All-order Resurgence from Complexified Path Integral in a Quantum Mechanical System with Integrability

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We discuss all-order transseries in one of the simplest quantum mechanical systems: a U(1) symmetric single-degree-of-freedom system with a first-order time derivative term. Following the procedure of the Lefschetz thimble method, we explicitly evaluate the path integral for the generating function of the Noether charge and derive its exact transseries expression. Using the conservation law, we find all the complex saddle points of the action, which are responsible for the non-perturbative effects and the resurgence structure of the model. The all-order power-series contributions around each saddle point are generated from the one-loop determinant with the help of the differential equations obeyed by the generating function. The transseries are constructed by summing up the contributions from all the relevant saddle points, which we identify by determining the intersection numbers between the dual thimbles and the original path integration contour. We confirm that the Borel ambiguities of the perturbation series are cancelled by the non-perturbative ambiguities originating from the discontinuous jumps of the intersection numbers. The transseries computed in the path-integral formalism agrees with the exact generating function, whose explicit form can be obtained in the operator formalism thanks to the integrable nature of the model. This agreement indicates the non-perturbative completeness of the transseries obtained by the semi-classical expansion of the path integral based on the Lefschetz thimble method.

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1. INTRODUCTION

Path integral formalism is one of the fundamental tools to formulate quantum systems. It is based on integration over infinite-dimensional functional spaces of fields. Although it is a general and intuitive formulation, path integrals can rarely be evaluated exactly. One can use the perturbative expansion to approximate a path integral as a power series of a coupling constant. Although such a perturbation series gives a good approximation when the coupling constant is small, it is usually an asymptotic series with a zero radius of convergence. Hence, the perturbation series truncated at a finite order has limited accuracy, particularly for a large expansion parameter. A possible prescription for such a divergent series is Borel resummation. It applies to asymptotic series with factorially divergent expansion coefficients and gives a closed form for the series if it is Borel summable, i.e. its Borel transform is non-singular on the positive real axis of the Borel plane. However, in many physical systems, perturbation series are non-Borel summable and associated with ambiguities depending on the regularization.

The remedy for such an ill-defined series is to construct the so-called transseries by appropriately summing up the contributions of saddle points, that is, classical solutions of the action. According to the resurgence theory [1] (see e.g. [2–6] for reviews on the application of the resurgence theory to field theories), all the ambiguities from the perturbative and non-perturbative sectors cancel in the transseries. As in the steepest descent (stationary phase) method for ordinary finite-dimensional integrals, we have to take into account complex saddle points which can be found by analytically continuing the action as a holomorphic functional of the fields (see e.g. [7–14] for examples of systems where complex saddle points called “bions” play important roles). Not all the saddle points are relevant, but a specific subset can contribute. Such a subset can be determined by the Lefschetz thimble method, which states that the relevant saddle points are those with steepest-ascent flows (dual thimbles) intersecting with the original path integration contour (the original configuration space). The contribution from each saddle point is given by the integral over the associated thimble (steepest descent flows). Its ambiguity is related to a Stokes phenomenon, a sudden change in the shape of the (dual) thimble, which occurs when the argument of the coupling constant is varied. Although the contribution from each saddle point can be ambiguous due to such a Stokes phenomenon, the transseries constructed through the Lefschetz thimble method is unambiguous thanks to the cancellation mechanism of the ambiguities. Such a resurgence structure enables us to find a well-defined closed form with the correct asymptotic expansion. The procedure for evaluating path integrals based on the Lefschetz thimble method can be summarized as follows:

1. The first step is to find saddle points by solving the complexified equations of motion derived from the classical action analytically continued to the complexified configuration space.

2. The second step is to determine the contribution of each saddle point. It is defined as the path integral over the Lefschetz thimble associated with each saddle point. It is, however, usually impossible to directly evaluate such a path integral. Instead, one can evaluate the saddle point contribution by applying the Borel resummation to the perturbation series around the saddle point configuration.

3. The third step is to identify the relevant saddle points by examining the intersection numbers be-
tween the original configuration space and the dual thimbles. The transseries can be constructed by summing up the contributions of all the saddle points using the intersection numbers as coefficients.

The question is whether such a transseries is exact or not. It would not be so difficult to see that the Lefschetz thimble method gives exact results for finite-dimensional integrals \[7\]. On the other hand, path integrals cannot be explicitly evaluated in almost all cases, and hence there is less chance to check the exactness of the transseries. If there exists a model with the following properties, it can serve as a testing ground for the Lefschetz thimble method:

1. All saddle point solutions can be found by solving the complexified equations of motion.
2. Expansion coefficients around each saddle point can be determined to all orders.
3. All intersection numbers can be determined.
4. Exact results can be obtained via another method.

It is natural to imagine that the set of the properties described above implies integrability. Resurgence structure of integrable fields theories has been discussed in \[15–23\] and it has been shown that integrability is a powerful tool for studying resurgence structure. In this paper, we focus on the case of quantum mechanics that is exactly solvable due to integrability. An integrable quantum mechanical system is a model with a finite number of degrees of freedom possessing a maximal set of commuting conserved charges. In such a system, Hamiltonian can be written by using the action-angle variables \((\nu, \theta)\) as a function depending only on the conserved charge \(H = V(\nu)\). The simplest class of such integrable quantum mechanical models is the single variable \(U(1)\) symmetric first-order time-derivative system.

This model can be viewed as a system of a particle on a 2D plane with a large rotationally invariant potential and a constant magnetic field.\(^1\) Compared to quantum mechanics with quadratic kinetic terms, where resurgence has been extensively discussed \[8–14, 24–44\], the first-order time derivative system has half the degrees of freedom and hence a single variable system is integrable if there is a conserved charge. Therefore, it provides a good playground where we can test the completeness of the Lefschetz thimble method. Another important property, which enables us to evaluate the perturbation series around each saddle point, is that the generating function \(Z\) for the conserved charge obeys a partial differential equation of the form

\[
\frac{\partial}{\partial g} Z = X(g, \partial \mu) Z, \tag{1.1}
\]

where \(g\) is the coupling constant (expansion parameter), \(\mu\) is the external source (imaginary chemical potential) for the conserved charge and \(X(g, \partial \mu)\) is a differential operator which depends on the Hamiltonian of the system. By using the power series ansatz on top of the saddle point value \(e^{-S_{\text{saddle}}/g^2}\), the differential equation (1.1) can be rewritten into a recursion relation for the expansion coefficients.

\(^1\) This system can be viewed as a dimensional reduction of the non-linear Schrödinger system in two dimensions, whose resurgence structure is yet to be elucidated from the viewpoint of the Lefschetz thimble method.
Starting from the initial term corresponding to the one-loop determinant around the saddle point, we can solve the recursion relation and determine the all-order power series around each saddle point. Another convenient property of our model is that the intersection numbers are accessible in a simple way. In particular, we will explicitly determine the intersection numbers by solving the gradient flow equation. Although the gradient flow equation is originally defined in the complexified configuration space, it is reduced to a finite-dimensional problem using a symmetry argument. Using these special properties, we will show that the transseries obtained in the path integral formalism agrees with the exact partition function obtained in the operator formalism.

The organization of this paper is as follows. In section 2 we discuss the resurgence structure in the first-order time derivative system with a $U(1)$ symmetric quartic potential. After defining the generating function for the conserved charge $Z(g)$ in section 2.1, we discuss the perturbation series for $Z(g)$ in section 2.2. All the coefficients of $Z(g)$ are determined by perturbatively solving the differential equation for $Z(g)$. We see that the perturbation series is non-Borel summable due to some singularities of its Borel transform. In section 2.3, we calculate the contributions of complex saddle point solutions and determine the relevant saddle points by examining the intersection numbers based on the Lefschetz thimble method in section 2.4. We see that the ambiguities of the saddle point contributions cancel those of the perturbative part. In section 2.5, we compare the generating function obtained in the path integral formalism with that calculated in the operator formalism. In section 3, we discuss the generalization to the case of generic $U(1)$ symmetric potential. Section 4 outlines a generalization to more general integrable quantum mechanical systems. Section 5 is devoted to conclusions and discussion. Appendix A is a brief review of the Lefschetz thimble method, and Appendix B is a supplement on the properties of the differential equation for the generating function.

2. FIRST-ORDER SYSTEM WITH A $U(1)$ SYMMETRIC QUARTIC POTENTIAL

In this section, we discuss the resurgence structure of the first-order time derivative system with a $U(1)$ symmetric quartic potential. This quantum mechanical system is one of the simplest examples of the models in which transseries for some quantities such as partition function can be exactly obtained in the path integral formalism.

2.1. Action, Hamiltonian and Generating Function

Let us consider the 1d system described by the action

$$S = \int dt L = \int dt \left[ i \phi \partial_t \phi - \frac{g}{2} |\phi|^4 \right],$$

(2.1)

where $\phi$ stands for a complex scalar degree of freedom and $g$ is a coupling constant. This model can be viewed as a system of a particle on the $(x, y)$-plane ($\phi \propto \sqrt{B}(x + iy)$) with a large magnetic field $B$ and a potential $V \propto B^2 |x + iy|^4$. Since the Lagrangian $L$ is linear in the time derivative, the canonical
conjugate of $\phi$ is identified with its complex conjugate $i \bar{\phi}$. The Hamiltonian of this system is given by

$$H = i \bar{\phi} \partial_t \phi - L = \frac{g}{2} |\phi|^4.$$  \hfill (2.2)

This is the conserved quantity corresponding to the time translation invariance. Another conserved quantity is the Noether charge for the phase rotation symmetry $\phi \to e^{i\alpha} \phi$

$$\mathcal{N} = |\phi|^2.$$  \hfill (2.3)

In this section, we discuss the resurgence structure of this model by investigating the weak coupling expansion of the generating function for the expectation value of $\mathcal{N}$

$$Z = \text{Tr} \left[ e^{-\beta (H + i\mu \hat{N})} \right],$$  \hfill (2.4)

where $\mu$ is the external source for $\hat{N}$ and can be interpreted as an imaginary chemical potential. Since the canonical commutation relation in this system is given by

$$[\hat{\phi}, \hat{\phi}^\dagger] = 1,$$  \hfill (2.5)

the operators $\hat{\phi}$ and $\hat{\phi}^\dagger$ do not commute with each other and hence we must specify the order of the operators to define the conserved charges. In this paper, we adopt the following ordering for the conserved charges

$$\hat{\mathcal{N}} = \hat{\phi}^\dagger \hat{\phi}, \quad \hat{H} = \frac{g}{2} (\hat{\phi}^\dagger \hat{\phi})^2.$$  \hfill (2.6)

With this convention, we can show that the generating function \[2.4\] satisfies the “heat equation”

$$\left[ \frac{\partial}{\partial g} - \frac{1}{2\beta} \frac{\partial^2}{\partial \mu^2} \right] Z = \text{Tr} \left[ -\frac{\beta}{2} \left\{ (\hat{\phi}^\dagger \hat{\phi})^2 - \hat{\mathcal{N}}^2 \right\} e^{-\beta (H + i\mu \hat{N})} \right] = 0.$$  \hfill (2.7)

As we will see, this differential equation enables us to determine the perturbation series to all orders in the coupling constant $g$.

