Constraining the Temperature of Astrophysical Black Holes through Ringdown Detection: Results of GW150914 Remnant

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Abstract
The ringdown of a black hole as a result of the merger of two black holes is a potent laboratory of the strong-field dynamics of spacetime. For example, it conveys information about the mass and spin of the remnant object, which can be related to the temperature of black holes. However, such relationships depend intimately on the assumption that general relativity is correct, and their capacity to test general relativity is restricted. We propose a novel method to measure the temperature of astrophysical black holes through detecting their quasi-normal modes, without assuming a specific dependence of the temperature on the mass and spin of black hole. In particular, we re-evaluate the emission of gravitational waves from the ringdown under the assumption that a black hole also radiates gravitational waves through Hawking radiation. We find that the resulting gravitational-wave signal has a temperature dependence that is independent of fixed relationships amongst the mass, spin and temperature. By re-analysing the gravitational-wave signal of GW150914, we set a constraint on the temperature of its remnant to be $T < 10^6 \text{K}$. Our results rule out the possibility of having detected anomalously strong quantum-gravity effects, but does not provide evidence of possible quantum-gravity signatures.

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I. INTRODUCTION

The detection of gravitational waves emitted by the merger of black holes with the Advanced LIGO and Advanced Virgo detectors [1, 2] has opened a new window into the strong-field dynamics of gravity [3–11]. In particular, measuring the ringdown of a black-hole merger opens up the possibility of probing the quantum nature of black holes [12–19]. During this ringdown phase, the distorted remnant black hole radiates gravitational waves in a discrete set of (complex) quasi-normal frequencies (see e.g. [20–23]). In the literature, these perturbations are typically regarded as purely ingoing at the event horizon (see, for example, [24, 25]). This choice of the boundary condition determines the ringdown waveform at spatial infinity.

However, the purely ingoing boundary condition may not be the full story if the intrinsic radiation of a black hole is taken into account. In particular, Hawking predicted that a black hole of mass $M$ radiates at a temperature of $T \sim 10^{-6} \frac{M}{M_\odot} \text{K}$ [26, 27]. Such radiation implies that there are waves that propagate away from the horizon, whose amplitudes depend on the temperature. Therefore, it is necessary that the ringdown of a black-hole merger contains information about the temperature of the black hole.

Existing tests of black-hole properties mainly rely on the measurement of the mass and spin, coupled with theoretical or computational relationships for the area or temperature [4–6, 28–31]. These include relationships amongst mass, spin, area and temperature and fits for the final mass and spin of a merger of two black holes. While these considerations may be valid ways to test black-hole properties, they necessarily depend on the correctness of general relativity, which limits their scope.

In this paper, we calculate the ringdown waveform by including outgoing waves into the boundary condition at the horizon. This results in two additional terms in the ringdown waveforms that depend on temperature, without relying on an explicit relationship amongst mass, spin and temperature. Importantly, these extra terms allow us to measure the temperature of astrophysical black holes independently of their mass and spin. Finally, we use this waveform to put a constraint on the temperature of the merger remnant of GW150914.

The rest of the paper is structured as follows. We first study the ringdown phase of so-called “quantum” black holes [32] by solving the Teukolsky equation subject to the extended boundary condition for quantum black hole in Sec. [II] In Sec. [III] we re-analyse the ringdown...
of GW150914 using our new ringdown waveforms. Finally, the implications of these results are discussed in Sec. [IV]. Throughout the paper, unless otherwise specified, $c = G = \hbar = 1$.

