Descent theory for semiorthogonal decompositions

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Abstract. We put forward a method for constructing semiorthogonal decompositions of the derived category of $G$-equivariant sheaves on a variety $X$ under the assumption that the derived category of sheaves on $X$ admits a semiorthogonal decomposition with components preserved by the action of the group $G$ on $X$. This method is used to obtain semiorthogonal decompositions of equivariant derived categories for projective bundles and blow-ups with a smooth centre as well as for varieties with a full exceptional collection preserved by the group action. Our main technical tool is descent theory for derived categories.

Bibliography: 12 titles.

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§ 1. Introduction

Our article is devoted to the study of the derived category of equivariant sheaves on an algebraic variety equipped with an action of an algebraic group. Our aim is to construct semiorthogonal decompositions of this category.

We describe a method for obtaining a semiorthogonal decomposition of the equivariant derived category starting from a semiorthogonal decomposition of the standard derived category. In this form the problem admits a natural generalization, namely, an investigation of the connections between the derived categories of the base and the covering variety. In the case of the derived category of $G$-equivariant sheaves on a variety $X$ with an action of $G$ the role of the covering variety is played by $X$ and the base is the quotient stack $X//G$ by the action of $G$. That is why we have to work in the category of stacks and not in the category of algebraic varieties or of schemes.

For a morphism of stacks $p: X \rightarrow S$, there is a standard way to describe the category of sheaves on $S$ in terms of the category of sheaves on $X$. Namely, if the morphism $p$ is faithfully flat, giving a sheaf on $S$ is equivalent to giving a sheaf $F$ on $X$ with an isomorphism $p_1^* F \rightarrow p_2^* F$ on $X \times_S X$ as gluing data satisfying the...
cocycle condition. A similar assertion is valid for an object of the derived category of sheaves on $S$, provided the following splitting condition holds: the sheaf $\mathcal{O}_S$ as a direct summand under the natural morphism $\mathcal{O}_S \to Rp_*\mathcal{O}_X$. In other words, under this condition the derived category of sheaves on the base $S$ is equivalent to some descent category associated with the morphism $p: X \to S$. This allows us to use the language of descent data to study the connection between semiorthogonal decompositions of derived categories of sheaves on the base and the covering variety. In the case of the morphism $X \to X//G$ the splitting condition reduces to the requirement that the group $G$ is linearly reductive, that is, the category of linear representations of $G$ is semisimple.

The descent category associated with a morphism $p: X \to S$ admits two equivalent definitions. In the first one, the classical descent category $\mathcal{D}(X)/p$ is formed by pairs consisting of an object $F$ of the unbounded derived category $\mathcal{D}(X)$ of quasi-coherent sheaves on a scheme $X$ and an isomorphism $p_1^*F \to p_2^*F$ satisfying the cocycle condition. In the second definition, the descent category is the category $\mathcal{D}(X)_{T_p}$ of comodules over the comonad $T_p$ on the category $\mathcal{D}(X)$ associated with the pair of adjoint functors $p^*$ and $p_*$. The language of comodules over a comonad is sometimes more convenient. In particular, Theorem 3.2 is formulated in these terms. It gives a semiorthogonal decomposition of the descent category provided there exists a semiorthogonal decomposition of the initial category compatible in a certain sense with the functor $T_p$.

Our main results on the connection between derived categories of the base and the covering variety are based on Theorem 3.2. These are Theorems 4.1, 4.3 and 4.4 for coverings of schemes and Theorems 6.1–6.3 on equivariant derived categories. Under the assumption that there exists a semiorthogonal decomposition of the category $\mathcal{D}^{perf}(X)$ of perfect complexes on $X$ preserved by an action of a linearly reductive group $G$ Theorem 6.2 produces a semiorthogonal decomposition of the derived category $\mathcal{D}^{perf,G}(X)$ of $G$-equivariant perfect complexes on $X$. The components of this decomposition can be described in terms of the descent category.

In this article we consider applications of Theorem 6.2 to the cases of an action of a group on the projectivization of an equivariant vector bundle and on the blow-up of a smooth subvariety. In both cases Theorem 6.2 is applied to semiorthogonal decompositions of the derived category on the corresponding varieties constructed by Orlov. We give an explicit description of the components of the decomposition as certain equivariant derived categories.

Another important application of Theorem 6.2 concerns an action of a linearly reductive group preserving a full exceptional collection on the variety. This is the case of the simplest semiorthogonal decomposition invariant under a group action. Here we also give an explicit description (Theorem 9.4) of components of the decomposition constructed in Theorem 6.2. They are equivalent to the derived categories of representations of the group twisted by different cocycles. A similar result was obtained earlier in [1] under the assumption that the exceptional collection is formed by sheaves. The descent technique allows us to drop this assumption.

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§ 2. Preliminaries

All schemes in this article are assumed to be quasi-projective and defined over an arbitrary field \( k \). By an algebraic group we mean an affine group scheme of finite type over \( k \). We shall denote the multiplication morphism of a group \( G \) by \( \mu: G \times G \to G \) and an action of a group \( G \) on a scheme \( X \) by \( a: G \times X \to X \). We denote the projections of products \( G \times X, G \times G \times X, \) and so on, onto their factors by \( p_1, p_2, p_{12}, \) etc.

We consider three versions of the derived category of sheaves on a scheme \( X \). The first is the unbounded derived category of quasi-coherent sheaves \( \mathcal{D}(X) = \mathcal{D}(\text{qcoh}(X)) \), the second is the bounded derived category of coherent sheaves \( \mathcal{D}^b(X) = \mathcal{D}^b(\text{coh}(X)) \), and the third is the category of perfect complexes \( \mathcal{D}^{\text{perf}}(X) \). By definition, the latter is a full subcategory in \( \mathcal{D}(X) \) formed by complexes that are locally quasi-isomorphic to a bounded complex of vectors bundles. We have \( \mathcal{D}^{\text{perf}}(X) \subset \mathcal{D}^b(X) \), and \( \mathcal{D}^{\text{perf}}(X) = \mathcal{D}^b(X) \) provided the scheme is smooth (see [2]). Let \( f: X \to Y \) be a morphism of schemes. Following Spaltenstein (see [3]), we consider the adjoint derived inverse and direct image functors between the categories \( \mathcal{D}(X) \) and \( \mathcal{D}(Y) \). We denote them by \( f^* \) and \( f_* \), when there is no confusion. Note that \( f^* \) sends \( \mathcal{D}^{\text{perf}}(Y) \) to \( \mathcal{D}^{\text{perf}}(X) \). If the morphism \( f \) has finite Tor-dimension, in particular, if \( f \) is flat, the functor \( f^* \) sends \( \mathcal{D}^b(Y) \) to \( \mathcal{D}^b(X) \). The functor \( f_* \) sends \( \mathcal{D}^b(X) \) to \( \mathcal{D}^b(Y) \) provided \( f \) is a proper morphism.

Definition 2.1. An equivariant sheaf on a scheme \( X \) with an action of a group \( G \) is a pair consisting of a sheaf \( F \) and an isomorphism \( \theta: p_2^*F \to a^*F \) of sheaves on \( G \times X \) such that the cocycle condition

\[
(1 \times a)^*\theta \circ p_{23}^*\theta = (\mu \times 1)^*\theta
\]

holds on \( G \times G \times X \).

A morphism of equivariant sheaves \( (F_1, \theta_1) \) and \( (F_2, \theta_2) \) is a morphism \( f: F_1 \to F_2 \) compatible with \( \theta \), that is, \( \theta_2 \circ p_2^*f = a^*f \circ \theta_1 \).

Coherent (quasi-coherent) \( G \)-equivariant sheaves on a scheme \( X \) form an abelian category \( \text{coh}^G(X) \) (\( \text{qcoh}^G(X) \), respectively). We let \( \mathcal{D}^G(X) = \mathcal{D}(\text{qcoh}^G(X)) \) denote the unbounded derived category of \( G \)-equivariant quasi-coherent sheaves, \( \mathcal{D}^b(\text{coh}^G(X)) \) the bounded derived category of \( G \)-equivariant coherent sheaves, and \( \mathcal{D}^{\text{perf}, G}(X) \) the category of \( G \)-equivariant perfect complexes. The latter is a full subcategory in \( \mathcal{D}^G(X) \) formed by complexes that lie in \( \mathcal{D}^{\text{perf}}(X) \) after forgetting the group action.

One can view \( G \)-equivariant sheaves on a scheme \( X \) as sheaves on the quotient stack \( X/G \) of \( X \) by the action of \( G \). The definition of the stack \( X/G \) can be found in [4]. Under our assumptions this is an algebraic stack of finite type over \( k \). There is a canonical flat morphism \( p: X \to X/G \) such that the inverse image with respect to \( p \) just involves forgetting the equivariant structure. This point of view allows us to describe the derived category of equivariant sheaves on a scheme in terms of the derived category of all sheaves. There are two essentially equivalent ways of giving this description, namely, via cosimplicial categories and via comonads. The latter will be discussed below.

Let \( \Delta \) denote the category whose objects are sets \([1, \ldots, n], \ n \in \mathbb{N}, \) and whose morphisms are nondecreasing maps between them.
A cosimplicial object of a category $\mathcal{C}$ (say, a cosimplicial set or a cosimplicial scheme) is a functor from $\Delta$ to $\mathcal{C}$. Taking the 2-category of categories $\text{Cats}$ for $\mathcal{C}$, we obtain a definition of a cosimplicial category.

**Definition 2.2.** A cosimplicial category is a covariant 2-functor $\Delta \to \text{Cats}$ to the 2-category of categories. More explicitly, a cosimplicial category $\mathcal{C}_\bullet$ consists of the following data:

1) a family of categories $\mathcal{C}_k$, $k = 0, 1, 2, \ldots$, indexed by objects of $\Delta$ (here the category $\mathcal{C}_k$ is associated with the set $[1, \ldots, k+1]$);
2) a family of functors $P^*_f : \mathcal{C}_m \to \mathcal{C}_n$ indexed by morphisms of $\Delta$, that is, nondecreasing maps $f : [1, \ldots, m+1] \to [1, \ldots, n+1]$;
3) a family of functor isomorphisms $\epsilon_{f,g} : P^*_f P^*_g \to P^*_{fg}$ indexed by pairs of maps $f, g$ such that the composition $f \circ g$ makes sense.

Isomorphisms in 3) should satisfy the following cocycle condition: the diagram

$$
\begin{array}{ccc}
P^*_f P^*_g P^*_h & \xrightarrow{\epsilon_{f,g}} & P^*_f P^*_g P^*_h \\
\downarrow \epsilon_{g,h} & & \downarrow \epsilon_{f,g,h} \\
P^*_f P^*_g P^*_h & \xrightarrow{\epsilon_{f,gh}} & P^*_f P^*_g P^*_h 
\end{array}
$$

is commutative for any triple of maps $f, g, h$ such that the composition $f \circ g \circ h$ makes sense.

We are interested in the following example of a cosimplicial category.

**Example 2.3.** Let $X \to S$ be a morphism of schemes or stacks. The schemes $X$, $X \times_S X$, $X \times_S X \times_S X$, $\ldots$ and the morphisms

$$p_f : \underbrace{X \times_S X \times \cdots \times X}_{n \text{ times}} \to \underbrace{X \times_S X \times \cdots \times X}_{m \text{ times}},$$

given by the rule

$$p_f(x_1, \ldots, x_n) = (x_{f(1)}, \ldots, x_{f(m)})$$

where $f \in \text{Hom}_\Delta([1, \ldots, m],[1, \ldots, n])$, form a simplicial scheme. The categories of sheaves (the abelian categories of coherent or quasi-coherent sheaves and their derived categories) on these schemes and the inverse image functors between them form a cosimplicial category.

We denote such cosimplicial categories by

$$[\mathcal{D}(X), \mathcal{D}(X \times_S X), \mathcal{D}(X \times_S X \times_S X), \ldots, p^*_\bullet].$$

It is natural to use notation for the functors $P^*_f$ similar to the notation for the inverse image functors between categories of sheaves. For example, the functor $P^*_f : \mathcal{C}_1 \to \mathcal{C}_2$ for the map $f : [1, 2] \to [1, 2, 3]$ such that $f(1) = 1$, $f(2) = 3$, is denoted by $P^*_{13}$.

For any cosimplicial category $\mathcal{C}_\bullet = [\mathcal{C}_0, \mathcal{C}_1, \ldots, P^*_\bullet]$ there is a well-defined descent category, see [5], §19.3. It is denoted by $\text{Kern}(\mathcal{C}_\bullet)$. 
Definition 2.4 (Classical Descent Category). An object of \( \text{Kern}(\mathcal{C}_\bullet) \) is a pair \((\mathcal{F}, \theta)\), where \( \mathcal{F} \in \text{Ob} \mathcal{C}_0 \) and \( \theta \) is an isomorphism \( P_1^\ast \mathcal{F} \to P_2^\ast \mathcal{F} \) satisfying the cocycle condition. The latter means that the following diagram is commutative:

\[
P_{12}^\ast P_1^\ast \mathcal{F} \sim P_{13}^\ast P_1^\ast \mathcal{F} \xrightarrow{P_{13}^\ast \theta} P_{13}^\ast P_2^\ast \mathcal{F} \sim P_{12}^\ast P_2^\ast \mathcal{F} \sim P_{23}^\ast P_1^\ast \mathcal{F} \xrightarrow{P_{23}^\ast \theta} P_{23}^\ast P_2^\ast \mathcal{F}.
\]

In this diagram isomorphisms of functors from the definition of a cosimplicial category are denoted by segments marked by \( \sim \). A morphism in \( \text{Kern}(\mathcal{C}_\bullet) \) from \((\mathcal{F}_1, \theta_1)\) to \((\mathcal{F}_2, \theta_2)\) is a morphism \( f \in \text{Hom}_{\mathcal{C}_0}(\mathcal{F}_1, \mathcal{F}_2) \) such that \( P_2^\ast f \circ \theta_1 = \theta_2 \circ P_1^\ast f \).

For the canonical morphism of stacks \( X \to X/\!/G \), the simplicial scheme from Example 2.3 has the form

\[
(\mathcal{X}/G)_\bullet = [X_0, X_1, X_2, \ldots, p_\bullet] = [X, G \times X, G \times G \times X, \ldots, p_\bullet], \tag{2.1}
\]

where the morphisms \( p_\bullet \) are defined as follows. For a nondecreasing map

\[
f: [1, \ldots, m] \to [1, \ldots, n]
\]

the morphism of schemes

\[
p_f: G \times \cdots \times G \times X \xrightarrow{n \text{ times}} G \times \cdots \times G \times X \xrightarrow{m \text{ times}}
\]

is given by the rule

\[(g_n, \ldots, g_2, x_1) \mapsto (g_f(m) \cdots g_f(m-1)+1, \ldots, g_f(2) \cdots g_f(1)+1, g_f(1) \cdots g_2 x_1).\]

For small \( n = m \pm 1 \) and strictly increasing \( f \) the morphisms \( p_\bullet \) have the form

\[
\begin{align*}
G \times G \times X & \xrightarrow{1 \times a} G \times X \xleftarrow{a} X, \\
G \times G \times X & \xrightarrow{\mu \times 1} G \times X \xleftarrow{e \times 1} X, \\
G \times G \times X & \xrightarrow{e \times 1} G \times X \xleftarrow{p_2} X, \\
G \times G \times X & \xrightarrow{p_{23}} G \times G \times X.
\end{align*}
\]

It is clear from the definition that for a cosimplicial category

\[
[q\text{coh}(X), q\text{coh}(G \times X), q\text{coh}(G \times G \times X), \ldots, p_\bullet^\ast],
\]

formed by abelian categories of quasi-coherent sheaves on (2.1), the descent category \( \text{Kern} \) is precisely the category of equivariant sheaves. Below we show that for a linearly reductive group \( G \) the descent category

\[
\mathcal{D}(X)^G = \text{Kern}([\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, p_\bullet^\ast]) \tag{2.2}
\]
is equivalent to the derived category of equivariant sheaves $\mathcal{D}^G(X)$. In other words, passage to the descent category commutes with passage to the derived category.

