Forgetting complex propositions

David Fernández–Duque\textsuperscript{1}, Ángel Nepomuceno–Fernández\textsuperscript{2},
Enrique Sarrión–Morrillo\textsuperscript{2}, Fernando Soler–Toscano\textsuperscript{2} and
Fernando R. Velázquez–Quesada\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Instituto Tecnológico Autónomo de México,
david.fernandez@itam.mx

\textsuperscript{2}Grupo de Lógica, Lenguaje e Información, Universidad de Sevilla,
{nepomuc,esarrion,fsoler,FRVelazquezQuesada}@us.es

*Corresponding author.

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Abstract

This paper uses possible-world semantics to model the changes that may occur in an agent’s knowledge as she loses information. This builds on previous work in which the agent may forget the truth-value of an atomic proposition, to a more general case where she may forget the truth-value of a propositional formula. The generalization poses some challenges, since in order to forget whether a complex proposition $\pi$ is the case, the agent must also lose information about the propositional atoms that appear in it, and there is no unambiguous way to go about this.

We resolve this situation by considering expressions of the form $[\exists\pi]\varphi$, which quantify over all possible (but ‘minimal’) ways of forgetting whether $\pi$. Propositional atoms are modified non-deterministically, although uniformly, in all possible worlds. We then represent this within action model logic in order to give a sound and complete axiomatization for a logic with knowledge and forgetting. Finally, some variants are discussed, such as when an agent forgets $\pi$ (rather than forgets whether $\pi$) and when the modification of atomic facts is done non-uniformly throughout the model.

Keywords: forgetting, dynamic epistemic logic, action models, theory contraction, knowledge representation.

1 Introduction

Epistemic notions such as knowledge and belief are subject to the effect of different epistemic actions, many of which have been studied in the literature. Just as beliefs can be affected by expansion \cite{23}, contraction \cite{11,10}, revision \cite{23,5,15,26,21,48}, merging \cite{15} and diverse forms of inference \cite{33,34} among others, knowledge can be affected by deductive inference \cite{31,32}, public \cite{21,11} and other forms of announcements \cite{2}. 
One action that has not received much attention is that of forgetting and its effect on an agent’s knowledge. One of the reasons for this is its similarities with belief contraction, an action that, when represented semantically, typically relies on Lewis’ system of spheres for conditionals [17]. This system of spheres uses an ordering among theories (the theories’ ‘plausibility ordering’) and thus provides a guideline for defining the new beliefs an agent will have when one of the current ones is discarded [12]. This is adequate for belief contraction, as a plausibility ordering is natural when defining beliefs: the collection of epistemically possible situations can be understood as having an order which not only defines this epistemic notion (as what is true in the most plausible situations) but also establishes a ranking among what is not believed but still has not been discarded. However, such an ordering is not natural when dealing with knowledge: there does not seem to be an ordering among the epistemically possible situations that are known to not be the case and hence have been discarded.

On the other hand, in the knowledge representation area there are approaches for forgetting a finite set of atoms. In such proposals knowledge is represented as a finite set of formulas (the knowledge set), and the typical definition of forgetting uses some notion of similarity between models: a knowledge set is the result of forgetting the atoms in \( A_t' \) if and only if every model of the resulting knowledge set is equivalent to a model of the original knowledge set when the atoms in \( A_t' \) are disregarded [18]. In modal contexts as in this paper, the used equivalence notion is that of bisimulation, which gives rise to systems [36, 37] similar to those that contain modalities for bisimulation quantification [9].

This work presents a logical treatment under possible worlds semantics of an action that represents the forgetting of propositional formulas, without relying on an ordering among theories or epistemic possibilities and without using any notion of similarity notion between models. It can be seen as an extension of [29], which deals only with forgetting the truth-value of atomic propositions. Several ways of forgetting a given formula are possible. We focus on two of them and give some hints about variants with different properties. The key intuition guiding our definitions is that an agent forgetting \( \pi \) will lose her (possible) previous knowledge of \( \pi \). But if she previously knew \( \neg \pi \), there are two possibilities after forgetting \( \pi \): her knowledge of \( \neg \pi \) may fail or not. We call the first option forgetting whether \( \pi \) and the second forgetting \( \pi \). While we will focus more on the first, we will also discuss the second possibility.

**Layout of the paper** Section 2 recalls some basic notions from propositional and epistemic logic which will be used throughout the text. Section 3 presents the notion of uniform forgetting whether which, being the main focus of the article, is discussed in some detail in Section 4. Section 5 introduces a simpler action of ‘forgetting’ (where a propositional formula is considered to be possibly false, but not necessarily possibly true) and compares it to the action of forgetting whether. Section 6 presents our main result, which is a sound and complete axiomatization for our logic of knowledge and forgetting. Finally, Section 7 discusses some alternate ways of modelling the action of forgetting.
2 Basic definitions

Our formalism for reasoning about forgetting will be based on epistemic, or more generally modal, logic. Throughout this text, $At$ denotes a designated countable non-empty set of atoms or propositional variables. Let us begin by reviewing the basic language of propositional modal logic.

**Definition 1.** The grammar of $L$ is given by

$$\varphi ::= \top \mid p \mid \neg \varphi \mid (\varphi \land \psi) \mid \Box \varphi$$

where $p \in At$. Formulas of the form $\Box \varphi$ are read as “the agent knows that $\varphi$ is the case”. The symbols $\bot$, $\lor$, $\rightarrow$, $\leftrightarrow$ and $\Box$ are defined as usual.

Modal logics are typically interpreted via their Kripke or possible worlds semantics, as described below:

**Definition 2.** A Kripke frame is a tuple $F = \langle W, R \rangle$ where $W$ is a non-empty set and $R \subseteq W \times W$ a binary relation; no assumptions are made a priori about $R$.

A model $M = \langle F, V \rangle$ is a frame $F$ equipped with a valuation $V : At \rightarrow P(W)$. A pointed model is a pair $(M, w)$ with $M$ a model and $w$ an element of its domain.

**Definition 3.** Let $M = \langle W, R, V \rangle$ be a model. The satisfaction relation $\models$ between pointed models and formulas is defined as follows:

- $M, w \models p$ iff $w \in V(p)$;
- $M, w \models \neg \varphi$ iff $M, w \not\models \varphi$;
- $M, w \models \varphi \land \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
- $M, w \models \Box \varphi$ iff for all $v \in W$, $wRv$ implies $M, v \models \varphi$.

Given a model $M$, define the function $[\varphi]^M : L \rightarrow P(W)$ as $w \in [\varphi]^M$ if and only if $M, w \models \varphi$. The notation $[\varphi]^M$ will be abbreviated as $\llbracket \varphi \rrbracket$ when this does not lead to confusion.

As usual, $M \models \varphi$ states that $[\varphi]^M = W$, and if $X$ is a class of models, $X \models \varphi$ states that $M \models \varphi$ for all $M \in X$. The formula $\varphi$ is valid when $M \models \varphi$ for every model $M$, a case denoted by $\models \varphi$.

When modelling knowledge, the class of models in which the relation is an equivalence relation, $S5$, is of particular interest. However, this paper will keep a more general discussion, only restricting its attention to models with particular properties when explicitly stated.

In order to formalize our notion of forgetting, it will be convenient to represent propositional formulas in conjunctive normal form using sets of clauses. Recall that a literal $\ell$ is an atom or its negation, and a clause $D$ is a finite (possibly empty) set of literals interpreted disjunctively, so that $D$ represents the formula $\lor D$. A clause $D$ is said to be a consequence of a propositional formula $\pi$ when $\models \pi \rightarrow \lor D$.

A propositional formula is in conjunctive normal form when it is given as a finite (possibly empty) set of clauses, interpreted conjunctively. More precisely, the set of clauses $\mathcal{C}$ is interpreted as the formula $\hat{\mathcal{C}}$ defined as

$$\hat{\mathcal{C}} := \bigwedge_{D \in \mathcal{C}} \lor D.$$  

$^1$As usual, $\lor \emptyset := \bot$ and $\land \emptyset := \top$. 



