Holography and Brane Cosmology in Domain Wall Backgrounds

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Abstract

We consider a class of domain-wall black hole solutions in the dilaton gravity with a Liouville-type dilaton potential. Using the surface counterterm approach we calculate the stress-energy tensor of quantum field theory (QFT) corresponding to the domain-wall black hole in the domain-wall/QFT correspondence. A brane universe is investigated in the domain-wall black hole background. When the tension term of the brane is equal to the surface counterterm, we find that the equation of motion of the brane can be mapped to the standard form of FRW equations, but with a varying gravitational constant on the brane. A Cardy-Verlinde-like formula is found, which relates the entropy density of the QFT to its energy density. At the moment when the brane crosses the black hole horizon of the background, the Cardy-Verlinde-like formula coincides with the Friedmann equation of the brane universe, and the Hubble entropy bound is saturated by the entropy of domain-wall black holes.

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1 Introduction

Holographic principle is perhaps one of fundamental principles of nature, which relates a theory with gravity in $D$ dimensions to a theory in $(D - 1)$ dimensions without gravity \[1, 2\]. The AdS/CFT correspondence \[3, 4, 5\] is a beautiful example for the realization of the holographic principle.

The brane world scenario of the Randall and Sundrum model \[6\] has been a new arena to further understand the holographic feature of gravity (see, for example, \[7, 8\] and references therein). In Ref. \[3\] it has been shown that a radiation-dominated flat ($k = 0$) Friedmann-Robertson-Walker (FRW) cosmology emerges as the induced metric on a codimension one hypersurface of constant extrinsic curvature in the background of a five-dimensional AdS Schwarzschild solution with a Ricci flat horizon. The radiation can be interpreted as the conformal field theory (CFT) corresponding to the AdS Schwarzschild black hole in the AdS/CFT correspondence. This holographic picture has been further studied recently by Savonije and Verlinde \[10\], based on an observation made by Verlinde \[11\] on the relation between the Cardy formula of CFTs and the Friedmann equation of a closed FRW universe. By considering a brane universe in the background of AdS Schwarzschild black holes in arbitrary dimensions, except that the induced geometry of the brane is exactly given by that of a standard radiation-dominated closed FRW universe, Savonije and Verlinde observed that when the brane crosses the horizon of background geometry, the entropy and temperature of the universe can be simply expressed in terms of the Hubble constant and its time derivative; the entropy formula of CFTs in arbitrary dimensions, called the Cardy-Verlinde formula \[11\], coincides with the Friedmann equation of the brane universe; and the Hubble entropy bound is just equal to the entropy of the AdS Schwarzschild black holes\[1\].

It would be of interest to see to what extent these observations on the holographic properties are universally valid. For this purpose, in this paper we investigate these holographic connections in the case of dilaton domain-wall black holes as the background, replacing the AdS Schwarzschild black holes. In this case, according to the domain-wall...
wall/QFT correspondence [25, 26, 27], corresponding to the domain-wall black hole is a general quantum field theory (QFT), rather than a CFT. Using the method of surface counterterm, in section II we first calculate the stress-energy tensor of the QFT and discuss the thermodynamics of the domain-wall black holes. The brane cosmology in the background of the dilaton domain-wall black holes and holographic properties of the brane universe are discussed in section III. Our results are summarized in section IV with a brief discussion.

2 Domain-wall black holes

We start with the action of an \((n+2)\)-dimensional dilaton gravity with a Liouville-type dilaton potential,

\[
S = \frac{1}{16\pi G_{n+2}} \int d^{n+2}x \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 + V_0 e^{-a\phi} \right),
\]

(2.1)

where \(G_{n+2}\) is the gravitational constant in \((n+2)\) dimensions, \(V_0\) and \(a\) are assumed to be two positive constants (when \(a\) is a negative constant, it can be changed to a positive one by replacing \(\phi\) by \(-\phi\)).

