Persistence of an active asymmetric rigid Brownian particle in two dimensions

Anirban Ghosh,* Sudipta Mandal, and Dipanjan Chakraborty†
Indian Institute of Science Education and Research Mohali, Sec. 81, S.A.S. Nagar, Knowledge City, Manauli, Punjab-140306, India.
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We have studied the persistence probability $p(t)$ of an active Brownian particle with shape asymmetry in two dimensions. The persistence probability is defined as the probability of a stochastic variable that has not changed its sign in the fixed given time interval. We have investigated two cases—diffusion of a free active particle and that of harmonically trapped particle. In our earlier work, Ghosh et al., Journal of Chemical Physics, 152, 174901, (2020), we had shown that $p(t)$ can be used to determine translational and the rotational diffusion constant of an asymmetric shape particle.

The route to calculating of the persistence probability is through the non-stationary two-time correlation function. The Lamperti transformation converts the non-stationary correlator to a stationary process. For a Gaussian-Markovian stochastic process, the persistence probability can be directly calculated using Slepian’s theorem. In contrast, when the process is non-Markovian, $p(t)$ is evaluated either using the Independent Interval Approximation (IIA) when the density of zero crossings stays finite or a perturbative expansion.

In our earlier work we investigated the effect of shape asymmetry on the persistence probability of a Brownian particle. We explicitly showed that the measured persistence probability could estimate the translational and the rotational diffusion constants of an asymmetric shape particle. The method has the advantage that the measurement of the rotational motion of the anisotropic particle is not required. In this paper, we extend the study to an active anisotropic particle and show how the persistence probability of an anisotropic particle is modified in the presence of a propulsion velocity. Further, we validate our analytical expression against the measured persistence probability from the numerical simulations of single particle Langevin dynamics and test whether the method proposed in our earlier work can distinguish between an active and a passive anisotropic particle.

I. INTRODUCTION

Persistence plays a very important role in describing a stochastic processes in nature, specifically the non-stationary dynamics of the system. The phenomenon of persistence is typically quantified through the persistence probability. For the last two decades, this has attracted quite significant attention in the scientific community. The persistence probability $p(t)$ of a stochastic variable is the probability that the variable has not changed its sign up to time $t$. In a wide range of non-equilibrium systems $p(t)$ is found to decay algebraically with an exponent $\theta$, $p(t) = t^{-\theta}$, where $\theta$ is a non-trivial exponent. As the temporal correlation of the non-Markovian stochastic process is highly non-local in behaviour, the exact calculation of persistence for even simple non-Markovian stochastic systems is often very difficult and exact analytical expression for $p(t)$ exists for very few cases. In spite of this, analytical and or numerical results for the persistence probability and the exponent $\theta$ exists for systems such as Brownian motion and diffusion process, reaction-diffusion systems, dynamical systems, phase ordered kinetics, fluctuating interfaces, critical dynamics, polymer dynamics, financial markets and many more. Even experimental results exist for persistence exponent in one-dimensional Ising model, diffusion process, fluctuating steps and interfaces. For a more comprehensive review, we invite the readers to the review by Bray et al. and Majumdar.

In our earlier work we investigated the effect of shape asymmetry on the persistence probability of a Brownian particle. We explicitly showed that the measured persistence probability could estimate the translational and the rotational diffusion constants of an asymmetric shape particle. The method has the advantage that the measurement of the rotational motion of the anisotropic particle is not required. In this paper, we extend the study to an active anisotropic particle. Our main interest lies in how the persistence probability of an anisotropic particle is modified in the presence of a propulsion velocity and whether the method proposed in our earlier work can distinguish between an active and a passive anisotropic particle.

This article is organized as follows: In the Section II we have presented the results for the two-time correlation function for the position of a free active Brownian particle with shape asymmetry and along with that the survival probability has been calculated from the two-time correlation. In the Section III, we have carried out a perturbative expansion for the position of an anisotropic particle.
active Brownian particle trapped in a harmonic potential. The two-time correlation has been calculated using the perturbative expansion method. Finally persistence probability is constructed from the two-time correlation function.

II. ACTIVE ASYMMETRIC PARTICLE IN TWO DIMENSIONS

We consider an self propelled asymmetric particle with velocity $v_0$ in two dimensions with mobilities $\Gamma_\parallel$ and $\Gamma_\perp$ along the longer and the shorter axes of the particle, respectively. We have fixed the body frame $x$ and $y$ directions as the long and the short axis, respectively. The particle has a single rotational mobility $\Gamma_\theta$. The particle is immersed in a bath of temperature $T$ so that the translational diffusion coefficients along the two directions are given by $D_\parallel = k_BT\Gamma_\parallel$ and $D_\perp = k_BT\Gamma_\perp$, and the rotational diffusion constant is $D_\theta = k_BT\Gamma_\theta$. At a given time $t$ the particle can be described by the position vector of its center of mass $\vec{r}(t)$ and the angle $\theta(t)$ between the $x$ axis of the lab-frame and the long axis of the particle. In this frame, the self-propulsion speed, which is taken along the long axis of the rod, is given by, $v_0\hat{n}(t)$, where $\hat{n}(t) \equiv (\cos(\theta(t)), \sin(\theta(t)))$ is a unit vector along the long axis of the particle. In the body frame, the equations of motion for the center of mass of the particle take the form

\[
\begin{align*}
\Gamma_1^{-1} \frac{\partial \vec{\xi}}{\partial t} &= F_x \cos(\theta(t)) + F_y \sin(\theta(t)) + \frac{v_0}{\Gamma_1} + \tilde{\xi}_x(t) \\
\Gamma_2^{-1} \frac{\partial \tilde{\xi}_y(t)}{\partial t} &= F_y \cos(\theta(t)) - F_x \sin(\theta(t)) + \tilde{\xi}_y(t) \\
\Gamma_3^{-1} \frac{\partial \tilde{\theta}(t)}{\partial t} &= \tau + \tilde{\theta}_\theta(t) 
\end{align*}
\]

Here $F_x$ and $F_y$ are the forces acting on the particle along the $x$ and $y$ axes (in the lab frame), respectively, and $\tau$ is the torque acting on the particle. The correlations of the thermal fluctuations in the body frame are given by

\[
\begin{align*}
\langle \tilde{\xi} \rangle &= 0 \\
\langle \tilde{\xi}_i(t) \tilde{\xi}_j(t') \rangle &= \frac{2k_BT}{\Gamma_i} \delta_{ij} \delta(t-t')
\end{align*}
\]

In the lab frame, the displacements are related to the body frame as

\[
\begin{align*}
\delta x &= \cos \theta \delta \hat{x} - \sin \theta \delta \hat{y} \\
\delta y &= \sin \theta \delta \hat{x} + \cos \theta \delta \hat{y}
\end{align*}
\]

Using the transformation in Eq. (3), the corresponding equations in the lab frame is given by

\[
\begin{align*}
\frac{\partial x}{\partial t} &= v_0 \cos \theta(t) + F_x' \Gamma + \frac{\Delta \Gamma}{2} \cos 2\theta(t) + \frac{\Delta \Gamma}{2} F_x \sin 2\theta(t) + \xi_x(t) \\
\frac{\partial y}{\partial t} &= v_0 \sin \theta(t) + F_y' \Gamma - \frac{\Delta \Gamma}{2} \cos 2\theta(t) + \frac{\Delta \Gamma}{2} F_x \sin 2\theta(t) + \xi_y(t) \\
\frac{\partial \theta}{\partial t} &= \Gamma_3 \tau + \xi_\theta(t)
\end{align*}
\]

The thermal fluctuations given in Eq. (3) are also transformed in the body frame. The correlations of the thermal fluctuations in the body frame are given by

\[
\langle \xi_\theta(t) \xi_\theta(t') \rangle = 2D_\theta \delta(t-t')
\]

and

\[
\langle \xi_i(t) \xi_j(t') \rangle_{\theta(t)} = 2k_BT \Gamma_{ij} \delta(t-t')
\]

Here $\Gamma = (\Gamma_\parallel + \Gamma_\perp)/2$ and $\Delta \Gamma = (\Gamma_\parallel - \Gamma_\perp)$, and mobility tensor can be written as $\Gamma_{ij} = \Gamma \delta_{ij} + \frac{\Delta \Gamma}{2} \delta_{ij} \tilde{R}_{ij}[\theta(t)]$, when the form of $\Delta \tilde{R}$ is written as

\[
\Delta \tilde{R} = \left( \begin{array}{cc} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{array} \right)
\]

FIG. 1. Representation of an ellipsoid in the $x - y$ lab frame and the $\hat{x} - \hat{y}$ body frame. The angle between two frames is $\theta$. The displacement $\tilde{R}$ can be decomposed as $\langle \delta \hat{x}, \delta \hat{y} \rangle$ or $\langle \delta x, \delta y \rangle$. 
A. Mean Square Displacement of the free active particle

We first take the case of free active ellipsoidal particle setting the external potential zero, the equation of motion takes the form

\[ x_i(t) = v_0 \int_0^t \cos \theta(t') dt' + \int_0^t \xi_i(t') dt' + x(0) \tag{7} \]

The mean \( \langle \Delta x(t) \rangle \), where \( \Delta x = x(t) - x(0) \) take the form

\[ \langle \Delta x(t) \rangle_{\xi, \theta} = v_0 \int_0^t \langle \cos \theta(t') \rangle dt' = v_0 \cos \theta_0 \left( \frac{1 - e^{-D\theta t}}{D\theta} \right) . \tag{8} \]

The mean square displacement of the particle is calculated from Eq. (7)

\[ \langle \Delta x^2(t) \rangle_{\xi, \theta} = v_0^2 \int_0^t \int_0^{t'} \langle \cos \theta(t') \cos \theta(t'') \rangle dt' dt'' \]

\[ + \int_0^t \xi_i(t') \xi_i(t'') dt' dt'' \tag{9} \]

