Abstract: We prove a version of the Selberg integral formula for local fields of characteristic zero.

Keywords: Selberg integral, Gamma factor, $p$-adic fields

MSC 2010: 33E15

Communicated by: Freydoon Shahidi

1 Introduction

Selberg introduced his beautiful integral formula in 1944 (see [9]) asserting

$$S_n(a, b, c) = \prod_{0 \leq i < j \leq n} |t_i - t_j|^{2c} dt_1 \cdots dt_n = \prod_{j=0}^{n-1} \frac{\Gamma(a + jc) \Gamma(b + jc) \Gamma(1 + (j + 1)c)}{\Gamma(a + b + (n + j - 1)c) \Gamma(1 + c)},$$

(1.1)

where $n$ is a positive integer and $a, b, c$ are complex numbers satisfying

$$\text{re } a > 0, \quad \text{re } b > 0, \quad \text{re } c > - \min\left\{ \frac{1}{n}, \text{re } \frac{a}{n-1}, \text{re } \frac{b}{n-1} \right\}.$$

We refer to Forrester and Warnaar’s exposition [7] for the history, generalizations and the applications of the Selberg integral. Evans [5] conjectured a finite field analog of the Selberg integral formula in 1980. Anderson [1] proved a major case of it in 1981 and his ideas were used to obtain the complete result [6]. On the other hand, Aomoto [4] proved an analog of the Selberg integral for the complex field $\mathbb{C}$ in 1987. In the present paper, we show that Anderson’s method [1, 2] can be extended to prove the Selberg integral formula for local fields of characteristic zero. To state our results, we first introduce some notations.

Let $F$ be a local field of characteristic zero, let $\psi$ be a non-trivial additive character, let $dx$ be the self-dual Haar measure with respect to $\psi$, i.e. the Fourier transform defined using $dx$. Then

$$\mathcal{F}f(x) = \int_F f(y) \psi(xy) \, dy$$

(1.2)

is an isometry. The absolute value $|a|_F$ of $a \in F$ is defined by the formula $\text{vol}(a U) = |a|_F \text{vol}(U)$. For a quasi-
character $c$ of $F^*$, its real part, denoted by $\text{re } c$, is the unique real number satisfying the condition

$$|c(x)| = |x|^c_F. \quad (1.3)$$

The $\rho$-factor $\rho(c)$ of $c$ is defined by

$$\rho(c) := \frac{\int f(x)c(x)|x|^{-1}_F \, dx}{\int |f(x)|^{-1}(x) \, dx},$$

where $f$ is a Schwartz function on $F$, both integrals converge when $0 < \text{re } c < 1$ and they are understood as analytic continuation for general $c$; see [12]. We also use the gamma function $\Gamma(c)$ as in [8]; it is related to $\rho(c)$ by $\Gamma(c) = (c(-1))\rho(c)$. We set $\chi_0(x) = |x|_F$.

The choice of $\psi$ induces an additive character $\psi_E$ on any finite extension $E$ by $\psi_E(x) = \psi(\text{Tr } x)$, and therefore the self-dual Haar measure $d_E$ on $E$. Then for any quasi-character $c$ of $E^*$ we can similarly define $\rho_E(c)$ and $\Gamma_E(c)$. The absolute value of $E$ is denoted by $|x|_E$, and we have $|x|_E = |N(x)|_F$. Note that for a quasi-character $c$ of $F^*$ its composition $N \circ c$ with the norm map has the property $\text{re}(N \circ c) = \text{re } c$.

Let $M_n$ be the space $\{f \in F[x] \mid f \text{ is monic of degree } n\}$ for $n \in \mathbb{Z}_{\geq 0}$. Equate $M_n$ with $F^n$ via the map

$$\eta : M_n \rightarrow F^n, \quad f(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \mapsto (b_{n-1}, \ldots, b_0).$$

Let $df$ be the measure inherited from the product Haar measure of $F^n$. Recall that the discriminant $\Delta(f)$ of $f \in M_n$ is defined by

$$\Delta(f) = (-1)^{\frac{1}{2}n(n-1)} \prod_{i \neq j}(a_i - a_j) = \prod_{1 \leq i < j \leq n} (a_i - a_j)^2,$$

where $a_i$ ($i = 1, \ldots, n$) are the roots of $f$. The empty product is considered to be equal to 1. For an irreducible polynomial $h(x)$ over $F$, we denote the field $F[x]/(h(x))$ by $F_h$, and if $\chi$ is a quasi-character of $F^*$, we denote by $\Gamma_h(\chi)$ the Gamma function $\Gamma_E(\chi \circ N).

For quasi-characters $\alpha$, $\beta$, $\gamma$ of $F^*$ in the region $R_n$ given by

$$\text{re } \alpha, \text{re } \beta, \text{re } \gamma > 0, \quad \text{re } \alpha + \text{re } \beta + 2(n-1) \text{re } \gamma < 1, \quad (1.4)$$

we define the Selberg integral over $F$ by

$$S_n(\alpha, \beta, \gamma) = \alpha(-1)^n\gamma(-1)^{n-1} \prod_{f \in M_n} \alpha \chi_0^{-1}(f(0))\beta \chi_0^{-1}(f(1))\gamma \chi_0^{-1}(\Delta(f)) \prod_{i=1}^{l} \frac{\Gamma_h(\gamma)}{\Gamma(\gamma)^{\deg h_i}} \, df, \quad (1.5)$$

where we write $f(x) = \prod_{i=1}^{l} h_i(x)$ with $h_i$ being a monic irreducible polynomial over $F$. Note that polynomials with zero discriminant have measure zero, so we may assume $f$ has no repeated roots.

We have the following theorem that generalizes the Selberg integral formula (1.1).

