Improved variational iteration method with the trapezoidal rule to solve fractional ordinary differential equations

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Abstract

In this work, a combined technique the Variation Iteration Method (VIM) with the Trapezoidal Rule (TR) was recommended to solve linear and nonlinear fractional ordinary differential equations (F.O.D.E.), where the results obtained from the Variation Iteration method were improved, and numerical results were obtained by determining the maximum absolute errors (MAE) and mean square error (MSE) for the given examples. As the results It is proved that that the proposed method is better than the default method.

1. Introduction

The trapezoidal rule considered as one of the old and famous methods in numerical analysis to approximating the definite integral. Since the sum of the series from the trapezoid has been approximated to the area below the function curve, which is why it is called by this name. This is what is represented by the results of calculating the left and right Riemann mean, and this integration can be better approximated by dividing the integral into several periods, and then we apply this rule to each period, and then we collect the results.[1].

The Variational iteration technique was first suggested by [2] and has been extensively applied by many authors in problems that deal with many mathematical models that result from the different sciences in engineering and physics. This technique solves the issues with none ought to discrete the variables. In applications, the method gives ease while preserving efficiency because no need is required calculation the round off errors, and it can dispense with the large size of computer working memory, so it has flourished due to its high flexibility [3].

Provides the VIM algorithmic program Flexible and outstanding formula for analytic approximate solutions and numeric simulations for practical applications in sciences in contrast to the Adomian decomposition methodology, wherever process algorithms area unit ordinarily accustomed alter the nonlinear terms. In this method, we do not need limiting assumptions for nonlinear terms used in other methods which may make analytical processes difficult. The (VIM) approaches linear and nonlinear issues directly during a like way[4].

We will solve the F.O.D.E. of the following type:

\[ D^\alpha v(k) + N v(k) = g(k) \quad b - 1 < \alpha \leq b, \]

(1)
where $D^\alpha$ denote to Caputo F.O.D.E of order $\alpha$, $N$ is operator linear or nonlinear functions, $g(k)$ is any given function.

Fractional equations have been used in many fields, including engineering and basic sciences, and were developed from by Spanier and Oldham [5]. Due to the importance of these equations, there are many modern applications and works, and these equations have been solved in Different ways, including analytical and numerical methods [6]. Derivation and fractional integration have many definitions, but the most important and most used of these definitions are (Riemann-Liouville) integration and (Caputo) derivation [7]. Leibniz, in his letter to L'Hospital in 1695, discussed F.O.D.E. (Caputo), where the value of $\alpha$ is between $[0,1]$ and is found in many physical and biological models [8-12]. As we mentioned earlier, there is a lot of research and work on this type of equations for their importance. See [13] , and also that one of the most important books that talked about this type of equations is Podlubny [14].

In this paper, the results of the (VIM) method for solving fractional equations were improved by using the trapezoid rule and obtaining better results as shown in the examples.

This work is organized as follows, in Part 2, Provide some definitions. In Part 3, the VIM method is explained. In Part 4, the proposed method is explained(VIM-TR). In Part 5, solve some examples with numerical results. Conclusions are presented in Part 6.

2. Definitions

In this section, we will show some definitions used in the research.

2.1. Riemann-Liouville fractional integer (RFI) [15]

The (RFI) operator of order $\alpha$ of a function $f(u) \in C_\eta, \eta \geq -1$ is defined as:

$$J^\alpha f(u) = \frac{1}{\Gamma(\alpha)} \int_0^u (u-k)^{\alpha-1} f(k) dk, \quad u > 0,$$

$$J^\alpha f(u) = f(u),$$

For $f(u) \in C_\eta, \eta \geq -1, \alpha, \beta \geq 0, \zeta \geq -1$ properties of the operator $J^\alpha$

$$J^\alpha J^\beta f(u) = J^\beta J^\alpha f(u)$$

$$J^\alpha u^\zeta = \frac{\Gamma(\zeta+1)}{\Gamma(\alpha+\zeta+1)} u^{\alpha+\zeta}$$

2.2. Derivative Caputo Fractional (DCF) [15].

The (DCF) of $f(u)$ defined as:

$$D^\alpha_u f(u) = J^{n-\alpha} D^n f(u) = \frac{1}{\Gamma(b-\alpha)} \int_0^u (u-k)^{b-\alpha-1} f^{(b)}(k) dk.$$

