THE RAINBOW ERDŐS-ROTHSCHILD PROBLEM FOR THE FANO PLANE

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Abstract. The Fano plane is the unique linear 3-uniform hypergraph on seven vertices and seven hyperedges. It was recently proved that, for all $n \geq 8$, the balanced complete bipartite 3-uniform hypergraph on $n$ vertices, denoted by $B_n$, is the 3-uniform hypergraph on $n$ vertices with the largest number of hyperedges that does not contain a copy of the Fano plane. For sufficiently large $r$ and $n$, we show that $B_n$ admits the largest number of $r$-edge colorings with no rainbow copy of the Fano plane.

1. Introduction

This paper contributes to a line of research about coloring problems on combinatorial structures that originated from a graph-theoretical question of Erdős and Rothschild [13]. Their question was motivated by the Turán problem [41].

As usual, for a fixed graph $F$, we say that a graph $G$ is $F$-free if it does not contain $F$ as a subgraph. The Turán problem associated with $F$ asks us to find the maximum number of edges among all $F$-free $n$-vertex graphs, which is denoted $\text{ex}(n, F)$, and to determine the $F$-free $n$-vertex graphs $G$ with this number of edges, known as the $F$-extremal graphs.

Here we consider a related problem, which deals with $r$-colorings of the edge set of $G$ that do not contain a copy of $F$ colored according to a fixed pattern. An $r$-edge-coloring (or simply $r$-coloring) $\Delta$ of a graph $G = (V, E)$ is a function $\Delta: E \rightarrow [r]$, where $[r] = \{1, \ldots, r\}$, and an $r$-pattern $P$ of a graph $F$ is a partition of the edge set of $F$ into at most $r$ classes. An $r$-coloring of $G$ is said to be $(F, P)$-free if it does not contain a copy of $F$ in which the partition of the edge set induced by the coloring is isomorphic to $P$. Given a graph $G$, one may consider the number $c_{r,(F,P)}(G)$ of $(F, P)$-free $r$-colorings of $G$ and define $c_{r,(F,P)}(n)$ as the maximum of this quantity over all $n$-vertex graphs. An $n$-vertex graph $G$ for which $c_{r,(F,P)}(G) = c_{r,(F,P)}(n)$ is said to be $(r, F, P)$-extremal.

The original Erdős-Rothschild question concerned the instance where $F$ is a complete graph $K_\ell$ and the pattern $P$ consists of a single class, i.e., Erdős and Rothschild were interested in edge-colorings of $G$ avoiding monochromatic copies of $K_\ell$. Later, Pikhurko, Staden and Yilma [36] proposed a generalization of the original monochromatic problem, while Balogh [2] and two of the current authors [21] studied problems leading to the more general version considered here. A non-monochromatic pattern that attracted considerable attention for various graphs $F$ is the rainbow pattern, namely the pattern where all partition classes are singletons.

This work was partially supported by CAPES and DAAD via Probral (CAPES Proc. 88881.143993/2017-01 and DAAD 57391132 and 57518130). The first author was supported by CAPES. The second author acknowledges the support of CNPq (Proc. 308054/2018-0).
Since the late 90’s, several researchers have obtained substantial progress both for the monochromatic case [1, 18, 37, 12] and for other patterns [3, 5, 7, 20, 23, 24, 26]. Typically, the sets of F-extremal graphs and \((r, F, P)\)-extremal graphs coincide if \(P\) is monochromatic and \(r \leq 3\) or if \(P\) is rainbow and \(r\) is sufficiently large. Versions of the Erdős-Rothschild problem have also been studied in the monochromatic setting for set systems [10, 19], linear spaces [10, 22], partial orders [12] and sum-free sets [17], for instance.

In this paper, we consider this problem for \(k\)-uniform hypergraphs, that is, for pairs \(H = (V, E)\) where \(V\) is a finite set called the vertex set of \(H\) and \(E \subseteq \{e: e \subseteq V, |e| = k\}\) is called the edge set of \(H\). For a fixed \(k\)-uniform hypergraph \(F\), the concepts of \(F\)-free hypergraph and of \(F\)-extremal \(k\)-uniform hypergraph with \(n\) vertices may be defined as in the graph case. The same holds for the Turán number \(\text{ex}(n, F)\). For a positive integer \(r\), an \(r\)-coloring of a hypergraph \(H = (V, E)\) is again a function associating each hyperedge in \(E\) with a color in \([r]\) and a color pattern \(P\) of a hypergraph \(F\) is a partition of its edge set. An \(r\)-coloring of a hypergraph \(H\) is said to be \((F, P)\)-free if there is no copy of \(F\) in \(H\) such that the partition of \(E(F)\) induced by the \(r\)-coloring of \(H\) is isomorphic to \(P\). For fixed integers \(n, r\) and \(k\), if \(\mathcal{H}_{n,k}\) denotes the set of \(n\)-vertex \(k\)-uniform hypergraphs, \(F\) is a fixed \(k\)-uniform hypergraph and \(P\) is a pattern of \(F\), let \(c_{r,(F,P)}(H)\) be the number of \((F, P)\)-free \(r\)-colorings of \(H\). The aim is to determine the function

\[
c_{r,(F,P)}(n) = \max \{c_{r,(F,P)}(H): H \in \mathcal{H}_{n,k}\} \tag{1}
\]

and the hypergraphs \(H \in \mathcal{H}_{n,k}\) that achieve equality in (1). The hypergraphs satisfying this are said to be \((r, F, P)\)-extremal. When \(P\) is the monochromatic pattern, exact results about this problem have been obtained, for sufficiently large \(n\), when \(F\) lies in some class of expanded graphs [29] and when \(F\) is the Fano plane [30]. The Fano plane is the unique linear 3-uniform hypergraph on seven vertices and with seven hyperedges. Results for the monochromatic pattern in more general hypergraphs may be found in [31].

A proof method that has been particularly useful for the monochromatic pattern in instances where the \(F\)-extremal configuration is dense (that is, where \(\text{ex}(n, F) = \Omega(n^k)\)) was developed by Alon, Balogh, Keevash and Sudakov [1]. Their approach consists of two steps: (i) Prove a stability result establishing that any counterexample \(H\) to the desired result would be similar to the actual \(n\)-vertex \(F\)-extremal structure. (ii) Assuming that there is an \(n\)-vertex graph other than the \(F\)-extremal graph with at least as many \(r\)-colorings as the extremal graph, prove that there is a sub-hypergraph whose number of colorings creates a ‘gap’ to the number of colorings of the \(F\)-extremal configuration that is even larger. A recursive application of this step leads to a counterexample whose number of colorings is too large to be feasible.

For instance, the results in [30] are an adaptation of this approach to the case when \(F\) is the Fano plane, \(P\) is the monochromatic pattern and \(r \in \{2, 3\}\). As a second example, the results in [31] give step (i) for arbitrary \(F\), where \(P\) is again the monochromatic pattern and \(r \in \{2, 3\}\). This is used in [29] to derive the exact result (namely step (ii)) when \(F\) is an expanded complete graph or a Fan hypergraph.

We show that, for the rainbow pattern \(P\) of the 3-uniform Fano plane and sufficiently large \(r\) and \(n\), the stability given in (i) holds. We then use the strategy in (ii) to prove
that this stability implies that the Fano-extremal and the \((r, \text{Fano}, P)\)-extremal \(n\)-vertex configurations coincide for large \(n\).

We finish the introduction with the precise statement of our main result. In the remainder of this paper, we shall write \(F^R\) for the rainbow pattern of the Fano plane. In particular, we shall refer to \(F^R\)-free \(r\)-colorings, to \((r, F^R)\)-extremal hypergraphs and to the function \(c_{r,F^R}(n)\) instead of \((\text{Fano}, P)\)-free \(r\)-colorings, \((r, \text{Fano}, P)\)-extremal hypergraphs and \(c_{r, (\text{Fano}, P)}(n)\), respectively.

Given a positive integer \(n\), let \(B_n\) be the \(n\)-vertex 3-uniform hypergraph defined as follows. There is a partition \(V(B_n) = V_1 \cup V_2\) of the vertex set of \(B_n\), with \(||V_1| - |V_2|| \leq 1\), such that \(E(B_n)\) consists of all triples having non-empty intersection with both classes. Füredi and Simonovits [16] and, independently, Keevash and Sudakov [27] proved that \(B_n\) is the unique \(F\)-extremal hypergraph for all \(n\) sufficiently large. Recently, Bellmann and Reiher [6] proved that this holds for every \(n \geq 8\).

We prove that \(B_n\) is \((r, F^R)\)-extremal if \(r\) and \(n\) are sufficiently large.

**Theorem 1.1.** The following holds for every \(r \geq r_0 = 6^{492^{64} \cdot (37 \cdot 16^3 \cdot 1406^9)^{63}}\).

There exists \(n_0 = n_0(r)\) such that for any \(n \geq n_0\) and any 3-uniform \(n\)-vertex hypergraph \(H\), the inequality

\[
c_{r,F^R}(H) \leq r_{\text{ex}}^{(n, \text{Fano})}
\]

holds. Moreover, for \(n \geq n_0\), equality holds in (2) if and only if \(H\) is isomorphic to \(B_n\).

Our paper is structured as follows. In Section 2 we introduce definitions and auxiliary results that will be useful for proving our main theorem. In Section 3 we will derive a colored stability result, which is Theorem 2.6. With this we prove an embedding result in Section 4, which implies Theorem 1.1. We conclude the paper with final remarks and open problems.

## 2. Notation and tools

In this section, we state auxiliary results that will be used in our proofs. In addition to fairly standard definitions and technical results, we shall consider a version of the regularity method for hypergraphs, known as the Weak Hypergraph Regularity Lemma, and embedding results related to it.

### 2.1. Regularity Lemma

A key tool in this paper is the so-called weak hypergraph regularity lemma. It is a natural extension of the Szemerédi Regularity Lemma [10].

Let \(H = (V, E)\) be a \(k\)-uniform hypergraph and let \(W_1, \ldots, W_k\) be mutually disjoint non-empty subsets of \(V\). Let \(E(W_1, \ldots, W_k) = \{e \in E(H) : |e \cap W_i| = 1 \forall i \in [k]\}\) and consider the density

\[
d_H(W_1, \ldots, W_k) = \frac{|E(W_1, \ldots, W_k)|}{|W_1| \cdots |W_k|}
\]

of \(H\) with respect to the sets \(W_1, \ldots, W_k\).

For fixed \(\varepsilon > 0\) and \(d > 0\), we say that \((V_1, \ldots, V_k)\) is \((\varepsilon, d)\)-regular, if

\[|d_H(W_1, \ldots, W_k) - d| \leq \varepsilon\]
for all $k$-tuples $(W_1, \ldots, W_k)$ of subsets $W_i \subseteq V_i$, $i \in [k]$, with $\prod_{i=1}^{k} |W_i| \geq \varepsilon \prod_{i=1}^{k} |V_i|$. Such an $k$-tuple $(V_1, \ldots, V_k)$ is said to be $\varepsilon$-regular if it is $(\varepsilon, d)$-regular for some $d \geq 0$, and $\varepsilon$-irregular otherwise.

