Mountain pass solutions to equations with subcritical Musielak-Orlicz-Sobolev growth

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Abstract
In this paper, we study the existence of solutions for equations driven by a new subcritical Musielak-Orlicz-Sobolev growth with homogeneous Dirichlet boundary conditions. These equations have a variational structure and we find a non-trivial solution for them using the Mountain Pass Theorem.

Keywords Musielak-Orlicz-Sobolev spaces · Generalized \( \Phi \)-function · Subcritical Musielak-Orlicz-Sobolev · Mountain pass solutions

Mathematics Subject Classification 35B38 · 35B30 · 35J60

1 Introduction
Since the 1970s, the theory of critical points has undergone rapid development in the branch of the calculus of variations because it has wide applications in other mathematical fields. In particular, to prove the existence of solutions to partial differential equations and dynamical systems.

The simplest quasilinear elliptic equation is expressible by the p-Laplacian operator

\[
\begin{aligned}
&- \text{div}(|\nabla u|^{p-2}\nabla u) = f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.1)

where \( 1 < p < \infty \) and \( \Omega \) is a bounded domain of \( \mathbb{R}^N \). Such problem has been extensively studied in the literature because there are some physical phenomena that such kinds of equations can model, see for example [1, 2, 5, 6, 13]. One of the approaches most used to prove the existence of solutions to this problem is based on the Mountain Pass theorem. For the history, Ambrosetti and Rabinowitz treated the semilinear case \((p = 2)\) [2]. Next,
Dinca, Jebelean, and Mawhin [5] used it to solve the problem for \( p > 1 \) under the following conditions: \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function that satisfies

- The \( p \)-subcritical growth condition:
  \[
  |f(x, t)| \leq C(1 + |t|^{q-1}) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega,
  \]
  where \( p < q < p^* = \frac{Np}{N-p} \) if \( p < N \) or \( p^* = \infty \) if \( p \geq N \).
- The \( p \)-superlinear at 0 condition:
  \[
  \limsup_{t \to 0} \frac{f(x, t)}{|t|^{p-2}t} < \lambda_1 \quad \text{uniformly with } x \in \Omega,
  \]
  where \( \lambda_1 \) is the first eigenvalue of \( p \)-Laplacian operator with homogeneous Dirichlet boundary on \( W^{1,p}_0(\Omega) \).
- The \( p \)-Ambrosetti-Rabinowitz type condition: there exist \( \theta > p \) and \( t_0 > 0 \) such that
  \[
  0 < \theta F(x, t) \leq tf(x, t) \quad \text{for all } t \in \mathbb{R} \text{ such that } |t| \geq t_0, \ x \in \Omega,
  \]
  where \( F(x, t) = \int_0^t f(x, s) \, ds \).

Note that, the restriction to \( p \)-subcritical case \( (p < q < p^*) \) ensures that the embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \) is compact. The Ambrosetti-Rabinowitz condition (AR) is important to ensure that the energy functional associated with the problem has the mountain pass geometry and verify the Palais-Smale condition (PS). Over the years, many researchers have tried to weaken the above condition in different situations.

Later on, the problem (1.1) has been extended in [15] to variable exponent case:

\[
\begin{cases}
- \text{div}(\mathbf{a}(x) \nabla u) = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

(1.2)

where \( x \to p(x) \) is a continuous function on \( \overline{\Omega} \) such that \( 1 < p^- := \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \sup_{x \in \Omega} p(x) < \infty \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function which satisfies

- The \( p(x) \)-subcritical growth condition:
  \[
  |f(x, t)| \leq C(1 + |t|^{q(x)-1}) \quad \text{for all } t \in \mathbb{R} \text{ and } x \in \Omega,
  \]
  where \( p^+ < q^- < q(x) < p^*(x) = \frac{Np(x)}{N-p(x)} \) if \( p(x) < N \) and \( p^*(x) = \infty \) if \( p(x) \geq N \).
- The \( p(x) \)-superlinear at 0 condition:
  \[
  \lim_{t \to 0} \frac{f(x, t)}{|t|^{p^*-1}} = 0 \quad \text{uniformly with } x \in \Omega.
  \]
- The \( p(x) \)-Ambrosetti-Rabinowitz condition: there exist \( \theta > p^+ \) and \( t_0 > 0 \) such that
  \[
  0 < \theta F(x, t) \leq tf(x, t) \quad \text{for all } t \in \mathbb{R} \text{ such that } |t| \geq t_0, \ x \in \Omega,
  \]
  where \( F(x, t) = \int_0^t f(x, s) \, ds \).

