A FRESH APPROACH TO CLASSICAL EISENSTEIN SERIES AND THE NEWER HILBERT–EISENSTEIN SERIES

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Dedicated to the memory of Godfrey Harold Hardy, a Discoverer and Mentor of Srinivasa Aiyangar Ramanujan

This paper is concerned with new results for the circular Eisenstein series $e_r(z)$ as well as with a novel approach to Hilbert-Eisenstein series $h_r(z)$, introduced by Michael Hauss in 1995. The latter turn out to be the product of the hyperbolic sinh–function with an explicit closed form linear combination of digamma functions. The results, which include differentiability properties and integral representations, are established by independent and different arguments. Highlights are new results on the Butzer–Flocke–Hauss Omega function, one basis for the study of Hilbert-Eisenstein series, which have been the subject of several recent papers.

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1. Eisenstein series

In order to introduce his method for constructing elliptic functions, Ferdinand Gotthold Max Eisenstein (1823–1852) first considered the simpler case of trigonometric functions, specifically the series

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k \in \mathbb{N}} \left(\frac{1}{z+k} + \frac{1}{z-k}\right),$$
originally discovered by Leonhard Euler in 1748, presented in [18, §178] Eisenstein introduced the series (later to be famously known as Eisenstein series, see e.g. Weil [37], [38] and Iwaniec [25])

\[ \varepsilon_r(z) := \sum_{k \in \mathbb{Z}} \frac{1}{(z+k)^r}, \quad (1.1) \]

which are defined for \( z \in \mathbb{C} \setminus \mathbb{Z} \) and all \( r \in \mathbb{N}_2 = \{2, 3, \cdots \} \), they being a normally convergent series of meromorphic functions in \( \mathbb{C} \). Since these Eisenstein series of order \( r \in \mathbb{N} \) do not exist for \( r = 1 \), one defines aesthetically

\[ \varepsilon_1(z) = \sum_{k \in \mathbb{Z}} \frac{1}{z+k} := \lim_{N \to \infty} \sum_{|k| \leq N} \frac{1}{z+k} = \frac{1}{z} + \sum_{k \in \mathbb{Z}} \left( \frac{1}{z+k} - \frac{1}{k} \right). \]

One observes that \( \varepsilon_1(z) = \pi \cot(\pi z) \) (Euler), and by differentiation

\[ \varepsilon_2(z) = \frac{\pi^2}{\sin^2(\pi z)}, \quad \varepsilon_3(z) = \frac{\pi^3 \cot(\pi z)}{\sin^2(\pi z)}; \quad (1.2) \]

this results in the intriguing relation [34, p. 299]

\[ \varepsilon_3(z) = \varepsilon_1(z) \cdot \varepsilon_2(z). \]

There immediately arises the question: "do there exist further \( r \in \mathbb{N}_2 \) such that \( \varepsilon_{r+2}(z) = \varepsilon_{r+1}(z) \cdot \varepsilon_r(z) \) is valid?" Our answer is the following result.

**Theorem 1.1.** The unique solution in \( r \in \mathbb{N} \) of the equation

\[ \varepsilon_{r+2}(z) = \varepsilon_{r+1}(z) \cdot \varepsilon_r(z), \quad (z \in \mathbb{C} \setminus \mathbb{Z}) \quad (1.3) \]

is \( r = 1 \).

**Proof.** Obviously \( r \) has to be odd. Indeed, setting \( z = \frac{1}{2} \) in (1.1), we can write \( \varepsilon_{r+2}(\frac{1}{2}) \) in terms of the Dirichlet’s Lambda–function

\[ \lambda(r) = \sum_{k \in \mathbb{N}_0} \frac{1}{(2k+1)^r}, \quad (r > 1) \]

in the form

\[ \varepsilon_r(\frac{1}{2}) = 2^r(1+(-1)^r) \lambda(r). \]

But by this result the initial equation (1.3) makes sense only for \( r \) odd, since for even \( r = 2\ell, \ell \in \mathbb{N} \), the relation (1.3) becomes a contradiction

\[ 2^{2\ell+3} \lambda(2\ell+2) = 0 \cdot 2^{2\ell+1} \lambda(2\ell) = 0. \]

*It is worth mentioning that it is regarded by Konrad Knopp [27, p. 207] as the "most remarkable expansion in partial fractions". Also, J. Elstrodt [16] nominated this partial fraction expansion for the most interesting formula involving elementary functions, see also [2, p. 149].
A new approach to Eisenstein series and Hilbert–Eisenstein series

On the other hand, bearing in mind the essential differentiability property \[37\] pp. 6–13, \[34\] p. 299, namely

\[ \varepsilon_r(z) = \frac{(-1)^{r-1}}{\Gamma(r)} \varepsilon_1^{(r-1)}(z), \quad (r \in \mathbb{N}) , \tag{1.4} \]

and accordingly

\[ \varepsilon''_r(z) = r(r+1) \varepsilon_{r+2}(z), \]

we deduce from (1.3) the nonlinear second order ODE:

\[ y'' + (r+1) y' y = 0, \quad (y = \varepsilon_r(z)). \tag{1.5} \]

Moreover, as the Eisenstein series is 1–periodic in the sense that \( \varepsilon_r(z + k) = \varepsilon_r(z) \)

for all \( z \in \mathbb{C} \setminus \mathbb{Z}, k \in \mathbb{Z} \), we are looking for a 1–periodic particular solution of the ordinary differential equation (1.5). It is

\[ y = \sqrt{\frac{2c_1}{r+1}} \tanh \left[ \frac{c_1(r+1)}{2} (z + c_2) \right], \]

where \( c_1, c_2 \) stand for integration constants. The \( \tanh \) function is \( i\pi \)–periodic, so

\[ \sqrt{\frac{c_1(r+1)}{2}} = i\pi, \]

accordingly

\[ \varepsilon_r(z) = -\frac{2\pi}{r+1} \tan \pi (z + c_2). \tag{1.6} \]

Now, we have

\[ \varepsilon'_r(z) = -\frac{2\pi^2}{r+1} \cdot \frac{1}{\cos^2 \pi (z + c_2)}, \]

\[ = -\frac{2\pi^2}{r+1} \left[ \tan^2 \pi (z + c_2) + 1 \right], \]

\[ = -\frac{r+1}{2} \varepsilon_2^2(z) - \frac{2\pi^2}{r+1}, \]

which coincides exactly for \( r = 1 \) with the Riccati–type differential identity \[34\] p. 268, Eq. (1)]

\[ \varepsilon'_1 = -\varepsilon_1^2 - \pi^2. \]

Also, we observe that (1.6) becomes the Eisenstein series \( \varepsilon_1(z) \) for \( c_2 = -\frac{1}{2} \).

The cotangent form of \( \varepsilon_1(z) \) and the examples (1.2) are best expressed and extended when one recalls the beautiful reflection formula for the more-practical digamma–function \( \psi(z) := \frac{d}{dz} \log \Gamma(z) = \Gamma'(z)/\Gamma(z) \), namely \[11\] p. 259, Eq. 6.3.7]

\[ \varepsilon_1(z) = \pi \cot(\pi z) = \psi(1-z) - \psi(z), \tag{1.7} \]
Therefore, letting \( \Re \) inside the vertical strip \( \Re \) we conclude assuming \( z \)

\[
\text{Theorem 1.3. There holds the integral representation}
\]

\[
\epsilon_r(z) = \frac{1}{\Gamma(r)} \psi_{r-1}(1-z) + (-1)^r \psi_r(z)
\]

Special attention is given to the case \( r = 0 \), that is

\[
\psi(z) := \psi_0(z) = \psi^{(0)}(z) = \sum_{k \in \mathbb{N}} \left( \frac{1}{k} - \frac{1}{z+k-1} \right) - \gamma,
\]

where \( \gamma = 0.5772156649... \) signifies the Euler–Mascheroni constant.

A first new, but simple result in this respect reads, noting (1.4),

**Proposition 1.2.** For all \( z \in \mathbb{C} \setminus \mathbb{Z} \) we have

\[
\epsilon_r(z) = \frac{1}{\Gamma(r)} \psi_{r-1}(1-z) + (-1)^r \psi_r(z)
\]

As to the proof, it follows immediately from (1.7) and (1.4).

Our first more important result is a new integral representation of \( \epsilon_r(z) \).

**Theorem 1.3.** There holds the integral representation

\[
\epsilon_r(z) = (z - [\Re(z)])^{-r} + \frac{1}{\Gamma(r)} \int_0^{\infty} \frac{t^{r-1}}{e^t - 1} \left( e^{-(z-[\Re(z)]^t + (-1)^r e^{(z-[\Re(z)]^t} \right) dt,
\]

for all \( r \in \mathbb{N} \) and for all \( z \in \mathbb{C} \setminus \mathbb{Z} \). Here \( [x] \) stands for the integer part of \( x \in \Re \).