In the operator formalism, the generating function $Z$ can be determined by using the number eigenstates. The Hamiltonian can be rewritten in terms of the number operator $\hat{\mathcal{N}}$ as

$$\hat{H} = \frac{g}{2} \hat{\mathcal{N}}^2.$$  \hfill (2.8)

---

2 The chemical potential $\mu$ can also be viewed as a constant background gauge field (holonomy) $A_0$ for the $U(1)$ symmetry, and hence it has periodicity $\mu \sim \mu + 2\pi/\beta$.

3 Throughout this paper, the operator corresponding to the classical variable $\bar{\phi}$ is denoted by $\hat{\phi}^\dagger$. 
This implies that the energy eigenstates are the number eigenstates

$$|n\rangle = \frac{1}{\sqrt{n!}} (\hat{\phi}^\dagger)^n |0\rangle, \quad \hat{\phi} |0\rangle = 0, \quad \hat{H} |n\rangle = \frac{g}{2} n^2 |n\rangle. \quad (2.9)$$

Therefore, the generating function can be written as

$$Z = \text{Tr} \left[ e^{-\beta (\hat{H} + i\mu \hat{N})} \right] = \sum_{n=0}^{\infty} \exp \left[ -\frac{\beta g}{2} n^2 - i\beta \mu n \right]. \quad (2.10)$$

We can confirm that this satisfies the differential equation (2.7). In the next section, we calculate the same quantity by applying the Lefschetz thimble method (see Appendix A for a brief review of the Lefschetz thimble method) in the path integral formalism and check that the nontrivial resurgence structure obtained through the Lefschetz thimble method leads to the exact result (2.10) with no ambiguity.

### a. Generating Function in Path Integral Formalism

Let us consider the generating function $Z$ from the viewpoint of the path integral formalism. Using the Weyl ordering, we can rewrite the operator $\hat{H} + i\mu \hat{N}$ as

$$\hat{H} + i\mu \hat{N} = \left[ \frac{g}{2} (|\phi|^4)_W + i\mu (|\phi|^2)_W \right] + \left[ -\frac{g}{2} (|\phi|^2)_W - \frac{i\mu}{2} \right], \quad (2.11)$$

where $(|\phi|^{2n})_W$ denotes the Weyl ordered operator

$$(|\phi|^{2n})_W = \frac{1}{(2n)!} \frac{\partial^n}{\partial a^n} \frac{\partial^n}{\partial b^n} (a\hat{\phi} + b\hat{\phi}^\dagger)^{2n}. \quad (2.12)$$

Therefore, in the path integral formalism, the generating function is given by

$$Z = \int \mathcal{D}\phi \exp \left( -S_E - S_W \right), \quad (2.13)$$

where $S_E$ is the (Wick rotated $t \to -i\tau$) action with the source term and $S_W$ is the part generated when the Hamiltonian is rewritten in terms of the Weyl ordered operators

$$S_E = \int_0^\beta d\tau \left[ \hat{\phi} \partial_\tau \phi + \frac{g}{2} |\phi|^4 + i\mu |\phi|^2 \right], \quad S_W = \int_0^\beta d\tau \left( -\frac{g}{2} |\phi|^2 - \frac{i\mu}{2} \right). \quad (2.14)$$

Corresponding to the trace in Eq. (2.4), the path integral should be carried out over the configurations satisfying the periodic boundary condition

$$\phi(\tau + \beta) = \phi(\tau). \quad (2.15)$$

It is convenient to rescale the variable as

$$\phi = \sqrt{g} \frac{\varphi}{\sqrt{g}}, \quad \hat{\phi} = \frac{\hat{\varphi}}{\sqrt{g}}. \quad (2.16)$$
Then, \( S_E \) and \( S_W \) become

\[
S_E = \frac{1}{g} \int_0^\beta d\tau \left( \bar{\varphi} \partial_\tau \varphi + \frac{1}{2} |\varphi|^4 + i \mu |\varphi|^2 \right), \quad S_W = \int_0^\beta d\tau \left( -\frac{1}{2} |\varphi|^2 - \frac{i \mu}{2} \right).
\] (2.17)

Thus, identifying the coupling constant \( g \) as the Planck constant, we regard \( S_E \) and \( S_W \) as a “classical action” and an “operator insertion”, respectively.

### 2.2. Perturbation Series

Let us first evaluate the path integral for the generating function \( Z \) in Eq. (2.13) by using the perturbative expansion. Although the standard diagrammatic perturbative expansion is possible, we can obtain the perturbation series to all orders in the coupling constant \( g \) more easily by using the differential equation for the generating function \( Z \):

\[
\left[ \frac{\partial}{\partial g} - \frac{1}{2\beta} \frac{\partial^2}{\partial \mu^2} \right] Z = 0.
\] (2.18)

This equation implies that the perturbation series can be obtained from the generating function of the free theory \( Z_0 = Z|_{g=0} \) as

\[
Z_{\text{pert}} = \sum_{n=0}^{\infty} \frac{g^n}{n!} \frac{\partial^n Z}{\partial g^n} \bigg|_{g=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{g}{2\beta} \frac{\partial^2}{\partial \mu^2} \right)^n Z_0.
\] (2.19)

To compute the generating function of the free theory \( Z_0 \), let us use the Fourier series expansion of the original variable

\[
\phi = \sum_{p=-\infty}^{\infty} c_p \exp \left( \frac{2\pi ip\tau}{\beta} \right).
\] (2.20)

In terms of the Fourier coefficients \( c_p \), the free theory action can be rewritten as

\[
\lim_{g \to 0} S_E = i\beta \sum_{p=-\infty}^{\infty} \omega_p |c_p|^2, \quad \lim_{g \to 0} S_W = -\frac{i\beta \mu}{2},
\] (2.21)

where we have defined

\[
\omega_p = \mu + \frac{2\pi p}{\beta} \quad (p \in \mathbb{Z}).
\] (2.22)

Performing the Gaussian integral for each mode, we obtain

\[
Z_0 = N e^{\frac{i \beta \mu}{\pi}} \prod_{p=-\infty}^{\infty} \int dc_p d\bar{c}_p e^{-i\beta\omega_p |c_p|^2} = N e^{\frac{i \beta \mu}{\pi}} \prod_{p=-\infty}^{\infty} \frac{\pi}{i(\beta \mu + 2\pi p)} = \frac{1}{1 - e^{-i\beta \mu}},
\] (2.23)
where we have chosen the normalization factor as
\[ N = \frac{1}{\pi} \prod_{p=1}^{\infty} (4p^2), \] (2.24)
and used the formula for the infinite product
\[ \prod_{p=1}^{\infty} \frac{1}{1 - z^2/p^2} = \frac{\pi z}{\sin \pi z}. \] (2.25)

One can easily show that this choice of the normalization and sign\(^4\) is consistent with the canonical quantization (see Sec. 2.5). Plugging \( Z_0 \) into Eq. (2.19), we obtain the perturbation series
\[ Z_{\text{pert}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{g^2}{2\beta} \frac{\partial^2}{\partial \mu^2} \right)^n \frac{1}{1 - e^{-i\beta \mu}}. \] (2.26)
This perturbation series is a divergent asymptotic series. To show this, let us expand the generating function of the free theory as
\[ Z_0 = \frac{1}{1 - e^{-i\beta \mu}} = \frac{1}{2} - i \sum_{p=-\infty}^{\infty} \frac{1}{\beta \omega_p}. \] (2.27)
Using this expanded form of \( Z_0 \), we can rewrite the perturbation series in Eq. (2.26) as
\[ Z_{\text{pert}} = \frac{1}{2} - i \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{g^2}{2\beta} \frac{\partial^2}{\partial \mu^2} \right)^n \sum_{p=-\infty}^{\infty} \frac{1}{\beta \omega_p} = \frac{1}{2} - \sum_{p=-\infty}^{\infty} \frac{i}{\beta \omega_p} \sum_{n=0}^{\infty} \left( \frac{2n!}{n!} \right) \left( \frac{g^2}{2\beta \omega_p^2} \right)^n. \] (2.28)
Since \( (2n)!/n! = 4^n \Gamma(n + 1/2)/\Gamma(1/2) \), this perturbation series is factorially divergent. Rewriting the series as
\[ \sum_{n=0}^{\infty} \left( \frac{2n!}{n!} \right) x^n = \int_0^{\infty} dt e^{-t} \sum_{n=0}^{\infty} \left( \frac{2n!}{(n!)^2} \right) (xt)^n = \int_0^{\infty} dt \frac{e^{-t}}{\sqrt{1 - 4xt}}, \] (2.29)
\(^4\) Note that the sign of the infinite product is ambiguous. Relabeling \( p = p' + q \) with an arbitrary integer \( q \), we find that the sign depends on the choice of \( q \)
\[ Z_0 = N e^{i\beta \mu} \prod_{p=-\infty}^{\infty} \frac{\pi}{i(\beta \mu + 2\pi p)} = N e^{i\beta \mu} \prod_{p'=\infty}^{\infty} \frac{\pi}{i(\beta \mu + 2\pi q + 2\pi p')} = \frac{(-1)^q}{1 - e^{-i\beta \mu}}. \]
This ambiguity is related to the "anomaly" of the periodicity (large gauge transformation) \( \mu \sim \mu + 2\pi/\beta \), which is canceled if \( S_W \) is appropriately taken into account.
\(^5\) Here, the summation should be interpreted as
\[ \sum_{p=-\infty}^{\infty} \frac{1}{\beta \omega_p} = \frac{1}{\beta \mu} + \sum_{p=1}^{\infty} \left( \frac{1}{\beta \omega_p} + \frac{1}{\beta \omega_{-p}} \right) = \frac{1}{\beta \mu} + \sum_{p=1}^{\infty} \frac{2\beta \mu}{(\beta \mu)^2 - (2\pi p)^2}. \]
we obtain the formal Borel resummation of the perturbation series

\[
Z_{\text{pert}} = \int_0^\infty dt \, e^{-t} \left[ \frac{1}{2} - \sum_{p=-\infty}^{\infty} \frac{i}{\beta \omega_p} \frac{1}{\sqrt{1 - \frac{2g}{\beta \omega_p^2}}} \right].
\]  

(2.30)

Since there are singularities at \( t = \frac{\beta \omega_p^2}{2g} \), we have to regularize the integral to obtain a finite value.