Clearly, a given propositional formula may have many equivalent conjunctive normal forms, but we wish to pick one canonically. In order to do so, first discard all tautological clauses, i.e. those clauses \( D \) in which there is an atom \( p \) such that \( \{p, \neg p\} \subseteq D \). A clause \( D \neq \emptyset \) which is non-tautological is called contingent. Then, within each clause, ‘unnecessary’ literals are removed: a clause \( D \) is said to be a minimal consequence of a propositional formula \( \pi \) if and only if \( |D| = |\pi \rightarrow \bigvee D| \) and there is no \( D' \subsetneq D \) such that \( |D'| = |\pi \rightarrow \bigvee D'| \). With this in mind, here is a formal definition:

**Definition 4.** Let \( \pi \) be a formula of propositional logic. Define the clausal form \( C(\pi) \) to be the set of all clauses that are minimal non-tautologica l consequences of \( \pi \). Figure 1 shows some examples.

The normal form \( C(\pi) \) is actually the set of prime implicates of \( \pi \) (cf. [21, 22]), and there are several algorithms for calculating it (e.g., [21, 6, 14, 25, 22]; see [4] for more). This concept has been already used for epistemic concerns, mainly on proposals following the AGM approach for belief revision [1] in which the agent’s beliefs are represented syntactically (e.g., [19, 38]). Here it will be used to simplify the model operation defined of the next section for representing the action of forgetting \( \pi \).

The next lemma is then straightforward.

**Lemma 1.** For any propositional formula \( \pi \), the set \( C(\pi) \) is finite, its elements are finite, and it satisfies \( |C(\pi)| = |\pi \leftrightarrow \bigwedge C(\pi)| \). Moreover, \( \pi_1 \equiv \pi_2 \) implies \( C(\pi_1) = C(\pi_2) \), and \( C(\top) = \emptyset \) while \( C(\bot) = \{\emptyset\} \).

### 3 Uniform forgetting

In order to reason about forgetting whether, the basic modal language will be extended with a new modality.

**Definition 5.** The language \( L_{\boxtimes} \) extends \( L_\Box \) with expressions of the form \( \langle \boxtimes \pi \rangle \phi \) with \( \pi \) a propositional formula, read as “after the agent forgets whether \( \pi \), \( \phi \) is the case”. The expression \( \langle \boxtimes \pi \rangle \phi \) is defined in the standard way as \( \neg \langle \boxtimes \neg \pi \rangle \neg \phi \).

It is worthwhile to emphasise that, as discussed before, this paper understands “forgetting whether \( \pi \)” simply as “forgetting \( \pi \)’s truth-value”; as such, the act of forgetting studied here does not involve other related actions (as, e.g., becoming unaware of atoms/formulas [27, 28]).

Observe how, since \( \Box \phi \) is the case when \( \phi \) holds in all of the agent’s epistemic alternatives, in order for her to forget (i.e., to not know anymore) that a given
propositional formula \( \pi \) is the case, she needs to consider as possible at least one situation where \( \pi \) fails. The first step is, then, to decide how to make a given \( \pi \) fail in a given world \( w \). Suppose that \( \pi \) is not a tautology (such case will be discussed later). When \( \pi \) is rewritten in its clausal form \( \mathcal{C}(\pi) = \{D_1, \ldots, D_n\} \), it is clear that in order to make \( \pi \) false at \( w \), at least one clause in \( \mathcal{C}(\pi) \) should be false in such a world; this would, in principle, give us a total of \( 2^n - 1 \) different forms of falsifying \( \pi \). However, falsifying an arbitrary non-empty set of such clauses would be problematic, both because of the combinatorial explosion and because the negations of different clauses might be mutually inconsistent.

A better alternative is to follow a minimal change approach, where \( \pi \) will be falsified by making only one of its clauses \( D_i \) false.

Now, given the clause that will be falsified, we need to decide not only how many worlds need to be introduced as part of the agent’s epistemic possibilities, but also which truth-value will be assigned, in such new worlds, to the atoms that do not appear in the given clause. Again, the minimal change approach suggests that the least intrusive way to change the agent’s knowledge is to make a copy of the current epistemic possibilities and then falsify the given clause in each one of them. In the resulting model, the original formula \( \pi \) has been uniformly falsified because the same clause \( D_i \in \mathcal{C}(\pi) \) has been falsified across all the worlds in the new copy of the set of epistemic possibilities.

The formalisation of this idea will be used to provide the semantic interpretation of formulas expressing the effect of the slightly different ‘forgetting whether \( \pi \)’, \( [\pi] \varphi \). In order for an agent to forget the truth-value of a given \( \pi \), she needs to consider not only a possibility that falsifies \( \pi \) (by falsifying one of the clauses of \( \pi \)’s clausal form) but also a possibility that falsifies \( \neg \pi \) (by falsifying one of the clauses of \( \neg \pi \)’s clausal form). Thus, a model operation representing this action takes two clauses and creates two copies of the current set of epistemic possibilities, with each copy falsifying one clause. The operation defined below is a generalisation that receives an epistemic model and a finite set of clauses \( C \), returning a model with a copy of the current set of epistemic possibilities falsifying each clause in \( C \).

**Definition 6.** Let \( \mathcal{M} = (W, R, V) \) be a model and \( C = \{D_i : i \in I\} \) a finite (possibly empty) set of non-tautological clauses, where without loss of generality \( 0 \notin I \). The new model \( \mathcal{M}_u^C = (W_u^C, R_u^C, V_u^C) \) is defined as follows:

1. \( W_u^C := W \times (\{0\} \cup I) \);
2. for all \( w, v \in W \) and \( i, j \in \{0\} \cup I \), \((w, i)R_u^C(v, j)\) if and only if \( wRv \);
3. for all \( w \in W \), \((w, 0) \in V_u^C(p)\) if and only if \( w \in V(p) \); and
4. for all \( w \in W \) and \( i \in I \), \((w, i) \in V_u^C(p)\) if and only if one of the following holds:
   - (a) \( \neg p \in D_i \); or
   - (b) both \( \{p, \neg p\} \cap D_i = \emptyset \) and \( w \in V(p) \).

\(^2\)Of course, such operation is not minimal with respect to the number of worlds that will be added; it is minimal with respect to the changes in the agent’s knowledge.
Thus, $W_u^C$ has two types of worlds. Worlds of the form $(w,0)$ preserve the original valuation: an atom $p$ is true on $(w,0)$ if and only if $p$ was already true on $w$. On the other hand, each world of the form $(w,i)$ with $i \in I$ falsifies all of the literals in $D_i$, leaving the remaining atoms as before. The relation in the new model simply follows the original relation, making a world $(v,j)$ accessible from a world $(w,i)$ when $v$ is accessible from $w$ in the original model.

Note also that we are modelling forgetting within the context of epistemic logic, where knowledge is represented semantically. This leads to several key differences from syntactic approaches of knowledge representation. Most notably, an agent cannot distinguish between semantically equivalent formulas (so, if she knows $\pi$, she also knows all its semantic equivalents). By using both $\pi$’s and $\neg \pi$’s minimal clausal forms, the forgetting whether action treats semantically equivalent formulas in the same way (so, afterwards, the agent has forgotten not only $\pi$’s truth value, but also that of all $\pi$’s semantic equivalents). Approaches that distinguish semantically equivalent formulas are possible, but would require a different framework for modelling the agent’s knowledge.

With this model operation it is possible to define the semantic interpretation of formulas of the form $[\sharp \pi] \varphi$ which, it is recalled, are intuitively read as “after the agent forgets the truth-value of $\pi$, $\varphi$ is the case”.

**Definition 7.** Let $\mathcal{M} = (W,R,V)$ be a model and $w$ a world of $W$. We extend Definition 6 to $L_{\mathcal{C}}^{\sharp}$ by defining $\mathcal{M} = [\sharp \pi] \varphi$ if and only if, for all $D_1 \in \mathcal{C}(\pi)$ and $D_2 \in \mathcal{C}(\neg \pi)$,

\[
\mathcal{M}_{(D_1,D_2)}^w, (w,0) \models \varphi.
\]

The set of formulas in $L_{\mathcal{C}}^{\sharp}$ valid under $\models$ will be denoted $\log_{\mathcal{C}}^\sharp$.

