Locally Repairable Codes with Multiple \((r_i, \delta_i)\)-Localities

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Abstract—In distributed storage systems, locally repairable codes (LRCs) are introduced to realize low disk I/O and repair cost. In order to tolerate multiple node failures, the LRCs with \((r, \delta)\)-locality are further proposed. Since hot data is not uncommon in a distributed storage system, both Zeh et al. and Kadhe et al. focus on the LRCs with multiple localities or unequal localities (ML-LRCs) recently, which said that the localities among the code symbols can be different. ML-LRCs are attractive and useful in reducing repair cost for hot data. In this paper, we generalize the ML-LRCs to the \((r, \delta)\)-locality case of multiple node failures, and define an LRC with multiple \((r_i, \delta_i)\)-localities \((s \geq 2)\), where \(r_1 \leq r_2 \leq \cdots \leq r_s\) and \(\delta_1 \geq \delta_2 \geq \cdots \geq \delta_s \geq 2\). Such codes ensure that some hot data can be repaired more quickly and have better failure-tolerance in certain cases because of relatively smaller \(r_i\) and larger \(\delta_i\). Then, we derive a Singleton-like upper bound on the minimum distance for the proposed LRCs by employing the regenerating-set technique. Finally, we obtain a class of explicit and structured constructions of optimal ML-LRCs, and further extend them to the cases of multiple \((r_i, \delta_i)\)-localities.

I. INTRODUCTION

Recently, locally repairable codes (LRCs) have attracted a lot of interest. Let \(F_q\) be a finite field with size \(q\). The \(i\)th symbol \(c_i\) of a \(q\)-ary \([n, k]\) linear code \(C\) said to have locality \(r\) if this code can be recovered by accessing at most \(r\) other symbols of \(C\). In distributed storage systems, \(r \ll k\) indicates that only a small number of storage nodes are involved in repairing a failed node, which means low disk I/O and repair cost. The code is called an \((n, k, r)\) LRC or an \(r\)-local LRC. It was shown that the minimum distance \(d\) is upper bounded by

\[
d \leq n - k - \lceil k/r \rceil + 2. \tag{1}
\]

An LRC meeting this Singleton-like bound is called optimal. Various constructions of optimal LRCs were obtained recently, e.g., \([1, 7]\).

In order to tolerate multiple node failures in a distributed storage system, an important extension to the \(r\)-local codes is the so-called LRC with \((r, \delta)\)-locality. \([8]\). According to \([8]\), the \(i\)th symbol \(c_i\) of a \(q\)-ary \([n, k]\) linear code \(C\) is said to have \((r, \delta)\)-locality \((\delta \geq 2)\) if there exists a punctured subcode of \(C\) with support containing \(i\), whose length is at most \(r + \delta - 1\), and whose minimum distance is at least \(\delta\), i.e., there exists a subset \(S_i \subseteq [n] \triangleq \{1, 2, \ldots, n\} \) such that \(i \in S_i\), \(|S_i| \leq r + \delta - 1\) and \(d_{\min}(C|S_i) \geq \delta\). The code \(C\) is said to have \((r, \delta)\)-locality or be a \((r, \delta)\)-LRC if all the symbols have \((r, \delta)\)-localities. And a Singleton-like bound was also obtained, which said that

\[
d \leq n - k + 1 - (\lceil k/r \rceil - 1)(\delta - 1). \tag{2}
\]

The codes meeting it are called optimal \((r, \delta)\)-LRCs. Note that it degenerates to \((1)\) when \(\delta = 2\). Many optimal constructions of \((r, \delta)\)-LRCs can be founded in \([7, 12]\).

Cows with multiple localities or unequal localities were firstly introduced in \([13]\) and \([14]\), which said that the locality among the code symbols can be different. Such an LRC with multiple localities is practically appealing in hot data (i.e., the data is accessed more frequently) that need to be repaired quickly and thus require smaller locality. More specifically, a code \(C\) with unequal information locality \([14]\) was interpreted as follows: the information locality profile of an \([n, k, d]\) linear code \(C\) is defined by a length-\(r\) vector \(k(C) = \{k_1, \ldots, k_r\}\), where \(k_j\) is the number of information symbols with locality \(j\) for \(j \in [r]\). Clearly, \(\forall j \in [r], 0 \leq k_j \leq k, k_r \geq 1\) and \(\sum_{j=1}^{r} k_j = k\). An upper bound on the minimum distance was obtained as

\[
d \leq n - k - \sum_{j=1}^{r} \lceil k_j/j \rceil + 2. \tag{3}
\]

