Eikonal analysis of Coulomb distortion in quasi-elastic electron scattering

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An eikonal expansion is used to provide systematic corrections to the eikonal approximation through order $1/k^2$, where $k$ is the wave number. Electron wave functions are obtained for the Dirac equation with a Coulomb potential. They are used to investigate distorted-wave matrix elements for quasi-elastic electron scattering from a nucleus. A form of effective-momentum approximation is obtained using trajectory-dependent eikonal phases and focusing factors. Fixing the Coulomb distortion effects at the center of the nucleus, the often-used ema approximation is recovered. Comparisons of these approximations are made with full calculations using the electron eikonal wave functions. The ema results are found to agree well with the full calculations.

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I. INTRODUCTION

Professor Manoj Banerjee was a collaborator and friend of both of the authors. It is an honor to contribute a paper that is dedicated to his memory. He exhibited an enthusiasm and passion for physics that has inspired many.

In this paper, we summarize some methods and results concerning the issue of Coulomb corrections in quasi-elastic scattering of electrons by nuclei. Electrons have been used extensively as an experimental probe of the internal structure of nuclei in the past few decades. Electron scattering is considered to be a precise tool in view of the strength of the e.m. interaction, so that as a result the scattering process can be treated in the one photon exchange approximation. In particular, quasi-elastic scattering of electrons by nuclei has been used to investigate properties like the validity of the Coulomb sum rule in nuclei. Experiments have been performed at the MIT Bates Laboratory [1, 2, 3, 4, 5, 6, 7, 8, 9], at the Saclay Laboratory [10, 11, 12, 13, 14, 15] and at SLAC [16, 17, 18]. Although the electron may in general be considered a weak probe, complications arise due to Coulomb distortion effects in the electron wave function owing to the nuclear charge distribution. As a result, in order to extract nuclear and nucleon structure information from these experiments, the Coulomb distortion contributions have to be accounted for in the theoretical analysis of the data. In addition there exists the complication of the presence of final state interaction. Studies have shown that in the quasi-free region at high momentum transfer these effects are expected to be small.

Neglecting the final state interaction, the Coulomb distortion can be handled in the so-called distorted wave Born approximation. Here the Coulomb potential is treated exactly by solving numerically the Dirac equation in the presence of the Coulomb potential. Exact solutions for the Dirac-Coulomb wave functions may be obtained as a sum over partial waves. [19, 20, 21]. As the electron energy increases, the partial-wave expansions converge more slowly in spite of the fact that the Coulomb corrections become smaller. These calculations are numerically complex and have the disadvantage of not allowing for a simple theoretical interpretation in terms of nuclear structure functions. So one important theoretical issue has been to investigate whether there is a simple yet reliable way to characterize the reaction process in terms of response functions, similar to what can be done in the absence of final state interactions and Coulomb distortions. Confining attention to high enough energy, a natural and reliable framework for the description is given by the use of the eikonal wave function.

The eikonal approximation clearly gets better at increasing energy and it allows a simpler analysis of the effects of Coulomb corrections. Some particularity transparent results have been obtained using the eikonal approximation to derive an effective-momentum approximation (ema) [22, 23] that produces results very similar to plane-wave results. It is important to include focusing factors such as those found in the WKB approximation [24] and revisited more recently in quasi-elastic scattering [26, 27]. However, the attempts to combine the eikonal analysis with focusing factors suffer from the lack of a systematic basis. Significant disagreements in the determination of nuclear response functions from experimental data [28, 29] have arisen at least in part owing to the use of different theoretical methods to remove the Coulomb corrections. Therefore, it is of interest to study a systematic expansion of the eikonal approach, where the various effects arise in a natural way.

In order to address the issue of Coulomb corrections, we have developed corrections to the eikonal approximation based upon a systematic expansion in the high-energy limit [30]. This eikonal expansion is shown to be rapidly convergent already at typical energies of few hundred MeV for targets used in quasi-elastic scattering. Corrections to the eikonal approximation have a long history. Work by Saxon and Schiff [31] showed how to correct the approximation to leading order in $1/k$. A systematic expansion for the scattering t-matrix was developed by Sugar and Blankenbe-
In this paper we consider electron scattering at intermediate energies, where the eikonal approach is expected to be reasonably accurate. The electron is assumed to be a Dirac particle. For the Dirac equation, the eikonal expansion is carried out in two stages. First we consider the Pauli spinor \( u(\mathbf{r}) \) that contains the two upper components of the Dirac wave function, i.e.,

\[
\psi(\mathbf{r}) = \begin{pmatrix} u(\mathbf{r}) \\ \ell(\mathbf{r}) \end{pmatrix}.
\]

It follows from the Dirac equation that the Pauli spinor \( \ell(\mathbf{r}) \) that contains the two lower components may be determined in a second stage, where the two lower components are determined in terms of \( u(\mathbf{r}) \).

