TIME-SPLITTING METHODS TO SOLVE THE HALL-MHD SYSTEMS WITH LÉVY NOISES

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Abstract. In this paper, we establish the existence of a martingale solution to the stochastic incompressible Hall-MHD systems with Lévy noises in a bounded domain. The proof is based on a new method, i.e., the time splitting method and the stochastic compactness method.

1. Introduction. The homogeneous incompressible Hall-MHD system in a bounded domain $D \subset \mathbb{R}^n$, $n = 2, 3$, during the time interval $[0, T]$ is described by the following set of equations

$$
\begin{aligned}
&\frac{\partial}{\partial t} u + (u \cdot \nabla) u + \nabla p = (\nabla \times B) \times B + \Delta u + f_1, \\
&\frac{\partial}{\partial t} B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = \Delta B + \nabla \times f_2, \\
&\nabla \cdot u = 0, \quad \nabla \cdot B = 0,
\end{aligned}
$$

(1.1)

where the $n$-dimensional vector fields $u$ and $B$ are the fluid velocity and the magnetic field, respectively. The scalar field $p$ is the pressure. $(u \cdot \nabla) = \sum_{i=1}^{n} u_i \partial_i$ and the symbol $\times$ stands for the usual $n$-dimensional cross-product. The given vector fields $f_1$ and $\nabla \times f_2$ are external forces on the magnetically charge fluid flows. The system (1.1) has been studied in the physics literature for decades (see e.g. [1], [15] and their references) and has application in a number of physics fields as geo-dynamo (see e.g. [17]), neutron stars (see e.g. [23]) and magnetic reconnection in plasmas (see e.g. [10]). The Hall-MHD system was first studied by Lighthill [15] in 1960. In comparison with the usual incompressible magnetohydrodynamic (MHD) system, we have the new term $\nabla \times ((\nabla \times B) \times B)$ (Hall term) which is due to Hall effect and prevents straightforward adaptations from arguments used in the mathematical analysis of Navier-Stokes and related models. This term is very important when the...
magnetic shear is large, where the magnetic reconnection happens. The magnetic reconnection means the change of the topology of magnetic field lines. We refer [10, 24] for the physical background of the magnetic reconnection and the Hall-MHD.

From several points of view it is reasonable to add the stochastic parts to the equation of motion.

- It can be understood as a turbulence in the fluid motion.
- It can be interpreted as a perturbation from the physical model.
- Apart from the force $f_1$ and $\nabla \times f_2$ we are observing there might be further quantities with a (usually small) influence on the motion.

We are therefore interested in the set of equations:

\[
\begin{aligned}
du + [(u \cdot \nabla)u + \nabla p - (\nabla \times B) \times B - \Delta u]dt \\
= \sigma_1(u, B)dW_1(t) + \int_Z g_1(u, z)\tilde{\pi}_1(dt, dz), \\
dB - [\nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) - \Delta B]dt \\
= \sigma_2(u, B)dW_2(t) + \int_Z g_2(u, z)\tilde{\pi}_2(dt, dz), \\
\nabla \cdot u = 0, \nabla \cdot B = 0,
\end{aligned}
\]

where $W_1, W_2$ are $\mathbb{H}$-valued Wiener processes with positive symmetric trace class covariance operator $Q$. $\int_Z g_1(u, z)\tilde{\pi}_1(dt, dz)$ and $\int_Z g_2(u, z)\tilde{\pi}_2(dt, dz)$ stand for the random forces where $(\tilde{\pi}_1(dt, dz), \tilde{\pi}_2(dt, dz)) := \tilde{\pi}(dt, dz) = \pi(dt, dz) - \mu(dz)dt$ are the compensated Poisson measures and $\pi(dt, dz)$ denotes the Poisson random measures associated to Poisson point processes on $Z$ and $\mu(dz)$ are the $\sigma$-finite Lévy measures on $(Z, \mathcal{F})$. The processes $W = (W_1, W_2)$ and $\pi$ are assumed to be independent.

By using vector identity, we can rewrite the equations (1.2) as follows:

\[
\begin{aligned}
du + [(u \cdot \nabla)u - (B \cdot \nabla)B + \nabla \left( p + \frac{|B|^2}{2} \right) - \Delta u]dt \\
= \sigma_1(t, u, B)dW_1(t) + \int_Z g_1(t, u, B, z)\tilde{\pi}_1(dt, dz), \\
dB + [(u \cdot \nabla)B - (B \cdot \nabla)u + \nabla \times ((\nabla \times B) \times B) - \Delta B]dt \\
= \sigma_2(t, u, B)dW_2(t) + \int_Z g_2(t, u, B, z)\tilde{\pi}_2(dt, dz).
\end{aligned}
\]

Then, the equations (1.3) and (1.4) can be written as the following integral form:

\[
\begin{aligned}
U(t) = & U(0) - \int_0^t [AU(s) + \mathcal{B}(U(s))]|ds + \int_0^t \sigma(s, U(s))dW \\
& + \int_0^t f(s)|ds + \int_0^t \int_Z g(s, U(s^-), z)\tilde{\pi}(dz, ds),
\end{aligned}
\]

where $U$ is the transpose of $(u, B)$, the operators $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ and the vectors $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, $\tilde{\pi} = \begin{pmatrix} \tilde{\pi}_1 \\ \tilde{\pi}_2 \end{pmatrix}$, and $f = \begin{pmatrix} -\nabla \times ((\nabla \times B) \times B) \end{pmatrix}$.

In the case of laminar flows where the shear is weak, one ignores the Hall term, and the system reduces to the usual MHD. The stochastic MHD system was considered in many papers. Barbu and Da Parto [3] obtained the existence of strong solutions to the two-dimensional MHD equations driven by random exterior forcing.
terms both in velocity field and in magnetic field, and they proved the existence and uniqueness of an invariant measure via the coupling methods. The stochastic magneto-hydrodynamic equations driven by multiplicative noise in two-dimensional domains were studied by Chueshov and Millet [8]. Concerning the MHD system perturbed by white noises in bounded domains, the analysis of the existence of solutions was provided by Sritharan and Sundar [25]. Sango [21] proved the existence of martingale solutions to the stochastic MHD equations by Galerkin approximation method with Prokhorov and Skorokhod’s compactness results in [9, 14]. As for the existence and uniqueness of strong solutions to the two-dimensional stochastic MHD equations driven by a multiplicative or an additive noise, we refer to Sundar [26]. Motyl [19] proved the existence of martingale solutions of the hydrodynamic-type equations driven by Lévy noise in 3D possibly unbounded domains. Tan, Wang and Wang [27] showed the global existence of strong solutions to the stochastic MHD equations driven by a multiplicative noise in probability if the initial data are sufficiently small.

To the best of our knowledge, the analysis of the global existence of martingale solutions to the stochastic Hall-MHD equations driven by Lévy noises is not solved yet. The aim of this paper is to prove the global existence of martingale solutions to the stochastic Hall-MHD equations. Comparing with the usual stochastic MHD, stochastic Hall-MHD will be more complex. Thanks to the effect of the Hall term, to complete our energy estimates and establish tightness property of the approximate solutions, we need to estimate the norm of the Hall term which is actually a nonlinear term. Furthermore, when we identify the convergence of the approximate solutions, we also need to identify the convergence of the Hall term. In the study of the existence of a martingale solution, one often use the classical Faedo-Galerkin approximation method, such as [3, 8, 25, 26, 19]. Here we perform a time splitting method which provide a new construction of the solutions. By using the time splitting method we can see how the stochastic part influence the Hall-MHD in the evolutionary process (see Remark 4.2).

2. Hypotheses and main result.

2.1. Hypotheses. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. To solve the Hall-MHD systems (1.2) in \(\Omega \times \mathbb{R}_+ \times D\), we assume that the initial data
\[
\begin{align*}
-u|_{t=0} &= u_0(x), \quad B|_{t=0} = B_0(x), \quad x \in \partial D
\end{align*}
\] (2.1)
are divergence-free and possess certain regularity and the following boundary conditions:
\[
\begin{align*}
u|_{\partial D} &= 0; \quad B \cdot n = 0, \quad \text{curl} B|_{\partial D} = 0,
\end{align*}
\] (2.2)
where \(n\) is the unit outer normal on \(\partial D\). Moreover, we give the precise assumptions on each of the stochastic term appearing in systems (1.2):

Assumption (A). Let us denote \(\mathcal{L}_{HS}(Z_W; \mathbb{H})\) the space of Hilbert-Schmidt operators from \(Z_W\) into \(\mathbb{H} = H_u \times H_B\) (see (3.2) and (3.5) for details). Assume that 
\[
\sigma_i : (0, T) \times \mathcal{V} \rightarrow \mathcal{L}_{HS}(Z_W; \mathbb{H}), \quad i = 1, 2
\]
are nonlinear mapping, which satisfies the following condition: there exists a positive constant \(C > 0\) such that
\[
\|\sigma_i(t, u, B) - \sigma_i(t, v, b)\|_{\mathcal{L}_{HS}(Z_W; \mathbb{H})} \leq C \left( \|u - v\|_{\mathcal{V}u} + \|B - b\|_{\mathcal{V}B} \right),
\]
for all \(u, v \in \mathcal{V}_u\) and \(B, b \in \mathcal{V}_B\).

\[
\|\sigma_i(t, u, B)\|_{\mathcal{L}_{HS}(Z_W; \mathcal{V})} \leq C \left( 1 + \|u\|_{\mathcal{H}_u} + \|B\|_{\mathcal{H}_B} \right).
\]
for all \( u \in H_u \) and \( B \in H_B. \) \( H_u, V_u, V_B \) and \( H_B \) are defined in Section 3. More precisely, see (3.2), (3.3), (3.5) and (3.6).

We define a continuous mapping from \( L^2(0, T; \mathbb{H}) \) into \( L^2([0, T]; \mathcal{L}_{H\delta}(Z\!W, \mathbb{R})) \) by

\[
(\tilde{\sigma}_\psi(u))(t) := \langle \sigma(t, U(t)), \psi \rangle,
\]

where \( u \in L^2(0, T; \mathbb{H}) \) and \( t \in [0, T]. \)

**Assumption (B).** For all \( t \in [0, T] \) and \( \forall u, v \in H_u, B, b \in H_B, \) there exists a positive constant \( C \) such that

\[
\int_Z \| g_i(t, u, B, z) - g_i(t, v, b, z) \|^2 d\mu(z) \leq C \| (u - v, B - b) \|^2_{\mathbb{H}}, \tag{2.3}
\]

for each \( p \geq 2 \) and all \( t \in [0, T], \) there exists a positive constant \( C \) such that

\[
\int_Z \| g_i(t, u, B, z) \|^p d\mu(z) \leq C (1 + \| (u, B) \|^p_{\mathbb{H}}). \tag{2.4}
\]

Here, we define a continuous mapping from \( L^2(0, T; \mathbb{H}) \) into \( L^2([0, T] \times Z; d\mu \otimes \mu; \mathbb{R}) \) by

\[
(\tilde{g}_\psi(U))(t, y) := (g(t, U(t^-)); y) \psi_{\mathbb{H}},
\]

for all \( u \in L^2(0, T; \mathbb{H}), (t, y) \in [0, T] \times Z. \)

First, we define the concept of solutions for the problems (1.5)-(2.2) as follows.

