In this work we extend the so-called Minimal Geometric Deformation method in 2+1 dimensional space–times with cosmological constant. We obtain that the conditions for gravitational decoupling of two circularly symmetric sources coincides with those found in the 3+1 dimensional case. As a particular example, we implement the method to generate an exterior charged BTZ solution starting from the BTZ vacuum as the isotropic sector.

I. INTRODUCTION

Needed to say, three dimensional gravity still represents an important source for theoretical developments: quantum models [1] which have allowed to get some insight about the nature of quantum gravity in (3+1) dimensions, and identification of cosmic strings solutions with topological defects in two dimensional condensed matter systems as graphene layers (see, for example, [2–4]), are some of the examples we can list. These are some of the reasons why it is important to find new solutions in three dimensional gravity. Recently, the so called Minimal Geometric Deformation (MGD) method [5–42] has been used to obtain new solutions in 2+1 dimensions with and without cosmological terms [33, 41, 42]. However, it is worth mentioning that the main limitation of the MGD approach is that the geometric deformation is performed only in the radial component of the metric. In order to overcome this limitation, an extension to the MGD method has been proposed very recently in 3+1 dimensions in Ref. [38]. In this work, the author introduced a modification in two components of the metric in a spherically symmetric space–time. The main result of that work is that the sources can be successfully decoupled as long as there is exchange of energy between them. As a particular case, the decoupling can be reached when either the isotropic sector corresponds to a barotropic fluid or to vacuum regions. Following Ref. [38], in this work we obtain the extended version of the MGD-decoupling in 2+1 dimensions with cosmological constant.

II. EXTENDED EINSTEIN EQUATIONS IN 2+1 SPACE–TIME DIMENSIONS

In Ref. [38] the MGD-method has been extended to deal with a more general geometrical deformation. In this section we obtain the corresponding extended case for 2+1 dimensions with cosmological constant. Let us consider the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}^{\text{tot}}, \]

and assume that the total energy-momentum tensor can be decomposed as

\[ T_{\mu\nu}^{\text{tot}} = T_{\mu\nu}^{(m)} + \theta_{\mu\nu}, \]

where \( T_{\mu\nu}^{(m)} = \text{diag}(-\rho, p, p) \) and \( \theta_{\mu\nu} = \text{diag}(-\rho^\theta, p^\theta, p^\theta) \). In what follows, we shall work with circularly symmetric space–times with a line element parametrized as

\[ ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\phi^2, \]

where \( \nu \) and \( \lambda \) are functions of the radial coordinate, \( r \), only. As it is well known, replacing Eq. (3) in (1) leads to a set of coupled non–linear second order differential equations. To be more precise, the decomposition \( T_{\mu\nu}^{(tot)} = T_{\mu\nu}^{(m)} + \theta_{\mu\nu} \) does not imply the decoupling of the Einstein field equations. However, a decoupling in the geometric sector can be successfully implemented, which allows to decouple the system. In the context of the MGD approach, the decomposition is performed in the radial component of the metric, namely,

\[ e^\lambda = e^{-\nu} + \alpha f \]
where \( \mu \), in combination with the metric function \( \nu \) of the original system, is regarded as a well known solution of the Einstein field equations. To be more precise, replacing Eq. 11 in [1] we obtain two sets of differential equations: i) a set of Einstein field equations for \( \{ \nu, \mu \} \) with matter sector given by \( T^{(m)}_{\mu\nu} = \text{diag}(-\rho, p, p) \) is the isotropic source and ii) a set of (quasi–)Einstein field equations in \( (3+1) \) \( 2+1 \) dimensional space–time sourced by \( \theta^{\nu}_{\mu} \). In this sense, given a solution of the system \( \{ \nu, \mu \} \), another solution can be found solving for the second set of equations involving the unknowns \( \{ f, \rho^0, p^0, p^\perp \} \). In spherical \( (3+1) \) dimensional space–times or circularly symmetric space–times \( (2+1) \) space–time dimensions, the method leads to three equations for four unknowns. In order to solve the equations, extra conditions have to be implemented. Some of the cases are listed below.