Eliminating the lower component spinor from the Dirac equation we find for the upper-component spinor the equation

\[
\left( E_1 - V_c - \sigma \cdot \mathbf{p} \frac{1}{E_2 - V_c} \sigma \cdot \mathbf{p} \right) u(\mathbf{r}),
\]

where \( E_1 = E - m \), \( E_2 = E + m \) and \( E \) is the energy of the incoming particle. For electron scattering it is generally the case that \( E \gg m \) and thus \( E_1 \approx E_2 \approx E \). Because the electron mass is much smaller than the energy, helicity is conserved and the lower components are given simply by \( \ell_\lambda(\mathbf{r}) = 2\lambda u_\lambda(\mathbf{r}) \), where \( \lambda = \pm 1/2 \) is the helicity.

For outgoing-wave boundary conditions, the Pauli spinor \( u(\mathbf{r}) \) is written in terms of a complex eikonal phase \( \chi(+) = \chi(+) + i\omega(+) \) and a complex spin-dependent phase \( \gamma(+) = \gamma(+) + i\delta(+) \) as follows

\[
u(+) = \left( 1 - \frac{V_c}{E_2} \right)^{1/2} e^{ikz} e^{i\chi(+)} e^{i\sigma \cdot \delta(+)}.
\]

where \( k = \sqrt{E^2 - m^2} \) is the momentum of the incoming wave. The wave propagates in the \( z \)-direction and an impact vector \( \mathbf{b} \) is defined as the part of \( \mathbf{r} \) that is perpendicular to the \( \hat{z} \)-direction, i.e., \( \mathbf{b} = \hat{z} \times (\mathbf{r} \times \hat{z}) \). Three orthogonal unit vectors are \( \hat{z}, \hat{b} = \mathbf{b}/|\mathbf{b}| \) and \( \hat{e} = \hat{b} \times \hat{z} \). The spin matrix in the eikonal phase is \( \sigma_\epsilon = \sigma \cdot \hat{e} \). The factor \( (1 - V_c/E_2)^{1/2} \) is introduced in order to sum up terms that otherwise arise in higher orders.

The eikonal expansion has been developed in Ref. \[30\]. The result is that the eikonal phases are expanded in a systematic fashion in powers of \( 1/k \) as

\[
\begin{align*}
\chi(+) &= \chi_0(+) + \chi_1(+) + \chi_2(+) + \cdots \\
\omega(+) &= \omega_1(+) + \omega_2(+) + \cdots \\
\gamma(+) &= \gamma_1(+) + \gamma_2(+) + \cdots \\
\delta(+) &= \delta_2(+) + \cdots,
\end{align*}
\]

In Section II we present the eikonal expansion for the Dirac wave function and show that the focusing effect can be obtained at order \( 1/k \) of the expansion. In particular, we focus on \( u(\mathbf{r}) \), which is a Pauli spinor containing the two upper components of the Dirac wave function. The lower components are simply \( \pm 1 \) times the upper components because of helicity conservation. Convergence of the eikonal expansion is shown to be fast in the few hundred MeV electron energy region. Because there is a spin-orbit interaction, spin-dependent terms arise in the eikonal expansion. They also are determined and their effects are found to be negligibly small. In Section III we summarize the basic formulae for quasi-elastic electron scattering. In Section IV we deal with effective-momentum approximations and discuss the original \( ema \) approximation. A natural modification (\( EMAr \)) to the \( ema \) approximation is proposed, where trajectory-dependent eikonal phases and focusing factors are included.

Specializing to the longitudinal response, section V discusses quasi-elastic scattering by use of a simple model of the nuclear response. Comparisons of the full calculations of the response functions are made with the two effective-momentum approximations. In the fits to the plane-wave impulse approximation with an effective momentum small deviations from unity of the normalization are found. However, overall reasonable agreement is found with the full calculations, lending support to the use of these effective-momentum approximations as a basis for the theoretical analysis of the quasi-elastic data. Some concluding remarks are made in Section VI.
where the subscript of each term denotes the power of $1/k$ that is involved. Explicit expressions can be found in Ref. [30]. The leading terms are given by

$$
\chi_0^{(+)}(r) = -\frac{1}{v} \int_{-\infty}^{z} dz' V_c(r'),
$$

$$
\omega_1^{(+)}(r) = \frac{1}{2k} \int_{-\infty}^{z} dz' \nabla^2 \chi_0^{(+)}(r'),
$$

$$
\gamma_1^{(+)}(r) = -\frac{1}{2k} \int_{-\infty}^{z} dz' \frac{\partial V_c(r)}{\partial b},
$$

$$
\delta_1^{(+)}(r) = 0,
$$

where higher order terms than $1/k$ have been dropped.

The upper-component spinor of the Dirac wave function for helicity $\lambda$ and outgoing-wave boundary conditions is given by

$$
u^{(+)\lambda}(r) = f^{D(+)}_i(r) e^{ikz} e^{i\chi^{(+)0}(r)} e^{i\sigma_\lambda \xi^{(+)i}(r)} \xi_\lambda,
$$

where $\xi_\lambda$ is a helicity eigenstate. The Dirac focusing factor $f^{D(+)}_i(r)$ is defined as

$$
f^{D(+)}_i(r) = \left(1 - \frac{V_c}{E_{2i}}\right)^{1/2} e^{-\omega^{(+)i}}.
$$

One may work at various orders of the eikonal expansion by truncating the expansions of Eq. (3). Similarly, the upper-component spinor for helicity $\lambda$ and incoming-wave boundary conditions can be obtained by replacing in Eqs. (4,6) the superscripts $(+)$ by $(-)$ and the integration ranges $\int_{-\infty}^{z}$ by $\int_{\infty}^{z}$.

