MODIFIED KÄHLER-RICCI FLOW ON PROJECTIVE BUNDLES

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ABSTRACT. On a compact Kähler manifold, a Kähler metric \( \omega \) is called Generalized Quasi-Einstein (GQE) if it satisfies the equation \( \text{Ric}(\omega) - H \text{Ric}(\omega) = L_X \omega \) for some holomorphic vector field \( X \), where \( H \text{Ric}(\omega) \) denotes the harmonic representative of the Ricci form \( \text{Ric}(\omega) \). GQE metrics are one of the self-similar solutions of the modified Kähler-Ricci flow: \( \frac{d\omega}{dt} = -\text{Ric}(\omega) + H \text{Ric}(\omega) \). In this paper, we propose a method of studying the modified Kähler-Ricci flow on special projective bundles, called admissible bundles, from the viewpoint of symplectic geometry. As a result, we can reduce the modified Kähler-Ricci flow to a simple PDE for a time-dependent function on the interval \([-1, 1]\). Moreover, we show that the solution of this evolution equation converges uniformly to the function corresponding to a GQE metric in exponential order under some assumptions.

1. INTRODUCTION

In Kähler geometry, Kähler-Einstein metrics are closely related to various types of stabilities, which have been studied by many experts. In order to find Kähler-Einstein metrics, Tian and Zhu [TZ07] studied the following Kähler-Ricci flow on an \( m \)-dimensional Fano manifold \( M \):

\[
\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \omega_t,
\]

where \( \omega_t \) is a \( t \)-dependent Kähler form and \( \text{Ric}(\omega_t) \) is its Ricci form, which are given by

\[
\begin{align*}
g_{ij} &= g \left( \frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j} \right) \\
\omega &= \sqrt{-1} \sum_{i,j} g_{ij} dw^i \wedge dw^j
\end{align*}
\]

and

\[
\begin{align*}
r_{ij} &= -\partial_i \partial_j \log(\det(g_{kl})) \\
\text{Ric}(\omega) &= \sqrt{-1} \sum_{i,j} r_{ij} dw^i \wedge dw^j
\end{align*}
\]

in local holomorphic coordinates \((w^1, \cdots, w^m)\). We assume that the initial metric \( \omega_0 \) is in \( 2\pi c_1(M) \). Then, we have \( \omega_t \in 2\pi c_1(M) \) under the evolution equation (1.1).

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A Kähler metric $g$ is called a Kähler-Ricci soliton if its Kähler form $\omega \in 2\pi c_1(M)$ satisfies the equation
\[ \text{Ric}(\omega) - \omega = L_X\omega, \]
where $L_X$ denotes the Lie derivative with respect to a holomorphic vector field $X$ on $M$. As usual, we denote a Kähler-Ricci soliton by a pair $(\omega, X)$. If $X = 0$, this is just a Kähler-Einstein metric. Kähler-Ricci solitons are one of the self-similar solutions of Kähler-Ricci flow. Actually, if we put $\omega_t = (\exp(-\text{Re}(X) \cdot t))^* \omega_0$ for any Kähler-Ricci soliton $(\omega_0, X)$, then $\omega_t$ satisfies the evolution equation (1.1). Tian and Zhu proved that if $M$ admits a Kähler-Ricci soliton $(\omega, X)$ and the initial Kähler metric is invariant under the action of the one-parameter subgroup generated by $\text{Im}(X)$, any solution of Kähler-Ricci flow (1.1) will converge to the Kähler-Ricci soliton $(\omega, X)$ in the sense of Cheeger-Gromov.

Tian and Zhu [TZ02] also defined a new holomorphic invariant, which is an obstruction to the existence of Kähler-Ricci solitons just as the Futaki invariant [F83] is an obstruction to the existence of Kähler-Einstein metrics. They also constructed the modified version of K-energy, which is a functional defined over the space of Kähler metrics, and Kähler-Ricci solitons are critical points of this functional. It is suggested that these play an important role in Geometric invariant theory.

For any polarized manifold, we can give a straightforward extension of Kähler-Ricci solitons. Let $M$ be a compact Kähler manifold and $\Omega$ a Kähler class on $M$. A Kähler metric $g$ is called a Generalized Quasi-Einstein (GQE) Kähler metric if its Kähler form $\omega \in \Omega$ satisfies the equation
\[ \text{Ric}(\omega) - \mathbb{H}\text{Ric}(\omega) = L_X\omega, \]
where $\mathbb{H}\text{Ric}(\omega)$ is the harmonic representative of the Ricci form $\text{Ric}(\omega)$ and $X$ is a holomorphic vector field on $M$. If $X = 0$, this is just a constant scalar curvature (CSC) Kähler metric. Examples of GQE metrics were calculated in [G95] and [MT11], however, the relations between the existence of GQE metrics and geometric stabilities are not found. We want to relate the existence problem of GQE metrics to some stabilities. For this, we will consider an analogue of Tian-Zhu’s convergence Theorem for Kähler-Ricci flow in general polarizations. Guan [G07] introduced the following modified Kähler-Ricci flow:
\[ \frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \mathbb{H}\text{Ric}(\omega_t), \]
which generalizes (1.1) to any polarizations, and GQE metrics are one of the self-similar solutions of (1.2). It is expected that if we assume that $M$ admits a GQE metric $(\omega, X)$ and the initial Kähler metric is invariant under the action of the one-parameter subgroup generated by $\text{Im}(X)$, the long time solution of (1.2) exists and will converge to the GQE metric $(\omega, X)$ in the sense of Cheeger-Gromov.
In this paper, we study the evolution equation (1.2) in a special case. Concretely, we study (1.2) on an admissible bundle [ACGT08], which is the total space of fiberwise projectivization of the direct sum of two projectively-flat holomorphic vector bundles over a compact Kähler manifold which has the universal covering written as a product of Kähler manifolds with CSC Kähler metrics. If we assume that $\Omega$ is an “admissible Kähler class” whose corresponding polynomial $P(t)$ has exactly one root in the interval $(-1, 1)$ and the initial Kähler metric is an “admissible Kähler metric” in $\Omega$, we can reduce (1.2) to the evolution equation

$$2\Theta_{\infty} \frac{d\varphi_t}{dt} = \Theta_{\infty} \Theta_t \varphi_t'' - (\Theta_{\infty} \varphi_t')^2 + \frac{P}{p_c} \cdot \Theta_{\infty} \varphi_t' + \left(\Theta_{\infty} \left(\frac{P'}{p_c}\right) - \left(\frac{P}{p_c}\right) \Theta'_{\infty}\right) (1 + \varphi_t) \varphi_t$$

(1.3)

for a $t$-dependent smooth function $\varphi_t$ on the interval $[-1, 1]$, where several words (admissible Kähler metric, admissible Kähler class, etc.) and notations ($P, p_c, etc.$) are defined in Sect. 3 and later. The important thing is that the evolution equation (1.3) is basically a heat equation which has one space variable. Applying the maximum principle to (1.3), we can show that

**Main theorem** (Theorem 5.11). Let $M$ be an $m = \sum_{a \in A} d_a + 1$-dimensional admissible bundle and $\Omega$ an admissible class on $M$ with the admissible data $\{x_a\}$. We assume that $P(t)$ has exactly one root in the interval $(-1, 1)$. Then, for any symplectic form defined by (3.3), the modified Kähler-Ricci flow (1.2) can be reduced to the evolution equation (1.3) for $\varphi_t$. Moreover, if $|x_a|$ is sufficiently small for all $a \in A$, the solution $\varphi_t$ of (1.3) converges uniformly to 0 in exponential order.

In Sect. 2, we review the modified Kähler-Ricci flow in [G07] and give a few remarks. In Sect. 3, we review the fundamental materials about admissible bundles [ACGT08] and define some notations that we will use in Sect. 4 and Sect. 5. In Sect. 4, we relate Maschler-Tønnesen’s invariant [MT11] to Tian-Zhu’s invariant [TZ02] on admissible bundles. Lastly, in Sect. 5, we propose a method of studying the modified Kähler-Ricci flow on admissible bundles via the $U(1)$-equivariant fiber-preserving diffeomorphisms. By this, we can reduce the equation of the modified Kähler-Ricci flow to the “symplectic version of the modified Kähler-Ricci flow” defined in the moduli space of Kähler metrics. Then, we give examples of Kähler classes which have good properties to prove the convergence of the solution.

