On the relationship between quantum entanglement and classical synchronization in open systems

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Abstract

We propose a simple model of classical open system consisting of two subsystems all stationary states of which correspond to phase synchronization between the subsystems. The model is generalized to quantum systems in a finite-dimensional Hilbert space. The analysis of the simplest two qubit version of the quantum model shows that all its stationary states are nonseparable.

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Entanglement of quantum states is a notable resource of quantum information science and plays a key role in its advanced applications such as superdense coding, quantum teleportation and quantum cryptography. The important question of whether there is any analogue of the entanglement in classical systems remains open. This problem has been considered by different approaches, but all of them focused on closed Hamiltonian systems only and tried to find a connection between the entanglement of ground state of the system and peculiarities in its phase portrait under the variation of parameters.

The goal of the present paper is to establish a relationship between the entanglement of stationary states of an open quantum system and such well-known classical nonlinear phenomenon as synchronization. As a first step on this way, we consider a simple model of phase synchronization of two subsystems which can be generalized to a quantum case. The analysis of the simplest two qubit version of the model leads to the main result of the paper: All stationary states of the system turn out to be entangled. Although this result is obtained only for a particular model and has not been generally proved, it suggests a new promising perspective for better understanding of mixed entangled states properties.

Consider a simple dynamical model of open system with three variables $l_x, l_y,$ and $l_z$ whose time evolution is governed by the following coupled equations

\[
\frac{dl_x}{dt} = 2 \left( l_y^2 + l_z^2 \right), \\
\frac{dl_y}{dt} = -2l_x l_y, \\
\frac{dl_z}{dt} = -2l_x l_z.
\]

This system may be considered as the model for phase synchronization of two subsystems for the following reasons. First, remind that synchronization in nonlinear dynamics is a phenomenon of rhythms adjustment of oscillating systems due to weak interaction between them. Here we consider only the simplest case of the phase synchronization when the phase difference of two subsystems vanishes with time. Using phases of subsystems $\varphi_1, \varphi_2$ and their amplitudes $r_1, r_2$ as dynamical variables, we can write the complete system of equations in the form: $d\varphi_i/dt = f_i(\varphi_1, \varphi_2, r_1, r_2)$ and $dr_i/dt = g_i(\varphi_1, \varphi_2, r_1, r_2)$, where $i = 1, 2$ and $f_i$ and $g_i$ are nonlinear functions providing the synchronization. Then, this system of four equations can be rewritten as a system of two equations by introducing complex variables $z_1 = r_1 \exp i\varphi_1$
and \( z_2 = r_2 \exp i\varphi_2 \). Finally, with new variables \( l_x, l_y, l_z \) connected to \( z_1, z_2 \) by the relations
\[
\begin{align*}
l_x & = \frac{z_1^* z_2 + z_1 z_2^*}{2}, \quad l_y = \frac{i(z_1 z_2^* - z_1^* z_2)}{2}, \quad l_z = \frac{|z_1|^2 - |z_2|^2}{2},
\end{align*}
\]
we eventually arrive at a system of nonlinear differential equations for \( l_x, l_y, \) and \( l_z \). Note that the transformation defined by Eq. (2) is a classic analogue of the Schwinger transformation in quantum mechanics which used for the description of two Bose oscillators in terms of angular momentum components. Since \( l_y \) can be expressed as \( l_y = r_1 r_2 \sin(\varphi_2 - \varphi_1) \) we assume that vanishing of \( l_y \) is the condition of the phase synchronization between the subsystems.

Returning to system (1), we note that it has two integrals of motion: \( l^2 = l_x^2 + l_y^2 + l_z^2 \) and \( k = l_y/l_z \). A simple analysis shows that under arbitrary value of \( l^2 \) the solution of Eq. (1) tends to the final state \( l_y = l_z = 0 \) and \( l_x = l \) Thus the presence of phase synchronization in the classical model represented by Eq. (1) is proved.

Now let us discuss the question about possible quantum analogues of model (1). The proper way to quantize Eq. (1) (at least in semiclassical approximation) has been proposed by the author in Ref. [6]. In the present case this method of quantization can be formulated as follows. First, we have to represent the classic equations for components \( l_x, l_y, l_z \) in the form allowing quantization:
\[
\frac{dl}{dt} = -\left(1 \times \frac{\partial H_0}{dl}\right) + 1 \times \left(iR\frac{\partial R^*}{dl} - iR^*\frac{\partial R}{dl}\right),
\]
where \( H_0 \) is real and \( R \) and \( R^* \) are complex functions of \( l_x, l_y, l_z \) (star means complex conjugation). Then, following Ref. [6], a self-consistent quantum version of Eq. (3) can be obtained if one writes down the Lindblad equation for density matrix of the system:
\[
\frac{d\hat{\rho}}{dt} = -i \left[\hat{H}_0, \hat{\rho}\right] + \left[\hat{R}\hat{\rho}, \hat{R}^+\right] + \left[\hat{R}, \hat{\rho}\hat{R}^+\right],
\]
where \( \hat{H}_0 \) and \( \hat{R}, \hat{R}^+ \) are operator analogues of functions \( H_0, R, R^* \) in Eq. (3).

