Total edge irregularity strength of some cycle related graphs

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Abstract

An edge irregular total $k$-labeling $f : V \cup E \rightarrow \{1, 2, ..., k\}$ of a graph $G = (V, E)$ is a labeling of vertices and edges of $G$ in such a way that for any two different edges $uv$ and $u'v'$, their weights $f(u) + f(uv) + f(v)$ and $f(u') + f(u'v') + f(v')$ are distinct. The total edge irregularity strength $tes(G)$ is defined as the minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling. In this paper, we determine the total edge irregularity strength of new classes of graphs $C_m \circ C_n$, $P_{m,n}^*$ and $C_{m,n}^*$ and hence we extend the validity of the conjecture $tes(G) = \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\}$ for some more graphs.

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1. Introduction

Throughout this paper, $G$ is a simple graph, $V$ and $E$ are the sets of vertices and edges of $G$, with cardinalities $|V|$ and $|E|$ respectively. A labeling of a graph is a map that carries graph elements to the numbers. A labeling is called a vertex labeling, an edge labeling or a total labeling, if the domain of the map is the vertex set, the edge set, or the union of vertex and edge sets respectively.
Baca et al. in [2] started to investigate the total edge irregularity strength of a graph, an invariant analogous to the irregularity strength for total labeling. For a graph $G = (V(G), E(G))$, the weight of an edge $e = xy$ under a total labeling $\xi$ is $wt_\xi(e) = \xi(x) + \xi(e) + \xi(y)$. For a graph $G$ we define a labeling $\xi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ to be an edge irregular total $k$-labeling of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ one has $wt_\xi(xy) \neq wt_\xi(x'y')$. The total edge irregular strength, $tes(G)$, is defined as the minimum $k$ for which $G$ has an edge irregular total $k$-labeling. In [3], we can find that

$$tes(G) \geq \max\left\{\left\lceil \frac{|E(G)| + 2}{3}\right\rceil, \left\lceil \frac{\Delta(G) + 1}{2}\right\rceil\right\},$$

where $\Delta(G)$ is the maximum degree of $G$, and also there are determined the exact values of the total edge irregularity strength for paths, cycles, stars, wheels and friendship graphs. Recently Ivanco and Jendrol [6] proved that for any tree $T$,

$$tes(T) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3}\right\rceil, \left\lceil \frac{\Delta(G) + 1}{2}\right\rceil\right\}.$$ 

Moreover, they posed a conjecture that for an arbitrary graph $G$ different from $K_5$ and the maximum degree $\Delta(G)$,

$$tes(G) = \max\left\{\left\lceil \frac{|E(G)| + 2}{3}\right\rceil, \left\lceil \frac{\Delta(G) + 1}{2}\right\rceil\right\}.$$ 

The Ivanco and Jendrol’s conjecture has been verified for complete graphs and complete bipartite graphs in [7], for categorical product of cycle and path in [1] and [12], for corona product of paths with some graphs in [11]. In[8], Jayanthi et al. verified the conjecture for disjoint union of double wheel graphs. In[5], Indra et al. verified the conjecture for generalized uniform theta graph.

Motivated by the papers [9, 10], we define three new classes of graphs and extend the validity for the conjecture for some more families of graphs. We define the graph $C_n \circ C_n$, $n \geq 3$, $m \geq 3$ as follows. Denote the vertex set of $C_m \circ C_n$ by $V(C_m \circ C_n) = \{i \mid 1 \leq i \leq n\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set of $C_m \circ C_n$ by $E(C_m \circ C_n) = \{e_{ij} = u_i u_{i+1} \mid 1 \leq i \leq n, u_{n+1} = u_1\} \cup \{e_{i1} = u_iv_i \mid 1 \leq i \leq n\} \cup \{e_{i(j+1)} = v_{ij} v_{i(j+1)} \mid 1 \leq i \leq n, 1 \leq j \leq m, v_{i(m+1)} = v_{i1}\}$. In $C_m \circ C_n$, $|V(C_m \circ C_n)| = n(m + 1)$ and $|E(C_m \circ C_n)| = n(m + 2)$. The graph $C_3 \circ C_9$ is shown in Figure 1.