The existence of a nontrivial solution to problem (1.2) has been studied in [15]. For more details in this direction, see [7, 15] and the references therein.
Although the operator $p(x)$-Laplacian is a natural generalization of the $p$-Laplacian ($p(x) = p$ constant), the variational problem (1.2) requires a more analysis which is performed to study the existence of nontrivial solutions. However, the generalizations of the conditions used in problem (1.1) to the exponent variable are not tricky. Nevertheless, it is not always the case as in the Orlicz situation. So, it is a natural question to study this problem in a general framework as the Musielak-Orlicz-Sobolev case. For this, we consider the following problem

$$
\begin{align*}
- \text{div} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \right) &= f(x, u) \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

(1.3)

where $g(x, \cdot) \in C^1(\mathbb{R}^+)$ (see Sect. 2) which satisfies the following structure condition: (SC)

There exist two constants $g^0 \geq g_0 > 1$ such that,

$$g_0 - 1 \leq \frac{tg'(x, t)}{g(x, t)} \leq g^0 - 1.$$ 

In the Musielak-Orlicz case, the difficulty is to provide a subcritical growth type condition. So, inspired by the compact embedding theorem given by Harjulehto and Hästö in their monograph [8] we introduce a new subcritical assumption as follows: for $g^0 < \min\{g^0, N\}$,

$$|f(x, t)| \leq C(1 + \psi(x, t)),$$

with $\psi$ is the density of a generalized $\Phi$-function $\Psi(\cdot)$ which satisfies $\Psi^{-1}(x, t) = t^{-\alpha}G^{-1}(x, t)$ with $\alpha \in \left(\frac{1}{g_0} - \frac{1}{g^0}, \frac{1}{N}\right)$ and $G(x, t) = \int_0^t g(x, s) \, ds$. Such condition coincides with the $p(x)$-subcritical growth condition for $G(x, t) = t^p(x)$. Indeed, the first thing, we have $g_0 = p^-$, $g^0 = p^+$ and $G^{-1}(x, t) = t^{p_0}$. Hence

$$\Psi^{-1}(x, t) = t^{-\alpha + \frac{1}{p(x)}} = t^{\frac{1 - ap(x)}{p(x)}}.$$ 

So,

$$\Psi(x, t) = t^{\frac{p(x)}{1 - ap(x)}} = t^{\frac{p(x)}{p^*(x)}},$$

where $q(x) := \frac{p(x)}{1 - ap(x)}$. As $\alpha < \frac{1}{N}$, then

$$q(x) < \frac{p(x)}{1 - \frac{p(x)}{N}} = \frac{Np(x)}{N - p(x)} = p^*(x).$$

But, we still have to $p^* < q^-$. For this, we determine the (SC) of $\Psi(\cdot)$. From the condition (SC) of $g(\cdot)$, we have

$$g_0 \leq \frac{tg(x, t)}{G(x, t)} \leq g^0.$$ 

Hence, we have the function $t \to \frac{G(x, t)}{p_0}$ is increasing and $t \to \frac{G(x, t)}{p^0}$ is decreasing. So, using Proposition 2.3.7 in [8] we get
As
\[
\frac{1}{g^0} \leq \frac{t(G^{-1})'(x, t)}{G^{-1}(x, t)} \leq \frac{1}{g_0}.
\]

Thus
\[
t(G^{-1})'(x, t) = -\alpha + \frac{t(G^{-1})'(x, t)}{G^{-1}(x, t)},
\]
then
\[
-\alpha + \frac{1}{g^0} \leq \frac{t(\Psi^{-1})'(x, t)}{\Psi^{-1}(x, t)} \leq -\alpha + \frac{1}{g_0}.
\]

Thus
\[
\frac{1}{-\alpha + \frac{1}{g_0}} \leq \frac{t\psi(x, t)}{\psi(x, t)} \leq \frac{1}{-\alpha + \frac{1}{g^0}}.
\]

So, an assumption of type \( g^0 < \min\{\frac{1}{-\alpha + \frac{1}{g_0}}, N\} \) is needed. Hence, we restrict that \( \alpha \in (\frac{1}{g_0} - \frac{1}{g^0}, \frac{1}{N}) \). But, for this last interval to be well defined, we suppose that \( g^0 < \min\{g_0^*, N\} \) which implies that \( \frac{1}{g_0} - \frac{1}{g^0} < \frac{1}{N} \). For the other two conditions, we can introduce them by the same way as in the variable exponent case (see Sect. 3). With these assumptions, we are able to establish the existence of nontrivial solutions to problem (1.3).

## 2 Preliminary results

We recall some definitions relating to Musielak-Orlicz spaces. A major synthesis of the functional analysis in these spaces is given in the monographs of Musielak [12] and Harjulehto, Hästö [8].

Throughout this work, let \( \Omega \) be a bounded domain, \( C \) be a generic constant whose value may change between appearances and \( g : \Omega \times [0, \infty) \to [0, \infty) \) such that for each \( t \in [0, \infty) \), the function \( g(\cdot, t) \) is measurable and for a.e. \( x \in \Omega \), \( g(x, \cdot) \) is a \( C^1([0, \infty)) \) which satisfies the condition:

- **(SC)** There exist two constants \( g^0 \geq g_0 > 1 \) such that,
  \[
g_0 - 1 \leq \frac{tg'(x, t)}{g(x, t)} \leq g^0 - 1.
  \]

- **(A_0)** There exists a constant \( C > 1 \) such that,
  \[
  \frac{1}{C} \leq g(x, 1) \leq C \text{ a.e. } x \in \Omega.
  \]

- **(A_1)** If there exists a positive constant \( C \) such that, for every ball \( B_R \) with \( R < 1 \) and \( x, y \in B_R \cap \Omega \), we have
  \[
  G(x, t) \leq CG(y, t) \text{ when } G^-(t) \in \left[ 1, \frac{1}{R^n} \right].
  \]
where $G(\cdot):\Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ is the generalized $\Phi$-function defined by

$$G(x,t) = \int_0^t g(x,s) \, ds,$$

and $G_B^-(t) = \inf_{B^r} G(x,t)$. We notice $G(\cdot) \in \Phi(\Omega)$.

