Convergence analysis of a Petrov-Galerkin method for fractional wave problems with nonsmooth data *

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Abstract

This paper analyzes the convergence of a Petrov-Galerkin method for time fractional wave problems with nonsmooth data. Well-posedness and regularity of the weak solution to the time fractional wave problem are firstly established. Then an optimal convergence analysis with nonsmooth data is derived. Moreover, several numerical experiments are presented to validate the theoretical results.

Keywords: fractional wave problem, regularity, Petrov-Galerkin, convergence analysis, nonsmooth data.

1 Introduction

Let $T > 0$ be a given time and $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ be a convex $d$-polytope. This paper considers the following time fractional wave problem:

$$
\begin{cases}
D_{0+}^\alpha (u - u_0 - tu_1) - \Delta u = f & \text{in } \Omega \times (0, T), \\
u = 0 & \text{on } \partial \Omega \times (0, T), \\
u(0) = u_0 & \text{in } \Omega, \\
u_t(0) = u_1 & \text{in } \Omega,
\end{cases}
$$

(1)

where $1 < \alpha < 2$, $D_{0+}^\alpha$ is a Riemann-Liouville fractional differential operator of order $\alpha$, and $u_0, u_1$ and $f$ are given data.

In recent years, the time fractional wave problem (1) has attracted much attention. It has been applied to model the anomalous process which may occur in anomalous transport or diffusion in heterogeneous media [31]. In addition, the solution to the time fractional wave problem governs the propagation of stress waves in viscoelastic media [13, 14]. For more details related to the applications of problem (1), we refer the reader to [3, 16].

Let us first summarize some regularity results of the fractional wave problem. In [11], Bazhlekov considered the Duhamel-type representation of the solution

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to the fractional wave equation by the Mittag-Leffler function; however, the author did not investigate the regularity of the solution. Later on, in [12], Bazhlekova obtained the maximal $L^p$-regularity estimate
\[ \|u\|_{L^p(0,T;L^q(\Omega))} + \|D_x^\alpha u\|_{L^p(0,T;L^q(\Omega))} + \|\Delta u\|_{L^p(0,T;L^q(\Omega))} \leq C \|f\|_{L^p(0,T;L^q(\Omega))}, \]
where $1 < p, q < \infty$. Sakamoto et al. [19] introduced a weak solution to the fractional wave equation by means of the eigenfunction expansions. They established the well-posedness of the weak solution and derived several regularity estimates in the continuous vector-valued spaces.

Then, let us review the numerical treatments for the fractional wave equation. In [20], two kinds of finite difference methods for the computation of fractional derivatives were presented: the first method, called $L$-type scheme, uses the Lagrange interpolation technique; the second one, called $G$-type method, is based on the Grünwald-Letnikov definition. Sun et al. [33] developed a Crank-Nicolson scheme by the $L1$-scheme for the fractional wave equation and derived the convergence order $O(\tau^{3-\alpha})$ for $C^3$ solutions. Jin et al. [4] analyzed the $G1$-method and the second-order backward difference method for fractional wave equations, and they obtained the accuracies $O(\tau)$ and $O(\tau^2)$, respectively. In our previous work [6], a time-spectral method for fractional wave problems was designed, which possesses exponential decay in temporal discretization, under the condition that the solution is smooth enough. Recently, to conquer the singularity in time variable, Li et al. [7] presented a space-time finite element method for problem (1), and proved that high-order temporal accuracy can still be achieved if appropriate graded temporal grids are adopted. Under some conditions, problem (1) is equivalent to an integro-differential model, and there are many works on the numerical methods for this model; see [9, 10, 18] and the references therein. To our knowledge, except for [4], no work available is devoted to the numerical analysis for problem (1) with nonsmooth data.

This motivates us to consider the numerical analysis for problem (1) with low regularity data. In this paper, we first introduce a weak solution of problem (1) by the variational approach and establish the regularity results of the weak solution in the case $u_0 = u_1 = 0$. Then by means of the famous transposition method [17], the weak solution and its regularity of problem (1) are also considered with more general data. Finally, under the condition that $u_0 = u_1 = 0$, for a Petrov-Galerkin method we obtain the following error estimates:

- If $f \in L^2(0,T;L^2(\Omega))$, then
  \[ \|(u - U)\|_{H^{1-1/2}(0,T;L^2(\Omega))} + \|u - U\|_{C([0,T];H^1(\Omega))} \leq C(\tau^{(\alpha-1)/2} + \eta_1(\alpha, \tau, h)) \|f\|_{L^2(0,T;L^2(\Omega))}, \]
  where
  \[ \eta_1(\alpha, \tau, h) := \begin{cases} h^{1-1/\alpha} & \text{if } 1 < \alpha \leq 3/2, \\ \tau^{-1/2}h & \text{if } 3/2 < \alpha < 2; \end{cases} \]

- If $f \in H^{2-\alpha}(0,T;L^2(\Omega))$, then
  \[ \|u - U\|_{C([0,T];H^1(\Omega))} \leq C(\tau^{(3-\alpha)/2} + \eta_2(\alpha, \tau, h)) \|f\|_{H^{2-\alpha}(0,T;L^2(\Omega))}, \]
  \[ \|u' - U'\|_{H^{(\alpha-1)/2}(0,T;L^2(\Omega))} \leq C(\tau^{(3-\alpha)/2} + \eta_3(\alpha, \tau, h)) \|f\|_{H^{2-\alpha}(0,T;L^2(\Omega))}. \]
where
\[ \eta_2(\alpha, \tau, h) := \begin{cases} h & \text{if } 1 < \alpha < 3/2, \\ (1 + |\log h|)h & \text{if } \alpha = 3/2, \\ h^{3/\alpha - 1} + \tau^{3/2 - \alpha}h & \text{if } 3/2 < \alpha < 2, \end{cases} \]
and
\[ \eta_3(\alpha, \tau, h) := \begin{cases} h^{3/\alpha - 1} & \text{if } 1 < \alpha \leq 3/2, \\ \tau^{3/2 - \alpha}h & \text{if } 3/2 < \alpha < 2. \end{cases} \]

We note that, if $1 < \alpha \leq 3/2$ then estimates (2) and (4) are optimal with respect to the regularity of $u$ and (3) is optimal and nearly optimal with respect to the regularity of $u$ for $1 < \alpha < 3/2$ and $\alpha = 3/2$, respectively. This is verified by our numerical experiments. If $3/2 < \alpha < 2$, then all the estimates (2), (3) and (4) are optimal with respect to the regularity of $u$ provided that $h \leq C\tau^{\alpha/2}$. However, numerical results also indicate the optimal accuracy with respect to the regularity without this requirement.

The remainder of this paper consists of five sections. Firstly, some conventions and Sobolev spaces are introduced in Section 2. Secondly, several fundamental properties of the fractional calculus operators are summarized in Section 3. Thirdly, the well-posedness and regularity of the weak solution to problem (1) are rigorously established in Section 4.1. Fourthly, the convergence of a Petrov-Galerkin method is derived in Section 5. Finally, in Section 6 numerical experiments are presented to verify the theoretical results and Section 7 provides some concluding remarks.

2 Preliminary

First of all, let us introduce some conventions: for a Lebesgue measurable set $\omega$ of $\mathbb{R}^l$ ($l = 1, 2, 3, 4$), $H^\gamma(\omega)$ ($\gamma \in \mathbb{R}$) and $H^0_0(\omega)$ ($\beta > 0$) denote two standard Sobolev spaces [21, Chapter 34] and the symbol $\langle p, q \rangle_\omega$ means $\int_\omega pq$ whenever $pq \in L^1(\omega)$; for a Banach space $X$, $X^*$ is the dual space of $X$ and $\langle \cdot, \cdot \rangle_X$ means the duality pairing between $X^*$ and $X$; if $X$ and $Y$ are two Banach spaces, then $[X, Y]_{\theta, 2}$ is the interpolation space constructed by the famous $K$-method [21, Chapter 22]; the symbol $C_\times$ denotes a generic positive constant depending only on its subscript(s) $\times$, and its value may differ at each occurrence.

Next, we form some Hilbert spaces on the eigenvectors of $-\Delta$ and present some basic properties of these spaces. It is well known that there exists an orthonormal basis $\{ \phi_n : n \in \mathbb{N} \}$ of $L^2(\Omega)$ such that
\[
\begin{cases} 
-\Delta \phi_n = \lambda_n \phi_n & \text{in } \Omega, \\
\phi_n = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\{ \lambda_n : n \in \mathbb{N} \}$ is a positive non-decreasing sequence and $\lambda_n \to \infty$ as $n \to \infty$. For any $\gamma \in \mathbb{R}$, define
\[
\dot{H}^\gamma(\Omega) := \left\{ \sum_{n=0}^{\infty} c_n \phi_n : \sum_{n=0}^{\infty} \lambda_n^\gamma c_n^2 < \infty \right\}
\]
and endow this space with the inner product

$$\left(\sum_{n=0}^{\infty} c_n \phi_n, \sum_{n=0}^{\infty} d_n \phi_n\right)_{H^\gamma(\Omega)} := \sum_{n=0}^{\infty} \lambda_n c_n d_n,$$

for all $\sum_{n=0}^{\infty} c_n \phi_n, \sum_{n=0}^{\infty} d_n \phi_n \in H^\gamma(\Omega)$. Denote by $\| \cdot \|_{H^\gamma(\Omega)}$ the induced norm with respect to this inner product. We see that $H^\gamma(\Omega)$ is a separable Hilbert space with an orthonormal basis $\{\lambda_n^{-\gamma/2} \phi_n : n \in \mathbb{N}\}$ and the space $H^{-\gamma}(\Omega)$ is the dual space of $H^\gamma(\Omega)$ in the following sense

$$\left(\sum_{n=0}^{\infty} c_n \phi_n, \sum_{n=0}^{\infty} d_n \phi_n\right)_{H^{-\gamma}(\Omega)} := \sum_{n=0}^{\infty} c_n d_n,$$

for all $\sum_{n=0}^{\infty} c_n \phi_n \in H^{-\gamma}(\Omega)$ and $\sum_{n=0}^{\infty} d_n \phi_n \in H^\gamma(\Omega)$. Furthermore, it is clear that $H^0(\Omega) = L^2(\Omega)$ and $H^1(\Omega)$ coincides with $H^1_0(\Omega)$ with equivalent norms. Hence, for $0 < \gamma < 1$, by the theory of interpolation spaces [21], $H^\gamma(\Omega)$ coincides with $H^\gamma_0(\Omega) = [L^2(\Omega), H^1_0(\Omega)]_{\gamma,2}$ with equivalent norms. As [30, Corollary 9.1.23] implies

$$\| v \|_{H^\gamma(\Omega)} \leq C_\Omega \| v \|_{H^{\gamma}(\Omega)} \quad \forall v \in \dot{H}^\gamma(\Omega),$$

the space $\dot{H}^\gamma(\Omega)$ is continuously embedded into $H^\gamma_0(\Omega) \cap H^\gamma(\Omega)$ if $1 < \gamma < 2$.

In the rest of this section, assume that $-\infty < a < b < \infty$. Now we introduce some Sobolev spaces as follows. For any $m \in \mathbb{N}$, define

$$0^m H^m(a, b) := \{ v \in H^m(a, b) : v^{(k)}(b) = 0, \ 0 \leq k < m, \ k \in \mathbb{N}\},$$

$$0^m H^m(a, b) := \{ v \in H^m(a, b) : v^{(k)}(a) = 0, \ 0 \leq k < m, \ k \in \mathbb{N}\},$$

where $v^{(k)}$ is the $k$-th weak derivative of $v$, and endow those two spaces with the following norms

$$\| v \|_{0^m H^m(a, b)} := \| v^{(m)} \|_{L^2(a, b)}, \quad \forall v \in 0^m H^m(a, b),$$

$$\| v \|_{0^m H^m(a, b)} := \| v^{(m)} \|_{L^2(a, b)}, \quad \forall v \in 0^m H^m(a, b),$$

respectively. For $k - 1 < \gamma < k$, $k \in \mathbb{N}_{>0}$, define

$$0^m H^\gamma(a, b) := [0^m H^{k-1}(a, b), 0^m H^k(a, b)]_{\gamma-k+1,2},$$

$$0^m H^\gamma(a, b) := [0^m H^{k-1}(a, b), 0^m H^k(a, b)]_{\gamma-k+1,2}.$$

By [17, Chapter 1], we have the following standard results: if $0 < \gamma < 1/2$, then $0^m H^\gamma(a, b), 0^m H^\gamma(a, b)$ and $H^\gamma(a, b)$ are equivalent; if $m + 1/2 < \gamma < m + 1, m \in \mathbb{N}$, then

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : v^{(m)}(b) = 0, \ (b-t)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \},$$

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : v^{(m)}(a) = 0, \ (t-a)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \},$$

with equivalent norms; if $m \leq \gamma \leq m + 1/2, m \in \mathbb{N}$, then

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : (b-t)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \},$$

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : (t-a)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \},$$

and

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : (b-t)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \},$$

$$0^m H^\gamma(a, b) = \{ v \in 0^m H^m(a, b) : (t-a)^{m-\gamma} v^{(m)}(t) \in L^2(a, b) \}.$$
in the sense of equivalent norms. For $\gamma \geq 0$, the spaces $0^H\gamma(a, b)$ and $\partial^H\gamma(a, b)$ can be defined equivalently as the domains of fractional power of second order differential operators (see [23, 25, 32]). For $\gamma > 0$, denote by $0^H-\gamma(a, b)$ and $\partial^H-\gamma(a, b)$ the dual spaces of $0^H\gamma(a, b)$ and $\partial^H\gamma(a, b)$, respectively. Since $0^H\gamma(a, b)$ and $\partial^H\gamma(a, b)$ are reflexive, they are the dual spaces of $0^H-\gamma(a, b)$ and $\partial^H-\gamma(a, b)$, respectively. Moreover, by [1, Theorems 1.18 and 1.23] and Theorems 12.2-12.6 of [17, Chapter 1], we readily conclude the following lemma.

**Lemma 2.1.** If $0 < \theta < 1$ and $\beta, \gamma \in \mathbb{R}$ then

$$\left[0^H\beta(a, b), 0^H\gamma(a, b)\right]_{\theta, 2} = 0^{H(1-\theta)\beta+\theta\gamma}(a, b),$$

$$\left[\partial^H\beta(a, b), \partial^H\gamma(a, b)\right]_{\theta, 2} = \partial^{H(1-\theta)\beta+\theta\gamma}(a, b),$$

(5)

with equivalent norms.

**Remark 2.1.** We will give more details about how to derive (5). As $(H^1(a, b))^\ast$ and $H^1_0(a, b)$ are continuously embedded in $0^{H^{-1}}(a, b)$ and $\partial^H1(a, b)$, respectively, by Theorem 12.3 of [17, Chapter 1] we have that $L^2(a, b)$ is continuously embedded in $[0^{H^{-1}}(a, b), H^1(a, b)]_{1/2, 2}$. Conversely, since $0^{H^{-1}}(a, b)$ and $\partial^H1(a, b)$ are continuously embedded in $H^{-1}(a, b)$ and $H^1(a, b)$, respectively, by Theorem 12.4 of [17, Chapter 1] we have that $[0^{H^{-1}}(a, b), \partial^H1(a, b)]_{1/2, 2}$ is continuously embedded in $L^2(a, b)$. Therefore, $[0^{H^{-1}}(a, b), \partial^H1(a, b)]_{1/2, 2} = L^2(a, b)$ with equivalent norms. Then by [1, Theorem 1.23] we obtain that, for any $0 < \theta < 1$,

$$\partial^0\theta(a, b) = [L^2(a, b), \partial^H1(a, b)]_{\theta, 2} = [0^{H^{-1}}(a, b), \partial^H1(a, b)]_{1/2, 2, a, b}$$

$$= [0^{H^{-1}}(a, b), \partial^H1(a, b)]_{(1+\theta)/2, 2},$$

with equivalent norms. The other cases are derived similarly.