Such an action (2.1) naturally arises as a consistent truncation of various supergravities. For example, in the decoupling limit of Dp-branes in type II supergravity, one can do a consistent sphere reduction in the dual frame, resulting in an effective action like Eq. (2.1) with

\[
V_0 = \frac{1}{2} (9 - p)(7 - p)N^{-2\lambda/p}, \quad a = \frac{\sqrt{2}(p - 3)}{\sqrt{p(9 - p)}}, \quad \lambda = \frac{2(p - 3)}{7 - p},
\]

(2.2)

where \(p = n\) and \(N\) is the number of Dp-branes. In this case, when \(p = 3\), the dilaton potential becomes a constant, and one then has a five-dimensional AdS Schwarzschild black hole solution in the action (2.1) with a constant dilaton. This is consistent with the fact that the bulk geometry of D3-branes is a five-dimensional AdS Schwarzschild solution (times a 5-sphere) in the decoupling limit.

In a general case, one may consider the following Lagrangian in \(D\) dimensions [29]

\[
\mathcal{L} = R - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2(D - n - 2)!} e^{a_1 \phi_1} F^{2}_{(D-n-2)},
\]

(2.3)
where \( F_{(D-n-2)} \) is a \((D-n-2)\)-form field strength and
\[
a_1^2 = \frac{4}{N_1} - \frac{2(n+1)(D-n-3)}{D-2}.
\]

\( N_1 \) is an integer in supergravity theories. The case \( N_1 = 1 \) can arise for all forms in supergravity. In particular, all the field strengths have \( N_1 = 1 \) in \( D = 10 \) and \( D = 11 \) supergravities. The case \( N_1 = 2 \) can appear for 2-forms in \( D < 9 \), and 3-forms in \( D < 6 \) in the non-maximal supergravities. The case \( N_1 = 3 \) can arise for 2-forms in \( D \leq 5 \) and \( N_1 = 4 \) for 2-forms in \( D \leq 4 \) \[29\]. After a consistent sphere reduction, one can obtain an effective action like (2.1) with \[29\]
\[
V_0 = \frac{2(D-n-3)^2b^2}{\triangle}, \quad \triangle = \frac{4(D-n-3)}{2(D-n-2) - (D-n-3)N_1},
\]
\[
a^2 = \frac{2(n+1)}{n} - \frac{4(D-n-3)}{2(D-n-2) - (D-n-3)N_1},
\]
where \( b^2 \) is a parameter.

There exists a class of domain wall solutions in the action (2.1), and the localization of gravity on the domain wall has been investigated in some detail in \[29\] (see also \[30\]). For our purposes, however, we need to look for solutions with black hole horizon. Solving the equations of motion of the action (2.1), we find a set of solutions as follows,
\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + R^2(r)d\tau^2_n,
\]
\[
R(r) = r^N,
\]
\[
\phi(r) = \phi_0 + \sqrt{2nN(1-N)} \ln r,
\]
\[
f(r) = \frac{V_0 e^{-a\phi_0}r^{2N}}{nN(N(n+2)-1)} - \frac{mN^{1-nN}}{\sqrt{2nN(1-N)}},
\]
where \( dx^2_n \) stands for the line element for a \( n \)-dimensional Ricci flat space, \( \phi_0 \) and \( m \) are two integration constants and the parameter \( N \) obeys
\[
a = \frac{\sqrt{2nN(1-N)}}{nN}.
\]
Let us first discuss some special cases of the solution (2.6). When \( m = 0 \), the solution (2.6) can be transformed to the form of domain-wall solution in \[29\] by dropping the ”1” of the
harmonic function $H$ there. When $m \neq 0$, the solution (2.6) also approaches the domain-wall solution as $r \to \infty$ if $1 - nN < 2N$. When $N = 1$, by redefining the integration constant $m$, we can see that in this case the solution is just the AdS Schwarzschild black hole solution with a Ricci flat horizon, and the constant $m$ is proportional to the mass of black hole. In a general case where $m > 0$ and $1 - nN < 2N$, the solution (2.6) has a black hole horizon $r_\pm$ satisfying the equation $f(r_\pm) = 0$. In this sense, we call the solution (2.6) domain-wall black hole. When $m < 0$, the singularity at $r = 0$ in the solution (2.6) becomes naked. Therefore, we will not further consider this case.