**2. Modified Kähler-Ricci flow**

Let $M$ be an $m$-dimensional compact Kähler manifold and $\Omega$ a Kähler class on $M$. We consider the following evolution equation:

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \mathbb{H}\text{Ric}(\omega_t),$$

(2.1)
where $\omega_t = \sqrt{-1} g_t \, dz^i \wedge dz^\bar{j} \in \Omega$ is a $t$-dependent Kähler form and $\mathbb{H} \text{Ric}(\omega_t) = \sqrt{-1} \gamma_t \, dz^i \wedge dz^\bar{j} \in 2\pi c_1(M)$ is the harmonic representative of the Ricci form $\text{Ric}(\omega_t) = -\sqrt{-1} \partial \bar{\partial} \log \det g_t = \sqrt{-1} r_{t \bar{i} j} dz^i \wedge dz^\bar{j} \in 2\pi c_1(M)$. This is called the “modified Kähler-Ricci flow”, which was first introduced in [G07, §11]. Because $\frac{\partial [\omega_t]}{\partial t} = -2\pi c_1(M) + 2\pi c_1(M) = 0$, it is clear that if the initial metric $\omega_0$ is in $\Omega$, then $\omega_t \in \Omega$ for all $t$. Thus, the cohomology class of the initial metric is preserved under (2.1). If a long time solution of (2.1) exists and converges to some Kähler metric, it must be a constant scalar curvature (CSC) Kähler metric.

However, it is difficult to estimate the behavior of the potential function of $\mathbb{H} \text{Ric}(\omega_t)$ for general polarizations, so we will study the contraction typed flow instead of (2.1):

$$
\frac{\partial}{\partial t} \log g_t = -\text{Scal}(g_t) + \overline{\text{Scal}},
$$

where $\text{Scal}(g_t) = r_{t \bar{i} j}^1$ is the scalar curvature of the Kähler metric $g_t$ and $\overline{\text{Scal}} = \gamma_{t \bar{i} j}^1 = 2\pi m c_1(M) \Omega^{m-1}$ is a topological invariant.

**Remark 2.1.** We call $\log g_t$ for a “local Ricci potential”, which is defined in each local coordinate neighborhood. Let $(w^1, \cdots, w^m)$ and $(\bar{w}^1, \cdots, \bar{w}^m)$ be $t$-independent local holomorphic coordinate systems. We define the transition map $h = (h^1, \cdots, h^m)$ by $w^i = h^i(\bar{w}^1, \cdots, \bar{w}^m)$ for $i = 1, \cdots, m$. If we put $g_{i \bar{j}} = g \left( \frac{\partial}{\partial w^i}, \frac{\partial}{\partial w^j} \right)$ and $\bar{g}_{i \bar{j}} = g \left( \frac{\partial}{\partial \bar{w}^i}, \frac{\partial}{\partial \bar{w}^j} \right)$, then we have

$$
\det (\bar{g}_{i \bar{j}}) = \left| \det \left( \frac{\partial h^i}{\partial \bar{w}^j} \right) \right|^2 \det (g_{i \bar{j}}).
$$

So, local Ricci potentials differ by a $t$-independent function if we change coordinate systems. Thus, $\frac{\partial}{\partial t} \log g_t$ is a function defined over $M$ as long as we treat it in a $t$-independent local coordinate neighborhood.

The evolution equation (2.2) is equivalent to (2.1). We can check it as follows: Let $\omega_t$ be the solution of (2.2), then there exists a $t$-dependent smooth function $f_t$ such that $-\text{Ric}(\omega_t) + \mathbb{H} \text{Ric}(\omega_t) = \sqrt{-1} \partial \bar{\partial} f_t$. After taking trace and using the assumption, we get $\Delta_f = -\frac{\partial}{\partial t} \log g_t$. On the other hand, if we set $g_{i \bar{j}} = g_{0 i \bar{j}} + u_{i \bar{j}}$ for some smooth function $u_t$, we have $\Delta_u f = -g^{i \bar{j}} \frac{\partial u}{\partial t} = \Delta_u \frac{\partial}{\partial t}$. By the maximum principle, we have $f = \frac{\partial}{\partial t}$ modulo some $t$-dependent constant. Hence, we have $\frac{\partial g_{i \bar{j}}}{\partial t} = -r_{i \bar{j}} + \gamma_{i \bar{j}}$ and this means that $g_t$ is the solution of (2.1).

**Definition 2.2.** A pair $(\omega, X)$ of a Kähler form $\omega \in \Omega$ and a holomorphic vector field $X$ is called a “Generalized Quasi-Einstein (GQE) Kähler metric” if it satisfies the equation

$$
\text{Ric}(\omega) - \mathbb{H} \text{Ric}(\omega) = L_X \omega,
$$

where $L_X$ denotes the Lie derivative with respect to $X$. 

If there exists a GQE metric with respect to a holomorphic vector field $X \neq 0$, a long time solution of (2.1) does not converge. Actually, for any GQE metric $(\omega_0, X)$, $\omega_t := (\exp (-\text{Re}(X) \cdot t))^* \omega_0$ is a solution of (2.1) and does not converge. In this case, we should add the term $L_X \omega_t$ to the right hand side of (2.1) and consider the evolution equation:

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \text{H}\text{Ric}(\omega_t) + L_X \omega_t. \tag{2.4}$$

If a long time solution of (2.4) exists and converges to some Kähler metric, it must be a GQE metric with respect to $X$. Generally, it is known that the evolution equation (2.4) has the unique short time solution [G07, §11].

3. Admissible bundles

In this section, we recall special projective bundles called “admissible bundles” [ACGT08, §1].

**Definition 3.1.** A projective bundle of the form $M = \mathbb{P}(E_0 \oplus E_{\infty}) \to S$ is called an “admissible bundle” if it satisfies the following conditions:

1. $S$ has the universal covering $\tilde{S} = \prod_{a \in A} S_a$ (for $A \subset \mathbb{N}$) of simply connected Kähler manifolds $(S_a, \pm g_a, \pm \omega_a)$ of complex dimensions $d_a$ with $(g_a, \omega_a)$ being pullbacks of tensors on $S$; here, “$\pm$” means that either $+\omega_a$ or $-\omega_a$ is a Kähler form which defines a Kähler metric denoted by $+g_a$ or $-g_a$ respectively.
2. $E_0$ and $E_{\infty}$ are holomorphic projectively-flat Hermitian vector bundles over $S$ of rank $d_0 + 1$ and $d_{\infty} + 1$ with $c_1(E_{\infty})/\text{rank} E_{\infty} - c_1(E_0)/\text{rank} E_0 = [\omega_S/2\pi]$ and $\omega_S = \sum_{a \in A} \omega_a$.

Let $M$ be an admissible bundle. We define several notations and give some remarks that we will use later:

- we set the index set $\hat{A} := \{ a \in \mathbb{N} \cup \{0, \infty\} | d_a > 0 \}$.
- $e_0 = \mathbb{P}(E_0 \oplus 0)$ (resp. $e_{\infty} = \mathbb{P}(0 \oplus E_{\infty})$) denotes a subbundle of $M$. Then, $e_0$ and $e_{\infty}$ are disjoint submanifolds of $M$.
- $\mathbb{P}(E_0) \to S$ (resp. $\mathbb{P}(E_{\infty}) \to S$) is equipped with the fiberwise Fubini-Study metric with the scalar curvature $d_0(d_0+1)$ (resp. $d_{\infty}(d_{\infty}+1)$), which is denoted by $(g_0, \omega_0)$ (resp. $(-g_{\infty}, -\omega_{\infty})$).
- Let $\hat{M}$ be the blow-up of $M$ along the set $e_0 \cup e_{\infty}$, and set $\hat{S} = \mathbb{P}(E_0) \times_S \mathbb{P}(E_{\infty}) \to S$. Then, $\hat{M} \to \hat{S}$ has a $\mathbb{P}^1$-bundle structure.
- We define a $U(1)$-action on $M$ by the canonical $U(1)$-action on $E_0$. Then, the Hermitian structures of $E_0$ and $E_{\infty}$ induce the fiberwise moment map $z : M \to [-1, 1]$ of this $U(1)$-action with critical sets $z^{-1}(1) = e_0$ and $z^{-1}(-1) = e_{\infty}$.
- $K$ denotes the infinitesimal generator of the $U(1)$-action on $M$.
- $e_0$ (resp. $e_{\infty}$) denotes the exceptional divisor corresponding to the submanifold $e_0$ (resp. $e_{\infty}$), and set $M^0 = M \setminus (e_0 \cup e_{\infty})$. Then, $M^0 \to \hat{S}$ has a $\mathbb{C}^*$-bundle structure.
If we regard $M^0$ as an open set of $\hat{M}$, the restriction of the canonical $U(1)$-action on $\hat{M}$ to $M^0$ coincides with the induced $U(1)$-action from $M$.

**Definition 3.2.** A Kähler class $\Omega$ on $M$ is called “admissible” if there are constants $x_a$, with $x_0 = 1$ and $x_\infty = -1$, such that the pullback of $\Omega$ to $\hat{M}$ has the form

$$\Omega = \sum_{a \in \hat{A}} \frac{[\omega_a]}{x_a} + \hat{\Xi},$$

where $\hat{\Xi}$ is the Poincaré dual to $2\pi [\hat{e}_0 + \hat{e}_\infty]$.

