Self-Duality, Four-Forms, and the Eight-Dimensional Yang-Mills/Dittmann-Bures Field over the Three-Level Quantum Systems

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Utilizing a number of results of Dittmann, we investigate the nature of the Yang-Mills field over the eight-dimensional convex set, endowed with the Bures metric, of three-level quantum systems. Paralleling the decompositions of eight-dimensional Euclidean fields by Corrigan, Devchand, Fairlie and Nuysts, as well as Figueroa-O’Farrill and others, we investigate the properties of self-dual (Ω+) and anti-self-dual (Ω−) four-forms corresponding specifically to our Bures/non-Euclidean context.

For any of a number of (nondegenerate) 3 × 3 density matrices, we are able to solve the pair of eigenequations, *F = ±(1/Λ) (Ω± ∧ F), where * is the Hodge operator with respect to the Bures metric and F a two-form. The resultant sets of (traceless) twenty-eight λ’s coincide with the sets of twenty-eight λ’s, the sets consisting of four singlets and three octets. The four-forms Ω± are found to exhibit quite simple behaviors, though we are not able to derive them in full generality.

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The Bures metric, defined on the nondegenerate density matrices, has been the object of considerable study [1, 2]. It is the minimal member of the nondenumerable family of monotone metrics [3]. Other members of this family of particular note are the “Bogoliubov-Kubo-Mori” (BKM) metric [4], the maximal monotone metric, as well as the “Morozova-Chentsov” [1] and “quasi-Bures” [11] ones, the last yielding the minimax/maximin asymptotic redundancy in universal quantum coding [12]. Interestingly, these (operator) monotone metrics correspond in a direct fashion to certain “measures of central tendency”, with, for example, the Bures metric corresponding to the arithmetic mean, (x + y)/2, of numbers x and y [13]. All these monotone metrics constitute various extensions to the quantum domain of the (unique) Fisher information metric in the classical realm where the objects of study are probability distributions (rather than density matrices) [13] (cf. [14]). However, the only one of these monotone metrics that can be extended to the boundary of pure states yielding the standard Fubini-Study metric on this boundary is the Bures (minimal monotone) one [1, sec. IV] (cf. [15]).

Dittmann has shown that “the connection form (gauge field) related to the generalization of the Berry phase to the mixed states proposed by Uhlmann satisfies the source-free Yang-Mills equation *D * Dω, where the Hodge operator is taken with respect to the Bures metric on the space of finite-dimensional density matrices” [16, eq. (9)]. These findings may be seen as extensions to mixed states of numerous examples relating the original Berry phase to Dirac eigenequations, *F = ±(1/Λ)(Ω± ∧ F), where * is the Hodge operator with respect to the Bures metric and F a two-form [15].

In [11, sec. 6], Figueroa-O’Farrill studied the eigenspaces of various endomorphisms (Ŷ), defined by four-forms (Υ) in E8, of the space of two-forms. “One can use Ŷ to define a generalised self-duality in eight dimensions by demanding that a two-form belong to a definite G-submodule of S8. This generalises self-duality in four dimensions, where we can take Ŷ = *1, and Ŷ = * itself. The eigenspaces of Ŷ are the subspaces of self-dual and anti-self-dual two-forms” [11].

In this work [11, eq. (9)], we seek to extend this type of analysis to our different (non-Euclidean/Bures), but still eight-dimensional setting.

To proceed, we exploit our recent work [20, 21] in determining the elements of the Bures metric using a certain Euler angle parameterization of the 3 × 3 density matrices [22]. (Earlier, Dittmann had noted [11] “that in affine coordinates (e. g. using the Pauli matrices for n = 2) the [Bures] metric becomes very complicated for n > 2 and no good parameterization seems to be available for general n.”.) This allows us, among other things, to employ as our parameter space four (noncontiguous) eight-dimensional hyperrectangles (each of the four having three sides of length π, three sides of length π, one side of length one cos−1 1/3), rather than less analytically convenient ones, such as that discussed by Bloore [23], Fig. 3, involving spheroids and “tetrapaks”. (The four hyperrectangles are not adjacent due to the fact that two of the six Euler angles used in the parameterization have disconnected ranges. This situation, that is a multiplicity of hyperrectangles—which leads to different normalization factors only—has only come to our attention relatively recently, and serves as an erratum to earlier analyses [23, 24], which had relied upon work of Marinov [25], without taking into account a correction [23] to [24] (cf. [28]).)

