THE GAUGED NAMBU-JONA LASINIO MODEL: A MEAN FIELD CALCULATION WITH NON MEAN FIELD EXPONENTS

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ABSTRACT

We analyse the phase diagram of the lattice gauged Nambu-Jona Lasinio model with the help of a mean field approximation plus numerical simulations. We find a phase transition line in the coupling parameters space separating the chirally broken phase from the symmetric phase, which is in good qualitative agreement with results obtained in the quenched-ladder approximation. The mean field approximation relates the critical exponents along the continuous phase transition line with the mass dependence of the chiral condensate in the Coulomb phase of standard noncompact $QED$. Our numerical results for noncompact $QED$ strongly suggest non mean field exponents along the critical line.
The gauged Nambu-Jona Lasinio (GNJL) model has become increasingly interesting in recent time, one of the reasons for this increasing interest being the possibility to define a strongly coupled QED with non trivial dynamics \[1\]. In fact if a non gaussian fixed point exists in non compact QED, the naive dimensional analysis does not applies. Therefore operators of dimension higher than four, which are non renormalizable in perturbation theory, could acquire anomalous dimensions and become renormalizable \[2\]. A good candidate is the four Fermi interaction, which when added to the standard QED lagrangian preserving the continuous chiral symmetry, gives us the GNJL model.

The lattice action for the GNJL model with noncompact gauge fields and staggered fermions reads

\[
S = \frac{\beta}{2} \sum_{n,\mu<\nu} \Theta_{\mu\nu}^2(n) + \bar{\chi} \Delta(\theta) \chi + m \bar{\chi} \chi - G \sum_{n,\mu} \bar{\chi}_n \chi_n \bar{\chi}_{n+\mu} \chi_{n+\mu}. \tag{1}
\]

where \( \Theta_{\mu\nu}^2 \) is the standard noncompact plaquette action, \( \beta \) the inverse square coupling, \( \Delta(\theta) \) the massless Dirac operator for Kogut-Susskind fermions and \( G \) the four fermion coupling.

In the chiral limit, \( m = 0 \), this action is invariant under the continuous transformations

\[
\chi_n \rightarrow \chi_n e^{i\alpha(-1)^{n_1+\cdots+n_d}} \quad \bar{\chi}_n \rightarrow \bar{\chi}_n e^{i\alpha(-1)^{n_1+\cdots+n_d}} \tag{2}
\]

which define a continuous chiral U(1) symmetry group.

The vacuum expectation value of the chiral condensate is given by the following ratio of path integrals over the Grassmann and gauge fields

\[
\langle \bar{\chi} \chi \rangle = \frac{\int [d\theta d\bar{\chi} d\chi] e^{-S} \frac{1}{V} \sum_n \bar{\chi}_n \chi_n}{\int [d\theta d\bar{\chi} d\chi] e^{-S}} \tag{3}
\]

The main technical difficulty when computing vacuum averages as (3) in the GNJL model comes from the fact that the action (1) is not a bilinear of the fermion fields. The standard procedure consists in the introduction of an auxiliary vector field which allows to bilinearize the fermion action. The prize to pay for that is that we have one more field to include in the numerical simulations of this model that besides the number of free parameters \( (\beta, m, G) \), makes it difficult to analyse this model with reasonable computer resources \[3\].
Alternatively, we can perform a standard mean field approximation which also bilinearizes the action (1). Following the mean field technique, we make in (1) the following substitution

\[ G \sum_{n,\mu} \bar{\chi}_n \chi_{n+\mu} + \mu \chi_n + \mu \rightarrow 2dG \langle \bar{\chi} \chi \rangle \sum_n \bar{\chi}_n \chi_n \]

where \(d\) is the space-time dimension. The action (1) becomes in this way a bilinear in the fermion fields and the path integral over the Grassmann variables can be done by means of the Matthews-Salam formula.

The v.e.v. of the chiral condensate (3) after the substitution of the mean field approximation (4) in the action (1) is given by

\[
\langle \bar{\chi} \chi \rangle = -\int [d\theta] e^{-\frac{d}{2} \sum \Theta_{\mu\nu}^{(n)}} \det[\Delta + (m - 8G \langle \bar{\chi} \chi \rangle)I] \frac{1}{V} \text{tr} \frac{1}{\Delta + (m - 8G \langle \bar{\chi} \chi \rangle)I},
\]

which after simple algebraic operations can be written as

\[
\langle \bar{\chi} \chi \rangle = -2(m - 8G \langle \bar{\chi} \chi \rangle) \left( \frac{1}{V} \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2 + (m - 8G \langle \bar{\chi} \chi \rangle)^2} \right)
\]

where the sum in (6) runs over all positive eigenvalues of the massless Dirac operator and the integration measure in the v.e.v. includes the fermionic determinant of standard noncompact QED, evaluated at the effective mass \(\bar{m} = m - 8G \langle \bar{\chi} \chi \rangle\). In the chiral limit \(m = 0\), equation (6) becomes

\[
\langle \bar{\chi} \chi \rangle = 16G \langle \bar{\chi} \chi \rangle \left( \frac{1}{V} \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2 + 64G^2 \langle \bar{\chi} \chi \rangle^2} \right)
\]

