Factorization and Topological States in $c=1$ Matter Coupled to 2-D Gravity

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Abstract

Factorization of the $N$-point amplitudes in two-dimensional $c=1$ quantum gravity is understood in terms of short-distance singularities arising from the operator product expansion of vertex operators after the Liouville zero mode integration. Apart from the tachyon states, there are infinitely many topological states contributing to the intermediate states.
It is very important to understand the non-perturbative results of matrix models [1,2] from the viewpoint of the usual continuum approach of the two-dimensional quantum gravity, i.e. the Liouville theory [3-6]. Recently correlation functions on the sphere topology have been computed in the Liouville theory [7-12]. These results are consistent with those of matrix models [13-15]. So far only conformal field theories with central charge $c \leq 1$ have been successfully coupled to quantum gravity.

The $c = 1$ case is the richest and the most interesting. It has been observed that this theory can be regarded effectively as a critical string theory in two dimensions, since the Liouville field zero mode provides an additional “time-like” dimension besides the obvious single spatial dimension given by the zero mode of the $c = 1$ matter [16]. Since there are no transverse directions, the continuous (field) degrees of freedom are exhausted by the tachyon field. In fact, the partition function for the torus topology was computed in the Liouville theory, and was found to give precisely the same partition function as the tachyon field alone [17,18]. However, there are evidences for the existence of other discrete degrees of freedom in the $c = 1$ quantum gravity. Firstly, the correlation functions obtained in the matrix model exhibit a characteristic singularity structure [14]. In the continuum approach of the Liouville theory, Polyakov has observed that special states with discrete momenta and “energies” can produce such poles, and has called these operators co-dimension two operators [19]. More recently, the two-loop partition function has been computed in a matrix model and evidence has been noted for the occurrence of these topological states [20]. It is clearly of vital importance to pin down the role played by these topological states as much as possible. In the critical string theory, the particle content of the theory and unitarity has been most clearly revealed through the factorization analysis of scattering amplitudes. On the other hand, the factorization and unitarity of the Liouville theory has not yet been well understood.

The purpose of this paper is to understand the factorization of $c = 1$ quantum gravity in terms of the short-distance singularities arising from the operator
product expansion (OPE) of vertex operators. Since we are interested in the short distance singularities, we consider correlation functions on a sphere topology only. We find that the singularities of the amplitudes can be understood as short-distance singularities of two vertex operators and that infinitely many discrete states contribute to the intermediate states of the factorized amplitudes, apart from the tachyon states. These are the co-dimension two operators of Polyakov [19] and presumably are topological in origin. We have also explicitly constructed some of the topological states.

Let us consider the $c = 1$ conformal matter realized by a single bosonic field (string variable) $X$ coupled to the two-dimensional quantum gravity. After fixing the conformal gauge $g_{\alpha\beta} = e^{\alpha\phi} \hat{g}_{\alpha\beta}$ using the Liouville field $\phi$, the $c = 1$ quantum gravity can be described by the following action on a sphere [3-6]

$$S = \frac{1}{8\pi} \int d^2 z \sqrt{\hat{g}} \left( \hat{g}^{\alpha\beta} \partial_\alpha X \partial_\beta X + \hat{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - Q \hat{R} \phi + 8\mu e^{\alpha\phi} \right), \quad (1)$$

where the parameters are given by $Q = 2\sqrt{2}$, $\alpha = -\sqrt{2}$. Since the correct cosmological term operator [4] in the $c = 1$ case should be $\phi e^{\alpha\phi}$ rather than $e^{\alpha\phi}$, the renormalized (correct) cosmological constant $\mu_r$ is given by the following procedure: one should replace $\mu$ and $\alpha$ by $\mu_r/(2\epsilon)$ and $(1-\epsilon)\alpha$ and take the $\epsilon \to 0$ limit [8,10]. We have also set the “Regge slope parameter” $\alpha' = 2$. The gravitationally dressed tachyon vertex operator with momentum $p$ has conformal weight $(1,1)$:

$$O_p = \int d^2 z \sqrt{\hat{g}} e^{ipX} e^{\beta(p)\phi}, \quad \beta(p) = -\sqrt{2} + |p|, \quad (2)$$

We have chosen the plus sign in front of $|p|$, following the argument of refs. [4,5]. We see that the Liouville zero mode can be regarded as an “imaginary time” and the exponent $\beta(p)$ as “energy”.