Now we recall necessary definitions and facts concerning comonads and comodules over a comonad. Details can be found in Barr and Wells [6], Ch. 3 and Mac Lane [7], Ch. 6.

Let $\mathcal{C}$ be a category.

Definition 2.5. A comonad $T = (T, \varepsilon, \delta)$ on a category $\mathcal{C}$ consists of a functor $T: \mathcal{C} \to \mathcal{C}$ and natural transformations of functors $\varepsilon: T \to \text{Id}_\mathcal{C}$ and $\delta: T \to T^2 = TT$ such that the following diagrams are commutative:

$$
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow \delta & & \downarrow T\varepsilon \\
T^2 & \xrightarrow{\varepsilon T} & T
\end{array}
$$

and

$$
\begin{array}{ccc}
T & \xrightarrow{\delta} & T^2 \\
\downarrow \delta & & \downarrow T\delta \\
T^2 & \xrightarrow{T\delta} & T^3.
\end{array}
$$

The main (and essentially unique) example of a comonad is the following one.

Example 2.6. Consider a pair of adjoint functors $P^*: \mathcal{B} \to \mathcal{C}$ (left) and $P_*: \mathcal{C} \to \mathcal{B}$ (right). Let $\eta: \text{Id}_\mathcal{B} \to P_*P^*$ and $\varepsilon: P^*P_* \to \text{Id}_\mathcal{C}$ be the natural adjunction morphisms. Define a triple $(T, \varepsilon, \delta)$ by taking $T = P^*P_*$ and $\varepsilon: P^*P_* \to \text{Id}_\mathcal{C}$, $\delta = P^*\eta P_*: P^*P_* \to P^*P_*P^*P_*$. Then $T = (T, \varepsilon, \delta)$ is a comonad on the category $\mathcal{C}$.

In fact, every comonad can be obtained in this way from a pair of adjoint functors. This follows from a construction due to Eilenberg and Moore which will be discussed below.

Definition 2.7. Let $T = (T, \varepsilon, \delta)$ be a comonad on a category $\mathcal{C}$. A comodule over $T$ (or a $T$-coalgebra) is a pair $(F, h)$, where $F \in \text{Ob} \mathcal{C}$ and $h: F \to TF$ is a morphism satisfying the following two conditions: the composition

$$
F \xrightarrow{h} TF \xrightarrow{\varepsilon F} F
$$

is the identity, and the diagram

$$
\begin{array}{ccc}
F & \xrightarrow{h} & TF \\
\downarrow h & & \downarrow T^h \\
TF & \xrightarrow{\delta F} & T^2 F
\end{array}
$$

is commutative.

A morphism between comodules $(F_1, h_1)$ and $(F_2, h_2)$ is a morphism $f: F_1 \to F_2$ in the category $\mathcal{C}$ such that $h_2 \circ f = Tf \circ h_1$.

All comodules over a given comonad $T$ on $\mathcal{C}$ form a category which is denoted by $\mathcal{C}_T$. We define the functor $P^*: \mathcal{C}_T \to \mathcal{C}$ as the forgetful functor $(F, h) \mapsto F$ and the functor $P_*: \mathcal{C} \to \mathcal{C}_T$ by $F \mapsto (TF, \delta F)$. One can show (see [6], §3.2) that the functors $P^*$ and $P_*$ are adjoint and the comonad $T$ is isomorphic to the comonad constructed from the pair $P^*, P_*$.

We will need a more general notion.
Definition 2.8. Let $T$ be a comonad on $\mathcal{C}$ and let $\mathcal{C}' \subset \mathcal{C}$ be a subcategory. We define $\mathcal{C}'_T$ to be a full subcategory in the category of comodules over $T$ formed by comodules $(F, h)$ such that $F \in \mathcal{C}'$.

Suppose that a cosimplicial category $[\mathcal{C}_0, \mathcal{C}_1, \ldots, P^\bullet]$ satisfies two conditions. First, the functors $P^f$ have right adjoint functors. We denote them by $P^f_*$. Second, the functors $P^f_\mathcal{C}$ and $P^f_\mathcal{C}'$ satisfy the condition which is an axiomatization of the theorem on flat base change. Namely, for any commutative square in the category $\Delta$

\[
\begin{array}{ccc}
1, \ldots, m + n - r & \xleftarrow{f'} & 1, \ldots, n \\
\uparrow g' & & \uparrow g \\
1, \ldots, m & \xleftarrow{f} & 1, \ldots, r
\end{array}
\]

with injective maps $f$, $f'$ and with $[1, \ldots, m + n - r] = \text{Im } f' \cup \text{Im } g'$, the natural base change morphism

$P^f_* P^g_* \to P^{g'}_* P^{f'}_*$

is an isomorphism of functors.

Definition 2.9. A cosimplicial category satisfying the two conditions given above is called a cosimplicial category with base change.

Cosimplicial categories formed by the categories of sheaves (abelian or derived) in Example 2.3 are cosimplicial categories with base change.

To any cosimplicial category with base change $\mathcal{C}_\bullet = [\mathcal{C}_0, \mathcal{C}_1, \ldots, P^\bullet]$ one can associate a comonad on the category $\mathcal{C}_0$. Let $P^\bullet \colon \text{Kern}(\mathcal{C}_\bullet) \to \mathcal{C}_0$ be the forgetful functor sending a pair $(F, \theta)$ to $F$. By [8], Proposition 2.9, it has the right adjoint functor $P_\bullet$.

Definition 2.10. We define $T_{\mathcal{C}_\bullet} = (P^\bullet P_\bullet, \varepsilon, \delta)$ as the comonad on $\mathcal{C}_0$ associated with the adjoint pair $(P^\bullet, P_\bullet)$.

For the cosimplicial categories in Example 2.3 this comonad coincides with the comonad associated with the pair of direct and inverse image functors between the categories of sheaves on $X$ and $S$.

Definition 2.11 (Comonad Descent Category). For a cosimplicial category with base change $\mathcal{C}_\bullet$ we define $\mathcal{C}_\bullet_T$ as the category of comodules over the comonad $T_{\mathcal{C}_\bullet}$ on $\mathcal{C}_0$ associated with $\mathcal{C}_\bullet$.

Proposition 2.12 (see [8], Proposition 4.2). For a cosimplicial category with base change $\mathcal{C}_\bullet$, the descent categories $\text{Kern}(\mathcal{C}_\bullet)$ and $\mathcal{C}_\bullet_T$ are equivalent.

It is easy to see that the category of comodules over a comonad $T = (T, \varepsilon, \delta)$ with an exact functor $T$ on an abelian category $\mathcal{C}$ is also abelian. At the same time, it is not clear whether the category $\mathcal{C}_T$ will be triangulated for a comonad $T = (T, \varepsilon, \delta)$ with an exact functor $T$ on a triangulated category $\mathcal{C}$. We can attempt to define a triangulated structure on $\mathcal{C}_T$ in the following way.
Definition 2.13. Let \( \mathcal{C} \) be a triangulated category and \( T = (T, \varepsilon, \delta) \) be a comonad on \( \mathcal{C} \). We define the shift functor on \( \mathcal{C}_T \) by \( (F, h)[1] = (F[1], h[1]) \), \( f[1] = f[1] \). A triangle \( (F', h') \to (F, h) \to (F'', h'') \to (F', h')[1] \) is called distinguished in \( \mathcal{C}_T \) if the triangle \( F' \to F \to F'' \to F'[1] \) is distinguished in \( \mathcal{C} \).

The cone operation is not functorial, so without additional assumptions we cannot verify that any morphism in \( \mathcal{C}_T \) fits into a distinguished triangle. But in the examples below which are of interest to us the definition given above does indeed introduce a triangulated structure on \( \mathcal{C}_T \).

Let \( p: X \to S \) be a morphism and let
\[
\mathcal{C}_\bullet = [\mathcal{D}(X), \mathcal{D}(X \times S X), \ldots, p^*_\bullet]
\]
be a cosimplicial category formed by the derived categories of sheaves on the cosimplicial scheme in Example 2.3. We denote the corresponding comonad on the category \( \mathcal{D}(X) \) by \( T_p \). There is a canonical functor
\[
\Phi: \mathcal{D}(S) \to \mathcal{D}(X)_{T_p}
\]
called a comparison functor. It sends a complex \( H \) to the pair \((p^*H, h)\), where \( h: p^*H \to p^*p_*p^*H \) is the canonical adjunction morphism.

We will now state the main results on the equivalence of the descent category and the base category.

Theorem 2.14 (see [8], Theorem 7.3). Assume that for a morphism of quasi-projective schemes \( p: X \to S \) over a field the canonical morphism \( \mathcal{O}_S \to Rp_*\mathcal{O}_X \) is a split embedding. Then the comparison functors
\[
\Phi: \mathcal{D}(S) \to \mathcal{D}(X)_{T_p}, \quad \Phi: \mathcal{D}^{perf}(S) \to \mathcal{D}^{perf}(X)_{T_p}, \quad \Phi: \mathcal{D}^b(S) \to \mathcal{D}^b(X)_{T_p}
\]
are equivalences.

To formulate the second result we require the following definition.

Definition 2.15 (see [9], Definition 1.4). A group scheme \( G \) over a field \( k \) is called linearly reductive if the category of finite-dimensional representations of \( G \) over \( k \) is semisimple.

Theorem 2.16 (see [8], Theorem 9.6). If a linearly reductive group scheme \( G \) over a field \( k \) acts on a quasi-projective scheme \( X \) over \( k \), then the comparison functors
\[
\Phi: \mathcal{D}^G(X) \to \mathcal{D}(X)_{T_G}, \quad \Phi: \mathcal{D}^{perf,G}(X) \to \mathcal{D}^{perf}(X)_{T_G}, \quad \Phi: \mathcal{D}^b(coh^G(X)) \to \mathcal{D}^b(X)_{T_G}
\]
are equivalences. (Here \( T_G \) is a comonad on \( \mathcal{D}(X) \) associated with (2.2).)

Using the equivalence between \( \mathcal{D}(X)_{T_p} \) and the triangulated category \( \mathcal{D}(S) \) obtained above we can transfer the triangulated structure from \( \mathcal{D}(S) \) to \( \mathcal{D}(X)_{T_p} \). This also works for \( \mathcal{D}^{perf}(X)_{T_p} \) and \( \mathcal{D}^b(X)_{T_p} \). The triangulated structure we thus obtain coincides with the structure in Definition 2.13 (see [8], Proposition 3.13).

The descent category operation is functorial. To make this more precise, we introduce morphisms of cosimplicial categories and morphisms of categories with a comonad.
**Definition 2.17.** A funtor between cosimplicial categories $\mathcal{C}_\bullet(1)$ and $\mathcal{C}_\bullet(2)$ is a family of functors 
$$\Psi_k: \mathcal{C}_k^{(1)} \rightarrow \mathcal{C}_k^{(2)}, \quad k = 0, 1, \ldots,$$

together with a family of isomorphisms of functors 
$$\beta_f: \Psi_n \circ P_f^{(1)*} \sim P_f^{(2)*} \circ \Psi_m$$

for any morphism $f: [1, \ldots, m+1] \rightarrow [1, \ldots, n+1]$ in $\Delta$, which are compatible with composition of morphisms in $\Delta$.

**Definition 2.18.** Let $\mathcal{C}_i$, $i = 1, 2$ be two categories and $T_i = (T_i, \varepsilon_i, \delta_i)$ be two comonads on them. We say that a functor $\Psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is compatible with $T_1$ and $T_2$ if there exists an isomorphism of functors $\beta: \Psi T_1 \rightarrow T_2 \Psi$ such that the diagrams

$$\begin{array}{ccc}
\Psi T_1 & \xrightarrow{\beta} & T_2 \Psi \\
\downarrow{\Psi \varepsilon_1} & & \downarrow{\varepsilon_2 \Psi} \\
\Psi & \xrightarrow{\Psi \delta_1} & \varepsilon_2 \Psi
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\Psi T_1 & \xrightarrow{\beta} & T_2 \Psi \\
\downarrow{\Psi \delta_1} & & \downarrow{\delta_2 \Psi} \\
\Psi T_1^2 & \xrightarrow{\beta T_1} & T_2 \Psi T_1^2 \\
\downarrow{T_2 \beta} & & \downarrow{T_2^2 \beta}
\end{array}$$

are commutative.

In fact, more accurately, we could include the isomorphism $\beta$ into the defining data and consider the pair $(\Psi, \beta)$ as a morphism in the 2-category of categories with comonad. For our purposes, this is not essential.

One can easily check the following fact.

**Lemma 2.19.** Let $(\Psi_k)$ be a funtor between cosimplicial categories $\mathcal{C}_\bullet(1)$ and $\mathcal{C}_\bullet(2)$ with base change, and let $T_1$ and $T_2$ be the comonads on $\mathcal{C}_0^{(1)}$ and $\mathcal{C}_0^{(2)}$ from Definition 2.10. Then $\Psi_0: \mathcal{C}_0^{(1)} \rightarrow \mathcal{C}_0^{(2)}$ is a funtor compatible with the comonads $T_1$ and $T_2$.

The following lemma will be used to describe the components of semiorthogonal decompositions.

**Lemma 2.20.** Let $T_1$, $T_2$ be comonads on the categories $\mathcal{C}_1$, $\mathcal{C}_2$ respectively, and let $\mathcal{C}_1' \subset \mathcal{C}_1$, $\mathcal{C}_2' \subset \mathcal{C}_2$ be subcategories. Let $\Psi: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a funtor sending $\mathcal{C}_1'$ to $\mathcal{C}_2'$. If $\Psi$ is compatible with $T_1$ and $T_2$, then $\Psi$ induces a funtor $\Psi_T: \mathcal{C}_1'T_{T_1} \rightarrow \mathcal{C}_2'T_{T_2}$. If $\Psi$ is a fully faithful funtor, which establishes an equivalence between $\mathcal{C}_1'$ and some subcategory $\mathcal{A}'$ in $\mathcal{C}_2'$, then $\Psi_T$ is an equivalence $\mathcal{A}'_{T_{T_1}} \subset \mathcal{C}_2'T_{T_2}$.

The proof is obvious.

§ 3. Semiorthogonal decompositions for categories of comodules

Let $T$ be a comonad on a triangulated category $\mathcal{C}$. In this section we will show that a semiorthogonal decomposition of the category $\mathcal{C}$ induces a semiorthogonal decomposition of the category of comodules $\mathcal{C}_T$ provided the category $\mathcal{C}_T$ is triangulated and the initial semiorthogonal decomposition is compatible with $T$. 