Finally, let us introduce some vector-valued spaces. Let $X$ be a separable Hilbert space with an orthonormal basis $\{e_n : n \in \mathbb{N}\}$. For any $\gamma \in \mathbb{R}$, define

$$0^H\gamma(a, b; X) := \left\{ \sum_{n=0}^{\infty} c_n e_n : \sum_{n=0}^{\infty} \|c_n\|_{0^H\gamma(a, b)}^2 < \infty \right\},$$

and endow this space with the norm

$$\left\| \sum_{n=0}^{\infty} c_n e_n \right\|_{0^H\gamma(a, b; X)} := \left( \sum_{n=0}^{\infty} \|c_n\|_{0^H\gamma(a, b)}^2 \right)^{1/2}.$$

The space $0^H\gamma(a, b; X)$ can be defined analogously. It is evident that both $0^H\gamma(a, b; X)$ and $\partial^H\gamma(a, b; X)$ are reflexive. In addition, the space $0^{H-\gamma}(a, b; X)$ is the dual space of $0^H\gamma(a, b; X)$ in the sense that

$$\left\langle \sum_{n=0}^{\infty} c_n e_n, \sum_{n=0}^{\infty} d_n e_n \right\rangle_{0^H\gamma(a, b; X)} := \sum_{n=0}^{\infty} \langle c_n, d_n \rangle_{0^H\gamma(a, b)},$$

for all $\sum_{n=0}^{\infty} c_n e_n \in 0^{H-\gamma}(a, b; X)$ and $\sum_{n=0}^{\infty} d_n e_n \in 0^H\gamma(a, b; X)$, and the space $0^{H-\gamma}(a, b; X)$ is the dual space of $0^H\gamma(a, b; X)$ in the sense that

$$\left\langle \sum_{n=0}^{\infty} c_n e_n, \sum_{n=0}^{\infty} d_n e_n \right\rangle_{0^H\gamma(a, b; X)} := \sum_{n=0}^{\infty} \langle c_n, d_n \rangle_{0^H\gamma(a, b)}.$$

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for all \( \sum_{n=0}^{\infty} c_n e_n \in \mathcal{H}^{-\gamma}(a, b; X) \) and \( \sum_{n=0}^{\infty} d_n e_n \in \mathcal{H}^{\gamma}(a, b; X) \). Moreover, we use \( C([a, b]; X) \) to denote the continuous \( X \)-valued space.

**Lemma 2.2.** Assume that \( s, r, \beta, \gamma \in \mathbb{R} \) and \( 0 < \theta < 1 \). If \( v \in \mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega)) \cap \mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega)) \), then

\[
\|v\|_{\mathcal{H}^{(1-\theta)\beta+\theta\gamma}(0, 1; \mathcal{H}^{(1-\theta)r+\theta s}(\Omega))} \leq C_{\beta, \gamma, \theta} \|v\|_{\mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega))}^{1-\theta} \|v\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^\theta .
\]

**Proof.** By definition, there exists a unique decomposition \( v = \sum_{n=0}^{\infty} v_n \phi_n \), such that

\[
\|v\|_{\mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega))}^2 = \sum_{n=0}^{\infty} \lambda_n^\beta \|v_n\|_{\mathcal{H}^\beta(0, 1)}^2 ,
\]

\[
\|v\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^2 = \sum_{n=0}^{\infty} \lambda_n^\gamma \|v_n\|_{\mathcal{H}^\gamma(0, 1)}^2 .
\]

Therefore, by [1, Corollary 1.7] and Lemma 2.1 we have

\[
\|v\|_{\mathcal{H}^{(1-\theta)\beta+\theta\gamma}(0, 1; \mathcal{H}^{(1-\theta)r+\theta s}(\Omega))}^2 = \sum_{n=0}^{\infty} \lambda_n^{(1-\theta)r+\theta s} \|v_n\|_{\mathcal{H}^{(1-\theta)\beta+\theta\gamma}(0, 1)}^2 
\leq C_{\beta, \gamma, \theta} \sum_{n=0}^{\infty} \lambda_n^{(1-\theta)r+\theta s} \|v_n\|_{\mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega))}^{2(1-\theta)} \|v_n\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^{2\theta} 
\leq C_{\beta, \gamma, \theta} \sum_{n=0}^{\infty} \lambda_n^{(1-\theta)r+\theta s} \|v_n\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^{2\theta} \|v_n\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^{2\theta} 
= C_{\beta, \gamma, \theta} \sum_{n=0}^{\infty} \left( \lambda_n^\beta \|v_n\|_{\mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega))} \right)^{1-\theta} \left( \lambda_n^\gamma \|v_n\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))} \right)^\theta 
\leq C_{\beta, \gamma, \theta} \|v\|_{\mathcal{H}^\beta(0, 1; \mathcal{H}^r(\Omega))}^{2(1-\theta)} \|v\|_{\mathcal{H}^\gamma(0, 1; \mathcal{H}^s(\Omega))}^{2\theta} ,
\]

which implies (6).

**3 Fractional Calculus Operators**

In this section, we firstly summarize several fundamental properties of fractional calculus operators, then we generalize the fractional integral operator and prove some useful results. Assume that \( -\infty < a < b < \infty \) and \( X \) is a separable Hilbert space.

**Definition 3.1.** For \( \gamma > 0 \), define

\[
(D_{a+}^\gamma v)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} v(s) \, ds, \quad t \in (a, b),
\]

\[
(D_{b-}^\gamma v)(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} v(s) \, ds, \quad t \in (a, b),
\]
for all \( v \in L^1(a, b; X) \), where \( \Gamma(\cdot) \) is the Gamma function and \( L^1(a, b; X) \) denotes the \( X \)-valued Bochner integrable space. In addition, let \( D^0_{a^+} \) and \( D^0_{b^-} \) be the identity operator on \( L^1(a, b; X) \). For \( j - 1 < \gamma \leq j \) with \( j \in \mathbb{N}_{>0} \), define
\[
D^\gamma_{a^+} v := D^j D^\gamma_{a^+} v, \\
D^\gamma_{b^-} v := (-D)^j D^\gamma_{b^-} v,
\]
for all \( v \in L^1(a, b; X) \), where \( D \) is the first-order differential operator in the distribution sense.

**Lemma 3.1 ([27]).** Let \( v \in L^1(a, b) \). If \( \gamma, \beta \geq 0 \), then
\[
D^-\gamma_{a^+} D^-\beta_{a^+} v = D^-\gamma-\beta_{a^+} v, \quad D^-\gamma_{b^-} D^-\beta_{b^-} v = D^-\gamma-\beta_{b^-} v.
\]
If \( \gamma \geq \beta > 0 \), then
\[
D^\gamma_{a^+} D^-\beta_{a^+} v = D^\gamma-\beta_{a^+} v, \quad D^\gamma_{b^-} D^-\beta_{b^-} v = D^\gamma-\beta_{b^-} v.
\]

**Lemma 3.2 ([3]).** Assume that \( \gamma \geq 0 \). If \( w, v \in L^2(a, b) \), then
\[
\langle D^-\gamma_{a^+} w, v \rangle_{(a, b)} = \langle w, D^-\gamma_{b^-} v \rangle_{(a, b)}.
\]
If \( v \in L^2(a, b) \), then
\[
\left\| D^-\gamma_{a^+} v \right\|_{L^2(a, b)} \leq \frac{(b - a)^\gamma}{\Gamma(\gamma + 1)} \left\| v \right\|_{L^2(a, b)}, \\
\left\| D^-\gamma_{b^-} v \right\|_{L^2(a, b)} \leq \frac{(b - a)^\gamma}{\Gamma(\gamma + 1)} \left\| v \right\|_{L^2(a, b)}.
\]

**Lemma 3.3.** If \( \gamma, \beta \geq 0 \), then
\[
\left\| D^\gamma_{a^+} v \right\|_{a^+H^{\beta+\gamma}(a, b)} \leq C_{\beta, \gamma} \left\| v \right\|_{a^+H^{\gamma}(a, b)} \quad \forall v \in a^+H^{\beta}(a, b), \tag{7}
\]
\[
\left\| D^\gamma_{b^-} v \right\|_{b^-H^{\beta+\gamma}(a, b)} \leq C_{\beta, \gamma} \left\| v \right\|_{b^-H^{\gamma}(a, b)} \quad \forall v \in b^-H^{\beta}(a, b). \tag{8}
\]

**Proof.** As the proof of (8) is analogous to that of (7) and the case \( \gamma, \beta \in \mathbb{N} \) is trivial, we only prove (7) for the case that \( \gamma \notin \mathbb{N} \) or \( \beta \notin \mathbb{N} \).

We first use the standard scaling argument to prove the case \( \beta = 0 \) and \( 0 < \gamma < 1 \). By definition we have
\[
a^+H^{\gamma}(a, b) = \left[ a^+H^0(a, b), a^+H^1(a, b) \right]_{\gamma/2},
\]
and this space is endowed with the following norm
\[
\left\| w \right\|_{a^+H^{\gamma}(a, b)} = \left( \int_0^\infty \left( t^{-\gamma} K(t, w) \right)^2 \frac{dt}{t} \right)^{1/2} \quad \forall w \in a^+H^{\gamma}(a, b),
\]
where
\[
K(t, w) := \inf_{w = w_0 + w_1} \left\| w_0 \right\|_{a^+H^0(a, b)} + t \left\| w_1 \right\|_{a^+H^1(a, b)}, \quad 0 < t < \infty,
\]
for all \( w \in a^+H^{\gamma}(a, b) \). For any \( v \in a^+H^0(a, b) \), define
\[
\hat{v}(s) := v(a + (b - a)s), \quad 0 < s < 1,
\]
then a direct computation gives
\[ K(t, D_{a+}^{-\gamma} v) = (b - a)^{1/2 + \gamma} K \left( t/(b - a), D_{a+}^{-\gamma} \hat{v} \right), \quad 0 < t < \infty. \]

Since using [7, Lemma A.4] gives
\[ \| D_{a+}^{-\gamma} w \|_{aH^{\gamma}(0,1)} \leq C_\gamma \| w \|_{aH^\gamma(0,1)} \quad \forall \ w \in aH^0(0,1), \]

it follows that
\[ \| D_{a+}^{-\gamma} \|_{aH^\gamma(a,b)} = (b - a)^{1/2} \| D_{\hat{a}+}^{-\gamma} \hat{v} \|_{aH^\gamma(0,1)} \]
\[ \leq C_\gamma (b - a)^{1/2} \| \hat{v} \|_{aH^0(0,1)} = C_\gamma \| v \|_{aH^\gamma(a,b)} . \]

This proves (7) for \( \beta = 0 \) and \( 0 < \gamma < 1. \)

Then we consider the case \( \beta \in \mathbb{N} \) and \( m < \gamma < m + 1, m \in \mathbb{N}. \) Since \( v \in aH^\beta(a, b), \) by Lemma 3.1, it is evident that
\[ D_{a+}^{m+\beta} D_{a+}^{-\gamma} v = D_{a+}^{m+\beta-\gamma} D_{a+}^{-\beta} v = D_{a+}^{m-\gamma} D_{a+}^{\beta} v. \]

Therefore, a direct calculation yields
\[ K(t, D_{a+}^{-\gamma} v) = \inf_{D_{a+}^{-\gamma} v = \gamma_0 + \gamma_1} \| u_0 \|_{aH^{m+\beta}(a,b)} + t \| u_1 \|_{aH^{m+\beta+1}(a,b)} \]
\[ = \inf_{D_{a+}^{m-\gamma} D_{a+}^{\beta} v = \gamma_0 + \gamma_1} \| v_0 \|_{aH^\gamma(a,b)} + t \| v_1 \|_{aH^\gamma(a,b)} \]
\[ = K(t, D_{a+}^{-\gamma} D_{a+}^{\beta} v), \]

for all \( 0 < t < \infty, \) which implies that
\[ \| D_{a+}^{-\gamma} v \|_{aH^{\beta+\gamma}(a,b)} = \| D_{a+}^{m-\gamma} D_{a+}^{\beta} v \|_{aH^{\gamma-m}(a,b)}. \]

Consequently, by the previous case, we have
\[ \| D_{a+}^{-\gamma} v \|_{aH^{\beta+\gamma}(a,b)} \leq C_{\beta, \gamma} \| D_{a+}^{\beta} v \|_{aH^\gamma(a,b)} = C_{\beta, \gamma} \| v \|_{aH^\beta(a,b)}. \]

This proves (7) for the case \( \beta \in \mathbb{N} \) and \( m < \gamma < m + 1, m \in \mathbb{N}. \)

Finally it remains to consider the case \( \gamma \geq 0 \) and \( n < \beta < n + 1, n \in \mathbb{N}. \)

Since we have proved that
\[ \| D_{a+}^{-\gamma} w \|_{aH^{\gamma+n}(a,b)} \leq C_{\beta, \gamma} \| w \|_{aH^\gamma(a,b)} \quad \forall \ w \in aH^n(a, b), \]
\[ \| D_{a+}^{-\gamma} w \|_{aH^{\gamma+n+1}(a,b)} \leq C_{\beta, \gamma} \| w \|_{aH^{n+1}(a,b)} \quad \forall \ w \in aH^{n+1}(a, b), \]

applying the theory of interpolation spaces [21, Lemma 22.3] gives
\[ \| D_{a+}^{-\gamma} v \|_{aH^{\beta+\gamma}(a,b)} \leq C_{\beta, \gamma} \| v \|_{aH^\beta(a,b)}, \]

for any \( v \in aH^\beta(a, b). \) This completes the proof of this lemma. ■
Remark 3.1. In \cite[Theorem 2.1]{32}, Lemma 3.3 has been proved for \( \beta = 0 \) and \( 0 \leq \gamma \leq 1 \).

Lemma 3.4. Let \( v \in L^2(a, b) \) and \( \beta \geq \gamma > 0 \). If \( D_{a+}^{\gamma} v \in aH^{\beta - \gamma}(a, b) \), then
\[
\|v\|_{aH^{\beta}(a, b)} \leq C_{\beta, \gamma} \|D_{a+}^{\gamma} v\|_{aH^{\beta - \gamma}(a, b)}.
\]  
(9)

If \( D_{b-}^{\gamma} v \in bH^{\beta - \gamma}(a, b) \), then
\[
\|v\|_{bH^{\beta}(a, b)} \leq C_{\beta, \gamma} \|D_{b-}^{\gamma} v\|_{bH^{\beta - \gamma}(a, b)}.
\]  
(10)

Proof. Let us first prove (9). Suppose that \( k < \gamma \leq k + 1, \ k \in \mathbb{N} \). By definition,
\[
D_{a+}^{\gamma} v = D^{k+1}_{a+} D_{a+}^{\gamma-k-1} v,
\]
then applying \( D_{a+}^{\gamma-k-1} \) on both sides of the above equation and using integral by parts yield that
\[
(D_{a+}^{\gamma-k-1} v)(t) = (D_{a+}^{\gamma-k-1} D_{a+}^{\gamma} v)(t) + \sum_{i=0}^{k} \frac{c_i(t-a)^i}{i!(i+1)}, \ \ a < t < b,
\]  
(11)

where \( c_i \in \mathbb{R} \). Moreover, since by Lemma 3.1
\[
D_{a+}^{\gamma} D_{a+}^{k+1-\gamma} D_{a+}^{\gamma-k-1} v = D_{a+}^{\gamma} v,
\]
\[
D_{a+}^{\gamma} D_{a+}^{k+1-\gamma} D_{a+}^{\gamma-k-1} v = D_{a+}^{\gamma} D_{a+}^{\gamma-k-1} v = D_{a+}^{\gamma} v,
\]
applying \( D_{a+}^{\gamma} D_{a+}^{k+1-\gamma} \) on both sides of (11) implies
\[
\sum_{i=0}^{k} \frac{c_i(t-a)^i}{i!(i+1)} = 0, \ \ a < t < b.
\]

Therefore, it follows that \( c_i = 0 \) for \( 0 \leq i \leq k \), which, together with (11), gives
\[
D_{a+}^{\gamma-k-1} v = D_{a+}^{\gamma-k} D_{a+}^{\gamma} v.
\]

By Lemma 3.1, applying \( D_{a+}^{k+1-\gamma} \) on both sides of the above equation yields that
\( v = D_{a+}^{\gamma} D_{a+}^{\gamma} v \). Hence, by Lemma 3.3,
\[
\|v\|_{aH^{\beta}(a, b)} = \|D_{a+}^{\gamma} D_{a+}^{\gamma} v\|_{aH^{\beta}(a, b)} \leq C_{\beta, \gamma} \|D_{a+}^{\gamma} v\|_{aH^{\beta - \gamma}(a, b)},
\]
which proves (9). As (10) can be proved similarly, this completes the proof. \( \blacksquare \)

Lemma 3.5. If \( \gamma > 0 \), then
\[
D_{a-}^{-\gamma} D_{a+}^{\gamma} v = v \quad \forall v \in aH^{\gamma}(a, b), \quad (12)
\]
\[
D_{b-}^{-\gamma} D_{b+}^{\gamma} v = v \quad \forall v \in bH^{\gamma}(a, b). \quad (13)
\]

Proof. Let \( k \in \mathbb{N} \) satisfy that \( k - 1 < \gamma \leq k \). For any \( v \in aH^{\gamma}(a, b) \), since Lemma 3.3 implies \( D_{a+}^{\gamma-k} v \in aH^{k}(a, b) \), a straightforward computation yields that
\[
D_{a+}^{\gamma} D_{a+}^{\gamma} v = D_{a+}^{\gamma} D_{a+}^{\gamma-k} v = D_{a+}^{\gamma-k} D_{a+}^{\gamma} v = D_{a+}^{\gamma} v = v,
\]
which proves (12). An analogous argument proves (13) and thus concludes the proof of this lemma. \( \blacksquare \)
Lemma 3.6. If $\beta \geq \gamma > 0$, then
\[
C_{\beta, \gamma} \|v\|_{aH^\beta(a,b)} \leq \|D^\gamma_{a^+} v\|_{aH^{\beta-\gamma}(a,b)} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(a,b)} \quad \forall v \in aH^\beta(a,b),
\] (14)
\[
C_{\beta, \gamma} \|v\|_{aH^\beta(a,b)} \leq \|D^{-\gamma}_{a^+} v\|_{aH^{\beta-\gamma}(a,b)} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(a,b)} \quad \forall v \in aH^\beta(a,b).
\] (15)

Proof. Since the proof of (15) is similar to that of (14), we only prove (14). If we can prove
\[
\|D^\gamma_{a^+} v\|_{aH^{\beta-\gamma}(a,b)} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(a,b)}
\] (16)
for all $v \in aH^\beta(a,b)$, then, by Lemma 3.4,
\[
\|v\|_{aH^\beta(a,b)} \leq C_{\beta, \gamma} \|D^\gamma_{a^+} v\|_{aH^{\beta-\gamma}(a,b)}.
\]
Hence it suffices to prove (16). By Lemmas 3.1 and 3.5,
\[
D^\gamma_{a^+} v = D^\gamma_{a^+} D^{-\beta}_{a^+} D^\beta_{a^+} v = D^\gamma_{a^+} D^\beta_{a^+} v,
\]
and using Lemma 3.3 gives
\[
\|D^\gamma_{a^+} v\|_{aH^{\beta-\gamma}(a,b)} = \|D^\gamma_{a^+} D^\beta_{a^+} v\|_{aH^{\beta-\gamma}(a,b)} \leq C_{\beta, \gamma} \|D^\beta_{a^+} v\|_{aH^\gamma(a,b)}.
\]
Let $k \in \mathbb{N}_{>0}$ satisfy that $k - 1 < \beta \leq k$. Invoking Lemma 3.3 again implies that
\[
\|D^\beta_{a^+} v\|_{aH^\beta(a,b)} = \|D^k D^{-k}_{a^+} v\|_{aH^\beta(a,b)} = \|D^\beta_{a^+} v\|_{aH^\beta(a,b)} \leq C_{\beta} \|v\|_{aH^\beta(a,b)},
\]
which, together with the previous inequality, proves (16). This finishes the proof of this lemma. ■

Remark 3.2. In [32, Theorem 2.1], an alternative proof of (14) has been given for $0 \leq \beta \leq 1$ and $\gamma = \beta$.

