Fractional matching number and spectral radius of nonnegative matrices of graphs

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ABSTRACT
A fractional matching of a graph $G$ is a function $f : E(G) \rightarrow [0, 1]$ such that for any $v \in V(G)$, $\sum_{e \in E(v)} f(e) \leq 1$ where $\overline{E(G)} = \{ e \in E(G) : e \text{ is incident with } v \}$ in $G$. The fractional matching number of $G$ is $\mu_f(G) = \max\{ \sum_{e \in E(G)} f(e) : f \text{ is a fractional matching of } G \}$. For any real numbers $a \geq 0$ and $k \in (0, n)$, it is observed that if $n = |V(G)|$ and $\delta(G) > \frac{n-k}{2}$, then $\mu_f(G) > \frac{n-k}{2}$. We determine a function $\varphi(a, n, \delta, k)$ and show that for a connected graph $G$ with $n = |V(G)|$, $\delta(G) \leq \frac{n-k}{2}$, spectral radius $\lambda_1(G)$ and complement $\overline{G}$, each of the following holds.

(i) If $\lambda_1(aD(G) + A(G)) < \varphi(a, n, \delta, k)$, then $\mu_f(G) > \frac{n-k}{2}$.
(ii) If $\lambda_1(aD(\overline{G}) + A(\overline{G})) < (a+1)(\delta-k-1)$, then $\mu_f(G) > \frac{n-k}{2}$.

As applications, we prove a relationship between $\mu_f(G)$ and $\lambda_1(aD(G) + A(G))$ for a graph $G$. Furthermore, sufficient spectral conditions for a graph to have a fractional perfect matching are also obtained.

1. Introduction
Graphs considered are simple and undirected. We generally follow [1] for undefined terms and notation. For a graph $G$, and a vertex $v \in V(G)$, define $N_G(v) = \{ u \in V(G) : uv \in E(G) \}$ and $\overline{E(G)} = \{ e \in E(G) : e \text{ is incident with } v \}$ in $G$. If $S \subseteq V(G)$, then define $N_G(S) = \bigcup_{v \in S} N_G(v)$. As in [1], we use $K_n$, $\delta(G)$, $\Delta(G)$, $\kappa(G)$ and $\overline{G}$ to denote the complete graph of order $n$, the minimum degree, the maximum degree, the connectivity and the complement of $G$, respectively. For a subset $S \subseteq V(G)$ or $S \subseteq E(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$. A cycle (or path, respectively) on $n$ vertices is often denoted by $C_n$ (or $P_n$, respectively). The cycle containing all vertices of a graph $G$ is called a Hamilton cycle of $G$. Let $G$ be a graph with vertex set $V(G) = \{ v_1, v_2, \ldots, v_n \}$ and edge set $E(G)$. The adjacency matrix of $G$ is the $n \times n$ matrix $A(G) := (a_{ij})$, where $a_{ij}$ is the number of edges joining $v_i$ and $v_j$ in $G$, and the diagonal degree matrix of $G$ is the $n \times n$ matrix...
$D(G) := (d_{ij})$, where $d_{ij} = 0$ if $i \neq j$ and $d_{ii} = d_G(v_i)$, for all $i$ with $1 \leq i \leq n$. For a subset $X \subseteq E(G)$, the characteristic function of $X$ is a function $f_X : E \to \{0, 1\}$ such that $f(e) = 1$ if and only if $e \in X$.

For any graph $G$ with the adjacency matrix $A(G)$ and the diagonal degree matrix $D(G)$ and for real numbers $a \geq 0$ and $b > 0$, define $J(G : a, b) = aD(G) + bA(G)$. As $J(G : a, b)$ is also real and symmetric, all its eigenvalues are real. Let $\lambda_i(J(G : a, b))$ be the $i$th largest eigenvalue of $J(G : a, b)$. In particular, $\lambda_1(J(G : 0, 1)) = \lambda_1(G)$ is known as the spectral radius of $G$; $\lambda_1(J(G : 1, 1)) = q_1(G)$ is the $Q$-index of $G$; and $\lambda_1(J(G : \alpha, 1 - \alpha)) = \lambda_1(A_\alpha(G))$ is called the $A_\alpha$-spectral radius of $G$ with $\alpha \in [0, 1]$, introduced by Nikiforov in [2].

As in [1], an edge subset $M \subseteq E(G)$ is a matching of a graph $G$ if $\Delta(G[M]) \leq 1$. The matching number of $G$ is $\alpha'(G) = \max\{|M| : M \text{ is a matching of } G\}$. A fractional matching of a graph $G$ is a function $f : E(G) \to [0, 1]$ such that for any $v \in V(G)$, $\sum_{e \in E_G(v)} f(e) \leq 1$. Thus if $f(e) \in [0, 1]$ for every edge $e \in E(G)$, then $f$ is the characteristic function of a matching in $G$. The fractional matching number of $G$ is $\mu_f(G) = \max\{\sum_{e \in E(G)} f(e) : f \text{ is a fractional matching of } G\}$. The following is known.

**Theorem 1.1:** Let $G$ be a graph with $n = |V(G)|$. Each of the following holds.

(i) (Lemma 2.1.2 of [3]) Any fractional matching $f$ satisfies $\mu_f(G) \leq n/2$.