In the AdS/CFT correspondence, the thermodynamics of AdS Schwarzschild black hole in the high temperature limit can be identified with that of boundary CFTs \[37\]. Naturally we can identify the thermodynamics of domain-wall black holes with that of QFTs residing on the domain walls \[27\]. In the traditional Euclidean path integral approach to black hole thermodynamics, one usually uses the background subtraction procedure, in which in order to get a finite Euclidean action of black hole and a finite quasilocal stress-energy tensor of gravitational field \[34\], one has to choose a suitable reference background. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. Furthermore, sometimes one may encounter situations in which there are no appropriate reference backgrounds. In asymptotically AdS spacetimes, this difficulty has been solved by adding suitable surface counterterms to the gravitational action \[35\]. In this way, one can obtain a well-defined boundary stress-energy tensor and a finite Euclidean action for the asymptotically AdS spacetimes.

Now we calculate the stress-energy tensor of the QFT corresponding to the bulk solution (2.6) using the surface counterterm approach. In \[27\] (see also \[28\]) we have already shown that for a class of solutions like (2.6) one can also obtain a well-defined boundary stress-energy tensor and a finite Euclidean action by adding an appropriate surface term to the bulk action, although the solution is not asymptotically AdS. For the solution (2.6),

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2This term comes from Ref. \[31\], in which a set of charged domain-wall black hole solutions with $N = 1/2$ has been found. Such solutions have been called black plane solutions in earlier references \[32\], there some charged black plane solutions have been obtained in four dimensions. Similar solutions have been found independently in \[33\].
we find that the suitable counterterm is
\[ S_{ct} = -\frac{1}{8\pi G_{n+2}} \int d^{n+1}x \sqrt{-h} \frac{c_0}{l_{\text{eff}}}, \]  
where
\[ c_0 = n \sqrt{\frac{N(n+1)}{N(n+2) - 1}}, \quad \frac{1}{l_{\text{eff}}} = \sqrt{\frac{V_0 e^{-\alpha \phi}}{n(n+1)}}. \]

With this counterterm, the quasilocal stress-energy tensor of gravitational field at the boundary \( r \) with an induced metric \( h \) is
\[ T_{ab} = \frac{1}{8\pi G_{n+2}} \left( K_{ab} - Kh_{ab} - \frac{c_0}{l_{\text{eff}}} h_{ab} \right), \]
where \( K \) is the extrinsic curvature of the boundary \( h \). Calculating the tensor yields
\[ 8\pi G_{n+2} T_{tt} = \frac{nN}{2} \left( \frac{V_0 e^{-\alpha \phi}}{nN(N(n+2) - 1)} \right)^{1/2} \frac{m}{\sqrt{2nN(1-N)}} \frac{1}{r^{(n-1)N}} + \cdots, \]
\[ 8\pi G_{n+2} T_{ij} = \delta_{ij} \frac{2N-1}{2} \left( \frac{V_0 e^{-\alpha \phi}}{nN(N(n+2) - 1)} \right)^{-1/2} \frac{m}{\sqrt{2nN(1-N)}} \frac{1}{r^{(n-1)N}} + \cdots, \]
where the ellipses denote higher order terms, which have no contributions when we move the boundary to the spatial infinity (\( r \to \infty \)).

Given a well-defined quasilocal stress-energy tensor of gravitational field, one can calculate some conserved charges associated with Killing vectors \[ [35] \]. The energy of gravitational field is a conserved charge associated with a timelike Killing vector. Applying this to the solution (2.6), we obtain the mass \( M \) of the domain-wall black holes
\[ M = \int_{r \to \infty} d^n x r^{nN} f^{-1/2} T_{tt} = \frac{nN}{\sqrt{2nN(1-N)}} \frac{mV_n}{16\pi G_{n+2}}, \]
where \( V_n \) denotes the volume of the domain wall, namely, the volume of line element \( dx_n^2 \) in the solution (2.6). We see that as we explained above, the integration constant \( m \) is indeed related to the mass of the domain-wall black holes.

