GEODESICS IN NONEXPANDING IMPULSIVE GRAVITATIONAL WAVES WITH $\Lambda$, II

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Abstract. We investigate all geodesics in the entire class of nonexpanding impulsive gravitational waves propagating in an (anti-)de Sitter universe using the distributional metric. We extend the regularization approach of part I (Geodesics in nonexpanding impulsive gravitational waves with $\Lambda$, Part I, Class. Quantum Grav., 33(11):115002, 2016) to a full nonlinear distributional analysis within the geometric theory of generalized functions. We prove global existence and uniqueness of geodesics that cross the impulsive wave and hence geodesic completeness in full generality for this class of low regularity spacetimes. This, in particular, prepares the ground for a mathematically rigorous account on the ‘physical equivalence’ of the continuous with the distributional ‘from’ of the metric.

Keywords: impulsive gravitational waves, geodesic completeness, nonlinear generalized functions, distributional geometry

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1. Introduction

Impulsive gravitational waves are exact radiative spacetimes of general relativity that provide theoretic models of short but violent bursts of gravitational radiation, which are also of significant interest to quantum theories of gravity. Originally introduced by R. Penrose (e.g. [32]) their construction and physical properties have been extensively studied by many authors, for an overview see e.g. [17] Chapter 20, as well as [3, 34]. Apart from their physical significance these models are also interesting from a purely mathematical point of view since they are described by metrics of low regularity. More precisely impulsive gravitational waves have been described by two ‘forms’ of the metric, one (locally Lipschitz-)continuous, the other one even distributional. In particular, the ‘physical equivalence’ of these two descriptions has been established in several families of these models in a formal way, leaving open some quite subtle issues in low regularity Lorentzian geometry.

In this work we are especially interested in nonexpanding impulsive gravitational waves which propagate on a cosmological background of constant curvature, that is on de Sitter (with cosmological constant $\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) universe. These models have come into focus with the pioneering work of Hotta and Tanaka [19], who performed an ultra-relativistic boost to the Schwarzschild–(anti-)de Sitter solution to obtain a nonexpanding spherical impulsive gravitational wave generated by a pair of null monopole particles. Since then many more such solutions have been found, see e.g. the review in [39] Section 2]. The corresponding continuous form of the metric is given by ([33, 36])

\begin{equation}
\begin{align*}
(1) \quad ds^2 = & \frac{2 |dZ + U_+(H_{ZZ}dZ + H_{Z\bar{Z}}d\bar{Z})|^2 - 2 dUdV}{[1 + \frac{1}{6}\Lambda(\bar{Z}Z - UV + U_+G)]^2}.
\end{align*}
\end{equation}

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Here $H(Z, \bar{Z})$ is an arbitrary real-valued function, $G(Z, \bar{Z}) = ZH,Z + \bar{Z}H,\bar{Z} - H$ and $U_+(U) = 0$ for $U \leq 0$ and $U_+(U) = U$ for $U \geq 0$ is the *kink function*. The metric (1) is most easily generated from the conformally flat form of the constant curvature background

$$ds^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V}}{[1 + \frac{1}{6} \Lambda (\eta \bar{\eta} - \mathcal{U} \mathcal{V})]^2},$$

where $\mathcal{U}, \mathcal{V}$ are the usual null and $\eta, \bar{\eta}$ the usual complex spatial coordinates. More precisely applying the transformation

$$U = U, \quad \mathcal{V} = \begin{cases} V & \text{for } \mathcal{U} < 0 \\ V + H + UH,ZH,\bar{Z} & \text{for } \mathcal{U} > 0 \end{cases}, \quad \eta = \begin{cases} Z & \text{for } \mathcal{U} < 0 \\ Z + UH,\bar{Z} & \text{for } \mathcal{U} > 0 \end{cases}$$

to (2) separately for negative and positive values of $\mathcal{U}$ one formally obtains (1). For all details see the introduction to part I, i.e., [42, Section 1]. The corresponding distributional form

$$ds^2 = \frac{2 d\eta d\bar{\eta} - 2 d\mathcal{U} d\mathcal{V} + 2H(\eta, \bar{\eta}) \delta(\mathcal{U}) d\mathcal{U}^2}{[1 + \frac{1}{6} \Lambda (\eta \bar{\eta} - \mathcal{U} \mathcal{V})]^2}$$

is formally derived by writing (3) in the form of a ‘discontinuous coordinate transform’ using the Heaviside function $\Theta$

$$\mathcal{U} = U, \quad \mathcal{V} = V + \Theta H + U_+ H,ZH,\bar{Z}, \quad \eta = Z + U_+ H,\bar{Z}$$

applying it to (1) and retaining all distributional terms. Clearly, a mathematically sound treatment of the transformation (5) is a delicate matter.

A first rigorous result in this realm has been established in [21] in the special case of impulsive pp-waves, i.e., nonexpanding impulsive waves propagating in a Minkowski background, hence $\Lambda = 0$ in the above metrics (1), (4). In particular, nonlinear distributional geometry ([16, Chapter 3]) based on algebras of generalized functions ([8]) has been employed to show the following: The ‘discontinuous coordinate change’ (3) (cf. [32] for the plane wave and [138] for the general pp-wave case) relating the distributional Brinkmann form of the metric, i.e., (1) with $\Lambda = 0$ to the continuous Rosen form, i.e., (1) with $\Lambda = 0$ is the distributional limit of a ‘generalized diffeomorphism’. For details on the latter concept see [11] [12]. This result rests on the nonlinear distributional analysis of the geodesics in the metric (4) with $\Lambda = 0$, providing an existence, uniqueness and completeness result ([15] [20]).

Observe that especially the completeness result is remarkable since it proves that the analytically ‘very singular’ distributional spacetime is nonsingular in view of the standard definition ([31]). More precisely, the metric (4) possesses a distributional coefficient hence lies outside the Geroch-Traschen class of metrics ([15]). Indeed this class is defined by the metric being of Sobolev regularity $W^{1,2}_{loc} \cap L^\infty_{loc}$ and is known to be the most general class which allows to stably define the curvature in distributions (see also [28] [47]). Concerning the use of geodesic completeness as a criterion for a spacetime to be nonsingular observe that the metric (4) is of regularity far below the class, which classically guarantees even the *local existence* of geodesics, which is $C^1$. Still more, it is outside the class of metrics which classically allows for unique local solvability of the geodesic equation ($C^{1,1}$, i.e., the first derivatives of the metric being locally Lipschitz continuous), which is the natural regularity assumption for the singularity theorems whose validity has recently been extended to this class ([26] [27]). From this point of view it becomes quite clear that in this low regularity scenario both the solution concept for the geodesic equation as well as the notion of geodesic completeness has to be
there is no global can give precise bounds on \(\varepsilon\) nately this fact obstructs a straight forward formulation of completeness results: Although we zone to the background region ‘behind’ it, \(U\) provided
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|\gamma| \leq 2 + |\delta| \leq 1.
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The main achievement of part I on the geodesics in \((M, g_0)\) can now be formulated: We consider any geodesic \(\gamma = (U, V, Z_0)\) in the regularized nonexpanding impulsive wave spacetime \((M, g_0)\) starting outside the wave zone \(|U| \leq \varepsilon\) and heading towards it. Before \(\gamma\) hits the wave zone (for the first time), i.e., \(U = -\varepsilon\) it is also a geodesic of the background, called seed geodesic, and we can fix some \((\varepsilon\text{-independent})\) initial data for \(\gamma\) such that its speed is normalized, i.e., \(|\dot{\gamma}| = e = \pm 1, 0\). When the seed geodesic hits the wave zone it continues at least for some small parameter time as a local solution \(\gamma_0\) of the regularized geodesic equations. Now [42, Theorems 3.1 and 3.2] guarantee that there exists \(\varepsilon_0\) such that \(\gamma_0\) passes through the wave zone to the background region ‘behind’ it, \(\varepsilon \leq \varepsilon_0\).

Here \(\varepsilon_0\) depends on the seed \(\gamma\) via its initial data, cf. the discussion in part I, p. 11. Unfortunately this fact obstructs a straight forward formulation of completeness results: Although we can give precise bounds on \(\varepsilon_0\) in terms of the seed geodesics’ data (see [42, Equation (A.14)]) there is no global \(\varepsilon\) such that all geodesics are complete. Consequently none of the spacetimes \((M, g_\varepsilon)\) is geodesically complete (in the usual sense of the definition). Even worse, geodesics in
the background spacetime with $\sigma e > 0$ (spacelike geodesics in de Sitter and timelike geodesics in anti-de Sitter space) are periodic. Consequently geodesics $\gamma_\epsilon$ in the regularized spacetime constructed from such seed geodesics $\gamma$ will cross the wave zone infinitely often and we have to reapply the above results repeatedly with changing initial data. While this allows us to specify an $\epsilon_0$ which will guarantee that $\gamma_\epsilon$ crosses the wave zone $N$ times for any fixed integer $N$, we cannot give a global $\epsilon_0$ for which the geodesic $\gamma_\epsilon$ is complete.

In this paper we remedy these defects by invoking the geometric theory of generalized functions ([16, Chapter 3]) which will allow us to formulate a ‘global’ completeness statement. We will show global existence and uniqueness of geodesics in this framework and, in particular, handle the case of infinitely many crossings thereby establishing completeness for all geodesics in nonexpanding impulsive gravitational waves with $\Lambda$, which is also a completely new result as compared to [42].

Geodesic completeness of (semi-)Riemannian manifolds in the nonlinear distributional setting has been first considered explicitly in [41], where also completeness of a different class of impulsive gravitational waves has been established. This class consists of impulsive versions of models considered in [6, 13, 7, 14] and called (general) plane-fronted waves (PFW) there. They generalize $pp$-waves by allowing for an arbitrary $n$-dimensional Riemannian manifold $N$ as wave surface and consequently may be called $N$-fronted waves with parallel rays (NPWs). Now the geodesics in impulsive NPWs (iNPWs) have been analyzed again using a regularization approach in [40], which was turned into a ‘global’ completeness result (in the sense discussed above) in [41]. Our current approach in some respect parallels the one of [40, 41], however the technical complexity is severely increased, cf. [42, p. 19].

