MORI DREAM SPACES AND BLOW-UPS OF WEIGHTED PROJECTIVE SPACES

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Abstract. For every $n \geq 3$, we find a sufficient condition for the blow-up of a weighted projective space $\mathbb{P}(a, b, c, d_1, \cdots, d_{n-2})$ at the identity point not to be a Mori Dream Space. We exhibit several infinite sequences of weights satisfying this condition in all dimensions $n \geq 3$.

1. Introduction

We study the question whether the blow-up of a projective, $\mathbb{Q}$-factorial toric variety over $\mathbb{C}$ of Picard number one, at the identity point $p$ of the open torus, is a Mori Dream Space (MDS).

Mori Dream Spaces were introduced by Hu and Keel in [HK00]. By [BCHM10], log Fano varieties over $\mathbb{C}$ are Mori Dream Spaces. Projective, $\mathbb{Q}$-factorial toric varieties, being log Fano, are MDS. The property of being a MDS is nevertheless not a birational invariant. In fact, the blow-up of $\mathbb{P}^n$ at $r$ very general points stops being a MDS if $r > 8$ for $\mathbb{P}^2$ and $\mathbb{P}^4$, $r > 7$ for $\mathbb{P}^3$, and $r > n + 3$ for $n \geq 5$ [Muk05]. One of the motivations to study blow-ups of toric varieties at the identity point comes from the proof by Castravet and Tevelev [CT15] that the moduli spaces of stable rational curves $\overline{M}_{0,n}$ are not MDS when $n > 133$, which was later improved to $n > 12$ by González and Karu [GK16] and to $n > 9$ by Hausen, Keicher and Laface [HKL16]. The proof of [CT15] used the examples of not MDS blow-ups of weighted projective planes (see 1.4 and 1.5) by Goto, Nishida and Watanabe [GNW94].

The discussion above prompts the question of searching for not MDS blow-ups of toric varieties of small Picard numbers, which was formulated in [Cas15]. Historically, much research work was done for surfaces. For a weighted projective plane $S = \mathbb{P}(a, b, c)$, let $p$ be the identity point of the open torus. If the anticanonical divisor $-K$ of the blow-up $\text{Bl}_p S$ of $S$ at $p$ is big, then $\text{Bl}_p S$ is a MDS [Cut91]. If one of $a, b, c$ is at most 4 or equals 6 then $\text{Bl}_p S$ is a MDS [Cut91][Sri91]. The first examples where $\text{Bl}_p S$ is not a MDS were given in [GNW94]. A generalization was achieved by González and Karu [GK16] for toric varieties of Picard number one whose corresponding polytope $\Delta$ has specific numbers of lattice points in its columns. The question can be formulated as an interpolation problem on the lattice points in $\Delta$ and leads to 3 families of new nonexamples [He17]. We note that for any weighted projective space $X$, $\text{Bl}_p X$ is a MDS if and only if the Cox ring of $\text{Bl}_p X$ is a finitely generated $\mathbb{C}$-algebra, which is also equivalent to the finite generation of the symbolic Rees algebra associated to $X$ [Cut91][GNW94], which is of independent interest.
In higher dimensions not much was known until the recent work [GK17]. In [GK17] González and Karu constructed higher dimensional toric varieties $X$ of Picard number one with $Bl_p X$ not a MDS, by exhibiting a nef but not semiample divisor on $Bl_p X$. Their examples include some weighted projective 3-spaces $X = \mathbb{P}(a, b, c, d)$ such that $Bl_p X$ is not a MDS.

In this paper, we give a sufficient condition (Theorem 1.2) so that the blow-up of the weighted projective $n$-space $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ at the identity $p$ is not a MDS. We show new examples of such $X$ in all dimensions $n \geq 3$.

We sum up our results below. We work over the complex numbers $\mathbb{C}$. Let $N = \mathbb{Z}^2$ and $M$ be the dual lattice of $N$. Let $S$ be a normal projective, $\mathbb{Q}$-factorial toric surface of Picard number one, with fan $\Sigma_S$ in $N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2$. Then a polarization $H = H_\Delta$ on $S$ is determined by a rational triangle $\Delta$ in $M \otimes_{\mathbb{Z}} \mathbb{R}$ whose normal fan is $\Sigma_S$. Let the sides of $\Delta$ have rational slopes $s_1 < s_2 < s_3$. We choose $\Delta$ so that after translating one vertex of $\Delta$ to $(0, 0)$, the opposite side passes through $(0, 1)$. Then the width of this $\Delta$ equals $w := 1/(s_2 - s_1) + 1/(s_3 - s_2)$. This $w$ is called the width of the polarized toric surface $(S, H_\Delta)$ (see [GK16, Thm 1.2]).

A weighted projective plane $S = \mathbb{P}(a, b, c)$ is an example of normal $\mathbb{Q}$-factorial toric surfaces of Picard number one. A triple $(e, f, -g)$ is called a relation between the weights $(a, b, c)$ if $e, f, g \in \mathbb{Z}_{>0}$ and $ae + bf = cg$ [GK16, Thm. 1.5]. Then there exists a polarization $H_\Delta$ such that the width $w$ of $(S, H_\Delta)$ is smaller than 1 if and only if there exists a relation $(e, f, -g)$ with $cg^2/ab = w < 1$. Such $(e, f, -g)$ is unique if it exists, even when permuting the weights $a, b, c$. Therefore for a relation $(e, f, -g)$ we define the width of $(e, f, -g)$ to be $cg^2/(ab)$.

Given $\xi = (e, f, -g)$ a relation with width $w < 1$, we can construct a fan $\Sigma_\xi$ of $S$ and the polytope $\Delta_\xi$ with width $w$ as follows: By [He17, Prop. 5.1], there exists a unique integer $r$ such that $1 \leq r \leq g$, $g \mid er - b$ and $g \mid fr + a$. Then the following vectors are primitive and span $\mathbb{Z}^2$:

\begin{equation}
(1)
\begin{align*}
u_0 &= \left(\frac{er - b}{g}, -e\right), \\
u_1 &= \left(\frac{fr + a}{g}, -f\right), \\
u_2 &= (-r, g).
\end{align*}
\end{equation}

Clearly $au_0 + bu_1 + cu_2 = 0$. Hence the fan $\Sigma_\xi$ with ray generators $u_0, u_1$ and $u_2$ is a fan of $\mathbb{P}(a, b, c)$. The triangle $\Delta_\xi$ has vertices

\begin{equation}
(2)
(0, 0), \quad \left(-\frac{eg}{b}, -\frac{er - b}{b}\right), \quad \left(\frac{fg}{a}, \frac{fr + a}{a}\right),
\end{equation}

which is normal to $\Sigma_\xi$ and has width $w = cg^2/(ab)$ (See Figure 1).

Throughout this paper, we always assume that the weights $q_0, q_1, \cdots, q_n$ of a weighted projective $n$-space $\mathbb{P}(q_0, q_1, \cdots, q_n)$ are well-formed, i.e., any $n$ weights are relatively prime.

For any weighted projective space $X$, let $p$ be the identity point of the open torus in $X$. For $S = \mathbb{P}(a, b, c)$, let $B$ be the pseudo-effective divisor on $S$ generating $Cl(S) \cong \mathbb{Z}$. Let $e$ be the exceptional divisor of the blow-up $\pi : Bl_p S \to S$. Our main result is:
Theorem 1.1. Let $X = \mathbb{P}(a, b, c, d_1, d_2, \ldots, d_{n-2})$ where $a, b, c$ are pairwise coprime. Let $S = \mathbb{P}(a, b, c)$. Suppose there is a negative curve $C$ on $\text{Bl}_p S$, different from $e$, with $C \sim \lambda \pi^* B - \mu e$ for some $\lambda, \mu \in \mathbb{Q}$. Suppose all the following hold:

(i) every $d_i$ lies in the semigroup generated by $a, b$ and $c$ (i.e., $d_i$ is a linear combination of $a, b, c$ with non-negative integer coefficients),

(ii) $d_i < \frac{abc \mu}{\lambda}$ for every $i$,

(iii) $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS.

Then $\text{Bl}_p X$ is not a MDS.

We show a special case of Theorem 1.1 when there is a relation $(e, f, -g)$ between the weights $(a, b, c)$ with $w < 1$. In this case, there exists a negative curve $C \sim cg \pi^* B - e$ on $\text{Bl}_p S$, and we have:

**Theorem 1.2.** Let $X = \mathbb{P}(a, b, c, d_1, d_2, \ldots, d_{n-2})$ be a weighted projective $n$-space where $a, b, c$ are pairwise coprime. Let $p$ be the identity point of the open torus in $X$. Suppose all the following hold:

(i) there is a relation between the weights $(a, b, c)$ such that the width satisfies $w < 1$.

(ii) every $d_i$ lies in the semigroup generated by $a, b$ and $c$.

(iii) $d_i^2 w < abc$ for every $i$.

(iv) $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS.

Then $\text{Bl}_p X$ is not a MDS.

In particular, if all $d_i = a$ and $a < b < c$ with $w < 1$, then $d_i^2 w = a^2 w < a^2 < abc$. Thus we have the following corollary:

**Corollary 1.3.** Assume that $a < b < c$ are pairwise coprime. Suppose $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS, and there is a relation between the weights $(a, b, c)$ such that the width satisfies $w < 1$. Then $\text{Bl}_p \mathbb{P}(a, b, c, a, \ldots, a)$ is not a MDS.
Example 1.4. By [GNW94], the Cox ring of the blow-up of \( \mathbb{P}(a, b, c) \) at the identity point is not finitely generated as a \( \mathbb{C} \)-algebra when \((a, b, c) = (7m - 3, 8m - 3, (5m - 2)m) \) for \( m \geq 4 \) and \( 3 \nmid m \). Equivalently, the blow-up at \( p \) is not a MDS. The sequence of weights has relation \((e, f, -g) = (m, m, -3)\) so that \( w < 1 \).

By Theorem 1.2, we conclude that \( \text{Bl}_p \mathbb{P}(7m - 3, 8m - 3, (5m - 2)m, d_1, \ldots, d_{n-2}) \) is not a MDS when

(i) \( m \geq 4 \) and \( 3 \nmid m \),
(ii) every \( d_i \) lies in the semigroup generated by \( 7m - 3, 8m - 3 \) and \( (5m - 2)m \), and
(iii) every \( d_i < (7m - 3)(8m - 3)/3 \).

By Corollary 1.3, \( \text{Bl}_p \mathbb{P}(7m - 3, 8m - 3, (5m - 2)m, 7m - 3, \ldots, 7m - 3) \) is not a MDS for \( m \geq 4 \) and \( 3 \nmid m \).

Example 1.5. Another infinite sequence given by [GNW94] where the blow-ups at \( p \) are not MDS is \((a, b, c) = (7m - 10, 8m - 3, 5m^2 - 7m + 1)\) for any \( m \geq 5 \) such that \( 3 \nmid 7m - 10 \) and \( m \not\equiv -7 \pmod{59} \) (By [GK16] the blow-up at \( p \) is also not a MDS when \( m = 3 \)). The sequence of weights has relation \((e, f, -g) = (m, m - 1, -3)\) so that \( w < 1 \).

We conclude by Theorem 1.2 that \( \text{Bl}_p \mathbb{P}(7m - 10, 8m - 3, 5m^2 - 7m + 1, d_1, \ldots, d_{n-2}) \) is not a MDS when

(i) \( m \geq 3 \), \( 3 \nmid 7m - 10 \) and \( m \not\equiv -7 \pmod{59} \),
(ii) every \( d_i \) lies in the semigroup generated by \( 7m - 10, 8m - 3 \) and \( 5m^2 - 7m + 1 \), and
(iii) every \( d_i < (7m - 10)(8m - 3)/3 \).

