Proof of the Wehrl-type Entropy Conjecture for Symmetric $SU(N)$ Coherent States

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Abstract

The Wehrl entropy conjecture for coherent (highest weight) states in representations of the Heisenberg group, which was proved in 1978 and recently extended by us to the group $SU(2)$, is further extended here to symmetric representations of the groups $SU(N)$ for all $N$. This result gives further evidence for our conjecture that highest weight states minimize group integrals of certain concave functions for a large class of Lie groups and their representations.

1 Introduction

With the aid of coherent states, A. Wehrl [22] introduced the idea of a ‘classical entropy’ associated to a quantum density matrix. He showed that it has the desirable feature of being positive and conjectured that the minimum entropy over all density matrices would be achieved by a one-dimensional projector onto a coherent state. He stated the conjecture only for Glauber coherent states on $L^2(\mathbb{R}^n)$, and this was proved shortly thereafter in [13], in which the conjecture was extended to $SU(2)$. This $SU(2)$ conjecture was finally settled by us 35 years later [15], although there were several special cases proved earlier [3, 8, 16, 17, 18, 19].

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We believe that the analog of Wehrl’s conjecture should hold true, at least for a wide class of Lie groups (see also [1, 21]). If so, this would presumably have some general significance for representation theory since there are not very many theorems about integrals over the group of finite dimensional representations. The progress reported here is a proof of the conjecture for all symmetric representations of $SU(N)$, i.e., the representations corresponding to one-row Young diagrams. In the case of $SU(2)$ there are no other representations. For $SU(N)$ our proof trivially generalizes to the conjugate of the symmetric representations which are unitarily equivalent to the representations with young diagrams having $N - 1$ rows of equal length. We note that the proof we give now is much improved over that in [15], which did not generalize to $SU(N)$ for $N \geq 3$. The improvement utilizes a combinatorial identity (see (8) below) that replaces some $SU(2)$-specific relations used before. Non-symmetric representations, corresponding to other multi-rowed Young diagrams, are not addressed.

As in [15] we extend the conjecture to all concave functions, not only to the entropy function $-x \ln x$. Again, we prove the conjecture for $SU(N)$ Wehrl entropy as the infinite dimensional limit of a sequence of finite dimensional theorems.

We begin with a very brief reminder of the conjecture. Let $\mathcal{H}$ be a Hilbert space for an irreducible representation of a suitable (e.g., compact, simple and connected) Lie group $G$, let $\Omega_I \in \mathcal{H}$ be a normalized highest weight vector, called a coherent state vector, let $\Omega_R = R\Omega_I$ for $R \in G$, and let $v \in \mathcal{H}$ be any other normalized vector. Naturally, $\Omega_R$ is also a highest weight vector. Next form the Husimi function [10] on $G$, which is defined by the inner product $|\langle v | \Omega_R \rangle|^2$. With $f(x) = -x \ln x$, we define Wehrl’s classical entropy $S(v)$ by the (normalized) Haar measure integral

$$S(v) = \int_G f(|\langle v | \Omega_R \rangle|^2) \, dR. \quad (1)$$

The integral above may be considered on the space of equivalence classes corresponding to $R \sim R'$ if $\Omega_R$ and $\Omega_{R'}$ are equal up to a phase. This space of equivalence classes (the co-adjoint orbit of highest weight vectors) has a natural symplectic structure for which the Liouville measure agrees with the measure inherited from the Haar measure (see, e.g., [20]). In this sense the Husimi function becomes a probability distribution on a classical phase space, and the number $S(v)$ is a classical entropy on that space, corresponding to the state $v$. The symplectic structure, however, plays no role for our purpose, and we will consider $S(v)$ as an integral over the group, as defined in (1).

The conjecture is that $S(v)$ is minimized when the normalized $v$ is any of the vectors $\Omega_R$. Note that $S(v) > 0$ since the Husimi function is less than one almost everywhere,
unlike the Boltzmann entropy which can be negative. The extended conjecture is that this minimization property also holds if \( f \) in (1) is any concave function.

The group considered in this paper is \( SU(N) \) and the irreducible representations are the totally symmetric ones (and their conjugates), defined in the next section.

The Husimi function that associates a classical distribution function to a quantum state can be generalized to certain maps associating quantum states on one representation space to states on another representation space. The maps in question are completely positive trace preserving and are called 'quantum channels'. As we shall discuss below the particular channels we study are sometimes referred to as the 'universal quantum cloning channels'. A further generalization of the entropy conjecture is that it has a natural extension to these channels, i.e., that highest weight vectors minimize the trace of concave functions of the channel output (see Theorem 4.1). In particular, we thus determine the minimal output entropy of the universal cloning channels. In the proof we use that the minimal output of cloning channels agrees with the minimal output of what is called the ‘measure-and-prepare channels’ (see \cite{6}). As a corollary we therefore also determine their minimal output entropy (See Theorem 4.6).

The original Wehrl-type conjecture for the Husimi function will be derived as the limit of the finite dimensional results. In a similar way, we showed in \cite{15} how to prove the original Glauber and \( SU(2) \) (Bloch) coherent state conjectures from the infinite dimensional limit of finite dimensional representations of \( SU(2) \). Although the story begins with the proof in \cite{13} of the Wehrl entropy conjecture for Glauber states, it is only in \cite{15} that the generalization to all concave functions was achieved for Glauber states.

