LONG TERM BEHAVIOR OF A RANDOM HOPFIELD NEURAL LATTICE MODEL

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Abstract. A Hopfield neural lattice model is developed as the infinite dimensional extension of the classical finite dimensional Hopfield model. In addition, random external inputs are considered to incorporate environmental noise. The resulting random lattice dynamical system is first formulated as a random ordinary differential equation on the space of square summable bi-infinite sequences. Then the existence and uniqueness of solutions, as well as long term dynamics of solutions are investigated.

1. Introduction. Neural networks have been gaining increasing attention of researchers, due to their wide range of applications such as character recognition (see, e.g., [17]), image compression (see, e.g., [9]), stock market prediction (see, e.g., [13, 16]), traveling saleman’s problem (see, e.g., [12]), etc. In particular, one of the most popular mathematical models for (artificial) neural network is the Hopfield neural model proposed by John Hopfield in 1984 [15], described by the following system of \( n \) ordinary differential equations (ODEs):

\[
\frac{du_i(t)}{dt} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=1}^{n} \lambda_{i,j} g_j(u_j(t)) + I_i, \quad i = 1, \ldots, n,
\]

where \( u_i \) represents the voltage on the input of the \( i \)th neuron at time \( t \); \( \mu_i > 0 \) and \( \gamma_i > 0 \) represents the neuron amplifier input capacitance and resistance of the \( i \)th neuron, respectively; and \( I_i \) is the constant external forcing on the \( i \)th neuron.

Here \( n \) is the total number of neurons coupled by an \( n \times n \) matrix \( (\lambda_{i,j})_{1 \leq i,j \leq n} \), where the entity \( \lambda_{i,j} \) represents the connection strength between the \( i \)th and the \( j \)th neuron. More precisely, for each pair of \( i, j = 1, \ldots, n \), \( \lambda_{i,j} \) is the synapse efficacy between neurons \( i \) and \( j \), and thus \( \lambda_{i,j} > 0 \) (\( \lambda_{i,j} < 0 \), resp.) means the output of neuron \( j \) excites (inhibits, resp.) neuron \( i \). The term \( \lambda_{i,j} g_j(u_j(t)) \) represents the electric current input to neuron \( i \) due to the present potential of neuron \( j \), in which the function \( g_j \) is neuron activation functions and assumed to be a sigmoid type function.

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We are interested in studying dynamics of the above Hopfield neural network model when its size becomes increasingly large, i.e., \( n \to \infty \). To this end, we extend the \( n \) dimensional ODE system (1) to an infinite dimensional lattice system, that models the dynamics of an infinite number of neurons indexed by \( i \in \mathbb{Z} \), in which each neuron is still connected with other neurons within its finite \( N \) neighborhood. More precisely, the \( i \)th neuron is connected to the \((i-N)\)th, \( \cdots \), \((i+N)\)th neurons through the strength matrix \((\lambda_{i,j})_{-N \leq j \leq i+N}\) and the activation functions \( g_j \) for \( j = i-N, \cdots , i+N \). In addition, to take into account random perturbations of the environment, we introduce a noise in the equations in (1) by replacing each constant input \( I_i \) by a random forcing \( I_i(\theta,\omega) \) represented by a measure-preserving dynamical system \( \{\theta_t\}_{t \in \mathbb{R}} \) acting on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). System (1) then becomes the following random lattice dynamical system, namely the random Hopfield neural lattice model:

\[
\mu_i \frac{du_i(t)}{dt} = -\frac{u_i(t)}{\gamma_i} + \sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j(t)) + I_i(\theta,\omega), \quad i \in \mathbb{Z}.
\]

Over the past two decades, extensive studies have been done on dynamics of lattice dynamical systems (see, e.g., [3, 6, 5, 19, 20, 21, 22, 23] and references therein). However, most of the existing works consider a simple linear diffusion described by a linear operator such as \( u_i = \sum_j u_j + q \). Very often the lattice dynamical system under consideration arises from discretization of a partial differential equation. Notice that the terms \( \sum_{j=i-q}^{i+q} \lambda_{i,j} g_j(u_j(t)) \) in system (2) model a nonlinear finite neighborhood connection structure, which is intrinsically discrete and does not arise from a continuous operator. Such models have not been studied in the literature of lattice dynamical systems.

The goal of this work is to investigate the long term dynamics of the random lattice dynamical system (2), in particular, the existence of random attractors. The main tool is the theory of random dynamical systems and random attractors (see, e.g., [1, 14, 8, 7, 18]). The paper is organized as follows. In Section 2 we provide necessary preliminaries on random dynamical systems and random attractors. In Section 3 we first reformulate system (2) as a random ordinary differential equation (RODE) on the space of bi-infinite sequences and then show that the resulting RODE has a unique solution that generates a random dynamical system (RDS). In Section 4 we investigate the existence of a random attractor for the RDS obtained in Section 3. Some closing remarks will be given in Section 5.

2. Preliminaries. In this section, we present some basic concepts and theory of random dynamical system (RDS) required in the sequel (see, e.g., [1, 2, 4, 11]).

Let \((X, \| \cdot \|_X)\) be a separable Banach space and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space where \(\mathcal{F}\) is the \(\sigma\)-algebra of measurable subsets of \(\Omega\) (called “events”) and \(\mathbb{P}\) is the probability measure. To connect the state \(\omega \in \Omega\) at time 0 with its state after a time of \(t\) elapses, define a flow \(\Theta = \{\theta_t\}_{t \in \mathbb{R}}\) on \(\Omega\) with each \(\theta_t\) being a mapping \(\theta_t : \Omega \to \Omega\) that satisfies

1. \(\theta_0 = Id_\Omega;\)
2. \(\theta_s \circ \theta_t = \theta_{s+t}\) for all \(s, t \in \mathbb{R};\)
3. the mapping \((t, \omega) \mapsto \theta_t(\omega)\) is measurable;
4. the probability measure \(\mathbb{P}\) is preserved by \(\theta_t\), i.e., \(\theta_t^\mathbb{P} = \mathbb{P}\).

Then \((\Omega, \mathcal{F}, \mathbb{P}, \theta)\) is called driving dynamical system [1].
Definition 2.1. A stochastic process \( \{ \varphi(t, \omega) \}_{t \geq 0, \omega \in \Omega} \) is said to be a continuous random dynamical system (RDS) over \( (\Omega, \mathcal{F}, \mathbb{P}, \Theta) \) with state space \( X \) if \( \varphi : [0, \infty) \times \Omega \times X \to X \) is \( \mathcal{B}[0, +\infty) \times \mathcal{F} \times \mathcal{B}(\mathcal{X}), \mathcal{B}(\mathcal{X}) \)-measurable, and for each \( \omega \in \Omega \),

(i) The mapping \( \varphi(t, \omega, \cdot) : X \to X, x \mapsto \varphi(t, \omega, x) \) is continuous for every \( t \geq 0 \);

(ii) \( \varphi(0, \omega, \cdot) \) is the identity operator on \( X \);

(iii) (cocycle property) \( \varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s \omega, \varphi(s, \omega, \cdot)) \) for all \( s, t \geq 0 \).