Lemma 3.7. If $\beta, \gamma \geq 0$, then
\[
\|v\|_{aH^\beta(a,b)} \leq C_{\beta, \gamma} \|D^\gamma_{a^+} v\|_{aH^{\beta+\gamma}(a,b)} \quad \forall v \in aH^\beta(a,b),
\]
\[
\|v\|_{aH^\beta(a,b)} \leq C_{\beta, \gamma} \|D^\gamma_{a^+} v\|_{aH^{\beta+\gamma}(a,b)} \quad \forall v \in aH^\beta(a,b).
\]

Proof. Combining Lemmas 3.5 and 3.6, we obtain
\[
\|v\|_{aH^\beta(a,b)} = \|D^\gamma_{a^+} D^{-\gamma}_{a^+} v\|_{aH^\beta(a,b)} \leq C_{\beta, \gamma} \|D^\gamma_{a^+} v\|_{aH^{\beta+\gamma}(a,b)} \quad \forall v \in aH^\beta(a,b),
\]
\[
\|v\|_{aH^\beta(a,b)} = \|D^\gamma_{a^+} D^{-\gamma}_{a^+} v\|_{aH^\beta(a,b)} \leq C_{\beta, \gamma} \|D^\gamma_{a^+} v\|_{aH^{\beta+\gamma}(a,b)} \quad \forall v \in aH^\beta(a,b).
\]
This concludes the proof. ■

Lemma 3.8 ([5, 29]). If $-1/2 < \gamma < 1/2$ and $v \in H^{\max\{0, \gamma\}}(a,b)$, then
\[
\cos(\gamma \pi) \|D^\gamma_{a^+} v\|_{L^2(a,b)}^2 \leq \langle D^\gamma_{a^+} v, D^\gamma_{b^-} v \rangle_{(a,b)} \leq \sec(\gamma \pi) \|D^\gamma_{a^+} v\|_{L^2(a,b)}^2,
\]
\[
\cos(\gamma \pi) \|D^\gamma_{b^-} v\|_{L^2(a,b)}^2 \leq \langle D^\gamma_{a^+} v, D^\gamma_{b^-} v \rangle_{(a,b)} \leq \sec(\gamma \pi) \|D^\gamma_{b^-} v\|_{L^2(a,b)}^2.
\]
Moreover, if $v, w \in H^\gamma(a,b)$ with $0 < \gamma < 1/2$, then
\[
\langle D^\gamma_{a^+} v, w \rangle_{H^\gamma(a,b)} = \langle D^\gamma_{a^+} v, D^\gamma_{b^-} w \rangle_{(a,b)} = \langle D^\gamma_{b^-} w, v \rangle_{H^\gamma(a,b)}.
\]

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Lemma 3.9. If \( \beta \geq 1 \) and \( \gamma < 1/2 \), then
\[
\|v\|_{C[0,1]} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(0,1)}^{(1/2-\gamma)/(\beta-\gamma)} \|v\|_{aH^1(0,1)}^{(\beta-1/2)/(\beta-\gamma)},
\] (17)
for all \( v \in aH^\beta(0,1) \).

Proof. Let \( s := \max\{0, \gamma\} \). Since \( v \in aH^\beta(0,1) \), by Lemmas 3.2, 3.5, 3.6 and 3.8, a direct calculation gives
\[
\frac{1}{2} v^2(t) = \langle v', v \rangle_{(0,1)} = \langle v', D_0^- v, D_0^- v \rangle_{(0,1)} = \langle D_0^- v, D_0^- v \rangle_{(0,1)}
\]
\[
\leq C_s \|D_0^- v\|_{L^2(0,1)}^2 \leq C_s \|D_0^- v\|_{L^2(0,1)}^2 \leq C_s \|D_0^- v\|_{L^2(0,1)} \|D_0^- v\|_{L^2(0,1)}
\]
\[
\leq C_s \|v\|_{aH^\beta(0,1)} \|v\|_{aH^1(0,1)}.
\]
for all \( 0 \leq t \leq 1 \). It follows that
\[
\|v\|_{C[0,1]} \leq C_s \|v\|_{aH^1(0,1)}^{1/2} \|v\|_{aH^\beta(0,1)}^{1/2}.
\]
Additionally, by Lemma 2.1 and [1, Corollary 1.7] we have
\[
\|v\|_{aH^\beta(0,1)} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(0,1)}^{(1-\gamma)/(\beta-\gamma)} \|v\|_{aH^1(0,1)}^{(\beta-s)/(\beta-\gamma)},
\]
\[
\|v\|_{aH^\beta(0,1)} \leq C_{\beta, \gamma} \|v\|_{aH^\beta(0,1)}^{(1-\gamma)/(\beta-\gamma)} \|v\|_{aH^\beta(0,1)}^{(\beta-s)/(\beta-\gamma)}.
\]
Consequently, combining the above three estimates proves (17).

Lemma 3.10. If \( v \in aH^\beta(0,1) \) with \( \beta > 1/2 \), then
\[
\|v\|_{C[0,1]} \leq C_{\beta, \epsilon} \|v\|_{aH^{1/2}(0,1)}^{1-\epsilon} \|v\|_{aH^\beta(0,1)}^\epsilon,
\] (18)
for all \( 0 < \epsilon \leq 1/ \max\{2, 2\beta\} \).

Proof. For any \( w \in L^2(0,1) \), extend \( w \) to \((-\infty, 0)\) by zero and denote this extension by \( \tilde{w} \). Let \( n \in \mathbb{N} \) satisfy that \( n-1 < \beta \leq n \). Following the proof of [17, Theorem 8.1], we define an extension operator \( E : L^2(0,1) \rightarrow L^2(\mathbb{R}) \) by that, for any \( w \in L^2(0,1) \),
\[
(Ew)(t) := \begin{cases}
\tilde{w}(t), & \text{if } t < 1, \\
\sum_{j=1}^{n+1} \gamma_j \tilde{w}(j + 1 - j t), & \text{if } t > 1,
\end{cases}
\]
where \( \gamma_j \) is defined by
\[
\sum_{j=1}^{n+1} (-j)^k \gamma_j = 1, \quad 0 \leq k \leq n.
\]
Since a straightforward computation gives
\[
\|Ew\|_{H^k(\mathbb{R})} \leq C_k \|w\|_{aH^\beta(0,1)} \forall w \in aH^\beta(0,1), \quad 0 \leq k \leq n,
\]
applying [21, Lemma 22.3] yields
\[ \|Ev\|_{H^{(1-\epsilon)/2+\beta}(R)} \leq C_\beta \|v\|_{aH^{1/2}(0,1),aH^\beta(0,1)} \cdot \]
In addition, [21, (23.11)] implies that
\[ \left( \int_R (1 + |\xi|^2)^{(1-\epsilon)/2+\beta} |(FEv)(\xi)|^2 \, d\xi \right)^{1/2} \leq C_\beta \|Ev\|_{H^{(1-\epsilon)/2+\beta}(R)}, \]
where \( F \) is the Fourier transform operator. Moreover, using [22, (1.2.4) and (1.2.5)] yields
\[ \|Ev\|_{L^\infty(R)} \leq \frac{C_\beta}{\sqrt{\epsilon}} \left( \int_R (1 + \xi^2)^{(1-\epsilon)/2+\beta} |(FEv)(\xi)|^2 \, d\xi \right)^{1/2}. \]
Therefore it follows that
\[ \|v\|_{C([0,1])} = \|Ev\|_{L^\infty(R)} \leq \frac{C_\beta}{\sqrt{\epsilon}} \|v\|_{aH^{1/2}(0,1),aH^\beta(0,1)} \cdot \]
Since borrowing the proof of [1, Corollary 1.7] gives
\[ \|v\|_{aH^{1/2}(0,1),aH^\beta(0,1)} \leq \frac{1}{\sqrt{\epsilon}} \|v\|_{X} \|v\|_{aH^\beta(0,1)}, \]
we finally obtain (18) by the above two inequalities. This concludes the proof of this lemma. \( \blacksquare \)

Now let us generalize the fractional integral operator as follows. Recall that in this section \( X \) denotes a separable Hilbert space. Assume that \( \beta, \gamma > 0 \) and \( v \in aH^{-\beta}(a, b; X) \). If \( 0 < \gamma \leq \beta \), then define \( \mathcal{D}_{a+}^{-\gamma} v \in aH^{-\beta}(a, b; X) \) by that
\[ \langle \mathcal{D}_{a+}^{-\gamma} v, \xi \rangle_{aH^{\beta-\gamma}(a,b,X)} := \langle v, \mathcal{D}_{a-}^{-\gamma} \xi \rangle_{aH^{\beta}(a,b,X)}, \quad \text{(19)} \]
for all \( w \in aH^{\beta-\gamma}(a, b; X) \); if \( \gamma > \beta \), then define \( \mathcal{D}_{a+}^{-\gamma} v \in aH^{-\beta}(a, b; X) \) by that
\[ \mathcal{D}_{a+}^{-\gamma} v := \mathcal{D}_{a-}^{-\gamma} \mathcal{D}_{a+}^{-\gamma} v \quad \text{(20)} \]
By Lemmas 3.3, 3.6 and 3.7, the generalized left-sided fractional integral operator
\[ \mathcal{D}_{a+}^{-\gamma} : aH^{-\beta}(a, b; X) \rightarrow aH^{-\beta}(a, b; X) \]
is well-defined for all \( \beta, \gamma > 0 \). Symmetrically, we can generalize the right-sided fractional integral operator as follows. Assume that \( \beta, \gamma > 0 \) and \( v \in aH^{-\beta}(a, b; X) \). Define \( \mathcal{D}_{a-}^{-\gamma} v \in aH^{-\beta}(a, b; X) \) by that
\[ \langle \mathcal{D}_{a-}^{-\gamma} v, \xi \rangle_{aH^{\beta-\gamma}(a,b,X)} := \langle v, \mathcal{D}_{a+}^{-\gamma} \xi \rangle_{aH^{\beta}(a,b,X)}, \quad \text{for all } w \in aH^{\beta-\gamma}(a, b; X). \]

**Lemma 3.11.** If \( \beta, \gamma > 0 \), then
\[ C_{\beta, \gamma} \|v\|_{aH^{-\beta}(a, b)} \leq \|\mathcal{D}_{a+}^{-\gamma} v\|_{aH^{-\beta}(a, b)} \leq C_{\beta, \gamma} \|v\|_{aH^{-\beta}(a, b)} \quad \forall v \in aH^{-\beta}(a, b), \quad \text{(21)} \]
\[ C_{\beta, \gamma} \|v\|_{aH^{-\beta}(a, b)} \leq \|\mathcal{D}_{a-}^{-\gamma} v\|_{aH^{-\beta}(a, b)} \leq C_{\beta, \gamma} \|v\|_{aH^{-\beta}(a, b)} \quad \forall v \in aH^{-\beta}(a, b). \quad \text{(22)} \]
Proof. Since the proofs of (21) and (22) are similar, we only give the proof of the former.

Let us first prove that

$$\|D_{a+}^{-\gamma} v\|_{H^{1-\beta}(a,b)} \leq C_{\beta,\gamma} \|v\|_{H^{1-\gamma}(a,b)} \quad \forall \ v \in _0 H^{-\beta}(a,b). \quad (23)$$

If $\gamma \leq \beta$, then by Lemma 3.3 and definition (19),

$$\langle D_{a+}^{-\gamma} v, w \rangle_{H^{1-\gamma}(a,b)} = \langle v, D_{b-}^{-\gamma} w \rangle_{H^{1-\gamma}(a,b)}$$

$$\leq \|v\|_{H^{1-\beta}(a,b)} \|D_{b-}^{-\gamma} w\|_{H^{1-\gamma}(a,b)}$$

$$\leq C_{\beta,\gamma} \|v\|_{H^{1-\beta}(a,b)} \|w\|_{H^{1-\gamma}(a,b)},$$

for all $w \in _0 H^{1-\gamma}(a,b)$, which proves (23) for $\gamma \leq \beta$. If $\gamma > \beta$, then by definition (20) and Lemma 3.3 and the previous case, we have

$$\|D_{a+}^{-\gamma} v\|_{H^{1-\gamma}(a,b)} = \|D_{a+}^{\beta-\gamma} D_{a+}^{-\beta} v\|_{H^{1-\gamma}(a,b)}$$

$$\leq C_{\beta,\gamma} \|D_{a+}^{-\beta} v\|_{L^2(a,b)} \leq C_{\beta,\gamma} \|v\|_{H^{1-\gamma}(a,b)},$$

This proves (23) for the case $\gamma > \beta$.

