Analytic Treatment of Positronium Spin Splittings in Light-Front QED

Billy D. Jones and Robert J. Perry

Department of Physics, The Ohio State University, Columbus, Ohio 43210-1106

Stanislaw D. Glazek

Institute of Theoretical Physics, Warsaw University, ul. Hoża 69, 00-861 Warsaw, Poland

We study the QED bound-state problem in a light-front hamiltonian approach. Starting with a bare cutoff QED Hamiltonian, $H_B$, with matrix elements between free states of drastically different energies removed, we perform a similarity transformation that removes the matrix elements between free states with energy differences between the bare cutoff, $\Lambda$, and effective cutoff, $\lambda$ ($\lambda < \Lambda$). This generates effective interactions in the renormalized Hamiltonian, $H_R$. These effective interactions are derived to order $\alpha$ in this work, with $\alpha \ll 1$. $H_R$ is renormalized by requiring it to satisfy coupling coherence. A nonrelativistic limit of the theory is taken, and the resulting Hamiltonian is studied using bound-state perturbation theory (BSPT). The effective cutoff, $\lambda^2$, is fixed, and the limit, $0 \leftarrow m^2 \alpha^2 \ll \lambda^2 \ll m^2 \alpha \rightarrow \infty$, is taken. This upper bound on $\lambda^2$ places the effects of low-energy (energy transfer below $\lambda$) emission in the effective interactions in the $|e\bar{e}\rangle$ sector. This lower bound on $\lambda^2$ insures that the nonperturbative scale of interest is not removed by the similarity transformation. As an explicit example of the general formalism introduced, we show that the Hamiltonian renormalized to $\mathcal{O}(\alpha)$ reproduces the exact spectrum of spin splittings, with degeneracies dictated by rotational symmetry, for the ground state through $\mathcal{O}(\alpha^4)$. The entire calculation is performed analytically, and gives the well known singlet-triplet ground state spin splitting of positronium, $7\alpha^2 \text{Ryd}$. We discuss remaining corrections other than the spin splittings and how they can be treated in calculating the spectrum with higher precision.

PACS number(s): 11.10.Ef, 11.10.Gh, 12.20.Ds

1email: bjones@mps.ohio-state.edu
I. INTRODUCTION

There is much effort being put into solving for the hadronic spectrum from first principles of QCD in (3+1)-dimensions using a light-front similarity hamiltonian approach [1-5]. However, low-energy QCD is challenging, and a realistic analytical calculation may be impossible. There is a need for exact analytical calculations that test and illustrate the approach. This paper is one such calculation. The calculation in this paper was discussed in Ref. [2], where the leading order calculation was completed. We start from the canonical QED Hamiltonian, and set up the general formalism for deriving the renormalized Hamiltonian, $H_R$, to some prescribed order in $\alpha$ (with $\alpha \ll 1$), and then using BSPT to compute the low energy observables of interest to some prescribed order in $\alpha$.

The general formulae are then applied explicitly to the spin splittings in positronium. Renormalization is carried out to order $\alpha$, and a nonrelativistic limit of the theory is taken. The limit, 

$$0 \leftarrow m^2 \alpha^2 \ll \lambda^2 \ll m^2 \alpha \rightarrow \infty,$$  \hspace{1cm} (1)$$

is taken, and with BSPT, a calculation of the singlet-triplet ground state spin splitting of positronium to order $\alpha^4$ is carried out. The entire calculation is performed analytically, and the well known result, $\frac{7}{5} \alpha^2 \text{Ryd}$, is obtained. Our results apply to all physical positronium states with arbitrary momentum $P$.

Kaluza and Pirner have calculated the singlet-triplet ground state spin splitting of positronium (neglecting annihilation channel contributions) to order $\alpha^4$ in a light-front hamiltonian approach [6], and they obtained correct results numerically, but were forced to make ad hoc assumptions because their Hamiltonian depended on the full eigenvalue of the problem. We avoid these assumptions in our approach, and perform the calculation analytically.

II. THE SIMILARITY HAMILTONIAN APPROACH

The starting point in the similarity hamiltonian approach is a bare cutoff continuum Hamiltonian of physical interest, $H_B$, with energy widths restricted to be below the bare cutoff, $\Lambda$. A similarity

\footnote{The “energy difference” between the free states in a matrix element of a Hamiltonian is defined to be its “energy width.”}
transformation (unitary here) is defined that acts on $H_B$ and restricts the energy widths in the final Hamiltonian, $H_R$, to be below the effective cutoff, $\lambda$. All the energy changes between $\Lambda$ and $\lambda$ are “integrated out” and replaced by effective interactions in $H_R$. The initial Hamiltonian must then be adjusted so that $H_R$ satisfies coupling coherence, which produces a renormalized Hamiltonian order by order in the running couplings, $e\lambda$ and $m\lambda$. Coupling coherence can be realized by requiring a Hamiltonian at one scale to equal a Hamiltonian at a new scale after changing the explicit scale dependence in the Hamiltonian and the implicit scale dependence in a finite number of independent running couplings. In addition, all the dependent couplings are required to vanish when the independent marginal couplings are taken to zero.

The second step of this similarity hamiltonian approach is the diagonalization of $H_R$. First, for this QED calculation, a nonrelativistic limit of $H_R$ is taken. This is reasonable because the bound-state electron momenta (equivalent formulae hold for the positron) satisfy:

\[
\frac{p^\perp_{\text{electron}}}{m} = O(\alpha), \quad (3)
\]

\[
\alpha \ll 1, \quad (4)
\]

where $m$ is the renormalized electron mass and $P^+$ is the total longitudinal momentum of positronium.\(^3\)

Next, the Hamiltonian is divided into a nonperturbative and perturbative part,

\[
H_R = H_o + (H_R - H_o) \equiv H_o + V. \quad (5)
\]

This is a standard trick for hamiltonian problems, and will work best if the lowest order spectrum of $H_R$ is well approximated by the spectrum of $H_o$. Phenomenological input can be used to determine $H_o$. The main point is that if the spectrum of $H_o$ differs too much from the lowest order spectrum of $H_R$, the subsequent BSPT will not converge rapidly (if at all).

\(^3\)In the initial setup of the similarity transformation, $\lambda$ and $\Lambda$ will be used as a shorthand for $\frac{\lambda^2}{P^+}$ and $\frac{\Lambda^2}{P^+}$ respectively, where $P^+$ is the total longitudinal momentum of the physical state of interest, and $\lambda^2$ and $\Lambda^2$ have dimension (mass)$^2$.

\(^4\)The reader unaccustomed to light-front coordinates should consult Appendix A.
We proceed to estimate a lower bound on $\lambda$ where the nonperturbative bound state effects of $H_\lambda$ enter. This lower bound naturally appears at the physical binding energy of interest:

$$\frac{\lambda^2}{P^+} \gg \frac{|M_N^2 - (2m)^2|}{P^+},$$

(6)

where $M_N$ is the mass of the physical state of interest.\(^5\) Given this lower bound we next proceed with an estimate of an upper bound on $\lambda$ that will allow us to obtain our approximate spectrum entirely within the $|e\bar{e}|$ sector. The fact that this upper bound arises at all is one of the utilities of the similarity hamiltonian approach. This upper bound is

$$\frac{\lambda^2}{P^+} \ll q_{\text{photon}},$$

(7)

where $q_{\text{photon}}$ is the dominant energy of emitted or absorbed photons.

Given this range of $\lambda$ that allows the mass eigenstates of the theory to be dominated by the few-body sector of interest, and at the same time, does not remove the nonperturbative bound state physics of interest, we proceed with the BSPT and calculate all relativistic corrections to the zeroth order spectrum to some specified order in $\alpha$. The utility of this paper is that the relativistic corrections to order $\alpha^4$ in the ground state spin splittings can be calculated analytically. In the following subsections we describe the similarity hamiltonian approach more explicitly.

A. Step one: derivation of $H_R$

A self-contained discussion of the derivation of the renormalized Hamiltonian, $H_R$, will now be given. The similarity transformation we use to derive our effective interactions was developed by Glazek and Wilson \(^3\), and separately by Wegner \(^9\). An early application of this approach \(^3\) was a weak-coupling treatment of QCD.

The approach starts with a definition of the bare Hamiltonian:

$$H_B \equiv h + v_\lambda,$$

(8)

\(^5\) $M_N^2 \equiv (2m + B_N)^2$, where $N$ labels all the quantum numbers and $-B_N$ is the binding energy of the physical state of interest.
\[ v_{\Lambda} \equiv f_{\Lambda} \overline{v}_{\Lambda}, \]  
\[ \overline{v}_{\Lambda} \equiv v_{\text{can}} + \delta v_{\Lambda}, \]  
\[ H_{\text{can}} \equiv h + v_{\text{can}}, \]

where \( h \) is the free Hamiltonian, \( H_{\text{can}} \) is the canonical Hamiltonian, \( f_{\Lambda} \) is a regulating function, and \( \delta v_{\Lambda} \) are counterterms defined through the process of renormalization. The canonical Hamiltonian, \( H_{\text{can}} \), is written in terms of renormalized parameters and will be specified at the end of this section. The counterterms, \( \delta v_{\Lambda} \), are fixed by coupling coherence. Coupling coherence will be explained further below.

The free Hamiltonian, \( h \), is given by:

\[ h \equiv \int \sum_{p} \left\{ (b_{s}^{\dagger}(p)b_{s}(p) + d_{s}^{\dagger}(p)d_{s}(p)) \left( \frac{p_{+}^2 + m^2}{p^{+}} \right) + a_{s}^{\dagger}(p)a_{s}(p) \left( \frac{p_{+}^2}{p^{+}} \right) \right\}, \]

\[ h|\i\rangle = \varepsilon_{i}|\i\rangle, \quad \sum_{i} |i\rangle\langle i| = 1, \]

where the sum over \( i \) implies a sum over all Fock sectors and spins, and integrations over all momenta in the respective free states. We use the shorthand \( \int_{p} = \int \frac{d^{2}p^{+} dp^{+} \theta(p^{+})}{16\pi^{3} p^{+}}. \) \( m \) is the renormalized fermion mass.

The regulating function, \( f_{\Lambda} \), is defined to act in the following way:

\[ \langle i|f_{\Lambda}\overline{v}_{\Lambda}|j\rangle \equiv f_{\Lambda ij}\langle i|\overline{v}_{\Lambda}|j\rangle \equiv f_{\Lambda ij}\overline{v}_{\Lambda ij}, \]

\[ f_{\Lambda ij} \equiv \theta(\Lambda - |\Delta_{ij}|), \quad \Delta_{ij} \equiv \varepsilon_{i} - \varepsilon_{j}. \]

Note that this choice of a step function is not necessary and can lead to pathologies, however it is useful for doing analytical calculations.

Next, a similarity transformation is defined that acts on \( H_{\Lambda} \) and restricts the energy widths in the renormalized Hamiltonian, \( H_{R} \), to be below the effective cutoff, \( \lambda \). This transformation allows recursion relationships to be set up for \( H_{R} \), which can be written in the following general form:

\[ H_{R} = h + v_{\Lambda}, \]
\[ v_{\Lambda} \equiv f_{\Lambda}\overline{v}_{\Lambda}, \]
\[ \overline{v}_{\Lambda} = \overline{v}_{\Lambda}^{(1)} + \overline{v}_{\Lambda}^{(2)} + \cdots, \]
where the superscripts imply the respective order in \( v_{\text{can}} \).

Now, starting with the above bare Hamiltonian, we will describe this procedure more explicitly. The similarity transformation is defined to act on a bare cutoff continuum Hamiltonian, \( H_B \), in the following way:

\[
H_R \equiv S(\lambda, \Lambda) H_B S(\lambda, \Lambda)^\dagger, \tag{19}
\]

\[
S(\lambda, \Lambda) S(\lambda, \Lambda)^\dagger \equiv 1. \tag{20}
\]

This transformation is unitary, so \( H_B \) and \( H_R \) have the same spectrum:

\[
H_B |\Psi_B\rangle = E |\Psi_B\rangle, \tag{21}
\]

\[
S(\lambda, \Lambda) H_B S(\lambda, \Lambda)^\dagger |\Psi_B\rangle = E S(\lambda, \Lambda) |\Psi_B\rangle. \tag{22}
\]

Therefore, \( E \) is independent of the effective cutoff, \( \lambda \), if an exact transformation is made. \( E \) is also independent of the bare cutoff, \( \Lambda \), after the Hamiltonian is renormalized.

To put the equations in a differential framework, note that Eq. (19) is equivalent to the following equation

\[
\frac{dH_R}{d\lambda} = [H_R, T_\lambda], \tag{23}
\]

with

\[
S(\lambda, \Lambda) \equiv \mathcal{T} \exp \left( \int_{\lambda}^{\Lambda} T_{\lambda^\prime} d\lambda^\prime \right), \tag{24}
\]

where ‘\( \mathcal{T} \)’ orders operators from left to right in order of increasing energy scale, \( \lambda^\prime \). Eq. (23) is a first order differential equation, thus one boundary condition must be specified to obtain its solution. This boundary condition is the bare Hamiltonian: \( H_R|_{\lambda \to \Lambda} \equiv H_B \). \( H_B \) is determined by coupling coherence. Now we must specify \( T_\lambda \), the anti-hermitian (\( T_\lambda^\dagger = -T_\lambda \)) generator of energy width transformations. To define \( T_\lambda \) note that it is enough to specify how \( v_\lambda \) and \( h \) change with the energy scale \( \lambda \). This is seen by writing out Eq. (23) more explicitly using Eq. (17):

\[
\frac{dh}{d\lambda} + \frac{d}{d\lambda} (f_\lambda v_\lambda) = [h, T_\lambda] + [v_\lambda, T_\lambda]. \tag{25}
\]
We solve this perturbatively in $v_{can}$, choosing the transformation so that $h$ is independent of $\lambda$. Also, we demand that $T_\lambda$ and $v_\lambda$ do not contain any small energy denominators. Thus we define:

$$\frac{dh}{d\lambda} \equiv 0,$$

$$\frac{dv_\lambda}{d\lambda} \equiv [v_\lambda, T_\lambda]. \quad (26)$$

Eq. (27) is a choice such that $T_\lambda$ and consequently $v_\lambda$ do not allow any small energy denominators. These additional constraints determine $T_\lambda$ and $v_\lambda$, which are given by the following equations:

$$[h, T_\lambda] = \mathcal{T}_\lambda \frac{df_\lambda}{d\lambda} - \mathcal{F}_\lambda [v_\lambda, T_\lambda], \quad (28)$$

$$\mathcal{V}_\lambda = v_{can} + \int_\lambda^\Lambda [v_\lambda', T_\lambda'] d\lambda', \quad (29)$$