We can see that any admissible class $\Omega$ has the form

$$\Omega = \sum_{a \in A} \frac{[\omega_a]}{x_a} + \Xi,$$

where the pullback of $\Xi$ to $\hat{M}$ is $[\omega_0] - [\omega_\infty] + \hat{\Xi}$, i.e., the cohomology class $[\omega_0] - [\omega_\infty] + \hat{\Xi}$ vanishes along the fiber of $\hat{e}_0 \to e_0$ and $\hat{e}_\infty \to e_\infty$. Since $\Omega$ is Kähler, one can also see that $0 < |x_a| < 1$ for all $a \in A$ and $x_a$ has the same sign as $g_a$. Since the blow-up $\hat{M} \to M$ induces an injective map on cohomology, admissible classes are uniquely determined by the parameters $\{x_a\}$. We call this for the “admissible data” of $\Omega$.

In this paper, we also assume that $(\pm g_a, \pm \omega_a)$ has constant scalar curvature $\text{Scal}_{\pm g_a}(\pm \omega_a) = \pm d_a s_a x_a$, where $s_a$ are constants defined in [ACGT08, §1.2].

**Definition 3.3.** Let $\Omega$ be an admissible class with the admissible data $\{x_a\}$. An “admissible Kähler metric” $g$ is the Kähler metric on $M$ which has the form

$$g = \sum_{a \in \hat{A}} \frac{1 + x_a z}{x_a} g_a + \frac{dz^2}{\Theta(z)} + \Theta(z) \theta^2, \quad \omega = \sum_{a \in \hat{A}} \frac{1 + x_a z}{x_a} \omega_a + dz \wedge \theta$$

on $M^0$, where, $\theta$ is the connection 1-form ($\theta(K) = 1$) with the curvature $d\theta = \sum_{a \in \hat{A}} \omega_a$, and $\Theta$ is a smooth function on $[-1, 1]$ satisfying

$$\Theta > 0 \text{ on } (-1, 1), \quad \Theta(\pm 1) = 0 \quad \text{and} \quad \Theta'(\pm 1) = \mp 2.$$

The form $\omega$ defined by (3.3) is a symplectic form, and the compatible complex structure $J$ of $(g, \omega)$ is given by the pullback of the base complex structure and the relation $J dz = \Theta \theta$.

**Remark 3.4.** Using the relation $d\theta = \sum_{a \in \hat{A}} \omega_a$, we can check that $\omega$ is closed and $\Omega = [\omega]$. So, $g$ is a Kähler metric whose Kähler form $\omega$ belongs to $\Omega$.

**Remark 3.5.** The definitional equation (3.3) is motivated by the representation of the canonical admissible metric $g_c$ in polar coordinates. In this case, the corresponding function $\Theta_c$ is given by $\Theta_c(z) = 1 - z^2$, where $g_c$ and $\Theta_c$ will be defined later in Sect. 3.
Here, the condition (3.4) is the necessary and sufficient condition to extend a metric \( g \) on \( M^0 \) which has the form (3.3) to a smooth metric defined on \( M \). We also use the function \( F(z) = \Theta(z) \cdot p_c(z) \), where \( p_c(z) = \prod_{a \in \hat{A}} (1 + x_a z)^{d_a} \) is a polynomial of \( z \). Then, from (3.4), \( F \) has the condition

\[
(3.5) \quad F > 0 \text{ on } (-1, 1), \quad F(\pm 1) = 0 \quad \text{and} \quad F' (\pm 1) = \mp 2p_c (\pm 1).
\]

This is an only necessary condition for \( F \), i.e. we can not restore \( \Theta \) from \( F \) satisfying (3.5) in general. However, it is possible if \( g \) is extremal or GQE (cf. [ACGT08, §2.4] and [MT11, §4]).

Conversely, for any cohomology class \( \Omega \) defined by (3.2), we can show that \( \Omega \) is Kähler if we assume \( 0 < |x_a| < 1 \) and \( x_a \) has the same sign as \( g_a \), and hence \( \Omega \) is admissible. We can prove this by constructing the “canonical admissible metric” \( g_c \) and the “canonical symplectic form” \( \omega_c \) belonging to \( \Omega \): Let \( r_0 \) and \( r_\infty \) be the norm functions induced by the Hermitian metrics on \( E_0 \) and \( E_\infty \). Then \( z_0 = \frac{1}{2} r_0^2 \) and \( z_\infty = \frac{1}{2} r_\infty^2 \) are fiberwise moment map for the \( U(1) \)-actions given by the scalar multiplication in \( E_0 \) and \( E_\infty \). Let us consider the diagonal \( U(1) \)-action on \( E_0 \oplus E_\infty \). Since \( U(1) \) acts freely on the level set \( z_0 + z_\infty = 2 \), the restricted metric on this level set descends to the fiberwise Fubini-Study metric on the quotient manifold \( M \), which we denote by \( (g_{M/S}, \omega_{M/S}) \). We extend \((g_{M/S}, \omega_{M/S})\) to a tensor on \( M \) by requiring that the horizontal distribution of the induced connection on \( M \) is degenerate. Hence, \((g_{M/S}, \omega_{M/S})\) is semi-positive. In order to get a (positive definite) metric on \( M \), we set

\[
g_c = \sum_{a \in A} \frac{1 + x_a z}{x_a} g_a + g_{M/S}, \quad \omega_c = \sum_{a \in A} \frac{1 + x_a z}{x_a} \omega_a + \omega_{M/S}.
\]

Then \((g_c, \omega_c)\) is a Kähler metric with respect to the canonical complex structure \( J_c \) on \( M \). We can see that this metric is admissible and the corresponding function \( \Theta_c \) is given by \( \Theta_c(z) = 1 - z^2 \) (cf. [ACGT08, Lemma 1]). We call this for the canonical admissible Kähler metric.

**Remark 3.6.** In the original paper [ACGT08, §1.3, §1.4], admissible classes and admissible metrics are defined by (3.2) and (3.3) "up to scale" respectively because several conditions for metrics (extremal, GQE, etc.) are preserved under scaling of metrics. However, in this paper, the argument of scaling metrics sometimes becomes essential. This is why we define them not up to scale.

Lastly, we will mention symplectic potentials [ACGT08, §1.4]. As is seen above, admissible metrics with a fixed symplectic form \( \omega \) define different complex structures. However, we can regard them as the same complex structure \( J \) via \( U(1) \)-equivariant fiber-preserving diffeomorphisms: a function \( u \in C^0([-1, 1]) \) is called a “symplectic potential” if \( u''(z) = 1/\Theta(z) \),
\[ u(\pm 1) = 0 \text{ and } u - u_c \text{ is smooth on } [-1, 1], \text{ where } u_c \text{ is the canonical symplectic potential defined by} \]

\[ u_c(z) = \frac{1}{2} \{(1 - z) \log(1 - z) + (1 + z) \log(1 + z) - 2 \log 2\}. \]

By de l’Hôpital’s rule, we can see that there is a one to one correspondence between \( u \) and \( \Theta \) satisfying (3.4) (cf. [ACGT08, Lemma 2]). We can write a Kähler potential of \( \omega \) by means of the symplectic potential \( u \) and its fiberwise Legendre transform over \( \hat{S} \). Actually, if we put

\[ (3.6) \quad y = u'(z) \quad \text{and} \quad h(y) = -u(z) + yz, \]

then we obtain \( dd^c_j y = \theta \) and \( dd^c_j h(y) = \omega - \sum_{a \in A} \omega_a / x_a \) on \( M^0 \). There are local 1-forms \( \alpha \) on \( \hat{S} \) such that \( \theta = dt + \alpha \), where \( t : M^0 \to \mathbb{R} / 2\pi \mathbb{Z} \) is an angle function locally defined up to an additive constant. Let \( y_c \) and \( h_c \) be the functions corresponding to \( u_c \). Since \( \exp (y + \sqrt{-1}t) \) and \( \exp (y_c + \sqrt{-1}t) \) give \( \mathbb{C}^* \) coordinates on the fibers, there exists \( U(1) \)-equivariant fiber-preserving diffeomorphism \( \Psi \) of \( M^0 \) such that

\[ (3.7) \quad \Psi^* y = y_c, \quad \Psi^* t = t \quad \text{and} \quad \Psi^* J = J_c. \]

As \( J_c \) and \( J \) are integrable complex structures, \( \Psi \) extends to a \( U(1) \)-equivariant diffeomorphism of \( M \) leaving fixed any point on \( e_0 \cup e_\infty \). Hence \( \Psi^* \omega \) is a Kähler form on \( M \) with respect to \( J_c \). As \( \Psi : (M, J_c) \to (M, J) \) is biholomorphic, we have \( dd^c_J h(y_c) = dd^c_{J_c} h(\Psi^* y) = \Psi^* dd^c_j h(y) = \Psi^* \omega - \sum_{a \in \hat{A}} \omega_a / x_a \)
and \( \Psi^* \omega - \omega = dd^c_{J_c} (h(y_c) - h_c(y_c)) \) on \( M^0 \), where we remark that the function \( h(y_c) - h_c(y_c) \) is extended smoothly on \( M \) [ACGT08, Lemma 3]. Let \( K^{adm}_\omega \) be the moduli space of admissible metrics with a fixed symplectic form \( \omega \). From the above, we have \( K^{adm}_\omega = \{ \Theta \text{ satisfying (3.4)} \} = \{ \text{symplectic potential } u \} \).

