Causality and Statistics on the Groenewold - Moyal Plane

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Abstract Quantum theories constructed on the noncommutative spacetime called the Groenewold-Moyal plane exhibit many interesting properties such as Lorentz and CPT noninvariance, causality violation and twisted statistics. We show that such violations lead to many striking features that may be tested experimentally. These theories predict Pauli forbidden transitions due to twisted statistics, anisotropies in the cosmic microwave background radiation due to correlations of observables in spacelike regions and Lorentz and CPT violations in scattering amplitudes.

Keywords Noncommutative QFT · Moyal Plane · Statistics · Lorentz violation · Cosmic microwave background · CPT

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1 Introduction

The connection between spin and statistics is established in local quantum field theories by the requirement of causality [1], [2], [3]. The condition of locality is generally expressed in such theoretical frameworks by the assumption that the observables of spacelike separated observers commute.

In the theory of response functions in physical systems, the Kramers-Kronig relations connect the real and imaginary parts of the response function by making use of the fact that causality implies analyticity and vice versa.

A physical system should not respond before the time at which it is disturbed. If \( R(t) \) is the response and disturbance of the system is zero for time \( t < 0 \),

\[
R(t) = 0, \quad t < 0,
\]

then its Fourier transform

\[
\tilde{R}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} R(t) = \int_{0}^{\infty} dt e^{i\omega t} R(t)
\]

being holomorphic in

\[
\text{Im } \omega > 0
\]

leads to Kramers-Kronig relations.

Causal set theory, a discrete and Lorentz invariant approach to quantum gravity, rests on the central hypothesis that spacetime is a causal set [4], [5], [6]. In a causal set \( C \), the binary (partial order) condition \( \succ \) between its two elements \( x \) and \( y \) reads:

“\( x \succ y \), if \( x \) is to future of \( y \).”

In local quantum field theories, two observables \( \rho(x), \eta(y) \) commute if \( x \) and \( y \) are spacelike separated:

\[
[\rho(x), \eta(y)] = 0
\]

if

\[
(x^0 - y^0)^2 - (x - y)^2 < 0 \quad \text{or} \quad x \sim y.
\]

For scalar fields the above relation takes the form of a commutator

\[
[\varphi(x), \chi(y)] = 0 \quad x \sim y,
\]

and for spinor fields it takes the form an anti-commutator

\[
\left[ \psi^{(1)}_{\alpha}(x), \psi^{(2)}_{\beta}(y) \right] = 0 \quad x \sim y.
\]

These relations also express statistics of the quantum fields. So causality and statistics are connected.

It is interesting to study how the connection between causality and statistics is affected in quantum field theories that exhibit features like nonlocality, Lorentz noninvariance etc. A quantum field theory based on the noncommutativity of spacetime shows these interesting features. There are indications
Fig. 1 The observer is at the spacetime point $O$. Events $C, D$ are in the future, and $A, B$ are in the past light cone of $O$. They are causally related to $O$. Events $Q, P$ are spacelike relative to $O$ and are not causally related to $O$.

both from theories of quantum gravity and string theory that spacetime is noncommutative with length scale of the order of Planck length. We can model such spacetime noncommutativity using the algebra of functions called the Groenewold-Moyal (GM) plane.

The GM plane describes noncommutative spacetime where commutation relations and hence causality and statistics are deformed.

2 The Groenewold-Moyal Plane

The GM plane is the algebra $\mathcal{A}_\theta$ of smooth functions on $\mathbb{R}^{d+1}$ with a twisted product:

$$ f \ast g(x) = f(x) e^{i \sum_{\mu, \nu} \theta_{\mu \nu} \partial_\mu x \partial_\nu y} g(y) \bigg|_{x=y}, $$

(8)

where $\theta_{\mu \nu} = -\theta_{\nu \mu} = \text{constant}$. It implies the commutation relation

$$(\hat{x}_\mu \ast \hat{x}_\nu - \hat{x}_\nu \ast \hat{x}_\mu) = [\hat{x}_\mu, \hat{x}_\nu]_\ast = i \theta_{\mu \nu}, \quad \mu, \nu = 0, 1, \cdots, d,$$

(9)

with

$\hat{x}_\mu = \text{coordinate functions}, \quad \hat{x}_\mu(x) = x_\mu.$

We will describe a particular approach to the formulation of quantum field theories on the GM plane and indicate its physical consequences.

