Example of quantum systems reduction

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Abstract

To solve the quantum-mechanical problem the procedure of mapping onto linear space $W$ of generators of the (sub)group violated by given classical trajectory is formulated. The formalism is illustrated by the plane H-atom model. The problem is solved noting conservation of the Runge-Lentz vector $n$ and reducing the 4-dimensional incident phase space $T$ to the 3-dimensional linear subspace $W = T^*V \times R^1$, where $T^*V$ is the (angular momentum ($l$) - angle ($\varphi$)) phase space and $R^1 = n$. It is shown explicitly that (i) the motion in $R^1$ is pure classical as the consequence of the reduction, (ii) motion in the $\varphi$ direction is classical since the Kepler orbits are closed independently from initial conditions and (iii) motion in the $l$ direction is classical since all corresponding quantum corrections are defined on the bifurcation line ($l = \infty$) of the problem. In our terms the H-atom problem is exactly quasiclassical and is completely integrable by this reasons.
1 Introduction

The mapping

\[ J : T \to W, \quad (1.1) \]

where \( T \) is the \( 2N \)-dimensional phase space and \( W \) is a linear space solves the mechanical problem iff

\[ J = \otimes_1^N J_i, \quad (1.2) \]

where \( J_i \) are the first integrals in involution, e.g. \([1]\) (the formalism of reduction \([1,1]\) in classical mechanics is described also in \([2]\)). The aim of this article is to adopt this procedure for quantum systems.

The mapping \((1.1)\) introduces integral manifold \( J_\omega = J^{-1}(\omega) \) in such a way that the classical phase space flaw with given \( v \in J_\omega \) belongs to \( J_\omega \) completely. We wish quantize the \( J_\omega \) manifold instead of flow in \( T \) noting that the quantum trajectory also should belong to \( J_\omega \) completely. This important conclusion was demonstrated in \([3]\) by the canonical transformation of the path-integral measure. New perturbation theory is extremely simple since \( W \) is the linear space.

The ‘direct’ mapping \((1.1)\) used in \([3]\) assumes that \( J \) is known. But it seems inconvenient having in mind the general problem of nonlinear waves quantization, when the number of degrees of freedom \( N = \infty \), or if the transformation is not canonical. We will consider by this reason the ‘inverse’ approach assuming that the classical flow is known. Then, since the flow belongs to \( J_\omega \) completely \([3]\), we would be able to find the quantum motion in \( W \). It is the main technical result illustrated in this paper.

The manifold \( J_\omega \) is invariant relatively to some subgroup \( G_\omega \) \([4]\) in accordance to topological class of classical flaw. This introduces the \( J_\omega \) classification and summation over all (homotopy) classes should be performed. Note, the classes are separated by the boundary bifurcation lines in \( W \) \([4]\). If the quantum perturbations switched on adiabatically then the homotopy group should stay unbroken. It is the ordinary statement for quantum mechanics. (But, generally speaking, this is not true for field theories.)

We will calculate the bound state energies in the Coulomb potential. This popular problem was considered by many authors, using various methods, see, e.g., \([5]\). The path-integral solution of this problem was offered in \([6]\). We will restrict ourselves by the plane problem. Corresponding phase space \( T = (p, l, r, \varphi) \) is 4-dimensional.

The classical flaw of this problem can be parametrized by the angular momentum \( l \), corresponding angle \( \varphi \) and by the normalized on total Hamiltonian Runge-Lentz vector length \( n \). So, we will consider the mapping (\( p \) is the radial momentum in the cylindrical coordinates):

\[ J_{l,n} : (p, l, r, \varphi) \to (l, n, \varphi) \quad (1.3) \]

to construct the perturbation theory in the \( W = (l, n, \varphi) \) space. I.e. \( W \) is not considered as the cotangent foliation on \( T \).

The mapping \((1.3)\) assumes additional reduction of the four-dimensional incident phase space up to three-dimensional linear subspace\(^1\). Just this reduction phenomena leads to

\(^1\)\( W \) would not have the symplectic structure.
corresponding stability of $n$ concerning quantum perturbations and will allow to solve our H-atom problem completely.