The $N$-point correlation function of the vertex operators (2) is given by a path
\langle O_{p_{1}} \cdots O_{p_{N}} \rangle = \int \frac{\mathcal{D}X \mathcal{D}\phi}{V_{SL(2,\mathbb{C})}} \, O_{p_{1}} \cdots O_{p_{N}} \, e^{-S}
\begin{align*}
&= \frac{\Gamma(-s)}{-\alpha} \int \prod_{i=1}^{N} [d^{2}z_{i} \sqrt{g}] \frac{1}{V_{SL(2,\mathbb{C})}} \left\langle \prod_{j=1}^{N} e^{ip_{j}X(z_{j})} \right\rangle_{X} \\
&\quad \times \left\langle \left( \frac{\mu}{\pi} \int d^{2}w \sqrt{g} e^{\alpha\phi(w)} \right)^{s} \prod_{j=1}^{N} e^{\beta_{j}\phi(z_{j})} \right\rangle_{\tilde{\phi}},
\end{align*}

(3)

where $V_{SL(2,\mathbb{C})}$ is the volume of the $SL(2, \mathbb{C})$ group and powers of the string coupling constant $g_{st}^{-2}$ for the sphere topology are omitted. The expectation value with $\tilde{\phi}$ denotes the path integral over the non-zero mode $\tilde{\phi}$ of the Liouville field $\phi = \phi_{0} + \tilde{\phi}$, after the zero mode $\phi_{0}$ integration. The defining formula for $s$ can be regarded as “energy-momentum conservation”

$$\sum_{j=1}^{N} p_{j} + s q + Q = 0,$$

(4)

where $p_{j} = (p_{j}, -i\beta_{j})$, $q = (0, -i\alpha)$ and $Q = (0, -iQ)$ are two-momenta for tachyons, “cosmological terms”, and the source. For a non-negative integer $s$, we can evaluate the non-zero mode $\tilde{\phi}$ integral by regarding the amplitude as a scattering amplitude of $N$-tachyons and $s$ “cosmological terms”. After fixing the $SL(2, \mathbb{C})$ gauge invariance ($z_{1} = 0, z_{2} = 1, z_{3} = \infty$), an integral representation for the $N$-tachyon amplitude is given by

$$\left\langle \prod_{j=1}^{N} O_{p_{j}} \right\rangle = 2\pi \delta \left( \sum_{j=1}^{N} p_{j} \right) \frac{1}{-\alpha} \Gamma(-s) \tilde{A}(p_{1}, \cdots, p_{N}),$$

(5)

$$\tilde{A} = \left( \frac{\mu}{\pi} \right)^{s} \int \prod_{i=4}^{N} d^{2}z_{i} \prod_{j=1}^{s} d^{2}w_{j} \prod_{i=4}^{N} (|z_{i}|^{2p_{1} \cdot p_{i}} |1 - z_{i}|^{2p_{2} \cdot p_{i}}) \prod_{4 \leq i < j \leq N} |z_{i} - z_{j}|^{2p_{1} \cdot p_{j}} \prod_{s \leq i < j \leq N} |z_{i} - z_{j}|^{2p_{1} \cdot p_{j}} \prod_{1 \leq j < k \leq s} |w_{j} - w_{k}|^{2q \cdot q_{j} \cdot q_{k}},$$

(6)
In spite of the non-analytic relation (2) between energy $\beta$ and momentum $p$, we need to continue analytically the formula into general complex values of momenta in order to explore the singularity structure. Hence we will define the tachyon to have positive (negative) chirality if $(\beta + \sqrt{2})/p = 1(-1)$ irrespective of the actual values of momentum [19]. It seems to us that the operators with $\beta < -\sqrt{2}$ in eq. (3) are free from the trouble noted in [4,5] since $\phi_0$ has already been integrated out. The physical values of momenta are reached by analytic continuation in $s$, since $s$ is related to other momenta through energy-momentum conservation (4). For generic physical values of momenta, one finds a finite result for the $N$-tachyon amplitudes. However, the result is different in different chirality configurations, since the amplitude is non-analytic in momenta.