**Proof.** By the 1-periodicity of Eisenstein’s functions \( \epsilon_r(z) \), it is sufficient to consider it inside the vertical strip \( \Re(z) \in (0, 1) \) of the complex plane. Indeed, otherwise, assuming \( z \neq 0 \), by the relation \( \epsilon_r(z) = \epsilon_r(z - [\Re(z)]), \) we have the same property. Therefore, letting \( \Re(z) \in (0, 1) \), by the Gamma–function formula

\[
\Gamma(r) A^s = \int_0^{\infty} t^{s-1} e^{-At} dt, \quad (\Re(s) > 0, \Re(A) > 0),
\]

we conclude

\[
\epsilon_r(z) = \sum_{k \in \mathbb{Z}} \frac{1}{(z+k)^r} - \frac{1}{z^r} + \frac{1}{\Gamma(r)} \int_0^{\infty} \left( \sum_{k \in \mathbb{N}} \frac{1}{(z+k)^r} + \frac{(-1)^r}{(k-z)^r} \right)
\]

\[
= \frac{1}{z^r} + \frac{1}{\Gamma(r)} \int_0^{\infty} t^{r-1} \left( \sum_{k \in \mathbb{N}} e^{-kt} \right) \left( e^{-zt} + (-1)^r e^{zt} \right) dt
\]

\[
= \frac{1}{\Gamma(r)} \int_0^{\infty} \frac{t^{r-1} \left( \sum_{k \in \mathbb{N}} e^{-kt} \right) \left( e^{-zt} + (-1)^r e^{zt} \right) dt}{1 - e^{-t} (e^{-(z+1)t} + (-1)^r e^{-(1-z)t})}
\]

The integral converges for \( r \geq 1 \), when \( |\Re(z)| < 1 \), as the integrand’s behavior is controlled near to the origin and at infinity. The rest is clear.

**Corollary 1.4.** For all \( r \in \mathbb{N} \) and \( z \in \mathbb{C} \setminus \mathbb{Z} \), we have

\[
\epsilon_r(z) = \frac{1}{(z - [\Re(z)]^r} + \frac{2}{\Gamma(r)} \int_0^{\infty} \frac{t^{r-1}}{e^t - 1} \left\{ \begin{array}{ll}
\cosh(z - [\Re(z)]^t) & \text{if } r \text{ is even} \\
\sinh(z - [\Re(z)]^t) & \text{if } r \text{ is odd}
\end{array} \right\}
\]

\[dt, \quad \left\{ \begin{array}{ll}
r \text{ even} \\
r \text{ odd}
\end{array} \right\}.
\]
2. Backgrounds to Hilbert–Eisenstein series

A basis to the Hilbert–Eisenstein series includes the classical Bernoulli numbers $B_n := B_n(0), n \in \mathbb{N}_0,$ defined in terms of the Bernoulli polynomials $B_n(x),\,$ defined, for example, via their exponential generating function

$$\sum_{n \in \mathbb{N}_0} B_n(x) \frac{z^n}{n!} = \frac{ze^{zx}}{e^x - 1} \quad (z \in \mathbb{C}, \, |z| < 2\pi, \, x \in \mathbb{R}).$$

(2.1)

We need some facts concerned with $B_n(x).$ Starting with the 1–periodic Bernoulli polynomials $B_n(x),$ defined as the periodic extension of $B_n(x) = B_n(x), \, x \in (0, 1],$ we need to introduce the 1–periodic conjugate functions $B_\sim n(x),\, x \in \mathbb{R} (x \not\in \mathbb{Z} \text{ if } n = 1)$ by

$$B_\sim n(x) := \mathcal{H}_1 [B_n(\cdot)](x), \quad (n \in \mathbb{N}).$$

Here $\mathcal{H}_1$ is the (periodic) Hilbert transform of the 1–periodic function $\varphi$ defined by

$$\mathcal{H}_1 [\varphi](x) = \text{PV} \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(x - u) \cot(\pi u) \, du,$$

so that

$$B_\sim n(x) = \text{PV} \int_{-\frac{1}{2}}^{\frac{1}{2}} B_n(x - u) \cot(\pi u) \, du,$$

with $B_\sim 0(x) = B_\sim 0(x) = 0$ for all $x \in \mathbb{R},$ since $B_0(x) = B_0(0) = 1.$ Written as a Fourier series, they then are to be [10]

$$B_{2n+1}(x) = -2(2n+1)! \sum_{k \in \mathbb{N}} \frac{\sin \left(\frac{2\pi kx - (2n+1)\pi}{2}\right)}{(2\pi k)^{2n+1}}, \quad (n \in \mathbb{N}_0).$$

(2.2)

These conjugate periodic functions $B_\sim n(x)$ are used to define the non–periodic functions $B_\sim n(x),$ which can be regarded as conjugate Bernoulli "polynomials" in a form such that their properties are similar to those of the classical Bernoulli polynomials $B_n(x).$ For details see Butzer and Hauss [11 p. 22] and Butzer [10 pp. 37-56]. The conjugate Bernoulli numbers needed, the $B_\sim 2m+1,$ are the $B_\sim 2m+1(0)\,(= B_\sim 2m+1(1))$ for which

$$B_{2m+1}^{\sim} (\frac{1}{2}) = (2^{-2m} - 1) \cdot B_{2m+1}^{\sim}(1), \quad B_{2m}^{\sim} (\frac{1}{2}) = 0.$$

Some values of the conjugate Bernoulli numbers are (see [10 p. 69])

$$B_{2m+1}^{\sim} (\frac{1}{2}) = \begin{cases} \frac{-\log 2}{\pi} & m = 0 \\ \frac{\log 2}{4\pi} - 2 \int_{0+}^{\frac{1}{2}} u^3 \cot(\pi u) \, du & m = 1 \\ \frac{11}{8} \int_{0+}^{\frac{1}{2}} u \cot \pi u \, du + \frac{5}{3} \int_{0+}^{\frac{1}{2}} u^3 \cot(\pi u) \, du & m = 2 \\ -2 \int_{0+}^{\frac{1}{2}} u^5 \cot(\pi u) \, du & m = 2 \end{cases},$$

(2.3)
and

\[ B_1^\sim(x) = -\frac{1}{\pi} \log(2 \sin(\pi x)) . \]

Of basic importance is also the exponential generating function of \( B_k^\sim(\frac{1}{2}) \), given for \(|z| < 2\) by

\[
\sum_{k \in \mathbb{N}_0} B_k^\sim(\frac{1}{2}) \frac{z^k}{k!} = -\frac{z e^z}{e^z - 1} \Omega(z) = -\frac{z}{2 \sinh \frac{z}{2}} \Omega(z) ,
\]

first established by M. Hauss [23, p. 91–95], [24] (see also [11, pp. 21–29] and [10, pp. 37–38, 78–80]). Above, \( \Omega(z) \), \( z \in \mathbb{C} \) is the so-called Butzer–Flocke-Hauss (complete) Omega function introduced in [12] in the form

\[
\Omega(z) := 2 \int_0^\frac{1}{2} \sinh(zu) \cot(\pi u) \, du, \quad (z \in \mathbb{C}) .
\]

It is the Hilbert transform \( \mathcal{H}_1[e^{-zx}](0) \) at zero of the 1–periodic function \( (e^{-zx})_1 \), defined by the periodic extension of the exponential function \( e^{-zx} \), \( |x| < \frac{1}{2} \), \( z \in \mathbb{C} \), thus

\[
\mathcal{H}_1[e^{-zx}](0) = \text{PV} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{zu} \cot(\pi u) \, du = \Omega(z) .
\]

As to the Omega function, we further need its basic partial fraction development for \( z \in \mathbb{C} \setminus i\mathbb{Z} \), namely

\[
\Omega(2\pi z) = \frac{1}{\pi} (e^{-\pi z} - e^{\pi z}) \sum_{k \in \mathbb{N}_0} (-1)^k k \frac{z^k}{z^2 + k^2} = -\frac{i \sinh(\pi z)}{\pi} \sum_{k \in \mathbb{Z}} (-1)^k \text{sgn}k \frac{z + ik}{z + k} ,
\]

the proof of which depends upon a new Hilbert–Poisson formula, introduced by Hauss; see [23] pp. 97–103 or [24].

A useful formula which will link Hilbert–Eisenstein series, Hilbert transform versions of the Bernoulli numbers and the Riemann Zeta function is given by (see [17, Eq. 1.17(7)] and [22, Eq. (54.10.3)])

\[
\sum_{k \in \mathbb{N}} \zeta(2k + 1) z^{2k} = -\frac{1}{2} [\psi(1 + z) + \psi(1 - z)] + \gamma, \quad (|z| < 1)
\]

or, replacing \( z \mapsto zi \), then

\[
\sum_{k \in \mathbb{N}} (-1)^{k-1} \zeta(2k + 1) z^{2k} = \frac{1}{2} [\psi(1 + iz) + \psi(1 - iz)] + \gamma, \quad (|z| < 1).
\]

A second formula in this respect reads [11, 6.3.17], [24, Eq. (54.3.5)]

\[
\sum_{k \in \mathbb{N}} (-1)^{k-1} \zeta(2k + 1) z^{2k} = \gamma + \Re[\psi(1 + iz)], \quad (z \in \mathbb{R}).
\]
3. Hilbert–Eisenstein series

Now, we come to the main sections of this article, dealing with Hilbert–Eisenstein series. A Hilbert (conjugate function) – type version of the Eisenstein series $\varepsilon_1(z)$ was first studied by Michael Hauss in his doctoral thesis [23].