The code with all-symbol multiple localities or unequal all-symbol locality was introduced respectively in \([13]\) and \([14]\). Two different forms of upper bounds on the minimum distance were also obtained according to different restrictive conditions. In this paper, we adopt the definition of the LRC with all-symbol multiple localities (ML-LRC) in \([13]\). Let \(s \geq 2\) and \(T_1, T_2, \ldots, T_s\) be a partition of \([n]\), i.e., \(\bigcup_{i \in [s]} T_i = [n]\) and \(T_i \cap T_j = \emptyset\). Let \(n_i = |T_i|\). A \(q\)-ary \([n, k, d]\) linear code is called \((n_1, n_2, r_2), \ldots, (n_s, r_s)\)-local \((r_1 < r_2 < \cdots < r_s)\) if each code symbols in a set \(T_i\) are a linear combination of at most \(r_i\) other code symbols within \(T_i\) for all \(i \in [s]\). In \([13]\), a Singleton-like upper bound for the code is obtained, i.e., if

\[
d \leq n - k + 2 - \sum_{i \in [s-1]} \left\lfloor \frac{n_i}{r_i + 1} \right\rfloor - \left\lfloor \frac{k - \sum_{i \in [s-1]} r_i \left\lfloor \frac{n_i}{r_i + 1} \right\rfloor}{r_s} \right\rfloor. \tag{4}
\]

It was proved that an optimal \(r\)-local LRC can be shortened to an optimal ML-LRC with respect to bound \((4)\). For the case of two localities \((s = 2)\), \([13]\) gave an explicit algorithm that described the structure of the parity-check matrix for an optimal ML-LRC. To the best of our knowledge, direct and structured constructions of optimal ML-LRCs with respect
to bound \( \Phi \) have not yet been obtained except the above shortening technique.

Just like that the \((r, \delta)\)-locality generalizes the \(r\)-local LRC, it is naturally to add similar features to the ML-LRCs. In this paper, we introduce the LRCs with all-symbol multiple \((r_i, \delta_i)\)-localities \((s \geq 2)\). Comparing with ML-LRCs, an LRC with multiple \((r_i, \delta_i)\)-localities could not only locally recover a single failed node, but also tolerate multiple nodes failures in other nodes among every \(n_i \) nodes. Moreover, the parameters satisfy \( r_1 \leq \cdots \leq r_s \) and \( \delta_1 \geq \cdots \geq \delta_s \geq 2 \), which make the code more useful and attractive in some practical scenarios, e.g., when the distributed storage system employs such a multiple \((r_i, \delta_i)\)-localities LRC, some hot data can be repaired quickly with the smaller locality \( r_i \) while having a better erasure-tolerance with the larger \( \delta_i \).

By employing the regenerating-set technique of Wang and Zhang \cite{15}, we derive a Singleton-like upper bound on the minimum distance for the LRCs with multiple \((r_i, \delta_i)\)-localities. Then, we construct a class of explicit and structured optimal ML-LRCs by employing the parity-splitting technique, and further extend them to the cases of multiple \((r_i, \delta_i)\)-localities.

The rest of this paper is organized as follows. In Section \( II \) the concept of regenerating sets in \cite{15} are recalled. In Section \( III \) we firstly deal with the case of two \((r_i, \delta_i)\)-localities, and then the general one of multiple \((r_i, \delta_i)\)-localities, where Singleton-like bounds are obtained. Section \( IV \) studies the optimal constructions. Finally, we conclude the paper in section \( V \).

\section{Preliminaries}

In this section, we give some preliminaries of regenerating sets, which was proposed by Wang and Zhang \cite{15} to prove some minimum distance bounds, e.g., the Singleton-like bounds and the integer programming-based bound \cite{16}. We will also use this technique to derive bounds in the next section.

Let \( \mathbb{F}_q \) be a finite field with size \( q \), where \( q \) is a prime power. An \([n, k, d]_q \) linear code \( C \) is a \( q \)-ary linear code with length \( n \), dimension \( k \) and minimum distance \( d \).

\begin{definition} \cite{15} \end{definition}

For an \([n, k, d]_q \) linear code \( C \), a regenerating set of the \( i \)-th coordinate, \( 1 \leq i \leq n \), is a subset \( R_i \subseteq [n] \triangleq \{1, 2, \ldots, n\} \) such that \( i \in R_i \) and \( \mathbf{g}_i \) is an \( \mathbb{F}_q \)-linear combination of \( \{\mathbf{g}_j\}_{j \in R_i \setminus \{i\}} \), where \( \mathbf{g}_i \) denotes the \( i \)-th column vector of the generator matrix \( \mathbf{G} \) of code \( C \).