Definition 3.1. We say that a functor $T: \mathcal{C} \to \mathcal{C}$ is upper triangular with respect to a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ if

$$T\mathcal{A}_k \subset \langle \mathcal{A}_1, \ldots, \mathcal{A}_k \rangle \quad \text{for all } k, \quad 1 \leq k \leq n. \quad (3.1)$$

Theorem 3.2. Let $T = (T, \varepsilon, \delta)$ be a comonad on a triangulated category $\mathcal{C}$ and let $\mathcal{C}' \subset \mathcal{C}$ be a triangulated subcategory. Assume that the functor $T$ is exact and $\mathcal{C}_T$ is a category triangulated in the sense of Definition 2.13. Assume also that the functor $T$ is upper triangular with respect to a semiorthogonal decomposition $\mathcal{C} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ and that this decomposition induces a semiorthogonal decomposition $\mathcal{C}' = \langle \mathcal{A}'_1, \ldots, \mathcal{A}'_n \rangle$, where $\mathcal{A}'_k = \mathcal{A}_k \cap \mathcal{C}'$.

Then the category $\mathcal{C}'_T$ is triangulated and admits a semiorthogonal decomposition $\mathcal{C}'_T = \langle \mathcal{A}'_1 T, \ldots, \mathcal{A}'_n T \rangle$.

Proof. The fact that the categories $\mathcal{C}'_T$ and $\mathcal{A}'_i T$ are triangulated follows directly from the definitions. We will prove our result by induction on $n$. First we treat the case $n = 2$.

Clearly, the subcategories $\mathcal{A}'_1 T$ and $\mathcal{A}'_2 T$ are semiorthogonal. We must check that any object $(F, h)$ in $\mathcal{C}'_T$ can be included in a distinguished triangle

$$(F_2, h_2) \to (F, h) \to (F_1, h_1) \to (F_2, h_2)[1], \quad (3.2)$$

where $F_i \in \mathcal{A}'_i$ and the morphisms lie in $\mathcal{C}'_T$. Since $\mathcal{C}' = \langle \mathcal{A}'_1, \mathcal{A}'_2 \rangle$, there exists a distinguished triangle in the category $\mathcal{C}'$,

$$F_2 \to F \to F_1 \to F_2[1], \quad (3.3)$$

where $F_i \in \mathcal{A}'_i$. Applying the exact functor $T$ to this triangle, we obtain a distinguished triangle

$$TF_2 \to TF \to TF_1 \to TF_2[1].$$

Since $T$ is upper triangular, $\text{Hom}(F_2, TF_1) = 0$, and the morphism $h: F \to TF$ can be extended to a morphism of triangles

$$\begin{array}{cccc}
F_2 & \to & F & \to & F_1 & \to & F_2[1] \\
\downarrow h_2 & & \downarrow h & & \downarrow h_1 & & \downarrow h_2[1] \\
TF_2 & \to & TF & \to & TF_1 & \to & TF_2[1].
\end{array} \quad (3.4)$$

We will show that the pairs $(F_i, h_i)$ are comodules over $T$. Indeed, taking the composition of (3.4) with

$$\begin{array}{cccc}
TF_2 & \to & TF & \to & TF_1 & \to & TF_2[1] \\
\downarrow Th_2 & & \downarrow Th & & \downarrow Th_1 & & \downarrow Th_2[1] \\
TTF_2 & \to & TTF & \to & TTF_1 & \to & TTF_2[1]
\end{array}$$

and

$$\begin{array}{cccc}
TF_2 & \to & TF & \to & TF_1 & \to & TF_2[1] \\
\downarrow \delta F_2 & & \downarrow \delta F & & \downarrow \delta F_1 & & \downarrow \delta F_2[1] \\
TTF_2 & \to & TTF & \to & TTF_1 & \to & TTF_2[1]
\end{array}$$
we obtain two morphisms of triangles

\[
\begin{array}{cccc}
F_2 & \rightarrow & F & \rightarrow & F_1 & \rightarrow & F_2[1] \\
\downarrow Th_2 \circ h_2 & & \downarrow \delta F_2 \circ h_2 & & \downarrow Th_1 \circ h_1 & & \downarrow \delta F_1 \circ h_1 [1] & \downarrow \delta F_2 \circ h_2 [1] \\
TTF_2 & \rightarrow & TTF & \rightarrow & TTF_1 & \rightarrow & TTF_2 [1].
\end{array}
\]

These morphisms coincide at the middle terms. Since Hom\((F_2, TTF_1) = 0\), there exists a unique morphism of triangles extending the given morphism of middle terms. Consequently, \(\delta F_i \circ h_i = Th_i \circ h_i\). In the same way we can check that \(\varepsilon F_i \circ h_i = \text{Id}_{F_i}\). It follows from diagram (3.4) that we have constructed the required triangle (3.2).

Now assume that the assertion is proved for some \(n \geq 2\). We will verify it for \(n + 1\). The decompositions \(\mathcal{A}_0 = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\) and \(\mathcal{C} = \langle \mathcal{A}_0, \mathcal{A}_{n+1} \rangle\) satisfy the assumptions of the theorem, and we obtain

\[
\mathcal{C}_T = \langle \mathcal{A}'_{0T}, \mathcal{A}'_{n+1T} \rangle = \langle \langle \mathcal{A}'_{1T}, \ldots, \mathcal{A}'_{nT} \rangle, \mathcal{A}'_{n+1T} \rangle = \langle \mathcal{A}'_{1T}, \ldots, \mathcal{A}'_{nT}, \mathcal{A}'_{n+1T} \rangle.
\]

\[\text{§ 4. Descent for semiorthogonal decompositions: coverings of schemes}\]

Let \(p: X \rightarrow S\) be a flat morphism of quasi-projective schemes such that \(\mathcal{O}_S\) is a direct summand in \(Rp_* \mathcal{O}_X\). In this case the derived category of sheaves on \(S\) is equivalent to the descent category associated with the derived category of sheaves on \(X\); see Theorem 2.14. Together with Theorem 3.2, this allows us to construct semiorthogonal decompositions of the derived category of sheaves on \(S\). The role of the category \(\mathcal{C}\) in Theorem 3.2, where a comonad is defined, is played by the unbounded derived category \(\mathcal{D}(X)\). Here the functor \(p^* p_*\) on the category \(\mathcal{D}(X)\) does not preserve ‘small’ subcategories, such as \(\mathcal{D}_{\text{perf}}(X)\) or \(\mathcal{D}^b(X)\). So to apply the results of the previous section we need to construct a semiorthogonal decomposition of the ‘big’ category.

We denote the comonad on \(\mathcal{D}(X)\) associated with the morphism \(p\) by \(T_p = (T_p, \varepsilon, \delta)\).

**Theorem 4.1.** Let \(X\) and \(S\) be quasi-projective schemes and let \(p: X \rightarrow S\) be a flat morphism. Assume that \(\mathcal{O}_S\) is a direct summand in \(Rp_* \mathcal{O}_X\). Let \(\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\) be a semiorthogonal decomposition such that the functor \(T_p = p^* p_*\) is upper triangular. Then the category \(\mathcal{D}(S)\) admits a semiorthogonal decomposition \(\langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle\), where \(\mathcal{B}_k \subset \mathcal{D}(S)\) denotes a full subcategory consisting of objects \(H\) such that \(p^* H \in \mathcal{A}_k\).

**Proof.** By Theorem 2.14, the category \(\mathcal{D}(S)\) is equivalent to \(\mathcal{D}(X)_{T_p}\). The assertion follows from Theorem 3.2 applied to \(\mathcal{C} = \mathcal{C}' = \mathcal{D}(X)\), the description of the equivalence \(\mathcal{D}(S) \cong \mathcal{D}(X)_{T_p}\), and Definition 2.8 of categories \(\mathcal{A}_{iT_p}\).

To prove similar statements for the category of perfect complexes and the bounded derived category of coherent sheaves on a scheme, we need to extend semiorthogonal decompositions of these categories to a semiorthogonal decomposition of the unbounded derived category. In [10] Kuznetsov showed that such an extension exists. We recall the construction.
Let \( \mathcal{A} \subset \mathcal{D}(X) \) be a subcategory. We define \( \mathcal{A} \oplus \infty \subset \mathcal{D}(X) \) to be the minimal triangulated subcategory in \( \mathcal{D}(X) \) containing \( \mathcal{A} \) and closed under direct sums.

**Lemma 4.2.** 1. Assume that the category \( \mathcal{D}^{\text{perf}}(X) \) admits a semiorthogonal decomposition \( \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \). Let \( \mathcal{A}_i = (\mathcal{A}_i^{\text{perf}}) \oplus \infty \). Then the categories \( \mathcal{A}_i \) form a semiorthogonal decomposition \( \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \) and \( \mathcal{A}_i^{\text{perf}} = \mathcal{A}_i \cap \mathcal{D}^{\text{perf}}(X) \).

2. Let \( \mathcal{A}^p \subset \mathcal{D}^{\text{perf}}(X) \) be a left admissible triangulated subcategory and let \( \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}^p, \mathcal{Z}^p \rangle \) be the corresponding semiorthogonal decomposition. Then \( \mathcal{A} = (\mathcal{A}^p) \oplus \infty \) is the right orthogonal to \( \mathcal{Z}^p \) in \( \mathcal{D}(X) \).

3. Let \( \mathcal{D}^b(X) = \langle \mathcal{A}_1', \ldots, \mathcal{A}_n' \rangle \) be a semiorthogonal decomposition into admissible subcategories. Then the categories \( \mathcal{A}_i' \) defined as \( \langle \mathcal{A}_1' \rangle \oplus \infty \) form a semiorthogonal decomposition \( \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \), and \( \mathcal{A}_i' = \mathcal{A}_i \cap \mathcal{D}^b(X) \).

**Proof.** 1. This follows from [10], Proposition 4.2.

2. By part 1, there is a semiorthogonal decomposition

\[
\mathcal{D}(X) = \langle (\mathcal{A}^p) \oplus \infty, (\mathcal{Z}^p) \oplus \infty \rangle.
\]

Consequently,

\[
(\mathcal{A}^p) \oplus \infty = ((\mathcal{Z}^p) \oplus \infty)^\perp = (\mathcal{Z}^p) ^\perp.
\]

3. For any \( i \) consider a semiorthogonal decomposition \( \mathcal{D}^b(X) = \langle \mathcal{A}_i', \mathcal{Z}_i' \rangle \) and take

\[
\mathcal{A}_i^p = \mathcal{A}_i' \cap \mathcal{D}^{\text{perf}}(X), \quad \mathcal{Z}_i^p = \mathcal{Z}_i' \cap \mathcal{D}^{\text{perf}}(X), \quad \mathcal{A}_i = (\mathcal{A}_i^p) \oplus \infty.
\]

By [10], Proposition 4.1 we obtain a decomposition \( \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_i^p, \mathcal{Z}^p \rangle \). Part 2 implies that \( \mathcal{A}_i' \subset (\mathcal{Z}_i')^\perp \subset (\mathcal{Z}^p)^\perp = \mathcal{A}_i \). Thus we have \( \mathcal{A}_i = (\mathcal{A}_i') \oplus \infty \). Finally, [10], Proposition 4.1 yields that there is a decomposition \( \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_1^p, \ldots, \mathcal{A}_n^p \rangle \).

By part 1, we obtain a semiorthogonal decomposition \( \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \). The final assertion follows from [10], Lemma 3.2.

**Theorem 4.3.** Let \( X, S \) and \( p \) be as in Theorem 4.1 and let \( p_1, p_2 \) be the projections \( X \times_S X \to X \). Assume that the category of perfect complexes \( \mathcal{D}^{\text{perf}}(X) \) admits a semiorthogonal decomposition \( \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \), and \( \text{Hom}(p_i^* F_j, p_j^* F_i) = 0 \) for all \( 1 \leq i < j \leq n \) and all objects \( F_i \in \mathcal{A}_i^{\text{perf}}, F_j \in \mathcal{A}_j^{\text{perf}} \). Then \( \mathcal{D}^{\text{perf}}(S) \) admits a semiorthogonal decomposition \( \langle \mathcal{B}_1^{\text{perf}}, \ldots, \mathcal{B}_n^{\text{perf}} \rangle \), where \( \mathcal{B}_k^{\text{perf}} \subset \mathcal{D}^{\text{perf}}(S) \) denotes a full subcategory formed by objects \( H \) such that \( p^* H \in \mathcal{A}_k^{\text{perf}} \).

**Proof.** By Theorem 2.14, the category \( \mathcal{D}^{\text{perf}}(S) \) is equivalent to \( \mathcal{D}^{\text{perf}}(X)_{T_p} \). Thus we can use Theorem 3.2 on semiorthogonal decompositions for descent categories.

We extend the semiorthogonal decomposition \( \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \) to a decomposition \( \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \) of the category \( \mathcal{D}(X) \) as in part 1 of Lemma 4.2. We are going to check that the functor \( T_p = p^* p_* \) is upper triangular with respect to the decomposition \( \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \). It can be shown that the category \( \mathcal{A}_i = (\mathcal{A}_i^{\text{perf}}) \oplus \infty \) is obtained from \( \mathcal{A}_i^{\text{perf}} \) by adding all possible direct sums (once) and successively adding cones of morphisms; see [10], Proposition 4.2. By assumption, with \( i < j \), \( F_i \in \mathcal{A}_i^{\text{perf}}, F_j \in \mathcal{A}_j^{\text{perf}} \), we have

\[
\text{Hom}(F_j, T_p F_i) = \text{Hom}(F_j, p^* p_* F_i) = \text{Hom}(F_j, p_1^* p_2^* F_i) = \text{Hom}(p_1^* F_j, p_2^* F_i) = 0,
\]
where the second equality is the base change formula. For arbitrary families
\( F_i^\alpha \in \mathcal{A}_i^{\text{perf}}, F_j^\beta \in \mathcal{A}_j^{\text{perf}} \) we obtain
\[
\text{Hom}\left( \bigoplus \beta F_j^\beta, T_p \left( \bigoplus \alpha F_i^\alpha \right) \right) = \prod \text{Hom}\left( F_j^\beta, \bigoplus \alpha T_p F_i^\alpha \right) = \prod \bigoplus \text{Hom}(F_j^\beta, T_p F_i^\alpha) = 0,
\]
because the objects \( F_j \) are compact and the functor \( T_p \) commutes with direct sums. Taking cones does not affect the orthogonality. Hence the equality
\[
\text{Hom}(F_j, T_p F_i) = 0
\]
holds for any \( F_i \in \mathcal{A}_i, F_j \in \mathcal{A}_j \). This means that \( T_p \mathcal{A}_i \subset \langle \mathcal{A}_1, \ldots, \mathcal{A}_i \rangle \), that is, \( T_p \) is upper triangular with respect to this decomposition.

To complete the proof of the theorem, we apply Theorem 3.2 to the categories \( \mathcal{C} = \mathcal{D}(X), \mathcal{C}' = \mathcal{D}^{\text{perf}}(X) \) and \( \mathcal{A}_i \) constructed above.

