Degenerations of Ruijsenaars–van Diejen operator and \(q\)-Painlevé equations

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It is known that the Painlevé VI is obtained by connection preserving deformation of some linear differential equations, and the Heun equation is obtained by a specialization of the linear differential equations. We investigate degenerations of the Ruijsenaars–van Diejen difference operators and show difference analogues of the Painlevé–Heun correspondence.

Keywords: Ruijsenaars system; degeneration; Painlevé equation; Heun equation.

1. Introduction

In this article, we investigate \(q\)-difference equations that are generalizations of the Heun equation and the Painlevé VI equation.

Heun’s differential equation is given by

\[
\frac{d^2y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-t)} y = 0,
\]

with the condition \(\gamma + \delta + \epsilon = \alpha + \beta + 1\), and it is a standard form of Fuchsian differential equation with four singularities \(\{0, 1, t, \infty\}\). Note that the Gauss hypergeometric equation is a standard form of Fuchsian differential equation with three singularities \(\{0, 1, \infty\}\). The Heun equation has an accessory parameter \(q\) which is independent from local exponents, although the hypergeometric equation does not have it.

It is known that the Heun equation admits an expression in terms of elliptic functions. Let \(\wp(x)\) be the Weierstrass elliptic function with basic periods \((2\omega_1, 2\omega_3)\). Put \(\omega_2 = -\omega_1 - \omega_3, \omega_0 = 0\) and \(e_i = \wp(\omega_i)\) \((i = 1, 2, 3)\). By setting

\[
z = \frac{\wp(x) - e_1}{e_2 - e_1}, \quad t = \frac{e_3 - e_1}{e_2 - e_1}
\]

and applying a gauge transformation, we obtain an elliptical representation of Heun’s differential equation (see [1]):

\[
\left(-\frac{d^2}{dx^2} + \sum_{i=0}^{3} l_i(l_i + 1)\wp(x + \omega_i)\right)f(x) = Ef(x).
\]
Here the coupling constants $l_0, \ldots, l_3$ correspond to the parameters $\alpha, \ldots, \epsilon$ in Equation (1.1) and the eigenvalue $E$ corresponds to the accessory parameter $q$.

The Painlevé VI equation is a non-linear ordinary differential equation given by

$$
\frac{d^2\lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} + \frac{\lambda(\lambda - 1)(\lambda - t)}{t^2(t - 1)^2} \left\{ \alpha + \frac{\beta t}{\lambda^2} + \gamma \frac{(t - 1)}{(\lambda - 1)^2} + \delta \frac{t(t - 1)}{(\lambda - t)^2} \right\}.
$$

(1.4)

See [2] for a review of the Painlevé equations. In particular, it is known that solutions of the Painlevé VI equation do not have movable singularities other than poles, that is called the Painlevé property. Painlevé VI is also obtained by monodromy preserving deformation of the $2 \times 2$ Fuchsian system of equations with four singularities $\{0, 1, t, \infty\}$. The Fuchsian system of equations is equivalent to the following Fuchsian equation

$$
\frac{d^2y_1}{dz^2} + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{1 - \theta_t}{z - t} - \frac{1}{\lambda - \lambda} \right) \frac{dy_1}{dz} + \left( \frac{\kappa_1(\kappa_2 + 1)}{z(z - 1)} + \frac{\lambda(\lambda - 1)\mu}{z(z - 1)(z - \lambda)} - \frac{t(t - 1)H}{z(z - 1)(z - t)} \right) y_1 = 0,
$$

(1.5)

$$
H = \frac{1}{t(t - 1)}[\lambda(\lambda - 1)(\lambda - t)\mu^2 - (\theta_0(\lambda - 1)(\lambda - t) + \theta_t(\lambda - t) + (\theta_t - 1)\lambda(\lambda - 1))\mu + \kappa_1(\kappa_2 + 1)(\lambda - t)].
$$

Note that the singularity $z = \lambda$ is apparent, which follows from the equality for $H$. The monodromy of the solution to Equation (1.5) is preserved as the parameter $t$ varies, if there exist rational functions $a_1(z, t)$ and $a_2(z, t)$ of the variable $z$ such that the equation

$$
\frac{\partial y}{\partial t} = a_1(z, t)y + a_2(z, t) \frac{\partial y}{\partial z}
$$

(1.6)

is compatible to Equation (1.5) (see [2]). It follows from a lengthy calculation that the compatibility condition is equivalent to

$$
\frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda},
$$

(1.7)

which is called the Painlevé VI system. By eliminating $\mu$, we obtain the Painlevé VI equation.

Recall that Equation (1.5) has five singularities $\{0, 1, t, \infty, \lambda\}$, and the singularity $z = \lambda$ is superfluous for the Heun equation. By specializing the point $z = \lambda$ to regular singularities $\{0, 1, t, \infty\}$, we may derive the Heun equation. For example, by setting $\lambda = t$ in Equation (1.5) we have

$$
\frac{d^2y_1}{dz^2} + \left( \frac{1 - \theta_0}{z} + \frac{1 - \theta_1}{z - 1} + \frac{-\theta_t}{z - t} \right) \frac{dy_1}{dz} + \frac{\kappa_1(\kappa_2 + 1)(z - t) + \theta_t(t - 1)\mu}{z(z - 1)(z - t)} y_1 = 0.
$$

(1.8)
Therefore, the Heun equation is related with the Painlevé VI equation through the linear differential equation given by Equation (1.5). We can also obtain the Heun equation by other specializations, and they are related with the space of initial conditions (see [3]). See also [4, 5] for other perspectives on relationship between the Heun equation and the Painlevé VI equation. Note that the Painlevé VI equation also admits elliptical representations [5–8], which were applied in various ways.

In this paper, we propose a difference analogue of the correspondence between the Heun equation and the Painlevé VI equation.

Sakai [9] investigated difference analogue of the Painlevé equation by using structures of some algebraic surfaces which are generalizations of the space of initial conditions, and proposed a list of the equations. There are three kinds of difference Painlevé equations, i.e. elliptic difference, $q$-difference (or multiplicative difference) and additive difference, and each difference equation is labelled by some affine root systems from its symmetry. The $q$-difference Painlevé equations of types $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)}$ are at issue in this article.

Before giving a difference analogue of the Heun equation, we discuss a multivariable generalization of the Heun equation. The quantum Inozemtsev system of type $BC_N$ is a quantum mechanical $N$-particle system whose Hamiltonian is given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2(l + 1) \sum_{1 \leq j < k \leq N} \left( \wp(x_j - x_k) + \wp(x_j + x_k) \right) + \sum_{j=1}^{N} \sum_{i=0}^{3} l_i (l_i + 1) \wp(x_j + \omega_i).$$

It is a generalization of the Calogero–Moser–Sutherland model, and the Inozemtsev model of type $BC_N$ is quantum Liouville integrable (see [10, 11]). By restricting to the case $N = 1$, we recover the elliptical representation of Heun’s equation (see Equation (1.3)).

A difference (relativistic) analogue of the Inozemtsev system of type $BC_N$ is known as the Ruijsenaars–van Diejen system [12, 13] (or the Ruijsenaars system of type $BC_N$), whose defining second order difference operator is given by

$$A(\mu; x) = \sum_{j=1}^{N} (V_j^+(x) \exp(\delta \partial_j) + V_j^-(x) \exp(-\delta \partial_j)) + V_0(x),$$

where

$$V_j^\pm(x) = \prod_{s=1}^{8} \frac{\theta(\pm x_j + \mu_s)}{\theta(\pm 2 x_j) \theta(\pm 2 x_j + \delta)} \prod_{k \neq j} \frac{\theta(\pm x_j + x_k + \kappa)}{\theta(\pm x_j + x_k) \theta(\pm x_j - x_k)},$$

where $\theta(x)$ is the theta function and we omit the expression of the function $V_0(x)$, instead we give another explicit expression in Equation (4.1). Note that

$$\exp(\pm \delta \partial_j)f(x_1, \ldots, x_j, \ldots, x_N) = f(x_1, \ldots, x_j \pm \delta, \ldots, x_N).$$

The system contains the parameters $\delta, \kappa, \mu_1, \ldots, \mu_8$. By a suitable limit as $\delta \to 0$, we obtain the Hamiltonian of the Inozemtsev system [12, 13]. It is known that commuting operators of the Ruijsenaars–van Diejen system exist as in the case of the Inozemtsev system [14]. We may regard the Ruijsenaars–van Diejen operator with one variable as a difference analogue of the Heun equation. It is known that the
Ruijsenaars–van Diejen operator has $E_8$ spectral symmetry [15]. On the other hand, the elliptic difference Painlevé equation admits $E_8^{(1)}$ symmetry [9]. We expect to clarify relationships between the one variable difference equation of Ruijsenaars–van Diejen type and the elliptic difference Painlevé equation.

In this article, we investigate degenerations of the Ruijsenaars–van Diejen operator and find correspondences with linear $q$-difference equations which are related with $q$-difference Painlevé equations. We find that we can take degenerations of the Ruijsenaars–van Diejen operator of $N$ variables four times, although it seems that the first two were essentially obtained by van Diejen [16]. The degenerations are still interesting in the setting of one variable. By taking degeneration four times, we obtain the following $q$-difference operator $A^{(4)}(x)$:

$$A^{(4)}(x)g(x) = x^{-1}(x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q)$$

$$- (l_3 + l_4)x + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2}(h_1^{1/2} + h_2^{-1/2})x^{-1})g(x)$$

$$+ x^{-1}l_3 l_4(x - l_1 q^{-1/2})g(qx).$$

Then we may regard the equation

$$A^{(4)}(x)g(x) = Eg(x) \quad (E: \text{eigenvalue})$$

as a $q$-deformation of Heun equation (1.1). On the other hand, Equation (1.13) is obtained as a special case of the linear $q$-difference equation by Jimbo and Sakai [17] which is related with the $q$-Painlevé VI equation by the connection preserving deformation. Note that Yousuke Ohyama kindly informed the author that Equation (1.13) had been discovered by Hahn [18] in 1971 as a direct $q$-deformation of Heun equation. The equations for eigenfunctions of the second degenerate operator and the third degenerate operator are also obtained as special cases of the linear $q$-difference equations obtained by Yamada [19] which are related with the $q$-Painlevé equations of type $E_6^{(1)}$ and type $E_7^{(1)}$.

This article is organized as follows. In Section 2, we apply degeneration of the Ruijsenaars–van Diejen operator with one variable four times. In Section 3, we review linear $q$-difference equations which are related with some $q$-Painlevé equations and obtain the degenerated Ruijsenaars–van Diejen operators with one variable by specializing the parameters. In Section 4, we extend the degeneration to the multivariable case. In Section 5, we propose some problems related with results in this article.