**Properties of generalized $\Phi(\cdot)$-functions.** The first lemma recall some useful inequalities that can be deduced easily from the condition $(SC)$, see Lemma 1.1 in [10] and Proposition 2.1 in [11].

**Lemma 2.1**

(a) $t \to g(x,t)$ is nondecreasing and

$$ag(x,b) \leq ag(x,a) + g(x,b)b. \quad (2.1)$$

(b) $G(\cdot)$ satisfies also the condition type $(SC)$

$$g_0 \leq \frac{tg(x,t)}{G(x,t)} \leq g^0.$$

(c) $\sigma^g G(x,t) \leq G(x,\sigma t) \leq \sigma^g G(x,t)$, for $x \in \Omega$, $t \geq 0$ and $\sigma \geq 1.$ \hspace{1cm} (2.2)

$\sigma^g G(x,t) \leq G(x,\sigma t) \leq \sigma^g G(x,t)$, for $x \in \Omega$, $t \geq 0$ and $\sigma \leq 1.$ \hspace{1cm} (2.3)

(d) There exists a constant $C > 0$ such that

$$G(x,a+b) \leq C(G(x,a) + G(x,b)). \quad (2.4)$$

**Definition 2.1** We define $G^*(\cdot)$ the conjugate $\Phi$-function of $G(\cdot)$, by

$$G^*(x,s) := \sup_{t \geq 0} (st - G(x,t)), \text{ for } x \in \Omega \text{ and } s \geq 0.$$ \hspace{1cm}

Note that $G^*(\cdot)$ is also a generalized $\Phi$-function and can be represented as

$$G^*(x,t) = \int_0^t g^{-1}(x,s) \, ds,$$

with $g^{-1}(x,s) := \sup\{t \geq 0 : g(x,t) \leq s\}$.

The next Lemma establishes some properties of the conjugate function of $G(\cdot)$, see Proposition 2.4.9, and Lemma 3.2.11 in [8].

**Lemma 2.2**

(a) The function $G^*(\cdot)$ satisfies also $(SC)$, as follows
The functions $G(\cdot)$ and $G^*(\cdot)$ satisfy the following Young inequality

$$st \leq G(x, t) + G^*(x, s), \quad \text{for } x \in \Omega \text{ and } s, t \geq 0.$$ 

Further, we have the equality if $s = g(x, t)$ or $t = g^{-1}(x, s)$.

(c) The functions $G(\cdot)$ and $G^*(\cdot)$ satisfy the Hölder inequality

$$\left| \int_{\Omega} u(x)v(x) \, dx \right| \leq 2\|u\|_{G(\cdot)} \|v\|_{G^*(\cdot)}, \quad \text{for } u \in L^{G(\cdot)}(\Omega) \text{ and } v \in L^{G^*(\cdot)}(\Omega).$$

Musielak-Orlicz-Sobolev spaces. The generalized Orlicz space, also called Musielak-Orlicz space, is defined as the set

$$L^{G(\cdot)}(\Omega) = \{ u \in L^0(\Omega) : \int_{\Omega} G(x, |u|) \, dx < \infty \},$$

equipped with the following norms:

- Luxembourg norm:

$$\|u\|_{G(\cdot)} := \inf\{ \lambda > 0 : \int_{\Omega} G\left(x, \frac{|u|}{\lambda}\right) \, dx \leq 1 \}.$$

- Orlicz norm:

$$\|u\|^0_{G(\cdot)} = \sup \left\{ \int_{\Omega} u(x)v(x) \, dx : v \in L^{G^*(\cdot)}(\Omega), \int_{\Omega} G^*(x, |v|) \, dx \leq 1 \right\}.$$ 

These norms are equivalent, precisely we have

$$\|u\|_{G(\cdot)} \leq \|u\|^0_{G(\cdot)} \leq 2\|u\|_{G(\cdot)}.$$

Then, by definition of Orlicz norm and Young inequality, we have

$$\|u\|_{G(\cdot)} \leq \|u\|^0_{G(\cdot)} \leq \int_{\Omega} G(x, |u|) \, dx + 1. \quad (2.5)$$

Next, we define the Musielak-Orlicz-Sobolev space by

$$W^{1,G(\cdot)}(\Omega) := \{ u \in L^{G(\cdot)}(\Omega) : |\nabla u| \in L^{G^*(\cdot)}(\Omega), \text{ in the distribution sense} \},$$
equipped with the norm

$$\|u\|_{1,G(\cdot)} = \|u\|_{G(\cdot)} + \|\nabla u\|_{G^*(\cdot)}.$$ 

The following lemma establishes some properties of convergent sequences in Musielak-Orlicz spaces, see Proposition 2.1 in [11].