II. PHENOMENOLOGICAL DESCRIPTION OF THE QUASI-NORMAL MODES OF QUANTUM BLACK HOLES

In the Boyer-Lindquist coordinates $(t, r, \theta, \phi)$, where $\theta$ is the angle between the line of sight and the spin of the black hole and $\phi$ is the azimuthal angle, the Teukolsky equation for vacuum perturbations is given by [33–36]

$$
\left[ \frac{\Delta}{\Delta} - M^2 a^2 \sin^2 \theta \right] \frac{\partial^2 \psi}{\partial t^2} + \frac{4M^2 a r}{\Delta} \frac{\partial^2 \psi}{\partial t \partial \phi} + \left[ \frac{M^2 a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right] \frac{\partial^2 \psi}{\partial \phi^2} - \Delta^{-s} \frac{\partial}{\partial r} \left( \Delta^{s+1} \frac{\partial \psi}{\partial r} \right) - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) - 2s \left[ \frac{M a (r - M)}{\Delta} + \frac{i \cos \theta}{\sin^2 \theta} \right] \frac{\partial \psi}{\partial \phi} - 2s \left[ \frac{M (r^2 - M^2 a^2)}{\Delta} - r - iMa \cos \theta \right] \frac{\partial \psi}{\partial t} + (s^2 \cot^2 \theta - s) \psi = 0,
$$

(1)

where $\Delta = (r - r_+)(r - r_-)$, $r_+$ and $r_-$ are the outer and inner event horizon respectively, $\psi = \rho^{-4} \psi_4$ is the perturbation function for gravitational perturbations, $\psi_4$ is the fourth Weyl scalar, $\rho^{-1} = r - iMa \cos \theta$, $M$ and $M^2 a$ are the mass and angular momentum of the black hole respectively, and $s$ is the spin weight of perturbation fields. In particular, at spatial infinity $(r \to \infty)$, $\psi_4 \propto \bar{h}_+ - i \bar{h}_x$, where $\bar{h}_+$ and $\bar{h}_x$ are the plus mode and cross mode polarization of gravitational waves. Since the Teukolsky equation is a separable differential equation, we let $\psi(t, r, \theta, \phi) = R_{nlm}(r)S_{lm}(\theta, \phi)e^{-i\omega t}$, where $S_{lm}(\theta, \phi)$ is the spheroidal function given $l$ and $m$ [37]. The purely ingoing boundary condition at the event horizon can be expressed as

$$
R(x \to -\infty) \propto e^{-ikx},
$$

(2)

where $x$ is the tortoise coordinate, $k = \omega - m \Omega_H$, $\Omega_H$ is the angular velocity of the event horizon.

However, when considering particle creation by a black hole at the horizon [26, 27], one should also consider an outgoing boundary condition where the created gravitons propagate
as waves. In particular, we consider the following boundary condition at the event horizon,

\[ R_{nlm}(x \to -\infty) = R_{nlm}^\text{in}(x) + R_{nlm}^\text{out}(x), \]

\[ R_{nlm}^\text{in}(x) = \mathcal{E}_{nlm}(\omega) \Delta^2 e^{-i k x}, \]

\[ R_{nlm}^\text{out}(x) = \mathcal{H}_{nlm}(\omega | \beta) \Delta^2 e^{+i k^\dagger x}, \]

where \( \mathcal{E}_{nlm}(\omega) \) and \( \mathcal{H}_{nlm}(\omega | \beta) \) (given an inverse temperature \( \beta = 1/T \)) are the spectra of the ingoing and outgoing gravitational waves with angular dependence \( S_{lm}(\theta, \phi) \) at the horizon.

We have assumed that the final black hole starts emitting gravitons after the start of its ringdown phase, when the perturbed black hole has well settled into a stationary black hole and its temperature can be defined.

The created gravitons are represented by the complex conjugate of the ingoing waves with corresponding frequencies because of the following two reasons. Firstly, in quantum mechanics, complex conjugation of wave functions corresponds to the time-reversal of quantum states \[38\]. This is consistent with our consideration that the emission of gravitons is the same as the time-reversal of their absorption \[39\]. Secondly, the complex conjugation ensures a time-decaying quasi-normal-mode response of quantum black holes. Otherwise, this boundary condition may lead to solutions growing exponentially with time, which violates conservation of energy.

