SOME REMARKS ON NON-SYMMETRIC POLARIZATION

FELIPE MARCECA

Abstract. Let $P : \mathbb{C}^n \to \mathbb{C}$ be an $m$-homogeneous polynomial given by

$$P(x) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1} \ldots x_{j_m}.$$  

Defant and Schlüters defined a non-symmetric associated $m$-form $L_P : (\mathbb{C}^n)^m \to \mathbb{C}$ by

$$L_P(x^{(1)}, \ldots, x^{(m)}) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1}^{(1)} \ldots x_{j_m}^{(m)}.$$  

They estimated the norm of $L_P$ on $(\mathbb{C}^n, \| \cdot \|)^m$ by the norm of $P$ on $(\mathbb{C}^n, \| \cdot \|)$ times a $(c \log n)^m$ factor for every 1-unconditional norm $\| \cdot \|$ on $\mathbb{C}^n$. A symmetrization procedure based on a card-shuffling algorithm which (together with Defant and Schlüters’ argument) brings the constant term down to $(cm \log n)^{m-1}$ is provided. Regarding the lower bound, it is shown that the optimal constant is bigger than $(c \log n)^{m/2}$ when $n \gg m$. Finally, the case of $\ell_p$-norms $\| \cdot \|_p$ with $1 \leq p < 2$ is addressed.

1. Introduction

Let $P : \mathbb{C}^n \to \mathbb{C}$ be an $m$-homogeneous polynomial. It is well-known that there is a unique symmetric $m$-linear form $B : (\mathbb{C}^n)^m \to \mathbb{C}$, such that $B(x, \ldots, x) = P(x)$ for all $x \in \mathbb{C}$. Moreover, the polarization formula gives an expression for the $m$-linear form $B$ in terms of $P$ (see e.g. [3, Section 1.1]). In fact, for every $x^{(1)}, \ldots, x^{(m)} \in \mathbb{C}$, we have

$$B(x^{(1)}, \ldots, x^{(m)}) = \frac{1}{2^m m!} \sum_{\varepsilon \in \{-1, 1\}^m} P\left(\varepsilon_1 x^{(1)} + \ldots + \varepsilon_m x^{(m)}\right).$$

It follows from this identity that

$$\sup_{\|x^{(k)}\| \leq 1} \|B(x^{(1)}, \ldots, x^{(m)})\| \leq e^m \sup_{\|x\| \leq 1} |P(x)|,$$

for any norm $\| \cdot \|$ in $\mathbb{C}^n$.

In [2], Defant and Schlüters defined a non-symmetric $m$-linear form $L_P$ arising from a given $m$-homogeneous polynomial $P$. More precisely, for an $m$-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ defined by

$$P(x) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1} \ldots x_{j_m},$$

its associated $m$-linear form $L_P : (\mathbb{C}^n)^m \to \mathbb{C}$ is given by

$$L_P(x^{(1)}, \ldots, x^{(m)}) = \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1}^{(1)} \ldots x_{j_m}^{(m)}.$$

This work has been supported by CONICET-PIP 11220130100329CO, ANPCyT PICT 2015-2299, UBACyT 20020130100474BA and a CONICET doctoral fellowship.
Assuming unconditionality of the norm $\| \cdot \|$ in $\mathbb{C}^n$, Defant and Schlüters proved that a similar estimate as in (1) holds for $L_P$. Before providing further details we introduce an ad hoc definition:

**Definition 1.1.** For $m, n \in \mathbb{N}$, we define $C(m, n)$ as the infimum of the constants $C > 0$ such that for every $m$-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ and every 1-unconditional norm $\| \cdot \|$ on $\mathbb{C}^n$ we have

$$\sup_{\|x^{(k)}\| \leq 1} |L_P (x^{(1)}, \ldots, x^{(m)})| \leq C \sup_{\|x\| \leq 1} |P(x)|.$$

Similarly, for $1 \leq p < 2$, we take $C_p(m, n)$ as the infimum of the constants $C > 0$ such that for every $m$-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ we have

$$\sup_{\|x^{(k)}\|_p \leq 1} |L_P (x^{(1)}, \ldots, x^{(m)})| \leq C \sup_{\|x\|_p \leq 1} |P(x)|.$$

The aforementioned result of [2] can be stated in terms of the previous definition.

**Theorem 1.2** [2, Theorem 1.1]. There exists a universal constant $c_1 \geq 1$ such that

$$C(m, n) \leq (c_1 \log n)^{m^2}.$$

Moreover, for $1 \leq p < 2$, there is a constant $c_2 = c_2(p) \geq 1$ for which

$$C_p(m, n) \leq c_2^{m^2}.$$

Note that by the uniqueness of the symmetric $m$-linear form $B$ we have

$$B (x^{(1)}, \ldots, x^{(m)}) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L_P (x_{\sigma(1)}, \ldots, x_{\sigma(m)}),$$

where $\Sigma_m$ is the group of permutations of $m$ elements. The proof of Theorem 1.2 consists of bounding the norm of $L_p$ by successive partial symmetrizations starting at $L_p$ and ending at the fully symmetrized $B$. Finally, applying (1) yields the result. Changing only the way in which this symmetrization is carried out and using the same arguments as in [2], we obtain improved bounds for the constants $C(m, n)$ and $C_p(m, n)$. Additionally, we provide lower bounds for these constants. Our main result is the following.

