Lax representations with non-removable parameter and exotic cohomology of symmetry algebras of PDEs

Oleg I. Morozov

Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, Cracow 30-059, Poland;

Institute of Control Sciences of Russian Academy of Sciences, Profsoyuznaya 65, Moscow 117997, Russia;

e-mail: morozov@agh.edu.pl

AMS classification scheme numbers: 35A30, 58J70, 35A27, 17B80

Abstract. This paper develops the technique of constructing Lax representations for PDEs via non-central extensions of their contact symmetry algebras. We show that the method is applicable to the Lax representations with non-removable spectral parameters.
1. Introduction

Lax representations, also known as zero-curvature representations, Wahlquist–Estabrook prolongation structures, inverse scattering transformations, or differential coverings [21, 22], are a key feature of integrable partial differential equations (PDEs) and a starting setting for a number of techniques of studying them such as Bäcklund transformations, Darboux transformations, recursion operators, nonlocal symmetries, and nonlocal conservation laws. Lax representations with non-removable (spectral) parameter are of special interest in the theory of integrable PDEs, see, e.g., [1, 12, 13, 41]. The challenging unsolved problem in this theory is to find conditions that are formulated in inherent terms of a PDE under study and ensure existence of a Lax representation for the PDE. Recently, an approach to this problem has been proposed in [34, 35], where it was shown that for some PDEs their Lax representations can be inferred from the second exotic cohomology of the contact symmetry algebras of the PDEs.

The present paper provides an important supplement to the technique of [34, 35]. Namely, we show that Lax representations with a non-removable parameter arise naturally from non-central extensions of the symmetry algebras generated by nontrivial second exotic cohomology groups. We consider here two equations: the hyper-CR Einstein–Weyl equation [24, 29, 39, 15]

\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy} \]  

(1)

with the Lax representation

\[
\begin{align*}
vt &= (\lambda^2 + \lambda u_x - u_y) vx, \\
v_y &= -(u_x + \lambda) vx,
\end{align*}
\]

and the four-dimensional equation

\[ u_{zz} = u_{tx} + u_x u_{yz} - u_z u_{xy} \]

(3)

which was introduced in [4]. The 3-dimensional reduction of (3) defined by substitution for \(u_t = 0\) produces the universal hierarchy equation [26, 27]

\[ u_{zz} = u_x u_{yz} - u_z u_{xy}, \]

therefore we refer equation (3) as the four-dimensional universal hierarchy equation. The Lax representation

\[
\begin{align*}
v_t &= \lambda^2 v_x + (\lambda u_x + u_z) v_y, \\
v_z &= \lambda v_x + u_x v_y
\end{align*}
\]

(4)

for (3) was found in [40].

The parameters \(\lambda\) in (2) and (4) are non-removable. This assertion can be proven by the method of [22] Sections 3.2, 3.6], [18, 17, 28, 9, 10, 11], see Remarks 1 and 3 below.

The following structure distinguishes the contact symmetry algebras for both equations: they are semi-direct products \(\mathfrak{s}_\infty \rtimes \mathfrak{s}_c\) of an (invariantly defined) infinite-dimensional ideal \(\mathfrak{s}_\infty\) and a non-Abelian finite-dimensional Lie algebra \(\mathfrak{s}_c\). The second
exotic cohomology groups of the finite-dimensional subalgebras \( \mathfrak{s}_\infty \) appear to be nontrivial for both equations, and the corresponding nontrivial 2-cocycles produce non-central extensions of their symmetry algebras \( \mathfrak{s}_\infty \times \mathfrak{s}_\infty \). We show that certain linear combination of the Maurer–Cartan forms of the extension of the symmetry algebra defines the Lax representation \((2)\) for equation \((1)\). For equation \((3)\) we apply the extension procedure twice and show that the Maurer–Cartan forms of the second extension produce the Lax representation \((4)\).

2. Preliminaries

All considerations in this paper are local. All functions are assumed to be real-analytic.

2.1. Symmetries and differential coverings

The presentation in this subsection closely follows [19, 20], see also [21, 22, 42]. Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \), be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \( (x^i, u^\alpha, u^\alpha_i) \), where \( I = (i_1, \ldots, i_n) \) are multi-indices, and for every local section \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m \) of \( \pi \) the corresponding infinite jet \( j_\infty(f) \) is a section \( j_\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi) \) such that \( u^\alpha_i(j_\infty(f)) = \frac{\partial^{|I|} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\cdots+i_n} f^\alpha}{(\partial x^1)^{i_1} \cdots (\partial x^n)^{i_n}} \). We put \( u^\alpha_i = u^{\alpha}_{(i_0, \ldots, 0)} \). Also, we will simplify notation in the following way, e.g., in the case of \( n = 4, m = 1 \): we denote \( x^1 = t, x^2 = x, x^3 = y, x^4 = z \) and \( u^1_{(i,j,k,l)} = u_{t \cdots x \cdots z} \) with \( i \) times \( t \), \( j \) times \( x \), \( k \) times \( y \), and \( l \) times \( z \).

The vector fields

\[
D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{#I \geq 0} \sum_{\alpha=1}^m u^\alpha_{I+1,k} \frac{\partial}{\partial u^\alpha_i}, \quad k \in \{1, \ldots, n\},
\]

\((i_1, \ldots, i_k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n)\), are called total derivatives. They commute everywhere on \( J^\infty(\pi) \): \([D_{x^i}, D_{x^j}] = 0\).