Convergence of the eikonal expansion has been studied for scattering of a 500 MeV electron. A rough estimate can be made of the higher order corrections of the expansion. Given that the electron mass is $m = .511$ MeV, it follows that $k \approx E$ and $v \approx 1$, both within a part per million. The Coulomb potential is approximately $V_c(0) = 25$ MeV at the center of the nucleus. The eikonal expansion introduces corrections that involve the nondimensional ratio $V_c/E \approx .05$, so we expect $\frac{\chi_0^{(+)0}}{\chi_0^{(+)0}} \approx .0025$ It should be noted, that the eikonal expansion is not convergent but is asymptotic, meaning that the error should be bounded by the first neglected term.

Figure 1 shows the eikonal phases for a charge $Z = 100$ and electron energy $E = 200$ MeV. The Coulomb potential is chosen to be

$$
V_c(r) = -\frac{V_0 R}{\sqrt{r^2 + R^2}},
$$

where $-V_0$ is the value of the potential at $r = 0$ and $R$ is a range parameter. This Coulombic potential corresponds to a charge density

$$
\rho(r) = \frac{3V_0}{4\pi e} \frac{R^3}{(r^2 + R^2)^{5/2}}.
$$

The above parameters were chosen in order to make the corrections visible. The corrections are much smaller for a 500 MeV electron and a smaller nuclear charge.

### III. QUASI-ELASTIC ELECTRON SCATTERING

Let us consider the quasi-elastic nucleon knock-out process $(e, e', N)$ from a nucleus. The cross section of quasi-elastic electron scattering can be expressed in terms of the transition matrix element $\mathcal{M}$:

$$
\frac{d\sigma}{d\Omega dE_f} = \int d\Omega_p \frac{4\alpha^2}{(2\pi)^3} |\mathcal{M}|^2 E_f^2 p E_p,
$$

where $p$ is the momentum of the knocked-out nucleon and $E_p = \sqrt{M^2 + p^2}$ is its energy. The bar denotes an average over initial helicities and a sum over final helicities. We have

$$
\mathcal{M} = \delta_{\lambda,\lambda'} \int d^4 q \frac{1}{(2\pi)^2} j^\mu_c \left(\frac{1}{q^2 - \omega^2}\right) J^{N\lambda'}(q, p)
$$

(10)
with \( j_{\mu}^e \) the electron current matrix element

\[
j_{\mu}^e = \int d^3r u_{\lambda_f}^{(-)}(r)\gamma_{\mu} e^{-i\mathbf{q} \cdot \mathbf{r}} u_{\lambda_i}^{(+)}(r),
\]

(11)

for emission of a photon of energy \( \omega = E_i - E_f \) and momentum \( \mathbf{q} = \mathbf{k}_i - \mathbf{k}_f \). In Eq. (11) \( u_{\lambda_i}^{(+/-)} \) are the electron wave functions corresponding to initial momentum \( \mathbf{k}_i \) and final momentum \( \mathbf{k}_f \) with respectively outgoing and ingoing boundary conditions. The helicity conservation factor \( \delta_{\lambda_f,\lambda_i} \) is produced by matrix elements that incorporate the lower components. Because the lower components are \( \pm 1 \) times the upper components, in what follows one needs only the upper components of the wave function.

In the plane-wave impulse approximation (PWIA), Coulomb distortion of the electron waves is neglected and the electron is described by a plane wave. As a result the integration over \( \mathbf{r} \) produces \( \delta^{(3)}(\mathbf{q} - \mathbf{Q}) \) with \( \mathbf{Q} = \mathbf{k}_i - \mathbf{k}_f \). We get

\[
\mathcal{M}_{\text{PWIA}} = \delta_{\lambda_f,\lambda_i} h_{\text{PWIA}}^\mu \frac{J_{\text{PWIA}}(\mathbf{Q}, \mathbf{p})}{Q^2},
\]

(12)

where \( h_{\text{PWIA}}^\mu \) denotes known helicity dependent factors \( [30] \). The PWIA cross section may be expressed in terms of longitudinal and transverse response functions, \( R_L \) and \( R_T \)

\[
\frac{d\sigma}{d\Omega_f dE_f} = \sigma_{\text{Mott}} \left\{ \frac{Q^4}{Q^2 R_L + \frac{Q^2}{2\epsilon} R_T} \right\},
\]

(13)

where

\[
\sigma_{\text{Mott}} = \frac{4\alpha^2 E_f^2 \cos^2 \frac{\theta_e}{2}}{Q^4},
\]

(14)

and

\[
\epsilon = \left[ 1 + \frac{2Q^2}{Q^2 \tan^2 \frac{\theta_e}{2}} \right]^{-1}.
\]

(15)

