Wong-Zakai Approximations of Backward Doubly Stochastic Doubly Backward Differential Equations

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Abstract

In this paper we obtain a Wong-Zakai approximation to solutions of backward doubly stochastic differential equations.

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1 Framework and Introduction

Let \{\(W_t, 0 \leq t \leq T\)\} and \{\(B_t, 0 \leq t \leq T\)\} be two independent standard Brownian motions on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\mathcal{N}\) denote the class of \(\mathbb{P}\)-null sets. For each \(t \in [0, T]\), we define

\[
\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T},
\]

where for any process \(\{\eta_t\}\), \(\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}\). Let \(f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) be a bounded measurable function satisfying

(H.1) \[
|f(y_1, z_1) - f(y_2, z_2)| \leq c(|y_1 - y_2| + |z_1 - z_2|)
\]

Let \(g \in C^1_b(\mathbb{R})\). For \(n \geq 1\), define the linear interpolation \(B^n\) of \(B\) as

\[
B^n_t = B_{\frac{k+1}{2^n}} + 2^n(t - \frac{k+1}{2^n})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}}), \quad \text{for} \quad t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right].
\]

Let \(\xi \in L^2(\Omega)\) be \(\mathcal{F}_T\)-measurable. Consider the following backward doubly stochastic differential equations (BDSDE):

\[
Y_t = \xi + \int_t^T f(Y_s, Z_s)ds + \int_t^T g(Y_s)dB_s + \frac{1}{2} \int_t^T gg'(Y_s)ds - \int_t^T Z_s dW_s.
\]

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\[ Y^n_t = \xi + \int_t^T f(Y^n_s, Z^n_s)ds + \int_t^T g(Y^n_s)dB^n_s \\
- \int_t^T Z^n_s dB^n_s. \quad (1.3) \]

Here \( dB_s dB^n_s \) stand for the backward integrals. In the sequel, we will use \( \overrightarrow{ds} \) to indicate the backward integral against the Lebesgue measure.

Backward doubly stochastic differential equations was first studied by Pardoux and Peng in [PP]. It is now a powerful tool to study stochastic partial differential equations with singular coefficients. Our purpose of this paper is to obtain the convergence of the Wong-Zakai approximation to the backward doubly stochastic differential equations, namely we will prove that \((Y^n, Z^n)\) converges to \((Y, Z)\) in \(L^2\). The convergence of Wong-Zakai approximations to stochastic differential equations is now well known, see e.g. [IW]. Because of the nature of the BDSDEs, the integrand \(Z^n, Z\) in the stochastic integral against Brownian motion are also part of the solutions. This makes the problem drastically different from the Wong-Zakai approximation for stochastic differential equations. Another difficulty comes from the fact that the Hölder type estimate

\[ E[|Y^n_t - Y^n_s|^p] \leq C|t - s|^\alpha \]

is no longer available. We overcome this by carefully exploiting the independence of the two Brownian motions \(B\) and \(W\).

The application of our results to stochastic partial differential equations will be discussed in a forthcoming paper.

# 2 Main results

The following is an priori estimate for the family \(\{(Y^n, Z^n), n \geq 1\}\).

**Proposition 2.1** There exists a constant \(C\) such that

\[ \sup_n \sup_{0 \leq t \leq T} \{E[(Y^n_t)^2] + E[\int_t^T (Z^n_s)^2 ds]\} \leq C. \quad (2.4) \]

**Proof.** By Ito’s formula, we have

\[ (Y^n_t)^2 + \int_t^T (Z^n_s)^2 ds = (\xi)^2 + 2 \int_t^T Y^n_s f(Y^n_s, Z^n_s) ds + 2 \int_t^T Y^n_s g(Y^n_s)dB^n_s \\
- 2 \int_t^T Y^n_s Z^n_s dB^n_s. \quad (2.5) \]
For $s \in \left[ \frac{k}{2n}, \frac{k+1}{2n} \right]$, set $s^+ = \frac{k+2}{2n}$ and $s^- = \frac{k-1}{2n}$. In view of (H.1), it is easy to see that there exists $C_1 > 0$ such that
\[
2 \int_t^T Y^n_s f(Y^n_s, Z^n_s) ds \leq C_1 \int_t^T (Y^n_s)^2 ds + \frac{1}{4} \int_t^T (Z^n_s)^2 ds + C_1.
\] (2.6)

Now, the third term on the right side of (2.5) can be written as
\[
2 \int_t^T Y^n_s g(Y^n_s) dB^n_s = 2 \int_t^T Y^n_s g(Y^n_s) dB^n_s
+ 2 \int_t^T (Y^n_s - Y^n_s^+) g(Y^n_s) dB^n_s
+ 2 \int_t^T Y^n_s^+ (g(Y^n_s) - g(Y^n_s^+)) dB^n_s
:= I_1 + I_2 + I_3.
\] (2.7)

As a stochastic integral, we have $E[I_1] = 0$. By the equation (1.3) it follows that
\[
I_2 = 2 \int_t^T (\int_s^{s^+} f(Y^n_u, Z^n_u) du) g(Y^n_s) dB^n_s
+ 2 \int_t^T (\int_s^{s^+} g(Y^n_u) dB^n_u) g(Y^n_s) dB^n_s
- 2 \int_t^T (\int_s^{s^+} Z^n_u dB^n_u) g(Y^n_s) dB^n_s
:= I_{2.1} + I_{2.2} + I_{2.3}.
\] (2.8)

By the boundedness of $f$ and $g$, we have
\[
E[I_{2.1}] \leq C \int_t^T (\int_s^{s^+} du) E[|\dot{B}^n_s|] ds
\leq C \left( \frac{1}{2n} \right)^{\frac{1}{2}},
\] (2.9)

and
\[
E[I_{2.2}] \leq CE[\int_t^T (\int_s^{s^+} |\dot{B}^n_u| du) |\dot{B}^n_s| ds]
\leq C \int_t^T ds \int_s^{s^+} du E[|\dot{B}^n_u| |\dot{B}^n_s|]
\leq C \int_t^T ds \int_s^{s^+} du E[|\dot{B}^n_u|^2]^{\frac{1}{2}} (E[|\dot{B}^n_s|^2])^{\frac{1}{2}}
\leq C \int_t^T ds \int_s^{s^+} du (2^n)^{\frac{1}{2}} (2^n)^{\frac{1}{2}} \leq C,
\] (2.10)
where $C$ is a constant independent of $n$. For the term $I_{2,3}$, we have

$$E[I_{2,3}] = 2 \int_t^T E[(\int_s^{s^+} Z^n_u dW_u)g(Y^n_s)\dot{B}_s^n]ds$$

$$= 2 \int_t^T E[g(Y^n_s)\dot{B}_s^n E[(\int_s^{s^+} Z^n_u dW_u)|\mathcal{F}_s]]ds$$

$$= 0. \quad (2.11)$$

Putting together (2.9), (2.10), (2.11) we get

$$\sup_n E[I_2] \leq C, \quad (2.12)$$

for some constant $C$. To bound $I_3$ in (2.7), we write

$$I_3 = 2 \int_t^T Y^n_{s^+} \int_0^1 d\lambda g'(Y^n_{s^+} + \lambda(Y^n_s - Y^n_{s^+}))(Y^n_s - Y^n_{s^+})dB^n_s$$

$$+ 2 \int_t^T Y^n_{s^+} \int_0^1 d\lambda g'(Y^n_{s^+} + \lambda(Y^n_s - Y^n_{s^+}))(\int_s^{s^+} f(Y^n_u, Z^n_u)du)dB^n_s$$

