The dynamics of monolithic suspensions for advanced detectors: a 3-segment model

F Piergiovanni\textsuperscript{1,2}, M Lorenzini\textsuperscript{2}, G Cagnoli\textsuperscript{2,3}, E Campagna\textsuperscript{1,2}, E Cesarini\textsuperscript{1,2}, G Losurdo\textsuperscript{2}, F Martelli\textsuperscript{1,2}, F Vetrano\textsuperscript{1,2}, A Vicerè\textsuperscript{1,2}

\textsuperscript{1} Università di Urbino, Via S.Chiara 27, 61029 Urbino (PU), Italy
\textsuperscript{2} INFN, Istituto Nazionale di Fisica Nucleare, Sez. di Firenze, via G. Sansone 1, 50019 Sesto Fiorentino (FI), Italy
\textsuperscript{3} SUPA, University of Glasgow, Department of Physics and Astronomy, Kelvin Building G12 8QQ Glasgow, Scotland

E-mail: piergiovanni@fi.infn.it

Abstract.

In order to reduce the suspension thermal noise, the second generation GW interferometric detectors will employ monolithic suspensions in fused silica to hold the mirrors. The fibres are produced by melting and pulling apart a fused silica rod, obtaining a long thin wire with two thicker heads. The dynamics of such a fibre is in principle different from that of a cylindrical, regular fibre, because most of the deformation energy is stored in the neck region where the diameter is variable. This is an advantage, since adjusting the neck tapering, a thermoelastic noise cancellation effect can be obtained. Therefore, a careful study of the suspensions behavior is necessary to estimate the overall noise and to optimize the control strategy. To simplify the control design, a simple three segment model for the silica fibres has been developed, fully equivalent to the beam equation at low frequencies. The model, analytically proved for a regular cylindrical fibre, can be extended to a fibre with tapered necks, provided that the equivalent bending length is suitably measured. We developed a tool to measure the position of the bending point for each fibre, thus allowing to experimentally check the validity of the model. A numerical code has been written to solve the beam equation for wires with varying diameter. This code confirms the validity of the three segment model. Moreover, it is possible to extend the solution to higher frequencies thus computing the transfer function and the energy distribution of the suspension system and estimating the thermal noise contribution.

1. Introduction

The modeling of the suspension system of GW interferometric detectors is necessary to verify the static equilibrium conditions and to design the best control strategies. Different approaches can be used to model the behavior of suspension fibres: elastic equations can be solved analytically or numerically, a finite element analysis can be performed or just simple pendulum can be considered, neglecting the elasticity. The elastic equations solution and the FEM analysis are computationally expensive and the simple pendulum approach is inaccurate for many purposes. The simplest model that describes with sufficient accuracy the low frequency elastic behavior of suspension fibers, is the so-called 3-segment model: each fiber is replaced by 3 rigid segments connected at the bending points\textsuperscript{[1]}. The model provides a great simplification of the equations for the elastic forces and torques at the ends of the fibre. The simplicity of this model allows
2. The 3-segment model
Considering the shape of a slightly bent fibre, the bending point is the intersection of two lines that are tangent to the first and the middle part of the fibre[2]. The distance of this point from the fibre end is the bending length ($\lambda$) (figure 1). The two bending lengths ($\lambda_1$ and $\lambda_2$) and the fibre total length ($L_{tot}$) are the only three parameters used by the model. A triple pendulum composed by the 3 segments of length $\lambda_1$, $L_{tot} - (\lambda_1 + \lambda_2)$, $\lambda_2$ is considered (figure 2). The deformation is described by the transverse coordinates $X$, $Y$ and by the tilt angles $\theta_x$, $\theta_y$, while the excitations are due to the force $F$ and the torque $\tau$. Since each fibre is linked to the rest of the suspension through its ends, the equivalence of the model with respect to the elastic fibre, means that the relationships between the coordinates and the excitations in the model at the fibre ends are the same obtained from the elastic equations at first order of approximation. The model is valid under the following conditions: the oscillation wavelength must be greater than the bending length (low frequencies), the transverse displacement is small compared with the total length (small deflections) and the bending length has to be smaller than the total length. These conditions are all largely satisfied in GW detector suspensions in a range of frequency below some tens of Hz.