4. GQE METRICS ON ADMISSIBLE BUNDLES

Let \( M \) be an \( m \)-dimensional compact Kähler manifold and \( \Omega \) a Kähler class on \( M \). Let \( g \) be a Kähler metric whose Kähler form \( \omega \) belongs to \( \Omega \). For any holomorphic vector field \( V \), we define a complex valued smooth function \( \theta_V \) on \( M \) by

\[ (4.1) \quad i_V \omega = (\text{harmonic (0,1)-form}) + \sqrt{-1} \partial \bar{\partial} \theta_V. \]

We call the function \( \theta_V \) is a “Killing potential” if \( \text{Im}(V) \) is a Killing vector field with respect to \( g \), where \( i_V \) means the inner product with respect to \( V \). The function \( \theta_V \) uniquely exists up to an additive constant. And, we define a real valued smooth function \( \kappa \) on \( M \) by

\[ (4.2) \quad \text{Ric}(\omega) - \mathbb{H} \text{Ric}(\omega) = \sqrt{-1} \partial \bar{\partial} \kappa. \]

The function \( \kappa \) is called the “Ricci potential”. Then we have

**Lemma 4.1.** A Kähler metric \( g \) is a GQE metric with respect to a holomorphic vector field \( X \) if and only if its Ricci potential \( \kappa \) satisfies the equation \( \kappa = \theta_X \) up to an additive constant.
Proof. Applying $d$ to the both hand sides of (4.1), we get $L_V \omega = \sqrt{-1} \partial \bar{\partial} \theta_V$. Combining this with (2.3) and (4.2), and using the maximum principle, we have the desired result.

Taking the trace of the both hand sides of (4.2), we have

(4.3) \[ \text{Scal}_g(\omega) - \overline{\text{Scal}} = -\Delta_\partial \kappa. \]

Now, we will consider the case when $\Omega = 2\pi c_1(M)$ for a moment. Since $\mathbb{H}\text{Ric} = \omega$, (2.3) becomes

(4.4) \[ \text{Ric}(\omega) - \omega = L_X \omega, \]

and we call the solutions of (4.4) for “Kähler-Ricci solitons”. Applying $L_V$ to the both hand sides of (4.2), we have

(4.5) \[ -\Delta_\partial \theta_V + \theta_V + V(\kappa) = (\text{const}). \]

The following function is known as the obstruction to the existence of Kähler-Ricci solitons:

**Theorem 4.2** ([TZ02]). The function $\text{TZ}_X$ defined over the space of all holomorphic vector fields on $M$ by

(4.6) \[ \text{TZ}_X(V) = \int \theta_V e^{\theta_X} \frac{\omega^m}{m!} = -\int V(\kappa - \theta_X) e^{\theta_X} \frac{\omega^m}{m!} \]

for a holomorphic vector field $X$ is independent of the choice of a Kähler form $\omega \in 2\pi c_1(M)$, here, for any $V$, $\theta_V$ is normalized by $-\Delta_\partial \theta_V + \theta_V + V(\kappa) = 0$.

In the proof of Theorem 4.2, (4.5) is the key equation. Actually, the Euler-Lagrange equation of (4.6) is given by (4.5). However, the equation (4.5) does not hold in general polarizations, which causes a lot of problems.

Now, let $M$ be an $m := \sum_{a \in \hat{A}} d_a + 1$-dimensional admissible bundle and $\Omega$ an admissible class on $M$. First, we will review the method of constructing GQE metrics over admissible bundles studied by Maschler and Tønnesen-Friedman [MT11]. Let $C^\infty([-1,1])$ be the space of smooth functions over the interval $[-1,1]$. According to Lemma 2.1 in [KS86], there is a one to one correspondence

$C^\infty([-1,1]) \rightarrow \{ \text{smooth function over } M \text{ depending only on } z \}$

given by $S \mapsto S(z) := S \circ z$ for $S \in C^\infty([-1,1])$.

**Lemma 4.3** (Proposition 3.1 in [MT11]). For any admissible metric and $S \in C^\infty([-1,1])$, we have

(4.7) \[ \Delta_\partial S = -\frac{[S'(z) \cdot F(z)]'}{2p_c(z)}. \]
According to [ACGT08, §2.2], we can calculate the scalar curvature of any admissible metric $g$ as

\begin{equation}
\text{Scal}_g(\omega) = \frac{1}{2} \left( \sum_{a \in \mathcal{A}} \frac{2d_a s_a x_a}{1 + x_a z} - \frac{F''(z)}{p_c(z)} \right). 
\end{equation}

By (4.3) and (4.8), $\Delta_\partial \kappa$ is a function depending only on $z$. Hence, Corollary 3.2 in [MT11] implies $\kappa$ depends only on $z$. We can write $\kappa$ as the composition of $z$ and an element of $C^\infty([-1, 1])$, which we also denote by $\kappa$. On the other hand, Theorem 4.4 in [K95] implies that a Kähler metric $g$ is GQE if and only if its Ricci potential $\kappa$ is a Killing potential. So, we have

**Lemma 4.4.** An admissible metric $g$ is GQE if and only if there exists $k \in \mathbb{R}$ such that $\kappa = k z$ up to an additive constant.

Put

\begin{equation}
P(t) = 2 \int_{-1}^{t} \left( \sum_{a \in \mathcal{A}} \frac{d_a s_a x_a}{1 + x_a s} \cdot p_c(s) - \frac{\beta_0}{\alpha_0} \cdot p_c(s) \right) ds + 2p_c(-1),
\end{equation}

where $\alpha_0$ and $\beta_0$ are constants defined by

\begin{equation}
\alpha_0 = \int_{-1}^{1} p_c(t) dt \quad \text{and} \quad \beta_0 = p_c(1) + p_c(-1) + \int_{-1}^{1} \left( \sum_{a \in \mathcal{A}} \frac{d_a s_a x_a}{1 + x_a t} \right) p_c(t) dt.
\end{equation}

We often use the following properties for $P(t)$:

**Lemma 4.5 (Lemma 4.3 in [MT11]).** For any given admissible data, $P(t)$ satisfies: If $d_0 = 0$, then $P(-1) > 0$, otherwise $P(-1) = 0$. If $d_\infty = 0$, then $P(1) < 0$, otherwise $P(1) = 0$. Furthermore, $P(t) > 0$ in some (deleted) right neighborhood of $t = -1$, and $P(t) < 0$ in some (deleted) left neighborhood of $t = 1$. Concretely, we see that if $d_0 > 0$, then $P^{(d_0)}(-1) > 0$ (and the lower order derivatives vanish), while if $d_\infty > 0$, then $P^{(d_\infty)}(1)$ has sign $(-1)^{d_\infty+1}$ (and the lower order derivatives vanish).

Combining (4.3), (4.7), (4.8), $\kappa = k z$ and $\text{Scal} = \beta_0 / \alpha_0$ (cf. [ACGT08, §2.2]), we have

**Lemma 4.6.** For any admissible GQE metric with the Ricci potential $k z$, the equation

\begin{equation}
F''(z) + k F'(z) = P'(z)
\end{equation}

holds.

We can give the explicit solution for (4.11) by

\begin{equation}
F(z) = e^{-k z} \int_{-1}^{z} P(t) e^{k t} dt
\end{equation}
under the boundary condition $F(-1) = 0$ and $F'(±1) = ±2p_c(±1)$. In order to get a GQE metric defined over $M$, $F$ must satisfy $F(1) = 0$, so,

\[(4.13) \quad \text{MT}(k) := \int_{-1}^{1} P(t)e^{kt}dt\]

is an obstruction to the existence of admissible GQE metrics with the Ricci potential $kz$.

**Remark 4.7.** Clearly, $α_0$, $β_0$, $P(t)$ and $\text{MT}(k)$ are independent of the choice of admissible metrics $(g, ω)$. These quantities depend only on $M$ and the admissible class $Ω$.

**Lemma 4.8.** For any admissible metric, the equation

\[(4.14) \quad F'(z) + κ'(z) \cdot F(z) = P(z)\]

holds.

