Trace Moments of the Sample Covariance Matrix with Graph-Coloring

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Abstract
Let $S_{p,n}$ denote the sample covariance matrix based on $n$ independent identically distributed $p$-dimensional random vectors in the null-case. The main result of this paper is an expansion of trace moments and power-trace covariances of $S_{p,n}$ simultaneously for both high- and low-dimensional data. To this end we develop a graph theory oriented ansatz of describing trace moments as weighted sums over colored graphs. Specifically, explicit formulas for the highest order coefficients in the expansion are deduced by restricting attention to graphs with either no or one cycle. The novelty is a color-preserving decomposition of graphs into a tree-structure and their seed graphs, which allows for the identification of Euler circuits from graphs with the same tree-structure but different seed graphs. This approach may also be used to approximate the mean and covariance to even higher degrees of accuracy.

Keywords sample covariance matrix · trace moment · colored graphs · trees

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1. Introduction

Let $S_{p,n} \in \mathbb{R}^{p \times p}$ denote the sample covariance matrix for a data set of $n$ independent $p$-dimensional random vectors $(Y_{1j}, \ldots, Y_{pj})^T$, $1 \leq j \leq n$, where each random vector consists of iid components with mean zero and variance one. If $X_{p,n} = (Y_{ij})$ is the corresponding $(p \times n)$ data-matrix, then the sample covariance matrix is given by $S_{p,n} = \frac{1}{n} X_{p,n} X_{p,n}^T$. Note that $\mathbb{E}[S_{p,n}] = \text{Id}_{p \times p}$.

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Throughout we will assume $p \leq n$. We can do this almost without loss of generality, since the cyclic property of the trace implies

$$
\text{tr}(S_{p,n}^l) = \frac{1}{n^l} \text{tr} \left((X_{p,n}X_{p,n}^T)^l\right) = \frac{1}{n^l} \frac{1}{l!} \text{tr} \left((X_{p,n}X_{p,n}^T)^l\right) = \frac{1}{n^l} \text{tr}(S_{n,p}^l) .
$$

For any two numbers $a, b \in \mathbb{R}$ use the notation $a \land b = \min(a, b)$ and $a \lor b = \max(a, b)$.

Our two main results are the following trace moment expansions.

**Theorem 1.** For any $l \in \mathbb{N}$, assume that $E[Y_{11}^{2l}] < \infty$. Then for $p \leq n$ we have

$$
E[\text{tr}(S_{p,n}^l)] = \sum_{b=1}^{l \lor p} \binom{\binom{l}{l+b}}{\binom{n-b}{l+b}} l! \binom{l-1}{b-1} + \sum_{b=1}^{l \lor p} \binom{\binom{l}{l+b}}{\binom{n-b}{l+b}} \frac{b!}{n^l} \left( \binom{2l}{2b} + (2b-1) \binom{l}{b}^2 \right) + \binom{E[Y_{11}^4] - 3}{b < l} b! \left( \binom{l}{b-1} \binom{l}{b+1} \right) + O \left( \frac{n^b}{p^{b+1}} \right) .
$$

Proof given in subsection 6.2.

**Theorem 2.** For any $l_1, l_2 \in \mathbb{N}$, assume that $E[Y_{11}^{2l_1+2l_2}] < \infty$. Then for $p \leq n$ we have

$$
\text{Cov}[\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2})] = \sum_{b=1}^{l_1+l_2 \lor p} \binom{\binom{l_1}{l_1+b}}{\binom{n-b}{l_1+b}} \left( C(l_1, l_2, b) + (E[Y_{11}^4] - 3) D(l_1, l_2, b) \right) + O \left( \frac{n^b}{p^{b+1}} \right) ,
$$

where

$$
C(l_1, l_2, b) := 2 b! \left( l_1 + l_2 - b \right) \sum_{k=0}^{b} \binom{l_1}{k} \binom{l_2}{b-k} \sum_{m=0}^{b-k} \binom{l_1}{k+m} \binom{l_2}{b-m-k} .
$$

and the ‘fourth moment correction term’ is given by

$$
D(l_1, l_2, b) := b! \left( l_1 + l_2 - b \right) \sum_{k=0}^{b-1} \binom{l_1}{k} \binom{l_1}{k+1} \binom{l_2}{b-1-k} \binom{l_2}{b-k} .
$$

Proof given in subsection 7.6.

While Bai and Silverstein were able to derive the limits of the mean and covariance $E[\text{tr}(S_{p,n}^l)]$ and $\text{Cov}[\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2})]$ based on the work of Jonsson (see (1.23) and (1.24) of [2]), they needed to assume $E[(X_{p,n})_{1j}^4] = 3$ and their methods can not be used
to make statements about the speed at which the mean and covariance converge to these limits. Our expansions allow for formulations of spectral central limit theorems (CLTs), where the exact mean-covariance structure of the limiting Gaussian distributions can be given in closed form.

A similar spectral CLT was first proved by Jonsson in 1982 (see Thm. 4.2 of [3]), however in that paper no closed expression for the mean and covariances of the limiting process were given. In their seminal Paper from 2004 Bai and Silverstein proved a general spectral CLT for the sample covariance matrix using the Stieltjes transform method (see [2]), by assuming that the fourth moment of the entries of the data-matrix is $E[(X_{p,n})^4_{i,j}] = 3$. They were later able to slightly extend their result in their book 'Spectral analysis of large dimensional random matrices' (see Thm. 9.10 of [1]). In all of these spectral CLTs the mean and covariance structure of the limiting Gaussian process is poorly understood.

**Strategy of the proofs**

As seen in Remark 3.7, the trace moment $E[\text{tr}(S^l_{p,n})]$ can be written as a weighted sum over closed walks on $2l$ vertices. For every odd step of a walk the visited vertex is colored black, while the others remain white. Since there exists little literature on coloring walks, but colored graphs are well known, we identify our closed walks with linearly ordered graphs (see 3.4). The trace moment $E[\text{tr}(S^l_{p,n})]$ is then a weighted sum over colored directed multigraphs and literature on Euler circuits may be applied to evaluate this sum.

We prove that the highest order coefficient of the polynomial $(p, n) \mapsto n^l E[\text{tr}(S^l_{p,n})]$ depends only on the number of Euler circuits through colored tree-graphs, where every undirected edge is replaced with one directed edge in each direction. We call such graphs 'balanced trees' (introduced in Definition 3.9). These balanced trees are uniquely defined by the property that edges come in pairs and the number of visited vertices is $l + 1$, while there are $2l$ edges. The next highest order coefficients is shown to depend on the number of Euler circuits through colored directed graphs where the $2l$ edges come in pairs, but this time only $l$ vertices are visited. It is shown that such graphs $G$ necessarily have a single circuit-equivalent, meaning that $U(G)$ in Definition 3.3 must have a single circuit. While counting such Euler circuits for the highest order coefficient is relatively straightforward due to existing literature on trees (where a particular challenge is to account for the black-and-white coloring of the vertices), we have to invent a new strategy for the second highest order coefficient.

To this end, we define the seed graph to a given colored directed multigraph. The seed graph is found by iteratively removing balanced leaves (vertices connected to the rest of the graph by one balanced edge pair) until there are either no more balanced leaves, or the graph consists of two edges. The remaining graph is then called the seed graph and the original graph is called a sprouting version of the seed graph. By decomposing the graph into its seed graph and its 'balanced tree structure', we derive an elegant formula...
for the number of sprouting graphs with a prescribed set of black vertices to a given seed graph. (see Proposition 5.2)

The paper is organized as follows. A brief motivation for the use of graph theory to find trace moments is given in the following Section 2. Section 3 then introduces the basic notation we will need throughout the paper. Section 4 defines the black-and-white coloring we have described above, while also containing some results on how the coloring works for some simple types of graphs. After this, Section 5 can be considered the backbone of the paper, as it establishes Proposition 5.2 which allows us to only sum over all possible seed graphs to evaluate the otherwise complicated sums over weighted graphs. This proposition will together with Propositions 6.1 and 7.5 allow us to establish our main results. Finally, Sections 6 and 7 deal with the categorization of seed graphs and its effects on their weight before applying Proposition 5.2 to get our main results on the trace moment and trace power covariance respectively.

Most proofs, other than those of the main results, have been moved to the appendix.

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2. Motivation for graph theory

2.1. Definition (Sample covariance matrix)

Let \((Y_{ij})_{i,j} \subset \mathbb{N}\) be iid random variables with \(\mathbb{E}[Y_{ij}] = 0\) and \(\mathbb{V}[Y_{ij}] = 1\). For \(p \leq n\) define the data-matrix

\[
X_{p,n} := (Y_{ij})_{1\leq p,j \leq n} = \begin{pmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{p1} & \cdots & Y_{pn} \end{pmatrix}
\]

and the corresponding \((p \times p)\) sample covariance matrix

\[
S_{p,n} := \frac{1}{n}X_{p,n}X_{p,n}^T = \begin{pmatrix} \frac{1}{n} \sum_{k=1}^{n} Y_{1k}Y_{1k} & \cdots & \frac{1}{n} \sum_{k=1}^{n} Y_{1k}Y_{pk} \\ \vdots & \ddots & \vdots \\ \frac{1}{n} \sum_{k=1}^{n} Y_{pk}Y_{1k} & \cdots & \frac{1}{n} \sum_{k=1}^{n} Y_{pk}Y_{pk} \end{pmatrix}.
\]

2.2. Remark (Trace moments as sums over weighted graphs)

For any \(l \in \mathbb{N}\) we can apply the simple identity from Lemma A.1 to the matrix \(S_{p,n}\), which yields

\[
\mathbb{E}[\text{tr}(S_{p,n}^l)] = \sum_{i_1,...,i_l=1}^{p} \mathbb{E}\left[\prod_{i=1}^{l} (S_{p,n})_{i_1,i_2} \cdots (S_{p,n})_{i_{l-1},i_l} (S_{p,n})_{i_l,i_1}\right]
\]

\[
\begin{aligned}
&= \sum_{i_1,...,i_l=1}^{p} \mathbb{E}\left[\left(\frac{1}{n} \sum_{k_1=1}^{n} Y_{i_1 k_1}Y_{i_2 k_1}\right) \cdots \left(\frac{1}{n} \sum_{k_{l-1}=1}^{n} Y_{i_{l-1} k_{l-1}}Y_{i_l k_{l-1}}\right) \left(\frac{1}{n} \sum_{k_1=1}^{n} Y_{i_l k_1}Y_{i_1 k_1}\right)\right] \\
&= \frac{1}{n^l} \sum_{i_1,...,i_l=1}^{p} \sum_{k_1,...,k_{l-1}=1}^{n} \mathbb{E}\left[(Y_{i_1 k_1}Y_{i_2 k_1}) \cdots (Y_{i_{l-1} k_{l-1}}Y_{i_l k_{l-1}})(Y_{i_l k_1}Y_{i_1 k_1})\right] \\
&= \frac{1}{n^l} \sum_{(\mathcal{I}) \leq (\mathcal{I})} \sum_{(\mathcal{K}) \leq (\mathcal{K})} \mathbb{E}\left[(Y_{i_{\mathcal{I}_1} k_{\mathcal{K}_1}}Y_{i_{\mathcal{I}_2} k_{\mathcal{K}_1}}) \cdots (Y_{i_{\mathcal{I}_{l-1}} k_{\mathcal{K}_{l-1}}}(Y_{i_l k_l}Y_{i_1 k_1})\right].
\end{aligned}
\]

Here \(\mathcal{I}\) denotes the set of indexes, which occur in \(\mathcal{i}\), i.e. \(\mathcal{I} := \{i_1, ..., i_l\}\). We have thus split up the sum by the number of different indexes, which occur in \(\mathcal{I}\) and \(\mathcal{K}\). We also use the fact that we can then rename these indexes without changing the mean

\[
\mathbb{E}\left[(Y_{i_{\mathcal{I}_1} k_{\mathcal{K}_1}}Y_{i_{\mathcal{I}_2} k_{\mathcal{K}_1}}) \cdots (Y_{i_{\mathcal{I}_{l-1}} k_{\mathcal{K}_{l-1}}}(Y_{i_l k_l}Y_{i_1 k_1})\right]
\]

to see...
\[ \mathbb{E}[\text{tr}(S_{p,n}^l)] = \sum_{r=1}^{(2l)\wedge n} \left( \frac{(2l)^r}{n^r} \right) \sum_{\substack{i \in \{1, \ldots, b\}^l \\cap \\{j\} \in \{1, \ldots, r\} \\cap \{b\} \in \{1, \ldots, r\}}} \mathbb{E} \left[ Y_{i_1,j_1} Y_{i_2,j_2} \cdots Y_{i_{l-1},j_{l-1}} Y_{i_l,j_l} Y_{i_1,j_1} \right] . \]

If we swap the two indexes \( i_\bullet \) and \( k_\bullet \) of each second \( Y \) in the mean \( W_{i,k} \) we get an index-sequence of the form

\[(i_1, k_1), (k_1, i_2), (i_2, k_2), \ldots, (i_{l-1}, k_{l-1}), (k_{l-1}, i_l), (i_l, k_l), (k_l, i_1), \]

which forms a 'chain-structure' in the sense that each index-tuple starts with the entry at which the previous tuple ended. By interpreting each of the above tuples \((i_\bullet, k_\bullet)\) (or \((k_\bullet, i_\bullet)\)) as a directed edge between the vertices \((v_{i_\bullet}, v_{k_\bullet})\) (or \((v_{k_\bullet}, v_{i_\bullet})\) respectively), the above 'chain' forms a closed walk on \( r \) vertices. In this paper we develop methods of counting the different types of such closed walks relevant to \( \mathbb{E}[\text{tr}(S_{p,n}^l)] \), by exploiting the rich literature on Euler circuits and trees.

While our main result only approximates \( \mathbb{E}[\text{tr}(S_{p,n}^l)] \) with an error of \( O \left( \frac{1}{n} \right) \), when \( \frac{p_n}{n} \to y \in (0,1] \) (see Corollary 6.3), the methods can be used to find further terms of the expansion and approximate with even lower errors of \( O \left( \frac{1}{n^2} \right) \) or \( O \left( \frac{1}{n^3} \right) \). This would however require certain categorizations of graphs with more than one cycle, which gets tedious quickly. More precisely one would have to find a version of Proposition 6.1 which also takes into account the case, where \( G_{i,k} \) has \( l-1 \) or \( l-2 \) (visited) vertices.
3. Graph theoretical notation and lemmas

3.1. Definition (Directed multigraph)

A directed multigraph $G$ is a triple $(V, E, f_G)$ consisting of a finite vertex set $V$, a finite edge set $E$ and a map $f_G : E \to V \times V$. We say the edge $e \in E$ has tail in $\text{tail}(e) = v \in V$ and has head in $\text{head}(e) = v' \in V$, if $f_G(e) = (v, v')$.

This definition of a multigraph is commonly known as a multigraph with edges with own identity, since edges $e, e' \in E$ can be distinct even if they both originate and terminate at the same vertices.

A directed multigraph $G = (V, E, f_G)$ will be called labeled, if $V$ has a canonical ordering. (We will mostly just be interested in $V \subset \mathbb{N}$.) We can then label these vertices by their ordering, meaning $V$ can be written as $V_r = \{v_1, ..., v_r\}$ for some $r \in \mathbb{N}$.

A linearly ordered directed multigraph $G$ is a labeled directed multigraph $(V_r, E_N, f_G)$, where $E$ also has a canonical ordering. We again label the edges by ordering and write $E$ as $E_N = \{e_1, ..., e_N\}$ for some $N \in \mathbb{N}$. We will often use $[N] := \{1, ..., N\}$ as $E_N$.

For fixed $V_r$ and $E_N$ let $G_{V_r,E_N}$ denote the set of all linearly ordered directed multigraphs $G = (V_r, E_N, f_G)$.

3.2. Definition (Visited vertices and exhaustive graphs)

For a directed multigraph $G = (V, E, f_G)$ a vertex $v \in V$ is called visited, if there exists an edge $e \in E$, for which $v$ is head or tail. Let $V(G)$ denote the set of all visited vertices in $G$. The graph $G$ is called exhaustive, if every vertex $v \in V$ is visited, i.e. $V(G) = V$.

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\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) [fill=black!10!white, draw] {$v_1$};
  \node (v2) at (1,1) [fill=black!10!white, draw] {$v_2$};
  \node (v3) at (0,-1) [fill=black!10!white, draw] {$v_3$};
  \node (v4) at (1,-1) [fill=black!10!white, draw] {$v_4$};

  \draw (v1) to [out=90, in=90] (v2);
  \draw (v1) to [out=-90, in=-90] (v3);
  \draw (v2) to [out=-90, in=-90] (v4);

  \node at (0.5,1.2) {2};
  \node at (0.5,-1.2) {\small an exhaustive $G \in G_{V_4,[4]}$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node (v1) at (0,0) [fill=black!10!white, draw] {$v_1$};
  \node (v2) at (1,1) [fill=black!10!white, draw] {$v_2$};
  \node (v3) at (0,-1) [fill=black!10!white, draw] {$v_3$};
  \node (v4) at (1,-1) [fill=black!10!white, draw] {$v_4$};

  \draw (v1) to [out=90, in=90] (v2);
  \draw (v1) to [out=-90, in=-90] (v3);
  \draw (v2) to [out=-90, in=-90] (v4);

  \node at (0.5,1.2) {1};
  \node at (0.5,-1.2) {\small a non-exhaustive $G \in G_{V_4,[4]}$};
\end{tikzpicture}
\end{center}

3.3. Definition (Undirected connection)

Let $G = (V, E, f_G)$ be a directed multigraph with more than one vertex. For two different vertices $v, w \in V$ we say there is an undirected connection between $v$ and $w$, if there exists an edge $e \in E$ with $f_G(e) \in \{(v, w), (w, v)\}$. The number of such edges does not play a role.

Let $U(G)$ describe the undirected simple graph (possibly with self-loops), which we get by replacing all undirected connections of $G$ with undirected edges.
The directed multigraph $G$ is called undirectedly connected, if $U(G)$ is connected.

3.4. Definition (Routes and circuit multigraphs)

For any fixed vertex set $V_r$ a route through $V_r$ of length $N$ is defined to be a sequence $i \subset V_r^N$ with the property

$$V_r = \{i\} := \text{set of all vertices occurring in } i.$$

For a given route $i \subset V_r^N$ let $E_N := [N]$, then we define a linearly ordered directed multigraph $G_i = (V_r, [N], f_i) \in \mathcal{G}_{V_r,E_N}$ by

$$f_i(k)_{e_k} := \begin{cases} (i_k, i_{k+1}) & \text{if } k < N \\ (i_N, i_1) & \text{if } k = N \end{cases}.$$

The set of circuit multigraphs on $V_r$ of length $N$ is then defined as

$$C_{V_r,N} := \{G_i \mid i \subset V_r^N \text{ route through } V_r \text{ of length } N\}.$$

Such graphs are by construction exhaustive and connected.

3.5. Definition (Zipped sequences)

For two sequences $i, k \in \mathbb{N}^l$ we define the zipped sequence

$$(i, k) := (i_1, k_1, i_2, k_2, ..., i_l, k_l).$$
3.6. Definition (Reversal operator)

For any \( r, N \in \mathbb{N} \) define the reversal operator

\[
R : \mathcal{C}_{V_r, N} \to \mathcal{G}_{V_r, [N]}
\]

by reversing the direction of every second edge \( e_k \in [N] \). In other words if \( G = (V_r, [N], f) \in \mathcal{C}_{V_r, N} \), then \( R(G) = (V_r, [N], f') \), where \( f' \) is given by

\[
f'(k) = \begin{cases} (v_i, v_j) & \text{for } f(k) = (v_i, v_j), \text{ if } k \text{ is odd} \\ (v_j, v_i) & \text{for } f(k) = (v_i, v_j), \text{ if } k \text{ is even} \end{cases}.
\]

3.7. Remark (Trace moments and circuit multigraphs)

At this point the link between circuit multigraphs and the trace moments \( \mathbb{E}[\text{tr}(S^l_{p,n})] \) becomes apparent. Recalling equality (2.2.1) we see that

\[
\mathbb{E}[\text{tr}(S^l_{p,n})] = \sum_{r=1}^{(2l)/n} \frac{\binom{2l}{r} \binom{n-b}{r-b}}{n!} \sum_{k \in \{1, \ldots, r\}^l \backslash \{\{i\}\}} \mathbb{E}\left[ Y_{i_1 k_1} Y_{i_2 k_1} \ldots Y_{i_{l-1} k_{l-1}} Y_{i_l k_l} Y_{i_1 k_l} \right] =: W_{i,k}.
\]

The mean \( W_{i,k} =: W(G_{(i,k)}) \) can now be interpreted as a ‘weight’ of the circuit multigraph \( G_{(i,k)} \). By construction, the edges of \( G_{(i,k)} \) (in order) are

\[
(v_{i_1}, v_{k_1}), (v_{k_1}, v_{i_2}), \ldots, (v_{k_{l-1}}, v_{i_l}), (v_{i_l}, v_{k_l}), (v_{k_l}, v_{i_1}),
\]

which means the edges of \( R(G_{(i,k)}) \) must be

\[
(v_{i_1}, v_{k_1}), (v_{k_1}, v_{i_2}), \ldots, (v_{k_{l-1}}, v_{i_l}), (v_{i_l}, v_{k_l}), (v_{k_l}, v_{i_1}).
\]

Hereby \( R(G_{(i,k)}) \) has one edge \((v_i, v_j)\) for each factor \( Y_{i,j} \) in the above mean and with the independence of the entries \((Y_{i,j})_{i,j \leq l}\) it follows that

\[
\mathbb{E}\left[ (Y_{i_1 k_1} Y_{i_2 k_1}) \ldots (Y_{i_{l-1} k_{l-1}} Y_{i_l k_l}) (Y_{i_l k_l} Y_{i_1 k_l}) \right] = \prod_{i,j=1}^{l} \mathbb{E}\left[ Y_{i,j}^{A(R(G_{(i,k)}))_{i,j}} \right].
\]

We have thus shown that \( W(G_{(i,k)}) \) depends only on the adjacency matrix of \( R(G_{(i,k)}) \) and, since \( \mathbb{E}[Y_{i,j}] = 0 \), that for \( W(G_{(i,k)}) \neq 0 \) to hold each edge in \( R(G_{(i,k)}) \) must have at least one other edge going in the same direction. This implies that \( R(G_{(i,k)}) \) (and
thus also $G_{(i,k)}$ can have at most $\frac{2l}{2} = l$ many undirected connections. As $G_{(i,k)}$ is connected, it can thus have no more than $l + 1$ many visited vertices. For the trace moment formula this means we may adjust the upper limit of the sums to be

$$E[\text{tr}(S^l_{p,n})] = \sum_{r=1}^{(l+1)^n} \sum_{b=1}^{(p)\binom{n-b}{r-b}} \frac{1}{n!} \sum_{G_{i,k} \in C_{V_{r+1},2l}} W(G_{i,k}) \cdot \binom{n}{r}.$$ 

In section $4$ we will give an interpretation for role of $b$ and the condition $\{i\} = \{1, \ldots, b\}$ in the above sum by defining a black-and-white coloring of circuit multigraphs. The rest of this section consists of definitions and lemmas that will help in classifying and counting the two most important types of circuit multigraphs.