In Sec.2 we will show how the mapping (1.3) can be performed for path-integral differential measure. In Sec.3 the consequence of reduction will be derived and in Sec.4 the perturbation theory in the $W$ space will be analyzed. The calculations are based on the formalism offered in [3].

## 2 Mapping

We will calculate the integral [3]:

$$
\rho(E) = \int_0^\infty dTe^{-i\hat{K}(j,e)} \int DM(p, l, r, \varphi)e^{-iV(r,e)},
$$

(2.1)

where $\rho(E)$ is the probability to find a particle with energy $E$, i.e. we should find [7] that normalized on the zero-modes volume

$$
\rho(E) = \pi \sum_n \delta(E - E_n),
$$

(2.2)

where $E_n$ are the bound states energies. For $H$-atom problem $E_n \leq 0$. This condition will define considered homotopy class.

Expansion over operator

$$
\hat{K}(j, e) = \frac{1}{2} \int_0^T dt(\hat{J}_r \hat{e}_r + \hat{J}_\varphi \hat{e}_\varphi),
$$

(2.3)

$$
\hat{X}(t) = \frac{\delta}{\delta X(t)},
$$

(2.4)

generates the perturbation series. It will be seen that in our case we may omit the question of perturbation theory convergence.

The differential measure

$$
DM(p, l, r, \varphi) = \delta(E - H_0) \prod_t dr(t)dp(t)dl(t)d\varphi(t)\delta(\dot{r} - \frac{\partial H_j}{\partial p}) \times
$$

$$
\times \delta(\dot{p} + \frac{\partial H_j}{\partial r})\delta(\dot{\varphi} - \frac{\partial H_j}{\partial l})\delta(\dot{l} + \frac{\partial H_j}{\partial \varphi}),
$$

(2.5)

with total Hamiltonian ($H_0 = H_j|_{j=0}$)

$$
H_j = \frac{1}{2}p^2 - \frac{l^2}{2r^2} - \frac{1}{r} - j_rr - j_\varphi \varphi
$$

allows perform arbitrary transformations because of its $\delta$-likeness. The functional

$$
V(r, e) = -s_0(r) + \int_0^T dt \left[ \frac{1}{((r + e_r)^2 + r^2e_\varphi^2)^{1/2}} - \frac{1}{((r - e_r)^2 + r^2e_\varphi^2)^{1/2}} + \frac{2e_r}{r} \right]
$$

(2.6)

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2 In other words, we would demonstrate that the hidden Bargman-Fock [3] $O(4)$ symmetry is stay unbroken concerning quantum perturbations.
describes the interaction between various quantum modes and $s_0(r)$ defines the nonintegrable phase factor \[7\]. The quantization of this factor determines the bound state energy (see below). Such factor will appear if the phase of amplitude cannot be fixed (as, for instance, in the Aharonov-Bohm case). Note that the Hamiltonian \[2.5\] contains the energy of radial $j_r r$ and angular $j_\varphi \varphi$ excitation independently.

Let

$$\Delta = \int \prod_t d^2 \xi d^2 \eta \delta(r - r_c(\xi, \eta)) \delta(p - p_c(\xi, \eta)) \delta(l - l_c(\xi, \eta)) \delta(\varphi - \varphi_c(\xi, \eta))$$

be the functional of known functions $(r_c, p_c, \varphi_c, l_c)(\xi, \eta)$. It is assumed that there are such functions $(\xi, \eta)(t)$ at given $(r, p, \varphi, l)(t)$ that the functional determinant

$$\Delta_c = \int \prod_t d^2 \xi d^2 \eta \delta(\frac{\partial r_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial r_c}{\partial \eta} \cdot \dot{\eta}) \delta(\frac{\partial p_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial p_c}{\partial \eta} \cdot \dot{\eta}) \times \delta(\frac{\partial \varphi_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial \varphi_c}{\partial \eta} \cdot \dot{\eta}) \delta(\frac{\partial l_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial l_c}{\partial \eta} \cdot \dot{\eta}) \neq 0.$$

Note that this is the condition for $(r_c, p_c, \varphi_c, l_c)(\xi, \eta)$ only since one can choose $(r, p, \varphi, l)(t)$ in eq.(2.7) in an arbitrary useful way.