(ii) (Theorem 2.1.3 of [3]) If $G$ is bipartite, then $\mu_f(G) = \alpha'(G)$.

(iii) (Theorem 2.1.5 of [3]) $2\mu_f(G)$ is an integer.

A fractional matching $f$ of $G$ is a fractional perfect matching if $\sum_{e \in E(G)} f(e) = n/2$. If a fractional perfect matching takes values only in $\{0, 1\}$, then it is the characteristic function of a perfect matching.

Let $i(G)$ be the number of isolated vertices of a graph $G$. In [3], the following are stated.

**Theorem 1.2:** (Scheinerman and Ullman [3]) Let $G$ be a graph. Each of the following holds.

(i) $\mu_f(G) \geq \alpha'(G)$.

(ii) (The fractional Berge–Tutte Formula)

$$\mu_f(G) = \frac{1}{2} \left(n - \max \{i(G - S) - |S| : \forall S \subseteq V(G)\}\right).$$

(iii) (The fractional Tutte’s 1-Factor Theorem) A graph $G$ has a fractional perfect matching if and only if

$$\forall S \subseteq V(G), \ i(G - S) \leq |S|. \quad (2)$$

The investigation on the relationship between the eigenvalues and the matching number of a graph was initiated by Brouwer and Haemers [4], in which sufficient conditions on $\lambda_3(G)$ to assure the existence of a perfect matching were discussed. Cioabă et al. in [5–7] improved and generalized these results to obtain a best possible upper bound on $\lambda_3(G)$ for the existence of a perfect matching. Furthermore, O and Cioabă [8] determined the relationship between the eigenvalues of a $t$-edge-connected $k$-regular graph and its matching number when $t \leq k - 2$. 
On the fractional matching number, O [9] studied the connections between the fractional matching number and the spectral radius of a connected graph with given minimum degree. Xue et al. [10] considered the relationship between the fractional matching number and the Laplacian spectral radius of a graph.

The current research is motivated by the results proven in [4–10]. The main goal of this study is to investigate the relationship between the fractional matching number and the largest eigenvalue of the nonnegative matrix $J(G : a, b)$ of a graph $G$. This provides a mechanism to have a unified approach to both adjacency eigenvalues and signless Laplacian eigenvalues, and is quite different from those using only the Laplacian matrix. Consequently, related results involving the $Q$-index and $A_\alpha$-spectral radius of a graph $G$ are also obtained.

Our approach is inspired by the idea used in [9] as well as the methods deployed in [11–15]. We first observe that (see Theorem 2.2 in the next section) for any real number $k \in (0, n)$ and a connected graph $G$ with $n = |V(G)|$ and $\delta = \delta(G)$, if $\delta > \frac{n-k}{2}$, then $\mu_f(G) > \frac{n-k}{2}$. Thus we focus on the investigation of $\mu_f(G)$ when $\delta \le \frac{n-k}{2}$.

For real numbers $a$, $k$ and an integer $\delta$, with $a \ge 0$, $k \in (0, n)$ and $1 \le \delta \le \frac{n-k}{2}$, define

$$\varphi(a, n, \delta, k) = \begin{cases} \sqrt{\frac{n+k}{n-k}} & \text{if } a = 0, \\ \frac{2a\delta n}{n-k} & \text{if } a \in (0, 1], \\ \frac{a\delta(n+k)}{n-k} & \text{if } a \in (1, +\infty). \end{cases}$$

Our main results are the following.

**Theorem 1.3:** Let $a$ be a real number with $a \ge 0$, $k \in (0, n)$ be a real number, and $G$ be a connected simple graph with $n = |V(G)|$ and $\delta = \delta(G) \le \frac{n-k}{2}$. If

$$\lambda_1(aD(G) + A(G)) < \varphi(a, n, \delta, k),$$

then $\mu_f(G) > \frac{n-k}{2}$.

The upper bound in Theorem 1.3 is best possible when $a \in \{0, 1\}$, as will be shown in Section 3. A sufficient condition to ensure $\mu_f(G) > \frac{n-k}{2}$ in terms of $\lambda_1(J(G : a, 1))$ is obtained in Theorem 1.4. The upper bound in Theorem 1.4 is also best possible in some sense.

**Theorem 1.4:** Let $a$ be a real number with $a \ge 0$, $k \in (0, n)$ be a real number, and $G$ be a connected simple graph with $n = |V(G)|$ and $\delta = \delta(G) \le \frac{n-k}{2}$. If

$$\lambda_1(aD(G) + A(G)) < (a + 1)(\delta + k - 1),$$

then $\mu_f(G) > \frac{n-k}{2}$.

The main results can be applied to obtain sufficient conditions in terms of $Q$-index $q_1(G)$ and $A_\alpha$-spectral radius $\lambda_1(A_\alpha(G))$ for a simple graph $G$ to have a large value of $\mu_f(G)$. Detailed justifications of the following corollaries will be presented in Section 4.

**Corollary 1.5:** Let $G$ be a connected simple graph with $n = |V(G)|$ and $\delta = \delta(G) \le \frac{n-k}{2}$. Each of the following holds.
(i) If \( q_1(G) < \frac{2\delta n}{n-k} \), then \( \mu_f(G) > \frac{n-k}{2} \).