Definition 2.2. A set-valued mapping \( K : \Omega \to 2^X \setminus \emptyset \) is said to be a random set if the mapping \( \omega \mapsto \text{dist}_X(x, K(\omega)) \) is measurable for any \( x \in X \). A random set \( K(\omega) \subset X \) is said to be tempered with respect to \( \Theta \) if for a.e. \( \omega \in \Omega \),

\[
\lim_{t \to \infty} e^{-\beta t} \sup_{x \in K(\theta_t \omega)} \| x \|_X = 0, \quad \forall \beta > 0.
\]

A random variable \( \omega \mapsto r(\omega) \in \mathbb{R} \) is said to be tempered if for a.e. \( \omega \in \Omega \),

\[
\lim_{t \to \infty} e^{-\beta t} \sup_{t \in \mathbb{R}} |r(\theta_t \omega)| = 0, \quad \forall \beta > 0.
\]

Throughout this paper, denote by \( \mathcal{A}(X) \) the set of all tempered random sets of \( X \).

Definition 2.3. Let \( \{ \varphi(t, \omega) \}_{t \geq 0, \omega \in \Omega} \) be an RDS over \( (\Omega, \mathcal{F}, \mathbb{P}, \Theta) \) with state space \( X \) and let \( \mathcal{A}(\omega)(\subset X) \) be a random set. Then \( \mathcal{A}(\omega) \) is called a global random \( \mathcal{D} \) attractor (or pullback \( \mathcal{D} \) attractor) for \( \{ \varphi(t, \omega) \}_{t \geq 0, \omega \in \Omega} \) if \( \omega \mapsto \mathcal{A}(\omega) \) satisfies

(i) (random compactness) \( \mathcal{A}(\omega) \) is a compact set of \( X \) for a.e. \( \omega \in \Omega \);

(ii) (invariance) for a.e. \( \omega \in \Omega \) and all \( t \geq 0 \), it holds

\( \varphi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t \omega) \);

(iii) (attracting property) for any \( K \in \mathcal{D}(X) \) and a.e. \( \omega \in \Omega \),

\[
\lim_{t \to \infty} \text{dist}_X(\varphi(t, \theta_{-t} \omega, K(\theta_{-t} \omega)), \mathcal{A}(\omega)) = 0,
\]

where

\[
\text{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|_X
\]

is the Hausdorff semi-metric for \( A, B \subseteq X \).

The existence of random attractors usually requires the existence of random absorbing set defined below.

Definition 2.4. A random set \( \Gamma(\omega) \subset X \) is called a random absorbing set in \( \mathcal{D}(X) \) if for any \( K \in \mathcal{D}(X) \) and a.e. \( \omega \in \Omega \), there is \( T_K(\omega) > 0 \) such that

\( \varphi(t, \theta_{-t} \omega, K(\theta_{-t} \omega)) \subset \Gamma(\omega) \) \( \forall t \geq T_K(\omega) \).

The following theorem is one of the most widely used results for the existence of random attractors.

Proposition 1 ([2, 4, 11]). Let \( \Gamma \in \mathcal{D}(X) \) be a closed absorbing set for the continuous RDS \( \{ \varphi(t, \omega) \}_{t \geq 0, \omega \in \Omega} \) and satisfies the asymptotic compactness condition for a.e. \( \omega \in \Omega \); i.e., each sequence \( x_n \in \varphi(t_n, \theta_{-t_n}, \Gamma(\theta_{-t_n} \omega)) \) has a convergent subsequence in \( X \) when \( t_n \to \infty \). Then the RDS \( \varphi \) has a unique global random attractor with component subsets

\[
\mathcal{A}(\omega) = \bigcap_{\tau \geq t(\omega)} \bigcup_{t \geq \tau} \varphi(t, \theta_{-t} \omega, \Gamma(\theta_{-t} \omega)).
\]
If the pullback absorbing set is positively invariant, i.e., $\varphi(t, \omega, \Gamma(t)) \subset \Gamma(t, \omega)$ for all $t \geq 0$, then
$$\mathcal{A}(\omega) = \bigcap_{t \geq 0} \varphi(t; \theta^t \omega, \Gamma(t, \omega)).$$

3. Basic properties of solutions. In this section we study the existence and uniqueness of solutions to the lattice system (2), and show that the solutions generate a random dynamical system. To this end, we consider the separable Hilbert space of square summable bi-infinite sequences:

$$\ell^2 = \left\{ u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} u_i^2 < \infty \right\}$$

equipped with inner product and norm

$$(u, v) = \sum_{i \in \mathbb{Z}} u_i v_i, \quad \|u\| = \left( \sum_{i \in \mathbb{Z}} u_i^2 \right)^{\frac{1}{2}}, \quad \forall u = (u_i)_{i \in \mathbb{Z}}, v = (v_i)_{i \in \mathbb{Z}} \in \ell^2.$$

For any $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, define the operators $Au = ((Au)_i)_{i \in \mathbb{Z}}$ and $\Lambda u = ((\Lambda u)_i)_{i \in \mathbb{Z}}$ by

$$(Au)_i = \frac{1}{\mu_i} u_i, \quad (\Lambda u)_i = \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} g_j(u_j).$$

Throughout this paper it is assumed that

(A1) the efficacy among neurons is finite, i.e., there exists $B_\lambda > 0$ such that

$$\sup_{i,j \in \mathbb{Z}} |\lambda_{i,j}| \leq B_\lambda.$$

In addition, it is assumed that

(A2) for each $i \in \mathbb{Z}$, $\mu_i \in [b_\mu, B_\mu]$, $\gamma_i \in [b_\gamma, B_\gamma]$ with $b_\gamma, b_\mu > 0$;

(A3) for each $i \in \mathbb{Z}$, $g_i \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $g_i(0) = 0$, and there exists a continuous non-decreasing function $L(r) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$\sup_{i \in \mathbb{Z}} \max_{s \in [0, r]} |g'(s)| \leq L(r) \quad \forall r \in \mathbb{R}_+.$$

It follows immediately from the assumption (A2) that $Au \in \ell^2$ for every $u \in \ell^2$. In addition, notice that the assumption (A3) implies that given any $u = (u_i)_{i \in \mathbb{Z}} \in \ell^2$, for each $i \in \mathbb{Z}$ there exists $\xi_i \in \mathbb{R}$ with $|\xi_i| \leq |u_i|$ such that

$$|g_i(u_i)| = |g'(\xi_i) u_i| \leq L(|u_i|)|u_i| \leq L(||u||)|u_i|. \quad (3)$$

Then by the assumptions (A1) and (A3) we have

$$\sum_{i \in \mathbb{Z}} (\Lambda u)_i^2 \leq (2N+1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \lambda_{i,j}^2 g_j^2(u_j) \leq (2N+1) B_\lambda^2 L^2(||u||) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} u_j^2 \leq (2N+1)^2 B_\lambda^2 \cdot L^2(||u||) \cdot ||u||^2,$$

and thus $\Lambda u \in \ell^2$ for every $u \in \ell^2$.