**Theorem 1.1.** The integral $S_n(\alpha, \beta, \gamma)$ converges for $(\alpha, \beta, \gamma) \in R_n$ and

$$S_n(\alpha, \beta, \gamma) = \prod_{j=0}^{n-1} \frac{\Gamma(ay)^{j}(\beta y^j)^{\Gamma(\gamma^{j+1})}}{\Gamma(a\beta y^{j+1})^{\Gamma(\gamma)}}. \quad (1.6)$$

In the case that $F$ is a finite field, we consider the Gauss sums for $F$ as an analog of the Gamma factors. Then the factor

$$\prod_{i=1}^{l} \frac{\Gamma_h(\gamma)}{\Gamma(\gamma)^{\deg h_i}}$$

is equal to 1 by the Hasse–Davenport relation, so this term does not appear in the finite field generalization of the Selberg integral in [1, 5] (see also [3]). For the case $F = \mathbb{C}$, since all $F_h = \mathbb{C}$, the above factor is also equal to 1, and (1.6) reduces to Aomoto’s generalization of the Selberg integral [4]; see the end of Section 3 for more details. While Aomoto [4] considers (1.5) for $F = \mathbb{C}$ as a pairing in certain twisted de Rham cohomology and
homology, our integral is just the ordinary integral. Our domain of convergence (1.4) for the case $F = \mathbb{C}$ is contained in Aomoto’s defining domain for (1.5). Note that only the unramified quasi-characters $\alpha, \beta, \gamma$ are considered in [4]. We also remark that for general $F$, $$f \mapsto \prod_{i=1}^{l} \frac{\Gamma_{h_i}(y)}{\Gamma(y)^{\deg h_i}}$$
is a locally constant function on the region $M_n - \{ f | \Delta(f) = 0 \}$, which is a disjoint unions of open sets of $f$’s with $F[x]/(f(x))$ isomorphic to a direct product of extensions of $F$ of fixed types.

We like to comment that (1.6) for the case $F = \mathbb{R}$ is not equivalent to the original Selberg integral (1.1), which should be understood as (1.6) for $(\mathbb{R}_{\geq 0}, +, \cdot)$, the set of non-negative real numbers with usual addition and multiplication.

We prove Theorem 1.1 by evaluating a double integral in two different ways, which gives a recursive formula relating $S_n$ to $S_{n-1}$. This method is due to Anderson [1, 2].

This paper is organized as follows. In Section 2, we prove an extension of the beta integral and two corollaries that are used in the proof of Theorem 1.1. In Section 3, we prove our main theorem and compare our formula in the case $F = \mathbb{C}$ with Aomoto’s.

2 Generalized beta integrals

Lemma 2.1. Let $V$ be a finite-dimensional vector space over $F$, let $dx$ be a Haar measure on $V$ as an additive group. If $N : V \to \mathbb{R}_{\geq 0}$ is a $F$-norm, then for $r_0 < -\dim_F V$,

$$\int_{|x| \geq r} N(x)^s dx$$

converges for any $r > 0$.

Proof. The cases $F = \mathbb{R}, \mathbb{C}$ are standard exercises in calculus. We assume $F$ to be non-Archimedean. Let $n = \dim_F V$. Since all the norms on $E$ are equivalent, we may assume that $E$ is a field extension of $F$ and $N(x) = |x|_E^{1/n}$. The result follows from the fact that

$$\int_{|x| > r} |x|_E^{s/n} dx$$

converges when $\frac{1}{n} \Re s < -1$. \hfill $\Box$

Recall that $\rho(c)$ in (1.3) for $\Re c > 0$ can be written as an integral

$$\rho(c) = c(-1) \int_{F} \psi(x)c(x)|x|_F^{-1} dx := c(-1) \lim_{r \to \infty} \int_{|x| \leq r} \psi(x)c(x)|x|_F^{-1} dx.$$  

Equivalently, the Gamma function of $c$ can be written as

$$\Gamma(c) = c(-1)\rho(c) = \int_{F} \psi(x)c(x)|x|_F^{-1} dx;$$  \hfill (2.1)

see [8]. This formula is used to prove the beta integral formula [8]:

$$\int_{F} c_1(x)|x|^{-1}c_2(1-x)|1-x|^{-1} dx = \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(c_1c_2)};$$  \hfill (2.2)

where the convergence region is $\Re c_1 > 0, \Re c_2 > 0, \Re c_1 + \Re c_2 < 1$; see [11, p. 61] for a detailed proof. We will need the following generalization of (2.2).
Lemma 2.2. Let $E_1, \ldots, E_k$ be finite extensions of $F$ of degrees $d_1, \ldots, d_k$ with $d = d_1 + \cdots + d_k \geq 2$. Let $\text{Tr}_i : E_i \to F$ be the trace map, and let $\phi : E_1 \times \cdots \times E_k \to F$ be the $F$-linear map given by

$$
\phi(x_1, \ldots, x_k) = \sum_{i=1}^{k} \text{Tr}_i(x_i).
$$

Let $S = \phi^{-1}(1)$ and let $ds$ be the measure on $S$ uniquely determined by the conditions that it is invariant under translations by $\phi^{-1}(0)$ and that the map

$$
F^* \times S \to E_1 \times \cdots \times E_k, \quad (a, (x_1, \ldots, x_k)) \mapsto (ax_1, \ldots, ax_k)
$$

changes the measure $|a|^{d-1} \, da$ to $dy_1 \cdots dy_k$ (where $da$ and $dy_i$ are self-dual measures defined in Section 1). If $c_1, \ldots, c_k$ are quasi-characters on $E_1^*, \ldots, E_k^*$ with $\text{re} \, c_i > 0$ for all $i$ and

$$
d_1 \text{re} \, c_1 + \cdots + d_k \text{re} \, c_k < 1,
$$

then the integral

$$
\int_S c_1(x_1)|x_1|_{E_1}^{-1} \cdots c_k(x_k)|x_k|_{E_k}^{-1} \, ds
$$

converges and is equal to

$$
\prod_{i=1}^{k} \frac{\Gamma_{E_i}(c_i)}{\Gamma(c)},
$$

where $c = (c_1|_{F^*}) \cdots (c_k|_{F^*})$.

