A Short Note on the Bruinier-Kohnen Sign Equidistribution Conjecture and Halász’ Theorem

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Abstract

In this note, we improve earlier results towards the Bruinier-Kohnen sign equidistribution conjecture for half-integral weight modular eigenforms in terms of natural density by using a consequence of Halász’ Theorem. Moreover, applying a result of Serre we remove all unproved assumptions.

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By using the celebrated Sato-Tate theorem for integral weight modular eigenforms and the Shimura lift, in [4] and in [1] (together with Sara Arias-de-Reyna), we prove results related to the Bruinier-Kohnen sign equidistribution conjecture for modular eigenforms of half integral weight. In this note we improve one of our main results to a formulation in terms of natural density. Moreover, a theorem of Serre’s allows us to remove all unproved assumptions.

The first improvement is due to the following application of Halász’ Theorem that one of us learned from Kaisa Matomäki.

**Theorem 1.** Let \( g : \mathbb{N} \to \{-1, 0, 1\} \) be a multiplicative function. If \( \sum_{p\leq x} \frac{1}{p} \) converges and \( \sum_{p, g(p)=\pm 1} \frac{1}{p} \) diverges then

\[
\lim_{x \to \infty} \frac{\sum_{n \leq x, g(n) \geq 0}}{\sum_{n \leq x, g(n) \neq 0}} = \frac{1}{2}.
\]

**Proof.** By Lemma 2.2 of [3], which is a consequence of Halász’ Theorem (see [3] for details), there exists an absolute positive constant \( C \) such that

\[
\sum_{n \leq x} g(n) \leq C \cdot x \exp \left( -\frac{1}{4} \sum_{p \leq x} \frac{1-g(p)}{p} \right)
\]

for all \( x \geq 2 \). By assumption, we have \( 1 - g(p) \geq 0 \) for all \( p \) and \( 1 - g(p) = 2 > 1 \) for any \( p \) with \( g(p) = -1 \). We conclude that for \( x \to \infty \), \( \exp \left( -\frac{1}{4} \sum_{p \leq x} \frac{1-g(p)}{p} \right) \) tends to 0. Hence for the average value of \( g \), we have

\[
\lim_{x \to \infty} \frac{\sum_{n \leq x, g(n)}}{x} = 0
\]

and therefore

\[
\lim_{x \to \infty} \frac{\sum_{n \leq x, g(n) \geq 0}}{\sum_{n \leq x, g(n) \neq 0}} = \frac{1}{2}.
\]

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\[ \sum_{n \leq x} g(n) = \left| \{ n \leq x \mid g(n) > 0 \} \right| - \left| \{ n \leq x \mid g(n) < 0 \} \right| = o(x). \]

Since \( \sum_{p, g(p)=0} \frac{1}{p} \) converges by assumption, we conclude that for \( x \to \infty \),

\[ \frac{\sum_{n \leq x} |g(n)|}{x} = \left| \{ n \leq x \mid g(n) > 0 \} \right| + \left| \{ n \leq x \mid g(n) < 0 \} \right| \]

tends to a positive limit, hence the assertion follows immediately.

In order to state and prove the results towards the Bruinier-Kohnen conjecture, we introduce some notation to be used throughout the note. Let \( k \geq 2 \) and \( 4 \mid N \) be integers and \( \chi \) be a quadratic Dirichlet character modulo \( N \). We denote the space of cusp forms of weight \( k+1/2 \) for the group \( \Gamma_1(N) \) with character \( \chi \) by \( S_{k+1/2}(N, \chi) \) in the sense of Shimura, as in the main theorem in [8] on p. 458. For Hecke operators \( T_p \) for primes \( p \nmid N \), let \( f = \sum_{n \geq 1} a(n)q^n \in S_{k+1/2}(N, \chi) \) be a non-zero cuspidal Hecke eigenform with real coefficients. For a fixed squarefree \( t \) such that \( a(t) \neq 0 \), denote by \( F_t \) the Shimura lift of \( f \) with respect to \( t \). It is a cuspidal Hecke eigenform of weight \( 2k \) for the group \( \Gamma_0(N/2) \) with trivial character. By normalising \( f \) we can and do assume \( a(t) = 1 \), in which case \( F_t \) is normalised.

As in our previous treatments, the following theorem, of which we only state a weak version, is in the core of our approach. Its proof is based on the Sato-Tate theorem, see [2].

**Theorem 2.** [2],[11] Assume the set-up above and define the set of primes

\[ \mathbb{P}_{>0} := \{ p : a(tp^2) > 0 \} \]

and similarly \( \mathbb{P}_{<0} \) and \( \mathbb{P}_{=0} \) (depending on \( f \) and \( t \)). Then the sets \( \mathbb{P}_{>0} \) and \( \mathbb{P}_{<0} \) have positive natural densities and the set \( \mathbb{P}_{=0} \) has natural density 0.