For the required connection form A, Dittmann has presented the general formula [16, eq. (9)],
Here the elements of $T$ lie in the tangent space to the principal $U(H)$-bundle, the manifold of invertible normalized $(\text{Tr}W^*W = 1)$ Hilbert-Schmidt operators, while $L$ is the operator (depending on $W$) of left multiplication by the density matrix $\rho = WW^*$ and $R$ the corresponding operator of right multiplication. Also, $\tilde{L}$ and $\tilde{R}$ are the counterpart operators for $\tilde{\rho} = W^*W$. Since the Euler-angle parameterization of the $3 \times 3$ density matrices presented in [22] is of the (“Schur-Schatten”) form,

$$\rho = UDU^*, \quad (2)$$

$U$ being unitary, $U^*$ its conjugate transpose, and $D$ the diagonal matrix composed of the three eigenvalues of $\rho$, one can immediately express $W$ as $UD^{1/2}U^*$, with $W = W^*$. Let us note here that Sjöqvist et al have addressed the issue of mixed state holonomy — first raised by Uhlmann \[1\] as a “purely mathematical” problem — developing “a new formalism of geometric phase for mixed states in the experimental context of quantum interferometry” \[30\]. In their notation,

$$\Omega = i\text{Tr}[\rho_0 W^d W] \quad (3)$$

is regarded as a gauge potential on the space of density operators, where

$$W(t) = \frac{\text{Tr}[\rho_0 U(t)\rho_0]}{|\text{Tr}[\rho_0 U(t)]|} U(t), \quad (4)$$

the unitary operator $U(t)$ being used to fix the parallel transport conditions. It would be of interest to examine the relation of their expressions to that of Dittmann \[4\], which we implement here (cf. \[31\]). (In the context of pure states, Sanders, de Guise, Bartlett, and Zhang have presented a scheme for producing and measuring an Abelian geometric phase shift in a three-level quantum system where states are invariant under a non-Abelian group \[32\]. They geodesically evolve $U(2)$-invariant states in a four-dimensional $SU(3)/U(2)$ space, using a three-channel interferometer (cf. \[33\]).)

In the non-trivial task (involving computations with modular operators \[7\]) of implementing formula \[1\] for the three-level quantum systems, we relied upon the implicit relation (equivalence) between two formulas for the Bures metric for $n$-level quantum systems \[34\], eqs. (2) and (16),

$$g = \frac{1}{2} \text{Tr}d\rho \frac{1}{L + R} d\rho, \quad (5)$$

and

$$g = \frac{1}{2} \sum_{i,j} a_{ij} \text{Tr}d\rho \rho^{-1} d\rho \rho^{j-1}. \quad (6)$$

(The somewhat involved formula that we used for the coefficients $a_{ij}$, functions of elementary invariants, is stated in Proposition 3 of \[34\].) In other words, we used the formula for the connection

$$A = \frac{1}{2} \sum_{i,j} a_{ij} \rho^{-1} S \rho^{j-1}, \quad (7)$$

where $S = W^*T - T^*W$.