Then it remains to prove that

$$\|v\|_{H^{1-\gamma}(a,b)} \leq C_{\beta,\gamma} \|D_{a+}^{-\gamma} v\|_{H^{1-\gamma}(a,b)} \quad \forall \ v \in _0 H^{-\beta}(a,b). \quad (24)$$

If $\gamma \leq \beta$, then by definition (19) and Lemmas 3.5 and 3.6

$$\langle v, w \rangle_{H^{\beta}(a,b)} = \langle v, D_{b-}^{-\beta} D_{b-}^{\gamma} w \rangle_{H^{\gamma}(a,b)} = \langle D_{a+}^{-\gamma} v, D_{b-}^{\beta} w \rangle_{H^{\beta}(a,b)}$$

$$\leq \|D_{a+}^{-\gamma} v\|_{H^{1-\beta}(a,b)} \|D_{b-}^{\beta} w\|_{H^{1-\gamma}(a,b)}$$

$$\leq C_{\beta,\gamma} \|D_{a+}^{-\beta} v\|_{H^{1-\gamma}(a,b)} \|w\|_{H^{1-\gamma}(a,b)},$$

for all $w \in _0 H^{\beta}(a,b)$, so that we have

$$\|v\|_{H^{1-\gamma}(a,b)} \leq C_{\beta,\gamma} \|D_{a+}^{-\gamma} v\|_{H^{1-\gamma}(a,b)}.$$

If $\gamma > \beta$, then by definition (20) and Lemmas 3.5 and 3.6, an evident calculation gives

$$\langle v, w \rangle_{H^{\beta}(a,b)} = \langle v, D_{b-}^{\beta} D_{b-}^{-\gamma} w \rangle_{H^{\gamma}(a,b)}$$

$$= \langle D_{a+}^{\beta} v, D_{b-}^{\gamma} w \rangle_{(a,b)} = \langle D_{a+}^{\gamma} D_{b-}^{\beta} v, D_{b-}^{\gamma} w \rangle_{(a,b)}$$

$$\leq \|D_{a+}^{\beta} v\|_{L^2(a,b)} \|D_{b-}^{-\gamma} w\|_{L^2(a,b)}$$

$$\leq C_{\beta,\gamma} \|D_{a+}^{\gamma} v\|_{H^{1-\gamma}(a,b)} \|w\|_{H^{1-\gamma}(a,b)},$$

for all $w \in _0 H^{\beta}(a,b)$, which implies that

$$\|v\|_{H^{1-\gamma}(a,b)} \leq C_{\beta,\gamma} \|D_{a+}^{-\gamma} v\|_{H^{1-\gamma}(a,b)}.$$

This proves (24) and thus completes the proof of this lemma. □
Lemma 3.12. If \( v \in H^\gamma(a, b) \) with \( 0 < \gamma < 1/2 \), then
\[
\left| \langle D_{a+}^{2\gamma} v, D_{a+}^{-2\gamma} v \rangle_{H^\gamma(a, b)} \right| \leq C_{\gamma} \| v \|_{aH^\gamma(a, b)} \| v \|_{0H^{-\gamma}(a, b)}. \tag{25}
\]

Proof. Since by Lemma 3.8,
\[
\| D_{b-}^{\gamma} D_{a+}^{-2\gamma} v \|_{L^2(a, b)} \leq C_{\gamma} \| D_{b-}^{\gamma} D_{a+}^{-2\gamma} v \|_{L^2(a, b)} = C_{\gamma} \| D_{a+}^{-\gamma} v \|_{L^2(a, b)},
\]

it follows that
\[
\left| \langle D_{a+}^{\gamma} v, D_{a+}^{-2\gamma} v \rangle_{H^\gamma(a, b)} \right| = \left| \langle D_{a+}^{\gamma} v, D_{b-}^{\gamma} D_{a+}^{-2\gamma} v \rangle_{(a, b)} \right| \\
\leq \| D_{a+}^{\gamma} v \|_{L^2(a, b)} \| D_{b-}^{\gamma} D_{a+}^{-2\gamma} v \|_{L^2(a, b)} \\
\leq C_{\gamma} \| D_{a+}^{\gamma} v \|_{L^2(a, b)} \| D_{a+}^{-\gamma} v \|_{L^2(a, b)} \\
\leq C_{\gamma} \| v \|_{aH^{\gamma}(a, b)} \| v \|_{0H^{-\gamma}(a, b)},
\]

by Lemmas 3.6 and 3.11. This completes the proof. \( \square \)

Lemma 3.13. Assume that \( \gamma < \beta + 1/2 \). If \( v \in aH^\beta(a, b) \), then \( D_{a+}^{\gamma} v \in aH^{\beta-\gamma}(a, b) \) and
\[
\langle D_{a+}^{\gamma} v, w \rangle_{aH^{\beta-\gamma}(a, b)} = \langle D_{a+}^{\beta} v, D_{a+}^{-\beta} w \rangle_{aH^\beta(a, b)} \tag{26}
\]
for all \( w \in aH^{\beta-\gamma}(a, b) \). If \( v \in aH^\beta(a, b) \), then \( D_{b-}^{\gamma} v \in aH^{\beta-\gamma}(a, b) \) and
\[
\langle D_{b-}^{\gamma} v, w \rangle_{aH^{\beta-\gamma}(a, b)} = \langle D_{b-}^{\beta} v, D_{b-}^{-\beta} w \rangle_{aH^\beta(a, b)} \tag{27}
\]
for all \( w \in aH^{\beta-\gamma}(a, b) \).

Proof. As the proof of (27) is similar to that of (26), we only prove (26).

Firstly, if \( 0 < \gamma \leq \beta \), then by Lemma 3.6, it is obvious that \( D_{a+}^{\gamma} v \in aH^{\beta-\gamma}(a, b) \), and (26) holds indeed by by Lemma 3.5 and definition (19).

Next we consider the case \( \gamma \leq 0 \) and \( \beta \geq \gamma \). By Lemmas 3.3 and 3.11, we have \( D_{a+}^{\gamma} v \in aH^{\beta-\gamma}(a, b) \). If \( \beta \geq 0 \), then by Lemmas 3.1 and 3.5, it is evident that
\[
D_{a+}^{\beta-\gamma} D_{a+}^{\gamma} v = D_{a+}^{\gamma-\beta} D_{a+}^{\beta} D_{a+}^{\gamma} v = D_{a+}^{\gamma} v.
\]

If \( \beta < 0 \), then by definition (20),
\[
\langle D_{b-}^{-\beta} w, D_{a+}^{\gamma} v \rangle_{aH^{\beta-\gamma}(a, b)} = \langle w, D_{a+}^{-\beta} D_{a+}^{\gamma} v \rangle_{aH^\beta(a, b)},
\]
for all \( w \in aH^{\beta-\gamma}(a, b) \), which proves (26) for \( \gamma \leq 0 \) and \( \beta \geq \gamma \).

Then let us consider the case \( \gamma \leq 0 \) and \( \gamma - 1/2 < \beta < \gamma \). By Lemma 3.11, we have \( D_{a+}^{\gamma} v \in aH^{\beta-\gamma}(a, b) \), and using Lemmas 3.1 and 3.5 and definition (19) gives
\[
\langle D_{a+}^{\gamma} v, w \rangle_{aH^{\beta-\gamma}(a, b)} = \langle v, D_{b-}^{-\beta} w \rangle_{aH^{\beta-\gamma}(a, b)} = \langle v, D_{b-}^{\beta} D_{b-}^{\gamma} w \rangle_{aH^\beta(a, b)} = \langle D_{a+}^{\beta} v, D_{b-}^{\gamma-\beta} w \rangle_{(a, b)},
\]
14
for all $w \in C_{0}^{s}(a, b)$. This proves (26) for $\gamma \leq 0$ and $\gamma - 1/2 < \beta < \gamma$.

Finally it remains to consider the case $\gamma > 0$ and $\gamma - 1/2 < \beta < \gamma$. Let $k \in \mathbb{N}$ satisfy $k - 1 < \gamma \leq k$. If $\beta \geq 0$, then by Lemmas 3.2 and 3.6, a direct manipulation implies that

$$
\langle D_{a+}^{\gamma} v, \phi \rangle = \langle D^{k} D_{a+}^{\gamma-k} v, \phi \rangle = (-1)^{k} \langle D_{a+}^{\gamma-k} v, D^{k} \phi \rangle_{(a,b)}
$$

$$
= (-1)^{k} \langle v, D_{b-}^{k-\gamma} D^{\gamma} \phi \rangle_{(a,b)} = \langle v, D_{b-}^{\gamma} \phi \rangle_{(a,b)}
$$

$$
= \langle v, D_{b-}^{\gamma-\beta} D^{\beta} \phi \rangle_{(a,b)} = \langle D_{a+}^{\beta} v, D_{b}^{\gamma-\beta} \phi \rangle_{(a,b)}
$$

$$
\leq C_{\beta, \gamma} \|v\|_{H^{\gamma}(a,b)} \|\phi\|_{H^{-\beta}(a,b)}.
$$

for all $\phi \in C_{0}^{\infty}(a, b)$. If $\beta < 0$, then $0 < \gamma < 1/2$. By Lemmas 3.6 and 3.11 and definition (20), for all $\phi \in C_{0}^{\infty}(a, b)$, a similar deduction gives

$$
\langle D_{a+}^{\gamma} v, \phi \rangle = \langle D D_{a+}^{\gamma-1} v, \phi \rangle = \langle D^{k} D_{a+}^{\gamma-k} v, D \phi \rangle_{(a,b)}
$$

$$
= \langle D^{k-1} D_{a+}^{\gamma-k} v, D \phi \rangle_{(a,b)} = \langle D_{a+}^{\gamma-k-\beta} D_{a+}^{\beta} v, D \phi \rangle_{(a,b)}
$$

$$
= \langle D_{a+}^{\beta} v, D_{b-}^{\gamma-k-1} D \phi \rangle_{(a,b)} = \langle D_{a+}^{\beta} v, D_{b-}^{\gamma-\beta} \phi \rangle_{(a,b)}
$$

$$
\leq C_{\beta, \gamma} \|v\|_{H^{\gamma}(a,b)} \|\phi\|_{H^{-\beta}(a,b)}.
$$

Above, we use $\langle \cdot, \cdot \rangle$ to denote the dual pair between the dual space of $C_{0}^{\infty}(a, b)$ and $C_{0}^{\infty}(a, b)$. Since $0 < \gamma - \beta < 1/2$, it is clear that $C_{0}^{\infty}(a, b)$ is dense in $C_{0}^{\infty}(a, b)$. Consequently, we conclude that $D_{a+}^{\gamma} v \in H^{s-\gamma}(a, b)$ and (26) holds indeed. This concludes the proof of this lemma.

**Lemma 3.14.** If $\max\{\beta, \beta + \gamma\} < s + 1/2$, then

$$
D_{a+}^{\gamma} D_{a+}^{\beta} v = D_{a+}^{\gamma+\beta} v \quad \forall v \in H^{s}(a, b),
$$

$$
D_{b-}^{\beta} D_{b-}^{\gamma} v = D_{b-}^{\gamma+\beta} v \quad \forall v \in H^{s}(a, b).
$$

**Proof.** Let us first prove that

$$
D_{a+}^{s-\beta} D_{a+}^{\beta} v = D_{a+}^{s} v \quad \forall v \in H^{s}(a, b).
$$

(30)

By Lemmas 3.6 and 3.11, it is evident that $D_{a+}^{s} v \in L^{2}(a, b)$. In addition, applying Lemma 3.13 yields that

$$
\langle D_{a+}^{s-\beta} D_{a+}^{\beta} v, \phi \rangle_{(a,b)} = \langle D_{a+}^{s} v, D_{a+}^{s-\beta} \phi \rangle_{(a,b)}
$$

$$
= \langle D_{a+}^{s} v, D_{b-}^{s-\beta} D_{b-}^{\beta} \phi \rangle_{(a,b)} = \langle D_{a+}^{s} v, \phi \rangle_{(a,b)}
$$

for any $\phi \in C_{0}^{\infty}(a, b)$, which proves (30).

Then we turn to the proof of (28). Since using Lemma 3.13 implies that $D_{a+}^{s} v \in H^{s-\gamma}(a, b)$ and both $D_{a+}^{\gamma} D_{a+}^{\beta} v$ and $D_{a+}^{\gamma+\beta} v$ belong to $H^{s-\gamma-\beta}(a, b)$, by (30), we have

$$
\langle D_{a+}^{\gamma} D_{a+}^{\beta} v, w \rangle_{H^{s+\gamma-\beta}(a,b)} = \langle D_{a+}^{s-\beta} D_{a+}^{\beta} v, D_{b-}^{s+\gamma-\beta} w \rangle_{(a,b)}
$$

$$
= \langle D_{a+}^{s} v, D_{b-}^{s+\gamma-\beta} w \rangle_{(a,b)} = \langle D_{a+}^{s} v, w \rangle_{H^{s+\gamma-\beta}(a,b)}.
$$

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for all \( w \in 0^{H^{\beta+\gamma}}(a, b) \), which proves (28). As (29) can be proved analogously, we finish the proof of this lemma.

\[ \square \]

4 Weak Solution and Regularity

4.1 The first definition

Define
\[
\hat{W} := 0^{H^{3\alpha/4}}(0, T; L^2(\Omega)) \cap 0^{H^{\alpha/4}}(0, T; \dot{H}^1(\Omega)),
\]
\[
W := 0^{H^{\alpha/4}}(0, T; L^2(\Omega)) \cap 0^{H^{-\alpha/4}}(0, T; \dot{H}^1(\Omega)),
\]
and endow these two spaces with the norms
\[
\| \cdot \|_{\hat{W}} := \left( \| \cdot \|_{0^{H^{3\alpha/4}}(0, T; L^2(\Omega))}^2 + \| \cdot \|_{0^{H^{\alpha/4}}(0, T; \dot{H}^1(\Omega))}^2 \right)^{1/2},
\]
\[
\| \cdot \|_W := \left( \| \cdot \|_{0^{H^{\alpha/4}}(0, T; L^2(\Omega))}^2 + \| \cdot \|_{0^{H^{-\alpha/4}}(0, T; \dot{H}^1(\Omega))}^2 \right)^{1/2},
\]
respectively. Assuming that
\[
D^{\alpha/2}_0(u_0 + tu_1) \in W^* \quad \text{and} \quad f \in \hat{W}^*,
\]
we call \( u \in W \) a weak solution to problem (1) if
\[
\langle D^{\alpha/2}_0 u, v \rangle_{0^{H^{\alpha/4}}(0, T; L^2(\Omega))} + \langle \nabla D^{\alpha/4}_0 u, \nabla D^{\alpha/4}_0 v \rangle_{0^{H^1}(\Omega) \times (0, T)} = \langle f, D^{\alpha/2}_0 u \rangle_{W^*} + \langle D^{\alpha/2}_0 (u_0 + tu_1), v \rangle_W
\]
for all \( v \in W \).

Remark 4.1. Notice that
\[
0^{H^{\alpha/4}}(0, T; L^2(\Omega)) = 0^{H^{\alpha/4}}(0, T; L^2(\Omega)),
\]
\[
0^{H^{-\alpha/4}}(0, T; \dot{H}^1(\Omega)) = 0^{H^{-\alpha/4}}(0, T; \dot{H}^1(\Omega)),
\]
with equivalent norms. Hence, by Lemmas 3.3 and 3.8, it is easy to verify that each term in (31) makes sense.

By Lemmas 3.3, 3.6 and 3.8 and the well-known Lax-Milgram theorem, a routine argument yields that the above weak solution is well-defined.

Theorem 4.1. Problem (31) admits a unique solution \( u \in W \) and
\[
\| u \|_W \leq C_\alpha \left( \| f \|_{\hat{W}^*} + \| D^{\alpha/2}_0 (u_0 + tu_1) \|_{W^*} \right).
\]

Remark 4.2. If \( u_0 \in L^2(\Omega) \) and \( u_1 \in \dot{H}^{-1}(\Omega) \), then a simple calculation yields that \( D^{\alpha/2}_0 (u_0 + tu_1) \in W^* \). Therefore the weak solution is well-defined by (31) for \( u_0 \in L^2(\Omega), u_1 \in \dot{H}^{-1}(\Omega) \) and \( f \in \hat{W}^* \).
Next, we employ the Galerkin method to investigate the regularity of the solution to problem (31) in the case \( u_0 = u_1 = 0 \). Let us first define \( y \in \phi H^{\alpha/4}(0, T) \) by that

\[
\langle D_{0+}^{\alpha/2} y, z \rangle_{H^{\alpha/4}(0, T)} + \lambda \langle D_{0+}^{-\alpha/2} y, z \rangle_{0, T} = \langle D_{0+}^{-\alpha/2} g, z \rangle_{H^{\alpha/4}(0, T)}
\]

(32)

for all \( z \in \phi H^{\alpha/4}(0, T) \), where \( g \in \phi H^{-3\alpha/4}(0, T) \) and \( \lambda \) is a positive constant. Similar to problem (31), problem (32) admits a unique solution \( y \in \phi H^{\alpha/4}(0, T) \) and

\[
\|y\|_{\phi H^{\alpha/4}(0, T)} + \lambda^{1/2} \|y\|_{\phi H^{-4/4}(0, T)} \leq C_\alpha \|g\|_{\phi H^{-3\alpha/4}(0, T)}.
\]

**Lemma 4.1.** If \( \gamma \geq -3\alpha/4 \) and \( g \in \phi H^\gamma(0, T) \), then

\[
D_{0+}^{\alpha+\gamma} y + \lambda D_{0+}^\gamma y = D_{0+}^\gamma g,
\]

(33)

\[
\|y\|_{\phi H^{\alpha+\gamma}(0, T)} + \lambda \|y\|_{\phi H^\gamma(0, T)} \leq C_{\alpha\gamma} \|g\|_{\phi H^\gamma(0, T)}.
\]

Moreover, if \( 1 - \alpha \leq \gamma < 1/2 \) then

\[
\lambda^{1+(\gamma-1/2)/\alpha} \|y\|_{C[0,T]} \leq C_{\alpha\gamma} T \|g\|_{\phi H^\gamma(0, T)},
\]

(34)

and if \( \gamma = 1/2 \) then

\[
\lambda^{1-\epsilon/2} \|y\|_{C[0,T]} \leq \frac{C_{\alpha\gamma} T}{\epsilon} \|g\|_{\phi H^{1/2}(0, T)},
\]

(35)

for all \( 0 < \epsilon \leq 2/(2\alpha + 1) \).

