IDEMPOTENTS IN NONASSOCIATIVE ALGEBRAS AND EIGENVECTORS OF QUADRATIC OPERATORS

YU. LYUBICH AND A. TSUKERMAN

Abstract. Let $F$ be a field, $\text{char}(F) \neq 2$. Then every finite-dimensional $F$-algebra has either an idempotent or an absolute nilpotent if and only if over $F$ every polynomial of odd degree has a root in $F$. This is also necessary and sufficient for existence of eigenvectors for all quadratic operators in finite-dimensional spaces over $F$.

1. Introduction

Let $F$ be a field, and let $A$ be an $F$-algebra, i.e. a vector space over $F$ endowed by a multiplication that is a bilinear mapping $(x, y) \mapsto xy$ from the Cartesian square $A \times A$ into $A$. The associativity, the commutativity and the unitality are not assumed in this general definition.

Let $\dim A = n$, $1 \leq n < \infty$, and let $(e_i)_{i=1}^n$ be a basis of $A$. Then the coordinates of the product $z = xy$ are

$$\zeta_j = \sum_{i,k=1}^n \alpha_{ik,j} \xi_i \eta_k, \quad 1 \leq j \leq n,$$

where $\xi_i, \eta_k$ are the coordinates of $x$ and $y$, respectively, and $\alpha_{ik,j} \in F$ are the uniquely determined coefficients. Every cubic matrix $[\alpha_{ik,j}]$ can be realized in this context but the algebra is commutative if and only if the square matrices $[\alpha_{ik,j}]_{i,k=1}^n$, $1 \leq j \leq n$, are symmetric.

Any algebra $A$ determines a quadratic operator $Vx = x^2$ in the underlying vector space. This mapping $A \to A$ is homogeneous of degree 2, i.e. $V(\alpha x) = \alpha^2 Vx$ for all $x \in A$ and all $\alpha \in F$. With $y = x$ in (1.1) we have

$$\zeta_j = \sum_{i,k=1}^n \alpha_{ik,j} \xi_i \xi_k, \quad 1 \leq j \leq n,$$

where $z_j$ are the coordinates of $Vx$. If $\text{char}(F) \neq 2$ then the coefficients $\alpha_{ik,j}$ in (1.2) can be symmetrized, thus in this case all quadratic operators come from the commutative algebras.

Similarly to the linear case, a vector $x \neq 0$ is called an eigenvector of $V$ if there exists an eigenvalue $\lambda \in F$ such that $Vx = \lambda x$, i.e.

$$\sum_{i,k=1}^n \alpha_{ik,j} \xi_i \xi_k = \lambda \xi_j, \quad 1 \leq j \leq n.$$
The eigenvectors of $V$ are just the basis vectors of the one-dimensional subalgebras of the algebra $A$.

The set $\sigma(V)$ of eigenvalues of $V$ can be called its spectrum. However, only two following cases are substantial.

1) $\lambda = 1$. In this case every eigenvector $x$ is a fixed point of $V$, so $x^2 = x$, i.e. $x$ is an idempotent in the generating algebra $A$.

2) $\lambda = 0$. In this case every eigenvector $x$ is a root of $V$, so $x^2 = 0$, i.e. $x$ is an absolute nilpotent.

Now consider any $\lambda \in \sigma(V)$, and let $x$ be a corresponding eigenvector. Then there exists $z \in \text{Span}(x) \setminus \{0\}$ which is either an idempotent or an absolute nilpotent in $A$.

Hence, the intersection $\sigma_p(V) = \sigma(V) \cap \{0, 1\}$ is not empty if and only if $\sigma(V) \neq \emptyset$.

Moreover, if $\sigma_p(V) = \{0, 1\}$ then $\sigma(V) = F$; if $\sigma_p(V) = \{1\}$ then $\sigma(V) = F \setminus \{0\}$; if $\sigma_p(V) = \{0\}$ then $\sigma(V) = \{0\}$.

We say that a field $F$ is of class $\Lambda$ if every finite-dimensional $F$-algebra contains either an idempotent or an absolute nilpotent. Equivalently, $F \in \Lambda$ means that in every finite-dimensional vector space over $F$ every quadratic operator has an eigenvector, i.e. its spectrum is not empty.