**Theorem 1.3.** There exists a universal constant $c_1 \geq 1$ such that

$$\left( \frac{\log \left( \frac{2m}{\pi} \right) - \pi}{\pi} \right)^{m/2} \leq C(m, n) \leq c_1^m m^m \log n)^{m-1}.$$

Moreover, for $1 \leq p < 2$, there is a constant $c_2 = c_2(p) \geq 1$ for which

$$m^\frac{m}{p} \leq C_p(m, n) \leq c_2^m m^m.$$

**Remark 1.4.** Defant and Schlüters achieved similar upper bounds by refining their original calculations from [2] as it was mentioned during a personal communication.

**Remark 1.5.** Scrutiny of the theorem’s proof suggests that the underlying reason which determines the magnitude of the constants $C(m, n)$ and $C_p(m, n)$ is the behaviour of the operator known as the main triangle projection. Roughly speaking, the main triangle projection is the operator which given a matrix in $\mathbb{C}^{n \times n}$ returns the same matrix with zeroes below the diagonal. Each norm on $\mathbb{C}^n$ induces an operator norm in $\mathbb{C}^{n \times n}$ and again this induces a norm for the main triangle projection. Estimations
of the latter norm are the ones that shape the upper and lower bounds of $C(m, n)$ and $C_p(m, n)$ that were obtained.

2. Symmetrization

The following may be deduced from (2).

$$B \left( x^{(1)}, \ldots, x^{(m)} \right) = \frac{1}{m!} \sum_{\sigma \in \Sigma_m} L_P \left( x^{\sigma(1)}, \ldots, x^{\sigma(m)} \right)$$

$$= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x^{\sigma(j_1)} \ldots x^{\sigma(j_m)}$$

$$= \frac{1}{m!} \sum_{\sigma \in \Sigma_m} \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1 \sigma(1)} \ldots x_{j_m \sigma(m)}$$

From a probabilistic point of view, this may be restated as

$$B \left( x^{(1)}, \ldots, x^{(m)} \right) = E \left[ \sum_{1 \leq j_1 \leq \ldots \leq j_m \leq n} c_{j_1 \ldots j_m} x_{j_1 \sigma(1)} \ldots x_{j_m \sigma(m)} \right],$$

where expectation is taken over $\sigma \sim P_k$. In particular, from (3) and the fact that the $(m-1)$-th step of the shuffle achieves equidistribution we have

$$B = S_{m-1} L_P.$$
However, it should be noticed that the intermediate shuffles are not partial symmetrizations since we are symmetrizing the monomials’ subindices rather than the variables.

In order to study the structure of \( S_k \), we define the \( k \)-th shuffling step \( T_k \) of an \( m \)-form \( L : (\mathbb{C}^n)^m \rightarrow \mathbb{C} \) by

\[
T_k L \left( x^{(1)}, \ldots, x^{(m)} \right) = \frac{1}{m-k+1} \sum_{l=k}^{m} L \left( x^{(1)}, \ldots, x^{(k-1)}, x^{(k+1)}, \ldots, x^{(l)}, x^{(k)}, x^{(l+1)}, \ldots, x^{(m)} \right).
\]

**Lemma 2.2.** For every \( 1 \leq k \leq m-1 \) we have that \( S_k = T_k \ldots T_1 \).

**Proof.** Since \( T_k \) and \( S_k \) are linear for every \( 1 \leq k \leq m-1 \), it is enough to check that the equality holds for monomials. Fix \( 1 \leq i_1, \ldots, i_m \leq n \), we have to prove that

\[
S_k \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right) = T_k \ldots T_1 \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right).
\]

We will proceed by induction. If \( k = 1 \), the random permutation \( \sigma \) is a cycle in \( \Sigma_m \). More precisely, using the cycle notation in \( \Sigma_m \) we have that \( \sigma \) takes the value \( (l \ l-1 \ldots 1) \) for some \( 1 \leq l \leq m \) with probability \( 1/m \). Therefore, we get

\[
S_1 \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right) = E \left[ x_{i_{\tau(1)}}^{(1)} \ldots x_{i_{\tau(m)}}^{(m)} \right] = \frac{1}{m} \sum_{l=1}^{m} x_{l_{\sigma(1)}}^{(1)} x_{l_{\sigma(2)}}^{(2)} \ldots x_{l_{\sigma(l)}}^{(l)} x_{l_{\sigma(l+1)}}^{(l+1)} \ldots x_{l_{\sigma(m)}}^{(m)} = T_1 \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right).
\]