**Definition 2.1.** A martingale solution of (1.5)-(2.2) is a system \(((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}), W, \pi, u, B), \) which satisfies

(1) \( \mathcal{U} = (\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) \) is a filtered probability space with a filtration \( \mathcal{F}_t, \) i.e., a set of sub \( \sigma \)-fields of \( \mathcal{F} \) with \( \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \) for \( 0 \leq s < t < \infty. \)

(2) \( W(t) \) is a cylindrical Wiener process with positive symmetric trace class covariance operator \( Q. \) The process \( W \) is independent of \( \pi. \)

(3) \( \pi \) are the time homogeneous Poisson random measures over \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) with the intensity measures \( \mu. \)

(4) For almost every \( t, u(t) \) and \( B(t) \) are \( \mathcal{F}_t \) measurable.

(5) \( u \in L^p(\Omega, \mathbb{D}([0, T]; H_{uw}))) \cap L^p(\Omega, L^2(0, T; V_u)) \) and \( B \in L^p(\Omega, \mathbb{D}([0, T]; H_{Bw})) \cap L^p(\Omega, L^2(0, T; V_B)), \) \( p \in [1, \infty). \) For all \( t \in [0, T], \) and \( \psi \in V \) (for details, see (3.1)-(3.6) and Remark 4.1), the following holds \( \mathbb{P} \)-a.s.

\[
\int_D u(t) \cdot \psi dx + \int_0^t \int_D [(u \cdot \nabla)u - (B \cdot \nabla)B - \Delta u] \cdot \psi x dx ds - \int_D u_0 \cdot \psi dx
\]

\[
= \int_0^t \int_D \sigma_1(s, u, B) \cdot \psi x dw_1 + \int_0^t \int_D g_1(s, u, B, z) \cdot \psi x d\pi_1(ds, dz),
\]

\[
\int_D B(t) \cdot \psi x dx + \int_0^t \int_D [(u \cdot \nabla)B - (B \cdot \nabla)u + \nabla \times ((\nabla \times B) \times B)] \cdot \psi x dx ds
\]

\[
= \int_D B_0 \cdot \psi x + \int_0^t \int_D \sigma_2(s, u, B) \cdot \psi x dw_2 + \int_0^t \int_D g_2(s, u, B, z) \cdot \psi x d\pi_2(ds, dz),
\]

and

\[
u|_{t=0} = u_0, B|_{t=0} = B_0.
\]

In the above, all stochastic integrals are defined in the sense of Itô, see [2, 9, 14, 20].
2.2. Main result. Now, we state the main result of this paper as follows.

**Theorem 2.2.** Let Assumptions (A) and (B) be satisfied and assume that \( u_0 \in H_2(D) \), \( B_0 \in H_B(D) \). Then there exists a martingale solution of problems (1.5)-(2.2) in the sense of Definition 2.1.

Theorem 2.2 will be proved through the following steps. First, we use the time splitting method to construct the approximate solutions to the problems (1.5)-(2.2). More precisely, on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with cylindrical Wiener process \(W\) and Poisson random measures \(\pi\), we solve alternatively the deterministic part of (1.5) on time intervals \([t_{2k}, t_{2k+1}]\) and the stochastic part of (1.5) on time intervals \([t_{2k+1}, t_{2k+2}]\). For details, see (4.1)-(4.4). Meanwhile, we can get the energy estimates by using the Itô formula to the function \(U\), and the Burkholder-Davis-Gundy inequality. Thanks to the orthogonality in \(L^2\) of the Hall term with \(B\), the energy estimates of the deterministic part, similar to the usual MHD case, hold true. However, for the stochastic part of (1.5) on time intervals \([t_{2k+1}, t_{2k+2}]\), we need to estimate it in the view of stochastic partial differential equations.

The second step is to take a limit as \(\tau \to 0\) and prove the existence of martingale solutions. From energy estimates, the approximate solutions \((W_\tau, \bar{\pi}_\tau, u_\tau, B_\tau)\) may converge on \([0, T]\). However, the convergence is too weak to guarantee that the limit is a solution on \([0, T]\). In the two-dimensional case, it can be shown by using certain monotonicity principle that the nonlinear terms converge to the right limit and hence a global strong solution can be obtained [16]. But when the space dimension is three, the monotonicity does not hold and to the best of our knowledge there is no result on the global strong solutions. This is why we pursue the martingale solutions instead. As is explained, the main issue is the convergence of the nonlinear terms.

To this end, we relax the restriction on the probability space and aim to prove a tightness result of the random variables \((W_\tau, \pi_\tau, u_\tau, B_\tau)\). Since we already have the energy estimates, this can be obtain by showing that \(U = (u_\tau, B_\tau)\) satisfies the Aldous condition. Then from the Jakubowski-Skorokhod Theorem there exist a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and random variables \((W_{\tau_j}, \bar{\pi}_{\tau_j}, u_{\tau_j}, B_{\tau_j}) \to (W_{\bar{\tau}}, \bar{\pi}_{\bar{\tau}}, u_{\bar{\tau}}, B_{\bar{\tau}})\) \(\mathbb{P}\)-a.s., with the property that the probability distribution of \((W_{\tau_j}, \bar{\pi}_{\tau_j}, u_{\tau_j}, B_{\tau_j})\) is the same as that of \((W_{\tau}, \bar{\pi}_{\tau}, u_{\tau}, B_{\tau})\). When passing to a limit as \(\tau \to 0\), the usual method is to show that the limit process of the stochastic integral is a martingale, and to identify its quadratic variation. Then they usually apply the representation theorem for martingales or the revised representation theorem (see [12]) to prove that it solves the equations. But here instead, we choose to prove Lemma 5.1 and Lemma 5.2 to obtain that \((W_{\bar{\tau}}, \bar{\pi}_{\bar{\tau}}, u_{\bar{\tau}}, B_{\bar{\tau}})\) satisfies the equations (1.5)-(2.2) by passing to the limit directly. Therefore it is a martingale solution of (1.5)-(2.2) in the sense of Definition 2.1.

The rest of the paper is organized as follows. We recall some analytic tools in Sobolev spaces and some basic theory of stochastic analysis in Section 3. In Section 4, we construct the solutions to an approximate scheme by the time splitting method and we prove the tightness property of the approximate solutions \((W_\tau, \pi_\tau, u_\tau, B_\tau)\). In Section 5, we pass to the limit as \(\tau \to 0\) to get a martingale solution of (1.5)-(2.2) in the sense of Definition 2.1.

**Notation.** Throughout this paper we drop the parameter \(\omega \in \Omega\). Moreover, we use \(C\) to denote a generic constant which may vary in different estimates. For simplicity, we will write \(A \lesssim B\) if \(A \leq CB\).
3. Preliminaries. Let $H^1(D)$ stand for the Sobolev space of all $\varphi \in L^2(D)$ for which there exist weak derivatives $\frac{\partial \varphi}{\partial x_i} \in L^2(D), i = 1, 2, \ldots, d$. Let $C^\infty_c(D)$ denote the space of all $\mathbb{R}^d$ valued functions of class $C^\infty$ with compact supports contained in $D$ and denote

$$\mathcal{Y}_u = \{u \in C^\infty_c(D) : \text{div} u = 0\},$$  \hspace{1cm} (3.1)$$
$$H_u = \text{the closure of } \mathcal{Y}_u \text{ in } L^2(D),$$  \hspace{1cm} (3.2)$$
$$V_u = \text{the closure of } \mathcal{Y}_u \text{ in } H^1(D).$$  \hspace{1cm} (3.3)$$

$V_B = \{B \in C^\infty_c(D) : \text{div} B = 0 \text{ and } (B \cdot n)|_{\partial D} = 0\},$  \hspace{1cm} (3.4)$$
$$H_B = \text{the closure of } \mathcal{Y}_B \text{ in } L^2(D),$$  \hspace{1cm} (3.5)$$
$$V_B = \text{the closure of } \mathcal{Y}_B \text{ in } H^1(D).$$  \hspace{1cm} (3.6)$$

In the space $H_u, H_B$ and $\mathbb{H} = H_u \times H_B$, we consider the scalar product and the norm inherited from $L^2(D)$ and (for brevity, we omit subscript “u” and “B”) denote them by $\langle \cdot, \cdot \rangle_H$ and $|\cdot|_H$ respectively, i.e.

$$\langle u, v \rangle_H = \langle u, v \rangle_{L^2(D)}, \quad |u|_H = \|u\|_{L^2(D)}, \quad \forall u, v \in H.$$  

In the space $V_u$ and $V_B$ we consider the scalar product inherited from $H^1(D)$, that is

$$\langle u, v \rangle_V = \langle u, v \rangle_H + \langle \nabla u, \nabla v \rangle_{L^2(D)},$$

and the norm $\|u\|^2 = \|u\|^2_V := \|\nabla u\|_{L^2}^2$. Let $V'$ be the dual space of $V = V_u \times V_B$. With this scalar product in $V$ we can associate the operator $A$ from $V$ into $V'$ defined by

$$\langle A\varphi|\psi \rangle = \langle A_u u|\psi \rangle + \langle A_B B|b \rangle = \langle \nabla u, \nabla v \rangle_{L^2} + \langle \nabla B, \nabla b \rangle_{L^2},$$

$\forall \varphi = (u, B), \psi = (v, b) \in V$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $V'$ and $V$. Moreover, we define a bilinear map $B$ by $B(\varphi, \psi) := \tilde{b}(\varphi, \psi, \cdot), \forall \varphi, \psi \in V$ where

$$\tilde{b}(\varphi_1, \varphi_2, \varphi_3) = b(u_1, u_2, u_3) - b(B_1, B_2, B_3) + b(u_1, B_2, B_3) - b(B_1, u_2, B_3)$$

for $\forall \varphi_i = (u_i, B_i) \in V$, $i = 1, 2, 3$, and $b(u, w, v) = \int_D (u \cdot \nabla w) v dx$. By the Sobolev embedding theorem and Hölder’s inequality, one has

$$|\tilde{b}(\varphi_1, \varphi_2, \varphi_3)| \leq \|\varphi_1\|_{L^4} \|\varphi_2\|_{V} \|\varphi_3\|_{L^4} \leq C \|\varphi_1\|_V \|\varphi_2\|_V \|\varphi_3\|_V,$$

$\forall \varphi_i \in V, i = 1, 2, 3$, and

$$\tilde{b}(\varphi_1, \varphi_2, \varphi_3) = -\tilde{b}(\varphi_1, \varphi_3, \varphi_2), \quad \forall \varphi_i \in V, \quad i = 1, 2, 3,$$

and in particular

$$\tilde{b}(\varphi_1, \varphi_2, \varphi_2) = 0, \quad \forall \varphi_1, \varphi_2 \in V.$$

Meanwhile, we infer that $B(\varphi, \psi) \in V'$ for all $\varphi, \psi \in V$ and we use the notation $B(\varphi) := B(\varphi, \varphi)$. For $B$, we have the following property (see Lemma 6.4 in [19]):

1. There exists a constant $C > 0$, such that

$$|B(\varphi, \psi)|_V \leq C \|\varphi\|_V \|\psi\|_V, \quad \forall \varphi, \psi \in V.$$  

2. In particular, the form $B : V \times V \rightarrow V'$ is bilinear and continuous. Moreover,

$$\langle B(\varphi, \psi), \theta \rangle = -\langle B(\varphi, \theta), \psi \rangle, \quad \forall \varphi, \psi, \theta \in V.$$  

3. The mapping $B$ is locally Lipschitz continuous on the space $V$, i.e. for every $R > 0$, if $\|\varphi\|_V, \|\psi\|_V \leq R$, there exists a constant $M > 0$, such that

$$|B(\varphi) - B(\psi)|_V \leq M \|\varphi - \psi\|_V, \quad \forall \varphi, \psi \in V.$$
(4) Let $\mathcal{V}_m :=$ the closure of $\mathcal{V}_a \times \mathcal{V}_B$ in the space $H^m(D, \mathbb{R}^n) \times H^m(D, \mathbb{R}^n)$. If $m > \frac{3}{2} + 1$, then $\mathcal{B}$ can be extended to the bilinear mapping from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{V}_m$ (still denoted by $\mathcal{B}$) such that
\[
|\mathcal{B}(\varphi, \psi)|_{\mathbb{V}_m} \leq C|\varphi|_{\mathbb{H}}|\psi|_{\mathbb{H}} \quad \forall \varphi, \psi \in \mathbb{H},
\]
where $C > 0$ is a constant.