In [Ly] it was proven that $\mathbb{R} \in \Lambda$. The proof is topological and very short. Let us reproduce it here.

Let $\| \cdot \|$ be an Euclidean norm in $\mathbb{R}^n$, and let $S = \{ x : \| x \| = 1 \}$. One can assume that $Vx \neq 0$ for all $x \neq 0$. Then $\hat{V}x = Vx/\| Vx \|$ is a smooth mapping $S \to S$. Its topological degree is even since $\hat{V}(-x) = Vx$. Therefore, $\hat{V}$ is not homotopic to the identity mapping $S \to S$. Hence, there are $x \in S$ and $\tau \in (0, 1)$ such that $(1 - \tau)\hat{V}x + \tau x = 0$. Thus, $x$ is an eigenvector of $V$.

Note that if a field $F$ is of class $\Lambda$ then every its finite (i.e., finite-dimensional) extension $\Phi$ is also of class $\Lambda$. Indeed, let $\Lambda$ be an $l$-dimensional $\Phi$-algebra. Then it is an $m$-dimensional $F$-algebra with $m = l(\Phi : F) \equiv l \dim_F \Phi$. Every idempotent or absolute nilpotent in $F$-setting remains as is when $F$ extends to $\Phi$.

In particular, $C \in \Lambda$ since $\mathbb{R} \in \Lambda$.

In the present paper we establish a necessary and a sufficient condition for $F \in \Lambda$. These conditions coincide if $\text{char}(F) \neq 2$.

2. Results

Denote by $\Gamma$ the class of fields $F$ such that every polynomial of odd degree with coefficients from $F$ has a root in $F$. For $F = \mathbb{R}$ this property is the only topological ingredient of the Gauss proof (1815) of the “Fundamental Theorem of Algebra”.

Theorem 2.1. Every algebraically closed field $F$ is of class $\Lambda$.

Our Main Theorem is

Theorem 2.2. $\Gamma' \subset \Lambda \subset \Gamma$ where $\Gamma' = \Gamma \cap \{ F : \text{char}(F) \neq 2 \}$.

Corollary 2.3. If $\text{char}(F) \neq 2$ then $F \in \Lambda$ if and only if $F \in \Gamma$.

With the additional condition $\text{char}(F) \neq 2$ Theorem 2.1 follows from Corollary 2.3. Also, the latter yields a “real” counterpart of Theorem 2.1.

Corollary 2.4. Every formally real and really closed field $F$ is of class $\Lambda$.

Indeed, in this case $F \in \Gamma'$ since $\text{char}(F) = 0$ and $F \in \Gamma$, see e.g. [La], Ch.11.
In several corollaries below we use the inclusion $\Lambda \subset \Gamma$. For instance, the field $\mathbb{Q}$ of rational numbers is not of class $\Lambda$ since the polynomial $\alpha^3 - 2$ has no roots in $\mathbb{Q}$. More generally, we have

**Corollary 2.5.** Every finite extension $F \supset \mathbb{Q}$ is not of class $\Lambda$.

*Proof.* Assume $F \in \Lambda$. Let $\Phi$ be a finite extension of $F$ normal over $\mathbb{Q}$. With $n = [\Phi : \mathbb{Q}]$ let $m$ be any odd number, $m > n$. By Eisenstein’s test there exists an irreducible polynomial $f$ of degree $m$ over $\mathbb{Q}$. This $f$ has no roots in $\Phi$, otherwise, all $m$ roots of $f$ belong to $\Phi$, so $m \leq n$. Since $f$ is a polynomial over $\Phi$, we see that $\Phi \in \Gamma$. Hence, $\Phi \notin \Lambda$, a fortiori, $F \notin \Lambda$, a contradiction. □

The same argument yields

**Corollary 2.6.** Let $p$ be a prime number, and let $\mathbb{Q}_p$ be the field of $p$-adic numbers. Every finite extension $F \supset \mathbb{Q}_p$ is not of class $\Lambda$.