Example 1.6. The infinite sequence \((a, b, c) = (7, 15 + 2t, 26 + 3t)\) for \( t \geq 0 \) has the relation \((e, f, -g) = (1, 3, -2)\). The weights \((a, b, c)\) are pairwise coprime if and only if \( 7 \nmid t - 3 \). They all satisfy the criterion of [GK16, Thm. 1.5], so \( \text{Bl}_p \mathbb{P}(a, b, c) \) is not MDS for every \( t \geq 0 \), where the width

\[
w = \frac{4(26 + 3t)}{7(15 + 2t)} = \frac{104 + 12t}{105 + 14t} < 1
\]

for \( t \geq 0 \). Theorem 1.2 (3) then gives the upper bound

\[
d < \sqrt{\frac{abc}{w}} = \frac{ab}{g} = \frac{7(15 + 2t)}{2}.
\]

Note that when \( t \geq 0 \), \( a + b = 2t + 22 < \frac{7(15 + 2t)}{2} \). Hence \( d = a + b \) is on the list. As a result, \( \text{Bl}_p \mathbb{P}(7, 15 + 2t, 26 + 3t, d_1, \ldots, d_{n-2}) \) is not a MDS when

(i) \( t \geq 0 \) and \( 7 \nmid t - 3 \),
(ii) every \( d_i \) lies in the semigroup generated by \( 7, 15 + 2t \) and \( 26 + 3t \), and
(iii) every \( d_i < 7(15 + 2t)/2 \).

The paper is organized as follows. In Section 2, we give a sufficient condition (Theorem 2.1) for the blow-up \( \text{Bl}_p X \) of a normal projective variety \( X \) with Picard number 1 not to be a MDS, with \( p \) a smooth point on \( X \). Such \( \text{Bl}_p X \) has a nef but not semiample divisor. Sections 3 and 4 consider weighted projective \( n \)-spaces \( X \) with properties described in
Theorem 1.1. We show that $X$ contains a closed subvariety isomorphic to $S = \mathbb{P}(a, b, c)$. Section 5 verifies the conditions in Theorem 2.1 for $X$ and $S$, applying a result of Fulton and Sturmfels [FS97, Lem. 3.4]. In particular, we prove that $\text{Bl}_p X$ is not a MDS.

In Section 6, we compare our results with the examples in [GK17]. Proposition 6.6 describes the overlap of list in dimension 3 with González and Karu’s in [GK17]. The only common examples are $X = \mathbb{P}(a, b, c, cg)$ where $(e, f, -g)$ is a relation between $(a, b, c)$, and $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS and satisfies the assumptions in [GK17, Cor. 2.5]. Note that we give more examples beyond the overlap (Examples 1.4, 1.5 and 1.6).

In Section 7, we apply Theorem 1.1 to the case when $X = \mathbb{P}(a, b, c, d_1, d_2, \ldots, d_{n-2})$ where $S = \mathbb{P}(a, b, c)$ being of the form considered in [GAGK17, Ex. 1.4]. Hence $\text{Bl}_p S$ is again not a MDS. This leads to new examples where $\text{Bl}_p X$ is not MDS in Corollary 7.1.

2. Blow-ups of varieties of Picard number one

Let $X$ be a normal, projective, $\mathbb{Q}$-factorial variety of Picard number 1 and dimension $n \geq 3$. Suppose $Y_1, \ldots, Y_{n-2}$ are prime Weil divisors of $X$ ($Y_i$ not necessary normal), such that the set-theoretic intersection $S := \cap_{i=1}^{n-2} Y_i$, with the reduced subscheme structure on $S$, is a normal, projective, $\mathbb{Q}$-factorial surface of Picard number 1. In addition, suppose both $\text{Pic}(X)$ and $\text{Pic}(S)$ are finitely generated.

Let us blow up $S$ and $X$ at a point $p \in S$ which is smooth in $X$, $S$ and each $Y_i$. Let $f : \text{Bl}_p S \to \text{Bl}_p X$ be the natural inclusion. Let $E$ be the exceptional divisor of the blow-up $\pi_X : \text{Bl}_p X \to X$ and $e$ be the exceptional divisor of $\pi : \text{Bl}_p S \to S$.

Theorem 2.1. Let $X, Y_i, S$ and $f$ be defined as above. Suppose there exists an irreducible curve $C$ in $\text{Bl}_p S$, different from the exceptional divisor $e$ in $\text{Bl}_p S$, with $C^2 < 0$, such that for every $i$, $(f_* C) \cdot \text{Bl}_p Y_i < 0$ in $\text{Bl}_p X$. Then if $\text{Bl}_p S$ is not a Mori Dream Space (MDS), then $\text{Bl}_p X$ is not a Mori Dream Space.

Proof. Here both $\text{Bl}_p S$ and $\text{Bl}_p X$ have Picard number 2. Since $C^2 < 0$ in $\text{Bl}_p S$, $C$ spans an extremal ray of the Mori cone $\overline{\text{NE}}(\text{Bl}_p S)$ [KM08, Lem. 1.22]. Since $e$ is numerically equivalent to a general line in the exceptional divisor $E$ of $\text{Bl}_p X$, $[e]$ spans an extremal ray in both $\overline{\text{NE}}(\text{Bl}_p X)$ and $\overline{\text{NE}}(\text{Bl}_p S)$.

Let $C_1$ be the image of $C$ in $\text{Bl}_p X$, and $e_1$ be the image of $e$ in $\text{Bl}_p X$. We show that $[C_1]$ spans the other extremal ray of $\overline{\text{NE}}(\text{Bl}_p X)$. Since $C$ is irreducible, $C_1$ is irreducible. Suppose towards a contradiction that $C_1$ is not extremal in $\overline{\text{NE}}(\text{Bl}_p X)$. Then $C_1 \equiv r_1 F_1 + s_1 e_1$ for some effective curve $F_1$ and some rational numbers $r_1, s_1 > 0$. Then there exists an irreducible component $F_2$ of $F_1$ such that $F_1 \equiv r_2 F_2 + s_2 e_1$ for some rational numbers $r_2 > 0$ and $s_2 \geq 0$. Therefore we can assume at the beginning that $F_1$ is irreducible. By assumption, $C_1 \cdot \text{Bl}_p Y_i < 0$ for every $i$. Since $\text{Bl}_p Y_i$ is isomorphic to the proper transform of $Y_i$ in $X$, and the class of $e_1$ is the class of a line in $E$, we have $e_1 \cdot \text{Bl}_p Y_i \geq 0$. Therefore $F_1 \cdot \text{Bl}_p Y_i < 0$. The irreducibility assumption of $F_1$ implies that $F_1 \subset \text{Bl}_p Y_i$. Run this for every $i$, and we have $F_1 \subset \cap_i \text{Bl}_p Y_i = \text{Bl}_p S$. Consider the pushforward $f_* : N_1(\text{Bl}_p S) \to N_1(\text{Bl}_p X)$ and the pullback $f^* : N^1(\text{Bl}_p X) \to N^1(\text{Bl}_p S)$. Since $N^1(\text{Bl}_p S)$ is spanned by $[f_* H]$ and $[e]$ where $H = \pi_X^* H_0$ is the total transform of a very ample divisor $H_0$ on $X$, and $e \equiv f^* E$, we have $f^*$ is surjective. The dual paring between $N^1(\text{Bl}_p X)$ and $N_1(\text{Bl}_p X)$
(respectively \(N^1(\text{Bl}_p S)\) and \(N_1(\text{Bl}_p S)\)) is perfect. Hence \(f_*\) is injective by the projection formula. Now \(f_*(C - r_1 F_1 - s_1 e) \equiv C_1 - r_1 F_1 - s_1 e_1 \equiv 0\). By injectivity, \(C - r_1 F_1 - s_1 e \equiv 0\). Then the ray \(\mathbb{R}_{\geq 0}[C]\) is not extremal in \(\overline{N}\mathcal{E}(\text{Bl}_p S)\), which is a contradiction. Hence the ray \(\mathbb{R}_{\geq 0}[C_1]\) is extremal in \(\overline{N}\mathcal{E}(\text{Bl}_p X)\).

Finally, suppose \(\text{Bl}_p X\) is a MDS. Since \(X\) is \(\mathbb{Q}\)-factorial, and \(p\) is smooth in \(X\). \(\text{Bl}_p X\) is also \(\mathbb{Q}\)-factorial. Then the nef cone of \(\text{Bl}_p X\) is generated by semiample divisors. In particular, there is a semiample divisor \(D\) such that \(D.C_1 = 0\). Therefore \(f^*D \cdot C = f_*(f^*D \cdot C) = D \cdot f_*C = D \cdot C_1 = 0\) by projection formula. Hence \([f^*D]\) spans an extremal ray of \(\text{Nef}(\text{Bl}_p S)\). Now \(f^*D\) is also semiample. This shows that \(\text{Bl}_p S\) is a MDS. \(\Box\)

### 3. Divisors on weighted projective spaces

In this section we construct the fan of the weighted projective \(n\)-space \(X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2})\) and define \(n - 2\) divisors \(Y_j\) on \(X\) for \(j = 3, 4, \ldots, n\), under the assumption (i) of Theorem 1.1. Then we show that the set-theoretic intersection of those \(Y_j\) equals the Zariski closure of a 2-dimensional subtorus in \(X\).

**Notation 3.1.** We list some notations and terminology for later use.

- For any integer \(n \geq 3\), let \(J := \{3, 4, \ldots, n\}\).
- Let \(N \cong \mathbb{Z}^n\) \((n \geq 3)\) be a lattice. Let \(T_N = N \otimes \mathbb{C}^*\). Then \(T_N\) is a torus of dimension \(n\). Let \(M = \text{Hom}(N, \mathbb{Z})\) be the dual lattice of \(N\). Then \(M = \text{Hom}(T_N, \mathbb{C}_m)\), so each \(u \in M\) defines a character \(\chi^u\) on \(T_N\).
- If \(e_1, e_2, \ldots, e_n\) form a basis of \(N\), then \(e_1^*, \ldots, e_n^*\) form the dual basis of \(M\). Write \(\chi_j := \chi^{e_j}\). Then \(T_N = \text{Spec} \mathbb{C}[\chi_1, \chi_1^{-1}, \ldots, \chi_n, \chi_n^{-1}]\).
- For any lattice \(L\), define \(L_{\mathbb{R}} := L \otimes \mathbb{R}\).
- Let \(N_1 := \mathbb{Z}\{e_1\}\) be the sublattice of \(N\) spanned by \(e_1\). Let \(N_{12} := \mathbb{Z}\{e_1, e_2\}\) be the sublattice spanned by \(e_1\) and \(e_2\). Let \(T_1 := N_1 \otimes \mathbb{C}^*\) and \(T_{12} := N_{12} \otimes \mathbb{C}^*\) be the corresponding subtori of \(T_N\). Let \(M_{12} := \text{Hom}(N_{12}, \mathbb{Z})\).
- Let \(L_j := \mathbb{Z}\{e_1, e_2, \ldots, e_j, \ldots, e_n\}\) for \(j \in J\). Let \(T_j := L_j \otimes \mathbb{C}^*\).
- Let \(\Sigma\) be a full dimensional fan in \(N_{\mathbb{R}}\). If \(X\) is the toric variety corresponding to the fan \(\Sigma\), then \(T_N\) is the open torus in \(X\). For any full dimensional cone \(\sigma \in \Sigma\), let \(U_\sigma := \text{Spec} \mathbb{C}[\sigma^\vee \cap M]\). Then \(\{U_\sigma \mid \sigma \in \Sigma\ \text{is full dimensional}\}\) is an affine open cover of \(X\).
- Write \(\tau \prec \sigma\) if \(\tau\) is a face of \(\sigma\). For any cone \(\tau \in \Sigma\), let \(O(\tau)\) be the \(T_N\)-orbit associated to \(\tau\) in \(X\). Then for a full dimensional cone \(\sigma\) and any cone \(\tau\) in \(\Sigma\), \(O(\tau) \subseteq U_\sigma\) if and only if \(\tau \prec \sigma\) (see [CLS11, 3.2.6c]).
- Let \(V(\tau)\) be the Zariski closure of \(O(\tau)\) in \(X\). Then \(V(\tau)\) is a torus-invariant closed subvariety of \(X\).
- A fan \(\Sigma\) is simplicial if any cone \(\sigma \in \Sigma\) is generated by linearly independent generators. Assume that \(\Sigma\) is a simplicial fan in \(\mathbb{R}^n\) with \(n + 1\) rays \(R_0, R_1, \ldots, R_n\), where every \(n\) of them are linearly independent. For every \(I \subseteq \{0, 1, \ldots, n\}\), let \(\sigma_I \in \Sigma\) be the cone spanned by \(\{R_i \mid i \in I\}\). Every cone \(\sigma \in \Sigma\) corresponds to a unique subset \(I\) in the way above. Let \(\Sigma(k)\) be the \(k\)-dimensional cones in \(\Sigma\). Then \(\Sigma(k) = \{\sigma_I \mid |I| = k\}\). We write \(V(\sigma_I)\) as \(V_I\), and \(O(\sigma_I)\) as \(O_I\). Then \(O_I\) is a torus of dimension \(n - |I|\). If \(I = \{i\}\), then we write the torus-invariant divisor \(V(\sigma_{\{i\}})\) as \(D_{ij}\).
We start with the fan of the weighted projective plane \( \mathbb{P}(a, b, c) \). The assumption and conclusion of Proposition 1.1 are symmetric about \( a, b \) and \( c \). Hence up to a permutation on \( (a, b, c) \), we can choose a fan \( \Sigma_S \) of \( S \) with ray generators \( u_i = (x_i, y_i) \) such that both \( y_0, y_1 < 0 \) and \( y_2 > 0 \). Note that we have \( au_0 + bu_1 + cu_2 = 0 \).