3.8. Definition (Balanced directed multigraphs)

We call a directed multigraph $G = (V, E, f_G)$ balanced, if the edges $E$ can be split into (balanced) edge pairs $(e, e')$ such that the head of edge $e$ is the tail of edge $e'$ and vice versa.

If $G$ is labeled, an equivalent definition would be to say that its adjacency matrix $A(G)$ is symmetric and has only even entries on the diagonal.

3.9. Definition (Balanced tree)

For any $l \in \mathbb{N}$ and $V_{l+1}$ the set

$$T_{V_{l+1}} := \{G \in C_{V_{l+1},2l} \mid G \text{ is balanced}\}$$

will be called the set of balanced trees of length $l$.

Since elements $G \in C_{V_{l+1},2l}$ are by construction exhaustive and connected, we know $U(G)$ to be connected with $l + 1$ vertices. As $U(G)$ can have at most $l$ many edges, Lemma $A.2$ then tells us that $U(G)$ can have no cycles and must have exactly $l$ edges, i.e. be a tree. Also each undirected edge of $U(G)$ must correspond to one balanced edge pair in $G$. 

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Since elements $G \in C_{V_{l+1},2l}$ are by construction exhaustive and connected, we know $U(G)$ to be connected with $l + 1$ vertices. As $U(G)$ can have at most $l$ many edges, Lemma $A.2$ then tells us that $U(G)$ can have no cycles and must have exactly $l$ edges, i.e. be a tree. Also each undirected edge of $U(G)$ must correspond to one balanced edge pair in $G$. 

3.8. Definition (Balanced directed multigraphs)

We call a directed multigraph $G = (V, E, f_G)$ balanced, if the edges $E$ can be split into (balanced) edge pairs $(e, e')$ such that the head of edge $e$ is the tail of edge $e'$ and vice versa.

If $G$ is labeled, an equivalent definition would be to say that its adjacency matrix $A(G)$ is symmetric and has only even entries on the diagonal.

3.9. Definition (Balanced tree)

For any $l \in \mathbb{N}$ and $V_{l+1}$ the set

$$T_{V_{l+1}} := \{G \in C_{V_{l+1},2l} \mid G \text{ is balanced}\}$$

will be called the set of balanced trees of length $l$.

Since elements $G \in C_{V_{l+1},2l}$ are by construction exhaustive and connected, we know $U(G)$ to be connected with $l + 1$ vertices. As $U(G)$ can have at most $l$ many edges, Lemma $A.2$ then tells us that $U(G)$ can have no cycles and must have exactly $l$ edges, i.e. be a tree. Also each undirected edge of $U(G)$ must correspond to one balanced edge pair in $G$. 

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If $G$ is labeled, an equivalent definition would be to say that its adjacency matrix $A(G)$ is symmetric and has only even entries on the diagonal.
3.10. Lemma (Number of balanced trees with the same adjacency matrix)

Let $A \in \mathbb{N}_0^{(l+1)\times(l+1)}$ be the adjacency matrix of a balanced tree $T \in \mathcal{T}_{V_{l+1}}$, i.e. the adjacency matrix of a labeled tree with $l+1$ vertices, then there are

$$2l \prod_{i=1}^{l+1} (\text{indeg}_T(v_i) - 1)!$$

many balanced trees $G \in \mathcal{T}_{V_{l+1}}$ with the same adjacency matrix.

Proof.

This is a simple application of the B.E.S.T. Theorem. The term $\prod_{i=1}^{l+1} (\text{indeg}_T(v_i) - 1)!$ gives the number of Euler circuits on $T$. Since $T$ has no self-loops the in-degree is always equal to the pair-degree, and since Euler circuits are counted modulo starting edge, there are $2l \prod_{i=1}^{l+1} (\text{indeg}_T(v_i) - 1)!$ different closed walks through $G_0$ such that each edge is walked only once. The bijection between $\{G \in \mathcal{T}_{V_{l+1}} | A(G) = A(T)\}$ and the set of such closed walks can be directly constructed by identifying both with their respective routes.

3.11. Definition (Ring-type graphs)

For $l_0 \in \mathbb{N}$ and a vertex set $V_{l_0}$ a circuit multigraph $G = G_4$ from $C_{V_{l_0}, 2l_0}$ will be called of ring-type, if $U(G)$ is a cycle graph and each undirected connection in $G$ consists of two edges. If $G$ is additionally balanced, it will be called a two-directional ring-type graph. Otherwise it will be called a one-directional ring type graph. Note that a one-directional ring type graph is only possible for $l_0 \geq 3$.

For any $G \in C_{V_{l_0}, 2l_0}$ we (with slight abuse of notation as $V_{l_0}$ is lost) write $G \in 1$-d-Ring$_{l_0}$, if $G$ is a one-directional ring-type graph with ring length $l_0$. Analogously we write $G \in 2$-d-Ring$_{l_0}$, if $G$ is a two-directional ring-type graph with ring length $l_0$.

For any $G \in C_{V_{l_0}, 2l_0}$ we (with slight abuse of notation as $V_{l_0}$ is lost) write $G \in 1$-d-Ring$_{l_0}$, if $G$ is a one-directional ring-type graph with ring length $l_0$. Analogously we write $G \in 2$-d-Ring$_{l_0}$, if $G$ is a two-directional ring-type graph with ring length $l_0$. 

\[ G_{v_1,v_2}, \text{ the only element in 2-d-Ring}_1 \] 

\[ G_{v_1,v_2,v_3}, \text{ and } G_{v_2,v_1,v_2,v_3}, \text{ the only elements in 2-d-Ring}_2 \]
3.12. Definition (Balanced leaves)

Let \( G = (V, E, f) \) be a directed multigraph with more than one vertex. We call a vertex \( v \in V \) a balanced leaf of \( G \), if there is exactly one edge with head \( v \), one other edge with tail \( v \) and both these edges are between \( v \) and one other vertex \( w \in V \). Accordingly, in circuit multigraphs \( G = G_i \in C_{V_r,N} \) a vertex \( v_j \) is a balanced leaf, if and only if \( v_j \) occurs only once in \( i \) and this occurrence’s left- and right-hand neighbors in \( i \) are equal. If the sequence \( i \) starts or ends in \( v_j \), we loop around the ends of \( i \) to find these neighbors.

3.13. Definition (Removing balanced leaves)

For any \( G_i \in C_{V_r,N} \) with a balanced leaf \( v_j \in V_r \) and \( N > 2 \) let \( v_s \) be the only neighbor of \( v_j \) in \( G_i \), then the sub-sequence \( (v_s, v_j, v_s) \) must occur in \( i \), though it might happen that the sub-sequence is interrupted by the end of the route, in which case the sub-sequence will continue at the beginning of the route. We define the modified route

\[
i' = \begin{cases} 
(v_1, ..., v_s, \hat{v}_j, v_s, ..., v_{2l}) & \text{if } v_{2l} \neq v_j \\
(v_s, v_1, ..., v_{2l-2}, \hat{v}_s, \hat{v}_j) & \text{if } v_{2l} = v_j 
\end{cases} \in \{v_1, ..., \hat{v}_j, ..., v_r\}^{N-2}
\]

by ignoring the singular occurrence of \( v_j \) in \( i \) together with the next entry \( v_s \) of \( i \), if it exists. If the next entry does not exists, then \( v_j \) must be the last entry in \( i \) and we instead ignore the entry \( v_s \) previous to the occurrence of \( v_j \). In both cases we have ignored the occurrence of \( v_j \) and an occurrence of \( v_s \). (This definition guarantees that in both cases the positions of the remaining entries stay the same modulo 2.) Let \( e, e' \in E_N \) be the two edges between \( v_j \) and \( v_s \) in \( G_i \), then we call the graph

\[
\tilde{G} := G' \in C_{V_{r-1}:V_r \setminus \{v_j\},N-2}
\]

the version of \( G_i \) with \( v_j \) removed.

We had assumed \( N > 2 \) in order to guarantee that \( \tilde{G} \) still has edges (and can thus have visited vertices). For \( N \leq 2 \), we say that \( G_i \) has no balanced leaves.
3.14. Definition (Seed graph)

For any $G \in \mathcal{C}_{V,2l}$ let the seed graph $S(G) \in \mathcal{C}_{V_0,2l_0}$ (for certain $r_0, l_0$ with $l_0 - r_0 = l - r$) be given by the following recursive definition. If $G$ has no balanced leaves (includes $l = 1$), we define $S(G) := G$. Otherwise let $v_j$ be the 'smallest' (equivalently 'lowest indexed') balanced leaf of $G$ and $\tilde{G}$ be the version of $G$ with $v_j$ removed. We recursively define $S(G) := S(\tilde{G})$ to be the seed graph of $\tilde{G}$.

In the opposite direction we say $G$ is sprouting from $S(G)$. Also the vertices in $S(G)$ are called seed vertices and vertices from $G \setminus S(G)$ are called sprouted vertices.

3.15. Remark (Seeds of trees)

By properties of the well known Prüfer-code algorithm, balanced trees are precisely the elements $G \in \mathcal{C}_{V,l+1,2l}$ where the seed graph consist of two vertices connected by a balanced edge pair. Balanced trees are also the only type of circuit multigraph for which the seed graph depends on the ordering of $V_r$. To avoid problems, that may stem from this, we will in Proposition 5.2 only look at balanced trees, where there is an edge pair between the two largest (highest indexed) vertices. This guarantees that their seed graph will consist of these two largest vertices.

In later applications we will not be examining balanced trees and we will not need to address these problems by making requirements to the order of the vertices.
4. Graph weight and coloring

4.1. Definition (Black-white coloring of circuit multigraphs)

For any $G \in C_{V_r,N}$ we call a vertex $v_i \in V_r$ white, if it is only tail of even numbered edges $e_k = k \in [N]$. Due to $G$ being a circuit multigraph, this is equivalent to $v_i$ only being head of odd numbered edges. We call a vertex of $G$ black, if it is not white.

In terms of the route $i_G$ this is equivalent to a vertex $v_j \in V_r$ being white, iff $j$ only appears in even numbered entries of $i_G \in V_r^N$.

Let $B(G) \subset V_r$ denote the set of all black vertices. By construction we have

$$B(G) := \{v_{i_1}, v_{i_2}, \ldots\} = \{v_t \mid t \leq N \text{ odd}\}.$$

Let $B(G) \subset V_r$ denote the set of all black vertices. By construction we have

$$B(G) := \{v_{i_1}, v_{i_2}, \ldots\} = \{v_{i_t} \mid t \leq N \text{ odd}\}.$$

The above definition is easily extended to non-exhaustive circuit multigraphs by calling all unvisited vertices white. This will only become necessary in Definition 7.1.

4.2. Remark (Coloring and zipped multi-indexes)

In (3.7.3) we had seen

$$E[\text{tr}(S_{p,n}^l)] = \left( \sum_{r=1}^{(l+1) \wedge n} \sum_{b=1}^{(l+1) \wedge p \wedge r} \binom{p}{r} \binom{n-b}{r-b} \right) \frac{1}{n^l} \sum_{G(i,k) \in C_r, 2l} W(G(i,k)) .$$

By the route-interpretation of our coloring, $B(G(i,k))$ is the set of all vertices whose indexes appear in odd numbered entries of $\langle i, k \rangle$, which by construction are the indexes in $\{i\}$. It follows that

$$B(G(i,k)) = \{i\} .$$

For the formula of the trace moments this implies

$$E[\text{tr}(S_{p,n}^l)] = \left( \sum_{r=1}^{(l+1) \wedge n} \sum_{b=1}^{(l+1) \wedge p \wedge r} \binom{p}{r} \binom{n-b}{r-b} \right) \frac{1}{n^l} \sum_{G(i,k) \in C_r, 2l} W(G(i,k)) .$$

(4.2.1)
The final sum is now a weighted sum over colored graphs, which is very hard to calculate for arbitrary \( r \) and \( l \). Fortunately the term \( \frac{(p^b)}{n^l} \) is of \( O \left( \frac{p^b}{n^{l+r}} \right) \) for \( r < l \), which means to approximate the trace moment \( \mathbb{E}[\text{tr}(S_{p,n}^l)] \) with an error of \( O \left( \frac{1}{n^l} \right) \) or less, we will only need to consider the cases \( r = l + 1 \) and \( r = l \). More precisely by changing the order of summation we have

\[
\mathbb{E}[\text{tr}(S_{p,n}^l)] = \sum_{b=1}^{(l+1)\wedge p} \left[ \sum_{r=3\wedge b}^{(l+1)\wedge n} \left( \frac{p^b}{n^l} \right)^{\frac{(r-b)}{n^l}} \sum_{G(i,k)\in C[r,2l]} W(G(i,k)) + O \left( \frac{p^b}{n^{l+b+1}} \right) \right].
\]

(4.2.2)

4.3. Lemma (Removing balanced leaves does not change the coloring)

For any \( G_i \in C_{V_l,E_2} \) with balanced leaf \( v_j \) (of arbitrary coloring) let \( \tilde{G} \in C_{V_r-1,2l-2} \) be the version of \( G_i \) with \( v_j \) removed as in Definition 3.13. The property

\[
B(G) \setminus \{v_j\} = B(\tilde{G})
\]

holds.

Proof.

By construction of the route \( i' \) in Definition 3.13 the positions of the entries in \( i' \) are the same as their positions in \( i \) modulo 2. We thus have

\[
B(G_i) \setminus \{v_j\} = \{v_{i_1}, v_{i_3}, \ldots, v_{2l-1}\} \setminus \{v_j\} = \{v_{i_1}', v_{i_3}', \ldots, v_{2l-3}'\} = B(\tilde{G}).
\]

4.4. Lemma (Coloring of balanced trees)

For any balanced tree \( G_i \in T_{V_{l+1}} \) each edge must be between a black and a white vertex.

4.5. Lemma (Only balanced trees have \( l + 1 \) vertices and positive weight)

For any vertex set \( V_{l+1} \) and \( i, k \in V_{l+1} \) with \( \{i\} \cup \{k\} = V_{l+1} \) let \( G(i,k) \) be the circuit multigraph in \( C_{V_{l+1},2l} \) with route \( \langle i, k \rangle \), then

\[
W(G(i,k)) := \mathbb{E} \left[ \left( Y_{i_1}k_1 Y_{i_2}k_1 \ldots Y_{i_{l-1}}k_{l-1} Y_{i_l}k_{l-1} \right) \left( Y_{i_1}k_1 Y_{i_2}k_1 \right) \right] = \mathbb{1}_{G(i,k) \in T_{V_{l+1}}}.
\]
4.6. Lemma (Number of balanced trees $T$ with given $B(T)$)

For any given set $B \subset V_{l+1} = \{v_1, \ldots, v_{l+1}\}$ the number of balanced trees $T \in T_{V_{l+1}}$ with $B(T) = B$ is zero, if $b := \#B \in \{0, l + 1\}$, and is otherwise given by

$$\#\{T \in T_{V_{l+1}} \mid B(T) = B\} = l! \binom{l - 1}{b - 1}.$$ 

4.7. Lemma (Coloring of one-directional ring-type graphs)

For even $l_0 \geq 4$ any one-directional ring-type graph $G \in 1$-d-Ring$_{l_0}$ will have alternating black and white vertices along its ring structure.

**Proof.**

By construction two edges in the same undirected connection have the same parity and this parity must toggle along the ring structure.

\[\] 

4.8. Lemma (Coloring of two-directional ring-type graphs)

For even $l_0 \geq 2$ any two-directional ring-type graph $G \in 2$-d-Ring$_{l_0}$ will have alternating black and white vertices along its ring structure. If $l_0$ is odd, then there exists one exception where two neighboring vertices are both black.

\[\]
5. Counting sprouting graphs

5.1. Lemma (Number of sprouting graphs only depends on \( l_0 \))

For any \( r_0, l_0 \in \mathbb{N} \) let \( G_0 \in C_{V_{r_0}, 2l_0} \) and \( \overline{G_0} \in C_{\overline{V}_{r_0}, 2l_0} \) be circuit multigraphs without balanced leaves. Further let \( V'_{l'} \) be a vertex set disjoint to both \( V_{r_0} \) and \( \overline{V}_{r_0} \). We call \( V'_{l'} \) the set of sprouting vertices, while \( V_{r_0} \) and \( \overline{V}_{r_0} \) are two possible choices of seed vertices.

If \( l_0 = 2 \) and \( G_0 \) is of the form \( G_{v_i, v_j} \) for \( \{v_i, v_j\} = \{v_1, v_2\} = V_2 \) we assume that \( v_1, v_2 \) are larger than all vertices in \( V'_{l'} \). The same goes for \( \overline{G_0} \). We do this in order to not run into the problems addressed in Remark 3.15.

Under these conditions for any \( B \subset V'_{l'} \) the number of circuit multigraphs sprouting from \( G_0 \) such that \( B \) is the set of black colored sprouting vertices is, precisely the number of circuit multigraphs sprouting from \( \overline{G_0} \) such that \( B \) is the set of black colored sprouting vertices. More precisely we have the equality

\[
\# \{ G \in C_{V_{r_0} \sqcup V'_{l'}, 2l_0 + 2l' | S(G) = G_0, B(G) \cap V'_{l'} = B \} = \# \{ G \in C_{\overline{V}_{r_0} \sqcup V'_{l'}, 2l_0 + 2l' | S(G) = \overline{G_0}, B(G) \cap V'_{l'} = B \}.
\]

We prove this via a bijection between the sets. Here an example of two graphs with different seed graphs, where our bijection would map one to the other.

5.2. Proposition (Counting sprouting graphs)

For any \( l_0 \in \mathbb{N} \) let \( G_0 \in C_{V_2, 2l_0} \) be the circuit multigraph with route

\[
i_0 = (v_1, v_2, v_1, v_2, ..., v_1, v_2) \quad \text{length: } 2l_0.
\]

For any finite sets \( B', W' \subset \mathbb{N} \) such that \( V_2 = \{v_1, v_2\}, B' \) and \( W' \) are disjoint define \( b' := \#B', w' := \#W', l' = b' + w' \) and \( V'_l = B' \sqcup W' \). Further assume that \( v_1, v_2 \) are the two largest vertices in \( V_2 \sqcup V'_l \) (see Remark 3.15). We then have
\[ \# \{ G \in C_{V_i \cup V'_i, 2l_0 + 2l'} \mid S(G) = G_0, B(G) \cap V'_i = B' \} = \frac{(l_0 + b' + w')!}{(l_0 + b')!(l_0 + w')!}. \]

By Lemma 5.3, this is also true for arbitrary other \( G_0 \in C_{V_i, 2l_0} \) without balanced leaves. Since for other \( G_0 \in C_{V_i, E_{2l_0}} \), the order of \( V_{r_0} \cup V'_i \) has no effect on the seed graph of a \( G \in C_{V_i \cup B' \cup W', 2l_0 + 2b' + 2w'} \), we in this case also don’t need to make any assumptions about elements of \( V_{r_0} \) being larger than those of \( V'_i \).

### 5.3. Corollary (Counting sprouting one-directional rings)

For even \( l_0 \geq 4 \) and any \( b', w' \geq 0 \) define \( l := l_0 + b' + w' \) and let \( B \) be a subset of \( V_i \) such that \( b := \#B = b' + \frac{l_0}{2} \). The number of \( G \in C_{V_i, 2l} \) with \( B(G) = B \) and the property that \( S(G) \in 1\text{-d-Ring}_{l_0} \) is given by

\[ b!w! \binom{l}{b'} \binom{l}{w'}. \]

Here \( b := b' + \frac{l_0}{2} \) denotes the total number of black vertices and \( w := w' + \frac{l_0}{2} = l - b \) denotes the total number of white vertices.