2. Degeneration of Ruijsenaars–van Diejen operator with one variable

2.1 Ruijsenaars–van Diejen operator

We describe the Ruijsenaars–van Diejen operator with one variable explicitly. Let $a_+, a_-$ be complex numbers whose real parts are positive and $R_{\pm}(z)$ be the functions defined by

$$R_{\pm}(z) = \prod_{k=1}^{\infty} (1 - q^2 e^{2\pi i z})(1 - q^2 e^{-2\pi i z}), \quad q_{\pm} = e^{-\pi a_{\pm}}.$$  \hspace{1cm} (2.1)

They are modified versions of theta functions with the half periods 1/2 and $ia_{\pm}/2$. The Ruijsenaars–van Diejen operator of one variable is given by

$$A_{+}(h; z) = V_{+}(h; z) \exp(-ia_{-} \partial_z) + V_{+}(h; -z) \exp(i a_{-} \partial_z) + U_{+}(h; z),$$  \hspace{1cm} (2.2)
where

\[
V_+(h; z) = \frac{\prod_{n=1}^{8} R_+(z - h_n - ia_-/2)}{R_+(2z + ia_+/2)R_+(2z - ia_- + ia_+/2)}.
\]

(2.3)

\[
U_+(h; z) = \frac{\sum_{t=0}^{3} p_t(h)[E_{t,+}(\mu; z) - E_{t,+}(\mu; \omega_{t,+})]}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)},
\]

and we are using

\[
\omega_{0,+} = 0, \quad \omega_{1,+} = 1/2, \quad \omega_{2,+} = ia_+/2, \quad \omega_{3,+} = -1/2 - ia_+/2,
\]

(2.4)

\[
p_{0,+}(h) = \prod_{n=1}^{8} R_+(h_n), \quad p_{2,+}(h) = e^{-2\pi a_+} \prod_{n=1}^{8} e^{-\pi h_n} R_+(h_n - ia_+/2),
\]

\[
p_{1,+}(h) = \prod_{n=1}^{8} R_+(h_n - 1/2), \quad p_{3,+}(h) = e^{-2\pi a_+} \prod_{n=1}^{8} e^{\pi h_n} R_+(h_n + 1/2 + ia_+/2),
\]

\[
E_{t,+}(\mu; z) = \frac{R_+(z + \mu - ia_+/2 - ia_-/2 - \omega_{t,+})R_+(z - \mu + ia_+/2 + ia_-/2 - \omega_{t,+})}{R_+(z - ia_+/2 - ia_-/2 - \omega_{t,+})R_+(z + ia_+/2 + ia_-/2 - \omega_{t,+})},
\]

(2.5) We adapt the expression in [15], which is slightly different from the one in [20] with an additive constant. Note that the function \(U_+(h; z)\) is independent from the parameter \(\mu\) in the case of one variable \(z\), which can be proved as the first part of Lemma 3.2 in [13]. Hence the operator \(A_+(h; z)\) is also independent from the parameter \(\mu\).

We can obtain an elliptical representation of the Heun equation (1.3) from the equation \(A_+(h; z)f(z) = Ef(z)\) (\(E\): eigenvalue) by taking a suitable limit as \(a_- \to 0\). For details see [12, 13].

2.2 First degeneration

We are going to take a trigonometric limit (\(q_+ \to 0\)) of the Ruijsenaars–van Diejen operator with one variable. The function \(R_+(z)\) satisfies

\[
R_+(z \mp ia_+) = -e^{\pi a_+} e^{\pm 2\pi i z} R_+(z)
\]

and we have the following expansion as \(q_+ \to 0\) (or \(a_+ \to +\infty\):

\[
R_+(z) = 1 - (e^{2\pi i z} + e^{-2\pi i z})q_+ + q_+^2 + O(q_+^3),
\]

(2.6)

\[
R_+(z \pm ia_+/2) = (1 - e^{\mp 2\pi i z})(1 - (e^{2\pi i z} + e^{-2\pi i z})q_+^2 + O(q_+^3)).
\]

We set \(h_n = \tilde{h}_n - ia_+/2\). Then the function \(V_+(h; z)\) admits the following limit as \(q_+ \to 0\):

\[
V_+(h; z) \to V^{(1)}(h; z) = \frac{\prod_{n=1}^{8}(1 - e^{-2\pi i z}e^{2\pi i h_n}e^{-\pi a_-})}{(1 - e^{-4\pi i z})(1 - e^{-4\pi i z}e^{-2\pi a_-})}.
\]

(2.7)

By considering the limit of the function \(U_+(h; z)\) as \(q_+ \to 0\), we have the following proposition.
Proposition 2.1 Let $A(h, q_+; z)$ be the Ruijsenaars–van Diejen operator defined in Equation (2.2). As $q_+ \to 0$, we have

$$\left( A(h, q_+; z) + \prod_{n=1}^{8} e^{\pi i \hbar n} (1 - e^{2\pi i \varepsilon})^2 e^{q_+^2} + C \right) f(z) \to A^{(1)}(h; z) f(z)$$

(2.8)

for any $f(z)$, where

$$A^{(1)}(h; z) = V^{(1)}(h; z) \exp(-ia_- \partial_z) + V^{(1)}(h; -z) \exp(ia_- \partial_z) + U^{(1)}(h; z),$$

(2.9)

$V^{(1)}(h; z)$ was defined in Equation (2.7),

$$U^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (e^{2\pi i \hbar n} - 1)}{2(1 - e^{2\pi i \varepsilon}) (1 - e^{-2\pi i \varepsilon})} + \frac{\prod_{n=1}^{8} (e^{2\pi i \hbar n} + 1)}{2(1 + e^{2\pi i \varepsilon}) (1 + e^{-2\pi i \varepsilon})}$$

(2.10)

$$+ e^{-\pi a_-} \prod_{n=1}^{8} e^{\pi i \hbar n} \left[ (e^{2\pi i \varepsilon} + e^{-2\pi i \varepsilon}) \sum_{n=1}^{8} (e^{2\pi i \hbar n} + e^{-2\pi i \hbar n}) - (e^{\pi a_-} + e^{-\pi a_-}) (e^{4\pi i \varepsilon} + e^{-4\pi i \varepsilon}) \right]$$

and

$$C = \prod_{n=1}^{8} e^{\pi i \hbar n} \left[ e^{-\pi a_-} (e^{\pi a_-} + e^{-\pi a_-}) \right. + 12 + \sum_{1 \leq n < n' \leq 8} (e^{2\pi i \hbar n} + e^{-2\pi i \hbar n})(e^{2\pi i \hbar n'} + e^{-2\pi i \hbar n'}) + \frac{1}{2} \left( \prod_{n=1}^{8} (e^{2\pi i \hbar n} - 1) + \prod_{n=1}^{8} (e^{2\pi i \hbar n} + 1) \right)$$

(2.11)

$$\times \frac{1}{(1 - e^{\pi a_-})^2}.$$ 

Proof. It follows from $R_+(z \pm \pm ia_+ / 2) = 1 - e^{2\pi i \varepsilon} + O(q_+^2)$ that

$$p_{0,+}(h) \mathcal{E}_{0,+}(\mu; z) = \frac{R_+(z + \mu - ia_+ / 2 - ia_- / 2) R_+(z - \mu + ia_+ / 2 + ia_- / 2)}{R_+(z - ia_+ / 2 - ia_- / 2) R_+(z + ia_+ / 2 + ia_- / 2)} \prod_{n=1}^{8} R_+(\hbar n - ia_+ / 2)$$

(2.12)

$$= \frac{(1 - e^{2\pi i (z + \mu - ia_+ / 2)}) (1 - e^{-2\pi i (z - \mu + ia_+ / 2)})}{(1 - e^{2\pi i (z - ia_+ / 2)}) (1 - e^{-2\pi i (z + ia_+ / 2)})} \prod_{n=1}^{8} (1 - e^{2\pi i \hbar n}) + O(q_+^2)$$

$$= \left\{ e^{2\pi i \mu} + \frac{(e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)}{(1 - e^{2\pi i \varepsilon} e^{\pi a_-})(1 - e^{-2\pi i \varepsilon} e^{\pi a_-})} \right\} \prod_{n=1}^{8} (e^{2\pi i \hbar n} - 1) + O(q_+^2),$$

and

$$p_{1,+}(h) \mathcal{E}_{1,+}(\mu; z) = \frac{e^{2\pi i \mu} + (e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)}{(1 + e^{2\pi i \varepsilon} e^{\pi a_-})(1 + e^{-2\pi i \varepsilon} e^{\pi a_-})} \prod_{n=1}^{8} (e^{2\pi i \hbar n} + 1) + O(q_+^2),$$

(2.13)

$$R_+(\mu - ia_+ / 2) R_+(\mu - ia_- - ia_+ / 2) = (1 - e^{2\pi i \mu})(1 - e^{2\pi i \mu} e^{2\pi a_-}) + O(q_+^2).$$
Hence
\[
\frac{p_{0,+}(h)(\mathcal{E}_{0,+}(\mu; z) - \mathcal{E}_{0,+}(\mu; \omega_{0,+}))}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)} + \frac{p_{1,+}(h)(\mathcal{E}_{1,+}(\mu; z) - \mathcal{E}_{1,+}(\mu; \omega_{1,+}))}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)} = \left\{ \frac{1}{2(1 - e^{2\pi i \epsilon_\mu a_-} - e^{-2\pi i \epsilon_\mu a_-})} - \frac{1}{2(1 - e^{\pi a_-})^2} \right\} \prod_{n=1}^{8}(e^{2\pi i \tilde{h}_n} - 1)
\]
\[
+ \left\{ \frac{1}{2(1 + e^{2\pi i \epsilon_\mu a_-} - e^{-2\pi i \epsilon_\mu a_-})} - \frac{1}{2(1 - e^{\pi a_-})^2} \right\} \prod_{n=1}^{8}(e^{2\pi i \tilde{h}_n} + 1) + O(q_+^2).
\]

Set
\[
\tilde{p}(h) = 8 + \sum_{1 \leq n < n' \leq 8} (e^{2\pi i \tilde{h}_n} + e^{-2\pi i \tilde{h}_n})(e^{2\pi i \tilde{h}_{n'}} + e^{-2\pi i \tilde{h}_{n'}}).
\]

It follows from Equations (2.5, 2.6) that
\[
p_{2,+}(h) = e^{-2\pi a_+} \prod_{n=1}^{8} e^{-\pi i (\tilde{h}_n - ia_+/2)} R_+(\tilde{h}_n - ia_+) = e^{2\pi a_+} \prod_{n=1}^{8} e^{\pi i \tilde{h}_n} R_+(\tilde{h}_n)
\]
\[
= e^{2\pi a_+} \prod_{n=1}^{8} e^{\pi i \tilde{h}_n} \{(1 - q_+(e^{2\pi i \tilde{h}_n} + e^{-2\pi i \tilde{h}_n}) + q_+^2 + O(q_+^3))
\]
\[
= q_+^2 \left[ 1 - \sum_{n=1}^{8} (e^{2\pi i \tilde{h}_n} + e^{-2\pi i \tilde{h}_n}) q_+ + \tilde{p}(h) q_+^2 + O(q_+^3) \right] \prod_{n=1}^{8} e^{\pi i \tilde{h}_n},
\]
\[
p_{3,+}(h) = e^{2\pi a_+} \prod_{n=1}^{8} e^{\pi i \tilde{h}_n} R_+(\tilde{h}_n + 1/2)
\]
\[
= q_+^2 \left[ 1 + \sum_{n=1}^{8} (e^{2\pi i \tilde{h}_n} + e^{-2\pi i \tilde{h}_n}) q_+ + \tilde{p}(h) q_+^2 + O(q_+^3) \right] \prod_{n=1}^{8} e^{\pi i \tilde{h}_n},
\]

and we have
\[
\mathcal{E}_{2,+}(\mu; z) = \frac{R_+(z + \mu - ia_-/2 - ia_+)R_+(z - \mu + ia_-/2)}{R_+(z - ia_-/2 - ia_+)R_+(z + ia_-/2)}
\]
\[
= e^{2\pi i \mu} \frac{R_+(z + \mu - ia_-/2)R_+(z - \mu + ia_-/2)}{R_+(z - ia_-/2)R_+(z + ia_-/2)}
\]
\[
= e^{2\pi i \mu} \frac{(1 - e^{2\pi i (z+\mu-ia_-/2)}) q_+ + q_+^2 + O(q_+^3))}{(1 - e^{2\pi i (z-ia_-/2)}) q_+ + q_+^2 + O(q_+^3))}
\]
\[
= e^{2\pi i \mu} [1 - (e^{2\pi i} - e^{-\pi i})q_+] (e^{\pi i} e^{\pi a_-} - e^{-\pi i} e^{-\pi a_-})(e^{2\pi i} - e^{-2\pi i}) q_+ - \mathcal{E}(\mu; z) q_+^2 + O(q_+^3),
\]

