ON $\mu$-SCALE INVARIANT OPERATORS

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Dedicated to the memory of M. Krein on the occasion of his one hundredth birthday anniversary

ABSTRACT. We introduce the concept of a $\mu$-scale invariant operator with respect to a unitary transformation in a separable complex Hilbert space. We show that if a nonnegative densely defined symmetric operator is $\mu$-scale invariant for some $\mu > 0$, then both the Friedrichs and the Krein-von Neumann extensions of this operator are also $\mu$-scale invariant.

1. INTRODUCTION

Given a unitary operator $U$ in a separable complex Hilbert space $H$ and a (complex) number $\mu \in \mathbb{C} \setminus \{0\}$, we introduce the concept of a $\mu$-scale invariant operator $T$ (with respect to the transformation $U$) as a (bounded) “solution” of the following equation

$$UTU^* = \mu T.$$  \hspace{1cm} (1.1)

Note, that in this case $U$ and $T$ commute up to a factor, that is,

$$UT = \mu TU,$$  \hspace{1cm} (1.2)

and then necessarily $|\mu| = 1$ (see [6]), provided that $T$ is a bounded operator and $\text{spec}(UT) \neq \{0\}$.

The search for pairs of unitaries $U$ and $T$ satisfying the canonical (Heisenberg) commutation relations (1.2) with $|\mu| = 1$ leads to realizations of the rotation algebra, the $C^*$-algebra generated by the monomials $T^mU^n$, $m, n \in \mathbb{Z}$ (see, e.g., [15]). The irreducible representations of this algebra play a crucial role in the study of the Hofstadter type models. For instance, the Hofstadter Hamiltonian $H = T + T^* + U + U^*$ typically has fractal spectrum that is rather sensitive to the algebraic properties of the “magnetic flux” $\theta$, $\mu = e^{i\theta}$, which is captured in the beauty of the famous Hofstadter butterfly (see [15] and references therein). We also note that self-adjoint realizations $U$ and $T$ of commutation relations (1.1) or (1.2) for $|\mu| = 1$ are obtained in [6] while the case of contractive (not necessarily self-adjoint) solutions $T$, and unitary $U$, has been discussed in [14].

To incorporate the case of $|\mu| \neq 1$, where unbounded solutions to (1.1) are of necessity considered, we extend the concept of the $\mu$-scale invariance to the case of unbounded operators $T$ by the requirement that $\text{Dom}(T)$ is invariant, that is,

$$U^*\text{Dom}(T) \subseteq \text{Dom}(T),$$  \hspace{1cm} (1.3)

and

$$UTU^* f = \mu T f \quad \text{for all } f \in \text{Dom}(T).$$  \hspace{1cm} (1.4)

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In this short Note we restrict ourselves to the case \( \mu > 0 \) and focus on the study of symmetric as well as self-adjoint unbounded solutions \( T \) of (1.3) and (1.4). Our main result (see Theorem 2.2) states that if a densely defined nonnegative (symmetric) operator \( T \) is \( \mu \)-scale invariant with respect to a unitary transformation \( U \), then the two classical extremal nonnegative self-adjoint extensions, the Friedrichs and the Krein-von Neumann extensions, are \( \mu \)-scale invariant as well.

The paper is organized as follows: In Section 2, based on a result by Ando and Nishio [3], we provide the proof of Theorem 2.2. Section 3 is devoted to further generalizations and a discussion of the \( \mu \)-scale invariance concept from the standpoint of group representation theory.

2. MAIN RESULT

Recall that if \( \dot{A} \) is a densely defined (closed) nonnegative operator, then the set of all nonnegative self-adjoint extensions of \( \dot{A} \) has the minimal element \( A_K \), the Krein-von Neumann extension (different authors refer to the minimal extension \( A_K \) by using different names, see, e.g., [2], [3], [4], [5]), and the maximal one \( A_F \), the Friedrichs extension. This means, in particular, that for any nonnegative self-adjoint extension \( \tilde{A} \) of \( \dot{A} \) the following operator inequality holds [11]

\[
(A_F + \lambda I)^{-1} \leq (\tilde{A} + \lambda I)^{-1} \leq (A_K + \lambda I)^{-1},
\]

for all \( \lambda > 0 \).

The following result characterizes the Friedrichs and the Krein-von Neumann extensions a form convenient for our considerations.