For $n = 3$ we have determined, for particular points in the eight-dimensional manifold, four-forms $\Omega_+$ for which $\ast\Omega_+ = \Omega_+$, where the Hodge star is with respect to the Bures metric \[21\], as well as four-forms $\Omega_-$ for which $\ast\Omega_- = \Omega_-$. We formulated this problem as a set of seventy linear simultaneous homogeneous equations in seventy unknowns (that is, unknown constants). Selecting various points in our eight-dimensional space (and making use of exact arithmetic), MATHEMATICA has consistently succeeded in expressing thirty-five of the unknowns in terms of the other thirty-five variables. However, we did not find a unique such solution valid for arbitrary points in the eight-dimensional space, with MATHEMATICA apparently indicating that such a general solution over the manifold with constant coefficients for $\Omega$ is not possible. We have obtained precisely the same type of findings in our search for anti-self-dual four-forms. (The Hodge $\ast$ operation squares to 1 on four-forms in eight dimensions. It has eigenvalues
\( +1 \) and \(-1\) with equal multiplicity. So one can always find a basis where \(*\) is diagonal with thirty-five \(+1\)'s and thirty-five \(-1\)'s. In all these calculations we have used the duality relation, \([20, \text{eq. (1.5.15)}]\) (making use of our explicit — though, in some cases somewhat cumbersome — formulas for the \(g^{ij}\)'s \([20,21]\))

\[ (*α)_{j_1\ldots j_n} = g^{i_1j_1}\ldots g^{i_nj_n} \epsilon_{j_1\ldots j_n} \sqrt{g/p!} \alpha_{i_1\ldots i_p}, \]

where we took \(n = p = 4\).

For the specific point \(q_1\), we set the eight variables, described in \([20]\), to be \(α = -\frac{π}{3}, \tau = \frac{2π}{3}, a = \frac{4π}{3}, b = -\frac{2π}{3}, β = \frac{2π}{3}, θ = -\frac{2π}{3}, θ_1 = -\frac{π}{3}, θ_2 = -\frac{π}{3}\) — the first six of these being Euler angles that parameterize the unitary matrix \(U\), while the last two parameterize the diagonal matrix of eigenvalues \(D\) in the decomposition \([3]\). In our eight-dimensional manifold of states, the point \(q_1\) gives us the specific \(3 \times 3\) density matrix,

\[
ρ_1 = \begin{pmatrix}
\frac{707}{1024} & -\frac{3(59i+25\sqrt{3})}{2048} & \frac{1024}{1024 - (1)^{11/12}(9 + 20i\sqrt{3})} \\
\frac{1024}{2048 - (1)^{11/12}(9 + 20i\sqrt{3})} & \frac{1024}{1024} & -\frac{(1)^{11/12}(21 + 25\sqrt{7})}{1024 - (1)^{11/12}(21 + 25\sqrt{7})} \\
\frac{1024}{1024} & \frac{(1)^{11/12}(21 + 25\sqrt{7})}{1024 - (1)^{11/12}(21 + 25\sqrt{7})} & \frac{1024}{1024 - (1)^{11/12}(9 + 20i\sqrt{3})}
\end{pmatrix}.
\]

This matrix is, of course, Hermitian, nonnegative definite — possessing eigenvalues \(\frac{9}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\) — and has trace 1. (It corresponds to a strictly impure or mixed state. Also, we should point out that the entries of the \(8 \times 8\) Bures metric tensor turn out, in fact, to be independent of the variables \(a\) and \(α\), as established in \([20]\).) For the solutions for \(λ_+\) at \(q_1\) of the eigenequation, \((F\) being a two-form),

\[ *F = \frac{1}{λ_+}(Ω_+ \wedge F), \]

(having substituted unity for each of the free thirty-five variables in the thirty-five-dimensional solution to the set of seventy linear homogeneous equations for \(Ω\), based upon \([8]\), which enforce self-duality), we obtained the twenty-eight eigenvalues (cf. \([20, \text{eq. (2.1)}]\))

\[ λ_+(q_1) = 6.15149, -6.06045, ±5.11128(\text{eightfold}), -4.16211, 4.07107, ±.994689(\text{eightfold}), ±.0455182(\text{eightfold}). \]

Of course, the three octets are each split into two quartets of opposite sign. (The adjoint representation of \(SU(3)\) is eight-dimensional, with its elements generically lying in \(SO(8)\) \([30, \text{sec. 7}]\).) The four singlets or isolated values are the roots of the quartic (biquadratic) equation,

\[ 17848517231861271296 + 718875λ_+(-72767012864 - 2131611323040λ_+ + 39304490625λ_+^2) = 0. \]

