Discriminated Belief Propagation

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2nd February 2008

Near optimal decoding of good error control codes is generally a difficult task. However, for a certain type of (sufficiently) good codes an efficient decoding algorithm with near optimal performance exists. These codes are defined via a combination of constituent codes with low complexity trellis representations. Their decoding algorithm is an instance of (loopy) belief propagation and is based on an iterative transfer of constituent beliefs. The beliefs are thereby given by the symbol probabilities computed in the constituent trellises. Even though weak constituent codes are employed close to optimal performance is obtained, i.e., the encoder/decoder pair (almost) achieves the information theoretic capacity. However, (loopy) belief propagation only performs well for a rather specific set of codes, which limits its applicability.

In this paper a generalisation of iterative decoding is presented. It is proposed to transfer more values than just the constituent beliefs. This is achieved by the transfer of beliefs obtained by independently investigating parts of the code space. This leads to the concept of discriminators, which are used to improve the decoder resolution within certain areas and defines discriminated symbol beliefs. It is shown that these beliefs approximate the overall symbol probabilities. This leads to an iteration rule that (below channel capacity) typically only admits the solution of the overall decoding problem. Via a GAUSS approximation a low complexity version of this algorithm is derived. Moreover, the approach may then be applied to a wide range of channel maps without significant complexity increase.

Keywords: Iterative Decoding, Coupled Codes, Information Theory, Complexity, Belief Propagation, Typical Decoding, Set Representations, Central Limit Theorem, Equalisation, Estimation, Trellis Algorithms

Decoding error control codes is the inversion of the encoding map in the presence of errors. An optimal decoder finds the codeword with the least number of errors. However, optimal decoding is generally computationally infeasible due to the intrinsic non linearity of the inversion operation. Up to now only simple codes can be optimally decoded, e.g., by a simple trellis representation. These codes generally exhibit poor performance or rate [11].

On the other hand, good codes can be constructed by a combination of simple constituent codes (see e.g., [14 pp.567ff]). This construction is interesting as then a trellis based inversion may perform almost optimally: BERROU et al. [2] showed that iterative turbo decoding leads to near capacity performance. The same holds true for iterative decoding of Low Density Parity Check (LDPC) codes [6]. Both decoders are conceptually similar and based on the (loopy) propagation of beliefs [10] computed in the constituent trellises. However, (loopy) belief propagation is often limited to idealistic situations. E.g., turbo decoding generally performs poorly for multiple constituent codes, complex channels, good constituent codes, and/or relatively short overall code lengths.

In this paper a concept called discrimination is used to generalise iterative decoding by (loopy) belief propagation. The generalisation is based on an uncertainty or distance discriminated investigation of the

*Technical Report TR-CSC-07-01 of the D_Max Project funded by the University of Luxembourg.
code space. The overall results of the approach are linked to basic principles in information theory such as typical sets and channel capacity [18][15][13].

**Overview:** The paper is organised as follows: First the combination of codes together with the decoding problem and its relation to belief propagation are reviewed. Then the concept of discriminators together with the notion of a common belief is introduced. In the second section local discriminators are discussed. By a local discriminator a controllable amount of parameters (or generalised beliefs) are transferred. It is shown that this leads to a practically computable common belief that may be used in an iteration. Moreover, a fixed point of the obtained iteration is typically the optimal decoding decision. Section 3 finally considers a low complexity approximation and the application to more complex channel maps.

1. **Code Coupling**

To review the combination of constituent codes we here consider only binary linear codes $C$ given by the encoding map

$$C : x = (x_1, \ldots, x_k) \mapsto c = (c_1, \ldots, c_n) = xG \mod 2$$

with $G$ the $(k \times n)$ generator matrix with $x_i, c_i,$ and $G_{i,j} \in \mathbb{Z}_2 = \{0, 1\}$.

The map defines for rank$(G) = k$ the event set $E(C)$ of $2^k$ code words $c$. The rate of the code is given by $R = k/n$ and it is for an error correcting code smaller than one.

The event set $E(C)$ is by linear algebra equivalently defined by a $((n - k) \times n)$ parity matrix $H$ with $HG^T = 0 \mod 2$ and thus

$$E(C) = \{c : Hc^T = 0 \mod 2\}.$$  

Note that the modulo operation is in the sequel not explicitly stated.

$E(C)$ is a subset of the set $S$ of all $2^n$ binary vectors of length $n$. The restriction to a subset is interesting as this leads to the possibility to correct corrupted words. However, the correction is a difficult operation and can usually only be practically performed for simple or short codes.

On the other hand long codes can be constructed by the use of such simple constituent codes. Such constructions are reviewed in this section.

**Definition 1** *(Direct Coupling)* The two constituent linear systematic coding maps

$$C^{(l)} : x \mapsto c^{(l)} = x \cdot G^{(l)} = x \cdot [IP^{(l)}]$$

with $l = 1, 2$ and $P^{(l)}$ the permutation of the input vector $x$, which significantly improves the overall code properties but does not affect the complexity of the constituent decoders. If the two codes have rate $1/2$ then the overall code will have rate $1/3$.

Another possibility is to concatenate two constituent codes as defined below.

**Definition 2** *(Concatenated Codes)* By

$$c^{(1)} = xG^{(1)} \text{ and } c^{(a)} = c^{(1)}G^{(2)} = xG^{(1)}G^{(2)}$$

(provided matching dimensions, i.e. a $(k \times n^{(1)})$ generator matrix $G^{(1)}$ and a $(n^{(1)} \times n)$ generator matrix $G^{(2)}$) a concatenated code is given.
Remark 1 (Generalised Concatenation) A concatenation can be used to construct codes with defined properties as usually a large minimum HAMMING distance. Note that generalised concatenated \([3,4]\) codes exhibit the same basic concatenation map. There distance properties are investigated under an additional partitioning of code \(G^{(2)}\).

Another possibility to couple codes is given in the following definition. This method will show to be very general, albeit rather non intuitive as the description is based on parity check matrices \(H\).

Definition 3 (Dual Coupling) The overall code
\[
C^{(a)} := E(C^{(a)}) = \left\{ c : \begin{bmatrix} H^{(1)}_s & H^{(1)}_r \end{bmatrix} c^T = 0 \right\}
\]
is obtained by a dual coupling of the constituent codes \(C^{(l)} := E(C^{(l)}) = \left\{ c : H^{(l)} c^T = 0 \right\} for l = 1, 2\).

By a dual coupling the obtained code space is obtained by the intersection \(C^{(a)} = C^{(1)} \cap C^{(2)}\) of the constituent code spaces.

Example 2 A dually coupled code construction similar to turbo codes is to use two mutually permuted rate \(2/3\) convolutional codes. The intersection of these two codes gives a code with rate at least \(1/3\). To obtain a larger rate one may employ puncturing (not transmitting certain symbols). However, the encoding of the overall code is not as simple as for direct coupling codes. A straightforward way is to just use the generator matrix representation of the overall code.

Remark 2 (LDPC Codes) LDPC codes are originally defined by a single parity check matrix with low weight rows (and columns). An equivalent representation is via a graph of check nodes (one for each column) and variables nodes (one for each row). This leads to a third equivalent representation with two dually coupled constituent codes and a subsequent puncturing [12]. The first constituent code is thereby given by a juxtaposition of repetition codes that represent the variable nodes (all node inputs need to be equal). The second one is defined by single parity check codes representing the check nodes. The puncturing at the end has to be done such that only one symbol per repetition code (code column) remains.

Theorem 1 Both direct coupling and concatenated codes are special cases of dual coupling codes.

Proof: The direct coupling code is equivalently described in the parity check form \(H^{(a)} G^{(a)}^T = 0\) by the parity check matrix
\[
H^{(a)} = \begin{bmatrix} H^{(s1)} & H^{(r1)} \\ H^{(s2)} & 0 \end{bmatrix} \text{ where } H^{(l)} = [H^{(sl)} H^{(rl)}] \text{ for } l = 1, 2
\]
is the parity check matrix of \(G^{(l)}\) consisting of systematic part \(H^{(sl)}\) and redundant part \(H^{(rl)}\). This is obviously a dual coupling. For a concatenated code with systematic code \(G^{(2)} = [I P^{(2)}]\) the equivalent description by a parity check matrix is
\[
H^{(a)} = \begin{bmatrix} H^{(1)} & 0 \\ H^{(s2)} & H^{(r2)} \end{bmatrix} \text{ with } H^{(1)} \text{ and } H^{(2)} = [H^{(s2)} H^{(r2)}]
\]
the parity check matrix of \(G^{(1)}\) respectively \(G^{(2)}\). For non-systematic concatenated codes a virtual systematic extension (punctured prior to the transmission) is needed [12]. Hence, a representation by a dual coupling is again possible.

It is thus sufficient to consider only dual code couplings. The “dual” is therefore mostly omitted in the sequel.

Remark 3 (Multiple Dual Codes) More than two codes can be dually coupled as described above: By
\[
C^{(a)} = C^{(1)} \cap C^{(2)} \cap C^{(3)}
\]
a coupling of three codes is given. The overall parity check matrix is there given by the juxtaposition of the three constituent parity check matrix. Multiple dual couplings are produced by multiple intersections. In the sequel mostly dual couplings with two constituent codes are considered.
1.1. Optimal Decoding

As stated above the main difficulty is not the encoding but the decoding of a corrupted word. This corruption is usually the result of a transmission of the code word over a channel.

**Remark 4 (Channels)** In the sequel we assume that the code symbols \( C \) are in \( \mathbb{B} = \{-1, +1\} \). This is achieved by the use of the “BPSK”-map

\[
B : x \mapsto y = \begin{cases} 
+1 & \text{for } x = 0 \\
-1 & \text{for } x = 1
\end{cases}
\]

prior to the transmission. As channel we assume either a Binary Symmetric Channel (BSC) with channel error probability \( p \) and

\[
P(r|s) = \prod_{i=1}^{n}(1-p)^{\langle s_i, r_i \rangle} p^{\langle \bar{s}_i, r_i \rangle} \propto \prod_{i=1}^{n}(1-p)^{s_i, r_i} = \prod_{i=1}^{n}\exp_2(s_i r_i \log_2(\frac{1-p}{p}))
\]

\[
\text{with } P(r|s) = \exp(\sum_{i=1}^{n}s_i r_i) \text{ with } K = \log_2(\frac{1-p}{p}) \text{ and } s_i, r_i \in \mathbb{B} = \{-1, +1\}
\]

or a channel with Additive White GAUSS Noise (AWGN) given by

\[
P(r|s) \propto \prod_{i=1}^{n}2^{-(r_i-s_i)^2} \propto \exp_2(\sum_{i=1}^{n}r_i s_i) \text{ and } s_i \in \mathbb{B}
\]

(this actually is the GAUSS probability density) and the by \( 2\sigma^2 = \log_2(e) \) normalised noise variance. The received elements \( r_i \) are in the AWGN case real valued, i.e., \( r_i \in \mathbb{R} \). Note that the normalised noise variance is obtained by \( r_i = K r_i^{(i)} \) and an appropriate constant \( K \). Moreover, then both cases coincide.

Overall this gives that decoding is based on

\[
\begin{cases}
1) \text{the knowledge of the code space } \mathbb{E}(C), \\
2) \text{the knowledge of the channel map given by } P(r|c), \text{ and} \\
3) \text{the received information represented by } r.
\end{cases}
\]

A decoding can be performed by a decision for some word \( \hat{c} \), which is in the Maximum Likelihood (ML) word decoding case

\[
\hat{c} = \arg \max_{c \in \mathbb{E}(C)} P(r|c)
\]

or decisions on the code symbols by ML symbol by symbol decoding

\[
\hat{c}_i = \arg \max_{x \in \mathbb{B}} P^{(c)}(x|r) = \arg \max_{x \in \mathbb{B}} \sum_{c \in \mathbb{E}(C), c_i = x} P(r|c).
\]

Here \( P^{(c)}(x|r) \) is the probability that \( c_i = x \) under the knowledge of the code space \( \mathbb{E}(C) \). If no further prior knowledge about the code map or other additional information is available then these decisions are obviously optimal, i.e., the decisions exhibit smallest word respectively Bit error probability.

**Remark 5 (Dominating ML Word)** If by \( P^{(a)}(\hat{c}^{(a)}|r) \rightarrow 1 \) a dominating ML word decision exists then necessarily holds that \( \hat{c}^{(a)} = \bar{c}^{(a)} \). The decoding problem is then equivalent to solving either of the ML decisions.

ML word decoding is for the BSC equivalent to find the code word with the smallest number of errors \( c_i \neq r_i \), respectively the smallest HAMMING distance \( d_H(c, r) \). For the AWGN channel the word \( c \) that minimises EUCLID’s quadratic distance \( d_E^2(c, r) = \| r - c \|^2 \) needs to be found.
For the independent channels of Remark 4 the ML decisions can be computed (see Appendix A.1) in the code trellis by the Viterbi or the BCJR algorithm. However, due to the generally large trellis complexity of the overall code these algorithms do there (practically) not apply.

On the other hand one may compute the “uncoded” word probabilities

\[ P(s|r) \propto P(r|s) \langle s \in \Sigma \rangle, \tag{1} \]

and for small constituent trellis complexities the constituent code word probabilities

\[ P^{(l)}(s|r) := P_{C^{(l)}|R}(s|r) = \frac{P(r|s) \cdot \langle s \in \mathbb{C}^{(l)} \rangle}{\sum_{s' \in \mathbb{S}} P(r|s') \cdot \langle s' \in \mathbb{C}^{(l)} \rangle} \propto P(r|s) \cdot \langle s \in \mathbb{C}^{(l)} \rangle \]

for \( l = 1, 2 \) with \( \mathbb{S} := \mathcal{E}(\mathbb{S}) \) the set of all words. This is interesting as the overall code word distribution

\[ P^{(a)}(s|r) := P_{C^{(a)}|R}(s|r) \propto P(r|s) \cdot \langle s \in \mathbb{C}^{(a)} \rangle \]

can be computed out of \( P^{(l)}(s|r) \) and \( P(s|r) \): It holds with Definition 3 that \( \mathbb{C}^{(a)} = \mathbb{C}^{(1)} \cap \mathbb{C}^{(2)} \) and thus

\[ P^{(1)}(s|r) \cdot P^{(2)}(s|r) \propto (P(r|s))^2 \cdot \langle s \in \mathbb{C}^{(1)} \rangle \cdot \langle s \in \mathbb{C}^{(2)} \rangle = (P(r|s))^2 \cdot \langle s \in \mathbb{C}^{(a)} \rangle, \]

which gives with (1) that

\[ P^{(a)}(s|r) \propto \frac{P^{(1)}(s|r) P^{(2)}(s|r)}{P(s|r)}. \tag{2} \]

If the constituent word probabilities are all known then optimal decoding decisions can be taken. I.e., one can compute the ML word decision by

\[ \hat{c}^{(a)} = \arg \max_{s \in \mathbb{S}} \frac{P^{(1)}(s|r) P^{(2)}(s|r)}{P(s|r)} \tag{3} \]

or the ML symbol decisions by

\[ \hat{c}^{(a)}_i = \arg \max_{x \in \mathbb{S}} P^{(a)}_{C_i}(x|r) = \arg \max_{x \in \mathbb{S}} \sum_{s \in \mathbb{S} \cap \mathbb{S}_i (x)} P^{(a)}(s|r) = \arg \max_{x \in \mathbb{S}} \sum_{s \in \mathbb{S} \cap \mathbb{S}_i (x)} \frac{P^{(1)}(s|r) P^{(2)}(s|r)}{P(s|r)} \tag{4} \]

with \( \mathbb{S}_i (x) := \{ s \in \mathbb{S} : s_i = x \} \).