Only the inductive step remains to be proven. Let \( 2 \leq k \leq m-1 \) and suppose the lemma holds for \( k-1 \). From the definition of the Fischer-Yates shuffle we may deduce that a random permutation with law \( \mathbb{P}_k \) can be written as the composition of two independent random permutations \( \tau \) and \( \sigma \) where \( \sigma \sim \mathbb{P}_{k-1} \) and \( \tau \) takes the value \( \tau_l = (l \ l-1 \ldots k) \) for some \( k \leq l \leq m \) with probability \( 1/(m-k+1) \). For a fixed \( \tau \), we may define new indices \( j_1, \ldots, j_m \) such that \( j_k = i_{\tau(k)} \) for every \( 1 \leq k \leq m \). So we obtain

\[
S_k \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right) = E_{r,\sigma} \left[ x_{i_{\tau(1)}}^{(1)} \ldots x_{i_{\tau(m)}}^{(m)} \right] = E_\tau \left[ E_{\sigma} \left( x_{j_{\sigma(1)}}^{(1)} \ldots x_{j_{\sigma(m)}}^{(m)} \right) \right] = E_\tau \left[ S_{k-1} \left( x_{j_1}^{(1)} \ldots x_{j_m}^{(m)} \right) \right] = \frac{1}{m-k+1} \sum_{l=k}^{m} S_{k-1} \left( x_{j_{\tau(1)}}^{(1)} \ldots x_{j_{\tau(m)}}^{(m)} \right)
\]

\[
= \frac{1}{m-k+1} \sum_{l=k}^{m} S_{k-1} \left( x_{i_1}^{(1)} \ldots x_{i_{k-1}}^{(k-1)} x_{i_k}^{(k)} x_{i_{k+1}}^{(k+1)} \ldots x_{i_{l-1}}^{(l)} x_{i_l}^{(l)} x_{i_{l+1}}^{(l+1)} \ldots x_{i_m}^{(m)} \right)
\]

\[
= \frac{1}{m-k+1} \sum_{l=k}^{m} S_{k-1} \left( x_{i_1}^{(1)} \ldots x_{i_{k-1}}^{(k-1)} x_{i_k}^{(k)} x_{i_{k+1}}^{(k+1)} \ldots x_{i_{l-1}}^{(l)} x_{i_l}^{(l)} x_{i_{l+1}}^{(l+1)} \ldots x_{i_m}^{(m)} \right)
\]

\[
= T_k S_{k-1} \left( x_{i_1}^{(1)} \ldots x_{i_m}^{(m)} \right),
\]

which completes the proof. \( \square \)
Following [2], we turn to study how the coefficients of the succesive shuffles of $L_P$ change. Let $L : (\mathbb{C}^n)^m \to \mathbb{C}$ be an $m$-linear form given by

$$L \left( x^{(1)}, \ldots, x^{(m)} \right) = \sum_{i \in I(m,n)} c_i x^{(1)}_{i_1} \cdots x^{(m)}_{i_m},$$

where $I(m,n) = \{1, \ldots, n\}^m$. We will denote its coefficients by $c_i(L) = c_i$.

**Lemma 2.3.** For $m, n \in \mathbb{N}$, $1 \leq k \leq m - 1$, $i \in I(m,n)$ and an $m$-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ we have

$$c_i \left( S_{k-1}L_P \right) = \begin{cases} (m - k + 1) \left( 1 + \sum_{u=1}^{m-k} \delta_{i_k,i_{k+u}} \left( \frac{1}{u+1} - \frac{1}{u} \right) \right) c_i \left( S_kL_P \right) & \text{if } i_k \leq i_{k+1}, \\ 0 & \text{otherwise} \end{cases},$$

where $\delta$ is the Kronecker delta and we take $S_0L_P = L_P$.

**Proof.** We begin the proof by calculating the coefficients $c_i \left( S_kL_P \right)$ in terms of the coefficients $c_i \left( S_{k-1}L_P \right)$. Observe that for an $m$-linear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ we have

$$T_kL \left( x^{(1)}, \ldots, x^{(m)} \right)$$

$$= \frac{1}{m - k + 1} \sum_{l=k}^{m} L \left( x^{(1)}, \ldots, x^{(l-1)}, x^{(l+1)}, \ldots, x^{(m)} \right)$$

$$= \frac{1}{m - k + 1} \sum_{l=k}^{m} \sum_{i \in I(m,n)} c_i(L)x^{(1)}_{i_1} \cdots x^{(l-1)}_{i_{l-1}} x^{(l+1)}_{i_l} \cdots x^{(m)}_{i_m}$$

$$= \sum_{i \in I(m,n)} \frac{1}{m - k + 1} \sum_{l=k}^{m} c_i(L)x^{(1)}_{i_1} \cdots x^{(l-1)}_{i_{l-1}} x^{(l+1)}_{i_l} \cdots x^{(m)}_{i_m}$$

$$= \sum_{i \in I(m,n)} \frac{1}{m - k + 1} \sum_{l=k}^{m} c_{(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_l, \ldots, i_{k+l}, \ldots, i_m)}(L)x^{(1)}_{i_1} \cdots x^{(m)}_{i_m}.$$