The ringdown of quantum black holes can be studied by solving the Teukolsky equation subject to the emission boundary condition in Eq. 3. By the Green’s function technique \[20, 40, 41\], we obtain a gravitational waveform of the ringdown of quantum black holes at the spatial infinity (see full derivation in Appendix A),

\[ h_+ - i h_x = \frac{M}{r} \sum_{nlm} A_{nlm} (1 + C(\tilde{\omega}_{nlm} | \beta)) S_{lm}(\theta, \phi) e^{-i \tilde{\omega}_{nlm} t} \]

\[ + \frac{M}{r} \sum_{nlm} S_{lm}(\theta, \phi) \int_0^{+\infty} \mathcal{H}_{nlm}(\omega | \beta) e^{-i \omega t} d\omega, \]

where \( A_{nlm} \) are coefficients related to the quasi-normal modes of “classical” black holes.

From Eq. 4, we see that the classical quasi-normal-mode solution (i.e. the first term, which is proportional to \( A_{lmn} \)) has been supplemented by an additional excitation factor, \( C(\tilde{\omega}_{nlm} | \beta) \), which are given by

\[ C(\tilde{\omega} | \beta) \approx \frac{\mathcal{H}(\omega_\text{Re} | \beta)}{8 M^4 (\omega_\text{Im})^5} \left( \frac{\omega_\text{Im}}{e} \right)^2 \left( \frac{4 M \omega_\text{Im}^2}{\Gamma(2 - 4 M \omega_\text{Im}^2)} \right)^2 \sum_{l=0}^{\infty} (\omega_\text{Im})^l Q_l, \]

(5)
where $Q_i$ are polynomials of $\omega^{\text{Im}}$ and $\omega^{\text{Re}}$,

$$
Q_0 = 2M \omega a^2 \left[ C_1 \omega^{\text{Im}} - 8(\omega^{\text{Im}})^2 + a^2 \omega_M^{\text{Im}} (\omega^{\text{Re}})^2 \right] \\
+ M \omega \left[ 29C_1 \omega^{\text{Im}} + 8C_2 (\omega^{\text{Im}})^2 + 2C_3 (\omega^{\text{Im}})^3 \\
- 111(\omega^{\text{Re}})^2 + \omega_M^{\text{Im}} (-3(2 + a^2)C_1 \omega^{\text{Im}} \\
- 2a^2 C_2 (\omega^{\text{Im}})^2 + 2(25 + 6a^2)(\omega^{\text{Re}})^2) \right] \\
Q_1 = 15C_1 + \omega^{\text{Im}} \left( 6C_2 + \omega^{\text{Im}} (3C_3 + 2C_4 \omega^{\text{Im}}) \right), \\
Q_2 = 45.
$$

The $C_i$ coefficients are given by

$$
C_1 = 4\omega^{\text{Re}}, \\
C_2 = \lambda^{\text{Im}} + 2Mam \omega^{\text{Im}} - 12M \omega^{\text{Re}}, \\
C_3 = 4Mam - 2\lambda^{\text{Im}} M, \\
C_4 = 16M^3 a^2 \omega^{\text{Re}} - 4M^2 am,
$$

and $\lambda^{\text{Im}}$ is the separation constant of the Teukolsky equation. These terms are extra excitation of quasi-normal modes due to particle creation. Since a quantum black hole can be regarded as a classical black hole with sources emitting gravitational waves at the event horizon (see Appendix A), these effective sources give rise to extra excitation of quasi-normal modes [40, 41].

Moreover, an additional term proportional to $\int_0^{+\infty} \mathcal{H}_{nlm}(\omega|\beta) e^{-i\omega t} d\omega$ is introduced in Eq. (4). This term represents the continuous creation of gravitons by quantum black hole at the event horizon. $\mathcal{H}_{nlm}$ depends on the emission rate of gravitons (the number emitted per unit time) at the horizon, which in turn depends on the nature of a hypothetical quantum gravity theory. In general, the emission rate takes the form of,

$$
\frac{dN}{dt} \propto \int_0^{+\infty} d\omega \frac{n(\omega)}{e^{\beta \omega} - 1},
$$