Proof. We prove the convergence by induction. We may assume all $c_i$ are $\mathbb{R}_{>0}$-valued, so $c_i(x) = |x|^{r_i}_{E_i}$, $r_i \in \mathbb{R}$. If $k = 1$, then $d_1 > 1$. Let $v \in E_1$ satisfy $\phi(v) = 1$. The integral is

$$
\int_S c_1(x_1)|x_1|_{E_1}^{-1} \, ds = \int_{\phi^{-1}(0)} |y + v|_{E_1}^{r_1-1} \, dy,
$$

where we change the variable $x_1 \to y + v$ and $dy$ is the Haar measure on $\phi^{-1}(0)$ induced from $ds$ on $S$. Since $y + v$ is never zero, the integrand has no finite singular point. It is enough to prove

$$
\int_{y \in \phi^{-1}(0) \mid |y| > r} |y + v|_{E_1}^{r_1-1} \, dy < \infty
$$

for any $r > 0$. Note

$$
|y + v|_{E_1}^{r_1-1} < C|y|_{E_1}^{\text{re} \, c_1-1}
$$

for $y$ with $|y|_{E_1} > r$ ($C$ is a constant depending on $r$). So it is enough to prove

$$
\int_{|y|_{E_1} \geq r} |y|_{E_1}^{r_1-1} < \infty.
$$

Notice that $y \mapsto |y|_{E_1}^{1/d_1}$ is an $F$-norm. The result follows from Lemma 2.1 for $V = \phi^{-1}(0)$.

The case $k = 2$ and $d_1 = d_2 = 1$ is (2.2). The case $k = 2$ and $d_1 + d_2 > 2$ can be proved using the induction on $d_1 + d_2$. For $k \geq 3$, we have

$$
\int_S c_1(x_1)|x_1|_{E_1}^{-1} \cdots c_k(x_k)|x_k|_{E_k}^{-1} \, dx
$$

$$
= \int_{\text{Tr} \, x_1 + \text{Tr} \, x_2 = 1} c_1(x_1)|x_1|_{E_1}^{-1} c_2(x_2)|x_2|_{E_2}^{-1} \, d(x_1, x_2)
$$

$$
\cdot \int_{\text{Tr} \, x_3 + \cdots + \text{Tr} \, x_k = 1} |d|^{d_3 r_3 + \cdots + d_k r_k - 1} c_3(x_3)|x_3|_{E_3}^{-1} \cdots c_k(x_k)|x_k|_{E_k}^{-1} \, d(a, x_3, \ldots, x_k),
$$
where \( d(x_2, x_3) \) is a certain measure on \( \text{Tr} x_1 + \text{Tr} x_2 = 1 \) invariant under the translations by \( \text{Tr} x_1 + \text{Tr} x_2 = 0 \), and \( d(a, x_3, \ldots, x_k) \) is a certain measure on \( a + \text{Tr} x_3 + \cdots + \text{Tr} x_k = 1 \) invariant under the translations by \( a + \text{Tr} x_3 + \cdots + \text{Tr} x_k = 0 \). Both integrals on the right-hand side converge by induction assumption. Finally, we have

\[
\Gamma(0) \int_S c_1(x_1)|x_1|_{E_1}^{-1} \cdots c_k(x_k)|x_k|_{E_k}^{-1} ds = \int_F \psi(a)c(a)|a|^{-1} da \int_S c_1(x_1)|x_1|_{E_1}^{-1} \cdots c_k(x_k)|x_k|_{E_k}^{-1} ds
\]

\[
= \int_{F \times S} \psi(\text{Tr}(ax_1 + \cdots + ax_k))|c_1(ax_1)|_{E_1}^{-1} \cdots c_k(ax_k)|ax_k|_{E_k}^{-1}|a|^{d-1} ds da
\]

\[
= \int_{E_1 \times \cdots \times E_k} \psi(\text{Tr}(y_1) + \cdots + \text{Tr}(y_k))|c_1(y_1)|_{E_1}^{-1} \cdots c_k(y_k)|y_k|_{E_k}^{-1} \prod_i dy_i
\]

\[
= \prod_{i=1}^k \Gamma_{E_i}(c_i),
\]

as desired. \( \square \)

We remark that if the map \( \phi \) in Lemma 2.2 is replaced by \( \phi(x_1, \ldots, x_k) = \sum_{i=1}^k \text{Tr}(a_i x_i) \) for \( a_i \in E_i^* \), the integral (2.3) is convergent under the same conditions on the \( c_i \) and the result is (2.4) times \( \prod_{i=1}^k c_i(a_i^{-1}) \). This can be proved using Lemma 2.2 and the change of variable \( a_i x_i \mapsto x_i \).

For any \( g \in F[x] \), let \( n = \deg g \), and denote \( F[x]/(g(x)) \) by \( F_g \). Equate \( F_g \) with \( F^n \) via the map

\[
\eta^g : F_g \to F^n, \quad f(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0 \mapsto (b_{n-1}, b_{n-2}, \ldots, b_0).
\]

Let \( d_g f \) be the measure inherited from the product Haar measure of \( F^n \).