At last, define $T(\theta t) = \left( \frac{1}{\mu_i} (\theta t) \right)_{i \in \mathbb{Z}}$ and assume that
\[ (A4) \] \( I_i(\theta \omega) \) is continuously differentiable for each \( i \in \mathbb{Z} \) and satisfies
\[
\sum_{i \in \mathbb{Z}} I_i^2(\theta \omega) < \infty \quad \forall \ t \in \mathbb{R}.
\]

Then \( \mathcal{I}(\theta \omega) \in C^1(\mathbb{R}, \ell^2) \) and the lattice system (2) can be rewritten as the random ordinary differential equation (RODE):
\[
\frac{du(t, \omega)}{dt} = -Au + \Lambda u + \mathcal{I}(\theta \omega).
\tag{4}
\]

Theorem 3.2 below states the existence and uniqueness of a global solution to the RODE (4). The proof needs the following general Gronwall type Lemma.

**Lemma 3.1.** Let \( a(t) \) be non-negative and non-decreasing, \( b(t) \) be non-negative and continuous, and \( c(t) \) be non-negative and integrable on \([t_0, T] \), and assume that the function \( w : [t_0, T] \to [0, \infty) \) satisfies
\[
w(t) \leq a(t) + \int_{t_0}^{t} b(s)w(s)ds + \int_{t_0}^{t} c(s)\ln[w(s) + 1]w(s)ds, \quad \forall \ t \in [t_0, T]
\]
Then
\[
w(t) \leq \max \left\{ 2a(t)e^{2\int_{t_0}^{t} b(s)ds} + 2e^{\int_{t_0}^{t} c(s)ds} - 1 \right\}, \quad \forall \ t \in [t_0, T].
\]

**Proof.** Since all functions and integrals are nonnegative, we have
\[
w(t) \leq 2a(t) + 2\int_{t_0}^{t} b(s)w(s)ds + \int_{t_0}^{t} c(s)\ln[w(s) + 1]w(s)ds
\]
for all \( t \in [t_0, T] \).

With \( w(t) \leq 2a(t) + 2\int_{t_0}^{t} b(s)w(s)ds \) and the assumption that \( a(t) \) is non-decreasing it follows directly from Gronwall’s inequality that
\[
w(t) \leq 2a(t)e^{2\int_{t_0}^{t} b(s)ds}.
\tag{6}
\]

With \( w(t) \leq 2\int_{t_0}^{t} c(s)\ln[w(s) + 1]w(s)ds \) and the assumption that \( c(t) \) is non-negative we have
\[
w(t) + 1 \leq 2\int_{t_0}^{t} c(s)\ln[w(s) + 1](w(s) + 1)ds + 1
\]
\[
\leq 2\left( \int_{t_0}^{t} c(s)\ln[w(s) + 1](w(s) + 1)ds + 1 \right).
\]

Then by using a generalized Gronwall-like integral inequality \( [10] \), we obtain
\[
w(t) + 1 \leq 2e^{2\int_{t_0}^{t} c(s)ds}.
\]

which means that
\[
w(t) \leq 2e^{2\int_{t_0}^{t} c(s)ds} - 1.
\tag{7}
\]

Inserting the inequalities (6) and (7) into (5) gives immediately the desired assertion. The proof is complete. \( \square \)

**Theorem 3.2.** Let assumptions \( (A1) - (A4) \) hold. In addition, assume that there exist \( \kappa_1, \kappa_2 > 0 \) such that \( L(r) \leq \kappa_1 \ln(r^2 + 1) + \kappa_2 \) for all \( r \geq 0 \). Then for any \( \omega \in \Omega \), \( t_0 \in \mathbb{R} \) and any initial data \( u_\omega = (u_{\omega,i})_{i \in \mathbb{Z}} \in \ell^2 \), the RODE (4) has a unique solution \( u(\cdot; t_0, \omega, u_\omega) \in C([t_0, \infty), \ell^2) \) with \( u(t_0; t_0, \omega, u_\omega) = u_\omega \). In addition, the solution \( u(\cdot; t_0, \omega, u_\omega) \) is continuous in \( u_\omega \in \ell^2 \).
Proof. Given \( \omega \in \Omega \) and \( u \in \ell^2 \), let
\[
F(u, \theta_\omega) = -Au + \Lambda u + \mathbb{L}(\theta_\omega).
\]
Then \( F \) is continuous in \( u \) and measurable in \( \omega \) from \( \ell^2 \times \Omega \) into \( \ell^2 \). First for any \( u, v \in \ell^2 \), we have
\[
\|Au - Av\|^2 = \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i \gamma_i} (u_i - v_i)^2 \leq \frac{1}{b_\mu^2 b_\gamma^2} \|u - v\|^2.
\]
(8) Second, by using assumptions (A2) and (A3) we have
\[
\|\Lambda u - \Lambda v\|^2 = \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i^2} \left( \sum_{j=i-N}^{i+N} \lambda_{i,j} (g_j(u_j) - g_j(v_j)) \right)^2 \\
\leq \frac{1}{b_\mu^2} (2N + 1) \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \lambda_{i,j}^2 (g_j(u_j) - g_j(v_j))^2 \\
\leq \frac{B_\lambda^2}{b_\mu^2} (2N + 1)^2 L^2 (2 \max \{|u_j|, |v_j|\}) \|u - v\|^2.
\]
(9) Hence by summing (8) and (9) we obtain
\[
\|F(u, \theta_\omega) - F(v, \theta_\omega)\|^2 \leq 2\|Au - Av\|^2 + 2\|\Lambda u - \Lambda v\|^2 \\
\leq \frac{2}{b_\mu} \left( \frac{1}{b_\gamma^2} + \frac{B_\lambda^2}{b_\mu^2} (2N + 1)^2 L^2 (2 \max \{|u|, |v|\}) \right) \|u - v\|^2,
\]
which implies that \( F(u, \theta_\omega) \) is locally Lipschitz in \( u \). Therefore the RODE (4) has a unique local solution.

