THE LAW OF THE HITTING TIMES TO POINTS
BY A STABLE LÉVY PROCESS WITH
NO NEGATIVE JUMPS

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Abstract
Let \( X = (X_t)_{t \geq 0} \) be a stable Lévy process of index \( \alpha \in (1, 2) \) with the Lévy measure \( \nu(dx) = \frac{c}{x^{1+\alpha}} I_{(0,\infty)}(x) dx \) for \( c > 0 \), let \( x > 0 \) be given and fixed, and let \( \tau_x = \inf\{t > 0 : X_t = x\} \) denote the first hitting time of \( X \) to \( x \). Then the density function \( f_{\tau_x} \) of \( \tau_x \) admits the following series representation:

\[
f_{\tau_x}(t) = \frac{x^{\alpha-1}}{\pi(c \Gamma(-\alpha) t)^{2-1/\alpha}} \sum_{n=1}^{\infty} \left[ (-1)^{n-1} \sin(\pi/\alpha) \frac{\Gamma(n-1/\alpha)}{\Gamma(n-1)} \left( \frac{x^\alpha}{c \Gamma(-\alpha) t} \right)^{n-1} \right.
\]

\[
- \sin \left( \frac{n \pi}{\alpha} \right) \frac{\Gamma(1+n/\alpha)}{n!} \left( \frac{x^\alpha}{c \Gamma(-\alpha) t} \right)^{(n+1)/\alpha-1} \left] \right.
\]

for \( t > 0 \). In particular, this yields \( f_{\tau_x}(0+) = 0 \) and

\[
f_{\tau_x}(t) \sim \frac{x^{\alpha-1}}{\Gamma(\alpha-1) \Gamma(1/\alpha)} \left( \frac{c \Gamma(-\alpha) t}{x^\alpha} \right)^{-2+1/\alpha}
\]

as \( t \to \infty \). The method of proof exploits a simple identity linking the law of \( \tau_x \) to the laws of \( X_t \) and \( \sup_{0 \leq s \leq t} X_s \) that makes a Laplace inversion amenable. A simpler series representation for \( f_{\tau_x} \) is also known to be valid when \( x < 0 \).

1 Introduction

If a Lévy process \( X = (X_t)_{t \geq 0} \) jumps upwards, then it is much harder to derive a closed form expression for the distribution function of its first passage time \( \tau_{(x,\infty)} \) over a strictly positive level \( x \), and
in the existing literature such expressions seem to be available only when $X$ has no positive jumps (unless the Lévy measure is discrete). A notable exception to this rule is the recent paper \cite{11} where an explicit series representation for the density function of $\tau_{(x,\infty)}$ was derived when $X$ is a stable Lévy process of index $\alpha \in (1,2)$ having the Lévy measure given by \( \nu(dx) = (c/x^{1+\alpha})I_{(0,\infty)}(x) \) \( dx \) with $c > 0$ given and fixed. This was done by performing a time-space inversion of the Wiener-Hopf factor corresponding to the Laplace transform of \( (t,y) \mapsto P(S_t > y) \) where $S_t = \sup_{0 \leq s \leq t} X_s$ for $t > 0$ and $y > 0$.

Motivated by this development our purpose in this note is to search for a similar series representation associated with the first hitting time $\tau_x$ of $X$ to a strictly positive level $x$ itself. Clearly, since $X$ jumps upwards and creeps downwards, $\tau_x$ will happen strictly after $\tau_{(x,\infty)}$, and since $X$ reaches $x$ by creeping through it independently from the past prior to $\tau_{(x,\infty)}$, one can exploit known expressions for the latter portion of the process and derive the Laplace transform for \( (t,y) \mapsto P(\tau_y > t) \). This was done in \cite{6} Theorem 1 and is valid for any Lévy process with no negative jumps (excluding subordinators). A direct Laplace inversion of the resulting expression appears to be difficult, however, and we show that a simple (Chapman-Kolmogorov type) identity which links the law of $\tau_x$ to the laws of $X_x$ and $S_\tau$ proves helpful in this context (due largely to the scaling property of $X$). It enables us to connect the old result of \cite{13} through an additive factorisation of the Laplace transform of \( (t,y) \mapsto P(\tau_y > t) \). This makes the Laplace inversion possible term by term and yields an explicit series representation for the density function of $\tau_x$.