The explicit evaluation of the two terms have been shown in Appendix A. The final expression for the mean-square displacement, using Eq. (A.1) and Eq. (A.6), take the form:

\[ \langle \Delta x^2(t) \rangle_{\xi, \theta} = 2k_B T \left[ \frac{\Gamma}{2} - \cos \theta_0 \left( \frac{1 - e^{-D\theta t}}{4D\theta} \right) \right] \]

\[ + \frac{v_0^2 \cos \theta_0}{12D\theta^2} (3 - 4e^{-D\theta t} + e^{-4D\theta t}) \]

\[ + \frac{v_0^2}{D\theta^2} (D\theta t + e^{-D\theta t} - 1) \tag{10} \]

\[ C_{\theta_0=0}^4(t) = \Delta D^2 \left[ \frac{3}{2} t\theta_0 - 3t\theta_0^2(t) - \frac{1}{2} t\theta_0 \tau_{16}(t) - \tau_{14}(t) \tau_\theta \right] \]

\[ + v_0^2 \left( 12D\tau_1(t) \tau_4(t) + t \left[ 24D\tau_1^2(t) + \frac{32}{3} D\tau_1(t) \tau_\theta - 8D\tau_4(t) \tau_\theta \right] - 16D\tau_\theta^2 \right) \tag{11} \]

The expression for \( \langle \Delta x^2(t) \rangle_{\xi, \theta} \) up to the order of \( v_0^2 \)

\[ \langle \Delta y^2(t) \rangle_{\xi, \theta} = 2k_B T \left[ \frac{\Gamma}{2} - \cos \theta_0 \left( \frac{1 - e^{-D\theta t}}{4D\theta} \right) \right] \]

\[ - \frac{v_0^2 \cos \theta_0}{12D\theta^2} (3 - 4e^{-D\theta t} + e^{-4D\theta t}) \]

\[ + \frac{v_0^2}{D\theta^2} (D\theta t + e^{-D\theta t} - 1) \tag{12} \]

In the absence of an active propulsion velocity, the position of the particle in the lab frame is a non-Gaussian stochastic variable. The non-Gaussianity parameter is defined as

\[ \phi(t, \theta_0) = \frac{\langle |\Delta x(t) - \langle \Delta x(t) \rangle|^4 \rangle - 3\langle \langle \Delta x(t) - \langle \Delta x(t) \rangle \rangle^2 \rangle^2 }{3\langle \langle \Delta x(t) - \langle \Delta x(t) \rangle \rangle^2 \rangle^2} \]

Defining \( \tau_\theta = 1/2D\theta \) and \( \tau_\tau(t) = 1 - e^{-D\theta t}/nD\theta \), the expressions in Eq. (8) and Eq. (10) take the form

\[ \langle \Delta x(t) \rangle = v_0 \cos \theta_0 \tau_1(t) \tag{13} \]

and

\[ \langle \Delta x(t)^2 \rangle_{\xi, \theta} = 2D\tau_4(t) + 2\tau_\theta v_0^2 \left( t - \tau_1 \right) + \frac{1}{3}(\tau_1 - \tau_4) \cos \theta_0 \tag{14} \]

Further, defining \( C_{\theta_0=0}^4(t) = \langle |\Delta x(t) - \langle \Delta x(t) \rangle_{\xi, \theta} |^4 \rangle_{\xi, \theta} - 3\langle \langle \Delta x(t) - \langle \Delta x(t) \rangle_{\xi, \theta} \rangle^2 \rangle_{\xi, \theta}^2 \), the non-Gaussian parameter is written as,

\[ \phi(t, \theta_0) = \frac{C_{\theta_0=0}^4(t)}{3\langle \langle \Delta x(t)^2 \rangle \rangle^2} \tag{15} \]

Since we evaluate the persistence probability keeping the initial angle \( \theta_0 \) fixed, specifically \( \theta_0 = 0 \), we estimate the non-Gaussian parameter at \( \theta_0 = 0 \). Further, we will also consider a weak asymmetry and weak propulsion velocity so that we evaluate \( \phi(t, \theta_0 = 0) \) only up to the order of \( v_0^2 \). The expression for \( C_{\theta_0=0}^4(t) \) takes the form

\[ \langle \Delta x^2(t) \rangle_{\xi, \theta} \] has the form
3\langle \Delta x^2(t) \rangle = 12\bar{D}^2t^2 + 12\bar{D}\Delta D\tau_0(t) + 3\Delta \bar{D}^2t^2 \\
+ v_0^2 \left( \left( -6\Delta \bar{D}\tau_1^2(t)\tau_4(t) - 8\Delta \bar{D}\tau_0\tau_1(t)\tau_4(t) - 4\Delta \bar{D}\tau_0\tau_2^2(t) \right) \\
+ \bar{D}( -12\bar{D}\tau_1^2(t) - 16\bar{D}\tau_1(t)\tau_0 - 8\bar{D}\tau_4(t)\tau_0 + 12\Delta \bar{D}\tau_4(t)\tau_0 + 24\bar{D}\tau_0^2 ) \right) \right)

(17)

Clearly from Eq. (16) and Eq. (17), the non-Gaussian parameter depends on the ratio \( \Delta D^2/\bar{D}^2 \) and \( v_0^2/\bar{D}^2 \). In the limit of weak asymmetry and small propulsion velocity, the non-Gaussian parameter remains small. The time-dependent \( \phi(t,0) \) exhibits a non-monotonic behaviour with a peak at \( D_0 t \approx 1 \).

### B. Persistence of the free particle

We now turn our attention to the persistence probability of a free asymmetrical active Brownian particle. Setting the external potential zero, the formal solution to the equation of motion becomes

\[ x_i(t) = x_i(0) + \int_0^t \xi(t') dt' + v_0 \int_0^t \cos \theta(t') dt', \quad (18) \]

with the initial condition \( x_i(0) = 0 \). The calculation of the two-time correlation function \( \langle x(t_1)x(t_2)\rangle_{\xi,\theta} \) can be achieved by

\[ \langle x(t_1)x(t_2)\rangle_{\xi,\theta} = v_0^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \cos \theta(t_1') \cos \theta(t_2') \]
\[ + \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \xi(t_1') \xi(t_2') \]

(19)

Considering \( t_1 > t_2 \), the explicit evaluation of the two terms in Eq. (19) has been shown in Appendix B. The final expression is obtained from Eq. (B.1) and Eq. (B.3) to give

\[ \langle x(t_1)x(t_2)\rangle_{\xi,\theta} = 2k_B T t_2 \left[ 1 + \frac{\Delta \Gamma}{2t} \cos \theta_0 \left( \frac{1 - e^{-4D_0 t_2}}{4D_0 t_2} \right) \right] + v_0^2 \left[ \cos \theta_0 \left( \frac{1 - e^{-D_0 t_2}}{6D_0^2} + \frac{1 - e^{-4D_0 t_2}}{12D_0^2} - \frac{1 - e^{-D_0 t_1 - 4D_0 t_2}}{2D_0^2} \right) \right] \]

(20)

We now set the initial angle \( \theta_0 = 0 \). The diffusion coefficients \( \bar{D} \) and \( \Delta D \) are renormalized by the active velocity. Furthermore, we note that the last term in Eq. (20) contains a stationary component which survives in the long time limit of \( t_1 \) and \( t_2 \) large but \( (t_1 - t_2) \) finite. This, of course, makes the conversion of this non-stationary correlator to a stationary one slightly problematic. In order to transform the non-stationary correlation into a stationary correlator, we make the approximation \( t_1 \gg t_2 \) so that both the terms \( 2v_0^2\tau_0\tau_3(t)e^{-D_0 t_1} \) and the last term

\[ v_0^2 e^{-D_0 t_1} (1 - e^{-4D_0 t_2})/D_0^2 \]

in Eq. (20) can be dropped.

\[ \langle x(t_1)x(t_2)\rangle_{\xi,\theta} = \left( 2k_B T t_2 + v_0^2/3D_0 \right) t_2 \]
\[ + \left( \Delta D + \frac{v_0^2}{3D_0} \right) \left( \frac{1 - e^{-4D_0 t_2}}{4D_0} \right) - v_0^2 \left( \frac{1 - e^{-D_0 t_2}}{3D_0^2} \right) \]

(21)

Dropping the second term is strictly valid only when \( t_1 \gg t_2 \). Nevertheless, even with this approximation, we want to figure out how well the analytical expression for \( p(t) \) compares with the numerical results. We use the Lamperti transformation and define \( \hat{X}(t) = x(t)/\sqrt{\langle \Delta x^2(t)\rangle_{\xi,\theta}} \). The two-time correlation function of the rescaled variable \( \langle \hat{X}(t_1)\hat{X}(t_2)\rangle_{\xi,\theta} \) becomes
\[ \langle \tilde{X}(t_1)\tilde{X}(t_2) \rangle = (t_2/t_1)^{1/2} \left[ 2D_{\text{eff}} + \Delta D_{\text{eff}} \left( \frac{1 - e^{-4D_{\theta}t_2}}{4D_{\theta}t_2} \right) - \frac{v_0^2}{6D_{\theta}} \left( \frac{1 - e^{-D_{\theta}t_2}}{D_{\theta}t_2} \right) \right]^{1/2} \]

\[ \left[ 2D_{\text{eff}} + \Delta D_{\text{eff}} \left( \frac{1 - e^{-4D_{\theta}t_1}}{4D_{\theta}t_1} \right) - \frac{v_0^2}{6D_{\theta}} \left( \frac{1 - e^{-D_{\theta}t_1}}{D_{\theta}t_1} \right) \right]^{-1/2} \]

where the effective diffusivity is given by \( D_{\text{eff}} = \bar{D} + v_0^2/2D_{\theta} \) and \( \Delta D_{\text{eff}} = \Delta \bar{D} + v_0^2/3D_{\theta} \). We now define the transformation in time as

\[ e^{T} = \sqrt{2D_{\text{eff}}t} \left[ 1 + \frac{\Delta D_{\text{eff}}}{2D_{\text{eff}}} \left( \frac{1 - e^{-4D_{\theta}t}}{4D_{\theta}t} \right) - \frac{v_0^2}{6D_{\theta}D_{\text{eff}}} \left( \frac{1 - e^{-D_{\theta}t}}{D_{\theta}t} \right) \right]^{-1/2} \]