Definition 3.1. The Hilbert–Eisenstein (HE) series $h_r(w)$ are defined for $z \in \mathbb{C} \setminus i\mathbb{Z}$ and $r \in \mathbb{N}_2$ by

$$h_r(z) := \sum_{k \in \mathbb{Z}} \frac{(-1)^k \text{sgn}(k)}{(z + ik)^r} = \sum_{k \in \mathbb{N}} (-1)^k \left( \frac{1}{(z + ik)^r} - \frac{1}{(z - ik)^r} \right),$$

and, for $r = 1$ recalling (2.5), by

$$h_1(z) := \sum_{k \in \mathbb{Z}} \frac{(-1)^k \text{sgn}(k)}{z + ik} = \frac{i\pi \Omega(2\pi z)}{\sinh(\pi z)} = i\Omega(2\pi z) \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{z + ik},$$

with $h_1(0) = 2i\log 2$, noting $\text{sgn}(0) = 0$.

In this respect recall that

$$\frac{\pi}{\sinh(\pi z)} = \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{z + ik} = \frac{1}{z} + 2z \sum_{k \in \mathbb{N}} \frac{(-1)^k}{z^2 + k^2}, \quad (z \in \mathbb{C} \setminus i\mathbb{Z}).$$

The basic properties of $h_r$ for $z \in \mathbb{C} \setminus i\mathbb{Z}$ and $r \in \mathbb{N}_2, s \in \mathbb{N}$, are

$$h_r(z) = -r h_{r+1}(z),$$

$$h_r(z) = (-1)^s (r)_s h_{r+s}(z), \quad (r > 0), \quad (s \in \mathbb{N}),$$

as well as their difference and symmetry property [23 Eq. (6.5.72)], [10 Eq. (9.7)]

$$h_r(z) + h_r(z + i) = z^{-r} - (z + i)^{-r}; \quad h_r(-z) = (-1)^{r+1} h_r(z), \quad (z \in \mathbb{C} \setminus i\mathbb{Z}).$$

Above

$$(\rho)_\sigma := \frac{\Gamma(\rho + \sigma)}{\Gamma(\rho)} = \begin{cases} 1, & (\sigma = 0; \rho \in \mathbb{C} \setminus \{0\}) \cr \rho(\rho + 1) \cdots (\rho + \sigma - 1), & (\sigma \in \mathbb{N}; \rho \in \mathbb{C}) \end{cases},$$

stands for the Pochhammer symbol (or shifted, rising factorial). Note that it being understood conventionally that $(0)_0 := 1$.

Proposition 3.2. a) For $z \in \mathbb{C}, \ |z| < 1$ one has

$$\sum_{k \in \mathbb{Z}} (-1)^k \frac{\text{sgn}(k)}{z + ik} = 2i \sum_{n \in \mathbb{N}_0} (-1)^n \eta(2n + 1) z^{2n}. \quad (3.2)$$

b) Moreover

$$B_{2n+1}^{2n+1}(\frac{1}{2}) = (-1)^{n+1}(2n + 1)!2^{-2n+1}\pi^{-2n} \eta(2n + 1) \quad (3.3)$$

$$= (-1)^n (2n + 1)! (4^{-2n} - 2^{-2n}) \pi^{-2n} \zeta(2n + 1), \quad (3.4)$$

where $\eta(s) = \sum_{n \in \mathbb{N}} (-1)^{n-1} n^{-s}, \ \Re(s) > 0$ stands for the Dirichlet’s Eta function.
Proof. We have

\[ \lim_{N \to \infty} \sum_{|k| \leq N} \frac{(-1)^k \text{sgn}(k)}{z + ik} = \lim_{N \to \infty} \sum_{k=1}^{N} (-1)^{k-1} \left\{ \frac{1}{z - ik} - \frac{1}{z + ik} \right\} = 2i \sum_{k \in \mathbb{N}} \frac{(-1)^{k-1}}{z^2 + k^2} = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k} \sum_{n \in \mathbb{N}_0} (-1)^n \left( \frac{z}{k} \right)^{2n} \]

the interchange of the summation order being possible on account of the Weierstraß double series theorem (see e.g. [27, p. 428]). This proves part a).

As to part b), on account of (2.4)

\[ \frac{1}{2z} \sum_{n \in \mathbb{N}_0} B_{2n}^\sim \left( \frac{1}{2} \right) \frac{(2\pi z)^n}{n!} = \frac{\pi}{e^{\pi z} - e^{-\pi z}} \cdot \Omega(2\pi z). \]

Comparing coefficients with (3.2), gives \( B_{2n}^\sim \left( \frac{1}{2} \right) = 0 \) and results in (3.3). As to (3.4), it follows from (3.3) by noting that \( \eta(n(2n + 1)) = (1 - 2^{-2n}) \zeta(2n + 1), \ n \in \mathbb{N} \).

Alternatively, (3.4) follows from (2.2), by setting \( x = \frac{1}{2} \), which yields

\[ B_{2n+1}^\sim \left( \frac{1}{2} \right) = 2(2n + 1)! \sum_{k \in \mathbb{N}} \frac{\sin \left( \left( k + \frac{1}{2} \right) \pi \right)}{2\pi k (2n+1)!} = (-1)^n \frac{2(2n + 1)!}{(2\pi)^{2n+1}} \eta(2n + 1). \]

This finishes the proof of proposition. \( \Box \)

Now, the generating function of \( B_{k}^\sim \left( \frac{1}{2} \right) \) can be expressed in terms of the digamma function. In fact,

**Theorem 3.3.** For \( z \in \mathbb{C} \setminus (1 + i)\mathbb{Z} \) with \(|z| < 2\pi\) there holds

\[ \sum_{k \in \mathbb{N}_0} B_{k}^\sim \left( \frac{1}{2} \right) \frac{z^k}{k!} = \begin{cases} \frac{z}{\pi} \left\{ \log 2 + \psi \left( 1 + \frac{i z}{4\pi} \right) - \psi \left( 1 + \frac{iz}{2\pi} \right) \right\} + \frac{i z}{2\pi} \left\{ \coth \frac{z}{4\pi} - \coth \frac{z}{2\pi} \right\} - i, & (|z| < 2\pi) \\ -\frac{z}{\pi} \left\{ \log 2 + \Re \psi \left( 1 + \frac{i z}{4\pi} \right) - \Re \psi \left( 1 + \frac{i z}{2\pi} \right) \right\}, & (-2\pi < z < 2\pi) \end{cases} \] (3.5)
Proof. Substituting formula (3.4) for $B_{2m+1} \left( \frac{1}{2} \right)$ into the series below, and observing (2.6), we have

\[
\sum_{k \in \mathbb{N}_0} B_{2k}(\frac{z}{2}) \frac{z^k}{k!} = -\frac{\log 2}{\pi} \cdot z + \sum_{k \in \mathbb{N}} B_{2k}(\frac{z}{2}) \frac{z^k}{k!} = -\frac{\log 2}{\pi} \cdot z + 4 \sum_{k \in \mathbb{N}} (-1)^k \left( \frac{z}{4\pi} \right)^{2k+1} \zeta(2k+1)
\]

This establishes the representation in (3.5) for complex $|z| < 2\pi$. That for real $z \in (-2\pi, 2\pi)$ follows from (2.8).

Observe that the proof of Theorem 3.3 has the same outward appearance as that of Theorem 7.3. in [10, p. 74], but it uses the correct formula (2.3), provided with two proofs in Proposition 3.2.

Now, the $h_1(z)$ can also be represented in terms of the classical digamma function.

Theorem 3.4. There holds

\[
h_1(z) = \begin{cases} 
2i \log 2 + i \left\{ \psi\left(1 + \frac{i z}{2}\right) + \psi\left(1 - \frac{i z}{2}\right) - \psi(1 + iz) - \psi(1 - iz) \right\} & (|z| < 2\pi) \\
2i \log 2 + 2i \Re \left\{ \psi\left(1 + \frac{i z}{2}\right) - \psi\left(1 + iz\right) \right\} & (-2\pi < z < 2\pi)
\end{cases}
\]

First proof. According to (3.2) of Proposition 3.2 and Definition 3.1, we have, noting $\eta(s) = (1 - 2^{1-s})\zeta(s)$, $\Re(s) > 1$,

\[
h_1(z) = 2i \sum_{k \in \mathbb{N}_0} (-1)^k \eta(2k+1) z^{2k} = 2i \sum_{k \in \mathbb{N}_0} (-1)^k \left( 1 - 2^{-2k} \right) \zeta(2k+1) z^{2k} = 2i \sum_{k \in \mathbb{N}_0} \left( (iz)^{2k} - \left( \frac{iz}{2} \right)^{2k} \right) \zeta(2k+1) = 2i \left\{ \lim_{h \to 0^+} \left( (iz)^{2h} - \left( \frac{iz}{2} \right)^{2h} \right) \zeta(2h+1) + \sum_{k \in \mathbb{N}} \left( (iz)^{2k} - \left( \frac{iz}{2} \right)^{2k} \right) \zeta(2k+1) \right\}.
\]
We now express the sum via the linear combination of digamma functions, recalling (2.7), that means

\[
\eta_1(z) = 2i \left\{ \lim_{h \to 0+} \left( (iz)^{2h} - \left(\frac{iz}{2}\right)^{2h} \right) \zeta(2h + 1) 
+ \frac{1}{2} \left[ \psi \left( 1 + \frac{iz}{2} \right) + \psi \left( 1 - \frac{iz}{2} \right) \right] - \frac{1}{2} \left[ \psi (1 + iz) + \psi (1 - iz) \right] \right\}.
\]

On the other hand

\[
\lim_{h \to 0+} \left( (iz)^{2h} - \left(\frac{iz}{2}\right)^{2h} \right) \zeta(2h + 1) = \lim_{h \to 0+} \left( (iz)^{2h} - \left(\frac{iz}{2}\right)^{2h} \right) \left( \frac{1}{2h} + \gamma + o(h) \right)
= \log(iz) - \log \left( \frac{iz}{2} \right) = \log 2.
\]

Now, obvious steps lead to the assertion of Theorem 3.4.

\[\Box\]

Observe that the real parts of Theorem 3.4 can also be expressed as integrals, noting

\[\Re \psi \left( 1 + \frac{iz}{2\pi} \right) = -\gamma + 2 \int_0^\infty e^{-u} \sin^2 \left( \frac{u}{2\pi} \right) \sinh(u) \, du.\]

Although Theorem 3.4 is to be found in [10, Eq. (7.8)], the above proof is a new approach to Hilbert–Eisenstein series.