Fig. 1: Contours on Borel plane

Giving a small imaginary part to the coupling constant \( g \) (\( \arg g = \epsilon \)), or equivalently, performing the Borel resummation along the contours \( C_{\pm} \) shown in Fig. 1, we can avoid the singularity and obtain a finite value. However, the Borel resummation gives different answers depending on the sign of \( \text{Im} \sqrt{\frac{\beta}{2g} \omega_p} \).

\[
Z_{\text{pert}} = \frac{1}{2} - \sqrt{\frac{\pi}{2 \beta g}} \sum_{p=-\infty}^{\infty} e^{-\frac{\beta}{2 g} \omega_p^2} \left[ \text{erf} \left( \frac{i}{2} \sqrt{\frac{\beta}{2g} \omega_p} \right) + \text{sign} \left( \text{Im} \sqrt{\frac{\beta}{2g} \omega_p} \right) \right],
\]  

(2.31)

where \( \text{erf}(z) \) is the error function defined by

\[
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z dx \exp(-x^2).
\]  

(2.32)

The discontinuity at \( \arg g = 0 \) is given by

\[
Z_{\text{pert}}^{(+)} - Z_{\text{pert}}^{(-)} = \sqrt{\frac{2\pi}{\beta g}} \sum_{p=-\infty}^{\infty} \text{sign}(\omega_p) \exp \left( -\frac{\beta \omega_p^2}{2g} \right),
\]  

(2.33)

where \( Z_{\text{pert}}^{(\pm)} \) are the perturbation series for \( \arg g > 0 \) and \( \arg g < 0 \), respectively. This discontinuity has non-perturbative factors, and hence it is expected to be related to non-perturbative effects. In the next section, we show that there are complex saddle points of the Euclidean action whose non-perturbative contributions cancel these ambiguities of the perturbation series.
In this section, we look for the saddle points responsible for the non-perturbative effects in this model. In the following, we use the rescaled variable $\varphi = \sqrt{g} \phi$ so that the action takes the form given in (2.17). In addition to the classical vacuum solution $\varphi = 0$, the classical action $S_E$ in Eq. (2.17) has non-trivial complex saddle point solutions. Such solutions can be found by complexifying the degree of freedom

$$S_E[\varphi, \bar{\varphi}] \rightarrow S_E[\varphi, \tilde{\varphi}],$$

where $S_E[\varphi, \tilde{\varphi}]$ is interpreted as a holomorphic functional of two independent complex variables $\varphi$ and $\tilde{\varphi}$. The saddle points can be found by solving the complexified equations of motion

$$0 = \left[ + \partial_\tau + i\mu + \tilde{\varphi}\varphi \right] \varphi, \quad (2.35)$$
$$0 = \left[ - \partial_\tau + i\mu + \bar{\varphi}\tilde{\varphi} \right] \tilde{\varphi}. \quad (2.36)$$

We can show by using the conservation laws that besides $\varphi = \tilde{\varphi} = 0$, there are infinitely many complex saddle points labeled by an integer $p \in \mathbb{Z}$

$$\varphi = \sqrt{-i\omega_p} \exp\left(\frac{2\pi ip\tau}{\beta} + i\theta\right), \quad \tilde{\varphi} = \sqrt{-i\omega_p} \exp\left(-\frac{2\pi ip\tau}{\beta} - i\theta\right),$$

where $\theta$ is an integration constant (moduli parameter) and $\omega_p = \mu + 2\pi p/\beta$ as in the previous section. Note that $\tilde{\varphi}$ is not the complex conjugate of $\varphi$ (see Fig. 2.3) and hence these solutions are complex saddle points which are not contained in the original configuration space before the complexification. The values of the action at these saddle points are given by

$$S_E = \frac{\beta \omega_p^2}{2g}.$$
We can confirm that these values agree with the non-perturbative exponents of the discontinuity of the perturbative part \(^{(2.33)}\). To compute the contributions from these complex saddle points, let us consider the integration over the thimble \(J_p\) associated with the \(p\)-th complex saddle point

\[
Z_p = \int_{J_p} \mathcal{D}\varphi \exp \left( - S_E[\varphi, \tilde{\varphi}] - S_W[\varphi, \tilde{\varphi}] \right). \tag{2.39}
\]

We first focus on the leading order contribution in the weak coupling limit \(g \to 0\). Let \(c_q\) and \(\tilde{c}_q\) be the Fourier coefficients of \(\varphi\) and \(\tilde{\varphi}\)

\[
\varphi = \sqrt{g} \sum_{q=-\infty}^{\infty} c_q \exp \left( \frac{2\pi i q \tau}{\beta} \right), \quad \tilde{\varphi} = \sqrt{g} \sum_{q=-\infty}^{\infty} \tilde{c}_q \exp \left( \frac{2\pi i q \tau}{\beta} \right). \tag{2.40}
\]

The \(p\)-th saddle point corresponds to the configuration with

\[
c_q = e^{i\theta} \times \begin{cases} \sqrt{-i\omega_p/g} & \text{for } q = p \\ 0 & \text{for } q \neq p \end{cases} \quad \tilde{c}_q = e^{-i\theta} \times \begin{cases} \sqrt{-i\omega_p/g} & \text{for } q = -p \\ 0 & \text{for } q \neq -p \end{cases}. \tag{2.41}
\]

Now let us consider the change of the integration variables from \((c_q, \tilde{c}_q)\) to a set of coordinates parameterizing the neighborhood of the saddle point in the configuration space. Choosing the new integration variables \(b_q, \tilde{b}_q (q = \pm 1, \pm 2, \cdots), a\) and \(\theta\) around the saddle point as

\[
c_q = e^{i\theta} \times \begin{cases} \sqrt{-i\omega_p/g + a} & \text{for } q = p \\ b_{q-p} & \text{for } q \neq p \end{cases}, \quad \tilde{c}_q = e^{-i\theta} \times \begin{cases} \sqrt{-i\omega_p/g + a} & \text{for } q = -p \\ \tilde{b}_{q+p} & \text{for } q \neq -p \end{cases}, \tag{2.42}
\]

we find that around the saddle point, the action \(S_E\) and \(S_W\) take the forms of

\[
S_E = \beta \left[ \frac{\omega^2_p}{2g} - 2i\omega_p a^2 - i \sum_{q=1}^{\infty} \tilde{B}_{-q} X_{p,q} B_q \right] + \mathcal{O}(g^{3/2}), \quad S_W = p\pi i + \mathcal{O}(g^{1/2}), \tag{2.43}
\]

with

\[
B_q = \begin{pmatrix} b_q \\ \tilde{b}_q \end{pmatrix}, \quad \tilde{B}_{-q} = \begin{pmatrix} \tilde{b}_{-q} & b_{-q} \end{pmatrix}, \quad X_{p,q} = \begin{pmatrix} \omega_{p-q} & \omega_p \\ \omega_{p} & \omega_{p+q} \end{pmatrix}. \tag{2.44}
\]

In the weak coupling limit, the integration measure takes the form of

\[
N \prod_{q=-\infty}^{\infty} \left( \frac{i}{2} dc_q \wedge d\tilde{c}_{-q} = N \sqrt{\frac{\omega_p}{ig}} \frac{da \wedge d\theta}{\prod_{q=1}^{\infty} \left( \frac{i}{2} db_q \wedge d\tilde{b}_{-q} \right) \wedge \left( \frac{i}{2} db_{-q} \wedge d\tilde{b}_q \right) + \mathcal{O}(g^0)}, \right. \tag{2.45}
\]

where the normalization factor \(N\) is the same as the one used to compute the perturbation series in Eq. \((2.24)\). Using this integration measure, one can evaluate the leading order contribution from the \(p\)-th

---

\(^{6}\) The tangent directions corresponding to \(b_q, \tilde{b}_q\) and \(a\) are chosen so that their tangent vectors are orthogonal to the direction of the zero mode. The moduli integration over \(0 < \theta < 2\pi\) compensates the missing zero mode direction.
saddle point as

$$Z_p = N \int_\mathbb{R}_q \prod_{q=-\infty}^{\infty} \left( \frac{i}{2} dc_q \wedge d\bar{c}_{-q} \right) \exp (-S_E - S_W) = N e^{-\frac{\beta \omega^2_p}{2\tau} + p\pi i} \prod_{q=0}^{\infty} I_{p,q} + \cdots, \quad (2.46)$$

with

$$I_{p,0} = \sqrt{\frac{\omega_p}{ig}} \int da d\theta \exp (2i\omega_p \beta a^2) = \frac{\pi \sqrt{2\pi}}{\beta g}, \quad (2.47)$$

$$I_{p,q} = \int db db_q db_{-q} db_{-q} \exp (i\beta \bar{B}_{-q} X_{p,q} B_q) = \frac{1}{4q^2} \quad (q \neq 0), \quad (2.48)$$

where we have determined the integration contours by the steepest descent method (Lefschetz thimble method). To evaluate the infinite product, let us consider the ratio between $Z_p$ and the leading order contribution around the perturbative vacuum $Z_0$

$$\frac{Z_p}{Z_0} = e^{-\frac{\beta \omega^2_p}{2\tau} - \frac{i \beta \mu}{2\tau}} \sqrt{\frac{2\pi}{\beta g}} i\beta \omega_p \prod_{q=1}^{\infty} \left[ 1 - \left( \frac{\beta \omega_p}{2\pi q} \right)^2 \right] + \cdots = e^{-\frac{\beta \omega^2_p}{2\tau}} \sqrt{\frac{2\pi}{\beta g}} \left( 1 - e^{-i\beta \mu} \right) + \cdots, \quad (2.49)$$

where we have used the fact that $Z_0$ in (2.23) can be rewritten as

$$Z_0 = N e^{i\beta \mu} \prod_{q=0}^{\infty} \tilde{I}_{p,q}, \quad \text{with} \quad \tilde{I}_{p,q} = \begin{cases} \frac{\pi}{(i\beta \omega_p)} & \text{for } q = 0 \\ \frac{\pi^2}{(2\pi q)^2} - (i\beta \omega_p)^2 & \text{for } q \neq 0 \end{cases}. \quad (2.50)$$

Since $Z_0 = (1 - e^{-i\beta \mu})^{-1}$, we find that the leading order contribution of the $p$-th saddle point is given by

$$Z_p = \sqrt{\frac{2\pi}{\beta g}} \exp \left( -\frac{\beta \omega^2_p}{2\tau} \right) \left[ 1 + \mathcal{O}(g) \right]. \quad (2.51)$$

The higher order part can be determined by using the differential equation (2.18). Since the leading order part takes the form of the “heat kernel”, it satisfies the differential equation (2.18). As shown in Appendix B, this is the unique solution that is regular at $\omega_p = 0$. Therefore, there is no higher order correction, i.e. $Z_p$ is one-loop exact

$$Z_p = \sqrt{\frac{2\pi}{\beta g}} \exp \left( -\frac{\beta \omega^2_p}{2\tau} \right). \quad (2.52)$$

### 2.4. Intersection Numbers

Although we have determined the integral along the thimbles associated with the non-perturbative saddle points, not all of them contribute to the generating function $Z$. In the Lefschetz thimble method, the generating function can be constructed by combining the perturbation series and the non-perturbative
Fig. 3: Intersection and symmetry. If two surfaces intersect at a point that is not a fixed point of a symmetry, the whole orbit is contained in the intersection of the surfaces (left panel). The intersection is a single point if it is a fixed point of the symmetry (right panel).

Contributions from the complex saddle points as

\[ Z = Z_{\text{pert}} + \sum_{p=-\infty}^{\infty} n_p Z_p, \]  

(2.53)

where \( n_p \) is the intersection number between the original path integration contour and the dual thimble of the \( p \)-th saddle point. The dual thimble is defined as the set of points that flow to the \( p \)-th saddle point under the gradient flow of \( S_E \)

\[ \overline{\partial_s \tilde{\varphi}} = \frac{1}{g} \left[ + \partial_s + i\mu + \tilde{\varphi} \varphi \right] \varphi, \]  

(2.54)

\[ \overline{\partial_s \varphi} = \frac{1}{g} \left[ - \partial_s + i\mu + \tilde{\varphi} \right] \tilde{\varphi}, \]  

(2.55)

in the limit \( s \to \infty \). The original integration contour is the subspace of the complexified configuration space specified by the condition \( \tilde{\varphi} = \tilde{\varphi} \) (complex conjugate of \( \varphi \)). If the original contour and the dual thimble intersect with each other, there exists a solution to the flow equation satisfying the following initial and final conditions with respect to the flow time \( s \in [s_0, \infty) \)

\[ \tilde{\varphi}(s_0, \tau) = \tilde{\varphi}(s_0, \tau), \quad \lim_{s \to \infty} (\varphi(s, \tau), \tilde{\varphi}(s, \tau)) = (\varphi_p(\tau), \tilde{\varphi}_p(\tau)), \]  

(2.56)

where \( (\varphi_p(\tau), \tilde{\varphi}_p(\tau)) \) denotes the \( p \)-th saddle points solution (2.37). In the following, we determine the intersection number by looking for a solution of the flow equation satisfying the condition (2.56).

