An Inherently Quantum Computation Paradigm: NP-complete=P Under the Hypothetical Notion of Continuous Uncomplete von Neumann Measurement

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(November 21, 2018)

The topical quantum computation paradigm is a transposition of the Turing machine into the quantum framework. Implementations based on this paradigm have limitations as to the number of: qubits, computation steps, efficient quantum algorithms (found so far). A new exclusively quantum paradigm (with no classical counterpart) is propounded, based on the speculative notion of continuous uncomplete von Neumann measurement. Under such a notion, NP-complete is equal to P. This can provide a mathematical framework for the search of implementable paradigms, possibly exploiting particle statistics.

I. INTRODUCTION

During the Helsinki meeting on quantum computation and communication (September 26-28, 1998), some participants explicitly asserted that full fledged quantum computation is completely out of the current course of research (in my understanding: N. Gisin, S. Harroche, R. Landauer, A. Zeilinger). Going beyond 10 qubits (even in the most optimistic estimates) and a very limited number of computation steps would be out of reach for today’s technology and paradigm. Therefore, new ideas and, in particular, new computation paradigms would be needed (A. Ekert, the author).

The following is an extended version of part of my speech. It concerns an alternative quantum computation paradigm, completely speculative for the time being.

This paradigm appears in former works of the author and others\cite{1-6}, where it is somewhat buried inside implementation attempts. Here I will explain its bare principle, by resorting to the speculative notion of continuous uncomplete von Neumann measurement.

II. A CRITIQUE OF TURING-QUANTUM COMPUTATION

Until now, the topical quantum computer is a transposition of the classical Turing machine into the quantum framework. On one side of the coin, there are the well-known revolutionary results, on the other side, severe limitations are being encountered. One could think that the very importance of the results obtained might trap research inside a limited horizon.

In the first place, we should clearly acknowledge the boundaries implicit in the notion of Turing machine computation.

Before going to this, it is useful to introduce a special perspective. All efficient quantum algorithms found so far (more efficient than their classical counterparts as far as we know) do nothing but solving systems of simultaneous Boolean equations. In fact all NP and NP-complete problems\cite{7-8} can be solved by a Turing machine in a number of elementary computation steps which has an upper bound. Because of this feature, they can be represented by a system of simultaneous Boolean equations, whose size is polynomial in problem size in the case of NP and NP-complete problems. It is convenient to have in mind one system of this kind and its network representation (fig. 1):

\[
\begin{align*}
  x_3 &= \text{Nand}(x_1, x_2) \\
  x_6 &= \text{Nand}(x_4, x_5) \\
  x_1 &= x_5 \\
  x_2 &= x_6 \\
  x_3 &= x_4
\end{align*}
\]

Fig. 1

\[^{1}\text{The search problem, belonging to P, will be discussed at the end.}\]
This system implicitly defines the solutions: \( x_1 = x_5 = 0, x_2 = x_6 = 1, x_3 = x_4 = 1, \) and two others. Of course, it does not describe a physical system that constructs the solutions. In other words, the system of simultaneous equations (i.e. the problem) is the definition of an object (the solutions) but it does not map directly the way of constructing the object. In order to construct it, one should scan the possible Boolean variable assignments and, for each assignment, check whether it satisfies all gates and wires, thus following a very indirect way. This is a sequential task that a Turing machine can perform by undergoing a suitable time evolution. Summing up, while the definition of the object is a-sequential and a-temporal, its classical construction needs to be sequential and temporal.

On the contrary, the computation paradigm I am going to expound will make a limited use of sequentiality. It will rely on the notion that, in the quantum framework, there could be identity between defining and constructing. By the way, this resembles the description-action identification asserted by D. Finkelstein: there are only actions\(^7\).

Going back to the boundaries of Turing machine quantum computation, one can see that:

1. A computational efficiency unknown in the classical framework is achieved by exploiting some exclusively quantum features — those involved in multiparticle interference. See for example ref. \(^8\). Whereas the basic computation paradigm remains a classical one.

   It is characterized by the logical and physical sequentiality of computation. For example, let us consider the task of computing a function given the argument. The argument is the input and the result of computation is the output, necessarily separated from the input by a non-zero time interval.

   An alternative characterization of sequentiality lies in the fact that a reversible Boolean network appears in the time-diagram of the computation process.

   I will show that such a classical notion of sequential computation, inherent in the Turing paradigm, must be in some measure given up, if one wants to achieve any computation speed up in the quantum framework.

