DIFFERENTIAL GRADED LIE ALGEBRAS CONTROLLING INFINITESIMAL DEFORMATIONS OF COHERENT SHEAVES

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Abstract. We use the Thom-Whitney construction to show that infinitesimal deformations of a coherent sheaf \( F \) are controlled by the differential graded Lie algebra of global sections of an acyclic resolution of the sheaf \( \text{End}^* (\mathcal{E}) \), where \( \mathcal{E} \) is any locally free resolution of \( F \). In particular, one recovers the well known fact that the tangent space to \( \text{Def}_F \) is \( \text{Ext}^1 (F, F) \), and obstructions are contained in \( \text{Ext}^2 (F, F) \).

The main tool is the identification of the deformation functor associated with the Thom-Whitney DGLA of a semicosimplicial DGLA \( g^{\Delta} \), whose cohomology is concentrated in nonnegative degrees, with a noncommutative \( \check{\text{C}} \v{e}c \text{h} \) cohomology-type functor \( H^1_{\text{sc}} (\text{exp} g^{\Delta}) \).

Introduction

The classical approach to deformation theory, starting with Kodaira and Spencer’s studies on deformations of complex manifolds, consists in deforming the objects locally and then glue back together these local deformations. During the last thirty years, another approach to deformation problems has been developed. The philosophy underlying it, essentially due to Quillen, Deligne, Drinfeld and Kontsevich, is that, in characteristic zero, every deformation problem is controlled by a differential graded Lie algebra, via solutions of Maurer-Cartan equation modulo gauge equivalence. The aim of this paper is to exhibit an explicit equivalence between the two approaches for the problem of infinitesimal deformations of coherent sheaves.

In the particular case of a locally free sheaf \( \mathcal{E} \) of \( \mathcal{O}_X \)-modules on a complex manifold \( X \), the Kodaira-Spencer’s description of deformations of \( \mathcal{E} \) is given in terms of the \( \check{\text{C}} \v{e}c \text{h} \) functor \( H^1 (X; \text{exp} \text{End} (\mathcal{E})) \), where \( \text{End} (\mathcal{E}) \) is the sheaf of endomorphism of \( \mathcal{E} \). Indeed, a locally free sheaf has only trivial local deformations and so a deformation of \( \mathcal{E} \) is reduced to a deformation of the gluing data of its local charts, and the compatibility conditions these gluing data have to satisfy is precisely expressed by the cocycle condition in the \( \check{\text{C}} \v{e}c \text{h} \) functor. On the other hand, it is well known that deformations of \( \mathcal{E} \) are controlled by the DGLA of global sections of an acyclic resolution of \( \text{End} (\mathcal{E}) \), e.g., by the DGLA \( A^*_{\mathcal{E}} (\text{End} (\mathcal{E})) \) of \((0, *)\)-forms on \( X \) with values in the sheaf of endomorphisms of the sheaf \( \mathcal{E} \).

The equivalence between these two descriptions is best understood by moving from set-valued to groupoid-valued deformation functors; see, e.g., [19, 20]. Associating with any open set \( U \) in \( X \) the groupoid \( \text{Def}_{\mathcal{E}|U} \) of infinitesimal deformations of \( \mathcal{E} \) over \( U \) (over a fixed base \( \text{Spec} A \), for some local Artin ring \( A \)) defines a stack over \( \text{Top}_X \); this is just a one-word way of saying that global deformations of \( \mathcal{E} \) are the same thing as the descent

1991 Mathematics Subject Classification. 18G30, 18G50, 18G55, 13D10, 17B70.
Key words and phrases. Differential graded Lie algebras, functors of Artin rings.
data for its local deformations:

$$\text{Def}_E \cong \text{holim}_{U \in \Delta_U} \text{Def}_{E|U},$$

where $\Delta_U$ is the semisimplicial object in $\text{Top}_X$ associated with an open cover $U$ of $X$. Next, one sees that locally the groupoid of deformations of $E|U$ is equivalent to the Deligne groupoid of $\text{End}(\mathcal{E})(U)$; since these equivalences are compatible with restriction maps, one has an equivalence of semisimplicial groupoids. Finally, Deligne groupoid commutes with homotopy limits of DGLA concentrated in positive degree (see [9]), so that

$$\text{Def}_E \cong \text{holim}_{U \in \Delta_U} \text{Del}_{\text{End}(\mathcal{E})(U)} \cong \text{Del}_{\text{holim}_{U \in \Delta_U} \text{End}(\mathcal{E})(U)}.$$

This shows that the problem of infinitesimal deformations of $E$ is controlled by the DGLA $\text{holim}_{U \in \Delta_U} \text{End}(\mathcal{E})(U)$. It is now a simple exercise in homological algebra showing that there is a quasi-isomorphism of DGLAs

$$\text{holim}_{U \in \Delta_U} \text{End}(\mathcal{E})(U) \cong A^{0,*}_X(\text{End}(\mathcal{E})).$$

The reader who prefers to not leave the peaceful realm of set-valued deformation functors can find a direct (but less enlightening) proof of the equivalence between the Kodaira-Spencer’s and the DGLA approach to infinitesimal deformation of locally free sheaves in [7], where the explicit Thom-Whitney model for $\text{holim}_{U \in \Delta_U} \text{End}(\mathcal{E})(U)$ is used.

We now turn our attention to deformations of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a complex manifold or an algebraic variety $X$. The classical approach to this deformation problem is based on a locally free resolution $\mathcal{E} \to \mathcal{F}$ of $\mathcal{F}$; then, the data of a deformation of $\mathcal{F}$ are the data of local deformations of $\mathcal{E}$ with appropriate gluing conditions. More precisely, the sheaf of differential graded Lie algebras $\text{End}^\ast(\mathcal{E})$ of the endomorphisms of the resolution $\mathcal{E}$ controls infinitesimal deformations of $\mathcal{F}$ via the Čech-type functor $H^{1}_{\text{Ho}}(X; \text{exp End}^\ast(\mathcal{E}))$; the subscript $\text{Ho}$ refers to the fact that cocycle conditions hold only up to homotopy. The functor $H^{1}_{\text{Ho}}(X; \text{exp End}^\ast(\mathcal{E}))$ is actually independent of the particular resolution chosen. And again, on the DGLA side, one proves that infinitesimal deformations of $\mathcal{F}$ are controlled by the DGLA of global sections of an acyclic resolution of $\text{End}^\ast(\mathcal{E})$; in particular, one recovers the well known fact that the tangent space to $\text{Def}_\mathcal{F}$ is $\text{Ext}^1(\mathcal{F}, \mathcal{F})$, and obstructions are contained in $\text{Ext}^2(\mathcal{F}, \mathcal{F})$.

To see why such a result should hold, one has to make a further step and go from groupoid-valued to $\infty$-groupoid-valued deformation functors, and to think the whole problem in terms of $\infty$-stacks [10, 16, 24]. Indeed, due to the presence of negative degree components in $\text{End}^\ast(\mathcal{E})$, the groupoids $\text{Def}_\mathcal{F}|_U$ are no more equivalent to the Deligne groupoids $\text{Def}_{\text{End}^\ast(\mathcal{E})}|_U$; yet from the $\infty$-groupoid point of view it is natural to expect that the stack $\text{Def}_\mathcal{F}$ is locally homotopy equivalent to the $\infty$-stack $\text{MC}_\bullet(\text{End}^\ast(\mathcal{E}))$. Then one reasons as in the locally free sheaf case, using the fact that the Kan complex-valued functor $\text{MC}_\bullet$ commutes with homotopy limits of DGLAs whose cohomology is concentrated in positive degree [8]:

$$\text{Def}_\mathcal{F} \cong \text{holim}_{U \in \Delta_U} \text{Def}_\mathcal{F}|_U \cong \text{holim}_{U \in \Delta_U} \text{MC}_\bullet(\text{End}^\ast(\mathcal{E})|_U) \cong \text{MC}_\bullet(\text{holim}_{U \in \Delta_U} \text{End}^\ast(\mathcal{E})|_U).$$

As above, the homotopy limit $\text{holim}_{U \in \Delta_U} \text{End}^\ast(\mathcal{E})|_U$ is quasiisomorphic to the DGLA of global sections of an acyclic resolution of $\text{End}^\ast(\mathcal{E})$, which therefore controls the infinitesimal deformations of $\mathcal{F}$. 

The aim of this paper is to give a direct proof of this fact at the level of set-valued deformation functors. The proof closely follows the argument in [7] and does not rely on the conjectural homotopy equivalence between DefT_W and MC_*(End^n*(E')(U)). More precisely, we associate with any semicosimplicial DGLA g^Δ a set-valued functor of Artin rings Z^1_sc(exp g^Δ) together with an equivalence relation ~ on it, such that the quotient functor H^1_sc(exp g^Δ) = Z^1_sc(exp g^Δ)/~ is an abstract version of H^1_Ho(X; exp End^n*(E')). The latter is obtained, as a particular case, by considering the Čech semicosimplicial Lie algebra End^n*(E')(U)

\[ \prod_i \text{End}^n(E)(U_i) \longrightarrow \prod_{i<j} \text{End}^n(E)(U_{ij}) \longrightarrow \prod_{i<j<k} \text{End}^n(E)(U_{ijk}) \longrightarrow \cdots. \]

Namely,

\[ H^1_Ho(X; \exp \text{End}^n(E')) = \lim_{\text{SetArt}_K} \text{End}^n(E') \]

and both sides coincide with H^1_sc(exp End^n*(E')(U)), for an End^n*(E')-acyclic cover of X.

Next, we consider the Thom-Whitney model Tot_T_W g^Δ for holim g^Δ and show that there exists a commutative diagram of functors

\[
\begin{array}{ccc}
\text{DGLA}_{H^{\geq 0}} & \xrightarrow{Tot_T_W} & \text{DGLA} \\
H^1_sc(\exp -) & \searrow & \text{Maurer-Cartan/gauge} \\
\downarrow \hspace{0.5cm} \text{SetArt}_K & & \\
\end{array}
\]

where DGLA_{H^{\geq 0}} is the category of semicosimplicial DGLAs with no negative cohomology. From the point of view of ∞-groupoids, this can be seen as an explicit description of the set π_0(MC_*(holim g^Δ)).

The paper is organized as follows: in Section 1 we discuss deformations of coherent sheaves from a classical perspective and show how deformation data can be conveniently encoded into a Čech homology group with coefficient in a sheaf of DGLAs. In Section 2, the functors H^1_sc(exp g^Δ) and H^1_Ho(X; exp L) are defined; next, in Sections 3 and 4, we recall the definition of the Thom-Whitney DGLA associated with g^Δ and with its truncations g^Δ^{[n,n]}. Sections 5 and 6 are rather technical; namely Section 5 is devoted to a technical lemma on Maurer-Cartan elements in the Thom-Whitney DGLAsTot_T_W(g^Δ^{[0,1]}) and Tot_T_W(g^Δ^{[0,2]}) and Section 6 to the proof of the isomorphism H^1_sc(exp g^Δ^{[0,1]}) and Def_Tot_T_W(g^Δ^{[0,1]}). Finally, in Section 7, we are able to prove our main result (Theorem 7.6): under the cohomological hypothesis H^{-1}(g_2) = 0 there is a natural isomorphism of functors Def_Tot_T_W(g^Δ^{[0,2]}) ≃ H^1_sc(exp g^Δ); moreover, if H^i(g_i) = 0 for all i ≥ 0 and j < 0, then there is a natural isomorphism of functors Def_Tot_T_W(g^Δ^{[0,2]}) ≃ H^1_sc(exp g^Δ). In the concluding Section 8, we use this isomorphism to prove that infinitesimal deformations of a coherent sheaf F are controlled by the DGLA of global sections of an acyclic resolution of End^n*(E'), where E' is a locally free resolution of F.