where $f_\lambda + \mathcal{F}_\lambda \equiv 1$. Eqs. (28) and (29) follow from Eqs. (25)-(27) and the boundary condition $H_{R}|_{\lambda=\Lambda} \equiv H_B$. Now we solve Eqs. (28) and (29) for $T_\lambda$ and $v_\lambda$. Given Eq. (16), we need to determine $v_\lambda$, and $H_B$ is known. The solution to Eqs. (28) and (29) is:

$$v_\lambda = \mathcal{V}_\lambda^{(1)} + \mathcal{V}_\lambda^{(2)} + \cdots, \quad (33)$$

$$T_\lambda = T_\lambda^{(1)} + T_\lambda^{(2)} + \cdots, \quad (34)$$

$$\delta v_\lambda = \delta v_\lambda^{(2)} + \delta v_\lambda^{(3)} + \cdots, \quad (35)$$

where the superscripts imply the respective order in the canonical interaction, $v_{can}$, and these quantities are given by:

$$\mathcal{V}_\lambda^{(1)} = v_{can}, \quad (36)$$

$$[h, T_\lambda^{(1)}] = v_{can} \frac{df_\lambda}{d\lambda}, \quad (37)$$

$$\mathcal{V}_\lambda^{(2)} = - \int_\lambda^\Lambda d\lambda' [v_\lambda', T_\lambda^{(1)}] + \delta v_\lambda^{(2)}, \quad (38)$$

$$[h, T_\lambda^{(2)}] = \mathcal{V}_\lambda^{(2)} \frac{df_\lambda}{d\lambda} - \mathcal{F}_\lambda [v_\lambda^{(1)}, T_\lambda^{(1)}], \quad (39)$$

$$\mathcal{V}_\lambda^{(3)} = - \int_\lambda^\Lambda d\lambda' \left( [v_\lambda', T_\lambda^{(2)}] + \left[ v_\lambda^{(2)}, T_\lambda^{(2)} \right] \right) + \delta v_\lambda^{(3)}, \quad (40)$$

$$[h, T_\lambda^{(3)}] = \mathcal{V}_\lambda^{(3)} \frac{df_\lambda}{d\lambda} - \mathcal{F}_\lambda \left( \left[ v_\lambda^{(1)}, T_\lambda^{(2)} \right] + \left[ v_\lambda^{(2)}, T_\lambda^{(1)} \right] \right), \quad (41)$$

$$\vdots$$
A general form of these effective interactions is:

\[
\mathfrak{v}_\lambda^{(i)} = - \sum_{j,k=1}^{\infty} \delta_{j+k,i} \int_{\Lambda} d\lambda' \left[ v_{\lambda'}^{(i)} T_{\lambda'}^{(i)} \right] + \delta v_{\lambda}^{(i)},
\]

for \( i = 2, 3, \cdots \), with \( \mathfrak{v}_\lambda^{(1)} = v_{\text{can}} \).

\( H_{\bar{\mu}} \) is renormalized by requiring it to satisfy coupling coherence [7]. A coupling coherent Hamiltonian satisfies:

\[
S(\lambda, \Lambda) H_{\bar{\mu}} (\Lambda, e_{\lambda}, m_{\Lambda}, c(e_{\lambda}, m_{\lambda})) S^\dagger (\lambda, \Lambda) = H_{\bar{\mu}} (\lambda, e_{\lambda}, m_{\lambda}, c(e_{\lambda}, m_{\lambda})) ,
\]

with the additional requirement that all dependent couplings (only one is shown in the argument of the Hamiltonians for simplicity) vanish when the independent marginal couplings are taken to zero. Note that there are only a finite number of independent couplings. This is a highly non-trivial constraint on the theory and to date has only been solved perturbatively. In this paper, Eq. (40) is solved to order \( e^2 \), which turns out to be fairly simple because \( e \) does not run until order \( e^3 \).

Now we write the solution to second order for \( \mathfrak{v}_{\lambda}^{(2)} \). From Eq. (35) we obtain:

\[
\mathfrak{v}_{\lambda ij}^{(2)} = \sum_k (v_{\text{can}})_{ik} (v_{\text{can}})_{kj} \left( g_{ikj}^{(\Lambda)} \frac{g_{ji}^{(\Lambda)}}{\Delta_{ik}} + \frac{g_{kj}^{(\Lambda)}}{\Delta_{jk}} \right) + \delta v_{\lambda ij}^{(2)},
\]

where

\[
g_{ikj}^{(\Lambda)} = \int_{\Lambda} d\lambda' n_{\lambda' k} \frac{d\lambda_{\lambda' i}}{d\lambda'} .
\]

\( \delta v_{\lambda ij}^{(2)} \) will be determined (in §III) by requiring the conditions of coupling coherence to be satisfied. These previous equations are valid for an arbitrary similarity function, \( f_\lambda \). In this work we will use \( f_{\lambda ij} = \theta(\lambda - |\Delta_{ij}|) \) (a convenient choice for doing analytical calculations). This gives:

\[
g_{ikj}^{(\Lambda)} = (f_{\lambda ik} - f_{\lambda jk}) \Theta_{ikj} ,
\]

\[
\Theta_{ikj} = \theta (|\Delta_{ik}| - |\Delta_{kj}|) .
\]

To complete this section we write the canonical QED Hamiltonian. We start by dividing \( H_{\text{can}} \) into free and interacting parts:

\[
H_{\text{can}} \equiv h + v_{\text{can}} ,
\]
where $h$ is given by Eq. (12). Starting with the QED lagrangian ($e > 0$, i.e. the charge of the electron is $-e$):

$$L_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} (i \not D - m) \psi , \quad \text{(46)}$$

with $i \not D \psi = \gamma^\mu (i \partial_\mu + e A_\mu) \psi , \quad \text{(46)}$

in a fixed gauge, $A^+ = 0$, the constrained degrees of freedom are removed explicitly, producing $v_{\text{can}}$. For details of the derivation see §IV.A of Ref. [1]. We use the two-component representation chosen by Zhang and Harindranath [10]. Below we write the resulting Hamiltonian completely. The field operator expansions and light-front conventions followed in this paper can be found in Appendix A. The canonical Hamiltonian is

$$P^-_{\text{can}} = H_{\text{can}} = h + v_{\text{can}} , \quad \text{(47)}$$

$$v_{\text{can}} = \int d^2 x^\perp dx^- \mathcal{H}_{\text{int}} , \quad \text{(48)}$$

where

$$\mathcal{H}_{\text{int}} = \mathcal{H}_{ee\gamma} + \mathcal{H}_{ee\gamma\gamma} + \mathcal{H}_{eeee} , \quad \text{(49)}$$

and

$$\mathcal{H}_{ee\gamma} = e \xi^\dagger \left\{ -2 \left( \frac{\partial^\perp}{\partial^+} \cdot A^+ \right) + \sigma \cdot A^+ \frac{\sigma \cdot \partial^\perp + m}{\partial^+} + \frac{\sigma \cdot \partial^\perp + m}{\partial^+} \sigma \cdot A^+ \right\} \xi , \quad \text{(50)}$$

$$\mathcal{H}_{ee\gamma\gamma} = -ie^2 \left\{ \xi^\dagger \sigma \cdot A^+ \frac{1}{\partial^+} (\sigma \cdot A^+ \xi) \right\} , \quad \text{(51)}$$

$$\mathcal{H}_{eeee} = 2e^2 \left\{ \left[ \frac{1}{\partial^+} (\xi^\dagger \xi) \right] \left[ \frac{1}{\partial^+} (\xi^\dagger \xi) \right] \right\} . \quad \text{(52)}$$

Note: $i = 1, 2$ only; e.g., $\sigma \cdot \partial^\perp = \sigma^i \partial^i = \sigma^1 (-\partial_1) + \sigma^2 (-\partial_2) ; \{ \sigma^i \}$ are the standard $2 \times 2$ Pauli matrices. Also, $h$ is given by Eq. (12).

B. Step two: diagonalization of $H_R$

The second step in the similarity hamiltonian approach is to solve for the spectrum of $H_R$. The Schrödinger equation for eigenstates of $H_R$ is:
\[
\sum_j \langle i | H_R | j \rangle \langle j | \Psi_{R,N}(\mathcal{P}) \rangle = E_N \langle i | \Psi_{R,N}(\mathcal{P}) \rangle .
\] (53)

See Eq. (13) and the comment immediately following it for an explanation of the notation. 'N' labels all the quantum numbers of the state, and is discrete for bound states and continuous for scattering states. \(E_N \equiv \frac{p_{\perp}^2 + M_N^2}{p_{\perp}^2} \), \(M_N^2 \equiv (2m + B_N)^2\), and '\(\mathcal{P}\)' is the total momentum of the state of physical interest (for this paper, positronium).

Solving this eigenvalue equation exactly is not feasible, because all sectors are still coupled:

\[
|\Psi_{R,N}(\mathcal{P})\rangle = \sum_i |i\rangle \langle i|\Psi_{R,N}(\mathcal{P})\rangle = \sum_{i'} |e\bar{e}(i')\rangle \langle e\bar{e}(i')|\Psi_{R,N}(\mathcal{P})\rangle + \\
+ \sum_{i'} |e\bar{e}\gamma(i')\rangle \langle e\bar{e}\gamma(i')|\Psi_{R,N}(\mathcal{P})\rangle + \sum_{i'} |e\bar{e}\bar{e}(i')\rangle \langle e\bar{e}\bar{e}(i')|\Psi_{R,N}(\mathcal{P})\rangle + \cdots .
\] (54)

We divide \(H_R\) into two pieces:

\[
H_R = H_o + (H_R - H_o) \equiv H_o + \mathcal{V},
\] (55)

diagonalize \(H_o\) exactly, and calculate corrections to the spectrum of \(H_o\) in BSPT with \(\mathcal{V}\). The \(H_o\) we choose for positronium is:

\[
H_o = h + \sum_{i'j'} |e\bar{e}(i')\rangle \langle e\bar{e}(i')|V_c|e\bar{e}(j')\rangle \langle e\bar{e}(j')|.
\] (56)

\(h\) is the free Hamiltonian given in Eq. (12). We are assuming photons couple perturbatively to the \(|e\bar{e}\rangle\) sector, which must be justified \textit{a posteriori}. \(V_c\) is the Coulomb interaction and will be written explicitly later. Note that the lowest order low-lying spectrum of the complete \(H_R\) is identical to that of \(h + V_c\) as long as the limit in Eq. (1) is taken.

We close this section by writing the standard BSPT Raleigh-Schrödinger formulae. For simplicity, we write the formulae for the non-degenerate case [11]:

\[
(H_o + \mathcal{V}) |\Psi_{R,N}(\mathcal{P})\rangle = E_N |\Psi_{R,N}(\mathcal{P})\rangle,
\] (57)

\[
H_o |\psi_N(\mathcal{P})\rangle = \mathcal{E}_N |\psi_N(\mathcal{P})\rangle,
\] (58)

\[
|\Psi_{R,N}(\mathcal{P})\rangle = |\psi_N(\mathcal{P})\rangle + \sum_{M \neq N} \frac{|\psi_M(\mathcal{P})\rangle \langle \psi_M(\mathcal{P})|\psi_N(\mathcal{P})\rangle}{\mathcal{E}_N - \mathcal{E}_M} + \mathcal{O}(V^2),
\] (59)

\[
E_N = \mathcal{E}_N + \frac{\langle \psi_N(\mathcal{P})|\psi_N(\mathcal{P})\rangle}{\langle \psi_N(\mathcal{P})|\psi_N(\mathcal{P})\rangle} + \sum_{M \neq N} \frac{|\psi_M(\mathcal{P})\rangle \langle \psi_M(\mathcal{P})|\psi_N(\mathcal{P})\rangle}{\mathcal{E}_N - \mathcal{E}_M} + \mathcal{O}(V^3),
\] (60)
where $\mathcal{P}$ is the total three-momentum of the state and “$N$” labels the total mass of the state. These general formulae will be used below in §III to solve for positronium’s spin structure. Note that for the light-front case: $E_N = \frac{p^2 + M^2_N}{2p^+}$ and $E'_N = \frac{p^2 + M^2_N}{2p^+}$.

III. POSITRONIUM’S SPIN STRUCTURE

Now we will apply the procedure outlined in §II to obtain positronium’s ground state spin splittings to order $\alpha^4$. We will now give a brief overview of this section. First, we derive $H_R$ to second order in $\alpha$. This includes a discussion of the effective fermion self-energy, but the photon self-energy and electromagnetic coupling do not run at this order. Then we move on to the diagonalization of $H_R$. This starts with a discussion of our zeroth order Hamiltonian, $\mathcal{H}_o$, which will be treated nonperturbatively. This includes a discussion of a coordinate change that takes $(x \in [0,1]) \rightarrow (\kappa_z \in [\infty, \infty])$, which allows easier identification of $\mathcal{H}_o$. We solve for the spectrum of $\mathcal{H}_o$ exactly, which among other things, fixes the $\alpha$-scaling of the momenta in the matrix elements in BSPT. Then we move on to a derivation of the perturbative effects coming from low-energy (energy transfer below $\lambda$) photon emission, absorption and annihilation at order $\alpha^2$, which includes a discussion of the full electron and positron self-energies and a derivation to order $\alpha^2$ of the complete exchange and annihilation interactions. Given this, we determine the range of $\lambda$ that allows the effects of low-energy (energy transfer below $\lambda$) photon emission and absorption to be transferred to the effective interactions in the $|e\bar{e}\rangle$ sector alone, and at the same time, does not cut into the nonperturbative features of the solutions of $\mathcal{H}_o$. Finally, we proceed with BSPT in $\mathcal{V}$ noting that all shifts appear in the few-body sector, $|e\bar{e}\rangle$, alone.