Let us assume that the dual thimble intersects with the original integration contour at an isolated point\(^7\). Then, we can show that the intersection point must be at the fixed point of the simultaneous

\(^7\) This is a natural assumption since both the original integration contour and the dual thimble are half-dimensional subspaces of the complexified configuration space. If they have a higher dimensional intersection, we need to continuously deform.
shift of the time and the angle variable
\[
\varphi(\tau) \to e^{\frac{2\pi ip\tau_0}{\beta}} \varphi(\tau - \tau_0), \quad \tilde{\varphi}(\tau) \to e^{-\frac{2\pi ip\tau_0}{\beta}} \tilde{\varphi}(\tau - \tau_0) \quad \text{with} \quad \tau_0 \in \mathbb{R}.
\] (2.57)

The reason why the intersection is the fixed point is because the \( p \)-th saddle point is a fixed point of this symmetry and hence if the intersection point were not invariant under the symmetry, we have a continuous family of flow lines that intersects with the original contour along the orbit of the symmetry action (see Fig. 3). This contradicts the assumption that the dual thimble intersects with the original integration contour at a single point. Therefore, we assume the following invariant ansatz for the flow connecting the \( p \)-th saddle point and the intersection point
\[
\varphi = \sqrt{\nu(s)} e^{\frac{2\pi ip\tau_0}{\beta} + i\theta}, \quad \tilde{\varphi} = \sqrt{\tilde{\nu}(s)} e^{-\frac{2\pi ip\tau_0}{\beta} - i\theta},
\] (2.58)

where \( \nu(s) \) and \( \tilde{\nu}(s) \) are function depending only on the flow parameter \( s \). By using the conservation law, we can show that \(|\nu| - |\tilde{\nu}|\) is a constant on the flow
\[
\partial_s \int d\tau (|\varphi|^2 - |\tilde{\varphi}|^2) = \beta \partial_s (|\nu| - |\tilde{\nu}|) = 0.
\] (2.59)

Since \( \nu(s) \) and \( \tilde{\nu}(s) \) converge to the \( p \)-th saddle point value for \( s \to \infty \)
\[
\lim_{s \to \infty} \nu = \lim_{s \to \infty} \tilde{\nu} = -i\omega_p,
\] (2.60)
the difference \(|\nu| - |\tilde{\nu}|\) vanishes for \( s \to \infty \) and hence
\[
|\tilde{\nu}(s)| = |\nu(s)| \quad \text{for all} \quad s.
\] (2.61)

Then, we find from the flow equations (2.54) and (2.55) that the difference of the arguments also vanishes
\[
\partial_s (\arg \nu - \arg \tilde{\nu}) = 0 \quad \implies \quad \arg \tilde{\nu}(s) = \arg \nu(s) \quad \text{for all} \quad s.
\] (2.62)

Using (2.61) and (2.62), we can eliminate \( \tilde{\nu} \) by setting \( \tilde{\nu} = \nu \) and then the flow equations (2.54) and (2.55) reduce to a single equation for \( \nu \)
\[
\nu'(s) = 2|\nu(s)| \left[ \frac{\nu(s) + i\omega_p}{g} \right].
\] (2.63)

This equation can be solved by using the conservation law
\[
\partial_s \text{Im} S_p(\nu) = 0 \quad \text{with} \quad S_p(\nu) = \frac{\beta}{g} \left( \frac{1}{2} \nu^2 + i\omega_p \nu \right),
\] (2.64)
where \( S_p(\nu) \) is the value of the action obtained by substituting the ansatz (2.58) with \( \tilde{\nu} = \nu \) into the

the model so that the intersection becomes an isolated point.
original action $S_E$ in (2.17). There are two solutions corresponding to the a pair of lines that flow to the saddle point from the opposite directions

$$
\nu(s) = -i\omega_p \left( 1 \pm \frac{e^{-i\epsilon \coth ms}}{\cosh ms \pm \cos \epsilon \sinh ms} \right),
$$

(2.65)

where we have defined

$$
\epsilon = \arg \omega_p - \frac{1}{2} \arg g, \quad m = \frac{|2\omega_p|}{g}.
$$

(2.66)

To examine if this flow intersects the original integration contour, it is convenient to see the orbit of the Noether charge $\nu = \varphi \bar{\varphi}$ in the complex plane. We can show that the following relation holds along the flow

$$
\text{Re} \left( e^{i\epsilon \nu/\omega_p} \right) = \sin \epsilon.
$$

(2.67)

Therefore, this flow is a straight line on the complex $\nu$-plane (see Fig. 4). Since $\text{Im} \nu = 0$ and $\text{Re} \nu > 0$ on the original integration contour ($\varphi = \bar{\varphi}$), an intersection point exists only when the line (2.67) intersects the positive real axis on the complex $\nu$-plane, that is, if the parameters satisfy the condition

$$
\text{Re} \nu \big|_{\text{Im} \nu = 0} = \frac{|\omega_p| \sin \epsilon}{\cos(\arg g/2)} > 0,
$$

(2.68)

there is an intersection between the original contour and the dual thimble. Therefore, the intersection number is given by

$$
n_p = \begin{cases} 
1 & \text{for } \sin \epsilon > 0 \\
0 & \text{for } \sin \epsilon < 0
\end{cases} = \frac{1}{2} \left[ 1 + \text{sign} \left( \text{Im} \sqrt{\frac{\beta}{2g} \omega_p} \right) \right],
$$

(2.69)

where we have assumed that $\arg g$ is small and hence $\cos(\arg g/2) > 0$. 

- (a) $\epsilon > 0$
- (b) $\epsilon < 0$

Fig. 4: Flows in complex $\nu$-plane ($\arg \omega_p > 0$).
2.5. Exact Generating Function and Comparison with Operator Formalism

Having determined the perturbation series (2.26), all the non-perturbative contributions (2.52) and the intersection numbers (2.69), we can construct the transseries for the generating function by combining them as

\[
Z = Z_{\text{pert}} + \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{g}{2\beta} \frac{\partial^2}{\partial \mu^2} \right)^n \frac{1}{1 - e^{-i\beta \mu}} + \sqrt{\frac{2\pi}{\beta g}} \sum_{n=0}^{\infty} n_p \exp \left( -\frac{\beta^2}{2g} \omega_p^2 \right). \tag{2.70}
\]

By applying the Borel resummation to the perturbation series as in Eq. (2.31) and taking into account the discontinuities of the intersection numbers (2.69), we can write down the unambiguous form of the full generating function as

\[
Z = \frac{1}{2} - \sqrt{\frac{\pi}{2\beta g}} \sum_{p=-\infty}^{\infty} e^{-\frac{\beta}{2\pi} \omega_p^2} \left[ \text{erf} \left( \frac{\sqrt{\beta}}{2\pi} \omega_p \right) - 1 \right]. \tag{2.71}
\]

This shows that the ambiguities of the perturbative and non-perturbative sectors completely cancel out each other in the transseries obtained through the Lefschetz thimble method.

We can show that the expression (2.71) is not only well-defined but also exact by comparing it with the generating function obtained in the operator formalism. By using the number eigenstates, the generating function can be written as

\[
Z = \text{Tr} \left[ e^{-\beta (\hat{H} + i\mu \hat{N})} \right] = \sum_{n=0}^{\infty} \exp \left( -\frac{\beta g}{2} n^2 - i\beta \mu n \right). \tag{2.72}
\]

To compare this with the result of the Lefschetz thimble method (2.71), let us use the relation

\[
\sum_{n=0}^{\infty} f(gn) = \frac{1}{2} f(0) + \frac{1}{g} \sum_{p=-\infty}^{\infty} \int_{0}^{\infty} d\nu f(\nu) e^{-\frac{2\pi ip\nu}{g}}. \tag{2.73}
\]

This relation can be regarded as a variant of the Poisson resummation. Applying this resummation method, we can rewrite the generating function as

\[
Z = \frac{1}{2} + \frac{1}{g} \sum_{p=-\infty}^{\infty} \int_{0}^{\infty} d\nu \exp \left[ -\mathcal{S}_p(\nu) \right] \quad \text{with} \quad \mathcal{S}_p = \frac{\beta}{g} \left( \frac{\nu^2}{2} + i\omega_p \nu \right). \tag{2.74}
\]

Evaluating the integrals by using the definition of the error function (2.32), we find the complete agreement of the generating functions obtained through the Lefschetz thimble method (2.71) and the operator formalism (2.74).

It is worth examining how the transseries for the generating function is obtained from the viewpoint of the operator formalism. To extract the perturbation series from (2.74), let us consider steepest ascent path \( \hat{C}_p \) of \( \mathcal{S}_p(\nu) \) starting from the origin in the complex \( \nu \)-plane and decompose the integral along the
positive real axis $\mathbb{R}_{\geq 0}$ as
\[\frac{1}{g} \int_{\mathbb{R}_{+}} d\nu \exp[-S_p(\nu)] = \frac{1}{g} \int_{\tilde{C}_p} d\nu \exp[-S_p(\nu)] + \frac{1}{g} \int_{\mathbb{R}_{+} - \tilde{C}_p} d\nu \exp[-S_p(\nu)], \quad (2.75)\]

where $\mathbb{R}_{+} - \tilde{C}_p$ is the path consisting of the positive real axis and the inverse path of $\tilde{C}_p$ connected at the origin (see Fig. 5). We can show that the first term gives the perturbative part by changing the variable as
\[\frac{1}{g} \int_{\tilde{C}_p} d\nu \exp[-S_p(\nu)] = -\frac{i}{\beta \omega_p} \int_{0}^{t} dt \frac{e^{-t}}{\sqrt{1 - 2g \beta \omega_p^2}} \quad \text{with} \quad t = S_p(\nu). \quad (2.76)\]

Summing over $p$, we find that the collection of these terms and $1/2$ in (2.74) correspond to the Borel resummation of the perturbation series (2.30). The second integral in (2.75) can be evaluated by applying the Lefschetz thimble method to this integral. The saddle point of $S_p(\nu)$ is located at $\nu = -i\omega_p$, which is nothing but the value of $\nu = \varphi \tilde{\varphi}$ for the $p$-th saddle point of the original action $S_E$ in Eq. (2.37). Evaluating the integral along the associated thimble, we find that the saddle contribution agrees with (2.52). The dual thimble is the path determined from $\text{Im} S_p(\nu) = \text{Im} S_p(-i\omega)$. This agrees with the flow determined by the equation (2.64) reduced from the original flow equations. Therefore, we obtain the same intersection numbers as Eq. (2.69). In this way, we can see the agreement of the transseries obtained from the path integral and operator formalism through the thimble analysis of the single variable functions $S_p(\nu)$. 

