THE LOWER BOUND OF THE PCM QUANTIZATION ERROR IN HIGH DIMENSION

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Abstract In this note, we investigate the performance of the PCM scheme with linear quantization rule for quantizing unit-norm tight frame expansions for $\mathbb{R}^d$ without the White Noise Hypothesis. In [4], Wang and Xu showed that for asymptotically equidistributed unit-norm tight frame the PCM quantization error has an upper bound $O(\delta^{(d+1)/2})$ and they conjecture the upper bound is sharp. In this note, we confirm the conjecture with employing the asymptotic estimate of the Bessel functions.

Key words and phrases Unit-norm tight frames; PCM quantization error; Bessel functions; Orthogonal polynomials

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1. Introduction

In signal processing, one of the primary goals is to find a digital representation for a given signal that is suitable for storage, transmission, and recovery. We assume that the signal $x$ is an element of a finite-dimensional Hilbert space $\mathbb{H} = \mathbb{R}^d$. One often begins to expand $x$ over a dictionary $\mathcal{F} = \{e_j\}_{j=1}^N$, i.e.,

$$ x = \sum_{j=1}^N c_j e_j, $$

where $c_j$ are real numbers. We say $\mathcal{F}$ is a tight frame of $\mathbb{R}^d$ if

$$ x = \frac{d}{N} \sum_{j=1}^N \langle x, e_j \rangle e_j $$

holds for all $x \in \mathbb{R}^d$. The tight frame is called unit-norm if $\|e_j\|_2 = 1$ holds for all $1 \leq j \leq N$. In the digital domain the coefficients $x_j = \langle x, e_j \rangle$ must be mapped to a discrete set of values $\mathcal{A}$ which is called the quantization alphabet. The simplest way for such a mapping is the Pulse Code Modulation (PCM) quantization scheme, which has $\mathcal{A} = \delta \mathbb{Z}$ with $\delta > 0$ and the mapping is done by the function

$$ Q_\delta(t) := \text{argmin}_{r \in \mathcal{A}} |t - r| = \delta \left\lfloor \frac{t}{\delta} + \frac{1}{2} \right\rfloor. $$

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Thus in practical applications we in fact have only a quantized representation
\[ q_j := Q_\delta(\langle x, e_j \rangle), \quad j = 1, \ldots, N \]
for each \( x \in \mathbb{R}^d \). The linear reconstruction is
\[ \tilde{x}_F = \frac{d}{N} \sum_{j=1}^{N} q_j e_j. \]

Naturally we are interested in the error for this reconstruction, i.e.
\[ E_\delta(x,F) := \| x - \tilde{x}_F \|, \]
where \( \| \cdot \| \) is \( \ell_2 \) norm. To simplify the investigation of \( E_\delta(x,F) \), one employs the *White Noise Hypothesis* (WNH) in this area (see [6, 8, 9, 10, 5]), which asserts that the quantization error sequence \( \{ x_j - q_j \}_{j=1}^{N} \) can be modeled as an independent sequence of i.i.d. random variables that are uniformly distributed on the interval \((-\delta/2, \delta/2)\). Under the WNH, one can obtain the mean square error \( \text{MSE} = \mathbb{E}(\| x - \tilde{x}_F \|^2) = \frac{d^2 \delta^2}{12N} \).

The result implies that the MES of \( E_\delta(x,F) \) tends to 0 with \( N \) tending to infinity. However, as pointed out in [11, 5], the WNH only asymptotically holds for fine quantization (i.e. as \( \delta \) tends to 0) under rather general conditions. So, for a fixed \( x \), one is interested in whether \( E_\delta(x,F) \) really tending to 0 without WNH. The result in [4] gives a solution for the case where \( d = 2 \) which shows that for some \( x \in \mathbb{R}^2 \) the quantization error \( E_\delta(x,F) \) does not diminish to 0 with \( N \) tending to infinity. Naturally, one would like to know whether it is possible to extend the result to higher dimension. In [4], Wang and Xu investigate the case where \( F \) is the *asymptotically equidistributed unit-norm tight frame* in \( \mathbb{R}^d \). A sequence of finite sets \( A_m \subset \mathbb{S}^{d-1} \) with cardinality \( N_m = \# A_m \) is said to be *asymptotically equidistributed* on \( \mathbb{S}^{d-1} \) if for any piecewise continuous function \( f \) on \( \mathbb{S}^{d-1} \) we have
\[ \lim_{m \to \infty} \frac{1}{N_m} \sum_{v \in A_m} f(v) = \int_{z \in \mathbb{S}^{d-1}} f(z) dv, \]
where \( f \) are piecewise continuous functions on \( \mathbb{S}^{d-1} \) and \( dv \) denotes the normalized Lebesgue measure on \( \mathbb{S}^{d-1} \). Then the following theorem presents an upper bound for \( \lim_{m \to \infty} E_\delta(x,F_m) \).