Thus, $[\sharp \pi] \varphi$ states that $\varphi$ is the case after the agent forgets the truth value of $\pi$, independently of the choice of the clauses $D_1 \in \mathcal{C}(\pi)$ and $D_2 \in \mathcal{C}(\neg \pi)$ that are falsified in the added worlds.

## 4 The effect of forgetting whether

The model $\mathcal{M}_u^C$ is the result of the agent considering new possibilities in which each clause in $\mathcal{C}$ fails. This is achieved by keeping a copy of the original model (the $(w,0)$-worlds, which preserve the original valuation) and adding, for each clause $D_i$, a copy of the original model (the $(w,i)$-worlds) in which $D_i$ is falsified by falsifying each of its literals in all of the worlds in the copy, keeping the remaining atoms as before. Thus, each clause $D_i$ is false at each world $(w,i)$, as the following lemma shows.

**Lemma 2.** Let $\mathcal{M} = (W,R,V)$ be a model and $\mathcal{C} = \{D_i : i \in I\}$ a non-empty finite set of non-tautological clauses (again $0 \notin I$). Then, for any $w \in W$ and any $i \in I$,

\[
\mathcal{M}_u^C, (w,i) \not\models \bigvee D_i.
\]

**Proof.** If $D_i = \emptyset$ the result is trivial, as $\bigvee D_i = \bot$. Otherwise take $D_i = \{l_1, \ldots, l_m\}$ (i.e., $D_i$ is contingent). Then, for any $l_k \in D_i$,

- if $l_k$ is an atom $p$, then since $D_i$ is contingent, $\neg p \notin D_i$; thus, by definition, $(w,i) \not\in V_u^C(p)$ and hence $(w,i) \not\in \llbracket l_k \rrbracket^\mathcal{M}_u^C$.


if \( l_k \) is an atom’s negation \( \neg p \), then \( \neg p \in D_i \); so, by definition, \( (w, i) \in V_u^c(p) \) and thus \( (w, i) \not\in [l_k]^{\mathcal{M}_w^c} \).

Hence, every literal in \( D_i \) fails at \( (w, i) \) and therefore so the disjunction \( \bigvee D_i \).

**Example 1.** Consider the following pointed model \( (\mathcal{M}, w_0) \) (with each world \( w \) containing \( V(w) \) and the evaluation double-circled) in which the agent knows \( p \) (i.e., \( \mathcal{M}, w_0 \models \square p \)):

\[
\begin{array}{c}
p \quad (w_0, 0) \\
\downarrow \\
p \quad (w_1, 0) \\
\downarrow \\
p \quad (w_0, 1) \\
\downarrow \\
p \quad (w_1, 1) \\
\end{array}
\]

Consider the action of forgetting whether \( p \). Given that \( C(p) = \{\{p\}\} \), there is only one clause to be chosen: \{\{p\}\}. Similarly, \( C(\neg p) = \{\{\neg p\}\} \), so the only clauses the agent will consider when forgetting whether \( p \) are \( D_1 = \{\{p\}\} \) and \( D_2 = \{\{\neg p\}\} \). The pointed model \( (\mathcal{M}_w^c, \{\{p\}\}, \{\{\neg p\}\}, (w_0, 0)) \) appears below, with the top row being the copy that results from making \{\{p\}\} false and the bottom row being the copy that results from making \{\{\neg p\}\} false (thus making \( p \) true in those worlds):

\[
\begin{array}{c}
p \quad (w_0, 0) \\
\downarrow \\
p \quad (w_0, 1) \\
\downarrow \\
p \quad (w_1, 0) \\
\downarrow \\
p \quad (w_1, 2) \\
\end{array}
\]

As a result of the action, the agent considers possible worlds where \( p \) holds as well as worlds where \( p \) fails. Thus, \( \mathcal{M}_u^c, \{\{p\}\}, \{\{\neg p\}\}, (w_0, 0) \models \neg \square p \land \neg \square \neg p \), and hence \( \mathcal{M}, w_0 \models \square p \land \square \neg p \) and \( \mathcal{M}, w_0 \models \square \square \perp \).

In the previous example, note how, if \( w_1 \) were the evaluation point at the initial model (and hence \( (w_1, 0) \) the evaluation point at the model after the operation), then the agent would know \( p \) before the action (by vacuity), but she would still know \( p \) afterwards (by vacuity too). The following proposition shows that this counterintuitive outcome of the forgetting whether action can only occur when the knowledge of the agent is inconsistent to begin with.

**Proposition 1.** Let \( \pi \) be a propositional formula that is neither a tautology nor a contradiction (so \( C(\pi) \) and \( C(\neg \pi) \) are both non-empty sets of contingent clauses). Then,

\[
\models [\sharp \pi](\square \neg \pi \lor \square \pi) \leftrightarrow \square \perp.
\]

**Proof.** Let \( (\mathcal{M}, w) \) be a pointed model with \( \mathcal{M} = (W, R, V) \).

From right to left, suppose \( \mathcal{M}, w \models \square \perp \). Then there is no \( v \) such that \( wRv \) and hence by \( R_u^c \)'s definition, and regardless of \( C \), there is no \( (v, i) \) such
that \((w,0)R_u^C(v,i)\). Hence \(\mathcal{M}_u^C,(w,0) \models \square \neg \pi \lor \Box \pi\) and therefore \(\mathcal{M},w \models [\sharp \pi](\square \neg \pi \lor \Box \pi)\).

The left-to-right direction is proved by contrapositive. Suppose that \(\mathcal{M},w \models \neg \Box \bot\); then there is \(v\) such that \(wRv\). By Definition \(\mathfrak{R}\) for any \(D_1 \in \mathcal{C}(\pi)\) and \(D_2 \in \mathcal{C}(\neg \pi), (w,0)R_u^{(D_1,D_2)}(v,1)\) and \((w,0)R_u^{(D_1,D_2)}(v,2)\). By the contingency of \(D_1\) and \(D_2\) and Lemma \(\mathfrak{L}\), \(\mathcal{M}^{(D_1,D_2)}_u,(v,i) \not\models \forall D_i\) for \(i \in \{1,2\}\) and hence both \(\mathcal{M}^{(D_1,D_2)}_u,(v,1) \not\models \mathcal{C}(\pi)\) and \(\mathcal{M}^{(D_1,D_2)}_u,(v,2) \not\models \mathcal{C}(\neg \pi)\). Then \(\mathcal{M}^{(D_1,D_2)}_u,(w,0) \models \square \neg \pi \land \Box \pi\) and therefore, since neither \(\mathcal{C}(\pi)\) nor \(\mathcal{C}(\neg \pi)\) is empty, \(\mathcal{M},w \models [\sharp \pi](\square \pi \land \Box \neg \pi)\), i.e., \(\mathcal{M},w \not\models [\sharp \pi](\square \pi \lor \Box \neg \pi)\).

As a special case, if \(\pi\) is an atom \(p\), both \(\mathcal{C}(\pi) = \{\{p\}\}\) and \(\mathcal{C}(\neg \pi) = \{\{\neg \pi\}\}\) are non-empty and both contain only contingent clauses, so from the above proposition it follows that \([\sharp \pi](\square \pi \lor \Box \neg \pi) \leftrightarrow \square \bot\) is valid. More interestingly, recall that an agent’s knowledge is consistent at \(w\) if and only if \(w\) has at least one accessible world. In the class of models where this consistency property holds, called serial and denoted by \(\text{Ser}\), we obtain a stronger version of Proposition \(\mathfrak{P}\).

**Corollary 1.** For any non-tautological and non-contradictory propositional formula \(\pi\),

\[\text{Ser} \models [\sharp \pi] \top \land [\sharp \pi](\neg \square \pi \land \neg \Box \neg \pi)\].