$$- 2 \int_t^T Y^n_{s^+} \int_0^1 d\lambda g'(Y^n_{s^+} + \lambda(Y^n_s - Y^n_{s^+}))(\int_s^{s^+} Z^n_u dW_u)dB^n_s$$

$$:= I_{3,1} + I_{3,2} + I_{3,3}. \quad (2.13)$$

For $I_{3,1}$, we have

$$E[I_{3,1}] \leq C \int_t^T E[|Y^n_{s^+}| \frac{1}{2^n} |\dot{B}_s^n||]ds$$

$$\leq C \int_t^T E[(Y^n_{s^+})^2]ds + C \int_t^T E[(\frac{1}{2^n})^2 |\dot{B}_s^n|^2]ds$$

$$\leq C \int_t^T E[(Y^n_{s^+})^2]ds + C. \quad (2.14)$$

Similarly we have

$$E[I_{3,2}] \leq C \int_t^T ds \int_s^{s^+} du E[|Y^n_{s^+}| |\dot{B}_u^n||\dot{B}_s^n||]ds$$

$$\leq C \int_t^T ds \int_s^{s^+} du (E[(Y^n_{s^+})^2])^{\frac{1}{2}} (E[|\dot{B}_u^n|^2 |\dot{B}_s^n|^2])^{\frac{1}{2}} ds$$

$$\leq C \int_t^T ds \int_s^{s^+} du (E[(Y^n_{s^+})^2])^{\frac{1}{2}} 2^n$$

$$\leq C \int_t^T E[(Y^n_{s^+})^2]ds + C. \quad (2.15)$$
By virtue of the independence of $Y^n_{s+}$ and $\dot{B}^n_{s}$, we have

$$
E[I_{3.3}] \leq CE[\int_t^T |Y^n_{s+}| |\dot{B}^n_s| \int_s^{s+} Z^n_u dW_u |ds]
\leq C \int_t^T (E[(Y^n_{s+})^2 |\dot{B}^n_s|^2])^{\frac{1}{2}} (E[|\int_s^{s+} Z^n_u dW_u|^2])^{\frac{1}{2}} ds
\leq C \int_t^T (E[(Y^n_{s+})^2])^{\frac{1}{2}} (E[|\int_s^{s+} (Z^n_u)^2 du|^2])^{\frac{1}{2}} ds
\leq \frac{1}{4} \int_t^T 2^n E[\int_s^{s+} (Z^n_u)^2 du] ds + C_2 \int_t^T E[(Y^n_{s+})^2] ds
= \frac{1}{4} E[\int_t^T (Z^n_u)^2 du 2^n (\int_{s-}^s ds)] + C_2 \int_t^T E[(Y^n_{s+})^2] ds
\leq \frac{1}{4} E[\int_t^T (Z^n_u)^2 du] + C_2 \int_t^T E[(Y^n_{s+})^2] ds
$$

(2.16)

(2.14) – (2.16) imply that

$$
E[I_3] \leq \frac{1}{4} E[\int_t^T (Z^n_u)^2 du] + C_2 \int_t^T E[(Y^n_{s+})^2] ds + C
$$

(2.17)

It follows from (2.5), (2.6), (2.12) and (2.17) that

$$
E[(Y^n_t)^2] + \frac{1}{2} E[\int_t^T (Z^n_s)^2 ds]
\leq E[(\xi)^2] + C \int_t^T E[(Y^n_{s+})^2] ds + C
$$

(2.18)

Applying the Gronwall’s inequality, we complete the proof of the Proposition.

The above result can be strengthened as

**Proposition 2.2** For any $p \geq 1$, there exists a constant $C$ such that

$$
\sup_n \{E[\sup_{0 \leq t \leq T} |Y^n_t|^p] + E[\int_t^T (Z^n_s)^2 ds]^p] \leq C.
$$

(2.19)

**Theorem 2.3**

$$
\lim_{n \to \infty} \sup_{0 \leq t \leq T} \{E[(Y^n_t - Y_t)^2] + E[\int_t^T (Z^n_s - Z_s)^2 ds] \} = 0.
$$

(2.20)
Proof. By Ito’s formula, we have

\[(Y^n_t - Y^n_t)^2 + \int_t^T (Z^n_s - Z_s)^2 ds\]

\[= 2 \int_t^T (Y^n_s - Y_s) (f(Y^n_s, Z^n_s) - f(Y_s, Z_s)) ds\]

\[+ 2 \int_t^T (Y^n_s - Y_s) g(Y^n_s) dB^n_s + \int_t^T g^2(Y_s) ds\]

\[- 2 \int_t^T (Y^n_s - Y_s) g(Y_s) dB_s - 2 \int_t^T (Y^n_s - Y_s) (Z^n_s - Z_s) dW_s\]

\[- \int_t^T (Y^n_s - Y_s) gg'(Y_s) ds\]

:= \text{I}_1^n + \text{I}_2^n + \text{I}_3^n + \text{I}_4^n + \text{I}_5^n + \text{I}_6^n. \quad (2.21)\]

Crucially we need to bound the term \(\text{I}_2^n\). We write

\[\text{I}_2^n = 2 \int_t^T (Y^n_s - Y_s) g(Y^n_s) dB^n_s\]

\[= 2 \int_t^T [(Y^n_s - Y_s) - (Y^n_s - Y^n_s+)] g(Y^n_s) dB^n_s\]

\[+ 2 \int_t^T (Y^n_s+ - Y^n_s) (g(Y^n_s) - g(Y^n_s+)) dB^n_s\]

\[+ 2 \int_t^T (Y^n_s+ - Y^n_s) g(Y^n_s) dB^n_s\]

:= \text{A} + \text{B} + \text{C}. \quad (2.22)\]

As a stochastic integral, we have \(E[C] = 0\). It is a long proof to establish the bounds for \(E[A]\) and \(E[B]\). We will split it into two lemmas for clarity.

Lemma 2.4 We have

\[E[A] \leq C \left( \frac{1}{2n} \right)^{\frac{1}{2} - \delta} - 2E \left[ \int_t^T g(Y^n_s) g(Y^n_s) ds \right]\]

\[+ E \left[ \int_t^T g^2(Y^n_s) ds \right]. \quad (2.23)\]
**Proof.** By the equations satisfied by $Y^n$ and $Y$ we have

$$A = 2 \int_t^T \left( \int_s^{s^+} g(Y^n_u)dB^n_u \right) g(Y^n_s)dB^n_s$$

$$- 2 \int_t^T \left( \int_s^{s^+} g(Y_u)dB_u \right) g(Y^n_s)dB^n_s$$

$$+ 2 \int_t^T \left( \int_s^{s^+} [f(Y^n_u, Z^n_u) - f(Y_u, Z_u)] du \right) g(Y^n_s)dB^n_s$$

$$- 2 \int_t^T \left( \int_s^{s^+} (Z^n_u - Z_u) dW_u \right) g(Y^n_s)dB^n_s$$

$$- \int_t^T \left( \int_s^{s^+} g(Y_u) du \right) g(Y^n_s)dB^n_s$$

$$:= A_1 + A_2 + A_3 + A_4 + A_5. \quad (2.24)$$

Clearly,

$$E[|A_5|] \leq C \frac{1}{2^n} \int_t^T E[|\dot{B}_{s^n}|] ds \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2}}, \quad (2.25)$$

also

$$E[|A_3|] \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2}}. \quad (2.26)$$