A. Derivation of $H_R$ to second order

From §II, the final renormalized Hamiltonian to second order is given by:

$$\langle i|H_R|j \rangle = f_{\lambda ij} \left\{ h_{ij} + (v_{can})_{ij} + \sum_k (v_{can})_{ik}(v_{can})_{kj} \left( g^{(\Lambda\Lambda)}_{ikj} \frac{\Delta_{ik}}{\Delta_{ik}} + g^{(\Lambda\Lambda)}_{jki} \frac{\Delta_{jk}}{\Delta_{jk}} \right) + \delta v^{(2)}_{\Lambda ij} + \mathcal{O}(\alpha^3) \right\} . \quad (61)$$

$g^{(\Lambda\Lambda)}_{ikj}$ is given in Eq. (42) and $v_{can}$ is given in Eq. (48).
1. Renormalization issues

The form of $\delta v^{(2)}_\Lambda$ follows from the constraint that $H_R$ satisfies coupling coherence. To order $e^2$ the fermion and photon masses run, but the coupling does not. First, we discuss the result for the electron self-energy coming from the second-order effective interactions in $H_R$. We skip the tedious but simple details of the calculation of the matrix element, but the interested reader should consult Appendix A, where we have collected our light-front conventions for our field expansions and commutation relations. We use the rules in Ref. [13] to calculate our matrix elements in this work. Specifically, we calculate a free matrix element of $H_R$ given in Eq. (61) in the electron self-energy channel. The results are summarized by the following equation which also shows how the running electron mass squared, $m^2_\lambda$, is defined:

$$\frac{\kappa^2 + m^2}{x} = \frac{\langle e(3)\pi(4)|H_R|e(1)\pi(2)\rangle_{\text{self energy}}}{\langle e(3)\pi(4)|e(1)\pi(2)\rangle} \mathcal{P}^+ - \mathcal{P}^{\perp,2} - \frac{\kappa^2 + m^2}{1 - x} x,$$

where

$$\delta \Sigma^{(2)}_{\lambda'} = \frac{\alpha}{2\pi} \left( -3\lambda'^2 x + m^2 \frac{2 + \lambda'^2 x}{2} + \frac{m^2 (m^2 + \lambda^2 x)}{m^2 + \lambda'^2 x} - 3m^2 \log \left( \frac{m^2 + \lambda'^2 x}{m^2} \right) \right)$$

$$+ \frac{\alpha}{2\pi} \left( 2\lambda^2 x \log \left( \frac{m^2 + \lambda^2 x}{\lambda'^2 x} \right) + 2\lambda'^2 x \log \left( \frac{x}{\epsilon} \right) \right).$$

(63)

In the respective $\delta \Sigma^{(2)}_{\lambda}$ terms of Eq. (62), $\lambda' = \Lambda$ and $\lambda$. In these formulae,

$$p_{\text{electron}} = p_1 = (xP^+, \kappa + xP^\perp) \quad \text{and} \quad P_{\text{positronium}} = \mathcal{P} = (\mathcal{P}^+, \mathcal{P}^\perp),$$

and $|e(1)\rangle$ ( or $|\pi(1)\rangle$ ) is a state of the free Hamiltonian, $h$, with spin and momenta coordinates labeled by “1.” Note that we are forced to introduce an infrared regulator, $\epsilon$. This is introduced by requiring all longitudinal momenta (real, virtual or instantaneous) to satisfy:

$$\frac{|p^-|}{p^+} \geq \epsilon = 0_+,$$

(64)

where $\mathcal{P}^+$ is the total longitudinal momentum of the physical state. The absolute value sign is required for instantaneous lines. Physical results can not depend on this infrared regulator, $\epsilon$, and in this QED

Note that the energy dependence on the electron’s relative transverse momentum, $\kappa$, does not change.
calculation we show that treating the photon perturbatively leads to an exact cancellation of this infrared divergence in the above running electron mass squared, $m_\lambda^2$.

We constrain the electron mass to run coherently with the cutoff which from Eq. (40) and Eq. (62) amounts to the requirement:

$$m_\lambda^2 = \left[ m^2 - (\delta \Sigma^{(2)}_\lambda - \delta \Sigma^{(2)}_\lambda') + \delta v^{(2)}_\lambda + \mathcal{O}(\epsilon^4) \right] = \left[ m^2 + \delta v^{(2)}_\lambda + \mathcal{O}(\epsilon^4) \right]_{\Lambda \to \lambda} .$$

This fixes the mass counterterm:\footnote{Actually, any finite $\mathcal{O}(\epsilon^2)$ scale independent term could be added to the counterterm, and Eq. (65) would still be satisfied. However, higher order renormalization reveals that this term would not be coherent, and so is excluded.}

$$\delta v^{(2)}_\lambda = \delta \Sigma^{(2)}_\lambda + \mathcal{O}(\epsilon^4) ,$$

and to second order the fermion mass renormalization is complete.

For arbitrary $\lambda$, the photon mass also runs at order $\epsilon^2$. The discussion follows that of the electron mass except for the fact that the running photon mass is infrared finite. For $\lambda^2 < (2m)^2$, the photon mass does not run because pair production is no longer possible. Thus, for $\lambda^2 \ll m^2$ the photon mass is zero to all orders in perturbation theory. There are additional difficulties with marginal operators that are encountered at $\mathcal{O}(\epsilon^3)$, but this is beyond the focus of this paper.

### 2. $H_R$ to order $\epsilon^2$: exchange and annihilation channels

To complete the derivation of $H_R$ to second order we need to write the coherent interactions for the exchange and annihilation channels in the $|e\bar{e}|$ sector. At second order, these come from tree level diagrams, with no divergences or running couplings, thus the results follow from:

$$\delta v^{(2)}_\lambda \equiv - \int_{\Lambda}^{\infty} [v^{(1)}_{\lambda'}, T^{(1)}_{\lambda'}] d\lambda' .$$

To show that $\delta v^{(2)}_\lambda$ produces a coherent interaction recall Eq. (23). We have:

$$\nabla_\lambda = v_{can} - \int_{\Lambda}^{\infty} [v^{(1)}_{\lambda'}, T^{(1)}_{\lambda'}] d\lambda' - \int_{\Lambda}^{\infty} [v^{(1)}_{\lambda'}, T^{(1)}_{\lambda'}] d\lambda' + \mathcal{O}(\epsilon^3)$$

$$= v_{can} - \int_{\Lambda}^{\infty} [v^{(1)}_{\lambda'}, T^{(1)}_{\lambda'}] d\lambda' + \mathcal{O}(\epsilon^3)$$

\(68\)
\[
\begin{align*}
\text{ Exchange Channel } \\
V_{\lambda, \text{exchange}} &= \left[ v_{\text{can}} - \int_{\Lambda}^{\infty} \left[ v_{\lambda'}^{(1)}(x) T_{\lambda'}^{(1)}(x) \right] d\lambda' + \mathcal{O}(e^3) \right] \bigg|_{\Lambda \to \lambda},
\end{align*}
\]

which satisfies the coupling coherence constraint, Eq. (69). At second order this seems trivial, but at higher orders the constraint that only \( e \) and \( m \) run independently with the cutoff places severe constraints on the Hamiltonian.

Given this second order interaction, the free matrix elements of \( H_{\hat{\pi}} \), shown in Eq. (51), in the exchange and annihilation channels are:

\[
\begin{align*}
V_{\lambda, \text{exchange}} &= \frac{\langle e(3)\pi(4)|H_{\hat{\pi}}|e(1)\pi(2)\rangle_{\text{exchange}}}{16\pi^3\delta^3(p_1 + p_2 - p_3 - p_4) \sqrt{x'x(1-x)(1-x')}} \\
&= V_1 + V_2 + \mathcal{O}(e^4),
\end{align*}
\]

where

\[
\begin{align*}
V_1 &= -e^2 N_1 \theta \left( \lambda^2 - \left| \mathcal{M}^2_o - \mathcal{M}^2_{o'} \right| \right) \\
&\quad \times \left( \frac{\theta(|\Delta_1| - |\Delta_2|)}{DEN_1} \theta(|\Delta_1| - \lambda^2) + \frac{\theta(|\Delta_2| - |\Delta_1|)}{DEN_2} \theta(|\Delta_2| - \lambda^2) \right),
\end{align*}
\]

\[
\begin{align*}
V_2 &= -e^2 \theta \left( \lambda^2 - \left| \mathcal{M}^2_o - \mathcal{M}^2_{o'} \right| \right) \left( \frac{4}{(x - x')^2} \delta_{s_1 s_2} \delta_{s_2 s_4} \right).
\end{align*}
\]

The variables are defined as follows (see Figure 1 also):

- \( p_1 = (x\mathcal{P} - \kappa + x\mathcal{P}' \perp) \), \( p_2 = (1 - x)\mathcal{P} - \kappa + (1 - x)\mathcal{P}' \perp \)
- \( p_3 = (x'\mathcal{P} + \kappa' + x'\mathcal{P}' \perp) \), \( p_4 = (1 - x')\mathcal{P} + \kappa' + (1 - x')\mathcal{P}' \perp \)
- \( N_1 = \delta_{s_1 s_2} \delta_{s_2 s_4} T_{1}^{\perp} \cdot T_{2}^{\perp} - 2m^2 \delta_{s_{1}s_{4}} \delta_{s_{2}s_{1}} \delta_{s_{2}s_{4}} \frac{(x - x')^2}{xx'(1 - x)(1 - x')} \)
  \quad + \text{im} \sqrt{2}(x - x') \left( \frac{s_1}{xx'} \delta_{s_{1}s_{4}} \delta_{s_{2}s_{4}} \epsilon_{s_{1}}^{\perp} \cdot T_{1}^{\perp} + \frac{s_2}{(1 - x')(1 - x')} \delta_{s_{2}s_{4}} \delta_{s_{2}s_{4}} \epsilon_{s_{2}}^{\perp} \cdot T_{2}^{\perp} \right)
- \kappa^i(s) = \kappa^i + i s \epsilon_{ij} \kappa^j (s = \pm 1 \text{ and } \overrightarrow{s} \equiv -s) ; \epsilon_{12} = -\epsilon_{21} = 1 \text{ , } \epsilon_{11} = \epsilon_{22} = 0
- \Delta_1 = \frac{DEN_1}{x' - x}, \text{ } \Delta_2 = \frac{DEN_2}{x' - x}
- DEN_1 = \frac{(\kappa \mathcal{P} - \kappa' x')^2}{xx'}, \text{ } DEN_2 = \frac{(\kappa \mathcal{P} - \kappa' x')^2}{xx'}
- \mathcal{M}^2_o = \frac{\kappa^2 + m^2}{x(1 - x)}, \text{ } \mathcal{M}^2_{o'} = \frac{\kappa'^2 + m^2}{x'(1 - x')}. 
\]

14
Annihilation Channel

\[ V_{\lambda, \text{annihil}} = \frac{\langle e(3)\bar{e}(4)|H_{R}|e(1)\bar{e}(2)\rangle_{\text{annihilation}}}{16\pi^3\delta^3(p_1 + p_2 - p_3 - p_4)\sqrt{xx'(1 - x)(1 - x')}} = V_3 + V_4 + O(e^4), \]  

(73)

where

\[ V_3 = e^2 N_2 \theta \left( \lambda^2 - |M_0^2 - M_o^2|^2 \right) \left( \frac{\theta \left( M_0^2 - M_o^2 \right)}{M_0^2} + \frac{\theta \left( M_o^2 - M_0^2 \right)}{M_o^2} \right), \]  

(74)

\[ V_4 = 4e^2 \theta \left( \lambda^2 - |M_0^2 - M_o^2|^2 \right) \delta_{s_1s_2}\delta_{s_3s_4}, \]  

(75)

and

\[
N_2 = \delta_{s_3s_4}\delta_{s_1s_2} \left( T_3^+ \cdot T_3^- + \delta_{s_1s_2}\delta_{s_3s_4} \frac{2m^2}{xx'(1 - x)(1 - x')} \right) \\
+ im\sqrt{2} \left( \frac{s_1}{x(1 - x)} \delta_{s_1s_2}\delta_{s_3s_4} \epsilon_{s_1} \cdot T_3^- \right) \delta_{s_1s_2}\delta_{s_3s_4} \epsilon_{s_4}^{*} \cdot T_4^+.
\]

\[
T_3^i = \frac{\kappa^i(s_3)}{1 - x'} - \frac{i\kappa^i(s_3)}{x'}, \quad T_4^i = \frac{i\kappa^i(s_1)}{1 - x} - \frac{\kappa^i(s_1)}{x}.
\]

V_2 and V_4 are canonical instantaneous exchange and annihilation interactions, respectively, with widths restricted by the regulating function, \( f_\lambda \). V_1 and V_3 are effective interactions that arise because photon emission and annihilation have vertices with widths restricted by the regulating function, \( f_\lambda \).

B. Diagonalization of \( H_R \)

First we discuss the lowest order spectrum of \( H_R \), after which we discuss BSPT, renormalization and a limiting procedure which allows the effects of low-energy (energy transfer below \( \lambda \)) emission to be transferred to the \( |e\bar{e}\rangle \) sector alone.

1. \( H_o \), a coordinate change and its exact spectrum

\( H_o \) in the \( |e\bar{e}\rangle \) sector is
\[ \mathcal{H}_o = h + V_C, \quad (76) \]

where \( h \) is the free Hamiltonian given in Eq. (12), and \( V_C \) is given by (using the same variables defined below Eq. (72); note, \( \kappa_z \) is defined below by Eq. (78))

\[ V_C \equiv \langle e(3)e(4)|V_C|e(1)e(2) \rangle \equiv -\frac{16\pi^3\delta_{s_1s_2}\delta_{s_3s_4}}{(\kappa - \kappa')^2 + (\kappa_z - \kappa'_z)^2}. \quad (77) \]

In all other sectors we choose \( \mathcal{H}_o = h. \) \( \mathcal{H}_o \) in the \( |e\bar{e}\rangle \) sector was motivated from the form of our second order renormalized Hamiltonian, \( \mathcal{H}_R, \) and arises from a nonrelativistic limit of the instantaneous photon exchange interaction combined with the two time orderings of the dynamical photon exchange interaction.

We choose it to simplify positronium bound-state calculations. Other choices are possible, and must be used to study problems such as photon emission. Later, in BSPT this choice is shown to produce the leading order contribution to positronium’s mass as long as the limit in Eq (1) is taken.

The coordinates \( \kappa_z \) and \( \kappa'_z \) in Eq. (77) follow from a standard coordinate transformation that takes the range of longitudinal momentum fraction, \( x \in [0, 1] \) to \( \kappa_z \in [-\infty, \infty]. \) This coordinate change is:

\[ x = \frac{1}{2} + \frac{\kappa_z}{2\sqrt{\kappa^2 + \kappa'^2 + m^2}}. \quad (78) \]

We introduce a new three-vector defined as:

\[ p = (\kappa, \kappa_z). \quad (79) \]

Note that

\[ \mathcal{M}_o^2 = \frac{\kappa^2 + m^2}{x(1-x)} = 4(m^2 + p^2) \quad (80) \]

is invariant with respect to rotations in the space of vectors \( p. \) The nonrelativistic assumption of Eqs. (2) and (3) in terms of this three-vector becomes:

\[ \frac{|p|}{m} = \mathcal{O}(\alpha). \quad (81) \]

Note the simple forms that our “exchange channel denominators” take in the nonrelativistic limit:

\[ DEN_1 = (p - p')^2 - \frac{(\kappa_z - \kappa'_z)(p^2 - p'^2)}{m} + \mathcal{O} \left[ \frac{|\mathbf{P}|^5}{m^2} \right], \quad (82) \]

\[ DEN_2 = (p - p')^2 + \frac{(\kappa_z - \kappa'_z)(p^2 - p'^2)}{m} + \mathcal{O} \left[ \frac{|\mathbf{P}|^5}{m^2} \right]. \quad (83) \]
Also note the form that the longitudinal momentum fraction transferred between the electron and positron
takes:

\[ x - x' = \frac{\kappa_z - \kappa'_z}{2m} + \frac{(p'^2 \kappa'_z - p^2 \kappa_z)}{4m^3} + O \left( \frac{p}{m} \right)^5. \]  

(84)

These formulae are used throughout this paper.