To perform the mapping we should insert

$$1 = \frac{\Delta}{\Delta_c}$$

into (2.1) and integrate over $r(t)$, $p(t)$, $\varphi(t)$ and $l(t)$. In result we find the measure:

$$DM(\xi, \eta) = \frac{1}{\Delta_c} \delta(E - H_0) \prod_t d^2 \xi d^2 \eta \delta(\dot{r}_c - \frac{\partial H_i}{\partial p_c}) \times \delta(\dot{p}_c + \frac{\partial H_j}{\partial r_c}) \delta(\dot{r}_c - \frac{\partial H_i}{\partial p_c}) \times \delta(\dot{p}_c + \frac{\partial H_j}{\partial r_c}) \delta(\dot{\varphi}_c - \frac{\partial H_j}{\partial l_c}) \delta(\dot{\varphi}_c - \frac{\partial H_j}{\partial l_c}).$$

Note that the functions $(r_c, p_c, \varphi_c, l_c)(\xi, \eta)$ was not specified.

A simple algebra gives:

$$DM(\xi, \eta) = \frac{\delta(E - H_0)}{\Delta_c} \prod_t d^2 \xi d^2 \eta \int \prod_t d^2 \xi d^2 \eta \times \delta^2(\dot{\xi} - (\dot{\xi} - \frac{\partial h_j}{\partial \eta})) \delta^2(\dot{\eta} - (\dot{\eta} + \frac{\partial h_j}{\partial \xi})) \times \delta(\frac{\partial r_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial r_c}{\partial \eta} \cdot \dot{\eta} + \{r_c, h_j\} - \frac{\partial H_i}{\partial p_c}) \times \delta(\frac{\partial p_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial p_c}{\partial \eta} \cdot \dot{\eta} + \{p_c, h_j\} + \frac{\partial H_i}{\partial r_c}) \times \delta(\frac{\partial \varphi_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial \varphi_c}{\partial \eta} \cdot \dot{\eta} + \{\varphi_c, h_j\} - \frac{\partial H_i}{\partial l_c}) \times \delta(\frac{\partial l_c}{\partial \xi} \cdot \dot{\xi} + \frac{\partial l_c}{\partial \eta} \cdot \dot{\eta} + \{l_c, h_j\} + \frac{\partial H_j}{\partial \varphi_c}).$$

(2.10)
The Poisson notation:

\[ \{ X, h_j \} = \frac{\partial X}{\partial \xi} \frac{\partial h_j}{\partial \eta} - \frac{\partial X}{\partial \eta} \frac{\partial h_j}{\partial \xi} \]

was introduced in (2.10).

We will define the ‘auxiliary’ quantity \( h_j \) by following equalities:

\[
\begin{align*}
\{ r_c, h_j \} - \frac{\partial H_j}{\partial p_c} &= 0, \\
\{ p_c, h_j \} + \frac{\partial H_j}{\partial r_c} &= 0, \\
\{ \varphi_c, h_j \} - \frac{\partial H_j}{\partial l_c} &= 0, \\
\{ l_c, h_j \} + \frac{\partial H_j}{\partial \varphi_c} &= 0.
\end{align*}
\]

(2.11)

Then the functional determinant \( \Delta_c \) is canceled and

\[
DM(\xi, \eta) = \delta(E - H_0) \prod_t d^2 \xi d^2 \eta \delta^2(\dot{\xi} - \frac{\partial h_j}{\partial \eta}) \delta^2(\dot{\eta} + \frac{\partial h_j}{\partial \xi}),
\]

(2.12)

It is the desired result of transformation of the measure for given generating functions \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\). In this case the ‘Hamiltonian’ \( h_j(\xi, \eta) \) is defined by four equations (2.11).

But there is another possibility. Let us assume that

\[
h_j(\xi, \eta) = H_j(r_c, p_c, \varphi_c, l_c).
\]

(2.13)

and the functions \((r_c, p_c, \varphi_c, l_c)(\xi, \eta)\) are unknown. Then eqs.(2.11) are the equations for this functions. It is not hard to see that the eqs.(2.11) simultaneously with equations fixed by \( \delta \)-functions in (2.12) are equivalent of incident equations if the equality (2.13) is hold. So, incident dynamical problem was divided on two parts. First one defines the trajectory in the W space through eqs.(2.11). Second one defines the dynamics, i.e. the time dependence, through the equations fixed by \( \delta \)-functions in the measure.