Therefore, since $S_k = T_kS_{k-1}$, we deduce the formula

$$c_i \left( S_kL_P \right) = \frac{1}{m - k + 1} \sum_{l=k}^{m} c_{(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_l, \ldots, i_{k+l}, \ldots, i_m)} \left( S_{k-1}L_P \right). \tag{4}$$

By the definition of $L_P$ if a coefficient $c_i \left( L_P \right)$ is not zero, then the index $i$ must satisfy that $1 \leq i_1 \leq \ldots \leq i_m \leq n$. We will prove inductively that for $0 \leq k \leq m - 1$, if the coefficient $c_i \left( S_kL_P \right)$ is not zero, then the index $i$ must satisfy that $1 \leq i_{k+1} \leq \ldots \leq i_m \leq n.$

Since $S_0L_P = L_P$, the case $k = 0$ is already proven. Now assume the assertion holds for $0 \leq k-1 \leq m-1$ and fix $i \in I(m,n)$ such that $i_s > i_{s+1}$ for some $k+1 \leq s \leq m-1$. Applying the inductive hypothesis we may deduce that

$$c_{(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_l, \ldots, i_{k+l}, \ldots, i_m)} \left( S_{k-1}L_P \right) = 0,$$

for every $k \leq l \leq m$. Hence, using (4) we get that $c_i \left( S_kL_P \right) = 0$ proving the inductive step. In particular, we have shown that $c_i \left( S_{k-1}L_P \right) = 0$ if $i_k > i_{k+1}$ as sought.

Now assume that $i_k \leq i_{k+1}$. If for some $k+1 \leq s \leq m-1$ we have that $i_s > i_{s+1}$, then by the previous argument we may deduce that $c_i \left( S_{k-1}L_P \right) = c_i \left( S_kL_P \right) = 0$ as
desired. Therefore, it remains to check the statement when \(1 \leq i_k \leq \ldots \leq i_m \leq n\).

Define \(s = \sup\{k \leq u \leq m : i_u = i_k\}\) and notice that

\[
c_i(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_l, i_{k+1}, \ldots, i_m) (S_{k-1} L_P) = \begin{cases}
c_i(S_{k-1} L_P) & \text{if } k \leq l \leq s \\
0 & \text{if } s < l \leq m.
\end{cases}
\]

Thus, we may push (4) further to get

\[
c_i(S_k L_P) = \frac{1}{m-k+1} \sum_{l=k}^m c_i(i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_l, i_{k+1}, \ldots, i_m) (S_{k-1} L_P)
= \frac{1}{m-k+1} \sum_{l=k}^s c_i(S_{k-1} L_P) = \frac{s-k+1}{m-k+1} c_i(S_{k-1} L_P).
\]

Since \(s \geq k\), we have that \(s-k+1 \neq 0\). Thus, we get

\[
c_i(S_{k-1} L_P) = \frac{m-k+1}{s-k+1} c_i(S_k L_P)
= (m-k+1) \left( 1 + \sum_{u=1}^{s-k} \left( \frac{1}{u+1} - \frac{1}{u} \right) \right) c_i(S_k L_P)
= (m-k+1) \left( 1 + \sum_{u=1}^{m-k} \delta_{i_k,i_{k+u}} \left( \frac{1}{u+1} - \frac{1}{u} \right) \right) c_i(S_k L_P).
\]

This concludes the proof. \(\square\)

As in [2], we will restate the previous lemma using Schur products. For \(A, B \in \mathbb{C}^{I(m,n)}\), the Schur product \(A \ast B\) is given by

\[
c_i(A \ast B) = c_i(A)c_i(B),
\]

where \(c_i(\cdot)\) denotes de \(i\)-th entry of a matrix. By identifying an \(m\)-linear form with its coefficients, we may compute the product between a matrix and an \(m\)-form. More precisely, for \(A \in \mathbb{C}^{I(m,n)}\) and an \(m\)-linear form \(L : (\mathbb{C}^n)^m \to \mathbb{C}\) we define \(A \ast L : (\mathbb{C}^n)^m \to \mathbb{C}\) by

\[
c_i(A \ast L) = c_i(A)c_i(L).
\]

With this notation Lemma 2.3 proves the formula

\[
S_{k-1} L_P = R_k \ast S_k L_P,
\]

(5)

where \(R_k \in \mathbb{C}^{I(m,n)}\) is given by

\[
c_i(R_k) = \begin{cases}
(m-k+1) \left( 1 + \sum_{u=1}^{m-k} \delta_{i_k,i_{k+u}} \left( \frac{1}{u+1} - \frac{1}{u} \right) \right) & \text{if } i_k \leq i_{k+1} \\
0 & \text{otherwise}
\end{cases}
\]