Due to its importance in the sequel, here we recall the following notion (Definition 2.2.1 of [11]).

**Definition 3.** Let \( S \) be a set of primes. It is called weakly regular if there is \( a \in \mathbb{R} \) (called the Dirichlet density of \( S \)) and a function \( g(z) \) which is holomorphic on \( \{ \text{Re}(z) > 1 \} \) and continuous (in particular, finite) on \( \{ \text{Re}(z) \geq 1 \} \) such that

\[ \sum_{p \in S} \frac{1}{p^2} = a \log \left( \frac{1}{z-1} \right) + g(z). \]

The second improvement of this paper is the observation that a result of Serre’s allows us to prove directly that the set \( \mathbb{P}_{=0} \) is always weakly regular. This approach avoids the use of Sato-Tate equidistribution and consequently does not depend on any unproved error terms for it. It only applies to \( \mathbb{P}_{=0} \) and hence does not seem to give us the weak regularity of the other sets \( \mathbb{P}_{>0} \) and \( \mathbb{P}_{<0} \).

**Proposition 4.** Assume the setup above. Then the set \( \mathbb{P}_{=0} \) is weakly regular.

**Proof.** Let \( F = F_t = \sum_{n=1}^{\infty} A(n)q^n \) be the Shimura lift of \( f \) with respect to \( t \). If \( F \) has CM, then the result has been proved in Theorem 4.1.1(c) of [11]. So let us assume that \( F \) has no CM. Due to the assumption \( a(t) = 1 \) we have the formula

\[ A(p) = a(tp^2) + \epsilon(p)p^{k-1} \]

for all primes \( p \), where \( \epsilon \) is an at most quadratic (due to the assumption that all coefficients are real) Dirichlet character of modulus \( 2tn^2 \) (see e.g. equation (4.1) of [11]). Consequently, we have the inclusion

\[ \{ p < x : p \nmid 2tn, p \in \mathbb{P}_{=0} \} = \{ p < x : p \nmid 2tn, a(tp^2) = 0 \} \]

\[ \subseteq \{ p < x : p \nmid 2tn, A(p) = p^{k-1} \} \cup \{ p < x : p \nmid 2tn, A(p) = -p^{k-1} \}. \]
By Corollaire 1 of Théorème 15 in [7] (with \( h(T) = T^{k-1} \) and \( h(T) = -T^{k-1} \)), it follows that

\[
\#\{ p < x : p \in \mathbb{P}_{=0} \} = o\left( \frac{x}{\log(x)^{9/8}} \right).
\]

Consequently, Corollary 2.2.4 of [1] implies that \( \mathbb{P}_{=0} \) is weakly regular.

We now use the application of Halász’ theorem and the weak regularity of \( \mathbb{P}_{=0} \) to prove the equidistribution result we are after in terms of natural density. In [1] we needed regularity to achieve this goal.

**Theorem 5.** Assume the setup above. Then the sets \( \{ n \in \mathbb{N} | a(tn^2) > 0 \} \) and \( \{ n \in \mathbb{N} | a(tn^2) < 0 \} \) have equal positive natural density, that is, both are precisely half of the natural density of the set \( \{ n \in \mathbb{N} | a(tn^2) \neq 0 \} \).

**Proof.** Let \( g(n) = \begin{cases} 1 & \text{if } a(tn^2) > 0, \\ 0 & \text{if } a(tn^2) = 0, \\ -1 & \text{if } a(tn^2) < 0. \end{cases} \) Due to the relations \( a(tn^2m^2)a(t) = a(tn^2)a(tm^2) \) for \( \gcd(n, m) = 1 \) (see p. 453 of [3]), it is clear that \( g(n) \) is multiplicative. Since \( \mathbb{P}_{=0} \) is weakly regular by Proposition 4 it follows that \( \sum_{p \in \mathbb{P}_{=0}} \frac{1}{p} \) is finite. Moreover, the fact that \( \mathbb{P}_{<0} \) is of positive density implies that \( \sum_{p \in \mathbb{P}_{<0}} \frac{1}{p} \) diverges. Thus the result follows from Theorem 1. \( \square \)

In [1] we obtained the same conclusion under the additional assumption of the Generalised Riemann Hypothesis (GRH) because we needed to achieve the regularity of \( \mathbb{P}_{=0} \), which we could derive from the very strong error term in Sato-Tate proved in [6], under the assumption of GRH.

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