We introduce another new class of graph $P^*_n$. The graph $P^*_n$, $m \geq 3$, $n \geq 2$ is defined as follows: denote the vertex set of $P^*_n$ by $V(P^*_n) = \{v_i \mid 1 \leq i \leq n + 1\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m - 2\}$ and the edge set of $P^*_n$ by $E(P^*_n) = \{e_{i1} = v_{i+1}v_{i+2} \mid 1 \leq i \leq n\} \cup \{e_{ij} = v_{i(j-1)}v_{i(j)} \mid 1 \leq i \leq n, 2 \leq j \leq m - 2\} \cup \{e_{i1} = v_{i+1}v_{i+2} \mid 1 \leq i \leq n\} \cup \{e_{i(m-1)} = v_{n(m-2)}v_{n1} \mid 1 \leq i \leq n\}$. In $P^*_n$, $|V(P^*_n)| = mn - n + 1$ and $|E(P^*_n)| = nm$. The graph $P^*_9$ is shown in Figure 2. In $P^*_n$, $m \geq 3$, $n \geq 2$, identifying the vertices $v_1$ and $v_{n+1}$ we obtain the new class of graph denoted by $C^*_n$. 

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In this paper, we determine the total edge irregularity strength of these new classes of graphs $C_m@C_n$, $P_m^*$ and $C_m^*$, and hence we extend the validity of the conjecture

$$tes(G) = \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}$$

for some more families of graphs.

2. Main Results

In the following theorem we describe an optimal edge irregular total labeling for the graph $C_m@C_n$.

**Theorem 2.1.** For any integers $m \geq 3$, $n \geq 3$, $tes(C_m@C_n) = \left\lceil \frac{n(m+2)+2}{3} \right\rceil$.

**Proof.** The vertex set of $C_m@C_n$ is $\{u_i | 1 \leq i \leq n\} \cup \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set of $C_m@C_n$ is $\{e_i = u_iu_{i+1} | 1 \leq i \leq n, u_{n+1} = u_1\} \cup \{e_{i1} = u_iv_{i1} | 1 \leq i \leq n\} \cup \{e_{ij+1} = v_{ij}v_{(j+1)} | 1 \leq i \leq n, 1 \leq j \leq m, v_{i(m+1)} = v_{i1}\}$. Since $tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\}$,

it is enough to prove that $tes(C_m@C_n) \leq \left\lceil \frac{n(m+2)+2}{3} \right\rceil$.

Let $k = \left\lceil \frac{n(m+2)+2}{3} \right\rceil$.

We construct an edge-irregular total labeling $l$ as follows:

\[
l(u_i) = \begin{cases} 
1 & \text{if } i = 1 \\
\left\lceil \frac{m+3}{2} \right\rceil & \text{if } i = 2 \\
m + 5 & \text{if } i = 3 \\
k - \left( i - 3 \right) m & \text{if } 4 \leq i < \frac{n}{2} \\
k & \text{if } \frac{n}{2} \leq i \leq \frac{n}{2} + \frac{1}{2} + 2 \\
k - \left( i - \frac{1}{2} \right) m & \text{if } \frac{n}{2} + 3 \leq i \leq n - 2 \\
\left\lceil \frac{3m+2}{2} \right\rceil & \text{if } i = n - 1 \\
3 + m & \text{if } i = n \\
\end{cases}
\]

\[
l(v_{1j}) = \left\lceil \frac{j+1}{3} \right\rceil \text{ for } 1 \leq j \leq m,
\]

\[
l(v_{2j}) = \left\lceil \frac{m+j+5}{3} \right\rceil \text{ for } 1 \leq j \leq m,
\]