This paper is organized as follows. To keep our presentation self contained we briefly recall the basic elements of nonlinear distributional geometry in Section 2 where we also transfer the five-dimensional formalism to the generalized setting. We prove global existence and uniqueness to the system of geodesic equations in generalized functions in Section 3. Our main result on completeness of nonexpanding impulsive gravitational waves in (anti-)de Sitter universe is given in Section 4. Finally we explicitly relate these complete geodesics to the geodesics of the cosmological background in section 5.

2. Nonlinear distributional geometry

To keep this presentation self-contained, we begin this section with a brief review of generalized semi-Riemannian geometry in the setting of Colombeau’s nonlinear generalized functions. Colombeau algebras of generalized functions ([8]) are differential algebras which contain the vector space of Schwartz distributions and at the same time display maximal consistency with classical analysis. The theory of semi-Riemannian geometry based on the so-called special Colombeau algebra $\mathcal{G}(M)$ has been developed in [22, 23], see also [16, Section 3.2]. The basic idea of the construction to be detailed below is regularization of distributions via nets of smooth functions combined with an elaborate bookkeeping of asymptotic estimates in terms of a regularization parameter.

Let $M$ be a smooth (second countable and Hausdorff) manifold and denote by $\mathcal{E}(M)$ the set of all nets $(u_\epsilon)_{\epsilon \in (0,1]} =: J$ in $C^\infty(M)^I$ depending smoothly on $\epsilon$. Smooth dependence on the parameter renders the theory technically more pleasant but was not assumed in earlier references, cf. the discussion in [4, Section 1]. The algebra of generalized functions on $M$ ([10]) is defined as the quotient $\mathcal{G}(M) := \mathcal{E}_M(M)/\mathcal{N}(M)$ of moderate modulo negligible nets
in $\mathcal{E}(M)$, which are defined via
\[
\mathcal{E}_M(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(M) : \forall K \subseteq M \ \forall P \in \mathcal{P} \ \exists N : \sup_{p \in K} |P u_\varepsilon(p)| = O(\varepsilon^{-N})\},
\]
\[
\mathcal{N}(M) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(M) : \forall K \subseteq M \ \forall m : \sup_{p \in K} |u_\varepsilon(p)| = O(\varepsilon^m)\},
\]
where $\mathcal{P}$ denotes the space of all linear differential operators on $M$ and $K \subseteq M$ means that $K$ is a compact subset of $M$. Elements of $\mathcal{G}(M)$ are denoted by $u = [(u_\varepsilon)_\varepsilon]$ and we call $(u_\varepsilon)_\varepsilon$ a representative of the generalized function $u$. Defining sum and product in $\mathcal{G}(M)$ componentwise (i.e., for $\varepsilon$ fixed) and the Lie derivative with respect to smooth vector fields $\xi \in \mathfrak{X}(M)$ via $L_\xi u := [(L_\xi u_\varepsilon)_\varepsilon]$, $\mathcal{G}(M)$ becomes a fine sheaf of differential algebras.

There exist embeddings $\iota$ of the space of distributions $\mathcal{D}'(M)$ into $\mathcal{G}(M)$ that are sheaf homomorphisms and preserve the product of $C^\infty(M)$-functions. A coarser way of relating generalized functions in $\mathcal{G}(M)$ to distributions is as follows: $u \in \mathcal{G}(M)$ is called associated with $v \in \mathcal{G}(M)$, $u \approx v$, if $u_\varepsilon - v_\varepsilon \to 0$ in $\mathcal{D}'(M)$. Moreover, $u \in \mathcal{D}'(M)$ is called associated with $u$ if $u \approx \iota(u)$.

The ring of constants in $\mathcal{G}(M)$ is the space $\overline{\mathbb{R}} = \mathcal{E}_M/\mathcal{N}$ of generalized numbers, which form the natural space of point values of Colombeau generalized functions. These, in turn, are uniquely characterized by their values on so-called compactly supported generalized points.

More generally the space $\Gamma_G(M, E)$ of generalized sections of the vector bundle $E$ is defined as $\Gamma_G(M, E) = \mathcal{G}(M) \otimes_{C^\infty(M)} \Gamma(M, E) = L_{C^\infty(M)}(\Gamma(M, E^*)).$ It is a fine sheaf of finitely generated and projective $\mathcal{G}$-modules. For generalized tensor fields of rank $r, s$ we use the notation $\mathcal{G}_s^r(M)$, i.e.
\[
\mathcal{G}_s^r(M) \cong L_{\mathcal{G}(M)}(\mathcal{G}_0^1(M)^s, \mathcal{G}_0^1(M)^r; \mathcal{G}(M)).
\]

Observe that this, in particular, allows the insertion of generalized vector fields and one-forms into generalized tensors, which is not possible in the distributional setting (cf. [29, 30]) but essential when dealing with generalized metrics. Here a generalized pseudo-Riemannian metric is a section $g \in \mathcal{G}_1^0(M)$ that is symmetric with determinant $\det g$ invertible in $\mathcal{G}$ (equivalently $|\det(g_\varepsilon)_{ij}| > \varepsilon^m$ for some $m$ on compact sets), and a well-defined index $\nu$ (the index of $g_\varepsilon$ equals $\nu$ for $\varepsilon$ small). By a “globalization Lemma” in ([25, Lemma 2.4, p. 6]) any generalized metric $g$ possesses a representative $(g_\varepsilon)_\varepsilon$ such that each $g_\varepsilon$ is a smooth metric globally on $M$. We call a pair $(M, g)$ consisting of a smooth manifold and a generalized metric a generalized semi-Riemannian manifold.

Based on this definition a convenient framework for non-smooth pseudo-Riemannian geometry has been developed. It in turn enables an analysis of spacetimes of low regularity in general relativity consistently extending the “maximal distributional” setting of Geroch and Traschen ([15]), see [47, 48]. In particular, any generalized metric induces an isomorphism between generalized vector fields and one-forms, and there is a unique Levi-Civita connection $\nabla$ corresponding to $g$.

Finally to discuss geodesics in generalized semi-Riemannian manifolds we have to introduce the space of generalized curves defined on an interval $J$ taking values in the manifold $M$ $\mathcal{G}[J, M]$. It is again a quotient of moderate modulo negligible nets $(\gamma_\varepsilon)_\varepsilon$ of smooth curves, where we call a net moderate (negligible) if $(\psi \circ \gamma_\varepsilon)_\varepsilon$ is moderate (negligible) for all smooth $\psi : M \to \mathbb{R}$. In addition, $(\gamma_\varepsilon)_\varepsilon$ is supposed to be $c$-bounded, which means that $\gamma_\varepsilon(K)$ is contained in a compact set for $\varepsilon$ small and all compact sets $K \subseteq J$. Observe that no distributional counterpart of such a space exists.
The induced covariant derivative of a generalized vector field $\xi$ on a generalized curve $\gamma = [(\gamma_\epsilon)_\epsilon] \in G[J,M]$ is defined componentwise and gives again a generalized vector field $\xi'$ on $\gamma$. In particular, a geodesic in a generalized pseudo-Riemannian manifold is a curve $\gamma \in G[J,M]$ satisfying $\gamma'' = 0$. Equivalently the usual local formula holds, i.e.,

\[
\left( \frac{d^2 \gamma^k_\epsilon}{dt^2} + \sum_{i,j} \Gamma^k_{\epsilon ij} \frac{d \gamma^i_\epsilon}{dt} \frac{d \gamma^j_\epsilon}{dt} \right) = 0,
\]

where $\Gamma^k_{ij} = \left( (\Gamma^k_{\epsilon ij})_\epsilon \right)$ denotes the Christoffel symbols of the generalized metric $g = [(g_\epsilon)_\epsilon]$. Now the following definition is natural.

**Definition 2.1.** ([41, Definition 2.1, p. 240]) Let $g \in G^2(\mathbb{R}, M)$ be a generalized semi-Riemannian metric. Then $(\mathbb{R}, g)$ is said to be geodesically complete if every geodesic $\gamma$ can be defined on $\mathbb{R}$, i.e., every solution of the geodesic equation

\[
\gamma'' = 0,
\]

is in $G[\mathbb{R}, M]$.

Now we set up the five-dimensional framework to deal with impulsive waves propagating in cosmological backgrounds. To begin with we recall that a model delta function is an element $D \in G(\mathbb{R})$ that has as a representative a model delta net $\delta_\epsilon(x) = 1/\epsilon \rho(x/\epsilon)$, where again $\rho \in C^\infty(\mathbb{R})$ with compact support in $[-1,1]$ and $\int_\mathbb{R} \rho = 1$.

Now let $H \in C^\infty(\mathbb{R}^3)$ and $\sigma := \text{sign}(\Lambda) = \pm 1$. Then we define the 5-dimensional generalized impulsive $pp$-wave manifold $(\bar{M} = \mathbb{R}^5, \bar{g})$ via

\[
d\bar{s}^2 = dZ_2^2 + dZ_3^2 + \sigma dZ_4^2 - 2dUdV + H(Z_2, Z_3, Z_4)D(U)dU^2,
\]

where $Z_2, Z_3, Z_4$ are global Cartesian coordinates on $\mathbb{R}^3$ and $U, V$ are global null coordinates. One easily checks that this defines a generalized metric in the sense above. At this point we specify the hypersurface $M$ in $(\bar{M}, \bar{g})$, which will be our main playground

\[
M := \{(U, V, Z_2, Z_3, Z_4) \in \bar{M} : F(U, V, Z_2, Z_3, Z_4) = 0\}, \quad \text{where}
\]

\[
F(U, V, Z_2, Z_3, Z_4) := -2UV + Z_2^2 + Z_3^2 + \sigma Z_4^2 - \sigma a^2.
\]

Note that $M$ is a (classical) smooth hypersurface. Finally we restrict the metric $\bar{g}$ to $M$ to obtain the generalized spacetime $(M, g)$ which we take as our model of nonexpanding impulsive waves propagating on an anti-de Sitter universe.

