,x}(\Theta)] - \mathbb{E}[F_{k,t,x}(\Theta)]| > u/2]
= \mathbb{E}_z \mathbb{P}_z[|F_{k,t,x}(\Theta) - \mathbb{E}[F_{k,t,x}(\Theta)]| > u/2] + \mathbb{P}_z[|\mathbb{E}[F_{k,t,x}(\Theta)] - \mathbb{E}[F_{k,t,x}(\Theta)]| > u/2]
\leq 2\exp\left(-\frac{n\epsilon^2}{16(M^4/\Delta + \epsilon M^2)}\right) + \exp\left(-\frac{n\epsilon^2}{32(M^4/\Delta + \epsilon M^2)^2}\right)
\] (157)
where the last inequality comes from Lemmas 6.1 and 6.2.

VII. A Stochastic Calculus Interpretation

We note that the proofs do not require any upper limit on $K$. This suggests that it is possible to formulate the stochastic interpolation method entirely in a continuum language. We show this explicitly for the simplest problem, namely symmetric rank-one matrix factorisation, and leave out the other cases which can be treated similarly.
It is helpful to first write down explicitly the \((k,t)\)-interpolating Hamiltonian (12) (leaving out the perturbation in (19) which is irrelevant for the argument here)

\[
\mathcal{H}_{k,t}(x; \theta) = \frac{1}{K\Delta} \sum_{k'=k+1}^{K} \sum_{i \leq j = 1}^{n} \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} - \sqrt{\frac{K\Delta}{n}} x_{i}x_{j}z_{ij}^{(k')} \right) \tag{158}
\]

\[
+ \frac{1}{K\Delta} \sum_{k'=1}^{k-1} m_{k'} \sum_{i = 1}^{n} \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} - \sqrt{\frac{K\Delta}{m_{k'}}} x_{i}z_{i}^{(k')} \right) \tag{159}
\]

\[
+ \frac{1-t}{K\Delta} \sum_{i \leq j = 1}^{n} \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} - \sqrt{\frac{K\Delta}{(1-t)n}} x_{i}x_{j}z_{ij}^{(k)} \right) \tag{160}
\]

\[
+ \frac{t m_{k}}{K\Delta} \sum_{i = 1}^{n} \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} - \sqrt{\frac{K\Delta}{tm_{k}}} x_{i}z_{i}^{(k)} \right), \tag{161}
\]

and to define the step-wise function \(m(u) = m_{k'}, k'/K < u < (k' + 1)/K, k' = 1, \ldots, K\).

Let us first look at the terms that do not involve Gaussian noise and become simple Riemann integrals. We have for the contribution coming from (158) and (160),

\[
\frac{1}{\Delta} \sum_{i \leq j = 1}^{n} \left\{ \frac{1}{K} \sum_{k'=k+1}^{K} \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} \right) + \frac{1-t}{K} \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} \right) \right\}
\]

\[
= \frac{1}{\Delta} \sum_{i \leq j = 1}^{n} \left\{ \int_{\frac{k+1}{K}}^{\frac{k+2}{K}} du \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} \right) + \int_{\frac{k+1}{K}}^{\frac{k+2}{K}} du \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} \right) \right\}
\]

\[
= \frac{1}{\Delta} \sum_{i \leq j = 1}^{n} \int_{\frac{k+1}{K}}^{\frac{k+2}{K}} du \left( \frac{x_{i}^{2}x_{j}^{2}}{2n} - \frac{x_{i}x_{j}s_{i}s_{j}}{n} \right). \tag{162}
\]

Similarly, we have for the terms coming from (159) and (161),

\[
\frac{1}{\Delta} \sum_{i = 1}^{n} \left\{ \frac{1}{K} \sum_{k'=1}^{k-1} m_{k'} \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} \right) + \frac{t m_{k}}{K} \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} \right) \right\} = \frac{1}{\Delta} \sum_{i = 1}^{n} \left\{ \int_{\frac{1}{K}}^{\frac{1}{K}} du m(u) \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} \right) + \int_{\frac{k-t}{K}}^{\frac{k+1}{K}} du m(u) \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} \right) \right\}
\]

\[
= \frac{1}{\Delta} \sum_{i = 1}^{n} \left\{ \int_{\frac{1}{K}}^{\frac{k-t}{K}} du m(u) \left( \frac{x_{i}^{2}}{2} - x_{i}s_{i} \right) \right\}. \tag{163}
\]

