Disordered random matrices

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Abstract. A revision is made of the effect, in the statistical properties of the matrices of random matrix theory, of dividing them by a positive random variable. As the method preserves unitary invariance analitical derivations are easily performed. It is shown that the same procedure applied to the adjacency matrices of random graphs gives rise to a family of graphs that interpolates between the Erdős-Renyi and the scale-free model. Previous applications and new ones are discussed.

1. Introduction
After being linked to the manifestations of chaos in quantum physical systems[1], the classes of random matrices introduced by Wigner in the 50s have found a great success. This success generated a great activity and extensions and generalizations of that ensemble have occurred. To the description of the spectral statistical fluctuations observed in compound nuclei, Wigner adapted the Wishart ensemble well known to statisticians. Some of the extensions of the Gaussian ensemble also can be considered as applications of common processes in statistics, for instance, models to describe symmetry breaking have been constructed by adding two random matrices, one block diagonal and, the other, its complement[2].

The concept of disorder has been introduced in random matrix theories[3] to denote ensembles constructed by superimposing an external source of randomness to the internal fluctuations of their matrices. A random process in which there is a competition between two types of random variables is typical of disordered systems or, in the case of Ising models, spin glasses[4]. As the two types of randomness are independent, one can be kept quenched while the fluctuations of the other continue to operate. Several families of ensembles fit into this category, for instance, for the Wigner Gaussian case see Ref. [5, 6]; for the case of Wishart matrices see [7]; and for the case of random graphs see [3]. Disordered ensembles can be considered as a special case of the so-called superstatistics[8].

Here, a revision is made of the procedure of introducing disorder in random matrices focussing in new results that go beyond the previous applications made in modelling regimes in growth process[9] and in quantum chaos[10].

2. Disordered Gaussian matrices
Generating a new random variable by taking the ratio of two other random ones is a common procedure known by statisticians and, sometimes called randomization[11]. By taking a random matrix $H_G$ of the Gaussian ensemble and dividing it by a positive random variable $\xi$, this
procedure has been extended in [3] to the case of random matrices. A new ensemble of matrices given by

\[ H = \frac{H_G}{\sqrt{\xi/\xi'}} \]  

(1)

is constructed in which \( H_G \) are matrices whose elements have joint density probability distribution \[ P_G(H) = \left( \frac{\beta}{2\pi} \right)^{f/2} \exp \left( -\frac{\beta}{2} \text{tr}H^2 \right), \]

(2)

while \( \xi \) is a positive random variable with density distribution \( w(\xi) \) and first moment \( \bar{\xi} \). In (2), \( f \) is the number \( f = N + \beta N(N-1)/2 \) of independent elements of Hermitian matrices of dimension \( N \) and \( \beta \) is the Dyson index \( \beta = 1, 2, 4 \) for GOE (real elements), GUE (complex elements) and GSE (quaternion elements). Here and in what follows the subindex \( G \) indicates Gaussian. In this way, an external source of randomness is superimposed to the fluctuations of the Gaussian matrix \( H_G \): Here the disorder is represented by \( \xi \) which is the quenched variable in opposition to the randomness of the Gaussian matrices. In physical terms, the randomization procedure expressed by Eq. (1) leads to a disordered ensemble.

It is straightforward to deduce from Eqs. (1) and (2) that the disordered ensemble has density distribution

\[ P(H) = \int_0^\infty d\xi w(\xi) \left( \frac{\beta \xi}{2\pi} \right)^{f/2} \exp \left( -\frac{\beta \xi}{2} \text{tr}H^2 \right), \]

(3)

an expression, Eq. (3), with characteristics of the so-called superstatistics[8], but, the fact that the two distributions appear normalized inside the integral, departs it from that scheme.