**Second proof.** According to (2.4), we have on the one hand

\[
\sum_{k \in \mathbb{N}_0} B_k \left( \frac{1}{2} \right) \frac{(2\pi z)^k}{k!} = -\frac{\pi z}{\sinh(\pi z)} \Omega(2\pi z), \quad (z \in \mathbb{C} \setminus i\mathbb{Z}),
\]

and, on the other hand

\[
\eta_1(z) = \frac{i\pi}{\sinh(\pi z)} \Omega(2\pi z).
\]

Thus, following the argument along the lines of the proof of Theorem 3.3,

\[
-\frac{z}{i} \eta_1(z) = \sum_{k \in \mathbb{N}_0} B_k \left( \frac{1}{2} \right) \frac{(2\pi z)^k}{k!}
= -2z \log 2 + 2 \sum_{k \in \mathbb{N}} (-1)^k \left( \frac{2\pi z}{4\pi} \right)^{2k+1} \zeta(2k+1)
- \sum_{k \in \mathbb{N}} (-1)^k \left( \frac{2\pi z}{2\pi} \right)^{2k+1} \zeta(2k+1)
= -2z \log 2 + 2 \cdot \frac{z}{4} \left\{ 2\psi(1) - \psi \left( 1 + \frac{iz}{2} \right) - \psi \left( 1 - \frac{iz}{2} \right) \right\}
- \frac{z}{2} \{ 2\psi(1) - \psi (1 + iz) - \psi (1 - iz) \}.
\]
Therefore
\[ h_1(z) = 2i \log 2 + i \left\{ \psi(1 + i \frac{z}{2}) + \psi(1 - i \frac{z}{2}) - \psi(1 + iz) - \psi(1 - iz) \right\}. \]

This completes the proof of the first formula of Theorem 3.4. The second one follows immediately by the mirror symmetry formula \( \psi(w) = \psi(w), w \in \mathbb{C} \).

It is important to mention that this representation of \( h_1(z) \) is not given in [10], but contained implicitly in a more complicated form in the proof of Proposition 6.4.1 in [24].

Corollary 3.5. The Omega function \( \Omega(z) \) has the representation
\[ \Omega(z) = \frac{1}{\pi} \sinh \left( \frac{z}{2} \right) \left\{ 2 \log 2 + \psi \left( 1 + i \frac{z}{4\pi} \right) + \psi \left( 1 - i \frac{z}{4\pi} \right) - \psi \left( 1 + \frac{iz}{2\pi} \right) - \psi \left( 1 - \frac{iz}{2\pi} \right) \right\}, \]
for \( z \in \mathbb{C} \setminus \mathbb{Z}^- \), \(|z| < 2\pi\).

The proof is immediate from Theorem 3.3 in view of
\[ \Omega(z) = -i \frac{1}{\pi} \sinh \left( \frac{z}{2} \right) \cdot h_1 \left( \frac{z}{2\pi} \right), \]
so it is omitted. However, we remark that this corollary could also be derived, just as simply, via Theorem 3.4.

Although, as observed in [10, p. 67], the Omega function is not an "elementary function", it is nevertheless expressible in terms of the hyperbolic sine function multiplied by a (simple) linear combination of digamma functions.

4. A novel alternative approach to Theorem 3.4

The representation of \( h_1(z) \) in terms of certain combinations of \( \psi \)-functions with the constant \( 2i \log 2 \) (Theorem 3.4), was established via the generating function of the conjugate Bernoulli numbers \( B_k^\sim \left( \frac{1}{2} \right) \), equation (2.4), plus arguments used in the proof of Theorem 3.3, which in turn were based fundamentally upon the representation of \( \zeta(2k+1) \) in terms of \( B_{2k+1}^{-} (0) \), thus
\[ \zeta(2m+1) = (-1)^m \frac{2^{2m} \pi^{2m-1} B_{2m+1}^{-} (0)}{(2m+1)!}, \quad (m \in \mathbb{N}), \]
as well as the delicate formulae (2.7) and (2.8) given in tables by Hansen [22] and Abramowitz–Stegun [1]. But the last four formulae are only to be found in tables. Thus one does not know of the possible difficulties of their proofs.

A further aim of this article is to present alternative proofs which are fully independent of these two formulae (2.7) and (2.8). Moreover, since two proofs of Theorem 3.4 presented are based on power series expansions in the open unit disc \( |z| < 1 \), we can extend the validity range to the whole \( (\mathbb{C} \setminus \mathbb{Z}) \cup \{0\} \) by our present approach.
There exists a vast literature concerning Mathieu series $S_r(x)$ and the more recent alternating Mathieu series $\tilde{S}_r(x)$, both of which are defined by

$$S_r(x) = \sum_{k \in \mathbb{N}} \frac{2k}{(k^2 + x^2)^r}, \quad (r > 1) \quad (4.1)$$

$$\tilde{S}_r(x) = \sum_{k \in \mathbb{N}} \frac{(-1)^{k-1}2k}{(k^2 + x^2)^r}, \quad (r > 0); \quad (4.2)$$

see among others [15, 31, 32] and the references therein. These articles are of interest, in particular since the series $\tilde{S}_r$ is connected to HE series $b_1$. We now come to a new third proof of Theorem 3.4, at the same time establishing the representation not only for $b_1$ but also for higher order $b_r$ on the extended range.

**Theorem 4.1.** For all $z \in (\mathbb{C} \setminus i\mathbb{Z}) \cup \{0\}$ there holds

$$b_1(z) = 2i \log 2 + i \left\{ \psi \left( 1 + iz \right) + \psi \left( 1 - iz \right) - \psi \left( 1 + i\frac{z}{2} \right) - \psi \left( 1 - i\frac{z}{2} \right) \right\}. \quad (4.3)$$

Moreover, for the same $z$–domain we have for $r \in \mathbb{N}_2$

$$b_r(z) = \frac{i^r}{\Gamma(r)} \left\{ 2^{-r+1} \psi_{r-1} \left( 1 - i\frac{z}{2} \right) + (-2)^{-r+1} \psi_{r-1} \left( 1 + i\frac{z}{2} \right) - \psi_{r-1} (1 - iz) - (-1)^{r-1} \psi_{r-1} (1 + iz) \right\}. \quad (4.4)$$

**Proof.** To perform the proof of the representation formula (4.3) we have to connect the Hilbert–Eisenstein series $b_1(z)$, which is to be understood in the sense of Eisenstein summation, with the normally convergent HE series $b_r(z)$, $r \geq 2$ (for the latter see Remmert [34, p. 290]), which is termwise integrable (see [33, p. 42]).

From Definition 3.1 and the series form (1.19) of the digamma function, using the straightforward representation of the alternating series, say

$$\sum_{k \in \mathbb{Z}} (-1)^{k-1}a_k = \sum_{k \text{ odd}} a_k - \sum_{k \text{ even}} a_k = \sum_{k \in \mathbb{Z}} a_k - 2 \sum_{k \in \mathbb{Z}} a_{2k},$$

we have for all $z \in (\mathbb{C} \setminus i\mathbb{Z}) \cup \{0\},$

$$b_2(z) = -\sum_{k \in \mathbb{N}_0} \left( \frac{1}{(z + ik)^2} - \frac{1}{(z - ik)^2} \right) + 2 \sum_{k \in \mathbb{N}_0} \left( \frac{1}{(z + 2ik)^2} - \frac{1}{(z - 2ik)^2} \right)$$

$$= -\sum_{k \in \mathbb{N}_0} \left( \frac{1}{(k + iz)^2} - \frac{1}{(k - iz)^2} \right) + \frac{1}{2} \sum_{k \in \mathbb{N}_0} \left( \frac{1}{(k + i\frac{z}{2})^2} - \frac{1}{(k - i\frac{z}{2})^2} \right)$$

$$= \frac{1}{2} \left[ \psi_1 \left( 1 + i\frac{z}{2} \right) - \psi_1 \left( 1 - i\frac{z}{2} \right) \right] - \psi_1 (1 + iz) + \psi_1 (1 - iz); \quad (4.5)$$

actually, we employ here the trigamma function $\psi_1(z) = \sum_{k \in \mathbb{N}_0} (k + z)^{-2}$, which normally converges in $(\mathbb{C} \setminus i\mathbb{Z}) \cup \{0\}$.