2. Until now, the quantum Turing machine exploits an uncomplete set of exclusively quantum features: there is no reason whatsoever to think that a quantum computer based on such features can efficiently simulate phenomena involving other exclusively quantum features, like, for example, particle (boson, fermion, anyon, ...) statistics. Interestingly, in his 1982 paper\(^9\), Feyman wrote: “I’m not sure whether Fermi particles could be described by such a system [in my understanding, today’s “universal” quantum computer]”. Parenthetically, the problem of the simulation of fermion statistics with today’s “universal” quantum computer is addressed in ref. \([10-11,\) among others].

   I will conjecture that a more radical detaching from sequentiality implies the exploitation of new exclusively quantum features, possibly particle statistics.

From a methodological standpoint, it is legitimate to think that the above inherent limitations of Turing quantum computation might have something to do with the severe limitations encountered by the implementations: limited number of qubits, computation steps, and efficient algorithms. Moreover, the Turing-quantum paradigm is suspected of being inherently unable to solve an NP-complete problem in polynomial time\(^{12}\). Perhaps we are trying to force the quantum nature into a scheme un-congenial to it.

This work explores the possibility of exploiting the quantum nature in an alternative and perhaps more congenial way. Although the computation paradigm propounded will be completely speculative, it might help in “thinking out of the box” of the Turing paradigm.

### III. THE RELAXATION OF CLASSICAL SEQUENTIALITY IN THE QUANTUM-TURING PARADIGM

Classical sequentiality implies that there must be a time interval between the input and the output and that the output is a function of the input alone. I will show that this is no more the case in quantum Turing computation.

Let us consider Simon’s algorithm\(^ {13-14}\). At some stage of it, at time say \( t_f \), the computer state is:

\[
|\psi (t_f)\rangle = \sum_x |x\rangle_a |f (x)\rangle_b ,
\]

(1)

\(^2\)If fig. 1 network were implemented as an electronic circuit, it would start oscillating like a flip-flop, because of the time delays and the lack of simultaneity of the classical world. In general, the probability of its settling down in a solution decreases exponentially with network size.
where \(a\) (or \(b\)) is the register containing the argument (the function). This function is periodic and defined over two periods. State (1) is naturally a function of the input (the preparation), which it follows in time.

At the next stage of the algorithm, it is convenient to think that we measure the content of register \(b\) in state (1), obtaining \(f(x) = k\), some constant value. There are two values of \(x\) such that \(f(x) = k\): \(x = \mathcal{I}\) and \(x = \mathcal{I} + p\), where \(p\) is the period. Consequently state (1) changes into

\[|\psi(t_f)| = (|\mathcal{I} \rangle_a + |\mathcal{I} + p \rangle_a) |k \rangle_b \]

Further ingenuity and operations allow to extract \(p\) out of the superposition (2). But the quantum trick, giving the “speed up”, has already been done\(^6\).

Now, the essential point is that state (2) is a function of both the input and the output, in this case the outcome of measurement.

The “wave function collapse”, from (1) to (2), is not necessarily located after \(t_f\). According to von Neumann and Wigner (among others) it can be retrodicted to any time \(t\) between preparation and measurement, provided that the deterministic evolution of the collapsed state at \(t\), gives eventually the state after measurement at \(t_f\). Therefore, we can place the collapse before \(t_f\). Thus \(|\psi(t_f)|\) is either (2) or (1) whether or not the collapse has been taken into account. In the former case, we have a deterministic evolution from the result of measurement to \(t_f\); this evolution undergoes back in time the unitary transformations of the conventional evolution. In the latter, we have the conventional evolution from preparation to \(t_f\). This means that measurement changes a quantum state which was a function of the input into a state which is a function of both the input and the output. This quantum violation of sequentiality justifies the “quantum speed up”, as shown in ref. [6].

The exclusively quantum computation paradigm I am going to expound is in a way an extrapolation of the quantum mechanism discussed above. We will be dealing with a quantum computation state that changes in function of conditions placed both in its immediate past and future, under a speculative, continuous uncomplete von Neumann measurement.

### IV. AN EXCLUSIVELY QUANTUM COMPUTATION PARADIGM

It is convenient to use another way of representing a general system of simultaneous Boolean equations: the feedback loops of fig. 1 are substituted by the condition that the output of an open Boolean network has a pre-assigned value, conventionally 1 (fig. 2).

\[
y = f(x_1, x_2, \ldots, k_1, k_2, \ldots)
\]

\[
k_1 = 0
\]

\[
k_2 = 1
\]

\[
y = 1
\]

**Fig. 2**

In fig. 2, part of the input, \(k_1, k_2, \ldots\), as the output \(y\), are pre-established, the other part of the input, \(x_1, x_2, \ldots\), is “unknown”. \(f\) is a general Boolean function. The problem is whether there is an assignment to the unknown part of the input, such that the output is \(y = 1\). This is the well-known SAT problem, which is NP-complete. Input and output have a purely logical meaning; they mean argument and function. As a matter of fact, they will be coexisting qubits.