Note that Proposition \(\mathfrak{P}\) is restricted to formulas \(\pi\) that are neither tautologies nor contradictions because otherwise the proof does not go through: in such cases either \(\mathcal{C}(\pi) = \emptyset\) or else \(\mathcal{C}(\neg \pi) = \emptyset\), and hence there are no clauses for falsifying one of \(\pi\) or \(\neg \pi\). As a consequence of this behaviour, both \([\sharp \top] \varphi\) and \([\sharp \bot] \varphi\) are valid for any formula \(\varphi\), and thus neither \([\sharp \top] \top\) nor \([\sharp \bot] \top\) is satisfiable.

**Example 2.** Consider the following pointed model in which the agent knows \(p \rightarrow q\).

Consider the act of forgetting whether \(p \rightarrow q\). Since \(\mathcal{C}(p \rightarrow q) = \{\{p\}, \{q\}\}\) and \(\mathcal{C}(\neg (p \rightarrow q)) = \{\{p\}, \{\neg q\}\}\), there are two possible outcomes:

**By using \{\neg p, q\} and \{p\}:**

\[
\begin{array}{c}
\text{p} \\
(w_0, 1) \\
\text{p} \\
(w_1, 1) \\
\text{q} \\
(w_0, 0) \\
\text{q} \\
(w_1, 0) \\
\text{q} \\
(w_2, 0) \\
\text{q} \\
(w_2, 2) \\
\end{array}
\]

**By using \{\neg p, q\} and \{\neg q\}:**

\[
\begin{array}{c}
\text{p} \\
(w_0, 1) \\
\text{p} \\
(w_1, 1) \\
\text{q} \\
(w_0, 0) \\
\text{q} \\
(w_1, 0) \\
\text{q} \\
(w_2, 0) \\
\text{q} \\
(w_2, 2) \\
\end{array}
\]\n
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Observe how $$\neg \Box(p \rightarrow q)$$ holds in the two pointed models, as in both the agent considers possible a world where $$p$$ holds but $$q$$ fails. Similarly, $$\neg \Box(\neg(p \rightarrow q))$$ holds in the two cases, as in both models the agent considers possible worlds where $$p \rightarrow q$$ holds.

Still, the two pointed models do not represent the same state of knowledge or, to be precise, they are not bisimilar. In the model on the left, the agent considers possible a world where $$\neg p \land q$$ holds and $$p \land q$$ is possible (the path $$(w_0, 0) \rightarrow (w_0, 2) \rightarrow (w_0, 0)$$), a possibility that does not exists in the model on the right (any transition from $$(w_0, 0)$$ to a $$\neg p \land q$$-world forces a move to the right-hand side of the diagram, from which there are no arrows back to the left-side). Thus,

$$\mathcal{M}_u\{\{\neg p, q\}, \{p\}\}, (w_0, 0) \models \Diamond(\neg p \land q \land \Diamond(p \land q))$$

but

$$\mathcal{M}_u\{\{\neg p, q\}, \{\neg q\}\}, (w_0, 0) \not\models \Diamond(\neg p \land q \land \Diamond(p \land q))$$

and hence

$$\mathcal{M}, w_0 \not\models [\exists(p \rightarrow q)]\Diamond(\neg p \land q \land \Diamond(p \land q))$$.

**Example 3.** Consider again the initial pointed model of Example 2. Observe how the agent knows neither $$p \land q$$ (she considers $$w_1$$ possible) nor $$\neg(p \land q)$$ (she considers $$w_0$$ possible). Since $$\mathcal{C}(p \land q) = \{\{p\}, \{q\}\}$$ and $$\mathcal{C}(\neg(p \land q)) = \{\{\neg p, \neg q\}\}$$, there are two possible outcomes for an action of forgetting the truth-value of the already ‘unknown’ $$p \land q$$:

*By using $$\{p\}$$ and $$\{\neg p, \neg q\}$$:*  

*By using $$\{q\}$$ and $$\{\neg p, \neg q\}$$:*

In both resulting pointed models the agent still knows neither $$p \land q$$ nor $$\neg(p \land q)$$. However, in both cases the action has an effect on the agent’s information: in the leftmost pointed model she considers possible a $$\neg p \land q$$-world, $$(w_0, 1)$$, from which there is an accessible $$p \land q$$-world, $$(w_0, 0)$$, something she did not consider possible before:

$$\mathcal{M}_u\{\{p\}, \{\neg p, \neg q\}\}, (w_0, 0) \models \Diamond(\neg p \land q \land \Diamond(p \land q)) \quad \text{but} \quad \mathcal{M}, w_0 \not\models \Diamond(\neg p \land q \land \Diamond(p \land q))$$.

Moreover, in the rightmost pointed model she considers possible a $$p \land \neg q$$-world, $$(w_0, 1)$$, something she did not consider possible before:

$$\mathcal{M}_u\{\{q\}, \{\neg p, \neg q\}\}, (w_0, 0) \models \Diamond(p \land \neg q) \quad \text{yet} \quad \mathcal{M}, w_0 \not\models \Diamond(p \land \neg q).$$

Thus, forgetting the truth-value of formulas whose truth-value is not known to begin with can affect the agent’s information by giving her ‘new reasons’ to not know the formula’s truth-value.
5 A simpler ‘forgetting’ action

In the current setting it is straightforward to define a simpler action that, instead of forgetting π’s truth value, simply forgets that π is the case. For this, it is enough to use the model operation of Definition 6 omitting the clause for C(¬π).

Here are the formal definitions:

Definition 8. The language $\mathcal{L}_{\Box\dag}$ extends $\mathcal{L}_{\Box}$ with operators of the form $[\dag\pi]$ for π a propositional formula, thus allowing the construction of formulas of the form $[\dag\pi]\varphi$, read as “after the agent forgets π, φ is the case”.

Definition 9. Let $\mathcal{M} = \langle W, R, V \rangle$ be a model, $w \in W$ and π be a propositional formula. We extend Definition 3 by setting $\mathcal{M}, w \models [\dag\pi]\varphi$ iff $\forall D \in C(\pi), \mathcal{M}^{D}_u, (w, 0) \models \varphi$.

This shows how the forgetting whether π action of before consists of simultaneously forgetting both π and ¬π. The question naturally arises of whether the action of forgetting π’s truth-value could instead be defined as forgetting π and then forgetting ¬π. Below it is shown that this is not the case.

Proposition 2. The expressions $[\dag\pi]\varphi$ and $[\dag\pi][\dag\neg\pi]\varphi$ are not equivalent, even over the class of S5 models.

Proof. Consider the following pointed model $(\mathcal{M}, w_0)$ with both p and q false at $w_0$:

\[
\begin{array}{c}
\circ \;
\end{array}
\]

Now, let π be ¬p ∨ ¬q. So, $C(\pi) = \{\{\neg p, \neg q\}\}$ and $C(\neg\pi) = \{\{p\}, \{q\}\}$. Then, by using first $\{\neg p, \neg q\}$ in $C(\pi)$ and then $\{q\}$ in $C(\neg\pi)$, we build $(\mathcal{M}_u^{\{\neg p, \neg q\}})_{\{q\}}$ in the following way:

\[
\begin{array}{c}
(w_0, 0) \quad (w_0, 0, 1) \\
(p, q) \quad (w_0, 1, 0) \quad (w_0, 1, 1)
\end{array}
\]

Observe that in the resulting model the agent can access the state $((w_0, 1), 1)$ where $p \land \neg q$ is true, so

$\mathcal{M}, w_0 \models [\dag\pi][\dag\neg\pi]\Diamond(p \land \neg q)$

But with forgetting whether π, starting at $\mathcal{M}$ it is not possible to produce a state where $p \land \neg q$ is true. With independence of the chosen clause in $C(\neg\pi)$, we arrive at the following model:
Then, $\mathcal{M}, w_0 \not\models (\sharp \pi) \diamond (p \land \neg q)$.

Another difference between forgetting and forgetting whether is that, while it is not possible to forget the truth-value of a contradiction, it is possible to forget a contradiction, as $C(\bot) \neq \emptyset$. In fact, $\models [\top \bot] \varphi \leftrightarrow \varphi$ and $\models [\bot \bot] \varphi \leftrightarrow (\bot \bot) \varphi$. Note, however, that if the agent knows a contradiction, the action of forgetting (the contradiction itself or any other formula) will not ‘fix’ this. In fact, the action cannot turn an agent’s knowledge contradictory or consistent if it was not that way before.