Term–wise integration then implies

$$\int_0^z b_2(t) \, dt = b_1(0) - b_1(z) = 2i \log 2 - b_1(z).$$
Integrating (4.5) directly on \([0, z]\) too, we obtain
\[
\int_0^z h_2(t) \, dt = -i \left[ \psi \left(1 + i \frac{z}{2} \right) + \psi \left(1 - i \frac{z}{2} \right) - \psi(1 + iz) - \psi(1 - iz) \right],
\]
which completes the proof of the theorem.

In view of the differentiation property (3.1) of the HE series (see also [4, p. 796, Eq. (41)]),
\[
h_r(z) = \frac{(-1)^r}{\Gamma(r)} h_2^{(r-2)}(z), \tag{4.6}
\]
where \(z \in \left(\mathbb{C} \setminus \mathbb{Z}\right) \cup \{0\}\) and \(r \in \mathbb{N}_2\), applied to (4.5), we obtain
\[
h_r(z) = \frac{i^r}{\Gamma(r)} \left\{ 2^{-r+1} \psi_{r-1} \left(1 + i \frac{z}{2} \right) + (-2)^{-r+1} \psi_{r-1} \left(1 + i \frac{z}{2} \right) \right.
\]
\[
- \psi_{r-1} (1 - iz) - (-1)^{r-1} \psi_{r-1} (1 + iz) \right\},
\]
which completes the proof of the theorem.

The restriction of (4.4) to \(\mathbb{R}\) yields

**Corollary 4.2.** For all \(x \in \mathbb{R}, r \in \mathbb{N}\) we have
\[
h_r(x) = \begin{cases} 
\frac{i(-1)^{r+1}}{\Gamma(r)} \Re \left\{ 2^{-r+1} \psi_{r-1} \left(1 + i \frac{x}{2} \right) - \psi_{r-1} (1 + x) \right\}, & r \text{ odd} \\
\frac{i(-1)^{r-1}}{\Gamma(r)} \Im \left\{ 2^{-r+1} \psi_{r-1} \left(1 + i \frac{x}{2} \right) - \psi_{r-1} (1 + x) \right\}, & r \text{ even}
\end{cases}
\]

Finally, let us observe that the HE-series may also be connected with the original Eisenstein series, the digamma function being the connecting link. In fact, for real \(z = x\) (not possible for \(z\) complex, because otherwise we cannot exploit the mirror symmetry formula for the digamma function, that is \(\psi(1 - iz) = \psi(1 + iz)\)).

**Theorem 4.3.** a) For all \(x \in \mathbb{R} \setminus \{0\}\)
\[
h_1(x) = 2i \log 2 + 2i \Re \left\{ \varepsilon_1 \left(i \frac{x}{2} \right) - \varepsilon_1 (ix) - \psi \left(i \frac{x}{2} \right) + \psi (ix) \right\}. \tag{4.7}
\]
b) Also, we have
\[
h_1(x) = 2i \log 2 + 2i \Re \left\{ \coth \left(i \frac{x}{2} \right) - \coth (x) - \psi \left(i \frac{x}{2} \right) + \psi (ix) \right\}. \tag{4.8}
\]
c) For all \(x \in \mathbb{R} \setminus \{0\}\) and \(r \in \mathbb{N}_2\) we have
\[
h_r(x) = i^r \left\{ 2^{-r+1} \left( \varepsilon_r (i \frac{x}{2}) + (-1)^{r-1} \varepsilon_r (-i \frac{x}{2}) \right) - \varepsilon_r (ix) - (-1)^{r-1} \varepsilon_r (-ix) \right\}
\]
\[
+ \frac{(-1)^{r-1} i^r}{\Gamma(r)} \left\{ 2^{-r+1} \left( \psi_{r-1} (i \frac{x}{2}) + (-1)^{r-1} \psi_{r-1} (-i \frac{x}{2}) \right) \right.
\]
\[
- \psi_{r-1} (ix) - (-1)^{r-1} \psi_{r-1} (-ix) \right\}.
\]
Proof. The reflection formula (4.7) gives an efficient tool connecting Eisenstein and Hilbert–Eisenstein series. Indeed, replacing here \( \pm i x / 2 \), \( \pm i x \) for \( z \), the asserted relation (4.7) follows from Theorem 4.1, (4.3).

Now, by the differentiation property (4.6) and (4.4) we connect the HE series \( h_r(z) \) and the Eisenstein series \( \varepsilon_r(z) \), yielding part c) for \( r \in \mathbb{N}_2 \).

5. The \( \Omega(z) \)–function and its properties

This section devoted to the \( \Omega \)–function begins with the cases \( r = 1 \) and \( r = 2 \) of Definition 3.1 thus

\[
\frac{i\pi \Omega(2\pi z)}{\sin(\pi z)} = h_1(z), \quad (z \in \mathbb{C} \setminus i\mathbb{Z})
\]

\[
h_2(z) = 2iz \sum_{k \in \mathbb{N}} \frac{(-1)^k - 1}{2k} \frac{1}{(k^2 + z^2)^2} = 2iz \tilde{S}_2(z),
\]

the latter following directly from its definition, having in mind, and noting that the \( h_2(z) \) and \( \tilde{S}_2(z) \) are connected via \( h_2(z) = 2iz \tilde{S}_2(z) \) (see (4.2) as well). An immediate consequence of Theorem 4.1 or also of Corollary 3.5 is

Proposition 5.1. For all \( x \in \mathbb{R} \), \( \Omega(x) \) has the representation

\[
\Omega(x) = \frac{1}{\pi} \sinh\left(\frac{x}{2}\right) \left\{ 2 \log 2 + \psi\left(1 + i \frac{x}{2\pi}\right) + \psi\left(1 - i \frac{x}{2\pi}\right) - \psi\left(1 + i \frac{x}{2\pi}\right) - \psi\left(1 - i \frac{x}{2\pi}\right) \right\}.
\]

The next basic property, essentially stated in [4], concerns \( \Omega(x) \) as a solution of ODE.

It would be of great interest to know whether one can express \( \psi(ix) - \psi(i\frac{x}{2}) \) in terms of the coth–function (or any another hyperbolic function).

Theorem 5.2. For all \( x \in \mathbb{R} \) the \( \Omega(x) \) function is a particular solution of the following linear ODE:

\[
y' = \frac{1}{2} \coth\left(\frac{x}{2}\right) y - \frac{x}{\pi^3} \sinh\left(\frac{x}{2}\right) E(x),
\]

where

\[
E(x) = \begin{cases} 
\tilde{S}_2(x) = \frac{1}{2x} \int_0^\infty \frac{u \sin(xu)}{e^u + 1} \, du, & x \neq 0, \\
2 \eta(3), & x = 0.
\end{cases}
\]

Proof. Differentiating \( h_1(x) \) (or which is the same \( \Omega(2\pi x) \)) of (5.1) with respect to \( x \neq 0 \), this results in

\[
\Omega'(2\pi x) - \frac{1}{2} \coth(\pi x) \Omega(2\pi x) = - \frac{2x \sinh(\pi x)}{\pi^2} \tilde{S}_2(x).
\]
Substituting $x \mapsto 2\pi x$ we confirm (5.2) for $x \neq 0$. It remains to prove the case $x = 0$, which follows by continuity argument is equivalent to the asserted ODE, because

$$E(0) = \lim_{h \to 0} \tilde{S}_2(h) = 2\eta(3).$$

The integral form of $E(x)$ is already reported e.g. in [31, Eq. (2.8)]. □

**Theorem 5.3.** [13] a) For all $x \geq 0$ the following two-sided bounding inequalities hold true:

$$\frac{1}{\pi} \sinh \left( \frac{x}{2} \right) \log \left( \frac{\zeta(3)x^2 + 8\pi^2}{3x^2 + 2\pi^2} \right) \leq \Omega(x) \leq \frac{1}{\pi} \sinh \left( \frac{x}{2} \right) \log \left( \frac{3x^2 + 8\pi^2}{\zeta(3)x^2 + 2\pi^2} \right).$$

Moreover, for $x < 0$ the two-sided inequality is reversed.

b) For the asymptotic behavior of $\Omega(x)$ for large values of $x$ we have

$$\left( \frac{1}{2\pi} \log \frac{\zeta(3)}{3} \right) e^\frac{x^2}{2} \leq \Omega(x) \leq \left( \frac{1}{2\pi} \log \frac{3}{\zeta(3)} \right) e^\frac{x^2}{2}, \quad (x \to \infty).$$

See Figure 1 for the graphs of part a).

![Figure 1](image_url)

See Figure 2 where graph of $\Omega(x)$ lies within horn-type bounds. As to the proof of b), also announced in [13, Theorem 4], we only have to apply the asymptotic of the lower bound in a):

$$\sinh \left( \frac{x}{2} \right) \log \left( \frac{\zeta(3)x^2 + 8\pi^2}{3x^2 + 2\pi^2} \right) \sim e^\frac{x^2}{2} \log \frac{\zeta(3)}{3}, \quad (x \to \infty);$$

the same procedure leads to the upper bound in Theorem 5.3. b).