(ii) If \( q_1(G) < 2(\delta + k - 1) \), then \( \mu_f(G) > \frac{n-k}{2} \).

**Corollary 1.6:** Let \( G \) be a connected simple graph with \( n = |V(G)| \) and \( \delta = \delta(G) \leq \frac{n-k}{2} \), and let \( \alpha \) be a real number with \( 0 \leq \alpha < 1 \). Each of the following holds.

(i) If \( \alpha = 0 \) and \( \lambda_1(A_\alpha(G)) < \delta \), then \( \mu_f(G) > \frac{n-k}{2} \).

(ii) If \( 0 < \alpha \leq \frac{1}{2} \) and \( \lambda_1(A_\alpha(G)) < \frac{2\alpha \delta n}{n-k} \), then \( \mu_f(G) > \frac{n-k}{2} \).

(iii) If \( \frac{1}{2} < \alpha < 1 \) and \( \lambda_1(A_\alpha(G)) < \frac{\alpha \delta (n+k)}{n-k} \), then \( \mu_f(G) > \frac{n-k}{2} \).

(iv) If \( \lambda_1(A_\alpha(G)) < \delta + k - 1 \), then \( \mu_f(G) > \frac{n-k}{2} \).

Preliminaries will be presented in the next section. Proofs of the main results and discussions of the consequences will be given in the last two sections.

**2. Preliminaries**

Throughout this section, we always assume that \( k \in (0, n) \) is a real number, and \( G \) is a connected simple graph with \( n = |V(G)| \) and \( \delta = \delta(G) \). A path with end vertices \( u \) and \( v \) is called a \((u, v)\)-path. We start with some of the well-known results of Dirac.

**Theorem 2.1:** (see also Bondy and Murty [1]) Let \( G \) be a simple graph with \( n = |V(G)| \) and \( \delta = \delta(G) > 0 \). Each of the following holds.

(i) (Dirac [16]) If \( \delta \geq n/2 \) and \( n \geq 3 \), then \( G \) is hamiltonian.

(ii) (Dirac [16]) If \( k(G) \geq 2 \) and \( \delta \leq n/2 \), then \( G \) contains a cycle of length at least \( 2\delta \).

(iii) \( G \) contains a path of length \( \delta \).

Theorem 2.1 can be applied to prove Theorem 2.2, which indicates that the focus of the research should be restricted to the cases when \( \delta \leq \frac{n-k}{2} \).

**Theorem 2.2:** Let \( s \geq 1 \) be an integer. Each of the following holds.

(i) If \( G \) has a subgraph \( L \) with \( \Delta(L) \leq 2 \) and \( |E(L)| \geq 2s \), then \( \mu_f(G) \geq s \). In particular, if \( G \) has a cycle or path with at least \( 2\delta \) edges, then \( \mu_f(G) \geq \delta \).

(ii) If \( \delta > \frac{n-k}{2} \), then \( \mu_f(G) > \frac{n-k}{2} \).

**Proof.** If \( \Delta(L) \leq 2 \), then \( f = \frac{1}{2|E(L)|} \) is a fractional matching, and so \( \mu_f(G) \geq \frac{|E(L)|}{2} \geq s \). This proves (i). To show (ii), we observe that \( \delta > 0 \). As \( n \geq n-k \), by Theorem 2.1(i) and Theorem 2.2(i), we may assume that \( \frac{n-k}{2} < \delta < \frac{n}{2} \). If \( k(G) \geq 2 \), then by Theorem 2.1(ii), \( G \) contains a cycle \( C_s \) with \( s = |E(C_s)| \geq 2\delta \). By Theorem 2.2(i), \( \mu_f(G) \geq \delta > \frac{n-k}{2} \). Hence we assume that \( G \) contains a cut vertex \( u \). Let \( G_1 \) and \( G_2 \) be two of connected components...
of $G-u$. Let $v_1, v_2 \in N_G(u)$ with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. As $G$ is simple, $\delta(G-u) \geq \delta(G) - 1$. In particular, $\delta(G_i) \geq \delta(G) - 1$, for $i \in \{1, 2\}$. By Theorem 2.1(iii), for $i \in \{1, 2\}$, each $G_i$ contains a $(v_i, v'_i)$-path $P_i$, for some $v'_i \in V(G_i - v_i)$, with $|E(P_i)| \geq \delta - 1$. Then $G$ contains a path $P = G[E(P_1) \cup v_1 \nu v_2 \cup E(P_2)]$ of length $2\delta$, and by Theorem 2.2(i), $\mu_f(G) \geq \delta > \frac{n-k}{2}$.

The main tool in our paper is the following eigenvalue interlacing technique. Given two non-increasing real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to interlace the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is tight if exists an integer $k \in [0, m]$ such that $\theta_i = \eta_i$ for $1 \leq i \leq k$ and $\theta_{n-m+i} = \eta_i$ for $k + 1 \leq i \leq m$.

Let $M$ be the following $n \times n$ matrix

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{pmatrix},$$

whose rows and columns are partitioned into subsets $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. Let $M_{ij}$ denote the submatrix (called a block) of $M$ by deleting the rows in $\{1, 2, \ldots, n\} - X_i$ and deleting the columns in $\{1, 2, \ldots, n\} - X_j$. The quotient matrix $R$ of the matrix $M$ (with respect to the given partition) is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{ij}$ of $M$. The partition is equitable if each block $M_{ij}$ of $M$ has constant row (and column) sum.