We next show that given any \( T > t_0 \) fixed the solution \( u(t) \) is bounded for all \( t \in [t_0, T] \). To this end, multiplying both sides of the equation (2) by \( u_i(t) \) and sum over all \( i \in \mathbb{Z} \) to give
\[
\frac{1}{2} \frac{d}{dt} \|u(t, \omega)\|^2 = -\sum_{i} \frac{u_i^2}{\mu_i \gamma_i} + \sum_{i} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) + \sum_{i \in \mathbb{Z}} \frac{u_i}{\mu_i} I_i(\theta_\omega).
\]
(10) First by Assumption (A2), we have
\[
-\sum_{i} \frac{u_i^2}{\mu_i \gamma_i} \leq -\|u\|^2 \frac{1}{B_\mu B_\gamma}.
\]
(11) Then by using the fact that \( xy \leq \frac{1}{2} (ax^2 + \frac{1}{a} y^2) \), for some \( a > 0 \) (to be determined later), it holds
\[
\sum_{i \in \mathbb{Z}} \frac{u_i}{\mu_i} I_i(\theta_\omega) \leq \sum_{i \in \mathbb{Z}} \frac{1}{2} \left( au_i^2 + \frac{I_i^2(\theta_\omega)}{a \mu_i^2} \right) \leq \frac{a}{2} \|u\|^2 + \frac{1}{2a} \sum_{i \in \mathbb{Z}} I_i^2(\theta_\omega).
\]
(12)
Then by assumptions (A1) and (A2),
\[
\sum_i \frac{1}{\mu_i} \sum_{j=1-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq \frac{B_\lambda}{b_\mu} \sum_i \sum_{j=1-N}^{i+N} |u_i g(u_j)| = \frac{B_\lambda}{b_\mu} \sum_i \sum_{j=1-N}^{i+N} |u_i u_j| \cdot |g'(\xi_j)|
\]
for some $\xi_j$ with $|\xi_j| \leq |u_j|$. Thus by Assumption (A2), (A3) and the fact that $xy \leq \frac{1}{2}(x^2 + y^2)$ we have
\[
\sum_i \frac{1}{\mu_i} \sum_{j=1-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq \frac{B_\lambda}{2b_\mu} L(\|u\|) \sum_i \sum_{j=1-N}^{i+N} (u_j^2 + u_i^2) \leq (2N + 1) \frac{B_\lambda}{b_\mu} L(\|u\|)\|u\|^2.
\]
Inserting estimations (11) – (13) into (10) we obtain
\[
\frac{d}{dt} \|u(t)\|^2 \leq 2 \left( -\frac{1}{B_\mu B_\gamma} + \frac{2(2N + 1)B_\lambda}{b_\mu} \right) \cdot \|u\|^2 + \frac{1}{ab_\mu^2} \sum_{i \in \mathbb{Z}} I_i^2(\theta_i) \omega.
\]
Pick $a > 0$ such that $-\frac{1}{B_\mu B_\gamma} + \frac{2}{a} < 0$. In particular, let $a = \frac{1}{B_\mu B_\gamma}$, and for simplicity denote
\[
B_1 := \frac{1}{B_\mu B_\gamma}, \quad B_2 := \frac{2(2N + 1)B_\lambda}{b_\mu}, \quad M(\theta_i) := \frac{B_\mu B_\gamma}{b_\mu^2} \sum_{i \in \mathbb{Z}} I_i^2(\theta_i).
\]
Then the above inequality becomes
\[
\frac{d}{dt} \|u(t)\|^2 \leq -B_1 \|u\|^2 + B_2 L(\|u\|)\|u\|^2 + M(\theta_1) \omega.
\]
Multiplying both sides of (14) by $e^{B_1 t}$ gives
\[
\frac{d}{dt} \|e^{B_1 t} u(t)\|^2 \leq B_2 e^{B_1 t} L(\|u(t)\|)\|u(t)\|^2 + M(\theta_1) e^{B_1 t}.
\]
Integrating the above inequality from $t_0$ to $t$ results in
\[
e^{B_1 t} \|u(t)\|^2 \leq \|u_0\|^2 e^{B_1 t_0} + B_2 \int_{t_0}^{t} e^{B_1 s} L(\|u(s)\|)\|u(s)\|^2 ds + \int_{t_0}^{t} M(\theta_1) e^{B_1 s} ds,
\]
and hence
\[
\|u(t)\|^2 \leq \|u_0\|^2 e^{-B_1(t-t_0)} + \int_{t_0}^{t} M(\theta_1) e^{-B_1(t-s)} ds
\]
\[
+ B_2 \int_{t_0}^{t} e^{-B_1(t-s)} L(\|u(s)\|)\|u(s)\|^2 ds.
\]
First notice that the assumption (A4) implies that $\int_{t_0}^{t} M(\theta_1) e^{-B_1(t-s)} ds < \infty$ for all $t \geq t_0$. Now using the assumption that $L(r) \leq \kappa_1 \ln(r^2 + 1) + \kappa_2$ in (15) we
obtain
\[\|u(t)\|^2 \leq \|u_0\|^2 + \int_{t_0}^{t} M(\theta_s \omega) e^{-B_1(t-s)} ds + B_2 \kappa_1 \int_{t_0}^{t} \ln(\|u(s)\|^2 + 1) \|u(s)\|^2 ds.\]

Since \(M(\theta_s \omega) \geq 0\) for all \(s \in \mathbb{R}\), \(\int_{t_0}^{t} M(\theta_s \omega) e^{-B_1(t-s)} ds\) is non-negative and non-decreasing. It then follows directly from Lemma 3.1 that
\[\|u(t)\|^2 \leq \max \left\{ 2\|u_0\|^2 + M(\theta_{t_0} \omega)) e^{2\kappa_2 B_2(t-t_0)} e^{2e^{2B_2(t-t_0)} - 1} \right\}. \tag{16}\]

for all \(t \in [t_0, T]\), where
\[\hat{M}(t, \omega) := \int_{t_0}^{t} M(\theta_s \omega) e^{-B_1(t-s)} ds < \infty \quad \forall \ t \geq t_0. \tag{17}\]

We have just shown that the solution exists for all \(t \geq t_0\). It remains to show the continuous dependence of solutions on initial data. To this end, let \(u_0, v_0 \in \ell^2\) and consider two solutions of system (4) with initial value \(u(t_0) = u_0\) and \(u(t_0) = v_0\), respectively, and write
\[X(t) = (X_i(t))_{i \in \mathbb{Z}} := u(t; t_0, \omega, u_0), \quad Y(t) = (Y_i(t))_{i \in \mathbb{Z}} := u(t; t_0, \omega, v_0).\]

Let \(h(t) = (h_i(t))_{i \in \mathbb{Z}} = X(t) - Y(t)\), then \(h(t)\) satisfies the random lattice system
\[\frac{d}{dt} h_i(t) = -\frac{1}{\mu_i} h_i + \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} (g_j(X_j) - g_j(Y_j)), \quad i \in \mathbb{Z}. \tag{18}\]

Multiplying the equation (18) by \(h_i(t)\) and summing over \(i \in \mathbb{Z}\) gives
\[\frac{1}{2} \frac{d}{dt} \|h(t)\|^2 \leq - \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} h_i^2 + \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} (g_j(X_j) - g_j(Y_j)) h_i \]
\[\leq - \frac{1}{B_\mu B_\gamma} \|h(t)\|^2 + \frac{B_\lambda}{b_\mu} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} |g_j(X_j) - g_j(Y_j)| \cdot |h_i|.\]