Let \( G(x) \) be a monic separable polynomial over \( F \). Assume \( G(x) = \prod_{i=1}^k g_i(x) \), with \( g_i(x) \) being different monic irreducible polynomials over \( F \). Then we have an isomorphism

\[
\varphi : E = F[x]/(G(x)) \to \prod_{i=1}^k F[x]/(g_i(x))
\]

such that

\[
\varphi(f) = (\varphi_1(f), \varphi_2(f), \ldots, \varphi_k(f)) = (f \mod g_1, f \mod g_2, \ldots, f \mod g_k).
\]

Let \( g_i(x) = \prod_{j=1}^{d_i} (x - a_{ij}) \) with \( a_{ij} \in F \) and \( d_i = \deg g_i, 1 \leq i \leq k \). Define the trace and norm maps on \( F_{g_i} \) by

\[
\text{Tr}_{g_i}(f) := \sum_{j=1}^{d_i} f(a_{ij}) \quad \text{and} \quad N_{g_i}(f) := \prod_{j=1}^{d_i} f(a_{ij}).
\]

They are just the usual trace and norm maps for the field extension \( F_{g_i} \) over \( F \). Let \( \psi_i : F_{g_i} \to S^1 \) be the additive character \( \psi_i(f) = \psi(\text{Tr}_{g_i} f) \). It defines the Fourier transform on \( S(F_{g_i}) \) by

\[
\mathcal{F}h(y) := \int_{F_{g_i}} h(x)\psi_i(xy) d_l x,
\]

where \( d_l x \) is the unique measure on \( F_{g_i} \) such that \( \mathcal{F} \) is an isometry. We need to know the relations of Haar measures \( d_l x \) and \( d_{g_i} x \). For this purpose, we prove the following lemma.

**Lemma 2.3.** Let \( D \) be a \( n \times n \) non-degenerate symmetric matrix over \( F \), and let \( d_{D} x \) be the unique Haar measure on \( F^n \) such that the Fourier transform

\[
\mathcal{F}_D f(y) = \int_{F^n} f(x)\psi(x^T D y) d_{D} x
\]

is an isometry. Then \( d_{D} x = |\det D|^{1/2} dx \), where \( dx = dx_1 \cdots dx_n \) is the product measure of the self-dual measure on \( F \) determined by \( \psi \).
Proof. By the uniqueness of the Haar measure, we have $d_0 x = C dx$ for some positive scalar $C$. We first prove the case that $F$ is non-Archimedean. Let $R$ be the ring of integers in $F$, let $\pi \in R$ be a local parameter and let $q = |R/\pi R|$. There exists $\delta \in \mathbb{Z}$ such that $\psi(\pi^{\delta} R) = 1$ and $\psi(\pi^{\delta - 1} R) \not= 1$. Then $\mathcal{F}_D 1_R(x) = q^{-\delta/2} 1_{\pi^{-\delta} R}(y)$, where $1_S$ denotes the characteristic function of the set $S$ and $\mathcal{F}$ is as in (1.2). It is easy to see that

$$\mathcal{F}_D 1_R^n(x) = C q^{-n\delta} 1_{\pi^{-n\delta} R}(x).$$

The condition that $\mathcal{F}_D$ is an isometry implies that $C = |\det D|^{1/2}$. For $F = \mathbb{R}$ or $\mathbb{C}$ we use Gaussian functions instead of $1_R$ to get the result.

From the lemma, it is easy to see that $d_R x = |\Delta(g)|^{1/2} d_R^G x$. For any quasi-character $\chi$ on $F^*$, we have that $\chi \circ N$ is a quasi-character of $F^*_\text{g}$, and we write $\Gamma_{\text{fg}}(\chi \circ N) = \Gamma_{\text{fg}}(\chi)$. The $F^*_\text{g}$-version of (2.1) reads as

$$\Gamma_{\text{fg}}(\chi) = \int_{F^*_\text{g}} \psi(\text{Tr}_{\text{fg}}(x)) \chi(N_{\text{fg}}(x)) |x|_{F^*_\text{g}}^{-1} d_R^G x,$$

where $|x|_{F^*_\text{g}}$ is the absolute value on the field $F^*_\text{g}$, and we have $|x|_{F^*_\text{g}} = |N_{\text{fg}}(x)|_F$.

Similarly, define trace and norm map on $G_F$ by

$$\text{Tr}_G(f) := \sum_{i=1}^{k} \text{Tr}_{\text{fg}}(\phi_i(f)) \quad \text{and} \quad N_G(f) := \prod_{i=1}^{k} N_{\text{fg}}(\phi_i(f)).$$

Let $\psi : G_F \to S^1$ be the additive character $\phi(f) = \psi(\text{Tr}_{\text{fg}} f)$. It defines the Fourier transform on $S(F_G)$ by

$$\mathcal{F}_G h(y) := \int_{F^*_\text{g}} h(x) \phi(xy) d_R^G x,$$

where $d_R^G x$ is the Haar measure such that $\mathcal{F}_G$ is an isometry. Using Lemma 2.3, we can prove that

$$d_R^G x = |\Delta(G)|^{1/2} d_R^G x. \quad (2.5)$$

It is also easy to see that $d_R^G x = \prod_{i=1}^{k} d_R^G \phi_i(x)$.

For $f(x), g(x) \in F[x]$, assume $f(x) = a \prod_{i=1}^{n} (x - \alpha_i)$ and $g(x) = b \prod_{j=1}^{m} (x - \beta_j)$, where $a, b \in F$ and $\alpha_i, \beta_j \in \mathbb{F}$, $1 \leq i \leq n$, $1 \leq j \leq m$. The resultant of $f$ and $g$ is defined by

$$R(f, g) := a^m b^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j) = a^m \prod_{i=1}^{n} g(\alpha_i) = (-1)^{mn} b^n \prod_{j=1}^{m} f(\beta_j).$$

In particular, for $g = b \in F^*$, we have $R(f, b) = b^{\deg f}$.

We will use the following properties of the resultant, which can be proved by definition:

$$R(f, f^*) = (-1)^{\frac{n(n-1)}{2}} \Delta(f),$$

$$R(f, g_1 g_2) = R(f, g_1) R(f, g_2),$$

$$R(f_1 f_2, g) = R(f_1, g) R(f_2, g).$$

Then we have the following two propositions.