Decoding decisions may therefore be taken by the constituent probabilities. However, one may by (2) only compute a value proportional to each single word probability. The representation complexity of the constituent word probability distribution remains prohibitively large. I.e., the decoding decisions by (3) and (4) do not reduce the overall complexity as all word probabilities have to be jointly considered, which is equivalent to investigating the complete code constraint.

### 1.2. Belief Propagation

The probabilities of the two constituent codes thus contain the complete knowledge about the decoding problem. However, the constituent decoders may not use this knowledge (with reasonable complexity) as then \( 2^n \) values need to be transferred. I.e., a realistic algorithm based on the constituent probabilities should transfer only a small number of parameters.

In (loopy) belief propagation algorithm this is done by transmitting only the constituent “believed” symbol probabilities but to repeat this several times. This algorithm is here shortly reviewed: One first uses a transfer vector \( \mathbf{w}^{(1)} \) to represent the believed \( P^{(1)}_{C_i}(x|r) \) of code 1. This belief representing transfer vector is then used together with \( r \) in the decoder of the other constituent code. I.e., a transfer vector \( \mathbf{w}^{(2)} \) is
computed out of \( P^{(2)}_{C_i}(x|r, w^{(1)}) \) that will then be reused for a new \( w^{(1)} \) by \( P^{(1)}_{C_i}(x|r, w^{(2)}) \) and so forth. The algorithm is stopped if the beliefs do not change any further and a decoding decision is emitted.

The beliefs \( P^{(h)}_{C_i}(x|r, w^{(l)}) \) for \( l, h \in \{1, 2\} \) and \( l \neq h \) are obtained by

\[
P(r, w^{(l)}|s) = P(w^{(l)}|s)P(r|s),
\]

which is a in \( w \) and \( r \) independent representation. Moreover, it is assumed that \( s_i \in \mathbb{B} = \{-1, +1\} \) and that

\[
P(w^{(l)}|s) = \prod_{i=1}^{n} P(w^{(l)}_i|s_i) \propto \exp_2(\sum_{i=1}^{n} w^{(l)}_i s_i) = \exp_2(w^{(l)}s^T)
\]

are of the form of \( P(r|s) \) in Remark 4.

**Remark 6** *(Distributions and Trellis)* Obviously many other choices for \( P(w^{(l)}|s) \) exist. However, the again independent description of the symbols \( C_i = S_i \) in (5) leads to (see Appendix A.1) the possibility to use trellis based computations, i.e., the symbol probabilities \( P^{(l)}_{C_i}(x|r, w^{(h)}) \) can be computed as before \( P^{(l)}_{C_i}(x|r) \).

The transfer vector \( w^{(h)} \) for belief propagation for given \( r \) and \( w^{(l)} \) with \( l, h \in \{1, 2\} \) and \( h \neq l \) is defined by

\[
P_{C_i}(x|r, w^{(1)}, w^{(2)}) = P^{(h)}_{C_i}(x|r, w^{(l)}) \text{ for all } i.
\]

I.e., the beliefs under \( r, w^{(1)}, w^{(2)} \), and no further set restriction are set such that they are equal to the beliefs under \( w^{(l)}, r \), and the knowledge of the set restriction of the \( h \)-th constituent code. This is always possible as shown below.

**Remark 7** *(Notation)* To simplify the notation we set in the sequel

\[
m = (r, w^{(1)}, w^{(2)}), \quad m^{(1)} = (r, w^{(2)}), \quad m^{(2)} = (r, w^{(1)}),
\]

and often \( w^{(0)} := r \).

For the *uncoded* beliefs \( P_{C_i}(x|m) \) it is again assumed that the information and belief carrying \( r, w^{(1)} \) and \( w^{(2)} \) are independent, i.e.,

\[
P(m|c) = P(r, w^{(1)}, w^{(2)}|c) = P(r|c)P(w^{(1)}|c)P(w^{(2)}|c).
\]

The computation of \( w^{(h)} \) for given \( w^{(l)} \) is then simple as the independence assumptions (5) and (6) give that

\[
P_{C_i}(x|m) = P_{C_i|m}(x|r_i + w^{(1)}_i + w^{(2)}_i).
\]

Moreover, the definition of the \( w^{(l)} \) is simplified by the use of logarithmic probability ratios

\[
L_i(m) = \frac{1}{2} \log_2 \frac{P_{C_i}(+1|m)}{P_{C_i}(-1|m)} \quad \text{and} \quad L_i^{(l)}(m^{(l)}) = \frac{1}{2} \log_2 \frac{P_{C_i}^{(l)}(+1|m^{(l)})}{P_{C_i}^{(l)}(-1|m^{(l)})}
\]

for \( l = 1, 2 \). This representation is handy for the computations as (5) directly gives

\[
L_i(m) = r_i + w^{(1)}_i + w^{(2)}_i
\]

and thus that Equation (6) is equivalent to

\[
w^{(l)}_i = L^{(l)}_i(m^{(l)}) - r_i - w^{(h)}_i \quad \text{for } l \neq h \text{ and all } i.
\]

This equation can be used as an *iteration* rule such that the uncoded beliefs are subsequently updated by the constituent beliefs. The transfer vectors \( w^{(1)} \) and \( w^{(2)} \) are thereby via (8) iteratively updated. The following definition further simplifies the notation.
Algorithm 1 Loopy Belief Propagation

1. Set $w^{(1)} = w^{(2)} = 0$, $l = 1$, and $h = 2$.
2. Swap $l$ and $h$.
3. Set $w^{(l)} = \tilde{L}^{(l)}(m^{(l)})$.
4. If $w^{(h)} \neq \tilde{L}^{(h)}(m^{(h)})$ then go to Step 2.
5. Set $\hat{c}_i = \text{sign}(r_i + w_i^{(1)} + w_i^{(2)})$ for all $i$.

Definition 4 (Extrinsic Symbol Probability) The extrinsic symbol probability of code $l$ is

$$\tilde{P}_{C_i}^{(l)}(x|m^{(l)}) \propto P_{C_i}^{(l)}(x|m^{(l)}) \exp(-x(w_i^{(h)} - r_i))$$

for $h \neq l$.

The extrinsic symbol probabilities are by (5) independent of $w_i^{(l)}$ for $l = 1, 2$ and $r_i$, i.e., they depend only on belief and information carrying $w_j^{(l)}$ and $r_j$ from with $j \neq i$ or “extrinsic” symbol positions. Moreover, one directly obtains the extrinsic logarithmic probability ratios

$$\tilde{L}_i^{(l)}(m^{(l)}) := \frac{1}{2} \log_2 \frac{P_{C_i}^{(l)}(+1|m^{(l)})}{P_{C_i}^{(l)}(-1|m^{(l)})} - r_i - w_i^{(h)} = L_i^{(l)}(m^{(l)}) - r_i - w_i^{(h)} \text{ for } l \neq h. \quad (9)$$

With Equation (8) this gives the iteration rule

$$w_i^{(l)} = \tilde{L}_i^{(l)}(m^{(l)})$$

and thus Algorithm 1. Note that one generally uses an alternative, less stringent stopping criterion in Step 4 of the algorithm.

If the algorithm converges then one obtains that

$$r_i + w_i^{(1)} + w_i^{(2)} = L_i^{(1)}(r, w^{(1)}(r)) = L_i^{(2)}(r, w^{(2)})$$

and

$$\hat{c}_i = \text{sign}(L_i(r) + \tilde{L}_i^{(1)}(m^{(1)}) + \tilde{L}_i^{(2)}(m^{(2)})) \quad (10)$$

with $L_i(r) = r_i$. This is a rather intuitive form of the fixed point of iterative belief propagation. The decoding decision $\hat{c}_i$ is defined by the sum of the (representations of the) channel information $r_i$ and the extrinsic constituent code beliefs $\tilde{L}_i^{(l)}(m^{(l)})$.

Remark 8 (Performance) If the algorithm converges then simulations show that the decoding decision is usually good. By density evolution [17] or extrinsic information transfer charts [19] the convergence of iterative belief propagation is further investigated. These approaches evaluate, which constituent codes are suitable for iterative belief propagation. This approach and simulations show that only rather weak codes should be employed for good convergence properties. This indicates that the chosen transfer is often too optimistic about its believed decisions.

1.3. Discrimination

The belief propagation algorithm uses only knowledge about the constituent codes represented by $w^{(l)}$. In this section we aim at increasing the transfer complexity by adding more variables and hope to obtain thereby a better representation of the overall information and thus an improvement over the propagation of only symbol beliefs.
Reconsider first the additional belief representation \( w^{(l)} \) given by the distributions \( P(s|w^{(1)}) \) and \( P(s|w^{(2)}) \) used for belief propagation. The overall distributions are

\[
P(s|m) = P(s|r, w^{(1)}, w^{(2)}) \propto P(r|s)P(u^{(1)}|s)P(u^{(2)}|s) \\
P^{(1)}(s|m^{(1)}) = P^{(1)}(s|r, u^{(1)}) \propto P(r|s)P(u^{(1)}|s) \\
P^{(2)}(s|m^{(2)}) = P^{(2)}(s|r, u^{(1)}) \propto P(r|s)P(u^{(1)}|s).
\]

The following lemma first gives that these additional beliefs do not change the computation of the overall word probabilities.

**Lemma 1** It holds for all \( w^{(1)}, w^{(2)} \) that

\[
P^{(a)}(s|r) \propto \frac{P^{(1)}(s|m^{(1)})P^{(2)}(s|m^{(2)})}{P(s|m)}.
\]

**Proof:** A direct computation of the equation with (11) gives as for (2) equality. The terms that depend on \( w^{(l)} \) vanish by the independence assumption (5).

To increase the transfer complexity now additional parameters are added to \( s \). This first seems counter intuitive as no new knowledge is added. However, with Lemma (1) the same holds true for the belief carrying \( w^{(l)} \) and optimal decoding.

**Definition 5** *(Word uncertainty)* The uncertainty augmented word probability \( P(h)(s, u|m^{(h)}) \) is

\[
P(h)(s, u|m^{(h)}) := P^{(h)}(s|m^{(h)}) \prod_{l=0}^{2} \langle u_l = w^{(l)} s^T \rangle
\]

with \( u = u(s) = (u_0, u_1, u_2) \).

This definition naturally extends to \( P(s, u|m) \) and to \( P^{(a)}(s, u|r) \).

**Remark 9** *(Notation)* The notation of \( P^{(a)}(s, u|r) \) does not reflect the dependency on \( m \). The same holds true for \( P^{(l)}(s, u|m^{(l)}) \) etc. A complete notation is for example \( P(s, u|m||r) \) or \( P^{(l)}(s, u|m||m^{(l)}) \). To maintain readability this dependency will *not* be explicitly stated in the sequel.

Under the assumption that code words with the same \( u \) do not need to be distinguished one obtains the following definition.

**Definition 6** *(Discriminated Distribution)* The distribution of \( u \) discriminated by \( m \) is

\[
P^\otimes(u|m) \propto \frac{P^{(1)}(u|m^{(1)})P^{(2)}(u|m^{(2)})}{P(u|m)}
\]

with \( \sum_{u \in U} P^\otimes(u|m) = 1 \),

\[
P^{(l)}(u|m^{(l)}) = \sum_{s \in \mathbb{S}} P^{(l)}(s, u|m^{(l)}),
\]

and \( U = \mathbb{E}(U) = \{ u : u_l = w^{(l)} s^T \forall l \} \).

**Remark 10** *(Discrimination)* Words \( s \) with the same \( u \) are not distinguished. As \( m \) and \( s \) define \( u \) the *discrimination* of words is steered by \( m \). The variables \( u_l \) are then used to relate to the *distances* \( ||c−w^{(l)}||^2 \) (see Remark 4). Words that do not share the same distances are discriminated. The choice of \( u \) and (5) is natural as all code words with the same \( u \) have the same probability, i.e., that

\[
P^{(l)}(s, u|m^{(l)}) \propto \exp(\sum_{k=0, k \neq l}^{2} u_k) \cdot \prod_{j=0}^{2} \langle u_j = w^{(j)} s^T \rangle \cdot \langle s \in \mathbb{C}^{(l)} \rangle
\]

(12)
and similar for $P(s, u|m)$ and $P^{(o)}(s, u|r)$. Generally it holds that $u$ is via
\[ \sum_{k=0}^{2} u_k = K + \log_2 P(s|m) = K - H(s|m) \]
(with $K$ some constant) related to the uncertainty $H(s|m)$. Note that any map of $s$ on some $u$ will define some discrimination. However, we will here only consider the correlation map, respectively the discrimination of the information theoretic word uncertainties.

In the same way one obtains the much more interesting (uncertainty) discriminated symbol probabilities.

**Definition 7** *(Discriminated Symbol Probabilities)* The symbol probabilities discriminated by $m$ are
\[ P_{C_i}^{\otimes}(x|m) = \sum_{u \in U} P_{C_i}^{\otimes}(x, u|m) \propto \sum_{u \in U} \frac{P_{C_i}^{(1)}(x, u|m^{(1)}) P_{C_i}^{(2)}(x, u|m^{(2)})}{P_{C_i}(x, u|m)} \]  \hspace{1cm} (13)
with
\[ P_{C_i}^{(l)}(x, u|m^{(l)}) = \sum_{s \in S(x)} P^{(l)}(s, u|m^{(l)}) \propto \sum_{s \in C^{(l)}, s_i = x} P(s, u|m^{(l)}). \]

**Remark 11** *(Independence)* Note that $P_{C_i}^{\otimes}(x|m)$ is by [5] independent of both $w^{(l)}_i$.

The discriminated symbol probabilities may be considered as *commonly* believed symbol probabilities under word uncertainties.

To obtain a first intuitive understanding of this fact we relate $P_{C_i}^{\otimes}(x|m)$ to the more accessible constituent symbol probabilities $P_{C_i}^{(l)}(x|m^{(l)})$. It holds by BAYES’ theorem that
\[ P_{C_i}^{\otimes}(x|m) \propto \frac{P_{C_i}^{(1)}(x|m^{(1)}) P_{C_i}^{(2)}(x|m^{(2)})}{P_{C_i}(x|m)} P_{C_i}^{\otimes}(x|m) \]
with (abusing notation as this is not a probability)
\[ P_{C_i}^{\otimes}(x|m) \propto \sum_{u \in U} \frac{P_{C_i}^{(1)}(u|x, m^{(1)}) P_{C_i}^{(2)}(u|x, m^{(2)})}{P_{C_i}(u|x, m)}. \]  \hspace{1cm} (14)

In the logarithmic notation this gives
\[ L_i^{\otimes}(m) = L_i^{(1)}(m^{(1)}) + L_i^{(2)}(m^{(2)}) - L_i(m) + L_i^{\otimes}(m) \]
or in the extrinsic notation of [9] that
\[ L_i^{\otimes}(m) = \tilde{L}_i^{(1)}(m^{(1)}) + \tilde{L}_i^{(2)}(m^{(2)}) + L_i(r) + L_i^{\otimes}(m). \]  \hspace{1cm} (15)

Note first the similarity with [10]. One has again a sum of the extrinsic beliefs, however, an additional value $L_i^{\otimes}(m)$ is added, which is by Remark [11] necessarily independent of $w^{(l)}_i$ and $l = 1, 2$. Overall the common belief joins the two constituent beliefs together with a “distance” correction term.