**Theorem 1.1.** [4] Assume that \( F_m \) are asymptotically equidistributed unit-norm tight frames in \( \mathbb{R}^d \). Then for any \( x \in \mathbb{R}^d \) we have
\[ \lim_{m \to \infty} E_\delta(x,F_m) \leq C_d \delta^{(d+1)/2} \frac{r}{r(d-1)/2}, \]
where \( r = \| x \| \) and \( C_d \) is a constant depending on \( d \).

A main tool for obtaining Theorem 1.1 is Euler-Maclaurin formula. However, it seems that it is difficult to extend the method to obtain the lower bound. In [4], Wang and Xu conjecture the bound \( O \left( \frac{\delta^{(d+1)/2}}{r^{(d-1)/2}} \right) \) is sharp. In this note, we employ the tools of Bessel function and hence confirm the conjecture. In particular, we have:
Theorem 1.2. Suppose that $d > 2$ is an integer. Assume that $x \in \mathbb{R}^d$ and that $F_m$ are asymptotically equidistributed unit-norm tight frames in $\mathbb{R}^d$. Set $r := \|x\|, R := r/\delta$, $\epsilon := R - \lfloor R \rfloor$, and

$$I := d \int_0^{2\pi} |(\sin(\theta_2))^{d-3} \cdots \sin \theta_{d-2}| d\theta_2 \cdots d\theta_{d-2} > 0.$$  

(i) If $d = 2n$ and $1/4 \leq \epsilon \leq 1/2$, then

$$\lim_{m \to \infty} E_\delta(x, F_m) \geq C_{1,d} \frac{\delta^{\frac{d+1}{2}}}{r^{\frac{d}{2}-1}}.$$  

provided that $R = \|x\|/\delta$ is big enough, where

$$C_{1,d} = \frac{(n-1)! \cdot (2n-2) \cdot M_1}{\pi^{n-1} 2^{2n-2}} \cdot I, \quad M_1 = \frac{4}{5} |\cos(2\pi \epsilon - \frac{3}{4}\pi)| - \frac{5}{4} \sum_{k=2}^{+\infty} \frac{1}{k^{\frac{d+1}{2}}} > 0.$$  

(ii) If $d = 2n + 1$ and $1/6 \leq \epsilon \leq 1/3$, then

$$\lim_{m \to \infty} E_\delta(x, F_m) \geq C_{2,d} \frac{\delta^{\frac{d+1}{2}}}{r^{\frac{d}{2}}}.$$  

provided that $R = \|x\|/\delta$ is big enough, where

$$C_{2,d} = \frac{(n-1)! \cdot M_2}{\pi^{n+1}} \cdot I, \quad M_2 = \frac{7}{8} |\cos(2\pi \epsilon - \frac{1}{2}\pi)| - \frac{8}{7} \sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} > 0.$$  

After introducing some necessary concepts and results to be used in our investigation in Section 2, we present the proof of Theorem 1.2 in Section 3.

2. Preliminaries

Bessel function. (see [12]) For $\alpha > 0$, the Bessel function $J_\alpha$ is defined by the series representation

$$J_\alpha(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \cdot \Gamma(k + \alpha + 1)} \left(\frac{x}{2}\right)^{2k+\alpha}.$$  

Particularly, when $\alpha \in \mathbb{N}$, we have

$$J_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos(\alpha \tau - x \sin(\tau)) d\tau.$$  

We also need an asymptotic estimate for the Bessel function $J_\alpha$ which is presented in [2].

Theorem 2.1. ([2]) For the Bessel function $J_\alpha$, we have

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \omega_\alpha) + \theta c \mu x^{-\frac{3}{2}},$$  

where $\omega_\alpha = \frac{\alpha \pi}{2} + \frac{\alpha}{4} \pi$, $\mu = |\alpha^2 - \frac{1}{4}|$, $|\theta| \leq 1$, and

$$c = \begin{cases} (2/\pi)^{3/2}, & x \geq 0, \quad |\alpha| \leq \frac{1}{2} \\ \sqrt{2}/2, & x \geq \sqrt{\mu}, \quad \alpha > \frac{1}{2} \\ 5/4, & 0 < x < \sqrt{\mu}, \quad \alpha > \frac{1}{2}. \end{cases}$$
A simple calculation shows that

\[ \sum_{m=0}^h (-1)^m \frac{(n+h)}{h-m} \left( \frac{n+h}{h+m} \right) = \frac{1}{2} \left( \frac{n+h}{h} \right) + \frac{1}{2} \left( \frac{n+h}{h} \right)^2 \]