### 5.4. Corollary (Counting sprouting two-directional rings)

For \( l_0 \neq 2 \) and any \( b', w' \geq 0 \) define \( l := l_0 + b' + w' \) and let \( B \) be a subset of \( V_i \) such that \( b := \#B = b' + \lfloor \frac{l_0}{2} \rfloor \). The number of \( G \in C_{V_i, 2l} \) with \( B(G) = B \) and the property that \( S(G) \in 2\text{-d-Ring}_{l_0} \) is given by

\[ l_0 b!w! \binom{l}{b'} \binom{l}{w'}. \]

Here \( b := b' + \lfloor \frac{l_0}{2} \rfloor \) denotes the total number of black vertices and \( w := w' + \lfloor \frac{l_0}{2} \rfloor = l - b \) denotes the total number of white vertices.

For \( l_0 = 2 \) there are only

\[ b!w! \binom{l}{b'} \binom{l}{w'} = b!w! \binom{b' + w' + 2}{b'} \binom{b' + w' + 2}{w'} \]

such \( G \), which is only half as many as expected by the above formula.

### 6. Approximating trace moments

#### 6.1. Proposition (Weight of \( G \in C_{V_i, E_{2l}} \))

For any \( l \in \mathbb{N} \) suppose \( V_i \subset \mathbb{N} \). For \( i \in V_i \), \( k \in V_i \) with \( \{i\} \cup \{k\} = V_i \) we have \( G_{(i,k)} \in C_{V_i, 2l} \) and

\[ W(G_{(i,k)}) := \mathbb{E} \left[ (Y_{i_1 k_1} Y_{i_2 k_1}) \ldots (Y_{i_{l-1} k_{l-1}} Y_{i_l k_{l-1}}) (Y_{i_l k_l} Y_{i_{l+1} k_l}) \right] \]
\[
\alpha, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_2
\]
\[
1, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_{l_0} \text{ for some } l_0 \neq 2
\]
\[
1, \text{ if } S(G_{i,k}) \in 1\text{-d-Ring}_{l_0} \text{ for even } l_0 \geq 4
\]
\[
0 \text{ else}
\]

\[\alpha, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_2\]
\[1, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_{l_0} \text{ for some } l_0 \neq 2\]
\[1, \text{ if } S(G_{i,k}) \in 1\text{-d-Ring}_{l_0} \text{ for even } l_0 \geq 4\]
\[0 \text{ else}\]

\[v_1 \quad v_2 \quad v_3 \quad v_4\]
\[v_1 \quad v_2 \quad v_3 \quad v_5\]
\[v_1 \quad v_4 \quad v_6 \quad v_5\]

\[\alpha, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_2\]
\[1, \text{ if } S(G_{i,k}) \in 2\text{-d-Ring}_{l_0} \text{ for some } l_0 \neq 2\]
\[1, \text{ if } S(G_{i,k}) \in 1\text{-d-Ring}_{l_0} \text{ for even } l_0 \geq 4\]
\[0 \text{ else}\]
6.2. Proof of Theorem 1

Proof.
In (4.2.2) we had seen
\[ \mathbb{E}[\text{tr}(S_{p,n}^l)] = \sum_{b=1}^{l+1} \left[ \frac{1}{n!} \left( \frac{(l+1)^n}{n!} \right)^2 \right] \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, b\}} W(G_{(i,k)}) + O \left( \frac{p^b}{n^{b+1}} \right). \]

The Lemmas 4.5 and 4.6 yield
\[ \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, b\}} W(G_{(i,k)}) = \# \{ T \in T_{[l+1]} \mid B(T) = [b] \} = \mathbb{1}_{b \leq l} l! \left( \frac{l-1}{b-1} \right). \]

Since this is zero for \( b = l + 1 \) and the sum \( \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, b\}} W(G_{(i,k)}) \) is clearly also zero for \( b = l + 1 \), because there are not enough vertices to be colored black, we can change the upper limit of the sum over \( b \) into \( l \wedge p \). We thus have
\[ \mathbb{E}[\text{tr}(S_{p,n}^l)] = \sum_{b=1}^{l \wedge p} \left[ \frac{1}{n!} \left( \frac{(l+1)^n}{n!} \right)^2 \right] \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, b\}} W(G_{(i,k)}) + O \left( \frac{p^b}{n^{b+1}} \right). \]

We now prove
\[ \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, b\}} W(G_{(i,k)}) = A(l,b) \]
\[ = \frac{b!(l-b)!}{2} \left( \frac{2l}{2b} \right) + (2b-1) \left( \frac{l}{b} \right)^2 \]
\[ + (\alpha - 3) \mathbb{1}_{b \leq l} b!(l-b)! \left( \frac{l}{b-1} \right) \left( \frac{l}{b+1} \right), \]

(6.2.1)

first for \( b = l \) and then for \( b < l \). By Proposition 6.1 and Lemmas 6.7 and 6.8 the only possible seed graph of \( G_{(i,k)} \) with positive weight for \( b = l \) is \( S(G_{(i,k)}) = G_{(i,v)} \in 2d\text{-Ring}_1 \) (for some \( v \in [l] \)), since there are no white vertices in \( G_{(i,k)} \) and thus no white vertices in the seed graph. By applying Corollary 5.3 for \( l_0 = 1, b' = l - 1 \) and \( w' = 0 \) we get
\[ \sum_{G_{(i,k)} \in C_I, 2l \mid B(G_{(i,k)}) = \{1, \ldots, l\}} W(G_{(i,k)}) = \# \{ G \in C_I, 2l \mid S(G) \in 2d\text{-Ring}_1, B(G) = [l] \} \]
\[
\begin{align*}
= 1! 0! \left( \binom{l}{l-1} \right) \binom{l}{0} = l! = A(l, l) + (\alpha - 3) B(l, l).
\end{align*}
\]

It remains to show \(0 \leq 2\) for \(b < l\). By Proposition 6.1 we have
\[
\sum_{G_{(i,k)} \in C_{l,2l}} W(G_{(i,k)}) = \sum_{G \in C_{l,2l}} W(G)
\]
\[
= \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring} \text{ for some } l_0 \neq 2, \ B(G) = [b] \} \\
+ \alpha \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}, \ B(G) = [b] \} \\
+ \# \{ G \in C_{l,2l} \mid S(G) \in 1\text{-}d\text{-Ring} \text{ for some even } l_0 \geq 4, \ B(G) = [b] \}.
\]

We add zero in the form of \(\pm \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}, \ B(G) = [b] \} \) three times. This will later make the expressions easier to handle. Heuristically this counteracts the exclusion of the case \(l_0 = 2\) from both other cardinalities. Write
\[
\sum_{G_{(i,k)} \in C_{l,2l}} W(G_{(i,k)})
\]
\[
= \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring} \text{ for odd } l_0, \ B(G) = [b] \} \\
+ \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring} \text{ for some even } l_0 \neq 2, \ B(G) = [b] \} \\
+ 2 \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}, \ B(G) = [b] \} \\
+ \# \{ G \in C_{l,2l} \mid S(G) \in 1\text{-}d\text{-Ring} \text{ for some even } l_0 \geq 4, \ B(G) = [b] \} \\
+ \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}, \ B(G) = [b] \} \\
+ (\alpha - 3) \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring} \text{ for even } l_0 \geq 4, \ B(G) = [b] \}.
\]

Let \(w := l - b\) be the number of white vertices in \(V_l\). We sum over all possible ring lengths \(l_0\), which are precisely those for which we have enough black and white vertices to satisfy the coloring from Lemmas 4.7 and 4.8 for \(S(G)\). For \(b < l\), i.e. \(w > 0\), we get
\[
\sum_{G_{(i,k)} \in C_{l,2l}} W(G_{(i,k)}) = \sum_{m=1}^{b} \sum_{m \leq w+1} \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}_{2m-1}, \ B(G) = [b] \} \\
+ \sum_{m=2}^{b} \sum_{m \leq w} \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}_m, \ B(G) = [b] \} \\
+ 2 \sum_{m=2}^{b} \sum_{m \leq w} \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}_m, \ B(G) = [b] \} \\
+ \sum_{m=2}^{b} \sum_{m \leq w} \# \{ G \in C_{l,2l} \mid S(G) \in 1\text{-}d\text{-Ring}_{2m}, \ B(G) = [b] \} \\
+ \sum_{m=2}^{b} \sum_{m \leq w} \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}_m, \ B(G) = [b] \} \\
+ \# \{ G \in C_{l,2l} \mid S(G) \in 2\text{-}d\text{-Ring}_m, \ B(G) = [b] \}.
\]

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+ (\alpha - 3) \# \{G \in C_{[l,2]} \mid S(G) \in 2\text{-d-Ring}_2, B(G) = [b]\}.

With $b' = b - \lceil \frac{b}{2} \rceil$, $w' = l - b - \lfloor \frac{b}{2} \rfloor$ and Corollaries 5.3 and 5.4 this becomes

$$\sum_{G_{(i,k)} \in C_{[l,2]} \atop L(G_{(i,k)}) = [b]} W(G_{(i,k)}) = b \wedge (w+1)$$

$$= \sum_{m=1}^{b \wedge (w+1)} (2m - 1) b! \cdot w! \left(\binom{l}{b-m} \left(\binom{l}{l} - \binom{l}{l-b-m+1}\right) + \binom{l}{b-1} \left(\binom{l}{l} - \binom{l}{l-b-1}\right)\right)$$

$$+ \sum_{m=1}^{b \wedge (l-b)} 2m b! \cdot w! \left(\binom{l}{b-m} \left(\binom{l}{l} - \binom{l}{l-b-m}\right) + \binom{l}{b-1} \left(\binom{l}{l} - \binom{l}{l-b-1}\right)\right)$$

$$+ \sum_{m=1}^{b \wedge (l-b+1)} b! \cdot w! \left(\binom{l}{b-m} \left(\binom{l}{l} - \binom{l}{l-b-m+1}\right) + \binom{l}{b-1} \left(\binom{l}{l} - \binom{l}{l-b-1}\right)\right)$$

$$b! \cdot w! \sum_{m=1}^{b \wedge (l-b+1)} (2m + 1) \left(\binom{l}{b-m} \left(\binom{l}{l} - \binom{l}{l-b+m}\right) + \binom{l}{b+1} \left(\binom{l}{l} - \binom{l}{l-b-1}\right)\right).$$

This can further be simplified thanks to Peter Taylor’s answer to our question on Math Overflow (see [9]). Peter Taylor shows that the upper two sums are together

$$\frac{b! \cdot w!}{2} \left(\frac{2l}{2b} + (2b - 1) \left(\binom{l}{b}\right)^2\right),$$

which means for $b < l$ we get

$$\sum_{G_{(i,k)} \in C_{[l,2]} \atop L(G_{(i,k)}) = [b]} W(G_{(i,k)}) = \frac{b! \cdot w!}{2} \left(\frac{2l}{2b} + (2b - 1) \left(\binom{l}{b}\right)^2\right) + (\alpha - 3) b! \cdot w! \left(\binom{l}{b-1} \left(\binom{l}{b+1}\right)\right).$$

$$=: A(l,b)$$

$$=: B(l,b)$$

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6.3. Corollary (When \( p_n \) goes to infinity)

Let \((p_n)_{n \in \mathbb{N}} \subset \mathbb{N}\) be a sequence with \( p_n \leq n \) and \( p_n \xrightarrow{n \to \infty} \infty \) such that

\[
\frac{p_n}{n} \xrightarrow{n \to \infty} y \in (0, 1].
\]

Then we have

\[
\mathbb{E}[\text{tr}(S_{p_n,n}^l)] = n \sum_{b=1}^{l} \left( \frac{p_n}{n} \right)^b \binom{l}{b-1} \left( \frac{l-1}{b-1} \right) \\
+ \sum_{b=1}^{l-1} \left( \frac{p_n}{n} \right)^b \left[ \frac{1}{2} \left( \frac{2l}{2b} \right) - \frac{1}{2} \left( \frac{l}{b} \right)^2 \right] + (\alpha - 3) \left( \frac{l}{b-1} \right) \left( \frac{l}{b+1} \right) + O\left( \frac{1}{n} \right).
\]

6.4. Corollary (When \( p \) is constant)

For constant \( p \in \mathbb{N} \) we can simplify the statement of Theorem 1 to

\[
\mathbb{E}[\text{tr}(S_{p,n}^l)] = p + \frac{pl}{2n} (\alpha(l-1) + lp - 2l - p + 2) + O\left( \frac{1}{n^2} \right).
\]
7. Approximating power-trace covariances

7.1. Definition (Double-circuit multigraph)

For any vertex set \( V_r \) we call a pair \((i_1, i_2)\) of sequences \( i_1 \in V_r^{N_1} \), \( i_2 \in V_r^{N_2} \) with \( \{i_1\} \cup \{i_2\} = V_r \) a \textit{double-route} over \( V_r \) with lengths \( N_1, N_2 \in \mathbb{N} \).

The pair of graphs \((G_{i_1}, G_{i_2})\), with both \( G_{i_1}, G_{i_2} \) as in Definition 3.4 is called a \textit{double-circuit multigraph} on \( V_r \) of lengths \( N_1, N_2 \). We interpret the pair as a single 'combined' directed multigraph defined by \((V_r, ([N_1] \times \{1\}) \sqcup ([N_2] \times \{2\}), f_{(i_1,i_2)})\), where

\[
f_{(i_1,i_2)}((\cdot, 1)) := f_{G_1} = f_{i_1} : [N_1] \to V_r \times V_r ; \ f_{(i_1,i_2)}((\cdot, 2)) := f_{G_2} = f_{i_2} : [N_2] \to V_r \times V_r .
\]

Let \( C_{V_l,N_l,N_2}^2 \) denote the set of all such double-circuit multigraphs.

The coloring of a double-circuit multigraph is then defined such that vertices are white, if they are white in both \( G_{i_1} \) and in \( G_{i_2} \), i.e.

\[
B((G_{i_1}, G_{i_2})) := B(G_{i_1}) \cup B(G_{i_2}) .
\]

Lemma A.4 allows us to canonically define the seed graph \( S((G_{i_1}, G_{i_2})) \) by removal of balanced leaves, since balanced leaves are only ever removed from either \( G_{i_1} \) or \( G_{i_2} \).

7.2. Definition (Double ring-type graphs)

For \( l_0 \geq 3 \) a double-circuit multigraph \((G_{i_1}, G_{i_2}) \in C_{V_l,N_l,N_2}^2\) will be called \textit{one-directional double ring-type} with ring-length \( l_0 \), if \( i_1 \) and \( i_2 \) are the same modulo starting position.

On the other hand for any \( l_0 \in \mathbb{N} \) a double-circuit multigraph \((G_{i_1}, G_{i_2}) \in C_{V_l,0}^2 \) will be called \textit{two-directional double ring-type} with ring-length \( l_0 \), if \( i_1 \) and the backward-sequence ((\(i_2\))_{l_0+1-i} \(i \leq l_0 \) are equal modulo starting position.

Both of these definitions already imply that each vertex also has in- and out-degree 1, that both graphs are exhaustive and that \( G_{i_1}, G_{i_2} \in C_{V_l,0} \). We could thus equally well have chosen \((G_{i_1}, G_{i_2})\) from \( C_{V_l,0} \times C_{V_l,0} \).

For even \( l_0 \geq 4 \) and any \((G_{i_1}, G_{i_2}) \in C_{V_l,0} \times C_{V_l,0} \) we say

\[
(G_{i_1}, G_{i_2}) \in \text{Double-1-d-Ring}_{l_0} ,
\]

if \((G_{i_1}, G_{i_2})\) is of one-directional double ring-type with ring-length \( l_0 \) and the starting positions of \( i_1 \) and \( i_2 \) are an even number of steps apart. Analogously for even \( l_0 \in \mathbb{N} \) we say

\[
(G_{i_1}, G_{i_2}) \in \text{Double-2-d-Ring}_{l_0} ,
\]

if \((G_{i_1}, G_{i_2})\) is of two-directional double ring-type with ring-length \( l_0 \) and the starting positions of \( i_1 \) and \( i_2 \) are also an even number of steps apart.
7.3. Lemma (Coloring of double ring-type graphs)

For even $l_0 \in \mathbb{N}$ and any $(G_{i_1}, G_{i_2}) \in C_{V_{l_0,0}, l_0, 2} \times C_{V_{l_0,0}, l_0, 2}$ with

$$(G_{i_1}, G_{i_2}) \in \text{Double-1-d-Ring}_{l_0} \quad \text{or} \quad (G_{i_1}, G_{i_2}) \in \text{Double-2-d-Ring}_{l_0}$$

the vertices will alternate between black and white along the ring-structure of $(G_{i_1}, G_{i_2})$.

Proof.

This trivially follows from the assumptions that $l_0$ is even and that the starting positions of the routes $i_1$ and $i_2$ are an even number of steps apart.

7.4. Lemma (Counting sprouting double ring-type graphs)

For even $l_0 \in \mathbb{N}$ and any $b_1', b_2', w_1', w_2' \geq 0$ define $l_1 := \frac{b_1'}{2} + b_1' + w_1'$, $l_2 := \frac{b_2'}{2} + b_2' + w_2'$ as well as $l = l_1 + l_2$. Let $B$ be a subset of $V_l$ such that $b := \#B = \frac{b_1'}{2} + b_1' + b_2'$, then the number of $(G_1 := G_{i_1}, G_2 := G_{i_2}) \in C_{V_{l_1,2l_1, 2l_2}}^{l_0}$ with $B(G) = B, S((G_1, G_2)) \in \text{Double-1-Ring}_{l_0}$ and the properties

$$b_1' = \#(V(G_1) \cap B) \setminus V(S(G_1)) \quad ; \quad b_2' = \#(V(G_2) \cap B) \setminus V(S(G_2))$$
$$w_1' = \#(V(G_1) \setminus B) \setminus V(S(G_1)) \quad ; \quad w_2' = \#(V(G_2) \setminus B) \setminus V(S(G_2))$$

(7.4.1)
is given by
\[
\mathbb{1}_{l_0 \geq 4} \frac{a}{2} b! w! \left( \begin{pmatrix} l_1 \\ b_1 \\ w_1 \\ \end{pmatrix} \left( \begin{pmatrix} l_2 \\ b_2 \\ w_2 \\ \end{pmatrix} \right) \right),
\]
where \( w = l - b \). Analogously the number of \((G_1, G_2) \in \mathbb{C}^2_{w, 2l_1, 2l_2}\) with \( B(G) = B \), \( S((G_1, G_2)) \in \text{Double-2-Ring}_{l_0}\) and \( (7.3.1)\) is also given by
\[
\frac{a}{2} b! w! \left( \begin{pmatrix} l_1 \\ b_1 \\ w_1 \\ \end{pmatrix} \left( \begin{pmatrix} l_2 \\ b_2 \\ w_2 \\ \end{pmatrix} \right) \right).
\]

7.5. Proposition (Covariance-weight of double graphs)

For any \( l_1, l_2 \in \mathbb{N} \) define \( l := l_1 + l_2 \) and a vertex set \( V_l \). For \( i, k \in V_{l_1}^l \) and \( j, m \in V_{l_2}^l \) with
\[
\#(\{i\} \cup \{k\} \cup \{j\} \cup \{m\}) \geq l
\]
we have
\[
\begin{align*}
E \left[ (Y_1 k_1 Y_{i_2 k_1}) \cdots (Y_1 k_1 Y_{i_1 k_1}) \cdot (Y_1 m_1 Y_{j_2 m_1}) \cdots (Y_1 m_1 Y_{j_1 m_1}) \right] \\
- E \left[ (Y_1 k_1 Y_{i_2 k_1}) \cdots (Y_1 k_1 Y_{i_1 k_1}) \right] E \left[ (Y_1 m_1 Y_{j_2 m_1}) \cdots (Y_1 m_1 Y_{j_1 m_1}) \right]
\end{align*}
\]
\[
= \begin{cases} 
0 & \text{if } \#(\{i\} \cup \{k\} \cup \{j\} \cup \{m\}) > l \\
\alpha - 1 & \text{if } S((G_{(i, k)}, G_{(j, m)})) \in \text{Double-2-d-Ring}_2 \\
1 & \text{if } S((G_{(i, k)}, G_{(j, m)})) \in \text{Double-1-d-Ring}_{l_0} \text{ for even } l_0 \geq 4 \\
1 & \text{if } S((G_{(i, k)}, G_{(j, m)})) \in \text{Double-2-d-Ring}_{l_0} \text{ for even } l_0 \geq 4 \\
0 & \text{else}
\end{cases}
\]

7.6. Proof of Theorem 2

Proof.