Now we describe the leading Schrödinger equation. We seek solutions of the following eigenvalue
equation:

\[ \mathcal{H}_o |\psi_N(P)\rangle = \mathcal{E}_N |\psi_N(P)\rangle, \]

(85)

where \( \mathcal{E}_N \equiv \frac{\mathcal{P}^2 + \mathcal{M}_N^2}{p^2} \). \( \mathcal{H}_o \) is diagonal with respect to the different particle sectors, thus we can solve
Eq. (84) sector by sector. In all sectors other than \(|e\bar{e}\rangle\), \( \mathcal{H}_o = \hbar \), and the solution is trivial. For the \(|e\bar{e}\rangle\) sector, a general \(|\psi_N(P)\rangle\) is:

\[ |\psi_N(P)\rangle = \sum_{s_1 s_2} \int_{p_1 p_2} \sqrt{p_1^+ p_2^+} 16\pi^3 \delta^3(P - p_1 - p_2) \tilde{\phi}_N(x_1 s_1 s_2) b_{s_1}^\dagger(p_1) d_{s_2}^\dagger(p_2) |0\rangle, \]

(86)

with norm:

\[ \langle \psi_N(P)|\psi_{N'}(P')\rangle \equiv \delta_{NN'} 16\pi^3 p^+ \delta^3(P - P') \]

\[ \Rightarrow \sum_{s_1 s_2} \int_{p_1} \int_{p_2} \frac{d^2 p}{16\pi^3} \tilde{\phi}_N(x_1 s_1 s_2) \tilde{\phi}_{N'}(x_2 s_2 s_2) = \delta_{NN'}. \]

The tilde on \( \tilde{\phi}_N \) will be notationally convenient below. In the \(|e\bar{e}\rangle\) sector, Eq. (85) becomes:

\[ \left( \mathcal{M}_N^2 - \frac{\kappa^2 + m^2}{x'(1 - x')} \right) \tilde{\phi}_N(x' \kappa' s_3 s_4) = \sum_{s_1 s_2} \int_{p_1} \int_{p_2} \frac{dx}{16\pi^3} V_C \tilde{\phi}_N(x_1 s_1 s_2). \]

(87)

After the above coordinate change, this becomes:

\[ \left( \mathcal{M}_N^2 - 4(m^2 + p'^2) \right) \phi_N(p' s_3 s_4) = \sum_{s_1 s_2} \int d^3p \frac{J(p)J(p')}{16\pi^3} V_C \phi_N(p s_1 s_2), \]

(88)

where the tilde on the wavefunction has been removed by redefining the norm in a convenient fashion:

\[ \delta_{NN'} = \sum_{s_1 s_2} \int_{p_1} \int_{p_2} \frac{dx}{16\pi^3} \tilde{\phi}_N^*(x_1 s_1 s_2) \tilde{\phi}_{N'}(x_2 s_2 s_2) = \sum_{s_1 s_2} \int d^3p \frac{J(p)}{16\pi^3} \tilde{\phi}_N^*(p s_1 s_2) \tilde{\phi}_{N'}(p s_1 s_2) \]

\[ \equiv \sum_{s_1 s_2} \int d^3p \phi_N^*(p s_1 s_2) \phi_{N'}(p s_1 s_2). \]

(89)
and the Jacobian of the transformation is given by:

\[ J(p) \equiv \frac{dx}{d\kappa_z} = \frac{\kappa^2 + m^2}{2(p^2 + m^2)^{3/2}}. \]  

Note that the Jacobian factor in Eq. (88) satisfies:

\[ \sqrt{J(p)J(p')} = \frac{1}{2m} \left( 1 - \frac{p^2 + 2\kappa^2 + p'^2 + 2\kappa'^2}{4m^2} + O \left( \frac{p^4}{m^4}, \frac{p'^4}{m^4}, \ldots \right) \right). \]  

Before defining \( H_\alpha \) in the \( |e\bar{e}\rangle \) sector we mention a subtle but important point in the definition of \( H_\alpha \). \( H_\alpha \) in the \( |e\bar{e}\rangle \) sector will not be defined by Eq. (88). Rather, it will be defined by taking the leading order nonrelativistic expansion of the Jacobian factor in Eq. (88). This gives

\[ (\mathcal{M}_N^2 - 4(m^2 + p'^2)) \phi_N(p's_3s_4) = \sum_{s_1s_2} \frac{d^3p}{10\pi^4} V_C \phi_N(p's_1s_2), \]  

where \( V_C \) is defined in Eq. (77). This \( H_\alpha \) will be diagonalized exactly, and the subsequent BSPT will be set up as an expansion in \( V \equiv H_\alpha - H_\alpha \). First, we discuss the exact diagonalization of \( H_\alpha \).

Putting the expression for \( V_C \) into Eq. (92) results in the following equation:

\[ \left( -B_N + \frac{p'^2}{m} \right) \phi_N(p's_3s_4) = \frac{\alpha}{2\pi^2} \int \frac{d^3p}{(p - p')^2} \phi_N(p's_3s_4). \]  

This is recognized as the familiar nonrelativistic Schrödinger equation for positronium. Note that we have defined a leading order binding energy, \( B_N \), as:

\[ \mathcal{M}_N^2 \equiv 4m^2 + 4mB_N. \]  

Note the difference in the definition of this leading order binding energy and the full binding energy as given by \( M_N^2 \equiv (2m + B_N)^2 \) (see Appendix D for further discussion of this difference).

To proceed with the solution of Eq. (93) note that there is no spin dependence in the operator so the spin part just factors out:

\[ \phi_{\mu,s_e,s_{\bar{e}}}(p's_3s_4) \equiv \phi_{\mu}(p')\delta_{s_e,s'}\delta_{s_{\bar{e}}s_4}. \]  

We rewrote \( N \) as \((\mu, s_e, s_{\bar{e}})\), where \((s_e, s_{\bar{e}})\) label the spin quantum numbers and \( \mu \) labels all other quantum numbers, which are discrete for the bound states and continuous for the scattering states.
The solutions to Eq. (93) are well known. For $B_N < 0$, following Fock [12], we change coordinates according to

$$mB_N \equiv -e_n^2,$$

$$u \equiv (u_0, u),$$

$$u_0 \equiv \cos(\omega) = \frac{e_n^2 - p^2}{e_n^2 + p^2},$$

$$u \equiv \frac{p}{p} \sin(\omega) = \sin(\omega) (\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)) = \frac{2e_n p}{e_n^2 + p^2}. $$

Useful relations implied by this coordinate change are in Appendix B. Note that in our notation we anticipate that $\mu$ will be given by $(n, l, m)$, the usual principal and angular momentum quantum numbers, and that the leading order binding will depend only on the principal quantum number, $n$. Given this, Eq. (93) becomes

$$\psi_\mu(\Omega') = \frac{\alpha}{2\pi^2} m \int \frac{d\Omega}{|u - u'|^2} \psi_\mu(\Omega),$$

where

$$\psi_\mu(\Omega) = \frac{(e_n^2 + p^2)^2}{4(e_n^2)^2} \phi_\mu(p).$$

Using Eq. (182) of Appendix B, this is seen to have the following solution:

$$\psi_\mu(\Omega) = Y_\mu(\Omega) \quad \text{with} \quad \frac{\alpha}{2\pi^2} \frac{m}{2e_n^2} \frac{2\pi^2}{n} = 1,$$

where $Y_\mu(\Omega)$ is a hyperspherical harmonic. Thus,

$$e_n = \frac{m\alpha}{2n} \quad \text{and} \quad B_N = -\frac{m\alpha^2}{4n^2}.$$

This is the standard nonrelativistic solution for the bound states of positronium to order $\alpha^2$. This completes the solution of $H_0$ for the bound states. The scattering $|e\bar{\gamma}\rangle$ states are also needed in our BSPT calculation. We use propagator techniques to include these scattering states where required (see Appendix C).
Here we use the BSPT formulae (appropriately generalized to the degenerate case) of §II.B to analyze positronium’s spin structure. The potential to be used in BSPT is:

\[ V = H_R - H_o, \]

where the eigenvalue equation for \( H_o \) is given by Eq. (92), and \( H_R \) to second order is given in §III.A. We will be perturbing about the nonperturbative eigenstates of \( H_o \).

First, we discuss electron mass renormalization. In second order BSPT there is an electron mass shift coming from the \( f_\lambda v_{\text{can}} \) part of \( H_R \), with \( v_{\text{can}} \) given by \( \int d^2x^+ dx^- H_{e\gamma} \) (see Eq. (50)). This is photon emission and absorption restricted by the regulating function, \( f_\lambda \). The calculation is similar to that of §III.A.1. Assuming \( \langle M_N^2 - M_o^2 \rangle = O(e^2) \), this electron mass-squared shift is

\[ \delta m^2 = -\delta \Sigma^{(2)}_\lambda + O(e^4). \]

(105)

\( \delta \Sigma^{(2)}_\lambda \) is the same function that was defined in Eq. (63). Using this result (Eq. (105)) one obtains:

\[ \langle M_N^2 - M_o^2 \rangle = \langle 4m^2 + 4mB_N - 4(m^2 + p^2) \rangle = O(e^4), \]

and our initial assumption is satisfied. When this is combined with the only other second order electron mass shift, \( m_\lambda^2 \), of Eq. (62) we have, for the full electron mass-squared, to second order,

\[ m_e^2 = m_\lambda^2 + \delta m^2 \]

\[ = \left[ m^2 - \left( \delta \Sigma^{(2)}_\lambda - \delta \Sigma^{(2)}_\Lambda \right) + \delta v^{(2)}_\Lambda \right] + \left[ -\delta \Sigma^{(2)}_\lambda \right] + O(e^4) \]

\[ = m^2 + O(e^4). \]

(106)

In this last step we recalled the result from Eq. (64). We see that to second order, the full electron mass is given by the electron mass in the free Hamiltonian, \( h \). Also, as promised below Eq. (14), we see that treating photons perturbatively has led to an exact cancellation of the infrared divergence in the running mass, \( m_\lambda^2 \): the full electron mass, \( m_e^2 \), to second order is infrared finite.

Now we move on to the discussion of BSPT. The only channels to order \( e^2 \) are exchange and annihilation. Parts of these effective interactions are given in §III.A.2. We also need to include the perturbative
mixing of the $|e\bar{e}\gamma\rangle$ and $|\gamma\rangle$ sectors with the $|e\bar{e}\rangle$ sector arising from $f_\lambda v_{\text{can}}$, with $v_{\text{can}} = \int d^2 x^\perp d x^- \mathcal{H}_{ee\gamma}$.

In second order BSPT this gives rise to the following effective interactions that must be added to $V_{\lambda,\text{exchange}}$ and $V_{\lambda,\text{annihil}}$ of Eqs. (70) and (73) respectively.

**Exchange Channel**

$$ V_5 = -e^2 N_1 \theta \left( \lambda^2 - |\Delta_1| \right) \theta \left( \lambda^2 - |\Delta_2| \right) \frac{1}{DEN_3}, $$

(107)

with

$$ DEN_3 = (\kappa - \kappa')^2 + \frac{1}{2}(x - x') A + |x - x'| \left( \frac{1}{2} (\mathcal{M}_o^2 + \mathcal{M}'_o^2) - \mathcal{M}_N^2 \right), $$

and

$$ A = \frac{\kappa^2 + m^2}{1-x} - \frac{\kappa'^2 + m^2}{1-x'} + \frac{\kappa'^2 + m^2}{x'} - \frac{\kappa^2 + m^2}{x}. $$

**Annihilation Channel**

$$ V_6 = e^2 N_2 \frac{\theta \left( \lambda^2 - M_o^2 \right) \theta \left( \lambda^2 - M'_o^2 \right)}{M_N^2}. $$

(108)

Note that in a nonrelativistic expansion (after the coordinate change of Eq. (78)), the above “exchange channel denominator” becomes

$$ DEN_3 = (p - p')^2 + |x - x'| \left( \frac{1}{2} (\mathcal{M}_o^2 + \mathcal{M}'_o^2) - \mathcal{M}_N^2 \right) + O \left[ \left( \frac{p}{m} \right)^6 m^2 \right]. $$

(109)

The full exchange and annihilation channel interactions to order $e^2$ are

$$ V_{\text{exchange}} = V_{\lambda,\text{exchange}} + V_5, $$

(110)

$$ V_{\text{annihil}} = V_{\lambda,\text{annihil}} + V_6, $$

(111)

where Eqs. (70) and (73) give $V_{\lambda,\text{exchange}}$ and $V_{\lambda,\text{annihil}}$, respectively.

One way to summarize the results, recalling the form of Eq. (88) and the norm in Eq. (89), is to state: the full order $e^2$ effective interactions give rise to the following first order BSPT shift of the bound-state mass-squared spectrum of $\mathcal{H}_o$: 

21
\[ \delta^{(1)} M^2(s_3, s_4; s_1, s_2) = \langle \phi_{n,l,m, s_3, s_4} | V | \phi_{n,l,m, s_1, s_2} \rangle = \int d^3p \, d^3p' \phi^*_{n,l,m_1}(p') V(p', s_3, s_4; p, s_1, s_2) \phi_{n,l,m_1}(p) , \quad (112) \]

where

\[ V(p', s_3, s_4; p, s_1, s_2) = \sqrt{\frac{J(p)}{16\pi^3}} \frac{1}{16\pi^3} (V_{\text{exchange}} + V_{\text{annihil}}) - \sqrt{\frac{1}{2m}} \frac{1}{16\pi^3} V_C (113) \]

The Dirac notation in Eq. (112) will be used in the remainder of this paper. See Eqs. (77), (110) and (111) for \( V_C, V_{\text{exchange}} \) and \( V_{\text{annihil}} \) respectively. The interaction \( V \) must be diagonalized in the degenerate spin space following the standard rules of degenerate BSPT. Note that \( V \) needs to be considered in second order BSPT in this paper also.

The diagonalization of \( V \) in the degenerate spin space follows shortly, but first recall from §II the range of \( \lambda \) that allows the effects of low-energy (energy transfer below \( \lambda \)) photon emission and absorption to be transferred to the effective interactions in the \( |eE| \) sector alone, and at the same time does not remove the nonperturbative bound-state physics of interest:

\[ \frac{|M_N^2 - (2m)^2|}{P^+} \ll \frac{\lambda^2}{P^+} \ll q_{\text{photon}} . \quad (114) \]

After the solutions of \( H_0 \) are known the \( \alpha \)-scaling in all BSPT matrix elements is known and the bounds in Eq. (114) become

\[ m^2 \alpha^2 \ll \lambda^2 \ll m^2 \alpha . \quad (115) \]

This is satisfied under the following limit:

\[ \lambda^2 \longrightarrow \text{a fixed number} , \quad (116) \]

\[ \frac{m^2 \alpha^2}{\lambda^2} \longrightarrow 0 , \quad (117) \]

\[ \frac{m^2 \alpha}{\lambda^2} \longrightarrow \infty . \quad (118) \]

Given the nonrelativistic limit:

\[ \alpha \longrightarrow 0 , \quad (119) \]

\[ \frac{m^2}{\lambda^2} \longrightarrow \infty , \quad (120) \]
this implies

\[ \frac{m^2}{\lambda^2} \propto \alpha^{-\frac{k}{2}}, \quad (121) \]

where

\[ 2 < k < 4 \quad . \quad (122) \]

Note that this “window of opportunity” is available to us because one, we have introduced an adjustable similarity hamiltonian energy scale, \( \frac{\lambda^2}{\alpha^2} \), into the theory, and two, \( QED \) is a theory with two dynamical energy scales, \( m^2/\alpha^2 \) and \( m^2/\alpha \), a fact known for a long time, and the reason that \( QED \) calculations have been so successful over the years.