To derive the geodesic equations in $(M, g)$ we recall that all the classical formulas hold componentwise (i.e., for fixed $\epsilon$) in nonlinear generalized functions. So we may use the classical condition that the geodesics’ $\bar{M}$-acceleration is normal to $M$, $\nabla_T T = -\sigma g(T, \nabla_T N)N/g(N, N)$ to derive the geodesic equations. Here $T$ is the geodesic tangent and $N$ denotes the (non-normalized) normal vector to $M$ defined via its representative $N^\alpha_\epsilon = g_\epsilon^{\alpha3} dF_\beta$, cf. [23, p. 6]. In this way we arrive for the representatives $\gamma_\epsilon = (U_\epsilon, V_\epsilon, Z_{pe})$ of a generalized geodesic...
\[\gamma = (U, V, Z_p) \text{ in } M \text{ precisely at } [12, (2.17)], \text{ which for } \gamma \text{ gives}
\]
\[
\begin{align*}
\dot{U} &= -\left(\epsilon + \frac{1}{2}\dot{U}^2 \tilde{G} - \dot{U} (H D U)\right) \frac{U}{\sigma a^2 - U^2 HD}, \\
\dot{V} - \frac{1}{2} H D \dot{U}^2 - \delta^m H_{,p} Z_q D \dot{U} &= -\left(\epsilon + \frac{1}{2}\dot{U}^2 \tilde{G} - \dot{U} (H D U)\right) \frac{V + H D U}{\sigma a^2 - U^2 HD}, \\
Z_i - \frac{1}{2} H_i D \dot{U}^2 &= -\left(\epsilon + \frac{1}{2}\dot{U}^2 \tilde{G} - \dot{U} (H D U)\right) \frac{Z_i}{\sigma a^2 - U^2 HD}, \\
Z_{i4} - \frac{\sigma}{2} H_{i4} D \dot{U}^2 &= -\left(\epsilon + \frac{1}{2}\dot{U}^2 \tilde{G} - \dot{U} (H D U)\right) \frac{Z_{i4}}{\sigma a^2 - U^2 HD}.
\end{align*}
\]

Here again \(\epsilon = |\dot{\gamma}| = \pm 1, 0\) for which it is natural to be fixed independently of \(\epsilon\).

3. **Unique Global Existence of Geodesics Crossing the Impulse**

In this section we prove existence and uniqueness of globally defined (generalized) solutions of the geodesic equations (14) with suitable initial data that enforces them to cross the wave impulse.

To begin with we recall the basic steps one has to take to solve an initial value problem (IVP) in generalized functions. We consider

\[
\begin{align*}
\dot{Z}(t) &= F(Z(t)), \\
Z(t_0) &= Z^0,
\end{align*}
\]

where we assume the right hand side to be a generalized function \(F \in \mathcal{G}(\mathbb{R}^n, \mathbb{R}^n)\) and the initial condition to be a generalized vector \(Z^0 \in \mathbb{R}^n\). We look for solutions \(Z \in \mathcal{G}(J, \mathbb{R}^n)\) on an interval \(J \subseteq \mathbb{R}\) containing \(t_0\). For all details we refer to [24].

To prove existence and uniqueness of (15) one basically solves the IVP componentwise, i.e., for each \(\epsilon\) one solves

\[
\begin{align*}
\dot{Z}_\epsilon(t) &= F_\epsilon(Z_\epsilon(t)), \\
Z_\epsilon(t_0) &= Z^0_\epsilon,
\end{align*}
\]

where \(F = [(F_\epsilon)_\epsilon], Z = [(Z_\epsilon)_\epsilon] \text{ and } Z^0 = [(Z^0_\epsilon)_\epsilon]\) and then derives the necessary asymptotic estimates. In some more detail one proceeds in the following three steps.

1. First, one proves existence of a so-called *solution candidate*, i.e., a net of smooth functions \(Z_\epsilon: J \rightarrow \mathbb{R}^n\) depending smoothly on the parameter \(\epsilon\) such that for all fixed (and small) \(\epsilon\) we have \(Z_\epsilon = F_\epsilon(Z_\epsilon), Z_\epsilon(t_0) = Z^0_\epsilon\). Note that the domain of definition \(J\) of the solution candidate has to be independent of \(\epsilon\) or at least to depend on \(\epsilon\) favorably (e.g. not shrinking to a point for \(\epsilon \rightarrow 0\)).

2. Second, one shows existence of a generalized solution by establishing c-boundedness and moderateness of the solution candidate, i.e., \(Z := [(Z_\epsilon)_\epsilon] \in \mathcal{G}(J, \mathbb{R}^n)\).

3. Third, to show uniqueness in \(\mathcal{G}\) one solves a negligibly perturbed version of (16), i.e., \(\dot{Z}_\epsilon = F_\epsilon(\tilde{Z}_\epsilon) + a_\epsilon, \tilde{Z}_\epsilon(t_0) = Z^0_\epsilon + b_\epsilon\) with negligible \((a_\epsilon)_\epsilon, (b_\epsilon)_\epsilon\) and shows that the corresponding net of solution \((\tilde{Z}_\epsilon)_\epsilon\) only differs negligibly from \(Z\), i.e., \([(\tilde{Z}_\epsilon)_\epsilon] = [(Z_\epsilon)_\epsilon] = Z\).

Note that a uniqueness result in \(\mathcal{G}\) amounts to an additional stability result, since it says that negligible perturbations of the initial value problem lead to only negligibly perturbed solutions.
In our present analysis we aim at establishing geodesic completeness of the impulsive wave spacetime \((M, g)\) with generalized metric \(g\). So we have to prove that all solutions of the geodesic equations \((14)\) are global. In this section we will prove global existence and uniqueness of solutions given any initial data that forces them to run into the impulsive wave. For convenience we will generate such arbitrary initial data from geodesics of the constant curvature background, cf. the discussion of seed geodesics in Section 1. All other geodesics, i.e., those which do not cross the impulsive wave are simple to deal with and we will briefly do so in the following section prior to presenting the completeness result.

The main issue in showing global existence is that we have to first construct a global solution candidate. To this end we will use the power of the theory of nonlinear generalized functions to overcome the obstacle that the regularization approach in combination with the fixed point argument detailed in part I does not produce a net of global \(\varepsilon\)-wise solutions, cf. the discussion on page 3. Indeed the \(\varepsilon_0\) from which on a geodesics is complete depends in its initial data. This, in particular, affects the case \(\sigma \varepsilon > 0\) where the geodesics cross the wave impulse arbitrarily often. Observe that in our approach it is a result that the geodesics preserve the norm of their tangent vector when crossing the impulse (cf. [42, Remark 3.3]) and hence their causal character. As discussed in the introduction (see also [42, p. 10f.]) the ‘pure’ regularization approach of part I does not allow to prove existence for infinitely many crossings. Here, however, we handle also this case when constructing a global solution candidate.

We will nevertheless start by first constructing a locally defined solution candidate, i.e., a net of geodesics that cross the wave impulse once and we will only later extend it to a global solution candidate.

3.1. Construction of a local solution candidate

Already in the construction of a local solution candidate, to be detailed in this subsection, we need to generalize the approach of [42] such as to allow the initial data to be generated by a family of seed geodesics rather than by a single seed geodesic.

To begin with consider a family of geodesics \(\gamma_{\varepsilon} = (U_{\varepsilon}, V_{\varepsilon}, Z_{\varepsilon}^p)\) of the background (anti-)de Sitter universe without impulsive wave but reaching \(U = 0\) (all other geodesics are not of interest now and will be dealt with separately in section 4). Without loss of generality we can assume that \(U^-_{\varepsilon}(0) = 0\) by choosing an affine parameter for \(\gamma_{\varepsilon}^-\) appropriately and we assume \(\dot{\gamma}_{\varepsilon}^-\) to be normalized by \(e = \pm 1, 0\), independently of \(\varepsilon\). Furthermore, again without loss of generality we can assume that the \(U\)-speed when reaching \(U = 0\), call it \(U_{\varepsilon}^-\), is positive (the case \(U_{\varepsilon}^- < 0\) can be treated in complete analogy) so that locally \(U_{\varepsilon}^-\) is increasing. It is thus most convenient to prescribe initial data at the affine parameter value 0, that is we set

\[
\gamma_{\varepsilon}^-(0) =: (0, V_{\varepsilon}^0, Z_{\varepsilon}^p), \quad \dot{\gamma}_{\varepsilon}^-(0) =: (\dot{U}_\varepsilon > 0, V_{\varepsilon}^0, \dot{Z}_{\varepsilon}^p),
\]

where the constants satisfy the constraints

\[
(Z_{2\varepsilon}^0)^2 + (Z_{3\varepsilon}^0)^2 + \sigma (Z_{4\varepsilon}^0)^2 = \sigma a^2, \quad Z_{2\varepsilon}^0 \dot{Z}_{2\varepsilon}^0 + Z_{3\varepsilon}^0 \dot{Z}_{3\varepsilon}^0 + \sigma Z_{4\varepsilon}^0 \dot{Z}_{4\varepsilon}^0 - V_{\varepsilon}^0 \dot{U}_\varepsilon^0 = 0,
\]

for every \(\varepsilon\), which follows from (6) (with \(U_{\varepsilon}^0 = 0\)). In addition the normalization condition

\[
-2 U_{\varepsilon}^0 \dot{V}_{\varepsilon}^0 + (\dot{Z}_{2\varepsilon}^0)^2 + (\dot{Z}_{3\varepsilon}^0)^2 + \sigma (\dot{Z}_{4\varepsilon}^0)^2 = e
\]

holds for every \(\varepsilon\). Now we assume that the net \((\gamma_{\varepsilon}^- (0))_\varepsilon\) of data at 0 converges, more precisely we assume

\[
\lim_{\varepsilon \searrow 0} \gamma_{\varepsilon}^- (0) =: (0, V^0, Z_p^0), \quad \lim_{\varepsilon \searrow 0} \dot{\gamma}_{\varepsilon}^- (0) =: (\dot{U}^0 > 0, \dot{V}^0, \dot{Z}_p^0),
\]
which automatically satisfy the constraint and the normalization. We will refer to a family $(\gamma^-_\varepsilon)_{\varepsilon}$ as described here as a family of seed geodesics. Note that $(\gamma^-_\varepsilon)_{\varepsilon}$ depends on $\varepsilon$ only via its data and that it converges locally uniformly together with its first derivative to a fixed background geodesic $\gamma^-$ with data $[20]$ by continuous dependence of solutions of ODEs on their data.