The six possible values assumed by the three octets are the roots of the sextic equation (a cubic equation in \(x^2\)),

\[ 82734971267961585664 - 18225\lambda_+^2(2195802859754043904 + 291144375\lambda_+^2(-7894856752 + 291144375\lambda_+^2)) = 0. \]

(Consistent with tracelessness, the twenty-eight eigenvalues sum to zero — as well as, therefore, the four isolated eigenvalues themselves.) We have (as one particular example) "simplified" the exact symbolic expression for the root of \([13]\) corresponding to the eigenvalue 5.11128, so that it is explicitly real-valued, obtaining thereby that

\[ (5.11128)^2 \approx \frac{1}{873433125}\left(16\left(493428547 + 2\sqrt{217739666231788507}\right)\tan^{-1}\left(\frac{19986057\sqrt{257834787813597115559383045701069731}}{101904855629270248323646732}\right)\right). \]

(The value of the cosine term here is close to unity, that is .999444. Any further simplification of \([14]\) does not seem possible.)

We should also note that the cardinality of the set of eigenvalues, that is twenty-eight, corresponds to the number of entries of the (antisymmetric) two-form \(F\), that is \(F_{ij}\) (\(i,j = 1,\ldots,8\)), for which \(i < j\) (having observed that \(F_{ij} = -F_{ji}\) and \(F_{ii} = 0\)). Proceeding in the exact same manner, but using the anti-self-dual form MATHEMATICA provided, and solving
we obtained precisely the same set of twenty-eight eigenvalues \( [11] \), despite that fact that \( \Omega_+ \neq \pm \Omega_- \).

We observe here that the well-known octonionic instanton (CDFN) equations \( [29] \) are obtained by choosing a particular constant (hence closed) four-form in \( \mathbb{R}^8 \). This form decomposes the bundle of two-forms into two sub-bundles of dimensions seven and twenty-one. Each of these sub-bundles corresponds to a different nonzero eigenvalue (-3 and 1, respectively) of the transformation (which is symmetric, hence diagonalizable, as well as traceless)

\[
F \mapsto * (\Omega \wedge F),
\]

so restricting a Lie-algebra valued two-form to lie in either of these two sub-bundles guarantees that it will satisfy the Yang-Mills equations of motion.

We have (denoting by \( \zeta_{ijkl} \) the four form \( dx_i \wedge dx_j \wedge dx_k \wedge dx_l \), where for our eight variables we take \( x_1 = \alpha, x_2 = \tau, x_3 = a, x_4 = \beta, x_5 = b, x_6 = \theta, x_7 = \theta_1, x_8 = \theta_2 \), and using lexicographic ordering of the indices), corresponding to the point \( q_1 \),

\[
\Omega_+(q_1) = \frac{378375}{1654016} \zeta_{1234} - \frac{14037}{127232} \zeta_{1235} \ldots + \frac{601}{71} \zeta_{1578} - \frac{59079}{284} \zeta_{1678} + \zeta_{2345} \ldots + \zeta_{5678},
\]

and

\[
\Omega_-(q_1) = \frac{6975}{23296} \zeta_{1234} - \frac{14187}{127232} \zeta_{1235} \ldots - \frac{647}{71} \zeta_{1578} + \frac{58745}{284} \zeta_{1678} + \zeta_{2345} \ldots + \zeta_{5678}.
\]

So we see that \( \Omega_+(q_1) \neq \pm \Omega_-(q_1) \) (nevertheless, clearly sharing certain features). When we used these same two four-forms, but at another point \( \lambda = \frac{2\pi}{3}, \tau = \frac{2\pi}{3}, a = \frac{2\pi}{3}, b = \frac{2\pi}{3}, \beta = \frac{2\pi}{3}, \theta = \frac{2\pi}{3}, \theta_1 = \frac{2\pi}{3}, \theta_2 = \frac{2\pi}{3} \) than that yielding \( \rho_1 \), the two sets of eigenvalues for the equations \( (10) \) and \( (11) \) were quite similar (e. g. listing the largest ones in absolute value, 9.83657 vs. 9.66359, -9.73817 vs. -9.59167,...) but not now strictly equal, and the twenty-eight eigenvalues in each set were all distinct from one another.