Finally, an equitable partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ of the vertex set of a $k$-uniform hypergraph $H = (V, E)$ is $\varepsilon$-regular if, for all but at most $\varepsilon \binom{s}{k}$ distinct $k$-element subsets $\{i_1, \ldots, i_k\} \subseteq [t]$, the $k$-tuple $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon$-regular.

The following is a colored version of the Weak Regularity Lemma that will be used in this paper, which may be easily derived from the work in [9, 15, 28, 39].

**Theorem 2.1 (Colored Regularity Lemma).** For all integers $r \geq 1$, $k \geq 2$ and $m_0 \geq 1$, and every $\varepsilon > 0$, there exist $M_0 = M_0(r, k, m_0, \varepsilon)$ and $N_0 = N_0(r, k, m_0, \varepsilon)$ with the following property. Every $k$-uniform hypergraph $H = (V, E)$ on $n \geq N_0$ vertices whose set of hyperedges is $r$-colored $E(H) = E_1 \cup \cdots \cup E_r$ admits an equitable partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ of $V$ with $m_0 \leq m \leq M_0$ that is simultaneously $\varepsilon$-regular for all sub-hypergraphs $H_i = (V, E_i)$, where $i \in [r]$.

This colored regularity lemma gives rise to a cluster hypergraph where, for each regular $k$-tuple, we record the colors that appear with density larger than a certain threshold $\eta$.

**Definition 2.2 (Multicolored cluster hypergraph).** Let $H = (V, E)$ be a $k$-uniform hypergraph, whose set of hyperedges is $r$-colored $E(H) = E_1 \cup \cdots \cup E_r$. Consider an equitable partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ of $V$ that is simultaneously $\varepsilon$-regular for all sub-hypergraphs $H_i = (V, E_i)$, where $i \in [r]$. For $\eta > 0$, the multicolored cluster hypergraph $\mathcal{R} = \mathcal{R}_H(\mathcal{V}, \eta)$ associated with this partition $\mathcal{V}$ and $\eta$ is the hypergraph with vertex set $[m]$ and edge set $E_\mathcal{R}$ defined as follows. A $k$-set $\{i_1, \ldots, i_k\} \in \binom{[m]}{k}$ belongs to $E_\mathcal{R}$ if the $k$-tuple $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon$-regular with respect to all colors and $d_{H_i}((V_{i_1}, \ldots, V_{i_k})) \geq \eta$ for some $i \in [r]$. Moreover, every hyperedge $\{i_1, \ldots, i_k\} \in E_\mathcal{R}$ is assigned a list $L_{i_1, \ldots, i_k} = \{i \in [r] : d_{H_i}((V_{i_1}, \ldots, V_{i_k})) \geq \eta\}$ of $\eta$-dense colors with respect to $(V_{i_1}, \ldots, V_{i_k})$.

A counting lemma (also known as the Key Lemma) in the context of the Weak Hypergraph Regularity Lemma was proved in [25]. However, it holds only for linear hypergraphs, that is, for hypergraphs $F$ for which the size of the intersection of any two hyperedges is at most one. Here we state this result in a colored form. We say that a hypergraph $F$ is a colored hypergraph if every hyperedge is assigned a color and $\mathcal{R}$ is a multicolored hypergraph if every hyperedge of $\mathcal{R}$ is assigned a nonempty list of colors.

**Definition 2.3 (Colored sub-hypergraph).** Let $\mathcal{R}$ be a multicolored hypergraph. A colored hypergraph $F$ is said to be a colored sub-hypergraph of $\mathcal{R}$ if the following holds:

(a) Ignoring colors and lists of colors, $F$ is a sub-hypergraph of $\mathcal{R}$.

(b) For every hyperedge $e \in E(F)$, the color of $e$ in $F$ belongs to the list $L_e$ of colors of $e$ in $\mathcal{R}$.

**Lemma 2.4 (Multicolored Embedding Lemma).** For all integers $r, k \geq 2$, and every $\eta > 0$ there exist a constant $\varepsilon = \varepsilon(r, k, \eta)$ and an integer $s_0 = s_0(r, k, \eta)$ such that the following holds for every positive integer $m$. Let $H = (V, E)$ be a hypergraph whose hyperedges are $r$-colored $E(H) = E_1 \cup \cdots \cup E_r$, and consider a partition $\mathcal{V} = \{V_1, \ldots, V_m\}$

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This is a partition that satisfies $||V_i| - |V_j|| \leq 1$ for all $i, j \in [t]$. 

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of $V$ that is simultaneously $\varepsilon$-regular for all sub-hypergraphs $H_i = (V, E_i)$, where $i \in [r]$. Further assume that $|V_j| \geq s_0$ for every $j \in [m]$. Fix a colored linear hypergraph $F$. If $F$ is a colored sub-hypergraph of $\mathcal{R}_H(V, \eta)$, then $H$ contains a copy of $F$.

2.2. The Fano plane. The Fano plane is the unique linear 3-uniform hypergraph on seven vertices and seven hyperedges. As mentioned in the introduction, Füredi and Simonovits [16] and, independently, Keevash and Sudakov [27] have proved that the unique extremal hypergraph for the Fano plane is $B_n$, for $n$ sufficiently large. Bellmann and Reiher [6] proved this for all $n \geq 8$, which is best possible. Note that, considering the parity of $n$, we have

$$\frac{n^3}{8} - \frac{n^2}{4} - \frac{n}{8} + \frac{1}{4} \leq |E(B_n)| = \text{ex}(n, \text{Fano}) \leq \frac{n^3}{8} - \frac{n^2}{4}. \quad (3)$$

For a 3-uniform hypergraph $H = (V, E)$ and a subset $A \subseteq V$, let $E_H(A)$ denote the set and $e_H(A)$ denote the number of hyperedges that are contained in $A$ (we write $E(A)$ or $e(A)$ if the hypergraph under consideration is obvious).

To derive the extremality of $B_n$, Keevash and Sudakov [27] (and Füredi and Simonovits [16]) established a stability result as follows.

**Theorem 2.5 (Stability).** For every $\delta > 0$ there exist $\varepsilon = \varepsilon(\delta) > 0$ and $n_0$ such that every $n$-vertex Fano-free hypergraph $H = (V, E)$ with $n > n_0$ containing at least $\text{ex}(n, \text{Fano}) - \varepsilon n^3$ hyperedges admits a partition $V(H) = A \cup B$ with $e(A) + e(B) \leq \delta n^3$.

A careful analysis of the proof of [27, Theorem 1.2] allows us to obtain the following quantitative version of Theorem 2.5, whose proof may be found in [25].

**Theorem 2.6.** For any fixed $0 < \delta \leq 1/36^8$, there exists $n_0$ such that the following holds for all $n \geq n_0$. If $H = (V, E)$ is a Fano-free 3-uniform hypergraph on $n \geq n_0$ vertices with $\text{ex}(n, \text{Fano}) - \delta n^3$ hyperedges, then there is a partition $V(H) = A \cup B$ so that $e_H(A) + e_H(B) < 2\delta^{1/64} n^3$.

Let $H$ be a hypergraph and let $X \cup Y$ be a bipartition of its vertex set. Hyperedges of $H$ that contain at least one vertex from each of the two classes are said to be *crossing*, and the set of all crossing hyperedges with respect to the given partition is denoted by $E_C(H)$. The set of all non-crossing hyperedges with respect to the given partition is denoted by $E_N(H)$.

**Lemma 2.7.** For every $\delta > 0$ and integer $n \geq \max\{8, 1/\sqrt{\delta}\}$, let $H = (V, E)$ be an $n$-vertex 3-uniform hypergraph such that there exists a partition $V = \{X, Y\}$ of $V$, for which

$$|E_C(H)| \geq |E(B_n)| - \delta n^3.$$

The following inequalities hold:

$$\frac{n}{2} - 2n\sqrt{\delta} \leq \min\{|X|, |Y|\} \leq \max\{|X|, |Y|\} \leq \frac{n}{2} + 2n\sqrt{\delta}.$$
Proof. Let \( |X| = a \) and \( |Y| = n - a \). With (3), we infer that
\[
\begin{align*}
    a \left( \frac{n-a}{2} \right) + (n-a) \left( \frac{a}{2} \right) &\geq \text{ex}(n, \text{Fano}) - \delta n^3 \\
    \Rightarrow \quad an^2 - 2an - a^2n + 2a^2 &\geq \frac{n^3}{4} - \frac{n^2}{2} - \frac{n}{4} - 2\delta n^3 \\
    \Rightarrow \quad |a - \frac{n}{2}| &\leq \sqrt{\frac{n}{4(n-2)} + \frac{2\delta n^3}{n-2}} \leq 2\sqrt{\delta n}.
\end{align*}
\]

We will use the entropy function \( h: [0, 1] \to [0, 1] \) given by
\[
h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \quad \text{for } 0 < x < 1 \quad \text{and} \quad h(0) = h(1) = 0.
\]
For \( 0 \leq \alpha \leq 1 \) the following inequality is well-known:
\[
\left( \frac{n}{\alpha n} \right) \leq 2^{h(\alpha)n}. \tag{4}
\]
We will also use the following upper bound on the entropy function for \( x \leq 1/8 \):
\[
h(x) \leq -2x \log_2 x. \tag{5}
\]
Namely, (5) is equivalent to \( g(x) = x \ln x - (1-x) \ln(1-x) \leq 0 \). Taking the derivative gives \( g'(x) = \ln x + 2 + \ln(1-x) \leq 0 \) for \( x \leq 1/8 \). With \( g(1/8) < 0 \) inequality (5) follows.

3. A stability result for \( F^R \)-free \( r \)-colorings

To prove Theorem 1.1, we first establish a stability result for colorings, which implies that, for \( n \) sufficiently large, any \( n \)-vertex hypergraph with a large number of \( F^R \)-free \( r \)-colorings must be structurally similar to the hypergraph \( B_n \).

Lemma 3.1. For any fixed \( 0 < \delta \leq 1/36^8 \), there exists \( r_0 = r_0(\delta) = \frac{6^{49764}}{\delta^{63}} \) such that the following holds for all \( r \geq r_0 \). There is \( n_1 \) such that, if \( n \geq n_1 \) and \( H = (V, E) \) is a 3-uniform \( n \)-vertex hypergraph satisfying \( c_{r,F^R}(H) \geq r^{\text{ex}(n, \text{Fano})} \), then there is a partition \( \mathcal{V} = \{V_1, V_2\} \) of \( V \) with \( e(V_1) + e(V_2) \leq \delta n^3 \).