At this point we start to think of $\gamma^-_\varepsilon$ as geodesics in the impulsive wave spacetime $[7]$, $[6]$ ‘in front’ of the impulse that is for $U^-_\varepsilon < 0$. Also, $\gamma^-_\varepsilon$ are geodesics in the regularized spacetime $[5]$, $[6]$ ‘in front’ of the sandwich wave, that is for $U^-_\varepsilon \leq -\varepsilon$. We will denote the affine parameter time when $\gamma^-_\varepsilon$ enters this regularization wave region by $\alpha_\varepsilon$,

$$U^-_\varepsilon(\alpha_\varepsilon) = -\varepsilon. $$

Observe that $\alpha_\varepsilon \to 0$ from below as $\varepsilon \to 0$.

Finally we come to set up the data for the solution candidate $\gamma_\varepsilon$ of the system $[14]$ by

$$\gamma_\varepsilon(\alpha_\varepsilon) = \gamma^-_\varepsilon(\alpha_\varepsilon), \quad \dot{\gamma}_\varepsilon(\alpha_\varepsilon) = \dot{\gamma}^-_\varepsilon(\alpha_\varepsilon),$$

i.e., as the data the family of seed geodesics assumes at $\alpha_\varepsilon$. We will frequently refer to these data $[22]$ as initial data constructed from the seed family with data $[17]$.

Now we will provide the local solution candidate by proving that we can extend the background geodesics not only into but even through the entire wave zone. Local existence of such an extension for one crossing is provided by a fixed point argument. More precisely it suffices to study the model system below. Note that compared to $[12]$ Equation (A.1), p. 20 we deal with a negligibly perturbed system which is necessary to prove uniqueness in Subsection 3.4 below. We consider

$$\dot{u}_\varepsilon = -\frac{e u_\varepsilon + \Delta u_\varepsilon}{N_\varepsilon} + a_\varepsilon, \quad \ddot{z}_\varepsilon - \frac{1}{2} DH \delta_\varepsilon \dot{u}_\varepsilon^2 = -\frac{e z_\varepsilon + \Delta z_\varepsilon}{N_\varepsilon} + c_\varepsilon,$$

$$u_\varepsilon(\alpha_\varepsilon) = -\varepsilon, \quad \dot{u}_\varepsilon(\alpha_\varepsilon) = \dot{u}_\varepsilon^0 + d_\varepsilon, \quad z_\varepsilon(\alpha_\varepsilon) = z_\varepsilon^0 + f_\varepsilon, \quad \ddot{z}_\varepsilon(\alpha_\varepsilon) = \ddot{z}_\varepsilon^0 + h_\varepsilon,$$

where

$$\Delta = \frac{1}{2} \dot{u}_\varepsilon^2 G_\varepsilon - \dot{u}_\varepsilon \left( H \delta_\varepsilon u_\varepsilon \right),$$

$$N_\varepsilon = \sigma a^2 - u_\varepsilon^2 H \delta_\varepsilon,$$

and $H = H(z_\varepsilon)$ is a smooth function on $\mathbb{R}^3$, $DH$ denotes its gradient, and $G_\varepsilon(u_\varepsilon, z_\varepsilon) := DH(z_\varepsilon) \delta_\varepsilon(u_\varepsilon) z_\varepsilon + H(z_\varepsilon) \delta'_\varepsilon(u_\varepsilon) u_\varepsilon$. Moreover we assume $(a_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}), (c_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}^3), (d_\varepsilon)_\varepsilon \in \mathcal{N}, (f_\varepsilon)_\varepsilon, (h_\varepsilon)_\varepsilon \in \mathcal{N}^3$. Finally, the initial conditions are specified in the spirit of the seed family approach as follows: We fix $u^0 > 0$ and $z^0, \dot{z}^0 \in \mathbb{R}^3$ and assume

$$x^0 := (-\varepsilon, z^0_\varepsilon) \to (0, z^0) := x^0, \quad \dot{x}^0 := (\dot{u}_\varepsilon^0, \dot{z}_\varepsilon^0) \to (\dot{u}^0, \dot{z}^0) := \dot{x}^0$$

as $\varepsilon \downarrow 0$. Furthermore let $\alpha_\varepsilon < 0$ such that $\alpha_\varepsilon \nrightarrow 0$ for $\varepsilon \downarrow 0$. Also we will frequently use the notation $x_\varepsilon := (u_\varepsilon, z_\varepsilon)$. 


At this point we set up the space of possible solutions. We will outline the basic steps of the construction extending the approach of [42, Appendix A]. Let $C_1 > 0$ and set

$$C_2 := 1 + \max \left( \frac{81}{2} \| DH \| \| \rho \| \| u \| \left( \frac{12}{a^2} \| z \| + C_1 \right), \frac{54}{a^2} \| z \| + C_1, \left( 3 \left( \frac{3}{2} \| u \| \left( \frac{12}{a^2} \| z \| + C_1 \right) + \frac{3}{2} \| H \| \| \rho \| \right) + 2 \| DH \| \| \rho \| + 3 \| H \| \| \rho \| \right),$$

where $\| H \| = \| H \|_{L^\infty(B_{C_1}(z))}$ and $\| DH \| = \| DH \|_{L^\infty(B_{C_1}(z))}$, the $L^\infty$-norm on the closed ball of radius $C_1$ around $z$.

Now we define $\eta > 0$ — the length of the solution interval (which is independent of $\varepsilon$) by

$$\eta := \min \left\{ 1, \frac{a^2}{4(1 + 9\| u \|)}, \frac{C_1}{6C_1}, \frac{C_1}{4(1 + 9\| u \|)}, \frac{C_1}{4(1 + 9\| u \|)}, \frac{C_1}{16(|z| + C_1)}, \frac{C_1}{16(|z| + C_1)} \right\} \times \left( \frac{9}{2} \| u \| \| z \| + C_1 \right) + \left( \frac{9}{2} \| u \| \| z \| + C_1 \right) + \left( \frac{9}{2} \| u \| \| z \| + C_1 \right) + \left( \frac{9}{2} \| u \| \| z \| + C_1 \right) + \left( \frac{9}{2} \| u \| \| z \| + C_1 \right).$$

Set $J_r := [\alpha, \alpha + \eta]$ and define a closed subset of the complete metric space $C^1(J_r, \mathbb{R}^4)$ by setting

$$X_\varepsilon := \left\{ x_\varepsilon := (u_\varepsilon, z_\varepsilon) \in C^1(J_r, \mathbb{R}^4) : x_\varepsilon(\alpha) = x_\varepsilon^0 + (0, f_\varepsilon), \dot{x_\varepsilon}(\alpha) = \dot{x_\varepsilon}^0 + (d_\varepsilon, h_\varepsilon), \right\} \leq C_1, \| z_\varepsilon - \dot{z_\varepsilon} \| \leq C_2, \| u_\varepsilon \| \leq \frac{1}{2} \| u_\varepsilon \| \left( \frac{1}{2} \| u_\varepsilon \| \right) \right\}.$$

Now we define the solution operator $A_\varepsilon = (A_\varepsilon^1, A_\varepsilon^2)$ on $X_\varepsilon$:

$$A_\varepsilon^1(x_\varepsilon)(t) := -\varepsilon + (\dot{u_\varepsilon}^0 + d_\varepsilon)(t - \alpha) - \int_{\alpha}^{t} \int_{\alpha}^{s} u_\varepsilon + \frac{\Delta \varepsilon u_\varepsilon}{N_\varepsilon} \, dr \, ds + \int_{\alpha}^{t} \int_{\alpha}^{s} a_\varepsilon \, dr \, ds,$$

$$A_\varepsilon^2(x_\varepsilon)(t) := \dot{z_\varepsilon}^0 + f_\varepsilon + (\dot{z_\varepsilon}^0 + h_\varepsilon)(t - \alpha) + \int_{\alpha}^{t} \int_{\alpha}^{s} \frac{1}{2} DH \dot{e_\varepsilon} u_\varepsilon^2 - \frac{\varepsilon \dot{z_\varepsilon} + \Delta \varepsilon z_\varepsilon}{N_\varepsilon} \, dr \, ds + \int_{\alpha}^{t} \int_{\alpha}^{s} c_\varepsilon \, dr \, ds.$$

Next we will show that the operator $A_\varepsilon$ possesses a unique fixed point in $X_\varepsilon$. The first steps are (cf. [42, Lemma A.1 and A.2, p. 21]):

**Lemma 3.1.**

(i) Let $z \in B_{C_1}(z^0)$ and $u \in \mathbb{R}$ then if $\varepsilon \leq \frac{a^2}{2 \| \rho \| \| H \|}$ we have

$$\left\| \frac{1}{\sigma a^2 - u^2 H \dot{e_\varepsilon}} \right\| \leq \frac{2}{a^2}.$$