For $f(s)$, by the Sobolev embedding theorem, we have $H^{m-1}(D, \mathbb{R}^n) \hookrightarrow L^\infty(D, \mathbb{R}^n)$, meanwhile, we obtain
\[
|f(s)|_{\mathbb{V}_m} = |\nabla \times ((\nabla \times B) \times B)|_{\mathbb{V}_m} = \sup_{\phi \in \mathbb{V}_m} \frac{\langle \nabla \times ((\nabla \times B) \times B), \phi \rangle}{\|\phi\|_{\mathbb{V}_m}} \leq C
\]
\[
= C\|\nabla \left( \frac{1}{2}B^2 \right) - (B \cdot \nabla)B\|_{\mathbb{V}_m} \leq C\|\nabla \left( \frac{1}{2}B^2 \right)\|_{\mathbb{V}_m} + C\|\nabla \cdot (B \cdot \nabla)B\|_{\mathbb{V}_m}
\]
\[
(3.7)
\]
\[
= C\sup_{\phi \in \mathbb{V}_m} \frac{\langle \nabla (B^2), \phi \rangle}{\|\phi\|_{\mathbb{V}_m}} + C\sup_{\phi \in \mathbb{V}_m} \frac{\langle (B \cdot \nabla)B, \phi \rangle}{\|\phi\|_{\mathbb{V}_m}} \leq C\sup_{\phi \in \mathbb{V}_m} \frac{\|\nabla \phi\|_{L^\infty} \cdot \|B\|_{H^2}^2}{\|\phi\|_{\mathbb{V}_m}} \leq C\|B\|_{H^2}^2.
\]

We now recall some preliminaries of stochastic analysis and useful tools for the sake of convenience and completeness. For details, we refer the readers to [2, 9, 14, 20] and the references therein. In particular, we shall state the definitions of Wiener process, time homogenous Poisson random measure, Lévy process, Itô’s formula and the BDG inequality and so on.

**Definition 3.1.** Let $Z_W$ be a Hilbert space. A stochastic process $\{W(t)\}_{0 \leq t \leq T}$ is said to be an $Z_W$-valued $\mathcal{F}_t$-adapted Wiener process with covariance operator $Q$ if

1. For each non-zero $h \in Z_W$, $|Q^{1/2}h|^{-1}\langle W(t), h \rangle$ is a standard one-dimensional Wiener process,

2. For any $h \in Z_W$, $\langle W(t), h \rangle$ is a martingale adapted to $\mathcal{F}_t$.

If $W$ is an $Z_W$-valued Wiener processes with covariance operator $Q$ with $\text{Tr} Q < \infty$, then $W$ is a Gaussian process on $Z_W$ and $\mathbb{E}(W(t)) = 0, \text{Cov}(W(t)) = tQ, \ t \geq 0$. Let $Z_{W0} = Q^{1/2}Z_W$, then $Z_{W0}$ is a Hilbert space equipped with the linear product $\langle \cdot, \cdot \rangle_0, \langle u, v \rangle_0 = \langle Q^{-1/2}u, Q^{-1/2}v \rangle, \forall u, v \in Z_{W0}$, where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$. Since $Q$ is a trace class operator, the imbedding of $Z_{W0}$ in $Z_W$ is Hilbert-Schmidt. Let $L_Q$ denote the space of linear operators $S$ such that $S Q^{1/2}$ is a Hilbert-Schmidt operator from $Z_{W0}$ to $Z_W$.

Denote $\tilde{N} := N \cup \{\infty\}, \mathbb{R}_+ := [0, \infty)$. Let $(Z, \mathcal{Z})$ be a measurable space. Then we denote the set of all real valued measures on $(Z, \mathcal{Z})$ by $M(Z)$, and $\mathcal{M}(Z)$ denotes the $\sigma$-field on $M(Z)$ generated by functions $i_B : \mu \mapsto \mu(B) \in \mathbb{R}$ for $\mu \in M(Z), B \in \mathcal{Z}$. Next, we denote the set of all non-negative measures on $Y$ by $M_+(Z)$, and $\mathcal{M}_+(Z)$ denotes the $\sigma$-field on $M_+(Z)$ generated by functions $i_B : M_+(Z) \ni \mu \mapsto \mu(B) \in \mathbb{R}_+, B \in \mathcal{Z}$. Finally, by $\mathcal{M}(\tilde{N}, (Z, \mathcal{Z}))$ we denote the family of all $\mathbb{N}$-valued measures on $(Z, \mathcal{Z})$, and $\mathcal{M}(\tilde{N}, (Z, \mathcal{Z}))$ denotes the $\sigma$-field on $\mathcal{M}(\tilde{N}, (Z, \mathcal{Z}))$ generated by functions $i_B : M(Z) \ni \mu \mapsto \mu(B) \in \tilde{N}, B \in \mathcal{Z}$.

**Definition 3.2.** Let $(Z, \mathcal{Z})$ be a measurable space and $\mu \in \mathcal{M}_+(Z)$. A measurable function $\pi : (\Omega, \mathcal{F}) \to (\mathcal{M}(\tilde{N}, (Z \times \mathbb{R}_+)), \mathcal{M}(\tilde{N}, (Z \times \mathbb{R}_+)))$ is called a time homogenous
Poisson random measure on \((Z, \mathcal{L})\) over \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) if and only if the following conditions are satisfied

1. for each \(A \in \mathcal{L} \otimes \mathcal{B}(\mathbb{R}_+)\), \(\pi(A) := i_A \circ \pi : \Omega \to \bar{N}\) is a Poisson random variable with parameter \(\mathbb{E}[\pi(A)]\) (if \(\mathbb{E}[\pi(A)] = \infty\), then \(\pi(A) = \infty\));
2. \(\pi\) is independently scattered, that is, if the sets \(A_j \in \mathcal{L} \otimes \mathcal{B}(\mathbb{R}_+), j = 1, 2, \ldots, n\) are pair-wise disjoint, then the random variables \(\pi(A_j), j = 1, 2, \ldots, n\) are pair-wise independent;
3. for all \(A \in \mathcal{L}\) and \(I \in \mathcal{B}(\mathbb{R}_+)\), \(\mathbb{E}[\pi(A \times I)] = \lambda(I)\mu(A)\), where \(\lambda\) is Lebesgue measure;
4. for each \(U \in \mathcal{L}\), the \(\bar{N}\)-valued process \((N(t, U))_{t \geq 0}\) defined by \(N(t, U) := \pi(U \times (0, t]), t \geq 0\) is \(\mathcal{F}_t\)-adapted and its increments are independent of the past, i.e. if \(t > s \geq 0\), then \(N(t, U) - N(s, U) = \pi(U \times (s, t))\) is independent of \(\mathcal{F}_s\).

Note that we can construct a corresponding Poisson random measure from a Lévy process. For example, given an \(E\)-valued Lévy process \(\{\xi_t \in \mathcal{B}(\mathbb{R})\times \mathcal{B}(\mathbb{R}_+)\}\), one can construct an integer valued random measure in the following: for each \((Y, I) \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}_+)\), define
\[
\pi_L(Y, I) := \sharp\{t \in I \mid \Delta_t L \in Y\} \in \bar{N}.
\]
where \(\Delta_t L(t) := L(t) - L(t^-) = L(t) - \lim_{s \uparrow t} L(s), t > 0\) and \(\Delta_0 L := 0\). If \(E = \mathbb{R}^d\), then \(\pi_L\) is a time homogeneous Poisson random measure, for details see Theorem 19.2 [22, Chapter 4]. Conversely, given a Poisson random measures, we can also construct a corresponding Lévy process. So we list the definition of a Lévy process.

**Definition 3.3.** Let \(E\) be a Banach space. A stochastic process \(L = \{L(t) : t \geq 0\}\) over \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) is called an \(E\)-valued Lévy process if the following conditions are satisfied.

1. \(L(t)\) is \(\mathcal{F}_t\)-measurable for any \(t \geq 0\);
2. the random variable \(L(t) - L(s)\) is independent of \(\mathcal{F}_s\) for any \(0 \leq s < t\);
3. \(L(0) = 0\) a.s.;
4. For all \(0 \leq s < t\), the law of \(L(t + s) - L(s)\) does not depend on \(s\);
5. \(L\) is stochastically continuous;
6. the trajectories of \(L\) are càdlàg in \(E\) \(\mathbb{P}\)-a.s., i.e. which are right-continuous with left limits.

Let us now recall the Itô formula for general Lévy-type stochastic integrals, see [2, 20]. We define \(\mathcal{P}_2(T, \mathcal{E})\) to be the set of all equivalence classes of mappings \(f : [0, T] \times \mathcal{E} \times \Omega \to \mathbb{R}\) which coincide almost everywhere with respect to \(\varrho \times \mathbb{P}\) and which satisfy the following conditions:

1. \(f\) is predictable;
2. \(\mathbb{P}\left(\int_0^T \int_{\mathcal{E}} |f(t, x)|^2 \varrho(dt, dx) < \infty\right) = 1\).

Here \(\varrho(t, A) = \varrho((0, t] \times A)\).

Now we are ready to give Itô’s formula for general Lévy-type stochastic integrals, let \(X\) be such a process in the following:
\[
dX(t) = G(t)dt + F(t)dW(t) + \int_{\mathcal{L}} H(t, z)\tilde{\pi}(dt, dz) + \int_{\mathcal{Z}} K(t, x)\pi(dt, dx),
\]
where for each \(t \geq 0, z \in Z, |G|^{\frac{1}{2}}, F \in \mathcal{P}_2(T)\) and \(H \in \mathcal{P}_2(T, \mathcal{E})\). Furthermore, we take \(K\) to be predictable and \(\mathcal{E} = \{z \in \mathbb{R}^d : 0 < |z| < 1\}\). Denote
\[
dX_c(t) = G(t)dt + F(t)dW(t),
\]
Lemma 3.4. Let $\Phi$ be a function of class $C^2$ on $\mathbb{R}^n$ and $X(t)$ a $n$-dimensional semi-martingale given as above. Assume that $H_i(t, z)H_j(t, z) = 0$, $i, j = 1, ..., n$. Then the stochastic process $\Phi(X(t))$ is also a semi-martingale and the following formula holds

$$
\Phi(X(t)) - \Phi(X(0)) = \int_0^t \partial_t \Phi(X(s^-))dX_t^0(s) + \int_0^t \int_Z [\Phi(X(s^-) + K(s, z)) - \Phi(X(s^-))] \pi(ds, dz) + \int_0^t \int_Z [\Phi(X(s^-) + H(s, z)) - \Phi(X(s^-))] \tilde{\pi}(ds, dz) + \int_0^t \int_Z [\Phi(X(s^-) + H(s, z)) - \Phi(X(s^-)) - H^i(s, z)\partial_i \Phi(X(s^-))] \mu(dz)ds.
$$

We now recall an important inequality in the stochastic analysis:

Lemma 3.5 (Burkholder-Davis-Gundy inequality). Let $T > 0$, for every fixed $p \geq 1$, there is a constant $C_p \in (0, \infty)$ such that for every real-valued square integrable càdlàg martingale $M_t$ with $M_0 = 0$, and for every $T \geq 0$,

$$
C_p^{-1} \mathbb{E} \left( \frac{(M_T)^p}{p} \right) \leq \mathbb{E} \left( \max_{0 \leq t \leq T} |M_t|^p \right) \leq C_p \mathbb{E} \left( \frac{(M_T)^p}{p} \right),
$$

where $(M)_T$ is the quadratic variation of $M_t$ and the constant $C_p$ does not depend on the choice of $M_t$.