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\[ \mathcal{E}_{3,+}(\mu; z) = \frac{R_+(z + \mu - ia_-/2 + 1/2)R_+(z - \mu + ia_-/2 + 1/2 + ia_+)}{R_+(z - ia_-/2 + 1/2)R_+(z + ia_-/2 + 1/2 + ia_+)} = \mathcal{E}_{2,+}(\mu; z + 1/2) \]
\[ = e^{2\pi i \mu}[1 + (e^{\pi i \mu} - e^{-\pi i \mu})(e^{\pi i \mu} e^{\pi a_-} - e^{-\pi i \mu} e^{-\pi a_-})(e^{2\pi i z} + e^{-2\pi i z})q_+ - \mathcal{E}(\mu; z)q_+^2 + O(q_+^3)]. \]

where
\[ \mathcal{E}(\mu; z) = (e^{\pi i \mu} - e^{-\pi i \mu})(e^{\pi i \mu} e^{\pi a_-} - e^{-\pi i \mu} e^{-\pi a_-})[(e^{4\pi i z} + e^{-4\pi i z})(e^{-\pi a_-} + e^{\pi a_-}) \]
\[ - (e^{\pi i \mu} - e^{-\pi i \mu})(e^{\pi i \mu} e^{\pi a_-} - e^{-\pi i \mu} e^{-\pi a_-})]. \]

Therefore
\[ p_{2,+}(h)\mathcal{E}_{2,+}(\mu; z) + p_{3,+}(h)\mathcal{E}_{3,+}(\mu; z) = 2e^{2\pi i \mu} \prod_{n=1}^{8} e^{\pi i \hbar_n} \cdot \left[ q_-^2 + \tilde{p}(h) - \mathcal{E}(\mu; z) \right] \]
\[ + (e^{\pi i \mu} - e^{-\pi i \mu})(e^{\pi i \mu} e^{\pi a_-} - e^{-\pi i \mu} e^{-\pi a_-})(e^{2\pi i z} + e^{-2\pi i z}) \sum_{n=1}^{8} (e^{2\pi i \hbar_n} + e^{-2\pi i \hbar_n}) + O(q_+). \]

For \( t = 0, 1, 2, 3, \) we have
\[ \mathcal{E}_{t,+}(\mu; \omega_{t,+}) = \frac{R_+(\omega_{t,+}/2 + ia_+)}{R_+(ia_-/2 + ia_+)/2} \]
\[ = \frac{(1 - e^{2\pi i \mu} e^{\pi a_-})^2}{(1 - e^{\pi a_-})^2} \left[ 1 + 2(e^{\pi a_-} - e^{-\pi a_-} e^{-2\pi i \mu})(1 - e^{2\pi i \mu})q_+^2 + O(q_+^4) \right]. \]

Then
\[ p_{2,+}(h)\mathcal{E}_{2,+}(\mu; \omega_{2,+}) + p_{3,+}(h)\mathcal{E}_{3,+}(\mu; \omega_{3,+}) \]
\[ = 2(1 - e^{2\pi i \mu} e^{\pi a_-})^2 \prod_{n=1}^{8} e^{\pi i \hbar_n} \cdot \left[ q_-^2 + \tilde{p}(h) + 2(e^{\pi a_-} - e^{-\pi a_-} e^{-2\pi i \mu})(1 - e^{2\pi i \mu}) + O(q_+) \right]. \]

By combining with
\[ R_+(\mu - ia_+/2)R_+(\mu - ia_/-ia_+/2) \]
\[ = (1 - e^{2\pi i \mu})(1 - e^{2\pi i \mu} e^{2\pi a_-})(1 - (e^{2\pi a_-} e^{2\pi i \mu} + e^{-2\pi a_-} e^{-2\pi i \mu} + e^{2\pi i \mu} + e^{-2\pi i \mu})q_+^2 + O(q_+^4)), \]

we have
\[ p_{2,+}(h)(\mathcal{E}_{2,+}(\mu; z) - \mathcal{E}_{2,+}(\mu; \omega_{2,+})) + p_{3,+}(h)(\mathcal{E}_{3,+}(\mu; z) - \mathcal{E}_{3,+}(\mu; \omega_{3,+})) \]
\[ = \frac{8}{2R_+(\mu - ia_+/2)R_+(\mu - ia_- - ia_+/2)} \]
\[ \prod_{n=1}^{8} e^{\pi i \hbar_n} \cdot \left[ (e^{2\pi i z} + e^{-2\pi i z}) \sum_{n=1}^{8} (e^{2\pi i \hbar_n} + e^{-2\pi i \hbar_n}) - (e^{-\pi a_-} + e^{\pi a_-})(e^{4\pi i z} + e^{-4\pi i z}) \right] \]
\[ - \frac{q_-^2}{(1 - e^{2\pi a_-})^2} - \tilde{p}(h) + 4 \frac{q_-^2}{(1 - e^{2\pi a_-})^2} - e^{-\pi a_-}(e^{-\pi a_-} + e^{\pi a_-}) + O(q_+). \]
Therefore

\[
\sum_{k=0}^{3} p_{k,+}(h)(\mathcal{E}_{k,+}(\mu; z) - \mathcal{E}_{k,+}(\mu; \omega_{k,+}))
\]

\[
\frac{2}{2R_+(\mu - i\alpha_+/2)R_+(\mu - i\alpha_- - i\alpha+/2)}
\]

\[
= \prod_{n=1}^{8} e^{i\pi h_n}(e^{\pi h_n} - e^{-\pi h_n}) + \prod_{n=1}^{8} e^{i\pi h_n}(e^{\pi h_n} + e^{-\pi h_n}) + \prod_{n=1}^{8} e^{2\pi i h_n} + 8 e^{-\pi a_-} (e^{-\pi a_-} + e^{\pi a_-}) (e^{2\pi i} + e^{-2\pi i})
\]

\[
- \frac{12 + \sum_{1 \leq n < n' \leq 8} (e^{2\pi i h_n} + e^{-2\pi i h_n})(e^{2\pi i h_n'} + e^{-2\pi i h_n'}) + \frac{1}{2} \left\{ \prod_{n=1}^{8} (e^{2\pi i h_n} - 1) + \prod_{n=1}^{8} (e^{2\pi i h_n} + 1) \right\}}{(1 - e^{\pi a_-})^2}.
\]

\[
\]

\[
\]

Note that the operator \(A^{(1)}(h; z)\) does not contain the parameter \(\mu\), and it is not surprising because the Ruijsenaars–van Diejen operator \(A(h, q_+; z)\) is independent from the parameter \(\mu\) despite it appears in the expression.

To obtain the second degeneration in Section 2.3, we apply a gauge transformation to the operator in Proposition 2.1 by using the function \(R_-(z)\) defined in Equation (2.1), which satisfies

\[
R_-(z \mp i\alpha_-) = -e^{\pi a_-} e^{\pm 2\pi i} R_-(z).
\]

By the gauge transformation

\[
\tilde{A}^{(1)}(h, z) = R_-(z)^{-2} \circ A^{(1)}(h, z) \circ R_-(z)^2,
\]

we have the following operator:

\[
\tilde{A}^{(1)}(h; z) = \tilde{V}^{(1)}(h; z) \exp(-i\alpha_- \partial_z) + \tilde{W}^{(1)}(h; z) \exp(i\alpha_- \partial_z) + U^{(1)}(h; z),
\]

where \(U^{(1)}(h; z)\) was defined in Equation (2.10) and

\[
\tilde{V}^{(1)}(h; z) = -\frac{\prod_{n=1}^{8} (1 - e^{-2\pi i h_n} e^{\pi a_-})}{e^{-2\pi a_-} e^{4\pi i} (1 - e^{-4\pi i} e^{-2\pi a_-})},
\]

\[
\tilde{W}^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (1 - e^{2\pi i h_n} e^{-\pi a_-})}{e^{-2\pi a_-} e^{4\pi i} (1 - e^{4\pi i} e^{-2\pi a_-})}.
\]

Note that this operator was essentially obtained by van Diejen [16] in the multivariable case.
2.3 Second degeneration

We investigate a degeneration of the operator given by Equation (2.27).

PROPOSITION 2.2 In Equation (2.27), we replace $z \mapsto z \mp R, \tilde{h}_n$ ($n = 1, 2, 3, 4$) by $h_n + iR, \tilde{h}_n$ ($n = 5, 6, 7, 8$) by $h_n - iR$ and take the limit $R \to +\infty$. Then we arrive at the operator

$$A^{(2)}(h; z) = V^{(2)}(h; z) \exp(-ia_1\partial_z) + W^{(2)}(h; z) \exp(ia_1\partial_z) + U^{(2)}(h; z),$$

where

$$V^{(2)}(h; z) = e^{4\pi i\bar{z}} \prod_{n=1}^{4} (1 - e^{-2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}) \prod_{n=5}^{8} e^{2\pi i h_n},$$

$$W^{(2)}(h; z) = e^{2\pi a_-} e^{-4\pi i\bar{z}} \prod_{n=5}^{8} (1 - e^{2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}),$$

$$U^{(2)}(h; z) = \prod_{n=5}^{8} e^{2\pi i h_n} \left[ \left( \prod_{n=1}^{4} e^{2\pi i h_n} + \prod_{n=5}^{8} e^{-2\pi i h_n} \right) e^{-\pi a_-} e^{2\pi i\bar{z}} - (1 + e^{-2\pi a_-}) e^{4\pi i\bar{z}} \right]$$

$$+ \prod_{n=1}^{4} e^{-2\pi i h_n} \left[ \left( \prod_{n=5}^{8} e^{2\pi i h_n} + \prod_{n=1}^{4} e^{2\pi i h_n} \right) e^{-\pi a_-} e^{2\pi i\bar{z}} - (1 + e^{-2\pi a_-}) e^{-4\pi i\bar{z}} \right].$$

Namely we have

$$e^{-4\pi R} \widetilde{A}^{(1)}(h + iR; z + iR) f(z) \to A^{(2)}(h; z) f(z)$$

as $R \to +\infty$ for any $f(z)$, where $\nu = (1, 1, 1, 1, -1, -1, -1, -1)$.