For $f(u) \in C^b_\eta, \eta \geq -1, \alpha, \beta \geq 0, \zeta \geq -1, b-1 < \alpha \leq b, b \in \mathbb{R}$ properties of the operator $D^\alpha_u$

$$D^\beta_u D^\alpha_u f(u) = D^\alpha_u D^\beta_u f(u) = D^\beta_u D^\alpha_u f(u)$$

$$D^\alpha_u u^\zeta = \frac{\Gamma(1+\zeta)}{\Gamma(1+\zeta-\alpha)} u^{\zeta-\alpha}, u > 0$$

3. Variational Iteration Method (VIM)

We will explain the method if we have the following differential equation:
\[ L \nu(k) + N \nu(k) = g(k), \]  
where \( L \) is a linear operator, \( N \) is Liner or nonlinear operator and \( g(k) \) is a known analytical function. We will construct a functional patch using (VIM) as follows

\[ \nu_{r+1}(k) = \nu_r(k) + \int_0^k \lambda(k,c) \left( L\nu_r(c) + N\tilde{\nu}_r(c) - g(c) \right) dc, \]

where \( \lambda \) is a General Lagrange Multiplier(GLM), It can be best obtained by the theory of variance, \( \nu_p \) is the \( r^{th} \) approximate solution and \( \tilde{\nu}_r \) It leads to a restricted difference, means \( \delta \tilde{\nu}_r = 0 \). Successive approximations, \( \nu_{r+1}(k) \), Which we will get by applying the Lagrange multiplier that we got from the first approximation \( \nu_0(t) \). So the solution is given by \( \nu = \lim_{r \to \infty \nu_r} \). So have VIM convergence and error estimation [16].

4. The Proposed Technique VIM-TR

Consider the F.O.D.E. (1), with condition \( v^i(0) = b_i, \; i = 0,1,\ldots, r - 1 \)

Now solve equation (1) using the (VIM) method is

\[ \nu_{r+1}(k) = \nu_r(k) + \int^k \lambda(k,c) \left( L\nu_r(c) + N\tilde{\nu}_r(c) - g(c) \right) dc, \]

To find duplicates \( \nu_{r+1} \) for each of \( r = 0,1,\ldots \), we must find the values of \( \nu_0 \) and \( \lambda \).

The value of \( \nu_0 = \nu_0 = \sum_{n=0}^{r-1} l^k a^m \)

To find \( \lambda \) multiply equation (1) by a (GLM), yields to:

\[ \lambda(c) \left( D^n \nu_r(k) + N\nu_r(k) - g(k) \right) = 0, \]

We take \( (l^m) \) to both sides of the equation (4) we get

\[ \lambda(c) \left( D^n \nu_r(k) + N\nu_r(k) - g(k) \right) = 0, \]

Therefore, the correction for the function in equation (1) will be as follows

\[ \nu_{r+1}(k) = \nu_r(k) + \int^k \lambda(c) \left( D^n \nu_r(c) + N\nu_r(c) - g(c) \right) dc, \]

In this way, we may not obtain a value \( (\lambda) \) easily from equation (6) and we will obtain fractional functions and integrals, so the approximation of the corrected function can be expressed as follows:

\[ \nu_{r+1}(k) = \nu_r(k) + \int^k \lambda(c) \left( D^n \nu_r(c) + N\nu_r(c) - g(c) \right) dc, \]

Thus, by taking the first variation with respect to the independent variable \( \nu_r \) and noticing that \( \delta \nu_r = 0 \) yields to:

\[ \delta \nu_{r+1}(k) = \delta \nu_r(k) + \int^k \lambda(c) \left( D^n \nu_r(c) + N\nu_r(c) - g(c) \right) dc, \]

Where \( \tilde{\nu}_r \) is So it's a restricted variance, and that leads to \( \delta \tilde{\nu}_r = 0 \) therefore equation (8) when \( m = 1 \) we have the following

\[ \delta \nu_{r+1}(k) = \delta \nu_r(k) + \lambda(c) \nu'(c) dc, \]

now, using the method of integration by parts on equation (9) will give the following formula:

\[ \delta \nu_{r+1}(k) = \delta \nu_r(k) + \lambda(c) \delta \nu_r(c) \bigg|_{c=k} - \int^k \lambda'(c) \delta \nu_r(c) dc, \]

and then:

\[ \delta \nu_{r+1}(k) = (1 + \lambda(c)) \delta \nu_r(k) - \int^k \lambda'(c) \delta \nu_r(c) dc, \]