**Corollary 2.7.** Every finite field $F$ is not of class $\Lambda$.

*Proof.* If $q = \text{card}(F)$ then $\alpha^q - \alpha + 1 \neq 0$ for all $\alpha \in F$. This polynomial is of odd degree if $\text{char}(F) \neq 2$. If $\text{char}(F) = 2$ then such a polynomial is $\alpha^{q+1} - \alpha^2 + 1$. □

**Corollary 2.8.** For every field $K$ and every $n \geq 1$ the field $F = K(t_1, \cdots, t_n)$ of rational functions of $n$ variables over $K$ is not of class $\Lambda$.

*Proof.* $\alpha^3 - t_1 \neq 0$ for all $\alpha \in F$. □

The class $\Gamma$ is closely related to the class $\Delta$ of fields such that the degrees of all their finite extensions are powers of 2.

**Theorem 2.9.** $\Gamma' \subset \Delta \subset \Gamma$.

**Corollary 2.10.** If $\text{char}(F) \neq 2$ then $F \in \Gamma$ if and only if $F \in \Delta$.

Thus, if $\text{char}(F) \neq 2$ then each of conditions $F \in \Gamma$ and $F \in \Delta$ is necessary and sufficient for $F \in \Lambda$.

3. Proofs

We start with the following

**Lemma 3.1.** A field $F$ is of class $\Gamma$ if and only if the degrees of all its nontrivial finite extensions are even.

*Proof.* Let $F \notin \Gamma$, so there exists a polynomial $f$ over $F$ of odd degree without roots in $F$. Then $f$ has an irreducible factor with the same properties. The corresponding extension of $F$ is of odd degree $d > 1$, a contradiction.

Now let $F \in \Gamma$, and let $\Phi \supset F$ be a finite extension of odd degree $d > 1$. For $x \in \Phi \setminus F$ the degree $d_x$ of the extension $F[x]$ is odd as a divisor of $d$. However, $d_x$ is the degree of the irreducible polynomial $f$ over $F$ such that $f(x) = 0$. This is a contradiction since $f$ has a root in $F$. □

This Lemma immediately implies $\Delta \subset \Gamma$ that is a part of Theorem 2.9.
Proof of \( \Lambda \subset \Gamma \). Let \( F \notin \Gamma \). By Lemma 3.1 there exists an extension \( \Phi \supset F \) of odd degree \( d > 1 \). Consider the projection \( \pi : \Phi \to \Phi \) onto the one-dimensional subspace \( F \). We introduce in \( \Phi \) the new multiplication

(3.1) \[
x \circ y = (x - \pi x)(y - \pi y).
\]

Then \( \Phi \) becomes a commutative \( F \)-algebra. The subspace \( F \) is an ideal in \( \Phi \) since \( x \circ y = 0 \) for \( x \in F \) and all \( y \in \Phi \). In the quotient algebra \( F \)-algebra \( A = \Phi/F \) we consider the equation

(3.2) \[
X \circ X = \lambda X
\]

with \( \lambda \in F \). To conclude that \( F \notin \Lambda \) it suffices to show that (3.2) has no solutions \( X \neq 0 \). Suppose to the contrary. Then there exists \( x \in \Phi \setminus F \) such that

\[
(x - \pi x)^2 - \lambda x - \mu = 0
\]

with a \( \mu \in F \). This is a quadratic equation over \( F \) since \( \pi x \in F \). Its second root is \( x' = 2\pi x + \lambda - x \equiv -x \pmod{F} \), so \( x' \notin F \). Hence, the extension \( F[x] \supset F \) is of degree 2. On the other hand, \( F[x] \subset \Phi \). Therefore, \( d = 2[\Phi : F[x]] \). Thus, \( d \) is even, a contradiction.

Now note that every field \( F \in \Gamma' \) is perfect. Indeed, if char\( (F) \) is an odd prime \( p \) then \( F^p = F \) because of \( F \in \Gamma \). Recall that if a field \( F \) is perfect then all its normal algebraic extensions are separable, so they are the Galois extensions that allows us to refer to the Galois theory, see e.g. [La], Ch. 8.