Given the above limit (Eqs. (116)-(118)),

\[ \theta \left( \lambda^2 - 4|p^2 - p'|^2 \right), \theta \left( |\Delta_1| - \lambda^2 \right), \theta \left( |\Delta_2| - \lambda^2 \right) \rightarrow 1 \quad (123) \]

\[ \theta \left( 4(p^2 + m^2) - \lambda^2 \right), \theta \left( 4(p'^2 + m^2) - \lambda^2 \right) \rightarrow 1 \quad (124) \]

\[ \theta \left( \lambda^2 - |\Delta_1| \right), \theta \left( \lambda^2 - |\Delta_2| \right) \rightarrow 0 \quad (125) \]

\[ \theta \left( \lambda^2 - 4(p^2 + m^2) \right), \theta \left( \lambda^2 - 4(p'^2 + m^2) \right) \rightarrow 0 . \quad (126) \]

Now we proceed with the diagonalization of \( V \) in the degenerate spin space (see Eqs. (112) and (113)).

We will calculate all corrections to order \( \alpha^4 \) that arise in the spin splitting structure of the ground state spectrum of \( \mathcal{H}_0 \) via BSPT. First, we write \( V \) more explicitly given the above limits in Eqs. (123)-(126):

\[ V(p', s_3, s_4; p, s_1, s_2) = \frac{1}{16\pi^3} \frac{1}{2m} \left( 1 - \frac{p^2 + 2\kappa_2^2 + p'^2 + 2\kappa'_2^2}{4m^2} + O \left( \frac{p^4}{m^4} \right) \right) \]

\[ \times \left( -\frac{e^2 N_1}{DEN_4} - \frac{4e^2}{(x-x')^2} \delta_{s_1s_3} \delta_{s_2s_4} + \frac{e^2 N_2}{DEN_5} + 4e^2 \delta_{s_1s_3} \delta_{s_2s_4} \right) \]

\[ - \frac{1}{16\pi^3} \frac{1}{2m} V_c , \quad (127) \]

where

\[ \frac{1}{DEN_4} \equiv \frac{\theta_{12}}{DEN_1} + \frac{\theta_{21}}{DEN_2} , \quad \theta_{12} \equiv \theta (DEN_1 - DEN_2) \quad (128) \]

\[ \frac{1}{DEN_5} \equiv \frac{\theta (M_0^2 - M_o^2)}{M_o^2} + \frac{\theta (M_o^2 - M_o'^2)}{M_o'^2} . \quad (129) \]
Note that we have expanded out the Jacobian factors as given by Eq. (91). Also, \( DEN_1 \) and \( DEN_2 \) are defined below Eq. (72) and written in their expanded version in Eqs. (82) and (83) respectively. Finally, \( N_1 \) and \( N_2 \) are written below Eqs. (72) and (75) respectively.

Now, since the eigenstate wavefunctions of \( H_o \) force \( p \) to scale as \( p \sim m\alpha \), it is useful to note the \( \alpha \)-scaling of specific terms in \( V \). Recalling that we are always assuming \( \alpha \to 0 \) (without which our matrix elements would not have a well-defined \( \alpha \) scaling), we see the following structure arising

\[
V = V^{(0)} + V^{(1)} + V^{(2)} + \cdots,
\]

where \( V^{(S)} \) scales as \( V^{(S)} \sim \alpha^S \). Thus in first-order BSPT these respective terms contribute

\[
\delta^{(4)} M_{NN'}^2 = \langle \phi_N | V^{(S)} | \phi_{N'} \rangle \sim \alpha^{3+S}.
\]

So, to be consistent to order \( \alpha^4 \) we need to look at all the matrix elements \( V^{(S)} \) with \( S \leq 1 \).

Before proceeding to write out these expressions for \( V^{(S)} \), we note the following facts:

- In this work we will only calculate the spin splittings, so any constants along the diagonal in spin space do not contribute.
- Since we are only working to order \( \alpha^4 \), obviously any splittings that can be shown to contribute at order \( \alpha^{4+k} \) with \( k > 0 \) need not be calculated.
- Symmetries of the integrand can be used to simplify expressions immensely.

One final discussion that we must have, before we write out these expressions for \( V^{(S)} \), is how we are going to deal with \( DEN_4 \) and \( DEN_5 \) defined above.\(^8\) These denominators are dealt with by noting the following formulae:

\[
\begin{align*}
\frac{\theta(a - b)}{a} + \frac{\theta(b - a)}{b} &= \frac{1}{2} \left( \frac{\theta(a - b)}{a} + \frac{\theta(b - a)}{b} + \frac{1}{2} \frac{\theta(a - b) + \theta(b - a)}{b} \right) \\
&+ \frac{1}{2} \left( \frac{-\theta(a - b) - \theta(b - a)}{a} - \frac{1}{2} \frac{-\theta(a - b) - \theta(b - a)}{b} \right)
\end{align*}
\]

\(^8\) For example, \( \frac{e^2 p}{(p-p')^2} \sim \frac{\alpha^2}{\alpha^2} \Rightarrow S = 0 \).

\(^9\) Actually the \( DEN_5 \) term is handled with analogous techniques as the \( DEN_4 \) term, and has even smaller corrections than those of \( DEN_4 \). Thus, we will just discuss the \( DEN_4 \) term in what follows and here state the result for the \( DEN_5 \) term: Take \( DEN_5 \to 4m^2 \); the corrections to this start shifting the bound state mass at order \( \alpha^6 \).
\[
\frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) + \frac{1}{2} \left( \theta(a-b) - \theta(b-a) \right) \left( \frac{1}{a} - \frac{1}{b} \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) + \frac{1}{2} \left| \frac{a-b}{a-b} \right| \left( \frac{1}{a} - \frac{1}{b} \right)
\]
\[
= \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} \right) - \frac{1}{2} \left| \frac{a-b}{a-b} \right| \cdot \frac{1}{a} \cdot \frac{1}{b}.
\]

(132)

To proceed it is useful to note

\[
D E N_1 = (\mathbf{p} - \mathbf{p}')^2 - \frac{(\kappa_z - \kappa'_z)(\mathbf{p}^2 - \mathbf{p}'^2)}{m} + O \left( \frac{\mathbf{p}}{m} \right)^5,
\]

(133)

\[
D E N_1 = D E N_2 - \frac{2(\kappa_z - \kappa'_z)(\mathbf{p}^2 - \mathbf{p}'^2)}{m} + O \left( \frac{\mathbf{p}}{m} \right)^5,
\]

(134)

\[
\frac{1}{2} \left( \frac{1}{D E N_1} + \frac{1}{D E N_2} \right) = \frac{1}{(\mathbf{p} - \mathbf{p}')^2} + \frac{(\kappa_z - \kappa'_z)^2(\mathbf{p}^2 - \mathbf{p}'^2)}{m^2(\mathbf{p} - \mathbf{p}')^6} + O \left( \frac{\mathbf{p}}{m} \right)^2 \left( \frac{1}{m} \right)^2.
\]

(135)

Especially note that this last equation scales as: \( \frac{1}{\alpha^2} + 1 + \alpha^2 + \cdots \), i.e. the corrections start at order 1 (not order \( \frac{1}{\alpha} \)); this implies that only the \( \frac{1}{(\mathbf{p} - \mathbf{p}')^2} \) term of Eq. (133) contributes to the spin splittings to order \( \alpha^4 \). But we still have to discuss the second term that arises in Eq. (132). This term is given by

\[
\frac{1}{D E N_4} \bigg|_{\text{second term}} = \frac{1}{2} \left| D E N_1 - D E N_2 \right| = \frac{1}{2} \left| \frac{2(\kappa_z - \kappa'_z)(\mathbf{p}^2 - \mathbf{p}'^2)}{m} \right| \left( \frac{(\mathbf{p} - \mathbf{p}')^4}{m} \right) + O \left( \alpha^0 \right).
\]

(136)

Including \( N_1 \), this starts out as an \( O(\alpha^3) \) spin conserving contribution. The next order contribution is \( O(\alpha^4) \) with spin structure, but is odd under \( \mathbf{p} \leftrightarrow \mathbf{p}' \), and thus integrates to zero. However, the \( O(\alpha^3) \) spin conserving term appears to lead to an order \( \alpha^4 \) shift to the spin splittings in second order BSPT when the cross terms with \( V^{(0)} \) of Eq. (138) are considered, however, these cross term contributions add to zero due to the facts that the \( O(\alpha^3) \) term including Eq. (136) conserves spin and the \( O(\alpha^3) \) term including Eq. (136) is even while the term from Eq. (138) is odd under \( \mathbf{p} \leftrightarrow \mathbf{p}' \).

To summarize the preceding discussion of \( D E N_4 \) and \( D E N_5 \), we can say that through order \( \alpha^4 \), for the spin splittings of positronium, there are no relativistic corrections to the following replacements:

\[
D E N_4 \rightarrow (\mathbf{p} - \mathbf{p}')^2 \quad \text{and} \quad D E N_5 \rightarrow 4m^2.
\]

(137)

This is valid for the ground and excited states, but in what follows we specialize to the ground state for simplicity.

\(^{10}V^{(0)} \) of Eq. (138) comes from the first term of Eq. (134) combined with the complete next to leading order term in \( N_1 \).
Given this general conclusion about $DEN_4$ and $DEN_5$, we list the pieces of $V$ that contribute to positronium’s ground state spin splittings through order $\alpha^4$. Explicitly, as far as the $\alpha$-scaling goes, we need to consider

$$V^{(0)}(p's_3s_4; p's_1s_2) = \frac{-c_{ex} e^2}{4\pi^3(p-p')^2} v^{(0)}(p's_3s_4; p's_1s_2), \quad (138)$$

where

$$v^{(0)}(p's_3s_4; p's_1s_2) = (\delta_{s_1} \pi_{s_1} f_1(p's_3s_4; p's_1s_2) + \delta_{s_1} \pi_{s_1} f_2(p's_3s_4; p's_1s_2)), \quad (139)$$

$$f_1(p's_3s_4; p's_1s_2) = s_1(\kappa_y - \kappa'_y) - i(\kappa_x - \kappa'_x), \quad (140)$$

$$f_2(p's_3s_4; p's_1s_2) = s_4(\kappa_y - \kappa'_y) + i(\kappa_x - \kappa'_x). \quad (141)$$

Recall that $s_i = \pm 1$ ($i = 1, 2, 3, 4$) only. The only other interaction we need to consider is

$$V^{(1)}(p's_3s_4; p's_1s_2) = \frac{e^2}{4m^3 \pi^3} (c_{ex} \delta_{s_1} \pi_{s_1} \pi_{s_1} + c_{ex} \delta_{s_2} \pi_{s_1} \pi_{s_1} +$$

$$+ \left( c_{ex} \frac{1}{2} - c_{ex} \frac{(\kappa - \kappa')^2}{(p-p')^2} \right) \delta_{s_1} \pi_{s_1} \pi_{s_1} ). \quad (142)$$

The constants $c_{ex}$ and $c_{ex}$ were introduced only to distinguish the terms that arise from the ‘exchange’ and ‘annihilation’ channels respectively, and $c_{ex} = c_{ex} = 1$ will be used in the remainder of this work.

Two simplifications were made in deriving $V^{(1)}$. First, we did not include terms that are a constant along the diagonal in spin space, because these do not contribute to the spin splittings to order $\alpha^4$. Second, we noted that terms of the following type integrate to zero:

$$\langle \psi_{1,0,0,\pi_1,\pi_2} | e^2 (\kappa_x \kappa'_{y_2} \kappa \times \kappa') (p-p')^2 | \psi_{1,0,0,\pi_1,\pi_2} \rangle = (0, 0, 0). \quad (143)$$

and thus were not included in the definition of $V^{(1)}$.

The ground state spin splitting to order $\alpha^4$ contains contributions from $V^{(1)}$ in first order BSPT ($V^{(0)}$ vanishes in first order BSPT) and $V^{(0)}$ in second order BSPT. We begin with the first order BSPT calculation. These results are shown in Figure 2. Then we perform the second order BSPT calculation. The combined results of first and second order BSPT are shown in Figure 3.

26
First Order BSPT:

The lowest order wavefunctions are given near the end of §III.B.1 (see Appendix B for the hyperspherical harmonics). \(V^{(1)}\) in first order BSPT contributes the following to positronium’s ground state mass squared:

\[
\delta M_1^2 \equiv \delta^{(1)} M^2(s_3, s_4; s_1, s_2) = N \int d^3p \, d^3p' \frac{1}{(e_1^2 + p^2)^2} \frac{1}{(e_1'^2 + p'^2)^2} V^{(1)}(p', s_3, s_4; p, s_1, s_2),
\]

where

\[
N = \frac{8e_1^5}{\pi^2} \quad \text{and} \quad e_1 = \frac{m\alpha}{2n} |_{n=1}.
\]

Using the rotational symmetry of the integrand we can make the substitution

\[
\frac{(\kappa - \kappa')^2}{(p - p')^2} \rightarrow \frac{2}{3} \frac{(\kappa_x - \kappa'_x)^2 + (\kappa_y - \kappa'_y)^2 + (\kappa_z - \kappa'_z)^2}{(p - p')^2} = \frac{2}{3}.
\]

After this, the remaining integrals are trivial and the splittings that arise from diagonalization of the \(\delta M_1^2\) matrix in spin space are:

\[
\langle 1 | \delta M_1^2 | 1 \rangle = -m^2\alpha^4,
\]
\[
\langle 2 | \delta M_1^2 | 2 \rangle = \frac{2}{3}m^2\alpha^4,
\]
\[
\langle 3 | \delta M_1^2 | 3 \rangle = m^2\alpha^4,
\]
\[
\langle 4 | \delta M_1^2 | 4 \rangle = m^2\alpha^4,
\]

where

\[
\left\{ |1\rangle \equiv \frac{|+\rangle - |-\rangle}{\sqrt{2}}, \ |2\rangle \equiv \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \ |3\rangle \equiv |\rangle, \ |4\rangle \equiv |+\rangle \right\}.
\]

Figure 2 shows these results, which taken alone do not produce the degeneracies required by rotational invariance.