We, then, studied a second density matrix (cf. \( [3] \)),

\[
\rho_2 = \begin{pmatrix}
\frac{41}{128} & \frac{151}{128} & -\frac{1}{128} & \frac{1}{128} \\
\frac{1}{128} & \frac{151}{128} & \frac{41}{128} & \frac{1}{128} \\
\frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128} \\
\frac{1}{128} & \frac{1}{128} & \frac{1}{128} & \frac{1}{128}
\end{pmatrix}
\]

having eigenvalues \( \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \text{ and } \frac{1}{8} \). It corresponds to a new point \( q_2 \) for which

\[
\alpha = \frac{\pi}{4}, \quad \tau = \frac{3\pi}{4}, \quad a = \frac{2\pi}{3}, \quad b = \frac{2\pi}{3}, \quad \beta = \frac{\pi}{4}, \quad \theta = \frac{\pi}{4}, \quad \theta_1 = \frac{\pi}{4}, \quad \theta_2 = \frac{\pi}{6}.
\]

Adopting the same analytical strategy as employed for \( \rho_1 \), we obtained for the solutions for \( \lambda_+ \) at \( q_2 \) of the equation \( (14) \), the twenty-eight eigenvalues (cf. \( [11] \)),

\[
\lambda_+(q_2) = 2.68934, -2.60397, \pm 2.14857(\text{eightfold}), -1.69317, 1.6078, \pm 4.98082(\text{eightfold}), 0.426838(\text{eightfold}).
\]

So again, fully analogously to \( (11) \), we have four singlets and three octets. (Tracelessness is again verifiable. Let us also note that for the analysis based on \( \rho_1 \), as well as that on \( \rho_2 \), the four singlet eigenvalues have the largest, second largest, and fourth and fifth largest of the seven possible distinct absolute values.) Computations of the analogues based on \( \rho_2 \) of the exact results \( (12), (13) \) and \( (14) \) proved now to be much more intractable, though. The self-dual four-form here employed was

\[
\Omega_+(q_2) = \frac{448 + 128\sqrt{3} + 27\sqrt{6}}{896} \zeta_{1234} \ldots + \frac{-3 + 224\sqrt{6}}{6} \zeta_{1678} + \zeta_{2345} \ldots + \zeta_{5678},
\]

while the anti-self-dual four-form was

\[
\Omega_-(q_2) = \frac{448 + 128\sqrt{3} - 27\sqrt{6}}{896} \zeta_{1234} \ldots + \frac{-3 - 224\sqrt{6}}{6} \zeta_{1678} + \zeta_{2345} \ldots + \zeta_{5678}.
\]
We found the same set of twenty-eight eigenvalues (21) relying upon \( \Omega_- (q_2) \), using the equation (15), as we did on \( \Omega_+ (q_2) \), using the equation (14).

For several additional density matrices (other than \( \rho_1 \) and \( \rho_2 \)), we have been able to obtain fully analogous results. In all these instances, we let the “second” (lexicographically speaking) set of thirty-five variables (that is, the coefficients of \( \zeta_{2345} \ldots \zeta_{5678} \)) be unity, and solved for the coefficients of the “first” set of thirty-five (\( \zeta_{1234} \ldots \zeta_{1678} \)).

We have also been able to derive univariate generalizations of the (constant) four-forms (22) and (23), allowing either the Euler angle \( \tau \), \( \beta \), \( b \) or \( \theta \) to be free, rather than fixed at particular values. It turns out — as numerical evidence firmly convinced us — that the twenty-eight eigenvalues (21) are completely unchanged as \( \beta \) is varied (even though the coefficients of the extended four-form themselves do explicitly depend on \( \beta \)), but do change with \( \tau \), \( b \) or \( \theta \). For example, for the point \( q_2 \) (20) we have obtained the four-form (22). The coefficient there of \( \zeta_{1234} \) is

\[
 c_{1234} = \frac{448 + 128\sqrt{3} + 27\sqrt{6}}{896} \approx .821249. \tag{24}
\]