**Proposition 3.** Let $\pi$ be any propositional formula that is neither a tautology nor a contradiction. Then,

\[ \models \Box \bot \leftrightarrow [\top \pi] \Box \bot. \]

**Proof.** For the left-to-right direction, take any pointed model $(\mathcal{M}, w)$. The antecedent $\Box \bot$ states that $w$ has no successors and hence, by Definition 8, neither does $(w, 0)$ regardless of the chosen clause $D \in C(\pi)$; thus, $[\top \pi] \Box \bot$.

For the other direction, argue by contrapositive. Assume that $\mathcal{M}, w \models \neg \Box \bot$. Then, $w$ has at least one successor and hence, by Definition 8 so does $(w, 0)$ regardless of the chosen clause $D \in C(\pi)$; thus, $[\top \pi] \neg \Box \bot$.

With respect to tautologies, forgetting behaves as forgetting whether: the clausal form of $\top$ is $\emptyset$, and therefore it is not possible to forget a tautology.

When compared with the action of forgetting whether, the action of forgetting is closer to the well-known action of belief contraction: both represent an epistemic action after which the agent does not know/believe a given formula, regardless of the epistemic attitude towards the formula’s negation. This allows a more accurate comparison with a key concept within belief contraction: that of recovery.

An action of forgetting a given $\pi$ might have side-effects: the agent might also forget a second formula $\varphi$. In such cases it seems desirable for an action of ‘remembering’ $\pi$ to make the agent to remember $\varphi$ too. The forgetting action of this section satisfies a form of recovery, restricted to cases in which $\pi$ was known to begin with. For describing this we will use the public announcement operation in public announcement logic [20, 11], represented syntactically with formulas of the form $!\pi$, as it matches closely the semantic nature of this approach.

**Proposition 4.** If $\pi$ is a propositional formula and $\varphi$ an arbitrary formula of $\mathcal{L}_\Box$ then

\[ (\Box \pi \land \varphi) \rightarrow [\top \pi]![\pi] \varphi \]

is valid over the class of transitive models.

\footnote{Still, within AGM, the recovery postulate is the most discussed, as there are examples showing that such behaviour is not always reasonable. See, e.g., [15, 16].}
can be represented by the specific action model described below.

\[\langle f, d \rangle\] is a pair defined as follows.

To each action in \(L\), assign a formula of \(\mathcal{C}\) \(\preceq \mathcal{E}\) \(\rightarrow \mathcal{L}\) a precondition function assigning a formula of \(\mathcal{L}\) to each action in \(E\), and \(\text{Post} : (E \times At) \rightarrow \mathcal{L}\) a postcondition function assigning a formula of \(\mathcal{L}\) to each pair of atom in \(At\) and action in \(E\). A pointed action model is a pair \((U, e)\) where \(U\) is an action model and \(e\) an element of its domain.

Action models are intended to be applied to relational models; such application produces a new relational model, defined as follows.

A model \(\mathcal{M} = (W, R, V)\) be a model and \(U = (E, R, Pre, Post)\) an action model. The new model \(\mathcal{M} \otimes U = (W', R', V')\) is given by:

- \(W' := \{(w, e) \in (W \times E) \mid \mathcal{M}, w \models Pre(e)\}\);

- \((w, e)R'(u, f)\) \(\iff\) \(wRu\) and \(eRf\); and

- for every \(p \in At\), \(V'(p) := \{(w, e) \in W' \mid \mathcal{M}, w \models Post(e, p)\}\).

In words, the new model's domain is the restricted Cartesian product between \(\mathcal{M}\)'s and \(U\)'s: \((w, e)\) is a world in \(\mathcal{M} \otimes U\) if and only if \(w\) satisfies \(e\)'s precondition. In this new model, the agent cannot distinguish world \((u, f)\) from world \((w, e)\) if and only if she did not distinguish \(u\) from \(w\) in \(M\) and could not distinguish \(f\) from \(e\) in \(U\). Finally, a world \((w, e)\) satisfies an atom \(p\) if and only if \(w\) satisfied \(p\)'s postcondition at \(e\) in \(M\).

Action models will be useful to us since the model operation of Definition 6 can be represented by the specific action model described below.
Definition 12. Let $\mathcal{C} = \{D_i : i \in I\}$ be a finite (possibly empty) set of non-tautological clauses. The action model $U_{\mathcal{C}} = \langle E, R, \text{Pre}, \text{Post} \rangle$ is given by

$$E := \{e_i \}_{i \in \{0\} \cup I}, \quad R := E \times E, \quad \text{Pre}(e_i) := \top \text{ for all } i \in \{0\} \cup I,$$

for every $p \in At$, $\text{Post}(e_0, p) := p$ and, for $i \in I$,

$$\text{Post}(e_i, p) := \begin{cases} p & \text{if } \{p, \neg p\} \cap D_i = \emptyset; \\ \top & \text{if } \neg p \in D_i; \\ \perp & \text{if } p \in D_i. \end{cases}$$

Example 4. Consider $\pi = p \land q$, and recall that $\mathcal{C}(\pi) = \{\{p\}, \{q\}\}$ and $\mathcal{C}(\neg \pi) = \{\{\neg p, \neg q\}\}$. Then, the action model $U_{\{\{p\}, \{\neg p, \neg q\}\}}$, defined using one clause in $\mathcal{C}(\pi)$ and one in $\mathcal{C}(\neg \pi)$, is given by:

Preconditions are represented inside the states. For states other than $e_0$, the postconditions are set up to falsify each state’s respective clause.

Note how, in every action model $U_{\mathcal{C}}$, the relation $E$ is the full Cartesian product. Thus, the upgrade operation of Definition 11 preserves many relational properties, including seriality, reflexivity, symmetry, transitivity and euclidean-ness. These action models give us an alternative representation of the models $M^\pi_u$.

Proposition 5. Let $\mathcal{M}$ be a model and $\mathcal{C}$ a finite (possibly empty) set of non-tautological clauses. Then, the models $M^\mathcal{C}_u$ from Definition 0 and $\mathcal{M} \otimes U_{\mathcal{C}}$ from Definitions 12 and 11 are isomorphic.

Proof. Take a model $\mathcal{M} = (W, R, V')$; it will be proved that $M^\mathcal{C}_u = (W^\mathcal{C}_u, R^\mathcal{C}_u, V^\mathcal{C}_u)$ and $\mathcal{M} \otimes U_{\mathcal{C}} = (W', R', V')$ are isomorphic, witness the function $f : W^\mathcal{C}_u \rightarrow W'$ given by $f(w, i) = (w, e_i)$.

First, note how $(w, i)R^\mathcal{C}_u(v, j)$ iff $(w, e_i)R(v, e_j)$. This is because, by Definition 11 $(w, i)R^\mathcal{C}_u(v, j)$ iff $wRv$. Moreover, by Definition 11 $(w, e_i)R(v, e_j)$ iff $wRv$ and $e_i \text{Re} j$. But, by Definition 12 $R$ is the total relation in $E$, so $(w, i)R^\mathcal{C}_u(v, j)$ iff $(w, e_i)R(v, e_j)$.