Consider \( N = \mathbb{Z}^n \). Fix a basis \( e_1, e_2, \ldots, e_n \) of \( N \). By assumption (ii), there exist nonnegative integers \( \{m_{ij}\} \) such that \( d_{j-2} = am_{0,j} + bm_{1,j} + cm_{2,j} \) for every \( j \in J \). Define the following vectors in \( N \):

\[
\begin{align*}
v_0 &= (x_0, y_0, -m_{0,3}, \ldots, -m_{0,n}), \\
v_1 &= (x_1, y_1, -m_{1,3}, \ldots, -m_{1,n}), \\
v_2 &= (x_2, y_2, -m_{2,3}, \ldots, -m_{2,n}), \\
v_j &= e_j, \text{ for } j \in J = \{3, 4, \ldots, n\}.
\end{align*}
\]

(3)

Note that for every \( j \in J \), at least one of the integers \( m_{0,j}, m_{1,j}, m_{2,j} \) is necessarily nonzero.

Those \( v_i \) satisfy the relation

\[ av_0 + bv_1 + cv_2 + d_1v_3 + \cdots + d_{i-2}v_{i-2} + \cdots + d_nv_n = 0. \]

Moreover, each \( v_i \) is primitive, and together they span the lattice \( N \). As a result, if we let \( \Sigma_X \) be the fan in \( \mathbb{R}^n \) spanned by the \( n + 1 \) rays along \( v_i \) \( (i = 0, 1, \ldots, n) \), then \( \Sigma_X \) is a fan of \( X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2}) \).

**Definition 3.2.** Let the fan \( \Sigma_X \) of \( X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2}) \) be defined as above. For every \( j \in J \), let \( Y_j \) be the Zariski closure of the subtorus \( T_j = L_j \otimes \mathbb{C}^* \) in \( X \). Define \( S \) to be the set-theoretic intersection \( \cap_{j=3}^n Y_j \). Let \( Z \) be the Zariski closure of the subtorus \( T_{12} = N_{12} \otimes \mathbb{C}^* \) in \( X \).

By definition, all the \( Y_j \) and \( Z \) are irreducible. We claim:

**Proposition 3.3.**

(i) The set-theoretic intersection \( S \) equals \( Z \).

(ii) With the reduced subscheme structure, \( S \) is isomorphic to \( \mathbb{P}(a, b, c) \). In particular, \( S \) is normal.

We prove (ii) of Proposition 3.3 in the next section. Here we prove (i) by showing that \( Z \) is the unique irreducible component of the intersection \( S \). We will reduce the question to the affine case and apply the following lemma.

**Lemma 3.4.** Let \( \sigma \) in \( \mathbb{N}_{\mathbb{R}} \) be a simplicial cone spanned by \( n \) linearly independent rays \( R_i, i = 1, \cdots, n \). Let \( U_\sigma := \text{Spec } \mathbb{C}[\sigma^\vee \cap M] \). For any \( u \in M \) such that \( u \) is primitive and \( u \neq 0 \), let \( T_u \) be the subtorus of \( T_N \) defined by \( \chi^n = 1 \), and take the Zariski closure \( \overline{T_u} \) in \( U_\sigma \). Then we have:

(i) If \( \tau \prec \sigma \) such that \( u \in \tau^\vee \cap (-\tau^\vee) \) and \( u \not\in \tau^\perp \), then the set-theoretic intersection \( \overline{T_u} \cap O(\tau) = \emptyset \). In particular:

(a) For \( \tau = R_i \), if \( u \not\in \tau^\perp \), then \( \overline{T_u} \cap O(\tau) = \emptyset \).

(b) If \( u \in \sigma^\vee \cap (-\sigma^\vee) \), then \( \overline{T_u} \cap O(\sigma) = \emptyset \).

(ii) If \( u \in \tau^\perp \) and \( u \in \sigma^\vee \cup (-\sigma^\vee) \), then \( \overline{T_u} \cap O(\tau) \) has codimension at least 1 in \( O(\tau) \).
Proof. When $\tau = R_i$ is a ray, $\tau^\vee \cup (-\tau^\vee) = M$. When $\tau = \sigma$, $\tau^\perp = \sigma^\perp = \{0\}$. Therefore the two special cases (a) and (b) of (i) follow from the general result. Now let $\tau$ be a $d$-dimensional face of $\sigma$ such that $u \in \tau^\vee \cup (-\tau^\vee)$, and $u \notin \tau^\perp$. Then $O(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap M]$ is a $(n-d)$-dimensional torus (see Notation 3.1). Let $V(\tau)$ be the closure of $O(\tau)$ in $U_\sigma$. Then $V(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M]$. Then the inclusions

$$O(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap M] \hookrightarrow V(\tau) \cong \operatorname{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \hookrightarrow U_\sigma \cong \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

correspond to the maps of $\mathbb{C}$-algebras

$$\mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\phi_\tau} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap M],$$

where $\phi_\tau$ sends $\chi^u$ to $\chi^u$ if $u \in \tau^\perp$, and 0 otherwise. To prove that $T_u$ does not intersect $O(\tau)$, it suffices to show that there is a regular function $f$ vanishing on $T_u$ but not vanishing anywhere on $O(\tau)$. There are two cases.

Case I. $u \in \sigma^\vee \cup (-\sigma^\vee)$ and $u \notin \tau^\perp$. Note that $\sigma^\vee \subseteq \tau^\vee$ since $\tau \prec \sigma$. Suppose $u \in -\sigma^\vee$. Then $-u \in \sigma^\vee$. By definition, $T_u = T_{-u}$, so we can assume $u \in \sigma^\vee$. Now $f := \chi^u - 1 = \chi^u - \chi^0 \in \mathbb{C}[\sigma^\vee \cap M]$ is a regular function on $U_\sigma$. Since $u \notin \tau^\perp$, $\phi_\tau(\chi^u) = 0$. Since $0 \in \tau^\perp$, $\phi_\tau(\chi^0) = 1$. Therefore $\phi_\tau(f) = -1$ is a regular function on $V(\tau)$ which does not vanish on $O(\tau)$.

Case II. $\tau \not\prec \sigma$ is a proper face, $u \in \sigma^\vee \cup (-\sigma^\vee)$ and $u \notin \sigma^\vee \cup (-\sigma^\vee)$ and $u \notin \tau^\perp$. For each $i = 1, \ldots, n$, let $r_i$ be the ray generator of the ray $R_i$. Without loss of generality, we can assume $\tau$ is the face spanned by $r_1, \ldots, r_d$, with $d < n$, and $u \in \tau^\vee$. Let $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ be the dual pairing. Then $\langle r_i, u \rangle \geq 0$ for $i = 1, \ldots, d$, with $\langle r_i, u \rangle > 0$ for some $i \leq d$, and $\langle r_j, u \rangle < 0$ for some $j \in \{d + 1, \ldots, n\}$. We claim there exist $p, q \in \sigma^\vee \cap M - \{0\}$ and $k \in \mathbb{Z}_{>0}$ such that $ku = p - q$ and $q \in \tau^\perp$. Indeed, since $\sigma$ is simplicial, $r_1, \ldots, r_n$ form a basis of $N \otimes \mathbb{Q}$. Let $r_1^*, \ldots, r_n^*$ be the dual basis of $M \otimes \mathbb{Q}$. Then $u = u_1 r_1^* + \cdots + u_n r_n^*$ for rational numbers $u_i$, $i = 1, \ldots, n$. Define

$$p' := \sum_{u_i > 0} u_i r_i^*, \quad q' := - \sum_{u_i < 0} u_i r_i^*.$$

Then $u = p' - q'$. Indeed both $p'$ and $q'$ are in $\sigma^\vee$. Since $\langle r_i, u \rangle > 0$ for some $i \leq d$, and $\langle r_j, u \rangle < 0$ for some $j \in \{d + 1, \ldots, n\}$, we have $p' \neq 0$ and $q' \neq 0$. Take any $k \in \mathbb{Z}_{>0}$ such that $k p'$ and $k q'$ are both in $M$. Let $p := k p'$ and $q := k q'$, then $ku = p - q$ and $p, q \in \sigma^\vee \cap M - \{0\}$, which proves the claim. Now let $f = \chi^p - \chi^q$. Then $f \in \mathbb{C}[\sigma^\vee \cap M]$. We have $f = \chi^d - \chi^p = -\chi^q (\chi^{ku} - 1)$. Since $\chi^u - 1$ divides $\chi^{ku} - 1$, and $\chi^q$ has no poles on $T_u$, $f$ must vanish everywhere $T_u$. On the other hand, since $u \notin \tau^\perp$ and $q \in \tau^\perp$, $p = ku + q \notin \tau^\perp$. Therefore $\phi_\tau(f) = 0$, and $\phi_\tau(f) = \phi_\tau(\chi^q) = \chi^q$. When restricted to $O(\tau)$, $\chi^q$ is a nonzero monomial in the coordinate functions on $O(\tau)$, therefore $\chi^q$ does not vanish anywhere on the torus $O(\tau)$. This proves (i).

By the symmetry between $u$ and $-u$, to prove (ii), we need only prove for the case when $u \in \tau^\perp \cap \sigma^\vee$. In this case, $\phi_\tau(\chi^u) = \chi^u$, so $\chi^u - 1$ is a regular function of $O(\tau)$. Now $T_u$ is contained in the zero locus of $\chi^u - 1$. By assumption, $u \neq 0$, so $\chi^u \neq 1$. Restricting to $O(\tau)$, $\chi^u \neq 1$ is a monomial of the coordinate functions on $O(\tau)$, so $\chi^u = 1$ defines a subtorus of codimension 1 in $O(\tau)$. This proves (ii). □
Proof of Proposition 3.3 (i). By Definition 3.2, $S$ is the set-theoretic intersection of $Y_j$, $j \in J$. Since each $Y_j$ has codimension one in $X$, the codimension of each irreducible component of $S$ in $X$ is at most $n - 2$. For every $j \in J$, since $T_{1j} \subseteq T_j$, $Z$ is contained in $Y_j$. Hence $Z$ is contained in $S$. Therefore it suffices to prove that $Z$ is the unique irreducible component of $S$ of dimension at least 2.