2 Result and proof

1. Let $X = (X_t)_{t \geq 0}$ be a stable Lévy process of index $\alpha \in (1,2)$ whose characteristic function is given by

\[
\mathbb{E} e^{i\lambda X_t} = \exp \left( t \int_0^\infty (e^{i\lambda x} - 1 - i\lambda x) \frac{dx}{\Gamma(-\alpha) x^{1+\alpha}} \right) = e^{t(\psi(-\lambda))\alpha}
\]

for $\lambda \in \mathbb{R}$ and $t \geq 0$. It follows that the Laplace transform of $X$ is given by

\[
\mathbb{E} e^{-\lambda X_t} = e^{t\lambda^\alpha}
\]

for $\lambda \geq 0$ and $t \geq 0$ (the left-hand side being $+\infty$ for $\lambda < 0$). From \cite{2} we see that the Laplace exponent of $X$ equals $\psi(\lambda) = \lambda^\alpha$ for $\lambda \geq 0$ and $\varphi(p) := \psi^{-1}(p) = p^{1/\alpha}$ for $p \geq 0$.

2. The following properties of $X$ are readily deduced from \cite{1} and \cite{2} using standard means (see e.g. \cite{2} and \cite{9}): the law of $(X_{ct})_{t \geq 0}$ is the same as the law of $(c^{1/\alpha} X_t)_{t \geq 0}$ for each $c > 0$ given and fixed (scaling property); $X$ is a martingale with $EX_t = 0$ for all $t \geq 0$; $X$ jumps upwards (only) and creeps downwards (in the sense that $P(X_{t+\Delta t} = x) = 1$ for $x < 0$ where $\tau_{(-\infty,x)} = \inf\{ t > 0 : X_t < x \}$ is the first passage time of $X$ over $x$); $X$ has sample paths of unbounded variation; $X$ oscillates from $-\infty$ to $+\infty$ (in the sense that $\liminf_{t \to \infty} X_t = -\infty$ and $\limsup_{t \to \infty} X_t = +\infty$ both a.s.); the starting point 0 of $X$ is regular (for both $(0,\infty)$ and $(0,0)$). Note that the constant $c = 1/\Gamma(-\alpha)$ in the Lévy measure $\nu(dx) = (c/x^{1+\alpha})dx$ of $X$ is chosen/fixed for convenience so that $X$ converges in law to $\sqrt{2}B$ as $\alpha \uparrow 2$ where $B$ is a standard Brownian motion, and all the facts throughout can be extended to a general constant $c > 0$ using the scaling property of $X$.

3. Letting $f_{X_1}$ denote the density function of $X_1$, the following series representation is known to be
valid (see e.g. (14.30) in [14, p. 88]):

\[ f_{S_1}(x) = \sum_{n=1}^{\infty} \frac{\sin(n \pi / \alpha)}{\pi} \frac{\Gamma(1+n/\alpha)}{n!} x^{n-1} \]  

(3)

for \( x \in \mathbb{R} \). Setting \( S_1 = \sup_{0 \leq t \leq 1} X_t \) and letting \( f_{S_1} \) denote the density function of \( S_1 \), the following series representation was recently derived in [11, Theorem 1]:

\[ f_{S_1}(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin(n \pi / \alpha)}{\pi} \frac{\Gamma(n-1/\alpha)}{\Gamma(\alpha n-1)} x^{\alpha n-2} \]  

(4)

for \( x > 0 \). Clearly, the series representations (3) and (4) extend to \( t \neq 1 \) by the scaling property of \( X \) since \( X_t = \text{law} t^{1/\alpha} X_1 \) and \( S_t := \sup_{0 \leq s \leq t} X_s = \text{law} t^{1/\alpha} S_1 \) for \( t > 0 \).

4. Consider the first hitting time of \( X \) to \( x \) given by

\[ \tau_x = \inf \{ t > 0 : X_t = x \} \]  

(5)

for \( x > 0 \). Then it is known (see (2.16) in [6]) that the time-space Laplace transform equals

\[ \int_0^\infty e^{-\lambda x} E(e^{-p \tau_x}) \, dx = \frac{1}{\lambda - \varphi(p)} + \frac{1}{\varphi'(p)(p-\psi(\lambda))} = \frac{1}{\lambda - p^{1/\alpha}} + \frac{\alpha}{p^{1+1/\alpha}(p-\lambda^\alpha)} \]  

(6)

for \( \lambda > 0 \) and \( p > 0 \). Note that this can be rewritten as follows:

\[ \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\lambda x} P(\tau_x > t) \, dx \, dt = \frac{1}{\lambda p} + \frac{1}{p(p^{1/\alpha} - \lambda)} - \frac{\alpha}{p^{1+1/\alpha}(p-\lambda^\alpha)} \]  

(7)

for \( \lambda > 0 \) and \( p > 0 \).