As a result, the following stationary conditions are obtained:

\[ \lambda'(c) = 0, \quad 1 + \lambda(c) \bigg|_{c=k} = 0, \]

Finally, by solving the ordinary differential equation, we will get the (GLM), which is as it comes

\[ \lambda(c) = -1, \]
By, substituting $\lambda(c) = -1$ into the correction functional (6), will give the following variational iteration formula:

$$v_{r+1}(k) = v_r(k) - I^a[D^a v_r(k) + Nv_r(k) - g(k)],$$  \hspace{1cm} (14)

finally, the solution is given by $\lim_{r \to \infty} v_r$

The proposed technique (VIM-TR) was hybridized by taking equation (14) obtained by the method (VIM) when $r = 3$ (the third iteration) and considering it the initial solution to the problem, then solving the problem again using the trapezoidal rule.

5. Illustrative Examples

In this section, we found a solution to some linear and nonlinear examples using the (VIM) and (VIM-TR) and find the numerical results for different values of alpha.

Example 1 [12]:

let's have a linear example with the initial conditions of F.O.D.E:

$$D^a v(k) + v(k) = k^2 + \frac{2k^{2.5}}{\Gamma(2.5)}, \quad 0 \leq k \leq 1, \quad 0 < a \leq 1 \text{ with condition } v(0) = 0.

v_{\text{Exact}}(k) = k^2.

By applying the technique mentioned in Part 4, we obtain the following iterations:

$$v_0(k) = 0$$

$$v_1(k) = -h \left( \frac{2k^{a+2}}{3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2} + \frac{2k^{a+1.5}}{\Gamma(2.5+a)} \right)$$

$$v_2(k) = -4hk^{a+2} - \frac{4hk^{a+1.5}}{\Gamma(2.5+a)} - \frac{2h^2k^{a+2}}{6h^2k^{a+2}\Gamma(a)\alpha} - \frac{2h^2k^{a+1.5}}{\Gamma(2.5+a)}$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{4(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha^3}{\sqrt{\pi}} + \frac{6(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{2(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{2h^2k^{2a+2}\Gamma(a)\alpha^3}{\sqrt{\pi}} + \frac{2h^2k^{2a+2}\Gamma(a)\alpha^2}{\sqrt{\pi}} + \frac{2h^2k^{2a+2}\Gamma(a)\alpha}{\sqrt{\pi}}\right)$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{4(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha^3}{\sqrt{\pi}} + \frac{6(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha^2}{\sqrt{\pi}} + \frac{2(2^{2k})\Gamma(a)\Gamma\left(a + \frac{1}{2}\right)\alpha}{\sqrt{\pi}}\right)$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{2h^2k^{2a+1.5}}{\Gamma(2a + 2.5)} \right)$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{6h^2k^{a+2}}{2h^2k^{a+2}\Gamma(a)\alpha^2} + \frac{6h^2k^{a+1.5}}{\Gamma(2.5+a)} \right)$$

$$\left(3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2\right) \left(\frac{6h^2k^{a+2}}{3\Gamma(a)\alpha^2 + 2\Gamma(a)\alpha + \Gamma(a)\alpha^2} - \frac{6h^2k^{a+1.5}}{\Gamma(2.5+a)} \right)$$