The matrix \(R_k \in \mathbb{C}^{I(m,n)}\) may be decomposed as sums and products of simpler matrices. For \(u, v \in \{1, \ldots, m\}\), let \(D^{u,v}, T^{u,v} \in \mathbb{C}^{I(m,n)}\) be such that for every \(i \in I(m,n)\) we have

\[
c_i(D^{u,v}) = \begin{cases}
1 & \text{if } i_u = i_v \\
0 & \text{otherwise}
\end{cases}
\]

\[
c_i(T^{u,v}) = \begin{cases}
1 & \text{if } i_u \leq i_v \\
0 & \text{otherwise}
\end{cases}
\]
Keeping Remark 1.5 in mind, we may observe that $T^{u,v}$ bears a close resemblance with the main triangle projection $T : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$. Indeed, note that $c_i(T^{u,v}) = c_{i,u,v}(T)$ for every $i \in I(m,n)$.

**Lemma 2.4.** For $1 \leq k \leq m - 1$, we have

$$R_k = (m - k + 1)T^{k,k+1} \ast \left(1 + \sum_{u=1}^{m-k} D^{k,k+u} \left(\frac{1}{u+1} - \frac{1}{u}\right)\right).$$

**Proof.** For $i \in I(m,n)$, we deduce that

$$c_i \left((m - k + 1) T^{k,k+1} \ast \left(1 + \sum_{u=1}^{m-k} D^{k,k+u} \left(\frac{1}{u+1} - \frac{1}{u}\right)\right)\right) =$$

$$= (m - k + 1) c_i \left(T^{k,k+1} \ast \left(1 + \sum_{u=1}^{m-k} c_i \left(D^{k,k+u} \left(\frac{1}{u+1} - \frac{1}{u}\right)\right)\right)\right)$$

$$= c_i \left(T^{k,k+1} \right) (m - k + 1) \left(1 + \sum_{u=1}^{m-k} \delta_{i,k+u} \left(\frac{1}{u+1} - \frac{1}{u}\right)\right)$$

$$= c_i (R_k),$$

which proves the statement.

\[\Box\]

3. **Upper bounds**

In this section we provide the upper bounds for Theorem 1.3. Let $\| \cdot \|$ be a norm on $\mathbb{C}^n$. For $A \in \mathbb{C}^I(m,n)$, we define $\mu_{\| \cdot \|}(A)$ as the infimum of the constants $C > 0$ such that for every $m$-linear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ we have

$$\sup_{\|x^{(k)}\| \leq 1} |A \ast L(x^{(1)}, \ldots, x^{(m)})| \leq C \sup_{\|x^{(k)}\| \leq 1} |L(x^{(1)}, \ldots, x^{(m)})|.$$  

Note that $(\mathbb{C}^I(m,n), \mu_{\| \cdot \|})$ is a Banach algebra.

We will use the following lemma by Defant and Schlüters.

**Lemma 3.1** [2, Lemma 3.2]. For every $n, m \in \mathbb{N}$, every $u, v \in \{1, \ldots, m\}$ and every 1-unconditional norm $\| \cdot \|$ on $\mathbb{C}^n$

$$\mu_{\| \cdot \|}(D^{u,v}) = 1,$$

$$\mu_{\| \cdot \|}(T^{u,v}) \leq \log_2(2n).$$

Moreover, for every $1 \leq p < 2$, there exists a constant $c = c(p)$ so that for every $n, m \in \mathbb{N}$

$$\mu_{\| \cdot \|_p}(T^{u,v}) \leq c.$$

As mentioned in Remark 1.5, the estimates for $T^{u,v}$ rely on bounds for the norm of the main triangle projection obtained by Kwapień and Pełczyński in [5] and Bennett in [1].

**Corollary 3.2.** For every $n, m \in \mathbb{N}$, every $1 \leq k \leq m - 1$ and every 1-unconditional norm $\| \cdot \|$ on $\mathbb{C}^n$ we have

$$\mu_{\| \cdot \|}(R_k) \leq 2(m - k + 1)\mu_{\| \cdot \|}(T^{k,k+1}).$$
Proof. From the last lemma we know that \( \mu_{\|\cdot\|}(D_{u,v}) = 1 \) for every \( u, v \in \{1, \ldots, m\} \). Since \((\mathbb{C}^{2(m,n)}, \mu_{\|\cdot\|})\) is a Banach algebra, we may deduce from Lemma 2.4 that

\[
\mu_{\|\cdot\|}(R_k) = \mu_{\|\cdot\|} \left( (m - k + 1)T^{k,k+1} \ast \left( 1 + \sum_{u=1}^{m-k} D^{k,k+u} \left( \frac{1}{u + 1} - \frac{1}{u} \right) \right) \right)
\]

\[
\leq (m - k + 1)\mu_{\|\cdot\|}(T^{k,k+1}) \left( 1 + \sum_{u=1}^{m-k} \mu_{\|\cdot\|}(D^{k,k+u}) \left( \frac{1}{u + 1} - \frac{1}{u} \right) \right)
\]

\[
\leq (m - k + 1) \left( 1 + \sum_{u=1}^{\infty} \left( \frac{1}{u} - \frac{1}{u + 1} \right) \right) \mu_{\|\cdot\|}(T^{k,k+1})
\]

\[
= 2(m - k + 1)\mu_{\|\cdot\|}(T^{k,k+1}),
\]

as required. \( \square \)

We are ready to prove the upper bounds for Theorem 1.3.