I shall give a preliminary outline of the model first. This will not be accurate, but gives the line of thinking. Let

\[
\mathcal{H} = \text{span}\left\{|0\rangle_{x_1}, |0\rangle_{x_2}, \ldots, |0\rangle_{k_1}, |0\rangle_{k_2}, \ldots, |0\rangle_y, \ldots, |1\rangle_{x_1}, |1\rangle_{x_2}, \ldots, |1\rangle_{k_1}, |1\rangle_{k_2}, \ldots, |1\rangle_y\right\},
\]

be the Hilbert space the independent qubits,

\[
\mathcal{H}^c = \text{span}\left\{|0\rangle_{x_1} |0\rangle_{x_2} \ldots |0\rangle_{k_1} |1\rangle_{k_2} \ldots |f(0, 0, \ldots, k_1 = 0, k_2 = 1, \ldots)\rangle_y, \ldots, |1\rangle_{x_1} |1\rangle_{x_2} \ldots |0\rangle_{k_1} |1\rangle_{k_2} \ldots |f(1, 1, \ldots, k_1 = 0, k_2 = 1, \ldots)\rangle_y\right\},
\]

be a constrained subspace of \(\mathcal{H}\), whose basis vectors satisfy the gate logical constraint and the pre-assigned input, and \(P_f \left(\mathcal{P}_f^2 = \mathcal{P}_f\right)\) be the projector from \(\mathcal{H}\) on \(\mathcal{H}^c\).

We suppress the constraint \(y = 1\) and give an arbitrary assignment to the unknown part of the input, obtaining (in polynomial time), say, \(y = 0\). Obtaining \(y = 1\) would mean solving the problem by sheer luck, which is disregarded.

We prepare the independent qubits in such an assignment. Then we rotate only qubit \(y\) from \(|0\rangle_y\) to \(|1\rangle_y\), under continuous uncomplete von Neumann \(P\) measurement of the entire set of qubits. We will see that, correspondingly, the
input changes to a value such that the output is \( y = 1 \) (provided there is a solution). This is in a way an extrapolation of Simon’s algorithm “trick” [changing from state (1) to state (2), because of measurement of the \( b \) register only].

In order to show this, it is useful to clarify first the notion of continuous measurement. This is better done by considering a simple (reversible) identity gate

\[
\mathcal{H}^c = \text{span} \left\{ |0\rangle_x |0\rangle_y , |1\rangle_x |1\rangle_y \right\},
\]

or to \( \overline{\mathcal{H}}^c = \text{span} \left\{ |0\rangle_x |1\rangle_y , |1\rangle_x |0\rangle_y \right\}. \]

The former subspace satisfies the gate, the latter does not.

We ask the question whether there is a way of applying \( P \) measurement to obtain that qubit \( x \) follows a rotation applied only to qubit \( y \). The answer is NO if measurement is *intermittent*, even in the limit of infinite frequency (Zeno effect). The answer is YES if measurement is *continuous* to start with.

We shall consider the former case first. We should start from a state satisfying the gate, say \( |0\rangle_x |0\rangle_y \) for simplicity. Time is split into small consecutive intervals \( \Delta t \) (fig. 3). During each interval \( k \) an \( \omega t \) rotation is applied, with \( k\Delta t < t < (k + 1)\Delta t \). At the end of the first interval we have the state:

\[
|\psi(0)\rangle = |0\rangle_x |0\rangle_y \rightarrow |\psi(\Delta t)\rangle = \cos \omega \Delta t |0\rangle_x |0\rangle_y + \sin \omega \Delta t |0\rangle_x |1\rangle_y.
\]

Then the two qubits are measured under \( P \). If \( \Delta t \) is infinitesimal, with certainty the outcome of measurement is \( |0\rangle_x |0\rangle_y \) back again. This holds for any time \( t \) (for any number of repetitions of the rotation-measurement process) in the limit \( \Delta t \rightarrow 0 \). This is of course the Zeno effect, which freezes the evolution in its initial state. It depends on the fact that, in the quantum framework, there are both probability amplitudes and probabilities, which are the squared modulus of the former ones. The probability that the outcome of measurement violates the gate is higher order infinitesimal \( \sin^2 \omega \Delta t \) and can be disregarded.

