Coupled Dispersionless and Generalized Heisenberg Ferromagnet Equations with Self-Consistent Sources: Geometry and Equivalence

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Abstract

We propose a new integrable coupled dispersionless equation with self-consistent sources (CDESCS). We obtain the Lax pair and the equivalent generalized Heisenberg ferromagnet equation (GHFE), demonstrating its integrability. Specifically, we explore the geometry of these equations. Last, we consider the relation between the motion of curves/surfaces and the CDESCS and the GHFE.

1 Introduction

The study of integrable systems or solvable nonlinear differential equations (NDE) is an active area of research since the discovery of the inverse scattering method. These equations are in a sense universal since they show up in many areas of physics and mathematics. As integrable systems we understand those which have infinite hierarchy of symmetries and conservation laws. In the theory of solvable nonlinear differential equations, one of the most important issues is a systematic method for construction of integrable systems. For integrable systems there exist several parallel schemes of construction.

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Besides the integrable NDEs, there is another important class of integrable partial differential equations: the so-called integrable equations of hydrodynamic type, often called dispersionless equations [1]-[7]. In some cases, these equations are the dispersionless (or semiclassical) limits of integrable soliton systems, or by construction, such as the system of hydrodynamic type. They often arise in various problems of physics and mathematics, for this reason, they are intensively studied in the recent years. Some of these equations are integrable in the Hamiltonian sense. Integrable dispersionless equations are equivalent to the commutation condition of vector fields Lax pairs, so that they can be in an arbitrary number of dimensions. For example, in [1]-[3], a new systematic method was introduced for construction of dispersionless systems in 3+1 dimensions using nonisospectral Lax pairs that involve contact vector fields.

In this paper we deal with the so-called Konno-Oono equation (KOE) [8]

\[
\begin{align*}
q_{xt} - \rho q &= 0, \\
\rho_x + 0.5(q^2)_t &= 0. 
\end{align*}
\]

The KOE (1) is also known as the coupled dispersionless equation (CDE). Later, Konno and Kakuhata proposed a generalization of this equation in the following form [9]

\[
\begin{align*}
q_{xt} - \rho q &= 0, \\
r_{xt} - \rho r &= 0, \\
\rho_x + 0.5(rq)_t &= 0
\end{align*}
\]

which is called the generalized Konno-Oono equation (GKOE) or the generalized CDE (GCDE). Its Lax representation (LR) can be expressed as

\[
\Psi_x = U_1 \Psi, \quad \Phi_t = V_1 \Psi,
\]

where

\[
V_1 = -i\lambda \begin{pmatrix} \rho & q_t \\ r_t & -\rho \end{pmatrix}, \quad U_1 = \begin{pmatrix} i/4\lambda & -0.5q \\ 0.5r & -i/4\lambda \end{pmatrix}.
\]

It were pointed that the KOE (1) is gauge equivalent to the sine-Gordon equation, whereas its complex version that corresponds to the GKOE (3)-(5) with the reduction \( r = \bar{q} \) is gauge equivalent to the Pohlmeyer-Lund-Regge equation [10]. It is interesting to note that there exist other gauge and geometrical equivalent equations to the GKOE (3)-(5) and to its reductions. These equivalent equations are some kind of generalizations of the famous Heisenberg ferromagnet equations (HFE) [61]-[62]

\[
iA_t = \frac{1}{2} [A, A_{xx}]
\]

and which are the subject of this paper.
The integrable systems with self-consistent sources (ISSCS) have attracted some attention (see, for example, [1]-[14]). The ISSCS can be solved by the inverse scattering transform (IST) method, and N-soliton solutions of some ISSCSs were obtained [1]-[15].

The present paper is organized as follows. In Sec. 2, we present the M-XIV equation in the matrix and the vector form. Specifically, we employ a Lax representation of this equation. In Sec. 3, we demonstrate the geometric representation of the M-XIV equation. In particular, we show that the M-XIV equation and the coupled dispersionless equation with self-consistent equation are closely related to each other. In the following Sec. 4 we establish the gauge equivalence between the M-XIV and M-XXXII equations. In Sec. 5, we analyze several reductions of the M-XXXII equation in detail and present their Lax representations. The scale transformations of the M-XXXII equation is given in Sec. 6. Integrable surface induced by the M-XIV equation is presented in Sec. 7. The corresponding surface equation is the M-XXXI equation. Section 8 is devoted to concluding remarks.