The evolutionary vector field associated to an arbitrary vector-valued smooth function \( \varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m \) is the vector field

\[
E_{\varphi} = \sum_{#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u^\alpha_i}
\]

with \( D_I = D_{(i_1, \ldots, i_n)} = D_{x^{i_1}} \circ \cdots \circ D_{x^{i_n}} \).

A system of PDEs \( F_r(x^i, u^\alpha_i) = 0 \) of the order \( s \geq 1 \) with \#I \leq s, r \in \{1, \ldots, R\} \) for some \( R \geq 1 \), defines the submanifold \( \mathcal{E} = \{(x^i, u^\alpha_i) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^\alpha_i)) = 0, \#K \geq 0\} \) in \( J^\infty(\pi) \).

A function \( \varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m \) is called a (generator of an infinitesimal) symmetry of equation \( \mathcal{E} \) when \( E_{\varphi}(F) = 0 \) on \( \mathcal{E} \). The symmetry \( \varphi \) is a solution to the defining system

\[
\ell_{\mathcal{E}}(\varphi) = 0,
\]

\((5)\).
where $\ell_E = \ell_F|_E$ with the matrix differential operator

$$\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_I^\alpha} D_I \right).$$

The symmetry algebra $\mathfrak{sym}(\mathcal{E})$ of equation $\mathcal{E}$ is the linear space of solutions to Equation (5) endowed with the structure of a Lie algebra over $\mathbb{R}$ by the Jacobi bracket $\{ \varphi, \psi \} = \mathbf{E}_{\varphi}(\psi) - \mathbf{E}_{\psi}(\varphi)$. The algebra of contact symmetries $\mathfrak{sym}_0(\mathcal{E})$ is the Lie subalgebra of $\mathfrak{sym}(\mathcal{E})$ defined as $\mathfrak{sym}(\mathcal{E}) \cap J^1(\pi)$.

Consider $\mathcal{W} = \mathbb{R}^\infty$ with coordinates $w^s$, $s \in \mathbb{N} \cup \{0\}$. Locally, an (infinite-dimensional) differential covering of $\mathcal{E}$ is a trivial bundle $\tau: J^\infty(\pi) \times \mathcal{W} \rightarrow J^\infty(\pi)$ equipped with extended total derivatives

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^\infty T_k^s(x^i, u_I^\alpha, w^j) \frac{\partial}{\partial w^s}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \neq j$ whenever $(x^i, u_I^\alpha) \in \mathcal{E}$. Define the partial derivatives of $w^s$ by $w^s_{x^k} = \tilde{D}_{x^k}(w^s)$. This yields the system of covering equations

$$w^s_{x^k} = T_k^s(x^i, u_I^\alpha, w^j),$$

which is compatible whenever $(x^i, u_I^\alpha) \in \mathcal{E}$.

Dually, the covering is defined by the Wahlquist–Estabrook forms

$$dw^s - \sum_{k=1}^m T_k^s(x^i, u_I^\alpha, w^j) \, dx^k$$

as follows: when $w^s$ and $u^\alpha$ are considered to be functions of $x^1$, ..., $x^n$, forms (7) are equal to zero whenever system (6) holds.

2.2. Exotic cohomology of Lie algebras

For a Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, its representation $\rho: \mathfrak{g} \rightarrow \text{End}(V)$, and $k \geq 1$ let $C^k(\mathfrak{g}, V) = \text{Hom}(\Lambda^k(\mathfrak{g}), V)$ be the space of all $k$–linear skew-symmetric mappings from $\mathfrak{g}$ to $V$. Then the Chevalley–Eilenberg differential complex

$$V = C^0(\mathfrak{g}, V) \xrightarrow{d} C^1(\mathfrak{g}, V) \xrightarrow{d} \ldots \xrightarrow{d} C^{k}(\mathfrak{g}, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}, V) \xrightarrow{d} \ldots$$

is generated by the differential $d: \theta \mapsto d\theta$ such that

$$d\theta(X_1, ..., X_{k+1}) = \sum_{q=1}^{k+1} (-1)^{q+1} \rho(X_q) (\theta(X_1, ..., \hat{X}_q, ..., X_{k+1})) + \sum_{1 \leq p < q \leq k+1} (-1)^{p+q} \theta([X_p, X_q], X_1, ..., \hat{X}_p, ..., \hat{X}_q, ..., X_{k+1}).$$

