Angular correlation between proton and neutron rotors

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Abstract. A brief review is given on the controversy and its solution about the fact that the angular momentum vector of protons and that of neutrons in well-deformed nuclei at low total angular momenta have a strong correlation that they are oriented in opposite directions. In a simple two-rotor model in 2-dimensional space, this fact is explained as originating from the quantum mechanical uncertainty relation between the angle and the angular momentum for the relative rotation of the two rotors. As the second topic, a more realistic model consisting of two triaxial rotors in 3-dimensional space coupled with a QQ interaction is employed to investigate a possible shears-band-like collective rotation predicted by T. Otsuka, in which the angle at which the angular momentum of protons and that of neutrons intersect changes continuously from 180° at spin zero toward 0° at high spins within the same rotational band. The probability distributions of the angle between the two angular momenta and the angle between the longest principal axes of two rotors are calculated to examine the participation of the scissors mode in the evolution of the ground rotational band versus spin.

1. Introduction
We have recently given careful consideration to the meaning of the antiparallel proton and neutron angular momenta at low spins [1]. This fact that the total angular momenta of protons and that of neutrons are oriented in the opposite directions has first been demonstrated by Otsuka et al. for the angular-momentum projected Nilsson wave functions[2, 3, 4]. A question was raised immediately [5] that it might be incorrect because it would mean an unphysical free contra-rotation of proton and neutron ellipsoids as depicted in Fig. 1, while only small-amplitude oscillations in the scissors mode as depicted in Fig. 2 are possible from a physical point of view. Actually, Ref. [2, 3, 4] has never described this opposite rotation as a free contra rotation. However, it would be indeed a plausible picture of this fact from a classical point of view.

Ref.[5] argued that the removal of the spurious center-of-mass motion could make the angular momenta of protons and neutrons parallel. Afterward, two papers [6, 7] have cited Ref.[5] as solving the controversy. In Ref.[1], Otsuka and we have shown that the elimination of the spurious center-of-mass motion does not substantially change the situation in a classical treatment of independent nucleon motions. By using a simple two-rotor model in 2-dimensional space, we have also given a correct interpretation of the opposite-direction correlation of the two angular momenta, as is discussed in Sec. 2.

The principal intention of Ref.[2] was to suggest the possibility that collective rotational excitations are accompanied with a narrowing of the angle between the proton and neutron angular momenta. Namely, along a rotational band, the angle changes from 180° at spin zero
Figure 1. The free contra-rotation. The blue and red ovals represent neutrons and protons, respectively.

Figure 2. The scissors mode oscillation.

toward $0^\circ$ at high spins. It was further suggested that this change is accompanied with a reduction of the overlap between the proton and neutron ellipsoids by means of the scissors-mode degree of freedom at intermediate spins. In order to investigate such a novel collective rotational excitation mechanism, we employ in sec. 3 a model in which two triaxial rotors in 3-dimensional space are coupled through an attractive quadrupole-quadrupole interaction. We calculate not only the expectation values but also the probability distribution of the angle between the angular momenta and the angle between the longest principal axes of rotors. These two angles have different kinds of information because of the quantum mechanical uncertainty in the directions of the angular momenta relative to the principal axes of the ellipsoids, and also owing to dynamically correlated rotational motions of the two rotors.

2. Analysis with a two-rotor model in 2-dimensional space

In Ref. [1] we have employed a model consisting of two rotors in 2-dimensional space to discuss the implication of the oppositely directed angular momenta of protons and neutrons. The Hamiltonian of the model is written as,

$$H = -\frac{\hbar^2}{2I_p} \frac{\partial^2}{\partial \varphi_p^2} - \frac{\hbar^2}{2I_n} \frac{\partial^2}{\partial \varphi_n^2} + V(\varphi_p - \varphi_n),$$

where $\varphi_p$ and $\varphi_n$ are the azimuths of proton and neutron rotors, respectively, as depicted in Fig. 3, $I_p$ and $I_n$ are the moment of inertia of protons and that of neutrons, respectively, and $V(\varphi_p - \varphi_n)$ is an attractive potential which favors alignments of the long axes of the proton and the neutron ellipses.