Now we treat the more interesting contributions involving the Gaussian noise. Let \(B(u)\) be the Wiener process defined by \(B(0) = 0, \mathbb{E}[B(u)] = 0, \mathbb{E}[B(u)B(v)] = \min(u,v)\) for \(u, v \in \mathbb{R}_+\). We introduce independent copies \(B_{ij}(u), i, j = 1, \ldots, n\) and consider the sum of increments (also written as an Itô integral)

\[
\{ B_{ij}\left( \frac{k+1}{K} \right) - B_{ij}\left( \frac{k+t}{K} \right) \} + \sum_{k'=k+1}^{K} \left\{ B_{ij}\left( \frac{k'+1}{K} \right) - B_{ij}\left( \frac{k'}{K} \right) \right\} = \int_{\frac{k-t}{K}}^{\frac{k+1}{K}} dB_{ij}(u). \tag{164}
\]

Since the increments are independent and \(\mathbb{E}[(B(u) - B(v))^2] = |u-v|\), this is a Gaussian random variable with zero mean and variance \((K+1-k-t)/K\). It is therefore equal in distribution to

\[
\frac{1}{\sqrt{K}} \sum_{k'=k+1}^{K} Z_{ij}^{(k')} + \sqrt{\frac{1-t}{K}} Z_{ij}^{(k)}, \tag{165}
\]

and the contribution of the (random) Gaussian noise in (158) and (160) becomes

\[
\frac{1}{\sqrt{\Delta n}} \sum_{i \leq j = 1}^{n} x_{i}x_{j} \left\{ \frac{1}{\sqrt{K}} \sum_{k'=k+1}^{K} Z_{ij}^{(k')} + \sqrt{\frac{1-t}{K}} Z_{ij}^{(k)} \right\} = \frac{1}{\sqrt{\Delta n}} \sum_{i \leq j = 1}^{n} \int_{\frac{k-t}{K}}^{\frac{k+1}{K}} dB_{ij}(u)x_{i}x_{j}. \tag{166}
\]

To represent the contributions of (159), (161) we introduce independent copies of the Wiener process \(\tilde{B}_{i}(u), i = 1, \ldots, n\) and form the Itô integral

\[
\sum_{k'=1}^{k-1} \sqrt{m_{k'}} \left\{ \tilde{B}_{i}\left( \frac{k'+1}{K} \right) - \tilde{B}_{i}\left( \frac{k'}{K} \right) \right\} + \sqrt{m_{k}} \left\{ \tilde{B}_{i}\left( \frac{k+t}{K} \right) - \tilde{B}_{i}\left( \frac{k}{K} \right) \right\} = \int_{\frac{k-t}{K}}^{\frac{k+1}{K}} \sqrt{m(u)}d\tilde{B}_{i}(u) \tag{167}
\]
which has the same variance than
\[
\frac{1}{\sqrt{K}} \sum_{k'=1}^{k-1} \sqrt{m_{k'}} \bar{Z}_i^{(k')} + \sqrt{\frac{t m_k}{K}} \bar{Z}_i^{(k)}.
\]  
(168)

Indeed
\[
\frac{1}{K} \sum_{k'=1}^{k-1} m_{k'} + \frac{t m_k}{K} = \sum_{k'=1}^{k-1} m_{k'} \left( \frac{k' + 1}{K} - \frac{k'}{K} \right) + m_k \left( \frac{k + t}{K} - \frac{k}{K} \right) = \frac{1}{K} \int_\tau^{\tau + \frac{k}{K}} du m(u). \tag{169}
\]

Therefore the contribution of (159) and (161) can be represented as
\[
\frac{1}{\sqrt{\Delta}} \sum_{i=1}^n x_i \int_\tau^{\tau + \frac{t}{K}} \sqrt{m(u)} dB_i(u). \tag{170}
\]