Turning now to eigenvalues and eigenvectors, we observe that Eq. (1) implies in \( H \) having the same eigenvectors of \( H_G \). Therefore Eq. (1) can be diagonalized to give

\[ D = \frac{D_G}{\sqrt{\xi/\xi'}}, \]

(4)

where the matrices \( D \) and \( D_G \) are diagonal matrices containing the respective eigenvalues, as a consequence, the relation

\[ P(E_1, \ldots E_N) = \int_0^\infty d\xi w(\xi) \left( \alpha \xi / \xi' \right)^N P_G(x_1, \ldots x_N) \]

(5)

follows, where \( x_i = \sqrt{\xi/\xi'} E_i \) and

\[ P_G(x_1, \ldots x_N) = K_N^{-1} \exp \left( -\frac{\beta}{2} \sum_{k=1}^N x_k^2 \right) \prod_{j>i} |x_j - x_i|^\beta \]

(6)

is the known normalized eigenvalue distribution of the Gaussian ensemble.

Eq. (5) means that measures of the generalized family can be calculated by weighting the corresponding measures of the Gaussian ensembles with the \( w(\xi) \) distribution. For instance the eigenvalue density is expressed in terms of the Wigner’s semi-circle law[12] as

\[ \rho(E) = \frac{1}{\pi} \int_0^{\xi_{max}} d\xi w(\xi) (\xi/\xi')^{1/2} \sqrt{2N - \xi E^2 / \xi}. \]

(7)

where \( \xi_{max} = 2\xi N/E^2 \). By making the substitution \( v = \xi/\xi_{max} \) the density becomes
\[ \rho(E) = \frac{(2N)^2}{\pi |E|^3} \int_0^1 d\xi w \left( \frac{2N\xi v}{E^2} \right) \sqrt{v(1-v)}, \] (8)

which, depending on the behavior of the function \( w(\xi) \) at the origin, implies in a power-law decay. Ensembles showing heavy-tailed densities occurs, for instance, when \( w(\xi) \) is the gamma distribution

\[ w(\xi) = \exp(-\xi)\xi^{\xi-1}/\Gamma(\xi). \] (9)

With variance \( \sigma_\xi = \sqrt{\xi} \) this distribution becomes more localized when \( \xi \) increases and, as a consequence, it should be expected that the Gaussian ensembles is recovered for large values of \( \xi \).

Another choice that has been investigated is the hyperbolic function

\[ w(a, b, \lambda, \xi) = \left( \frac{b}{a} \right)^{\lambda/2} \xi^{\lambda-1} \frac{1}{2K_\lambda(\sqrt{ab})} \exp \left[ -\frac{1}{2} \left( \frac{a}{\xi} + b\xi \right) \right], \] (10)

which, in contrast to Eq. (9), has all moments, and should produce a more regular behavior. The introduction in the Gaussian matrices of this disorder lead to a model that gives a good description of the intermediate statistics[13] exhibited by some quantum physical systems[10].

3. A random walk

![Figure 1. Random walk with correlated gaussian steps with \( \xi = 0.5 \).](image)

Substituting Eq. (9) in Eq. (3), the integral in the external variable \( \xi \) can be performed to give

\[ P(H, \xi) = \left( \frac{\beta}{2\pi \xi} \right)^{\frac{1}{\xi}} \frac{\Gamma \left( \xi + \frac{\xi}{2} \right)}{\Gamma \left( \frac{\xi}{2} \right) \left( 1 + \frac{\beta}{2\xi^2} H^2 \right)^{\xi + \frac{1}{2}}} \] (11)
for the joint density distribution of matrix elements. Defining
\[ \frac{1}{q - 1} = \xi + \frac{f}{2}, \quad \text{with } q > 1. \] (12)

Eq. (11) is just Eq. (4) of [5, 6] which was derived using a generalized maximum entropy principle[14] with \( q \) being identified with Tsallis entropic parameter.