3. Analytical proof of the model equivalence for cylindrical fibres
For a cylindrical fibre with a constant cross section, it is possible to verify analytically the validity of the model solving the elastic equations for a slightly deflected cylindrical rod stretched by a tension $T$[3]. For harmonic excitation a Fourier transform can be performed:

$$EIX^{IV}(z) - TX^{II}(z) - \rho S \omega^2 X(z) = 0 \quad (1)$$

where $E$ and $\rho$ are the Young modulus and the density of the material, $I$ the momentum of inertia, $S$ the cross section, $T$ the fibre tension, $\omega$ the angular frequency and $X$ the Fourier transform of the deflection $x$ in x-z plane. The Roman numerals in suffix mean the order of the derivative with respect to the $z$ coordinate. Obviously the equation is the same for the deflection $y$ in plane y-z. A general solution of the equation (1) is:

$$X(z) = A \exp(-z/\lambda) + B \exp(-L-z/\lambda) + C \cos(p\lambda) + D \sin(p\lambda) \quad (2)$$

where:

$$\lambda \equiv \left( \frac{T + \sqrt{T^2 + 4EI\rho S\omega^2}}{2EI} \right)^{-1/2}$$

and

$$p \equiv \left( \frac{T + \sqrt{T^2 + 4EI\rho S\omega^2}}{2EI} \right)^{-1/2} \quad (3)$$

Boundary conditions at the ends ($X(0), X'(0), X(L), X'(L)$) fix the values of the constants $A, B, C, D$. In the low frequency regime, equations (3) show that $p$ goes to zero and $\lambda \approx \sqrt{EI/T}$, where $\lambda$ is the above mentioned bending length. Forces and torques can be obtained by boundary
The bending energy for a real suspension fibre is concentrated in a region around the bending
rod value to the fibre one (usually few hundred µm) are called necks[14][8]. The length of the
necks ranges from few millimeters to tens of millimeters. Size and shape of the neck can be
optimized to further depress the thermoelastic contribution to thermal noise[15].

4.1. A numerical validation of the model for fibres with necks: a finite element model

The bending energy for a real suspension fibre is concentrated in a region around the bending
points, close to the fibre’s necks[3]. For the presence of a variable cross section the dynamical
behavior of these fibres is different from that of the perfectly cylindrical ones, even if cylindrical
symmetry is preserved. In this case no analytical solution of the elastic equations can be found.
To test the validity of the 3-segment scheme, a finite element model of the fibre was realized
and the results were compared. Meshing is obtained by dividing the fibre into N 1-dimensional
elements of length \( h = L/N \) separated by nodes (figure 3)[16][17]. For each element \( j \) a local
non-dimensional coordinate \( ζ = j - z/h \) is defined and four cubic interpolating functions \( φ_{1,4}(ζ) \)
are introduced. The displacement inside the element \( j \) (\( x(ζ) \)), can then be written in terms of
slopes and displacement at the nodes \( x_{j-1}, \theta_{j-1}, x_j \) and \( θ_j \):

\[
x(ζ) = φ_1(ζ)x_{j-1} + φ_2(ζ)hθ_{j-1} + φ_3(ζ)x_j + φ_4(ζ)hθ_j
\]

Equation (4) contains the relationships between displacements and excitations. Identical
relationships can be much more easily obtained from the 3-segment model. A scheme of
computation for a \( X(L)=δ \) displacement in x-z plane is shown in figure 2. Restoring forces
are the x-components of the tension \( T \), while torques are evaluated using \( λ \) as lever arm. The
same results are obtained by putting \( X(0) = 0, X(L) = δ, θ_y(0) = 0, θ_y(L) = 0 \) in equation (4).