With Coulomb corrections included, the longitudinal matrix element of interest must take a gauge invariant form. This requires that the electron current must be conserved in the sense that

\[
\int d^3r \psi_{k_f}^{(-)*}(r) \left( \omega_j^0 - \mathbf{q} \cdot \mathbf{j}_e \right) e^{-i\mathbf{q} \cdot \mathbf{r}} \psi_{k_i}^{(+)}(r) = 0,
\]

(16)
and that the nuclear current should separately be conserved,

$$\int d^3r \Psi_p(-)^*(r)e^{iq\cdot r}\left(\omega J_0^N - q \cdot J_N\right)\psi(r) = 0,$$

(17)

where q is the photon three momentum. With Coulomb distorted waves, the photon momentum q differs from the electron’s momentum transfer $Q = k_i - k_f$ and the longitudinal current is defined with respect to the direction of the photon that is exchanged, not with respect to the difference of asymptotic electron momenta.

In view of current conservation, the longitudinal current matrix element can be simplified to

$$j_0^N J_0^N - (\hat{q} \cdot J_N)(\hat{q} \cdot J_N) = j_0^N J_0^N\left(1 - \frac{\omega^2}{q^2}\right),$$

(18)

Using Eq. (5), we get for the longitudinal contribution to $M$

$$M_L = = \delta_{\lambda_f \lambda_i} \int d^3r \int \frac{d^3q}{(2\pi)^3} e^{i(Q-q)\cdot r} e^{i\chi(r)} f^D(-)(r) f^D(+) (r) \hat{n}_0^N(\chi) \hat{J}^N_{0}(q,p),$$

(19)

where Eq. (18) has been used to include the components of $j_e$ and $J_N$ that are parallel to q. The electron’s momentum transfer is $Q = k_i - k_f$ and $\chi = \gamma_i^0 + \gamma_f^0$ includes the phases of initial and final electron states. Note that $\gamma_i^0$, $\omega_i^0$, and $\gamma_f^0$ are obtained from Eq. (4) with the z-axis parallel to initial momentum $k_i$. In passing, we note that the Glauber approximation is obtained when the terms $r^{\hat{q}}(k_i + k_f)$, and only the leading-order phases, $\lambda_0^0$ and $\chi^0$, are retained. This approximation omits the focusing factors.

The longitudinal response function is obtained by dividing the cross section integrated over the angles of the knocked-out nucleon by the Mott cross section,

$$R_L = \frac{Q^4}{\sigma_{Mott}Q^4} \int d\Omega_p \frac{4\alpha^2}{(2\pi)^3} |M_L|^2 E_p^2 E_p,$$

(20)

where $M_L$ is the longitudinal amplitude of Eq. (19). The full calculation thus involves a six-dimensional integration in order to obtain the amplitude $M_L$. Two more integrations over the angles of the knocked-out nucleon are required in order to obtain the response function. Results based on the eight-dimensional integration are called “full calculations” in the following sections. The transverse transition matrix element, which will not be considered further in this work, is simply the difference of Eqs. (19) and (20).

In the actual calculations we mostly use a very simple model for the nuclear current

$$J^N_{0}\left(q,p\right) = \left(\frac{p_i^\mu + p_f^\mu}{\sqrt{4E_p(E_p - \omega)}}\right)\hat{\psi}(q - p),$$

(21)

where $\hat{\psi}(k)$ is a gaussian wave function for a bound nucleon,

$$\hat{\psi}(k) = (2\pi \beta^2)^{3/4} e^{-\beta^2 k^2}/4,$$

(22)

normalized such that $\int d^3k |\hat{\psi}(k)|^2/(2\pi)^3 = 1$.

This simple gaussian model is used because the Coulomb corrections should depend mainly on the electron wave functions. In order to get some idea how a more realistic model of nuclear structure would affect the results we considered also shell-model wave functions for $^{56}$Fe and $^{208}$Pb nuclei. In those cases, the Coulomb potential was calculated based on the empirical charge densities of Ref. 38 and the range parameter $R$ of the Coulomb potential was determined so that the average Coulomb potential matched the empirical one in the sense that $\int d^3r r \rho_{\text{expt}}(r) V_c(r) = \int d^3r r \rho_{\text{expt}}(r) V_{\text{expt}}(r)$. See Table III for the parameters used. For the shell-model wave functions, the gaussian parameter $\beta$ was selected such that the charge radius of the nuclei agreed with the empirical charge radius and when the higher orbitals are included, they are assumed to be described by harmonic-oscillator wave functions in coordinate space as follows,

$$\psi_{nlm}(r) = N Y_{nlm}(r) r^l F_1\left((-n - l)/2, l + 3/2, r \sqrt{2}/\beta\right) e^{-\left(r/\beta\right)^2},$$

