CRITICAL HYSTERESIS

Sourendu Gupta
HLRZ, c/o KFA Jülich, D-5170 Jülich, Germany.

ABSTRACT

Hysteresis is observed at second order phase transitions. Universal scaling formulæ for the areas of hysteresis loops are written down. Critical exponents are defined, and related to other exponents for static and dynamic critical phenomena. These relations are verified with Langevin dynamics in both the critical and tricritical mean-field models. A finite-size scaling relation is tested in the two-dimensional Ising model with heat-bath dynamics.
Hysteresis has been studied quantitatively for one hundred years now [1]. This phenomenon is well-known at first-order phase transitions. We find that it can also be observed at a second-order transition. Such critical hysteresis is characterised by certain scaling laws, i.e., universal exponents and scaling functions. The exponents are related to well-known static and dynamical critical exponents. Critical hysteresis is thus seen to be entirely consistent with the phenomenon of critical slowing down. Furthermore, when the critical point is a limit of a line of first order phase transitions, the scaling function for the hysteresis loop area is a smooth limit of those obtained along the line of transitions.

The characteristic observation in hysteresis is that when a system is subjected to a cyclic external field, $h$, its response, $m$, lags behind the field. Thus the curve $m(h)$ forms a loop. Although many different kinds of systems show hysteretic behaviour, in this letter we shall use a language appropriate to magnetic systems. Thus, the external field will be a periodic magnetic field, with period $T = 2\pi/\omega$, and an amplitude $h_0$. The response of the system will be characterised by an induced time-dependent magnetisation. All the statements we shall make have their analogues for other kinds of phase transitions, provided the external field is linearly coupled to the order parameter, which forms the response.

Of interest in most experiments is the area of the hysteresis loop, defined as

$$A(\omega, h_0) = \frac{1}{4\pi} \oint m(t) dh,$$

where the integral is over one period of the applied field. This is the energy dissipated by the system. Also of interest are the shapes of the hysteresis loops. There have been many early studies of the dependence of $A$ on $\omega$ and $h_0$ [2]. Recent theoretical work has focussed on the low-temperature phase of $O(N)$ models[2,3]. In these models power laws have been derived for the scaling of hysteresis loop areas with $h_0$ and $\omega$.

The essential physics of hysteresis is the lag of the response behind the force. This is due to finite autocorrelation times. Whenever the inverse frequency of the external force is comparable to an appropriate autocorrelation time of the system, one expects to see hysteresis. Due to the phenomenon of critical slowing down, one must therefore see hysteresis also at a second-order phase transition. We shall first outline the dynamics that we use. Next, assuming the existence of this phenomenon, we shall present scaling laws for the loop area and define critical exponents. Finally we shall present some examples of critical hysteresis. The scaling laws are seen to be satisfied with appropriate exponents. Further details will be presented elsewhere [4].

In this letter we shall take the dynamics to be given by a Langevin equation without conservation laws. We shall consider real one-component fields $\phi_i$ on the
sites, $i$, of a $d$-dimensional lattice, whose interactions are specified by a Hamiltonian $H$. In terms of a Gaussian white noise $\nu_i(t)$, with width $\Gamma$, the Langevin equation can be written as

$$\frac{d\phi_i}{dt} = \Gamma \frac{\delta H}{\delta \phi_i} + \nu_i(t).$$

(2)

Note that this dynamics has no conserved quantities. We shall restrict our attention to the zero mode or magnetisation, $\phi = \sum_i \phi_i$. The argument is that this Fourier mode has the largest autocorrelation time, and therefore the long-time behaviour of the system is due to this. It is known [5,6] that all correlations functions arising from this one-mode dynamics are exactly equal to those obtained from the Euclidean quantum mechanics specified by the Euclidean action

$$S = \frac{1}{\hbar} \int dt \int \left\{ \frac{1}{2} \partial_t \phi^2 + \frac{1}{2} W^2 - \frac{\hbar}{2} W' \right\},$$

(3)

where $W$ stands for $\Gamma \delta H/\delta \phi$, and the prime denotes a derivative with respect to $\phi$. The ‘Planck’s constant’, $\hbar$, is equal to $\Gamma/L^d$. The infinite volume limit of the theory is therefore obtained as the ‘classical limit’. This is investigated by the Euler-Lagrange equations arising from the Euclidean action above, and reduces to the Langevin equation with the noise term dropped. This method has previously been used [6] to obtain the instanton solution in the dynamics.

