NOISE STABILITY OF WEIGHTED MAJORITY

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Abstract. Benjamini, Kalai and Schramm (2001) showed that weighted majority functions of \( n \) independent unbiased bits are uniformly stable under noise: when each bit is flipped with probability \( \epsilon \), the probability of the weighted majority changes is at most \( C\epsilon^{1/4} \). They asked what is the best possible exponent that could replace \( 1/4 \). We prove that the answer is \( 1/2 \). The upper bound obtained for \( p_\epsilon \) is within a factor of \( \sqrt{\pi/2 + o(1)} \) from the known lower bound when \( \epsilon \to 0 \) and \( n\epsilon \to \infty \).

1. Introduction

In their study of noise sensitivity and stability of Boolean functions, Benjamini, Kalai and Schramm [2] showed that weighted majority functions of \( n \) independent unbiased \( \pm 1 \)-valued variables are uniformly stable under noise: when each variable is flipped with probability \( \epsilon \), the weighted majority changes with probability at most \( C\epsilon^{1/4} \). They asked what is the best possible exponent that could replace \( 1/4 \). In this note we prove that the answer is \( 1/2 \). Denote \( \text{sgn}(u) = u/|u| \) for \( u \neq 0 \) and \( \text{sgn}(0) = 0 \), and let \( N_\epsilon : \mathbb{R}^n \to \mathbb{R}^n \) be the noise operator that flips each variable in its input independently with probability \( \epsilon \). Formally, given a random vector \( X = (X_1, \ldots, X_n) \), the random vector \( N_\epsilon(X) \) is defined as \( (\sigma_1X_1, \ldots, \sigma_nX_n) \) where the i.i.d. random variables \( \sigma_i \) are independent of \( X \) and take the values \( 1, -1 \) with probabilities \( 1 - \epsilon, \epsilon \) respectively.

Theorem 1.1. Let \( X = (X_1, \ldots, X_n) \) be a random vector uniformly distributed over \( \{-1, 1\}^n \). Given nonzero weights \( w_1, \ldots, w_n \in \mathbb{R} \) and a threshold \( t \in \mathbb{R} \), consider the weighted majority function \( f : \mathbb{R}^n \to \{-1, 0, 1\} \) defined by

\[
f(x) = \text{sgn}\left(\sum_{i=1}^n w_ix_i - t\right)
\]
Then for \( \epsilon \leq \frac{1}{2} \),

\[
p_\epsilon(n, w, t) = P\left(f(X) \neq f(N_\epsilon(X))\right) \leq 2\epsilon^{1/2}.
\] (1.2)

Moreover, \( p^*_\epsilon = \limsup_{n \to \infty} \sup_{w, t} p_\epsilon(n, w, t) \) satisfies

\[
\limsup_{\epsilon \to 0} \frac{p^*_\epsilon}{\sqrt{\epsilon}} \leq \sqrt{\frac{2}{\pi}}.
\] (1.3)

In the statement of the theorem we opted for a simple formulation: Our proof yields the following sharper, but more involved estimate:

\[
p_\epsilon(n, w, t) \leq \frac{2}{m} E|B_m - \frac{m}{2}| + \left[1 - \left(1 - \epsilon\right)^n\right]\left(\frac{n}{\lfloor n/2 \rfloor}\right)^2 - n,
\] (1.4)

where \( m = \lfloor \epsilon^{-1} \rfloor \) and \( B_m \) is a Binomial\((m, 1/2)\) variable.