Fig. 5: Thimbles in complex $\nu$-plane ($\arg \omega_p > 0$). (a) The positive real axis $\mathbb{R}_{\geq 0}$ can be decomposed into the steepest ascent path $\tilde{C}_p$ and the thimble associated with the saddle point at $\nu = -i\omega_p$. (b) The positive real axis $\mathbb{R}_{\geq 0}$ can be continuously deformed to $\tilde{C}_p$. 

Evaluating the integral along the associated thimble, we find that the saddle contribution agrees with (2.52). The dual thimble is the path determined from $\text{Im} S_p(\nu) = \text{Im} S_p(-i\omega)$. This agrees with the flow determined by the equation (2.64) reduced from the original flow equations. Therefore, we obtain the same intersection numbers as Eq. (2.69). In this way, we can see the agreement of the transseries obtained from the path integral and operator formalism through the thimble analysis of the single variable functions $S_p(\nu)$.
3. GENERALIZATION TO ARBITRARY U(1) SYMMETRIC POTENTIAL

3.1. Preliminary

In the previous section, we have seen that the Lefschetz thimble method gives exact results in the case of the quartic potential. It is also possible to generalize the discussion to the case of an arbitrary U(1) symmetric potential

\[ L = i \dot{\varphi} \partial_t \varphi + \frac{1}{g} V(g|\varphi|^2) = \frac{1}{g} \left[ i \varphi \partial_t \varphi + V(|\varphi|^2) \right], \quad (\varphi = \sqrt{g} \varphi). \]  

(3.1)

In the following, we assume that the potential \( V(|\varphi|^2) \) has its minimum at \( \varphi = 0 \) and can be expanded as

\[ V(|\varphi|^2) = \sum_n \kappa_n |\varphi|^{2n} = \kappa_1 |\varphi|^2 + \kappa_2 |\varphi|^4 + \cdots. \]  

(3.2)

As in the previous case, we consider the generating function

\[ Z = \text{Tr} \left[ e^{-\beta (\hat{H} + i\mu \hat{N})} \right], \]  

(3.3)

where the Hamiltonian \( \hat{H} \) and the conserved charge \( \hat{N} \) are given by

\[ \hat{H} = \frac{1}{g} V(g\hat{N}), \quad \hat{N} = \hat{\varphi}^\dagger \hat{\varphi}. \]  

(3.4)

We can show that the generating function satisfies the differential equation

\[ \left[ -\frac{1}{\beta} \frac{\partial}{\partial g} + \frac{1}{g^2} V \left( g \frac{i}{\beta} \frac{\partial}{\partial \mu} \right) - \frac{1}{g} \frac{i}{\beta} \frac{\partial}{\partial \mu} V' \left( g \frac{i}{\beta} \frac{\partial}{\partial \mu} \right) \right] Z = 0. \]  

(3.5)

We will use this differential equation to determine the perturbation series in the following.

From the viewpoint of operator formalism, the generating function can be calculated by using the number eigenstates \( \hat{N}|n\rangle = n|n\rangle \) as

\[ Z = \sum_{n=0}^{\infty} \exp \left[ -\frac{\beta}{g} V(gn) - i\beta \mu n \right]. \]  

(3.6)

On the other hand, the path integral expression for the generating function is given by

\[ Z = \int D\phi \exp \left( -S_E - S_W \right), \]  

(3.7)

where \( S_E \) is the classical Wick-rotated action

\[ S_E = \frac{1}{g} \int_0^\beta d\tau \left[ \phi \partial_\tau \varphi + V(|\varphi|^2) + i\mu |\varphi|^2 \right], \]  

(3.8)
and $S_W$ is the part generated when the original Hamiltonian is rewritten in terms of the Weyl ordered operators

$$S_W = \int_0^{\beta} d\tau \left[ -\frac{1}{2} V'(|\varphi|^2) - \frac{i\mu}{2} + \mathcal{O}(g) \right]. \quad (3.9)$$

To derive this expression, we have used

$$\hat{\mathcal{N}}_W^n = (\hat{\mathcal{N}}_W^n)_{W} - \frac{n}{2} (\hat{\mathcal{N}}_W^{n-1}) + \mathcal{O}(g), \quad (3.10)$$

where $(\hat{\mathcal{N}}_W^n)$ stands for the Weyl ordered operator defined in (2.12). We will not use the details of the higher order terms since their contributions can be determined through the differential equation.

### 3.2. Perturbation Series

Let us first consider perturbative expansion of the generating function $Z$ with respect to the coupling constant $g$. Substituting the power series ansatz

$$Z_{\text{pert}} = \sum_{k=0}^{\infty} C_k g^k, \quad (3.11)$$

into the differential equation (3.5), we obtain a recursion relation from which the coefficients $C_k$ can be determined order-by-order as

$$C_k = -\beta \sum_{l=1}^{k} \frac{1}{k^{l+1}} \left( \frac{i}{\beta} \frac{\partial}{\partial \mu} \right)^{l+1} C_{k-l}, \quad (3.12)$$

where the initial term is given by the generating function in the free theory

$$C_0 = \frac{1}{1 - e^{-\beta(\kappa_1 + \mu)}}. \quad (3.13)$$

We can show that the Borel resummation of the perturbation series is given by

$$Z_{\text{pert}} = \int_{0}^{\infty} dt \ e^{-t} \left( \frac{1}{2} + \frac{1}{\beta} \sum_{p=-\infty}^{\infty} \frac{1}{V'(\nu_p(gt)) + i\omega_p} \right), \quad (3.14)$$

where $\omega_p = \mu + \frac{2\pi p}{\beta}$ and $\nu_p(gt)$ is the solution of the equation

$$S_p(\nu) \equiv \frac{\beta}{g} [V(\nu) + i\omega_p \nu] = t \quad \text{with} \quad \nu(0) = 0. \quad (3.15)$$

We can check that Eq. (3.14) gives the correct perturbation series by confirming that it satisfies the differential equation (3.5). For this purpose it is convenient to change the integration variable from $t$ to
\[ Z_{\text{pert}} = \frac{1}{2} + \sum_{p = -\infty}^{\infty} \int_{c_p} \frac{d\nu}{g} \exp \left( -\mathcal{S}_p(\nu) \right), \]  

where the integration contour \( c_p \) is the image of the positive real axis under the map from the \( t \)-plane to the \( \nu \)-plane, that is, the ascending flow of \( \mathcal{S}_p(\nu) \) emanating from the origin on the complex \( \nu \)-plane (see the examples in Fig. 6). Substituting into (3.5), we find that (3.16) satisfies the differential equation

\[ \left[ -\frac{1}{\beta} \frac{\partial}{\partial g} + \frac{1}{g^2} V \left( \frac{i}{\beta} \frac{\partial}{\partial \mu} \right) - \frac{1}{g} \frac{i}{\beta} \frac{\partial}{\partial \mu} V' \left( \frac{i}{\beta} \frac{\partial}{\partial \mu} \right) \right] Z_{\text{pert}} = \sum_{p = -\infty}^{\infty} \int_{c_p} \frac{d\nu}{g} \frac{\nu}{g^2} e^{-S_p} = 0. \]  

Furthermore, (3.16) satisfies the initial condition (3.13) and hence \( Z_{\text{pert}} \) in (3.14) gives the correct perturbation series.

The formal Borel resummation (3.14) of the perturbation series is non-Borel summable if the integrand (Borel transform) has singularities along the positive real axis in the Borel plane (complex \( t \)-plane). This occurs when one of the contours \( \tilde{C}_p \) in Eq. (3.16) connects the origin and a saddle point of \( \mathcal{S}_p(\nu) \), that is, a point at which \( \nu \) satisfies

\[ \mathcal{S}_p' (\nu) \propto V' (\nu) + i\omega_p = 0. \]  

Although such singularities can be avoided by complexifying the coupling constant \( g \rightarrow |g|e^{\pm 0} \), the Borel resummation of the perturbation series has ambiguities of the form

\[ \Delta Z_{\text{pert}} = Z_{\text{pert}}^{(+)} - Z_{\text{pert}}^{(-)} = \sum_{p = -\infty}^{\infty} \left[ \int_{\tilde{c}_p^{(+)}} \frac{d\nu}{g} \exp \left( -\mathcal{S}_p(\nu) \right) - \int_{\tilde{c}_p^{(-)}} \frac{d\nu}{g} \exp \left( -\mathcal{S}_p(\nu) \right) \right], \]  

where \( \tilde{c}_p^{(\pm)} \) are the ascending flows of \( \mathcal{S}_p(\nu) \) for \( \arg g = \pm 0 \). Noting that \( \tilde{c}_p^{(+)} - \tilde{c}_p^{(-)} \) is the thimble associated with the saddle point connected to the origin by the flow \( \tilde{c}_p \mid_{\arg g = 0} \) (see Fig. 6-(b)), we can rewrite the discontinuity as

\[ \Delta Z_{\text{pert}} = \sum_{p, \sigma} m_{p, \sigma} (\Delta Z_{\text{pert}})_{p, \sigma} = \sum_{p, \sigma} m_{p, \sigma} \int_{J_{p, \sigma}} \frac{d\nu}{g} \exp \left( -\mathcal{S}_p(\nu) \right), \]  

where \( \sigma \) is the label of the saddle points of \( \mathcal{S}_p(\nu) \), \( J_{p, \sigma} \) is the thimble\footnote{The orientation of the thimble is chosen so that
\[ \int_{J_{p, \sigma}} \frac{d\nu}{g} \exp \left( -\mathcal{S}_p(\nu) \right) = \sqrt{\frac{2\pi}{\beta g S_p'(\nu_p, \sigma)}} \exp \left( -\mathcal{S}_p(\nu_p, \sigma) \right) \left[ 1 + \mathcal{O}(g) \right]. \]}

associated with the saddle point.
\( \nu_{p,\sigma} \) and the coefficient \( m_{p,\sigma} \) is given by

\[
m_{p,\sigma} = \begin{cases} 
\text{sign}(\text{Im} S'_p(0)) & \text{if } \nu_{p,\sigma} \in \tilde{C}_p \\
0 & \text{if } \nu_{p,\sigma} \notin \tilde{C}_p 
\end{cases}
\]  

(3.21)

Note that each \((\Delta Z_{\text{pert}})_p,\sigma\) satisfies the differential equation (3.5). In the next section, we will see that these ambiguities are canceled by the contributions from complex saddle point solutions.