The collection of all regenerating sets of the \( i \)-th coordinate is denoted by \( \mathcal{R}_i \). Furthermore, a sequence of regenerating sets \( R_1, R_2, \ldots, R_m \), where \( R_i \subseteq \mathcal{R}_i \) and \( i \in [n] \) for \( 1 \leq i \leq m \), is said to have a nontrivial union if \( l_j \not\subseteq \bigcup_{i=1}^{j-1} R_i \) for \( 1 \leq j \leq m \).

Remark 1: For the regenerating set \( R_i \) of the \( i \)-th coordinate, it is called \emph{minimal} if there is no proper subset \( R' \subset R_i \setminus \{i\} \) such that \( \mathbf{g}_i \) is an \( \mathbb{F}_q \)-linear combination of \( \{\mathbf{g}_j\}_{j \in R'} \). In the rest of this paper, without the loss of generality, we always assume that a regenerating set is minimal, and under this stricter definition, the set \( R \) is a regenerating set of each of its elements. Moreover, on one hand, the regenerating sets \( R_1, R_2, \ldots, R_m \) with a nontrivial union implies that \( R_j \not\subseteq \bigcup_{i=1}^{j-1} R_i \) for \( 1 \leq j \leq m \); and on the other hand, if \( R_j \not\subseteq \bigcup_{i=1}^{j-1} R_i \) for \( 1 \leq j \leq m \), it is clear that there exist \( l_j \in R_i \), \( i = 1, \ldots, m \), such that \( l_j \not\in \bigcup_{i=1}^{j-1} R_i \) for \( 1 \leq j \leq m \).

For a linear code \( C \), define the function \( \Phi(x) = \min \{ \bigcup_{i=1}^{j-1} R_i : R_i \in R_i, \text{ and } R_1, \ldots, R_x \text{ have a nontrivial union} \} \). It is easy to see that \( \Phi(x+1) \geq \Phi(x) + 1 \), which implies that \( \Phi(x+1) - (x+1) \geq \Phi(x)-x \), or \( \Phi(x)-x \) is increasing by \( x \). The following theorem gives a general upper bound of the minimum distance \( d \).

\begin{proposition} \cite{15} \end{proposition}

For an \([n, k, d]_q \) linear code, \( d \leq n - k + 1 - \rho \), where \( \rho = \max \{ x : \Phi(x) - x < k \} \).

Next we give an alternative proof of Proposition 1 by employing a parity-check matrix approach \cite{5}. By Definition 1 and Remark 1, it is clear that \( R_i \) is a \( (n, k, d) \)-locality, \( R_i \) satisfies \( \Phi(x) \geq \Phi(x)-x \), and \( R_1, \ldots, R_x \) have a non-trivial union and \( \Phi(x) = \bigcup_{i=1}^{j-1} R_i \). Let \( e_1, \ldots, e_x \) be their corresponding parity-check equations. Since \( R_j \not\subseteq \bigcup_{i=1}^{j-1} R_i \) for \( 1 \leq j \leq m \), it is clear that \( e_1, \ldots, e_x \) are linearly independent, which implies that \( x \leq n - k \). Let \( H \) be an \((n-k) \times n \) parity-check matrix of \( C \), where \( e_1, \ldots, e_x \) form its first \( x \) rows. By deleting the first \( x \) rows and the columns in \( \bigcup_{i=1}^{j-1} R_i \) of \( H \), we obtain an \((n-k-x) \times (n-\Phi(x)) \) submatrix \( H' \). Let \( C' \) be the \([n', k', d']_q \) linear code with the parity-check matrix \( H' \). Clearly, \( n' = n - \Phi(x) \), \( k' \geq k + x - \Phi(x) > 0 \) and \( d' \geq d \). Therefore, defining the largest possible minimum distance of an \([n', k', d']_q \) linear code by \( d_{\text{opt}}(n', k', d') \), we have the following result \cite{17}.

\begin{proposition} \end{proposition}

For an \([n, k, d]_q \) linear code, \( d \leq \min \{ d_{\text{opt}}(n - \Phi(x), k, x - \Phi(x)) : i \leq x \leq \rho, \} \), where \( \rho = \max \{ x : \Phi(x) - x < k \} \).

By invoking the well known Singleton bound for \( x = \rho \) in the right-hand side of (5), we have that \( d \leq n - k + 1 - \rho \). Hence, Proposition 1 is a naturally corollary of Proposition 2.