The cohomology groups of the complex $(C^*(\mathfrak{g}, V), d)$ are referred to as the cohomology groups of the Lie algebra $\mathfrak{g}$ with coefficients in the representation $\rho$. For the trivial representation $\rho_0: \mathfrak{g} \rightarrow \mathbb{R}$, $\rho_0: X \mapsto 0$, the cohomology groups are denoted by $H^*(\mathfrak{g})$. 
Consider a Lie algebra \( \mathfrak{g} \) over \( \mathbb{R} \) with non-trivial first cohomology group \( H^1(\mathfrak{g}) \) and take a closed 1-form \( \alpha \) on \( \mathfrak{g} \) such that \( [\alpha] \neq 0 \). Then for any \( c \in \mathbb{R} \) define new differential \( d_{c\alpha}: C^k(\mathfrak{g}, \mathbb{R}) \to C^{k+1}(\mathfrak{g}, \mathbb{R}) \) by the formula
\[
d_{c\alpha} = d\theta - c\alpha \wedge \theta.
\]
From \( d\alpha = 0 \) it follows that \( d^2_{c\alpha} = 0 \). The cohomology groups of the complex
\[
C^1(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} \ldots \xrightarrow{d_{c\alpha}} C^k(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} C^{k+1}(\mathfrak{g}, \mathbb{R}) \xrightarrow{d_{c\alpha}} \ldots
\]
are referred to as the exotic cohomology groups \([36, 37]\) of \( \mathfrak{g} \) and denoted by \( H^*_\alpha(\mathfrak{g}) \).

3. Hyper-CR Einstein–Weyl equation

3.1. Contact symmetries

Direct computations\[†\] show that the Lie algebra of the contact symmetries \( \mathfrak{sym}_0(\mathcal{E}_1) \) of the hyper-CR Einstein–Weyl equation \( \mathcal{E}_1 \) is generated by the functions
\[
\begin{align*}
\phi_0(A) &= -A u_t - \left( x A' + \frac{1}{2} y^2 A'' \right) u_x - y A' u_y + u A' + x y A'' + \frac{1}{6} y^3 A''', \\
\phi_1(A) &= -y A' u_x - A u_y + x A' + \frac{1}{2} y^2 A'', \\
\phi_2(A) &= -A u_x + y A', \\
\phi_3(A) &= A, \\
\psi_0 &= -2 x u_x - y u_y + 3 u, \\
\psi_1 &= -y u_x + 2 x
\end{align*}
\]
where \( A = A(t) \) and \( B = B(t) \) below are arbitrary functions of \( t \). The commutators of the generators are given by equations
\[
\begin{align*}
\{\phi_i(A), \phi_j(B)\} &= \phi_{i+j}(A B' - B A'), \\
\{\psi_i, \phi_k(A)\} &= -k \phi_{k+i}(A), \\
\{\psi_0, \psi_1\} &= -\psi_1,
\end{align*}
\]
(8)
where \( \phi_k(A) = 0 \) for \( k > 3 \). From equations (8) it follows that the contact symmetry algebra of equation (11) is the semi-direct product \( \mathfrak{sym}_0(\mathcal{E}_1) = \mathfrak{b}_\infty \rtimes \mathfrak{b}_o \) of the two-dimensional non-Abelian Lie algebra \( \mathfrak{b}_o = \langle \psi_0, \psi_1 \rangle \) and the infinite-dimensional ideal \( \mathfrak{b}_\infty = \mathcal{D}(\mathfrak{sym}_0(\mathcal{E}_1)) = \langle \phi_k(A) \mid 0 \leq k \leq 3 \rangle \), which, in its turn, is isomorphic to the tensor product \( \mathbb{R}_3[h] \otimes \mathfrak{w} \) of the (commutative associative) algebra of truncated polynomials \( \mathbb{R}_3[h] = \mathbb{R}[h]/(h^4 = 0) \) and the Lie algebra \( \mathfrak{w} = \langle t^n \partial_t \mid n \in \mathbb{Z}_+ \rangle \).

**Remark 1.** The spectral parameter \( \lambda \) in the covering (2) is non-removable, that is, it cannot be eliminated from (2) by a transformation of the covering that is identical on the
\[†\] We carried out computations of generators of contact symmetries and their commutator tables in the Jets software [2].
\[‡\] Here and below we use notation \( \mathcal{D}(\mathfrak{g}) = \mathcal{D}^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \), \( \mathcal{D}^{k+1}(\mathfrak{g}) = [\mathcal{D}^k(\mathfrak{g}), \mathcal{D}^k(\mathfrak{g})] \) for the derived series of a Lie algebra \( \mathfrak{g} \).
base equation $E_1$. In accordance with \cite{22, 18, 17}, to prove this claim it suffices to note that it is impossible to lift the symmetry $\psi_1$ of equation (11) to a symmetry of system (2). Then for the vector field $V(\psi_1) = y \partial_x + 2x \partial_u$ associated to the generator $\psi_1$ and the Wahlquist–Estabrook form $\omega = dv + u_y v_x dt - v_x dx + u_x v_x dy$ that defines covering (2) with $\lambda = 0$ we have $e^{-\lambda V(\psi_1)} \omega = dv - (\lambda^2 + \lambda u_x - u_y) v_x dt - v_x dx + (u_x + \lambda) v_x dy$. This is the Wahlquist–Estabrook form of covering (2) with $\lambda \in \mathbb{R}$.

Consider the basis of $\text{sym}_0(E_1)$ given by generators $\psi_0$, $\psi_1$, and $\phi_k(t^n)$ with $k \in \{0, \ldots, 3\}, n \in \mathbb{Z}_+$. Define the Maurer–Cartan forms $\alpha_0, \alpha_1, \theta_{k,n}$ for $\text{sym}_0(E_1)$ as the dual forms to this basis: $\alpha_i(\psi_j) = \delta_{ij}$, $\alpha_i(\phi_k(t^n)) = 0$, $\theta_{k,n}(\psi_i) = 0$, $\theta_{k,n}(\phi_l(t^n)) = \delta_{kl} \delta_{nm}$.