### 3.3. Complex Saddle Points

Let us look for the saddle points that cancel the ambiguities of the perturbation series in Eq. (3.20). The complexified equations of motion \( \delta S_E/\delta \varphi = \delta S_E/\delta \tilde{\varphi} = 0 \) are given by

\[
0 = \left[ + \partial_\tau + i\mu + V'(\tilde{\varphi}) \right] \varphi, \\
0 = \left[ - \partial_\tau + i\mu + V'(%(\tilde{\varphi}) \right] \tilde{\varphi}. 
\]

(3.22)  

(3.23)

Using the conservation law, we can show that the solution takes the form

\[
\varphi = \sqrt{2} \exp \left( \frac{-2\pi i\tau}{\beta} + i\theta \right), \\
\tilde{\varphi} = \sqrt{2} \exp \left( -\frac{2\pi i\tau}{\beta} - i\theta \right),
\]

(3.24)

where \( \nu \) is a constant satisfying the condition

\[
V'(\nu) + i\omega_p = 0. 
\]

(3.25)

This is nothing but the condition in Eq. (3.18) that determines the locations of the singularities in the Borel plane for the perturbation series in Eq. (3.14). Suppose that \( \nu_{p,\sigma} \) is a solution of (3.25). Then, we can show that the value of action for the solution corresponding to \( \nu_{p,\sigma} \) is given by

\[
S_p(\nu_{p,\sigma}) = \frac{\beta}{g} \left[ V(\nu_{p,\sigma}) + i\omega_p \nu_{p,\sigma} \right]. 
\]

(3.26)

The leading order contributions from these saddle points can be calculated similarly to the previous case. For example, we can show that the one-loop determinant can be obtained by replacing \( g \) in the previous section to \( gV''(\nu_{p,\sigma}) \)

\[
(-1)^p \sqrt{\frac{2\pi}{\beta g}} \rightarrow (-1)^p \sqrt{\frac{2\pi}{\beta gV''(\nu_{p,\sigma})}}. 
\]

(3.27)

The leading order part of \( S_W \) is given by

\[
S_W = -\frac{1}{2} \left[ \beta V'(\nu) + i\beta \mu \right] + O(g) = \pi pi + O(g),
\]

(3.28)
and hence the leading order part of the saddle point contribution takes the form of

$$Z_{p,\sigma} = \sqrt{\frac{2\pi}{\beta g V''(\nu_{p,\sigma})}} e^{-S_p(\nu_{p,\sigma}) \left[ 1 + \mathcal{O}(g) \right]}.$$  \hfill (3.29)

This leading order contribution is identical to that of the corresponding ambiguity of the perturbation series in Eq. (3.20). Since the higher order part can be uniquely determined from the leading part by the differential equation, the agreement of the leading order parts implies that the saddle point contribution $Z_{p,\sigma}$ and the corresponding ambiguity $(\Delta Z_{\text{pert}})_{p,\sigma}$ in Eq. (3.20) agree to all orders in the coupling constant $g$

$$Z_{p,\sigma} = (\Delta Z_{\text{pert}})_{p,\sigma} = \int_{\mathbb{J}_{p,\sigma}} \frac{d\nu}{g} \exp\left(-S_p(\nu)\right).$$  \hfill (3.30)

Therefore, the transseries

$$Z = \int_0^\infty dt e^{-t} \left( \frac{1}{2} + \frac{1}{\beta} \sum_{p=\infty}^{\infty} \frac{1}{V''(\nu_{p}(gt)) + i\omega_p} \right) + \sum_{p,\sigma} n_{p,\sigma} Z_{p,\sigma},$$  \hfill (3.31)

do not have ambiguities if the intersection numbers $n_{p,\sigma}$ have appropriate discontinuities $\Delta n_{p,\sigma} = \pm 1$ at $\arg g = 0$. In the next section, we determine the intersection numbers $n_{p,\sigma}$ by using the flow equation.

### 3.4. Intersection numbers

To determine the intersection numbers, let us consider the flow equations

$$\overline{\partial_s \hat{\varphi}} = \frac{1}{g} \left[ + \partial_\tau + i\mu + V'(\hat{\varphi}) \right] \varphi,$$  \hfill (3.32)

$$\overline{\partial_s \varphi} = \frac{1}{g} \left[ - \partial_\tau + i\mu + V'(\varphi) \right] \hat{\varphi}. \hfill (3.33)$$

Let us look for flows connecting the saddle points and some points on the original integration contour. The same argument as in the case of $V(\hat{\varphi}\varphi) = (\hat{\varphi}\varphi)^2$ discussed in subsection 2.4 leads to the following ansatz for the flow

$$\varphi = \sqrt{\nu(s)} e^{\frac{2\pi i \sigma}{\beta} + i\theta}, \quad \hat{\varphi} = \sqrt{\tilde{\nu}(s)} e^{-\frac{2\pi i \tau}{\beta} - i\theta}. \hfill (3.34)$$

As in the previous case, we can show by using the conservation law for the $U(1)$ symmetry that

$$\nu(s) = \tilde{\nu}(s).$$  \hfill (3.35)

Then, the flow equation reduces to that for $\nu$

$$\nu'(s) = 2\nu(s) \left[ \frac{V''(\nu(s)) + i\omega_p}{g} \right].$$  \hfill (3.36)
The orbit of the flow obeying this equation can be determined through the conservation law

$$\text{Im} \left[ \frac{\beta}{g} \{V(\nu) + i\omega_p \nu\} \right] = \text{Im} S_p(\nu_{p,\sigma}) = \text{const.} \quad (3.37)$$

By solving this conservation law, we can draw a flow line from each saddle point on the complex \(\nu\)-plane and determine the intersection number by checking if the flow intersects the positive real axis in the complex \(\nu\)-plane corresponding to the original integration contour \(\tilde{\varphi} = \tilde{\varphi}\). We can also rephrase the condition for the intersection number as

$$n_{p,\sigma} = \begin{cases} 1 & \text{if } \nu_{p,\sigma} \in \tilde{D}_p \\ 0 & \text{if } \nu_{p,\sigma} \notin \tilde{D}_p \end{cases}, \quad (3.38)$$

where \(\tilde{D}_p\) is the region in the \(\nu\)-plane surrounded by \(\tilde{C}_p\) and the positive real axis, that is, the orbit of the positive real axis under the ascending flow (see examples in Fig. 6). If the saddle point \(\nu_{p,\sigma}\) is on the boundary of \(\tilde{D}_p\), that is, \(\tilde{C}_p\), the intersection number has discontinuity at \(\text{arg } g = 0\). From the facts that

- \(\text{sign}(\text{Im} S_p(\nu_{p,\sigma})) = \mp 1\) for \(\text{arg } g = \pm 0\), \(\because S_p(\nu_{p,\sigma}) = \frac{\beta}{g} [V(\nu_{p,\sigma}) + i\omega_p \nu_{p,\sigma}] \in \mathbb{R}_{\geq 0}\) at \(\text{arg } g = 0\) by assumption,

- \(\text{sign}(\text{Im} S_p(\nu)) = \text{sign}(\text{Im} S'_p(0))\) in the neighborhood of \(\tilde{C}_p\) in \(\tilde{D}_p\),

we conclude that the discontinuity of the intersection number is given by

$$\Delta n_{p,\sigma} = n_{p,\sigma}^{(+) - n_{p,\sigma}^{(-)} = \begin{cases} -\text{sign}(\text{Im} S'_p(0)) & \text{if } \nu_{p,\sigma} \in \tilde{C}_p \\ 0 & \text{if } \nu_{p,\sigma} \notin \tilde{C}_p \end{cases}. \quad (3.39)$$

This completely cancels the discontinuity of the perturbation series \((3.20)\) and hence the transseries \((3.31)\) obtained through the Lefschetz thimble method has no ambiguity.

### 3.5. Operator formalism

So far, we have seen from the viewpoint of the path integral formalism that the transseries expression for the generating function \(Z\) takes the form in Eq. \((3.31)\) with the intersection numbers determined through Eq. \((3.37)\). Here, we confirm that the transseries in Eq. \((3.31)\) is consistent with that obtained from the viewpoint of operator formalism.

By using the number eigenstate \(\hat{N}|n\rangle = n|n\rangle\), the generating function \((3.3)\) can be rewritten as

$$Z = \sum_{n=0}^{\infty} \exp \left[-\frac{\beta}{g} V(gn) - i\beta \mu n\right]. \quad (3.40)$$

This expression can be further rewritten by using the Poisson resummation \((2.73)\) as

$$Z = \frac{1}{2} + \sum_{p=-\infty}^{\infty} \int_{0}^{\infty} d\nu \exp (-S_p(\nu)) \quad \text{with} \quad S_p = \frac{\beta}{g} \{V(\nu) + i\omega_p \nu\}. \quad (3.41)$$
This full generating function has the same form as the perturbative part (3.16) except for the integration contour: the ascending flow \( \tilde{C}_p \) of \( S_p(\nu) \) for the perturbative part and the positive real axis for the full generating function. By the change of integration variable from \( \nu \) to \( t = S_p(\nu) \) given in (3.15), the generating function can be rewritten as

\[
Z = \frac{1}{2} + \frac{1}{\beta} \sum_{p=-\infty}^{\infty} \int_{C_p} dt \frac{e^{-t}}{V'(\nu(gt)) + i\omega_p}. \tag{3.42}
\]

The integration contour \( C_p \) on the complex \( t \)-plane is the image of the positive real axis on the complex \( \nu \)-plane under the map \( \nu \rightarrow t = S_p(\nu) \). By deforming the integration contour \( C_p \), we can decompose the full generating function (3.42) into perturbative and non-perturbative parts as

\[
Z = \left[ \frac{1}{2} + \frac{1}{\beta} \sum_{p=-\infty}^{\infty} \int_{0}^{\infty} dt \frac{e^{-t}}{V'(\nu(gt)) + i\omega_p} \right] + \sum_{p,\sigma} n_{p,\sigma} \left[ \frac{1}{\beta} \int_{C_{p,\sigma}} dt \frac{e^{-t}}{V'(\nu(gt)) + i\omega_p} \right], \tag{3.43}
\]

where \( C_{p,\sigma} \) is the contour surrounding each singularity and associated branch cut. If \( t = t_{p,\sigma} \) is a singularity, the corresponding intersection number is given by

\[
n_{p,\sigma} = \begin{cases} 
1 & \text{if } t_{p,\sigma} \in \mathcal{D}_p \\
0 & \text{if } t_{p,\sigma} \notin \mathcal{D}_p
\end{cases}, \tag{3.44}
\]

where \( \mathcal{D}_p \) is the region surrounded by the contour \( C_p \) and the positive real axis on the complex \( t \)-plane.

This agrees with the intersection number (3.38) obtained through the analysis of the flow equation since \( \nu_{p,\sigma} \) and \( \tilde{\mathcal{D}}_p \) are mapped to \( t_{p,\sigma} \) and \( \mathcal{D}_p \) under the change of variable \( \nu \rightarrow t = S_p(\nu) \). Furthermore, each non-perturbative part in Eq. (3.43) is related to \( Z_{p,\sigma} \) in Eq. (3.30) by the change of variable \( t = S_p(\nu) \). Thus, we conclude that the transseries obtained through the Lefschetz thimble method (3.31) is non-perturbatively complete and agrees with the exact result obtained in the operator formalism (3.43).

### 3.6. Example

To illustrate the discussion in this section, let us consider the monomial potential as an example

\[
V(|\varphi|^2) = |\varphi|^l, \quad l \in \mathbb{Z}_{\geq 0}. \tag{3.45}
\]

In this case the generating function is given by

\[
Z = \sum_{n=-\infty}^{\infty} \exp\left(-\beta g^{l-1} n^l - i\beta \mu n\right). \tag{3.46}
\]

We can easily verify that this generating function satisfies the differential equation

\[
\left[ \frac{\partial}{\partial g} + (l-1)\beta g^{l-2} \left( \frac{i}{\beta} \frac{\partial}{\partial \mu} \right) \right] Z = 0. \tag{3.47}
\]
Fig. 6: Lefschetz thimbles for $S_p(\nu) = \frac{\beta}{g} (\nu^4 + i\omega_p \nu)$ ($l = 4$, left panel) and for $S_p(\nu) = \frac{\beta}{g} (\nu^6 + i\omega_p \nu)$ ($l = 6$, right panel). For $l = 4$, the integration contour $\mathbb{R}_+$ can be decomposed to the ascending path $\tilde{C}_p$ from the origin and the thimble associated with the saddle point in the region $\tilde{D}_p$ (forth quadrant). For $l = 6$, the the ascending path $\tilde{C}_p$ depends on arg $g$. The difference $\tilde{C}_p^+ - \tilde{C}_p^-$ is the thimble associated with the saddle point on the negative imaginary axis, whose intersection number is $n = 1$ for $\arg g < 0$ and $n = 0$ for $\arg g > 0$.