Proof. We define the equivalence $a \sim b$ by $\lim_{R \to +\infty} a/b = 1$. Then

$$\tilde{V}^{(1)}(h + iR; z + iR)$$

$$= \frac{\prod_{n=1}^{4} (1 - e^{-2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}) \prod_{n=5}^{8} (1 - e^{-2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-})}{e^{-2\pi a_-} e^{-4\pi i\bar{z}} e^{4\pi R} (1 - e^{-4\pi i\bar{z}} e^{4\pi R}) (1 - e^{-4\pi i\bar{z}} e^{-4\pi R})}$$

$$\sim e^{4\pi i\bar{z}} e^{4\pi R} \prod_{n=1}^{4} (1 - e^{-2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}) \prod_{n=5}^{8} e^{2\pi i h_n},$$

$$\tilde{W}^{(1)}(h + iR; z + iR)$$

$$= \frac{\prod_{n=1}^{4} (1 - e^{2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}) \prod_{n=5}^{8} (1 - e^{2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-})}{e^{-2\pi a_-} e^{4\pi i\bar{z}} e^{4\pi R} (1 - e^{4\pi i\bar{z}} e^{4\pi R}) (1 - e^{4\pi i\bar{z}} e^{-4\pi R})}$$

$$\sim e^{2\pi a_-} e^{-4\pi i\bar{z}} e^{4\pi R} \prod_{n=5}^{8} (1 - e^{2\pi i\bar{z}} e^{2\pi i h_n} e^{-\pi a_-}),$$

$$\tilde{U}^{(1)}(h + iR; z + iR)$$

$$= \prod_{n=5}^{8} e^{2\pi i h_n} \left[ \left( \prod_{n=1}^{4} e^{2\pi i h_n} + \prod_{n=5}^{8} e^{2\pi i h_n} \right) e^{-\pi a_-} e^{2\pi i\bar{z}} - (1 + e^{-2\pi a_-}) e^{4\pi i\bar{z}} \right]$$

$$+ \prod_{n=1}^{4} e^{-2\pi i h_n} \left[ \left( \prod_{n=5}^{8} e^{2\pi i h_n} + \prod_{n=1}^{4} e^{2\pi i h_n} \right) e^{-\pi a_-} e^{2\pi i\bar{z}} - (1 + e^{-2\pi a_-}) e^{-4\pi i\bar{z}} \right].$$
\[ \tilde{U}^{(1)}(h + iRv; z + iR) \]
\[ = -e^{6\pi R} e^{-\pi a-} e^{2\pi iz} \prod_{n=5}^{8} e^{2\pi ih_n} \prod_{n=1}^{4} (1 - e^{2\pi ih_n} e^{-2\pi R}) \prod_{n=5}^{8} (1 - e^{-2\pi R} e^{-2\pi R})(1 - e^{-\pi a-} e^{2\pi ic} e^{-2\pi R}) \]
\[ + e^{6\pi R} e^{-\pi a-} e^{2\pi iz} \prod_{n=5}^{8} e^{2\pi ih_n} \prod_{n=1}^{4} (1 + e^{2\pi ih_n} e^{-2\pi R}) \prod_{n=5}^{8} (1 + e^{-2\pi R} e^{-2\pi R})(1 + e^{-\pi a-} e^{2\pi ic} e^{-2\pi R}) \]
\[ + e^{-\pi a-} \prod_{n=1}^{8} e^{\pi ih_n} \left\{ \left( \sum_{n=1}^{4} e^{2\pi ih_n} e^{-2\pi R} + e^{-2\pi ih_n} e^{2\pi R} \right) + \sum_{n=5}^{8} (e^{2\pi ih_n} e^{2\pi R} + e^{-2\pi ih_n} e^{-2\pi R}) \right\} \]
\[ \cdot (e^{2\pi iz} e^{-2\pi R} + e^{-2\pi iz} e^{2\pi R}) - (e^{-\pi a-} + e^{\pi a-})(e^{4\pi iz} e^{-4\pi R} + e^{-4\pi iz} e^{4\pi R}) \]
\[ \sim 0 \cdot e^{6\pi R} + e^{4\pi R} \prod_{n=5}^{8} e^{2\pi ih_n} \left\{ \left( \sum_{n=1}^{4} e^{2\pi ih_n} + \sum_{n=5}^{8} e^{-2\pi ih_n} \right) e^{-\pi a-} e^{2\pi iz} - (1 + e^{-2\pi a-}) e^{4\pi iz} \right\} \]
\[ + e^{4\pi R} e^{-\pi a-} \prod_{n=1}^{8} e^{\pi ih_n} \left\{ \left( \sum_{n=1}^{4} e^{-2\pi ih_n} + \sum_{n=5}^{8} e^{2\pi ih_n} \right) e^{-2\pi iz} - (e^{-\pi a-} + e^{\pi a-}) e^{-4\pi iz} \right\}. \]

Thus the proposition is obtained. \[\square\]

Set \( l_n = -h_{n+4} \) \((n = 1, 2, 3, 4)\). By the multiplication and the gauge transformation given by

\[\tilde{A}^{(2)}(h, l; z) = e^{\pi iz} \prod_{n=5}^{8} e^{-2\pi ih_n} \cdot e^{\pi iz} \circ A^{(2)}(h, z) \circ e^{-\pi iz}, \] (2.35)

we have the following operator:

\[\tilde{A}^{(2)}(h, l; z) = \tilde{V}^{(2)}(h; z) \exp(-ia_-.\partial_z) + \tilde{W}^{(2)}(l; z) \exp(ia_.\partial_z) + \tilde{U}^{(2)}(h, l; z), \] (2.36)

where

\[\tilde{V}^{(2)}(h; z) = e^{4\pi iz} \prod_{n=1}^{4} (1 - e^{2\pi ih_n} e^{-2\pi iz} e^{-2\pi ic}), \]
\[\tilde{W}^{(2)}(l; z) = e^{4\pi iz} \prod_{n=1}^{4} (1 - e^{2\pi ilh_n} e^{\pi a-} e^{-2\pi iz}), \] (2.37)

\[\tilde{U}^{(2)}(h, l; z) = \sum_{n=1}^{4} (e^{2\pi ih_n} + e^{2\pi ilh_n}) e^{2\pi iz} - (e^{\pi a-} + e^{-\pi a-}) e^{4\pi iz} \]
\[+ \prod_{n=1}^{4} e^{\pi ih_n + ilh_n} \left[ \sum_{n=1}^{4} (e^{-2\pi ih_n} + e^{-2\pi ilh_n}) e^{-2\pi iz} - (e^{-\pi a-} + e^{-\pi a-}) e^{-4\pi iz} \right]. \]

Note that this operator was also essentially obtained in [16].
Set $x = e^{2\pi i z}$, $q = e^{-2\pi a -}$, and replace $e^{2\pi i h_n}$ and $e^{2\pi i l_n}$ by $h_n$ and $l_n$. Then the difference operator $\tilde{A}^{(2)}(h, l; z)$ is written as

$$\tilde{A}^{(2)}(x)g(x) = x^{-2} \prod_{n=1}^{4} (1 - h_n q^{1/2}) g(x/q) + x^{-2} \prod_{n=1}^{4} (1 - l_n q^{-1/2}) g(qx) + U(x)g(x),$$

(2.38)

where

$$U(x) = -(q^{1/2} + q^{-1/2}) x^2 + \sum_{n=1}^{4} (h_n + l_n)x
+ \prod_{n=1}^{4} h_n^{1/2} l_n^{1/2} \cdot [-(q^{1/2} + q^{-1/2}) x^2 + \sum_{n=1}^{4} (h_n^{-1} + l_n^{-1}) x^{-1}].$$

The equation $A^{(2)}(x)g(x) = Eg(x)$ is also obtained as a specialization of the linear difference equation which is related with the $q$-Painlevé equation of type $E_7^{(1)}$ in [19]. We discuss it in Section 3.3.

### 2.4 Third degeneration

**Proposition 2.3** In Equation (2.36), we replace $z$ by $z - iR$, $h_n$ ($n = 1, 2$) by $h_n - iR$, $h_n$ ($n = 3, 4$) by $h_n + iR$, $l_n$ ($n = 1, 2, 3, 4$) by $l_n - iR$ and take the limit $R \to +\infty$. Then we arrive at the operator

$$A^{(3)}(h, l; z) = V^{(3)}(h; z) \exp(-ia_- \partial_z) + W^{(3)}(l; z) \exp(ia_- \partial_z) + U^{(3)}(h, l; z),$$

(2.39)

where

$$V^{(3)}(h; z) = e^{4\pi i z} \prod_{n=1}^{2} (1 - e^{2\pi i h_n} e^{-\pi a_-} e^{-2\pi i z}),$$

(2.40)

$$W^{(3)}(l; z) = e^{4\pi i z} \prod_{n=1}^{4} (1 - e^{2\pi i l_n} e^{\pi a_-} e^{-2\pi i z}),$$

$$U^{(3)}(h, l; z) = \left( \prod_{n=1}^{2} e^{2\pi i h_n} + \prod_{n=1}^{4} e^{2\pi i l_n} \right) e^{2\pi i z} - (e^{\pi a_-} + e^{-\pi a_-}) e^{4\pi i z}
+ e^{\pi i h_1} e^{\pi i h_2} (e^{\pi i (h_3 - h_4)} + e^{\pi i (h_4 - h_3)}) \prod_{n=1}^{4} e^{\pi i l_n} e^{-2\pi i z}.$$

Namely, as $R \to +\infty$ we have

$$e^{-4\pi R} \tilde{A}^{(2)}(h - iRv_1, l - iRv_2; z - iR) f(z) \to A^{(3)}(h, l; z) f(z)$$

(2.41)

for any $f(z)$, where $v_1 = (1, 1, -1, -1), v_2 = (1, 1, 1, 1)$.

**Proof.** We have

$$\tilde{V}^{(2)}(h - iRv_1; z - iR)$$

(2.42)
Thus the proposition is obtained. □

Set $x = e^{2\pi i c}, q = e^{-2\pi a-},$ and replace $e^{2\pi i h_a}$ and $e^{2\pi i l_a}$ by $h_n$ and $l_n$. Then the difference operator $A^{(3)}(h, l; z)$ is written as

$$A^{(3)}(x)g(x) = \prod_{n=1}^{2}(x - h_n q^{1/2})g(x/q) + x^{-2}\prod_{n=1}^{4}(x - l_n q^{-1/2})g(qx) + U(x)g(x), \quad (2.45)$$

$$U(x) = \left(\sum_{n=1}^{2} h_n + \sum_{n=1}^{4} l_n\right)x - (q^{1/2} + q^{-1/2})x^2 + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2}(h_3^{1/2} h_4^{-1/2} + h_3^{-1/2} h_4^{1/2})x^{-1}. \quad (2.46)$$

Let $E$ be a constant. The equation $A^{(3)}(x)g(x) = Eg(x)$ is written as

$$\prod_{n=1}^{2}(x - h_n q^{1/2})g(x/q) + x^{-2}\prod_{n=1}^{4}(x - l_n q^{-1/2})g(qx) + \{- (q^{1/2} + q^{-1/2})x^2 + \left(\sum_{n=1}^{2} h_n + \sum_{n=1}^{4} l_n\right)x - E + (l_1 l_2 l_3 l_4 h_1 h_2)^{1/2}(h_3^{1/2} h_4^{-1/2} + h_3^{-1/2} h_4^{1/2})x^{-1}\}g(x) = 0. \quad (2.46)$$
This equation is also obtained as a specialization of the linear difference equation which is related with the $q$-Painlevé equation of type $E_6^{(1)}$ in [19]. We discuss it in section 3.2. Set

$$\bar{A}^{(3)}(x) = v(x) \circ A^{(3)}(h, l; z) \circ v(x)^{-1}. \quad (2.47)$$

$$v(x) = (x^{-1}l_4q^{1/2}; q)_\infty = \prod_{k=0}^{\infty} (1 - x^{-1}l_4q^{1/2}q^k).$$

