Heavy quark production near the threshold in QCD

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Abstract
Theoretical results for the cross section of heavy quark production near the threshold at NNLO of NRQCD are briefly overviewed.

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1 Introduction

The heavy quark production near thresholds will be thoroughly investigated at future accelerators, e.g. [1]. One can study the $q\bar{q}$ systems near threshold in $e^+e^-$ annihilation [2,3] and $\gamma\gamma$ collisions [4,5]. In $e^+e^-$ annihilation the production vertex is local (the electromagnetic current in case of photon and/or neutral weak current in case of $Z$-boson), the basic observable is a production cross section which is saturated by S-wave (for the vector current). In $\gamma\gamma$ collisions the production vertex is nonlocal and both S- and P-waves can be studied for different helicity photons, the number of observables is larger (cross sections $\sigma_S$, $\sigma_P$, S-P interference). In $e^+e^-$ annihilation high precision data are already available for $b\bar{b}$ and expected for $t\bar{t}$. The $\gamma\gamma \to q\bar{q}$ experiments are planned.

2 Features of theoretical description

Heavy quarks near the production threshold moves slowly that justifies the use of the nonrelativistic quantum mechanics for their description [6-8]. Being much simpler than the comprehensive relativistic treatment of the bound state problem with Bethe-Salpeter amplitude [9], the nonrelativistic approach allows one to take into account exactly such an essential feature of the near-threshold dynamics as Coulomb interaction [10]. For unstable heavy quarks with a large decay width it is possible to compute the cross section near threshold point-wise in energy because the large decay width suppresses the long distance effects of strong interaction [11]. The $t\bar{t}$-pair near the production threshold is just a system that satisfies the requirement of being nonrelativistic. Therefore the description of $t\bar{t}$-system near the production threshold $\sqrt{s} \approx 2m_t$ ($\sqrt{s}$ is a total energy of the pair, $m_t$ is the top quark mass) is quite precise within nonrelativistic QCD (NRQCD). Reasons for this accuracy are related to the large mass of the top:

- The top quark is very heavy $m_t = 175$ GeV [12] and there is an energy region of about $8 - 10$ GeV near the threshold where the nonrelativistic approximation for the kinematics is very precise. For $\sqrt{s} = 2m_t + E$ with $|E| < 5$ GeV the quark velocity is small

$$v = \sqrt{1 - \frac{4m_t^2}{s}} = \sqrt{1 - \frac{4m_t^2}{(2m_t + E)^2}} \approx \frac{E}{m_t} < 0.15 \ll 1.$$  \hspace{1cm} (1)

Relativistic effects are small and can be taken into account perturbatively in $v$ (even in $v^2$).

- The strong coupling constant at the high energy scale is small $\alpha_s(m_t) \approx 0.1$ that makes the perturbative mapping of QCD onto the low energy effective theory (NRQCD) numerically precise.

- The decay width of top quark is large, $\Gamma_t = 1.43$ GeV; infrared (small momenta) region is suppressed and PT calculation for the cross section near the threshold is reliable point-wise in energy.

Because $\alpha_s \sim v$ and the ratio $\alpha_s/v$ is not small, the Coulomb interaction is enhanced. The ordinary perturbation theory for the cross section (with free quarks as the lowest order approximation) breaks down and all terms of the order $(\alpha_s/v)^n$ should be summed. The expansion for a generic observable $f(E)$ in this kinematical region has the form

$$f(E) = f_0(\alpha_s/v) + \alpha_s f_1(\alpha_s/v) + \alpha_s^2 f_2(\alpha_s/v) + \ldots$$  \hspace{1cm} (2)
where \( f_i(\alpha_s/v) \) are some (not polynomial) functions of the ratio \( \alpha_s/v \), \( f_0(\alpha_s/v) \) is a result of the pure Coulomb approximation (or a kind of its improvement). The expansion in \( \alpha_s \) in eq. (2) takes into account the perturbative QCD corrections to the parameters of NRQCD and relativistic corrections (in the regime \( v \sim \alpha_s \)).