Definition of the \((r, \delta)\)-locality proposed in \cite{18} could be redefined in regenerating-set language as follows.

\begin{definition} \cite{15} \end{definition}

The \( i \)-th coordinate, \( 1 \leq i \leq n \), of an \([n, k, d]_q \) linear code \( C \) is said to have \((r, \delta)\)-locality if there exists a subset \( S_i \subseteq [n] \) satisfying

\begin{enumerate}
  \item \( i \in S_i \), \( \delta \leq |S_i| \leq r + \delta - 1 \); and
  \item For any \( E \subseteq S_i \) with \(|E| = \delta - 1 \), and for any \( j \in E \), it has \( \rho(S_i - E) \cup \{j\} \in \mathcal{R}_j \).
\end{enumerate}
III. UPPER BOUNDS FOR CODES WITH MULTIPLE $(r_i, \delta_i)_{i \in [s]}$-LOCALITIES

In this section, we firstly define LRCs with two $(r_i, \delta_i)_{i \in \{1,2\}}$ localities and provide an upper bound on the minimum distance $d$ by employing the regenerating set technique. Then, we extend the bound to LRCs with $(r_i, \delta_i)_{i \in [s]}$ ($s \geq 2$)-localities similar to [13]. The definition of LRCs with two $(r_i, \delta_i)_{i \in \{1,2\}}$ localities follows.

Definition 3. Let $T_i \subseteq [n]$ and $T_2 = [n] \setminus T_1$ be two distinct sets with $|T_i| = n_i$ for $i = 1, 2$. Let $r_1, r_2, \delta_1, \delta_2$ be integers such that $r_1 \leq r_2, \delta_1 \leq \delta_2 \geq 2$. An $[n, k, d]$ linear code $C$ is said to have two $(r_i, \delta_i)_{i \in \{1,2\}}$ localities if for $i = 1, 2$ and each coordinate $i \in T_i$, there exist a subset $S_i \subseteq T_i$ satisfying

1. $i \in S_i$, $\delta_i \leq |S_i| \leq r_i + \delta_i - 1$; and
2. For any $E \subseteq S_i$ with $|E| = \delta_i - 1$, and for any $j \in E$, it has $(S_i - E) \cup \{j\} \in R_j$.

Lemma 1. For any $[n, k, d]$ LRC $C$ with two $(r_i, \delta_i)_{i \in \{1,2\}}$ localities, if $r_1 = n_i / (r_1 + \delta_1 - 1) \leq k - 1$ and $r_1 [(\Delta - 1) / (\delta_1 - 1)] + (\Delta - 1) < n_1$, then

$$\Phi(x) \leq \begin{cases} r_1 \left\lceil \frac{x - \frac{1}{r_1}}{\delta_1 - 1} \right\rceil + x, & \text{if } 0 \leq x \leq \Delta; \\ r_1 \left\lceil \frac{n_1 - 1}{r_1 + \frac{1}{r_1}} \right\rceil + r_2 \left\lceil \frac{x - \Delta}{\delta_2 - 1} \right\rceil + x, & \text{if } \Delta \leq x \leq \rho + 1, \end{cases}$$

where $\Delta \triangleq \left\lceil \frac{n_1}{(r_1 + \delta_1 - 1) (\delta_1 - 1)} \right\rceil$.

Proof: We prove the first part by induction on $x$ as Lemma 2 in [13]. It holds trivially for $x = 0$. Assume it also holds for $x \leq x_0 \leq \Delta - 1$. Denote $x_0 + 1 = a(\delta_1 - 1) + b$, $a \in \mathbb{Z}$, $b \in [\delta_1 - 1]$. Let

$T_{a(\delta_1 - 1)} = R_1 \cup \ldots \cup R_{a(\delta_1 - 1)}$ be a nontrivial union of $a(\delta_1 - 1)$ regenerating sets such that $\Phi(a(\delta_1 - 1)) = \left| T_{a(\delta_1 - 1)} \right|$.

Note that $a(\delta_1 - 1) \leq x_0 \leq \Delta - 1$, thus

$$\Phi(a(\delta_1 - 1)) = \left| T_{a(\delta_1 - 1)} \right| \leq r_1 \left\lceil \frac{\Delta - 1}{\delta_1 - 1} \right\rceil + (\Delta - 1) < n_1,$$

which implies $T_1 \setminus T_{a(\delta_1 - 1)} \neq \emptyset$. There are two cases:

- There exists $t_0 \in T_1 \setminus T_{a(\delta_1 - 1)}$ such that $|S_{t_0} \setminus T_{a(\delta_1 - 1)}| \geq \delta_1 - 1 \geq b$. Let $E = S_{t_0} \setminus T_{a(\delta_1 - 1)}$ with $|E| = \delta_1 - 1$. Suppose $E = \{t_1, \ldots, t_{b-1}\}$. Let