Now by taking equation (15) and considering it the initial solution to the example(1) then solving by the trapezoidal rule.
\[ TR_{\alpha=0.25} = 0.816981x^{1.75} - 0.216000x^2 + 0.536626x^{2.25} - 0.129989x^{2.5} - 0.077005x^{2.75} + 0.021627x^3 - 0.049610x^{3.25} + 0.062839x^{3.5} - 0.024796x^{3.75} + 0.030475x^4 - 0.011688x^{4.25} + 0.002484x^{4.5} - 0.000206x^{4.75} \]
\[ TR_{\alpha=0.5} = 0.999000x^2 + 0.016248x^{2.5} - 0.181024x^3 + 0.175628x^{3.5} + 0.066115x^4 - 0.008694x^{4.5} + 0.001220x^5 - 0.001527x^{5.5} \]
\[ TR_{\alpha=0.75} = 0.789117x^{2.25} + 0.454820x^{2.75} - 0.297045x^3 - 0.072952x^{3.25} - 0.101547x^{3.5} + 0.161656x^{3.75} + 0.047396x^4 + 0.076133x^{4.25} + 0.010248x^{4.5} - 0.020247x^{4.75} + 0.004474x^5 - 0.005357x^{5.25} + 0.001866x^{5.5} - 0.002188x^{5.75} - 0.000859x^{6.25} \]
\[ TR_{\alpha=1} = 0.672603x^{2.5} + 0.372549x^3 - 0.027900x^{3.5} - 0.013528x^4 + 0.085252x^{4.5} + 0.037170x^5 - 0.022789x^{5.5} - 0.009122x^6 - 0.003561x^{6.5} - 0.001324x^7 \]

Numerical results of the (VIM) and (VIM_TR) method for the third iteration

Table 1: Comparison of MAE for example.1

| \( \alpha \) | MAE of (VIM_TR) | MAE of (VIM) |
|------------|----------------|-------------|
| 0.25       | 0.0384         | 0.6629      |
| 0.5        | 0.0069         | 0.1719      |
| 0.75       | 0.0555         | 0.0911      |
| 1          | 0.0893         | 0.2652      |

Table 2: Comparison of MSE for example.1

| \( \alpha \) | MSE of (VIM_TR) | MSE of (VIM) |
|------------|----------------|-------------|
| 0.25       | 0.0003         | 0.0981      |
| 0.5        | 0.0000         | 0.0048      |
| 0.75       | 0.0013         | 0.0043      |
| 1          | 0.0035         | 0.0238      |

Figure (1)
Numerical results of the (VIM) and (VIM_TR) method with the exact solution A at \( \alpha = 0.25 \) and B at \( \alpha = 0.5 \)
Example 2 [11]:
let's have a nonlinear example with the initial conditions F.O.D.E
\[ D^\alpha v(k) = v^2(k) + 1, 0 \leq k \leq 1, 0 < \alpha \leq 1 \]
with condition \( v(0) = 0 \).

By applying the technique mentioned in Part 4, we obtain the following iterations:

\[
v_0(k) = 0
\]
\[
v_1(k) = -\frac{hk^\alpha}{\Gamma(\alpha)\alpha}
\]
\[
v_2(t) = -\frac{2hk^\alpha}{\Gamma(\alpha)\alpha} - \frac{h^2k^\alpha}{\Gamma(\alpha)\alpha} - \frac{2h^3\sqrt{\pi}2^{-2\alpha}\Gamma\left(\alpha + \frac{1}{2}\right)k^{3\alpha}\sqrt{3}}{\alpha^2\Gamma(\alpha)2^{-3\alpha}\Gamma\left(\alpha + \frac{2}{3}\right)}
\]
\[
v_3(k) = -\frac{3hk^\alpha}{\Gamma(\alpha)\alpha} - \frac{3h^2k^\alpha}{\Gamma(\alpha)\alpha} - \frac{10h^3\sqrt{\pi}2^{-2\alpha}\Gamma\left(\alpha + \frac{1}{2}\right)k^{3\alpha}\sqrt{3}}{3\alpha^2\Gamma(\alpha)2^{-3\alpha}\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)} - \frac{h^3k^\alpha}{\Gamma(\alpha)\alpha}
\]
\[
- \frac{10h^4\sqrt{\pi}2^{-2\alpha}\Gamma\left(\alpha + \frac{1}{2}\right)k^{3\alpha}\sqrt{3}}{3\alpha^2\Gamma(\alpha)2^{-3\alpha}\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)} - \frac{2h^5\sqrt{\pi}2^{-2\alpha}\Gamma\left(\alpha + \frac{1}{2}\right)k^{3\alpha}\sqrt{3}}{3\alpha^2\Gamma(\alpha)2^{-3\alpha}\Gamma\left(\alpha + \frac{1}{3}\right)\Gamma\left(\alpha + \frac{2}{3}\right)}
\]
\[
- \cdots \quad (16)
\]

Now by taking equation (16) and considering it the initial solution to the example(2) then solving by the trapezoidal rule.

