Abstract

We consider a pairwise interacting quantum 3-body system in 3-dimensional space with finite masses and the interaction term $V_{12} + \lambda(V_{13} + V_{23})$. The pair interaction of the particles \{1, 2\} is tuned to make them have a zero energy resonance and no negative energy bound states. The coupling constant $\lambda > 0$ is allowed to take the values for which the particle pairs \{1, 3\} and \{2, 3\} have no bound states with negative energy. Let $\lambda_{cr}$ denote the critical value of the coupling constant such that $E(\lambda) \to -\infty$ for $\lambda \to \lambda_{cr}$, where $E(\lambda)$ is the ground state energy of the 3-body system.

We prove the theorem, which states that near $\lambda_{cr}$ one has $E(\lambda) = C(\lambda - \lambda_{cr})[\ln(\lambda - \lambda_{cr})]^{-1} + h.t.$, where $C$ is a constant and h.t. stands for “higher terms”. This behavior of the ground state energy is universal (up to the value of the constant $C$), meaning that it is independent of the form of pair interactions.
I. INTRODUCTION

Universality plays an important role in physics. The interest to it is inspired by the striking similarity in behavior near the critical point among systems that are otherwise quite different in nature. For example, various substances, which exhibit liquid-gas phase transition, near the critical point obey the universal law $\rho_{\text{gas}} - \rho_c \rightarrow -A(T_c - T)^\beta$. Here $\rho_{\text{gas}}, \rho_c$ denote the density of gas and critical density respectively, $T, T_c$ are temperature and critical temperature and $\beta$ is the so-called critical exponent [1]. The value of $\beta \approx 0.325$ is the same for many substances, which are completely different on the atomic level. Amazingly, it has the same value, when one consider magnetization as a function of temperature near the critical point in ferromagnetic materials. The values of the critical-point exponents describing the quantitative nature of the singularities are identical for large groups of apparently diverse physical systems. Another example of universality can be found in the ground state energy of the Bose gas as a function of density. In the low density limit it approaches an expression, which does not depend on the form of pair interactions [2].

Small quantum systems also exhibit universal features [1]. One example of universality in the two-particle case concerns the behavior of the energy depending on the coupling constant near the threshold. Suppose that $E(\lambda)$ is the energy of an isolated non-degenerate state of the Hamiltonian $h(\lambda) = T + \lambda V_{12}$ in 3-dimensional space and $E(\lambda) \rightarrow 0$ for $\lambda \rightarrow \lambda_{cr}$. Then universally for $\lambda$ near $\lambda_{cr}$ one has $E(\lambda) = c(\lambda - \lambda_{cr})^2 + \text{h.t.}$ or $E(\lambda) \approx c(\lambda - \lambda_{cr}) + \text{h.t.}$ depending on whether zero energy is an eigenvalue of $h(\lambda_{cr})$ or not, see [3]. Universal in this context means that this behavior up to a constant is true for all short range interactions independently of their form. “h.t.” is the shorthand notation for “higher terms” and $E(\lambda) = f(\lambda) + \text{h.t.}$ for $f(\lambda) \rightarrow 0$ always implies that $E(\lambda) = f(\lambda) + o(f(\lambda))$.

If two particles are put into an n-dimensional space there exist various scenarios [3]: for example, in the 2-dimensional flatland the energy of the ground state (g.s.) energy $E(\lambda)$ can approach zero exponentially fast $E(\lambda) = \exp(-c(\lambda - \lambda_{cr})^{-1}) + \text{h.t.}$, and in 4 dimensions the g.s. energy approaches zero very slow, namely, $E(\lambda) = c(\lambda - \lambda_{cr})|\log(\lambda - \lambda_{cr})|^{-1} + \text{h.t.}$. For a full account of possible scenarios see Table I in [3]. Another universality associated with 2-body system in 3-dimensional space relates to the wave function near the threshold. When the energy of a non-degenerate bound state near the threshold satisfies $E(\lambda) = c(\lambda - \lambda_{cr})^2 + \text{h.t.}$ then the wave function of this bound state $\psi(\lambda, x)$ approaches
spherically symmetric expression \[4\], that is
\[
\left| \psi(\lambda, x) - |E(\lambda)|^{1/4} e^{-|E(\lambda)|^{1/4}|x|} \right| \rightarrow 0,
\]
see Eq. (8) in \([5]\), where we have omitted the phase factor. For well-behaved short-range interactions Eq. (1) holds irrespectively of the form of pair potential.

For 3-particle systems the notorious example of universality is the Efimov effect. Efimov’s striking and counterintuitive prediction \([6]\) was that just by tuning coupling constants of the short-range interactions in the 3-body system one can bind an infinite number of levels, even though the two-body subsystems bind none. The form of the pair interactions does not play a role and the infinite sequence of levels is universally scaled \([1, 6, 7]\), the scaling factor for bosons being \(e^{2\pi/s_0}\). Here \(s_0 \simeq 1.006\) is a transcendental number, which solves the equation
\[
s_0 = \frac{8}{\sqrt{3}} \frac{\sinh(\pi s_0/6)}{\cosh(\pi s_0/2)}.
\]
Efimov’s prediction was later confirmed mathematically \([8–10]\), and Efimov states were experimentally observed in ultracold gases \([11]\). In the case of 3 bosons the Efimov effect takes place when the pair interactions are tuned to have a zero energy resonance and no negative energy bound states. The so-called 4-body universality \([12]\) holds only approximately \([13]\) and the question of finite range corrections is still being debated \([14]\).