Certainly the question regarding the order of operators \( l_i \) arrangement in \( H_0 \) and \( R \) exists because of their noncommutativity, but in semiclassical approximation this problem does not arise. It is easy to check by direct verification that system (1) can be represented in the form required for quantization of Eq. (3) by substitution \( H = f(l^2), R = l_z - il_y \) (the first term in r.h.s. of Eq. (3) vanishes in this case). Thus we see that possible quantum analogues of classical model of synchronization (1) can be described by the Lindblad equation
\[
\frac{d\hat{\rho}}{dt} = \left[\hat{R}\hat{\rho}, \hat{R}^+\right] + \left[\hat{R}, \hat{\rho}\hat{R}^+\right],
\]
where  $\hat{R} = \hat{l}_z - i\hat{l}_y$.

It should be noted that using the evolution equation for the density matrix, given Eq. (5), we can write equations of motion for the average value $\langle A \rangle$ of an arbitrary observable $A$. For example, if one takes $l_x$ as $A$, the result is

$$\frac{d}{dt} \langle \hat{l}_x \rangle = \langle [\hat{l}_x, \hat{R}] \rangle + \text{c.c.} = \langle \left( \hat{l}_z + i\hat{l}_y \right) \left( \hat{l}_z - i\hat{l}_y \right) \rangle + \text{c.c.}$$  \hspace{1cm} (6)$$

We see that in semiclassical approximation Eq. (6) coincides with the first from Eqs. (1) as it should be. Our next task is to find stationary solutions of Eq. (5) and to evaluate their entanglement. The simplest situation in which this can be done exactly is a two qubit realization of Eq. (4). In this case operators $l_x, l_y, l_z$ have the following representation:

$$l_x = \frac{1}{2} (\sigma_x \otimes 1 + 1 \otimes \sigma_x), \quad l_y = \frac{1}{2} (\sigma_y \otimes 1 + 1 \otimes \sigma_y), \quad l_z = \frac{1}{2} (\sigma_z \otimes 1 + 1 \otimes \sigma_z),$$

where $\sigma_x, \sigma_y, \sigma_z$ are ordinary Pauli matrices. Operator $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$ has the form $\hat{l}^2 = \frac{1}{2} (3 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$ and commutes with all $\hat{l}_i$. Operator $\hat{R} = \hat{l}_z - i\hat{l}_y$ has the following matrix representation:

$$\hat{R} = \frac{1}{2} \begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -2 \end{pmatrix}.$$  \hspace{1cm} (7)$$

With the help of Eq. (7) we can find that equation $\hat{R} |\psi\rangle = 0$ has two linearly independent solutions:

$$|\Psi_1\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad |\Psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$  

Therefore in the two qubit case the general stationary solution of Eq. (5) can be written as

$$\hat{\rho}_{st} = a |\Psi_1\rangle \langle \Psi_1| + b |\Psi_2\rangle \langle \Psi_2| + c |\Psi_1\rangle \langle \Psi_2| + \text{h.c.}$$  \hspace{1cm} (8)$$

Coefficients $a, b, c$ in Eq. (8) must satisfy two conditions: $a + b = 1$ and $ab \geq c^2$ which correspond to normalization and positivity of matrix $\hat{\rho}_{st}$. Using Eq. (8) we can write the
desired expression for $\hat{\rho}_{st}$ as follows

$$
\begin{align*}
\hat{\rho}_{st} = \left(\begin{array}{cccc}
\frac{a}{4} & \frac{a + c}{4 + 2\sqrt{2}} & \frac{a - c}{4 - 2\sqrt{2}} & \frac{a}{4} \\
\frac{a + c}{4 + 2\sqrt{2}} & \frac{a + b}{4 + 2\sqrt{2}} & \frac{a - b}{4 - 2\sqrt{2}} & \frac{a + c}{4} \\
\frac{a}{4} & \frac{a - c}{4 - 2\sqrt{2}} & \frac{a + b}{4 + 2\sqrt{2}} & \frac{a - c}{4} \\
\frac{a - c}{4} & \frac{a + c}{4} & \frac{a}{4} & \frac{a + c}{4}
\end{array}\right).
\end{align*}
\tag{9}
$$