By conditioning on $\mathcal{F}_s$, we find that

$$E[A_4] = -2 \int_t^T E \left[ \left( \int_s^{s^+} (Z^n_u - Z_u) dW_u \right) g(Y^n_s)\dot{B}_s^n \right] ds$$

$$= -2 \int_t^T E \left[ E[\left( \int_s^{s^+} (Z^n_u - Z_u) dW_u \right) |\mathcal{F}_s] g(Y^n_s)\dot{B}_s^n \right] ds$$

$$= 0. \quad (2.27)$$

To bound $A_1$, we write it as

$$A_1 = 2 \int_t^T \left( \int_s^{s^+} g(Y^n_u)dB^n_u \right) g(Y^n_s)dB^n_s$$

$$= 2 \int_t^T \left[ \int_s^{s^+} g(Y^n_u)dB^n_u \right] g(Y^n_s)dB^n_s$$

$$+ 2 \int_t^T g(Y^n_s) (B^n_s - B^n_{s^+}) (g(Y^n_s) - g(Y^n_{s^+})) dB^n_s$$

$$+ 2 \int_t^T g(Y^n_s) (B^n_s - B^n_{s^+}) g(Y^n_{s^+}) dB^n_s$$

$$:= A_{11} + A_{12} + A_{13}. \quad (2.28)$$
Splitting the interval \([t, T]\) into subintervals \([\frac{k}{2^n}, \frac{k+1}{2^n}]\) we see that

\[
A_{13} = 2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g^2(Y^n_{\frac{k+1}{2^n}})2^n(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})ds
\]

\[+ 2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g^2(Y^n_{\frac{k+1}{2^n}})(2^n)^2(s - \frac{k + 1}{2^n})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2ds \]

\[:= A_{13,1} + A_{13,2}. \tag{2.29}\]

Conditioning on \(\mathcal{F}_{\frac{k+2}{2^n}}\) we have

\[
E[A_{13,1}] = -2 \sum_k E[g^2(Y^n_{\frac{k+1}{2^n}})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})E[(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})|\mathcal{F}_{\frac{k+2}{2^n}}]] = 0. \tag{2.30}\]

Integrating with respect to \(s\), we get that

\[
E[A_{13}] = E[A_{13,2}]
\]

\[= E[\sum_k g^2(Y^n_{\frac{k+1}{2^n}})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2] = E[\sum_k g^2(Y^n_{\frac{k+1}{2^n}})((B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2 - \frac{1}{2^n})]
\]

\[+ E[\int_T^t g^2(Y^n_s)ds] = E[\int_T^t g^2(Y^n_s)ds], \tag{2.31}\]

where the fact that the sequence \\(((B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2 - \frac{1}{2^n}, k \geq 0)\) is a martingale has been used. For the term \(A_{11}\) in (2.28), we have

\[
A_{11} = 2 \int_0^1 d\lambda \int_T^t \int_s^{s+} g'(Y^n_{s+} + \lambda(Y^n_u - Y^n_{s+}))(Y^n_u - Y^n_{s+})dB^n_u g(Y^n)dB^n_s
\]

\[= 2 \int_0^1 d\lambda \int_T^t \int_s^{s+} g'(Y^n_{s+} + \lambda(Y^n_u - Y^n_{s+}))\int_s^{s+} f(Y^n_v, Z^n_v)dv]dB^n_u g(Y^n)dB^n_s
\]

\[+ 2 \int_0^1 d\lambda \int_T^t \int_s^{s+} g'(Y^n_{s+} + \lambda(Y^n_u - Y^n_{s+}))\int_s^{s+} g(Y^n_v)dB^n_u dB^n_s
\]

\[+ 2 \int_0^1 d\lambda \int_T^t \int_s^{s+} g'(Y^n_{s+} + \lambda(Y^n_u - Y^n_{s+}))\int_s^{s+} Z^n_v dW_v]dB^n_u g(Y^n)dB^n_s
\]

\[:= A_{11,1} + A_{11,2} + A_{11,3}. \tag{2.32}\]

The first two terms on the right can be bounded as follows.

\[
E[A_{11,1}] \leq C \frac{1}{2^n} \int_T^t ds \int_s^{s+} E[|\dot{B}^n_u||\dot{B}^n_s|]du
\]

\[\leq C \frac{1}{2^n} (2^n)^{\frac{1}{2}} (2^n)^{\frac{1}{2}} \frac{1}{2^n} \leq C \frac{1}{2^n}. \tag{2.33}\]
\[ E[A_{11,2}] \leq C \int_t^T ds \int_s^{s^+} du \int_u^{s^+} d\lambda E[|\hat{B}_u^n||\hat{B}_{s^+}^n||\hat{B}_s^n|] \leq C(2^n)^\frac{1}{2} \leq C(\frac{1}{2^n})^\frac{1}{2}. \] (2.34)

The last term \( A_{11,3} \) can be estimated as follows.

\[ E[A_{11,3}] \leq CE\left[ \int_t^T ds \int_s^{s^+} |\hat{B}_u^n||\hat{B}_{s^+}^n||\int_u^{s^+} Z_{s^+}^n dW_v|du \right] \leq C \int_t^T ds \int_s^{s^+} du (E[|\hat{B}_u^n|^2|\hat{B}_{s^+}^n|^2])^{\frac{1}{2}} (E[\int_u^{s^+} Z_{s^+}^n dW_v]^2)^{\frac{1}{2}} \leq C2^n \int_t^T ds \int_s^{s^+} du (E[\int_u^{s^+} (Z_{s^+}^n)^2 dv])^{\frac{1}{2}} \leq C \left( \int_t^T ds E[\int_s^{s^+} (Z_{s^+}^n)^2 dv] \right)^{\frac{1}{2}} \leq C \left( E[\int_t^T (Z_v^n)^2 dv \int_v^{s^+} ds] \right)^{\frac{1}{2}} \leq C(\sup_n E[\int_t^T (Z_v^n)^2 dv])^{\frac{1}{2}}(\frac{1}{2^n})^{\frac{1}{2}}. \] (2.35)

Putting together (2.33)–(2.35) together we get

\[ E[A_{11}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \] (2.36)

Similarly the term \( A_{12} \) can be decomposed as

\[ A_{12} = 2 \int_0^1 d\lambda \int_t^T g(Y_v^n) \int_s^{s^+} dB^n_u g'(Y_v^n + \lambda(Y_v^n - Y_{s^+}^n))(Y_v^n - Y_{s^+}^n) dB^n_s + A_{12,1} + A_{12,2} + A_{12,3}. \] (2.37)

Using the similar arguments as for (2.33) and (2.34) we can show that

\[ E[A_{12,j}] \leq C \frac{1}{2^n}, j = 1, 2, 3. \] (2.38)

Hence,

\[ E[A_{12}] \leq C(\frac{1}{2^n})^{\frac{1}{2}}. \] (2.39)
Combining (2.31), (2.36) and (2.39) we get

\[ E[A_1] \leq C\left(\frac{1}{2^n}\right) + E\left[\int_t^T g^2(Y_s^n)ds\right]. \]

(2.40)

Now we turn to \( A_2 \) which can be written as

\[ A_2 = -2 \int_t^T \left( \int_s^{s^+} g(Y_u)dB_u \right) g(Y_s^n)dB_s^n \]

\[ = -2 \int_t^T \left[ \int_s^{s^+} (g(Y_u) - g(Y_{s^+}))dB_u \right] g(Y_s^n)dB_s^n \]

\[ - 2 \int_t^T g(Y_{s^+})(B_s - B_{s^+})(g(Y_s^n) - g(Y_{s^+}^n))dB_s^n \]

\[ - 2 \int_t^T g(Y_{s^+})(B_s - B_{s^+})g(Y_{s^+}^n)dB_s^n \]

\[ := A_{21} + A_{22} + A_{23}. \]

(2.41)

