ECCENTRIC BEHAVIOR OF DISK GALAXIES

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ABSTRACT

A theory is developed for the dynamics of eccentric perturbations ($\propto \exp \pm i\phi$) of a disk galaxy residing in a spherical dark matter halo and including a spherical bulge component. The disk is represented as a large number $N$ of rings with shifted centers and with perturbed azimuthal matter distributions. Account is taken of the dynamics of the shift of the matter at the galaxy’s center which may include a massive black hole. The gravitational interactions between the rings and between the rings and the center is fully accounted for, but the halo and bulge components are treated as passive gravitational field sources. Equations of motion and a Lagrangian are derived for the ring+center system, and these lead to total energy and total angular momentum constants of the motion.

We first study the eccentric motion of a disk consisting of two rings of different radii but equal mass, $M_d/2$. For small $M_d$ the two rings are stable, but for $M_d$ larger than a threshold value the rings are unstable with a dynamical timescale growth. For $M_d$ sufficiently above this threshold, the instability acts to decrease the angular momentum of the inner ring, while increasing that of the outer ring. The instability results from the merging positive and negative energy modes with increasing $M_d$. Second, we analyze the eccentric motion of one ring interacting with a radially shifted central mass. In this case instability sets in above a threshold value of the central mass (for a fixed ring mass), and it acts to increase the angular momentum of the central mass (which therefore rotates in the direction of the disk matter), while decreasing the angular momentum of the ring.

Third, we study the eccentric dynamics of a disk with an exponential surface density distribution represented by a large number of rings. The inner part of the disk is found to be strongly unstable. Angular momentum of the rings is transferred outward and to the central mass if present, and a trailing one-armed spiral wave is formed in the disk. Fourth, we analyze a disk with a modified exponential density distribution where the density of the inner part of the disk is reduced. In this case we find much slower, linear growth of the eccentric motion. A trailing one-armed spiral wave forms in the disk and becomes more tightly wrapped as time increases. The motion of the central mass if present is small compared with that of the disk.

Subject headings: celestial mechanics, stellar dynamics — galaxies: kinematics and dynamics — galaxies: structure

1. INTRODUCTION

Although studies of spiral galaxies commonly assume an axisymmetric equilibrium state, possibly perturbed by spiral arms, there is growing evidence that many galaxies lack this symmetry. Based on optical appearance, approximately 30% of disk galaxies exhibit significant “lopsidedness” (Rix & Zaritsky 1995; Kornreich, Haynes, & Lovelace 1998), which supports the early findings of Baldwin, Lynden-Bell, & Sancisi (1980). As many as ~50% of spiral galaxies show departures from the expected symmetric two-horned global H I line profile (Richter & Sancisi 1994; Haynes et al. 1998). Furthermore, H I maps of several galaxies, for example, NGC 3631 (Knapen 1997), NGC 5474 (Rownd, Dickey, & Helou 1994), and NGC 7217 (Buta et al. 1995), have revealed offsets between the optical centers of light and their kinematic centers. Two examples of kinematically lopsided galaxies have recently been discussed by Swaters et al. (1998). Other recent observations such as the Hubble Space Telescope observations of the nucleus of M31 (Lauer et al. 1993) and the Kitt Peak 0.9 m observations of NGC 1073 (Kornreich et al. 1998) indicate that even the optical center of light may be displaced from the center of the optical isophotes of the major part of the galaxy in a significant fraction of cases.

The origin of the lopsidedness in disk galaxies remains uncertain. Baldwin et al. (1980) proposed a simple kinematic model in which different rings of the galaxy, assumed noninteracting, are initially shifted from their centered equilibrium positions. The shifted rings precess in the overall gravitational potential in a direction opposite to that of the mass motion. Because the precession rate decreases in general with radial distance, an initial disturbance tends to “wind up” into a leading spiral arm in a time appreciably less than the Hubble time. Miller & Smith (1988, 1992) have made extensive computer simulation studies of the unstable eccentric motion of matter in the nuclei of galaxies which they suggest is pertinent to the off-center nuclei observed in a number of galaxies, for example, M31, M33, and M101.

Understanding the origin of the observed disk asymmetries is important because it provides clues to ongoing
accretion of gas and the distribution of dark, unseen mass in the halo. Schoenmakers, Franx, & de Zeeuw (1997) interpret optical asymmetry as an indicator of asymmetry in the overall galactic potential and, therefore, an indicator of the spatial distribution of the dark matter in a galaxy which may have a triaxial distribution. By analyzing the spiral components present in the surface brightness or H1 distribution, the velocity gradient, and therefore, the shape of the gravitational potential, may be uncovered. Jog (1997) has studied the orbits of stars and gas in a lopsided potential and shows that lopsided potentials arising from disks alone are not self-consistent; rather, a stationary lopsided disk may be responding to asymmetries in the halo.

Zaritsky & Rix (1997) proposed that optical lopsidedness arises from tidal interactions and/or minor mergers. Such mergers are often suggested as the most likely contributors to galaxy asymmetry, even when no interacting companions are evident. While galaxies such as NGC 5474 are well-known to be under the tidal influence of neighbors, the apparently long-lived kinematic offsets of other relatively isolated objects, and the common asymmetries in flocculent (as opposed to tidally induced “grand design”) spiral galaxies, are not explained by simple tidal interaction models, which produce only transient asymmetric features.

A further possibility is that an optical disk may be in a quasi-stationary lopsided state in a symmetric potential, as discussed by Syer & Tremaine (1996). In this model, gaseous and stellar matter swirl about the minimum of the halo potential in a state not fully relaxed. The result is a lopsided flow within a symmetric mass distribution. Numerical simulations of this situation have been done by Levine & Sparke (1998) using a gravitational N-body tree-code method (see Barnes & Hut 1989) for disk galaxies shifted from the center of the main halo potential. The results are suggestive of lopsidedness with large lifetimes. An N-body simulation study of a rotating spheroidal stellar system including the dynamics of a massive central object by Taga & Iye (1998a) indicates that the central object goes into a long-lasting oscillation similar to those found earlier by Miller & Smith (1988, 1992) and which may explain asymmetric structures observed in M31 and NGC 4486B. A linear stability analysis of a self-gravitating fluid disk including a massive central object also by Taga & Iye (1998b) indicates a linear instability. We comment on the relation of this work to the present study in the conclusions section of this work.

Here we develop a theory of the dynamics of eccentric perturbations of a disk galaxy residing in a spherical dark matter halo. We represent the disk as a large number $N$ of rings as suggested by Baldwin et al. (1980) (and Lovelace 1998 for the treatment of disk warping). In contrast to Baldwin et al., the gravitational interactions between the rings is fully accounted for. We show that for general eccentric perturbations, the centers of the rings are shifted and the azimuthal distribution of matter in the rings is perturbed. The ring representation is analogous to the approach of Contopoulos & Grosbøl (1986, 1988), where self-consistent galaxy models are constructed from a finite set of stellar orbits.

Section 2 develops a theory for treating eccentric perturbations of a disk galaxy. The assumed equilibrium is first discussed (§ 2.1), and a description of the disk perturbations is developed (§ 2.2). The representation of the disk in terms of a finite number $N$ of rings is presented (§ 2.3), and the ring equations of motion are derived (§ 2.4). We renormalize the ring equations so as to reduce the nearest ring interactions (§ 2.5). The dynamics and influence of the displacement of the center of the galaxy, which may include a massive black hole, is discussed separately (§ 2.6). We obtain an energy constant of the motion for the dynamical equations (§ 2.7), the Lagrangian, and the conserved total canonical angular momentum (§ 2.8).

We discuss the nature of the precession of a single ring in § 3. In § 4 we study the eccentric motion of a disk consisting of two rings and show that this motion is unstable for sufficiently large ring masses. In § 5 we study the eccentric motion of one ring including the radial shift of the central mass and show that this situation is unstable for sufficiently large mass of the center and/or of the ring. Section 6 presents numerical results for the eccentric dynamics of disk of many rings including the radial shift of the central mass. Section 7 summarizes the conclusions of this work.

2. THEORY

2.1. Equilibrium

The equilibrium galaxy is assumed to be axisymmetric and to consist of a thin disk of stars and gas and a spheroidal distributions consisting of a bulge component and a halo of dark matter. We use inertial cylindrical $(r, \phi, z)$ and Cartesian $(x, y, z)$ coordinate systems with the disk and halo equatorial planes in the $z = 0$ plane. The total gravitational potential is written as

$$\Phi(r, z) = \Phi_d + \Phi_b + \Phi_h,$$

where $\Phi_d$ is the potential due to the disk, $\Phi_b$ is due to the bulge, and $\Phi_h$ is that for the halo. The galaxy may have a central massive black hole of mass $M_{bh}$, in which case a term $\Phi_{bh} = -GM_{bh}/r^2 + z^2$ is added to the right-hand side of equation (1). The particle orbits in the equilibrium disk are approximately circular with angular rotation rate $\Omega(r)$, where

$$\Omega^2(r) = \frac{1}{r} \frac{\partial \Phi}{\partial r} \bigg|_{z=0} = \Omega_d^2 + \Omega_b^2 + \Omega_h^2.$$

The equilibrium disk velocity is $v = r\Omega(r)\phi$. A central black hole is accounted for by adding the term $\Omega_{bh}^2 = GM_{bh}/r^3$ to the right-hand side of equation (2).

The surface mass density of the (optical) disk is taken to be $\Sigma_d = \Sigma_{d0} \exp(-r/r_d)$ with $\Sigma_{d0}$ and $r_d$ constants and $M_d = 2\pi r_d^2 \Sigma_{d0}$ the total disk mass. The potential due to this disk matter is

$$\Phi_d(r, 0) = -\frac{GM_d}{r_d} R[I_0(R)K_1(R) - I_1(R)K_0(R)],$$

and the corresponding angular velocity is

$$\Omega_d^2 = \frac{1}{2} \frac{GM_d}{r_d^2} [I_0(R)K_0(R) - I_1(R)K_1(R)],$$

where $R \equiv r/(2r_d)$ and the $I$’s and $K$’s are the usual modified Bessel functions (Freeman 1970; Binney & Tremaine 1987, p. 77). Typical values are $M_d = 6 \times 10^10 M_\odot$ and $r_d = 4$ kpc. For these values, $\kappa \equiv (GM_d/r_d)^{1/2} \approx 255$ km s$^{-1}$.