Here the fan $\Sigma_X$ is simplicial, spanned by ray generator $v_i$. By Notation 3.1, $\Sigma_X = \{\sigma_I \mid I \subseteq \{0, 1, \ldots, n\}\}$. To prove that $Z$ is the unique irreducible component of $S$ of dimension at least 2, we need only show that $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n - 2$. Indeed, suppose $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n - 2$. Then $X \setminus T_N$ is a disjoint union of $T_N$-orbits $O_I$ for $1 \leq |I| \leq n - 2$, with $\dim O_I = n - |I|$. Therefore, if we assume there is some irreducible component $S'$ of $S$ disjoint from $Z$, then $S'$ is contained in $X \setminus T_N$, hence $\dim S' \leq 1$. This proves that $Z$ is the unique irreducible component of $S$ of dimension at least 2.

It remains to show $S \cap O_I$ is contained in a curve for every $1 \leq |I| \leq n - 2$. By Notation 3.1, $\{U_\sigma \mid \sigma \in \Sigma_X(n)\}$ is a torus-invariant open affine cover of $X$. For every $T_N$-orbit $O_I$ with $1 \leq |I| \leq n - 2$, we choose some $\sigma' \in \Sigma_X(n)$ such that $\sigma_I \prec \sigma'$. Then $O_I \subseteq U_{\sigma'}$. By definition, $Y_j$ is the Zariski closure of $O_{\sigma'}$ in $X$. Indeed, $O_{\sigma'} \cap T_N \subseteq U_{\sigma'}$. Let $Y_j'$ be the restriction of $Y_j$ to this $U_{\sigma'}$. Then $Y_j'$ equals the Zariski closure of $O_{\sigma'}$ in $U_{\sigma'}$. We apply Lemma 3.4 to $\sigma = \sigma'$, $\tau = \sigma_I$ and $u = e_j^\sigma$. Recall (3) that $-m_{ij} \leq 0$ is the $j$-th entry of $v_i$ for $i = 0, 1, 2$, $j \geq 3$. Define the following index sets:

\[
I_+ := I \cap \{0, 1, 2\}, \\
J_- := \{j \in J \mid m_{ij} > 0 \text{ for some } i \in I_+\}, \\
I_0 := \{j \in I \cap J \mid m_{ij} = 0 \text{ for all } i \in I_+\}.
\]

There are 4 possible cases: (a) $I_+ = \emptyset$; (b) $I_+ \neq \emptyset$ and $J_- \neq \emptyset$; (c) $I_+ \neq \emptyset$ and $I_0 \neq \emptyset$; and (d) $I_+ \neq \emptyset$ and $J_- = I_0 = \emptyset$.

In Cases (a) (b) and (c), we apply Lemma 3.4 (i) to show that there exists $j \in J$ such that $Y_j' \cap O_I = \emptyset$ for some $j \in J$ and for every choice of $\sigma_I \prec \sigma'$. Hence $S \cap O_I = \emptyset$. For (d), we apply Lemma 3.4 (ii) to show that $S \cap O_I$ is contained in a curve by choosing a specific $\sigma'$.

(a) $I_+ = \emptyset$. Choose any $j \in I$. Then $e_j^\sigma \notin \sigma_I^+$ and $e_j^\sigma \notin \sigma_J^\gamma$. Apply Lemma 3.4 (i) to any full dimensional $\sigma'$ such that $\sigma_I \prec \sigma'$, $\tau = \sigma_I$ and $u = e_j^\sigma$. Then $Y_j' \cap O_I = \emptyset$.

(b) $I_+ \neq \emptyset$ and $J_- \neq \emptyset$. Then choose any $j \in J_-$. We have $\langle v_i, e_j^\sigma \rangle = -m_{ij} < 0$ for some $i \in I_+$, and $\langle v_i, e_j^\sigma \rangle \leq 0$ for all $i \in I$. Hence $e_j^\sigma \in -\sigma_I^\gamma$ and $e_j^\sigma \notin -\sigma_I^\tau$. Therefore $Y_j' \cap O_I = \emptyset$.

(c) $I_+ \neq \emptyset$ and $I_0 \neq \emptyset$. Choose any $j \in I_0$. Then $\langle v_i, e_j^\sigma \rangle = 1 > 0$. If $i \in I$ and $i \neq j$, then either $i \in J$ or $i \in I_+$. If $i \in J$, then $v_i = e_i$ and $i \neq j$, so $\langle v_i, e_j^\sigma \rangle = 0$. If $i \in I_+$, then $\langle v_i, e_j^\sigma \rangle = -m_{ij} = 0$ since $j \in I_0$. Hence $e_j^\sigma \in \sigma_I^\gamma$ and $e_j^\sigma \notin \sigma_I^\tau$, so $Y_j' \cap O_I = \emptyset$.

(d) $I_+ \neq \emptyset$ and $J_- = I_0 = \emptyset$. Since $|I| \leq n - 2$, and $I_+ \neq \emptyset$, it must be that $J \subseteq I$. Therefore $I_+ \neq \{0, 1, 2\}$ (otherwise for every $j \in J \setminus I$, there exists an $m_{ij} > 0$, so $j \in J_-$), so $|I_+| = 1$ or 2. Fix some $j \in J \setminus I$. Since $J_- = \emptyset$, $m_{ij} = 0$ for all $i \in I_+$. Therefore
e_j^* \in \sigma^t_I. For this j \in J/I, define I' = \{0, 1, 2, \ldots, \hat{j}, \ldots, n\} and let \sigma' := \sigma_{I'}. Define Y'_j to be the restriction of Y_j to U_{I'} as discussed above. Then U_{I'} contains O_I, with e_j^* \in -(\sigma')^\vee. In Lemma 3.4 (ii), let \sigma = \sigma', \tau = \sigma_I and u = e_j^*. Then Y'_j \cap O_I is of codimension at least one in O_I and is contained in the zero locus of \chi_j - 1, regarded as a regular function on O_I. Now the number of such j equals |J/I| = n - 2 - |I \cap J| = n - 2 - (|I| - |I_+|). Since n - |I| = \dim O_I, we have |J/I| = \dim O_I - (2 - |I_+|). Recall that M = \Z\{e_1^*, \ldots, e_n^*\} and O_I = \Spec \C[\sigma_I^t \cap M]. Since |I_+| = 1 or 2, the semigroup \sigma_I^t \cap M is generated by \{e_i^* \mid i \in J/I\} if |I_+| = 2, or by \{e_i^* \mid i \in J/I\} together with some \xi \in \Z\{e_1^*, e_2^*\} if |I_+| = 1. Therefore each \chi_j, j \in J/I restricts to different coordinate functions on O_I. Hence, the intersection of the zero loci of all those \chi_j - 1 (j \in J/I) has dimension exactly 2 - |I_+|, which is either 1 or 0. Therefore S \cap O_I is contained in a curve. This finishes Case (d) and the proof.

4. Normality of the closure of subtori

In this section we prove (ii) of Proposition 3.3, namely that the surface S is normal and isomorphic to the weighted projective plane \P(a, b, c).

We recall the following construction in [CLS11, §2.1] of a projective toric variety \X_A out of a finite set of lattice points A \subset M. Let N = \Z^n and M = \Hom(N, \Z). Then each m \in M gives a character \chi^m of the torus T_N. Any list of k lattice points A = (m_1, \ldots, m_k) \subset M defines a morphism \phi_A from T_N to \P^{k-1}:

\begin{equation}
\phi_A : T_N \rightarrow T_k \xrightarrow{\mu} \P^{k-1},
\end{equation}

\begin{equation}
t \mapsto (\chi^{m_1}(t), \ldots, \chi^{m_k}(t)) \mapsto [\chi^{m_1}(t) : \cdots : \chi^{m_k}(t)].
\end{equation}

where \T_k \cong (\C^*)^k and \mu : T_k \rightarrow \P^{k-1} maps \T_k to the open torus \{[x_0 : \cdots : x_{k-1}] \mid all \ x_i \neq 0\} of \P^{k-1}.

Definition 4.1. [CLS11, Definition 2.1.1] We denote by \X_A the not necessarily normal toric variety given by the Zariski closure of the image \phi_A(T_N) in \P^{k-1}.

Remark 4.2. Up to isomorphism, the definition of \X_A only depends on the set of points appearing in A. So up to isomorphism we can ignore the order of the points in A, and can remove possible duplicates from A.

We note that by definition, \X_A is projective. However \X_A need not be normal. One of the ways to obtain normal toric varieties is from polytopes. Let P be a full dimension polytope in M_\R. Call P a lattice polytope if the vertices of P are in M. Now consider a semigroup S \subset M, with the addition inherited from M. Recall that S is said to be saturated if for every m \in M, every k \in \Z - \{0\}, km \in S implies m \in S.

Definition 4.3. [CLS11, Definition 2.2.17] A lattice polytope M is very ample if for every vertex m \in P, the semigroup S_{P,m} generated by the set P \cap M - m is saturated in M.

Lemma 4.4. [CLS11, Cor. 2.2.19] If P is a full dimensional lattice polytope, then kP is very ample if k \geq \dim P - 1. In particular, if P is a lattice polygon in \R^2 then P is very ample.
**Definition 4.5.** [CLS11, Definition 2.3.14] Suppose that $P \subset M_\mathbb{R}$ is a full dimensional lattice polytope. Then define the toric variety $X_P$ to be $X_A$ with $A = kP \cap M$, for any integer $k > 0$ such that $kP$ is very ample.

The toric variety $X_P$ is well defined since $X_{kP \cap M}$ and $X_{\ell P \cap M}$ are isomorphic when both $kP$ and $\ell P$ are very ample (see [CLS11, §2.3]).

**Lemma 4.6.** If $P$ is a full dimensional very ample lattice polytope, then $X_{P \cap M}$ is a normal projective toric variety, whose fan in $N$ is the normal fan $\Sigma$ of $P$.

*Proof.* This follows from [CLS11, Thm. 2.3.1, Thm. 1.3.5]. □

Now we are ready to prove that $S$ is normal and isomorphic to $\mathbb{P}(a, b, c)$.

*Proof of Proposition 3.3 (ii).* Let $M_{12} = \mathbb{Z}\{e_1^*, e_2^*\}$. We first show that $S$ is a normal projective variety. By Lemma 4.6, we need only show $S \cong X_{Q \cap M_{12}}$ for some full dimensional very ample lattice polytope $Q$ in $(M_{12})_\mathbb{R}$. Consider $X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2})$, with the fan $\Sigma_X$ defined by generators $v_i$ in (3). Choose any lattice polytope $P$ in $M_\mathbb{R}$ whose normal fan is $\Sigma_X$. By replacing $P$ with some multiple $kP$, we can assume $P$ is very ample. By Lemma 4.6, we have $X = X_P = X_{P \cap M}$. Let $m_0, m_1, \ldots, m_u$ be the distinct lattice points of $P \cap M$. Let $\psi := \phi_{P \cap M}$ be the map defined in (4). Then

$$
\psi = \phi_{P \cap M} : T_N \to T_{u+1} \to \mathbb{P}^u,
$$

$$
t \mapsto (\chi^{m_0}(t), \chi^{m_1}(t), \ldots, \chi^{m_u}(t)) \mapsto [\chi^{m_0}(t) : \chi^{m_1}(t) : \ldots : \chi^{m_u}(t)].
$$

Then $X$ equals the Zariski closure of $\psi(T_N)$ in $\mathbb{P}^u$. Let $\rho : M \to M_{12}$ be the projection map. If $t \in T_{12}$, then $\chi^{m_i}(t) = \chi^{\rho(m_i)}(t)$ for every $i$. Therefore, the restriction of $\psi$ on $T_{12}$ equals

$$
\psi|_{T_{12}} : T_{12} \to T_{u+1} \to \mathbb{P}^u,
$$

$$
t \mapsto (\chi^{\rho(m_0)}(t), \ldots, \chi^{\rho(m_u)}(t)) \mapsto [\chi^{\rho(m_0)}(t) : \ldots : \chi^{\rho(m_u)}(t)].
$$

By Proposition 3.3 (i), $S$ equals to the Zariski closure of $\psi(T_{12})$ in $X$. Since $X$ is closed in $\mathbb{P}^u$, we have $S$ equals to the Zariski closure of $\psi(T_{12})$ in $\mathbb{P}^u$.