Definition 3.6. Let $\mathcal{X}$ be a separable Banach space and let $\mathcal{B}(\mathcal{X})$ be its Borel sets. A family of probability measures $\mathbb{P}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is tight if for any $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \mathcal{X}$ such that $\Pi(K_\varepsilon) \geq 1 - \varepsilon$ for all $\Pi \in \mathbb{P}$. A sequence of measures $\{\Pi_n\}$ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ is weakly convergent to a measure $\Pi$ if for all continuous and bounded functions $h$ on $\mathcal{X}$,

$$
\lim_{n \to \infty} \int_{\mathcal{X}} h(x)\Pi_n(dx) = \int_{\mathcal{X}} h(x)\Pi(dx).
$$

Lemma 3.7 (Jakubowski-Skorokhod Theorem [12]). Let $\mathcal{X}$ be a topological space such that there exists a sequence $\{h_m\}$ of continuous functions $h_m : \mathcal{X} \to \mathbb{R}$ that separate points of $\mathcal{X}$. Denote by $\mathcal{F}$ the $\sigma$-algebra generated by the maps $\{h_m\}$. Then

(1) every compact subset of $\mathcal{X}$ is metrizable.

(2) every Borel subset of a $\sigma$-compact set in $\mathcal{X}$ belongs to $\mathcal{F}$.

(3) every probability measure supported by a $\sigma$-compact set in $\mathcal{X}$ has a unique Radon extension to the Borel $\sigma$-algebra on $\mathcal{F}$.

(4) if $\Pi_m$ is a tight sequence of probability measures on $(\mathcal{X}, \mathcal{F})$, there exist a subsequence $\Pi_{m_k}$ converging weakly to a probability measure $\Pi$, and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}$ valued Borel measurable random variables $X_k$ and $X$ such
where $\Pi_{mk}$ is the distribution of $X_k$, and $X_k \to X$ a.s. on $\Omega$. Moreover, the law of $X$ is a Radon measure.

4. **Time splitting method and energy estimates.** In order to solve (1.5)-(2.2) with given cylindrical Wiener process $W$ and Poisson random measures $\pi$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we first consider $X$ which is a Hilbert space and satisfy

$$X \ni \nu \rightarrow \nu \leftrightarrow \mathbb{H}.$$ 

In particular, $X$ is compactly embedded into the space $\mathbb{H}$.

**Remark 4.1.** Here we consider the following functional spaces being the counterparts in our framework of the spaces.

- $\mathcal{D}([0, T], \mathbb{R}^n)$ := the space of càdlàg functions $\xi : [0, T] \rightarrow \mathbb{R}^n$ with the topology induced by the Skorohod metric $\delta_T$. The Skorohod metric $\delta_T$ is defined by:

$$\delta_T(u, v) := \inf_{\lambda \in \Lambda_T} \left[ \sup_{t \in [0, T]} \left| u - v \lambda \right| + \sup_{t \in [0, T]} \left| t - \lambda(t) \right| + \sup_{s \neq t} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right],$$

where $\Lambda_T$ is the set of increasing homeomorphisms of $[0, T]$.

- $L^2_w(0, T; \mathcal{V}) :=$ the space $L^2(0, T; \mathcal{V})$ with the weak topology.

- $L^2(0, T; L^2_{loc}(D)) :=$ the space of measurable functions $u : [0, T] \times D \rightarrow \mathbb{R}^n$ with the topology generated by the seminorms $p_T$, where

$$p_T(u) := \left( \int_0^T \int_D |u(t, x)|^2 dx dt \right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{D}([0, T]; \mathbb{H}_w) :=$ the space of weakly càdlàg paths from $[0, T]$ into $\mathbb{H}$, where $\mathbb{H}_w$ is a Hilbert space $\mathbb{H}$ endowed with the weakest topology, such that for all $h \in \mathbb{H}$ the mappings $\mathcal{D}([0, T]; \mathbb{H}_w) \ni u \mapsto (u(\cdot), h)_\mathbb{H} \in \mathbb{D}([0, T]; \mathbb{R})$. In particular, $u_n \to u$ in $\mathcal{D}([0, T]; \mathbb{H}_w)$ if and only if for all $h \in \mathbb{H}$, $(u_n(\cdot), h)_\mathbb{H} \to (u(\cdot), h)_\mathbb{H}$ in the space $\mathcal{D}([0, T]; \mathbb{R})$.

Let

$$h^\tau_{dt}(s) = \sum_{j=0}^{T-1} 1_{(t_{2j}, t_{2j+1})}(s) = 1 - h^\tau_{dt}(s),$$

where $T$ is assumed to be an even integer. We partition the time interval $[0, T]$ into $T$ time intervals of length $\tau$. Meanwhile, we denote $t_j = j \tau$. We shall look for the pair of sequences $U_\tau$ which satisfy the following equation:

- for $t_{2k} \leq t < t_{2k+1}$,

$$dU_\tau + 2AU_\tau dt + 2B(U_\tau)dt = 2f(s)dt \text{ in } D \times (0, T), \quad (4.1)$$

$$U_\tau(t_{2k}) = U_\tau(t_{2k-}) \text{ in } D, \quad (4.2)$$

- for $t_{2k+1} \leq t < t_{2k+2}$,

$$dU_\tau = \sqrt{2}\sigma_\tau(t, U_\tau) dW(t) + 2 \int_{\mathbb{Z}} g_\tau(t, U_\tau(t^-)) \tilde{\pi}(dt, dz) \text{ in } D \times (0, T), \quad (4.3)$$

$$U_\tau(t_{2k+1}) = U_\tau(t_{2k+1-}) \text{ in } D. \quad (4.4)$$
**Remark 4.2.** In the time-splitting scheme, since the deterministic equation and the stochastic equation just “run” in half the time, in order to approximate the equation (1.5), we need to “speed up” them in their “running” time. Here notice that we took care to speed up the deterministic equation (4.1) by a factor 2 and the stochastic equation (4.3) by a factor $(\sqrt{2}, 2)$, and these rescaling procedures should yield a solution $U_\tau$ is consistent with the solution $U$ to (1.5) when $\tau \to 0$. Actually in the stochastic equation (4.3), we can see the term $\sigma_\tau(t, U_\tau) dW(t)$ is “faster” than the other term caused by the Wiener process $W$, and this phenomenon can not be seen if we use the classical Faedo-Galerkin approximation method. In (4.3) we have also regularized the coefficients $\sigma_\tau$ and $g_\tau$, where $\sigma_\tau$ and $g_\tau$ are globally Lipschitz continuous and satisfies Assumption (A) and Assumption (B) uniformly in $\tau$ and converge punctually to $\sigma$ and $g$ when $\tau \to 0$, respectively. We note that the derivatives of the coefficients $\sigma_\tau$ and $g_\tau$ are also Lipschitz continuous. For example, $[\sigma_\tau e_k](x) = \Psi_\tau \ast (\sigma e_k) 1_{k \leq \tau}$, $g_\tau = \Phi_\tau \ast g$, where $\Psi_\tau = \frac{1}{\tau^2} \Psi(\frac{z}{\tau}) \Psi(\frac{x}{\tau})$, $\Phi_\tau = \frac{1}{\tau^2} \Phi(\frac{z}{\tau}) \Phi(\frac{x}{\tau})$, and $\Psi \Phi$ is the non-negative smooth density of a probability measure or a mollifier.

Let us introduce the stopping times

$$\varsigma_N = \left\{ \inf\left\{ t > 0 : |U_\tau(t)| + \int_0^t \|U_\tau(s)\|^2 ds \geq N \right\} \right\}.$$

Applying Itô’s formula in Lemma 3.4 to the function $\Phi(X) = |X|^p_H$, where $X = U_\tau(t \wedge \varsigma_N) \in \mathbb{H}$, one deduces that for all $t \in [0, T]$

$$|U_\tau(t \wedge \varsigma_N)|^p_H + \int_0^{t \wedge \varsigma_N} h^*_d p|U_\tau(s)|^{p-2} (2A U_\tau(s) + 2B(U_\tau(s), U_\tau(s))) ds$$

$$\leq |U_\tau(0)|^p_H + \int_0^{t \wedge \varsigma_N} \phi h^*_d (2f, U_\tau)|U_\tau(s)|^{p-2} ds$$

$$+ \sqrt{2p} \int_0^{t \wedge \varsigma_N} h^*_d |U_\tau|^{p-2} (\sigma(s, U_\tau(s)), U_\tau(s))dW$$

$$+ p(|p| - 1) \int_0^{t \wedge \varsigma_N} h^*_d |U_\tau|^{p-2} |U_\tau(s), U_\tau(s))Q \sigma_\tau(s, U_\tau(s))) ds$$

$$+ \int_0^{t \wedge \varsigma_N} \int_Z 2h^*_d p(g_\tau(U_\tau(s^-, z), U_\tau(s))) |U_\tau|^{p-2} \pi(ds, dz)$$

$$+ \int_0^{t \wedge \varsigma_N} \int_Z 2h^*_d [p(U_\tau(s)^{p-2} g_\tau(U_\tau(s)), U_\tau(s))] \pi(ds, dz)$$

$$+ \int_0^{t \wedge \varsigma_N} \int_Z 2h^*_d [U_\tau(s^-) + g_\tau(U_\tau(s^-), z)|U_\tau|^{p-2} - |U_\tau(s^-)|^{p-2}] \mu(dz) ds$$

For all $a, b \in \mathbb{H}$ and $p \geq 2$, from Taylor’s formula, it holds that

$$||a + b||_H^p - |a||_H^p \leq C(p) (|a||_H^p + |b||_H^p),$$

$$||a + b||_H^p - |a||_H^p - p|a||_H^{p-2}(a, b) \leq C(p) \left( |a||_H^{p-2}|b||_H^2 + |b||_H^p \right). \quad \text{(4.5)}$$

By using these above inequalities (4.5), when $p = 2$, we have

$$|U_\tau(t \wedge \varsigma_N)|^2_H + \int_0^{t \wedge \varsigma_N} 2h^*_d |U_\tau|_H^2 ds$$
\[ \leq |U_\tau(0)|^2_{\mathbb{H}} + \int_0^{T \wedge \varsigma_N} 4(h^{de}_{de} f_n, U_\tau) \, ds \]  \quad (4.6)

\[ + \int_0^{T \wedge \varsigma_N} 2\sqrt{2} h^{T}_{st} \sigma_\tau(s, U_\tau(s)) \, dW \]  \quad (4.7)

\[ + \int_0^{T \wedge \varsigma_N} 2h^{T}_{st} Tr(\sigma_\tau(s, U_\tau(s)) Q \sigma^*_\tau(s, U_\tau(s))) \, ds \]

\[ + \int_0^{T \wedge \varsigma_N} \int_Z 4h^{T}_{st}(g_\nu(U_\tau(s^{-}), z), U_\tau(s)) \tilde{\pi}(ds, dz) \]

\[ + \int_0^{T \wedge \varsigma_N} \int_Z 2h^{T}_{st}|g_\nu(s, U_\tau(s))|^2_{H^2} \mu(dz). \]