**Proposition 2.4.** Let $G(x) \in F[x]$ be such that $G(x) = \prod_{i=1}^{k} g_i(x)$, with $g_i(x)$ being different monic irreducible polynomials over $F$ such that $g_i(x) = \prod_{j=1}^{d_i} (x - \alpha_{ij})$ with $d_i = \deg g_i$, $1 \leq i \leq k$ and $n = \sum_{i=1}^{k} d_i$. Then we have

$$\int_{M_{n-1}} \chi \chi_0^{-1} \frac{R(G, f)}{\Delta(G)} \, df = \chi_0 \left( \frac{\Delta(G)}{\Delta(G')} \right)^{-\frac{1}{2}} \chi \left( \frac{R(G, G')}{\Delta(G')} \right) \prod_{i=1}^{k} \Gamma_{\text{fg}}(\chi), \quad (2.6)$$

where $\chi_0(x) = |x|_F$, and $\chi$ is a quasi-character of $F^*$ such that $0 < n \Re \chi < 1$.

**Proof.** View $M_{n-1}$ as a subset of $F_G$. For any $f \in F_G$, we have

$$f(x) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} f(\alpha_{ij}) \prod_{l \neq i} (x - \alpha_{lk}) \prod_{l \neq i, j} (\alpha_{lj} - \alpha_{lk}).$$
by the Lagrange interpolation formula. Hence,

\[ f \in M_{n-1} \iff \sum_{i=1}^{k} \sum_{j=1}^{d_i} \frac{f(a_{ij})}{G'(a_{ij})} = \sum_{i=1}^{k} \text{Tr}_G \left( \frac{\phi_i(f)}{\phi_i(G')} \right) = 1. \]

Let \( \phi(g) = \sum_{i=1}^{k} \text{Tr}_G (\phi_i(g)/\phi_i(G')) \). Then \( M_{n-1} = \phi^{-1}(1) \) and we have

\[ \int_{f \in M_{n-1}} \chi x_0^{-1} (R(G, f)) \, df = \int_{f \in M_{n-1}} \prod_{i=1}^{k} \chi(N_G \phi_i(f) \phi_i(f)_{G'}^{-1}) \, df. \]

By Lemma 2.2 and the remark after its proof, we see the integral converges in the region \( 0 < n \Re \chi < 1 \). At this point, we may change the variable \( \phi_i(f)/\phi_i(G') \mapsto x_i \) and use Lemma 2.2 to prove (2.6), but it is messy to compute the change of various Haar measures, so we choose to proceed directly as follows. We have a one-to-one map \( \delta : F \times M_{n-1} \to F^n \) such that for \( a \in F \) we obtain \( f(x) = x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_0 = M_{n-1} \) and \( \delta(a, f) = h = a f(x) = a x^{n-1} + ab_{n-2}x^{n-2} + \cdots + ab_0 = (a, ab_{n-2}, \ldots, ab_0) \). The Jacobian of \( \delta \) is equal to \( a^{n-1} \).

Note that \( F^n \setminus \delta(M_{n-1} \times F) \) has measure zero in \( F^n \). So we have

\[ \Gamma(\chi') \int_{f \in M_{n-1}} \chi x_0^{-1} (R(G, f)) \, df \]

\[ = \int_{F} \sum_{i=1}^{k} \psi(a) \chi(R(G, a)) |a|^{-1} \, da \int_{F} \chi x_0^{-1} (R(G, f)) \, df \]

\[ = \int_{F} \psi(a) \chi x_0^{-1} (R(G, h)) \, d_x h \text{ (where } a \text{ is the highest coefficient of the polynomial } h) \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \int_{F} \psi(a) \chi x_0^{-1} (R(G, h)) \, d_x h \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \int_{F} \psi(a) \chi x_0^{-1} (R(G, h)) \, d_x h \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \chi x_0^{-1} \left( \prod_{i=1}^{k} \psi(a_{ij}) G'(a_{ij}) \right) \int_{F} \prod_{i=1}^{k} \psi \left( \text{Tr}_G \phi_i(h) \phi_i(G') \right) \prod_{i=1}^{k} \chi x_0^{-1} \left( N_G \phi_i(h) \phi_i(G') \right) \prod_{i=1}^{k} d_i(h) \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \chi x_0^{-1} \left( R(G, G') \right) \prod_{i=1}^{k} \Gamma_{G_i}(\chi) \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \chi x_0^{-1} \left( R(G, G') \right) \prod_{i=1}^{k} \Gamma_{G_i}(\chi) \]

\[ = \chi_0(\Delta(G))^{-\frac{1}{2}} \chi x_0^{-1} \left( R(G, G') \right) \prod_{i=1}^{k} \Gamma_{G_i}(\chi) \]

where we used the relation \( d_x h = \chi_0(\Delta(G))^{-1/2} \, d_x h \) in the third equation, which is just (2.5), a corollary of Lemma 2.3. We used the Lagrange interpolation in the fourth equation:

\[ h(x) = \sum_{i=1}^{k} \sum_{j=1}^{d_i} h(a_{ij}) \prod_{l \neq i, j} (x - a_{kl}) / \prod_{l \neq i, j} (a_{ij} - a_{kl}) \quad \implies \quad a = \sum_{i=1}^{k} \sum_{j=1}^{d_i} h(a_{ij}) / G'(a_{ij}) \]