We replace $h_3, h_4$ and $l_4$ by $\tilde{h}, 1$ and $h_3$ in Equation (2.46). Then the gauge transformed equation $\bar{A}^{(3)}(x)g(x) = Eg(x)$ is written as

$$\prod_{n=1}^{3}(x - h_nq^{1/2})g(x/q) + \prod_{n=1}^{3}(x - l_nq^{-1/2})g(qx) + \left\{ - (q^{1/2} + q^{-1/2})x^3 \right\}$$

$$+ \sum_{n=1}^{3}(h_n + l_n)x^2 - Ex + (l_1l_2l_3h_1h_2h_3)^{1/2}(\tilde{h}^{1/2} + \tilde{h}^{-1/2}) \right\}g(x) = 0.$$

To obtain the fourth degeneration, we apply the gauge transformation

$$\tilde{A}^{(3)}(h, l; z) = R^-(z)^2 \circ A^{(3)}(h, l; z) \circ R^-(z)^{-2}. \quad (2.49)$$

Then we have

$$\tilde{A}^{(3)}(h, l; z) = \tilde{V}^{(3)}(h; z) \exp(-ia_\cdot \partial_z) + \tilde{W}^{(3)}(l; z) \exp(ia_\cdot \partial_z) + U^{(3)}(h, l; z), \quad (2.50)$$

where $U^{(3)}(h, l; z)$ was defined in Equation (2.40) and

$$\tilde{V}^{(3)}(h; z) = e^{-2\pi a - 2} \prod_{n=1}^{2}(1 - e^{2\pi it_n}e^{-2\pi iz}), \quad (2.51)$$

$$\tilde{W}^{(3)}(l; z) = e^{-2\pi a - 2} \prod_{n=1}^{4}(1 - e^{2\pi il_n}e^{-2\pi iz}).$$

2.5 Fourth degeneration and $q$-Heun equation

Proposition 2.4 In Equation (2.50), we replace $z$ by $z + iR, h_n (n = 1, 2, 3, 4)$ by $h_n + iR, l_n (n = 1, 2, 3, 4)$ by $l_n - iR$ and take the limit $R \to +\infty$. Then we arrive at the operator

$$A^{(4)}(h, l; z) = V^{(4)}(h; z) \exp(-ia_\cdot \partial_z) + W^{(4)}(l; z) \exp(ia_\cdot \partial_z) + U^{(4)}(h, l; z), \quad (2.52)$$

where

$$V^{(4)}(h; z) = e^{-2\pi a - 2} \prod_{n=1}^{2}(1 - e^{2\pi it_n}e^{-2\pi iz}), \quad (2.53)$$
The proposition follows from the asymptotics

\[ W^{(4)}(l; z) = e^{4\pi iz} e^{2\pi i(l_3 + l_4)} \prod_{n=1}^{2} (1 - e^{2\pi i l_n} e^{\pi a_+} e^{-2\pi i z}), \]

\[ U^{(4)}(h, l; z) = (e^{2\pi i l_3} + e^{2\pi i l_4}) e^{2\pi iz} + e^{\pi i h_1} e^{\pi i h_2} (e^{\pi i(h_3 - h_4)} + e^{\pi i(h_4 - h_3)}) \prod_{n=1}^{4} e^{\pi i l_n} e^{-2\pi i z}. \]

Namely, as \( R \to +\infty \) we have

\[ \tilde{A}^{(3)}(h + iRv_2, l + iRv_1; z) \to A^{(4)}(h, l; z)(z) \] (2.54)

for any \( f(z) \), where \( v_1 = (1, 1, -1, -1) \), \( v_2 = (1, 1, 1, 1) \).

**Proof.** The proposition follows from the asymptotics

\[ \tilde{V}^{(3)}(h + iRv_2; z + iR) = e^{-2\pi a_-} \prod_{n=1}^{2} (1 - e^{2\pi i l_n} e^{-\pi a_-} e^{-2\pi i z}), \] (2.55)

\[ \tilde{W}^{(3)}(l + iRv_1; z + iR) = e^{-8\pi R} e^{-2\pi a_-} \prod_{n=1}^{2} (1 - e^{2\pi i l_n} e^{\pi a_+} e^{-2\pi i z}). \] (2.56)

\[ \tilde{U}^{(3)}(h + iRv_2, l + iRv_1; z + iR) = \left( \sum_{n=1}^{2} e^{2\pi i l_n} e^{-2\pi R} + \sum_{n=1}^{2} e^{2\pi i l_n} e^{-2\pi R} + \sum_{n=3}^{4} e^{2\pi i l_n} e^{2\pi R} \right) e^{2\pi iz} e^{-2\pi R} \]

\[ - (e^{\pi a_-} + e^{-\pi a_-}) e^{4\pi iz} e^{-4\pi R} + \prod_{n=1}^{4} e^{\pi i l_n} e^{\pi i h_1} e^{\pi i h_2} e^{-2\pi R} (e^{\pi i(h_3 - h_4)} + e^{\pi i(h_4 - h_3)}) e^{-2\pi iz} e^{2\pi R} \]

\[ \sim (e^{2\pi i l_3} + e^{2\pi i l_4}) e^{2\pi iz} + \prod_{n=1}^{4} e^{\pi i l_n} e^{\pi i h_1} e^{\pi i h_2} (e^{\pi i(h_3 - h_4)} + e^{\pi i(h_4 - h_3)}) e^{-2\pi iz}. \]

By the multiplication and the gauge transformation given by

\[ \tilde{A}^{(4)}(h, l; z) = -R^-(z)^{-1} e^{-\pi iz} \circ A^{(4)}(h, l; z) \circ R^-(z) e^{\pi iz}, \] (2.58)

we have

\[ \tilde{A}^{(4)}(h, l; z) = \tilde{V}^{(4)}(h; z) \exp(-ia_- \partial_z) + \tilde{W}^{(4)}(l; z) \exp(ia_- \partial_z) - U^{(4)}(h, l; z), \] (2.59)
which was defined in Equation (2.53) and

\[ \tilde{V}^{(4)}(h; z) = e^{2\pi i z} \prod_{n=1}^{2} (1 - e^{2\pi i h} e^{-\pi a} e^{-2\pi i c}), \]  

\( (2.60) \)

\[ \tilde{W}^{(4)}(l; z) = e^{2\pi i l} e^{2\pi i d} e^{2\pi i c} \prod_{n=1}^{2} (1 - e^{2\pi i h} e^{\pi a} e^{-2\pi i c}). \]

Set \( x = e^{2\pi i z}, \ q = e^{-2\pi a}, \) replace \( e^{2\pi i h} \) and \( e^{2\pi i l} \) by \( h_n \) and \( l_n \), and set \( h_4 = 1 \). Then the difference operator is written as

\[ A^{(4)}(x)g(x) = x^{-1}(x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q) \]

\[ - \{(l_3 + l_4)x + (l_1 l_2 l_3 l_4 l_1 h_1 h_2)\} x^{-1}g(x) \]

\[ + x^{-1}l_3 l_4 (x - l_1 q^{-1/2})(x - l_2 q^{-1/2})g(qx). \]  

Let \( E \) be a constant. The equation \( A^{(4)}(x)g(x) = Eg(x) \) is written as

\[ (x - h_1 q^{1/2})(x - h_2 q^{1/2})g(x/q) + l_3 l_4(x - l_1 q^{-1/2})(x - l_2 q^{-1/2})g(qx) \]

\[ - \{(l_3 + l_4)x + (l_1 l_2 l_3 l_4 l_1 h_1 h_2)\} x^{-1}g(x) = 0. \]  

This equation is also obtained as a specialization of the linear difference equation which is related with the \( q \)-Painlevé VI equation [17]. We discuss it in section 3.1.

We call Equation (2.62) the \( q \)-Heun equation, because it has a limit to the Heun equation, which we show in this subsection. We rewrite Equation (2.62) as

\[ (x - t_1 q^{-1/2})(x - t_2 q^{1/2})g(x/q) + q^{-1} x^{-1}g(x) + q^{-1} x^{-1}g(qx) \]

\[ - \{(q^{1/2} + q^{-1/2})x^2 - (2(t_1 + t_2) + (q - 1)E_1 + (q - 1)^2 E)x \]

\[ + t_1 t_2 q^{l_1 + l_2 + l_3 + l_4 + h_1 + h_2/2} (q^{h_3/2} + q^{-h_3/2}) \} g(x) = 0, \]

where

\[ E_1 = (l_3 + l_4)(t_1 + t_2) + (l_1 + h_1)t_1 + (l_2 + h_2)t_2. \]  

Set \( q = 1 + \varepsilon \). We divide Equation (2.63) by \( \varepsilon^2 \). By using Taylor’s expansion

\[ g(x/q) = g(x) + (-\varepsilon + \varepsilon^2)xg'(x) + \varepsilon^2 x^2 g''(x)/2 + O(\varepsilon^3), \]

\[ g(qx) = g(x) + \varepsilon xg'(x) + \varepsilon^2 x^2 g''(x)/2 + O(\varepsilon^3), \]

we find the following limit as \( \varepsilon \to 0 \):

\[ x^2(x - t_1)(x - t_2)g''(x) \]

\[ + [(1 + h_2 - l_2)x(x - t_1) + (1 + h_1 - l_1)x(x - t_2) + (h_1 - 1)(x - t_1)(x - t_2)]xg'(x) \]

\[ + [l_3 l_4 x^2 + B x + t_1 t_2 (h_2/2 - 1 + h_3/2)(h_2/2 - 1 - h_3/2)]g(x) = 0, \]
where
\[
\tilde{l} = l_1 + l_2 + l_3 + l_4 - h_1 - h_2,
\]
and
\[
\tilde{B} = \tilde{E} - \frac{t_1}{2}\left[ h_1^2 + (l_3 + l_4 + l_1 - 1)^2 - \frac{1}{2}\right] + \frac{t_2}{2}\left[ h_2^2 + (l_3 + l_4 + l_2 - 1)^2 - \frac{1}{2}\right].
\]

This is a Fuchsian differential equation with four singularities \(\{0, t_1, t_2, \infty\}\) and the local exponents are given by the following Riemann scheme:
\[
\begin{pmatrix}
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1
\end{pmatrix}
\]
\[
(2.68)
\]
By setting \(z = x/t_1\) and \(g(x) = x^{1-i/2-h_3/2} \tilde{g}(z)\), the function \(y = \tilde{g}(z)\) satisfies Heun’s differential equation;
\[
\frac{d^2 y}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z - t}\right) \frac{dy}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - t)} y = 0,
\]
where \(t = t_2/t_1, \gamma = 1 - h_3, \delta = 1 + h_1 - l_1, \epsilon = 1 + h_2 - l_2, \{\alpha, \beta\} = \{l_3 + 1 - \tilde{l}/2 - h_3/2, l_4 + 1 - \tilde{l}/2 - h_3/2\}\) and \(t = t_2/t_1\). Therefore it is reasonable to call Equation (2.62) the \(q\)-Heun equation.

### 3. Linear \(q\)-difference equations related with \(q\)-Painlevé equations

It is widely known that the Lax formalism is a powerful method for studying soliton equations, and a similar method is applied for Painlevé-type equations. In particular, Jimbo and Sakai [17] obtained \(q\)-Painlevé VI (or the \(q\)-Painlevé equation of type \(D_5^{(1)}\)) by finding “Lax forms.” We may regard the Lax forms for difference Painlevé equations as a linear difference equation and an associated deformation equation where the difference Painlevé equation is written in the form of compatibility of them (see [21]).