Taking supremum \( t \in [0, T \wedge \varsigma_N] \) in the above inequality (4.6) and expectation on the interval \([0, t \wedge \varsigma_N] \) where \( t \wedge \varsigma_N = \min\{t, \varsigma_N\} \), we obtain

\[ \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \varsigma_N} |U_\tau|^2_{\mathbb{H}} \right) + \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 2h^{T}_{de} |U_\tau|^2_{H^2} \, ds \right] \]

\[ \leq \mathbb{E} \left( |U_\tau(0)|^2_{\mathbb{H}} \right) + \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 4h^{T}_{de}(f_\tau, U_\tau) \, ds \right] \]

\[ + \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \varsigma_N} \int_0^{T \wedge \varsigma_N} 2\sqrt{2} h^{T}_{st} (\sigma_\tau(s, U_\tau(s)), U_\tau(s)) \, dW \right] \]

\[ + \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 2h^{T}_{st} Tr(\sigma_\tau(s, U_\tau(s)) Q \sigma^*_\tau(s, U_\tau(s))) \, ds \right] \]

\[ + \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \varsigma_N} \int_0^{T \wedge \varsigma_N} \int_Z 4h^{T}_{st}(g_\nu(U_\tau(s^{-}), z), U_\tau(s)) \tilde{\pi}(ds, dz) \right] \]

\[ + \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} \int_Z 2h^{T}_{st}|g_\nu(s, U_\tau(s))|^2_{H^2} \mu(dz) \, ds \right]. \]  \quad (4.8)

For the second term on the right hand side of (4.8), integrating by parts, one has

\[ \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 4h^{T}_{de}(f_\tau, U_\tau) \, ds \right] = \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 4h^{T}_{de}(- \nabla \times (\nabla \times B_\tau) \times B_\tau, B_\tau) \, ds \right] \]

\[ = \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 4h^{T}_{de}(\nabla \times B_\tau) \times B_\tau, \nabla \times B_\tau) \, ds \right] = 0. \]  \quad (4.9)

For the last term on the right hand side of (4.8), by using (2.4) in Assumption (B), we have

\[ \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} \int_Z 2h^{T}_{st}|g_\nu(s, U_\tau(s))|^2_{H^2} \mu(dz) \, ds \right] \leq C \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} (1 + |U_\tau(s)|^2_{H^2})^2 \, ds \right]. \]  \quad (4.10)

For the fourth term on the right hand side of (4.8), thanks to Assumption (A), one has

\[ \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} 2h^{T}_{st} Tr(\sigma_\tau(s, U_\tau(s)) Q \sigma^*_\tau(s, U_\tau(s))) \, ds \right] \leq C \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} |\sigma_\tau(s, U_\tau(s))|^2 \, ds \right] \]

\[ \leq C \mathbb{E} \left[ \int_0^{T \wedge \varsigma_N} (1 + |U_\tau(s)|^2_{H^2})^2 \, ds \right]. \]  \quad (4.11)
For the stochastic terms in (4.8), by the Burkholder-Davis-Gundy inequality and Young's inequality, we can get

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \xi_N} \int_0^{t \wedge \xi_N} \left| 2\sqrt{2} h_{st}^\tau \langle \sigma_r(s, U_r(s)), U_r(s) \rangle dW_s \right| \right] \leq \mathbb{E} \int_0^{t \wedge \xi_N} (\sigma_r(s, U_r(s)), U_r(s))^2 ds \right]^{\frac{1}{2}} \\
\leq \mathbb{E} \int_0^{t \wedge \xi_N} |\sigma_r(s, U_r(s))|^2 |U_r(s)|_{H^2}^2 ds \right]^{\frac{1}{2}} \right]
\leq \mathbb{E} \int_0^{t \wedge \xi_N} (1 + |U_r(s)|_{H^2})^2 |U_r(s)|_{H^2}^2 ds \right]^{\frac{1}{2}} \\
\leq \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \xi_N} |U_r(t \wedge \xi_N)|_{H^2}^2 \right) + C \mathbb{E} \left[ \int_0^{t \wedge \xi_N} (1 + |U_r(s)|_{H^2})^2 ds \right],
\]
and

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \xi_N} \int_0^{t \wedge \xi_N} \int_Z 4h_{st}^\tau (g_r(U_r(s^\tau), z), U_r(s)) \tilde{\pi}(ds, dz) \right] \\
\leq C \mathbb{E} \int_0^{t \wedge \xi_N} \int_Z (g_r(U_r(s^\tau), z), U_r(s))^2 \mu(dz) ds \right]^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \xi_N} |U_r(t \wedge \xi_N)|_{H^2}^2 \right) + C \mathbb{E} \left[ \int_0^{t \wedge \xi_N} (1 + |U_r(s)|_{H^2})^2 ds \right].
\]

Meanwhile, it holds that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \xi_N} \int_0^{t \wedge \xi_N} 2\sqrt{2} h_{st}^\tau |U_r(s)|_{H^2}^{p-2} (\langle \sigma_r(s, U_r(s)), U_r(s) \rangle dW_s \right] \right]^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} \int_0^{t \wedge \xi_N} |U_r(s)|_{H^2}^{2(p-2)} (\langle \sigma_r(s, U_r(s)), U_r(s) \rangle^2 ds \right]^{\frac{1}{2}} \\
\leq C \mathbb{E} \left( \sup_{0 \leq t \leq T} |U_r(t \wedge \xi_N)|_{H^2}^{p-1} \left( \int_0^{t \wedge \xi_N} |\sigma_r(s, U_r(s))|^p ds \right)^{\frac{1}{2}} \right] \right]
\leq \mathbb{E} \left( \sup_{0 \leq t \leq T \wedge \xi_N} |U_r(t \wedge \xi_N)|_{H^2}^p \right) + C \mathbb{E} \left[ \int_0^{t \wedge \xi_N} (1 + |U_r(s)|_{H^2}^p) ds \right].
\]

From (4.5), one has

\[
\mathbb{E} \left[ \int_0^{t \wedge \xi_N} \int_Z 2h_{st}^\tau \left( |U_r(s^-) + g_r(U_r(s^-), z)|_{H^2}^p - |U_r(s^-)|_{H^2}^p \mu(dz) \right) ds \right]
\]
and equality, then we have

\[ -E \left[ \int_0^{T \wedge \varsigma_N} \int_Z p|U_\tau(s^-)|_{H^1}^{p-2}(g_\tau(s, U_\tau(s^-)), U_\tau(s)) \mu(dz)ds \right] \]

\[ \leq CE \left[ \int_0^{T \wedge \varsigma_N} \int_Z \left( |U_\tau(s)|_{H^1}^{p-2}|g_\tau(U_\tau(s^-), z)|^2 + |g_\tau(U_\tau(s^-), z)|^p \right) \mu(dz)ds \right] \quad (4.15) \]

\[ \leq CE \left[ \int_0^{T \wedge \varsigma_N} (1 + |U_\tau(s)|_{H^1}^p) ds \right]. \]

In summary, by using (4.8)-(4.15), Gronwall’s Lemma and Fatou’s lemma (passing to the limit as \( N \to \infty \)), we have the following Proposition:

**Proposition 1.** Let \( p \in [1, \infty) \), there exists a constant \( C \) such that

\[ E \left( \sup_{0 \leq t \leq T} |U_\tau|_{H^1}^{2p} \right) + E \left( \int_0^T h_{de}^\tau |U_\tau|^2 ds \right) \leq C \left[ 1 + E \left( |U_\tau(0)|_{H^1}^{2p} \right) \right]. \quad (4.16) \]

After getting the energy estimates of the deterministic part, we need to estimate the stochastic part.

**Proposition 2.** Let \( p \in [1, \infty) \), there exists a constant \( C \) such that

\[ E \left( \int_0^T h_{st}^\tau |U_\tau|^2 ds \right) \leq C \left[ 1 + E \left( |U_\tau(0)|_{H^1}^{2p} \right) \right]. \quad (4.17) \]

**Proof.** For short, Let

\[ \Sigma_1(s) = \int_{t_{2k+1}}^s \sqrt{2} \sigma_\tau(\zeta, U_\tau(\zeta))dW(\zeta), \]

and

\[ \Sigma_2(s) = \int_{t_{2k+1}}^s \int_Z 2g_\tau(U_\tau(\zeta^-), z) \hat{\pi}(d\zeta, dz). \]

By using Assumption (A), Assumption (B) and the Burkholder-Davis-Gundy inequality, then we have

\[ E \left( \int_0^T h_{st}^\tau |U_\tau|^2 ds \right) = E \left( \sum_{k=0}^{T} \int_{t_{2k+1}}^{t_{2k+2}} |U_\tau(t_{2k+1}) + \Sigma_1(s) + \Sigma_2(s)|^2 ds \right)^p \]

\[ \leq E \left( \sum_{k=0}^{T} \int_{t_{2k+1}}^{t_{2k+2}} \left( |U_\tau(t_{2k+1})|^2 + \int_{t_{2k+1}}^s |U_\tau(\zeta)|^2 d\zeta \right) ds \right)^p \]

\[ \leq E \left( \sum_{k=0}^{T} \int_{t_{2k+1}}^{t_{2k+2}} \left( |U_\tau(t_{2k+1})|^2 + \int_{t_{2k+1}}^s |U_\tau(\zeta)|^2 d\zeta \right) ds \right)^p \]

\[ \leq CE \left( \sum_{k=0}^{T} \sup_{t \in [0, T]} |U_\tau|_{H^1}^2 \int_{t_{2k+1}}^{t_{2k+2}} (1 + s - t_{2k+1}) ds \right)^p \]

\[ \leq C(T)E \left( \sup_{t \in [0, T]} |U_\tau|_{H^1}^{2p} \right) \leq C \left[ 1 + E \left( |U_\tau(0)|_{H^1}^{2p} \right) \right]. \]

\( \square \)

We combine Proposition 1 and Proposition 2 to get the following estimate:
Proosition 3. Let \( p \in [1, \infty) \), there exists a constant \( C > 0 \) such that
\[
E \left( \sup_{0 \leq t \leq T} |U_t|^p \right) + E \left( \int_0^T \|U_t\|^2 dt \right)^p \leq C \left[ 1 + E \left( |U_0|^p \right) \right].
\] (4.18)

Let \( \mathcal{L}(U_t) \) be probability measures of the solutions \( U_t \) of the splitting equations on the measurable space \( (G, \mathcal{F}) \), where \( G = L^2_w(0, T; V) \cap L^2(0, T; L^2_{loc}(D)) \cap D([0, T]; X') \). Before proving the tightness of \( \mathcal{L}(U_t) \), we recall the following lemma.

Lemma 4.1. Let \( (U_t)_{t \in (0,1)} \) be a sequence of càdlàg \( (\mathcal{F}_t)_{t \geq 0} \)-adapted \( X' \)-valued processes such that
\((i)\) there exist a positive constant \( C > 0 \) such that
\[
\sup_{\tau} \mathbb{E} \left( \sup_{0 \leq t \leq T} |U_t|^{2p} \right) + \sup_{\tau} \mathbb{E} \left( \int_0^T \|U_t\|^2 dt \right)^p \leq C.
\]

\((ii)\) \( (U_t) \) satisfies the Aldous condition in \( X' \).

Let \( \mathbb{P} \) be the law of \( U_t \) on \( G \). Then for every \( \varepsilon > 0 \) there exists a compact subset \( K_\varepsilon \) of \( G \) such that
\[
\mathbb{P}(K_\varepsilon) \geq 1 - \varepsilon.
\]

Let us recall the Aldous condition in the form given by Métivier (see [13]).

