PRODUCTS OF GENERALIZED NEVANLINNA FUNCTIONS
WITH SYMMETRIC RATIONAL FUNCTIONS

S. HASSI AND H. L. WIETSMA

Abstract. New classes of generalized Nevanlinna functions, which under multiplication with an arbitrary fixed symmetric rational function remain generalized Nevanlinna functions, are introduced. Characterizations for these classes of functions are established by connecting the canonical factorizations of the product function and the original generalized Nevanlinna function in a constructive manner. Also a detailed functional analytic treatment of these classes of functions is carried out by investigating the connection between the realizations of the product function and the original function. The operator theoretic treatment of these realizations is based on the notions of rigged spaces, boundary triplets, and associated Weyl functions.

1. Introduction

In the fifties M.G. Kreın introduced the classes of Stieltjes and inverse Stieltjes functions, denoted here by $\mathcal{S}$ and $\mathcal{S}^{-1}$, as subclasses of the class of Nevanlinna functions. These functions were introduced in connection with investigations on the theory of the generalized resolvents and the theory of spectral functions of strings. M.G. Kreın showed that these classes have a simple characterization:

$$f(z) \in \mathcal{S} \iff f(z), zf(z) \in \mathcal{N}_0,$$

with a similar characterization for the class of inverse Stieltjes functions. Here $\mathcal{N}$ denotes the class of (ordinary) Nevanlinna functions. In the seventies the class of Nevanlinna functions was generalized by M.G. Kreın and H. Langer to a class of generalized Nevanlinna functions with $\kappa$ negative squares, denoted by $\mathcal{N}_\kappa$; see [23, 24]. One specific subclass of $\mathcal{N}_\kappa$, which was initially introduced for solving indefinite analogues of the Stieltjes moment problem, is the class $\mathcal{N}_\kappa^+$ defined as

$$f(z) \in \mathcal{N}_\kappa^+ \iff f(z) \in \mathcal{N}_\kappa, zf(z) \in \mathcal{N}_0,$$

see [24]. This class extends the class of Stieltjes functions; it is known to be connected with the theory of spectral functions of a generalized string, see [27].

Further generalizations of the classes of Stieltjes (and inverse Stieltjes) classes are due to V.A. Derkach and M.M. Malamud:

$$f(z) \in \mathcal{S}_\kappa^{+\kappa}(\alpha, \beta) \iff f(z) \in \mathcal{N}_\kappa, \frac{z - \beta}{z - \alpha}f(z) \in \mathcal{N}_\kappa,$$

where $-\infty \leq \alpha < \beta < \infty$ (if $\alpha = -\infty$, then $z + \infty$ should be interpreted as being one). These classes were introduced to describe the selfadjoint extensions

1991 Mathematics Subject Classification. Primary 30E20, 47A45, 47B25; Secondary 47A06, 47A10, 47B50.

Key words and phrases. Generalized Nevanlinna function, Hilbert space, Pontryagin space, selfadjoint operator, boundary triplet, Weyl function, realization.
of a symmetric operator which have spectral gaps in their spectra, see [8]. V.A. Derkach extended the Stieltjes (and inverse Stieltjes) classes also in a different direction:

\[ f(z) \in \mathcal{N}_k^+ \iff f(z) \in \mathcal{N}_k, zf(z) \in \mathcal{N}_k, \]

see [3]. All the preceding classes of complex functions can be seen as special cases of the classes \( \mathcal{N}_k^\kappa(r) \), which will be investigated in the present paper. The definition of the class \( \mathcal{N}_k^\kappa(r) \) involves an arbitrary symmetric rational function \( r \) and two indices \( \kappa, \tilde{\kappa} \), which are used to describe the number of negative squares of the Nevanlinna kernels associated with the functions \( \mathcal{Q} \) and \( rQ \):

\[ \mathcal{N}_k^\kappa(r) = \{ Q \in \mathcal{N}_k : rQ \in \mathcal{N}_\kappa \}. \]

All the above mentioned special cases involve a simple symmetric rational function \( r \) of degree one; namely either \( r \) or \( 1/r \) has the form \( \frac{z-b}{z-a} \) or \( z-b, a, b \in \mathbb{R} \).

The aim of this paper is to describe the characteristic properties of functions in the classes \( \mathcal{N}_k^\kappa(r) \) and to investigate relevant functional analytic connections they have in the area of operator and spectral theory. The applicability of these functions is covering topics, which occur in the literature for some elementary rational functions \( r \). The results extend various previously known facts, which has been established for some special cases of the classes \( \mathcal{N}_k^\kappa(r) \).

The methods used in the present paper differ from those appearing in the above mentioned papers in some special cases; the approach is constructive and a basic tool here is the canonical factorization of generalized Nevanlinna functions, see [11] (cf. also [7] or Proposition 2.4 below). Although from a general point of view the framework involves notions from indefinite inner product spaces, the approach used here reduces the main problems to analogous problems for the classes \( \mathcal{N}_0^\kappa(\tilde{r}) \), where no indefiniteness occurs. Here \( \tilde{r} \) can be taken to be a symmetric rational function with real zeros and poles of order at most two (see Theorem 4.12).

In this paper systematically track is kept off the connections between the canonical factorization of the functions \( \mathcal{Q} \) and \( rQ \) (see Theorems 4.5 and 4.9). This gives a basis for the investigations concerning the realizations for the functions \( \mathcal{Q} \) and \( rQ \) and the functional analytic connections between these realizations. The key observation in establishing realizations for the functions \( \mathcal{Q} \) and \( rQ \) is based on some local integrability properties of these functions, involving so-called Kac-Donoghue classes of Nevanlinna functions (see Propositions 3.1 and 4.17). This motivates a study of certain local versions of rigged Hilbert spaces associated with selfadjoint relations (see in particular Sections 5.3, 5.4). The functional analytic connection is first made precise in the abstract case (see Theorems 6.2, 6.3 and Proposition 6.6) and thereafter it is made more accessible by means of (the minimal) \( L^2(d\sigma) \)-models (Theorem 6.17).

The contents of the paper are now outlined. In Section 2 some preliminary results on (generalized) Nevanlinna functions and their realization are stated. In Section 3 Kac-Donoghue classes of Nevanlinna functions are introduced and Nevanlinna functions with spectral gaps are characterized. Using these results the classes \( \mathcal{N}_k^\kappa(r) \) are characterized in Section 4. Local versions of rigged spaces are introduced in Section 5 and they are used to obtain realizations for functions in the Kac-Donoghue classes. Finally, Section 6 uses all the preparations to connect the realizations of the functions \( \mathcal{Q} \) and \( rQ \) when \( Q \in \mathcal{N}_k^\kappa(r) \).
2. Preliminary results

Some facts on scalar (generalized) Nevanlinna functions and linear relations are recalled, and a realization for Nevanlinna functions by boundary triplets is given.

2.1. Nevanlinna functions. Recall that a (scalar) complex function \( f(z) \) is called symmetric if \( f(\overline{z}) = f(z) \). A symmetric function \( Q \) belongs to the class of (ordinary) Nevanlinna functions, i.e. \( Q \in \mathcal{N} \), if \( Q \) is holomorphic on \( \mathbb{C} \setminus \mathbb{R} \) and \( \operatorname{Im}(Q(z))/\operatorname{Im}(z) \geq 0 \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \). Equivalently, \( Q \) is a Nevanlinna function if and only if there exists \( \alpha \in \mathbb{R} \), \( \beta > 0 \), and a nonnegative Borel measure \( d\sigma \) such that \( Q \) has the following integral representation:

\[
Q(z) = \alpha + \beta z + \int_{\mathbb{R} \setminus \rho(Q)} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), \quad \int_{\mathbb{R}} \frac{d\sigma(t)}{1+t^2} < \infty,
\]

where \( \rho(Q) \) denoted the domain of holomorphy of \( Q \). The set \( \sigma(Q) \) is defined as

\[
\sigma(Q) = \mathbb{R} \setminus \rho(Q), \quad \text{if } \beta = 0; \quad \sigma(Q) = (\mathbb{R} \setminus \rho(Q)) \cup \{\infty\}, \quad \text{if } \beta > 0.
\]

Here \( \beta > 0 \) is to be interpreted as a point mass at \( \infty \). In particular, if \( \sigma_p(Q) \) denotes the point spectrum of \( Q \), then \( \infty \in \sigma_p(Q) \) if and only if \( \beta > 0 \). If \( Q \) is holomorphic on an interval \((c, d) \subset \mathbb{R} \), then it follows from (2.1) that for \( \lambda \in (c, d) \),

\[
Q'(\lambda) = \beta + \int_{(-\infty,c] \cup [d,\infty)} \frac{d\sigma(t)}{(t-\lambda)^2}.
\]

In particular, if \( Q \) is not a constant function, then it is strictly increasing on the interval \((c, d) \); such an interval is called a spectral gap of \( d\sigma \).

The characterizing objects in the integral representation of a Nevanlinna function are its spectral measure \( d\sigma \) and \( \beta \). The spectral measure \( d\sigma \) of a Nevanlinna function \( Q \) can be recovered from the integral representation by means of the generalized Stieltjes inversion formula: if \( \varphi \) is real-valued on \([c, d] \subset \mathbb{R} \) and holomorphic on an open neighborhood of \([c, d] \) in \( \mathbb{C} \), then

\[
\lim_{\epsilon \downarrow 0} \int_{[c,d]} \operatorname{Im} \left( \varphi(\lambda + i\epsilon)Q(\lambda + i\epsilon) \right) \frac{d\lambda}{\pi} = \int_{[c,d]} \varphi(t) \left( 1_{(c,d)} + \frac{1_{(c)} + 1_{(d)}}{2} \right) d\sigma(t).
\]

This formula for sign varying functions \( \varphi \) is easily derived from [20, (S1.2.6)]. In particular, since the spectral measure of Nevanlinna functions is nonnegative, \( \sigma \)

yields the following statement.

Lemma 2.1. Let \( Q_0 \) and \( Q_1 \) be Nevanlinna functions and assume that there exists functions \( \varphi_0 \) and \( \varphi_1 \) holomorphic on a neighborhood \( D \) of \([c, d] \) in \( \mathbb{C} \), which are real-valued, nonzero and have opposite, but fixed, sign on \([c, d] \), such that

\[
\varphi_0(z)Q_1(z) = \varphi_1(z)Q_1(z), \quad z \in D \setminus \mathbb{R}.
\]

Then \((c, d) \subset \rho(Q_0) \cap \rho(Q_1) \).

The next lemma contains some further information about the local behavior of Nevanlinna functions on the real line: these facts can easily be deduced from the integral representations of Nevanlinna functions; see e.g. [12, 20].

\footnote{The spectral measures of Nevanlinna functions will always be assumed to be normalized in the standard manner: \( \sigma(t) = (\sigma(t+) + \sigma(t-))/2 \) and \( \sigma(0) = 0 \).}
Lemma 2.2. Let $c \in \mathbb{R}$ and let $Q \in \mathcal{N}$ have the integral representation \((2.1)\). Then
\[
\lim_{z \to c}(c - z)Q(z) = \int_{\mathbb{R}} \mathbf{1}_{\{c\}} \, d\sigma(t) \quad \text{and} \quad \lim_{z \to \infty} \frac{Q(z)}{z} = \beta.
\]
In particular, both limits are finite and nonnegative. Moreover,
\[
\lim_{z \to \infty} \frac{Q(z)}{z - c} = \begin{cases} \beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - c)^2}, & \text{if } \lim_{z \to c} Q(z) = 0 \text{ and } \int_{\mathbb{R}} \frac{d\sigma(t)}{(t - c)^2} < \infty; \\ \infty, & \text{otherwise}; \end{cases}
\]
and
\[
\lim_{z \to \infty} zQ(z) = \begin{cases} -\int_{\mathbb{R}} d\sigma(t), & \text{if } \beta = 0, \lim_{z \to \infty} Q(z) = 0 \text{ and } \int_{\mathbb{R}} d\sigma(t) < \infty; \\ \infty, & \text{otherwise}. \end{cases}
\]

Remark 2.3. The notation $z \to c$ stands for the non-tangential limit from the upper (or lower) half-plane, if $c \in \mathbb{R}$, or a sectorial limit (with $|\arg(z) - \pi/2| \geq \alpha > 0$) if $c = \infty$. In particular, $\lim_{z \to c} f(z) = \infty$ indicates that $\lim_{z \to c} |f(z)| = +\infty$. If the limit of $f(x)$ (with $x \in \mathbb{R}$) exists as an improper limit the notation $\lim_{x \to c} f(x) = +\infty$ or $\lim_{x \to c} f(x) = -\infty$ is used.

2.2. Generalized Nevanlinna functions. A symmetric function $Q$ belongs to the class of **generalized Nevanlinna functions**, denoted by $\mathcal{N}_\kappa$ for $\kappa \in \mathbb{N}$, if $Q$ is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and its kernel $(Q(z) - Q(w))/(z - w)$ has $\kappa$ negative squares for $z, w \in \rho(Q)$; see \([22, 23]\). In particular, $\mathcal{N} = \mathcal{N}_0$ is the class of (ordinary) Nevanlinna functions. For a generalized Nevanlinna function $Q \neq 0$ define $\pi_\alpha$ with $\alpha \in \mathbb{R} \cup \{\infty\}$ to be the largest nonnegative integer such that
\[
Q(z) < 0 \quad \text{if } \lim_{z \to c}(z - c)^{2\pi_\alpha - 1} Q(z) < \infty.
\]
Likewise, define $\kappa_\beta$ with $\beta \in \mathbb{R} \cup \{\infty\}$ to be the smallest nonnegative integer such that
\[
Q(z) < 0 \quad \text{if } \lim_{z \to \infty}(z - \beta)^{2\kappa_\beta + 1} Q(z) < \infty.
\]
If $\pi_\alpha > 0$ ($\kappa_\beta > 0$) then $\alpha$ ($\beta$) is said to be a **generalized zero (pole) of non-positive type** (GZNT) (or (GPNT)) of $Q$ with multiplicities $\pi_\alpha$ ($\kappa_\beta$, respectively). Let $\{\alpha_i\}_{i=1}^m$ be the collection of all zeros in $\mathbb{C}_+$ and all generalized zeros of non-positive type (GZNT) in $\mathbb{R}$ of $Q$ with multiplicities $\pi_{\alpha_i}$, $1 \leq i \leq m$. Similarly, let $\{\beta_i\}_{i=1}^n$ be the collection of all poles in $\mathbb{C}_+$ and generalized poles of non-positive type (GPNT) in $\mathbb{R}$ of $Q$ with multiplicities $\kappa_{\beta_i}$, $1 \leq i \leq n$. The following result characterizes the class of generalized Nevanlinna functions; see \([11]\) (cf. also \([7]\)).

Proposition 2.4. $Q \in \mathcal{N}_\kappa$ if and only if there exists a unique $Q_0 \in \mathcal{N}$ such that
\[
Q(z) = \frac{\prod_{i=1}^m (z - \alpha_i)^{\pi_{\alpha_i}} (z - \overline{\alpha_i})^{\pi_{\overline{\alpha_i}}} \prod_{i=1}^n (z - \beta_i)^{\kappa_{\beta_i}} (z - \overline{\beta_i})^{\kappa_{\overline{\beta_i}}}}{Q_0(z)},
\]
where $\kappa = \max\left\{\sum_{i=1}^m \pi_{\alpha_i}, \sum_{i=1}^n \kappa_{\beta_i}\right\}$.

The unique factorization $\phi Q_0$, $Q_0 \in \mathcal{N}$, of any generalized Nevanlinna function $Q$ provided by the above proposition is called the **canonical factorization** of $Q$. Moreover, note that $\pi_\infty + \sum_{i=1}^m \pi_{\alpha_i} = \kappa_\infty + \sum_{i=1}^n \kappa_{\beta_i}$. In particular, a generalized Nevanlinna function is an ordinary Nevanlinna function if it does not have any GZNT’s or GPNT’s in $\mathbb{C} \cup \{\infty\}$.

The canonical factorization allows an easy proof of the next composition result.
Lemma 2.5. Let $Q \in \mathcal{N}_k$ and let $\tau \in \mathcal{N}$ be a rational Nevanlinna function of degree $k$. Then $Q \circ \tau \in \mathcal{N}_{kn}$ and, furthermore, the canonical factorizations of $Q$ and $Q \circ \tau$ are connected by

$$Q = \phi Q_0, \quad Q \circ \tau = (\phi \circ \tau)(Q_0 \circ \tau).$$

Proof. For rational functions $r_1$ and $r_2$ the degree of the composition $r_1 \circ r_2$ is given by $\deg(r_1 \circ r_2) = \deg(r_1) \deg(r_2)$, see [28]. Let $Q = \phi Q_0$, $Q_0 \in \mathcal{N}$, be the canonical factorization of $Q$ in Proposition 2.4. Then $\phi \circ \tau$ is a rational function of degree $kn$. Since $\tau$ and $\phi$ are symmetric, so is $\phi \circ \tau$. Moreover, this composition is nonnegative on the real line, because $\phi$ is nonnegative on the real line and $\tau$ maps $\mathbb{R} \cup \{\infty\}$ into $\mathbb{R} \cup \{\infty\}$. Since $Q_0 \circ \tau \in \mathcal{N}$, the statement follows from Proposition 2.4. \qed

2.3. Symmetric and selfadjoint relations. Let $\mathfrak{H}_i$ be a Hilbert space with inner product $(\cdot, \cdot)_i$, $i = 1, 2$. Then $H$ is called a (linear) relation from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ if its graph is a subspace of $\mathfrak{H}_1 \times \mathfrak{H}_2$. In particular, $H$ is closed if and only if its graph is closed (as a subset of $\mathfrak{H}_1 \times \mathfrak{H}_2$); in what follows $H$ is usually identified with its graph. The symbols $\text{dom } H$, $\text{ran } H$, $\ker H$, and $\text{mul } H$ stand for the domain, range, kernel, and the multi-valued part of $A$, respectively.

The adjoint $H^*$ of $H$ is defined by

$$H^* = \{(f, f') \in \mathfrak{H}_2 \times \mathfrak{H}_1 : (f', g)_1 = (f, g')_2, \; \forall g, g' \in \mathfrak{H}_2\},$$

A relation $H$ in the Hilbert space $\mathfrak{H}$ (i.e., relation from $\mathfrak{H}$ to $\mathfrak{H}$) is said to be symmetric or selfadjoint if $H \subset H^*$ or $H = H^*$, respectively. For a relation $H$ in $\mathfrak{H}$ the eigenspaces of $H$ are denoted by

$$\mathfrak{M}_\lambda(H) = \ker (H - \lambda) \quad \text{and} \quad \hat{\mathfrak{M}}_\lambda(H) = \{(f, \lambda f) : f \in \mathfrak{M}_\lambda(H)\}, \quad \lambda \in \mathbb{C}.$$

Recall that for a symmetric relation $S$ in $\mathfrak{H}$, $n_+(S) = \dim \mathfrak{M}^{-1}_+(S)$ and $n_-(S) = \dim \mathfrak{M}^{-1}_-(S)$, $\lambda \in \mathbb{C}_+$ denote its deficiency indices. If $S$ has equal defect numbers, then $S$ allows selfadjoint extensions. If $A$ is a selfadjoint extension of $S$, then

$$S^* = A \oplus \hat{\mathfrak{M}}_\lambda(S^*), \quad \lambda \in \rho(A).$$

Here $\oplus$ indicates the componentwise sum (linear span) of the subspaces.

The multi-valued part of a selfadjoint relation $A$ reduces the relation:

(2.8) \hspace{1cm} $A = A_o \oplus \{0\} \times \text{mul } A$.

Here $A_o = P_\infty A \upharpoonright \text{ran } P_\infty$, where $P_\infty$ stands for the orthogonal projector onto $\text{dom } A = \mathfrak{H} \oplus \text{mul } A$, the so-called operator part of $A$ is a selfadjoint operator in $\text{dom } A$. Using the above decomposition define $|A|^\alpha$, $\alpha > 0$, as

(2.9) \hspace{1cm} $|A|^\alpha = |A_o|^\alpha \oplus \{0\} \times \text{mul } A$,

where $|A_o|$ is the modulus of $A_o$, and set $|A|^\alpha = |A^{-1}|^{-\alpha}$ for $\alpha < 0$. Then $|A|^\alpha$ is a selfadjoint relation in $\mathfrak{H}$.

2.4. Realizations of Nevanlinna functions as Weyl functions. Boundary triplets for symmetric operators in Hilbert spaces, which were introduced in [2, 26], can be used to realize Nevanlinna functions. Here the definition of a boundary triplet is given for symmetric relations with defect numbers $n_+(S) = n_-(S) = 1$ together with a definition for its associated $\gamma$-field and Weyl function; see [8, 9].

Definition 2.6. Let $S$ be a closed symmetric relation in a Hilbert space $\mathfrak{H}_\delta(\cdot, \cdot)$. Then $\{\mathbb{C}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$ if
The mappings $\Gamma_0, \Gamma_1 : S^* \to \mathbb{C}$ satisfy the abstract Green's identity:

$$(f', g) - (f, g') = (\Gamma_1 \{f, f'\}, \Gamma_0 \{g, g'\}) - (\Gamma_0 \{f, f'\}, \Gamma_1 \{g, g'\})$$

for all $\{f, f'\}, \{g, g'\} \in S^*$;

(ii) $\Gamma : S^* \to \mathbb{C} \times \mathbb{C}, \{f, f'\} \mapsto \{\Gamma_0 \{f, f'\}, \Gamma_1 \{f, f'\}\}$ is surjective.

With $A_0 := \ker \Gamma$, the $\gamma$-field $\gamma_\lambda$ and the Weyl function $M(\lambda)$ associated with $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ are the vector function and scalar function defined by

$$\gamma_\lambda = \pi_1(\Gamma_0 | \hat{\mathcal{R}}_\lambda(S^*))^{-1}, \quad M(\lambda) = \Gamma_1(\Gamma_0 | \hat{\mathcal{R}}_\lambda(S^*))^{-1}, \quad \lambda \in \rho(A_0).$$

Here $A_0 := \ker \Gamma_0$ and $A_1 := \ker \Gamma_1$ are selfadjoint extension of $S$, and $\pi_1$ denotes the orthogonal projection in $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H} \times \{0\}$. The $\gamma$-field and the Weyl function satisfy the formulas

$$(2.10) \quad \gamma_\lambda = (I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma_\mu, \quad \lambda, \mu \in \rho(A_0),$$

and

$$(2.11) \quad M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma^*_\mu \gamma_\lambda, \quad \lambda, \mu \in \rho(A_0);$$

see [8]. This shows that $\gamma$ is a holomorphic vector-function on $\rho(A_0)$ and that $M(\lambda)$ is a Nevanlinna function. In particular, if $M$ is not a constant function, then $0 \in \rho(\text{Im} \, M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Nevanlinna functions which have this additional property are called uniformly strict. Notice that $S$ can be recovered from any of its selfadjoint extensions $A$ by means of the $\gamma$-field via

$$(2.12) \quad S = \{\{f, f'\} \in A : (f' - \bar{\lambda}f, \gamma_\lambda) = 0\}, \quad \lambda \in \rho(A);$$

see [13]. In the terminology of boundary triplets one can formulate the following realization result for uniformly strict Nevanlinna functions; cf. [8 Theorem 1.1].

**Theorem 2.7.** Let $Q$ be a uniformly strict scalar (i.e., nonconstant) Nevanlinna function. Then there exist a closed symmetric relation $S$ in a Hilbert space and a boundary triplet $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ for $S^*$ such that $Q$ is the corresponding Weyl function.

Conversely, let $S$ be a closed symmetric relation in a Hilbert space and let $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, then the Weyl function associated with the boundary triplet is a uniformly strict scalar Nevanlinna function.

The realization in Theorem 2.7 is unique up to unitary equivalence under the minimality condition

$$(2.13) \quad \mathcal{H} = \text{span} \{\gamma_\lambda : \lambda \in \mathbb{C} \setminus \mathbb{R}\},$$

in which case $\rho(A_0) = \rho(Q)$. If the realization is not minimal, then $\rho(A_0)$ may be a proper subset of $\rho(Q)$; see (2.10), (2.11), and e.g. (2.14) below.

**Remark 2.8.** It is a consequence of (2.10), (2.11), and Theorem 2.7 that for every Nevanlinna function $Q$ there exists a selfadjoint relation $A$ in a Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ and an element $v \in \mathcal{H}$, such that

$$(2.14) \quad Q(\lambda) = Q(\lambda_0)^* + (\lambda - \lambda_0)(I + (\lambda - \lambda_0)(A - \lambda)^{-1})v, \quad \lambda, \lambda_0 \in \rho(A).$$

This type of realization of Nevanlinna functions, as so-called $Q$-functions, was developed by M.G. Krein and H. Langer (see e.g. [22] [23]). In this case the $\gamma$-field associated to the $Q$ function is defined as

$$(2.15) \quad \gamma_\lambda = (I + (\lambda - \lambda_0)(A - \lambda)^{-1})v, \quad \lambda \in \rho(A).$$
An application of the resolvent identity shows that $\gamma_\lambda$ satisfies (2.10). This discussion shows that each Nevanlinna function $Q$ can be realized by means of a selfadjoint relation $A$ in a Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ and a function $\gamma_\lambda$ which satisfies (2.15); in what follows, such a realization is called a \{A, $\gamma_\lambda$\}-realization of $Q$.

For a Nevanlinna function $Q$ and $\xi \in \mathbb{R}$, define the transform
\begin{equation}
Q_\xi(\lambda) = -Q((\lambda - \xi)^{-1}), \quad \xi \in \mathbb{R}.
\end{equation}

Clearly, $Q_\xi$ is a Nevanlinna function (cf. Lemma 2.5). The following result gives a connection between the realizations of $Q$ and $Q_\xi$; see [16, Lemma 2.4].

\textbf{Lemma 2.9.} Let $Q \in \mathcal{N}$ have the representation
\begin{equation}
Q(z) = Q(z_0)^* + (z - z_0)(A - z_0)^{-1}v, v,
\end{equation}
for $z, z_0 \in \rho(A)$, where $A$ is a selfadjoint relation in the Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ and $v \in \mathcal{H}$. Then $Q_\xi$, see (2.10), has the representation
\begin{equation}
Q_\xi(\lambda) = Q_\xi(\lambda_0)^* + (\lambda - \lambda_0)(A - \lambda_0)^{-1}\tilde{v}, \tilde{v},
\end{equation}
where $z = (\lambda - \xi)^{-1}$, $z_0 = (\lambda_0 - \xi)^{-1}$, and $\tilde{v} = (\lambda_0 - \xi)^{-1}v$.

If the $\gamma$-fields associated to $Q$ and $Q_\xi$ are denoted by $\gamma_\lambda$ and $\gamma_\lambda^\xi$, respectively, then it follows from Lemma 2.9 that they are connected by
\begin{equation}
(\lambda - \xi)\gamma_\lambda^\xi = \gamma_{(\lambda - \xi)^{-1}}.
\end{equation}

Furthermore, Lemma 2.9 shows that if \{A, $\gamma_\lambda$\} is a realization for $Q$, see Remark 2.8, then $Q_\xi$ is realized by \{A$^{-1}$ + $\xi$, $\gamma_\lambda^\xi$\}.

3. Nevanlinna functions having spectral gaps

Characteristic properties of Nevanlinna functions having gaps in their spectral measure are studied. This involves use of local variants of so-called Kac-Donoghue subclasses and the limit values of $Q$ at the endpoints of the spectral gaps.