**Lemma 2.3** Let $G(\cdot) \in \Phi(\Omega)$ satisfy (SC), then the following relations hold true
(a) \( \|u\|_{G(\cdot)}^{g_0} \leq \int_{\Omega} G(x, |u|) \, dx \leq \|u\|_{G(\cdot)}^{g_0}, \forall u \in L^{G(\cdot)}(\Omega) \) with \( \|u\|_{G(\cdot)} > 1 \).
(b) \( \|u\|_{G(\cdot)}^{g_0} \leq \int_{\Omega} G(x, |u|) \, dx \leq \|u\|_{G(\cdot)}^{g_0}, \forall u \in L^{G(\cdot)}(\Omega) \) with \( \|u\|_{G(\cdot)} < 1 \).
(c) For any sequence \( \{u_i\} \) in \( L^{G(\cdot)}(\Omega) \), we have
\[
\|u_i\|_{G(\cdot)} \to 0 \text{ (resp.} 1; \infty) \iff \int_{\Omega} G(x, |u_i(x)|) \, dx \to 0 \text{ (resp.} 1; \infty).\]

The following lemma gives the continuous embedding in Musielak-Orlicz spaces, see Theorem 3.2.6 in [8].

**Lemma 2.4** Let \( G_1(\cdot), G_2(\cdot) \in \Phi(\Omega) \). Then \( L^{G_1(\cdot)}(\Omega) \hookrightarrow L^{G_2(\cdot)}(\Omega) \) if and only if there exist \( C > 0 \) and \( h \in L^1(\Omega) \) with \( \|h\|_1 \leq 1 \) such that
\[
G_2(x, t) \leq C(G_1(x, t) + h(x)),
\]
for all \( x \in \Omega \) and all \( t \geq 0 \).

To study boundary value problems, we need the concept of weak boundary value problems. We define \( W^{1, G(\cdot)}_0(\Omega) \) as the closure of \( C_c^\infty(\Omega) \) in \( W^{1, G(\cdot)}(\Omega) \). Next, we recall the norm version of the Poincaré inequality used in this work, see Theorem 6.2.8 in [8].

**Lemma 2.5** Let \( \Omega \) be a bounded set of \( \mathbb{R}^N \) and \( G(\cdot) \in \Phi(\Omega) \) satisfy (SC), \((A_0)\) and \((A_1)\). For every \( u \in W^{1, G(\cdot)}_0(\Omega) \), we have
\[
\|u\|_{G(\cdot)} \leq C \|\nabla u\|_{G(\cdot)}.
\]
In particular, \( \|\nabla u\|_{G(\cdot)} \) is a norm on \( W^{1, G(\cdot)}_0(\Omega) \) and it is equivalent to the norm \( \|u\|_{1, G(\cdot)} \).

Harjulehto and Hästö in [8] (see Theorem 6.3.8) give the following compact embedding theorem for Musielak-Sobolev spaces.

**Lemma 2.6** Let \( G(\cdot) \in \Phi(\mathbb{R}^N) \) satisfy (SC), \((A_0)\) and \((A_1)\) for \( g^0 < N \). Let \( \psi \in \Phi(\mathbb{R}^N) \) such that \( t^{-\alpha}G^{-1}(x, t) \approx \psi^{-1}(x, t) \), for some \( \alpha \in [0, \frac{1}{N}) \). Then
\[
W^{1, G(\cdot)}_0(\Omega) \hookrightarrow \hookrightarrow L^{\psi(\cdot)}(\Omega).
\]

**Mountain Pass Theorem in Banach spaces.** We recall here a version of the Mountain Pass theorem, which was discussed by [2, 4, 14]. We shall apply this theorem to establish critical points for finding solutions to problem (1.3).

**Definition 2.2** Let \( X \) be a Banach space and let \( J \in \mathcal{C}^1(X, \mathbb{R}) \). We say that \( J \) satisfies Palais-Smale condition \((PS)\) in \( X \) if any sequence \( \{u_i\} \) in \( X \) such that
(i) \( \{J(u_i)\} \) is bounded,
(ii) \( J'(u_i) \to 0 \) as \( i \to \infty \),
has a convergence subsequence.
Theorem 2.1 Let $X$ be a Banach space and let $J \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition. Assume that $J(0) = 0$, and,

- $(MP)_1$: There exist two positive real numbers $\rho$ and $r$ such that $J(u) \geq r$ with $\|u\| = \rho$,
- $(MP)_2$: There exists $u_1 \in X$ such that $\|u_1\| > \rho$ and $J(u) < 0$.

Put

$$A = \{ f \in C([0, 1], X) : f(0) = 0, f(1) = u_1 \}.$$  

Set

$$\beta = \inf \{ \max J(f([0, 1])) : f \in A \}.$$  

Then $\beta \geq r$ and $\beta$ is a critical value of $J$.

3 The main result and proof

Throughout this section, we assume that, for each $x \in \mathbb{R}^n$, $g(x, \cdot) \in C^1(\mathbb{R}^+)$ and satisfies $(SC)$ with $g^0 < \min \{ g^*, N \}$. We consider the following problem

\[
\begin{aligned}
- \nabla \cdot \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \right) &= f(x, u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(3.1)

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

- $(f_a)$: The $G(\cdot)$-subcritical growth condition

$$|f(x, t)| \leq C(1 + \psi(x, |t|)) \quad \text{for } x \in \Omega, \ t \in \mathbb{R},$$

where $\psi$ is the density of a generalized $\Phi$-function $\Psi(\cdot)$ satisfying $\Psi^{-1}(x, t) = t^{-a}G^{-1}(x, t)$ with $a \in (\frac{1}{s_0} - \frac{1}{g^0}, \frac{1}{N})$.