**Theorem 4.4.** Let \( X, S \) and \( p \) be as in Theorem 4.1. Assume that the bounded derived category \( \mathcal{D}^b(X) \) admits a semiorthogonal decomposition \( \langle \mathcal{A}_1', \ldots, \mathcal{A}_n' \rangle \) into admissible subcategories and
\[
\text{Hom}(p^*_i F_j, p^*_j F_i) = 0
\]
for all \( 1 \leq i < j \leq n \) and all objects \( F_i \in \mathcal{A}_i', F_j \in \mathcal{A}_j' \). Then the category \( \mathcal{D}^b(S) \) admits a semiorthogonal decomposition \( \langle \mathcal{B}_1', \ldots, \mathcal{B}_n' \rangle \), where \( \mathcal{B}_k' \subset \mathcal{D}^b(S) \) denotes the full subcategory consisting of objects \( H \) such that \( p^* H \in \mathcal{A}_k' \).

The proof is identical with the proof of the previous theorem; here we use part 3 of Lemma 4.2.

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**§ 5. Decompositions invariant under group actions**

In this section we introduce invariant semiorthogonal decompositions of the category \( \mathcal{D}^{\text{perf}}(X) \) under an action of an algebraic group \( G \) on a scheme \( X \). We show that the functor \( p^* p_* \), where \( p \) is the morphism \( X \to X/G \), is upper triangular with respect to the decomposition \( \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \) if and only if
\[
p_2^* \mathcal{A}_i^{\text{perf}} = a^* \mathcal{A}_i^{\text{perf}}
\]
holds for all \( i \) in notation of the following lemma. As above, we denote by \( a \) and \( p_2 : G \times X \to X \) the morphism of the action and the projection to the second factor, respectively.

**Lemma 5.1.** Let \( X \) be a quasi-projective scheme and \( Y \) an affine scheme over \( k \). Assume that the category of perfect complexes \( \mathcal{D}^{\text{perf}}(X) \) admits a semiorthogonal decomposition \( \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \). Then we have the following semiorthogonal decompositions:
\[
\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle,
\mathcal{D}(Y \times X) = \langle p_2^* \mathcal{A}_1, \ldots, p_2^* \mathcal{A}_n \rangle,
\mathcal{D}^{\text{perf}}(Y \times X) = \langle p_2^* \mathcal{A}_1^{\text{perf}}, \ldots, p_2^* \mathcal{A}_n^{\text{perf}} \rangle,
\]
where \( \mathcal{A}_i = (\mathcal{A}_i^{\text{perf}}) \oplus \infty \) and \( p_2^* \mathcal{A}_k \) is generated as a triangulated subcategory in \( \mathcal{D}(Y \times X) \) by objects of the form \( p_2^* F_k, F_k \in \mathcal{A}_k \).
Proof. Kuznetsov’s paper [10] on base change for semiorthogonal decomposition covers most of the assertions in the lemma. We recall some constructions in order to prove the following fact which is specific to the case of an affine scheme $Y$: the category $p^*_A \mathcal{A}_k$ is generated by objects of the form $p^*_A F_k$, $F_k \in \mathcal{A}_k$.

We define $p^*_A \mathcal{A}_i^{\text{perf}}$ to be the subcategory in $\mathcal{D}^{\text{perf}}(Y \times X)$ generated by objects of the form $p^*_A H \otimes p^*_A F_i$, where $H \in \mathcal{D}^{\text{perf}}(Y)$ and $F_i \in \mathcal{A}_i^{\text{perf}}$, using shifts, cones and taking direct summands. By [10], § 5.1 there is a semiorthogonal decomposition

$$\mathcal{D}^{\text{perf}}(Y \times X) = \langle p^*_A \mathcal{A}_1^{\text{perf}}, \ldots, p^*_A \mathcal{A}_n^{\text{perf}} \rangle.$$

We define $\mathcal{A}_i$ to be $(\mathcal{A}_i^{\text{perf}})^{\boxplus \infty}$ and $p^*_A \mathcal{A}_i$ to be $(p^*_A \mathcal{A}_i^{\text{perf}})^{\boxplus \infty}$. By [10], Proposition 4.2 these categories form semiorthogonal decompositions

$$\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \quad \text{and} \quad \mathcal{D}(Y \times X) = \langle p^*_A \mathcal{A}_1, \ldots, p^*_A \mathcal{A}_n \rangle.$$

Now we are going to check that the category $p^*_A \mathcal{A}_i$ is generated as a triangulated category by objects of the form $p^*_A F_i$, $F_i \in \mathcal{A}_i$. Firstly, such objects lie in $p^*_A \mathcal{A}_i$. Secondly, consider a triangulated subcategory $\mathcal{D}'$ in $\mathcal{D}(Y \times X)$ generated by objects $p^*_A F_i$, $F_i \in \mathcal{A}_i$, for all $1 \leq i \leq n$. This category is closed under direct sums because all the $\mathcal{A}_i$ are of this form and the functor $p^*_A$ commutes with direct sums. Consequently, $\mathcal{D}'$ is closed under taking direct summands because they can be expressed in terms of countable direct sums and cones. The category $\mathcal{D}'$ contains all objects of the form $p^*_A F_i$ for $F_i \in \mathcal{A}_i^{\text{perf}}$ and thus all objects of the form $p^*_A F = p^*_A \mathcal{O}_Y \otimes p^*_A F$ for $F \in \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle$. The right orthogonal to $\mathcal{O}_Y$ in $\mathcal{D}(Y)$ equals zero. So by a result of Ravenel and Neeman’s (see [2], 2.1.2), the object $\mathcal{O}_Y$ generates the category $\mathcal{D}^{\text{perf}}(Y)$ by taking shifts, cones and direct summands. The category $\mathcal{D}'$ is closed under direct summands, and the functor $p^*_A (\_ \otimes p^*_A F$ preserves shifts, cones and direct summands. So all objects of the form $p^*_A H \otimes p^*_A F$ with $H \in \mathcal{D}^{\text{perf}}(Y)$, $F \in \mathcal{D}^{\text{perf}}(X)$ lie in $\mathcal{D}'$. Therefore [10], Lemma 5.2 implies that $\mathcal{D}'$ contains $\mathcal{D}^{\text{perf}}(Y \times X)$. Finally, since the category $\mathcal{D}'$ is closed under direct sums, it coincides with $\mathcal{D}(Y \times X)$. Now [10], Lemma 3.2 yields that the category $p^*_A \mathcal{A}_i$ is generated by objects of the form $p^*_A F_i$, $F_i \in \mathcal{A}_i$.

Assume that an algebraic group $G$ acts on a scheme $X$ and $p$ is the canonical morphism $X \to X/G$. Note that the morphisms $p_2$ and $a: G \times X \to X$ are isomorphic in the sense that there is a commutative diagram

$$G \times X \xrightarrow{(g,x) \mapsto (g^{-1}g_x)} G \times X \xrightarrow{p_2} X \xrightarrow{a} G \times X$$

Hence the previous lemma can be applied to each of the morphisms $p_2$ and $a: G \times X \to X$. For the same reasons, this lemma holds for morphisms $\mu: G \times G \to G$, $p_{23}, \mu \times 1, 1 \times a: G \times G \times X \to G \times X$ and $p_3, a p_{23}, a(1 \times a) = a(\mu \times 1): G \times G \times X \to X$.

Lemma 5.1 motivates the following definition.

Definition 5.2. A subcategory $\mathcal{A}^{\text{perf}} \subset \mathcal{D}^{\text{perf}}(X)$ is invariant under an action of a group $G$ on a scheme $X$ if the subcategories $p^*_G \mathcal{A}^{\text{perf}}$ and $a^* \mathcal{A}^{\text{perf}}$ in $\mathcal{D}^{\text{perf}}(G \times X)$ coincide. We also say that an action of a group preserves a subcategory.
Proposition 5.3. Assume that the category of perfect complexes on \(X\) admits a semiorthogonal decomposition \(\langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\), and let \(\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\) be the corresponding decomposition of the unbounded derived category. Then the following conditions are equivalent:

1) for all \(1 \leq i < j \leq n\) and \(F_i \in \mathcal{A}_i^{\text{perf}}, F_j \in \mathcal{A}_j^{\text{perf}}\) we have

\[
\text{Hom}(p_2^* F_j, a^* F_i) = 0;
\]

2) the functor \(T_p = p^* p_*\) is upper triangular with respect to the decomposition \(\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle\);

3) \(p_2^* \mathcal{A}_i = a^* \mathcal{A}_i\) for all \(i\);

4) \(p_2^* \mathcal{A}_i^{\text{perf}} = a^* \mathcal{A}_i^{\text{perf}}\) for all \(i\).

If these conditions are satisfied, we say that the action of the group preserves the semiorthogonal decomposition.

Proof of Proposition 5.3. Implication 1) \(\implies\) 2) is contained in the proof of Theorem 4.3, if we replace \(p_2\) by \(p_1\) and \(a\) by \(p_2\).

2) \(\implies\) 3). Take an object \(F_k \in \mathcal{A}_k\). For any \(i < k\) and \(F_i \in \mathcal{A}_i\) we have

\[
\text{Hom}(p_2^* F_k, a^* F_i) = \text{Hom}(F_k, p_2^* a^* F_i) = \text{Hom}(F_k, T_p F_i) = 0,
\]

because \(T_p F_i \in \langle \mathcal{A}_1, \ldots, \mathcal{A}_i \rangle = \langle \mathcal{A}_{i+1}, \ldots, \mathcal{A}_n \rangle^\perp\). By Lemma 5.1, the objects \(a^* F_i\) generate \(a^* \mathcal{A}_i\) as a triangulated category. This means that \(p_2^* F_k \in \perp a^* \mathcal{A}_i\). Similarly, for any \(i > k\) and any \(F_i \in \mathcal{A}_i\) we have

\[
\text{Hom}(a^* F_i, p_2^* F_k) = \text{Hom}(F_i, a_* p_2^* F_k) = \text{Hom}(F_i, T_p F_k) = 0.
\]

As above, we obtain that \(p_2^* F_k \in a^* \mathcal{A}_i^\perp\). Finally,

\[
p_2^* F_k \in \perp a^* \mathcal{A}_1, \ldots, a^* \mathcal{A}_{k-1}) \cap \langle a^* \mathcal{A}_{k+1}, \ldots, a^* \mathcal{A}_n \rangle^\perp = a^* \mathcal{A}_k.
\]

By Lemma 5.1, the objects \(p_2^* F_k\) generate the category \(p_2^* \mathcal{A}_k\), and thus \(p_2^* \mathcal{A}_k \subset a^* \mathcal{A}_k\). In the same way we can prove that \(a^* \mathcal{A}_k \subset p_2^* \mathcal{A}_k\).

3) \(\implies\) 4). By Lemma 4.2, part 1, we obtain

\[
p_2^* \mathcal{A}_i^{\text{perf}} = p_2^* \mathcal{A}_i \cap \mathcal{D}^{\text{perf}}(G \times X) = a^* \mathcal{A}_i \cap \mathcal{D}^{\text{perf}}(G \times X) = a^* \mathcal{A}_i^{\text{perf}}.
\]

4) \(\implies\) 1). We have \(p_2^* F_j \in p_2^* \mathcal{A}_j^{\text{perf}} = a^* \mathcal{A}_j^{\text{perf}}\) and \(a^* F_i \in a^* \mathcal{A}_i^{\text{perf}}\). Since \(a^* \mathcal{A}_i^{\text{perf}}\) and \(a^* \mathcal{A}_j^{\text{perf}}\) are semiorthogonal to each other, we obtain

\[
\text{Hom}(p_2^* F_j, a^* F_i) = 0.
\]

Remark 5.4. Conditions 1) and 2) refer to the whole semiorthogonal decomposition, while Conditions 3) and 4) express the invariance of a decomposition through the invariance of its components.

Remark 5.5. Analogues of Lemma 5.1 and Proposition 5.3 hold for bounded derived categories instead of perfect complexes; see [10], Theorem 5.6 and Lemma 4.2, part 3. However, the components of semiorthogonal decompositions of bounded derived categories must be admissible.
Remark 5.6. An action of a finite group $G$ preserves the decomposition
\[ \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_i^{p}, \ldots, \mathcal{A}_n^{p} \rangle, \]
if and only if $g^* \mathcal{A}_i^{p} = \mathcal{A}_i^{p}$ holds for every element $g \in G$ and all $i$. Indeed, in this case the product $G \times X$ is a disjoint union $\bigcup_{g \in G} X$ of several copies of $X$, and the projections $p_2, a: G \times X \to X$ have the form $(1_X)_{g \in G}$ and $(g)_{g \in G}$, respectively. The subcategories $p_2^* \mathcal{A}_i^{p}$ and $a^* \mathcal{A}_i^{p}$ in $\mathcal{D}^{\text{perf}}(G \times X)$ are the categories $\bigoplus_{g \in G} \mathcal{A}_i^{p}$ and $\bigoplus_{g \in G} g^* \mathcal{A}_i^{p}$. The fact that they coincide means that $g^* \mathcal{A}_i^{p} = \mathcal{A}_i^{p}$ for all $g$ and $i$.

§ 6. Descent for semiorthogonal decompositions: equivariant categories

In this section we prove the main theorems which allow us to construct a semiorthogonal decomposition of the derived category of equivariant sheaves on a scheme starting from a semiorthogonal decomposition of the derived category of sheaves that is preserved by the action of the group.

For any object $F \in \mathcal{D}^G(X)$ we let $\overline{F} \in \mathcal{D}(X)$ denote the object obtained from $F$ by forgetting the equivariant structure.

**Theorem 6.1.** Let $X$ be a quasi-projective scheme over a field $k$ equipped with an action of an affine group scheme $G$ of finite type over $k$. Assume that $G$ is linearly reductive and there is a semiorthogonal decomposition

\[ \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \]

with $\text{Hom}(p_2^* F_j, a^* F_i) = 0$ for any $1 \leq i < j \leq n$ and $F_i \in \mathcal{A}_i$, $F_j \in \mathcal{A}_j$. Let $\mathcal{B}_i$ denote the full subcategory in $\mathcal{D}^G(X)$ consisting of objects $F$ such that $\overline{F} \in \mathcal{A}_i$:

\[ \mathcal{B}_i = \{ F \in \mathcal{D}^G(X) \mid \overline{F} \in \mathcal{A}_i \}. \]

Then there is a semiorthogonal decomposition

\[ \mathcal{D}^G(X) = \langle \mathcal{B}_1, \ldots, \mathcal{B}_n \rangle. \]

**Proof.** Let $p: X \to X/G$ be the natural morphism of stacks. By Theorem 2.16, $\mathcal{D}^G(X)$ is equivalent to the descent category $\mathcal{D}(X)_{T_G}$ of comodules over the comonad $T_G = (p^* p_*, \varepsilon, \delta)$ on $\mathcal{D}(X)$. It follows from the proof of Theorem 4.3 that the functor $T_G = p^* p_* = p_2^* a^*$ is upper triangular with respect to the initial semiorthogonal decomposition. Theorem 3.2 applied to the categories $\mathcal{C} = \mathcal{C}' = \mathcal{D}(X)$, the decomposition $\mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$ and the comonad $T_G$ completes the proof.