3.1. Kac-Donoghue classes of Nevanlinna functions. A Nevanlinna function $Q$ is said to belong to the Kac class $\mathcal{N}(\infty, 1)$, see [19], or to the Kac-Donoghue class $\mathcal{N}(\xi, 1)$ with $\xi \in \mathbb{R}$, cf. [17], if
\begin{equation}
\int_{[1, \infty)} \frac{\text{Im} Q(iy)}{y} dy < \infty \quad \text{or} \quad \int_{(0, 1]} \frac{\text{Im} Q(iy)}{y} dy < \infty,
\end{equation}
respectively. The classes $\mathcal{N}(\xi, 1), \xi \in \mathbb{R}$, are connected to the Kac class $\mathcal{N}(\infty, 1)$ by means of transformation $Q_\xi, \xi \in \mathbb{R}$, in (2.10). In fact, the identities
\begin{equation}
\int_0^1 \frac{\text{Im} Q_\xi(iy)}{y} dy = \int_0^1 -\frac{\text{Im} Q(\frac{i}{\xi}y)}{y} dy = \int_1^\infty -\frac{\text{Im} \{ Q(\frac{-it}{y}) \}}{t} dt = \int_1^\infty -\frac{\text{Im} \{ Q(it/y) \}}{t} dt,
\end{equation}
show that if $Q \in \mathcal{N}$ and $Q_\xi$ is defined by (2.10), then
\begin{equation}
Q \in \mathcal{N}(\infty, 1) \quad \text{if and only if} \quad Q_\xi \in \mathcal{N}(\xi, 1);
\end{equation}

cf. [17] Proposition 3.3. It should be also noted that if $Q$ has the integral representation (2.1), then the spectral measure $d\sigma_\xi$ of $Q_\xi$ is given by
\begin{equation}
d\sigma_\xi(t) = -(t - \xi)^2 d\sigma(1/(t - \xi)), \quad t \in \mathbb{R} \setminus \{\xi\}, \quad \sigma_\xi(\xi) = \beta;
\end{equation}

\textbf{PRODUCTS OF GENERALIZED NEVANLINNA FUNCTIONS 7}
Lemma 3.2. Following lemma, which will also be used in later subsections, is stated. If either of the above equivalent conditions holds, then \( \lim_{z \to \xi} Q(z) \in \mathbb{R} \). Therefore the classes \( \mathcal{N}(\xi, 1) \) can also be characterized via the spectral measures; for the case \( \xi = \infty \), see [17, Theorem 3.2], [15, Proposition 2.2] and for the case \( \xi \in \mathbb{R} \), see [17, Lemma 3.5].

Proposition 3.1. Let \( Q \in \mathcal{N} \) with the integral representation (2.1). Then \( Q \in \mathcal{N}(\xi, 1) \) if and only if

\[
\int_{t \to -\xi < 1} \frac{d\sigma(t)}{|t - \xi|} < \infty, \quad \xi \in \mathbb{R}, \quad \text{or} \quad \int_{\mathbb{R}} \frac{d\sigma(t)}{|t| + 1} < \infty \quad \text{and} \quad \beta = 0, \quad \xi = \infty.
\]

If either of the above equivalent conditions holds, then \( \lim_{z \to \xi} Q(z) \in \mathbb{R} \).

If \( \xi \) is the endpoint of an interval contained in \( \rho(Q) \), then the class \( \mathcal{N}(\xi, 1) \) can be characterized by means of the limit of the function at the point \( \xi \). For this the following lemma, which will also be used in later subsections, is stated.

Lemma 3.2. Let \( Q \in \mathcal{N} \) have the integral representation (2.1). Then the following statements hold:

(i) if \( (\infty, c) \subset \rho(Q) \), then with \( x \in (\infty, c) \) and \( \Delta_c = [c, \infty) \)

\[
\lim_{x \to -\infty} Q(x) = \begin{cases} 
\alpha - \int_{\Delta_c} \frac{t d\sigma(t)}{1 + t^2} & \in \mathbb{R}, \quad \text{if} \quad \int_{\Delta_c} \frac{d\sigma(t)}{|t| + 1} < \infty \quad \text{and} \quad \beta = 0; \\
\alpha - \int_{\Delta_c} \frac{t d\sigma(t)}{1 + t^2} & \in \mathbb{R}, \quad \text{otherwise}; 
\end{cases}
\]

(ii) if \( (c, c+) \subset \rho(Q) \), then with \( x \in (c, c+) \) and \( \Delta_c = [c, \infty) \setminus (c, c+) \)

\[
\lim_{x \to c_\pm} Q(x) = \begin{cases} 
\alpha + \beta c_\pm + \int_{\Delta_c} \frac{1 + ct}{(t - c_\pm)(1 + t^2)} d\sigma(t) & \in \mathbb{R}, \quad \text{if} \quad \int_{|t - c_\pm| \leq 1} \frac{d\sigma(t)}{|t - c_\pm|} < \infty; \\
\alpha + \beta c_\pm + \int_{\Delta_c} \frac{1 + ct}{(t - c_\pm)(1 + t^2)} d\sigma(t) & \in \mathbb{R}, \quad \text{otherwise}. 
\end{cases}
\]

Proof. By monotonicity, see (2.2), the limit in (i) exists in \( \mathbb{R} \cup \{ -\infty \} \). Furthermore, it follows from Lemma 2.2 that for the limit in (ii) to be finite it is necessary that \( \beta = 0 \). If \( \beta = 0 \), then the monotone convergence theorem implies that

\[
\lim_{x \to -\infty} Q(x) = \alpha - \int_{(\infty, c)} \frac{td\sigma(t)}{1 + t^2}
\]

for \( x \in (\infty, c) \). This limit is finite if and only if \( d\sigma \) satisfies the integrability condition in (i). The other statement can be proven with similar arguments. \( \square \)

Combining Lemma 3.2 with Proposition 3.1 yields the following result.

Corollary 3.3. Let \( Q \in \mathcal{N} \), let \( \xi \in \mathbb{R} \cup \{ \infty \} \) and assume that there exists \( c \in \mathbb{R} \) such that \( (\xi, c) \) or \( (c, \xi) \) belongs to \( \rho(Q) \), if \( \xi \in \mathbb{R} \), or that \( (\infty, c) \) or \( (c, \infty) \) belongs to \( \rho(Q) \) if \( \xi = \infty \). Then

\[ Q \in \mathcal{N}(\xi, 1) \quad \text{if and only if} \quad \lim_{z \to \xi} Q(z) \in \mathbb{R}. \]

3.2. Nevanlinna functions holomorphic on (the complement of) a compact interval. Nevanlinna functions with a gap in their spectrum are characterized by means of their limits at the endpoints of this gap. Two cases are considered: the case where \( \rho(Q) \) contains a bounded interval (finite spectral gap) and the case where \( \rho(Q) \) contains the complement of a compact interval.

Proposition 3.4. Let \( Q \in \mathcal{N} \) have the integral representation (2.1). Then the following statements are equivalent for \( c, d \in \mathbb{R} \) with \( c < d \):

(i) \( (c, d) \subset \rho(Q) \) and \( \lim_{x \to d} Q(x) \in \mathbb{R}; \)
(ii) the integral representation of $Q$ is given by

$$Q(z) = \eta + \frac{z - d}{z - c} \left( \beta z - \gamma + \int_{\mathbb{R}\setminus(c,d)} \left( \frac{1}{t - z} - \frac{t}{1 + t^2} \right) d\sigma(t) \right),$$

where $\gamma, \eta \in \mathbb{R}$, $d\sigma(t) = \frac{(t-c)d\sigma(t)}{(t-d)}$, and $\int_{\mathbb{R}\setminus(c,d)} \frac{d\sigma(t)}{t+1} < \infty$;

(iii) $\exists \tilde{Q} \in \mathcal{N}$ and $\tilde{\eta} \in \mathbb{R}$ such that $Q(z) = \tilde{\eta} + \frac{z - d}{z - c} \tilde{Q}(z)$.

In particular, in (ii) $\eta = \lim_{x \uparrow d} Q(x)$ and in (iii) $\tilde{\eta} \geq \eta$.

Proof. (i) ⇒ (ii) If $(c,d) \subset \rho(Q)$, then $Q$ has the integral representation (2.1) with $\Delta = \mathbb{R} \setminus (c,d)$. Moreover, by Lemma 3.2 (ii) the assumption in (i) implies that $\int_{|t-d| \leq 1} |t - d|^{-1} d\sigma(t) < \infty$. Therefore $Q$ can be rewritten as

$$Q(z) = \eta + Q(z) - \eta = \eta + (z - d) \left( \beta + \int_{\mathbb{R}\setminus(c,d)} \frac{d\sigma(t)}{(t - z)(t - d)} \right),$$

where $\eta := \lim_{x \uparrow d} Q(x) \in \mathbb{R}$, see Lemma 3.2. Observe that

$$Q(z) = \eta + Q(z) - \eta = \eta + (z - d) \left( \beta + \int_{\mathbb{R}\setminus(c,d)} \frac{d\sigma(t)}{(t - z)(t - d)} \right),$$

where the integral terms converge (for $z \in \rho(Q)$) as a consequence of the stated integrability condition that $d\sigma$ satisfies. Substituting (3.3) into (3.2) yields the integral representation in (ii).

(ii) ⇒ (iii) This is evident.

(iii) ⇒ (i) This follows from Lemma 2.1 and Lemma 2.2.

As to the last statement, observe that the measure $d\sigma(t)$ in (ii) does not involve a point mass at $x = d$, in which case $\eta = \lim_{x \uparrow d} Q(x)$ by Lemma 2.2. On the other hand, if $Q$ is as in (iii) then by applying Lemma 2.2 to the function $\tilde{Q}$ one obtains $\lim_{x \uparrow d} Q(x) = \tilde{\eta} + \lim_{z \uparrow \gamma,\eta} (z - d)Q(z)/(d - c) \leq \eta$.

If (iii) in Proposition 3.4 holds for some $\tilde{\eta} \geq \eta$, then it holds also for $\eta = \lim_{x \uparrow d} Q(x)$. Moreover, if (iii) holds for some $\tilde{\eta} \geq \eta$, then it holds for every $\tilde{\eta} \geq \eta$.

Next the case that $\rho(Q) \cap \mathbb{R}$ contains the complement of a compact interval is characterized. The proof for this statement is similar to the proof of Proposition 3.4 and is therefore omitted.

**Proposition 3.5.** Let $Q \in \mathcal{N}$ have the integral representation (2.1). Then the following statements are equivalent for $c,d \in \mathbb{R}$ with $c < d$:

(i) $(-\infty, c) \cup (d, \infty) \subset \rho(Q)$, $\beta = 0$, and $\lim_{x \downarrow c} Q(x) \in \mathbb{R}$;

(ii) the integral representation of $Q$ is given by

$$Q(z) = \eta + \frac{d - z}{z - c} \left( -\gamma + \int_{[c,d]} \frac{d\sigma(t)}{t - z} \right),$$

where $\gamma, \eta \in \mathbb{R}$, $d\sigma(t) = \frac{(t-c)d\sigma(t)}{(d-t)}$, and $\int_{[c,d]} d\sigma(t) < \infty$;

(iii) $\exists \tilde{Q} \in \mathcal{N}$ and $\tilde{\eta} \in \mathbb{R}$ such that $Q(z) = \tilde{\eta} + \frac{z - d}{z - c} \tilde{Q}(z)$.

In particular, in (ii) $\eta = \lim_{x \downarrow c} Q(x)$ and in (iii) $\tilde{\eta} \leq \eta$. 
Proposition 3.8. and only if an interval of the form \((a,b)\) are investigated. The statements below are formulated explicitly only in case that section Nevanlinna functions having a semibounded interval as their spectral gap Nevanlinna functions holomorphic on a semibounded interval.

\[
\tilde{\eta} \in \mathcal{N} \text{ if and only if } (c,d) \subset \rho(Q) \text{ and } -\infty < \lim_{x \to d} Q(x) \leq 0;
\]

\[
\tilde{\eta} \in \mathcal{N} \text{ if and only if } (c,d) \subset \rho(Q) \text{ and } 0 \leq \lim_{x \to c} Q(x) < \infty;
\]

\[
\tilde{\eta} \in \mathcal{N} \text{ if and only if } (c,d) \subset \rho(Q), \beta = 0, \text{ and } 0 \leq \lim_{x \to c} Q(x) < \infty;
\]

\[
\tilde{\eta} \in \mathcal{N} \text{ if and only if } (c,d) \subset \rho(Q), \beta = 0, \text{ and } -\infty < \lim_{x \to d} Q(x) \leq 0.
\]

Proof. (i) Assume that \((c,d) \subset \rho(Q)\) and \(-\infty < \lim_{x \to d} Q(x) \leq 0\). Then it follows from Proposition 3.4 that

\[
\frac{z - c}{z - d} Q(z) = \frac{z - c}{z - d} \eta + \tilde{Q}(z),
\]

where \(\tilde{Q} \in \mathcal{N}\) and \(\eta = \lim_{x \to d} Q(x)\). Since \(\frac{z - c}{z - d} \eta \in \mathcal{N}\) if \(\eta \leq 0\), one concludes that \(\tilde{\eta} \in \mathcal{N}\). Conversely, if \(\frac{z - c}{z - d} Q(z) \in \mathcal{N}\), then item (iii) in Proposition 3.4 holds with \(\tilde{\eta} = 0\). Hence item (i) together with the last statement in Proposition 3.4 shows that \((c,d) \subset \rho(Q)\) and \(\eta = \lim_{x \to d} Q(x) \in \mathbb{R}\) with \(\eta \leq \tilde{\eta} = 0\).

The equivalence in (ii) is obtained easily from the equivalence in (i) by passing to the inverses. Finally, (iii) and (iv) follow from Proposition 3.5 in the same manner as (i) and (ii) follow from Proposition 3.4.

Remark 3.7. If \(Q_1(z) := \frac{z - c}{z - d} Q(z) \in \mathcal{N}\), then by Lemma 2.2 \(\lim_{x \to c} Q_1(x) > 0\) if and only if \(c \in \sigma_p(Q)\). Furthermore, \(Q_1\) admits a holomorphic continuation to the point \(c \in \mathbb{C}\) if and only if either \(c \in \rho(Q)\), or \(c \in \sigma_p(Q)\) is a separated pole of \(Q\), i.e., \(\sigma(Q) \cap (c - \varepsilon, c + \varepsilon) = \{c\}\) for some \(\varepsilon > 0\).

Similarly, by Lemma 2.2 \(\lim_{x \to d} Q_1(x) < \infty\) if and only if \(\lim_{x \to d} Q(x) = 0\) and \(\int_{\mathbb{R}} d\sigma(t)/(t - d)^2 < \infty\). Moreover, \(Q_1\) admits a holomorphic continuation to the point \(d \in \mathbb{R}\) if and only if \(d \in \rho(Q)\) and \(\lim_{x \to d} Q(x) = 0\).

3.3. Nevanlinna functions holomorphic on a semibounded interval. In this section Nevanlinna functions having a semibounded interval as their spectral gap are investigated. The statements below are formulated explicitly only in case that an interval of the form \((-\infty, c)\) belongs to \(\rho(Q)\). Using the equivalence \(Q(z) \in \mathcal{N}\) if and only if \(-Q(-z) \in \mathcal{N}\), these results are easily modified to the case \((c, \infty) \subset \rho(Q)\).

Proposition 3.8. Let \(Q \in \mathcal{N}\) have the integral representation \((2.1)\). Then with \(c \in \mathbb{R}\) the following statements are equivalent:

(i) \((-\infty, c) \subset \rho(Q)\) and \(\lim_{x \to c} (Q(x) - \beta x) = 0\);

(ii) the integral representation of \(Q\) is given by

\[
Q(z) = \eta + \beta z + \int_{[c,\infty)} \frac{d\sigma(t)}{t - z}.
\]
where \( \eta \in \mathbb{R} \) and \( \int_{[c, \infty)} \frac{d\sigma(t)}{t+z} < \infty \);

(iii) \( \exists \tilde{Q} \in \mathcal{N} \) and \( \tilde{\eta} \in \mathbb{R} \) such that \( Q(z) = \tilde{\eta} + \beta z + \frac{\tilde{Q}(z)}{z-c} \).

In particular, in (ii) \( \eta = \lim_{x \downarrow -\infty} (Q(x) - \beta x) \) and in (iii) \( \tilde{\eta} \leq \eta \).

Proof. (i) \( \Rightarrow \) (ii) If \((-\infty, c) \subset \rho(Q)\), then \( Q \) has the representation (2.1) with \( \Delta = [c, \infty) \). If the limit in (i) is finite, then Lemma 3.2 implies that \( \int_{[c, \infty)} \frac{d\sigma(t)}{1+t|t|} < \infty \).

Therefore \( Q \) has the representation given in (ii).

(ii) \( \Rightarrow \) (iii) This implication follows from

\[
(z-c) \int_{(c, \infty)} \frac{d\sigma(t)}{t-z} = \int_{(c, \infty)} (z-t+c) d\sigma(t) \\
= \int_{(c, \infty)} \left( \frac{1}{t-z} - \frac{t}{1+t^2} \right) (t-c) d\sigma(t) - \int_{(c, \infty)} \left( 1 - \frac{t(t-c)}{1+t^2} \right) d\sigma(t)
\]

where the integrals converge as consequence of the assumptions; in particular one has \( \int_{(c, \infty)} (t-c) d\sigma(t)/(1+t^2) < \infty \).

(iii) \( \Rightarrow \) (i) If (iii) holds, then \((-\infty, c) \subset \rho(Q)\) by Lemma 2.1 and by Lemma 2.2

\[
\lim_{x \downarrow -\infty} (Q(x) - \beta x) = \lim_{x \downarrow -\infty} \left( \tilde{\eta} + \frac{\tilde{Q}(x)}{x-c} \right) \in \mathbb{R}
\]

The last statement is clear by monotone convergence on \((-\infty, c)\), see (2.2). \( \square \)

Next consider the case that the limit of \( Q \) at the finite endpoint of the semibounded spectral gap is finite.

Proposition 3.9. Let \( Q \in \mathcal{N} \) have the integral representation (2.1). Then with \( c \in \mathbb{R} \) the following statements are equivalent:

(i) \((-\infty, c) \subset \rho(Q)\) and \( \lim_{x \uparrow c} Q(x) \in \mathbb{R} \);

(ii) the integral representation of \( Q \) is given by

\[
Q(z) = \eta + (z-c) \left( \beta + \int_{(c, \infty)} \frac{d\tilde{\sigma}(t)}{t-z} \right),
\]

where \( \eta \in \mathbb{R} \), \( d\tilde{\sigma}(t) = \frac{d\sigma(t)}{1+c-t} \), and \( \int_{(c, \infty)} \frac{d\tilde{\sigma}(t)}{1+t|t|} < \infty \);

(iii) \( \exists \tilde{Q} \in \mathcal{N} \) and \( \tilde{\eta} \in \mathbb{R} \) such that \( Q(z) = \tilde{\eta} + (z-c) \tilde{Q}(z) \).

In particular, in (ii) \( \eta = \lim_{x \uparrow c} Q(x) \) and in (iii) \( \tilde{\eta} \geq \eta \).

Proof. (i) \( \Rightarrow \) (ii) For this implication see the proof of the implication form (i) to (ii) in Proposition 3.4, especially (3.2).

(ii) \( \Rightarrow \) (iii) This implication is evident.

(iii) \( \Rightarrow \) (i) If (iii) holds, then by Lemma 2.1 \((-\infty, c) \subset \rho(Q)\) and by Lemma 2.2 applied to the spectral measure \( d\tilde{\sigma} \) of \( \tilde{Q} \), one obtains

\[
\lim_{x \uparrow c} Q(x) = \tilde{\eta} + \lim_{x \uparrow c} (x-c) \tilde{Q}(x) = \tilde{\eta} - \int_{(c, \infty)} 1_{(c)} d\tilde{\sigma}(t) \in \mathbb{R}
\]

Finally, Lemma 2.2 applied to the integral representation in part (ii) gives the stated limit expression for \( \eta \). \( \square \)
Remark 3.10. Recall that the class of Stieltjes (inverse Stieltjes) functions consists of those Nevanlinna functions which are holomorphic and nonnegative (nonpositive) on $\mathbb{R}_-$. These functions were introduced and characterized by Kreĭn, see [20] and the references therein. In fact, since Nevanlinna functions $Q$ are nondecreasing on $\rho(Q) \cap \mathbb{R}$, it follows that the class of Stieltjes (inverse Stieltjes) functions consists of those Nevanlinna functions which are holomorphic on $\mathbb{R}_-$ and satisfy $0 \leq \lim_{x \downarrow -\infty} Q(x) < \infty$ ($-\infty < \lim_{x \uparrow c} Q(x) \leq 0$). Therefore, Proposition 3.8 and Proposition 3.9 with $c = 0$ and $\eta \geq 0$ or $\eta \leq 0$, respectively, contain, in particular, a characterization for the classes of Stieltjes and inverse Stieltjes functions.

The previous two propositions give rise an analogue of Corollary 3.6.

Corollary 3.11. Let $Q \in \mathcal{N}$ have the integral representation (2.1). Then with $c \in \mathbb{R}$ the following statements are equivalent:

(i) $(z-c)Q(z) \in \mathcal{N}$ if and only if $(-\infty, c) \subset \rho(Q)$ and $0 \leq \lim_{x \downarrow -\infty} Q(x) < \infty$;

(ii) $\frac{1}{z-c}Q(z) \in \mathcal{N}$ if and only if $(-\infty, c) \subset \rho(Q)$ and $-\infty < \lim_{x \uparrow c} Q(x) \leq 0$.

4. Classes of generalized Nevanlinna functions associated with symmetric rational functions

The characterizations of Nevanlinna functions holomorphic on some interval of the real line in Section 3.2 and 3.3 show that the product of these Nevanlinna functions is defined by the formula

$$Q \in \mathcal{N} \Rightarrow \kappa \in \mathcal{N} \Rightarrow \kappa Q \in \mathcal{N}$$

and let $\kappa$.

Definition 4.1. Let $r$ be a nonconstant symmetric rational function ($\deg r > 0$), and let $\kappa$ and $\tilde{\kappa}$ be nonnegative integers. Then the class $\mathcal{N}_\kappa^\tilde{\kappa}(r)$ of generalized Nevanlinna functions is defined by the formula

$$\mathcal{N}_\kappa^\tilde{\kappa}(r) = \{ Q \in \mathcal{N}_\kappa : rQ \in \mathcal{N}_{\tilde{\kappa}} \}.$$

If a generalized Nevanlinna function $Q$ belongs to $\mathcal{N}_\kappa^\tilde{\kappa}(r)$ for some symmetric rational function $r$, whose range meets the negative real line, then $Q$ necessarily has gaps in its spectrum.

Lemma 4.2. Let $Q \in \mathcal{N}_\kappa^\tilde{\kappa}(r)$ for a symmetric rational function $r$ and let $\phi_0Q_0$, $Q_0 \in \mathcal{N}$, be the canonical factorization of $Q$. Then $0 \leq r \leq \infty$ on $\sigma(Q_0)$ except for finitely many poles of $Q_0$.

Proof. Let $\phi_1Q_1$, $Q_1 \in \mathcal{N}$, be the canonical factorization of $rQ$. Then $r\phi_0Q_0 = \phi_1Q_1$. The rational factors $\phi_0$ and $\phi_1$ admit only finitely many zeros and poles, and between them $\phi_0(x), \phi_1(x) > 0$. Hence one concludes from Lemma 2.1 that there can exist only finitely many separated poles (hence also zeros) of $Q_0$ and $Q_1$ on each open interval where $r < 0$. $\square$

The main results in this section give characterizations for functions belonging to the classes $\mathcal{N}_\kappa^\tilde{\kappa}(r)$ and also identify their canonical factorizations. In particular, in Theorem 4.9 an inverse statement to Lemma 4.2 will be proved. The approach is constructive and, in particular, the connection between the canonical factorizations of the functions $Q$ and $rQ$, $Q \in \mathcal{N}_\kappa^\tilde{\kappa}(r)$, is made explicit. From these results one immediately gets, for instance, factorized integral representations for the functions $Q$ and $rQ$, from which also Kreĭn-Langer type integral representations for functions in $\mathcal{N}_\kappa^\tilde{\kappa}(r)$ can be obtained along the lines of [7] Corollary 3.5.
4.1. Factorization of symmetric rational functions. An arbitrary symmetric rational function \( r \) is a generalized Nevanlinna function. In fact, the Nevanlinna kernel for \( r \) can be expressed explicitly by means of a Bezoutian; for details see e.g. [4]. The aim of this subsection is to derive in simple terms the canonical factorization in Proposition 4.3 for symmetric rational functions. This factorization is used to simplify later considerations.

If \( r \) is a symmetric rational function, then it admits a factorization of the form

\[
(4.1) \quad r(z) = \gamma \prod_{i=1}^{m_1}(z - \alpha_i)^{\nu_i}(z - \bar{\alpha}_i)^{\nu_i} \prod_{i=1}^{m_2}(z - \beta_i)^{\nu_i}(z - \bar{\beta}_i)^{\nu_i} \prod_{i=1}^{m_3}(z - \bar{\beta}_i) \prod_{i=1}^{m_4}(z - b_i),
\]

where \( \alpha_i, \beta_i \in \mathbb{C}, m_1, m_2, n_1, n_2, \mu_i, \nu_i \in \mathbb{N} \) and \( a_i, b_i, \gamma \in \mathbb{R} \). The factorization (4.1) is unique if no cancelations occur; this is assumed in the rest of this subsection. The canonical factorization for \( r \) can be obtained from (4.1) by describing the canonical factorization for the simple symmetric rational function

\[
(4.2) \quad s(z) = \gamma \prod_{i=1}^{m_1}(z - a_i) \prod_{i=1}^{m_2}(z - b_i), \quad a_i, b_i, \gamma \in \mathbb{R};
\]

here a rational function is called simple if all its zeros and poles are real and of order one. Clearly, only a zero or pole of \( s(z) \) can be its GZNT or GPNT, respectively. Next observe that for any zero \( a \in \mathbb{R} \) of \( s(z) \) the multiplicity \( \pi_a(s) \) is given by

\[
(4.3) \quad \pi_a(s) = \begin{cases} 0, & \text{if } 0 < \lim_{z \to \gamma} \frac{s(z)}{z - a} \leq \infty \Leftrightarrow s'(a) > 0; \\ 1, & \text{if } -\infty < \lim_{z \to \gamma} \frac{s(z)}{z - a} \leq 0 \Leftrightarrow s'(a) < 0, \end{cases}
\]

and, similarly, for any pole \( b \in \mathbb{R} \) of \( s(z) \) the multiplicity \( \kappa_b(s) \) is given by

\[
(4.4) \quad \kappa_b(s) = \begin{cases} 0, & \text{if } -\infty < \lim_{z \to \gamma} (z - b)s(z) \leq 0 \Leftrightarrow (-1/s)'(b) > 0; \\ 1, & \text{if } 0 < \lim_{z \to \gamma} (z - b)s(z) \leq \infty \Leftrightarrow (-1/s)'(b) < 0, \end{cases}
\]

see Remark 2.3. Note that, since no cancelations occur in (4.2), the limits

\[
\lim_{z \to a} \frac{s(z)}{z - a} \quad \text{and} \quad \lim_{z \to b} (z - b)s(z)
\]

are actually finite and nonzero for every zero \( a \in \mathbb{R} \) and every pole \( b \in \mathbb{R} \) of \( s \). The multiplicities \( \pi_a \) and \( \kappa_b \) only depend on the sign of \( \gamma \) and the location of the poles and zeros of \( s \) in (4.2). Therefore, associate with \( s \) and \( c \in \mathbb{R} \) the integer \( \eta_c(s) \) by

\[
(4.6) \quad \eta_c(s) = \{ \text{number of poles and zeros of } s \text{ greater than } c \}.
\]

Then the canonical factorization of \( s \) can be stated in an explicit form as follows.