Recently a new type of universality in 3-body systems has been discovered in \([5]\). Consider the Hamiltonian of the 3–particle system in \(\mathbb{R}^3\)
\[
H(\lambda) = H_0 + v_{12} + \lambda(v_{13} + v_{23}),
\]
where \(H_0\) is the kinetic energy operator with the center of mass removed, \(\lambda > 0\) is the coupling constant and none of the particle pairs has negative energy bound states. All particles are supposed to have a finite mass. Suppose that the interaction between the particles \(\{1, 2\}\) is tuned to make them have a zero energy resonance and no negative energy bound states. This implies that in the absence of particle 3 the particles \(\{1, 2\}\) are “almost” bound, meaning that a bound state with negative energy appears if and only if the interaction \(v_{12}\) is strengthened by a negligibly small amount. The coupling constant \(\lambda > 0\) is allowed to take the values for which the particle pairs \(\{1, 3\}\) and \(\{2, 3\}\) have neither zero energy resonances nor bound states with negative energy. Let \(\lambda_{cr}\) denote the critical value of the coupling constant such that \(E(\lambda) \rightarrow -0\) for \(\lambda \rightarrow \lambda_{cr}\), where \(E(\lambda)\) is the g.s. energy of the
3-body system. In [16] it was shown that for \( \lambda \) just above \( \lambda_{cr} \) there exists a bound state with negative energy \( \psi_{\lambda} \), which totally spreads for \( \lambda \to \lambda_{cr} \), and zero is not an eigenvalue for \( H(\lambda_{cr}) \). In [5] it was proved that \( \psi_{\lambda} \) for \( \lambda \to \lambda_{cr} \) approaches in norm a universal expression, that is

\[
\psi_{\lambda} \to \frac{1}{\sqrt{2\pi^{3/2}}} \frac{1}{\ln |E(\lambda)|^{1/2}} \exp(-|E(\lambda)|^{1/2}|x|)
\]

for all admissible pair interactions. In (4) \( x, y \in \mathbb{R}^3 \) are Jacobi coordinates, see [5] for notations. In the limit the wave function \( \psi_{\lambda} \) describes the state, in which average distances between all three particles go to infinity. This is the reason why the actual form of pair interactions becomes unimportant. By analogy with the 2-particle case it is thus natural to assume that \( E(\lambda) \) would exhibit universal behavior around \( \lambda_{cr} \). In this paper we shall prove Theorem 1 which states that it is indeed so and universally one has

\[
E(\lambda) = C(\lambda - \lambda_{cr})[\ln(\lambda - \lambda_{cr})]^{-1} + \text{h.t.},
\]

where \( C > 0 \). Let us remark that this result does not follow directly from (4). If one tries to calculate the average of the Hamiltonian using the universal expression for the wave function one misses the fact that universal expression contains error terms that in spite of going to zero in norm can still affect resulting quantity. The obtained behavior of \( E(\lambda) \) remarkably mimics that of the g.s. energy of 2 particles in 4-dimensional space. The experimental observation of this type of universality can possibly be obtained in ultracold gas mixtures, see [5] for discussion. When the pair interaction \( V_{12} \) is not tuned one has 3 possibilities, which are listed in Theorem 2.

Throughout the paper we use the following notation. For an operator \( A \) acting on a Hilbert space \( D(A) \), \( \sigma(A) \) and \( \sigma_{ess}(A) \) denote the domain, the spectrum, and the essential spectrum of \( A \) respectively [18]. \( A \geq 0 \) means that \( (f, Af) \geq 0 \) for all \( f \in D(A) \), while \( A \nless 0 \) means that there exists \( f_0 \in D(A) \) such that \( (f_0, Af_0) < 0 \). \( \mathcal{B}(\mathcal{H}) \) denotes the set of bounded linear operators on the Hilbert space \( \mathcal{H} \).

II. MAIN RESULT

Consider the Hamiltonian (3), where the pair–interactions \( v_{ik} \) are operators of multiplication by real \( V_{ik}(r_i - r_k) \leq 0 \) and \( r_i \in \mathbb{R}^3 \) are particle position vectors. For pair potentials we require like in [5] that

\[
\gamma_0 := \max_{i=1,2} \left[ \int d^3 r |V_{i3}(r)|^2, \int d^3 r |V_{i3}(r)|(1 + |r|^{2\delta}) \right] < \infty,
\]

(5)
where $0 < \delta < 1/8$ is a fixed constant. And

$$-b_1 e^{-b_2 |r|} \leq V_{12}(r) \leq 0,$$

(6)

where $b_{1,2} > 0$ are constants. As an operator $H(\lambda)$ is self–adjoint on $D(H_0) = \mathcal{H}^2(\mathbb{R}^6) \subset L^2(\mathbb{R}^6)$, where $\mathcal{H}^2(\mathbb{R}^6)$ denotes the corresponding Sobolev space [20, 21]. The pair interaction between particles $\{1, 2\}$ is tuned so that they have a zero energy resonance [16] (equivalently the Hamiltonian $H_0 + v_{12}$ is at critical coupling in the sense of Definition 1 in [15]).

Let $\lambda_{cr}$ be the value of the coupling constant such that $H(\lambda_{cr}) \geq 0$ but $H(\lambda_{cr} + \epsilon) \not\equiv 0$ for all $\epsilon > 0$. Let $\lambda'_{1,2}$ be the values of the coupling constants such that $H_0 + \lambda'_1 v_{13}$ and $H_0 + \lambda'_2 v_{23}$ are at critical coupling in the sense of Def. 1 in [15]. Then by the analysis in Sec. 6 in [16] $\lambda > \lambda_{cr}$, where $\lambda := \min[\lambda'_1, \lambda'_2]$. By the HVZ theorem (see [18], Vol. 4 and [20]) for $\lambda \in (\lambda_{cr}, \lambda]$ one has $\sigma_{ess}(H(\lambda)) = [0, \infty)$ and the Hamiltonian $H(\lambda)$ has a g.s. level with the energy $E(\lambda) < 0$ and the normalized g.s. wave function $\psi_\lambda \in D(H_0)$. By Theorems 1, 3 in [16] and Theorem 2 in [3] for $\lambda \searrow \lambda_{cr}$ one has $E(\lambda) \nearrow 0$, $\psi_\lambda$ totally spreads (see the definition in [16]) and approaches in norm a universal expression given by Eq. (20) in [3]; zero is not an eigenvalue of $H(\lambda_{cr})$. Our aim in this paper is to prove

**Theorem 1.** Suppose that $E(\lambda) := \inf \sigma(H(\lambda))$, then for $\lambda \searrow \lambda_{cr}$ one has

$$E(\lambda) = C_0 \frac{(\lambda - \lambda_{cr})}{\ln(\lambda - \lambda_{cr})} + o\left(\frac{(\lambda - \lambda_{cr})}{\ln(\lambda - \lambda_{cr})}\right),$$

(7)

where $C_0 > 0$ is a finite constant.