Splitting the interval \([t, T]\) into subintervals \([\frac{k}{2^n}, \frac{k+1}{2^n}]\),

\[ A_{23} = -2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)(B_s - B_{\frac{k+1}{2^n}})2^n(B_{\frac{k+1}{2^n}} - B_{\frac{k+1}{2^n}})ds \]

\[ = -2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)(B_s - B_{\frac{k+1}{2^n}})2^n(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})ds \]

\[ - 2 \sum_k g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2 \]

\[ := A_{23,1} + A_{23,2}. \]

(2.42)

Using the independence of the increments of \( B \) and conditioning on \( \mathcal{F}_{\frac{k+1}{2^n}} \) it follows that

\[ E[A_{23,1}] \]

\[ = -2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} 2^n ds E[g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)E[(B_s - B_{\frac{k+1}{2^n}})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})|\mathcal{F}_{\frac{k+2}{2^n}}]] \]

\[ = -2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} 2^n ds E[g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)E[(B_s - B_{\frac{k+1}{2^n}})(B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})] \]

\[ = 0. \]

(2.43)

Now,

\[ A_{23,2} = -2 \sum_k g(Y_{\frac{k+1}{2^n}})g(Y_{\frac{k}{2^n}}^n)((B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})^2 - \frac{1}{2^n}) \]

\[ - 2 \int_t^T g(Y_{s^+})g(Y_{s^+}^n)ds \]

(2.44)
By conditioning on $F_{k+2}$ we see that the expectation of the first term of the above equation vanishes. Hence,

$$E[A_{23}] = E[A_{23,2}] = -2E\left[\int_t^T g(Y_{s+})g(Y^n_{s+})ds\right]. \tag{2.45}$$

For the term $A_{21}$ we have

$$A_{21} = -2\int_0^1 d\lambda \int_t^T \int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))(Y_u - Y_{s+})dB_ug(Y^n_s)dB^n_s$$

$$= -2\int_0^1 d\lambda \int_t^T \int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))\int_u^{s+} f(Y_v, Z_v)dv|dB_ug(Y^n_s)dB^n_s$$

$$- 2\int_0^1 d\lambda \int_t^T \int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))\int_u^{s+} g(Y_v)dB_v|dB_ug(Y^n_s)dB^n_s$$

$$+ 2\int_0^1 d\lambda \int_t^T \int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))\int_u^{s+} Z_vdB_v|dB_ug(Y^n_s)dB^n_s$$

$$:= A_{21,1} + A_{21,2} + A_{21,3}. \tag{2.46}$$

We will estimate each of the terms on the right. First we have

$$E[A_{21,1}] \leq C\int_0^1 d\lambda \int_t^T E\left[\int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))\int_u^{s+} f(Y_v, Z_v)dv|dB_u||\dot{B}_s^n|^2\right]ds$$

$$\leq C\int_0^1 d\lambda \int_t^T (E[\int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))])^{1/2} (E[|\dot{B}_s^n|^2])^{1/2} ds$$

$$\leq C(2^n)^{1/2} \int_0^1 d\lambda \int_t^T (E[\int_s^{s+} g'(Y_{s+} + \lambda(Y_u - Y_{s+}))^2 (\int_u^{s+} f(Y_v, Z_v)dv)^2du])^{1/2} ds$$

$$\leq C(2^n)^{1/2} \int_t^T [(s^+ - s)(s^+ - s)^2]^{1/2} ds$$

$$\leq C(\frac{1}{2^n}). \tag{2.47}$$
Similarly,

\[
E[A_{21,2}] \\
\leq C \int_0^1 d\lambda \int_t^T \left[ \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} g(Y_v) dB_v \right) \right] ds \\
\leq C \int_0^1 d\lambda \int_t^T \left( E\left[ \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} g(Y_v) dB_v \right)^2 \right] \right)^{1/2} \lambda^{1/2} ds \\
\leq C(2^n)^{1/2} \int_0^1 d\lambda \int_t^T \left( \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} g(Y_v) dB_v \right)^2 \right)^{1/2} ds \\
\leq C(2^n)^{1/2} \int_t^T \left( \int_{s^+} g(Y_{s^+}) dB_v \right)^{1/2} \left( \int_{s^+} g(Y_v) dB_v \right)^{1/2} ds \\
\leq C \left( \frac{1}{2^n} \right)^{1/2}. \tag{2.48}
\]

By Hölder inequality and Ito isometry, we have

\[
E[A_{21,3}] \\
\leq C \int_0^1 d\lambda \int_t^T \left[ \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} Z_v dw_v \right) \right] ds \\
\leq C \int_0^1 d\lambda \int_t^T \left( E\left[ \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} Z_v dw_v \right)^2 \right] \right)^{1/2} \lambda^{1/2} ds \\
\leq C(2^n)^{1/2} \int_0^1 d\lambda \int_t^T \left( \int_{s^+} g'(Y_{s^+} + \lambda(Y_u - Y_{s^+})) \left( \int_u^{s^+} Z_v dw_v \right)^2 \right)^{1/2} ds \\
\leq C(2^n)^{1/2} \int_t^T \left( \int_{s^+} \left( \int_u^{s^+} \left( \int_{s^+} Z_v dw_v \right)^2 \right) ds \right)^{1/2} ds \\
\leq C \left( \int_t^T \left( \int_{s^+} Z_v^2 dv \right) ds \right)^{1/2} \\
\leq C \left( \int_t^T Z_v^2 dv \int_{s^+} ds \right)^{1/2} \\
\leq C \left( \frac{1}{2^n} \right)^{1/2}. \tag{2.49}
\]

It follows from (2.47), (2.48) and (2.49) that

\[
E[A_{21}] \leq C \left( \frac{1}{2^n} \right)^{1/2}. \tag{2.50}
\]
Let us turn to the term $A_{22}$. We have

\[
A_{22} \leq C \int_t^T |B_s - B_{s+}| ||Y^n_s - Y^n_{s+}| ||\hat{B}^n_s| ds \\
\leq C \int_t^T |B_s - B_{s+}| \int_s^{s+} f(Y^n_u, Z^n_u) du ||\hat{B}^n_s| ds \\
+ C \int_t^T |B_s - B_{s+}| \int_s^{s+} g(Y^n_u) dB^n_u ||\hat{B}^n_s| ds \\
+ C \int_t^T |B_s - B_{s+}| \int_s^{s+} Z^n_u dW_u ||\hat{B}^n_s| ds \\
:= A_{22,1} + A_{22,2} + A_{22,3}. \tag{2.51}
\]

Now,

\[
E[A_{22,1}] \leq C \frac{1}{2^n} \int_t^T \left( E[||B_s - B_{s+}|^2] \right)^{\frac{1}{4}} \left( E[||\hat{B}^n_s|^4] \right)^{\frac{1}{2}} ds \\
\leq C \frac{1}{2^n}, \tag{2.52}
\]

and

\[
E[A_{22,2}] \leq C \frac{1}{2^n} \int_t^T \left( E[||B_s - B_{s+}|^2] \right)^{\frac{1}{4}} \left( E[\sup_u ||\hat{B}^n_u|^4] \right)^{\frac{1}{2}} ds \\
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} (2^n)^2 \int_t^T \left( E[ \sup_{|r-v| \leq \frac{1}{2^n}} |B_r - B_v|^4] \right)^{\frac{1}{2}} ds \\
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}. \tag{2.53}
\]

By Hölder inequality,

\[
E[A_{22,3}] \leq C \int_t^T \left( E[||B_s - B_{s+}|^4] \right)^{\frac{1}{4}} \left( E[||\hat{B}^n_s|^4] \right)^{\frac{1}{4}} \left( E[\int_s^{s+} Z^n_u dW_u|^2] \right)^{\frac{1}{2}} ds \\
\leq C \left( \int_t^T E[\int_s^{s+} (Z^n_u)^2 du]^2 ds \right)^{\frac{1}{2}} \\
\leq C \frac{1}{2^n}. \tag{2.54}
\]