where $\beta = 1/T$ is the inverse temperature and $n(\omega)$ is known as the greybody factor, which depends on the type of particle considered and the specific emission mechanism. For black holes in general relativity, we choose $n(\omega) \propto \omega^6$, which is obtained by computing the absorption cross section of gravitational waves by a black hole via solving the Teukolsky equation [42]. Correspondingly, the power due to graviton creation is

$$
\frac{dE}{dt} \propto \int_0^{+\infty} d\omega \frac{\omega n(\omega)}{e^{\beta \omega} - 1} = \int_0^{+\infty} d\omega \frac{\omega^7}{e^{\beta \omega} - 1}.
$$
FIG. 1. The magnitude of $\frac{M}{r} |\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ and $\frac{M}{r} |C(\tilde{\omega}_{022}|\beta)|$ as a function of temperature, with $\tilde{\omega}_{022}$, $M$ and $r$ taken to be the estimated 022-mode frequency, final mass and luminosity distance of the GW150914 remnant. These two terms correspond to the extra quasi-normal-mode excitation term and continuous-emission term in Eq. 4 at the start of the ringdown phase. The horizontal dashed line plots the value of $\frac{M}{r}$ of the GW150914 remnant. From the plot, we conclude that $|\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ is much larger than $|C(\tilde{\omega}_{022}|\beta)|$. Thus, the continuous emission is comparatively more visible than the extra excitation of quasi-normal modes.

If we pick the following form

$$\mathcal{H}_{nlm}(\omega|\beta) = A_{nlm} \sqrt{\frac{G\hbar}{c^5}} \left( \frac{GM}{c^3} \right)^2 \omega^2 (e^{\beta\omega} - 1)^{-\frac{1}{2}},$$

then the total power due to the term proportional to $\int_0^{+\infty} \mathcal{H}_{nlm}(\omega|\beta)e^{-i\omega t} d\omega$ in Eq. 4 will be approximately equal to the power associated to graviton emission (i.e. Eq. 9), because $\sum_{nlm} |A_{nlm}|^2 \approx 1$ [31].

Fig. 1 plots the magnitude of $\frac{M}{r} |\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ and $\frac{M}{r} |C(\tilde{\omega}_{022}|\beta)|$ as a function of $T \in [1,10^8] K$, with $\tilde{\omega}_{022}$, $M$ and $r$ taken to be the estimated 022-mode frequency, final mass and luminosity distance of the GW150914 remnant. The plot gives an estimation of
the orders of the extra quasi-normal-mode excitation term and continuous-emission term in Eq. 4 at the start of the ringdown phase. For $T \in [1, 10^8] \text{K}$, we see that $|\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ and $|C(\bar{\omega}_{022}|\beta)|$ are both increasing with $T$. In general, $|\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ is larger than $|C(\bar{\omega}_{022}|\beta)|$. Specifically, $|\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega| \sim T^3$ and $|C(\bar{\omega}_{022}|\beta)| \sim T^{1/2}$, which are their asymptotic behaviour as $\beta \to 0$. Since $|\bar{\omega}|$ of other overtones are of similar order as the dominant mode, we expect similar order of $|C(\bar{\omega}|\beta)| \sim |C(\bar{\omega}_{022}|\beta)| \ll |\int_0^{+\infty} \mathcal{H}(\omega|\beta) d\omega|$ for other overtones in the ringdown phase. From Fig. 1, we conclude that the continuous emission of gravitons should dominate over the other contributions at a particular value of the temperature $T$.

III. RESULTS OF GW150914 REMNANT

We use the waveform in Eq. 4 with $\mathcal{H}_{nlm}(\omega|\beta)$ given by Eq. 10 to re-analyse 4096 seconds of data around GW150914 [43]. The signal is band-passed in the band [20,2038] Hz interval.