For technical reasons, we shall assume that the amplitude of the applied magnetic field is small and linear response theory is applicable. A fluctuation-dissipation theorem [7] relates the energy dissipation to the the absorptive part of the complex susceptibility, $\chi''(\omega)$, through the relation

$$A(\omega, h_0) = \frac{1}{2} h_0^2 \omega \chi''(\omega).$$

(4)

Along with the definition

$$\chi''(\omega) = \text{Im} \int_0^\infty dt e^{-i\omega t} \langle \phi(t)\phi(0) \rangle,$$

(5)

where the angular brackets denote averaging over the equilibrium ensemble, we now have the technical machinery to compute quantities of interest. Note that the average over the equilibrium ensemble in the formula above is a reflection of the fact that we are working in linear response theory.

As an example, consider the most trivial of models—a Gaussian integral. The Hamiltonian is

$$H = \frac{1}{2} \omega_0^2 \phi^2.$$

(6)
By a variety of methods one can construct the autocorrelation function in Langevin dynamics
\[ C(t) = e^{-t/\tau}, \quad (\tau = 1/\omega_0^2). \] (7)

This yields hysteresis loop areas
\[ A(\omega, h_0) = \frac{1}{2} \frac{h_0^2 \omega \tau^2}{1 + \omega^2 \tau^2}. \] (8)

The claim made earlier, that hysteresis vanishes only when the autocorrelation time does, is explicitly demonstrated in this model. Note also the 1/\omega fall at large \omega. This behaviour has been observed, for example, in O(N) models in dimensions \( d > 1 \) [3]. It will be seen again in critical hysteresis. Such behaviour is obtained whenever a pole in the spectral density determines the behaviour of A at large frequencies.

Now we write a scaling relation at second-order phase transitions for the hysteresis loop area as a function of \( \omega \) and \( h_0 \). The necessary presence of an external magnetic field implies that we can tune \( \theta = (T - T_c)/T_c \) to zero, and let \( h_0 \) control the approach to criticality. As a result, exponents corresponding to \( \theta \) do not appear in the scaling relations. We write
\[ A(\omega, h_0, \theta) = h_0^\lambda F(\omega h_0^\mu). \] (9)

In the limit \( \omega \to 0 \), this must yield the usual relation between the magnetic field and the magnetisation. This implies
\[ \lambda = 1 + \frac{1}{\delta}. \] (10)

Time-dependent phenomena are characterised by the scaling of the field autocorrelation time \( \tau \) with increasing correlation length \( \xi \). The relation between these two quantities, \( \tau \sim \xi^z \), defines a dynamical critical exponent \( z \). As one approaches the critical point by tuning \( h_0 \), the scaling of \( \omega \) required to keep the hysteresis loop area fixed must be given by the relation between \( \tau \) and \( \xi \) above. This gives the remaining critical exponent
\[ \mu = -\frac{z\nu}{\beta\delta}. \] (11)

The scaled variable \( \omega h_0^\mu \) shall be called \( \Omega \). For non-zero \( \theta \), there will be, of course, corrections to the scaling form given above which become apparent as \( h_0 \to 0 \).

A trivial illustration of the scaling formula is provided by the Gaussian integral analysed before. Coupling a magnetic field linearly to \( \phi \), it is easy to see that \( \delta = 1 \) and, since \( \omega_0 \) is independent of \( h \), \( \nu/\delta\beta = 0 \). As a result, \( \lambda = 2 \), consistent with the
overall factor of $h_0^2$, and $\mu = 0$, consistent with the fact that the only $h_0$ dependence is in this factor.

The scaling laws illuminate the origin of critical hysteresis. Provided that the scaling function $F(\Omega)$ is non-zero, one must have critical hysteresis. This is a consequence of critical slowing down. We anticipate the subsequent discussion leading up to the fact that $F(\Omega)$ has a maximum at (say) $\Omega_0$, decays for $\Omega > \Omega_0$ as $1/\Omega$, and for small $\Omega$ increases as a power of $\Omega$ (see Fig. 1). As a consequence, the scaling form above predicts that as $h_0$ decreases, the hysteresis loop area increases at any fixed arbitrarily small frequency. At large frequencies, on the other hand, the system cannot follow changes in the driving field, and the magnetisation remains zero. In this qualitative sense, hysteretic behaviour at a critical point closely resembles that at a first order transition.