$R_{j} = (S_{t_0} - E) \cup \{j\}$ for $j \in [\delta_1 - 1]$. Then $R_{j} \in R_{E}$ and $T_{a(\delta_1 - 1)} \cup \cup_{j=1}^{b} R_{j}$ is a nontrivial union. We have

$$\Phi(x_0 + 1) \leq \left| T_{a(\delta_1 - 1)} \cup R_{t_0} \right| + \ldots + \left| R_{t_b} \right| \leq \Phi(a(\delta_1 - 1)) + |S_{t_0} - E| + b$$

$$\leq \Phi(a(\delta_1 - 1)) + |S_{t_0} - E| + b$$

$$= r_1 \left\lceil \frac{x_0 + 1}{\delta_1 - 1} \right\rceil + x_0 + 1.$$

- If for any $t \in T_1 \setminus T_{a(\delta_1 - 1)}$, $|S_t \setminus T_{a(\delta_1 - 1)}| < \delta_1 - 1$. Let $R_t = (S_t \cap T_{a(\delta_1 - 1)}) \cup \{t\}$, we have $R_t \in R_{E}$. If $n_1 - \left| T_{a(\delta_1 - 1)} \right| \geq b$, we can choose $t_1, \ldots, t_b \in T_1 \setminus T_{a(\delta_1 - 1)}$ such that $T_{a(\delta_1 - 1)} \cup \cup_{j=1}^{b} R_{t_j}$ is a nontrivial union. Thus

$$\Phi(x_0 + 1) \leq \left| T_{a(\delta_1 - 1)} \cup R_{t_1} \right| + \ldots + \left| R_{t_b} \right| \leq \Phi(a(\delta_1 - 1)) + b \leq r_1 \left\lceil \frac{x_0 + 1}{\delta_1 - 1} \right\rceil + x_0 + 1.$$
Theorem 1. For an \([n, k, d] \) LRC \(C\) with two \((r_i, \delta_i) \in \{1, 2\}\) localities, if \(r_1 \left\lfloor n_1/(r_1 + \delta_1 - 1) \right\rfloor \leq k - 1\) and \(r_1 \left(\Delta - 1\right)/\left(\delta_1 - 1\right) + \left(\Delta - 1\right) < n_1\). Then
\[
d \leq n - k + 1 - \left\lfloor n_1/(r_1 + \delta_1 - 1) \right\rfloor \left(\delta_1 - 1\right) - \left(\left\lceil \frac{k - r_1 n_1/(r_1 + \delta_1 - 1)}{r_2} \right\rceil - 1\right) \left(\delta_2 - 1\right).
\]

Proof: By the definition of \(\rho\) and Lemma 1
\[
k \leq k \leq \Phi(\rho - 1) - (\rho + 1).
\]
It follows that
\[
\left\lceil \frac{k - r_1 n_1/(r_1 + \delta_1 - 1)}{r_2} \right\rceil \leq \left(\left(\frac{\rho - (\Delta + 1)}{\delta_2 - 1}\right) - 1\right) \left(\delta_2 - 1\right)
\]
thus
\[
\left(\left\lfloor \frac{k - r_1 n_1/(r_1 + \delta_1 - 1)}{r_2} \right\rfloor - 1\right) \left(\delta_2 - 1\right) \leq \left(\frac{\rho - \Delta - 1}{\delta_2 - 1}\right) \left(\delta_2 - 1\right) \leq \rho - \Delta,
\]
or
\[
\rho \geq \Delta + \left(\left\lfloor \frac{k - r_1 n_1/(r_1 + \delta_1 - 1)}{r_2} \right\rfloor - 1\right) \left(\delta_2 - 1\right).
\]

Hence, we have the desired bound (6) by Proposition 1.

Remark 2: For \(\delta_1 = \delta_2 = 2\), the condition \(r_1 \left\lfloor (\Delta - 1)/(\delta_1 - 1) \right\rfloor + (\Delta - 1) < n_1\) is naturally satisfied, and the bound (6) reduces to the bound (2) in [13]. For \(\delta_1 > 2\), the condition \(r_1 \left\lfloor (\Delta - 1)/(\delta_1 - 1) \right\rfloor + (\Delta - 1) < n_1\), i.e., \(r_1 + \delta_1 - 1 \mid n_1\). Note that an LRC with two \((r_1, \delta_1) \in \{1, 2\}\) localities is also an \((r_2, \delta_2)\)-locality LRC, it is easy to verify that the bound (6) is usually tighter than the bound (2) for \(r = r_2\) and \(\delta = \delta_2\).