Below we show that this new common belief is – under again practically prohibitively high complexity – just the real overall “belief”, i.e., the correct symbol probabilities obtained by optimal symbol decoding.

**Definition 8** *(Globally Maximal Discriminator)* The discriminator $m$ is globally maximal (for $S$) if $|\mathcal{S}(u|m)| = 1$ for all $u \in U$. I.e., for globally maximal discriminators exists a one-to-one correspondence between $s$ and $u$ and thus $|\mathcal{S}| = |U|$. 

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Lemma 2 For a globally maximal discriminator $m$ it holds that
\[ P^\otimes(u|m) = P^{(a)}(u|r) \text{ and } P_{C_i}^\otimes(x|m) = P_{C_i}^{(a)}(x|r). \]

I.e., the by $m$ discriminated symbol probabilities are correct.

Proof: Lemma $\|$ and Definition $\|$ give
\[
P^{(a)}(s, u|r) \propto \frac{P^{(1)}(s, u|m^{(1)}) P^{(2)}(s, u|m^{(2)})}{P(s, u|m)}
\]
as $u$ follows directly from $s$. For a globally maximal discriminator $m$ exists a one-to-one correspondence between $s$ and $u$ This gives that one can omit for any probability either $u$ or $s$. This proves the optimality of the discriminated distribution.

For the overall symbol probabilities holds
\[
P_{C_i}^{(a)}(x|r) = \sum_{s \in S_i(x)} P^{(a)}(s|r) = \sum_{s \in S_i(x)} \frac{P^{(1)}(s, u|m^{(1)}) P^{(2)}(s, u|m^{(2)})}{P(s, u|m)}.
\]

With $P_{C_i}^{(l)}(x, s, u|m^{(l)}) = P^{(l)}(s, u|m^{(l)})$ for $s \in S_i(x)$ and $P_{C_i}^{(l)}(x, s, u|m^{(l)}) = 0$ for $s \not\in S_i(x)$ the right hand side becomes
\[
\sum_{s \in S_i(x)} \frac{P^{(1)}(s, u|m^{(1)}) P^{(2)}(s, u|m^{(2)})}{P(s, u|m)} = \sum_{s \in S_i(x)} \frac{P_{C_i}^{(1)}(x, s, u|m^{(1)}) P_{C_i}^{(2)}(x, s, u|m^{(2)})}{P_{C_i}(x, s, u|m)}.
\]

By the one-to-one correspondence one can replace the sum over $s$ by a sum over $u$ to obtain
\[
P_{C_i}^{(a)}(x|r) = \sum_{u \in U} \frac{P_{C_i}^{(1)}(x, s, u|m^{(1)}) P_{C_i}^{(2)}(x, s, u|m^{(2)})}{P_{C_i}(x, s, u|m)}.
\]

which is ($s$ can be omitted due to the one-to-one correspondence) the optimality of the discriminated symbol probabilities.

A globally maximal discriminator $m$ thus solves the problem of ML symbol by symbol decoding. Likewise by
\[
\arg \max_{u \in U} P^\otimes(u|m) = \arg \max_{u \in U} P^{(a)}(u|r) = \arg \max_{s \in S} P^{(a)}(s|r)
\]
the problem of ML word decoding is solved (provided the one-to-one correspondence of $u$ and $s$ can be easily inverted).

This is not surprising as a globally maximal discriminator has by the one-to-one correspondence of $s$ and $u$ the discriminator complexity $|U| = |S|$. The transfer complexity is then just the complexity of the optimal decoder based on constituent probabilities.

Remark 12 (Globally Maximal Discriminators) The vector $m = (r, w^{(1)}, 0)$ and $w_i^{(1)} = 2^i$ is an example of a globally maximal discriminator as $u_1(s) = \sum_{i=1}^{n} s_i 2^i$ is different for all values of $s$. I.e., there exists a one-to-one correspondence between $s$ and $u$. Generally it is rather simple to construct a globally maximal discriminator. E.g., the $r$ received via an AWGN channel is usually already maximizing the probability that two words $s^{(1)}, s^{(2)} \in S$ share the same real valued distance to the received word is generally Zero.
2. Local Discriminators

In the last section the coupling of error correcting codes was reviewed and different decoders were discussed. It was shown that an optimal decoding is due to the large representation complexity practically not feasible, but that a transfer of beliefs may lead to a good decoding algorithm. A generalisation of this approach led to the concept of discriminators and therewith to a new overall belief. The complexity of the computation of this belief is depending on $|U|$, i.e., the number of different outcomes $u$ of the discrimination. Finally it was shown that the obtained overall belief leads to the optimal overall decoding decision if the set is with $|U| = |S|$ maximally large. (However, then the overall decoding complexity is not reduced.)

In this section we consider local discriminators with $|U| \ll |S|$. Then only a limited number of values need to be transferred to compute by (13) a new overall belief $P_C^0(u|m)$. These discriminated beliefs $P_C^0(u|m)$ may then be practically employed to improve iterative decoding. To do so we first show that local discriminators exist.

**Example 3** The $r$ obtained by a transmission over a BSC is generally a local discriminator. The map $U(r) : s \mapsto u = (u_0, 0, 0)$ is then only dependent on the HAMMING distance $d_H(r, s)$, i.e.,

$$U_0(r) : s \mapsto u_0 = rs^T = n - 2d_H(r, s)$$

and thus $U = \mathbb{E}(U_0) \subseteq \{-n, -n+2, \ldots, n-2, n\}$, which gives $|U| \leq n + 1$. This furthermore gives that an additional “hard decision” choice of the $w^{(l)}$ will continue to yield a local “HAMMING” discriminator $m$.

To investigate local discrimination now reconsider the discriminated distributions. With Remark [10] one obtains the following lemma.

**Lemma 3** The distributions of $u$ given $m$ are

$$P(u|m) \propto |S(u|m)| \exp_2(u_0 + u_1 + u_2)$$

and

$$P^{(l)}(u|m^{(l)}) \propto |C^{(l)}(u|m)| \exp_2(\sum_{k=0, k\neq l}^2 u_k)$$

where the sets $S(u|m)$ and $C^{(l)}(u|m)$ are defined by $M(u|m) := \{ s \in M : u_l = w^{(l)}s^T \forall l \}$.

**Proof:** By (12) follows that the probability of all words $s \in S(u|m)$ with the same $u$ is equal and proportional to $\exp_2(\sum_{k=0}^2 u_k)$. As $|S(u|m)|$ words are in $S(u|m)$ this gives the first equation. The second equation is obtained by adding the code constraint.

**Remark 13** (Overall Distribution) In the same way follows (see Remark [9]) that

$$P^{(a)}(u|r) \propto |C^{(a)}(u|m)| \exp_2(u_0).$$

(16)

More general restrictions (see below) can always be handled by imposing restrictions on the considered sets. One thus generally obtains for the distributions of $u$ a description via on $u$ dependent sets sizes.

**Example 4** With the concept of set sizes Example [3] is continued. Assume again that the discriminator is given by $m = (r, 0, 0)$. In this case no discrimination takes place on $u_1$ and $u_2$ as one obtains $u_1 = u_2 = 0$ for all $s$. One first obtains the overall distribution $P^{(a)}(u|r)$ to be with Remark [13] the multiplication of $\exp_2(u_0)$ with the distribution of the correlation $cr^T$ with $c \in C^{(a)}$ given by $|C^{(a)}(u|m)|$.

Assume furthermore that the overall maximum likelihood decision $\hat{c}^{(a)}$ is with

$$P^{(a)}(\hat{c}^{(a)}|r) \rightarrow 1$$
With Lemma 3 one obtains that the discriminated symbol probabilities defined by (13) are

\[ P^{(0)}(u_0|r) = P^{(0)}(u_0|m) \approx \begin{cases} 1 & \text{for } u_0 = u_0(\hat{c}^{(a)}) = n - 2d_H(r, \hat{c}^{(a)}) \\ 0 & \text{else.} \end{cases} \]

I.e., \( P^{(0)}(u|r) \) consists of one peak.

For the other probabilities \( P(u_0|m) \) and \( P^{(l)}(u_0|m^{(l)}) \) with Lemma 3 again a multiplication of correlation distributions with \( \exp_2(u_0) \) is obtained. These distributions will, however, due to the much larger spaces

\[ |S| \gg |C^{(l)}| \gg |C^{(a)}| \]

usually not be in the form of a single peak. Other words with \( u_0 \geq \hat{c}^{(a)}r^T \) may appear. The same then holds true for \( P^{(a)}(u_0|m) \). These considerations are exemplary depicted in Figure 1. Note that the distributions can all be computed (see Appendix A.1) in the constituent trellises.

For a local discrimination a computation in the constituent trellises produces by (13) symbol probabilities \( P^{(a)}_{C_i}(x|m) \). In equivalence to (loopy) belief propagation these probabilities should lead to the definition of some \( w \) and thus to some iteration rule. Before considering this approach we evaluate the quality of the discriminated symbol probabilities.

### 2.1. Typicality

With Lemma 3 one obtains that the discriminated symbol probabilities defined by (13) are

\[
P^{(a)}_{C_i}(x|m) \propto \sum_{u \in U} \frac{|C^{(1)}_i(x, u|m)|\exp_2(u_0 + u_2)|C^{(2)}_i(x, u|m)|\exp_2(u_0 + u_1)}{|S_i(x, u|m)|\exp_2(u_0 + u_1 + u_2)}
\]

\[
= \sum_{u \in U} \frac{|C^{(1)}_i(x, u|m)||C^{(2)}_i(x, u|m)|}{|S_i(x, u|m)|} \exp_2(u_0)
\]

with the sets \( C^{(l)}_i(x, u|m) \) defined by \( s \in C^{(l)} \) and \( s_i = x \). Hence, \( P^{(a)}_{C_i}(x|m) \) only depends on the discriminated set sizes \( C^{(l)}_i(x, u|m), S_i(x, u|m) \), and the word probabilities \( P^{(a)}(s|s) \propto \exp_2(u_0(s)) \).

The discriminated symbol probabilities should approximate the overall probabilities, i.e.,

\[ P^{(a)}_{C_i}(x|m) \approx P^{(a)}(x|r). \]

With Remark 13 and (17) this approximation is surely good if

\[
\frac{|C^{(1)}_i(x, u|m)||C^{(2)}_i(x, u|m)|}{|S_i(x, u|m)|} \approx |C^{(a)}_i(x, u|m)|.
\]

Figure 1: Hard Decisions
Intuitively, the approximation thus uses the knowledge how many words of the same correlation values \( u \) and decision \( c_i = x \) are in both codes simultaneously. Moreover, depending on the discriminator \( m \) the quality of this approximation will change.

An average consideration of the approximations (18) is related to the following lemma.

**Lemma 4** If the duals of the (linear) constituent codes do not share common words but the zero word then

\[
|C^{(1)}| |C^{(2)}| = |S||C^{(a)}|. 
\]  
(19)

**Proof:** With Definition 3 and by assumption linearly independent \( H^{(1)} \) and \( H^{(2)} \) it holds that the dual code dimension of the coupled code is just the sum of the dual code dimension of the constituent codes, i.e.,

\[
n - k = (n - k^{(1)}) + (n - k^{(2)}). 
\]

This is equivalent to \( k^{(1)} + k^{(2)} = n + k \) and thus the statement of the lemma.

This lemma extends to the constrained set sizes \( |C^{(l)}(x)| \) as used in (18). The approximations are thus in the mean correct.

For random coding and independently chosen \( m = (r, w^{(1)}, w^{(2)}) \) this consideration can be put into a more precise form.

**Lemma 5** For random (long) codes \( C^{(1)} \) and \( C^{(2)} \) and independently chosen \( m \) holds the asymptotic equality

\[
|C_i^{(a)}(x, u|m)\rangle \approx \frac{|C_i^{(1)}(x, u|m)||C_i^{(2)}(x, u|m)|}{|S_i(x, u|m)|}. 
\]  
(20)

**Proof:** The probability of a random choice in \( S \) to be in \( S(u|m) \) is just the fraction of the set sizes \( |S(u|m)| \) and \( |S| \).

For a random coupled code \( |C^{(a)}| \) the codewords are a random subset of the set \( |S| \). For \( |C^{(a)}| \gg 1 \) the law of large numbers thus gives the asymptotic equality

\[
\frac{|C_i^{(a)}(x, u|m)|}{|C^{(a)}|} \approx \frac{|S_i(x, u|m)|}{|S|}. 
\]  
(21)

The same holds true for the constituent codes

\[
\frac{|C_i^{(l)}(x, u|m)|}{|C^{(l)}|} \approx \frac{|S_i(x, u|m)|}{|S|}. 
\]

A multiplication of the equality of code 1 with the one of code 2 gives the asymptotic equivalence

\[
\frac{|S|}{|C^{(1)}||C^{(2)}|} \frac{|C_i^{(1)}(x, u|m)||C_i^{(2)}(x, u|m)|}{|S_i(x, u|m)|} \approx \frac{|S_i(x, u|m)|}{|S|}. 
\]  
(22)

Combining (21) and (22) then leads to

\[
\frac{|S|}{|C^{(1)}||C^{(2)}|} \frac{|C_i^{(1)}(x, u|m)||C_i^{(2)}(x, u|m)|}{|S_i(x, u|m)|} \approx \frac{|C_i^{(a)}(x, u|m)|}{|C^{(a)}|}. 
\]

With (19) this is the statement of the lemma.

**Remark 14 (Randomness)** The proof of the lemma indicates that the approximation is rather good for code choices that are independent of \( m \). I.e., perfect randomness of the codes is generally not needed. This can be understood by the concept of random codes in information theory. A random code is generally a good code. Conversely a good code should not exhibit any structure, i.e., it behaves as a random code.
2.2. Distinguished Words

The received vector \( r \) is obtained from the channel and the encoding. The discriminator \( m \) is due to the dependent \( r \) thus generally not independent of the encoding. This becomes directly clear by reconsidering Example 3 and the assumptions that a distinguished word \( \hat{c}^{(a)} \) with

\[
P^{(a)}(\hat{c}^{(a)}|r) \rightarrow 1
\]

exists. In this case the constituent distributions and thus likewise the discriminator distribution \( P^{\otimes}(u|r) \) will be large in a region where a “typical” number of errors \( \ell \) occurred, i.e., \( u_0 = r e^T \approx n - 2\ell \).

For an independent \( m \), however, this would not be the case: Then \( P^{\otimes}(u|m) \) would with Lemma 5 be large in the vicinity of a typical minimal overall code word distance. This distance is generally larger than the expected number of errors \( \ell \) under a distinguished word. Hence, \( P^{\otimes}(u|m) \) would then be large at a smaller \( u_0 \) than under a dependent \( m \).

Remark 15 (Channel Capacity and Typical Sets) The existence of a distinguished word is equivalent to assuming a long random code of rate below capacity [18]. The word sent is then the only one in the typical set, i.e., it has a small distance to \( r \). The other words of a random code will typically exhibit a large distance to \( r \).

To describe single words one needs to describe how well certain environments in \( u \) given \( m \) are discriminated. The precision of the approximation of \( C^{(a)}_i(x,u|m) \) by (18) hereby obviously depends on the set size \( |S_i(x,u|m)| \). This leads to the following definition.