Now, to prove that $(w, i) \in V^\mathcal{C}_u(p)$ iff $f(w, i) \in V'(p)$, observe that, by Definition 6 $(w, 0) \in V^\mathcal{C}_u(p)$ iff $w \in V(p)$. By Definition 11 $(w, e_i) \in V'(p)$ iff $\mathcal{M}, w \models \text{Post}(e_0, p)$, and by Definition 12 Post$(e_0, p) = p$, so $(w, e_i) \in V'(p)$ iff $\mathcal{M}, w \models p$ iff $w \in V(p)$ iff $(w, 0) \in V^\mathcal{C}_u(p)$. For $i \neq 0$, by Definition 6 $(w, i) \in V^\mathcal{C}_u(p)$ iff

either $\{p, \neg p\} \cap D_i = \emptyset$ and $w \in V(p)$, or else $\neg p \in D_i$. \hspace{1cm} (1)

By Definition 11

$(w, e_i) \in V'(p)$ iff $\mathcal{M}, w \models \text{Post}(e_i, p)$; \hspace{1cm} (2)
but by Definition 12, \((M, w)\) may only satisfy \(\text{Post}(e, p)\) in two cases, when it is \(p\) and \(\top\), as \(\bot\) is unsatisfiable. Then, recalling (2), \((w, e_i) \in V'(p)\) iff

\[ M, w \models p \quad \text{and} \quad \{p, \neg p\} \cap D_i = \emptyset, \quad \text{or else} \quad M, w \models \top \quad \text{and} \quad \neg p \in D_i. \]

(3)

But by Definition 12, \((M, w)\) may only satisfy \(\text{Post}(e, p)\) in two cases, when it is \(p\) and \(\top\), as \(\bot\) is unsatisfiable. Then, recalling (2), \((w, e_i) \in V'(p)\) iff

\[ M, w \models p \quad \text{and} \quad \{p, \neg p\} \cap D_i = \emptyset, \quad \text{or else} \quad M, w \models \top \quad \text{and} \quad \neg p \in D_i. \]

Note the equivalence of (1) and (3), which proves that \((w, i) \in V'_C(p)\) iff \((w, e_i) \in V'(p)\).

This correspondence of our model operation with the effect of \(U_C\) allows the use of the action models machinery for obtaining an axiom system for the language \(\mathcal{L}_{\pi}^\mathcal{C}\) with respect to our semantic models. First, recall the definition of the satisfaction relation for action model logic.

**Definition 13.** Let \(\mathcal{A}\) be a set of finite pointed \(\mathcal{L}_G\)-action models, that is, a set containing pointed \(\mathcal{L}_G\)-action models (thus with precondition and postconditions functions returning formulas in \(\mathcal{L}_G\)) whose domain is finite (non-empty) and in which each action affects the truth-value of at most a finite number of atomic propositions.\(^4\) The language \(\mathcal{L}_{\pi A}\) extends \(\mathcal{L}_{\pi}\) with new formulas of the form \([U, e]\phi\) with \((U, e) \in \mathcal{A}\). Let \(\mathcal{M} = (W, R, V)\) and \(w \in W\). The satisfaction relation of Definition \(\mathcal{M}\) is extended by setting \(\mathcal{M}, w \models [U, e]\phi\) if and only if

\[ \mathcal{M}, w \models \text{Pre}(e) \quad \Rightarrow \quad \mathcal{M} \otimes U, (w, e) \models \phi. \]

The set of valid formulas of \(\mathcal{L}_{\pi A}\) is denoted by \(\text{Log}_{\mathcal{L}_{\pi A}}\).

The following result is based on the action model axiomatization provided in Section 6.6 of [30] together with the remarks of [35] (the latter in the context of public announcements). Recall that the logic \(K\) contains propositional tautologies, modus ponens, the distribution axiom \(\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)\) and the necessitation rule that derives \(\vdash \Box \varphi\) from \(\vdash \varphi\).

**Theorem 1.** Let \(\Lambda\) be any of the logics \(K, T, K4, K5, S4, S5\) and let \(\mathcal{A}\) be a set of \(\Lambda\)-complying finite pointed \(\mathcal{L}_G\)-action models. The logic \(\text{Log}_{\mathcal{L}_{\pi A}}\) is axiomatized by the modal logic \(\Lambda\) together with the following axioms and rules for all \((U, e) \in \mathcal{A}\):

\[
\begin{align*}
\vdash [U, e]p & \iff (\text{Pre}(e) \rightarrow \text{Post}(e, p)) \\
\vdash [U, e]\Box \varphi & \iff (\text{Pre}(e) \rightarrow \varphi) \\
\vdash [U, e](\varphi \land \psi) & \iff ([U, e]\varphi \land [U, e]\psi) \\
\vdash \Box \varphi & \text{From } \vdash \varphi \text{ infer } \vdash [U, e]\varphi.
\end{align*}
\]

This result can be used to obtain an axiom system for our particular modality \(\{\pi\}\). Given clauses \(D_1 \in C(\pi)\) and \(D_2 \in C(\neg \pi)\), our axiom system will use the auxiliary modalities \([D_1, D_2]\), \([D_1, D_2]\) and \([D_1, D_2]\), whose semantic interpretation is as follows:

\[
\begin{align*}
\mathcal{M}, w \models [D_1, D_2] \varphi & \iff \mathcal{M}(D_1, (D_2), (w, 0)) \models \varphi; \\
\mathcal{M}, w \models [D_1, D_2] \varphi & \iff \mathcal{M}(D_1, (D_2), (w, 1)) \models \varphi; \quad \text{and} \\
\mathcal{M}, w \models [D_1, D_2] \varphi & \iff \mathcal{M}(D_1, (D_2), (w, 2)) \models \varphi.
\end{align*}
\]

\(^4\)This finiteness condition is required to allow each pointed action model \([U, e]\) to be associated to a syntactic object and thus to be used as a modality within formulas. For more details, the reader is referred to Section 6.1 of [30].
Observe how \([D_1, D_2], [D_1, D_2], [D_1, D_2]\) correspond, respectively, to \([U(D_1, D_2), e_i], [U(D_1, e_i), e_1]\) and \([U(D_1, D_2), e_2]\), with \(U_c\) the \(L_c\)-action model of Definition 12. Moreover, note how the use of \(U_c\) within a modality is proper, as it is a finite action model: it has a non-empty finite domain and, given that both \(D_1\) and \(D_2\) are finite (Lemma 1), each one of its actions changes the truth-value of at most a finite number of atomic propositions. Finally, note how \([\pi]_\varphi\) is equivalent to \(\bigwedge_{D_1 \in C(\pi)} \bigwedge_{D_2 \in C(-\pi)} [D_1, D_2] \varphi\). Our axiomatization is, then, as follows:

**Definition 14** (The axiom system \(Ax_{\text{C}1}\)). Let \(D\) be the set of all pointed action models of the form \((U(D_1, D_2), e_i)\) with \(i \in \{0, 1, 2\}\) (recall that \(\text{Pre}(e_1) = \text{Pre}(e_1) = \text{Pre}(e_2)\)) and \(D_j\) a finite non-tautological clause; let \(L_{\text{C}1}\) be the extension of \(L_{\text{C}}\) with expressions of the form \([U, e]_\varphi\) with \((U, e) \in D\).

The set of axioms \(Ax_{\text{C}1}\) is defined by the following schemas:

\[
\vdash [\pi]_\varphi \iff \bigwedge_{D_1 \in C(\pi)} \bigwedge_{D_2 \in C(-\pi)} [D_1, D_2] \varphi
\]

\[
\vdash [D_1, D_2] p \iff p \quad \text{for all } p \in \text{At}
\]

\[
\vdash [D_1, D_2] p \iff p \quad \text{if } \{p, \lnot p\} \cap D_1 = \emptyset
\]

\[
\vdash [D_1, D_2] p \iff \top \quad \text{if } \lnot p \in D_1
\]

\[
\vdash [D_1, D_2] p \iff \bot \quad \text{if } p \in D_1
\]

\[
\vdash [D_1, D_2] p \iff p \quad \text{if } \{p, \lnot p\} \cap D_2 = \emptyset
\]

\[
\vdash [D_1, D_2] p \iff \top \quad \text{if } \lnot p \in D_2
\]

\[
\vdash [D_1, D_2] p \iff \bot \quad \text{if } p \in D_2
\]

\[
\vdash \lnot \varphi \iff \lnot [\_\_] \varphi
\]

\[
\vdash [(\varphi \land \psi)] \iff ([\_\_] \varphi \land [\_\_] \psi)
\]

\[
\vdash [\square \varphi] \iff [\square ([D_1, D_2] \varphi \land [D_1, D_2] \varphi \land [D_1, D_2] \varphi)]
\]

\[
\vdash [(\varphi \rightarrow \psi)] \iff ([\_\_] \varphi \rightarrow [\_\_] \psi)
\]

From \(\vdash \varphi\) infer \(\vdash [\_\_] \varphi\)

where \([\_\_]\) is any of \([D_1, D_2], [D_1, D_2]\) and \([D_1, D_2]\) and \(\pi\) is propositional.