Fig. 7: Singularities of $B_p(t)$ for $l = 4$ (left panel) and $l = 6$ (right panel). The integration along the positive real axis gives the perturbative part. For $l = 2 \mod 4$, there is a singularity on the positive real axis, and hence the perturbation series is non-Borel-summable. The ambiguity is given by the integration along the branch cut emanating from the singularity on the positive real axis. The contour $C_p$ corresponds to the full contribution. The difference between the perturbative and full contributions is given by the integration along the branch cuts emanating from the singularities contained in the region $D_p$.

By using the power series ansatz, we can determine the perturbative part as

$$Z_{\text{pert}} = \frac{1}{2} + \sum_{n=0}^{\infty} \sum_{p=-\infty}^{\infty} \frac{1}{i\beta \omega_p} \frac{\Gamma((n+1))}{\Gamma(n+1)} \left( i \frac{g}{\omega_p} \right)^n \left( i \frac{\beta}{\omega_p} \right)^{n(l-1)}. \quad (3.48)$$
Since this perturbation series is factorially divergent, let us consider the Borel resummation
\[
Z_{\text{pert}} = \frac{1}{2} + \int_0^\infty dt e^{-t} \sum_{n=0}^\infty \sum_{p=-\infty}^\infty \frac{1}{i\beta\omega_p} \frac{\Gamma(n+1)}{\Gamma(n+1)\Gamma(n(\ln+1)+1)} \left( \frac{t}{\omega_p} \right)^n \left( \frac{tg}{i\beta\omega_p} \right)^{n(l-1)}
\]
\[
= \int_0^\infty dt e^{-t} B(tg).
\] (3.49)

The Borel transform \( B(tg) \) takes the form of
\[
B(tg) = \frac{1}{2} + \sum_{p=-\infty}^\infty B_p(tg),
\] (3.50)
where \( B_p(tg) \) are given by the hypergeometric functions
\[
B_p(tg) = \frac{1}{i\beta\omega_p} \left\{ \begin{array}{c} 1 \ 2 \ l-1 \\ \frac{1}{l-1} \ \cdots \ \frac{l-2}{l} \end{array} \right\} \left\{ \begin{array}{c} 1 \ 2 \ l-1 \\ \frac{1}{l-1} \ \cdots \ \frac{l-2}{l-1} \end{array} \right\},
\] (3.51)
with
\[
z_p = \frac{l^l}{(l-1)^{l-1}} \frac{i}{\omega_p} \left( \frac{tg}{i\beta\omega_p} \right)^{l-1}.
\] (3.52)

The function \( B(tg) \) becomes singular when \( z_p = 1 \), i.e. it has singularities at
\[
t = \frac{l-1}{l} \frac{i\beta\omega_p}{g} \nu_{p,\sigma} \quad \text{with} \quad \nu_{p,\sigma} = \left( \frac{-i\omega_p}{l} \right)^{\frac{l}{l-1}} \exp \left( \frac{2\pi\sigma i}{l} \right), \quad (\sigma = 1, \cdots, l-1).
\] (3.53)

The complex saddle points corresponding to this singularity on the Borel plane is given by
\[
\varphi = \sqrt{\nu_{p,\sigma}} \exp \left( \frac{2\pi p i}{\beta} \tau + i\theta \right), \quad \tilde{\varphi} = \sqrt{\nu_{p,\sigma}} \exp \left( -\frac{2\pi p i}{\beta} \tau - i\theta \right).
\] (3.54)

Note that \( \nu_{p,\sigma} \) satisfies the saddle point condition (3.25)
\[
S'(\nu_{p,\sigma}) = \frac{\beta}{g} \left[ V'(\nu_{p,\sigma}) + i\omega_p \right] = \frac{\beta}{g} \left[ l\nu_{p,\sigma}^{l-1} + i\omega_p \right] = 0.
\] (3.55)

We can check that the value of the action for this saddle point agrees with the location of the singularity (3.53)
\[
S_p(\nu_{p,\sigma}) = \frac{\beta}{g} \left( \nu_{p,\sigma}^l + i\omega_p \nu_{p,\sigma} \right) = \frac{l-1}{l} \frac{i\beta\omega_p}{g} \nu_{p,\sigma}.
\] (3.56)

We can show that the saddle points which contribute to the generating function are those with
\[
\sigma = l - 1 - \left[ \frac{l-2}{4} \right], \cdots, l-1.
\] (3.57)
Examples of the thimble structure of $S_p(\nu)$ for $l = 4$ and $l = 6$ are shown in Fig. 6. To see this, let us rewrite the exact generating function as

$$Z = \sum_{n=-\infty}^\infty \exp\left(-\beta g^{l-1}n^l - i\beta \mu n\right)$$

$$= \frac{1}{2} + \frac{1}{g} \sum_{p=-\infty}^{\infty} \int d\nu \exp\left[-\frac{\beta}{g} (\nu^l + i \omega_p \nu)\right]$$

$$= \frac{1}{2} + \sum_{p=-\infty}^{\infty} \int_{C_p} dt \ e^{-t B_p(tg)}, \quad (3.58)$$

where the functions $B_p(tg)$ are the same functions as those which appeared in the Borel transform (3.51). The integration contours $C_p$ are the image of the positive real axis on $\nu$-plane under the change of the variable $t = \frac{\beta}{g} (\nu^l + i \omega_p \nu)$

$$C_p = \left\{ t \in \mathbb{C} \left| \text{Re} \ t = \frac{\beta}{g} \nu^l, \ \text{Im} \ t = \frac{\omega_p \nu}{g}, \ \nu \in \mathbb{R}_{\geq 0} \right\} \right. \quad (3.59)$$

The saddle points (3.57) are enclosed by the curve $C$ and the positive real axis on the Borel plane, and hence they have contributions to the generating function. In particular, for $l = 2 \mod 4$, the singularity with $q = l-1-(l-2)/4$ is on the positive real axis and hence gives rise to an ambiguity of the perturbative part.

4. GENERALIZATION TO QUANTUM MECHANICS WITH INTEGRABILITY

The analysis in this paper can also be generalized to the multi-variable cases. In particular, it would be possible to obtain exact results if there exist the same number of conserved charges as degrees of freedom. For example, in the $N$-variable system described by the Lagrangian

$$L = \frac{1}{g} \sum_{i=1}^{N} \left[ i \tilde{\varphi}_i \partial_t \varphi_i + V(\vert \varphi_1 \vert^2, \cdots, \vert \varphi_N \vert^2) \right], \quad (4.1)$$

there are $N$ conserved charges $\mathcal{N} = (\vert \varphi_1 \vert^2/g, \cdots, \vert \varphi_N \vert^2/g)$ corresponding to the phase rotations $\varphi_i \rightarrow e^{i\alpha_i} \varphi_i$ ($i = 1, \cdots, N$) and hence some exact results can be obtained. For example, the generating function

$$Z(\mu) = \text{Tr} \left[ \exp \left( -\beta \hat{H} - i\beta \mu \cdot \hat{\mathcal{N}} \right) \right], \quad \left( \hat{H} = \frac{1}{g} V(g\hat{\mathcal{N}}) \right) \quad (4.2)$$

satisfies the differential equation

$$\left[ -\frac{1}{\beta} \frac{\partial}{\partial g} + \frac{1}{g^2} V(\tilde{\nu}) - \tilde{\nu} \cdot \frac{\partial}{\partial \tilde{\nu}} V(\tilde{\nu}) \right] Z(\mu) = 0, \quad \left( \tilde{\nu} = g \frac{i}{\beta} \frac{\partial}{\partial \mu} \right), \quad (4.3)$$
where $\mu = (\mu_1, \cdots, \mu_N)$ are chemical potentials for the conserved charges. The perturbation series can be determined from this differential equation with the initial condition $Z_{g=0} = \prod_{i=1}^{N} (1 - e^{-i\beta \mu_i})^{-1}$. In general, the Borel transform of the perturbation series has singularities corresponding to non-perturbative saddle points. Such saddle point solutions of the Wick rotated equation of motion can be obtained by using the conservation laws

$$\varphi_i = \sqrt{\nu_i} \exp (i \omega_p \tau + i \theta)_i, \quad \tilde{\varphi}_i = \sqrt{\nu_i} \exp (-i \omega_p \tau - i \theta)_i,$$

(4.4) where $\theta = (\theta_1, \cdots, \theta_N)$ are moduli parameters (integration constants) and we have defined

$$\omega_p = \mu + \frac{2\pi}{\beta} p, \quad p = (p_1, \cdots, p_N) \in \mathbb{Z}^N.$$

(4.5)

The values of $\nu = (\nu_1, \cdots, \nu_N)$ are determined from the conditions

$$\nu_i \frac{\partial}{\partial \nu_i} S_p(\nu) = 0 \quad (i = 1, \cdots, N),$$

(4.6)

where $S_p(\nu)$ is the $N$-variable function

$$S_p(\nu) = \beta V(\nu) + i \beta \omega_p \cdot \nu.$$

(4.7)

The contribution from these saddle points can also be determined from the one-loop determinant by solving the differential equation. The intersection number can be determined by the flow equation. Using the ansatz

$$\varphi_i = \sqrt{\nu_i(s)} \exp (i \omega_p \tau + i \theta)_i, \quad \tilde{\varphi}_i = \sqrt{\nu_i(s)} \exp (-i \omega_p \tau - i \theta)_i,$$

(4.8)

we can reduce the flow equation for $(\varphi, \tilde{\varphi})$ to that for $\nu_i$

$$\frac{1}{2 |\nu_i|} \frac{\partial}{\partial s} = \frac{1}{\beta} \frac{\partial S_p}{\partial \nu_i}.$$

(4.9)

Combining the saddle point contributions and the intersection numbers, we can construct the transseries for the generating function.

In the operator formalism, the generating function is given by

$$Z = \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \exp \left[ -\frac{\beta}{g} V(gn_1, \cdots, gn_N) - i\beta \mu_i n_i \right].$$

(4.10)

By applying the Poisson resummation formula (2.73) to each summation, the generating function can be rewritten as

$$Z = Z_0 + Z_1 + \cdots + Z_N,$$

(4.11)
where $\mathcal{Z}_m$ is given by integrals over $m$-face of the region $\nu_i \geq 0$ ($i = 1, \cdots, N$)

$$
\mathcal{Z}_m = \frac{1}{g^{m2^{N-m}}} \sum_{p_1 \in \mathbb{Z}} \cdots \sum_{p_m \in \mathbb{Z}} \int_0^\infty d\nu_1 \cdots d\nu_m \exp \left(-S_p\right)_{\nu_m+1=\cdots=\nu_N=0} + \{\text{permutations}\}. \quad (4.12)
$$

Applying the Lefschetz thimble method to each integral, we can confirm the correspondence between the path integral and operator formalisms. For example, each solution of the saddle point conditions (4.6) in the path integral formalism is a saddle point of one of $\mathcal{Z}_m$. In this way, we can check the correspondence of saddle points, gradient flows, and turning points in the path integral and operator formalisms. Note that, in general, the saddle point configurations satisfying (4.6) are complex saddle points. This shows that the complexification of the path integral is indispensable for obtaining exact transseries.