If the condition of Theorem 1 is not satisfied, or \(r_1 n_1/(r_1 + \delta_1 - 1) \geq k > k - 1\), then
\[
\left\lfloor k - 1/(r_1) \right\rfloor \left(\delta_1 - 1\right) < n_1/(r_1 + \delta_1 - 1) \left(\delta_1 - 1\right) = \Delta.
\]

Hence, by Lemma 1 we have
\[
\Phi(\left\lfloor (k - 1)/(r_1) \right\rfloor \left(\delta_1 - 1\right) - \left\lfloor (k - 1)/r_1 \right\rfloor \left(\delta_1 - 1\right)) \leq \left\lceil \frac{k - 1}{r_1} \right\rceil \leq k - 1 < k.
\]
By the definition of \(\rho\), we obtain \(\rho \geq \left\lceil (k - 1)/r_1 \right\rceil \delta_1 - 1 = \left\lceil k/r_1 \right\rceil \left(\delta_1 - 1\right)\). Therefore, by Proposition 1 we have
\[
d \leq n - k + 1 - \left\lceil k/r_1 \right\rceil \left(\delta_1 - 1\right),
\]
which corresponds to the bound (2) for a code with \((r_1, \delta_1)\)-locality. Note that if \(r_1 n_1/(r_1 + \delta_1 - 1) = k - 1\), the bound (2) is identical with the bound (6).

Definition 3 can be easily generalized to a code with \((r_i, \delta_i) \in [s]\) \((s \geq 2)\) localities, and the Singleton-like bound (6) can also be generalized as follows.

Definition 4. Let \(T_1, T_2, \ldots, T_s\) be a partition of \([n]\), where \(s \geq 2\) and \(|T_i| = n_i, i \in [s]\). Let \(r_1, r_2, \ldots, r_s\) and \(\delta_1, \delta_2, \ldots, \delta_s\) be integers such that \(r_1 \leq r_2 \leq \cdots \leq r_s\), \(\delta_1 \geq \delta_2 \geq \cdots \geq \delta_s \geq 2\). An \([n, k, d]_s\) linear code \(C\) is said to have multiple \((r_i, \delta_i) \in [s]\)-localities if for \(i = 1, 2, \ldots, s\) and each coordinate \(i \in T_i\), there exist a subset \(S_i \subseteq T_i\) satisfying
\[
i \in S_i, \delta_i \leq |S_i| \leq r_i + \delta_i - 1; \text{ and}
\]
(2) For any \(E \subseteq S_i\) with \(|E| = \delta_i - 1\), and for any \(j \in E\), it has \((S_i - E) \cup \{j\} \in R_j\).

Lemma 2. For an \([n, k, d] \) LRC \(C\) with multiple \((r_i, \delta_i) \in [s]\) \((s \geq 2)\) localities, if \(\sum_{i=1}^{s-1} r_i \left\lfloor n_i/(r_i + \delta_i - 1) \right\rfloor \leq k - 1\) and
\[
r_j \left(\Delta_j - 1\right)/\delta_j + (\Delta_j - 1) < n_j,
\]
where \(\Delta_0 \equiv 0\) and \(\Delta_j \equiv \sum_{i=1}^{j} n_i/(r_i + \delta_i - 1)\). Then
\[
\Phi(x) \leq \sum_{i=1}^{j-1} r_i \left\lfloor n_i/(r_i + \delta_i - 1) \right\rfloor + r_j \left(\Delta_j - 1\right)/\delta_j + x;
\]
• For \(\Delta_{j-1} \leq x \leq \Delta_j, j = 1, 2, \ldots, s - 1\),
\[
\Phi(x) \leq \sum_{i=1}^{s-1} r_i \left\lfloor n_i/(r_i + \delta_i - 1) \right\rfloor + r_s \left(\Delta_s - 1\right)/\delta_s + x;
\]
Proof: For \(\Delta_{j-1} \leq x \leq \Delta_j, j \in [s - 1]\), we can easily prove the results by employing the method of induction as Lemma 1. For \(\Delta_{s-1} \leq x \leq \rho + 1\); note that the condition \(\sum_{i=1}^{s-1} r_i n_i/(r_i + \delta_i - 1) \leq k - 1\) also ensures that \(\rho + 1 \geq \Delta_{s-1}\). Therefore, we can similarly obtain the result by induction on \((x - \Delta_{s-1})\) as Lemma 1.