**Theorem 2.3:** (Brouwer and Haemers [17, 18]) Let $M$ be a real symmetric matrix. Then the eigenvalues of every quotient matrix of $M$ interlace the ones of $M$. Furthermore, if the interlacing is tight, then the partition is equitable.

**Theorem 2.4:** (Haynsworth [19], You et al. [20]) Let $M$ be a partitioned matrix, and $R$ be its equitable quotient matrix. Then the eigenvalues of the quotient matrix $R$ are eigenvalues of $M$. Furthermore, if $M$ is a nonnegative matrix, then the spectral radius of the quotient matrix $R$ equals to the spectral radius of $M$.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices. Define $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i$ and $j$; and $A < B$ if $A \leq B$ and $A \neq B$.

**Theorem 2.5:** (Berman and Plemmons [21], Horn and Johnson [22]) Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ matrices with the spectral radii $\lambda_1(A)$ and $\lambda_1(B)$. If $0 \leq A \leq B$, then $\lambda_1(A) \leq \lambda_1(B)$. Furthermore, if $B$ is irreducible and $0 \leq A < B$, then $\lambda_1(A) < \lambda_1(B)$.

### 3. Proof of Theorems 1.3 and 1.4

Following the notation in [1], for $S, T \subseteq V(G)$, define $E_G([S, T]) = \{uv \in E(G) : u \in S$ and $v \in T\}$. We will use (3) as the definition of the function $\varphi(a, n, \delta, k)$. Throughout this section, we always assume that $n$ is a positive integer, $a$ and $k$ are real numbers satisfying $a \geq 0$ and $0 < k < n$, and that $G$ is a connected simple graph with $n = |V(G)|$, $\delta = \delta(G) \leq \frac{n-k}{2}$.
Before proceeding further, we present a lemma for the spectral radius of a bipartite graph.

**Lemma 3.1:** Let $H$ be the bipartite graph with bipartition $(S, T)$. If $|S| = s$, $|T| = t$ and $|E(H)| = r$, then

$$
\lambda_1(aD(H) + A(H)) \geq \frac{1}{2} \left( a \left( \frac{r}{s} + \frac{r}{t} \right) + \sqrt{a^2 - 1} \left( \frac{r}{s} - \frac{r}{t} \right)^2 + \left( \frac{r}{s} + \frac{r}{t} \right)^2 \right).
$$

**Proof.** Let $R(aD(H) + A(H))$ be the quotient matrix of $aD(H) + A(H)$ with respect to the partition $(S, T)$. One can see that

$$
R(aD(H) + A(H)) = \left( \begin{array}{cc} \frac{a r}{s} & \frac{r}{t} \\ \frac{a r}{s} & \frac{r}{t} \end{array} \right).
$$

A direct computation shows that the characteristic polynomial of $R(aD(H) + A(H))$ is

$$
\lambda^2 - a \left( \frac{r}{s} + \frac{r}{t} \right) \lambda + (a^2 - 1) \frac{r^2}{st} = 0,
$$

which yields

$$
\lambda_1(R(aD(H) + A(H))) = \frac{1}{2} \left( a \left( \frac{r}{s} + \frac{r}{t} \right) + \sqrt{(a^2 - 1) \left( \frac{r}{s} - \frac{r}{t} \right)^2 + \left( \frac{r}{s} + \frac{r}{t} \right)^2} \right).
$$

The result follows from Theorem 2.3.

### 3.1. Proof of Theorem 1.3

By contradiction, we assume that $\mu_f(G) \leq \frac{n-k}{2}$. By (1), there exists a vertex subset $S \subseteq V(G)$ satisfying $i(G - S) - |S| \geq k$. Let $T$ be the set of isolated vertices in $G - S$, $|S| = s$ and $|T| = t$. Then $s + t \leq n$ and $t = i(G - S) \geq s + k$, and so $s \leq \frac{n-k}{2}$. As $T$ is the set of isolated vertices in $G - S$, we observe that $N_G(T) \subseteq S$, and so $s \geq \delta$.

Let $H$ be the bipartite graph with vertex set $S \cup T$ and edge set $E_G[S, T]$. Clearly, $H$ is a bipartite subgraph of $G$. Set $r = |E(H)|$. Then $r \geq t\delta$. According to Lemma 3.1, we have

$$
\lambda_1(aD(H) + A(H)) \geq \frac{1}{2} \left( a \left( \frac{r}{s} + \frac{r}{t} \right) + \sqrt{(a^2 - 1) \left( \frac{r}{s} - \frac{r}{t} \right)^2 + \left( \frac{r}{s} + \frac{r}{t} \right)^2} \right). \quad (4)
$$

By Theorems 2.5, to reach a contradiction to the assumption of Theorem 1.3, it suffices to show that in each of the following cases, we always have

$$
\lambda_1(aD(G) + A(G)) \geq \varphi(a, n, \delta, k). \quad (5)
$$

**Case 1** $a = 0$. 

\[\lambda_1(aD(G) + A(G)) \geq \lambda_1(aD(H \cup (n-s-t)K_1) + A(H \cup (n-s-t)K_1))\]
\[= \lambda_1(aD(H) + A(H)) \geq r\sqrt{\frac{1}{st}} \geq \delta \sqrt{\frac{t}{s}} \geq \delta \sqrt{1 + \frac{k}{s}} \geq \delta \sqrt{\frac{n+k}{n-k}}.\]