Proof of the inclusion \( \Gamma' \subset \Delta \). Let \( \Phi \supset F \) be a finite extension of \( F \), and let \( \Psi \supset \Phi \) be a Galois extension of \( F \). If \( [\Phi : F] = 2^i(2k + 1) \) then \( [\Psi : F] = 2^i(2l + 1) \) where \( j \geq i \) and \( 2l + 1 \) is divisible by \( 2k + 1 \). Consider the Galois group \( G = \text{Gal}(\Psi/F) \) and its Sylow 2-subgroup \( H \). Let \( \Omega \) be the subfield of \( \Psi \) consisting of the fixed points of \( H \). The index of \( H \) in \( G \) is \( 2l + 1 \), hence \( [\Omega : F] = 2l + 1 \). Since \( F \in \Gamma \), we conclude that \( l = 0 \) by Lemma 3.1. A fortiori, \( k = 0 \), hence \( [\Phi : F] = 2^i \). \[ \]

Theorem 2.4 is already proven. As to Theorem 2.2 we have to prove the inclusion \( \Gamma' \subset \Lambda \). To this end (and also to prove Theorem 2.1) we return to (1.3) and consider it as a system of \( n \) homogeneous equations of second degree

(3.3) \[
g_j(x, \lambda) = \sum_{i,k=1}^{n} \alpha_{ik,j} \xi_i \xi_k - \lambda \xi_j = 0, \quad 1 \leq j \leq n, \quad x = (\xi_1, \ldots, \xi_n) \in F^n,
\]

with \( n + 1 \) unknowns \( \xi_1, \ldots, \xi_n, \lambda \). With \( \lambda \in F \) this is a system of equations in the projective space \( FP^n \). The solution \((0:1) \equiv (0 : \cdots : 0 : 1)\) is called trivial. In all nontrivial solutions \((x, \lambda) \in FP^n\) the scalar component \( \lambda \) is an eigenvalue of the quadratic operator under consideration. Thus, our aim is to prove the existence of a nontrivial solution in \( FP^n \).

Let us extend \( FP^n \) to the projective space \( \overline{FP^n} \) over the algebraic closure \( \overline{F} \). We call the system (3.3) generic over \( F \) if the set of its solutions in \( \overline{FP^n} \) is finite. Then the number of solutions (i.e., the sum of its multiplicities) is equal to \( 2^n \) by the Bezout Theorem. A solution is called simple if its multiplicity is equal to 1.

Lemma 3.2. If the system (3.3) is generic over \( F \) then the trivial solution is simple.
Proof. The affine space $F^n$ can be identified with the projective Zariski neighborhood of $(0:1)$ in $FP^n$ consisting of the points $(x:1)$, $x \in F^n$. The corresponding restrictions of the quadratic forms $g_j(x, \lambda)$ are the polynomials $f_j(x) = g_j(x, 1)$.

Let $R$ be the local ring of the point $0 \in F^n$, and let $M$ be its maximal ideal. In the quotient $\mathcal{F}$-vector space $M/M^2$ the images of $(-\xi_j) \in M$, $1 \leq j \leq n$, constitute a basis. The images of $f_j$ are the same since $f_j + \xi_j \in M^2$. By Nakayama’s Lemma the system $(f_j)_{j=1}^n$ generates the ideal $M$. Hence, the intersection index of the corresponding divisors is 1.

Let us emphasize that Lemma 3.2 is true irrespective to char$(F)$.

Proof of Theorem 2.1. If $F = F^*$ then either the set of solutions in $FP^n$ is infinite or the number of these solutions is $2^n > 1$. By Lemma 3.2 there exists a nontrivial solution of the system (3.3) in $FP^n$.

Later on $F \supseteq F^*$. Let the system (3.3) be generic over $F$. For every solution we fix the projective coordinates and take them from $F$ whenever the solution belongs to $FP^n$. Now we consider the Galois extension $\Psi \supseteq F$ containing all these quantities. By Theorem 2.9 we have $[\Psi : F] = 2^i$ with an $i \geq 1$. Accordingly, the group $G = \text{Gal}(\Psi/F)$ is of order $2^i$. Since the coefficients of all $g_j(x, \lambda)$ belong to $F$, the group $G$ naturally acts on the set of solutions preserving their multiplicities. A solution is $G$-invariant if and only if it is from $FP^n$.