## 2 Symmetric Irreducible Representations of \( SU(N) \)

The symmetric irreducible representations (irrep) of \( SU(N) \) are obtained by taking \( M \) symmetric copies of the fundamental representation. We consider totally symmetric tensor products of \( N \)-dimensional complex space, i.e., for \( M \in \mathbb{N} \), \( \mathcal{H}_M = \bigotimes^M \mathbb{C}^N \). Here, \( P_M : \bigotimes^M \mathbb{C}^N \to \bigotimes^M \mathbb{C}^N \) is the projection onto the symmetric subspace, i.e., for \( u_i \in \mathbb{C}^N \),

\[
P_M u_1 \otimes \ldots \otimes u_M = \frac{1}{M!} \sum_{\sigma \in S_M} u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(M)}.
\]

The group \( SU(N) \) acts on \( \bigotimes^M \mathbb{C}^N \) equally on each factor, i.e., \( R \in SU(N) \) acts as \( R \otimes \ldots \otimes R \). This action commutes with \( P_M \) and hence acts on \( \mathcal{H}_M \). It is well-known that this is an irreducible representation, as explained in the appendix.
The highest weight vectors are the ones in which all the $u_i$ are the same normalized vector $u$, i.e., a highest weight vector is of the form $\otimes^M u$, and projectors onto such vectors are called coherent states.

Recall that a density matrix on a Hilbert space is a positive semi-definite operator of unit trace. Our notation here is that $\langle u | v \rangle$ is the inner product of vectors $u$ and $v$, while $\langle u | \rho | v \rangle$ is the inner product of $u$ with $\rho v$. The projector onto a normalized vector $|u\rangle$ is denoted $|u\rangle\langle u|$.  

**2.1 THEOREM (Generalized Wehrl Inequality).** Let $f : [0,1] \to \mathbb{R}$ be a concave function. Then for any density matrix $\rho$ on $\mathcal{H}_M$ we have

$$\int_{SU(N)} f(|\otimes^M (Ru)| \otimes^M (Ru)) dR \geq \int_{SU(N)} f(|\otimes^M (Ru)| \otimes^M u|^2) dR. \tag{3}$$

Here $u$ is any (by $SU(N)$ invariance) normalized vector in $\mathbb{C}^N$. In other words the integral on the left is minimized for $\rho = |\otimes^M u\rangle\langle \otimes^M u|$, i.e., $\rho$ is a coherent state.

This theorem is proved in Section 5.

The classical phase space discussed in the introduction is, in this case of the symmetric irreps of $SU(N)$, the space of pure quantum states on the one-body space $\mathbb{C}^N$. This is the space of unit vectors in $\mathbb{C}^N$ modulo a phase, i.e., the complex projective space $\mathbb{C}P^{N-1} = \{u \in \mathbb{C}^N \mid |u| = 1\} / \sim$, where two vectors are equivalent under $\sim$ if they agree up to multiplication by a complex phase. The complex projective space $\mathbb{C}P^{N-1}$ is a classical phase space, i.e., a symplectic manifold. For any function $h$ defined on $\mathbb{C}P^{N-1}$ we have the equivalence of the normalized integrations

$$\int_{SU(N)} h(Ru) dR = \int_{\{u \in \mathbb{C}^N : |u| = 1\}} h(u) du = \int_{\mathbb{C}P^{N-1}} h(u) du, \tag{4}$$

where we have abused notation and identified unit vectors $u$ with their equivalence class in $\mathbb{C}P^{N-1}$. The first integration above is over the $N(N+1)/2$ dimensional real manifold $SU(N)$, the middle integration is over the $2N-1$ dimensional real sphere, and the last integral is over the $2N-2$ dimensional real manifold $\mathbb{C}P^{N-1}$. In the special case $N = 2$ we have that $\mathbb{C}P^1$ is the 2-sphere $S^2$ (the Bloch sphere). For $N \geq 3$, the compact manifold $\mathbb{C}P^{N-1}$ is not a sphere.

\footnote{As already stated, we will not be concerned with the symplectic 2-form on $\mathbb{C}P^{N-1}$, known as the Fubini-Study form. We will only need the corresponding volume form, which corresponds to the normalized Haar measure on $SU(N)$.}
3 The Quantum Channels

In order to prove the main theorem we introduce maps from operators on the space \( \mathcal{H}_M \) to operators on \( \mathcal{H}_{M+k} \) for some \( k \geq 0 \).

If \( \Gamma \) is an operator on the symmetric tensor product \( \mathcal{H}_M \) we can extend \( \Gamma \) uniquely to an operator which we also call \( \Gamma \) on the larger space \( \bigotimes^M \mathbb{C}^N \) such that

\[
\Gamma = P_M \Gamma = \Gamma P_M = P_M \Gamma P_M.
\]

That is, \( \Gamma = 0 \) on vectors of non-symmetric symmetry type.

We can write \( \Gamma \) using second quantization as

\[
\Gamma = \frac{1}{M!} \sum_{i_1=1}^N \cdots \sum_{i_M=1}^N \sum_{j_1=1}^N \cdots \sum_{j_M=1}^N \Gamma_{i_1, \ldots, i_M; j_1, \ldots, j_M} a_{i_1}^* \cdots a_{i_M}^* a_{j_M} \cdots a_{j_1},
\]

where we have introduced the matrix elements for the extended \( \Gamma \) as

\[
\Gamma_{i_1, \ldots, i_M; j_1, \ldots, j_M} = \langle u_{i_1} \otimes \cdots \otimes u_{i_M} | \Gamma | u_{j_M} \otimes \cdots \otimes u_{j_1} \rangle
\]

using an orthonormal basis \( \{ u_i \}_{i=1}^N \) for \( \mathbb{C}^N \) and the corresponding Fock space creation and annihilation operators \( a_i^* = a^*(u_i) \), \( a_i = a(u_i) \) satisfying the well known canonical commutation relations \([a_i, a_j] = 0 = [a_i^*, a_j^*] \) and \([a_i, a_j^*] = \delta_{i,j} \). A more extensive discussion of creation and annihilation operators can be found in Section 4 of our earlier \( SU(2) \) paper [15].