**Theorem 2.1** ([1], [3]). Let \( \dot{A} \) be a (closed) densely defined nonnegative symmetric operator. Denote by \( a \) the closure\(^1\) of the quadratic form

\[
\dot{a}[f] = (\dot{A}f, f), \quad \text{Dom}(\dot{a}) = \text{Dom}(\dot{A}).
\]

Then,

(i) the Friedrichs extension \( A_F \) of \( \dot{A} \) coincides with the restriction of the adjoint operator \( \dot{A}^* \) on the domain

\[
\text{Dom}(A_F) = \text{Dom}(\dot{A}^*) \cap \text{Dom}(\dot{a});
\]

(ii) the Krein-von Neumann extension \( A_K \) of \( \dot{A} \) coincides with the restriction of the adjoint operator \( \dot{A}^* \) on the domain \( \text{Dom}(A_K) \) which consists of the set of elements \( f \) for which there exists a sequence \( \{f_n\}_{n \in \mathbb{N}}, f_n \in \text{Dom}(\dot{A}) \), such that

\[
\lim_{n,m \to \infty} \dot{a}[f_n - f_m] = 0 \quad \text{and} \quad \lim_{n \to \infty} \dot{A}f_n = \dot{A}^* f.
\]

We now state the main result of this Note:

**Theorem 2.2.** Assume that \( \mu > 0 \) and that a densely defined (closed) nonnegative symmetric operator \( \dot{A} \) is \( \mu \)-scale invariant with respect to a unitary transformation \( U \); that is,

\[
U^* \text{Dom}(\dot{A}) \subseteq \text{Dom}(\dot{A})
\]

and that

\[
U \dot{A} U^* = \mu \dot{A} \quad \text{on Dom}(\dot{A}).
\]

Then

(i) the adjoint operator \( \dot{A}^* \),
(ii) the Friedrichs extension $A_F$ of $\dot{A}$, and
(iii) the Krein-von Neumann extension $A_K$ of $\dot{A}$
are $\mu$-scale invariant with respect to the unitary transformation $U$.

Proof. Clearly, it is sufficient to prove (i) followed by the proof of the fact that the domains of both the Friedrichs and the Krein-von Neumann extensions are invariant with respect to the operator $U^*$.

(i). Given $f \in \text{Dom}(\dot{A})$ and $h \in \text{Dom}(\dot{A}^*)$, one obtains
\[
(\dot{A}f, U^*h) = (U\dot{A}f, h) = (U\dot{A}U^*Uf, h) = (\mu\dot{A}Uf, h) = (f, U^*\mu\dot{A}^*h),
\]
thereby proving the inclusion $U^*\text{Dom}(\dot{A}^*) \subseteq \text{Dom}(\dot{A}^*)$ as well as the equality
\[
(2.2)\quad \dot{A}^*U^*h = \mu U^*\dot{A}^*h, \quad h \in \text{Dom}(\dot{A}).
\]
The proof of (i) is complete.

(ii). First we show that the domain of the closure of the quadratic form $(2.1)$ is invariant with respect to the operator $U^*$.

Recall that $f \in \text{Dom}[a]$ if and only if there exists a sequence $\{f_n\}_{n \in \mathbb{N}}, f_n \in \text{Dom}(\dot{A})$, such that
\[
\lim_{n,m \to \infty} a[f_n - f_m] = 0 \quad \text{and} \quad \lim_{n \to \infty} f_n = f.
\]
Take an $f \in \text{Dom}[a]$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ satisfying the properties above. Clearly
\[
(2.3)\quad \lim_{n \to \infty} U^*f_n = U^*f,
\]
with $U^*f_n \in \text{Dom}(\dot{A})$. Moreover,
\[
a[U^*f_n - U^*f_m] = (\dot{A}U^*f_n - U^*f_m) = (\dot{A}A^*U^*)U(f_n - f_m) = (\mu \dot{A}U^*)U(f_n - f_m) = a[f_n - f_m].
\]
Since $\lim_{n,m \to \infty} a[f_n - f_m] = 0$, one proves that
\[
\lim_{n,m \to \infty} a[U^*f_n - U^*f_m] = 0
\]
which together with (2.3) implies that $U^*f \in \text{Dom}[a]$. Hence, we have proven the inclusion
\[
(2.4)\quad U^*\text{Dom}[a] \subseteq \text{Dom}[a].
\]

Next, by (i) the domain $\text{Dom}(\dot{A}^*)$ is invariant with respect to $U^*$. This combined with (2.4) and Theorem 2.1(i) proves that the domain of the Friedrichs extension $A_F$ of $\dot{A}$ is invariant with respect to the operator $U^*$. Therefore, $A_F$ is $\mu$-scale invariant as a restriction of the $\mu$-scale invariant operator $\dot{A}^*$ onto a $U^*$-invariant domain.

