Unbreakable $\mathcal{PT}$ symmetry of exact solitons supported by transversally modulated nonlinearity acting as a pseudopotential

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We demonstrate analytically and numerically the existence of exact solitons in the form of double-kink and fractional-transform supported by a symmetric transversally modulated defocusing nonlinearity acting as a pseudopotential combined with an antisymmetric gain-loss profile. We explicate here that the $\mathcal{PT}$ symmetry is never broken in the dynamical system under study, even in the absence of any symmetric modulation of linear refractive-index, and the transversally modulated defocusing nonlinearity comes in the way as a requirement to establish the ensuing $\mathcal{PT}$ symmetry.

I. INTRODUCTION

Much attention has been paid to the study of light propagation in parity-time ($\mathcal{PT}$) symmetric optical media. As demanded by quantum mechanical notion that every physical observable associated with a real spectrum must be Hermitian, these $\mathcal{PT}$-symmetric optical structures deliberately exploit this quantum mechanical notion of parity and time reversal symmetry of non-Hermitian Hamiltonian. Indeed, a non-Hermitian Hamiltonian having unbroken $\mathcal{PT}$-symmetry possesses entirely real and positive energy eigenvalues and may describe a physically viable system without violating any of the postulates of quantum mechanics.

In order to implement these $\mathcal{PT}$-symmetric settings in optics, it amounts to combine spatially symmetric refractive-index landscapes with mutually balanced spatially separated gain and loss. These ideas were proposed in Refs. [1-8] and demonstrated in Refs. [9-11]. Subsequently, these works had drawn a great deal of attention to models of optical systems featuring the $\mathcal{PT}$-symmetry [12]. A majority of such models actually include Kerr nonlinearity, and they are modeled by nonlinear Schrödinger equation with a complex potential, whose real and imaginary parts are said to be spatially even and odd. Some of the interesting situations occur when the underlying evolution equations contain only nonlinear $\mathcal{PT}$-symmetric terms [13, 14] or mixed linear-nonlinear lattices [15-17].

In this paper, we propose to explicate the existence of exact chirped solitons in the form of double-kink and fractional-transform supported by a symmetric transversally modulated defocusing nonlinearity acting as a pseudopotential combined with an antisymmetric gain-loss profile. We demonstrate here that the $\mathcal{PT}$ symmetry is never broken in the dynamical system under study, even in the absence of any symmetric modulation of linear refractive-index. For a homogeneous nonlinearity, the $\mathcal{PT}$ symmetry is always broken, but for the transversally modulated defocusing nonlinearity we demonstrate analytically that the ensuing $\mathcal{PT}$ symmetry is never broken. We corroborate this fact for two specific examples of nonlinearity and the gain-loss profiles. Recently, for a special case of $\mathcal{PT}$-symmetric Hamiltonian system, the phenomenon of unbreakable symmetry was demonstrated for a dimer [18]. In the present work, we show that in the presence of the nonlinearity modulation, the symmetry is said to become unbreakable, as it holds at arbitrarily large strengths of the balanced gain and loss. Furthermore, we observe that the chirp associated with each of these exact solutions is dependent on the intensity of the wave quadratically. We begin our analysis by considering optical wave propagation in self-defocusing Kerr nonlinear $\mathcal{PT}$ symmetric potential. In this case, the beam evolution is governed by the following normalized nonlinear Schrödinger equation,

$$i \frac{\partial \psi}{\partial z} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial \eta^2} + \sigma(\eta)|\psi|^2 + iG(\eta)\psi,$$

where $\psi$ is the scaled amplitude and is proportional to the electric field envelope; $z$ and $\eta$ are the propagation distance normalized to the diffraction length $k\sigma_0^2$ and transverse coordinate normalized to the characteristic transverse scale $x_0$ respectively; the function $\sigma(\eta) > 0$, which is assumed to be even, describes the profile of a self-defocusing nonlinearity; and the function $G(\eta)$, assumed to be odd, stands for the gain-loss profile. Recently, Kartashov et al [19] adopted a different variety of nonlinearity and gain-loss profiles and examined unbreakable $\mathcal{PT}$ symmetry of solitons numerically and analytically. Motivated by this work and others [20, 21], we explicate here, this phenomenon of unbreakable $\mathcal{PT}$ symmetry for two special cases of nonlinearity function and the gain-loss profiles: Case(i):— $\sigma(\eta) = \frac{m^2 \sinh^2(\eta \eta)}{e + \sinh^2(\eta \eta)}$ and $G(\eta) = \ldots$
pressed in a generic form by means of the Weierstrass
as periodic, kink and solitary wave type solutions. In
localized. For solutions like double-kink and fractional-transform solitons
parameter conditions and obtained chirped soliton solu-
tional profiles are specifically chosen because they create
tightly bound localized solitons, which are convenient for
the exact and numerical analysis alike.