While revising this paper, we became aware of [25] where a similar construction is developed and investigated.

Throughout this paper we work on a fixed algebraically closed field K of characteristic zero; the symbol Art_K denotes the category of local Artinian K-algebras (A, m_A), with residue field K.
Acknowledgement. We thank Marco Manetti for stimulating discussions on the subject and for useful comments and suggestions on the first version of this paper; d.i. thanks the Mathematical Department “Guido Castelnuovo”, Sapienza Università di Roma for the hospitality.

1. Infinitesimal deformations and sheaves of DGLAs

In this section, we study infinitesimal deformations of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a smooth projective variety $X$ and explain how these deformations can be naturally described in terms of a sheaf of differential graded Lie algebras on $X$.

An infinitesimal deformation of the coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ over $A \in \text{Art}_\mathbb{K}$ is given by a coherent sheaf $\mathcal{F}_A$ of $\mathcal{O}_X \otimes A$-modules on $X \times \text{Spec} A$, flat over $A$, with a morphism of sheaves $\pi : \mathcal{F}_A \to \mathcal{F}$ inducing an isomorphism $\mathcal{F}_A \otimes_A \mathbb{K} \cong \mathcal{F}$.

Two deformations $\mathcal{F}_A, \mathcal{F}'_A$ of the coherent sheaf $\mathcal{F}$ over $A$ are isomorphic if there exists an isomorphism of sheaves $f : \mathcal{F}_A \to \mathcal{F}'_A$, that commutes with the morphisms to $\mathcal{F}$. We denote by $\text{Def}_\mathcal{F} : \text{Art}_\mathbb{K} \to \text{Set}$ the functor of infinitesimal deformations of the sheaf $\mathcal{F}$.

We start by studying infinitesimal deformations of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on an affine variety $X$. Let $X = \text{Spec} R$, where $R$ is a Noetherian $\mathbb{K}$-algebra and let $\mathcal{F}$ be the coherent sheaf associated with a finitely generated $R$-module $M$; in this simple case, deformations of the sheaf $\mathcal{F}$ reduce to deformations of the $R$-module $M$.

An infinitesimal deformation of the $R$-module $M$ over $A \in \text{Art}_\mathbb{K}$ is given by a $R \otimes A$-module $M_A$, flat over $A$, with a morphism $\pi : M_A \to M$ inducing an isomorphism $M_A \otimes_A \mathbb{K} \cong M$. Two deformations $M_A$ and $M'_A$ of the module $M$ over $A$ are isomorphic if there exists an isomorphism of $R \otimes A$-modules $f : M_A \to M'_A$, that commutes with the morphisms to $M$.

Next, let

$$\cdots \xrightarrow{d} R^{n_1} \xrightarrow{d} R^{n_0} \xrightarrow{d} M \longrightarrow 0$$

be a presentation of $M$ as $R$-module. If $M_A$ is a deformation of $M$ over $A$, then it is an $A$-flat $R \otimes A$-module; therefore, flatness allows to lift relations between generators and to construct the exact sequence

$$\cdots \xrightarrow{d_A} R^{n_1} \otimes A \xrightarrow{d_A} R^{n_0} \otimes A \xrightarrow{d_A} M_A \longrightarrow 0,$$

that reduces to (1) when tensored by $\mathbb{K}$ over $A$. On the other hand, the datum of such an exact sequence assures flatness of the $R \otimes A$-module $M_A$ and so it defines a deformation of $M$ over $A$ (see [21] par. 3, or [23] Theorem A.31) for details of these correspondences). Moreover, if $M_A$ and $M'_A$ are isomorphic deformations of $M$ over $A$, the isomorphism between them lifts to an isomorphism between the correspondent deformed complexes and viceversa.

Next, we return to the global case of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a smooth projective variety $X$. Let

$$0 \longrightarrow \mathcal{E}^{-m} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0 \xrightarrow{d} \mathcal{F} \longrightarrow 0$$

be a global syzygy for $\mathcal{F}$, and denote by $\mathcal{E}$ the complex of locally free sheaves

$$(\mathcal{E}, d) : 0 \longrightarrow \mathcal{E}^{-m} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^{-1} \xrightarrow{d} \mathcal{E}^0 \longrightarrow 0.$$ 

Let $U = \{U_i\}_{i \in I}$ be an affine open cover of $X$, such that every sheaf of $\mathcal{E}$ is free on each $U_i$.\footnote{or Stein, if we work in the complex analytic category.}
The Kodaira-Spencer approach to infinitesimal deformations of $\mathcal{F}$ consists in deforming the sheaf $\mathcal{F}$ locally in such a way that local deformations glue together to a global sheaf, or equivalently, in view of the above discussion of the affine case, in deforming the complex $(\mathcal{E}^i, d)$ on every open set $U_i$ in such a way that these data glue together in cohomology.

Following this approach, let us make explicit the deformation data: the first datum is an element $l = \{l_i\} \in \prod_i \mathcal{E}nd^1(\mathcal{E})(U_i) \otimes m_A$ defining, on every open set $U_i$, a complex $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$ which is a deformation of the complex $(\mathcal{E}^i|_{U_i}, d)$. Note that the condition for $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$ to be a complex is the Maurer-Cartan equation:

$$dl_i + \frac{1}{2}[l_i, l_i] = 0,$$

for all $i \in I$.

Also note that, by upper semicontinuity of cohomology, the complex $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$ is exact except possibly at zero level. To glue together the deformed local complexes $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$, we need to specify isomorphisms between the deformed complexes on the double intersections of open sets of the cover $\mathcal{U}$. Since these isomorphisms will have to be deformations of the identity, they will be of the form

$$e^{m_{ij}} : (\mathcal{E}^i|_{U_{ij}} \otimes A, d + l_j) \rightarrow (\mathcal{E}^i|_{U_{ij}} \otimes A, d + l_i),$$

with $m = \{m_{ij}\}_{i < j} \subset \prod_{i < j} \mathcal{E}nd^0(\mathcal{E})(U_{ij}) \otimes m_A$. The compatibility with the differentials, i.e., the commutativity of the diagrams

$$\begin{array}{ccc}
\mathcal{E}^i|_{U_{ij}} \otimes A & \xrightarrow{e^{m_{ij}}} & \mathcal{E}^i|_{U_{ij}} \otimes A \\
\downarrow{d + l_j|_{U_{ij}}} & & \downarrow{d + l_i|_{U_{ij}}} \\
\mathcal{E}^i|_{U_{ij}} \otimes A & \xrightarrow{e^{m_{ij}}} & \mathcal{E}^i|_{U_{ij}} \otimes A
\end{array}$$

can be written as $d + l_j|_{U_{ij}} = e^{m_{ij}}(d + l_j|_{U_{ij}})e^{-m_{ij}}$, i.e., as

$$l_i|_{U_{ij}} = e^{m_{ij}} \cdot l_j|_{U_{ij}},$$

for all $i < j$.

Finally, the above isomorphisms have to satisfy the cocycle condition up to homotopy. Indeed, in order to obtain a deformation of $\mathcal{F}$, we actually do not want to glue together the complexes $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$, but rather their cohomology sheaves. In other words, we require $e^{m_{jk}}e^{-m_{ik}}e^{m_{ij}}$ to be homotopic to the identity on triple intersections. Taking logarithm, what we require is that $m_{jk} \bullet -m_{ik} \bullet m_{ij}$ is homotopy equivalent to zero, i.e.,

$$m_{jk}|_{U_{ijk}} \bullet -m_{ik}|_{U_{ijk}} \bullet m_{ij}|_{U_{ijk}} = [d + l_j|_{U_{ijk}}, n_{ijk}],$$

for some $n = \{n_{ijk}\}_{i < j < k} \subset \prod_{i < j < k} \mathcal{E}nd^{-1}(\mathcal{E})(U_{ijk})$. This homotopy cocycle equation is conveniently rewritten as

$$m_{jk}|_{U_{ijk}} \bullet -m_{ik}|_{U_{ijk}} \bullet m_{ij}|_{U_{ijk}} = d_{\mathcal{E}nd^*(\mathcal{E})} n_{ijk} + [l_j|_{U_{ijk}}, n_{ijk}].$$

Next, let explain how the data introduced above are concretely linked with deformations of the coherent sheaf $\mathcal{F}$ over $A$. As the homotopy cocycle equation is satisfied, the local $A$-flat sheaves of $\mathcal{O}_X|_{U_i} \otimes A$-modules $\mathcal{F}_A|_{U_i} := \mathcal{H}^*(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$ glue together to give a global coherent sheaf $\mathcal{F}_A$ which is a deformation of $\mathcal{F}$. On the other hand, every deformation $\mathcal{F}_A$ of the sheaf $\mathcal{F}$ can be obtained in this way. Indeed, the resolution $(\mathcal{E}^i, d)$ locally extends to projective resolutions $(\mathcal{E}^i|_{U_i} \otimes A, d + l_i)$ of $\mathcal{F}_A|_{U_i}$; these deformed local resolutions are linked each other on double intersections by isomorphisms of complexes lifting the identity of $\mathcal{F}_A$ and the compositions of these isomorphisms on triple intersections are homotopy to the identity, since they lift the identity of $\mathcal{F}_A$ and liftings are unique up to homotopy.
Let now $\mathcal{F}_A$ and $\mathcal{F}'_A$ be isomorphic deformations of the sheaf $\mathcal{F}$, associated with deformation data $(l, m)$ and $(l', m')$, respectively. The restriction to every open set $U_i$ of the isomorphism between $\mathcal{F}_A$ and $\mathcal{F}'_A$ lifts to local isomorphisms between the corresponding deformed complexes. Since these isomorphisms specialize to identities of $(\mathcal{E}|_{U_i}, d)$, they are of the form $e^{a_i} : (\mathcal{E}|_{U_i} \otimes A, d + l_i) \to (\mathcal{E}|_{U_i} \otimes A, d + l'_i)$, where $a = \{a_i\}_i \in \prod_i \text{End}^0(\mathcal{E})(U_i) \otimes m_A$. As above, compatibility with the differentials translates into the equations

$$e^{a_i} \star l_i = l'_i,$$

for all $i \in I$.

Finally, since the local isomorphisms $e^{a_i}$ lift a global isomorphism in cohomology, the diagrams

$$\begin{array}{ccc}
(\mathcal{E}|_{U_{ij}} \otimes A, d + l_j|_{U_{ij}}) & \xrightarrow{e^{m_{ij}}|_{U_{ij}}} & (\mathcal{E}|_{U_{ij}} \otimes A, d + l_i|_{U_{ij}}) \\
\downarrow e^{a_i|_{U_{ij}}} & & \downarrow e^{a_i|_{U_{ij}}} \\
(\mathcal{E}|_{U_{ij}} \otimes A, d + l'_j|_{U_{ij}}) & \xrightarrow{e^{m_{ij}'|_{U_{ij}}}} & (\mathcal{E}|_{U_{ij}} \otimes A, d + l'_i|_{U_{ij}}),
\end{array}$$

expressing compatibility with the gluing morphisms, commute in cohomology. Moreover, since the compositions $e^{-m_{ij} e^{-a_i} e^{m_{ij}'}}$ lift the identity of $\mathcal{F}_A$ on double intersections and liftings are unique up to homotopy, these compositions are homotopy to identity and, reasoning as above, we find

$$-m_{ij} \cdot -a_i|_{U_{ij}} \cdot m_{ij}' \cdot a_j|_{U_{ij}} = d\text{End}^*\mathcal{E}b_{ij} + [l_j|_{U_{ij}}, b_{ij}],$$

for some $b = \{b_{ij}\}_{i < j} \in \prod_{i < j} \text{End}^{-1}(\mathcal{E})(U_{ij}) \otimes m_A$. Viceversa, if for the deformation data $(l, m)$ and $(l', m')$ there exist $a = \{a_i\}_i \in \prod_i \text{End}^0(\mathcal{E})(U_i) \otimes m_A$ and $b = \{b_{ij}\}_{i < j} \in \prod_{i < j} \text{End}^{-1}(\mathcal{E})(U_{ij}) \otimes m_A$ that satisfy equations above, the local isomorphisms $e^{a_i}$ glue together in cohomology to give a global isomorphism of the correspondent deformed sheaves $\mathcal{F}_A$ and $\mathcal{F}'_A$.