About 10 years later from the discovery of \(q\)-Painlevé VI, Sakai [21] presented a problem to find Lax forms for difference Painlevé equations in his list [9], and Yamada made a significant contribution for the problem. Namely, Yamada discovered Lax forms for the \(q\)-difference Painlevé equations of types \(D_5^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}\) [19] and the elliptic difference Painlevé equation [22]. For type \(D_5^{(1)}\), it essentially coincides with the one by Jimbo and Sakai [17].

In this section, we observe that our degenerate operators of Ruijsenaars–van Diejen appear by restricting the parameters in the linear \(q\)-differential equations related to \(q\)-Painlevé equations of types \(D_5^{(1)}, E_6^{(1)}, E_7^{(1)}\).

#### 3.1 Linear \(q\)-difference equation related with \(q\)-Painlevé VI

Jimbo and Sakai [17] found a Lax form for the \(q\)-Painlevé VI by considering connection preserving deformation of the linear system of \(q\)-differential equations
\[
Y(qx, t) = A(x, t)Y(x, t).
\]
\[
(3.1)
\]
To derive $q$-Painlevé VI, they rewrote the condition for connection preserving deformation as compatibility of Equation (3.1) with a deformation equation of the form $Y(x, qt) = B(x, t)Y(x, t)$, which is a Lax form of $q$-Painlevé VI.

We focus on Equation (3.1). The $2 \times 2$ matrix $A(x, t)$ is taken in the form Equations (9)-(11) in [17], i.e.

$$
A(x, t) = A_0(t) + A_1(t)x + A_2x^2 = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad (3.2)
$$

$$
A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad A_0(t) \text{ has eigenvalues } t\theta_1, t\theta_2, \\
\text{det } A(x, t) = \kappa_1 \kappa_2(x - ta_1)(x - ta_2)(x - a_3)(x - a_4).
$$

Note that we have the relation $\kappa_1 \kappa_2 a_1 a_2 a_3 a_4 = \theta_1 \theta_2$. Define $\lambda, \mu_1, \mu_2$ by

$$
\mu_1 = a_{11}(\lambda)/\kappa_1, \quad \mu_2 = a_{22}(\lambda)/\kappa_2 \quad (3.3)
$$

so that $\mu_1 \mu_2 = (\lambda - ta_1)(\lambda - ta_2)(\lambda - a_3)(\lambda - a_4)$ (see Equation (14) in [17]) and introduce $\mu$ by

$$
\mu = (\lambda - ta_1)(\lambda - ta_2)/(q\kappa_1 \mu_1).
$$

Then the matrix elements can be parametrized by these variables and the gauge freedom $w$ (see page 149 of [17]).

The first component of $Y(x)$ satisfies the following equation:

$$
Y_1(q^2 x) - \left( a_{11}(qx) + \frac{a_{12}(qx)}{a_{12}(x)} a_{22}(x) \right) Y_1(qx)
$$

$$
+ \frac{a_{12}(qx)}{a_{12}(x)} (a_{11}(x) a_{22}(x) - a_{12}(x) a_{21}(x)) Y_1(x) = 0.
$$

In our parametrization, we have

$$
\frac{a_{12}(qx)}{a_{12}(x)} (a_{11}(x) a_{22}(x) - a_{12}(x) a_{21}(x)) = \frac{qx - \lambda}{x - \lambda} \kappa_1 \kappa_2 (x - ta_1)(x - ta_2)(x - a_3)(x - a_4),
$$

$$
a_{11}(qx) + \frac{a_{12}(qx)}{a_{12}(x)} a_{22}(x) = \frac{q(qk_1 + \kappa_2)x^3 + c_2 x^2 + c_1 x - \lambda t(\theta_1 + \theta_2)}{x - \lambda},
$$

$$
c_2 = \frac{q^2 \kappa_1 \kappa_2 (\lambda - a_3) (\lambda - a_4) \mu}{\lambda} - (q + 1)(qk_1 + \kappa_2)\lambda - \frac{q(\theta_1 + \theta_2)}{\lambda} + \frac{(\lambda - a_1 t)(\lambda - a_2 t)}{\mu},
$$

$$
c_1 = -q \kappa_1 \kappa_2 (\lambda - a_3) (\lambda - a_4) \mu + (q k_1 + \kappa_2) \lambda^2 + (q + 1) t(\theta_1 + \theta_2) - \frac{(\lambda - a_1 t)(\lambda - a_2 t)}{\mu}.
$$

Note that there are two accessory parameters $\lambda$ and $\mu$, which play the role of dependent variables in the $q$-Painlevé VI equation (see Equations (19), (20) in [17]). We may regard Equation (3.4) with Equation (3.5) as a $q$-difference analogue of Equation (1.5) in the differential equations. We impose a restriction on the accessory parameters as we have done for Equation (1.5) to obtain the Heun equation. Here we require $\lambda = a_3$. Then we have

$$
Y_1(q^2 x) - \{ q(qk_1 + \kappa_2)x^2 + d_1 x + t(\theta_1 + \theta_2) \} Y_1(qx)
$$

$$
+ \kappa_1 \kappa_2 (qx - a_3) (x - ta_1)(x - ta_2)(x - a_4) Y_1(x) = 0, \quad (3.6)
$$
where
\[ d_1 = \frac{(a_1 t - a_3)(a_2 t - a_3)}{a_3 \mu} - a_3(q \kappa_1 + \kappa_2) - \frac{gt(\theta_1 + \theta_2)}{a_3}. \] (3.7)

Let \( u(x) \) be the function which satisfies \( u(qx) = (x - ta_1)(x - ta_2)u(x) \), e.g. \( u(x) = x^{(\log q a_1)/\log q}(x/ta_1); q_{\infty}(x/(ta_2); q_{\infty}) \). Then the function \( f(x) = Y_1(qx)/u(qx) \) satisfies
\[
(x - ta_1)(x - ta_2)f(qx) - ((q \kappa_1 + \kappa_2)/q)x^2 + (d_1/q)x + t(\theta_1 + \theta_2)f(x) \]
\[ + (\kappa_1 \kappa_2/q)(x - a_3)(x - qa_4)f(x/q) = 0. \] (3.8)

Note that there is the relation \( \kappa_1 \kappa_2 a_1 a_2 a_3 a_4 = \theta_1 \theta_2 \). In Equation (2.62), we set
\[
l_1 = a_1 t q^{1/2}, \quad l_2 = a_2 t q^{1/2}, \quad h_1 = a_3 q^{-1/2}, \quad h_2 = a_4 q^{1/2}, \]
\[ l_3 = 1/\kappa_1, \quad l_4 = q/\kappa_2, \quad \theta_1 = (a_1 a_2 a_3 a_4 \kappa_1 \kappa_2)^{-1/2}, \quad E = d_1/(\kappa_1 \kappa_2). \] (3.9)

Then we have \( h_3^{1/2} = \theta_2 (a_1 a_2 a_3 a_4 \kappa_1 \kappa_2)^{-1/2} \) and Equation (3.8).

Hence Equation (3.8) is equivalent to the fourth degeneration of the Ruijsenaars–van Diejen operator. We may regard \( d_1 \) as an accessory parameter.

3.2 Linear q-difference equation related with q-Painlevé equation of type \( E_6^{(1)} \)

Yamada [19] derived a q-difference Painlevé equation of type \( E_6^{(1)} \) by Lax formalism, i.e. the compatibility condition for two linear q-difference equations. One of the linear difference equations is written as Equation (40) in [19], i.e.
\[
\left( \frac{b_1 q - z}{q(f q - z)z^4} \right) y(z/q) - \frac{gz}{t^2(g z - q)} y(z) + \left( \frac{b_3 g - 1)(b_2 g - 1)(b_4 g - 1)}{g(f g - 1)z^2(g z - q)} - \frac{b_6(b_7 g - t)(b_8 g - t)}{fg^2} \right) y(z) = 0. \] (3.10)

We may regard \( f, g \) as accessory parameters. The other linear equation (deformation equation) contains the difference on the variable \( t \) as well as the variable \( x \) (see Equation (40) in [19]). By compatibility condition for the two linear difference equations, we have q-Painlevé equation of type \( E_6^{(1)} \) for the dependent variables \( f \) and \( g \) and the independent variable \( t \) (see Equation (39) in [19]).

We now specialize the parameters \( f \) and \( g \) to \( f = b_1 \) in Equation (3.10). Then we obtain
\[
\left( \frac{b_2 q - z}{q z^4} \right) y(z/q) + \frac{c(z)}{z^{2q^{1/2}}(b_1 - z)} y(z) + \frac{(z - b_1 t)(z - b_6 t)}{(b_1 - z)z^{2q^{1/2}}} y(qz) = 0, \] (3.11)
\[
c(z) = -(q^{1/2} + q^{-1/2})z^2 + (b_1 q^{-1/2} + b_2 q^{1/2} + b_3 q^{1/2} + b_4 q^{1/2} + b_5 t q^{1/2} + b_6 t q^{1/2})z
\[ + c_1 + \frac{q^{1/2} b_3 b_6 (b_7 + b_6)}{z} , \]
\[ d_1 = \frac{(a_1 t - a_3)(a_2 t - a_3)}{a_3 \mu} - a_3(q \kappa_1 + \kappa_2) - \frac{gt(\theta_1 + \theta_2)}{a_3}. \] (3.7)
and the term $c_1$ contains the accessory parameter $g$. Note that there is a relation $qb_ib_2b_3b_4 = b_5b_6b_7b_8$ in Yamada’s article. We apply a gauge transformation by $y(z) = z^{-1/2 + 2(\log t)/\log q} \tilde{y}(z)$. Then we have

$$
\frac{(z - b_1)(z - b_2)(z - b_3)(z - b_4)}{z^2} \tilde{y}(z/q) + c(z)\tilde{y}(z) + (z - b_5)(z - b_6)\tilde{y}(qz) = 0. 
$$

(3.12)

By replacing $q$ to $q^{-1}$, it turns out that Equation (3.12) is the third degeneration of the Ruijsenaars–van Diejen operator in Equation (2.46). The eigenvalue $E$ in Equation (2.46) essentially corresponds to the accessory parameter $g$ in $c_1$.

### 3.3 Linear $q$-difference equation related with $q$-Painlevé equation of type $E_7^{(1)}$

Yamada [19] also derived a $q$-difference Painlevé equation of type $E_7^{(1)}$ by Lax formalism, i.e. the compatibility condition for two linear $q$-difference equations. Set

$$
B_1(z) = (1 - b_1)(1 - b_2)(1 - b_3)(1 - b_4)(1 - b_5), \quad B_2(z) = (1 - b_1)(1 - b_2)(1 - b_3)(1 - b_4)(1 - b_5). 
$$

(3.13)

Then one of the $q$-difference equations is written as Equation (37) in [19], i.e.

$$
\frac{B_2(t/z)}{t^2(f - z)} \left[ y(qz) - \frac{t^2(1 - gz)}{t^2 - gz} y(z) \right] + \frac{B_1(q/z)}{q(f - z)} \left[ y(z/q) - \frac{qt^2 - gz}{t^2(q - gz)} y(z) \right] 
$$

$$
+ \frac{(1 - t^2)}{gz^2} \left[ \frac{qB_1(g)}{(fg - 1)(gz - q)} - \frac{t^2B_2(g/t)}{(fg - t^2)(gz - t^2)} \right] y(z) = 0. 
$$

(3.14)

We also obtain the $q$-Painlevé equation of type $E_7^{(1)}$ (see Equation (36) in [19]) by a compatibility condition of Equation (3.14) with the deformation equation written as Equation (38) in [19].