**Theorem 6.2.** Let $X$ and $G$ be as in Theorem 6.1. Assume that in the semiorthogonal decomposition

\[ \mathcal{D}^{\text{perf}}(X) = \langle \mathcal{A}_1^{\text{perf}}, \ldots, \mathcal{A}_n^{\text{perf}} \rangle \]

for all $1 \leq i \leq n$ we have $p_2^* \mathcal{A}_i^{\text{perf}} = a^* \mathcal{A}_i^{\text{perf}}$. Let

\[ \mathcal{B}_i^{\text{perf}} = \{ F \in \mathcal{D}^{\text{perf}, G}(X) \mid \overline{F} \in \mathcal{A}_i^{\text{perf}} \}. \]

Then there is a semiorthogonal decomposition

\[ \mathcal{D}^{\text{perf}, G}(X) = \langle \mathcal{B}_1^{\text{perf}}, \ldots, \mathcal{B}_n^{\text{perf}} \rangle. \]
Proof. Following Lemma 4.2, part 1, we extend the initial semiorthogonal decomposition to a semiorthogonal decomposition \( \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \) such that \( \mathcal{A}_i^{\text{perf}} = \mathcal{A}_i \cap \mathcal{D}^{\text{perf}}(X) \). By Proposition 5.3, the functor \( p^*p_* \) is upper triangular with respect to the decomposition thus obtained. To prove the theorem, we apply Theorem 3.2 to the categories \( C = \mathcal{D}(X) \), \( C' = \mathcal{D}^{\text{perf}}(X) \), the decomposition \( \mathcal{D}(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle \) and the comonad \( T_G \).

Theorem 6.3. Let \( X \) and \( G \) be as in Theorem 6.1. Assume that in the semiorthogonal decomposition into admissible components

\[
\mathcal{D}^b(X) = \langle \mathcal{A}'_1, \ldots, \mathcal{A}'_n \rangle
\]

for any \( 1 \leq i \leq n \) we have \( p^* \mathcal{A}'_i = a^* \mathcal{A}'_i \). Let

\[
\mathcal{B}'_i = \{ F \in \mathcal{D}^b(\text{coh}^G(X)) \mid F \in \mathcal{A}'_i \}.
\]

Then there is a semiorthogonal decomposition

\[
\mathcal{D}^b(\text{coh}^G(X)) = \langle \mathcal{B}'_1, \ldots, \mathcal{B}'_n \rangle.
\]

The proof is similar to the proof of Theorem 6.2; we use an analogue of Proposition 5.3 for a bounded derived category.

§ 7. Derived descent theory for twisted equivariant sheaves

In this section we generalize the results on descent for derived categories obtained in [8] to twisted equivariant sheaves. We use the notion of a cocycle on an algebraic group and of twisted equivariant sheaves introduced in [1]. We recall the necessary definitions and background.

Let \( G \) be a group scheme of finite type over a field \( k \) with multiplication morphism \( \mu: G \times G \to G \).

Definition 7.1. A cocycle on \( G \) is a pair \((\mathcal{L}, \alpha)\) consisting of a line bundle \( \mathcal{L} \) on \( G \) and an isomorphism of line bundles \( \alpha: p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \to a^* \mathcal{L} \) on \( G \times G \) such that the following associativity condition holds: the diagram of isomorphisms of sheaves on the product \( G \times G \times G \)

\[
p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* \mathcal{L} \xrightarrow{1 \otimes p_2^* \alpha} p_1^* \mathcal{L} \otimes (p_2 p_3)^* \mathcal{L} \xrightarrow{(1 \times 1)^* \alpha} (\mu \times 1)^* \mathcal{L} \sim (\mu(1 \times 1))^* \mathcal{L}
\]

is commutative.

Let \((\mathcal{L}, \alpha)\) be a cocycle on \( G \) and let the group \( G \) act on a scheme \( X \). Denote the morphism of the action by \( a: G \times X \to X \).

Definition 7.2. A pair \( \mathcal{F} = (F, \theta) \) formed by a sheaf \( F \) on \( X \) and an isomorphism

\[
\theta: p_1^* \mathcal{L} \otimes p_2^* F \to a^* F
\]
of sheaves on $G \times X$ is called an $(\mathcal{L}, \alpha)$-$G$-equivariant sheaf on $X$ if the following compatibility condition holds: the diagram of sheaves on $G \times G \times X$

\[
p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \otimes p_3^* F \xrightarrow{1 \otimes p_2^* \theta} p_1^* \mathcal{L} \otimes (ap_{23})^* F
\]

\[
p_1^* \alpha \otimes 1
\]

\[
(\mu p_{12})^* \mathcal{L} \otimes p_3^* F \xrightarrow{(\mu \times 1)^* \theta} (a(\mu \times 1))^* F \xrightarrow{(1 \times a)^* \theta} (a(1 \times a))^* F
\]

is commutative. A morphism of $(\mathcal{L}, \alpha)$-equivariant sheaves from $(F_1, \theta_1)$ to $(F_2, \theta_2)$ is a homomorphism of sheaves $F_1 \to F_2$ compatible with the structure morphisms $\theta_1$ and $\theta_2$.

In particular, if $X$ is a point, we obtain the definition of an $(\mathcal{L}, \alpha)$-representation of the group $G$.

We denote the abelian category of quasi-coherent $(\mathcal{L}, \alpha)$-$G$-equivariant sheaves on $X$ by $\text{qcoh}_{G, \mathcal{L}, \alpha}(X)$, and the category of coherent $(\mathcal{L}, \alpha)$-$G$-equivariant sheaves on $X$ by $\text{coh}_{G, \mathcal{L}, \alpha}(X)$.

A (tensor) multiplication of cocycles on a group $G$ can be defined in a natural way. It turns the set of cocycles on $G$ into an abelian group. We denote tensor powers of a cocycle $(\mathcal{L}, \alpha)$ by $(\mathcal{L}^r, \alpha^r)$. As in the case of finite groups, cocycles classify central extensions of the group by the multiplicative group $\mathbb{G}_m$ of the ground field. More precisely, there is a bijection between isomorphy classes of cocycles on $G$ and isomorphism classes of central extensions

\[
1 \to \mathbb{G}_m \to \tilde{G} \xrightarrow{\pi} G \to 1.
\]

Twisted representations and equivariant sheaves can be described by means of the group extension corresponding to a given cocycle. The following proposition holds.

**Proposition 7.3** (see [1], Proposition 1.9). *In the notation given above for any integer $r$ we have an equivalence of categories*

\[
\text{qcoh}^{G, \mathcal{L}^r, \alpha^r}(X) \cong \text{qcoh}^{\tilde{G}}_{(r)}(X),
\]

*where $\text{qcoh}^{\tilde{G}}_{(r)}(X)$ is a full subcategory in $\text{qcoh}^{\tilde{G}}(X)$ formed by sheaves of weight $r$ with respect to the action of the subgroup $\mathbb{G}_m \subset \tilde{G}$. The same assertion holds for coherent sheaves.*

Since the subgroup $\mathbb{G}_m \subset \tilde{G}$ is central and acts on $X$ identically, there is a decomposition over characters of $\mathbb{G}_m$:

\[
\text{coh}^{\tilde{G}}(X) = \bigoplus_{r \in \mathbb{Z}} \text{coh}^{\tilde{G}}_{(r)}(X), \quad \text{qcoh}^{\tilde{G}}(X) = \prod_{r \in \mathbb{Z}} \text{qcoh}^{\tilde{G}}_{(r)}(X).
\]

This shows that the category of twisted $G$-equivariant sheaves can be described as a full subcategory (and even a direct factor) in the category of equivariant sheaves with respect to some extension of the group $G$ by $\mathbb{G}_m$.

As in the case of ordinary equivariant sheaves, twisted equivariant sheaves can be considered as objects of a descent category associated with some cosimplicial category.
Definition 7.4. Consider simplicial scheme (2.1)

\[(X/G)_\bullet = [X, G \times X, G \times G \times X, \ldots, p_\bullet]\]

and a cosimplicial category

\[[\text{qcoh}(X), \text{qcoh}(G \times X), \text{qcoh}(G \times G \times X), \ldots, p_\bullet^*]\]

consisting of categories of quasi-coherent sheaves and functors of inverse image between them. We twist the functors \(p_j^*\) in the following way. Let \(F\) be a sheaf on \(G \times \cdots \times G \times X\) and \(f: [1, \ldots, m] \rightarrow [1, \ldots, n]\) be a map. We consider \(m-1\) times

\[P_f^* F = p_1^* \mathcal{L} \otimes \cdots \otimes p_{n-f(m)}^* \mathcal{L} \otimes p_f^* F.\]  

(7.1)

Here \(p_i, i = 1, \ldots, n - f(m)\), denotes the projection \(G \times \cdots \times G \times X\) onto the \(i\)th factor \(G\).

We define isomorphisms \(\epsilon_{f,g}': P_{f'}^* P_g^* \simto P_{f g}^*\) for any pair of maps \(f, g\) admitting composition. Let \(f: [1, \ldots, n] \rightarrow [1, \ldots, k]\) and \(g: [1, \ldots, m] \rightarrow [1, \ldots, n]\) be morphisms in \(\Delta\). We have

\[P_{f'}^* P_g^* (-) = P_f^* (p_1^* \mathcal{L} \otimes \cdots \otimes p_{n-g(m)}^* \mathcal{L} \otimes p_g^* (-))\]
\[= p_1^* \mathcal{L} \otimes \cdots \otimes p_{k-f(n)}^* \mathcal{L} \otimes p_f^* p_1^* \mathcal{L} \otimes \cdots \otimes p_f^* p_{n-g(m)}^* \mathcal{L} \otimes p_g^* (-)\]
\[= p_1^* \mathcal{L} \otimes \cdots \otimes p_{k-f(n)}^* \mathcal{L} \otimes (\mu p_{k-f(n)+1}, \ldots, k-f(g(m)))^* \mathcal{L}\]
\[\quad \otimes \cdots \otimes (\mu p_{k-f(g(m)+1)}, \ldots, k-f(g(m)))^* \mathcal{L} \otimes p_f^* p_g^* (-),\]

\[P_{f g}^* (-) = p_1^* \mathcal{L} \otimes \cdots \otimes p_{k-f(g(m))}^* \mathcal{L} \otimes p_{f g}^* (-),\]

where \(p_{i,j+1,\ldots,j-1,j}\) denotes the projection of \(G \times \cdots \times G \times X\) to the product of \(i, \ldots, j\) factors, and \(\mu\) denotes the multiplication map \(G \times \cdots \times G \rightarrow G\). We define an isomorphism \(\epsilon_{f,g}'\) by means of an isomorphism \(\epsilon_{f,g}: p_f^* p_g^* \rightarrow p_{f g}^*\) and the isomorphisms

\[(\mu p_{i+1, \ldots, j-1, j})^* \mathcal{L} \rightarrow p_i^* \mathcal{L} \otimes \cdots \otimes p_j^* \mathcal{L},\]

obtained by iterations of the structure isomorphism \(\alpha: p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \rightarrow \mu^* \mathcal{L}\) on \(G \times G\).

The cosimplicial categories obtained this way

\[[\text{qcoh}(X), \text{qcoh}(G \times X), \text{qcoh}(G \times G \times X), \ldots, P_\bullet^*],\]  

(7.2)

\[[\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P_\bullet^*]\]  

(7.3)

and similar ones will be called cosimplicial categories associated with the action of \(G\) on \(X \times G\) twisted by the cocycle \((\mathcal{L}, \alpha)\).

Proposition 7.5. The categories (7.2) and (7.3) are cosimplicial categories with base change in the sense of Definition 2.9.

Proof. We have to verify two conditions: right adjoint functors to the functors \(P_j^*\) exist and the canonical morphisms of base change for exact Cartesian squares are isomorphisms.
It is easy to see that a right adjoint to

$$P_{f^*}^* = p_1^*L \times \cdots \times p_{n-f(m)}^*L \times p_f^*(-)$$

is the functor

$$P_{f}^* = p_f^*(p_1^*L^{-1} \times \cdots \times p_{n-f(m)}^*L^{-1} \times -).$$

Indeed,

$$\text{Hom}(p_1^*L \times \cdots \times p_{n-f(m)}^*L \times p_f^*F_1, F_2)$$

$$\cong \text{Hom}(p_f^*F_1, p_1^*L^{-1} \times \cdots \times p_{n-f(m)}^*L^{-1} \times F_2)$$

$$\cong \text{Hom}(F_1, p_f^*(p_1^*L^{-1} \times \cdots \times p_{n-f(m)}^*L^{-1} \times F_2)).$$

The second condition states that for a commutative square of nondecreasing maps of finite sets

$$[1, \ldots, m + n - r] \xleftarrow{f'} [1, \ldots, n]$$

$$\xrightarrow{g'} [1, \ldots, m] \xleftarrow{f} [1, \ldots, r]$$

where $f$ and $f'$ are injective and $[1, \ldots, m + n - r] = \text{Im } f' \cup \text{Im } g'$, the base change morphism $P_{f^*}^*P_{g^*}^* \to P_{f'}^*P_{g'}^*$ is an isomorphism of functors. Decomposing $f$ and $g$ into a composition of elementary maps which correspond to faces and degenerations, we can reduce the proof to several easy cases. Namely, we can assume that $m = r+1$, $f = \delta_i$ (where $\delta_i(x) = x$ for $x < i$, $\delta_i(x) = x+1$ for $x \geq i$) and that $n = r+1$, $g = \delta_j$ or $n = r-1$, $g = s_j$ (where $s_j(x) = x$ for $x \leq j$ and $s_j(x) = j-1$ for $x > j$). The different cases appear depending on whether or not $i$ and $j$ are equal to 1. In each of the cases the verification is elementary. It reduces to standard properties of adjoint functors, applications of the isomorphism $\alpha$ and standard base change formulae. For example, assume that $f = g = \delta_1$. Then either $f' = \delta_1$, $g' = \delta_2$, or $f' = \delta_2$, $g' = \delta_1$. We consider the first case.

By Definition 7.4 and the first part of the proof, we have

$$P_{g^*}^*P_{f^*}^* = p_{g^*}^*(p_1^*L \times p_f^*(-)) \cong p_{g^*}^*(p_1^*L \times p_2^*L \times p_2^*L^{-1} \times p_f^*(-))$$

(1) $$\cong p_{g^*}^*((\mu p_{12})^*L \times p_f^*p_{12}^*L^{-1} \times p_f^*(-))$$

(2) $$\cong p_{g^*}^*(p_{g^*}^*p_1^*L \times p_f^*(p_1^*L^{-1} \times -))$$

(3) $$\cong p_1^*L \times p_{g^*}^*p_f^*(p_1^*L^{-1} \times -) \cong p_1^*L \times p_f^*p_{g^*}(p_1^*L^{-1} \times -) = P_{f}^*P_{g^*}^*.$$
We let $T_{G,\mathcal{L},\alpha}$ denote the comonads on $\text{qcoh}(X)$ and $\mathcal{D}(X)$ associated with cosimplicial categories (7.2) and (7.3); see Definition 2.10. By Propositions 2.12 and 7.5, the descent categories

$$\text{qcoh}^{G,\mathcal{L},\alpha}(X) = \text{Kern}([\text{qcoh}(X), \text{qcoh}(G \times X), \text{qcoh}(G \times G \times X), \ldots, P^*_\bullet])$$

and

$$\mathcal{D}(X)^{G,\mathcal{L},\alpha} = \text{Kern}([\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P^*_\bullet])$$

are equivalent to the categories of comodules $\text{qcoh}(X)_{T_{G,\mathcal{L},\alpha}}$ and $\mathcal{D}(X)_{T_{G,\mathcal{L},\alpha}}$, respectively.