Let \( \mathcal{L}^{-1}_p \) denote the inverse Laplace transform with respect to \( p \). Using that \( 1/(p(p^{1/\alpha} - \lambda)) = \sum_{n=1}^{\infty} \lambda^n/p^{1+n/\alpha} \) and \( \mathcal{L}^{-1}_p[1/p^a] = t^{a-1}/\Gamma(a) \) for \( a > 0 \), it is easily verified that

\[ \mathcal{L}^{-1}_p \left[ \frac{1}{p(p^{1/\alpha} - \lambda)} \right](t) = \frac{1}{\lambda} E_{1/\alpha}(\lambda t^{1/\alpha}) - 1 \]  

(8)

for \( t > 0 \) where \( E_{1/\alpha}(x) = \sum_{n=0}^{\infty} x^n/\Gamma(an+1) \) denotes the Mittag-Leffler function. On the other hand, by (3) in [8, p. 238] we find

\[ \mathcal{L}^{-1}_p \left[ \frac{1}{p^{1/\alpha}(p-\lambda^\alpha)} \right](t) = \frac{1}{\Gamma(1/\alpha)} \frac{e^{\lambda^\alpha t}}{\lambda} \gamma(1/\alpha, \lambda^\alpha t) \]  

(9)

for \( t > 0 \) where \( \gamma(a,x) = \int_0^x y^{a-1} e^{-y} \, dy \) denotes the incomplete gamma function. Combining (7) with (8) and (9) we get

\[ \int_0^\infty e^{-\lambda x} P(\tau_x > t) \, dx = \frac{1}{\lambda} E_{1/\alpha}(\lambda t^{1/\alpha}) - \frac{\alpha}{\Gamma(1/\alpha)} \frac{e^{\lambda^\alpha t}}{\lambda} \gamma(1/\alpha, \lambda^\alpha t) \]

\[ = \frac{\alpha}{\Gamma(1/\alpha)} \frac{e^{\lambda^\alpha t}}{\lambda} \gamma(1/\alpha, \lambda^\alpha t) + \frac{1}{\alpha} E_{1/\alpha}(\lambda t^{1/\alpha}) \]  

(10)
for \( \lambda > 0 \) and \( t > 0 \).

The first and the third term on the right-hand side of (10) may now be recognised as the Laplace transforms of particular functions considered in [11] and [13] respectively (recall also (2.2) above). The proof of the following theorem provides a simple probabilistic argument (of Chapman-Kolmogorov type) for this additive factorisation (see Remark 1 below).

**Theorem 1.** Let \( X = (X_t)_{t \geq 0} \) be a stable Lévy process of index \( \alpha \in (1, 2) \) with the Lévy measure \( \nu(dx) = (c/x^{1+\alpha})I_{(0,\infty)}(x)\,dx \) for \( c > 0 \), let \( x > 0 \) be given and fixed, and let \( \tau_x \) denote the first hitting time of \( X \) to \( x \). Then the density function \( f_{\tau_x} \) of \( \tau_x \) admits the following series representation:

\[
f_{\tau_x}(t) = \frac{x^{\alpha-1}}{\pi(c \Gamma(-\alpha)t)^{2-1/\alpha}} \sum_{n=1}^{\infty} \left[ (-1)^{n-1} \sin(\pi/\alpha) \frac{\Gamma(n-1/\alpha)}{\Gamma(\alpha n-1)} \left( \frac{x^{\alpha}}{c \Gamma(-\alpha)t} \right)^{n-1} \right. \\
- \left. \sin \left( \frac{n \pi}{\alpha} \right) \frac{\Gamma(1+n)}{n!} \left( \frac{x^{\alpha}}{c \Gamma(-\alpha)t} \right)^{(n+1)/\alpha-1} \right]
\]

for \( t > 0 \). In particular, this yields:

\[
f_{\tau_x}(t) = o(1) \quad \text{as} \quad t \downarrow 0; \tag{12}
\]

\[
f_{\tau_x}(t) \sim \frac{x^{\alpha-1}}{\Gamma(1/\alpha)} (c \Gamma(-\alpha)t)^{-2+1/\alpha} \quad \text{as} \quad t \uparrow \infty. \tag{13}
\]

**Proof.** It is no restriction to assume below that \( c = 1/\Gamma(-\alpha) \) as the general case follows by replacing \( t \) in (11) with \( c \Gamma(-\alpha)t \) for \( t > 0 \).