1. The phase diagram

Equation (7) is always verified if \(\langle \bar{\chi} \chi \rangle = 0\), and this is the only solution in the symmetric phase. In the broken phase where \(\langle \bar{\chi} \chi \rangle \neq 0\), the v.e.v. of the chiral condensate will be given by the solution of the following equation

\[
1 = 16G \left( \frac{1}{V} \sum_{j=1}^{V/2} \frac{1}{\lambda_j^2 + 64G^2 \langle \bar{\chi} \chi \rangle^2} \right)
\]

which gives for the critical line, where the chiral condensate vanishes continuously, the following expression
The existence of this critical line in the GNJL model was discovered some time ago \[4\] in the continuum formulation using the quenched-ladder approximation.

Let us discuss qualitatively the phase diagram. For $\beta < \beta_c^0$, where $\beta_c^0$ is the critical coupling at $G = 0$, the symmetry is always spontaneously broken since in this case equation (8) has a non vanishing solution for any $G \neq 0$. On the other side, the symmetry is also spontaneously broken in the $G \to \infty$ limit since in this limit equation (8) can be written as

$$1 = 16G \frac{1}{64G^2\langle \bar{\chi}\chi \rangle^2} + O\left(\frac{1}{G^2}\right)$$

from which it follows that

$$\langle \bar{\chi}\chi \rangle \sim \frac{1}{2\sqrt{G}}$$

At $\beta = \infty$ the theory can be solved analytically in this approximation. The value of the critical four fermion coupling is in this limit $G_c = 0.2017$. For $G$ values smaller than this value, the symmetry is restored.

In Fig. 1 we present our numerical results for the phase diagram in the $\beta, G$ plane. The critical line has been obtained by computing numerically the v.e.v. of the sum of the inverse square eigenvalues (eq. (9)), which is proportional to the chiral transverse susceptibility of the standard noncompact QED in the chiral limit. The numerical simulations where performed using the MFA approach \[5\], which allows to do computations in the chiral limit. We refer the interested reader to the extended bibliography on this subject \[6\] and especially to the ref. \[7\] where the computation of the chiral susceptibility and the determination of the critical coupling in noncompact QED is discussed in detail.

**2. The critical exponents**

The phase diagram of Fig. 1 is in good qualitative agreement with the corresponding phase diagram obtained in the quenched-ladder approximation \[4\]. Using this analytical approach, a line of critical points with continuously varying critical exponents was found in \[7\], the intersection point of this line with the $G = 0$ axis corresponding to an essential singularity \[8\].

$$G_c(\beta) = \frac{1}{\frac{16}{V} \langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j} \rangle}$$  \hspace{1cm} (9)
Later on, numerical simulations of noncompact QED disproved the essential singularity behavior [1], putting in evidence the limitations of the quenched-ladder approximation. Since our approach contains weaker approximations, we do hope to get more reliable results for the critical exponents.

In order to extract the critical exponents, we will start from the key equation of state (eq. (6)) relating the order parameter with the "external magnetic field" \( m \) and the gauge and four fermion couplings. Using the previous notation we can write equation (6) as

\[
\langle \bar{\chi}\chi \rangle = -2\bar{m}F(\beta, \bar{m})
\]

where the right hand side in (12) is just the chiral condensate in full noncompact QED evaluated at the gauge coupling value \( \beta \) and fermion mass \( \bar{m} \). Concerning critical exponents the interesting physical region, as follows from the phase diagram of Fig. 1, is \( \beta > \beta_c^0 \) (Coulomb phase of noncompact QED).