3. PROOF OF THEOREM 1.2

To this end, we first introduce several lemmas:

**Lemma 3.1.** For all \( n \in \mathbb{N}^+ \) and \( h \in \mathbb{N} \) we have

\[ \sum_{m=0}^h (-1)^m (2m+1) \left( \frac{n+h}{h-m} \right) \left( \frac{n+h}{h+m+1} \right) = n \left( \frac{n+h}{n} \right). \]

and

\[ \sum_{m=0}^h (-1)^m (2m+1) \left( \frac{2h+1}{h-m} \right) \left( \frac{m+l}{2l} \right) = 0, \quad l = 0, 1, \ldots, h-1. \]

**Proof.** We prove (3) by induction. To state conveniently, set

\[ A^h_n := \sum_{m=0}^h (-1)^m (2m+1) \left( \frac{n+h}{h-m} \right) \left( \frac{n+h}{h+m+1} \right). \]

A simple observation is that (3) holds when \( n \in \mathbb{N}^+, h = 0 \) and when \( n = 0, h \in \mathbb{N}^+ \).

Assume that \( n_0, h_0 \in \mathbb{N}^+ \). For the induction step, we assume that (3) is true both for \( n \leq n_0 \in \mathbb{N}^+, h \in \mathbb{N}^+ \) and for \( n = n_0 + 1, h \leq h_0 - 1 \in \mathbb{N}^+ \). To this end, we just need prove that the result holds for \( n = n_0 + 1, h = h_0 \). We have

\[ A^h_{n_0+1} = \sum_{m=0}^{h_0} (-1)^m (2m+1) \left( \frac{n_0 + h_0 + 1}{h_0 - m} \right) \left( \frac{n_0 + h_0 + 1}{h_0 + m + 1} \right) \]

\[ = \sum_{m=0}^{h_0} (-1)^m (2m+1) \left( \frac{n_0 + h_0}{h_0 - m} \right) + \left( \frac{n_0 + h_0}{h_0 + m - 1} \right) \left( \frac{n_0 + h_0}{h_0 + m + 1} \right) + \left( \frac{n_0 + h_0}{h_0 + m} \right) \]

\[ = A^{h_0}_{n_0} + A^{h_0-1}_{n_0+1} + \sum_{m=0}^{h_0} (-1)^m (2m+1) \left( \frac{n_0 + h_0}{h_0 - m} \right) \left( \frac{n_0 + h_0}{h_0 + m} \right) + \left( \frac{n_0 + h_0}{h_0 + m - 1} \right) \left( \frac{n_0 + h_0}{h_0 + m + 1} \right) \]

\[ = A^{h_0}_{n_0} + A^{h_0-1}_{n_0+1} + 2 \sum_{m=0}^{h_0} (-1)^m \left( \frac{n_0 + h_0}{h_0 - m} \right) \left( \frac{n_0 + h_0}{h_0 + m} \right) - \left( \frac{n_0 + h_0}{h_0} \right)^2 \]

\[ = A^{h_0}_{n_0} + A^{h_0-1}_{n_0+1} + \left( \frac{n_0 + h_0}{n_0} \right) \left( \frac{n_0 + h_0 + 1}{n_0 + 1} \right), \]

where the last equality uses the identity (2) and the induction assumption.

We now turn to (4). Set

\[ g_m := (-1)^{m+1} (h + m + 1) \left( \frac{2h+1}{h-m} \right) \left( \frac{m+1}{2l} \right). \]

A simple calculation shows that

\[ g_{m+1} - g_m = (-1)^m \left( \frac{2h+1}{h-m} \right) \left( \frac{m+1}{2l} \right) \left( (h-m)(l+1) + (h+m+1)(m-l) \right) \]
\[ = (-1)^m (2m+1) \left( \frac{2h+1}{h-m} \right) \left( \frac{m+1}{2l} \right). \]
Then
\[
\sum_{m=l}^{h} (-1)^m (2m + 1) \left( \frac{2h + 1}{h - m} \right) \left( \frac{m + l}{2l} \right) = \sum_{m \geq l} (-1)^m (2m + 1) \left( \frac{m + l}{2l} \right) = \sum_{m \geq l} g_{m+1} - g_m = g_l = 0.
\]

Here, the first equality holds since \( \binom{n}{k} = 0 \) provided \( k < 0 \).

**Remark 1.** A key step to prove (4) is to construct the sequence \( g_m \) which satisfies (6). In the proof of Lemma 3.1, we obtain \( g_m \) using Gosper algorithm [7]. However, it is also simple to verify (6) by hand.

