Kinematics of stock prices

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Abstract

We investigate the general problem of how to model the kinematics of stock prices without considering the dynamical causes of motion. We propose a stochastic process with long-range correlated absolute returns. We find that the model is able to reproduce the experimentally observed clustering, power law memory, fat tails and multifractality of real financial time series. We find that the distribution of stock returns is approximated by a Gaussian with log-normally distributed local variance and shows excellent agreement with the behavior of the NYSE index for a range of time scales.

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Remarkable progress has been made in quantitatively describing non-stationary and non-Gaussian phenomena, including those observed in economic and social systems. The behavior of financial markets has recently become a focus of interest to physicists as well as an area of active research because of its rich and complex dynamics. The daily returns $r_t$ for a given stock can be defined as

$$r_t = \log_{10} \frac{S_{t+1}}{S_t},$$

at time $t$ (Fig. 1), where $S_t$ is the price of an asset or the value of an index or a currency exchange rate. In this paper we address the important yet unsolved problem of how to model the kinematics of stock prices without handling the problem of the causes of motion. Our point of departure is the family of ARCH-GARCH models. These families of stochastic kinematic models neglect the causes underlying the price variations and focus only on the equations of motion governing the fluctuating returns. Specifically, the interacting buying and selling processes are not considered. ARCH models, introduced by Engle in 1982, are stochastic Auto-Regressive Conditional Heteroscedasticity models, characterized by zero mean and non-constant variances dependent on the past. Such models can simultaneously have global stationarity as well as local non-stationary behavior. The simplest ARCH model of stock returns can be defined by

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2,$$

with returns defined as

$$r_t = \sigma_t \omega_t,$$

where the random Gaussian variable $\omega_t$ has unitary variance and zero mean, while $\alpha_0$ and $\alpha_1$ are tunable parameters. Hence, the returns $r_t$ are Gaussian distributed with zero mean and local variance $\sigma_t^2$. Only very recently has the physics community begun to study such models, although they are common in the economics literature. In more general ARCH processes, the local variance can depend not only on the previous value $r_{t-1}$, but on any finite number $n$ of previous values $r_{t-1} \ldots r_{t-n}$. GARCH models are a further generalization in which the local variance can depend not only on previous values of the returns but also on the previous values of the locally measured variances.

ARCH-GARCH models have succeeded in capturing important features (such as volatility clustering) of real financial data. They are, therefore, widely used, both in the financial
community and in econophysics to model the kinematics of price variations. Nevertheless, several characteristic features of the real data are not well described. The three most important (and difficult to model) features not well accounted in ARCH-GARCH models are (i) the time scaling of the probability density distribution of the returns, (ii) the known long-range power law correlations in the volatility (the latter being a measure of the local standard deviation of the returns), and (iii) the multifractality of the returns, i.e., volatility power law correlation exponents are non-unique and depend on the magnitude of the events considered. As an alternative model, Lévy [21] and truncated (see ref. [22]) Lévy distributions have been proposed to fit the observed fat tailed distributions. While Lévy processes can correctly reproduce the time scaling of distributions, they cannot explain one of the most relevant and characteristic phenomena: the multifractal long-range correlations in volatility which has been found to be responsible for clustering of volatility and the persistence of fat tails for long time lags [4, 23].

Here we propose a new model that does not suffer from these drawbacks. In order to test the model, we compare the model with a typical dataset: the New York Exchange (NYSE) daily composite index price closes from January 1966 to September 2001 (a total of some 9000 data points).

The model is inspired in the recent finding that the probability density distribution of the local variance $\sigma_t$ is similar to a log-normal [4] . Other key findings include the long-range correlations found in the absolute value of the returns and the multifractal behavior of the returns [7]. Our proposal, based on these findings, is the following map for the evolution of the volatility:

$$\sigma_{t+1} = e^{[a+b\tilde{\omega}_{t+1}]} (\sigma_t)^d ,$$  \hspace{1cm} (1)

with returns defined as

$$r_t = \sigma_t \tilde{\omega}_t ,$$  \hspace{1cm} (2)

where $\tilde{\omega}_t$ and $\omega_t$ are independently distributed unitary Gaussian variables with zero mean that are also independent from each other. Note that the independence of these two variables contrasts with ARCH models, for which they coincide. We discuss why this is preferable below. The tunable constants $a$, $b$ and $d$ are related to the nearly log-normal form of $\sigma_t$. The value of $d$ must be close but smaller than 1 to guaranty the stationarity of the returns; $a$ is related to the typical size of the daily volatility, and $b$ to the typical size of its fluctuations.
The term $\sigma_T$ represents the average with respect to last $T$ days, from $t - T$ to $t$. (We have used below a simple unweighted average but, in principle, we could have used any type of moving average, for instance a power law or exponentially weighted moving average.)