\[
l(v_{ij}) = \begin{cases} 
\left\lceil \frac{m+3+2i-3(m+2)}{3} \right\rceil & \text{for } 1 \leq j \leq m, 3 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \\
\left\lceil \frac{m+4+(n-1)(m+2)}{3} \right\rceil & \text{for } 1 \leq j \leq m, i = \left\lceil \frac{n}{2} \right\rceil + 1 \\
\left\lceil \frac{m+3+2n-2i+2(m+2)}{3} \right\rceil & \text{for } 1 \leq j \leq m, \left\lceil \frac{n}{2} \right\rceil + 2 \leq i \leq n,
\end{cases}
\]
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\[
l(e_i) = \begin{cases} 
  m + 3 - \left\lfloor \frac{m+3}{2} \right\rfloor & \text{if } i = 1 \\
  2m + 3 - \left\lfloor \frac{m+3}{2} \right\rfloor & \text{if } i = 2 \\
  3m + 7 - k + \left\lfloor \frac{n-6}{3} \right\rfloor m = 4m + 4i - 2k + 2\left\lfloor \frac{n-6}{2} \right\rfloor m & \text{if } i = 3 \\
  m + \left\lfloor \frac{n}{2} \right\rfloor m + \left\lfloor \frac{n-6}{2} \right\rfloor m + 4(\left\lfloor \frac{5}{2} \right\rfloor - 2) - 2k & \text{if } 4 \leq i < \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
  2\left\lfloor \frac{m}{2} \right\rfloor m + 4(\left\lfloor \frac{5}{2} \right\rfloor - m) - 2k & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor - 1 \\
  mn + 2n - m + 1 - 2k & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor \\
  2mn - m + 4n - 2\left\lfloor \frac{n}{2} \right\rfloor m - 4\left\lfloor \frac{n}{2} \right\rfloor - 2k - 2 & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
  2mn + 4n + 2m + 2n - 4i + 6 - 2k - \left\lfloor \frac{i - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor m + \left\lfloor \frac{i + 1 - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor m & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\
  6m + 14 - k + \left\lfloor \frac{n-2}{3} \right\rfloor - \left\lfloor \frac{3m+7}{2} \right\rfloor m - \left\lfloor \frac{3m+7}{2} \right\rfloor & \text{if } i = n - 2 \\
  3m + 7 - \left\lfloor \frac{3m+7}{2} \right\rfloor & \text{if } i = n - 1 \\
  m + 2 & \text{if } i = n,
\end{cases}
\]

\[
l(e_{ij}) = \begin{cases} 
  1 & \text{if } j = 1 \\
  j + 2 - \left\lfloor \frac{j+3}{3} \right\rfloor - \left\lfloor \frac{j+4}{3} \right\rfloor & \text{if } 2 \leq j \leq m \\
  m + 2 - \left\lfloor \frac{m+4}{3} \right\rfloor & \text{if } j = m + 1,
\end{cases}
\]

\[
l(e_{ij}) = \begin{cases} 
  j + 2 + (2i - 3)(m + 2) - 2\left\lfloor \frac{m + 3 + (2i - 3)(m + 2)}{3} \right\rfloor & \text{if } 2 \leq j \leq m + 1, 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  j + 3 + (n - 1)(m + 2) - 2\left\lfloor \frac{m + 4 + (n - 1)(m + 2)}{3} \right\rfloor & \text{if } 2 \leq j \leq m + 1, i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
  j + 2 + (2n - 2i + 2)(m + 2) - 2\left\lfloor \frac{m + 3 + (2n - 2i + 2)(m + 2)}{3} \right\rfloor & \text{if } 2 \leq j \leq m + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n,
\end{cases}
\]

\[
l(e_{2j}) = \begin{cases} 
  m + 4 + j - \left\lfloor \frac{m + j + 4}{3} \right\rfloor - \left\lfloor \frac{m + 5 + j}{3} \right\rfloor & \text{if } 2 \leq j \leq m \\
  2m + 5 - \left\lfloor \frac{2m+5}{3} \right\rfloor - \left\lfloor \frac{m+6}{3} \right\rfloor & \text{if } j = m + 1.
\end{cases}
\]