For the \( e^+e^- \rightarrow t\bar{t} \) process mediated by the photon the NNLO analysis is well known. The basic quantity is the vacuum polarization function

\[
\Pi(E) = i \int \langle Tj_{em}(x)j_{em}(0) \rangle e^{iqx} dx, \quad q^2 = (2m_t + E)^2.
\]

Near the threshold (for small energy \( E \)) NRQCD is used. The cross section is saturated with S-wave scattering. In this approximation the polarization function near the threshold to the NNLO accuracy in NRQCD is given by

\[
\Pi(E) = \frac{2\pi}{m_t^2} C_h(\alpha_s) C_O(E/m_t) G(E; 0, 0).
\]

The pole mass definition is used for \( m_t \) (e.g. [13]), \( \alpha_s \) is the strong coupling constant. \( C_h(\alpha_s) \) is the high energy coefficient. \( G(E; 0, 0) \) is the nonrelativistic Green function (GF). The quantity \( C_O(E/m_t) \) describes the contributions of higher dimension operators within the effective theory approach. These contributions have, in general, a different structure than the leading term. To the NNLO of NRQCD the contribution of higher dimension operators can be written as a total factor \( C_O(E/m_t) \) for the leading order GF, \( C_O(E/m_t) = 1 - 4E/3m_t \). The polarization function near the threshold (4) contains expansions in small parameters \( \alpha_s \) and/or \( v \), cf. eq. (3). The leading order approximation of the low energy part is the exact Coulomb solution for the Green function.

The nonrelativistic Green’s function \( G(E) = (H - E)^{-1} \) is determined by the nonrelativistic Hamiltonian

\[
H = \frac{p^2}{m_t} + V(r)
\]

(describing dynamics of the \( t\bar{t} \)-pair near the threshold. The most complicated part of Hamiltonian (5) to find is the heavy quark static potential \( V_{pot}(r) \) entering into the potential \( V(r) \). The static potential \( V_{pot}(r) \) is computed in perturbation theory and can be written in the form

\[
V_{pot}(q) = -\frac{4\alpha_v(r)}{3r}
\]

that gives a definition of the effective charge \( \alpha_v \) related to the \( \overline{\text{MS}} \)-scheme coupling constant

\[
\alpha_v(\mu) = \alpha_s(\mu)\left(1 + a_1\alpha_s(\mu) + a_2\alpha_s(\mu)^2\right).
\]

Coefficients \( a_{1,2} \) are known [14-15]. The effective coupling \( \alpha_v \) is nothing but a coupling constant in some special subtraction scheme. The coefficient \( a_2 \) allows one to find the effective \( \beta \)-function \( \beta_v \) for the evolution of the coupling \( \alpha_v \) at NNLO.

High energy coefficient \( C_h(\alpha_s) \) is given by the expression

\[
C_h(\alpha) = 1 - \frac{16}{3} \alpha_s + \left( \frac{\alpha_s}{\pi} \right)^2 \left( -\frac{140}{27}\pi^2 \ln \frac{\mu_f}{m_t} + c_2 \right).
\]

The NLO result for \( C_h(\alpha_s) \) has been known since long ago [17-18]. At NNLO there appears a term proportional to the logarithm of the factorization parameter \( \mu_f \) that separates long and short distances.
(or large and small momenta) within the effective theory approach. The finite ($\mu_f$ independent) coefficient $c_2$ is known \[20, 21\]. An explicit dependence of high and low energy quantities on the factorization scale $\mu_f$ is a general feature of effective theories which are valid only for a given region of energy. Physical quantities are factorization scale independent. In NRQCD the $\mu_f$ dependence cancels between Green’s function and the high energy coefficient $C_h$.

The main dynamical quantity in description of the $t\bar{t}$ system near the threshold is the nonrelativistic Green’s function $G = (H - E)^{-1}$. The Hamiltonian is represented in the form \[22-24\]

$$ H = H_C + \Delta H, \quad H_C = \frac{p^2}{m_t} - \frac{4\alpha_s}{3r} $$

with $\Delta H$ describing high order corrections. Constructing the Green’s function is straightforward and can be done analytically within perturbation theory near Coulomb Green’s function $G_C(E)$ or numerically for complex values of $E$ that can be used to describe the production of particles with nonzero width \[25-33\].

The analytical solution for Green’s function is perturbative in $\Delta H$

$$ G = G_C - G_C \Delta H G_C + G_C \Delta H G_C \Delta H G_C - \ldots $$

Results are presented basically as an expansion in consecutive orders

$$ G = G_0 + \Delta G_1 + \Delta G_2 $$

to check the convergence of the approximations. The leading order is given by Coulomb approximation, $G_0 = G_C$. At NLO the quantity $\Delta G_1$ takes into account the corrections from the static potential $V_{pot}(r)$ related to $a_1$ coefficient in eq. (7). At the NNLO the quantity $\Delta G_2 \sim O(\alpha_s^2)$ in a sense of eq. (2) accounts for $\alpha_s^2$ terms in the static potential $V_{pot}(r)$ ($a_2$ coefficient in eq. (7)) and relativistic $v^2$ corrections. It also contains a second iteration of the $O(\alpha_s)$ term from $V_{pot}(r)$. The $\overline{\text{MS}}$-scheme for the static potential $V_{pot}(r)$ is mainly used in the solution. The numerical results for GF obtained by different authors agree with each other \[34\].