Obviously, by observing Figure 1 we conclude that the elegant but not so precise bilateral bounding inequality exposed in Theorem 5.3 could perhaps be improved.
Having in mind connections (see e.g. Theorem 5.2 and its proof) between the Omega function and the alternating Mathieu series $\tilde{S}_2$, various estimates for $\Omega$ and their efficiency have been considered in [32]; these approximants are consequences of the Čaplygin differential inequality. However, we will not consider this question here, since it deserves a separate retrospect.

Further, power series characterization of the complete Omega function are established in the sequel.

**Theorem 5.4.** For the complete Omega–function

$$\Omega(z) := 2 \int_{0+}^{z} \sinh(zu) \cot(\pi u) \, du, \quad (z \in \mathbb{C}),$$

there hold the properties:

(i) *(Partial fraction expansion)*

$$\Omega(z) = 2i \sinh \left( \frac{z}{2} \right) \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1} \text{sgn}(k)}{z + 2\pi ik};$$

(ii) *(Taylor–series expansion I.)*

$$\Omega(z) = \sum_{k \in \mathbb{N}_0} \Omega_{2k+1} \frac{z^{2k+1}}{(2k+1)!}, \quad (z \in \mathbb{C}), \quad (5.3)$$

where $\Omega_k$ are the moments of $2 \cot(\pi u)$, thus

$$\Omega_k = 2 \int_{0+}^{\frac{1}{2}} u^k \cot(\pi u) \, du = D_{\pi}^k \Omega(z) \bigg|_{z=0};$$
(iii) (Taylor–series expansion II.)

\[
\Omega(z) = \sum_{k \in \mathbb{N}_0} \frac{1}{2^{2k}} \left\{ \sum_{n=0}^{k} \frac{(-1)^n \eta(2n+1)}{\pi^{2n+1} (2(k-n)+1)!} \right\} z^{2k+1}, \quad (|z| < 2\pi);
\]

(5.4)

(iv) (Mirror symmetry formula)

\[
\Omega(\overline{z}) := \Omega(x + iy) = \overline{\Omega(z)}, \quad (z \in \mathbb{C});
\]

(v) (Reflexivity properties)

\[
\Re \Omega(x + iy) = \Re \Omega(x - iy),
\]
\[
\Im \Omega(x + iy) = -\Im \Omega(x - iy),
\]
\[
\Im \Omega(x) = 0, \quad \Re \Omega(\pm iy) = 0.
\]

Proof. Whereas property (i) is a basic property of the paper, (ii) follows by expanding \(\sinh(uz)\) into its power series, and substituting it into the definition of \(\Omega(z)\). Indeed,

\[
\Omega(z) = 2 \int_{0^+}^{\pi} \sum_{k \in \mathbb{N}_0} \left( \frac{uz}{2k + 1} \right)^{2k+1} \cot(\pi u) \, du = \sum_{k \in \mathbb{N}_0} \Omega_{2k+1} \frac{z^{2k+1}}{(2k + 1)!},
\]

where the legitimate exchange of summation and integration order is applied. Note that \(\Omega_k = D_k^+ \Omega(z)|_{z=0}\).

As to (iii), firstly let us remark that the Taylor series of the HE–series \(h_1(z)\) has been established in Proposition 3.2 a), precisely

\[
h_1(z) = 2i \sum_{n \in \mathbb{N}_0} (-1)^n \eta(2n+1)z^{2n}, \quad (|z| < 1).
\]

By (i), it is

\[
\Omega(z) = -\frac{i}{\pi} \sinh \left( \frac{z}{2} \right) h_1 \left( \frac{z}{2\pi} \right),
\]

where, in turn, the convergence is assured inside the disk \(|z| < 2\pi\). Accordingly

\[
\Omega(z) = \frac{2}{\pi} \sum_{m \in \mathbb{N}_0} \frac{\left( \frac{z}{2\pi} \right)^{2m+1}}{(2m+1)!} \sum_{n \in \mathbb{N}_0} (-1)^n \eta(2n+1) \left( \frac{z}{2\pi} \right)^{2n} \sum_{m,n \in \mathbb{N}_0} (-1)^n \eta(2n+1) z^{2(m+n)+1} 2^{(m+n)+1} (2m + 1)!.
\]

Eliminating \(m\) in the double sum, which becomes a simple one with respect to \(k = m + n; k \in \mathbb{N}_0, 0 \leq n \leq k\) we get

\[
\Omega(z) = \sum_{k \in \mathbb{N}_0} \frac{1}{2^{2k}} \left\{ \sum_{n=0}^{k} \frac{(-1)^n \eta(2n+1)}{\pi^{2n+1} (2(k-n)+1)!} \right\} z^{2k+1}.
\]
As to (iv) and (v), observe that

\[ \Omega(x \pm iy) = 2 \int_{0+}^{\pm} \sinh u(x \pm iy) \cot(\pi u) \, du = 2 \int_{0+}^{\pm} \sinh(ux) \cos(uy) \cot(\pi u) \, du \pm 2i \int_{0+}^{\pm} \cosh(ux) \sin(uy) \cot(\pi u) \, du. \]

In particular there follows

\[ \Re \Omega(x \pm iy) = 2 \int_{0+}^{\pm} \sinh u(x \pm iy) \cot(\pi u) \, du \]

\[ \Im \Omega(x \pm iy) = \pm 2 \int_{0+}^{\pm} \cosh(ux) \sin(uy) \cot(\pi u) \, du. \]

Now (iv) and (v) follow readily.

Since inside the disk \(|z| < 2\pi\) the function \(\Omega(z)\) possesses a unique Taylor expansion, both expansions (5.3) and (5.4) coincide there. So, equating the coefficients, we deduce the finite closed form expression for the moments of \(\cot(\pi x)\), namely

**Corollary 5.5.** The moments \(\Omega_k\) can be expressed as

\[ \Omega_{2k+1} = 2 \int_{0+}^{\pm} u^{2k+1} \cot(\pi u) \, du = \frac{(2k + 1)!}{2^{2k}} \sum_{n=0}^{k} \frac{(-1)^{n} \eta(2n + 1)}{\pi^{2n+1} (2(k-n) + 1)!}, \quad (5.5) \]

for all \(k \in \mathbb{N}_0\); thus in particular,

\[ \Omega_1 = \frac{\eta(1)}{\pi}, \quad \Omega_3 = \frac{1}{4} \left( \frac{\eta(1)}{\pi} - \frac{6\eta(3)}{\pi^3} \right), \quad \Omega_5 = \frac{1}{16} \left( \frac{\eta(1)}{\pi} - \frac{20\eta(3)}{\pi^3} + \frac{120\eta(5)}{\pi^5} \right). \]

The authors could not find representations of these important moments as a finite sum in tables of sums and integrals. However, there does exist a representation of \(\Omega_{2k+1}\) in terms of an infinite series, namely

\[ \Omega_{2k+1} = \frac{1}{2^{2k}\pi} \left\{ \frac{1}{2k + 1} + \sum_{n \in \mathbb{N}} \frac{(-1)^{n} B_{2n} \pi^{2n}}{(2n)! (2k + 2n + 1)!} \right\}, \quad (k \in \mathbb{N}_0), \]

see [21] p. 333, Eq. 13 d] and [20] p. 428, Eq. 3.748 2.

**6. Early ideas of Bernoulli, Euler and Ramanujan; some conjectures**

In order to observe to contribution of the innovative Ramanujan but also of the great Jacob I. Bernoulli, we first need to observe the famous representation of the
Riemann Zeta function, actually due to Euler (1735/39) [18], with the Bernoulli numbers. Indeed, we have shown (see [10, p. 62 et seq.])

**Theorem 6.1.** Let $\alpha \in \mathbb{C}$. Then

a) There holds for all $\alpha \neq 2m + 1$, $n \in \mathbb{N}$

$$
\zeta(\alpha) = -\sec \left( \frac{\alpha \pi}{2} \right) \cdot 2^{\alpha-1} \pi^{\alpha} \frac{B_\alpha}{\Gamma(\alpha + 1)};
$$

b) There holds for all $\Re(\alpha) > 1$

$$
\zeta(\alpha) = \csc \left( \frac{\alpha \pi}{2} \right) \cdot 2^{\alpha-1} \pi^{\alpha} \frac{B_\sim}{\Gamma(\alpha + 1)};
$$

c) **Counterpart of Euler’s formula.** There holds for odd arguments $2m + 1$, $m \in \mathbb{N}$

$$
\zeta(2m + 1) = (-1)^m 2^{2m} \pi^{2m-1} \frac{B_{2m+1}^\sim}{(2m + 1)!};
$$

d) **Euler’s closed form representation of $\zeta(2m)$.** There holds for even arguments

$$
\zeta(2m) = (-1)^{m+1} 2^{2m-1} \pi^{2m} \frac{B_{2m}}{(2m)!}.
$$

Firstly, the numbers $B_\alpha = B_\alpha(0)$ occurring in Euler’s representation d), defined in terms of the Bernoulli polynomials $B_n(x)$ via their exponential generating function (2.1), where introduced by Jacob I. Bernoulli – prior to 1695 – published posthumously in 1713 in the second chapter of his *Ars Conjectandi* [8].