In particular, we estimate the parameters associated with the ringdown of the remnant of GW150914, with the base-10 log of black-hole temperature, log $T$, used as a free parameter. According to Bayes’s theorem, the posterior is proportional to the product of the likelihood and the prior

$$p(\log T|d, H, I) \propto p(d|\log T, H, I)p(\log T|H),$$

where $p(\log T|H)$ is the prior on the temperature and $p(d|\log T, H, I)$ is the likelihood of a quantum black hole with temperature $\log T$ producing a signal $d$. The prior $p(\log T|H)$ is set to be uniform between $\log T \in [0, 10]$, which covers a range that include estimates of the black-hole temperature from studies on accretion disks [44]. The natural log of the likelihood is given by [8]

$$\ln p(d|\log T, H, I) = -\frac{1}{2} \int dt \int d\tau (d(t) - h(t|\log T))C^{-1}(\tau)(d(t+\tau) - h(t+\tau|\log T))$$

where $d$ denotes the detected signals, $h(t|\log T)$ denotes the template waveform (i.e. Eq. 4) and $C(\tau)$ is the two-point autocovariance function of noise, defined by

$$C(\tau) = \int dt n(t)n(t+\tau),$$

where $n(t)$ denotes the noises. To ensure that there is no contamination from the merger into ringdown signal, we choose the lower bound of the prior of the ringdown start time.
to be $15M$ after the merger\cite{7}, with $M \sim 68M_\odot$ is the estimated final mass of GW150914 remnant \cite{1 45}. For this work, we only include the $nl|m| = 022$ modes, because these modes have been found to be dominant \cite{31}, and the sole inclusion of these are sufficient for accurate ringdown spectroscopy \cite{8}. Finally, we make use of the pyRing package, which is introduced to perform the analysis presented in \cite{8}, to sample the posterior by the Bayesian Nested-sampling algorithm CPNest \cite{46}.

The top left panel of Fig. 2 shows the posterior of $\log T$ for the final black hole of GW150914. The posterior shows no support for $\log T > 6$, setting a constraint on the temperature of the remnant of GW150914. The obtained upper bound is consistent with the predicted temperature of black holes of this range of masses.

The top right panel of Fig. 2 shows the two-dimensional posterior of ringdown parameters $M_f$ and $a$ for the final black hole of GW150914. On the two-dimensional posterior, the solid line and the dashed line respectively enclose the 90 % and 65 % confidence intervals, with a dot denoting the $M_f$ and $a$ reported by the LIGO-Virgo scientific Collaboration \cite{45}. Our test recovers values of $M_f$ and $a$ that are consistent with various studies on GW150914 \cite{4,7,8,11,45} while giving a reasonable posterior of $\log T$. We conclude that our test can measure the mass and spin of the final black hole accurately while putting a reasonable constraint on the temperature.

Finally, the posterior of the black-hole temperature is not degenerate with other ringdown parameters. The bottom right and left panel of Fig. 2 respectively show the two-dimensional posteriors of $M_f - \log T$ and $a - \log T$. The solid line and dashed line, corresponding to the 90 % and 65 % confidence contours respectively, show no significant inclination on the parameter plane. These two-dimensional posteriors suggest that there are no strong correlations amongst the ringdown parameters and $\log T$. Therefore, our test is not expected to incorrectly attribute the quantum signature as signatures of another classical black hole of alternative mass and spin.

\section*{IV. CONCLUDING REMARKS}

In summary, we have studied the ringdown waveform of a quantum black hole by solving the Teukolsky equation subject to the emissive boundary condition at the event horizon. Re-analysing the remnant of GW150914 using this waveform, we set a constraint on the rem-
FIG. 2. (Top left panel) The posterior of log of the temperature of the final black hole of GW150914, \( p(\log T|d, H, I) \), where \( T \) is measured in Kelvin. Our test finds no support for temperatures \( \log T > 6 \). (Top right panel) The two-dimensional posterior of \( M_f \) and \( a \). The solid line and the dashed line respectively denote the 90% and 65% confidence regions on the \( M_f - a \) plane. The dot denotes the values of \( M_f \) and \( a \) reported by the LIGO-Virgo Collaboration [45] (labeled as "LVC measurement"). Our estimation of \( M_f \) and \( a \) are consistent with the results reported by the LIGO-Virgo Collaboration and other relevant studies. (Bottom right panel) The two-dimensional posterior of \( \log T \) and \( M_f \). (Bottom left panel) The two-dimensional posterior of \( \log T \) and \( a \). The solid lines and the dashed lines on the bottom panels respectively enclose 90% and 65% confidence regions of the two-dimensional posteriors. As the confidence contours do not significantly inclined to a particular direction, \( \log T \) has no strong correlation with \( M_f \) and \( a \).