Using this map, we generate a data set of 9000 returns (Fig. 1(b)) with the following choice of constants: $a = -0.1$, $b = 0.1$, $d = 0.98$ and $T = 10$. We then compare the statistical and scaling properties of the time series generated with the real data set. Note that the above model can reproduce very well the clustering of volatility, as is evident comparing Fig. 1(a) with Fig. 1(b). Nevertheless, we proceed with a quantitative comparison of both the scaling of the distribution and the multifractal power law correlations.

We next study the probability density of returns for the proposed model. We estimate the probability density of returns using

$$P(r) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sqrt{2\pi \Delta}} \exp\left\{ -\frac{(r - r_i)^2}{2\Delta^2} \right\}$$

with a smoothing window of $\Delta = 0.001$ (Fig 2a). The $r_i$ can be either the real or artificial data and $N$ is their number (about 9000 for both). The dashed and dotted lines in Fig. 2(a) show the distributions of daily (one business day) returns, for both real and artificial data. Typical of financial times series is their invariance under rescaling of time. Therefore, we estimate with the same smoothing window the distribution for monthly (25 business days) returns (Fig 2b)

$$\frac{\sum_{i=1}^{25} r_{i+t}}{(25)^{\delta}} = \frac{1}{(25)^{\delta}} \log \frac{S_{t+25}}{S_t},$$

using the measured value $\delta = 0.53$ of the scaling exponent both for real and artificial data. Note that the theoretical curve (solid line, see discussion below) is exactly the same in the two figures, therefore both real and artificial data exhibit almost perfect time scale invariance of the return distributions (see also [23]). The lack of agreement for small monthly returns is due to short-range (1 day) correlations in the signs of the returns that are neglected in our model, but that could easily be incorporated [23]. Compared to other models of stock returns, this model shows remarkably good agreement with real data.

We next compare these probability distributions of real and artificial data with a “theoretical” prediction (solid line in Figs. 2). The model proposed leads to a distribution that is a Gaussian $P_t$ with log-normally distributed local variance. We can find $P_t$ by convolving
Gaussians of varying widths:

\[ P_t(r) = \int \rho(\sigma) \frac{e^{-\frac{r^2}{2\sigma^2}}}{\sqrt{2\pi\sigma}} d\sigma \]  

(3)

\[ \rho(\sigma) = \frac{1}{\sqrt{2\pi s\sigma}} e^{-\frac{1}{2} \left( \log \sigma - m \right)^2} \]  

(4)

We have used the empirically found values \( s = 0.41 \) and \( m = -0.34 \). The solid lines in Figs. leave no doubt that this distribution is one of the best candidates—if not the best—for describing the distribution of price variations. The agreement found is exceptionally strong.

It is known that daily returns have no auto-correlations for lags larger than a single day, consistent with the efficient market hypothesis. This fact can be also checked by using Detrended Fluctuation Analysis (DFA) and related methods. Consider the cumulative returns \( \phi_t(L) \), defined as the sum of \( L \) successive returns divided by \( L \):

\[ \phi_t(L) = \frac{1}{L} \sum_{i=1}^{L} r_{t+i} \]  

(5)

One can define \( N/L \) non overlapping variables of this type, and compute the associated variance \( \sigma^2(\phi(L)) \), where \( N \approx 9000 \) is the number of data points. According to the central limit theorem, uncorrelated (or short-range correlated) \( r_t \) would lead to power-law behavior:

\[ \sigma^2(\phi(L)) \sim L^{-\alpha} , \]  

(6)

with exponent \( \alpha = 1 \) for large \( L \). The exponent \( \alpha \) both for the NYSE index and the model proposed here is near 1 (see Fig. 3), confirming that returns are uncorrelated.

The lack of correlations does not hold true for quantities related to the absolute returns. In order to perform the appropriate scaling analysis, we introduce the generalized cumulative absolute returns defined as the sum of \( L \) successive absolute return powers \( |r_t|^{\gamma}, ..., |r_{t+L-1}|^{\gamma} \), divided by \( L \)

\[ \phi_t(L, \gamma) = \frac{1}{L} \sum_{i=1}^{L} |r_{t+i}|^{\gamma} \]  

(7)

where \( \gamma \) is a real exponent (noting that these quantities do not overlap.) Using this method, if the autocorrelation for powers of absolute returns exhibits a power-law with exponent \( \alpha(\gamma) \leq 1 \) for large \( L \), then we expect

\[ \sigma^2(\phi(L, \gamma)) \sim L^{-\alpha(\gamma)} . \]  

(8)
(Note, however, that if the $|r_t|^\gamma$ are short-range correlated or power-law correlated with an exponent $\alpha(\gamma) > 1$, then we would not detect anomalous scaling in the analysis of variance, because it is not possible to detect $\alpha > 1$ using the method discussed here).