Put $\Theta = \sum_{k=0}^3 \sum_{m=0}^\infty \frac{h_m^0}{m!} h^2_k \theta_{k,m}$, where $h_1$ and $h_2$ are (formal) parameters such that $dh_i = 0$, and denote $\nabla_i = \partial_{h_i}$. Then the commutator table \cite{8} gives the Maurer–Cartan structure equations

$$\begin{align*}
\left\{
\begin{array}{l}
\frac{d\alpha_0}{dt} = 0, \\
\frac{d\alpha_1}{dt} = \alpha_0 \wedge \alpha_1, \\
\frac{d\Theta}{dt} = \nabla_1(\Theta) \wedge \Theta + (h_2 \alpha_0 + h_2^2 \alpha_1) \wedge \nabla_2(\Theta)
\end{array}\right.
\tag{9}
\end{align*}$$

of $\text{sym}_0(E_1)$.

### 3.2. Second exotic cohomology group and non-central extension

From the structure equations \cite{3} it follows that $H^1(\text{sym}_0(E_1)) = \mathbb{R}\alpha_0$ and

$$H^2_{\text{conn}}(b_\circ) = \left\{ \begin{array}{ll}
\langle [\alpha_0 \wedge \alpha_1] \rangle, & c = 1, \\
\{ [0] \}, & c \neq 1.
\end{array} \right. $$

Moreover, we have $H^2_{\alpha_0}(b_\circ) \subseteq H^2_{\text{conn}}(\text{sym}_0(E_1))$. Hence the nontrivial 2-cocycle $\alpha_0 \wedge \alpha_1$ of the differential $d_{\alpha_0}$ defines a non-central extension $\tilde{b}_\circ$ of the Lie algebra $b_\circ$ and thus a non-central extension $b_\circ \rtimes \tilde{b}_\circ$ of the Lie algebra $\text{sym}_0(E_1)$. The additional Maurer–Cartan form $\sigma$ for the extended Lie algebra is a solution to $d_{\alpha_0} \sigma = \alpha_0 \wedge \alpha_1$, that is, to equation

$$
\frac{d\sigma}{dt} = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1. \tag{10}
$$

This equation is automatically compatible with the structure equations \cite{3} of the Lie algebra $\text{sym}_0(E_1)$.

### 3.3. Maurer–Cartan forms and Lax representation

For the purposes of the present paper we need explicit expressions for the Maurer–Cartan forms $\alpha_i, \theta_{k,0}$, and $\sigma$. We can compute them via two approaches. The first one is to integrate equations \cite{3}, \cite{11} step by step. Each integration gives certain number of new coordinates (the ‘integration constants’) to express the new form, while it is not clear how these coordinates relate to the coordinates of $E_1$. For example, integrating the first two equations from system \cite{3} and then equation \cite{11} we obtain

$$\begin{align*}
\alpha_0 &= \frac{dq}{q}, & \alpha_1 &= q \, ds, & \sigma &= q \left( dv + \ln q \, ds \right),
\end{align*}$$
where $q$, $s$, and $v$ are free parameters and $q > 0$ (we put $\alpha_0 = dq/q$ instead of the natural choice $\alpha_0 = dq$ to simplify the further computations). The second approach to computing the Maurer–Cartan forms is to use Cartan’s method of equivalence, see details and examples of applying the method to symmetry pseudo-groups to PDEs in [30, 31, 32]. In case of the structure equations (9) this technique shows that

(i) $\theta_{0,0}$ is a multiple of $dt$, $\theta_{1,0}$ is a linear combination of $dy$, $dt$, $\theta_{2,0}$ is a linear combination of $dx$, $dy$, $dt$;

(ii) $\theta_{3,0}$ is a multiple of the contact form $du - u_t dt - u_x dx - u_y dy$.

Using (i) we have $\theta_{0,0} = a_0 dt$, $\theta_{1,0} = a_0 q (dy + a_1 dt)$, $\theta_{2,0} = a_0 q^2 (dx + (a_1 + s) dy + a_2 dt)$, with new parameters $a_0 \neq 0$, $a_1$, $a_2$, while (ii) then gives $a_1 = -u_x - 2s$, $a_2 = -u_y + s u_x + s^2$, and $\theta_{3,0} = a_0 q^3 (du - u_t dt - u_x dx - u_y dy)$.

Consider the linear combination

$$\sigma - \theta_{2,0} = q \left( dv + \ln q ds - a_0 q (dx - (s + u_x) dy + (s^2 + s u_x - u_y) dt) \right)$$

and assume that $u$ and $v$ are functions of $t$, $x$, $y$. Then $\sigma - \theta_{2,0} = 0$ implies $q = \exp(-v_s)$, $a_0 = v_x \exp(v_s)$. After this change of notation we obtain the Wahlquist–Estabrook form

$$\sigma - \theta_{2,0} = e^{-v_s} \left( dv - v_x ds - v_x dx + (s + u_x) dy - (s^2 + s u_x - u_y) dt \right)$$

of the covering

$$\begin{cases}
  v_t &= (s^2 + s u_x - u_y) v_x, \\
  v_y &= -(u_x + s) v_x.
\end{cases}$$

This system differs from (2) by notation.