As all formulas in the formulation of Theorem 2 are symmetric in \( l_1 \) and \( l_2 \), without loss of generality assume \( l_1 \leq l_2 \). We again begin by using Lemma \( \text{A.3} \) to see
\[
E[\text{tr}(S_{p,n}^{l_1}) \text{ tr}(S_{p,n}^{l_2})] \\
= \sum_{i_1, \ldots, i_k = 1}^{l_1} \sum_{j_1, \ldots, j_k = 1}^{l_2} E \left[ (S_{p,n})_{i_1, j_2} \cdots (S_{p,n})_{i_{k-1}, j_1} (S_{p,n})_{i_k, j_k} \cdot (S_{p,n})_{j_1, j_2} \cdots (S_{p,n})_{j_{k-1}, j_k} (S_{p,n})_{j_k, j_1} \right] \\
= \frac{1}{n^{l_1+2}} \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{j \in \{1, \ldots, p\}^{l_2}} E \left[ (Y_{i_1, k_1} Y_{i_2, k_1}) \cdots (Y_{i_k, k_1} Y_{i_1, k_1}) \cdot (Y_{j_1, m_1} Y_{j_2, m_1}) \cdots (Y_{j_k, m_1} Y_{j_1, m_1}) \right] =: W_{i, k, j, m}.
\]
Using the same calculation as in Remark 2.2 we get
\[
\mathbb{E}[\text{tr}(S_{p,n}^l)] \mathbb{E}[\text{tr}(S_{p,n}^{l_2})] = \left( \frac{1}{n^{l_1}} \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{k \in \{1, \ldots, n\}^{l_1}} \mathbb{E} \left[ (Y_{i_1,k_1} Y_{i_2,k_1}) \ldots (Y_{i_{l_1},k_1}) \right] \right) \cdot \left( \frac{1}{n^{l_2}} \sum_{j \in \{1, \ldots, p\}^{l_2}} \sum_{m \in \{1, \ldots, n\}^{l_2}} \mathbb{E} \left[ (Y_{j_1,m_1} Y_{j_2,m_1}) \ldots (Y_{j_{l_2},m_2}) \right] \right) =: W_{i,k} W_{j,m},
\]
which implies
\[
\text{Cov}[\text{tr}(S_{p,n}^l), \text{tr}(S_{p,n}^{l_2})] = \frac{1}{n^{l_1+l_2}} \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{k \in \{1, \ldots, n\}^{l_1}} \sum_{j \in \{1, \ldots, p\}^{l_2}} \sum_{m \in \{1, \ldots, n\}^{l_2}} (W_{i,k,j,m} - W_{i,k} W_{j,m}).
\]
We once more split up the sum by number of occurring indexes and use the fact that renaming the indexes does not change the weights to similarly get
\[
\text{Cov}[\text{tr}(S_{p,n}^l), \text{tr}(S_{p,n}^{l_2})] = \frac{1}{n^{l_1+l_2}} \sum_{r=1}^{(2l_1+2l_2)\wedge n} \left( \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{k \in \{1, \ldots, n\}^{l_1}} \sum_{j \in \{1, \ldots, p\}^{l_2}} \sum_{m \in \{1, \ldots, n\}^{l_2}} (W_{i,k,j,m} - W_{i,k} W_{j,m}) \right)
\]
\[
= \frac{1}{n^{l_1+l_2}} \sum_{r=1}^{(2l_1+2l_2)\wedge n} \sum_{b=1}^{r \wedge p} \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{k \in \{1, \ldots, n\}^{l_1}} \sum_{j \in \{1, \ldots, p\}^{l_2}} \sum_{m \in \{1, \ldots, n\}^{l_2}} (W_{i,k,j,m} - W_{i,k} W_{j,m})
\]
\[
= \mathcal{O} \left( \frac{p^b}{n^{l_1+l_2}} \right), \text{ for } r < l_1 + l_2
\]
\[
\mathcal{O} \left( \frac{p^b}{n^{l_1+l_2}} \right) \sum_{r=1}^{(l_1+l_2)\wedge n} \sum_{b=1}^{r \wedge p} \sum_{i \in \{1, \ldots, p\}^{l_1}} \sum_{k \in \{1, \ldots, n\}^{l_1}} \sum_{j \in \{1, \ldots, p\}^{l_2}} \sum_{m \in \{1, \ldots, n\}^{l_2}} (W_{i,k,j,m} - W_{i,k} W_{j,m})
\]
\[
= \mathcal{O} \left( \frac{p^b}{n^{l_1+l_2}} \right), \text{ for } r < l_1 + l_2
\]
\[
\sum_{b=1}^{(l_1+l_2)\wedge p} \left[ \frac{{l_1 \choose b}{n-b \choose (l_1-l_2)-b}}{n^{l_1+l_2}} \sum_{i\in [l_1]^1, k\in [l_1+l_2]^1} \sum_{j\in [l_2]^2, m\in [l_1+l_2]^2} (W_{i,k,j,m} - W_{i,k}W_{j,m}) + O\left( \frac{p^b}{n^{b+1}} \right) \right].
\]

It remains to show
\[
\sum_{i\in [l_1]^1, k\in [l_1+l_2]^1} \sum_{j\in [l_2]^2, m\in [l_1+l_2]^2} (W_{i,k,j,m} - W_{i,k}W_{j,m}) = C(l_1, l_2, b) + (\alpha - 3)D(l_1, l_2, b)
\]
for all \( b \leq l_1 + l_2 \). We see by Proposition 7.5 that
\[
\sum_{i\in [l_1]^1, k\in [l_1+l_2]^1} \sum_{j\in [l_2]^2, m\in [l_1+l_2]^2} (W_{i,k,j,m} - W_{i,k}W_{j,m}) = (\alpha - 1)\#\{ (G_1, G_2) \in C^2_{[l_1+l_2],[l_1],[l_2]} \mid B((G_1, G_2)) = [b], \\
(S(G_1), S(G_2)) \in \text{Double-1-d-Ring}_2 \}
+ \#\{ (G_1, G_2) \in C^2_{[l_1+l_2],[l_1],[l_2]} \mid B((G_1, G_2)) = [b], \\
(S(G_1), S(G_2)) \in \text{Double-1-d-Ring}_{10} \text{ for even } l_0 \geq 4 \}
+ \#\{ (G_1, G_2) \in C^2_{[l_1+l_2],[l_1],[l_2]} \mid B((G_1, G_2)) = [b], \\
(S(G_1), S(G_2)) \in \text{Double-2-d-Ring}_{10} \text{ for even } l_0 \geq 4 \}.
\]

Next sum over all choices of \( b'_1, b'_2, w'_1, w'_2 \) from Lemma 7.3
\[
\sum_{i\in [l_1]^1, k\in [l_1+l_2]^1} \sum_{j\in [l_2]^2, m\in [l_1+l_2]^2} (W_{i,k,j,m} - W_{i,k}W_{j,m}) = (\alpha - 1) \sum_{b'_1=0\vee (b-1)}^{(l_1-1)\wedge (b-1)} \#\{ (G_1, G_2) \in C^2_{[l_1+l_2],[l_1],[l_2]} \mid B((G_1, G_2)) = [b], \\
(l_1-l_0) \wedge (b-\frac{1}{2}) \},
\]
\[
\sum_{4 \leq l_0 \leq 11 \wedge 2l_2 \leq 2l_1} \sum_{b'_1=0\vee (b-12)}^{(l_1-1)\wedge (b-12)} \#\{ (G_1, G_2) \in C^2_{[l_1+l_2],[l_1],[l_2]} \mid B((G_1, G_2)) = [b], \\
(l_1-l_0) \wedge (b-\frac{1}{2}) \}.
\]

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\[(\mathbb{R}^1)\) holds, \((S(G_1), S(G_2)) \in \text{Double-1-d-Ring}_{l_0}\}
\[+
\sum_{l_0 \text{ even}} \sum_{4 \leq l_0 \leq 2l_1 \land 2l_2 = 2l_1}
\mathbb{1}\{(l_1 \cdot \mathbb{R}) + (b - \frac{b}{2})\}
\# \{(G_1, G_2) \in C_{l_1+l_2, l_1}[l_1, l_2] \mid B(G_1, G_2) = [b]\},
\]

\[\text{(\mathbb{R}^2) holds, } (S(G_1), S(G_2)) \in \text{Double-2-d-Ring}_{l_0}\}\]

\[(\alpha - 1) b! w! \sum_{l_1, b', w} (l_1, b', w) \left( \begin{array}{ccc}
\frac{l_0}{2} + b' + w', & l_2 &= \frac{l_0}{2} + b' + w' , \quad b = \frac{l_0}{2} + b' + b' ,
\end{array} \right)
\]

The three equalities

\[l_1 = \frac{l_0}{2} + b' + w' , \quad l_2 = \frac{l_0}{2} + b' + w' , \quad b = \frac{l_0}{2} + b' + b' ,
\]

where the left hand sides and \(l_0\) are fixed, show that \(b', w', w'\) are already uniquely determined by \(b'\). More precisely we have

\[w' = l_1 - \frac{l_0}{2} - b' , \quad w' = l_2 - b + b' , \quad b' = b - \frac{l_0}{2} - b' .
\]

In addition to plugging in these equalities we also add zero twice in the form of

\[\pm b! w! \sum_{l_1, b', w} (l_1, b', w) \left( \begin{array}{ccc}
\frac{l_0}{2} + b' + w', & l_2 &= \frac{l_0}{2} + b' + w' , \quad b = \frac{l_0}{2} + b' + b' ,
\end{array} \right)
\]

As in the proof of Theorem 1 this heuristically counteracts the exclusion of the case \(l_0 = 2\) from the other two sums. The above formula becomes

\[
\sum_{\{i \in \mathbb{R}, k \in l_1 + l_2, m \in l_1 + l_2 \mid \{i, k\} \neq \{j, m\} = [l_1 + l_2] \mid i \neq j \neq k \}} (W_{i, k, j, m} - W_{i, k} W_{j, m})
\]

\[= (\alpha - 3) b! w! \sum_{l_1, b', w} (l_1, b', w) \left( \begin{array}{ccc}
\frac{l_0}{2} + b' + w', & l_2 &= \frac{l_0}{2} + b' + w' , \quad b = \frac{l_0}{2} + b' + b' ,
\end{array} \right)
\]
\[ + 2 b! w! \sum_{l_0 \text{ even}} \sum_{l_0 \leq 2l_1} \frac{(l_1 \wedge b)}{2} l_0 \left( \frac{l_1}{b'_1} \right) \left( l_1 - l_1 - \frac{l_0}{b'_1} \right) \left( b - \frac{l_0}{b'_1} - b'_1 \right) \left( l_2 - b + b'_1 \right) \]

\[ = (\alpha - 3) b! w! \sum_{b'_1 = 0 \vee (b - 2l_2)} (l_1 \wedge b') \left( l_1 \right) \left( l_1 - 1 - b'_1 \right) \left( b - 1 - b'_1 \right) \left( l_2 - b + b'_1 \right) \]

\[ + 2 b! w! \sum_{m=1}^{l_1} (l_1 \wedge m) \sum_{b'_1 = 0 \vee (b - 2l_2)} l_1 b'_1 \left( l_1 \right) \left( l_1 - m - b'_1 \right) \left( l_2 - m - b'_1 \right) \left( l_2 - b + b'_1 \right) . \]

where \( w = l_1 + l_2 - b \) denotes the total number of white vertices. Using the fact that the generalized binomial coefficient \( \binom{n}{k} \) is zero when \( k > n \), this expression must be equal to

\[ (\alpha - 3) b! w! \sum_{k=0}^{b-1} \binom{l_1}{k} \left( \frac{l_1}{k+1} \right) \left( \frac{l_2}{b-1-k} \right) \left( \frac{l_2}{b-k} \right) \]

\[ + 2 b! w! \sum_{k=0}^{b-1} \binom{l_1}{k} \left( \frac{l_1}{k-b} \right) \sum_{m=0}^{b-k} m \left( \frac{l_1}{k+m} \right) \left( \frac{l_2}{b-m-k} \right) . \]

\[ 7.7. \text{ Corollary (When } p_n \text{ goes to infinity)} \]

Let \( (p_n)_{n \in \mathbb{N}} \subset \mathbb{N} \) be a sequence with \( p_n \leq n \) and \( p_n \xrightarrow{n \to \infty} \infty \) such that

\[ \frac{p_n}{n} \xrightarrow{n \to \infty} y \in (0, 1] . \]

Then we have

\[ \text{Cov}[\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2})] = \sum_{b=1}^{l_1 + l_2} \left[ \frac{(\frac{p_n}{n})^b}{b!(l_1 + l_2 - b)!} (C(l_1, l_2, b) + (\alpha - 3)D(l_1, l_2, b)) + O \left( \frac{1}{n^2} \right) \right] . \]

\[ 7.8. \text{ Corollary (When } p \text{ is constant)} \]

For constant \( p \in \mathbb{N} \) we can simplify the statement of Theorem 2 to

\[ \text{Cov}[\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2})] = \frac{pl_1 l_2}{n} (\alpha - 1) + \frac{pl_1 l_2}{n^2} \left[ (p - 1)(l_1 - 1)(l_2 - 1) - (\alpha - 1)l_1 l_2 \right] + O \left( \frac{1}{n^3} \right) . \]
A. Simple lemmas

This property is fundamental to the link between trace moments and graph theory. It is needed in Remark 2.2.

A.1. Lemma (Expressing traces as sums over routes)

For any matrix $M \in \mathbb{R}^{p \times p}$ and any $l \in \mathbb{N}$ the property

$$\text{tr}(M^l) = \sum_{i_1, \ldots, i_l=1}^p M_{i_1, i_2} \cdots M_{i_{l-1}, i_l} M_{i_l, i_1}$$

holds.

Proof. For $l > 1$ we simply calculate

$$(M^l)_{i_1, i_1} = \sum_{i_2=1}^p M_{i_1, i_2} (M^{l-1})_{i_2, i_1} = \cdots = \sum_{i_2, \ldots, i_l=1}^p M_{i_1, i_2} \cdots M_{i_{l-1}, i_l} M_{i_l, i_1}.$$

This implies the formula

$$\text{tr}(M^l) = \sum_{i_1, \ldots, i_l=1}^p M_{i_1, i_2} \cdots M_{i_{l-1}, i_l} M_{i_l, i_1}.$$

Note that this is also true for $l = 1$.

The following lemma is used within many of our graph theoretical considerations.

A.2. Lemma (Visited vertices and cycles)

For any simple connected graph $G$ with $m$ edges and $n$ vertices we have $m + 1 = n$, if and only if $G$ has no cycles, i.e. is a tree.

Further we have $m = n$ if and only if $G$ has one cycle.

Proof.

This is a basic exercise in graph theory. The first property is easily seen by induction over $n$ or $m$ and the second property follows from the fact that adding any edge to a tree creates a graph with a single cycle, which works in the inverse direction as well: Removing an edge, which was part of the single cycle, yields a tree.
This lemma will be needed for all of our lemmas and propositions linking the weight of graphs to their cycle-structure, i.e. Lemma 4.5 and Propositions 6.1 and 7.5

A.3. Lemma (Leaves are balanced leaves)

For any \( G_i \in \mathcal{C}_{V,N} \) a leaf of \( U(G) \), where the undirected connection consists of less than 4 edges, is a balanced leaf of \( G \).

Proof.

If the number of edges to the connection were odd, then the in-degree and out-degree of the balanced leaf can not be equal and \( G_i \) can not be an Eulerian graph, which stands in opposition to its construction (see 3.4). It follows that there must be two edges to the undirected connection and by the same argument these two edges must point in opposite directions, making them a balanced edge pair.

The following lemma allows us to define seed graphs of pairs \((G_1, G_2)\) of graphs defined in Definition 7.1.

A.4. Lemma (Balanced leaves of double-circuit graphs)

For any \((G_1, G_2) \in \mathcal{C}_{V_1,2l_1,2l_2}^2\), where \( l = l_1 + l_2 \), the following two statements are equivalent.

a) \( v_j \) is a balanced leaf of \((G_1, G_2)\)

b) \( v_j \) is a balanced leaf of one of the graphs \( G_1, G_2 \) and is not visited by the other graph

Proof.

The direction \( a \Rightarrow b \) follows directly from the fact that the two edges in the edge pair of a balanced leaf \( v_j \) of \((G_1, G_2)\) must belong to the same graph, either \( G_1 \) or \( G_2 \). Since there are no other edges of \((G_1, G_2)\) connected to \( v_j \), the vertex \( v_j \) is not visited by the other graph. The direction \( b \Rightarrow a \) follows easily from the definitions.

This is needed in the proof of Proposition 7.5

A.5. Lemma (Characterizing sprouting double ring-type graphs)

For any \((G_1, G_2) \in \mathcal{C}_{V_1,2l_1,2l_2}^2\), where \( l = l_1 + l_2 \), and \( l_0 > 2 \) the following two statements are equivalent.

a) \( S((G_1, G_2)) \in \text{Double-1-Ring}_{l_0} \)

b) \( (S(G_1), S(G_2)) \in \text{Double-1-Ring}_{l_0} \) and the sets

\[ V'_1 := V(G_1) \setminus V(S(G_1)) \quad \text{and} \quad V'_2 := V(G_2) \setminus V(S(G_2)) \]

are disjoint.
The same holds for Double-2-Ring\textsubscript{0} instead of Double-1-Ring\textsubscript{0}.

**Proof.**
In order to prove the direction \( a \Rightarrow b \) we show beforehand that \( (a) \) implies \( V'_1 \cup V'_2 = \emptyset \). If there were a \( v_j \in V'_1 \cup V'_2 \), then \( v_j \) would be a sprouting vertex of both \( G_1 \) and \( G_2 \). In this case \( v_j \) can never be a balanced leaf when trimming \( (G_1, G_2) \) down to \( S((G_1, G_2)) \) and thus must still be in \( S((G_1, G_2)) \) together with all four edges (one balanced edge pair from each graph \( G_1, G_2 \)) connected to \( v_j \). Since we had assumed \( l_0 > 2 \), this is not possible when \( S((G_1, G_2)) \) is supposed to be a ring-type graph of length \( l_0 \).

We have thus shown \( (a) \Rightarrow V'_1 \cup V'_2 = \emptyset \).

For the equivalence of \( (a) \) and \( (b) \) it now suffices to show that under the assumption \( V'_1 \cap V'_2 = \emptyset \) the double-circuit multigraphs \( S((G_1, G_2)) \) and \( (S(G_1), S(G_2)) \) are equal. This follows in part from the fact that Lemma A.4 tells us that we can not trim \( (G_1, G_2) \) any further down than \( (S(G_1), S(G_2)) \), since balanced leaves will always either belong to the left or right graph and \( S(G_1), S(G_2) \) both have no balanced leaves. And also in part from the fact that Lemma A.4 tells us that any vertices from \( V' \) will also be removed when trimming \( (G_1, G_2) \) down to \( S((G_1, G_2)) \). The same holds for vertices from \( V'_2 \). \( \Box \)

**B. Technical lemmas**

This rather involved technical lemma is only needed for its final statement that knowledge of the tree structure and the seed graph already contains all the knowledge of the graph, i.e. its route. We apply this in the proof of Lemma 5.1.

**B.1. Lemma (Routes of sprouting graphs)**

For any \( G_{i_0} \in \mathcal{C}_{V_0, 2l_0} \) and \( G_i \in \mathcal{C}_{V_0 \cup V'_1 \cup V'_2, 2l_0 + 2r} \) such that \( S(G_i) = G_{i_0} \), the route \( i \) can be written in the form

\[
\begin{align*}
&\text{length: } m_0 \quad \text{length: } m_1 \quad \text{length: } m_{2l_0} \\
= &((i_1, \ldots, i_k), (i_0)_1, (i_0)_2, \ldots, (i_0)_2l_0, (i_1, \ldots, i_{2l_0 + 2r})), \\
= &:(j_1^1, \ldots, j_{m_0}^1), =:f_1 =:(j_1^2, \ldots, j_{m_1}^2), =:f_2 =:=f_{2l_0} =:(j_1^{2l_0}, \ldots, j_{m_{2l_0}}^{2l_0})
\end{align*}
\]

where \( j_k^q \) are the entries of \( i \), which are removed in the trimming of \( G_i \) down to \( G_{i_0} \) (not just the sprouting vertices). The lengths \( m_k \) can clearly also be zero. For this form the following properties hold:

1) \( m_k \) is even for all \( 0 \leq k \leq 2l_0 \)

2) \( j_k^q \in V'_1 \cup \{ (i_0)_k \} \) for all \( 0 < k < 2l_0 \) and \( q \leq m_k \)

3) \( j_k^q \in V'_1 \cup \{ (i_0)_1, (i_0)_2l_0 \} \) for \( k \in \{0, 2l_0 \} \) and all \( j \leq m_k \)

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4) In $(f_{2l_0}, j_1^{2l_0}, \ldots, j_m^{2l_0}, j_1^0, \ldots, j_m^0, f_1)$ there exists exactly one neighboring pair $((i_0)_{2l_0}, (i_0)_1)$ and all $j^*_l$ left of this pair are from $V'_l \cup \{(i_0)_{2l_0}\}$, while all $j^*_l$ right of this pair are from $V'_l \cup \{(i_0)_1\}$. This is also true, when $(i_0)_1 = (i_0)_{2l_0}$ or $m_0 = 0 = m_{2l_0}$.

These four properties together imply that knowledge of $i_0 = (f_1, \ldots, f_{2l_0})$ as well as the positions of all $V'_l$-elements already uniquely determines $i$.

Proof.

1) Removal of a balanced leaf from $i$ always removes two neighboring entries and, since the $f_1, f_2, \ldots$ are not removed, the two removed entries must always be in the same $(j_1^1, \ldots, j_m^1)$-block.