**Theorem 3.3.** There exists a universal constant \( c_1 \geq 1 \) such that

\[
C(m, n) \leq c_1^m m^m (\log n)^{m-1}.
\]

Moreover, for \( 1 \leq p < 2 \), there is a constant \( c_2 = c_2(p) \geq 1 \) for which

\[
C_p(m, n) \leq c_2^m m^m.
\]

**Proof.** Using (5), the definition of \( \mu_{\|\cdot\|} \) and the previous corollary we get

\[
\sup_{\|x^{(k)}\| \leq 1} |S_{k-1}L_P(x^{(1)}, \ldots, x^{(m)})| = \sup_{\|x^{(k)}\| \leq 1} |R_k \ast S_kL_P(x^{(1)}, \ldots, x^{(m)})|
\]

\[
\leq \mu_{\|\cdot\|}(R_k) \sup_{\|x^{(k)}\| \leq 1} |S_kL_P(x^{(1)}, \ldots, x^{(m)})|
\]

\[
\leq 2(m - k + 1)\mu_{\|\cdot\|}(T^{k,k+1}) \sup_{\|x^{(k)}\| \leq 1} |S_kL_P(x^{(1)}, \ldots, x^{(m)})|,
\]

for every \( 1 \leq k \leq m - 1 \). Taking \( \mu = \sup_{1 \leq k \leq m - 1} \mu_{\|\cdot\|}(T^{k,k+1}) \) and linking the previous inequalities together, we deduce that

\[
\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \ldots, x^{(m)})| \leq 2m\mu \sup_{\|x^{(k)}\| \leq 1} |S_1L_P(x^{(1)}, \ldots, x^{(m)})|
\]

\[
\leq 2^2m(m - 1)\mu^2 \sup_{\|x^{(k)}\| \leq 1} |S_2L_P(x^{(1)}, \ldots, x^{(m)})|
\]

\[
\leq \ldots \leq 2^{m-1}m!\mu^{m-1} \sup_{\|x^{(k)}\| \leq 1} |S_{m-1}L_P(x^{(1)}, \ldots, x^{(m)})|.
\]

Using the identity \( S_{m-1}L_P = B \) and applying (1), we obtain

\[
\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \ldots, x^{(m)})| \leq 2^{m-1}m!\mu^{m-1} \sup_{\|x^{(k)}\| \leq 1} |B(x^{(1)}, \ldots, x^{(m)})|
\]

\[
\leq 2^{m-1}e^m m!\mu^{m-1} \sup_{\|x\| \leq 1} |P(x)|.
\]

The theorem follows by applying Stirling’s formula to estimate \( m! \) and Lemma 3.1 to estimate \( \mu \). \( \square \)
4. Lower Bounds

Firstly, we provide a lower bound for \( C_p(m,n) \).

**Lemma 4.1.** For every \( n \geq m \) and every \( 1 \leq p < 2 \), we have that \( C_p(m,n) \geq m^{\frac{m}{p}} \).

**Proof.** Let \( P : \mathbb{C}^m \to \mathbb{C} \) be the \( m \)-homogeneous polynomial defined by

\[
P(x) = x_1 \ldots x_m.
\]

So, its associated \( m \)-linear form \( L_P : (\mathbb{C}^m)^m \to \mathbb{C} \) is given by

\[
L_P(x^{(1)}, \ldots, x^{(m)}) = x_1^{(1)} \ldots x_m^{(m)}.
\]

Observe that

\[
\sup_{\|x^{(k)}\|_p \leq 1} |L_P(x^{(1)}, \ldots, x^{(m)})| = \sup_{\|x^{(k)}\|_p \leq 1} |x_1^{(1)} \ldots x_m^{(m)}| = 1. \tag{6}
\]

where equality is achieved by taking \( x^{(i)} \) to be the \( i \)-th canonical vector of \( \ell^m_p \).

On the other hand, a straightforward computation using Lagrange multipliers gives

\[
\sup_{\|x\|_p \leq 1} |P(x)| = |P\left(m^{-\frac{1}{p}}(1, \ldots, 1)\right)| = m^{-\frac{m}{p}}. \tag{7}
\]

Applying (6) and (7) together with the definition of \( C_p(m,n) \) we get

\[
1 = \sup_{\|x^{(k)}\|_p \leq 1} |L_P(x^{(1)}, \ldots, x^{(m)})| \leq C_p(m,n) \sup_{\|x\|_p \leq 1} |P(x)| = m^{-\frac{m}{p}} C_p(m,n),
\]

as desired. \( \square \)

Secondly, we estimate \( C(m,n) \) from below. In order to do this we will need the following special case of a theorem proved by Pełczyński.