**Case 2** \(0 < a \leq 1.\)

This implies that \(a^2 - 1 \leq 0,\) and so by (4),
\[\lambda_1(aD(H) + A(H)) \geq \frac{1}{2} \left( a \left( \frac{r}{s} + \frac{r}{t} \right) + \sqrt{(a^2 - 1) \left( \frac{r}{s} + \frac{r}{t} \right)^2 + \left( \frac{r}{s} + \frac{r}{t} \right)^2} \right) = a \left( \frac{r}{s} + \frac{r}{t} \right).\]

It follows that
\[\lambda_1(aD(G) + A(G)) \geq \lambda_1(aD(H) + A(H)) \geq a \left( \frac{r}{s} + \frac{r}{t} \right) \geq a\delta \left( 1 + \frac{t}{s} \right) \geq a\delta \left( 2 + \frac{k}{s} \right) \geq \frac{2a\delta n}{n-k}.\] \hfill (6)

**Case 3** \(1 < a < +\infty.\)

This implies that \(a^2 - 1 > 0,\) and so by (4),
\[\lambda_1(aD(H) + A(H)) \geq \frac{1}{2} \left( a \left( \frac{r}{s} + \frac{r}{t} \right) + \sqrt{(a^2 - 1) \left( \frac{r}{s} - \frac{r}{t} \right)^2 + \left( \frac{r}{s} - \frac{r}{t} \right)^2} \right) = \frac{ar}{s}.\]

It follows that
\[\lambda_1(aD(G) + A(G)) \geq \lambda_1(aD(H) + A(H)) > \frac{ar}{s} \geq a\delta \frac{t}{s} \geq a\delta \left( 1 + \frac{k}{s} \right) \geq \frac{a\delta n}{n-k}.\]

As in all the possible values of \(a,\) (5) is always obtained, Theorem 1.3 is now justified.

### 3.2. Sharpness discussion of Theorem 1.3

Following [1], for disjoint vertex sets \(X\) and \(Y,\) \(G[X, Y]\) denotes a bipartite graph with a vertex bipartition \((X, Y).\) Example 3.2 below, originally from [9] by O, also indicates that Theorem 1.3 is best possible when \(a \in \{0, 1\}.\)

**Example 3.2:** Let \(X\) and \(Y\) be disjoint vertex sets, and \(k, n\) and \(\delta\) be positive integers satisfying \(n > k > 0\) and \(\delta \leq \min\{|X|, |Y|\}.\) Define \(B(\delta, k)\) to be the family of bipartite graphs of the form \(G[X, Y]\) with \(n = |X| + |Y|,\) satisfying each of the following:
(H1) for any \( v \in X, d(v) \) is a constant,

(H2) for any \( v \in Y, d(v) = \delta \),

(H3) \( |Y| = |X| + k \).

By definition, the complete bipartite graph \( K_{\delta, \delta + k} \in \mathcal{B}(\delta, k) \).

**Lemma 3.3:** For any \( H^* \in \mathcal{B}(\delta, k) \), each of the following holds.

(i) \( \mu_f(H^*) = \alpha'(H^*) = |X| = \frac{n-k}{2} \).

(ii) \( a = 0 \) or \( 1 \), \( \lambda_1(aD(H^*) + A(H^*)) = \varphi(a, n, \delta, k) \).

(iii) \( a \in (0, 1) \) or \( (1, +\infty) \), \( \lambda_1(aD(H^*) + A(H^*)) > \varphi(a, n, \delta, k) \).

**Proof.** Let \( d \) be the degree of each vertex in \( X \). By (H1) and (H2), \( d|X| = |E(H^*)| = \delta|Y| = \delta(|X| + k) \), leading to \( d > \delta > 1 \). For any subset \( S \subseteq X \), \( |N_{H^*}(S)| \geq \frac{\delta}{\delta} |S| > |S| \). It follows by Hall’s Theorem (Theorem 16.4 of [1]) that \( H^* \) has a matching \( M \) of size \( |X| \). Since \( H^* \) is bipartite, by Theorem 1.1(ii), we conclude that \( \mu_f(H^*) = \alpha'(H^*) = |X| = \frac{n-k}{2} \), and so (i) follows.

We prove both (ii) and (iii) simultaneously. As in Lemma 3.1, the quotient matrix \( R(aD(H^*) + A(H^*)) \) of \( aD(H^*) + A(H^*) \) with respect to the partitions \( X \) and \( Y \) of \( V(H^*) \) is

\[
R(aD(H^*) + A(H^*)) = \begin{pmatrix}
\frac{a\delta}{|X|} & \frac{\delta}{|X|} \\
\delta & \alpha
\end{pmatrix},
\]

and its spectral radius is

\[
\lambda_1(R(aD(H^*) + A(H^*)))
= \frac{1}{2} \left( a\delta \left( \frac{|Y|}{|X|} + 1 \right) + \sqrt{(a^2 - 1)\delta^2 \left( \frac{|Y|}{|X|} - 1 \right)^2 + \delta^2 \left( \frac{|Y|}{|X|} + 1 \right)^2} \right).
\] (7)

As the partition is equitable and \( aD(H^*) + A(H^*) \) is nonnegative, it follows from Theorem 2.4 that \( \lambda_1(aD(H^*) + A(H^*)) = \lambda_1(R) \). This, together with (7), leads to the following observations.