A wave-packet wavefunction of the form,

$$f(\varphi) = \sum_{m=-\infty}^{\infty} f_m e^{im\varphi}, \quad f_m = Ne^{-a^2m^2}$$

represents a state of each ellipse whose long axis is aligned to the space-fixed $x$-axis with a quantum mechanical fluctuation of $\sqrt{\langle \varphi'^2 \rangle} \approx a$. The products of such states, $f(\varphi_p) f(\varphi_n)$, can be viewed as an intrinsic state of a deformed nucleus. In order to gain energy from the interaction term $V(\varphi_p - \varphi_n)$, the size of the fluctuation ($a$) in the angles $\varphi_p$ and $\varphi_n$ should be decreased, which inevitably increases the width of the spreading of the angular momenta ($m$) of each rotor since $\sqrt{\langle m^2 \rangle} \approx (2a)^{-1}$. One can construct a state of given total angular momentum $M$ by means of the angular momentum projection method as

$$\Psi = \hat{P}_M f(\varphi_p) f(\varphi_n) = \sum_{m_p=-\infty}^{\infty} \sum_{m_n=-\infty}^{\infty} \delta_{m_p+m_n,M} f_{m_p} f_{m_n} e^{im_p\varphi_p} e^{im_n\varphi_n}. $$

As depicted in Fig. 4, many combinations of angular momenta $(m_p, m_n)$ in the line $m_p + m_n = M$ are mixed in the state of Eq. (3). When $M$ is large, $m_p$, $m_n$, and $M$ have the same sign.
However, when $M$ is smaller than the spreading $(2a)^{-1}$, $m_p$ and $m_n$ most often have different signs. Therefore, as far as this 2-dimensional model of deformed nuclei concerns, the opposite senses of the rotations of the proton and neutron rotors do not mean an unphysical free contra rotation, but a close binding of the two ellipses.

3. Analysis of a two-rotor model in 3-dimensional space

In this paper, we extend the two-rotor model to the 3-dimensional space. We assume the following Hamiltonian,

$$H = H_1 + H_2 - \kappa Q^2, \quad H_q = \sum_{i=1}^{3} \frac{\hbar^2}{2T_i} J_{q,i}^2, \quad T_i = 4B_1\beta^2 \sin^2\left(\frac{2\pi}{3}i\right),$$

where $H_q$ is the Hamiltonian for a triaxial rotor (the Davydov model\cite{8}) with the irrotational-flow moment of inertia $T_i$. The index $q (=1,2)$ is introduced to distinguish the two rotors of this model. (It may be labeled as $q=p,n$ as in the 2-dimensional model).

The two rotors are coupled through a quadrupole-quadrupole interaction. ($Q_\mu = \alpha_2\mu$). In the principal-axes frame, $\alpha_{2,0} = \beta \cos \gamma$, $\alpha_{2,\pm 1} = 0$, $\alpha_{2,\pm 2} = \frac{1}{\sqrt{2}} \beta \sin \gamma$, where $\beta$ and $\gamma$ are the quadrupole deformation parameters.)

We express the eigenstates of the coupled system in the laboratory frame as,

$$|IM\alpha\rangle = \sum_{J_1K_1J_2K_2} C_{J_1K_1J_2K_2}^{\alpha} \sum_{M_1M_2} \langle J_1M_1J_2M_2|IM\rangle |J_1M_1K_1\rangle_1 |J_2M_2K_2\rangle_2.$$  

Wavefunction of the Euler angles $\Omega = (\alpha, \beta, \gamma)$ of each rotor’s orientation is given by

$$\langle \Omega |JK\rangle = \sqrt{\frac{2J+1}{16\pi^2}} \cdot \frac{1}{\sqrt{1+\delta_{K0}}} \left[ D_{M,K}^J(\alpha, \beta, \gamma) + (-1)^J D_{M,-K}^J(\alpha, \beta, \gamma) \right]$$

with $K = 0, 2, 4, \cdots \leq J$, because the signature quantum numbers are naturally restricted to $(r_1, r_2, r_3) = (+, +, +)$. 