In the counterpart c) for all arguments, the $B_\alpha$ has been replaced by the conjugate Bernoulli numbers $B_{2m+1}^\sim$, defined in terms of the Hilbert transform.

As to Ramanujan, he introduced the ”sign–less” fractional Bernoulli numbers $B_\alpha^*$ in terms of

$$
B_\alpha^* = \frac{2\Gamma(\alpha + 1)}{(2\pi)^\alpha} \zeta(\alpha),
$$

so that $B_{2m} = (-1)^{m+1} B_{2m} > 0$ for $\alpha = 2m, m \in \mathbb{N}$. Thus, he avoided to find substitute for $(-1)^{m+1}$ in part d) of the previous theorem (see Berndt [10, p. 125]).

There exists a short contribution by J.W.L. Glaisher [19] who defined fractional sign–less Bernoulli numbers $B_\alpha^*$, similarly as Ramanujan did, *via* the ”Euler” formula

$$
B_\alpha^* = \frac{2\Gamma(2\alpha + 1)}{(2\pi)^\alpha} \zeta(2\alpha).
$$

In fact, Euler himself (c.f. [5, p. 351]) had already proceeded in this way for the particular case $\alpha = \frac{3}{2}$ and $\alpha = \frac{5}{2}$. He set

$$
p = \frac{3}{2\pi^3} \left\{ 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots \right\} \approx 0.05815227,
$$

$$
q = \frac{15}{2\pi^5} \left\{ 1 + \frac{1}{2^5} + \frac{1}{3^5} + \cdots \right\} \approx 0.025413275,
$$
and defined
\[ B_{\frac{1}{2}} = p, \quad B_{\frac{3}{2}} = q. \]

The problem of evaluating \( \zeta(2m + 1) \), at odd integer values, first formulated by P. Mengoli in 1650 (see [5, p. 125]), cannot be solved by replacing the \( B_{2m} \) in part d) by \( B_{2m+1} \) as \( B_{2m+1} = B_{2m+1}(0) = 0 \) for all \( m \in \mathbb{N} \). There exist further articles published more recently, namely by Böhmer [9], Sinocov [36], Jonquière [26] and Musés [28].

At Aachen, we discovered the structural closed form solution of Mengoli’s question by replacing \( B_{2m+1} \) by the conjugate Bernoulli numbers \( B_{2m+1}(0) \), which do not vanish for all odd integer values.

The question arises as to what did Ramanujan (and Glaisher) actually mean with \( B_1 \) (and \( B_2 \))? For this purpose at first some words concerning the \( B_\alpha \) with fractional indices, not discussed in Section 2. At Aachen the Bernoulli functions \( B_\alpha(x) \) with index \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \) were defined first for \( x \in [0, 1) \) by
\[ B_\alpha(x) = \mathcal{B}_\alpha(x), \]
where \( \mathcal{B}_\alpha(x) \) is the periodic Bernoulli function, given by
\[ \mathcal{B}_\alpha(x) = -2\Gamma(\alpha + 1) \sum_{k \in \mathbb{N}} \frac{\cos(\frac{2\pi k x - \frac{\pi}{2}}{(2\pi k)^\alpha})}{(2\pi k)^\alpha}, \quad (x \in \mathbb{R}), \quad (6.1) \]
with \( x \neq 0 \) if \( \Re(\alpha) \in (0, 1] \). In addition, for \( \Re(\alpha) > 1 \), \( \mathcal{B}_\alpha := \mathcal{B}_\alpha(0) \), is called \( \alpha \)-th Bernoulli number.

Now, \( \mathcal{B}_\alpha(x), x \in \mathbb{R} \) is a holomorphic function of \( \alpha \) for \( \Re(\alpha) > 1 \), even holomorphic for \( \Re(\alpha) > 0 \) when \( x \in \mathbb{R} \setminus \mathbb{Z} \). The periodic functions \( \mathcal{B}_\alpha(x), x \in [0, 1) \) were then extended to \( \mathbb{R} \) such that \( \mathcal{B}_\alpha(x) \) interpolates the classical Bernoulli polynomials \( B_n(x) \) for all \( \alpha = n \in \mathbb{N} \), and were then denoted by \( B_{\alpha}(x) \), for all \( x \in \mathbb{R} \). They were then extended to \( B_{\alpha}(z) \), with \( \mathbb{C} \setminus \mathbb{R}_0^- \), for arbitrary \( \alpha \in \mathbb{C} \) (see below). The \( B_{\alpha}(z) \) led the way to the assertions a) and d) of Theorem 6.1, the former for \( \alpha \neq 2m + 1 \), the latter for \( \alpha = 2m \).

In order to solve parts b) and c), we introduced the Hilbert transform of \( \mathcal{B}_\alpha(x) \),
\[ \mathcal{B}_\alpha^\pi(x) = \mathcal{H}_1[\mathcal{B}_\alpha(\cdot)](x), \] set up the Fourier series of \( \mathcal{B}_\alpha^\pi(x) \), and proceeded to \( B_{\alpha}(z) \), their periodic extensions to \( \mathbb{R} \).

The Bernoulli functions, first defined for all \( \Re(\alpha) > 0 \) and \( x \in \mathbb{R} \), were then extended by analytic continuation to all \( \alpha \in \mathbb{C} \) and \( x = z \in \mathbb{C} \setminus \mathbb{R}_0^- \) by the contour integral representation
\[ B_{\alpha}(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{\mathcal{C}} \frac{ue^{uz}}{e^u - 1} \frac{du}{u^{\alpha+1}}, \quad (\Re(z) > 0, \alpha \in \mathbb{C}); \]
here \( \mathcal{C} \) denotes the positively oriented loop around the negative real axis \( \mathbb{R}^- \), which is composed of a circle \( C(0; 2r) \) centered at the origin and of radius \( 2r \) (\( 0 < r < \pi \)) together with the lower and upper edges \( C_1 \) and \( C_2 \) of the complex plane cut along the negative real axis \( \mathbb{R}^- \) (see Figure 3.).

The \( B_{\alpha}(x) \) coincide with the classical Bernoulli polynomials in the case \( \alpha = n \in \mathbb{N}_0 \). Indeed, according to the Cauchy integral formula for derivatives, noting that
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$C_1 = -C_2$, one has for $\Re(z) > 0$

\[
\frac{\Gamma(\alpha + 1)}{2\pi i} \int e^{u^z} \frac{dz}{e^u - 1} = \left( \frac{du}{e^u - 1} \right)^n \left( \frac{e^{u^z}}{e^u - 1} \right) = B_n(z),
\]

the last equality following by the defining generating function.

This definition was then extended to $z \in \mathbb{C} \setminus \mathbb{R}_0^-$, and it turned out to be consistent with the classical $B_n(z)$, with which they coincide for $\alpha = n$.

It is our conjecture that the Bernoulli functions $B_\sim^\alpha(z)$ can also be defined for $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \mathbb{R}_0^-$ in terms of the contour integral

\[
B_\sim^\alpha(z) = -\frac{\Gamma(\alpha + 1)}{2\pi i} \int e^{u^z} \Omega(u) \frac{du}{e^u - 1},
\]

(6.2)

It is based upon the fact that, formally,

\[
\mathcal{H}_1 \left[ \frac{ue^{u^z}}{e^u - 1} \right](z) = \frac{u}{e^u - 1} \text{PV} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{(z-\frac{1}{2})u} \cot(\pi y) \, dy = -\frac{ue^{u^z}}{e^u - 1} \Omega(z).
\]

Let us finally add that given the definition of $B_\sim^\alpha(z)$ via (6.2) for $\alpha = 2j + 1$, and assuming that it is correct, one can surely deduce the Fourier expansion of $B_\sim^\alpha(z)$ (found in [10 p. 32]), using the calculus of residues (see e.g. Saalschütz [35] p. 27, also see [33] p. 331)). In that case, the validity of part b) of Theorem 6.1 could be extended from $\Re(\alpha) > 1$ to all $\alpha \in \mathbb{C}$ (just as for part a)).

Hopefully the values of (6.2) at $z = 0$, that is $B_\sim^\alpha(0)$, could then give us more information concerning the possible irrationalities of $\zeta(5), \zeta(7), \zeta(9), \cdots$.