2) Since $k < 2l_0$, when removing two entries by removal of a balanced leaf, an entry of the neighbor of the balanced leaf will remain on the left hand side of the two removed entries. By an inductive argument over the iterative removal of leaves (starting at $i_0$ when all leaves are removed) it is seen that $j_1^k, \ldots, j_m^k$ can only consist of elements of $V'_l$ (which is the set of leaves removed while trimming) and of occurrences of $(i_0)_k$, since this is the entry on the very left of the block $(j_1^1, \ldots, j_m^1)$.

3) Consider $(j_1^{2l_0}, \ldots, j_m^{2l_0}), (j_1^0, \ldots, j_m^0)$ a single block and repeat the argument from (2). This time the remaining neighbor of a removed leaf can be on the left or right hand side.

4) The pair $((i_0)_{2l_0}, (i_0)_1)$ must be in the block $(f_{2l_0}, j_1^{2l_0}, \ldots, j_m^{2l_0}, j_1^0, \ldots, j_m^0, f_1)$, since the corresponding edge is in $G_{i_0}$ and thus also in the sprouting graph $G_i$. Analogously there must only be a single pair of the form $((i_0)_*, (i_0)_*)$ in the block, since there is just a single such pair in the corresponding part of the seed graph $(f_{2l_0}, f_1)$ and edges between vertices of the seed graph are not removed when trimming.

Since neither $(i_0)_{2l_0}$ nor $(i_0)_1$ can be a balanced leaf and the pair $((i_0)_{2l_0}, (i_0)_1)$ can be found in the block through the entire process of trimming the route, the pair acts as a mobile block-delimiter in the sense of (2) or (3) meaning all non-$V'_l$-elements right of this delimiter must be $(i_0)_1$ and all left of this delimiter must be $(i_0)_{2l_0}$. \hfill \□

This next property is needed in the proof of the highly influential Proposition 3.2.

B.2. Lemma (Spanning trees of $K_{b+1,w+1}$ with the edge $(a_1, c_1)$)

For any $b, w \geq 0$ and given $d_1, \ldots, d_{b+1}, e_1, \ldots, e_{w+1} \in \mathbb{N}$ with the properties

$$d_1 + \ldots + d_{b+1} = b + w + 1 = e_1 + \ldots + e_{w+1}$$

\[\text{[B.2]}\]
let $S$ denote the set of spanning trees $t$ of $K_{b+1,w+1}$ such that $t$ contains the edge $(a_1, c_1)$ and

$$\forall i \leq b + 1 : \ deg_t(a_i) = d_i$$
$$\forall j \leq w + 1 : \ deg_t(c_j) = e_j.$$  \hfill (B.2.2)

The cardinality of $S$ is

$$\#S = \binom{b}{e_1 - 1, \ldots, e_{w+1} - 1} \binom{w}{d_1 - 1, \ldots, d_{b+1} - 1} \times \begin{cases} 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{bw}, & \text{if } w, b > 0 \\ 1, & \text{if } w = 0 \text{ or } b = 0 \end{cases}.$$  

By symmetry the same holds if we prescribe any edge $(a_k, c_l)$ instead of $(a_1, c_1)$. We only need to replace $d_1$ and $e_1$ in our formulas with $d_k$ and $e_l$.

**Proof.**

We split this proof into three cases.

1) Suppose $b = 0$ or $w = 0$:

In this case one side of $K_{b+1,w+1}$ has only a single vertex, meaning there is only one possible spanning tree. This spanning tree contains the edge $(a_1, c_1)$ and thus the formula holds.

2) Suppose either $d_1 = 1$ or $e_1 = 1$, while $b, w > 0$:

Without loss of generality assume $d_1 = 1$, then $a_1$ must be a leaf. Let $c_j$ denote its neighbor. The tree is only counted, if $j = 1$. Removal of the leaf $a_1$ (and relabeling the vertices $a_2, \ldots, a_{b+1}$ into $a_1, \ldots, a_b$) then defines a bijection between the set of all spanning trees $t$ of $K_{b+1,w+1}$ with $(B.2.2)$, where the leaf $a_1$ is connected to $c_1$, and the set of all spanning trees $\hat{t}$ of $K_{b,w+1}$ with

$$\forall i \leq b : \ deg_{\hat{t}}(a_i) = d_{i+1}$$
$$\forall 1 < j \leq w + 1 : \ deg_{\hat{t}}(c_j) = e_j$$

and $\deg_{\hat{t}}(c_1) = e_1 - 1$. By (2.2) of [7] there are

$$= \binom{b - 1}{e_1 - 1, \ldots, e_{w+1} - 1} \binom{w}{d_1 - 1, \ldots, d_{b+1} - 1} \times \begin{cases} 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{bw}, & \text{if } w, b > 0 \\ 1, & \text{if } w = 0 \text{ or } b = 0 \end{cases}$$

such trees.
3) Suppose both $d_1$ and $e_1$ are larger than 1 and $b, w > 0$:
Without loss of generality assume $b \geq w$, it is then easily seen that at least one element of $d_1, \ldots, d_{b+1}$ must have value 1 and by assumption this element is not $d_1$. Without loss of generality assume $d_{b+1} = 1$ and let $S_q$ denote the set of all spanning trees $t$ of $K_{b+1,w+1}$ with (B.2.2), where the edge $(a_1, c_1)$ exists and $c_q$ is the only neighbor of $a_{b+1}$, which is a leaf since $d_{b+1} = 1$.

In order to use an inductive argument over the total number of vertices $N = w + b + 2$, we first need to show that removing the vertex $a_{b+1}$ can only land us in the cases (2) or (3) and not in case (1). We could only land in case (1), if $b = 1$ and thus also $w = 1$ hold, as we had assumed both $b, w > 0$ and $b \geq w$. However for $b = 1 = w$, we can not have $d_1 > 1$ and $w_1 > 1$ simultaneously, since then (B.2.1) could not hold. It follows that we can not land in case (1) by removing the leaf $a_{b+1}$.

We now inductively prove the formula
\[
\#S = \left( \begin{array}{c} b \\ e_1 - 1, \ldots, e_{w+1} - 1 \end{array} \right) \left( \begin{array}{c} w \\ d_1 - 1, \ldots, d_{b+1} - 1 \end{array} \right) \left( 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{bw} \right),
\]
which is consistent with case (2), but not case (1). The above argument allows us to use case (2) as the start to the induction (effectively assume $b \geq 2$) and we only need to do the inductive step.

For each $q \leq w + 1$ we similarly as in (2) have a bijection between $S_q$ and the set of spanning trees $\hat{t}$ of $K_{b,w+1}$ with connection $(a_1, c_1)$,
\[
\forall i \leq b : \text{deg}_t(a_i) = d_i
\]
\[
\forall j \leq w + 1, j \neq q : \text{deg}_t(c_j) = e_j
\]
and $\text{deg}_t(c_q) = e_q - 1$ by removing the leaf $a_{b+1}$. By inductive assumption we thus know the cardinality of $S_q$ to be
\[
\#S_q = \left( \begin{array}{c} b - 1 \\ e_1 - 1, \ldots, e_q - 2, \ldots, e_{w+1} - 1 \end{array} \right) \left( \begin{array}{c} w \\ d_1 - 1, \ldots, d_b - 1 \end{array} \right) \left( 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{(b - 1)w} \right)
\]
for $q \neq 1$ and
\[
\#S_1 = \left( \begin{array}{c} b - 1 \\ e_1 - 2, \ldots, e_{w+1} - 1 \end{array} \right) \left( \begin{array}{c} w \\ d_1 - 1, \ldots, d_b - 1 \end{array} \right) \left( 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{(b - 1)w} \right).
\]

It follows that
\[
\#S = \sum_{q=1}^{w+1} \#S_q = S_1 + \sum_{q=2}^{w+1} \#S_q
\]
\[
= \left( \begin{array}{c} b - 1 \\ e_1 - 2, \ldots, e_{w+1} - 1 \end{array} \right) \left( \begin{array}{c} w \\ d_1 - 1, \ldots, d_b - 1 \end{array} \right) \left( 1 - \frac{(b - e_1 + 1)(w - d_1 + 1)}{(b - 1)w} \right).
\]
between a black and a white vertex. We prove, by induction over

\[ \text{Proof.} \]

\[ \text{C.1. Proof of Lemma 4.4} \]

\[ \text{C. Proofs} \]

\[ 2) \]

\[ 1) \]

\[ \text{For } l = 1 \text{ there are only two vertices } \{v_1, v_2\} = V_2 = V_{l+1} \text{ and two edges } \{e_1, e_2\} = [2] = \{1, 2\}. \text{ The only two possible balanced trees are then } G_{(v_1, v_2)} \text{ and } G_{(v_2, v_1)}. \text{ In both cases the sets } \{i_1\} \text{ and } \{i_2\} \text{ are disjoint.} \]

\[ 2) l - 1 \rightarrow l; \]

\[ \text{For } l > 1 \text{ any balanced tree } G_i \in \mathcal{T}_{V_{l+1}} \text{ has a balanced leaf } v_j. \text{ Let } \tilde{G} = G_{i'} \in \mathcal{T}_{\tilde{V}_l} \text{ be the version of } G_i \text{ with } v_j \text{ removed. By inductive assumption we know that the sets } \{i'_1, i'_3, ..., i'_{2l-3}\} \text{ and } \{i'_2, i'_4, ..., i'_{2l-2}\} \text{ are disjoint. Let } s \in \{1, 2, ..., l + 1\} \text{ be the index of } v_s, \text{ the only neighbor of the balanced leaf } v_j \text{ in } G_i, \text{ then by construction of } \tilde{i}' \text{ from } i \text{ in Definition 5.13 we know that } \{i_1, i_3, ..., i_{2l-1}\} \text{ and } \{i_2, i_4, ..., i_{2l}\} \text{ must also be disjoint, where } v_j \text{ is in the set, in which } v_s \text{ is not}. \]

By \[ \text{1) } \] the sum \[ \sum_{q=2}^{w+1} e_q \] must be equal to \[ b + w + 1 - e_1 \] and the expression in the square bracket is calculated to be \[ 1 - \frac{(b-e_1+1)(w-d_1+1)}{b w}. \]
C.2. Proof of Lemma 4.5

Proof.
In (3.7.2) we had seen that
\[ W(G_{(i,k)}) = \prod_{i,j=1}^{l} E\left[Y_{i,j}^{A(R(G_{(i,k)}))_{i,j}}\right], \]
which means that for \( W(G_{(i,k)}) \) to not be zero, each \( Y_{i,j} \) in \( W(G_{(i,k)}) \) must occur at least twice (or not at all). Analogously to our argument in Definition 3.9 we can use Lemma A.2 to see that \( U(G_{(i,k)}) \) must be a tree on the vertices \( V_{l+1} \). Since \( G_{(i,k)} \) then has \( l \) many undirected connections with at least two of the \( 2l \) total edges, each undirected connection must have exactly two edges and Lemma A.3 then already implies that \( G_{(i,k)} \) is a balanced tree. We have thus shown that \( W(G_{(i,k)}) \neq 0 \) for \( G_{(i,k)} \in C_{l+1, 2l} \) already implies that \( G_{(i,k)} \) is a balanced tree. It remains to show that all balanced trees have weight 1.

Lemma 4.4 tells us that each vertex is either tail of only odd numbered edges (black) or tail of only even numbered edges (white). Since each balanced edge-pair \( (e_k, e_k') \) of the balanced tree \( G_{(i,k)} \) must be between a black and a white vertex, it consist of an even and an odd numbered edge. It follows that all edges of the reversed \( G = R(G_{(i,k)}) \) occur precisely twice, when \( G_{(i,k)} \) is a tree, meaning
\[ W(G_{(i,k)}) = E[Y_{11}^2] = V[Y_{11}]^l = 1^l = 1. \]

C.3. Proof of Lemma 4.6

Proof.
Let \( w := l + 1 - b \) denote the number of white vertices. We start by renaming the vertices \( v_1, ..., v_{l+1} \) into \( a_1, ..., a_b \) and \( c_1, ..., c_w \), such that \( B = \{a_1, ..., a_b\} \). This renaming is uniquely defined, if we require the ordering of both \( a_1, ..., a_b \) and \( c_1, ..., c_w \) to remain the same as their \( v_i \)-counterparts. From Lemma 4.4 we know that each balanced tree has only edges between a black and a white vertex. This means the adjacency matrix \( A(G) \) is also the adjacency matrix of a spanning tree of the (undirected) complete bipartite graph \( K_{b,w} \), where the vertices \( a_1, ..., a_b \) are interpreted as being on the left hand side and \( c_1, ..., c_w \) on the right hand side. We now count the number of balanced trees \( T \in T_{V_{l+1}} \) with \( B(T) = B \) by summing over all possible adjacency matrices and then using Lemma 3.10 to count the number of trees per matrix. Let \( S \) denote the set of adjacency matrices of spanning trees of \( K_{b,w} \), where the enumeration of the vertices \( \{a_1, ..., a_b, c_1, ..., c_w\} \) can be taken from their old names \( \{v_1, ..., v_{l+1}\} \). We then get
\[ \#\{T \in T_{V_{l+1}} \mid B(T) = B\} = \sum_{A \in S} \#\{T \in T_{V_{l+1}} \mid A(T) = A, B(T) = B\}. \]
In Lemma 3.10 we had seen, that the number of \( T \in T_{V_{l+1}} \) with \( A(T) = A \) is given by
\[ 2l \prod_{i=1}^{l+1} (\text{indeg}(A(v_i)) - 1)! \cdot \]
However not every such balanced tree has the right coloring. If the starting edge goes from right to left, then only the right hand vertices will be black and otherwise only the left hand vertices will be black. Since we are looking for the trees where the left hand side is black, only half of the $2l \prod_{i=1}^{l+1} (\text{indeg}_A(v_i) - 1)!$ many trees with adjacency matrix $A$ will be counted. We have thus shown

$$\#\{T \in \mathcal{T}_{l+1} \mid A(T) = A, B(T) = B\} = l \prod_{i=1}^{l+1} (\text{indeg}_A(v_i) - 1)! .$$

Since this number is only dependent on the (in-)degree of each vertex and not the adjacency matrix itself, we can rewrite our original sum into

$$\#\{T \in \mathcal{T}_{l+1} \mid B(T) = B\} = \frac{(w-1)!}{b!} \prod_{i=1}^{b} (d_i - 1)! \prod_{j=1}^{w} (e_j - 1)! \times \#\{A \in S \mid \text{indeg}_A(a_i) = d_i, \forall i \leq b, \text{indeg}_A(c_j) = e_j, \forall j \leq w\} .$$

A formula for the right most expression is given in equation (2.2) of [7] and it fortunately has a very useful form:

$$\#\{A \in S \mid \text{indeg}_A(a_i) = d_i, \forall i \leq b, \text{indeg}_A(c_j) = e_j, \forall j \leq w\} = \frac{(w-1)!}{b!} \prod_{i=1}^{b} (d_i - 1)! \prod_{j=1}^{w} (e_j - 1)! .$$

This allows us to calculate

$$\#\{G \in \mathcal{T}_{l+1} \mid B(G) = B\} = l (w-1)! (b-1)! \left( \sum_{d_1, \ldots, d_b \geq 1} \frac{1}{d_1 + \ldots + d_b = l} \right) \left( \sum_{e_1, \ldots, e_w \geq 1} \frac{1}{e_1 + \ldots + e_w = l} \right)$$

$$= l (w-1)! (b-1)! \#\{(d_1, \ldots, d_b) \in \mathbb{N}^b \mid d_1 + \ldots + d_b = l\} \times \#\{(e_1, \ldots, e_w) \in \mathbb{N}^w \mid e_1 + \ldots + e_w = l\}$$

$$= l (w-1)! (b-1)! \left( \frac{l-1}{b-1} \right) \left( \frac{l-1}{w-1} \right) !^{b+w+l+1} \left( b-1 \right) .$$

### C.4. Proof of Lemma 4.8

**Proof.**

The case $l_0 = 1$ is trivial, since there is only one vertex and this vertex must be black.
For $l_0 = 2$ the only two choices elements of 2-d-Ring$_2$ are $G_{(v_1,v_2,v_3,v_2)}$ and $G_{(v_2,v_1,v_2,v_1)}$, which both satisfy the property, since one of the two vertices is black and the other white. From now on assume $l_0 \geq 3$.

Any routes $i \in V_{2l_0}^2$ of two-directional ring-type graphs must be of the form that there is some ordering $(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(l_0)})$ of the vertices $V_{l_0}$ and a ’turning point’ $k \in [l_0]$ such that

\[ i = (v_{\sigma(1)}, \ldots, v_{\sigma(k)}) + (v_{\sigma(k-1)}, \ldots, v_{\sigma(1)}) + (v_{\sigma(l_0)}, \ldots, v_{\sigma(k)}) + (v_{\sigma(k+1)}, \ldots, v_{\sigma(l_0)}) \]

walking forward
walking backward
walking forward

where + denotes concatenation of the sequences and the ordering is the order of the vertices in the ring structure starting at the first vertex of the route. For $k > 1$ the order is in the direction of the first edge and for $k = 1$ the order is in precisely the opposite direction. The grayed out starting vertex at the end is not part of the route, but is just to illustrate which edges (between vertices) are pointing forward along the ring structure and which are pointing backward.

Finding all black colored vertices $B(G_i)$ is now as simple as listing every vertex in $i$, which occurs at an odd position.

1. case) $l_0$ and $k$ are even:
   The first two blocks of $i$ are
   \[ (v_{\sigma(1)}, \ldots, v_{\sigma(k)}) + (v_{\sigma(k-1)}, \ldots, v_{\sigma(1)}) \]
   and clearly contribute $v_{\sigma(1)}, v_{\sigma(3)}, \ldots, v_{\sigma(k-1)}$ to $B(G_i)$. Since the first two blocks together always have an odd length and the next two blocks are
   \[ (v_{\sigma(l_0)}, \ldots, v_{\sigma(k)}) + (v_{\sigma(k+1)}, \ldots, v_{\sigma(l_0)}) \]
   these two blocks must contribute $v_{\sigma(l_0-1)}, v_{\sigma(l_0-3)}, \ldots, v_{\sigma(k+1)}$ to $B(G_i)$. We have thus shown $B(G_i)$ to contain exactly all vertices with odd positions in the ring structure.

2. case) $l_0$ is even and $k$ odd:
   With the same arguments as in the first case we get
   \[ B(G_i) = \{v_{\sigma(1)}, v_{\sigma(3)}, \ldots, v_{\sigma(k)}\} \cup \{v_{\sigma(l_0-1)}, v_{\sigma(l_0-3)}, \ldots, v_{\sigma(k)}\} \]
   and have again shown $B(G_i)$ to contain exactly all vertices with odd positions in the ring structure.

3. case) $l_0$ and $k$ are odd:
   With the same arguments as in the first case we get
   \[ B(G_i) = \{v_{\sigma(1)}, v_{\sigma(3)}, \ldots, v_{\sigma(k)}\} \cup \{v_{\sigma(l_0-1)}, v_{\sigma(l_0-3)}, \ldots, v_{\sigma(k+1)}\} \]
   and see that this time there is exactly one pair $v_{\sigma(k)}, v_{\sigma(k+1)}$ of neighboring vertices, which are both black.
4. case) $l_0$ is odd and $k$ even:

With the same arguments as in the first case we get

$$B(G_i) = \{v_{\sigma(1)}, v_{\sigma(3)}, \ldots, v_{\sigma(k-1)}\} \cup \{v_{\sigma(l_0-1)}, v_{\sigma(l_0-3)}, \ldots, v_{\sigma(k)}\}$$

and see that this time there is exactly one pair $v_{\sigma(k-1)}, v_{\sigma(k)}$ of neighboring vertices, which are both black.

\[\square\]

C.5. Proof of Lemma 5.1

Proof.

We define a bijection $\Phi$ from the sprouting graphs of $G_0 = G_{t_0}$ to the sprouting graphs of $\overline{G}_0 = G_{\overline{t}_0}$, which preserves coloring of the sprouting vertices $V'_i$. In Lemma [B.1] we see, that for any $G_i \in C_{V_r \cup V'_i, 2l_0 + 2l'}$ such that $S(G_i) = G_{t_0}$ the route $i \in (V_r \cup V'_i)^{2l + 2l'}$ is of the form

$$\sigma = (i_1, \ldots, i_s, (i_0)_1, \ldots, i_s, (i_0)_2, \ldots, (i_0)_m, i_s, \ldots, i_2l + 2l')$$

where $j^*_k \in V_r \cup V'_i$ are the entries of $i$, which are removed in the trimming of $G_i$ down to $G_{t_0}$. The rules (2-4) of Lemma [B.1] show that we can uniquely determine the value of all those $j^*_k \in V_{t_0}$, which are seed vertices, by their position in $i$ and knowledge of $f_1, f_2, \ldots \in V_{t_0}$. Define the new route $\Psi_{t_0, \overline{t}_0}(i) \in (\overline{V}_r \cup \overline{V}'_i)^{2l + 2l'}$ by inserting the new seed vertices $(\overline{i}_0)_1, \ldots, (\overline{i}_0)_m \in \overline{V}_{t_0}$ into the positions of $f_1, \ldots, f_m$ and by replacing all $j^*_k \in V_{t_0}$, which are seed vertices by elements of $\overline{V}_{t_0}$ such that rules (2-4) of Lemma [B.1] still apply to $\Psi_{t_0, \overline{t}_0}(i)$. Since entries with values in $V'_i$ are not changed and equality-relations between entries were not changed in the modification of the route, we know that $\Psi_{t_0, \overline{t}_0}(i)$ must be the route of a sprouting graph of $G_{\overline{t}_0}$, i.e. $S(G\Psi_{t_0, \overline{t}_0}(i)) = G_{\overline{t}_0}$.

