Black hole entropy: 
departures from area law

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Abstract

The thermodynamic and euclidean functional integral approaches to black hole entropy are discussed. The existence of some freedom in the definition of the entropy is pointed out and the possibility of a departure from the semiclassical expression discussed in the light of quantum corrections. The semiclassical area dependence of the entropy of matter in the background of a black hole is also reviewed and shown to break down in the case of extremal black holes. The cutoff dependence is shown to be different for the extreme dilatonic and Reissner-Nordstrom black holes.

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1 Introduction

It is now widely known that the area of the horizon of a black hole can be interpreted as an entropy and satisfies all the thermodynamical laws. This is not completely understood in terms of the usual formulation of entropy as a measure of the number of states available, but the naïve Lagrangian path integral does lead to a partition function from which the area formula for entropy can be obtained by neglecting quantum fluctuations.

That formula is supposed to describe the gravitational entropy corresponding to a black hole. There have also been investigations on the entropy of quantum fields in black hole backgrounds. The values thus obtained may be considered to be additional contributions to the entropy of the black hole - field system, but the gravitational entropy itself has occasionally been envisaged to arise in this way. The calculations produce divergences, with the area of the horizon appearing as a factor. This has been interpreted to mean that the gravitational constant gets renormalized in the presence of the quantum fields.

Of great topical interest is the case of extremal black holes, which possess peculiarities not present in the corresponding nonextremal cases. For extremal dilatonic black holes, the temperature is nonzero, but the area vanishes. For extremal Reissner - Nordstrom black holes, the temperature is zero, but the area is nonzero. Topological arguments have been presented in the context of Euclidean quantum gravity to suggest that these extremal black holes have zero gravitational entropy in spite of the nonvanishing area and, what is more surprising, no definite temperature. What has actually been shown, however, concerns the classical action of extremal Euclidean Reissner - Nordstrom black hole configurations, and it has been argued to lead semiclassically to a vanishing entropy.

In this talk I shall critically reexamine (a) the connection between the action and the entropy and (b) the appearance of the area in the expression for the entropy of a scalar field in the background of an extremal black hole. Both non-extremal and extremal black holes will be considered. New expressions for entropy will be given in several cases on the basis of our work.

To set the stage, let me start by reminding you of the two extremal black hole solutions mentioned above. The metric of the Reissner - Nordstrom
spacetime is given by

\[ ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2d\Omega^2 \]  

(1)

in general, with \( M \) and \( Q \) denoting the mass and the charge respectively. This is a solution of the Einstein - Maxwell equations. There are apparent singularities at

\[ r_{\pm} = M \pm \sqrt{M^2 - Q^2} \]  

(2)

provided \( M \geq Q \). Cosmic censorship dictates that this inequality holds and then there is a horizon at \( r_+ \). The limiting case when \( Q = M \) and \( r_+ = r_- \) is referred to as the extremal case.

There is also an extremal case of the dilatonic black hole. The usual four-dimensional model [10] is

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g}(R - 2g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi), \]  

(3)

where \( \phi \) is the massless dilaton field, \( R \) the scalar curvature and \( g_{\mu\nu} \) the metric. Electromagnetic interactions are brought in by including the term

\[ -\frac{1}{32\pi} \int d^4x \sqrt{-g}e^{-2\phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} \]  

(4)

in the action. Exact black hole solutions of this model have been found with non-zero magnetic charge and angular momentum.

The black hole solution with zero angular momentum strongly resembles the Schwarzschild solution of standard general relativity.

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu \]

\[ = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r(r - a)d\Omega^2 \]

\[ e^{-2\phi} = e^{-2\phi_0}(1 - \frac{a}{r}) \]

\[ F_{\theta\phi} = Q \sin \theta \]  

(5)

where \( M \) is the mass of the black hole, \( Q \) its magnetic charge, the parameter \( a \) is defined by

\[ a = \frac{Q^2}{2M} e^{-2\phi_0} \]  

(6)
and \( \phi_0 \) is an arbitrary constant. This black hole has as usual a horizon at \( r = 2M \). An interesting feature is that a curvature singularity occurs at \( r = a \). The so-called extremal solution corresponds to the coincidence of these two regions and thus has \( a = 2M \). This extremal limit is interesting from the point of view of entropy because the area \( 4\pi 2M(2M - a) \) of the horizon vanishes.