nant’s temperature of \( T < 10^6 \) K. While constraining the temperature, our test accurately estimates the final mass and spin of GW150914. Moreover, the posterior of the temperature is not degenerate with those of other ringdown parameters. Our work serves as a proof of principle that gravitational-wave detection may constrain the strength of quantum-gravity effects of astrophysical black holes.
Although our constraint is several orders away from the predicted temperature by quantum gravity, the constraint is still relevant considering the current knowledge of the temperature of black holes. For example, our constraint is lower than the temperature of the accretion disk for some black holes\cite{44}. Moreover, unlike existing literature\cite{30,47}, our work is the first method to quantitatively study quantum gravity-effects of astrophysical black holes without assuming a dependence of temperature on the mass and spin of black hole. This lays a foundation toward model-independent tests of black-hole thermodynamics and observational studies of quantum gravity with astrophysical black holes.

Our constraint can be considered in light of an order-of-magnitude estimation. For GW150914, the amplitude of the detected signals is of the order of $O(10^{-19})$. The continuous emission tail is approximately given by

$$
\int_{0}^{\infty} \omega^2 (e^{\beta \omega} - 1)^{-\frac{1}{2}} e^{-i\omega t} d\omega \propto \beta^{-3}.
$$

By requiring that the continuous emission should be comparable to the classical ringdown strain $s_{RD}$, we can relate the black-hole temperature up to the distance, mass and the detectable ringdown strain

$$
T \approx \frac{\hbar}{k} \left( \frac{c^8}{G^3 t_p M^3} \right)^{1/3} s_{RD}^{1/3},
$$

where $t_p$ is the Planck time. For GW150914, the amplitude of the detected ringdown signals is of the order of $s_{RD} \sim 10^{-19}$. Eq. 15 gives $T \leq 10^6$ K, which is consistent with Figs. 1 and 2. Future gravitational-wave detectors, such as the Laser Interferometer Space Antenna (LISA), will instead be capable of detecting the ringdown signals due to black holes of $10^6 M_\odot$ within a luminosity distance of $10^4$ Mpc\cite{48} with a ringdown strain of $s_{RD} \sim 10^{-17}$. This means that these future detectors can potentially lower the constraints down to $T \lesssim 10^2$ K.

Finally, it should be noted that the constraints on black-hole temperature should be interpreted on a case-by-case basis for different emission-models. In particular, the assumption of the functional form of $n(\omega)$ is encoded in the likelihood for our parameter estimation of log $T$. If different emission models are considered, the likelihood will have different dependence on log $T$. Therefore, by Eq. 11 our test will obtain a different posterior of log $T$ for the same event if different greybody factors are used (see e.g. 49,52).
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Appendix A: Derivation of the Ringdown Waveform of Quantum Black Holes

The Teukolsky equation is a separable linear second order differential equation. For gravitational perturbations, we let \( u(r, t) = \sqrt{r^2 + M^2 a^2} \Delta^{-1} R(r, t) \). The radial Teukolsky equation in the time-domain becomes,