- **Interior solutions.** In this case, the matching conditions for the interior of a Schwarzschild for \( 3+1 \) dimensions (see, for example, Ref. [22]) or BTZ in \( 2+1 \) dimensions (see Ref. [12] for details) vacuum lead to the mimetic constraint of the radial pressure, namely \( p = \rho^0 \).
- **Hairy Black Hole.** In this case the imposition of equations of state for the components of \( \theta^{\nu}_{\mu} \) is mandatory. Examples of such equations are the barotropic, perfect polytropic fluids. The reader is referred to Refs. [36, 39, 41] and source [33] for the \( 2+1 \) case.
- **Inverse problem.** In this case, the constraint that allows the application of the method is simply \( \tilde{p}_\perp - \tilde{p}_r = p^0 - \rho^\perp \), where \( \tilde{p}_\perp, \tilde{p}_r \) corresponds to the components of \( T^{(\text{tot})}_{\mu\nu} \). In the inverse problem, a solution of Eq. 11 is assumed to be known and the task is to explore the decoupler sectors. This problem has been worked out in \( 3+1 \) and \( 2+1 \) dimensions in Refs. [30, 38, 41].

However, the MGD method is not the most general method to decouple Einstein’s equations. Very recently, in Ref. 38 has been successfully performed a geometric deformation in two components of the metric in \( 3+1 \) dimensions, more precisely

\[
\nu = \xi + \alpha g,
\]  
\[
e^{-\lambda} = e^{-\mu} + \alpha f. 
\]  

In this case the method allows to decouple the sources \( T^{(m)}_{\mu\nu} \) and \( \theta_{\mu\nu} \) of Eq. 11 with exchange of energy between them. As a particular case, the decoupling can be obtained without exchange of energy when: i) \( T^{(m)}_{\mu
u} \) is a barotropic fluid whose equation of state is \( \rho = -p \) and ii) for those regions where \( T^{(m)}_{\mu\nu} = 0 \), which corresponds to the case of a region \( r > R \) filled by a source \( \theta_{\mu\nu} \) surrounding a self–gravitating system of radius \( R \) and source \( T^{(m)}_{\mu\nu} \). In this work, our main goal is to propose a decomposition in the whole geometric sector which could allow us to decouple (the system in \( 2+1 \) dimensional space–times and explore the conditions under which the decoupling of the sources without exchange of energy–momentum can be reached.

Let us start by considering Eq. 8 as a solution of the Einstein Field Equations, namely

\[
\kappa^2 \tilde{\rho} = -\Lambda + \frac{e^{-\lambda} \lambda'}{2r} \]  
\[
\kappa^2 \tilde{p}_r = \Lambda + \frac{e^{-\lambda} \nu'}{2r} \]  
\[
\kappa^2 \tilde{p}_\perp = \Lambda + \frac{e^{-\lambda}}{4} \left( -\lambda' \nu' + 2 \nu'' + \nu'^2 \right), 
\]  

where the prime denotes derivation with respect to the radial coordinate and we have defined

\[
\tilde{\rho} = \rho + \rho^0, \]  
\[
\tilde{p}_r = p + p^0, \]  
\[
\tilde{p}_\perp = p + p^\perp, 
\]

The next step consists in decoupling the Einstein Field Equations 8, 9 and 10 by performing the decomposition

\[
\nu = \xi + \alpha g, 
\]
\[
e^{-\lambda} = \mu + \alpha f, 
\]

where \( g \) and \( f \) are the geometric deformation undergone by \( \xi \) and \( \mu \), “controlled” by the free parameter \( \alpha \). By doing so, we obtain two sets of differential equations: one describing a system sourced by the conserved energy–momentum tensor of a fluid \( T^{(m)}_{\mu\nu} \) and the other set corresponding to the equation of motion of the source \( \theta_{\mu\nu} \). After taking into account that the cosmological constant can be interpreted as some kind of isotropic fluid, we include the \( \Lambda \)–term in the isotropic sector and we obtain

\[
\kappa^2 \rho = -\Lambda + \frac{e^{-\nu} \mu'}{2r} \]  
\[
\kappa^2 p = \Lambda + \frac{e^{-\nu} \xi'}{2\kappa^2 r} \]  
\[
\kappa^2 p = \Lambda - \frac{e^{-\nu} (\mu' \xi' - 2 \nu'' - \xi'^2)}{4}, 
\]

for \( T^{(m)}_{\mu\nu} \) and

\[
\kappa^2 \rho^0 = -\frac{\alpha f'}{2r} \]  
\[
\kappa^2 p^0_r - \alpha Z_1 = \frac{\alpha f'}{2r} \]  
\[
\kappa^2 p^0_\perp - \alpha Z_2 = \frac{\alpha}{4} f' \nu + \frac{\alpha}{4} \frac{f'}{r} (2 \nu'' + \nu'^2), 
\]

where

\[
Z_1 = \frac{e^{-\nu} g'}{2r}, 
\]
\[
Z_2 = \frac{1}{4} e^{-\mu} (2g'' + g' (\alpha g' - \mu' + 2 \xi')). 
\]
for the source $\theta_{\mu \nu}$ [38]. We would like to emphasize that the addition of the cosmological constant only affects the isotropic sector because Eqs. (18), (19) and (20) remain unchanged.