As mentioned above, two different contributions to entropy will be discussed in this talk, and their relation can be understood as follows. As argued in [11] the partition function for the system can be defined by a Euclidean path integral for the gravitational action coupled with matter fields. The dominant contribution will come from the classical solutions of the action. So one expands the Euclidean action, which involves the metric, additional fields (the dilaton and/or the electromagnetic field) and an external scalar matter field, as

\[
S_E[g, \Phi, \varphi] = S_E[g_0, \Phi_0] + S_2[\delta g, \delta \Phi, \delta \varphi; g_0, \Phi_0] + \cdots, \tag{7}
\]

where

\[
g_{\mu\nu} = g_{0\mu\nu} + \delta g_{\mu\nu}, \quad \Phi = \Phi_0 + \delta \Phi, \quad \varphi = \varphi_0 + \delta \varphi. \tag{8}
\]

Here the subscript 0 denotes the classical value and \( \delta \) denotes the fluctuations. \( S_2 \) is the part quadratic in the fluctuations. The background matter field is taken as \( \varphi_0 = 0 \) i.e. the contributions of these fields are subtracted out while calculating the quantum corrections due to the matter fields. \( S_2 \) can be decoupled into the gravitational \((\delta g, \delta \Phi)\) and the matter \((\delta \varphi)\) sectors. The entropy arising from the partition function defined by ignoring all fluctuations is the semiclassical entropy. Consideration of the gravitational part of \( S_2 \) leads to quantum corrections to the gravitational entropy. Inclusion of the matter part of \( S_2 \) adds the entropy of the scalar field in the background of the black hole.

## 2 Gravitational entropy

For the simplest black hole, namely the one discovered by Schwarzschild, the Hawking temperature is given by

\[
T = \frac{1}{8\pi M}, \tag{9}
\]
where $M$ is the mass of the black hole. Accordingly, the first law of thermodynamics can be written as

$$dM = TdS = (8\pi M)^{-1}dS,$$

which shows that the entropy must be $4\pi M^2$ up to an additive constant, i.e., essentially a quarter of the area. No analogy is involved here, and the standard result is obtained directly from thermodynamics. There is no scope for ambiguity at this level of approximation defined by the Hawking temperature. Unfortunately this is no longer the case if we go on to more complicated black holes. We shall demonstrate below that thermodynamics allows some freedom in the expression for the entropy in the case of black holes depending on extra parameters like charge or angular momentum. Furthermore, the expression used for the Hawking temperature is a semiclassical one and therefore subject to higher quantum corrections. This can be expected to lead to quantum corrections to the entropy. We shall indicate how all these extra terms can be obtained, at least in principle, by calculating some functional integrals.

A generalized Smarr formula \cite{12, 13} can be written down for charged black holes as

$$M = \frac{TA}{2} + \Phi Q.$$  \hspace{1cm} (11)

Here $A$ is the area of the horizon and $\Phi$ is the analogue of the chemical potential for the electric or magnetic charge given by

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_A.$$  \hspace{1cm} (12)

The formula (11) is the integrated form of the first law of black hole physics \cite{13}

$$Td(A/4) = dM - \Phi dQ.$$  \hspace{1cm} (13)

By comparing this with the first law of thermodynamics

$$TdS = dM - \Phi dQ$$  \hspace{1cm} (14)

one may be tempted to conclude that the entropy is simply $A/4$ up to an additive constant \cite{5}. 