Before we proceed with the proof let us remark that 1) Eq. (7) is universal, meaning that up to a constant it does not depend on the details of pair interaction; 2) the function $E(\lambda)$ at $\lambda = \lambda_{cr}$ cannot be Taylor expanded in powers of $(\lambda - \lambda_{cr})^\alpha$ for any $\alpha > 0$; 3) the three-body g. s. energy in the 3-dimensional case has the same behavior near $\lambda_{cr}$ as the 2-body g. s. energy in the 4-dimensional case (we do not have an explanation for this finding); 4) the method of the proof is different from [3, 17]: the method in [3, 17], which uses the low energy expansions of the Birman-Schwinger operator, is not applicable here.

The proof below hinges on Theorem 3 in Section III whose technical proof is based on the results and methods in [3].

**Proof of Theorem 1.** For $\lambda \in (\lambda_{cr}, \lambda]$ there exists $\psi_\lambda \in D(H_0)$, $\|\psi_\lambda\| = 1$ such that $H(\lambda)\psi_\lambda = E(\lambda)\psi_\lambda$, besides we can assume that $\psi_\lambda > 0$ because it is the ground state. $E(\lambda)$ is smooth
and monotone increasing on \((\lambda_{cr}, \lambda)\). Using perturbation theory \([18, 19]\), we obtain
\[
- \frac{dE(\lambda)}{d\lambda} = \|v_{13}|^{1/2}\psi_{\lambda}\|^2 + \|v_{23}|^{1/2}\psi_{\lambda}\|^2. \tag{8}
\]

By Theorem 3 for \(\lambda\) close enough to \(\lambda_{cr}\) there exists a constant \(C_0 > 0\) such that
\[
- \frac{C_0 + \varepsilon}{\ln(-E(\lambda))} \geq - \frac{dE(\lambda)}{d\lambda} \geq - \frac{C_0 - \varepsilon}{\ln(-E(\lambda))} \tag{9}
\]
for any given \(\varepsilon > 0\). The last inequality can be equivalently rewritten as
\[
C_0 + \varepsilon \geq \frac{d}{d\lambda} \left( E(\lambda) \ln(-E(\lambda)) - E(\lambda) \right) \geq C_0 - \varepsilon. \tag{10}
\]

Integrating (10) we obtain
\[
(C_0 + \varepsilon)(\lambda - \lambda_{cr}) \geq E(\lambda) \ln(-E(\lambda)) - E(\lambda) \geq (C_0 - \varepsilon)(\lambda - \lambda_{cr}). \tag{11}
\]

Let us set
\[
E(\lambda) = -f(\lambda)(\lambda - \lambda_{cr}), \tag{12}
\]
where \(f(\lambda) > 0\). From (11) we get
\[
(C_0 + \varepsilon) \geq f(\lambda)[- \ln(-E(\lambda)) + 1] \geq (C_0 - \varepsilon). \tag{13}
\]

Using that \(E(\lambda) \rightarrow 0\) from (13) we conclude that \(\lim_{\lambda \rightarrow \lambda_{cr}} f(\lambda) = 0\). Again substituting (12) into (13) we obtain
\[
(C_0 + \varepsilon) \geq -f(\lambda) \ln(\lambda - \lambda_{cr}) - [f(\lambda) \ln(f(\lambda))] - f(\lambda) \geq (C_0 - \varepsilon) \tag{14}
\]

The term in square brackets in the last inequality goes to zero for \(\lambda \rightarrow \lambda_{cr}\). Thus for \(\lambda\) close to \(\lambda_{cr}\) we have
\[
- \frac{C_0 + \varepsilon/2}{\ln(\lambda - \lambda_{cr})} \geq f(\lambda) \geq - \frac{C_0 - \varepsilon/2}{\ln(\lambda - \lambda_{cr})}. \tag{15}
\]

Now (8) follows from (12), (15) since \(\varepsilon > 0\) is arbitrary small. \(\square\)

Theorem 1 considers the case when the particles \(\{1, 2\}\) have a zero energy resonance. Now let us consider a more general situation and assume that the 3-particle system is described by the Hamiltonian \(H \tag{3}\), where for simplicity we require that \(V_{ik} \leq 0\) are bounded and have a compact support. We still require that \(\lambda < 3\), \(i.e.\ \lambda\) takes the values for which the subsystems \(\{1, 3\}\) and \(\{2, 3\}\) have no bound states with negative energy and no zero energy
resonances. However, we do not impose restrictions on the spectrum of the particles \( \{1, 2\} \), which means that this pair determines the energy of the dissociation threshold \( E_{\text{thr}} \), that is

\[
E_{\text{thr}} := \inf \sigma_{\text{ess}}(H(\lambda)) = \inf \sigma(H_0 + v_{12}).
\]  

The critical coupling constant \( \lambda_{cr} \) is the value of \( \lambda \) for which the 3-body bound state, whose energy lies below \( E_{\text{thr}} \), is about to be formed. Mathematically speaking

\[
\lambda_{cr} = \sup \{\lambda | \inf \sigma(H(\lambda)) = E_{\text{thr}} \}.
\]  

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\[
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\]  

By the methods similar to [22], one can prove that \( \lambda_{cr} < \lambda \).