2 Myrzakulov-XIV equation

In this section, we present the different forms of the Myrzakulov-XIV (M-XIV) equation as well as its Lax representation, reduction and simplest conservation law.

2.1 Equation

2.1.1 The matrix M-XIV equation

Consider the following M-XIV equation

\begin{align*}
    iA_t &= i(fA)_x + \frac{1}{4\alpha}[A, A_t]_x + \frac{1}{\alpha}[A, W], \quad (9) \\
    W_x &= i(\alpha - \omega)[A, W] \quad (10)
\end{align*}

or

\begin{align*}
    iA_t &= ifA_x + \frac{1}{4\alpha}[A, A_{12}] + \frac{1}{\alpha}[A, W], \quad (11) \\
    f_x &= \frac{1}{4i\alpha}tr(A[A_t, A_x]), \quad (12) \\
    W_x &= i(\alpha - \omega)[A, W]. \quad (13)
\end{align*}

Here

\begin{align*}
    A &= \begin{pmatrix} A_3 & \sigma A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I, \quad A^+ = A_1 \pm iA_2, \quad A_1^2 + A_2^2 + A_3^2 = 1, \quad (14) \\
    W &= \begin{pmatrix} W_3 & \sigma W^- \\ W^+ & -W_3 \end{pmatrix}, \quad W^2 = I, \quad W^+ = W_1 \pm iW_2, \quad W_1^2 + W_2^2 + W_3^2 = 1. \quad (15)
\end{align*}
2.1.2 The vector M-XIV equation

In the vector form the M-XIV equation looks like

\[ A_t = (fA)_x + \frac{1}{2\alpha}(A \wedge A_t)_x + \frac{2}{\alpha}A \wedge W, \]  
\[ W_x = 2(\omega - \alpha)A \wedge W \]  

or

\[ A_t = fA_x + \frac{1}{2\alpha}A \wedge A_xt + \frac{2}{\alpha}A \wedge W, \]  
\[ f_x = \frac{1}{2\alpha}A \cdot (A_t \wedge A_x), \]  
\[ W_x = 2(\omega - \alpha)A \wedge W, \]

where

\[ A = (A_1, A_2, A_3), \quad A^2 = 1, \quad f = \frac{1}{\alpha^2}a + \frac{1}{\alpha(\alpha - \omega)}\eta, \]  
\[ W = (W_1, W_2, W_3), \quad W_1^2 + W_2^2 + W_3^2 = const = 1, \quad A \cdot W_x = 0. \]

2.1.3 The M-XIV equation as the equation with self-consistent sources

We can rewrite the M-XIV equation as the equation with self-consistent sources. Let us introduce the following representation for the components of the matrix function \( W \):

\[ W_1 = \phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2, \]  
\[ W_2 = i(\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2), \]  
\[ W_3 = |\phi_1|^2 - |\phi_2|^2 \]

or

\[ W^+ = 2\bar{\phi}_1\phi_2, \]  
\[ W^- = 2\phi_1\bar{\phi}_2, \]  
\[ W_3 = |\phi_1|^2 - |\phi_2|^2, \]

so that the matrix \( W \) is given by

\[ W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix} = \begin{pmatrix} |\phi_1|^2 - |\phi_2|^2 & 2\phi_1\bar{\phi}_2 \\ 2\bar{\phi}_1\phi_2 & |\phi_2|^2 - |\phi_1|^2 \end{pmatrix}. \]

Here \( \phi_j \) obey the following set of equations

\[ \phi_{1x} = -i\zeta(A_3\phi_1 + A^-\phi_2), \]  
\[ \phi_{2x} = -i\zeta(A^+\phi_1 - A_3\phi_2), \]
where $\zeta = \omega - \alpha$. Thus the M-XIV equation (as the equation with self-consistent sources) can be written as

$$iA_t = i fA_x + \frac{1}{4\alpha}[A, A_{tx}] + F, \quad (32)$$

$$f_x = \frac{1}{4i\alpha} tr(A[A_t, A_x]), \quad (33)$$

$$\phi_{1x} = -i\zeta(A_3\phi_1 + A^-\phi_2), \quad (34)$$

$$\phi_{2x} = -i\zeta(A^+\phi_1 - A_3\phi_2), \quad (35)$$

where

$$F = \frac{2}{\alpha} \left( \begin{array}{cc} A^-\bar{\phi}_1\phi_2 - A^+\phi_1\bar{\phi}_2 & 2A_3\phi_1\bar{\phi}_2 - A^-(|\phi_1|^2 - |\phi_2|^2) \\ A^+|\phi_2|^2 - 2A_3\bar{\phi}_2\phi_2 & A^+\phi_1\bar{\phi}_2 - A^-\bar{\phi}_1\phi_2 \end{array} \right). \quad (36)$$