\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - V(r) u = 0 \tag{A1}
\]

where \( x \) is the tortoise coordinate defined by \( \frac{dx}{dr} = \frac{\Delta}{r^2 + a^2} \) and \( V(r) \) is the complex effective potential. If we consider the Laplace transform of \( u(r, t) \),

\[
\tilde{u}(r, \omega) = \int_0^{+\infty} u(r, t) e^{i\omega t} dt, \tag{A2}
\]

then \( u(r, \omega) \) satisfies the equation of

\[
\frac{\partial^2 \tilde{u}}{\partial x^2} + (\omega^2 - V(r)) \tilde{u} = \mathcal{I}(x, \omega), \tag{A3}
\]
where \( \mathcal{I} = i\omega u_0 - \partial_t u_0 \) is the initial data of the quasi-normal-mode excitation, \( \partial_t u_0 = \partial_t u(t = 0, x) \) and \( u_0 = u(t = 0, x) \). When \( \mathcal{I} = 0 \), Eq. A3 admits two linearly independent solutions: the down mode,

\[
\tilde{u}_{\text{down}}(x, \omega) \approx \begin{cases} 
\Delta e^{-ikx}, & x \to r_+ \\
A_1(\omega)r^{-2}e^{-i\omega r} + A_2(\omega)r^2e^{i\omega r}, & x \to +\infty
\end{cases}
\]

(A4)

which is purely ingoing at the event horizon, and the up mode

\[
\tilde{u}_{\text{up}}(x, \omega) \approx \begin{cases} 
B_1(\omega)\Delta e^{-ikx} + B_2(\omega)\Delta^{-1}e^{ikx}, & x \to r_+ \\
r^2e^{i\omega r}, & x \to +\infty
\end{cases}
\]

(A5)

which is purely outgoing at spatial infinity, where \( A_1(\omega), A_2(\omega), B_1(\omega) \) and \( B_2(\omega) \) are functions of \( \omega \). In particular, \( A_1(\omega) \) and \( B_2(\omega) \) are proportional to the Wronskian of the differential equation.

When \( \mathcal{I} \neq 0 \), Eq. A3 can be solved by the Green’s function technique. The desired Green’s function can be constructed by Wronskian \( W(\omega) \), a function of frequency, of Eq. A4 and Eq. A5. By definition, quasi-normal-mode frequencies \( \tilde{\omega}_{nlm} \) are complex zeroes of \( W(\omega) \) (for this reason, \( A_1(\omega) \propto B_2(\omega) \propto W(\omega) \) contribute no amplitude to the ringdown waveform). Thus, for a given set of \( l \) and \( m \),

\[
\frac{1}{W(\omega)} \approx \frac{1}{2\pi i M^4} \sum_n \frac{\zeta_{nlm}}{\omega - \tilde{\omega}_{nlm}},
\]

(A6)

where \( M^4 \) is a factor for dimensional consistency and \( \zeta_{nlm} \) are constants which can be determined by the classical quasi-normal-mode solutions at spatial infinity (corresponding to the given \( l, m \),

\[
\tilde{u}(x \to +\infty, t) \approx Mr^2 \sum_n (\tilde{\omega}_{nlm})^2 A_{nlm} e^{-i\tilde{\omega}_{nlm}t},
\]

(A7)

where \( A_{nlm} = \sum_j a_j e^{i\delta j} \eta^j \) are function of mass, symmetric mass ratio and spins of the parental black holes \[31, 53\]. If we choose

\[
\mathcal{I}(x, \omega) = \lim_{x_0 \to -\infty} M\omega^2 \Delta^{-1}e^{ikx}\delta(x - x_0),
\]

(A8)

and

\[
\zeta_{nlm} = A_{nlm},
\]

(A9)
then upon performing the inverse Laplace transform,

$$u(x, t) = \frac{1}{2\pi i} \lim_{\epsilon \to 0} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} d\omega \tilde{u}(x, \omega) e^{-i\omega t}, \quad (A10)$$

with the integration contour chosen to be closed by a semicircle centred at $\omega^{\text{Im}} = 0$ covering negative imaginary axis [20, 40], the form of Wronksian Eq. [A6] recovers the ringdown waveform of classical black holes.