**Example 7.6.** For small values of $n, m$ and maps $f$ corresponding to faces and degenerations, the functors $P^*_f$ from the definition have the form

$$\begin{align*}
\text{qcoh}(G \times G \times X) &\xleftarrow{(1 \times a)^*} \text{qcoh}(G \times G \times X) \xrightarrow{(\mu \times 1)^*} \text{qcoh}(G \times X) \xrightarrow{(e \times 1)^*} \text{qcoh}(X) \\
&\xleftarrow{p_1^* L \otimes p_2^*} \xrightarrow{(e \times 1)^*} \text{qcoh}(G \times X) \xrightarrow{(e \times 1)^*} \text{qcoh}(X).
\end{align*}$$

With $g: [1] \rightarrow [1, 2]$, $g(1) = 1$ and $f: [1, 2] \rightarrow [1, 2, 3]$, $f(1) = 1$, $f(2) = 3$ the isomorphism between the functors $P^*_f P^*_g$ and $P^*_f P^*_g$ has the following form:

$$P^*_f P^*_g(-) = (\mu \times 1)^*(p_1^* L \otimes p_2^*(-)) \xrightarrow{(e \times 1)^*} p_{12}^* \mu^* L \otimes p_3^*(-) \xrightarrow{p_{12}^*(\alpha)^{-1}} p_{12}^*(p_1^* L \otimes p_2^* L) \otimes p_3^*(-) \xrightarrow{(e \times 1)^*} p_1^* L \otimes p_2^* L \otimes p_3^*(-) = P^*_f P^*_g(-).$$

The definition implies that the descent category $\text{Kern}$ associated with (7.2) is precisely the category of twisted $(\mathcal{L}, \alpha)$-equivariant quasi-coherent sheaves on $X$.

Notice that the cosimplicial category (7.2) is not obtained by the standard construction of Example 2.3 associated with a morphism of stacks.

Now we turn to the derived version of descent for twisted sheaves. Let $\mathcal{D}^{G,\mathcal{L},\alpha}(X) = \mathcal{D}(\text{qcoh}^{G,\mathcal{L},\alpha}(X))$ be the unbounded derived category of $(\mathcal{L}, \alpha)$-equivariant quasi-coherent sheaves. We let $\mathcal{D}^b(\text{coh}^{G,\mathcal{L},\alpha}(X))$ denote the bounded derived category of $\text{coh}^{G,\mathcal{L},\alpha}(X)$ and $\mathcal{D}^{\text{perf},G,\mathcal{L},\alpha}(X)$ the category of $(\mathcal{L}, \alpha)$-equivariant perfect complexes. It is a full subcategory in $\mathcal{D}^{G,\mathcal{L},\alpha}(X)$ formed by objects which are perfect complexes on $X$ after forgetting the equivariant structure.

As in the nontwisted case, we are going to show that if $G$ is linearly reductive, then the derived category of twisted equivariant sheaves is equivalent to the descent category

$$\mathcal{D}(X)^{G,\mathcal{L},\alpha} = \text{Kern}([\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P^*_\bullet])$$

associated with the simplicial category of Definition 7.4. We deduce this from the fact that the comparison functor is an equivalence for the derived category of sheaves on $X$ which are equivariant with respect to an extension of $G$. Let

$$1 \rightarrow G_m \rightarrow \tilde{G} \xrightarrow{\pi} G \rightarrow 1$$

(7.4)
be the central extension of the group $G$ by $\mathbb{G}_m$ represented by a cocycle $(\mathcal{L}, \alpha)$. The category of quasi-coherent $\bar{G}$-equivariant sheaves on $X$ can be decomposed into a direct product of categories

$$\text{qcoh} \bar{G}(X) \cong \prod_{i \in \mathbb{Z}} \text{qcoh}_{(i)} \bar{G}(X) \cong \prod_{i \in \mathbb{Z}} \text{qcoh} \bar{G}^i, \alpha^i(X), \quad (7.5)$$

where $\text{qcoh}_{(i)} \bar{G}$ denotes the subcategory consisting of $\bar{G}$-equivariant sheaves of weight $i$ with respect to the action of the subgroup $\mathbb{G}_m \subset \bar{G}$; see Proposition 7.3.

**Theorem 7.7.** Let $X$ be a quasi-projective scheme over a field $k$ with an action of a group scheme $G$ of finite type over $k$. Assume that the category of representations of $G$ is semisimple. Then the comparison functors

$$\mathcal{D}^G, \mathcal{L}, \alpha(X) \to \mathcal{D}(\mathcal{X})^G, \mathcal{L}, \alpha,$$

$$\mathcal{D}^\text{perf}, G, \mathcal{L}, \alpha(X) \to \mathcal{D}^\text{perf}(\mathcal{X})^G, \mathcal{L}, \alpha,$$

$$\mathcal{D}^b(\text{coh} G, \mathcal{L}, \alpha(X)) \to \mathcal{D}^b(\mathcal{X})^G, \mathcal{L}, \alpha$$

are equivalences. Here $\mathcal{D}^\text{perf}(X)^G, \mathcal{L}, \alpha$ and $\mathcal{D}^b(X)^G, \mathcal{L}, \alpha$ denote the subcategories in the descent category

$$\mathcal{D}(X)^G, \mathcal{L}, \alpha = \text{Kern}([\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P^*_i])$$

associated with the subcategories $\mathcal{D}^\text{perf}(X), \mathcal{D}^b(X) \subset \mathcal{D}(X)$.

**Proof.** We let

$$\mathcal{D}(i) \subset \mathcal{D}(\mathcal{X}) = \text{Kern}([\mathcal{D}(X), \mathcal{D}(\mathcal{G} \times X), \mathcal{D}(\mathcal{G} \times \mathcal{G} \times X), \ldots, \Phi^*])$$

denote the full subcategory formed by objects of weight $i$ with respect to the action of the subgroup $\mathbb{G}_m \subset \mathcal{G}$. It is easy to see that for different $i$ these subcategories are orthogonal, that is, all morphisms between objects of different subcategories are zero. The group extension (7.4) induces a fully faithful functor on the descent categories $\mathcal{D}(X)^G, \mathcal{L}, \alpha^i \to \mathcal{D}(\mathcal{X})^G$. Moreover, its image lies in the subcategory $\mathcal{D}(i)$, see the proof of Proposition 1.9 in [1]. We denote this functor $\mathcal{D}(X)^G, \mathcal{L}, \alpha^i \to \mathcal{D}(i)$ by $\Psi_i$. Then (7.5) yields the decomposition

$$\mathcal{D}^G(X) \cong \prod_{i \in \mathbb{Z}} \mathcal{D}^G, \mathcal{L}, \alpha^i(X)$$

of derived categories. It is included into the commutative diagram of categories and functors

$$\begin{array}{ccc}
\prod_i \mathcal{D}(X)^G, \mathcal{L}, \alpha^i(X) & \sim & \mathcal{D}^G(X) \\
\downarrow \Pi \Phi_i & & \downarrow \Phi_{\bar{G}} \\
\prod_i \mathcal{D}(X)^G, \mathcal{L}, \alpha^i & \sim & \prod_i \mathcal{D}(i) \\
\end{array}$$

$$\begin{array}{ccc}
\prod_i \mathcal{D}(X)^G, \mathcal{L}, \alpha^i \Pi \Psi_i & \rightarrow & \prod_i \mathcal{D}(i) \\
\downarrow \Phi_{\bar{G}} & & \downarrow \mathcal{D}(\mathcal{X})^G.
\end{array}$$
The group $\tilde{G}$ as an extension of $G$ by a torus is linearly reductive, so the comparison functor $\Phi_{\tilde{G}}$ is an equivalence (Theorem 2.14). Since the subcategories $\mathcal{D}_{(i)} \subset \mathcal{D}(X)^{\tilde{G}}$ are orthogonal, the natural functor $\prod_i \mathcal{D}_{(i)} \to \mathcal{D}(X)^{\tilde{G}}$ is fully faithful. This implies that any composition of functors

$$
\mathcal{D}^{G,\mathcal{L},\alpha}(X) \xrightarrow{\Phi_i} \mathcal{D}(X)^{G,\mathcal{L},\alpha} \xrightarrow{\Psi_i} \mathcal{D}_{(i)}
$$

is an equivalence as well. Since $\Psi_i$ is fully faithful, we conclude that $\Phi_i$ is an equivalence.

The assertions concerning $\mathcal{D}_{\text{perf}}^{G,\mathcal{L},\alpha}(X)$ and $\mathcal{D}^b(\text{coh}^{G,\mathcal{L},\alpha}(X))$ follow.

§ 8. Functors between descent categories

The results in §6 allow us to construct semiorthogonal decompositions of equivariant derived categories. Below we describe the components of these decompositions explicitly in three interesting examples. In this section we describe the technical tools we need.

As above, schemes are assumed to be quasi-projective schemes over a field $k$. By a group we always mean an affine group scheme of finite type over $k$.

Let $X$ and $Y$ be schemes equipped with actions of a group $G$, let $(\mathcal{L}_X, \alpha_X)$, $(\mathcal{L}_Y, \alpha_Y)$ be cocycles on $G$, and $\Psi: \mathcal{D}(X) \to \mathcal{D}(Y)$ a functor.

**Definition 8.1.** Assume that there exists a functor $\Psi_\bullet$ between the cosimplicial categories $[\mathcal{D}(X), \mathcal{D}(G \times X), \ldots, P^*_X]$ and $[\mathcal{D}(Y), \mathcal{D}(G \times Y), \ldots, P^*_Y]$ associated with the action of $G$ on $X$ twisted by the cocycle $(\mathcal{L}_X, \alpha_X)$, and with the action of $G$ on $Y$ twisted by the cocycle $(\mathcal{L}_Y, \alpha_Y)$ (see Definition 7.4), such that $\Psi_0 = \Psi$. In this case we say that the functor $\Psi$ is compatible with the (twisted) actions of $G$ on $X$ and $Y$. Similarly we define functors between other versions of a derived category or between abelian categories.

By Lemma 2.19, if the functor $\Psi: \mathcal{D}(X) \to \mathcal{D}(Y)$ is compatible with the actions of $G$ on $X$ and $Y$ twisted by cocycles $(\mathcal{L}_X, \alpha_X)$ and $(\mathcal{L}_Y, \alpha_Y)$, then it is compatible with the comonads $T_{G,\mathcal{L}_X,\alpha_X}$ and $T_{G,\mathcal{L}_Y,\alpha_Y}$. By Lemma 2.20, in this case the functor $\Psi$ induces a functor on the descent categories:

$$
\mathcal{D}(X)^{G,\mathcal{L},\alpha_X} \to \mathcal{D}(Y)^{G,\mathcal{L},\alpha_Y}.
$$

**Lemma 8.2.** Let $f: X \to Y$ be an equivariant morphism of schemes over $k$ with actions of the group $G$, and let $(\mathcal{L}, \alpha)$ be a cocycle on $G$. Then the inverse image functors $\text{qcoh}(Y) \to \text{qcoh}(X)$, $\mathcal{D}(Y) \to \mathcal{D}(X)$, $\mathcal{D}_{\text{perf}}(Y) \to \mathcal{D}_{\text{perf}}(X)$ are compatible with the action of $G$ twisted by $(\mathcal{L}, \alpha)$. Consequently, the following inverse image functors between descent categories are well-defined:

$$
\text{qcoh}^{G,\mathcal{L},\alpha}(Y) \xrightarrow{f^*} \text{qcoh}^{G,\mathcal{L},\alpha}(X),$$

$$
\mathcal{D}(Y)^{G,\mathcal{L},\alpha} \xrightarrow{f^*} \mathcal{D}(X)^{G,\mathcal{L},\alpha},$$

$$
\mathcal{D}_{\text{perf}}(Y)^{G,\mathcal{L},\alpha} \xrightarrow{f^*} \mathcal{D}_{\text{perf}}(X)^{G,\mathcal{L},\alpha}.
$$
and if a morphism $f$ has finite Tor-dimension,

$$\mathcal{D}^b(Y)^{G,\mathcal{L},\alpha} \xrightarrow{f^*} \mathcal{D}^b(X)^{G,\mathcal{L},\alpha}.$$ 

Analogous assertions hold for direct image functors

$$\text{qcoh}(X) \to \text{qcoh}(Y) \quad \text{and} \quad \mathcal{D}(X) \to \mathcal{D}(Y).$$

**Proof.** It is easy to see that the functors

$$\Psi_k = (1 \times 1 \times \cdots \times 1 \times f)^*$$

and the canonical isomorphisms of the form $s^*t^* \cong (ts)^*$ define the inverse image functor on the cosimplicial categories

$$[\mathcal{D}(Y), \mathcal{D}(G \times Y), \ldots, P^*_Y], \quad [\mathcal{D}(X), \mathcal{D}(G \times X), \ldots, P^*_X]$$

(8.1)

associated with the twisted action of $G$ on $X$ and $Y$. For the categories qcoh and $\mathcal{D}^\text{perf}$ and the direct image functors the arguments are similar.