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It has been tacitly assumed that the $\Phi$-s entering the above two equations are the same. Whereas the definition of $\Phi$ given above is applicable to the first law of black hole physics, the first law of thermodynamics should really be written as

$$\tilde{T}dS = dM - \tilde{\Phi}dQ$$  \hspace{1cm} (15)

where $\tilde{\Phi}$ is

$$\tilde{\Phi} = \left( \frac{\partial M}{\partial Q} \right)_S.$$  \hspace{1cm} (16)

In assuming $\tilde{\Phi}$ to be the same as $\Phi$ one in effect assumes that differentiation at constant entropy is the same as differentiation at constant area, which makes the derivation of the area formula for the entropy somewhat circular. We use different symbols to allow for a possible difference. One should also allow for higher quantum corrections to the temperature and therefore write $\tilde{T}$.

Subtracting (13) from (15) we find

$$d\left(S - \frac{A}{4}\right) = \beta_q dM - \left( \frac{\tilde{\Phi}}{\tilde{T}} - \frac{\Phi}{T} \right)dQ,$$  \hspace{1cm} (17)

where

$$\beta_q = \frac{1}{\tilde{T}} - \frac{1}{T}.$$  \hspace{1cm} (18)

We set

$$S = \frac{A}{4} - F(Q, M),$$  \hspace{1cm} (19)

where the function $F$ is undetermined except for the requirement that

$$\frac{\partial F}{\partial M} = -\beta_q.$$  \hspace{1cm} (20)

Furthermore,

$$\frac{\partial F}{\partial Q} = \frac{\tilde{\Phi}}{\tilde{T}} - \frac{\Phi}{T}.$$  \hspace{1cm} (21)

But this equation is not a restriction on $F$ because $\tilde{\Phi}$ is not known. It can be used to fix the latter quantity if $F$ can be found.

$F$ can be sought to be determined from the partition function

$$Z = e^{-\frac{1}{kL}}$$  \hspace{1cm} (22)
where $I_E$ is an effective action defined from the Euclidean functional integral \[2\]. $Z$ can also be interpreted as the grand canonical partition function

$$Z = e^{-W/\tilde{T}}, \quad (23)$$

where

$$W = M - \tilde{T}S - \tilde{\Phi}Q. \quad (24)$$

In the stationary phase approximation the functional integral is taken to be dominated by the classical configuration and $I_E$ is approximately equal to $A/4$ \[3\] \[14\]. So the full effective action including fluctuations can be written as

$$I_E = A/4 + I_{corr} = \frac{1}{2\tilde{T}}(M - \Phi Q) + I_{corr}. \quad (25)$$

From these equations we can see that

$$\frac{M - \tilde{T}S - \tilde{\Phi}Q}{\tilde{T}} = \frac{1}{2\tilde{T}}(M - \Phi Q) + I_{corr} \quad (26)$$

By substituting the expressions for $S$ and $\tilde{\Phi}$ we find

$$M(\beta_q + \frac{1}{2\tilde{T}}) - A/4 - \frac{\Phi Q}{2\tilde{T}} = I_{corr} + Q \frac{\partial F}{\partial Q} - F. \quad (27)$$

Comparison with the Smarr formula \[11\] and \[20\] yields the result

$$F - Q \frac{\partial F}{\partial Q} - M \frac{\partial F}{\partial M} = I_{corr}. \quad (28)$$

In the semiclassical approximation $I_{corr} \approx 0$. Then $F$ need not vanish, but can be zero, yielding the usual expression for the entropy. Generally, if $I_{corr}$ is calculated from the Euclidean functional integral, \[28\] can be formally solved as

$$F = (1 - Q \frac{\partial}{\partial Q} - M \frac{\partial}{\partial M})^{-1}I_{corr} + F_0, \quad (29)$$

where $F_0$ is a solution of the homogeneous equation. It cannot be an arbitrary solution of the homogeneous equation because of the restriction imposed
by (20). $F_0$ has to be independent of $M$, and to satisfy the homogeneous equation it has to be a linear function of $Q$. This piece of $F$ exposes the freedom in the definition of the entropy, whereas the first piece provides quantum corrections to both the entropy and the temperature. If $F$ is to be proportional to the area, which is a homogeneous function of $M$ and $Q$, it is clear that $I_{\text{corr}}$ itself must be proportional to the area.