II. CHIRPED LOCALIZED MODES IN PT SYMMETRIC OPTICAL MEDIA

To start with, we have chosen the following form for complex field
\[ \psi(z, \eta) = \rho(\eta) e^{i(\phi(\eta) + \gamma z)}, \] (2)
where \( \rho \) and \( \phi \) are the real functions of \( \eta \) and \( \gamma \) is the
propagation constant. The corresponding chirp is given by
\( \delta \omega(\eta) = -\frac{\gamma}{\rho}(\phi'(\eta) + \gamma \eta) = -\phi_0(\eta). \) Substituting Eq. (2) into Eq. (1) and separating out the real and imaginary
parts of the equation, we arrive at the following coupled equations in \( \rho \) and \( \phi \),
\[ \rho_{\eta\eta} - 2\gamma \rho - \phi_0^2 \rho + 2\sigma(\eta)\rho^3 = 0 \] (3)
and
\[ 2\phi_\eta \rho_\eta + \phi_\eta \rho_\eta + 2G(\eta)\rho = 0. \] (4)
To solve these coupled equations, we choose the following ansatz
\[ \phi_\eta = c_1 + c_2 \rho^2. \] (5)
Hence, chirping is given as \( \delta \omega(\eta) = -(c_1 + c_2 \rho^2) \), where
\( c_1 \) and \( c_2 \) denote the constant and nonlinear chirp parameters, respectively. It means chirping of wave is directly
proportional to the intensity of wave. Using the ansatz Eq. (5), Eqs. (3) and (4) reduce to
\[ \rho_{\eta\eta} - (2\gamma + c_1^2)\rho - 2c_1 c_2 \rho^3 - 2\sigma(\eta)\rho^3 - c_2^2 \rho^5 = 0, \] (6)
\[ G(\eta) = c_1 \frac{\rho_\eta}{\rho} + 2c_2 \rho \rho_\eta. \] (7)
The elliptic equation given by Eq. (6) can be mapped to \( \phi^8 \) field equation to obtain a variety of solutions such as
periodic, kink and solitary wave type solutions. In general, all traveling wave solutions of Eq. (6) can be expressed
in a generic form by means of the Weierstrass \( \phi \) function. In this work, we report only those soliton-like solutions for which gain profile, given by Eq. (7), remains
localized. For \( c_2 = 0 \), Eq. (6) reduces to well known cubic elliptic equation for which non-chirped soliton solutions
can be easily found. Here, we studied Eq. (6) for different
parameter conditions and obtained chirped soliton solutions. The reported solutions consist various soliton solutions like double-kink and fractional-transform solitons that respect \( PT \) symmetry. For double-kink solitons, we consider the case \( c_1 = 0 \) because for non-zero values of \( c_1 \) the corresponding gain will no longer be localized.