**Lemma 8.3.** Let $E$ be an object of the category $\mathcal{D}^{G,\mathcal{L}_0,\alpha_0}(X)$. The tensor product by $E$ is compatible with the action of $G$ on $X$ twisted by a cocycle $(\mathcal{L}, \alpha)$ and the action of $G$ on $X$ twisted by a cocycle $(\mathcal{L} \otimes \mathcal{L}_0, \alpha \otimes \alpha_0)$. It induces the functor

$$\mathcal{D}(X)^{G,\mathcal{L},\alpha} \to \mathcal{D}(X)^{G,\mathcal{L} \otimes \mathcal{L}_0,\alpha \otimes \alpha_0}.$$ 

If, in addition, $E \in \mathcal{D}^\text{perf},G,\mathcal{L}_0,\alpha_0(X)$, then the following functors are induced

$$\mathcal{D}^\text{perf}(X)^{G,\mathcal{L},\alpha} \to \mathcal{D}^\text{perf}(X)^{G,\mathcal{L} \otimes \mathcal{L}_0,\alpha \otimes \alpha_0},$$

$$\mathcal{D}^b(X)^{G,\mathcal{L},\alpha} \to \mathcal{D}^b(X)^{G,\mathcal{L} \otimes \mathcal{L}_0,\alpha \otimes \alpha_0}.$$ 

**Proof.** Consider the cosimplicial categories

$$[\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P^*_X],$$

$$[\mathcal{D}(X), \mathcal{D}(G \times X), \mathcal{D}(G \times G \times X), \ldots, P''^*_X],$$

associated with the action of $G$ on $X$ twisted by the cocycles $(\mathcal{L}, \alpha)$ and $(\mathcal{L} \otimes \mathcal{L}_0, \alpha \otimes \alpha_0)$ respectively, see Definition 7.4. Let $E$ be an object in $\mathcal{D}(X)$ obtained from $E$ by forgetting the equivariant structure. We define the functors

$$\Psi_k : \mathcal{D}(G \times \cdots \times G \times X)^{k \text{ times}} \to \mathcal{D}(G \times \cdots \times G \times X)^{k \text{ times}}$$

by the formula

$$\Psi_k(-) = p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_k^*\mathcal{L}_0 \otimes (-) \otimes p_{k+1}^*E.$$ 

We will define the isomorphisms of functors

$$\beta_f : \Psi_n \circ P^*_f \to P''^*_f \circ \Psi_m$$
for any map \( f : [1, \ldots, m + 1] \to [1, \ldots, n + 1] \) in \( \Delta \). We have

\[
\Psi_n \circ P_f^*(-) = p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_n^*\mathcal{L}_0 \otimes P_f^*(-) \otimes p_{n+1}^*E
\]

\[
= p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_n^*\mathcal{L}_0 \otimes p_1^*\mathcal{L}
\]

\[
\otimes \cdots \otimes p_{n+1-f(m+1)}^*\mathcal{L} \otimes p_f^*(-) \otimes p_{n+1}^*E,
\]

\[
P_f^* \circ \Psi_m(-) = p_1^*(\mathcal{L} \otimes \mathcal{L}_0) \otimes \cdots \otimes p_{n+1-f(m+1)}(\mathcal{L} \otimes \mathcal{L}_0)
\]

\[
\otimes p_f^*(p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_m^*\mathcal{L}_0 \otimes (-) \otimes p_{m+1}^*E)
\]

\[
= p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_{n+1-f(m+1)}^*\mathcal{L}_0 \otimes (\mu p_{n+1-f(m+1)+1,...,n+1-f(m)})^*\mathcal{L}_0
\]

\[
\otimes \cdots \otimes (\mu p_{n+1-f(2)+1,...,n+1-f(1)})^*\mathcal{L}_0 \otimes p_1^*\mathcal{L}
\]

\[
\otimes \cdots \otimes p_{n+1-f(m+1)}^*\mathcal{L} \otimes p_f^*(-) \otimes (ap_{n+1-f(1)+1,...,n+1})^*E,
\]

where \( p_{i,i+1,...,j-1,j} \) denotes the projection \( G \times \cdots \times G \times X \) to the product of \( i, \ldots, j \) factors, \( \mu \) is the multiplication map \( G \times \cdots \times G \to G \), and \( a \) denotes the iterated action \( G \times \cdots \times G \times X \to X \). To define \( \beta_f \) we use the isomorphisms

\[
(\mu p_{i,i+1,...,j-1,j})^*\mathcal{L}_0 \to p_1^*\mathcal{L}_0 \otimes \cdots \otimes p_j^*\mathcal{L}_0,
\]

obtained by iterating the structure isomorphism \( \alpha_0 : p_1^*\mathcal{L}_0 \otimes p_2^*\mathcal{L}_0 \to \mu^*\mathcal{L}_0 \) on \( G \times G \), and the isomorphism

\[
(ap_{n+1-f(1)+1,...,n+1})^*E \to p_{n+1-f(1)+1}^*\mathcal{L}_0 \otimes \cdots \otimes p_n^*\mathcal{L}_0 \otimes p_{n+1}^*E,
\]

which is obtained by multiple applications of the structure morphism

\[
\theta : p_1^*\mathcal{L}_0 \otimes p_2^*E \to a^*E
\]

on \( G \times X \). The cocycle condition for \( \alpha_0 \) and the compatibility of \( \alpha_0 \) and \( \theta \) yield that the functors \( \Psi_k \) with the isomorphisms \( \beta_f \) define a functor between the cosimplicial categories. Thus \( \Psi_0 = - \otimes E : \mathcal{D}(X) \to \mathcal{D}(X) \) is compatible with twisted actions of \( G \) on \( X \).

Using the arguments given at the beginning of the section, we obtain a functor \( \mathcal{D}(X)^G,\mathcal{L},\alpha \to \mathcal{D}(X)^G,\mathcal{L} \otimes \mathcal{L}_0,\alpha \otimes \alpha_0 \) on the descent categories. For the categories \( \mathcal{D}^{\text{perf}}(X) \) and \( \mathcal{D}^b(X) \) the arguments are similar.

**Proposition 8.4.** Let \( X \) and \( Y \) be quasi-projective schemes over \( k \) equipped with an action of a group scheme \( G \) of finite type over \( k \). Let \( \Psi_E : \mathcal{D}(X) \to \mathcal{D}(Y) \) be a functor defined by the kernel \( E \in \mathcal{D}(X \times Y) \):

\[
\Psi_E(-) = p_2^*(p_1^*(-) \otimes E).
\]

Assume that there exists an object \( \mathcal{E} \in \mathcal{D}^G,\mathcal{L}_0,\alpha_0(X \times Y) \) which is sent to \( E \) under forgetting the action, and \( (\mathcal{L}, \alpha) \) is a cocycle.

1. The functor \( \Psi_E \) is compatible with the action of \( G \) on \( X \) twisted by \( (\mathcal{L}, \alpha) \) and the action of \( G \) on \( Y \) twisted by \( (\mathcal{L} \otimes \mathcal{L}_0, \alpha \otimes \alpha_0) \).

2. Let \( T_{G,\mathcal{L},\alpha} \) and \( T_{G,\mathcal{L} \otimes \mathcal{L}_0,\alpha \otimes \alpha_0} \) be comonads on \( \mathcal{D}(X) \) and \( \mathcal{D}(Y) \) associated with the action of \( G \) on \( X \) twisted by \( (\mathcal{L}, \alpha) \) and the action of \( G \) on \( Y \) twisted by \( (\mathcal{L} \otimes \mathcal{L}_0, \alpha \otimes \alpha_0) \), respectively. Then the functor \( \Psi_E \) induces a functor on the
descent categories \(\mathcal{D}(X)_{T_{G,\mathcal{L},\alpha}} \to \mathcal{D}(Y)_{T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}}\). If \(\Psi_E\) is fully faithful, then the induced functor on the descent categories is fully faithful as well.

3. Assume that \(\Psi_E\) is fully faithful and sends \(\mathcal{D}^{\text{perf}}(X)\) to \(\mathcal{D}^{\text{perf}}(Y)\). Assume further that the image of the functor

\[
\Psi_E: \mathcal{D}^{\text{perf}}(X) \to \mathcal{D}^{\text{perf}}(Y)
\]

is the subcategory \(\mathcal{A} \subset \mathcal{D}^{\text{perf}}(Y)\). Then \(\Psi_E\) induces a fully faithful functor from \(\mathcal{D}^{\text{perf}}(X)_{T_{G,\mathcal{L},\alpha}}\) to \(\mathcal{D}^{\text{perf}}(Y)_{T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}}\) whose image is the subcategory

\[
\mathcal{A}_{T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}} \subset \mathcal{D}^{\text{perf}}(Y)_{T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}}.
\]

In this case the subcategory \(\mathcal{A}\) is invariant under the action of \(G\).

**Proof.**

1. That \(\Psi_E\) is compatible with the twisted actions of \(G\) on \(X\) and \(Y\) follows from Lemmas 8.2 and 8.3.

2. By the definition of a functor compatible with twisted actions of \(G\) on \(X\) and \(Y\), the functor \(\Psi_E\) can be extended to a functor between cosimplicial categories which are associated with twisted actions of \(G\) on \(X\) and \(Y\). By Lemma 2.19, the functor \(\Psi_E\) is compatible with comonads \(T_{G,\mathcal{L},\alpha}\) and \(T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}\). By Lemma 2.20, the functor \(\Psi_E\) induces a functor on the descent categories

\[
\mathcal{D}(X)_{T_{G,\mathcal{L},\alpha}} \to \mathcal{D}(Y)_{T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}},
\]

which is fully faithful provided \(\Psi_E\) is fully faithful.

3. The first assertion follows from Lemma 2.20 applied to the categories \(\mathcal{D}(X)\), \(\mathcal{D}(Y)\), their subcategories \(\mathcal{D}^{\text{perf}}(X)\), \(\mathcal{D}^{\text{perf}}(Y)\), the comonads \(T_{G,\mathcal{L},\alpha}\) and \(T_{G,\mathcal{L}\otimes L_0,\alpha\otimes\alpha_0}\) and the subcategory \(\mathcal{A} \subset \mathcal{D}^{\text{perf}}(Y)\). We will check that the subcategory \(\mathcal{A}\) is invariant under the action. Consider the commutative diagram

\[
\begin{array}{ccc}
G \times X & \xleftarrow{p_{12}} & G \times X \times Y \\
\downarrow p_X & & \downarrow p_{23} \\
X & \xleftarrow{p_1} & X \times Y
\end{array}
\]

where \(p_X\) and \(p_Y\) are the projections onto \(X\) and \(Y\), respectively. We are going to verify condition 4) of Proposition 5.3. In \(\mathcal{D}^{\text{perf}}(G \times Y)\) the subcategories \(p_Y^* \mathcal{A}\) and \(a^* \mathcal{A}\) coincide. By definition, the category \(p_Y^* \mathcal{A}\) is generated by objects of the form \(p_Y^* F' \otimes p_Y^* p_{23}(p_1^* F \otimes E)\), where \(F' \in \mathcal{D}^{\text{perf}}(G)\), \(F \in \mathcal{D}^{\text{perf}}(X)\). We have

\[
p_1^* F' \otimes p_Y^* p_{23}(p_1^* F \otimes E) = p_1^* F' \otimes p_{13}^* p_2^* (p_3^* F \otimes E)
\]

\[
= p_1^* F' \otimes p_{13}^* (p_{12}^* p_X^* F \otimes p_1^* L_0^* \otimes a^* E)
\]

\[
\in p_1^* F' \otimes p_{13}^* (p_{12}^* a^* \mathcal{D}^{\text{perf}}(X) \otimes p_1^* L_0^* \otimes a^* E)
\]

\[
= p_1^* F' \otimes p_{13}^* (a^*(p_1^* \mathcal{D}^{\text{perf}}(X) \otimes E) \otimes p_{13}^* p_3^* L_0^*)
\]

\[
= p_1^* F' \otimes p_1^* L_0^* \otimes p_{13}^* a^* (p_1^* \mathcal{D}^{\text{perf}}(X) \otimes E)
\]

\[
= p_1^* (F' \otimes L_0^*) \otimes a^* p_{23}(p_1^* \mathcal{D}^{\text{perf}}(X) \otimes E) = p_1^* (F' \otimes L_0^*) \otimes a^* \mathcal{A} = a^* \mathcal{A}.
\]
Here we use the fact that
\[ p_X^* \mathcal{D}_{\text{perf}}(X) = \mathcal{D}_{\text{perf}}(G \times X) = a^* \mathcal{D}_{\text{perf}}(X) \]
(see Lemma 5.1). Consequently, \( p_Y^* \mathcal{A} \subset a^* \mathcal{A} \). The reverse inclusion can be verified in the same way.

§ 9. Semiorthogonal decompositions for varieties with an invariant exceptional collection

In this section we describe the components of the semiorthogonal decomposition in Theorem 6.2 in the case when the invariant semiorthogonal decomposition of the derived category of coherent sheaves on \( X \) is generated by a full exceptional collection.

Let \( X \) be a quasi-projective scheme over a field \( k \). In this section by a group we mean a reduced affine group scheme of finite type over \( k \).

Assume that \( E \) is an exceptional object in the category \( \mathcal{D}_{\text{perf}}(X) \). It generates a subcategory \( \langle E \rangle \subset \mathcal{D}_{\text{perf}}(X) \). We will show that invariance of this subcategory under the action of \( G \) on \( X \) (see Definition 5.2) is connected with invariance of the object \( E \) in the sense of Definition 9.1.

If \( G \) is finite, it is natural to say that a sheaf \( F \) on \( X \) is preserved by the action if \( g^* F \cong F \) for all \( g \in G \). For arbitrary algebraic groups this definition does not work well because the group may have too few rational points. When we talk about invariant objects, we keep the following definition in mind.

**Definition 9.1.** An action of an algebraic group \( G \) on a scheme \( X \) preserves an object \( F \in \mathcal{D}(X) \) of the derived category if there is a line bundle \( \mathcal{L} \) on \( G \) such that the objects \( p_1^* \mathcal{L} \otimes p_2^* F \) and \( a^* F \) on \( G \times X \) are quasi-isomorphic.

**Proposition 9.2.** The following conditions on an exceptional object \( E \in \mathcal{D}_{\text{perf}}(X) \) on a projective scheme \( X \) over a field \( k \) with an action of a reduced affine group scheme \( G \) of finite type over \( k \) are equivalent.

1) For a suitable line bundle \( \mathcal{L} \) on \( G \) there exists an isomorphism \( p_1^* \mathcal{L} \otimes p_2^* E \cong a^* E \) in the category \( \mathcal{D}_{\text{perf}}(G \times X) \).

2) For any closed point \( g \) of the scheme \( G \) with the residue field \( k(g) \) we have \( g^* E' \cong E' \), where the object \( E' \) is obtained from \( E \) by extension of scalars \( k \rightarrow k(g) \).

3) The subcategories \( p_2^* \langle E \rangle \) and \( a^* \langle E \rangle \) in \( \mathcal{D}_{\text{perf}}(G \times X) \) coincide.

**Proof.** 2) \( \implies \) 1) is proved in [1], in Proposition 2.17.

1) \( \implies \) 3). By definition, the subcategory \( p_2^* \langle E \rangle \) in \( \mathcal{D}_{\text{perf}}(G \times X) \) is generated by objects of the form \( p_1^* F \otimes p_2^* F' \), where \( F \in \mathcal{D}_{\text{perf}}(G) \), \( F' \in \langle E \rangle \). In particular, \( p_2^* \langle E \rangle \) contains the object \( a^* E \cong p_1^* \mathcal{L} \otimes p_2^* E \). Hence \( p_2^* \langle E \rangle \supset a^* \langle E \rangle \). The inverse inclusion can be proved in the same way.

3) \( \implies \) 2). Let \( g \) be a closed point of the scheme \( G \). Let us restrict the equal subcategories \( p_2^* \langle E \rangle \) and \( a^* \langle E \rangle \) in \( \mathcal{D}_{\text{perf}}(G \times X) \) to the fibre \( X' = g \times X \). We obtain subcategories in \( \mathcal{D}_{\text{perf}}(X') \) generated by exceptional objects \( E' \) and \( g^* E' \), respectively. We conclude that \( E' \) and \( g^* E' \) are isomorphic.

It should be noted that an exceptional object \( E \) on \( X \), preserved by the group action, carries an equivariant structure with respect to some cocycle on \( G \).
**Proposition 9.3.** Let $E$ be an exceptional object in $\mathcal{D}(X)$ preserved by an action of a group $G$. Then for some cocycle $(L', \alpha)$ on $G$ there exists an $(L, \alpha)$-equivariant structure on $E$, that is, an isomorphism $\theta: p_1^* L \otimes p_2^* E \rightarrow a^* E$ compatible with $\alpha$ in the sense of Definition 7.2.