This general discussion about charged black holes relates to the non-extremal case. The extremal case is special because of the newly discovered properties of extremal black holes [6, 7, 8] which cannot be continuously deformed into nonextremal black holes and which, moreover, can be in equilibrium at arbitrary temperatures. We shall now investigate the possibility of an ambiguity in the definition of the semiclassical entropy for such black holes, i.e., the analogue of the $F_0$ noted above.

The first law of thermodynamics is (15), as before. In this extremal case, $Q = \alpha M$, where $\alpha$ is a constant depending on the type of the black hole (dilatonic or Reissner-Nordstrom). Then

$$\tilde{T} dS = (1 - \alpha \tilde{\Phi}) dM.$$  \hfill (30)

We now appeal to the vanishing Euclidean action [8]. The partition function can be approximately written as

$$Z = e^{-I},$$  \hfill (31)

with $I = 0$. Now note that this is the grand canonical partition function [4] because the charge $Q$ is variable. It is not an independent variable if we consider black holes which remain extremal, but it has to change in processes where the energy $M$ changes. So we write the usual formula

$$Z = e^{-W/\tilde{T}},$$  \hfill (32)

with

$$W = M - \tilde{T} S - \tilde{\Phi} Q.$$  \hfill (33)

Noting that $W$ has to vanish, we arrive at the relation

$$\tilde{T} S = (1 - \alpha \tilde{\Phi}) M.$$  \hfill (34)

Comparing with (30), and assuming the factors to be nonvanishing, we conclude that

$$\frac{dS}{S} = \frac{dM}{M}.$$  \hfill (35)
\( i.e., \)
\[ S = kM, \]
where \( k \) is an undetermined constant. If \( k \) vanishes, we get a vanishing entropy \([4, 5]\), but \([36]\) is more generally valid. It is crucial here to abandon the naive result \( \alpha \Phi = 1 \) which might be obtained by assuming continuity of non-extremal and extremal black holes. As has been argued in \([7]\), the temperature cannot be fixed. The potential too cannot be fixed, but it is related to the temperature by the formula
\[ \alpha \Phi = 1 - kT. \]

The only definite result is that the thermodynamical entropy, in contradistinction to the action, is given by the mass of the extremal black hole up to a constant which may, however, be zero. Note that this is at the semiclassical level itself.

Perhaps it is more interesting to look at quantum corrections, but these are not reliably known in 3+1 dimensional cases. We consider therefore a 2+1 dimensional black hole \([15]\). The rotating version is described by the metric
\[ ds^2 = -f^2 dt^2 + f^{-2} dr^2 + r^2 (d\phi - \frac{J}{2r^2} dt)^2, \]
where
\[ f^2 = -8GM + \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}, \]
and \( l^{-2} \) is a cosmological constant. The constant \( G \) is retained here in contrast to the rest of the talk because of the unusual units used in \([13]\). The outer horizon is at \( r_+ \), where
\[ r_+^2 = 4GMl^2 \left[ 1 + \sqrt{1 - \frac{J^2}{M^2l^2}} \right]. \]

The Hawking temperature is
\[ T = \frac{r_+^2 - 4GMl^2}{\pi r_+ l^2}. \]

The first law of black hole physics can be written in the form
\[ Td\frac{\pi r_+}{2G} = dM - \Omega dJ. \]
The area is $2\pi r_+$ because of the 1-dimensional nature of the horizon here. This equation can be directly checked from the expression for $r_+$. The analogue of the chemical potential for the angular momentum is

$$\Omega = \left( \frac{\partial M}{\partial J} \right)_A = \frac{4GJ}{r_+^2}. \quad (43)$$

Comparing (42) with the first law of thermodynamics

$$\tilde{T} dS = dM - \tilde{\Omega} dJ, \quad (44)$$

we can write

$$d(S - \frac{\pi r_+}{2G}) = \beta_q dM - \left( \frac{\tilde{\Omega}}{T} - \frac{\Omega}{T} \right) dJ. \quad (45)$$