If $p_1$ has negative chirality and the rest $p_2, \cdots, p_N$ positive chirality, the amplitude is given by [7-11]

$$\tilde{A}(p_1, \cdots, p_N) = \frac{\pi^{N-3}[\mu\Delta(-\rho)]^s}{\Gamma(N + s - 2)} \prod_{j=2}^{N} \Delta(1 - \sqrt{2}p_j),$$  \hspace{1cm} (7)

where $\Delta(x) = \Gamma(x)/\Gamma(1-x)$. The regularization parameter $\rho$ is given by $\rho = -\alpha^2/2$ and is eventually set equal to $-1$ after the analytic continuation (in the central charge $c$). We should replace the combination $\mu\Delta(-\rho)$ by the renormalized cosmological constant $\mu_r$, since the correct cosmological term is $\phi \, e^{\alpha\phi}$. The amplitude exhibits singularities at $p_j = (n + 1)/\sqrt{2}$, $n = 0, 1, 2, \cdots$, but has no singularities in other combinations of momenta contrary to the dual amplitudes in the critical string theory. These poles for $n = 1, 2, \cdots$ will be shown to correspond to topological states as argued by several people [14,19].

The amplitudes with one tachyon of positive chirality and the rest negative are given by changing the sign of $p_j$. On the other hand, if each chirality has two or more tachyons, $\tilde{A}$ is finite for generic momenta but has the factor $1/\Gamma(-s)$. Hence $\tilde{A}$ vanishes for more than two tachyons in each chirality, when we consider non-negative integer $s$ in the following. This property has been explicitly demonstrated
for the four- and five-tachyon amplitudes in the Liouville theory [8,19], and has been argued to be a general property using the matrix model [14]. Therefore we take it for granted that the tachyon scattering amplitudes \( \tilde{A} \) vanish for non-negative integer \( s \), unless there is only one tachyon in either one of the chiralities. Let us note that our assertion is consistent with the argument for vanishing S-matrix in ref. [14]: they absorbed the \( \Delta(1 \pm \sqrt{2} \rho) \) factor in the amplitude to a renormalization factor of vertex operators, which becomes infinite if there is only one tachyon in either one of the chiralities [8-10]. Because of this infinite renormalization, their renormalized amplitudes vanish even if there is only one tachyon in either one of the chiralities.

In order to understand the poles of the amplitudes in terms of short-distance singularities in the OPE, we shall consider the case of \( s = \) non-negative integers by choosing the momentum configuration appropriately. These amplitudes at non-negative integer \( s \) represent so-called “bulk” or “resonant” interactions [14,19]. Here we shall take the case of \( s = 0 \) for the \( N \)-tachyon amplitude with only one negative chirality tachyon \( (p_1) \), and examine the \( s = \) positive integers case at the end.

First we shall illustrate the origin of short-distance singularities in the simplest context by expanding the integrand of the four tachyon scattering amplitude with \( s = 0 \) (we fix \( z_2 = 0, z_3 = 1, z_4 = \infty \) and call \( z_1 = z \))

\[
\tilde{A}(p_1, \cdots, p_4) = \int d^2z \left| z \right|^{2p_1 \cdot p_2} \left| 1 - z \right|^{2p_1 \cdot p_3}
\approx \int d^2z \left| z \right|^{-2\sqrt{2}p_2} \left| \sum_{n=0}^{\infty} \left( \frac{\Gamma(1 - \sqrt{2}p_3)}{n! \Gamma(-\sqrt{2}p_3 - n + 1)} \right) (-z)^n \right|^2
\approx \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{\pi}{n + 1 - \sqrt{2}p_2} \prod_{j=3}^{4} \Delta(1 - \sqrt{2}p_j).
\]