Let $z = (x, \lambda) \neq 0$ be a solution of a multiplicity $m$, and let $Z$ be its $G$-orbit. All points from $Z$ are solutions of the same multiplicity $m$. The contribution of $Z$ into the number of solutions equals $\tilde{m} = \text{card}(Z)m$. In this product the second factor coincides with the index of the stabilizer of $z$. This index is $2^i$ with some $j \geq 1$ if $z$ is not a fixed point, i.e. if this solution is not from $FP^n$. Hence, the number of such solutions is even. Since the number of all solutions is $2^n$, the number of those solutions which are from $FP^n$ is even. By Lemma 3.2 there exists a nontrivial solution in $FP^n$. The generic case in Theorem 2.2 is settled.

To complete the proof of Theorem 2.2 we show that an “infinitesimal” perturbation of the system (3.3) is generic over an extension of $F$.

Lemma 3.3. Let a field $K \supseteq F$ be endowed by a non-Archimedean valuation, let $A$ be the valuation ring, and let $A^0$ be its maximal ideal. Then for $1 \leq j \leq n$ there are some linear forms $\varphi_j(x)$ with coefficients from $A$ and some $\varepsilon_j \in A^0$ such that the system

\begin{equation}
(3.4)
g_j(x, \lambda) - \varepsilon_j(\varphi_j(x))^2 = 0, \quad 1 \leq j \leq n,
\end{equation}

is generic over $K$.

Proof. Inductively on $m$, $1 \leq m \leq n$, we construct the variety

$$V_m = \{(x, \lambda) \in \overline{K}P^n : g_j(x, \lambda) - \varepsilon_j(\varphi_j(x))^2 = 0, 1 \leq j \leq m\}$$

of dimension $\leq n - m$. Then $\dim V_n = 0$, thus the system (3.3) is generic over $K$.

Since $g_1 \neq 0$, we can take $\varepsilon_1 = 0$ to get $\dim V_1 = n - 1$ with any $\varphi_1$. Now let $1 \leq m < n$, and let $\dim V_m \leq n - m$. Consider the decomposition $V_m = \bigcup_{1 \leq i \leq r} X_i$ into irreducible components, and choose any point $(x_i, \lambda_i) \in X_i$, $1 \leq i \leq r$. Let $g_{m+1}(x_i, \lambda_i) = 0$ for $1 \leq i \leq s$, while $g_{m+1}(x_i, \lambda_i) \neq 0$ for $s + 1 \leq i \leq r$. Since the field $K$ is infinite, there exists a linear form $\varphi_{m+1}(x)$ on $\overline{K}^n$ with coefficients from
A such that \( \varphi_{m+1}(x_i) \neq 0, 1 \leq i \leq r \). With \( \varepsilon_{m+1} \in A^0 \) different from 0 and from all fractions
\[
\frac{g_{m+1}(x_i, \lambda_i)}{(\varphi_{m+1}(x_i))^2}, \quad s + 1 \leq i \leq r,
\]
the form \( g_{m+1}(x, \lambda) - \varepsilon_{m+1}(\varphi_{m+1}(x))^2 \) does not vanish on each component of \( V_m \). Therefore, \( \dim V_{m+1} < \dim V_m \), hence \( \dim V_{m+1} \leq n - m - 1 \). □

For our purposes we need to get \( K \in \Gamma' \) in Lemma 3.1. The first step in this direction is the extension of \( F \) to the field \( L = F((t)) \) whose nonzero elements are the formal Laurent series
\[
a(t) = t^\nu \sum_{k=0}^{\infty} \alpha_k t^k
\]
where \( \alpha_k \in F, \alpha_0 \neq 0, \nu = \nu(a) \in \mathbb{Z} \). On \( L \) we have the standard non-Archimedean valuation \( v_0(a) = \exp(-\nu(a)), a \neq 0 \). The ground field \( F \) is embedded in \( L \) as the field of constants, \( v_0(\alpha) = 1 \) for \( \alpha \in F \setminus \{0\} \).

