Falling Toward Charged Black Holes

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Abstract

The growth of the “size” of operators is an important diagnostic of quantum chaos. In [1] it was conjectured that the holographic dual of the “size” of an operator is the radial component of the momentum of the particle created by the operator. Thus the growth of operators in the background of a black hole is nothing but the acceleration of the particle as it falls toward the horizon, and vice versa.

In this note we will use the momentum-size correspondence as a tool to study scrambling in the field of a near-extremal charged black hole. The agreement with previous work provides a non-trivial test of the momentum-size relation, as well as an explanation of a paradoxical feature of scrambling previously discovered by Leichenauer [2]. Naively Leichenauer’s result says that only the non-extremal entropy participates in scrambling. The same feature is also present in the SYK model.

In this paper we find a quite different interpretation of Leichenauer’s result which does not have to do with any decoupling of the extremal degrees of freedom. Instead it has to do with the buildup of momentum as a particle accelerates through the long throat of the Reissner-Nordström geometry.
1 Two Puzzles

All horizons are locally the same; namely they are Rindler-like\(^1\). Therefore one might expect their properties as scramblers and complexifiers to be universal. For example, the rate of growth of complexity for all neutral static black holes scales as

\[
\frac{dC}{dt} \sim \frac{S}{R_s},
\]

(1.1)

where \(S\) and \(R_s\) are the entropy and Schwarzschild radius of the black hole [3]. Similarly there is a universal formula for the scrambling time [4],

\[
t_* = \frac{\beta}{2\pi} \log S.
\]

(1.2)

It is therefore surprising that charged black holes behave differently. For charged black holes, 1.1 and 1.2 are modified to [3, 2]

\[
\frac{dC}{dt} \sim \frac{S - S_0}{R_s}
\]

(1.3)

\[
t_* = \frac{\beta}{2\pi} \log (S - S_0).
\]

(1.4)

\(^1\)Extremal black holes are an exception. In this paper we consider the limit in which the non-extremality parameter \((r_+ - r_-)/r_+\) is arbitrary but fixed as \(r_+\) becomes large. In this limit the horizon is Rindler-like.
Here $R_+$ is the area-radius of the horizon (otherwise known as $r_+$) and $S_0$ is the entropy of the extremal black hole with the same charge.

A simple explanation would be that the extremal degrees of freedom are somehow decoupled from the chaotic behavior, leaving only the non-extremal component to actively “compute”. But given the fact that all horizons are Rindler-like, it is hard to understand why this should be so.

In what follows we will see a correct interpretation of [1.3] and [1.4] that has nothing to do with any decoupling of extremal degrees of freedom. Horizons do indeed have universal computational properties in which all $S$ degrees of freedom actively compute. Neutral and charged black hole horizons compute in exactly the same way.

For simplicity, in this paper we will work with asymptotically flat black holes. Our results would apply equally to black holes of small or intermediate size in AdS.

## 2 Complexity Growth

The explanation of [1.3] for the rate of complexity growth is simple. For near-extremal black holes, the entropy above extremality is linear in the temperature $T$

$$\frac{S - S_0}{S} \sim r_+ T \equiv \frac{r_+}{\beta}, \quad (2.5)$$

where $r_+$ is the area-radius of the outer horizon. Thus we may write [1.3] in the form,

$$\frac{dC}{dt} \sim TS. \quad (2.6)$$

We can get more insight into the meaning of [2.6] by replacing the usual Schwarzschild time $t$ by the dimensionless Rindler time $\tau = \frac{2nt}{\beta}$ (i.e. the hyperbolic angle), which gives

$$\frac{dC}{d\tau} \sim S \quad (2.7)$$

Equation [2.7] expresses a universal property of horizons, charged and uncharged: they all compute at a rate $\sim$ one gate per Rindler time per bit of entropy. All degrees of freedom participate; extremal degrees of freedom do not decouple.
3 Scrambling = Falling

Consider a black hole whose holographic dual is perturbed by a simple operator $W$. In the boundary theory, this perturbation then grows with time: $W(t) \equiv U(t)^\dagger W U(t)$ becomes an increasingly complicated operator. We can track how complicated the operator has become by its “size”—roughly speaking the size of a generic $k$-local operator is $k$. In fermionic systems the size can be calculated from the out-of-time-order correlator \[^5\], and can be shown to grow exponentially for some period \[^4, 5, 6\].

In the bulk, the perturbation $W$ can create a particle that then falls in. As it falls towards the black hole, the particle accelerates. According to \[^1\] the growth of the radial momentum of the particle is dual to the growth of the size of the operator in the boundary theory. Both the particle’s momentum and the operator’s size grow exponentially, and with exactly the same Lyapunov exponent.