Finally, collecting (162), (163), (166), (170), setting \(\tau := (t+k)/K\) and \(K \to \infty\), we obtain a continuous form of the random \((k, t)\)-interpolating Hamiltonian,
\[
\mathcal{H}_\tau(x; s, B) = \frac{1}{\Delta} \sum_{i \leq j = 1}^n \int_\tau^{\tau + \frac{1}{K}} \left\{ \left( \frac{x_i^2 x_j^2}{2n} - \frac{x_i x_j s_i s_j}{n} \right)(1 - \tau) - x_i x_j Z_i \sqrt{\frac{\Delta(1 - \tau)}{n}} \right\} \mu(u) du
\]
\[+ \frac{1}{\Delta} \sum_{i=1}^n \left\{ \left( \frac{x_i^2}{2} - x_i s_i \right) \int_0^\tau m(u) du - x_i \tilde{Z}_i \sqrt{\Delta \int_0^\tau m(u) du} \right\}. \tag{171}
\]

Clearly, the usual Guerra-Toninelli interpolation appears as a special case where one choose a constant trial function \(m(u) = m\) constant. When we go from (171) to (172) we eliminate completely the Wiener process, however we believe it is useful to keep in mind the point of view expressed by (171) which may turn out to be important for more complicated problems.

Starting from (171) or (172) it is possible to evaluate the free energy change along the interpolation path. We define the free energy
\[
f(\tau) = -\frac{1}{n} \mathbb{E}_{S, B} \left[ \ln \mathbb{E}_{X} \left[ e^{-\mathcal{H}_\tau(X; S, B)} \right] \right]. \tag{173}
\]

For \(\tau = 0\) using we recover the original Hamiltonian \(\mathcal{H}_{k=1, t=0}\) (see (23)) and \(f(0) = f\) given in (6). For \(\tau = 1\) setting \(\int_0^1 du m(u) = m_{\text{mf}}\) we recover the mean-field Hamiltonian \(\mathcal{H}_{k=K, t=1}\) (see (27)) and \(f(1) = f_{\text{den}}(\Sigma(\int_0^1 du m(u)); \Delta)\). Then proceeding similarly to sec. II-G1 one finds the identity
\[
f = f_{\text{RS}} \left( \int_0^1 d\tau m(\tau); \Delta \right) + \left\{ \int_0^1 d\tau m(\tau)^2 - \left( \int_0^1 d\tau m(\tau) \right)^2 \right\} - \frac{1}{4\Delta} \int_0^1 d\tau \mathbb{E}_{S, B} \left[ (\langle q_{X, S} - m(\tau) \rangle)^2 \right] + \mathcal{O}(n^{-1}) \tag{174}
\]
where \(\langle \cdot \rangle_\tau\) is the Gibbs average w.r.t (171).

Of course this immediately gives the upper bound in Proposition 2.5. The matching lower bound is obtained by the same ideas used in the discrete version. We briefly review them informally in the continuous language. One first introduces the \(\epsilon\)-perturbation term (19) and proves a concentration property for the overlap analogous to Lemma 2.6. Starting with the continuous version of the interpolating Hamiltonian the proof of the free energy concentration is essentially identical (even simpler) than in sec. VI, which implies the overlap concentration through sec. V that is unchanged. Then, the square in the remainder term is approximately equal to \(\langle \mathbb{E}_{S, B} [(q_{X, S})_{\tau, \epsilon} - m(\tau)]^2 \rangle\) and we make it vanish by choosing
\[
m(\tau) = \mathbb{E}_{S, B} \left[ (q_{X, S})_{\tau, \epsilon} \right]. \tag{175}
\]

This continuous setting thus allows to avoid proving Lemma 2.8. This then easily yields the lower bound in Proposition 2.5. One must still check that (175) has a solution. The right hand side is a function \(G_{\text{mf}}(\tau; \int_0^1 du m(u))\) so setting \(x(\tau) = \int_0^1 du m(u), dx/d\tau = m(\tau)\), we recognize that (175) is a first order differential equation with initial condition \(x(0) = 0\). One then has to prove (using standard theorems) that there exists a unique global solution on \(\tau \in [0, 1]\). This last step of the analysis replaces Lemma 2.9.
Similarly, $E$ has not been performed yet. For the general case of tensor estimation one expects similar results but to the best of our knowledge the detailed analysis has not been performed yet.