Summing up, we have shown that in the Kodaira-Spencer approach, infinitesimal deformations of the coherent sheaf $\mathcal{F}$ are controlled by the sheaf of DGLAs $\text{End}^*(\mathcal{E})$, via the equations above. At the end of Section 7, we will apply techniques of semicosimplicial DGLAs developed in this paper to recover the classical well known fact that the functor of infinitesimal deformations of $\mathcal{F}$ has $\text{Ext}^1(\mathcal{F}, \mathcal{F})$ as tangent space and its obstructions are contained in $\text{Ext}^2(\mathcal{F}, \mathcal{F})$.

Remark 1.1. The above description of the functor of infinitesimal deformations of $\mathcal{F}$ is actually independent of the resolution chosen. Indeed, the DGLAs of the endomorphisms of any two locally free resolutions of $\mathcal{F}$ are quasi-isomorphic (see, e.g., [22, Lemma 4.4]).

Remark 1.2. If the sheaf $\mathcal{F}$ is locally free, then we can take its trivial resolution $0 \to \mathcal{F} \to \mathcal{F} \to 0$; thus, we recover the well known fact that the infinitesimal deformations of $\mathcal{F}$ are controlled by the sheaf $\text{End}(\mathcal{F})$ of the endomorphism of $\mathcal{F}$, via the Čech functor $H^1(X, \text{End}(\mathcal{F}))$.

Remark 1.3. Note that the results of this section actually hold under the hypothesis that $\mathcal{F}$ admits a global syzygy. This hypothesis is always satisfied, but in the general case the resolution is less obvious. Indeed, following Illusie [12, Section 1.5], for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a topological space $X$, one can construct the standard free resolution of $\mathcal{F}$:

$$\cdots \to \mathcal{R}(\mathcal{F})^2 \xrightarrow{D^2} \mathcal{R}(\mathcal{F})^1 \xrightarrow{D^1} \mathcal{R}(\mathcal{F})^0 \to \mathcal{F} \to 0.$$
Its terms are defined by recurrence: $R(F)^0$ is the free sheaf of $O_X$-modules associated with the presheaf $U \mapsto O_X(U)^F(U)$, given on every open set $U \subset X$ by the free $O_X(U)$-module generated by $F(U)$; $R(F)^j$ is the free sheaf of $O_X$-modules associated with the presheaf $U \mapsto O_X(U)^R(F)^{j-1}(U)$, given on every open set $U \subset X$ by the free $O_X(U)$-module generated by $R(F)^{j-1}(U)$.

To define morphisms $D^j$, let’s write explicitly elements in $R(F)^j(U)$. An element in $R(F)^0(U)$ is of the form $a^{i_0} \circ f_{i_0}$, where $a^{i_0} \in O_X(U)$, $f_{i_0} \in F(U)$, and we used the $\circ$ to denote the action of $O_X(U)$ on the free $O_X(U)$-module generated by $F(U)$, in order to distinguish it from the action of $O_X(U)$ on the $O_X(U)$-module $F(U)$. Recursively, an element in $R(F)^j(U)$ is of the form

$$a^{i_j} \circ a^{i_{j-1}} \circ \cdots \circ a^{i_0} \circ f_{i_0}$$

where $a^{i_k} \in O_X(U)$, $f_{i_0} \in F(U)$. The differential of the resolution is defined as $D^j = \sum_{k=0}^j (-1)^k d^j_k$, where $d^j_k : R(F)^j \rightarrow R(F)^{j-1}$ is defined by

$$a^{i_j} \circ \cdots \circ a^{i_k+1} \circ a^{i_k} \circ \cdots \circ a^{i_1} \circ f_{i_0} \rightarrow a^{i_j} \circ \cdots \circ a^{i_k+1} \circ a^{i_k} \circ \cdots \circ a^{i_1} \circ f_{i_0}$$

The relevant fact is that the sequence of free sheaves of $O_X$-modules $(R(F), D) \rightarrow F$ is a resolution of $F$ [12, Theorem 1.5.3]. This construction can be done for every sheaf $F$ of $O_X$-modules on a topological space $X$; Illusie obtains it as an example of the even more general construction of the standard simplicial resolution of a pair of adjoint functors [13, Section 1.5].

2. Semicosimplicial DGLAs and the functor $H^1_{sc}(\exp g^\Delta)$

A *semicosimplicial differential graded Lie algebra* is a covariant functor $\Delta_{mon} \rightarrow$ DGLA, from the category $\Delta_{mon}$, whose objects are finite ordinal sets and whose morphisms are order-preserving injective maps between them, to the category of DGLAs. Equivalently, a semicosimplicial DGLA $g^\Delta$ is a diagram

$$g_0 \longleftarrow \ldots \longleftarrow g_1 \longleftarrow g_2 \longleftarrow \cdots ,$$

where each $g_i$ is a DGLA, and for each $i > 0$ there are $i + 1$ morphisms of DGLAs

$$\partial_{k,i} : g_{i-1} \rightarrow g_i, \quad k = 0, \ldots, i,$

such that $\partial_{k+1,i+1} \partial_{l,j} = \partial_{l,i+1} \partial_{k,j}$, for any $k \geq l$.

A classical example is the following: given a sheaf $L$ of DGLAs on a topological space $X$, and an open cover $U$ of $X$, one has the Čech cosimplicial DGLA $L(U)$,

$$\prod U_i \longleftarrow \prod_{i < j} L(U_{ij}) \longleftarrow \cdots ,$$

where the morphisms $\partial_{k,i}$ are the restriction maps.

**Definition 2.1.** Let $g^\Delta$ be a semicosimplicial DGLA. The functor

$$Z^1_{sc}(\exp g^\Delta) : \text{Art}_K \rightarrow \text{Set}$$

is defined, for all $A \in \text{Art}_K$, by

$$Z^1_{sc}(\exp g^\Delta)(A) = \begin{cases} (l, m) \in (g_0^1 \oplus g_0^0) \otimes m_A & dl + \frac{1}{2}[l, l] = 0, \\
\partial_{1,1}l = e^m \ast \partial_{0,1}l, \\
\partial_{0,2}m \ast -\partial_{1,2}m \ast \partial_{2,2}m = dn + [\partial_{2,2}\partial_{0,1}l, n] & \text{for some } n \in g_2^{-1} \otimes m_A \end{cases}.$$
Remark 2.2. In DGLA theory, given a DGLA $L$ and a Maurer-Cartan element $x$ in $MC_L(A)$, the set

$$\text{Stab}(x) = \{dh + [x, h] \mid h \in L^{-1} \otimes m_A\}$$

is called the irrelevant stabilizer of $x$. Note that $\text{Stab}(x) \subseteq \text{stab}(x)$, where $\text{stab}(x) = \{a \in L^0 \otimes m_A \mid e^a \ast x = x\}$ is the stabilizer of $x$ under the gauge action of $L^0 \otimes m_A$ on $MC_L(A)$. Also note that, for any $a \in L^0 \otimes m_A$, $e^a e^{\text{Stab}(x)} e^{-a} = e^{\text{Stab}(y)}$, with $y = e^a \ast x$.

We now introduce an equivalence relation on the set $Z_{sc}^1(\exp g^\Delta)(A)$ as follows: we say that two elements $(l_0, m_0)$ and $(l_1, m_1) \in Z_{sc}^1(\exp g^\Delta)(A)$ are equivalent under the relation $\sim$ if and only if there exist elements $a \in g_0^0 \otimes m_A$ and $b \in g_1^{-1} \otimes m_A$ such that

$$\begin{align*}
e^a \ast l_0 &= l_1 \\
-m_0 \ast -\partial_{1,1} a \ast m_1 \ast \partial_{0,1} a &= db + [\partial_{0,1} l_0, b].
\end{align*}$$

Remark 2.3. The relation $\sim$ is actually an equivalence relation on $Z_{sc}^1(\exp g^\Delta)(A)$. First note that the set $Z_{sc}^1(\exp g^\Delta)(A)$ is closed under $\sim$. Indeed, let $(l_0, m_0)$ and $(l_1, m_1) \in (g_0^0 \otimes g_1^1) \otimes m_A$ be equivalent under $\sim$ via elements $a \in g_0^0 \otimes m_A$ and $b \in g_1^{-1} \otimes m_A$, and suppose that $(l_0, m_0) \in Z_{sc}^1(\exp g^\Delta)(A)$. Then $l_1 = e^a \ast l_0$ satisfies the Maurer Cartan equation and

$$e^{m_1} \ast \partial_{0,1} l_1 = e^{\partial_{1,1} a \ast m_0 \ast (db + [\partial_{0,1} l_0, b])} \ast -\partial_{0,1} a \ast \partial_{0,1} l_0 = e^{\partial_{1,1} a} \ast \partial_{0,1} l_0 = \partial_{0,1} l_1.$$ 

Moreover, an easy calculation, using relations between maps $\partial_{1,k}$ and Remark 2.2, shows that $\partial_{0,1} m_1 \ast -\partial_{1,2} m_1 \ast \partial_{2,2} m_1$ is an element of the irrelevant stabilizer of $\partial_{0,1} \partial_{0,1} l_1$.

Secondly $\sim$ is an equivalent relation. Reflexivity is trivial; for symmetry, let $(l_0, m_0)$ and $(l_1, m_1)$ be equivalent via elements $a \in g_0^0 \otimes m_A$ and $b \in g_1^{-1} \otimes m_A$, then $e^{-a} \ast l_1 = l_0$ and $-m_1 \ast \partial_{1,1} (a) \ast m_0 \ast -\partial_{0,1} a = \partial_{0,1} a \ast (db + [\partial_{0,1} l_0, b]) \ast -\partial_{0,1} a$ is an element of the irrelevant stabilizer of $\partial_{0,1} l_1$, by Remark 2.2. Next, let $(l_0, m_0) \sim (l_1, m_1)$ via $a \in g_0^0 \otimes m_A$ and $b \in g_1^{-1} \otimes m_A$, and $(l_1, m_1) \sim (l_2, m_2)$ via $\alpha \in g_0^0 \otimes m_A$ and $\beta \in g_1^{-1} \otimes m_A$; then, $e^{\alpha \ast a} \ast l_0 = l_2$ and

$$-m_0 \ast \partial_{1,1} (-b \ast a) \ast m_2 \ast \partial_{0,1} (b \ast a) = -m_0 \ast -\partial_{1,1} (b) \ast \partial_{0,1} (a) = -m_0 \ast -\partial_{0,1} (a) \ast m_1 \ast (db + [\partial_{0,1} l_0, b]) \ast \partial_{0,1} (a),$$

by Remark 2.2 it is an element of the irrelevant stabilizer of $\partial_{0,1} l_0$, therefore $\sim$ is transitive.