(23)
Eq. (29) has obviously the form of an effective momentum approximation except that the full integral for the quasi-elastic matrix element \( eikonal \) phase and the focusing factors is retained. It is considerably simpler to calculate than the full six-dimensional integrals from around this point. Expanding the photon propagator around \( q \approx \sqrt{\omega} \), where \( \omega \) and \( q \) are the photon's energy and momentum. Because of energy conservation, \( E_p = M + \omega - B \), where \( B \approx .008GeV \) is a typical binding energy of a nucleon. For the PWIA response function the angular integration in Eq. (20) can easily be done. We find for the gaussian model

\[
R_L^{PWIA}(\omega, Q) = \frac{1}{\sqrt{2\pi}} \frac{(2E_p - \omega)^2}{4(E_p - \omega)} \beta |Q| \left( e^{-\beta^2(|Q|-p)/2} - e^{-\beta^2(|Q|+p)/2} \right),
\]

\( R_L^{PWIA} \) is normalized so that at fixed \( Q, \int d\omega R_L(Q, \omega) \approx 1. \)

### IV. EFFECTIVE-MOMENTUM APPROXIMATIONS

Let us consider the electron current matrix element for emission of a photon using spinors corresponding to initial and final helicity \( \lambda_i \) and \( \lambda_f \). Using Eq. (5) and a similar relation for the incoming electron wave function, the electron current can be rewritten as

\[
j^\mu_e = \int d^3r u^{(+)}\xi_{\lambda_f}(r) \gamma^\mu e^{-i\mathbf{q} \cdot \mathbf{r}} u^{(-)}\xi_{\lambda_i}(r)
\]

Using Eq. (27) the quasi-elastic transition matrix element (10) takes the form

\[
\mathcal{M} = \delta_{\lambda_f \lambda_i} \int d^3r \int \frac{d^3q}{(2\pi)^2} e^{i(\mathbf{Q} - \mathbf{q}) \cdot \mathbf{r}} e^{i\mathbf{q} \cdot \mathbf{r}} J^{D(-)}(r) f_i^{D(+)}(r) h^\mu_e(r) \left( \frac{1}{q^2} \right) J^N_n(q, p),
\]

with

\[
\delta_{\lambda_f \lambda_i} h^\mu_e = \xi^\dagger_{\lambda_f} e^{i\sigma_\alpha \gamma_i^{(-)} + i\sigma_\alpha \gamma_i^{(+)}},
\]

We may now use a stationary-phase-like argument to calculate Eq. (28). We see that for large \( Q \) the integrand of (28) has a rapidly changing phase except when \( Q - q + \nabla \chi(\mathbf{r}) = 0 \). So we expect that the dominant contribution in the integrals comes from around this point. Expanding the photon propagator around \( q = Q_{eff} \equiv Q + \nabla \chi(\mathbf{r} = 0) \) we can explicitly factor the photon propagator out of the integral over \( q \). Hence we expect that Eq. (28) can in a good approximation for large \( Q \) be determined by

\[
\mathcal{M}^{EMArm} = \delta_{\lambda_f \lambda_i} \frac{1}{Q_{\text{eff}}} \int \frac{d^3r}{2\pi^2} e^{i(Q - q) \cdot r} e^{i\mathbf{q} \cdot \mathbf{r}} J^{D(-)}(r) f_i^{D(+)}(r) h^\mu_e(r) J^N_n(r, p),
\]

with

\[
J^N_n(r, p) = \int d^3q e^{i\mathbf{q} \cdot \mathbf{r}} J^N_n(q, p).
\]

Eq. (29) has obviously the form of an effective momentum approximation except that the full \( \mathbf{r} \)-dependence of the eikonal phase and the focusing factors is retained. It is considerably simpler to calculate than the full six dimensional integral for the quasi-elastic matrix element \( \mathcal{M} \).
A further approximation can be made by approximating the eikonal phase by \( \chi \approx \chi(0) + r \cdot \nabla \chi(0) \) but keeping the \( r \)-dependence of the focusing factors, as follows,

\[
\mathcal{M}^{EMAr} = \delta_{\lambda f \lambda i} \left( \frac{1}{Q_{\text{eff}}^2} \right) e^{i\chi(0)} \int \frac{d^3r}{(2\pi)^3} e^{i(Q_{\text{eff}} - q \cdot r)} f_f^{D(-)}(r) f_i^{D(+)}(r) h^\mu_\nu(r, p) j^N(r, p)
\]

(30)

This is called the EMAr\(^*\) approximation. Finally one may take both the eikonal phase and focusing factors at the central value \( r = 0 \). In so doing we get the often-used \( ema \) approximation. This approximation usually is based on expanding the eikonal phase in a Taylor's series about \( r = 0 \) and keeping the first two terms. Moreover, the focusing factors are approximated by their values at \( r = 0 \) and the helicity matrix elements are approximated by the plane-wave values. Integration over \( r \) then gives \( \delta^{(0)}(q - Q_{\text{eff}}) \), so the longitudinal amplitude simplifies to the PWIA form

\[
\mathcal{M}_L^{ema} = 2\pi \delta_{\lambda f \lambda i} e^{i\chi(0)} h_{\text{PWIA}} f_i^0(Q_{\text{eff}}, p) \frac{f_f^{D(-)}(0) f_i^{D(+)}(0)}{Q_{\text{eff}}^2}.
\]

(31)

Combining the \( e^{-\omega r^2} \approx 1 - V_c(0)/(2E_i) \) factor of the eikonal correction with the \((1 - V_c/E_{2i})^{1/2} \) yields a focusing factor \( f_i^{D(+)} \approx 1 - V_c/E_i \) in the Dirac wave function, thus reproducing at \( r = 0 \) the expected factor \( 1 - V_c(0)/E_i \) that has been derived by Yennie, Boos and Ravenhall \([24]\) based on a WKB analysis of the Dirac-Coulomb wave function. A similar result holds for the final-state focusing factor, \( f_f^{D(-)} \), which is approximately \( 1 - V_c/E_f \). Thus, the overall focusing effect in the matrix element is approximately equal to \((1 - V_c(0)/E_f)(1 - V_c(0)/E_i)\).