By specializing to $f = b_1$ in Equation (3.14) and applying a gauge transformation, we have

$$
(z - b_5)(z - b_6)(z - b_7)(z - b_8)y(qz) - c(z)y(z) 
$$

$$
+ (z - b_1)(z - b_2)(z - b_3)(z - b_4)(z - b_5)(z - b_6)(z - b_7)(z - b_8)y(qz/q) = 0,
$$

(3.15)

where

$$
c(z) = q^{-1/2}((1 + q)^2 + c_3q^3 + c_2q^2 + c_1q + (b_5b_6b_7b_8 + q^2b_1b_2b_3b_4)q),
$$

$$
c_3 = -(b_1 + b_2q + b_3q + b_4q + b_5q + b_6q + b_7q + b_8q),
$$

$$
c_1 = -q(b_2b_3b_4q^2 + b_1b_2b_3q + b_1b_3b_4q + b_1b_2b_3q^2 + b_1b_2b_4q + b_1b_3b_4q + b_2b_3b_4q + b_1b_2b_3q^2 + b_1b_2b_3q + b_1b_2b_4q) + b_6b_7b_8 + b_5b_7b_8 + b_5b_6b_8 + b_5b_6b_7 + b_5b_6b_7 + b_5b_6b_7),
$$

and the term $c_2$ contains the accessory parameter $g$. There is a relation $qb_1b_2b_3b_4 = b_5b_6b_7b_8$ in Yamada’s article. Equation (3.15) is the second degeneration of Ruijsenaars–van Diejen operator in Equation (2.38) by setting

$$
h_1 = b_1q^{-1/2}, \quad h_2 = b_2q^{1/2}, \quad h_3 = b_3q^{1/2}, \quad h_4 = b_4q^{1/2},
$$

$$
l_1 = b_5q^{1/2}, \quad l_2 = b_6q^{1/2}, \quad l_3 = b_7q^{1/2}, \quad l_4 = b_8q^{1/2}. 
$$

(3.17)
Note that we used the relation \((h_1 h_2 h_3 h_4 l_1 l_2 l_3 l_4)^{1/2} = q t_2 b_3 b_5 b_7 b_8 = q^2 t^2 b_1 b_2 b_3 b_4\), which follows from \(q b_1 b_2 b_3 b_4 = b_3 b_5 b_7 b_8\).

4. Degeneration of Ruijsenaars–van Diejen operator with \(N\) variables

We describe Ruijsenaars–van Diejen operator \((1.10)\) of \(4\) variables and investigate degenerations of the operator. By using the notation in Section 2, the Ruijsenaars–van Diejen operator of \(N\) variables is given by

\[
A_+(h; z) = \sum_{j=1}^{N} (V_{j,+}(h; z) \exp(-ia_- \partial_{z_j}) + V_{j,-}(h; -z) \exp(ia_- \partial_{z_j})) + V_{h,+}(h; z),
\]

where

\[
V_{j,+}(h; z) = \frac{\prod_{n=1}^{8} R_+(z_j - h_n - ia_-/2)}{R_+(2z_j + ia_/2)R_+(2z_j - ia_- + ia_/2)} \cdot \prod_{k \neq j} \frac{R_+(z_j - z_k - \mu + ia_/2)R_+(z_j + z_k - \mu + ia_/2)}{R_+(z_j - z_k + ia_/2)R_+(z_j + z_k + ia_/2)},
\]

\[
V_{h,+}(h; z) = \frac{\sum_{n=0}^{3} p_{i,+}(h)[\prod_{j=1}^{N} \mathcal{E}_{i,+}(\mu; z_j) - \mathcal{E}_{i,-}(\mu; \omega e)^N]}{2R_+(\mu - ia_-/2)R_+(\mu - ia_- - ia_/2)}.
\]

We set \(h_n = \tilde{h}_n - ia_/2\). By the limit \(q_+ \to 0\), we have the following proposition, whose proof is similar to the case of one variable.

**Proposition 4.1** As \(q_+ \to 0\), the Ruijsenaars–van Diejen operator in Equation (4.1) corresponds to the operator

\[
A^{(1)}(h; z) = \sum_{j=1}^{N} (V^{(1)}_{j}(h; z) \exp(-ia_- \partial_{z_j}) + V^{(1)}_{j}(h; -z) \exp(ia_- \partial_{z_j})) + U^{(1)}(h; z),
\]

up to some additive constant, where

\[
V^{(1)}_{j}(h; z) = \prod_{n=1}^{8} \frac{(1 - e^{-2\pi i \xi_j} e^{2\pi i h_n} e^{-\pi a_-})}{(1 - e^{-4\pi i \xi_j}) (1 - e^{-2\pi i \xi_j} e^{-2\pi a_-})} \prod_{k \neq j} \frac{(1 - e^{2\pi i (z_k - z_j)} e^{2\pi i \mu}) (1 - e^{-2\pi i (z_k + z_j)} e^{2\pi i \mu})}{(1 - e^{2\pi i (z_k - z_j)}) (1 - e^{-2\pi i (z_k + z_j)})},
\]

\[
U^{(1)}(h; z) = \frac{\prod_{n=1}^{8} (e^{2\pi \xi h_n} - 1)}{2(e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)} \prod_{j=1}^{N} \left\{ e^{2\pi i \mu} + \frac{(e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)}{(1 - e^{2\pi i \xi_j} e^{\pi a_-})(1 - e^{-2\pi i \xi_j} e^{\pi a_-})} \right\}
\]

\[
+ \frac{\prod_{n=1}^{8} (e^{2\pi \xi h_n} + 1)}{2(e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)} \prod_{j=1}^{N} \left\{ e^{2\pi i \mu} + \frac{(e^{2\pi i \mu} - 1)(e^{2\pi i \mu} e^{2\pi a_-} - 1)}{(1 + e^{2\pi i \xi_j} e^{\pi a_-})(1 + e^{-2\pi i \xi_j} e^{\pi a_-})} \right\}
\]
By applying the gauge transformation with respect to the function $(R_-(z_1) \ldots R_-(z_N))^2$, we arrive at the following operator

$$\tilde{A}^{(1)}(h; z) = \sum_{j=1}^{N} (\tilde{V}^{(1)}_j(h; z) \exp(-ia_\partial z_j) + \tilde{W}^{(1)}_j(h; z) \exp(ia_\partial z_j)) + U^{(1)}(h; z),$$

where $U^{(1)}(h; z)$ was defined in Equation (4.5) and

$$\tilde{V}^{(1)}_j(h; z) = \frac{\prod_{n=1}^{8} (1 - e^{-2\pi i\mu} e^{2\pi i\mathbf{h}_n} e^{-\pi a_-})}{e^{-2\pi a_-} e^{-4\pi i\mu j} (1 - e^{-4\pi i\mu j}) (1 - e^{2\pi i\mu j} e^{-2\pi a_-})} \prod_{k \neq j} \frac{(1 - e^{2\pi i(z_k - z_j)} e^{2\pi i\mu}) (1 - e^{-2\pi i(z_k - z_j)} e^{-2\pi i\mu})}{(1 - e^{2\pi i(z_k - z_j)}) (1 - e^{-2\pi i(z_k - z_j)})},$$

$$\tilde{W}^{(1)}_j(h; z) = \frac{\prod_{n=1}^{8} (1 - e^{2\pi i\mu} e^{2\pi i\mathbf{h}_n} e^{-\pi a_-})}{e^{-2\pi a_-} e^{4\pi i\mu j} (1 - e^{4\pi i\mu j}) (1 - e^{-4\pi i\mu j} e^{2\pi a_-})} \prod_{k \neq j} \frac{(1 - e^{-2\pi i(z_k - z_j)} e^{2\pi i\mu}) (1 - e^{2\pi i(z_k - z_j)} e^{-2\pi i\mu})}{(1 - e^{-2\pi i(z_k - z_j)}) (1 - e^{2\pi i(z_k - z_j)})}. $$

Note that this operator was essentially obtained by van Diejen [16].

We apply the second degeneration.

**Proposition 4.2** In Equation (4.6), we replace $z$ by $z + iR, \tilde{h}_n$ ($n = 1, 2, 3, 4$) by $h_n + iR, \tilde{h}_n$ ($n = 5, 6, 7, 8$) by $h_n - iR$ and take the limit $R \to +\infty$. Then we have the operator

$$A^{(2)}(h; z) = \sum_{j=1}^{N} (V^{(2)}_j(h; z) \exp(-ia_\partial z_j) + W^{(2)}_j(h; z) \exp(ia_\partial z_j)) + U^{(2)}(h; z),$$

where

$$V^{(2)}_j(h; z) = e^{4\pi i\mu j} e^{2(N-1)\pi i\mu} \prod_{n=1}^{4} (1 - e^{-2\pi i\mu} e^{2\pi i\mathbf{h}_n} e^{-\pi a_-}) \prod_{n=5}^{8} e^{2\pi i\mathbf{h}_n} \prod_{k \neq j} \frac{(1 - e^{2\pi i(z_k - z_j)} e^{2\pi i\mu})}{(1 - e^{2\pi i(z_k - z_j)})},$$

$$W^{(2)}_j(h; z) = e^{2\pi a_-} e^{-4\pi i\mu j} \prod_{n=5}^{8} (1 - e^{2\pi i\mu} e^{2\pi i\mathbf{h}_n} e^{-\pi a_-}) \prod_{k \neq j} \frac{(1 - e^{-2\pi i(z_k - z_j)} e^{2\pi i\mu})}{(1 - e^{-2\pi i(z_k - z_j)})}.$$
In Equation (4.10), we replace where

\[
\begin{align*}
A\left( h; z \right) &= e^{2(N-1)\pi i\mu} \prod_{n=1}^{N} e^{2\pi i\eta_{n}} \left[ \left( \sum_{n=1}^{N} e^{2\pi i\eta_{n}} + \sum_{n=5}^{8} e^{-2\pi i\eta_{n}} \right) e^{-\pi a_{-}} \sum_{j=1}^{N} e^{2\pi i\eta_{j}} \right] \\
- \left( 1 + e^{-2\pi a_{-}} \right) \sum_{j=1}^{N} e^{4\pi i\eta_{j}} + e^{-2\pi i\mu} e^{2\pi a_{-}} \left( e^{2\pi i\mu} e^{2\pi a_{-}} - 1 \right) \sum_{1 \leq j < k \leq N}^{N} e^{2\pi i\eta_{j} e^{2\pi i\eta_{k}}} \\
+ e^{2(N-1)\pi i\mu} \prod_{n=1}^{N} e^{\pi i\eta_{n}} \left[ \left( \sum_{n=1}^{N} e^{-2\pi i\eta_{n}} + \sum_{n=5}^{8} e^{2\pi i\eta_{n}} \right) e^{-\pi a_{-}} \sum_{j=1}^{N} e^{-2\pi i\eta_{j}} \right] \\
- \left( 1 + e^{-2\pi a_{-}} \right) \sum_{j=1}^{N} e^{-4\pi i\eta_{j}} + e^{-2\pi i\mu} e^{-2\pi a_{-}} \left( e^{2\pi i\mu} e^{2\pi a_{-}} - 1 \right) \sum_{1 \leq j < k \leq N}^{N} e^{-2\pi i\eta_{j} e^{-2\pi i\eta_{k}}} 
\end{align*}
\]