**Theorem 4.2** [8, Theorem 1]. For a finite index set \( J \), let \((a_j)_{j \in J}\) and \((b_j)_{j \in J}\) be sequences of characters on compact abelian groups \( S \) and \( T \) respectively. Suppose there are constants \( c_1, c_2 > 0 \) such that

\[
\frac{1}{c_1} \left\| \sum_{j \in J} \alpha_j a_j \right\|_{C(S)} \leq \left\| \sum_{j \in J} \alpha_j b_j \right\|_{C(T)} \leq c_2 \left\| \sum_{j \in J} \alpha_j a_j \right\|_{C(S)}, \tag{8}
\]

for every sequence of scalars \( (\alpha_j)_{j \in J} \subseteq \mathbb{C} \). Then, for every Banach space \( E \) and every sequence of vectors \((v_j)_{j \in J} \subseteq E \) we have

\[
\frac{1}{c_1 c_2} \int_S \left\| \sum_{j \in J} v_j a_j(s) \right\|_E \, ds \leq \int_T \left\| \sum_{j \in J} v_j b_j(t) \right\|_E \, dt \leq c_1 c_2 \int_S \left\| \sum_{j \in J} v_j a_j(s) \right\|_E \, ds. \tag{9}
\]

We are ready to provide the lower bound for \( C(m,n) \) stated in Theorem 1.3.

**Lemma 4.3.** For \( n, m \in \mathbb{N} \) such that \( \log \left(\frac{2m}{n}\right) \geq \pi \), we have

\[
C(m,n) \geq \left(\frac{\log \left(\frac{2m}{n}\right) - \pi}{\pi}\right)^{m/2}.
\]
Proof. Consider the norm \( \| \cdot \|_\infty \) on \( \mathbb{C}^n \). Since \( P(x) = L_P(x, \ldots, x) \), we deduce that

\[
\sup_{\| x \|_\infty \leq 1} |P(x)| \leq \sup_{\| x^{(k)} \|_\infty \leq 1} |L_P(x^{(1)}, \ldots, x^{(m)})| \leq C(m, n) \sup_{\| x \|_\infty \leq 1} |P(x)|,
\]

for every \( m \)-homogeneous polynomial \( P : \mathbb{C}^n \to \mathbb{C} \). Equivalently, by the maximum modulus principle we get

\[
\sup_{x \in \mathbb{T}^n} |P(x)| \leq \sup_{x^{(k)} \in \mathbb{T}^n} |L_P(x^{(1)}, \ldots, x^{(m)})| \leq C(m, n) \sup_{x \in \mathbb{T}^n} |P(x)|, \tag{10}
\]

where \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \).

Thus, the conditions of Pelczyński’s theorem are satisfied. Indeed, denote the compact abelian groups \( \mathbb{T}^n \) and \( (\mathbb{T}^n)^m \) by \( S \) and \( T \) respectively and consider the index set \( J = \{ j \in \mathcal{I}(m, n) : 1 \leq j_1 \leq \ldots \leq j_m \leq n \} \). For every \( j \in J \), define the characters \( a_j : S \to \mathbb{T} \) and \( b_j : T \to \mathbb{T} \) by

\[
a_j(x) = x_{j_1} \ldots x_{j_m} \quad \text{and} \quad b_j(x^{(1)}, \ldots, x^{(m)}) = x^{(1)}_{j_1} \ldots x^{(m)}_{j_m}.
\]

If we restate (10) with this notation we get (8), with \( c_1 = 1 \) and \( c_2 = C(m, n) \). Therefore, we deduce from Pelczyński’s theorem that

\[
\frac{1}{C(m, n)} \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x_{j_1} \ldots x_{j_m} \right\|_E dx \leq \int_{\mathbb{T}^n} \ldots \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x^{(1)}_{j_1} \ldots x^{(m)}_{j_m} \right\|_E dx^{(1)} \ldots dx^{(m)} \\
\leq C(m, n) \int_{\mathbb{T}^n} \left\| \sum_{j \in J} v_j x_{j_1} \ldots x_{j_m} \right\|_E dx. \tag{11}
\]

for every Banach space \( E \) and every sequence of vectors \( (v_j)_{j \in J} \subseteq E \). Choosing the space \( E \) and the vectors \( (v_j)_{j \in J} \subseteq E \) adequately will yield the estimate we seek.