**Proof.** Firstly, let us consider the following problem: find \( w \in \phi H^{\alpha/4}(0, T) \) such that

\[
\langle D_{0+}^{\alpha/2} w, z \rangle_{H^{\alpha/4}(0, T)} + \lambda \langle D_{0+}^{-\alpha/2} w, z \rangle_{0, T} = \langle D_{0+}^{\gamma} g, z \rangle_{0, T}
\]

(36)

for all \( z \in \phi H^{\alpha/4}(0, T) \). Since using Lemma 3.6 yields

\[
\|D_{0+}^{\gamma} g\|_{L^2(0, T)} \leq C_{\gamma} \|g\|_{\phi H^\gamma(0, T)},
\]

(37)

by Lemmas 3.3, 3.6 and 3.8 and the Lax-Milgram theorem, we claim that problem (37) admits a unique solution \( w \in \phi H^{\alpha/4}(0, T) \). In addition, inserting \( z = w \) into (37) gives

\[
\|w\|_{\phi H^{\alpha/4}(0, T)} + \lambda \|w\|_{\phi H^{-\alpha/4}(0, T)} \leq C_{\alpha} \|D_{0+}^{\gamma} g\|_{L^2(0, T)} \|w\|_{L^2(0, T)}.
\]

(38)

Since Lemma 2.1 and [1, Corollary 1.7] imply

\[
\|w\|_{L^2(0, T)} \leq C_{\alpha} \|w\|_{\phi H^{\alpha/4}(0, T)}^{1/2} \|w\|_{\phi H^{-\alpha/4}(0, T)}^{1/2},
\]

a simple calculation gives, by (39), that

\[
\lambda^{1/4} \|w\|_{\phi H^{\alpha/4}(0, T)} + \lambda^{3/4} \|w\|_{\phi H^{-\alpha/4}(0, T)} \leq C_{\alpha} \|D_{0+}^{\gamma} g\|_{L^2(0, T)}.
\]

(39)

Observe that by (37)

\[
D_{0+}^{\alpha/2} w + \lambda D_{0+}^{-\alpha/2} w = D_{0+}^{\gamma} g \quad \text{in} \ L^2(0, T),
\]

(40)
and then multiplying $D_{0+}^{\alpha/2} w$ on both sides of the above equality and integrating on $(0,T)$ yields that
\[
\| D_{0+}^{\alpha/2} w \|^2_{L^2(0,T)} = \langle D_{0+}^{\alpha/2} g, D_{0+}^{\alpha/2} w \rangle_{(0,T)} - \lambda \langle D_{0+}^{\alpha/2} w, D_{0+}^{-\alpha/2} w \rangle_{(0,T)} \\
\leq \| D_{0+} g \|_{L^2((0,T))} \| D_{0+}^{\alpha/2} w \|_{L^2(0,T)} + C_\alpha \lambda \| w \|_{\alpha H^{-\alpha/4}(0,T)} \| w \|_{\alpha H^{-\alpha/4}(0,T)},
\]
by Lemma 3.12. Thus using (40) and Young’s inequality with $\epsilon$ yields
\[
\| D_{0+}^{\alpha/2} w \|_{L^2(0,T)} \leq C_{\alpha} \| D_{0+} g \|_{L^2(0,T)},
\]
and it follows from (41) that
\[
\lambda \| D_{0+}^{\alpha/2} w \|_{L^2(0,T)} = \| D_{0+}^{\alpha/2} w - D_{0+}^{\alpha/2} g \|_{L^2(0,T)} \leq C_{\alpha} \| D_{0+}^{\alpha/2} g \|_{L^2(0,T)}.
\]
Hence, by Lemmas 3.4 and 3.11, combining (38) and the above two estimates yields
\[
\| w \|_{\alpha H^{\alpha/2}(0,T)} + \lambda \| w \|_{\alpha H^{-\alpha/2}(0,T)} \leq C_{\alpha,\gamma} \| g \|_{\alpha H^{\gamma}(0,T)}.
\]
(42)

Secondly, let us prove (33). By the facts $w \in \alpha H^{\alpha/2}(0,T)$ and Lemma 3.14, applying $D_{0+}^{-\alpha/2-\gamma}$ on both sides of (41) yields
\[
(D_{0+}^{\alpha/2} + \lambda D_{0+}^{-\alpha/2}) D_{0+}^{-\alpha/2-\gamma} w = D_{0+}^{-\alpha/2} g.
\]
Therefore, $y := D_{0+}^{-\alpha/2-\gamma} w$ is the solution to problem (32), and using Lemmas 3.3 and 3.11 and (42) proves (34). Hence by the fact $y \in \alpha H^{\alpha+\gamma}(0,T)$, the relation $w = D_{0+}^{\alpha/2+\gamma} y$, Lemma 3.14 and (41), we obtain that
\[
D_{0+}^{\alpha/2+\gamma} y + \lambda D_{0+} y = D_{0+}^{\alpha/2} g,
\]
which proves (33).

Finally, it remains to prove (35) and (36). If $1 - \alpha \leq \gamma < 1/2$, then by Lemma 3.9 and (34),
\[
\lambda^{1+(\gamma-1/2)/\alpha} \| g \|_{C[0,T]} \\
\leq C_{\alpha,\gamma,T} \| g \|_{\alpha H^{\alpha+\gamma}(0,T)} \left( \lambda \| g \|_{\alpha H^{\gamma}(0,T)} \right)^{1+(\gamma-1/2)/\alpha} \\
\leq C_{\alpha,\gamma,T} \left( \| g \|_{\alpha H^{\alpha+\gamma}(0,T)} + \lambda \| g \|_{\alpha H^{\gamma}(0,T)} \right) \\
\leq C_{\alpha,\gamma,T} \| g \|_{\alpha H^{\gamma}(0,T)},
\]
which proves (35). If $\gamma = 1/2$, then using Lemma 3.10 and (34) gives
\[
\lambda^{1-\epsilon/2} \| g \|_{C[0,T]} \leq C_{\alpha,T} \left( \lambda \| g \|_{\alpha H^{1/2}(0,T)} \right)^{1-\epsilon/2} \| g \|_{\alpha H^{1/2}(0,T)} \\
\leq C_{\alpha,T} \| g \|_{\alpha H^{1/2}(0,T)},
\]
for all $0 < \epsilon \leq 2/(2\alpha + 1)$. This proves (36) and thus completes the proof of this lemma. ■
Lemma 4.2. If \( u_0 = u_1 = 0 \) and
\[
f \in \dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega)) \cup \dot{a}H^{-\alpha/4}(0, T; \dot{H}^{-1}(\Omega)),
\]
then the solution to problem (31) is given by that
\[
u(t) = \sum_{n=0}^{\infty} y_n(t) \phi_n, \quad 0 \leq t \leq T,
\]
where \( y_n \) satisfies
\[
\langle \mathcal{D}_{0+}^{\alpha/2} y_n, z \rangle_{\dot{a}H^{\alpha/4}(0, T)} + \lambda_n \langle \mathcal{D}_{0+}^{-\alpha/2} y_n, z \rangle_{\dot{a}H^{-\alpha/4}(0, T)} = \langle \mathcal{D}_{0+}^{-\alpha/2} \langle f, \phi_n \rangle_{H^{1}(\Omega)}, z \rangle_{(0, T)} (44)
\]
for all \( z \in \dot{a}H^{\alpha/4}(0, T) \).

Proof. Let us first consider the case that \( f \in \dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega)) \). Similar to problem (32), problem (44) admits a unique solution \( y_n \in \dot{a}H^{\alpha/4}(0, T) \), and
\[
\| y_n \|_{\dot{a}H^{\alpha/4}(0, T)} + \lambda_n \| y_n \|_{\dot{a}H^{-\alpha/4}(0, T)} \leq C_{\alpha} \| f \|_{\dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega))}.
\]
Since
\[
\sum_{n=0}^{\infty} \| \langle f, \phi_n \rangle_{H^{1}(\Omega)} \|^2_{\dot{a}H^{-3\alpha/4}(0, T)} = \sum_{n=0}^{\infty} \| \langle f, \phi_n \rangle_{L^2(\Omega)} \|^2_{\dot{a}H^{-3\alpha/4}(0, T)} = \| f \|^2_{\dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega))},
\]
it follows that
\[
\sum_{n=0}^{\infty} \left( \| y_n \|^2_{\dot{a}H^{\alpha/4}(0, T)} + \lambda_n \| y_n \|^2_{\dot{a}H^{-\alpha/4}(0, T)} \right) \leq C_{\alpha} \| f \|^2_{\dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega))}.
\]
Therefore, \( u \) defined by (43) belongs to \( W \) and satisfies that
\[
\| u \|_{W} \leq C_{\alpha} \| f \|_{\dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega))}.
\]

Next, let us verify that \( u \) is the solution to problem (31). For any \( v \in W \), there exists a unique decomposition \( v = \sum_{n=0}^{\infty} v_n \phi_n \), and
\[
\sum_{n=0}^{\infty} \left( \lambda_n \| v_n \|^2_{\dot{a}H^{-\alpha/4}(0, T)} + \| v_n \|^2_{\dot{a}H^{\alpha/4}(0, T)} \right) = \| v \|^2_{W}.
\]
It is evident that
\[
\langle \mathcal{D}_{0+}^{\alpha/2} u, v \rangle_{H^{\alpha/4}(0, T; L^2(\Omega))} = \sum_{n=0}^{\infty} \langle \mathcal{D}_{0+}^{\alpha/2} y_n, v_n \rangle_{H^{\alpha/4}(0, T)},
\]
\[
\langle \nabla \mathcal{D}_{0+}^{-\alpha/4} u, \nabla \mathcal{D}_{T-}^{-\alpha/4} v \rangle_{\Omega \times (0, T)} = \sum_{n=0}^{\infty} \lambda_n \langle \mathcal{D}_{0+}^{-\alpha/2} y_n, v_n \rangle_{(0, T)}.
\]
Since \( f \in \dot{a}H^{-3\alpha/4}(0, T; L^2(\Omega)) \subset \dot{W}^{*} \), we also have
\[
\langle f, \mathcal{D}_{T-}^{-\alpha/2} v \rangle_{\dot{W}^{*}} = \sum_{n=0}^{\infty} \langle \mathcal{D}_{0+}^{-\alpha/2} \langle f, \phi_n \rangle_{H^{1}(\Omega)}, v_n \rangle_{(0, T)}.
\]
Combining the above three equations and (44) proves that \( u \) given by (43) fulfills (31) for all \( v \in W \), and therefore it is indeed the solution to problem (31). Since the proof of the case that \( f \in H^{-\alpha/4}(0, T; \dot{H}^{-1}(\Omega)) \) is similar. This completes the proof of the lemma. ■

Combining Lemmas 4.1 and 4.2, we readily conclude the following regularity results for the weak solution \( u \) to problem (1).

**Theorem 4.2.** Assume that \( \beta \geq 0, \gamma \geq -3\alpha/4 \) or \( \beta \geq -1, \gamma \geq -\alpha/4 \). If \( u_0 = u_1 = 0 \) and \( f \in \dot{H}^\gamma(0, T; \dot{H}^\beta(\Omega)) \), then

\[
D_0^{\alpha+\gamma} u - \Delta D_0^\gamma u = D_0^\gamma f \quad \text{in} \quad L^2(0, T; \dot{H}^\beta(\Omega)),
\]

and

\[
\|u\|_{\dot{H}^{\alpha+\gamma}(0, T; \dot{H}^\beta(\Omega))} + \|u\|_{\dot{H}^\gamma(0, T; \dot{H}^{2+\beta}(\Omega))} \leq C_{\alpha, \gamma} \|f\|_{\dot{H}^\gamma(0, T; \dot{H}^\beta(\Omega))}.
\]

Moreover, if \( 1 - \alpha \leq \gamma < 1/2 \) then

\[
\|u\|_{\dot{C}([0, T], \dot{H}^{2+\beta+(2\gamma-1)/\alpha}(\Omega))} \leq C_{\alpha, \gamma, \epsilon} \|f\|_{\dot{H}^\gamma(0, T; \dot{H}^\beta(\Omega))},
\]

and if \( \gamma = 1/2 \) then

\[
\|u\|_{\dot{C}([0, T], \dot{H}^{2+\beta-(1/\alpha)}(\Omega))} \leq \frac{C_{\alpha, \epsilon}}{\epsilon} \|f\|_{\dot{H}^{1/2}(0, T; \dot{H}^\beta(\Omega))},
\]

for all \( 0 < \epsilon \leq 2/(2\alpha + 1) \).

**Remark 4.3.** By Lemmas 4.1 and 4.2, if \( 1 - \alpha \leq \gamma < 1/2 \), then the series in (43) converges to \( u \) in \( \dot{H}^{2+\beta+(2\gamma-1)/\alpha}(\Omega) \) uniformly on \([0, T]\), so that \( u \in \dot{C}([0, T]; \dot{H}^{2+\beta+(2\gamma-1)/\alpha}(\Omega)) \). Similarly, if \( \gamma = 1/2 \), then \( u \in \dot{C}([0, T]; \dot{H}^{2+\beta-\epsilon}(\Omega)) \) for all \( 0 < \epsilon \leq 2/(2\alpha + 1) \).

**Remark 4.4.** We observe that [12, Theorem 4.21] has already contained the regularity estimate

\[
\|u\|_{L^p(0, T; L^q(\Omega))} + \|D_0^{\alpha+\gamma} u\|_{L^p(0, T; L^q(\Omega))} + \|\Delta u\|_{L^p(0, T; L^q(\Omega))} \leq C_{\alpha, p, q} \|f\|_{L^p(0, T; L^q(\Omega))},
\]

where \( 1 < p, q < \infty \). In the case \( p = q = 2 \), this result implies

\[
\|u\|_{\dot{H}^\gamma(0, T; L^2(\Omega))} + \|u\|_{L^2(0, T; \dot{H}^\beta(\Omega))} \leq C_\alpha \|f\|_{L^2(0, T; L^2(\Omega))}.
\]

**Remark 4.5.** Let us introduce a simple example to demonstrate that a smooth source term \( f \) with \( f(0) \neq 0 \) can not guarantee a smooth solution. For any \( \beta > 0 \), define the Mittag-Leffler function \( E_{\alpha, \beta}(z) \) by

\[
E_{\alpha, \beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad z \in \mathbb{C},
\]

and this function admits a growth estimate that \([16]\)

\[
|E_{\alpha, \beta}(-t)| \leq \frac{C_{\alpha, \beta}}{1+t}, \quad t > 0.
\]
For any $t \geq 0$, let

$$w(t) := t^\alpha / \Gamma(\alpha + 1),$$

$$y(t) := -\lambda^{-1} E_{\alpha,1}(-\lambda t^\alpha) + \lambda^{-1},$$

where $\lambda$ is a positive constant, then a straightforward calculation yields

$$D_{0^+}^\alpha y + \lambda y = 1, \quad t \geq 0.$$

If $1 < \alpha < 3/2$, then by (46) we obtain

$$\lambda^{2-3/\alpha} \| (y - w)^{''} \|_{L^2(0,T)}^2 = \int_0^T \lambda^{4-3/\alpha} t^{4\alpha-4} E_{\alpha,2\alpha-1}(-\lambda t^\alpha) \, dt$$

$$\leq C_\alpha \int_0^T \lambda^{4-3/\alpha} t^{4\alpha-4} (1 + \lambda t^\alpha)^{-2} \, dt$$

$$\leq C_\alpha \left( \int_0^{\lambda^{-1/\alpha}} \lambda^{4-3/\alpha} t^{4\alpha-4} \, dt + \int_{\lambda^{-1/\alpha}}^\infty \lambda^{2-3/\alpha} t^{4\alpha-4} \, dt \right)$$

$$\leq C_\alpha.$$

If $3/2 \leq \alpha < 2$, then an analogous deduction gives

$$\lambda^{2-5/\alpha} \| (y - w)^{'''} \|_{L^2(0,T)}^2 \leq C_\alpha.$$

Now, assume that $u_0 = u_1 = 0$. If $1 < \alpha < 3/2$ and $f(t) = v \in \dot{H}^{3/\alpha-2}(\Omega)$, $0 \leq t \leq T$, then by the techniques used in the proof of Theorem 4.2 we obtain

$$\| u - t^\alpha / \Gamma(\alpha + 1)v \|_{H^2(0,T;L^2(\Omega))} \leq C_\alpha \| v \|_{H^{3/\alpha-2}(\Omega)}.$$

Similarly, if $3/2 \leq \alpha < 2$ and $f(t) = v \in \dot{H}^{5/\alpha-2}(\Omega)$, $0 \leq t \leq T$, then

$$\| u - t^\alpha / \Gamma(\alpha + 1)v \|_{H^2(0,T;L^2(\Omega))} \leq C_\alpha \| v \|_{H^{3/\alpha-2}(\Omega)}.$$

Consequently, the temporal regularity of $u$ is essentially determined by $t^\alpha v$, and its temporal regularity can not exceed $H^{\alpha+1/2}$.

Analogously to Theorem 4.2, we have the following theorem.