Next we calculate the stress-energy tensor of the thermal QFT corresponding to the domain-wall black hole solution (2.6). The boundary metric \( \gamma_{ab} \) of the spacetime, in which the boundary QFT resides, can be determined as follows,
\[ \gamma_{ab} = \lim_{r \to \infty} \frac{1}{r^{2N}} h_{ab} = -\frac{V_0 e^{-\alpha \phi}}{nN(N(n+2) - 1)} dt^2 + dx_n^2. \]
Rescaling the time coordinate \( t \), one has
\[
\gamma_{ab} = -d\tau^2 + dx_n^2.
\] (2.14)

The boundary stress-energy tensor \( \tau_{ab} \) of the QFT can be achieved through the relation
\[
\sqrt{-\gamma} \gamma^{ab} \tau_{bc} = \lim_{r \to \infty} \sqrt{-h} h^{ab} T_{bc}.
\] (2.15)

Substituting (2.11) into the above formula, we finally arrive at
\[
\tau_{\tau\tau} = M \frac{cV}{eN}, \quad c = \left( \frac{V_0 e^{-a\phi_0}}{nN(N(n+2)-1)} \right)^{1/2},
\tau_{ij} = \delta_{ij} \frac{M(2N-1)}{cnNV_n}.
\] (2.16)

This can be explained as the vacuum expectation value of the stress-energy tensor of the QFT residing in the spacetime (2.14). From (2.16) we obtain the equation of state of the QFT
\[
p = \frac{2N-1}{nN} \rho,
\] (2.17)
relating the pressure \( p \) and the energy density \( \rho \) of the QFT. We can see clearly from (2.17) that when \( N = 1 \), one has \( p = \rho/n \), an equation of state for CFTs. This is consistent with the fact that when \( N = 1 \), the solution (2.6) describes an \((n+2)\)-dimensional AdS Schwarzschild black hole, to which there is a \((n+1)\)-dimensional CFT corresponding. When \( N = 1/2 \), the pressure \( p \) vanishes. The D5-branes and NS5-branes are just this case [27]. When \( N < 1/2 \), the pressure becomes negative and in this case, the domain-wall/QFT correspondence might be invalid. The Dp-branes with \( p > 5 \) belong to this case [27], for which we know gravity does not decouple in the usual decoupling limit.

Before closing this section, let us calculate the Hawking temperature \( T_{\text{HK}} \) and the entropy \( S \) of the domain-wall black holes (2.6). A simple calculation yields
\[
T_{\text{HK}} = \frac{N(n+2)-1}{4\pi} c_2 r_+^{2N-1},
S = \frac{r_+^{nN}}{4G_{k+2}} V_n,
\] (2.18)
where \( c_2 \) is the one in (2.16) and \( r_+ \) is the horizon radius of the black hole. It is easy to check that these quantities satisfy, \( dM = T_{\text{HK}} dS \), the first law of black hole thermodynamics.
3 Brane cosmology in the domain-wall background

In this section following [10, 9], we discuss the dynamics of a brane universe in the background of the dilaton domain-wall black holes (2.6). The dynamics of the \((n+1)\)-dimensional brane is governed by the action

\[
S_b = -\frac{1}{8\pi G_{n+2}} \int d^{n+1} x \sqrt{-h} K - \int d^{n+1} x \sqrt{-h} \sigma, \tag{3.1}
\]

where the first term is the Gibbon-Hawking boundary term and \(\sigma\) is the tension of the brane. Viewing the brane as the edge at a finite \(r\) of the bulk (2.6), one has the equation of motion of the brane

\[
K_{ab} = \frac{8\pi G_{n+2}}{n} \sigma h_{ab}. \tag{3.2}
\]

Furthermore, in order for the variation principle to be well-defined for the total action, which is sum of the bulk action (2.1) and the brane action (3.1), the tension \(\sigma\) of the brane must depend on the dilaton and satisfies [33]

\[
n^a \partial_a \phi = 16\pi G_{n+2} \frac{\partial \sigma}{\partial \phi}, \tag{3.3}
\]

where \(n^a\) is a unit normal vector to the brane.