(8)

This shows that all the singularities in \( p_2 \) in the full amplitude are correctly accounted for by these short-distance singularities near \( z_1 \sim z_2 \). Furthermore we find
that successive poles are due to successive terms in the OPE

\[ :e^{i p_1 \cdot X(z_1)} : : e^{i p_2 \cdot X(z_2)} : \sim \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^2 |z_1 - z_2|^2 p_1 \cdot p_2 + 2n : e^{i p_2 \cdot X(z_2)} \bar{\partial}^n \partial^n e^{i p_1 \cdot X(z_2)} : , \]

with \( X = (X, \phi) \). The pole at \( p_2 = 1/\sqrt{2} \ (n = 0) \) is due to the tachyon intermediate state. The higher level poles \((n \geq 1)\) are due to the topological states which we discuss shortly.

We next examine short-distance singularities in amplitudes with five or more tachyons. By the same token, we consider the short-distance singularities due to the OPE (9) of two vertex operators \( p_1 \) and \( p_2 \) at \( z_1 \) and \( z_2 \). Because of the kinematical constraint, these singularities give poles in \( p_2 \) at \((n + 1)/\sqrt{2}, n = 0, 1, 2, \cdots \). The residues of these poles are given by kinds of dual amplitudes with \( N - 2 \) tachyons \( p_3, \cdots, p_N \), and an intermediate particle of two-momentum \( p = p_1 + p_2 \) (Fig. 1).

Similarly to the four tachyon case, the \( p_2 = 1/\sqrt{2} \ (n = 0) \) pole is due to the tachyon intermediate state with negative chirality. In fact we find that the residue of the pole \( p_2 = 1/\sqrt{2} \) is precisely given by the \( N - 1 \) tachyon amplitude with a single (intermediate state) tachyon \( p \) having negative chirality and the rest \( p_3, \cdots, p_N \) having positive chirality

\[ \tilde{A}(p_1, p_2, p_3, \cdots, p_N) \approx \frac{\pi}{(N - 3)(1 - \sqrt{2}p_2)} A(p, p_3, \cdots, p_N). \]  

This shows that the factorization is valid similarly to critical string theory. By symmetry, we can explain the lowest poles in each individual momentum \( p_j = 1/\sqrt{2} \) as the tachyon intermediate state in the OPE of \( p_1 \) and \( p_j \).

For higher level poles, we explicitly evaluate the residue of the short-distance singularities up to \( p = 3/\sqrt{2} \) and up to \( N = 5 \). For instance, the five tachyon amplitude has short-distance singularities at \( p_2 = 2/\sqrt{2} \) and \( 3/\sqrt{2} \)

\[ \tilde{A}(p_1, \cdots, p_5) \approx \left[ -\frac{\pi^2}{2(2 - \sqrt{2}p_2)} + \frac{\pi^2}{8(3 - \sqrt{2}p_2)} \right] \prod_{j=3}^{5} \Delta(1 - \sqrt{2}p_j). \]  

The residues of these poles in fact correctly reproduce the residues of the poles in
the full amplitude. It is rather difficult to compute the short-distance singularities explicitly to an arbitrary level except for the four-point amplitude that we have already worked out in eq. (8). Therefore we content ourselves with the computation of lower level singularities in explicitly demonstrating that the singularities of the amplitudes all come from the short-distance singularities of $p_1$ and $p_j$.