Keeping \( n \) matrix elements fixed and integrating the others, it is found that the probability of a set of \( n \leq f \) matrix elements is given by an expression with the same structure of Eq. (11) that is
\[ p(h_1, h_2, \ldots, h_n) = \left( \beta \right)^{n/2} \frac{\Gamma(\xi + \frac{n}{2})}{\Gamma(\xi) \left(1 + \beta n \frac{1}{\Gamma(\xi)} \sum_{i} h_i^2 \right)^{\xi + n/2}} \] (13)

where \( h_i = H_{ii} \) for the diagonal and \( h_i = \sqrt{2} H_{ij} \) for the off-diagonal elements. Eq. (13) makes evident that the matrix elements are correlated.

Eq. (1) makes straightforward to do numerical simulations in terms of the Gaussian matrices. However, matrices of the ensemble also can be generated by taking into account the correlations among their elements. To do this, consider the identity
\[ p(h_1, \ldots, h_f) = p(h_1) \prod_{n=2}^{f} \frac{p(h_1, \ldots, h_n)}{p(h_1, \ldots, h_{n-1})} \] (14)

where each fraction gives the conditional probability \( P(h_n|h_1, \ldots, h_{n-1}) \) for the \( n \)th element once the \( n-1 \) previous ones are given. Since
\[ P(h_n|h_1, \ldots, h_{n-1}) = \sqrt{\frac{\beta}{2B_{n-1} \pi \xi}} \frac{\Gamma(\xi + \frac{n}{2})}{\Gamma(\xi + \frac{n-1}{2})} \frac{1}{\left(1 + \frac{\beta h_n^2}{2B_{n-1}} \right)^{\xi + n/2}}, \] (15)

where
\[ B_{n-1} = 1 + \frac{\beta}{2\xi} \sum_{i=1}^{n-1} h_i^2, \] (16)

the variables \( h_i \) sequentially can be sorted using, at each step, an univariate distribution. The sum of the elements thus generated, defines the random walk
\[ S_n(\xi) = \sum_{i=1}^{n} h_i. \] (17)

Although, the sequence \( h_1, h_2, \ldots, h_n \) is made of correlated variables, by Eq. (1), every matrix of the disordered ensemble should belong to a Gaussian distribution defined by a particular value of the external variable \( \xi \). This means that the random walk is made of a set of correlated steps taken from a Gaussian distribution.

In Fig. 1 and Fig. 2, plots of calculated \( S_n(\xi) \) shows the behavior of the random walk for the values \( \xi = 0.5 \) and 100 respectively.

4. Fluctuations of the largest eigenvalue

The probability \( E_{G, \beta} \) that the infinite interval \( (t, \infty) \) has no eigenvalues, that is, that the largest eigenvalue \( x_{\text{max}} \) be less than \( t \) has been calculated for the three classes of the Gaussian ensemble
Figure 2. Random walk with Gaussian correlated steps for $\xi = 100$.

[15] and found many applications[16]. At the edge of the spectrum, it is necessary to introduce the scaling

$$x = (2\sqrt{N} + \frac{s}{N^{1/6}})\sigma$$

(18)

where $\sigma$ is the variance $\sigma = 1/\sqrt{2\beta}$ of the off-diagonal matrix elements. In this new scaled variable, $s$, the probability $E_{G,2}(s)$, for the unitary case, $\beta = 2$, is

$$E_{G,2}(s) = \exp\left[-\int_s^\infty (x-s)q^2(x)dx\right],$$

(19)

where $q(s)$ satisfies the Painlevé II equation

$$q'' = sq + 2q^3$$

(20)

with boundary condition

$$q(s) \sim \text{Ai}(s) \text{ when } s \to \infty,$$

(21)

where $\text{Ai}(s)$ is the Airy function. For GOE ($\beta = 1$) and GSE ($\beta = 4$) are

$$[E_{G,1}(s)]^2 = E_{G,2}(s) \exp[-\mu(s)]$$

(22)

and

$$[E_{G,4}(s)]^2 = E_{G,2}(s) \cosh^2 \frac{\mu(s)}{2}$$

(23)

where

$$\mu(s) = \int_s^\infty q(x)dx.$$  

(24)
The above equations give a complete description of the fluctuations of the eigenvalues at the edge of the spectra of the Gaussian ensembles.