Therefore, we should consider \( r_c, p_c, \varphi_c, l_c \) as the solutions in the \( \xi, \eta \) parametrization. The desired parametrization of classical orbits has the form (one can find it in arbitrary textbook of classical mechanics):

\[
\begin{align*}
r_c &= \frac{\eta^2_1 (\eta^2_1 + \eta^2_2)^{1/2}}{(\eta^2_1 + \eta^2_2)^{1/2} + \eta_2 \cos \xi_1}, \\
p_c &= \frac{\eta_2 \sin \xi_1}{\eta_1 (\eta^2_1 + \eta^2_2)^{1/2}}, \\
\varphi_c &= \xi_1, \\
l_c &= \eta_1.
\end{align*}
\]

(2.14)

At the same time,

\[
h_j = \frac{1}{2(\eta^2_1 + \eta^2_2)^{1/2}} - j_r r_c - j_\varphi \xi_1 \equiv h(\eta) - j_r r_c - j_\varphi \xi_1.
\]

(2.15)

Noting that the derivatives over \( \xi_2 \) are equal to zero\(^3\) we find that

\[
DM(\xi, \eta) = \delta(E - h(T)) \prod_t d^2 \xi d^2 \eta \delta(\dot{\xi}_1 - \omega_1 + j_r \frac{r_c}{\partial \eta_1})
\times \delta(\dot{\xi}_2 - \omega_2 + j_r \frac{r_c}{\partial \eta_2}) \delta(\dot{\eta}_1 - j_r \frac{\partial r_c}{\partial \xi_1} - j_\varphi \delta(\eta_2)),
\]

(2.16)

\(^3\)To have the condition \((2.8)\) we should assume that \( \partial r_c/\partial \xi_2 \sim \varepsilon \neq 0 \). We put \( \varepsilon = 0 \) completing the transformation.
where
\[ \omega_i = \partial h / \partial \eta_i \] (2.17)
are the conserved in classical limit \( j_r = j_\varphi = 0 \) ‘velocities’ in the W space.

3 Reduction

We see from (2.16) that the length of Runge-Lentz vector is not perturbated by the quantum forces \( j_r \) and \( j_\varphi \). To investigate the consequence of this fact it is useful to project this forces on the axis of W space. This means splitting of \( j_r, j_\varphi \) on \( j_\xi, j_\eta \). The equality
\[
\prod_t \delta (\dot{\xi}_1 - \omega_1 + j_r \frac{r_c}{\partial \eta_1}) = e^{\frac{1}{2} \int_0^T dt \dot{j}_\xi_1 \dot{\xi}_1} e^{2\int_0^T dt j_r e \xi_1 \partial r_c / \partial \eta_1} \prod_t \delta (\dot{\xi}_1 - \omega_1 + j_\xi_1)
\]
becomes evident if the Fourier representation of \( \delta \)-function is used (see also [3]). The same transformation of arguments of other \( \delta \)-functions in (2.16) can be applied. Then, noting that the last \( \delta \)-function in (2.16) is source-free, we find the same representation as (2.1) with
\[
\hat{K}(j, e) = \int_0^T dt (\dot{j}_{\xi_1} \dot{\xi}_1 + \dot{j}_{\xi_2} \dot{\xi}_2 + \dot{j}_m \dot{\eta}_m),
\] (3.1)
where the operators \( \hat{j} \) are defined by the equality:
\[
\hat{j}_X(t) = \int_0^T dt \theta(t - t') \dot{X}(t')
\] (3.2)
and \( \theta(t - t') \) is the Green function of our perturbation theory [3].