Figure 2. A 3-segment model scheme for a deflection
\( X(L)=δ \) in x-z plane. The relationships obtained by the
model, between displacement and force and torque are the
same as those in equation (4)
Matching conditions at the nodes determine the four constants of each cubic polynomial $\phi_{1,4}$.

The kinetic and potential energies of the fibre transverse motion are:

$$E^\text{kin} = \frac{1}{2} \int_0^L \rho(z) [\dot{x}(z)]^2 \, dz; \quad E^\text{pot} = \frac{1}{2} \int_0^L \left( EI [x''(z)]^2 + T [x'(z)]^2 \right) \, dz$$

(6)

Using the defined interpolating functions the integrals can be decomposed in a sum of terms:

$$E^\text{kin}_j = \frac{1}{2} \dot{X}_j^T M_j X_j; \quad E^\text{pot}_j = \frac{1}{2} X_j^T K_j X_j$$

(7)

where $X_j = (x_{j-1}, h\theta_{j-1}, x_j, h\theta_j)^T$. The mass and the stiffness matrix, respectively $M_j$ and $K_j$, are fully defined by the integrals in equation(6), i.e.

$$M_j^{nn} = h \rho_j \int_0^1 \phi_m(\zeta)\phi_n(\zeta) \, d\zeta; \quad K_j^{nn} = \frac{EI_j}{h^3} \int_0^1 \phi_m''(\zeta)\phi_n''(\zeta) \, d\zeta + \frac{T}{h} \int_0^1 \phi_m'(\zeta)\phi_n'(\zeta) \, d\zeta$$

(8)

Variable profile of the fibre is obtained specifying different $\rho_j$ and $I_j$ for each segment. For each element the equation of motion can be written as $M_j \ddot{X}_j = -K_j X_j + F^\text{ext}_j$ where $F^\text{ext}_j = (F_{j-1}, \tau_{j-1}, F_j, \tau_j)$. Introducing vectors $Q$ and $F^\text{ext}$ whose elements are displacements, slopes, forces and torques at each node, the equations of motion for the whole fibre are $MQ + KQ = F^\text{ext}$. $M$ and $K$ are built from $M_j$ and $K_j$ keeping into account the coincidence of the lower and the upper end of two contiguous elements.

Some typical fibre diameter profiles are shown in figure 4 where a gaussian neck shape is considered. When a shift $X(L) = \delta$ is imposed, the bending length can be fitted from the bent fibre shape obtained by the FEM (figure 5). Forces and the torques predicted by the simulation for various neck profiles are in good agreement with the results of the 3-segment model. The relative error is of the order of $10^{-5}$ for forces and $10^{-4}$ for torques for any typical value of $\sigma$ (figure 6).

![Figure 3. The mesh of the fibre. The element $j$ is limited by the nodes $j$ and $j-1$.](image)

![Figure 4. Diameter profiles for gaussian necks with different width $\sigma$.](image)

![Figure 5. Bending of the firsts mm's of the fibre. The bending length increases with $\sigma$.](image)
Neck shape gaussian ($\sigma [m]$)

| Force at the end of the fibre [N] | FEM model | 3-segment model fitted $\lambda$ from the mode shape | 3-segment model calculated $\lambda$ for rod’s diameter | 3-segment model calculated $\lambda$ for fibre’s diameter | 1-segment model (simple pendulum) |
|---------------------------------|-----------|-----------------------------------------------------|-----------------------------------------------------|-----------------------------------------------------|----------------------------------|
| | -0.146 | -0.145 | -0.144 | -0.143 | -0.142 | -0.141 | -0.14 |

Figure 6. The line shows the prediction of the FEM for the force, for a displacement $X(L) = \delta$. Crosses are the results of a 3-segment model with $\lambda$ length that is fitted from the bent shape or calculated from equation (3) for rod’s diameter (plus) or fibre diameter (triangle). Diamond dots are the results for a simple pendulum.