This can be seen with an inductive argument over the iterative removal of balanced leaves, since balanced leaves of $G_i$ are also balanced leaves of $G\Psi_{t_0, \overline{t}_0}(i)$. We can now define

$$\Phi_{G_0, \overline{G}_0} : \text{Sp}_{V'_i, V'_{2l'}}(G_0) \to \text{Sp}_{\overline{V}'_i, \overline{V}'_{2l'}}(\overline{G}_0)$$

$$G_i \mapsto G_{\Psi_{t_0, \overline{t}_0}(i)}$$

This $\Phi_{G_0, \overline{G}_0}$ must be injective, since its inverse is $\Phi_{\overline{G}_0, G_0}$. By construction $\Phi_{G_0, \overline{G}_0}$ does not change the positions of sprouting vertices $V'_i$ and thus does not change their color.  \[\square\]
C.6. Proof of Proposition 5.2

Proof.
For any $G \in C_{V_2 \cup V'_2,2l_0+2l'}$ with $S(G) = G_0$ the vertex $v_1$ must be black and $v_2$ must be white, since this is their coloring in $G_0$ and Lemma 4.3 tells us that the coloring does not change when trimming $G$ down to $G_0$. This means $G$ as described above has a total of $b' + 1$ black and $w' + 1$ white vertices.

The reason we have assumed $v_1, v_2$ to be the largest vertices, is because for $l_0 = 1$ any $G$ as above will be a balanced tree, which means the ordering of its vertices influences the seed graph of the tree. Assuming $v_1, v_2$ to be the largest vertices guarantees that $S(G)$ will be $G_{(v_1,v_2)}$.

In the case for $l_0 = 1$ we are looking for the number of balanced trees $T \in T_{V_2 \cup V'_2}$ with $B(T) = B' \sqcup \{v_1\}$ and an edge pair between $v_1$ and $v_2$.

We use the same method as in Lemma 4.6 to count the number of such $G$ for arbitrary $l_0$. Let

$$P := \{ A(G) \in \mathbb{N}_0^{(b'+1) \times (w'+1)} \mid \ldots \}
\quad G \in C_{V_2 \cup V'_2,2l_0+2l'}, S(G) = G_0 \}

be the set of all adjacency matrices of sprouting versions of $G_0$ (note that we for this ignore the coloring), then we clearly have a bijection to the set

$$S := \{ A(T) \mid T \in T_{V_2 \cup V'_2}, A(T)_{v_1,v_2} = 1 \} ,$$

where the bijective mapping $\Phi : P \to S$ simply subtracts $l_0 - 1$ from the entries $A_{v_1,v_2}$ and $A_{v_2,v_1}$. Similarly to the proof of Lemma 4.6, we now define a renaming of the vertices $\{v_1\} \sqcup B'$ into $a_1, \ldots, a_{b'+1}$ and $\{v_1\} \sqcup W'$ into $c_1, \ldots, c_{w'+1}$, where the ordering of the two sub-sets is preserved. Since $v_1, v_2$ were assumed to be the largest vertices, we have $a_{b'+1} := v_1$ and $c_{w'+1} := v_2$. With this renaming we may (as in the proof of Lemma 4.6) use Lemma 4.4 to see that a matrix $A \in S$ is an adjacency matrix of a balanced tree $T$, with $B(T) = \{v_1\} \sqcup B'$, if and only if it is also the adjacency matrix of an (undirected) spanning tree of the complete bipartite graph $K_{b'+1,w'+1}$. We thus get that all $A \in S$ are adjacency matrices of spanning trees of $K_{b'+1,w'+1}$ with an edge between $a_{b'+1}$ and $c_{w'+1}$, i.e.

An element of $\text{Spr}_{B',W'}(G_0)$ for $B' = \{v'_3, v'_5, v'_6, v'_7\}$, $W' = \{v'_1, v'_2, v'_4\}$ and $G_0$ as above with $l_0 = 3$. (edge labels not drawn)

Spanning tree of $K_{b'+1,w'+1}$ with adjacency matrix $\Phi(A(G))$, where $G$ is the sprouting graph above.

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\[ S_{B'} := \{ A(T) \mid T \in \mathcal{T}_{v_2 \cup B' \cup W'}, \ A(T)_{v_1, v_2} = 1, \ B(T) = \{ v_1 \} \cup B' \} \]
\[ = \{ A(t) \mid t \text{ spanning tree of } K_{v'+1, w'+1}, \ A(t)_{a_{v'+1}, c_{w'+1}} = 1 \} . \]

Since we also have \[ S_{B'} = \{ \Phi(A(G)) \mid G \in \text{Spr}_{B', W'}(G_0) \} , \]
we can find all adjacency matrices of such \( G \) as we are counting by finding all spanning trees of \( K_{v'+1, w'+1} \) with a connection between \( a_{v'+1} \) and \( c_{w'+1} \). For this to be useful, we first calculate the number of valid \( G \), with the same adjacency matrix.

For any fixed \( G' \in \text{Spr}_{B', W'}(G_0) \) with \( A(G') = A \) the number of Euler circuits through \( G' \) is by the B.E.S.T. Theorem equal to

\[
\frac{l}{l_0!} \prod_{v \in V_{G'} \cup B' \cup W'} (\deg_{G'}(v) - 1)! .
\]

As Euler circuits are counted modulo starting edge and only every second edge in \( G' \) originates from a black vertex, giving the wanted coloration, there are

\[
(l_0 + l') l \prod_{v \in V_{G'} \cup B' \cup W'} (\deg_{A}(v) - 1)!
\]

many closed Euler walks through \( G' \) starting at a black vertex. The routes of closed Euler walks through \( G' \) correspond to all possible routes \( i \) such that \( A(G_i) \in S_{B'} \) and \( B(G_i) = \{ v_1 \} \cup B' \). Since we have \( l_0 \) many edges in both directions between \( v_1 \) and \( v_2 \) and closed Euler walks through \( G' \) maintain edge-identity, there must be \( l_0!l_0! \) many closed Euler walks through \( G' \) with the same route, for each route as above. It follows that

\[
\# \{ G \in \text{Spr}_{B', W'}(G_0) \mid A(G) = A \}
= \frac{(l_0 + l') l_0}{(l_0 - 1)! l_0!} \prod_{v \in V_{G'} \cup B' \cup W'} (\deg_{A}(v) - 1)!
= \frac{l_0 + l'}{(l_0 - 1)! l_0!} (\deg_{\Phi(A)}(v_1) + l_0 - 2)! (\deg_{\Phi(A)}(v_2) + l_0 - 2)! \prod_{v \in B' \cup W'} (\deg_{\Phi(A)}(v) - 1)!
\]

for each \( A \in \Phi^{-1}(S_{B'}) \). Since \( \Phi \) is a bijection and this expression depends only on the degrees of the vertices, and not the adjacency matrix itself, we can sum over all possible degrees. Define \( d_j := \deg_{\Phi(A)}(a_j) \) for \( j \leq b' + 1 \) and \( e_j := \deg_{\Phi(A)}(c_j) \) for \( j \leq w' + 1 \), then we get

\[
\# \text{Spr}_{B', W'}(G_0)
= \sum_{A \in \Phi^{-1}(S_{B'})} \frac{l_0 + l'}{(l_0 - 1)! l_0!} \prod_{v \in \{ v_1, v_2 \} \cup B' \cup W'} (\deg_{A}(v) - 1)!
\]

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\[
= \frac{l_0 + l'}{(l_0 - 1)!l_0!} \sum_{A \in S_{l_0'}} (\deg_A(v_1) + l_0 - 2)! (\deg_A(v_2) + l_0 - 2)! \prod_{v \in B \cup W} (\deg_A(v) - 1)!
\]

\[
= \frac{l_0 + l'}{(l_0 - 1)!l_0!} \sum_{d_1, \ldots, d_{l_0'} \in \mathbb{N}} \sum_{e_1, \ldots, e_{l_0'} \in \mathbb{N}} (d_{l_0' + 1} + l_0 - 2)! (e_{l_0' + 1} + l_0 - 2)! \prod_{j=1}^{l_0'} (d_j - 1)! \prod_{j=1}^{l_0'} (e_j - 1)!
\]

\[
\times \#\{A \in S_{l_0'} \mid \forall j \leq b' + 1 : \deg_A(a_j) = d_j, \forall j \leq w' + 1 : \deg_A(c_j) = e_j, A_{a_{l_0'} + 1, c_{w' + 1}} = 1\}
\]

\[
= \frac{l_0 + l'}{(l_0 - 1)!l_0!} \sum_{d_1, \ldots, d_{l_0'} \in \mathbb{N}} \sum_{e_1, \ldots, e_{l_0'} \in \mathbb{N}} (d_{l_0' + 1} + l_0 - 2)! (e_{l_0' + 1} + l_0 - 2)! \prod_{j=1}^{l_0'} (d_j - 1)! \prod_{j=1}^{l_0'} (e_j - 1)!
\]

\[
\times \# \{t \text{ spanning tree of } K_{b' + 1, w' + 1} \mid \forall j \leq b' + 1 : \deg_t(a_j) = d_j, \\
\forall j \leq w' + 1 : \deg_t(c_j) = e_j, A_{t_{a_{l_0'} + 1, c_{w' + 1}} = 1}\}
\]

\[
= \frac{(l_0 + l')b'!w'}{(l_0 - 1)!l_0!} \sum_{d_{l_0'+1} \in \mathbb{N}} \sum_{e_{l_0'+1} \in \mathbb{N}} (d_{l_0' + 1} + l_0 - 2)! (e_{l_0' + 1} + l_0 - 2)! \prod_{j=1}^{l_0'} (d_j - 1)! (e_j - 1)!
\]

\[
\times \left(1 - \prod_{b', w' > 0} (\frac{(b' - e_{w' + 1} + 1)(w' - d_{b' + 1} + 1)}{w'b'})\right) \left(b' - 1\right) \left(w' - 1\right)
\]

\[
= \frac{(l_0 + b' + w')b'!w'}{(l_0 - 1)!l_0!} \sum_{m=1}^{w' + 1} \sum_{n=1}^{w' + 1} \frac{(m + l_0 - 2)!}{(m - 1)!} (\frac{n + l_0 - 2)!}{(n - 1)!} (\frac{b' + w' - m}{b' - 1}) (\frac{b' + w' - n}{w' - 1})
\]

\[
\times \left(1 - \prod_{b', w' > 0} (\frac{(b' + w' - m + 1)(w' - m + 1)}{w'b'})\right)
\]

\[
= \frac{(l_0 + b' + w')b'!w'}{(l_0 - 1)!l_0!} \sum_{m=1}^{w' + 1} \sum_{n=1}^{w' + 1} \frac{(m + l_0 - 2)!}{(m - 1)!} (\frac{n + l_0 - 2)!}{(n - 1)!} (\frac{b' + w' - m}{b' - 1}) (\frac{b' + w' - n}{w' - 1})
\]

\[
\times \left(1 - \prod_{b', w' > 0} (\frac{(b' + w' - n + 1)(w' - m + 1)}{w'b'})\right)
\]

\[
= \frac{(l_0 + b' + w')b'!w'}{(l_0 - 1)!l_0!} \sum_{m=1}^{w' + 1} \sum_{n=1}^{w' + 1} \frac{(m + l_0 - 2)!}{(m - 1)!} (\frac{n + l_0 - 2)!}{(n - 1)!} (\frac{b' + w' - m}{b' - 1}) (\frac{b' + w' - n}{w' - 1})
\]

\[
\times \left(1 - \prod_{b', w' > 0} (\frac{(b' + w' - n + 1)(w' - m + 1)}{w'b'})\right)
\]
By Lemma 4.8 there will be \(|b| w! – m\) choices for the ordering of the white vertices of \(S(G)\) are uniquely defined and have cardinalities \(w\) many choices for which vertices make up \(S(G)\). Since the sets \(B' = B \setminus S(G)\) and \(W' = V \setminus ((S(G) \cup B')\) are uniquely defined and have cardinalities \(b'\) and \(w'\) respectively, we may now apply Proposition 5.2 for fixed \(S(G)\) to see that there are

\[
\frac{(l_0 + b' + w')^2}{(l_0 + b')!(l_0 + w')!}
\]

many \(G\) to each chosen \(S(G)\). In total we have counted

\[
\left(\begin{array}{c}
\frac{b'}{2} \\
\frac{w}{2}
\end{array}\right) \left(\begin{array}{c}
w' + \frac{w_0}{2} \\
\frac{w}{2}
\end{array}\right) \left(\begin{array}{c}
\frac{l_0}{2} \\
\frac{b'}{2}
\end{array}\right) \frac{l_0 l_0}{2} \frac{(l_0 + b' + w')^2}{(l_0 + b')!(l_0 + w')!} = b! w! \left(\begin{array}{c}
l' \\
b'
\end{array}\right) \left(\begin{array}{c}
l' \\
w'
\end{array}\right)
\]

many \(G\) with the properties described above. \(\square\)

C.8. Proof of Corollary 5.4

Proof. By Lemma 4.8 there will be \(\left\lceil \frac{b'}{2} \right\rceil\) black and \(\left\lfloor \frac{w}{2} \right\rfloor\) white vertices in \(S(G)\), meaning there are \(\left\lceil \frac{b'}{2} \right\rceil \left\lfloor \frac{w}{2} \right\rfloor\) many choices for which vertices make up \(S(G)\). The form of the route of \(S(G)\) was discussed in the proof of Lemma 4.8. We have \(\left\lceil \frac{w}{2} \right\rceil\) choices for the ordering of the black vertices in the route, \(\left\lfloor \frac{w}{2} \right\rfloor\) choices for the ordering of the white vertices of the route and \(l_0\) choices where to initially change direction. Since the sets
\[ B' = B \setminus S(G) \text{ and } W' = V_i \setminus (S(G) \cup B') \text{ are uniquely defined and have cardinalities } b' \text{ and } w' \text{ respectively, we may now apply Proposition 5.2 to see that there are} \]
\[
\frac{(l_0 + b' + w')^2}{(l_0 + b')!(l_0 + w')!}
\]
many \( G \) to each chosen \( S(G) \). In total we have counted
\[
\left( \binom{b'}{\frac{l_0}{2}} \right) \left( \binom{w'}{\frac{l_0}{2}} \right) l_0 \left[ \frac{l_0}{2}! \right] \frac{(l_0 + b' + w')^2}{(l_0 + b')!(l_0 + w')!} = l_0 b! w! \binom{l}{b} \binom{l}{w'}
\]
many \( G \) with the properties described above.

For \( l_0 = 2 \) the only change in our argument is that there are not \( l_0 \left[ \frac{l_0}{2}! \right] \left[ \frac{l_0}{2}! \right] = 2 \) many choices for \( S(G) \) but only 1 choice.

C.9. Proof of Proposition 6.1

Proof.

In (3.7.2) we had already seen that
\[
W(G_{(i,k)}) := \mathbb{E} \left[ (Y_{i_1} k_1 Y_{i_2} k_1) \cdots (Y_{i_{l-1}} k_{l-1} Y_{i_l} k_l) (Y_{i_l} k_l Y_{i_1} k_1) \right] = \prod_{i,j=1}^{l} \mathbb{E} \left[ Y_{v_i, v_j}^{A(R(G_{(i,k)}))_{v_i, v_j}} \right]
\]
and we thus need to show the equality
\[
W(G) := \prod_{i,j=1}^{l} \mathbb{E} \left[ Y_{v_i, v_j}^{A(R(G))_{v_i, v_j}} \right] = \begin{cases} \alpha, & \text{if } S(G) \in 2\text{-d-Ring} \\ 1, & \text{if } S(G) \in 2\text{-d-Ring}_{l_0} \text{ for some } l_0 \neq 2 \\ 1, & \text{if } S(G) \in 1\text{-d-Ring}_{l_0} \text{ for even } l_0 \geq 4 \\ 0, & \text{else} \end{cases} \tag{6.1.1}
\]
for all \( G \in \mathcal{C}_{V_l, 2l} \). Since the two edges in an edge pair belonging to a balanced leaf must be of opposing parity, they will point in the same direction in the reversed graph \( R(G) \) and thus not make a difference for the left hand side, as \( \mathbb{E}[Y_{v_i, v_j}^2] = 1 \). The right hand side is clearly also invariant under the removal of balanced leaves, which means it suffices to show property (6.1.1) for all \( G \in \mathcal{C}_{V_l, 2l} \) without balanced leaves (thus implying \( G = S(G) \) and \( l = l_0 \)).

For this we first observe that \( W(G) \neq 0 \) is only possible, if no edge in \( R(G) \) occurs only twice. This means each undirected connection in \( G \) must consist of at least two edges and, since there are only \( 2l \) edges in \( G \), there must be no more than \( l \) undirected edges in \( U(G) \). Lemma A.2 tells us that there are then less than 2 cycles in \( U(G) \) and we make a distinction between the number of cycles. It is worth keeping in mind that we have presumed \( W(G) \neq 0 \).
1. case) There are 0 cycles in $U(G)$:
Since $U(G)$ has no cycles, it is a tree. However, $G$ is not allowed to have balanced leaves, which by Lemma A.3 means every undirected connection in $G$ must consist of at least 4 edges. The $2l_0$ edges of $G$ can then only make up at most $\left\lfloor \frac{l_0}{2} \right\rfloor$ many undirected connections, meaning $U(G)$ is a tree with $l_0$ vertices and $\left\lfloor \frac{l_0}{2} \right\rfloor$ edges, which is only possible for $l_0 = 2$. It now easily follows that $G$ is a two-directional ring-type graph of length 2.
In this case all 4 edges of $R(G)$ point in the same direction and we have $W(G) = E[Y_{v_i v_j}^4] = \alpha$.

2. case) There is 1 cycle in $U(G)$:
This is, as already seen by Lemma A.2, only possible, if there are $l_0$ edges in $U(G)$, meaning each undirected connection in $G$ must consist of exactly two edges, as none may consist of one edge, when $W(G) \neq 0$. It follows by Lemma A.3 that leaves of $U(G)$ are balanced leaves of $G$ and, as we had assumed $G$ to have no balanced leaves, $U(G)$ must also have no leaves. The only possibility for $U(G)$, which has no leaves and one cycle, is then a cycle graph of length $l_0 \neq 2$. (For $l_0 = 2$ there are no simple cycle graphs and $l_0 = 1$ ia allowed since we count a self-loop as a cycle.)
It is easily seen that the only way for $U(G)$ to be a cycle graph of length $l_0 \neq 2$ is for $G$ to be a one- or two-directional ring-type graph of length $l_0 \neq 2$. However not all such ring-type graphs have $W(G) \neq 0$. Analogously to Lemmas 4.7 and 4.8 we (for $l_0 \neq 2$) see that $W(G) = 1$, if and only if $G$ is either a one-directional ring-type graph of even length $l_0$ or $G$ is a two-directional ring-type graph of any length. Otherwise the weight is 0.

\[ \text{Proof of Corollary 6.3} \]

The map $(p, n) \mapsto \binom{n}{b} \binom{n-b}{r-b}$ is a polynomial of the form
\[
\binom{p}{b} \binom{n-b}{r-b} = \frac{1}{b!(r-b)!} p^b n^{r-b} + c_1(b, r) p^{b-1} n^{r-b} + c_2(b, r) p^b n^{r-b-1}
\]
\[ + \sum_{s+t \leq r-2 \atop s,b \leq n} c_{s,t}(b, r) p^s n^t
\]
\[ = \mathcal{O}(n^{r-2}) \text{ for } p \leq n
\]
for
\[
c_1(b, r) = \frac{1}{b!(r-b)!} \sum_{i=0}^{b-1} -i = -\frac{b(b-1)}{2b!(r-b)!}
\]
\[
c_2(b, r) = \frac{1}{b!(r-b)!} \sum_{i=0}^{r-b-1} -b - i = -\frac{(r-b)(b+r-1)}{2b!(r-b)!}.
\]

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We plug this into the result of Theorem 1 to see

\[
E[\text{tr}(S_{p_n, n}^l)] = \sum_{b=1}^{l \wedge p_n} \left( \frac{p_n}{n} \right) \frac{1}{b!} \frac{(l - 1)}{n^l} b \left( \frac{n - b}{l - b} \right) l! \left( \frac{1}{b!} \right) (A(l, b) + (\alpha - 3)B(l, b)) + O \left( \frac{p_n}{n^{b+1}} \right)
\]

\[
= \sum_{b=1}^{l} \left[ \frac{1}{b!(l + 1 - b)!} b_n^l l^{l+1-b} + c_1(b, l + 1) p_n^{b-1} b_n l^{l+1-b} + c_2(b, l + 1) p_n^b l^b \right] l! \left( \frac{1}{b!} \right) (A(l, b) + (\alpha - 3)B(l, b)) + O \left( \frac{1}{n} \right)
\]

\[
= n \sum_{b=1}^{l} \frac{p_n^b}{b} \left( \frac{l}{b} \right) \left( \frac{l - 1}{b - 1} \right) + \sum_{b=1}^{l} c_1(b, l + 1) \left( \frac{p_n}{n} \right)^{b-1} l! \left( \frac{l - 1}{b - 1} \right)
\]

\[
+ \sum_{b=1}^{l} c_2(b, l + 1) \left( \frac{p_n}{n} \right)^b l! \left( \frac{l - 1}{b - 1} \right)
\]

\[
+ \sum_{b=1}^{l} \frac{p_n^b}{b!(l - b)!} (A(l, b) + (\alpha - 3)B(l, b)) + O \left( \frac{1}{n} \right)
\]

It now remains to show that

\[
\sum_{b=1}^{l} c_1(b, l + 1) \left( \frac{p_n}{n} \right)^{b-1} l! \left( \frac{l - 1}{b - 1} \right) + \sum_{b=1}^{l} c_2(b, l + 1) \left( \frac{p_n}{n} \right)^b l! \left( \frac{l - 1}{b - 1} \right)
\]

\[
+ \sum_{b=1}^{l} \frac{p_n^b}{b!(l - b)!} A(l, b)
\]  \hspace{1cm} \text{(6.3.1)}

is equal to

\[
\sum_{b=1}^{l-1} \left( \frac{p_n^b}{n} \right) \left[ \frac{1}{2} \left( \frac{2l}{2b} \right) - \frac{1}{2} \left( \frac{l}{b} \right)^2 \right].
\]  \hspace{1cm} \text{(6.3.2)}

By shifting the index of the first sum and plugging in the formulas for \(c_1, c_2\) and \(A(l, b)\), the expression (6.3.1) is computed to be

\[
- \sum_{b=1}^{l} \left( \frac{p_n}{n} \right)^b \left( \frac{l}{b} \right)^2 b + \frac{1}{2} \sum_{b=1}^{l} \left( \frac{p_n}{n} \right)^b \left( \frac{2l}{2b} \right) + (2b - 1) \left( \frac{l}{b} \right)^2)
\]

It is now easily verified that the \(b = l\) parts of both sums cancel each other out and this is precisely (6.3.2).
C.11. Proof of Corollary 6.4

Proof.