We introduce the following results for Bessel functions

**Lemma 3.2.** Set
\[
L_m := \int_{-\pi}^{\pi} \cos(2m + 1) \theta \cos(\sin \theta)^{2n-2} d\theta
\]
\[
D_m := \int_{0}^{\pi} \cos(2m + 1) \theta \cos(\sin \theta)^{2n-1} d\theta.
\]

Then we have
\[
\sum_{m=0}^{n-1} (-1)^m L_m J_{2m+1}(x) = L_0 2^{n-1} n! \frac{1}{x^{n-1}} J_n(x),
\]
\[
\sum_{m=0}^{+\infty} (-1)^m D_m J_{2m+1}(x) = \sqrt{\pi} 2^{n-\frac{3}{2}} (n-1)! \frac{1}{x^{n-\frac{1}{2}}} J_{n+\frac{1}{2}}(x).
\]

**Proof.** To this end, we first calculate the value of \( L_m \). Using the expansion
\[
\sin^{2n-2} \theta = \frac{1}{2^{2n-2}} \binom{2n-2}{n-1} + \frac{2}{2^{2n-2}} \sum_{k=0}^{n-2} (-1)^{n-1-k} \binom{2n-2}{k} \cos((2n-2-2k)\theta),
\]
we can obtain that
\[
L_m = (-1)^m \frac{\pi}{2^{2n-2}} \binom{2n-2}{n+m-1} \frac{2m+1}{n+m}.
\]

Recall that the series representation of the Bessel function \( J_\alpha \)
\[
J_\alpha(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k+\alpha+1)} \left( \frac{x}{2} \right)^{2k+\alpha}.
\]

Substituting (10) into (7) we obtain that
\[
\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \sum_{m=0}^{n-1} (-1)^m L_m \frac{1}{\Gamma(2m+k+2)} \left( \frac{x}{2} \right)^{2k+2m+1} = L_0 n! \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k+n+1)} \left( \frac{x}{2} \right)^{2k+1}.
\]
To this end, we just need prove (11). Comparing the coefficients of the powers of $x$ on the both sides of (11), we only need prove

$$(12) \quad \sum_{m=0}^{h} L_m \frac{1}{(h-m)! \cdot \Gamma(h+m+2)} = L_0 \cdot n! \cdot \frac{1}{h! \cdot \Gamma(h+n+1)}.$$ 

which is equivalent to

$$(13) \quad \sum_{m=0}^{h} (-1)^m (2m+1) \begin{pmatrix} n+h \\ h-m \end{pmatrix} \begin{pmatrix} n+h \\ h+m+1 \end{pmatrix} = n \begin{pmatrix} n+h \\ n \end{pmatrix}.$$

Here, we use (9). According to Lemma 3.1, (13) holds which in turn implies (7).

We next turn to (10). Substitute (10) into (8) and compare the coefficients of the powers of $x$ on the two sides of this equation, we only need to prove

$$(14) \quad \sum_{m=0}^{h} \frac{D_m}{(h-m)! \cdot (h+m+1)!} = \frac{(n-1)!}{4} \cdot \frac{2^{2h+2n+3} \cdot (h+n+1)!}{h! \cdot (2h+2n+2)!}, \quad h = 0, 1, \ldots$$

Using

$$\cos nx = \frac{x}{2} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^{k} \begin{pmatrix} n-k \\ k \end{pmatrix} \frac{(2\cos x)^{n-2k}}{n-k},$$

and

$$\int_{0}^{\frac{\pi}{2}} (\sin t)^{x} (\cos t)^{y} dt = \frac{\pi}{2^{x+y+1} \left(\frac{\pi}{2}\right)! \cdot \left(\frac{x}{2}\right)! \cdot \left(\frac{x+y}{2}\right)!},$$

where $x! = \Gamma(x+1)$ for $x > 0$, we have

$$(15) \quad D_m = (2m+1) \sum_{k=0}^{m} (-1)^{k} \begin{pmatrix} 2m+1-k \\ k \end{pmatrix} \frac{\sqrt{\pi} (n-1)!}{4(2m+1-k) \cdot (m+1-k)!} \cdot \frac{(2m+2-2k)!}{(2m+2n-2k+1)!}.$$

Substituting (15) into (14), we can rewrite (14) as

$$(16) \quad \sum_{m=0}^{h} \frac{2m+1}{(h-m)! \cdot (h+m+1)!} \sum_{k=0}^{m} (-1)^{k} \begin{pmatrix} 2m+1-k \\ k \end{pmatrix} \frac{1}{2m+1-k} \frac{(2m+2-2k)!}{(m+1-k)! \cdot (2m+2n-2k+1)!}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{2^{2h+2n+3} (h+n+1)!}{h! \cdot (2h+2n+2)!}.$$