For \( n \) is odd,

\[
l(e_1) = \begin{cases} 
  m + 5 - \left\lfloor \frac{m+6}{3} \right\rfloor - \left\lfloor \frac{m+3}{2} \right\rfloor & \text{if } i = 2 \\
  2m + 4 - \left\lfloor \frac{4m+9}{3} \right\rfloor & \text{if } i = 3 \\
  mi + 4i - 3 - k + \left\lfloor \frac{n-6}{2} \right\rfloor m - \left\lfloor \frac{m+3+(2i-3)(m+2)}{3} \right\rfloor & \text{if } 4 \leq i < \left\lfloor \frac{n}{2} \right\rfloor \\
  2mi + 4i - 3m - 3 - k - \left\lfloor \frac{m+3+(2i-3)(m+2)}{3} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
  4 + (n - 1)(m + 2) - k - \left\lfloor \frac{m+4+(n-1)(m+2)}{3} \right\rfloor & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\
  3 + (2n - 2i + 2)(m + 2) - k + \left\lfloor \frac{i - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor m - \left\lfloor \frac{m+3+(2n-2i+2)(m+2)}{3} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n - 2 \\
  4m + 11 - \left\lfloor \frac{3m+7}{2} \right\rfloor - \left\lfloor \frac{m+3+4(m+2)}{3} \right\rfloor & \text{if } i = n - 1 \\
  m + 4 - \left\lfloor \frac{3m+7}{2} \right\rfloor & \text{if } i = n.
\end{cases}
\]

For \( n \) is even,

\[
l(e_1) = \begin{cases} 
  m + 5 - \left\lfloor \frac{m+6}{3} \right\rfloor - \left\lfloor \frac{m+3}{2} \right\rfloor & \text{if } i = 2 \\
  2m + 4 - \left\lfloor \frac{4m+9}{3} \right\rfloor & \text{if } i = 3 \\
  mi + 4i - 3 - k + \left\lfloor \frac{n-6}{2} \right\rfloor m - \left\lfloor \frac{m+3+(2i-3)(m+2)}{3} \right\rfloor & \text{if } 4 \leq i < \frac{n}{2} \\
  2mi + 4i - 3m - 3 - k - \left\lfloor \frac{m+3+(2i-3)(m+2)}{3} \right\rfloor & \text{if } i = \frac{n}{2} \\
  4 + (n - 1)(m + 2) - k - \left\lfloor \frac{m+4+(n-1)(m+2)}{3} \right\rfloor & \text{if } i = \frac{n}{2} + 1 \\
  3 + (2n - 2i + 2)(m + 2) - k - \left\lfloor \frac{i - \left\lfloor \frac{n}{2} \right\rfloor}{2} \right\rfloor m - \left\lfloor \frac{m+3+(2n-2i+2)(m+2)}{3} \right\rfloor & \text{if } \frac{n}{2} + 2 \leq i \leq n - 2 \\
  4m + 11 - \left\lfloor \frac{3m+7}{2} \right\rfloor - \left\lfloor \frac{m+3+4(m+2)}{3} \right\rfloor & \text{if } i = n - 1 \\
  m + 4 - \left\lfloor \frac{3m+7}{2} \right\rfloor & \text{if } i = n.
\end{cases}
\]

Now \( \max\{l(v) \mid v \in V(C_m \circ C_n)\} \cup \{l(e) \mid e \in E(C_m \circ C_n)\} = k \) and \( l \) is a function
from \( V(C_m \circ C_n) \cup E(C_m \circ C_n) \) into \( \{1, 2, ..., k\} \).