5. Experimental check of the model for real fibres

A good agreement between the 3-segment model and the FEM is shown in figure 7. Such comparison is made after the bending length of the fibre has been worked out through a fitting analysis of the bent fibre as it comes from the FEM model. If $\lambda$ value is computed from equation (3) with rod or fibre’s diameter, the comparison is worse. For a real fibre, it is not possible to calculate analytically the bending length, hence a direct measurement is needed.

The so called $\lambda$ machine is used to determine the positions of the $\lambda$ points along the fibre[1]. The working principle is to rotate slightly and slowly the upper part of a loaded fibre and to detect the horizontal displacement of the lower part. When the rotation axis is adjusted to the bending point, the load does not shift horizontally. In figure 7, the working principle of the lambda machine is shown. The machine was realized at INFN Firenze and it is operational in EGO (European Gravitational Observatory). A typical fused silica suspension fibre was clamped and loaded as in figure 8. A computation of the proper angular frequencies with the 3-segment model gives:

$$\omega^2_{\pm} = \frac{9}{2LI_c} \left( Ml^2 + I_c + MlL \pm \sqrt{(Ml^2 + I_c + MlL)^2 - 4MI_cI_l} \right)$$

(9)

where $L$ is the distance between the bending points, $d$ the distance between the end fibre and the load’s center of mass, $M$ and $I_c$ are the mass and the momentum of inertia of the load, $\lambda_t$ and $\lambda_b$ the upper and lower bending lengths and $l = \lambda_b + d$. The load mass is designed to allow the variation of the position of the center of mass ($d$) with a threaded rod and the momentum of inertia ($I_c$) adding extra-loads (figure 9). Measurements of resonant frequencies (figure 10) are consistent with the expected values for each oscillation mode, even though a more complete characterization would require heavier extra-loads, to appreciate the effects of $I_c$ variations.
Figure 8. The scheme of a 3-segment modelized fibre, clamped at one end and loaded at the other with a mass $M$ and a momentum of inertia $I_c$.

Figure 9. The load mass. A threaded rod gives the possibility to shift the center of mass position and $I_c$ can be modified adding some extra loads.

Figure 10. Experimental results (triangles and circles) are consistent with the expected lines computed by equation (9) for $\omega_+$ (blue) and $\omega_-$ (red). Three $\omega_+$ lines correspond to different values of $I_c$. The bending point shift in x axis is the decrement, from a reference value, of the parameter $d$ cited in the text.

Figure 11. Fibre thermal noise predictions. Circles and squares refers to cylindrical and optimized fibres, respectively. Results for gaussian neck fibres are shown as triangles and diamonds. Each fibre is loaded with a quarter of the AdVirgo mirror mass (40 kg) and it has distance of 700 mm between the bending points.

6. An estimation of the thermal noise of fibres with necks.

Because most of the elastic energy is stored in the neck regions, the necks determine the dynamical behavior of a suspension fibre[3][9]. Hence the mechanical impedance strongly depends on the neck profile rather than on the middle part of the fibre. This means that the thermal noise contribution of a fibre with neck is different from that of a perfectly cylindrical fibre with stepped transition to larger diameter. To estimate thermal noise up to some hundreds of Hz, the FEM described in section 4.1 can be used. To compute the mechanical impedance, an oscillating force is applied to the mirror and thereby to one fibre’s end. The mirror is suspended by four fibres hence the angular displacement is suppressed, this means that the tilts $X^I(0)$ and $X^I(L)$ are zero and the mirror only shifts horizontally. For a harmonic excitation, the amplitude of the shift results proportional to the amplitude of the force. The ratio between
the time derivative of the shift (the speed) and the force, multiplied by the four suspension fibres, is the admittance of the system. As usual, the dissipation is evaluated adding the loss angle as an imaginary part to the Young modulus. The total loss angle is the sum of bulk, surface and thermoelastic contributions. The bulk loss angle is a property of the material, while surface and thermoelastic losses depend on the geometry, in this case on the diameter at each position along the fiber. Thus, the losses are computed separately for each element of the model mesh, then the total sum is weighted with the bending energy derived from the FEM analysis. Finally, thermal noise can be calculated from the resulting complex mechanical admittance, using the fluctuation-dissipation theorem[18]. Thermal noise contribution is plotted in figure 11, for a fibre with a diameter of 400 µm and gaussian neck’s profile, loaded with the AdVirgo mirrors[12]. The figure also shows thermal noise analytically computed for a cylindrical fibre (with a diameter of 400 µm) and for an optimized fibre (named tapered or dumbbell-shaped[19]) designed to minimize thermal noise (two end sections, of diameter 800 µm, and a middle part of 400 µm). All fibres present a distance between the bending points of 700 mm. For neck profiles similar to the real ones, thermal noise turns out to be smaller than for cylindrical fibre and not so far from the values given by optimized fibre, less than a factor two above.