Numerical results of the (VIM) and (VIM_TR) method for the third iteration
Table 3: Comparison of MAE for example.2

| $\alpha$ | MAE of (VIM_TR) | MAE of (VIM) |
|----------|----------------|--------------|
| 0.25     | 0.4458         | 4.1439       |
| 0.5      | 0.2560         | 2.3027       |
| 0.75     | 0.1238         | 0.7419       |
| 1        | 0.0049         | 0.0748       |

Table 4: Comparison of MSE for example.2

| $\alpha$ | MSE of (VIM_TR) | MSE of (VIM) |
|----------|----------------|--------------|
| 0.25     | 0.0854         | 6.9480       |
| 0.5      | 0.0357         | 1.4961       |
| 0.75     | 0.0091         | 0.1505       |
| 1        | 5.06E-06       | 6.10E-05     |

Figure (3)

Numerical results of the (VIM) and (VIM_TR) method with the exact solution A at $\alpha = 0.25$ and B at $\alpha = 0.5$
Example 3 [17]

let's have a nonlinear example Riccati F. O.D.E.

\[ D^\alpha v(k) + v^2(k) = 1 , 0 \leq k \leq 1 , 0 < \alpha \leq 1 \]
with condition \( v(0) = 0 \).

Let \( v_{Exact}(k) = \frac{a^{2k-1}}{a^{2k+1}} \).

By applying the technique mentioned in Part 4, we obtain the following iterations:

\( v_0(k) = 0 \)

\( v_1(k) = -\frac{hk^\alpha}{\Gamma(\alpha)\alpha} \)

\( v_2(k) = -\frac{2hk^\alpha}{\Gamma(\alpha)\alpha} - \frac{h^2k^\alpha}{\Gamma(\alpha)\alpha} + \frac{2h^3\sqrt{\pi}2^2\alpha^2 \Gamma(\alpha + \frac{1}{2}) k^{3\alpha} \sqrt{3}}{\alpha^2 \Gamma(\alpha)^2 3^{3\alpha} \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3})} \)

\( v_3(k) = -\frac{3hk^\alpha}{\Gamma(\alpha)\alpha} - \frac{3h^2k^\alpha}{\Gamma(\alpha)\alpha} + \frac{10h^3\sqrt{\pi}2^2\alpha^2 \Gamma(\alpha + \frac{1}{2}) k^{3\alpha} \sqrt{3}}{3\alpha^2 \Gamma(\alpha)^2 3^{3\alpha} \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3})} - \frac{h^3k^\alpha}{\Gamma(\alpha)\alpha} + \frac{10h^4\sqrt{\pi}2^2\alpha^2 \Gamma(\alpha + \frac{1}{2}) k^{3\alpha} \sqrt{3}}{3\alpha^2 \Gamma(\alpha)^2 3^{3\alpha} \Gamma(\alpha + \frac{1}{3}) \Gamma(\alpha + \frac{2}{3})} \)

\[ \pm \cdots \] \( (17) \)
Now by taking equation (17) and considering it the initial solution to the example(3) then solving by the trapezoidal rule.

Numerical results of the (VIM) and (VIM_TR) method for the third iteration

| α    | MAE of (VIM_TR) | MAE of (VIM) |
|------|----------------|--------------|
| 0.25 | 0.2608         | 0.4270       |
| 0.5  | 0.1430         | 0.2365       |
| 0.75 | 0.0479         | 0.1137       |
| 1    | 0.0015         | 0.0225       |

Table 6: Comparison of MSE for example.3

| α    | MSE of (VIM_TR) | MSE of (VIM) |
|------|----------------|--------------|
| 0.25 | 0.0337         | 0.1317       |
| 0.5  | 0.0084         | 0.0396       |
| 0.75 | 0.0009         | 0.0070       |
| 1    | 5.30E-07       | 6.24E-06     |

Figure (5)
Numerical results of the (VIM) and (VIM_TR) method with the exact solution
A at $\alpha = 0.25$ and B at $\alpha = 0.5$
6. Conclusions

In this work, the (VIM) method was hybridized with the trapezoidal rule, where the third iteration of the (VIM) method was used as a primary solution in the trapezoidal rule, and we obtained better results than a default method by calculating (MAE) and (MSE) as it is shown in Tables (1, 2, ...., 6) and also through the drawing one can see that the proposed method was more accurate and closer to the exact solution.

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