In what follows we shall explore the Bianchi identities in order to study the conditions for the gravitational decoupling. The conservation $\nabla_\mu T^\nu_\nu = 0$ leads to

$$
\left( p' + \frac{1}{2}(p + \rho)\xi' \right) + \frac{1}{2}\alpha g' (p + \rho) + \alpha \left( \frac{1}{2}(p_r^\theta + \rho^\theta)\nu' + \frac{-p^\theta_r + \rho^\theta_r + r \rho^\theta_r}{r} \right) = 0
$$

In the above equation, the first bracketed term corresponds to the conservation of $T^{(m)}_\nu$ computed with the metric $(\xi, \mu)$, namely $\nabla_\rho T^{(m)}_\nu$. Indeed, this expression corresponds to the linear combination of equations [15], [16] and [17] which implies that

$$
\nabla^{(\xi, \mu)} T^{(m)}_\nu = p' + \frac{1}{2}(p + \rho)\xi' = 0
$$

The third bracketed term corresponds to $\nabla_\mu \theta^\mu_\nu$ calculated with metric $(\nu, \lambda)$, namely

$$
\nabla_\rho \theta^\rho_\nu = \rho_r^\nu \nu + \frac{1}{2}(p^\theta_r + \rho^\theta)\nu' + \frac{\rho^\theta - \rho_\nu^\theta}{r},
$$

and it is a linear combination of Eqs. (18), (19) and (20). Now, with the previous notation, the conservation of the total energy momentum tensor can be written as

$$
\nabla_\rho T^{(\text{tot})}_\nu = \nabla_\rho T^{(m)}_\nu + \nabla_\rho \theta^\rho_\nu + \frac{1}{2}\alpha g' (p + \rho) \delta^1_\nu = 0
$$

Note that the total energy momentum tensor is conserved whenever

$$
\nabla_\rho \theta^\rho_\nu = - \frac{1}{2}\alpha g' (p + \rho) \delta^1_\nu
$$

which means decoupling with exchange of energy–momentum. However, a decoupling without energy–momentum exchange can be reached either imposing $g' = 0$ or $p + \rho = 0$. The former requirement corresponds to the standard MGD where only $g_r^{-1}$ undergoes a geometrical deformation. The latter entails a barotropic equation of state in the isotropic sector. What is more, if the isotropic sector is vacuum (the exterior of a star), the barotropic condition is trivially fulfilled and the decoupling without exchange of energy–momentum is straightforward. We conclude this section pointing out that the conditions for the decoupling of the sources $T^{(m)}_\nu$ and $\theta^\mu_\nu$ coincides with those found for the $3 + 1$ case reported in Ref. [38].

In the next section, we shall implement the extended MGD protocol in order to test it by generating an exterior charged BTZ solution starting form a BTZ vacuum.

### III. BTZ-MAXWELL SYSTEM

Let us consider the static BTZ metric as the solution for the isotropic sector given by Eqs. (15), (16) and (17),

$$
\xi = \log(-M + \frac{r^2}{L^2}),
$$

$$
\mu = -\log(-M + \frac{r^2}{L^2}),
$$

The decoupling matter content and the Maxwell energy momentum tensor are given, respectively, by

$$
\theta^\mu_\nu = \frac{1}{4\pi} \left( F_{\mu \sigma} F^{\nu \sigma} - \frac{1}{4} g^{\nu \sigma} F_{\tau \sigma} F^{\tau \sigma} \right),
$$

where $F_{\mu \nu}$ satisfies

$$
\nabla_\nu F^{\mu \nu} = 4\pi j^\mu,
$$

$$
\partial_\gamma F_{\mu \nu} = 0.
$$

In the previous equations, $F_{\mu \nu}$ and $j^\mu$ are the Maxwell tensor and the four-current, respectively. Now, in the static and circularly symmetric case, the only non-vanishing components of the Maxwell tensor are $F^{01} = F^{10}$ and the four current reads