3.1 The Neutral Case

If the particle was created at time $t = 0$ with energy $E_0$ of order $1/R_s$ then the operator $W(0)$ has size \[^1\]

$$s(0) = 1. \quad (3.8)$$

As time passes, the size $s(t)$ of the operator $W(t)$ grows. The basic conjecture of \[^1\] is that the size of the operator is dual to the radial momentum of the infalling particle (as measured by a static observer at fixed radius),

$$s(t) \sim R_s |P(t)| \quad (3.9)$$

The arguments for this identification were given in \[^1\] and will not be repeated here.

In the Rindler region close to the event horizon ($r \lesssim 2R_s$) the momentum of an infalling particle grows according to

$$P(\tau) \sim E_0 e^\tau, \quad (3.10)$$

where again $\tau$ is the Rindler time $\tau \equiv \frac{2\pi t}{\beta}$. It was noted in \[^1\] that \[^3, 10\] defines a Lyapunov exponent that saturates the chaos bound of \[^4\].

The size of an operator cannot exceed the entropy $S$ of the black hole. The definition of the scrambling time $t_*$ is how long it takes to saturate this maximum size. According to
this means the scrambling time for an initially unit-sized operator is how long it takes for the momentum to grow by an overall multiplicative factor of $S$. Equation (3.10) tells us this occurs when
\[
\exp \left( \frac{2\pi t_*}{\beta} \right) = S. \tag{3.11}
\]
Thus one finds
\[
t_* = \frac{\beta}{2\pi} \log S, \tag{3.12}
\]
or expressed in Rindler units,
\[
\tau_* = \log S. \tag{3.13}
\]
Had the initial operator $W$ been of size greater than one, say $s_i$, then (3.11) would have been replaced by
\[
s_i \exp \left( \frac{2\pi t_*}{\beta} \right) = S, \tag{3.14}
\]
and therefore the time required to scramble the perturbation reduced to
\[
\tau_* = \log \left( \frac{S}{s_i} \right). \tag{3.15}
\]

### 3.2 The Charged Case

Consider the 3+1-dimensional Reissner-Nordström black hole,
\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2
\]
\[
f(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{r_-}{r} \right). \tag{3.16}
\]
The inner (-) and outer (+) horizons are at $r_\pm \equiv GM \pm \sqrt{G^2M^2 - GQ^2}$, and the Hawking temperature is
\[
T = \frac{1}{4\pi} \left( \frac{r_+ - r_-}{r_+^2} \right). \tag{3.17}
\]
All Reissner-Nordström black holes must have $Q^2 \leq GM^2$ and $r_- \leq r_+$, and when these inequalities are saturated the black hole is said to be ‘extremal’. In this paper, we will be interested in near-extremal black holes, so that $r_+ - r_- \ll r_+$. In this limit, the temperature is small ($\beta \gg r_+$) and the near-horizon region develops a ‘throat’.

3.2.1 The geometry of the throat

The exterior of a near-extremal black hole can be divided into three regions, as in Fig. [1]

- The innermost region is the Rindler region, defined by

\[
\begin{align*}
  r_+ < r & \lesssim 2r_+ - r_- \\
  0 < \Delta \rho & \lesssim r_+,
\end{align*}
\]

where $\Delta \rho = \int dr/\sqrt{f(r)}$ is the proper distance from the outer horizon. The gravitational field (i.e. the proper acceleration $\alpha = \partial_r \sqrt{f(r)}$ required to remain static at fixed $r$) grows rapidly near the horizon. While the quantity $(1 - \frac{r_+}{r})^{-1}$ varies significantly in the Rindler region, $(1 - \frac{r_-}{r})^{-1}$ is essentially constant.

- The next region out is the throat, defined by

\[
\begin{align*}
  2r_+ - r_- \lesssim r & \lesssim 2r_+ \\
  r_+ \lesssim \Delta \rho & \lesssim r_+ \log \left[ \frac{r_+}{r_+ - r_-} \right].
\end{align*}
\]

The throat is long and of approximately constant width (it resembles AdS$_2 \times S^2$) and the gravitational field is approximately constant. We have $(1 - \frac{r_+}{r})^{-1} \sim (1 - \frac{r_-}{r})^{-1}$ and both vary significantly through the throat. The throat is unique to charged black holes—in uncharged black holes the Rindler region connects directly onto the Newtonian region.

(Later we will comment on the relation between near-extremal RN black holes and the SYK model. For now we note that the dynamical boundary of the AdS$_2$ dual of SYK should be identified with the end of the RN throat adjacent to the Newtonian region.)