**Lemma 4.3.** Let \( s \) be a simple symmetric rational function of the form (4.2) with \( \gamma \neq 0 \) and \( a_i \neq b_j \), and let \( \eta_c(s) \) be defined by (4.6). Then the canonical factorization for \( s \in \mathcal{N}_\kappa \), \( \kappa \in \mathbb{N} \), is given by \( s = \psi \varphi_0 \), where \( \varphi_0 \in \mathcal{N} \) and \( \psi \) is given by

\[
\varphi_0(z) = \prod_{i=1}^{m_1}(z - a_i)^{-2\eta_a(s)} \prod_{i=1}^{m_2}(z - b_i)^{-2\eta_b(s)}.
\]

Here \( \pi_a(s), \kappa_b(s) \), and \( \kappa = \max\{\sum_{i=1}^{m_1} \pi_a(s), \sum_{i=1}^{m_2} \kappa_b(s)\} \) are determined by

\[
(4.7) \quad \pi_a(s) = \frac{1 - (-1)^{\eta_a(s)} \text{sgn}(\gamma)}{2} \quad \text{and} \quad \kappa_b(s) = \frac{1 + (-1)^{\eta_b(s)} \text{sgn}(\gamma)}{2}.
\]
Proof: Since \( a_i \neq b_j \), the limits in (4.5) are finite and the signs of these limits are given by \((-1)^{\nu_k(s)}\text{sgn}(\gamma)\) and \((-1)^{\nu_k(s)}\text{sgn}(\gamma)\), respectively. This gives the expressions (4.7) for \(\pi_{a_i}(s)\) and \(\kappa_{b_i}(s)\) in (4.3) and (4.4), respectively. If \(\pi_{a_i}(s) = 0\) and \(\kappa_{b_i}(s) = 0\), then the corresponding factors \((z - a_i)\) and \(1/(z - b_i)\) are included in \(s_0\). If \(\pi_{a_i}(s) = 1\) \((\kappa_{b_i}(s) = 1)\), then \((z - a_i)^2\) \((\text{resp. } 1/(z - b_i)^2)\) is included as a factor in \(\psi\), while \(1/(z - a_i)\) \((\text{resp. } (z - b_i))\) becomes a factor for \(s_0\). This gives the stated formulas for \(s_0\) and \(\psi\). Observe, that the new terms do not change \(s\) and the limits in (4.5). In terms of \(s_0\) this means that
\[
\lim_{z \to c} \frac{s_0(z)}{z - c} > 0, \quad \lim_{z \to d} (z - d)s_0(z) < 0
\]
for every finite zero \(c\) and every finite pole \(d\) of \(s_0\). Clearly, the above conditions alone imply that \(s_0 \in \mathcal{N}\). \(\square\)

The proof of Lemma 4.3 can be used to obtain an independent and constructive proof for the existence of the canonical factorization for simple symmetric rational functions \(s\) of the form (4.2). Combining the factorization of \(s\) in Lemma 4.3 with the factorization of \(r\) in (4.1) the existence of the canonical factorization for an arbitrary symmetric rational function \(r\) is obtained. To express that factorization along the lines of Lemma 4.3 define \(\eta_c(r)\) for \(c \in \mathbb{R}\) by
\[
(4.8) \quad \eta_c(r) = \{\text{number of poles and zeros of odd order of } r \text{ greater than } c\}.
\]

**Proposition 4.4.** Let \(r\) be a symmetric rational function with the (unique) factorization \(r = \phi s\) as in (4.1) and let \(s = \psi s_0, s_0 \in \mathcal{N}\), be the canonical factorization of \(s\) given by Lemma 4.3. Then \(r \in \mathcal{N}_r, \kappa \in \mathbb{N}\), and the canonical factorization of \(r\) is given by \(r = \phi_0 r_0, r_0 \in \mathcal{N}\), where
\[
\phi_0(z) = \phi(z)\psi(z) \quad \text{and} \quad r_0(z) = s_0(z).
\]
In particular, if \(\mu(a)\) \((\nu(b))\) denotes the order of a \((\text{pole})\) of \(r\), then for every nonreal zero \(\alpha_i\) and nonreal pole \(\beta_i\) of \(r\) one has \(\pi_{a_i}(r) = \mu(\alpha_i)\) and \(\kappa_{b_i}(r) = \nu(\beta_i)\). For every real zero \(c_i\) and pole \(d_i\) of \(r\) of even order one has \(\pi_{c_i}(r) = \mu(c_i)/2\) and \(\kappa_{d_i}(r) = \nu(d_i)/2\), while for real zeros \(a_i\) and poles \(b_i\) of \(r\) of odd order one has
\[
(4.9) \quad \pi_{a_i}(r) = \frac{\mu(a_i) - (-1)^{\nu_{a_i}(r)}\text{sgn}(\gamma)}{2} \quad \text{and} \quad \kappa_{b_i}(r) = \frac{\nu(b_i) + (-1)^{\nu_{b_i}(r)}\text{sgn}(\gamma)}{2}.
\]

**Proof.** The factorization for \(r\) is obtained by combining (4.1) with Lemma 4.3. The formulas for \(\pi_{a_i}(r)\) and \(\kappa_{b_i}(r)\) and obtained from (4.1) and (4.7), since clearly 
\((-1)^{\nu_{a_i}(r)} = (-1)^{\nu_{b_i}(s)}\) if \(c\) is a zero or pole of \(r\) of odd order. \(\square\)

In view of (4.9) and Lemma 4.3 the poles and zeros of \(r_0 = s_0\) originate only from odd order real zeros or odd order real poles of \(r\). Moreover, it is easy to decide from the canonical factorization \(s = \psi s_0\) in Lemma 4.3 that the poles and zeros of \(s_0\) have the interlacing property, since in view of (4.6) the signs of the limits in (4.5) are alternating for consecutive poles and zeros of \(s\).

### 4.2. The case of simple rational functions of degree one.

To describe the class \(\mathcal{N}_r^{(0)}(r)\) for simple symmetric rational functions \(r\) of degree one, first some observations are made on the product of a general symmetric rational function with a Nevanlinna function.
Let \( r \) be an arbitrary symmetric rational function and let \( Q_0 \in \mathcal{N}, Q_0 \neq 0 \). Then it is clear from Sections 3.2 and 3.3 that the product \( rQ_0 \) is not in general a Nevanlinna function; the product \( rQ_0 \) need not even belong to the class of generalized Nevanlinna functions. However, if \( rQ_0 \in \mathcal{N}_\kappa \) for some \( \kappa \in \mathbb{N} \), then the multiplicities of generalized zeros and poles of nonpositive type of \( rQ_0 \) are well defined and given by (2.4) and (2.5), respectively. In the case of a symmetric rational function \( r \) it follows from Lemma 2.2 that for any simple zero \( a \in \mathbb{R} \) of \( r \) one has

\[
\pi_a(rQ_0) = \begin{cases} 
0, & \text{if } 0 < \lim_{z \to a} \left\{ \frac{r(z)Q_0(z)}{\hat{z} - a} \right\} \leq \infty; \\
1, & \text{if } -\infty < \lim_{z \to a} \left\{ \frac{r(z)Q_0(z)}{\hat{z} - a} \right\} \leq 0,
\end{cases}
\]

and similarly for any simple pole \( b \in \mathbb{R} \) of \( r \) one has

\[
\kappa_b(rQ_0) = \begin{cases} 
0, & \text{if } -\infty < \lim_{z \to b} \left\{ (z - b)r(z)Q_0(z) \right\} \leq 0; \\
1, & \text{if } 0 < \lim_{z \to b} \left\{ (z - b)r(z)Q_0(z) \right\} \leq \infty.
\end{cases}
\]

In fact, if \( r \) is a symmetric rational function and \( rQ_0 \in \mathcal{N}_\kappa \) then (4.10) and (4.11) imply that for every zero \( a \) and pole \( b \) of \( r \) of order one the limits \( \lim_{z \to a} Q_0(z) \) and \( \lim_{z \to b} Q_0(z) \) exist in \( \mathbb{R} \cup \{ \pm \infty \} \); see Section 2.2. Furthermore, in this case \( r \) changes its sign at \( a \) and \( b \), which implies the existence of the usual (improper) limits \( \lim_{x \to a} Q_0(x) \) and \( \lim_{x \to b} Q_0(x) \) for \( x \in \rho(Q_0) \cap \{ y \in \mathbb{R} : r(y) < 0 \} \); see Lemma 4.6. Observe that the signs of \( \frac{r(z)}{\hat{z} - a} \) and \( (z - b)r(z) \) remain constant around \( a \) and \( b \), respectively. Consequently, the multiplicities \( \pi_a(rQ_0) \) and \( \kappa_b(rQ_0) \) can be determined easily from the limit values of \( Q_0 \) at \( a \) and \( b \).

The following theorem gives a characterization for the class \( \mathcal{N}_\kappa^\sigma(s) \) in the case of a simple symmetric rational function \( s \) of degree one.

**Theorem 4.5.** Let \( Q \in \mathcal{N}_\kappa \) have the canonical factorization \( Q = \phi Q_0 \), \( Q_0 \in \mathcal{N} \), and let \( s \) be a symmetric rational function of degree one as in (4.2). Then the following statements are equivalent:

(i) \( Q \in \mathcal{N}_\kappa^\sigma(s) \) for some \( \kappa \in \mathbb{N} \);

(ii) \( 0 \leq s \leq \infty \) on \( \sigma(Q) \) except for finitely many poles of \( Q \);

(iii) \( 0 \leq s \leq \infty \) on \( \sigma(Q_0) \) except for finitely many poles of \( Q_0 \).

Furthermore, if any of the above equivalent statements holds, then \( sQ_0 \in \mathcal{N}_{\kappa_0} \) for some \( \kappa_0 \in \mathbb{N} \) and the canonical factorization of \( sQ \) is given by \( \overline{\phi}Q_0 \), \( \overline{Q}_0 \in \mathcal{N}_\kappa \), where

\[
\overline{\phi} = \phi \psi, \quad \overline{Q}_0 = \frac{s}{\psi} Q_0, \quad \psi(z) = \frac{(z - a)^2 \pi_a(sQ_0) \prod_{i=1}^{n_1}(z - \alpha_i)^2}{(z - b)^2 \kappa_b(sQ_0) \prod_{i=1}^{n_2}(z - \beta_i)^2}.
\]

Here the factors \((z - a)\) and \((z - b)\) with \( \pi_a(sQ_0) \) and \( \kappa_b(sQ_0) \) as in (4.10) and (4.11) appear only for the finite zero \( a \) and pole \( b \) of \( s \). Moreover, \( \alpha_i \in \mathbb{R}, 1 \leq i \leq n_1 \), are the zeros of \( Q_0 \) which satisfy \(-\infty < s(\alpha_i) < 0 \) and \( \beta_i \in \mathbb{R}, 1 \leq i \leq n_2 \), are the poles of \( Q_0 \) which satisfy \(-\infty < s(\beta_i) < 0 \).

For the proof of Theorem 4.5 the following lemma will be used. The lemma itself is proved by making use of Lemma 2.5, which allows to prove the statement via rational functions \( s \) of specific form.

**Lemma 4.6.** Let \( Q = \phi Q_0 \), \( s, \pi_a(sQ_0), \) and \( \kappa_b(sQ_0) \) be as in Theorem 4.5. Then \( 0 \leq s \leq \infty \) on \( \sigma(Q) \) implies that \( Q \in \mathcal{N}_\kappa^\sigma(s) \) for some \( \kappa \in \mathbb{N} \).
In this case \( sQ_0 \in \mathcal{N}_{\kappa_0} \) for some \( \kappa_0 \in \mathbb{N} \) and the canonical factorization of \( sQ \in \mathcal{N}_\frac{\gamma}{\kappa} \) is given by \( sQ = \phi Q_0, \ Q_0 \in \mathcal{N}, \) where

\[
\tilde{\phi} = \phi \psi, \quad \tilde{Q}_0 = \frac{s}{\psi} Q_0, \quad \psi(z) = \frac{(z - a)^{2\pi_n(sQ_0)} (z - a)^{2j_\alpha}}{(z - b)^{2\kappa_0(sQ_0)}}.
\]

Here \( j_\alpha = 1 \) if \( \alpha \in \mathbb{R} \) is a zero of \( Q_0 \) for which \( -\infty < s(\alpha) < 0 \) (there is at most one such zero \( \alpha \) of \( Q_0 \)), and \( j_\alpha = 0 \) otherwise.

**Proof.** Observe that if \( sQ \in \mathcal{N}_{\frac{\gamma}{\kappa}} \), then \(-1/(sQ) = (1/s) \cdot (-1/Q) \in \mathcal{N}_{\frac{\gamma}{\kappa}} \). Hence by considering either \( s \) or \( 1/s \) we can restrict ourselves, without loss of generality, to the case that \( s \in \mathcal{N} \). In this case \( s \) takes the form

\[
s(z) = \gamma \frac{z - a}{z - b}, \quad \gamma(b - a) < 0; \quad s(z) = \gamma(z - a), \quad \gamma > 0; \quad \text{or} \quad s(z) = \frac{\gamma}{z - b}, \quad \gamma < 0.
\]

**Case \( \gamma > 0 \):** Then \( s \) is given by

\[
s(z) = \gamma \frac{z - a}{z - b}, \quad b < a; \quad \text{or} \quad s(z) = \gamma(z - a).
\]

Define \( b = -\infty \) in the second case, so that in both cases \( b < a \).

The assumption \( 0 \leq s \leq \infty \) on \( \sigma(Q_0) \) implies that \((b, a) \subset \rho(Q_0)\). Moreover, since \( Q_0 \in \mathcal{N} \), monotonicity of \( Q_0 \) (see (2.2)) implies that there exists at most one zero \( \alpha \in (b, a) \) of \( Q_0 \) and, furthermore, the limits \( \lim_{z \to \alpha} Q_0(z) \) and \( \lim_{z \to a} Q_0(z) \) exist in \( \mathbb{R} \cup \{\pm \infty\} \), see Lemma 3.2.

Next observe that by Corollary 3.6 (\( b \) finite) and 3.11 (\( b \) infinite) \( sQ_0 \in \mathcal{N} \) if and only if \( 0 \leq \gamma \lim_{z \to \alpha} Q_0(z) < \infty \), in which case \( \kappa_0(sQ_0) = \pi_n(sQ_0) = j_\alpha = 0 \). Hence the statements hold in this case with \( \psi = 1 \).

If \(-\infty \leq \gamma \lim_{z \to a} Q_0(z) < 0 \) and \( Q_0 \) does not change its sign between \( a \) and \( b \), so that \( j_\alpha = 0 \) and \( \gamma \lim_{z \to a} Q_0(z) < 0 \), then by Corollary 3.6 (\( b \) finite) and 3.11 (\( b \) infinite) \( (1/s)Q_0 \in \mathcal{N} \). Since \( sQ_0 = (s)^2 (1/s)Q_0 \), Proposition 2.4 shows that the statements hold in this case with \( \pi_n(sQ_0) = \kappa_0(sQ_0) = 1, j_\alpha = 0, \tilde{Q}_0 = \gamma^2 (1/s)Q_0 \) and

\[
(4.12) \quad \psi(z) = \frac{(z - a)^2}{(z - b)^2}, \quad b \in \mathbb{R}, \quad \text{or} \quad \psi(z) = (z - a)^2, \quad b = \infty.
\]

If \(-\infty \leq \gamma \lim_{z \to a} Q_0(z) < 0 \) and \( Q_0 \) changes its sign between \( a \) and \( b \), then there exists a (unique) zero \( \alpha \) of \( Q_0 \) between \( a \) and \( b \) with \(-\infty < s(\alpha) < 0 \). Hence, \( 0 < \gamma \lim_{z \to \alpha} Q_0(z) \leq \infty \) and the sign of \( Q_0 \) is constant between \( b \) and \( \alpha \) and between \( \alpha \) and \( a \). Define \( s_1(z) = \frac{s(z) - b}{z - a} \) if \( b \) is finite and \( s_1(z) = \gamma \frac{z - b}{z - a} \) if \( b \) is infinite and in both cases define \( s_2(z) = \frac{s(z) - b}{z - a} \). Now Corollary 3.6 (\( b \) finite) and 3.11 (\( b \) infinite) imply that \( s_1Q_0 \in \mathcal{N} \). In addition, \( s_1Q_0 \) has a constant (positive) sign between \( \alpha \) and \( a \), in particular, \( 0 \leq \lim_{z \to a} s_1(z)Q_0(z) < \infty \), see Lemma 2.2. Thus, Corollary 3.6 shows that \( s_2(s_1Q_0) \in \mathcal{N} \). Define \( \psi \) as

\[
(4.13) \quad \psi(z) = \frac{(z - \alpha)^2}{(z - b)^2}, \quad b \in \mathbb{R}, \quad \text{or} \quad \psi(z) = (z - \alpha)^2, \quad b = \infty.
\]

Then \( sQ_0 = \psi s_2 s_1 Q_0 \) and, hence, Proposition 2.4 shows that the statements hold in this case with \( \kappa_0(sQ_0) = j_\alpha = 1, \pi_n(sQ_0) = 0, \tilde{Q}_0 = s_2(s_1 Q_0) \) and \( \psi \) as in (4.13).

**Case \( \gamma < 0 \):** Using Lemma 2.5, the statement can be reduced to the cases where \( \gamma > 0 \). If \( s(z) = \gamma \frac{z - a}{z - b} \in \mathcal{N}, \gamma < 0 \), then consider the functions \( s \circ \tau \) and \( Q \circ \tau \).
where $\tau(\lambda) = (a + b)/2 - 1/\lambda \in \mathcal{N}$. Then

$$(s \circ \tau)(\lambda) = -\frac{\lambda - 2/(b-a)}{\lambda - 2/(a-b)}$$

and

$$(Q \circ \tau)(\lambda) \in \mathcal{N}. $$

Clearly $\tau$ maps the interval with the endpoints $2/(b-a), 2/(a-b)$ to the complement of the interval between $a, b$ (in $\mathbb{R} \cup \{\infty\}$). Moreover, when transforming back to get the factorization for the initial function $rQ$ from (4.12), (4.13) note that

$$\frac{(\lambda - 2/(b-a))^2}{(\lambda - 2/(a-b))^2} = \frac{(z - a)^2}{(z - b)^2}$$

and

$$\frac{(\lambda - \alpha)^2}{(\lambda - 2/(a-b))^2} = \left\{ \begin{array}{ll}
\frac{\alpha^2 (a-b)^2}{4} (z - \frac{a+b-1}{2})^2, & 0 < |\alpha| < \frac{2}{|b-a|}, \\
\frac{1}{4(2-z-b)^2}, & \alpha = 0.
\end{array} \right.$$ 

Here $\alpha = 0$ corresponds to the GZNT of $sQ$ at $\infty$ (also to the zero of $Q_0$ at $\infty$).

Similarly, the case $\gamma/(z-b) \in \mathcal{N}$ can be reduced to the proven case by composing it with the transformation $\tau(\lambda) = (b-1) - 1/\lambda \in \mathcal{N}$. \hfill \Box

**Proof of Theorem 4.3** (i) $\Rightarrow$ (iii) This holds by Lemma 4.2

(ii) $\Leftrightarrow$ (iii) The factorization $Q = \phi Q_0$ implies that the sets $\sigma(Q)$ and $\sigma(Q_0)$ can differ from each other only at the zeros or poles of the rational function $\phi$; see also (2.8).

(iii) $\Rightarrow$ (i) Without loss of generality assume that $\gamma > 0$; compare the proof of Lemma 4.6. Then by the assumption the open interval where $s$ is negative contains finitely many $(n_2)$ separated poles $\beta_i \in \sigma_0(Q_0)$. Since $Q_0 \in \mathcal{N}$, there are also finitely many $(n_1)$ zeros $\alpha_i$ of $Q_0$, and the zeros $\alpha_i$ and the poles $\beta_i$ of $Q_0$ between $a$ and $b$ are interlacing, in particular, $|n_1 - n_2| \leq 1$. By considering either $Q_0$ or $-1/Q_0 \in \mathcal{N}$ one can assume that the zeros and poles of $Q_0 \in \mathcal{N}$ on the open interval between $a$ and $b$ are ordered as follows: $(\alpha_0 \leq \beta_1 \leq \alpha_1 \leq \ldots \leq \beta_{n_2} \leq \alpha_{n_2}$ (here $\alpha_0$ is excluded if $n_1 = n_2$). In particular, if $n_2 = 0$ then (i) follows immediately from Lemma 4.6.

Now assume that $n_1 \geq n_2 \geq 1$. Because of the disjointness of the intervals $[\beta_i, \alpha_i], 1 \leq i \leq n_2$, define $s_i(z) := \frac{z - \alpha_i}{z - \beta_i}, 1 \leq i \leq n_2$. Then Corollary 3.6 shows that $Q_0/s_i \in \mathcal{N}$. Furthermore, by Remark 3.7 this function admits a holomorphic continuation to the endpoints of the interval $(\beta_i, \alpha_i)$ and $Q_0(x)/s_i(x) > 0$ for all $x \in [\beta_i, \alpha_i]$, while outside the interval $[\beta_i, \alpha_i]$ the values of $Q_0$ and $Q_0/s_i$ have the same sign. Consequently, $Q_1 = Q_0/(\prod_{i=1}^{n_2} s_i) \in \mathcal{N}$ admits a holomorphic continuation to the open interval where $s$ is negative, and on this interval $Q_0(x) > 0$ (with $x > \alpha_0$ if $n_1 > n_2$). Furthermore,

$$s(z)Q(z) = s(z)\phi(z)Q_0(z) = \left( \prod_{i=1}^{n_2} s_i(z) \right) s(z)\phi(z)Q_1(z).$$

Now $s$ and $Q_1$ satisfy the condition of Lemma 4.9 and therefore

$$s(z)Q_1(z) = \frac{(z-a)^{2\pi_0(sQ_1)}(z-\alpha_0)^{2\pi_0}}{(z-b)^{2\pi_0(sQ_1)}}Q_2(z),$$

where $\tau(\lambda) = (a + b)/2 - 1/\lambda \in \mathcal{N}$. Then
where $Q_2 \in \mathcal{N}$ and $\pi_{\alpha_0} = |n_1 - n_2|$. Here $Q_2$ has no zeros where $s$ is negative, and $\alpha_0$ is a pole of $Q_2$ only if $\pi_{\alpha_0} = 1$. In particular, $Q_1$ and $Q_2$ have different signs on the interval where $s$ is negative. Applying Lemma 4.6 \((n_2 \text{ times})\) shows that

\[
\prod_{i=1}^{n_2} s_i(z) Q_2(z) = \left( \prod_{i=1}^{n_2} s_i(z) \right)^2 \frac{Q_2(z)}{\prod_{i=1}^{n_2} s_i(z)} = \prod_{i=1}^{n_2} (s_i(z))^2 Q_3(z),
\]

where $Q_3 \in \mathcal{N}$. Now (4.16) together with (4.14) and (4.15) implies (i). Then, equivalently, $sQ_0 \in \mathcal{N}_{\kappa_0}$ for some $\kappa_0 \in \mathbb{N}$; see (4.14). Moreover, it is clear that $\pi_a(sQ_1) = \pi_a(sQ_0)$ and $\kappa_b(sQ_1) = \kappa_b(sQ_0)$, since $\prod_{i=1}^{n_2} s_i(z)$ is positive if $z$ tends to the finite zero or pole of $s$. This proves the canonical factorization $sQ = \phi Q$ with $\phi$ given in the statement.

Note that Theorem 4.3 contains the analogous results without factorizations on the classes $\mathcal{N}_{a,b}^\pm(\mathbb{R})$ from [24] as well as the results (in the scalar case) on the classes $S_{-1} S_{1}(\mathbb{R})$ and on the classes $\mathcal{N}_{a,b}^\pm(C)$ from [3].

Remark 4.7. Theorem 4.3 gives in particular a characterization for the classes $\mathcal{N}_{a}^\pm(s)$, when $s$ is a simple rational function of degree one. For instance, if $a \in \mathbb{R} \cup \{\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$, $a \neq b$, denote the zero and pole of $s$, then the class $\mathcal{N}_{a}^\pm(s)$ consists of all Nevanlinna functions $Q$ such that either $Q$ has $\tilde{\kappa}$ zeros (poles) where $s$ is negative (on $\mathbb{R} \cup \{\infty\}$) and $\pi_a(sQ) = 0$ ($\kappa_b(sQ) = 0$) or $Q$ has $\tilde{\kappa} - 1$ zeros (poles) where $s$ is negative (on $\mathbb{R} \cup \{\infty\}$) and $\pi_a(sQ) = 1$ ($\kappa_b(sQ) = 1$).

Let $s$ be a simple symmetric rational function of degree one and let $Q \in \mathcal{N}_{a}^\pm(s)$ have the canonical factorization $\phi Q_0$, $Q_0 \in \mathcal{N}$. Then by Theorem 4.3 all the zeros $\alpha_i$ and poles $\beta_i$ of $Q_0$ which satisfy $-\infty < s(\alpha_i) < 0$ or $-\infty < s(\beta_i) < 0$ become GZNT’s and GPNT’s of $sQ_0$ (with multiplicity one), respectively, and the only other points which can become a GZNT and a GPNT of $sQ_0$ (with multiplicity one) are the pole and zero of $s$ (including $\infty$), respectively. Consequently, only the zeros of $\phi$, the zero of $s$, and $\alpha_i$ can become GZNT’s of $sQ$ and only the poles of $\phi$, the pole of $s$, and $\beta_i$ can become GPNT’s of $sQ$.

Furthermore, by means of (2.4) or (4.10) it can be decided whether the (finite) zero of $s$ becomes a GZNT of $sQ_0$ and, similarly, by means of (2.5) or (4.11) it can be decided whether the (finite) pole of $s$ becomes a GPNT of $sQ_0$. To decide whether $\infty$ is a GZNT or GPNT of $sQ_0$ (and hence possible a GZNT or GPNT of $sQ$) one can use (2.4) and (2.5): $\infty$ is a GZNT or GPNT of $sQ_0$ if and only if

\[
0 \leq \lim_{z \to \infty} \{zs(z)Q_0(z)\} < \infty \quad \text{or} \quad -\infty \leq \lim_{z \to \infty} \left\{ \frac{s(z)Q_0(z)}{z} \right\} < 0.
\]

4.3. Characterization of the classes $\mathcal{N}_{\kappa}^\pm(r)$. The results obtained in the previous subsection are extended to obtain a characterization for the classes $\mathcal{N}_{\kappa}^\pm(r)$, when $r$ is an arbitrary symmetric rational function.

The following lemma is used to prove the main result via Theorem 4.5.