**Remark 2.** The non-removable parameter $s$ in the above system has appeared during computation of form $\alpha_1$, which is dual to the unlietable symmetry $\psi_1$, cf. Remark 1.

### 4. The four-dimensional universal hierarchy equation

#### 4.1. Contact symmetries

The generators of the Lie algebra $\mathfrak{sym}_0(\mathcal{E}_2)$ of the contact symmetries of equation (3) are

$$\begin{align*}
  \phi_0(A) &= -A u_y + A_y u - A_t z, \\
  \phi_1(A) &= A, \\
  \psi_1 &= -t u_t + x u_x - u, \\
  \psi_2 &= -u_t, \\
  \psi_3 &= -2 x u_x - z u_z + u, \\
  \psi_4 &= -\frac{1}{2} z u_x - t u_z, \\
  \psi_5 &= -u_z, \\
  \psi_6 &= -u_x.
\end{align*}$$
Remark 3. The spectral parameter \( \lambda \) in the covering (\ref{4}) is non-removable. Indeed, the symmetry \( \psi_4 \) of equation (\ref{3}) is unliptable to a symmetry of the covering (\ref{4}). For the associated vector field \( V(\psi_4) = \frac{1}{z} z \partial_z + t \partial_x \) the action of \( \exp(2\lambda V(\psi_4)) \) to the Wahlquist–Estabrook form \( \omega = dv - u_x v_y dt - v_x dx - v_y dy - u_x v_y dz \) that defines covering (\ref{4}) with \( \lambda = 0 \) gives \( e^{2\lambda V(\psi_4)} \omega = dv - (\lambda^2 v_x + (\lambda u_x + u_x) v_x) dt - v_x dx - v_y dy - (\lambda v_x + u_x v_y) dz \). This is the Wahlquist–Estabrook form of covering (\ref{4}) with \( \lambda \in \mathbb{R} \).

Define the Maurer–Cartan forms \( \beta_i, \theta_{k,m,n}, i \in \{1,\ldots,6\}, k \in \{0,1\}, m,n \in \mathbb{Z}_+ \), of the Lie algebra of \( \mathfrak{sym}(\mathcal{E}_2) \) as dual forms to its basis \( \psi_i, \phi_k(t^m y^n) \), that is, put \( \beta_i(\psi_j) = \delta_{ij}, \beta_i(\phi_k(t^m y^n)) = 0, \theta_{k,m,n}(\psi_i) = 0, \theta_{k,m,n}(\phi_k(t^m y^n)) = \delta_{kp}\delta_{mq}\delta_{nr} \). Denote \( \Theta_k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{h_1^m h_2^n}{m! n!} \theta_{k,m,n} \) for the formal parameters \( h_1, h_2 \) such that \( dh_1 = 0 \). Then the Maurer-Cartan structure equations of \( \mathfrak{sym}(\mathcal{E}_2) \) read

\[
\begin{align*}
d\beta_1 &= 0, \\
d\beta_2 &= \beta_1 \wedge \beta_2, \\
d\beta_3 &= 0, \\
d\beta_4 &= (\beta_3 - \beta_1) \wedge \beta_4, \\
d\beta_5 &= \beta_3 \wedge \beta_5 - \beta_2 \wedge \beta_4,
\end{align*}
\]
Lax representations and exotic cohomology of symmetry algebras

4.2. Non-central extensions, Maurer–Cartan forms and Lax representation

From the structure equations (11) – (18) it follows that $H^1(\text{sym}(\mathcal{E}_2)) = \langle \beta_1, \beta_3 \rangle$ and

$$H^2_{c_1, c_2, c_3}(\mathfrak{c}_o) = \begin{cases} 
\langle [\beta_2 \wedge \beta_3] \rangle, & c_1 = 1, 
\langle [\beta_1 \wedge [\beta_2, [\beta_2 \wedge \beta_3]]] \rangle, & c_1 = 1, 
\langle [\beta_1 \wedge [\beta_3, [\beta_3, \beta_4]]] \rangle, & c_1 = -1, 
\langle [\beta_4 \wedge \beta_6] \rangle, & c_1 = -2, 
\{ [0] \}, & \text{otherwise.}
\end{cases}$$

Moreover, all the nontrivial exotic 2-cocycles of $\mathfrak{c}_o$ are nontrivial exotic 2-cocycles of $\text{sym}_0(\mathcal{E}_2)$ as well. Therefore they define a non-central extension $\widehat{\mathfrak{c}}_o$ of the Lie algebra $\mathfrak{c}_o$ and hence a non-central extension $\mathfrak{c}_\infty \times \widehat{\mathfrak{c}}_o$ of the Lie algebra $\text{sym}_0(\mathcal{E}_2)$. The additional Maurer–Cartan forms $\beta_7, \ldots, \beta_{12}$ for the extended Lie algebra are solutions to the system

$$d\beta_7 = (\beta_1 + \beta_3) \wedge \beta_7 + \beta_2 \wedge \beta_5, \quad (19)$$
$$d\beta_8 = \beta_1 \wedge \beta_8 + \beta_1 \wedge \beta_2, \quad (20)$$
$$d\beta_9 = \beta_1 \wedge \beta_9 + \beta_2 \wedge \beta_3, \quad (21)$$
$$d\beta_{10} = (\beta_3 - \beta_1) \wedge \beta_{10} + \beta_1 \wedge \beta_4, \quad (22)$$
$$d\beta_{11} = (\beta_3 - \beta_1) \wedge \beta_{11} + \beta_3 \wedge \beta_4, \quad (23)$$
$$d\beta_{12} = (3 \beta_3 - 2 \beta_1) \wedge \beta_{12} + \beta_4 \wedge \beta_6. \quad (24)$$

This system is automatically compatible with equations (11) – (18).