Second Order BSPT:

\(V^{(2)}\) gives rise to the following contribution to positronium’s ground state mass squared in second order BSPT:
\[
\delta M^2_2 \equiv \delta^{(2)} M^2(s_3, s_4; s_1, s_2) = \sum_{s_e, s_e \neq (1, 0, 0, 0)} \frac{\langle \phi_{1,0,0,s_3,s_4}|V^{(0)}|\phi_{1,0,s_e,s_e}\rangle \langle \phi_{1,0,s_e,s_e}|V^{(0)}|\phi_{1,0,0,s_1,s_2}\rangle}{M^2_1 - M^2_n}. \tag{152}
\]

Recall that \(\mu = (n, l, m)\), the usual principal and angular momentum quantum numbers of nonrelativistic positronium. The calculation of \(\delta M^2_2\) is tedious, but can be done analytically. This calculation is performed in Appendix C. The result is (see Eq. (248) in Appendix C):

\[
\delta M^2_2 = -\frac{m^2}{24}\alpha^4(3g_1 + g_2), \tag{153}
\]

where \(g_1\) and \(g_2\) are given in Appendix C in Eqs. (210) and (211) respectively.

Now we combine the \(\delta M^2_1\) and \(\delta M^2_2\) matrices and diagonalize the result. The combined matrix is given by:

\[
\frac{\delta M^2_1 + \delta M^2_2}{2m^2\alpha^4} = \frac{1}{2}\delta_{s_1 s_2}\delta_{s_3 s_4} - \frac{1}{12}\delta_{s_2 s_4}\delta_{s_3 s_3} + \frac{1}{2}\delta_{s_2 s_4}\delta_{s_3 s_2} \delta_{s_3 s_3} - \frac{1}{48}(3g_1 + g_2). \tag{154}
\]

The eigenvalues are:

\[
\langle 1 | \delta M^2_1 + \delta M^2_2 | 1 \rangle = -\frac{5}{3}m^2\alpha^4, \tag{155}
\]

\[
\langle 2 | \delta M^2_1 + \delta M^2_2 | 2 \rangle = \frac{2}{3}m^2\alpha^4, \tag{156}
\]

\[
\langle 3 | \delta M^2_1 + \delta M^2_2 | 3 \rangle = \frac{2}{3}m^2\alpha^4, \tag{157}
\]

\[
\langle 4 | \delta M^2_1 + \delta M^2_2 | 4 \rangle = \frac{2}{3}m^2\alpha^4, \tag{158}
\]

and the corresponding eigenvectors are:

\[
\left\{ |1\rangle \equiv \frac{\langle + -\rangle - \langle - +\rangle}{\sqrt{2}}, |2\rangle \equiv \frac{\langle + -\rangle + \langle - +\rangle}{\sqrt{2}}, |3\rangle \equiv \langle - -\rangle, |4\rangle \equiv \langle + +\rangle \right\}. \tag{159}
\]

Figure 3 displays these results. These results lead to the well-known result, \(\frac{7}{6}\alpha^2 R_\text{Yd}\), as detailed in Appendix D. We see rotational invariance in the degeneracies of the ground state \(n = 1\) levels exactly maintained at order \(\alpha^4\).
IV. REMAINING CORRECTIONS TO THE POSITRONIUM SPECTRUM

We have shown how one obtains spin splittings at $O(\alpha^4)$, starting with a light-front Hamiltonian renormalized through $O(\alpha)$. Even though the spin splittings are correct through $O(\alpha^4)$, there are corrections starting at $O(\alpha^3)$ in the actual position of the levels computed using $H_R$ renormalized to $O(\alpha)$ and $m^2\alpha^2 \ll \lambda^2 \ll m^2\alpha$. We will outline these corrections and argue that higher order counterterms, obtained from renormalizing the Hamiltonian through $O(\alpha^2)$ and beyond, fix any errors appearing in the physical spectrum computed with $H_R$ renormalized to $O(\alpha)$.

The dressing of electrons by two photon loops in all allowed time-orderings shift the electron self-energy by $O(\alpha^2\lambda^2 \log^2(1/\epsilon))$. In an analytic calculation this contribution can be cancelled by a one-body counterterm that is exactly what one expects from higher order renormalization when the Hamiltonian is computed to $O(\alpha^2)$. The infrared divergences must cancel leaving a free electron energy $\frac{p^2-m^2}{p^2}$.

The leading correction to the spectrum we have computed in this paper may be $O(\alpha^2\lambda^2 \log^2(1/\epsilon))$, coming from the above $O(\alpha^2)$ correction to the electron self-energy. However, with $H_R$ renormalized through $O(\alpha)$, there are indications that all the infrared divergences arising in BSPT cancel due to the fact that large wavelength photons decouple from neutral positronium states [2], leading only to residual long-range two-body potentials that may still correct the spectrum. Assuming this holds at higher orders too, the leading correction from higher order self-energy shifts is $O(\alpha^2\lambda^2)$. With the cutoffs used above the corrections to the electron self-energy can depend on the longitudinal momentum fraction of the electron, while if cutoffs involving the momenta of the connected pieces of the matrix elements are used, the relevant counterterm is a constant.

From Eq. (121) in the paper, the constraint placed on $\lambda^2$ is

$$\lambda^2 \propto m^2\alpha^k,$$

where $2 < k < 4$. So the leading order shift to the spectrum arising from higher order self-energy corrections is

$$\delta M^2_{c1} \sim m^2\alpha^2 \alpha^k.$$

(161)
Another correction to the spectrum from the $O(\alpha)$ Hamiltonian arises from the interaction containing the leading order term of $N_1$ and the term from Eq. (137). This shifts the spectrum an amount

$$\delta M_{12}^2 \sim m^2 \alpha^3.$$ (162)

So, after considering the possible corrections, we see that our choice of $H_o$ has produced eigenvalues that are correct through $O(m^2 \alpha^2)$. That is, the nonperturbative physics was correctly chosen and is not changing. However, fourth order perturbative effective interactions are required to obtain $O(m^2 \alpha^4)$ precision for the actual position of the levels—the spin splittings are correct through order $\alpha^4$ as detailed in the paper.

To be complete, we must also consider coupling and photon mass renormalization. However, there are no further corrections to the photon mass if $\lambda < 2m$ because pair production is no longer possible. This also affects coupling renormalization. If $\lambda < 2m$, no electron loop appears to modify the electron-photon vertex. Since this contribution is entirely responsible for renormalization of the marginal part of this operator (i.e., the running coupling), we do not expect the charge to renormalize. There will be modifications to the irrelevant part of this vertex, but they should not contribute to leading order.

There are additional counterterms that are not typically encountered in a lagrangian calculation. For example, there are tree level counterterms that modify the electron-photon and positron-photon coupling at $O(e^3)$. These counterterms should produce cancellations with photon exchange contributions to the mass, but they must be present for this to occur. Such counterterms may be necessary for all the infrared divergences to cancel.

We offer no full solution to these problems other than renormalizing the Hamiltonian to higher orders using a similarity transformation and coupling coherence, as outlined and illustrated in earlier sections.

V. SUMMARY AND DISCUSSION

If $H_o$ is derived approximately to a finite order in $\alpha$, to what order in $\alpha$ is the nonperturbative spectrum correct? The precise connection between the approximate renormalized Hamiltonian and the physical spectrum is only qualitatively understood at present. The results of this paper make it clear that
in light-front QED to obtain positronium’s spin structure through order $\alpha^5$, the renormalized Hamiltonian must be derived either through at least second order in $\alpha$ or else via a nonperturbative similarity transformation which uses Coulomb states instead of free states in the perturbative expansion for the effective interactions.

We have calculated the order $\alpha^4$ spin splittings for the $n = 1$ levels. This restriction leads to a great simplification because constants along the diagonal in spin space do not contribute to the splittings. Some of the terms that are not calculated because they cancel in the difference are quite complicated and a future analytical calculation including these terms would be quite complex, involving complicated sums over 9-J symbols for example. Even though we do not calculate these $\alpha^4$ “radial” shifts in this paper, the fact that our spin splittings are correct and can be computed analytically from first principles gives us much hope that the procedure outlined in this work is well-defined and at least for $QED_{3+1}$ leads to consistent bound state calculations. It becomes very interesting to think about the Lamb shift.

**ACKNOWLEDGMENTS**

The authors wish to acknowledge useful discussions with Brent H. Allen, Martina M. Brisudová, James B. White and Kenneth G. Wilson. In particular, we would like to thank Brent H. Allen for the use of his matrix element rules [13]. Research reported in this paper has been supported by the National Science Foundation under grant PHY-9409042 and the Maria Skłodowska-Curie Foundation under grant MEN/NSF-94-190.
APPENDIX A: LIGHT FRONT CONVENTIONS

In this appendix we present our light-front conventions. In this initial paragraph, let $A$ and $B$ be arbitrary 4-vectors. An “ET” label implies an equal-time vector; the absence of a label implies a light-front vector. Two vectors by definition are related by

- $A^\pm = A^0_{ET} \pm A^3_{ET}$
- $A^i = A^i_{ET}$, $i = 1, 2$
- $A^\mu = (A^+, A^-, A^i)$
- $A^\mu_{ET} = (A^0_{ET}, A^3_{ET}, A^i)$.

Then scalars are required to agree by adjusting the metric tensor so that this is so. This fixes the light-front metric tensor:

$$A^\mu B_\mu = g^{\mu \nu} A^\mu B_\nu = A^0_{ET} A^0_{ET} B^0_{ET} + A^3_{ET} A^3_{ET} B^3_{ET} - A^i_{ET} A^i_{ET} B^i_{ET}.$$

Thus:

$$1 = 2 g_{+-} = 2 g_{-+} = -g_{11} = -g_{22}.$$

Another relevant scalar is of course

$$g^{\mu \nu} g^\mu_{ET} g^\nu_{ET} = 4.$$

Thus:

$$1 = \frac{g^{+-}}{2} = \frac{g^{-+}}{2} = -g^{11} = -g^{22}.$$

The components of the light-front metric tensor not mentioned are zero. Thus, there are factors of two in places like

$$d^4x = \frac{1}{2} dx^+ dx^- d^2x^\perp, \quad A_+ = g_{+-} A^+ = \frac{1}{2} (A^0 + A^3) = \frac{1}{2} (A_0 - A_3), \quad etc.$$

Conventionally, $x^+$ is chosen to be the light-front time coordinate. This fixes $x^-$ to be the light-front longitudinal space coordinate. Also, from the $p \cdot x$ scalar:

$$p_\mu x^\mu = g_+ p^- x^+ + g_- p^+ x^- - p^i x^i = \frac{1}{2} p^- x^+ + \frac{1}{2} p^+ x^- - p^i x^i,$$
we see that $p^-$ is fixed to be the light-front energy coordinate, and $p^+$ is fixed to be the light-front longitudinal momentum coordinate.

All constituents in the forward light-cone in light-front coordinates have $p^+ \geq 0$. This can be seen from the following relation (which also shows why particle lines in Hamiltonian diagrams are all on mass shell):

$$\int \frac{d^4 p}{(2\pi)^4} 2\pi \delta(p^2 - m^2) \theta(p^0) f(p) = \int \frac{d^3 p}{16\pi^3 p_0} f(p) \bigg|_{p_0=\sqrt{p^2 + m^2}}$$

$$= \int \frac{d^2 p^- dp^+ \theta(p^+)}{16\pi^3 p^+} f(p) \bigg|_{p^- = \sqrt{p^2 + m^2}} = \int \frac{f(p)}{p^-} \bigg|_{p^- = \sqrt{p^2 + m^2}}.$$

Especially note this last definition of $\int_p$, which is a shorthand used in the paper.

In momentum space the field operators are expanded as (at $x^+=0$)

$$A^i(x) = \sum_{s=\pm 1} \int_q (\epsilon^*_s a_s(q) e^{-iq \cdot x} + h.c.),$$

$$\xi(x) = \sum_{s=\pm 1} \chi_s \int_p \sqrt{p^+} (b_s(p) e^{-ip \cdot x} + d^s_+(p) e^{ip \cdot x}),$$

with $$\epsilon^i_1 = \frac{-1}{\sqrt{2}} (\delta_{i,1} + i \delta_{i,2}) , \quad \epsilon^i_{-1} = \frac{1}{\sqrt{2}} (\delta_{i,1} - i \delta_{i,2}),$$

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The fermion helicity can only take on the values $\pm 1/2$, however we define $h_3 = s/2$; therefore, “$s$” can only take on the values $\pm 1$. Note that $\bar{s} \equiv -s$. Examples of the commutation (anti-commutation) relations and free Fock states are

$$[a_\lambda(q), a^\dagger_{\lambda'}(q')] = 16\pi^3 q^+ \delta^3(q-q')\delta_{\lambda\lambda'}, \quad (\delta^3(p) \equiv \delta^2(p^-)\delta(p^+) ),$$

$$\{b_s(p), b^\dagger_{s'}(p')\} = \{d_s(p), d^\dagger_{s'}(p')\} = 16\pi^3 p^+ \delta^3(p-p')\delta_{ss'},$$

$$\langle p_1 s_1 | p_2 s_2 \rangle = 16\pi^3 p^+_1 \delta^3(p_1 - p_2)\delta_{s_1 s_2}, \quad |p_1 s_1\rangle = b^\dagger_{s_1}(p_1)|0\rangle , \quad etc.$$
the following could be used too):
\[
\frac{1}{\partial^+} f(x^-) = \frac{1}{4} \int_{-\infty}^{+\infty} dy^- \epsilon(x^- - y^-) f(y^-),
\]
\[
\partial^+ - 2\partial_- = 2 \frac{\partial}{\partial x^-} ,
\]
\[
\partial_- \epsilon(x^- - y^-) = 2\delta(x^- - y^-) ,
\]
\[
\epsilon(x) = \theta(x) - \theta(-x) .
\]

Notice that this is non-local in the longitudinal direction.

