“Quantiles Equivariance”

Reza Hosseini, University of British Columbia
333-6356, Agricultural Road, Vancouver,
BC, Canada, V6T1Z2
reza1317@gmail.com

1 Abstract

It is widely claimed that the quantile function is equivariant under increasing transformations. We show by a counterexample that this is not true (even for strictly increasing transformations). However, we show that the quantile function is equivariant under left continuous increasing transformations. We also provide an equivariance relation for continuous decreasing transformations. In the case that the transformation is not continuous, we show that while the transformed quantile at $p$ can be arbitrarily far from the quantile of the transformed at $p$ (in terms of absolute difference), the probability mass between the two is zero. We also show by an example that weighted definition of the median is not equivariant under even strictly increasing continuous transformations.

Keywords: Quantile, quantile function, distribution function, equivariance, continuous transformation, increasing transformation

2 Introduction

The traditional definition of quantiles for a random variable $X$ with distribution function $F$,

$$lq_X(p) = \inf\{x|F(x) \geq p\},$$

appears in classic works as [4]. We call this the “left quantile function”. In some books (e.g. [4]) the quantile is defined as

$$rq_X(p) = \inf\{x|F(x) > p\} = \sup\{x|F(x) \leq p\},$$

this is what we call the “right quantile function”. Also in robustness literature people talk about the upper and lower medians which are a very specific case of
these definitions. Hosseini in [2] considers both definitions, explore their relation and show that considering both has several advantages.

It is widely claimed that (e.g. Koenker in [3] or Hao and Naiman in [1]) the traditional quantile function is equivariant under monotonic transformations. We show that this does not hold even for strictly increasing functions. However, we prove that the traditional quantile function is equivariant under non-decreasing left continuous transformations. We also show that the right quantile function is equivariant under non-decreasing right continuous transformations. A similar neat result is found for continuous decreasing transformations using the Quantile Symmetry Theorem also proved by Hosseini in [2]. We state this theorem later when we need it. Hosseini in [2], proved the two following useful lemmas using the definition of quantiles. We will use some of the items in these lemmas in our proofs.

**Lemma 2.1:** (Quantile Properties Lemma) Suppose $X$ is a random variable on the probability space $(\Omega, \Sigma, P)$ with distribution function $F$:

a) $F(lq_F(p)) \geq p$.

b) $lq_F(p) \leq rq_F(p)$.

c) $p_1 < p_2 \Rightarrow rq_F(p_1) \leq lq_F(p_2)$.

d) $rq_F(p) = \sup\{x|F(x) \leq p\}$.

e) $P(lq_F(p) < X < rq_F(p)) = 0$. i.e. $F$ is flat in the interval $(lq_F(p), rq_F(p))$.

f) $P(X < rq_F(p)) \leq p$.

h) If $lq_F(p) < rq_F(p)$ then $F(lq_F(p)) = p$ and hence $P(X \geq rq_F(p)) = 1 - p$.

i) $lq_F(p)$ and $rq_F(p)$ are non-decreasing functions of $p$.

j) If $P(X = x) > 0$ then $lq_F(F(x)) = x$.

k) $x < lq_F(p) \Rightarrow F(x) < p$ and $x > rq_F(p) \Rightarrow F(x) > p$.

**Lemma 2.2:** (Quantile Value Criterion Lemma)
a) \( lq_F(p) \) is the only \( a \) satisfying (i) and (ii), where
(i) \( F(a) \geq p \),
(ii) \( x < a \Rightarrow F(x) < p \).

b) \( rq_F(p) \) is the only \( a \) satisfying (i) and (ii), where
(i) \( x < a \Rightarrow F(x) \leq p \),
(ii) \( x > a \Rightarrow F(x) > p \).

**Proof**

a) Both properties hold for \( lq_F(p) \) by the previous lemma. If both \( a < b \) satisfy them, then \( F(a) \geq p \) by (i). But since \( b \) satisfies the properties and \( a < b \), by (ii), \( F(a) < p \) which is a contradiction.

b) Both properties hold for \( rq_F(p) \) by the previous lemma. If both \( a < b \) satisfy them, then we can get a contradiction similar to above.

It is customary to use weighted procedures to define the quantiles of a data vector. The most widely used example is the definition of median when for the sorted data vector \( x = (x_1, \ldots, x_n) \), \( n \) is even, in which case the median is defined to be \( x_{\frac{n}{2}} + x_{\frac{n+2}{2}} \). We start by an example that shows with this definition the median is not equivariant even under continuous strictly increasing transformation (a continuous re-scaling of data).