**Theorem 2.** One can distinguish 3 cases: (A) the pair \( \{1, 2\} \) has no negative energy bound states and no zero energy resonance; (B) the pair \( \{1, 2\} \) has no negative energy bound states but has a zero energy resonance; (C) the pair \( \{1, 2\} \) has at least one bound state with negative energy. Suppose that \( E(\lambda) := \inf \sigma(H(\lambda)) \), then for \( \lambda \downarrow \lambda_{cr} \) in each case one has

\[
\begin{align*}
(A) \quad & E(\lambda) - E_{\text{thr}} = E(\lambda) = -c(\lambda - \lambda_{cr}) + \text{h.t.} \\
(B) \quad & E(\lambda) - E_{\text{thr}} = E(\lambda) = c(\lambda - \lambda_{cr})[\ln(\lambda - \lambda_{cr})]^{-1} + \text{h.t.} \\
(C) \quad & E(\lambda) - E_{\text{thr}} = -c(\lambda - \lambda_{cr})^2 + \text{h.t.}
\end{align*}
\]

where \( c > 0 \) is a finite constant.

**Proof.** Case (B) follows from Theorem 1 and case (C) was proved in Theorem 3.2 in [17]. In case (A) let \( \psi_\lambda \) denote the eigenfunction of \( H(\lambda) \), which corresponds to the eigenvalue \( E(\lambda) \). As follows from the proof of Theorem 2 in [16], \( \psi_\lambda \to \psi_0 \) in norm, where \( \psi_0 \in D(H_0) \) is the eigenfunction corresponding to the zero eigenvalue of \( H(\lambda_{cr}) \). Thus \( \|v_{13}|^{1/2}\psi_\lambda\|^2 + \|v_{23}|^{1/2}\psi_\lambda\|^2 \to c \), where \( c \in (0, \infty) \) and, hence, \( dE/d\lambda \to -c \). The rest of the proof is trivial.

**III. ASYMPTOTICS OF POTENTIAL ENERGY TERMS**

The aim of this section is to prove

**Theorem 3.** Suppose that the interaction between the particles \( \{1, 2\} \) is tuned to make them have a zero energy resonance and no negative energy bound states. Let \( \lambda_n \in (\lambda_{cr}, \lambda) \) be any
sequence such that \( \lambda_n \to \lambda_{cr} \) and \( \psi_n \in D(H_0) \) be the g.s. wave functions of \( H(\lambda_n) \). Then there exists a unique \( C_0 \in (0, \infty) \) such that

\[
\lim_{n \to \infty} |\ln k_n| \left\{ \left| v_{13}^\frac{1}{2} \psi_n \right|^2 + \left| v_{23}^\frac{1}{2} \psi_n \right|^2 \right\} = C_0/2,
\]

where \( k_n := |E(\lambda_n)|^{\frac{1}{2}} \).

Note that all requirements R1-R3 in \cite{5} are satisfied and the sequence \( \psi_n \) totally spreads (see Sec. Theorems 1,3 in \cite{16} for the proof). Since the proof of Theorem \cite{3} uses the notations of \cite{5}, it does not make sense to list them again, the current section can be regarded as a prolongation of the text in \cite{5}. The proof is given later in this section. For a shorter notation let us denote

\[
M_n := \left\{ \left| v_{13}^\frac{1}{2} \psi_n \right|^2 + \left| v_{23}^\frac{1}{2} \psi_n \right|^2 \right\}^{1/2}.
\]

Similar to Eqs. (39)-(40) in \cite{5} let us introduce the operator function

\[
\tilde{B}_{12}(k_n) := \mathcal{F}_{{12}}^{-1} \xi_n(p_y) \mathcal{F}_{12},
\]

where

\[
\xi_n(p_y) := \begin{cases} 
|p_y|^\delta/8 + (k_n)^{\delta/8} & \text{if } |p_y| \leq 1 \\
1 + (k_n)^{\delta/8} & \text{if } |p_y| \geq 1.
\end{cases}
\]

The partial Fourier transform \( \mathcal{F}_{12} \) is defined in Eq. (17) in \cite{16}.

**Lemma 1.** The sequences \( \varphi_n^{(1)} := M_n^{-1} v_{13}^\frac{1}{2} \psi_n \) and \( \varphi_n^{(2)} := M_n^{-1} v_{23}^\frac{1}{2} \psi_n \), where \( \psi_n \) is defined in Theorem \cite{3} converge in norm. The sequence \( \varphi_n^{(3)} := M_n^{-1} \tilde{B}_{12}(k_n) v_{12}^\frac{1}{2} \psi_n \) is uniformly norm-bounded.

**Proof.** From the Schrödinger equation for \( \psi_n \) it follows that

\[
\lambda_n^{-1} \begin{pmatrix} \varphi_n^{(1)} \\ \varphi_n^{(2)} \end{pmatrix} = \mathcal{K}(k_n^2) \begin{pmatrix} \varphi_n^{(1)} \\ \varphi_n^{(2)} \end{pmatrix},
\]

where \( \mathcal{K}(z) \) for \( z > 0 \) is a bounded operator on \( L^2(\mathbb{R}^6) \oplus L^2(\mathbb{R}^6) \) defined as

\[
\mathcal{K}(z) := \begin{pmatrix} |v_{13}|^\frac{1}{2} (H_0 + v_{12} + z)^{-1} |v_{13}|^\frac{1}{2} & |v_{13}|^\frac{1}{2} (H_0 + v_{12} + z)^{-1} |v_{23}|^\frac{1}{2} \\
|v_{23}|^\frac{1}{2} (H_0 + v_{12} + z)^{-1} |v_{13}|^\frac{1}{2} & |v_{23}|^\frac{1}{2} (H_0 + v_{12} + z)^{-1} |v_{23}|^\frac{1}{2} \end{pmatrix}.
\]