Define $A := \rho(P \cap M)$. Then $A$ is the set of distinct elements in the list $A' = (\rho(m_1), \ldots, \rho(m_u))$. By Remark 4.2, we can remove the duplicates in $A'$, so that $S \cong X_A$.

Now we only need to show that $\rho(P \cap M) = \rho(P) \cap M_{12}$ and $Q := \rho(P)$ is a full dimensional very ample lattice polytope in $M_{12}$. We first show that $Q$ is a lattice triangle in $(M_{12})_\mathbb{R}$. Recall that $P$ has the following facet presentation:

$$
P = \{ z \in M_\mathbb{R} \mid \langle v_i, z \rangle \leq a_i \text{ for } i = 0, 1, \ldots, n \}
$$

for some $a_i \in \mathbb{Z}$ (See [Ful93, p. 66], [CLS11, 2.2.1]). Since the normal fan of $P$ is $\Sigma_X$, $P$ has exactly $n+1$ facets $F_i$ whose outer normal vectors are $v_i$, $i = 0, \ldots, n$ respectively. The reason that $a_i \in \mathbb{Z}$ is as follows: Fix $i \in \{0, 1, \ldots, n\}$. Let $m$ be a vertex of the facet $F_i$. Then $m$ is a vertex of $P$, so $m \in M$. Since $m \in F_i$, we in fact have $\langle v_i, m \rangle = a_i$. Thus $a_i \in \mathbb{Z}$ since $v_i \in N$.

Let $z = (z_1, \ldots, z_n) \in M_\mathbb{R}$. Then $\rho(z) = (z_1, z_2)$. By definition of $u_i$ and $v_i$ in (3), we have $\langle v_i, z \rangle = \langle u_i, \rho(z) \rangle - (z_{3+i} m_{1,3} + \cdots + z_n m_{1,n})$ for $i = 0, 1, 2$, and $\langle v_j, z \rangle = z_j$ for $j \in J = ...
\{3, 4, \ldots, n\}. Therefore \(z \in P\) if and only if \(\langle u_i, \rho(z) \rangle \leq a_i + (3m_{i,3} + \cdots + z_n m_{i,n})\) for \(i = 0, 1, 2\) and \(z_j \leq a_j\) for \(j \in J\). Recall that every \(m_{i,j} \geq 0\). As a result, \(y \in Q\) if and only if \(\langle u_i, y \rangle \leq a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})\) for \(i = 0, 1, 2\). Define \(q_i := a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})\) for \(i = 0, 1, 2\). Then

\[Q = \{y \in (M_{12})_\mathbb{R} \mid \langle u_i, y \rangle \leq q_i, \text{ for } i = 0, 1, 2\}.\]

Indeed (5) is a facet presentation of \(Q\). Thus \(Q\) is a triangle in \((M_{12})_\mathbb{R}\).

It remains to show that \(Q\) is a lattice triangle. A point \(z \in P\) (or \(Q\)) is a vertex of \(P\) (or \(Q\)) if and only if \(z\) lives in all but one facets. By the facet presentation (5) of \(P\), \(m\) is a vertex of \(P\) if and only if \(\langle v_i, m \rangle = a_i\) for all \(v_i\) but one. Suppose that \(\xi_0, \xi_1, \xi_2\) are the vertices of \(P\) where \(\xi_i\) lives in the \(n\) facets except \(F_i\). We claim that \(\rho(\xi_0), \rho(\xi_1)\) and \(\rho(\xi_2)\) are the three vertices of \(Q\). Indeed, we need only to prove this for \(\xi_0\). Let \(\xi_0 = (z_1, \ldots, z_n)\). Then \(a_j = \langle v_j, \xi_0 \rangle = z_j\) for \(j \in J\), and \(a_k = \langle v_k, \xi_0 \rangle = \langle u_k, \rho(\xi_0) \rangle = (3m_{k,3} + \cdots + z_n m_{k,n})\) for \(k = 1, 2\). By definition, this shows that \(\langle u_k, \rho(\xi_0) \rangle = g_k\) for \(k = 1, 2\). Let \(F_1^0\) be the fac of \(Q\) normal to \(u_i\) for \(i = 0, 1, 2\) (see (6)). Then \(\rho(\xi_0) = F_1^0 \cap F_2^0\) is a vertex of \(Q\). Since \(P\) is a lattice polytope, \(\xi_0 \in M\), so \(\rho(\xi_0) \in M_{12}\). Repeat this argument for \(\xi_1\) and \(\xi_2\). Then \(\rho(\xi_0), \rho(\xi_1)\) and \(\rho(\xi_2)\) are distinct vertices of \(Q\). Therefore \(Q\) is a lattice triangle.

By Lemma 4.4, any lattice triangle in \(M_{12}\) is very ample, so \(Q\) is very ample. Hence we verified that \(Q\) is a full dimensional very ample lattice polytope.

It remains to show \(\rho(P \cap M) = \rho(P) \cap M_{12}\). By definition, \(\rho(P \cap M) \subseteq \rho(P) \cap M_{12}\). Conversely, suppose \(y = (z_1, z_2) \in \rho(P) \cap M_{12}\). Then \(y = \rho(z)\) where \(z := (z_1, z_2, a_3, \ldots, a_n)\). By (6), we have \(\langle u_i, y \rangle \leq q_i\) for \(i = 0, 1, 2\). Hence \(\langle u_i, \rho(z) \rangle \leq q_i = a_i + (a_3 m_{i,3} + \cdots + a_n m_{i,n})\) for \(i = 0, 1, 2\). The argument preceding (6) shows that \(z \in P\). Since \(z_1, z_2\), all \(a_i\) and all \(m_{i,j}\) are integers, we have \(z \in M\). Thus \(\rho(P) \cap M_{12} \subseteq \rho(P \cap M)\). We conclude that \(\rho(P \cap M) = \rho(P) \cap M_{12}\). Therefore, \(S = \bigcap_{\rho(P) \cap M_{12}}\) is normal. Furthermore, by Proposition 4.6, the fan of \(S\) in \(N_{12}\) is the normal fan of \(Q\) with respect to \(N_{12}\), hence is spanned by \(u_0, u_1\) and \(u_2\). By (3), the fan spanned by \(u_0, u_1\) and \(u_2\) is a fan of \(\mathbb{P}(a, b, c)\). As a conclusion, \(S \cong \mathbb{P}(a, b, c)\). \(\square\)

5. Intersection products on weighted projective spaces

We prove Theorem 1.1 and Theorem 1.2 in this section. In Section 3 we constructed a fan \(\Sigma_X\) for \(X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2})\), under the assumption (i) of Theorem 1.1. Recall that \(S\) is defined as the intersection of \(Y_J^j\) for \(j \in J\), where \(J = \{3, 4, \ldots, n\}\). By Lemma 3.3 (ii), \(S\) is isomorphic to \(\mathbb{P}(a, b, c)\).

We start with a review of the intersection products of various torus-invariant divisors on \(X\) and \(S\). Let \(A_d(X)\) be the Chow group of \(d\)-dimensional cycles in \(X\). Since \(X\) is a complete simplicial toric variety, by [CLS11, Lem. 12.5.1], \(A_d(X)\) is generated by the classes of torus-invariant subvarieties \([V_I]\) where \(|I| = n - d\). In particular, \(A_{n-1}(X)\) is generated by the classes of torus-invariant Weil divisors \([D_i] \mid i = 0, 1, 2, \ldots, n\). The divisor class group \(\text{Cl}(X)\) of \(X\) is isomorphic to \(\mathbb{Z}\) by [CLS11, Ex. 4.1.5]. Let \(A\) be a pseudo-effective Weil divisor on \(X\) which generates \(\text{Cl}(X)\). Then in \(A_{n-1}(X) = \text{Cl}(X)\) we have

\[D_0 = a[A], \quad D_1 = b[A], \quad D_2 = c[A], \quad D_j = d_{j-2}[A], \text{ for } j \geq 3.\]
Now $\Sigma_X$ is simplicial (Notation 3.1). By [CLS11, Lem. 12.5.2], we have the following intersection products:

$$[A]^n = \frac{1}{abcd_1 \cdots d_{n-2}},$$

$$[D_3] \cdot [D_4] \cdots [D_n] = [V_j],$$

$$[V_j] \cdot [D_i] = [V_{j,i}], \text{ for } i = 0, 1, 2,$$

$$[D_1] \cdot [D_2] \cdot [V_j] = \frac{1}{a}, \quad [D_0] \cdot [D_2] \cdot [V_j] = \frac{1}{b}, \quad [D_0] \cdot [D_1] \cdot [V_j] = \frac{1}{c}.$$  

By Notation 3.1, $N_{12} = \mathbb{Z}\{e_1, e_2\}$. Let $\Sigma_S$ in $(N_{12})_\mathbb{R}$ be the fan of $S$ generated by ray generators $u_0, u_1$ and $u_2$ (See (3)). Define $B_i := V(\sigma_{i\{i\}})$ to be the torus-invariant divisors of $S$ corresponding to $u_i$. By [CLS11, Ex. 4.1.5], Cl($S$) $\cong$ $\mathbb{Z}$. Let $B$ be a pseudo-effective Weil divisor on $S$ which generates Cl($S$). Then

$$[B_0] = a[B], \quad [B_1] = b[B], \quad [B_2] = c[B], \quad [B]^2 = \frac{1}{abc}.$$  

Next we recall a result by Fulton and Sturmfels [FS97]. Let $W$ be a toric variety of a fan $\Sigma \subset N = \mathbb{Z}^n$. As in [FS97], define $N_{\sigma}$ as $\mathbb{Z}(N \cap \sigma)$, the sublattice spanned by $\sigma$ in $N$. Let $L$ be a saturated $d$-dimensional sublattice of $N$. Let $Y$ be the Zariski closure of the subtorus $T_L = L \otimes \mathbb{Z} \mathbb{C}^*$ in $W$. For every lattice point $w \in N$, define

$$\Sigma(w) := \{ \sigma \in \Sigma : L_{\mathbb{R}} + w \text{ meets } \sigma \text{ in exactly one point} \}.$$  

Here $L_{\mathbb{R}} + w := \{ x + w \mid x \in L_{\mathbb{R}} \}$.

**Definition 5.1.** [FS97, §3] $w$ is called *generic* (with respect to $L$) if $\dim \sigma = n - d$ for all $\sigma \in \Sigma(w)$.

**Lemma 5.2.** [FS97, Lem. 3.4] Let $W$, $L$ and $Y$ be defined as above. If $w \in N$ is a generic point with respect to $L$, then

$$[Y] = \sum_{\sigma \in \Sigma(w)} m_\sigma [V(\sigma)] \in A_d(W),$$

where $m_\sigma := [N : L + N_{\sigma}]$ is the index of the lattice sum $L + N_{\sigma}$ in $N$.

For simplicity, when there are no ambiguity of the choice of $L$, and when the toric variety $W$ has a simplicial fan $\Sigma$ spanned by rays $r_0, r_1, \cdots, r_n$, we write $m_{\sigma_I} = [N : L + N_{\sigma_I}]$ as $m_I$, for $I \subset \{0, 1, \cdots, n\}$. When $I = \{i\}$, we write $m_{\sigma_i}$ as $m_i$.

**Lemma 5.3.** Let $X$, $Y_j$ and $S$ be defined as in Definition 3.2. Then $[Y_j] = [D_j]$ for all $j \in J$, and $[S] = [V_j]$.