Theorem 2. For an \([n, k, d] \) LRC \(C\) with multiple \((r_i, \delta_i) \in [s]\) \((s \geq 2)\) localities satisfying the condition stated in Lemma 2 then
\[
d \leq n - k + 1 - \sum_{i=1}^{s-1} \left(\left\lfloor n_i/(r_i + \delta_i - 1) \right\rfloor \left(\delta_i - 1\right) - \left(\left\lceil \frac{k - \sum_{i=1}^{s-1} r_i n_i/(r_i + \delta_i - 1)}{r_s} \right\rceil - 1\right) \left(\delta_s - 1\right)\right)
\]
Proof: By the definition of \(\rho\) and Lemma 1
\[
k \leq k \leq \Phi(\rho + 1) - (\rho + 1)
\]
By a little calculation, we have that
\[
\rho \geq \Delta_{s-1} + \left(\left\lfloor \frac{k - \sum_{i=1}^{s-1} r_i n_i/(r_i + \delta_i - 1)}{r_s} \right\rceil - 1\right) \left(\delta_s - 1\right),
\]
which implies the bound (9) by Proposition 1.

Remark 3: For \(\delta_1 = \delta_2 = \cdots = \delta_{s-1} = \delta_s = 2\), the condition (8) is naturally satisfied, and the bound (9) reduces to the bound (3) in [13]. Otherwise, for \(\delta_{i_0} > 2\), where \(i_0 = \max\{i \in [s - 1] | \delta_i > 2\}\), the condition (8), i.e., \(r_i + \delta_i - 1 \mid n_i, i = 1, \ldots, i_0\). Moreover, the bound (9) is also tighter than the \((r_s, \delta_s)\)-bound (2).
Corollary 3. Assume the condition in Theorem 2 is satisfied and \( \delta_1 = \delta_2 = \cdots = \delta_\ell = \delta \geq 2 \), then bound (9) reduces to
\[
d \leq n - k + 1 - (\Gamma - 1) (\delta - 1),
\]
where \( \Gamma = \sum_{i=1}^{s-1} \left( \frac{n_i}{r_i + \delta - 1} \right) + \left[ \frac{k - \sum_{i=1}^{s-1} r_i n_i / (r_i + (\delta - 1))}{r_s} \right] \).

IV. OPTIMAL CONSTRUCTIONS OF ML-LRCs AND LRCs WITH MULTIPLE \((r_i, \delta)_{i \in [s]}\)-LOCALITIES

In this section, based on the parity-splitting technique of Reed-Solomon (RS) codes in [8], we firstly give an explicit and structured optimal ML-LRCS meeting the bound (4). Then the proposed constructions are generalized to the LRCs with multiple \((r_i, \delta)_{i \in [s]}\)-localities, which are optimal with respect to the bound (10).

Theorem 3. Let \( n = \sum_{i \in [s]} n_i, s \geq 2 \). Let \( r_1, r_2, \ldots, r_s \) be integers such that \( 2 \leq r_1 \leq r_2 \leq \cdots \leq r_s, r_i + 1 \mid n_i, i \in [s] \) and \( n_i = \left\lceil \frac{n}{r_i + \delta - 1} \right\rceil \). Then \( q > n \), there then exists an explicit and optimal ML-LRC over \( \mathbb{F}_q \).

Proof: Let \( H' \) be the parity check matrix of an \([n, k', d']\) RS code over \( \mathbb{F}_q \), where
\[
k' = k + \sum_{i \in [s-1]} \frac{n_i}{r_i + 1} - 1 - \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right],
\]
\[
d' = n - k' + 1 - \sum_{i \in [s-1]} \frac{n_i}{r_i + 1} - \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right].
\]

Note that if \( q > n \), such RS code always exists, and \( H'_{(n-k') \times n} \) can be equivalently transformed into a Vandermonde matrix, i.e.,
\[
H' = \begin{bmatrix}
1, 1, \ldots, 1_{\times n} \\
A_{(n-k') \times n}
\end{bmatrix}.
\]

Consider the code \( C \) whose parity check matrix \( H \) obtained by splitting the first row of \( H' \) as follows:
\[
H = \begin{bmatrix}
A_1 \\
\vdots \\
A_s
\end{bmatrix}
\]
\[
A_i = \begin{bmatrix}
1, 1, \ldots, 1_{\times n_i} \\
r_i + 1 \\
\vdots \\
r_i + 1 \\
r_i + 1 \\
r_i + 1
\end{bmatrix}.
\]