(iii). Analogously, in order to show that the Krein-von Neumann extension $A_K$ is $\mu$-scale invariant with respect to the transformation $U$, it is sufficient to show that its domain is invariant with respect to $U^*$.

Take $f \in \text{Dom}(A_K)$. By Theorem 2.1(ii) there exists an $a$-Cauchy sequence $\{f_n\}_{n \in \mathbb{N}}$, $f_n \in \text{Dom}(\dot{A})$, such that
\[
(2.5)\quad \lim_{n \to \infty} \dot{A}f_n = \dot{A}^*f.
\]
\footnote{in the “metric” generated by the form $a$}
From (2.2) it follows that
\[(2.6) \quad \hat{A}U^* f_n = \hat{A}^* U^* f_n = \mu U^* \hat{A} f_n = \mu U^* \hat{A} f_n \quad \text{and} \quad \hat{A}^* U^* f = \mu U^* \hat{A}^* f.\]
Combining (2.5) and (2.6), for the \(T\) is the maximal operator on the Sobolev space there-fore, the Krein-von Neumann extension \(S\) is the maximal operator on the Sobolev space extension of a semi-bounded relation, applying Theorem 2.2(ii) proves the existence of \(\mu\)-scale invariant operators can immediately be extended to the case of linear relations: we say that a linear relation \(S\) is \(\mu\)-scale invariant with respect to the unitary transformation \(U\) if its domain is \(U^*\)-invariant and \((f, g) \in S\) implies \((U^* f, \mu U^* g) \in S\).

Remark 2.3. We remark that the concept of \(\mu\)-scale invariant operators can immediately be extended to the case of linear relations: we say that a linear relation \(S\) is \(\mu\)-scale invariant with respect to the unitary transformation \(U\) if its domain is \(U^*\)-invariant and \((f, g) \in S\) implies \((U^* f, \mu U^* g) \in S\).

Recall that the Friedrichs extension \(S_F\) of a semi-bounded from below relation \(S\) is defined as the restriction of \(S^*\) onto the domain of the closure of the quadratic form associated with the operator part of \(S\) and the Krein-von Neumann extension \(S_K\) is defined by
\[(2.7) \quad S_K = ((S^{-1})_F)^{-1},\]
provided that \(S\) is, in addition, nonnegative (no care should be taken about inverses, for they always exist).

Assume that a nonnegative linear relation \(S\) is \(\mu\)-scale invariant. Almost literally repeating the arguments of the proof of Theorem 2.2(i) one concludes that the adjoint relation \(S^*\) is also \(\mu\)-scale invariant. Given the above characterization of the Friedrichs extension of a semi-bounded relation, applying Theorem 2.2(ii) proves the \(\mu\)-scale invariance of \(S_F\). As it follows from (2.7), a simple observation that \(S\) is \(\mu\)-scale invariant if and only if the inverse relation \(S^{-1}\) is \(\mu^{-1}\)-scale invariant ensures that the Krein-von Neumann extension \(S_K\) of \(S\) is also \(\mu\)-scale invariant. Thus, Part (iii) of Theorem 2.2 is a direct consequence of Parts (i) and (ii) up to the representation theorem that states that Krein-von Neumann extension \(A_K\) of a nonnegative densely defined symmetric operator \(\hat{A}\) can be “evaluated” as
\[(2.8) \quad A_K = \left((\hat{A}^{-1} F)^{-1},\right]\]
with \(\hat{A}^{-1}\) being understood as a linear relation (for the proof of (2.8) we refer to [8], also see [3] and [4]).

Remark 2.4. Note without proof that if the symmetric nonnegative operator \(\hat{A}\) referred to in Theorem 2.2 has deficiency indices \((1, 1)\) the Friedrichs and the Krein-von Neumann extensions of \(\hat{A}\) are the only ones \(\mu\)-scale invariant self-adjoint extensions.