On the other hand, we can rewrite the left side of (16) as

$$(17) \quad \sum_{m=0}^{h} \frac{2m+1}{(h-m)! \cdot (h+m+1)!} \sum_{k=0}^{m} (-1)^{k} \begin{pmatrix} m+l+1 \\ m-l \end{pmatrix} \frac{1}{m+l+1} \frac{(2l+2)!}{(l+1)! \cdot (2l+2n+1)!}$$

$$= \sum_{l=0}^{h} \frac{1}{\binom{2l+2n+1}{2}} \sum_{m=l}^{h} \frac{2m+1}{(h-m)! \cdot (h+m+1)!} \frac{(-1)^{m-l}}{m+l+1} \frac{1}{(l+1)! \cdot (2l+2)!}.$$
Here, in the first equality, we set a new variable \( l := m - k \). To this end, we consider the second term on the right side of the last equality in \((17)\). Note that, for \( l = 0, \ldots, h - 1, \)

\[
\frac{\sum_{m=l}^{h} \frac{2m+1}{(h-m)!} \cdot (h+m+1)! \cdot (m+l+1)!}{(2l+2)!} = \frac{(-1)^l (2l+2)!}{(2l+1)(2h+1)! \cdot (l+1)!} \sum_{m=l}^{h} (2m+1) \left( \frac{2h+1}{h-m} \right) (-1)^m \left( \frac{m+l}{2l} \right)
\]

\[
= 0.
\]

Here, the last equality follows from \((19)\) in Lemma 3.1. Hence the last summation in \((17)\) is reduced to

\[
\frac{1}{(\frac{2h+2n+1}{2})! \cdot (h+h+1)!} (-1)^{h-h} \left( \frac{h+h+1}{h-h} \right) \frac{1}{h+h+1} \left( \frac{2h+2}{h+1} \right)!
\]

\[
= \frac{1}{(\frac{2h+2n+1}{2})!} = \frac{1}{\sqrt{\pi}} \frac{2^{2h+2n+3} (h+n+1)!}{h! \cdot (2h+2n+2)!}.
\]

Here, the last equality uses

\[
\left( \frac{2h+2n+1}{2} \right)! = \Gamma \left( h+n+1 + \frac{1}{2} \right) = \frac{(2h+2n+2)!}{2^{2h+2n+2} (h+n+1)!} \sqrt{\pi}.
\]

We arrive at the conclusion. \( \square \)

Now we can give an estimation for the integrals \( \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-2} d\theta \) and \( \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-1} d\theta \).

Lemma 3.3. Set \( R := \frac{\pi}{\delta} \) and \( \epsilon := R - \lfloor R \rfloor \). Then when \( 1/4 \leq \epsilon \leq 1/2 \)

\[
\frac{(n-1)! \cdot (\frac{2n-2}{2n-2})^2}{2^{2n-2} \cdot \pi^n} M_1 \delta^{2n+1} \leq \left| \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-2} d\theta \right|
\]

\[
\leq \frac{5}{4} \frac{(n-1)! \cdot (\frac{2n-2}{2n-2})^2}{2^{2n-2} \cdot \pi^n} \sum_{k=1}^{\infty} \frac{\delta^{2n+1}}{\delta^{2n+1}},
\]

provided that \( R = \frac{\pi}{\delta} \) is big enough, where \( M_1 = \frac{3}{2^3} \left| \cos(2\pi \epsilon - \frac{3}{2} \pi) \right| - \frac{3}{2} \sum_{k=2}^{\infty} \frac{1}{k^2} > 0 \) and \( n \geq 2 \). When \( 1/6 \leq \epsilon \leq 1/3 \), we have

\[
\frac{M_2(n-1)! \cdot \delta^{n+1}}{\pi^{n+1} \cdot \pi^n} \leq \left| \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-1} d\theta \right|
\]

\[
\leq \frac{8}{7} \frac{(n-1)! \cdot (\frac{2n-2}{2n-2})^2}{\pi^{n+1} \cdot \pi^n} \delta^{n+1},
\]

provided that \( R = \frac{\pi}{\delta} \) is big enough, where \( M_2 = \frac{7}{8} \left| \cos(2\pi \epsilon - \frac{1}{2} \pi) \right| - \frac{8}{7} \sum_{k=2}^{\infty} \frac{1}{k^{n+1}} > 0 \) and \( n \geq 1 \).