The effective momentum involves the gradient of the eikonal phase shift \( \chi = \chi_f(-) + \chi_i(+) \) at the origin. Because of cylindrical symmetry of \( \chi_i(+) \) about the direction \( \hat{k}_i \), \( \nabla \chi_i(+) \) at the origin is nonzero only along the direction \( \hat{k}_i \), and similarly \( \nabla \chi_f(-) \) at the origin is nonzero only along the direction \( \hat{k}_f \). With \( v_i = v_f \approx 1 \), we find the same result as Traini,

\[
Q_{\text{eff}} = \hat{k}_i \left[ k_i - \delta k \right] - \hat{k}_f \left[ k_f - \delta k \right],
\]

(32)

where \( \delta k = V_c(0) \). It is correct up to first order in the eikonal expansion because the contribution from the gradient of eikonal correction \( \chi \) vanishes at the origin.

As shown by Rosenfelder \([25]\) and Traini \([22]\), there are significant cancellations in the Coulomb corrections when response functions are evaluated in this effective-momentum approximation (\( ema \)) using the approximate focusing factors, \( f_i^{D(+)} \approx 1 - V_c(0)/E_i \) and \( f_f^{D(-)} \approx 1 - V_c(0)/E_f \). Coulomb effects in the focusing factors and the effective photon propagator cancel if one considers the photon propagator of the transverse amplitude, which is \( 1/(Q_{\text{eff}}^2 - \omega^2) = 1/[4(k_i - V_c(0)][k_f - V_c(0)]sin^2 \theta c] \), i.e.,

\[
\frac{f_f^{D(-)}(0) f_i^{D(+)}(0)}{Q_{\text{eff}}^2 - \omega^2} = \frac{1}{Q_{\text{eff}}^2}.
\]

(33)

These factors produce the same result as in the plane-wave case, Eq. (12), but the momentum transfer argument in the nuclear structure function is shifted.

Evidence has been presented that the momentum shift as predicted by the \( ema \) is too large and that a smaller value should be taken for the Coulomb potential at the origin. This is done in view of the plausible classical argument that the Coulomb potential which is felt by the electron is not the central value of the potential, but rather is the average potential along the electron trajectory. Based on this argument the momentum shift \( \delta k \) in the eikonal wave function is weakened by a factor \( f_{ema} \)

\[
\delta k = f_{ema} V_c(0),
\]

(34)

where \( f_{ema} \) is determined by fitting the experimental quasi-elastic peak value. In practice one finds a reduction factor of typically \( f_{ema} \approx 0.7 \) to 0.8.

It should be noted that although the \( Q_{\text{eff}} \) is modified by the factor \( f_{ema} \), in the actual analysis one assumes that the cancellation \([33]\) still holds. In general, this cancellation is clearly expected not to be complete. Using a gauge-invariant response function as obtained from Eq. (130) leads to deviations, which are of the order of one percent. Another source of deviation is a more precise treatment of the \( r \)-dependence of the focusing factor as is done in the EMAr approximation, given by Eq. (29). The break down of the cancellation in the effective-momentum approach is reflected in allowing for an additional overall normalization \( A^{ema} \) in the response function as given in Eq. (35).
TABLE I: Parameters used in calculations: $\beta$ is the harmonic oscillator parameter; $V_0$ and $R$ are the Coulomb potential parameters. The gaussian model refers to Eq. (22).

| Nucleus | $\beta$ (fm) | $V_0$ (GeV) | $R$ (fm) |
|---------|--------------|-------------|----------|
| gaussian | 2.0          | 0.0273      | 2.0      |
| $^{208}$Pb | 3.564        | 0.0256      | 7.10     |
| $^{56}$Fe  | 2.854        | 0.0124      | 3.97     |

V. RESULTS

In this paper, we have described the eikonal expansion for relativistic wave functions in the presence of a Coulomb potential based on the Dirac equation. In the considered $1/k$ expansion, focusing factors are obtained in a systematic manner by use of the eikonal expansion. Although focusing factors take somewhat different forms using the Klein-Gordon wave function, equivalent results are found for the current matrix elements for the two cases [30].

Calculations of the longitudinal response function are performed for four cases: PWIA, ema, EMAr and the full calculation using distorted waves based on the Dirac equation. Eikonal phases are evaluated through second order, i.e., $\chi = \chi_0 + \chi_1 + \chi_2$ and $\omega = \omega_1 + \omega_2$. It should be noted however, that the expansion converges rapidly for the parameters and energies used and results based on $\chi_0 + \chi_1$ and $\omega_1$ differ by about 0.3% at the quasi-elastic peak.