**Figure 3.** Two rotor model in 2-dimensional space.

**Figure 4.** Mixture of $(m_p, m_n)$ components in eigenstates of a system of two rotors in 2-dimensional space.
We use two identical rotors, each of which has \( \gamma = 8^\circ \), which determines two important eigenenergies of \( \epsilon_{z^+} = 1.04 \) and \( \epsilon_{z^+} = 26.16 \) in units of \( \hbar/\beta \). The eigenstates of the total Hamiltonian are obtained by matrix diagonalizations with the strength of the QQ coupling of \( \kappa_\beta^2 = 1000 \times 0.3^2 = 90 \) in units of \( \hbar^2/\beta^2 \). The basis includes \( J_1 \leq 40 \) and all (practically) possible values of \( K_2 \). Obtained eigenenergies of the coupled system are shown in Fig. 5. Two important excitation energies are \( E_{2_+}^+ = 0.52 \simeq \frac{1}{2} \epsilon_{z^+} \), and \( E_{1_+}^+ = 13.14 \) (scissors mode) in units of \( \hbar^2/\beta^2 \).

The angle between the angular momenta of two rotors, \( J_1 \) and \( J_2 \), is naturally defined by

\[
\hat{\theta}_{JJ} = \arccos \frac{\hat{J}_1 \cdot \hat{J}_2}{\sqrt{\hat{J}_1^2 \hat{J}_2^2}},
\]

where \( \hat{J}_1 \) and \( \hat{J}_2 \) are given as

\[
\hat{\theta}_{JJ} |J_1 \alpha_1 J_2 \alpha_2; IM \rangle = \arccos \frac{I(I+1) - J_1 (J_1 + 1) - J_2 (J_2 + 1)}{\sqrt{J_1(J_1+1)J_2(J_2+1)}} |J_1 \alpha_1 J_2 \alpha_2; IM \rangle,
\]

for \( J_1 \neq 0, J_2 \neq 0 \). The average value of \( \theta_{JJ} \) can be defined in a few different ways, e.g.,

\[
\hat{\theta}_{JJ}^{(1)} = \left\langle \arccos \frac{\hat{J}_1 \cdot \hat{J}_2}{\sqrt{\hat{J}_1^2 \hat{J}_2^2}} \right\rangle', \quad \hat{\theta}_{JJ}^{(2)} = \arccos \left\langle \frac{\hat{J}_1 \cdot \hat{J}_2}{\sqrt{\hat{J}_1^2 \hat{J}_2^2}} \right\rangle', \quad \hat{\theta}_{JJ}^{(3)} = \arccos \left\langle \frac{\hat{J}_1 \cdot \hat{J}_2}{\sqrt{\hat{J}_1^2 \hat{J}_2^2}} \right\rangle,
\]

where \( \langle \cdots \rangle' \) are calculated after projecting out components having \( J_1 = 0 \) and/or \( J_2 = 0 \) and renormalizing the remaining part of the state vector. As shown in Fig. 6, \( \hat{\theta}_{JJ}^{(2)} \) is larger than \( \hat{\theta}_{JJ}^{(1)} \) (\( \approx \hat{\theta}_{JJ}^{(2)} \)) by \( \sim 0.3 \) radians. However, the decreasing behavior is common to all the definitions.