Proof. Let \( 0 < \delta \leq 1/36^8 \) be given. Let \( X = 1/432^{64} < 1/36^8 \). Define
\[
r_0 = \max \left\{ 6^{\frac{1}{12}X^2} + 2, 6^{\frac{49764}{\delta^{63}}} \right\}. \tag{6}
\]
In fact, it is easy to see that the second of the above terms is larger, but we write it in this way to avoid distracting calculations later in the proof. Let \( r \geq r_0 \). Fix a positive number \( \eta \) with
\[
\eta < \frac{\delta}{4r} \tag{7}
\]
that satisfies
\[
\frac{\delta}{2} \leq 243(4h(\eta r) + 4r\eta) < \frac{3\delta}{4}. \tag{8}
\]
For this value of $\eta$, Lemma 2.4 gives us constants $\varepsilon = \varepsilon(r, 3, \eta)$ and $s_0 = s_0(r, 3, \eta)$ which we further assume to satisfy

$$\varepsilon < \min\{3\eta/2, r\eta/4\}. \quad (9)$$

Fix $m_0$ with

$$m_0 \geq \frac{1}{\varepsilon}. \quad (10)$$

Moreover, we shall also consider that $m_0$ is sufficiently large to ensure that some of the upcoming equations are satisfied (all such equations will be marked by $m_0 \gg 1$).

Let $N_0 = N_0(r, 3, m_0, \varepsilon)$ and $M_0 = M_0(r, 3, m_0, \varepsilon)$ be given by Theorem 2.1 and fix $n_1 \geq \max\{N_0, s_0M_0\}$, such that the equations marked by $n \gg 1$ are satisfied for $n \geq n_1$.

For $F$ being the Fano plane, let $H$ be an $n$-vertex 3-uniform hypergraph satisfying $c_{r,F^R}(H) \geq r^{ex(n,Fano)}$ and fix one of its $F^R$-free $r$-colorings. By Theorem 2.1 there exists an equitable partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ of $V$, where $m_0 \leq m \leq M_0$, that is $\varepsilon$-regular simultaneously with respect to each subhypergraph $H_i = (V_i, E_i)$, for all $i \in [r]$. Let $\mathcal{R} = \mathcal{R}_H(\mathcal{V}, \eta)$ be the multicolored cluster hypergraph with vertex set $[m]$ associated with this partition. For a hyperedge $e$ in $\mathcal{R}$, let $L_e$ be its list of colors, as in Definition 2.2. Given a subset $\{i_1, i_2, i_3\}$ of $[m]$, we say that a hyperedge $e$ in $\mathcal{R}$ lies in a triple $(V_{i_1}, V_{i_2}, V_{i_3})$ if $|e \cap V_{i_j}| = 1$ for all $j \in [3]$.

Our aim is to find an upper bound on the number of $F^R$-free $r$-colorings of $H$. To do this, we sum over all possible $\varepsilon$-regular partitions $\mathcal{V} = \{V_1, \ldots, V_m\}$ and all multicolored cluster hypergraphs $\mathcal{R}$, and we find an upper bound on the number of $r$-colorings (of sub-hypergraphs of $H$) for which $\mathcal{V}$ is an $\varepsilon$-regular partition (with respect to all colors) associated with the multicolored cluster hypergraph $\mathcal{R}$.

Fix $\mathcal{V}$ and $\mathcal{R}$. In the following, we assume that $m$ divides $n$ to avoid technicalities, but the same conclusion holds for other values of $m$ and $n$. There are at most $r \varepsilon \binom{m}{3}$ subsets $\{i_1, i_2, i_3\}$ of $[m]$ for which the triple $(V_{i_1}, V_{i_2}, V_{i_3})$ is not $\varepsilon$-regular with respect to the partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ for at least one of the colors. At most

$$r \varepsilon \binom{m}{3} \left( \frac{n}{m} \right)^3 \leq r \varepsilon \frac{n^3}{6} \quad (11)$$

hyperedges lie in a triple of this type. Because (11) holds, there are at most

$$m \left( \frac{n}{m} \right)^2 n \leq \varepsilon n^3 \quad (12)$$

hyperedges with at least two elements in a same class with respect to $\mathcal{V}$. Next, consider hyperedges $e$ that lie in an $\varepsilon$-regular triple $(V_{i_1}, V_{i_2}, V_{i_3})$ for which the set of hyperedges with the same color as $e$ has density less than $\eta$ with respect to $(V_{i_1}, V_{i_2}, V_{i_3})$. The number of such hyperedges is bounded above by

$$r \eta \binom{m}{3} \left( \frac{n}{m} \right)^3 \leq r \eta \frac{n^3}{6} \quad (13)$$

Using (11), (12), (13) and (9), we conclude that there are at most

$$\left( \frac{r \varepsilon}{6} + \varepsilon + \frac{r \eta}{6} \right) n^3 \leq r \eta n^3$$
hyperedges in $H$ that do not lie in an $\varepsilon$-regular triple $(V_i, V_{i_1}, V_{i_2})$ for which their color is dense. There are at most $\binom{n^3}{\varepsilon \eta m^3}$ ways to fix these hyperedges in $H$, and they can be colored in at most $r^{\varepsilon \eta m^3}$ ways. The remaining hyperedges of $H$ can be colored in at most
\[
\left( \prod_{e \in E(\mathcal{R})} |L_e| \right)^3
\]
ways. Thus, the total number of $r$-colorings of $H$ that give rise to the partition $\mathcal{V} = \{V_1, \ldots, V_m\}$ and the multicolored cluster hypergraph $\mathcal{R}$ is bounded above by
\[
\binom{n^3}{\varepsilon \eta m^3} \cdot \left( \prod_{e \in E(\mathcal{R})} |L_e| \right)^3.
\]

Let $e_j(\mathcal{R}) = |\{e \in \mathcal{R} : |L_e| = j\}|$, $j \in [r]$. We write a sum over a set $\mathcal{P}$ of pairs to denote a sum over all equitable partitions $\mathcal{V}$ and all possible multicolored cluster hypergraphs $\mathcal{R}$ associated with $\mathcal{V}$. We have
\[
\begin{align*}
c_{r,F_R}(H) & \leq \sum_{(\mathcal{V},\mathcal{R}) \in \mathcal{P}} \binom{n^3}{\varepsilon \eta m^3} \left( \prod_{e \in E(\mathcal{R})} |L_e| \right)^3 \\
& \leq \sum_{(\mathcal{V},\mathcal{R}) \in \mathcal{P}} 2^{h(\varepsilon \eta)n^3} r^{\varepsilon \eta m^3} \left( \prod_{j=1}^{r} \sum_{j=7}^{6} e_j(\mathcal{R}) \right)^3 \\
& \leq \sum_{(\mathcal{V},\mathcal{R}) \in \mathcal{P}} 2^{h(\varepsilon \eta)n^3} r^{\varepsilon \eta m^3} \left( 6 \sum_{j=1}^{r} e_j(\mathcal{R}) \right)^3 \left( \prod_{j=7}^{r} e_j(\mathcal{R}) \right)^3. \tag{14}
\end{align*}
\]
Assume that there is a copy of the Fano plane $F$ in $\mathcal{R}$ so that every hyperedge $e$ in this copy satisfies $|L_e| \geq 7$. Then we may greedily assign a color from each list to produce a rainbow coloring of $F$, which would lead to a rainbow copy of $F$ in the original coloring because of Lemma 2.4. As there is no such copy, we must have
\[
\sum_{j=7}^{r} e_j(\mathcal{R}) \leq \text{ex}(m, \text{Fano}).
\]

For a multicolored cluster hypergraph $\mathcal{R} = \mathcal{R}(\eta)$, let
\[
\beta(\mathcal{R}) = \frac{1}{m^3} \left( \text{ex}(m, \text{Fano}) - \sum_{j=7}^{r} e_j(\mathcal{R}) \right).
\]

We consider two cases.

(a) For every multicolored cluster hypergraph $\mathcal{R}$, we have $\beta(\mathcal{R}) \geq 4h(\varepsilon \eta) + 4\varepsilon \eta$.
(b) There exists a multicolored cluster hypergraph $\mathcal{R}$ for which $\beta(\mathcal{R}) < 4h(\varepsilon \eta) + 4\varepsilon \eta$.

Case (a) Our aim is to show that this case cannot apply by proving that the number of $F_R$-free $r$-colorings of $H$ is less than $r^{\text{ex}(n, \text{Fano})}$. We first consider all those terms
\((\mathcal{V}, \mathcal{R}) \in \mathcal{P}\) in the sum (14) where the multicolored cluster hypergraph \(\mathcal{R}\) satisfies \(\beta(\mathcal{R}) \geq X\). We denote the set of all these terms by \(\mathcal{P}^*\). This leads to the upper bound

\[
\sum_{(\mathcal{V}, \mathcal{R}) \in \mathcal{P}^*} 2^{h(n\eta)n^3 \cdot \rho \eta m^3} \left( \frac{6 \beta(\mathcal{R})}{\beta(\mathcal{R})} \cdot \left( \frac{6 \beta(\mathcal{R})}{\beta(\mathcal{R})} \right)^n \right) \frac{m^3}{\rho \eta m^3} \leq \left( \frac{6 \beta(\mathcal{R})}{\beta(\mathcal{R})} \right)^n \leq 1.
\]

However, as \(r \geq r_0 \geq 6^{1/(12X) + 2}\) by (6), we have

\[
\left( \frac{6 \frac{1}{2r} + \beta(\mathcal{R})}{r \frac{1}{2r}} \right)^n \leq \left( \frac{6 \frac{1}{2r} + \beta(\mathcal{R})}{r \frac{1}{2r} + 2} \right)^n = \left( \frac{6 \frac{1}{2r} \cdot \beta(\mathcal{R})}{6 \frac{1}{2r} + \beta(\mathcal{R})} \right)^n \leq 1.
\]

Then, (15) is at most

\[
\sum_{(\mathcal{V}, \mathcal{R}) \in \mathcal{P}^*} 2^{h(n\eta)n^3 \cdot \rho \eta m^3} \cdot \frac{m^3}{\rho \eta \cdot m^3} \leq \frac{m^3}{\rho \eta \cdot m^3}.
\]

Since the number of classes is \(m \leq M_0\), we have at most \(M_0^m\) partitions \(\mathcal{V} = \{V_1, \ldots, V_m\}\) and at most \(2^{rM_0^m}\) multicolored cluster hypergraphs. Moreover, as we are in case (a) where \(\beta(\mathcal{R}) \geq 4h(n\eta) + 4\eta\), the expression in (16) is at most

\[
M_0^m \cdot 2^{rM_0^m} \cdot 2^{h(n\eta)n^3 \cdot \rho \eta m^3} \cdot \frac{m^3}{\rho \eta \cdot m^3} \leq 1 \leq 2^{\beta(\mathcal{R})m^3}.
\]