**Proof.** By (4.3), (4.7) and (4.8), we have

\[(4.15) \quad \frac{1}{2} \left( \sum_{a \in \hat{A}} \frac{2d_as_ax_a}{1 + x_az} - \frac{F''(z)}{p_c(z)} \right) - \frac{β_0}{α_0} = \frac{[κ'(z) \cdot F(z)]'}{2p_c(z)}.\]

Multiplying $2p_c(z)$ to the both hand sides of (4.15) and integrating on $[-1, z]$, we get

\[F'(z) + κ'(z) \cdot F(z) = P(z) + (\text{const}).\]

Since $F'(z)$ and $P(z)$ have the same boundary condition, we have $(\text{const}) = 0$. □

For any $k \in \mathbb{R}$, let $X^k_j$ be a holomorphic vector field with the potential function $kz$, i.e. $X^k_j$ satisfies $i_{X^k_j}ω = \sqrt{-1}∂Jkz$, where $J$ is the compatible complex structure induced by an admissible metric. Since $K$ is the infinitesimal generator of the $U(1)$-action on $M$ and the function $z$ is the moment map of this action, we get $i_Kω = -dz$. Hence, $X^2_j = -JK + \sqrt{-1}K$ and $X^0_j = \frac{k}{2} \cdot X^2_j = -\frac{k}{2}(JK + \sqrt{-1}K)$.

**Theorem 4.9.** Let $M$ be an admissible bundle and $Ω$ an admissible class with the admissible data $\{x_a\}$. We assume that $Ω$ coincides with $2πc_1(M)$ up to a multiple positive constant. (Hence, $M$ is Fano and $c_1(M)$ becomes admissible up to scale automatically.) Then, the following statements hold:

1. If we set $Ω = 2πλ^{-1}c_1(M)$ for some (positive) constant $λ$, then we have $λ = d_0 + d_∞ + \frac{2}{2}$.

2. Tian-Zhu’s holomorphic invariant (4.6) and Maschler-Tønnesen’s invariant (4.13) have a relation

\[(4.16) \quad \text{TZ}_{λ^{-1}X^k_j}(X^2_j) = -2πλ^n \exp \left( -\frac{kC}{2λ} \right) \text{Vol} \left( S, \prod_{a \in \hat{A}} \frac{ω_a}{x_a} \right) \text{MT}(k)\]

as a function of $k$, where $C = d_0 - d_∞$. 


Proof. In this proof, we consider a fixed admissible metric $g$ whose Kähler form $\omega$ belongs to $\Omega$.

(1) Put $g^\prime = \lambda g$ and $\omega^\prime = \lambda \omega$, then $(g^\prime, \omega^\prime)$ defines a Kähler structure and $\omega^\prime \in 2\pi c_1(M)$. Let $\kappa$ be the Ricci potential of $\omega$. Since the Ricci form is preserved under scaling of $\omega$, $\kappa$ is also the Ricci potential of $\omega^\prime$. In this proof, we promise that $\theta_V$ denotes the potential function of a holomorphic vector field $V$ with respect to $g^\prime$, which is normalized by $-\Delta_{\partial, g^\prime} \theta_V + \theta_V + V(\kappa) = 0$, where $\Delta_{\partial, g^\prime}$ is the $\partial$-Laplacian with respect to $g^\prime$. We set $\theta_{X^2_J} = 2\lambda z - C$ for some constant $C$, then $C$ is calculated by

\begin{align}
C &= -2\Delta_{\partial, g^\prime} \lambda z + 2\lambda z + \kappa'(z) \cdot \Theta(z) \\
&= -2\Delta_{\partial, g} z + 2\lambda z + \kappa'(z) \cdot \Theta(z) \\
&= \frac{F'(z)}{p_c(z)} + 2\lambda z + \kappa'(z) \cdot \Theta(z),
\end{align}

(4.17)

where we used (4.7) and $X^2_J(\kappa(z)) = -JK(\kappa(z)) = -d(\kappa(z))(JK) = \kappa'(z)Jdz(K) = \kappa'(z) \cdot \Theta(z)$, and denoted the $\partial$-Laplacian with respect to $g$ by $\Delta_{\partial, g}$. In order to find $C$ as above, we take the limit of $z$ to the boundary. Since

\begin{align}
\lim_{z \to 1} \frac{F'(z)}{p_c(z)} &= \Theta'(z) + \Theta(z) \cdot \frac{p'_c(z)}{p_c(z)} = \Theta'(z) + \Theta(z) \cdot \sum_{a \in A} \frac{x_ad_a}{1 + x_a z},
\end{align}

(4.18)

using the boundary condition (3.4) and de l'Hôpital’s rule, we get

\begin{align}
\lim_{z \to 1} \frac{F'(z)}{p_c(z)} &= -2 + \lim_{z \to 1} \Theta(z) \cdot \frac{-d_\infty}{1 - z} \\
&= -2 - \lim_{z \to 1} \frac{\Theta'(z)d_\infty}{-1} = -2 - 2d_\infty.
\end{align}

Similarly,

\begin{align}
\lim_{z \to -1} \frac{F'(z)}{p_c(z)} &= 2 + 2d_0.
\end{align}

Therefore, combining with (4.17), we have

\begin{align}
C &= -2 - 2d_\infty + 2\lambda = 2 + 2d_0 - 2\lambda.
\end{align}

Hence, we get $C = d_0 - d_\infty$ and $\lambda = \frac{d_0 + d_\infty + 2}{2}$.

(2) From the argument in (1), we have $\theta_{X^2_J} = 2\lambda z - C$ and $\theta_{X^2_J} = k\lambda z - \frac{kC}{2}$. Hence, by (4.14) and (4.17), we have

\begin{align}
2\lambda z p_c(z) - C p_c(z) + P(z) = 0.
\end{align}

(4.19)
So, the direct computation shows that
\[
\mathbf{T}Z_{\lambda^{-1}X^k_{\lambda}}(X^{k}_{\lambda}) = \int (2\lambda z - C)e^{kz - \frac{kC}{2\lambda}} \frac{\lambda^m}{m!} m
\]
\[
= \int (2\lambda z - C)e^{kz - \frac{kC}{2\lambda}} \lambda^m \cdot \psi(z) \left( \sum_{a \in A} \frac{\omega_a/x_a d_a}{d_a!} \right) dz \wedge \theta
\]
\[
= 2\pi \lambda^m \exp \left( -\frac{C}{2\lambda} \right) \text{Vol} \left( S, \prod_{a \in \hat{A}} \omega_a/x_a \right) \int_{-1}^{1} (2\lambda z - C \cdot \psi(z))e^{kz}dz
\]
\[
= -2\pi \lambda^m \exp \left( -\frac{C}{2\lambda} \right) \text{Vol} \left( S, \prod_{a \in \hat{A}} \omega_a/x_a \right) \int_{-1}^{1} P(z)e^{kz}dz
\]
\[
= -2\pi \lambda^m \exp \left( -\frac{C}{2\lambda} \right) \text{Vol} \left( S, \prod_{a \in \hat{A}} \omega_a/x_a \right) \text{MT}(k),
\]
where we used the equation \( \omega^m/m! = \psi(z) \left( \sum_{a \in \hat{A}} \frac{\omega_a/x_a d_a}{d_a!} \right) dz \wedge \theta \) (cf. [ACGT08, §2.2]).

**Corollary 4.10.** We assume the same as above. Then, \( \Omega = 2\pi c_1(M) \) holds if and only if \( d_0 = d_\infty = 0 \), i.e. a blow-down occurs. In this case, we have
\[
(4.20) \quad \mathbf{T}Z_{X^k_{\lambda}}(X^{k}_{\lambda}) = -2\pi \text{Vol} \left( S, \prod_{a \in \hat{A}} \omega_a/x_a \right) \text{MT}(k)
\]
for any admissible metrics.

**Proof.** \( \Omega = 2\pi c_1(M) \) holds if and only if \( \lambda = 1 \) if and only if \( d_0 = d_\infty = 0 \).

---

### 5. Modified Kähler-Ricci flow on Admissible bundles

Let \( M \) be an \( m \) \((:= \sum_{a \in \mathcal{A}} d_a + 1)\)-dimensional admissible bundle and \( \Omega \) an admissible class. We assume that \( P(t) \) has exactly one root in the interval \((-1, 1)\). Then, we have the following properties:

**Lemma 5.1** (Lemma 4.4 in [MT11]). If the function \( P(t) \) has exactly one root in the interval \((-1, 1)\), then there exists a unique \( k_0 \in \mathbb{R} \) such that \( \text{MT}(k_0) = 0 \). Moreover, for this \( k_0 \), the function \( F(z) \) defined by (4.12) satisfies \( F > 0 \) on \((-1, 1)\), and an admissible GQE metric is naturally constructed from \( F \).