3 Physical results

For the $b\bar{b}$ system an accurate description of the spectrum in terms of moments

$$ M_n = \int_{4m_b^2}^{\infty} \frac{\rho(s)ds}{s^n} $$

is obtained in the near-threshold Coulomb PT calculations. This analysis gave the best determination of the numerical value for the $b$ quark mass

$$ m_b = 4.80 \pm 0.06 \, \text{GeV}. $$

For the $t\bar{t}$ system the top quark width $\Gamma_t$ plays a crucial role in the calculation of the production cross section near the threshold. At the formal level the width is taken into account by a shift of the energy variable $E$. The mass operator of the top quark is approximated by the expression $M = m_t - i\Gamma_t/2$. 

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Then the kinematical variable $s - 4m^2$ relevant to the near-threshold dynamics is substituted with $s - 4M^2$ ($\sqrt{s} = E + 2m_t$) and one finds

$$s - 4M^2 = 4m_t(E + i\Gamma_t) + E^2 + \Gamma_t^2.$$  

Neglecting higher orders in $E$ and $\Gamma_t$ one obtains a recipe for taking into account the width $\Gamma_t$ by the shift $E \to E + i\Gamma_t$. The dispersion relation for the vacuum polarization function $\Pi(E)$ has the form

$$\Pi(E) = \int \frac{\rho(E')dE'}{E' - E}.$$  

With the shift recipe one finds

$$\sigma(E) \sim \text{Im} \Pi(E + i\Gamma_t) = \text{Im} \int \frac{\rho(E')dE'}{E' - E - i\Gamma_t} = \Gamma_t \int \frac{\rho(E')dE'}{(E' - E)^2 + \Gamma_t^2}. \quad (12)$$

Because the point $E + i\Gamma_t$ lies sufficiently far from the positive semiaxis (and the origin) in the complex energy plane the cross section eq. (12) is calculable point-wise in energy. The hadronic cross section $\sigma(E)$ was obtained by many authors [34]. The normalized cross section for typical values $m_t = 175$ GeV, $\Gamma_t = 1.43$ GeV, $\alpha_s(MZ) = 0.118$ has the characteristic points which are usually considered as basic observables. They are: $E_p$ – the position of the peak in the cross section and $H_p$ – its height. In the limit of the small $\Gamma_t$ (at least for $\Gamma_t$ that is smaller than the spacing between the first two Coulomb poles) one would have $E_p \sim E_0$ and $H_p \sim |\psi_0(0)|^2$. The actual value of $\Gamma_t = 1.43$ GeV is larger than the spacing $|E_0 - E_1| \sim m_t\alpha_s^2/3 \approx 0.6$ GeV therefore the peak position and height are not determined by the first resonance only. The convergence for $E_p$ and $H_p$ in the consecutive orders of perturbation theory near the Coulomb solution is not fast in the $\overline{\text{MS}}$-scheme. For the typical numerical values of the theoretical parameters $m_t$, $\Gamma_t$ and $\alpha_s(MZ)$ one finds [31]

$$E_p = E_0(1 + 0.58 + 0.38 + \ldots)$$

$$H_p = H_0(1 - 0.15 + 0.12 + \ldots) \quad (13)$$

(see also [35, 36]). Important contributions that affect the quality of convergence are the local term ($\sim \alpha_sV_0\delta(\vec{r})$ which is related to $1/r^2$ non-Abelian term [37]) and higher order PT corrections to $V_{pot}(r)$.  