Now we consider those terms \((\mathcal{V}, \mathcal{R})\) in the sum (14) for which \(\mathcal{R}\) satisfies \(4h(n\eta) + 4\eta \leq \beta(\mathcal{R}) \leq X\). Fix one such pair \((\mathcal{V}, \mathcal{R})\) and let \(\mathcal{R}'\) be the hypergraph obtained from \(\mathcal{R}\) by deleting all hyperedges \(e\) satisfying \(|L_e| \leq 6\). Since \(\text{ex}(m, \text{Fano}) - \sum_{j=7} e_j(\mathcal{R}) = 6\cdot \beta(\mathcal{R})\cdot m^3 - 2\cdot \beta(\mathcal{R})\cdot m^3 \geq 0\), and \(\beta(\mathcal{R}) \leq X = 1/432^{64} < 1/36^8\), and since there is no copy of \(F^R\) in \(\mathcal{R}'\). Theorem 2.16 produces a partition \(U_1 \cup U_2\) of \(V(\mathcal{R}') = [m]\) with respect to which

\[
\text{ex}(\mathcal{R}', U_1) + \text{ex}(\mathcal{R}', U_2) \leq 2\beta(\mathcal{R}) \cdot m^3.
\]

Let \(\mathcal{R}''\) be the subhypergraph of \(\mathcal{R}'\) obtained by removing all hyperedges in \(E_{\mathcal{R}'}(U_1) \cup E_{\mathcal{R}'}(U_2)\), thus

\[
\text{ex}(\mathcal{R}'') \geq \text{ex}(m, \text{Fano}) - \beta(\mathcal{R})m^3 - 2\beta(\mathcal{R}) \cdot m^3.
\]

Let \(K = K(U_1, U_2)\) be the complete bipartite 3-uniform hypergraph with partition \([m] = U_1 \cup U_2\).

For every hyperedge \(f \in E_{\mathcal{R}}(U_1) \cup E_{\mathcal{R}}(U_2)\), let \(C(f)\) be the set of all copies of a Fano plane \(F\) in \(K + f\). Let \(C = \min\{|C(f)| : f \in E_{\mathcal{R}}(U_1) \cup E_{\mathcal{R}}(U_2)\}\).

On the one hand we have

\[
C \cdot |E_{\mathcal{R}}(U_1) \cup E_{\mathcal{R}}(U_2)| \leq \sum_{f \in E_{\mathcal{R}}(U_1) \cup E_{\mathcal{R}}(U_2)} \sum_{F \in C(f)} 1.
\]
On the other hand, consider $\overline{E} = E(K) \setminus E(\mathcal{R}''')$ which satisfies
\[
|\overline{E}| \leq |E(B_m)| - |E(\mathcal{R}''')| \leq \text{ex}(m, \text{Fano}) - \left( \text{ex}(m, \text{Fano}) - \beta(\mathcal{R})m^3 - 2\beta(\mathcal{R})\overline{\beta}m^3 \right) = \beta(\mathcal{R})m^3 + 2\beta(\mathcal{R})\overline{\beta}m^3 < 3\beta(\mathcal{R})\overline{\beta}m^3. \tag{21}
\]

We claim that for every hyperedge $f \in E_R(U_1) \cup E_R(U_2)$ and $F \in \mathcal{C}(f)$, the Fano plane $F$ contains a hyperedge in $\overline{E}$. Assuming this, and using the fact that every hyperedge in $\overline{E}$ belongs to at most $m^4$ copies of $F$ in the complete $k$-uniform hypergraph with vertex set $[m]$, we obtain
\[
\sum_{f \in E_R(U_1) \cup E_R(U_2)} \sum_{F \in \mathcal{C}(f)} 1 \leq \sum_{g \in \overline{E}} \sum_{f \in E_R(U_1) \cup E_R(U_2)} 1 \leq m^4|\overline{E}| < 3\beta(\mathcal{R})\overline{\beta}m^7. \tag{22}
\]

In this equation, $F \in \bigcup \mathcal{C}(f)$ denotes $F \in \bigcup_{f \in E_R(U_1) \cup E_R(U_2)} \mathcal{C}(f)$. Combining (22) with (20), we derive
\[
e_R(U_1) + e_R(U_2) \leq \frac{3\beta(\mathcal{R})\overline{\beta}m^7}{C}. \tag{23}
\]

Before finding a lower bound on $C$, we show that, for every $f \in E_R(U_1) \cup E_R(U_2)$ and $F \in \mathcal{C}(f)$, the Fano plane $F$ indeed contains a hyperedge in $\overline{E}$. Given a hyperedge $f \in E_R(U_1) \cup E_R(U_2)$ and $F \in \mathcal{C}(f)$, we know that $E(F) \setminus \{f\} \subseteq E(K)$. Moreover, because $f \in E(\mathcal{R})$, we know that its list has at least one color. From this we derive that $E(F) \setminus \{f\} \nsubseteq E(\mathcal{R}'')$, as otherwise the hyperedges of $E(F) \setminus \{f\}$ would have at least seven colors in their list of colors, which would lead to a rainbow copy of $F$ in $\mathcal{R}$ where every hyperedge is assigned a color of its own list. Lemma 24 produces the desired contradiction.

**Claim 3.2.** Given a hyperedge $f \in E_R(U_1) \cup E_R(U_2)$, we have $|\mathcal{C}(f)| \geq (m/3)^4$.

**Proof.** Given a hyperedge $f \in E_R(U_1) \cup E_R(U_2)$, say $f \in E_R(U_1)$ with vertices $v_1, v_2, v_3$, we may create copies of a Fano plane $F$ in $K + f$ as follows. Fix four vertices $v_4, \ldots, v_7$ in $U_2$ or one vertex $v_4$ in $U_1$ and three vertices $v_5, v_6, v_7$ in $U_2$, and choose six hyperedges $e_1, \ldots, e_6$ on the seven vertices to form a Fano plane. The number of ways to do this is exactly
\[
2 \left( \frac{4}{2} \right) \left( \frac{|U_2|}{4} \right) + 2 \left( \frac{4}{2} \right) (|U_1| - 3) \left( \frac{|U_2|}{3} \right) \geq u^4,
\]
where $u = \min\{|U_i|: i \in [2]|$.

By Lemma 27 with (21) applied to $\mathcal{R}'''$, we must have
\[
u \geq \frac{m}{2} - 2\sqrt{3\beta(\mathcal{R})\overline{\beta}m}. \tag{24}
\]

Then,
\[
|\mathcal{C}(f)| \geq \left( \frac{m}{2} - 2\sqrt{3\beta(\mathcal{R})\overline{\beta}m} \right)^4 \geq \left( \frac{m}{3} \right)^4.
\]

$\square$
By Claim 3.2 and (23), we have
\[ e_R(U_1) + e_R(U_2) \leq 243\beta(R)\frac{1}{\alpha} m^3. \]  
(25)

Thus,
\[ \sum_{j=1}^{6} e_j(R) \leq 243\beta(R)\frac{1}{\alpha} m^3 + 3\beta(R)\frac{2}{\alpha} m^3 \leq 246\beta(R)\frac{1}{\alpha} m^3. \]  
(26)

Consider all those terms \((V, R)\) in the sum (14) where the multicolored cluster hypergraph \(R\) satisfies \(4h(\eta) + 4r\eta \leq \beta(R) \leq X \leq 1/24^{64}\). Denoting the set of all these terms by \(P^{**}\), we have
\[ \sum_{(V, R) \in P^{**}} 2^{h(\eta)n^3} r^{\eta m^3} \left( 6\sum_{j=1}^{6} e_j(R)_{ex(m,Fano) - \beta(R)m} \right) \left( \frac{\mu}{m} \right)^3 \]
\[ \leq M_{64}^0 \cdot 2^{n^3} \cdot 2^{h(\eta)n^3} \cdot r^{\eta m^3} \left( 6^{246\beta(R)\frac{1}{\alpha} m^3} r_{ex(m,Fano) - \beta(R)m} \right) \left( \frac{\mu}{m} \right)^3 \]
\[ \leq \frac{1}{2} r^{h(\eta)n^3 + \eta m^3} r_{ex(n,Fano) - \frac{\beta(R)}{2} m} \]
(27)
provided that
\[ 6^{246\beta(R)\frac{1}{\alpha} m^3} < r^{\frac{\beta(R)}{2}}. \]  
(28)

This holds if \(6^{492} < r^{\beta(R)\frac{63}{64}}\). Since \(\beta(R) \geq 4h(\eta) + 4r\eta\), and by the lower bound in (3), namely
\[ \left( \frac{\delta}{492} \right)^{64} \leq 4h(\eta) + 4r\eta \leq \beta(R), \]
inequality (28) holds if
\[ 6^{492} < r^{\left( \frac{\delta}{492} \right)^{64}} \Rightarrow 6^{\frac{492^{64}}{492}} < r, \]
which is precisely our assumption.

For \(\beta(R) \geq 4h(\eta) + 4r\eta\) inequality (27) yields
\[ r^{-\eta(h(\eta)+\eta) n^3} r_{ex(n,Fano)} \leq \frac{1}{2} r_{ex(n,Fano)}. \]  
(29)

Combining (17) and (29), we have less than \(r_{ex(n,F)}\) distinct \(FR\)-free \(r\)-colorings when case (a) applies, so that it cannot happen.

Case (b) In this case, there is a partition \(V\) of the vertex set of \(H = (V, E)\) with \(m\) classes that is associated with a multicolored cluster hypergraph \(R\) for which \(\beta(R) < 4h(\eta) + 4r\eta\). Again, consider \(R'\) obtained from \(R\) by removing all hyperedges with less than six colors in their list of colors. As in case (a), Theorem 2.6 tells us that \(R'\) admits a partition \(U = \{U_1, U_2\}\) satisfying
\[ e_{R'}(U_1) + e_{R'}(U_2) \leq 2\beta(R)\frac{1}{\alpha} m^3. \]

Consider the partition \(W = \{W_1, W_2\}\) of the vertex set of \(H\) where, for all \(i \in [2]\), \(W_i = \bigcup_{j \in U_i} V_j\). We want to find an upper bound on the cardinality of the set \(E_H(W_1) \cup E_H(W_2)\) with respect to \(W\). Such hyperedges are either one of the hyperedges counted by equations (11), (12) and (13) or are associated with a non-crossing hyperedge of \(R\) with respect to \(U\). Clearly, each non-crossing hyperedge of \(R\) with respect to \(U\)
generates at most \((n/m)^3\) non-crossing hyperedges of \(H\) with respect to \(W\). Using the upper bound on \(e_R(U_1) + e_R(U_2)\) given in (25), we conclude that
\[
e_H(W_1) + e_H(W_2) \leq r\eta n^3 + \left(\frac{n}{m}\right)^3 (e_R(U_1) + e_R(U_2)) \\
\leq \left( r\eta + 243\beta(\mathcal{R})\frac{1}{3}\right) n^3. \tag{30}
\]
For \(\beta(\mathcal{R}) < 4h(r\eta) + 4r\eta\) expression (30) is at most
\[
\left( r\eta + 243(4h(r\eta) + 4r\eta)\frac{1}{3}\right) n^3 \leq \left( \frac{\delta}{4} + \frac{3\delta}{4} \right) n^3 = \delta n^3
\]
as desired. \(\square\)

4. Proof of Theorem 1.1

In this section, we use the stability result of the previous section to prove that Theorem 1.1 holds.