(A) If \( a = 0 \), then \( \lambda_1(A(H^*)) = \delta \sqrt{\frac{|Y|}{|X|}} = \delta \sqrt{1 + \frac{k}{|X|}} = \delta \sqrt{1 + \frac{2k}{n-k}} = \varphi(0, n, \delta, k) \).

(B) If \( a = 1 \), then \( \lambda_1(D(H^*) + A(H^*)) = \delta \left( \frac{|Y|}{|X|} + 1 \right) = \frac{2n\delta}{n-k} = \varphi(1, n, \delta, k) \).

(C) If \( 0 < a < 1 \), then \( a^2 - 1 < 0 \), and so

\[
\lambda_1(aD(H^*) + A(H^*))
> \frac{1}{2} \left( a\delta \left( \frac{|Y|}{|X|} + 1 \right) + \sqrt{(a^2 - 1)\delta^2 \left( \frac{|Y|}{|X|} + 1 \right)^2 + \delta^2 \left( \frac{|Y|}{|X|} + 1 \right)^2} \right)
= a\delta \left( \frac{|Y|}{|X|} + 1 \right) = \frac{2a\delta n}{n-k} = \varphi(a, n, \delta, k) \cdot
\]
(D) If 1 < a < +∞, then a² − 1 > 0, and so
\[
\lambda_1(aD(H^*) + A(H^*)) > \frac{1}{2} \left( a\delta \left( \frac{|Y|}{|X|} + 1 \right) + \sqrt{(a^2 - 1)\delta^2 \left( \frac{|Y|}{|X|} - 1 \right)^2 + \delta^2 \left( \frac{|Y|}{|X|} - 1 \right)^2} \right) 
= a\delta \left( \frac{|Y|}{|X|} + 1 \right) = \frac{a\delta(n + k)}{n - k} = \varphi(a, n, \delta, k).
\]

These observations (A)–(D) complete the proof of the lemma.

**Remark 3.4:** By Lemma 3.3, there exist graphs H* ∈ B(δ, k) with minimum degree δ such that λ₁(A(H*)) = ϕ(0, n, δ, k), μ₁(H*) = \frac{n-k}{2} and λ₁(D(H*) + A(H*)) = ϕ(1, n, δ, k), μ₁(H*) = \frac{n-k}{2}. Hence, the upper bound in Theorem 1.3 is best possible when a ∈ {0, 1}.

### 3.3. The proof and discussion of Theorem 1.4

By contradiction, assume that μ₁(G) ≤ \frac{n-k}{2}. By (1), there exists a vertex subset S ⊆ V(G) such that i(G − S) − |S| ≥ k. Let T be the set of isolated vertices in G − S. Let |S| = s and |T| = t. Then s + t ≤ n and t = i(G − S) ≥ s + k. As \( \bigcup_{v \in T} N_{G}(v) \subseteq S \), we observe that s ≥ δ, and so t ≥ s + k ≥ δ + k. As K₁ ∪ (n − t)K₁ is a spanning subgraph of G, it follows by Theorem 2.5 that
\[
λ₁(aD(G) + A(G)) = λ₁(aD(K₁ ∪ (n − t)K₁) + A(K₁ ∪ (n − t)K₁)) = (a + 1)(t − 1) ≥ (a + 1)(δ + k − 1),
\]
leading to a contradiction to the hypothesis. This completes the proof of Theorem 1.4.

**Example 3.5:** By definition, the complete bipartite graph Kₙ,δ+δ⁺∈ B(δ, k). Direct computation yields that λ₁(aD(Kₙ,δ+δ⁺) + A(Kₙ,δ+δ⁺)) = (a + 1)(δ + k − 1), and μ₁(Kₙ,δ+δ⁺) = μ(Kₙ,δ+δ⁺) = δ = \frac{n-k}{2}. In this sense that the upper bound in Theorem 1.4 is best possible.

### 4. Corollaries of Theorems 1.3 and 1.4

Throughout this section, we assume that a, b and k are real numbers with a ≥ 0, b > 0 and k ∈ (0, n). Sometimes, it is more convenient to consider matrices of the form aD(G) + bA(G), in which case we get the following rescaling of our results:

**Corollary 4.1:** Let G be a connected simple graph with n = |V(G)| and δ = δ(G) ≤ \frac{n-k}{2}. If λ₁(aD(G) + bA(G)) < bϕ(a, n, δ, k), then μ₁(G) > \frac{n-k}{2}.

**Corollary 4.2:** Let G be a connected simple graph of order n with n = |V(G)| and δ = δ(G) ≤ \frac{n-k}{2}. If λ₁(aD(G) + bA(G)) < (a + b)(δ + k − 1), then μ₁(G) > \frac{n-k}{2}.