Using this transformation in time the two-time correlation function \( \langle \tilde{X}(T_1)\tilde{X}(T_2) \rangle \) from Eq. (22) takes the simple form of \( \langle \tilde{X}(T_1)\tilde{X}(T_2) \rangle = e^{-(T_1-T_2)^2/4t} \). Since the stationary correlation function now decays exponentially for all times, following Slepian\(^1\), the asymptotic form of the persistence probability is found as

\[ P(T) = e^{-\lambda T} \quad \text{(24)} \]

Transforming back to real-time \( t \), we get the persistence probability for the free particle as

\[ p(t, \theta_0 = 0) = \frac{1}{\sqrt{2D_{\text{eff}}t}} \left[ 1 + \frac{\Delta D_{\text{eff}}}{2D_{\text{eff}}} \left( \frac{1 - e^{-4D_{\theta}t}}{4D_{\theta}t} \right) - \frac{v_0^2}{6D_{\theta}D_{\text{eff}}} \left( \frac{1 - e^{-D_{\theta}t}}{D_{\theta}t} \right) \right]^{-1/2} \quad \text{(25)} \]

Rearranging the above expression, we get

\[ t^{1/2} p(t, \theta_0 = 0) = \frac{1}{\sqrt{2D_{\text{eff}}} \left[ 1 + \frac{\Delta D_{\text{eff}}}{2D_{\text{eff}}} \left( \frac{1 - e^{-4D_{\theta}t}}{4D_{\theta}t} \right) - \frac{v_0^2}{6D_{\theta}D_{\text{eff}}} \left( \frac{1 - e^{-D_{\theta}t}}{D_{\theta}t} \right) \right]^{-1/2} \quad \text{(26)} \]

In the absence of the propulsion velocity \( v_0 = 0 \), we recover the persistence probability of free anisotropic particle.\(^3\)

In order to validate the expression for the persistence probability we performed numerical simulations of Eq. (4). The initial condition was chosen from a Gaussian distribution with a very small width, so the sign of \( \vec{r}(0) \) is clearly defined. The trajectories were evolved in time with an integration time-step of \( \delta t = 0.001 \). At every instant, the survival of the particle trajectory was checked by looking at the sign of \( \vec{r}(t) \). Fraction of trajectories for which the position did not change its sign up to time \( t \) gave the survival probability \( p(t) \). We ensure that the asymmetry of the particles is picked up as expected from our earlier work,\(^3\); for small propulsion velocity \( t^{1/2} p(t) \) is unable to pick up the activity of the particle. In the case when the activity of the particle is comparatively large, the \( t^{1/2} p(t) \) indeed picks up the activity of the particle. When compared with the analytical expression of Eq. (26), for the small propulsion velocity, the expression compares quite well with the simulation results with the overall constant as the only fit parameter (the dotted lines in the figures). When the data is fitted to Eq. (26) with the overall constant fixed and \( \bar{D} \) and \( \Delta \bar{D} \) as fit parameters, it yields the correct values of \( \bar{D} \) and \( \Delta \bar{D} \). However, for comparatively larger values of \( v_0 \), when the expression is plotted against the numerical data with the overall constant as the only fit parameter, the data matches only asymptotically with the analytical expression. On the other hand, when the data is fitted with Eq. (26) with \( \bar{D} \) and \( \Delta \bar{D} \) as fit parameters, the fit yields a slightly lower value of \( \bar{D} \) and a slightly higher value of \( \Delta \bar{D} \). For example, in the case of \( \bar{D} = 0.975 \) and \( \Delta \bar{D} = 0.05 \) (Fig. 2 open triangles), we obtain from the fit a value of \( \bar{D} \approx 0.96 \) and \( \Delta \bar{D} \approx 0.08 \). In the case of \( \bar{D} = 0.95 \) and \( \Delta \bar{D} = 0.1 \) (Fig. 2 open triangles), we obtain from the fit a value of \( \bar{D} \approx 0.93 \) and \( \Delta \bar{D} \approx 0.14 \).
III. HARMONICALLY TRAPPED ASYMMETRIC PARTICLE

We now consider the case when such an active anisotropic particle is trapped in a harmonic trap. This situation typically occurs in the trapping and tracking of colloids in experiments. The harmonic potential is taken as isotropic potential with no preferred directional alignment. The potential is taken as $U(x, y) = \kappa(x^2 + y^2)/2$. Using Eq. (4) the corresponding Langevin equation becomes

$$
\begin{align*}
\frac{\partial x}{\partial t} &= -\kappa x(\bar{\Gamma} + \frac{\Delta \Gamma}{2} \cos 2\theta(t)) - \kappa y \frac{\Delta \Gamma}{2} \sin 2\theta(t) + v_0 \cos \theta(t) + \xi_1(t) \\
\frac{\partial y}{\partial t} &= -\kappa x \frac{\Delta \Gamma}{2} \sin 2\theta(t) - \kappa y(\bar{\Gamma} - \frac{\Delta \Gamma}{2} \cos 2\theta(t)) + v_0 \sin \theta(t) + \xi_2(t) \\
\frac{\partial \theta}{\partial t} &= \Gamma_3 \tau + \xi_3(t)
\end{align*}
$$

The correlations of the thermal fluctuations follow Eq. (5).

A. Perturbative expansion

Let us define the vector space $\mathbf{R} \equiv (x, y)^T$, and the equation takes the general form as

$$
\dot{\mathbf{R}} = -\kappa \bar{\Gamma} \mathbf{1} + \frac{\Delta \Gamma}{2} \bar{\mathbf{R}}(t) \mathbf{R}(t) + v_0 \mathbf{n} + \xi(t) \tag{28}
$$

In order to solve this equation we use the perturbative expansion

$$
\mathbf{R}(t) = \mathbf{R}_0(t) - \left(\frac{\kappa \Delta \Gamma}{2}\right) \mathbf{R}_1(t) + \left(\frac{\kappa \Delta \Gamma}{2}\right)^2 \mathbf{R}_2(t) + \mathcal{O}\left(\frac{\kappa \Delta \Gamma}{2}\right)^3 \tag{29}
$$

Substituting Eq. (29) in Eq. (28) and equalizing both sides we get the equations for $\mathbf{R}_0(t)$ and $\mathbf{R}_1(t)$ as

$$
\dot{\mathbf{R}}_0(t) = -\kappa \bar{\Gamma} \mathbf{R}_0(t) + v_0 \mathbf{n} + \xi(t) \\
\dot{\mathbf{R}}_1(t) = -\kappa \bar{\Gamma} \mathbf{R}_1(t) + \bar{\mathbf{R}}(t) \mathbf{R}_0(t) \tag{30}
$$

The formal solutions for Eq. (30) with the initial con-
The explicit form of the correlation matrix

\[
\langle R_{0,i}(t)R_{0,j}(t)\rangle_{\xi,\theta} = \int_0^t dt' \int_0^t dt'' e^{-\kappa\Gamma(t-t')} e^{-\kappa\Gamma(t-t'')} \langle \xi(t')\xi(t'') \rangle + \int_0^t dt' \int_0^t dt'' e^{-\kappa\Gamma(t-t')} e^{-\kappa\Gamma(t-t'')} v_0^2 \langle n_i(t)n_j(t'') \rangle
\]

The calculations of the time integrals in Eq. (33) have been explicitly shown in Appendix C (from Eq. (C.1) to Eq. (C.6)) and the final result for \(\langle x_0^2(t)\rangle\) is found as

\[
\langle x_0^2(t)\rangle_{\xi,\theta} = \frac{k_B T}{\kappa} (1 - e^{-2\Gamma t}) + \Delta D \cos 2\theta_0 \left( e^{-4D_0 t} - e^{-2\Gamma t} \right)
\]

\[
+ \frac{v_0^2 \cos 2\theta_0}{2} \left[ \frac{2D_0 (e^{-4D_0 t} - e^{-2\Gamma t})}{(2\Gamma - 4D_0)(\kappa\Gamma - 3D_0)(\kappa\Gamma - D_0)} \right.
\]

\[
+ \frac{e^{-4D_0 t} - 2e^{-(\kappa\Gamma + D_0) t} + e^{-2\Gamma t}}{(\kappa\Gamma - D_0)(\kappa\Gamma - 3D_0)}
\]

\[
+ \frac{v_0^2}{2} \left[ \frac{1 - 2e^{-(\kappa\Gamma + D_0) t} + e^{-2\Gamma t}}{(\kappa\Gamma - D_0)(\kappa\Gamma + D_0)} - \frac{D_0 (1 - e^{-2\Gamma t})}{\kappa\Gamma - D_0}(\kappa\Gamma + D_0) \right]
\]

(34)

In the limit of \(\kappa \rightarrow 0\), Eq. (34) reproduces Eq. (10), the correct result for the free diffusion of active anisotropic particles.