Note that the outgoing wave $\Delta e^{ik^+x}$ does not satisfy the radial Teukolsky equation Eq. [A3]. Instead, it satisfies the complex conjugate of the equation. To simplify the calculations, we construct an auxiliary function $z = \tilde{u} - (\omega^+)^2 \mathcal{H}(\omega|\beta)\Delta e^{ik^+x}$, which satisfies

$$\frac{\partial^2 z}{\partial x^2} + \left(\omega^2 - V(r)\right) z = 2i \left(V_{\text{Im}}(r) - \omega_{\text{Im}}^2\right) \mathcal{H}(\omega|\beta)(\omega^+)^2 \Delta e^{ik^+x}, \quad (A11)$$

where $V_{\text{Im}}(r)$ is the imaginary part of the effective potential,

$$V_{\text{Im}}(r) \approx \frac{4\omega^\text{Re} r + \lambda^\text{Im} + 2Ma\omega^\text{Im} m - 12M\omega^\text{Re}}{r^2 + M^2 a^2} + \frac{(4Man - 2\lambda^\text{Im} M)r + 16M^3 a^2 \omega^\text{Re} - 4M^2 am}{(r^2 + M^2 a^2)^2}, \quad (A12)$$

where $\omega^\text{Re}$ and $\omega^\text{Im}$ are respectively the real part and imaginary part of $\omega$, $m$ is the azimuthal number of the quasi-normal mode, $a$ is the dimensionless spin of the black hole and $\lambda^\text{Im}$ is the imaginary part of the separation constant $\lambda$.

The boundary condition of Eq. [A11] is purely ingoing at the event horizon: $z(x \to -\infty) \propto \Delta e^{-ikx}$. Thus, its particular solution can be obtained by applying the Green’s function technique. Using the Green’s function technique, we have

$$z(x, \omega) = 2i\tilde{u}_{\text{up}}(x, \omega) \frac{\mathcal{H}(\omega|\beta) C(\omega)}{W(\omega)}, \quad (A13)$$

where $C(\omega)$ is the excitation factor given by

$$C(\omega) \approx i \int_{-\infty}^{+\infty} dx' \tilde{u}_{\text{down}}(x') \left(V_{\text{Im}}(r') - \omega_{\text{Im}}^2\right) (\omega^+)^2 \Delta' e^{ik^+x'}, \quad (A14)$$

where $r' = r(x')$, $\Delta' = (r' - r_+)(r' - r_-)$. The above integral is approximately given by Eq. [5].

We then perform the inverse Laplace transform to Eq. [A13] yielding

$$z(r \to +\infty, t) = Mr^2 \sum_{nlm} A_{nlm}(\tilde{\omega}_{nlm}^+)^2 \mathcal{H}(\tilde{\omega}_{nlm}|\beta) C(\tilde{\omega}_{nlm}) r^2 e^{-i\tilde{\omega}_{nlm} t}. \quad (A15)$$
If we redefine \( C(\tilde{\omega}|\beta) = \mathcal{H}(\tilde{\omega}|\beta)C(\tilde{\omega}) \), then the time-domain ringdown waveform Eq. [4] follows from the above calculations.

The above calculations can similarly be done for different \( l \) and \( m \). Adding the contribution from all quasi-normal modes at the spatial infinity where we detect gravitational waves, we obtain the waveform in Eq. [4].

**Appendix B: Numerical Computation of the Ringdown Waveform of Quantum Black Holes**

When numerically evaluating the integral \( \int_{0}^{+\infty} H_{nlm}(\omega|\beta)e^{-i\omega t}d\omega \), we approximate the integral by the following. We first expand \( H(\omega|\beta) \) as a power series of \( e^{\beta \omega} \) and then perform the integration. Up to the leading order, we have

\[
\int_{0}^{\infty} \omega^{2}(e^{\beta \omega} - 1) - \frac{1}{2} e^{-i\omega t} d\omega \approx \frac{16}{(\beta + 2it)^3}.
\]  

(B1)

In particular, we assume that gravitons are created right after the start of the ringdown phase when the final black hole has just been formed. Since no calculations have been done to estimate the exact time difference, as an ad hoc approach, we assume that the final black hole starts to radiate at the next sampling point after the start of the ringdown phase.

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