The potential due to the bulge component is taken as an Plummer model

$$\Phi_b = -\frac{GM_b}{(r_b^2 + r^2 + z^2)^{1/2}},$$

where...
where $M_\text{b}$ is the mass of the bulge and $r_\text{b}$ is its characteristic radius (Binney & Tremaine 1987, p. 42). We have
\[
\Omega_\text{b}^2 = \frac{GM_\text{b}}{(r_\text{b}^2 + r^2)^{3/2}}.
\] (4)

Typical values are $M_\text{b} = 10^{10} M_\odot$ and $r_\text{b} = 1 \text{ kpc}$, and for these values $v_\text{b} = (GM_\text{b}/r_\text{b})^{1/2} \approx 208 \text{ km s}^{-1}$.

The potential of the halo is taken to be
\[
\Phi_\text{h} = \frac{1}{2} v_\text{h}^2 \ln (r_\text{h}^2 + r^2 + z^2),
\]
where $v_\text{h} = \text{const}$ is the circular velocity at large distances and $r_\text{h} = \text{const}$ is the core radius of the halo. We have
\[
\Omega_\text{h}^2 = \frac{v_\text{h}^2}{r_\text{h}^2 + r^2}.
\] (5)

Typical values are $v_\text{h} \sim 200$–300 km s$^{-1}$ and $r_\text{h} \sim 2$–20 kpc. Figure 1 shows an illustrative rotation curve.

2.2. Perturbations

We treat the disk as fluid and use a Lagrangian representation for the perturbation as developed by Frieman & Rotenberg (1960). The position vector $r$ of a fluid element which at $t = 0$ was at $r_0$ is given by
\[
r = r_0 + \xi(r_0, t).
\] (6)

That is, $r_0(t)$ is the unperturbed orbit and $r(t)$ the perturbed orbit of a fluid element. This description is applicable to both the disk gas and the disk stars which are in approximately laminar motion with circular orbits. The perturbations of the halo and bulge are assumed negligible compared with that of the disk. For these approximately spheroidal components, the particle motion is highly non-laminar, with crisscrossing orbits with the result that their response “averages out” the disk perturbation.

Further, the perturbations are assumed to consist of small in-plane displacements or shifts of the disk matter,
\[
\xi = \xi_r \hat{r} + \xi_\phi \hat{\phi},
\] (7)

with azimuthal mode number $m = 1$. That is, $\xi_r$ and $\xi_\phi$ have $\phi$-dependences proportional to $\exp(i\phi)$.

From equation (6), we have $v(r, t) = v_0(r_0, t) + \partial \xi_r / \partial t + (v \cdot \nabla) \xi$. The Eulerian velocity perturbation is $\delta v(r, t) = v(r, t) - v_0(r, t)$. Therefore,
\[
\delta v(r, t) = \frac{\partial \xi}{\partial t} + (v \cdot \nabla) \xi - (\xi \cdot \nabla) v.
\] (8)

The components of this equation are
\[
\delta v_r = D \xi_r, \\
\delta v_\phi = D \xi_\phi - r \Omega^2 \xi_r,
\] (9)

where
\[
D = \frac{\partial}{\partial t} + \Omega(r) \frac{\partial}{\partial \phi}, \\
\Omega = \partial \Omega / \partial r.
\]

The main equation of motion is
\[
\frac{d\delta v}{dt} = -\delta F = -\nabla \delta \Phi,
\] (10)

where $\delta F$ is the perturbation in the gravitational force (per unit mass) and $\delta \Phi$ is the perturbation of the gravitational potential. The pressure force contribution is small compared to $\delta F$ by a factor $(v_{th}/v_\text{h})^2 \ll 1$ and is neglected, where $v_{th}$ is the “thermal” spread of the velocities of the disk matter. Also, note that
\[
\frac{d\delta v}{dt} = \frac{\partial \delta v}{\partial t} + (v \cdot \nabla) \delta v + (\delta v \cdot \nabla) v.
\] (11)

The components of equation (11) give
\[
\begin{align*}
\left(\frac{d\delta v_r}{dt}\right)_r & = D \delta v_r - 2 \Omega \delta v_\phi, \\
& = (D^2 + 2 \Omega \Omega) \xi_r - 2 \Omega D \xi_\phi, \\
\left(\frac{d\delta v_\phi}{dt}\right)_\phi & = D \delta v_\phi + \left(\frac{\kappa^2}{2 \Omega}\right) \delta v_r, \\
& = D^2 \xi_\phi + 2 \Omega D \xi_r,
\end{align*}
\] (12)

where $\kappa^2 \equiv (1/r^3) (dr^2 \Omega^2)/dr$ is the radial epicyclic frequency (squared).

The perturbation of the surface mass density of the disk obeys
\[
\frac{\partial \delta \Sigma}{\partial t} = -\nabla \cdot (\Sigma \delta v + \delta \Sigma v),
\]
where $\Sigma(r)$ is the surface density of the equilibrium disk. Because $\nabla \cdot (\Sigma v) = 0$, this equation implies
\[
\delta \Sigma = -\nabla \cdot (\Sigma \xi).
\] (13)

The perturbation of the gravitational potential is given by
\[
\delta \Phi(r, t) = -G \int d^2r' \frac{\delta \Sigma(r', t)}{|r - r'|},
\] (14)

where the integration is over the surface area of the disk.

2.3. Ring Representation

We represent the disk by a finite number $N$ of radially shifted plane circular rings. The matter distribution around each ring is also perturbed. An elliptical distortion of a ring corresponds to $m = \pm 2$, which is not considered here. This
description is general for small shifts \((dr/r)^2 \ll 1\), where the linearized equations are applicable (Lovelace 1998). Of course, the orbit of a single perturbed particle is not in general closed in the inertial frame used. However, the orbit is closed in an appropriately rotating frame, and this rotation rate is simply the angular precession frequency of the ring \(\omega\) discussed below (Baldwin et al. 1980). The disk is assumed geometrically thin.

For the equilibrium disk we take

\[
\Sigma(r) = \frac{M_j}{2\pi r} \frac{\exp \left[ -\frac{(r - r_j)^2}{2\Delta r_j^2} \right]}{\Delta r_j^2},
\]

where \(M_j\) is the mass of the \(j\)th ring, \(r_j\) is its radius with \(0 < r_1 < r_2, \ldots, r_N\), and \(\Delta r_j \ll r_j\) is its width. The motion of the central part of the disk \((r < r_1)\) is treated separately in §2.6.

A physical choice for the rings distribution will have the ring spacing of the order of the disk thickness, \(r_{j+1} - r_j = O(\Delta r)\). Furthermore, we assume \((r_{j+1} - r_j)^2 \ll r_{j+1} - r_j\) in order to simplify the calculation of the ring interaction as discussed in the Appendix. For example, a possible choice is \(r_j = r_1 + (j - 1)\Delta r\) with \(r_1 = 1 \text{ kpc}, \Delta r = 0.5 \text{ kpc}, \text{ and } M_j = 2\pi r_j \delta r \Sigma(r_j)\). For \(\Delta r_j = \delta r(2n 1/2) \approx \delta r/1.177\), the profile of a ring falls to half its maximum value at \(\delta r\) so that equation (15) gives a fairly smooth representation of \(\Sigma(r)\).

The perturbation in the disk’s surface density is

\[
\delta \Sigma(r, \phi, t) = \sum_j \delta \Sigma_j = -\nabla \cdot (\Sigma_j \hat{\xi}_j),
\]

where \(\Sigma_j \equiv [M_j/2\pi r(2n 1/2)\Delta r_j] \exp [-(r - r_j)^2/(2\Delta r_j^2)]\).

We can express different moments of the perturbed disk in terms of the rings. For example, the center of mass of the disk is

\[
\langle r \rangle = \sum_j \frac{M_j \langle r_j \rangle}{\Sigma M_j},
\]

where

\[
M_j \langle r_j \rangle = \int d^2 r \delta \Sigma_j
= M_j \int \frac{d\phi}{2\pi} \xi_j,
\]

where an integration by parts has been made.

We can write in general

\[
\xi_{\mu} = \epsilon_{\mu} \cos \phi + \epsilon_{\nu} \sin \phi,
\]

\[
\xi_{\nu} = -\delta_{\mu} \sin \phi + \delta_{\nu} \cos \phi.
\]

Here \(\epsilon_{\mu, \nu}\) and \(\delta_{\mu, \nu}\) are the ring displacement amplitudes: \(\epsilon_{\mu, \nu}\) represents the shift of the ring’s center, and \(\delta_{\mu, \nu}\) represents in general both the shift of the ring’s center and the azimuthal displacement of the ring matter. First, notice that for \(\epsilon_{\mu, \nu} = 0\) and \(\delta_{\mu, \nu} \neq 0\), there is no shift of the ring’s center but rather an azimuthal displacement of the ring matter. In this case \(\delta \Sigma_j = -(\Sigma_j/\partial r)(\partial \xi_{\nu}/\partial \phi)\). Second, notice that \(\delta_{\mu, \nu} = \epsilon_{\mu, \nu}\) corresponds to a rigid shift of the ring without azimuthal displacement of the ring matter. For example, a rigid shift in the \(x\)-direction has \(\epsilon_{\mu} = \delta_{\mu}\) and \(\epsilon_{\nu} = 0 = \delta_{\nu}\), so that \(\xi_{\mu} = \epsilon_{\mu} \cos \phi + \epsilon_{\nu} \sin \phi = -\epsilon_{\mu} \sin \phi\). In this case, \(\nabla \cdot \xi = 0\), so that \(\delta \Sigma_j = -\nabla \cdot (\Sigma_j \xi_j) = -\xi_{\nu}(\partial \Sigma_j/\partial r)\). Figure 2 shows the nature of perturbations with a shift of the ring’s center and with an azimuthal displacement of the ring matter.

Equation (18) now gives

\[
\langle r_{\mu} \rangle = \frac{1}{2} (\epsilon_{\mu} + \delta_{\mu}),
\]

\[
\langle r_{\nu} \rangle = \frac{1}{2} (\epsilon_{\nu} + \delta_{\nu}).
\]

For the case of a rigid shift of a ring, \(\epsilon_{\mu, \nu} = \delta_{\mu, \nu}\), the center-of-mass position is simply \(\langle r_{\mu, \nu} \rangle = \epsilon_{\mu, \nu}\) as expected.

Similarly, the velocity perturbation of the disk can be written as

\[
\langle \delta v \rangle = \sum_j \frac{M_j \langle \delta v_j \rangle}{\Sigma M_j},
\]

where

\[
\langle \delta v_j \rangle = \frac{1}{M_j} \int d^2 r (\delta \Sigma_j v + \Sigma_j \delta v_j).
\]

Evaluation of equation (22) gives

\[
\langle \delta v_{\mu} \rangle = \frac{1}{2} (\epsilon_{\mu} + \delta_{\mu}),
\]

\[
\langle \delta v_{\nu} \rangle = \frac{1}{2} (\epsilon_{\nu} + \delta_{\nu}).
\]

The influence of the eccentric motion on the rotation curves is discussed at the end of §6.3.