(ii) For $x_\varepsilon \in X_\varepsilon$ define $\Gamma_\varepsilon(x_\varepsilon) := \{ t \in J_r : |u_\varepsilon(t)| \leq \varepsilon \}$ to be the parameter interval, where $u_\varepsilon$ is in the regularized wave zone, then

$$\operatorname{diam}(\Gamma_\varepsilon(x_\varepsilon)) \leq \frac{4\varepsilon}{\dot{u_\varepsilon}}.$$

Now we can quantify how small $\varepsilon$ has to be. To begin with let

$$\varepsilon_0 := \min \left\{ \frac{1}{12}, \frac{1}{12 a^2} \| u \| \| \rho \| \left( \frac{3}{2} \| DH \| (|z| + C_1) + 2 \| DH \| (|z| + C_1) + \frac{3}{2} \| H \| + \frac{3}{2} \| H \| \| \rho \| \right) \right\},$$

$$\eta := \frac{C_1}{16}, \frac{C_2}{6}, \frac{C_1 \eta}{2} \left( \frac{\eta}{2} \right),$$

$$\eta_0 := \frac{\eta}{6} \left( \frac{\eta}{6} \right).$$
and let $0 < \varepsilon_0 \leq \varepsilon_0'$ such that for all $0 < \varepsilon \leq \varepsilon_0$

$$|d_\varepsilon| \leq \varepsilon, \|f_\varepsilon\| \leq \varepsilon, \|h_\varepsilon\| \leq \varepsilon, |a_\varepsilon| \leq \varepsilon (on \ [-1, 1]), \|c_\varepsilon\| \leq \varepsilon (on \ [-1, 1]),$$

(30)  $$|\dot{u}_\varepsilon^0 - \dot{a}_0^0| \leq \frac{1}{8}, \|\dot{z}_\varepsilon^0 - \dot{z}_0^0\| \leq \frac{C_1}{8}, \text{ and } \|\ddot{z}_\varepsilon^0 - \ddot{z}_0^0\| \leq \min\left(\frac{C_1}{16}, \frac{C_2}{6}\right).$$

With these preparations one shows that $A_\varepsilon$ maps $X_\varepsilon$ to $X_\varepsilon$ (cf. [42, Proposition A.3, p. 22]) as well as the results analogous to [42, Lemma A.4, p. 25, Proposition A.5, p. 25] (in fact, with modified constants) providing the necessary estimates which in turn allow to show that $A_\varepsilon$ possesses a fixed point in $X_\varepsilon$ and hence that (23) has a local solution candidate (cf. [42, Theorem A.6, p. 27]).

**Proposition 3.2** (Local solution candidate). Consider the IVP (23) with $\varepsilon \leq \varepsilon_0$ where $\varepsilon_0$ is given by (29), (30). Then there is a unique smooth solution $(u_\varepsilon, z_\varepsilon)$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$, depending smoothly on $\varepsilon$. Moreover, $u_\varepsilon$ and $z_\varepsilon$ as well as their first order derivatives are uniformly bounded in $\varepsilon$.

**Proof:** The above construction allows the application of Weissinger’s fixed point theorem ([42]) for fixed $\varepsilon \leq \varepsilon_0$ and $\eta$ as above, providing thus a unique fixed point for the operator $A_\varepsilon$ on the space $X_\varepsilon$ which in turn gives a unique $C^1$-solution $x_\varepsilon = (u_\varepsilon, z_\varepsilon)$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ to the IVP (23). Moreover, since the right hand side of the system is smooth the solution is smooth as well, and in addition depends smoothly on $\varepsilon$.

The solution obtained via the fixed point argument is unique in the space $X_\varepsilon$ and thereby unique among all smooth solutions assuming this data by the usual argument from ODE-theory.

Finally, $u_\varepsilon$, $\dot{u}_\varepsilon$, $z_\varepsilon$, and $\dot{z}_\varepsilon$ are bounded uniformly in $\varepsilon$ on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ by the very definition of $X_\varepsilon$.

\[\square\]

Going back from the model IVP (23) to the geodesic equations (14) on the level of representatives

$$\ddot{U}_\varepsilon = \left(e + \frac{1}{2} \dot{U}_\varepsilon G_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)\right) \frac{U_\varepsilon G_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H_\varepsilon},$$

$$\ddot{V}_\varepsilon - \frac{1}{2} H \delta_\varepsilon \dot{U}_\varepsilon^2 - \delta^{pq} H_\varepsilon \delta_\varepsilon \ddot{Z}_\varepsilon^p U_\varepsilon = -\left(e + \frac{1}{2} \dot{U}_\varepsilon G_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)\right) \frac{V_\varepsilon + H \delta_\varepsilon U_\varepsilon}{\sigma a^2 - U_\varepsilon^2 H_\varepsilon},$$

(31)  $$\ddot{Z}_{le} - \frac{1}{2} H_\varepsilon \delta_\varepsilon \dot{U}_\varepsilon^2 = -\left(e + \frac{1}{2} \dot{U}_\varepsilon G_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)\right) \frac{Z_{le}}{\sigma a^2 - U_\varepsilon^2 H_\varepsilon},$$

$$\ddot{Z}_{le} - \frac{1}{2} H_\varepsilon \delta_\varepsilon \dot{U}_\varepsilon^2 = -\left(e + \frac{1}{2} \dot{U}_\varepsilon G_\varepsilon - \dot{U}_\varepsilon (H \delta_\varepsilon U_\varepsilon)\right) \frac{Z_{le}}{\sigma a^2 - U_\varepsilon^2 H_\varepsilon},$$

(which at the moment we only need for $a_\varepsilon$, $c_\varepsilon$, $d_\varepsilon$, $f_\varepsilon$, $h_\varepsilon$ vanishing) Proposition 3.2 gives solution candidates $U_\varepsilon, Z_{le}$ of the corresponding components of (31) on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$. Furthermore, we are able to solve the $V$-equation on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$ since it is linear (using the already constructed solution candidates $U_\varepsilon, Z_{le}$). Moreover, $(V_\varepsilon)_{\varepsilon, le}$ is locally bounded uniformly in $\varepsilon$, as can be seen as in [42, Proposition 4.1, p. 12f.]. This gives a unique net of solutions of (31)

(32)  $$\gamma_\varepsilon := (U_\varepsilon, V_\varepsilon, Z_{le})$$

defined on $[\alpha_\varepsilon, \alpha_\varepsilon + \eta]$, with data constructed from the seed family with data (17), see (24).

This net hence constitutes a solution candidate for the system (14) of geodesic equations in the generalized spacetime $(M, g)$ with data constructed from the seed data (17) and defined
on the interval $J_\varepsilon = [\alpha_\varepsilon, \alpha_\varepsilon + \eta]$. First, for the purpose of constructing a local solution candidate we assume the data \((17)\) to be constant in $\varepsilon$ and to be given by \((20)\). That is we assume the data to be given by the single seed geodesic $\gamma^-$ with $\gamma^-(0) = (0, V^0, Z_{p_0}^0)$ and $\dot{\gamma}^-(0) = (\dot{U}^0, \dot{V}^0, \dot{Z}_{p_0}^0)$, which leads to the data

$$\gamma_\varepsilon(\alpha_\varepsilon) = \gamma^-(\alpha_\varepsilon), \quad \dot{\gamma}_\varepsilon(\alpha_\varepsilon) = \dot{\gamma}^-(\alpha_\varepsilon)$$

for the net $\gamma_\varepsilon$ at $\alpha_\varepsilon$. Next observe that the solution interval $J_\varepsilon$ is depending on $\varepsilon$ but in a favorable way. Indeed we have by the definition \((25)\) of $X_\varepsilon$ that

$$U_\varepsilon(\alpha_\varepsilon + \eta) = -\varepsilon + \int_{\alpha_\varepsilon}^{\alpha_\varepsilon + \eta} \dot{U}_\varepsilon(s) \, ds \geq -\varepsilon + \frac{\eta}{2} \dot{U}^0 \geq -\varepsilon + 3\varepsilon \geq \varepsilon,$$

where in the one but last inequality we have used $\varepsilon \leq \eta \dot{U}^0/6$ according to \((29)\).

Summing up we have found a net of solutions $$(\gamma_\varepsilon)_\varepsilon$$ to \((31)\) with data \((33)\) which (for $\varepsilon$ small enough) crosses the regularized wave zone. More precisely call the parameter value when $\gamma_\varepsilon$ leaves the wave zone $\beta_\varepsilon$, i.e., $U_\varepsilon(\beta_\varepsilon) = \varepsilon$, then $\gamma_\varepsilon$ is defined on an interval that contains $[\alpha_\varepsilon, \beta_\varepsilon]$ in its interior since outside this interval $\gamma_\varepsilon$ reaches the region of spacetime coinciding with the background (anti-)de Sitter space.