Substituting the Taylor series representation (5.3) of $\Omega(u)$ into $B_\sim^\alpha(z)$ of (6.2) for $\alpha = 2j + 1, j \in \mathbb{N}_0$, and interchanging the integral and sum, there results the
representation

\[ B_{2j+1}(z) = \frac{\Gamma(2j + 2)}{2\pi i} \sum_{k \in \mathbb{N}_0} \frac{\Omega_{2k+1}}{(2k + 1)!} \int e^{uz} \frac{u^{2k}}{1 - e^u} du, \quad (6.3) \]

the latter integral being easier to evaluate than \( B_{2j+1}(z) \) in terms of the original (6.2). In turn, as inside \(|u| < 2\pi\) the Bernoulli polynomials’ generating function

\[ \frac{ue^{uz}}{e^u - 1} = \sum_{n \in \mathbb{N}_0} B_n(z) \frac{u^n}{n!}, \]

by the Cauchy’s integral formula we have

\[ J_{j,k}(z) = \int e^{uz} \frac{u^{2k+1}}{e^u - 1} \frac{du}{u^{2j+1}} = \frac{2\pi i}{\Gamma(2j + 1)} \left( \frac{d}{du} \right)_{u=0}^{2j} \left( u^{2k} \frac{ue^{uz}}{e^u - 1} \right) \]

\[ = \frac{2\pi i}{\Gamma(2j + 1)} \left( \frac{d}{du} \right)_{u=0}^{2j} \left( \sum_{n \in \mathbb{N}_0} B_n(z) \frac{u^{2k+n}}{n!} \right) \]

\[ = \frac{2\pi i}{(2j)!} \sum_{n \in \mathbb{N}_0} B_n(z) \frac{(2k+n)!}{(2k + n - 2j)!} \left( u^{2k+n-2j} \right)_{u=0} \]

\[ = \frac{2\pi i}{(2j - 2k)!} B_{2j-2k}(z). \]

So, employing Corollary 5.5, we conjecture that \( B_{2j+1}(z) \) can be represented as the following double finite sum, involving the \( B_{2j-2k} \) and the \( \eta(2n + 1) \), as

\[ B_{2j+1}(z) = -\frac{\Gamma(2j + 2)}{2\pi i} \sum_{k \in \mathbb{N}_0} \frac{\Omega_{2k+1}}{(2k + 1)!} J_{j,k}(z) \]

\[ = -\frac{(2j + 1)!}{\pi} \sum_{k = j}^{j} \frac{B_{2j-2k}(z)}{4^k (2j - 2k)!} \sum_{n = 0}^{k} \frac{(-1)^n \eta(2n + 1)}{\pi^{2n} (2(k - n) + 1)!}. \quad (6.5) \]

A portrait of Godfrey Harold Hardy (1877–1947)

Born February 7, 1877 in Cranleigh, Surrey, Hardy graduated from Trinity College, Cambridge in 1899, became a fellow at Trinity in 1900, and lectured in mathematics there from 1906 to 1919, since 1914 as Cayley Lecturer. In 1919 he was appointed to the Savilian Chair of Geometry at the University of Oxford, spent 1928–29 as visiting professor at Princeton, returned back to Oxford, and finally became Sadleirian Professor of Pure Mathematics in Cambridge in 1931. There he remained until his death on December 1, 1947, after having retired in 1942.

Among Hardy’s early works are his popular eleven books, among them Integration of Functions of a Single Variable, CUP (1905); A Course of Pure Mathematics, CUP (1908); (10th Edit. 2008, with T. Korner); followed by The General
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Theory of Dirichlet’s Series, CUP (1915), with M. Riesz; Inequalities, CUP (1934), with J.E. Littlewood and G.Pólya (reprinted in 1952); An Introduction to the Theory of Numbers, OUP (1938), with E.M. Wright (6th Edit. in 2008 with D.R. Heath–Brown and J.H. Silverman); A Mathematician’s Apology, CUP (1940, 2004); Fourier Series, CUP (1944), with W.W. Rogosinski; Divergent Series, OUP (1949). Hardy was the author or coauthor of more than 300 papers and the recipient of numerous honours. His doctoral students included L. Bosanquet, M. Cartwright, U. Haslam-Jones, A.C. Offord, R. Rado, R. Rankin, K.A. Rau, D. Spencer and E. Titchmarsh. Hardy’s collaboration with J.E. Littlewood, which set in 1911 and extended over 30 years, brought fresh impetus into his work. Their collaboration is among the most famous as such in mathematics history.

The sequence of essays by his former students in [J. London Math. Soc. 25 (1950) 81–101] is an excellent source on Hardy’s place in mathematics. See also G.H. Hardy, Collected Papers, 7 vols. (Clarendon Press, Oxford, 1966-1979).

According to MacTutor’s article on Hardy: "... Hilbert was so concerned that Hardy was not being properly treated (while living at Cambridge) that he wrote to the Master of the College pointing out that the best mathematician in England should have the best rooms."

Harald Bohr, who stood in close contact with Hardy, assessing the leadership of Hardy and Littlewood in English research (1947), wrote "I may report what an excellent colleague once jokingly said: "Nowadays, there are only three really great English mathematicians: Hardy, Littlewood, and Hardy–Littlewood."

The following quotations give an interesting view of Hardy’s thoughts: "I am obliged to interpolate some remarks on a very difficult subject: PROOF and its importance in mathematics. All physicists, and a good many quite reputable mathematicians, are contemptuous about proof. I heard Professor Eddington, for example, mention that proof, as pure mathematicians understand it, is really quite uninteresting and unimportant, and that no one who is really certain that he has found something good should waste his time looking for proof." While reviewing the question of the reality of nature from a mathematical viewpoint, G.H. Hardy stated: "... I will state my own position dogmatically in order to avoid minor misapprehensions. I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that theorems which we prove, and which we describe grandiloquently as our "creations" are simply our notes of our observations. This view has been held in some form or another, from Plato onwards."

According to the appraisal of the Trinity College Chapel, Hardy was universally recognized as pre-eminent among the world’s best mathematicians"; "As his most important influence Hardy cited the self-study of Cours d’Analyse de l’Ecole Polytechnique by the French mathematician Camille Jordan ... Hardy is credited with reforming British mathematics by bringing rigour into it, which was previously a characteristic of French, Swiss and German mathematics. British mathematics had remained largely in the tradition of applied mathematics, in thrall to the reputation
of Isaac Newton (see Cambridge Mathematical Tripos). Hardy was more in tune with the *cours d’analyse* methods dominant in France, and aggressively promoted his conception of pure mathematics, in particular against the hydrodynamics which was an important part of Cambridge mathematics.”

More significant than Hardy’s collaboration with Littlewood was that with Srinivasa Aiyangar Ramanujan, born 1887 at Erode, Madras Presidency *(of a Tamil Brahmin family)*, a self-taught and obsessive shipping clerk from Madras (see [3]). In 1913 he sent a nine-page paper to Hardy, dealing with two remarkable, novel infinite series of hypergeometric type (related to research of Euler and Gauss), and continued fractions. Hardy was so amazed that he commented to Littlewood that Ramanujan was "a mathematician of highest quality, a man of altogether exceptional originality and power". Hardy brought him to Cambridge (the well-established H.F. Baker and E.W. Hobson had returned the papers without comment), made him aware of modern mathematics and so provided a solid foundation to Ramanujan’s inventiveness. They became friends, collaborated (called ”the one romantic incident in my life” by Hardy) and wrote five remarkable papers together.

Ramanujan made extraordinary contributions to mathematical analysis, number theory, continued fractions and infinite series, rediscovered known theorems of Bernoulli, Euler, Gauss and Riemann. He conjectured or proved nearly 3900 theorems, identities and equations. Hardy regarded the "discovery of Ramanujan" as "his greatest contribution to mathematics", and assailed Ramanujan’s natural genius, being on the same league as Euler and Gauss. In his book *Ramanujan’s twelve lectures on subjects suggested by his life and work* (Cambridge, 1940), Hardy placed an everlasting monument for Ramanujan who had died too young. Since Bernoulli polynomials and numbers play an essential role in the present paper, and Ramanujan wrote his first formal paper (at age 17, but published it later) [Some properties of Bernoulli’s numbers, *J. Indian Math. Soc.* 3 (1911) 219–234] this paper is dedicated to Hardy, his discoverer. For an overview of Bruce Berndt’s excellent work in matters Ramanujan, see his *An overview of Ramanujan’s notebooks* [6] and *Ramanujan, his lost notebooks, its importance* [7]; also see [3].

For his lecture tour of ten universities in UK, which was kindly organized by Lionel Cooper (1915–1979), P.L.B. offered five possible topics, one being "On some theorems of Hardy, Littlewood and Titchmarsh". Surprisingly, no university in the England or Scotland picked it. At a special dinner at Chelsea College at the beginning of the tour, in the presence of about a dozen of London’s major analysts, one remarked that a reason could be that Hardy is/was no longer regarded as Britain’s top mathematician at the time. Even students (whom P.L.B. also regarded as great British analysts) were assigned to the same category. Peculiarly, no one present reacted to the astonishing assertion or countered it.

Both authors are grateful to Maurice Dodson for inviting them to his conference Fourier Analysis and Applications, held at York in 1993, where both experienced [8]
three of Britain’s best analysts at the time, namely James (Jim) Gourlay Clunie (1926–2013) [14, pp. 108–109], Walter Hayman and Frank Bonsall (1920–2011). The former two, spent their "retirement" at York University, the latter at the spa town of Harrogate.

* * *

In memory of Pater Wilhelm Brabender, OMI (1879–1945), a paternal great-uncle of P.L.B., an Oblate missionary in Saakatchevan and British Columbia (Canada) from 1905 to 1931, who preached in French, German and English, was an authority on Cree ethnography, and Rev. Pater Julius Pogany, S.J. (1909-1986), paternal uncle of T.K.P., who was a Jesuit missionary and teacher at St. Aloysius’ College in Galle (Sri Lanka) from 1949 to 1963 (teaching in both English and Sinhalese), had received his MSc in Theoretical Mathematics and Physics at Royal Yugoslav University in Zagreb in 1936, and had completed his Jesuit education at the Pontifical Gregorian University, Rome, in 1944.

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