**Proof.** Fix $j \in J$. By Notation 3.1, $L_j := \mathbb{Z}\{e_1, e_2, \cdots, \tilde{e}_j, \cdots, e_n\}$. By Definition 3.2, $Y_j$ is the Zariski closure of $T_j = L_j \otimes \mathbb{Z} \mathbb{C}^*$ in $X$. We apply Lemma 5.2 to $W = X$, $Y = Y_j$ and $L = L_j$. First, $e_j$ is generic with respect to $L_j$. Indeed if $j \not\in I$, then $(L_j)_{\mathbb{R}} + e_j$ does not meet $\sigma_I$. If $j \in I$, then $\sigma_I$ intersects $(L_j)_{\mathbb{R}} + e_j$ at a single point if and only if $I = \{j\}$. Hence $\Sigma(e_j) = \{\sigma_{I\{j\}}\}$. Since $\sigma_{I\{j\}}$ is a 1-dimensional cone, $e_j$ is generic. By Lemma 5.2, $[Y_j] = m_j [D_j]$, and $m_j$ equals the index of $L_j + N_{\sigma_{I\{j\}}}$ in $N$, which equals to 1, so $[Y_j] = [D_j]$. 


Similarly, $N_{12} := \mathbb{Z}\{e_1, e_2\}$, and $S$ is the Zariski closure of $T_{12} := N_{12} \otimes_{\mathbb{Z}} \mathbb{C}^*$. The same argument above shows that $\Sigma(\omega) = \{\sigma_j\}$, where $\omega = (0, 0, 1, \ldots, 1) \in N$ is generic with respect to $N_{12}$. Apply Lemma 5.2 to $W = X, Y = S$ and $L = N_{12}$. Then we have $[S] = m_J[V_J]$. Here $m_J = 1$ since $N_{12} + N_{\sigma_j} = N$. \hfill \Box

**Definition 5.4.** Let $N_1 = \mathbb{Z}\{e_1\}$ and $T_1 := N_1 \otimes_{\mathbb{Z}} \mathbb{C}^*$. Let $C_1$ be the Zariski closure of the subtorus $T_1$ in $S$.

**Lemma 5.5.** Let $C_1$ be defined as above. Then

(i) The irreducible curve $C_1$ equals the closure of the subtorus $T_1$ in $X$.

(ii) The class $[C_1] = -y_0[V_{J_{\cup}(0)}] - y_1[V_{J_{\cup}(1)}] \in A_1(X)$.

(iii) The class $[C_1] = y_2[V_J] \cdot [D_2] \in A_1(X)$.

(iv) The class $[C_1] = y_2[B_2] = cy_2[B] \in A_1(S)$.

**Proof.** Let $\overline{T_1}$ be the closure of $T_1$ in $X$. By definition, $T_1$ is contained in $S$. Since $S$ is closed in $X$, $\overline{T_1}$ is contained in $S$. Therefore $C_1 = \overline{T_1}$. Hence, both $C_1$ and $C$ are irreducible. This proves (i). For (ii), we work in $N = \mathbb{Z}^n$. Define $w = (w_1, w_2, 1, \ldots, 1) \in N$ such that $(w_1, w_2)$ lies in the interior of the cone spanned by $w_0$ and $w_1$. We claim that $w$ is generic with respect to $N_1$. Indeed, by the definition of $u_i$ (see (3)), the second coordinates of $u_0$ and $u_1$ are negative and the second coordinate of $w_1$ is positive. Hence $w_2 < 0$. Suppose the line $\ell : = (N_1)_R + w$ intersects $\sigma_I$. Then $J \subset I$. Since $w_2 < 0$, $\ell$ misses $\sigma_J$ and $\sigma_{J_{\cup}(2)}$, and meets $\sigma_{J_{\cup}(0)}$ and $\sigma_{J_{\cup}(1)}$ at a unique point. In the remaining case, $I = J \cup \{i_1, i_2\}$ with distinct $i_1, i_2 \in \{0, 1, 2\}$, so $\ell$ intersects $\sigma_I$ at infinitely many points. As a conclusion, $\Sigma(w) = \{\sigma_{J_{\cup}(0)}, \sigma_{J_{\cup}(1)}\}$, so $w$ is generic.

Apply Lemma 5.2 to $W = X, Y = C_1$ and $L = N_1$. We have

$$[C_1] = m_{J_{\cup}(0)}[V_{J_{\cup}(0)}] + m_{J_{\cup}(1)}[V_{J_{\cup}(1)}].$$

By definition, $m_{J_{\cup}(0)} = [N : N_1 + N_{\sigma_{J_{\cup}(0)}}]$. Since $N_1 + N_{\sigma_{J_{\cup}(0)}}$ is spanned by $e_1, e_3, \ldots, e_n$ together with $v_0$, the index equals to the absolute value of the second coordinate of $v_0$. That is, $m_{J_{\cup}(0)} = |y_0|$. Recall our assumption in Section 3 that $y_0, y_1 < 0$ and $y_2 > 0$. Hence $m_{J_{\cup}(0)} = -y_0$. Similarly we have $m_{J_{\cup}(1)} = -y_1$. This proves (ii). Now use formulas (7) and (8):

$$[C_1] = -y_0[V_{J_{\cup}(0)}] - y_1[V_{J_{\cup}(1)}] = -y_0[V_J] \cdot [D_0] - y_1[V_J] \cdot [D_1] = [V_J] \cdot [-y_0a[A] - y_1b[A]] = cy_2[V_J] \cdot [A] = y_2[V_J] \cdot [D_2].$$

This proves (iii).

Finally consider $C_1$ as a curve on $S$. The fan $\Sigma_S$ lives in $(N_{12})_R$ (See Notation 3.1). We have $\Sigma(e_2) = \{B_2\}$. Therefore $e_2 = (0, 1)$ is generic with respect to $N_1$. Apply Lemma 5.2 to $W = S, Y = C_1$ and $L = N_1$. Then $[C_1] = m_2[B_2] \in A_1(S)$ where $m_2 = [\mathbb{Z}^2 : (N_1)_R + zw_2] = |y_2| = y_2$. This proves (iv). \hfill \Box

**Lemma 5.6.** Consider the class $[B] \in A_1(X)$. Then we have $[B][Y_J] = \frac{d_{j-2}}{abc}$, for $j \in J$. 
Proof. By Lemma 5.5, \([C_1] = cy_2[V_j] \cdot [A] \in A_1(X)\), and \([C_1] = cy_2[B] \in A_1(S)\). Therefore \(cy_2[B] = cy_2[V_j] \cdot [A] \in A_1(X)\), so \([B] = [V_j] \cdot [A] = \frac{1}{a} [V_j] \cdot [D_0] \in A_1(X)\). Then

\[
[B] \cdot [Y_j] = \frac{1}{a} [V_j] \cdot [D_0] \cdot \frac{d_j - 2}{b} [D_1] = \frac{d_j - 2}{abc}.
\]

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By definition \(X = \mathbb{P}(a, b, c, d_1, \ldots, d_{n-2})\) is a weighted projective \(n\)-space. By Proposition 3.3, \(S = \mathbb{P}(a, b, c)\) is a weighted projective plane. Hence both \(X\) and \(S\) are normal projective \(\mathbb{Q}\)-factorial varieties, with finitely generated Picard groups. By Proposition 3.3, \(S = \cap_{j=3}^n Y_j\). By assumption, \(C\) is a negative curve on \(Bl_p S\) and \(C \neq e\).

To apply Theorem 2.1 to \(X, Y_j, S\) and \(C\), we need only verify that \((f_s C) \cdot Bl_p Y_j < 0\) for \(j = 3, 4, \ldots, n\). Here \((f_s C) \cdot Bl_p Y_j = f_s C \cdot (\pi_X^* Y_j - E)\), and \(C \sim_\mathbb{Q} \lambda \pi^* B - \mu E\). Hence by Lemma 5.6 and projection formula:

\[
f_s C \cdot (\pi_X^* Y_j - E) = (\pi_X^*)_* (f_s C) \cdot [Y_j] - f_s (C) \cdot [E] = \lambda [B] \cdot [Y_j] - \mu = \frac{\lambda d_j - 2}{abc} - \mu < 1.
\]

By Theorem 2.1, \(Bl_p X\) is not a MDS. This proves the theorem.

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. Suppose there is a relation \((e, f, -g)\) between the weights \((a, b, c)\) such that the width \(w = cg^2/(ab) < 1\).

We need only show that there exists a non-exceptional negative curve \(C\) on \(Bl_p S\) satisfying the assumption in Theorem 1.1 with \(\lambda = cg\) and \(\mu = 1\), and \(d_i < abc\mu/\lambda = ab/g\) for all \(i = 0, 1, \ldots, n - 2\). We first choose a specific fan \(\Sigma_S\) and use \(\Sigma_S\) to define \(\Sigma_X\). Indeed, by [He17, Prop. 5.1], there exists a unique integer \(r\) with \(1 \leq r \leq g\), \(g | er - b\) and \(g | fr + a\). Let \(u_i = (x_i, y_i)\) be given by (1):

\[
u_0 = \left(\frac{er - b}{g}, -e\right), \quad u_1 = \left(\frac{fr + a}{g}, -f\right), \quad u_2 = (-r, g).
\]

Then \(u_i\) span a fan of \(S\). Let this fan be \(\Sigma_S\). We check that \(y_0 = -e < 0\), \(y_1 = -f < 0\) and \(y_2 = g > 0\), so all the assumptions in Section 3 are satisfied. Then we can use \(u_i\) to define \(v_i\) and the fan \(\Sigma_X\) as in (3). Consider the curve \(C_1\) in Definition 5.4. Let \(C\) be the proper transform of \(C_1\) on \(Bl_p S\). Then \(C \sim \pi^* C_1 - e\) on \(Bl_p S\). By Lemma 5.5 (iv), \(C \sim cg\pi^* B - e\).

Hence \(\lambda = cg\) and \(\mu = 1\). By (9), \([B]^2 = 1/abc\). Hence \([C_1]^2 = g^2 ce^2/abc = cg^2/ab = w\), and \([C]^2 = [C_1]^2 + 1 = w + 1 < 0\). Since \(\pi(C) = C_1\) is not a point, \(C\) is not \(e\). As a result, \(C\) is a non-exceptional negative curve on \(Bl_p S\). Finally by assumption (ii) of Theorem 1.2, for every \(i\), \(d_i^2 w < abc\). Therefore \(d_i^2 cg^2/(ab) < abc\). That is, \(d_i < ab/g\). By Theorem 1.1, we conclude that \(Bl_p X\) is not a MDS.

6. Comparison with González and Karu’s examples

We compare the 3-dimensional case of Theorem 1.2 with [GK17, Thm. 2.3, Cor. 2.5].
**Definition 6.1.** Consider a \( n \)-dimensional convex polytope \( \Delta \) in \( \mathbb{R}^n \) such that all its vertices have rational coordinates.

(i) For \( n = 3 \), we say such a polytope is of González-Karu type if the vertices of \( \Delta \) are \((0,0,1), (0,1,0), P_L \) and \( P_R \), with \( P_L \) and \( P_R \) and 0 collinear, and \( x(P_L) < 0 \leq x(P_R) \leq x(P_L) + 1 \), where \( x(P_R) \) and \( x(P_L) \) are the \( x \)-coordinates. (see [GK17, §2.2])

(ii) For \( n = 2 \), we say such a polytope is of González-Karu type if \( \Delta \) is a triangle with vertices \((0,0), P_L \) and \( P_R \), with \( P_L \) and \( P_R \) and \((0,1) \) collinear, and \( x(P_L) < 0 < x(P_R) < x(P_L) + 1 \).

(iii) In both dimension 2 and 3, define the width of a polytope of González-Karu type to be \( x(P_R) - x(P_L) \).

By definition, 3-dimensional polytope \( \Delta \) of González-Karu type has some evident properties:

(a) The cross sections of \( \Delta \) at \( x = i \in \mathbb{N} \) are isosceles right triangles.

(b) Projecting \( \Delta \in \mathbb{R}^3 \) of González-Karu type and of width \( < 1 \) to \( xy \)-plane or \( xz \)-plane, and then translating by the vector \((0,-1)\) will give a triangle of González-Karu type with the same width.