A supervisor asked 2 graduate students to summarize the following data regarding the intensity of the earthquakes in a specific region:
Earthquake intensity is usually measured in $M_L$ scale, which is related to $A$ by the following formula:

$$M_L = \log_{10} A.$$ 

In the data file handed to the students (Table 1), the data is sorted with respect to $M_L$ in increasing order from top to bottom. Hence the data is arranged decreasingly with respect to $A$ from top to bottom.

The supervisor asked two graduate students to compute the center of the intensity of the earthquakes using this dataset. One of the students used $A$ and the usual definition of median and so obtained

$$\frac{16.39759 \times 10^4 + 19.47287 \times 10^4}{2} = 17.93523 \times 10^4.$$ 

The second student used the $M_L$ and the usual definition of median to find

$$\frac{5.21478 + 5.28943}{2} = 5.252105.$$ 

When the supervisor saw the results he figured that the students must have used different scales. Hence he tried to make the scales the same by transforming the second student’s result

$$10^{5.252105} = 17.86920 \times 10^4.$$
To his surprise the results were not quite the same. He was bothered to notice that the definition of median is not equivariant under the change of scale which is continuous strictly increasing.

3 Equivariance property of quantile functions

(Counter example for Koenker–Hao claim) Suppose $X$ is distributed uniformly on $[0,1]$. Then $lq_X(1/2) = 1/2$. Now consider the following strictly increasing transformation

$$
\phi(x) = \begin{cases} 
    x & -\infty < x < 1/2 \\
    x + 5 & x \geq 1/2
\end{cases}.
$$

Let $T = \phi(X)$ then the distribution of $T$ is given by

$$
P(T \leq t) = \begin{cases} 
0 & t \leq 0 \\
t & 0 < t \leq 1/2 \\
1/2 & 1/2 < t \leq 5 + 1/2 \\
t - 5 & 5 + 1/2 < t \leq 5 + 1 \\
1 & t > 5 + 1
\end{cases}.
$$

It is clear from above that $lq_T(1/2) = 1/2 \neq \phi(lq_X(1/2)) = \phi(1/2) = 5 + 1/2$.

We start by defining

$$
\phi^\leq(y) = \{x|\phi(x) \leq y\}, \ \phi^*(y) = \sup \phi^\leq(y),
$$

and

$$
\phi^\geq(y) = \{x|\phi(x) \geq y\}, \ \phi_*(y) = \inf \phi^\geq(y).
$$

Then we have the following lemma.

Lemma 3.1: Suppose $\phi$ is non-decreasing.

a) If $\phi$ is left continuous then

$$
\phi(\phi^*(y)) \leq y.
$$

b) If $\phi$ is right continuous then

$$
\phi(\phi_*(y)) \geq y.
$$
Proof

a) Suppose $x_n \uparrow \phi^*(y)$ a strictly increasing sequence. Then since $x_n < \phi^*(y)$, we conclude $x_n \in \phi^< (y) \Rightarrow \phi(x_n) \leq y$. Hence $\lim_{n \to \infty} \phi(x_n) = \phi(\phi^*(y))$.

b) Suppose $x_n \downarrow \phi_*(y)$ a strictly decreasing sequence. Then since $x_n > \phi_*(y)$, we conclude $x_n \in \phi^>(y) \Rightarrow \phi(x_n) \geq y$. Hence $\lim_{n \to \infty} \phi(x_n) = \phi(\phi_*(y))$.

Theorem 3.1: (Quantile Equivariance Theorem) Suppose $\phi : \mathbb{R} \to \mathbb{R}$ is non-decreasing.

a) If $\phi$ is left continuous then

$$l_{q_{\phi(X)}}(p) = \phi(l_{q_X}(p)).$$

b) If $\phi$ is right continuous then

$$r_{q_{\phi(X)}}(p) = \phi(r_{q_X}(p)).$$

Proof

a) We use Lemma 2.2 to prove this. We need to show (i) and (ii) in that lemma for $\phi(l_{q_X}(p))$. First note that (i) holds since

$$F_{\phi(X)}(\phi(l_{q_X}(p))) = P(\phi(X) \leq \phi(l_{q_X}(p))) \geq P(X \leq l_{q_X}(p)) \geq p.$$ 

For (ii) let $y < \phi(l_{q_X}(p))$. Then we want to show that $F_{\phi(X)}(y) < p$. It is sufficient to show $\phi^*(y) < l_{q_X}(p)$. Because then

$$P(\phi(X) \leq y) \leq P(X \leq \phi^*(y)) < p.$$ 

To prove $\phi^*(y) < l_{q_X}(p)$, note that by the previous lemma

$$\phi(\phi^*(y)) \leq y < \phi(l_{q_X}(p)).$$
b) We use Lemma 2.2 to prove this. We need to show (i) and (ii) in that lemma for $\phi(rq_X(p))$. To show (i) note that if $y < \phi(rq_X(p))$, 

$$P(\phi(X) \leq y) \leq P(\phi(X) < \phi(rq_X(p))) \leq P(X < rq_X(p)) \leq p.$$ 

To show (ii), suppose $y > \phi(rq_X(p))$. We only need to show $\phi_*(y) > rq_X(p)$ because then 

$$P(\phi(X) \leq y) \geq P(X < \phi_*(y)) > p.$$ 

But by previous lemma $\phi(\phi_*(y)) \geq y > \phi(rq_X(p))$. Hence $\phi_*(y) > rq_X(p)$.