Lemma 4.8. Let $s$ with $n = \deg s \geq 1$ be a simple symmetric rational function, whose poles and zeros (all of order one) are real and interlacing. Then $s$ admits a factorization $s = \prod_{i=1}^{n} s_i$, where $s_i$ are rational functions of degree one, such that

\[
D^- \cap D^- = \emptyset, \quad i \neq j, \quad \text{where} \quad D^- = \text{cl} \{ x \in \mathbb{R} : s_i(x) \leq 0 \}.
\]
In the second case, define

\[ s(z) = \gamma \prod_{i=1}^{l_1} \frac{z - a_i}{z - b_i}, \quad l_1 = l_2; \quad \text{or} \quad s(z) = \gamma \prod_{i=1}^{l_1} \frac{z - a_i}{z - b_0}, \quad l_2 = l_1 + 1; \]

where \( \gamma = \lim_{z \to \infty} s(z) \), if \( l_1 = l_2 \), and \( \gamma = \lim_{z \to \infty} z s(z) \), if \( l_2 = l_1 + 1 \). In the first case, define

\[
\begin{align*}
\phi_1(z) &= \frac{z - a_1}{z - b_1}, & \phi_i(z) &= \frac{z - a_i}{z - b_{i-1}}, & 2 \leq i \leq l_1, & \text{if } \gamma < 0; \\
\psi_1(z) &= \frac{z - a_1}{z - b_1}, & \psi_i(z) &= \frac{z - a_i}{z - b_i}, & 2 \leq i \leq l_1, & \text{if } \gamma > 0.
\end{align*}
\]

In the second case, define

\[
\begin{align*}
\phi_i(z) &= \frac{z - a_i}{z - b_{i-1}}, & 1 \leq i \leq l_1, & \text{and} \quad \phi_{l_2}(z) = \frac{\gamma}{z - b_0}, & \text{if } \gamma < 0; \\
\psi_i(z) &= \frac{z - a_i}{z - b_i}, & 1 \leq i \leq l_1, & \text{and} \quad \psi_{l_2}(z) = \frac{\gamma}{z - b_0}, & \text{if } \gamma > 0.
\end{align*}
\]

Then clearly \( s = \prod_{i=1}^{n} \phi_i \) (with \( n = l_2 \)) and it is easy to check that in each case the factors \( s_i \) satisfy the stated properties. \( \square \)

**Theorem 4.9.** Let \( Q \in \mathcal{N}_e, \ Q \neq 0 \), have the canonical factorization \( Q = \phi Q_0, \ Q_0 \in \mathcal{N} \), and let \( r \) be a symmetric rational function. Then the following statements are equivalent:

(i) \( Q \in \mathcal{N}_e^k(r) \) for some \( \kappa \in \mathbb{N} \);

(ii) \( 0 \leq r \leq \infty \) on \( \sigma(Q) \) except for finitely many poles of \( Q \);

(iii) \( 0 \leq r \leq \infty \) on \( \sigma(Q_0) \) except for finitely many poles of \( Q_0 \).

Furthermore, if \( r = \psi Q_0 \) is the canonical factorization of \( r \), then \( s_0 Q_0 \in \mathcal{N}_{\kappa_0} \) for some \( \kappa_0 \in \mathbb{N} \) and the canonical factorization of \( rQ \) is given by \( \phi Q_0 \in \mathcal{N} \), where

\[ \phi = \phi_0 \psi, \quad \phi_0 = \frac{s_0}{\psi} Q_0, \quad \psi(z) = \prod_{i=1}^{n_1} \left( \frac{z - a_i}{z - b_i} \right)^{2\pi i \kappa_i (s_0 Q_0)} \prod_{i=1}^{n_2} \left( \frac{z - \beta_i}{z - b_i} \right)^{2\pi i \kappa_i (s_0 Q_0)}. \]

Here \( \pi_i (s_0 Q_0) \) and \( \kappa_i (s_0 Q_0) \) are as in (4.10) and (4.11) for the finite zeros \( a_i \) and poles \( b_i \) of \( s_0 \), respectively. Moreover, \( \alpha_i \in \mathbb{R}, 1 \leq i \leq n_1 \), are the zeros of \( Q_0 \) which satisfy \( -\infty < s_0 (\alpha_i) < 0 \) and \( \beta_i \in \mathbb{R}, 1 \leq i \leq n_2 \), are the poles of \( Q_0 \) which satisfy \( -\infty < s_0 (\beta_i) < 0 \).

**Proof.** (i) \( \Rightarrow \) (iii) Again this holds by Lemma 4.2.

(ii) \( \Leftrightarrow \) (iii) As in the proof of Theorem 4.5, this follows from \( Q = \phi Q_0 \).

(iii) \( \Rightarrow \) (i) Let \( r = \psi s_0, \ s_0 \in \mathcal{N} \), be the canonical factorization of \( r \) given by Proposition 4.4. Then \( s_0 \) satisfies the assumptions in Lemma 4.8. Hence, there exists a factorization \( s_0 = \prod_{i=1}^{n} s_i \), where \( s_i, 1 \leq i \leq n \), are simple rational functions of degree one satisfying the properties (4.8) in Lemma 4.8. Now, as a consequence of the properties (4.8), Theorem 4.5 can be applied inductively (\( n \) times) to show that \( s_0 Q \in \mathcal{N}_e \), for some \( \kappa \in \mathbb{N} \). Consequently, \( rQ = \psi s_0 Q \in \mathcal{N}_e \), for some \( \kappa \in \mathbb{N} \) by Proposition 2.3. This proves (i).
Finally, since the rational functions $s_1, \ldots, s_n$ satisfy the properties (4.8), one can produce the canonical factorization for the product $rQ$ stepwise by applying the canonical factorization in Theorem 4.5 to the products $\prod_{i=1}^{k} s_i Q_0$, $1 \leq k \leq n$. □

In the canonical factorization of $rQ$ in Theorem 4.9 there can occur many cancel-ations in the products $\tilde{\phi} = \phi \tilde{\psi} \tilde{\psi}$ and $s_0 \tilde{\psi}$. The canonical factorization $r = \psi s_0$ of $r$ does not take into account the structure of $Q_0$, and hence there can occur already cancel-ations in the product $\psi \tilde{\psi}$. It is easy to see that the canonical factorization of $rQ$ in Theorem 4.9 can be reformulated via an arbitrary (not necessarily canonical) factorization $r = \psi s$ of $r$, where $\psi$ is a nonnegative rational function and $s$ is a simple symmetric rational function. In this case the formulas for $\phi$ and $Q_0$ appear as in Theorem 4.10 where the multiplicities $\pi_0(s Q_0)$ and $\kappa_0(\phi Q_0)$ can still be calculated as in (4.10) and (4.11) by the simplicity of $s$. In fact, the factorization of $rQ$ in Theorem 4.9 shows that it is possible to factorize $r$ with respect to $Q_0$ as $r = \psi' s'$ such that $\psi'$ is nonnegative and $s' Q_0 \in \mathcal{N}$; take $\psi' = \psi \tilde{\psi}$ and $s' = s_0 / \tilde{\psi}$.

Theorem 4.9 together with the discussion after Remark 4.7 yields the following result.

**Proposition 4.10.** Let $Q \in \mathcal{N}_{\kappa}^0(r)$ and let $\phi Q_0$, $Q_0 \in \mathcal{N}$, be its canonical factorization. Then the only points which can become GZNT’s and GPNT’s of $rQ$ are

(i) the zeros and poles of $r$ in $\mathbb{R} \cup \{\infty\}$;

(ii) the GZNT’s and GPNT’s of $Q$;

(iii) all the zeros $\alpha_i$ and the poles $\beta_i$ of $Q_0$ in $\mathbb{R} \cup \{\infty\}$ for which $-\infty < r(\alpha_i) < 0$ or $-\infty < r(\beta_i) < 0$.

In particular, if $Q \in \mathcal{N}_{\kappa}^0(r)$ then all the points $\alpha_i$ and $\beta_i$ in (iii) become GZNT’s and GPNT’s of $rQ$, respectively, and the only other possible GZNT’s and GPNT’s of $rQ$ are the zeros and poles of $r$, respectively.

Here is an example of the type of results which is obtained if $\mathcal{N}_{\kappa}$-function are multiplied with non-simple symmetric rational functions. This corollary contains as a special case [21, Corollary 4.9].

**Corollary 4.11.** Let $Q \in \mathcal{N}_{\kappa}$ have the canonical factorization $\phi Q_0$, $Q_0 \in \mathcal{N}$, and let $r(z) = \gamma(z - a)^{2m+1}$, $a, \gamma \in \mathbb{R}$ and $m \in \mathbb{N}$. Then $rQ \in \mathcal{N}_{\kappa}$ if and only if the interval $(\infty, a)$, $\gamma > 0$, or $(a, \infty)$, $\gamma < 0$, contains only finitely many poles of $Q_0$.

Furthermore, $rQ_0 \in \mathcal{N}_{\kappa_0}$ for some $\kappa_0 \in \mathbb{N}$ and the canonical factorization of $rQ$ is given by $\tilde{\phi} Q_0$, $\tilde{Q}_0 \in \mathcal{N}$, where

$$
\tilde{\phi}(z) = \frac{\phi(z)(z - a)^{2(\pi_a((z-a)Q_0(z)) + m)}}{\prod_{i=1}^{n} (z - \alpha_i)^2} \text{ and } \tilde{Q}_0 = \frac{r}{\phi} Q.
$$

Here $\pi_a((z-a)Q_0(z))$ is as in (4.10). Moreover, $\alpha_i$, $1 \leq i \leq n_1$, are the zeros of $Q_0$ satisfying $\alpha_i \in (-\infty, a)$, $\gamma > 0$, or $\alpha_i \in (a, \infty)$, $\gamma < 0$, and $\beta_i$, $1 \leq i \leq n_2$, are the poles of $Q_0$ satisfying $\beta_i \in (-\infty, a)$, $\gamma > 0$, or $\beta_i \in (a, \infty)$, $\gamma < 0$.

4.4. Characterization of the class $\mathcal{N}_{\kappa}^0(r)$. Assume that $Q \in \mathcal{N}_{\kappa}^0(r)$ and let $Q = \phi Q_0$ and $\tilde{Q} = \tilde{\phi} \tilde{Q}_0$ with $Q_0, \tilde{Q}_0 \in \mathcal{N}$ be the (unique) canonical factorizations of $Q$ and $rQ$. Then

$$
\tilde{Q}_0(z) = \frac{\tilde{r}(z) \phi(z)}{\phi(z)} Q_0(z),
$$

where $\tilde{r}(z) = \frac{r(z) \phi(z)}{\phi(z)}$. □
i.e. $Q_0 \in \mathcal{N}_0^\kappa(r)$. This shows that of all the classes $\mathcal{N}_0^\kappa(r)$, the classes $\mathcal{N}_0^0(r)$ play a key role. The next theorem gives a characterization for these classes.

**Theorem 4.12.** Let $Q_0 \in \mathcal{N}$, $Q_0 \neq 0$, and let $r$ (deg $r > 0$) be a symmetric rational function. Then $Q_0 \in \mathcal{N}_0^0(r)$ if and only if

(i) $0 \leq r \leq \infty$ on $\sigma(Q_0)$ and if $-\infty < r(x) < 0$, $x \in \mathbb{R}$, then $Q_0(x) \neq 0$;

(ii) every finite zero and pole $\gamma$ of $r$ is real and of order at most two:

(a) if $\gamma$ is a zero (pole) of $r$ of order two, then $\gamma$ is an isolated pole of $Q_0$

\[\lim_{z \to \gamma} \frac{r(z)}{(z - \gamma)^2} < 0 \quad \text{or} \quad -\infty < \lim_{z \to \gamma} (z - \gamma)^2 r(z) < 0;\]

(b) if $\gamma$ is a zero (pole) of $r$ of order one and $\nu = \lim_{z \to \gamma} \text{sgn} \frac{r(z)}{z - \gamma}$ (or $\nu = \lim_{z \to \gamma} \text{sgn} ((z - \gamma)r(z))$), then, respectively,

\[0 < \lim_{z \to \gamma} \nu Q_0(z) \leq \infty \quad \text{or} \quad -\infty < \lim_{z \to \gamma} \nu Q_0(z) \leq 0.

**Proof.** If $Q_0 \in \mathcal{N}_0^0(r)$, then (i) holds by Theorem 4.10, see also Proposition 4.10.

Now let $r = \psi \tilde{\psi}$ be the canonical factorization of $r$ as in Proposition 4.14. Since $rQ_0 \in \mathcal{N}_0$, the factorization in Theorem 4.10 shows that $\psi \tilde{\psi} \equiv 1$ and hence the order of every finite zero and pole of $r$ is at most two.

If $\gamma$ is a zero (pole) of $r$ of order two, then in view of $\psi \tilde{\psi} \equiv 1$ the factorization in Theorem 4.10 shows that $Q_0$ should have a pole (zero) at $\gamma$ and that $-\infty < s(\gamma) < 0$; thus $\gamma$ is an isolated pole (zero) of $Q_0$ by Lemma 4.2. This gives (ii)(a).

If $\gamma$ is a simple zero (or pole) of $r$, then the limit value $\nu$ as defined in the statement belongs to $\mathbb{R} \setminus \{0\}$. It follows from $rQ_0 \in \mathcal{N}_0$ that the (improper) limit $\lim_{z \to \gamma} \nu Q_0(z)$ exists and satisfies the given inequalities (with $\gamma$ a zero or pole of $r$, respectively); see Lemma 4.2. Thus (ii)(b) holds.

Conversely, if the condition (i) holds, then by Theorem 4.10 $rQ \in \mathcal{N}_\kappa$ for some $\kappa \in \mathbb{N}$. Moreover, if $-\infty < r(x) < 0$ then $x \in \rho(Q_0)$ by Lemma 4.13 and $Q_0(x) \neq 0$ by the assumption in (i). According to Proposition 4.10 only the zeros and poles of $r$ can produce GZNT and GPNT for $rQ_0$. The assumptions in (ii)(a) and (ii)(b) imply that if $\gamma$ is a zero (pole) of $r$, then $\pi_\gamma(rQ_0) = 0$ ($\kappa_\gamma(rQ_0) = 0$); see (2.4), (2.7). Therefore, $rQ \in \mathcal{N}$.

Let $Q_0 \in \mathcal{N}_0^0(r)$ for a symmetric rational function $r$. By Theorem 4.12, $r$ can have only real zeros and poles of order at most two. Clearly, such rational functions can be written as the product of degree one symmetric rational functions. The following results show how rational functions of degree one can be chosen in such a way that the stepwise products with $Q_0$ also stays in the class of Nevanlinna functions.

**Proposition 4.13.** Let $Q_0 \in \mathcal{N}_0^0(r)$. Then there exist rational functions $s_i$, $1 \leq i \leq n$, of degree one, such that

\[\left(\prod_{i=1}^{j} s_i\right) Q_0 \in \mathcal{N}, \quad 1 \leq j \leq n, \quad \text{and} \quad r = \prod_{i=1}^{n} s_i,

and there are no cancelations between the factors $s_i$. Moreover, the even order poles and zeros of $r$ are interlacing on intervals, where $r$ is not positive.
Proof. Let $\phi$ be a decomposition of $r$, where $s$ contains all the simple zeros and poles of $r$ and, hence, $\phi$ is nonnegative, see Theorem 4.12. The rational factors $s_i$ of $r$ of degree one will be constructed in two steps. First it is shown that on each finite maximal interval $(a, b)$, where $s$ is negative a suitable factorization can be defined involving all the poles and zeros of $r$ contained in $[a, b]$. Then the desired factorization is obtained inductively by considering all such negative intervals of $s$.

Step 1. Let $(a, b)$ with $a, b \in \mathbb{R}$ be a finite maximal interval, where $s$ is negative. By the maximality assumption the endpoints of the interval $(a, b)$ are zeros or poles of $s$, and hence also of $r$, which by Theorem 4.12 must be of order one. Let $r_{[a,b]}$ be the factor of $r$, which contains all the finite zeros and poles of $r$ on the interval $[a, b]$. Decompose $r_{[a,b]} = \phi_{[a,b]} s_{[a,b]}$, where $\phi_{[a,b]}$ is nonnegative and $s_{[a,b]}$ is simple with $s_{[a,b]}(x) = s(x) < 0$ for $x \in (a, b)$; note that the only finite zeros and poles of $s_{[a,b]}$ are $a$ and $b$. On the other hand, by Theorem 4.12 (ii)(a) all the finite poles and zeros of $\phi$, thus also of $\phi_{[a,b]}$, are of order two and they are necessarily zeros and poles of $Q_0$ on the interval $(a, b)$. Furthermore, by part (i) of Theorem 4.12, $Q_0$ cannot have any other poles or zeros on the interval $(a, b)$, since if $-\infty < r(x) < 0$, then $x \in \rho(Q_0)$ and $Q_0(x) \neq 0$. It follows that the ordered zeros $\alpha_j$ (poles of $\phi_{[a,b]}$) and poles $\beta_j$ (zeros of $\phi_{[a,b]}$) of $Q_0 \in \mathcal{N}$ in $(a, b)$ are interlacing:

$$(a_0 \leq) \beta_1 \leq \alpha_1 \leq \ldots \leq \beta_n \leq \alpha_n (\leq \beta_{n+1}),$$

where $\alpha_0$ or $\beta_{n+1}$ need not occur (due to ordering). Consequently, $\phi_{[a,b]}$ is given by

$$\phi_{[a,b]}(z) = \left(1 \left( \frac{1}{z - \alpha_0} \right) \right) \prod_{i=1}^{n} \frac{z - \beta_i^2}{(z - \beta_i - \alpha_i)^2},$$

where the first and last term in the brackets is excluded if, respectively, $\alpha_0$ or $\beta_{n+1}$ does not exist. Furthermore, by Theorem 4.12 (ii)(b) $a$ is a zero or pole of $s_{[a,b]}$ if

$$-\infty \leq \lim_{z \to a} Q_0(z) < 0 \quad \text{or} \quad 0 \leq \lim_{z \to a} Q_0(z) < \infty,$$

respectively, and $b$ is a zero or pole of $s_{[a,b]}$ if

$$0 < \lim_{z \to b} Q_0(z) \leq \infty \quad \text{or} \quad -\infty < \lim_{z \to b} Q_0(z) \leq 0,$$

respectively. Now, let $\bar{s}_i$ be defined by

$$\bar{s}_i(z) = \frac{z - \beta_i}{z - \alpha_i}, \quad i = 1, \ldots, n, \quad \bar{s}_{n+1}(z) = 1; \quad \text{or} \quad \bar{s}_i(z) = \frac{z - \beta_i}{z - \alpha_{i-1}}, \quad i = 1, \ldots, n+1,$$

if $\alpha_0$ or $\beta_{n+1}$ does not exist, respectively, if $\alpha_0$ and $\beta_{n+1}$ both exist. Moreover, let $s_1$ and $s_2$ be defined by

$$s_1(z) = \frac{z - a}{z - b}, \quad s_2(z) = 1; \quad s_1(z) = \frac{z - \alpha_0}{z - \alpha}, \quad s_2(z) = 1;$$

$$s_1(z) = \frac{z - a}{z - \alpha_0^2}, \quad s_2(z) = \frac{z - b}{z - \alpha_0}; \quad \text{or} \quad s_1(z) = \frac{z - \beta_{n+1}}{z - a}, \quad s_2(z) = \frac{z - \beta_{n+1}}{z - b},$$

if $\alpha_0$ and $\beta_{n+1}$ do exist, if $\alpha_0$ and $\beta_{n+1}$ do not exist, if only $\alpha_0$ exists, or if only $\beta_{n+1}$ exists, respectively. Then by applying Theorem 4.12 (or Corollary 3.6) it is seen that

$$\left( \prod_{i=1}^{j} \bar{s}_i \right) Q_0 \in \mathcal{N} \quad \text{for all} \quad 1 \leq j \leq n+1.$$
has a pole nor a zero on \((a, b)\), then \(Q_1\) is either positive (\(\alpha_0\) and \(\beta_{n+1}\) do not exist) or negative (\(\alpha_0\) and \(\beta_{n+1}\) both exist) on \((a, b)\); in the first case \(a\) is a pole and \(b\) is a zero of \(s\), and in the second case \(a\) is a zero and \(b\) is a pole of \(s\). In all cases \(\prod_{i=1}^{k} s_i\) \(Q_1 \in \mathcal{N}, 1 \leq k \leq 2\), by Theorem 4.12 (or Corollary 3.6) and, hence, \(Q_2 := \prod_{i=1}^{k} s_i\) \(Q_1 \in \mathcal{N}\). Finally, by construction, for \(x \in (a, b)\) one has \(Q_2(x) < 0\) \((Q_2(x) > 0)\) if and only if \(Q_1(x) > 0\) \((Q_1(x) < 0)\) and \(\alpha_0\) is a pole and \(\beta_{n+1}\) is a zero of \(Q_2\) (when they exist). Since \(Q_1\) and \(Q_2\) have opposite signs on the interval, it follows again from Theorem 4.12 (or Corollary 3.6) that \(\prod_{i=1}^{k} \tilde{s}_i\) \(Q_2 \in \mathcal{N}, 1 \leq j \leq n + 1\).

As a conclusion, the above construction shows that \(r_{[a, b]} = (\prod_{i=1}^{n+1} \tilde{s}_i) (\prod_{i=1}^{2} s_i)\) and, furthermore, that for all \(1 \leq j \leq n + 1\) and \(1 \leq k \leq 2\), the products

\[
(4.20) \left(\prod_{i=1}^{j} \tilde{s}_i\right) Q_0, \quad \left(\prod_{i=1}^{k} s_i\right) \left(\prod_{i=1}^{n+1} \tilde{s}_i\right) Q_0, \quad \text{and} \quad \left(\prod_{i=1}^{j} \tilde{s}_i\right) \left(\prod_{i=1}^{2} s_i\right) \left(\prod_{i=1}^{n+1} \tilde{s}_i\right) Q_0
\]

are Nevanlinna functions. In particular, \(r_{[a, b]} Q_0 \in \mathcal{N}_0\).

Step 2. The result (4.20) in Step 1 can be applied to every maximal bounded interval on which \(s\) is negative, and if \(s\) is negative on an interval \((−∞, b), (a, ∞)\) or on \((−∞, \infty)\), then it can be transformed to an interval of the type \((a, b)\) by means of a rational function of degree one using Lemma 2.5 (cf. the proof of Lemma 4.6). Now proceed inductively by applying Step 1 to each maximal interval where \(s\) is negative. Note that the closures of these intervals do not overlap, since \(s(x)\) changes its sign, when \(x\) passes a pole or zero of \(s\). After one interval \((a, b)\), where \(s\) is negative, is considered, the new functions to which (4.20) in Step 1 is applied are \(\bar{Q}_0 = r_{[a, b]} Q_0 \in \mathcal{N}_0\) and \(r/r_{[a, b]}\), whose maximal negative intervals coincide with those of \(r\), apart from the interval \((a, b)\), since \(0 \leq r_{[a, b]}(x) \leq \infty\) if \(x \notin [a, b]\) and \(-\infty \leq r_{[a, b]}(x) \leq 0\) if \(x \in [a, b]\). This gives the factorization of \(r\) as the product of \(s_i\), without cancelations. This complete the proof.

The order of the rational functions \(s_i\) in Proposition 4.13 is essential; they cannot be reordered in an arbitrary manner, since then some of the products may produce functions which are not Nevanlinna functions; see Example 4.13 below. In particular, in (4.20) the factors \(s_i\) and \(\tilde{s}_i\) can be reordered within the products \(\prod_{i=1}^{k} \tilde{s}_i\) and \(\prod_{i=1}^{k} s_i\), but in general it is not possible to interchange factors between the three product terms that appear (4.20).

Example 4.14. The function \(\log z\) (the principle branch of the logarithm) is a Nevanlinna function; its integral representation is given by

\[
\log(z) = \int_{(−\infty, 0)} \left(\frac{1}{t} - \frac{t}{1 + t^2}\right) dt,
\]

see [12, p. 27]. Now consider the function \(Q\) defined as

\[
Q(z) = \log(z) + (2 - z)^{-1} - 1.
\]

Then \(Q\) has a zero at 1 and a pole at 2. Moreover, \(Q\) has pole also at 0 and one further zero at a point \(c_0 > 2\). Applying Theorem 4.12 (or Corollary 3.6) one concludes that \(r_1 Q, r_2 r_1 Q, r_1 r_2 r_1 Q \in \mathcal{N}\), where

\[
r_1(z) = \frac{z - 2}{z - 1} \quad \text{and} \quad r_2(z) = \frac{z - a}{z - b}, \quad a \in [0, 1), b \in (2, c_0].
\]
In particular, $rQ \in N_0$ if $r(z) = \frac{(z-2)^2}{(z+1)(z+2)}$ with $a$ and $b$ as indicated. Note that $r^2Q \in N_1$; the only GZNT is at $z = 1$ and the only GPNT is at $z = 2$; see Theorem 4.15. Similarly $r^2Q \in N_1$ with a GZNT at $z = 2$ and a GPNT at $z = 1$; see Proposition 2.3. Note that if (4.20) is applied to $Q \in N^0_0(r)$, it produces the factors $r_1$, $r_2$, and $r_1$ in this order.

Note that one cannot use simpler factors of $r$ in this connection. For instance, the function $(z-2)Q(z)$ cannot belong to the class $N_\kappa$ for any $\kappa \in \mathbb{N}$, since $x < 2$ for $x < 2$; see Theorem 4.15. However, since $Q(c_0) = 0$ and $Q(x) > 0$ for $x > c_0$, one concludes from Theorem 4.12 that

$$Q_c(z) = \frac{Q(z)}{c-z} \in N_0, c \geq c_0, \quad \text{and} \quad Q_2(z) = (2-z)Q_{c_0}(z) = \frac{z-2}{z-c_0} Q_0(z) \in N_0.$$  

Proposition 4.13 shows that the class $N^0_0(r)$ is in particular interesting when $r$ is a symmetric rational function which has only simple zeros and poles (including $\infty$). In that case the following theorem holds; it generalizes the corollaries established in Sections 3.2 and 3.3.

**Theorem 4.15.** Let $Q_0 \in N$, $Q_0 \neq 0$, and let $s$ be a symmetric rational function, whose zeros and poles are real and simple (including $\infty$). Then the following statements are equivalent:

(i) $Q_0 \in N^0_0(s)$;

(ii) (a) if $-\infty < s(x) < 0$ then $Q_0$ is holomorphic at $x$ and $Q_0(x) \neq 0$;

(b) $\pi_{a_i}(sQ_0) = 0$ and $\kappa_{b_i}(sQ_0) = 0$, see (4.10) and (4.11), for every finite zero $a_i$ and every finite pole $b_i$ of $s$;

(iii) (a) if $(\alpha, \beta)$, $\alpha, \beta \in \mathbb{R} \cup \{\pm \infty\}$, is a maximal interval such that $-\infty < s(x) < 0$ for all $x \in (\alpha, \beta)$, then $Q_0$ is holomorphic and either $Q_0(x) > 0$ or $Q_0(x) < 0$ on $(\alpha, \beta)$.

(b) if $(\alpha, \beta)$ is a maximal interval as in (a) and, moreover, if $0 < Q_0(x) < \infty$ for $\alpha < x < \beta$ and $\alpha \in \mathbb{R}$ (or $\beta \in \mathbb{R}$), then $\alpha$ is a pole of $s$ ($\beta$ is a zero of $s$); if $-\infty < Q_0(x) < 0$ for $\alpha < x < \beta$ and $\alpha \in \mathbb{R}$ (or $\beta \in \mathbb{R}$), then $\alpha$ is a zero of $s$ ($\beta$ is a pole of $s$).