It is interesting to see how noncommutative structure of spacetime emerges at very small length scales from the arguments based on Heisenberg’s uncertainty principle and Einstein’s theory of classical gravity. Doplicher, Fredenhagen and Roberts \[7\] give the following arguments.
In order to probe physics at the Planck scale $L$, the Compton wavelength $\hbar/Mc$ of the probe must fulfill

$$\frac{\hbar}{Mc} \leq L \text{ or } M \geq \frac{\hbar}{Lc} \simeq \text{Planck mass.}$$

Such high mass in the small volume $L^3$ will strongly affect gravity and can cause black holes and their horizons to form. This suggests a fundamental length limiting spatial localization indicating space-space noncommutativity.

Similar arguments can be made about time-space noncommutativity. Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta t \Delta [\mathcal{F}] \geq L^2, \quad L = \text{a fundamental length.}$$

The GM plane models above spacetime uncertainties.

3 The Twisted Coproduct

If there is a symmetry group $G$ with elements $g$ and it acts on single particle Hilbert spaces $\mathcal{H}_i$ by unitary representations $g \rightarrow U_i(g)$, then conventionally it acts on $\mathcal{H}_1 \otimes \mathcal{H}_2$ by the representation

$$g \rightarrow [U_1 \otimes U_2](g \times g).$$

The homomorphism

$$\Delta : G \rightarrow G \times G,$$

$$g \rightarrow \Delta(g) := g \times g$$

underlying these equations is said to be a coproduct on $G$.

The action of $G$ on multiparticle states involves more than just group theory. It involves the coproduct $\Delta$.

The $\star$-multiplication between two functions $f$ and $g$ on the noncommutative algebra can be expressed in terms of the twist element $[8], [9], [10], [11], [12]$

$$\mathcal{F}_\theta = e^{\frac{i}{2} \theta_\mu \otimes \theta^{\mu \nu} \partial_\nu},$$

as follows

$$f \star g = m_0 \cdot \mathcal{F}_\theta(f \otimes g),$$

where $m_0$ is the point-wise multiplication map of the commutative algebra $\mathcal{A}_0$:

$$m_0(\alpha \otimes \beta)(x) = \alpha(x)\beta(x)$$

Let $A$ be an element of the connected component of the Poincaré group $\mathcal{P}_+^1$. Then for $x \in \mathbb{R}^N$ we have

$$A : x \rightarrow Ax \in \mathbb{R}^N.$$
It acts on functions on $\mathbb{R}^N$ by pull-back:

$$A : \alpha \rightarrow A^* \alpha, \quad (A^* \alpha)(x) = \alpha[A^{-1}x].$$

(15)

The work of Aschieri et al. [13] and Chaichian et al. [11] based on Drinfel’d’s original work [9], [10] shows that $P_+^1$ acts on $A_0(\mathbb{R}^N)$ compatibly with $m_\theta$ if its coproduct is “twisted” to $\Delta_\theta$ where

$$\Delta_\theta(A) = F^{-1}_\theta(A \otimes A)F_\theta. \quad (16)$$

4 The Twisted Statistics

The action of the twisted coproduct is not compatible with standard statistics. Statistics also should be twisted in quantum theory.

A two-particle system for commutative case $(\theta_{\mu\nu} = 0)$ is a function of two sets variables and it lives in $A_0 \otimes A_0$. It transforms according to the usual coproduct $\Delta_0$.

Similarly in noncommutative case, the wavefunction lives in $A_\theta \otimes A_\theta$ and transforms according to the twisted coproduct $\Delta_\theta$.

For $\theta_{\mu\nu} = 0$ we require that the physical wave functions describing identical particles are either symmetric (bosons) or antisymmetric (fermions).

That is we work with either the symmetrized or antisymmetrized tensor product

$$\phi \otimes_{S,A} \chi \equiv \frac{1}{2} (\phi \otimes \chi \pm \chi \otimes \phi) \quad (17)$$

In a Lorentz-invariant theory, these relations have to hold in all frames of reference. The twisted coproduct action of the Lorentz group is not compatible with the usual symmetrization/antisymmetrization.