- $(f_0)$: The $G(\cdot)$-superlinear at 0 condition

$$f(x, t) = o(t^{g^0-1}), \ t \to 0 \quad \text{for } x \in \Omega.$$

- $(AR)$: The $G(\cdot)$-Ambrosetti-Rabinowitz type condition: there exist $t_0 > 0$ and $\theta > g^0$ such that

$$0 < \theta F(x, t) \leq tf(x, t) \quad \text{for } x \in \Omega \quad \text{if } |t| \geq t_0,$$

with $F(x, t) := \int_0^t f(x, s) \, ds$.

Weak solutions. We say that a function $u \in W^{1, G(\cdot)}_0(\Omega)$ is a weak solution of the equation (3.1) in $\Omega$ if
whenever $\varphi \in W_0^{1, G^{(+)}}(\Omega)$.

Note that, if $f(x, t) = 0$ the existence of weak solutions to problem (3.1) have been proved in [9] and, in [3] the authors studied the case $f(x, t) = g(x, t)$. We denote by $X$ the Musielak-Orlicz-Sobolev space $W_0^{1, G^{(+)}}(\Omega)$ and we define the functional energy corresponding to problem (3.1) by $J : X \to \mathbb{R}$

$$J(u) := \int_{\Omega} G(x, |\nabla u|) \, dx - \int_{\Omega} F(x, u) \, dx.$$ 

It’s well known that standard arguments imply that $J \in C^1(X, \mathbb{R})$ and its derivative is given by

$$\langle J'(u), v \rangle = \int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot \nabla v \, dx - \int_{\Omega} f(x, u) v \, dx.$$ 

The lack of homogeneity is a source of difficulty in proving the (PS) condition in our situation. So, we developed a method inspired by Lieberman’s article [10], which will allow us to overcome this problem.

**Proposition 3.1** Suppose $(f_u)$ and (AR) hold, then $J$ satisfies condition (PS).

**Proof** Let us assume that there exists a sequence $\{u_i\} \subset X$ such that

$$|J(u_i)| \leq C \quad J'(u_i) \to 0.$$ 

Then, by the condition (AR), we have

$$C \geq J(u_i) = \int_{\Omega} G(x, |\nabla u_i|) \, dx - \int_{\Omega} F(x, u_i) \, dx$$

$$= \int_{\Omega} G(x, |\nabla u_i|) \, dx - \int_{\{|u_i| < t_0\}} F(x, u_i) \, dx - \int_{\{|u_i| \geq t_0\}} F(x, u_i) \, dx. \quad (3.2)$$

We will estimate separately the last two integrals. By the condition $(f_u)$, we have

$$\int_{\{|u_i| < t_0\}} F(x, u_i) \, dx \leq C \left( \int_{\{|u_i| < t_0\}} |u_i| \, dx + \int_{\{|u_i| < t_0\}} \Psi(x, |u_i|) \, dx \right)$$

$$\leq C \left( t_0 + \int_{\{|u_i| < t_0\}} \Psi(x, t_0) \, dx \right).$$

Or, from the structural condition of $\Psi$ (see introduction)

$$\frac{1}{-\alpha + \frac{1}{\delta_0}} \leq \frac{\nu(t, x)}{\Psi(x, t)} \leq \frac{1}{-\alpha + \frac{1}{\delta}}$$

and Lemma 2.1, we have
\[ \Psi(x, t_0) \leq \max \left\{ t_0^{-a + \theta \frac{1}{n}}, t_0^{-a + \frac{1}{n}} \right\} \Psi(x, 1). \]

As \( G(\cdot) \) satisfies the condition \((A_0)\), then \( G^{-1}(\cdot) \) and \( \psi(\cdot) \) also satisfy \((A_0)\). Hence
\[ \int_{\{u_i < t_0\}} F(x, u_i) \, dx \leq C. \tag{3.3} \]

For the second integral, using the conditions \((AR)\) and \((f_\alpha)\), we get
\[
\int_{\{|u_i| \geq t_0\}} F(x, u_i) \, dx \leq \frac{1}{\theta} \int_{\{|u_i| \geq t_0\}} f(x, u_i) u_i \, dx \\
\leq \frac{1}{\theta} \left( \int_{\Omega} f(x, u_i) u_i \, dx + \int_{\{|u_i| < t_0\}} |f(x, u_i)| |u_i| \, dx \right) \\
\leq \frac{1}{\theta} \left( \int_{\Omega} f(x, u_i) u_i \, dx + C \int_{\{|u_i| < t_0\}} (1 + \psi(x, t_0) t_0) \, dx \right). \]

By the same method of inequality (3.3), we obtain
\[ \int_{\{|u_i| < t_0\}} (1 + \psi(x, t_0) t_0) \, dx \leq C. \]