(iv) (a) $\sigma(Q_0) \cap \{x \in \mathbb{R} \cup \{\infty\} : -\infty < s(x) < 0\} = \emptyset$;

(b) for every pole $b$ of $s$

$$-\infty < \lim_{z \to b \in \mathbb{R}} \{s(x)Q_0(z)\} \leq 0, \quad 0 \leq \lim_{z \to b = \infty} \left\{\frac{s(z)Q_0(z)}{z}\right\} < \infty.$$

**Proof.** (i) $\iff$ (ii) This is clear by Theorem 4.12.

(i), (ii) $\Rightarrow$ (iv) This is immediate, see also Lemma 2.2.

(iv) $\Rightarrow$ (i) This implication follows from the fact that a generalized Nevanlinna function is a Nevanlinna function if and only if it has no generalized poles.

(ii) $\iff$ (iii) The equivalence of (ii)(a) and (iii)(a) is obvious. Clearly, every $a_i$ or $b_i$ as in (ii)(b) is the endpoint of a unique maximal interval $(\alpha, \beta)$ where $s$ is negative, and, conversely, every finite endpoint of a maximal interval $(\alpha, \beta)$ is a zero $a_i$ or a pole $b_i$ of $s$. Recall that by (4.10), (4.11) $\pi_{a_i}(sQ_0) = 0$ or $\kappa_{b_i}(sQ_0) = 0$ if and only if

$$0 < \lim_{z \to a_i} \left\{\frac{s(z)Q_0(z)}{z-a_i}\right\} \leq \infty \quad \text{or} \quad -\infty < \lim_{z \to b_i} \{(z-b_i)s(z)Q_0(z)\} \leq 0,$$  

respectively. Hence, the equivalence of (ii)(b) and (iii)(b) follows from the observation that $\lim_{z \to a_i} \frac{s(z)}{z-a_i} \in \mathbb{R}$ (or $\lim_{z \to b_i} (z-b_i)s(z) \in \mathbb{R}$) is positive if and only if $s$
is positive on a neighborhood to the right of \( a_i \) or, equivalently, negative on a neighborhood to the left of \( a_i \).

\[ \square \]

Theorem 4.15 (iii)(b) gives some further information about the location of zeros and poles of \( s \), because by simplicity of \( s \) the sign of \( s(x) \) changes, when \( x \) passes a pole or zero of \( s \); compare Proposition 4.13.

**Remark 4.16.** Let \( Q_0 \in \mathcal{N}_0^0(s), Q_0 \neq 0 \), with a simple symmetric rational function \( s \). Then the poles and zeros of \( s \) satisfy the following (interlacing type) property: if all the finite zeros and poles are ordered on the real line, then next to (i.e. before or after) each zero of \( s \) there is at least one pole of \( s \), and next to each pole of \( s \) there is at least one zero of \( s \).

The location property stated in Remark 4.16 includes the case where the poles and zeros of \( s \) are interlacing. Hence, Theorem 4.15 generalizes in particular the class \( S(E_m) \) introduced in [25], p. 396, which in the present notation would be \( \mathcal{N}_0^0(s) \) for a simple symmetric rational function whose zeros and poles are interlacing (starting with a finite pole).

If \( -\infty < \lim_{x \to -\infty} s(x) < 0 \) then \( Q_0 \) is holomorphic also at \( -\infty \) and, moreover, \( \lim_{x \to -\infty} Q_0(x) \neq 0 \); see Lemmas 2.1, 2.2. Hence, in Theorem 4.15 (ii) (a) one can also include the point \( x = \pm \infty \). Similarly if, for instance, \( s(x) < 0 \) for \( (0, \beta) \) and this interval is maximal in the sense that \( -\infty \) is either a pole or zero of \( s \), then in Theorem 4.15 (iii) (b), \( 0 < Q(x) < \infty \) implies that \( -\infty \) is actually a pole of \( s \), while \( -\infty < Q_0(x) < 0 \) for \( -\infty < x < \beta \) implies that \( -\infty \) is a zero of \( s \).

Some further information on the class \( \mathcal{N}_0^0(r) \), now related to the functions \( Q_0 \) and \( rQ_0 \), is given in the next result. This information will be used for constructing models for functions belonging to the class \( \mathcal{N}_0^0(r) \).

**Proposition 4.17.** Let \( Q_0 \in \mathcal{N}_0^0(r) \) and let \( a_i \in \mathbb{R} \cup \{ \infty \}, 1 \leq i \leq n_1 \), and \( b_i \in \mathbb{R} \cup \{ \infty \}, 1 \leq i \leq n_2 \), be some sets of zeros and poles of \( r \). Then

\[ Q_0 \in \bigcap_{i=1}^{n_2} \mathcal{N}(b_i, 1) \quad \text{and} \quad rQ_0 \in \bigcap_{i=1}^{n_1} \mathcal{N}(a_i, 1). \]

**Proof.** If \( b \) is a simple pole of \( r \), then \( \lim_{z \to b} (z - b)r(z) \in \mathbb{R} \setminus \{ 0 \}, b \in \mathbb{R} \), and \( \lim_{z \to \beta} \frac{r(z)}{z} \in \mathbb{R} \setminus \{ 0 \}, \beta = \infty \). Hence, if \( Q_0 \in \mathcal{N}_0^0(r) \), then Theorem 4.15 (iv)(b) shows that \( \lim_{z \to b} Q_0(z) \in \mathbb{R} \). Moreover, since \( r(x) \) changes its sign when \( x \) passes a simple pole of \( r \), \( Q_0 \) is holomorphic on an interval with a pole of \( r \) as its endpoint; see Theorem 4.15 (ii)(a). Now, according to Corollary 3.3 \( Q_0 \in \mathcal{N}(b, 1) \).

If \( b \in \mathbb{R} \) is a pole of \( r \) of order greater than one, then its order is two and by Theorem 4.12 (ii)(a) \( \beta \) is an isolated zero of \( Q_0 \), in particular, \( \beta \in \rho(Q_0) \). Hence, again \( Q_0 \in \mathcal{N}(b, 1) \) by Corollary 3.3.

The statement concerning the Nevanlinna function \( rQ_0 \) is now obtained e.g. by means of the equivalence \( Q_0 \in \mathcal{N}_0^0(r) \) if and only if \( rQ_0 \in \mathcal{N}_0^0(\frac{1}{r}) \).

**Proposition 4.17** implies that the function \( Q_0 \in \mathcal{N}_0^0(r) \) does not have poles at the poles of \( b_i \) of \( r \); see Proposition 3.1. However, the function \( rQ_0 \) may have poles at these points. In fact, there is a pole of \( rQ_0 \) at \( b_i \) when \( b_i \) is a pole of order 2 of \( r \) or when \( b_i \) is a pole of order 1 of \( r \) and \( \lim_{z \to b_i} Q_0(z) \neq 0 \); cf. Lemma 2.2.
5. Realizations of a subclass of Nevanlinna functions

In this section local versions of rigged Hilbert spaces are associated to a selfadjoint relation. These rigged spaces are used to construct realizations for functions in the Kac-Donoghue classes, cf. Proposition 4.17.

5.1. Rigged spaces. Let $A$ be a selfadjoint relation in the Hilbert space $\{\mathfrak{H}, (\cdot, \cdot)\}$. Define the inner products $(\cdot, \cdot)_{\infty,+1}$ via

$$
(f, g)_{\infty,+1} = (f, g) + \langle |A_o|^{\frac{1}{2}} P_\infty f, |A_o|^{\frac{1}{2}} P_\infty g \rangle, \quad f, g \in \text{dom} \ |A_o|^{\frac{1}{2}} \oplus \text{mul} \ A;
$$

$$
(f, g)_{\infty,-1} = ((I - P_\infty) f, g) + ((I + |A_o|) P_\infty f, P_\infty g), \quad f, g \in \mathfrak{H},
$$

see (5.1). Then $\mathfrak{H}_{+1}(A, \infty) := \{\text{dom} \ |A_o|^{\frac{1}{2}} \oplus \text{mul} \ A, (\cdot, \cdot)_{\infty,+1}\}$ is a Hilbert space and also the completion of $\mathfrak{H}$ with respect to $(\cdot, \cdot)_{\infty,-1}$, denoted by $\mathfrak{H}_{-1}(A, \infty)$, is a Hilbert space; it is the dual space of $\mathfrak{H}_{+1}(A, \infty)$. The rigging of $\mathfrak{H}$ with respect to $A$ at $\infty$ means the space triplet $\mathfrak{H}_{+1}(A, \infty) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A, \infty)$; for the case of operators, see [1].

In the rigging $\mathfrak{H}_{+1}(A, \infty) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A, \infty)$ the topology is changed on the part of the space $\mathfrak{H}$ connected with the behavior of $A$ close to $\infty$. The topology on the other parts of the space is not changed (though the norm in general is).

**Proposition 5.1.** Let $A$ be a selfadjoint relation in the Hilbert space $\{\mathfrak{H}, (\cdot, \cdot)\}$ and let $\mathfrak{H}_{+1}(A, \infty) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A, \infty)$ be the rigging of $\mathfrak{H}$ with respect to $A$ at $\infty$. Then $\ker (A - \xi), \xi \in \mathbb{R}$, and $\text{mul} \ A$ are closed in $\mathfrak{H}_{+1}(A, \infty)$ and $\mathfrak{H}_{-1}(A, \infty)$.

**Proof.** Clearly, $\text{mul} \ A$ is closed in all the topologies by definition of the rigging. If $f \in \ker (A - \xi), \xi \in \mathbb{R}$, then by (5.1)

$$
||f||_{\infty,+1} = (1 + |\xi|)||f|| \quad \text{and} \quad ||f||_{\infty,-1} = (1 + |\xi|)^{-1}||f||.
$$

This shows that the statement holds.

By definition,

$$(f, f)_{\infty,+1} \geq (f, f) \geq (f, f)_{\infty,-1}, \quad f \in \mathfrak{H}_{+1}(A, \infty).$$

It is clear from (5.1) that $||f||_{\infty,+1} = ||f||$ holds if and only if $P_\infty f \in \ker |A_o|^{\frac{1}{2}} = \ker A_o$. Similarly, (5.1) implies that $||f||_{\infty,-1} = ||f||$ holds if and only if $P_\infty f \in \ker A_o$. Hence the decomposition $\mathfrak{H} = (\text{dom} A \oplus \ker A) \oplus \ker A \oplus \text{mul} A$ implies corresponding decompositions for the rigged spaces $\mathfrak{H}_{\pm 1}(A, \infty)$:

$$
\mathfrak{H}_{\pm 1}(A, \infty) = \mathfrak{H}_{\pm 1}(A_o| \quad \text{mul} A_o, \infty) \oplus \ker A \oplus \text{mul} A.
$$

Furthermore, all the norms $||f||_{\infty,+1}, ||f||$ and $||f||_{\infty,-1}$ coincide on $\ker A \oplus \text{mul} A$.

If the operator part $A_o$ of $A$ is bounded, then the topologies on $\mathfrak{H}_{+1}(A, \infty)$ and $\mathfrak{H}_{-1}(A, \infty)$ are equal to the original topology on $\mathfrak{H}$, implying that the rigging $\mathfrak{H}_{+1}(A, \infty) \subset \mathfrak{H} \subset \mathfrak{H}_{-1}(A, \infty)$ collapses to a single (topological) space, equipped in general with different but equivalent norms.

Observe that $A_o$ and $A_o + \lambda I, \lambda \in \mathbb{C}$, when treated as operators from $\mathfrak{H}_{+1}(A, \infty) \oplus \text{mul} A$ to $\mathfrak{H}_{-1}(A, \infty) \oplus \text{mul} A$, are bounded, and by continuity can be uniquely extended to everywhere defined mappings $(A_o)_\infty$ and $(A_o + \lambda I)_\infty = (A_o)_\infty + \lambda I$ from $\mathfrak{H}_{+1}(A, \infty) \oplus \text{mul} A$ to $\mathfrak{H}_{-1}(A, \infty) \oplus \text{mul} A$. Consequently, define

$$
A_\infty = (A_o)_\infty \oplus \{0\} \times \text{mul} A.
$$

For $\lambda \in \rho(A)$, the resolvent operator $(A - \lambda I)^{-1}$ is bounded as a mapping from $\mathcal{H}_{-1}(A, \infty)$ to $\mathcal{H}_{+1}(A, \infty)$ and, hence, it also admits a unique extension by continuity:

$$
(5.4) \quad ((A - \lambda I)^{-1})_\infty = ((A - \lambda I)_\infty)^{-1} = (A_\infty - \lambda I)^{-1}, \quad \lambda \in \rho(A).
$$

The resolvent identity implies that

$$
(5.5) \quad (A_\infty - \lambda)^{-1} - (A_\infty - \mu)^{-1} = (\lambda - \mu)(A - \lambda)^{-1}(A_\infty - \mu)^{-1}, \quad \lambda, \mu \in \rho(A).
$$

Since $(I + |A_\infty|)^{-1} \oplus I_{\text{mul} A}$ is an isometric operator from $\mathcal{H}_{-1}(A, \infty)$ to $\mathcal{H}_{+1}(A, \infty)$, its closure, $V_\infty : \mathcal{H}_{-1}(A, \infty) \to \mathcal{H}_{+1}(A, \infty)$, is everywhere defined and unitary. This mapping is called the Riesz operator associated with the rigging. Clearly, the Riesz operator decomposes with respect to (5.2):

$$
(5.6) \quad V_\infty = \text{clos} \left( V_\infty \mid \mathop{\text{ran} A} \right) \oplus I_{\text{ker} A} \oplus I_{\text{mul} A}.
$$

The Riesz operator can be used to introduce the duality $(\cdot, \cdot)_\infty$ between $\mathcal{H}_{+1}(A, \infty)$ and $\mathcal{H}_{-1}(A, \infty)$ by

$$
(5.7) \quad (f, g)_\infty = (f, V_\infty g)_{\infty, +1} = (V_\infty^{-1} f, g)_{\infty, -1}, \quad (g, f)_\infty = (f, g)_{\infty},
$$

for $f \in \mathcal{H}_{+1}(A, \infty)$ and $g \in \mathcal{H}_{-1}(A, \infty)$.

**Remark 5.2.** If $f \in \mathcal{H}_{+1}(A, \infty)$ and $g \in \mathcal{H}_{-1}$, then as a consequence of (5.1) the duality reduces to the inner product on $\mathcal{H}$; i.e. $(f, g)_\infty = (f, g)$. Moreover, note that the form $(\cdot, \cdot)_\infty$ is continuous on $\mathcal{H}_{+1}(A, \infty) \times \mathcal{H}_{-1}(A, \infty)$; see (5.7).

Using the duality introduce for a relation $H$ from $\mathcal{H}_{+1}(A, \infty)$ to $\mathcal{H}_{-1}(A, \infty)$ the dual relation, denoted by $H^*$, along the lines of (2.6):

$$
(5.8) \quad H^* = \{ (f', f) \in \mathcal{H}_{+1}(A, \infty) \times \mathcal{H}_{-1}(A, \infty) : (f', g)_\infty = (f', g)_{\infty}, \forall g \in H \}.\n$$

Then $H^*$ is a closed subspace of $\mathcal{H}_{+1}(A, \infty) \times \mathcal{H}_{-1}(A, \infty)$; see Remark 5.2. The dual $H^*$ is connected with the Hilbert space adjoint $H^*$ from $\mathcal{H}_{-1}(A, \infty)$ to $\mathcal{H}_{+1}(A, \infty)$, see (2.6), by means of the Riesz operator:

$$
(5.9) \quad H^* = V_\infty H^* V_\infty = \{ (f, V_\infty f') : \{ V_\infty f, f' \} \in H^* \}.
$$

In particular, $H$ is a bounded operator from $\mathcal{H}_{+1}(A, \infty)$ to $\mathcal{H}_{-1}(A, \infty)$ if and only if its dual mapping $H^* : \mathcal{H}_{+1}(A, \infty) \to \mathcal{H}_{-1}(A, \infty)$ is bounded and, moreover, $\|H^*\| = \|H^*\| = \|H\|$. Finally, the relation $H$ is said to be symmetric (with respect to the duality) or self-dual if $H \subset H^*$ or $H = H^*$, respectively.

If $A$ is considered as a relation from $\mathcal{H}_{+1}(A, \infty)$ to $\mathcal{H}_{-1}(A, \infty)$, then, as in the case where $A$ is an operator, the dual relation $A^*$ coincides with $A_\infty$.

**Proposition 5.3.** Let $A$ be a selfadjoint relation in the Hilbert space $\mathcal{H}_{\{\cdot, \cdot\}}$. Then the dual relation $A^*$ (of $A$ as a relation from $\mathcal{H}_{+1}(A, \infty)$ to $\mathcal{H}_{-1}(A, \infty)$) coincides with $A_\infty$. Moreover, $A_\infty$ is self-dual, i.e., $(A_\infty)^* = A_\infty$.

**Proof.** Since $A_\infty$ is a bounded operator from $\mathcal{H}_{+1}(A, \infty)$ to $\mathcal{H}_{-1}(A, \infty)$ and by Remark 5.2 satisfies $(f, A_\infty g)_\infty = (A_\infty f, g)_\infty$ for all $f, g \in \text{dom} A_\infty$, $(A_\infty)^*$ is self-dual as a mapping from $\mathcal{H}_{+1}(A, \infty) \oplus \text{mul} A$ to $\mathcal{H}_{-1}(A, \infty) \oplus \text{mul} A$. Now it follows from the decompositions (5.2), (5.8), and (5.6) that $A^* = \text{clos} A = A_\infty$. This implies that $(A_\infty)^* = (A^*)^* = \text{clos} A = A_\infty$. $\square$
5.2. Local versions of rigged spaces. The rigging of the space $\mathcal{H}$ presented in the previous subsection is connected with the behavior of $A$ in the neighborhood of $\infty$. Here riggings determined by the behavior of $A$ in a neighborhood of a point on the real line are considered.

Definition 5.4. Let $A$ be a selfadjoint relation in the Hilbert space $\{\mathcal{H}, \langle \cdot, \cdot \rangle\}$ and let $\xi \in \mathbb{R}$. Define the Hilbert spaces $\mathcal{H}_{\pm 1}(A, \xi)$ by $\mathcal{H}_{\pm 1}(A, \xi) = \mathcal{H}_{\pm 1}((A - \xi^{-1})^{-1}, \infty)$. Then $\mathcal{H}_{\pm 1}(A, \xi) \subset \mathcal{H} \subset \mathcal{H}_{\pm 1}(A, \xi)$ is called the rigging of $\mathcal{H}$ with respect to $A$ at $\xi$.

Note that the decompositions in (5.2) take for the rigged spaces $\mathcal{H}_{\pm 1}(A, \xi)$, $\xi \in \mathbb{R}$, the following form:

\[(5.10) \quad \mathcal{H}_{\pm 1}(A, \xi) = \mathcal{H}_{\pm 1}((A - \xi^{-1})^{-1}, \infty) \oplus \text{mul } A \oplus \ker (A - \xi).\]

The rigging of $\mathcal{H}$ with respect to $A$ at $\xi \in \mathbb{R}$ gives rise to the rigged space closure of the selfadjoint relation $(A - \xi^{-1})^{-1}$:

\[(5.11) \quad ((A - \xi^{-1})^{-1})_{\xi} : \mathcal{H}_{\pm 1}(A, \xi) \to \mathcal{H}_{-1}(A, \xi);\]

in what follows it will be denoted by $(A - \xi^{-1})^{-1}$ for short. Note that its operator part is a bounded everywhere defined operator on $\mathcal{H}_{\pm 1}(A, \xi) \ominus \ker (A - \xi)$. Moreover, the resolvent $((A - \xi^{-1})^{-1} - \lambda)^{-1}$ extends to a bounded operator $R_{\xi}(\lambda) : \mathcal{H}_{\pm 1}(A, \xi) \to \mathcal{H}_{\pm 1}(A, \xi)$ with $R_{\xi}(\lambda) = \ker (A - \xi)$ and it satisfies the analog of (5.5):

\[R_{\xi}(\lambda) - R_{\xi}(\mu) = (1 - \lambda)^{-1} - (1 - \lambda)^{-1}R_{\xi}(\mu), \quad \lambda, \mu \in \rho((A - \xi^{-1})^{-1}).\]

Furthermore, the corresponding Riesz operator $V_{\xi}$ from $\mathcal{H}_{\pm 1}(A, \xi)$ onto $\mathcal{H}_{\pm 1}(A, \xi)$ gives rise to a duality:

\[(f, g)_{\xi} = (f, V_{\xi}g)_{\xi+1} = (V_{\xi}^{-1}f, g)_{\xi-1}, \quad f \in \mathcal{H}_{\pm 1}(A, \xi), g \in \mathcal{H}_{\pm 1}(A, \xi).\]

Let $K_{\xi} = \ker (A - \xi)$, $\xi \in \mathbb{R}$, and let $K_{\infty} = \text{mul } A$, then for $\xi \in \mathbb{R}$ denote by

\[(5.12) \quad P_{\xi} : \mathcal{H} \to \mathcal{H} \ominus K_{\xi} \quad \text{and} \quad P_{\xi}^\pm : \mathcal{H}_{\pm 1}(A, \xi) \to \mathcal{H}_{\pm 1}(A, \xi) \ominus K_{\xi}\]

the orthogonal projections from the rigged space members onto their subspaces as indicated. Similarly, define the following orthogonal projections for $\xi \in \mathbb{R} \cup \{\infty\}$:

\[(5.13) \quad P_{\infty} : \mathcal{H} \to \mathcal{H} \ominus K_{\infty} \quad \text{and} \quad P_{\pm}^\pm : \mathcal{H}_{\pm 1}(A, \xi) \to \mathcal{H}_{\pm 1}(A, \xi) \ominus K_{\xi}.\]

Note that, $P_{\xi}^\pm = P_{\xi}^\pm |_{\mathcal{H}_{\pm 1}(A, \xi)}$ and $P_{\xi}^\pm = P_{\xi}^\pm |_{\mathcal{H}_{\pm 1}(A, \xi)}$. Using the introduced projections, the Riesz operator $V_{\xi}$, and the spectral families $\{E_{t}\}_{t \in \mathbb{R}}$ of $A$, the inner products $\langle \cdot, \cdot \rangle_{-1, \xi}$ and $\langle \cdot, \cdot \rangle_{+1, \xi}$ can be made explicit. From (5.1) and (5.10) one obtains for $f, g \in \mathcal{H}_{\pm 1}(A, \xi)$, $\xi \in \mathbb{R}$,

\[(f, g)_{+1, \xi} = (((I - P_{\infty})f, g) + ((I - P_{\xi})f, g) + \int_{\mathbb{R} \setminus \{\xi\}} \left(1 + \frac{1}{|t - \xi|}\right) d(E_{t}P_{\infty}f, g),\]

and a similar formula for $f, g \in \mathcal{H}_{-1}(A, \xi)$ is obtained via $(f, g)_{-1, \xi} = (V_{\xi}f, V_{\xi}g)_{+1, \xi}$. Finally, for $R_{\xi}(\lambda)$ the following extended functional calculus result holds:

\[(5.14) \quad (R_{\xi}(\lambda)f, g)_{\xi} = -\frac{1}{\lambda}((I - P_{\infty})f, (I - P_{\infty})g)\]

\[+ \int_{\mathbb{R} \setminus \{\xi\}} \frac{(1 + |t - \xi|) d(E_{t}P_{\infty}V_{\xi}f, V_{\xi}g)}{(1 - \lambda|t - \xi|)(t - \xi)},\]

where $f, g \in \mathcal{H}_{-1}(A, \xi)$ and $\lambda \in \rho((A - \xi^{-1})^{-1})$, $\xi \in \mathbb{R}$.
5.3. Realization of Kac-Donoghue classes. The following proposition gives some characterizations for functions in the class \( \mathcal{N}(\xi, 1) \); for \( \xi = \infty \) the result has been established in [13 Theorem 3.1], [15 Proposition 2.2]; for \( \xi = 0 \) see also [14 Theorem 7.1].

**Proposition 5.5.** Let \( Q \) be a Nevanlinna function and let \( A \) be a selfadjoint extension in the Hilbert space \( \{ \mathcal{S}_{\gamma}(\cdot, \cdot) \} \) such that (2.14) holds for some \( v \in \mathcal{S} \). Moreover, let \( \gamma \) and \( S \) with \( n_{+}(S) = n_{-}(S) = 1 \) as in (2.15) and (2.12), respectively. Then the following statements are equivalent for \( \xi \in \mathbb{R} \):

(i) \( Q \in \mathcal{N}(\xi, 1) \);
(ii) \( \gamma_{\lambda} \in \text{ran} \{ A - \xi \}^{1/2} \), \( \lambda \in \rho(A) \);
(iii) there exists \( \omega \in \mathcal{S}_{\gamma}(A, \xi) \), \( \omega \notin \ker (A - \xi) \), such that

\[
S = \{ (f, f') \in A : (f' - \xi f, \omega) = 0 \}
\]

(iv) there exists \( \omega \in \mathcal{S}_{\gamma}(A, \xi) \), \( \omega \notin \ker (A - \xi) \), and \( \eta_{\xi} \in \mathbb{R} \) such that \( Q \) has the representation

\[
Q(\lambda) = \eta_{\xi} + (\lambda - \xi)(\gamma_{\lambda}, \omega), \quad \lambda \in \rho(A),
\]

where \( \eta_{\xi} = \lim_{\lambda \to \xi} Q(\lambda) \) and

\[
\gamma_{\lambda} = \left( I + (\xi - \lambda)(A - \xi)^{-1} \right)^{-1} \omega, \quad \lambda \in \rho(A).
\]

The function \( \gamma_{\lambda} \) in (iv) is the \( \gamma \)-field for \( Q \).

**Proof.** If \( Q \in \mathcal{N}(\xi, 1) \), then \( Q_{\infty}(z) := -Q(\xi + 1/z) \in \mathcal{N}(\infty, 1) \); see (3.1). Let \( \{ B, \gamma_{\lambda}^{\infty} \} \) be a realization for \( Q_{\infty} \), see Remark 2.8 Then by Lemma 2.9 and the discussion following it, \( \{ A, \gamma_{\lambda} \} \) is a realization for \( Q \), where \( A = B^{-1} + \xi \) and

\[
(\lambda - \xi)\gamma_{\lambda} = \gamma_{(\lambda - \xi)^{-1}}^{\infty}.
\]

Consequently, the equivalence of (i) and (ii) follows from the fact that \( Q_{\infty} \in \mathcal{N}(\infty, 1) \) if and only if \( \gamma_{\lambda}^{\infty} \in \text{dom} \{ B \}^{1/2} \), see [15 Proposition 2.2]. Furthermore, according to the characterizations for \( \xi = \infty \) in [13 Theorem 3.1] (see also [15 Section 4]) one has the representations \( Q_{\infty}(\lambda) = \eta + ((B_{\infty} - \lambda)^{-1} \omega, \omega) \) and \( \gamma_{\lambda}^{\infty} = (B_{\infty} - \lambda)^{-1} \omega \) for some \( \omega \in \mathcal{S}_{\gamma}(B, \infty) = \mathcal{S}_{\gamma}(A, \xi) \), under the additional minimality assumption on \( S \) (cf. (2.13)) or the assumption that \( (S - \xi)^{-1} \) is an operator. Combining this with the formula (5.15) gives

\[
\gamma_{\lambda}^{\xi} = \frac{(A - \xi)^{-1} - (\lambda - \xi)^{-1}}{\lambda - \xi} \omega = -(I - (\lambda - \xi)(A - \xi)^{-1})^{-1} \omega,
\]

and Lemma 2.9 then yields

\[
Q(\lambda) = -Q_{\infty}((\lambda - \xi)^{-1}) = -\eta - (\gamma_{(\lambda - \xi)^{-1}}^{\infty}, \omega) = -\eta - (\lambda - \xi)(\gamma_{\lambda}, \omega)
\]

with \( \omega \in \mathcal{S}_{\gamma}(A, \xi) \); cf. Definition 5.4. It is easy to see that in (iii) and (iv) the minimality assumption on \( S \), or the assumption that \( (S - \xi)^{-1} \) is an operator, can be replaced by the condition \( \omega \notin \text{mul} \{ A - \xi \}^{-1} \); in the opposite case \( S = A \) in (iii) and \( Q \) is constant in (iv). This yields all the desired conclusions. \( \square \)

**Remark 5.6.** The element \( \omega \) in Proposition 5.5 is not unique: if \( \omega' \in \mathcal{S}_{\gamma}(A, \xi) \) is such that \( \omega' - \omega \in \ker (A - \xi) \), then the Nevanlinna functions associated to them in part (iv) are the same. This non-uniqueness of \( \omega \) can be used to express \( \eta_{\xi} \) in terms of \( \omega \) e.g. in the form \( \eta_{\xi} = c_{\xi} \| (I - P_{\xi}^{-})\omega \|_{\xi^{-1}}^{2} \), \( c_{\xi} = \pm 1 \); see Theorem 6.2.
Furthermore, by the assumptions in Proposition 5.5, $S$ has defect numbers $(1, 1)$. However, all the statements (i)-(iv) in Proposition 5.5 remain equivalent also if $S = A$; in this case the function $Q$ in (i) is constant, $\gamma_\lambda = 0$ in (ii), and $\omega \in \ker (A - \xi)$ in (iii) and (iv). In what follows, such constant functions may occur.