Proof. Firstly we consider \( \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-2} d\theta \). Using the Fourier expansion for \( [x] \) with \( x \in \mathbb{R} \setminus \mathbb{Z} \),

\[
|x| = x - \frac{1}{2} + \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k},
\]
we have
\[
\int_{0}^{\pi} \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-2} d\theta \\
= \frac{1}{2} \int_{-\pi}^{\pi} \Delta_\delta(r \cos \theta) \cos \theta (\sin \theta)^{2n-2} d\theta \\
= \frac{\delta}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^{+\infty} \frac{\sin(2k\pi \frac{r}{\delta}) + k\pi}{k} \cos \theta (\sin \theta)^{2n-2} d\theta \\
= -\frac{\delta}{2\pi} \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \int_{-\pi}^{\pi} \sin(2k\pi \frac{r}{\delta}) \cos \theta (\sin \theta)^{2n-2} d\theta \\
= -\frac{\delta}{2\pi} \cdot L_0 \cdot 2^{n-1} n! \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \frac{1}{(2k\pi \frac{r}{\delta})^{n-1}} \cdot J_n(2k\pi \frac{r}{\delta}) \\
= -\frac{1}{\pi^n} \cdot \frac{\delta^n}{r^{n-1}} \cdot L_0 \cdot n! \cdot \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \cdot J_n(2k\pi \frac{r}{\delta}) \\
= -\frac{1}{\pi^n} \cdot \frac{\pi}{2n-2} \cdot \frac{\pi}{2n-2} \cdot \left( 2n - 2 \right) \cdot (n - 1)! \cdot \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \cdot J_n(2k\pi \frac{r}{\delta}).
\]

In the fourth equality, we use the formula
\[
\sin(x \cos \theta) = 2 \sum_{m=0}^{+\infty} (-1)^m \cos((2m+1)\theta)J_{2m+1}(x)
\]
and the orthogonality of the systems \(\{\cos(kx)\}_{k=0}^{+\infty}\) on the interval \([-\pi, \pi]\). We use (7) in the fifth equality. To this end, according to (20), we only need to estimate
\[
\left| \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \cdot J_n(2k\pi \frac{r}{\delta}) \right|.
\]
In fact, note that
\[
(21) \quad |J_n(2\pi \frac{r}{\delta})| - \sum_{k=1}^{+\infty} \frac{1}{k} \cdot |J_n(2k\pi \frac{r}{\delta})| \leq \left| \sum_{k=1}^{+\infty} \frac{(1)^k}{k} \cdot J_n(2k\pi \frac{r}{\delta}) \right| \leq \sum_{k=1}^{+\infty} \frac{1}{k} \cdot |J_n(2k\pi \frac{r}{\delta})|.
\]
We first consider \(|J_n(2\pi \frac{r}{\delta})| - \sum_{k=1}^{+\infty} \frac{1}{k} \cdot |J_n(2k\pi \frac{r}{\delta})|\). Using the asymptotic estimate for \(J_n(x)\) in Theorem (23), we have
\[
|J_n(2\pi \frac{r}{\delta})| = \left| \frac{1}{\pi} \sqrt{\frac{\delta}{r}} \cos(2\pi \frac{r}{\delta} - \omega_n) + \theta c\mu(2\pi \frac{r}{\delta})^{-3/2} \right| \\
= \left| \frac{1}{\pi} \sqrt{\frac{\delta}{r}} \cos(2\pi \epsilon - \frac{(2n+1)\pi}{4}) + \theta c\mu(\frac{1}{2\pi \frac{r}{\delta}})^{3/2} \right|
\]
which implies that
\[
|J_n(2\pi \frac{r}{\delta})| \geq \frac{4}{5} \left| \frac{1}{\pi} \sqrt{\frac{\delta}{r}} \cos(2\pi \epsilon - \frac{3\pi}{4}) \right|
\]
provided that \( R = \frac{\pi}{\delta} \) is big enough. On the other hand,

\[
\sum_{k=2}^{+\infty} \frac{1}{k^n} |J_n(2k\pi r/\delta)| = \sum_{k=2}^{+\infty} \frac{1}{k^n} \left| \frac{\delta}{k^n} \cos(2\pi k r/\delta - \omega_n) + \theta c \mu(2\pi k r/\delta)^{-3/2} \right|
\]

\[
= \sum_{k=2}^{+\infty} \frac{1}{k^n} \left| \frac{\delta}{k^n} \cos(2\pi k r - \frac{(2n+1)\pi}{4}) + \theta c \mu(2\pi k r/\delta)^{-3/2} \right|
\]