To the past of $\alpha_\varepsilon$ the net $\gamma_\varepsilon$ smoothly extends to the seed geodesic $\gamma^-$ as long as the latter is defined. Being a background geodesic it will be past complete and it will also be a complete solution in the regularized impulsive wave spacetime as long as it does not reach the regularized wave zone (again). This will be for all negative values of the parameter in case $\sigma \varepsilon \leq 0$ and at least for some finite negative parameter value in case $\sigma \varepsilon > 0$. In any case the net $\gamma_\varepsilon$ extends smoothly to the single seed geodesic $\gamma^-$ in the region ‘before’ the wave zone. To the future of $\beta_\varepsilon$ the situation is slightly more complicated. Indeed $\gamma_\varepsilon$ extends smoothly to a family of background geodesics $\gamma_\varepsilon^+$ which are determined by the data of $\gamma_\varepsilon$ at $\beta_\varepsilon$, i.e., we have

$$\gamma_\varepsilon^+(\beta_\varepsilon) = \gamma_\varepsilon(\beta_\varepsilon) =: (\varepsilon, V^0_\varepsilon, Z_{p_{\varepsilon}}^0), \quad \dot{\gamma}_\varepsilon^+(\beta_\varepsilon) = \dot{\gamma}_\varepsilon(\beta_\varepsilon) =: (\dot{U}^0_\varepsilon, \dot{V}^0_\varepsilon, \dot{Z}_{p_{\varepsilon}}^0).$$

Now again in case $\sigma \varepsilon \leq 0$ each $\gamma_\varepsilon^+$ is (forward) complete and we have already found a global solution candidate. However, in case $\sigma \varepsilon > 0$ we will again only have some finite parameter value before $\gamma_\varepsilon^+$ reenters the wave zone. So in total we have at least found a local solution candidate that crosses the wave zone once and extends as a single seed geodesic into the background ‘before’ the impulse and as a family of background geodesics into part of the (anti-)de Sitter spacetime ‘behind’ the impulse. This situation is depicted in the left half of Figure 1 and we summarize it as follows:

**Proposition 3.3** (Extension of the solution candidate). The unique smooth geodesics $(\gamma_\varepsilon)_\varepsilon$ of \((32)\) with initial data \((33)\) extend to geodesics of the background (anti-)de Sitter spacetime ‘before’ and ‘behind’ the wave zone. Hence it provides a local solution candidate that crosses the impulse once.

### 3.2. Construction of a global solution candidate

Now we are going to globalize the above result, in the sense that we handle the case of infinitely many crossings of the wave impulse, thereby extending the maximal interval of definition of these solution candidates to all of $\mathbb{R}$. As discussed above this is only necessary if $\sigma \varepsilon > 0$, which will be assumed throughout this subsection. To this end we will employ advanced construction methods for (Colombeau) generalized functions. We make use of the
following globalization Lemma, c.f. [18] Lemma 4.3, p. 12] (see also [25] Lemma 2.4, p. 6] but note that actually point (ii) there is redundant).

**Lemma 3.4.** Let $M$, $N$ be manifolds, and $J := (0, 1)$. Let $u: J \times M \to N$ be a smooth map and let $(P)$ be a property attributable to values $u(\varepsilon, p)$, satisfying: For any $K \Subset M$ there exists some $\varepsilon_K > 0$ such that $(P)$ holds for all $p \in K$ and $\varepsilon < \varepsilon_K$. Then there exists a smooth map $\tilde{u}: J \times M \to N$ such that $(P)$ holds for all $\tilde{u}(\varepsilon, p)$ ($\varepsilon \in J$, $p \in M$) and for each $K \Subset M$ there exists some $\varepsilon_K \in J$ such that $\tilde{u}(\varepsilon, p) = u(\varepsilon, p)$ for all $(\varepsilon, p) \in (0, \varepsilon_K) \times K$.

**Theorem 3.5** (Global solution candidate). There exists a net of solutions $(\gamma_\varepsilon)_{\varepsilon} \in [31]$, with data (33), defined on all of $\mathbb{R}$ for all $\varepsilon \in I$, thus constituting a global solution candidate for the system (14).

**Proof:** We will only prove forward completeness, i.e., existence on $[0, \infty)$, the case of backward completeness is completely analogous. We aim at applying Lemma 3.4. To this end we need to establish that we can apply Proposition 3.2 again after we already crossed the impulse once. That is we have to show that the geodesics $\gamma_\varepsilon$ which ‘behind’ the wave zone coincide with the background geodesics $\gamma_\varepsilon^+$, see (55) qualify as a seed family (as defined in the beginning of Subsection 3.1).

To formalize this denote the parameter value when $\gamma_\varepsilon = \gamma_\varepsilon^+$ (re)enters the regularization strip by $\alpha_\varepsilon^+$, i.e., $U_\varepsilon(\alpha_\varepsilon^+) = \varepsilon$ (of course there is now a sign change). Then $\gamma_\varepsilon = \gamma_\varepsilon^+$ on $[\beta_\varepsilon, \alpha_\varepsilon^+]$ and we have to prove that $\gamma_\varepsilon(\alpha_\varepsilon^+) = \gamma_\varepsilon^+(\alpha_\varepsilon^+)$ and $\gamma_\varepsilon(\alpha_\varepsilon^+) = \gamma_\varepsilon^+(\alpha_\varepsilon^+)$ converge.

First observe that on $[\beta_\varepsilon, \alpha_\varepsilon^+]$ being a background geodesic we can explicitly calculate the $U$ component of $\gamma_\varepsilon = \gamma_\varepsilon^+$ to be (c.f. e.g. [12] Equation (3.2), p. 8])

$$U_\varepsilon(t) = a\tilde{U}_\varepsilon^0 \sin \left( \frac{t}{a} \right).$$

So the next crossing of the impulse happens at $t = a\pi$ hence $\alpha_\varepsilon^+ \to \pi a$, see Figure 1 for an illustration.

Now by continuous dependence of solutions on the data, on every compact subinterval $[t_1, t_2]$ of $[0, \pi a]$ the geodesics $\gamma_\varepsilon = \gamma_\varepsilon^+$ convergence uniformly together with their first derivatives to the background geodesic $\gamma^+$ with initial data at $t = 0$ given by the limit of (55) (cf. [12] Theorem 5.1, p. 15]). The latter indeed exists and has been explicitly related to the corresponding seed data in [12] Proposition 5.3, p. 18). Moreover this convergence is uniform with respect to $t_2$, i.e., we have for fixed $t_2$ and any $t_2 \leq \pi a$

$$\sup_{t_1 \leq t \leq t_2} \left( |\gamma_\varepsilon(t) - \gamma^+(t)|, |\gamma_\varepsilon(t) - \dot{\gamma}^+(t)| \right)$$

$$\leq \max \left( |\gamma_\varepsilon(t_1) - \gamma^+(t_1)|, |\gamma_\varepsilon(t_1) - \dot{\gamma}^+(t_1)| \right) e^{\pi a L} =: A(\varepsilon) C$$

where $L$ is a Lipschitz constant for the right hand side of the geodesic equation of the background on a suitable compact set, $C$ is a constant and $A(\varepsilon) \to 0$ for $\varepsilon \to 0$. Hence we obtain

$$\max \left( |\gamma_\varepsilon(\alpha_\varepsilon^+) - \gamma^+(\pi a)|, |\gamma_\varepsilon(\alpha_\varepsilon^+) - \dot{\gamma}^+(\pi a)| \right)$$

$$\leq \max \left( |\gamma_\varepsilon(\alpha_\varepsilon^+) - \gamma^+(\alpha_\varepsilon^+)| + |\gamma^+(\alpha_\varepsilon^+) - \gamma^+(\pi a)|, |\gamma_\varepsilon(\alpha_\varepsilon^+) - \dot{\gamma}^+(\alpha_\varepsilon^+)| + |\dot{\gamma}^+(\alpha_\varepsilon^+) - \dot{\gamma}^+(\pi a)| \right)$$

$$\leq A(\varepsilon) C \max \left( |\gamma^+(\alpha_\varepsilon^+) - \gamma^+(\pi a)|, |\dot{\gamma}^+(\alpha_\varepsilon^+) - \dot{\gamma}^+(\pi a)| \right).$$

Now the latter term converges to zero by smoothness of $\gamma^+$ and we have established that $\gamma_\varepsilon^+(\alpha_\varepsilon^+) = \gamma_\varepsilon(\alpha_\varepsilon^+) \to \gamma^+(\pi a)$ and $\gamma_\varepsilon^+(\alpha_\varepsilon^+) = \gamma_\varepsilon(\alpha_\varepsilon^+) \to \gamma^+(\pi a)$. In conclusion, the net $(\gamma_\varepsilon^+)_{\varepsilon}$ is
Figure 1. The construction of the solution candidate for the first two crossings of the impulse at parameter values \( t = 0 \) and \( t = L = \alpha \pi \).

a seed family and we can (re)apply the machinery of the previous subsection, in particular, Proposition 3.2. However, the corresponding \( \epsilon_0 \) now depends on (the limit of) (35).

To iterate this construction set \( K_n := [-\alpha \pi^2, n \alpha \pi^2] \) for \( 1 \leq n \in \mathbb{N} \) which gives a compact exhaustion of \( [-\alpha \pi^2, \infty) \) (i.e., \( K_n \subseteq K^0_{n+1} \) and \( \bigcup_{n \in \mathbb{N}} K_n = [-\alpha \pi^2, \infty) \)), such that \( K_n \) yields exactly \( n \) crossings (\( n \geq 1 \)). Then by the above for every \( n \in \mathbb{N} \) there is an \( \epsilon_n > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_n \) there is a unique solution \( \gamma_{\epsilon} \) of (31) with initial data (33) on \( K_n \).

At this point we apply Lemma 3.4 by setting \( M := [-\alpha \pi^2, \infty) \) and \( N := \mathbb{R}^5 \) and define \( \gamma: (0, \infty) \times [-\alpha \pi^2, \infty) \to \mathbb{R}^5 \) by

\[
(\epsilon, t) \mapsto \gamma_{\epsilon}(t), \text{ whenever } t \in K_n, \text{ where } n \text{ is minimal and } 0 < \epsilon \leq \epsilon_n
\]

and extend it arbitrary but smoothly to bigger values of \( t \) and \( \epsilon \). Then from the above (in particular, the uniqueness of the solutions of (31), (33)) it is clear that if \( \gamma_{\epsilon} \) is a solution on \( K_n \), then \( \gamma_{\epsilon} \) is also a solution on \( K_m \) for \( m \leq n \). At this point we define the property (P) by

\[
(P) \text{ holds at } \gamma_{\epsilon}(t) \text{ if } \gamma_{\epsilon}(t) \text{ solves (31) at } t.
\]

Now the assumptions of the lemma hold since for any \( K \subseteq [0, \infty) \) there is a minimal \( n \in \mathbb{N} \) such that \( K \subseteq K_n \). Thus there is an \( \epsilon_n > 0 \) and there are unique solutions \( (\gamma_{\epsilon})_{\epsilon \leq \epsilon_n} \) on \( K_n \), hence on \( K \). So Lemma 3.4 provides \( \tilde{\gamma}_{\epsilon} \) (again called \( \gamma_{\epsilon} \) in the statement of the theorem) defined on \([0, \infty)\) that is a solution of the geodesic equations (31) with data (33) for all \( \epsilon \). □

3.3. Global existence of solutions

In this section we establish existence of solutions of the geodesic equations (14). Indeed, by showing c-boundedness and moderateness of the solution candidate given in Theorem (3.5) we actually prove existence of global solutions.