With the distance \( v_0(a - b) \) the set \( L \) is a complete metric space. The closed unit ball \( A_L = \{ a \in L : v_0(a) \leq 1 \} \), i.e. the set of regular series, is just the valuation ring of the field \( L \). Its unique maximal ideal is the open ball \( A_L^0 = \{ a : v_0(a) < 1 \} \). The residue field \( R_L = A_L/A_L^0 \) is isomorphic to \( F \).

The valuation \( v_0 \) can be extended to a non-Archimedean valuation \( v \) of the algebraic closure \( \overline{L} \). The corresponding ring is \( A_{\overline{L}} = \{ x \in \overline{L} : v(x) \leq 1 \} \), its maximal ideal is \( A_{\overline{L}}^0 = \{ x \in \overline{L} : v(x) < 1 \} \). Denote by \( \rho \) the natural epimorphism from \( A_{\overline{L}} \) to the residue field \( R_{\overline{L}} = A_{\overline{L}}/A_{\overline{L}}^0 \).

Now let \( F \) be the family of fields \( E \) such that \( L \subset E \subset \overline{L} \), and let \( F \) be partially ordered by inclusion. On every \( E \in F \) we have the valuation \( v|E \). The corresponding residue field is isomorphic to \( R_E = \text{Im}(\rho|A_E) \) where \( A_E = A_{\overline{L}} \cap E \) is the valuation ring of the field \( E \).

Obviously, if \( E_1, E_2 \in F \) and \( E_1 \subset E_2 \) then \( A_{E_1} \subset A_{E_2} \) and \( R_{E_1} \subset R_{E_2} \). In particular, \( R_L \subset R_E \subset R_{\overline{L}} \) for all \( E \in F \). By Zorn’s Lemma there exists a field \( K \in F \) which is maximal among \( E \in F \) with \( R_E = R_L \). Indeed, let \( \{ E_i \} \) be a linearly ordered subfamily of \( F \), and let all \( R_{E_i} = R_L \). Then \( \cup E_i \) is a field \( E \in F \), and \( A_E = A_{\overline{L}} \cap (\cup E_i) = \cup (A_{\overline{L}} \cap E_i) = \cup A_{E_i} \), hence \( R_E = \cup R_{E_i} = R_L \).

**Lemma 3.4.** \( K \in \Gamma' \).

**Proof.** Since \( \text{char}(K) = \text{char}(F) \neq 2 \), we actually have to prove that \( K \in \Gamma \). Suppose to the contrary. Then by Lemma 3.1 there exists a finite extension \( E \supset K \) of odd degree \( d > 1 \). One can take \( E \subset \overline{L} \) since \( K \subset \overline{L} \). We have \( E \in F \), but \( R_E \neq R_L \) by maximality of \( K \) in \( F \). Hence, there exists \( x \in E \setminus K \) such that \( \rho x \notin R_L \). Let \( f \) be an irreducible polynomial over \( K \) such that \( f(x) = 0 \). Its degree \( d_f \) is odd since \( d_f \) divides \( d \).

Let us extend the initial field \( L \) to a field \( L_f \subset K \) by joining of all coefficients of \( f \). Obviously, \( L_f \in F \). The valuation \( v|L_f \) is discrete and \( L_f \) is complete. Furthermore, \( R_{L_f} = R_L \) since \( R_L \subset R_{L_f} \subset R_K = R_L \). Hence, \( R_{L_f} \) is isomorphic to \( F \), thus it is a perfect field of class \( \Gamma \).

Now we consider the field \( L_f[x] \in F \). The degree \( [L_f[x] : L_f] \) coincides with \( d_f \) since the polynomial \( f \) is determined and irreducible over \( L_f \). Hence, this degree is odd. The residue field of \( L_f[x] \) is a finite extension of \( R_{L_f} \). Its degree \( \delta \) divides
Lemma 3.5. The ring $A_K$ is the direct sum of the subrings $F$ and $A_K^0$.

