$\text{SL}_n(\mathbb{Z}[t])$ is not $FP_{n-1}$

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Abstract

We prove the result from the title using the geometry of Euclidean buildings.

1 Introduction

Little is known about the finiteness properties of $\text{SL}_n(\mathbb{Z}[t])$ for arbitrary $n$.

In 1959 Nagao proved that if $k$ is a field then $\text{SL}_2(k[t])$ is a free product with amalgamation [Na]. It follows from his description that $\text{SL}_2(\mathbb{Z}[t])$ and its abelianization are not finitely generated.

In 1977 Suslin proved that when $n \geq 3$, $\text{SL}_n(\mathbb{Z}[t])$ is finitely generated by elementary matrices [Su]. It follows that $H_1(\text{SL}_n(\mathbb{Z}[t]), \mathbb{Z})$ is trivial when $n \geq 3$.

More recent, Krstić-McCool proved that $\text{SL}_3(\mathbb{Z}[t])$ is not finitely presented [Kr-Mc].

It’s also worth pointing out that since $\text{SL}_n(\mathbb{Z}[t])$ surjects onto $\text{SL}_n(\mathbb{Z})$, that $\text{SL}_n(\mathbb{Z}[t])$ has finite index torsion-free subgroups.

In this paper we provide a generalization of the results of Nagao and Krstić-McCool mentioned above for the groups $\text{SL}_n(\mathbb{Z}[t])$.

Theorem 1. If $n \geq 2$, then $\text{SL}_n(\mathbb{Z}[t])$ is not of type $FP_{n-1}$.

Recall that a group $\Gamma$ is of type $FP_m$ if if there exists a projective resolution of $\mathbb{Z}$ as the trivial $\mathbb{Z}\Gamma$ module

$P_m \rightarrow P_{m-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0$

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where each \( P_i \) is a finitely generated \( \mathbb{Z} \Gamma \) module.

In particular, Theorem 1 implies that there is no \( K(\text{SL}_n(\mathbb{Z}[t]), 1) \) with finite \((n-1)\)-skeleton, where \( K(G, 1) \) is the Eilenberg-Mac Lane space for \( G \).

### 1.1 Outline of paper

The general outline of this paper is modelled on the proofs in \([Bu-Wo 1]\) and \([Bu-Wo 2]\), though some important modifications have to be made to carry out the proof in this setting.

As in \([Bu-Wo 1]\) and \([Bu-Wo 2]\), our approach is to apply Brown’s filtration criterion \([Br]\). Here we will examine the action of \( \text{SL}_n(\mathbb{Z}[t]) \) on the locally infinite Euclidean building for \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \). In Section 2 we will show that the infinite groups that arise as cell stabilizers for this action are of type \( FP_m \) for all \( m \), which is a technical condition that is needed for our application of Brown’s criterion.

In Section 3 we will demonstrate the existence of a family of diagonal matrices that will imply the existence of a “nice” isometrically embedded codimension 1 Euclidean space in the building for \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \). In \([Bu-Wo 1]\) analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because \( \mathbb{Q}((t^{-1})) \) is not locally compact, our treatment in Section 3 is quite a bit more hands on.

Section 4 contains the main body of our proof. We use translates of portions of the codimension 1 Euclidean subspace found in Section 3 to construct spheres in the Euclidean building for \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \) (also of codimension 1). These spheres will lie “near” an orbit of \( \text{SL}_n(\mathbb{Z}[t]) \), but will be nonzero in the homology of cells “not as near” the same \( \text{SL}_n(\mathbb{Z}[t]) \) orbit. Theorem 1 will then follow from Brown’s criterion.

### 2 Stabilizers

**Lemma 2.** If \( X \) is the Euclidean building for \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \), then the \( \text{SL}_n(\mathbb{Z}[t]) \) stabilizers of cells in \( X \) are \( FP_m \) for all \( m \).

**Proof.** Let \( x_0 \in X \) be the vertex stabilized by \( \text{SL}_n(\mathbb{Q}[t^{-1}]) \). We denote a diagonal matrix in \( \text{GL}_n(\mathbb{Q}((t^{-1}))) \) with entries \( s_1, s_2, \ldots, s_n \in \mathbb{Q}((t^{-1}))^\times \) by \( D(s_1, s_2, \ldots, s_n) \), and we let \( \mathcal{G} \subseteq X \) be the sector based at \( x_0 \) and containing
vertices of the form $D(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \ldots \geq m_n$.

The sector $\mathcal{S}$ is a fundamental domain for the action of $\text{SL}_n(\mathbb{Q}[t])$ on $X$ (see [So]). In particular, for any vertex $z \in X$, there is some $h_z' \in \text{SL}_n(\mathbb{Q}[t])$ and some integers $m_1 \geq m_2 \geq \ldots \geq m_n$ with $z = h_z'D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$. We let $h_z = h_z'D_z(t^{m_1}, t^{m_2}, ..., t^{m_n})$.