- Outermost is the Newtonian region, where $(1 - \frac{r_+}{r})^{-1} \sim (1 - \frac{r_-}{r})^{-1} \sim 1.$
3.2.2 Falling into a charged black hole

Now consider a particle of initial energy $E_0 = 1/r_+$ falling into the black hole. We will track the momentum of this particle as a function of time. The momentum $P$ is to be measured by a static (fixed $r$) observer\(^2\); since the particle is moving ultrarelativistically, the momentum is equal to the energy $E_0\sqrt{-g^{tt}}$ so

$$P = \frac{E_0}{\sqrt{(1 - \frac{r_+}{r})(1 - \frac{r_+}{r})}}. \quad (3.22)$$

The ‘time’ will not be the proper time of the particle, but instead the Schwarzschild time $t$. The time for an ultrarelativistic particle to reach a depth $r$ is

$$\Delta t = \int \frac{dr}{-f(r)} = \Delta \left[ -r + \frac{r_+^2 \log[r - r_-] - r_-^2 \log[r - r_+]}{r_+ - r_-} \right]. \quad (3.23)$$

Upon arriving from the Newtonian region, the particle first encounters the throat. Even though the throat is long, it is traversed in just one thermal time

$$\Delta t \bigg|_{\text{throat}} \sim \beta \quad \leftrightarrow \quad \Delta \tau \bigg|_{\text{throat}} \sim 1. \quad (3.24)$$

---

\(^2\)This definition coincides with the one used in [1]: it is the canonical momentum conjugate to $\rho$. 

Figure 1: The three regions outside a near-extremal charged black hole. Unlike for uncharged black holes, there is now a ‘throat’ separating the Rindler and Newtonian regions.
During this short time, the momentum grows linearly with a large coefficient, exiting the throat into the Rindler region with momentum

$$P_{\Delta \rho = r_+} \sim E_0 \frac{\beta}{r_+} \gg E_0.$$ 

(3.25)

According to the momentum-size correspondence of Eq. 3.9, this implies that the size of $W(t)$ has grown by a factor $\beta/r_+$ in just a single thermal time.

Now let us follow the particle as it falls through the Rindler region. The momentum calculation is the same as for a neutral black hole, Eq. 3.10, with the same exponential growth

$$P_{\text{Rindler region}} \sim E_0 \left( \frac{\beta}{r_+} \right) e^{\tau}.$$ 

(3.26)

The only difference from the uncharged case is the prefactor $\beta/r_+$, which is the additional momentum that is picked up traversing the throat. To calculate the scrambling time we follow the procedure that led to 3.14 and see how long it take for the size of the perturbation to reach its saturation value $S$. Up to $O(1)$ additive terms, the scrambling time is

$$S \sim \left( \frac{\beta}{r_+} \right) e^{\tau} \rightarrow \tau_* = \log \frac{r_+ S}{\beta}.$$ 

(3.27)

Using 2.5, we can express this in terms of the extremal entropy to recover 1.2

$$\tau_* = \log (S - S_0).$$ 

(3.28)

To summarize, the scrambling time for a near-extremal Reissner-Nordström black hole is smaller than might have been expected: it is proportional to $\log (S - S_0)$ rather than $\log S$. One possible explanation would have been that the extremal entropy is somehow frozen out of the scrambling process. But our analysis suggests a different reason. We saw that a particle falling through the throat is rapidly accelerated by a factor $\beta/r_+$. By the time the particle reaches the Rindler region ($\Delta \rho = r_+$), the momentum is therefore much larger than it would have been for an uncharged black hole, which reduces the time taken for the momentum to grow by a total factor of $S$. This suggests that the correct explanation for the reduced scrambling time of a charged black hole is not that the extremal degrees of freedom decouple, but instead that the size of an operator grows rapidly
at early times; it will be interesting to see this supported by a direct calculation in the dual theory. The success in explaining the reduced scrambling time of charged black holes provides a non-trivial confirmation of the connection between size and momentum.

4 Comment about GR=QM and SYK

A final point worth noting is that Eq. 1.3 was derived for the SYK model [7] on purely quantum mechanical grounds, without any assumption of a dual geometry. It is extremely interesting that a whole class of very generic quantum systems reproduce a formula whose interpretation is both geometric and gravitational: the existence of a long throat (geometry) through which a particle will accelerate as it falls toward the horizon (gravity). This is another example of “GR=QM”, i.e. the view that the origins of gravity are to be found in the generic behavior of complex quantum systems [8].

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