A. Double-kink solitons

Eq. (6) admits double-kink solutions of the form
\[ \rho(\eta) = \frac{m \sinh(n\eta)}{\sqrt{\epsilon + \sinh^2(n\eta)}}, \] (8)
where \( m = \left( -\frac{2\gamma}{2 + c_2^2} \right)^{1/4} \), \( n = \sqrt{-2\gamma} \), \( c_2 = 1 \) and \( c_1 = 0 \).
The choice \( c_1 = 0 \) has been made in order to avoid the singularities in gain profile. Here, \( \epsilon \) is a free parameter
which controls the width of soliton solutions. The interesting double-kink feature of the solution is prominent
for large values of \( \epsilon \). Now, for \( m, n \) and \( c_2 \) to be real numbers, \( \epsilon \) should be always greater than 1. For solution
given in Eq. (8), the gain will be of the following form
\[ G(\eta) = \frac{2c_2 m^2 \rho^2 \sinh(n\eta)}{(\epsilon + \sinh^2(n\eta))^{2}} \] (9)
and the chirping will be
\[ \delta \omega(\eta) = -\frac{c_2 m^2 \sinh^2(n\eta)}{\epsilon + \sinh^2(n\eta)}. \] (10)
The variation of gain, amplitude and chirp for different
values of \( \epsilon \) is shown in Fig. 1. The parameter used in the plot is \( \gamma = -10 \). One can point out that as the value of \( \epsilon \) changes, there is a small change in the gain profile which in turn have significant effect on the amplitude and chirp of double-kink solution. Hence, one can observe that the double-kink feature of the wave is more prominent for a gain medium with large values of \( \epsilon \), and different gain medium effects only the width of the double-kink wave whereas the amplitude of the wave always remains same. From the plot of chirp, it is clear that it has a maximum at the center of the wave and saturates at some finite value of \( \eta \).

III. FRACTIONAL-TRANSFORM SOLITONS

For the parametric condition \( c_1 = 0 \), Eq. (6) can be solved for very interesting fractional-transform solitons [22, 23]. To accomplish this, one can substitute \( \rho^2 = y \) in Eq. (6) to obtain the following equation:
\[ y'' + p y^2 + q y^3 + ry + c_0 = 0, \] (11)
where \( p = -b/2, \ q = -4 - 2c_2^2, \ r = -a - 4\gamma \) and \( c_0 \) is
integration constant. For illustration purposes, here the values of \( a, \ b \) and \( c_0 \) are chosen as \(-84, \ 5.334 \) and \(-1 \) respectively. In order to find localized soliton solutions of Eq. (11), we use a fractional transformation
\[ y(\eta) = A + Bf^2(\eta), \] (12)
where the determinant \( AD - B \neq 0 \).
Our main aim is to study the localized solutions, we consider the case where \( f = \text{cn}(\eta, m) \) with modulus parameters \( m = 1 \) and \( m = 0 \). By substituting Eq. (12) into Eq. (11) and equating the coefficients of equal powers of Jacobian cn to zero, we obtain the following consistency conditions for Jacobian elliptic modulus \( m = 1 \):

\[
qA^3 + pA^2 + rA - c_0 = 0, \quad (13)
\]

\[
4(B - AD) + 2pB + pA^2D + 3A^2B
+ rB + 2rAD - 3Dc_0 = 0, \quad (14)
\]

\[
6(AD - B) + 4D(AD - B)pB^2 + 2pBD
+ 3AB^2 + rAD^2 + 2rBD - 3D^2c_0 = 0, \quad (15)
\]

\[
2D(B - AD) + pB^2D + qB^3 + rBD^2 - D^3c_0 = 0. \quad (16)
\]

The set of Eqs. (13) to (16) can be solved consistently for the unknown parameters \( A, B, D \) and for a particular value of \( c_0 \).

### A. Soliton solution

The generic profile of the solution reads

\[
y(\eta) = \frac{A + B \text{sech}^2 \eta}{1 + D \text{sech}^2 \eta}.
\]

(17)

And, \( \rho(t) \) can be written as

\[
\rho(\eta) = \sqrt{\frac{A + B \text{sech}^2 \eta}{1 + D \text{sech}^2 \eta}}.
\]

Since the analytical form of solution is known, a simple maxima-minima analysis can be done to distinguish parameter regimes supporting dark and bright soliton solutions. In this case, when \( AD < B \) one gets a bright soliton, whereas if \( AD > B \) then a dark soliton exists.

For soliton solution given in Eq. (15), the chirping is given by

\[
\delta \omega(\eta) = -c_2 \frac{A + B \text{sech}^2 \eta}{1 + D \text{sech}^2 \eta},
\]

(19)

The amplitude, chirp and gain profiles for fractional-transform soliton are shown in Figs. 2(a), 2(b) and 2(c) respectively for \( c_0 = -1, c_2 = 1 \) and \( \gamma = 10 \). For these values, the various unknown parameters have been found to be \( A = 2.56124, B = 0 \) and \( D = 5.12248 \). Here, solution is of the form of dark soliton and has a small amplitude over a finite background. For this case, chirping is maximum at the center of the wave and is dominant away from the center.