For any $N \in \mathbb{N}$, let $W_N$ be the $(N + 1)$-dimensional vector space $W_N = \{ p(t) \in \mathbb{C}[t] \mid \deg(p(t)) \leq N \}$ which is endowed with the obvious $\mathbb{Q}$–structure. If $N_1, \ldots, N_n$ in $\mathbb{N}$ are arbitrary then let

$$G_{\{N_1, \ldots, N_n\}} = \{ x \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(x) = 1 \}$$

where $\det(x)$ is a polynomial in the coordinates of $x$. To be more precise this is obtained from the usual determinant function when one considers the usual $n \times n$ matrix presentation of $x$, and calculates the determinant in $\text{Mat}_n(\mathbb{C}[t])$.

For our choice of vertex $z \in X$ above, the stabilizer of $z$ in $\text{SL}_n(\mathbb{Q}((t^{-1})))$ equals $h_z\text{SL}_n(\mathbb{Q}[[t^{-1}]]h_z^{-1}$. And with our fixed choice of $h_z$, there clearly exist some $N_i^z \in \mathbb{N}$ such that the stabilizer of the vertex $z$ in $\text{SL}_n(\mathbb{Q}[t])$ is $G_{\{N_1^z, \ldots, N_n^z\}}(\mathbb{Q})$. Furthermore, conditions on $N_i^z$ force a group structure on $G_z = G_{\{N_1^z, \ldots, N_n^z\}}$. Therefore, the stabilizer of $z$ in $\text{SL}_n(\mathbb{Q}[t])$ is the $\mathbb{Q}$–points of the affine $\mathbb{Q}$–group $G_z$, and the stabilizer of $z$ in $\text{SL}_n(\mathbb{Z}[t])$ is $G_z(\mathbb{Z})$.

The action of $\text{SL}_n(\mathbb{Q}[t])$ on $X$ is type preserving, so if $\sigma \subset \mathcal{S}$ is a simplex with vertices $z_1, z_2, \ldots, z_m$, then the stabilizer of $\sigma$ in $\text{SL}_n(\mathbb{Z}[t])$ is simply

$$(G_{z_1} \cap \ldots \cap G_{z_m})(\mathbb{Z})$$

That is, the stabilizer of $\sigma$ in $\text{SL}_n(\mathbb{Z}[t])$ is an arithmetic group, and Borel-Serre proved that any such group is $FP_m$ for all $m$ [Bo-Se].

\[\square\]

### 3 Polynomial points of tori

This section is devoted exclusively to a proof of the following

**Proposition 3.** There is a group $A \leq \text{SL}_n(\mathbb{Z}[t])$ such that
(i) $A \cong \mathbb{Z}^{n-1}$

(ii) There is some $g \in \text{SL}_n(\mathbb{Q}((t^{-1})))$ such that $gAg^{-1}$ is a group of diagonal matrices

(iii) No nontrivial element of $A$ fixes a point in the Euclidean building for $\text{SL}_n(\mathbb{Q}((t^{-1})))$.

The proof of this proposition is modelled on a classical approach to finding diagonalizable subgroups of $\text{SL}_n(\mathbb{Z})$. The proof will take a few steps.

### 3.1 A polynomial over $\mathbb{Z}[t]$ with roots in $\mathbb{Q}((t^{-1}))$

Let $\{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\}$ be the sequence of prime numbers. Let $q_i = 1$. For $2 \leq i \leq n$, let $q_i = p_{i-1} + 1$.

Let $f(x) \in \mathbb{Z}[t][x]$ be the polynomial given by

$$f(x) = \left[\prod_{i=1}^{n}(x + q_i t)\right] - 1$$

It will be clear by the conclusion of our proof that $f(x)$ is irreducible over $\mathbb{Q}(t)$, but we will not need to use this directly.

**Lemma 4.** There is some $\alpha \in \mathbb{Q}((t^{-1}))$ such that $f(\alpha) = 0$.

**Proof.** We want to show that there are $c_i \in \mathbb{Q}$ such that if $\alpha = \sum_{i=0}^{\infty} c_i t^{1-i n}$ then $f(\alpha) = 0$.

To begin let $c_0 = -1$. We will define the remaining $c_i$ recursively. Define $c_{i,k}$ by $\alpha + q_k t = \sum_{i=0}^{\infty} c_{i,k} t^{1-i n}$. Thus, $c_{i,k} = c_i$ when $i \geq 1$, each $c_{0,k}$ is contained in $\mathbb{Q}$, and $c_{0,1} = 0$.

That $\alpha$ is a root of $f$ is equivalent to

$$1 = \prod_{k=1}^{n}(\alpha + q_k t) = \prod_{k=1}^{n}\left(\sum_{i=0}^{\infty} c_{i,k} t^{1-i n}\right)$$

$$= \sum_{i=0}^{\infty} \left(\sum_{\sum_{k=1}^{n} i_k = i} \left(\prod_{k=1}^{n} c_{i,k}\right)\right) t^{n(1-i)}$$

Our task is to find $c_m$'s so that the above is satisfied. Note that for the above equation to hold we must have

$$0 \cdot t^n = \sum_{\sum_{k=1}^{n} i_k = 0}^{n} \left(\prod_{k=1}^{n} c_{i,k}\right) t^{n(1-0)}$$
That is
\[ 0 = \prod_{k=1}^{n} c_{0,k} \]
which is an equation we know is satisfied because \( c_{0,1} = 0 \). Now assume that we have determined \( c_0, c_1, \ldots, c_{m-1} \in \mathbb{Q} \). We will find \( c_m \in \mathbb{Q} \).