Definition 2.4. Let $g^\Delta$ be a semicosimplicial DGLA, the functor

$$H^1_{sc}(\exp g^\Delta) : \mathbf{Art}_K \to \mathbf{Set}$$

is defined, for all $A \in \mathbf{Art}_K$, by

$$H^1_{sc}(\exp g^\Delta)(A) = \frac{Z_{sc}^1(\exp g^\Delta)(A)}{\sim}.$$ 

Remark 2.5. Note that, if $g^\Delta$ is a semicosimplicial Lie algebra, i.e., if all the DGLAs $g_i$ are concentrated in degree zero, then the functor $H^1_{sc}(\exp g^\Delta)$ reduces to the one defined in [7].

Lemma 2.6. The projection $\pi : Z_{sc}^1(\exp g^\Delta) \longrightarrow H^1_{sc}(\exp g^\Delta)$ is a smooth morphism of functors.

Proof. Let $\beta : B \longrightarrow A$ be a surjection in $\mathbf{Art}_K$, we prove that the map

$$Z_{sc}^1(\exp g^\Delta)(B) \longrightarrow H^1_{sc}(\exp g^\Delta)(B) \times_{H^1_{sc}(\exp g^\Delta)(A)} Z_{sc}^1(\exp g^\Delta)(A),$$

induced by
A simple computation shows that it is enough to choose \( 2.8 \).

Remark \( a \) \( \rho \) \( \pi \) \( Z \) \( X \) \( U \) \( \beta \).

Both associated with the Thom-Whitney DGLA of the truncation, then the functor of Artin rings \( H \).

Let \( L \) be a sheaf of DGLAs on a topological space \( X \) and \( U \) an open cover. Considering the \( \check{\text{C}} \)ech cosimplicial DGLA \( H \), the \( \check{\text{C}} \)ech cosimplicial DGLA \( \pi \) \( F \) \( \phi \) \( \pi \).

Next, let \( L \) be a sheaf of DGLAs on a topological space \( X \) and \( U \) an open cover. Considering the \( \check{\text{C}} \)ech cosimplicial DGLA \( H \), we can define the functor \( H \).

Lemma 2.7. Let \( U = \{ U_\alpha \}_{\alpha \in I} \) and \( U' = \{ U'_\alpha \}_{\alpha \in I'} \) be open covers of \( X \) with \( U \) refinement of \( U \) and let \( \phi, \psi : I' \to I \) two refinement maps. Then, the induced morphisms \( \rho_\phi, \rho_\psi : H^1_{sc}(\exp L(U)) \to H^1_{sc}(\exp L(U')) \) coincide.

Proof. Both \( \phi \) and \( \psi \) induce, for all \( A \in \text{Art}_{sc} \), a morphism \( Z^1_{sc}(\exp L(U))(A) \to Z^1_{sc}(\exp L(U'))(A) \), defined sending \( (l_i, m_{ij}) \) to \( (l_\phi(l_i), m_{\phi i}) \) and \( \rho_\psi(l_i, m_{ij}) = (l_\psi, m_{\phi i}) \).

Therefore, it remains to prove that \( \rho_\phi(l_i, m_{ij}) \sim \rho_\psi(l_i, m_{ij}) \).

A simple computation shows that it is enough to choose \( a_i := m_{\phi i} \).

Remark 2.8. Having introduced the limit \( H \).

Remark \( \phi \) \( \pi \) \( U \) \( \phi \).

INFINITESIMAL DEFORMATIONS OF COHERENT SHEAVES

\[
\begin{array}{ccc}
Z^1_{sc}(\exp g^A)(B) & \xrightarrow{\rho} & Z^1_{sc}(\exp g^A)(A) \\
\pi & & \pi \\
H^1_{sc}(\exp g^A)(B) & \xrightarrow{\rho} & H^1_{sc}(\exp g^A)(A),
\end{array}
\]

is surjective. Let \( (|l, m|, (l_0, m_0)) \in H^1_{sc}(\exp g^A)(B) \), \( H^1_{sc}(\exp g^A)(A) \).

Then \( \beta \).

Remark \( a \) \( \rho \) \( \pi \) \( Z \) \( X \) \( U \) \( \beta \).

Having introduced the limit \( H \).

The example of coherent sheaves on projective manifolds together with the DGLA approach to deformation theory suggests that the functors of Artin rings \( H^1_{sc}(\exp g^A) \) could actually be isomorphic to functors \( \text{Def}_{L(g^A)} \) for some DGLA \( L(g^A) \) canonically associated with \( g^A \).

We are going to show that, under the cohomological hypothesis \( H^{-1}(g_2) = 0 \), it is indeed so. More precisely, we are going to prove that, if \( H^{-1}(g_2) = 0 \), then the functor of Artin rings \( H^1_{sc}(\exp g^A) \) is isomorphic to the deformation functor associated with the Thom-Whitney DGLA of the truncation \( g^{[0,2]} \).
3. The Thom-Whitney DGLA $\text{Tot}_{TW}(g^\Delta)$

Let $g^\Delta$ be a semicosimplicial DGLA. The maps
\[ \partial_i = \partial_0, i - \partial_1, i + \cdots + (-1)^i \partial_i \]
endow the vector space $\bigoplus_i g_i$ with the structure of a differential complex. Moreover, being a DGLA, each $g_i$ is in particular a differential complex
\[ g_i = \bigoplus_j g_i^j; \quad d_i : g_i^j \to g_i^{j+1} \]
and since the maps $\partial_{k,i}$ are morphisms of DGLAs, the space
\[ g = \bigoplus_i g_i \]
has a natural bicomplex structure. The associated total complex
\[ (\text{Tot}(g^\Delta), d_{\text{Tot}}) \quad \text{where} \quad \text{Tot}(g^\Delta) = \bigoplus_i g_i[-i], \quad d_{\text{Tot}} = \sum_i \partial_i + (-1)^j d_j \]
has no natural DGLA structure. Yet there is another bicomplex naturally associated with a semicosimplicial DGLA, whose total complex is naturally a DGLA.

For every $n \geq 0$, denote by $\Omega_n$ the differential graded commutative algebra of polynomial differential forms on the standard $n$-simplex $\Delta^n$:
\[ \Omega_n = \mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n] / \left( \sum t_i - 1, \sum dt_i \right) \]
Denote by $\delta^{k,n} : \Omega_n \to \Omega_{n-1}$, $k = 0, \ldots, n$, the face maps; then, one has natural morphisms of bigraded DGLAs
\[ \delta^{k,n} : \Omega_n \otimes g_n \to \Omega_{n-1} \otimes g_n, \quad \partial_{k,n} : \Omega_{n-1} \otimes g_{n-1} \to \Omega_{n-1} \otimes g_n, \]
for every $0 \leq k \leq n$.

The Thom-Whitney bicomplex is defined as
\[ C^{i,j}_{TW}(g^\Delta) = \{(x_n)_{n \in \mathbb{N}} \in \bigoplus_n \Omega^n_i \otimes g^n_j \mid \delta^{k,n} x_n = \partial_{k,n} x_{n-1} \quad \forall \quad 0 \leq k \leq n \}, \]
where $\Omega^n_i$ denotes the degree $i$ component of $\Omega_n$. Its total complex is a DGLA, called the Thom-Whitney DGLA, and it is denoted by $\text{Tot}_{TW}(g^\Delta)$; denote by $d_{TW}$ the differential of the Thom-Whitney DGLA. It is a remarkable fact that the integration maps
\[ \int_{\Delta^n} \otimes \text{Id} : \Omega_n \otimes g_n \to \mathbb{K}[n] \otimes g_n = g_n[n] \]
give a quasi-isomorphism of differential complexes
\[ I : (\text{Tot}_{TW}(g^\Delta), d_{TW}) \to (\text{Tot}(g^\Delta), d_{\text{Tot}}). \]
Moreover, Dupont has described in [3, 4] an explicit morphism of differential complexes
\[ E : \text{Tot}(g^\Delta) \to \text{Tot}_{TW}(g^\Delta) \]
and an explicit homotopy
\[ h : \text{Tot}_{TW}(g^\Delta) \to \text{Tot}_{TW}(g^\Delta)[-1] \]
such that
\[ IE = \text{Id}_{\text{Tot}(g^\Delta)}; \quad EI - \text{Id}_{\text{Tot}_{TW}(g^\Delta)} = [h, d_{TW}]. \]
We also refer to the papers [2, 8, 19] for the explicit description of $E, h$ and for the proof of the above identities. Here, we point out that $E$ and $h$ are defined in terms of
integration over standard simplexes and multiplication with canonical differential forms: in particular, the construction of $\text{Tot}_{\text{TW}}(g^\Delta)$, $\text{Tot}(g^\Delta)$, $I$, $E$ and $h$ is functorial in the category $\text{DGLA}^{\Delta_{\text{mon}}}$ of semicosimplicial DGLAs.

Recall that with a DGLA $L$ there is a canonically associated deformation functor $\text{Def}_L$, defined as the solutions of Maurer-Cartan equation modulo gauge action (or, equivalently, modulo homotopy equivalence). Moreover, the tangent space to $\text{Def}_L$ is $H^1(L)$ and obstructions live in $H^2(L)$. Thus, with a semicosimplicial DGLA $g^\Delta$ is also associated the deformation functor $\text{Def}_L$; its tangent space is

$$T \text{Def}_{\text{Tot}_{\text{TW}}(g^\Delta)} \cong H^1(\text{Tot}_{\text{TW}}(g^\Delta)) \cong H^1(\text{Tot}(g^\Delta))$$

and obstructions live in

$$H^2(\text{Tot}_{\text{TW}}(g^\Delta)) \cong H^2(\text{Tot}(g^\Delta))$$

Let $\Delta^+_{\text{mon}}$ the category obtained by adding the empty set $\emptyset$ to the category $\Delta_{\text{mon}}$. An augmented semicosimplicial differential graded Lie algebra is a covariant functor $\Delta^+_{\text{mon}} \to \text{DGLA}$, from the category $\Delta^+_{\text{mon}}$ to the category of DGLAs. Equivalently, an augmented semicosimplicial DGLA $g^\Delta$ is a diagram

$$
\begin{array}{cccccc}
g_{-1} & \longrightarrow & g_0 & \longrightarrow & g_1 & \longrightarrow & \cdots,
\end{array}
$$

where the truncated diagram $g^\Delta$

$$
\begin{array}{cccc}
g_0 & \longrightarrow & g_1 & \longrightarrow & \cdots
\end{array}
$$

is a semicosimplicial DGLA and

$$\partial_{0,0} : g_{-1} \to g_0$$

is a DGLA morphism such that $\partial_{0,1}\partial_{0,0} = \partial_{1,1}\partial_{0,0}$.

Remark 3.1. There is a morphism of DGLAs

$$g_{-1} \to \text{Tot}_{\text{TW}}(g^\Delta)$$

$$x \mapsto (\partial_{0,0}x, \partial_{1,1}\partial_{0,0}x, \partial_{2,2}\partial_{1,1}\partial_{0,0}x, \ldots);$$

the image of $x$ is an element in $\text{Tot}_{\text{TW}}(g^\Delta)$ because of equations $\partial_{1,1}\partial_{0,0} = \partial_{0,1}\partial_{0,0}$ and $
\partial_{k+1,i+1}\partial_{l,i} = \partial_{l,i+1}\partial_{k,i}$, for any $k \geq l$. This morphism is obtained as the composition of the natural inclusion $g_{-1} \hookrightarrow \text{Tot}(g^\Delta)$ with the morphism $E : \text{Tot}(g^\Delta) \to \text{Tot}_{\text{TW}}(g^\Delta)$. The existence of the DGLA morphism $g_{-1} \to \text{Tot}_{\text{TW}}(g^\Delta)$ is not surprising; indeed, it is induced by the natural morphism $\lim g^\Delta \to \text{holim} g^\Delta$.