As before,

$$S = \frac{\pi r_+}{2G} - F(J, M), \quad (46)$$

with $F(J, M)$ as yet undetermined except for (20). The connection between the unknown function $F$ and the effective action $I_E$ is given by

$$I_E = F - J \frac{\partial F}{\partial J} - M \frac{\partial F}{\partial M} - \frac{\pi r_+}{4G}. \quad (47)$$

in view of the Smarr formula which in this case reads

$$M = \frac{\pi r_+ T}{4G} + \Omega J. \quad (48)$$

If $F$ is to be proportional to $r_+$, which is a homogeneous function of $M$ and $J$, it is clear that $I_E$ itself must be proportional to $r_+$. Indeed, such an expression for a quantum correction has been given in [15]:

$$I_E \approx -\frac{\pi r_+}{2G} - \frac{2\pi r_+}{l} + \frac{M}{T} - \frac{4GJ^2}{r_+^2 T}. \quad (49)$$

It follows that $F$ must satisfy the equation

$$F - J \frac{\partial F}{\partial J} - M \frac{\partial F}{\partial M} = -\frac{2\pi r_+}{l}. \quad (50)$$
This equation can be formally solved as
\[
F = \left( J \frac{\partial}{\partial J} + M \frac{\partial}{\partial M} - 1 \right)^{-1} \frac{2\pi r_r}{l} + F_0.
\]
Accordingly, a possible answer for the entropy is
\[
S = \frac{\pi r_r}{2G} + \frac{4\pi r_r}{l} - F_0,
\]
which is similar to, though not identical with, the expression suggested in [15] by comparison with the quantum action. They in effect ignored not only the possibility of the solution \(F_0\) of the homogeneous equation, but also the presence of the derivatives of \(F\) in the equation.

If \(\hbar\) is reinstated and \(F_0\) ignored, we can write
\[
S = \frac{2\pi r_r}{4\hbar G} + \frac{4\pi r_r}{l}.
\]
Using (20), one can also find the corrected temperature:
\[
\tilde{T}^{-1} = \frac{\pi r_r^2 l^2}{r_r^2 - 4GMl^2} \left( \frac{1}{8\hbar G} + \frac{1}{l} \right).
\]
To sum up, we have demonstrated that only the simplest black holes have their entropies uniquely determined by the first law of thermodynamics at the semiclassical level of approximation. More generally, an expression for the entropy can be written down in terms of a function of the mass, charge, angular momentum and so on. Differential equations relate these functions to the Euclidean functional integrals for the black holes, but there is no unique solution. However, if one knows higher quantum corrections to these functional integrals, the equations may be used to determine the corresponding corrections to both the entropy and the temperature.

### 3 Matter outside black hole

To study the entropy of a scalar field in the background of a black hole we employ the brick-wall boundary condition [3]. In this model the wave
function is cut off just outside the horizon. Mathematically,

$$\varphi(x) = 0 \quad \text{at } r = r_h + \epsilon$$  \hspace{1cm} (55)

where $\epsilon$ is a small, positive, quantity and signifies an ultraviolet cut-off. There is also an infrared cut-off

$$\varphi(x) = 0 \quad \text{at } r = L$$  \hspace{1cm} (56)

with $L \gg r_h$.

We consider a static, spherically symmetric black hole spacetime with the metric

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{\theta\theta}(r)d\Omega^2,$$  \hspace{1cm} (57)

and study spinless particles bounded by the brick wall at $r = r_h + \epsilon$ and the long distance cutoff at $r = L$. An $r$-dependent radial wave number can be introduced for particles with mass $m$, energy $E$ and orbital angular momentum $l$ by

$$k_r^2(r, l, E) = g_{rr}[-g_{tt}E^2 - l(l + 1)g_{\theta\theta} - m^2].$$  \hspace{1cm} (58)

Only such values of $E$ are to be considered here that the above expression is nonnegative. The values are further restricted by the semiclassical quantization condition

$$n_r \pi = \int_{r_h + \epsilon}^{L} dr \, k_r(r, l, E),$$  \hspace{1cm} (59)

where the radial quantum number $n_r$ has to be a nonnegative integer.