Definition 4.2. A sequence \( (U_t)_{t \in (0,1)} \) satisfies the Aldous condition in the space \( X' \) if and only if for all \( \varepsilon > 0 \), \( \forall \eta > 0 \), \( \exists \delta > 0 \) such that for every sequence \( (\varpi_t)_{t \in (0,1)} \) of \( (\mathcal{F}_t)_{t \geq 0} \)-stopping times with \( \varpi_t \leq T \), one has
\[
\sup_{\tau} \mathbb{E} \left( \sup_{0 \leq \theta \leq \delta} \mathbb{P}\{ |U_t(\varpi_t + \theta) - U_t(\varpi_t)|_{X'} \geq \eta \} \right) \leq \varepsilon\]

Now, we are going to prove the following lemma.

Lemma 4.3. The set of measures \( \{\mathcal{L}(U_t)\} \) is tight on \( (G, \mathcal{F}) \).

Proof. In order to prove the tightness of measures \( \{\mathcal{L}(U_t)\} \) on \( (G, \mathcal{F}) \), since we have already proved Proposition 1, applying Lemma 4.1, it is sufficient to prove that the sequence \( (U_t) \) satisfies the Aldous condition in Definition 4.2. Note that
\[
U_t(t) = U_t(0) - 2 \int_0^t h_{de}^\tau A_U(s) + B(U(s)) \, ds + \int_0^t 2h_{de}^\tau f(\tau(s)) \, ds
\]
\[+ \int_0^t \sqrt{2} h_{de}^\tau \sigma(s, U_t) \, dW(s) + \int_0^t \int_Z 2h_{st}^\tau g_{n}(s, U_t(s^-), z) \, \bar{n}(ds, dz)
\]
\[=: \sum_{i=1}^6 R_i(t).
\]

For the term \( R_2 \), since \( A_n : \mathcal{V} \to \mathcal{V}' \) and \( |A_n U_n|_{\mathcal{V}'} \leq \|U_n\| \) and the embedding \( \mathcal{V}' \to X' \) is continuous, by using inequality (4.18), we have
\[
\mathbb{E}[|R_2(\varpi_t + \theta) - R_2(\varpi_t)|_{X'}] = 2\mathbb{E} \left[ \int_{\varpi_t}^{\varpi_t + \theta} h_{de}^\tau A_U(s) ds \right] \lesssim \mathbb{E} \left[ \int_{\varpi_t}^{\varpi_t + \theta} h_{de}^\tau |A_U(s)|_{\mathcal{V}'} ds \right]
\]
For the term $R_3$, since $X \hookrightarrow V_m$ and by the property of $B$, we get
\[
\mathbb{E}[|R_3(\varphi_\tau + \theta) - R_3(\varphi_\tau)|_{X'}] = 2\mathbb{E} \left[ \left| \int_{\varphi_\tau}^{\varphi_\tau + \theta} h_{de}^\tau B U_\tau(s) ds \right|_{X'} \right] \\
\lesssim \mathbb{E} \left[ \int_{\varphi_\tau}^{\varphi_\tau + \theta} h_{de}^\tau |B U_\tau(s)|_{V_m} ds \right] \\
\lesssim \mathbb{E} \left[ \int_{\varphi_\tau}^{\varphi_\tau + \theta} |B| \cdot |U_\tau(s)|_{H}^2 ds \right] \\
\lesssim \|B\| \mathbb{E} \sup_{s \in [0,T]} |U_\tau(s)|_H^2 \cdot \theta \\
\lesssim \|B\| \left[ 1 + \mathbb{E} \left( |U_\tau(0)|^2 \right) \right] \cdot \theta = C \cdot \theta,
\]
where $\|B\|$ stands for the norm of $B : \mathbb{H} \times \mathbb{H} \rightarrow V'_m$, and we have used the inequality (4.18) in the last inequality of (4.21).

For the term $R_4$, we use the fact that $X \hookrightarrow V_m$ and (3.7) to obtain
\[
\mathbb{E}[|R_4(\varphi_\tau + \theta) - R_4(\varphi_\tau)|_{X'}] = 2\mathbb{E} \left[ \left| \int_{\varphi_\tau}^{\varphi_\tau + \theta} h_{de}^\tau f_\tau(s) ds \right|_{X'} \right] \\
\lesssim \mathbb{E} \left[ \int_{\varphi_\tau}^{\varphi_\tau + \theta} h_{de}^\tau |f_\tau(s)|_{V_m'} ds \right] \\
\lesssim \mathbb{E} \left[ \int_{\varphi_\tau}^{\varphi_\tau + \theta} |U_\tau(s)|_{H}^2 ds \right] \lesssim \mathbb{E} \left[ \sup_{s \in [0,T]} |U_\tau(s)|_{H}^2 \right] \cdot \theta \\
\lesssim \left[ 1 + \mathbb{E} \left( |U_\tau(0)|^2 \right) \right] \cdot \theta = C \cdot \theta.
\]

For the term $R_5$, by the Itô isometry, Assumption (A), continuity of the embedding $V' \hookrightarrow X'$ and (4.18), we obtain the following inequality
\[
\mathbb{E}[|R_5(\varphi_\tau + \theta) - R_5(\varphi_\tau)|_{X'}^2] = \mathbb{E} \left[ \left| \int_{\varphi_\tau}^{\varphi_\tau + \theta} h_{st}^\tau \sigma_\tau(s, U_\tau) dW(s) \right|_{X'}^2 \right] \\
\lesssim \mathbb{E} \left[ \int_{\varphi_\tau}^{\varphi_\tau + \theta} (1 + |U_\tau(s)|_{H}^2) ds \right] \\
\lesssim \theta (1 + \mathbb{E} \left[ \sup_{s \in [0,T]} |U_\tau|^2 \right]) \\
\lesssim \theta \cdot \left[ 1 + \mathbb{E} \left( |U_\tau(0)|^2 \right) \right] = C \theta.
\]
Finally, we consider the term $R_6$. By Assumption (B), and (4.18), one has

$$
\mathbb{E}[|R_6(\varpi_\tau + \theta) - R_6(\varpi_\tau)|^2] = 2\mathbb{E}\left[ \int_{\varpi_\tau}^{\varpi_\tau + \theta} \int_Z h^\tau_s g_\tau(s, U_\tau(s^-), z) d\pi(ds, dz) \right]^2 
\leq \mathbb{E}\left[ \int_{\varpi_\tau}^{\varpi_\tau + \theta} \int_Z g_\tau(s, U_\tau(s^-), z) d\pi(ds, dz) \right]^2
\leq \mathbb{E}\left[ \int_{\varpi_\tau}^{\varpi_\tau + \theta} \int_Z |g_\tau(s, U_n(s^-), z)|^2 \mu(dy)ds \right]
\leq (1 + \mathbb{E}[\sup_{s \in [0, T]} |U_\tau(s)|]) \cdot \theta
\leq [1 + \mathbb{E}(|U_\tau(0)|^2)] \cdot \theta = C \cdot \theta.
$$

Combining (4.19)-(4.24), we know that there exist a constant $C > 0$ such that for every sequence $\varpi_\tau$ of $\mathcal{F}$-stopping times with $\varpi_\tau < T$ and for every $\tau \in (0, 1)$ and $\theta \geq 0$ there holds

$$
\mathbb{E}[|U_\tau(\varpi_\tau + \theta) - U_\tau(\varpi_\tau)|^2] \leq C\theta^2.
$$

Now, we can apply Lemma 9 in [18] to infer that the sequence $(U_\tau)_{\tau \in (0, 1)}$ satisfies the Aldous condition in the space $\mathcal{X}$. And from lemma 4.1 we complete the proof. \(\Box\)

5. Proof of Theorem 2.2. Let now $(\tau_n)$ be a sequence decreasing to 0. For simplicity, we will keep the notation $\tau$ for $\tau_n$ and eventual subsequence. After we proved the tightness of $\{\mathcal{L}(U_\tau)\}$ on $(G, \mathcal{G})$, we will apply the Lemma 3.7 for the sequence of laws of $(U_\tau, \tilde{\pi}_\tau, W_\tau)$, where $\tilde{\pi}_\tau = \hat{\pi}$ and $W_\tau = W$. Before that, we need to define the space: $C([0, T]; Z_W)$-the space of $Z_W$-valued continuous functions with the standard supremum-norm. $M_\mathbb{F}([0, T] \times Z)$ is defined by Definition 3.2. Then we know the set $\{\mathcal{L}(U_\tau, \tilde{\pi}_\tau, W_\tau)\}$ is tight on $G \times M_\mathbb{F}([0, T] \times Z) \times C([0, T]; Z_W)$. Now, we apply Corollary C.1 and Remark C.2 in [19] to infer that there exist a subsequence $(\tau_n)_{k \in \mathbb{N}}$ still denoted by $\tau$, a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and $G \times M_\mathbb{F}([0, T] \times Z) \times C([0, T]; Z_W)$-valued random variables $(U_0, \tilde{\pi}_0, W_0), (U_\tau, \tilde{\pi}_\tau, W_\tau)$, such that:

1. $\mathcal{L}((\tilde{U}_\tau, \tilde{\pi}_\tau, \tilde{W}_\tau)) = \mathcal{L}((U_{\tau_n}, \tilde{\pi}_{\tau_n}, W_{\tau_n}))$ for all $\tau \in (0, 1)$;
2. $(\tilde{U}_\tau, \tilde{\pi}_\tau, \tilde{W}_\tau) \rightarrow (U_0, \tilde{\pi}_0, W_0)$ in $G \times M_\mathbb{F}([0, T] \times Z) \times C([0, T]; Z_W)$ with probability 1 on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as $\tau \rightarrow 0$;
3. $(\tilde{\pi}_\tau(\tilde{\omega}), \tilde{W}_\tau(\tilde{\omega})) = (\tilde{\pi}_0(\tilde{\omega}), W_0(\tilde{\omega}))$ for all $\tilde{\omega} \in \tilde{\Omega}$.

We still denote these sequences by $(U_\tau, \tilde{\pi}_\tau, W_\tau)_{\tau \in (0, 1)}$ and $(\tilde{U}_\tau, \tilde{\pi}_\tau, \tilde{W}_\tau)_{\tau \in (0, 1)}$, respectively. Here, $\tilde{\pi}_\tau, \tau \in (0, 1)$ and $\tilde{\pi}_0$ are time homogeneous Poisson random measures on $(Z, \mathcal{Z})$ with the intensity measure $\mu$ and $\tilde{W}_\tau, \tau \in (0, 1)$ and $\tilde{W}_0$ are cylindrical Wiener processes, see Section 9 of [7]. By the definition of the space $G$, we have

$$
\tilde{U}_\tau \rightarrow U_0 \quad in \quad G \quad \tilde{\mathbb{P}} \quad a.s.
$$

(5.1)
Since the random variables $\bar{U}_\tau$ and $U_\tau$ are identically distributed, we have the following inequality:

$$\sup_{\tau \in (0, 1)} \mathbb{E} \left( \sup_{0 \leq t \leq T} |\bar{U}_\tau| \right) + \sup_{\tau \in (0, 1)} \mathbb{E} \left( \int_0^T \|\bar{U}_\tau\|^2 \, ds \right) \leq C(p), \quad (5.2)$$

for every $p \in [1, \infty)$.