Let $\tau_0$ be the statistics (flip) operator associated with exchange for $\theta_{\mu\nu} = 0$:

$$\tau_0(\phi \otimes \chi) = \chi \otimes \phi. \quad (18)$$

For $\theta_{\mu\nu} = 0$, we have the axiom that $\tau_0$ is superselected. In particular, for Lorentz group action, $\Delta_0(A) = A \otimes A$, must and does commute with the statistics operator:

$$\tau_0 \Delta_0(A) = \Delta_0(A)\tau_0. \quad (19)$$

Given an element $\phi \otimes \chi$ of the tensor product, the physical Hilbert spaces can be constructed from the elements

$$\left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi). \quad (20)$$

Now

$$\tau_0 F_\theta = F^{-1}_\theta \tau_0 \quad (21)$$

so that

$$\tau_0 \Delta_\theta(A) \neq \Delta_\theta(A)\tau_0.$$
It shows that the usual statistics is not compatible with the twisted coproduct.

But the new statistics operator [14]

\[ \tau_\theta \equiv F_{\theta}^{-1} \tau_0 F_{\theta}, \quad \tau_\theta^2 = 1 \otimes 1 \]  

(22)
does commute with the twisted coproduct \( \Delta_\theta \):

\[ \Delta_\theta(A) = F_{\theta}^{-1} A \otimes A F_{\theta}. \]  

(23)

The states constructed according to

\[ \phi \otimes_{S_\theta} \chi \equiv \left( \frac{1 + \tau_\theta}{2} \right) (\phi \otimes \chi), \]  

(24)

\[ \phi \otimes_{A_\theta} \chi \equiv \left( \frac{1 - \tau_\theta}{2} \right) (\phi \otimes \chi) \]  

(25)

form the physical two-particle Hilbert spaces of (generalized) bosons and fermions and obey twisted statistics.

5 The Pauli Principle

In Ref. [15] the statistical potential \( V_{\text{STAT}} \) between two identical fermions at inverse temperature \( \beta \) has been computed:

\[ \exp \left( -\beta V_{\text{STAT}}(\mathbf{x}_1, \mathbf{x}_2) \right) = \langle \mathbf{x}_1, \mathbf{x}_2 | e^{-\beta H} | \mathbf{x}_1, \mathbf{x}_2 \rangle, \]

\[ H = \frac{1}{2m} (\mathbf{p}_1^2 + \mathbf{p}_2^2). \]

Here \( | \mathbf{x}_1, \mathbf{x}_2 \rangle \) has twisted antisymmetry:

\[ \tau_\theta | \mathbf{x}_1, \mathbf{x}_2 \rangle = - | \mathbf{x}_2, \mathbf{x}_1 \rangle. \]

It is explicitly shown not to have an infinitely repulsive core, establishing the violation of Pauli principle, as we had earlier suggested.

This result has phenomenological consequences such as Pauli forbidden transitions (on which there are stringent limits). For example, in the Borexino and Super Kamiokande experiments, the forbidden transitions from \( O^{16}(C^{12}) \) to \( \tilde{O}^{16}(\tilde{C}^{12}) \) where the tilde nuclei have an extra nucleon in the filled 1S_{1/2} level are found to have lifetimes greater than \( 10^{27} \) years. There are also experiments on forbidden transitions to filled K-shells of crystals done in Maryland which give branching ratios less than \( 10^{-25} \) for such transitions. The consequences of these results to noncommutative models are yet to be studied.
5.1 Bounds from NEMO experiment

There have been experiments done by NEMO-2 [16], where searches for non-Paulian atoms and transitions to non-Paulian states have been conducted. Data from these experiments may be used to obtain a bound for the non-commutativity parameter $\theta$.

Non-Paulian atoms are those whose atomic orbitals are filled violating the Pauli Exclusion Principle. As an example, non-Paulian Carbon has as its atomic configuration, $1s^22s^22p^1$. The presence of just a single electron in the outermost shell of non-Paulian Carbon makes it behave chemically like Boron, whose atomic configuration is $1s^22s^22p^1$. Thus searching for non-Paulian Carbon atoms in samples of Boron, we can get the concentration of the former in the latter. Bounds on these values were found by the NEMO experiments.