Like in Sec. II in \cite{23} one proves that \( \mathcal{K}(z) \) for \( z \to +0 \) has a norm limit \( \mathcal{K}(0) \), besides \( \mathcal{K}(z) \) for \( z \geq 0 \) is a positivity preserving self-adjoint operator and \( \sigma_{ess}(\mathcal{K}(z)) \subseteq [0, 1/\lambda] \), where
\( \chi \) was defined above. \((\varphi_n^{(1)}; \varphi_n^{(2)})\) is a normalized eigenvector of \(K(k_n^2)\) corresponding to the eigenvalue \(\lambda_n^{-1}\). Due to the location of the essential spectrum \(\|K(k_n^2)\|\) equals the maximal eigenvalue of \(K(k_n^2)\). Because \(\varphi_n^{(1)}, \varphi_n^{(2)} \geq 0\) we conclude due to the positivity preserving property that \(\|K(k_n^2)\| = \lambda_n^{-1}\) (see Theorem XIII.43 in Vol. 4 of [18]). Therefore, due to the norm convergence \(\lambda_n^{-1} = \|K(0)\|\) is the maximal eigenvalue of \(K(0)\), which is isolated and non-degenerate. Let \((\varphi_\infty^{(1)}; \varphi_\infty^{(2)}) \) with \(\varphi_\infty^{(1)}, \varphi_\infty^{(2)} \geq 0\) be the eigenvector of \(K(0)\), which corresponds to \(\lambda_\infty^{-1}\). Again by the norm convergence of \(K(k_n^2)\) we have that \(\varphi_n^{(1)} \to \varphi_\infty^{(1)}\) and \(\varphi_n^{(2)} \to \varphi_\infty^{(2)}\) in norm.

To prove that \(\sup_n \|\varphi_n^{(3)}\| < \infty\) note that by Eq. (67) in [5]
\[
\varphi_n^{(3)} = \chi_{[\rho, \rho_0]}(\sqrt{p_y^2 + k_n^2}) \hat{\varphi}_n^{(4)} + \mathcal{O}(1),
\]
where \(\mathcal{O}(1)\) denotes the terms, which are uniformly norm-bounded (c.f. Eqs. (71), (72) in [5]). Now using Lemma 11 in [16] (the value of \(\rho_0\) is also defined there) and Eqs. (73)-(74) in [5] we obtain
\[
\chi_{[\rho, \rho_0]}(\sqrt{p_y^2 + k_n^2}) \hat{\varphi}_n^{(4)} = \chi_{[\rho, \rho_0]}(\sqrt{p_y^2 + k_n^2}) a^{-1} |p_0| v_{12} |^{\frac{3}{2}}
\times (|p_y|^2 + k_n^2)^{-\frac{1}{2}} \xi_n(p_y) [-\Delta_x + p_y^2 + k_n^2]^{-1} |v_{13}|^{\frac{1}{2}} \hat{\varphi}_n^{(1)} + \mathcal{O}(1),
\]
where \(|v_{13}|^{\frac{3}{2}} := \mathcal{F}_{12} |v_{13}|^{\frac{1}{2}} \mathcal{F}_{12}^{-1}\). Thus
\[
\|\varphi_n^{(4)}\| \leq a^{-1} \left| |v_{12}|^{\frac{3}{2}} \chi_{[\rho, \rho_0]}(|p_y|) (|p_y|^2 + k_n^2)^{-\frac{1}{2}} \xi_n(|p_y|) [-\Delta_x + p_y^2 + k_n^2]^{-1} |v_{13}|^{\frac{1}{2}} \right| + \mathcal{O}(1)
\]
(28) It remains to prove that the operator norm on the right-hand side (rhs) of (28) is uniformly bounded. This can be trivially estimated through its Hilbert-Schmidt norm (c.f. proof of Lemma 9 in [16])
\[
\left| |v_{12}|^{\frac{3}{2}} \chi_{[\rho, \rho_0]}(|p_y|) (|p_y|^2 + k_n^2)^{-\frac{1}{2}} \xi_n(|p_y|) [-\Delta_x + p_y^2 + k_n^2]^{-1} |v_{13}|^{\frac{1}{2}} \right|^2
\leq c \int_{|p_y| \leq \rho_0} d^3 p_y \xi_n^2(|p_y|) \frac{\xi_n^2(|p_y|)}{(p_y^2 + k_n^2)^{3/2}},
\]
(29)
where $c > 0$ is a constant. The integral on the rhs of (29) is clearly convergent and uniformly bounded.

Proof of Theorem 3. From Eq. (90) in [5] and the fact that $\|\hat{f}_n - \hat{\psi}_n\| \to 0$ together with $\|\hat{\psi}_n\| = 1$ it follows that for $n \to \infty$

$$|\hat{g}_n(0)| \ln k_n^{1/2} \to \frac{\sqrt{2a}}{R(0)} > 0.$$  

(30)

Our aim is to show that the sequence $|\hat{g}_n(0)|/M_n$ converges and $\lim_{n \to \infty} |\hat{g}_n(0)|/M_n = C_1 > 0$; then (18) follows with $C_0 = 4a^2[C_1 R(0)]^{-2}$. Note that due to the positivity preserving property of $[H_0 + k_n^2]^{-1}$

$$\frac{|\hat{g}_n(0)|}{M_n} = \frac{\|g^{(1)}_n\|_1}{M_n} + \frac{\|g^{(2)}_n\|_1}{M_n},$$

(31)

where $g^{(1,2)}(y) \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ are defined in Eqs. (93), (94) in [5]. It suffices to prove the convergence of the first term on the rhs of (31) (the convergence of the second term is proved analogously). For all $R > 0$ we have

$$M_n^{-1} \|g^{(1)}_n\|_1 = \|\chi_{[0,R]}(|\eta|)\phi_0\mathcal{X}_n\varphi_n^{(1)}\|_1 + M_n^{-1} \|\chi_{(R,\infty)}(|\eta|)\phi_0|v_{12}|^{1/2} [H_0 + k_n^2]^{-1} |v_{13}|\varphi_n\|_1,$$

(32)

where we have defined

$$\mathcal{X}_n := |v_{12}|^{1/2} [H_0 + k_n^2]^{-1} |v_{13}|^{1/2}.$$  

(33)