Hence,
\[ \int_{\{|u_i| \geq t_0\}} F(x, u_i) \, dx \leq \frac{1}{\theta} \int_{\Omega} f(x, u_i) u_i \, dx + C. \tag{3.4} \]

By combining inequalities (3.2), (3.3) and (3.4), we get
\[ C \geq \int_{\Omega} G(x, |\nabla u_i|) \, dx - \frac{1}{\theta} \int_{\Omega} f(x, u_i) u_i \, dx. \tag{3.5} \]

In the other hand, we have \( J'(u_i) \to 0 \) then, we can choose \( i_0 \in \mathbb{N} \) such that for all \( i \geq i_0 \), we have
\[ |\langle J'(u_i), v \rangle| \leq ||v||_{1,G(\cdot)}. \]

So, for \( v = u_i \) and the condition \((SC)\), we obtain
\[
-||u_i||_{1,G(\cdot)} \leq \int_{\Omega} \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \cdot \nabla u_i \, dx - \int_{\Omega} f(x, u_i) u_i \, dx \\
\leq g^0 \int_{\Omega} G(x, |\nabla u_i|) \, dx - \int_{\Omega} f(x, u_i) u_i \, dx.
\]

Therefore, by the inequality (3.5) and Lemma 2.3, we have
\[
C \geq \left( 1 - \frac{g^0}{\theta} \right) \int_{\Omega} G(x, |\nabla u_i|) \, dx - \frac{1}{\theta} ||u_i|| \\
\geq \left( 1 - \frac{g^0}{\theta} \right) \min\{ ||u_i||^{g^0}_{1,G(\cdot)}, ||u_i||^{g^0}_{1,G(\cdot)} \} - \frac{1}{\theta} ||u_i||.
\]
As \( \theta > g^0 \), then \( \{u_i\} \) is bounded in \( X \). Which implies \( u_i \rightharpoonup u \) in \( X \). As \( \Psi^{-1}(x, t) = t^{-\alpha}G^{-1}(x, t) \) with \( \alpha \in \left( \frac{1}{g^0} - 1, \frac{1}{N} \right) \), then \( X \hookrightarrow L^{\Psi} (\Omega) \), so \( u_i \rightarrow u \) in \( L^{\Psi^*} (\Omega) \). Thus, by Hölder inequality in \( L^{\Psi^*} (\Omega) \), we have
\[
\int_{\Omega} f(x, u_i)(u_i - u) \, dx \leq C \int_{\Omega} (1 + \psi(x, |u_i|)|u_i - u| \, dx \\
\leq C ||1 + \psi(x, |u_i|)||_{\Psi^*} ||u_i - u||_{\Psi^*}.
\]
So, by the inequality (2.5), the Young equality, the condition (SC) and Lemma 2.3, we have
\[
||1 + \psi(x, |u_i|)||_{\Psi^*} \leq 1 + ||\psi(x, |u_i|)||_{\Psi^*} \\
\leq C \int_{\Omega} \Psi(x, \psi(x, |u_i|)) \, dx + 2 \\
\leq C \int_{\Omega} \Psi(x, |u_i|) \, dx + 2 \\
\leq C \max \left\{ ||u_i||_{\Psi^*}, ||u_i||_{\Psi^*} \right\} + 2 \\
\leq C.
\]

Then
\[
\int_{\Omega} f(x, u_i)(u_i - u) \, dx \leq C ||u_i - u||_{\Psi^*} \rightarrow 0.
\]
As \( |\langle J'(u_i), u_i - u \rangle| \rightarrow 0 \), then
\[
\int_{\Omega} \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \cdot (\nabla u - \nabla u_i) \, dx \rightarrow 0.
\]
Furthermore, as \( (L^{G^*}(\Omega))' \approx L^{G^*}(\Omega) \), we have
\[
\int_{\Omega} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u \cdot (\nabla u - \nabla u_i) \, dx \rightarrow 0.
\]
Hence
\[
\int_{\Omega} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \, dx \rightarrow 0
\]
Using the condition (SC) and Cauchy-Schwarz inequality, we have for \( \theta_i = tu + (1 - t)u_i \), with \( t \in (0, 1) \)
\[
\left( \frac{g(x, \nabla u)}{\nabla u} - \frac{g(x, \nabla u_i)}{\nabla u_i} \right) \cdot (\nabla u - \nabla u_i) \\
= \left( \int_0^1 \frac{\partial}{\partial t} \left( \frac{g(x, \nabla \theta_i)}{\nabla \theta_i} \right) \, dt \right) \cdot (\nabla u - \nabla u_i) \\
= |\nabla u - \nabla u_i|^2 \int_0^1 \frac{g(x, \nabla \theta_i)}{\nabla \theta_i} \, dt \\
+ \int_0^1 g(x, \nabla \theta_i) \left( \frac{|\nabla \theta_i g'(x, \nabla \theta_i)|}{g(x, \nabla \theta_i)} - 1 \right) \left( \nabla \theta_i \cdot (\nabla u - \nabla u_i)^2 \right) \, dt \\
\geq \min(1, g_0 - 1)|\nabla u - \nabla u_i|^2 \int_0^1 \frac{g(x, \nabla \theta_i)}{\nabla \theta_i} \, dt.
\]