It follows from (2.51)–(2.54) that

\[
E[A_{22}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}. \tag{2.55}
\]

Collecting (2.45), (2.50) and (2.51) we obtain

\[
E[A_2] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta} - 2E[\int_t^T g(Y^n_s) g(Y^n_{s+}) ds]. \tag{2.56}
\]

The proof is now completed by putting (2.25), (2.26), (2.27), (2.40) and (2.56) together.
Lemma 2.5 We have

\[ E[B] \leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}} + E\left[ \int_t^T (Y_{s+}^n - Y_s^n) g\left( Y_{s+}^n \right) ds \right]. \tag{2.57} \]

Proof. Write

\[ B = 2 \int_t^T (Y_{s+}^n - Y_s^n)(g(Y_{s+}^n) - g(Y_{s+}^n))dB_s^n \]

\[ = 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n + \lambda(Y_s^n - Y_{s+}^n))(Y_s^n - Y_{s+}^n)dB_s^n \]

\[ = 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n + \lambda(Y_s^n - Y_{s+}^n))\int_s^{s+} f(Y_v^n, Z_v^n)dv dB_s^n \]

\[ + 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n + \lambda(Y_s^n - Y_{s+}^n))\int_s^{s+} g(Y_v^n)dB_v^n dB_s^n \]

\[ - 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n + \lambda(Y_s^n - Y_{s+}^n))\int_s^{s+} Z_v^n dB_v^n dB_s^n \]

\[ := B_1 + B_2 + B_3. \tag{2.58} \]

By Proposition 2.2, we have

\[ E[B_1] \leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}} E\left[ \int_t^T |Y_{s+}^n - Y_s^n| |\dot{B}_s^n| ds \right] \]

\[ \leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}} \int_0^1 \left( E[|Y_{s+}^n - Y_s^n|^2] \right)^{\frac{1}{2}} (E[|\dot{B}_s^n|^2])^{\frac{1}{2}} ds \]

\[ \leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}}. \tag{2.59} \]

\[ B_2 \]

\[ = 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)[g'(Y_{s+}^n + \lambda(Y_s^n - Y_{s+}^n)) - g'(Y_{s+}^n)]\int_s^{s+} g(Y_v^n)dB_v^n dB_s^n \]

\[ + 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n)\int_s^{s+} (g(Y_v^n) - g(Y_{s+}^n))dB_v^n dB_s^n \]

\[ + 2 \int_0^1 d\lambda \int_t^T (Y_{s+}^n - Y_s^n)g'(Y_{s+}^n)g(Y_{s+}^n)(B_v^n - B_{s+})dB_s^n \]

\[ := B_{21} + B_{22} + B_{23}. \tag{2.60} \]
By the Lipschitz continuity of \(g'\), it follows that

\[
B_{21} \leq C \int_t^T |Y^n_s - Y^n_t||Y^n_s - Y^n_t| \int_s^t g(Y^n_u)dB^n_u||\dot{B}_s^n|ds
\]

\[
\leq C \int_t^T |Y^n_s - Y^n_t| \int_s^t f(Y^n_v, Z^n_v)dv ||\int_s^t g(Y^n_u)dB^n_u||\dot{B}_s^n|ds
\]

\[
+ C \int_t^T |Y^n_s - Y^n_t| \int_s^t g(Y^n_u)dB^n_u^2|\dot{B}_s^n|ds
\]

\[
+ C \int_t^T |Y^n_s - Y^n_t| \int_s^t Z^n_v dW_v ||\int_s^t g(Y^n_u)dB^n_u||\dot{B}_s^n|ds
\]

\[
:= B_{21.1} + B_{21.2} + B_{21.3}. \quad (2.61)
\]

The following two inequalities will be used frequently in sequel.

\[
\sup_u |\dot{B}_u^n| \leq 2^n \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s| \qquad (2.62)
\]

For any \(\delta > 0\) and \(p \geq 1\), there exists a constant \(C_{p,\delta}\) such that

\[
E[ \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^p ] \leq C_{p,\delta}(\frac{1}{2n})^{\frac{p}{2} - \delta}. \quad (2.63)
\]

By Hölder’s inequality and \((2.62), (2.63)\), we have

\[
E[B_{21.1}] \leq C \frac{1}{2^n} E[ \int_t^T |Y^n_s - Y^n_t| \int_s^t |\dot{B}_u^n|du ||\dot{B}_s^n|ds ]
\]

\[
\leq C \frac{1}{2^n} (2^n)^2 E[ \int_t^T |Y^n_s - Y^n_t| \int_s^t du \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^2ds ]
\]

\[
\leq CE[ \int_t^T |Y^n_s - Y^n_t| \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^2ds ]
\]

\[
\leq C \int_t^T (E[|Y^n_s - Y^n_t|^2])^{\frac{p}{2}} (E[ \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^4])^{\frac{1}{2}} ds
\]

\[
\leq C(\frac{1}{2n})^{1-\delta}. \quad (2.64)
\]

Similarly, in view of \((2.63)\), we have

\[
E[B_{21.2}] \leq CE[ \int_t^T |Y^n_s - Y^n_t| \int_s^t du |\dot{B}_u^n|^3ds ]
\]

\[
\leq C(\frac{1}{2^n})^2(2^n)^3 E[ \int_t^T |Y^n_s - Y^n_t| \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^3ds ]
\]

\[
\leq C2^n \int_t^T (E[|Y^n_s - Y^n_t|^2])^{\frac{1}{2}} (E[ \sup_{|r-s|\leq \frac{1}{2n}} |B_r - B_s|^6])^{\frac{1}{2}} ds
\]

\[
\leq C(\frac{1}{2n})^{\frac{6}{p} - \delta}, \quad (2.65)
\]
and

\[ E[B_{21,3}] \leq CE\int_t^T |Y^n_{s+} - Y^n_s| |\int_s^{s+} Z^n_u dW_v| (\int_s^{s+} |\dot{B}^n_u| du) |\dot{B}^n_s| ds \]
\[ \leq C(2^n)^2 E\int_t^T |Y^n_{s+} - Y^n_s| |\int_s^{s+} Z^n_v dW_v| \sup_{|r-s| \leq \frac{1}{2^n}} |B_r - B_s|^2 ds \]
\[ \leq C2^n \int_t^T (E[|Y^n_{s+} - Y^n_s|^4])^{\frac{1}{4}} (E[\int_s^{s+} Z^n_v dW_v]^2)^{\frac{3}{4}} (E[\sup_{|r-s| \leq \frac{1}{2^n}} |B_r - B_s|^8])^{\frac{1}{4}} ds \]
\[ \leq C\left(\frac{1}{2^n}\right)^{1-\delta} 2^n \int_t^T (E[|Y^n_{s+} - Y^n_s|^4])^{\frac{1}{4}} (E[\int_s^{s+} (Z^n_v)^2 dv]^2)^{\frac{3}{4}} ds \]
\[ \leq C\left(\frac{1}{2^n}\right)^{1-\delta} 2^n (\int_t^T (E[|Y^n_{s+} - Y^n_s|^4])^{\frac{1}{4}} ds)^{\frac{1}{4}} (\int_t^T E[\int_s^{s+} (Z^n_v)^2 dv] ds)^{\frac{3}{4}} \]
\[ \leq C\left(\frac{1}{2^n}\right)^{1-\delta} 2^n (\int_t^T (Z^n_v)^2 dv \int_t^T dv)^{\frac{1}{2}} \]
\[ \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2}-\delta}, \quad (2.66) \]

where the a priori estimate \((2.19)\) has been used. \((2.64)-(2.61)\) yields

\[ E[B_{21}] \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2}-\delta}. \quad (2.67) \]