We should change also
\[
e_r \to e_c = e_\eta \frac{\partial r_c}{\partial \xi_1} - e_{\xi_1} \frac{\partial r_c}{\partial \eta_1} - e_{\xi_2} \frac{\partial r_c}{\partial \eta_2}, \quad e_\varphi \to e_{\xi_1}
\] (3.3)
in the eq.(2.6). The differential measure takes the simplest form:
\[
DM(\xi, \eta) = \delta(E - h(T)) \prod_t d^2 \xi d^2 \eta \delta (\dot{\xi}_1 - \omega_1 - j_\xi_1) \delta (\dot{\xi}_2 - \omega_2 - j_\xi_2)
\]
\[
\times \delta (\dot{\eta}_1 - j_\eta_1) \delta (\dot{\eta}_2).
\] (3.4)

Note now that the \( \xi, \eta \) variables are contained in \( r_c \) only:
\[
r_c = r_c(\xi_1, \eta_1, \eta_2).
\]
This means that the action of the operator \( \hat{j}_{\xi_2} \) gives identical to zero contributions into perturbation theory series. And, since \( \hat{\dot{\xi}}_2 \) and \( j_{\xi_2} \) are conjugate operators, see (3.1), we can put
\[
j_{\xi_2} = e_{\xi_2} = 0.
\]
This conclusion ends the reduction:

\[ \hat{K}(j, e) = \int_0^T dt (\hat{j}_\xi \hat{e}_\xi + \hat{j}_\eta \hat{e}_\eta), \quad (3.5) \]

\[ e_c = e_m \frac{\partial r_c}{\partial \xi_1} - e_{\xi_1} \frac{\partial r_c}{\partial \eta_1}. \quad (3.6) \]

The measure has the form:

\[ DM(\xi, \eta) = \delta(E - h(T))d\xi_2 d\eta_2 \prod_t d\xi_1 d\eta_1 \delta(\bar{\xi}_1 - \omega_1 - j_\xi) \delta(\bar{\eta}_1 - j_\eta) \quad (3.7) \]

since \( V = V(r_c, e_c, \xi_1) \) is \( \xi_2 \) independent and

\[ \int \prod_t dX(t) \delta(\bar{X}) = \int dX(0). \]

4 Perturbations

One can see from (3.7) that the reduction can not solve the H-atom problem completely: there are nontrivial corrections to the orbital degrees of freedom \( \xi_1, \eta_1 \). By this reason we should consider the expansion over \( \hat{K} \).

Using last \( \delta \)-functions in (3.7) we find, see also [3] (normalizing \( \rho(E) \) on the integral over \( \xi_2 \)):

\[ \rho(E) = \int_0^\infty dTe^{-i\hat{K}(j, e)} \int dM e^{-iV(r_c, e)}; \quad (4.1) \]

where

\[ dM = \frac{d\xi_1 d\eta_1}{\omega_2(E)}. \quad (4.2) \]

The operator \( \hat{K}(j, e) \) was defined in (3.3) and

\[ V(r, e) = -s_0(r) + \int_0^T dt \left[ \frac{1}{((r_c + e_c)^2 + r_c e_{\xi_1}^2)^{1/2}} - \frac{1}{((r_c - e_c)^2 + r_c e_{\xi_1}^2)^{1/2}} + \frac{e_c}{r_c} \right] \quad (4.3) \]

with \( e_c, e_{\xi_1} \) defined in (3.6, 3.3) and

\[ r_c(t) = r_c(\eta_1 + \eta(t), \bar{\eta}_2(E, T), \xi_1 + \omega_1(t) + \xi(t)), \quad E \equiv h(\eta_1 + \eta(T), \bar{\eta}_2), \quad (4.4) \]

where \( \bar{\eta}_2(E, T) \) is the solution of equation \( E = h \).

The integration range over \( \xi_1 \) and \( \eta_1 \) is as follows:

\[ 0 \leq \xi_1 \leq 2\pi, \quad -\infty \leq \eta_1 \leq +\infty. \quad (4.5) \]

First inequality defines the principal domain of the angular variable \( \varphi \) and second ones take into account the clockwise and anticlockwise motions of particle on the Kepler orbits.
We can write:

$$\rho(E) = \int_0^\infty dT \int dM : e^{-iV(r_c, \hat{e})} :$$  \hspace{1cm} (4.6)

since the operator $\hat{K}$ is linear over $\hat{e}_{\xi_1}, \hat{e}_{\eta_1}$. The colons means ‘normal product’ with operators staying to the left of functions and $V(r_c, \hat{e})$ is the functional of operators:

$$2i\hat{e}_c = \hat{j}_n \frac{\partial r_c}{\partial \xi_1} - \hat{j}_\xi \frac{\partial r_c}{\partial \eta_1}, \quad 2i\hat{e}_{\xi_1} = \hat{j}_{\xi_1}. \hspace{1cm} (4.7)$$