Now let us specify the location of the brane as \(r = r(t)\). If we introduce a cosmic time \(\tau\) so that \(t = t(\tau)\) and \(r = r(\tau)\) and satisfies

\[
f(r) \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{f(r)} \left( \frac{dr}{d\tau} \right)^2 = 1, \tag{3.4}
\]

from Eq. (2.6), one can see that the induced metric on the brane then takes the form

\[
ds^2 = -d\tau^2 + R^2(\tau) dx_n^2. \tag{3.5}
\]

Note that here \(R\) is a function of \(\tau\) through the relation \(R = r^N\). The equation (3.5) is a standard form of the metric for an \((n+1)\)-dimensional flat \((k = 0)\) FRW universe.

From Eqs. (3.2) and (3.4) we obtain the equation of motion of the scale factor \(R\)

\[
H^2 = \left( \frac{8\pi G_{n+2} \sigma}{n} \right)^2 - \frac{N^2}{R^{2/N} f(R)}, \tag{3.6}
\]

\(\text{The motion of a brane (domain wall) in the AdS black hole background has also been discussed in [38, 39], but in those papers the brane is embedded to the black hole background, and does not act as the edge of the bulk.}\)
where \( H = \dot{R}/R \) is the Hubble constant and the overdot stands for derivative with respect to the cosmic time \( \tau \). We find from Eqs. (3.2) and (3.3) that the tension \( \sigma \) is a function of \( \phi \) as (see also [33])

\[
\sigma = \sigma_0 e^{-a\phi/2},
\]

(3.7)

where \( \sigma_0 \) is a constant. Substituting the tension into Eq. (3.6) yields

\[
H^2 = \left( \frac{8\pi G_{n+2}\sigma_0}{n} \right)^2 e^{-a\phi_0} R^{2-2/N} - \frac{V_0 N^2 e^{-a\phi_0} R^{2-2/N}}{nN(N(n+2)-1)} + \frac{mN^2}{\sqrt{2nN(1-N)R^{n+1/N}}}. \]

(3.8)

Now we choose the constant \( \sigma_0 \) so that the first two terms in Eq. (3.8) cancel with each other. That is, we take

\[
\sigma_0 = \frac{1}{8\pi G_{n+2}} \sqrt{\frac{nNV_0}{N(n+2)-1}}.
\]

(3.9)

This choice corresponds to the fine-tuning of the brane in the case of AdS Schwarzschild black holes [10, 9]. In such a choice as (3.9), we note that the brane tension term in (3.1) is just equal to the surface counterterm (2.8). The surface counterterm (2.8) makes the quasilocal stress-energy tensor of gravitational field (2.10) be well-defined, while here the effective cosmological constant on the brane vanishes under the choice (3.9).

Given this choice, the Friedmann equation (3.8) then reduces to

\[
H^2 = \frac{16\pi G_{n+2}MN}{nV_n} \frac{1}{R^{n+1/N}},
\]

(3.10)

where the integration constant \( m \) has been replaced by the mass \( M \) of the domain-wall black hole. Further, differentiating the equation (3.10) with respect to the cosmic time \( \tau \), one can have the equation of the time derivative of \( H \),

\[
\dot{H} = - \left( n + \frac{1}{N} \right) \frac{8\pi G_{n+2} MN}{nV_n} \frac{1}{R^{n+1/N}}.
\]

(3.11)

The evolution of the brane universe is easily determined by integrating the equation (3.10), which gives

\[
R = \left( \frac{nN + 1}{2N} \right)^{2N/(nN+1)} e^{2N/(nN+1)}.
\]

(3.12)

Here an integration constant has been set to zero, for one can always do so by shifting the cosmic time \( \tau \). The solution (3.12) describes an expanding flat \( (k = 0) \) universe.

We now turn to studying the holographic properties associated with the brane universe. Note that the mass \( M \) of the domain-wall black holes, namely, the energy of thermal QFT,
is measured with respect to the coordinate time $t$ in the metric (2.13). Measured in the coordinates (3.5), the energy of the QFT is

$$E = \frac{1}{cR}M,$$

(3.13)

where $c$ is given by Eq. (2.16). Let us further note that the gravitational constant $G_{n+2}$ is the one in $(n + 2)$ dimensions. That is, it is the bulk gravitational constant. We find that the gravitational constant $G_{n+1}$ on the brane is

$$G_{n+1} = \frac{(n-1)Nc}{R^{(1-N)/N}}G_{n+2},$$

(3.14)

which has been obtained by adding a small amount of stress energy on the brane and then comparing the equation of motion of the brane with the standard Friedmann equation in $(n+1)$ dimensions\(^4\). It is interesting to note that the brane gravitational constant depends on the scale factor $R$. It implies that one can view the brane universe in the background of the dilaton domain-wall black holes as a universe model of varying gravitational constant.