We study the operator

\[
\mathcal{T}^k(\Gamma) = \sum_{i_1, \ldots, i_k} a_{i_1}^* \cdots a_{i_k}^* \Gamma a_{i_k} \cdots a_{i_1}
\]

on \( \mathcal{H}_{M+k} \). The map \( \mathcal{T}^k \), which maps operators on \( \mathcal{H}_M \) to operators on \( \mathcal{H}_{M+k} \), is completely positive. Note that we have not normalized it to be trace preserving, in fact,

\[
\text{Tr}_{M+k} \mathcal{T}^k(\Gamma) = \frac{(M+k+N-1)!}{(M+N-1)!} \text{Tr}_M \Gamma,
\]

where \( \text{Tr}_M \) refers to the trace in \( \mathcal{H}_M \). Thus

\[
\hat{\mathcal{T}}^k = \frac{(M+N-1)!}{(M+k+N-1)!} \mathcal{T}^k
\]

is completely positive and trace preserving. We may also write the map \( \mathcal{T}^k \) as

\[
\mathcal{T}^k(\Gamma) = \frac{(M+k)!}{M!} P_{M+k} \left( \bigotimes^k I_{\mathbb{C}^N} \otimes \Gamma \right) P_{M+k}.
\]

In this form we recognize the map \( \hat{\mathcal{T}}^k \) as the universal \( M \)-to-\( M+k \) cloning channel [4, 6, 7, 23].

The No-Cloning Theorem states that exact cloning of a quantum state is impossible. The universal cloning channels achieve the best degree of cloning for general input states. We thank Kamil Brádler for pointing out the relation to cloning channels.
4 The Main Theorem for Quantum Channels

Our main result on the cloning channels $\hat{T}_k$ is that coherent states minimize the trace of concave functions of the channel output. If the concave function is $f(x) = -x \ln(x)$ this theorem says that coherent states, which are pure states, give the minimal output von Neumann entropy. Since we can prove the optimality of coherent states for all concave functions we refer to this as the generalized minimal output entropy.

4.1 THEOREM (Main Theorem: Generalized minimal output entropy of $T_k$). For any concave function $f : [0,1] \rightarrow \mathbb{R}$ and any density matrix $\rho$ on $\mathcal{H}_M$ we have

$$\text{Tr}_{M+k} f(\hat{T}^k(\rho)) \geq \text{Tr}_{M+k} f(\hat{T}^k(| \otimes^M u\rangle\langle \otimes^M u|)),$$

For $M \geq 2$ and $k > 0$ and $f$ strictly concave equality holds if and only if $\rho = | \otimes^M u\rangle\langle \otimes^M u|$ for some unit vector $u \in \mathbb{C}^N$.

We will prove this by establishing that the sequence of ordered eigenvalues of $\hat{T}^k(| \otimes^M u\rangle\langle \otimes^M u|)$ majorizes the ordered sequence of eigenvalues of $\hat{T}^k(\rho)$ for any density matrix $\rho$.

The fact that majorizing eigenvalues is equivalent to minimizing traces of concave functions is not difficult to prove and is known as Karamata’s Theorem (see [14]). We used it also in our earlier paper [15] for $SU(2)$.

Recall that one ordered sequence of numbers $x_1 \geq x_2 \geq \cdots \geq x_{\mu}$ majorizes another, $y_1 \geq y_2 \geq \cdots \geq y_{\mu}$, if, for each $1 \leq k \leq \mu$, $\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j$, and with equality for $k = \mu$. If $X$ and $Y$ are Hermitian matrices we write $X \succ Y$ if the ordered eigenvalue sequence of $X$ majorizes the ordered eigenvalue sequence of $Y$. Our main theorem above is thus a consequence of the following majorization theorem.

4.2 THEOREM (Coherent States Majorize). Let $\Gamma$ be a positive semi-definite operator on $\mathcal{H}_M$. The ordered sequence of eigenvalues of $\hat{T}^k(\Gamma)$ is majorized by the ordered sequence of eigenvalues of $\text{Tr}_M(\Gamma)\hat{T}^k(| \otimes^M u\rangle\langle \otimes^M u|)$ for any unit vector $u \in \mathbb{C}^N$. Moreover, for $M \geq 2$ and $k > 0$ strict majorization holds unless $\Gamma = \text{Tr}_M(\Gamma)| \otimes^M u\rangle\langle \otimes^M u|$. The norm $\text{Tr}_M(\Gamma)$ occurs since we have not assumed $\Gamma$ to have unit trace.

Proof. We will use induction on $k$. The case $k = 0$ is trivial as $\hat{T}^0(\Gamma) = \Gamma$ and any positive semidefinite operator is majorized by a rank 1 operator with the same trace.

More generally, we may assume that $\Gamma$ is a rank one projection. The assumption that $\text{Tr}_M(\Gamma) = 1$ is, trivially, no loss of generality. That the rank may be assumed to be one...
follows from the fact that if $A$ and $B$ are both majorized by $C$ then for $0 \leq \lambda \leq 1$ we have that $\lambda A + (1 - \lambda)B$ is majorized by $C$. Alternatively, we have that the partial sum of eigenvalues is a convex function. Now simply write the spectral decomposition of $\Gamma$, i.e.,

$$\Gamma = \sum_{p=1}^{r} \lambda_p |\psi_p\rangle\langle\psi_p|,$$

where $0 \leq \lambda_p \leq 1$ with $\sum_{p=1}^{r} \lambda_p = 1$. If we can show that each $T^k(|\psi_p\rangle\langle\psi_p|)$ is majorized as claimed then the result follows for $\Gamma$.