Since the short-distance singularities should come from terms in the OPE, we next examine the operators responsible for these singularities. For higher level poles, it has been pointed out that there are only null states at generic values of momenta [19,14]. However, there are exceptional values of momenta where the null states degenerate and new primary states emerge as a result. These new primary states are called co-dimension two operators by Polyakov [19], and special states or topological states by other people [14,18]. We can construct vertex operators for these topological states in the following way. The Virasoro generators $L_m$ for the $c = 1$ quantum gravity are the sum of the generators of the free scalar $X$ and those of the Liouville field $\phi$. The condition for the existence of the pole at level $n$ is given by

$$p \cdot (p + Q) = 2(1 - n).$$

(12)

We should note that there are two branches of the solution for the condition (12)

$$\beta = -\sqrt{2} \pm \sqrt{p^2 + 2n}.$$  

(13)

Although Seiberg has noted trouble with the lower sign due to the zero mode $\phi_0$ [4,5], we consider both cases here, since we are considering OPE of vertex operators consisting of non-zero mode $\tilde{\phi}$ only ($\phi_0$ integration gave the s “cosmological terms”). We shall call the upper sign solution S- (Seiberg) type and the lower A- (anti-Seiberg) type. We should construct the field of conformal weight $(1,1)$ by taking linear combinations of monomials of derivatives of $X$ multiplied by $e^{i\mathbf{p} \cdot \mathbf{X}}$. For instance, at level $n = 1$ we find only one field with weight $(1,1)$ at generic
values of momentum, i.e. $p \neq 0$

$$V^{(1)} = p \cdot \partial X \cdot \bar{p} \cdot \partial X e^{i p \cdot X} = -L_{-1} \bar{L}_{-1} e^{i p \cdot X}. \quad (14)$$

The above state is clearly null.

However, the situation changes at $p = 0$. For the S-type, the operator vanishes at $p = 0$. Therefore we can construct a new operator by a limit

$$V_{(1,1)} = \lim_{p \to 0} \frac{V^{(1)}}{p^2} = \partial X \bar{\partial} X. \quad (15)$$

We easily find that this field is primary and not null. This kind of a peculiar operator exists only at a discrete momentum and hence it is called co-dimension two. This is precisely the “graviton”, i.e. the first topological state which gives rise to the pole at $p_2 = 2/\sqrt{2}$ in eq. (8). As for the A-type at $p = 0$, we find that the $(1, 1)$ operator condition does not constrain the polarization tensor multiplying the operator $\partial X \bar{\partial} X e^{i p \cdot X}$. Hence we again obtain a new primary field

$$V'_{(1,1)} = \partial X \bar{\partial} X e^{-2\sqrt{2}\phi}. \quad (16)$$

At level two, we find two independent fields of weight one for the holomorphic part. The two fields for the holomorphic part are

$$V^{(2)} = \left( L_{-2} + \frac{3}{2} L_{-1}^2 \right) e^{i p \cdot X},$$

$$V^{(3)} = L_{-1} \left( \frac{1}{4} i \left[ (8 - p \cdot Q)p - 2Q \right] \cdot \partial X e^{i p \cdot X} \right). \quad (17)$$

Both fields are null. These two operators are linearly dependent at a special value of the momentum and we obtain a topological state. For instance, the (2,1) topo-
logical state of S-type is given by

$$V_{(2,1)} = \lim_{p \to \frac{1}{\sqrt{2}}} \frac{6\sqrt{2}}{p - \frac{1}{\sqrt{2}}}(V^{(2)} - V^{(3)})$$

$$= (13 \partial X \partial X - \partial \phi \partial \phi - 6 i \partial X \partial \phi - \sqrt{2} i \partial^2 X - \sqrt{2} \partial^2 \phi) e^{\frac{1}{\sqrt{2}} i (X - i \phi)}.$$  \hspace{1cm} (18)

We find exactly the same situation for the antiholomorphic part. We can continue to explore (1, 1) operators at higher levels similarly. We expect that these (1, 1) operators are null fields for generic values of momenta, and that, at special values of momenta, these null states are not linearly independent, namely they degenerate. Then we obtain a new primary state from a limit of an appropriate linear combination of these null states. We expect to have both S-type and A-type topological states.