The corresponding gain will be of the following form

\[
G(\eta) = \frac{2c_2(AD - B)\text{sech}(\eta)\tanh(\eta)}{(1 + D\text{sech}^2(\eta))^2}.
\]

(20)

### B. Trigonometric solution

For the Jacobian elliptic modulus \( m = 0 \), we obtain a very interesting trigonometric solution. Following are the consistency conditions we obtain for this Jacobian elliptic modulus:

\[
qA^3 + pA^2 + rA - (c_0 + 1) = 0, \quad (21)
\]

\[
4(AD - B)(2 + 3D) + 2pB + pA^2D + 3qA^2B
+ rB + 2rAD - 3Dc_0 = 0, \quad (22)
\]

\[
4D(AD - B) + pB^2 + 2pBD
+ 3AB^2 + rAD^2 + 2rBD - 3D^2c_0 = 0, \quad (23)
\]

\[
pB^2D + qB^3 + rBD^2 - D^3c_0 = 0. \quad (24)
\]
Chirp is given by
\[
\delta \omega(\eta) = -c_2 \frac{A + B \cos^2 \eta}{1 + D \cos^2 \eta},
\]
(27)

The corresponding gain will be of the following form
\[
G(\eta) = \frac{2c_2(AD - B) \cos(\eta) \sin(\eta)}{(1 + D \cos^2(\eta))^2}.
\]
(28)

FIG. 2: (a) Amplitude, (b) Chirp and (c) Gain profiles for fractional-transform dark solitons for \(c_2 = 1\), \(c_0 = -1\) and \(\gamma = 10\).

\[
y(\eta) = \frac{A + B \cos^2 \eta}{1 + D \cos^2 \eta}.
\]
(25)

And, \(\rho(t)\) can be written as
\[
\rho(\eta) = \sqrt{\frac{A + B \cos^2 \eta}{1 + D \cos^2 \eta}}.
\]
(26)

FIG. 3: (a) Amplitude, (b) Chirp and (c) Gain profiles for fractional-transform dark solitons for \(c_2 = 1\), \(c_0 = -1\) and \(\gamma = 10\).
FIG. 4: Intensity evolution of nonlinear mode for (a) fractional-transform trigonometric solution and (b) fractional-transform soliton solution.

IV. STABILITY OF THE LOCALIZED MODES IN $\mathcal{PT}$ SYMMETRIC OPTICAL MEDIA

In this section, we study the stability of the $\mathcal{PT}$ symmetric periodic and hyperbolic fractional-transform solitons under the evolution of Eq. (1), numerically using the Crank-Nicolson finite-difference method, which is unconditionally stable. Figures 4(a) and 4(b), depict the perturbed solution with the initial condition $\psi(\eta, z = 0) = \psi(\eta, z = 0) + \epsilon$, where $\epsilon$ is a function which assumes a random value at each point. It is found that the maximum value of $\epsilon$ is 10% of the peak value of the intensity profile. One can clearly see from Fig. 4(a) and Fig. 4(b), which are the $z$-evolution of periodic and hyperbolic fractional-transform solitons, that perturbation doesn’t destabilize the solutions. We have checked addition of random noise up to 30% of the peak value of the intensity profile; the solitons are found to be quite stable. In the case of periodic soliton, as evidenced from Fig.4 (a), the fractional-transform soliton is found to be quite stable, although the peak of the intensity oscillates. The step size $d\eta$ and $dz$ were taken as 0.01 and 0.0001, respectively, in these simulations.

V. CONCLUSION

In conclusion, we have demonstrated the existence of exact unbreakable $\mathcal{PT}$ symmetric chirped double-kink solitons and fractional-transform solitons in nonlinear optical media for the propagation of a laser beam. We have exemplified this phenomenon of unbreakable $\mathcal{PT}$ symmetry for two specific cases of nonlinearity function and gain-loss profiles. Toward the end, we have conducted numerical stability experiments of the $\mathcal{PT}$ symmetric localized excitations—periodic and hyperbolic fractional-transform solitons—using a semi-implicit Crank-Nicolson finite difference algorithm and found that they are quite stable, against finite perturbations. Even though there can be other exact solutions found out for this dynamical system, all of them may not satisfy the $\mathcal{PT}$ symmetry.

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