Turning now to the disordered case, the probability that the largest eigenvalue $E_{\text{max}}$ is smaller than a given value $t$ is obtained by averaging the above expression with the weight distribution $w(\xi)$ as

$$
E_\beta (E_{\text{max}} < t) = \int_0^\infty d\xi w(\xi) E_{G,\beta}[S(\xi, t)],
$$

where the argument of $S(\xi, t)$ is obtained by plugging in Eq. (18) the variance $\sigma(\xi) = \sqrt{\xi/2\beta\xi}$ such that

$$
S(\xi, t) = N^{1/6} \left(t \sqrt{\frac{2\beta\xi}{\xi}} - 2\sqrt{N}\right).
$$

One successful application of the Tracy-Widom distributions was to growth processes\cite{17}. Considering the evolution of a surface generated by an automata cellular model, analytically was found that the maximum point at the frontier follows the Tracy-Widom distribution. It was then also found that, in the presence of a random media, a transition occurs from the Tracy-Widom to a Gaussian regime\cite{18}. Our disordered ensemble seems well fitted to model this transition. In fact, by introducing a scaling of the average $\bar{\xi}$ of Eq. (9) with the matrix size $N$, it was shown, in Ref. [9], that the largest eigenvalue, in the limit $N \to \infty$, undergoes the transition from Tracy-Widom to Gaussian. Moreover, a critical regime described by the convolution of the two limiting distributions has also been observed.

The fluctuations described by the Tracy-Widom distributions and its disordered counterpart, are well localized around the edge of the semi-circle density. Another kind of fluctuations would be very large and, in principle, very rare events in which deviations are of the order of the size of the spectrum. This kind of fluctuations have been calculated\cite{19} including the probability that all eigenvalues be negative (or positive). In order to address the effect of disorder in these large fluctuations, we can put in Eq. (25) the distribution derived in Ref. [19] instead of the Tracy-Widom ones. Letting this approach to a future work, here we consider what happens in the limit $N \to \infty$ when the scaling of $\bar{\xi}$ is removed.

In this limit, compared with the function $w(\xi)$, and keeping the ratio $t/\sqrt{2N}$ fixed, the function $E_{G,\beta}$ approaches a step function centered at $\xi = 2N\bar{\xi}/t^2$. As a consequence, the probability distribution for the largest eigenvalue converges to

$$
E_\beta (E_{\text{max}} < t) = \int_0^\infty d\xi w(\xi)
$$

with density distribution

$$
\frac{dE_\beta (t)}{dt} = \frac{4N\bar{\xi}w(2N\bar{\xi}/t^2)}{t^3}.
$$

Using Eq. (9), Eq. (28) becomes

$$
\frac{dE_\beta (t)}{dt} = \frac{2\bar{\xi} \exp(-\bar{\xi}/t^2)}{\Gamma(\bar{\xi}) t^{2\bar{\xi}+1}}.
$$

Notice that in (28) the variable $t$ is the eigenvalue itself without any edge scaling. This means that, in this case, fluctuations become of the order of the size of the average ensemble spectrum. Eq. (29) resembles a Fréchet distribution, that is the distribution that describes extreme values at the edge of uncorrelated sequences with a power-law decay. However, the fact
that the power of \( t \) in the exponent is fixed at the value two makes it a different distribution except for \( \xi = 1 \) when it coincides with a Fréchet distribution. Therefore, Eq. (29) defines a long-tailed distribution of extreme values of a correlated set of points. The comparison with Fréchet distribution is shown in Fig. 3.