It is now possible to state this paper’s main result:

**Theorem 2.** Let \(\Lambda\) be any of the logics \(K, T, K4, K5, S4, S5\). A formula \(\varphi \in L_{\text{C}1}\) is valid over the class of \(\Lambda\)-complying models if and only if \(\Lambda + Ax_{\text{C}1} \vdash \varphi\).

**Proof.** Soundness is immediate since all axioms are true and all rules preserve validity, where we appeal to Theorem 1 for those axioms involving action models.

For completeness, let \(\varphi \in L_{\text{C}1}\) be a valid formula. Then, by the first axiom of \(Ax_{\text{C}1}\), the formula \(\varphi\) can be replaced by a provably equivalent formula \(\hat{\varphi} \in L_{\text{C}1}\). By Theorem 1, \(\hat{\varphi}\) is derivable, hence so is \(\varphi\).

Since the action models in \(D\) were auxiliary, it might be useful to restate this result in terms of our original language:

**Corollary 2.** Let \(\Lambda\) be any of the logics \(K, T, K4, K5, S4, S5\). A formula \(\varphi \in L_{\text{C}}\) is valid over the class of \(\Lambda\)-complying models if and only if it is derivable in \(\Lambda + Ax_{\text{C}1}\).
7 Alternative forgetting operators

There are many possibilities when modelling the action of forgetting. Our aim has been to present a semantic approach, rather than a syntactic one, but even then there are several routes one can take. In this section we mention some variations and discuss how they relate to our proposal.

7.1 Conditionally forgetting

The defined actions of forgetting $\pi$ (Definition 9) and forgetting whether $\pi$ (Definition 7) do not have any precondition, and thus they can take place regardless of whether the agent knows $\pi$ (for forgetting $\pi$) or knows either $\pi$ or else $\neg \pi$ (for forgetting whether $\pi$). This choice has been made because, technically, there is no reason to restrict the respective operations: they can be applied to any model, regardless of the agent’s epistemic attitude towards $\pi$.5 As a result, even though anomalies may occur in ‘abnormal’ situations, the defined actions work as expected in the intended cases (the agent knows / knows whether the –non tautological and non contradictory– formula she is forgetting is the case).

However, it is also interesting to assume that the agent would not act unless she is in an intended situation. Let us focus on the action of forgetting $\pi$. An interesting possibility is working as in Definition 9 when the agent knows $\pi$ but doing nothing otherwise.

Definition 15. For a propositional formula $\pi$, define a modal operator $[\hat{\mathbf{I}}']\pi$ and extend the semantics in Definition 3 by setting

$$M, w \models [\hat{\mathbf{I}}']\pi \varphi \iff \left\{ \begin{array}{ll}
\forall D \in \mathcal{C}(\pi), M_u^{(D)}, (w, 0) \models \varphi & \text{if } M, w \models [\square] \pi \\
M, w \models \varphi & \text{otherwise.}
\end{array} \right.$$

Thus, $\varphi$ should be always evaluated, and the precondition only determines where: in $M_u^{(D)}$ for every $D \in \mathcal{C}(\pi)$ when the precondition holds, and only in $M$ otherwise.6

This new operator has several properties that are interesting when compared to other approaches as belief contraction. For example, a vacuity principle is immediate:

Proposition 6. If $\pi$ is propositional and $\varphi$ is an arbitrary formula then

1. $\neg \square \pi \rightarrow (\varphi \leftrightarrow [\hat{\mathbf{I}}']\pi \varphi)$ is valid, but

2. $\neg \square \pi \rightarrow (\varphi \leftrightarrow [\hat{\mathbf{I}}]\pi \varphi)$ is not necessarily valid.

5Compare this with the precondition for public announcements. In order to be announced, a formula needs to be true, not only because of the interpretation of the operation (public and truthful announcements), but also for technical reasons: if the formula is false, then the evaluation point will be removed, and thus it is not possible to evaluate formulas in it after the operation.

6Note how this is different from other alternatives, as the one used in public announcement logic when evaluating $[!\pi]\varphi$: if the announced formula $\pi$ is not true, then $\varphi$ does not need to be evaluated, and $[!\pi]\varphi$ holds by vacuity.
Proof. The first claim is immediate from Definition 15.

As a counterexample for the second claim we may take \( \pi = p \land q \) and \( \varphi = \Box p \), then consider a model with a single reflexive world \( w \) satisfying \( p \land \neg q \). Then, \( \mathcal{M}, w \models \Box p \), but \( \mathcal{M}_w(\{p\}), (w, 0) \models \neg \Box p \), and hence
\[
\mathcal{M}, w \models \neg \Box p \land \neg \varphi \iff [\Diamond \varphi] \varphi.
\]

Thus the two notions of forgetting behave differently. However, in order to study them together, it is not necessarily to have both as primitives: the version with precondition can be defined in terms of its more general counterpart.

**Proposition 7.** If \( \pi \) is propositional and \( \varphi \) is arbitrary, then
\[
\models [\Diamond \pi] \varphi \iff (\neg \Box \pi \land \varphi) \lor (\Box p \land [\Diamond \pi] \varphi).
\]

A similar variant could be defined of \( [\pi] \). We will not go into details, but the treatment here would be more nuanced, as we would have to consider three cases, depending on whether \( \Box \pi \), \( \Box \neg \pi \), or neither one of them holds.

### 7.2 Strongly forgetting whether

As another natural alternative, a deterministic version of our operator can be explored which would entail that the agent loses all information regarding \( \pi \). According to Definition 7, in order to check \( [\pi] \varphi \), we need to check \( \varphi \) in several models, one for each element in \( C(\pi) \times C(\neg \pi) \). The number of models to check can be exponential. An alternative deterministic forgetting operator could create just one model by appending a new copy of each world for each clause in \( C(\pi) \cup C(\neg \pi) \).

**Definition 16.** For a propositional formula \( \pi \), define a modal operator \( [\Diamond^* \pi] \) and extend the semantics in Definition 3 by setting
\[
\mathcal{M}, w \models [\Diamond^* \pi] \varphi \iff \mathcal{M}_w^\pi(\pi) \cup C(\neg \pi) \models \varphi.
\]

So now we need only check one model, but the price to pay is that the new model may be exponentially larger than the original. As before, the alternative operator \( [\Diamond^* \pi] \varphi \) would have different properties to \( [\pi] \varphi \). For example, we have the following:

**Proposition 8.** Over the class of serial models,

1. \( [\Diamond^* (p \land q)] (\neg \Box p \land \neg \Box q) \) is valid, but
2. \( [\Diamond (p \land q)] (\neg \Box p \land \neg \Box q) \) is not valid.

**Proof.** Suppose that \( \mathcal{M} \) is a serial model, \( w \) is a world of \( \mathcal{M} \) and \( v \) a world accessible from \( w \). The clausal form of \( p \land q \) is \( \{p\}, \{q\} \), while the clausal form of \( \neg (p \land q) \) is \( \{\neg p, \neg q\} \). Thus \( C(p \land q) \cup C(\neg (p \land q)) = \{p\}, \{q\}, \{\neg p, \neg q\} \) and in \( \mathcal{M}_w^\pi(p \land q) \cup C(\neg (p \land q)) \) from \((w, 0)\) is accessible a world \((v, \{p\})\) satisfying \( \neg p \) and another \((v, \{q\})\) satisfying \( \neg q \). It follows that
\[
\mathcal{M}, w \models [\Diamond^* (p \land q)] (\neg \Box (p \land \neg \Box q)).
\]

Since \( \mathcal{M} \) and \( w \) were arbitrary, the first claim follows.