The assumption for \( P(t) \) is always satisfied when \(|x_a|\) is sufficiently small for all \( a \in \mathcal{A} \) (cf. [MT11, §5]) or \( \Omega = 2\pi \lambda^{-1} c_1(M) \) for a positive constant \( \lambda \) determined by Theorem 4.9. Actually, we have
Lemma 5.2. If we assume $\Omega = 2\pi \lambda^{-1} c_1(M)$ for a positive constant $\lambda$, we have $P(t) = (C - 2\lambda t) p_c(t)$, where $\lambda$ and $C$ are constants determined by Theorem 4.9. Hence, $P(t)$ has exactly one root $t = \frac{C}{2\lambda}$ in the interval $(-1, 1)$, and there exists an admissible Kähler-Ricci soliton.

Proof. This follows directly from (4.19) and Lemma 5.1. □

Example 5.3 (Koiso’s Example (cf. Example 4.1 in [TZ02])). We consider an admissible bundle $M := \mathbb{C}P^d + \mathbb{C}P^d + 1 \rightarrow \mathbb{C}P^d$ for $d \geq 1$. Since $b_2(\mathbb{C}P^d) = 1$, every Kähler class on $M$ is admissible up to scale (cf. [ACGT08, Remark 2]), so $c_1(M)$ is admissible up to scale. Hence, Corollary 4.10 implies that there exists an admissible class $\Omega$ with the admissible data $x \in (-1, 1)$ ($x \neq 0$) such that $\Omega = 2\pi c_1(M)$. Then, we have

$$MT(0) = -2 \int_{-1}^{1} t(1 + xt)^d dt = -4 \sum_{i=1}^{[d/2]} \left( \frac{l}{2i - 1} \right) \frac{x^{2i - 1}}{2i + 1} \neq 0.$$ 

This shows that there exists an admissible Kähler-Ricci soliton with respect to a non-trivial holomorphic vector field.

As is seen in Sect. 3, for any $\Theta \in K^\omega_{adm}$, there exists a unique fiber-preserving $U(1)$-equivariant diffeomorphism $\Psi$ satisfying (3.7). Thus, we can define an inclusion map

$$(5.1) \quad K^\omega_{adm} \hookrightarrow \{ \text{Kähler form in } (\Omega, J_c) \}$$

by $\Theta \mapsto \Theta^* \omega$, where $(\Omega, J_c)$ denotes the Dolbeault cohomology class with respect to $J_c$. In this section, we propose a method of studying the modified Kähler-Ricci flow as a PDE for a $t$-dependent function $\Theta_t \in K^\omega_{adm}$ via the inclusion map (5.1).

First, we consider the case of $MT(0) = 0$ for simplicity. In this case, there exists an admissible CSC Kähler metric in $\Omega$. So, we consider the equation

$$(5.2) \quad \frac{\partial}{\partial t} \Psi_t^* \omega = -\text{Ric}(\Psi_t^* \omega) + H\text{Ric}(\Psi_t^* \omega),$$

where $\Psi_t$ is a $t$-dependent diffeomorphism defined by (3.7). Let $g_t$ be a $t$-dependent admissible metric and $J_t$ be the compatible complex structure corresponding to $\Psi_t$ for each $t$. Taking the trace of the both hand sides of (5.2), we have

$$(5.3) \quad \frac{\partial}{\partial t} \log \det(\Psi_t^* g_t) = -\text{Scal}(\Psi_t^* g_t) + \overline{\text{Scal}}.$$ 

Since $\Psi_t : (M, J_c, \Psi_t^* g_t, \Psi_t^* \omega) \rightarrow (M, J_t, g_t, \omega)$ is a biholomorphic isometry, $\Psi_t$ commutes with log det $g_t$ and Scal($g_t$). Thus, we have

$$(5.4) \quad \frac{\partial}{\partial t} \Psi_t^* \log \det g_t = \Psi_t^* (-\overline{\text{Scal}}(g_t) + \overline{\text{Scal}}).$$

Now we will calculate a local Ricci potential $\log \det g_t$ by a local trivialization of $M^0$. This calculation is a special case of (77) in [ACG06] and essentially the same as Lemma 1.2 in [KS86]: We take a local trivialization...
\[ \{w^a\}_{a \in \hat{A}}; w \} \) of \( C^*\)-bundle \( M^0 \to \hat{S} \) such that \( w^a = (w^{a,1}, \ldots, w^{a,d_a}) \) is a local coordinate system of \( S_a \) for each \( a \in \hat{A} \) and \( \frac{\partial}{\partial w} = -J_\tau K - \sqrt{-1}K \).

Then we have

\[
g_t \left( \frac{\partial}{\partial w}, \frac{\partial}{\partial \bar{w}} \right) = 2 \Theta_t(z), \quad g_t \left( \frac{\partial}{\partial w^{a,i}}, \frac{\partial}{\partial \bar{w}} \right) = 2 \frac{\partial z}{\partial w^{a,i}},
\]

and

\[
g_t \left( \frac{\partial}{\partial w^{a,i}}, \frac{\partial}{\partial w^{a,j}} \right) = \frac{1}{x_a} \frac{g_a}{x_a} \left( \frac{\partial}{\partial w^{a,i}}, \frac{\partial}{\partial w^{a,j}} \right) + \frac{2}{\Theta_t(z)} \frac{\partial z}{\partial w^{a,i}} \frac{\partial z}{\partial w^{a,j}},
\]

where \( i, j = 1, \ldots, d_a; a \in \hat{A} \) and \( a, b \in \hat{A} \). Hence, we can compute \( \log \det g_t \) as

\[
\log \det g_t = \log \left( 2 \Theta_t(z) \cdot p_c(z) \cdot \prod_{a \in \hat{A}} \det(g_a/x_a) \right)
\]

\[
= \log \Theta_t(z) + \log p_c(z) + \sum_{a \in \hat{A}} \log \det(g_a/x_a).
\]

Let \( V_t \) be the \( t \)-dependent real vector field corresponding to the \( t \)-dependent diffeomorphism \( \Psi_t \). Then the left hand side of (5.4) is calculated as

\[
\frac{\partial}{\partial t} \Psi_t^* \log \det g_t = \frac{\partial}{\partial t} \Psi_t^* \left( \log \Theta_t(z) + \log p_c(z) + \sum_{a \in \hat{A}} \log \det(g_a/x_a) \right)
\]

\[
= \Psi_t^* \left( V_t(\log \Theta_t(z)) + \frac{\partial}{\partial t} \log \Theta_t(z) + V_t(\log p_c(z)) \right)
\]

\[
= \Psi_t^* \left( \frac{\Theta_t(z)}{\Theta_t(z)} \cdot V_t(z) + \frac{1}{\Theta_t(z)} \frac{d\Theta_t}{dt}(z) + \frac{p'(z)}{p_c(z)} \cdot V_t(z) \right),
\]

where \( \frac{d}{dt} \) denotes the partial derivative in \( t \) for a function of \( z \) and \( t \).

**Lemma 5.4.** The equation

\[
V_t(z) = -\Theta_t(z) \cdot \frac{dy_t}{dt}(z)
\]

holds, where \( y_t \) is the function with respect to \( \Theta_t \) defined by (3.7).

**Proof.** Differentiating (3.7) in \( t \) implies

\[
V_t(y_t(z)) + \frac{dy_t}{dt}(z) = 0.
\]
Since $d^t_{y_t}(y_t(z)) = y'_t(z)J_t dz = \theta$, we obtain $dz = -\Theta_t(z)J_t \theta = \Theta_t(z)d(y_t(z))$.

Hence, we have $V_t(z) = dz(V_t) = \Theta_t(z)V_t(y_t(z)) = \Theta_t(z) \cdot \left(-\frac{dy_t}{dt}(z)\right) = -\Theta_t(z) \cdot \frac{dy_t}{dt}(z)$.