It would also be possible to generalize the discussions to general integrable systems, where the action can be rewritten by using the action-angle variables $(\nu, \vartheta)$ as

$$
S_E = \int d\tau \left[i\nu \cdot \partial_\tau \vartheta - H(\nu) - i\mu \cdot \nu \right]. \quad (4.13)
$$

Assuming that $\vartheta$ is on the invariant torus $\vartheta = \frac{2\pi}{\beta} p + \theta$ ($p \in \mathbb{Z}^N$), the saddle point condition and flow equation for $S_E$ reduces to those for the function

$$
S_p = \beta H + i\beta \omega_p \cdot \nu, \quad \left(\omega_p = \mu + \frac{2\pi}{\beta} p\right). \quad (4.14)
$$

On the other hand, in the operator formalism, the generating function can be rewritten into a form similar to (4.12) depending on the details of the quantization conditions of the conserved charges $\nu$. Then, applying the Lefschetz thimble method, we can confirm the correspondence between the path integral and operator formalisms. In this way, it would be possible to show that the Lefschetz thimble formalism gives exact results which are consistent with the operator formalism in general integrable systems.

5. CONCLUSIONS AND DISCUSSION

In this paper, we have discussed the resurgence structure of the generating function for the conserved charge in the $U(1)$ symmetric first-order time derivative systems. We have explicitly evaluated the path integral for the generating function by following the Lefschetz thimble method with the help of the differential equation which enables us to determine the all-order perturbation series around each saddle point. We have checked that the results obtained through the Lefschetz thimble method were consistent with the exact expressions obtained in the operator formalism. This fact indicates the non-perturbative completeness of the Lefschetz thimble method.

We have seen that the resurgence structure of the quantum mechanical system considered in this paper can be correctly captured by the Lefschetz thimble method. It would be interesting to generalize the discussion to the more general quantum mechanical systems with explicit analytic solutions. The key point that enables us to analyze exact results explicitly is integrability, i.e., the property that the number...
of degrees of freedom is the same as that of conserved charges. It would be possible to generalize our discussion to general integrable quantum mechanical systems. The explicit analysis of thimbles of the action written in terms of the action-angle variables (4.13) is important future work. Quantum mechanics with a single degree of freedom is one of the simplest classes of models where the action can be rewritten into the form (4.13) by using the conserved energy. Therefore, we can apply the analysis in this paper to such systems. It would be interesting to analyze the relationship between the method discussed in this paper and the exact WKB analysis.

It is also important to generalize the thimble analysis to the integrable quantum field theories. The non-linear Schrödinger system in two dimensions, whose 1d reduction is the model discussed in Sec. 2 is one of the examples of integrable field theories. It is more non-trivial to correctly determine the resurgence structure of field theories due to the existence of so-called renormalons [17], whose relation to saddle point configurations has not yet been well understood. Understanding the resurgence structure, in particular, the renormalons in the path integral formalism of exactly solvable models is important future work.

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Appendix A: Lefschetz thimble method

In this appendix, we recapitulate the Lefschetz thimble method. Suppose that we are interested in a path integral of the form of

\[ Z = \int_C D\phi \exp(-S[\phi]). \]  

(A.1)

By deforming the integration contour \( C = \{ \phi \in \mathbb{R} \} \), this path integral can be decomposed as

\[ Z = \sum_{\sigma \in \mathcal{G}} n_{\sigma} Z_{\sigma}, \]  

(A.2)

where \( \mathcal{G} \) denotes the set of all the saddle points of \( S[\phi] \), that is, the solutions of the complexified equation of motion

\[ \frac{\delta S}{\delta \phi} = 0. \]  

(A.3)
The contribution associated with each saddle point is given by the path integral over the Lefschetz thimble $J_\sigma$:

$$Z_\sigma = \int_{J_\sigma} D\varphi \exp(-S).$$  \hspace{1cm} (A.4)

The thimble $J_\sigma$ associated with the saddle point $\phi_\sigma$ is the set of points in the complexified configuration space which can be reached from the saddle point by the flow

$$\frac{d\varphi}{ds} = \frac{\delta S}{\delta \varphi}, \quad \lim_{s \to -\infty} \varphi = \phi_\sigma,$$

where $s$ is a formal flow parameter. Note that $\text{Re} \ S$ is strictly increasing and $\text{Im} \ S$ is constant along the upward flow

$$\frac{d}{ds} \text{Re} \ S > 0, \quad \frac{d}{ds} \text{Im} \ S = 0.$$  \hspace{1cm} (A.6)

The coefficients $n_\sigma$ indicate how the original integration contour $C$ is decomposed:

$$C = \sum_{\sigma \in \mathcal{S}} n_\sigma J_\sigma.$$  \hspace{1cm} (A.7)

They can also be defined as the intersection numbers between $C_R$ and “the dual thimble $K_\sigma$” defined as the set of points which flows to the saddle point $\sigma$:

$$\frac{d\varphi}{dt} = \frac{\delta S}{\delta \varphi}, \quad \lim_{t \to \infty} \varphi = \varphi_{\text{sol}, \sigma}.$$  \hspace{1cm} (A.8)

Since the thimble $J_\sigma$ and its dual $K_\sigma$ are defined in terms of the flow, it follows that the real and imaginary parts of the complexified action satisfy

$$\text{Re} \ S|_{J_\sigma} \geq \text{Re} \ S|_{\text{sol}, \sigma} \geq \text{Re} \ S|_{K_\sigma}, \quad \text{Im} \ S|_{J_\sigma} = \text{Im} \ S|_{\text{sol}, \sigma} = \text{Im} \ S|_{K_\sigma}.$$  \hspace{1cm} (A.9)

These properties imply that $J_\sigma$ and $K_\sigma$ intersect exactly once at the saddle point $\sigma$, and $J_\sigma$ cannot intersect with $K_{\sigma'}$ ($\sigma' \neq \sigma$) since $\text{Im} \ S|_{J_\sigma} \neq \text{Im} \ S|_{K_{\sigma'}}$ for a generic action. Therefore, the intersection pairing of $J_\sigma$ and $K_{\sigma'}$, regarded as middle dimensional relative homology cycles, is given by

$$\langle J_\sigma, K_{\sigma'} \rangle = \delta_{\sigma \sigma'}.\hspace{1cm} (A.10)$$

Using this pairing, we can calculate the coefficients $n_\sigma$ as the intersection number of the original contour $C_R$ and the dual thimble $K_\sigma$:

$$n_\sigma = \langle C_R, K_\sigma \rangle.$$  \hspace{1cm} (A.11)

The perturbative part of the partition function corresponds to $Z_0$ defined as the path integral over
the thimble \( \mathcal{J}_0 \) emanating from the trivial vacuum configuration. Non-perturbative contributions are
given by the path integral over thimbles associated with non-trivial saddle points \( \sigma \). It is often the case
that the partition function for a real positive coupling constant \( g \) is on the Stokes line, i.e., the line
on which the thimbles \( \mathcal{J}_\sigma \) and the coefficients \( n_\sigma \) change discontinuously when we vary the coupling
constant in the complex \( g \) plane. If \( \mathcal{J}_0 \) jumps on the real axis (\( \text{Im} \, g = 0 \)), the perturbative part \( Z_0 \) has
an ambiguity depending on how we take the limit \( \text{Im} \, g \to \pm 0 \). However, the original partition function \( Z \) has
no ambiguity since it is defined independently of \( \mathcal{J}_\sigma \) and \( n_\sigma \). Therefore, the ambiguity of \( Z_0 \) has
to be canceled by those associated with other non-trivial saddle points. In the case of \( \mathbb{C}P^1 \) quantum
mechanics, such saddle points correspond to the bion configurations [45–48], and their contributions have
ambiguities as can also be seen in the result of the Gaussian approximation (B.1). We will see below
that the ambiguity of the bion contribution originates from the discontinuous change of the intersection
number \( n_\sigma \) associated with the bion saddle points.

Appendix B: Differential equation for generating function

Here we determine the higher-order correction around the non-perturbative saddle point in the case
of the quartic potential. The leading order contribution from the \( p \)-th saddle point is given by

\[
Z_p = \sqrt{\frac{2\pi}{\beta g}} \exp \left( -\frac{\beta}{2g} \omega_p^2 \right) \left[ 1 + \mathcal{O}(g) \right].
\]  

(B.1)

To determine the higher order corrections, let us solve the differential equation (2.18) by assuming the
power series ansatz

\[
Z_p = \sqrt{\frac{2\pi}{\beta g}} \exp \left( -\frac{\beta}{2g} \omega_p^2 \right) \left[ 1 + a_1 g + a_2 g^2 + \cdots \right],
\]  

(B.2)

where the coefficients \( a_n \) \( (n = 1, 2, \cdots) \) are functions of \( \mu \). Substituting into the differential equation
(2.18), we obtain the following recursive differential equation

\[
(\omega_p \partial_\mu + n) a_n = \frac{1}{2\beta} \partial_\mu^2 a_{n-1}.
\]  

(B.3)

The general solution is given by

\[
a_n = \sum_{m=1}^{n} \frac{c_m}{\omega_p^m} \left( -\frac{1}{2\beta \omega_p^2} \right)^{n-m} \frac{(2n-m-1)!}{(n-m)!(m-1)!},
\]  

(B.4)

where \( c_m \) are arbitrary constants. We can show from the path integral expression in the quartic potential
model that the coefficients \( a_n \) are non-singular at \( \omega_p = 0 \). Thus, we conclude that \( c_n = 0 \) and there
is no correction to the leading order contribution (B.1), that is, the non-perturbative contributions are
one-loop exact.
In general, the non-perturbative contributions of non-trivial saddle points take the form of

\[ Z_{p,\sigma} = \sqrt{\frac{2\pi}{\beta g V''(\nu_{p,\sigma})}} \exp \left( \frac{-S_p(\nu_{p,\sigma})}{g} \right) \left[ 1 + a_1 g + a_2 g^2 + \cdots \right]. \]  

(B.5)

The coefficients \( a_n \) can be determined by the differential equation (3.5), which reduces to the recursive differential equation of the form of

\[ (X\partial_\mu + n) a_n = Y_n(\mu), \]  

(B.6)

where \( Y \) is a function of \( \mu \) determined once the solutions \( a_i \) \( (i = 1, \cdots, n - 1) \) are given and \( X(\omega_p) \) is a function of \( \omega_p \) such that

\[ \lim_{\omega_p \to 0} X(\omega_p) = 0. \]  

(B.7)

This implies that the general solution of the homogeneous equation \((X\partial_\mu + n)a_n = 0\) is singular in the limit \( \omega_p \to 0 \). Therefore, we can uniquely fix the solution of the differential equation (B.6) by requiring that it is regular in the limit \( \omega_p \to 0 \). Thus, all the coefficients around the saddle points can be uniquely determined by solving the differential equation (3.5).

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