### 2.2 Lax representation

The M-XIV equation (9)-(10) is integrable. Its LR reads as

$$\Phi_x = U_2\Phi, \quad (37)$$

$$\Phi_t = V_2\Phi, \quad (38)$$

where

$$U_2 = -i(\lambda - \alpha)A, \quad (39)$$

$$V_2 = -i(\lambda - \alpha) \left( \alpha^2fA - \frac{i\alpha}{4}[A, A_t] - \frac{\alpha}{\alpha - \omega}W \right) - \frac{i(\lambda - \alpha)}{(\alpha - \omega)(\lambda - \omega)}W. \quad (40)$$

### 2.3 Reduction

Let $W = 0$. Then the M-XIV equation takes the form

$$iA_t = i(fA)_x + \frac{1}{4\alpha}[A, A_t]_x \quad (41)$$

or

$$iA_t = i fA_x + \frac{1}{4\alpha}[A, A_{tx}], \quad (42)$$

$$f_x = \frac{1}{4i\alpha} tr(A[A_t, A_x]), \quad (43)$$

and which is the so-called M-XIII equation [94].
2.4 Conservation laws

As the integrable system, the M-XIV equation admits the infinity number of conservation laws. The simplest one that we can get, for example, from Eqs. (18)- (20) has the following form

\[(A_x^2)_t + (8A \cdot W + 8\alpha^2 f)_x = 0.\]  \hspace{1cm} (44)

In \(W = 0\) case, this equation takes the form

\[(A_x^2)_t + 8\alpha^2 f_x = 0\]

so that

\[(A_x^2)_t = 4\alpha A \cdot (A_x \land A_t).\]  \hspace{1cm} (46)

3 Integrable motion of space curves. The equation Lakshmanan (geometrical) equivalent to the M-XIV equation

In this section, we establish the link between the M-XIV equation and the motion of space curves. Then using this link we can find the Lakshmanan (geometrical) equivalent counterpart of the M-XIV equation. Consider a family of smooth space curve in \(R^3\) which we define as

\[
\gamma(x, t) : [0, X] \times [0, T] \to R^3, \hspace{1cm} (47)
\]

where \(x\) is the arc length of the curve at each time \(t\). In this case, the unit tangent vector \(e_1\), principal normal vector \(e_2\) and binormal vector \(e_3\) are defined as

\[
e_1 = \gamma_x, \quad e_2 = \frac{\gamma_{xx}}{\gamma_{xx}}, \quad e_3 = e_1 \land e_2, \hspace{1cm} (48)
\]

respectively. The corresponding Frenet-Serret equation reads as

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_x = C \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix},
\hspace{1cm} (49)
\]

where \(\tau, \kappa_1\) and \(\kappa_2\) are torsion, geodesic curvature and normal curvature of the curve, respectively.
It is well-known that between some integrable systems there take place the geometrical (Lakshmanan) and gauge equivalences. The Frenet-Serret equation and the general temporal evolution equation are given by

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_x = C \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}, \quad \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_t = G \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix},
\]

(50)

where

\[
C = \begin{pmatrix}
0 & \kappa_1 & \kappa_2 \\
-\kappa_1 & 0 & \tau \\
-\kappa_2 & -\tau & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
0 & \omega_3 & \omega_2 \\
-\omega_3 & 0 & \omega_1 \\
-\omega_2 & -\omega_1 & 0
\end{pmatrix}.
\]

(51)

The compatibility condition of these equations is written as

\[
C_t - G_x + [C, G] = 0
\]

(52)

or in elements

\[
\begin{align*}
\kappa_{1t} - \omega_{3x} - \kappa_2 \omega_1 + \tau \omega_2 &= 0, \\
\kappa_{2t} - \omega_{2x} + \kappa_1 \omega_1 - \tau \omega_3 &= 0, \\
\tau_t - \omega_{1x} - \kappa_1 \omega_2 + \kappa_2 \omega_3 &= 0.
\end{align*}
\]