Proof of Theorem 1.1. Let \(\gamma, \xi = \xi(\gamma), \delta = \delta(\xi), r_0 = r_0(\delta)\) and \(r \geq r_0\) be positive numbers satisfying
\[
\gamma \leq \frac{1}{1406} \tag{31}
\]
\[
\xi \leq \frac{\gamma^3}{16} \tag{32}
\]
\[
\delta < \min\left\{ \frac{1}{400^2 \cdot 4 \cdot 9^2 \cdot 36}, \frac{\gamma^2}{4}, \frac{\xi^3}{36} \right\} \tag{33}
\]
\[
r > \max\{r_0, 21^{64}\}, \tag{34}
\]
hence we may have equality in (31) and (32) and fix
\[
\delta = \frac{1}{37 \cdot 16^3 \cdot 1406^5},
\]
and thus
\[
r_0 = r_0(\delta) = 6^{492^{64} \cdot (37 \cdot 16^3 \cdot 1406^5)^{64}}
\]
comes from Lemma 3.1.

Let \(n_1 = n_1(r, \delta)\) defined in Lemma 3.1 for fixed \(r\) and set \(n_0 \geq n_1 + 3\binom{n_1}{3}\). Consider \(n \geq n_0\) large enough to ensure that (39), (11), (13), (44), (16) and (17) hold. Assume that we are given a hypergraph \(H\) on \(n \geq n_0\) vertices and with at least \(r^{\text{ex}(n,\text{Fano})+m}\) distinct \(F^R\) colorings, for some \(m \geq 0\).

Claim 4.1. If \(H\) is a hypergraph with at least \(r^{\text{ex}(n,\text{Fano})+m}\) distinct \(F^R\)-free \(r\)-colorings, for some \(m \geq 0\), and \(H \neq B_n\), then there exists an induced sub-hypergraph \(H'\) on \(n' \geq n - 3\) vertices and at least \(r^{\text{ex}(n',\text{Fano})+m+1}\) distinct \(F^R\)-free \(r\)-colorings.
If Claim 4.1 is true, we inductively arrive at some sub-hypergraph $H_1$ with $n' \geq n_1$ vertices that allows more than $r^{n'}$ feasible colorings, which is impossible and yields the desired contradiction.

To prove Claim 4.1 let $H = (V, E) \not= B_n$ be a 3-uniform hypergraph on $n$ vertices and with at least $r^{\text{ex}(n, \text{Fano})+m}$ feasible colorings, with $m \geq 0$. As $B_n$ can be colored arbitrarily without producing a rainbow Fano plane, we have that $|E(H)| \geq |E(B_n)|$. Let $\delta_1(H)$ be the minimum degree of $H$. If $\delta_1(H) < \delta_1(B_n)$, let $v$ be a vertex of minimum degree in $H$ and consider the sub-hypergraph $H' = H - v$. Since

$$|E(B_{n-1})| = |E(B_n)| - \delta_1(B_n) \leq |E(B_n)| - \delta_1(H) - 1,$$

we conclude that the number of $(F, P)$-free $r$-colorings of $H'$ is at least

$$\frac{r^{|E(B_n)|+m}}{r^{\delta_1(H)}} = r^{|E(B_n)|+m-\delta_1(H)} \geq r^{|E(B_{n-1})|+m+1},$$

as desired. So let us assume that $\delta_1(H) \geq \delta_1(B_n) \geq 3n^2/8 - n$.

Consider a partition of $\mathcal{V} = \{X, Y\}$ of $\mathcal{V}$ that minimizes $e_H(X) + e_H(Y)$. Let $E_C(H)$ and $E_N(H)$ denote the number of crossing hyperedges and non-crossing hyperedges in $H$ with respect to $\mathcal{V}$, respectively. By Lemma 4.1, we have $e_H(X) + e_H(Y) \leq \delta n^3$, and hence

$$|E(H)| \leq |E(B_n)| + \delta n^3. \quad (35)$$

It follows from $|E(H)| \geq |E(B_n)|$ that

$$|E_C(H)| \geq |E(B_n)| - \delta n^3,$$

thus by Lemma 2.7 we have

$$n/2 - 2\sqrt{\delta n} \leq \min\{|X|, |Y|\} \leq \max\{|X|, |Y|\} \leq n/2 + 2\sqrt{\delta n} \quad (36)$$

For a vertex $v$ of $H$, define its link graph $\mathcal{L}(v)$ with vertex set $V(H) - v$ and edge set $L(v) = \{\{u, v\} : \{v, u, w\} \in E(H)\}$. Every coloring of the hyperedges incident with $v$ naturally induces a coloring of $L(v)$, so that when we have a coloring of the hyperedges incident with $v$, we may view it as an edge-coloring of $\mathcal{L}(v)$. Given an $r$-coloring $\Delta$ of $H$, we say that a color $\alpha$ is abundant with respect to a vertex $v$ and a class $Z \in \{X, Y\}$ when the set of hyperedges $e$ of color $\alpha$ such that $v \in e$ and $e - v \subseteq Z$ generate a matching of size at least $\xi n$ in $\mathcal{L}(v)$, otherwise the color is called rare.

Our argument is divided into two cases. First we shall assume that there exists a vertex $v$ with at least $\gamma n^2$ link edges in its "own" partition class.

**Case 1.** $H$ has the property that there is a vertex $v$, without loss of generality $v \in Y$, such that $|L(v) \cap \binom{Y}{2}| \geq \gamma n^2$.

The minimality of the number $|E_N(H)|$ of non-crossing hyperedges in $H$ implies that $|L(v) \cap \binom{X}{2}| \geq \gamma n^2$, as otherwise we could move $v$ from $Y$ to $X$ to achieve a smaller number of non-crossing hyperedges.

Let $\Delta$ be an $r$-coloring of $H$. For class $Z \in \{X, Y\}$, let $A_Z = A_Z(\Delta)$ be the set of abundant colors with respect to vertex $v$ and $Z$, and let $J_Z = J_Z(\Delta)$ be the set of edges in $L(v) \cap \binom{Z}{2}$ whose color is not in $A_X \cup A_Y$.

We split the set $C$ of feasible colorings of $H$ into two disjoint classes $C_1$ and $C_2 = C \setminus C_1$, where $C_1$ is the set of colorings for which either
(a) $|A_X \cup A_Y| \not\in \{1, 2\}$

or

(b) $|J_Z| \geq 4\sqrt[3]{\xi n^2}$, for some class $Z \in \{X, Y\}$.

Claim 4.2. In every coloring $\Delta \in \mathcal{C}_1$, there exists a matching $M$ in $\mathcal{L}(v)$ with the following property. There is a set $T(v)$ of 3-colored triples $(f_1, f_2, f_3) \in M^3$ such that:

(a) $f_1 \in L(v) \cap \left(\binom{X}{2}\right), f_2 \in L(v) \cap \left(\binom{Y}{2}\right), f_3 \in L(v) \cap \left(\binom{Z}{2}\right)$, with $Z \in \{X, Y\}$.

(b) distinct triples $(f_1, f_2, f_3), (f_1', f_2', f_3') \in T(v)$ satisfy $\{f_1, f_2, f_3\} \neq \{f_1', f_2', f_3\}$.

(c) $|T(v)| \geq \xi^3 n^3/9$.

Proof. Consider a coloring $\Delta \in \mathcal{C}_1$. We split the argument into three cases depending on the cardinalities of $A_X(\Delta)$ and $A_Y(\Delta)$.

Case 1) $|A_X \cup A_Y| \geq 3$. Without loss of generality, assume that $|A_X| \geq |A_Y|$, which implies that $|A_X| \geq 2$. We consider two cases.

First assume that $A_Y \neq \emptyset$, so that there are three distinct colors $\alpha_1, \alpha_2 \in A_X$ and $\alpha_3 \in A_Y$. This implies that there exist vertex disjoint matchings $M_1, M_2, M_3$ in $\mathcal{L}(v)$, for which every edge in $M_i$ has color $\alpha_i$, $i = 1, 2, 3$ and $|M_1| \geq \xi n/3$, $|M_2| \geq \xi n/3$, $|M_3| \geq \xi n$. To see why this is true, note that, if we let $M_1$ be an arbitrary matching of size $\xi n/3$ whose elements have color $\alpha_1$, each edge in $M_1$ is incident with at most two edges in a maximum matching of color $\alpha_2$, so that at least $\xi n - 2 \cdot \xi n/3 = \xi n/3$ edges in this matching are not incident with edges in $M_1$, which allows us to construct $M_2$. Set $M = M_1 \cup M_2 \cup M_3$. It is a matching with the property that all triples in $M_1 \times M_2 \times M_3$ are 3-colored. The number of triples is at least

$$\left(\frac{\xi n}{3}\right)^2 \xi n = \frac{\xi^3 n^3}{9}.$$  

If $A_Y = \emptyset$, then consider colors $\alpha_1, \alpha_2 \in A_X$ with vertex disjoint matchings $M_1, M_2$ having sizes $|M_1| \geq \xi n/3$, $|M_2| \geq \xi n/3$. As $A_Y = \emptyset$, the size of a maximum monochromatic matching of color $\alpha_1$ or $\alpha_2$ in $L(v) \cap \left(\binom{Y}{2}\right)$ is at most $\xi n$. Note that the number of edges of color $\alpha_1$ in $L(v) \cap \left(\binom{Y}{2}\right)$ is at most $2|Y|n$, as every edge in $L(v) \cap \left(\binom{Y}{2}\right)$ shares a vertex with at most $2(|Y| - 2) \leq 2|Y|$ other edges in this set. This implies that the number of edges in $L(v) \cap \left(\binom{Y}{2}\right)$ with a rare color not in $\{\alpha_1, \alpha_2\}$ is at least

$$\left|L(v) \cap \left(\binom{Y}{2}\right)\right| - 4\xi n|Y| \geq \gamma n^2 - 4\xi n^2 \geq \frac{\gamma n^2}{2}.$$  

Then we may greedily find a matching $M_3$ in $L(v) \cap \left(\binom{Y}{2}\right)$ whose edges have colors in $[\gamma] \setminus \{\alpha_1, \alpha_2\}$ of size at least

$$\frac{\gamma n^2/2}{2|Y|} \geq \frac{\gamma n}{4}.$$  

Set $M = M_1 \cup M_2 \cup M_3$. It is a matching with the property that all triples in $M_1 \times M_2 \times M_3$ are 3-colored. The number of triples is at least

$$\left(\frac{\xi n}{3}\right)^2 \cdot \frac{\gamma n}{4} \geq \frac{\xi^3 n^3}{9}.$$  