2.4. Ring Equations of Motion

Equations (12) and (19) give the ring equations of motion,

\[
\xi_x + 2\Omega \xi_y - \Omega^2 \epsilon_x - 2\Omega (\delta_y - \Omega \delta_x) = \langle \delta F_{\xi} \rangle,
\]

\[
\xi_y + 2\Omega \xi_x - \Omega^2 \epsilon_y + 2\Omega (\delta_x + \Omega \delta_y) = \langle \delta F_{\xi} \rangle,
\]

\[
\delta_x + 2\Omega \delta_y - \Omega^2 \delta_x - 2\Omega (\epsilon_x - \Omega \epsilon_y) = - \langle \delta F_{\delta} \rangle,
\]

\[
\delta_y + 2\Omega \delta_x - \Omega^2 \delta_y + 2\Omega (\epsilon_x + \Omega \epsilon_y) = \langle \delta F_{\delta} \rangle.
\]
where the $j$ subscripts are implicit, where the angular brackets indicate the average over the ring $\langle \ldots \rangle$ \equiv $2\pi \int \langle \ldots \rangle \Sigma_k(r)/M_j$, where $\Omega^2 \equiv \Omega^2 - 2\Omega\Omega$, and

$$
\delta F^{e,s}_a \equiv \int \frac{d\phi}{\pi} [\cos \phi, \sin \phi] \delta F^a,
$$

with $a = r, \phi$.

We now evaluate the different force terms on the right-hand side of equation (24). For this it is useful to write $\delta \Sigma = \delta \Sigma_a + \delta \Sigma_b$, where

$$
\delta \Sigma_a = -\frac{1}{r} \frac{d}{dr} \delta (r \Sigma r), \quad \delta \Sigma_b = -\frac{1}{r} \frac{d}{dr} \delta (\Sigma \phi),
$$

from equation (13). The corresponding contributions to the potential, $\delta \Phi = \delta \Phi_a + \delta \Phi_b$, evaluated at $(r, \phi)$ are from equation (13),

$$
\delta \Phi_a(r, \phi) = G \sum_k M_k (\epsilon_{kx} \cos \phi + \epsilon_{ky} \sin \phi)
\times \int_0^\infty r' dr' \left[ \frac{1}{r'} \frac{d}{dr'} \left[ r' S(r'|r_k) \right] \right]
\times \int_0^{2\pi} d\psi' \cos \psi',
$$

$$
\delta \Phi_b(r, \phi) = -G \sum_k M_k (\delta_{kx} \cos \phi + \delta_{ky} \sin \phi)
\times \int_0^\infty r' dr' \frac{S(r'|r_k)}{r'} \int_0^{2\pi} d\psi' \cos \psi',
$$

where

$$
R^2(r, r') \equiv r^2 + (r')^2 - 2rr' \cos \psi',
$$

$$
S(r|r_k) = \frac{2\pi \Sigma(r)}{M_k} = \exp \left[ -\frac{2(r-r_k)^2/2}{\Delta r_k^2} \right],
$$

and $\Psi \equiv \phi' - \phi$.

Evaluation of the force on the $j$th ring due to the other rings gives

$$
M_j \langle \delta F^a_{r_j} \rangle = \sum_k (C_{jk} \epsilon_{kx} + D_{jk} \delta_{kx}),
$$

$$
M_j \langle \delta F^a_{\psi_j} \rangle = \sum_k (C_{jk} \epsilon_{ky} + D_{jk} \delta_{ky}),
$$

$$
- M_j \langle \delta F^a_{\phi_j} \rangle = \sum_k (E_{jk} \delta_{kx} + D_{jk} \epsilon_{kx}),
$$

$$
M_j \langle \delta F^a_{\phi_j} \rangle = \sum_k (E_{jk} \delta_{ky} + D_{jk} \epsilon_{ky}),
$$

where the "tidal coefficients" are

$$
C_{jk} = -GM_j M_k \int \langle \ldots \rangle \Sigma_j(r) \Sigma_j(r')\langle \ldots \rangle
\times S(r|r_j) \frac{\partial}{\partial r} \frac{\partial S(r'|r_j)}{\partial r'} \frac{\partial \mathcal{F}^{a}(r, r')}{\partial r},
$$

$$
D_{jk} = GM_j M_k \int \langle \ldots \rangle \Sigma_j(r) \Sigma_j(r')\langle \ldots \rangle
\times S(r|r_j) S(r'|r_j) \frac{\partial \mathcal{F}^{a}(r, r')}{\partial r},
$$

and this shows that $D_{jk} = D_{jk}^e$. Expressions for the tidal coefficients in terms of elliptic integrals are given in the Appendix.

Following the approach of Lovelace (1998), we introduce the complex displacement amplitudes

$$
\delta_j \equiv \epsilon_{jx} - iE_{jx} = \epsilon_j(t) \exp \left[ -i\psi_j(t) \right],
$$

$$
\Delta_j \equiv \delta_{jx} = \delta_j(t) \exp \left[ -i\psi_j(t) \right].
$$

Here $\epsilon_j \geq 0$ is the amplitude of the shift of the ring’s center, and $\varphi_j$ is the angle of the shift with respect to the x-axis; $\delta_j \geq 0$ is the amplitude of the azimuthal displacement of the ring matter, and $\psi_j$ is the angle of the average motion also with respect to the x-axis. If $\varphi(t)$ and $\psi(t)$ increase with time, the ring precesses in the same sense as the particle motion, and we refer to this as “forward precession”. The opposite case, with $\varphi(t)$ and $\psi(t)$ decreasing with time, is termed “backward precession”.

We now combine equations (24) and (28) to obtain the ring equations of motion,

$$
M_j \left[ \delta_j + 2i\Omega_j \delta_j - \bar{\Omega}_j^2 \delta_j - 2\Omega_j \Delta_j + 2\Omega_j^2 \Delta_j \right]
= \sum_k (C_{jk} \delta_k + D_{jk} \Delta_k),
$$

$$
M_j \left[ \Delta_j + 2i\Omega_j \Delta_j - \bar{\Omega}_j^2 \Delta_j - 2\Omega_j \delta_j + 2\Omega_j^2 \delta_j \right]
= \sum_k (E_{jk} \delta_k + D_{jk} \delta_j),
$$

where $j = 1 \ldots N$, $\Omega_j \equiv \Omega(r_j)$, and $\bar{\Omega}_j \equiv \bar{\Omega}(r_j)$.

2.5. Renormalization of Ring Equations

Here we redo the ring equations of motion so as to diminish the strong tidal interactions of nearest neighbor rings
due to the terms $\propto C_{j+k}$. First, we rewrite the right-hand side of equation (38) as
\[
\sum_k [C_{jk}(\delta_k - \delta_j) + D_{jk}(\Delta_k - \Delta_j)] + \delta_j \sum_k (C_{jk} + D_{jk}). \tag{40}
\]
Similarly, we rewrite the right-hand side of equation (39) as
\[
\sum_k [E_{jk}(\Delta_k - \Delta_j) + D'_{jk}(\delta_k - \Delta_j)] + \Delta_j \sum_k (E_{jk} + D'_{jk}). \tag{41}
\]
We define
\[
\Omega^2_{\delta j} = \frac{1}{M_j} \sum_k (C_{jk} + D_{jk}). \tag{42}
\]
Using the relations $\tilde{\Omega}^2 = \Omega^2 - r(d\Omega^2/dr)$, $\Omega^2 = \Omega^2 + \Omega^2$, and equation (A14) of the Appendix gives
\[
\Omega^2_{\delta j} = \Omega^2 - \left[ \frac{\partial(\Omega^2 + \Omega^2)}{\partial r} \right]_j + \frac{1}{M_j} \sum_k (D'_{jk} + E_{jk}). \tag{43}
\]
(If there is a central black hole $M_{bh}$, then the right-hand side of eq. [43] also has the term $- [r(\partial \Omega^2/\partial r)]_j = 3GM_{bh}/r_k^3$. Similarly, we define
\[
\Omega^2_{\Lambda} = \frac{1}{M} \sum_j (D'_{jk} + E_{jk}). \tag{44}
\]
The ring equations of motion now become
\[
M_j \delta_j + 2\Omega_{\delta j} \delta_j - \Omega^2_{\delta j} \delta_j - 2\Omega_{\Lambda j} \Delta_j + 2\Omega^2_{\Delta j} \Delta_j = \sum_k [C_{jk}(\delta_k - \delta_j) + D_{jk}(\Delta_k - \Delta_j)], \tag{45}
\]
\[
M_j \Delta_j + 2\Omega_{\Lambda j} \Delta_j - \Omega^2_{\Lambda} \Delta_j - 2\Omega_{\delta j} \delta_j + 2\Omega^2_{\delta} \delta_j = \sum_k [E_{jk}(\Delta_k - \Delta_j) + D_{jk}(\delta_k - \Delta_j)]. \tag{46}
\]
In the large $N$, limit, the sums in equations (45) and (46) go over to bounded integrals. The diagonal elements, $C_{jj}$ (and $E_{jj}$), are absent from equations (45) and (46). Note also that the self-interaction of a ring vanishes as it should for the case of a rigid shift where $\delta_j = \Delta_j$.

2.6. Dynamics and Influence of “Center”

The dynamical equations (45) and (46) do not account for the part of the disk inside the innermost ring $r_1$. Also, there may be a massive black hole $M_{bh}$ near the galaxy center. We treat this central region separately as a point mass $M_0$,
\[
M_0 = M_{bh} + 2\pi \int_{r_1}^{r_1'} r \, dr \Sigma_d(r), \tag{47}
\]
where $r_1' \equiv r_1 - \delta r_1/2$.

The horizontal displacement of the “center” is
\[
\epsilon_0(t) = \epsilon_{ox} \hat{x} + \epsilon_{oy} \hat{y}. \tag{48}
\]
The equation of motion for $\epsilon_0$ taking into account the eccentric displacements of the rings is
\[
M_0 \left( \frac{d^2 \epsilon_0}{dt^2} + \Omega^2_0 \epsilon_0 \right) = - \sum_{k=1}^{N} F_{0k} (\delta_k - \frac{1}{2} \Delta_k), \tag{49}
\]
where
\[
\delta_0 \equiv \epsilon_{ox} - i \epsilon_{oy} = \epsilon_0 \exp(-i\varphi_0) \tag{50}
\]
is the complex displacement amplitude of the “center”; $F_{0k} \equiv GM_0 M_k/r_k^3$, $k = 1, \ldots, N$, are the tidal coefficients between the “center” and the rings; and
\[
\Omega^2_0 = \left( \frac{1}{r} \frac{\partial \Phi_0}{\partial r} \right) = \left[ \frac{1}{r} \left( \epsilon(\Phi_0 + \Phi_h) \right) \right]_0 \tag{51}
\]
is the angular oscillation frequency of a particle at the galaxy center. From § 2.1, we have $[\{1/r(\partial \Phi_0/\partial r)\}]_0 = GM_0/r_k^3$ and $[\{1/r(\partial \Phi_h/\partial r)\}]_0 = \epsilon_0/r_k$. For all of the considered conditions, we find $\Omega^2_0 > 0$.