Next Proposition 5.5 is augmented by giving a structured realization for the class $N(\xi, 1)$, $\xi \in \mathbb{R} \cup \{\infty\}$, as Weyl functions of boundary triplets. First observe the following lemma which is a consequence of Proposition 5.5 and [7, Theorem 6.2].

**Lemma 5.7.** Let $A$ be a selfadjoint relation in the Hilbert space $H, (\cdot, \cdot)$ and let $\omega \in H_{-1}(A, \infty)$, $\omega \notin \mul A$. Then

$$S_\omega = \{(f, f') \in A : (f, \omega)_\infty = 0\}$$

is a closed symmetric relation with $n_+(S_\omega) = n_-(S_\omega) = 1$ and its adjoint is

$$S_\omega^* = \{(f, f' - \omega cf) \in H \times H : (f, f') \in A_\infty, c_f \in \mathbb{C}\}.$$

A boundary triplet for $S_\omega^*$ is given by $\{\mathbb{C}, \Gamma_0^{\omega, \eta}, \Gamma_1^{\omega, \eta}\}$, where

$$\Gamma_0^{\omega, \eta} f = c_f \quad \text{and} \quad \Gamma_1^{\omega, \eta} f = \eta_\infty c_f + (f, \omega)_\infty$$

for $f = (f, f' - \omega cf) \in S_\omega^*$ and $\eta_\infty \in \mathbb{R}$. The corresponding $\gamma$-field and Weyl function are given by

$$\gamma_\lambda^\omega = (A_\infty - \lambda)^{-1} \omega \quad \text{and} \quad M_\omega(\lambda) = \eta_\infty + (\gamma_\lambda^\omega, \omega)_\infty, \quad \lambda \in \rho(A).$$

In particular, the Weyl function $M_\omega$ belongs to $N(\infty, 1)$ and $\eta_\infty = \lim_{\lambda \to \infty} M_\omega(\lambda)$.

The Weyl function associated to the boundary triplet in Lemma 5.7 has, see [13, (2.7)], the integral representation

$$M_\omega(\lambda) = \eta_\infty + \int_{\mathbb{R}} \frac{(1 + |t|^2)}{t - \lambda} d(E_\lambda P_\infty V_\infty \omega, V_\infty \omega).$$

In Lemma 5.7, the formulas for $S_\omega$, $S_\omega^*$, and the boundary mappings $\Gamma_0^{\omega, \eta}$, $\Gamma_1^{\omega, \eta}$ remain the same when $\omega \in H_{-1}(A, \xi)$ is replaced by $\omega' \in H_{-1}(A, \xi)$ with $\omega' - \omega \in \mul A$; cf. Remark 5.9. If $\omega \in \mul A$, then $S_\omega = A = S_\omega^*$, $\Gamma = \Gamma_0 \times \Gamma_1$ becomes a so-called boundary relation and its Weyl function is $\eta \in \mathbb{R}$, see [5, Example 6.1] or Section 5.3 below.

The next result is the analog of Lemma 5.7 for $N(\xi, 1)$, $\xi \in \mathbb{R}$. Recall, that $(A - \xi)_\xi := ((A - \xi)^{-1})^{-1}$ denotes the closure of $(A - \xi)$ in $H_{-1}(A, \xi) \times H_{+1}(A, \xi)$.

**Lemma 5.8.** Let $A$ be a selfadjoint relation in the Hilbert space $H, (\cdot, \cdot)$ and let $\omega \in H_{-1}(A, \xi)$, $\xi \in \mathbb{R}$, $\omega \notin \ker (A - \xi)$. Then

$$S_\omega = \{(f, f') \in A : (f' - \xi f, \omega)_\xi = 0\}$$

is a closed symmetric relation with $n_+(S_\omega) = n_-(S_\omega) = 1$ and its adjoint is

$$S_\omega^* = \{(f + \omega cf, f' + \xi f + \xi \omega cf) \in H^2 : (f, f') \in (A - \xi)_\xi, c_f \in \mathbb{C}\}.$$

A boundary triplet for $S_\omega^*$ is given by $\{\mathbb{C}, \Gamma_0^{\omega, \xi}, \Gamma_1^{\omega, \xi}\}$, where

$$\Gamma_0^{\omega, \xi} f = c_f \quad \text{and} \quad \Gamma_1^{\omega, \xi} f = \eta_\xi c_f + (f', \omega)_\xi$$

for $f = (f + \omega cf, f' + \xi f + \xi \omega cf) \in S_\omega^*$ and $\eta_\xi \in \mathbb{R}$. The corresponding $\gamma$-field and Weyl function are given by

$$\gamma_\lambda^\omega = (I + (\xi - \lambda)(A - \xi)^{-1})^{-1} \omega, \quad M_\omega(\lambda) = \eta_\xi + (\lambda - \xi)(\gamma_\lambda^\omega, \omega)_\xi, \quad \lambda \in \rho(A).$$
In particular, the Weyl function $M_\omega$ belongs to $\mathcal{N}(\xi, 1)$ and $\eta_\xi = \lim_{\lambda \to \xi} M_\omega(\lambda)$.

**Proof.** The formulas for $S_\omega$ and $S_\omega^*$ are obtained from Lemma 5.7 by means of the transformation $H \mapsto (H - \xi)^{-1}$. Next observe that the mapping $U_\xi : \mathcal{D} \to \mathcal{D}$

$$U_\xi \{f, f'\} = \{f' - \xi f, f\}, \quad f, f' \in \mathcal{D},$$

is $J_\mathcal{D}$-unitary where $J_\mathcal{D} \{f, f'\} = \{-if', if\}$. Therefore the composition $\Gamma^{\omega, \omega} \circ U_\xi$, where $\Gamma^{\omega, \omega} = \{\Gamma^{\omega, \omega}_0, \Gamma^{\omega, \omega}_1\}$ is the boundary triplet from Lemma 5.7, defines a boundary triplet for $S^\omega_\omega$, cf. [6] Theorem 2.10, it coincides with $\{\Gamma^{\omega, \omega}_0, \Gamma^{\omega, \omega}_1\}$. By Proposition 5.8, $\gamma^\omega_\lambda$ spans the defect space ker $(S^\omega_\omega - \lambda)$. A direct calculation shows that $\{\gamma^\omega_\lambda, \lambda \gamma^\omega_\lambda\}$ can be written as

$$\{(\lambda - \xi)P_\xi^\omega (A - \xi)\xi^{-1}\gamma^\omega_\lambda + P_\xi^\omega \omega, (\lambda - \xi)\gamma^\omega_\lambda + \xi(\lambda - \xi)P_\xi^\omega (A - \xi)\xi^{-1}\gamma^\omega_\lambda + \xi P_\xi^\omega \omega\}.$$

Since $\{(I - P_\xi^\omega)\omega, \xi(I - P_\xi^\omega)\omega\} \in \ker \Gamma^{\omega, \omega}$, applying the boundary mappings to $\{\gamma^\omega_\lambda, \lambda \gamma^\omega_\lambda\}$ yields the stated $\gamma$-field and Weyl function.

Note that the resolvents of $S_\omega$ and its adjoint $S^*_\omega$ at the point $\xi$ are given by

$$S_\omega - \xi)^{-1} = \{(f', f) \in (A - \xi)\xi : (f', \omega)\xi = 0\},$$

$$S_\omega - \xi)^{-1} = \{(f', f + \omega c f) \in \mathcal{D} : \{f, f'\} \in (A - \xi)\xi, c f \in \mathbb{C}\}.$$

These formulas are analogous to those appearing in Lemma 5.7 and this gives an alternative method to derive various facts appearing e.g. in Lemma 5.7 from the riggings of $A$ at $\xi$ to the riggings of $A$ at a finite point $\xi \in \mathbb{R}$.

As a consequence of (5.14) the Weyl function associated to the boundary triplet in Lemma 5.8 has the integral representation

$$M_\omega(\lambda) = \eta_\xi + (\lambda - \xi)((I - P_\infty)\omega, (I - P_\infty)\omega)$$

$$+ (\lambda - \xi) \int_{\mathbb{R} \setminus \{\xi\}} \frac{(1 + |t - \xi|)^2 d(E_t P_\infty^\omega V_t \omega, V_t \omega)}{(t - \lambda)(t - \xi)}.$$  

**Remark 5.9.** To see the connection to the integral representation for $M_\omega$ in (2.11), recall from [15] Proposition 2.1 that $d\sigma(t) = (t^2 + 1)d(E_t P_\infty \gamma_t, P_\infty \gamma_t)$. Incorporating the representation of $\gamma_\lambda$ in Lemmas 5.7 or 5.8 at $\lambda = i$ and using functional calculus it is easy to check that

$$d\sigma(t) = \left\{ \begin{array}{ll}
(1 + |t - \xi|)^2 d(E_t P_\infty^\omega V_t \omega, V_t \omega), & \xi \in \mathbb{R}, \\
(1 + |t|)^2 d(E_t P_\infty^\omega V_t \omega, V_t \omega), & \xi = \infty.
\end{array} \right.$$  

Combining Lemmas 5.7 and 5.8 with Proposition 5.3 the following realization result for $\mathcal{N}(\xi, 1)$ is obtained. Note that a realization for the case $\xi = \infty$ was given in [7] Theorem 4.4 & 6.2]; in fact, there also the Pontryagin space case was allowed.

**Theorem 5.10.** For each nonconstant $Q \in \mathcal{N}(\xi, 1)$, $\xi \in \mathbb{R} \cup \{\infty\}$, there exist a closed symmetric relation $S$ in a Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ and a boundary triplet $\{\mathcal{C}, \Gamma_0, \Gamma_1\}$ for $S^*$ such that $Q$ is its associated Weyl function.

In fact, there exist a selfadjoint relation $A$ in $\{\mathcal{H}, (\cdot, \cdot)\}$ and an element $\omega \in \mathcal{H}_{-1}(A, \xi)$ such that the symmetry and boundary triplet can be taken to be $S_\omega$ and $\{\mathcal{C}, \Gamma_0^\omega, \Gamma_1^\omega\}$ as in Lemma 5.7 or 5.8 respectively. Here $\eta_\xi = \lim_{\lambda \to \xi} Q(\lambda)$. 


5.4. Realizations for intersections of Kac-Donoghue classes. Let $Q$ be a Nevanlinna function from the Kac-Donoghue class $\mathcal{N}(\xi, 1)$. By Theorem 5.10 $Q$ can be realized with a selfadjoint relation $A$ in a Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ and an element $\omega \in \mathcal{H}_{-1}(A, \xi)$. If $Q$ is also contained in another Kac-Donoghue class, say $\mathcal{N}(\xi')$, then the following results show how this is reflected by $\omega$.

Lemma 5.11. Let $Q \in \mathcal{N}(\xi, 1)$, $\xi \in \mathbb{R} \cup \{\infty\}$, and let $\omega \in \mathcal{H}_{-1}(A, \xi)$ for a selfadjoint relation $A$ in the Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ such that $Q$ is realized via Lemma 5.4 or 5.8. Then $Q \in \mathcal{N}(\xi, 1) \cap \mathcal{N}(\xi', 1)$, $\xi \neq \xi' \in \mathbb{R} \cup \{\infty\}$, if and only if $(I - P_{\xi})\omega = 0$ and $V\omega \in \mathcal{H}_{+1}(A, \xi) \cap \mathcal{H}_{+1}(A, \xi')$.

Proof. The assumption $Q \in \mathcal{N}(\xi, 1)$ implies the integral representation (5.18) for $Q$. On the other hand, by Proposition 5.4 $Q \in \mathcal{N}(\xi', 1)$ if and only if the measure $d\sigma(t)$ satisfies the integrability condition in Proposition 3.7 with $\xi'$, where $\beta = 0$ if $\xi' = \infty$. In view of Remark 5.7 this means that $P_{\infty}V\omega \in \text{ran}|A - \xi'|^{1/2}$ if $\xi' \in \mathbb{R}$, and $P_{\infty}V\omega \in \text{dom}|A|^{1/2}$ and $(I - P_{\infty})\omega = 0$ (see (5.18)) if $\xi' = \infty$. Due to (5.10) this is equivalent to $V\omega \in \mathcal{H}_{+1}(A, \xi') \ominus \text{ker}(A - \xi')$ if $\xi' \in \mathbb{R}$, and to $V\omega \in \mathcal{H}_{+1}(A, \xi) \ominus \text{null} A$ if $\xi' = \infty$. This implies the statement of the lemma. □

Let $Q$ be a Nevanlinna function which is in the intersection of two Kac-Donoghue classes $\mathcal{N}(\xi, 1)$ and $\mathcal{N}(\xi', 1)$, and let $\{A, \gamma\}$ be a realization for $Q$, see Remark 2.8. Then Proposition 5.5 implies that $\gamma_{\lambda} \in \mathcal{H}_{+1}(A, \xi) \cap \mathcal{H}_{+1}(A, \xi')$. Hence there exists a $\omega \in \mathcal{H}_{-1}(A, \xi)$ and a $\omega' \in \mathcal{H}_{-1}(A, \xi')$ such that $Q$ is realized by the model associated to $A$ and $\omega$, and $A$ and $\omega'$ as in Lemma 5.7 or 5.8.

To connect these two realization of $Q$ a connection between $\omega$ and $\omega'$ is needed. Therefore observe that the $\gamma$-field, which is unique for a Nevanlinna function, can be expressed by means of $\omega$ and $\omega'$:

$$\gamma_{\lambda} = \left( I + (\xi - \lambda)(A - \xi)^{-1} \right)^{-1} \omega = \left( I + (\xi' - \lambda)(A' - \xi')^{-1} \right)^{-1} \omega',$$

if $\xi, \xi' \in \mathbb{R}$, see Lemma 5.8. This discussion motivates the following lemma in which the notation $\gamma_{\lambda, \xi}(A)$, $\lambda \in \text{sp}(A)$, is used for the mapping on $\mathcal{H}_{-1}(A, \xi)$ defined as

(5.19) $\gamma_{\lambda, \xi}(A)f = \begin{cases} (I + (\xi - \lambda)(A - \xi)^{-1})^{-1}f, & \xi \in \mathbb{R}; \\ (A_{\infty} - \lambda)^{-1}f, & \xi = \infty, \end{cases}$

i.e. $\gamma_{\lambda, \xi}f$ is the $\gamma$-field of a function in $\mathcal{N}(\xi, 1)$. Note that $\text{ker} \gamma_{\lambda, \xi}(A) = \text{ker}(A - \xi)$.

Lemma 5.12. Let $\gamma_{\lambda, \xi}(A)$ and $\gamma_{\lambda, \xi'}(A)$, $\xi, \xi' \in \mathbb{R} \cup \{\infty\}$, $\xi \neq \xi'$, be given by (5.19) and let

$$\rho_{\xi, \xi}(A) = P_{\xi}^{-1}(\gamma_{\lambda, \xi}(A))^{-1} \gamma_{\lambda, \xi'}(A), \quad \lambda \in \text{sp}(A).$$

Then $\rho_{\xi, \xi}(A)$ defines a bounded operator from $\mathcal{H}_{-1}(A, \xi) \cap P_{\xi}^{-1}V_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi')$ onto $P_{\xi}^{-1}V_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi) \cap P_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi')$, which does not depend on $\lambda \in \text{sp}(A)$. Moreover, $\rho_{\xi, \xi}$ is a bounded extension of the mapping $P_{\xi}^{-1}(A - \xi)^{-1}(A - \xi')^{-1}$, which (A - $\xi'$) is to be interpreted as $I$ if $\xi = \infty$ or $\xi' = \infty$, respectively.

Proof. From Section 5.2 and Proposition 5.5 it is known that $\gamma_{\lambda, \xi}(A)$ with $\lambda \in \text{sp}(A)$ is a bounded operator, which maps $\mathcal{H}_{-1}(A, \xi)$ onto $P_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi)$ with $\text{ker} \gamma_{\lambda, \xi}(A) \equiv \text{ker} P_{\xi}^{-1}$. Furthermore, it follows from Lemma 5.11 that $\gamma_{\lambda, \xi}(A)$ maps $\mathcal{H}_{-1}(A, \xi) \cap P_{\xi}^{-1}V_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi')$ onto $P_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi) \cap P_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi')$. With $\lambda \in \text{sp}(A)$ this implies that $\rho_{\xi, \xi}(A)$ is a bounded operator from $\mathcal{H}_{-1}(A, \xi) \cap P_{\xi}^{-1}V_{\xi}^{-1}\mathcal{H}_{+1}(A, \xi')$ onto
Finally, by means of functional calculus it is straightforward to check that $\rho_{\xi,\xi}(A)$ is independent of $\lambda \in \rho(A)$ and extends $P_{\xi}^{-1}(A - \xi')^{-1}(A - \xi)_{\xi}$.

\[ (5.20) \]

Proposition 5.13. Let $Q \in \mathcal{N}(\xi, \xi') \cap \mathcal{N}(\xi', \xi)$, $\xi, \xi' \in \mathbb{R} \cup \{\infty\}$ with $\xi \neq \xi'$ and let $A$ be a selfadjoint relation in a Hilbert space $\{\mathcal{H}, (\cdot, \cdot)\}$ associated to a realization of $Q$. Moreover, let $\omega \in \mathfrak{H}_{-1}(A, \xi)$ and $\eta_{\xi}$ be such that the boundary triplet $\{\mathcal{C}, \Gamma_0^{\omega, \eta_{\xi}}, \Gamma_1^{\omega, \eta_{\xi}}\}$ associated to $S^*_\omega$ is in Lemma 5.7 or 5.8 realizes $Q$. Then with $\omega':= \lim_{\lambda \equiv \xi'} Q(\lambda)$ also the boundary triplet $\{\mathcal{C}, \Gamma_0^{\omega', \eta_{\omega'}}, \Gamma_1^{\omega', \eta_{\omega'}}\}$ associated to $S^*_\omega$, as in Lemma 5.7 or 5.8 realizes $Q$.

In particular, $\Gamma_j^{\omega', \eta_{\omega'}} f = \Gamma_j^{\omega', \eta_{\omega'}} f$ for $f \in S^*_{\omega'} = S^*_\omega$, and $j = 1, 2$.

Proof. Lemma 5.11 shows that $S^*_{\omega'} = S^*_\omega$ in Lemma 5.12 $\omega' \in P_{\xi}^{-1} \mathfrak{H}_{-1}(A, \xi')$. Moreover, by definition of $\rho_{\xi,\xi}(A)$ the $\gamma$-fields $\gamma_\omega$ and $\gamma_{\omega'}$ associated to $\{\mathcal{C}, \Gamma_0^{\omega, \eta_{\omega}}, \Gamma_1^{\omega, \eta_{\omega}}\}$ and $\{\mathcal{C}, \Gamma_0^{\omega', \eta_{\omega'}}, \Gamma_1^{\omega', \eta_{\omega'}}\}$, respectively, satisfy $\gamma_\omega = \gamma_{\omega'}$ for all $\lambda \in \rho(A)$. This fact together with the observation that $A \subset S^*_\omega$ and $A \subset S^*_\omega'$ shows that $S^*_\omega = S^*_\omega'$, see (2.7).

Furthermore, if $M(\lambda)$ is the Weyl function associated to $\{\mathcal{C}, \Gamma_0^{\omega', \eta_{\omega'}}, \Gamma_1^{\omega', \eta_{\omega'}}\}$, then $M(\lambda) = Q(\lambda) \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, because both functions have the same $\gamma$-field. Since both function have by definition the same non-tangential limit $n_{\omega'}$ at $\xi'$ they must coincide and, hence, so do the boundary mappings.

Remark 5.14. If $\omega, \tilde{\omega} \in \mathfrak{H}_{-1}(A, \xi)$ are such that the boundary triplets associated to them (and $\eta_{\omega}$) via Lemma 5.7 or 5.8 have the same Weyl function, then $\tilde{\omega} - \omega \in \ker (A - \xi')$. Namely, in this case also the corresponding $\gamma$-fields $\gamma_\lambda(\omega)$ and $\gamma_\lambda(\tilde{\omega})$ coincide, and hence $\tilde{\omega} - \omega \in \ker \gamma_{\lambda, \xi}(A) = \ker (A - \xi)$; see (5.19).

5.5. Boundary triplets in rigged spaces. The relations $S_\omega$ and $S^*_\omega$ in Lemma 5.7 can be extended to relations from $\mathfrak{H}_{+1}(A, \infty)$ to $\mathfrak{H}_{-1}(A, \infty)$ by the formulas:

\[ (5.20) \]

where $S^*_\omega = A_{\infty}^{-1} S^*_\omega$ if $\omega \in \text{mul} A$. It follows from Proposition 5.3 that $S^*_\omega = S^*_\omega$, since $A_{\infty}^{-1} = A_{\infty}$; cf. (5.8), (5.9). Also the boundary mappings in Lemma 5.7 can be extended by the same formulas to mappings on $S^*_\omega$. For this reason the definition of boundary triplets as given in (2.6) is extended to the situation of riggings of Hilbert spaces. Here this definition is stated just in the case of the rigging $\mathfrak{H}_{+1}(A, \infty) \times \mathfrak{H}_{-1}(A, \infty)$.

Definition 5.15. Let $S$ be a symmetric relation in $\mathfrak{H}_{+1}(A, \infty) \times \mathfrak{H}_{-1}(A, \infty)$ with the dual relation $S^+$. Then $\{N, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^+$ if

(i) for all $\tilde{f} = \{f, f'\}, \tilde{g} = \{g, g'\} \in S^+$ the following Green’s identity holds:

\[ (f', g)_\infty - (f, g')_\infty = (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 \tilde{f}, \Gamma_1 \tilde{g}); \]
similar to the case of ordinary boundary triplets the formula
\[ (5.21) \]
\[ \tilde{A}_\Theta = \{ \tilde{f} \in \tilde{S}_+^\gamma : \tilde{f} \tilde{\in} \Theta \} = \ker (\tilde{\Gamma}_1 - \Theta \tilde{\Gamma}_0) \]
establishes a bijective correspondence between all self-dual extensions \( \tilde{A}_\Theta \) of \( S \) in \( \tilde{\mathcal{H}}_+^1(A, \infty) \times \tilde{\mathcal{H}}_-^1(A, \infty) \) and all selfadjoint relations \( \Theta \) on \( \mathcal{N} \).

It is also possible to extend the notion of boundary relation introduced in [3] to the present setting; cf. [5, Definition 3.1]. With a boundary triplet (or relation) for \( S^+ \) one can associate the \( \gamma \)-field and the Weyl function as in Definition 2.6 or as in [3, Definition 3.3, Section 4.2]. The only difference in these definitions is the change of the state space: \( \tilde{\mathcal{H}} \times \tilde{\mathcal{H}} \) is replaced by \( \tilde{\mathcal{H}}_+^1(A, \infty) \times \tilde{\mathcal{H}}_-^1(A, \infty) \). This offers a different, and also simpler, realization for functions belonging to the classes \( \mathcal{N}(\xi, 1), \xi \in \mathbb{R} \cup \{\infty\} \), since in rigged spaces the self-dual extensions \( \tilde{A}_\Theta \) of \( S \) appear as perturbations. This is made explicit in the next proposition for the class \( \mathcal{N}(\xi, 1) \).

**Proposition 5.16.** Let \( A \) be a selfadjoint relation in the Hilbert space \( \{ \tilde{\mathcal{H}}, (\cdot, \cdot) \} \), let \( \omega \in \tilde{\mathcal{H}}_-(A, \infty) \), and let \( \tilde{S}_\omega \) be given by (5.20). Then \( \{ \overline{C}, \tilde{\Gamma}_0^{\omega, \eta_\omega}, \tilde{\Gamma}_1^{\omega, \eta_\omega} \} \) with
\[ (5.22) \]
\[ \tilde{\Gamma}_0^{\omega, \eta_\omega} \tilde{f} = c_f \quad \text{and} \quad \tilde{\Gamma}_1^{\omega, \eta_\omega} \tilde{f} = \eta_\omega c_f + (f, \omega)_\infty, \]
\[ \tilde{f} = \{ f, f' - \omega c_f \}, \tilde{g} = \{ g, g' - \omega c_g \} \in \tilde{S}_+^\gamma, \]
is a boundary triplet for \( \tilde{S}_+^\gamma \), if \( \omega \notin \text{mul} \, A \), and a boundary relation for \( \tilde{S}_\omega \) with \( \text{mul} \tilde{\Gamma}_0^{\omega, \eta_\omega} = \text{ran} \tilde{\Gamma}_1^{\omega, \eta_\omega} = \{ \{ c, \eta_\omega c \} : c \in \overline{C} \} \), if \( \omega \in \text{mul} \, A \). Moreover, (5.21) with \( \omega \notin \text{mul} \, A \) gives a bijective correspondence between \( \tau \in \mathbb{R} \cup \{\infty\} \) and the self-dual extensions \( \tilde{A}_\tau \) of \( \tilde{S}_\omega \) of the form:
\[ (5.23) \]
\[ \tilde{A}_\tau = \tilde{S}_\omega \oplus \text{span} (\{0\} \times \{\omega\}). \]
The corresponding \( \gamma \)-field and Weyl function coincide with the \( \gamma \)-field and Weyl function given in Lemma 5.4.