(53-55)

Our next step is the following identification

\[
A \equiv e_1
\]

(56)

and the assumption

\[
\kappa_1 = -2\zeta, \quad \kappa_2 = r - q, \quad \tau = -i(r + q),
\]

(57)

where \(q = -0.5(\kappa_2 - i\tau)\) and \(r = 0.5(\kappa_2 + i\tau)\) are some functions, \(\zeta = \text{const}\). Then we have

\[
\begin{align*}
\omega_1 &= \frac{0.5r_t - 0.5q_t - n - p}{\zeta} + \frac{n + p}{\zeta - \omega}, \\
\omega_2 &= \frac{i(0.5r_t + 0.5q_t - n + p)}{\zeta} + \frac{n - p}{\zeta - \omega}, \\
\omega_3 &= \frac{2\eta}{\zeta - \omega} + \frac{2a}{\zeta}.
\end{align*}
\]

(58-60)

The Eqs. (53)-(55) give us the following equations for \(q, r, \rho = 4a, n, p, \eta\):

\[
\begin{align*}
q_x - \rho q + 4p_x &= 0, \\
r_x - \rho r - 4n_x &= 0, \\
\rho_x - 0.5(rq)_x + 2(qn - rp) &= 0, \\
\eta_x + 0.5(rp - qn) &= 0, \\
p_x + 2i\omega p + \eta q &= 0, \\
n_x - 2i\omega n - \eta r &= 0.
\end{align*}
\]

(61-66)
which is the so-called Myrzakulov-XXXII (M-XXXII) equation. Note that for the reductions $r = \sigma \bar{q}$ and $n = -\sigma \bar{p}$, the M-XXXII equation (44)-(49) takes the form

\begin{align*}
q_{xt} - 4aq + 2p_x &= 0, \\
\sigma x - \sigma [0.5(|q|^2)_t + q\bar{p} + \bar{q}p] &= 0, \\
\eta x + \sigma (\bar{q}p + q\bar{p}) &= 0, \\
p_x + 2i\omega p + 2\eta q &= 0.
\end{align*}

(67)

(68)

(69)

(70)

So, we have proved the Lakshmanan (geometrical) equivalence between the M-XIV equation (9)-(10) and the M-XXXII equation (67)-(70).

4 On the equation gauge equivalent to the M-XIV equation

Taking into account the results of the previous section, we expect that there exists gauge equivalence between the M-XIV equation (9)-(10) and the M-XXXII equation (44)-(49). To check it, consider the following transformation $\Phi = g\Psi$, where $\Phi$ is some solutions of the set (37)-(38) and $g = \Psi|_{\lambda=\alpha}$. Then the function $\Psi$ obeys the following set of linear equations

\begin{align*}
\Psi_x &= U_5 \Psi, \\
\Phi_t &= V_5 \Psi.
\end{align*}

(71)

(72)

Here

\begin{align*}
U_5 &= -i\lambda \sigma_3 + Q, \\
V_5 &= \frac{i}{\lambda} F + \frac{i}{\lambda - \omega} G,
\end{align*}

(73)

where

\begin{align*}
Q &= \begin{pmatrix} 0 & q \\ \sigma \bar{q} & 0 \end{pmatrix}, \\
F &= \begin{pmatrix} a & -0.5q_x - p \\ 0.5\sigma (\bar{q}_x + \bar{p}) & -a \end{pmatrix}, \\
G &= \begin{pmatrix} \eta & p \\ -\sigma \bar{p} & -\eta \end{pmatrix}.
\end{align*}

(74)

It can be easily shown that the compatibility condition $U_5t - V_5x + [U_5, V_5] = 0$ gives the M-XXXII equation (61)-(66).

5 The M-XXXII equation

For our convenience, here we collect the main formulas of the M-XXXII equation (61)-(66).
5.1 Equation

The M-XXXII equation (44)-(49) reads as

\[
\begin{align*}
q_{xt} - 4aq + 2p_x &= 0, \quad (75) \\
r_{xt} - 4ar - 2n_x &= 0, \quad (76) \\
a_x - 0.5(rq)_t + qn - rp &= 0, \quad (77) \\
\eta_x + rp - qn &= 0, \quad (78) \\
p_x + 2i\omega p + 2\eta q &= 0, \quad (79) \\
n_x - 2i\omega n - 2\eta r &= 0. \quad (80)
\end{align*}
\]