Firstly, we have \( \dim(C) \geq k \), which is due to the structure of \( H \) and
\[
\dim(C^\perp) = n - \dim(C) \leq n - k' - 1 + \sum_{i \in [s]} \frac{n_i}{r_i + 1} = n - k.
\]

Then we have
\[
d(a) = d' = n - k + 2 - \sum_{i \in [s-1]} \frac{n_i}{r_i + 1} - \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right],
\]
and
\[
d(b) = n - \dim(C) + 2 - \sum_{i \in [s-1]} \frac{n_i}{r_i + 1} - \left[ \frac{\dim(C) - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right].
\]

where (a) follows from the fact that \( C \) is a subcode of the RS code, (b) follows because the structure of \( H \) ensures the code \( C \) has multiple localities and the condition \( \frac{n_i}{r_i + 1} = \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right] \) implies \( \sum_{i \in [s-1]} r_i n_i / (r_i + 1) \leq k - 1 \). Combining (13) and (14), we have \( \dim(C) \leq k \).

Thus \( \dim(C) = k \). Combining (13) and (14), the optimality of \( C \) can also be obtained.

Based on the construction above, we can easily obtain the following general construction.

Theorem 4. Let \( n = \sum_{i \in [s]} n_i, s \geq 2 \). Let \( r_1, r_2, \ldots, r_s, \) be integers such that \( 2 \leq r_1 \leq r_2 \leq \cdots \leq r_s, r_i + \delta - 1 \mid n_i, i \in [s] \) and \( \frac{n_i}{r_i + \delta - 1} = \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right] \). If \( q > n \), then there exists an explicit and optimal multiple \((r_i, \delta)_{i \in [s]}\)-localities LRC with respect to bound (10).

Proof: Similar to Theorem 8 in [8], let \( H' \) be the parity check matrix of an \([n, k', d']\) RS code over \( \mathbb{F}_q \), where
\[
k' = k + (\Gamma - 1) (\delta - 1) = k + \left( \sum_{i \in [s]} \frac{n_i}{r_i + \delta - 1} - 1 \right) (\delta - 1),
\]
and \( \Gamma = \sum_{i \in [s]} \frac{n_i}{r_i + \delta - 1} + \left[ \frac{k - \sum_{i \in [s-1]} r_i n_i / (r_i + \delta - 1)}{r_s} \right] \).

Thus \( d' = n - k' + 1 = n - k + 1 - (\Gamma - 1) (\delta - 1) \).

We also take \( H' \) to be a Vandermonde matrix as follow:
\[
H' = \begin{bmatrix}
Q_{11} & Q_{12} & \cdots & Q_{1s} \\
Q_{21} & Q_{22} & \cdots & Q_{2s} \\
\vdots & \vdots & \ddots & \vdots \\
Q_{s1} & Q_{s2} & \cdots & Q_{ss}
\end{bmatrix}
\]
where \( Q_{ij}, i \in [s], j \in [n_i / (r_i + \delta - 1)] \) are matrices of size \((\delta - 1) \times (r_i + \delta - 1)\).

By employing the parity-splitting technique, we consider the code \( C \) whose parity check matrix \( H \) as follow:
\[
H = \begin{bmatrix}
A_1 \\
\vdots \\
A_s
\end{bmatrix}
\]

where

$$A_i = \begin{bmatrix} Q_1^{(i)} & \cdots & Q_{n_i}^{(i)} \end{bmatrix}, \quad i \in [s].$$

Clearly, the structure of $H$ ensures the code $C$ has multiple $(r_i, \delta_i)_{i \in [s]}$-localities and $C$ is a subcode of the RS code, it is not hard to prove that $\dim(C) = k$ and the optimality of $d$ with respect to the bound (10) similar to Theorem 3.

V. Conclusion

We introduced LRCs with multiple $(r_i, \delta_i)_{i \in [s]}$-localities, which is useful and attractive in some practical scenarios, especially for the hot data in distributed storage systems. An upper bound on the minimum distance was obtained for LRCs with multiple $(r_i, \delta_i)_{i \in [s]}$-localities, which extended the bound of ML-LRCs in [13]. By employing the parity-splitting technique of [8], we gave optimal constructions with $r_1 \leq r_2 \leq \cdots \leq r_s$ and $\delta_1 = \delta_2 = \cdots = \delta_1 = \delta$, $\delta \geq 2$. Future works might focus on constructing more explicit and structured optimal LRCs with multiple $(r_i, \delta_i)_{i \in [s]}$-localities, especially for the parameters $r_1 < r_2 < \cdots < r_s$, $\delta_1 > \delta_2 > \cdots > \delta_s$.

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