Therefore, according to (22),

\[
\sum_{k=2}^{+\infty} \frac{1}{k^n} |J_n(2k\pi r/\delta)| \leq \frac{5}{4\pi} \sqrt{\frac{2}{\pi}} \sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} \cdot \frac{5}{4}\frac{n}{4}\sum_{k=2}^{+\infty} \frac{1}{k^{n+1}}
\]

provided that \( R = \frac{\pi}{\delta} \) is big enough. Combining above results, we obtain that

\[
|J_n(2\pi r/\delta)| - \sum_{k=2}^{+\infty} \frac{1}{k^n} |J_n(2k\pi r/\delta)| \geq \frac{1}{\pi} \sqrt{\frac{2}{\pi}} \left( \sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} - \frac{5}{4}\sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} \right)
\]

provided that \( R = \frac{\pi}{\delta} \) is big enough. When \( 1/4 \leq \epsilon \leq 1/2 \) and \( n \geq 2 \),

\[
M_1 := \frac{4}{5} \left| \cos(2\pi \epsilon - \frac{3\pi}{4}) \right| - \frac{3}{4}\sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} \geq \frac{4}{5} \cdot \frac{5}{4}\sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} \approx 0.138 > 0.
\]

Combining (24), (24) and (24), we obtain the left side of (18). Similarly, based on (23), we have

\[
\sum_{k=1}^{+\infty} \frac{(-1)^k}{k^n} J_n(2k\pi r/\delta) \leq \sum_{k=1}^{+\infty} \frac{1}{k^n} |J_n(2k\pi r/\delta)| \leq \frac{5}{4\pi} \sqrt{\frac{2}{\pi}} \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}}
\]

provided that \( R = \frac{\pi}{\delta} \) is big enough, which implies the right side of (18).

Now let us turn to \( \int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta(\sin \theta)^{2n-1} d\theta \). Similar with the above, we have

\[
\int_0^\pi \Delta_\delta(r \cos \theta) \cos \theta(\sin \theta)^{2n-1} d\theta
\]

\[
= -\frac{\delta}{\pi} \sum_{k=1}^{+\infty} \sin(2k\pi \frac{r \cos \theta}{\delta} + k\pi) \cos \theta(\sin \theta)^{2n-1} d\theta
\]

\[
= -\frac{\delta}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^n} \int_0^\pi \sin(2k\pi \frac{r \cos \theta}{\delta} \cos \theta(\sin \theta)^{2n-1} d\theta)
\]

\[
= -\frac{2\delta}{\pi} \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^n} \sin(2k\pi \frac{r}{\delta} n \theta)
\]

\[
= -\frac{2\delta}{\pi} \sqrt{\pi} n^{2n-3} (n-1)! \sum_{m=0}^{+\infty} \frac{(-1)^k}{k^n} \frac{1}{(2k\pi \frac{r}{\delta})^n} J_{n+\frac{1}{2}}(2k\pi \frac{r}{\delta})
\]

\[
= \frac{(n-1)! \delta^{n+\frac{1}{2}}}{\pi^n} \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^{n+\frac{1}{2}}} J_{n+\frac{1}{2}}(2k\pi \frac{r}{\delta}),
\]

where we use (8) in Lemma 3.2 for the fourth equality. Using the asymptotic estimate for \( J_{n+\frac{1}{2}}(x) \) in Theorem 2.1 similarly with the above, we can show that

\[
\frac{1}{\pi} \sqrt{\frac{\delta}{r}} \left( \frac{7}{8} \cos(2\pi \epsilon - \frac{1}{2} \pi) - 8 \sum_{k=2}^{+\infty} \frac{1}{k^{n+1}} \right) \leq \left| \sum_{k=1}^{+\infty} \frac{(-1)^k}{k^{n+\frac{1}{2}}} J_{n+\frac{1}{2}}(2k\pi \frac{r}{\delta}) \right| \leq \frac{8}{7} \frac{1}{\pi} \sqrt{\frac{\delta}{r}} \sum_{k=1}^{+\infty} \frac{1}{k^{n+1}}
\]
provided that $R = \frac{\xi}{\delta}$ is big enough. When $1/6 \leq \epsilon \leq 1/3$ and $n \geq 1$,

$$M_2 := \frac{7}{8}\left| \cos(2\pi\epsilon - \frac{1}{2}\pi) \right| - \frac{8}{7} \sum_{k=2}^{\infty} \frac{1}{k^{n+1}} \geq \frac{7}{8} \frac{\sqrt{3}}{2} - \frac{8}{7} \sum_{k=2}^{\infty} \frac{1}{k^{2}} \approx 0.02 > 0.$$ 

Combing (25) and (26), we arrive at (19). □

We now can state the proof of the main theorem.

**Proof of Theorem 1.2.** The idea to prove Theorem 1.2 is similar to one of proving Theorem 1.1 in [4] with using Lemma 3.3 to estimate $\lim_{m \to \infty} E_\delta(x, F_m)$. We state the proof of (i) for the completeness. In fact, (ii) can be proved using a similar method.