As a consistency check, it is easy to see that when $N = 1$, i.e., the bulk geometry is the AdS Schwarzschild solution, the brane gravitational constant $G_{n+1}$ becomes a true constant and the relation $G_{n+1} = (n-1)G_{n+2}/L$ can be recovered \[9, 10\].

Using the gravitational constant $G_{n+1}$ and energy $E$ on the brane, we find that the equation (3.10) can be rewritten as

$$H^2 = \frac{16\pi G_{n+1} E}{n(n-1) V},$$

(3.15)

a standard form of the Friedmann equation for an $(n+1)$-dimensional flat ($k = 0$) universe, where $V = R^n V_n$ is the volume of the universe. So we see that the energy of brane universe is provided by the thermal QFT corresponding to the domain-wall black holes. For the time derivative of the Hubble constant $H$, we have

$$\dot{H} = -\frac{(nN+1)}{nN} \frac{8\pi G_{n+1} E}{(n - 1) V}.$$  

(3.16)

Comparing this with its standard form, $\dot{H} = -\frac{8\pi G_{n+1}}{(n-1)} \left( \frac{E}{V} + p \right)$, we obtain the equation of state of matter filling the brane universe

$$p = \frac{1}{nN} \rho, \quad \rho = \frac{E}{V}.$$  

(3.17)

\(^4\)For the equation of motion of brane with localized matter see \[38, 39\].
Comparing this with the equation (2.17) of state of the QFT, obtained in the previous section, one can see that they can be matched only when \( N = 1 \), namely, the bulk is the AdS Schwarzschild black holes (recall that in this case the gravitational constant on the brane is a true constant). This result seems strange, since naively one might expect that the equation of motion of the brane should be the Friedmann equation with the equation (2.17) of state of the QFT. This mismatch might be caused by the varying gravitational constant on the brane when \( N \neq 1 \), since we derived Eq. (3.17) from Eqs. (3.15) and (3.16) by comparing them with corresponding equations in the Einstein gravity. The fact, however, is that the gravity on the brane is a Brans-Dicke-like gravity theory: The gravitational constant is a dynamical one. Therefore we should expect that the dynamical gravitational constant makes some contributions to the pressure and energy density on the brane. This might be one of reasons why there is a difference between (3.17) and (2.17). Another possible cause is that there might be some contribution of the dilaton field. In Ref. [40], the gravitational equations on the brane have been given when the bulk action includes a dilaton potential. In our bulk solution (2.6), the dilaton field \( \phi \) is static in the coordinates (2.6), but is not static in the coordinates on the brane (3.5) through the relation \( r = r(\tau) \). From the gravitational equations in [40] one can see that the dynamical dilaton field will also make contributions to the energy density and pressure on the brane. Obviously, it is quite necessary to further clarify the origin of the difference between Eq. (2.17) and Eq. (3.17), which is currently under investigation.

Following Ref. [11], we define the Hubble entropy bound \( S_H \), the Bekenstein-Verlinde entropy bound \( S_{BV} \), and the Bekenstein-Hawking entropy bound \( S_{BH} \),

\[
S_H = (n - 1) \frac{HV}{4G_{n+1}}, \quad S_{BV} = \frac{2\pi}{n} ER, \quad S_{BH} = (n - 1) \frac{V}{4G_{n+1} R}.
\]  

(3.18)

The Friedmann equation (3.13) can then be expressed as

\[
S_H^2 = 2S_{BH}S_{BV}.
\]  

(3.19)

If one uses the Hubble entropy density, \( s_H = S_H/V \), and the energy density \( \rho \), the Friedmann equation (3.13) can be further rewritten as

\[
s_H^2 = \left( \frac{4\pi}{n} \right)^2 \frac{n(n - 1)}{16\pi G_{n+1}} \rho.
\]  