We use augmentation to link the Thom-Whitney DGLA of the Čech semicosimplicial DGLA of a sheaf of DGLAs with the DGLA of global sections of an acyclic resolution of the sheaf. This result is a translation of Theorem 7.2 in [7] in terms of the Thom-Whitney DGLA.

We recall that if $\mathcal{L}$ is a sheaf of DGLAs on a topological space $X$ and $\mathcal{U}$ is an open cover of $X$, the associated Čech semicosimplicial differential graded Lie algebra is:

$$\mathcal{L}(\mathcal{U}) : \prod_{i} \mathcal{L}(U_i) \longrightarrow \prod_{i<j} \mathcal{L}(U_{ij}) \longrightarrow \prod_{i<j<k} \mathcal{L}(U_{ijk}) \longrightarrow \cdots.$$ 

A morphism $\varphi : \mathcal{L} \to \mathcal{A}$ of sheaves of DGLAs is a quasi-isomorphism if it is a quasi-isomorphism of sheaves of differential complexes, i.e., if it induces linear isomorphisms
between the cohomology sheaves,
\[ H^\ast(\varphi) : H^\ast(L) \xrightarrow{\sim} H^\ast(A). \]
Moreover, if \( \mathcal{A}^k \) is an acyclic sheaf for any \( k \), then \( \varphi : \mathcal{L} \rightarrow \mathcal{A} \) is called an acyclic resolution of \( \mathcal{L} \).

**Theorem 3.2.** Let \( X \) be a paracompact Hausdorff topological space, \( \mathcal{L} \) a sheaf of differential graded Lie algebras on \( X \), and \( \varphi : \mathcal{L} \rightarrow \mathcal{A} \) an acyclic resolution. Also let \( A = A(X) \) be the DGLA of global sections of \( \mathcal{A} \). Then, if \( \mathcal{U} \) is an open cover of \( X \) which is acyclic with respect to both \( \mathcal{L} \) and \( \mathcal{A} \), the DGLA \( \text{Tot}_{TW}(\mathcal{L}(\mathcal{U})) \) is naturally quasi-isomorphic to the DGLA \( A \).

**Proof.** The natural inclusion \( A \rightarrow A(\mathcal{U}) \) gives an augmented semicosimplicial DGLA, and so it induces a morphism of DGLAs \( A \rightarrow \text{Tot}_{TW}(A(\mathcal{U})) \), that is the composition of the natural inclusion \( A \rightarrow \text{Tot}(A(\mathcal{U})) \) with the quasi-isomorphism \( E : \text{Tot}(A(\mathcal{U})) \rightarrow \text{Tot}_{TW}(A(\mathcal{U})) \), by Remark 3.1. Since the sheaves \( \mathcal{A}^k \) are acyclic and \( \mathcal{U} \)-acyclic, and \( A^k = H^0(X; \mathcal{A}^k) \), the inclusion \( A \rightarrow \text{Tot}(A(\mathcal{U})) \) is a quasiisomorphism. Indeed, we have a natural identification \( H^\ast(\text{Tot}(A(\mathcal{U}))) = H^\ast(X; A) \), and the spectral sequence abutting to the hypercohomology of \( X \) with coefficients in \( A \) degenerates at \( E_2 \), giving
\[ H^k(X; A) = \bigoplus_{p+q=k} E_2^{p,q} = E_2^{k,0} = H^k(A). \]

Then, \( A \rightarrow \text{Tot}_{TW}(A(\mathcal{U})) \) is a quasi-isomorphism of DGLAs.

The morphism \( \varphi : \mathcal{L} \rightarrow \mathcal{A} \) induces a morphism of semicosimplicial DGLAs
\[ \varphi : \mathcal{L}(\mathcal{U}) \rightarrow A(\mathcal{U}), \]
and a morphism of complexes
\[ \varphi : \text{Tot}_{TW}(\mathcal{L}(\mathcal{U})) \rightarrow \text{Tot}_{TW}(A(\mathcal{U})). \]
Since the open cover \( \mathcal{U} \) is \( \mathcal{L} \)-acyclic, the cohomology of the total complex \( \text{Tot}(\mathcal{L}(\mathcal{U})) \) is naturally identified with the hypercohomology of \( X \) with coefficients in \( \mathcal{L} \),
\[ H^\ast(\text{Tot}(\mathcal{L}(\mathcal{U}))) \cong H^\ast(X; \mathcal{L}), \]
and the induced linear map
\[ H^\ast(\varphi) : H^\ast(\text{Tot}(\mathcal{L}(\mathcal{U}))) \rightarrow H^\ast(\text{Tot}(A(\mathcal{U}))) \]
is identified with the linear map
\[ \mathbb{H}^\ast(\varphi) : \mathbb{H}^\ast(X; \mathcal{L}) \rightarrow \mathbb{H}^\ast(X; A) \]
induced in hypercohomology. Since, by hypothesis, \( \varphi \) is a quasi-isomorphism of sheaves of DGLAs, the induced map in hypercohomology is an isomorphism, and so the morphism \( \varphi : \text{Tot}(\mathcal{L}(\mathcal{U})) \rightarrow \text{Tot}(A(\mathcal{U})) \) is a quasi-isomorphism of complexes.

Via the composition with quasi-isomorphisms \( E \) and \( I \) between the total complex and the Thom-Whitney total complex of a semicosimplicial DGLA, the morphism \( \varphi \) induces a quasi-isomorphism of DGLAs
\[ \text{Tot}_{TW}(\mathcal{L}(\mathcal{U})) \rightarrow \text{Tot}_{TW}(A(\mathcal{U})). \]
Therefore, we have the chain of quasi-isomorphisms of DGLAs
\[ \text{Tot}_{TW}(\mathcal{L}(\mathcal{U})) \xrightarrow{\sim} \text{Tot}_{TW}(A(\mathcal{U})) \xrightarrow{\sim} A. \]
4. Truncations

Let
g^{\Delta} : \quad g_0 \longrightarrow g_1 \longrightarrow g_2 \longrightarrow g_3 \longrightarrow \cdots

be a semicosimplicial DGLA. Let \( m_1 \in \mathbb{N} \) and \( m_2 \in \mathbb{N} \cup \{\infty\} \) with \( m_1 \leq m_2 \), we denote by \( g^{\Delta[m_1,m_2]} \) the truncated between levels \( m_1 \) and \( m_2 \) semicosimplicial DGLA defined by

\[
(g^{\Delta[m_1,m_2]})_n = \begin{cases} g_n & \text{for } m_1 \leq n \leq m_2 \\ 0 & \text{otherwise,} \end{cases}
\]

with the obvious maps \( \partial^{[m_1,m_2]}_{k,i} = \partial_{k,i} \), for \( m_1 < i \leq m_2 \), and \( \partial^{[m_1,m_2]}_{k,i} = 0 \), otherwise. For any positive integers \( m_1, m_2, r_1, r_2 \), such that \( r_i \leq m_i \), the map \( \text{Id}_{[m_1,r_2]} : g^{\Delta[m_1,m_2]} \to g^{\Delta[r_1,r_2]} \) given by

\[
\text{Id}_{[m_1,r_2]}|_{(g^{\Delta[m_1,m_2]})_n} = \begin{cases} \text{Id}_{g_n} & \text{if } m_1 \leq n \leq r_2 \\ 0 & \text{otherwise.} \end{cases}
\]

is a morphism of semicosimplicial DGLAs; it induces the natural morphism of complexes \( \phi : \text{Tot}(g^{\Delta[m_1,m_2]}) \to \text{Tot}(g^{\Delta[r_1,r_2]}) \) and the natural morphism of DGLAs \( \psi : \text{Tot}_{TW}(g^{\Delta[m_1,m_2]}) \to \text{Tot}_{TW}(g^{\Delta[r_1,r_2]}) \). Notice that we have an homotopy commutative diagram of complexes

\[
\begin{array}{ccc}
\text{Tot}(g^{\Delta[m_1,m_2]}) & \xrightarrow{\phi} & \text{Tot}(g^{\Delta[r_1,r_2]}) \\
\text{Tot}_{TW}(g^{\Delta[m_1,m_2]}) & \xrightarrow{\psi} & \text{Tot}_{TW}(g^{\Delta[r_1,r_2]}). \\
\end{array}
\]

**Proposition 4.1.** Let \( g^{\Delta} \) be a semicosimplicial DGLA such that \( H^j(g_t) = 0 \), for all \( i \geq 0 \) and \( j < 0 \). Then, the morphism \( \text{Id}_{[0,2]} \) induces a natural isomorphism of functors:

\[
\text{Def}_{\text{Tot}_{TW}(g^{\Delta})} \cong \text{Def}_{\text{Tot}_{TW}(g^{\Delta[0,2]})}.
\]

**Proof.** It is a well known fact (see, e.g., [17] for a proof), that a DGLA morphism which is surjective on \( H^0 \), bijective on \( H^1 \) and injective on \( H^2 \) induces an isomorphism between the associated deformation functors. Since the above homotopy commutative diagram identifies \( H^*(\psi) \) with \( H^*(\phi) \), it is enough to prove that \( H^0(\phi) \) is surjective, \( H^1(\phi) \) is bijective and \( H^2(\phi) \) is injective. This is easily checked by looking at the spectral sequences associated with double complexes of \( g^{\Delta} \) and \( g^{\Delta[0,2]} \). \( \square \)

**Remark 4.2.** Observe that, for any semicosimplicial DGLA \( g^{\Delta} \), we have \( Z^1_{sc}(\exp g^{\Delta}) = Z^1_{sc}(\exp g^{\Delta[0,2]}) \) and \( H^1_{sc}(\exp g^{\Delta}) = H^1_{sc}(\exp g^{\Delta[0,2]}) \). Moreover, the inclusion \( Z^1_{sc}(\exp g^{\Delta}) \hookrightarrow Z^1_{sc}(\exp g^{\Delta[0,1]}) \) induces an injective map \( H^1_{sc}(\exp g^{\Delta}) \hookrightarrow H^1_{sc}(\exp g^{\Delta[0,1]}) \).

**Remark 4.3.** For later use, we point out that, if \( g^{\Delta} \) is a semicosimplicial DGLA with \( H^{-1}(g_2) = 0 \), then \( \text{Def}_{\text{Tot}_{TW}(g^{\Delta[2,2]})} \) is trivial. Indeed, \( H^1(\text{Tot}_{TW}(g^{\Delta[2,2]})) = H^{-1}(g_2) = 0 \).

**Remark 4.4.** Note that, by the definition of \( H^1_{sc}(\exp g^{\Delta}) \) it follows that, if \( H^{-1}(g_2) = 0 \), then

\[
TH^1_{sc}(\exp g^{\Delta}) = H^1(\text{Tot}(g^{\Delta[0,2]})).
\]
Hence, the two functors of Artin rings $H^1_{sc}(\exp g^D)$ and $\text{Def}_{\text{Tot}_{TW}(g^{\Delta[0,0]})}$ have naturally isomorphic tangent spaces when $H^{-1}(g_2) = 0$. We will show in Section [7] that in this case these two functors are actually isomorphic.