By the Lipschitz continuity of \(g\), we have

\[ B_{22} \leq C\int_t^T |Y^n_{s+} - Y^n_s|(|\int_s^{s+} f(Y^n_v, Z^n_v) dv| |\dot{B}^n_u| du) |\dot{B}^n_s| ds \]
\[ \leq C\int_t^T |Y^n_{s+} - Y^n_s|(|\int_s^{s+} f(Y^n_v, Z^n_v) dv| |\dot{B}^n_u| du) |\dot{B}^n_s| ds \]
\[ \leq C\int_t^T |Y^n_{s+} - Y^n_s|(|\int_s^{s+} g(Y^n_v) dB^n_v| |\dot{B}^n_u| du) |\dot{B}^n_s| ds \]
\[ \leq C\int_t^T |Y^n_{s+} - Y^n_s|(|\int_s^{s+} Z^n_v dW_v| |\dot{B}^n_u| du) |\dot{B}^n_s| ds \]
\[ := B_{22,1} + B_{22,2} + B_{22,3}. \quad (2.68) \]

By the similar arguments as above, we have

\[ E[B_{22,j}] \leq C\left(\frac{1}{2^n}\right)^{1-\delta}, j = 1, 2, 3. \quad (2.69) \]

Hence,

\[ E[B_{22}] \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2}-\delta}. \quad (2.70) \]
For the term $B_{23}$, we split the interval $[t, T]$ into subintervals $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ to get

$$B_{23} = 2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} (Y_{\frac{k+2}{2^n}} - Y_{\frac{k}{2^n}}) gg'(Y_{\frac{k+2}{2^n}}) 2^n (B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}}) (B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}}) ds$$

$$+ 2 \sum_k \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \sum (Y_{\frac{k+2}{2^n}} - Y_{\frac{k}{2^n}}) (B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}}) ds$$

$$- 2 \sum_k (Y_{\frac{k+2}{2^n}} - Y_{\frac{k+1}{2^n}}) gg'(Y_{\frac{k+2}{2^n}}) (B_{\frac{k+2}{2^n}} - B_{\frac{k+1}{2^n}})$$

$$+ \sum_k \int_t^T (Y_{s+} - Y_{s^+}) gg'(Y_{s^+}) ds$$

(2.71)

Conditioning on $\mathcal{F}_{\frac{k+2}{2^n}}$ it is easy to see that the expectation of the first two terms on the right vanishes. Hence,

$$E[B_{23}] = E[\int_t^T (Y_{s+} - Y_{s^+}) gg'(Y_{s^+}) ds].$$

(2.72)

Collect the terms in (2.67), (2.70) and (2.72) to obtain

$$E[B_2] \leq C\left(\frac{1}{2^n}\right)^{\frac{1}{2} - \delta} + E[\int_t^T (Y_{s+} - Y_{s^+}) gg'(Y_{s^+}) ds].$$

(2.73)

Now we turn to the term $B_3$ in (2.58). We further split it as

$$B_3 = -2 \{ \int_0^1 d\lambda \int_t^T [(Y_{s+} - Y_{s^+}) g'(Y_{s^+} + \lambda(Y_{s+} - Y_{s^+}))$$

$$- (Y_{s+} - Y_s) g'(Y_s) \{ \int_s^+ Z^n_v dB^n_s \} ds$$

$$- 2 \int_t^T (Y_{s+} - Y_s) g'(Y_{s^+}) (\int_s^+ Z^n_v dB^n_s) ds$$

$$:= B_{31} + B_{32}. $$

(2.74)

First we notice that

$$E[B_{32}] = -2 \int_t^T E[(Y_{s+} - Y_s) g'(Y_{s^+}) \hat{B}^n_s E[(\int_s^+ Z^n_v dB^n_v) | \mathcal{F}_s]] ds$$

$$= 0.$$ 

(2.75)
By the Lipschitz continuity of $g'$, we have

$$B_{31} \leq C \int_t^T |Y^n_s - Y^n_t| \int_s^{s+} Z^n_v dW_v ||\dot{B}^n_s|| ds$$

$$+ C \int_t^T |Y^n_s - Y^n_t| \int_s^{s+} Z^n_v dW_v ||\dot{B}^n_s|| ds$$

$$+ C \int_t^T |Y^n_s - Y^n_t||Y^n_{s+} - Y^n_{s-}| \int_s^{s+} Z^n_v dW_v ||\dot{B}^n_s|| ds$$

$$:= B_{31,1} + B_{31,2} + B_{31,3}. \quad (2.76)$$

Furthermore,

$$B_{31,1} \leq C \int_t^T \int_s^{s+} f(Y^n_v, Z^n_v) dv \int_s^{s+} Z^n_v dW_v ||\dot{B}^n_s|| ds$$

$$+ C \int_t^T \int_s^{s+} g(Y^n_v) dB^n_v \int_s^{s+} Z^n_v dW_v ||\dot{B}^n_s|| ds$$

$$+ C \int_t^T \int_s^{s+} Z^n_v dW_v [2] ||\dot{B}^n_s|| ds$$

$$:= B_{31,11} + B_{31,12} + B_{31,13}. \quad (2.77)$$

Now, interchanging the order of integration, we have

$$E[B_{31,11}] \leq C \frac{1}{2^n} \int_t^T (E[\int_s^{s+} (Z^n_v)^2 dv]^2) \frac{1}{2} (E(||\dot{B}^n_s||^2))^\frac{1}{2} ds$$

$$\leq C \frac{1}{2^n}. \quad (2.78)$$

In view of (2.62), (2.63), we have

$$E[B_{31,12}] \leq C \frac{1}{2^n} \int_t^T E[\sup_{s \leq \frac{1}{2^n}} |\dot{B}^n_s|^2] \int_s^{s+} Z^n_v dW_v ||ds$$

$$\leq C \frac{1}{2^n} (2^n)^2 \int_t^T (E[\sup_{|r-v| \leq \frac{1}{2^n}} |B_r - B_v|^4]) \frac{1}{2} (E[\int_s^{s+} Z^n_v dW_v|^2]) \frac{1}{2} ds$$

$$\leq C \frac{1}{2^n} (2^n)^2 \left( \frac{1}{2^n} \right)^{1-\delta} \left( \int_t^T E[\int_{s}^{s+} (Z^n_v)^2 dv] ds \right)^\frac{1}{2}$$

$$\leq C \frac{1}{2^n} (2^n)^2 \left( \frac{1}{2^n} \right)^{1-\delta} \left( \int_t^T E[(Z^n_v)^2] dv \right)^\frac{1}{2}$$

$$\leq C \frac{1}{2^n} \frac{1}{2^n} - \delta. \quad (2.79)$$
Using the fact that \(|\dot{B}_n^s|\) is \(F_s\)-measurable we have

\[
E[B_{31,13}] \leq CE\left[\int_t^T \left( \int_s^{s+} \left( |\dot{B}_n^s| \right)^2 Z_u^v dW_u \right)^2 |\dot{B}_s^n| ds \right]
\]

\[
= CE\left[\int_t^T \left( \int_s^{s+} \left( |\dot{B}_n^s| \right)^2 Z_u^v dW_u \right)^2 |\dot{B}_s^n| ds \right]
\]

\[
= CE\left[\int_t^T \int_s^{s+} \left( |\dot{B}_n^s| \right)(Z_u^n)^2 dv ds \right]
\]

\[
\leq C(2^n)E\left[ \sup_{|r-v| \leq \frac{1}{2^n}} |B_r - B_v| \right] \int_t^T \int_s^{s+} ((Z_u^n)^2 dv ds) \right]^\frac{1}{2}
\]

\[
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}, \quad (2.80)
\]

where (2.62),(2.63) again were used. (2.78)-(2.80) implies that

\[
E[B_{31,1}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}. \quad (2.81)
\]