Since we are interested in the critical region \( (m \to 0, \langle \bar{\chi}\chi \rangle \to 0) \), we will analyze the behavior of \( F(\beta, \bar{m}) \) in the \( \bar{m} \to 0 \) limit. In this limit we can write

\[
F(\beta, \bar{m}) = F(\beta, 0) + B\bar{m}^\omega + \ldots
\]

The second term in (13) possibly contains also logarithmic contributions and \( F(\beta, 0) \) is half the massless transverse susceptibility in noncompact QED. Therefore we can write

\[
F(\beta, 0) = \frac{1}{V}\sum \frac{1}{\lambda_j^2}\langle \bar{\chi}\chi \rangle = 1/16G_c(\beta)
\]

where \( G_c(\beta) \) in (14) stands for a generic point of the critical line in Fig. 1. Equation (13), after the substitution of \( \bar{m} \) by \( m - 8G(\bar{\chi}\chi) \), implies the following behavior for the chiral condensate in the \( m \to 0 \) limit

\[
\langle \bar{\chi}\chi \rangle \sim m^{-1+\omega}
\]

and therefore the \( \omega \) and \( \delta \) exponents are related by the equation

\[
\delta = \omega + 1
\]

A straightforward calculation allows to compute also the magnetic \( \beta_m \) and susceptibility \( \gamma \) exponents, the final result being
The determination of the critical exponents of the order parameter in our mean field approach reduces therefore to the determination of the $\omega$ exponent which controls the mass dependence of the chiral condensate in the Coulomb phase of noncompact \( QED \). In the $\beta \to \infty$ limit of noncompact \( QED \), the theory is free and the chiral condensate can be analytically computed. The well known result in this case ($\omega = 2$ plus logarithmic corrections) implies mean field exponents for the end point of the phase transition line, with the following behavior for $\langle \bar{\chi}\chi \rangle_{\beta=\infty,G=G^\infty}$

$$m \sim \langle \bar{\chi}\chi \rangle^3 \log \langle \bar{\chi}\chi \rangle$$

(18)

In the general case, the chiral condensate in the Coulomb phase of noncompact \( QED \) ($\langle \bar{\chi}\chi \rangle_{NCQED}$) can be parameterized as follows

$$\langle \bar{\chi}\chi \rangle_{NCQED} = A(\beta)m + B(\beta)m^{\omega+1} + \ldots$$

(19)

The first contribution in (19) is linear in $m$, as follows from the fact that the massless transverse susceptibility is finite in the Coulomb phase of noncompact \( QED \). The next contribution can possibly have logarithmic corrections, as happens in the $\beta \to \infty$ limit where it becomes $m^3 \log m$. In order to extract the $\omega$ exponent from the numerical simulations, we can use the results for the massless chiral transverse susceptibility \[6\] to fix $A(\beta)$ in (19) and fit the numerical results with eq. (19). This procedure has the inconvenient that higher order contributions in (19) can induce systematic errors in the determination of $\omega$. A better strategy is to measure the massless nonlinear susceptibility, defined as the third mass derivative of the chiral condensate. In this case we get only one contribution in the $m \to 0$ limit which is logarithmically divergent in the free field theory against a power divergence, which will appear if $\omega < 2$.

Of course in a finite lattice, the nonlinear susceptibility is always finite. However simple finite size scaling arguments tell us that the nonlinear susceptibility should diverge logarithmically with the lattice size in the free field case whereas a power divergence with the lattice size is expected in the case $\omega < 2$. In Fig. 2 we have plotted our results for the nonlinear susceptibility $\chi_{nl}$ of noncompact \( QED \) against the lattice size at $\beta = 0.237$, a value which is unambiguously in the Coulomb phase of this model \[6\]. This is a log-log
plot and the four points correspond to lattice sizes 4, 6, 8 and 10. As it is shown in the figure, the four points are very well fitted by a straight line, this implying that $\omega < 2$.

Due to the potentialities of the MFA method, we have computed vacuum expectation values of other operators, which can be considered as generalizations of a term contributing to the massless nonlinear susceptibility. More precisely we have defined $\chi_q$ by the expression

$$\chi_q = \frac{1}{V} \left\langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j^q} \right\rangle \quad (20)$$

When $q = 4$, we get one of the contributions to the standard massless nonlinear susceptibility. In the general case we can write this vacuum expectation value as an integral over the spectral density of eigenvalues in the following way

$$\frac{1}{V} \left\langle \sum_{j=1}^{V/2} \frac{1}{\lambda_j^q} \right\rangle = \int \frac{\rho(\lambda)}{\lambda^q} d\lambda \quad (21)$$

and if the density of eigenvalues $\rho(\lambda)$ behaves like $\lambda^p$ near the origin, $\chi_q$ will diverge when $q > p + 1$. In such a case and for lattices of finite size, we expect for $\chi_q$ the following behaviour with the lattice size $L$

$$\chi_q \sim L^{\alpha(q-p-1)} \quad (22)$$

where $\alpha$ in (22) is some positive number.