7. Conclusions
In future detectors, test masses will be suspended with fused silica fibres. The statics and the low frequencies dynamical behavior of the fibres can be simulated with a simple 3-segment model. The model validity was analytically proved for cylindrical fibre’s profile. A finite element modelization and experimental results lead to extend the validity of the 3-segment model for real fibre that presents necks at the ends. Moreover, for real fibres with typical diameter profiles, the FEM predicts a lower thermal noise with respect to perfectly cylindrical ones.

References
[1] Lorenzini M 2007 Suspension thermal noise issues for advanced GW interferometric detectors PhD thesis, Università di Firenze, available at http://www.infn.it/thesis/PDF/getfile.php?filename=2460-Lorenzini-dottorato.pdf
[2] Cagnoli G, Hough J, DeBra D, Fejer M M, Gustafson E, Rowan S and Mitrofanov V 2000 Phys. Lett. A 272 39–45
[3] Landau L D and Lifshitz E M 1970 Theory of Elasticity 2nd Edition (Oxford: Pergamon Press)
[4] Braginsky V B, Mitrofanov V P and Tokmakov K V 1995 Phys. Dokl. 40 564
[5] Rowan S, Hutchins R, McLaren A, Robertson N A, Twyford S M and Hough J 1997 Phys. Lett. A 227 153
[6] Gretarsson A M and Harry G M 1999 Rev. Sci. Instrum. 70 4081
[7] Wilkie B et al 2004 Class. Quantum Grav. 21 S417
[8] Lorenzini M et al 2009 The monolithic suspension for the interferometer Virgo. Proc. 8th Amaldi Conf. on Gravitational Waves (New York) in preparation
[9] Piergiovanni F, Punturo M and Puppo P 2009 The thermal noise of the Virgo+ and Virgo Advanced Last Stage Suspension (The PPP effect) Virgo Document VIR-015A-09
[10] Robertson N et al 2006 Advanced LIGO Suspension System Conceptual Design LIGO Document T010103-05
[11] Barton M et al 2008 Proposal for baseline change from ribbons to fibres in AdvLIGO test mass suspension monolithic stage LIGO Document T080091-00
[12] The Virgo Collaboration 2009 Advanced Virgo Baseline Design Virgo Document VIR-027A-09
[13] Cagnoli G et al 2006 J. Phys.: Conf. Series 32 386–392
[14] Heptonstall A, Martin I, Cumming A, Cantley C A, Cagnoli G, Jones R and Crooks D R M 2005 Production and characterization of synthetic fused silica ribbons for Advanced LIGO suspensions LIGO Document T050206-00
[15] Cagnoli G and Willems P A 2002 Phys. Rev. B 65 174111
[16] Meirovitch L 1997 Principles and techniques of vibrations (Upper Saddle River, New Jersey: Prentice Hall)
[17] Viceré A 1999 Proc. Int. Summer School on Experimental physics of gravitational waves (Urbino) (Singapore: World Scientific) p 349
[18] Callen H B and T A Welton 1951 Phys. Rev. 83 35
[19] Willems P 2002 Phys. Lett. A 300 162–168