The weights of the edges are given by

\[
\begin{align*}
\text{weight of edges } & \text{are given by} \\
w(e_{i,j}) = \begin{cases} 
    m + 4 & \text{if } i = 1 \\
    2 + (2i - 1)(m + 2) & \text{if } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
    3 + (n - 1)(m + 2) & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
    2 + (2n - 2i + 2)(m + 2) & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{weight of edges } & \text{are given by} \\
w(e_i) = \begin{cases} 
    j + 2 & \text{if } i = 1 \\
    j + 2 + (2i - 3)(m + 2) & \text{if } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
    j + 3 + (n - 1)(m + 2) & \text{if } i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\
    j + 2 + (2n - 2i + 2)(m + 2) & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n. 
\end{cases}
\end{align*}
\]

The weights of the edges of \( E \) under total labeling \( l \) form a set of consecutive integers from 3 to \( n(m + 2) + 2 \) and no two edges have the same weight. Hence \( tes(C_m \circ C_n) = \left\lceil \frac{n(m+2)+2}{3} \right\rceil \).

In the following theorem we describe an optimal edge irregular total labeling for the graph \( P_{m,n}^* \).

**Theorem 2.2.** For any integers \( m \geq 3, n \geq 2 \), \( tes(P_{m,n}^*) = \left\lceil \frac{nm+2}{3} \right\rceil \).

**Proof.** The vertex set of \( P_{m,n}^* \) is \( \{v_i \mid 1 \leq i \leq n + 1\} \cup \{v_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m - 2\} \) and the edge set of is \( \{e_i = v_i v_{i+1} \mid 1 \leq i \leq n\} \cup \{e_{ij} = v_{ij} v_{ij+1} \mid 1 \leq i \leq n, 2 \leq j \leq m - 2\} \cup \{e_{i1} = v_{i1} v_{i2} \mid 1 \leq i \leq n\} \cup \{e_{i(m-1)} = v_{i(m-2)} v_{i+1} \mid 1 \leq i \leq n\} \). Since \( tes(G) \geq \max \left\{ \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\} \), it is enough to prove that \( tes(P_{m,n}^*) \leq \left\lceil \frac{nm+2}{3} \right\rceil \).

We construct an edge-irregular total labeling \( l \) as follows:

\[
\begin{align*}
l(v_i) = \begin{cases} 
    1 & \text{if } i = 1, i = n + 1 \\
    \left\lceil \frac{m+5}{3} \right\rceil & \text{if } i = 2 \\
    \left\lceil \frac{2i-2}{3} \right\rceil m + 1 & \text{if } 3 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor \\
    \left\lceil \frac{nm+2}{3} \right\rceil & \text{if } i = \left\lfloor \frac{n+2}{2} \right\rfloor + 1, \left\lfloor \frac{n+2}{2} \right\rfloor + 2 \\
    \left\lceil \frac{(2n-2i+5)m+1}{3} \right\rceil & \text{if } \left\lfloor \frac{n+2}{2} \right\rfloor + 3 \leq i \leq n, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
l(v_{ij}) = \begin{cases} 
    \left\lceil \frac{n+2}{3} \right\rceil & \text{if } i = 1, 1 \leq j \leq m - 2 \\
    \left\lceil \frac{m+5}{3} \right\rceil & \text{if } i = 2, 1 \leq j \leq m - 2 \\
    \left\lceil \frac{(2n-2i+1)m+1}{3} \right\rceil & \text{if } 3 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor, 1 \leq j \leq m - 2 \\
    \left\lceil \frac{(2n-2i+3)m+1}{3} \right\rceil & \text{if } \left\lfloor \frac{n+2}{2} \right\rfloor + 1 \leq i \leq n, 1 \leq j \leq m - 2, 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
l(e_i) = \begin{cases} 
    m + 1 - \left\lceil \frac{m+5}{3} \right\rceil & \text{if } i = 1 \\
    2m + 2 - \left\lceil \frac{m+5}{3} \right\rceil - \left\lceil \frac{4m+1}{3} \right\rceil & \text{if } i = 2 \\
    2im - 2m + 2 - \left\lfloor \frac{(2i-2)m+1}{3} \right\rfloor - \left\lfloor \frac{2im+1}{3} \right\rfloor & \text{if } 3 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor - 1 \\
    (2\left\lfloor \frac{n+2}{2} \right\rfloor - 4)m + 2 - \left\lceil \frac{nm+2}{3} \right\rceil - \left\lceil \frac{(2\left\lfloor \frac{n+2}{2} \right\rfloor - 4)m+1}{3} \right\rceil & \text{if } i = \left\lfloor \frac{n+2}{2} \right\rfloor \\
    (2n - 2\left\lfloor \frac{n+2}{2} \right\rfloor + 1)m + 2 - \left\lceil \frac{(2n-2)(\frac{n+2}{2})+1)m+1}{3} \right\rceil & \text{if } i = \left\lfloor \frac{n+2}{2} \right\rfloor + 1 \\
    (2n - 2i + 3)m + 1 - \left\lfloor \frac{(2n-2i+5)m+1}{3} \right\rceil - \left\lfloor \frac{(2n-2i+3)m+1}{3} \right\rceil & \text{if } \left\lfloor \frac{n+2}{2} \right\rfloor + 2 \leq i \leq n, 
\end{cases}
\end{align*}
\]
Now \( \max \{ \{ l(v) \mid v \in V(P_{m,n}^*) \} \cup \{ l(e) \mid e \in E(P_{m,n}^*) \} \} = \left\lceil \frac{mn+2}{3} \right\rceil \) and \( l \) is a function from \( V(P_{m,n}) \cup E(P_{m,n}) \) into \( \{ 1, 2, \ldots, \left\lceil \frac{mn+2}{3} \right\rceil \} \).