There are other procedures to obtain topological states. These states were found to originate from the gravitational dressing of the primary states in the $c = 1$ conformal field theory which create the null descendants at level $n$ [15, 18]. The momentum $p$ of the initial primary state and the level $n$ are specified by two positive integers $(r, t)$ and thus the energy $\beta$ of the topological state is also given by

$$p = \frac{r - t}{\sqrt{2}}, \quad n = rt, \quad \beta = \frac{-2 \pm (r + t)}{\sqrt{2}}.$$  \hspace{1cm} (19)

The upper (lower) sign corresponds to the S-(A-)type solution. We find that these operators with $(r, t) = (1, 1)$ and $(2, 1)$ differ from our operators (15) and (18) respectively, only by a certain amount of null operators.

The OPE suggests that there may be other short-distance singularities in other combinations of momenta if one considers other combinations of vertex operators approaching to the same point. For instance, short-distance singularities corresponding to $k$ vertex operators approaching each other, say $z_1, \ldots, z_k$, should give poles in $p_2 + \cdots + p_k$. It is most convenient to fix reduced variables $u_j = (z_j - z_2)/(z_1 - z_2)$ ($j = 1, \ldots, k$) to take the short-distance limit $z_1 \to z_2$. 

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The amplitude exhibits short-distance singularities whose residues are given by a product of two dual amplitudes (Fig. 2)

\[ \tilde{A}(p_1, \cdots, p_N) \approx \frac{1}{V_{SL(2,C)}} \int_{|z_1-z_2| \leq \epsilon} d^2 z_1 d^2 z_2 \prod_{i=3}^{k} d^2 u_i \prod_{j=k+1}^{N} d^2 z_j |z_1 - z_2|^{p \cdot p + Q \cdot p - 4} \]

\[ \times \prod_{1 \leq i < j \leq k} |u_i - u_j|^2 p_i \cdot p_j \prod_{i=1}^{k} \prod_{j=k+1}^{N} \left| 1 + \frac{z_1 - z_2}{z_2 - z_j} u_i \right|^2 p_i \cdot p_j \]

\[ \times \prod_{i=k+1}^{N} |z_2 - z_i|^2 p_i \cdot p_i \prod_{k+1 \leq i < j \leq N} |z_i - z_j|^2 p_i \cdot p_j. \]

\( (20) \)

The dual amplitude with the original variables \( z_i (i = 2, k + 1, \cdots, N) \) has \( N - k \) positive chirality tachyons \( p_{k+1}, \cdots, p_N \) and the intermediate state particle \( p \) (right side blob in Fig. 2), whereas the dual amplitude with the reduced variables \( u_j \) \((j = 3, \cdots, k)\) has the intermediate state particle \(-p - Q\) and \( k \) tachyons \( p_1, \cdots, p_k \) whose chiralities are positive except \( p_1 \) (left side blob in Fig. 2). If \( p \) is the intermediate state momentum flowing into the right side blob, the corresponding momentum for the dual amplitude of the left side blob can be regarded as \(-p - Q\).

This implies that the chirality of the tachyon intermediate state is the same for both dual amplitudes. In the case of the intermediate topological state, the type (S or A) of the intermediate state for one dual amplitude turns out to be opposite to the other dual amplitude. In the present case of pinching together the single negative chirality tachyon with the positive chirality tachyons, energy-momentum conservation (4) dictates that the intermediate state tachyon has negative chirality. On the other hand, the tachyon amplitudes are non-vanishing only if a single tachyon has one of the chiralities and the rest have opposite chirality. Therefore the dual amplitude with the reduced variables vanishes except when it is the three-point function \((k = 2)\). This is precisely the case we have evaluated already in eq. (10). As for the intermediate topological states, kinematics dictates that it is of S-type (A-type) for the dual amplitude with the original variables (reduced variables). The three-point function with the A-type topological state \((k = 2)\) is nothing but
the OPE coefficient (9) which we have seen non-vanishing. Four- and more- point functions with the A-type topological state \((k \geq 3)\) are more difficult to compute.