Case 2) $|A_X \cup A_Y| = 0$. In this case, for every color $\alpha \in [\gamma]$, the maximum size of a matching in $X$ or $Y$ of color $\alpha$ is at most $\xi n$. With a greedy construction we obtain
that both \( L(v) \cap \binom{X}{2} \) and \( L(v) \cap \binom{Y}{2} \) contain matchings of size at least

\[
\min \left\{ \frac{|L(v) \cap \binom{X}{2}|}{2|X|}, \frac{|L(v) \cap \binom{Y}{2}|}{2|Y|} \right\} \geq \frac{\gamma n}{2}.
\]

Fix matchings \( M_X \) in \( X \) and \( M_Y \) in \( Y \) for which

\[
\frac{\gamma n}{2} \leq \min\{|M_X|, |M_Y|\} \leq \max\{|M_X|, |M_Y|\} \leq \frac{n}{2}.
\]

Clearly, \( M = M_X \cup M_Y \) is a matching in \( L(v) \), which implies \(|M_X| + |M_Y| \leq n/2\). For every color \( \alpha \in [r] \), let \( c_\alpha \) be the number of edges assuming color \( \alpha \) in \( M_X \) and \( d_\alpha \) be the number of edges assuming color \( \alpha \) in \( M_Y \). Note that \( c_\alpha, d_\alpha \leq \xi n \) and \( \sum_{\alpha=1}^{r} c_\alpha \leq n/2 \), \( \sum_{\alpha=1}^{r} d_\alpha \leq n/2 \) and \( \sum_{\alpha=1}^{r} (c_\alpha + d_\alpha) \leq n/2 \).

The number of choices of triples \((f_1, f_2, f_3)\) with at most two colors where \( f_1 \in M_X \), \( f_2 \in M_Y \) and \( f_3 \in M_X \cup M_Y \), is at most

\[
\sum_{\alpha=1}^{r} c_\alpha d_\alpha (|M_X| + |M_Y|) + \sum_{\alpha=1}^{r} c_\alpha |M_Y| (c_\alpha + d_\alpha) + \sum_{\alpha=1}^{r} |M_X| d_\alpha (c_\alpha + d_\alpha)
\]

\[
\leq \frac{n}{2} \sum_{\alpha=1}^{r} c_\alpha d_\alpha + \frac{n}{2} \sum_{\alpha=1}^{r} c_\alpha (c_\alpha + d_\alpha) + \frac{n}{2} \sum_{\alpha=1}^{r} d_\alpha (c_\alpha + d_\alpha)
\]

\[
(\xi n^2) \sum_{\alpha=1}^{r} c_\alpha + \frac{\xi n^2}{2} \sum_{\alpha=1}^{r} (c_\alpha + d_\alpha) + \frac{\xi n^2}{2} \sum_{\alpha=1}^{r} (c_\alpha + d_\alpha)
\]

\[
\leq \frac{\xi n^3}{4} + \frac{\xi n^3}{4} + \frac{\xi n^3}{4} < \xi n^3.
\]

As a consequence, the number of choices of 3-colored triples \((f_1, f_2, f_3)\) is at least

\[
\left(\frac{\gamma n}{2}\right)^3 - \xi n^3 = \frac{\gamma^3 n^3}{8} - \xi n^3 \geq \frac{\gamma^3 n^3}{16} \geq \frac{2\xi^3 n^3}{3}.
\]

Each set \( \{f_1, f_2, f_3\} \) may appear in at most \( 3! = 6 \) triples, so that we may define a set
\( \mathcal{T}(v) \) as in the statement of the claim with size at least \( \xi^3 n^3 / 9 \), as required.

Case 3) \(|A_X \cup A_Y| \in \{1, 2\}\), and for some \( Z \in \{X, Y\} \), \(|J_Z| \geq 4\sqrt{\xi} n^2 \). Without loss of generality we assume \( Z = Y \), so that \(|J_Y| \geq 4\sqrt{\xi} n^2 \). First we look into the case where \( A_X \neq \emptyset \). Then there exists a matching \( M' \) in \( J_Y \) of size at least

\[
\frac{4\sqrt{\xi} n^2}{2|Y|} \geq \frac{3\xi n^2}{2} \geq 2\sqrt{\xi} n + 1,
\]

and hence the number of distinct pairs \( \{f_2, f_3\} \) in \( M' \) is at least

\[
\left(\frac{2\sqrt{\xi} n + 1}{2}\right) \geq \frac{(2\sqrt{\xi} n)^2}{2} = 2\xi n^2.
\]

For every color \( \alpha \in [r] \), let \( d_\alpha \) be the number of edges assuming color \( \alpha \) in \( M' \). Recall that all hyperedges in \( J_Y \) are assigned colors in \( [r] \setminus (A_X \cup A_Y) \). Since the number of pairs in \( M' \subseteq J_Y \) with the same color is at most

\[
\sum_{\alpha=1}^{r} d_\alpha^2 \leq \xi n \sum_{\alpha=1}^{r} d_\alpha \leq \frac{\xi n^2}{2},
\]
we have that the number of pairs \( \{f_2, f_3\} \) as above such that \( f_2 \) and \( f_3 \) have different colors is at least

\[
2\xi n^2 - \frac{\xi n^2}{2} \geq \xi n^2.
\]

Let \( J \) denote this set of pairs.

Since there is a color \( \alpha \in A_X \), there exists a matching \( M'' \) of size \( |M''| \geq \xi n \) in \( X \) for which every edge in \( M'' \) assumes color \( \alpha \). Clearly, \( M = M' \cup M'' \) is a matching. Form a set \( T(v) \subseteq M'' \times M'' \) of triples given by one edge from \( M'' \) and a pair of edges in \( J \), so that each triple is 3-colored. The cardinality of \( T(v) \) is at least

\[
\xi n \cdot \xi n^2 = \xi^2 n^3 \geq \frac{\xi^3 n^3}{9}.
\]

If \( A_X = \emptyset \), then \( |A_Y| \geq 1 \). Let \( \alpha \in A_Y \), and let \( M' \) be a matching in \( L(v) \cap \binom{Y}{2} \) of size \( |M'| \geq \xi n \) for which every edge in \( M' \) assumes color \( \alpha \). Since \( A_X = \emptyset \), we know that there are at least

\[
\left| L(v) \cap \binom{X}{2} \right| - \xi n \cdot 2|X| \geq \gamma n^2 - 2\xi n^2 \geq 4\sqrt{\xi n^2}
\]

edges in \( L(v) \cap \binom{X}{2} \) assuming a rare color different from \( \alpha \). Thus, with calculations as the ones leading to (37), we obtain a set \( J \) of at least \( \xi n^2 \) mutually vertex disjoint pairs of edges contained in a matching \( M'' \) in \( X \) in different colors, which are different from \( \alpha \). Set \( M = M' \cup M'' \) and define \( T(v) \) by forming triples with a pair of edges in \( J \) and one edge in \( M'' \). The set \( T(v) \) contains at least

\[
\xi n^2 \cdot \xi n \geq \frac{\xi^3 n^3}{9}
\]

3-colored triples \( (f_1, f_2, f_3) \). \( \square \)

We wish to prove that \( |C_1| \leq r^{|E(R_n)| - 1} \). For any triple \( (f_1, f_2, f_3) \) in \( T(v) \), let \( t_1, t_2, t_3, t_4 \in \binom{V}{3} \) be four 3-element sets (not necessarily hyperedges from \( H \)) such that \( \{\{v\} \cup f_i : i = 1, 2, 3\} \cup \{t_1, t_2, t_3, t_4\} \) forms a Fano plane. Note that each of the 3-element sets \( t_1, t_2, t_3, t_4 \) contains precisely one vertex from each \( f_i \). (In fact, there are two different sets of four 3-element sets \( t_1, t_2, t_3, t_4 \) with this property for any given \( f_1, f_2, f_3 \) and we just fix one of those two sets arbitrarily.) Furthermore, note that for two different choices of \( f_1, f_2, f_3 \) and \( f_1', f_2', f_3' \), there is at least one \( i \in \{1, 2, 3\} \) with \( f_i \cap (f_i' \cup f_2 \cup f_3) = \emptyset \). Therefore the corresponding sets \( \{t_1, t_2, t_3, t_4\}, \{t_1', t_2', t_3', t_4'\} \) are disjoint.

Fix any \( r \)-coloring \( \Delta \in C_1 \). By Claim 4.2, the coloring of \( L(v) \) induced by this coloring leads to at least \( \frac{\xi^3 n^3}{9} \) distinct 3-colored triples in \( T(v) \). Fix one of these 3-colored triples \( (f_1, f_2, f_3) \). Since \( \{v\} \cup f_i \) are colored in different colors, either one of the 3-element sets \( t_i \) must be missing from \( H \), or altogether at most six colors are assigned to the seven edges \( \{f_1, f_2, f_3, t_1, t_2, t_3, t_4\} \), because there is no rainbow Fano plane in \( H \). This leads to at most

\[
Q = 3 \cdot 4 \cdot r^3 + \binom{4}{2} r^3 = 18r^3
\]
ways to extend the coloring of \( \{ f_1, f_2, f_3 \} \) to \( t_1, t_2, t_3, t_4 \), because one of these four hyperedges may be assigned one of the colors used for \( f_1, f_2, f_3 \) or two of \( t_1, t_2, t_3, t_4 \) use the same new color.