We will also use the following well known, simple observation.

4.3 LEMMA. If $v_1, \ldots, v_m$ are vectors in a Hilbert space $\mathcal{H}$ then the operator $\sum_{i=1}^{m} |v_i\rangle\langle v_i|$ has the same non-zero eigenvalues as the $m \times m$ Gram matrix with entries $\langle v_i|v_j \rangle$.

Proof. Let $A : \mathbb{C}^m \to \mathcal{H}$ be the linear map

$$(z_1, \ldots, z_m) \mapsto z_1|v_1\rangle + \ldots + z_m|v_m\rangle.$$ 

Its adjoint is the map $A^* : \mathcal{H} \to \mathbb{C}^m$ given by

$$A^*|v\rangle = (\langle v_1|v\rangle, \ldots, \langle v_m|v\rangle).$$

Then $AA^* = \sum_{i=1}^{m} |v_i\rangle\langle v_i|$ is an operator from $\mathcal{H}$ to itself and $A^*A$ is the linear map on $\mathbb{C}^m$ corresponding to the Gram matrix. The non-zero eigenvalues of $AA^*$ are always the same as those of $A^*A$, for any $A$. \hfill \square

We assume now that the main theorem has been proved for all values $0, 1, 2, \ldots, k - 1$. As explained above we may assume that $\Gamma = |\psi\rangle\langle \psi|$, where $\psi$ is a unit vector in $\mathcal{H}_M$. According to the lemma $T^k(|\psi\rangle\langle \psi|)$ has the same non-zero eigenvalues as the matrix

$$W^\psi_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \langle \psi|a_{i_1} \cdots a_{i_k} a^*_{j_k} \cdots a^*_{j_1}|\psi\rangle.$$

This matrix represents the operator on the space $\mathcal{H}_k$, given in second quantization by

$$W_k(|\psi\rangle\langle \psi|) = \frac{1}{k!} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sum_{j_1=1}^{N} \cdots \sum_{j_k=1}^{N} \langle \psi|a_{i_1} \cdots a_{i_k} a^*_{j_k} \cdots a^*_{j_1}|\psi\rangle a_{i_1} \cdots a_{i_k} a^*_{j_k} \cdots a^*_{j_1}.$$ (6)

The map $W_k$ from operators on $\mathcal{H}_M$ to operators on $\mathcal{H}_k$ is again a completely positive map, i.e., if we normalize it to be trace preserving the resulting map $\hat{W}_k$ would be a quantum channel. We thus want to prove that

$$W_k(|\otimes^M u\rangle\langle \otimes^M u|) \succ W_k(|\psi\rangle\langle \psi|).$$ (7)
By normal ordering the creation and annihilation operators (which means utilizing the 
commutation relations to switch the creation operators to the left of the annihilation op-
erators), inside the expectation value, we can express \( \mathcal{W}_k(\langle \psi \rangle \langle \psi \rangle) \) in terms of the reduced 
density matrices. These are the operators \( \gamma^\ell_\psi \), \( \ell = 0, \ldots \) defined on \( \mathcal{H}_\ell \) by

\[
\gamma^\ell_\psi = \frac{1}{\ell!} \sum_{i_1=1}^N \cdots \sum_{i_\ell=1}^N \sum_{j_\ell=1}^N \sum_{j_1=1}^N \langle \psi | a_{i_1}^* \ldots a_{i_\ell}^* \cdots a_{j_\ell}^* \cdots a_{j_1}^* | \psi \rangle a_{j_\ell}^* \cdots a_{j_1}^* a_{i_\ell} \cdots a_{i_1},
\]

with the normalization convention \( \text{Tr} \gamma^{(k-\ell)}_\psi \psi = \frac{M!}{(M-k+\ell)!} \) (they vanish if \( k - \ell > M \)). In fact,
as we commute creation operators \( a_{i_\ell}^* \) to the left of annihilation operators \( a_{j_\ell} \), we will create 
delta functions \( \delta_{ij} \) with positive coefficients, and it is thus evident that there will be positive 
constants \( C_\ell, \ell = 0, \ldots, k \) (the exact values are not important to us) such that

\[
\mathcal{W}_k(\langle \psi \rangle \langle \psi \rangle) = \sum_{\ell=0}^k C_\ell \sum_{i_1=1}^N \cdots \sum_{i_\ell=1}^N a_{i_1}^* \cdots a_{i_\ell}^* \gamma^{(k-\ell)}_\psi \gamma^\ell_\psi a_{i_\ell} \cdots a_{i_1} = \sum_{\ell=0}^k C_\ell T^\ell(\gamma^{(k-\ell)}_\psi). \tag{8}
\]

The explicit constants in this formula were derived in [6] where \( \hat{\mathcal{W}}_k \) was called the ‘universal 
measure-and-prepare’ channel. See also [11] and [9] (Theorem 7) for alternative calculations 
of the constants, which we recall are not important for our application of the formula.

From the induction hypothesis we see that for all \( \ell \leq k - 1 \)

\[
T^\ell(\gamma^{(k-\ell)}_\otimes M_u) \succ T^\ell(\gamma^{(k-\ell)}_\psi).
\]

For \( \ell = k \), however, this is trivial since \( \gamma^0_\otimes M_u = \gamma^0_\psi = 1 \) and thus

\[
T^k(\gamma^0_\otimes M_u) = T^k(\gamma^0_\psi) = \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1}^* \cdots a_{i_k}^* a_{i_k} \cdots a_{i_1} = k! I_{H_k}.
\]

We now use the following simple observation.