The free energy $F$ at inverse temperature $\beta$ is given by a sum over single particle states

$$\beta F = \sum_{n_r, l, m_i} \log(1 - e^{-\beta E})$$

$$\approx \int dl \, (2l + 1) \int dn_r \log(1 - e^{-\beta E})$$

$$= -\int dl \, (2l + 1) \int d(\beta E) \, (e^{\beta E} - 1)^{-1} n_r$$

$$= -\frac{\beta}{\pi} \int dl \, (2l + 1) \int dE \, (e^{\beta E} - 1)^{-1} \int_{r_h + \epsilon}^{L} dr \, g_{rr}^{1/2}$$

$$\sqrt{-g_{tt}E^2 - l(l + 1)g_{\theta\theta} - m^2}$$

$$\approx \int dl \, (2l + 1) \int dE \, (e^{\beta E} - 1)^{-1} \int_{r_h + \epsilon}^{L} dr \, g_{rr}^{1/2}$$

$$\sqrt{-g_{tt}E^2 - l(l + 1)g_{\theta\theta} - m^2}.$$
\[
\frac{1}{\beta} = \frac{1}{2\pi} (g_{rr})^{-1/2} \frac{\partial}{\partial r} (-g_{tt})^{1/2} \big|_{r=r_h}
\]

To put this expression in a more familiar form, we write the Hawking temperature as

\[
\frac{1}{\beta} = \frac{1}{2\pi} (g_{rr})^{-1/2} \frac{\partial}{\partial r} (-g_{tt})^{1/2} \big|_{r=r_h}
\]

The entropy due to the black hole can be obtained from the formula

\[
S = \beta^2 \frac{\partial F}{\partial \beta}. \tag{63}
\]

This gives

\[
S_{\text{sing}} = \frac{8\pi^3}{45\beta^3 \epsilon} [(r-r_h)g_{rr}]^{1/2} (-\frac{g_{tt}}{r-r_h})^{-3/2} g_{\theta\theta} \big|_{r=r_h}. \tag{64}
\]

Here the limits of integration for \(l, E\) are such that the arguments of the square roots are nonnegative. The \(l\) integration is straightforward and has been explicitly carried out. The \(E\) integral can be evaluated only approximately.

Because of the asymptotically flat nature of the metric of the spacetime containing the black hole, the contribution to the \(r\) integral from large values of \(r\) corresponds to the expression for the free energy valid in flat spacetime:

\[
F_0 = -\frac{2L}{9\pi} \int_{m}^{\infty} dE \frac{(E^2 - m^2)^{3/2}}{e^{\beta E} - 1}. \tag{61}
\]

This piece is not relevant. The contribution of the black hole is singular in the limit \(\epsilon \to 0\). The leading singularity is obtained by taking the metric coefficients, multiplied by appropriate powers of \((r-r_h)\) to make them finite at the horizon, outside the integral. For a non-extremal black hole, \(g_{rr}\) has a linear singularity and \(g_{tt}\) a linear zero at \(r = r_h\), while \(g_{\theta\theta}\) is regular there. Thus,

\[
F_{\text{sing}} \approx -\frac{2\pi^3}{45\epsilon \beta^4} \left[(r-r_h)g_{rr}\right]^{1/2} \left(-\frac{g_{tt}}{r-r_h}\right)^{-3/2} \left.g_{\theta\theta}\right|_{r=r_h}. \tag{62}
\]
\[
\frac{1}{4\pi} (g_{rr})^{-1/2} (-g_{tt})^{-1/2} \left. \frac{\partial}{\partial r} (-g_{tt}) \right|_{r=r_h} \\
= \frac{1}{4\pi} \left[ (r-r_h) g_{rr} \right]^{-1/2} \left. \left( -\frac{g_{tt}}{r-r_h} \right)^{1/2} \right|_{r=r_h}.
\]  
(65)