5. A lemma on Maurer-Cartan elements

We will now give an explicit description of the solutions of Maurer-Cartan equation for the DGLAs $\text{Tot}_{TW}(g^{\Delta[0,1]})$ and $\text{Tot}_{TW}(g^{\Delta[0,2]})$. Our main tool will be the following general result [6, Proposition 7.2]:

**Lemma 5.1.** Let $(L,d,[\ , \ ])$ be a differential graded Lie algebra such that:

1. $L = M \oplus C \oplus D$ as graded vector spaces,
2. $M$ is a differential graded subalgebra of $L$.
3. $d: C \rightarrow D[1]$ is an isomorphism of graded vector spaces.

Then, for every $A \in \text{Art}_{K}$ there exists a bijection

$$\alpha: \text{MC}_M(A) \times (C^0 \otimes m_A) \cong \text{MC}_L(A), \quad (x,c) \mapsto e^c \ast x.$$ 

As almost immediate corollaries we obtain:

**Proposition 5.2.** Let $g^D$ be a semicosimplicial DGLA. Then, for every $A \in \text{Art}_{K}$, the solutions of the Maurer-Cartan equation for the Thom-Whitney DGLA $\text{Tot}_{TW}(g^{\Delta[0,1]}) \otimes m_A$ are of the form $(x, e^p(t) \ast \partial_{0,1}x)$, where $x \in \text{MC}_0(A)$ and $p(t) \in (g_1^0[t] \cdot t) \otimes m_A$. The elements $x, p$ are uniquely determined, and they satisfy

$$\partial_{1,1}x = e^{p(1)} \ast \partial_{0,1}x.$$ 

**Proof.** Notice that $\text{Tot}_{TW}(g^{\Delta[0,1]})$ is a sub-DGLA of $g_0 \otimes \Omega_1 \otimes g_1$. Then, Lemma 5.1 with the decomposition of $\Omega_1 \otimes g_1$ given by

$$M = g_1, \quad C = g_1[t] \cdot t, \quad D = dC$$

tells us that every solution of the Maurer-Cartan equation for $\text{Tot}_{TW}(g^{\Delta[0,1]}) \otimes m_A$ is of the form specified above.

**Proposition 5.3.** Let $g^D$ be a semicosimplicial DGLA. Then, for every $A \in \text{Art}_{K}$, the solutions of the Maurer-Cartan equation for the Thom-Whitney DGLA $\text{Tot}_{TW}(g^{\Delta[0,2]}) \otimes m_A$ are of the form

$$(x, e^p(t) \ast \partial_{0,1}x, e^{q(s_0,s_1)} \ast r(s_0,s_1,ds_0,ds_1) \ast \partial_{0,2}\partial_{0,1}x),$$

where $x \in \text{MC}_0(A)$, $p(t) \in (g_1^0[t] \cdot t) \otimes m_A$, $q(s_0,s_1) \in (g_2^0[s_0,s_1] \cdot s_0 + g_2^0[s_0,s_1] \cdot s_1) \otimes m_A$ and $r(s_0,s_1,ds_0,ds_1) \in (g_2^{-1}[s_0,s_1] \cdot s_0 ds_1) \otimes m_A$. The elements $x, p, q, r$ are uniquely determined, and they satisfy

$$\begin{cases}
\partial_{1,1}x = e^{p(1)} \ast \partial_{0,1}x, \\
\partial_{0,2}p(t) = q(0,t), \\
\partial_{1,2}p(t) = q(t,0), \\
e^{-\partial_{2,2}p(t)}(q(1-t,1-t,dt) \ast (-q(0,1))) \ast \partial_{2,2}\partial_{0,1}x = \partial_{2,2}\partial_{0,1}x.
\end{cases}$$

**Proof.** Since $\text{Tot}_{TW}(g^{\Delta[0,2]})$ is a sub-DGLA of $g_0 \otimes \Omega_1 \otimes g_1 \otimes \Omega_2 \otimes g_2$, applying Lemma 5.1 with the decomposition of $\Omega_2 \otimes g_2$ given by

$$M = g_2, \quad C = g_2[s_0,s_1] \cdot s_0 + g_2[s_0,s_1] \cdot s_1 + g_2[s_0,s_1] \cdot s_0 ds_1, \quad D = dC$$

we obtain that every solution of the Maurer-Cartan equation for $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}}) \otimes m_A$ is of the form

$$(x, e^{p(t)} \ast y, e^{q(s_0,s_1)+r(s_0,s_1,d_{s_0,d_{s_1}})} \ast z),$$

with the face conditions

$$y = \partial_{0,1} x; \quad z = \partial_{0,2} \partial_{0,1} x.$$  

The first relations in (3) are a direct consequence of face conditions and uniqueness. The last one is obtained as follows. The last face condition is

$$\partial_{2,2}(e^{p(t)} \ast \partial_{0,1} x) = e^{q(t,1-t) + r(t,1-t,dt)} \ast \partial_{0,2} \partial_{0,1} x;$$

using the other face conditions and relations between maps $\partial_{k,i}$, we obtain that

$$\partial_{2,2} \partial_{0,1} x = \partial_{0,2} \partial_{1,1} x = \partial_{0,2}(e^{p(1)} \ast \partial_{0,1} x) = e^{q(0,1)} \ast \partial_{0,2} \partial_{0,1} x.$$

Then, the above equation becomes

$$e^{\partial_{2,2}p(t)} \ast \partial_{2,2} \partial_{0,1} x = e^{(q(t,1-t) + r(t,1-t,dt)) \ast (-q(0,1))} \ast \partial_{2,2} \partial_{0,1} x.$$

\[\square\]

6. The isomorphism $H^1_{sc}(\exp \mathfrak{g}^{\Delta_{0,1}}) \cong \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})}$

**Proposition 6.1.** Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial DGLA. The map

$$\Phi_{[0,1]} : \text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})}(A) \to (\mathfrak{g}_0 \oplus \mathfrak{g}_1) \otimes m_A,$$

given by

$$(x, e^{p(t)} \ast \partial_{0,1} x) \mapsto (x, p(1)),$$

induces a natural transformation of functors of Artin rings

$$\text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})} \to H^1_{sc}(\exp \mathfrak{g}^{\Delta_{0,1}}).$$

**Proof.** Clearly, if $(x, e^{p(t)} \ast \partial_{0,1} x) \in \text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})}(A)$, then $(x, p(1)) \in Z^1_{sc}(\exp \mathfrak{g}^{\Delta_{0,1}})$. We have to show that if two elements $\eta_0 = (x_0, e^{p_0(t)} \ast \partial_{0,1} x_0)$ and $\eta_1 = (x_1, e^{p_1(t)} \ast \partial_{0,1} x_1)$ in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})}(A)$ are homotopy equivalent, then $\Phi_{[0,1]}(\eta_0) \sim \Phi_{[0,1]}(\eta_1)$ in $Z^1_{sc}(\exp \mathfrak{g}^{\Delta_{0,1}})$. Let $z(\xi, d\xi)$ be an homotopy between $\eta_0$ and $\eta_1$. Therefore, $z(\xi, d\xi)$ is a Maurer-Cartan element for $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{0,1}})[\xi, d\xi]$ and so, reasoning as in the proof of Proposition 5.2 we find

$$z(\xi, d\xi) = (e^{T(\xi)} \ast u, e^{U(t,dt;\xi)} \ast v),$$

with $T(0) = U(t, dt; 0) = 0$. Since $z(0) = \eta_0$, we get

$$z(\xi, d\xi) = (e^{T(\xi)} \ast x_0, e^{U(t,dt;\xi)} \ast e^{p_0(t)} \ast \partial_{0,1} x_0).$$

The face conditions for $z(\xi, d\xi)$ and uniqueness imply

$$U(0; \xi) = \partial_{0,1} T(\xi) \quad \text{and} \quad U(1; \xi) = \partial_{1,1} T(\xi).$$

Moreover, $z(1) = \eta_1$, and so

$$e^{T(1)} \ast x_0, e^{U(t,dt;1)} \ast e^{p_0(t)} \ast \partial_{0,1} x_0 = (x_1, e^{p_1(t)} \ast \partial_{0,1} x_1);$$

by uniqueness again, we have

$$e^{T(1)} \ast x_0 = x_1.$$

Furthermore, $e^{U(t,dt;1)} \ast e^{p_0(t)} \ast \partial_{0,1} x_0 = e^{p_1(t)} \ast \partial_{0,1} x_1$, so, using the face conditions for $\eta_0$ and $\eta_1$, we obtain

$$\partial_{0,1} x_0 = e^{-p_0(t)} \ast U(t, dt; 1) \ast e^{p_1(t)} \ast \partial_{0,1} x_0.$$
Next, we recall [11] Lemma 6.15] that if $L$ is a DGLA, $x(t, dt)$ is a Maurer-Cartan element for $L[t, dt]$ and $\mu(t, dt) \in L[t, dt]^0$ is such that $e^{\mu(t, dt)} \ast x(t, dt) = x(t, dt)$, then $\mu(1)$ is an element of the irrelevant stabilizer of $x(1)$. Therefore, in our case we get

$$-p_0(1) \cdot -\partial_{1,1} T(1) \cdot p_1(1) \cdot \partial_{0,1} T(1) \in \text{Stab}(\partial_{0,1} x_0).$$

\[\square\]

**Proposition 6.2.** Let $\mathfrak{g}^\Delta$ be a semicosimplicial DGLA. The map

$$\Phi_{[0,1]} : \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})} \to H_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})$$

is an isomorphism of functors of Artin rings. In particular, $H_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})$ is a deformation functor.

**Proof.** Let $\Psi_{[0,1]} : Z_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})(A) \to \text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]}) \otimes m_A$ be the map given by $(l, m) \mapsto (l, e^{tm} \ast \partial_{0,1} l)$; it is immediate to check that $\Phi_{[0,1]}$ actually takes its values in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})}(A)$. Moreover, $\Psi_{[0,1]}$ induces a map

$$H_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})(A) \to \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})}(A),$$

which is the inverse of $\Phi_{[0,1]}$. Indeed, if $(l_0, m_0) \sim (l_1, m_1)$ in $Z_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})(A)$, then there exist elements $a \in \mathfrak{g}^0_0 \otimes m_A$ and $b \in \mathfrak{g}^{-1}_1 \otimes m_A$ such that

$$\begin{cases} e^a \ast l_0 = l_1 \\ -m_0 \cdot -\partial_{1,1} a \cdot m_1 \cdot \partial_{0,1} a = db + [\partial_{0,1} l_0, b]. \end{cases}$$

Therefore, the images $(l_0, e^{tm_0} \ast \partial_{0,1} l_0)$ and $(l_1, e^{tm_1} \ast \partial_{0,1} l_1)$ are homotopic via the element

$$z(\xi, d\xi) = (e^{\xi a} \ast l_0, e^t(\partial_{0,1}(\xi a) \ast m_0 \ast (d\xi a + [\partial_{0,1} l_0, b] \ast -\partial_{0,1}(\xi a)) \ast m_1 \ast \partial_{0,1}(\xi a) \ast \partial_{0,1} l_1).$$

The composition $\Phi_{[0,1]} \circ \Psi_{[0,1]} : Z_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})(A) \to Z_{sc}^1(\exp \mathfrak{g}^{\Delta[0,1]})(A)$ is clearly the identity, whereas the composition $\Psi_{[0,1]} \circ \Phi_{[0,1]} : \text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})}(A) \to \text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})}(A)$ is homotopic to the identity. Indeed, $(x, e^{\xi t} \ast \partial_{0,1} x)$ and $(x, e^{\xi t} \ast \partial_{0,1} x)$ are homotopic in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,1]})}(A)$ via the element $z(\xi, d\xi) = (x, e^{\xi t}p(t) + (1-\xi)p(t) \ast \partial_{0,1} x)$. \[\square\]

**Remark 6.3.** A particular case of Proposition 6.2 with an almost identical proof, has been considered by one of the authors in [11]. Namely, given three DGLAs $L$, $M$ and $N$ and two DGLA morphisms $h : L \to M$ and $g : N \to M$, one can consider the semicosimplicial DGLA

$$L \oplus N \overset{(0,g)}{\overset{(h,0)}{\longrightarrow}} M \longrightarrow 0 \longrightarrow \cdots$$

to reobtain [11] Theorem 6.17].