**4.4 LEMMA.** Consider Hermitean operators \( A_1, \ldots, A_k \) that can be diagonalized in the 
same basis and such that the eigenvalues are simultaneously ordered decreasingly. In other 
words the operators are all non-decreasing functions of the same operator, i.e., \( A_i = f_i(A) \), 
\( i = 1, 2, \ldots, k \) where \( A \) is Hermitean and \( f_1, f_2, \ldots, f_k : \mathbb{R} \to \mathbb{R} \) are non-decreasing. If the 
Hermitean operators \( B_1, \ldots, B_k \) satisfy \( A_i \succ B_1, \ldots, A_k \succ B_k \) then

\[
A_1 + \ldots + A_k \succ B_1 + \ldots + B_k.
\]
Proof. To see this assume that \( u_1, \ldots, u_q \) are the first \( q \) eigenvectors in the basis diagonalizing \( A \) and hence \( A_1, \ldots, A_k \). If \( v_1, \ldots, v_q \) are orthonormal eigenvectors for \( B_1 + \ldots + B_k \) corresponding to the top \( q \) eigenvalues \( \mu_1, \ldots, \mu_q \) then (by the min-max principle)

\[
\mu_1 + \ldots + \mu_q = \sum_{j=1}^{q} \langle v_j | B_1 + \ldots + B_k | v_j \rangle = \sum_{j=1}^{q} \langle v_j | B_1 | v_j \rangle + \ldots + \sum_{j=1}^{q} \langle v_j | B_k | v_j \rangle \\
\leq \sum_{j=1}^{q} \langle u_j | A_1 | u_j \rangle + \ldots + \sum_{j=1}^{q} \langle u_j | A_k | u_j \rangle \\
= \sum_{j=1}^{q} \langle u_j | A_1 + \ldots + A_k | u_j \rangle = \nu_1 + \ldots + \nu_q,
\]

where \( \nu_1, \ldots, \nu_q \) are the top \( q \) eigenvalues of \( A_1 + \ldots + A_k \). This proves the lemma. 

To finish the proof of the majorization in Theorem 4.2 we now show that the operators \( A_\ell = \mathcal{T}^{\ell}(\gamma_{\otimes M u}^{(k-\ell)}) \) indeed satisfy the simultaneously diagonalization property, i.e., are monotone functions of the same operator. On \( \mathcal{H}_{k-\ell} \) we have, in terms of second quantization,

\[
\gamma_{\otimes M u}^{(k-\ell)} = \gamma_{\otimes M u}^{(k-\ell)} = \frac{M! \langle \otimes^{k-\ell} u | \otimes^{k-\ell} u \rangle}{(M-k+\ell)!} = \frac{M!}{(M-k+\ell)!(k-\ell)!} a^*(u)^{(k-\ell)} a(u)^{(k-\ell)}.
\]

It follows that if \( v_2, \ldots, v_N \) are chosen so that they form an orthonormal basis of \( \mathbb{C}^N \) together with \( u \) then on \( \mathcal{H}_k \) we have

\[
\mathcal{T}^{\ell}(a^*(u)^{(k-\ell)} a(u)^{(k-\ell)}) = \\
\sum_{j=0}^{\ell} \binom{\ell}{j} a^*(u)^{(k-\ell+j)} a(u)^{(k-\ell+j)} \prod_{i_1=2}^{N} \cdots \prod_{i_{\ell-j}=2}^{N} a^*(v_{i_1}) \cdots a^*(v_{i_{\ell-j}}) a(v_{i_{\ell-j}}) \cdots a(v_{i_1}) \\
= \sum_{j=0}^{\ell} \frac{\ell!(k-\ell+j)!}{j!} 1_{a^*(u) a(u) = k-\ell+j}.
\]

This simply says that

\[
\mathcal{T}^{\ell}(a^*(u)^{(k-\ell)} a(u)^{(k-\ell)}) = f_\ell(a^*(u) a(u)),
\]

where

\[
f_\ell(m) = \begin{cases} 
\frac{\ell! m!}{(m-(k-\ell))!}, & \text{if } m \geq k - \ell; \\
0, & \text{if } m < k - \ell,
\end{cases}
\]

i.e., they are increasing functions. Hence using Lemma 4.4 and 3 we find that

\[
\mathcal{W}_k(\otimes^M u) \langle \otimes^M u | = \sum_{\ell=0}^{k} C_\ell \mathcal{T}^{\ell}(\gamma_{\otimes M u}^{(k-\ell)}) \otimes \sum_{\ell=0}^{k} C_\ell \mathcal{T}^{\ell}(\gamma_{\psi}^{(k-\ell)}) = \mathcal{W}_k(\langle \psi | \langle \psi |).
\]

(9)
which shows (7).

To show that majorization is strict for \( k > 0 \) unless \( \psi \) is a coherent state vector \( \otimes^M u \) we shall consider \( 0 < k \leq q(M - 1) \) and do induction on \( q = 1, 2, \ldots \). For \( q = 1 \) we are assuming that \( 0 < k \leq M - 1 \). If the eigenvalues of \( \mathcal{W}_k(|\psi\rangle\langle\psi|) \) are equal to the eigenvalues of \( \mathcal{W}_k(|\psi\rangle\langle\psi|) \) we conclude from (9), in particular, that the eigenvalues of \( \mathcal{T}^0(\gamma^k_\psi) = \gamma^k_\psi \) are the same as the eigenvalues of \( \mathcal{T}^0(\gamma^k_\otimes M u) = \gamma^k_\otimes M u \) which is rank 1. Hence \( \gamma^k_\psi \) is rank one. The result now follows from the following lemma.