And we made a change of variables \( \phi_i(h) \mapsto \frac{\phi_i(h)}{\phi_i(G')} \), \( 1 \leq i \leq k \).
Proposition 2.5. Consider any $G(x) \in F[x]$ such that $G(x) = \prod_{i=1}^{k} g_i(x)$, with $g_i(x)$ being different monic irreducible polynomials over $F$ such that $g_i(x) = \prod_{j=1}^{d_i} (x - a_{ij})$ with $a_{ij} \in \overline{F}$, $d_i = \deg g_i$, $1 \leq i \leq k$ and $\sum_{i=1}^{k} d_i = n - 1$. Assume $G(0)G(1) \neq 0$, let $S = x(1 - x)G$ and let $\alpha, \beta, \gamma$ be quasi-characters of $F'$ such that $\text{re } \alpha > 0$, $\text{re } \beta > 0$, $\text{re } \gamma > 0$ and $\text{re } \alpha + \text{re } \beta + (n - 1) \text{re } \gamma < 1$. Then we have

$$\int_{f \in M_n} a\chi_0^{-1}(f(0))\beta\chi_0^{-1}(f(1))\gamma\chi_0^{-1}(R(G, f)) \, df$$

$$= a(-1)\chi_0(\Delta(G))^{-\frac{1}{2}} a\gamma\chi_0^{-1}(G(0))\beta\gamma\chi_0^{-1}(G(1))\gamma(R(G, G')) \frac{\Gamma(\alpha)\Gamma(\beta)\prod_{i=1}^{k} \Gamma_{g_i}(\gamma)}{\Gamma(a\beta\gamma^{-1})}. $$

Proof. The proof is similar to the one of Proposition 2.4. View $M_n$ as a subset of $F_S$. For any $f \in F_S$, we have

$$f(x) = \prod_{i=1}^{k} \prod_{j=1}^{d_i} f(a_{ij}) \frac{x(x - 1)\prod_{i \neq j, \neq \overline{i}, \neq \overline{j}} (x - a_{ij})}{a_{ij}(a_{ij} - 1)\prod_{i \neq j, \neq \overline{i}, \neq \overline{j}} (a_{ij} - a_{\overline{i} \overline{j}})} + f(0) \frac{x(1 - x)G(x)}{S'(0)} + f(1) \frac{xG(x)}{S'(1)}$$

by the Lagrange interpolation formula. Hence,

$$f \in M_n \iff \prod_{i=1}^{k} \prod_{j=1}^{d_i} f(a_{ij}) \frac{x(x - 1)\prod_{i \neq j, \neq \overline{i}, \neq \overline{j}} (x - a_{ij})}{a_{ij}(a_{ij} - 1)\prod_{i \neq j, \neq \overline{i}, \neq \overline{j}} (a_{ij} - a_{\overline{i} \overline{j}})} + f(0) \frac{x(1 - x)G(x)}{S'(0)} + f(1) \frac{xG(x)}{S'(1)} = 1.$$

Let $\phi(f) = \sum_{i=1}^{k} \text{Tr}_{g_i}(\varphi_i(f)/\varphi_i(S')) + f(0)/S'(0) + f(1)/S'(1)$. Then $M_n = \phi^{-1}(1)$, and we have

$$\int_{f \in M_n} a\chi_0^{-1}(f(0))\beta\chi_0^{-1}(f(1))\gamma\chi_0^{-1}(R(G, f)) \, df$$

$$= \int_{f \in M_n} a\chi_0^{-1}(f(0))\beta\chi_0^{-1}(f(1))\gamma\chi_0^{-1}(R(G, f)) \, df$$

By Lemma 2.2 and the remark after its proof, we see the integral converges in the region: $\text{re } \alpha, \text{re } \beta, \text{re } \gamma > 0$, $\text{re } \alpha + \text{re } \beta + (n - 1) \text{re } \gamma < 1$. Similar to Proposition 2.4, we compute the integral as follows:

$$\Gamma(a\beta\gamma^{-1}) \int_{f \in M_n} a\chi_0^{-1}(f(0))\beta\chi_0^{-1}(f(1))\gamma\chi_0^{-1}(R(G, f)) \, df$$

$$= \int_{F} \psi(a)\alpha(a)\gamma(R(G, a)) |a|_{F}^{-1} \, da \int_{f \in M_n} a\chi_0^{-1}(f(0))\beta\chi_0^{-1}(f(1))\gamma\chi_0^{-1}(R(G, f)) \, df$$

$$= \int_{M_n \times F} \psi(a)\alpha\chi_0^{-1}(af(0))\beta\chi_0^{-1}(af(1))\gamma\chi_0^{-1}(R(G, af)) |a|_{F}^{-1} \, df \, da$$

$$= \int_{F_S} \psi(a)\alpha\chi_0^{-1}(h(0))\beta\chi_0^{-1}(h(1))\gamma\chi_0^{-1}(R(S, h)) \, d_h \text{ (where } a \text{ is the highest coefficient of } h)$$

$$= \chi_0(\Delta(S))^{-\frac{1}{2}} \int_{F_S} \psi(a)\alpha\chi_0^{-1}(h(0))\beta\chi_0^{-1}(h(1))\gamma\chi_0^{-1}(R(S, h)) \, d_h \text{ (Lemma 2.3)}$$

$$= \chi_0(\Delta(S))^{-\frac{1}{2}} \int_{F_S} \psi\left(\prod_{i=1}^{k} \prod_{j=1}^{d_i} \frac{h(a_{ij})}{a_{ij}(a_{ij} - 1)G(a_{ij}) + \frac{h(0)}{G(0)} + \frac{h(1)}{G(1)}}\right)$$

$$\times \alpha\chi_0^{-1}(h(0))\beta\chi_0^{-1}(h(1))\gamma\chi_0^{-1}\left(h(0)h(1) \prod_{i=1}^{k} \prod_{j=1}^{d_i} h(a_{ij})\right) \, d_h.$$