We observe that Corollary 1.5 follows by letting a = b = 1 in Corollaries 4.1 and 4.2, and Corollary 1.6 can be obtained from Corollaries 4.1 and 4.2 by setting a = α and b = 1 − α.
4.1. Fractional matching number and eigenvalues

We present an application of Theorem 1.3 in this subsection by showing a relationship between $\mu_f(G)$ and $\lambda_1(AD(G) + A(G))$ for a graph $G$. Theorem 4.3(i) was proved by O in [9], and is included here for the sake of completeness.

**Theorem 4.3:** Let $G$ be a connected simple graph with $n = |V(G)|$ and $\delta = \delta(G) \leq \frac{n-k}{2}$. Each of the following holds.

(i) (O [9]) If $a = 0$, then $\mu_f(G) \geq \frac{n^2}{\lambda_1(G)^2+\delta^2}$, where the equality holds if and only if $G$ is a regular graph or $G \in \mathcal{B}(\delta, k)$, where $k = \frac{n(\lambda_1(G)^2-\delta^2)}{\lambda_1(G)^2+\delta^2}$ is an integer.

(ii) If $a = 1$, then $\mu_f(G) \geq \frac{\delta n}{\lambda_1(AD(G)+A(G))}$, where the equality holds if and only if $G$ is a regular graph or $G \in \mathcal{B}(\delta, k)$, where $k = n - \frac{2\delta n}{\lambda_1(AD(G)+A(G))}$ is an integer.

(iii) If $0 < a < 1$, then $\mu_f(G) \geq \frac{a\delta n}{\lambda_1(AD(G)+A(G))}$, where the equality holds if $G \in \mathcal{B}(\delta, k)$ and $k = n - \frac{2\delta n}{\lambda_1(AD(G)+A(G))}$ is an integer.

(iv) If $1 < a < +\infty$, then $\mu_f(G) \geq \frac{a\delta n}{\lambda_1(AD(G)+A(G))-a\delta}$, where the equality holds if $G \in \mathcal{B}(\delta, k)$ and $k = n - \frac{2\delta n}{\lambda_1(AD(G)+A(G))+a\delta}$ is an integer.

**Proof.** We only prove Theorem 4.3 (ii)–(iv). Throughout the proof, we denote $\lambda_1(AD(G)+A(G))$ and $\mu_f(G)$ by $\lambda_1$ and $\mu_f$, respectively. Theorem 1.3 says that if $\lambda_1 < \varphi(a, n, \delta, k)$, then $\mu_f > \frac{n-k}{2}$. Theorem 4.3 follows by applying Theorem 1.3 to each of the following possible values of the number $a$.

Assume that $0 < a \leq 1$. Define $k_1 = n - \frac{2a\delta n}{\lambda_1}$. As $\frac{1}{n-x}$ is an increasing function of $x$ on $(0, n)$, $\frac{2a\delta n}{n-k}$ decreases towards $\lambda_1$ as $k$ decreases towards $k_1$, and $\lim_{k \to k_1} \frac{2a\delta n}{n-k} = \lambda_1$. By Theorem 1.3, $\mu_f > \frac{n-k}{2}$ for each value of $k \in (k_1, n)$. It follows that $\mu_f \geq \lim_{k \to k_1} \frac{n-k}{2} = \frac{n-k_1}{2} = \frac{a\delta n}{\lambda_1}$. Finally, if $G$ is a regular graph, then $\mu_f(G) = \frac{n}{2} = \frac{\delta n}{\lambda_1(AD(G)+A(G))}$. If $G \in \mathcal{B}(\delta, k)$ and $k = n - \frac{2a\delta n}{\lambda_1(AD(G)+A(G))}$ is an integer, then direct computation yields $\mu_f(G) = \alpha'(G) = \frac{n-k}{2} = \frac{a\delta n}{\lambda_1(AD(G)+A(G))+a\delta}$. This proves Theorem 4.3(iii) and lower bound part of Theorem 4.3(ii).

Now suppose that $a = 1$ and that $\mu_f(G) = \frac{\delta n}{\lambda_1(AD(G)+A(G))}$. To simplify the notation and terms, we shall adopt the definition of the vertex sets $S$ and $T$, and the definition of the graph $H$ in Subsection 3.1, with $s = |S|$ and $t = |T|$. As $\mu_f(G) = \frac{\delta n}{\lambda_1(AD(G)+A(G))}$, every inequality in the arguments of the previous paragraph must be an equality, and so $k = k_1 = n - \frac{2a\delta n}{\lambda_1(AD(G)+A(G))}$. If $k = 0$, then $\lambda_1(AD(G)+A(G)) = 2\delta$, and thus $G$ is a regular graph. Next we consider $k > 0$. Furthermore, every inequality in (6) in Subsection 3.1 must also be an equality. Hence in (6) we must have $a\delta(1 + \frac{t}{s}) = a\delta(2 + \frac{k}{s})$, leading to $k = t-s$, and so $k$ is a positive integer. Likewise, in (6) we must also have $2 + \frac{k}{s} = \frac{2n}{n-k}$. This, together with $k = t-s$, implies $n = s + t$. It follows that $r = t\delta$, $t = k + s$. By Theorem 2.3, the partition is equitable, and thus $G \cong H \in \mathcal{B}(\delta, k)$ from Theorem 2.5. This completes the proof of Theorem 4.3(ii).
Finally, we assume that \( a \in (1, +\infty) \). Define \( k_2 = \frac{n(\lambda_1 - a\delta)}{\lambda_1 + a\delta} \). As \( \frac{n+1}{n-1} \) is an increasing function of \( x \) on \((0, n)\), \( \frac{a\delta(n+k)}{n-k} \) decreases towards \( \lambda_1 \) as \( k \) decreases towards \( k_2 \) and 
\[
\lim_{k \to k_2} \frac{n-k}{n-k} = \lambda_1.
\]
By Theorem 1.3, \( \mu_f > \frac{n-1}{2} \) for each value of \( k \in (k_2, n) \), and so \( \mu_f \geq \frac{n-k}{2} = \frac{a\delta n}{\lambda_1 + a\delta} \). If \( G \in B(\delta, k) \) with \( k = \frac{n(\lambda_1(aD(G)+A(G))-a\delta)}{\lambda_1(aD(G)+A(G))+a\delta} \) being an integer, then direct computation yields that \( \mu_f(G) = \alpha'(G) = \frac{n-k}{2} = \frac{a\delta n}{\lambda_1(aD(G)+A(G))+a\delta} \). This justifies Theorem 4.3(iv), and completes the proof of the theorem.