Combining direct integration of the structure equations with Cartan’s method of equivalence we get the explicit expressions for the Maurer–Cartan forms

$$\beta_1 = \frac{dq}{q}, \quad \beta_2 = q \, dt, \quad \beta_3 = \frac{dr}{r}, \quad \beta_4 = \frac{r \, ds}{q}, \quad \beta_5 = r \, (dz + 2s \, dt),$$
$$\beta_6 = \frac{r^2}{q} \, (dx + s \, dz + s^2 \, dt), \quad \theta_{0,0,0} = a_0 \, (dy + a_1 \, dt),$$
$$\theta_{1,0,0} = \frac{a_0 r}{q} \, (du - a_1 \, dz - a_2 \, dy - a_3 \, dt), \quad \beta_{10} = \frac{r}{q} \, (da_4 + \ln q \, ds),$$

$$\cdots.$$

$$\cdots.$$
as well as the requirement for the linear combination
\[
\theta_{1,0,0} - \beta_6 = \frac{aq}{q} (du - \frac{r}{a_0} dx - a_2 dy - \left( a_1 + \frac{rs}{a_0} \right) dz - \left( a_3 + \frac{rs^2}{a_0} \right) dt)
\]
to be a multiple of the contact form \( du - u_t dt - u_x dx - u_y dy - u_z dz \). This gives the change of notation \( r = a_0 u_x, a_1 = u_z - s u_x, a_2 = u_y, a_3 = u_t - s^2 u_x \), which yields
\[
\theta_{1,0,0} - \beta_6 = \frac{a^2 u_x}{q} (du - u_t dt - u_x dx - u_y dy - u_z dz).
\]

Our attempts to find a linear combination of the Maurer–Cartan forms \( \beta_1, \ldots, \beta_{12} \), \( \theta_{k,m,n} \) have not given a Wahlquist–Estabrook form of any covering over equation (3). Therefore we have extended the Lie algebra \( \hat{\mathfrak{c}}_\circ \) with the structure equations (11) – (24) via the same procedure, that is, by finding nontrivial cocycles from \( H^2_{c_1 \beta_1 + c_2 \beta_3}(\hat{\mathfrak{c}}_\circ) \). The additional Maurer–Cartan forms \( \beta_{13}, \ldots, \beta_{30} \) of the resulting 18-dimensional non-central extension of \( \hat{\mathfrak{c}}_\circ \) and of \( \mathfrak{c}_\infty \times \hat{\mathfrak{c}}_\circ \) are solutions to the structure equations
\[
d\beta_{13} = 2 (\beta_3 - \beta_1) \wedge \beta_{13} + \beta_4 \wedge \beta_{10}, \tag{25}
d\beta_{14} = (2 \beta_1 + \beta_3) \wedge \beta_{14} + \beta_2 \wedge \beta_7, \\
d\beta_{15} = 2 \beta_1 \wedge \beta_{15} + \beta_2 \wedge \beta_8, \\
d\beta_{16} = 2 \beta_1 \wedge \beta_{16} + \beta_2 \wedge \beta_9, \\
d\beta_{17} = \beta_1 \wedge \beta_{17} + \beta_1 \wedge \beta_8, \\
d\beta_{18} = \beta_1 \wedge \beta_{18} + \beta_1 \wedge \beta_9 + \beta_3 \wedge \beta_8, \\
d\beta_{19} = \beta_1 \wedge \beta_{19} + \beta_3 \wedge \beta_9, \\
d\beta_{20} = 2 \beta_3 \wedge \beta_{20} + 2 \beta_2 \wedge \beta_6 + \beta_4 \wedge \beta_7, \\
d\beta_{21} = \beta_3 \wedge \beta_{21} + \beta_1 \wedge \beta_5 + \beta_4 \wedge \beta_8, \\
d\beta_{22} = \beta_3 \wedge \beta_{22} + \beta_2 \wedge \beta_{10} + \beta_4 \wedge \beta_8, \\
d\beta_{23} = \beta_3 \wedge \beta_{23} + \beta_2 \wedge \beta_{11} + \beta_4 \wedge \beta_9, \\
d\beta_{24} = \beta_3 \wedge \beta_{24} + \beta_3 \wedge \beta_5 + \beta_4 \wedge \beta_9, \\
d\beta_{25} = (3 \beta_3 - \beta_1) \wedge \beta_{25} + \beta_2 \wedge \beta_{12} - \beta_5 \wedge \beta_6, \\
d\beta_{26} = (3 \beta_3 - \beta_1) \wedge \beta_{26} + \beta_1 \wedge \beta_{11} + \beta_3 \wedge \beta_{10}, \\
d\beta_{27} = (3 \beta_3 - \beta_1) \wedge \beta_{27} + \beta_3 \wedge \beta_{11}, \\
d\beta_{28} = (3 \beta_3 - \beta_1) \wedge \beta_{28} + \beta_1 \wedge \beta_{10}, \\
d\beta_{29} = 2 (\beta_3 - \beta_1) \wedge \beta_{29} + \beta_4 \wedge \beta_{11}, \\
d\beta_{30} = (4 \beta_3 - 3 \beta_1) \wedge \beta_{30} + \beta_4 \wedge \beta_{12}. \]
These equations are compatible with equations (11) – (24). Integration of equation (25) gives $\beta_{13} = a_0^2 u_x^2 q^{-2} (dv - a_4 ds)$. Then the linear combination