Notice that the first coefficient in our Laurent series expansion above which involves \( c_m \) is the coefficient for the \( t^{-nm} \) term. This follows from the fact that each \( i_k \) is nonnegative.

Since
\[ \sum_{\sum_{k=1}^{n} i_k = m} \left( \prod_{k=1}^{n} c_{i_k,k} \right) \]
is the coefficient of the \( t^{-nm} \) term in the expansion of 1, we have
\[ 0 = \sum_{\sum_{k=1}^{n} i_k = m} \left( \prod_{k=1}^{n} c_{i_k,k} \right) \]

The above equation is linear over \( \mathbb{Q} \) in the single variable \( c_m \) and the coefficient of \( c_m \) is nonzero. Indeed, \( \sum_{k=1}^{n} i_k = m \), each \( i_k \geq 0 \), and \( c_0, \ldots, c_{m-1} \in \mathbb{Q} \) are assumed to be known quantities. Thus, \( c_m \in \mathbb{Q} \).

\[ \square \]

### 3.2 Matrices representing ring multiplication

By Lemma 4 we have that the field \( \mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1})) \) is an extension of \( \mathbb{Q}(t) \) of degree \( d \) where \( d \leq n \). It follows that \( \mathbb{Z}[t][\alpha] \) is a free \( \mathbb{Z}[t] \)-module of rank \( d \) with basis \( \{1, \alpha, \alpha^2, \ldots, \alpha^{d-1}\} \).

For any \( y \in \mathbb{Z}[t][\alpha] \), the action of \( y \) on \( \mathbb{Q}(t)(\alpha) \) by multiplication is a linear transformation that stabilizes \( \mathbb{Z}[t][\alpha] \). Thus, we have a representation of \( \mathbb{Z}[t][\alpha] \) into the ring of \( d \times d \) matrices with entries in \( \mathbb{Z}[t] \). We embed the ring of \( d \times d \) matrices with entries in \( \mathbb{Z}[t] \) into the upper left corner of the ring of \( n \times n \) matrices with entries in \( \mathbb{Z}[t] \).

By Lemma 4
\[ \prod_{i=1}^{n} (\alpha + q_i t) = 1 \]
so each of the following matrices are invertible:
\[ \alpha + q_1 t, \ \alpha + q_2 t, \ \ldots, \ \alpha + q_n t \]
(We will be blurring the distinction between the elements of $\mathbb{Z}[t][\alpha]$ and the matrices that represent them.)

For $1 \leq i \leq n-1$, we let $a_i = \alpha + q_i + 1 t$. Since $a_i$ is invertible, it is an element of $GL_n(\mathbb{Z}[t])$, and hence has determinate $\pm 1$. By replacing each $a_i$ with its square, we may assume that $a_i \in SL_n(\mathbb{Z}[t])$ for all $i$. We let $A = \langle a_1, \ldots, a_{n-1} \rangle$ so that $A$ is clearly abelian as it is a representation of multiplication in an integral domain. This group $A$ will satisfy Proposition 3.

### 3.3 $A$ is free abelian on the $a_i$

To prove part (i) of Proposition 3 we have to show that if there are $m_i \in \mathbb{Z}$ with

$$ \prod_{i=1}^{n-1} a_i^{m_i} = 1 $$

then each $m_i = 0$. But the first nonzero term in the Laurent series expansion for $\alpha$ is $-t$, which implies that the first nonzero term in the Laurent series expansion for each $a_i$ is $-t + q_i + 1 t = p_i t$. Hence, the first nonzero term of

$$ \prod_{i=1}^{n-1} a_i^{m_i} = 1 $$

is

$$ \prod_{i=1}^{n-1} (p_i t)^{m_i} = t^0 $$

Thus

$$ \prod_{i=1}^{n-1} p_i^{m_i} = 1 $$

and it follows by the uniqueness of prime factorization that $m_i = 0$ for all $i$ as desired.

Thus, part (i) of Proposition 3 is proved.

### 3.4 $A$ is diagonalizable

Recall that $\alpha$ is a $d \times d$ matrix with entries in $\mathbb{Z}[t]$ where $d$ is the degree of the minimal polynomial of $\alpha$ over $\mathbb{Q}(t)$. Let that minimal polynomial be
Because the characteristic of $Q(t)$ equals 0, $q(x)$ has distinct roots in $Q(t)(\alpha)$.

Let $Q(x)$ be the characteristic polynomial of the matrix $\alpha$. The polynomial $Q$ also has degree $d$ and leading coefficient $\pm 1$ with $Q(\alpha) = 0$. Therefore, $q = \pm Q$. Hence, $Q$ has distinct roots in $Q(t)(\alpha)$ which implies that $\alpha$ is diagonalizable over $Q(t)(\alpha) \leq Q((t^{-1}))$. That is to say that there is some $g \in \text{SL}_n(Q((t^{-1})))$ such that $g\alpha g^{-1}$ is diagonal.