It easy to see, and classical [9, 4], that for simple majority (when all weights are equal) we have

\[
\lim_{n \to \infty} P(\text{sgn} \sum_{i=1}^{n} X_i \neq \text{sgn} \sum_{i=1}^{n} (N_\epsilon X)_i) = \frac{1}{\pi} \arccos(1 - 2\epsilon) = \frac{2}{\pi} \sqrt{\epsilon} + O(\epsilon^{3/2}).
\] (1.5)

For the reader’s convenience we include a brief argument:

Since \( \text{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} (N_\epsilon X)_i\right) = n(1 - 2\epsilon) \), the central limit theorem implies that as \( n \to \infty \),

\[
\frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} (N_\epsilon X)_i\right) \Rightarrow (Z_1, Z_1^\ast) \text{ in law,}
\]

where \( Z_1, Z_1^\ast \) are standard normals with covariance \( 1 - 2\epsilon \). We can write \( Z_1^\ast = Z_1 \cos \alpha - Z_2 \sin \alpha \) where \( Z_1, Z_2 \) are i.i.d. standard normals and \( \alpha \in (0, \pi) \) satisfies \( \cos \alpha = 1 - 2\epsilon \). Rotating the random vector \( (Z_1, Z_2) \) by the angle \( \alpha \) yields a vector with first coordinate \( Z_1^\ast \). Since \( (Z_1, Z_2) \) has a rotationally-symmetric law, the rotation changes the sign of the first coordinate with probability \( \alpha/\pi \). This verifies the left-hand side of (1.5); the right-hand side follows from Taylor expansion of cosine.

Thus the estimate (1.2) is sharp (up to the value of the constant). Moreover, the ratio between the upper bound in (1.3) and the value for simple majority in (1.5) tends to \( \sqrt{\pi/2} < 1.26 \) as \( \epsilon \to 0 \). We remark that the stability result in theorem 1.1 is stronger than an assertion about stability of half-spaces, \( \{x : \sum_i w_i x_i > \theta\} \), because we consider the weighted majority as taking three values, rather than two.
2. Proof of Theorem 1.1

Using symmetry of $X_i$, we may assume that $w_i > 0$ for $i = 1, \ldots, n$. Let $\langle w, X \rangle = \sum_{i=1}^{n} w_i X_i$. We first consider the threshold $t = 0$. Later, we will extend the argument to thresholds $t \neq 0$.

We will need the following well-known fact from [3]:

$$
P(\langle w, X \rangle = 0) \leq \left( \frac{n}{\lfloor n/2 \rfloor} \right)^2 - n.
$$

Indeed, the collection $\mathcal{D}(w)$ of sets $D \subset \{1, \ldots, n\}$ such that $\sum_{i \in D} w_i = \sum_{k \notin D} w_k$ forms an anti-chain with respect to inclusion, so Sperner's theorem (see [4], Ch. 11) implies that the cardinality of $\mathcal{D}(w)$ is at most $\left( \frac{n}{\lfloor n/2 \rfloor} \right)$. Finally, observe that a vector $x \in \{-1, 1\}^n$ satisfies $\langle w, x \rangle = 0$ iff \{ $i : x_i = 1$ \} is in $\mathcal{D}(w)$.

Let $m = \lfloor \epsilon^{-1} \rfloor$ and let $\tau$ be a random variable taking the values $0, 1, \ldots, m$ with $P(\tau = j) = \epsilon$ for $j = 1, \ldots, m$ and $P(\tau = 0) = 1 - m\epsilon$. We use a sequence $\tau_1, \tau_2, \ldots, \tau_n$ of i.i.d. random variables with the same law as $\tau$, to partition $[n] = \{1, \ldots, n\}$ into $m + 1$ random sets

$$
A_j = \{ i \in [n] : \tau_i = j \} \text{ for } 0 \leq j \leq m. \tag{2.2}
$$

Denote $S_j = \sum_{i \in A_j} w_i X_i$ and let $Y_1 = \sum_{i \notin A_j} w_i X_i = \langle w, X \rangle - S_1$. Observe that $Y_1 - S_1$ has the same law, given $X$, as $\langle w, N_\epsilon(X) \rangle$. Therefore,

$$
p_{\epsilon}(n, w, 0) = P(\text{sgn}(\langle w, X \rangle) \neq \text{sgn}(\langle w, N_\epsilon(X) \rangle)) \tag{2.3}
$$