Definition 9 (Maximally Discriminated Region) The by \( m \) maximally discriminated region

\[
\mathbb{D}(m) := \bigcup_{|S(u|m)|=1} S(u|m)
\]

consists of all words \( s \) that uniquely define \( u \) with \( u_l = sw^{(l)T} \) for \( l = 0, 1, 2 \).

Theorem 2 For independent constituent codes and \( a \) by \( \hat{c}^{(a)} \in \mathbb{D}(m) \) maximally discriminated distinguished event is

\[
P^{\otimes}(x|m) \approx P^{(a)}(x|r) \text{ and } P^{\otimes}(u|m) \approx P^{(a)}(u|r).
\]

Proof: It holds with (17) that

\[
P^{\otimes}(x|m) \propto \sum_{u \in U} \frac{|C^{(1)}_i(x,u|m)| |C^{(2)}_i(x,u(\hat{c}^{(a)})|m)|}{|S_i(x,u(\hat{c}^{(a)}))|} \exp(u_0).
\]

For the distinguished event \( \hat{c}^{(a)} \in \mathbb{C}^{(a)} \) it follows that

\[
\frac{|C^{(1)}(x,u(\hat{c}^{(a)}))| |C^{(2)}(x,u(\hat{c}^{(a)}))|}{|S_i(x,u(\hat{c}^{(a)}))|} = |C^{(a)}_i(x,u(\hat{c}^{(a)}))| = 1 \text{ for } \hat{c}^{(a)} = x
\]

as by assumption \( \hat{c}^{(a)} \in \mathbb{D}(m) \) is maximally discriminated, which gives by definition and \( \hat{c}^{(a)} \in \mathbb{C}^{(l)} \) for \( l = 1, 2 \) that

\[
|S_i(x,u(\hat{c}^{(a)}))| = |C^{(1)}_i(x,u(\hat{c}^{(a)}))| = |C^{(2)}_i(x,u(\hat{c}^{(a)}))| = 1.
\]

I.e., the term with \( u = u(\hat{c}^{(a)}) \) in (22) is correctly estimated.

The other terms in (23) represent non distinguished words and can (with the assumption of independent constituent codes) be considered to be independent of \( m \). This gives that they can be assumed to be obtained by random coding. I.e., for

\[
u \neq u(\hat{c}^{(a)}) \text{ with } u_l(\hat{c}^{(a)}) = w^{(l)} e^{(a)}
\]
produces discriminated symbol beliefs close to the overall symbol probabilities. The discriminator computes
\[
\frac{\left| \mathbb{C}_i(x, u|m) \right| \mathbb{C}_i^{(2)}(x, u|m)}{\left| S_i(x, u|m) \right|} \approx \left| \mathbb{C}_i^{(a)}(x, u|m) \right|
\]
of Lemma 5. Hence the other words are (asymptotically) correctly estimated, too.

Moreover, with (23) one obtains for \( \hat{c}^{(a)} \) a probability value proportional to \( \exp_2(r \hat{c}^{(a)}T) \). The other terms of (23) are much smaller: An independent random code typically does not exhibit code words of small distance to \( r \). As the code rate is below capacity then \( P^\otimes(u(\hat{c}^{(a)}))m \) exceeds the sum of the probabilities of the other words. Asymptotically by (17) both the overall symbol probabilities and the overall distribution of correlations follow.

**Remark 16 (Distance)** Note that the multiplication with \( \exp_2(u_0) \) in (23) excludes elements that are not in the distinguished set (\( \equiv \) with large distance to \( r \)). These words can – as shown by information theory – not dominate (a random code) in probability. I.e., a maximal discrimination of non typical words will not significantly change the discriminated symbol probabilities \( P^\otimes_{C_i}(x|m) \). This indicates that a random choice of the \( w(l) \) for \( l = 1, 2 \) will typically lead to similar beliefs \( P^\otimes_{C_i}(x|m) \) as under \( w(1) = w(2) = 0 \). Conversely it holds that if one code word at a small distance is maximally discriminated then its probability typically dominates the probabilities of the other terms in (23).

**Example 5** We continue the example above. The discriminator \( m = (r, \hat{c}^{(a)}, 0) \) maximally discriminates the distinguished word \( \hat{c}^{(a)} \) at
\[
u = u(\hat{c}^{(a)}) = (n - 2d_H(r, \hat{c}^{(a)}), n, 0).
\]
The discriminator complexity \( U \) is maximally \( (n + 1)^2 \) as only this many different values of
\[
u = (n - 2d_H(r, c), n - 2d_H(\hat{c}^{(a)}, c), 0)
\]
exist. The complexity is then given by the computation of maximally \( (n + 1)^2 \) elements. As this has to be done \( n \) times in the trellis (see Appendix A.1) the asymptotic complexity becomes \( O(n^3) \) (for fixed trellis state complexity). The computation will give by Theorem 3 that
\[
P^\otimes_{C_i}(x|m) \approx P^\otimes_{C_i}(x|r) \text{ with } \hat{c}_i^{(a)} = \text{sign}(L_i^\otimes(m))
\]
as \( \hat{c}^{(a)} \) is distinguished and as all other words can be assumed to be chosen independently.
I.e., \( P^\otimes_{C_i}(u|m) \) exhibits a peak of height 1 and the \( P^\otimes_{C_i}(x|m) \) give the asymptotically correct symbol probabilities.

### 2.3. Well Defined Discriminators

**Example 5** shows that for the distinguished event \( \hat{c}^{(a)} \) the hard decision discriminator
\[
m = (r, \hat{c}^{(a)}, 0) \text{ with } \hat{c}_i^{(a)} = \text{sign}(L_i^\otimes(m))
\]
produces discriminated symbol beliefs close to the overall symbol probabilities. The discriminator complexity \( |U| \leq (n + 1)^2 \) is thus sufficient to obtain the asymptotically correct decoding decision.

**Remark 17** *(Equivalent Hard Decision Discriminators)* By (23) the hard decision discriminators
\[
m = (r, w, 0), m = (r, 0, w), \text{ and } m = (r, w, w)
\]
are equivalent: For the three cases the same \( \mathbb{C}_i^{(l)}(x|m) \) and \( S_i^{(l)}(x|m) \) and thus \( P^\otimes_{C_i}(x|m) \) follow. In the sequel of this section we will (for symmetry reasons) only consider the discriminators \( m = (r, w, w) \).

The discussion above shows that a discriminator with randomly chosen \( w \) should give almost the same \( L_i^\otimes(m) \) as \( L_i^\otimes(r, 0, 0) \). If, however, the discriminator is strongly dependent on the distinguished solution, i.e., \( w = \hat{c}^{(a)} \) then the correct solution is found via \( L_i^\otimes(m) \). This gives the following definition and lemma.
Definition 10  (Well Defined Discriminator) A well defined discriminator $m = (r, w, w)$ fulfils

$$w_i = \text{sign}(L_i^\otimes(m))$$

for all $i$. (24)

Lemma 6  For a BSC and distinguished $\hat{c}^{(a)}$ exists a well defined discriminator $m = (r, w, w)$ with $w_i, r_i \in B$ such that $\hat{c}_i^{(a)} = w_i$.

Proof:  Set $m = (r, \hat{c}^{(a)}, \hat{c}^{(a)})$. For this choice holds $\hat{c}^{(a)}(\mathbb{D}(m))$ and thus with Theorem 2 asymptotic equality. Moreover, holds for a distinguished element that

$$P_{\hat{C}_i}(\hat{c}_i^{(a)} | m) = P_{\hat{C}_i}(\hat{c}_i^{(a)} | r) \approx 1$$

and thus $\hat{c}_i^{(a)} = \text{sign}(L_i^a(r)) = \text{sign}(L_i^\otimes(m))$.

The definition of a well defined discriminator (24) can be used as an iteration rule, which gives Algorithm 2. The iteration thereby exhibits by Lemma 6 a fixed point, which provably represents the distinguished solution. Note that the employment of $w(1) = w(2) = w$ is here handy as by $L_i^\otimes(m)$ only one common belief is available. This is contrast to Algorithm 1 where the employment of the two constituent beliefs generally give that $w(1) \neq w(2)$.

Algorithm 2 Iterative Hard Decision Discrimination

1. Set $m = (r, 0, 0)$ and $w = 0$.
2. Set $v = w$ and $w_i \leftarrow \text{sign}(L_i^\otimes(m))$ for all $i$.
3. If $v \neq w$ then $m = (r, w, w)$ and go to 2.
4. Set $c = w$.

To understand the overall properties of the algorithm one needs to consider its convergence properties and the existence of other fixed points. A first intuitive assessment of the algorithm is as follows. The decisions taken by $w_i = \text{sign}(L_i^\otimes(r, 0, 0))$ should by (15) lead to a smaller symbol error probability than the one over $r$. Overall these decisions are based on $P^\otimes(u | r, 0, 0)$. This distribution is necessarily large in the vicinity of $\hat{u}_i = n - 2t$ with $t$ the expected number of errors.

The subsequent discrimination with $w$ and $r$ will consider the vicinity of $c$ more precisely if $wce^T$ is larger than $rce^T$: In this vicinity less words exist, which gives that the $|S(u|m)|$ are smaller there. Smaller error probability in $w$ is thus with (17) typically equivalent to a better discrimination in the vicinity of $\hat{c}^{(a)}$. This indicates that the discriminator $(r, w, w)$ is better than $(r, 0, 0)$. Hence, the new $w_i \leftarrow \text{sign}(L_i^\otimes(r, w, w))$ should exhibit again smaller error probability and so forth. If the iteration ends then a stable solution is found. Finally, the solution $w = \hat{c}^{(a)}$ is stable. This behaviour is exemplary depicted in Figure 2 where

![Figure 2: Hard Decision Discrimination](image)

the density of the squares represent the probability $P^\otimes(u|m)$ of $u = (u_0, u_1)$. 

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2.4. Cross Entropy

To obtain a quantitative assessment of Algorithm\ref{alg:2} we use the following definition.

**Definition 11 (Cross Entropy)** The cross entropy

\[ H(C|w|r) := E_C[H(s|w)|r] = - \sum_{s \in C} P_C(s|r) \log_2(P(s|w)) \]

is the expectation of the uncertainty \( H(s|w) = - \log_2 P(s|w) \) under \( r \) and \( c \in E(C) \).

The cross entropy measures as the KULLBACK-LEIBLER Distance

\[ D(C|w||r) := H(C|w||r) - H(C|r) \]

with

\[ H(C|r) := E_C[H_C(s|r)|r] = - \sum_{s \in C} P_C(s|r) \log_2(P_C(s|r)). \]

the similarity between the distributions \( P(c|r) \) and \( P(s|w) \). By JENSEN’s inequality it is easy to show \[15\] that \( D(C|w||r) \geq 0 \) and thus

\[ H(C|w||r) \geq H(C|r) \geq 0. \]

The entropy \( H(C|r) \) is an information theoretic measure of the number of probable words in \( E(C) \) under \( r \). To better explain the cross entropy \( H(C|w||r) \) we shortly review some results regarding the entropy.

The typical set \( A_{n\varepsilon}(C|r) \) is given by the typical region

\[ A_{n\varepsilon}(C|r) = \{ c \in E(C) : |H(c|r) - H(C|r)| \leq n\varepsilon \} \]

of word uncertainties

\[ H(c|r) = - \log_2 P(c|r). \]

This definition directly gives

\[ 1 \geq \sum_{A_{n\varepsilon}(C|r)} P(c|r) = \sum_{A_{n\varepsilon}(C|r)} \exp_2(-H(c|r)) \geq \exp_2(-H(C|r) - n\varepsilon) \sum_{A_{n\varepsilon}(C|r)} 1, \]

respectively,

\[ P_C(A_{n\varepsilon}(C|r)|r) = \sum_{A_{n\varepsilon}(C|r)} P(c|r) = \sum_{A_{n\varepsilon}(C|r)} \exp_2(-H(c|r)) \leq \exp_2(-H(C|r) + n\varepsilon) \sum_{A_{n\varepsilon}(C|r)} 1. \]

With

\[ \sum_{A_{n\varepsilon}(C|r)} 1 = |A_{n\varepsilon}(C|r)| \]

this leads to the bounds on the logarithmic set sizes

\[ H(C|r) + n\varepsilon \geq \log_2 |A_{n\varepsilon}(C|r)| \geq H(C|r) + \log_2(P_C(A_{n\varepsilon}(C|r)|r)) - n\varepsilon \]

by the entropy. For many independent events in \( r \) the law of large numbers gives for \( \varepsilon > 0 \) that

\[ P_C(A_{n\varepsilon}(C|r)|r) \approx 1 \]

and thus \( H(C|r) \approx \log_2(|A_{n\varepsilon}(C|r)|) \).

We investigate if a similar statement can be done for the cross entropy. To do so first a cross typical set \( A_{n\varepsilon}(C|w||r) \) is defined by the region of typical word uncertainties:

\[ \min(H(S|w), H(C|w||r)) - n\varepsilon \leq H(s|w) \leq \max(H(S|w), H(C|w||r)) + n\varepsilon. \]
I.e., the region spans the typical set in $w$ but includes more words if $H(S|w) \neq H(C|w||r)$. As the typical set in $w$ is included this gives for large $n$ that

$$P_S(\Lambda_{n\varepsilon}(C|w||r)|w) \approx 1$$

and then in the same way as above the bounds on the logarithmic set size

$$\max(H(S|w), H(C|w||r)) + n\varepsilon \geq \log_2 |\Lambda_{n\varepsilon}(C|w||r)| \geq \min(H(S|w), H(C|w||r)) - n\varepsilon.$$

Moreover, holds by the definition of the cross entropy and the law of large numbers that typically

$$P_C(\Lambda_{n\varepsilon}(C|w||r)|r) \approx 1$$

is true, too. This gives that the cross typical set includes the typical sets $\Lambda_{n\varepsilon}(S|w)$ and $\Lambda_{n\varepsilon}(C|w||r)$, i.e.,

$$\Lambda_{n\varepsilon}(C|w||r) \supseteq \Lambda_{n\varepsilon}(S|w) \text{ and } \Lambda_{n\varepsilon}(C|w||r) \supseteq \Lambda_{n\varepsilon}(C|r). \tag{26}$$

If one wants to define a transfer vector $w$ based on $r$ one is thus interested to obtain a representation in $w$ such that the logarithmic set size

$$\log_2 |\Lambda_{n\varepsilon}(C|w||r)| \leq \max(H(S|w), H(C|w||r)) + n\varepsilon$$

is as small as possible.

In the sequel we consider $P(s|w) \propto P(w|s)$ defined by (5). This probability is given by

$$P(s|w) = \frac{\prod_{i=1}^{n} P(w_i|s_i)}{n \sum_{s \in S} P(w|s)} = \frac{\prod_{i=1}^{n} P(s_i|w_i)}{P(+1|w_1) + P(-1|w_1)} = \prod_{i=1}^{n} \frac{2^{s_i w_i}}{2^{w_i} + 2^{-w_i}}. \tag{27}$$

The cross entropy thus becomes

$$H(C|w||r) = \sum_{s \in C} P_C(s|r) \sum_{i=1}^{n} \left( \log_2 (2^{w_i} + 2^{-w_i}) - s_i w_i \right)$$

$$= \sum_{i=1}^{n} \log_2 (2^{w_i} + 2^{-w_i}) - \sum_{i=1}^{n} \sum_{s \in C} P_C(s|r) s_i w_i \tag{28}$$

and

$$\sum_{i=1}^{n} \sum_{s \in C} P_C(s|r) s_i w_i = \sum_{i=1}^{n} E_{C,C_i}[x|r]w_i = \sum_{i=1}^{n} w_i \left( P_{C_i}^{(c)}(+1|r) - P_{C_i}^{(c)}(-1|r) \right).$$

This definition almost directly defines an optimal transfer.