By (5.2) with $p = 1$, we know that there exists a subsequence of $(\bar{U}_\tau)$, still denoted by $(\bar{U}_\tau)$, convergent weakly in the space $L^2([0, T] \times \bar{\Omega}, \mathcal{V})$ and also convergent weakly in the space $L^2(\bar{\Omega}, L^\infty(0, T; \mathbb{H}))$. From (5.1), we can infer that $U_\circ \in L^2([0, T] \times \bar{\Omega}, \mathcal{V}) \cap L^2(\bar{\Omega}, L^\infty(0, T; \mathbb{H}))$, i.e.,

$$\mathbb{E} \left( \sup_{0 \leq t \leq T} |U_\circ| \right) + \mathbb{E} \left( \int_0^T \|U_\circ\|^2 \, ds \right) < \infty. \quad (5.3)$$

Analogously to [7], we define:

$$\Gamma_{\tau}(\bar{U}_\tau, \bar{\pi}_\tau, W_\tau, \psi)(t)(t) = (\bar{U}_\tau(0), \psi)_\mathbb{H} - \int_0^t \langle 2h_{de}^\tau A\bar{U}_\tau(s), \psi \rangle \, ds$$
$$- \int_0^t \langle 2h_{de}^\tau B_{\tau}(\bar{U}_\tau(s)), \psi \rangle \, ds - \int_0^t \langle 2h_{de}^\tau f_{\tau}(s), \psi \rangle \, ds$$
$$+ \left\langle \int_0^t \sqrt{2h_{de}^\tau} \sigma_{\tau}(s, \bar{U}_\tau) dW_\tau, \psi \right\rangle$$
$$+ \int_0^t \int_{\mathbb{R}^2} 2h_{de}^\tau (g_{\tau}(s, \bar{U}_\tau(s)^-), y) \, \bar{\pi}_{\tau}(ds, dy), \quad (5.4)$$

and

$$\Gamma(U_\circ, \pi_\circ, W_\circ, \psi)(t)(t) = (U_\circ(0), \psi)_\mathbb{H} - \int_0^t \langle AU_\circ(s), \psi \rangle \, ds$$
$$- \int_0^t \langle B(U_\circ(s)), \psi \rangle \, ds - \int_0^t \langle f_\circ(s), \psi \rangle \, ds$$
$$+ \left\langle \int_0^t \sigma(s, U_\circ) dW_\circ, \psi \right\rangle$$
$$+ \int_0^t \int_{\mathbb{R}^2} (g(s, U_\circ(s)^-), y) \, \bar{\pi}_\circ(ds, dy), \quad (5.5)$$

where $f_\circ(s) = \left( \begin{array}{c} 0 \\ -\nabla \times ((\nabla \times B_\circ) \times B_\circ) \end{array} \right)$, $t \in [0, T]$ and $\psi \in \mathbb{X}$.

**Lemma 5.1.** For all $\psi \in \mathbb{X}$, we have the following equalities:

1. $\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T |(\bar{U}_\tau(t) - U_\circ(t), \psi)_\mathbb{H}| \, dt \right] = 0,$
2. $\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T |(\bar{U}_\tau(0) - U_\circ(0), \psi)_\mathbb{H}| \, dt \right] = 0,$
3. $\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T \left\langle 2h_{de}^\tau AU_\circ(s), \psi \right\rangle \, ds \right] = 0,$
4. $\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T \left\langle 2h_{de}^\tau BU_\circ(s), \psi \right\rangle \, ds \right] = 0,$
5. $\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T \left\langle 2h_{de}^\tau f_{\tau}(s) - f_\circ(s), \psi \right\rangle \, ds \right] = 0.$
Proof. (1) From (5.1), we know that \( \bar{U} \) is continuous at \( \tau \) and \( \bar{U}(\tau) \) is right-continuous at \( t = 0 \). Hence, this ends the proof of the assertion (1) by using the Vitali theorem.

(2) From (5.1), we know that \( \bar{U} \) is continuous at \( \tau \) and \( \bar{U}(\tau) \) is right-continuous at \( t = 0 \). Hence, this ends the proof of the assertion (2) by using the Vitali theorem.

(3) It follows from (5.1) that \( \bar{U} \) is continuous at \( \tau \) and \( \bar{U}(\tau) \) is right-continuous at \( t = 0 \). Hence, this ends the proof of the assertion (2) by using the Vitali theorem.

(4) Here we deal with the first nonlinear term. It follows from (5.2) that the sequence \( (\bar{U}_\tau) \) is bounded in \( L^2(0,T;\mathbb{H}) \). Meanwhile, we obtain \( \bar{U}_\tau \to \bar{U}_o \) in \( L^2(0,T;L^2_{\text{loc}}(D)) \), \( \bar{P} \)-a.s. from (5.1). This together with the boundedness and convergence implies that for all \( t \in [0,T] \) and \( \psi \in \mathbb{V}_m \)

\[
\lim_{\tau \to 0} \int_0^t (2h_{de}^T \mathcal{B}(\bar{U}_\tau(s)) - \mathcal{B}(\bar{U}_o(s)), \psi) ds = 0.
\]
Indeed, for all \( \psi \in X \subset V_m, \psi \in X \) and \( t \in [0, T] \), we have

\[
\lim_{\tau \to 0} \int_0^t \langle 2\tilde{h}_{de} B(\tilde{U}_t(s)) - B(U_\circ(s)), \psi \rangle ds = 0, \quad \bar{P} \text{ - a.s.} \tag{5.10}
\]

By applying the property of \( B \), Hölder’s inequality and (5.2), we have

\[
\bar{E} \left[ \left\| \int_0^t \langle 2\tilde{h}_{de} B(\tilde{U}_t(s)), \psi \rangle ds \right\|^p \right] \leq \bar{E} \left[ \left( \int_0^t |B(\tilde{U}_t(s))|_{V_m'} \| \psi \|_{V_m} ds \right)^p \right] \tag{5.11}
\]

\[
\lesssim \| \psi \|_{V_m}^p \bar{E} \left[ \int_0^t |\tilde{U}_t(s)|_{H_2}^{2p} ds \right] \lesssim \bar{E} \left[ \sup_{s \in [0, T]} |\tilde{U}_t(s)|_{H_2}^{2p} \right] \leq C(p),
\]

for all \( t \in [0, T], p \in (1, \infty) \) and \( \tau \in (0, 1) \). Combining (5.10) and (5.11) and using the Vitali theorem, we can get

\[
\lim_{\tau \to 0} \bar{E} \left[ \left| \int_0^t \langle 2\tilde{h}_{de} B(\tilde{U}_t(s)) - B(U_\circ(s)), \psi \rangle ds \right| \right] = 0, \tag{5.12}
\]

for all \( t \in [0, T] \).

By (5.2), we can infer that for all \( \tau \in (0, 1) \) and \( t \in [0, T] \),

\[
\bar{E} \left[ \left| \int_0^t \langle 2\tilde{h}_{de} B(\tilde{U}_t(s)), \psi \rangle ds \right| \right] \leq \bar{E} \left[ \sup_{s \in [0, T]} |\tilde{U}_t(s)|_{H_2}^2 \right] \leq C.
\]

According to (5.12) and the dominated convergence theorem, we complete the proof of assertion (4).

(5) Here we deal with the second nonlinear term (the Hall-term). By using (3.7) and (5.2) we know that the sequence \( \tilde{U}_t \) is bounded in \( L^2(0, T; H) \). Meanwhile, it follows from (5.1) that \( \tilde{U}_t \to U_\circ \) in \( L^2(0, T; L^{10}_m(D)) \cap L^2_\circ (0, T; V), \bar{P}\text{-a.s.} \). We apply the boundedness and convergence to infer that for all \( t \in [0, T] \) and \( \psi \in V_m \subset V \)

\[
\lim_{\tau \to 0} \int_0^t (2\tilde{h}_{de} \nabla \times ((\nabla \times B_\tau) \times B_\tau), \psi) ds = \lim_{\tau \to 0} \int_0^t 2\tilde{h}_{de} \langle (\nabla \times B_\tau) \times B_\tau, \nabla \times \psi \rangle ds
\]

\[
= \int_0^t \langle (\nabla \times B_\circ) \times B_\circ, \nabla \times \psi \rangle ds
\]

\[
= \int_0^t \langle (\nabla \times B_\circ) \times B_\circ, \psi \rangle ds.
\tag{5.13}
\]

By using (3.7), Hölder’s inequality and (5.2), we have

\[
\bar{E} \left[ \left| \int_0^t 2\tilde{h}_{de} \langle \nabla \times ((\nabla \times B_\tau) \times B_\tau), \psi \rangle ds \right|^p \right] \leq \bar{E} \left[ \left| \int_0^t \langle 2\tilde{h}_{de} (\nabla \times B_\tau) \times B_\tau, \nabla \times \psi \rangle ds \right|^p \right] \leq \bar{E} \left[ \left( \int_0^t 2\tilde{h}_{de} |B_\tau(U_\circ(s))|_{V_m'} \| \psi \|_{V_m} ds \right)^p \right] \tag{5.14}
\]
\begin{align}
\lesssim & \|\nabla \times \psi\|_{L^p}^p \mathbb{E} \left[ \int_0^t |\bar{U}_\tau(s)|_{L^p}^p \, ds \right] \\
\lesssim & \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{U}_\tau(s)|_{L^p}^2 \right] \leq C(p),
\end{align}

for all \( t \in [0,T], \ p \in (1, \infty) \) and \( \tau \in (0,1) \). Combining (5.13) and (5.14) and applying the Vitali theorem, we obtain

\[
\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^t \langle 2h^\tau_{ds} f_\tau(s) - f_\circ(s), \nabla \times \psi \rangle \, ds \right] = 0,
\]

(5.16) for all \( t \in [0,T] \). By (5.2), we can infer that for all \( \tau \in (0,1) \) and \( t \in [0,T] \)

\[
\mathbb{E} \left[ \int_0^t \langle 2h^\tau_{ds} (\nabla \times B_{\tau}) \times B_{\tau}, \nabla \times \psi \rangle \, ds \right] \lesssim \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{U}_\tau(s)|_{L^p}^2 \right] \leq C.
\]

By using (5.16), the definition of \( f_\tau \) and the dominated convergence theorem, we complete the proof of assertion (5).

(6) Here we notice that the independent processes \( X_1, X_2, \ldots \), defined by

\[
X_i(t) = \sqrt{2} \int_0^t h_{st}(s) \, \zeta_i(s),
\]

and

\[
W_\circ = \sum_{i \geq 1} X_i e_i.
\]

The random variable \( X_i(t) \) is Gaussian, with mean 0 and

\[
|\text{Var}(X_i(t)) - t| = |2t - t_{2n+1} - t| = |t - t_{2n+2}| \leq \tau,
\]

for \( t \in [t_{2n+1}, t_{2n+2}] \)

where \( \text{Var}(t) \) is variance. It is easy to check that \( X_i \to \beta_i \) in law for all \( i = 1, 2, 3, \ldots \) (see Section 3.2.7 in [5]). By using Assumption (A) and the definition of \( \bar{\sigma}_\psi \), we have

\[
\int_0^t \| \langle \sigma(s, \bar{U}_\tau(s)) - \sigma(s, U_\circ(s)), \psi \rangle \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^2 \, ds \\
= \int_0^t \| \bar{\sigma}_\psi(\bar{U}_\tau)(s) - \bar{\sigma}_\psi(U_\circ)(s) \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^2 \, ds \\
\leq & \| \bar{\sigma}_\psi(\bar{U}_\tau) - \bar{\sigma}_\psi(U_\circ) \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^2 \cdot \mathbb{E} \left[ \int_0^T (1 + |\bar{U}_\tau(s)|_{L^p}^{2p} + |U_\circ(s)|_{L^p}^{2p}) \, ds \right].
\]

Thanks to (5.1), \( \bar{U}_\tau \to U_\circ \) in \( L^2(0,T; L^2_{loc}(D)) \), \( \mathbb{P} \)-a.s. and Assumption (A), we get

\[
\lim_{\tau \to 0} \int_0^t \| \langle \sigma(s, \bar{U}_\tau(s)) - \sigma(s, U_\circ(s)), \psi \rangle \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^2 \, ds = 0.
\]

(5.17)

According to Assumption (A) and (5.2), we have

\[
\mathbb{E} \left[ \int_0^t \| \langle \sigma(s, \bar{U}_\tau(s)) - \sigma(s, U_\circ(s)), \psi \rangle \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^2 \, ds \right]^{p'} \\
\lesssim \mathbb{E} \left[ \| \psi \|_{L_{H^s}^{2}}^{2p} \cdot \int_0^t \left( \| \sigma(s, \bar{U}_\tau(s)) \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^{2p} + \| \sigma(s, U_\circ(s)) \|_{L_{H^s}^{2}(Z_{W;\mathbb{R}})}^{2p} \right) \, ds \right] \\
\lesssim \mathbb{E} \left[ \int_0^T (1 + |\bar{U}_\tau(s)|_{L^p}^{2p} + |U_\circ(s)|_{L^p}^{2p}) \, ds \right].
\]
By using (5.1), \( \bar{C} > 0 \) follows from (2.4) and (5.2) that for some constant \( C(p) > 0 \). By using (5.17), (5.18) and the Vitali theorem, we can get

\[
\lim_{t \to 0} \mathbb{E} \left[ \int_0^t \| \langle \sigma(s, \bar{U}_r(s)) - \sigma(s, U_o(s)), \psi \rangle \|_{L^{H_S}(Z_W, \mathbb{R})}^2 \right] = 0, \tag{5.19}
\]

for all \( \psi \in \mathbb{V} \).