Here, it is important to note that during the entire time interval \( \Delta t \) the gate is violated because of the presence of \( \sin \omega t |0\rangle_x |1\rangle_y \), with \( 0 < t < \Delta t \). The percentage measure of time during which it is violated is 100%: it is not violated in two points in time, it is violated during the entire time interval comprised between the two points. Naturally such a percentage measure remains unaltered in the limit \( \Delta t \rightarrow 0 \). To sum up, the time measure during which the gate is violated coincides with any elapsed time \( t \).

Let us now consider *continuous* measurement. If measurement is continuous to start with, *not* in the limit \( \Delta t \rightarrow 0 \), the gate can never be violated. If we start from an initial state that satisfies the gate, continuous measurement must always keep the state inside \( \mathcal{H}^c \). The evolution under continuous measurement must satisfy the simultaneous conditions\(^3\):

- i) \( |\psi(0)\rangle = |0\rangle_x |0\rangle_y \),
  \( \forall t > 0 \):
- ii) \( P|\psi(t)\rangle = |\psi(t)\rangle \), where \( P \) is the projector from \( \mathcal{H} \), the Hilbert space of the two independent qubits, on \( \mathcal{H}^c \) given by eq. (5),
- iii) the distance between the vectors before and after measurement is minimum,
- iv) \( Tr_x(|\psi(t)\rangle \langle \psi(t)| = \cos^2 \omega t |0\rangle_y \langle 0|_y + \sin^2 \omega t |1\rangle_y \langle 1|_y \), where \( Tr_x \) means partial trace over \( x \).

\(^3\)When gate input and output qubits coexist, one can use indifferently a logically reversible or irreversible gate. We should note that in sequential computation there can be states which map a logically irreversible gate between the coexisting qubits of the register, which, at some stage, contains both the output of computation and the memory of the input.
At any time $t$, $|\psi(t)\rangle$ is a generic normalized vector of $\mathcal{H}$; if $t_1 \neq t_2$, $|\psi(t_1)\rangle$ and $|\psi(t_2)\rangle$ are two independent generic vectors of $\mathcal{H}$. See also ref. [5].

It should be noted that the above conditions yield a very direct mapping of the gate, under any possible transformation of its state. We can see that sort of identity between definition and construction, discussed in Section I. However, we are of course in a speculative situation.

It can be seen that conditions (i), (ii), and (iv) yield

$$|\psi(t)\rangle = \cos \omega t |0\rangle_x |0\rangle_y + e^{i\delta} \sin \omega t |1\rangle_x |1\rangle_y. \quad (6)$$

Condition (iii) keeps the phase $\delta$, arbitrarily chosen once for all if the initial state is $|0\rangle_x |0\rangle_y$, frozen throughout the evolution, thus for $t > 0$.

It can be seen from equation (6) that qubit $x$ identically follows the rotation of qubit $y$, under this continuous uncomplete measurement.

Interestingly, evolution (6) is driven by both the initial condition (i) and the final condition that the result of $P$ measurement is a vector belonging to $\mathcal{H}^c$ and satisfying conditions (iii) and (iv). Conditions (i) through (iv) make up a peculiar variational problem. In a sense, it is Simon’s “trick” brought to the continuum.

If instead we started from the initial state

$$|\psi(t)\rangle = \cos \varphi |0\rangle_x |0\rangle_y + e^{i\delta} \sin \varphi |1\rangle_x |1\rangle_y, \quad (7)$$

with $\delta$ pre-established as an initial condition, the evolution would have been

$$|\psi(t)\rangle = \cos (\varphi + \omega t) |0\rangle_x |0\rangle_y + e^{i\delta} \sin (\varphi + \omega t) |1\rangle_x |1\rangle_y. \quad (8)$$

Evolution (8) avoids the difficulty of evolution (7), namely the appearance “out of the blue”, of the term $|1\rangle_x |1\rangle_y$, with an infinitesimal amplitude and a random phase. This resembles a state vector reduction reversed in time, and may not be palatable. Evolution (8) will serve our needs as well.