For the term \(B_{31,2}\) we have

\[
B_{31,2} \leq C \int_t^T \left| \int_s^{s+} f(Y_v, Z_v) dv \right| \int_s^{s+} Z_u^n dW_u |\dot{B}_s^n| ds
\]

\[
+ C \int_t^T \left| \int_s^{s+} g(Y_v) dB_v \right| \int_s^{s+} Z_u^n dW_u |\dot{B}_s^n| ds
\]

\[
+ C \int_t^T \left| \int_s^{s+} Z_v dW_v \right| \int_s^{s+} Z_u^n dW_u |\dot{B}_s^n| ds
\]

\[
:= B_{31,21} + B_{31,22} + B_{31,23}. \quad (2.82)
\]

By a similar argument as for (2.78), we have

\[
E[B_{31,21}] \leq C \left( \frac{1}{2^n} \right). \quad (2.83)
\]

As for \(B_{31,22}\), we have

\[
E[B_{31,22}] \leq C \int_t^T \left( E\left[ \left( \int_s^{s+} Z_u^n (|\dot{B}_s^n|)^\frac{1}{2} dW_u \right)^2 \right] \right)^\frac{1}{2} \left( E\left[ \left( \int_s^{s+} g(Y_v) dB_v \right)^2 \right] \right)^\frac{1}{2} ds
\]

\[
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \int_t^T \left( E\left[ \left( \int_s^{s+} (Z_u^n)^2 \right)^\frac{1}{2} dW_u \right] \right)^\frac{1}{2} ds
\]

\[
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}. \quad (2.84)
\]

From here following the same arguments as for (2.80) we obtain

\[
E[B_{31,22}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}. \quad (2.85)
\]
The term $B_{31,23}$ is bounded as

$$
B_{31,23} \leq C \int_t^T (\int_s^{s^+} Z_v^n dW_v)^2 |\dot{B}_s^n| ds
+ C \int_t^T (\int_s^{s^+} Z_v dW_v)^2 |\dot{B}_s^n| ds,
$$
(2.86)

which, together with the same arguments as for (2.103), yields

$$
E[B_{31,23}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}.
$$
(2.87)

(2.83) - (2.87) gives that

$$
E[B_{31,2}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta}.
$$
(2.88)

Finally we need to find an upper bound for $E[B_{31,3}]$. Notice that

$$
B_{31,3} \leq C \int_t^T |Y^n_s - Y_s| \int_s^{s^+} f(Y^n_v, Z^n_v) dv \int_s^{s^+} Z^n_v dW_v |\dot{B}_s^n| ds
+ C \int_t^T |Y^n_s - Y_s| \int_s^{s^+} g(Y^n_v) dB_v^n \int_s^{s^+} Z^n_v dW_v |\dot{B}_s^n| ds
+ C \int_t^T |Y^n_s - Y_s| \int_s^{s^+} Z^n_v dW_v |\dot{B}_s^n| ds
:= B_{31,31} + B_{31,32} + B_{31,33}.
$$
(2.89)

We have

$$
E[B_{31,31}] \leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} (E[|Y^n_s - Y_s|^2])^{\frac{1}{2}} (E[\int_s^{s^+} (|\dot{B}_s^n|)^{\frac{1}{2}} Z^n_v dW_v |^2])^{\frac{1}{2}} ds
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \int_t^T E[|Y^n_s - Y_s|^2] ds \right)^{\frac{1}{2}} \left( \int_t^T E[|\dot{B}_s^n| (Z^n_v)^2 dv] ds \right)^{\frac{1}{2}}
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \int_t^T E[(\sup_{|r-v| \leq \frac{s-v}{2^n}} |B_r - B_v|)] \int_t^T \int_s^{s^+} (Z^n_v)^2 dv ds \right)^{\frac{1}{2}}
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \int_t^T \int_s^{s^+} (Z^n_v)^2 dv ds \right)^{\frac{1}{2}}
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2}} \left( \int_t^T (Z^n_v)^2 dv ds \right)^{\frac{1}{2}}
\leq C \left( \frac{1}{2^n} \right)^{\frac{1}{2} - \delta},
$$
(2.90)
where the a priori bounds (2.4), (2.19) have been used. Noticing that \(|Y^n_s - Y_s|, |\hat{B}^n_s|ds\) are \(\mathcal{F}_s\) measurable we have

\[
E[B_{31.33}] = C \int_t^T E[|Y^n_s - Y_s|] \int_s^{s^+} Z^n_v dW_v|\hat{B}^n_s|ds
\]

\[
= C \int_t^T E[|Y^n_s - Y_s|] (\int_s^{s^+} (Z^n_v)^2 dv)|\hat{B}^n_s|ds
\]

\[
\leq C2^n E[(\sup_{0 \leq s \leq T} |Y^n_s - Y_s|)(\sup_{|r-v| \leq \frac{1}{n}} |B_r - B_v|^2)] \int_t^T (\int_s^{s^+} (Z^n_v)^2 dv)ds]
\]

\[
\leq C2^n (E[\sup_{0 \leq s \leq T} |Y^n_s - Y_s|^4])^{\frac{1}{4}} (E[\sup_{|r-v| \leq \frac{1}{n}} |B_r - B_v|^4])^{\frac{1}{4}}
\]

\[
\times (E[(\int_t^T (\int_s^{s^+} (Z^n_v)^2 dv)ds)^2])^\frac{1}{2}
\]

\[
\leq C2^n \left( \frac{1}{2^n} \right)^{\frac{1}{2}} - \delta (E[(\int_t^T (Z^n_v)^2 (\int_v^v ds)dv)^2])^\frac{1}{2}
\]

\[
\leq C2^n \left( \frac{1}{2^n} \right)^{\frac{1}{2}} - \delta \left( \frac{1}{2^n} \right) (E[(\int_t^T (Z^n_v)^2 dv)^2])^\frac{1}{2}
\]

\[
\leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}} - \delta.
\]

(2.91)

As for the term \(B_{31.32}\) we have

\[
E[B_{31.32}] \leq C \int_t^T E[|Y^n_s - Y_s|] \int_s^{s^+} Z^n_v dW_v|\hat{B}^n_s|ds
\]

\[
+ C \int_t^T E[|Y^n_s - Y_s|] \int_s^{s^+} g(Y^n_v)dB^n_v|\hat{B}^n_s|ds.
\]

(2.92)

Furthermore, we have

\[
\int_t^T E[|Y^n_s - Y_s|] \int_s^{s^+} g(Y^n_v)dB^n_v|\hat{B}^n_s|ds
\]

\[
\leq C(2^n)^3 E[(\sup_{|r-v| \leq \frac{1}{n}} |B_r - B_v|^3)] \int_t^T |Y^n_s - Y_s| (\int_s^{s^+} |g(Y^n_v)|dv)^2 ds]
\]

\[
\leq C(2^n) (E[(\sup_{|r-v| \leq \frac{1}{n}} |B_r - B_v|^6])^\frac{1}{2} (E[(\int_t^T |Y^n_s - Y_s|ds)^2])^\frac{1}{2}
\]

\[
\leq C\left( \frac{1}{2^n} \right)^{\frac{1}{2}} - \delta.
\]

(2.93)

It follows now from (2.91), (2.92) and (2.93) that

\[
E[B_{31.32}] \leq C(\frac{1}{2^n})^{\frac{1}{2}} - \delta.
\]

(2.94)
(2.90) - (2.94) yields that
\[ E[B_{3,1.3}] \leq C(\frac{1}{2n})^{\frac{3}{2}-\delta}. \]  
(2.95)

It follows from (2.81), (2.88) and (2.95) that
\[ E[B_{31}] \leq C(\frac{1}{2n})^{\frac{3}{2}-\delta}. \]  
(2.96)

This together with (2.75) yields
\[ E[B_{3}] \leq C(\frac{1}{2n})^{\frac{3}{2}-\delta}. \]  
(2.97)

The lemma now follows from (2.59), (2.73) and (2.97).