**Proof.** Fix an isomorphism $\theta: p_1^* L \otimes p_2^* E \rightarrow a^* E$. We have a commutative diagram on the triple product $G \times G \times X$

$$p_1^* L \otimes p_2^* L \otimes p_3^* E \xrightarrow{1 \otimes p_{23}^* \theta} p_1^* L \otimes (ap_{23})^* E \xrightarrow{(1 \times a)^* \theta} (a(1 \times a))^* E$$

$$\alpha' \downarrow$$

$$\quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow$$

$$(\mu p_{12})^* L \otimes p_3^* E \xrightarrow{(\mu \times 1)^* \theta} (a(\mu \times 1))^* E,$$

where

$$\alpha': p_1^* L \otimes p_2^* L \otimes p_3^* E \rightarrow (\mu p_{12})^* L \otimes p_3^* E$$

is some isomorphism. We will check that $\alpha'$ has the form $p_{12}^* \alpha \otimes 1$, where $\alpha$ is an isomorphism $p_1^* L \otimes p_2^* L \rightarrow \mu^* L$ on $G \times G$. Indeed, since $E$ is exceptional, we have

$$\text{Hom}(p_1^* L \otimes p_2^* L \otimes p_3^* E, (\mu p_{12})^* L \otimes p_3^* E)$$

$$= \text{Hom}(p_3^* E, (p_1^* L \otimes p_2^* L)^{-1} \otimes (\mu p_{12})^* L \otimes p_3^* E)$$

$$= \text{Hom}(E, p_{3*}( (p_1^* L \otimes p_2^* L)^{-1} \otimes (\mu p_{12})^* L \otimes p_3^* E))$$

$$= \text{Hom}(E, p_{3*}( (p_1^* L \otimes p_2^* L)^{-1} \otimes (\mu p_{12})^* L) \otimes E)$$

$$= \text{Hom}(E, H^0(G \times G, (p_1^* L \otimes p_2^* L)^{-1} \otimes \mu^* L) \otimes E)$$

$$= \text{Hom}(p_1^* L \otimes p_2^* L, \mu^* L).$$

The next to last equality holds by the theorem on flat base change. We consider sheaves and their morphisms on $G \times G \times G \times X$ and obtain the associativity condition: the isomorphisms on $G \times G \times G$

$$(1 \times \mu)^* \alpha \circ (1 \otimes p_{23}^* \alpha) \quad \text{and} \quad (\mu \times 1)^* \alpha \circ (p_{12}^* \alpha \otimes 1)$$

between the sheaves $p_1^* L \otimes p_2^* L \otimes p_3^* L$ and $(\mu(1 \times \mu))^* L$ are equal. Thus the pair $(L, \alpha)$ is a cocycle on the group $G$ in the sense of Definition 7.1, and $\mathcal{E} = (E, \theta)$ is an equivariant object.

Assume that there is a full exceptional collection $(E_1, \ldots, E_n)$ in $\mathcal{D}^{\text{perf}}(X)$. Such a collection defines a semiorthogonal decomposition of the category of perfect complexes

$$\mathcal{D}^{\text{perf}}(X) = \langle \langle E_1, \ldots, \langle E_n \rangle \rangle$$

into subcategories equivalent to $\mathcal{D}^b(k-\text{mod})$. Assume that the objects $E_i$ are invariant under an action of a group $G$. Proposition 9.2 implies that components of this decomposition are preserved by the action. By Proposition 9.3, we can introduce a twisted equivariant structure on the object $E_i$ for some cocycle $(L_i, \alpha_i)$ on $G$. We denote this twisted object in the category $\mathcal{D}^{\text{perf}}(X)^{G, L_i, \alpha_i}$ by $\mathcal{E}_i$. Suppose that the group $G$ is linearly reductive. By Theorem 7.7, $\mathcal{E}_i$ corresponds to an object of the derived category $\mathcal{D}^{\text{perf}}_{G, L_i, \alpha_i}(X)$, which also is denoted by $\mathcal{E}_i$. 

Theorem 9.4. The category $\mathcal{D}^{\text{perf},G}(X)$ admits a semiorthogonal decomposition

$$\mathcal{D}^{\text{perf},G}(X) = \langle \mathcal{E}_1 \otimes \mathcal{D}^b(\text{rep}(G, \mathcal{L}_1^{-1}, \alpha_1^{-1})), \ldots, \mathcal{E}_n \otimes \mathcal{D}^b(\text{rep}(G, \mathcal{L}_n^{-1}, \alpha_n^{-1})) \rangle.$$

Remark 9.5. This fact was proved in [1], Theorem 2.11 under slightly different assumptions: the objects of the original exceptional collections were sheaves (this means the technique of descent for derived categories does not have to be used) and the group was not assumed to be linearly reductive.

Proof of Theorem 9.4. By Theorem 2.14, the category $\mathcal{D}^{\text{perf},G}(X)$ is equivalent to the descent category $\mathcal{D}^{\text{perf}}(X)_{T_G}$, where $T_G$ denotes the comonad on the category $\mathcal{D}(X)$ associated with the morphism of stacks $X \to X/G$. Theorem 6.2 yields that $\mathcal{D}^{\text{perf}}(X)_{T_G}$ admits a semiorthogonal decomposition $\langle \langle E_1 \rangle_{T_G}, \ldots, \langle E_n \rangle_{T_G} \rangle$. Here $\langle E_i \rangle_{T_G}$ is the descent category associated with the subcategory $\langle E_i \rangle \subset \mathcal{D}^{\text{perf}}(X)$ (see Definition 2.8). Using Lemma 2.20, we will describe the categories $\langle E_i \rangle_{T_G}$ more explicitly.

Consider the functor

$$\Psi_i = - \otimes E_i : \mathcal{D}^b(k-\text{mod}) = \mathcal{D}^{\text{perf}}(\text{Spec } k) \to \mathcal{D}^{\text{perf}}(X).$$

Its image is the subcategory $\langle E_i \rangle \subset \mathcal{D}^{\text{perf}}(X)$. This functor can be extended to a fully faithful functor

$$\mathcal{D}(k-\text{Mod}) \cong \mathcal{D}(\text{Spec } k) \to \mathcal{D}(X),$$

which we denote also by $\Psi_i$. Clearly, $\Psi_i$ is a kernel functor defined by the kernel $E_i \in \mathcal{D}^{\text{perf}}(\text{Spec } k \times X)$. The kernel admits a structure of an equivariant object twisted by a cocycle $(\mathcal{L}_i, \alpha_i)$. By Proposition 8.4, the functor $\Psi_i$ induces a fully faithful functor

$$\mathcal{D}^b(\text{rep}(G, \mathcal{L}_i^{-1}, \alpha_i^{-1})) \cong \mathcal{D}^{\text{perf}}(\text{Spec } k)_{T_G, \mathcal{L}_i^{-1}, \alpha_i^{-1}} \to \mathcal{D}^{\text{perf}}(X)_{T_G} \cong \mathcal{D}^{\text{perf},G}(X),$$

and its image is the subcategory $\langle E_i \rangle_{T_G}$.

§ 10. Semiorthogonal decompositions for projective bundles and blow-ups

In this section we apply Theorem 6.2 to two particular cases of group actions on schemes and describe the components of the resulting semiorthogonal decomposition explicitly. The first case is the equivariant derived category of a projective bundle, which was studied in [11] for actions of finite groups. We start from the semiorthogonal decomposition of the derived category of a projective bundle obtained by Orlov in [12]. We consider the simplest case, namely, when the action on a projective bundle is induced by an equivariant structure on a vector bundle on the base.

Let $S$ be a quasi-projective scheme over a field $k$, let $E$ be a vector bundle of rank $r$ on $S$, and $X = \mathbb{P}_S(E)$ its projectivization. We denote by $\pi : X \to S$ the natural projection. Then the semiorthogonal decomposition constructed by Orlov has the following form:

$$\mathcal{D}^{\text{perf}}(X) = \langle \pi^* \mathcal{D}^{\text{perf}}(S), \mathcal{O}_{X/S}(1) \otimes \pi^* \mathcal{D}^{\text{perf}}(S), \ldots, \mathcal{O}_{X/S}(r-1) \otimes \pi^* \mathcal{D}^{\text{perf}}(S) \rangle.$$ (10.1)
Let us assume that a group $G$ (or, more precisely, a group scheme of finite type over $k$) acts on $S$ and there is a $G$-equivariant structure on $E$ twisted with respect to the cocycle $(\mathcal{L}, \alpha)$. We denote the corresponding $(\mathcal{L}, \alpha)$-$G$-equivariant bundle by $\mathcal{E}$. Then there is an action of the group $G$ on $X$ such that the projection $\pi$ is equivariant. Suppose that the group $G$ is linearly reductive.

**Theorem 10.1.** Under the assumptions stated above we have a semiorthogonal decomposition

$$\mathcal{D}^\text{perf, }G(X) = \langle \pi^* \mathcal{D}^\text{perf, }G(S), \mathcal{O}_{X/S}(1) \otimes \pi^* \mathcal{D}^\text{perf, }G, \mathcal{L}, \alpha(S), \ldots$$

$$\ldots, \mathcal{O}_{X/S}(r-1) \otimes \pi^* \mathcal{D}^\text{perf, }G, \mathcal{L}^{r-1}, \alpha^{r-1}(S) \rangle.$$ 

**Proof.** First we note that $\pi^* \mathcal{E}$ is an $(\mathcal{L}, \alpha)$-equivariant bundle on $X$ and $\mathcal{O}_{X/S}(-1)$ is its linear $(\mathcal{L}, \alpha)$-equivariant subbundle. This means that $\mathcal{O}_{X/S}(k)$ is a twisted $(\mathcal{L}, \alpha)^{-k}$-$G$-equivariant bundle. For any integer $k$, $0 \leq k \leq r-1$, we consider a fully faithful functor

$$\Psi_k = \mathcal{O}_{X/S}(k) \otimes \pi^*(-): \mathcal{D}(S) \rightarrow \mathcal{D}(X).$$

This is a kernel functor with the sheaf $\mathcal{O}_{X/S}(k)$ located on the graph of the map $\pi: X \rightarrow S$ as the kernel.

Here $\Psi_k$ sends perfect complexes to perfect complexes, and the category $\Psi_k(\mathcal{D}^\text{perf}(S))$ is the subcategory $\mathcal{O}_{X/S}(k) \otimes \pi^* \mathcal{D}^\text{perf}(S)$ in (10.1). Let $T_{G, \mathcal{L}^k, \alpha^k}$ and $T_G$ be comonads on the categories $\mathcal{D}(S)$ and $\mathcal{D}(X)$ respectively, associated with the action of $G$ on $S$ (twisted by the cocycle $(\mathcal{L}^k, \alpha^k)$) and on $X$ (the standard one). By Proposition 8.4, the functor $\Psi_k$ induces a fully faithful functor

$$\mathcal{D}^\text{perf, }G, \mathcal{L}^k, \alpha^k(S) \cong \mathcal{D}^\text{perf}(S)_{T_{G, \mathcal{L}^k, \alpha^k}} \rightarrow \mathcal{D}^\text{perf}(X)_{T_G} \cong \mathcal{D}^\text{perf, }G(X),$$

with its image equal to the descent category $(\mathcal{O}_{X/S}(k) \otimes \pi^* \mathcal{D}^\text{perf}(S))_{T_G}$. To complete the proof it suffices to note that the subcategories $\mathcal{O}_{X/S}(k) \otimes \pi^* \mathcal{D}^\text{perf}(S)$ in (10.1) are invariant under the group action, and to apply Theorem 6.2 to (10.1).

The second case is a blow-up of a smooth invariant subvariety in a smooth variety.

Let $Z \subset X$ be a smooth subscheme of codimension $r$ of a smooth quasi-projective scheme over a field $k$, and let $\sigma: \overline{X} \rightarrow X$ be the blow-up of $X$ along $Z$. We denote the preimage of $Z$ with respect to $\sigma$ by $\overline{Z}$. Then $\overline{Z}$ is the projectivization of the normal bundle $\mathcal{N}_{Z/X}$ on $Z$. Denote the restriction of $\sigma$ to $\overline{Z}$ by $\sigma_Z$. As in the previous case we start from the semiorthogonal decomposition for blow-up constructed by Orlov in [12]:

$$\mathcal{D}^\text{perf}(\overline{X}) = \langle \mathcal{O}_{\overline{Z}/Z}(-r+1) \otimes \sigma_Z^* \mathcal{D}^\text{perf}(Z), \ldots$$

$$\ldots, \mathcal{O}_{\overline{Z}/Z}(-1) \otimes \sigma_Z^* \mathcal{D}^\text{perf}(Z), \sigma^* \mathcal{D}^\text{perf}(X) \rangle. \quad (10.2)$$

Assume that a linearly reductive group scheme $G$ of finite type over $k$ acts on a scheme $X$ and the subscheme $Z$ is invariant.
Theorem 10.2. Under the assumptions stated above there is a semiorthogonal decomposition
\[ \mathcal{D}^{\text{perf},G}(X) = \langle \mathcal{O}_{Z/Z}(-r+1) \otimes \sigma_Z^* \mathcal{D}^{\text{perf},G}(Z), \ldots, \mathcal{O}_{Z/Z}(-1) \otimes \sigma_Z^* \mathcal{D}^{\text{perf},G}(Z), \sigma^* \mathcal{D}^{\text{perf},G}(X) \rangle, \]

such that the components \( \mathcal{O}_{Z/Z}(-i) \otimes \sigma_Z^* \mathcal{D}^{\text{perf},G}(Z) \) are equivalent to the category \( \mathcal{D}^{\text{perf},G}(Z) \), and the component \( \sigma^* \mathcal{D}^{\text{perf},G}(X) \) is equivalent to the category \( \mathcal{D}^{\text{perf},G}(X) \).

Proof. We will sketch the proof. Its details can easily be reconstructed by analogy with the proof of Theorem 10.1.

The component \( \sigma^* \mathcal{D}^{\text{perf}}(X) \) in (10.2) is the image of the fully faithful functor \( \sigma^*: \mathcal{D}^{\text{perf}}(X) \to \mathcal{D}^{\text{perf}}(X) \).

This functor is compatible with the action of \( G \) on \( X \) and \( X \). Consequently, the subcategory \( \sigma^* \mathcal{D}^{\text{perf}}(X) \subset \mathcal{D}^{\text{perf}}(X) \) is preserved by the action and the corresponding component of the semiorthogonal decomposition of \( \mathcal{D}^{\text{perf},G}(X) \) is equivalent to \( \mathcal{D}^{\text{perf},G}(X) \).

Similarly, the component \( \mathcal{O}_{Z/Z}(-i) \otimes \sigma_Z^* \mathcal{D}^{\text{perf}}(Z) \) in (10.2) is the image of the fully faithful functor
\[ \Psi_i = j_*(\mathcal{O}_{Z/Z}(-i) \otimes \sigma_Z^*(-)) : \mathcal{D}^{\text{perf}}(Z) \to \mathcal{D}^{\text{perf}}(X), \]

where \( j \) denotes the embedding of \( Z \) into \( X \). The normal bundle \( N_{Z/X} \) on \( Z \) is equivariant, so the bundles \( \mathcal{O}_{Z/Z}(i) \) on \( Z \) are equivariant as well. The functors \( \Psi_i \) are compatible with the action of \( G \) on \( Z \) and \( X \). Hence the components \( \mathcal{O}_{Z/Z}(-i) \otimes \sigma_Z^* \mathcal{D}^{\text{perf}}(Z) \) in (10.2) are preserved by the action and the components \( \mathcal{O}_{Z/Z}(-i) \otimes \sigma_Z^* \mathcal{D}^{\text{perf},G}(Z) \) are equivalent to the category \( \mathcal{D}^{\text{perf},G}(Z) \).

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