We now proceed to calculate the correction term to this correlation and restrict ourselves only to the first order correction. To this end, we first restructure the solution of \(R_1(t)\) as

\[
R_{1,i}(t) = \int_0^t dt' e^{-\kappa\Gamma(t-t')} \sum_j \mathcal{R}_{ij}(t') R_{0,j}(t')
\]

(35)

The correlation function \(\langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta}\) is then given by

\[
\langle R_{0,i}(t_1)R_{1,j}(t_2)\rangle_{\xi,\theta} = \left\langle R_{0,i}(t_1) \int_0^{t_2} dt' e^{-\kappa\Gamma(t_1-t')} \sum_k \mathcal{R}_{jk}(t_2') R_{0,k}(t_2') \right\rangle
\]

\[
= \left\langle \int_0^{t_2} dt' e^{-\kappa\Gamma(t_1-t')} \sum_k \mathcal{R}_{jk}(t_2') R_{0,i}(t_1) R_{0,k}(t_2') \right\rangle
\]

The final expression for \(\langle x_0(t)x_1(t)\rangle\) is given by

\[
\langle R_i(t)R_j(t)\rangle_{\xi,\theta} = \langle R_{0,i}(t)R_{0,j}(t)\rangle_{\xi,\theta} - (\kappa\Delta\Gamma) \langle R_{0,i}(t)R_{1,j}(t)\rangle_{\xi,\theta}
\]

\[
+ \left( \frac{\kappa\Delta\Gamma}{2} \right)^2 \left[ \langle R_{1,i}(t)R_{1,j}(t)\rangle_{\xi,\theta} + 2\langle R_{0,i}(t)R_{2,j}(t)\rangle_{\xi,\theta} \right]
\]

\[
+ O \left( \frac{\kappa\Delta\Gamma}{2} \right)^3
\]

(32)

Here we have considered the fact that \(\langle R_{0,i}R_{1,j}\rangle = \langle R_{0,j}R_{1,i}\rangle\). We now start to calculate the different terms of the correlation matrix. The correlation matrix for \(R_0(t)\) averaged over the translational and the rotational noise is given as

\[
\langle R_i(t_1)R_j(t_2)\rangle_{\xi,\theta} = \langle R_{0,i}(t_1)R_{0,j}(t_2)\rangle_{\xi,\theta}
\]

\[
- (\kappa\Delta\Gamma) \langle R_{0,i}(t_1)R_{1,j}(t_2)\rangle_{\xi,\theta}
\]

\[
+ \left( \frac{\kappa\Delta\Gamma}{2} \right)^2 \left[ \langle R_{1,i}(t_1)R_{1,j}(t_2)\rangle_{\xi,\theta} + 2\langle R_{0,i}(t_1)R_{2,j}(t_2)\rangle_{\xi,\theta} \right]
\]

\[
+ O \left( \frac{\kappa\Delta\Gamma}{2} \right)^3
\]

(33)
\begin{align}
\langle x_0(t)x_1(t)\rangle_{\xi,\theta} &= \left(\frac{k_B T}{\kappa}\right) \cos 2\theta_0 \left( e^{-4D_\theta t} - e^{-4\kappa \Gamma t} \right) + \left(\frac{k_B T}{\kappa}\right) \left(\frac{\Delta \Gamma}{2\Gamma}\right) \frac{1 - e^{-2\kappa \Gamma t}}{2\kappa \Gamma + 4D_\theta} \\
&\quad - 2\kappa \Gamma e^{-2\kappa \Gamma t} - e^{-2(2\kappa \Gamma + 4D_\theta)t} \right) \\
&\quad \left(-\frac{3v_0^2 D_\theta e^{-\kappa \Gamma t} \sinh \kappa \Gamma t}{2\kappa \Gamma (\kappa \Gamma + D_\theta)(\kappa \Gamma - D_\theta)} \right) \\
&\quad \left(\frac{\kappa \Gamma + 4D_\theta}{4D_\theta}\right)
\end{align}

Following Eq. (32), the mean-square displacement \(\langle x^2(t)\rangle_{\xi,\theta}\) up to the first order correction is given by \(\langle x^2(t)\rangle_{\xi,\theta} = \langle x_0^2(t)\rangle_{\xi,\theta} - (\kappa \Delta \Gamma)\langle x_0(t)x_1(t)\rangle_{\xi,\theta}\). From Eqs. (34) and (39) it is clear that the second term in Eq. (34) cancels with the first term in Eq. (39). Further, since we are interested in the expression for the mean square displacement up to the first order, the expression for \(\langle x^2(t)\rangle\) becomes

\begin{align}
\langle x^2(t)\rangle_{\xi,\theta} &= \left(\frac{k_B T}{\kappa}\right) \left[ (1 - e^{-2\kappa \Gamma t}) + \kappa \Delta \Gamma \cos 2\theta_0 \left( e^{-2\kappa \Gamma t} - e^{-4D_\theta + 2\kappa \Gamma t} \right) \right] \\
&\quad + \frac{v_0^2 \cos 2\theta_0}{2} \left[ \frac{2D_\theta(e^{-4D_\theta t} - e^{-2\kappa \Gamma t})}{(2\kappa \Gamma - 4D_\theta)(\kappa \Gamma - 3D_\theta)} + \frac{e^{-4D_\theta t} - 2e^{-(\kappa \Gamma + D_\theta)t} + e^{-2\kappa \Gamma t}}{(\kappa \Gamma - D_\theta)(\kappa \Gamma - 3D_\theta)} \right] \\
&\quad + \frac{v_0^2}{2} \left[ 1 - 2e^{-(\kappa \Gamma + D_\theta)t} + e^{-2\kappa \Gamma t} \right] - \frac{D_\theta(1 - e^{-2\kappa \Gamma t})}{\kappa \Gamma(D_\theta + \kappa \Gamma)} - \frac{\kappa \Delta \Gamma}{2\kappa \Gamma(\kappa \Gamma + D_\theta)(\kappa \Gamma - D_\theta)} \right] \\
&\quad \left(\frac{3v_0^2 D_\theta e^{-\kappa \Gamma t} \sinh \kappa \Gamma t}{2\kappa \Gamma(\kappa \Gamma + D_\theta)(\kappa \Gamma - D_\theta)} \right)
\end{align}

IV. PERSISTENCE PROBABILITY

We now want to calculate an expression for the persistence probability of an active anisotropic particle in the presence of a harmonic trap. As before, we first calculate the two-time correlation function of the \(x\)-coordinate of the position vector. Keeping up to the first order correction in \(\kappa \Delta \Gamma/2\), the expression for the two time correlation function becomes \(\langle x(t_1)x(t_2)\rangle_{\xi,\theta} = \langle x_0(t_1)x_0(t_2)\rangle_{\xi,\theta} - \kappa \Delta \Gamma \langle x_0(t_1)x_1(t_2)\rangle_{\xi,\theta}\), with \(t_1 > t_2\). Note that the quantities \(\langle x_0(t_1)x_1(t_2)\rangle\) and \(\langle x_0(t_2)x_1(t_1)\rangle\) are only equal in the asymptotic limit, which is also the limit under consideration when evaluating the persistence probability. In Appendix D and Appendix E we have explicitly shown the calculations of the two terms that appear in the expression for the two-time correlation function \(\langle x(t_1)x(t_2)\rangle_{\xi,\theta}\). We merely quote the final expression here:

\begin{align}
\langle x_0(t_1)x_0(t_2)\rangle_{\xi,\theta} &= \left(\frac{k_B T}{\kappa}\right) e^{-\kappa \Gamma t_1} \left[ e^{\kappa \Gamma t_2} - e^{-\kappa \Gamma t_2} \right] \\
&\quad + \frac{k_B T}{\kappa} \left(\frac{\kappa \Delta \Gamma}{2\Gamma}\right) \cos 2\theta_0 \left[ e^{(\kappa \Gamma - 4D_\theta)t} - e^{-\kappa \Gamma t_2} \right] \\
&\quad + \frac{v_0^2 \cos 2\theta_0}{2} \left[ 2D_\theta(e^{-\kappa \Gamma t_1} e^{(\kappa \Gamma - 4D_\theta)t} - e^{-\kappa \Gamma (t_1 + t_2)}) - e^{-\kappa \Gamma t_1} e^{-3D_\theta t_2} - e^{-\kappa \Gamma (t_1 + t_2)} \right] \\
&\quad + \frac{v_0^2}{2} \left[ 2D_\theta(2e^{-\kappa \Gamma (t_1 + t_2)} - e^{-\kappa \Gamma (t_1 - t_2)}) - e^{-\kappa \Gamma t_1} e^{-3D_\theta t_2} - e^{-\kappa \Gamma (t_1 + t_2)} \right] \\
&\quad + \frac{v_0^2}{2} \left[ 2D_\theta e^{-\kappa \Gamma (t_1 + t_2)} - e^{-\kappa \Gamma (t_1 - t_2)} \right] \\
&\quad + \frac{v_0^2}{2} \left[ 2D_\theta(2e^{-\kappa \Gamma (t_1 - t_2)} - e^{-\kappa \Gamma (t_1 + t_2)}) - e^{-\kappa \Gamma t_1} e^{-3D_\theta t_2} - e^{-\kappa \Gamma (t_1 + t_2)} \right] \\
&\quad + \frac{v_0^2}{2} \left[ 2D_\theta e^{-\kappa \Gamma (t_1 + t_2)} - e^{-\kappa \Gamma (t_1 - t_2)} \right]
\end{align}
and

\[ \langle x_0(t_1)x_1(t_2) \rangle_{\xi,\theta} = \left( \frac{kbT}{\kappa} \right) \cos 2\theta_0 e^{-\kappa T t_1} \left( \frac{e^{(e\kappa T - 4D\theta)t_2} - e^{-\kappa T t_2}}{2\kappa T - 4D\theta} \right) \\
- e^{-\kappa T t_2} - e^{-(4D\theta + \kappa T)t_2} \right) \\
+ \left( \frac{kbT}{\kappa} \right) \left( \frac{\Delta T}{2T} \right) e^{-\kappa T t_1} \left[ \frac{e^{\kappa T t_2} - e^{-\kappa T t_2}}{2\kappa T + 4D\theta} \right] \\
- \frac{2\kappa T e^{-\kappa T t_2} - e^{-(\kappa T + 4D\theta)t_2}}{4D\theta} \left( \frac{\kappa + 4D\theta}{\kappa + 4D\theta} \right) + \frac{3t_0^2 D\theta e^{-\kappa T t_2} \sinh \kappa T t_2}{2\kappa \Gamma + D\theta} \left( \kappa - D\theta \right) \left( \kappa + 2D\theta \right) \right] \tag{42} \]

Choosing initial angle \( \theta_0 = 0 \), all the terms in both the expressions for \( \langle x_0(t_1)x_0(t_2) \rangle_{\xi,\theta} \) and \( \langle x_0(t_1)x_1(t_2) \rangle_{\xi,\theta} \) survive. However, since we are interested in the asymptotic limit with \( t_1 \gg t_2 \), we drop the terms which are higher order exponentials in time and therefore decay faster. Further, since we are interested in the first order correction, we drop the second bracketed term in Eq. (42).