4.5 LEMMA. If, for \( 0 < k < M \), the \( k \)-particle reduced density matrix \( \gamma^{(k)}_\psi \) for a \( \psi \in \mathcal{H}_M \) is rank one then \( \psi \) must be a coherent state vector \( \otimes^M u \) for some unit vector \( u \).

Proof. Since \( \psi \in \otimes^M \mathbb{C}^N \) we can think of \( \gamma^k_\psi \) (up to an overall multiplicative factor) as the partial trace of \( |\psi\rangle\langle\psi| \) over the first \( M - k \) factors. If this is rank one so is \( \gamma^{M-k}_\psi \) and moreover \( |\psi\rangle\langle\psi| \) must be a tensor product of these two rank one operators, i.e., there is \( \psi_{M-k} \in \mathcal{H}_{M-k} \) and \( \psi_k \in \mathcal{H}_k \) such that \( \psi_M = \psi_{M-k} \otimes \psi_k \). By taking repeated partial traces we easily see that \( |\psi\rangle\langle\psi| = M^{-M} \otimes^M \gamma^1_\psi \), which implies that \( \gamma^1_\psi \) must be rank one as claimed.

To do the induction step we assume that we have proved the claim for all \( k \leq (q-1)(M - 1) \) we consider \( k \leq q(M - 1) \). In (9) all terms in the sum on the left must have the same eigenvalues as the terms in the sum on the right. If we consider the term for \( l = (q-1)(M - 1) \) we see that \( \mathcal{T}^l(\gamma^{k-l}_\psi) \) has the same eigenvalues as \( \mathcal{T}^l(\gamma^{k-l}_\otimes M u) \). Since \( \gamma^{k-l}_\otimes M u \) is proportional to \( |\otimes^{k-l} u \rangle\langle \otimes^{k-l} u| \) it follows from the induction assumption that \( \gamma^{(k-l)}_\psi \) has to be proportional to a coherent state, in particular, rank one. Note that \( k-l \leq q(M-1)-(q-1)(M-1) = M-1 \) and the result again follows from the lemma above.

One of the key observations in our proof was that the two maps \( \mathcal{W}_k \) and \( \mathcal{T}_k \) acting on the same pure states will output operators with the same eigenvalues. In particular this implies that we have also determined the minimal output entropy of the channel \( \mathcal{W}_k \) defined as the trace preserving normalization of the map \( \mathcal{W}_k \) given in (6).

4.6 COROLLARY (Minimal Output Entropy of \( \mathcal{W}_k \)). For any concave function \( f : [0, 1] \rightarrow \mathbb{R} \) and any density matrix \( \rho \) on \( \mathcal{H}_M \) we have

\[
\text{Tr}_{M+k} f(\mathcal{W}_k(\rho)) \geq \text{Tr}_k f(\mathcal{W}_k(|\otimes^M u\rangle\langle \otimes^M u|)).
\]

For \( M \geq 2 \) and \( k > 0 \) equality holds if and only if \( \rho = |\otimes^M u\rangle\langle \otimes^M u| \) for some unit vector \( u \in \mathbb{C}^N \).
5 The Semiclassical Limit

In this section we study the limit of the maps $T^k$ as $k$ tends to infinity with the goal of proving the generalized Wehrl inequality Theorem 2.1. The limit $k \to \infty$ will turn out to be a semiclassical limit, where the limiting object is a map from operators $\Gamma$ on $H_M$ to functions on a classical phase space. As we have discussed the classical phase space is the space of pure quantum states on the one-body space $\mathbb{C}^N$, i.e., the complex projective space $\mathbb{CP}^{N-1}$ of unit vectors in $\mathbb{C}^N$. This space will not play a role in our analysis as we will use (4) and simply work on $\{u \in \mathbb{C} : |u| = 1\}$, i.e., a $(2N - 1)$-dimensional real sphere.

Since $SU(N)$ acts irreducibly on the symmetric space $H_M$ (see appendix A) we have the usual coherent states decomposition on $H_M$.

$$\dim(H_M) \int_{u \in \mathbb{CP}^{N-1}} | \otimes^M u \rangle \langle \otimes^M u | d\mathbb{CP}^{N-1} u = I_{H_M}. \quad (10)$$

That the operator on the right is proportional to the identity follows from Schur’s Lemma and the identity then follows from the fact that both sides have the same trace. Recall, that the measure on the sphere is assumed to be normalized.

Using the Berezin-Lieb inequality [2, 12], which states that traces of concave functions are bounded above and below by analogous semiclassical expressions, we can compare the finite dimensional traces to integrals.