It is easy to check that $\hat{\rho}_{st}$ has two null eigenvalues and two positive ones, $\lambda_1$ and $\lambda_2$, which satisfy to the following equation: $\lambda^2 - \lambda + ab - c^2 = 0$. Let us find the values of coefficients $a, b, c$, for which density matrix $\hat{\rho}_{st}(a, b, c)$ corresponds to entangled states of two qubits. A simple way to accomplish this is to invoke the Peres criterium [7]. As well known [8], in the two qubit case this criterium is a necessary and sufficient condition of the mixed states separability. According to it density matrix $\hat{\rho}$ is separable if matrix $\hat{\rho}_{PT}$ obtained from $\hat{\rho}$ by operation of partial transposition (which corresponds to permutations of indices of one of the subsystems only) is nonnegative. Using Eq. (9) for $\hat{\rho}_{st}$ we can write $(\hat{\rho}_{st})_{PT}$ as

$$
(\hat{\rho}_{st})_{PT} = \left(\begin{array}{cccc}
\frac{a}{4} & \frac{a + c}{4 + 2\sqrt{2}} & \frac{a - c}{4 - 2\sqrt{2}} & \frac{a - b}{4} \\
\frac{a + c}{4 + 2\sqrt{2}} & \frac{a + b}{4 + 2\sqrt{2}} & \frac{a - b}{4 - 2\sqrt{2}} & \frac{a + c}{4} \\
\frac{a}{4} & \frac{a - c}{4 - 2\sqrt{2}} & \frac{a + b}{4 + 2\sqrt{2}} & \frac{a - c}{4} \\
\frac{a - c}{4} & \frac{a + c}{4} & \frac{a}{4} & \frac{a + c}{4}
\end{array}\right).
\tag{10}
$$

It is easy to check that $(\hat{\rho}_{st})_{PT}$ has eigenvector

$$
|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},
$$

with eigenvalue $\lambda_1 = b/2$. The remaining eigenvalues are found from characteristic equation

$$
\begin{align*}
\begin{vmatrix}
\frac{a - b}{2} - \lambda & \frac{a + c}{4 + 2\sqrt{2}} & \frac{a - c}{4 - 2\sqrt{2}} \\
\frac{a + c}{4 + 2\sqrt{2}} & \frac{a}{4} & \frac{a + b}{4} - \lambda \\
\frac{a}{4} & \frac{a - c}{4} & \frac{a}{4} + \frac{b}{2} - \lambda
\end{vmatrix} = 0.
\tag{11}
\end{align*}
$$
Computing this determinant explicitly we get the following cubic equation for $\lambda_1, \lambda_2,$ and $\lambda_3$

$$\lambda^3 - \left( a + b \frac{b}{2} \right) \lambda^2 - \left( \frac{b^2}{4} + c^2 - \frac{ab}{2} \right) \lambda + \frac{b^3}{8} = 0. \quad (12)$$

We see that two roots of Eq. (12) are positive but the third one is negative (when $b > 0$). Thus for all $b > 0$ density matrix $\hat{\rho}_{st}$ corresponds to nonseparable (entangled) state. This result obtained for the simple model of synchronization together with some qualitative reasons suggests that a relationship between the phase synchronization in classical open system and the entanglement in its quantum analogue exists in more general situations as well.

In conclusion, we want to examine our original model (1) from somewhat different point of view. Let us consider two functions of state $H = (l_x^2 + l_y^2 + l_z^2)/2$ and $S = 2l_x$. As follows from Eq. (1) the evolution of $H$ and $S$ satisfies two general conditions:

$$\frac{dH}{dt} = 0 \quad \text{and} \quad \frac{dS}{dt} \geq 0. \quad (13)$$

It should be noted that system (1) can be represented as:

$$\frac{dl_i}{dt} = \varepsilon_{ikl} \frac{\partial H}{\partial l_k} A_l, \quad (14)$$

where $A_l = \varepsilon_{lmn} \frac{\partial S}{\partial l_m} \frac{\partial H}{\partial l_n}$ and $\varepsilon_{lmn}$ is the antisymmetric tensor of Levi-Civita.

As shown by the author [9], two conditions (13) (in the general case of $n$ variables $x_1, \ldots, x_n$ describing the evolution of the system) define a class of nonhamiltonian systems called quasithermodynamic with interesting dynamical and statistical properties [9]. But the possibility of quantum quasithermodynamic systems existence was not discussed in Ref. [9]. The analysis presented above shows, in particular, that a quantum analogue of system (1) exists and it is a quasithermodynamical system as well. One can see it directly from the fact that average values of $\hat{H} = \hat{l}_x^2/2$ and $\hat{S} = 2\hat{l}_x$ obviously satisfy to conditions (13). Sure, more detail analysis of quasithermodynamic quantum systems is required, which is the subject of future work.

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[1] M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000).

[2] A.P. Hines, R.H. McKenzie, and G.J. Milburn, Phys. Rev. A **71**, 042303 (2005).
[3] C. Emary, N. Lambert, and T. Brandes, Phys. Rev. A 71, 062302 (2005).

[4] M.S. Santhanam, V.B. Sheorey, A. Lakshminarayan, Phys. Rev. E 77, 026213 (2008).

[5] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization. An universal concept in nonlinear sciences (Cambridge University Press, Cambridge, 2002).

[6] E.D. Vol, Phys. Rev. A 73, 062113 (2006).

[7] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).

[8] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).

[9] E.D. Vol, physics.class-ph 0812.3738.