The influence of the displaced “center” on the rings is included by adding the force terms, due to the “center’s” displacement, to the right-hand sides of equations (45) and (46),
\[
M_j (\delta_j + \cdots) = -2F_{0j} \epsilon_0 + \cdots, \tag{52}
\]
\[
M_j (\Delta_j + \cdots) = +F_{0j} \epsilon_0 + \cdots, \tag{53}
\]
where the ellipses denote terms in equations (45) and (46).

2.7. Energy Conservation

An energy constant of the motion of the ring system can be obtained by multiplying equation (52) by $\delta^*_j$ and equation (53) by $\Lambda^*_j$ (with $[\cdots]^*$ denoting the complex conjugate), adding the two equations, summing over $j$, and dividing by 2. (This factor of 2 makes the kinetic energy of a rigidly shifted ring with $\epsilon_j = \delta_j$ equal to $\frac{1}{2} M_j \dot{\epsilon}_j^* \dot{\epsilon}_j$.) Furthermore, we multiply equation (49) by $\epsilon^*_0$ and add the result to the previous sum. In this way we find $dE/dt = 0$, where
\[
E = \frac{1}{2} \sum_j M_j (\dot{\epsilon}_j^2 + \dot{\lambda}_j^2 + \dot{\delta}_j^2 + \dot{\Omega}_j^2 \delta_j^2 + 4\Omega_0^2 \epsilon_0^2) + \frac{1}{2} \sum_{k \neq j} \left[ C_{jk}(\epsilon_j - \epsilon_k)^2 + 2D_{jk}(\delta_j - \delta_k)^2 \right]
\]
\[
+ E_{jk}(\delta_j - \delta_k)^2 + \frac{1}{2} M_0 (\epsilon_0^2 + \Omega_0^2 \epsilon_0^2)
\]
\[
+ \sum_j F_{0j}(\epsilon_0 \cdot \dot{\epsilon}_j - \frac{1}{2} \epsilon_0 \cdot \dot{\delta}_j), \tag{54}
\]
and where the real vectors $\epsilon = \epsilon_\times \hat{x} + \epsilon_\times \hat{y}$ and $\delta = \delta_\times \hat{x} + \delta_\times \hat{y}$ are useful here.

2.8. Lagrangian

By inspection, we find the Lagrangian for the ring system $\mathcal{L}(\dot{\epsilon}_{jx}, \dot{\epsilon}_{jy}, \cdots)$,
\[
\mathcal{L} = \frac{1}{2} \sum_j M_j (\dot{\epsilon}_{jx}^2 + \dot{\lambda}_{jx}^2 + \dot{\delta}_{jx}^2 + \dot{\Omega}_{jx}^2 \delta_{jx}^2)
\]
\[
- 4\Omega_0^2 \epsilon_0 \cdot \dot{\lambda}_{jy} - 2\Omega_0 [\dot{\epsilon}_j - \delta_j] \times (\dot{\epsilon}_j - \delta_j) \cdot \hat{z}
\]
\[
+ \frac{1}{2} M_0 (\epsilon_0^2 + \Omega_0^2 \epsilon_0^2) \n\]
\[
+ \frac{1}{2} \sum_{k \neq j} \left[ C_{jk}(\epsilon_j - \epsilon_k)^2 + 2D_{jk}(\delta_j - \delta_k)^2 \right]
\]
\[
+ \sum_j F_{0j}(\epsilon_0 \cdot \dot{\epsilon}_j - \frac{1}{2} \epsilon_0 \cdot \dot{\delta}_j). \tag{55}
\]
Because $\partial \mathcal{L}/\partial t = 0$, the Hamiltonian,
\[
\mathcal{H} \equiv \sum_j \left( \dot{\epsilon}_{jx} \frac{\partial \mathcal{L}}{\partial \dot{\epsilon}_{jx}} + \cdots \right) - \mathcal{L}, \tag{56}
\]
is a constant of the motion. It is readily verified that $\mathcal{H} = E$.

We can make a canonical transformation $(\dot{\epsilon}_{jx}, \dot{\epsilon}_{jy}) \leftrightarrow (\epsilon_{jx}, \varphi_j)$, $(\delta_{jx}, \delta_{jy}) \leftrightarrow (\delta_{jx}, \psi_j)$ to obtain the Lagrangian as $\mathcal{L} = \mathcal{L}(\epsilon_{jx}, \delta_{jx}, \epsilon_{jy}, \delta_{jy})$. Note for example that $\dot{\epsilon}_j^2 \rightarrow \dot{\epsilon}_{jx}^2 + \dot{\epsilon}_{jy}^2 \delta_{jx}^2$. It is then easy to adopt the azimuthal symmetry of the equilibrium that $\mathcal{L}$ is invariant.
under the simultaneous changes \( \varphi_j \to \varphi_j + \theta, \psi_j \to \psi_j + \theta \) for \( j = 1, \ldots, N \), where \( \theta \) is an arbi-
trary angle. Thus
\[
\sum_j \left[ \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_j} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_j} \right] = 0 ,
\]
and consequently the total canonical angular momentum of
the ring system,
\[
\mathcal{P} \equiv \sum_j \left( \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_j} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}_j} \right) ,
\] (57)
is another constant of the motion. Evaluating equation (57)
gives
\[
\mathcal{P} = \frac{1}{2} \sum_j M J^2 \left( \varphi_j - \Omega_j \right) + \delta_j^2 (\psi_j - \Omega_j)
+ 2\Omega_j \epsilon_j \delta_j \cos (\varphi_j - \psi_j) + M_0 \epsilon_0^2 \phi_0 ,
\] (58)
where the last term represents the angular momentum of
the galaxy center. The last term within the square brackets
can also be written as \( 2\Omega_j (\epsilon_j \cdot \delta_j) \).

The constants of the motion \( \mathcal{H} \) and \( \mathcal{P} \) are valuable for
checking numerical integrations of the equations of motion
(52) and (53).

3. ECCENTRIC MOTION OF A SINGLE RING

Consider the eccentric motion of a particular ring with
the other rings not excited. This is not a self-consistent limit
because gravitational interactions will in general excite all of
the rings. However this limit is informative. With \( \epsilon_j \propto \Delta_j \propto \exp (-i\omega t), j = 1, \ldots, N, \) where \( \omega \) the ring precession
frequency, we get
\[
[(\omega - \Omega) + D] \epsilon + [2\Omega (\omega - \Omega) + d] \Delta = 0 ,
\]
\[
[2\Omega (\omega - \Omega) + d] \epsilon + [(\omega - \Omega)^2 + D] \Delta = 0 ,
\] (59)
where the \( j \) subscripts are implicit, \( E \equiv \Omega_j^2 - \Omega_j^2 - D_{jj}/M_j, \)
\( D \equiv \Omega_j^2 - \Omega_j^2 - D_{jj}/M_j, \) and \( d \equiv D_{jj}/M_j, \) With \( w \equiv \omega - \Omega, \) we get
\[
w^4 + (D + E - 4\Omega^2) w^2 - 4D\Omega w + ED - d^2 = 0 .
\] (60)

For the limit of many rings, the terms involving \( D \) become
negligible, and equation (60) can be readily solved to give
four real roots if \( 0 < ED < 4(\Omega^2 - D - E)^2/4 \).

Figure 3a shows the radial dependence of the four ring
precession frequencies \( \omega_j (\alpha = 1, \ldots, 4) \) corresponding to the
four modes of oscillation. These modes are the analogs of
the normal modes of vibration of a nonrotating system
(see Lovelace 1998).

Three of the modes in Figure 3a have positive frequencies
(\( \omega_j > 0 \) for \( \alpha = 2, 3, 4 \)) so that they have forward precession
with
\[
\varphi_j = \omega_j t + \text{const} , \quad \psi_j = \omega_j t + \text{const} .
\] (61)

These relations follow from equations (36) and (37) because
\( |\epsilon| = \epsilon \) and \( |\Delta| = \delta \) are constants. The other mode (\( \alpha = 1 \)) has
\( \omega_1 \approx 0 \) and therefore backward precession.

Figure 3b shows that the two modes \( \alpha = 2, 3 \) have \( \epsilon/\Delta \) small
compared to unity near the inner radius of the disk,
\( r_1 \). Note that \( \epsilon/\Delta = 0 \) corresponds to a pure azimuthal
shift of the ring matter without a shift of the ring center. The
mode \( \alpha = 1 \) with backward precession has \( \epsilon/\Delta \approx 1 \). As
mentioned, \( \epsilon/\Delta = 1 \) corresponds to a rigid shift of the ring
center.

Evaluation of the ring energy for the four modes using
equation (54) shows that the modes \( \alpha = 1, 2 \) have negative
energy whereas the modes \( \alpha = 3, 4 \) have positive energy.
The negative energy modes are unstable in the presence of
dissipation, for example, the force due to dynamical friction
(see, for example, Lovelace 1998).

Note that for vanishing ring mass \( (D \propto d \propto M_j \to 0) \),
the middle two roots approach \( \omega = \Omega \pm [DE(4\Omega^2 - E)]^{1/2} \).
Thus for \( M_j \to 0 \), there are only three different roots of
equation (60), \( \omega = \Omega \) and \( \omega = \Omega \pm (4\Omega^2 - E)^{1/2} \).

Outside of the central region of a galaxy we have \( \Omega \sim 1/r \).
Consequently, the radial dependences of the mode fre-
quencies \( \omega_j (r) \propto \pm 1/r \) will tend to “wrap up” an initially
coherent asymmetry into a tightly wrapped spiral (in the
absence of ring interactions). The forward precessing modes
(\( \alpha = 2, 3 \)) will give a trailing spiral wave, \( \varphi_2 \ll 1/r \) (with
respect to the azimuthal motion \( \psi_2 > 0 \)), whereas the back-
ward precessing mode (\( \alpha = 1 \)) will give a leading spiral wave,
\( \varphi_1 \ll -1/r \). The case of mode \( \alpha = 1 \) for noninteracting rings
was discussed earlier by Baldwin et al. (1980).