**Theorem 4.3.** Assume that $\beta \geq 0, \gamma \geq -3\alpha/4$ or $\beta \geq -1, \gamma \geq -\alpha/4$. If $q \in 0H^\gamma(0,T;\dot{H}^\beta(\Omega))$, then there exists a unique

$$w \in 0H^{\alpha+\gamma}(0,T;\dot{H}^\beta(\Omega)) \cap 0H^\gamma(0,T;\dot{H}^{2+\beta}(\Omega))$$

such that

$$D_{T^-}^{\alpha+\gamma} w - \Delta D_{T^-}^{\gamma} w = D_{T^-}^{\gamma} q \quad \text{in} \quad L^2(0,T;\dot{H}^\beta(\Omega)),$$

and

$$\| w \|_{0H^{\alpha+\gamma}(0,T;\dot{H}^\beta(\Omega))} + \| w \|_{0H^\gamma(0,T;\dot{H}^{2+\beta}(\Omega))} \leq C_{\alpha,\gamma} \| q \|_{0H^\gamma(0,T;\dot{H}^\beta(\Omega))}.$$
4.2 Transposition method

In this subsection, we use the transposition method [17] to investigate the regularity of problem (1) with more general data. Let

\[ G := \mathcal{H}^0(0, T; L^2(\Omega)) \cap L^2(0, T; \dot{H}^2(\Omega)), \]

and endow this space with the norm

\[ \| \cdot \|_G := \max \left\{ \| \cdot \|_{\mathcal{H}^0(0, T; L^2(\Omega))}, \| \cdot \|_{L^2(0, T; \dot{H}^2(\Omega))} \right\}. \]

In addition, define

\[ G_0 := \{(D^2_{\alpha, \beta} v)(0) : v \in G\}, \quad G_1 := \{(D^2_{\alpha, \beta} v)(0) : v \in G\}, \]

and we equip them respectively with the norms

\[ \| v_0 \|_{G_0} := \inf_{v \in G} \| v \|_G, \quad \| v_1 \|_{G_1} := \inf_{v \in G} \| v \|_G. \]

Assuming that \( f \in G^* \), \( u_0 \in G_0 \) and \( u_1 \in G_1 \), we call \( u \in L^2(0, T; L^2(\Omega)) \) a weak solution to problem (1) if

\[
\langle u, D^2_{\alpha, \beta} v - \Delta v \rangle_{\mathcal{H}^0(0, T)} = \langle f, v \rangle_G + \langle u_0, (D^2_{\alpha, \beta} v)(0) \rangle_{G_0} + \langle u_1, (D^2_{\alpha, \beta} v)(0) \rangle_{G_1}, \tag{47}
\]

for all \( v \in G \).

**Theorem 4.4.** Problem (47) admits a unique solution \( u \), and

\[ \| u \|_{L^2(0, T; L^2(\Omega))} \leq C_0 \left( \| f \|_{G^*} + \| u_0 \|_{G_0} + \| u_1 \|_{G_1} \right). \tag{48} \]

**Proof.** For any \( v \in L^2(0, T; L^2(\Omega)) \), Theorem 3.3 implies that there exists a unique \( w \in G \) such that \( D^2_{\alpha, \beta} w - \Delta w = v \) and

\[ \| w \|_G \leq C_0 \| v \|_{L^2(0, T; L^2(\Omega))}. \]

Moreover, for any \( w \in G \), Lemma 3.6 implies that

\[ \| D^2_{\alpha, \beta} w \|_{L^2(0, T; L^2(\Omega))} \leq C_0 \| w \|_{\mathcal{H}^0(0, T; L^2(\Omega))}. \]

Therefore, applying the Babuška-Lax-Milgram theorem [15] yields that there exists a unique \( u \in L^2(0, T; L^2(\Omega)) \) such that (47) and (48) hold. This completes the proof of this theorem. \( \square \)

**Remark 4.6.** If \( u_0 \in \dot{H}^{-1/2}(\Omega) \) and \( u_1 \in \dot{H}^{-3/2}(\Omega) \), then an evident computation implies that \( u_0 \in G_0 \) and \( u_1 \in G_1 \). Hence, the weak solution is well-defined by (47) for \( u_0 \in \dot{H}^{-1/2}(\Omega), u_1 \in \dot{H}^{-3/2}(\Omega) \) and \( f \in G^* \). Moreover, if \( f \in \dot{W}^* \) and \( D^2_{\alpha, \beta}(u_0 + u_1) \in \dot{W}^* \), then the weak solutions defined by (31) and (47) are essentially the same.
5 A Petrov-Galerkin Method

Given \( J \in \mathbb{N}_{>0} \), let \( 0 = t_0 < t_1 < \ldots < t_J = T \) be a partition of \([0, T]\). Set \( \tau_j := t_j - t_{j-1} \) and \( I_j := (t_{j-1}, t_j) \) for each \( 1 \leq j \leq J \), and define \( \tau := \max_{1 \leq j \leq J} \tau_j \).

Let \( K_h \) be a shape-regular triangulation of \( \Omega \) consisting of \( d \)-simplexes, and we use \( h \) to denote the maximum diameter of the elements in \( K_h \). Define

\[
\begin{align*}
S_h := & \{ v_h \in H^1_0(\Omega) : v_h|_K \in P_1(K), \ \forall K \in K_h \}, \\
\tilde{W}_\tau := & \{ \tilde{u}_\tau \in L^2(0, T) : \tilde{u}_\tau|_{I_j} \in P_0(I_j), \ \forall 1 \leq j \leq J \}, \\
W_\tau := & \{ w_\tau \in H^1(0, T) : w_\tau|_{I_j} \in P_1(I_j), \ \forall 1 \leq j \leq J \}.
\end{align*}
\]

Above and throughout, \( P_h(\mathcal{O})(k = 0, 1) \) denotes the set of polynomials defined on \( \mathcal{O} \) with degree \( \leq k \), where \( \mathcal{O} \) is either an interval or an element of \( K_h \).

Moreover, define

\[
\begin{align*}
\tilde{W}_\tau \otimes S_h := & \text{span}\{ \tilde{u}_\tau v_h : \tilde{u}_\tau \in \tilde{W}_\tau, \ v_h \in S_h \}, \\
W_\tau \otimes S_h := & \text{span}\{ w_\tau v_h : w_\tau \in W_\tau, \ v_h \in S_h \}.
\end{align*}
\]

Assuming that \( u_0, u_1 \in S_h^* \) and \( f \in (\tilde{W}_\tau \otimes S_h)^* \), we define an approximation \( U \in W_\tau \otimes S_h \) to problem (1) by that \( U(0) = u_{0,h} \) and

\[
\begin{align*}
\langle D_0^{\alpha/2} U, V \rangle_{\Omega \times (0,T)} + \langle \nabla U, \nabla V \rangle_{\Omega \times (0,T)} \\
= \langle f, V \rangle + \langle u_1, \langle D_0^{\alpha/2} t, V \rangle_{\Omega \times (0,T)} \rangle_{S_h}
\end{align*}
\] (49)

for all \( V \in \tilde{W}_\tau \otimes S_h \), where \( u_{0,h} \) is the Ritz projection of \( u_0 \) onto \( S_h \) if \( u_0 \in H^1_0(\Omega) \) and \( u_{0,h} \) is the \( L^2(\Omega) \)-orthogonal projection of \( u_0 \) onto \( S_h \) if \( u_0 \in S_h \setminus H^1_0(\Omega) \).

Here the Ritz projection \( R_h : H^1_0(\Omega) \to S_h \) is defined by that

\[
\langle \nabla (v - R_h v), \nabla v \rangle_{\Omega} = 0 \quad \forall v \in S_h,
\]

for all \( v \in H^1_0(\Omega) \).

Similar to [6, Theorem 4.1], we have the following stability result.

**Theorem 5.1.** Problem (49) admits a unique solution \( U \). Moreover, if \( u_0 \in H^1(\Omega), u_1 \in L^2(\Omega) \) and \( f \in \alpha H^{1-\alpha/2}(0, T; L^2(\Omega)) \), then

\[
\| U' \|_{\alpha H^{1-\alpha/2}(0, T; L^2(\Omega))} + \| U \|_{C([0,T]; H^1(\Omega))} \\
\leq C_{\alpha,T} \left( \| u_0 \|_{H^1(\Omega)} + \| u_1 \|_{L^2(\Omega)} + \| f \|_{\alpha H^{1-\alpha/2}(0, T; L^2(\Omega))} \right).
\]

**Remark 5.1.** By (31), it is natural to develop another numerical algorithm for problem (1) as follows: given \( u_0, u_1 \in S_h^* \) and \( f \in \tilde{W}_\tau^* \), seek \( U \in \tilde{W}_\tau \otimes S_h \) such that

\[
\begin{align*}
\langle D_0^{\alpha/2} U, V \rangle_{\Omega \times (0,T)} + \langle \nabla D_0^{\alpha/2} U, \nabla V \rangle_{\Omega \times (0,T)} \\
= \langle f, D_0^{\alpha/2} V \rangle_{\tilde{W}} + \langle u_0, \langle D_0^{\alpha/2} t, V \rangle_{\Omega \times (0,T)} \rangle_{S_h} + \langle u_1, \langle D_0^{\alpha/2} t, V \rangle_{\Omega \times (0,T)} \rangle_{S_h}
\end{align*}
\] (50)

for all \( V \in \tilde{W}_\tau \otimes S_h \). Analogous to Theorem 4.1, if \( D_0^{\alpha/2}(u_0 + t u_1) \in \tilde{W}_\tau^* \), then

\[
\| U \|_{\alpha H^{1-\alpha/2}(0, T; L^2(\Omega))} + \| U \|_{\alpha H^{1-\alpha/2}(0, T; H^1(\Omega))} \\
\leq C_{\alpha} \left( \| f \|_{\tilde{W}_\tau^*} + \| D_0^{\alpha/2}(u_0 + t u_1) \|_{\tilde{W}_\tau} \right).
\]
Hence, by Theorem 5.1 this algorithm is more robust than algorithm (49); however, the computational cost of this algorithm is larger than that of the latter. To the author’s knowledge, this algorithm has never been proposed before. We will pay more attention to this algorithm in future works.

5.1 Convergence analysis

This subsection considers the convergence analysis of the Petrov-Galerkin method with \( u_0 = u_1 = 0 \). Let

\[
\sigma := \max_{1 \leq i,j \leq J} \{\tau_i / \tau_j\} \quad \text{and} \quad \rho := \max_{1 \leq i \leq J} \max_{1 \leq j \leq J} \{|\tau_i - \tau_j|\}.
\]

In what follows, \( a \lesssim b \) means that there exists a positive constant \( C \), depending only on \( \alpha, \rho, T, \Omega \) and the shape-regular parameter of \( K_h \), such that \( a \leq Cb \). In addition, \( a \sim b \) means that \( a \lesssim b \lesssim a \). The main results are the following two theorems.

**Theorem 5.2.** If \( f \in C(0,T;H^1(\Omega)) \), then

\[
\|u - U\|_{C([0,T];H^1(\Omega))} \lesssim \left( t^{(3-\alpha)/2} + \epsilon_1(\alpha, \tau, h) \right) \|f\|_{C(0,T;L^2(\Omega))},
\]

(51)

\[
\|u' - U'\|_{H^{(\alpha-1)/2}(0,T;L^2(\Omega))} \lesssim \left( t^{(3-\alpha)/2} + \epsilon_2(\alpha, \tau, h) \right) \|f\|_{C(0,T;L^2(\Omega))},
\]

(52)

where

\[
\epsilon_1(\alpha, \tau, h) := \begin{cases} \frac{h}{1 + \log h} & \text{if } 1 < \alpha < 3/2, \\ h^{3/\alpha - 1} + C_\sigma \tau^{3/2 - \alpha} h & \text{if } 3/2 < \alpha < 2, \end{cases}
\]

and

\[
\epsilon_2(\alpha, \tau, h) := \begin{cases} h^{3/\alpha - 1} & \text{if } 1 < \alpha < 3/2, \\ C_\sigma \tau^{3/2 - \alpha} h & \text{if } 3/2 < \alpha < 2. \end{cases}
\]

(53)

**Theorem 5.3.** If \( f \in L^2(0,T;L^2(\Omega)) \), then

\[
\|(u - U)\|_{H^{(\alpha-1)/2}(0,T;L^2(\Omega))} + \|u - U\|_{C([0,T];H^1(\Omega))} \lesssim \left( t^{(\alpha-1)/2} + \epsilon_3(\alpha, \tau, h) \right) \|f\|_{L^2(0,T;L^2(\Omega))},
\]

(54)

where

\[
\epsilon_3(\alpha, \tau, h) := \begin{cases} h^{1-1/\alpha} & \text{if } 1 < \alpha < 3/2, \\ C_\sigma \tau^{-1/2} h & \text{if } 3/2 < \alpha < 2. \end{cases}
\]

(55)

**Remark 5.2.** If \( 1 < \alpha < 3/2 \), then by Lemma 2.2 and Theorem 4.2, estimates (52) and (54) are optimal with respect to the regularity of \( u \); moreover, (51) is optimal and nearly optimal with respect to the regularity of \( u \) for \( 1 < \alpha < 3/2 \) and \( \alpha = 3/2 \), respectively. If \( 3/2 < \alpha < 2 \), then all the estimates (51), (52) and (54) are optimal with respect to the regularity of \( u \) provided that the temporal grid is quasi-uniform and \( h \leq C\tau^{\alpha/2} \) for some positive constant \( C \). However, numerical results indicate that the requirement \( h \leq C\tau^{\alpha/2} \) is unnecessary.
To prove the above two theorems, we need several interpolation operators as follows. Define

\[ W_0^j := \{ w_\tau \in \mathcal{A}^1(0,T) : w_\tau|_{I_j} \in P_1(I_j), \forall 1 \leq j \leq J \}. \]

For \( 0 \leq j \leq J \), set

\[ \omega_j := \begin{cases} (t_0, t_1), & j = 0, \\ (t_{j-1}, t_j), & j = J, \\ (t_{j-1}, t_{j+1}), & 1 \leq j < J, \end{cases} \]

and let \( \Pi_j \) be the \( L^2 \)-orthogonal projection operator onto \( P_1(\omega_j) \). We introduce the Clément interpolation operator \( Q_\tau : L^2(0,T) \to W_0^j \) by that, for all \( v \in L^2(0,T) \),

\[(Q_\tau v)(t_j) = (\Pi_j v)(t_j), \quad 1 \leq j \leq J.\]

For each \( 1 \leq j \leq J \), define

\[ \tilde{W}_{\tau,j} := \{ \tilde{w}_\tau \in L^2(0,T) : \tilde{w}_\tau|_{I_j} \in P_0(I_j), \forall I_j \in T_j \}, \]

where

\[ T_j := \begin{cases} \{ \omega_{2i} : 0 \leq i < j/2 \} \cup \{ I_j : j < i \leq J \}, & j \text{ is odd}, \\ \{ \omega_{2i-1} : 1 \leq i < j/2 \} \cup \{ I_j : j < i \leq J \}, & j \text{ is even}. \end{cases} \]

Let \( P_{\tau,j} \) be the \( L^2 \)-orthogonal projection operator onto \( \tilde{W}_{\tau,j} \), and define a family of modified Clément interpolation operators \( Q_{\tau,j} : L^2(0,T) \to W_0^j \) by that, for any \( v \in L^2(0,T) \),

\[ \begin{cases} \langle v - Q_{\tau,j} v, 1 \rangle_{\omega_i} = 0 & \text{ if } 1 \leq i < j \text{ and } |i - j| \text{ is odd}, \\ \langle Q_{\tau,j} v \rangle(p) = (Q_\tau v)(p) & \text{ if } p \text{ is a node of the partition } T_j. \end{cases} \]

By definition, it is evident that \( Q_{\tau,1} = Q_\tau \).

For the above interpolant operators and the Ritz projection operator \( R_h \), we have the following standard results [2, 24, 28], which will be used implicitly in our proofs. If \( v \in H^{\gamma}(0,T) \) with \( 1/2 < \gamma \leq 2 \), then

\[ \| (I - Q_\tau) v \|_{C[0,T]} \leq C_{P,\tau} \gamma T^{\gamma-1/2} \| v \|_{H^{\gamma}(0,T)}; \]

if \( v \in H^{\gamma}(0,T) \) with \( 0 \leq \gamma \leq 2 \), then

\[ \| (I - Q_\tau) v \|_{L^2(0,T)} \leq C_{P,\gamma,\tau} \gamma \| v \|_{H^{\gamma}(0,T)}. \]

If \( v \in H^{\gamma}(0,T) \) with \( 0 \leq \gamma \leq 1 \), then

\[ \| (I - P_{\tau,j}) v \|_{L^2(0,T)} \leq C_{\tau,\gamma} \gamma \| v \|_{H^{\gamma}(0,T)}; \]

for all \( 1 \leq j \leq J \). If \( v \in \dot{H}^r(\Omega) \) with \( 1 \leq r \leq 2 \), then

\[ \| (I - R_h) v \|_{L^2(\Omega)} + h \| (I - R_h) v \|_{\dot{H}^r(\Omega)} \lesssim h^r \| v \|_{\dot{H}^r(\Omega)} . \]

Except for those well-known results, we also need to establish some nonstandard error estimates of the interpolation operator \( Q_{\tau,j} \).
Lemma 5.1. If \( v \in L^2(0, T) \) and \( z \in _{0}H^{\gamma}(0, T) \) with \( 0 \leq \gamma \leq 1 \), then
\[
\langle (I - Q_{\tau,j})v, z \rangle_{(0,t_j)} \leq C_{\gamma} \tau^{\gamma} \| (I - Q_{\tau,j})v \|_{L^2(0,t_j)} \| z \|_{_{0}H^{\gamma}(0,t_j)}
\]
for all \( 1 \leq j \leq J \).