7. Proof of the main theorem

In this section, we prove the existence of a natural isomorphism of functors of Artin rings $H_{sc}^1(\exp \mathfrak{g}^{\Delta}) \cong \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,2]})}$ of any semicosimplicial DGLA $\mathfrak{g}^{\Delta}$ such that $H^2(\mathfrak{g}_2) = 0$. As an immediate consequence we obtain a natural isomorphism of deformation functors $H_{sc}^1(\exp \mathfrak{g}^{\Delta}) \cong \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta})}$, for any semicosimplicial DGLA $\mathfrak{g}^{\Delta}$, such that $H^0(\mathfrak{g}_i) = 0$ for $i \geq 0$ and $j < 0$.

The proof is considerably harder than in the case $\mathfrak{g}^{\Delta[0,1]}$ considered in the previous section. Indeed, we are still able to define a map $\Phi : \text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta[0,2]})} \to Z_{sc}^1(\exp \mathfrak{g}^{\Delta})$.
inducing a natural transformation \( \text{Def}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})} \to H^1_{\text{sc}}(\exp \mathfrak{g}^\Delta) \), but we will not be able to explicitly define an homotopy inverse to \( \Phi \), so we will have to directly check that the map \( \text{Def}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})} \to H^1_{\text{sc}}(\exp \mathfrak{g}^\Delta) \) is an isomorphism.

**Proposition 7.1.** Let \( \mathfrak{g}^\Delta \) be a semicosimplicial DGLA. The map

\[
\Phi : \text{MC}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})}(A) \to (\mathfrak{g}_0 \oplus \mathfrak{g}_1^0) \otimes \mathfrak{m}_A,
\]

given by

\[
(x, e^{p(t)} \partial_{0,1} x, e^{q(s_0, s_1)+r(s_0, s_1, ds_1, ds_1)} \partial_{0,2} \partial_{0,1} x) \mapsto (x, p(1)),
\]

induces a natural transformation of functors of Artin rings

\[
\text{Def}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})} \to H^1_{\text{sc}}(\exp \mathfrak{g}^\Delta).
\]

**Proof.** First we check that \( \Phi \) takes its values in \( Z^1_{\text{sc}}(\exp \mathfrak{g}^\Delta)(A) \). The only nontrivial point consists in showing that \(-\partial_{2,2}p(1) \bullet \partial_{1,2}p(1) \bullet -\partial_{0,2}p(1) \) is an element of the irrelevant stabilizer of \( \partial_{2,2} \partial_{0,1} x \). This follows by the face condition

\[
e^{(-\partial_{2,2}p(t)) \bullet (q(t,1-t)+r(t,1-t,dt)) \bullet (-q(0,1))} \partial_{2,2} \partial_{0,1} x = \partial_{2,2} \partial_{0,1} x,
\]

applying [11, Lemma 6.15] once again. Next, we notice that the equivalence relation \( \sim \) on \( Z^1_{\text{sc}}(\exp \mathfrak{g}^\Delta)(A) \) only involves the DGLAs \( \mathfrak{g}_0 \) and \( \mathfrak{g}_1 \); hence, we can conclude verbatim following the proof of Proposition 6.1.

\[ \square \]

**Proposition 7.2.** The map \( \Phi : \text{Def}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})}(A) \to H^1_{\text{sc}}(\exp \mathfrak{g}^\Delta)(A) \) is surjective.

**Proof.** Let \((l, m) \in Z^1_{\text{sc}}(\exp \mathfrak{g}^\Delta)(A) \) and \( n \in \mathfrak{g}_2^{-1} \otimes \mathfrak{m}_A \), such that \( \partial_{0,2} m \bullet -\partial_{1,2} m \bullet \partial_{2,2} m = dn + \frac{1}{2} \mathfrak{g}_2 \otimes \mathfrak{m}_A \). Consider the element \( w(t) = d(tn) + \frac{1}{2} \mathfrak{g}_2 \otimes \mathfrak{m}_A \) in the irrelevant stabilizer of \( \partial_{2,2} \partial_{0,1} l \) and

\[
R(s_0, s_1) = s_0 s_1 \frac{s_0 \partial_{2,2} m \bullet -w(s_0) \bullet s_0 \partial_{0,2} m \bullet -s_0 \partial_{1,2} m}{s_0 (1-s_0)} - s_0 \partial_{1,2} m \bullet s_1 \partial_{0,2} m.
\]

Then,

\[
(l, e^{tm} \partial_{0,1} l, e^{R(s_0, s_1)} \partial_{0,2} \partial_{0,1} l)
\]

is an element in \( \text{MC}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})}(A) \) in the fiber of \( \Phi \) over \((l, m)\). Indeed, clearly it satisfies the Maurer-Cartan equation in \( \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \otimes \mathfrak{m}_A \); the first face conditions follow easily noticing that \( R(0, t) = t \partial_{0,2} m \) and \( R(t, 0) = t \partial_{1,2} m \); for the last one, we have:

\[
e^{R(t, 1-t)} \partial_{0,2} \partial_{0,1} l = e^{t \partial_{2,2} m \bullet -w(t) \bullet \partial_{0,2} m} \partial_{0,2} \partial_{0,1} l =
\]

\[
e^{t \partial_{2,2} m \bullet -w(t) \bullet \partial_{0,2} l \partial_{1,1} l} = e^{t \partial_{2,2} m \bullet -w(t) \bullet \partial_{2,2} \partial_{0,1} l} = e^{t \partial_{2,2} m} \partial_{2,2} \partial_{0,1} l.
\]

\[ \square \]

We will prove that the map \( \Phi : \text{Def}_{\text{Tot}T^W(\mathfrak{g}^{[0,2]})}(A) \to H^1_{\text{sc}}(\exp \mathfrak{g}^\Delta)(A) \) is injective, under the hypothesis \( H^{-1} \mathfrak{g}_2 = 0 \). For this we need two remarks.

**Remark 7.3.** Let \((L, d, [\ , \]) \) be a DGLA, \( A \in \text{Art}_K \) and \( x \in L^1 \otimes \mathfrak{m}_A \). The linear endomorphism \( d_x = d + [x, \ ] \) of \( L \otimes \mathfrak{m}_A \) is a differential if and only if \( x \in \text{MC}_L(A) \),
and in this case \((L \otimes m_A, d_x, [\cdot, \cdot])\) is a DGLA. So, we can define the set of the Maurer-Cartan elements \(MC_x^\pi(A)\) and the gauge action of \((L^0 \otimes m_A, d_x, [\cdot, \cdot])\) on it. We denote by \(\text{Def}_L^x(A)\) the quotient of \(MC_x^\pi(A)\) with respect to the gauge action. The affine map

\[
L \otimes m_A \to L \otimes m_A
\]

\[
v \mapsto v - x.
\]

induces an isomorphism \(\text{Def}_L(A) \cong \text{Def}_L^x(A)\) with obvious inverse \(v \mapsto v + x\).

Next, let \(M \subseteq L\) be a sub-DGLA and let \(x \in \text{MC}_L(A)\). If \(M \otimes m_A\) is closed under the differential \(d_x\), then we can consider the set of Maurer-Cartan elements \(MC_x^\pi_M(A)\), and its quotient \(\text{Def}_M^x(A)\). The tangent space to \(\text{Def}_M^x(A)\) is \(H^1(M \otimes m_A, d_x)\); so, by upper semicontinuity of cohomology, \(H^1(M, d) = 0\) implies that \(\text{Def}_M^x(A)\) is trivial, for all \(x \in \text{MC}_L(A)\) such that \(d_x(M \otimes m_A) \subseteq M \otimes m_A\).

_**Remark 7.4.**_ For any semicosimplicial DGLA \(g^\Delta\), the truncation morphism

\[
\text{Tot}_T^0(W(g^{\Delta^{[0,2]}_g})) \to \text{Tot}_T^0(W(g^{\Delta^{[0,1]}_g}))
\]

is surjective, i.e., for any \((a_0, a_1) \in \text{Tot}_T^0(W(g^{\Delta^{[0,2]}_g}))\) there exist \(a_2 \in (g_2 \otimes \Omega_2)^0\) such that \((a_0, a_1, a_2) \in \text{Tot}_T^0(W(g^{\Delta^{[0,2]}_g}))\). To see this, write \(a_1(t, dt) = a_0^0(t) + a_1^{-1}(t)dt\); then a possible choice for \(a_2\) is

\[
a_2(s_0, s_1, ds_0, ds_1) = a_2^0(s_0, s_1) + a_2^{-1}(s_0, s_1)ds_0 + a_2^{-1}(s_0, s_1)ds_1 + a_2^{-2}(s_0, s_1)ds_0ds_1,
\]

with

\[
a_2^0(s_0, s_1) = \partial_{1,2}a_1^0(s_0) + \partial_{0,2}a_1^0(s_1) - \partial_{1,2}a_1^0(0) + s_1 \frac{\partial_{2,2}a_1^0(s_0) - \partial_{1,2}a_1^0(1 - s_0) + \partial_{0,2}a_1^0(0)}{1 - s_0};
\]

\[
a_2^{-1}(s_0, s_1) = \partial_{1,2}a_1^{-1}(s_0) + s_1 \frac{1}{1 - s_0} \left( \partial_{2,2}a_1^{-1}(s_0) - \partial_{1,2}a_1^{-1}(s_0) + \partial_{0,2}a_1^{-1}(1 - s_0) - s_0 \partial_{0,2}a_1^{-1}(0) \right);
\]

\[
a_2^{-1}(s_0, s_1) = \partial_{0,2}a_1^{-1}(s_1)ds_1 - s_0 \partial_{0,2}a_1^{-1}(0);
\]

\[a_2^{-2}(s_0, s_1) = 0.
\]

It is an easy computation to verify that the element \((a_0, a_1, a_2)\) actually satisfies the face conditions.

_**Proposition 7.5.**_ Let \(g^\Delta\) be a semicosimplicial DGLA, such that \(H^{-1}(g_2) = 0\). The map \(\Phi : \text{Def}_{\text{Tot}_T^0(W(g^{\Delta^{[0,2]}_g}))}(A) \to H^1_{\text{sc}}(\exp g^\Delta)(A)\) is injective.