By the property of Itô’s integral, one has

\[
\lim_{t \to 0} \mathbb{E} \left[ \left| \int_0^t \left[ \sqrt{2} h_s \sigma(s, \bar{U}_r(s)) - \sigma(s, U_o(s)) \right] dW_o, \psi \right|^2 \right] = 0. \tag{5.20}
\]

Furthermore, by applying the Itô isometry, Assumption (A) and (5.2), we can infer that

\[
\mathbb{E} \left[ \left| \int_0^t \left[ \sqrt{2} h_s \sigma(s, \bar{U}_r(s)) - \sigma(s, U_o(s)) \right] dW_o, \psi \right|^2 \right] \leq C \mathbb{E} \left[ \left( 1 + \sup_{s \in [0,T]} |\bar{U}_r|^2 + \sup_{s \in [0,T]} |U_o|^2 \right) \right] \leq C(1 + 2C(2)),
\]

for all \( t \in [0, T] \) and all \( \tau \in (0, 1) \) and some \( C > 0 \). By using (5.20), (5.21), \( \sqrt{2} \int_0^t h_s d\beta_i(t) \to \beta_i(t) \) and the dominated convergence theorem, we obtain

\[
\lim_{t \to 0} \int_0^T \mathbb{E} \left[ \left| \int_0^t \left[ \sqrt{2} h_s \sigma(s, \bar{U}_r(s)) - \sigma(s, U_o(s)) \right] dW_o, \psi \right|^2 \right] dt = 0. \tag{5.22}
\]

(7) For all \( \psi \in \mathbb{H} \) and \( t \in [0, T] \), we conclude that

\[
\int_0^t \int_Z |g(s, U_r(s); z) - g(s, U_o(s); z)|^2 d\mu(z) ds \leq 2 \left\| \bar{U}_r - U_o \right\|_{L^2([0,T] \times Z; \mathbb{R})}^2.
\]

By using (5.1), \( \bar{U}_r \to U_o \) in \( L^2(0, T; L^2_{loc}(D)) \), \( \mathbb{P} \)-a.s., and Assumption (B), we obtain

\[
\lim_{t \to 0} \int_0^t \int_Z |g(s, U_r(s); z) - g(s, U_o(s); z)| d\mu(z) ds = 0. \tag{5.23}
\]

It follows from (2.4) and (5.2) that for some constant \( C > 0 \)

\[
\mathbb{E} \left[ \left( \int_0^t \int_Z |g(s, U_r(s); z) - g(s, U_o(s); z)|^2 d\mu(z) ds \right)^p \right] \leq 2^p \left\| \psi \right\|_{L^p}^p \mathbb{E} \left[ \left( \int_0^t \int_Z |g(s, U_r(s); z)|^2 + |g(s, U_o(s); z)|^2 d\mu(z) ds \right)^p \right] \leq |\psi|_{L^p}^p \mathbb{E} \left[ \left( \int_0^t (2 + |\bar{U}_r(s)|^2 + |U_o(s)|^2) ds \right)^p \right].
\]
\[
\leq \left( 1 + \mathbb{E} \left[ \sup_{s \in [0,T]} |\bar{U}_t(s)|_{\mathbb{H}}^2 \right] \right) \leq C(1 + C(p)), \tag{5.24}
\]
for every \( p \in [1, \infty) \), \( t \in [0,T] \) and \( \tau \in (0,1) \). By (5.23), (5.24) and the Vitali theorem, we have
\[
\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T \int_Z |g(s, \bar{U}_t(s); z) - g(s, U_\omega(s); z), \psi|_{\mathbb{H}}^2 \, d\mu(z) \, ds \right] = 0,
\]
for all \( \psi \in \mathbb{H} \). By the properties of the integral with respect to the compensated Poisson random measure, the weak convergence of \( h_{st} \) (\( h_{st} \to \frac{1}{2} \) in the sense of distribution) and the fact that \( \bar{\pi}_\tau = \pi_\omega \), we infer that
\[
\lim_{\tau \to 0} \mathbb{E} \left[ \int_0^T \int_Z (2h_{st} g(s, \bar{U}_t(s); z) - g(s, U_\omega(s); z), \psi) \bar{\pi}_\omega(ds, dz) \right] = 0. \tag{5.25}
\]
Furthermore, similar to (5.24) with \( p = 1 \), we have
\[
\mathbb{E} \left[ \int_0^T \int_Z (2h_{st} g(s, \bar{U}_t(s); z) - g(s, U_\omega(s); z), \psi) \bar{\pi}_\omega(ds, dz) \right]^2 = \mathbb{E} \left[ \int_0^T \int_Z (2|2h_{st} g(s, \bar{U}_t(s); z) - g(s, U_\omega(s); z), \psi|_{\mathbb{H}}^2 \, d\mu(z) \, ds \right] \leq C(1 + C(2)). \tag{5.26}
\]
By (5.25), (5.26) and the dominated convergence theorem, we obtain assertion (7). Then we complete the proof of Lemma 5.1. \( \square \)

The following Lemma can be implied by Lemma 5.1.

**Lemma 5.2.** For every \( \psi \in \mathbb{H} \), we have
\[
\lim_{\tau \to 0} \| (\bar{U}_t(\cdot), \psi)_{\mathbb{H}} - (U_\omega(\cdot), \psi)_{\mathbb{H}} \|_{L^2([0,T] \times \bar{\Omega})} = 0. \tag{5.27}
\]
and
\[
\lim_{\tau \to 0} \| \Gamma(\bar{U}_t, \bar{\pi}_\tau, \bar{W}_\tau, \psi) - \Gamma(U_\omega, \pi_\omega, W_\omega, \psi) \|_{L^1([0,T] \times \bar{\Omega})} = 0. \tag{5.28}
\]

**Proof.** From Lemma 5.1 (1), we have
\[
\lim_{\tau \to 0} \| (\bar{U}_t(\cdot), \psi)_{\mathbb{H}} - (U_\omega(\cdot), \psi)_{\mathbb{H}} \|_{L^2([0,T] \times \bar{\Omega})} = \mathbb{E} \left[ \int_0^T |(\bar{U}_t(t) - U_\omega(t), \psi)_{\mathbb{H}}|^2 \, dt \right] = 0.
\]
Now, we prove (5.28). The Fubini theorem yields that
\[
\| \Gamma(\bar{U}_t, \bar{\pi}_\tau, \bar{W}_\tau, \psi) - \Gamma(U_\omega, \pi_\omega, W_\omega, \psi) \|_{L^1([0,T] \times \bar{\Omega})} = \int_0^T \mathbb{E}[|\Gamma(\bar{U}_t, \bar{\pi}_\tau, \bar{W}_\tau, \psi)(t) - \Gamma(U_\omega, \pi_\omega, W_\omega, \psi)(t)|] \, dt.
\]
By using assertions (2)-(7) of Lemma 5.1, we can obtain that each term on the right hand side of (5.4) tends to the corresponding term in (5.5) at least in \( L^1([0,T] \times \bar{\Omega}) \). \( \square \)

**Proof of Theorem 2.2.** Since \( \bar{U}_t \) is a solution of (4.1)-(4.4), for all \( t \in [0,T] \), we obtain
\[
(U_\omega(t), \psi)_{\mathbb{H}} = \Gamma(U_\omega, \pi_\omega, W_\omega, \psi)(t), \quad \mathbb{P} - \text{a.s.}
\]
In particular, 
\[ \mathbb{E}[[U_{\tau}(t), \psi]]_{\mathbb{H}} - \Gamma_\tau(U_{\tau}, \pi_{\tau}, W_{\tau}, \psi)(t)] = 0. \]
Since \( \mathcal{L}(U_{\tau}, \pi_{\tau}, W_{\tau}) = \mathcal{L}(\bar{U}_{\tau}, \bar{\pi}_{\tau}, \bar{W}_{\tau}) \), we have 
\[ \mathbb{E}[[\bar{U}_{\tau}(t), \psi]]_{\mathbb{H}} - \Gamma_\tau(\bar{U}_{\tau}, \bar{\pi}_{\tau}, \bar{W}_{\tau}, \psi)(t)] = 0. \]
Moreover, it follows from (5.27) and (5.28) that 
\[ \mathbb{E}[[U_{\tau}(t), \psi]]_{\mathbb{H}} - \Gamma(U_{\tau}, \pi_{\tau}, W_{\tau}, \psi)(t)] = 0, \]
for details, see [6]. Hence, for \( l \)-almost all \( t \in [0, T] \) and \( \mathbb{P} \)-almost all \( \omega \in \Omega \), we have 
\[ (U_{\tau}(t), \psi)]_{\mathbb{H}} - \Gamma(U_{\tau}, \pi_{\tau}, W_{\tau}, \psi)(t) = 0. \]
Then we can infer that 
\[ (U_{\tau}(t), \psi)]_{\mathbb{H}} + \int_0^t \mathcal{A}U_{\tau}(s), \psi]ds + \int_0^t \mathcal{B}(U_{\tau}(s), \psi]ds \\
= (U_{\tau}(0), \psi)]_{\mathbb{H}} + \int_0^t f(s, \psi]ds + \int_0^t \sigma(s, U_{\tau}(s))dW_{\tau}(s), \psi \rangle \\
\quad + \int_0^t \int_Z (g(s, U_{\tau}(s); z), \psi)]_{\mathbb{H}}\pi_{\tau}(ds, dz). \]
Since \( U_{\tau} \) is \( G \)-valued random variable, in particular \( U_{\tau} \in \mathbb{D}([0, T]; \mathbb{H}_{\mathbb{C}}) \), i.e. \( U_{\tau} \) is weakly càdlàg, we infer that equality (5.29) holds for all \( t \in [0, T] \) and all \( \psi \in \mathbb{X} \). Since \( \mathbb{X} \) is dense in \( \mathbb{V} \), equality (5.29) holds for all \( \psi \in \mathbb{V} \), as well. Putting \( \mathbb{U} := (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F}) \), \( \mathbb{P} := \pi_{\tau}, W = W_{\tau} \) and \( \bar{U} = U_{\tau} \), we conclude that the system 
\[(\mathbb{U}, \mathbb{P}, \bar{W}, \bar{U}) \]
is a martingale solution of equation (1.5). Hence, the proof of Theorem 2.2 is completed. \( \square \)

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