Because every element of $Z[t][\alpha]$ is a linear combination of powers of $\alpha$, we have that $g(Z[t][\alpha])g^{-1}$ is a set of diagonal matrices. In particular, we have proved part (ii) of Proposition 3.

3.5 A has trivial stabilizers

To prove part (iii) of Proposition 3 we begin with the following

**Lemma 5.** If $\Gamma \leq \text{SL}_n(Q[[t]])$ is bounded under the valuation for $Q((t^{-1}))$, then the eigenvalues for any $\gamma \in \Gamma$ lie in $\mathfrak{Q}$.

**Proof.** We let $X$ be the Euclidean building for $\text{SL}_n(Q((t^{-1})))$. By assumption, $\Gamma z = z$ for some $z \in X$.

Let $x_0 \in X$ be the vertex stabilized by $\text{SL}_n(Q[[t^{-1}]])$. We denote a diagonal matrix in $\text{GL}_n(Q((t^{-1})))$ with entries $s_1, s_2, ..., s_n \in Q((t^{-1}))^\times$ by $D(s_1, s_2, ..., s_n)$, and we let $\mathcal{S} \subseteq X$ be the sector based at $x_0$ and containing vertices of the form $D(t^{m_1}, t^{m_2}, ..., t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq ... \geq m_n$.

The sector $\mathcal{S}$ is a fundamental domain for the action of $\text{SL}_n(Q[[t]])$ on $X$ which implies that there is some $h \in \text{SL}_n(Q[[t]])$ with $hz \in \mathcal{S}$.

Clearly we have $(h\Gamma h^{-1})hz = hz$, and since eigenvalues of $h\Gamma h^{-1}$ are the same as those for $\Gamma$, we may assume that $\Gamma$ fixes a vertex $z \in \mathcal{S}$.

Fix $m_1, ..., m_n \in \mathbb{Z}$ with $m_1 \geq ... \geq m_n \geq 0$ and such that $z = D(t^{m_1}, ..., t^{m_n})x_0$. Without loss of generality, there is a partition of $n$ — say $\{k_1, ..., k_\ell\}$ — such that

$$\{m_1, ..., m_n\} = \{q_1, ..., q_1, q_2, ..., q_2, ..., q_\ell, ..., q_\ell\}$$

where each $q_i$ occurs exactly $k_i$ times and

$$q_1 > q_2 > ... > q_\ell$$
We have that \( D(t_{m_1}, \ldots, t_{m_n})^{-1} \Gamma D(t_{m_1}, \ldots, t_{m_n}) x_0 = x_0 \). That gives us, \( D(t_{m_1}, \ldots, t_{m_n})^{-1} \Gamma D(t_{m_1}, \ldots, t_{m_n}) \subset \text{SL}_n(\mathbb{Q}[[t]]) \). Furthermore, a trivial calculation of resulting valuation restrictions for the entries of \( D(t_{m_1}, \ldots, t_{m_n}) \text{SL}_n(\mathbb{Q}[[t]]) \) shows that \( \Gamma \) is contained in a subgroup of \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \) that is isomorphic to

\[
\prod_{i=1}^\ell \text{SL}_{k_i}(\mathbb{Q}) \ltimes U
\]

where \( U \leq \text{SL}_n(\mathbb{Q}((t^{-1}))) \) is a group of upper-triangular unipotent matrices.

The lemma is proved.

Our proof of Proposition 3 will conclude by proving

**Lemma 6.** No nontrivial element of \( A \) fixes a point in the Euclidean building for \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \).

**Proof.** Suppose \( a \in A \) fixes a point in the building. We will show that \( a = 1 \).

Let \( F(x) \in \mathbb{Z}[t][x] \) be the characteristic polynomial for \( a \in \text{SL}_n(\mathbb{Z}[t]) \). Then

\[
F(x) = \pm \prod_{i=1}^n (x - \beta_i)
\]

where each \( \beta_i \in \mathbb{Q}((t^{-1})) \) is an eigenvalue of \( a \). By the previous lemma, each \( \beta_i \in \overline{\mathbb{Q}} \). Hence, each \( \beta_i \in \mathbb{Q} = \overline{\mathbb{Q}} \cap \mathbb{Q}((t^{-1})) \). It follows that \( F(x) \in \mathbb{Z}[x] \) so that each \( \beta_i \) is an algebraic integer contained in \( \mathbb{Q} \). We conclude that each \( \beta_i \) is contained in \( \mathbb{Z} \).

Recall, that \( a \) has determinate 1, and that the determinate of \( a \) can be expressed as \( \prod_{i=1}^n \beta_i \). Hence, each \( \beta_i \) is a unit in \( \mathbb{Z} \), so each eigenvalue \( \beta_i = \pm 1 \). It follows – by the diagonalizability of \( a \) – that \( a \) is a finite order element of \( A \cong \mathbb{Z}^{n-1} \). That is, \( a = 1 \).

We have completed our proof of Proposition 3.
4 Body of the proof

Let \( P \leq \text{SL}_n(\mathbb{Q}((t^{-1}))) \) be the subgroup where each of the first \( n - 1 \) entries along the bottom row equal 0. Let \( R_n(P) \leq P \) be the subgroup of elements that contain a \((n - 1) \times (n - 1)\) copy of the identity matrix in the upper left corner. Thus \( R_n(P) \cong \mathbb{Q}((t^{-1}))^{n-1} \) with the operation of vector addition.