Another important point is that the distribution Eq. (29), and supposedly also the general one, Eq. (28), vanishes at the origin that is for \( t = 0 \). Therefore, the above approach cannot reproduce the results of Ref. [19]. In particular, it is remarkable that the introduction of disorder does not affect the probability of having all eigenvalues negative (or positive). In fact, putting \( t = 0 \) in Eq. (25), one concludes from Eq. (26) that the disordered distribution of the largest eigenvalue coincides with the Tracy-Widom distribution.

5. Bernoulli disordered random variable

One convenient way to characterize a Bernoulli variable of probability \( p \) is by the Logit function\[20\]

\[
\alpha = \text{Logit}(p) = \log \left( \frac{p}{1-p} \right),
\]

where, as \( p \) is defined in the interval \([0, 1]\), the domain of variation of \( \alpha \) is \( ] - \infty, \infty[ \). In terms of \( \alpha \), the probability is expressed as

\[
p = \text{Logit}^{-1}(\alpha) = \frac{1}{1 + \exp(-\alpha)}. \tag{31}
\]

Considering a set \( a = (a_1, a_2, ..., a_f) \) of \( f \) independent Bernoulli variables with values 1(0), their joint density distribution is expressed in terms of the Logit function \( \alpha \) as

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**Figure 3.** For indicated values of \( \xi \) the distributions of the largest eigenvalue of the disordered ensemble are compared with the Fréchet distribution with the same decay.
\[ P(a_1, a_2, ..., a_f) = \frac{1}{[1 + \exp(\alpha)]^f} \exp \left( \alpha \sum_{i=1}^{f} a_i \right). \quad (32) \]

In particular, the probability that \( k \) elements of the set have value one while the others vanish is

\[ \text{Prob} \left( \sum_{i=1}^{f} a_i = k \right) = \frac{f!}{k!(f-k)!} \frac{\exp(\alpha k)}{[1 + \exp(\alpha)]^f}. \quad (33) \]

If the Logit function is multiplied by the ratio \( \xi/\overline{\xi} \) between a positive random variable \( \xi \) of density distribution \( w(\xi) \) and its average, then the set \( a \) becomes a correlated set of disordered Bernoulli variables with joint density distribution

\[ P(a_1, a_2, ..., a_f) = \int_{0}^{\infty} \frac{d\xi w(\xi)}{[1 + \exp(\xi \alpha / \overline{\xi})]} \exp \left( \frac{\alpha \xi}{\overline{\xi}} \sum_{i=1}^{f} a_i \right). \quad (34) \]

The probability that \( k \) elements of the set have value one while the others vanish is, in this case, given by

\[ \text{Prob} \left( \sum_{i=1}^{f} a_i = k \right) = \frac{f!}{k!(f-k)!} \int_{0}^{\infty} \frac{d\xi w(\xi)}{[1 + \exp(\xi \alpha / \overline{\xi})]} \exp \left( \frac{\alpha \xi}{\overline{\xi}} k \right). \quad (35) \]

A graph is an array of \( N \) points (nodes) connected by edges, it can be represented by its adjacency matrix \( A \) whose elements \( A_{ij} \) have value 1(0) if the pair \((ij)\) of nodes is connected (disconnected). The diagonal elements are taken equal to zero, i.e. \( A_{ii} = 0 \). A classical graph model was proposed by Erdős-Renyi (ER) in which there is a fixed probability \( p \) that a given pair of nodes is connected, independently of the others\[22\]. In this case, the adjacency matrix becomes a random matrix whose joint density distribution of elements is given as in Eq. (32) with \( f = \frac{N(N-1)}{2} \). A disordered Erdős-Renyi then follows in which elements of the adjacency matrix has density distribution as in Eq. (34).