We first recall the following numerical criteria from [GK16], [GK17] for the weights for \( P(a,b,c,d) \) or \( P(a,b,c) \) to have a polytope of González-Karu type. We rephrase the criteria as follows:

**Lemma 6.2.** (i) Given \( w \in \mathbb{Q} \cap (0,1) \). Consider \( P(a,b,c) \) with \( a,b,c \) pairwise coprime. Then \( P(a,b,c) \) has a polytope \( \Delta \) of González-Karu type of width \( w \) if and only if there exist a relation \((e,f,-g)\) with \( ae + bf = cg \) (up to a permutation of the weights \( a,b,c \)) and \( w = cg^2/ab \). Furthermore, up to switching \( a \) with \( b \), and up to a shear transformation \((x,y) \mapsto (x,y + kx)\) for some \( k \in \mathbb{Z} \), \( \Delta \) has vertices given by \((2)\), i.e.,

\[
(0,0), \quad \left( -\frac{eg}{b} - \frac{er - b}{b}, \frac{fg}{a}, \frac{fr + a}{a} \right),
\]

where \( r \) is the unique integer such that \( 1 \leq r \leq g \), \( g \mid er - b \) and \( g \mid fr + a \) [He17, Prop. 5.1], and \( \Delta \) is normal to the fan with ray generators given in \((1)\). In particular, when \( w < 1 \), the numbers of lattice points on slices of \( \Delta \) are determined by \( a,b,c \).

(ii) Given \( W \in \mathbb{Q} \cap (0,1) \). Consider \( P(a,b,c,d) \) with every 3 weights relatively prime. Then \( P(a,b,c,d) \) has a polytope \( \Delta \) of González-Karu type of width \( W \) if and only if there exist positive integers \( e,f,g_1,g_2 \) such that up to a permutation of the weights \( a,b,c \) and \( d \), we have 

\[
ae + bf = eg_1 = dg_2, \quad W = (dg_2)^3/(abcd), \quad \text{gcd}(e,f,g_1) = \text{gcd}(e,f,g_2) = \text{gcd}(g_1,g_2) = 1.
\]

The following definition is from [GK17]:

**Definition 6.3.** [GK17, §2.2] Suppose \( \Delta \) is a 2 or 3-dimensional polytope of González-Karu type. Suppose \( m \) is a positive integer such \( m\Delta \) is a lattice polytope. For any integer \( i \) such that \( m \cdot x(P_L) \leq i \leq m \cdot x(P_R) \), the slice at \( x = i \) is the set of lattice points in
$m\Delta$ with $x$-coordinates $i$. When $\dim \Delta = 2$, a slice of $m\Delta$ consists of consecutive lattice points on a line. When $\dim \Delta = 3$, a slice of $m\Delta$ forms a right triangle with the same number $n$ of lattice points on each side. Then say the slice at $x = i$ has size $n$.

To avoid ambiguity, in the following we use $\Gamma$ to represent a 2-dimensional polytope of of González-Karu type. We recall the following criteria in [GK16] and [GK17] for $\text{Bl}_p X$ to be not a MDS where $X$ is a toric surface or toric 3-fold with a polytope of González-Karu type.

**Theorem 6.4.** [GK16, Thm. 1.2] Suppose $S$ is a toric surface with fan $\Sigma$ in $\mathbb{R}^2$. Suppose $\Gamma \subset \mathbb{R}^2$ is a triangle of González-Karu type with width $w$ and normal fan $\Sigma$. Let $m > 0$ be a sufficiently large and divisible integer so that $m\Gamma$ is a lattice triangle. Then $\text{Bl}_p S$ is not a MDS if the following hold:

1. Let the slice at $m \cdot x(P_L) + 1$ of $m\Gamma$ have exactly $n$ elements. Then the slice at $m \cdot x(P_R) - n + 1$ of $m\Gamma$ has exactly $n$ elements.
2. $ns_2 \not\in \mathbb{Z}$, where $s_2 := (y(P_R) - y(P_L))/w$ is the slope of the line through $P_L$ and $P_R$.

**Theorem 6.5.** [GK17, Cor. 2.5] Suppose $X$ is a toric 3-fold with fan $\Sigma$ in $\mathbb{R}^3$. Suppose $\Delta \subset \mathbb{R}^3$ is a polytope of González-Karu type with width $W$ and normal fan $\Sigma$. Let $m > 0$ be a sufficiently large and divisible integer so that $m\Delta$ is a lattice polytope. Then $\text{Bl}_p X$ is not a MDS if the following hold:

1. Let the slice at $m \cdot x(P_L) + 1$ of $m\Delta$ have size $n$. Then the slice at $m \cdot x(P_R) - n + 1$ of $m\Delta$ has size $n$.
2. $n(s_y, s_z) \not\in \mathbb{Z}^2$, where $s_y := (y(P_R) - y(P_L))/W$ and $s_z := (z(P_R) - z(P_L))/W$ are the $y, z$-slopes of the line through $P_L$ and $P_R$.

Now a natural question is that whether there are examples of $\mathbb{P}(a, b, c, d)$ meeting assumptions in Theorem 1.2 and [GK17, Cor. 2.5]. The following proposition provides a precise answer on the overlap:

**Proposition 6.6.** Suppose $\mathbb{P}(a, b, c, d)$ has a polytope $\Delta$ of González-Karu type and satisfies the assumptions including (i) - (iv) of Theorem 1.2. Then $d = cg$, where $(e, f, -g)$ is the unique relation between $(a, b, c)$ with $w < 1$.

Conversely, every weighted projective 3-space $\mathbb{P}(a, b, c, cg)$ such that $(a, b, c)$ has a relation $(e, f, -g)$ with $w < 1$, and $\mathbb{P}(a, b, c)$ has a polytope satisfying the conditions in [GK16, Thm. 1.2] with width $w$, will satisfy the assumptions in both Theorem 1.2 and [GK17, Cor. 2.5].

**Remark 6.7.** In the proof of Theorem 1.2, we in fact showed that weighted projective spaces $\mathbb{P}(a, b, c, d)$ meeting the conditions of the theorem must contain the weighted projective plane $S = \mathbb{P}(a, b, c)$ where $\text{Bl}_p S$ is not a MDS. Recall Theorem 3.3 that $S$ is the Zariski closure of the subtorus $T_{12} = L_{12} \otimes \mathbb{C}^*$, where $(L_{12})_R$ is the $xy$-plane.

**Question:** Is there any $\mathbb{P}(a, b, c, d)$ such that $\text{Bl}_p \mathbb{P}(a, b, c, d)$ is not a MDS, but for any 2-dimensional subtorus $T'$ of the open torus $T_N$, the blow-up $\text{Bl}_p \overline{\overline{T'}}$ of the Zariski closure of $T'$ is a MDS?
Lemma 6.8. (See [GK16, §1]) Suppose $a, b, c$ are pairwise coprime positive integers. Then there exist at most one relation $(e, f, -g)$ of $(a, b, c)$ with $eg^2 < ab$, even when permuting $a, b, c$. 

Proof of Lemma 6.2. First we prove (i). Suppose $\mathbb{P}(a, b, c)$ has a relation of weight $w < 1$, then the polytope in (2) is of González-Karu type with width $w$. Conversely, suppose $S = \mathbb{P}(a, b, c)$ has a polytope $\Delta$ of González-Karu type with width $w < 1$. Then $S$ has a fan $\Sigma$ normal to $\Gamma$. Say the ray generators of $\Sigma$ are the outer normal vectors of the four faces of $\Delta$. Therefore the four rays $R_1, \cdots, R_4$ of $\Sigma$ are the outer normal vectors of the four faces of $\Delta$. Direct calculation shows that $R_i$ is spanned by the vector $r_i$:

$$
\begin{align*}
12 & 
\end{align*}
$$

Now let $r_i'$ be the first lattice point in the ray $R_i$. Because $x > 0$ and $\lambda < 0$, there must exist positive integers $e, f, g_1, g_2$ and integers $R, S, T, U$ such that

$$
\begin{align*}
13 & 
\end{align*}
$$

Since $\Sigma$ is the fan of $\mathbb{P}(a, b, c, d)$, up to a permutation of the weights, we have $ar_1' + br_2' + cr_3' + dr_4' = 0$. Take the last two components, we have $ae + bf = cg_1 = dg_2$. Since $\Sigma$ is a fan of $\mathbb{P}(a, b, c, d)$, the weights $(a, b, c, d)$ equal to the $3 \times 3$ minors of the matrix with

Note that the Zariski closure $\mathcal{T}$ may have Picard number 1 or 2.
rows $r'_1, \ldots, r'_4$. For any 3 vectors $v_1, v_2$ and $v_3$ in $\mathbb{R}^3$, we denote by $\det(v_1, v_2, v_3)$ the determinant of the square matrix with row vectors $v_1, v_2$ and $v_3$. Then we have

$$a = |\det(r'_2, r'_3, r'_4)| = \frac{g_1 g_2}{x} |Sx + fy + fz| = \frac{g_1 g_2}{x} \left| \frac{(\lambda y + \lambda z - 1)f}{-\lambda} + fy + fz \right| = \frac{fg_1 g_2}{\lambda x},$$

$$b = |\det(r'_1, r'_3, r'_4)| = \frac{g_1 g_2}{x} |Rx + ey + ez| = \frac{g_1 g_2}{x} (1 - y - z)e + ey + ez| = \frac{eg_1 g_2}{x},$$

where we used that each $r'_i$ is a scalar multiple of $r_i$. Note that the other two equations of $c$ and $d$ do not give new algebraic relations. As a result,

$$(13) \quad x = \frac{eg_1 g_2}{b}, \quad \lambda = -\frac{bf}{ae},$$

$$(14) \quad W = x(P_R) - x(P_L) = x - \lambda x = \frac{eg_1 g_2}{b} \left( 1 + \frac{bf}{ae} \right) = \frac{eg_1 g_2}{b} \cdot \frac{dg_2}{ae} = \frac{cg_1 \cdot dg_2 \cdot dg_2}{abcd} = \frac{(dg_2)^3}{abcd}.$$

At last, the coprime conditions follow from the assumption that every 3 of $a, b, c, d$ are relatively prime, and the expression of $a, b, c, d$ as the determinants of $r'_i$ with $R, S, T$ and $U$ are integers. This proves the ‘only if’ direction. Conversely, suppose $W = (dg_2)^3/(abcd)$. We can always choose integers $T$ and $U$ such that $\gcd(T, g_1) = \gcd(U, g_2) = 1$. Let $y = T x/g_1$ and $z = U x/g_2$, with $x$ and $\lambda$ given above in (13). The parameters $x, y, z, \lambda$ determine a fan $\Sigma'$ with rays $r_i$ from (12), and a polytope $\Delta'$ with $P_R = (x, y, z), x > 0$ and $P_L = \lambda(x, y, z)$. Then it is straightforward that $\Sigma'$ is a fan of $\mathbb{P}(a, b, c, d)$, and $\Delta'$ is of González-Karu type with width $W$, whose normal fan is $\Sigma'$. This proves the ‘if’ direction. \[\square\]

Finally we prove Proposition 6.6.

**Proof of Proposition 6.6.** Suppose $\mathbb{P}(a, b, c, d)$ has a polytope $\Delta$ of González-Karu type and the assumptions of Theorem 1.2. Then by Lemma 6.2, there exist $e, f, g_1, g_2 \in \mathbb{Z}_{>0}$ such that $ae + bf = cg_1 = dg_2$ (up to a permutation of the weights $a, b, c$ and $d$), and the width $W$ of $\Delta$ equals $(dg_2)^3/(abcd) \leq 1$. In this equation, $a$ and $b$ are symmetric. The weights $c$ and $d$ are also symmetric. Hence up to symmetry either $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS or $\text{Bl}_p \mathbb{P}(b, c, d)$ is not a MDS.