**Lemma 7** Equal logarithmic symbol probability ratios

$$w_i = L_i(w) = L_i^{(c)}(r) = \frac{1}{2} \log_2 \frac{P_{C_i}^{(c)}(+1|r)}{P_{C_i}^{(c)}(-1|r)}$$

and $P(s|w) \propto P(w|s)$ defined by (5) minimise cross entropy $H(C|w||r)$ and **KULLBACK-LEIBLER** distance $D(C|w||r)$.

**Proof:** First it holds by (5) that

$$w_i = L_i(w) = \frac{1}{2} \log_2 \frac{\exp_2 (+w_i)}{\exp_2 (-w_i)}.$$

A differentiation of (28) leads to

$$\frac{\partial}{\partial w_i} H(C|w||r) = \sum_{s \in C} P_C(s|r)(\tanh_2(w_i) - x_i) = \tanh_2(w_i) - \sum_{s \in C} c_i P_C(s|r) = 0$$

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with \( \tanh_2(x) = (2^x - 2^{-x})/(2^x + 2^{-x}) \). This directly gives that

\[
\tanh_2(w_i) = \sum_{s \in C} c_i P_C(s|r) = P_{C_i}^{(c)}(+1|r) - P_{C_i}^{(c)}(-1|r).
\]

As

\[
\tanh_2(L_i^{(c)}(r)) = P_{C_i}^{(c)}(+1|r) - P_{C_i}^{(c)}(-1|r)
\]

and \( \frac{\partial}{\partial w_i} H(C|r) = 0 \) this is equivalent to the statement of the lemma.

I.e., the definition of \( w \) by \( L_i(w) = L_i^{(c)}(r) \) is a consequence of the independence assumption (5). Especially interesting is that \( L_i(w) = L_i^{(c)}(r) \) directly implies that

\[ H(S|w) = H(C|w|r), \]

which gives with (26) that

\[ A_{nz}(C|w|r) = A_{nz}(S|w) \quad \text{and thus} \quad A_{nz}(S|w) \supseteq A_{nz}(C|r). \]

A belief representing transfer vector \( w \) thus typically describes all probable codewords. By reconsidering the definition of the cross typical set in (25) the in \( r \) and \( C \) typical set \( A_{nz}(C|r) \) is (in the mean) contained in the set of in \( w \) probable words \( s \in S \) if

\[ H(S|w) \geq H(C|w|r). \]

Hereby the set of probable words is defined by only considering the right hand side inequality of (25).

### 2.5. Discriminator Entropy

In this section the considerations are extended to the discrimination. To do so we use in equivalence to (28) the following definition.

**Definition 12** *(Discriminated Cross Entropy)* The discriminated cross entropy is

\[
H(C^\otimes|w|m) := -\sum_{u \in U} P^\otimes(u|m) \log_2 P(s|w) := \sum_{i=1}^{n} \log_2(2^{u_i} + 2^{-u_i}) - \sum_{u \in U} w_is_i \cdot P_{C_i}^\otimes(s_i,u|m)
\]

\[
= \sum_{i=1}^{n} \log_2(2^{u_i} + 2^{-u_i}) - w_i E_U^\otimes(c_i|m)
\]

with (27) and \( E_U^\otimes(c_i|m) = P_{C_i}^\otimes(+1|m) - P_{C_i}^\otimes(-1|m) \).

Note that this definition again uses the correspondence of \( u \) and \( s \). Even though by a discrimination not all words are independently considered, a word uncertainty consideration is still possible by attributing appropriate probabilities. Lemma 7 directly gives that the discriminated cross entropy is always larger than or equal to the discriminated symbol entropy \( H(C^\otimes|m) \), i.e.,

\[
H(C^\otimes|w|m) \geq -\sum_{u \in U} P^\otimes(u|m) \log_2 P(s|L^\otimes(m)) =: H(C^\otimes|m)
\]

The discriminator entropy measures the uncertainty of the discriminated decoding decision, i.e., the number of words in \( S \) that need to be considered. This directly gives the following theorem.

**Theorem 3** The decoding problem for a distinguished word is equivalent to the solution of

\[
w_i = \text{sign}(L_i^\otimes(m))
\]

with the discriminated symbol entropy \( H(C^\otimes|m) < 1 \) and \( m = (r, w, w) \).
Proof: For \( w_i = \hat{c}_i^{(a)} \) is \( \hat{c}^{(a)} \in \mathcal{D}(m) \). This gives with Lemma 6 for the discriminated distribution that
\[
P^\otimes(u|m) \propto P^{(a)}(u|r).
\]
As \( \hat{c}^{(a)} \) is a distinguished solution this gives \( P^{(a)}(u(\hat{c}^{(a)}))|r| \approx 1 \) or equivalently \( H(C\otimes|m) \approx 0 \).
The discriminated symbol entropy \( H(C\otimes|m) \) estimates by \( \exp_2 H(C\otimes|m) \) the logarithmic number of elements in the set of probable words in \( S \). Any solution \( m \) with
\[
H(C\otimes|m) < 1
\]
thus exhibits one word \( s \) with \( P^\otimes(u|m) \approx 1 \). I.e., one has a discriminated distribution \( P^\otimes(u|m) \) that contains just one peak of height almost one at \( \hat{u} \). As only one word distributes the decisions by (29) give this word, or equivalently that \( \hat{u} = u(w) \). Hence, \( \hat{c} = w \) is maximally discriminated.

This directly implies that the obtained \( \hat{c} \) needs to be a codeword of the coupled code: Both distributions \( P^{(i)}(u|m) \) are used for the single word description \( P^\otimes(u|m) \neq 0 \). Hence, both codes contain the in \( u \) maximally discriminated word \( \hat{c} \), which gives (by the definition of the dually coupled code) that this word is an overall codeword.

Assume that \( \hat{c} \neq \hat{c}^{(a)} \) represents a non distinguished word. With Remark [16] this word needs to exhibit a large distance to \( r \). Typically many words \( c \in C^{(a)} \) exist at such a large distance. By (21) these words are considered in the computation of \( P^\otimes(u|m) \). Thus \( P^\otimes(u|m) \) is not in the form of a peek, which gives that \( H(C\otimes|m) > 1 \). As this is a contradiction no other solution of (29) \( w \) but \( w = \hat{c}^{(a)} \) may exhibit a discriminated symbol entropy \( H(C\otimes|m) < 1 \).

Remark 18 (Typical Decoding) The proof of the theorem indicates that any code word \( c \in C^{(a)} \) with small distance to \( r \) may give rise to a well defined discriminator \( m \) with \( H(C\otimes|m) < 1 \) and \( w = c \). Hence, a low entropy solution of the equation is not equivalent to ML decoding. However, if the code rate is below capacity and a long code is employed only one distinguished word exists.

Theorem [3] gives that Algorithm [2] fails in finding the distinguished word if either the stopping criterion is never fulfilled (it runs infinitely long) or the solution exhibits a large discriminated symbol entropy. To investigate these cases consider the following Lemma.

Lemma 8 It holds that
\[
w_i \leftarrow \text{sign}(L_i^\otimes(m)) \text{ for all } i
\]
minimises the cross entropy \( H(C\otimes|w|m) \) under the constraint \( w_i \in \mathbb{B} \).

Proof: The cross entropy \( H(C\otimes|w|m) \) is given by
\[
H(C\otimes|w|m) = \sum_{i=1}^{n} \log_2(2^{w_i} + 2^{-w_i}) - w_i \cdot \tanh(L_i^\otimes(m)).
\]
The cross entropy is under constant \( |w_i| \) or \( w_i \in \mathbb{B} \) obviously minimal for
\[
\text{sign}(w_i \cdot \tanh(L_i^\otimes(m))) = 1,
\]
which is the statement of the Lemma.

The algorithm fails if the iteration does not converge. However, the lemma gives that (30) minimises in each step of the iteration the cross entropy towards \( w \). This is equivalent to
\[
H(C\otimes|m|m) \geq H(C\otimes|w|m).
\]
This cross entropy is with \( H(C\otimes|w|m) \geq \min_v H(C\otimes|v|m) = H(C\otimes|m) \) always larger than the overall discriminated symbol entropy. Furthermore, holds by the optimisation rule that
\[
H(S|w) \geq H(C\otimes|w|m) \geq H(C\otimes|m),
\]

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\[
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\]
The cross entropy is under constant \( |w_i| \) or \( w_i \in \mathbb{B} \) obviously minimal for
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\[
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\]
This cross entropy is with \( H(C\otimes|w|m) \geq \min_v H(C\otimes|v|m) = H(C\otimes|m) \) always larger than the overall discriminated symbol entropy. Furthermore, holds by the optimisation rule that
\[
H(S|w) \geq H(C\otimes|w|m) \geq H(C\otimes|m),
\]
which gives that the typical set under the discrimination remains included. The subsequent step will therefore continue to consider this set. If the discriminated cross entropy does not further decrease one thus obtains the same \( w \), which is a fixed point.

This observation is similar to the discussion above. A discriminator \( m \) describes environments with words close to \( r \). A minimisation of the cross entropy can be considered as an optimal description of this environment under the independence assumption (and the imposed hard decision constraint). If this knowledge is processed iteratively then these environments should be better and better investigated. The discriminated symbol entropy \( H(C^0 \parallel m) \) will thus typically decrease. For an infinite loop this is not fulfilled, i.e., such a loop is unlikely or non typical.

Moreover, the iterative algorithm fails if a stable solution \( w_i = \text{sign}(L_i^0(m)) \) with \( w \neq \hat{c} \) is found. These solutions exhibit with the proof of Theorem 3 large discriminated symbol entropy \( H(C^0 \parallel m) \) (many words are probable) and thus small \( |L_i^0(m)| \). However, solutions with small \( |L_i^0(m)| \) seem unlikely as these values are usually for \( w = 0 \) already relatively large and Lemma 8 indicates that these values will become larger in each step.

**Remark 19 (Improvements)** If the algorithm fails due to a well defined discriminator of large cross entropy then an appropriately chosen increase of the discriminator complexity should improve the algorithm. To increase the discrimination complexity under hard decisions one may use discriminators \( u_i^{(1)} \neq w^{(2)} \).

One possibility is hereby to reuse the old transfer vector by

\[
\begin{align*}
\hat{w}^{(2)} &= \hat{w}^{(1)} \quad \text{and} \quad \hat{w}_i^{(1)} = \text{sign}(L_i^0(m)),
\end{align*}
\]

in Step 2 of the iterative algorithm. The complexity of the algorithm will then, however, increase to \( O(n^4) \).

On the other hand the complexity can be strongly decreased without loosing the possibility to maximally discriminate the distinguished word. First holds that only (distinguished) words up to some distance \( t \) from the received word contribute to \( L_i^0(m) \). One may thus decrease (if full discrimination of the distance to \( w \) is used) the complexity \( |U| \leq t \cdot (n + 1) \) if only those values are computed.

A further reduction is obtained by the use of erasures in \( w \), i.e., by \( w_i \in \{-1, 0, +1\} \) and

\[
\begin{align*}
w_i = \frac{\text{sign}(L_i^0(m)) - r_i}{2}
\end{align*}
\]

in Step 2 of the algorithm. Note that this discrimination has only complexity \( |U| \leq t^2 \) as \( u_1(\hat{c}) \leq \hat{w}^T \hat{w} \leq t \) is typically fulfilled.

It remains to show that the distinguished solution is stable. We do this here with the informal proof: For \( \hat{c}_i = \text{sign}(L_i^0(m)) \) one obtains that \( w_i = \hat{c}_i \) if \( \hat{c}_i \neq r_i \) and 0 else. Hence one obtains that \( u_1(\hat{c}) = r \hat{c}^T \) and \( u_1(\hat{c}) = w \hat{w}^T \). If only one word \( s \) exists for these values \( u_1 \) then this discriminator \((r, w, \hat{w})\) is surely maximally discriminating. First holds that if \( u_1(\hat{c}) = w \hat{w}^T \) that then \( s_i = w_i \) for is uniquely defined \( w_i \neq 0 \). Under \( u_1(\hat{c}) \) then the other symbols are uniquely defined to \( s_i = r_i \) as \( u_1(\hat{c}) = r \hat{c}^T \) is the unique maximum of \( u_1(s) \) under \( s_i = w_i \) for \( w_i \neq 0 \).

The overall complexity of this algorithm is thus smaller than \( O(n \cdot t^2) \) respectively \( O(n \cdot t^3) \) for a discrimination with \( w^{(1)} \neq w^{(2)} \).

### 3. Approximations

The last section indicates that an iterative algorithm with discriminated symbol probabilities should outperform the iterative propagation of only the constituent beliefs. However, the discriminator approach was restricted to problems with small discriminator complexity \( |U| \).

In this form and Remark 12 the algorithm does not apply for example to AWGN channels. In this section discriminator based decoding is generalised to real valued \( w^{(1)} \) and \( r \), and hence generally \( |U| = |\mathbb{S}| \).
For a prohibitively large discriminator complexity \([U]\) the distributions \(P_{C_i}^{(1)}(x, u|m^{(l)})\) can not be practically computed; only an approximation is feasible. This approximation is usually done via a probability density, i.e.,

\[
p_{C_i}^{(l)}(x, u|m^{(l)})du = P_{C_i}^{(l)}(x, u|m^{(l)})
\]

where \(P_{C_i}^{(l)}(x, u|m^{(l)})\) is described by a small number of parameters.

**Remark 20 (Representation and Approximation)** The use of an approximation changes the premise compared to the last section. There we assumed that the representation complexity of the discriminator is limited but that the computation is perfect. In this section we assume that the discriminator is generally globally maximal but that an approximation is sufficient.

An estimation of a distribution may be performed by a histogram given by the rule

\[
\mathbb{U}_\varepsilon(u|m) := \bigcup_{|v-u|<\varepsilon} \mathbb{U}(v|m)
\]

and the quantisation \(\varepsilon\). These values can be approximated (see Appendix [A.1]) with an algorithm that exhibits a comparable complexity as the one for the computation of the hard decision values. For a sufficiently small \(\varepsilon\) one obviously obtains a sufficient approximation. Here the complexity remains of the order \(O(n^3)\). It may, however, be reduced as in Remark [19].

**Remark 21 (Uncertainty and Distance)** The approach with histograms is equivalent to assuming that words with similar \(u\) do not need to be distinguished; a discrimination of \(s^{(1)}\) and \(s^{(2)}\) is assumed to be not necessary if the “uncertainty distance”

\[
d_H(u^{(1)}, u^{(2)}) = \frac{2}{\varepsilon} \sum_{l=0}^2 \|H(s^{(1)}|w^{(l)}) - H(s^{(2)}|w^{(l)})\| = \frac{2}{\varepsilon} \|u^{(1)} - u^{(2)}\|
\]

of \(s^{(1)}\) and \(s^{(2)}\) is smaller than some \(\varepsilon\). The error that occurs by

\[
P_{C_i}^\circ(x|m) \propto \int_{\mathbb{U}} \frac{p_{C_i}^{(1)}(x, u|m^{(l)})p_{C_i}^{(2)}(x, u|m^{(l)})}{p_{C_i}(x, u|m)} du
\]

(31)

can for sufficiently small \(\varepsilon\) usually be neglected.