4. ECCENTRIC MOTION OF A “DISK” OF TWO RINGS

Here we consider the eccentric motion of a “disk” con-
sisting of two interacting rings, one of mass \( M_1 \) and radius
\( r_1 \), and the other of mass \( M_2 \), radius \( r_2 \). The values of \( \Omega_1^2, \Omega_2^2, \Omega_3^2, \Omega_4^2 \) \( (j = 1, 2) \) are given by equations (A11), (43), and
(44) with the bulge and halo potentials as given in \$ 2.1. Thus
the present treatment is self-consistent (in contrast to
the previous subsection). With $\delta_j$ and $\Lambda_j$ proportional to $\exp -i\omega t$, equations (45) and (46) give
\[
[-(\omega - \Omega_1)^2 + \Omega_1^2 - \Omega_2^2 + D_{11}]\delta_1
+ [2\Omega_1(\Omega_1 - \omega) - D_{11}]\Lambda_1
= [C_{12}(\delta_2 - \delta_1) + D_{12}(\Lambda_2 - \Lambda_1)]/M_1,
\]
\[
[-(\omega - \Omega_2)^2 + \Omega_1^2 - \Omega_2^2 + D_{22}]\delta_2
+ [2\Omega_2(\Omega_2 - \omega) - D_{22}]\Lambda_2
= [C_{12}(\delta_2 - \delta_2) + D_{12}(\Lambda_1 - \Lambda_2)]/M_2,
\]
\[
[-(\omega - \Omega_2)^2 + \Omega_2^2 - \Omega_2^2 + D_{22}]\delta_2
+ [2\Omega_2(\Omega_2 - \omega) - D_{22}]\Lambda_2
= [C_{12}(\delta_2 - \delta_2) + D_{12}(\Lambda_1 - \Lambda_2)]/M_2.
\]

For a nonzero solution, the determinant of the $4 \times 4$ matrix multiplying ($\delta_1$, $\Lambda_1$, $\delta_2$, $\Lambda_2$) must be zero. This leads to an eighth-order polynomial in $\omega$ which can be readily solved (with Maple V Release 5) for the frequencies of the eight modes ($\alpha = 1, \ldots, 8$).

Figure 4 shows the behavior, including instability, of a system of two rings of equal mass $M_1 = M_2$ so that the “disk” mass is $M = 2M_1$. For $M \to 0$, the modes $\alpha = 3, 4$ approach $\Omega_2$ and the modes $\alpha = 5, 6$ approach $\Omega_1$ which agrees with the behavior found in § 3. Note that for small $M_2$ modes $\alpha = 1 – 4$ are associated with ring 2 while modes $\alpha = 5 – 8$ are associated with ring 1. In the absence of interactions, the rings are stable independent of their mass. As the disk mass increases, the modes $\alpha = 4, 5$ approach each other and merge at $M_2 \approx 0.276 \times 10^{10} M_\odot$ and give instability with $\Im \omega = 0.03$ as shown in Figure 4b. We refer to this as the “first instability.” Notice that the onset of instability corresponds to a merging of the positive energy mode $\alpha = 4$ of ring 2 with the negative energy mode $\alpha = 5$ of ring 1 (see § 3). The interaction of positive and negative energy modes is a well-known instability mechanism (see, for example, Lovelace, Jure, & Haynes 1997). At the instability threshold, a dimensionless measure of the ring self-gravity is $GM_2/(r^2\Omega^2) \approx 0.1$, where $r = 4$ kpc and $\Omega \approx 0.0458/t_0$. The dependence of the growth rate is well fitted by $\omega t_0 \approx 0.000842(M_2 - M_{c1})^{0.65}$, with the masses in units of $10^{10} M_\odot$ and $t_0 = 10^9$ yr. For $M_2 > 3.12 \times 10^{10}$ $M_\odot$, there is a “second instability” with growth rate $\omega t_0 \approx 0.019(M_2 - M_{c2})^{0.55}$.

The ratios of the complex perturbation amplitudes can readily be obtained from equations (62)–(65) once the eight, possibly complex, frequencies are known. For the “first instability,” for example, for $M_2 = 10^{10} M_\odot$ and the same conditions as for Figure 4, we find $\delta \approx 0.254 - 0.011i$, $\Lambda \approx 1$ (by choice), $\delta_2 \approx 0.0108 - 0.365i$, and $\Lambda_2 \approx 0.673 + 1.30i$. These values correspond to $\phi_1 \approx 247^\circ$, $\phi_2 \approx 917^\circ$, $\psi_1 = 0$ (by choice), and $\psi_2 \approx 297^\circ$. Thus the azimuthal density enhancement in the outer ring trails the density enhancement of the inner ring. On the other hand, the radial shift of the outer ring leads the shift of the inner ring.

As a second example, for $M_2 = 4 \times 10^{10} M_\odot$ both the “first” and “second” instabilities occur. For the “first” instability, we again find that the azimuthal density enhancement of the outer ring trails that of the inner ring, whereas the radial shift of the outer ring leads that of the inner ring. For the “second” instability the situation is different in that both the azimuthal density enhancement and the radial shift of the outer ring trail those of the inner ring. Specifically, we find $\phi_1 \approx 187^\circ$, $\phi_2 \approx 89^\circ$, $\psi_1 = 0$ (by choice), and $\psi_2 \approx 283^\circ$. Note that for both rings, the angle of the radial shift is roughly 180° displaced from the azimuthal density enhancement. The displacement of the center of mass of the ring (eq. [20]) is dominated by the azimuthal displacement $|\delta_1| \approx 0.70|\Lambda_1|$ and $|\delta_2| \approx 0.67|\Lambda_2| \approx 0.22|\Lambda_1|$. Thus having $\phi$ and $\psi$ about 180° out of phase allows the center of mass of each ring to be closer to the origin, and this is a lower energy configuration.

Consider now the angular momentum of the perturbed two ring system which from equation (58) is $\mathcal{P}_\phi = \mathcal{P}_{\phi 1} + \mathcal{P}_{\phi 2} = \mathcal{P}_{\phi j}$. The evaluation of $\mathcal{P}_\phi$ reveals that for each of the eight modes, $\Lambda_j = -i\omega_1 \Delta_j$ and $\delta_j = -i\omega_1 \delta_j$, where
\[ x = 1, \ldots, 8 \] labels the mode. This implies that
\[ \omega_x = \psi_j + i(\delta_j/\delta_x) = \dot{\phi}_j + i(\epsilon_j/\epsilon_x). \] (66)

Thus, for the “first instability,” for \( M_d = 10^{10} M_\odot \) where \( \omega t_0 \approx 0.0500 + 0.00719i \), we have \( \psi_j = \dot{\phi}_j = 0.05/t_0 \) (which corresponds to forward precession) and \( \delta_j/\delta_x = \epsilon_j/\epsilon_x = 0.00719/t_0 \). For an unstable mode, the coefficients of the six terms in \( \mathcal{P}_\phi \) are all grow exponentially. The only possible way in which \( \mathcal{P}_\phi = \text{const} \) can be maintained is to have \( \mathcal{P}_\phi = 0 = \mathcal{P}_\phi_1 + \mathcal{P}_\phi_2 \). This relation provides a useful check on the correctness of the calculations.

For the “first instability” (\( M_d > M_{c1} \)), we find by evaluating equation (58) that \( \mathcal{P}_\phi = -\mathcal{P}_\phi_2 > 0 \). This means that the angular momentum of the inner ring increases while that of the outer ring decreases. Thus the instability acts to transfer angular momentum inward. In contrast, for the “second instability” (\( M_d > M_{c2} \)), we find that \( \mathcal{P}_\phi = -\mathcal{P}_\phi_2 < 0 \). Thus, the “second instability” acts to transfer angular momentum outward. If \( \mathcal{P}_\phi \) decreases and \( \mathcal{P}_\phi_2 \) increases, then the average radius of ring 1 must decrease and that of ring 2 must increase. Therefore, the “second instability” may be important for the accretion of matter in a gravitating disk.

5. ECCENTRIC MOTION OF ONE RING AND “CENTER”

Here we consider the eccentric motion of a “disk” of one ring including the influence of the eccentric motion of the “center” \( M_0 \) which is located at \( r = 0 \) in equilibrium. The mass \( M_0 \) includes the mass of a central black hole \( M_{bh} \) if it is present. The ring perturbation is described by \( \delta(t) \) and \( \Delta(t) \) as given by equations (52) and (53), while the “center” is described by \( \delta_0(t) \) which is given by equation (49). With the perturbations \( \propto \exp(-i\omega t) \), we find
\[
[(\omega - \Omega^2)^2 + E]\delta + [2\Omega(\omega - \Omega) + d]\Delta = 2\Omega^2 \delta_0, \\
[2\Omega(\omega - \Omega) + d]\delta + [(\omega - \Omega^2) + D]\Delta = \Omega^2 \delta_0, \\
(\omega^2 - \Omega^2)\delta_0 = \Omega^2 \left( \delta - \frac{\Delta}{2} \right),
\] (67)

where \( \Omega^2 \equiv GM_0/r_0^3 \) and \( \Omega^2 \equiv GM_1/r_1^3 \), with \( M_1 \) and \( r_1 \) the mass and radius of the ring. For a non-zero solution, the determinant of the \( 3 \times 3 \) matrix multiplying \( (\delta, \Delta, \delta_0) \) must be zero. This leads to a sixth-order polynomial in \( \omega \) or \( w = \omega - \Omega \) which can readily be solved (with Maple V Release 5) for the frequencies of the six modes \( \omega_x (x = 1, \ldots, 6) \). We obtain
\[
[(w + \Omega^2)^2 - \Omega^2]((w^2 + 2D + E - 2\Omega w + d)^2) \\
- \Omega^2 \left( 5 \frac{w^2}{2} + 2D + \frac{E}{2} + 2\Omega w + d \right) = 0,
\] (68)

where the strength of the interaction between the ring and the “center” is measured by
\[
\Omega^2_{ab} = \Omega_a \Omega_b = \frac{G(M_0 M_1)}{r_1^3}.
\] (69)

Here \( D, E, d, \) and \( \Omega = \Omega_1 \) are defined in equation (59), and \( \Omega_0 \) is defined in equation (51).

Figure 5 shows the dependence of the growth rate \( \omega = \Im(\omega_x) \) on the mass of the “center” \( M_0 \) with the mass of the ring held fixed. The associated real part of the frequency is positive. The onset of instability corresponds to the point where two of the six modes with frequencies given by equation (68) merge. The merging is again of positive and negative energy modes. The growth rate has to a good approximation the dependence \( \omega t_0 \approx 0.076(M_0 - 2.61)^{1/2} \), with \( M_0 \) in units of \( 10^9 M_\odot \) and \( t_0 = 10^6 \) yr.