Now we go back to the general gate of fig. 2. Each unknown input $i$ is prepared in the superposition $\frac{1}{\sqrt{2}}(|0\rangle_i + |1\rangle_i)$, unlike the “inaccurate” outline. This can be propagated in polynomial time to the (unconstrained) output by using a “conventional” quantum Boolean network, such that it keeps, in its output, the memory of the input. Such coexisting output qubits are of course the logical input and output appearing in fig. 2. We remain with an independent set of qubits prepared in the state:

$$|\Psi(0)\rangle = \cos \vartheta \sum_i \alpha_i |u_i\rangle_x |0\rangle_y + \sin \vartheta \sum_j \alpha_j |u_j\rangle_x |1\rangle_y. \quad (9)$$

with $\sum_i |\alpha_i|^2 = \sum_j |\alpha_j|^2 = 1$; $|u_i\rangle_x$, $|u_j\rangle_x$ denote tensor products of all qubit eigenstates but qubit $y$. $i$ ($j$) ranges over the set of input arguments whose function is $y = 0$ ($y = 1$). If the problem is hard to solve, $\sin \vartheta$ is exponentially smaller (in qubits number) than $\cos \vartheta$.

Now we apply the same manipulations already applied to the identity gate: the initial state becomes the right hand of eq. (9), $P$ becomes the projector from $\mathcal{H}$ given in (3) on $\mathcal{H}^c$ given in (4), pertaining the general Boolean gate. \footnote{By the way, any Boolean gate can be made of elementary Nand gates and wires (labeled by $h$). Interestingly, $P$ is the product of the projectors from $\mathcal{H}$ on $\mathcal{H}_h^c$, the Hilbert space spanned by the basis vectors satisfying gate or wire $h : P = \Pi_h \Pi_h$. All such projectors are diagonal in the qubit basis, thus pairwise commuting. We could speak as well of continuous uncomplete measurement under the commuting projectors $P_h$.}

With an $\omega t$ rotation of qubit $y$ ranging from 0 to $\pi/2$, state (9), submitted to (updated) conditions (i) through (iv), changes into

$$|\Psi(\pi/2)\rangle = \sin \vartheta \sum_i \alpha_i |u_i\rangle_x |0\rangle_y + \cos \vartheta \sum_j \alpha_j |u_j\rangle_x |1\rangle_y, \quad (10)$$

as it can be readily checked. Now the probability of measuring $|0\rangle_y$ is exponentially small. A solution is found almost certainly, provided there is one. However, if the result of measurement is not a solution, this can be checked in polynomial time. By repeating the overall process for a sufficient number of times, one can ascertain with any desired confidence level whether the network admits a solution.

Therefore, it is NP-complete=P under this continuous uncomplete von Neumann measurement. Whether continuous measurement is physical and, possibly, what are the resources required to implement it, remain open problems.
V. DISCUSSION

The notion of continuous measurement is physically plausible, in the usual sense that it is in agreement with some physical laws and does not seem to contradict any other laws. Therefore it can be argued that we have given a plausible physical scheme under which NP-complete=P. The time required to find a solution (if any) is the time required to rotate one independent qubit by \( \pi/2 \). Conventionally, this is just one computation step. Such a scheme might help in thinking out of the box of the conventional Turing-quantum paradigm.

In former work\(^3-^5\), it has been conjectured that continuous measurement appears in nature in the form of particle (fermion, boson, anyon, ...) statistics. Particle statistics can as well be seen as continuous measurement under the projector on the symmetric Hilbert subspace.

From being a passive constant of motion that does nothing to a unitary evolution, particle statistics should assume an active role; it should “shape” the evolution of which it becomes a constant of motion. One should look for coupling qubits by means of symmetries purely induced by particle statistics. The exotic kinds like anyon statistics, would provide a broader range of investigation. For example, we plan to investigate the possibility that the NOT and AND gate given in ref. \(^2\), where qubit coupling is obtained by (respectively) fermion and anyon statistics, behave as foreseen in Section IV, under rotation of one qubit.

It has been observed that finding a solution by amplifying its amplitude, as we did by transforming state (9) into state (10), resembles Grover search algorithm\(^{14,15}\), which has been demonstrated to be optimal\(^{16}\), and gives no exponential speed up. However, our (hypothetical) exponential speed up applies to an NP-complete problem, whereas the search problem is P (polynomial). The system of Boolean equations specifying it\(^5\) has the same dimension of the classical search process and is already solved. Therefore, the current scheme gives no advantage in a completely explicit search problem.

Thanks are due to A. Ekert, D. Finkelstein, M. Rasetti, and V. Vedral for useful discussions.

\(^{5}\) \( x_0 = 0, x_1 = 0, \ldots, x_k = 1, \ldots, x_{N-1} = 0, x_N = 0 \).
Fig. 1

Fig. 2

Fig. 3