5.1 LEMMA. If $f$ is a concave function and $\Gamma$ is a positive semi-definite operator on $H_M$ we have

$$\frac{1}{\dim H_{M+k}} \text{Tr}_{H_{M+k}} \left[ f \left( \frac{M!}{(M+k)!} T^k(\Gamma) \right) \right] \leq \int_{\{u \in \mathbb{C}^N : |u| = 1\}} f \left( \langle \otimes^M u | \Gamma | \otimes^M u \rangle \right) du. \quad (11)$$

Proof. Using the decomposition (10) to rewrite traces, we find from Jensen’s inequality and the concavity of $f$ that

$$\frac{1}{\dim H_{M+k}} \text{Tr}_{H_{M+k}} \left[ f \left( \frac{M!}{(M+k)!} T^k(\Gamma) \right) \right] = \int_{\{u \in \mathbb{C}^N : |u| = 1\}} \langle \otimes^{M+k} u | f \left( \frac{M!}{(M+k)!} T^k(\Gamma) \right) | \otimes^{M+k} u \rangle du$$

$$\leq \int_{\{u \in \mathbb{C}^N : |u| = 1\}} f \left( \frac{M!}{(M+k)!} \langle \otimes^{M+k} u | T^k(\Gamma) | \otimes^{M+k} u \rangle \right) du$$

$$= \int_{\{u \in \mathbb{C}^N : |u| = 1\}} f \left( \langle \otimes^M u | \Gamma | \otimes^M u \rangle \right) du.$$
The inequality from the first to the third line is the upper bound in the Berezin-Lieb inequalities. In the last step we used that
\[
\langle \otimes^{M+k} u | T^k(\Gamma) | \otimes^{M+k} u \rangle = \frac{(M+k)!}{M!} \langle \otimes^M u | \Gamma | \otimes^M u \rangle.
\]

\(\square\)

According to Theorem 4.1 we have if \(\text{Tr}_M \Gamma = 1\) that
\[
\text{Tr}_{\mathcal{H}_{M+k}} \left[ f \left( \frac{M!}{(M+k)!} T^k(\Gamma) \right) \right] \geq \text{Tr}_{\mathcal{H}_{M+k}} \left[ f \left( \frac{M!}{(M+k)!} T^k(\otimes^M v) \langle \otimes^M v \rangle \right) \right]
\]
(12)

We will now study the limit \(k \to \infty\) of the right side above. We have already seen at the end of the last section that the eigenvalues of
\[
T^k(\otimes^M v) \langle \otimes^M v \rangle = M!^{-1} T^k(a^* v)^M a(v)^M
\]
are
\[
\frac{k!m!}{M!(m-M)!}.
\]
The multiplicity of this eigenvalue is the number of ways we can choose \(N-1\) non-negative integers summing up to \(M+k-m\), i.e.,
\[
\binom{M+k-m+N-2}{N-2}.
\]
Similarly the dimension of \(\mathcal{H}_M\) is the number of ways in which we can choose \(N\) non-negative integers to sum up to \(M\), i.e.,
\[
\dim \mathcal{H}_M = \binom{M+N-1}{N-1}.
\]

We find that
\[
\frac{1}{\dim \mathcal{H}_{M+k}} \text{Tr}_{\mathcal{H}_{M+k}} \left[ f \left( \frac{M!}{(M+k)!} T^k(\otimes^M v) \langle \otimes^M v \rangle \right) \right]
= \left( \frac{M+k+N-1}{N-1} \right)^{-1} \sum_{m=M}^{M+k} f \left( \frac{k!m!}{(M+k)! M!(m-M)!} \right) \binom{M+k-m+N-2}{N-2}
= \frac{(N-1)(M+k)!}{(M+k+N-1)!} \sum_{m=M}^{M+k} f \left( \frac{k!m!}{(M+k)! (m-M)!} \right) \frac{(M+k-m+N-2)!}{(M+k-m)!}
= (N-1) \sum_{m=M}^{M+k} f \left( \frac{k!m!}{(M+k)! (m-M)!} \right) \frac{(M+k)! (M+k-m+N-2)!}{(M+k-m)! (M+k+N-2)!}
\times \frac{1}{(M+k+N-1)}.
It is straightforward to check that as $k \to \infty$ this converges for continuous $f$ to the integral
\[(N-1) \int_0^1 f(s^M)(1-s)^{N-2}ds. \tag{13}\]
Theorem 2.1 follows from Lemma 5.1, (12), and the above calculation if we can show that
\[(N-1) \int_0^1 f(s^M)(1-s)^{N-2}ds = \int_{\{u \in \mathbb{C}^N : |u|=1\}} f\left(\left|\langle \otimes^M u | \otimes^M v \rangle\right|^2\right)du.\]
This is simple. We choose $v = (1, 0, \ldots, 0)$ then $\left|\langle \otimes^M u | \otimes^M v \rangle\right|^2 = |u_1|^{2M}$, where $u_1$ is the first coordinate of $u \in \mathbb{C}^N$. The $(2N-2)$-dimensional measure of the set $\{u \in \mathbb{C}^N : |u|=1, |u_1| = \cos(t)\}$ is proportional to $\cos(t) \sin^{2N-3}(t)$. Hence
\[
\int_{\{u \in \mathbb{C}^N : |u|=1\}} f\left(\left|\langle \otimes^M u | \otimes^M v \rangle\right|^2\right)du = \int_{\{u \in \mathbb{C}^N : |u|=1\}} f(|u_1|^{2M})du
= 2(N-1) \int_0^{\pi/2} f(\cos^{2M}(t)) \cos(t) \sin^{2N-3}(t)dt,
\]
which is equal to the integral (13).

Since our proof of the generalized Wehrl inequality in Theorem 2.1 is based on a limiting argument it does not establish that coherent states are the only minimizers. In the case of Glauber states this was proved by Carlen in [5]. In contrast, for finite $k$, the uniqueness is established in Theorem 4.2.