The scaling relations for a thermodynamically large system imply, in the usual way, finite-size scaling (FSS) relations. Well-known arguments [8] can be adapted to obtain various scaling formulæ. Of particular interest is the finite-size scaling of the frequency, $\omega_m$, at which the hysteresis loop area reaches its maximum. When systems of different sizes, $L$, are studied, each at some appropriately defined pseudo-critical coupling, and $h_0$ is independent of $L$ but smaller than the value at which finite-size rounding sets in for the largest system, then

$$\omega_m(L) \sim L^z. \quad (12)$$

Note that this relation gives a method for the measurement of $z$. The usual FSS methods for dynamics only measure the combination of exponents $z/\nu$.

Our first illustrative example is of the time-dependent Landau-Ginzburg model at its critical point. The model is defined by the Langevin equation in Eq. (2) along with the zero-mode Hamiltonian

$$H = \frac{r}{2} \phi^2 + \frac{g}{4!} \phi^4 - h\phi. \quad (13)$$

A (possibly time-dependent) magnetic field $h$ has been introduced. The critical theory corresponds to $r = 0$. We investigate this theory for large volume systems. As explained earlier, this requires the solution of the ordinary differential equation obtained by dropping the noise term in the Langevin equation. After the scalings

$$t \to a^2 t, \quad \phi \to a\phi, \quad h \to h/a, \quad a = \left(\frac{6}{1g}\right)^{1/4}, \quad (14)$$

we obtain the equation

$$\frac{d\phi}{dt} = -\phi^3 - c\theta \phi + h(t). \quad (15)$$
We used a harmonic form, $h_0 \cos \omega t$, for the driving field $h(t)$. Note that the scaling of $t$ implies one for the frequency $\omega$. We have assumed that $r \sim \theta$, and that $c$ includes a factor $\sqrt{6\Gamma/g}$, whose temperature-dependence can be neglected for sufficiently small $\theta$. The presence of hysteresis makes this a stiff equation. It was solved using Gear’s method, starting from an arbitrary initial value and integrating until transients died out to the precision of the arithmetic. The subsequent time history of the system gave the stable limit cycles corresponding to hysteresis. Scaling was checked by varying $h_0$ over three decades and $\omega$ over four.

The solutions showed hysteretic behaviour quite clearly. At very high frequencies, the system could not follow the applied magnetic field; and the magnetisation remained zero and time independent. As $\omega$ decreased, the hysteresis loop $m(h)$ opened into an ellipse with its major axis along $h$. With further decrease in $\omega$, the major axis turned, and the ellipse began to get dented, until it resembled familiar hysteresis loops. Eventually, with decreasing $\omega$ these loops became thinner. We calculated the areas of the hysteresis loops as a function of $\omega$ and $h_0$ and found that the scaling law given in Eq. (9) is obtained with high precision. We found the exponents

$$\lambda = \frac{4}{3}, \quad \text{and} \quad \mu = -\frac{2}{3}. \quad (16)$$

These are completely consistent with the values $\delta = 3$, $\nu = \beta = 1/2$ and $z = 2$. The scaling function can be identified by our computation. It is shown in Fig. 1. It has a single maximum at $\Omega \approx 0.93$.

These results were obtained for $c\theta = 0$. Positive values for this parameter correspond to approaching the critical point from the disordered side. As the value of this parameter increased, distortions in $F(\Omega)$ could be seen at higher and higher $\Omega$. For small $\Omega$ these tended to decrease $F$, and push $\Omega_0$ towards larger values. For $c\theta$ less than zero, we approach criticality from the ordered side, where there are coexisting phases. Varying $c\theta$ in this region, for any fixed $h_0$, we found that $A(\omega, h_0)$ approached its value at criticality smoothly.