Theorem 3.6 (Global Existence). The global solution candidate \((\gamma_{\epsilon})_{\epsilon \in \mathbb{I}}\) of (14), given by Theorem 3.5, is moderate and c-bounded. Hence it defines a global solution \((\gamma_{\epsilon})_{\epsilon \in \mathbb{I}} \in G[\mathbb{R}, M]\) to the geodesic equation (14) with data (33) (derived from a single seed geodesic with data (20)).
Proof: First observe that we have to prove the asymptotic estimates on compact time intervals only. So we can restrict ourselves to such intervals that contain one single crossing of the impulse or to such which contain no crossing at all.

To deal with the first case without loss of generality we only consider the first crossing of the impulse at \( t = 0 \). Let \( (\gamma_\varepsilon)_\varepsilon \) be given by Theorem 3.5 then \( \gamma_\varepsilon \) is a solution of (31) on \([\alpha_\varepsilon, \alpha_\varepsilon + \eta]\), where \( \eta \) is independent of \( \varepsilon \). On this interval the \( U- \) and \( Z- \)components (i.e., \( X_\varepsilon = (U_\varepsilon, Z_\varepsilon) \)) and their first order derivatives are are bounded uniformly in \( \varepsilon \) by Proposition 3.2. So they are \( c \)-bounded together with their first order derivatives. Moreover by iteratively using the differential equation (31) (with coefficients at worst of order \( 1/\varepsilon \)) the derivatives of order \( k \) satisfy an \( O(\varepsilon^{1-k}) \) estimate. Also, \( V_\varepsilon \) is bounded, uniformly in \( \varepsilon \), on the interval \([\alpha_\varepsilon, \beta_\varepsilon]\) (cf. the discussion above equation (32)) and so by integration, \( \dot{V}_\varepsilon \) satisfies an \( O(\varepsilon^{-1}) \) estimate. Inductively, the higher order estimates again follow from (31).

Now let \( K \) be a compact time interval disjoint from any crossing of the impulse, i.e., \( k \pi \notin K \) for all \( k \in \mathbb{Z} \). There \( (\gamma_\varepsilon)_\varepsilon \) are background geodesics depending on \( \varepsilon \) only via their data. Since the latter converges we obtain moderateness and \( c \)-boundedness between impulses as follows. Via a mean value argument one sees that \( \gamma_\varepsilon \) and \( \dot{\gamma}_\varepsilon \) are uniformly bounded (cf. [42, Equation (3.10), p. 9]). Differentiating the differential equation (31) \( k - 2 \)-times and using again continuous dependence on initial conditions yields that the \( k \)-th derivative satisfies an \( O(\varepsilon^{-k+1}) \) estimate. This establishes moderateness, \( c \)-boundedness is clear by the uniform boundedness properties of \( \gamma_\varepsilon \) as noted above.

\[ \square \]

3.4. Global uniqueness of solutions

Our final step is to show that the global geodesics obtained above are in fact the unique solutions of (14) with corresponding data (33). As remarked in the beginning of this section this can be viewed as an additional stability result.

First note that uniqueness for (14), (33) does not follow from the uniqueness part of Proposition 3.2 since this only provides unique solvability of (23) respectively (31), (33), that is uniqueness 'on the \( \varepsilon \)-level'. Here we have, however, to provide uniqueness of (14), (33) in \( G[\mathbb{R}, M] \), that is we have to show that negligibly perturbed data and right-hand-side of the equations only lead to negligibly perturbed solutions.

We will even show uniqueness in \( G[\mathbb{R}, \mathbb{R}^5] \) by using the result for the (perturbed) model IVP (23). In fact, this is stronger than needed, since uniqueness in \( G[\mathbb{R}, M] \) would restrict the possible negligible perturbations (so as to stay on the hyperboloid \( M \)).

Moreover note that for uniqueness in \( G[J, \mathbb{R}^5] \) we would need also to perturb the \( U- \)initial data, i.e., \( u_\varepsilon(\alpha_\varepsilon) = -\varepsilon + n_\varepsilon \), where \( (n_\varepsilon)_\varepsilon \in \mathcal{N} \). However, by shifting the parameter (e.g., \( u_\varepsilon(\alpha_\varepsilon + A_\varepsilon) = -\varepsilon \)) we can without loss of generality assume that there is no negligible perturbation in the \( U \)-initial data.

**Theorem 3.7** (Uniqueness). The global solution \( [(\gamma_\varepsilon)_\varepsilon] \in G[\mathbb{R}, M] \) of Theorem 3.6 to the geodesic equation (14) with data (33) is unique.

**Proof:** To begin with we observe that it is sufficient to consider the model system (23) since the \( V \)-equation can simply be integrated once \( U \) and \( Z_\varepsilon \) are known. This gives uniqueness by the basic fact that integration is well defined in generalized functions.

So let \( x_\varepsilon \) be a solution of the initial value problem (23) with all negligible terms set to zero and let \( \tilde{x}_\varepsilon := (\tilde{u}_\varepsilon, \tilde{z}_\varepsilon) \) solve (23) with all the negligible terms on the r.h.s. and the data in effect. Generally we will denote quantities depending on \( \tilde{x}_\varepsilon \) also with a tilde.
Again we have to show the asymptotic estimates only on compact time intervals $J$ containing at most one crossing of the impulse. Without loss of generality we deal with the first crossing at $t = 0$. Let $q \in \mathbb{N}$.

We start with the $U$-component: Since $a$ and $d$ are negligible, there is $C > 0$ and such that for all $\varepsilon$ small enough and for all $t \in J$

$$|u_\varepsilon(t) - \tilde{u}_\varepsilon(t)| \leq C \varepsilon^q + \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |\frac{u_\varepsilon}{N_\varepsilon} - \frac{\tilde{u}_\varepsilon}{N_\varepsilon}| \, ds \, dr + \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |\Delta_\varepsilon u_\varepsilon - \Delta_\varepsilon \tilde{u}_\varepsilon| \, ds \, dr .$$

Using Lemma 3.1(i) the first integrand can be bounded by $\frac{4}{a^2} |u_\varepsilon| |\tilde{u}_\varepsilon \tilde{H} \tilde{\delta}_\varepsilon - u_\varepsilon^2 H \delta_\varepsilon| + \frac{2}{a^2} |u_\varepsilon - \tilde{u}_\varepsilon|$. Keeping the second term and estimating the first one further we obtain

$$|\tilde{u}_\varepsilon^2 \tilde{H} \tilde{\delta}_\varepsilon - u_\varepsilon^2 H \delta_\varepsilon| \leq |u_\varepsilon - \tilde{u}_\varepsilon| |\tilde{u}_\varepsilon \tilde{H} \tilde{\delta}_\varepsilon| + |\tilde{u}_\varepsilon u_\varepsilon \tilde{\delta}_\varepsilon| |H - \tilde{H}| + |H u_\varepsilon| |\tilde{u}_\varepsilon \tilde{\delta}_\varepsilon - u_\varepsilon \delta_\varepsilon| ,$$

where we can use $|H - \tilde{H}| \leq \text{Lip}(H) |z_\varepsilon - \tilde{z}_\varepsilon|$ for the second term.

For the third term we use a generalization of [42, Lemma A.4(i), p. 25]. Note that here we have $x_\varepsilon \in \mathcal{X}_\varepsilon \neq \tilde{\mathcal{X}}_\varepsilon \ni \tilde{z}_\varepsilon$ since they have different initial conditions. Nevertheless the estimate in Lemma 3.1(ii) on the diameter of $\Gamma_\varepsilon(u_\varepsilon)$ is independent of $d_\varepsilon$ and so

$$\text{diam}(\Gamma_\varepsilon(u_\varepsilon) \cup \Gamma_\varepsilon(\tilde{u}_\varepsilon)) \leq \frac{4 \varepsilon}{\tilde{u}^0}$$

and the proof of [42, Lemma A.4(i), p. 25] remains intact. Hence we obtain

$$\int_{\alpha_\varepsilon}^t |\tilde{u}_\varepsilon \tilde{\delta}_\varepsilon - u_\varepsilon \delta_\varepsilon| \, ds \leq C |\tilde{u}_\varepsilon(t) - u_\varepsilon(t)| .$$

Summing up we have

$$\int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s |\frac{u_\varepsilon}{N_\varepsilon} - \frac{\tilde{u}_\varepsilon}{N_\varepsilon}| \, ds \, dr \leq C \psi(t) ,$$

where $\psi(t) := |u_\varepsilon(t) - \tilde{u}_\varepsilon(t)| + |\tilde{u}_\varepsilon(t) - \tilde{\tilde{u}}_\varepsilon(t)| + |z_\varepsilon(t) - \tilde{z}_\varepsilon(t)| + |\tilde{z}_\varepsilon(t) - \hat{z}_\varepsilon(t)|$.