Case I. $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS, with relations $(E, F, -G)$ such that the width $w < 1$. By the argument above,

$$1 \geq W = \frac{(dg_2)^3}{abcd} = \frac{c g_1^2 g_2}{ab}.$$

We claim $W < 1$. Otherwise $W = 1$. Then $c g_1^2 g_2 = ab$, so $c \mid ab$, which contradicts the assumption of Theorem 2.1 that $a, b, c$ are pairwise coprime.

Hence $c g_1^2 / ab < 1/g_2 \leq 1$. By Lemma 6.2, $\gcd(e, f, g_1) = 1$. Now $(e, f, -g_1)$ is a relation between $(a, b, c)$ with $\gcd(e, f, -g_1) = 1$ and width $c(g_1)^2/(ab) = c g_1^2 / (ab) < 1$. By Lemma 6.8, we must have $e = E, f = F$ and $g_1 = G, ae + bf = cg_1$, and the width of $(e, f, -g_1)$ is

$$w = \frac{c G^2}{ab} = \frac{c g_1^2}{ab} < \frac{1}{g_2} \leq 1.$$

Suppose $g_2 \geq 2$. Then $w \leq 1/2$. By Theorem 2.5 and 2.6 of [He17], if $w \leq 1/2$, then $\text{Bl}_p \mathbb{P}(a, b, c)$ is a MDS, which contradicts the assumption. Therefore $g_2 = 1$, and $d = cg_1$. 
Case II. $\text{Bl}_p \mathbb{P}(b, c, d)$ is not a MDS, and $\gcd(b, c, d) = 1$. This together with $cg_1 = dg_2$ implies that $g_1 = kd$ and $g_2 = kc$ for some $k \in \mathbb{Z}_{>0}$. Now

$$1 \geq W = \frac{cg_1^2g_2}{ab} = \frac{k^3c^2d^2}{ab}.$$  

Hence $k^3c^2d^2 \leq ab$. On the other hand, $kcd = cg_1 = ae + bf \geq a + b \geq 2\sqrt{ab}$. Hence $k^3c^2d^2 \geq k \cdot (4ab) > ab$, so we reached a contradiction. This shows Case II does not happen and proves the first half of Proposition 6.6.

Next we prove the second half of Proposition 6.6. Consider any $S = \mathbb{P}(a, b, c)$ such that $a, b, c$ are pairwise coprime, $(e, f, -g)$ is a relation between $(a, b, c)$ of width $w < 1$ and $S$ satisfies the assumptions in [GK16, Thm. 1.2]. Then $\text{Bl}_p \mathbb{P}(a, b, c)$ is not a MDS.

Now $X := \mathbb{P}(a, b, c, cg)$ satisfies conditions (i), (ii) and (iv) of Theorem 1.2. Since $d = cg$, we have $d^2w/(abc) = cg^2w/(ab) = w^2 < 1$. This verifies condition (iii). Hence $X = \mathbb{P}(a, b, c, cg)$ is an example of Theorem 1.2.

It remains to show that $X = \mathbb{P}(a, b, c, cg)$ satisfies the two assumptions in [GK17, Cor. 2.5]. Indeed, here $ae + bf = cg = d \cdot 1$ with $cg^2/ab < 1$. By Lemma 6.2, $X$ and $S = \mathbb{P}(a, b, c)$ have polytopes $\Delta$ and $\Gamma$ of González-Karu type. Let $r$ be the unique integer such that $1 \leq r \leq g$, $g \mid er - b$ and $g \mid fr + a$. Recall the proof of Lemma 6.2. By setting $T = -r$ and $U = 0$, we can determine the parameters $x, y, z$ and $\lambda$ to give

$$P_L = \left(-\frac{fg}{a}, \frac{fr}{a}, 0\right), \quad P_R = \left(\frac{eg}{b}, -\frac{er}{b}, 0\right).$$

This gives a polytope $\Delta$ of González-Karu type. The fan $\Sigma$ of $X$ can be chosen as the fan with ray generators

$$r_1' = \left(\frac{er - b}{g}, e, e\right), \quad r_2' = \left(\frac{fr + a}{g}, f, f\right), \quad r_3' = (-r, -g, 0), \quad r_4' = (0, 0, -1).$$

Define $\Gamma$ to be the projection of $\Delta$ to the $xy$-plane, after translating $(0, 1)$ to $(0, 0)$ and a reflection about $y$-axis. Then $\Gamma$ is the triangle given by (2), which is a polytope of $S = \mathbb{P}(a, b, c)$.

Now let $\Gamma'$ be the reflection of $\Gamma$ about the $y$-axis. By the hypothesis and Lemma 6.2 (i), either $(S, \Gamma)$ or $(S, \Gamma')$ meets the assumptions of [GK16, Thm. 1.2]. By symmetry we can assume the case $(S, \Gamma)$. Then [GK16, Thm. 1.2] (i) says that for some $m > 0$, the slice at $m \cdot x(P_L) + 1$ of $m \Gamma$ has exactly $n$ elements, and the slice at $m \cdot x(P_R) - n + 1$ of $m \Gamma$ has exactly $n$ elements too. By Definition 6.3, every slice of $\Delta$ forms a right triangle with the same number of lattice points on each right side. Hence, both slices of $m \Delta$ at $m \cdot x(P_L) + 1$ and $m \cdot x(P_R) - n + 1$ of $m \Delta$ have size $n$. This shows that (i) of [GK17, Cor. 2.5] holds. For (ii) of [GK17, Cor. 2.5], we have $s_y$ equals $s_2$ of the triangle $\Gamma$ in $xy$-plane. If $\Gamma$ meets the assumption (ii) of [GK16, Thm. 1.2], then $ns_y = ns_2 \notin \mathbb{Z}$, so $\Delta$ meets the assumption (ii) of [GK17, Cor. 2.5]. Therefore, $X$ satisfies the two assumptions in [GK17, Cor. 2.5].

Remark 6.9. Consider $X = \mathbb{P}(a, b, c, cg)$ in the overlap described in Proposition 6.6. A comparison with [GK17, Lem. 5.1, 5.2] shows that the curve $C \subset \text{Bl}_p X$ we constructed in Definition 5.4, whose class is extremal in the Mori cone $\overline{\text{NE}}(\text{Bl}_p X)$ (by Theorem 2.1), is the same curve $C$ constructed in [GK17, Lem. 5.1, 5.2].
Example 6.10. An example in such family of $\mathbb{P}(a, b, c, e_g)$ is $\mathbb{P}(7, 15, 26, 52)$. By [GK16], $\text{Bl}_p \mathbb{P}(7, 15, 26)$ is not a MDS. The relation is $(e, f, -g) = (1, 3, -2)$. Both Theorem 1.2 and [GK17, Cor. 2.5] apply to $\mathbb{P}(7, 15, 26, 52)$, so $\text{Bl}_p \mathbb{P}(7, 15, 26, 52)$ is not a MDS.

7. Application

We apply Proposition 1.1 to the following examples in [GAGK17]. By [GAGK17, Ex. 1.4], the blow-up $\text{Bl}_p S$ of the following $S = \mathbb{P}(a, b, c)$ at the identity point $p$ is not a MDS:

$$(a, b, c) = ((m + 2)^2, (m + 2)^3 + 1, (m + 2)^3(m^2 + 2m - 1) + m^2 + 3m + 1),$$

where $m$ is a positive integer.

We briefly review the geometry on those $\text{Bl}_p S$. By [GAGK17, Thm. 1.1], for every positive integer $m \geq 1$, there exists an irreducible polynomial $\xi_m \in \mathbb{C}[x, y]$ such that $\xi_m$ has vanishing order $m$ at $(1, 1)$ and the Newton polygon of $\xi_m$ is a triangle with vertices $(0, 0), (m - 1, 0)$ and $(m, m + 1)$. Now the weighted projective plane $S$ above satisfies the conditions of [GAGK17, Thm. 1.3]. Then by [GAGK17, Thm. 1.3] and its proof, the polynomial $\xi_m$ above defines a curve $H$ in $S$, passing through $p$ with multiplicity $m$, such that the proper transform $C$ of $H$ in $\text{Bl}_p S$ is a negative curve. Then $C \neq e$. The proof of [GAGK17, Thm. 1.3] in fact shows that $H$ is the polarization given by the triangle $\Delta$ with vertices $(-\alpha, 0), (m - 1 + \beta, 0), (m, m + 1)$, with

$$\alpha = \frac{1}{(m + 2)^2}, \quad \beta = \frac{(m + 2)^2 + 1}{(m + 2)^3 + 1}.$$ 

Therefore on $S$ we have

$$H^2 = 2\text{Area}(\Delta) = \frac{(m + 1)^2c}{ab}.$$ 

Let $B$ be the pseudo-effective divisor on $S$ generating $\text{Cl}(S) \cong \mathbb{Z}$. Then $H \sim rB$ for some $r \in \mathbb{Q}_{>0}$. Since $B^2 = 1/abc$ and $H^2 = r^2B^2$, we have $r = c(m + 1)$, so $|H| = c(m + 1)[B] \in \text{Cl}(S)$. Therefore $C \sim c(m + 1)\pi^*B - me$.

When $m \geq 2$, those $S$ above have width $w \geq 1$, so Theorem 1.2 does not apply to $S$. Nevertheless, by Proposition 1.1, we have the following examples:

Corollary 7.1. Let $X = \mathbb{P}(a, b, c, d_1, d_2, \cdots, d_{n-2})$ where

$$(a, b, c) = ((m + 2)^2, (m + 2)^3 + 1, (m + 2)^3(m^2 + 2m - 1) + m^2 + 3m + 1),$$

such that $m \in \mathbb{Z}_{>0}$, every $d_i$ lies in the semigroup generated by $a, b$ and $c$, and that every $d_i < abm/(m + 1)$. Let $p$ be the identity point of the open torus in $X$. Then $\text{Bl}_p X$ is not a MDS.

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REFERENCES

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *Journal of the American Mathematical Society*, 23(2):405–468, 2010.

[Cas15] Ana-Maria Castravet. Mori Dream Spaces and blow-ups. *Proceedings of the AMS Summer Institute in Algebraic Geometry 2015*, pages 143–168, 2015.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Society, 2011.

[CT15] Ana-Maria Castravet and Jenia Tevelev. $\overline{M}_{0,n}$ is not a Mori dream space. *Duke Mathematical Journal*, 164(8):1641–1667, 06 2015.

[Cut91] Steven Dale Cutkosky. Symbolic algebras of monomial primes. *J. Reine Angew. Math*, 416:71–89, 1991.

[FS97] William Fulton and Bernd Sturmfels. Intersection theory on toric varieties. *Topology*, 36(2):335–353, 1997.

[Ful93] William Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.

[GAGK17] Javier González-Anaya, José Luis González, and Kalle Karu. On a family of negative curves, 2017, arxiv:1712.04635.

[GK16] José Luis González and Kalle Karu. Some non-finitely generated Cox rings. *Compositio Mathematica*, 152(5):984–996, 2016.

[GK17] José Luis González and Kalle Karu. Examples of non-finitely generated Cox rings, 2017, arxiv:1708.09064.

[GNW94] Shiro Goto, Koji Nishida, and Kei-ichi Watanabe. Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik’s question. *Proceedings of the American Mathematical Society*, 120(2):383–392, 1994.

[He17] Zhuang He. New examples and non-examples of MDS when blowing up toric surfaces, 2017, arxiv:1703.00819.

[HK00] Yi Hu and Sean Keel. Mori dream spaces and GIT. *The Michigan Mathematical Journal*, 48(1):331–348, 2000.

[HKL16] Jürgen Hausen, Simon Keicher, and Antonio Laface. On blowing up the weighted projective plane, 2016, arXiv:1608.04542 [math.AG].

[KM08] Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge University Press, 2008.

[Muk05] Shigeru Mukai. Finite generation of the Nagata invariant rings in A-D-E cases. In *RIMS Preprint 1502*, 2005.

[Sri91] Hema Srinivasan. On finite generation of symbolic algebras of monomial primes. *Communications in Algebra*, 19(9):2557–2564, 1991.