**Proof of Theorem 2.3 (continued).**

We are ready to complete the proof of Theorem 2.3. Taking expectation in (2.21) we obtain
\[
E[(Y^n_t - Y_t)^2] + E\left[\int_t^T (Z^n_s - Z_s)^2 ds\right]
\]
\[ = 2E\left[\int_t^T (Y^n_s - Y_s)(f(Y^n_s, Z^n_s) - f(Y_s, Z_s))ds\right]
\]
\[ + 2E\left[\int_t^T (Y^n_s - Y_s)g(Y^n_s)dB^n_s\right] + E\left[\int_t^T g^2(Y_s)ds\right]
\]
\[ - E\left[\int_t^T (Y^n_s - Y_s)g'(Y_s)ds\right]
\]
(2.98)

Taking into account of the estimates in Lemma 2.4 and Lemma 2.5 we deduce from (2.98) that
\[
E[(Y^n_t - Y_t)^2] + E\left[\int_t^T (Z^n_s - Z_s)^2 ds\right]
\]
\[ \leq C(\frac{1}{2n})^{\frac{3}{2}-\delta} + CE\left[\int_t^T |Y^n_s - Y_s|^2 ds\right]
\]
\[ + \frac{1}{2}E\left[\int_t^T |Z^n_s - Z_s|^2 ds\right] + E\left[\int_t^T g^2(Y_s)ds\right]
\]
\[ - E\left[\int_t^T (Y^n_s - Y_s)g'(Y_s)ds\right]
\]
\[ + E\left[\int_t^T (Y^n_{s+} - Y_{s+})g'(Y^n_{s+})ds\right]
\]
\[ - 2E[\int_t^T g(Y_{s+})g(Y^n_{s+})ds] + E[\int_t^T g^2(Y^n_{s+})ds]. \]  
(2.99)
To proceed with the proof, we claim that there is a constant $C$ such that

$$E[\int_0^T |Y^n_{s+} - Y^n_s|^2 ds] \leq C(\frac{1}{2^n})^{1-\delta}.$$ \hspace{1cm} (2.100)

$$E[\int_0^T |Y^+_s - Y_s|^2 ds] \leq C(\frac{1}{2^n})^{1-\delta}.$$ \hspace{1cm} (2.101)

Let us prove (2.100). The proof of (2.101) is similar. Indeed, we have

$$E[\int_0^T |Y^n_{s+} - Y^n_s|^2 ds]$$

$$\leq CE[\int_0^T |\int_s^{s+} f(Y^n_u, Z^n_u) du|^2 ds]$$

$$+ CE[\int_0^T |\int_s^{s+} g(Y^n_u) dB^n_u|^2 ds]$$

$$+ CE[\int_0^T |\int_s^{s+} Z^n_u dW_u|^2 ds]$$

$$\leq C\frac{1}{2^n} + CE[ \sup_{|r-v| \leq \frac{1}{2^n}} |B_r - B_v|^2]$$

$$+ CE[\int_0^T \int_s^{s+} (Z^n_u)^2 duds]$$

$$\leq C(\frac{1}{2^n})^{1-\delta} + \sup_n (E[\int_0^T (Z^n_u)^2 du]) \frac{1}{2^n}$$

$$\leq C(\frac{1}{2^n})^{1-\delta}. \hspace{1cm} (2.102)$$

By Hölder inequality it follows immediately from (2.100) and (2.101) that

$$E[\int_0^T |Y^n_{s+} - Y^n_s| ds] \leq C(\frac{1}{2^n})^{\frac{1}{2} - \delta}.$$ \hspace{1cm} (2.103)

$$E[\int_0^T |Y^+_s - Y_s| ds] \leq C(\frac{1}{2^n})^{\frac{1}{2} - \delta}. \hspace{1cm} (2.104)$$
Because of (2.100) and (2.101), we now replace \( s^+ \) by \( s \) on the right side of (2.99) to obtain

\[
E[(Y^n_t - Y_t)^2] + E \left[ \int_t^T (Z^n_s - Z_s)^2 ds \right] \\
\leq C \left( \frac{1}{2n} \right)^{\frac{1}{2} - \delta} + CE \left[ \int_t^T |Y^n_s - Y_s|^2 ds \right] \\
+ \frac{1}{2} E \left[ \int_t^T |Z^n_s - Z_s|^2 ds \right] + E \left[ \int_t^T g^2(Y_s) ds \right] \\
- E \left[ \int_t^T (Y^n_s - Y_s) gg'(Y_s) ds \right] \\
+ E \left[ \int_t^T (Y^n_s - Y_s)(g(Y^n_s) - g'(Y^n_s)) ds \right] \\
- 2E \left[ \int_t^T g(Y_s)g(Y^n_s) ds \right] + E \left[ \int_t^T g^2(Y^n_s) ds \right].
\]  

(2.105)

Remark that the constant \( C \) in front of \( \left( \frac{1}{2n} \right)^{\frac{1}{2} - \delta} \) is different from that in (2.99). Completing the square in (2.105) we get that

\[
E[(Y^n_t - Y_t)^2] + \frac{1}{2} E \left[ \int_t^T (Z^n_s - Z_s)^2 ds \right] \\
\leq C \left( \frac{1}{2n} \right)^{\frac{1}{2} - \delta} + CE \left[ \int_t^T |Y^n_s - Y_s|^2 ds \right] \\
+ \frac{1}{2} E \left[ \int_t^T |Z^n_s - Z_s|^2 ds \right] + E \left[ \int_t^T (g(Y_s) - g(Y^n_s))^2 ds \right] \\
+ E \left[ \int_t^T (Y^n_s - Y_s)(g(Y^n_s) - g'(Y^n_s)) ds \right].
\]  

(2.106)

By the Lipschitz continuity of \( gg' \) and \( g \), it follows from (2.106) that

\[
E[(Y^n_t - Y_t)^2] + \frac{1}{2} E \left[ \int_t^T (Z^n_s - Z_s)^2 ds \right] \\
\leq C \left( \frac{1}{2n} \right)^{\frac{1}{2} - \delta} + CE \left[ \int_t^T |Y^n_s - Y_s|^2 ds \right].
\]  

(2.107)

Application of the Gronwall’s inequality completes the proof of Theorem 2.3.

References

[ES] L. C. Evans and D. W. Stroock: An approximation scheme for reflected stochastic differential equations, Stochastic Processes and Their applications 121 (2011) 1464-1491.
[IW] N. Ikeda and S. Watanable: Stochastic Differential Equations and Diffusion Processes. North-Holland, Amsterdam, 1981.

[Z] Tusheng Zhang: Wong-Zakai approximations to SDEs with reflection, to appear in Potential Analysis.

[PP1] E. Pardoux and S. Peng, Adapted solution of a backward stochastic differential equation. Systems Control Letter 14(1990) 55-61.

[PP] E. Pardoux, S. Peng: Backward doubly SDEs and systems of quasilinear SPDEs, PTRF 98(1994) 209-227.

[WZ] E. Wong and M. Zakai: On the convergence of ordinary integrals to stochastic integrals, Ann. Math. Statist. 36(1965)1560-1564.

[WZ1] E. Wong and M. Zakai: On the relation between ordinary and stochastic differential equations, Internat. J. Engrg. Sci. 3 (1965) 213-229.