We will build upon an example provided by Bourgain (unpublished) and included in a paper by McConnell and Taqqu [7, Example 4.1] (see also [6, Section 6.9]). Consider the Banach space \( F = \mathcal{L}(\ell_2) \). For every \( 1 \leq i \neq j \leq n \), define vectors \( v_{ij} \in F \) by

\[
v_{ij} = \frac{1}{i-j} e_i \otimes e_j + \frac{1}{j-i} e_j \otimes e_i.
\]

Using complex Steinhaus variables instead of Bernoulli random variables and proceeding as in [6] we get

\[
\int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x_i x_j \right\|_E dx \leq \pi \quad \text{and} \quad \tag{12}
\]

\[
\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x_i^{(1)} x_j^{(2)} \right\|_E dx^{(1)} dx^{(2)} \geq \log n - \pi. \tag{13}
\]

Note that by the previous estimations we obtain the desired result for \( m = 2 \) since we have

\[
\log n - \pi \leq \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x_i^{(1)} x_j^{(2)} \right\|_E dx^{(1)} dx^{(2)} \leq C(2, n) \int_{\mathbb{T}^n} \left\| \sum_{1 \leq i < j \leq n} v_{ij} x_i x_j \right\|_E dx \leq C(2, n) \pi.
\]
Moreover, this together with Theorem 3.3 shows that the asymptotic behaviour of $C(2, n)$ is logarithmic.

To conclude our argument it remains to extend this 2-variable example to $m$ variables. Assume $m$ is even and let $E = \bigotimes_{k=1}^{m/2} F$ be the projective tensor product of $m/2$ copies of $F$. Consider the $m$-homogeneous vector-valued polynomial $P : \mathbb{C}^n \to E$ defined by

$$P(x) = \sum_{2m/(m-1) < j_2k-1 < j_2k < 2m} v_j x_j,$$

where $v_j = v_{j_1j_2} \otimes v_{j_3j_4} \otimes \ldots \otimes v_{j_{m-1}j_m}$.

Notice that

$$P(x) = \bigotimes_{k=1}^{m/2} \sum_{2m/(m-1) < j_2k-1 < j_2k < 2m} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}} x_{j_{2k}}.$$

Applying (12) we get

$$\int_{T^n} \| P(x) \| \, dx = \int_{T^n} \prod_{k=1}^{m/2} \left\| \sum_{2m/(m-1) < j_2k-1 < j_2k < 2m} v_{j_{2k-1}j_{2k}} x_{j_{2k-1}} x_{j_{2k}} \right\| \, dx$$

$$\leq \prod_{k=1}^{m/2} \pi = \pi^{m/2}.$$

On the other hand, from (13) we deduce

$$\int_{T^n} \ldots \int_{T^n} \| L_p \left( x^{(1)}, \ldots, x^{(m)} \right) \| \, dx^{(1)} \ldots dx^{(m)} =$$

$$\geq \prod_{k=1}^{m/2} \left( \log \left( \frac{2m}{m} \right) - \pi \right) = \left( \log \left( \frac{2m}{m} \right) - \pi \right)^{m/2}.$$

Finally, using (11) together with these estimates we obtain

$$C(n, m) \geq \left( \frac{\log \left( \frac{2m}{m} \right) - \pi}{\pi} \right)^{m/2},$$

as desired. \qed

Note that Lemmas 4.1 and 4.3 together with Theorem 3.3 prove Theorem 1.3.

**Remark 4.4.** Tracing back the argument to obtain (13), we find that Bourgain’s example is based on a lower estimate of the main triangle projection’s norm on $L(\ell_2)$. 
In other words, the lower bound for $C(m, n)$ was obtained by studying the behaviour of the main triangle projection as mentioned in Remark 1.5. Although $C(m, n)$ and $C_p(m, n)$ were not completely characterized, it seems that the main triangle projection plays a crucial role in determining their asymptotic behaviour.

**Acknowledgements**

I thank my supervisor Daniel Carando for his guidance and fruitful discussions and Sunke Schlüters for his helpful comments.

**References**

[1] Bennett, G. “Unconditional convergence and almost everywhere convergence”. In: Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 34.2 (1976), pp. 135-155.

[2] Defant, A. and Schlüters, S. “Non-symmetric polarization”. In: Journal of Mathematical Analysis and Applications 445.2 (2017), pp. 1291-1299.

[3] Dineen, S. Complex analysis on infinite dimensional spaces. London: Springer-Verlag, 1999.

[4] Fisher, R. A. and Yates, F. Statistical tables for biological, agricultural and medical research. Edinburgh: Oliver and Boyd, 1938, p. 285.

[5] Kwapień, S. and Pełczyński, A. “The main triangle projection in matrix spaces and its applications”. In: Studia Mathematica 34.1 (1970), pp. 43-67.

[6] Kwapień, S. and Woyczynski, W. Random series and stochastic integrals: single and multiple. Birkhäuser Basel, 1992.

[7] McConnell, T. R. and Taqqu, M. S. “Decoupling of Banach-valued multilinear forms in independent symmetric Banach-valued random variables”. In: Probability Theory and Related Fields 75.4 (1987), pp. 499-507.

[8] Pełczyński, A. “Commensurate Sequences of Characters”. In: Proceedings of the American Mathematical Society 104.2 (1988), pp. 525-531.