Proof. If \( j \) is even then, by the definitions of \( P_{\tau,j} \) and \( Q_{\tau,j} \),
\[
\langle (I - Q_{\tau,j})v, z \rangle_{(0,t_j)} = \langle (I - Q_{\tau,j})v, (I - P_{\tau,j})z \rangle_{(0,t_j)}
\]
\[
\leq \| (I - Q_{\tau,j})v \|_{L^2(0,t_j)} \| (I - P_{\tau,j})z \|_{L^2(0,t_j)}
\]
\[
\leq C_{\gamma} \tau^{\gamma} \| (I - Q_{\tau,j})v \|_{L^2(0,t_j)} \| z \|_{_{0}H^{\gamma}(0,t_j)},
\]
which proves (56). If \( j \) is odd then, also by the definitions of \( P_{\tau,j} \) and \( Q_{\tau,j} \),
\[
\langle (I - Q_{\tau,j})v, z \rangle_{(0,t_j)}
\]
\[
= \langle (I - Q_{\tau,j})v, z \rangle_{(0,t_j)} + \langle (I - Q_{\tau,j})v, (I - P_{\tau,j})z \rangle_{(t_{j-1}, t_j)}
\]
\[
\leq \| (I - Q_{\tau,j})v \|_{L^2(0,t_j)} \| (I - P_{\tau,j})z \|_{L^2(t_{j-1}, t_j)}
\]
\[
\leq \| (I - Q_{\tau,j})v \|_{L^2(0,t_j)} \| z \|_{L^2(t_{j-1}, t_j)} + C_{\gamma} \tau^{\gamma} \| z \|_{_{0}H^{\gamma}(0,t_j)}.
\]

Since Lemma 3.5 implies that \( z = D_{0+}^{\gamma} D_{0+}^{\gamma} z \), by Lemmas 3.2 and 3.6 we have
\[
\| z \|_{L^2(0,t_1)} = \| D_{0+}^{\gamma} D_{0+}^{\gamma} z \|_{L^2(0,t_1)}
\]
\[
\leq C_{\gamma} t_1^{\gamma} \| D_{0+}^{\gamma} z \|_{L^2(0,t_1)} \leq C_{\gamma} t_1^{\gamma} \| z \|_{_{0}H^{\gamma}(0,t_1)}.
\]

Therefore, combining the above inequality and (57) proves (56) in the case that \( j \) is odd. This completes the proof of this lemma.

Lemma 5.2. If \( v \in _{0}H^{\gamma}(0, T) \) with \( 0 \leq \gamma \leq 1 \), then
\[
\| (I - Q_{\tau,j})v \|_{L^2(0,T)} \leq C_{\rho, \gamma} \tau^{\gamma} \| v \|_{_{0}H^{\gamma}(0,T)}
\]
for all \( 1 \leq j \leq J \).

Proof. If \( j = 1 \) then (58) is standard, and so we assume that \( 2 \leq j \leq J \). By the definition of \( Q_{\tau,j} \), \( (Q_{\tau,j})v(0) = (Q_{\tau})v(0) \), \( (Q_{\tau,j})v(t_{j-1}, T) = (Q_{\tau})v(t_{j-1}, T) \), and \( (Q_{\tau,j})v(t_{j-1}) = (Q_{\tau})v(t_{j-1}) \) if \( 1 \leq i \leq j \) and \( j - i \) is even. If \( 1 \leq i \leq j - 1 \) and \( j - i \) is odd, then a straightforward calculation gives
\[
(Q_{\tau,j}v - Q_{\tau}v)(t_i) = \frac{2}{|\omega_i|} \int_{\omega_i} (v - Q_{\tau}v)(t) \, dt,
\]
and hence
\[
\| (Q_{\tau,j}v - Q_{\tau}v) \|_{L^2(\omega_i)} = \sqrt{\frac{|\omega_i|}{3}} \| (Q_{\tau,j}v - Q_{\tau}v)(t_i) \| \leq \frac{2}{\sqrt{3}} \| (I - Q_{\tau})v \|_{L^2(\omega_i)},
\]
which implies
\[
\| Q_{\tau,j}v - Q_{\tau}v \|_{L^2(0,T)} \leq \frac{2}{\sqrt{3}} \| (I - Q_{\tau})v \|_{L^2(0,T)}.
\]

It follows that
\[
\| (I - Q_{\tau,j})v \|_{L^2(0,T)} \leq \frac{2 + \sqrt{3}}{\sqrt{3}} \| (I - Q_{\tau})v \|_{L^2(0,T)} \leq C_{\rho, \gamma} \tau^{\gamma} \| v \|_{_{0}H^{\gamma}(0,T)}.
\]

This proves (58) and thus concludes the proof.
Lemma 5.3. Assume that $0 < \beta < 1/2$. If $v \in aH^\gamma(0,T)$ with $\beta + 1 \leq \gamma \leq 2$, then
\begin{equation}
\| (v - Q_{\tau,j}v) \|_{aH^\gamma(0,T)} \leq C_{p,\beta,\gamma,T}{\tau^{-1-\beta}}\|v\|_{aH^\gamma(0,T)},
\end{equation}
for all $1 \leq j \leq J$.

Proof. For simplicity, set $g := (v - Q_{\tau,j}v)$. Following the proof of [8, Lemma 4.1], we obtain
\begin{equation}
\|g\|^2_{aH^\gamma(0,T)} \leq C_{\beta} \|g\|^2_{H^\gamma(0,T)} \leq C_{\beta,T}(I_1 + I_2),
\end{equation}
where
\begin{align*}
I_1 & := \sum_{i=1}^J \int_{I_i} \int_{I_i} \frac{|g(s) - g(t)|^2}{|s - t|^{1+2\beta}} \, ds \, dt, \\
I_2 & := \sum_{i=1}^J \int_{I_i} g^2(t) \left( (t_i - t)^{-2\beta} + (t - t_{i-1})^{-2\beta} \right) \, dt.
\end{align*}
As $I_1$ can be estimated by that
\begin{align*}
I_1 & \leq \sum_{i=1}^J \tau_i^{2(\gamma-1-\beta)} \int_{I_i} \int_{I_i} \frac{|g(s) - g(t)|^2}{|s - t|^{1+2(\gamma-1)}} \, ds \, dt \\
& = \sum_{i=1}^J \tau_i^{2(\gamma-1-\beta)} \int_{I_i} \int_{I_i} \frac{|v'(s) - v'(t)|^2}{|s - t|^{1+2(\gamma-1)}} \, ds \, dt \\
& \leq C_{\gamma,T}{\tau^{2(\gamma-1-\beta)}} \|v'\|^2_{L^{\gamma-1}(0,T)} \\
& \leq C_{\gamma,T}{\tau^{2(\gamma-1-\beta)}} \|v\|^2_{aH^\gamma(0,T)},
\end{align*}
it remains to prove
\begin{equation}
I_2 \leq C_{p,\beta,\gamma,T}{\tau^{2(\gamma-1-\beta)}} \|v\|^2_{aH^\gamma(0,T)}.
\end{equation}
By [17, Theorem 11.2] and the fact that $H^{\gamma-1}(0,1)$ is continuously embedded in $H^\beta(0,1)$, a simple calculation yields
\begin{align*}
\int_0^1 |z(t)|^2 \left( t^{-2\beta} + (1 - t)^{-2\beta} \right) \, dt & \leq C_{\beta} \|z\|^2_{H^\gamma(0,1)} \leq C_{\beta,\gamma} \|z\|^2_{H^{\gamma-1}(0,1)} \\
& \leq C_{\beta,\gamma} \left( \|z\|^2_{L^2(0,1)} + \int_0^1 \int_0^1 \frac{|z(s) - z(t)|^2}{|s - t|^{1+2(\gamma-1)}} \, ds \, dt \right),
\end{align*}
where $z \in H^{\gamma-1}(0,1)$. Hence, a standard scaling argument gives
\begin{align*}
I_2 & \leq C_{\beta,\gamma} \sum_{i=1}^J \left( \tau_i^{-2\beta} \|g\|^2_{L^2(I_i)} + \tau_i^{2(\gamma-1-\beta)} \int_{I_i} \int_{I_i} \frac{|v'(s) - v'(t)|^2}{|s - t|^{1+2(\gamma-1)}} \, ds \, dt \right) \\
& \leq C_{\beta,\gamma,T} \left( \sum_{i=1}^J \tau_i^{-2\beta} \|g\|^2_{L^2(I_i)} + \tau^{2(\gamma-1-\beta)} \|v'\|^2_{H^{\gamma-1}(0,T)} \right) \\
& \leq C_{\beta,\gamma,T} \left( \sum_{i=1}^J \tau_i^{-2\beta} \|g\|^2_{L^2(I_i)} + \tau^{2(\gamma-1-\beta)} \|v\|^2_{aH^\gamma(0,T)} \right).
\end{align*}
To prove (61), it suffices to prove
\[
\sum_{i=1}^{J} \tau_i^{-2\beta} \|g\|_{L^2(I_i)}^2 \leq C_{\rho,\beta,\gamma,T} T^{2(\gamma-1-\beta)} \|v\|_{aH^{\gamma}(0,T)}^2.
\] (62)
By (59), we have
\[
\|(Q_\tau v - Q_{\tau,j} v')\|_{L^2(\omega_i)} \leq C_{\rho} |\omega_i|^{-1} \|(I-Q_\tau)v\|_{L^2(\omega_i)},
\]
for all \(1 \leq i < j - 1\) such that \(j - i\) is odd. Hence it follows that
\[
\sum_{i=1}^{J} \tau_i^{-2\beta} \|(Q_\tau v - Q_{\tau,j} v')\|_{L^2(I_i)}^2 \leq C_{\rho} \sum_{i=1}^{J} \tau_i^{-2\beta - 2} \|(I-Q_\tau)v\|_{L^2(I_i)}^2,
\]
which yields the inequality
\[
\sum_{i=1}^{J} \tau_i^{-2\beta} \|g\|_{L^2(I_i)}^2 \leq C_{\rho} \sum_{i=1}^{J} \tau_i^{-2\beta - 2} \left( \|(I-Q_\tau)v\|_{L^2(I_i)}^2 + \tau_i^2 \|(v - Q_\tau v')\|_{L^2(I_i)}^2 \right).
\]
Therefore, by the standard estimates
\[
\sum_{i=1}^{J} \tau_i^{-2\beta} \|(v - Q_\tau v')\|_{L^2(I_i)}^2 \leq C_{\rho,\beta,\gamma,T} T^{2(\gamma-1-\beta)} \|v\|_{aH^{\gamma}(0,T)}^2,
\]
\[
\sum_{i=1}^{J} \tau_i^{-2\beta - 2} \|(I-Q_\tau)v\|_{L^2(I_i)}^2 \leq C_{\rho,\beta,\gamma,T} T^{2(\gamma-1-\beta)} \|v\|_{aH^{\gamma}(0,T)}^2,
\]
we obtain (62). This completes the proof of this lemma. \[\blacksquare\]

Lemma 5.4. If \(w \in W_r\), then
\[
\|w\|_{aH^{\beta+1}(0,T)} \leq C_{\sigma,\beta,\gamma,T} T^{\gamma-\beta-1} \|w\|_{aH^{\gamma}(0,T)}
\] (63)
for all \(0 \leq \beta < 1/2\) and \(0 \leq \gamma \leq \beta + 1\).

Proof. If \(\beta = 0\), then (63) is trivial for \(\gamma = 1\) and standard for \(\gamma = 0\), and hence applying [21, Lemma 22.3] yields (63) for \(0 < \gamma < 1\). It remains therefore to prove (63) for \(0 < \beta < 1/2\) and \(0 \leq \gamma \leq \beta + 1\). To this end, we assume \(0 < \beta < 1/2\). For any \(1 \leq \gamma \leq \beta + 1\), following the proof of [8, Lemma 4.1], we obtain
\[
\|w\|_{aH^{\beta+1}(0,T)} \leq C_{\beta,T} \|w\|_{aH^{\gamma}(0,T)}^2
\]
\[
\leq C_{\beta,T} \sum_{i=1}^{J} \int_{t_{i-1}}^{t_i} |w'(t)|^2 \left( (t_i - t)^{-2\beta} + (t - t_{i-1})^{-2\beta} \right) dt
\]
\[
\leq C_{\beta,T} \sum_{i=1}^{J} \tau_i^{2\gamma-2\beta-2} \int_{t_{i-1}}^{t_i} |w'(t)|^2 \left( (t_i - t)^{2-2\gamma} + (t - t_{i-1})^{2-2\gamma} \right) dt
\]
\[
\leq C_{\sigma,\beta,\gamma,T} T^{2\gamma-2\beta-2} \|w\|_{aH^{\gamma-1}(0,T)}^2.
\]
Hence, by the estimates
\[
\|w\|_{aH^{\beta+1}(0,T)} \leq C_{\beta} \|w\|_{aH^{\gamma}(0,T)} \quad \text{and} \quad \|w\|_{aH^{\gamma-1}(0,T)} \leq C_{\gamma} \|w\|_{aH^{\gamma}(0,T)},
\]

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which can be easily proved by Lemma 3.6, we have
\[
\|w\|_{aH^{\gamma+1}(0,T)} \leq C_\beta \|w\|_{aH^\gamma(0,T)} \leq C_{\sigma,\beta,\gamma,T} T^{-\beta} \|w\|_{aH^{\gamma-1}(0,T)} \leq C_{\sigma,\beta,\gamma,T} T^{-\beta} \|w\|_{aH^\gamma(0,T)},
\]
namely (63) holds for \(1 \leq \gamma \leq \beta + 1\). In addition, for any \(0 \leq \gamma < 1\), since we have already proved
\[
\|w\|_{aH^\gamma(0,T)} \leq C_{\sigma,\gamma,T} T^{-\gamma} \|w\|_{aH^\gamma(0,T)},
\]
\[
\|w\|_{aH^{\gamma+1}(0,T)} \leq C_{\sigma,\beta,\gamma,T} T^{-\beta} \|w\|_{aH^\gamma(0,T)},
\]
it is clear that (63) holds. This completes the proof. \(\blacksquare\)

**Lemma 5.5.** If \(f \in aH^{2-\alpha}(0,T;L^2(\Omega))\), then
\[
\|U - Q_{\tau,j} R_h u\|_{aH^{(\alpha-1)/2}(0,T;L^2(\Omega))} + \|U - Q_{\tau,j} R_h u\|_{H^1(\Omega)} \lesssim \tau^{(3-\alpha)/2} \varepsilon_2(\alpha, \tau, h) \|f\|_{aH^{2-\alpha}(0,T;L^2(\Omega))},
\]
for all \(1 \leq j \leq J\), where \(\varepsilon_2(\alpha, \tau, h)\) is defined by (53).