By Assumption (A3) again, we have
\[|g_j(X_j) - g_j(Y_j)| \leq L(\max\{|X_j|, |Y_j|\}) |h_j| \leq L_T |h_j|,\]
where \(L_T\) is a constant depends on \(T\) according to (16). Therefore,
\[\frac{d}{dt} \|h(t)\|^2 \leq - \frac{2}{B_\mu B_\gamma} \|h(t)\|^2 + \frac{4B_\lambda L_T}{b_\mu} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} h_i^2 \]
\[\leq \left( - \frac{2}{B_\mu B_\gamma} + \frac{4(2N+1)B_\lambda L_T}{b_\mu} \right) \|h(t)\|^2.\]

Integrating the above inequality gives
\[\|h(t)\|^2 \leq e^{\left( - \frac{2}{B_\mu B_\gamma} + \frac{4(2N+1)B_\lambda L_T}{b_\mu} \right) (t-t_0)} \cdot \|h(0)\|^2,\]
which implies that
\[\sup_{t \in [t_0, T]} \|X(t) - Y(t)\|^2 \leq \max \left\{ e^{\left( - \frac{2}{B_\mu B_\gamma} + \frac{4(2N+1)B_\lambda L_T}{b_\mu} \right) (T-t_0)} \cdot \|u_0 - v_0\|^2, \right\} \cdot \|u_0 - v_0\|^2.\]
The solution depends continuously on the initial data. The proof is now complete.

It is straightforward to check that
\[ u(t + t_0; t_0, \omega, u_0) = u(t; 0, \theta_{t_0} \omega, u_0), \quad \forall t \geq 0, \quad u_0 \in \ell^2, \quad \omega \in \Omega, \]
and this allows us to define a continuous random dynamical system \( \varphi(t, \omega, \cdot) \) by
\[ \varphi(t, \omega, u_0) = u(t; 0, \omega, u_0), \quad \forall t \geq 0, \quad u_0 \in \ell^2, \quad \omega \in \Omega. \]

From now on, we will write \( u(t; \omega, u_0) \) instead of \( u(t; 0, \omega, u_0) \).

4. Existence of random attractors. In this section we investigate the existence of attractors for the random dynamical system \( \varphi(t, \omega) \) defined by the solutions to the RODE (4). To this end, we first construct a closed and bounded absorbing set \( \Gamma(\omega) \) and then prove the asymptotic compactness of the absorbing set. For simplification of exposition, throughout this section we assume that that \( L(r) \equiv L > 0 \) in Assumption (A3). In addition, assume that the functions \( g_i \) satisfy the dissipative condition

(A5) there exists \( \alpha \geq 0 \) and \( \beta = (\beta_i)_{i \in \mathbb{Z}} \in \ell^2 \) such that
\[ s g_j(s) \leq -\alpha s^2 + \beta_j^2, \quad \forall s \in \mathbb{R}, \quad j \in \mathbb{Z}. \]

Lemma 4.1. Assume that assumptions (A1) - (A5) hold. Then the continuous random dynamical system \( \varphi(t, \omega) \) generated by the RODE (4) has a random absorbing set \( \Gamma(\omega) \) provided

\[ b_\lambda := \inf_{i \in \mathbb{Z}, j = i - N, \cdots, i + N} \min_{\lambda_{i,j} \neq 0} |\lambda_{i,j}| > 0; \] (19)
\[ \sigma := \frac{1}{B_\mu B_\gamma} + \frac{\alpha(2N + 1)b_\lambda}{B_\mu} - (2N + 1)\frac{L B_\lambda}{b_\mu} \left( \frac{9}{2} + \frac{L}{\alpha} \right) > 0. \] (20)

Proof. We will start from the equation (10) and still use the inequalities (11) and (12). It then remains to estimate \( \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \). To this end first note that
\[ \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) = \sum_{j=i-N}^{i=N} \lambda_{i,j} u_i g_j(u_j) + \sum_{j=i-N}^{i=N} \lambda_{i,j} u_i g_j(u_j). \] (21)

First by Assumption (A5),
\[ u_i g_j(u_j) \leq -\alpha u_j^2 + \beta_j^2, \quad \forall j \in \mathbb{Z}. \]

Then by Assumption (A3) with \( L(r) \equiv L \) and the fact that \( xy \leq \frac{1}{2} \left( \frac{x^2}{\alpha} + ay^2 \right) \) we get
\[ g_j(u_j) \cdot (u_i - u_j) = g_j(\xi_j) u_j(u_i - u_j) \quad \text{for some} \quad \xi_j \quad \text{with} \quad |\xi_j| \leq |u_j| \]
\[ \leq L \left( \frac{u_j^2}{2\alpha} + \frac{\alpha u_j^2}{2} + u_j^2 \right) \]
for some \( a > 0 \) (to be determined later). Therefore
\[ u_i g_j(u_j) = (u_i - u_j) g_j(u_j) + u_j g_j(u_j) \]
\[ \leq \left( \frac{a}{2} L - \alpha \right) u_j^2 + L u_j^2 + \frac{L u_j^2 + \beta_j^2}{2a}, \quad \forall i, j \in \mathbb{Z}. \] (22)
It then follows from the assumption (19) and a shift of index that

\[ \sum_{j=i-N,\ldots,i+N} \lambda_{i,j} u_i g_j(u_j) \leq -\frac{\alpha}{2} \sum_{j=i-N,\ldots,j+N} \lambda_{i,j} u_j^2 \]

Using (21), (23) and (24) we obtain

\[ \sum_{j=i-N,\ldots,i+N} \lambda_{i,j} u_i g_j(u_j) \leq B_\lambda \left( \sum_{j=i-N,\ldots,j+N} \lambda_{i,j} u_j^2 + \frac{(2N+1)L^2}{2\alpha} u_i^2 + \sum_{j=i-N}^{i+N} \beta_j^2 \right) \]

Next with \( \lambda_{i,j} < 0 \) we have

\[ \sum_{j=i-N,\ldots,i+N} \lambda_{i,j} u_i g_j(u_j) \leq B_\lambda \sum_{j=i-N,\ldots,i+N} |u_i||g_j(u_j)| \]

\[ \leq B_\lambda \sum_{j=i-N}^{i+N} L |u_i| |u_j| \]

\[ \leq B_\lambda L \sum_{j=i-N}^{i+N} u_j^2 + \frac{B_\lambda L}{4} (2N+1)u_i^2. \] (24)

Using (21), (23) and (24) we obtain

\[ \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq -\frac{\alpha b_\lambda}{2} \sum_{j=i-N}^{i+N} u_j^2 + 2B_\lambda L \sum_{j=i-N}^{i+N} u_j^2 \]

\[ + B_\lambda \left( \frac{2N+1}{4\alpha} (2L + \alpha) u_i^2 + B_\lambda \sum_{j=i-N}^{i+N} \beta_j^2 \right). \] (25)