Set \( l_{n} = -h_{n+4} (n = 1, 2, 3, 4) \). By a multiplication and a gauge transformation, we have

\[
\tilde{A}^{(2)} ( h, l; z ) = \sum_{j=1}^{N} ( \tilde{V}_{j}^{(2)} ( h; z ) \exp (-i a_{-} \partial_{j} ) + \tilde{W}_{j}^{(2)} ( l; z ) \exp (i a_{-} \partial_{j} ) ) + \tilde{U}^{(2)} ( h, l; z ),
\]

where

\[
\begin{align*}
\tilde{V}_{j}^{(2)} ( h; z ) &= e^{4\pi i\eta_{j}} \prod_{n=1}^{N} ( 1 - e^{2\pi i\eta_{n}} e^{-\pi a_{-}} e^{-2\pi i\eta_{j}} ) \prod_{k \neq j} \frac{1 - e^{2\pi i\mu} e^{2\pi i(\eta_{k} - \eta_{j})}}{1 - e^{2\pi i\mu} e^{2\pi i(\eta_{k} - \eta_{j})}}, \\
\tilde{W}_{j}^{(2)} ( l; z ) &= e^{4\pi i\eta_{j}} \prod_{n=1}^{N} ( 1 - e^{2\pi i\eta_{n}} e^{\pi a_{-}} e^{-2\pi i\eta_{j}} ) \prod_{k \neq j} \frac{1 - e^{-2\pi i\mu} e^{2\pi i(\eta_{k} - \eta_{j})}}{1 - e^{-2\pi i\mu} e^{2\pi i(\eta_{k} - \eta_{j})}}, \\
\tilde{U}^{(2)} ( h, l; z ) &= \sum_{n=1}^{N} ( e^{2\pi i\eta_{n}} + e^{2\pi i\eta_{n}} ) \sum_{j=1}^{N} e^{2\pi i\eta_{j}} - ( e^{\pi a_{-}} + e^{-\pi a_{-}} ) \sum_{j=1}^{N} e^{2\pi i\eta_{j}} \\
&\quad + e^{-2\pi i\mu} e^{-\pi a_{-}} ( e^{2\pi i\mu} e^{2\pi a_{-}} - 1 ) \sum_{1 \leq j < k \leq N}^{N} e^{2\pi i\eta_{j} e^{-2\pi i\eta_{k}}} \\
&\quad + \sum_{n=1}^{4} e^{\pi i(\eta_{n} + i\eta_{n})} \left[ \sum_{n=1}^{N} e^{-2\pi i\eta_{n}} \sum_{j=1}^{N} e^{-2\pi i\eta_{j}} - ( e^{\pi a_{-}} + e^{-\pi a_{-}} ) \sum_{j=1}^{N} e^{-4\pi i\eta_{j}} \\
&\quad + e^{-2\pi i\mu} e^{-\pi a_{-}} ( e^{2\pi i\mu} e^{2\pi a_{-}} - 1 ) \sum_{1 \leq j < k \leq N}^{N} e^{-2\pi i\eta_{j} e^{-2\pi i\eta_{k}}} \right].
\end{align*}
\]

Note that this operator was also essentially obtained in [16]. We apply the third degeneration.

**Proposition 4.3** In Equation (4.10), we replace \( z \) by \( z - iR, h_{n} \) (\( n = 1, 2 \)) by \( h_{n} - iR, h_{n} \) (\( n = 3, 4 \)) by \( h_{n} + iR, l_{n} \) (\( n = 1, 2, 3, 4 \)) by \( l_{n} - iR \) and take the limit \( R \to +\infty \). Then we have the operator

\[
A^{(3)} ( h, l; z ) = \sum_{j=1}^{N} ( V_{j}^{(3)} ( h; z ) \exp (-i a_{-} \partial_{j} ) + W_{j}^{(3)} ( l; z ) \exp (i a_{-} \partial_{j} ) ) + U^{(3)} ( h, l; z ),
\]

where...
where

\[ V_j^{(3)}(h; z) = e^{4\pi i z_j} \prod_{n=1}^{2} (1 - e^{2\pi i h_n} e^{-\pi a_-} e^{-2\pi i z_j}) \prod_{k \neq j} \frac{1 - e^{2\pi i \mu} e^{2\pi i (\epsilon_k - z_j)}}{1 - e^{2\pi i (\epsilon_k - z_j)}}, \] (4.14)

\[ W_j^{(3)}(l; z) = e^{4\pi i z_j} \prod_{n=1}^{4} (1 - e^{2\pi i h_n} e^{\pi a_-} e^{-2\pi i z_j}) \prod_{k \neq j} \frac{1 - e^{-2\pi i \mu} e^{2\pi i (\epsilon_k - z_j)}}{1 - e^{2\pi i (\epsilon_k - z_j)}}, \] (4.15)

\[ U^{(3)}(h, l; z) = \left( \sum_{n=1}^{2} e^{2\pi i h_n} + \sum_{n=1}^{4} e^{2\pi i h_n} \right) \sum_{j=1}^{N} e^{2\pi i z_j} - (e^{\pi a_-} + e^{-\pi a_-}) \sum_{j=1}^{N} e^{4\pi i z_j} \] (4.15)

By applying a gauge transformation, we have

\[ \tilde{A}^{(3)}(h, l; z) = \sum_{j=1}^{N} (V_j^{(3)}(h; z) \exp(-i\alpha_\partial z_j) + W_j^{(3)}(l; z) \exp(i\alpha_\partial z_j)) + U^{(3)}(h, l; z), \] (4.16)

where

\[ \tilde{V}_j^{(3)}(h; z) = e^{-2\pi a_-} \prod_{n=1}^{2} (1 - e^{2\pi i h_n} e^{-\pi a_-} e^{-2\pi i z_j}) \prod_{k \neq j} \frac{1 - e^{2\pi i \mu} e^{2\pi i (\epsilon_k - z_j)}}{1 - e^{2\pi i (\epsilon_k - z_j)}}, \] (4.17)

\[ \tilde{W}_j^{(3)}(l; z) = e^{-2\pi a_-} e^{4\pi i z_j} \prod_{n=1}^{4} (1 - e^{2\pi i h_n} e^{\pi a_-} e^{-2\pi i z_j}) \prod_{k \neq j} \frac{1 - e^{-2\pi i \mu} e^{2\pi i (\epsilon_k - z_j)}}{1 - e^{2\pi i (\epsilon_k - z_j)}}. \]

We apply the fourth degeneration.

**Proposition 4.4** In Equation (4.16), we replace \( z \) by \( z + iR, h_n (n = 1, 2, 3, 4) \) by \( h_n + iR, \) \( l_n (n = 1, 2) \) by \( l_n + iR, \) \( l_n (n = 3, 4) \) by \( l_n - iR \) and take the limit \( R \to +\infty. \) Then we have the operator

\[ A^{(4)}(h, l; z) \equiv \sum_{j=1}^{N} (V_j^{(4)}(h; z) \exp(-i\alpha_\partial z_j) + W_j^{(4)}(l; z) \exp(i\alpha_\partial z_j)) + U^{(4)}(h, l; z), \] (4.18)

where

\[ V_j^{(4)}(h; z) = e^{-2\pi a_-} \prod_{n=1}^{2} (1 - e^{2\pi i h_n} e^{-\pi a_-} e^{-2\pi i z_j}) \prod_{k \neq j} \frac{1 - e^{2\pi i \mu} e^{2\pi i (\epsilon_k - z_j)}}{1 - e^{2\pi i (\epsilon_k - z_j)}}, \] (4.19)
Degenerations of Ruijsenaars–van Diejen Operator

\[ W_j^{(4)}(l; z) = e^{4\pi i l} e^{2\pi i (l_3+l_4) \sum_{n=1}^2} \prod_{n=1}^2 \prod_{k \neq j} \frac{1 - e^{-2\pi i l_n} e^{2\pi i (l_k-z_j)}}{1 - e^{2\pi i (l_k-z_j)}}, \]

\[ U^{(4)}(h, l; z) = \left( e^{2\pi i h} + e^{2\pi i l} \right) \sum_{j=1}^N e^{2\pi i z_j} + e^{\pi i (h_1+h_2)} (e^{\pi i (h_3-h_4)} + e^{\pi i (h_4-h_3)}) \prod_{n=1}^4 \prod_{j=1}^N e^{2\pi i l_n} e^{-2\pi i z_j}. \]

By a multiplication and a gauge transformation we have

\[ \tilde{A}^{(4)}(h, l; z) = \sum_{j=1}^N \left( \tilde{V}_j^{(4)}(h; z) \exp(-ia_+ \partial_{z_j}) + \tilde{W}_j^{(4)}(l; z) \exp(ia_+ \partial_{z_j}) \right) - U^{(4)}(h, l; z), \]

where

\[ \tilde{V}_j^{(4)}(h; z) = e^{2\pi i l} \prod_{n=1}^2 \prod_{k \neq j} \frac{1 - e^{2\pi i l_n} e^{-2\pi i z_j}}{1 - e^{2\pi i z_j}}, \]

\[ \tilde{W}_j^{(4)}(l; z) = e^{2\pi i (l_3+l_4)} e^{2\pi i l} \prod_{n=1}^2 \prod_{k \neq j} \frac{1 - e^{-2\pi i l_n} e^{2\pi i z_j}}{1 - e^{2\pi i z_j}}. \]

5. Discussion

We have found out that the degenerated Ruijsenaars–van Diejen operators of one variable appear by specializations of the linear $q$-difference equations related with the $q$-Painlevé equations of types $D_8^{(1)}$, $E_6^{(1)}$ and $E_7^{(1)}$. Our results should be extended to the case of the $q$-Painlevé equation of type $E_8^{(1)}$ and the elliptic-difference Painlevé equation. Note that Yamada and his collaborators found Lax pairs of the $q$-Painlevé equations of type $E_8^{(1)}$ and the elliptic-difference Painlevé equation [19, 22, 23]. On Lax pairs of the elliptic-difference Painlevé equation, see also the articles by Rains and Ormerod [24, 25].

We propose other related problems. Komori and Hikami [14] proved existence of the commuting operators for the multivariable Ruijsenaars–van Diejen operator. The commuting operators of the multivariable degenerate operators should be clarified. Note that a four parameter specialization of the operator in Equation (4.20) was already obtained by van Diejen [12] in Equation (2.25) in his article, and the commuting operators were also obtained there. Recently the spectral problem for this four-parameter specialization was discussed in [26].

It is known that the Ruijsenaars–van Diejen operator of one variable admits $E_8$ symmetry [15]. A kernel function plays important roles in [15], because the Hilbert–Schmidt operator of the kernel function is used to build up Hilbert space features and is also used to establish the invariance of the discrete spectra under the $E_8$ Weyl group. The symmetry of the degenerate operators should also be studied well. In particular, the kernel functions for the degenerate operators should be established.

The fourth degeneration of Ruijsenaars–van Diejen operator corresponds to Heun equation by the limit $q \to 1$, and it is known that Heun equation has a family of degenerate equations (e.g. confluent Heun equation) [4, 27]. On the other hand, degenerations of the $q$-Painlevé equations of types $D_8^{(1)}$ are known in Sakai’s table [9] and Lax pairs of those equations were obtained by Murata [28]. Then further
degenerations of Ruijsenaars–van Diejen operator and the $q$-Painlevé Heun correspondences should be clarified.

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