For a not too rarefied graph, the eigenvalue density of the eigenvalues of the adjacency matrix \( A \) of the ER model can be obtained from the moments of the trace of the powers of the matrix and one finds that it obeys the Wigner semi-circle law\[21\]

\[ \rho_{ER}(E, \alpha) = \begin{cases} \frac{1}{2\pi\sigma^2} \sqrt{4N \sigma^2 - E^2}, & \text{if } |E| < \sqrt{4N \sigma^2} \\ 0, & \text{if } |E| > \sqrt{4N \sigma^2} \end{cases} \quad (36) \]

where \( \sigma^2 \) is the variance of the matrix elements given by

\[ \sigma^2 = p(1-p) = \frac{1}{4 \cosh^2(\alpha/2)}. \quad (37) \]

The above argument fails if \( p \sim 1/N \) in which case deviations from the semi-circle appear\[23, 24\].

Immediately, the eigenvalue density of the ER disordered random graph is obtained by multiplying \( \alpha \) by the external random variable \( \xi \) and integrating it, that is

\[ \rho(E; \alpha) = \frac{2}{\pi} \int_{0}^{\xi_m} d\xi w(\xi) \cosh(\frac{\alpha \xi}{2}) \sqrt{N - \cosh^2(\frac{\alpha \xi}{2})E^2}, \quad (38) \]

where
\[ \xi_m = \frac{2}{\alpha} \cosh^{-1}(\sqrt{\frac{N}{E}}). \] (39)

To proceed, we choose the weight distribution \( w(\xi) \) to be given by Eq. (9). In Ref. [3] it is shown that with this choice the generalized model has features of a scale-free graph. In particular, the eigenvalue density exhibits a crossover from the Wigner semi-circle law to a distribution highly peaked with exponential tails.

In the same spirit, the degree distribution is given by Eq. (35)

\[ P_k(\alpha, \xi) = \frac{(N-1)!}{k!(N-1-k)\Gamma(\xi)} \int_0^\infty d\xi \exp(-\xi) \xi^{\lambda-1} \frac{\exp\left(\frac{k\alpha\xi}{\xi}\right)}{[1 + \exp\left(\frac{\alpha\xi}{\xi}\right)]^{N-1}}. \] (40)

The above integral is hard to evaluate when the number of nodes is large, however, as one is usually interested in large graphs that is, large values of \( N \), the integral in (40) becomes suitable to an asymptotic evaluation. In fact, if we define

\[ F(\xi) = \xi - (\lambda - 1) \ln \xi - k \ln p(\xi) - (N-1-k) \ln[1-p(\xi)] \] (41)

where

\[ p(\xi) = \frac{1}{1 + \exp\left(\frac{-\alpha\xi}{\xi}\right)} \] (42)

we deduce that

\[ P_k(\alpha, \xi) = \frac{(N-1)!}{k!(N-1-k)!} w(\xi_s) [p(\xi_s)]^k [1-p(\xi_s)]^{N-1-k} \sqrt{\frac{2\pi}{F''(\xi_s)}} \] (43)

where \( \xi_s \) is the solution of the transcendental equation \( F'(\xi_s) = 0 \). This asymptotic expression allows an accurate way to calculate the degree distribution of large graphs.

In Fig. 4, degree distributions using the asymptotic calculations are shown for the values \( \bar{\xi} = 1, 5, 10 \). The scaling \( p = N^{-z} \) has been used. It is clear the power-law behavior for \( \bar{\xi} = 1 \) while, for the other cases, there is a cut-off of the large connectivities.

6. Conclusions
In summary, a revision is made of the introduction of an external source of randomness superimposed to the internal fluctuations of random matrices of an ensemble. The method introduces correlations among the matrix ensembles and, as a consequence, the ergodic property (equivalence of spectral and ensemble averages) has to be abandoned. Therefore the new ensembles generated are more pertinent to cases in which only ensemble averages make sense as, for instance, the behavior of extreme eigenvalues. Another interesting application is to a random walk made of correlated Gaussian steps. Also in the case of random graph, the correlations introduced has an effect similar to that of the preferential attachment of the scale-free graphs[21].

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Figure 4. Asymptotic degree distributions for $\bar{\xi} = 1, 5, 10$ assuming the scaling $p = N^{-z}$.

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