Let \( L \leq P \) be the copy of \( \text{SL}_{n-1}(\mathbb{Q}((t^{-1}))) \) in the upper left corner of \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \). We apply Proposition 3 to \( L \) (notice that the \( n \) in the proposition is now an \( n - 1 \)) to derive a subgroup \( A \leq L \) that is isomorphic to \( \mathbb{Z}^{n-2} \). By the same proposition, there is a matrix \( g \in L \) such that \( gAg^{-1} \) is diagonal.

Let \( b \in \text{SL}_n(\mathbb{Q}((t^{-1}))) \) be the diagonal matrix given in the notation from the proofs of Lemmas 2 and 5 as \( D(t, t, ..., t, t-(n-1)) \). Note that \( b \in P \) commutes with \( L \), and therefore, with \( A \). Thus the Zariski closure of the group generated by \( b \) and \( A \) determines an apartment in \( X \), namely \( g^{-1}A \) where \( A \) is the apartment corresponding to the diagonal subgroup of \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \).

4.1 Actions on \( g^{-1}A \).

If \( x_* \in g^{-1}A \), then it follows from Proposition 3 that the convex hull of the orbit of \( x_* \) under \( A \) is an \((n - 2)\)-dimensional affine space that we will name \( V_{x_*} \). Furthermore, the orbit \( Ax_* \) forms a lattice in the space \( V_{x_*} \).

We let \( g^{-1}A(\infty) \) be the visual boundary of \( g^{-1}A \) in the Tits boundary of \( X \). The visual image of \( V_{x_*} \) is clearly an equatorial sphere in \( g^{-1}A(\infty) \). Precisely, we let \( P^- \) be the transpose of \( P \). Then \( P \) and \( P^- \) are opposite vertices in \( g^{-1}A(\infty) \). It follows that there is a unique sphere in \( g^{-1}A(\infty) \) that is realized by all points equidistant to \( P \) and \( P^- \). We call this sphere \( S_{P,P^-} \).

Lemma 7. The visual boundary of \( V_{x_*} \) equals \( S_{P,P^-} \).

Proof. Since \( g \in P \cap P^- \), it suffices to prove that \( gV_{x_*} \) is the sphere in the boundary of \( A \) that is determined by the vertices \( P \) and \( P^- \).

Note that \( gV_{x_*} \) is a finite Hausdorff distance from any orbit of a point in \( A \) under the action of the diagonal subgroup of \( L \). The result follows by observing that the inverse transpose map on \( \text{SL}_n(\mathbb{Q}((t^{-1}))) \) stabilizes diagonal matrices while interchanging \( P \) and \( P^- \).
We let $R_1, R_2, \ldots, R_{n-1}$ be the standard root subgroups of $R_u(P)$. Recall that associated to each $R_i$ there is a closed geodesic hemisphere $H_i \subseteq \mathcal{A}(\infty)$ such that any nontrivial element of $R_i$ fixes $H_i$ pointwise and translates any point in the open hemisphere $\mathcal{A}(\infty) - H_i$ outside of $\mathcal{A}(\infty)$. Note that $\partial H_i$ is a codimension 1 geodesic sphere in $\mathcal{A}(\infty)$.

We let $M \subseteq g^{-1}\mathcal{A}(\infty)$ be the union of chambers in $g^{-1}\mathcal{A}(\infty)$ that contain the vertex $P$. There is also an equivalent geometric description of $M$:

**Lemma 8.** The union of chambers $M \subseteq g^{-1}\mathcal{A}(\infty)$ can be realized as an $(n-2)$-simplex. Furthermore,

$$M = \bigcap_{i=1}^{n-1} g^{-1}H_i$$

and, when $M$ is realized as a single simplex, each of the $n-1$ faces of $M$ is contained in a unique equatorial sphere $g^{-1}\partial H_i = \partial g^{-1}H_i$.

**Proof.** Let $M' \subseteq \mathcal{A}(\infty)$ be the union of chambers in $\mathcal{A}(\infty)$ containing the vertex $P$. Since $M = g^{-1}M'$, it suffices to prove that $M'$ is an $(n-2)$-simplex with $M' = \bigcap_{i=1}^{n-1} H_i$ and with each face of $M'$ contained in a unique $\partial H_i$.

For any nonempty, proper subset $I \subseteq \{1, 2, \ldots, n\}$, we let $V_I$ be the $|I|$-dimensional vector subspace of $\mathbb{Q}(((t^{-1}))^n$ spanned by the coordinates given by $I$, and we let $P_I$ be the stabilizer of $V_I$ in $\text{SL}_n(\mathbb{Q}(((t^{-1})))$. For example, $P = P_{\{1, 2, \ldots, n-1\}}$.

Recall that the vertices of $\mathcal{A}(\infty)$ are given by the parabolic groups $P_I$, that edges connect $P_I$ and $P_{I'}$ exactly when $I \subseteq I'$ or $I' \subseteq I$, and that the remaining simplicial description of $\mathcal{A}(\infty)$ is given by the condition that $\mathcal{A}(\infty)$ is a flag complex.