Here we are using the Lagrange interpolation in the last equality to get the expression of $a$ in terms of $h$. Now via the isomorphism

$$F_S \equiv F \times F \times G, \quad h \mapsto (h(0), h(1), h \mod G), \quad d, h \mapsto d, h(0) \cdot d, h(1) \cdot d \cdot c \cdot h,$$
where \(d_*h\) is the Haar measure on \(F_G\) such that \(\mathcal{F}_G\) is an isometry, we have

\[
\int_{F_G} \psi(a) \alpha y^{-1}(h(0)) \beta y^{-1}(h(1)) \chi_0^{-1}(R(S, h)) \, d_* h
\]

\[
= \left\{ \psi \left( \sum_{i=1}^{k} \sum_{j=1}^{d_i} \frac{h(a_{ij})}{a_i} (a_{ij} - 1) G'(a_{ij}) \right) \chi_0^{-1} \left( \prod_{i=1}^{k} \prod_{j=1}^{d_i} h(a_{ij}) \right) \right\} \int_{F_G} \psi \left( \frac{h(0)}{G(O)} \right) \alpha \chi_0^{-1}(h(0)) \, d_* h(0) \int_{F_G} \psi \left( \frac{h(1)}{G(1)} \right) \beta \chi_0^{-1}(h(1)) \, d_* h(1)
\]

\[
= \int_{\mathbb{F}^{\mathbb{Z}}_{+} F_n} \psi \left( \frac{\operatorname{Tr}_{\mathbb{F}_i}}{\varphi_i(x(x-1)G')} \right) \chi_0^{-1} \left( N_{G} \left( \varphi_i(h) \right) \prod_{i=1}^{k} \right) \int_{F_G} \psi \left( \frac{h(0)}{G(O)} \right) \alpha \chi_0^{-1}(h(0)) \, d_* h(0) \int_{F_G} \psi \left( \frac{h(1)}{G(1)} \right) \beta \chi_0^{-1}(h(1)) \, d_* h(1),
\]

where \(\varphi_i(x(x-1)G')\) should be interpreted as \(\varphi_i(x(x-1)G' \mod G)\). Now make several changes of variables:

\[
\begin{align*}
\varphi_i(h) & \mapsto \varphi_i(h)/\varphi_i(x(x-1)G'), \quad 1 \leq i \leq k, \\
\varphi_i(h) & \mapsto -h(0)/G(0), \\
\varphi_i(h) & \mapsto h(1)/G(1).
\end{align*}
\]

We get

\[
\Gamma(\alpha \beta y^{n-1}) \int_{f \in M_n} \alpha \chi_0^{-1}(f(0)) \beta \chi_0^{-1}(f(1)) \gamma \chi_0^{-1}(R(G, f)) \, df
\]

\[
= \chi_0(\Delta(S))^{-\frac{1}{2}} \prod_{i=1}^{k} \chi_0^{-1}(N_{G_i} \left( \varphi_i(x(x-1)G') \right) \prod_{i=1}^{k} \Gamma_{G_i}(y) \int_{F_G} \chi_0^{-1}(-G(O)) \prod_{i=1}^{k} \chi_0^{-1}(G(1)) \Gamma(\alpha) \Gamma(\beta)
\]

\[
= \chi_0(\Delta(S))^{-\frac{1}{2}} \chi_0^{-1} \left( \prod_{i=1}^{k} \prod_{j=1}^{d_i} a_i(a_{ij} - 1) G'(a_{ij}) \right) \Gamma(\alpha) \Gamma(\beta)
\]

\[
= a(-G(0)) \beta(G(1)) \Gamma(\alpha) \Gamma(\beta)
\]

which completes the proof of the proposition.

\[
\square
\]

### 3 Proof of Theorem 1.1

**Proof.** The proof is by induction. When \(n = 1\), formula (1.6) is just (2.2). Now assume \(S_{n-1}(\alpha, \beta, y)\) converges for any \(\alpha, \beta, y\) in the region \(R_{n-1}\) and we have

\[
S_{n-1}(\alpha, \beta, y) = \prod_{j=0}^{n-2} \Gamma(\alpha y^j) \Gamma(\beta y^j) \Gamma(y^{n-1}) \Gamma(\alpha y^{n-1}).
\]

Consider the double integral

\[
T_n := \int_{P \in M_{n-1}} \int_{Q \in M_{n}} \alpha \chi_0^{-1}(Q(0)) \beta \chi_0^{-1}(Q(1)) \gamma \chi_0^{-1}(R(P, Q)) \, dQ \, dP.
\]
By Proposition 2.5, we get

\[
T_{n,P} := \int_{Q \in M_n} \left( \int_{P \in M_{n-1}} a \chi_0^{-1}(Q(0)) \beta \chi_0^{-1}(Q(1)) \gamma \chi_0^{-1}(R(P, Q)) \, dQ \right) \, dP
\]

\[
= \int_{Q \in M_n} \left( \int_{P \in M_{n-1}} a(-1) \chi_0^{-1}(\Delta(P)) a \beta \gamma \chi_0^{-1}(R(P, P')) \, \frac{\Gamma(a) \Gamma(\beta) \prod_{k=1}^{n} \Gamma_q(y)}{\Gamma(a \beta y^{n-1})} \, dP \right) \, dQ
\]

\[
= a^n(-1) S_{n-1}(a, \beta, y) \frac{\Gamma(\beta) \Gamma(y)^{n-1}}{\Gamma(a \beta y^{n-1})}
\]

valid in the region

\[
\Re a, \Re \beta, \Re y > 0, \\
\Re a + \Re \beta + (n - 1) \Re y < 1, \\
\Re a + 1 + \Re \beta + 1 + 2(n - 2) \Re y < 1,
\]

which is exactly the region \( R_n \). Note that the absolute value of the integrand is the same as replacing \( a, \beta, y \) by \( \chi_0^{\Re a}, \chi_0^{\Re \beta}, \chi_0^{\Re y} \). Hence the we have \( T_n = T_{n,P} = T_{n,Q} \) in \( R_n \) by the Fubini–Tonelli theorem, where \( T_{n,Q} \) is defined by