Note that another approach is to approximate only in \(u_0\) and continue to use a limited discriminator complexity in \(w\) (by for example hard decision \(w_i \in \mathbb{B}\)), which gives an exact discrimination in \(u_1\).

### 3.1. Gauss Discriminators

Distributions are usually represented via parameters defined by expectations. This is done as the law of large numbers shows that these expectations can be computed out of a statistics. Given these values then the unknown distributions may be approximated by maximum entropy [9] densities.

**Example 6** The simplest method to approximate distributions by probability densities is to assume that no extra knowledge is available over \(u\). This leads to the maximal entropy “distributions” (In BAYES’ estimation theory this is equivalent to a non proper prior) with stripped \(u\)

\[
P_{C_i}^{(1)}(x|m^{(l)}) \approx P_{C_i}^{(l)}(x, u|m^{(l)})\) and \(P_{C_i}(x|m) \approx P_{C_i}(x, u|m),
\]

which is equivalent to \(L_i^\circ(m) = 0\) as then \(P_{C_i}^\circ(u|x, m) = 1\) or

\[
L_i^\circ(m) = r_i + \tilde{L}_i^{(1)}(m^{(1)}) + \tilde{L}_i^{(2)}(m^{(2)})
\]

and thus implicitly to Algorithm [1]. Note that the derived tools do not give rise to a further evaluation of this approach: A discrimination in the sense defined above does not take place.
The additional expectations considered here are the mean values \( \mu_1 \) and the correlations \( \phi_{l,k} \). These are for the given correlation map
\[
u(t) = \sum_{i=1}^{n} w_i(t) s_i
\]
and \(|s_i| = 1\) defined by
\[
\mu_i^{(h)} = E_{C^{(h)}}[u_i|m^{(h)}] = \sum_{i=1}^{n} \sum_{c \in C^{(h)}} w_i^{(l)} c_i P^{(h)}(c|m^{(h)})
\]
and
\[
\phi_{l,k}^{(h)} = E_{C^{(h)}}[u_l u_k|m^{(h)}] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{e \in C^{(h)}} w_i^{(l)} w_j^{(k)} c_i c_j P^{(h)}(c|m^{(h)})
\]
and
\[
\left[\phi_{l,k}^{(h)}\right]^2 + \mu_{l,k}^{(h)} = E_{C^{(h)}}[u_l u_k|m^{(h)}] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{e \in C^{(h)}} w_i^{(l)} w_j^{(k)} c_i c_j P^{(h)}(c|m^{(h)})
\]
and
\[
\left[\phi_{l,k}^{(h)}\right]^2 = E_{C^{(h)}}[u_l u_k|m^{(h)}] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{e \in C^{(h)}} w_i^{(l)} w_j^{(k)} E_{C^{(h)}}[c_i c_j|m^{(h)}].
\]

The complexity of the computation of each value \( \mu_i^{(h)} \) and \( \phi_{l,k}^{(h)} \) is here (see Appendix A.1) comparable to the complexity of the BCJR Algorithm, i.e., for fixed trellis state complexity \( O(n) \).

For known mean values and variances the maximum entropy density is the GAUSS density. This density is with the following Lemma especially suited for discriminator based decoding.

**Lemma 9** For long codes with small trellis complexity one obtains asymptotically a GAUSS density for \( P^{(l)}(u|m^{(l)}) \) and \( P(u|m) \).

**Proof:** The values \( u_l(c) \) are obtained by the correlation given in (32). For \( P(u|m) \) this is equivalent to a sum of independent random values. I.e., \( P(u|m) \) is by the central limit theorem GAUSS distributed. For long codes with small trellis state complexity and many considered words the same holds true for \( P^{(h)}(u|m^{(h)}) \). In this case the limited code memory gives sufficiently many independent regions of subsequent code symbols. I.e., the correlation again leads to a sum of many independent random values.

**Remark 22** (Notation) The GAUSS approximated symbol probability distributions are here denoted by a hat, i.e., \( \hat{p}^{(l)}_{C^{(h)}}(x, u|m) \) and \( \hat{p}^{(l)}_{C^{(h)}}(x, u|m) \). The same is done for the approximated logarithmic symbol probability ratios.

The constituent GAUSS approximations then imply the approximation of \( P^{(h)}(u|m) \) by
\[
\hat{p}^{(h)}(u|m) \propto \frac{\hat{p}^{(2)}(u|m^{(2)}) \hat{p}^{(1)}(u|m^{(1)})}{\hat{p}(u|m)}
\]
and thus approximated discriminated symbol probabilities (for the computation see Appendix A.2)
\[
\hat{P}^{(h)}_{C^{(h)}}(x|m) \propto \int \frac{\hat{p}^{(1)}_{C^{(h)}}(x, u|m^{(1)}) \hat{p}^{(2)}_{C^{(h)}}(x, u|m^{(2)})}{\hat{p}^{(h)}_{C^{(h)}}(x, u|m)} du.
\]

This approximation is obtained via other approximations. Its quality can thus not be guaranteed as before. To use the approximated discriminated symbol probabilities in an iteration one therefore first has to check the validity of (33).
By some choice of \( w^{(1)} \) and \( w^{(2)} \) the approximations of the constituent distribution are performed in an environment \( A_{\text{ne}}(C^{(l)}|m^{(l)}) \) where \( \hat{p}(t|u|m^{(l)}) \) is large. The overall considered region is given by \( A_{\text{ne}}(S|m) \) defined by \( \hat{p}(u|m) \). This overall region represents the possible overall words. The approximation is surely valid if the possible code words of the \( l \)-th constituent code under \( m^{(l)} \) are included in this region. I.e., the conditions

\[
A_{\text{ne}}(C^{(l)}|m^{(l)}) \subseteq A_{\text{ne}}(S|m) \tag{34}
\]

for \( l = 1, 2 \) have to be fulfilled. In this case the description of the last section applies as then the approximation is typically good.

**Remark 23** That this consideration is necessary becomes clear under the assumption that the constituent GAUSS approximations do not consider the same environments. In this case their mean values strongly differ. The approximation of the discriminated distribution, however, will therefore consider regions with a large distance to the mean. The obtained results are then not predictable as a GAUSS approximation is only good for the words assumed to be probable, i.e., close to its mean value. Under (34) this can not happen.

The condition (34) is – in respect to the set sizes – fulfilled if

\[
H(S|m) \geq H(C^{(l)}|m^{(l)})
\]

as this is equivalent to

\[
A_{\text{ne}}(S|m) \supseteq A_{\text{ne}}(C^{(l)}|m^{(l)}) \tag{26}
\]

which gives with (26) that

\[
A_{\text{ne}}(S|m) \supseteq (A_{\text{ne}}(C^{(l)}|m^{(l)}) \cap C^{(l)}) \supseteq A_{\text{ne}}(C^{(l)}|m^{(l)}).
\]

However, by (34) not only the set sizes but also the words need to match. With

\[
H(C^{(l)}|m^{(l)}) = \sum_{i=0}^{n} H(C^{(l)}_i|v_i + w^{(1)}_i + w^{(2)}_i|m^{(l)})
\]

we therefore employ the symbol wise conditions

\[
H(C^{(l)}_i|v_i + w^{(1)}_i + w^{(2)}_i) \geq H(C^{(l)}_i|v_i + w^{(1)}_i + w^{(2)}_i|m^{(l)}). \tag{35}
\]

As all symbols are independently considered the conditions (34) are then typically fulfilled.

A decoding decision is again found if \( \hat{H}(C^{(l)}|m) < 1 \) under (35). To find such a solution we propose to minimise in each step \( \hat{H}(C^{(l)}|v|m) \) under the condition (35) of code \( l \).

As then (35) is fulfilled the obtained set of probable words remains in the region of common beliefs, which guarantees the validity of the subsequent approximation. This optimised \( v \) is then used to update \( w^{(l)} \) under fixed \( w^{(h)} \) and \( h \neq l \).

This gives Algorithm 3. Consider first the constrained optimisation in Step 3 of Algorithm 3. The definition of the cross entropy

\[
H(C^{(l)}_i|v_i|m^{(l)}) = \log_2(2^{v_i} + 2^{-v_i}) - v_i \tanh_2(L^{(l)}_{i}(m^{(l)}))
\]

transforms the constraint to \( v_i \tanh_2(v_i) \leq v_i \tanh_2(L^{(l)}_{i}(m^{(l)})) \), which is equivalent to

\[
|v_i| \leq |L^{(l)}_{i}(m^{(l)})| \text{ and } \text{sign}(v_i) = \text{sign}(L^{(l)}_{i}(m^{(l))}).
\]

Moreover, the optimisation \( \hat{H}(C^{(l)}|v|m) \rightarrow \min \) without constraint gives \( v_i = \hat{L}^{(l)}_{i}(m) \).
Algorithm 3 Iteration with Approximated Discrimination

1. Set \( w^{(1)} = w^{(2)} = 0 \). Set \( l = 2 \) and \( h = 1 \).
2. Swap \( l \) and \( h \). Set \( z = w^{(l)} \).
3. Set \( v \) such that
   \[
   \hat{H}(C^\otimes|v|m) \rightarrow \min
   \]
   under \( H(C_i|v_i) \geq \hat{H}(C_i^\otimes|m^{(l)}|) \) for all \( i \).
4. Set \( w^{(l)} = v - w^{(h)} - r \).
5. If \( w^{(l)} \neq z \) then go to Step 2.
6. Set \( \hat{c}_i = \text{sign}(v_i) \) for all \( i \).

This consideration directly gives the following cases:

- If this \( v_i \) does not violate the constraint then it is already optimal.
- It violates the constraint if
  \[
  \text{sign}(L_i^{(l)}(m^{(l)})) \neq \text{sign}(\hat{L}_i^\otimes(m)).
  \]
  In this case one has to set \( v_i = 0 \) to fulfil the constraint.
- For the remaining case that \( \text{sign}(L_i^{(l)}(m^{(l)})) = \text{sign}(\hat{L}_i^\otimes(m)) \) but that the constraint is violated by
  \[
  |L_i^{(l)}(m^{(l)})| < |\hat{L}_i^\otimes(m)|
  \]
  the optimal solution is
  \[
  v_i = L_i^{(l)}(m^{(l)})
  \]
  as the cross entropy \( \hat{H}(C_i^\otimes|v_i|m) \) is between \( v_i = 0 \) and \( v_i = \hat{L}_i^\otimes(m) \) a strictly monotonous function.

The obtained \( v_i \) are thus given by either \( \hat{L}_i^\otimes(m) \), \( L_i^{(l)}(m^{(l)}) \), or Zero. The zero value is hereby obtained if the two estimated symbol decisions mutually contradict each other, which is a rather intuitive result. Moreover, note that the constrained optimisation is symmetric, i.e., it is equivalent to
\[
H(C_i^{(l)}|v_i|m^{(l)}) \rightarrow \min \text{ under } H(C_i|v_i) \geq \hat{H}(C_i^\otimes|m|) \text{ for all } i.
\]

Remark 24 (Higher Order Moments) By the central limit theorem higher order moments do not significantly improve the approximation. This statement is surprising as the knowledge of all moments leads to perfect knowledge of the distribution and thus to globally maximal discrimination. However, the statement just indicates that one would need a large number of higher order moments to obtain additional useful information about the distributions.

3.1.1. Convergence

At the beginning of the algorithm many words are considered and a Gauss approximation surely suffices. I.e., in this case an approximation by histograms would not produce significantly different results. The convergence properties should thus at the beginning be comparable to an algorithm that uses a discrimination via histograms. However, there a sufficiently small \( \varepsilon \) should give good convergence properties.
At the end of the algorithm typically only few words remain to be considered. For this case the GAUSS approximation is surely outperformed by the use of histograms. Note, that this observation does not contradict the statement of Lemma 9 as we there implicitly assumed “enough” entropy. Intuitively, however, this case is simpler to solve, which implies that the GAUSS approximation should remain sufficient.

This becomes clear by reconsidering the region $\mathcal{A}_m(S|m)$ that is employed in each step of the algorithm. An algorithm that uses histograms will outperform an algorithm with a GAUSS approximation if different independent regions in $\mathcal{A}_m(S|m)$ become probable. A GAUSS approximation expects a connected region and will thus span over these regions. I.e., the error of the approximation will lead to a larger number of words that need to be considered. However, this should not have a significant impact on the convergence properties.

Typically the number of words to be considered will thus become smaller in any step: The iterative algorithm gives that in every step the discriminated cross entropy (see Definition 12) is smaller than the discriminated symbol entropy $H(C^\otimes|m)$. Hence the additionally imposed constraints seem not less restrictive than the use of histograms and $w^{(1)} = w^{(2)}$ as $w^{(1)} \neq w^{(2)}$ implies a better discrimination. However, solutions with large discriminated symbol entropy $H(C^\otimes|m)$ will even for the second case typically not exist. Moreover, the discrimination uses continuous values, which should be better than the again sufficient hard decision discrimination. I.e., the constraint should have only a small (negative) impact on the intermediate steps of the algorithm.

Usually $H(C^\otimes|m)$ is already at the start ($w^{(1)} = w^{(2)} = 0$) relatively small. The subsequent step will thus become smaller in any step: The iterative algorithm gives that in every step the discriminated cross entropy (see Definition 12)

$$\hat{H}(C^\otimes|m) = \mathcal{A}_m(S|m)$$

is smaller than the discriminated symbol entropy $\hat{H}(C^\otimes|m)$ under the assumed prior discrimination $m^{(o)}$. Hence the algorithm should converge to some fixed point.

### 3.1.2. Fixed Points

The considerations above give that the algorithm will typically not stay in an infinite loop and thus end at a fixed point. Moreover, at this fixed point the additional constraints will be fulfilled. It remains to consider whether the additional constraints introduce solutions of large discriminated symbol entropy $\hat{H}(C^\otimes|m)$.

Intuitively the additionally imposed constraints seem not less restrictive than the use of histograms and $w^{(1)} = w^{(2)}$ as $w^{(1)} \neq w^{(2)}$ implies a better discrimination. However, solutions with large discriminated symbol entropy $\hat{H}(C^\otimes|m)$ will even for the second case typically not exist. Moreover, the discrimination uses continuous values, which should be better than the again sufficient hard decision discrimination. I.e., the constraint should have only a small (negative) impact on the intermediate steps of the algorithm.

Usually $\hat{H}(C^\otimes|m)$ is already at the start ($w^{(1)} = w^{(2)} = 0$) relatively small. The subsequent step will thus become smaller in any step: The iterative algorithm gives that in every step the discriminated cross entropy (see Definition 12)

$$\hat{H}(C^\otimes|m^{(n)}|m^{(o)}) = \int_{U} \hat{p}^\otimes(u|m^{(o)}) \log_2 P(s|m^{(n)})du$$

is smaller than the discriminated symbol entropy $\hat{H}(C^\otimes|m^{(o)})$ under the assumed prior discrimination $m^{(o)}$. Hence the algorithm should converge to some fixed point.