(The coordinates $\eta, \zeta$ are defined in Eqs. (36)-(37) in [5]). By Lemma 7 in [16] the operators $\mathcal{X}_n : L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$ are norm-bounded and have a norm limit for $n \to \infty$, which we denote as $\mathcal{X}_0$. Thus by Lemma 1 $\mathcal{X}_n\varphi_n^{(1)} \to \mathcal{X}_0\varphi_n^{(1)}$ in $L^2$ sense. Then $\chi_{[0,R]}(|\eta|)\phi_0\mathcal{X}_n\varphi_n^{(1)}$ converges to $\chi_{[0,R]}(|\eta|)\phi_0\mathcal{X}_0\varphi_n^{(1)}$ in $L^1$ sense because by the Cauchy-Schwarz inequality

$$\|\chi_{[0,R]}(|\eta|)\phi_0(\mathcal{X}_n\varphi_n^{(1)} - \mathcal{X}_0\varphi_n^{(1)})\|_1 \leq \|\chi_{[0,R]}(|\eta|)\phi_0\|_2 \|\mathcal{X}_n\varphi_n^{(1)} - \mathcal{X}_0\varphi_n^{(1)}\|_2,$$

(34)

where $\|\chi_{[0,R]}(|\eta|)\phi_0\|_2$ is finite. Hence, the first term on the rhs of (32) converges for all $R > 0$. Now the convergence of the sequence on the left-hand side (lhs) of (32) follows from Lemmas 2, 3. We have proved that the sequence on the lhs of (31) converges to $C_1 \in [0, \infty)$. Since it converges for any choice of the sequence $\lambda_n$, which satisfies $\lambda_n < \lambda$ and $\lambda_n \to \lambda_{cr}$, the value of $C_1$ must be the same for any such sequence. It remains to show that $C_1 \neq 0$. This follows from the fact that $\varphi_n^{(1)}, \varphi_n^{(2)} \geq 0$ and besides $\|\varphi_n^{(1)}\|^2 + \|\varphi_n^{(2)}\|^2 = 1$, so at least one of the terms on the rhs of (31) converges to a positive value.
Lemma 2. For $R \to \infty$

$$\sup_{n} M_n^{-1} \left\| \chi_{(R,\infty)}(|\eta|)\phi_0 |v_{12}|^2 \left[ H_0 + k_n^2 \right]^{-1} |v_{13}| \psi_n \right\|_1 \to 0. \quad (35)$$

Proof. The proof largely repeats that of Lemma 4 in [5]. Repeating the steps in Eqs. (95)-(107) in [5] we obtain the inequality

$$M_n^{-1} \left\| \chi_{(R,\infty)}(|\eta|)\phi_0 |v_{12}|^2 \left[ H_0 + k_n^2 \right]^{-1} |v_{13}| \psi_n \right\|_1 \leq \frac{b_1}{2^{3/2} R^{5/2}} \sum_{i=1}^{3} M_n^{-1} \left\| \Psi_n^{(i)} \right\| \left\{ \int d^3 \eta' \int d^3 p_c |V_{13}(\alpha' \eta')| \tilde{I}_n^2(p_c) \tilde{J}^2(\eta', p_c) \right\}^{1/2}, \quad (36)$$

where

$$\tilde{J}(\eta', p_c) := \int_{|\eta| > R} d^3 \eta \int d^3 \zeta \frac{e^{-\sqrt{p^2_\zeta + k^2_n} / |\eta - \eta'|}}{|\eta - \eta'|} e^{-b_2 |m_{x_n} \eta + m_{x_c} \zeta|}. \quad (37)$$

By Lemma 4 we only need to prove that $\sup_n I_n \to 0$ for $R \to \infty$, where we defined

$$I_n := \int d^3 \eta' \int d^3 p_c |V_{13}(\alpha' \eta')| \tilde{I}_n^2(p_c) \tilde{J}^2(\eta', p_c). \quad (38)$$

Let us split the last integral as $I_n = I_{n}^{(1)} + I_{n}^{(2)}$, where

$$I_{n}^{(1)} := \int_{|\eta'| \leq R/2} d^3 \eta' \int d^3 p_c |V_{13}(\alpha' \eta')| \tilde{I}_n^2(p_c) \tilde{J}^2(\eta', p_c), \quad (39)$$

$$I_{n}^{(2)} := \int_{|\eta'| > R/2} d^3 \eta' \int d^3 p_c |V_{13}(\alpha' \eta')| \tilde{I}_n^2(p_c) \tilde{J}^2(\eta', p_c). \quad (40)$$

Clearly, we can write

$$\tilde{J}(\eta', p_c) \leq c_1 \int_{|\eta| > R} d^3 \eta \frac{e^{-\sqrt{p^2_\zeta + k^2_n} / |\eta - \eta'|}}{|\eta - \eta'|}, \quad (41)$$

where $c_1 > 0$ is a constant. For $|\eta'| \leq R/2$ and $|\eta| > R/2$ we have $|\eta - \eta'| \geq |\eta| - |\eta'| > |\eta|/2$, which gives the estimate

$$\tilde{J}(\eta', p_c) \leq c_2 \frac{e^{-\sqrt{p^2_\zeta + k^2_n} R/2}}{\sqrt{p^2_\zeta + k^2_n}} \left( R + \frac{2}{\sqrt{p^2_\zeta + k^2_n}} \right), \quad (42)$$

where $c_2 > 0$ is a constant. Substituting this estimate into (39) and using that $V_{13}(x) \in L^1(\mathbb{R}^3)$ we obtain

$$I_{n}^{(1)} \leq c_3 \int d^3 p_c \tilde{I}_n^2(p_c) \frac{e^{-\sqrt{p^2_\zeta + k^2_n} R}}{p^2_\zeta + k^2_n} \left[ R^2 + \frac{4}{p^2_\zeta + k^2_n} \right], \quad (43)$$

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where $c_3 > 0$ is a constant. Substituting Eq. (40) from [5] we get
\[
    I_n^{(1)} \leq c_4 \int_0^1 s^{2-2\delta} e^{-sR} \left[ R^2 + \frac{4}{s^2} \right] ds + c_4 \int_1^\infty e^{-sR} \left[ R^2 + \frac{4}{s^2} \right] ds
\]
where $c_4 > 0$ is another constant. Let us set $A_\beta := \sup_{x \geq 0} x^\beta e^{-x}$, where $A_\beta$ is finite and depends only on $\beta$. Then
\[
    I_n^{(1)} \leq c_4 R^{-\delta} (A_{2+\delta} + 4A_\delta) \int_0^1 s^{-3\delta} ds + c_4 R^2 e^{-R/2} \int_1^\infty e^{-sR/2} [1 + 4s^{-2} R^{-2}] ds = o(R) \tag{45}
\]