_**Proof.**_ Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Def}_{\text{Tot}_T^0(W(g^{\Delta^{[0,2]}_g}))}(A) & \xrightarrow{\Phi} & \text{Def}_{\text{Tot}_T^0(W(g^{\Delta^{[0,1]}_g}))}(A) \\
\downarrow & & \downarrow \\
H^1_{\text{sc}}(\exp g^\Delta)(A) & \xrightarrow{i} & H^1_{\text{sc}}(\exp g^{\Delta^{[0,1]}_g})(A)
\end{array}
\]
since the map $\Phi_{[0,1]}$ is an isomorphism by Proposition 6.2 it is sufficient to prove that $\text{Id}_{[0,1]}$ is injective. Let $(x_0, x_1, x_2)$ and $(x'_0, x'_1, x'_2)$ be two Maurer-Cartan elements for $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$, such that $(x_0, x_1)$ and $(x'_0, x'_1)$ are gauge equivalent elements in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,1]}})}(A)$. Let $(a_0, a_1) \in \text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,1]}}) \otimes \mathfrak{m}_A$ be an element realizing the gauge equivalence between $(x'_0, x'_1)$ and $(x_0, x_1)$, and let $(a_0, a_1, a_2)$ be a lift of $(a_0, a_1)$ in $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}) \otimes \mathfrak{m}_A$ (see Remark 7.3). Then $(x'_0, x'_1, x'_2)$ is gauge equivalent via $(a_0, a_1, a_2)$ to the Maurer-Cartan element $(x_0, x_1, e^{a_2} * x'_2)$ and we are left to prove that $(x_0, x_1, e^{a_2} * x'_2)$ is gauge equivalent to $(x_0, x_1, x_2)$.

To see this, consider the DGLA $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}) \otimes \mathfrak{m}_A$ and modify its differential with the Maurer-Cartan element $(x_0, x_1, x_2)$, as in Remark 7.3. Translation by $(x_0, x_1, x_2)$ gives an isomorphism

$$\text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A) \cong \text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A);$$

hence $(x_0, x_1, x_2)$ and $(x_0, x_1, e^{a_2} * x'_2)$ will be gauge equivalent in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A)$ if and only if $(0, 0, 0)$ and $(0, 0, e^{a_2} * x'_2 - x_2)$ are gauge equivalent in $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A)$.

Next, observe that the sub-DGLA $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}) \otimes \mathfrak{m}_A$ of $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}) \otimes \mathfrak{m}_A$ is closed under the modified differential $d_{(x_0,x_1,x_2)}$, so we can consider the deformation functor $\text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A)$. Since $H^1(\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}}), d_{\text{Tot}}) = H^1(\text{Tot}(\mathfrak{g}^{\Delta_{[0,2]}})) = 0$, this deformation functor is trivial (see Remark 7.3). Therefore $(0, 0, e^{a_2} * x'_2 - x_2)$ is gauge equivalent to $(0, 0, 0)$ as an element of $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A)$, and so, a fortiori, as an element of $\text{MC}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A)$.

Summing up, and recalling Proposition 6.1, we have proved:

**Theorem 7.6.** Let $\mathfrak{g}^{\Delta}$ be a semicosimplicial DGLA, and let $\text{Tot}_{TW}(\mathfrak{g}^{\Delta})$ and $\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})$ be the Thom-Whitney DGLAs associated with $\mathfrak{g}^{\Delta}$ and $\mathfrak{g}^{\Delta_{[0,2]}}$, respectively. Assume that $H^{-1}(\mathfrak{g}) = 0$; then, there is a natural isomorphism of functors $\text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta_{[0,2]}})}(A) \cong H^1_{\text{sc}}(\exp \mathfrak{g}^{\Delta})$.

If moreover $H^j(\mathfrak{g}) = 0$ for all $i \geq 0$ and $j < 0$, then there is a natural isomorphism of functors $\text{Def}_{\text{Tot}_{TW}(\mathfrak{g}^{\Delta})}(A) \cong H^1_{\text{sc}}(\exp \mathfrak{g}^{\Delta})$. In particular, in this case, the tangent space to $H^1_{\text{sc}}(\exp \mathfrak{g}^{\Delta})$ is $H^1(\text{Tot}(\mathfrak{g}^{\Delta}))$ and obstructions are contained in $H^2(\text{Tot}(\mathfrak{g}^{\Delta}))$.

**Theorem 7.7.** Let $X$ be a paracompact Hausdorff topological space, and let $\mathcal{L}$ be a sheaf of differential graded Lie algebras on $X$, such that the DGLAs $\mathcal{L}(U_{i_0, \ldots, i_k})$ has no negative cohomology. Then, every refinement $\mathcal{V} \geq \mathcal{U}$ of open covers of $X$ induces a natural morphism of deformation functors $\text{Def}_{\text{Tot}_{TW}(\mathcal{L}(\mathcal{U}))} \rightarrow \text{Def}_{\text{Tot}_{TW}(\mathcal{L}(\mathcal{V}))}$. In particular, the direct limit

$$\text{Def}_{[\mathcal{L}]} = \lim_{\mathcal{U}} \text{Def}_{\text{Tot}_{TW}(\mathcal{L}(\mathcal{U}))}$$

is well defined and there is a natural isomorphism of functors of Artin rings

$$H^1_{\text{Ho}}(X; \exp \mathcal{L}) \cong \text{Def}_{[\mathcal{L}]}.$$

Moreover, if acyclic open covers for $\mathcal{L}$ are cofinal in the directed family of all open covers of $X$, then

$$H^1_{\text{Ho}}(X; \exp \mathcal{L}) \cong H^1_{\text{sc}}(\exp \mathcal{L}(\mathcal{U})) \quad \text{and} \quad \text{Def}_{[\mathcal{L}]} \cong \text{Def}_{\text{Tot}_{TW}(\mathcal{L}(\mathcal{U}))},$$

for every $\mathcal{L}$-acyclic open cover $\mathcal{U}$ of $X$. 
Proof. Let $\mathcal{V} \geq \mathcal{U}$ be a refinement of open covers of $X$, and let $\tau$ be a refinement function, it induces a natural morphism of semicosimplicial Lie algebras $\mathcal{L} (\mathcal{U}) \rightarrow \mathcal{L} (\mathcal{V})$ and so a commutative diagram of natural transformations

$$
\text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{U}))} \xrightarrow{\sim} H^1_{\text{sc}}(\exp \mathcal{L} (\mathcal{U})) \\
\downarrow \\
\text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{V}))} \xrightarrow{\sim} H^1_{\text{sc}}(\exp \mathcal{L} (\mathcal{V})).
$$

Horizontal arrows are isomorphisms by Theorem 7.6 and the right vertical arrow is independent of the refinement function $\tau$, as observed in Lemma 2.7. Hence, also the left morphism is independent of $\tau$, then the direct limit

$$
\text{Def}_{\mathcal{L}} = \lim_{\mathcal{U}} \text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{U}))}
$$

is well defined and we have a natural isomorphism $\text{Def}_{\mathcal{L}} \cong H^1_{\text{Ho}} (X; \exp \mathcal{L})$. Assume now that acyclic open covers for $\mathcal{L}$ are cofinal in the family of all open covers of $X$. Then, for any refinement $\mathcal{V} \geq \mathcal{U}$ of acyclic open covers, the DGLAs-morphism $\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{U})) \rightarrow \text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{V}))$ is a quasi-isomorphism by Leray’s theorem. Therefore, we have a commutative diagram of natural transformations

$$
\text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{U}))} \xrightarrow{\sim} H^1_{\text{sc}}(\exp \mathcal{L} (\mathcal{U})) \\
\downarrow \\
\text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{V}))} \xrightarrow{\sim} H^1_{\text{sc}}(\exp \mathcal{L} (\mathcal{V})),
$$

where also the right vertical arrow is forced to be an isomorphism. Taking the direct limit over $\mathcal{L}$-acyclic covers, we obtain that, if $\mathcal{U}$ is an $\mathcal{L}$-acyclic open cover of $X$, then $H^1_{\text{Ho}} (X; \exp \mathcal{L}) \cong H^1_{\text{sc}}(\exp \mathcal{L} (\mathcal{U}))$ and $\text{Def}_{\mathcal{L}} \cong \text{Def}_{\text{Tot} \mathcal{T} \mathcal{W} (\mathcal{L} (\mathcal{U}))}$. □

8. Conclusions and Further Developments

We can now sum up our results to obtain a DGLA description of infinitesimal deformations of a coherent sheaf. In Section 1 we analysed infinitesimal deformations of a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$. If $\mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$ is a locally free resolution of $\mathcal{F}$ on $X$, we showed how infinitesimal deformations of $\mathcal{F}$ can be expressed in terms of the sheaf of DGLAs $\mathcal{E}nd^* (\mathcal{E})$. More precisely, in Section 2 we showed that the functor of infinitesimal deformations of $\mathcal{F}$ is isomorphic to $H^1_{\text{Ho}} (X; \exp \mathcal{E}nd^* (\mathcal{E}))$.

Since negative Ext-groups between coherent sheaves are always trivial, all terms in the semicosimplicial DGLA $\mathcal{E}nd^* (\mathcal{E}) (\mathcal{U})$ have zero negative cohomology. Therefore, Theorem 7.6 applies and we obtain that the functor of infinitesimal deformations of $\mathcal{F}$ is isomorphic to $\text{Def}_{\mathcal{E}nd^* (\mathcal{E})}$; in particular, we recover the well known fact that the tangent space to $\text{Def}_{\mathcal{F}}$ is $\text{Ext}^1 (\mathcal{F}, \mathcal{F})$ and that its obstructions are contained in $\text{Ext}^2 (\mathcal{F}, \mathcal{F})$.

Moreover, if $X$ is a smooth complex variety, then the DGLA controlling infinitesimal deformations of $\mathcal{F}$ turns out to be not at all mysterious. Indeed, let $\mathcal{E}nd^* (\mathcal{E}) \rightarrow \mathcal{A}^{0,*}_X (\mathcal{E})$ be the Dolbeault resolution of $\mathcal{E}nd^* (\mathcal{E})$. Since this resolution is fine, by Theorem 3.2 the functor of infinitesimal deformations of $\mathcal{F}$ is isomorphic to the deformation functor associated with the DGLA $\mathcal{A}^{0,*}_X (\mathcal{E})$ of global sections of $\mathcal{A}^{0,*}_X (\mathcal{E})$. We can also give an explicit description of this isomorphism of deformation functors. Indeed, a natural isomorphism

$$
\text{Def}_{\mathcal{A}^{0,*}_X (\mathcal{E})} (B) \rightarrow \text{Def}_{\mathcal{F}} (B), \quad \text{for } B \in \text{Art}_X
$$
is defined by associating with every Maurer-Cartan element $\xi$ of the DGLA $A_X^{0,*}(\mathcal{E}nd^*(\mathcal{E}))$ the cohomology sheaf of $(A_X^{0,*}(\mathcal{E}) \otimes B, \overline{\partial} + d_\mathcal{E} + \xi)$. Note that, by semicontinuity, this cohomology sheaf is concentrated in degree zero.

The techniques developed in this paper apply to a wide range of other geometric examples. More explicitly, we can use them in all cases when local deformations admit a simple DGLA description in terms of a resolution of the object to be deformed, for instance, in the case of infinitesimal deformations of a singular variety. Namely, let $X$ be a singular variety, $\mathcal{O}_X$ the sheaf of regular function of $X$ and $\mathcal{R} \to \mathcal{O}_X$ its standard free resolution \[\text{[12, Section 1.5]}\]. Then, the deformation functor of infinitesimal deformations of $X$ is isomorphic to $H^1(\mathcal{L}_X, \mathcal{O}_X)$; see \[\text{[5]}\] for details. From this, we also recover the classical result that the tangent space to deformations of $X$ is $\text{Ext}^1(\mathcal{L}_X, \mathcal{O}_X)$, and that obstructions are contained in $\text{Ext}^2(\mathcal{L}_X, \mathcal{O}_X)$, where $\mathcal{L}_X$ is the cotangent complex of $X$.

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