**Proof.** A straightforward calculation using \( (A_\infty)^+ = A_\infty \) (see Proposition 5.15) shows that the Green’s identity (i) in Definition 5.15 with the mappings in (5.22) holds for all \( \tilde{f}, \tilde{g} \in \tilde{S}_+^\gamma \):
\[ (f' - \omega c_f, g)_\infty - (f, f' - \omega c_g)_\infty = (\tilde{\Gamma}_1^{\omega, \eta_\omega} \tilde{f}, \tilde{\Gamma}_0^{\omega, \eta_\omega} \tilde{g}) - (\tilde{\Gamma}_0^{\omega, \eta_\omega} \tilde{f}, \tilde{\Gamma}_1^{\omega, \eta_\omega} \tilde{g}). \]
Surjectivity with \( \omega \notin \text{mul} \, A \) is clear, since by Lemma 5.7 \( \tilde{\Gamma}_0^{\omega, \eta_\omega} \) is surjective on \( S_\omega^\gamma \) and \( S_\omega^\gamma \subset \tilde{S}_\omega^\gamma \). Since \( (f, \omega)_\infty = 0 \) when \( \omega \notin \text{mul} \, A \), the stated formula for \( \text{mul} \tilde{\Gamma}_0^{\omega, \eta_\omega} \), as well as for \( \text{ran} \tilde{\Gamma}_1^{\omega, \eta_\omega} \), is obtained from (5.22). In particular, if \( \omega \notin \text{mul} \, A \) then the mapping \( \tilde{\Gamma}_0^{\omega, \eta_\omega} : \tilde{S}_+^\gamma \to \mathbb{C} \times \mathbb{C} \) is bounded, and if \( \omega \in \text{mul} \, A \), then \( \tilde{S}_\omega = A_\infty = \tilde{S}_+^\gamma \) and \( \tilde{\Gamma}_0^{\omega, \eta_\omega} \) is a boundary relation of the form discussed in [3, Example 6.1].

The formulas for \( \tilde{A}_\tau \) in (5.23) are obtained by incorporating the definitions (5.22) into (5.21). Finally, by taking closure of \( S_\omega^\gamma = A^+ \{ \gamma_\lambda, \lambda \in \rho(A) \} \), in \( \tilde{\mathcal{H}}_+^1(A, \infty) \times \tilde{\mathcal{H}}_-^1(A, \infty) \) gives \( \tilde{S}_+^\gamma = A_\infty^+ \{ \gamma_\lambda \} \). Hence, by applying the usual definitions of the \( \gamma \)-field and Weyl function (see Definition 2.6) with the mappings \( \tilde{\Gamma}_0^{\omega, \eta_\omega} \) and \( \tilde{\Gamma}_1^{\omega, \eta_\omega} \) gives the formulas for \( \tilde{\gamma}_\lambda \) and \( \tilde{M}_\lambda \) as in Lemma 5.7. \( \Box \)
Note that $\tilde{A}_r$ is the closure (in $\mathcal{H}_{+1}(A, \infty) \times \mathcal{H}_{-1}(A, \infty)$) of the corresponding selfadjoint extensions $A_\tau$ of $S_\omega$, in particular, $\tilde{A}_\infty = A_\infty$. Equivalently, $\tilde{A}_r \cap \mathcal{H}_0 = A_\tau$ and similarly $\tilde{S}_\omega \cap \mathcal{H}_0 = S_\omega$ and $\tilde{S}_0^+ \cap \mathcal{H}_0 = S_0^*$.

6. Realization of $\mathcal{N}_r^N(r)$-functions

Let $Q$ be a generalized Nevanlinna function in the class $\mathcal{N}_r^N(r)$. In this section the realizations for the functions $Q$ and $rQ$ as Weyl function are studied and, in particular, and the connection between their realizations is established. Recall that if $\phi Q_0$ and $\phi^* Q_0$ are the canonical factorizations of $Q$ and $rQ$, then $Q_0 \in \mathcal{N}_0^0(\phi^*/\phi)$; see [4.17]. Therefore, first the realizations and their connection is established for the class $\mathcal{N}_0^0(r)$. From these results the general case can then be derived.

To establish the connection between the realizations of $Q_0$ and $rQ_0$ in the case that $Q_0 \in \mathcal{N}_0^0(r)$ some further facts are needed. An important observation here is the result stated in Proposition 4.17 if $Q_0 \in \mathcal{N}_0^0(r)$ then $Q_0 \in \mathcal{H}(b_1, 1)$, while $rQ_0 \in \mathcal{H}(b_1, 1)$, where $b_1 \in \mathbb{R} \cup \{\infty\}$ and $a_i \in \mathbb{R} \cup \{\infty\}$ are the poles and zeros of $r$. Hence the realization of the subclasses $\mathcal{N}(\xi, 1)$, $\xi \in \mathbb{R} \cup \{\infty\}$, given in Section 5.3 will naturally appear here.

6.1. Rational transformations of selfadjoint relations. For $a, b \in \mathbb{R} \cup \{\infty\}$, $a \neq b$, and $\gamma \in \mathbb{R} \setminus \{0\}$, let $r_{\gamma, a, b}$ be the symmetric rational function given by

$$(6.1) \quad r_{\gamma, a, b}(\lambda) = \frac{\gamma \lambda - a}{\gamma \lambda - b},$$

where, for notational convenience, $\lambda - \infty$ should be interpreted as being 1. With the notations introduced in (5.12) and (5.13) associate with the selfadjoint relation $A$ the operator $r_{\gamma, a, b}(A) : \mathcal{H} \to \mathcal{H}$ by setting

$$(6.2) \quad r_{\gamma, a, b}(A)f = P_{K_{a} + \mathcal{K}_{b}} f + \begin{cases} \gamma(I + P_{b}(b - a)(A - b)^{-1})P_{b} f, & a, b \in \mathbb{R}; \\ \gamma P_{b}(A - a)P_{b} f, & a \in \mathbb{R}, b = \infty; \\ \gamma P_{b}(A - b)^{-1}P_{b} f, & a = \infty, b \in \mathbb{R}. \end{cases}$$

Furthermore, define its extension $\tilde{r}_{\gamma, a, b}(A) : \mathcal{H}_{+1}(A, b) \to \mathcal{H}_{-1}(A, b)$ by

$$(6.3) \quad \tilde{r}_{\gamma, a, b}(A)f = P_{K_{a} + \mathcal{K}_{b}} f + \begin{cases} \gamma(I + P_{b}^{-1}(b - a)(A - b)^{-1})P_{b} f, & a, b \in \mathbb{R}; \\ \gamma P_{b}(A_{\infty} - a)P_{b} f, & a \in \mathbb{R}, b = \infty; \\ \gamma P_{b}^{-1}(A - b)^{-1}P_{b} f, & a = \infty, b \in \mathbb{R}. \end{cases}$$

where $A_{\infty}$ and $(A - b)^{-1}$ are as defined in (5.3) and (5.11). Observe, that

$$(6.4) \quad (I - P_{a})r_{\gamma, a, b}(A) = I - P_{a}, \quad (I - P_{b})r_{\gamma, a, b}(A) = I - P_{b}, \quad (I - P_{a})\tilde{r}_{\gamma, a, b}(A) = I - P_{\tilde{b}}, \quad (I - P_{\tilde{b}})\tilde{r}_{\gamma, a, b}(A) = I - P_{b}. $$

If $r_{\gamma, a, b}(A)$ is a nonnegative operator, then the square root of $r_{\gamma, a, b}(A)$, denoted by $(r_{\gamma, a, b}(A))^\frac{1}{2}$, is an everywhere defined bounded operator from $\mathcal{H}_{+1}(A, b)$ onto $\mathcal{H}_{-1}(A, a)$. Moreover, in this case $(I + r_{\gamma, a, b}(A))$ is also an everywhere defined bounded operator from $\mathcal{H}_{+1}(A, b)$ onto $\mathcal{H}_{-1}(A, b)$. The next lemma identifies an isomorphism between the rigged spaces $\mathcal{H}_{-1}(A, b)$ and $\mathcal{H}_{-1}(A, a)$.

**Lemma 6.1.** Assume that the operator $r_{\gamma, a, b}(A)$ defined in (6.2) is nonnegative. Then $(\rho_{\gamma, a, b})(A) : \mathcal{H}_{-1}(A, b) \to \mathcal{H}_{-1}(A, a)$, defined by

$$(6.5) \quad (\rho_{\gamma, a, b})(A)f = |\gamma| \left( I + \tilde{r}_{\gamma, a, b}(A) \right) (r_{\gamma, a, b}(A))^\frac{1}{2} (I + \tilde{r}_{\gamma, a, b}(A))^{-1} f,$$
the dual mapping of \( (r_{a,b}^\gamma)^\sharp(A) \), which is an isomorphism. Furthermore, the following formula holds

\[
(6.6) \quad P_a^-(A-a)^{-1} P_a P_b = \text{sgn}(\gamma)(\rho_{a,b}^\gamma)^\sharp(A)P_b^-(A-b)^{-1} P_b P_a \left( (r_{a,b}^\gamma)^\sharp(A) \right)^{-1}.
\]

**Proof.** Let the operator \( r_{a,b}^\gamma(A) \) be nonnegative. Note in particular that, if \( a, b \in \mathbb{R} \) and \( \text{mul} A \neq \{0\} \) then necessarily \( \gamma > 0 \) (and the whole spectrum of \( A \) lies outside the open interval with endpoints \( a \) and \( b \)); see (6.2). Since \( (r_{a,b}^\gamma(A))^{-1} = r_{b,a}^{1/\gamma}(A) \), the operators \( r_{a,b}^\gamma(A) = \gamma^2 r_{b,a}^{1/\gamma}(A) \) and \( (r_{a,b}^\gamma)^\sharp(A) \) are also nonnegative.

Next observe that for all \( f \in \mathcal{H}_{\gamma+1}(A,a) \) and \( g \in \text{ran}(A-a) \) one has

\[
\left( g, (r_{a,b}^\gamma)^\sharp(A)f \right)_a = \left( r_{b,a}^{1/\gamma}(A)g, f \right)_a,
\]

since \( r_{b,a}^{1/\gamma}(A) \) is obtained as a continuous extension of the selfadjoint operator \( r_{b,a}^{1/\gamma}(A) \) and the above identity clearly holds for all \( f, g \in \text{ran}(A-a) \); see Remark 5.2. Now for all \( h \in \mathcal{H} \) and \( g \in \text{ran}(A-a) \) one obtains

\[
\left( g, (\rho_{a,b}^\gamma)^\sharp(h) \right)_a = |\gamma| \left( \left( I + r_{b,a}^{1/\gamma}(A) \right) g, (r_{a,b}^\gamma)^\sharp(A) \left( I + r_{a,b}^\gamma(A) \right)^{-1} h \right) = \left( (r_{a,b}^\gamma)^\sharp(A)g, h \right)_b,
\]

where the second equality holds by functional calculus. The first assertion of the lemma now follows from the above identity by boundedness of the mappings \( (r_{a,b}^\gamma)^\sharp(A) : \mathcal{H}_{\gamma+1}(A,a) \to \mathcal{H}_{\gamma+1}(A,b) \) and \( (\rho_{a,b}^\gamma)^\sharp(A) : \mathcal{H}_{\gamma-1}(A,b) \to \mathcal{H}_{\gamma-1}(A,a) \); see also Remark 5.2.

The identity (6.6) follows from the fact that the left- and righthand side are continuous operators on \( \mathcal{H}_{\gamma+1}(A,a) \) which coincide on the dense subset \( \text{ran}(A-a) \cap \ker(A-b) \) of \( \mathcal{H}_{\gamma+1}(A,a) \); cf. Proposition 5.1; see also (6.8).

6.2. **Realizations in the case of a rational function of degree one.** Let the rational function \( r_{a,b}^\gamma \) be given by (6.1) and assume that \( Q \in \mathcal{N}_b^0(r_{a,b}^\gamma) \). Then Theorem 4.15 implies that \( Q \) is holomorphic at \( x \in \mathbb{R} \) if \( r(x) < 0 \) and, moreover, Proposition 4.17 shows that \( Q \in \mathcal{N}(b,1) \). Hence, there exist a selfadjoint relation \( A \) in a Hilbert space \( \{\mathcal{H}, (\cdot, \cdot)\} \) and an element \( \omega \in \mathcal{H}_{-1}(A,b) \) such that \( Q \) can be realized as the Weyl function associated to \( A \) and \( \omega \); see Theorem 5.10.

If this realization is minimal, then \( \rho(A) = \rho(Q) \) and in this case the transform \( r_{a,b}^\gamma(A) \) is automatically a nonnegative operator. As noted in Remark 5.6 it is always possible to add to the selfadjoint relation \( A \) a nontrivial point spectrum at \( b \) and replace \( \omega \) by \( \omega' = \omega + e_b \), where \( e_b \in \ker(A-b) \), to produce another (necessarily non-minimal) realization for \( Q \). The transform \( r_{a,b}^\gamma(A) \) of the extended \( A \) still remains a nonnegative operator; cf. (6.4). The addition of a nontrivial eigenspace to \( A \) at \( b \) simplifies the expressions connecting the realizations of \( Q \) and \( r_{a,b}^\gamma Q \); see Remark 6.2 below.

**Theorem 6.2.** Let \( r_{a,b}^\gamma \) be given by (6.1) and assume that \( Q_b \in \mathcal{N}_b^0(r_{a,b}^\gamma) \). Let \( A \) be a selfadjoint relation in the Hilbert space \( \{\mathcal{H}, (\cdot, \cdot)\} \) such that \( b \in \sigma_f(A) \) and

\[
\sigma_f(A) = \sigma_f(A_0) \cup \{\infty\} \quad \text{and} \quad \sigma_f(A) = \sigma_f(A_0) \cup \{\infty\} \quad \text{if} \ \text{mul} A \neq \{0\}. \tag{6.10}
\]

3For a selfadjoint relation \( A \) with operator part \( A_0 \) (see (2.5)) \( \sigma(A) = \sigma(A_0) \) and \( \sigma_f(A) = \sigma_f(A_0) \) if \( \text{mul} A = \{0\} \), and \( \sigma(A) = \sigma(A_0) \cup \{\infty\} \) and \( \sigma_f(A) = \sigma_f(A_0) \cup \{\infty\} \) if \( \text{mul} A \neq \{0\} \).
\( r_{a,b}^\gamma(A) \geq 0 \). Moreover, let \( \omega_b \in \mathcal{S}_{-1}(A,b) \) be such that the model in Lemma 5.7 or 5.8 realizes \( Q_b \) with

\[
Q_b(b) = \begin{cases} 
\gamma(a-b)((I-P_b^-)\omega_b, (I-P_b^-)\omega_b), & a, b \in \mathbb{R}; \\
\gamma((I-P_b^-)\omega_b, (I-P_b^-)\omega_b), & a \in \mathbb{R}, b = \infty; \\
-\gamma((I-P_b^-)\omega_b, (I-P_b^-)\omega_b), & a = \infty, b \in \mathbb{R}.
\end{cases}
\]

Then \( \omega_a := (\rho_{\mu,a}^\gamma)^\lambda(A)\omega_b \in \mathcal{S}_{-1}(A,a) \) and the Weyl function \( Q_a \) associated with \( A, \omega_a, \) and

\[
\eta_a = \begin{cases} 
\frac{b-a}{2}((I-P_a^-)\omega_a, (I-P_a^-)\omega_a), & a, b \in \mathbb{R}; \\
-\frac{b-a}{2}((I-P_a^-)\omega_a, (I-P_a^-)\omega_a), & a \in \mathbb{R}, b = \infty; \\
\frac{b-a}{2}((I-P_a^-)\omega_a, (I-P_a^-)\omega_a), & a = \infty, b \in \mathbb{R},
\end{cases}
\]

via Lemma 5.7 or 5.8 coincides with \( r_{a,b}^\gamma Q_b \). In particular, \( Q_a = r_{a,b}^\gamma Q_b \in \mathcal{N}(a,1) \).

**Proof.** Only the case that \( a, b \in \mathbb{R} \) is proved in detail; the other cases can be treated by similar arguments. Using the assumption \( b \in \sigma_p(A) \) and Remark 5.6 one can assume that \( \omega_b \) is such that (6.7) holds, because

\[
\frac{Q_b(b)}{\gamma(a-b)} = \lim_{z \to b}(b-z)r_{a,b}^\gamma(z)Q_b(z) \geq 0,
\]

see Theorem 4.15 (iv)(b). Since \( r_{a,b}^\gamma(A) \) is a nonnegative operator, the transformation (6.5) in Lemma 6.1 is well defined and it maps \( \omega_b \in \mathcal{S}_{-1}(A,b) \) to an element \( \omega_a = (\rho_{\mu,a}^\gamma)^\lambda(A)\omega_b \in \mathcal{S}_{-1}(A,a) \). The equation (6.18) with \( \xi = a \) implies that \( Q_a \) can be written as

\[
Q_a(\lambda) = \frac{b-a}{\gamma}((I-P_a^-)\omega_a, (I-P_a^-)\omega_a) + (\lambda-a)((I-P_a^-)\omega_a, (I-P_a^-)\omega_a) + (\lambda-a) \int_{(b)} \frac{(1+|t-a|)^2 d(E(t)P_{\infty}V_\omega a, V_\omega a)}{(t-\lambda)(t-a)} + (\lambda-a) \int_{[1,a]} \frac{(1+|t-a|)^2 d(E(t)P_{\infty}V_\omega a, V_\omega a)}{(t-\lambda)(t-a)}.
\]

The formulas (6.4) and (6.5) imply the identities \( (I-P_a^-)\omega_a = |\gamma|(I-P_a^-)\omega_b \) and \( (I-P_b^-)\omega_a = |\gamma|(I-P_b^-)\omega_b \). This gives

\[
Q_a(a) = \frac{b-a}{\gamma}((I-P_a^-)\omega_a, (I-P_a^-)\omega_a) = \gamma(b-a)((I-P_a^-)\omega_b, (I-P_a^-)\omega_b)
\]

\[
= r_{a,b}^\gamma(\lambda)(\lambda-b) \int_{(a)} \frac{t-b}{t-\lambda} d(E(t)P_{\infty}\omega_b, P_{\infty}\omega_b)
\]

\[
= r_{a,b}^\gamma(\lambda)(\lambda-b) \int_{(a)} \frac{(1+|t-b|)^2 d(E(t)P_{\infty}V_\omega b, V_\omega b)}{(t-\lambda)(t-b)}
\]

and by a similar calculation one obtains

\[
(\lambda-a) \int_{(b)} \frac{(1+|t-a|)^2 d(E(t)P_{\infty}V_\omega a, V_\omega a)}{(t-\lambda)(t-a)} = r_{a,b}^\gamma(\lambda)Q_b(b).
\]
Moreover, if \( \text{mul} A \neq \{0\} \) then \( r_{a,b}^\gamma(A) \geq 0 \) implies that \( \gamma > 0 \) (see (6.2)) and a direct calculation shows that \((I - P_\infty)\omega_a = \gamma \frac{1}{2} (I - P_\infty)\omega_b \) and

\[
(\lambda - a)(I - P_\infty)\omega_a, (I - P_\infty)\omega_b) = r_{a,b}^\gamma(\lambda)(\lambda - b)((I - P_\infty)\omega_b, (I - P_\infty)\omega_b).
\]

Finally, using extended functional calculus one obtains

\[
\int_{\mathbb{R} \setminus \{a,b\}} \frac{t - a}{t - \lambda} d \left( E_t P_\infty V_a (\rho_{b,a}^\gamma) \frac{1}{2} (A) V_b^{-1} V_b \omega_b, V_a (\rho_{b,a}^\gamma) \frac{1}{2} (A) V_b^{-1} V_b \omega_b \right)
\]

\[
= \int_{\mathbb{R} \setminus \{a,b\}} \frac{t - a}{t - \lambda} \phi(t) (1 + |t - b|^{-1})^2 d \left( E_t P_\infty V_b \omega_b, V_b \omega_b \right),
\]

where for \( t \neq a, b \)

\[
\phi(t) = \gamma^2 \left( 1 + \frac{1}{\gamma} \frac{t - b}{t - a} \right)^2 \left( \gamma \frac{t - a}{t - b} \left( 1 + \gamma \frac{t - a}{t - b} \right)^{-2} \right) \gamma \frac{t - b}{t - a}.
\]

cf. Lemma 6.1. Hence, the last term in (6.8) can be rewritten in the form

\[
(\lambda - a) \int_{\mathbb{R} \setminus \{a,b\}} \frac{t - a}{t - \lambda} \phi(t) (1 + |t - b|^{-1})^2 d \left( E_t P_\infty V_b \omega_b, V_b \omega_b \right)
\]

\[
= r_{a,b}^\gamma(\lambda)(\lambda - b) \int_{\mathbb{R} \setminus \{a,b\}} (1 + |t - b|^{-1})^2 d \left( E_t P_\infty V_b \omega_b, V_b \omega_b \right) \frac{(t - \lambda)(t - b)}{(t - \lambda)(t - b)}.
\]

Combining all the above calculations yields the desired identity \( Q_a = r_{a,b}^\gamma Q_b \); see (6.18) with \( \xi = b \). Since \( \omega_a \in \mathcal{N}_{[-1]}(A, a) \), the inclusion \( Q_a \in \mathcal{N}(a, 1) \) is clear; cf. Proposition 4.17. □

The formulas for the limit value \( Q_b(b) \) in Theorem 6.2 indicate the use of a (non-minimal) selfadjoint relation \( A \) with \( b \in \sigma_p(A) \); note also the similar formulas for \( Q_a(a) \). Since \( Q_b \in \mathcal{N}(b, 1) \) this function does not have a pole at \( b \), see Lemma 3.3. Thus in a minimal realization of \( Q_b \) the selfadjoint relation, say \( A_{min}(Q_b) \) (see Remark 6.3 below), does not have \( b \) in its point spectrum. However, in general the product \( r_{a,b}^\gamma Q_b \) has a pole at \( b \), unless \( \lim_{z \to b} Q_b(z) = 0 \). Therefore, in a minimal realization for \( r_{a,b}^\gamma Q_b \) one typically has \( b \in \sigma_p(A_{min}(r_{a,b}^\gamma Q_b)) \). By symmetry, similar situation holds for \( a \): \( a \notin \sigma_p(A_{min}(r_{a,b}^\gamma Q_b)) \), but it is possible that \( a \in \sigma_p(A_{min}(Q_b)) \). Minimal realizations for \( Q_b \) and \( r_{a,b}^\gamma Q_b \) include the information about the possible point masses at \( b \) and \( a \); consequently the explicit formulas connecting minimal realizations would get longer and less readable, especially when the degree of \( r \) increases. On the other hand, it is easy to produce minimal realization for \( Q_b \) and \( r_{a,b}^\gamma Q_b \) from the general connection established in Theorem 6.2 this is explained in the next remark.

Furthermore, in Section 6.4 this connection between the minimal realizations will be made explicit in specific model spaces; see Theorem 6.7.

**Remark 6.3.** For \( Q_b \) the minimal realization is obtained via the corresponding \( \gamma \)-field using \( \mathcal{H}_{\text{min}} := \text{span} \{ \gamma_{\lambda} \omega : \lambda \in \mathbb{C} \setminus \{ \Re \} \} \), cf. (6.13). The subspace \( \mathcal{H}_{\text{min}} \) reduces \( A \) and consequently, the rigged spaces \( \mathcal{H}_{\pm 1}(A, b) \) and \( \mathcal{H}_{\pm 1}(A, b) \) get decomposed accordingly. Automatically, \( \omega_b \in \mathcal{H}_{\text{min}, -1}(A_{min}, b) \). The transforms \( r_{b,a}^\gamma(A) \) and \( (\rho_{b,a}^\gamma)^{-1} \frac{1}{2} (A) \) decompose accordingly.
Remark 6.4. Let $S_b$ and $S_a$ be the closed symmetric relations associated to $A$ and $\omega_b$, and for $A$ and $\omega_a := (\rho^\gamma_{a, a})^\sharp (A) \omega_b$, respectively, via Lemma 5.7 and 5.8. Then these symmetries can be extended to $\hat{S}_b$ and $\hat{S}_a$ in the corresponding rigged spaces by means of the transforms in (5.17): for instance, if $a, b \in \mathbb{R}$, then

\[
(S_b - b)^{-1} = \{ (f, f') \in (A - b)_b^{-1} : (f, \omega_b)_b = 0 \};
\]

\[
(S_a - a)^{-1} = \{ (f, f') \in (A - a)_a^{-1} : (f, \omega_a)_a = 0 \}.
\]

Taking into account the connection (6.6) between $(A - a)_a^{-1}$ and $(A - b)_b^{-1}$ in Lemma 6.1, it follows that

\[
P_a (S_a - a)^{-1} P_a P_b = \text{sgn} (\gamma) (\rho^\gamma_{b, a})^\sharp (A) P_b^{-1} (S_b - b)^{-1} P_a P_b \left( (\gamma^\gamma_{b, a})^\sharp (A) \right)^{-1};
\]

\[
P_a^* (S_a^* - a)^{-1} P_a P_b = \text{sgn} (\gamma) (\rho^\gamma_{a, b})^\sharp (A) P_b^{-1} (S_b^* - b)^{-1} P_a P_b \left( (\gamma^\gamma_{a, b})^\sharp (A) \right)^{-1}.
\]

6.3. Realizations in the case of symmetric rational functions. Assume that $Q \in \mathcal{N}^0_0 (r)$. Then according to Proposition 6.13, the symmetric rational function $r$ can be factorized as $r = \prod_{i=1}^n r_{a_i, b_i}$, see (6.1), where

\[
\left( \prod_{i=1}^j r_{a_i, b_i} \right) Q \in \mathcal{N}, \quad j = 1, \ldots, n.
\]

The realizations for the functions $Q$ and $rQ$ are constructed in this section by applying Theorem 6.2 combined with Proposition 6.13 inductively to the product functions $\left( \prod_{i=1}^j r_{a_i, b_i} \right) Q \in \mathcal{N}$ for $j = 1, \ldots, n$. 

Recall that $\rho_{a, c}(A)$ with $a, c \in \mathbb{R} \cup \{ \infty \}$, $a \neq c$, defined in Lemma 5.12, maps $\mathcal{H}_1 (A, a)$ into $\mathcal{H}_1 (A, c) \ominus \ker (A - c)$. In particular, if $Q_a \in \mathcal{N} (a, 1)$ corresponds to $\omega_a \in \mathcal{H}_1 (A, a)$ (cf. Theorem 6.2) and if, in addition, $Q_a \in \mathcal{N} (c, 1)$ then the limit value $Q_a (c) \in \mathbb{R}$ exists and, as noted in Remark 5.14, one can replace the vector $\omega_c = \rho_{a, c}(A) \omega_a \in \mathcal{H}_1 (A, c)$ by $\tilde{\omega}_c = \omega_c + e_c$, where $e_c \in \ker (A - c)$, in the realization of $Q_a \in \mathcal{N} (c, 1)$. In the main theorem of this section the vector $\tilde{\omega}_c$ is selected such that the limit value $Q_a (c)$ satisfies an analog of (6.7) with $c$ instead of $b$; such a selection of $\tilde{\omega}_c$ is expressed shortly by using the notation

\[
\tilde{\omega}_c = \rho_{a, c} \omega_a \in \mathcal{H}_1 (A, c),
\]

cf. Lemma 5.12.

Theorem 6.5. Assume that $Q \in \mathcal{N}^0_0 (r)$ and let $\prod_{i=1}^n r_{a_i, b_i}^\gamma$ be a factorization of $r$ such that (6.9) holds. Let $A$ be a selfadjoint relation in the Hilbert space $\mathcal{H}_1 (\cdot, \cdot)$ such that $\sigma (A) = \sigma (Q) \cup \{ b_i \in \mathbb{R} \cup \{ \infty \} : b_i \in \sigma_p (A), i = 1, \ldots, n \}$ and let $\omega \in \mathcal{H}_1 (A, b_1)$ be such that the model in Lemma 5.4 or 5.8 realizes $Q$.