$$\beta_{13} - \theta_{0,0,0} - \beta_5 - \beta_6 =$$

$$\frac{a_0^2 u_x^2}{q^2} \left( dv - a_4 ds - q \, dx - \frac{q^2}{a_0 u_x^2} \, dy - \frac{q + a_0 s u_x}{a_0 u_x} \, dz - \frac{q (q u_x + s q u_x + a_0 s^2 u_x^2)}{a_0 u_x^2} \, dt \right)$$

after the change of notation $q = v_x, a_0 = v_x^2 u_x^2 v_y^{-1}, a_4 = v_y$ gives

$$\beta_{13} - \theta_{0,0,0} - \beta_5 - \beta_6 =$$

$$\frac{v_x^2}{v_y^2 u_x^2} \left( dv - v_y ds - v_x dx - v_y dy - (s v_x + u_x v_y) \, dz - (s^2 v_x + (u_z + s u_x) \, dt \right),$$

which is the Wahlquist–Estabrook form of the covering

$$\begin{cases}
   v_t &= s^2 v_x + (s u_x + u_z) v_y, \\
   v_z &= s v_x + u_x v_y.
\end{cases}$$

This system differs from (4) by the notation.

**Remark 4.** The non-removable parameter $s$ in the above system has appeared during computation of form $\beta_4$, which is dual to the unliftable symmetry $\psi_4$, cf. Remark 3.

5. **Conclusion**

In the present paper we have shown that the method of [34, 35] is applicable to Lax representations with non-removable parameters, in particular, the Lax representations for equations (1) and (3) can be derived from the non-central extensions of contact symmetry algebras of these equations. In both examples the symmetry algebras have the specific structure of the semi-direct product of an infinite-dimensional ideal and a non-Abelian finite-dimensional Lie subalgebra. The cohomological properties of the finite-dimensional subalgebras appear to be sufficient to reveal non-central extensions that define the Lax representations.

It is natural to ask whether the method can produce the known as well as new Lax representations for other equations. Also, we expect that the proposed technique will be helpful in describing multi-component integrable generalizations of integrable PDEs, [14, 25, 3, 33, 23]. We intend to address these issues in the further study.

**Acknowledgments**

This work was partially supported by the Faculty of Applied Mathematics of AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education.

I am very grateful to I.S. Krasil’schik for useful discussions. I thank L.V. Bogdanov and P. Zusmanovich for important remarks.
References

[1] M.J. Ablowitz, P.A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press, Cambridge, 1991

[2] H. Baran, M. Marvan. Jets: A software for differential calculus on jet spaces and diffieties. Available on-line at http://jets.math.slu.cz

[3] L.V. Bogdanov. Non-Hamiltonian generalizations of the dispersionless 2DTL hierarchy. J. Phys. A: Math. Theor. 43 (2010), 434008

[4] L.V. Bogdanov, M.V. Pavlov. Linearly degenerate hierarchies of quasiclassical SDYM type. J. Math. Phys. 58 (2017), 093505

[5] Cartan É. Sur la structure des groupes infinités de transformations. In: Cartan É., Œuvres Complètes, Part II, 1953

[6] Cartan É. Les sous-groupes des groupes continus de transformations. In: Cartan É., Œuvres Complètes, Part II, 1953

[7] Cartan É. Les problèmes d’équivalence. In: Cartan É., Œuvres Complètes, Part II, 1953

[8] Cartan É. La structure des groupes infinités. In: Cartan É., Œuvres Complètes, Part II, 1335–1384. Gauthier - Villars, Paris, 1953

[9] J. Cieśliński. Lie symmetries as a tool to isolate integrable symmetries. In: Boiti, M., et al. (eds.) *Nonlinear Evolution Equations and Dynamical Systems*. World Scientific, Singapore, 1992

[10] J. Cieśliński. Non-local symmetries and a working algorithm to isolate integrable equations. J. Phys. A, Math. Gen. 26 (1993), L267–L271

[11] J. Cieśliński. Group interpretation of the spectral parameter in the case of nonhomogeneous, nonlinear Schrödinger system. J. Math. Phys. 34 (1993), 2372–2384

[12] A. Das. *Integrable Models*. World Scientific, Singapore, 1989.