The ratios of the complex amplitudes follow from equation (67), and for the unstable mode for \( M_0 = 2.75 \times 10^9 M_\odot \) we find \( \omega t_0 \approx 0.155 + 0.00289i \), \( \Omega t_0 \approx 0.131 \), \( \Omega t_0 \approx 0.142 \), \( \Omega t_0 \approx 0.0465 \), \( \delta \approx -0.817 + 0.033i \), \( \Delta = 1 \) (by choice), and \( \delta_0 \approx -3.38 + 0.916i \), which give \( |\delta| \approx 0.817 \), \( |\delta_0| \approx 3.51 \), \( \varphi \approx 182^\circ \), and \( \varphi_0 \approx 195^\circ \), where \( t_0 = 10^6 \) yr. Thus, the radial shift of the ring is roughly 180° away from the maximum of the density enhancement which has \( \psi = 0 \) (by choice). The shift of the center of mass of the ring (eq. [20]) is dominated by the azimuthal density enhancement. Thus the radial shift and azimuthal displacements are such that the center of mass of the ring moves closer to the origin which is a lower energy configuration. Note that the radial shift of the “center” trails the ring center of mass by an angle 360° - \( \varphi_0 \approx 165^\circ \). Hence, the torque of the “center” on the ring acts to reduce the angular momentum of the ring as verified below. At the same time the ring acts to increase the angular momentum of the “center.”

Consider now the angular momentum of the perturbed ring plus “center” system which is given by equation (58). Following the arguments of the prior section, the sum of the angular momentum of the “center” and that of the ring must be zero for a growing mode. The angular momentum of the “center” is simply \( M_0 \epsilon_0 \dot{\phi}_0 \). For a pure mode, we have \( \text{Re}(\omega) = \dot{\phi}_0 \). As mentioned, the real part of the frequency is positive for the unstable mode, and therefore the angular momentum of the “center” increases while the angular momentum of the ring decreases. Thus, there is a transfer of angular momentum from the ring to the “center.” Because of the loss of angular momentum the average radius of the ring will decrease. Thus the instability may be important for accretion of matter to the galaxy center.

![Figure 5](image-url)
6. ECCENTRIC MOTION OF N RINGS

For the results presented here, the rings are taken to be uniformly spaced in \( r \) with radii \( r_j = 1 + (j - 1)\delta r \) kpc with \( \delta r = 0.5 \) kpc and with \( j = 1, \ldots, N = 31 \). The value \( N = 31 \) gives good spatial resolution over all but the inner part of the disk. The outer radius \( r_N \) (in the range, say, 10–20 kpc) has little influence on the eccentric motion described here, as verified by comparing results with \( r_N = 16 \) kpc those obtained with significantly larger \( r_N \). Also, the eccentric motion of the outer disk, say, \( r \geq 3 \) kpc, is essentially independent of \( \delta r \). We first consider in § 6.1 the case where the ring masses correspond to the exponential distribution discussed in § 2.1. The inner part of the disk is found to be strongly unstable to eccentric motions, and therefore in § 6.2 we consider a disk with the mass of the innermost three rings reduced. In § 6.3 we consider disks with a smooth reduction in \( \Sigma(r) \) in the inner part of the disk, \( r \leq r_J \).

We solve equations (52) and (53) numerically as eight first-order equations for \( \epsilon_{xj}, \epsilon_{xp}, \epsilon_{yj}, and \delta_{xj}, \delta_{yj}, and \delta_{xy}, j = 1, \ldots, N \). At the same time, we solve the two additional equations,

\[
\begin{align*}
\frac{d\varphi_j}{dt} & = \frac{\epsilon_{xj} \hat{\epsilon}_{xy} - \epsilon_{yj} \hat{\epsilon}_{xy}}{\epsilon_{xj}^2 + \epsilon_{yj}^2}, \\
\frac{d\psi_j}{dt} & = \frac{\delta_{xj} \hat{\delta}_{xy} - \delta_{yj} \hat{\delta}_{xy}}{\delta_{xj}^2 + \delta_{yj}^2},
\end{align*}
\]

(70) (71)

to give \( \varphi_j(t) \), which is the angle to the maximum of the radial shift, and \( \psi_j(t) \), which is the angle to the maximum of the azimuthal density enhancement. These angles are analogous to the line-of-nodes angles for the tilting of the rings of a disk galaxy (Lovelace 1998). Thus, we solve 10N first-order equations. In all cases, the total energy (eq. [54]) and total canonical angular momentum (eq. [58]) are accurately conserved. The different frequencies \( \Omega_j, \Omega_{yj}, and \Omega_{xy} \), and the tidal coefficients \( \{C_{jk}\} \), etc., are evaluated using the equations of the Appendix.

6.1. Exponential Disk

Here we consider the eccentric motion of the rings for the case where \( M_j = 2\pi r_J \delta r \Sigma_j(r) \) with \( \Sigma_j(r) = \Sigma_0 \exp(-r/r_0) \).

The mass of the center is assumed given by equation (47), which gives \( M_0 \approx 1.06 \times 10^9 M_\odot \) for the parameters of Figure 1. Alternatively, this value of \( M_0 \) could be due in part to a central black hole.

Figure 6 shows the dependences of the radial shifts \( \epsilon_j(\varphi_j) \) and azimuthal displacements \( \delta_j(\psi_j) \) of the rings (\( j = 1–31 \)) and the radial shift of the “center” \( \epsilon_0(\varphi_0) \) at a short time, 100 Myr after an initial perturbation. This type of plot is related to the plots emphasized by Briggs (1990) for characterizing the warps of galactic disks (see also Lovelace 1998). The angles \( \varphi \) and \( \psi \) are analogous to the line-of-nodes angle the warp.

From Figure 6 note that the azimuthal displacements \( \delta_j \) are larger than the radial displacements \( \epsilon_j \), so that the displacement of the center of mass of a ring is dominated by \( \delta_j \). The eccentric motion of the inner rings, say, \( r_1 = 1 \) to \( r_2 = 2.5 \) kpc, show the most rapid, exponential growth. For these rings the angles \( \psi_j \) and \( \varphi_j \) are approximately 180° out of phase, and this agrees with the behavior found for the “second” instability of two rings discussed in § 4. As mentioned, this allows the center of mass of each ring to move closer to the origin, which is a lower energy configuration. Note that with increasing \( j \), both \( \varphi_j \) and \( \psi_j \) decrease for the inner rings, which corresponds to a trailing pattern the same as found for the second instability of two rings. Note also that the shift of the “center” \( \epsilon_0 \) trails the shift of the center of mass of the \( j = 1 \) ring, in agreement with § 5. Thus the torque of the first ring on the center acts to increase the angular momentum of the center while the torque of the center on the first ring decreases the rings angular momentum.

Figure 7 shows the exponential growth of the azimuthal displacement \( \epsilon_0 \) and radial shift \( \epsilon_1 \) of the first ring and the simultaneous growth of the radial shift of the center \( \epsilon_0 \). The e-folding time is about 29 Myr. For comparison, the period

\[
\text{Fig. 6.—Polar plot of the radial shift } \epsilon_j \text{ and azimuthal displacement } \delta_j \text{ of ring matter as a function of the angles } \varphi_j \text{ and } \psi_j \text{ (}\ j = 1–31\text{)} \text{ at time } t = 100 \text{ Myr}. \text{ The radial shift of the “center” } \epsilon_0 \text{ is indicated by the solid dot. The conditions correspond to the galaxy parameters of Fig. 1. The rings have radii } r_j = 1 + \delta r (j - 1) \text{ kpc and } \delta r = 0.5 \text{ kpc for } j = 1, \ldots, 31, \text{ masses } M_j = 2\pi r_J \delta r \Sigma_j(r) \text{ with } \Sigma_j \text{ given in § 2.1 and } M_0 \text{ given by eq. (47), which gives } M_0 \approx 1.06 \times 10^9 M_\odot. \text{ The initial values of the shifts and displacements are } \epsilon_j = 0.1 \epsilon_0 \text{ and } \psi_j = 0 = \varphi_j \text{ and } \epsilon_0 = 10^{-6} \text{ and } \epsilon_0 = 0. \text{ The units of } \epsilon_j \text{ and } \delta_j \text{ are arbitrary in that the equations are linear.}
\]

\[
\text{Fig. 7.—The plot shows the exponential growth of center shift } \epsilon_0 \text{ and radial shift } \epsilon_1 \text{ and azimuthal displacement } \delta_1 \text{ of the first ring at } r_1 = 1 \text{ kpc for the same conditions as for Fig. 6.}
\]
of oscillation of the center is \( T_0 = \frac{2\pi}{\Omega_0} \approx 46 \ \text{Myr} \) for the conditions shown, where \( \Omega_0 \) is given by equation (51). The growth of the eccentric motion of the inner rings is reduced somewhat if the mass of the center is reduced to \( M_0 = 10^8 M_\odot \); the e-folding time for in this case is about 38 Myr for ring 1. Figure 8 shows the perturbations of the angular momentum of the center \( P_0 \) and the rings \( P_j \) at \( t = 100 \ \text{Myr} \) obtained from equation (58). In agreement with § 5 and the above-mentioned direction of the torque, the angular momentum of the center increases while that of the first and second rings decrease. The decrease in angular momenta of these rings will result in their radii shrinking. Note that the center rotates in the same direction as the disk matter.

The present linear theory does not address the issue of saturation of growth of the eccentric motion. One possibility is that the strong instability of the inner rings of the disk leads to the destruction of this part of the disk.

6.2. Exponential Disk with Rings 1–3 Reduced

Here we consider the eccentric motion of the rings for the case where the ring masses are the same as in § 6.1 except that \( M_1 \to 10^{-2} M_1, M_2 \to 0.1 M_2, \) and \( M_3 \to 0.3 M_3, \) which are the rings with radii \( r_1 = 1, r_2 = 1.5, \) and \( r_3 = 2 \ \text{kpc}. \) The mass of the center is the same as in § 6.1, \( M_0 \approx 1.06 \times 10^9 M_\odot. \) The disk mass is reduced by a factor \( 0.84 \) compared with an exponential disk. The aim of reducing \( M_1, M_2, \) and \( M_3 \) is to reduce the growth rate of the eccentric motion of this part of the disk.

Figure 9 shows the essential behavior in a polar plot of the displacements and shifts at two times. The radial shift of the center is negligible on the scale of this figure. The e-folding time for ring 3 is about 49 Myr. Notice that the curves traced out by \( \delta_j \) and by \( \epsilon_j \) are approximated straight lines from the origin which rotate rigidly in the direction of motion of the matter for \( j = 4 \) to about \( j = 11, \) which corresponds to \( r_3 = 2.5 \) to \( r_{11} = 6 \ \text{kpc}. \) The instantaneous period of rotation of this pattern is \( \approx 60 \ \text{Myr}, \) which is longer that the oscillation period at the center, \( T_0 \approx 46 \ \text{Myr}. \) This case is an example of the phase locking of the eccentric motion of these rings due to the self-gravity between the rings. This phase locking is analogous to that which occurs in the tilting motion of the rings representing a disk galaxy due to self-gravity (Lovelace 1998). In the case of tilting of rings the phase locking results in the line-of-node angles of the rings in the inner part of the disk becoming the same.