We consider the tensor product of two-electron states, each of which is an eigenvector of a hydrogen atom Hamiltonian. We then see that the two-electron states we consider are eigenvectors of the co-product of $H$:

$$\Delta(H) = H_1 \otimes 1 + 1 \otimes H_2 \quad (26)$$

where $H_1$ and $H_2$ are hydrogen atom Hamiltonians.

Consider two such states, $|1s1s\rangle_\theta$ and $|1s2s\rangle_\theta$, the $\theta$ in the suffix indicates these states have been deformed according to twisted statistics as we have seen in section (4). We know the hydrogen atom is invariant under rotations and time translations, which implies that the twist element formed from the generators of rotation and time translations, commutes with this Hamiltonian and so it also commutes with the co-product of this Hamiltonian. Then, these states are eigenvectors of the above Hamiltonian (26) with two different energy eigenvalues namely $2E_1$ and $E_1 + E_2$ respectively. This means that they must be orthogonal. However, for the existence of non-Paulian atoms we expect to observe such transitions. We will indeed do so, if we include an interaction part in the above Hamiltonian. So we consider the new Hamiltonian to be:

$$H = H_1 \otimes 1 + 1 \otimes H_2 + H_{\text{int}}. \quad (27)$$

We then consider matrix elements of the interaction, $\langle 1s, 1s | H_{\text{int}} | 1s, 2s \rangle_\theta$, and find them to be non-zero. $H_{\text{int}}$ arises from the Coulomb interaction between the electrons.

The results of this work will appear in a forthcoming paper [17].

6 Cosmic Microwave Background (CMB)

The COBE satellite, in 1992, detected anisotropies in the CMB radiation, which led to the conclusion that the early universe was not smooth: There were small perturbations in the photon-baryon fluid.

The perturbations could be due to the quantum fluctuations in the inflaton (the scalar field driving inflation). These fluctuations act as seeds for the primordial perturbations over the smooth universe. Thus according to
these ideas, the early universe had inhomogeneities and we observe them today in the distribution of large scale structure and anisotropies in the CMB radiation.

The temperature field in the sky can be expanded in spherical harmonics:

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}).$$  \hspace{1cm} (28)

The $a_{lm}$ can be written in terms of perturbations to Newtonian potential $\Phi$

$$a_{lm} = 4\pi (-i)^l \int \frac{d^3k}{(2\pi)^3} \Phi(k) \Delta^T(k) Y^*_{lm}(\hat{k}).$$ \hspace{1cm} (29)

where $\Delta^T(k)$ are called the transfer functions.

In a noncommutative spacetime the quantum corrections to the inflaton are modified. The angular correlation functions acquire nondiagonal elements indicating rotational symmetry breaking in the universe.

The noncommutative angular correlation takes the form \[13\]

$$\langle a_{lm} a^{*}_{l'm'} \rangle_\theta = 8\pi^2 \int dk \sum_{l'\nu=0, l''\nu: even} \frac{i^{l+l'} (-1)^{l+m} (2l''+1) k^2 \Delta_\nu(k) \Delta_\nu(k) P_\Phi(k)}{\langle H \rangle} \sqrt{(2l+1)(2l''+1)} \left( \begin{array}{c} l \ l' \\ 0 \ 0 \end{array} \right) \left( \begin{array}{c} l \ l' \ l'' \\ -m \ m' \ 0 \ 0 \end{array} \right).$$ \hspace{1cm} (30)

where $\theta^0 = \theta(0,0,1)$ and $P_\Phi(k)$ is the power spectrum for mode $k$

$$P_\Phi(k) = \frac{16\pi G H^2}{9\epsilon} \bigg|_{aH=k}.$$ \hspace{1cm} (31)

$H$ is the Hubble parameter, $a$ is the cosmological scale factor and $\epsilon$ is the slow-roll parameter.