5.2 Lax representation

The M-XXXII equation is integrable and its LR is

\[
\begin{align*}
\Psi_x &= U_3 \Psi, \quad (81) \\
\Psi_t &= V_3 \Psi. \quad (82)
\end{align*}
\]

Here

\[
U_3 = -i\lambda \sigma_3 + Q, \quad V_3 = \frac{i}{\lambda} F + \frac{i}{\lambda - \omega} G, \quad (83)
\]

where

\[
Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t - p \\ 0.5r_t - n & -a \end{pmatrix}, \quad G = \begin{pmatrix} \eta & p \\ n & -\eta \end{pmatrix}. \quad (84)
\]

Note that from Eqs. (78)-(80) we get the following important formula

\[
\eta^2 + np = \text{const} \quad (85)
\]

or, for simplicity,

\[
\eta^2 + np = 1. \quad (86)
\]

5.3 Reductions

5.3.1 Case 1: \( r = \sigma q \)

First we consider the particular case when

\[
r = \sigma q. \quad (87)
\]
Then we have
\[ q_{xt} - 4aq + 2p_x = 0, \]  
\[ \eta_x + 2\sigma qp = 0, \]  
\[ p_x + 2i\omega p + 2\eta q = 0, \]
where \( n = -\sigma p \) and \( \omega = \bar{\omega} \). This equation is also integrable and its LR is
\[ \Psi_x = U_4 \Psi, \]  
\[ \Psi_t = V_4 \Psi. \]

Here
\[ U_4 = -\lambda \sigma_3 + Q, \quad V_4 = \frac{i}{\lambda} F + \frac{i}{\lambda - \omega} G, \]
where
\[ Q = \begin{pmatrix} 0 & q \\ \sigma q & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t - p \\ 0.5\sigma q_t + \sigma p & -a \end{pmatrix}, \quad G = \begin{pmatrix} \eta & p \\ \sigma p & -\eta \end{pmatrix}. \]

5.3.2 Case 2: \( r = \sigma \bar{q} \)

In this case that is when
\[ r = \sigma \bar{q}, \quad n = -\sigma \bar{p}, \]
we get the following set of equations
\[ q_{xt} - 4aq + 2p_x = 0, \]  
\[ a_x - \sigma [0.5(|q|^2)_t + \bar{q}p + \bar{p}q] = 0, \]  
\[ \eta_x + \sigma (\bar{q}p + q\bar{p}) = 0, \]  
\[ p_x + 2i\omega p + 2\eta q = 0. \]

This reduction of the M-XXXII equation is integrable with the following LR
\[ \Psi_x = U_5 \Psi, \]  
\[ \Phi_t = V_5 \Psi. \]

Here
\[ U_5 = -\lambda \sigma_3 + Q, \quad V_5 = \frac{i}{\lambda} F + \frac{i}{\lambda - \omega} G, \]
where
\[ Q = \begin{pmatrix} 0 & q \\ \sigma \bar{q} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t - p \\ 0.5\sigma (\bar{q}t + \bar{p}) & -a \end{pmatrix}, \quad G = \begin{pmatrix} \eta & p \\ -\sigma \bar{p} & -\eta \end{pmatrix}. \]
5.3.3 Case 3: \( p = n = \eta = 0 \)

In this particular case when

\[
\begin{align*}
p &= n = \eta = 0, \\
p = n &= \eta = 0,
\end{align*}
\]

we have

\[
\begin{align*}
q_{xt} - 4aq &= 0, \\
r_{xt} - 4ar &= 0, \\
as - 0.5(rq)_t &= 0,
\end{align*}
\]

which is, in fact, the GKOE (3)-(5).

5.4 The M-XXXII equation as the equation with self-consistent sources

We can rewrite the M-XXXII equation as the equation with self-consistent sources. Let us consider the following representation for the functions \( p, n \) and \( \eta \):

\[
p = 2\psi_1 \bar{\psi}_2, \quad n = -2\sigma \bar{\psi}_1 \psi_2, \quad \eta = |\psi_1|^2 + \sigma |\psi_2|^2;
\]

where \( |\psi_1|^2 - \sigma |\psi_2|^2 = \text{const} \) and \( \Psi = (\psi_1, \psi_2)^T \) satisfy the equations (81)-(82). Then the M-XXXII equation takes the form