**Proof.** By (45) and (49),
\[
\langle D_{0+}^{\alpha-1}(u - U), \xi_j \rangle_{\Omega \times (0,t_j)} + \langle \nabla (u - U), \nabla \xi_j \rangle_{\Omega \times (0,t_j)} = 0,
\]
where \(\xi_j = U - Q_{\tau,j} R_h u\). As \(\xi_j(0) = 0\), using integration by parts gives
\[
2 \langle \nabla \xi_j, \nabla \xi_j \rangle_{\Omega \times (0,t_j)} = \|\xi_j(t_j)\|_{H^1(\Omega)}^2,
\]
and a simple calculation then yields
\[
\langle D_{0+}^{\alpha-1} \xi_j, \xi_j \rangle_{\Omega \times (0,t_j)} + \frac{1}{2} \|\xi_j(t_j)\|_{H^1(\Omega)}^2 = \langle \nabla (u - Q_{\tau,j} R_h u), \nabla \xi_j \rangle_{\Omega \times (0,t_j)} + \langle D_{0+}^{\alpha-1} (u - Q_{\tau,j} R_h u), \xi_j \rangle_{\Omega \times (0,t_j)}.
\]
It follows from Lemmas 3.6 and 3.8 that
\[
\|\xi_j\|_{aH^{(\alpha-1)/2}(0,T;L^2(\Omega))}^2 + \|\xi_j(t_j)\|_{H^1(\Omega)}^2 \lesssim E_1 + E_2 + E_3,
\]
where
\[
E_1 := \langle D_{0+}^{\alpha-1} (u - Q_{\tau,j} u), \xi_j \rangle_{\Omega \times (0,t_j)},
\]
\[
E_2 := \langle \nabla (u - Q_{\tau,j} R_h u), \nabla \xi_j \rangle_{\Omega \times (0,t_j)};
\]
\[
E_3 := \langle D_{0+}^{\alpha-1} (Q_{\tau,j}(u - R_h u)), \xi_j \rangle_{\Omega \times (0,t_j)}.
\]

Next, let us estimate \(E_1, E_2\) and \(E_3\) one by one. Since applying Lemma 5.3 indicates
\[
\|U - Q_{\tau,j} R_h u\|_{aH^{(\alpha-1)/2}(0,T;L^2(\Omega))} \lesssim \tau^{(3-\alpha)/2} \|u\|_{aH^2(0,T;L^2(\Omega))},
\]

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by Lemmas 3.6 and 3.8 and Theorem 4.2 we obtain

\[ E_1 \preceq \| D_0^{(\alpha - 1)/2} (u - Q_{\tau,j} u) \|_{L^2(0,t_j,L^2(\Omega))} + \| D_0^{(\alpha - 1)/2} \xi_j \|_{L^2(0,t_j,L^2(\Omega))} \]

\[ \preceq \| (u - Q_{\tau,j} u) \|_{H^{(\alpha - 1)/2}(0,T;L^2(\Omega))} \| \xi_j \|_{H^{(\alpha - 1)/2}(0,T;L^2(\Omega))} \]

\[ \preceq \tau^{(3 - \alpha)/2} \| f \|_{H^{2 - \alpha}(0,T;L^2(\Omega))} \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} . \]

By Lemma 5.1 and the definition of \( R_h \),

\[ E_2 = \langle \nabla (u - Q_{\tau,j} u), \nabla \xi_j \rangle_{\Omega \times (0,t_j)} = \langle (Q_{\tau,j} - I) \Delta u, \xi_j \rangle_{\Omega \times (0,t_j)} \]

\[ \preceq \tau^{(\alpha - 1)/2} \| (I - Q_{\tau,j}) u \|_{L^2(0,T;H^2(\Omega))} \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} . \]

so that using Theorem 4.2 and Lemma 5.2 gives

\[ E_2 \preceq \tau^{(3 - \alpha)/2} \| f \|_{H^{2 - \alpha}(0,T;L^2(\Omega))} \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} . \]

If \( 1 < \alpha \leq 3/2 \), then applying Lemma 5.3 indicates

\[ \| (Q_{\tau,j}(u - R_h u)) \|_{H^{(\alpha - 1)/2}(0,T;L^2(\Omega))} \]

\[ \preceq \| (I - R_h) u \|_{H^{(\alpha + 1)/2}(0,T;L^2(\Omega))} \]

\[ \preceq C_\alpha \tau^{3/2 - \alpha} \| (Q_{\tau,j}(u - R_h u)) \|_{H^{1 - \alpha/2}(0,T;L^2(\Omega))} \]

\[ \preceq C_\alpha \tau^{3/2 - \alpha} \| (I - R_h) u \|_{H^{2 - \alpha/2}(0,T;L^2(\Omega))} \]

\[ \preceq C_\alpha \tau^{3/2 - \alpha} \| h \|_{H^{2 - \alpha/2}(0,T;H^1(\Omega))} . \]

and if \( 3/2 < \alpha < 2 \), then by Lemmas 5.3 and 5.4,

\[ \| (Q_{\tau,j}(u - R_h u)) \|_{H^{(\alpha - 1)/2}(0,T;L^2(\Omega))} \]

\[ \preceq \tau^{(3 - \alpha)/2} \| (Q_{\tau,j}(u - R_h u)) \|_{H^{1 - \alpha/2}(0,T;L^2(\Omega))} \]

\[ \preceq \tau^{(3 - \alpha)/2} \| (I - R_h) u \|_{H^{2 - \alpha/2}(0,T;L^2(\Omega))} \]

\[ \preceq \tau^{(3 - \alpha)/2} \| h \|_{H^{2 - \alpha/2}(0,T;H^1(\Omega))} . \]

Therefore, by Lemmas 2.2, 3.6 and 3.8 and Theorem 4.2 we get

\[ E_3 \preceq \| D_0^{(\alpha - 1)/2} (Q_{\tau,j}(u - R_h u)) \|_{L^2(0,t_j,L^2(\Omega))} + \| D_0^{(\alpha - 1)/2} \xi_j \|_{L^2(0,t_j,L^2(\Omega))} \]

\[ \preceq \| (Q_{\tau,j}(u - R_h u)) \|_{H^{(\alpha - 1)/2}(0,T;L^2(\Omega))} + \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} \]

\[ \preceq \varepsilon_2(\alpha, \tau, h) \| f \|_{H^{2 - \alpha}(0,T;L^2(\Omega))} \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} \].

Finally, combining the estimates of \( E_1, E_2 \) and \( E_3 \) and the Young’s inequality with \( \varepsilon \), we obtain that

\[ \| \xi_j \|_{H^{(\alpha - 1)/2}(0,t_j,L^2(\Omega))} + \| \xi_j (t_j) \|_{H^1(\Omega)} \]

\[ \preceq (\tau^{(3 - \alpha)/2} + \varepsilon_2(\alpha, \tau, h)) \| f \|_{H^{2 - \alpha}(0,T;L^2(\Omega))} , \]

for all \( 1 \leq j \leq J \). This proves (64) and thus concludes the proof.

\[ \blacksquare \]

Remark 5.3. Assume that \( f \in H^{2 - \alpha}(0,T;L^2(\Omega)) \) with \( 3/2 < \alpha < 2 \). By Lemma 2.2 and Theorem 4.2 we have

\[ u \in H^{(\alpha + 1)/2}(0,T;H^{3/2 - \alpha - 1}(\Omega)) \cap H^{2 - \alpha/2}(0,T;H^1(\Omega)). \]
Therefore, \((R_h u)' = R_h u' \in H^{(\alpha - 1)/2}(0, T; \dot{H}^{3/\alpha - 1}(\Omega))\) may not make sense since \(3/\alpha - 1 < 1\), but \((R_h u)' = R_h u' \in H^{1-\alpha/2}(0, T; \dot{H}^1(\Omega))\) makes sense indeed. This is the reason why we use

\[ \tau^{3/2 - \alpha} \|(Q_{\tau,j}(u - R_h u))'\|_{H^{1-\alpha/2}(0, T; L^2(\Omega))} \]

to bound

\[ \|(Q_{\tau,j}(u - R_h u))'\|_{H^{(\alpha - 1)/2}(0, T; L^2(\Omega))}, \]

when estimating the term \(E_3\) in the proof of the above lemma.

Analogously to Lemma 5.5, we have the following lemma.

**Lemma 5.6.** If \(f \in L^2(0, T; L^2(\Omega))\), then

\[
\|(U - Q_{\tau,j}R_h u)'\|_{H^{(\alpha - 1)/2}(0, t_j, L^2(\Omega))} + \|(U - Q_{\tau,j}R_h u)(t_j)\|_{H^1(\Omega)}
\leq \left(\tau^{(\alpha - 1)/2} + C_\tau \tau^{-1/2} h\right) \|f\|_{L^2(0, T; L^2(\Omega))},
\]

for all \(1 \leq j \leq J\).

**Proof of Theorem 5.2.** By Lemma 2.2 and Theorem 4.2, we have

\[
\|(I - Q_{\tau,j})R_h u\|_{C([0, T]; H^1(\Omega))} \leq \tau^{(3 - \alpha)/2} \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))},
\]

\[
\|(I - R_h)u\|_{C([0, T]; H^1(\Omega))} \leq \begin{cases} h \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))} & \text{if } 1 < \alpha < 3/2, \\ h^{1+\alpha} \|f\|_{H^{1/2}(0, T; L^2(\Omega))} & \text{if } \alpha = 3/2, \\ h^{3/\alpha - 1} \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))} & \text{if } 3/2 < \alpha < 2, \end{cases}
\]

where \(0 < \epsilon \leq 1/2\). Therefore, if \(\alpha = 3/2\), then letting \(\epsilon = 1/(2 + |\log h|)\) yields

\[
\|(I - R_h)u\|_{C([0, T]; H^1(\Omega))} \leq \left(1 + |\log h|\right) h \|f\|_{H^{1/2}(0, T; L^2(\Omega))}.
\]

Since \((Q_{\tau,j}R_h u)(t_j) = (Q_{\tau}R_h u)(t_j)\) for all \(1 \leq j \leq J\), we obtain

\[
\|Q_{\tau,j}R_h u - U\|_{C([0, T]; H^1(\Omega))} = \max_{1 \leq j \leq J} \|(Q_{\tau,j}R_h u - U)(t_j)\|_{H^1(\Omega)}
= \max_{1 \leq j \leq J} \|(Q_{\tau,j}R_h u - U)(t_j)\|_{H^1(\Omega)} \leq \left(\tau^{(3 - \alpha)/2} + c_2(\alpha, \tau, h)\right) \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))},
\]

by Lemma 5.5, and hence,

\[
\|R_h u - U\|_{C([0, T]; H^1(\Omega))}
\leq \|(I - Q_{\tau})R_h u\|_{C([0, T]; H^1(\Omega))} + \|Q_{\tau}R_h u - U\|_{C([0, T]; H^1(\Omega))}
\leq \left(\tau^{(3 - \alpha)/2} + c_2(\alpha, \tau, h)\right) \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))}, \quad (65)
\]

Therefore, using the triangle inequality

\[
\|u - U\|_{C([0, T]; H^1(\Omega))} \leq \|u - R_h u\|_{C([0, T]; H^1(\Omega))} + \|R_h u - U\|_{C([0, T]; H^1(\Omega))}
\]

proves (51). Since the proof of Lemma 5.5 yields that

\[
\|(u - Q_{\tau,j}R_h u)'\|_{H^{(\alpha - 1)/2}(0, T; L^2(\Omega))}
\leq \|(u - Q_{\tau,j}u)'\|_{H^{(\alpha - 1)/2}(0, T; L^2(\Omega))} + \|(Q_{\tau,j}(u - R_h u))'\|_{H^{(\alpha - 1)/2}(0, T; L^2(\Omega))}
\leq \left(\tau^{(3 - \alpha)/2} + c_2(\alpha, \tau, h)\right) \|f\|_{H^{2 - \alpha}(0, T; L^2(\Omega))},
\]

proves (51).
(52) follows from the above inequality and Lemma 5.5. This concludes the proof of the theorem. ■

**Proof of Theorem 5.3.** In view of Lemma 5.6, the proof of the case $3/2 < \alpha < 2$ is completely analogous to that of Theorem 5.2. Therefore, we only give the proof for $1 < \alpha \leqslant 3/2$ using the theory of interpolation space. As Theorems 4.2 and 5.1 imply

$$\| (u - U)^{'} \|_{\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega))} \lesssim \| f \|_{\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega))},$$

by (52) and the fact

$$L^2(0; T; L^2(\Omega)) = [\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega)), \alpha H^{2-\alpha}(0, T; L^2(\Omega))]_{(\alpha-1)/(3-\alpha), 2},$$

applying [21, Lemma 22.3] yields

$$\| (u - U)^{'} \|_{\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega))} \lesssim (\tau^{(\alpha-1)/2} + h^{3/\alpha - 1})^{(\alpha-1)/2} \| f \|_{L^2(0, T; L^2(\Omega))}.$$  

Hence, from the inequality

$$\left(\tau^{(\alpha-1)/2} + h^{3/\alpha - 1}\right)^{(\alpha-1)/2} < \tau^{(\alpha-1)/2} + h^{1-1/\alpha},$$

it follows that

$$\| (u - U)^{'} \|_{\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega))} \lesssim \left(\tau^{(\alpha-1)/2} + h^{1-1/\alpha}\right) \| f \|_{L^2(0, T; L^2(\Omega))}. \quad (66)$$

In addition, Theorems 4.2 and 5.1 imply

$$\| R_h u - U \|_{C([0, T]; H^1(\Omega))} \lesssim \| f \|_{\alpha H^{(1-\alpha)/2}(0, T; L^2(\Omega))},$$

and then, by this estimate and (65), proceeding as in the proof of (66) yields

$$\| R_h u - U \|_{C([0, T]; H^1(\Omega))} \lesssim \left(\tau^{(\alpha-1)/2} + h^{1-1/\alpha}\right) \| f \|_{L^2(0, T; L^2(\Omega))}. \quad (67)$$

Therefore, since Theorem 4.2 gives

$$\| u - R_h u \|_{C([0, T]; H^1(\Omega))} \lesssim h^{1-1/\alpha} \| f \|_{L^2(0, T; L^2(\Omega))},$$

we obtain

$$\| u - U \|_{C([0, T]; H^1(\Omega))} \lesssim \| u - R_h u \|_{C([0, T]; H^1(\Omega))} + \| R_h u - U \|_{C([0, T]; H^1(\Omega))} \lesssim \left(\tau^{(\alpha-1)/2} + h^{1-1/\alpha}\right) \| f \|_{L^2(0, T; L^2(\Omega))}.$$  

Finally, combining (66) and (67) proves (54) and thus concludes the proof of this theorem. ■

### 6 Numerical Experiments

In this section, we present some numerical examples to validate Theorems 5.2 and 5.3 in one dimensional case. We set $\Omega := (0, 1)$, $T := 1$, and use the uniform temporal and spatial grids. Define

$$E_1(U) := \| \tilde{u} - U \|_{C([0, T]; H^1(\Omega))},$$

$$E_2(U) := \| D_0^{(\alpha-1)/2} (\tilde{u} - U) \|_{L^2(0, T; L^2(\Omega))},$$
where $\tilde{u}$ is the numerical solution with $\tau = 2^{-17}$ and $h = 2^{-10}$. Here we observe that Lemma 3.6 implies

$$\| (\tilde{u} - U)' \|_{0, H^{(\alpha-1)/2}(0,T;L^2(\Omega))} \sim \| D_0^{(\alpha-1)/2} (\tilde{u} - U)' \|_{L^2(0,T;L^2(\Omega))}.$$ 

It is easy to see that (49) yields a block triangular Toeplitz-like with tri-diagonal block system, so that we can apply a fast direct $O(h^{-1}J(\log J)^2)$ solver based on the divide-and-conquer strategy [26] to solve this system efficiently. Additionally, the calculation of $\mathcal{E}_2(U)$ involves only the matrix-vector multiplication of a block triangular Toeplitz-like matrix, which can be completed within computational cost of $O(h^{-1}J\log J)$ by fast Fourier transform.

**Example 1.** This example adopts

$$f(x,t) = t^{-0.49}x^{-0.49}, \quad (x,t) \in \Omega \times (0,T).$$

The relationship between the spatial errors and the spatial step sizes are displayed in Fig. 1. These numerical results indicate that

$$\mathcal{E}_1(U) \approx O(h^{1-1/\alpha}), \quad \mathcal{E}_2(U) \approx O(h^{1-1/\alpha}).$$

The relationship between the errors and the temporal step sizes are plotted in Fig. 2, which demonstrate that

$$\mathcal{E}_1(U) \approx O(\tau^{(\alpha-1)/2}), \quad \mathcal{E}_2(U) \approx O(\tau^{(\alpha-1)/2}).$$

Therefore, if $1 < \alpha \leq 3/2$ then numerical results coincide well with Theorem 5.3. However, for $3/2 < \alpha < 2$, numerical results also show the optimal accuracy of $\mathcal{E}_1(U)$ and $\mathcal{E}_2(U)$ with respect to the regularity, without the restriction that $h \leq C_{\tau}^{\alpha/2}$.

![Figure 1: Spatial errors of Example 1, $\tau = 2^{-17}$.](image)
Example 2. This example employs
\[ f(x, t) = t^{1.51 - \alpha} x^{-0.49}, \quad (x, t) \in \Omega \times (0, T), \]
and the spatial errors and temporal errors are plotted in Figs. 3 and 4, respectively. These numerical results demonstrate that
\[
\begin{align*}
\mathcal{E}_1(U) &\approx \begin{cases} 
\mathcal{O}(\tau^{(3-\alpha)/2} + h) & \text{if } 1 < \alpha \leq 3/2, \\
\mathcal{O}(\tau^{(3-\alpha)/2} + h^{3/\alpha - 1}) & \text{if } 3/2 < \alpha < 2,
\end{cases} \\
\mathcal{E}_2(U) &\approx \mathcal{O}(\tau^{(3-\alpha)/2} + h^{3/\alpha - 1}).
\end{align*}
\]
Hence, if \(1 < \alpha \leq 3/2\), then numerical results verify the theoretical predictions of Theorem 5.2. But for \(3/2 < \alpha < 2\), numerical results also indicate that the convergence rates of \(\mathcal{E}_1(U)\) and \(\mathcal{E}_2(U)\) are optimal with respect to the regularity, without the requirement \(h \leq C \tau^{\alpha/2}\).
7 Concluding remarks

This paper concerns the convergence of a Petrov-Galerkin method for fractional wave problems with nonsmooth data. The weak solution and its regularity are studied by the variational approach. Optimal error estimate with respect to the regularity of the solution under the norm $C([0, T]; \dot{H}^1(\Omega))$ is derived if $f \in L^2(0, T; L^2(\Omega))$ and $1 < \alpha \leq 3/2$, and numerical results validate this theoretical result. For $3/2 < \alpha < 2$, similar optimal error estimate is also derived under the restriction that the temporal grid is quasi-uniform and $h \leq C\tau^{\alpha/2}$; however, numerical results demonstrate that the restriction $h \leq C\tau^{\alpha/2}$ is unnecessary.

In addition, optimal error estimates with respect to the regularity of the solution or the degrees of polynomials used in the discretization under the norm $C([0, T]; L^2(\Omega))$ have not been established, and this is our ongoing work.

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