The purpose of this appendix is to prove the identity (112). It actually follows from the exact formula

$$\mathbb{E}[\langle \mathcal{L}_e \rangle - \mathbb{E}[\langle \mathcal{L}_e \rangle]^2] = \frac{1}{4n^2} \sum_{i,j=1}^{n} \{ \mathbb{E}[\langle X_i, X_j \rangle^2] - \mathbb{E}[\langle X_i \rangle^2] \mathbb{E}[\langle X_j \rangle^2] \}$$

$$+ \frac{1}{2n^2} \sum_{i,j=1}^{n} \{ \mathbb{E}[\langle X_i, X_j \rangle^2] - \mathbb{E}[\langle X_i, X_j \rangle \langle X_i \rangle \langle X_j \rangle] \}$$

$$+ \frac{1}{4n^2 \epsilon} \sum_{i=1}^{n} \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2]$$

(176)

that we derive next. But before doing so, let us show how (176) implies (112). We first express the first two terms in terms of the overlap $q_{x_s}$. From (47) we have $\mathbb{E}[\langle X_i, X_j \rangle^2] = \mathbb{E}[S_i S_j \langle X_i, X_j \rangle]$ and therefore

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i, X_j \rangle^2] = \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[S_i S_j \langle X_i, X_j \rangle] = \mathbb{E}[\langle q_{x_s}^2 \rangle].$$

(177)

Similarly $\mathbb{E}[\langle X_i \rangle^2] = \mathbb{E}[S_i \langle X_i \rangle]$, so

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i \rangle^2] \mathbb{E}[\langle X_j \rangle^2] = \mathbb{E}[\langle q_{x_s} \rangle]^2,$$

(178)

and $\mathbb{E}[\langle X_i, X_j \rangle \langle X_i \rangle \langle X_j \rangle] = \mathbb{E}[S_i S_j \langle X_i \rangle \langle X_j \rangle]$ which implies

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \mathbb{E}[\langle X_i, X_j \rangle \langle X_i \rangle \langle X_j \rangle] = \mathbb{E}[\langle q_{x_s} \rangle^2].$$

(179)

These three last identities plugged in (176) leads (112).

We now summarise the main steps leading to the formula (176), using the identity (47) and integrations by parts w.r.t the Gaussian noise. This formula follows by summing the two following identities

$$\mathbb{E}[\langle \mathcal{L}_e \rangle] - \mathbb{E}[\langle \mathcal{L}_e \rangle]^2 = \frac{1}{2n^2} \sum_{i,j=1}^{n} \{ \mathbb{E}[\langle X_i, X_j \rangle^2] - 2 \mathbb{E}[\langle X_i, X_j \rangle \langle X_i \rangle \langle X_j \rangle] + \mathbb{E}[\langle X_i \rangle^2 \langle X_j \rangle^2] \} + \frac{1}{4n^2 \epsilon} \sum_{i=1}^{n} \mathbb{E}[\langle X_i^2 \rangle - \langle X_i \rangle^2],$$

(180)

$$\mathbb{E}[\langle \mathcal{L}_e \rangle^2] - \mathbb{E}[\langle \mathcal{L}_e \rangle]^2 = \frac{1}{4n^2} \sum_{i,j=1}^{n} \{ \mathbb{E}[\langle X_i, X_j \rangle^2] - \mathbb{E}[\langle X_i \rangle^2] \mathbb{E}[\langle X_j \rangle^2] \} + \frac{1}{2n^2} \sum_{i,j=1}^{n} \{ \mathbb{E}[\langle X_i \rangle \langle X_j \rangle \langle X_i, X_j \rangle] - \mathbb{E}[\langle X_i \rangle^2 \langle X_j \rangle^2] \}.$$

(181)