If we let the Euler angle \( \tau \) vary (its full range being \([0, \pi]\)) rather than being fixed at \( 3\pi/4 \) (as it is, of course, for \( q_2 \) itself), we obtain instead of (24) the more general expression,

\[
 c_{1234}(\tau) = \frac{28\sqrt{2}(42 - 5\sqrt{3}) \cos 2\tau + 119\sqrt{6} \cos 4\tau - 4480(7 + 2\sqrt{3}) \sin 2\tau + \sqrt{6}(2009 - 10 \sin 4\tau) + 84\sqrt{2} \sin 4\tau}{62720}. \tag{25}
\]

Similarly, letting \( \beta, b \) and \( \theta \) vary in turn we derive,

\[
 c_{1234}(\beta) = \frac{1}{996} (448 + 128\sqrt{3} + 27\sqrt{6}) \sin 2\beta, \tag{26}
\]

\[
 c_{1234}(b) = \frac{1}{2} + \frac{\sqrt{3}(128 + 27\sqrt{2}) \sin 2b}{128(7 + \cos 2b)}, \tag{27}
\]

and

\[
 c_{1234}(\theta) = \cos \theta + \frac{2 \sin \theta}{7} + \frac{9 \cos \theta \sin^3 \theta}{28\sqrt{2}}. \tag{28}
\]

Also,

\[
 c_{1234}(\theta_1) = \frac{1600 + 256(7 + 8\sqrt{3}) \cos 2\theta_1 + 64(11 + 14\sqrt{3}) \cos 4\theta_1 + 3\sqrt{3}(384 + 26 \sin \theta_1 + 35 \sin 3\theta_1 + 25 \sin 5\theta_1)}{128(25 + 28 \cos 2\theta_1 + 11 \cos 4\theta_1)}. \tag{29}
\]

(We have not so far — due to increased computational burdens — been able to similarly analyze the scenarios in which \( \theta_2 \) is free, nor in which pairs of the parameters can simultaneously vary.) In Figures 1 - 5 we plot these five simple well-behaved univariate functions (24)-(29), which generalize the constant coefficient (24) over the ranges of their respective variables.
FIG. 1. Univariate generalization (27), incorporating the Euler angle $\tau$, of the constant coefficient (26) of $\zeta_{1234}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

FIG. 2. Univariate generalization (27), incorporating the Euler angle $\beta$, of the constant coefficient (26) of $\zeta_{1234}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

FIG. 3. Univariate generalization (27), incorporating the Euler angle $b$, of the constant coefficient (26) of $\zeta_{1234}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).
FIG. 4. Univariate generalization (28), incorporating the Euler angle $\theta$, of the constant coefficient (24) of $\zeta_{1234}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

FIG. 5. Univariate generalization (29), incorporating the angle $\theta_1$, of the constant coefficient (24) of $\zeta_{1234}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

The maximum of Fig. 3 is precisely the value (24), that is .821249. (We already noted that the twenty-eight eigenvalues associated with the point $q_2$ are unchanged as $\beta$ varies, in contrast with the other four variables utilized here.) The maxima for the other four figures are $\tau = .821472, b = .83522, \theta = 1.04598, \text{ and } \theta_1 = 1.38035$, all of which exceed .821249. This last value thus falls (as it must) within the ranges of values assumed by the functions above of the form $c_{1234}()$. (Let us also remark that the specific values of the eight parameters employed to derive $\rho_1$ do not all strictly fall within the relatively narrow ranges — devised so as to avoid duplication — designated for them in [20]. We have so far been unable to fully determine the corresponding eight parameters, within these designated ranges, yielding $\rho_1$. The point corresponding to $\rho_2$, on the other hand, does lie within the domain of parameters specified in [20].)