A way to understand the mechanism which gives rise to this behavior of \( \theta_{JJ} \) is to examine the probability distribution of the combination of angular momentum components \( J_1, J_2 \) because \( \theta_{JJ} \) is a simple function of \( (J_1, J_2) \) as shown in Fig. 7. The distributions at \( I=10 \) and 40 are shown in Fig. 8. One can observe the following points.

i) The spreading in \( J_1 - J_2 \) is only weakly dependent on \( I \). (The tail is longer for larger \( I \) because of a trivial reason of the triangle condition.) By denoting the sum of two rotor’s energies by \( E(J_1, J_2) \), one can relate this approximate independence to the fact that

\[
\Delta E = E \left( J_1 = \frac{1}{2} I + \Delta J, J_2 = \frac{1}{2} I - \Delta J \right) - E(J_1 = J_2 = \frac{1}{2} I) = \frac{\hbar^2}{27} (\Delta J)^2
\]
fact that

ii) The width of the spreading in $J$ is an increasing function of $I$

is independent of $I$.

The tight binding of the two rotors is created by mixing both $J_1 - J_2$ and $J_1 + J_2$ at low $I$ and by mixing mainly $J_1 - J_2$ at high $I$. The pattern of mixture at low spins appears to be essentially different from that of the 2-dimensional case (Fig. 4). It may be related to a possible novel excitation modes such as the collective shears band[2].

Probability distributions of $\theta_{JJ}$ are shown in Fig.9. Instead of plotting original discrete-line spectra, we have replaced the delta functions representing the lines of the spectrum with a normalized Gaussian of width 3°. Around $I$=6-10 in the figure, there are sizeable components of $\theta_{JJ} \approx 90°$. We are interested in what takes place when the angular momentum vectors intersect at angles around $90°$. Is it only for the angular momentum vectors? Or, do the principal axes of rotors also intersect at large angles in spite of the attractive QQ interaction?

The angle between the 3rd (longest) axis of rotor-1 with its orientation specified by the Euler angles $\alpha_1, \beta_1, \gamma_1$ and the 3rd axis of rotor-2 with $\alpha_2, \beta_2, \gamma_2$ is given by

$$\theta_{33} = \arccos \left[ \sin \beta_1 \sin \beta_2 \cos(\alpha_1 - \alpha_2) + \cos \beta_1 \cos \beta_2 \right]. \quad (12)$$

Its probability distribution, $\rho(\theta) = \langle \text{IM} \alpha | \delta(\theta_{33} - \theta) | \text{IM} \alpha \rangle$, can be calculated in principle via $\langle \text{IM} \alpha | P_i(\cos \theta_{33}) | \text{IM} \alpha \rangle$ where $P_i$ is the Legendre polynomial. Instead, we have roughly estimated it by the Monte Carlo integral in which two sets of Euler angles are sampled randomly and uniformly over $\alpha_i, \gamma_i \in [0, 2\pi]$ and $\cos \beta_i \in [-1, 1]$ ($i=1, 2$).

Fig.10 shows the probability distribution of $\theta_{33}$ for the yrast states at $I=0$ and $I=40$. One can see that the distribution of this angle is practically independent of $I$. Small discrepancy can be ascribed to the error due to the Monte-Carlo method.

The Hamiltonian given by Eq. (1) for 2-dimensional rotors is formally identical to the Hamiltonian for translational motions of two interacting objects. Hence, there holds the “Galilean” invariance so that the total and the relative rotational motions of two rotors are exactly decoupled and the probability distribution of the relative angle $\varphi_R - \varphi_n$ is independent of the total angular momentum $M$. From Fig.10, there probably also holds the “Galilean”
invariance for two rotors in 3-dimensional space. It is a plausible conjecture within the classical mechanics, in which rotational motions of rigid bodies having a space fixed pivot are not affected by centrifugal and Coriolis forces because those forces are perpendicular to the velocity of the material points composing the rigid bodies.

However, this result does not exclude the possibility of the collective shears band [2] because in actual nuclei, i) the relative direction of angular momenta must affect the internal structure of the nucleus so that it cannot be represented by inert rotors, ii) the actual space is finite unlike the ideal rotor which has no band termination. We plan to include those effects in our model to obtain new insights into the dynamics of the relative orientations between partial angular momenta and the principal axes of deformations in nuclear collective rotations.

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