It is also necessary to measure the width of the brick wall in terms of the proper radial variable \( \tilde{r} \) defined by

\[
d\tilde{r}^2 = g_{rr} \, dr^2.
\]

\[
\tilde{\epsilon} = \tilde{r} (r_h + \epsilon) - \tilde{r} (r_h) \approx 2 \epsilon^{1/2} \left[ (r - r_h) g_{rr} \right]^{1/2}|_{r=r_h}.
\]
(66)

On making these substitutions, we find

\[
S_{\text{sing}} = \frac{1}{90\tilde{\epsilon}^2} g_{\theta\theta}|_{r=r_h} = \frac{1}{360\pi \tilde{\epsilon}^2} \text{Area},
\]
(67)
in agreement with known results [3].

The above derivation crucially depends on the behaviour of the metric coefficients near the horizon and is valid only for non-extremal black holes. For extremal cases, the behaviour is different and the area formula does not emerge.

For the extremal dilatonic black hole, \( F \) has a logarithmic singularity. This is present in the non-extremal case as well, but it is in general ignored because of the presence of the linearly divergent term. However, the linear term vanishes when \( a = 2M \), i.e., when the black hole becomes extremal. In this case, the logarithmic term is the dominant one. It arises because \( g_{\theta\theta} \) vanishes linearly at the horizon and has to be kept inside the \( r \) integral in the last line of (60). One obtains

\[
F_{\text{dil}} \approx -\frac{\pi^3}{45M} \log \left( \frac{2M}{\epsilon} \right) \left( \frac{2M}{\beta} \right)^4
\]
(68)
in the same approximation as above. Correspondingly,

\[
S_{\text{dil}} = \frac{1}{360} \log \left( \frac{(4M)^2}{\epsilon^2} \right).
\]
(69)

As the area of the horizon vanishes, one might have expected the entropy to vanish altogether. What does happen is that the linear divergence vanishes, but the logarithmic divergence, which is of course weaker, stays on. A similar logarithmic divergence is known to occur if the theory is truncated to \((1+1)\)
dimensions [4]. Our calculation shows that this is already present in (3+1) dimensions.

For an extremal Reissner - Nordstrom black hole, the singularity of the \( r \) integral in the last line of (60) becomes stronger because \( g_{rr} \) has now a quadratic singularity and \( g_{tt} \) a quadratic zero at the horizon. One obtains

\[
F_{RN} \approx -\frac{2\pi^3 r^2}{135 \epsilon^3} \left( \frac{r_+}{\beta} \right)^4.
\]  

(70)

The contribution to the entropy due to the presence of the black hole is

\[
S_{RN} = \frac{8\pi^3}{135} \left( \frac{r_+}{\beta} \right)^3 \left( \frac{r_+}{\epsilon} \right)^3.
\]

(71)

The formula for the Hawking temperature of the Reissner - Nordstrom black hole is

\[
T = \frac{r_+ - r_-}{4\pi r_+^2}.
\]

(72)

This expression vanishes in the extremal case where \( r_+ = r_- \). If the vanishing temperature is inserted, the expression (71) for the entropy also vanishes and this has been the understanding about this entropy until recently. However, the temperature may be nonvanishing, because of the new observation [7] that the Euclidean solution can be identified with an arbitrary period \( \beta \). For a general \( \beta \) (71) is nonzero and nominally cubically divergent, whereas nonextremal black holes have only a linear divergence in terms of the cutoff \( \epsilon \). In this extremal case, the cutoff in the proper radial variable \( \tilde{r} \) defined as above goes like -\( \log \epsilon \), so that the true divergence is exponential, i.e., much stronger than in the non-extremal case.

This is a rather surprising result. Whereas the previous section suggests that the gravitational entropy of an extremal black hole of the Reissner - Nordstrom type is essentially given by the mass, the entropy of the scalar field in this background is even more singular than in the nonextremal case. The background of an extreme dilatonic black hole also gives a nonzero result when a zero might have been expected, but there the leading linear singularity does drop out and only a milder, logarithmic term remains.

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