4 Discussion and conclusion

Slow convergence for the peak characteristics of the cross section given in eq. (13) has been actively discussed. The suggestions of the redefinition of the top quark mass have been made (e.g. [34, 38, 39]) as the use of the pole mass for a description of the cross section near the threshold is criticized on the ground of its infrared instability [40]. This approach is based on introduction of an effective mass that partly accounts for interaction [41-43]. In this talk I only discuss some possible ways of optimizing the convergence for the Green’s function with the pole mass as a theoretical parameter [44]. Actual calculations near the threshold have been performed within the pole mass scheme. For optimizing the convergence one can use methods of exact summation of some contributions in all orders and renormalization scheme invariance of PT series e.g. [25]. The Hamiltonian can be written in the form

$$H = H_{LO} + \Delta V_{PT} + \alpha_sV_0\delta(\vec{r}), \quad V_0 = -\frac{70\pi}{9m_t^2} \quad (14)$$
where corrections are given by the perturbation theory corrections to $V_{\text{pot}}(r)$ ($\Delta V_{\text{PT}}$-part) and by the local term ($\alpha_s V_0 \delta(\vec{r})$-part). The $\delta(\vec{r})$-part is a separable potential and can be taken into account exactly \[13\]. The solution reads

\[ G(E; 0, 0) = \frac{G_{ir}(E; 0, 0)}{1 + \alpha_s V_0 G_{ir}(E; 0, 0)} \]  \hfill (15)

with

\[ G_{ir}(E) = (H_{LO} + \Delta V_{PT} - E)^{-1} \]  \hfill (16)

being the irreducible Green’s function. The PT expansion of the static potential in NRQCD is important for getting stable results for the cross section near the threshold because the static potential is the genuine quantity which is computed in high order of PT in the strong coupling constant \[44\]. The convergence in the $\overline{\text{MS}}$ scheme is not fast which reflects the physical situation that the observables represented by the cross section curve (for instance, $E_p$ and $H_p$) are sensitive to different scales.

The finite-order perturbation theory expansion of the static potential cannot handle several distinct scales with the same accuracy. Indeed, the PT expansion of the static potential is done near some (arbitrary) scale (or distance) which can be considered simply as a normalization point. The farther a given point lies from this normalization point the worse the precision of the PT expansion for the static potential is. The PT expressions in the $\overline{\text{MS}}$ scheme are not directly sensitive to physical scales because subtractions are made in a mass independent way (for instance, massive particles do not decouple automatically in the $\overline{\text{MS}}$ scheme e.g. \[17, 48\]). One can rewrite the static potential through some physical parameters which is similar to the use of the momentum subtraction scheme instead of the $\overline{\text{MS}}$-scheme

\[ V_{\text{pot}}(r) = -\frac{4\alpha_0}{3r} \left(1 + \alpha_0 b_1 \ln r/r_0 + \alpha_0^2 (b_1^2 \ln^2 r/r_0 + b_2^2 \ln r/r_0 + c)\right) \]  \hfill (17)

Here $r_0$ and $c$ are the parameters of the renormalization scheme freedom in NNLO and $\alpha_0$ is the corresponding coupling in the $\{r_0, c\}$-scheme \[44\]. They parameterize the center of the expansion (a normalization point) and the derivative (respective $\beta$-function) of the static potential. The parameters ($r_0, c$) can be chosen such in order to minimize the higher order corrections to a particular observable (e.g. \[23\] where NLO analysis has been done). In such a case $r_0$ can be understood as a typical distance to which a chosen observable is sensitive. Note that the best approximation of the static potential $V_{\text{pot}}(r)$ for different scales would be provided by the use of the running coupling constant $\alpha_s(r)$. The analytical calculation of GF becomes technically impossible in this case. The numerical calculation of GF requires some regularization at large distances where the IR singularity (Landau pole) can occur in $\alpha_s(r)$. The singularity can be dealt with if an IR fixed point appears in the evolution for the effective coupling constant (e.g. \[50\]) or with some nonPT regularization for the potential (e.g. \[24\]). For the top quark production the contribution of the large $r$ region into the cross section is small because of the large decay width of the top quark. In the finite-order PT analysis the parameters $r_0$ and $c$ can be chosen to minimize higher order corrections either to $E_p$ or to $H_p$ but not to both simultaneously because $E_p$ and $H_p$ are sensitive to different distances. One finds the difference of scales minimizing corrections to the first Coulomb resonance in NLO to be

\[ \ln(r_E/r_\psi) = \frac{1}{3} + \frac{\pi^2}{9} \]  \hfill (18)

Because of the large top quark width many states contribute into the position and height of the peak in the cross section. Therefore the characteristic distance estimates are not so transparent (the
NNLO peak position, for instance, is not exactly the ground state energy in the zero width limit. The relation (18) can serve just as a basic guide. In practical analysis one can choose the particular numerical values for the parameters \(r_0, c\) which stabilize either \(E_p\) or \(H_p\).

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