We first easily see

\[ E[\text{tr}(S_{p,n}^l)] = \sum_{b=1}^{l/p} \left[ \frac{p}{n^l} \binom{n}{b-1} l! \left( \binom{l-1}{b-1} + \frac{p}{n^l} \binom{n}{l-b} \left( A(l, b) + (\alpha - 3)B(l, b) \right) + \mathcal{O} \left( \frac{1}{n^{b+1}} \right) \right] \]

\[ = \binom{n-1}{l-1} \frac{1}{n^l} l! \left( \binom{l-1}{0} + \frac{1}{n^l} \binom{n-1}{l-1} \left( A(l, 1) + (\alpha - 3)B(l, 1) \right) \right) + \mathcal{O} \left( \frac{1}{n^2} \right). \]

Since the polynomial \( n \mapsto \binom{n}{l} \) is of the form

\[ \frac{1}{l!} (n-1)(n-2) \ldots (n-l) = \frac{1}{l!} n^l - \frac{l(l+1)}{2!} n^{l-1} + \mathcal{O}(n^{l-2}), \]

this becomes

\[ E[\text{tr}(S_{p,n}^l)] = p - \frac{pl(l+1)}{2n} + \frac{p}{(l-1)!n} \left( A(l, 1) + (\alpha - 3)B(l, 1) \right) \]

\[ + \frac{p(p-1)l!(l-1)}{2n(l-1)!} + \mathcal{O} \left( \frac{1}{n^2} \right). \]

By plugging in

\[ A(l, 1) = \frac{1!(l-1)!}{2!} \left( \binom{2l}{2} + (2-1) \binom{l}{1} \right) = \frac{1}{2}(3l-1) \]

and

\[ B(l, 1) = \frac{1}{l-1} l! (l-1)! \binom{l}{l-1} \binom{l}{1+1} = \frac{1}{2}(l-1) \]

we see

\[ E[\text{tr}(S_{p,n}^l)] = p - \frac{pl(l+1)}{2n} + \frac{n}{(l-1)!n} \left( (3l-1) + (\alpha - 3)(l-1) \right) \]

\[ + \frac{p(p-1)l!(l-1)}{2n(l-1)!} + \mathcal{O} \left( \frac{1}{n^2} \right) \]

\[ = p + \frac{pl}{2n} \left[ - (l+1) + ((3l-1) + (\alpha - 3)(l-1)) + (p-1)(l-1) \right] + \mathcal{O} \left( \frac{1}{n^2} \right) \]

\[ = p + \frac{pl}{2n} \alpha(l-1) + lp - 2l - p + 2 \] + \mathcal{O} \left( \frac{1}{n^2} \right). \]

\[ \square \]
C.12. Proof of Lemma 7.4

Proof.

For \( l_0 > 2 \) Lemma 7.3 tells us that we can instead count the number of \((G_1, G_2) \in C_{V_1,2l_1,w_1}^2\) with \( B((G_1, G_2)) = B \) and \((S(G_1), S(G_2)) \in \text{Double-1-Ring}_{l_0}\) such that

\[
V'_1 := V(G_1) \setminus V(S(G_1)) \quad \text{and} \quad V'_2 := V(G_2) \setminus V(S(G_2))
\]

are disjoint. This allows us to first choose the sets \( V'_1, V'_2 \) and the order the vertices of \((S(G_1), S(G_2))\) are walked through before counting the possibilities for \(G_1\) and \(G_2\) separately.

By Lemma 7.3 there will be \( \frac{l_0}{2} \) many black and white vertices respectively in \( V(S(G_1)) = V(S(G_2)) \). Let \( B'_1 := (V(G_1) \cap B) \setminus V(S(G_1)) \) and \( B'_2 := (V(G_2) \cap B) \setminus V(S(G_2)) \) denote the sets of black sprouting vertices (from \( V'_1 \cup V'_2 \)) and analogously let \(W'_1 := (V(G_1) \setminus V(S(G_1))) \) and \( W'_2 := (V(G_2) \setminus V(S(G_2))) \) denote the sets of white sprouting vertices (from \( V'_1 \cup V'_2 \)). By construction we have \( V'_1 = B'_1 \cup W'_1 \) and \( V'_2 = B'_2 \cup W'_2 \).

For given \( b'_1 := \# B'_1, w'_1 := \# W'_1; \ i \in \{1, 2\} \) there are clearly \( (\frac{l_0}{w}, b'_1, b'_2) \) many ways to distribute the total number of \( b \) black vertices and \( (\frac{l_0}{w}, w'_1, w'_2) \) ways to distribute the total number of \( w \) white vertices. After the vertices are distributed, there are \( \frac{l_0}{w!} \) choices for the order of the black vertices \( B \cap V(S(G_1)) \) in the route of \( S(G_1) \) and likewise \( \frac{l_0}{w!} \) choices for the order of the white vertices \( V(S(G_1)) \setminus B \) in the route of \( S(G_1) \). After the choice of the route of \( S(G_1) \), there are also \( \frac{l_0}{w!} \) choices for the starting point of the route of \( S(G_2) \).

The routes of \( S(G_1) \) and \( S(G_2) \) with \((S(G_1), S(G_2)) \in \text{Double-1-Ring}_{l_0}\) are then uniquely determined. Lastly from Proposition 5.2 we know that there are

\[
\frac{(\frac{l_0}{w} + b'_1 + w'_1)!^2}{(\frac{l_0}{w} + b'_1)!(\frac{l_0}{w} + w'_1)!}
\]

many choices for \(G_1\) given \(S(G_1)\) and

\[
\frac{(\frac{l_0}{w} + b'_2 + w'_2)!^2}{(\frac{l_0}{w} + b'_2)!(\frac{l_0}{w} + w'_2)!}
\]

many choices for \(G_2\) given \(S(G_2)\). With \( l_1 = \frac{l_0}{w} + b'_1 + w'_1 \) and \( l_2 = \frac{l_0}{w} + b'_2 + w'_2 \) we count

\[
\left( \frac{b}{b'_1, b'_2, \frac{l_0}{w}} \right) \left( \frac{w}{w'_1, w'_2, \frac{l_0}{w}} \right) \left( \frac{l_0}{2} \right) \left( \frac{l_0}{2} \right) \left( \frac{\frac{l_0}{w} + b'_1 + w'_1)!^2}{(\frac{l_0}{w} + b'_1)!(\frac{l_0}{w} + w'_1)!} \right) \left( \frac{\frac{l_0}{w} + b'_2 + w'_2)!^2}{(\frac{l_0}{w} + b'_2)!(\frac{l_0}{w} + w'_2)!} \right)
\]

\[
= \frac{l_0}{2} \frac{b! w!}{b'_1! b'_2! w'_1! w'_2!} \left( \frac{l_1}{l'_1} \right) \left( \frac{l_1}{l'_1} \right) \left( \frac{l_2}{l'_2} \right) \left( \frac{l_2}{l'_2} \right)
\]

many \((G_1, G_2)\) as described above.
For \( l_0 = 2 \) we need to use a different way of counting the graphs, since Lemma 3.6 does not hold for \( l_0 = 2 \). The reasons for this are addressed in Remark 3.13 as the \((G_1, G_2)\) we are looking for are pairs of balanced trees. Instead we can use Lemma 4.6 and a simple symmetry argument.

As before there are \( \left( \begin{smallmatrix} b \vspace{1mm} \end{smallmatrix} b' \vspace{1mm} \end{smallmatrix} \right) \) many ways of distributing the black vertices and \( \left( \begin{smallmatrix} w' \vspace{1mm} \end{smallmatrix} w' \vspace{1mm} \end{smallmatrix} \right) \) ways of distributing the white vertices. Let \( \tau_1 \) denote the black vertex, which is in neither \( B'_1 \) nor \( B'_2 \). Analogously let \( \tau_2 \) denote the white vertex, which is in neither \( W'_1 \) nor \( W'_2 \). The seed graph is then necessarily \( S((G_1, G_2)) = (G_{(\tau_1, \tau_2)}, G_{(\tau_1, \tau_2)}) \), since this is the only option from Double-2-d-Ring2 with the right coloring. The set of possible choices for \((G_1, G_2)\) with this seed graph and \( B'_1, B'_2, W'_1, W'_2 \) as defined in (Lemma 1) is then

\[
\{G_1 \in \mathcal{T}(\tau_1, \tau_2) \cup B'_1 \cup W'_1 \mid B(G_1) = \{\tau_1\} \cup B'_1, \exists \text{ connection between } \tau_1, \tau_2\}
\times \{G_2 \in \mathcal{T}(\tau_1, \tau_2) \cup B'_2 \cup W'_2 \mid B(G_2) = \{\tau_1\} \cup B'_2, \exists \text{ connection between } \tau_1, \tau_2\}.
\]

In Lemma 4.6 we had seen that

\[
\#\{G_1 \in \mathcal{T}(\tau_1, \tau_2) \cup B'_1 \cup W'_1 \mid B(G_1) = \{\tau_1\} \cup B'_1\} = l_1! \left( \frac{b'_1 - 1}{b'_1} \right).
\]

A simple symmetry argument now yields that the proportion of these with a connection between \( \tau_1 \) and \( \tau_2 \) is \( \frac{l_1}{(w_1 + 1)(w_2 + 1)} \), i.e. the number of connections in \( G_1 \) divided by the number of edges in the complete bipartite graph \( K_{b'+1,w'+1} \). With the identities

\[
b = b'_1 + b'_2 + 2 \quad ; \quad w = w'_1 + w'_2 + 1
\]

\[
l_1 = b'_1 + w'_1 + 1 \quad ; \quad l_2 = b'_2 + w'_2 + 1 \quad ; \quad l = l_1 + l_2 = b + w
\]
it follows that the cardinality we are looking for is

\[
\left( \begin{smallmatrix} b \vspace{1mm} \end{smallmatrix} b' \vspace{1mm} 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} w \vspace{1mm} \end{smallmatrix} w' \vspace{1mm} 1 \end{smallmatrix} \right) \frac{l_1}{b'_1 + 1 + 1} \frac{l_1!}{b'_1} \frac{l_2}{(b'_2 + 1)(w'_2 + 1)} \frac{l_2!}{b'_2} \frac{b'_2 - 1}{b'_2} = b! w! \left( \begin{smallmatrix} l_1 \vspace{1mm} \end{smallmatrix} b'_1 \end{smallmatrix} \right) \left( \begin{smallmatrix} l_2 \vspace{1mm} \end{smallmatrix} b'_2 \end{smallmatrix} \right).
\]

C.13. Proof of Proposition 7.5

\textbf{Proof.}

We first make three important observations.

\textbf{i)} If \( G_{(i,k)} \) and \( G_{(j,m)} \) do not share an undirected connection, then \( R(G_{(i,k)}) \) and \( R(G_{(j,m)}) \) do not share an edge, which implies

\[
W_{i,j,k,m} = \prod_{i,j=1}^{l} \mathbb{E} \left[ Y_{i,j}^{A(R(G_{(i,k)}))_{i,j} + A(R(G_{(j,m)}))_{i,j}} \right].
\]
Thus for

\[ W((G_{i,k}, G_{j,m})) := W_{i,k,j,m} - W_{i,k} W_{j,m} \neq 0 \]

to hold, we know that \( U(G_{i,k}) \) and \( U(G_{j,m}) \) must share an edge, which also implies that \( U((G_{i,k}), G_{j,m})) \) is connected.

ii) Since balanced leaves of \((G_1, G_2)\) can only ever be visited by one of the two graphs \(G_1\) and \(G_2\), removing balanced leaves of does not change the weight-difference, i.e.

\[ W((G_1, G_2)) = W(\tilde{G_1}, \tilde{G_2}) = ... = W(S((G_1, G_2))) \]

where \(S((G_1, G_2))\) is the version with balanced leaf removed.

iii) The supposition \( W((G_{i,k}, G_{j,m})) \neq 0 \) implies that each undirected connection in \((G_{i,k}, G_{j,m}))\) must consist of two or more edges, since otherwise \( W_{i,k,j,m} = 0 \) and also \( W_{i,k} W_{j,m} = 0 \), since one of the graphs \(G_{i,k}, G_{j,m})\) must then too have an undirected connection with only one edge.

As we had assumed \(#\{(i) \cup \{k\} \cup \{j\} \cup \{m\} \geq l\), there must be \(l\) or more vertices in \(U((G_{i,k}, G_{j,m}))\), which is by (i) connected and by (iii) has at most \(l\) edges, since there are \(2l\) edges in \((G_{i,k}, G_{j,m}))\). Lemma \[\ref{lem:1}\] then yields that this is only possible for \(#\{(i) \cup \{k\} \cup \{j\} \cup \{m\} \leq l+1\) and that for \(l+1\) vertices in \(U((G_{i,k}, G_{j,m}))\) there must be 0 cycles, while for \(l\) vertices there can be 0 or 1 cycles. We make a case distinction based on the number of vertices and cycles, while still assuming \( W((G_{i,k}, G_{j,m})) \neq 0 \).

1) \( U((G_{i,k}, G_{j,m})) \) has \(l + 1\) vertices and thus 0 cycles:

Each edge in \(U((G_{i,k}, G_{j,m}))\) corresponds to two edges in \((G_{i,k}, G_{j,m}), \) since there must be \(l\) many undirected connections with a total of \(2l\) edges and no undirected connection may have only one edge.

This implies that a leaf of the tree \(U((G_{i,k}, G_{j,m}))\) corresponds to a balanced leaf of \((G_{i,k}, G_{j,m}), \) which then (by a simple inductive argument over leaves) must be a balanced tree. From (ii) we know that

\[ W((G_{i,k}, G_{j,m})) = W(S((G_{i,k}, G_{j,m}))) \]

which by (i) is zero, since \(S((G_{i,k}, G_{j,m}))\) consist of two vertices connected by two edges and these two edges must both come form either the left or right graph of \(S((G_{i,k}, G_{j,m})))\). We have thus shown that this case is not possible, when we assume \( W((G_{i,k}, G_{j,m})) \neq 0 \).
2) \( U((G_{(i,k)}, G_{(j,m)})) \) has \( l \) vertices and 1 cycle:

By Lemma A.2 there must be \( l \) many undirected connections and by the same arguments as in (1) each undirected connection of \( (G_{(i,k)}, G_{(j,m)}) \) must consist of two edges and we have

\[
W((G_{(i,k)}, G_{(j,m)})) = W(S((G_{(i,k)}, G_{(j,m)}))).
\]

Since every undirected connection consists of two edges, every leaf of the undirected graph \( U((G_{(i,k)}, G_{(j,m)})) \) must correspond to a balanced leaf of \( (G_{(i,k)}, G_{(j,m)}) \). This implies that \( U(S((G_{(i,k)}, G_{(j,m)}))) \) is a cycle graph, as only the one cycle in \( U((G_{(i,k)}, G_{(j,m)})) \) can not be removed while trimming balanced leaves. When \( U(S((G_{(i,k)}, G_{(j,m)}))) \) is a cycle graph and each undirected connection consist of two edges, the only way for (i) to hold is for \( S((G_{(i,k)}, G_{(j,m)})) \) to be a one- or two-directional double ring-type graph with length \( l_0 \neq 2 \).

It remains to show that, of all the possible choices for \( S((G_{(i,k)}, G_{(j,m)})) \) as double ring-type graphs, the weight is only 1 (and otherwise zero), if \( S((G_{(i,k)}, G_{(j,m)})) \) is in Double-1-d-Ring\(_{l_0} \) or Double-2-d-Ring\(_{l_0} \) for even \( l_0 \geq 4 \). Firstly we see that \( l_0 \) must be even, since \( l_0 \) is the number of edges in either side of \( S((G_{(i,k)}, G_{(j,m)})) \), the number of edges in both \( G_{(i,k)}, G_{(j,m)} \) was even (\( 2l_1 + 2l_2 \) respectively) and removal of balanced leaves always removes two edges from only one side. Remember, that \( l_0 = 2 \) was not allowed, since then \( U((G_{(i,k)}, G_{(j,m)})) \) does not have a cycle. Analogously to (3.7.2) we have

\[
W(S((G_{(i,k)}, G_{(j,m)}))) = (G_{i_1}, G_{i_2})
\]

\[
= \prod_{i,j=1}^{l} \left( Y_{i,j}^{A(R(G_{i_1}'))_{i,j} + A(R(G_{i_2}'))_{i,j}} - \prod_{i,j=1}^{l} \left( Y_{i,j}^{A(R(G_{i_1}'))_{i,j}} \right) \right) \prod_{i,j=1}^{l} \left( Y_{i,j}^{A(R(G_{i_2}'))_{i,j}} \right)
\]

\[
= 0, \text{ since } (G_{i_1}, G_{i_2}) \text{ is of double ring-type}
\]

\[
= 1_{l_1 \leq j \\ A(R(G_{i_1}'))_{i,j} + A(R(G_{i_2}'))_{i,j} \in \{0,2\}} = 1_{(G_{i_1}, G_{i_2}) \in \text{Double-1-d-Ring}_{l_0} \cup \text{Double-2-d-Ring}_{l_0}}.
\]

We have thus shown that the starting points of the routes \( i_1' \) and \( i_2' \) must also be an even number of steps apart.