The weights of the edges are given by

\[
w(e) = \begin{cases} 
  m + 2 & \text{if } i = 1 \\
  (2i - 2)m + 2 & \text{if } 2 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor \\
  (2n - 2i + 3)m + 2 & \text{if } \left\lceil \frac{n+2}{2} \right\rceil + 1 \leq i \leq n,
\end{cases}
\]

\[
w(e_{ij}) = \begin{cases} 
  j + 2 & \text{if } i = 1, 1 \leq j \leq m - 1 \\
  (2i - 3)m + 2 + j & \text{if } 2 \leq i \leq \left\lfloor \frac{n+2}{2} \right\rfloor, 1 \leq j \leq m - 1 \\
  (2n - 2i + 2)m + 2 + m - j & \text{if } \left\lceil \frac{n+2}{2} \right\rceil + 1 \leq i \leq n, 1 \leq j \leq m - 1.
\end{cases}
\]

The weights of the edges of \( E \) under total labeling \( l \) form a set of consecutive integers from 3 to \( nm + 2 \) and no two edges have the same weight. Hence \( \text{tes}(P_{m,n}^*) = \left\lceil \frac{nm+2}{3} \right\rceil \).

Consider \( n \) copies of the graph \( C_m \) and label the vertices in the \( i \)th copy of \( C_m \) as \( v_i, v_i, v_{i+1}, \ldots, v_{i(m-2)}, v_{i+1} \) for \( 1 \leq i \leq n \). Then using the labelings as in \( P_{m,n}^* \), we get the following result.

**Corollary 2.1.** [2] For any integers \( m \geq 3, n \geq 3 \), \( \text{tes}(nC_m) = \left\lceil \frac{nm+2}{3} \right\rceil \).

In the graph \( P_{m,n}^* \cup P_{m,r}^* \), labeling the vertices in \( P_{m,r}^* \) by

\[
\{ v_i \mid n + 1 \leq i \leq n + r \} \cup \{ v_{ij} \mid n + 1 \leq i \leq n + r - 1, 1 \leq j \leq m - 2 \},
\]
we obtain the following result.
Corollary 2.2. For any integers \( m \geq 3, n \geq 3 \), \( \text{tes}(P_{m,n}^* \cup P_{m,r}^*) = \lceil \frac{m(n+r)+2}{3} \rceil \).

3. Open Problem

In \( C_m \circ C_n \), we take \( n \) copies of \( C_m \). Instead of considering same cycles, consider \( n \) cycles with different lengths \( m_1, m_2, \ldots, m_n \) and denote the new graph by \( C' \). Prove that

\[
\text{tes}(C') = \left\lceil \sum_{i=1}^{n} m_i + 2 \right\rceil.
\]

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