[13] R. Dodd, A. Fordy. The prolongation structures of quasi-polynomial flows. Proc. R. Soc. London A 385 (1983), 389–429

[14] M. Dunajski. Anti-self-dual fourmanifolds with a parallel real spinor. Proc. Roy. Soc. Lond. A 458 (2002), 1205–1222

[15] M. Dunajski. A class of Einstein–Weil spaces associated to an integrable system of hydrodynamic type. J. Geom. Phys. 51 (2004) 126–137

[16] M. Fels, P.J. Olver. Moving coframes. I. A practical algorithm. Acta. Appl. Math. 51 (1998), 161–213

[17] S. Igonin, J. Krasil’shchik J. On one-parametric families of Bäcklund transformations. In: T. Morimoto, H. Sato, K. Yamaguchi (eds.), *Lie Groups, Geometric Structures and Differential Equations — One Hundred Years After Sophus Lie*. Advanced Studies in Pure Mathematics, 37, pp. 99–114. Math. Soc. Japan, Tokyo, 2002

[18] J. Krasil’shchik. On one-parametric families of Bäcklund transformations. The Diffiety Institute Preprint Series. — 2000. — Preprint DIPS–1/2000. Available on-line at diffiety.ac.ru

[19] J. Krasil’shchik, A. Verbovetsky. Geometry of jet spaces and integrable systems// J. Geom. Phys. 61 (2011), 1633–1674

[20] J. Krasil’shchik, A. Verbovetsky, R. Vitolo. A unified approach to computation of integrable structures. Acta Appl. Math. 120 (2012), 199–218

[21] I.S. Krasil’shchik, A.M. Vinogradov. Nonlocal symmetries and the theory of coverings, Acta Appl. Math. 2 (1984), 79–86

[22] I.S. Krasil’shchik, A.M. Vinogradov. Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations. Acta Appl. Math. 15 (1989), 161–209

[23] B.S. Kruglikov, O.I. Morozov. Integrable dispersionless PDEs in 4D, their symmetry pseudogroups and deformations. Lett. Math. Phys. 105 (2015), 1703–1723

[24] G.M. Kuz’mina. On a possibility to reduce a system of two first-order partial differential equations
Lax representations and exotic cohomology of symmetry algebras

to a single equation of the second order. Proc. Moscow State Pedagog. Inst. 271 (1967), 67–76 (in Russian)
[25] S.V. Manakov, P.M. Santini. The Cauchy problem on the plane for the dispersionless Kadomtsev-Petviashvili equation. JETP Lett. 83 (10) (2006), 462–466
[26] L. Martínez Alonso, A.B. Shabat. Energy-dependent potentials revisited: A universal hierarchy of hydrodynamic type. Phys. Lett. A 299 (2002), 359–365
[27] L. Martínez Alonso, A.B. Shabat. Hydrodynamic reductions and solutions of a universal hierarchy. Theor. Math. Phys. 140 (2004), 1073–1085
[28] M. Marvan. On the horizontal gauge cohomology and nonremovability of the spectral parameter. Acta Appl. Math. 72 (2002), 51–65
[29] V.G. Mikhalev. On the Hamiltonian formalism for Kortewegde Vries type hierarchies. Functional Analysis and Its Applications, 26 No 2 (1992), 140–142
[30] O.I. Morozov. Moving coframes and symmetries of differential equations. J. Phys. A 35 (2002), 2965–2977
[31] O.I. Morozov. Contact-equivalence problem for linear hyperbolic equations. J. Math. Sci. 135 (2006), 2680–2694
[32] O.I. Morozov. Contact integrable extensions of symmetry pseudo-groups and coverings of (2+1) dispersionless integrable equations. J. Geom. Phys. 59 (2009), 1461 – 1475
[33] O.I. Morozov. A two-component generalization of the integrable rdDym equation. SIGMA 8 (2012), 051
[34] O.I. Morozov. Deformed cohomologies of symmetry pseudo-groups and coverings of differential equations. J. Geom. Phys. 113 (2017), 215–225
[35] O.I. Morozov. Deformations of infinite-dimensional Lie algebras, exotic cohomology, and integrable nonlinear partial differential equations. J. Geom. Phys. 128 (2018), 20–31
[36] S.P. Novikov. On exotic De-Rham cohomology. Perturbation theory as a spectral sequence. arXiv:math-ph/0201019, 2002
[37] S.P. Novikov. On metric-independent exotic homology. Proc. Steklov Inst. Math. 251 (2005), 206–212
[38] P.J. Olver. Equivalence, Invariants, and Symmetry Cambridge University Press, Cambridge, 1995
[39] M.V. Pavlov. Integrable hydrodynamic chains. J. Math. Phys. 44 (2003) 4134–4156
[40] M.V. Pavlov, N. Stoilov. Three dimensional reductions of four-dimensional quasilinear systems. J. Math. Phys. 58 (2017), 111510
[41] L.A. Takhtadzhyan, L.D. Faddeev. Hamiltonian Methods in the Theory of Solitons Springer, Berlin, 1987
[42] A.M. Vinogradov, I.S. Krasil’shchik (eds.) Symmetries and Conservation Laws for Differential Equations of Mathematical Physics [in Russian], Moscow: Factorial, 2005; English transl. prev. ed.: I.S. Krasil’shchik, A.M. Vinogradov (eds.) Symmetries and Conservation Laws for Differential Equations of Mathematical Physics. Transl. Math. Monogr., 182, Amer. Math. Soc., Providence, RI, 1999