We first derive the second identity which requires somewhat longer calculations.
Derivation of (181): From (105) we have
\[
\langle L_c \rangle^2 = \frac{1}{n^2} \sum_{i,j=1}^{n} \left\{  \frac{1}{4} \langle X_i^2 \rangle \langle X_j^2 \rangle - \frac{1}{2} \langle X_i^2 \rangle \langle X_j \rangle s_j - \frac{1}{4\sqrt{\epsilon}} \langle X_i^2 \rangle \langle X_j \rangle \bar{Z}_j \\
- \frac{1}{2} \langle X_i \rangle s_i \langle X_j^2 \rangle + \langle X_i \rangle s_i \langle X_j \rangle s_j + \frac{1}{2\sqrt{\epsilon}} \langle X_i \rangle s_i \langle X_j \rangle \bar{Z}_j \\
- \frac{1}{4\sqrt{\epsilon}} \langle X_i \rangle \langle X_j^2 \rangle \bar{Z}_i + \frac{1}{2\sqrt{\epsilon}} \langle X_i \rangle \langle X_j \rangle s_j \bar{Z}_i + \frac{1}{4\epsilon} \langle X_i \rangle \langle X_j \rangle \bar{Z}_i \bar{Z}_j \right\}.
\]
(182)

Taking the expectation and using (47) we find
\[
E[\langle L_c \rangle^2] = \frac{1}{n^2} \sum_{i,j=1}^{n} \left\{ \frac{1}{4} E[\langle X_i^2 \rangle \langle X_j^2 \rangle] - \frac{1}{2} E[\langle X_i^2 \rangle \langle X_j \rangle] + \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j \rangle] \right\}
- \frac{1}{2} E[\langle X_i \rangle^2] + E[\langle X_i \rangle^2] + \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle^2] Z_j
- \frac{1}{4\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j^2 \rangle Z_i] + \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j \rangle Z_i] + \frac{1}{4\epsilon} E[\langle X_i \rangle \langle X_j \rangle Z_i Z_j].
\]
(183)

In order to simplify this expression we first integrate by parts the last term
\[
\frac{1}{4\epsilon} E[\langle X_i \rangle \langle X_j \rangle Z_i Z_j] = \frac{1}{4\epsilon} E \left[ \frac{\partial}{\partial Z_j} \langle X_i \rangle \langle X_j \rangle Z_i \right]
= \frac{1}{4\epsilon} \left( E[\langle X_i^2 \rangle \langle X_j \rangle Z_j] - E[\langle X_I \rangle Z_j \langle X_j \rangle Z_j] + E[\langle X_i \rangle \langle X_j \rangle Z_j] Z_j - E[\langle X_i \rangle \langle X_j \rangle Z_j] \right).
\]
(184)

Replacing in \(E[\langle L_c \rangle^2]\) we find
\[
E[\langle L_c \rangle^2] = \frac{1}{n^2} \sum_{i,j=1}^{n} \left\{ \frac{1}{4} E[\langle X_i^2 \rangle \langle X_j^2 \rangle] - \frac{1}{2} E[\langle X_i^2 \rangle \langle X_j \rangle] + \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j \rangle] + E[\langle X_i \rangle^2] \right\}
- \frac{1}{4\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j^2 \rangle Z_i] + \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j \rangle Z_i] + \frac{1}{4\epsilon} E[\langle X_i \rangle \langle X_j \rangle Z_i Z_j].
\]
(185)

Now, we integrate by parts the three terms still involving Gaussian variables. One finds
\[
- \frac{1}{4\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j^2 \rangle Z_i] = \frac{1}{4} \left( - E[\langle X_i^2 \rangle \langle X_j^2 \rangle] + E[\langle X_i \rangle \langle X_j^2 \rangle] + E[\langle X_i \rangle \langle X_j^2 \rangle] \right),
\]
(186)
\[
\frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \langle X_j \rangle^2 Z_i] = \frac{1}{2} \left( E[\langle X_i^2 \rangle \langle X_j \rangle^2] - \frac{1}{2} E[\langle X_i \rangle \langle X_j^2 \rangle] + E[\langle X_i \rangle \langle X_j \rangle \langle X_j \rangle] - E[\langle X_i \rangle \langle X_j \rangle^2] \right),
\]
(187)
\[
\frac{1}{4\epsilon} E[\langle X_i \rangle \langle X_j \rangle Z_i \bar{Z}_j] = \frac{1}{4} \left( E[\langle X_i \rangle \langle X_j \rangle Z_i \bar{Z}_j] - E[\langle X_i \rangle \langle X_j \rangle \langle X_j \rangle] + E[\langle X_i \rangle \langle X_j \rangle Z_i] - E[\langle X_i \rangle \langle X_j \rangle \langle X_j \rangle] \right).
\]
(188)