\[
T_{n,Q} := \int_{Q \in M_n} \left( \int_{P \in M_{n-1}} a \chi_0^{-1}(Q(0)) \beta \chi_0^{-1}(Q(1)) \gamma \chi_0^{-1}(R(Q, Q')) \, dP \right) \, dQ
\]

\[
= \int_{Q \in M_n} \left( a \chi_0^{-1}(Q(0)) \beta \chi_0^{-1}(Q(1)) \chi_0(\Delta(Q)) \, \frac{\prod_{k=1}^{n} \Gamma_q(y)}{\Gamma(y)} \right) \, dQ
\]

\[
= a^n(-1) S_n(a, \beta, y) \frac{\Gamma(y)^n}{\Gamma(y^n)}
\]

where we are using Proposition 2.4 in the second equation. Thus \( S_n(a, \beta, y) \) converges in the region \( R_n \), and \( T_{n,P} = T_{n,Q} \) gives us

\[
S_n(a, \beta, y) = S_{n-1}(a, \beta, y) \frac{\Gamma(a) \Gamma(\beta) \Gamma(y^n)}{\Gamma(a \beta y^{n-1}) \Gamma(y)} = \prod_{j=0}^{n-1} \frac{\Gamma(a y^j) \Gamma(\beta y^{n-j}) \Gamma(y^{n-j})}{\Gamma(a \beta y^{n-j-1}) \Gamma(y)}
\]

which completes the proof of the main theorem.

As a special case, we consider \( F = \mathbb{C} \). We take \( \psi(z) = \psi(x + iy) = e^{inx} \) as in [12]. The self-dual Haar measure \( dx \) is then twice the usual Lebesgue measure on \( \mathbb{C} \). Note that \( |z|_\mathbb{C} = |z|^2 \). The map

\[
\Phi_n : \mathbb{C}^n \to M_n, \quad (z_1, \ldots, z_n) \mapsto f(x) = \prod_{i=1}^{n} (x - z_i) = x^n + b_{n-1}x^{n-1} + \cdots + b_0
\]

is surjective and generically \( n! \) to 1. It is known that the Jacobian of this map is (see [10])

\[
\text{Jac } \Phi_n(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j).
\]

Thus

\[
|\text{Jac } \Phi_n(z_1, \ldots, z_n)|_\mathbb{C} = \left| \prod_{i<j} (z_i - z_j) \right|_\mathbb{C} = |\Delta(f)|_\mathbb{C}^{1/2}.
\]

So we have

\[
S_n(a, \beta, y) = a^n \frac{n-1}{n} \int_{f \in M_n} \alpha \chi_0^{-1}(f(0)) \beta \chi_0^{-1}(f(1)) \gamma \chi_0^{-1}(\Delta(f)) \, df
\]

\[
= \frac{1}{n!} \int_{\mathbb{C}^n} \alpha \chi_0^{-1} \left( \prod_{i=1}^{n} z_i \right) \beta \chi_0^{-1} \left( \prod_{i=1}^{n} (1 - z_i) \right) \gamma \left( \prod_{i<j} (z_i - z_j) \right) \prod_{i=0}^{n} dz_i.
\]
Then by Theorem 1.1 we get
\[
\int \prod_{i=1}^{n} \alpha_i(z_i) \beta_i(z_i) \prod_{i \neq j} (z_i - z_j) \prod_{i=0}^{n-1} \frac{\Gamma_C(ay^i) \Gamma_C(by^{i+1}) \Gamma_C(y^{i+1})}{\Gamma_C(a \beta y^{i+1})} \, dz_i = n! \prod_{j=0}^{n-1} \frac{\Gamma_C(a + j c) \Gamma_C(b + j c) \Gamma_C((j + 1)c)}{\Gamma_C(a + b + (n + j - 1)c) \Gamma_C(c)} \prod_{i=0}^{n-1} \frac{2 \sin(\pi a + j c) \sin(\pi b + j c) \sin(\pi (j + 1)c)}{\sin(\pi (a + b + (n + j - 1)c))} S_n(a, b, c)^2, \tag{3.2}
\]
valid in the region $R_n$. If $\alpha, \beta, y$ are all unramified, i.e. $\alpha = | \cdot |^a_C, \beta = | \cdot |^b_C, y = | \cdot |^c_C$ for some $a, b, c \in \mathbb{C}$, then formula (3.1) becomes
\[
\int \prod_{i=1}^{n} |z_i|^{2a-2} |z_i|^{2b-2} \prod_{1 \leq i < j \leq n} |z_i - z_j|^{a+c} \prod_{i=0}^{n-1} \frac{\Gamma_C(a + j c) \Gamma_C(b + j c) \Gamma_C((j + 1)c)}{\Gamma_C(a + b + (n + j - 1)c) \Gamma_C(c)} \prod_{i=0}^{n-1} \frac{2 \sin(\pi a + j c) \sin(\pi b + j c) \sin(\pi (j + 1)c)}{\sin(\pi (a + b + (n + j - 1)c))} S_n(a, b, c)^2,
\]
where for any $s \in \mathbb{C}$ with $0 < \text{re} \, s < 1$,
\[
\Gamma_C(s) := \Gamma_C(| \cdot |^s) = \frac{(2\pi)^{1-s} \Gamma(s)}{(2\pi)^s \Gamma(1-s)} = 2^{1-2s} \pi^{-s} \Gamma(s)^2 \sin(\pi s)
\]
(cf. [12]) and $S_n(a, b, c)$ is defined in (1.1). Formula (3.2) is the same as the formula obtained by Aomoto [4] (note that our measure is twice of the usual Lebesgue measure).

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