We denote the number of the non-zero entries in $x$ by $\|x\|_0$, i.e.,

$$\|x\|_0 := \#\{j : x_j \neq 0\}.$$ 

The proof is by induction on $\|x\|_0$. Note that

$$\lim_{m \to \infty} E_\delta(x, F_m) = \lim_{m \to \infty} \frac{d}{N_m} \sum_{j=1}^{N_m} \Delta_\delta(x \cdot e_j)e_j \|$$

$$= d \int_{z \in \mathbb{S}^d} \Delta_\delta(x \cdot z)z d\omega.$$ 

We begin with $\|x\|_0 = 1$. Without loss of generality, we suppose $x = [x_1, 0, \ldots, 0]^T \in \mathbb{R}^d$ and consider $\lim_{m \to \infty} E_\delta(x, F_m)$. By the sphere coordinate system, each $z = [z_1, \ldots, z_d] \in \mathbb{S}^{d-1}$ can be written in the form of

$$[\cos \theta_1, \sin \theta_1 \cos \theta_2, \sin \theta_1 \sin \theta_2 \cos \theta_3, \ldots, \sin \theta_1 \cdots \sin \theta_{d-1}]^T,$$

where $\theta_1 \in [0, \pi)$ and $\theta_j \in [-\pi, \pi), 2 \leq j \leq d - 1$. To state conveniently, we set

$$\Theta := [0, \pi) \times [-\pi, \pi) \times \cdots \times [-\pi, \pi),$$

$$S_i(\theta) := \prod_{j=1}^{i} \sin \theta_j$$

and

$$J_d(\theta) := \|(\sin \theta_1)^{d-2}(\sin \theta_2)^{d-3} \cdots (\sin \theta_{d-2})\|.$$

Noting that

$$d\omega = J_d(\theta)d\theta_1 \cdots d\theta_{d-1} \quad \text{and} \quad \int_{z \in \mathbb{S}^{d-1}} \Delta_\delta(x_1 z_1)z_j d\omega = 0, \quad 2 \leq j \leq d - 1,$$

we have

$$\lim_{m \to \infty} E_\delta(x, F_m) = d \left| \int_{z \in \mathbb{S}^{d-1}} \Delta_\delta(x \cdot z)z d\omega \right|$$

$$= d \left| \int_{z \in \mathbb{S}^{d-1}} \Delta_\delta(x_1 z_1)z_1 d\omega \right|$$

$$= d \left| \int_{\theta \in \Theta} \Delta_\delta(x_1 \cos \theta_1) \cos \theta_1 (\sin \theta_1)^{d-2} \right|$$

$$\left| (\sin \theta_2)^{d-3} \cdots (\sin \theta_{d-2}) |d\theta_1 \cdots d\theta_{d-1}| \right|$$

$$\geq C_1 d \cdot \xi^{(d+1)/2}/|x_1|^{(d-1)/2}$$

where the last inequality follows from Lemma 3.3.
For the induction step, we suppose that the conclusion holds for the case where \( \|x\|_0 \leq k \). We now consider \( \|x\|_0 \leq k + 1 \). Without loss of generality, we suppose \( x \) is in the form of \([0, \ldots, 0, x_{d-k}, \ldots, x_d] \in \mathbb{R}^d \). We can write \([x_{d-1}, x_d]\) in the form of \((r_0 \cos \varphi_0, r_0 \sin \varphi_0)\), where \( r_0 \in \mathbb{R}_+ \) and \( \varphi_0 \in [0, 2\pi) \). Then

\[
x \cdot z = \sum_{j=d-k}^{d-2} x_j S_j(\theta) \cos \theta_j + r_0 \sin \theta_1 \cdots \sin \theta_{d-2} \cos(\theta_{d-1} - \varphi_0) =: T(\varphi_0).
\]

A simple observation is

\[
\parallel x \parallel_2 \leq \sum_{j=d-k}^{d-2} x_j S_j(\theta) \cos \theta_j + \sum_{j=d-k}^{d-2} x_j S_j(\theta) \sin \theta_j \leq \sqrt{2} \parallel x \parallel_2.
\]

Then we have

\[
\lim_{m \to \infty} E_3(x, F_m) = d \parallel \int_{z \in S^{d-1}} \Delta_3(x \cdot z) z d\omega \parallel
\]

\[
= d \left( \sum_{t=d-k}^{d-1} \left( \int_{\theta \in \Theta} \Delta_3(T(\varphi_0)) S_{d-t}(\theta) J_d(\theta) \cos \theta d\theta \right)^2 + \left( \int_{\theta \in \Theta} \Delta_3(T(\varphi_0)) S_{d-t}(\theta) J_d(\theta) \sin \theta d\theta \right)^2 \right)^{1/2}
\]

\[
\geq C_{1,d} \cdot \delta^{(d+1)/2} / \mu^{(d-1)/2}
\]

where the last inequality follows from the fact \( \|x\|_0 \leq k \) provided \( \varphi_0 = 0 \). □

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