It is interesting that the hysteresis loops themselves are scaling functions. At $\theta = 0$ it was found that

$$m(h; \omega, h_0) = h_0^{1/\delta} L(h/h_0; \Omega), \quad (17)$$

where the loop shape $L$ depended on $\Omega$. This is a form of a dynamical principle of corresponding states. The shapes of the loops are shown in Fig. 1 at the appropriate points below the scaled frequency axis. For $\Omega \to \infty$, $L = 0$ identically, and for $\Omega \to 0$, $L(x) = x^{1/\delta}$. 5
The exponents and scaling functions enjoy a high degree of universality. We checked this by changing the form of the cyclic driving field \( h(t) \). For a variety of forms, all with time-reversal invariance, we found the same values for the exponents. Whether we used triangular waves, or piecewise constant functions did not matter. Furthermore, the scaling function obtained for each shape of the driving field was simply related to that obtained for the harmonic form. We found that all the scaling functions mapped on to each other with multiplicative redefinitions of \( h_0 \) and \( \omega \). This leads to the form \( c_1 F(c_2 \Omega) \), with universal \( F(\Omega) \), where \( c_1 \) and \( c_2 \) are constants independent of \( h_0 \), \( \omega \) and \( \theta \), dependent only on the shape of the driving force.

A second example is provided by the Langevin dynamics of the \( \phi^6 \) model at its tricritical point. After appropriate scalings, we have, for the large-volume limit of the Langevin equation

\[
\frac{d\phi}{dt} = -\phi^5 + h(t).
\]  

(18)

Numerical solutions were obtained for this equation, varying \( h_0 \) and \( \omega \) as before. The critical exponents are well-known [9] — \( \delta = 5, \beta = 1/4, \nu = 1/2 \) and \( z = 2 \). These yield \( \lambda = 6/5 \) and \( \mu = -4/5 \). These described the numerical results extremely well. The universal scaling function thus obtained is shown in Fig. 1. As for the \( \phi^4 \) model, this function is universal over cyclic driving fields. It is also interesting that the scaling functions for critical and tricritical dynamics are identical for \( \Omega \gg \Omega_m \).

Universal hysteresis loop shapes were also observed, and found to be qualitatively similar to those for the \( \phi^4 \) theory.

We checked the FSS relation in Eq. (12) through heat-bath simulations of the two-dimensional Ising model. We simulated 32\(^2\), 40\(^2\) and 64\(^2\) lattices at the pseudo-critical coupling, \( \beta^c_L \), defined by the maximum of the specific heat. Equilibration at zero field was achieved by \( 10^6 \) sweeps with the Swendson-Wang cluster algorithm. Hysteresis runs were performed changing the external field cyclically between \( \pm h_0 \) in steps of \( \Delta h \). On all the lattices we chose \( h_0 = 8 \times 10^{-4} \) and \( \Delta h = 10^{-4} \). At each value of the field we ran \( N \) steps of the heat-bath algorithm. The frequency is given by \( \omega = 2\pi \Delta h/Nh_0 \). Expectation values and errors for hysteresis loop areas were obtained by jack-knifing measurements from 500 loop traversals at each \( \omega \) and \( L \) into 10 blocks. A series of runs on each size of lattice located \( \omega_m \) with fairly high precision. The data are shown in in Fig. 2 along with the scaling law obtained using the dynamical exponent \( z = 2 \) for heat-bath dynamics. The hysteresis loop shapes were seen to be universal, and qualitatively similar to the time-dependent Landau-Ginzburg model.

In conclusion, we have observed hysteresis at a critical point. Scaling relations and critical exponents for hysteresis loop areas have been identified and related to
other static and dynamic critical exponents. Finite-size scaling relations following from these have been written down. Since the observation of hysteresis is no guarantee that a first-order phase transition is seen, these relations are important in deciding the nature of the transition. Scaled hysteresis loop shapes and areas are also universal, characterised by functions which do not depend on the precise form of the driving force. The scaling functions identified here should be interesting objects of study, since they contain information on the complex structure of the underlying spectral density in an easily accessible fashion.
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The universal scaling function $F(\Omega) = h_0^{-\lambda} A(\omega h_0^\mu)$ describing hysteresis in the time-dependent Landau-Ginzburg theory. The result for the critical $\phi^4$ theory is shown with a full line and that for the tricritical $\phi^6$ theory with a dashed line. The universal loop shapes $L(h/h_0; \Omega)$ in the $\phi^4$ model are shown below the main figure.
Finite-size scaling tests for hysteresis in the two-dimensional Ising model. The data have been obtained at couplings corresponding to the maximum of the specific heat. The line shows the FSS prediction using the dynamical exponent $z = 2$. 