We find that both the real data and the model show similar multifractal behavior with non-unique scaling exponents, i.e., $\alpha(\gamma) \neq \text{constant}$. Note that the function $\alpha(\gamma)$ is not universal [4, 7] but depends on the particular asset considered. Moreover, the values $\alpha(\gamma)$ we compute do not coincide for real and artificial data, nevertheless this is not a drawback if one takes into account the non-universality. For completeness, we note that in order to obtain full agreement for the multifractal exponents of any specific economic series one must generalize the volatility as follows:

$$x_{t+1} = e^{a+b\omega_{t+1}} \left(\bar{x}_t\right)^d. \tag{9}$$

where the exponent $c$ in the volatility average would allow for a finer control of the effects caused by extremal events. Also the resulting returns could be written as

$$r_t = (\alpha + x_t) \cdot \tilde{\omega}_t \equiv \sigma_t \tilde{\omega}_t. \tag{10}$$

with $\alpha$ representing an intrinsic constant contribution to daily volatility.

We now comment on the motivation for using two independent Gaussian variables, one for returns and one for volatility. Indeed, this choice separates the temporal evolution of volatility and returns. Besides giving reasonable agreement with experimental data, there is a deeper underlying motivation for this approach. In ARCH-GARCH models the present-day volatility depends on the previous day’s absolute return. Any trader knows that today’s sentiment about volatility does not depend on the previous day’s variation of price but rather on other things like previous day intraday volatility, implicit volatility in derivative products and public (or less public) information. Therefore, the previous day’s absolute return (which can be small after a fearful market day with enormous variation of prices) does not directly influence the present day volatility and so the evolution of the latter should be kept separate, exactly as we have done in the model proposed.

In conclusion, the proposed stochastic multifractal model of long-range correlated absolute returns is able to reproduce features of real financial data that are not well accounted for by existing models. Specifically, our model is able to overcome the inability of ARCH-GARCH and Lévy models to adequately explain (i) the fat tailed distribution of the returns
and its time scaling, (ii) the long-range power law volatility correlations, or (iii) the multifractality of the returns. The model presented here also represents an advance due to the exceptionally improved agreement with real data. We hope that this advance in modeling the kinematics of financial time series further contributes to the emerging study of econophysics and towards a better understanding of wider classes of complex systems.

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FIG. 1: Real (a) and artificial data (b) obtained from the maps in Eqs. 1 and 2 with constants $a = -0.1$, $b = 0.1$, $d = 0.98$ and $T = 10$. The striking resemblance between the time series generated by the kinematic model and the real data is further verified by quantitative analysis, as explained in the text.
FIG. 2: Symmetrized probability density distribution $p(r)$ of the returns $r$ measured over lags (a) $\tau = 1 \, \text{d}$ and (b) $\tau = 25 \, \text{d}$ (about one business month), shown for the artificial data (dashed line), the real data (long dashed) and compared with the “theoretical” distribution (solid line)). For $\tau = 1 \, \text{d}$ all three distributions are almost identical while for $\tau = 25$ a small discrepancy appears for real data for small returns. This discrepancy is due to short range correlations in real data not included in the model, as explained in the text.
FIG. 3: Variance $\sigma^2(\phi(L, \gamma)) \sim L^{-\alpha(\gamma)}$ of the generalized cumulative absolute returns as a function of $L$ on double log scales for (a) the NYSE index and (b) the proposed kinematic model. Data for the absolute moments with $\gamma = 2$ (square) and $\gamma = 4$ (diamond) are compared with the variance $\sigma^2(\phi(L)) \sim L^{-\alpha}$ of the cumulative returns (circles). Both data and model clearly show multifractal behavior, since there is no unique scaling exponent. The exponents of the best fitting straight lines (dashed lines) are, respectively: $\alpha(2) = 0.734 \pm 0.013, \alpha(4) = 0.979 \pm 0.003$ and $\alpha = 1.007 \pm 0.01$ for the NYSE index; $\alpha(2) = 0.386 \pm 0.008, \alpha(4) = 0.556 \pm 0.009$ and $\alpha = 1.025 \pm 0.008$ for the model. The ability of this kinematic model to generate multifractal behavior distinguishes it from the well-known families of models.