This implies
\[
\int_{\Omega} \left( \frac{g(x, \nabla u)}{\nabla u} - \frac{g(x, \nabla u_i)}{\nabla u_i} \right) \cdot (\nabla u - \nabla u_i) \, dx \\
\geq \min(1, g_0) \int_{\Omega} \int_0^1 \frac{g(x, \nabla \theta_i)}{\nabla \theta_i} |\nabla u - \nabla u_i|^2 \, dt \, dx.
\]

Now we write \( S_1 = \{ x \in \Omega : |\nabla u - \nabla u_i| \leq 2|\nabla u| \} \) and \( S_2 = \{ x \in \Omega : |\nabla u - \nabla u_i| > 2|\nabla u| \} \). Then \( S_1 \cup S_2 = \Omega \) and
\[
\frac{1}{2} |\nabla u| \leq |\nabla \theta_i| \leq 3|\nabla u| \quad \text{on} \ S_1 \ \text{for} \ t \geq \frac{3}{4}, \\
\frac{1}{4} |\nabla u - \nabla u_i| \leq |\nabla \theta_i| \leq 3|\nabla u - \nabla u_i| \quad \text{on} \ S_2 \ \text{for} \ t \leq \frac{1}{4}.
\]

Therefore
\[
\int_{\Omega} \left( \frac{g(x, \nabla u)}{\nabla u} - \frac{g(x, \nabla u_i)}{\nabla u_i} \right) \cdot (\nabla u - \nabla u_i) \, dx \\
\geq C \left( \int_{S_1} \frac{g(x, \nabla u)}{|\nabla u|} |\nabla u - \nabla u_i|^2 \, dx + \int_{S_2} G(x, |\nabla u - \nabla u_i|) \, dx \right).
\]

Hence
\[
\int_{S_2} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \int_{\Omega} \left( \frac{g(x, \nabla u)}{\nabla u} - \frac{g(x, \nabla u_i)}{\nabla u_i} \right) \cdot (\nabla u - \nabla u_i) \, dx.
\]

To estimate the integrals over \( S_1 \), using the condition \((SC)\) and \( t \to g(x, t) \) is a nondecreasing function, we have
\[
\int_{S_1} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \int_{S_1} g(x, |\nabla u - \nabla u_i|)|\nabla u - \nabla u_i| \, dx \\
\leq C \int_{S_1} g(x, |\nabla u|)|\nabla u - \nabla u_i| \, dx.
\]

As
Mountain pass solutions to equations with subcritical…

\[
\int_{S_1} g(x, |\nabla u|) |\nabla u - \nabla u_i| \, dx = \int_{S_1} g(x, |\nabla u|)^{\frac{3}{2}} |\nabla u - \nabla u_i| g(x, |\nabla u|)^{\frac{1}{2}} \, dx
\]

\[
= \int_{S_1} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{3}{2}} |\nabla u - \nabla u_i| \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{1}{2}} \, dx
\]

\[
= \int_{S_1} \left( \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{3}{2}} |\nabla u - \nabla u_i| \right) \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{1}{2}} \, dx.
\]

Then

\[
\int_{S_1} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \int_{S_1} \left( \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{3}{2}} |\nabla u - \nabla u_i| \right) \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \right)^{\frac{1}{2}} \, dx.
\]

So, by Hölder inequality in \(L^2(S_1)\), we obtain

\[
\int_{S_1} G(x, |\nabla u - \nabla u_i|) \, dx \leq C \left( \int_{S_1} \frac{g(x, |\nabla u|)}{|\nabla u|} |\nabla u - \nabla u_i|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{S_1} G(x, |\nabla u|) \, dx \right)^{\frac{1}{2}}.
\]

Hence, from the inequality (3.6), we have

\[
\int_{S_1} G(|\nabla u - \nabla u_i|) \, dx
\]

\[
\leq C \left( \int_{S_1} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \, dx \right)^{\frac{1}{2}} \left( \int_{S_1} G(x, |\nabla u|) \, dx \right)^{\frac{1}{2}}.
\]

Collecting the inequalities (3.7) and (3.8), we have

\[
\int_{S_1} \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \, dx \cdot (\nabla u - \nabla u_i) \, dx \]

\[
+ \int_{S_1} \left( \frac{g(x, |\nabla u|)}{|\nabla u|} \nabla u - \frac{g(x, |\nabla u_i|)}{|\nabla u_i|} \nabla u_i \right) \cdot (\nabla u - \nabla u_i) \, dx \rightarrow 0.
\]

Therefore, by Lemma (2.3) and Poincaré inequality, we have \(u_i \rightarrow u\) in \(X\).

**Theorem 3.1** Assume that \(f\) satisfy (AR), \((f_0)\) and \((f_a)\) with \(\alpha \in \left( \frac{1}{s_0} - \frac{1}{2}, \frac{1}{2} \right)\). Then problem (3.1) has nontrivial weak solution.