**APPENDIX B: HYPERSPHERICAL HARMONICS/FOCK COORDINATE CHANGE**

In this appendix we will list some useful mathematical relations used in the paper. The conventions followed in this paper are given in Ref. [14]. These hyperspherical harmonics are given by:

\[
Y_\mu(\Omega) \equiv Y_{n,l,m}(\Omega) \equiv f_{n,l}(\omega) Y_{l,m}(\theta,\phi),
\]

where

\[
0 \leq |m| \leq l \leq n - 1.
\]

These quantum numbers are the standard “hydrogen” quantum numbers. These 3D spherical harmonics, $Y_{l,m}(\theta,\phi)$, are given by [15]:

\[
Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \frac{(-1)^m}{2^l!} (1-x^2)^l e^{im\phi},
\]

where $x = \cos\theta$. These other functions, $f_{n,l}(\omega)$, are given by [16]:

\[
f_{n,l}(\omega) = (-i)^l \sqrt{\frac{2n(n-l-1)!}{\pi(n+l)!}} \sin^l \omega \frac{d^l}{(d\cos\omega)^l} C_{n-1}(\cos\omega),
\]

where $C_{n-1}(\cos\omega)$ are Gegenbauer polynomials. For example [16]:

\[
C_0(y) = 1, \ C_1(y) = 2y, \ C_2(y) = 4y^2 - 1, \ C_3(y) = 8y^3 - 4y,
\]

\[
C_4(y) = 16y^4 - 12y^2 + 1, \ C_5(y) = 32y^5 - 32y^3 + 6y.
\]
The orthogonality and phase relations of these above functions are:

\[ Y_{n,l,m} = (-1)^{l+m} Y_{n,-l,-m}^*, \quad Y_{l,m} = (-1)^m Y_{l,-m}^* , \quad f_{n,l} = (-1)^l f_{n,l}^* , \]

\[ d\Omega^3 = d\Omega \equiv d\Omega^{(3)} d\omega \sin^2 \omega , \]

\[ \int d\Omega^{(3)} Y_{\mu}^* Y_{\mu'} = \delta_{\mu\mu'} , \quad \int d\omega \sin^2 \omega f_{n,l}^* f_{n',l} = \delta_{nn'} , \quad \int d\Omega^{(3)} Y_{l,m}^* Y_{l',m'} = \delta_{ll'} \delta_{mm'} . \]

The first few hyperspherical harmonics are:

\[ Y_{1,0,0} = \frac{1}{\sqrt{2\pi^2}} , \quad Y_{2,0,0} = \frac{\sqrt{2} \cos \omega}{\pi} , \]

\[ Y_{2,1,-1} = -ie^{-i\phi} \sin \omega \sin \theta \quad \frac{\pi}{\pi} , \quad Y_{2,1,0} = -i\sqrt{2} \sin \omega \cos \theta \quad \frac{\pi}{\pi} , \]

\[ Y_{2,1,1} = i e^{i\phi} \sin \omega \sin \theta \quad \frac{\pi}{\pi} . \]

The first few hyperspherical harmonics are:

For further harmonics we refer the interested reader to Appendix 2 of Judd’s text [17], where this is done quite nicely.

For the coordinate change in the Coulomb Schrödinger equation (see Eq. (93) and the discussion that follows it), for \( B_N < 0 \), we define

\[ mB_N \equiv -e_n^2 , \quad \beta \equiv (u_0, u) , \quad \beta_0 \equiv \cos(\omega) \equiv e_n^2 - p^2 \quad \frac{e_n^2 + p^2}{e_n^2 + p^2} , \]

\[ \beta \equiv \frac{p}{p} \sin(\omega) \equiv \sin(\omega) (\sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta)) \equiv \frac{2e_n p}{e_n^2 + p^2} . \]

Note that \( u_0^2 + u^2 = 1 \). Conversely this coordinate change gives:

\[ p = \frac{e_n}{1 + u_0} u , \quad e_n^2 + p^2 = \frac{2e_n^2}{1 + u_0} . \]

We also have:

\[ d\Omega_p = \sin^2 \omega d\omega d\Omega^{(3)}_p = \left( \frac{2e_n}{e_n^2 + p^2} \right)^3 d^3 p , \]
\[
\delta^3(p - p') = \frac{(2e_n)^3}{(e_n^2 + p^2)^3} \delta(\Omega_p - \Omega_{p'}) = \frac{(1 + u_\mu)^3}{e_n^3} \delta(\Omega_p - \Omega_{p'}) ,
\]
(179)

\[
0 \leq \omega \leq \pi , \ 0 \leq \theta \leq \pi , \ 0 \leq \phi \leq 2\pi .
\]
(180)

Finally, a most useful relation is:

\[
|p - p'|^2 = \frac{(e_n^2 + p^2)(e_n^2 + p'^2)}{4e_n^2} |u - u'|^2 .
\]
(181)

This is useful because we can expand \(|u - u'|^2\) as follows:

\[
\frac{1}{|u - u'|^2} = \sum_\mu \frac{2\pi^2}{n} Y_\mu(\Omega_p)Y^{*}_\mu(\Omega_{p'}) .
\]
(182)

This completes the discussion of the hyperspherical harmonic mathematical relations used in this work; actually, as shown in the next appendix, the \(\delta M^2\) calculation requires some further formulae, which are given as they are needed.

**APPENDIX C: THE CALCULATION OF \(\delta M^2\)**

In this appendix we perform the following sum analytically:

\[
\delta M^2 = \sum_{s_e,s_e,\mu \neq (1,0,0)} \frac{\langle \phi_{1,0,0,s_3,s_4}|V(0)|\phi_{\mu,s_e,s_e}\rangle \langle \phi_{\mu,s_e,s_e}|V(0)|\phi_{1,0,0,s_1,s_2}\rangle}{M^2 - M_n^2} .
\]
(183)

Recall that \(\mu = (n,l,m_l)\), the usual principal and angular momentum quantum numbers of nonrelativistic positronium. Also recall that the spin factored completely out of our lowest order Schrödinger equation, so to proceed the following notation is useful:

\[
|\phi_{\mu,s_e,s_\mu}\rangle = |\phi_\mu\rangle \otimes |s_e,s_\mu\rangle ,
\]
(184)

\[
1 = \sum_{s_e,s_\mu,\mu} |\phi_\mu\rangle \langle \phi_\mu| \otimes |s_e,s_\mu\rangle \langle s_e,s_\mu|
\]
(185)

To proceed, define the following Green’s function for arbitrary \(E\):

\[
G_E = \sum_\mu \frac{|\phi_\mu\rangle \langle \phi_\mu|}{E - M^2_n} .
\]
(186)
The factor $\frac{1}{4m}$ will turn out to be useful. This Green’s function satisfies the familiar Coulomb Green’s function equation:

$$\delta^3(p - p') = (\tilde{E} - \frac{p'^2}{m})G_E(p', p) + \frac{\alpha}{2\pi^2} \int d^3p'' \frac{G_E(p'', p)}{(p' - p'')^2}, \quad (187)$$

where

$$\langle p' | G_E | p \rangle \equiv G_E(p', p), \quad (188)$$

and $\tilde{E} \equiv \frac{E - 4m^2}{4m}$. \quad (189)

Hostler and Schwinger independently obtained the solution for this Coulomb Green’s function in 1964 [18]. We find Schwinger’s form useful; the equation he solves is exactly the above equation with the following shifts in notation:

$$\left(\frac{Ze^2}{\alpha}\right)_{\text{Schwinger}} \rightarrow \alpha, \quad (190)$$

$$m_{\text{Schwinger}} \rightarrow \frac{m}{2}, \quad (191)$$

$$E_{\text{Schwinger}} \rightarrow \tilde{E}. \quad (192)$$

His result is amended because the sum we need has $E = M_1^2$ and does not include $\mu = (1, 0, 0)$. This subtraction of the $\mu = (1, 0, 0)$ term amounts to the term “$-\frac{1}{C}$” in $G_{III}$ below. The details of how this arises can be seen in Eqs. (233)-(235) below. With this amendment, Schwinger’s result is:

$$G'_{M1}(p, p') \equiv G_I + G_{II} + G_{III}, \quad (193)$$

$$G_I = \frac{\delta^3(p - p')}{E - T}, \quad (194)$$

$$G_{II} = -\frac{\alpha}{2\pi^2} \frac{1}{E - T} \frac{1}{(p - p')^2} \left(\frac{1}{E - T'}\right), \quad (195)$$

$$G_{III} = -\frac{\alpha}{2\pi^2} \frac{4e_1^2}{E - T} \left[\int_0^1 \left(\frac{1}{4e_1^2 \rho (p - p')^2 + C(1 - \rho)^2 - \frac{1}{C}}\right) \frac{1}{E - T'}\right], \quad (196)$$

where

$$T = \frac{p^2}{m}, \quad T' = \frac{p'^2}{m}, \quad e_1 = \frac{m\alpha}{2}, \quad (197)$$

$$C = (e_1^2 + p^2)(e_1^2 + p'^2), \quad (198)$$

$$\tilde{E} = \frac{M_1^2 - 4m^2}{4m} = -\frac{e_1^2}{m}. \quad (199)$$

37
The prime on $G'_{M1}^2$ denotes the fact that we have subtracted the $\mu = (1, 0, 0)$ part of $G_E$ as required by the sum that we have in $\delta M^2$. Note that this Green’s function is symmetric under $p \leftrightarrow p'$ and also $(p_x, p'_x) \leftrightarrow (p_y, p'_y)$, symmetries that will be used in later simplifications of the integrand of $\delta M^2$.

$\delta M^2$ is now:

$$\delta M^2 = \sum_{s, s_\omega} \int d^3p d^3k d^3p' d^3k' \langle \phi_{1, 0, 0} | k \rangle V^{(0)}(k, s_3, s_4; p, s_e, s \omega) \times \left( \frac{G_I + G_{II} + G_{III}}{4m} \right) V^{(0)}(p', s_e, s \omega; k', s_1, s_2) \langle k' | \phi_{1, 0, 0} \rangle$$

$$\equiv \delta M^2(I) + \delta M^2(II) + \delta M^2(III) \text{ respectively .} \tag{200}$$

Now we rewrite this in terms of hyperspherical harmonics and perform the integrations analytically. The variables are defined as:

- $\Omega_p \leftrightarrow [u \equiv (u_o, u)] \leftrightarrow [e_1, p]$, $\Omega_{p'} \leftrightarrow [u' \equiv (u'_o, u')] \leftrightarrow [e_1, p']$ \hspace{1cm} \tag{202}
- $\Omega_k \leftrightarrow [v \equiv (v_o, v)] \leftrightarrow [e_1, k]$, $\Omega_{k'} \leftrightarrow [v' \equiv (v'_o, v')] \leftrightarrow [e_1, k']$ \hspace{1cm} \tag{203}

See the previous appendix on hyperspherical harmonics for a summary of the mathematical relations that we use. The symbols appearing in Eqs. (202) and (203) are explained there. Note that we use $e_1$ in these variable definitions, a choice that is completely general and turns out to be useful because we are taking expectation values of $n = 1$ states in this work. The relations we use are:

- $\langle k' | \phi_{1, 0, 0} \rangle = \frac{4e_1^5}{(e_1^2 + k'^2)^2} \frac{1}{\sqrt{2\pi}}$ \hspace{1cm} \tag{204}
- $\frac{1}{(e_1^2 + k'^2)^2} = \frac{(1 + v'_o)^2}{4e_1^4}$ \hspace{1cm} \tag{205}
- $d^3k' = \frac{(e_1^2 + k'^2)^3}{8e_1^3} d\Omega_{k'} = \frac{e_1^3}{(1 + v'_o)^2} d\Omega_{k'}$. \hspace{1cm} \tag{206}

Given these, $\delta M^2$ becomes:

$$\delta M^2 = -\frac{m^4 e_0^5}{32\pi^2} \int d\Omega_p d\Omega_{p'} d\Omega_k d\Omega_{k'} \left[ \langle \tilde{E} - T \rangle (G_I + G_{II} + G_{III}) \right] \times S \sum_{\mu \mu'} \frac{1}{m \omega} \mathbf{Y}_\mu(\Omega_p) \mathbf{Y}_\mu'(\Omega_{p'}) \mathbf{Y}_\mu^*(\Omega_k) \mathbf{Y}_{\mu'}^*(\Omega_{k'}) \text{,} \tag{207}$$

where
\[ S \equiv \sum_{s_3, s_4} v^{(0)}(k, s_3, s_4; p, s_e, s_e^{(0)})(p', s_e, s_e^{(0)}; k', s_1, s_2). \] (208)

Recall Eq. (139) for the definition of \( v^{(0)} \). Using the symmetries of the integrand, the sum over spins \( s_e \) and \( s_e^{(0)} \) can be performed and a simplification is seen to arise. The spin completely factors out of the momenta integrations. In other words, we have:

\[ S = \frac{1}{6} (3g_1 + g_2) (p \cdot p' + k \cdot k' - 2p \cdot k') , \] (209)

where

\[ g_1 \equiv s_1 s_3 + s_2 s_4 , \] (210)

\[ g_2 \equiv 1 + s_1 s_2 - s_2 s_3 - s_1 s_4 + s_3 s_4 + s_1 s_2 s_3 s_4 . \] (211)

Recall that \( s_i = \pm 1, (i = 1, 2, 3, 4) \); i.e., the \( \frac{1}{2} \) has been factored out of these spins.\(^{11}\) So, in other words, instead of having to do sixteen twelve dimensional integrals because the spin and momenta are coupled together, we just have to do one twelve dimensional integral that is independent of spin and then diagonalize the result in the \( 4 \times 4 \) dimensional spin space with the spin dependence given by Eq. (209).

We define the following integral:

\[ \chi \equiv \frac{m_\alpha}{8\pi^2} \xi , \] (212)

where

\[ \xi \equiv \int \frac{d\Omega_p d\Omega_{p'}}{(1 + u_\alpha)(1 + u'_\alpha)^2} \left[ (E - T)(G_I + G_{II} + G_{III}) \right] \]
\[ \times \left( p_a \cdot p'_b + k_c \cdot k'_c - 2p \cdot k' \right) \sum_{\mu \nu} \frac{1}{nn'} Y_\mu(\Omega_p)Y_{\mu'}(\Omega_{p'})Y_\nu^*(\Omega_k)Y_{\nu'}^*(\Omega_{k'}) , \] (213)

and

\[ \delta M^2_z = -\frac{m^2_\alpha^4}{24} (3g_1 + g_2) \chi . \] (214)

For the quantities \( \xi, \chi \) and \( \delta M^2_z \), the labels \( I, II \) and \( III \) imply the respective terms with \( G_I, G_{II} \) and \( G_{III} \) above (see Eq. (193)). Also, the terms \( a, b \) and \( c \) above correspond to the respective superscripts in what follows. This integration will now be performed analytically.