![FIG. 2: Longitudinal response function versus the electron’s energy loss, $\omega$, calculated using the gaussian model for $e^-$ scattering at $E = 500$ MeV and $\theta_e = 60^\circ$. Dotted line shows PWIA, solid line shows ema based on $f_{ema} = 0.7$ and x’s show full calculations based on Eq. (19).](image)

In the figures 2 and 3 we have used the gaussian model of Table I and have considered electron and positron scattering. Calculations have omitted final-state interactions of the knocked-out nucleon. Figure 2 shows the longitudinal response function for 500 MeV electrons and electron scattering angle $\theta_e = 60^\circ$. Here the full calculations are plotted as x’s, while the ema calculation based on $\delta k = 0.7 V_c(0)$ is shown by the solid line. The PWIA prediction is shown by the dotted line. Figure 3 shows similar results for the longitudinal response function for $e^+$ scattering at 540 MeV using $f_{ema} = 0.7$.

In general the ema is seen in Figures 2 and 3 to produce a significant shift of $R_L$ away from the PWIA result and towards the full calculation of $R_L$. In both cases, the full calculations are reproduced quite well by the ema using $f_{ema} = 0.7$. This reconfirms the findings of Refs. [27, 28] that a smaller value of the $\delta k$ than $V_c(0)$ produces better agreement with the full results. From the effective-momentum approximation results we see that the shift in momentum due to the Coulomb distortion is predicted to be opposite in $e^+$ to that of $e^-$ scattering. There is indeed good agreement between the response functions for $e^-$ and $e^+$ scattering at the energies that make $Q_{eff}$ close to the same for both.

As discussed in Ref. [30], it is possible to fit the response functions more precisely if the momentum shift is allowed to be a function of energy loss, $\omega$. For the gaussian model, it is found that near the quasi-free peak the momentum shift is well described by $f_{ema} = 0.7$. The momentum shift can be significantly larger in magnitude, corresponding to $f_{ema} > 1$, when $\omega$ is significantly away from the value at the quasi-free peak. The reason is that the Coulomb distortions tend to alter the shape of the response functions away from the quasi-free peak. However, simply using a constant $f_{ema}$ does not incur large errors. The response integrated over $\omega$, as in the Coulomb sum rule, is expected
FIG. 3: Longitudinal response function versus the positron's energy loss, $\omega$, calculated using the gaussian model for $e^+$ scattering at $E = 540 MeV$ and $\theta_e = 60^\circ$. The dotted line shows PWIA, solid line shows $ema$ using $f_{ema} = 0.7$ and $\times$’s show full calculations based on Eq. (19).

TABLE II: The full calculation of the response function for the gaussian model at various $\omega$ values for $e^-$ scattering at 500 MeV, $\theta_e = 60^\circ$, together with the effective-momentum approximation results.

| $\omega$ | Full | $ema$ | EMAr | EMA$r'$ |
|----------|------|-------|------|--------|
| 0.060    | 1.616| 1.678 | 1.692| 1.756 |
| 0.080    | 4.548| 4.555 | 4.603| 4.655 |
| 0.100    | 7.626| 7.644 | 7.672| 7.717 |
| 0.120    | 9.122| 9.083 | 9.066| 9.122 |
| 0.140    | 8.236| 8.292 | 8.280| 8.326 |
| 0.160    | 6.124| 6.144 | 6.191| 6.194 |
| 0.180    | 3.854| 3.847 | 3.955| 3.909 |
| 0.200    | 2.124| 2.100 | 2.231| 2.160 |
| 0.220    | 1.055| 1.025 | 1.143| 1.075 |

to be accurate within one or two percent.

In Table II we show the numerical results for $e^-$ scattering at $E = 500 MeV$ and $\theta_e = 60^\circ$ of the full calculation together with the various effective-momentum approximations, using the gaussian model. From this we see that there is a close agreement between EMAr and the approximation EMA$r'$, obtained from EMAr on replacing the eikonal phase by $\chi(r) \approx f_{ema} \cdot \nabla \chi(0)$. This illustrates that the r-dependence of the phase shift can indeed well be approximated by the linearized form. Moreover, both $ema$ and EMAr are in good agreement with the full result. It should be noted that the assumed complete cancellation of the focus factor (33) in the $ema$ approximation, which is found to hold in this case, may be accidental.

The results for the simple gaussian model suggest that the $ema$ can reproduce the results of the more elaborate EMAr analysis quite well. In order to test this proposition for a more realistic model of the quasi-free scattering, calculations have also been made for the $^{56}Fe$ and $^{208}Pb$ nuclei using shell-model wave functions and a Coulomb potential that is based on the empirically determined charge density. The Dirac nucleon current Eq. (24) is used. Figures 4 and 5 show the results.