**Proof** We show that \(J\) satisfies all assumptions of Theorem 2.1. We start with condition (MP)_1. For this, we have

\[
J(u) = \int_{S_1} G(x, |\nabla u|) \, dx - \int_{S_1} F(x, u) \, dx.
\]

Let \(|u|_{1,G(\cdot)} < 1\). Then

\[
J(u) \geq ||\nabla u||_{G(\cdot)}^{\theta} - \int_{S_1} F(x, u) \, dx.
\]
From conditions \((f_0), (f_a)\) and condition \((A_0)\), we have for \(x \in \Omega\) and \(t \in \mathbb{R}\)

\[
|F(x, t)| \leq e|t|^{\gamma \frac{\mu}{\nu}} \mathbb{1}_{\{|x| \leq 1\}}(t) + C(t + \Psi(x, t))\mathbb{1}_{\{|x| > 1\}}(t) \\
\leq e|t|^{\gamma \frac{\mu}{\nu}} \mathbb{1}_{\{|x| \leq 1\}}(t) + C\Psi(x, t)\mathbb{1}_{\{|x| > 1\}}(t).
\]

Then

\[
\int_{\Omega} F(x, u) \, dx \leq e \int_{\{|u| \leq 1\}} |u|^{\gamma \frac{\mu}{\nu}} \, dx + C \int_{\{|u| > 1\}} \Psi(x, |u|) \, dx.
\]

By the condition \((A_0)\), we have \(t^{\gamma \frac{\mu}{\nu}} \leq CG(x, t)\) for \(t \leq 1\). Then, using Lemma 2.4, we get

\[
\int_{\{|u| < 1\}} |u|^{\gamma \frac{\mu}{\nu}} \, dx \leq ||u||_{1, G(\cdot)}^{\gamma \frac{\mu}{\nu}} \leq C||\nabla u||_{G(\cdot)}^{\gamma \frac{\mu}{\nu}}.
\]

Furthermore, by Lemma 2.3 and compact embedding in the Musielak-Orlicz-Sobolev spaces, we have

\[
\int_{\Omega} \Psi(x, |u|) \, dx \leq C||u||_{\Psi_{(\cdot)}}^{\frac{1}{\gamma \frac{\mu}{\nu} \frac{\nu - 1}{\nu}} \leq C||\nabla u||_{G(\cdot)}^{\frac{1}{\gamma \frac{\mu}{\nu} \frac{\nu - 1}{\nu}}}
\]

Therefore

\[
J(u) \geq ||\nabla u||_{G(\cdot)}^{\gamma \frac{\mu}{\nu}} - eC||\nabla u||_{G(\cdot)}^{\gamma \frac{\mu}{\nu}} - ||\nabla u||_{G(\cdot)}^{\frac{1}{\gamma \frac{\mu}{\nu} \frac{\nu - 1}{\nu}}}. 
\]

So, choosing \(e = \frac{1}{2C}\), we obtain

\[
J(u) \geq \frac{1}{2} ||\nabla u||_{G(\cdot)}^{\gamma \frac{\mu}{\nu}} - ||\nabla u||_{G(\cdot)}^{\frac{1}{\gamma \frac{\nu - 1}{\nu}}}.
\]

As \(\alpha > \frac{1}{g_0} - \frac{1}{g_0} \) which implies that \(g^0 < \frac{1}{-\alpha + \frac{1}{g_0}}\), then there exist two constants \(\eta, r\) such that \(J(u) \geq r > 0\) with \(||\nabla u||_{G(\cdot)} = \eta \in (0, 1)\).

For the second condition \((MP)\), We use an important consequence of the condition \((AR)\):

\[
F(x, t) \geq C|t|^\theta \text{ for } |t| \geq t_0.
\]

So, for \(\omega \in X \setminus \{0\}, \omega \geq 0\) and \(t > 1\), let us denote

\[
M_t(\omega) = \{x \in \Omega : t\omega(x) \geq t_0\}.
\]

We choose \(u \in X \setminus \{0\}, u \geq 0\) with \(|M_t(u)| > 0\). It is clear that \(M_1(u) \subset M_t(u)\) and hence \(|M_1(u)| \leq |M_t(u)|\) for all \(t > 1\). Then

\[
\int_{M_t(u)} F(x, tu) \, dx \geq t^\theta C \int_{M_t(u)} u^\theta \, dx.
\]

Furthermore, by the condition \((f_a)\) and the condition \((A_0)\), we have
\[
\int_{M(\omega)'\cup L(\omega)'} F(\omega, t\omega) \, dx \leq C \int_{M(\omega)'} tu + \Psi(x, t\omega) \, dx \\
\leq C \int_{M(\omega)'\cup L(\omega)'} t_0 + \Psi(x, t_0) \, dx \\
\leq C.
\]

Hence
\[
J(t\omega) \geq \rho^{\theta_0} \int_\Omega G(x, |\nabla u|) \, dx - \rho^{\theta} C \int_{M(\omega)'} u^\theta \, dx - C.
\]

Since \( \theta > \theta^0 \) then \( J(t\omega) \to -\infty \) when \( t \to \infty \). The fact \( J(0) = 0 \), \( J \) satisfies the all assumptions of Theorem 2.1. Therefore \( J \) has at least one nontrivial critical point, i.e problem (3.1) has a nontrivial weak solution. The proof is complete. \( \square \)

**Declarations**

**Conflicts of interest** The authors declared that they have no conflict of interest.

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