\(^{11}\) In order to get these simple forms for \( g_1 \) and \( g_2 \) and to see this spin/momentum decoupling it was useful to note the following simple relation: \( \delta_{s,s'} = \frac{1}{2} s(s + s') \) (true because \( s^2 = 1 \)).
First the three $G_I$ pieces. The mathematical relations used here are:

\begin{align}
\delta^3(p - p') &= \frac{8e_1^4}{(e_1^2 + p^2)^3} \delta(\Omega_p - \Omega_{p'}) = \frac{(1 + u_o)^3}{e_1^3} \delta(\Omega_p - \Omega_{p'}) \ , \quad (215) \\
p^2 &= \frac{e_1^2}{1 + u_o} (1 - u_o) \ , \quad (216) \\
k \cdot k' &\rightarrow 3k_z k'_z \ , \quad (217) \\
p \cdot k' &\rightarrow 3p_z k'_z \ . \quad (218)
\end{align}

Note that these last two relations are possible due to the rotational symmetry of the integrand. Then we expand these $z$-components of momenta upon the hyperspherical harmonic basis using the following simple relation (e.g. the $p_z$ case):

\begin{align}
p_z &= \frac{e_1}{1 + u_o} \left( \frac{\pi i}{\sqrt{2}} Y_{2,1,0}(\Omega_p) \right) \ . \quad (219)
\end{align}

Now we recall the hyperspherical harmonics (see the appendix on hyperspherical harmonics for details) that we will be using and their orthogonality and phase relationships:

\begin{align}
Y_{\mu}(\Omega) &\equiv Y_{n,l,m}(\Omega) \equiv f_{n,l}(\omega) Y_{l,m}(\theta, \phi) \ , \quad (220) \\
Y_{n,l,m} &= (-1)^{l+m} Y_{n,l,-m} \ , \ Y_{l,m} = (-1)^{m} Y_{l,-m} \ , \ f_{n,l} = (-1)^{l} f_{n,l}^* \ , \quad (221) \\
d\Omega^{(4)} &\equiv d\Omega \equiv d\Omega^{(3)} d\omega \sin^2 \omega \ , \quad (222) \\
\int d\Omega Y_{\mu}^* Y_{\mu'} &= \delta_{\mu\mu'} \ , \ \int d\omega \sin^2 \omega f_{n,l}^* f_{n',l} = \delta_{nn'} \ , \ \int d\Omega^{(3)} Y_{l,m}^* Y_{l',m'} = \delta_{ll'} \delta_{mm'} \ . \quad (223)
\end{align}

After straight-forward application of these relations we obtain:

\begin{itemize}
    \item \( \xi_I^a = \frac{4\pi}{c_1} \int_0^\pi d\omega \sin^2 \omega \frac{(1 - \cos \omega)}{(1 + \cos \omega)} = \frac{6\pi^2}{e_1} \) \quad (224) \\
    \item \( \xi_I^b = \frac{3\pi^2}{2e_1} \sum_{n=2}^{\infty} \frac{1}{n^2} \left( \int_0^\pi d\omega \sin^2 \omega \frac{f_{2,1}(\omega)}{1 + \cos \omega} f_{n,1}(\omega) \right)^2 \) \quad (225) \\
    \item \( \xi_I^c = -\frac{3\pi^2}{e_1} \sum_{n=2}^{\infty} \frac{1}{n} \left( \int_0^\pi d\omega \frac{f_{2,1}(\omega)}{1 + \cos \omega} f_{n,1}(\omega) \right)^2 \) \quad (226)
\end{itemize}

For the $G_{II}$ terms, we use the following relations:

\begin{align}
\frac{1}{E - T'} &= \frac{1}{c_1^2 - \frac{p^2}{m}} = -\frac{m}{2e_1^2} (1 + u_o') \ , \quad (227) \\
\frac{1}{(p - p')^2} &= \frac{(1 + u_o)(1 + u'_o)}{e_1^2} \sum_{\mu} \frac{2\pi^2}{n} Y_{\mu}(\Omega_p) Y_{\mu}^*(\Omega_{p'}) \ . \quad (228)
\end{align}
These give:

\[
(\tilde{E} - T)G_{II} = \frac{\alpha m}{2e_1} (1 + u_o')^2 (1 + u_o) \sum_{\mu} \frac{1}{n} Y_\mu(\Omega_p) Y^*_\mu(\Omega_{p'}) .
\]  

(229)

We use the rotational symmetry of the integrand and expand the integrand on the hyperspherical harmonic basis as was done for the three \(G_I\) terms. Then we have:

- \(\xi^a_{II} = \frac{3\pi^2}{2e_1} \sum_{n=2}^{\infty} \frac{1}{n^2} \left( \int_0^\pi \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 \) 

(230)

- \(\xi^b_{II} = \frac{3\pi^2}{2e_1} \sum_{n=2}^{\infty} \frac{1}{n^3} \left( \int_0^\pi \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 \) 

(231)

- \(\xi^c_{II} = -\frac{3\pi^2}{e_1} \sum_{n=2}^{\infty} \frac{1}{n^2} \left( \int_0^\pi \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 \). 

(232)

We use the same relations for the \(G_{III}\) terms and for the three \(G_{II}\) terms, and we use the rotational symmetry of the integrand to rewrite the appropriate pieces of the integrand in terms of \(Y_{2,1,0}\) as we did for the \(G_I\) and \(G_{II}\) terms. However, we need to discuss one additional relation that allows the remaining \(\delta M_2^2(III)\) calculation to be done analytically. In Schwinger’s 1964 paper [18] he gives the following formula:

\[
\frac{1}{2\pi^2} \left( \frac{1}{1 - \rho^2} + \rho (u - u')^2 \right) = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{n} \sum_{l,m} Y_{n,l,m}(\Omega) Y^*_{n,l,m}(\Omega') ,
\]  

(233)

where \(u\) and \(u'\) are of unit length and \(0 < \rho < 1\).\footnote{This is easily derivable from a more general standard formula that Schwinger gives,}

\[
\frac{1}{4\pi^2} \frac{1}{(u - u')^2} = \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{\rho_{n}^{1/2}} \frac{1}{2n} \sum_{l,m} Y_{n,l,m}(\Omega) Y^*_{n,l,m}(\Omega') .
\]  

(233)
Now, since \( n \geq 2 \) in this sum we can do the integral over \( \rho \):

\[
\int_0^1 d\rho \rho^{n-2} = \rho^{n-1} \bigg|_0^1 = \frac{1}{n-1},
\]

and we obtain:

\[
(\tilde{E} - T)G_{III} = \alpha m (1 + u_0) (1 + u'_0)^2 \sum_{\mu \neq (1,0,0)} \frac{1}{n(n-1)} Y_\mu (\Omega_p) Y^{*}_\mu (\Omega_{p'}) .
\]

For terms in \( \xi \) which contain \( G_{III} \), one obtains:

- \( \xi^a_{III} = \frac{3\pi^2}{2e_1} \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 \)

- \( \xi^b_{III} = \frac{3\pi^2}{2e_1} \sum_{n=2}^{\infty} \frac{1}{n^3(n-1)} \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 \)

- \( \xi^c_{III} = -\frac{3\pi^2}{e_1} \sum_{n=2}^{\infty} \frac{1}{n^2(n-1)} \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 . \)

Now recall \( \chi \equiv \frac{m}{8\pi^2} \xi \), and also notice that all the summands are the same, thus putting it all together we have:

\[
\chi = \frac{3}{2} + \frac{3}{8} \sum_{n=2}^{\infty} \left( \frac{1}{n} + \frac{1}{n(n-1)} + \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^3(n-1)} - \frac{2}{n} - \frac{2}{n^2(n-1)} \right) \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2
\]

\[
= \frac{3}{2} - \frac{3}{8} \sum_{n=2}^{\infty} \left( \frac{1}{n} \right) \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 .
\]

The remaining sum can be done analytically. To see this, first define two integrals:

\[
I_1 \equiv \int \frac{d\Omega_p}{1 + u_0} \frac{d\Omega_{p'}}{(1 + u'_0)^2} = (4\pi)^2 \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} \right)^2 = 16\pi^4 , \]

\[
I_2 \equiv \int \frac{d\Omega_p}{1 + u_0} \frac{d\Omega_{p'}}{(1 + u'_0)^2 (\mathbf{p} - \mathbf{p}')^2} = \frac{2}{3} I_1 .
\]

The last equality followed from rotational symmetry of the integrand. We can also calculate \( I_2 \) a hard way which gives:

\[
I_2 = 16\pi^4 - 4\pi^4 \sum_{n=2}^{\infty} \frac{1}{n} \left( \int_0^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 = \frac{32\pi^4}{3} .
\]

\[\text{We use Eq. (215),} \ p_x = \frac{\omega_2}{1 + u_0} \frac{\omega_2}{2} (Y_{2,1,-1}(\Omega_p) - Y_{2,1,1}(\Omega_p)) \text{ and } p_y = -\frac{\omega_2}{1 + u_0} \frac{\omega_2}{2} (Y_{2,1,-1}(\Omega_p) + Y_{2,1,1}(\Omega_p)) .\]
This last equality followed from Eqs. \((243)\) and \((244)\) (the easy way to calculate \(I_2\)). Thus, we have:

\[
\sum_{n=2}^{\infty} \frac{1}{n} \left( \int_{0}^{\pi} \frac{d\omega \sin^2 \omega}{1 + \cos \omega} f_{2,1}(\omega) f_{n,1}(\omega) \right)^2 = \frac{4}{3}.
\]

(246)

Combining this result with Eq. \((242)\) gives:

\[
\chi = \frac{3}{2} - \frac{3}{8} \left( \frac{4}{3} \right) = 1.
\]

(247)

Thus, recalling Eq. \((214)\), we have:

\[
\delta M^2 = -\frac{m^2 \alpha^4}{24} (3g_1 + g_2),
\]

(248)

where \(g_1\) and \(g_2\) are given in Eqs. \((210)\) and \((211)\) respectively.

**APPENDIX D: \(M_N^2\) -VS- \(B_N\)**

In this appendix we will invert the equation

\[
M_N^2 \equiv (2m + B_N)^2,
\]

(249)

and obtain the \(\alpha\)-expansion for the binding energy, \(B_N\). In this work we set up a procedure to calculate \(M_N^2\). This gave

\[
M_N^2 = M_N^2 + b_4 m^2 \alpha^4 + \mathcal{O} (\alpha^5),
\]

(250)

with

\[
M_N^2 \equiv 4m^2 + 4mB_N.
\]

(251)

For the lowest order spectrum of \(H_R\) we obtained

\[
B_N = -\frac{1}{4} \frac{m \alpha^2}{n^2}.
\]

(252)

Taking a square root of \(M_N^2\) gives

\[
B_N = B_N + \frac{m \alpha^4}{2} \left( \frac{b_4}{2} - \frac{1}{32n^4} \right) + \mathcal{O} (\alpha^5).
\]

(253)
Recall that $\frac{m^4}{2} = \alpha^2 \text{Ryd}$. Now, in this work, Eqs. (155)-(158) are the results of our calculation of the spin splittings of $M_2^2$ in the ground state of positronium. These were derived in the form of Eq. (250) with result

$$b_4(\text{triplet}) - b_4(\text{singlet}) = \frac{7}{3}.$$  \hspace{0.5cm} (254)

Given Eq. (253) this implies

$$B_{\text{triplet}} - B_{\text{singlet}} = \frac{7}{6} \alpha^2 \text{Ryd} + \mathcal{O}(\alpha^5).$$ \hspace{0.5cm} (255)

This we recognize as the well known result for the positronium system. A final note is that if the physical values of the fine structure constant and Rydberg energy ($\frac{1}{137.0}$ and 13.60 eV respectively) are applied to this previous formula, the result agrees with experiment to one-half of a percent \([19]\).
References

[1] K.G. Wilson, T. S. Walhout, A. Harindranath, W. M. Zhang, R. J. Perry and S. D. Glazek, Phys. Rev. D 49, 6720 (1994).

[2] R.J. Perry, in Proceedings of Hadrons 94, edited by V. Herscovitz and C. Vasconcellos (World Scientific, Singapore, 1995), hep-th/9407056.

[3] R. J. Perry, in Proc. Fourth Int. Workshop on Light-Front Quantization and Non-Perturbative Dynamics, edited by S. D. Glazek (World Scientific, Singapore, 1995), p. 56, hep-th/9411037. K. G. Wilson and D. G. Robertson, *ibid.*, p. 15, hep-th/9411007. K. G. Wilson and M. M. Brisudová, *ibid.*, p. 166, hep-th/9411008.

[4] W. M. Zhang, IP-ASTP-19-95, 1995, hep-ph/9510428.

[5] M. M. Brisudová and R. J. Perry, Phys. Rev. D 54, 1831 (1996); M. M. Brisudová, 1996, hep-ph/9604386; M. M. Brisudová, R. J. Perry and K. G. Wilson, 1996, hep-ph/9607280.

[6] M. Kaluza and H. J. Pirner, Phys. Rev. D 47, 1620 (1993).

[7] R. J. Perry and K. G. Wilson, Nucl. Phys. B 403, 587 (1993); R. J. Perry, Ann. Phys. 232, 116 (1994), p. 123.

[8] S. D. Glazek and K. G. Wilson, Phys. Rev. D 48, 5863 (1993); S. D. Glazek and K. G. Wilson, Phys. Rev. D 49, 4214 (1994).

[9] F. Wegner, Ann. Physik 3, 77 (1994).

[10] W. M. Zhang and A. Harindranath, Phys. Rev. D 48, 4868 (1993); *ibid.* 4881; *ibid.* 4903.

[11] G. Baym, Lectures on Quantum Mechanics (Addison-Wesley Pub. Co., California, 1990).

[12] V. Fock, Z. Phys. 98, 145 (1935). Further references on this subject are: J. Avery, Hyperspherical Harmonics: Applications in Quantum Theory (Kluwer Academic Publishers, The Netherlands, 1989) and M. Sawicki, Phys. Rev. D 32, 2666 (1985).
[13] B. H. Allen, “Light-Front Time-Ordered Perturbation Theory Rules,” OSU technical report, 1997 (unpublished).

[14] B. R. Judd, *Angular Momentum Theory for Diatomic Molecules* (Academic Press, New York, 1975), p. 32.

[15] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York, 1975), pp. 98-100.

[16] B. R. Judd, *Angular Momentum Theory for Diatomic Molecules* (Academic Press, New York, 1975), pp. 32, 37.

[17] *ibid.*, p. 222.

[18] L. Hostler, Journal of Math. Phys. 5, 591 (1964); L. Hostler, Journal of Math. Phys. 5, 1235 (1964); J. Schwinger, Journal of Math. Phys. 5, 1606 (1964).

[19] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), p. 493.

Figure captions

Figure 1: This illustrates the spin and momenta label conventions used in this paper.

Figure 2: $\delta M_1^2$ is the part of the ground state mass spin splittings from Eq. (144). $m$ is the electron mass and $\alpha$ is the fine structure constant. The state labels 1, 2, 3 and 4 are explained in Eq. (151). The two upper most levels should coincide in a rotationally invariant theory.

Figure 3: The combined ground state mass spin splitting in positronium to order $\alpha^4$ is illustrated using the same notation as in Figure 2. $\delta M_2^2$ is given by Eq. (152) and is calculated in Appendix C. The final combined result (on the right) corresponds to a rotationally invariant theory.
Figure 1
Figure 2

\[ \delta M_1^2 \]
\[ \frac{2}{2 m^2 \alpha^4} \]

\[ \begin{array}{c}
|1> \\
\frac{1}{2} \\
\frac{1}{3} \\
0 \\
\frac{\delta M_1^2}{2 m^2 \alpha^4} \\
\frac{3}{2} \\
|1> \\
|2> \\
|3>, |4> \\
\end{array} \]