The angular correlator $\langle a_{lm} a^{*}_{l'm'} \rangle_\theta$ is $\theta$ dependent indicating a preferred direction. Correlation functions are not invariant under rotations. They are not Gaussian either. It clearly breaks the statistical isotropy of the CMB radiation.

On fitting data \[19\], one finds an upper bound for the length scale associated with spacetime noncommutativity,

$$\sqrt{\theta} \lesssim 10^{-17} \text{cm}$$ \hspace{1cm} (32)

or a lower bound for the energy scale, $E$

$$E \gtrsim 10^{3} \text{GeV}.$$ \hspace{1cm} (33)
7 Causality, Lorentz invariance and CPT

7.1 Causality and Lorentz invariance

The $S$-matrix of quantum theories constructed on the GM plane is not Lorentz invariant. The reason is nothing but loss of causality.

Let $\mathcal{H}_I$ be the interaction Hamiltonian density in the interaction representation of the quantum theory. The interaction representation $S$-matrix is

$$ S = T \exp \left( -i \int d^4x \mathcal{H}_I(x) \right). \quad (34) $$

Bogoliubov and Shirkov [1] and then Weinberg [3] long ago deduced from causality (locality) and relativistic invariance that $\mathcal{H}_I$ must a local field:

$$ [\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0, \quad x \sim y. \quad (35) $$

But noncommutative theories are nonlocal and violate this condition; this is the essential reason for Lorentz noninvariance.

The effect Lorentz noninvariance on scattering amplitudes is striking. They depend on total incident momentum $\vec{P}_{\text{inc}}$ through the term $\theta_i P_i^{\text{inc}}$.

$$ e^{\frac{1}{4} \vec{P} \wedge \theta} $$

(36)

The effects of $\theta^{\mu\nu}$ disappear in the center-of-mass system, or more generally if

$$ \theta_i P_i^{\text{inc}} = 0. \quad (37) $$

But otherwise there is dependence on $\theta_i$.

The decay $Z^0 \rightarrow 2\gamma$ is forbidden even with noncommutativity in the approach of Aschieri et al. More generally, a massive particle of spin $j$ does not decay into two massless particles of same helicity if $j$ is odd.

7.2 CPT

The noncommutative $S$-matrix transforms under CPT in the following way [20],

$$ S_{\theta}^{M,G} = T \exp \left[ -i \int d^4x \mathcal{H}_{I0}^{M,G}(x) e^{\frac{1}{2} \vec{P} \wedge \theta} \right] $$

$$ \rightarrow T \exp \left[ i \int d^4x \mathcal{H}_{I0}^{M,G}(x) e^{-\frac{1}{2} \vec{P} \wedge \theta} \right] = (S_{-\theta}^{M,G})^{-1}, \quad (38) $$

where $\mathcal{H}_{I0}^{M,G}$ is the matter-gauge interaction hamiltonian density for $\theta^{\mu\nu} = 0$.

After performing the spatial integration we can reduce $e^{\frac{1}{2} \vec{P} \wedge \theta}$ in the $S$-matrix to $e^{\frac{1}{2} \vec{P}_{\theta} \wedge \theta^{\mu} P_i}$. Thus the effect of P and CPT is to reverse the sign of $\theta_i$:

$$ P \text{ or CPT} : \theta_i \rightarrow -\theta_i. $$
The $\theta^{0i}$ contributes to $P$, and more strikingly, to CPT violation.

The particle-antiparticle life times can differ to order $\theta^{0i}$:

$$\tau_{\text{particle}} - \tau_{\text{antiparticle}} \cong \theta^{0i} P_{i}^{\text{inc}}. \quad (39)$$

It can give rise to interesting effects such as mass difference in $K^0 - \bar{K}^0$ system and the $(g-2)$ difference of $\mu^+ - \mu^-$. (See [21] for bounds on $\theta$ estimated from these effects.)

8 Conclusions

Spacetime noncommutativity deforms statistics and so generically violate causality in noncommutative quantum theories. Such violations lead to many interesting features such as (i.) modification of Pauli principle causing forbidden atomic transitions, (ii.) correlations of observables in spacelike regions giving rise to anisotropies in the CMB radiation, (iii.) Lorentz and CPT violations in scattering amplitudes.