Since \( X \) creeps downwards, we can apply the strong Markov property of \( X \) at \( \tau_x \), use the additive character of \( X \), and exploit the scaling property of \( X \) to find

\[
P(S_1 > x) = P(S_1 > x, X_1 > x) + P(S_1 > x, X_1 \leq x)
\]

\[
= P(X_1 > x) + \int_0^1 P(X_1 \leq x \mid \tau_x = t) F_{\tau_x}(dt)
\]

\[
= P(X_1 > x) + \int_0^1 P(x + X_{1-t} \leq x) F_{\tau_x}(dt)
\]

\[
= P(X_1 > x) + \int_0^1 P((1-t)^{1/\alpha}X_1 \leq 0) F_{\tau_x}(dt)
\]

\[
= P(X_1 > x) + (1/\alpha) P(\tau_x \leq 1)
\]

where we also use that \( P(X_1 \leq 0) = 1/\alpha \) and \( F_{\tau_x} \) denotes the distribution function of \( \tau_x \). Note that the second equality in (14) represents a Chapman-Kolmogorov equation of Volterra type (see [11] Section 2) for a formal justification and a brief historical account of the argument). Since \( \tau_x \simlaw x^\alpha \tau_1 \) by the scaling property of \( X \), we find that (14) reads

\[
P(S_1 > x) = P(X_1 > x) + (1/\alpha) F_{\tau_x}(1/x^\alpha)
\]

(15)

for \( x > 0 \). Hence we see that \( F_{\tau_x} \) is absolutely continuous (cf. [10] for a general result on the absolute continuity) and by differentiating in (15) we get

\[
f_{\tau_x}(1/x^\alpha) = x^{1+\alpha} [f_{S_1}(x) - f_{X_1}(x)]
\]

(16)
for $x > 0$. Letting $t = 1/x^a$ we find that
\[
f_{\tau_x}(t) = t^{-1-1/a} [f_{S_1}(t^{-1/a}) - f_{X_1}(t^{-1/a})]
\]
for $t > 0$. Hence (11) with $x = 1$ follows by (3) and (4) above. Moreover, since $\tau_x = \text{law } x^\alpha \tau_1$ we see that $f_{\tau_x}(t) = t^{-\alpha} f_{\tau_1}(t^{-\alpha})$ and this yields (11) with $x > 0$.

It is known that $f_{X_1}(x) \sim c x^{-1-\alpha}$ as $x \to \infty$ (see e.g. (14.34) in [14] p. 88) and likewise $f_{S_1}(x) \sim c x^{-1-\alpha}$ as $x \to \infty$ (see [1] Corollary 3 and [7] for a proof). From (16) we thus see that $f_{\tau_1}(0+) = 0$ and hence $f_{\tau_1}(0+) = 0$ for all $x > 0$ as claimed in (12). The asymptotic relation (13) follows directly from (11) using the reflection formula $\Gamma(1-x)\Gamma(x) = \pi/\sin \pi x$ for $x \in \mathbb{C}\setminus\mathbb{Z}$. This completes the proof. \hfill \Box

Remark 1. Note that (14) can be rewritten as follows:
\[
(1/a) P(\tau_x > 1) = 1/a + F_{S_1}(x) - F_{X_1}(x) = F_{S_1}(x) - (F_{X_1}(x) - F_{X_1}(0))
\]
for $x > 0$, and from (2.30) in [11] we know that
\[
\int_0^\infty e^{-\lambda x} f_{S_1}(x) \, dx = e^{\lambda a} \int_0^\infty e^{-\alpha x^a} \, dz
\]
for $\lambda > 0$. In view of (10) this implies that
\[
\int_0^\infty e^{-\lambda x} f_{S_1}(x) \, dx = e^{\lambda a} - \frac{1}{\alpha} E_{1/a}(\lambda)
\]
for $\lambda > 0$. Recalling (2) we see that (20) is equivalent to
\[
\int_{-\infty}^0 e^{-\lambda x} f_{X_1}(x) \, dx = \frac{1}{\alpha} E_{1/a}(\lambda)
\]
for $\lambda > 0$. An explicit series representation for $f$ in place of $f_{S_1}$ in (21) was found in [13] (see also [12]) and this expression coincides with (3) above when $x < 0$. (Note that (21) holds for all $\lambda \in \mathbb{R}$ and substitute $y = -x$ to connect to [13].) This represents an analytic argument for the additive factorisation addressed following (11) above.