Also in this case the growth of the eccentric instability of the inner rings is sufficiently fast that it probably further disrupts the inner part of the disk.

6.3. Reduced Inner Disk

Here we study the eccentric motion of a disk the inner part of which is attenuated relative to an exponential disk. Specifically, the rings masses are \( M_j = 2\pi r_j \delta r \Sigma_d(r), \) where

\[
\dot{\Sigma}_d(r) = \Sigma_{d0} \exp \left( -\beta \frac{r}{\sigma} \right) \exp \left( -\frac{r}{r_d} \right),
\]

with \( \beta = \text{const}. \) We consider \( \beta = 1. \) The mass of the center is the same as in § 6.1, \( M_0 \approx 1.06 \times 10^9 M_\odot. \) However, the motion of the center is negligible and value \( M_0 \) has little influence on the eccentric motion of the disk described below. The disk mass is smaller than for an exponential disk by a factor \( \approx 0.47. \)

Figure 10 shows the essential behavior in a polar plot of the shifts and displacements at two times. The shift of the center is negligible on the scale of the plot. The magnitude of azimuthal displacement of the outer ring \( \delta_N \) exhibits an approximately linear growth with time, \( \delta_N \approx \text{const} + t/660 \ \text{Myr}, \) for the considered initial conditions and \( t \leq 1 \ \text{Gyr}. \) In contrast, the magnitude of the radial shifts \( \epsilon_j \) remains bounded by its initial maximum value \( \epsilon_j(t = 0). \)

The patterns formed by both \( \delta_j \) and \( \epsilon_j \) in Figure 10 are trailing spirals. This is different from the proposal of Baldwin et al. (1980) that leading spirals should form. The rotation of the outer point on the spiral \( \Delta_{\text{out}}(r_N = 16 \ \text{kpc}) \) in Figure 10 is in the direction of rotation of the disk matter.
First, we have a plot for an exponential disk. The momentum of the central mass is negligible. This figure in ring angular momenta. The perturbation of the angular momentum of the central mass is negligible on the scale of this plot. The rings, the center, and the initial values of the shifts and displacements are the same as in Fig. 6 except that the ring masses are obtained from eq. (72). The mass of the center is the same as in Fig. 6, \( M_0 \approx 1.06 \times 10^9 M_\odot \). Thus the conditions correspond to the galaxy parameters of Fig. 1 except that the disk mass is reduced to \( \approx 2.82 \times 10^{10} M_\odot \). The shifts and displacements of the rings \( j = 1, 2 \) are dynamically unimportant and are not shown.

Its pattern speed \( \Omega_p \) corresponds to a period \( 2\pi/\Omega_p \approx 460 \) Myr, which is a factor \( \approx 1.24 \) longer than the rotation period of matter at this radius (\( \approx 370 \) Myr). The pattern period of, say, \( \Delta_1(r = 8 \text{ kpc}) \approx 188 \) Myr, which is a less than the rotation period of the matter at this radius (\( \approx 206 \) Myr). Thus, it is evident that the spiral is “wrapping up” as time increases. At the same time, the spiral pattern propagates radially outward. The outward speed is about 10 km \( s^{-1} \) at \( r \sim 8 \) kpc for \( t \sim 300 \) Myr.

Figure 11 shows a polar plot of the radius to the maximum of the azimuthal displacement \( r(\psi_j) \) at two times. The curve is a trailing spiral with an approximate fit given by

\[
\psi = A(t) \exp \left[ -\frac{r}{a(t)} \right], \tag{73}
\]

where \( A \approx 0.065 \) (t/Myr) rad, and \( a \approx 7.0 + 0.0044 \) (t/Myr) kpc for \( t \leq 1 \) Gyr. Thus the radial spacing between spiral arms is \( \lambda_r \approx (2\pi a/A) \exp (r/a) \) for \( \lambda_r \ll a \). For validity of the ring representation, we must have \( \lambda_r \geq 2\delta r = 1 \) kpc, where \( \delta r \) the separation between rings.

Figure 12 shows the radial variation of the perturbations in ring angular momenta. The perturbation of the angular momentum of the central mass is negligible. This figure should be compared with Figure 8, which gives the same plot for an exponential disk.

Some simplification of equations is possible in the present case at long times due to the fact that \( \delta_j \gg \epsilon_j \). First, we have \( \delta v \approx \delta v_\phi \phi \) with

\[
\delta v_\phi(r, \phi) \approx -(\delta_x + \Omega \delta_j) \sin \phi + (\delta_y - \Omega \delta_j) \cos \phi. \tag{74}
\]

Second,

\[
\frac{\delta \Sigma(r, \phi)}{\Sigma(r)} \approx \frac{1}{r} (\delta_x \cos \phi + \delta_y \sin \phi), \tag{75}
\]

for \( \delta_j \gg r \epsilon_j / r \).

Figure 13 shows the profiles along the x-axis through the galaxy center of the fractional change in the surface density \( \delta \Sigma / \Sigma \) and the change in the azimuthal velocity \( \delta v_\phi \) obtained from equations (74) and (75). The opposite signs of \( \delta v_\phi \) on the two sides of the galaxy would of course make the rotation curves on the two sides different, as observed in some cases (Swaters et al. 1998). Note that in some regions the changes \( \delta \Sigma \) and \( \delta v_\phi \) are correlated and in other regions they are anticorrelated. Figure 14 shows two-dimensional appearance of the fractional surface density variations from equation (75).

For long times \( t > 1 \) Gyr, the azimuthal displacements and shifts of the inner rings (2 and 3) start to become large compared with the values in the outer disk (\( r > 4 \) kpc), even...
though these rings have very small masses. At the same time, the displacement of the center, which has mass $M_0 = 1.06 \times 10^9 M_\odot$, grows, and at $t = 1$ Gyr it is $\epsilon_0 \approx 0.057$ for the conditions of Figures 9–13. If the mass of the center is $M_0 = 10^6 M_\odot$, then the displacements and shifts of rings 2 and 3 at $t = 1$ Gyr are significantly reduced as is the shift of the center which is $\epsilon_0 \approx 0.014$.

7. CONCLUSIONS

The paper develops a theory of eccentric $(m = \pm 1)$ linearized perturbations of an axisymmetric disk galaxy residing in a spherical dark matter halo and with a spherical bulge component. The disk is represented by a large but finite number $N$ of rings with shifted centers and with perturbed azimuthal matter distributions. This description is appropriate for a disk with small “thermal” velocity spread $v_{th}$, where the matter is in approximately laminar circular motion. The spread for a thin disk has $(v_{th}/v_\theta)^2 < 1$, but it is sufficient to give a Toomre $Q(r) \gtrsim 1$. Earlier, Baldwin et al. (1980) discussed asymmetries in disk galaxies in terms of shifted rings but without interactions between the rings and without azimuthal displacements of the ring matter. Account is taken of the shift of the matter at the galaxy's center, which may include a massive black hole. The gravitational interactions between the rings and between the rings and the center is fully accounted for, but the halo and bulge components are treated as passive gravitational field sources. Equations of motion are derived for the ring and the center, and from these we obtain the Lagrangian for the rings + center system. For this system we derive an energy constant of the motion and a total canonical angular momentum constant of the motion.

![Diagram](image-url)
We first discuss the nature of the precession of a single ring with the other rings fixed; this case, although not self-consistent, is informative. There are four modes, analogs to the normal modes of a nonrotating system, and two have negative energy and two positive energy. Negative energy modes are unstable in the presence of dissipation such as that due to dynamical friction. We go on to study the eccentric motion of a disk consisting of two rings of different radii but equal mass $M_d/2$. Above a threshold value of $M_d$ the two rings are unstable with instability due merging of positive and negative energy modes. This result is obtained by solving the eighth-order polynomial for the frequencies of the eight modes. Above a second, somewhat larger threshold value of $M_d$, a second instability appears, and in this case the ring motion is such that the angular momentum of the inner ring decreases while that of the outer ring increases. For the unstable motion, the maximum of the azimuthal density enhancement of a ring occurs at an angle about $180^\circ$ from the direction of the radial shift. This allows the center of mass of the ring to move closer to the center of mass of the other ring and to the origin.

We also analyze the eccentric motion of a disk of one ring interacting with a radially shifted central mass. This system has six modes, the frequencies of which are obtained by solving a sixth-order polynomial. In this case, instability sets in above a threshold value of the central mass (for a fixed ring mass), and it acts to increase the angular momentum of the central mass (which therefore rotates in the direction of the disk matter), while decreasing the angular momentum of the ring. The instability is again due to the merging of positive and negative energy modes.

We study the eccentric dynamics of a disk with an exponential surface density distribution represented by a large number $N = 31$ of rings and a central mass $M_0 \sim 10^9 M_\odot$ which may include the mass of a black hole. The outer radius of the disk is $r_N = 16$ kpc; we have checked that this value has negligible affect on the reported results. In this case, we numerically integrate the equations of motion. A check on the validity of the integrations is provided by monitoring the mentioned total energy and total canonical angular momentum, which are found to be accurately constant in all presented results. The inner part of the disk $r \lesssim 2.5$ kpc is found to be strongly unstable with $\varepsilon$-folding time $\sim 30$ Myr for the conditions considered. The $\varepsilon$-folding time is somewhat longer if $M_0 = 0$. Angular momentum of the rings is transferred outward and to the central mass if it is present. A trailing one-armed spiral wave is formed in the disk. This differs from the prediction of Baldwin et al. (1980) of a leading one-armed spiral. The outer part of the disk $r \gtrsim r_d$ is stable, and in this region the angular momentum is transported by the wave. Thus our results appear compatible with the theorem of Goldreich & Nicholson (1989) regarding angular momentum in stable rotating fluids. The instability found here appears qualitatively similar to that found by Taga & Iye (1998b) for a fluid Kuzmin disk with surface density $\Sigma \propto 1/(1 + r)^{3/2}$ with a point mass at the center where unstable trailing one-armed spiral waves are found.

The present linear theory does not address the issue of saturation of growth of the eccentric motion. One possibility is that the strong instability of the inner rings of the disk leads to the destruction of this part of the disk. For this reason we have studied a disk with a modified exponential density distribution where the surface density of the inner part of the disk is reduced. However, the mass of the center of the galaxy was kept the same as in the case of an exponential disk, $M_0 \sim 10^9 M_\odot$. In this case we find much slower, linear—as opposed to exponential—growth of the eccentric motion of the disk for times $t \lesssim 1$ Gyr. A trailing one-armed spiral wave forms in the disk and becomes more tightly wrapped as time increases. Angular momentum is transferred outward. The motion of the central mass if present is small compared with that of the disk for $t \lesssim 1$ Gyr.