Multiplying both sides of (25) by \( \frac{1}{\mu_i} \) and summing over \( i \in \mathbb{Z} \) we obtain

\[ \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq \left( -\frac{\alpha b_\lambda}{2B_\mu} \sum_{i \in \mathbb{Z}, j=i-N}^{i+N} u_j^2 + \frac{2B_\lambda}{b_\mu} L \sum_{i \in \mathbb{Z}, j=i-N}^{i+N} u_j^2 \right. \]

\[ + \left. \frac{B_\lambda}{b_\mu} \left( \frac{2N+1}{4\alpha} (2L + \alpha) \right) \sum_{i \in \mathbb{Z}} u_i^2 + \frac{B_\lambda}{b_\mu} \sum_{i \in \mathbb{Z}, j=i-N}^{i+N} \beta_j^2 \right. \]

It then follows from the assumption (19) and a shift of index that

\[ \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq (2N+1) \left( \left( -\frac{\alpha b_\lambda}{2B_\mu} + \frac{B_\lambda L}{4\alpha b_\mu} (2L + \alpha) \right) \right. \]

\[ + \left. \left( \frac{2B_\lambda L}{b_\mu} \right) \|u(t)\|^2 + (2N+1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 \right). \] (26)
Now inserting the inequality (11), the inequality (12) with \(a = \frac{1}{\mu_\ell \beta_\ell}\), and the inequality (20) into (10) results in

\[
\frac{d}{dt}\|u(t, \omega)\|^2 \leq -\sigma\|u(t, \omega)\|^2 + 2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 + M(\theta, \omega),
\]

where \(\sigma\) is defined as in (20) and \(M(\theta, \omega) = \frac{B_\lambda B_\kappa}{b_\mu} \sum_{i \in \mathbb{Z}} T_i^2(\theta, \omega)\).

Integrating (27) from 0 to \(t\) gives

\[
\|\varphi(t, \omega, u_o)\|^2 \leq e^{-\sigma t}\|u_o\|^2 + e^{-\sigma t}2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 \int_0^t e^{\sigma s}ds 
+ \int_0^t M(\theta, \omega)e^{-\sigma(t-s)}ds
\leq e^{-\sigma t}\|u_o\|^2 + 2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 + \int_0^t M(\theta, \omega)e^{-\sigma(t-s)}ds.
\]

Then replacing \(\omega\) by \(\theta - \omega\) in the above inequality we finally obtain

\[
\|\varphi(t, \theta - \omega, u_o)\|^2 \leq e^{-\sigma t}\|u_o\|^2 + 2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 + \int_{-t}^0 M(\theta, \omega)e^{\sigma t}dt.
\]

Note that \(\int_{-\infty}^0 M(\theta, \omega)e^{\sigma t}dt\) is a tempered random variable due to Assumption (A4) and the integrability of \(e^{\sigma t}\) for \(\tau < 0\). Define

\[
\Gamma(\omega) := \left\{ u \in \ell^2 : \|u\| \leq \left( 2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 + \int_{-\infty}^0 M(\theta, \omega)e^{\sigma t}dt + 1 \right)^{1/2} \right\}.
\]

Then for any \(K(\omega) \in \mathcal{D}(\ell^2)\) and given \(u_0(\omega) \in K(\omega)\), there exists \(T_K(\omega) > 0\) such that

\[
\|\varphi(t, \theta - \omega, u_o(\theta - \omega))\|^2 \leq 2(2N + 1) \frac{B_\lambda}{b_\mu} \|\beta\|^2 + \int_{-\infty}^0 M(\theta, \omega)e^{\sigma t}dt + 1
\]

for all \(t \geq T_K(\omega)\), i.e., \(\varphi(t, \theta - \omega, K(\theta - \omega)) \in \Gamma(\omega)\). The proof is complete.

**Remark 1.** The dissipativity condition (A5) on \(g_i's\) is not necessary. In fact, even if \(\alpha < 0\), an absorbing set can still exist provided \(\frac{1}{\mu_\ell \beta_\ell}\) is large enough compared to \(L\). But with \(g_i's\) being dissipative, the convergence rate of solutions toward the absorbing set is faster.

Following the techniques first introduced in [3], we next show that the absorbing set defined in (29) is asymptotically compact under the RDS \(\varphi(t, \omega)\). This will be done by a tail estimate of solutions, presented in the lemma below. Note that as a particular case of Lemma 4.1, there exists \(T(\omega)\) such that

\[
\varphi(t, \theta - \omega, \Gamma(\theta - \omega)) \in \Gamma(\omega), \quad \forall t \geq T(\omega).
\]

**Lemma 4.2.** Assume that assumptions (A1) - (A5) and (19) - (20) hold. Then for any \(\epsilon > 0\) there exist \(\hat{T}(\epsilon, \omega, \Gamma) \geq T(\omega)\) and \(i(\epsilon, \omega) \in \mathbb{N}\) such that the solution \(\varphi(t, \omega, u_o) = u(t; \omega, u_o) = (u_i(t; \omega, u_o))_{i \in \mathbb{Z}}\) of the RODE (4) with \(u_o = (u_o,i)_{i \in \mathbb{Z}} \in \Gamma(\theta - \omega)\) satisfies

\[
\sum_{|i| \geq i(\epsilon, \omega)} \|(\varphi(t, \theta - \omega, u_o))_i\|^2 \leq \sum_{|i| \geq i(\epsilon, \omega)} |u_i(t, \theta - \omega, u_o)|^2 \leq \epsilon \quad \forall t \geq \hat{T}(\epsilon, \omega, \Gamma).
\]
Proof. Given \( k \in \mathbb{N} \) fixed (to be determine later), let \( \rho_k : \mathbb{R}_+ \to [0, 1] \) be a smooth, increasing and sub-additive function such that

\[
\rho_k(|i|) = \begin{cases} 
0, & |i| < k, \\
\in [0, 1], & |i| \in [k, 2k] \quad i \in \mathbb{Z}, \\
1, & |i| \geq 2k.
\end{cases}
\]

For any solution \( u(t; \omega, u_0) = (u_i(t; \omega, u_0))_{i \in \mathbb{Z}} \) of (4), set \( v = (v_i)_{i \in \mathbb{Z}} \) where \( v_i = \rho_k(|i|)u_i \), then \( v \in \ell^2 \). Taking the inner product of \( v \) with (4) we get

\[
\frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_k(|i|)u_i^2(t; \omega, u_0) = -\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{u_i^2}{\mu_i \gamma_i} + \sum_{i \in \mathbb{Z}} \rho_k(|i|)u_i \frac{1}{\mu_i} I_i(\theta_i \omega) + \sum_{i \in \mathbb{Z}} \frac{1}{\mu_i} \sum_{j=-N}^{i+N} \lambda_{i,j} \rho_k(|i|)u_i g_j(u_j). \tag{30}
\]

First by Assumption (A2),

\[
-\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{u_i^2}{\mu_i \gamma_i} \leq -\frac{1}{B_\mu B_\gamma} \sum_{i \in \mathbb{Z}} \rho_k(|i|)u_i^2. \tag{31}
\]

Second, similar to (12) there exists some \( a > 0 \) such that

\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{u_i}{\mu_i} I_i(\theta_i \omega) \leq \frac{a}{2} \sum_{i \in \mathbb{Z}} \rho_k(|i|)u_i^2 + \frac{1}{2ab_\mu^2} \sum_{i \in \mathbb{Z}} \rho_k(|i|)I_i^2(\theta_i \omega). \tag{32}
\]