As a further illustration of the apparent simple characteristics of the self-dual four-forms in our analytical context, we give the counterparts of (24)-(29) and Figs. 1-5, but based on the coefficient of $\zeta_{1678}$ rather than of $\zeta_{1234},$

$$c_{1678} = \frac{-3 + 224\sqrt{6}}{6} \approx 90.9476,$$  

$$c_{1678}(\tau) = 112\sqrt{\frac{2}{3}} + \left(\frac{1}{7} + 16\sqrt{2}\right) \cos 2\tau \cos \tau \sin \tau,$$  

$$c_{1678}(\beta) = \cos 2\beta + \frac{1}{6}(-3 + 224\sqrt{6}) \sin 2\beta,$$
\[ c_{1678}(b) = \frac{1}{2} + 56 \sqrt{\frac{2}{3}} \csc b \sec b. \]  
(33)

\[ c_{1678}(\theta) = \cos \theta + 28 \sqrt{2} \csc \theta \sec \theta. \]  
(34)

Also,

\[ c_{1678}(\theta_1) = \frac{1}{2} + \frac{16(13 \sin \theta_1 + \sin 3\theta_1)}{\sqrt{3(1 + 7 \cos 2\theta_1)^2}}. \]  
(35)

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**FIG. 6.** Univariate generalization (31), incorporating the Euler angle \( \tau \), of the constant coefficient \( \kappa \) of \( \zeta_{1678} \) in the self-dual four-form \( \Omega_+(q_2) \) (22), corresponding to the density matrix \( \rho_2 \).

**FIG. 7.** Univariate generalization (32), incorporating the Euler angle \( \beta \), of the constant coefficient \( \kappa \) of \( \zeta_{1678} \) in the self-dual four-form \( \Omega_+(q_2) \) (22), corresponding to the density matrix \( \rho_2 \).
FIG. 8. Univariate generalization (33), incorporating the Euler angle $b$, of the constant coefficient (30) of $\zeta_{1678}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

FIG. 9. Univariate generalization (34), incorporating the Euler angle $\theta$, of the constant coefficient (30) of $\zeta_{1678}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).

FIG. 10. Univariate generalization (35), incorporating the angle $\theta_1$ of the constant coefficient (30) of $\zeta_{1678}$ in the self-dual four-form $\Omega_+(q_2)$ (22), corresponding to the density matrix $\rho_2$ (19).
In all this line of work, we hope to further simplify the existing formulas \[^{20}\] for the elements of the Bures metric over the $3 \times 3$ density matrix and of its inverse, so that the challenging computations of the type reported above might be facilitated and further advanced to include the simultaneous consideration of additional variables.

By way of comparison/contrast of our results, let us note that for the well-known (self-dual) “Cayley calibration” \[^{19,29}\],

\[
\Upsilon = \zeta_{1234} + \zeta_{1258} - \zeta_{1267} + \zeta_{1368} - \zeta_{1456} + \zeta_{1478} 
+ \zeta_{2356} - \zeta_{2378} + \zeta_{2457} + \zeta_{2468} - \zeta_{3458} + \zeta_{3467} + \zeta_{5678},
\]

which is invariant under a Spin$_7$ subgroup of SO$_8$, the $\mathbf{28}$ or adjoint representation $\mathfrak{so}_8$ of SO$_8$ breaks up as

\[
\mathbf{28} \rightarrow \mathbf{7} \oplus \mathbf{21},
\]

where the $\mathbf{21}$ corresponds to the adjoint representation spin$_7$ $\in \mathfrak{so}_8$. The endomorphism $\hat{\Upsilon}$ of the space of two-forms $(F)$ obeys the characteristic polynomial,

\[
(\hat{\Upsilon} - I)(\hat{\Upsilon} + 3I) = 0.
\]

The eigenvalues are, therefore, 1 and -3, and (consistent with tracelessness) have multiplicities 21 and 7, respectively.

“Therefore there are two possible extensions of self-duality, and hence two possible extensions of the notion of instanton to eight dimensions” \[^{19}\]. (The analogous characteristic polynomials of a number of other possible generalized self-dualities — complex, special lagrangian, complex lagrangian, quaternionic, and sub-quaternionic — in eight-dimensional Euclidean space are also given in \[^{19}, \text{sec. 6}.\] Of course, it would be desirable to develop explanations comparable to these of the results we have presented above.

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