$$
\mathbf{j}^\mu = (j^0, 0, 0).
$$

After integration of the Maxwell equation (see Eq. (31)) we obtain

$$
F^{01} = \left( \frac{-\nu + \lambda}{2} \right) q(r)
$$

where $q(r)$ is the electric charge of a spherical system of radius $r$, defined as

$$
q(r) = \int_0^r 4\pi e^{(\nu + \lambda)/2} j^0 R dR.
$$

With these results at hand, Eqs. (18), (19) and (20) become

$$
-E^2 = \frac{\alpha f'}{2r},
$$

$$
E^2 - \alpha Z_1 = \frac{\alpha f' \nu}{2r},
$$

$$
-E^2 + \alpha Z_2 = \frac{\alpha f' \nu + \alpha}{4} f (2\nu' + \nu^2).
$$

From the conservation equation, the electric field turns to be

$$
E = \frac{Q}{r},
$$

where $Q$ is the total charge of the black hole. After replacing Eq. (39) in (29), (37) and (38) we obtain

$$
f(r) = c_1 + \frac{2Q^2 \log(r)}{\alpha}
$$

$$
g(r) = \frac{1}{\alpha} \log \left( \frac{L^2 (\alpha c_1 - M)}{r^2 - L^2 M} + 2L^2 Q^2 \log(r) + r^2 \right) + c_2.$$

(40)
Now, using the decoupling constraints given by Eqs. (43), we obtain
\[
\nu = \alpha c_2 + \log \left( \frac{L^2 (\alpha c_1 - M) + 2L^2 Q^2 \log(r) + r^2}{L^2 - M} \right) + \log \left( \frac{L^2}{L^2 - M} \right) - \log \left( r^2 - L^2 M \right)
\]
(42)
\[
\lambda = -\log \left( \alpha c_1 + \frac{r^2}{L^2} - M + 2Q^2 \log(r) \right), \quad (43)
\]
which, after some manipulation can be written as
\[
e^\nu = -M + \frac{r^2}{L^2} + 2Q^2 \log \left( \frac{r}{L} \right)
\]
(44)
\[
e^{-\lambda} = -M + \frac{r^2}{L^2} + 2Q^2 \log \left( \frac{r}{L} \right),
\]
(45)
which corresponds to the well known charged BTZ black hole solution.

To conclude this section, a couple of comments are in order. First, it is worth mentioning that we have extended our previous work concerning the MGD approach in 2+1 [33] to the case of two deformations instead of one. In this sense, more complex situations can be tackled in three-dimensional space-times with the improvement here presented. Second, we note that we have been able to reconstruct the charged–BTZ solution starting from the corresponding vacuum, which is pure BTZ. In this sense, our findings are equivalent to that of the 3 + 1–dimensional case presented in Ref. [38]. Third, we would like emphasize that the study of the BTZ–Maxwell case is advantageous with respect to considering an arbitrary source because, although the source \( \theta_{\mu\nu} \) is anisotropic, all the ignorance can be encoded in the electric field which is constrained by the Maxwell equations. Therefore, the system can be integrated without imposing any extra condition. To be more precise, the implementation of the method assuming an arbitrary source \( \theta_{\mu\nu} \) to obtain an exterior solution requires the knowledge of suitable equations of state which could depend on the nature of the system under study.

IV. CONCLUSIONS

In this work we have successfully extended the Minimal Geometric Deformation method in circularly symmetric 2 + 1–dimensional space–times with cosmological constant. This extended method allows the introduction of deformations in two of the components of the metric tensor, which results in the decoupling of the sources of the Einstein field equations. We found that this decoupling could be obtained without exchange of energy between the sources as far as the perfect fluid satisfies a barotropic equation of state or in situations where the isotropic sector corresponds to a vacuum solution. As an example, we have implemented the extended protocol to generate an exterior solution starting from a static BTZ vacuum resulting in a charged–BTZ system, in agreement with its corresponding 3 + 1–dimensional counterpart recently obtained in Ref. [38] in the sense that the vacuum sector leads to a charged one if the anisotropic source is the Maxwell energy–momentum tensor. We conclude this work by remarking that the extended method can be applied to obtain new solutions after an appropriate choice for the source is done but, in this case, the implementation of a suitable equation of state is mandatory.

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