[Proof]

In order to find an equivariance under decreasing transformations we need the Quantile Symmetry Theorem proved by Hosseini in [2].

**Theorem 3.2:** (Quantile Symmetry Theorem) Suppose $X$ is a random variable and $p \in [0, 1]$. Then 

$$lq_X(p) = -rq_{-X}(1 - p).$$

**Theorem 3.3:** (Decreasing transformation equivariance) 

a) Suppose $\phi$ is non-increasing and right continuous on $\mathbb{R}$. Then 

$$lq_{\phi(X)}(p) = \phi(rq_X(1 - p)).$$

b) Suppose $\phi$ is non-increasing and left continuous on $\mathbb{R}$. Then 

$$rq_{\phi(X)}(p) = \phi(lq_X(1 - p)).$$

**Proof**  

a) By the Quantile Symmetry Theorem, we have 

$$lq_{\phi(X)}(p) = -rq_{-\phi(X)}(1 - p).$$

But $-\phi$ is non-decreasing right continuous, hence the above is equal to 

$$-(-\phi(rq_X(1 - p))) = \phi(rq_X(1 - p)).$$

b) By the Quantile symmetry Theorem 

$$rq_{\phi(X)}(p) = -lq_{-\phi(X)(1-p)} = -(-\phi(lq_X(1-p))) = \phi(lq_X(p)),$$

since $-\phi$ is non-decreasing and left continuous.
4 The non-continuous case

We showed by an example that the equivariance property does not hold for increasing transformations that are not continuous. However we show here that the transformed quantile is not that much off at the end in a specific sense. We start by a lemma.

Lemma 4.1: Let $X$ be a random variable. Then

$$[lq_X(p), rq_X(p)] = \{y| F_X^o(y) \leq p, F_X(y) \geq p\},$$

where $F_X^o(x) = P(X < x)$, $F_X(x) = P(X \leq x)$.

Proof By the Quantile Property Lemma (a)

$$F(a) \geq F(lq_X(p)) \geq p,$$

for all $a \geq lq_X(p)$. Now note that by the Quantile Value Criterion Lemma, Part (b), we have

$$F_X^o(a) \leq F_X^o(p) = \lim_{x \to rq_X(p)^+} F(x) \leq p,$$

for all $a \leq rq_X(p)$, which shows

$$[lq_X(p), rq_X(p)] \subset \{y| F_X^o(y) \leq p, F_X(y) \geq p\}.$$

To prove the converse, suppose $y < lq_X(p)$ then $F(y) < p$ by Quantile Value Criterion Lemma, Part (a) and hence $y \notin \{y| F_X^o(y) \leq p, F_X(y) \geq p\}$. Similarly for $y > rq_X(p)$ take $y > z > rq_X(p)$ by the Part (b) of the lemma

$$F_X^o(y) \leq F_X(y) > p.$$

Hence $y \notin \{y| F_X^o(y) \leq p, F_X(y) \geq p\}$. ■

Lemma 4.2: (Equivariance under non-decreasing transformations) Suppose $X$ is a random variable with distribution function $F$ and $\phi : \mathbb{R} \to \mathbb{R}$ a non-decreasing transformation on $\mathbb{R}$. Also let $Y = \phi(X)$. Then

a) $\phi(lq_X(p)) \in [lq_Y(p),rq_Y(p)]$

b) $\phi(rq_X(p)) \in [lq_Y(p),rq_Y(p)]$. 
Proof. Note that
\[ F_Y'(\phi(lq_X(p))) = P(\phi(X) < \phi(lq_X(p))) \leq P(X < lq_X(p)) \leq p, \]
and
\[ F_Y(\phi(lq_X(p))) = P(\phi(X) \leq \phi(lq_X(p))) \geq P(X \leq lq_X(p)) \geq p. \]
Hence proving a) by the previous lemma and b) is similar.

Remark. If we consider the “probability loss function” defined as
\[ \delta_Y(a, b) = P(a < Y < b) + P(b < Y < a), \]
then the above lemma states that
\[ \delta_Y(\phi(lq_X(p)), lq_Y(p)) = 0, \]
and
\[ \delta_Y(\phi(rq_X(p)), rq_Y(p)) = 0. \]
Hosseini in [2] studied this loss function and used it in approximating quantiles in large datasets.

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