If the process stalls for

$$\hat{H}(C^\otimes|m) > 1$$

then the by $m$ investigated region either exhibits no or multiple typical words. As (typically) the distinguished word is the only code word in the typical set and as the typical set is (typically) included this typically does not occur.

Finally, for $\hat{H}(C^\otimes|m) \approx 0$ the distinguished solution is found. At the end of the algorithm (and the assumption of a distinguished solution) the obtained GAUSS discriminated distribution then mimics a GAUSS approximation of the overall distribution, i.e.,

$$\hat{p}^\otimes(u|m) \propto \frac{\hat{p}^{(1)}(u|m^{(1)})\hat{p}^{(2)}(u|m^{(2)})}{\hat{p}(u|m)} \approx \hat{p}^{(o)}(u|r).$$

(37)

Without the constraints many solutions $m$ exist. It only has to be guaranteed that the constituent sets intersect at the distinguished word. By the addition of the constraints the solution becomes unique and is defined such that the number of by $\hat{p}(u|m)$ considered words is as small as possible.

This generally implies that then both constituent approximated distributions need to be rather similar. This is the desired behaviour as the considered environment is defined by a narrow peak of $\hat{p}^{(o)}(u|r)$ around $u(\hat{c})$. Hence, the additional constraints seem needed for a defined fixed point and the limitations of the GAUSS approximation. This emphasises the statement above: Without the constraint non predictable behaviour may occur.
Remark 25 (Optimality) The values \( w_l^{(i)} \) are continuous. Thus, one can search for the optimum of

\[
\hat{H}(C^\otimes || m) \rightarrow \min
\]

by a differentiation of \( \hat{H}(C^\otimes | m) \). For the differentiation holds

\[
2 \frac{\partial \hat{H}(C^\otimes || m)}{\partial w_l^{(i)}} = \tanh L_i(m) - \tanh \hat{L}_i^\otimes (m) - 2 \int U \frac{\partial \hat{p}_i^\otimes (u|m)}{\partial w_l^{(i)}} \log_2 P(s|m) du.
\]

For the first term see Lemma 7 and the definition of the discriminated symbol probabilities. The second term is the derivation of the discriminated probability density. For the case of a maximal discrimination of the distinguished word it will consist of this word with probability of almost one. A differential variation of the discriminator should remain maximally discriminating, which gives that the second term should be almost zero. Hence, one obtains that

\[
L_i(m) \approx \hat{L}_i^\otimes (m)
\]

holds at the absolute minimum of \( \hat{H}(C^\otimes || m) \), which is a (soft decision) well defined discriminator. Note, furthermore, that for (37) and similar constituent distributions the distribution \( \hat{p}(u|m) \) will necessarily be similar to \( \hat{p}^{(a)}(u|m) \), which is a similar statement as in (38).

Remark 26 (Complexity) The decoding complexity is under the assumption of fast convergence of the order \( O(n) \). I.e., the complexity only depends on the BCJR decoding complexity of the constituent codes. Moreover, Algorithm 3 can still be considered as an algorithm where parameters are transferred between the codes. Hereby the number of parameters is increased by a factor of nineteen (for each \( i \) additionally to \( w_l^{(i)} \) for \( x = \pm 1 \) three means and six correlations).

Note, finally, that the original iterative (constituent) belief propagation algorithm is rather close to the proposed algorithm. Only by (36) an additional constraint is introduced. Without the constraint apparently too strong beliefs are transmitted. Algorithm 3 cuts off excess constituent code belief.

3.2. Multiple Coupling

Dually coupled codes constructed by just two constituent codes (with simple trellises) are not necessarily good codes. This can be understood by the necessity of simple constituent trellises. This gives that the left-right (minimal row span) \([11]\) forms of the (permuted) parity check matrices have short effective lengths. This gives that the codes cannot be considered as purely random as this condition strongly limits the choice of codes. However, to obtain asymptotically good codes one generally needs that the codes can be considered as random.

If – as in Remark 3 – more constituent codes are considered, then the dual codes will have smaller rate and thus a larger effective length. This is best understood in the limit, i.e., the case of \( n - k = 1 \). These codes can then be freely chosen without changing the complexity, which leaves no restriction on the choice of the overall code.

For a setup of a dual coupling with \( N \) codes the discriminated distribution of correlations is generalised to

\[
P^\otimes (u|m) \propto \prod_{l=1}^{N} \frac{P_l(u|m^{(l)})}{P_l(u|m)}^{N-1},
\]

with

\[
m = (r, w^{(1)}, \ldots, w^{(N)}), m^{(l)} = (r, w^{(1)}, \ldots, w^{(l-1)}, w^{(l+1)}, \ldots, w^{(N)}),
\]

and an independence assumption as in (11) and (5).
The definition of the discriminated symbol probabilities then becomes

\[ P_{C_i}^\otimes(x|m) \propto \sum_u \prod_{l=1}^N P_{C_i}(x,u|m^{(l)}) \left( P_{C_i}(x,u|m) \right)^{N-1}. \]

Moreover, for globally maximal discriminators

\[ P_{C_i}^\otimes(x|m) = P_{C_i}(x|a) \quad \text{and} \quad P_{u|m} = P_u(a) \]

remains true. The others lemmas and theorems above can be likewise generalised. Hence, discriminator decoding by GAUSS approximations applies to multiple dually coupled codes, too.

**Remark 27 (Iterative Algorithm)** The generalisation of Algorithm 3 may be done by using

\[ v_i = \arg \min_v H(C_i^\otimes|v_i|m) \quad \text{under} \quad H(C_i|v_i) \geq H(C_i^{(l)}|v_i|m^{(l)}) \quad \text{for all} \quad i \]

\[ w^{(l)} \leftarrow v - \sum_{h \neq l} w^{(l)} \]

as constituent code dependent update.

Overall this gives – provided the distinguished well defined solution is found – that discriminator decoding asymptotically performs as typical decoding for a random code. I.e., with dually coupled codes and (to the distinguished solution convergent) GAUSS approximated discriminator decoding the capacity is attained.

**Remark 28 (Complexity)** The complexity of decoding is of the order of the sum of the constituent trellis complexities and thus generally increases with the number of codes employed. For a fixed number of constituent codes of fixed trellis state complexity and GAUSS approximated discriminators the complexity thus remains of the order \( O(n) \).

**Remark 29 (Number of Solutions)** For a coupling with many constituent codes one obtains a large number of non linear optimisations that have to be performed simultaneously. The non linearity of the common problem should thus increase with the number of codes. Another explanation is that then many times typicality is assumed. The probability of some non typical event then increases. This may increase the number of stable solutions of the algorithm or introduce instability. This behaviour may be mitigated by the use of punctured codes. The punctured positions define beliefs, too, which gives that the transfer vector \( w \) is generally longer than \( n \). The transfer complexity is thus increased, which should lead to better performance. Note that this approach is implicitly used for LDPC codes.

### 3.3. Channel Maps

In the last sections only memory-less channel maps as given in Remark 4 were considered. A general channel is given by a stochastic map

\[ \mathcal{K} : S \to R \text{ defined by } P_{R|S}(r|s). \]

We will here only consider channels where signal and “noise” are independent. In particular we assume that the channel \( \mathcal{K} \) is given by some known deterministic map

\[ \mathcal{H} : s \mapsto v = (v_1, \ldots, v_n) \]

and \( r = v + e \) with the additive noise \( E \) defined by \( P_E(e) \).
A code map $C$ prior to the transmission together with the map $H$ may then be considered as a concatenated map. The concatenation is hereby (for the formal representation by dually coupled code see the proof of Theorem 1) equally represented by the dual coupling of the “codes”

$$C^{(1)} := \{c^{(1)} = (e, z) : c \in C\} \text{ and } C^{(2)} := \{c^{(2)} = (s, v) : s \in \mathbb{S} \text{ and } H : s \mapsto v\}$$

where $z = (z_1, \ldots, z_n)$ is undefined, i.e., no restriction is imposed on $z$. Moreover, $e$ is punctured prior to transmission and only $v + e$ is received. Discriminator based decoding thus applies and one obtains

$$P_{C_i}(x|m) \propto \sum_{w \in \mathbb{U}} \frac{P_{C_i}(x, u|w(2))P_{C_i}(x, u|r, w(1))}{P_{C_i}(x, u|w(1), w(2))}$$

as by the definition of the dually coupled code

$$P_{C_i}(x, u|m^{(1)}) = P_{C_i}(x, u|w^{(2)})P(u|r)$$

and by the independence assumption $P_{C_i}(x, u|m) = P_{C_i}(x, u|w^{(1)}, w^{(2)})P(u|r)$ are independent of the channel.

**Remark 30 (Trellis)** If a trellis algorithm exists to compute $P_{C_i}(x|r)$ then one may compute the symbol probabilities $P_{C_i}(x|r, w^{(1)})$, the mean values and variances of $u$ under $P_{C_i}(x, u|r, w^{(1)})$ with similar complexity.

**Example 7** A linear time invariant channel with additive white GAUSSian noise $E(t)$ is given by the map

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} g(t - \tau) h(\tau) d\tau + e(t).$$

Here, we assume a description in the equivalent base band. I.e., the signals $r(t)$ and $g(t)$ as well as the noise may be complex valued – indicated by the underbar. The noise is assumed to be white and thus exhibits the (stationary) correlation function $E[E(t)E^*(t + \tau)] = \sigma_E^2 \delta(\tau)$.

For amplitude shift keying modulation one employs the signal

$$g(t) = \sum_{i=-\infty}^{\infty} s_i w(t - iT) \text{ with } w(\tau) \text{ being the waveformer.}$$

With a matched filter and well chosen whitening filter one obtains an equivalent (generally complex valued) discrete channel

$$Q : s \mapsto r \text{ with } r_i = \sum_{j=0}^{M} s_{i-j} q_j + e_i$$

defined by $q = (q_0, \ldots, q_M)$ and independent GAUSS noise $E[E[e_i e_j^*]] = \sigma_E^2 \delta_{i-j}$.

For binary phase shift keying one has $s_i = Ax_i$ and $x_i \in \mathbb{B}$.

For quaternary phase shift keying the map is given by

$$s_i = \frac{A}{\sqrt{2}}(x_{2i} + jx_{2i+1}),$$

for $j^2 = -1$, and $x_i \in \mathbb{B}$. In both cases a trellis for $S$ may be constructed with logarithmic complexity proportional to the memory $M$ of the channel $q = (q_0, q_1, \ldots, q_M)$ times the number of information Bits per channel symbol $S_i$. Note, moreover, that a time variance of the channel does not change the trellis complexity.
3.4. Channel Detached Discrimination

Overall one obtains for a linear modulation and linear channels with additive noise the discrete probabilistic channel map

\[ K : s \mapsto r = sQ + e. \]

For uncorrelated Gaussian noise \( E \) this gives the probabilities (without prior knowledge about the code words)

\[ P(c|r) \propto \exp_2(-\frac{\log_2(e)}{2\sigma^2_E} ||r - cQ||^2). \]

If the channel has large memory \( M \) and/or if a modulation scheme with many Bits per symbol \( s_i \) is used then the trellis complexity of a trellis equalisation becomes prohibitively large. To use the channel map as a constituent code will then not give a practical algorithm.

Reconsider therefore the computation of the channel moments, i.e., the moments de-

\[ \text{compute} \left\{ E \left[ Q(x) \right] \right\}_{x \in \mathcal{X}}. \]

This is similar to the computation of the variance. Generally holds that the moments and correlations can be computed for linear channels with complexity that increases only linearly with the channel memory \( M \). This result follows as the expectations for the channels remain computations of moments, but now with vector operations. The computation of the variance of \( u_0 \) (for a channel with memory) is, e.g., equivalent to the computation of a fourth moments in the independent case.
Generally holds that the mean values $E_{\hat{C}_i}^{(l)}[u_0|x, w^{(h)}]$ are only computable up to a constant. This is under a GAUSS assumption and (41) equivalent to a shift of $u_0$ in $\exp_2(u_0)$ by this constant. However, this will lead to a proportional factor, which vanishes in the computation of $P_{\hat{C}_i}^{(l)}(x|m)$. This unknown constant may thus be disregarded.

**Remark 31** *(Constituent Code)* This approach applies by

$$P_{\hat{C}_i}^{(l)}(x|m^{(l)}) \propto \sum_{u \in \mathbb{U}} P_{\hat{C}_i}^{(l)}(x, u|w^{(h)}) \exp_2(u_0) \quad \text{for} \quad l \neq h$$

to the constituent codes $C^{(l)}$, too: One may likewise compute the constituent beliefs via the moments and a GAUSS approximation and thus apply Algorithm 3.

The GAUSS approximation for $u_0$ surely holds true if the channel is short compared to the overall length as then many independent parts contribute. With (41) one can thus apply the iterative decoder based on GAUSS approximated discriminators for linear channels with memory without much extra complexity.

**Remark 32** *(Matched Filter)* Note that one obtains by (42) for the initialisation $w^{(l)} = 0$, $l = 1, 2$ that $L^u(m)$ is proportional to the “matched filter output” given by $q_r^H$. Moreover, in all steps of the algorithm only $L^u(m)$ is directly affected by the channel map.

### 3.5. Estimation

In many cases the transmission channel is unknown at the receiver. This problem is usually mitigated by a channel estimation prior to the decoding. However, an independent estimation needs – especially for time varying channels [7] – considerable excess redundancy. The optimal approach would be to perform decoding, estimation, and equalisation simultaneously.

**Example 8** Assume that it is known that the channel is given as in (39), but that the channel parameters $q = (q_0, \ldots, q_L)$ are unknown. Moreover, assume that the transmission is in the base band, which gives that the $q_i$ are real valued. The aim is to determine these values together with the code symbol decisions. To consider them in the same way, i.e., by decisions one needs to reduce the (infinite) description entropy. We therefore assume a quantisation of $q$ by a binary vector $b$. This may, e.g., be done by

$$q_i = q_i \sum_{j=0}^{B_i-1} b_{i(j)+j} \exp_2(j), \quad l(i) = l(i-1) + B_{i-1}, \quad l(0) = 0, \quad \text{and} \quad b_i \in \mathbb{B}.$$  

Note that one uses the additional knowledge $|q_i| < q \exp_2(B_i)$ under this quantisation. Moreover, the quantisation error tends to zero with the quantisation step size $q$. Finally, surely a better quantisation can be found via rate distortion theory.

The example shows that one obtains with an appropriate quantisation additional binary unknowns $b_j$. Thus one needs additional parameters $w_{n+j}^{(l)}$ that discriminate these Bits. Moreover, again a probability distribution is needed for these $w_{n+j}^{(l)}$. Here it is assumed that the distribution given in (5) is just extended to these parameters. Note that this is equivalent to assuming that code Bits $c_i$ and “channel Bits” $b_j$ are independent.

The code symbol discriminated probabilities remain under the now longer $w$ as in (41). Additionally one obtains discriminated channel symbol probabilities given by

$$P_{\hat{B}_i}^{(l)}(x|m) \propto \sum_{u \in \mathbb{U}} P_{\hat{B}_i}^{(l)}(x, u|w^{(2)}) P_{\hat{B}_i}^{(l)}(x, u|w^{(1)}) \exp_2(u_0).$$
A GAUSS approximated discrimination is thus as before, however, one needs to compute new and more general expectations. E.g., for the general linear channel of (40) one needs to compute the expectation given by

$$E_{C_i}^{(l)}[u_0|x, w^{(h)}] = \text{const} - \frac{\log_2(e)}{2\sigma_E^2} E_{C_i}^{(l)}[||r - cQ(b)||^2 | x, w^{(h)}]$$

and equivalently for $E_{B_i}^{(l)}[u_0|x, w^{(h)}]$.