\[
\begin{align*}
q_{xt} - 4aq + 4(\psi_1 \bar{\psi}_2)_x &= 0, \\
as - 0.5\sigma(|q|^2)_t - 2\sigma(q \bar{\psi}_1 \psi_2 + \bar{q} \psi_1 \bar{\psi}_2) &= 0, \\
\psi_{1x} + i \xi \psi_1 - q \psi_2 &= 0, \\
\psi_{2x} - i \xi \psi_2 - \sigma \bar{q} \psi_1 &= 0,
\end{align*}
\]

or

\[
\begin{align*}
q_{xt} - 4aq - 8i \xi \psi_1 \bar{\psi}_2 + 4q(\sigma |\psi_1|^2 + |\psi_2|^2) &= 0, \\
as - 0.5\sigma(|q|^2)_t - 2\sigma(q \bar{\psi}_1 \psi_2 + \bar{q} \psi_1 \bar{\psi}_2) &= 0, \\
\psi_{1x} + i \xi \psi_1 - q \psi_2 &= 0, \\
\psi_{2x} - i \xi \psi_2 - \sigma \bar{q} \psi_1 &= 0.
\end{align*}
\]

This is the desired form of the M-XXXII equation written as the equation with self-consistent sources.
6 Scale transformation

Consider the following scale transformation

\[ a \to 0.25\rho, \quad (q, r) \to 0.5(q, r). \]  

(117)

Under this transformation, the M-XXXII equation (44)-(49) becomes

\[
q_{xt} - \rho q + 4p_x = 0, \\
r_{xt} - \rho r - 4n_x = 0, \\
\rho_x - 0.5(rq)_t + 2(qn - rp) = 0, \\
\eta_x + 0.5(rp - qn) = 0, \\
p_x + 2i\omega p + \eta q = 0, \\
n_x - 2i\omega n - \eta r = 0.
\]

(118) \quad (119) \quad (120) \quad (121) \quad (122) \quad (123)

When \( p = n = \eta = 0 \), it looks like

\[
q_{xt} - \rho q = 0, \\
r_{xt} - \rho r = 0, \\
\rho_x - 0.5(rq)_t = 0,
\]

(124) \quad (125) \quad (126)

which is a more standard form of the GKO (3)-(5) with \((r) \to (-r)\).

7 Integrable surface induced by the M-XIV equation

In this section, we present a surface which is induced by the M-XIV equation. Let us consider a surface in three-dimensional Euclidean space \( \mathbb{R}^3 \) that is parametrized by position vector \( \mathbf{r}(x, t) \) of the surface. To construct the surface, let us do the following identification

\[ \mathbf{A} \equiv \mathbf{r}_x, \]

(127)

where \( \mathbf{r}_x^2 = 1 \). After that from the vector M-XIV equation (16)-(17) we get the following Myrzakulov-XXXI (M-XXXI) equation

\[
\mathbf{r}_t = f\mathbf{r}_x + \frac{1}{2\alpha}\mathbf{r}_x \wedge \mathbf{r}_{xt} + \frac{1}{\alpha(\omega - \alpha)}\mathbf{W},
\]

(128)

\[
f_x = \frac{1}{2\alpha}\mathbf{r}_x \cdot (\mathbf{r}_{xt} \wedge \mathbf{r}_{xx}),
\]

(129)

\[
\mathbf{W}_x = 2(\omega - \alpha)\mathbf{r}_x \wedge \mathbf{W},
\]

(130)
where \( r = (r_1, r_2, r_3) \). The M-XXXI equation (128)-(130) is integrable. Its LR looks as

\[
\Phi_x = U_7 \Phi, \quad \Phi_t = V_7 \Phi,
\]

where

\[
U_7 = -i(\lambda - \alpha) r_x, \quad V_7 = -i(\lambda - \alpha) \frac{a^2}{\alpha \lambda} \left( \alpha^2 f_{rx} - i \frac{a}{4} [r_{rx}, r_{xt}] - \frac{a}{\alpha - \omega} W \right) - \frac{i(\lambda - \alpha)}{(\alpha - \omega)(\lambda - \omega)} W.
\]