We let $\mathcal{V}$ be the set of vertices in $\mathcal{A}(\infty)$ of the form $P_J$ where $\emptyset \neq J \subseteq \{1, 2, \ldots, n-1\}$. Note that $M'$ is exactly the set of vertices $\mathcal{V}$ together with the simplices described by the incidence relations inherited from $\mathcal{A}(\infty)$. Thus, $M'$ is easily seen to be isomorphic to a barycentric subdivision of an abstract $(n-2)$-simplex. Indeed, if $\overline{M'}$ is the abstract simplex on vertices $P_{\{1\}}, P_{\{2\}}, \ldots, P_{\{n-1\}}$, then a simplex of dimension $k$ in $\overline{M'}$ corresponds to a unique $P_J \in \mathcal{V}$ with $|J| = k + 1$. So we have that $M'$ can be topologically realized as an $(n-2)$-simplex.

Let $F_i$ be a face of the simplex $\overline{M'}$. Then there is some $1 \leq i \leq n-1$ such that the set of vertices of $F_i$ is exactly $\{P_{\{1\}}, P_{\{2\}}, \ldots, P_{\{n-1\}}\} - P_{\{i\}}$. 

Note that $R_i V_I = V_I$ exactly when $n \in I$ implies $i \in I$. It follows that $R_i$ fixes $M'$ pointwise, and thus $M' \subseteq H_i$ for all $1 \leq i \leq n - 1$. Furthermore, if $P_I \in H_i$ for all $1 \leq i \leq n - 1$, then $R_i P_I = P_I$ for all $i$ so that $n \in I$ implies $i \in I$ for all $1 \leq i \leq n - 1$. As $I$ must be a proper subset of $\{1, 2, ..., n\}$, we have $P_I \in \mathcal{V}$, so that $M' = \cap_{i=1}^{n-1} H_i$.

All that remains to be verified for this lemma is that $F_i \subseteq \partial H_i$. For this fact, recall that $F_i$ is comprised of $(n - 3)$-simplices in $\mathcal{A}(\infty)$ whose vertices are given by $P_J$ where $J \subseteq \{1, 2, ..., n - 1\} - \{i\}$. Hence, if $\sigma \subseteq \mathcal{A}(\infty)$ is an $(n - 3)$ simplex of $\mathcal{A}(\infty)$ with $\sigma \subseteq F_i$, then $\sigma$ is a face of exactly 2 chambers in $\mathcal{A}(\infty)$: $\mathcal{C}_P$ and $\mathcal{C}_{P',J}$ where $\mathcal{C}_P$ contains $P$ and thus $\mathcal{C}_P \subseteq M'$, and $\mathcal{C}_{P,J'}$ contains $P,J'$ where $J' = \{1, 2, ..., n\} - \{i\}$ and thus $\mathcal{C}_{P,J'} \notin M'$. Furthermore, $\sigma = \mathcal{C}_P \cap \mathcal{C}_{P',J}$.

Since $R_i V_J' \neq V_J$, it follows that $\mathcal{C}_{P,J'}$ is not fixed by $R_i$. Since $\mathcal{C}_{P,J}$ is fixed by $R_i$ we have that $\sigma = \mathcal{C}_P \cap \mathcal{C}_{P',J} \subseteq \partial H_i$. Therefore, $F_i \subseteq \partial H_i$. \hfill \Box

For any vertex $y \in X$, we let $C_y \subseteq X$ be the union of sectors based at $y$ and limiting to a chamber in $M$. Thus, $C_y$ is a cone. Note also that because any chamber in $g^{-1}\mathcal{A}(\infty)$ has diameter less than $\pi/2$, it follows that $M \cap S_{P,P'} = \emptyset$. Therefore, if we choose $x_s, y \in g^{-1}\mathcal{A}$ such that $x_s$ is closer to $P$ than $y$, then $C_y \subseteq g^{-1}\mathcal{A}$ and $V_{x_s} \cap C_y$ is a simplex of dimension $n - 2$.

We will set on a fixed choice of $y$ before $x_s$, and we will choose $y$ to satisfy the below

Lemma 9. There is some $y \in g^{-1}\mathcal{A}$ such that the $\mathcal{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $C_y$ pointwise.

Proof. Let $x_0$ be the point in $X$ stabilized by $\text{SL}_n(\mathbb{Q}[[t^{-1}]]).$ Recall that $R_u(P)M = M$ so that the $\mathcal{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $C_{x_0}$ pointwise.

Because $M \subseteq g^{-1}\mathcal{A}(\infty)$, there is a $y \in C_{x_0} \cap g^{-1}\mathcal{A}$. Any such $y$ satisfies the lemma. \hfill \Box

Choose $e$ such that with $x_s = e$ as above and with $y$ as in Lemma 9 there exists a fundamental domain $D_e$ for the action of $A$ on $V_e$ that is contained in $C_y$. The choice of $e$ can be made by travelling arbitrarily far from $y$ along a geodesic ray in $g^{-1}\mathcal{A}$ that limits to $P$.

By the choice of $D_e$ we have that

$$AD_e = V_e$$

and that the $\mathcal{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $D_e$. 