Remark 2. In contrast to (12) note that
\[
f_{\tau_{x,\infty}}(0+) = \frac{c}{\alpha x^a}
\]
for $x > 0$. This is readily derived from $P(\tau_{x,\infty} \leq t) = P(S_t \geq x)$ using $S_t = \text{law } t^{1/a} S_1$ and $f_{S_1}(x) \sim c x^{-1-\alpha}$ for $x \to \infty$ as recalled in the proof above.

Remark 3. If $x < 0$ then applying the same arguments as in (14) above with $I_t = \inf_{0 \leq s \leq t} X_s$ we find that
\[
P(I_t \leq x) = P(I_t \leq x, X_t \leq x) + P(I_t \leq x, X_t > x)
\]
\[
= P(X_t \leq x) + \int_0^t P(X_{t-s} > x) f_{\tau_x}(ds)
\]
\[
= P(X_t \leq x) + (1-1/a) P(\tau_x \leq t)
\]
for \( t > 0 \). In this case, moreover, we also have \( \mathbb{P}(I_t \leq x) = \mathbb{P}(\sigma_x \leq t) \) since \( X \) creeps through \( x \), so that (23) yields

\[
P(\tau_x \leq t) = \alpha \mathbb{P}(X_t \leq x)
\]

(24) for \( x < 0 \) and \( t > 0 \). Since \( X_t = \text{law} \ t^{1/\alpha}X_1 \) this implies

\[
f_{\tau_x}(t) = -x t^{-1-1/\alpha} F_{\alpha x}(x t^{-1/\alpha}) = -\sum_{n=1}^{\infty} \frac{\sin(n \pi / \alpha)}{\pi} \frac{\Gamma(1+n/\alpha)}{n!} \left( \frac{x}{t} \right)^n
\]

(25) for \( t > 0 \) upon using (3) above. Replacing \( t \) in (25) by \( c \Gamma(-\alpha) t \) we get a series representation for \( f_{\tau_x} \) in the case when \( c > 0 \) is a general constant. The first identity in (25) is known to hold in greater generality (see [4] and [2] p. 190 for different proofs).

**Remark 4.** If \( c = 1/2 \Gamma(-\alpha) \) and \( \alpha \uparrow 2 \) then the series representations (11) and (25) with \( t/2 \) in place of \( t \) reduce to the known expressions for the density function \( f_{\tau_x} \) of \( \tau_x = \inf \{ t > 0 : B_t = x \} \) where \( B = (B_t)_{t \geq 0} \) is a standard Brownian motion:

\[
f_{\tau_x}(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-x^2/2t} = \frac{|x|}{\sqrt{2\pi t^3}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n}}{t^n}
\]

(26) for \( t > 0 \) and \( x \in \mathbb{R} \setminus \{0\} \).

**Remark 5.** Duality theory for Markov/Lévy processes (see [3] Chap. VI and [2] Chap. II and Corollary 18 on p. 64]) implies that

\[
\mathbb{E}e^{-\rho \tau} = \frac{\int_0^\infty e^{-\rho t} f_{\tau_x}(x) \, dt}{\int_0^\infty e^{-\rho t} f_{\tau_x}(0) \, dt}
\]

(27) from where the following identity can be derived (see [2] Lemma 13, p. 230]):

\[
P(\tau_x \leq t) = \frac{1}{\Gamma(1-1/\alpha) \Gamma(1/\alpha)} \int_0^t \frac{f_{\tau_x}(x)}{(t-s)^{1-1/\alpha}} \, ds
\]

(28) for \( x \in \mathbb{R} \) and \( t > 0 \) (being valid for any stable Lévy process). By the scaling property of \( X \) we have \( f_{\tau_x}(x) = s^{-1/\alpha} f_{\tau_x}(xs^{-1/\alpha}) \) for \( s \in (0, t) \) and \( x \in \mathbb{R} \). Recalling the particular form of the series representation for \( f_{\tau_x} \) given in (3), we see that it is not possible to integrate term by term in (28) in order to obtain an explicit series representation.

**Remark 6.** The density function \( f_{X_1} \) from (3) can be expressed in terms of the Fox functions (see [15]), and the density function \( f_{\tau_x} \) from (4) can be expressed in terms of the Wright functions (see [5] Sect. 12 and the references therein). In view of the identity (17) and the fact that \( f_{\tau_x}(t) = x^{-\alpha} f_{\tau_x}(tx^{-\alpha}) \), these facts can be used to provide alternative representations for the density function \( f_{\tau_x} \) from (11) above. We are grateful to an anonymous referee for bringing these references to our attention.

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