For each triple \( \{ f_1, f_2, f_3 \} \) in \( T(v) \), we choose a 4-element set \( \{ t_1, t_2, t_3, t_4 \} \) of crossing hyperedges in \( H \) with respect to partition \( V \) that form a Fano plane with the triple \( \{ \{ v \} \cup f_1, \{ v \} \cup f_2, \{ v \} \cup f_3 \} \), if such a set exists (recall that there may be at most two). Let \( T = T(v) \) be the family of pairs obtained in this way. We wish to find a lower bound on the cardinality of \( T \). Every time a triple \( \{ f_1, f_2, f_3 \} \) cannot be paired with a 4-element set, it means that some crossing hyperedge must be missing from \( H \). Moreover, for any two triples \( \{ f_1, f_2, f_3 \} \) and \( \{ f_1', f_2', f_3' \} \) that cannot be paired with 4-element sets, the missing hyperedges are different. Recall that the number of non-crossing hyperedges in \( H \) with respect to \( V \) is at most \( \delta n^3 \) and that \( H \) has at least \( \text{ex}(n, \text{Fano}) \) hyperedges. This means that the number of triples that would be crossing hyperedges with respect to \( V \), but that do not lie in \( H \), is at most

\[
\delta n^3 \leq \frac{\xi^3 n^3}{36}.
\]

We conclude that

\[
|T| \geq \frac{\xi^3 n^3}{9} - \delta n^3 \geq \frac{\xi^3 n^3}{9} - \frac{\xi^3 n^3}{36} = \frac{\xi^3 n^3}{12}.
\]

To obtain an upper bound on \( |C_1| \), we have at most \( r^{|E(H)|} \) ways to color hyperedges containing vertex \( v \) and there are at most \( Q^{T_1} \) ways to color the Fano planes in \( H \) that extend 3-colored triples \( \{ f_1, f_2, f_3 \} \) in \( T(v) \), and finally at most \( r^{|E(H)|-4|T|} \) ways to color the remaining hyperedges of \( H \). Therefore,

\[
|C_1| \leq Q^{T_1} r^{|E(H)|-4|T|} \leq (18r^3)^{|T|} r^{|E(H)|-4|T|}.
\]  

(38)

The right-hand side of (38) increases as \( |T| \) decreases. Since \( |T| \geq \frac{\xi^3 n^3}{12} \), we have

\[
|C_1| \leq (18r^3) \frac{\xi^3 n^3}{12} r^{|E(H)|-4|T|} \frac{\xi^3 n^3}{12} \leq 18 r^3 \frac{\xi^3 n^3}{12} r^{|E(B_n)|+\delta n^3 - \frac{\xi^3 n^3}{12}} \leq \left( r^{\log_3 18} \right) \frac{\xi^3 n^3}{12} r^{|E(B_n)|-\frac{\xi^3 n^3}{18}} = r^{|E(B_n)|+\frac{\xi^3 n^3}{18}} \leq r^{|E(B_n)|-1}.
\]  

(39)

We recall that \( C_2 = C \setminus C_1 \) is the class of colorings \( \Delta \) satisfying \( |A_Y(\Delta) \cup A_Z(\Delta)| \in \{1,2\} \) and \( |J_Z| < 4\sqrt{\xi} n^2 \) for every class \( Z \in \{X,Y\} \). By our bound on \( |C_1| \), we derive

\[
|C_2| = |C| - |C_1| \geq r^{|E(B_n)|+m} - r^{|E(B_n)|-1} \geq r^{|E(B_n)|+m-1}.
\]  

(40)

Next we estimate the number of colorings of the set of hyperedges incident to vertex \( v \) that can be extended to a coloring in \( C_2 \).

By definition of \( C_2 \), in each class, there are at most \( 4\sqrt{\xi} n^2 \) edges assuming colors in \( [r] \setminus (A_X \cup A_Y) \). To count the number of colorings that can be extended to a coloring from \( C_2 \), we first choose at most two colors for \( A_X \cup A_Y \), which may be done in at most
$r^2$ ways. For each class, there are at most $\binom{n^2}{4\sqrt{\xi n^2}}$ ways to choose hyperedges assuming colors in $[r] \setminus (A_X \cup A_Y)$, which may be colored in at most $r^{4\sqrt{\xi n^2}}$ ways. There are at most $r^{(X||Y)}$ ways to color hyperedges containing $v$ and one vertex from each class. Finally, each of the other hyperedges containing $v$ must assume a color in $A_X \cup A_Y$, so that we have at most $2^{\left|L(v)\right| - 2 - 4\sqrt{\xi n^2} - |X||Y|}$. We conclude that, for $n$ sufficiently large, the number of colorings of the set of hyperedges incident with vertex $v$ that can be extended to a coloring in $C_2$ is at most

$$r^2 \left( \frac{n^2}{4\sqrt{\xi n^2}} \right)^2 r^{8\sqrt{\xi n^2} + |X||Y|} \left( 2^{\left|L(v)\right| - 8\sqrt{\xi n^2} - |X||Y|} \right)$$

Inequality (*) can be seen as follows. We claim that

$$\left( \log_2 \left( 2^{\left(2\left(4\sqrt{\xi} - 8\sqrt{\xi} + \frac{3}{4}\right) + 8\sqrt{\xi} + \frac{1}{4}\right) + \frac{26}{100}} \right) \right)$$

because the derivative of the function

$$f(x) = \left( \log_2 \left( 2^{\left(2\left(4\sqrt{\xi} - 8\sqrt{\xi} + \frac{3}{4}\right) + 8\sqrt{\xi} + \frac{1}{4}\right) + \frac{26}{100}} \right) \right)$$

satisfies

$$f'(x) = \frac{1}{\log_2(2)} \left( 2 \log \left( \frac{1-x}{x} \right) - 2 \log 2 \right) + 2,$$

hence is increasing for $0 < x < 1/2$ and $r \geq 2$. As $\xi \leq r/16 \leq 1/(1006^3 \cdot 16)$, to obtain (42), inserting $x = 4\sqrt{\xi} \leq (1/1006)^{3/2}$, gives with (41) for $r \geq 21^{64}$:

$$f((1/1006)^{3/2}) < (\log_2(2) \left( 2 \cdot 0.001 - 2 \left( \frac{1}{1006} \right)^{3/2} + \frac{3}{4} \right) + 2 \left( \frac{1}{1006} \right)^{3/2} + \frac{1}{4})$$

$$< (\log_2(2)0.76 + 0.251 < \frac{26}{100}.$$

Setting $H' = H - v$, we obtain that the number of feasible colorings of $H'$ is at least

$$\frac{|C_2|}{r^{26/100} n^2} \geq \frac{r^{\left|E(B_n)\right| + m - 1}}{r^{26/100} n^2} = r^{\left|E(B_n)\right| + m - 1 - \frac{26}{100} n^2}.$$

which proves Claim 41 for hypergraphs $H$ satisfying the assumptions of Case 1.

**Case 2.** $H$ has the property that for every class $Z \in \{X, Y\}$ and every vertex $v \in Z$, the inequality $|L(v) \cap (\frac{Z}{2})| \leq \gamma n^2$ holds.
Since \( H \neq B_n \), there exists a hyperedge \( e = \{ v_1, v_2, v_3 \} \) with all vertices in the same class, say \( e \subseteq Y \). Let \( \mathcal{L} \) be the graph with vertex set \( X \) and edge set \( L = \bigcap_{i=1}^{3} L(v_i) \cap \left( \binom{X}{2} \right) \). In particular, for any \( f \in \left( \binom{X}{2} \right) \), \( f \) lies in \( L \) if and only if \( f \in L(v_i) \) for every \( i \in [3] \). As \( |L(v_i) \cap \binom{X}{2}| \leq \gamma n^2 \) and \( \delta_1(H) \geq \delta_1(B_n) \geq 3n^2/8 - n \) we have

\[
|L(v_i) \cap \binom{X}{2}| \geq \frac{3}{8}n^2 - n - \gamma n^2 - |X||Y|.
\]

This implies that

\[
\left| \binom{X}{2} \setminus L(v_i) \right| \leq \left( \frac{|X|}{2} \right) - \frac{3}{8}n^2 + \gamma n^2 + |X||Y| + n,
\]

so that

\[
|L| = \left( \frac{|X|}{2} \right) - \left| \bigcup_{i=1}^{3} \left( \binom{X}{2} \setminus L(v_i) \right) \right|
\]

\[
\geq \left( \frac{|X|}{2} \right) - 3 \left( \left( \frac{|X|}{2} \right) - \frac{3}{8}n^2 + \gamma n^2 + |X||Y| + n \right)
\]

\[
= \frac{9}{8}n^2 - 2 \left( \left( \frac{|X|}{2} \right) - 3|X||Y| - 3\gamma n^2 - 3n \right)
\]

\[
\geq \frac{9}{8}n^2 - |X|^2 - 3|X||Y| - 3\gamma n^2 - 3n.
\]

**Claim 4.3.** There are at least

\[
\frac{1}{6} \left( \frac{2 - 240\gamma}{80} \right) n^2
\]

mutually edge-disjoint copies of the complete graph \( K_4 \) in \( \mathcal{L} \).

**Proof.** By Turán’s theorem \[41\], a graph with \( |X| \) vertices and more than \( |X|^2/3 \) edges contains a \( K_4 \). Because of this, if \( |L| > |X|^2/3 \), then there is a copy of \( K_4 \) in \( \mathcal{L} \). Removing the six edges of this copy from \( \mathcal{L} \), provided that \( |L| - 6 \geq |X|^2/3 \), we may find another copy of \( K_4 \) that is edge-disjoint from the first one. Repeating this argument, the number of such copies of \( K_4 \) that we find is at least

\[
\frac{1}{6} \left( |L| - \frac{|X|^2}{3} \right) \geq \frac{1}{6} \left( \frac{9}{8}n^2 - \frac{4}{3}|X|^2 - 3|X||Y| - 3\gamma n^2 - 3n \right).
\]

Since \( |X| + |Y| = n \), and \[36\] holds, we have that \( 4|X|^2/3 + 3|X||Y| \) is, without loss of generality, maximum for \( |X| = n/2 + 2\sqrt{\delta}n \) and \( |Y| = n/2 - 2\sqrt{\delta}n \), i.e.,

\[
\frac{4}{3}|X|^2 + 3|X||Y| + 3n
\]

\[
\leq \left( \frac{4}{3} \left( \frac{1}{2} + 2\sqrt{\delta} \right)^2 + 3 \left( \frac{1}{4} - 4\delta \right) \right) n^2 + 3n \leq \frac{11}{10} n^2. \tag{44}
\]

Thus, we have at least

\[
\frac{1}{6} \left( |L| - \frac{|X|^2}{3} \right) \geq \frac{1}{6} \left( \frac{9}{8} - \frac{11}{10} - 3\gamma \right) n^2 = \frac{1}{6} \left( \frac{2 - 240\gamma}{80} \right) n^2
\]

mutually edge-disjoint copies of \( K_4 \) in \( \mathcal{L} \). \(\square\)
Let $K^1, \ldots, K^q$ be the mutually edge-disjoint copies of $K_4$ in $\mathcal{L}$ given by Claim 4.3 where
\[ q \geq \frac{1}{6} \left( \frac{2 - 240\gamma}{80} \right) n^2. \] (45)

Since $E(K^j) \subseteq L$ for every $j \in [q]$, every such $K^j$ forms a Fano plane together with the hyperedge $e$. Fixing a color for $e$, we can color the six hyperedges that correspond to the edges of every $K^j$ in less than $6r^5 + (\frac{6}{r}) r \cdot r^4 = 21r^5$ ways.