After a little algebra, the two-time correlation function for the \( x \)-coordinate of the position vector, with initial angle \( \theta_0 = 0 \) becomes

\[ \langle x(t_1)x(t_2) \rangle_{\xi,\theta,\theta_0=0} = e^{-\kappa T t_1} \left[ \frac{2kbT}{\kappa'} \sinh \kappa T t_2 + \frac{v_0^2 \left( e^{-\kappa T t_2} - e^{-D\theta t_2} \right)}{\left( \kappa' - 3D\theta \right) \left( \kappa' + D\theta \right)} \right] \\
+ \left( \frac{2kbT}{\kappa} \right) \left( \frac{\kappa \Delta T}{2} \right) e^{-\kappa T t_2} \left( \frac{1 - e^{-4D\theta t_2}}{4D\theta} \right) \tag{43} \]

where the effective trap constant \( \kappa' \) is defined as \( \kappa'^{-1} = \frac{1}{\kappa^{-1}} \left[ 1 - \frac{v_0^2 D\theta}{2D\left( \kappa' - 3D\theta \right) \left( \kappa' + D\theta \right)} \right] \). As before, we define the variable \( X(t) = x(t)/\sqrt{\langle x^2 \rangle_{\xi,\theta}} \) and the correlation function of \( \langle X(t_1)X(t_2) \rangle_{\xi,\theta} \) is given by

\[ \langle X(t_1)X(t_2) \rangle_{\xi,\theta} = e^{-\kappa T t_1/2} \left[ \frac{2kbT}{\kappa'} \sinh \kappa T t_2 + \frac{v_0^2 \left( e^{-\kappa T t_2} - e^{-D\theta t_2} \right)}{\left( \kappa' - 3D\theta \right) \left( \kappa' + D\theta \right)} + \frac{2kbT}{\kappa} \left( \frac{\kappa \Delta T}{2} \right) e^{-\kappa T t_2} \left( \frac{1 - e^{-4D\theta t_2}}{4D\theta} \right) \right]^{1/2} \tag{44} \]

Using the time transformation for an imaginary time variable \( T \), such that

\[ e^T = e^{-\kappa T t} \left[ \frac{2kbT}{\kappa'} \sinh \kappa T t + \frac{v_0^2 \left( e^{-\kappa T t} - e^{-D\theta t} \right)}{\left( \kappa' - 3D\theta \right) \left( \kappa' + D\theta \right)} + \frac{2kbT}{\kappa} \left( \frac{\kappa \Delta T}{2} \right) e^{-\kappa T t} \left( \frac{1 - e^{-4D\theta t}}{4D\theta} \right) \right] \]

the two time correlation function in Eq. (44) transforms into a stationary correlator of the form \( C(T_1 - T_2) = e^{-\langle T_1 - T_2 \rangle/2} \) and the persistence probability in the asymptotic limit in the imaginary variable \( T \) is given by \( p(T) \sim e^{-T/2} \). Transforming back into real-time, the persistence probability becomes

\[ p(t, \theta_0 = 0) = e^{-\kappa T t} \left[ \frac{2kbT}{\kappa'} \sinh \kappa T t + \frac{v_0^2 \left( e^{-\kappa T t} - e^{-D\theta t} \right)}{\left( \kappa' - 3D\theta \right) \left( \kappa' + D\theta \right)} + \frac{2kbT}{\kappa} \left( \frac{\kappa \Delta T}{2} \right) e^{-\kappa T t} \left( \frac{1 - e^{-4D\theta t}}{4D\theta} \right) \right]^{-1/2} \tag{46} \]

In the limit of \( v_0 \to 0 \), the equation correctly recombines the result for a passive anisotropic particle.\(^{32}\). In order to validate the equation, we performed numerical simulations of Eq. (27) with the initial condition chosen from a Gaussian distribution with a very small width, so that the sign of \( \tilde{r}(0) \) is well defined. The trajectories were evolved in time with an integration time step of \( \delta t = 0.001 \). The persistence probability was determined from the fraction of trajectories for which \( x(t) \) did not change its sign. A comparison of the measured persistence probability is
FIG. 4. Plot of $p(t)$ for different choices of stiffness of the potential $\kappa$ of the harmonically trapped anisotropic particle (blue square for $\kappa = 0.01$ and orange triangle for $\kappa = 0.1$) for self-propelled velocity $v_0 = 0.05$: the colours representing different stiffness of the potential are written above the plot. The rotational diffusion constant and initial angle $\theta_0$ were fixed at $D_\theta = 1$ and $\theta_0 = 0$, translational diffusivities are fixed as $D_I = 1$, $D_\perp = 0.5$. The solid lines are plots of Eq. $(46)$ with the appropriate values of $\kappa$, $D_I$, $D_\perp$, and $D_\theta$.

V. CONCLUSION

In brief, we have calculated the persistence probability along the $x$-axis of an active anisotropic particle in two dimensions in the absence of any potential and in the presence of a harmonic potential. The two-time correlation function has been calculated in the both cases. In the case of the harmonic trapped particle, we have used a perturbative solution for calculating the correlation functions. The persistence probability has been calculated with suitable space and time transformations. The analytic expressions have been validated against numerically measured persistence probability.

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Appendices

A. MEAN-SQUARE DISPLACEMENT OF A ACTIVE ANISOTROPIC FREE PARTICLE

\[ I_1 = \int_0^t \langle \cos \theta(t') \cos \theta(t'') \rangle dt' dt'' \]
\[ = \frac{1}{2} \int_0^t dt' \int_0^t dt'' \left\langle \cos (\theta(t') + \theta(t'')) + \cos (\theta(t') - \theta(t'')) \right\rangle \]
\[ = \frac{1}{2} \cos 2\theta_0 \int_0^t e^{-D_b |t' + t'' - 2\min(t',t'')|} dt' dt'' + \frac{1}{2} \int_0^t e^{-D_b |t' + t'' - 2\min(t',t'')|} dt' dt'' \]

Integral \( I_1 \) is having two separate integrals and solving these two separately

\[ I' = \int_0^t e^{-D_b |t' + t'' + 2\min(t',t'')|} dt' dt'' \]
\[ = \int_0^t dt' \int_0^t dt'' e^{-D_b (t' + 3t'')} + \int_0^t dt'' \int_0^t dt' e^{-D_b (3t' + t'')} \]
\[ = \int_0^t e^{-D_b t'} dt' \int_0^t e^{-3D_b t''} dt'' + \int_0^t e^{-D_b t''} dt'' \int_0^t e^{-3D_b t'} dt' \]
\[ = \frac{1}{6D_b} (3 - 4e^{-D_b t} + e^{-4D_b t}) \]
and

\[
I'' = \int_0^t dt' \int_0^{t''} dt'' e^{-D_\theta [t' + t'' - 2 \min(t', t'')]} \\
= \int_0^t dt' \int_0^{t'} dt'' e^{-D_\theta (t' - t'')} + \int_0^t dt' \int_0^{t''} dt'' e^{-D_\theta (t'' - t')} \\
= \int_0^t e^{-D_\theta t'} dt' \int_0^{t'} e^{D_\theta t''} dt'' + \int_0^t e^{-D_\theta t''} dt'' \int_0^{t'} e^{D_\theta t'} dt' \\
= \frac{2}{D_\theta^2} (D_\theta t + e^{-D_\theta t} - 1) \tag{A.3}
\]

So values of Eq. (18) and Eq. (19) is added to get \( I_1 \) as

\[
I_1 = \frac{\cos 2\theta_0}{12D_\theta^2} (3 - 4e^{-D_\theta t} + e^{-4D_\theta t}) + \frac{1}{D_\theta^2} (D_\theta t + e^{-D_\theta t} - 1) \tag{A.4}
\]

Now the second integral of Eq. (9) is calculated as

\[
I_2 = \int_0^t dt' \int_0^t dt'' \langle \xi_i(t') \xi_i(t'') \rangle \\
= 2k_BT \int_0^t dt' \int_0^t dt'' \langle \Gamma_{ii}[\theta(t')] \rangle \xi_\theta \delta(t - t') \\
= 2k_BT \int_0^t dt' \langle \Gamma_{ii}[\theta(t')] \rangle \xi_\theta \tag{A.5}
\]