The second integrand of (39) can be controlled as follows

$$\frac{|u_\varepsilon \Delta_\varepsilon|}{N_\varepsilon} - \frac{\tilde{u}_\varepsilon \tilde{\Delta}_\varepsilon}{N_\varepsilon} \leq \frac{2}{a^2} |u_\varepsilon \Delta_\varepsilon - \tilde{u}_\varepsilon \tilde{\Delta}_\varepsilon| + \frac{4}{a^4} |\tilde{u}_\varepsilon \Delta_\varepsilon| |\tilde{N}_\varepsilon - N_\varepsilon| ,$$

where the first term on the right-hand-side can be bounded as in [42, (A.35), p. 27] using the above generalization of [42, Lemma A.4(i), p. 25]. The second term is proportional to the one estimated in (40) and we obtain

$$|u_\varepsilon(t) - \tilde{u}_\varepsilon(t)| \leq C \varepsilon^q + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \psi(r) \, ds \, dr .$$

The estimates for $|\tilde{u}_\varepsilon(t) - \hat{u}_\varepsilon(t)|$ are obtained in complete analogy and give

$$|\tilde{u}_\varepsilon(t) - \hat{u}_\varepsilon(t)| \leq C \varepsilon^q + C \int_{\alpha_\varepsilon}^t \psi(s) \, ds .$$
The estimates for $|z_\varepsilon(t) - \tilde{z}_\varepsilon(t)|$ and its derivative are obtained along the same lines now using an analogous generalization of [42, Lemma A.4(ii), p. 25] and give

$$|z_\varepsilon(t) - \tilde{z}_\varepsilon(t)| \leq C\varepsilon^q + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \psi(r)drds,$$

$$|\dot{z}_\varepsilon(t) - \dot{\tilde{z}}_\varepsilon(t)| \leq C\varepsilon^q + C \int_{\alpha_\varepsilon}^t \psi(s)ds + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \psi(r)drds + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \int_{\alpha_\varepsilon}^{\tau} \psi(\tau)drd\tau ds.$$ 

Here we have used that $x_\varepsilon$ and $\tilde{x}_\varepsilon$ are the fixed points of the corresponding operators (26) $A_\varepsilon$ and $\tilde{A}_\varepsilon$, respectively. Thus $z_\varepsilon = A_\varepsilon^2(x_\varepsilon)$ and $\tilde{z}_\varepsilon = \tilde{A}_\varepsilon^2(\tilde{x}_\varepsilon)$ and we can eliminate any $1/\varepsilon$-terms, in contrast to [42 Proposition A.5, p. 26ff.].

So finally,

$$\psi(t) \leq C\varepsilon^q + C \int_{\alpha_\varepsilon}^t \psi(s)ds + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \psi(r)drds + C \int_{\alpha_\varepsilon}^t \int_{\alpha_\varepsilon}^s \int_{\alpha_\varepsilon}^{\tau} \psi(\tau)drd\tau ds,$$

and consequently $\sup_{t \in J} \psi(t) \leq C\varepsilon^q$ by Bykov’s inequality (a generalization of Gronwall’s inequality, cf. [2 Theorem 1.11, p. 98f.]). Using [16 Lemma 1.2.3, p. 11] we obtain that $(u_\varepsilon - \tilde{u}_\varepsilon)_e$ and $(z_\varepsilon - \tilde{z}_\varepsilon)_e$ are negligible. \(\square\)

4. Geodesic completeness

We now address the issue of completeness and begin with a discussion of the geodesics neglected so far, i.e., those whose initial data is not given by a seed geodesic crossing $U = 0$. These remaining geodesics $\gamma = (U, V, Z_\rho)$ all have a trivial $U$-component hence propagate parallel to the impulsive wave located at $\{U = 0\}$. In case $U = \text{const.} \neq 0$ such geodesics will never enter the regularized wave zone for $\varepsilon$ small enough. Hence they are background geodesics and therefore complete. Finally to treat geodesics with $U \equiv 0$, observe that the surface $\{U = 0\}$ is totally geodesic (not only in the background but also) in the generalized spacetime, which can be seen from the $U$-component of the geodesic equations [14] (see also [39], the discussion prior to Theorem 3.6). Hence such geodesics have trivial $U$-components and consequently the system [14] reduces to the background geodesic equations which again leads to completeness.

So the only real issue are the geodesics crossing the shock which we have already dealt with in Section 3 above. Recall that there we did not put any restrictions on the cosmological constant $\Lambda$ or the causal character $e$ of the geodesics. So we obtain without further effort our main result, the geodesic completeness of all non-expanding impulsive gravitational waves propagating in the (anti-)de Sitter universe.

**Corollary 4.1 (Completeness).** The generalized impulsive wave spacetime $(M, g)$ given by [13], [12] is geodesically complete.

This result is the desired generalization of the completeness result for impulsive pp-waves in [20] to the case $\Lambda \neq 0$. Also it is related to the completeness result [43 Thm. 5.3], where the latter is formulated in the spirit of [12] without the use of nonlinear generalized functions. This result applies to a more general class of impulsive waves including gyratonic terms and allowing for a non-flat wave surface, but necessarily have vanishing $\Lambda$. Observe, however, that the techniques developed in the present paper rely heavily on the 5-dimensional embedding formalism and hence differ significantly from the ones used in the $\Lambda = 0$-cases of [20], [12].
5. Associated geodesics

In this final section we briefly demonstrate how we can recover the limiting geodesics of the background spacetime across the impulsive wave surface. For simplicity we only deal with the first crossing of the impulse at \( t = 0 \). Let \([ (\gamma_\varepsilon)_\varepsilon ] = [(U_\varepsilon, V_\varepsilon, Z_{\text{pe}})_\varepsilon] \) be the (global) solution of (31), (33), given by Theorem 3.5 and set

\[
(42) \quad \tilde{\gamma}(t) = (\tilde{U}, \tilde{V}, \tilde{Z}_p)(t) := \begin{cases} 
\gamma^-(t), & t \leq 0 \\
\gamma^+(t), & t > 0,
\end{cases}
\]

where \( \gamma^-(U^-V^-, Z^-) \) is the seed geodesic with data (20), that is

\[
(43) \quad \gamma^-(0) = (0, V^0, Z^0_p), \quad \text{and} \quad \tilde{\gamma}^-(0) = (\tilde{U}^0 > 0, \tilde{V}^0, \tilde{Z}_p^0),
\]

(cf. the discussion prior to (33)). Furthermore \( \gamma^+ = (U^+, V^+, Z^+_p) \) is the background geodesic crossing \( U = 0 \) at \( t = 0 \) with data given by the limit of (35), that is more precisely

\[
(44) \quad \gamma^+(0) = (0, B, Z^0_p), \quad \text{and} \quad \tilde{\gamma}^+(0) = (\tilde{U}^0, C, A_p),
\]

where

\[
(45) \quad A_p := \lim_{\varepsilon \to 0} \tilde{Z}_{\text{pe}}(\beta_\varepsilon), \quad B = \lim_{\varepsilon \to 0} V_\varepsilon(\beta_\varepsilon), \quad C = \lim_{\varepsilon \to 0} \tilde{V}_\varepsilon(\beta_\varepsilon).
\]

Again \( \beta_\varepsilon \) is defined to be the time when \( \gamma_\varepsilon \) leaves the regularization strip, i.e., \( U_\varepsilon(\beta_\varepsilon) = \varepsilon \). We also remark that the constants \( A_p, B, \) and \( C \) have been explicitly computed in terms of the seed data (43) and the values of \( H \) and its first order derivatives at the impulsive surface in [42, Proposition 5.3].

Now [42, Theorem 5.1, p. 14ff.] translates into the following statement.

**Theorem 5.2** (Associated geodesics). The solution \( \gamma = (U, V, Z_p) = [(\gamma_\varepsilon)_\varepsilon] \) of (14), (33) is associated with the limiting geodesic \( \tilde{\gamma} \) of (12). Moreover we have \( U \approx_1 \tilde{U} \) and \( Z_p \approx_0 \tilde{Z}_p \).

Note that these convergences are optimal in the light of \( \tilde{V} \) and \( \tilde{Z}_p \) being discontinuous across \( t = 0 \), i.e., the limiting geodesics being refracted geodesics of the background suffering a jump in the \( V \)-position and \( V \)-velocity as well as in the \( Z_p \)-velocity, cf. [42, Section 5].

Finally we discuss our completeness result in the light of the standard result (e.g. [5, Rem. 3.5]) in smooth semi-Riemannian geometry which says that for an incomplete geodesic \( \lambda \)
necessarily $\lambda'$ leaves any compact subset of the tangent bundle. On the other hand even if $\lambda$ is complete and stays in a compact set in the manifold, $\lambda'$ need not be contained in a compact subset of the tangent bundle. The situation in the generalized setting is more subtle as can be seen from the spacetime at hand.

First note that while in the background (anti-)de Sitter spacetime the spacelike (respectively timelike) geodesics are closed this is not the case in the impulsive spacetime due to the jump of the limiting geodesic $\tilde{\gamma}$ in the $V$-position on crossing the impulse. Moreover the jump will in general depend on the specific profile function $H$ and $\tilde{\gamma}$ will in general not stay in a compact set $K \Subset M$. Consequently, one cannot guarantee that the points $\gamma(t_n)$ when the generalized geodesic $\gamma = [(\gamma_\varepsilon)_{\varepsilon}]$ hits the impulse are confined to a compact subset of the hypersurface on which the wave is supported.

Now we concentrate our attention to just one crossing of the impulse by a generalized geodesic $\gamma$. By the boundedness properties established in [42, Prop. 4.1(i)-(iii)] $\gamma_\varepsilon$ stays in a compact subset of $M$ uniformly in $\varepsilon$. However, one sees that by the jump in $V$ the boundedness results established in [42, Prop. 4.1(iv)] on $\dot{V}_\varepsilon$ are optimal in the sense that one can only bound $\dot{V}_\varepsilon(\beta_\varepsilon)$, i.e., $\dot{V}_\varepsilon$ at the instant it exits the wave zone but not throughout the entire wave zone. Hence $\gamma_\varepsilon$ will not be confined to a compact set of $TM$ uniformly in $\varepsilon$ even on one single crossing. So one can rather think of the situation already locally of being similar to the above, where one has complete geodesics in a compact set with their tangents not contained in a compact set.

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