Set $H' = H - e$ (that is, the vertices in $e$ are deleted from $H$). Let $E_e$ denote the set of hyperedges of $H$ that contain at least one vertex from $e = \{v_1, v_2, v_3\}$. Obviously, $|E_e| \leq 3\gamma n^2 + 3(\binom{|X|}{2}) + 3|X||Y|$. From the assumption $|E(X)| + |E(Y)| \leq \delta n^3$ and the fact that $|E(H)| \geq |E(B_n)|$, it follows that, for $n$ sufficiently large,
\[
|E_e| \leq 3\left( \frac{|X|}{2} \right) + 3|X||Y| + 3\gamma n^2 \]
\[
\leq \frac{9}{2} \left( \frac{1}{2} + 2\sqrt{\delta} \right)^2 n^2 + 3\gamma n^2 \]
\[
\leq \frac{9}{8} n^2 + 4\gamma n^2 \]
\[
\leq \delta_1(B_n) + \delta_1(B_{n-1}) + \delta_1(B_{n-2}) + 5\gamma n^2 \]
\[
= |E(B_n)| - |E(B_{n-3})| + 5\gamma n^2,
\]
which implies that
\[
|E(B_n)| - |E_e| \geq |E(B_{n-3})| - 5\gamma n^2. \tag{46}
\]

We can color hyperedges in $E_e$ in at most
\[
r^{|E_e|} \left( \frac{21r^5}{r^6} \right)^q = 21^q r^{|E_e| - q}
\]
ways.

Consequently, for $n$ sufficiently large, the number of feasible colorings of $H'$ is at least
\[
\frac{r^{|E(B_n)| + m}}{21^q r^{|E_e| - q}} \geq 21^{-q} r^{|E(B_n)| + m - |E_e| + q} \geq r^{|E(B_{n-3})| + m - 5\gamma n^2 + q(1 - \log r, 21)} \geq r^{|E(B_{n-3})| + m + 1}, \tag{47}
\]
where the last inequality can be seen as follows. From (31) and (33) we obtain that
\[-5\gamma + \frac{1}{6} \left( \frac{2 - 240\gamma}{80} \right) (1 - \log r, 21) > 0,
\]
because $\gamma \leq 1/1406$ and $r > 21^{64}$ implies that $(1 - \log r, 21) > 63/64$.

This concludes Case 2, finishes the proof of Claim 4.1 and consequently the proof of Theorem 1.1. \qed
5. Final remarks and open problems

In this paper, we have shown that, for sufficiently large \( r \) and \( n \), the hypergraph \( B_n \) is the unique \( n \)-vertex 3-uniform hypergraph admitting the largest number of \( r \)-edge colorings with no rainbow copy of the Fano plane. A natural question would be to ask for the best possible values of \( r \) and \( n \) for which this holds. With respect to \( r \), it is clear that our result cannot possibly be extended to \( r \leq 10 \), as the complete \( n \)-vertex hypergraph \( K_n^{(3)} \) admits at least \( \max\{r\binom{n}{3}, 6\binom{n}{3}\} \) distinct \( r \)-colorings in which at most six colors are used, and this is larger than \( r^{\text{ex}(n,F)} \) for \( r \leq 10 \). On the other hand, we are convinced that the value of \( r_0 \) provided in Theorem 1.1 is far from optimal. To the best of our knowledge, it may even be that \( r_0 = 11 \). In fact, any improvement on the range and influence of \( \delta \) in Theorem 2.6 would immediately be translated into a better value of \( r_0 \) in Lemma 3.1. This lemma is the only significant hurdle for better bounds on \( r_0 \) using the current approach, that is, it is possible to adapt our proof of Theorem 1.1 to lesser values of \( r \) such that Lemma 3.1 holds.

Regarding the value of \( n_0 = n_0(r) \) given in the proof of Theorem 1.1, we believe that it is by no means optimal and we have made no particular effort to optimize it. Indeed, our proof is based on the weak regularity lemma [9, 15, 28, 39], which requires very large values on \( n_0 \) in the worst case. In the graph case, better bounds have been obtained in [3, 18] using strategies based on the container method [4, 38] (see also [14] for a colored version).

We would also like to mention that our proof of Lemma 3.1 applies in more general contexts, as we now describe. For fixed positive integers \( \ell \) and \( k \), let \( I_{\ell,k} \) be the set of nonnegative integral solutions to \( x_1 + \cdots + x_\ell = k \).

**Definition 5.1.** For integers \( n, \ell \) and \( k \), a vector \( I = (x_1, \ldots, x_\ell) \in I_{\ell,k} \) and a partition \( \mathcal{V} = \{V_1, \ldots, V_\ell\} \) of \([n]\), let \( H_{I,\mathcal{V}}(n) \) be the \( k \)-uniform hypergraph with vertex set \([n]\) and edge set given by all \( k \)-element subsets \( e \subseteq [n] \) such that there is a permutation \( \pi \) of \([\ell]\) for which \( |e \cap V_{\pi(1)}|, \ldots, |e \cap V_{\pi(\ell)}| \rangle = I \). The hypergraph \( H_{I,\mathcal{V}}(n) \) is called the complete multipartite hypergraph with respect to \( I \) and \( \mathcal{V} \). We say that \( I \) and \( \mathcal{V} \) are the intersection vector and the partition of \( H_{I,\mathcal{V}}(n) \), respectively. Moreover, if \( H = K_n^{(k)} \) is the complete \( k \)-uniform hypergraph on \([n]\), we say that \( B_{I,\mathcal{V}}(H) = E(H) \setminus E(H_{I,\mathcal{V}}(n)) \) is the set of bad hyperedges of \( H \) with respect to \( \mathcal{V} \) and \( I \).

For example, when \( \ell = 2, k = 3, I = (2, 1) \) and \( \mathcal{V} = \{V_1, V_2\} \) is a balanced partition of \([n]\), we have \( H_{I,\mathcal{V}}(n) = B_n \). Moreover, if \( H \) is any 3-uniform hypergraph on \([n]\), then \( B_{I,\mathcal{V}}(H) \) is the set of hyperedges of \( H \) that are entirely contained in \( V_1 \) or in \( V_2 \).

**Definition 5.2** (Well-behaved multipartite extremal hypergraph). A \( k \)-uniform hypergraph \( F \) has a well-behaved multipartite extremal hypergraph if there exist positive integers \( \ell \) and \( n_0 \), a constant \( \xi > 0 \), and an intersection vector \( I \in I_{\ell,k} \) satisfying the following properties.

(a) For every positive integer \( n \) and for every partition \( \mathcal{V} = \{V_1, \ldots, V_\ell\} \) of \([n]\), the hypergraph \( H_{I,\mathcal{V}}(n) \) is \( F \)-free;

(b) For any integer \( n \geq n_0 \), there is a partition \( \mathcal{V} = \{V_1, \ldots, V_\ell\} \) of \([n]\) for which \( \text{ex}(n, F) = |E(H_{I,\mathcal{V}}(n))| \). Moreover, this partition satisfies \( |V_i| \geq \xi n \) for all \( i \in [\ell] \).

(c) For every \( \delta > 0 \), there exist an integer \( n_1 \geq n_0 \) and a constant \( \varepsilon_\delta = \varepsilon_\delta(\delta) \) such that every \( F \)-free hypergraph \( H = (V, E) \) on \( n \geq n_1 \) vertices with at least
ex(n, F) − εan^k hyperedges admits a partition \( V = \{V_1, \ldots, V_\ell\} \) of \( V \) such that
\[
|B_{I,V}(H)| \leq \delta n^k.
\]

To the best of our knowledge, the structural description of a well-behaved multipartite extremal hypergraph applies to all linear hypergraphs \( F \) whose extremal hypergraph is known and is dense. Examples include expanded complete graphs (see Pikhurko [35] and Mubayi [32] for the extremal and the stability result, respectively), fan hypergraphs (see Mubayi and Pikhurko [33]) and for more general classes generalizing these instances (see Brandt, Irwin and Jiang [3] and Norin and Yepremyan [34]).

A few changes in our proof of Lemma 3.1 yield the following stability result. A proof of this result may be found in the first author’s doctoral thesis [11].

**Theorem 5.3.** Let \( k \geq 2 \) be an integer. For every \( \delta > 0 \) and every linear \( k \)-uniform hypergraph \( F \) that has a well-behaved multipartite extremal hypergraph with intersection vector \( I \in \mathcal{I}_{\ell,k} \), there is \( r_0 = r_0(\delta, F) \) with the following property. For all integers \( r \geq r_0 \) there exists \( n_0 = n_0(r) \) such that, if \( n \geq n_0 \) and \( H = (V, E) \) is an \( n \)-vertex \( k \)-uniform hypergraph with at least \( r \cdot \text{ex}(n, F) \) distinct \((F, R)\)-free \( r \)-colorings, where \( R \) is the rainbow pattern of \( F \), then there is a partition \( V = \{V_1, \ldots, V_\ell\} \) of \( V \) for which
\[
|B_{I,V}(H)| \leq \delta n^k.
\]

On the other hand, the part of our proof that uses the stability of Lemma 3.1 to show that \( B_n \) is the single \( n \)-vertex \((r, F^R)\)-extremal graph for \( r \geq r_0 \) and sufficiently large \( n \) (see Claim 4.1) uses ad-hoc arguments that rely heavily on the structure of the Fano plane and on \( B_n \). We have not been able to generalize it to other hypergraphs \( F \). However, we believe that the following general statement is true.

**Conjecture 5.4.** Given an integer \( k \geq 2 \) and a linear \( k \)-uniform hypergraph \( F \) such that \( \text{ex}(n, F) = \Omega(n^k) \), there exists \( r_0 \) with the following property. For every \( r \geq r_0 \), there is \( n_0 \) such that any \( n \)-vertex \( k \)-uniform hypergraph \( H \), where \( n \geq n_0 \), satisfies
\[
c_{r, (F, P)}(H) \leq r \cdot \text{ex}(n, F), \quad (48)
\]
where \( P \) is the rainbow pattern of \( F \). Moreover, equality holds in (48) for \( n \geq n_0 \) if and only if \( H \) is \( F \)-extremal.

We should point out that we do not expect the statement of Conjecture 5.4 to hold for \( k \)-uniform hypergraphs \( F \) whose \( F \)-extremal hypergraph is sparse (see Conjecture 5.5 below). Moreover, as done in [24] Remark 4.1], one may show that there exist \( k \)-uniform hypergraphs \( F \) such that the statement of Conjecture 5.4 does not hold for any non-rainbow pattern of \( F \).

For more general patterns, we also deem the following to be true, strengthening [24] Theorem 1.1]. Here, we let \( K_n^{(k)} \) denote the complete \( k \)-uniform hypergraph on \( n \) vertices.

**Conjecture 5.5.** Fix integers \( r, k \geq 2 \) and a linear \( k \)-uniform hypergraph \( F \) such that \( \text{ex}(n, F) = o(n^k) \). Let \( P \) be any pattern of \( F \) on \( t \geq 3 \) classes. Then there exists \( n_0 \) such that any \( n \)-vertex \( k \)-uniform hypergraph \( H \), where \( n \geq n_0 \), satisfies
\[
c_{r, (F, P)}(H) \leq c_{r, (F, P)}(K_n^{(k)}), \quad (49)
\]
Moreover, equality holds in (49) for \( n \geq n_0 \) if and only if \( H = K_n^{(k)} \).
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