It remains to estimate the last term of (30). To this end, multiplying (25) by \( \frac{1}{\mu_i} \rho_k(|i|) \) and summing over \( i \in \mathbb{Z} \) to get

\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{1}{\mu_i} \sum_{j=-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \tag{33}
\]

\[
\leq -\frac{ab_\lambda}{2B_\mu} \sum_{i \in \mathbb{Z}} \rho_k(|i|) \sum_{j=i-N}^{i+N} u_j^2 + \frac{2B_\lambda}{b_\mu} L \sum_{i \in \mathbb{Z}} \rho_k(|i|) \sum_{j=-N}^{i+N} u_j^2
\]

\[
+ \frac{B_\lambda(2N + 1)(2L + \alpha)}{4b_\mu \alpha} \sum_{i \in \mathbb{Z}} \rho_k(|i|)u_i^2 + \frac{B_\lambda}{b_\mu} \sum_{i \in \mathbb{Z}} \rho_k(|i|) \sum_{j=-N}^{i+N} \beta_j^2. \tag{34}
\]

Choose \( k > N \). Notice that since for any \( j = i - N, \ldots, i + N, |i - j| \leq N \), thus \( \rho_k(|i - j|) = 0 \). Then using the increasingness and sub-additivity of \( \rho_k \) we obtain

\[
\rho_k(|i|) \leq \rho_k(|i - j| + |j|) \leq \rho_k(|i - j|) + \rho_k(|j|) = \rho_k(|j|).
\]

For the first term of (34) we also need

\[
\rho_k(|j|) \leq \rho_k(|i| + |j - i|) \leq \rho_k(|i|) + \rho_k(|j - i|) = \rho_k(|i|).
\]
Apply the two above relations in the inequality (34) results in

\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) 
\leq \frac{\alpha b_{\lambda}}{2B_{\mu}} \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_k(|j|) u_j^2 + \frac{2B_{\lambda}}{b_{\mu}} L \sum_{i \in \mathbb{Z}} \sum_{j=i-N}^{i+N} \rho_k(|j|) u_j^2 
\]

\[
\frac{B_{\lambda} L (2N + 1)(2L + \alpha)}{4b_{\mu} \alpha} \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2 + \frac{B_{\lambda}}{b_{\mu}} \sum_{i \in \mathbb{Z}} \rho_k(|i|) \sum_{j=i-N}^{i+N} \beta_j^2.
\]

Then it follows from a shift of index that

\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) \frac{1}{\mu_i} \sum_{j=i-N}^{i+N} \lambda_{i,j} u_i g_j(u_j) \leq \frac{B_{\lambda}}{b_{\mu}} \sum_{i \in \mathbb{Z}} \left( \sum_{j=i-N}^{i+N} \rho_k(|j|) \right) \beta_i^2 
\]

\[
+ (2N + 1) \left( \frac{\alpha b_{\lambda}}{2B_{\mu}} + \frac{2B_{\lambda} L}{b_{\mu}} + \frac{B_{\lambda}(2L^2 + \alpha L)}{4b_{\mu} \alpha} \right) \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2. \tag{35}
\]

Inserting estimations (31), (32) with \(a = \frac{1}{B_{\mu} B_{\gamma}}\) and (35) in (30) we get

\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2(t) \leq -\sigma \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2 + \frac{2B_{\lambda}}{b_{\mu}} \sum_{i \in \mathbb{Z}} \left( \sum_{j=i-N}^{i+N} \rho_k(|j|) \right) \beta_i^2 
\]

\[
+ \frac{B_{\mu} B_{\gamma}}{b_{\mu}^2} \sum_{i \in \mathbb{Z}} \rho_k(|i|) I_i^2(\theta_i \omega),
\]

where \(\sigma\) is as defined in (20). Integrating the above inequality from 0 to \(t\) then replacing \(\omega \) by \(\theta_{-i} \omega\) gives

\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2(t; \theta_{-i} \omega, u_o) \leq e^{-\sigma t} \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_{o,i}^2(\theta_{-i} \omega) 
\]

\[
+ \frac{2B_{\lambda}}{\sigma b_{\mu}} \sum_{i \in \mathbb{Z}} \left( \sum_{j=i-N}^{i+N} \rho_k(|j|) \right) \beta_i^2 + \frac{B_{\mu} B_{\gamma}}{b_{\mu}^2} \int_{-t}^{0} \sum_{i \in \mathbb{Z}} \rho_k(|i|) I_i^2(\theta_{-i} \omega) e^{\sigma \tau} d\tau. \tag{36}
\]

First notice that \(\sum_{i \in \mathbb{Z}} \rho_k(|i|) u_{o,i}^2(\theta_{-i} \omega) \leq ||u_o(\theta_{-i} \omega)||^2\). Then for any \(\epsilon > 0\), there exists \(\hat{T}(\epsilon, \omega) > 0\) such that

\[
e^{-\sigma t} \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_{o,i}^2(\theta_{-i} \omega) < \frac{\epsilon}{3}, \quad \forall t \geq \hat{T}(\epsilon, \omega). \tag{37}
\]

Second, since \(\beta \in \ell^2\), given any \(\epsilon > 0\) there exists \(N_1(\epsilon) > 0\) such that

\[
\sum_{|i| \geq N_1(\epsilon)} \beta_i^2 < \frac{\sigma b_{\mu}}{6B_{\lambda}(2N + 1)} \epsilon.
\]

Pick \(k\) such that \(k > N_1(\epsilon) + N\). Then

\[
\sum_{|j| \geq k} \beta_j^2 < \frac{\sigma b_{\mu}}{6B_{\lambda}(2N + 1)} \epsilon, \quad \text{for each } j = i - N, \cdots, i, \cdots, i + N,
\]
and consequently
\[
\frac{2B_\lambda}{\sigma b_\mu} \sum_{i \in \mathbb{Z}} \left( \sum_{j=-N}^{i+N} \rho_i(|j|) \right) \beta_i^2 \leq \frac{2B_\lambda}{\sigma b_\mu} \left( \sum_{|i-N| \geq k} \beta_i^2 + \cdots + \sum_{|i+N| \geq k} \beta_i^2 \right) \\
\leq \frac{2B_\lambda}{\sigma b_\mu} (2N + 1) \cdot \frac{\sigma b_\mu}{6B_\lambda (2N + 1)} \epsilon = \frac{\epsilon}{3}. \tag{38}
\]

Next by using Assumption (A4), for any \( \epsilon > 0 \) there exists \( N_2(\epsilon, \omega) > 0 \) such that
\[
\sum_{i \in \mathbb{Z}} \rho_i(|i|) I^2_i(\theta_i \omega) \leq \sum_{|i| \geq N_2} I^2_i(\theta_i \omega) < \frac{\sigma b^2_\mu}{3B_\mu B_\gamma} \epsilon, \quad \forall t \in \mathbb{R}.
\]