The compatibility condition of the equations (131)-(132) gives the following equation

\[
ir_t = i f_{rx} + \frac{1}{4\alpha} [r_x, r_t] + \frac{i}{\alpha(\alpha - \omega)} W,
\]

\[
f_x = \frac{1}{4i \alpha} tr(r_x[r_{rx}, r_{xx}]), \quad W_x = i(\alpha - \omega)[r_x, W]
\]

which is the matrix form of the M-XXXI equation. Here

\[
\begin{pmatrix} r_1^- & r_2^- & r_3^- \\ r_1^+ & r_2^+ & r_3^+ \end{pmatrix}, \quad r_x^2 = I, \quad r_{1x}^2 + r_{2x}^2 + r_{3x}^2 = 1, \quad r^\pm = r_1 \pm ir_2.
\]

In our case, the first fundamental form of the surface is given by

\[
I = dx^2 + 2(r_x \cdot r_t) dx dt + r_t^2 dt^2,
\]

where

\[
r_x^2 = 1, \quad r_x \cdot r_t = f + \frac{1}{\alpha(\omega - \alpha)} r_x \cdot W,
\]

\[
r_t^2 = f^2 + \frac{1}{\alpha^2(\omega - \alpha)^2} + 2f(r_x \cdot W) + \frac{|r_x \wedge r_{xt}|^2}{4\alpha^2} + \frac{(r_x \wedge r_{xt}) \cdot W}{\alpha^2(\omega - \alpha)}.
\]

Finally, we note that the M-XXXI equation (128)-(130) can be rewritten as

\[
r_{xt} = 2\alpha r_t \wedge r_x + \frac{2}{\alpha - \omega} r_x \wedge W, \quad W_x = 2(\omega - \alpha) r_x \wedge W,
\]

or

\[
r_{xt} = 2\alpha r_t \wedge r_x - \frac{1}{(\alpha - \omega)^2} W_x, \quad W_x = 2(\omega - \alpha) r_x \wedge W.
\]
The above obtained results are enough to define the surface in three-dimensional Euclidean space $\mathbb{R}^3$ parametrized by the position vector $\mathbf{r}(x, t)$ of the surface. This surface is integrable as its equation that is the equation for $\mathbf{r}$ (128)-(130) admits LR. So we have shown that the M-XXXI equation induces the some integrable surface. Lastly, we present the reduction of the M-XXXI equation. If $W = 0$, then it turns to the following equation

$$r_t = fr_x + \frac{1}{2\alpha} r_x \wedge r_{xt}$$

or

$$r_{xt} = 2\alpha r_t \wedge r_x,$$

where

$$f = r_t \cdot r_x.$$ 

Finally, let us present the M-XXXI equation (128)-(130) as the equation with self-consistent sources. It has the form

$$r_{1t} = fr_{1x} + \frac{1}{2\alpha}(r_{2x}r_{3xt} - r_{2xt}r_{3x}) + \frac{1}{\alpha(\omega - \alpha)}(\phi_1\bar{\phi}_2 + \bar{\phi}_1\phi_2),$$

$$r_{2t} = fr_{2x} + \frac{1}{2\alpha}(r_{3x}r_{1xt} - r_{3xt}r_{1x}) + \frac{i}{\alpha(\omega - \alpha)}(\phi_1\bar{\phi}_2 - \bar{\phi}_1\phi_2),$$

$$r_{3t} = fr_{3x} + \frac{1}{2\alpha}(r_{1x}r_{2xt} - r_{1xt}r_{2x}) + \frac{1}{\alpha(\omega - \alpha)}(|\phi_1|^2 - |\phi_2|^2),$$

$$f_x = \frac{1}{4i\alpha}tr(r_x[r_{xt}, r_{xx}]),$$

$$\phi_{1x} = -i\zeta(r_{3x}\phi_1 + r_{\bar{3}x}\bar{\phi}_2),$$

$$\phi_{2x} = -i\zeta(r_x^+\phi_1 - r_{3x}\bar{\phi}_2).$$

It is the M-XXXI equation written in the form of the equation with self-consistent sources.

8 Conclusions

In this paper, we have shown that the HFE equation admits an integrable generalization with the self-consistent sources - the M-XIV equation. In particular, the integrability of the M-XIV equation has been established by constructing its Lax pair. We have also demonstrated that the M-XIV equation is equivalent to the M-XXXII equation. At the level, some reductions of the M-XXXII equation are found. Another interesting issue is to generalize these results to the real and complex short pulse equations [104]. Although in this paper, we have restricted our consideration to the mathematical aspects of the proposed integrable two equations, the relevance of these equations as models capable of describing the dynamics of ultra-short pulses in optical fibers is an important issue to be studied in a future work.
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