The expectations are generalised because $Q$ is a map of the random variables $b_j$. With the quantisation of the example above this map is linear in $b$. This first gives that $cQ(b)$ can be considered as a quadratic function in the binary random variables $x$ and $b$. The computation of the means is thus akin to the one of fourth moments and a known independent channel.

Overall this gives that the complexity for the computation of the means and variances is for unknown channels “only” twice as large as for a known channel (of the same memory). It may, however, still be computed with reasonable complexity. Hence, again an iteration based on GAUSS approximated discriminators can be performed.

**Remark 33 (Miscellaneous)** Note that without some known “training” sequence in the code word the iteration will by the symmetry usually stay at $w^{(l)} = 0$. Note, moreover, that this approach is easily extended to time variant channels as considered in [7] or even to more complex, i.e., non linear channel maps. The complexity then remains dominated by the complexity of the computation of the means and correlations.

### 4. Summary

In this paper first (dually) coupled codes were discussed. A dually coupled code is given by a juxtaposition of the constituent parity check matrices. Dually coupled codes provide a straightforward albeit prohibitively complex computation of the overall word probabilities $P^{(s)}(s|r)$ by the constituent probabilities $P^{(l)}(s|r)$. However, for these codes a decoding by belief propagation applies.

The then introduced concept of discriminators is summarised by augmenting the probabilities by additional (virtual) parameters $w^{(l)}$ and $u$ to $P(s, u|r, w^{(1)}, w^{(2)})$. This is similar to the procedure used for belief propagation but there the parameter $u$ is not considered. Such carefully chosen probabilities led (in a globally maximal form) again to optimum decoding decisions of the coupled code. However, the complexity of decoding with globally maximal discriminators remains in the order of a brute force computation of the ML decisions.

It was then shown that local discriminators may perform almost optimally but with much smaller complexity. This observation then gave rise to the definition of well defined discriminators and therewith again an iteration rule. It was then shown that this iteration theoretically admits any element of the typical set of the decoding problem as fixed point.

In the last chapter the central limit theorem then led to a GAUSS approximation and a low complexity decoder. Finally (linear) channel maps with memory were considered. It was shown that under additional approximations equalisation and estimation may be accommodated into the iterative algorithm with only little impact on the complexity.
A. Appendix

A.1. Trellis Based Algorithms

The trellis is a layered graph representation of the code space \( \mathbb{E}(C) \) such that every code word \( c = (c_1, \ldots, c_n) \) corresponds to a unique path through the trellis from left to right. For a binary code every layer of edges is labelled by one code symbol \( c_i \in \mathbb{Z}_2 = \{0, 1\} \). The complexity of the trellis is generally given by the maximum number of edges per layer.

As example the trellis of a “single parity check” code of length 5 with \( H = (11111) \) is depicted in the figure to the right. Each of the \( 2^5 \) paths in the trellis defines \( c_1 \) to \( c_5 \) of a code word \( c \) of even weight.

Here only the basic ideas needed to perform the computations in the trellis are presented. A formal description will be given in another paper [8]. The description here reflects the operations performed in the trellis. I.e., only the lengthening (extending one path) and the junction (combining two incoming paths of one trellis node) are considered.

This is first explained for the Viterbi [5] algorithm that finds the code word with minimal distance. The “lengthening” is given by an addition of the path correlations as depicted in Figure 4 (a). For the combination – the “join” operation – only the path of maximum value is kept. This is equivalent to a minimisation operation for the distances. This is reflected in the name of the algorithm, which is often called min-sum algorithm.

On the other hand, the BCJR [1] algorithm (to compute \( P(c_i | x, r) \)) is often called sum-product algorithm as the lengthening is performed by the product of the path probabilities. The combination of two paths is given by a sum. These operations are summarised in Figure 4 (b).

\[
\begin{align*}
X_s & \quad x_i, r_i \quad X_s + x_i \cdot r_i = X_e^{(L)} \\
X_s^{(1)} & \quad X_s^{(2)} \quad \text{max}(X_s^{(1)}, X_s^{(2)}) = X_e^{(L)} \\
X_s & \quad P_s \quad P_s \cdot P(x_i | r_i) = P_e^{(L)} \\
P_s^{(1)} + P_s^{(2)} = P_e^{(J)} \\
P_s^{(1)} & \quad P_s^{(2)} \quad P_e^{(L)}
\end{align*}
\]

(a) Viterbi Algorithm  (b) BCJR Algorithm

Figure 4: Basic Operations in the Viterbi and BCJR Algorithms

Remark 34 (Forward-Backward Algorithm) For the Viterbi algorithm the ML code word is found by following the selected paths (starting from the end node) in backward direction.

The operations of the BCJR algorithm (in forward direction) give at the end directly the probabilities \( P(c_i | x, r) \). To compute all \( P(c_i | x, r) \) the BCJR algorithm has be performed into both directions.

The same holds true for the algorithms below. This is here not considered any further – but keep in mind that only by this two way approach symbol based distributions or moments can be computed with low complexity.

In the following we shall reuse the notation of Figure 4 and use the indexes \( s \) and \( e \) before respectively after the lengthen or join operation.
A.1.1. Discrete Sets

To compute a hard decision distribution one can just count the number of words of a certain distance to \( r \). Let this number be denoted \( D(t) \) for weight \( t \in \mathbb{Z} \).

This can also be done directly in the trellis by using for the lengthening operation from \( D_s(t) \) to \( D_c^{(L)}(t) \) by

\[
D_c^{(L)}(t) = \begin{cases} 
D_s(t-1) & \text{for } c_i \neq r_i \\
D_s(t) & \text{for } c_i = r_i.
\end{cases}
\]

The junction of paths becomes just

\[
D_c^{(J)}(t) = D_c^{(1)}(t) + D_c^{(2)}(t).
\]

Given \( D(t) \) and a BSC with error probability \( p \) one may compute the probability of having words of distance \( t \) by

\[
P(t|r) \propto D(t) \cdot p^t(1-p)^{n-t}.
\]

This can also be done directly in the trellis by

\[
p_c^{(J)}(t) = p_c^{(1)}(t) + p_c^{(2)}(t) \quad \text{and} \quad p_c^{(L)}(t) = \begin{cases} 
p \cdot p_s(t-1) & \text{for } c_i \neq r_i \\
(1-p) \cdot p_s(t) & \text{for } c_i = r_i.
\end{cases}
\]

A.1.2. Moments

For the mean value

\[
\mu = E[re^T] = \sum_{i=1}^{n} E[r_i c_i]\]

holds

\[
E[\sum_{j=1}^{i} c_j r_j | c_i] = E[\sum_{j=1}^{i-1} c_j r_j] + c_i r_i.
\]

This directly gives that one obtains for the lengthening

\[
P_c^{(L)} = P_s \cdot 2^r c_i \quad \text{and} \quad \mu_c^{(L)} = \mu_s + r_i c_i.
\]

The junction is just the probability weighted sum of the prior computed input means given by

\[
P_c^{(J)} = P_c^{(1)} + P_c^{(2)} \quad \text{and} \quad \mu_c^{(J)} = \frac{P_c^{(1)} \mu_s^{(1)}}{P_c^{(L)}} + \frac{P_c^{(2)} \mu_s^{(2)}}{P_c^{(L)}}.
\]

Hence, the BCJR algorithm for the probabilities needs to be computed at the same time. Note that the obtained mean values are then readily normalised.

To compute the “energies” \( S = E[(\sum_{j=1}^{i} c_j r_j)^2] \) one uses in the same way that

\[
S = E[(\sum_{j=1}^{i} c_j r_j)^2 | c_i] = E[\sum_{j=1}^{i-1} c_j r_j]^2] + 2 c_i r_i \cdot E[\sum_{j=1}^{i-1} c_j r_j] + (c_i r_i)^2.
\]

This additionally gives – to the then necessary computation of means and probabilities – that lengthening and junction are now given by

\[
S_c^{(L)} = S_s + 2 r_i c_i \cdot \mu_s + (r_i c_i)^2 \quad \text{and} \quad S_c^{(J)} = \frac{P_s^{(1)} S_s^{(1)}}{P_c^{(L)}} + \frac{P_s^{(2)} S_s^{(2)}}{P_c^{(L)}}.
\]

Here again the normalisation is already included. Correlation and higher order moment trellis computations are derived in the same way. However, for an \( l \)-th moment all \( l-1 \) lower moments and the probability need to be additionally computed. Moreover, the description gives that these moment computations may be performed likewise for any linear operation \( eQ \) (defined over the field of real or complex numbers) then using vector operations.
A.1.3. Continuous Sets

Another possibility to use the trellis is to compute (approximated) histograms for \( u = wc^T \) with \( w_i \in \mathbb{R} \) and \( c_i \in \mathbb{B} \). It is here proposed (other possibilities surely exist) to use – as in the hard decision case above – a vector function \((h(t), \mu)\) with \( t \in \mathbb{Z} \) and \(|t| \leq Q\) and the mean value \( \mu \). I.e., the values of \( u \) with non-vanishing probability are assumed to be in a vicinity the mean value \( \mu \) (computed above) or

\[
p(u|m) = 0 \quad \text{for} \quad |u - \mu| > Q\varepsilon.
\]

Thus \((h(t), \mu)\) is defined to be the approximation of

\[
h(t) \approx \int_{t \varepsilon}^{(t+1)\varepsilon} p(u - \mu|m)du.
\]

Here, densities are used to simplify the notation. It is now assumed that the mean values are computed as above, which gives that the lengthening is the trivial operation

\[
(h^{(L)}(t), \mu^{(L)}) = (h_s(t), \mu_s + c_i w_i).
\]

The junction, however, cannot be easily performed as usually the mean values do not fit on each other. Here, it is assumed that the density has for any interval the form of a rectangle. Note that this is again a maximum entropy assumption.

This gives the approximation of the histogram \( h^{(j)}(t) \) by the junction operation to be

\[
(h^{(j)}(t), \mu^{(j)}) = \left( \frac{P_s^{(1)}}{P_c^{(L)}} h_s^{(1)}(t) + \frac{P_s^{(2)}}{P_c^{(L)}} h_s^{(2)}(t), \frac{P_s^{(1)}}{P_c^{(L)}} \mu_s^{(1)}(t) + \frac{P_s^{(2)}}{P_c^{(L)}} \mu_s^{(2)} \right).
\]

and

\[
h_s^{(j)}(t - \left\lfloor \left[ \frac{\mu_s^{(j)} - \mu_c^{(L)} \varepsilon} \right] \right) = a(\mu_s^{(j)}, \mu_c^{(L)}) \cdot h_s^{(j)}(t) + b(\mu_s^{(j)}, \mu_c^{(L)}) \cdot h_s^{(j)}(t + 1),
\]

with \([z]\) the integer part, \(\text{trunc}(z) := z - \lfloor z \rfloor\),

\[
a(\mu_s^{(j)}, \mu_c^{(L)}) + b(\mu_s^{(j)}, \mu_c^{(L)}) = 1, \quad \text{and} \quad b(\mu_s^{(j)}, \mu_c^{(L)}) = \text{trunc}((\mu_s^{(j)} - \mu_c^{(L)})\varepsilon).
\]

A.2. Computation of \( \hat{L}_i^\oplus(m) \)

Equation (44) gives the logarithmic probability ratio

\[
\hat{L}_i^\oplus(m) = r_i + \hat{L}_i^{(1)}(m^{(1)}) + \hat{L}_i^{(2)}(m^{(1)}) + \hat{L}_i^\ominus(m).
\]

The first three terms can be computed as before. For the computation of \( \hat{L}_i^\ominus(m) \) use that

\[
\hat{P}_{C_i}(x|m) \propto \int \frac{\hat{p}_C^{(1)}(u|x, m^{(1)}) \cdot \hat{p}_C^{(2)}(u|x, m^{(2)})}{\hat{p}_C(u|x, m)} du =: \int \hat{p}_C^\ominus(x, u|m) du. \tag{43}
\]

To compute (43) a multiplication of multivariate Gauss distributions has to be performed. The moments of the multivariate distributions \( \hat{p}_C^{(1)}(u|x, m^{(l)}) \) and \( \hat{p}_C(u|x, m) \) are defined by

\[
\mu_{i,j}^{(l)}(x) = E_{C^{(l)}|C_i}[u_j|x, m^{(l)}] \quad \text{and} \quad A_{i,j,k}^{(l)}(x) = E_{C^{(l)}|C_i}[(u_j - \mu_{i,j}^{(l)})(u_k - \mu_{i,k}^{(l)})|x, m^{(l)}]
\]

and likewise for \( \mu_{i,j}(x) \) and \( A_{i,j,k}(x) \).

The multivariate GAUSS distributions are of the form

\[
\hat{p}_C(u|x, m) = \frac{1}{\sqrt{[2\pi A_i(x)]}} \exp \left(- (u - \mu_i(x))^T [2A_i(x)]^{-1} (u - \mu_i(x)) \right).
\]
Set
\[ B_i^{(1)}(x) = \left[ A_i^{(l)}(x) \right]^{-1} \quad \text{and} \quad B_i(x) = [A_i(x)]^{-1}. \]

The operation in (43) then leads to
\[
\hat{p}_i^\Theta(x, u|m) = \frac{\exp \left( \hat{C}_i^\Theta(x, m) - (u - \hat{\mu}_i^\Theta(x)) \left[ 2 \hat{A}_i^\Theta(x) \right]^{-1} (u - \hat{\mu}_i^\Theta(x))^T \right)}{\sqrt{2\pi \hat{A}_i^\Theta(x)}}
\]

with
\[
\left[ \hat{A}_i^\Theta(x) \right]^{-1} = B_i^{(1)}(x) + B_i^{(2)}(x) - B_i(x)
\]
by a comparison of the terms \( u(.) u^T \),
\[
\hat{\mu}_i^\Theta(x) = \left( \mu_i^{(1)}(x) B_i^{(1)}(x) + \mu_i^{(2)}(x) B_i^{(2)}(x) - \mu_i(x) B_i(x) \right) \hat{A}_i^\Theta(x),
\]
by a comparison of the in \( u \) linear terms, and
\[
2\hat{C}_i^\Theta(x, m) = \hat{\mu}_i^\Theta(x) \left[ \hat{A}_i^\Theta(x) \right]^{-1} \hat{\mu}_i^{\Theta T}(x) + \mu_i(x) B_i(x) \mu_i^T(x) - \mu_i^{(1)}(x) B_i^{(1)}(x) \mu_i^{(1) T}(x) - \mu_i^{(2)}(x) B_i^{(2)}(x) \mu_i^{(2) T}(x) - \log \frac{|A_i^{(1)}(x)||A_i^{(2)}(x)|}{|A_i^\Theta(x)||A_i(x)|}
\]
by a consideration of the remaining constant.

From the definition of the multivariate distributions then follows that
\[
\hat{P}_i^\Theta(x|m) \propto \int \hat{p}_i^\Theta(x, u|m) du = \exp(\hat{C}_i^\Theta(x|m)),
\]
respectively \( \hat{L}_i^\Theta(m) = \frac{1}{2} \log_2(e) \cdot (\hat{C}_i^\Theta(+1|m) - \hat{C}_i^\Theta(-1|m)) \).

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