Therefore
\[
\int_{-t}^{0} \sum_{i \in \mathbb{Z}} \rho_i(|i|) I^2_i(\theta_i \omega) e^{\sigma \tau} \leq \frac{B_\mu B_\gamma}{b_\mu^2} \cdot \frac{\sigma b^2_\mu}{3B_\mu B_\gamma} \epsilon \cdot \int_{-t}^{0} e^{\sigma \tau} d\tau < \frac{\epsilon}{3}. \tag{39}
\]

In summary, letting \( k := \max \{ N_1(\epsilon) + N, N_2(\epsilon, \omega) \} \), and applying (37) – (39) to (36) we obtain
\[
\sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2(t; \theta_{-t} \omega, u_o) < \epsilon, \quad \forall t \geq \hat{T}(\epsilon, \omega),
\]
which implies that
\[
\sum_{|i| \geq 2k} u_i^2(t; \theta_{-t} \omega, u_o) \leq \sum_{i \in \mathbb{Z}} \rho_k(|i|) u_i^2(t; \theta_{-t} \omega, u_o) < \epsilon \quad \forall t \geq \hat{T}(\epsilon, \omega).
\]

The proof is complete. \( \square \)

**Lemma 4.3.** Assume that assumptions (A1) - (A5) and (19) – (20) hold. Then the absorbing set \( \Gamma(\omega) \) defined in (29) is asymptotically compact under the RDS \( \varphi(t, \omega) \) defined by solutions of the RODE (4).

**Proof.** For a sequence \( \{ t_n \} \) with \( \lim_{n \to \infty} t_n = \infty \), let \( x_n(\omega) \in \Gamma(\theta_{-t_n} \omega) \in \mathcal{D}(\ell^2) \) and
\[
p_n(\omega) = \varphi(t_n, \theta_{-t_n} \omega, x_n), \quad n = 1, 2, \cdots,
\]
where \( p_{n,i} = (\varphi(t_n, \theta_{-t_n} \omega, x_n)) \), for \( i \in \mathbb{Z} \).

First since \( \lim_{n \to \infty} t_n = \infty \) there exists \( N_1(\omega, \Gamma) \in \mathbb{N} \) such that \( t_n \geq T_1(\omega) \) if \( n \geq N_1(\omega, \Gamma) \) and hence
\[
p_n(\omega) = \varphi(t_n, \theta_{-t_n} \omega, x_n) \in \Gamma(\omega), \quad \forall n \geq N_1(\omega, \Gamma).
\]

Then there is a subsequence of \( \{ p_n \} \) (still denoted by \( \{ p_n \} \)), and \( u^{*} \in \ell^2 \) such that
\[
p_n = \varphi(t_n, \theta_{-t_n} \omega, x_n) \rightharpoonup u^{*} \text{ weakly in } \ell^2.
\]

We now show that this convergence is actually strong. Given any \( \epsilon > 0 \), by Lemma 4.2 there exists \( \iota_1(\epsilon, \omega) > 0 \) and \( N_2(\epsilon, \omega) > 0 \) such that
\[
\sum_{|i| \geq \iota_1(\epsilon, \omega)} \left| (\varphi(t_n, \theta_{-t_n} \omega, u_0)) \right|_i^2 \leq \frac{\epsilon^2}{8}, \quad \forall n \geq N_2(\epsilon, \omega). \tag{40}
\]

Also, since \( u^{*} = (u^*_i)_{i \in \mathbb{Z}} \in \ell^2 \), there exists \( \iota_2(\epsilon) > 0 \) such that
\[
\sum_{|i| \geq \iota_2(\epsilon)} |u^*_i|^2 \leq \frac{\epsilon^2}{8}. \tag{41}
\]
Set \( \iota(\epsilon, \omega) := \max\{\iota_1(\epsilon, \omega), \iota_2(\epsilon, \omega)\} \) and since \( \varphi(t_n, \theta_{-t_n} \omega, x_n) \to u^* \) in \( \ell^2 \) then
\[
(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i \to u^*_i \quad \text{for} \quad |i| \leq \iota(\epsilon, \omega), \quad \text{as} \quad n \to \infty.
\]
Therefore \( \exists N_3(\epsilon, \omega) > 0 \) such that
\[
\sum_{|i| \leq \iota(\epsilon, \omega)} |(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i - u^*_i|^2 \leq \frac{\epsilon^2}{2}, \quad \forall \ n \geq N_3(\epsilon, \omega).
\] (42)
Set \( \hat{N}(\epsilon, \omega) := \max\{N_1(\epsilon, \omega), N_2(\epsilon, \omega), N_3(\epsilon, \omega)\} \), then using (40) – (42) we have for \( n \geq \hat{N}(\epsilon, \omega) \)
\[
\|(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i - u^*_i\|^2 = \sum_{|i| \leq \iota(\epsilon, \omega)} |(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i - u^*_i|^2 + \sum_{|i| > \iota(\epsilon, \omega)} |(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i - u^*_i|^2 \leq \frac{\epsilon^2}{2} + \sum_{|i| > \iota(\epsilon, \omega)} |(\varphi(t_n, \theta_{-t_n} \omega, x_n))_i|^2 + |u^*_i|^2 \leq \epsilon^2.
\]
Hence \( p_n \) (the subsequence) have strongly converge in \( \ell^2 \), and therefore \( \Gamma(\omega) \) is asymptotically compact. The proof is complete.  

The following theorem follows directly from Lemma 4.1, Lemma 4.3 and Proposition 1.

**Theorem 4.4.** Assume that assumptions (A1) - (A5) and (19) – (20) hold. The random dynamical system \( \{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) generated by the RODE (4) possesses a unique global random attractor with component subsets
\[
\mathcal{A}(\omega) = \bigcap_{\tau \geq T_\omega} \bigcup_{t \geq \tau} \varphi(t, \theta_{-t} \omega, \Gamma(\theta_{-t} \omega)),
\]
where \( \Gamma \) is defined as in (29).

5. **Closing remarks.** This work was motivated by the increasing size of a Hopfield neural network under the influence of random forcing. In the context of Hopfield models, there are two novelties in this work: (1) the underlying system is an infinite dimensional system of ODEs, i.e., lattice dynamical system; and (2) the forcing at each neuron is general random process. In the context of lattice dynamical systems, the major novelty as well as difficulty is the consideration of a nonlinear interconnection structure described by the nonlinear operator \( \sum_{j=-N}^{N} \lambda_{i,j} g_j(u_j(t)) \). Such systems have never been studied in the past. The main contribution of this work is showing the existence of a global random attractor for the random Hopfield lattice system. Along with the construction of the random absorbing set, another interesting discovery is that inhibit relation among neurons contribute more to the convergence of solutions towards the attractor. These results serve as an important initial step towards the study on more interesting dynamical behavior of neural networks, such as upper semi continuity of the attractors and synchronization behavior of discrete neural networks under random perturbation.
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