It is shown that there are specific predictions that may be observable. Bounds on noncommutativity parameter are given in the context of different experimental measurements.

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References

1. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields, Interscience Publishers, New York, (1959).
2. R. Haag, Local Quantum Physics, Springer Verlag, Berlin, (1992).
3. S. Weinberg, The Quantum theory of fields. Vol. 1: Foundations, Cambridge University Press, UK (1995).
4. J. Henson, in Approaches to Quantum Gravity: Towards a New Understanding of Space, Time and Matter, edited by D. Oriti, Cambridge University Press, (2009), [arXiv:gr-qc/0611121].
5. F. Dowker, Causal sets as discrete spacetime, Contemp. Phys. 47, 1 (2006).
6. R. D. Sorkin, in Lectures on Quantum Gravity, Proceedings of the Valdivia Summer School, Valdivia, Chile, January 2002, edited by A. Gomberoff and D. Marolf, Plenum (2005), [arXiv:gr-qc/0309009].
7. S. Doplicher, K. Fredenhagen and J. E. Roberts, Spacetime quantization induced by classical gravity, Phys. Lett. B 331, 33-44 (1994).
8. A. P. Balachandran, A. Pinzul, B. A. Qureshi and S. Vaidya, S-Matrix on the Moyal Plane: Locality versus Lorentz Invariance, Phys. Rev. D 77, 025020 (2008) [arXiv:0708.1379 [hep-th]].
9. V. G. Drinfel’d, Almost cocommutative Hopf algebras, Leningrad Math. J. 1 (1990), 321-332.
10. V. G. Drinfel’d, Quasi-Hopf algebras, Leningrad Math. J. 1, 1419-1457 (1990).
11. M. Chaichian et al., On a Lorentz invariant interpretation of noncommutative space-time and its implications on noncommutative QFT, Phys. Lett. B 604, 98 (2004), [arXiv:hep-th/0408069].
12. M. Chaichian, P. Presnajder and A. Tureanu, New concept of relativistic invariance in NC space-time: Twisted Poincaré symmetry and its implications, Phys. Rev. Lett. 94, 151602 (2005), [arXiv:hep-th/0409096].
13. P. Aschieri et al., A gravity theory on noncommutative spaces, Class. Quant. Grav. 22, 3511 (2005), [arXiv:hep-th/0504183].
14. A. P. Balachandran, T. R. Govindarajan, G. Mangano, A. Pinzul, B. A. Qureshi and S. Vaidya, Statistics and UV-IR mixing with twisted Poincare invariance, Phys. Rev. D 75, 045009 (2007) [arXiv:hep-th/0608179].
15. B. Chakraborty, S. Gangopadhyay, A. G. Hazra and F. G. Scholtz, Twisted Galilean symmetry and the Pauli principle at low energies, J. Phys. A 39, 9557 (2006) [arXiv:hep-th/0601121].
16. A. S. Barabash et al., Search for anomalous carbon atoms evidence of violation of the Pauli principle during the period of nucleosynthesis, JETP Lett. 68 (1998) 112.
17. A.P. Balachandran, Anosh Joseph, Gianpiero Mangano, Pramod Padmanabhan, In Preparation.
18. E. Akofor, A. P. Balachandran, S. G. Jo, A. Joseph and B. A. Qureshi, Direction-Dependent CMB Power Spectrum and Statistical Anisotropy from Noncommutative Geometry, JHEP 0805, 092 (2008) [arXiv:0710.5897 [astro-ph]].
19. E. Akofor, A. P. Balachandran, A. Joseph, L. Pekowsky and B. A. Qureshi, Constraints from CMB on Spacetime Noncommutativity and Causality Violation, Phys. Rev. D 79, 063004 (2009) [arXiv:0806.2158 [astro-ph]].
20. E. Akofor, A. P. Balachandran, S. G. Jo and A. Joseph, Quantum Fields on the Groenewold-Moyal Plane: C, P, T and CPT, JHEP 0708, 045 (2007) [arXiv:0706.1259 [hep-th]].
21. A. Joseph, Particle phenomenology on noncommutative spacetime, arXiv:0811.3972 [hep-ph].