The following simple example illustrates the statement of Theorem 2.2

Example 2.5. Assume that \(\mu > 0, \mu \neq 1\), and that \(U\) is the unitary scaling transformation on the Hilbert space \(\mathcal{H} = L^2(0, \infty)\) defined by
\[(U f)(x) = \mu^{-\frac{1}{2}} f(\mu^{-\frac{1}{2}} x), \quad f \in L^2(0, \infty).\]
\(T\) is the maximal operator on the Sobolev space \(H^{2,2}(0, \infty)\) defined by
\[T = -\frac{d^2}{dx^2}, \quad \text{Dom}(T) = H^{2,2}(0, \infty).\]
Let $A_F$ and $A_K$ be the restrictions of $T$ onto the domains
\[
\text{Dom}(A_F) = \{ f \in \text{Dom}(T) \mid f(0) = 0 \}
\]
and
\[
\text{Dom}(A_K) = \{ f \in \text{Dom}(T) \mid f'(0) = 0 \}
\]
respectively. Denote by $\dot{A}$ the restriction of $T$ onto the domain
\[
\text{Dom}(\dot{A}) = \text{Dom}(A_F) \cap \text{Dom}(A_K).
\]
It is well known that $\dot{A}$ is a closed nonnegative symmetric operator with deficiency indices $(1, 1)$ and that $A_F$ and $A_K$ are the Friedrichs and the Krein-von Neumann extensions of $\dot{A}$ respectively and $T = \dot{A}^*$. A straightforward computation shows that all the operators $\dot{A}$, $A_F$, $A_K$ and $T$ are $\mu$-scale invariant with respect to the transformation $U$. Moreover, note that any other nonnegative self-adjoint extensions of $\dot{A}$ different from the extremal ones, $A_F$ and $A_K$, can be obtained by the restriction of $T$ onto the domain (see, e.g., [13], also see [9] and [10])
\[
\text{Dom}(\tilde{A}_s) = \{ f \in \text{Dom}(T) \mid f'(0) = sf(0) \}, \quad \text{for some } s > 0,
\]
which is obviously not $U^*$-invariant. Thus, the operator $\dot{A}$ admits the only two $\mu$-scale invariant extensions, the Friedrichs and the Krein-von Neumann extensions (cf. Remark 2.4).

3. Concluding remarks

We remark that any $\mu$-scale invariant operator $T$ with respect to a unitary transformation $U$ is also $\mu^n$-scale invariant with respect to the (unitary) transformations $U^n$, $n = 0, 1, \ldots$. That is,
\[
U^nTU^{-n} = \mu^nT, \quad \text{for all } n \in \{0\} \cup \mathbb{N}.
\]
If, in addition,
\[
U^*\text{Dom}(T) = \text{Dom}(T),
\]
then relation (3.1) holds for all $n \in \mathbb{Z}$. Thus, we naturally arrive at a slightly more general concept of scale invariance with respect to a one-parameter unitary representation of the additive group $\mathbb{G}$ ($\mathbb{G} = \mathbb{N}$ or $\mathbb{G} = \mathbb{R}$): *Given a character $\mu$, $\mu : \mathbb{G} \to \mathbb{C}$, of the group $\mathbb{G}$ and its one-parameter unitary representation $g \mapsto U_g$, a densely defined operator $T$ is said to be $\mu$-character-scale invariant with respect to the representation $U_g$ if*
\[
U_g\text{Dom}(T) = \text{Dom}(T), \quad g \in \mathbb{G},
\]
and
\[
U_gTU_{-g} = \mu(g)T, \quad \text{on } \text{Dom}(T), \quad g \in \mathbb{G}.
\]
Clearly, an appropriate version of Theorem 2.2 can almost literally be restated in this more general setting. It is also worth mentioning that upon introducing the representation $V_g = \mu^gU_g$, $g \in \mathbb{G}$, one can rewrite (3.2) in the form
\[
U_gT = TV_g, \quad g \in \mathbb{G},
\]
and we refer the interested reader to the papers [12] and [14] where commutation relations (3.3) for general groups $\mathbb{G}$ with not necessarily unitary representations $U_g$ and $V_g$, $g \in \mathbb{G}$, of the group $\mathbb{G}$ are discussed.

Note that an infinitesimal analog of the commutation relation in (3.2) is also available provided that $\mathbb{G} = \mathbb{R}$ and the unitary representation $U_t$, $t \in \mathbb{R}$, is strongly continuous. In
this case infinitesimal version of (3.2) can heuristically be written down as the following commutation relation

\[(3.4) \quad [B, T] = i\hbar T,\]

with \([\cdot, \cdot]\) the usual commutator and

\[(3.5) \quad \hbar = -\log \mu,\]

the structure constant of the simplest noncommutative two-dimesional Lie algebra (3.4) and (3.5). Here \(B\) is the infinitesimal generator of the group \(U_t\), so that \(U_t = e^{iBT}\), \(t \in \mathbb{R}\). And in conclusion, note that Theorem 2.2 paves the way for realizations of the Lie algebra by self-adjoint operators, provided that some “trial” symmetric realizations of the Lie algebra are available.

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