A Irreducibility of the Symmetric Representation

We explain here that the Hilbert space $\mathcal{H}_M$ of totally symmetric products (see [2]) gives an irreducible unitary representation of $SU(N)$. That it is a unitary representation is clear. If $V$ is an invariant subspace for the action of the group it is also invariant for the action of the representation of the Lie-algebra. The Lie-algebra of $SU(N)$ is the real vector space of traceless anti-Hermitian $N \times N$ matrices. If $X$ is any such matrix it is represented on $\mathcal{H}_M$ by
\[\pi(X) = X_1 + \ldots + X_M,\]
where $X_i$ is $X$ acting on the $i$-th tensor factor. Such matrices hence leave $V$ invariant. Taking complex linear combinations of matrices of the form $\pi(X)$ and the identity $I$ we find that $X_1 + \ldots + X_M$ leaves $V$ invariant for all $N \times N$ matrices $X$ (not only those that are traceless

\footnote{Since $f$ is assumed to be concave it is continuous except possibly at the endpoints. Discontinuity at the endpoints is not a problem.}
Hermitean). In particular, \( V \) is left invariant by \( E_{ij} = X_1 + \ldots + X_M \) with \( X = |e_i\rangle\langle e_j| \) for \( e_i, e_j \) elements in an orthonormal basis \( e_1, \ldots, e_N \) in \( \mathbb{C}^N \). The matrices \( E_{ii}, i = 1, \ldots, N \) can be simultaneously diagonalized on \( V \), which hence must contain at least one common eigenvector, i.e., a vector of the form

\[
P_M(\otimes^{n_1} e_1) \otimes (\otimes^{n_2} e_2) \cdots (\otimes^{n_{N-1}} e_{N-1}) \otimes (\otimes^{n_N} e_N),
\]

where \( n_1 + \cdots + n_N = M \). Multiple applications of the matrices \( E_{ij} \) on one vector of the form (14) can yield (multiples) of all vectors of the form (14). Consequently, they must all belong to \( V \), and thus \( V \) is all of \( \mathcal{H}_M \).

\[\square\]

References

[1] Bengtsson, Ingemar and Zyczkowski, Karol, Geometry of Quantum States an Introduction to Quantum Entanglement, Cambridge University Press, 2008.

[2] Berezin, F. A., Covariant and contravariant symbols of operators. (Russian) Izv. Akad. Nauk SSSR Ser. Mat., 36, (1972), 1134–1167

[3] Bodmann, Bernhard G., A lower bound for the Wehrl entropy of quantum spin with sharp high-spin asymptotics. *Comm. Math. Phys.*, 250 (2004), 287–300

[4] Brádler, Kamil, An infinite sequence of additive channels: the classical capacity of cloning channels, *IEEE Trans. on Inf. Theory* 57, (2011), 5497–5503

[5] Carlen, Eric A., Some integral identities and inequalities for entire functions and their application to the coherent state transform. *J. Funct. Anal.*, 97, (1991), 231–249

[6] Chiribella, G., On quantum estimation, quantum cloning and finite quantum de Finetti theorems, in Theory of Quantum Computation, Communication, and Cryptography, vol. 6519 of Lecture Notes in Computer Science, Springer, 2011

[7] Gisin, N. and Massar, S., Optimal Quantum Cloning Machines, *Phys. Rev. Lett.*, 79, (1997), 2153–2156

[8] Gnutzmann, S. and Zyczkowski, K., Rényi-Wehrl entropies as measures of localization in phase space. *J. Phys. A* 34, (2001), no. 47, 10123?–10139.

[9] Harrow, Aram, The church of the symmetric subspace, arXiv1308.6595, preprint (2013).
[10] Husimi, Kôdi, Some Formal Properties of the Density Matrix. *Proc. Phys. Math. Soc. Jpn.*, 22, (1940), 264–314

[11] Lewin, Mathieu and Nam, Phan Thanh and Rougerie, Nicolas, Remarks on the quantum de Finetti theorem for bosonic systems. *Appl. Math. Res. Express. AMRX*, no. 1, (2015) 48–763.

[12] Lieb, Elliott H., The classical limit of quantum spin systems. *Comm. Math. Phys.*, 31, (1973), 327–340

[13] Lieb, Elliott H., Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys.*, 62, (1978), no. 1, 35–41

[14] Lieb, Elliott H. and Seiringer Robert, The Stability of Matter in Quantum Mechanics, Cambridge Univ. Press, 2010

[15] Lieb, Elliott H. and Solovej, Jan Philip, Proof of an Entropy Conjecture for Bloch Coherent Spin States and its Generalizations, *Acta Math.* 212, 379-398 (2014). DOI: 10.1007/s11511-014-0113-6. [arXiv:1208.3632](https://arxiv.org/abs/1208.3632).

[16] Luo, S., A simple proof of Wehrl’s conjecture on entropy, *J. Phys. A*, 33, (2000), 3093–3096

[17] Schupp, Peter, On Lieb’s conjecture for the Wehrl entropy of Bloch coherent states. *Comm. Math. Phys.*, 207 (1999), 481–493

[18] Scutaru, Horia, On Lieb’s conjecture. *Romanian Jour. of Phys.*, 47, (2002), 189–198

[19] Sugita, Ayumu, Proof of the generalized Lieb-Wehrl conjecture for integer indices larger than one, *J. Phys. A: Math. Gen.* 35, (2002), 621–626

[20] Simon, Barry, The classical limit of quantum partition functions. *Comm. Math. Phys.*, 71, (1980), 247–276.

[21] Siomczynski, Wojciech and Zyczkowski, Karol, Mean Dynamical Entropy of Quantum Maps on the Sphere Diverges in the Semiclassical Limit, *Phys. Rev. Lett.*, 80, (1998), 1880–1883.

[22] Wehrl, Alfred, On the relation between classical and quantum-mechanical entropy. *Rep. Math. Phys.*, 16, (1979), no. 3, 353–358.
[23] Werner, Reinhard, Optimal cloning of pure states, *Phys. Rev. A.* **58**, (1998) 1827–1832.