Using the explicit form of \( \Gamma_{xx} \) from Eq. (6) the mean-square displacement along the \( x \)-direction becomes

\[
I_2 = 2k_BT \left[ \bar{\Gamma} t + \frac{\Delta \Gamma}{2} \cos 2\theta_0 \left( \frac{1 - e^{-4D_\theta t}}{4D_\theta} \right) \right] \tag{A.6}
\]

### B. Calculation of Two-Time Correlation for a Free Active Anisotropic Particle

\[
I_3 = v_0^2 \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \langle \cos \theta(t_1') \cos \theta(t_2') \rangle \\
= \frac{v_0^2}{2} \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \cos 2\theta_0 e^{-D_\theta [t_1' + t_2' + 2 \min(t_1', t_2')]} + \frac{v_0^2}{2} \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' e^{-D_\theta [t_1' + t_2' - 2 \min(t_1', t_2')]} \tag{B.1}
\]

for \( t_1' < t_2' \)

\[
t_1' + t_2' + 2t_1' = 3t_1' + t_2' \\
= -t_1' + t_2'
\]

for \( t_1' > t_2' \)

\[
t_1' + t_2' + 2t_2' = t_1' + 3t_2' \\
= t_1' - t_2'
\]
There are two integrals, lets say $I_3$ and $I_3'$. The integral $I_3'$ can be calculated in general terms like,

$$I_3' = \int_0^t dt' \int_0^{t_2} d\alpha \int_0^{t_2} d\beta e^{-\alpha(\alpha t_1 + \beta t_2)} \int_0^{t_1} dt' e^{-\alpha(\alpha t_1 + \beta t_2)}$$

for integral $I_3'$, $(\alpha, \beta) \equiv (3, 1)$

$$I_3' = \frac{1 - e^{-\alpha t_2}}{\alpha \beta D_\theta^2} - \frac{1 - e^{-\alpha(\alpha + \beta) D_\theta t_2}}{\alpha(\alpha + \beta) D_\theta^2} - \frac{1 - e^{-\beta D_\theta t_2} \alpha - e^{-\alpha D_\theta t_2} \beta}{\alpha \beta D_\theta^2}$$

and for integral $I_3''$, $(\alpha, \beta) \equiv (-1, 1)$

$$I_3'' = \frac{1 - e^{-\alpha t_2}}{2D_\theta^2} + \frac{t_2}{D_\theta} - \frac{1 - e^{-\alpha t_2}}{2D_\theta^2}$$

So final form of $I_3$ becomes

$$I_3 = v_0^2 \left[ \cos 2\theta_0 \left( \frac{1 - e^{-\alpha t_2}}{6D_\theta^2} + \frac{1 - e^{-\alpha(\alpha + \beta) D_\theta t_2}}{12D_\theta^2} - \frac{1 - e^{-\beta D_\theta t_2} \alpha - e^{-\alpha D_\theta t_2} \beta}{6D_\theta^2} \right) - \frac{1 - e^{-\alpha t_2}}{2D_\theta^2} + \frac{t_2}{D_\theta} - \frac{1 - e^{-\alpha t_2}}{2D_\theta^2} \right]$$

The second integral $I_4$ of Eq. (9) is solved as

$$I_4 = \int_0^{t_2} dt' \int_0^{t_2} d\alpha \int_0^{t_2} d\beta e^{-\alpha(\alpha t_1 + \beta t_2)} \xi_x(t_x(t'))$$

$$= 2k_B T \Gamma t_2 \left[ \frac{\Delta \Gamma}{2 \Delta \Gamma} \cos 2\theta_0 \left( \frac{1 - e^{-\alpha(\alpha + \beta) t_2}}{4D_\theta t_2} \right) \right]$$

**C. CALCULATION OF $\langle R_{0,i}(t)R_{0,j}(t) \rangle$ FOR A HARMONICALLY TRAPPED PARTICLE**

$$\langle R_{0,i}(t)R_{0,j}(t) \rangle = \int dt' \int_0^t dt'' e^{-\kappa \Gamma(t'-t)} e^{-\kappa \Gamma(t'-t'')} \langle \xi(t')\xi(t'') \rangle + \int dt' \int_0^t dt'' e^{-\kappa \Gamma(t'-t)} e^{-\kappa \Gamma(t'-t'')} v_0^2 \langle \hat{n}(t')\hat{n}(t'') \rangle$$

(C.1)

There are two integrals, lets say $I_5$ and $I_6$ respectively. Lets calculate these two separately

$$I_5 = \int dt' \int_0^t dt'' e^{-\kappa \Gamma(t'-t)} e^{-\kappa \Gamma(t'-t'')} \langle \xi(t')\xi(t'') \rangle$$

$$= \int dt' \int_0^t dt'' e^{-\kappa \Gamma(t'-t)} e^{-\kappa \Gamma(t'-t'')} \left[ \hat{\Gamma} \Gamma + \frac{\Delta \Gamma}{2} \bar{R}(t') \right] \delta(t' - t'')$$

$$= 2k_B T e^{-2\kappa \Gamma t} \int_0^t dt'' e^{2\kappa \Gamma t'} \left[ \hat{\Gamma} \Gamma + \frac{\Delta \Gamma}{2} \bar{R}(t') \right]$$

(C.2)
For \( x \)-direction

\[
I_5 = \frac{k_B T}{\kappa} (1 - e^{-2\kappa \Gamma t}) + \Delta D \cos 2\theta_0 \left( \frac{e^{-4D_{\theta} t} - e^{-2\kappa \Gamma t}}{2\kappa \Gamma - 4D_{\theta}} \right)
\]  
(C.3)

\[
I_6 = \int_0^t dt' \int_0^{t'} dt'' e^{-\kappa \Gamma (t-t')} e^{-\kappa \Gamma (t-t'')} \nu_0^2 \langle \hat{n}(t') \hat{n}(t'') \rangle
= v_0^2 e^{-2\kappa \Gamma t} \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} \cos \theta(t') \cos \theta(t'')
= \frac{v_0^2}{2} e^{-2\kappa \Gamma t} \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} \left[ \cos 2\theta_0 e^{-D_D [t'+t''+2\min(t',t'')]} + e^{-D_D [t'+t''-2\min(t',t'')]} \right]
\]  
(C.4)

Let's solve the integrals separately

\[
I_6' = \frac{v_0^2}{2} e^{-2\kappa \Gamma t} \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} \cos 2\theta_0 e^{-D_D [t'+t''+2\min(t',t'')]} \]
\[
= \frac{v_0^2 \cos 2\theta_0}{2} e^{-2\kappa \Gamma t} \left[ \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} e^{-D_D (t'+3t'')} + \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} e^{-D_D (3t'+t'')} \right]
\]
\[
= \frac{v_0^2 \cos 2\theta_0}{2} e^{-2\kappa \Gamma t} \left[ \int_0^t e^{(\kappa \Gamma - D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma - 3D_D) t''} dt'' + \int_0^t e^{(\kappa \Gamma - 3D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma - D_D) t''} dt'' \right]
\]  
(C.5)

\[
I_6'' = \frac{v_0^2}{2} e^{-2\kappa \Gamma t} \int_0^t dt' \int_0^{t'} dt'' e^{-D_D [t'+t''-2\min(t',t'')]} \]
\[
= \frac{v_0^2}{2} e^{-2\kappa \Gamma t} \left[ \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} e^{-D_D (t'-t'')} + \int_0^t dt' \int_0^{t'} dt'' e^{\kappa \Gamma (t'+t'')} e^{-D_D (t'-t'')} \right]
\]
\[
= \frac{v_0^2}{2} e^{-2\kappa \Gamma t} \left[ \int_0^t e^{(\kappa \Gamma - D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma + D_D) t''} dt'' + \int_0^t e^{(\kappa \Gamma + D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma - D_D) t''} dt'' \right]
\]  
(C.6)

\[
\frac{v_0^2}{2} e^{-2\kappa \Gamma t} \left[ \int_0^t e^{(\kappa \Gamma - D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma + D_D) t''} dt'' + \int_0^t e^{(\kappa \Gamma + D_D) t'} dt' \int_0^{t'} e^{(\kappa \Gamma - D_D) t''} dt'' \right]
\]

D. Calculation of \( \langle R_{0,i}(t_1)R_{0,j}(t_2) \rangle \)

\[
\langle R_{0,i}(t_1)R_{0,j}(t_2) \rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma (t_1-t'_i)} e^{-\kappa \Gamma (t_2-t'_j)} \langle \xi(t'_i) \xi(t'_j) \rangle + v_0^2 \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma (t_1-t'_i)} e^{-\kappa \Gamma (t_2-t'_j)} \langle \xi(t'_i) \xi(t'_j) \rangle + v_0^2 \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma (t_1-t'_i)} e^{-\kappa \Gamma (t_2-t'_j)} \langle \cos \theta(t'_i) \cos \theta(t'_j) \rangle
\]  
(D.1)