It is easy to see that the terms on the rhs of $(45)$ vanish for $R \to \infty$. Let us estimate $I_n^{(2)}$. From $(41)$ we get
\[
    \bar{J}(\eta', p_\zeta) \leq c_1 \int d^3 \eta e^{-\sqrt{\eta_1^2 + k_2^2} |\eta - \eta'|} \leq \frac{c_1'}{p_\zeta^2}, \tag{46}
\]
where $c_1' > 0$ is a constant. Substituting this into $(40)$ we obtain
\[
    I_n^{(2)} \leq c_1' \left\{ \int_{|\eta'| > R/2} d^3 \eta' |V_{13}(\alpha', \eta')| \right\} \int d^3 p_\zeta \bar{J}(\eta', p_\zeta)^4 = o(R), \tag{47}
\]
where the integral in curly brackets goes to zero because $V_{13}(x) \in L^1(\mathbb{R}^3)$ and the second integral is uniformly bounded for all $n$.

**Lemma 3.** Suppose that the sequence $a_n \in \mathbb{C}$ is such that $a_n = b_n(R) + c_n(R)$, where $b_n(R), c_n(R) \in \mathbb{C}$ depend on a parameter $R > 0$. Additionally assume that $b_n(R)$ is convergent for all $R > 0$ and $\lim \sup_{n \to \infty} |c_n(R)| \to 0$ for $R \to \infty$. Then $a_n$ converges.

**Proof.** The proof is a trivial application of the Cauchy convergence criterion. For any $\varepsilon > 0$ fix $N_1, R > 0$ so that $|c_n(R)| < \varepsilon/3$ for all $n > N_1$. Choose $N_2$ so that $|b_n(R) - b_m(R)| < \varepsilon/3$ for all $n, m > N_2$. Then
\[
    |a_n - a_m| \leq |b_n(R) - b_m(R)| + |c_n(R)| + |c_m(R)| < \varepsilon \tag{48}
\]
for all $n, m > \max(N_1, N_2)$.

**Lemma 4.** $\sup_n M_n^{-1} ||\Psi_n^{(i)}|| < \infty$, where $\Psi_n^{(i)}$ for $i = 1, 2, 3$ are defined in Eqs.(43), (44), (48) in [3].

**Proof.** We have $M_n^{-1} \Psi_n^{(2)} = \lambda_n \mathcal{T}_n^{(2)} \varphi_n^{(1)}$, where $\mathcal{T}_n^{(2)}$ defined in Eq. (63) in [3] are uniformly norm–bounded operators. By Lemma [4] $\sup_n M_n^{-1} ||\Psi_n^{(2)}|| < \infty$. From definition of $\Psi_n^{(1)}$ we have
\[
    M_n^{-1} \Psi_n^{(1)} = \mathcal{T}_n \mathcal{D}_n \varphi_n^{(3)}, \tag{49}
\]
we define operator functions \( \mathfrak{T}_n, \mathfrak{D}_n \) on \( L^2(\mathbb{R}^6) \). From (19) \( \sup_n M_n^{-1}||\Psi_n^{(1)}|| < \infty \) follows from \( \mathfrak{T}_n, \mathfrak{D}_n \) being uniformly norm-bounded. Let us start with estimating the norm of \( \mathfrak{T}_n \). Let us define operator functions \( \mathfrak{T}_n^{(1)}, \mathfrak{T}_n^{(2)} : \mathbb{R}^3 \to \mathfrak{B}(L^2(\mathbb{R}^3)) \), which act on \( h(\eta) \in L^2(\mathbb{R}^3) \) as follows

\[
\mathfrak{T}_n^{(1)}(p_\zeta) = |V_{13}(\alpha' \eta)|^{\frac{1}{2}} \left[ \xi_n \left( m_{yn}(i\nabla \eta) + m_{y\zeta} p_\zeta \right) \right]^{-1} \left[ -\Delta_\eta + p_\zeta^2 + k_n^2 \right]^{\frac{3+\delta}{4}},
\]

\[\times \chi_{[0,1]}(|-i\nabla \eta|)h,\]

\[
\mathfrak{T}_n^{(2)}(p_\zeta) = |V_{13}(\alpha' \eta)|^{\frac{1}{2}} \left[ \xi_n \left( m_{yn}(i\nabla \eta) + m_{y\zeta} p_\zeta \right) \right]^{-1} \left[ -\Delta_\eta + p_\zeta^2 + k_n^2 \right]^{\frac{3+\delta}{4}},
\]

\[\times \chi_{(1,\infty)}(|-i\nabla \eta|)h.
\]

The operators like \( \chi_{[0,1]}(|-i\nabla \eta|) \) act in the sense described in Chapter 4 in [24]. It is easy to see that

\[
[\mathcal{F}_{13} \mathfrak{T}_n f](\eta, p_\zeta) = \mathfrak{T}_n^{(1)}(p_\zeta) \hat{f}(\eta, p_\zeta) + \mathfrak{T}_n^{(2)}(p_\zeta) \hat{f}(\eta, p_\zeta),
\]

where \( \hat{f}(\eta, p_\zeta) \equiv \mathcal{F}_{13} f \). Thus

\[
||\mathfrak{T}_n|| \leq \sup_{p_\zeta} ||\mathfrak{T}_n^{(1)}(p_\zeta)|| + \sup_{p_\zeta} ||\mathfrak{T}_n^{(2)}(p_\zeta)||.
\]

The operator norms on the rhs of (55) can be bounded by the trace ideals norms, which in turn can be bounded by Theorem 4.1 in [24].