For long times $t > 1$ Gyr, the azimuthal displacements and shifts of the inner rings start to become large compared with the values in the outer disk. At the same time, the radial shift of the center grows. This shift is significantly reduced if the mass of the center is changed from $\sim 10^9$ to $10^8 M_\odot$.

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APPENDIX

TIDAL COEFFICIENTS

For $|r_j - r_k| \gg (\Delta r_j \Delta r_k)^{1/2}$, the ring profiles can be treated as delta functions, $S(r | r_j) \to \delta(r - r_j)/r$, and consequently the "tidal coefficients" of equations (29)–(32) can be simplified to give

$$C_{jk} \approx GM_j M_k \frac{\partial^2 \mathcal{X} (r_j, r_k)}{\partial r_j \partial r_k} = -\frac{2GM_j M_k}{\pi (r_j + r_k)(r_j - r_k)^2} E(k_{jk}),$$

(A1)

$$D_{jk} \approx GM_j M_k \frac{\partial \mathcal{X} (r_j, r_k)}{r_k \partial r_j} = -\frac{GM_j M_k}{\pi r_j^2 (r_j - r_k)} \left[ (r_j + r_k)E(k_{jk}) + (r_j - r_k)K(k_{jk}) \right],$$

(A2)

$$D'_{jk} \approx GM_j M_k \frac{\partial \mathcal{X} (r_j, r_k)}{r_j \partial r_k} = D_{kj},$$

(A3)
\[ E_{jk} \approx GM_j M_k \frac{\mathcal{X}(r_j, r_k)}{r_j r_k} = \frac{GM_j M_k}{r_j^2 r_k} [(r_j^2 + r_k^2)K(k_j) - (r_j + r_k)^2 E(k_j)] \]  

(A4)

(with Mathematica, Version 3), where \( k_j \equiv 2(r_j r_k)/(r_j + r_k) \), where \( \mathcal{X} \) is defined by equation (31), and where

\[
K(k) \equiv \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad E(k) \equiv \int_0^{\pi/2} d\phi \sqrt{1 - k^2 \sin^2 \phi}
\]

are complete elliptic integrals of the first and second kinds, respectively. Note that for \( r_k \gg r_j \),

\[
C_{jk} \approx -GM_j M_k/r_k^3, \quad D_{jk} \approx GM_j M_k/(2r_k^3),
\]

\[
D'_{jk} \approx -GM_j M_k/r_k^3, \quad E_{jk} \approx GM_j M_k/(2r_k^3).
\]

In the opposite limit where \( |r_j - r_k| \ll (r_j r_k)^{1/2} \), equations (27)–(30) can be evaluated approximately as

\[
C_{jk} \approx \frac{GM_j M_k}{8\sqrt{\pi}(\Delta r)^{3}} \int_{-\infty}^{\infty} dy \exp \left( -\frac{y^2}{4} \right) \left\{ 2\bar{r} - (r_j - r_j + y \Delta r) \ln \frac{8\bar{r}}{|r_j - r_j + y \Delta r|} \right\},
\]

\[
\approx \frac{GM_j M_k}{2\pi(\Delta r)^{3}} \left[ 1 - \sqrt{\pi} \frac{2}{u} \exp \left( -\frac{u^2}{4} \right) \text{erfi} \left( \frac{u}{2} \right) + O \left( \frac{\Delta r}{\bar{r}} \right) \right], \quad (A5)
\]

where

\[
\bar{r} \equiv (r_j + r_j)/2, \quad \Delta r \equiv [(\Delta r_j^2 + \Delta r_k^2)/2]^{1/2}, \quad u \equiv (r_j - r_j)/\Delta r,
\]

\[
\text{erfi}(x) \equiv \text{erf}(ix)/i = (2/\sqrt{\pi}) \int_0^x dy \exp (y^2).
\]

The integral in equation (A5) is a principal value integral of the form occurring in the plasma dispersion function \( W \) (Ichimaru 1973). Also,

\[
D_{jk} \approx \frac{GM_j M_k}{4\sqrt{\pi(\Delta r)^3}} \int_{-\infty}^{\infty} dy \exp \left( -\frac{y^2}{4} \right) \left\{ 2\bar{r} - (r_j - r_j + y \Delta r) \ln \frac{8\bar{r}/(\Delta r)}{|u + y|} \right\},
\]

\[
\approx -\frac{GM_j M_k}{4\sqrt{\pi(\Delta r)^3}} \left\{ \int_{-\infty}^{\infty} dy \exp \left( -\frac{y^2}{4} \right) \ln \frac{8\bar{r}/(\Delta r)}{|u + y|} - 2\pi \left( \frac{\bar{r}}{\Delta r} \right) \exp \left( -\frac{u^2}{4} \right) \text{erfi} \left( \frac{u}{2} \right) \right\}, \quad (A6)
\]

and

\[
E_{jk} \approx \frac{GM_j M_k}{2\sqrt{\pi(\Delta r)^3}} \int_{-\infty}^{\infty} dy \exp \left( -\frac{y^2}{4} \right) \ln \left( \frac{1.0827(\bar{r}/(\Delta r))}{|u + y|} \right). \quad (A7)
\]

From these expressions we obtain

\[
C_{jj} \approx \frac{GM_j^2}{2\pi r_j (\Delta r_j)^3}, \quad D_{jj} \approx -\frac{GM_j^2}{2\pi r_j^3} \ln \left( \frac{10.68 r_j}{\Delta r_j} \right), \quad E_{jj} \approx \frac{GM_j^2}{\pi r_j^3} \ln \left( \frac{1.445 r_j}{\Delta r_j} \right). \quad (A8)
\]

Equations (A1)–(A8) are valuable for numerical evaluation of the tidal coefficients.

In the following we derive some useful relations involving the tidal coefficients. From equation (12) we have

\[
\Omega_d^2(r) = \frac{1}{r} \frac{\partial \Phi_d}{\partial r} = 2\pi G \int_0^{\infty} \frac{r'}{d'} \delta \Sigma_d(r') \int_0^\infty dr' \frac{(1 - r' \cos \Psi/r)}{[r'^2 + (r')^2 - 2rr' \cos \Psi]^3/2}. \quad (A9)
\]

The \( \Psi \) integral in this case is

\[
\frac{1}{\pi r^4 (r^2 - (r')^2)} \left[ (r + r') E + (r - r') K \right]. \quad (A10)
\]

Comparison with equation (A2) shows that shows that

\[
\Omega_d^2 \equiv \int_0^\infty r dr S(r) \Omega_d^2(r) = -\frac{1}{M_j} \sum_{k=1}^N D_{jk}. \quad (A11)
\]

This expression does not include the disk mass within the inner ring. As discussed in § 2.6, this part of the disk is treated as a point mass \( M_0 \) with unperturbed position \( r = 0 \). If there is a central black hole \( M_{bh} \), its mass is included in \( M_0 \). To account for the influence of \( M_0 \), we simply add the term \( GM_0/r_j^3 \) to the right-hand side of equation (A11). The resulting expression for \( \Omega_{dij} \) is useful for the numerical calculations. For \( N \gtrsim 30 \) and \( r_j = 1 \) to 10–20 kpc, we find that equation (A11) gives accurate agreement with the analytic expression (3).
An alternative expression for $\Omega_2^2$ can be obtained by integration by parts,

\[
\Omega_2^2(r) = -\frac{2\pi G}{r} \int_0^\infty r' dr' \frac{\partial \Sigma}{\partial r} \mathcal{X}(r, r') .
\]  

(A12)

Thus

\[
M \left( \frac{\partial \Omega_2}{\partial r} \right)_j = \int r dr S(r | r_j) \frac{\partial \Omega_2}{\partial r} = GM \sum_k M_k \int r dr' dr' S(r | r_j) \frac{\partial S(r' | r_j)}{\partial r} \mathcal{X}(r, r') - \frac{\partial \mathcal{X}(r, r')}{\partial r} ,
\]

or

\[
M \left( \frac{\partial \Omega_2}{\partial r} \right)_j = GM \sum_k M_k \int r dr' dr' S(r | r_j) \left[ \frac{1}{r} \frac{\partial [r' S(r' | r_j)]}{\partial r'} - \frac{S(r' | r_j)}{r} \right] \mathcal{X}(r, r') - \frac{\partial \mathcal{X}(r, r')}{\partial r} .
\]  

(A13)

In view of equations (27)–(30), this equation can be written as

\[
M \left( \frac{\partial \Omega_2}{\partial r} \right)_j = \sum_k (C_{jk} + D_{jk}) - \sum_k (D_{jk} + E_{jk}) .
\]  

(A14)

This relation is useful in § 2.5.

We also evaluate the disk gravitational potential in the ring representation,

\[
\Phi_d(r) = -G \int d^2r' \frac{\Sigma(r')}{|r - r'|} = -2\pi G \int_0^\infty r' dr' \Sigma_d(r') \int \frac{d\Psi}{2\pi} \frac{1}{r^2 + (r')^2 - 2rr' \cos \Psi} \]

\[
= -2\pi G \int r' dr' \Sigma_d(r') \frac{2K(k)}{\pi(r + r')} ,
\]  

(A15)

with

\[
\Phi_{dj} \equiv \int_0^\infty r dr S(r | r_j) \Phi_d(r) = \frac{1}{M_j} \sum_k \Lambda_{jk} ,
\]  

(A16)

where

\[
\Lambda_{jk} = -GM_j M_k \int r dr' dr' S(r | r_j) S(r' | r_k) \frac{2K(k)}{\pi(r + r')} .
\]  

(A17)

For $|r_j - r_k| \gg (\Delta r_j \Delta r_k)^{1/2}$,

\[
\Lambda_{jk} \approx -GM_j M_k \frac{2K(k)}{\pi(r_j + r_k)} ,
\]  

(A18)

whereas for $|r_j - r_k| \ll (r_j r_k)^{1/2}$,

\[
\Lambda_{jk} \approx -GM_j M_k \frac{1}{2\pi \sqrt{r_j r_k}} \int_{-\infty}^{\infty} dy \exp \left( -\frac{y^2}{4} \right) \ln \left[ \frac{8r_j}{\Delta r} \right] ,
\]  

(A19)

where $u \equiv (r_k - r_j) / \Delta r$ as above. Note that $\Lambda_{jk} = \Lambda_{kj}$ and that $\Lambda_{jj} \approx -GM_j^2 / (1/\pi r_j) \ln(10.68r_j / \Delta r_j)$.

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