4.2. Fractional perfect matchings and eigenvalues

The following Corollaries 4.4 and 4.5 are immediate consequences of Theorems 1.3 and 1.4, respectively. Letting \( k = 1 \) in Theorems 1.3 and 1.4, respectively, we conclude that the hypotheses of Corollaries 4.4 and 4.5, respectively, imply that \( \mu_f(G) > \frac{n-1}{2} \). By Theorem 1.1, \( 2\mu_f(G) \) is an integer and \( \mu_f(G) \leq \frac{n}{2} \). This forces that \( \mu_f(G) = \frac{n}{2} \), and so \( G \) has a fractional perfect matching.

**Corollary 4.4:** Let \( a \) be a real number with \( a \geq 0 \), and \( G \) be a simple connected graph of order \( n \) with minimum degree \( \delta \leq \frac{n-1}{2} \). If 
\[
\lambda_1(aD(G) + A(G)) < \phi(a, n, \delta, 1),
\]
then \( G \) has a fractional perfect matching.

**Corollary 4.5:** Let \( a \) be a real number with \( a \geq 0 \), \( G \) be a connected graph of order \( n \) with minimum degree \( \delta \leq \frac{n-1}{2} \), and \( G \) be the complement of \( G \). If 
\[
\lambda_1(aD(G) + A(G)) < (a+1)\delta,
\]
then \( G \) has a fractional perfect matching.

As it is shown in Example 3.5, the upper bound in Corollary 4.5 is best possible. This upper bound in Corollary 4.5 can be slightly improved by excluding some graphs. Following [1], the join of two graphs \( G_1 \) and \( G_2 \), denoted \( G_1 \cup G_2 \), is the graph formed from the disjoint union of \( G_1 \) and \( G_2 \) by adding new edges joining every vertex of \( G_1 \) to every vertex of \( G_2 \).

**Theorem 4.6:** Let \( a \) be a real number with \( a \geq 0 \), \( G \) be a connected graph with \( n = |V(G)| \), and \( \delta = \delta(G) \leq \frac{n-1}{2} \). If
\[
\lambda_1(aD(G) + A(G)) < (a+1)(\delta+1), \tag{8}
\]
then \( G \) has a fractional perfect matching unless \( G \cong (\delta+1)K_1 \cup H_\delta \), where \( H_\delta \) is a simple graph of order \( \delta \).

**Proof.** By contradiction, assume that \( G \) has no fractional perfect matchings. By (2), there exists a vertex set \( S \subseteq V(G) \) such that \( i(G-S) - |S| > 0 \). Let \( T \) be the set of isolated
vertices in $G - S$. Then $\bigcup_{v \in T} N_G(v) \subseteq S$, and so $|S| \geq \delta$. This implies that $|T| \geq |S| + 1 \geq \delta + 1$.

We claim that $V(G) = S \cup T$. By contradiction, assume that there exists a vertex $v \in V(G) - S \cup T$. Then we have $K_{\delta+2} \subseteq \overline{G}[T \cup \{v\}] \subseteq \overline{G}$. By Theorem 2.5,

$$\lambda_1(aD(\overline{G}) + A(\overline{G})) \geq \lambda_1(aD(K_{\delta+2} \cup (n - \delta - 2)K_1) + A(K_{\delta+2} \cup (n - \delta - 2)K_1))$$

$$= \lambda_1(aD(K_{\delta+2}) + A(K_{\delta+2})) = (a + 1)(\delta + 1),$$

contrary to (8).

Furthermore, we claim that $|T| = \delta + 1$. Assume to the contrary that $|T| \geq \delta + 2$. Then $\overline{G}$ contains a clique of order $\delta + 2$, and so $\lambda_1(aD(\overline{G}) + A(\overline{G})) \geq (a + 1)(\delta + 1)$, contrary to (8) again. Hence $|T| = \delta + 1$, and so $|S| = \delta$. It follows that $G \cong (\delta + 1)K_1 \lor H_\delta$. As there exists a vertex subset $V(H_\delta) \subseteq V(G)$ such that $i(G - V(H_\delta)) > |V(H_\delta)|$, by (2), $(\delta + 1)K_1 \lor H_\delta$ has no fractional perfect matchings. This completes the proof.

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