Expanding $V(r_c, \hat{e})$ over $\hat{e}_c$ and $\hat{e}_{\eta_1}$ we find:

$$V(r_c, \hat{e}) = -s_0(r_c) + 2 \sum_{n+m \geq 1} C_{n,m} \int_0^T dt \frac{\hat{e}_c^{2n+1} \hat{e}_{\eta_1}^m}{r_c^{2n+2}}, \hspace{1cm} (4.8)$$

where $C_{n,m}$ are the numerical coefficients. We see that the interaction part presents expansion over $1/r_c$ and, therefore, the expansion over $V$ generates an expansion over $1/r_c$.

In result,

$$\rho(E) = \int_0^\infty dT \int dM \{e^{is_0(r_c)} + B_{\xi_1}(\xi_1, \eta_1) + B_{\eta_1}(\xi_1, \eta_1)\}. \hspace{1cm} (4.9)$$

The first term is the pure quasiclassical contribution and last ones are the quantum corrections. Using results of \[3\] functionals $B$ are the total derivatives:

$$B_{\xi_1}(\xi_1, \eta_1) = \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1), \quad B_{\eta_1}(\xi_1, \eta_1) = \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1). \hspace{1cm} (4.10)$$

This means that the mean value of quantum corrections in the $\xi_1$ direction are equal to zero:

$$\int_0^{2\pi} d\xi_1 \frac{\partial}{\partial \xi_1} b_{\xi_1}(\xi_1, \eta_1) = 0 \hspace{1cm} (4.11)$$

since $r_c$ is the closed trajectory independently from initial conditions, see (2.14).

In the $\eta_1$ direction the motion is classical:

$$\int_{-\infty}^{+\infty} d\eta_1 \frac{\partial}{\partial \eta_1} b_{\eta_1}(\xi_1, \eta_1) = 0 \hspace{1cm} (4.12)$$

since (i) $b_{\eta_1}$ is the series over $1/r_c^2$ and (ii) $r_c \to \infty$ when $|\eta_1| \to \infty$. Therefore,

$$\rho(E) = \int_0^\infty dT \int dMe^{is_0(r_c)}. \hspace{1cm} (4.13)$$

This is the desired result.

Noting that

$$s_0(r_c) = kS_1(E), \quad k = \pm 1, \pm 2, ..., \quad (4.14)$$

7
where $S_1(E)$ is the action over one classical period $T_1$:

$$\frac{\partial S_1(E)}{\partial E} = T_1(E),$$

and using the identity 7:

$$\sum_{-\infty}^{+\infty} e^{inS_1(E)} = 2\pi \sum_{-\infty}^{+\infty} \delta(S_1(E) - 2\pi n),$$

we find:

$$\rho(E) = \pi\Omega \sum_n \delta(E + 1/2n^2) \quad (4.14)$$

where $\Omega$ is the zero-modes volume.

## 5 Concluding remarks

Our result (4.13) essentially uses the fact that the quantum corrections are defined by the topology of the $G_\omega$. Considering $E \leq 0$ $G_\omega$ has the topology of torus $G_\omega = S^3 \times S^1$ in the $(l_0, h_0)$ plane 4 and at $|l_0| = \infty$ this torus degenerate to the circle with infinite radii. Therefore, because of property (4.12) the mean value of quantum corrections lie on a bifurcation line in the $(l_0, h_0)$ plane.

Absence of the angular corrections are due from the fact that the classical trajectory in the Coulomb potential is closed independently from $l$ and $h$. This property reflects conservation of Runge-Lentz vector, i.e. is the consequence of the hidden $O(4)$ symmetry 5.

### Acknowledgement

I would like to thank my colleagues in the Institute of Physics (Tbilisi) for interesting discussions. The work was supported in part by Georgian Academy of Sciences.
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