(3.20)

\(^5\)For a closed FRW universe, the corresponding form is \( S_H^2 = S_{BH}(2S_{BV} - S_{BH}) \). See [11].
We now consider a special moment when the brane crosses the black hole horizon of the background \(2.6\), \(i.e., R = r_+^n\). It is easy to show that the Hubble entropy bound is just saturated by the entropy of the domain-wall black hole,

\[
S_H = (n - 1) \frac{H V}{4 G_{n+1}} = \frac{V_n r_+^n N}{4 G_{n+2}} = S, \quad \text{at} \quad R = r_+^n,
\]

(3.21)
as the case of AdS Schwarzschild black holes. The temperature on the brane (that is, the temperature of brane universe) is

\[
T = \frac{1}{c R} T_{HK} = \frac{N(n + 2) - 1}{4 \pi} c r_+^{N-1}, \quad \text{at} \quad R = r_+^N.
\]

(3.22)
As in [11], if we define the limiting temperature \(T_H\) in terms of the derivative of the Hubble constant, \(T_H \equiv -\dot{H}/(2\pi H)\), it is then

\[
T_H = \frac{n N + 1}{4 \pi} c r_+^{N-1}, \quad \text{at} \quad R = r_+^N.
\]

(3.23)
Comparing (3.23) with (3.22), one can see that when the brane crosses the horizon of the domain-wall black holes, the two temperatures \(T\) and \(T_H\) are equal only if \(N = 1\). In other words, that the limiting temperature \(T_H\) is saturated by the temperature of the domain-wall black holes holds only when the bulk is the AdS Schwarzschild geometry [10]. This can be related to the fact that one has a varying gravitational constant on the brane when \(N \neq 1\).

The total entropy of the brane universe is the entropy of the domain-wall black hole, which is a constant during the evolution of the universe. But the entropy density is not a constant, which is

\[
s = \frac{r_+^{nN}}{4 G_{n+2} R^n};
\]

(3.24)
and the energy density of the brane universe is

\[
\rho \equiv \frac{E}{V} = \frac{c n N r_+^{(n+2)N-1}}{16 \pi G_{n+2} R^{n+1}}.
\]

(3.25)
We find that the entropy density can be related to its energy density as

\[
s^2 = \left(\frac{4\pi}{n}\right)^2 \gamma \rho, \quad \gamma = \frac{n(n - 1) r_+^{(n-2)N+1}}{16 \pi G_{n+1} R^{n-2+1/N}},
\]

(3.26)
which can be viewed as a deformed form of the Cardy-Verlinde formula of CFTs\footnote{The corresponding Cardy-Verlinde formula of CFTs in the closed FRW universe is \( s^2 = \left( \frac{4\pi}{\alpha} \right)^2 \gamma (\rho - \frac{\gamma}{R^2}) \). Here the term \( \gamma/R^2 \) is the contribution of Casimir effect. See \[10\].}. Since we are considering a flat universe, there is no contribution of Casimir effect. This relation\,(3.26) always holds during the evolution of the brane universe. But at the moment when the brane crosses the background horizon, we note that this equation\,(3.26) coincides with the Friedmann equation\,(3.20), as the case of the AdS Schwarzschild black holes\,[10]. Thus the same can be stated here: The Friedmann equation\,(3.20) somehow knows the entropy formula of QFTs filling the brane universe.