In these figures, we show the EMAr calculations as solid lines and two-parameter fits to them using the $ema$-fitting

TABLE III: Parameters used to fit the EMAr response functions using Eq. (35).

| Nucleus  | $E$ | $q$ | $\delta k$ | $f_{ema}$ | $A$ |
|----------|----|----|-----------|----------|----|
| $^{58}Fe$| 0.5| 0.55| -8.8      | 0.71     | 0.99 |
| $^{208}Pb$| 0.5| 0.55| -21.0     | 0.82     | 0.98 |
FIG. 4: EMAr longitudinal response function for $^{56}$Fe at $E_i = 500$ Mev and $q = 550$ MeV/c (solid line). The corresponding PWIA response functions without Coulomb effects included is shown by the dotted line. A fit of the EMAr response function using Eq. (35) is shown by the $\times$ symbols and values of the fitting parameters are given in Table III.

FIG. 5: EMAr longitudinal response function for $^{208}$Pb at $E_i = 500$ Mev and $q = 550$ MeV/c (solid line). The corresponding PWIA response function without Coulomb effects included is shown by the dotted line. A fit of the EMAr response function using Eq. (35) is shown by the $\times$ symbols and the parameters of the fit are given in Table III.

The obtained parameters are given in Table III. The main fitting parameter is the value of $f_{ema}$ that is used to determine the momentum shift in $Q_{eff}$. Using the shell-model wave functions, we find that response functions are fit by $f_{ema} = 0.71$ for $^{56}$Fe and by $f_{ema} = 0.82$ for $^{208}$Pb. Because of small distortions of the shape of the EMAr response function relative to the shape of the PWIA response function, a minor change of normalization is used also, as given by the $A$ parameter. The Coulomb effects are larger for the $^{208}$Pb nucleus because of the larger Coulomb potential, however the results can be fit using the $ema$ formula with an appropriate value of $f_{ema}$. The fact that similar values
of $f_{ema}$ are found for the gaussian model, for $^{56}Fe$ and for $^{208}Pb$ demonstrates that the Coulomb corrections are not very sensitive to the nuclear model.

Figure 6 shows partial response functions for individual shells of $^{208}Pb$ at 500 MeV electron energy. Three shells, the 1p, 1d and 1g shells of $^{208}Pb$, are shown based on three calculations: the EMAr, $ema$ using $f_{ema} = 0.82$ and PWIA. The results for individual shells show that the $ema$ results are close to the EMAr response functions that take into account the $r$-dependence of eikonal and shell wave functions. Differences are somewhat larger than for response functions summed over all shells. This is expected because the radial wave functions for the shells with $\ell > 0$ are suppressed near the origin where the Coulomb potential is largest. The $ema$ uses an average value for the Coulomb potential that works well for the sum over all shells and is less accurate for individual shells.

VI. CONCLUDING REMARKS

Within the eikonal approach we have studied how well the Coulomb distortion effects are described by effective-momentum approximations in electron scattering on nuclei at intermediate energies. This is expected to be a reliable description at increasing high energy. We have shown using a systematic eikonal expansion that already in the few hundred $MeV$ region the convergence of the expansion is indeed very fast and that the leading orders of the eikonal phase and focus factors in the electron wave function are sufficient to describe the Coulomb distortion in an accurate way. Moreover, the effective-momentum approximations are found to agree well with the full eikonal based calculations.

In this paper we have focused on the longitudinal response function. The transverse response contribution has also been calculated together with the spin-dependent terms occurring in the eikonal wave function \cite{Rho}. From the present study we find strong support for the conjecture that the effective-momentum approach can be used as the basis for analysis of the inclusive experimental data. In particular, the $ema$ approximation as used in the actual analysis of the experimental data is found in our model studies to do well.

A more precise form of the effective-momentum approximation would be useful for removing Coulomb corrections from experimental data in a straightforward manner and has been suggested in Ref. \cite{Pal}. In order to have a precise result, one could determine appropriate values of the momentum-shift function $\delta k(k_i, \omega, \theta_e)$ from which the appropriate $Q_{e ff}$ may be calculated as in Eq. \cite{Pal}. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{Solid lines show the partial EMAr longitudinal response function for the 1p, 1d and 1g shells of $^{208}Pb$ at $E_i = 500$ MeV and $q = 550$ MeV/c. The corresponding $ema$ response functions based on $f_{ema} = 0.82$ are shown as dash lines and the PWIA response functions are shown as dotted lines. The upper group is the 1g shell, the middle group is the 1d shell and the lower group is the 1p shell.}
\end{figure}
In the ema approximation a weakening factor $f_{ema}$ for the momentum shift is usually introduced, which takes into account in a phenomenological way the trajectory dependence of the phase shift. We have studied in this paper the EMAr approximation, which includes explicitly this r-dependence in the eikonal shift and focus factor without the introduction of $f_{ema}$. Up to a possibly small overall normalization constant correction, this is found to be in good agreement with the full calculation.

Our results for response functions omit final state interactions of the knocked-out nucleon. Their inclusion would affect the shape of the response functions but not the value of the Coulomb sum rule, which involves a sum over a complete set of states. The Coulomb corrections found in this work are not large enough to explain the differences that have been reported for the Coulomb sum rule by different experimental groups. [9, 11, 12]

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