The case of \( i = j = 1 \) corresponds to \( \langle x_0(t_1) x_0(t_2) \rangle \) and the explicit form of the correlation becomes

\[
\langle R_{0,1}(t_1)R_{0,1}(t_2) \rangle = \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma (t_1-t'_1)} e^{-\kappa \Gamma (t_2-t'_1)} \langle \xi(t'_1) \xi(t'_1) \rangle + v_0^2 \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma (t_1-t'_1)} e^{-\kappa \Gamma (t_2-t'_1)} \langle \cos \theta(t'_1) \cos \theta(t'_1) \rangle
\]  
(D.2)
It can be calculated as two separately $I_9$ and $I_{10}$ integrals

$$I_9 = \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{-\kappa \Gamma(t_1-t')} e^{-\kappa \Gamma(t_2-t'')} (\xi(t'_1) \xi(t'_2))$$

$$= 2k_B T e^{-\kappa \Gamma(t_1+t_2)} \int_0^{t_1} dt' \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} [\bar{\Gamma} \delta_{ij} + \frac{\Delta \Gamma}{2} (R_{ij}(t'_1))] \delta(t'_1 - t'_2)$$

$$= 2k_B T e^{-\kappa \Gamma(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\kappa \Gamma t} dt'_1 + k_B T \Delta \Gamma e^{-\kappa \Gamma(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\kappa \Gamma t} (R_{ij}(t'_1)) dt'_1$$

$$= k_B T \frac{\kappa}{\kappa - \kappa \Gamma(t_1-t_2) - e^{-\kappa \Gamma(t_1+t_2)}} + \frac{2k_B T}{\kappa} \frac{(\kappa \Gamma)}{2} \cos 2\theta_0 e^{-\kappa \Gamma(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\kappa \Gamma t} - \frac{1}{2} \frac{2 \Gamma}{2 \Gamma - 4 D \theta}

$$= k_B T \frac{\kappa}{\kappa - \kappa \Gamma(t_1-t_2) - e^{-\kappa \Gamma(t_1+t_2)}} + k_B T \frac{2 \Gamma}{\kappa} \frac{(\kappa \Gamma)}{2} \cos 2\theta_0 e^{-\kappa \Gamma(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\kappa \Gamma t} - \frac{1}{2} \frac{2 \Gamma}{2 \Gamma - 4 D \theta}

$$= k_B T \frac{\kappa}{\kappa - \kappa \Gamma(t_1-t_2) - e^{-\kappa \Gamma(t_1+t_2)}} + k_B T \frac{2 \Gamma}{\kappa} \frac{(\kappa \Gamma)}{2} \cos 2\theta_0 e^{-\kappa \Gamma(t_1+t_2)} \int_0^{\min(t_1,t_2)} e^{2\kappa \Gamma t} - \frac{1}{2} \frac{2 \Gamma}{2 \Gamma - 4 D \theta}

$$I_{10} = \frac{v_0^2}{2} e^{-\kappa \Gamma(t_1+t_2)} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} (\cos \theta(t'_1) \cos \theta(t'_2))$$

$$= \frac{v_0^2}{2} e^{-\kappa \Gamma(t_1+t_2)} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} \left[ \cos 2\theta_0 e^{-D \theta} e^{\kappa \Gamma(t'_1+t'_2)2 \min(t_1,t_2)} + e^{-D \theta}[t'_1+t'_2-2 \min(t_1,t_2)] \right]$$

Lets calculate integrals separately

$$I''_{10} = \frac{v_0^2}{2} \cos 2\theta_0 e^{-\kappa \Gamma(t_1+t_2)} \int_0^{t_1} dt'_1 \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} e^{-D \theta}[t'_1+t'_2+2 \min(t_1,t_2)]$$

$$= \frac{v_0^2}{2} e^{-\kappa \Gamma(t_1+t_2)} \left[ \int_0^{t_2} dt'' \int_0^{t_1} dt'_1 e^{\kappa \Gamma(t'_1+t'_2)} e^{-D \theta}(3t'_1+t'_2) + \int_0^{t_1} dt'_1 \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} e^{-D \theta}(t'_1+3t'_2) \right]$$

$$= \frac{v_0^2}{2} e^{-\kappa \Gamma(t_1+t_2)} \left[ \int_0^{t_2} dt'' \int_0^{t_1} dt'_1 e^{\kappa \Gamma(t'_1+t'_2)} e^{-D \theta}(3t'_1+t'_2) + \int_0^{t_1} dt'_1 \int_0^{t_2} dt'' e^{\kappa \Gamma(t'_1+t'_2)} e^{-D \theta}(t'_1+3t'_2) \right]$$

$$= \frac{v_0^2}{2} \cos 2\theta_0 \left[ \frac{2 D \theta e^{-\kappa \Gamma t_1} e^{-\kappa \Gamma(t_1+t_2)}}{(2 \kappa \Gamma - 4 D \theta)(\kappa \Gamma - D \theta)} + \frac{e^{-D \theta} t_1 e^{-D \theta} t_2 - e^{-\kappa \Gamma t_1} e^{-\kappa \Gamma t_1} e^{-D \theta} t_2 + e^{-\kappa \Gamma(t_1+t_2)}}{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)} \right]$$

$$\langle x_0(t_1)x_0(t_2) \rangle = k_B T \frac{\kappa}{\kappa} e^{-\kappa \Gamma t_1} + k_B T \frac{\kappa}{\kappa} \frac{(\kappa \Gamma)}{2} \cos 2\theta_0 e^{-\kappa \Gamma t_1} \left[ \frac{e^{\kappa \Gamma t_1} e^{-D \theta} t_2 - e^{-\kappa \Gamma t_1}}{2 \kappa \Gamma - 4 D \theta} \right]$$

$$+ \frac{v_0^2}{2} \cos 2\theta_0 \left[ \frac{2 D \theta e^{-\kappa \Gamma t_1} e^{-\kappa \Gamma(t_1+t_2)}}{(2 \kappa \Gamma - 4 D \theta)(\kappa \Gamma - D \theta)} + \frac{e^{-D \theta} t_1 e^{-D \theta} t_2 - e^{-\kappa \Gamma t_1} e^{-D \theta} t_2 + e^{-\kappa \Gamma(t_1+t_2)}}{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)} \right]$$

$$+ \frac{v_0^2}{2} \left[ \frac{2 D \theta e^{-\kappa \Gamma(t_1+t_2)} - e^{-\kappa \Gamma(t_1-t_2)}}{(2 \kappa \Gamma - 4 D \theta)(\kappa \Gamma - D \theta)} + \frac{e^{-D \theta} t_1 e^{-D \theta} t_2 - e^{-\kappa \Gamma(t_1+t_2)}}{\kappa \Gamma - D \theta} \right]$$

$$\frac{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)}{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)}$$

$$\frac{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)}{(\kappa \Gamma - D \theta)(\kappa \Gamma - D \theta)}$$

(D.7)
It is quite easy to see that substituting \( t_1 = t_2 = t \) in the above equation reproduces the result \( \langle x_0^2(t) \rangle \) that has been explicitly calculated in Eq. (C.3), Eq. (C.5) and Eq. (C.6).

### E. Calculation of \( \langle R_{0,i}(t_1)R_{1,j}(t_2) \rangle \)

We now turn our attention to the first order correction that enter the correlation matrix. Using the formal solution for \( R_{1,j}(t) \) we write down

\[
\langle R_{0,i}(t_1)R_{1,j}(t_2) \rangle = \left\langle R_{0,i}(t_1) \int_0^{t_2} dt_2' e^{-\kappa \Gamma(t_2'-t_1')} \sum_k R_{jk}(t_2') R_{0,k}(t_2') \right\rangle
\]

\[
= \left\langle \int_0^{t_2} dt_2' e^{-\kappa \Gamma(t_2'-t_1')} \sum_k R_{jk}(t_2') \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_1'-t_1')} e^{-\kappa \Gamma(t_2''-t_2')} \left[ \xi_i(t_1') \xi_k(t_2'') + v_0^2 n_i(t_1') n_k(t_2'') \right] \right\rangle
\]

We evaluate the two terms in the integral separately. The first term involves an average over the thermal fluctuations in the position and we denote it by \( I_{11} \). The explicit calculation of \( I_{11} \) becomes

\[
I_{11} = \int_0^{t_2} dt_2' e^{-\kappa \Gamma(t_2'-t_1')} \left\langle \sum_k R_{jk}(t_2') \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_1'-t_1')} e^{-\kappa \Gamma(t_2''-t_2')} \xi_i(t_1') \xi_k(t_2'') \right\rangle
\]

\[
= \int_0^{t_2} dt_2' e^{-\kappa \Gamma(t_2'-t_1')} \left\langle \sum_k R_{jk}(t_2') \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_1'-t_1')} e^{-\kappa \Gamma(t_2''-t_2')} \xi_i(t_1') \xi_k(t_2'') \right\rangle
\]

\[
= \frac{k_B T}{\kappa} e^{-\kappa \Gamma(t_1'+t_2')} \int_0^{t_2} dt_2' e^{\kappa \Gamma(t_2'-t_1')} \left( \sum_k R_{jk}(t_2') \left( \delta_{k_1}(e^{-\kappa \Gamma(t_1'-t_2')}) + k_B T \Delta \Gamma e^{-\kappa \Gamma(t_1'+t_2')} \int_0^{\min(t_1,t_2')} dt'' e^{2\kappa \Gamma t'' R_{kk}(t'')} \right) \right)
\]

\[
+ k_B T \Delta \Gamma e^{-\kappa \Gamma(t_1'+t_2')} \int_0^{\min(t_1,t_2')} e^{2\kappa \Gamma t''} \sum_k \left\langle R_{jk}(t_2') R_{kk}(t'') \right\rangle
\]