Replacing these three identities in (185) we get
\[
E[\langle L_c \rangle^2] = \frac{1}{n^2} \sum_{i,j=1}^{n} \left\{ \frac{1}{4} E[\langle X_i \rangle^2] \right\}.
\]
(189)

In the derivation of (180) we computed \(E[\langle L_c \rangle]\), and taking the square we have
\[
E[\langle L_c \rangle]^2 = \frac{1}{4n^2} \sum_{i,j=1}^{n} E[\langle X_i \rangle^2] E[\langle X_j \rangle^2].
\]
(190)

Subtracting (189) and (190) we finally find (181).

Derivation of (180): From (105) we have
\[
E[\langle L_c \rangle] = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{1}{4} E[\langle X_i \rangle^2] - E[\langle X_i \rangle s_i] - \frac{1}{2\sqrt{\epsilon}} E[\langle X_i \rangle \bar{Z}_i] \right\}.
\]
(191)

From (47) we have \(E[\langle X_i \rangle s_i] = E[\langle X_i \rangle^2]\) and by an integration by parts
\[
\frac{1}{\sqrt{\epsilon}} E[\langle X_i \rangle \bar{Z}_i] = \frac{1}{\sqrt{\epsilon}} E \left[ \frac{\partial}{\partial Z_i} \langle X_i \rangle \right] = E[\langle X_i^2 \rangle] - \langle X_i \rangle^2.
\]
(192)
Thus we find
\[ E[\langle L \rangle] = -\frac{1}{2n} \sum_{i=1}^{n} E[\langle X_i \rangle^2] \] (193)

which is formula (119). Acting with \( n^{-1} d/d\epsilon \) on both sides of (193) we find
\[ -E[\langle L^2 \rangle - \langle L \rangle^2] + \frac{1}{n} E \left[ \left( \frac{dL}{d\epsilon} \right) \right] = \frac{1}{n} \sum_{i=1}^{n} E[(L_i L_j) - \langle L_i \rangle \langle L_j \rangle]. \] (194)

Computing the derivative of \( L \), and using (192) we find that (194) is equivalent to
\[ E[\langle L^2 \rangle - \langle L \rangle^2] = -\frac{1}{n} \sum_{i=1}^{n} E[(\langle L_i \rangle \langle L_j \rangle) - \langle L_i \rangle \langle L_j \rangle] + \frac{1}{4n^2 \epsilon} \sum_{i=1}^{n} E[(\langle X_i \rangle^2) - \langle X_i \rangle^2] \] (195)

Now we compute the terms in the first sum. We have
\[ \langle X_i \rangle (\langle L_i \rangle - \langle L \rangle) = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{2} \langle X_i \rangle \langle X_j^2 \rangle - \langle X_i \rangle \langle X_j \rangle s_j - \frac{1}{2 \sqrt{\epsilon}} \langle L_i \rangle \langle X_j \rangle \hat{\alpha}_j \right\} \]

Then from (47),
\[ E[\langle X_i \rangle (\langle L_i \rangle - \langle L \rangle) = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{1}{2} E[(\langle X_i \rangle \langle X_j^2 \rangle) - E[(\langle X_i \rangle \langle X_j \rangle)] s_j - E[(\langle L_i \rangle \langle X_j \rangle \hat{z}_j) \right\]} \]

It remains to integrate by parts the two terms involving the explicit \( \hat{Z}_j \) dependence. For \( E[\langle X_i \rangle \langle X_j \rangle \hat{Z}_j] \) we can use (188) and for \( E[\langle X_i \rangle^2 \langle X_j \rangle \hat{Z}_j] \) we can use (187) with \( i, j \) exchanged. This leads to
\[ \frac{1}{n} \sum_{j=1}^{n} E[\langle X_i \rangle (\langle L_i \rangle - \langle L \rangle)] = -\frac{1}{2n^2} \sum_{i, j=1}^{n} \{ E[(\langle X_i \rangle^2 \hat{Z}_j)] - 2E[(\langle X_i \rangle \langle X_j \rangle)] + E[(\langle X_j \rangle^2 \hat{Z}_j)] \}. \] (198)

The formula (180) then follows from (195) and (198).

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