Differentiating (5.7) in $z$, we have

\[
(V_t(z))' = -\Theta_t'(z) \cdot \frac{dy_t}{dt}(z) - \Theta_t(z) \cdot \frac{d}{dt} \left(\frac{1}{\Theta_t(z)}\right)(z)
\]

\[
= -\Theta_t'(z) \cdot \frac{dy_t}{dt}(z) + \frac{1}{\Theta_t(z)} \cdot \frac{d\Theta_t}{dt}(z).
\]

From (5.6), (5.7) and (5.8), we obtain

\[
\frac{\partial}{\partial t} \Psi_t^* \log \det g_t = \Psi_t^* \left((V_t(z))' + \frac{p'(z)}{p_c(z)} \cdot V_t(z)\right).
\]

From (4.8), (5.4), (5.9) and $\text{scal} = \beta_0/\alpha_0$, we get

\[
(V_t(z))' + V_t(z) \cdot \frac{p'(z)}{p_c(z)} = -\frac{1}{2} \left(\sum_{a \in \mathcal{A}} \frac{2d_a s_a x_a}{1 + x_a z} - \frac{F_t''(z)}{p_c(z)}\right) + \frac{\beta_0}{\alpha_0},
\]

where $F_t(z) = \Theta_t(z) \cdot p_c(z)$. Multiplying $2p_c(z)$ and using (4.9), we have

\[
2[V_t(z)p_c(z)]' = -P'(z) + F_t''(z).
\]

Integrating on the interval $[-1, z]$, this can be written as

\[
2V_t(z)p_c(z) = -P(z) + F_t'(z) + (\text{const}).
\]

Since, $\Psi_t$ preserves each fiber and fixes any point on the critical set $e_0 \cup e_{\infty}$, we have $V_t(z) \equiv 0$ on $e_0 \cup e_{\infty}$. Moreover, $P$ and $F_t'$ have the same boundary condition, so we get

\[
2V_t(z)p_c(z) = -P(z) + F_t'(z).
\]

This is a PDE for a $t$-dependent function $\Theta_t \in \mathcal{K}^{\text{adm}}_\omega$ defined on $[-1, 1]$, which is equivalent to (5.2).

Now, we consider the general case. Let $\kappa_0$ be a real constant such that $\text{MT}(\kappa_0) = 0$. Then, there exists an admissible GQE metric $\Theta_\infty$ with respect to a holomorphic vector field $X^\kappa_{\infty} = -\frac{\kappa_0}{\kappa} (J_\infty K + \sqrt{-1}K)$, where $J_\infty$ denotes the compatible complex structure with $\Theta_\infty$. We will also use the notation “$\infty$” for the quantities corresponding to $\Theta_\infty$ ($\Psi_\infty, g_\infty, F_\infty$, etc.). Then, $\Psi_\infty^* g_\infty$ is a GQE metric with respect to a holomorphic vector field $X^\kappa_{\infty} := \Psi_\infty^{-1} \cdot X^\kappa_{\infty} = -\frac{\kappa_0}{\kappa} (J_\infty K + \sqrt{-1}K)$. So, in this case, we should consider the equation

\[
\frac{\partial}{\partial t} \Psi_t^* \omega = -\text{Ric}(\Psi_t^* \omega) + \mathbb{H}\text{Ric}(\Psi_t^* \omega) + L_{X^\kappa_{\infty}} \Psi_t^* \omega.
\]
Taking the trace of both hand sides of (5.11) with respect to $\Psi_t^*g_t$ yields
\[
\frac{\partial}{\partial t} \log \det \Psi_t^*g_t = -\text{Scal}(\Psi_t^*g_t) + \overline{\text{Scal}} + \left( L_{X_{Jt}^{k_0}}^*\Psi_t^*\omega, \Psi_t^*\omega \right)_{\Psi_t^*g_t}.
\]
So, we have
\[
\frac{\partial}{\partial t} \log \det g_t = \Psi_t^* \left( -\text{Scal}(g_t) + \overline{\text{Scal}} + \left( L_{X_{Jt}^{k_0}}\omega, \omega \right) \right)_{g_t},
\]
where we put $X_{Jt}^{k_0} := \Psi_t^*X_{Jt}^{k_0} = -\frac{k_0}{2}(J_tK + \sqrt{-1}K)$. On the other hand,
\[
L_{X_{Jt}^{k_0}}\omega = -\frac{k_0}{2}d_iJ_t\kappa_\omega = \frac{k_0}{2}d(\Theta_t(z) \cdot \theta)
= \frac{k_0}{2}(\Theta_t'(z)dz \wedge \theta + \Theta_t(z) \cdot d\theta).
\]
Hence, we can calculate $\left( L_{X_{Jt}^{k_0}}\omega, \omega \right)_{g_t}$ as
\[
\left( L_{X_{Jt}^{k_0}}\omega, \omega \right)_{g_t} = \frac{k_0}{2}(\Theta_t'(z)dz \wedge \theta + \Theta_t(z) \cdot d\theta, \omega)_{g_t}
= \frac{k_0}{2}\Theta_t'(z)(dz \wedge \theta, \omega)_{g_t} + \frac{k_0}{2}\Theta_t(z) \left( \sum_{a \in A} \omega_a, \omega \right)_{g_t}
= \frac{k_0}{2}\Theta_t'(z) + \frac{k_0}{2}\Theta_t(z) \cdot \sum_{a \in A} \frac{x_a d_a}{1 + x_a z}
= \frac{k_0}{2}\Theta_t'(z) + \frac{k_0}{2}\Theta_t(z) \cdot \frac{p_c'(z)}{p_c(z)}.
\]
Combining (5.9), (5.12), and (5.13), we obtain
\[
2[V_t(z)p_c(z)]' = -P'(z) + F_t''(z) + k_0 F_t'(z).
\]
Since $F_t(\pm 1) = 0$, we get the equation
\[
2V_t(z)p_c(z) = -P(z) + F_t'(z) + k_0 F_t(z).
\]
Summarizing the above, we obtain the following theorem:

**Theorem 5.5.** Let $M$ be an $m := \sum_{a \in A} d_a + 1$-dimensional admissible bundle and $\Omega$ an admissible class on $M$. We assume that $P(t)$ has exactly one root in the interval $(-1, 1)$. Then, for any fixed symplectic form $\omega$ defined by (3.3), the modified Kähler-Ricci flow (5.11) can be reduced to
\[
2V_t(z)p_c(z) = -P(z) + F_t'(z) + k_0 F_t(z)
\]
for $\Theta_t \in \mathcal{K}^{\text{adm}}$. Here, $F_t(z) = \Theta_t(z) \cdot p_c(z)$ and $V_t$ is the $t$-dependent real vector field corresponding to the $t$-dependent fiber-preserving $U(1)$-equivariant diffeomorphism $\Psi_t$ defined by (3.7), and $V_t(z)$ is calculated by (5.7).
We define a $t$-dependent function $\varphi_t$ by $\Theta_t = (1 + \varphi_t)\Theta_\infty$. Combining (5.14) with $F'_\infty(z) + k_0 F_\infty(z) = P(z)$, we get

$$2\Theta_\infty \frac{d\varphi_t}{dt} = \Theta_\infty \Theta_t \varphi''_t - (\Theta_\infty \varphi'_t)^2 + \frac{P'}{p_c} \cdot \Theta_\infty \varphi'_t$$

where we remark that $\frac{P}{p_c} = -2\Delta g + \kappa' \cdot \Theta$ is smooth on $[-1, 1]$. Guan [G07, §11] studied the modified Kähler-Ricci flow on a certain class of completions of $\mathbb{C}^*$-bundles introduced by Koiso and Sakane [KS86], and derived the evolution equation of the same type as (5.15). On the other hand, Koiso [K90] showed that the condition

$$(5.16) \Theta_\infty \left( \frac{P}{p_c} \right)' - \left( \frac{P}{p_c} \right) \Theta'_{\infty} < 0 \text{ on } [-1, 1]$$

is automatically satisfied when $\Omega = 2\pi c_1(M)$. Then, using (5.16), he also showed that the solution of (5.15) converges uniformly to 0 in exponential order. So, Guan suggested that for any $\Omega$ satisfying the condition (5.16), the long time solution of (5.15) exists and converges uniformly to 0 in exponential order. Actually, we can prove the desired result by the maximum principle as in [K90]. However, it is a difficult problem to check whether $\Omega$ satisfies (5.16) or not in general cases. So, we consider this problem only in some special situations.

**Lemma 5.6.** We assume that $P(t)$ has exactly one root in the interval $(-1, 1)$ and $(\log |P|)' < 0$ on the complement of the zero-set of $P$ in $(-1, 1)$. Then, (5.16) holds.