It is interesting to note that the $p$ exponent which controls the small $\lambda$ behavior of the spectral density $\rho(\lambda)$, can be related to the $\omega$ exponent by the following equations

$$\omega = p - 1 (p \leq 3)$$

$$\omega = 2 (p > 3) \quad (23)$$

These relations allow to extract the $\omega$ exponent from the finite size behavior of the generalized nonlinear susceptibility $\chi_q$.

In Fig. 3 we have plotted our results for the inverse of the generalized nonlinear susceptibility $\chi_q$ against the inverse lattice size for $q$ values running from 2 to 4 and $\beta = 0.237$. The solid lines in this figure correspond to a fit of all the points at any fixed $q$ with the function
\[ \chi_q^{-1}(L) = a_q + b_q L^{-c} \] (24)

The results reported in this figure show that, in the infinite volume limit, the inverse generalized nonlinear susceptibility vanishes at large \( q \) and is different from zero at small \( q \), as expected. Fig. 4 is a plot of the extrapolated values of \( \chi_q^{-1} \) (thermodynamical limit) against \( q \). The critical value of \( q \) at which \( \chi_q^{-1} \) vanishes can be estimated from these results. Hence we get \( q_c \sim 2.5 \) at \( \beta = 0.237 \), which implies \( p \sim 1.5 \) and \( \omega \sim 0.5 \). Using now the relations (16), (17) the following results for the order parameter critical exponents can be derived

\[ \delta \sim 1.5, \quad \beta_m \sim 2, \quad \gamma = 1, \] (25)

values which are clearly outside the range of the mean field exponents.

Non mean filed exponents within a mean field approximation might seem rather surprising at first sight. There is however no real contradiction, since we have applied the mean field approximation to the fermion field, while fluctuations of the gauge field are fully taken into account in our numerical simulations. In the infinite \( \beta \) limit, where the gauge field is frozen to the free field configuration, we get mean field exponents. However fluctuations of the gauge field at finite \( \beta \) seem to play a fundamental role in driving critical exponents to non mean field values.

The picture which emerges from this calculation is that the critical exponents change continuously along the critical line of Fig. 1 from their mean field values (end point of the critical line) to some (non mean field) values at the critical point of noncompact \( QED \). The \( \delta \) exponent approaches its mean field value (\( \delta = 3 \)) from below whereas the magnetic exponent approaches its mean field value (\( \beta_m = 0.5 \)) from above. Our results for several values of the gauge coupling \( \beta \) suggest also that the value of \( \delta \) increases systematically along the critical line with increasing \( \beta \), in contrast with the magnetic exponent results which are systematically decreasing with \( \beta \).

In spite of the mean field approach for the fermion field, we believe that our qualitative picture is realistic. It is in fact hard to imagine that non mean field exponents in a mean field approach will become mean field exponents after removing the mean field approximation, i.e. that restoring the full fluctuations of the fermion fields would drive back the critical exponents to mean field values.

A numerical analysis of the fermion-gauge-scalar model with compact \( U(1) \) gauge symmetry done in [8], has shown the existence of a critical line
separating a chirally broken phase from a symmetric phase. In the infinite
gauge coupling limit the model is effectively described by the Nambu-Jona
Lasinio model \cite{10} and therefore critical exponents at this point of the criti-
cal line are gaussian. However strong evidence for non mean field exponen-
tes has been found in \cite{9} near the tricritical point separating the second or-
der line from the first order one. Contrary to our results for the Gauged
Nambu-Jona Lasinio model, the critical behaviour along the critical line
of the fermion-gauge-scalar model seems to be well described by the pure
Nambu-Jona Lasinio model except near the tricritical point. However a slow
variation of the critical exponents with increasing inverse gauge coupling $\beta$
in this model, like the one we have found in the Gauged Nambu-Jona Lasinio
model, can not be excluded, we believe.

One important point which deserves further investigation in the $(GNJL)$
model is the physical origin of the non mean field behavior. In the $\beta \to \infty$
limit of noncompact $QED$ the second contribution to the chiral condensate
behaves like $m^3 \log m$ and this result will probably be true also in perturba-
tion theory. Therefore some important role of gauge configurations topolog-
ically non equivalent to the free field configuration is suggested by our result
$\omega < 2$.

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Figure captions

Figure 1. Phase diagram of the GNJL model in the $\beta, G$ plane.

Figure 2. Logarithm of the nonlinear susceptibility against the logarithm of the lattice size for lattice sizes 4, 6, 8, 10 and $\beta = 0.237$.

Figure 3. Inverse generalized nonlinear susceptibility against the inverse lattice volume at $\beta = 0.237$.

Figure 4. Infinite volume limit of the generalized nonlinear susceptibility against $q$ at $\beta = 0.237$. 