The topological state of level \(n\) consists of a linear combination of monomials of derivatives of \(X\) multiplied by a vertex operator \(e^{i p \cdot X}\). Both the number of \(\partial\) and the number of \(\bar{\partial}\) should be \(n\) for each monomial. The two momentum \(p\) is given by \(((r-t)/\sqrt{2}, -i(-2-r-t)/\sqrt{2})\) for the \((r,t)\) topological state of type A. If we do not specify the coefficients of the monomials, we obtain an operator containing the \((r,t)\) topological state together with certain amount of null states. We have taken such an operator as a substitute for the \((r,t)\) topological state of A-type at the level \(n = rt\), and have explicitly evaluated the dual amplitude with the topological state for the case of four-point function. We have found it to vanish. This amplitude arises as the left side blob in Fig. 2 contributing to the pole of level \(n = rt\) in the case of \(k = N - r - 1 = 3\). We conjecture in general that the A-type topological state gives vanishing dual amplitude except for the three-point function (\(k = 2\)). This property is presumably related to the Seiberg's finding that only the S-type is physical. Only in the three-point dual amplitude (\(k = 2\)), we can simply regard the factor for the blob of particles pinched together (left side blob in Fig. 2) as the coefficient of the OPE rather than the dual amplitude.

Other possibilities are short-distance singularities from the pinching of \(k\) tachyons all with positive chirality, say \(p_2, \cdots, p_{k+1}\). We again find that the chirality of the intermediate state tachyon is negative. Hence the dual amplitude with the original variables contains two tachyons with negative chirality \(p_1, p\) besides \(N - k - 1\) positive chirality tachyons \(p_{k+2}, \cdots, p_N\). Hence the amplitude vanishes except for the three-point function (\(N = k + 2\)). The non-zero result of the three-point function gives rise to poles in the momentum \(p_N\) of the single tachyon with the positive chirality. We can regard these poles to be the same short-distance singularities as obtained in \(z_N \to z_1\). Kinematics shows that the intermediate topological states in this case is of A-type (S-type) for the dual amplitude with the original variables (reduced variables). According to our conjecture above, the dual amplitude with the original variables again vanishes except for the three-point function (\(N = k + 2\)).
The non-zero result can be interpreted in the same way as the tachyon intermediate state.

These observations explain why there are only singularities in the individual $p_j$, and none in any combinations of momenta, although the factorization of the $N$-tachyon amplitudes is valid through the OPE as we have seen.

Let us finally discuss the case of $s =$ positive integer. The amplitudes with $s =$ positive integer can be obtained from the $s = 0$ case as follows: we consider the $N + s$ tachyon scattering amplitude and take a limit of vanishing momenta for $s$ tachyons and multiply by $(\mu/\pi)^s$. There is one subtlety: at the vanishing momenta, the chirality is ill-defined [8-10]. We define the vanishing momenta taking limit from the positive chirality tachyon. In the limit, we obtain an $s$-th power of a singular factor $\mu \Delta(0)$, which should be replaced by the renormalized cosmological constant $\mu_r$. In this way we find that the short-distance singularities of the amplitudes with non-vanishing $s$ can be obtained correctly once the short-distance singularities in the $s = 0$ amplitude are correctly obtained. Using the previous argument, we find that the only non-vanishing short-distance singularities are from the OPE of two vertex operators for tachyons. Short-distance singularities from one or more “cosmological term operators” approaching the tachyon vertex operators give a vanishing value for the residue.

Acknowledgements

One of the authors (NS) thanks Y. Kitazawa and D. Gross for a discussion on the Liouville theory. We would like to thank Patrick Crehan for a careful reading of the manuscript. This work is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture (No.01541237).
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Figure Captions

1) The factorization of the $N$-tachyon amplitude by the OPE of the operators 1 and 2. The signs $+$ and $-$ denote the chirality of the tachyons.

2) The factorization of the $N$-tachyon amplitude by the OPE of the operators $1, \cdots, k$. The signs $+$ and $-$ denote the chirality of the tachyons.