**Proof.** Put

$$\xi(t) = P(t)e^{k_0 t} \quad \text{and} \quad \eta(t) = \int_{-1}^{t} \xi(s)ds.$$ 

Then, we have $\Theta_\infty = e^{-k_0 z/p_c} \cdot \eta$ and

$$\left( \Theta_\infty \left( \frac{P}{p_c} \right)' - \left( \frac{P}{p_c} \right) \Theta'_{\infty} \right) \cdot e^{k_0 z/p_c} = -\left( \xi^2 - \eta \cdot \xi ' \right) \cdot e^{-k_0 z/p_c}.$$ 

By de l'Hôpital’s rule, we obtain $\Theta_\infty \left( \frac{P}{p_c} \right)' - \left( \frac{P}{p_c} \right) \Theta'_{\infty} = -4(d_{\infty} + 1) < 0$ at $t = 1$ and $\Theta_\infty \left( \frac{P}{p_c} \right)' - \left( \frac{P}{p_c} \right) \Theta'_{\infty} = -4(d_0 + 1) < 0$ at $t = -1$. So, it suffices to prove that $\xi^2 - \eta \xi ' > 0$ on $(-1, 1)$. Let $t = t_0$ be the unique root of $P(t)$ in $(-1, 1)$, then $\xi(t_0) = 0$. Since $P(t)$ has exactly one root in $(-1, 1)$, we have $\xi '(t_0) < 0$, $\eta > 0$ on $(-1, 1)$ and $\eta = 0$ at $t = \pm 1$. Hence we obtain $\xi^2 - \eta \xi ' > 0$ at $t = t_0$. So, we may consider only on the interval $(t_0, 1)$ (a similar proof works on $(-1, t_0)$). Thereafter, one can prove the desired result by the same argument as in Lemma 3.1 in [K90].
Remark 5.7. If \( P(t) \) is a product of polynomials of first order, clearly we have \( (\log |P|)'' < 0 \) on the complement of the zero-set of \( P \) in \((-1, 1)\).

Now, we give examples of admissible classes which satisfies (5.16).

Example 5.8. We assume that \( \Omega := 2\pi\lambda^{-1}c_1(M) \) is admissible. Then, by Lemma 5.2, we have \( P(t) = (C - 2\lambda t)p_\epsilon(t) \) and \( (\log |P|)'' < 0 \) holds on the complement of the zero-set of \( P \) in \((-1, 1)\). Hence, \( \Omega \) satisfies (5.16) by Lemma 5.6.

Example 5.9. Let \( \Omega \) be an admissible class on \( M \) with the admissible data \( \{x_a\} \). Then, \( \Omega \) satisfies (5.16) if \( |x_a| \) is sufficiently small for all \( a \in A \).

This statement follows from Lemma 5.6 and the next Lemma.

Lemma 5.10. Let \( \Omega \) be an admissible class on \( M \) with the admissible data \( \{x_a\} \). Then, \( (\log |P|)'' \) is negative on the complement of the zero-set of \( P \) in \((-1, 1)\) if \( |x_a| \) is sufficiently small for all \( a \in A \).

Proof. We denote the limit \( x_a \to 0 \) for all \( a \in A \) by “lim” for simplicity. We remark that lim and the derivatives of arbitrary order for \( P \) are commutative because the \( i \)-th derivative \( P^{(i)} \) converges uniformly on any closed interval in \( \mathbb{R} \) for all \( i \geq 1 \). From the argument in [MT11, §5], we can write \( \lim P(t) \) as

\[
\lim P(t) = -(2 + d_0 + d_\infty)(t - t_0)(1 + t)^{d_0}(1 - t)^{d_\infty}
\]

for some \( t_0 \in (-1, 1) \). This is a product of polynomials of first order, so we obtain

\[
\frac{(\lim P)'' \cdot \lim P - (\lim P')^2}{(\lim P)^2} = (\log |\lim P|)'' < 0
\]

on \((-1, t_0) \cup (t_0, 1)\). Moreover, \( \lim P(t_0) = 0 \) and \( \lim P'(t_0) < 0 \) yield \( (\lim P)'' \cdot \lim P - (\lim P')^2 < 0 \) at \( t = t_0 \). So, we get \( \lim(P''P - (P')^2) = (\lim P)'' \cdot \lim P - (\lim P')^2 < 0 \) on \((-1, 1)\).

We want to show that \( P''P - (P')^2 < 0 \) on \((-1, 1) \) if \( |x_a| \) is sufficiently small for all \( a \in A \). To do this, we observe the behavior of the function \( P''P - (P')^2 \) near the boundary as \( x_a \to 0 \) for all \( a \in A \).

Case 1 : \( d_0 = 0 \)

In this case, \( \lim P(t) \) has the form

\[
\lim P(t) = -(2 + d_\infty)(t - t_0)(1 - t)^{d_\infty}
\]

From the boundary condition \( \lim P(-1) = 2^{d_\infty+1} \), \( t_0 \) is determined by the equation \((2 + d_\infty)(1 + t_0) = 2\). So, the direct computation shows that

\[
\lim(P''P - (P')^2) = -(1 + d_\infty)(4 + d_\infty)2^{2d_\infty} < 0
\]

at \( t = -1 \). Thus, \( P''P - (P')^2 \) is negative near \( t = -1 \) if \( |x_a| \) is sufficiently small for all \( a \in A \).

Case 2 : \( d_0 = 1 \)
In this case, we have \( \lim_{t \to -1} P = 0 \) and \( \lim_{t \to -1} P' > 0 \). Hence, \( \lim_{t \to -1} (P'' - (P')^2) \) is negative at \( t = -1 \). This implies that \( P'' - (P')^2 \) is negative near \( t = -1 \) if \( |x_a| \) is sufficiently small for all \( a \in A \).

Case 3 : \( d_0 \geq 2 \)

In this case, we have \( \lim_{t \to -1} (P'' - (P')^2) = 0 \) at \( t = -1 \). However, we can see that \( P'' - (P')^2 \) is negative in some (deleted) right neighborhood of \( t = -1 \) if \( |x_a| \) is sufficiently small for all \( a \in A \) because \( t = -1 \) is a zero point of \( P'' - (P')^2 \) fixed as \( x_a \) changes.

A similar observation for \( P'' - (P')^2 \) near \( t = 1 \) follows in the similar way. As above, we conclude that \( P'' - (P')^2 \) is negative on \((-1, 1)\) if \( |x_a| \) is sufficiently small for all \( a \in A \) and this completes the proof of Lemma 5.10. \( \square \)

From the above, we conclude that

**Theorem 5.11.** Let \( M \) be an \( m \) \((:= \sum_{a \in \hat{A}} d_a + 1)\)-dimensional admissible bundle and \( \Omega \) an admissible class on \( M \) with the admissible data \( \{x_a\} \). We assume that \( P(t) \) has exactly one root in the interval \((-1, 1)\). Then, for any symplectic form defined by (3.3), the modified Kähler-Ricci flow (5.11) can be reduced to the evolution equation (5.15) for \( \varphi_t \). Moreover, if \( |x_a| \) is sufficiently small for all \( a \in A \), the solution \( \varphi_t \) of (5.15) converges uniformly to 0 in exponential order.

From Theorem 5.11 and the definition of \( \varphi_t \), we see that \( \Theta_t \) converges uniformly to \( \Theta_\infty \) in exponential order. Here, we remark that the convergence of the function \( \varphi_t \) does not directly indicate the convergence of the metric \( g_t \) in \( C^\infty \)-topology. However, it seems that the same argument as in [K90] works well in our case and one can prove that \( g_t \) converges to \( g_\infty \) in \( C^1 \)-topology. In order to estimate the higher order derivatives for \( g_t \), we need a new argument which substitutes for Cao’s estimate for complex Monge-Ampère equation (cf. [C85])

**References**

[ACG06] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, *Hamiltonian 2-forms in Kähler geometry, I General theory*, J. Differential Geom., **73** (2006), 359–412.
[ACGT08] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. W. Tønnesen-Friedman, *Hamiltonian 2-forms in Kähler geometry, III Extremal Metrics and Stability*, Invent. Math., **173** (2008), 547–601.
[C85] H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math., **81** (1985), 359–372.
[F83] A. Futaki, *An obstruction to the existence of Kähler-Einstein metrics*, Invent. Math., **73** (1983), 437–443.
[G95] D. Guan, *Quasi-Einstein Metrics*, Int. Journal of Math., **6** (1995), 371–379.
[G07] D. Guan, *Extremal-solitons and exponential \( C^\infty \) convergence of the modified Calabi flow on certain \( \mathbb{CP}^1 \) bundles*, Pacific Journal of Mathematics, **233** (2007), 91–124.
[K90] S. Kobayashi, *Transformation groups in differential geometry*, Springer-Verlag, Berlin (1995), p.94.
[KS86] N. Koiso and Y. Sakane, *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds*, Lecture Notes in Math., 1201 (1986), Springer-Verlag, Berlin, 165–179.

[K90] N. Koiso, *On rotationally symmetric Hamilton’s equations for Kähler-Einstein metrics*, Adv. Studies in Pure Math., 18-1 (1990), Academic Press, Kinokuniya, Tokyo, 327–337.

[MT11] G. Maschler and C. W. Tønnesen-Friedman, *Generalizations of Kähler-Ricci solitons on projective bundles*, Math. Scand., 108 (2011), 161–176.

[TZ02] G. Tian and X. H. Zhu, *A new holomorphic invariant and uniqueness of Kähler-Ricci solitons*, Comm. Math. Helv., 77 (2002), 297–325.

[TZ07] G. Tian and X. H. Zhu, *Convergence of Kähler-Ricci flow*, J. Amer. Math. Soc., 20 (2007), 675–699.

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