\[
||\mathfrak{T}_n^{(1)}(p_\zeta)|| \leq ||\mathfrak{T}_n^{(1)}(p_\zeta)||_2 \leq (2\pi \alpha')^{-\frac{3}{4}} ||V_{13}||_1 [J_n^{(1)}(p_\zeta)]^{\frac{1}{2}},
\]

where

\[
J_n^{(1)}(p_\zeta) := \int_{|s| \leq 1} \frac{d^3s}{|s|^{3-\delta}} \leq \int_{|s| \leq 1} \frac{d^3s}{|m_{yn} s + m_{y\zeta} p_\zeta|^{\frac{3}{2}}} \left[ \int_{|s| \leq 1} \frac{d^3s}{|m_{yn} s + m_{y\zeta} p_\zeta|^{\frac{3}{2}}} \right]^{\frac{3-\delta}{4}} + \int_{|s| \leq 1} \frac{d^3s}{|s|^{3-\delta}},
\]

\[
\leq \left[ \int_{|s| \leq 1} \frac{d^3s}{|s|^{3-\delta/2}} \right]^{\frac{3-\delta}{3-\delta/2}} \left[ \int_{|s| \leq 1} \frac{d^3s}{|m_{yn} s|^{\frac{3}{2}-\frac{\delta}{4}}} \right]^{\frac{\delta}{4-\delta}} + \int_{|s| \leq 1} \frac{d^3s}{|s|^{3-\delta}}.
\]
In (57) we have used Hölders inequality. The integrals on the rhs of (57) are convergent and independent of \( p_\zeta \) and \( n \), hence, \( \sup_{p_\zeta} \| \mathfrak{T}_n^{(1)}(p_\zeta) \| < \infty \). Similarly
\[
\| \mathfrak{T}_n^{(2)}(p_\zeta) \| \leq \| \mathfrak{T}_n^{(2)}(p_\zeta) \|_3 \leq (2\pi\alpha')^{-1}\| V_{13} \|_{3/2}[J_n^{(2)}(p_\zeta)]^{\frac{1}{4}}, \tag{58}
\]
where
\[
J_n^{(2)}(p_\zeta) := \int_{|s|>1} \frac{d^3s}{\xi_n^3(m_{y\zeta} s + m_{y\zeta} p_\zeta)|s|^{9-3\delta}} \leq \int_{|s|>1} \frac{d^3s}{m_{y\zeta} s + m_{y\zeta} p_\zeta} \left[ \frac{\alpha}{2} \right] s^{-9+3\delta}.
\]
(59)
The integrals on the rhs of (59) obviously converge. Thus we find that \( \sup_{p_\zeta} \| \mathfrak{T}_n^{(2)}(p_\zeta) \| < \infty \) and \( \| \mathfrak{T}_n \| \) is uniformly bounded (note that \( \| V_{13} \|_{3/2} \) in (59) is bounded because \( V_{13} \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \)).

Now we pass to estimating \( \| \mathfrak{D}_n \| \) and use the same method. Calculations similar to above ones give
\[
\| \mathfrak{D}_n \| \leq (2\pi\alpha)^{-\frac{3}{2}} \| V_{12} \|_1 \left[ \sup_{p_y} J_n^{(3)}(p_y) \right]^{\frac{1}{4}} + \sup_{p_y} \left[ \left| V_{12}(\alpha x) \right|^\frac{1}{4} \left[ \int \frac{d^3s}{m_{x\zeta} + m_{y\zeta} p_y} \right]^{\frac{1}{4}} \right], \tag{60}
\]
where
\[
J_n^{(3)}(p_y) := \int_{|s|\leq 1} \frac{d^3s}{\tilde{t}_n^2(m_{x\zeta} s + m_{y\zeta} p_y)|s|^{1+\delta}} \leq \int_{|s|\leq 1} \frac{d^3s}{m_{x\zeta} s + m_{y\zeta} p_y} \left[ \frac{2-2\delta}{3-\delta} \right] |s|^{1+\delta}
\]
(61)
The integrals on the rhs of (61) converge and it remains to estimate the operator norm in (60).
\[
\left\| V_{12}(\alpha x) \right\|^\frac{1}{4} \left[ \int \frac{d^3s}{m_{x\zeta} + m_{y\zeta} p_y} \right]^{\frac{1}{4}} \leq \left\| V_{12}(\alpha x) \right\|^\frac{1}{4} \left[ \int \frac{d^3s}{m_{x\zeta} + m_{y\zeta} p_y} \right]^{\frac{1}{4}} \leq \left\| V_{12} \right\|_{\infty} + (2\pi\alpha)^{-\frac{3}{2}} \| V_{12} \|_1 \left[ \sup_{p_y} \int \frac{d^3s}{m_{x\zeta} + m_{y\zeta} p_y} \right]^{\frac{1}{4}} \leq \left\| V_{12} \right\|_{\infty} + (2\pi\alpha m_{x\zeta})^{-\frac{3}{2}} \| V_{12} \|_1 \left[ \int \frac{d^3s}{|s|^{2-2\delta}} \right]^{\frac{1}{4}}, \tag{62}
\]

where we have again used Theorem 4.1 in [24]. Thus we find that \( \sup_n \| \mathbf{D}_n \| < \infty \) and, hence, \( \sup_n M_n^{-1}\|\Psi_n^{(1)}\| < \infty \). Note that because all potentials are non-positive we have

\[
M_n^{-1}\Psi_n^{(3)} = \lambda_n \mathcal{T}_n^{(1)} Q_n \left[ M_n^{-1}\Psi_n^{(1)} + M_n^{-1}\Psi_n^{(2)} \right],
\]

(63)

where \( \mathcal{T}_n^{(1)}, Q_n \) are defined in Eqs. (62) and (46) in [5]. In [5] it was proved that \( \sup_n \| \mathcal{T}_n^{(1)} \| < \infty \) and \( \sup_n \| Q_n \| < \infty \). Thus \( \sup_n M_n^{-1}\|\Psi_n^{(3)}\| < \infty \) as claimed.

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