$$
= P(\text{sgn}(Y_1 + S_1) \neq \text{sgn}(Y_1 - S_1)).
$$

Denote $\xi_j = \text{sgn}(S_j)$. A key step in the proof is the pointwise identity

$$
1_{\{\text{sgn}(Y_1 + S_1) \neq \text{sgn}(Y_1 - S_1)\}} \tag{2.4}
$$

$$
= 2 \cdot 1_{\{S_1 \neq 0\}} \mathbb{E}\left( \frac{1}{2} - 1_{\{\text{sgn}(S_1 + Y_1) = -\xi_1\}} \bigg| Y_1, |S_1| \right).
$$

To verify this, we consider three cases:

\textbf{(i)} Clearly both sides vanish if $S_1 = 0$.

\textbf{(ii)} Suppose that $0 < |S_1| < |Y_1|$ and therefore $\text{sgn}(Y_1 + S_1) = \text{sgn}(Y_1)$. The conditional distribution of $S_1$ given $Y_1$ and $|S_1|$ is uniform over $\{-|S_1|, |S_1|\}$, whence the conditional probability that $\text{sgn}(S_1 + Y_1) = -\xi_1$ is $1/2$. Thus both sides of (2.4) also vanish in this case.
Finally, suppose that \( S_1 \neq 0 \) and \(|S_1| \geq |Y_1|\). In this case \( \text{sgn}(S_1 + Y_1) \neq -\xi_1 \), so both sides of (2.3) equal 1.

Taking expectations in (2.4) and using (2.3), we deduce that

\[
p_\epsilon(n, w, 0) = 2 \mathbb{E}\left[ 1\{S_1 \neq 0\} \left( \frac{1}{2} - 1\{\text{sgn}(w, X) = -\xi_1\}\right) \right] \\
= \frac{2}{m} \mathbb{E} \sum_{j \in \Lambda} \left( \frac{1}{2} - 1\{\text{sgn}(w, X) = -\xi_j\}\right),
\]

where \( \Lambda = \{j \in [1, m] : S_j \neq 0\} \).

The random variable \( B_\Lambda = \#\{j \in \Lambda : \xi_j = 1\} \) has a Binomial\(\#\Lambda, \frac{1}{2}\) distribution given \( \Lambda \), and satisfies the pointwise inequality

\[
\sum_{j \in \Lambda} \left( \frac{1}{2} - 1\{\text{sgn}(w, X) = -\xi_j\}\right) \leq \left| B_\Lambda - \frac{\#\Lambda}{2} \right| + \frac{1}{2} \mathbb{1}\{\langle w, X \rangle = 0\} \sum_{j = 1}^{m} \mathbb{1}\{A_j \neq \emptyset\}.
\]

To see this, consider the three possibilities for \( \text{sgn}\langle w, X \rangle \). Taking expectations and using (2.5), we get

\[
p_\epsilon(n, w, 0) \leq \frac{2}{m} \mathbb{E} \left| B_\Lambda - \frac{\#\Lambda}{2} \right| + \mathbb{P}(A_1 \neq \emptyset) \mathbb{P}(\langle w, X \rangle = 0).
\]

Let \( B_\ell \) denote a Binomial\(\ell, \frac{1}{2}\) random variable. Since for any martingale \( \{M_\ell\}_{\ell \geq 1} \) the absolute values \( |M_\ell| \) form a submartingale, the expression \( \mathbb{E} |B_\ell - \frac{\ell}{2}| \) is increasing in \( \ell \). By averaging over \( \Lambda \), we see that \( \mathbb{E} |B_\Lambda - \frac{\#\Lambda}{2}| \leq \mathbb{E} |B_m - \frac{m}{2}| \). In conjunction with (2.6) and (2.1), this implies

\[
p_\epsilon(n, w, 0) \leq \frac{2}{m} \mathbb{E} |B_m - \frac{m}{2}| + \left[1 - (1 - \epsilon)^n\right] \left(\frac{n}{n/2}\right)^2 - n.
\]