3) \( U((G_{(i,k)}, G_{(j,m)})) \) has \( l \) vertices and 0 cycles:

By Lemma A.2 there must be \( l - 1 \) edges in \( U((G_{(i,k)}, G_{(j,m)})) \), which corresponds to there being \( l - 1 \) undirected connections in \( (G_{(i,k)}, G_{(j,m)}) \). By (iii) each undirected connection must have at least two edges and we thus have 2 'free' edges, which can be distributed among the undirected connections. An undirected connection with 3 edges is in this case not possible, since the tree-structure of \( U((G_{(i,k)}, G_{(j,m)})) \) would not allow the routes of \( G_{(i,k)} \) and \( G_{(j,m)} \) to return where they started from. This implies that there must be one undirected connection.
with 4 edges while all others have 2 edges. From Lemma A.3 it follows that the seed graph \( S((G(i,k), G(j,m)) \) consists of this undirected connection with 4 edges and the two vertices it connects. Again by (iii) the only choice for such a seed graph with positive weight is \( (G(v,v'), G(v,v')) \) for some \( v \neq v' \in V_i \), which precisely describes all elements of Double-2-d-Ring\(_2\). (As in (2) the starting points must be an even number of steps apart, which here means 0 steps.) The weight of this seed graph is given by

\[
\prod_{i,j=1}^{l} E(Y_{i,j}^{A(R(G(v,v'))_{i,j})}) - \prod_{i,j=1}^{l} E(Y_{i,j}^{A(R(G(v,v'))_{i,j})}) \prod_{i,j=1}^{l} E(Y_{i,j}^{A(R(G(v,v'))_{i,j})})
\]

\[
\alpha = 1.
\]

With (ii) we have shown that, under the above assumptions, the only graphs with weight different from zero are \( (G(i,k), G(j,m)) \) with

\[
S((G(i,k), G(j,m)) \in \text{Double-2-d-Ring}_2
\]

and that these have weight \( \alpha - 1 \).

\[\square\]

\section*{C.14. Proof of Corollary 7.7}

\textbf{Proof.}

The polynomial \( (p, n) \mapsto \binom{l}{p} \binom{n-b}{t_1+t_2-b} \) is of the form

\[
\binom{l}{p} \binom{n-b}{t_1+t_2-b} = \frac{1}{b!(l_1 + l_2 - b)!} p^n b^{n_1 + l_2 - b} + \mathcal{O}(p^{n_1})
\]

and the statement of Theorem 2 becomes

\[
\text{Cov}(\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2}))
\]

\[
= \sum_{b=1}^{(l_1+l_2) \land p_n} \left( \binom{l}{b} \binom{n-b}{t_1+t_2-b} \right) (C(l_1, l_2, b) + (\alpha - 3)D(l_1, l_2, b)) + \mathcal{O} \left( \frac{p^n b^{n_1}}{n+1} \right)
\]

\[
= \sum_{b=1}^{(l_1+l_2) \land p_n} \left( \binom{l}{b} \binom{n-b}{t_1+t_2-b} \right) (C(l_1, l_2, b) + (\alpha - 3)D(l_1, l_2, b)) + \mathcal{O} \left( \frac{1}{n^2} \right). \quad \square
\]

\section*{C.15. Proof of Corollary 7.8}

\textbf{Proof.}

From Theorem 2 we know

\[
\text{Cov}(\text{tr}(S_{p,n}^{l_1}), \text{tr}(S_{p,n}^{l_2})) = \sum_{p=1}^{(l_1+l_2) \land p_n} \left( \binom{l}{b} \binom{n-b}{t_1+t_2-b} \right) (C(l_1, l_2, b) + (\alpha - 3)D(l_1, l_2, b)) + \mathcal{O} \left( \frac{p^n b^{n_1}}{n+1} \right)
\]

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Theorem 2, we calculate the values of

\[ \frac{(p)}{n^{l_{1}+l_{2}-1}} (C(l_{1}, l_{2}, 1) + (\alpha - 3)D(l_{1}, l_{2}, 1)) \]

\[ + \frac{(p)}{n^{l_{1}+l_{2}-2}} (C(l_{1}, l_{2}, 2) + (\alpha - 3)D(l_{1}, l_{2}, 2)) + O\left(\frac{1}{n^{2}}\right). \]

Since the polynomial \( n \mapsto \binom{n-1}{l_{1}+l_{2}-1} \) is of the form

\[ \frac{1}{(l_{1} + l_{2} - 1)!} (n-1)(n-2) \ldots (n-(l_{1}+l_{2})+1) \]

\[ = \frac{1}{(l_{1} + l_{2} - 1)! n^{l_{1}+l_{2}-1} - \frac{(l_{1} + l_{2} - 1)(l_{1} + l_{2})}{2(l_{1} + l_{2} - 1)!} n^{l_{1}+l_{2}-2} + O(n^{l_{1}+l_{2}-3}), \]

this becomes

\[
\text{Cov}[\text{tr}(S_{p,n}^{l_{1}}), \text{tr}(S_{p,n}^{l_{2}})] = \left( \frac{p}{n} \frac{1}{l_{1} + l_{2} - 1}! - \frac{p}{n^{2}} \frac{(l_{1} + l_{2} - 1)(l_{1} + l_{2})}{2(l_{1} + l_{2} - 1)!} \right) (C(l_{1}, l_{2}, 1) + (\alpha - 3)D(l_{1}, l_{2}, 1))
\]

\[ + \frac{1}{n^{2} 2(l_{1} + l_{2} - 2)!} (C(l_{1}, l_{2}, 2) + (\alpha - 3)D(l_{1}, l_{2}, 2)) + O\left(\frac{1}{n^{3}}\right). \]  \[(7.8.1)\]

Using the non-simplified forms of \( C(l_{1}, l_{2}, b) \) and \( D(l_{1}, l_{2}, b) \) from the end of the proof of Theorem 2, we calculate the values of \( C(l_{1}, l_{2}, b) \) and \( D(l_{1}, l_{2}, b) \) for \( b \{1, 2\} \) to be

\[
C(l_{1}, l_{2}, 1) = 2! (l_{1} + l_{2} - 1)! \sum_{m=1}^{l_{1}} \sum_{k=0 \vee (1-l_{2})}^{(l_{1}, 1)-m} \binom{l_{1}}{k} \binom{l_{1}}{m+k} \binom{2}{l_{2}} \binom{2}{l_{2}}
\]

\[ = 2 (l_{1} + l_{2} - 1)! \binom{l_{1}}{0} \binom{l_{1}}{1+0} \binom{2}{l_{2}} \binom{2}{l_{2}}
\]

and

\[
C(l_{1}, l_{2}, 2) = 2! (l_{1} + l_{2} - 2)! \sum_{m=1}^{l_{1}} \sum_{k=0 \vee (2-l_{2})}^{(l_{1}, 2)-m} \binom{l_{1}}{k} \binom{l_{1}}{m+k} \binom{2-m}{l_{2}} \binom{2-m}{l_{2}}
\]

\[
= \begin{cases} 
4 \sum_{m=1}^{l_{1}} \sum_{k=1}^{1-m} \ldots, & \text{for } l_{1} = 1 = l_{2} \\
4(l_{2} - 1)! \sum_{m=1}^{l_{1}} \sum_{k=0}^{1-m} \ldots, & \text{for } l_{1} = 1, l_{2} \geq 2 \\
4(l_{1} - 1)! \sum_{m=1}^{l_{1}} \sum_{k=1}^{2-m} \ldots, & \text{for } l_{1} \geq 2, l_{2} = 1 \\
4(l_{1} + l_{2} - 2)! \sum_{m=1}^{l_{1}} \sum_{k=0}^{2-m} \ldots, & \text{for } l_{1} \geq 2, l_{2} \geq 2 
\end{cases}
\]

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\[
\begin{align*}
&= \begin{cases}
0, & \text{for } l_1 = 1 = l_2 \\
4(l_2 - 1)!\binom{l_1}{1} \binom{l_1}{2-1} \binom{l_2}{2-0} & \text{for } l_1 = 1, l_2 \geq 2 \\
4(l_1 - 1)!\binom{l_1}{1} \binom{l_2}{2-1} & \text{for } l_1 \geq 2, l_2 = 1 \\
4(l_1 + l_2 - 2)!\binom{l_1}{1} \binom{l_1}{2-1} \binom{l_2}{2-0} + \binom{l_1}{1} \binom{l_2}{2-1} & \text{for } l_1 \geq 2, l_2 \geq 2
\end{cases} \\
&= 4(l_1 + l_2 - 2)! \left( \frac{l_1}{l_2} \frac{l_2(l_2 - 1)}{2} + \frac{l_1}{l_2} \frac{l_1(l_1 - 1)}{2} + \frac{l_1}{l_2} \frac{l_2(l_2 - 1)}{2} \right) \\
&= 2(l_1 + l_2 - 2)!l_1 l_2 \left( l_1(l_1 - 1) + l_2(l_2 - 1) + (l_1 - 1)(l_2 - 1) \right)
\end{align*}
\]

and
\[
D(l_1, l_2, 1) = 1!(l_1 + l_2 - 1)! \sum_{k=0}^{(l_1+1)-1} \binom{l_1}{k} \binom{l_1}{1+k} \binom{l_2}{1-1-k} \binom{l_2}{1-k} = 2(l_1 + l_2 - 1)! \binom{l_1}{0} \binom{l_1}{1+0} \binom{l_2}{1-1-0} \binom{l_2}{1-0} = (l_1 + l_2 - 1)!l_1 l_2
\]

and
\[
D(l_1, l_2, 2) = 2!(l_1 + l_2 - 2)! \sum_{k=0}^{(l_1+2)-1} \binom{l_1}{k} \binom{l_2}{2-1-k} \binom{l_2}{2-k} = 4(l_1 - 1)! \sum_{k=1}^{2-1} \frac{2(l_1 + l_2 - 2)! \sum_{k=0}^{2-1} \binom{l_1}{k} \binom{l_2}{2-1-k} \binom{l_2}{2-k}}{2l_1}
\]
Finally we may plug these values into \((7.8.1)\) to get

\[
\text{Cov}[\text{tr}(S_{p,n}^{(1)}), \text{tr}(S_{p,n}^{(2)})] = \left(\frac{p}{n} \frac{1}{(l_1 + l_2 - 1)!} - \frac{p}{n^2} \frac{(l_1 + l_2 - 1)(l_1 + l_2)}{2(l_1 + l_2 - 1)!}\right) \times \left(2 (l_1 + l_2 - 1)! l_1 l_2 + (\alpha - 3)(l_1 + l_2 - 1)! l_1 l_2\right)
\]

\[
+ \frac{1}{n^2} \frac{p(p-1)}{2(l_1 + l_2 - 2)!} \left[2(l_1 + l_2 - 2)! l_1 l_2 \left(l_1(l_1 - 1) + l_2(l_2 - 1) + (l_1 - 1)(l_2 - 1)\right)
\]

\[
+ (\alpha - 3)(l_1 + l_2 - 2)! l_1 l_2 \left(l_1(l_1 - 1) + l_2(l_2 - 1)\right)\right) + O\left(\frac{1}{n^3}\right)
\]

\[
= (\alpha - 1) \left(\frac{p}{n} \frac{1}{(l_1 + l_2 - 1)!} - \frac{p}{n^2} \frac{(l_1 + l_2 - 1)(l_1 + l_2)}{2(l_1 + l_2 - 1)!}\right) (l_1 + l_2 - 1)! l_1 l_2
\]

\[
+ \frac{1}{n^2} \frac{p(p-1)}{2(l_1 + l_2 - 2)!} (l_1 + l_2 - 2)! l_1 l_2
\]

\[
\times \left(2(l_1 - 1)(l_2 - 1) + (\alpha - 1) \left(l_1(l_1 - 1) + l_2(l_2 - 1)\right)\right)
\]

\[
+ O\left(\frac{1}{n^3}\right)
\]

\[
= (\alpha - 1) \left(\frac{p}{n} - \frac{p}{n^2} \frac{(l_1 + l_2 - 1)(l_1 + l_2)}{2}\right) l_1 l_2
\]

\[
+ \frac{1}{n^2} \frac{p(p-1)}{2} l_1 l_2 \left(2(l_1 - 1)(l_2 - 1) + (\alpha - 1) \left(l_1(l_1 - 1) + l_2(l_2 - 1)\right)\right) + O\left(\frac{1}{n^3}\right)
\]

\[
= \frac{pl_1 l_2}{n} (\alpha - 1) + \frac{pl_1 l_2}{n^2} \left[\frac{p-1}{2} \left(2(l_1 - 1)(l_2 - 1) + (\alpha - 1) \left(l_1(l_1 - 1) + l_2(l_2 - 1)\right)\right)
\]

\[- (\alpha - 1) \frac{(l_1 + l_2 - 1)(l_1 + l_2)}{2}\right] + O\left(\frac{1}{n^3}\right)
\]

\[
= \frac{pl_1 l_2}{n} (\alpha - 1) + \frac{pl_1 l_2}{n^2} \left[(p-1)(l_1 - 1)(l_2 - 1) - (\alpha - 1)l_1 l_2\right] + O\left(\frac{1}{n^3}\right).
\]
D. Comparison with results of Bai and Silverstein

For any $l \in \mathbb{N}_0$ let $\varphi_l(x) := x^l$ be the $l$-th monomial and let $X_{\varphi_l}^n$ be the linear spectral statistic

$$X_{\varphi_l}^n := \frac{1}{p_n} \sum_{j=1}^{p_n} \varphi_l(\lambda(S_{p_n,n})) = \frac{1}{p_n} \text{tr}(S_{p_n,n}^l).$$

We can easily use this definition to linearly extend the process $X^n$ onto all polynomials. Further let $\frac{y_n}{n} \xrightarrow{n \to \infty} y > 0$ and assume $\alpha = 3$. On page 565 of [2] Bai and Silverstein give a formula for the mean and covariance structure of the limiting Gaussian process $(X_{\varphi_l})_{l \in \mathbb{N}}$ of the spectral CLT

$$p_n \left( X_{\varphi_l}^n - Q_{\varphi_l} \right)_{l \in \mathbb{N}} \xrightarrow{n \to \infty} \text{fdd}, \mathcal{D} ( (X_{\varphi_l})_{l \in \mathbb{N}}),$$

where $Q_{\varphi_l}$ denotes the Marčenko-Pastur distribution (as, for example, defined on page 40 of [1]) with parameters $y_n = \frac{p_n}{n}$ and $\sigma^2 = 1$. Lemma 3.1 of [1] shows that

$$Q_{\varphi_l} = \sum_{r=0}^{l-1} \frac{1}{l+1} \binom{l}{r} \left( \frac{p_n}{n} \right)^r = \sum_{b=1}^{l} \frac{(\frac{p_n}{n})^{b-1}}{b} \binom{l}{b-1} \binom{l-1}{b-1}.$$

In equations (1.23) and (1.24) Bai and Silverstein derive the formulas

$$\mathbb{E}[X_{\varphi_l}] = \frac{1}{4} \left( (1 - \sqrt{y})^{2l} + (1 + \sqrt{y})^{2l} \right) - \frac{1}{2} \sum_{j=0}^{l} \binom{l}{j} y^j$$

and

$$\text{Cov}[X_{\varphi_{l_1}}, X_{\varphi_{l_2}}] = 2y^{l_1+l_2} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2} \binom{l_1}{k_1} \binom{l_2}{k_2} \left( \frac{1-y}{y} \right)^{k_1+k_2} \times \sum_{m=1}^{l_1-k_1} m \left( 2l_1 - 1 - (k_1 + m) \right) \left( 2l_2 - 1 - k_2 + m \right).$$

As the expression (D.1) is easily seen to be

$$\mathbb{E}[X_{\varphi_l}] = \frac{1}{2} \sum_{j=0}^{l} \binom{2l}{2j} y^j - \frac{1}{2} \sum_{j=0}^{l} \binom{l}{j} y^j = \sum_{j=1}^{l-1} y^j \left[ \frac{1}{2} \binom{2l}{2j} - \frac{1}{2} \binom{l}{j} \right],$$

this is already consistent with our Corollary [3]. The case $y > 1$ is easily covered with the fact $\text{tr}(S_{p,n}^l) = \frac{p_n^l}{n} \text{tr}(S_{n,p}^l)$.  

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For the expression (D.2) we would like to show this to be equal to

$$\sum_{b=1}^{l_1+l_2} \frac{y^b}{b!(l_1 + l_2 - b)!} C(l_1, l_2, b),$$

which would be the prediction of Corollary (C.4). In order to verify that our formula is equal to theirs, we first bring (D.2) into a form, which allows us to read the (D.2) as a polynomial in $y$ such that we can check equality of the coefficients. We calculate

$$\text{Cov}[X_{\varphi_{l_1}}, X_{\varphi_{l_2}}]$$

$$= 2y^{l_1+l_2} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \binom{l_1}{k_1} \binom{l_2}{k_2} \left( \frac{1 - y}{y} \right)^{k_1+k_2}$$

$$\times \sum_{m=1}^{l_1-k_1} m \left( \frac{2l_1 - 1 - (k_1 + m)}{l_1 - 1} \right) \left( \frac{2l_2 - 1 - k_2 + m}{l_2 - 1} \right)$$

$$= 2y^{l_1+l_2} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \binom{l_1}{k_1} \binom{l_2}{k_2} \left( \sum_{r=0}^{k_1+k_2} \binom{k_1+k_2}{k_1+k_2-r} y^{-r} (-1)^{k_1+k_2-r} \right)$$

$$\times \sum_{m=1}^{l_1-k_1} m \left( \frac{2l_1 - 1 - (k_1 + m)}{l_1 - 1} \right) \left( \frac{2l_2 - 1 - k_2 + m}{l_2 - 1} \right)$$

$$= \sum_{r=0}^{l_1-l_2} 2y^{l_1+l_2-r} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \binom{l_1}{k_1} \binom{l_2}{k_2} \mathbb{1}_{l_1+k_2 \leq r} \binom{k_1+k_2}{k_1+k_2-r} (-1)^{k_1+k_2-r}$$

$$\times \sum_{m=1}^{l_1-k_1} m \left( \frac{2l_1 - 1 - (k_1 + m)}{l_1 - 1} \right) \left( \frac{2l_2 - 1 - k_2 + m}{l_2 - 1} \right)$$

$$= y^b 2^{l_1-1} \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \binom{l_1}{k_1} \binom{l_2}{k_2} \mathbb{1}_{l_1+l_2-b \leq k_1+k_2} \binom{k_1+k_2}{k_1+k_2-(l_1+l_2-b)} (-1)^{k_1+k_2-(l_1+l_2-b)}$$

$$\times \sum_{m=1}^{l_1-k_1} m \left( \frac{2l_1 - 1 - (k_1 + m)}{l_1 - 1} \right) \left( \frac{2l_2 - 1 - k_2 + m}{l_2 - 1} \right).$$

We have numerically checked the equality

$$2 \sum_{k_1=0}^{l_1-1} \sum_{k_2=0}^{l_2-1} \binom{l_1}{k_1} \binom{l_2}{k_2} \mathbb{1}_{l_1+l_2-b \leq k_1+k_2} \binom{k_1+k_2}{k_1+k_2-(l_1+l_2-b)} (-1)^{k_1+k_2-(l_1+l_2-b)}$$

$$\times \sum_{m=1}^{l_1-k_1} m \left( \frac{2l_1 - 1 - (k_1 + m)}{l_1 - 1} \right) \left( \frac{2l_2 - 1 - k_2 + m}{l_2 - 1} \right)$$

$$= \frac{1}{b!(l_1 + l_2 - b)!} C(l_1, l_2, b)$$

for all $1 \leq l_1, l_2 \leq 100$ and $b \leq l_1 + l_2$ and believe the identity to hold generally.
List of symbols

$A(G)$ Adjacency matrix: a matrix with one row and column per vertex of $G$, where $A(G)_{v_i,v_j}$ (slight abuse of notation) counts the number of edges from $v_i$ to $v_j$ (if $G$ is undirected, edges are counted in both directions.)

$B(G)$ set of black colored vertices in $G$ (see 4.1 and 7.1)

Cov covariance of two random variables

$C_{VR,N}$ set of circuit multigraphs (see 3.4)

$C_{VR,N_1,N_2}^2$ set of double-circuit multigraphs (see 7.1)

deg Degree: the number of edges of a vertex in an undirected (multi-)graph

$E$ mean of a random variable

$e_k$ an edge of a directed multigraph (see 3.1)

$E_N$ linearly oriented edge set of a directed multigraph (see 3.1)

$f_G$ map from $E_N$ to $V_r \times V_r$, which defines the graph $G$ (see 3.1)

$G_i$ uniquely defined circuit multigraph to the route $i$ (see 3.4)

$G_{r,N}$ set of directed multigraphs $G = (V_r, E_N, f)$ (see 3.1)

head a graph-dependent map assigning the termination vertex of an edge (see 3.1)

$i$ route of a circuit multigraph (see 3.4)

indeg In-Degree: number of directed edges with a given vertex as their head

$K_{b,w}$ complete bipartite graph with $b$ left-hand-vertices and $w$ right-hand-vertices

outdeg Out-Degree: number of directed edges with a given vertex as their tail

$R(G)$ reversed graph of $G$ (see 3.6)

$S_{p,n}$ sample covariance matrix (see 2.1)

$S(G)$ seed graph of $G$ (see 3.14 and 7.1)

tail a graph-dependent map assigning the origin vertex of an edge (see 3.1)

$\mathcal{T}_{V_{t+1}}$ set of balanced trees with vertex set $V_{t+1}$ (see 3.9)

tr trace of a matrix
\( U(G) \) undirected simple graph constructed by replacing undirected connections of \( G \) with edges (see 3.3)

\( v_i \) a vertex of a directed multigraph (see 3.1)

\( V_r \) ordered vertex set of a directed multigraph with \( r \) elements (see 3.1)

\( V(G) \) the set of visited vertices of \( G \) (see 3.2)

\( X_{p,n} \) random data matrix with \( n \) data-points and \( p \) features (see 2.1)

\( Y_{i,j} \) entry of the random data matrix \( X_{p,n} \) (see 2.1)

\( \wedge \) gives the minimum of two numbers, i.e. \( a \wedge b := \min(a, b) \)

\( \vee \) gives the maximum of two numbers, i.e. \( a \vee b := \max(a, b) \)

\( \{ \} \) set of vertices occurring in a sequence, i.e. \( i = \{i_1, ..., i_N\} \) (see 3.4)

\( [ ] \) set of positive integers up to a given integer, i.e. \( [N] = \{1, ..., N\} \)

\( [ ] \) gives the nearest lower whole number

\( [ ] \) gives the nearest higher whole number

\( \langle \cdot, \cdot \rangle \) zipped sequence of two sequences of equal length, i.e. \( \langle \mathbf{i}, \mathbf{k} \rangle = (i_1, k_1, ..., i_N, k_N) \) (see 3.5)

1-d-Ring\(_{l_0}\) set of one-dir. ring-type graphs of length \( l_0 \) (see 3.11)

2-d-Ring\(_{l_0}\) set of two-dir. ring-type graphs of length \( l_0 \) (see 3.11)

Double-1-d-Ring\(_{l_0}\) set of one-dir. double ring-type graphs of length \( l_0 \) (see 7.2)

Double-2-d-Ring\(_{l_0}\) set of two-dir. double ring-type graphs of length \( l_0 \) (see 7.2)
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Data Availability

Data sharing is not applicable to this article as no datasets are analysed.
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