Then with a vector $\omega_e$ of the form

\[
\omega_e = (\rho^\gamma_{b_n, a_n})^\sharp (A) \prod_{i=1}^{n-1} \left( \rho_{a_i, b_{i+1}} (A) (\rho^\gamma_{b_i, a_i})^\sharp (A) \right) \omega,
\]

where $\tilde{\rho}_{a_i, b_{i+1}} (A)$ and $(\rho^\gamma_{b_i, a_i})^\sharp (A)$ are defined as in (6.10) and (6.5), the model in Lemma 5.4 or 5.8 associated to $A$, $\omega_e \in \mathcal{H}_1 (A, a_n)$, and $\eta_c = \lim_{\lambda \to a_n} r (\lambda) Q (\lambda)$ realizes the function $rQ$. In particular, $rQ \in \mathcal{N}^0_0 (a_1, 1)$.
Proof. By the assumption on \( A \) all the products \( \prod_{i=1}^{\overline{i}} r_{a_i,b_i}^\gamma (A) \) are nonnegative operators. An application of Theorem 6.2 with \( \omega_{a_1} = (\rho_{b_1,a_1})^\frac{P}{2} (A) \omega \) gives a realization for the function \( r_{a_1,b_1}^\gamma Q \in \mathcal{N}(a_1, 1) \) and, by the selection of the factorization for \( r \), one actually has \( r_{a_1,b_1}^\gamma Q \in \mathcal{N}_0^0 (r_{a_2,b_2}^\gamma) \). In particular, \( r_{a_2,b_2}^\gamma Q \in \mathcal{N}(1, b_2) \) and one can define \( \omega_{b_2} = \tilde{\rho}_{a_2,b_2} \omega_{a_1} \in \mathcal{H}_{-1}(A, b_2) \), such that \( r_{a_1,b_1}(b_2)Q(b_2) \) satisfies the formula (6.7) with \( r_{a,b}^\gamma \) replaced by \( r_{a_2,b_2}^\gamma \); see (6.10). Now, one can apply Theorem 6.2 to the product \( r_{a_2,b_2}^\gamma \left( r_{a_1,b_1}^\gamma Q \right) \) to get a desired realization for this product. Now by proceeding inductively one gets the stated realization for the product function \( rQ \in \mathcal{N}(a_n, 1) \) with \( \omega_c \in \mathcal{H}_{-1}(A, a_n) \) of the form (6.11) and \( \eta_c = \lim_{\lambda \to a_n} r(\lambda)Q(\lambda) \in \mathbb{R} \). Finally, the fact that \( rQ \in \cap_{i=1}^{n} \mathcal{N}(a_i, 1) \) is a direct consequence of Proposition 4.17. \( \square \)

The integral representation of the functions \( Q \) and \( rQ \) can be obtained by means of \( \omega \) and \( \omega_c \), due to their connection to the underlying spectral measures \( d\sigma(t) \) and \( d\sigma_c(t) \) stated in Remark 5.9. Therefore note that in case that \( b_i \) and \( a_i \) are finite the spectral measure \( d\sigma_c \) of \( Q_c \) is given by

\[
d\sigma_c(t) = (1 + |t - a_i|^2) d(E_i P_{a_i} P_{\infty} V_{a_i} \omega_c, V_{a_i} \omega_c),
\]

see Remark 5.9. Using extended functional calculus this can be rewritten as

\[
d\sigma_c(t) = |r(t)|(1 + |t - b_i|^2) d(E_i P_{b_i} \cdots P_{b_1} P_{a_i} V_{b_i} \omega, V_{b_i} \omega) + \sum_{i=1}^{n} \zeta_i \delta_{b_i}(t)
\]

where \( d\sigma \) is the spectral measure of \( Q \) and \( \zeta_i \in \mathbb{R}_+ \), \( 1 \leq i \leq n \). See also Section 6.3 where \( L^2(d\sigma) \)-realizations of the functions \( Q \) and \( rQ \) are considered.

Theorem 6.3 contains the main result for describing the realization for the functions belonging to the class \( \mathcal{N}_0^0 (r) \) with some symmetric rational function \( r \). This result can be used to describe for \( Q \in \mathcal{N}_0^\kappa (r) \) the connection between the realizations for the functions \( Q \) and \( rQ \). Therefore let the canonical factorizations of \( Q \in \mathcal{N}_\kappa \) and \( rQ \in \mathcal{N}_{\tilde{\kappa}} \) be written in the form

\[
(6.12) \quad Q = \varphi \varphi^* Q_0, \quad rQ = \tilde{\varphi} \tilde{\varphi}^* \tilde{Q}_0; \quad \varphi^*(\lambda) = \overline{\varphi(\lambda)}, \quad \tilde{\varphi}^*(\lambda) = \overline{\tilde{\varphi}(\lambda)}.
\]

Then

\[
(6.13) \quad \tilde{Q}_0 = \tilde{r} Q_0, \quad \tilde{r} = \frac{r \varphi \varphi^*}{\tilde{\varphi} \tilde{\varphi}^*},
\]

and consequently \( Q_0 \in \mathcal{N}_0^\kappa (\tilde{r}) \); cf. (4.17). Now Theorem 6.3 guarantees that the functions \( Q_0 \) and \( \tilde{Q}_0 \) can be realized by means of the elements \( \omega \in \mathcal{H}_{-1}(A, b) \) and \( \tilde{\omega} \in \mathcal{H}_{-1}(A, a) \) for some \( a, b \in \mathbb{R} \cup \{ \infty \} \) using the same selfadjoint relation \( A \) in a Hilbert space \( \{ \mathcal{H}, (\cdot, \cdot) \} \). The realizations for the original functions \( Q \) and \( rQ \) can be obtained using the model based on their canonical factorizations in (6.12). To describe this consider the matrix functions

\[
(6.14) \quad \Phi(\lambda) = \begin{pmatrix} 0 & \varphi(\lambda) \\ \varphi^*(\lambda) & 0 \end{pmatrix}, \quad \tilde{\Phi}(\lambda) = \begin{pmatrix} 0 & \tilde{\varphi}(\lambda) \\ \tilde{\varphi}^*(\lambda) & 0 \end{pmatrix}.
\]

Then \( \Phi \in \mathcal{N}_\kappa \) with \( \kappa = \deg \varphi \) and \( \tilde{\Phi} \in \mathcal{N}_{\tilde{\kappa}} \) with \( \tilde{\kappa} = \deg \tilde{\varphi} \). Let \( A_\Phi \) and \( A_{\tilde{\Phi}} \) be selfadjoint relations in the reproducing kernel Pontryagin spaces \( \mathcal{H}_\Phi \) and \( \mathcal{H}_{\tilde{\Phi}} \), respectively.
which realize the functions $\Phi$ and $\bar{\Phi}$ with the $\gamma$-fields $\gamma_0^\Phi$ and $\gamma_0^\bar{\Phi}$, respectively. Now the realization for the functions $Q$ and $\bar{Q} = rQ$ can be obtained via the next result, which is formulated by means of $\{A, \gamma_\lambda\}$-realizations, see Remark 2.8, using extensions of the orthogonal sums $S_0 \oplus S_b$ and $\bar{S}_0 \oplus \bar{S}_b$ of the corresponding symmetric restrictions determined by (2.12).

**Proposition 6.6.** Let $Q \in \mathcal{N}_\Phi^g(r)$, let the canonical factorizations of $Q$ and $rQ$ be given by (6.12), and let $Q_0$ and $\bar{Q}_0 = \bar{r}Q_0$ be realized as in Theorem 6.3 with the $\gamma$-fields $\gamma_0^\Phi$ and $\gamma_0^{\bar{\Phi}}$, respectively. Moreover, let $\Phi$ and $\bar{\Phi}$ be realized via the pairs $\{A_\Phi, \gamma_\lambda^\Phi\}$ and $\{\bar{A}_\Phi, \gamma_\lambda^\bar{\Phi}\}$ in the Pontryagin spaces $S_{\Phi}$ and $\bar{S}_\Phi$, respectively.

Then the functions $Q$ and $\bar{Q} = rQ$ can be realized in the Pontryagin spaces $S_0 \oplus S_b$ and $\bar{S}_0 \oplus \bar{S}_b$ via the pairs $\{A_0, \gamma_\lambda\}$ and $\{\bar{A}_0, \gamma_\lambda\}$, where the $\gamma$-fields $\gamma_\lambda$ and $\bar{\gamma}_\lambda$ are given by

\begin{equation}
\gamma_\lambda = \begin{pmatrix}
\gamma_0^\Phi \\
\gamma_0^{\bar{\Phi}}
\end{pmatrix} = \begin{pmatrix}
\gamma_0^\Phi \\
\gamma_0^{\bar{\Phi}}
\end{pmatrix},
\end{equation}

and where $A_0$ and $\bar{A}_0$ are selfadjoint extensions of $S_0 \oplus S_b$ and $\bar{S}_0 \oplus \bar{S}_b$ satisfying the associated $\gamma$-field formulas in (2.10), respectively.

**Proof.** By symmetry it suffices to prove the statement for $Q$. It follows from the orthogonal sum construction that $\gamma_\lambda$ satisfies the formula (6.12) with some selfadjoint relation $\bar{A}_0$, which is a selfadjoint extension of $S_0 \oplus S_b$; see [4, Theorem 3.3]. On the other hand, a straightforward calculation using the matrix formula for $\gamma_\lambda$ in (6.15) and the definition of $\Phi(\lambda)$ in (6.14) yields the identity

\[
(\gamma, \gamma') = \frac{Q_0(\lambda) - Q_0(\bar{\mu})}{\lambda - \bar{\mu}} \varphi(\lambda) + \frac{Q_0(\mu)\varphi(\mu)}{\lambda - \bar{\mu}} \frac{\Phi(\lambda) - \Phi(\bar{\mu})}{\lambda - \bar{\mu}} \varphi(\lambda)
\]

for $\lambda, \mu \in \rho(A + A_b)$. Therefore, $\{A_\Phi, \gamma_\lambda^\Phi\}$ realizes the function $Q = \varphi Q_0 \varphi$. $\square$

The realizations used in Proposition 6.6 rely on the canonical factorizations in (6.12). The idea of factorization models and the above orthogonal sum approach via matrix functions as in (6.14) goes back to [4]. In particular, the construction of the $\gamma$-fields in (6.15) is based on [4, Theorem 3.3]. Also the selfadjoint realizations $A_0$ and $\bar{A}_0$ can be made explicit by means of boundary triplets: If, for instance, $\{C, \Gamma_0^\Phi, \Gamma_1^\Phi\}$ and $\{C^2, \Gamma_b^\Phi, \Gamma_1^\Phi\}$ are boundary triplets for $S_0^\Phi$ and $S_b^\Phi$, with Weyl-functions $Q_0$ and $\Phi$, then $A_0$ is given by the following boundary conditions:

\[
A_0 = \{\hat{f}_0 \oplus \bar{F} \in S_0^\Phi \oplus S_b^\Phi : \Gamma_0^\Phi \hat{f}_0 = (\Gamma_1^\Phi \bar{F})_1, \Gamma_1^\Phi \hat{f}_0 = (\Gamma_0^\Phi \bar{F})_1, (\Gamma_0^\Phi \bar{F})_2 = 0\},
\]

where e.g. $(\Gamma_0^\Phi \bar{F})_1$ and $(\Gamma_0^\Phi \bar{F})_2$ stand for the components of $\Gamma_0^\Phi \bar{F} \in \mathbb{C}^2$; cf. [4, Theorem 3.3].

The non-minimality of the realizations in Proposition 6.6 can only be due to a finite number of possible point masses in the underlying spectral measure of the functions (or, equivalently, eigenvalues of the associate symmetric relations determined by (2.12)), which are canceled due to multiplications by rational functions appearing in various factorizations. This means that minimality of models can be easily obtained by checking if the functions $Q$ and $rQ$ actually have poles or zeros at points, where the rational factors admit poles and zeros.
6.4. \(L^2(\sigma)\)-realizations. The abstract realizations from the previous section will be supplemented by explicit (and minimal) realizations in the form of \(L^2(\sigma)\)-models for Nevanlinna functions. With the help of these models the connection between the realizations for \(Q\) and \(rQ\), \(Q \in \mathcal{N}_0^0(r)\), as given in Theorem 6.3 is made more accessible. The reason is that the associated riggings as well as the mappings \((\rho_{\xi,\xi'})_1(A)\) and \((\rho_{\beta,\alpha})_1(A)\) appearing in Lemma 5.12 and 6.1 are easily expressed in \(L^2(\sigma)\)-spaces (without regularizations). For instance, the formula for \((\rho_{\xi,\xi'})_1(A)\) \((\xi,\xi' \in \mathbb{R} \cup \{\infty\}, \xi \neq \xi')\) in the \(L^2(\sigma)\) setting is given by the expression

\[
\rho_{\xi,\xi'}(t)f(t) = 1^e_{\xi}(t) \frac{t - \xi}{t - \xi'} f(t),
\]

with domain being described now by proper integrability conditions. Here, and in what follows, for a subset \(E \subseteq \mathbb{R} \cup \{\infty\}\) the notations \(1_E\) and \(1^e_E\) stand for the characteristic functions of the sets \(E \cap \mathbb{R}\) and \(E \cap \mathbb{R}\setminus \{E \cap \mathbb{R}\}\), respectively.

Let \(\beta \geq 0\) and let \(d\sigma\) be a nonnegative measure satisfying \(\int g \, d\sigma(t)/\sqrt{(1+t^2)} < \infty\). Associate with \(\beta\) and \(\sigma\) the Hilbert space \(\mathcal{H}_{\sigma,\beta} := (L^2(\sigma) \times \mathbb{C})/\langle \cdot,\cdot \rangle\), where

\[
(f \times f_{\infty},g \times g_{\infty}) = g_{\infty}^* \beta f_{\infty} + \int g(t)^* f(t) d\sigma(t), \quad f \times f_{\infty}, g \times g_{\infty} \in L^2(\sigma) \times \mathbb{C}.
\]

Thus \(\mathcal{H}_{\sigma,\beta}\) is \(L^2(\sigma) \times \mathbb{C}\) if \(\beta \neq 0\) and \(\mathcal{H}_{\sigma,\beta}\) is equivalent to \(L^2(\sigma)\) if \(\beta = 0\). In this space the multiplication operator with the independent variable \(A_{\sigma,\beta}\),

\[
(6.16) \quad A_{\sigma,\beta} = \{\{f(t) \times 0, tf(t) \times f_{\infty}\} \in \mathcal{H}_{\sigma,\beta}^2\},
\]

is a selfadjoint relation. Note that \(\sigma(A_{\sigma,\beta}) = \text{supp}(\sigma) + \{\infty\}\) if \(\beta \neq 0\). Let \(\mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi) \subset \mathcal{H}_{\sigma,\beta} \subset \mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi)\) be the rigging of \(\mathcal{H}_{\sigma,\beta}\) with respect to \(A_{\sigma,\beta}\) at \(\xi\) as defined in Definition 5.4 see also [18] \((\xi = \infty)\) and [14] \((\xi = 0)\). Then the spaces \(\mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi)\) and \(\mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi)\) are weighted \(L^2(\sigma)\) spaces and the duality \(\langle \cdot,\cdot \rangle_\xi\) associated with the above rigging takes for \(f \times f_{\infty} \in \mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi)\) and \(g \times g_{\infty} \in \mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\xi)\) the value

\[
(6.17) \quad (f \times f_{\infty},g \times g_{\infty})_\xi = g_{\infty}^* \beta f_{\infty} + \int g(t)^* f(t) d\sigma(t).
\]

The introduced rigging can be used to give explicit realizations for the subclass \(\mathcal{N}(\xi,1), \xi \in \mathbb{R} \cup \{\infty\}\) by interpreting Lemma 5.8 and 5.9. In particular, if for a Nevanlinna function \(\sigma\) is taken to be its spectral measure, \(\beta\) its non-tangential limit at \(\infty\) as in Lemma 2.2 and \(\omega\) is taken to be \(1/(t-\xi) \times 1\), if \(\xi \in \mathbb{R}\), and \(1 \times 1\), if \(\xi = \infty\), then the corresponding realization is minimal. Note that the case \(\xi = \infty\) has been studied in the \(L^2(\sigma)\) setting in [13] and [18], and that in [9] [15] a realization by means of \(L^2(\sigma)\)-models for all Nevanlinna functions is given.

**Theorem 6.7.** Let \(Q \in \mathcal{N}_0^0(r)\) and denote the zeros and poles of \(r\) by \(a_i\) and \(b_i\), \(1 \leq i \leq n\), where \(a_n = \infty\) or \(b_n = \infty\) if \(\infty\) is a zero or pole of \(r\).

Let \(Q\) be realized by the boundary triplet \(\{C,1^e_{\sigma/m},1_{\sigma/m}\}\) associated with \(S^\omega_{\sigma}\) in the Hilbert space \(\mathcal{H}_{\sigma,\beta}\), where \(\omega \times 1 = 1/(t - b_1) \times 1 \in \mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},b_1)\) if \(b_1 \in \mathbb{R}\) and \(\omega \times 1 = 1 \times 1 \in \mathcal{H}_{\sigma,\beta}(A_{\sigma,\beta},\infty)\) if \(b_1 = \infty\); cf. Theorem 5.10. With \(\zeta_0 = \lim_{\lambda \rightarrow b_1}(\beta \lambda - r(\lambda)Q(\lambda)/Q(\lambda))/\lambda\), define \(d\sigma,\beta_\epsilon\) and \(\omega_\epsilon\) as

\[
\begin{align*}
d\sigma_e(t) &= 1^e_{\xi}(a_1,...,a_n) d\sigma(t) + \sum_{i=1}^n \delta_{b_i}, \quad \beta_\epsilon = \gamma/\beta, \quad b_n, a_n \in \mathbb{R}; \\
d\sigma_\epsilon(t) &= 1_{\xi}(a_1,...,a_n) d\sigma(t) + \sum_{i=1}^{n-1} \delta_{b_i}, \quad \beta_\epsilon = \zeta_0, \quad b_n = \infty, a_n \in \mathbb{R}; \\
d\sigma_e(t) &= 1_{\xi}(a_1,...,a_{n-1}) d\sigma(t) + \sum_{i=1}^n \delta_{b_i}, \quad \beta_\epsilon = 0, \quad b_n \in \mathbb{R}, a_n = \infty;
\end{align*}
\]
and

\[ \omega_e(t) = \begin{cases} \frac{1}{t - a_n} \sqrt{|r(t)|} + \sum_{i=1}^{n} \frac{1}{b_i - a_n} \sqrt{b_i}, & b_n, a_n \in \mathbb{R}; \\ \frac{1}{t - a_n} \sqrt{|r(t)|} + \sum_{i=1}^{n-1} \frac{1}{b_i - a_n} \sqrt{b_i}, & b_n = \infty, a_n \in \mathbb{R}; \\ \frac{1}{t - a_n} \sqrt{|r(t)|} + \sum_{i=1}^{\infty} \frac{1}{b_i - a_n} \sqrt{b_i}, & b_n \in \mathbb{R}, a_n = \infty. \end{cases} \]

Then \( \omega_e \times 1 \in \mathcal{H}_1(A_{\sigma, \beta, a_n}) \) and, with \( \eta_{an} = \lim_\lambda \omega_{an} r(\lambda)Q(\lambda) \), the Nevanlinna function \( rQ \) is realized by the boundary triplet \( \{ \mathbb{C}, \Gamma_0^{\omega_e}, \Gamma_1^{\omega_e}, \Gamma_{\omega_e} \} \) associated with \( S_\omega \) in the Hilbert space \( \mathcal{H}_\sigma \), cf. Theorem 6.10.

**Sketch of the proof.** Let \( d\sigma \) be a nonnegative measure satisfying \( \int_\mathbb{R} d\sigma(t)/(1 + t^2) < \infty \), let \( \beta \geq 0 \) and let \( \omega \times 1 \in \mathcal{H}_1(A_{\sigma, \beta, \xi}) \), where \( A_{\sigma, \beta} \) is as in (6.16) and \( \xi \in \mathbb{R} \cup \{ \infty \} \). Then the spectral measure \( d\sigma_\omega \) of the Weyl function realized by \( A_{\sigma, \beta} \) and \( \omega \times 1 \) via Lemma 5.7 (\( \xi = \infty \)) or Lemma 5.8 (\( \xi \in \mathbb{R} \)) is given by

\[ d\sigma_\omega(t) = \begin{cases} (t - \xi)^2 |\omega(t)|^2 d\sigma(t), & \xi \in \mathbb{R}, \\ |\omega(t)|^2 d\sigma(t), & \xi = \infty, \end{cases} \quad (t \neq \xi). \]

By means of this observation, it can be easily seen that the spectral measure of the Weyl function associated with the boundary triplet or boundary relation corresponding to \( \omega_e \times 1 \) and \( A_{\sigma_e, \beta_e} \) (at \( a_n \)) is the spectral measure of \( rQ \), which outside the poles of \( r \) is given by \( r(t)d\sigma(t) = |r(t)|d\sigma(t) \); see (2.3).

\[ \square \]

Note that the zeros \( a_i \) and the poles \( b_i \) of \( r \) in Theorem 6.7 can be either of order one or of order two, cf. Theorem 4.12.

The models in Theorem 6.7 are minimal, see the discussion following (6.17), and, hence, different from Theorem 6.5 the spaces used in the realizations of \( Q \) and \( rQ \) will, in general, differ from each other. The equivalent of the above minimal model could also be constructed in the abstract case. In that case the coupling method should be used to properly treat point masses; here this is reflected by the fact that the measure changes due to the point masses as indicated in Theorem 6.7.

**References**

[1] Ju. M. Berezans'kii, Expansions in eigenfunctions of selfadjoint operators (Russian), Naukova Dumka, Kiev, 1965. English translation: Transl. Math. Monographs, vol. 17, Amer. Math. Soc., 1968.

[2] V.M. Bruk, “A certain class of boundary value problems with a spectral parameter in the boundary condition (Russian)”, Mat. Sb. (N.S.), 100 (142), no. 2 (1976), 210–216.

[3] V.A. Derkach, “On Weyl function and generalized resolvents of a Hermitian operator in a Krein space”, Integr. Eq. Oper. Th., 23 (1995), 387–415;

[4] V.A. Derkach and S. Hassi, “A reproducing kernel space model for \( \mathcal{N}_\omega \)-functions”, Proc. Amer. Math. Soc., 131 (2003), 3795–3806.

[5] V.A. Derkach, S. Hassi, M.M. Malamud and H.S.V. de Snoo, “Boundary relations and their Weyl families”, Trans. Amer. Math. Soc., 358 (2006), 5351–5400.

[6] V.A. Derkach, S. Hassi, M.M. Malamud, and H.S.V. de Snoo, “Boundary relations and generalized resolvents of symmetric operators”, Russian Journal of Mathematical Physics, 16 no. 1 (2009), 17–60.

[7] V.A. Derkach, S. Hassi and H.S.V. de Snoo, “Operator models associated with Kac subclasses of generalized Nevanlinna functions”, Methods of Funct. Anal. and Top., 5 (1999), 65–87.

[8] V.A. Derkach and M.M. Malamud, “Generalized resolvents and the boundary value problems for Hermitian operators with gaps”, J. Funct. Anal., 95 (1991), 1–95.

[9] V.A. Derkach and M.M. Malamud, “The extension theory of Hermitian operators and the moment problem”, J. Math. Sci., 73, no. 2 (1995), 141–242.
[10] V.A. Derkach and M.M. Malamud, “On some classes of holomorphic operator functions with nonnegative imaginary part”, Operator theory, operator algebras and related topics (Timişoara, 1996), Theta Found., Bucharest (1997), 113–147.

[11] A. Dijksma, H. Langer, A. Luger and Yu Shondin, “A factorization result for generalized Nevanlinna function of the class $N_\kappa$”, Integral Equations Operator Theory, 36 (2000), 121–125.

[12] W.F. Donoghue, Monotone matrix functions and analytic continuation, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[13] S. Hassi, M. Kaltenbäck and H.S.V. de Snoo, “Triplets of Hilbert spaces and Friedrichs extensions associated with the subclass $N_1$ of Nevanlinna functions”, J. Operator Theory, 37 (1997), 155–181.

[14] S. Hassi, M. Kaltenbäck and H.S.V. de Snoo, “Generalized Krein-von Neumann extensions and associated operator models”, Acta Sci. Math. (Szeged), 64, (1998), 627–655.

[15] S. Hassi, H. Langer and H.S.V. de Snoo, “Selfadjoint extensions for a class of symmetric operators with defect numbers (1,1)”, 15th OT Conference Proceedings, 1995, pp. 115–145.

[16] S. Hassi and A. Luger, “Generalized zeros and poles of $N_\kappa$-functions: on the underlying spectral structure”, Methods Funct. Anal. Topology, 12, no. 2 (2006), 131–150.

[17] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “One-dimensional perturbations, asymptotic expansions, and spectral gaps”, Oper. Theory Adv. Appl., 188 (2008), 149–173.

[18] S. Hassi and H.S.V. de Snoo, “One-dimensional graph perturbations of selfadjoint relations”, Ann. Acad. Sci. Fenn. A.I. Math., 22 (1997), 123–164.

[19] I.S. Kac, “On integral representations of analytic functions mapping the upper half-plane onto a part of itself”, Uspekhi Mat. Nauk, 11 (1956), 139–144.

[20] I.S. Kac and M.G. Krein, “$\Re$-functions - analytic functions mapping the upper halfplane into itself”, Amer. Math. Soc. Transl., 103 (1974), 1–18.

[21] M. Kaltenbäck, H. Winkler and H. Woracek, “Generalized Nevanlinna functions with essentially positive spectrum”, J. Operator Theory, 55 (2006), 17–48.

[22] M.G. Krein and H. Langer, “The defect subspaces and generalized resolvents of Hermitian operator in Pontryagin space”, Funkts. Anal. i Prilozhen, 5, no. 2 (1971), 59–71; ibid. 5, no. 3 (1971), 54–69 (Russian). English translation: Funkt. Anal. Appl. 5 (1971), 136–146; ibid. 5 (1971), 217–228.

[23] M.G. Krein and H. Langer, “Über die Q-Funktion eines π-Hermiteschen Operators im Raume $\Pi_\kappa$ (German)”, Acta Sci. Math. (Szeged), 34 (1973), 191–230.

[24] M.G. Krein and H. Langer, “Über einige Fortsetzungsprobleme, die eng mit der Theorie hermitescher Operatoren im Raume $\Pi_\kappa$ zusammenhängen. I. Einige Funktionsklassen und ihre Darstellungen”, Math. Nachr., 77 (1977), 187–236.

[25] M.G. Krein and A.A. Nudelman, The Markov moment problem and extremal problems, “Nauka”, Moscow, 1973 (Russian). English translation: Translation of Mathematical monographs AMS, vol. 40, 1977.

[26] A.N. Kochubei, “Characteristic functions of symmetric operators and their extensions”, (Russian) Izv. Akad. Nauk Armyan. SSR Ser. Mat. 15, no. 3 (1980), 219-232, 247.

[27] H. Langer and H. Winkler, “Direct and inverse spectral problems for generalized strings”, Integr. equ. oper. theory, 30 (1998), 409–431.

[28] B. van der Waerden, Modern Algebra, Fredrick Ungar, NY, 1964.

Department of Mathematics and Statistics, University of Vaasa, P.O. Box 700, FI-65101 Vaasa, Finland

E-mail address: sha@uwasa.fi; rwietsma@uwasa.fi