11
4.2 The filtration

We let

\[ X_0 = \SL_n(\mathbb{Z}[t])D_e \]

and for any \( i \in \mathbb{N} \) we choose an \( \SL_n(\mathbb{Z}[t]) \)-invariant and cocompact space \( X_i \subseteq X \) somewhat arbitrarily to satisfy the inclusions

\[ X_0 \subseteq X_1 \subseteq X_2 \subseteq ... \subseteq \bigcup_{i=1}^{\infty} X_i = X \]

In our present context, Brown’s criterion takes on the following form [Br]

**Brown’s Filtration Criterion 10.** By Lemma 2, the group \( \SL_n(\mathbb{Z}[t]) \) is not of type \( FP_{n-1} \) if for any \( i \in \mathbb{N} \), there exists some class in the homology group \( \tilde{H}_{n-2}(X_0, \mathbb{Z}) \) which is nonzero in \( \tilde{H}_{n-2}(X_i, \mathbb{Z}) \).

4.3 Translation to \( P \) moves away from filtration sets

The following is essentially Mahler’s compactness criterion.

**Lemma 11.** Given any \( i \in \mathbb{N} \), there is some \( k \in \mathbb{N} \) such that \( b^k e \notin X_i \).

**Proof.** The lemma follows from showing that the sequence

\[ \{\SL_n(\mathbb{Z}[t])b^k e\}_k \subseteq \SL_n(\mathbb{Z}[t]) \setminus X \]

is unbounded.

Since stabilizers of points in \( X \) are bounded subgroups of \( \SL_n(\mathbb{Q}((t^{-1}))) \), the claim above follows from showing that the sequence

\[ \{\SL_n(\mathbb{Z}[t])b^k\}_k \subseteq \SL_n(\mathbb{Z}[t]) \setminus \SL_n(\mathbb{Q}((t^{-1}))) \]

is unbounded.

But bounded sets in \( \SL_n(\mathbb{Z}[t]) \setminus \SL_n(\mathbb{Q}((t^{-1}))) \) do not contain sequences of elements \( \{\SL_n(\mathbb{Z}[t])g_\ell \}_\ell \) such that \( 1 \in g_\ell^{-1}(\SL_n(\mathbb{Z}[t]) - \{1\})g_\ell \). And clearly \( b^k \)'s contract some root groups to 1. Thus none of the sequences above is bounded. \( \square \)
4.4 Applying Brown’s criterion

As is described by Brown’s criterion, we will prove Theorem \( \text{I} \) by fixing \( X_i \) and finding an \((n - 2)\)-cycle in \( X_0 \) that is nontrivial in the homology of \( X_i \).

Recall that we denote the standard root subgroups of \( R_u(P) \) by \( R_1, ..., R_{n-1} \). Each group \( g^{-1}R_jg \) determines a family of parallel walls in \( g^{-1}A \). By Lemma 8, each face of the cone \( C_y \) is contained in a wall of one of these families.

Choose \( r_j \in g^{-1}R_jg \) for all \( j \) such that \( b^ke \) is contained in the wall determined by \( r_j \) where \( k \) is determined by \( i \) as in Lemma 11. In particular, \( r_jb^ke = b^ke \).

The intersection of the fixed point sets in \( g^{-1}A \) of the elements \( r_1, ..., r_{n-1} \) determine a cone that we name \( Z \). Note that \( Z \) is contained in – and is a finite Hausdorff distance from – the cone \( C_y \).

Let \( Z^- \subseteq g^{-1}A \) be the closure of the set of points in \( g^{-1}A \) that are fixed by none of the \( r_j \). The set \( Z^- \) is a cone based at \( b^ke \), containing \( y \), and asymptotically containing the vertex \( P^- \).

As the walls of \( Z^- \) are parallel to those of \( Z^- \) and hence of \( C_y \), we have that \( Z^- \cap V_e \) is an \((n - 2)\)-dimensional simplex. We will name this simplex \( \sigma \).

The component of \( Z^- - V_e \) that contains \( b^ke \) is an \((n - 1)\)-simplex that has \( \sigma \) as a face. Call this \((n - 1)\) simplex \( Y \).

For any \( \ell \in \mathbb{N} \), there are exactly \( 2^{n-1} \) possible subsets of the set \( \{r_1^\ell, ..., r_{n-1}^\ell\} \). For each such subset \( S_\ell \), we let

\[
Y_{S_\ell} = \left( \prod_{g \in S_\ell} g \right) Y
\]

and

\[
\sigma_{S_\ell} = \left( \prod_{g \in S_\ell} g \right) \sigma
\]

Notice that the product of group elements in the equations above are well-defined regardless of the order of the multiplication since \( R_u(P) \) is abelian. In the degenerate cases, \( \prod_{g \in \emptyset} g = 1 \), so \( Y_\emptyset = Y \) and \( \sigma_\emptyset = \sigma \).