Next we make some remarks about the result\,(3.26). The Cardy formula\,[41] is supposed to hold only for\,(1 + 1)-dimensional CFTs. After some appropriate identifications, however, Verlinde\,[11] argued that the Cardy formula also holds for CFTs in arbitrary dimensions, resulting in the so-called Cardy-Verlinde formula. The Cardy-Verlinde formula has been checked to be valid for CFTs with AdS duals (for example, see\,[11,15,16] and\,[17]). Here we found that a Cardy-Verlinde-like formula\,(3.26) holds for a QFT corresponding to the domain-wall black holes\,(2.6). At first sight, this result seems strange since one does not expect the thermodynamics of QFTs obeys a Cardy-Verlinde-like formula. At this point, it seems helpful to recall the recent claim that the entropy of asymptotically flat black holes can also be expressed by a modified Cardy formula\,[15,24], and in particular the Carlip’s result that for black holes in any dimension the Bekenstein-Hawking entropy can be reproduced using the Cardy formula\,[12]. Carlip obtained his result by considering general relativity on a manifold with boundary. He found that the constraint algebra of general relativity may acquire a central extension, which can be calculated using covariant phase space techniques. When the boundary is a (local) Killing horizon, a natural set of boundary conditions leads to a Virasoro subalgebra with a calculable central charge. He then used conformal field theory methods to determine the density of states at the boundary, which yields the expected entropy of black holes. Obviously, the Carlip’s method is quite different from the Verlinde’s argument. And the Carlip’s method is applicable for a wider class of black holes. But both methods lead to the same conclusion that the entropy of black holes can be expressed by a modified Cardy formula. These results strongly imply that the thermodynamics of black holes is of some common features.
of thermodynamics of (1 + 1)-dimensional CFTs [15]. Therefore, our result (3.26) looks strange from the point of view of thermodynamics of QFTs, but does not from the point of view of black hole thermodynamics and is consistent with the argument advocated by Carlip that the entropy of all black holes can be expressed in terms of a modified Cardy formula [42].

4 Conclusions

We have found a class of domain-wall black hole solutions (AdS Schwarzschild black hole is included as a special case) in a dilaton gravity with a Liouville-type dilaton potential. Using the method of surface counterterm we have obtained the stress-energy tensor of the boundary QFT corresponding the domain-wall black holes in the domain-wall/QFT correspondence. The entropy and Hawking temperature of the domain-wall black holes have been also calculated.

We have considered a brane universe in the background of the domain-wall black holes. With the choice that the tension term of the bane is just the surface counterterm, which is introduced in order to acquire the finite boundary stress-energy tensor of the QFT, we have found that the equation of motion of the brane can be mapped to a standard form of the Friedmann equation for a flat ($k = 0$) FRW universe. The gravitational constant on the brane depends on the scale factor of the brane universe. It means that the brane universe in the background of dilaton domain-wall black holes can be viewed as a universe model with a varying gravitational constant. By comparing the FRW equations of brane universe with the standard FRW equations in Einstein gravity, we have found that the resulting equation of state of matter filling the brane universe cannot be matched to the one of the QFT obtained from the surface counterterm approach unless $N = 1$. That is, they can be identified with each other only when the bulk is the AdS Schwarzschild black hole (recall in this case that the gravitational constant on the brane is a true constant). This might be caused by the varying gravitational constant on the brane. Because the gravity on the brane in fact is a Brans-Dicke-like gravity theory, the dynamical gravitational constant will also make some contributions to the pressure and energy density on the brane. The contribution of the dilaton field might be also one of reasons of the mismatch. Because
of this mismatch, the limiting temperature expressed in terms of the Hubble constant and its time derivative is not equal to the Hawking temperature of the domain-wall black holes when the brane crosses the horizon of black holes. However, a Cardy-Verlinde-like formula (3.26) has been found, which relates the entropy density of the QFT to its energy density. At the moment when the brane crosses the black hole horizon of the background, the Cardy-Verlinde-like formula coincides with the Friedmann equation of the brane, and the Hubble entropy bound is still saturated by the entropy of the domain-wall black hole. These are the same as those in the case of AdS Schwarzschild black hole.

Note that the Cardy-Verlinde formula can express the entropy of any dimensional black holes in asymptotically AdS spacetimes. It has been claimed recently in [15, 24] that the entropy of asymptotically flat black holes can also be expressed in terms of a modified Cardy formula. We have found here that the Cardy-Verlinde-like formula (3.26) can describe the entropy of a class of dilaton domain-wall black holes (2.6), which is neither asymptotically AdS, nor asymptotically flat. These results altogether support the argument advocated by Carlip that the entropy of all black holes can be expressed in terms of a modified Cardy formula. It seemingly implies that the thermodynamics of black holes has a close connection with that of (1 + 1)-dimensional CFTs.

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