For the second, consider a model consisting of a single reflexive point \( w \) satisfying \( p \land q \). Then, \( (\mathcal{M}_w^\pi(\{\neg p, \neg q\}), w) \) clearly satisfies \( \Box p \), so \( \mathcal{M}, w \not\models [\Diamond (p \land q)] (\neg \Box (p \land \neg \Box q)) \), and hence this formula is not valid. \( \square \)
7.3 Dependent forgetting

In uniform forgetting (Definition 7), an agent forgets a formula \( \pi \) by falsifying a fixed clause of \( \pi \)'s clausal form in one copy of the initial model and a fixed clause of \( \neg \pi \)'s clausal form in another. However, it may be that at every point in the model, she forgets \( \pi \) 'for a different reason'. This gives an alternative way to model forgetting, which behaves in a different way from uniform forgetting as presented above. Let us give the definitions.

**Definition 17.** Let \( \mathcal{M} = \langle W, R, V \rangle \) be a model and \( (C_1, C_2) \) be two sets of clauses. A forgetting function pair is a pair of functions \((f_1, f_2)\) such that \( f_1: W \to C_1 \) and \( f_2: W \to C_2 \).

The model \( \mathcal{M}_d^{f_1, f_2} = \langle W_d^{f_1, f_2}, R_d^{f_1, f_2}, V_d^{f_1, f_2} \rangle \) is given by

\[
W_d^{f_1, f_2} = W \times \{0, 1, 2\}, \quad (w, i) R_d^{f_1, f_2} (v, j) \iff wR_v
\]

and, for each \( p \in \mathcal{M} \):

1. \((w, 0) \in V_d^{f_1, f_2}(p) \iff w \in V(p)\); and
2. for \( i = 1, 2 \), \((w, i) \in V_d^{f_1, f_2}(p) \iff \{p, \neg p\} \cap f_i(w) = \emptyset \) and \( w \in V(p) \), or else \( \neg p \in f_i(w) \).

Thus, the model \( \mathcal{M}_d^{f_1, f_2} \) contains three copies of \( \mathcal{M} \). The elements of the first, the \((w, 0)\) worlds, have the original atomic valuation; each element of the second, the \((w, 1)\) worlds, falsifies a particular clause in \( C_1 \), as indicated by the forgetting function \( f_1 \); finally, each element of the third copy, the \((w, 2)\) worlds, falsifies a particular clause in \( C_2 \), as indicated by the forgetting function \( f_2 \).

**Definition 18.** Let \( \mathcal{M} \) be a model. Define the dependent satisfaction relation \( \models_d \) on \( \mathcal{M} \) by extending Definition 3 with \( \mathcal{M}, w \models_d \langle \langle \pi \rangle \varphi \) if and only if \( \forall f_1: W \to C(\pi), \forall f_2: W \to C(\neg \pi), \mathcal{M}_d^{f_1, f_2}, (w, 0) \models_d \varphi \).

The set of formulas in \( \mathcal{L}_{C4} \) valid under \( \models_d \) will be denoted \( \text{d-Log}_{C4} \).

It turns out that our dependent and uniform interpretations give rise to different logics.

**Proposition 9.** The logics \( \text{Log}_{C4} \) (for uniform forgetting) and \( \text{d-Log}_{C4} \) (for dependent forgetting) are different. In particular, if

\[
\varphi = \square (p \land q) \rightarrow [\langle \langle p \land q \rangle \rangle (\square p \lor \square q)],
\]

then \( \varphi \in \text{Log}_{C4} \setminus \text{d-Log}_{C4} \); this is true even if we restrict to the class of S5 models.

**Proof.** We argue semantically that \( \varphi \in \text{Log}_{C4} \). Let \( \mathcal{M} = \langle W, R, V \rangle \) be a model and \( w \in W \) satisfy \( \square (p \land q) \). We have that \( C(p \land q) = \{\{p\}, \{q\}\} \) and \( C(\neg (p \land q)) = \{\{\neg p, \neg q\}\} \), so to check that \( \langle \langle p \land q \rangle \rangle (\square p \lor \square q) \) holds in \( w \) it is enough to see that \( M_a^{\langle \langle p \land q \rangle \rangle, \{\neg p, \neg q\}}(w, 0) \) and \( M_a^{\langle \langle p \land q \rangle \rangle, \{\neg p, \neg q\}}(w, 0) \) both satisfy \( \square p \lor \square q \).

Let \( \mathcal{M}_d^{\langle \langle p \rangle \rangle, \{\neg p, \neg q\}} = \langle W', R', V' \rangle \) and suppose that \( (w, 0) R' (v, i) \); we claim that independently of \( i, (v, i) \in V'(q) \). If \( i = 0 \), this follows from the assumption
that \( w \in [p \land q]^{\mathcal{M}} \). If \( i = 1 \), then \( q \) does not occur in \( \{p\} \) and then \((v, 1) \in V'(q)\). Finally, if \( i = 2 \), then \( \neg q \in \{\neg p, \neg q\} \) so \((v, 2) \in V'(q)\). In all three cases, \((v, i) \in V'(q)\) and since \((v, i)\) was arbitrary, \((w, 0) \in \llbracket \Box p \lor \Box q \rrbracket_{M^{\mathcal{U}(\{p\},\{\neg p, \neg q\})}}\).

A symmetric argument shows that \( \mathcal{M}^{\mathcal{U}(\{q\},\{\neg p, \neg q\})}, (w, 0) \models_d \Box p \), so that \( \mathcal{M}, w \models [\Diamond (p \land q)](\Box p \lor \Box q) \), as claimed.

It remains to check that \( \varphi \notin d\text{-Log}_{\mathcal{U}} \). For this, consider the model \( \mathcal{M} = \langle W, R, V \rangle \) shown below, where \( W = \{w, v\} \), \( R \) is the full relation on \( W \) and \( V(p) = V(q) = W \); observe that \( \mathcal{M} \) is an \( S5 \) model.

\[
\begin{array}{cccc}
\text{p} & \rightarrow & \text{q} & \downarrow \\
| & \uparrow & | & \\
\text{pq} & (w, 1) & (v, 1) & \\
| & \uparrow & | & \\
\text{pq} & (w, 0) & (v, 0) & \\
| & \uparrow & | & \\
\text{pq} & (w, 2) & (v, 2) & \\
\end{array}
\]

Observe how \( R'' \) is also the full relation. Moreover, \((w, 1) \notin V''(p)\) and \((v, 1) \notin V''(q)\), so \((w, 0) \notin \llbracket \Box p \lor \Box q \rrbracket_{M^{\mathcal{U}(f_1, f_2)}}\) and hence \( \mathcal{M}, w \models [\Diamond (p \land q)](\Box p \lor \Box q) \), as desired.

Thus, the notion of dependent forgetting leads to a different logic of uniform forgetting. Our example above suggests that, with depending forgetting, more information is lost in the act of forgetting, and this may be desirable in applications. However, the technique of representing forgetting in terms of action models does not work in this setting (at least not in a straightforward way); a further exploration of this notion of forgetting is left for future work.

**8 Conclusions**

The present paper uses the possible world semantics to model the changes that occur in an agent’s information when she forgets the truth-value of a propositional formula as represented by its ‘minimal’ conjunctive normal form. Besides introducing a uniform forgetting whether model operation representing such action and its correspondent modality for expressing its effects, the paper has discussed several properties of the operation as well as provided a sound and
complete axiom system for it. Two variations of this uniform forgetting whether action have been explored: a simpler uniform forgetting action which simply forgets that a given formula is the case, and a more complex dependent forgetting whether under which the agent might forget the given formula’s truth-value for different reasons in different parts of the model. It has been proved not only that uniform forgetting whether cannot be defined in terms of the simpler uniform forgetting, but also that the uniform forgetting whether and the dependent forgetting whether give raise to different logics.

Several directions are left for further study. First, the axiomatization of the dependent forgetting logic is still an open issue. It is possible that the action models axioms can be used for dependent forgetting if some restrictions are introduced to the forgetting functions. Second, and possibly more interesting, is an action of forgetting modal formulas, which would allow the agents to forget their own or other agents’ epistemic states. Finally, our model of forgetting differs in several ways from the belief contraction approach. We leave a more comprehensive comparison of the two, along with a possible unification, for future work.

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