In order to proceed further with the calculation, we consider the correlation function \( \langle x_0(t_1)x_1(t_2) \rangle \) so that the above expression becomes

\[
I_{11} = \frac{k_B T}{\kappa} e^{-\kappa \Gamma(t_1'+t_2')} R_{ji}(\theta_0) \int_0^{t_2} dt_2' e^{-4 D_\theta t_2'} (e^{2\kappa \Gamma t_2'} - 1)
\]

\[
+ k_B T \Delta \Gamma e^{-\kappa \Gamma(t_1'+t_2')} \int_0^{t_2} dt_2' e^{2\kappa \Gamma t''} e^{-4 D_\theta (t_2'+t''-2 \min(t_2,t''))}
\]

\[
= \left( \frac{k_B T}{\kappa} \right) \cos 2\theta_0 e^{-\kappa \Gamma t_2} \left( \frac{e^{(\kappa \Gamma-4 D_\theta) t_2} - e^{-\kappa \Gamma t_2}}{2 \kappa \Gamma - 4 D_\theta} - \frac{e^{-\kappa \Gamma t_2} - e^{-(4 D_\theta+\kappa \Gamma) t_2}}{4 D_\theta} \right)
\]

\[
+ \left( \frac{k_B T}{\kappa} \right) \left( \frac{\Delta \Gamma}{2 \kappa \Gamma} \right) \left( \frac{\sin 2 \theta_0 e^{-\kappa \Gamma t_2}}{\cos 2 \theta_0} \left( \frac{\kappa \Gamma t_2 - e^{-\kappa \Gamma t_2}}{2 \kappa \Gamma + 4 D_\theta} - \frac{2 \kappa \Gamma e^{-\kappa \Gamma t_2} - e^{-(4 D_\theta+\kappa \Gamma) t_2}}{\kappa \Gamma + 4 D_\theta} \right) \right)
\]

The second term of Eq. (1.1) is denoted by \( I_{12} \) and is given as

\[
I_{12} = v_0^2 \int_0^{t_2} dt_2' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_2'-t_1')} \left\langle \sum_k R_{jk}(t_2') \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_1'-t_1')} e^{-\kappa \Gamma(t_2''-t_2')} n_i(t_1') n_j(t_2'') \right\rangle
\]

so that for the correlation function \( \langle x_0(t_1)x_1(t_2) \rangle \) Eq. (E.4) transforms as

\[
I_{12} = \int_0^{t_2} dt_2' e^{-\kappa \Gamma(t_2'-t_1')} \int_0^{t_1} dt_1' \int_0^{t_2} dt_2'' e^{-\kappa \Gamma(t_1'-t_1')} e^{-\kappa \Gamma(t_2''-t_2')} v_0^2 \left( \langle \cos 2 \theta(t_2') \cos \theta(t_1') \cos \theta(t_2'') \rangle + \langle \sin 2 \theta(t_2') \cos \theta(t_1') \cos \theta(t_2'') \rangle \right)
\]
We use the following trigonometric identities

\[
\cos 2\theta_1 \cos 2\theta_2 \cos \theta_3 = \frac{1}{4} \left[ \cos(2\theta_1 - \theta_2 - \theta_3) + \cos(2\theta_1 + \theta_2 - \theta_3) + \cos(2\theta_1 - \theta_2 + \theta_3) + \cos(2\theta_1 + \theta_2 + \theta_3) \right]
\]

\[
\sin 2\theta_1 \cos 2\theta_2 \cos \theta_3 = \frac{1}{4} \left[ \sin(2\theta_1 - \theta_2 - \theta_3) + \sin(2\theta_1 + \theta_2 + \theta_3) + \sin(2\theta_1 - \theta_2 + \theta_3) + \sin(2\theta_1 + \theta_2 + \theta_3) \right]
\]

(E.6)

to rewrite the triple product of the trigonometric functions, averaged over the rotational noise as

\[
\langle \cos(2\theta_1 - \theta_2 - \theta_3) \rangle = e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} + 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

\[
\langle \cos(2\theta_1 + \theta_2 - \theta_3) \rangle = \cos \theta_0 e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} + 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) - 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

\[
\langle \cos(2\theta_1 - \theta_2 + \theta_3) \rangle = \cos \theta_0 e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} + 4 \min(t_2^{t_1}, t_1^{t_2}) + 4 \min(t_2^{t_1}, t_2^{t_1}) - 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

\[
\langle \sin(2\theta_1 - \theta_2 - \theta_3) \rangle = e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} - 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

\[
\langle \sin(2\theta_1 + \theta_2 - \theta_3) \rangle = \sin \theta_0 e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} + 4 \min(t_2^{t_1}, t_1^{t_2}) + 4 \min(t_2^{t_1}, t_2^{t_1}) - 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

\[
\langle \sin(2\theta_1 - \theta_2 + \theta_3) \rangle = \sin \theta_0 e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} + 4 \min(t_2^{t_1}, t_1^{t_2}) + 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

(E.7)

Note that in the above set of equations, the time dependence in the right hand side of the set of equations (a)–(d) is identical to the set of equations (e)–(h). Let us calculate Eq.(E.2) by using Eq.(E.7), at first we take the first term \(\langle \cos(2\theta_1 - \theta_2 - \theta_3) \rangle\)

\[
\frac{\pi}{4} e^{-\kappa \Gamma \Gamma(t_1 + t_2)} \int_0^{t_2} dt_0 \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \cos \kappa \Gamma t_1 e^{\kappa \Gamma t_2} e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} - 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})} \right)
\]

(E.8)

Here in the above integral always \(t_2^{t_1} > t_2^{t_1}\) and in the first case let us take \(t_2^{t_1} > t_2^{t_1}\) we get

Case(1), \(t_2^{t_1} > t_2^{t_1}\)

\[
\int_0^{t_2} dt_0 \int_0^{t_2} dt_2 \int_0^{t_1} dt_1 \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \cos \kappa \Gamma t_1 e^{\kappa \Gamma t_2} e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} - 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

(E.9)

Case(2), \(t_2^{t_1} < t_2^{t_1}\)

\[
\int_0^{t_1} dt_1 \int_0^{t_1} dt_1 \int_0^{t_1} dt_1 \int_0^{t_1} dt_1 \int_0^{t_1} dt_2 \cos \kappa \Gamma t_1 e^{\kappa \Gamma t_2} e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} - 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})}
\]

Adding Eq.(E.9) and Eq.(E.10) terms we get the term Eq.(E.8) as,

\[
I_a = \frac{\pi}{4} e^{-\Delta \theta_1(4t_2^{t_1} + t_2^{t_1} - 4 \min(t_2^{t_1}, t_1^{t_2}) - 4 \min(t_2^{t_1}, t_2^{t_1}) + 2 \min(t_1^{t_2}, t_2^{t_1})} \right)
\]

(E.11)
The way first term has been calculated similarly other terms are calculated to find the exact expression of Eq. (E.5). The results of Integrals due to terms (b), (c), and (d) are as follows

\[
I_b = \frac{\nu_0^2 e^{-\kappa t_1}}{8\kappa (\kappa + 5D_\theta)} \left[ e^{(\kappa - 4D_\theta)t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{4D_\theta} \right] - 4\kappa (\kappa + 5D_\theta) (\kappa - 5D_\theta) \left( e^{(\kappa - 4D_\theta)t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{4D_\theta} \right)
\]

\[
I_c = \frac{\nu_0^2 e^{-\kappa t_1}}{4(\kappa - 3D_\theta)} \left[ e^{(\kappa - 4D_\theta)t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{2D_\theta} \right] + \frac{\nu_0^2 e^{-\kappa t_1}}{4(\kappa - 3D_\theta)} \left( e^{-D_\theta t_2} - e^{-\kappa t_2} \right)
\]

\[
I_d = \frac{\nu_0^2 e^{-\kappa t_1}}{4(\kappa - 5D_\theta)} \left[ e^{(\kappa - 16D_\theta)t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{2D_\theta} \right] + \frac{\nu_0^2 e^{-\kappa t_1}}{4(\kappa - 5D_\theta)} \left( e^{-9D_\theta t_2} - e^{-\kappa t_2} \right)
\]

Integral values to due term (e) will be same of (a) similarly others. Terms due to (f), (g), (h) of Eq. (E.7) will be zero for the initial orientational angle \( \theta_0 = 0 \). Now for the simplification we are taking only the term associated with \( \sinh \kappa t_2 \) of Eq. (E.11). Similarly another same term of \( \sinh \kappa t_2 \) will arise due to the contribution of (e) of Eq. (E.7).

After adding the relevant terms, the final expression for the two-time correlation term becomes

\[
\langle x_0(t_1)x_1(t_2) \rangle = \left( \frac{k_B T}{\kappa} \right) \cos \theta_0 e^{-\kappa t_1} \left( e^{(\kappa - 4D_\theta)t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{2D_\theta} \right) + \left( \frac{k_B T}{\kappa} \right) \left( \frac{\Delta \Gamma}{2\kappa} \right) e^{-\kappa t_1} \left[ e^{\kappa t_2} - e^{-\kappa t_2} - \frac{e^{-\kappa t_2} - e^{-(\kappa + 4D_\theta)t_2}}{2\kappa + 4D_\theta} \right] \left( \frac{3\nu_0^2 D_\theta e^{-\kappa t_1} \sinh \kappa t_2}{2\kappa (\kappa + D_\theta)(\kappa - D_\theta)(\kappa + 2D_\theta)} \right)
\]

From the above equation, the first order correction in \( \langle x^2(t) \rangle \) is given by

\[
\langle x_0(t)x_1(t) \rangle = \left( \frac{k_B T}{\kappa} \right) \cos \theta_0 \left( e^{-(4D_\theta t_2) - e^{-2\kappa t_2}} - \frac{e^{2\kappa t_2} - e^{-(2D_\theta + \kappa) t_2}}{2D_\theta} \right) + \left( \frac{k_B T}{\kappa} \right) \left( \frac{\Delta \Gamma}{2\kappa} \right) \left[ 1 - e^{-2\kappa t_2} - \frac{2\kappa e^{-\kappa t_2} - e^{-(\kappa + 2D_\theta) t_2}}{2\kappa + 4D_\theta} \right] \left( \frac{3\nu_0^2 D_\theta e^{-\kappa t_1} \sinh \kappa t_2}{2\kappa (\kappa + D_\theta)(\kappa - D_\theta)(\kappa + 2D_\theta)} \right)
\]