For any \( \ell \in \mathbb{N} \), we let \( Y_\ell = \cup S_\ell Y_{S_\ell} \). Because the wall in \( g^{-1}A \) determined by \( r_\ell^j \) is the same as the wall determined by \( r_j \), the space \( Y_\ell \) is a closed ball containing \( b^ke \) whose boundary sphere is \( \cup S_\ell \sigma_{S_\ell} \). Indeed the simplicial decomposition of \( Y_\ell \) described above is isomorphic to the simplicial decomposition of the unit ball in \( \mathbb{R}^{n-1} \) that is given by the \( n - 1 \) hyperplanes defined by setting a coordinate equal to 0.
Let \( \omega_\ell = \cup S_\ell \sigma_{S_\ell} \). Thus \( \omega_\ell = \partial Y_\ell \). Furthermore, the building \( X \) is \((n-1)\)-dimensional and contractible, so any \((n-1)\)-chain with boundary equal to \( \omega_\ell \) must contain \( Y_\ell \) and thus \( b^k e \). That is for all \( \ell \in \mathbb{N} \)

\[
[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X - b^k e, \mathbb{Z})
\]

If we can show that \( \omega_\ell \subseteq X_0 \) for some choice of \( \ell \), then we will have proved our main theorem by application of Brown’s criterion since we would have

\[
[\omega_\ell] \neq 0 \in \tilde{H}_{n-2}(X, \mathbb{Z})
\]

by Lemma 11.

**Lemma 12.** There exists some \( \ell \in \mathbb{N} \) such that \( \omega_\ell \subseteq X_0 \).

**Proof.** For any \( u \in R_u(P) \) there is a decomposition \( u = u'u'' \) where the entries of \( u' \in R_u(P) \) are contained in \( \mathbb{Q}[t] \) and the entries of \( u'' \in R_u(P) \) are contained in \( \mathbb{Q}[[t^{-1}]] \).

For any \( a \in A \) and \( u \in R_u(P) \) there is a power \( \ell(a, u) \in \mathbb{N} \) such that

\[
(a^{-1}u^\ell(a, u))' = ((a^{-1}ua')^\ell(a, u)) \in SL_n(\mathbb{Z}[t])
\]

(For the above equality recall that \( A \leq L \) normalizes \( R_u(P) \) and the group operation on \( R_u(P) \) is vector addition.)

There are only finitely many \( a \in A \) such that \( aD_e \cap \sigma \neq \emptyset \) (or equivalently, such that \( aD_e \cap Z^- \neq \emptyset \)). Call this finite set \( \mathcal{D} \subseteq A \).

At this point we fix

\[
\ell = \prod_{a \in \mathcal{D}} \prod_{i=1}^{n-1} \ell(a, r_i)
\]

Thus,

\[
[a^{-1}(\prod_{g \in S_\ell} g)a']' \in SL_n(\mathbb{Z}[t])
\]

for any \( a \in \mathcal{D} \) and any \( S_\ell \subseteq \{r_i\}_{i=1}^{n-1} \).

Because \( \omega_\ell = \cup_{S_\ell} \sigma_{S_\ell} \) and \( \sigma_{S_\ell} = (\prod_{g \in S_\ell} g)\sigma = (\prod_{g \in S_\ell} g)(AD_e \cap Z^-) \), we can finish our proof of this lemma by showing

\[
(\prod_{g \in S_\ell} g)aD_e \subseteq X_0
\]
for each $a \in D \subseteq A \leq \text{SL}_n(\mathbb{Z}[t])$ and each $S_t \subseteq \{n_i[t_i]^{n-1}\}$. For this, recall that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix $D_e$ and thus

\[
(\prod_{g \in S_t} g)aD_e = a[a^{-1}(\prod_{g \in S_t} g)a]D_e \\
= a[a^{-1}(\prod_{g \in S_t} g)a][a^{-1}(\prod_{g \in S_t} g)a]'D_e \\
= a[a^{-1}(\prod_{g \in S_t} g)a]'D_e \\
\subseteq \text{SL}_n(\mathbb{Z}[t])D_e \\
= X_0
\]

\[
\square
\]

**References**

[Bo-Se] A. Borel, J-P. Serre, *Corners and arithmetic groups*. Commentarii Mathematici Helvetici 48 (1973) p. 436–491.

[Br] Brown, K., *Finiteness properties of groups*. J. Pure Appl. Algebra 44 (1987), 45-75.

[Bu-Wo 1] Bux, K.-U., and Wortman, K., *Finiteness properties of arithmetic groups over function fields*. Invent. Math. 167 (2007), 355-378.

[Bu-Wo 2] Bux, K.-U., and Wortman, K., *Geometric proof that $\text{SL}_2(\mathbb{Z}[t, t^{-1}])$ is not finitely presented*. Algebr. Geom. Topol. 6 (2006), 839-852.

[Kr-Mc] Krstić, S., and McCool, J., *Presenting $\text{GL}_n(k\langle T \rangle)$*. J. Pure Appl. Algebra 141 (1999), 175-183.

[Na] Nagao, H., *On $\text{GL}(2, K[X])$*. J. Inst. Polytech. Osaka City Univ. Ser. A 10 (1959), 117-121.

[Su] Suslin, A. A. *The structure of the special linear group over rings of polynomials*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 2, 235-252, 477.
[So] Soulé, C., *Chevalley groups over polynomial rings*. Homological Group Theory, LMS 36 (1977), 359-367.

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