Proof. First, note that $F \subset A_K$ since $v(x) = 1$ for $x \in F$. For the same reason $F \cap A_K^0 = 0$. Now let $x \in A_K$. Then $px \in R_K = R_L$. Hence, $px = \rho a$ where $a \in L$, whence $x - a \in \ker(\rho|_{A_K}) = A_K^0$. In turn, $a = \alpha + \omega$ where $\alpha \in F$ and $\omega \in A_L^0 \subset A_K^0$. As a result, $x = \alpha + (x - a) + \omega \in F + A_K^0$. □

Since Theorem 2.9 is true in the generic case, the system (3.4) has a nontrivial solution $(x, \lambda) \in K P^n$ by Lemmas 3.3 and 3.4. Let $x = (\xi_i)_{i=1}^n \in K^n \setminus \{0\}$, and let $\|x\| = \max_i v(\xi_i)$. Show that $v(\lambda) \leq \|x\|$. Indeed, in the opposite case the division on $\lambda^2$ in (5.3) yields

$$\zeta_j = h_j(z), \quad 1 \leq j \leq n,$$

where $h_j$ are some quadratic forms with coefficients from $A_K$ and $z = (\zeta_j)_{j=1}^n = (\zeta_j/\lambda)_{j=1}^n \in A_K^0 \setminus \{0\}$. However, from (3.5) it follows that $\|z\| \leq \|z\|^2$, hence $\|z\| \geq 1$, a contradiction.

Thus, $\max(\|x\|, v(\lambda)) = \|x\|$. Let $\|x\| = v(\xi_1)$ for definiteness. By division on $\xi_1$ we get a solution $(x_1, \lambda_1) \in K P^n$ with $\|x_1\| = 1$ and $v(\lambda_1) \leq 1$. By Lemma 3.3, $(x_1, \lambda_1) = (\hat{x}, \hat{\lambda}) + (y, \omega)$, where the first summand belongs to $F^{n+1}$ and the second one belongs to $(A_K^0)^{n+1}$. In addition, $\hat{x} \neq 0$, otherwise $x_1 = y$, so $\|x_1\| < 1$. Since $A_K^0$ is an ideal in $A_K$, the system (3.4) yields $g_j(\hat{x}, \hat{\lambda}) \in F \cap A_K^0$, $1 \leq j \leq n$. This implies $g_j(\hat{x}, \hat{\lambda}) = 0$, $1 \leq j \leq n$, by Lemma 3.3 again. Theorem 2.9 is proven completely.

4. Some remarks

Remark 4.1. Over any field any power-associative finite-dimensional algebra has an idempotent or a nilpotent of some degree, see [S], Proposition 3.3. In the latter case if $x^r = 0$, $r \geq 2$, but $x^{r-1} \neq 0$, then $x^s$ is an absolute nilpotent for $s = r - \lfloor r/2 \rfloor$. Thus, if we restrict the definition of class $\Lambda$ to the power-associative algebras then $\Lambda$ extends to the class of all fields, so our problem disappears. The same happens trivially under the restriction to the unital algebras.

Remark 4.2. The infinite-dimensional version of $\Lambda$ is empty. Indeed, over any field $F$ there are no idempotents neither absolute nilpotents in the algebra of polynomials $f(t)$ such that $f(0) = 0$.

Remark 4.3. The algebra $A$ from the proof of the inclusion $\Lambda \subset \Gamma$ is commutative. Therefore, this inclusion remains valid if in the definition of $\Lambda$ we consider the commutative algebras only.

Remark 4.4. In the case $\text{char}(F) = 0$ the Corollary 2.1 follows from $C \subset \Lambda$ by the metamathematical Lefshets Principle.
REFERENCES

[La] S.Lang, Algebra. Addison-Wesley Publ.Co., 1965.

[Ly] Yu.I.Lyubich, The nilpotency of the Engel commutative algebras of dimension $n \leq 4$ over $\mathbb{R}$. Uspechi Mat.Nauk, 32, no.1 (1977), 195-196 (In Russian).

[S] R.D.Schafer, An Introduction to Nonassociative Algebras. Acad. Press, New York -London, 1966.

Corresponding Author: Yu. Lyubich, Department of Mathematics, Technion, Haifa 32000, Israel

E-mail address: lyubich@tx.technion.ac.il

A. Tsukerman, 30 Port Royal, Foster City, CA 94404