Next, suppose that \( f(x) = \text{sgn}\left( \sum_{i=1}^{n} w_i x_i - t \right) \), where \( t \neq 0 \) is a given threshold. Let \( X_{n+1} \) be a \( \pm 1 \) valued symmetric random variable, independent of \( X = (X_1, \ldots, X_n) \), and define \( w_{n+1} = t \). Then

\[
p_\epsilon(n, w, t) = \mathbb{P}\left(f(X) \neq f(N_\epsilon(X))\right)
\]

\[
= \mathbb{P}\left(\text{sgn} \sum_{i=1}^{n+1} w_i X_i \neq \text{sgn} \left[ \sum_{i=1}^{n} w_i (N_\epsilon X_i) + w_{n+1} X_{N+1}\right]\right),
\]

and the argument used above to establish the bound (2.7) for \( p_\epsilon(n, w, 0) \), yields the same bound for \( p_\epsilon(n, w, t) \). This proves (1.4).
To derive (1.2), we may assume that \( \epsilon \leq \frac{1}{4} \). Use Cauchy-Schwarz to write

\[
E|B_m - \frac{m}{2}| \leq \sqrt{\text{Var}(B_m)} = \sqrt{m/4}
\]

and apply the elementary inequalities

\[
\left( \frac{n}{|n/2|} \right)^2 \leq \sqrt{3/4} n^{-1/2},
\]

(see, e.g., [8], Section 2.3) and \([1 - (1 - \epsilon)^n] \leq \min\{n\epsilon, 1\} \leq n\epsilon\), to obtain

\[
p_\epsilon(n, w, t) \leq m^{-1/2} + \sqrt{n\epsilon} \cdot \sqrt{3/4} n^{-1/2}.
\]

(2.9)

Since \( m = [\epsilon^{-1}] \geq 4/(5\epsilon) \) for \( \epsilon \leq 1/4 \), we conclude that

\[
p_\epsilon(n, w, t) \leq \left( \sqrt{5/4} + \sqrt{3/4} \right) \epsilon^{1/2} < 2\epsilon^{1/2},
\]

and this proves (1.2).

Finally, the central limit theorem implies that

\[
\lim_{m \to \infty} E \frac{|2B_m - m|}{\sqrt{m}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |u| e^{-u^2/2} du = \sqrt{2/\pi}.
\]

This proves (1.3).

\(\Box\)

Remark.

1. Our proof of Theorem 1.1 was found in 1999, and was mentioned in [2]. We present it here, with more attention to the constants, in view of the recent interest in related “converse” inequalities, see [5]. The randomization idea which is crucial to the proof was inspired by an argument of Matthews [7] to bound cover times for Markov chains. See also [10] for related random walk estimates.

2. After I described the proof of Theorem 1.1 to R. O’Donnell, he found (jointly with A. Klivans and R. Servedio) some extensions and applications of the argument to learning theory, see [6] for this and many other results.

3. The proof of Theorem 1.1 extends verbatim to the case where \( X_i \) are independent symmetric real-valued random variables with \( P(X_i = 0) = 0 \) for all \( i \). However, this extension reduces to Theorem 1.1 by conditioning on \( |X_i| \). A more interesting extension would be to replace the symmetry assumption on \( X_i \) by the assumption \( E X_i = 0 \).

4. Is simple majority the most noise sensitive of the weighted majority functions (asymptotically when \( \epsilon \to 0 \) and \( n\epsilon \to \infty \))?

In particular, is it possible to replace the right-hand side of (1.3) by \( 2/\pi \)?

Acknowledgement. I am grateful to I. Benjamini, G. Kalai and O. Schramm for suggesting the problem, and to E. Mossel, R. Peled, O. Schramm, R. Siegmund-Schultze and H. V. Weizsäcker for useful discussions.
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