Variational characterization of the speed of reaction diffusion fronts for gradient dependent diffusion

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Abstract We study the asymptotic speed of travelling fronts of the scalar reaction diffusion for positive reaction terms and with a diffusion coefficient depending nonlinearly on the concentration and on its gradient. We restrict our study to diffusion coefficients of the form $D(u, u_x) = mu^{m-1}u_x^{m(p-2)}$ for which existence and convergence to travelling fronts has been established. We formulate a variational principle for the asymptotic speed of the fronts. Upper and lower bounds for the speed valid for any $m \geq 0, p \geq 1$ are constructed. When $m = 1, p = 2$ the problem reduces to the constant diffusion problem and the bounds correspond to the classic Zeldovich–Frank–Kamenetskii lower bound and the Aronson-Weinberger upper bound respectively. In the special case $m(p - 1) = 1$ a local lower bound can be constructed which coincides with the aforementioned upper bound. The speed in this case is completely determined in agreement with recent results.

Keywords Variational principles · reaction–diffusion equation · gradient dependent diffusion · p–Laplacian

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1 Introduction

In this work we study the asymptotic propagation of fronts of the scalar reaction diffusion equation,

\[ \partial_t u = \partial_x (|\partial_x u|^p u^{-2} \partial_x u^m) + f(u), \quad f(0) = f(1) = 0, f(u) > 0 \quad \text{in} \quad (0,1), \]

which reduces to the classical problem \[13\] when \( m = 1, p = 2 \). The diffusion term can be seen either as the scalar version of the \( p \)-Laplacian acting on \( u^m \) or as reaction diffusion equation with nonlinear diffusion coefficient \( D(u, u_x) = mu^m - 1|u_x|^{m(p-2)} \). Such diffusion coefficients are encountered, for example, in hot plasmas \[12,14\] and the corresponding processes are referred to as doubly nonlinear diffusion processes \[4\].

The classical problem \( m = 1, p = 2 \), is fully understood \[1,13\]. When nonlinear diffusion is included several scenarios may arise depending on the precise form of the diffusion coefficient. The case of a power of concentration diffusion coefficient of the form \( D(u) = u^s \) has been studied extensively beginning with the analytical solution found for \( s = 2 \). Existence and convergence results are known for all \( s \). A distinctive feature of density dependent diffusion is the appearance of a finite wave at the asymptotic speed. This is true even in the simpler \( p = 2 \) case when \( m > 1 \) (see, e.g., \[2,5\], and references therein). In recent work \[4\] the more general case of doubly nonlinear diffusion is considered. It is shown that for all \( m > 0, p > 1 \) such that \( \gamma = m(p-1) - 1 > 0 \), a unique monotonic increasing travelling wave joining the equilibria \( u = 0 \) and \( u = 1 \) exists for speeds \( c \geq c_\star(m, p) \) and none if \( 0 < c < c_\star(m, p) \). For \( c = c_\star(m, p) \) the travelling wave (TW) is finite, whereas for \( c > c_\star(m, p) \) the TW is positive (see \[4\], Theorem 2.1). In the case \( \gamma = 0 \) a unique monotonic increasing travelling wave joining the equilibria \( u = 0 \) and \( u = 1 \) exists for speeds \( c \geq c_\star(m, p) \) and none if \( 0 < c < c_\star(m, p) \). For \( c = c_\star(m, p) \) the travelling wave (TW) is positive (see \[4\], Theorem 2.2). Moreover, when \( \gamma = 0 \), an explicit expression for the minimal speed is given,

\[ c_\star(m, p)\big|_{\gamma=0} \equiv c_0(m, p) = (m^2p^{m+1}f'(0))^{1/(m+1)}. \]  

The convergence of suitable initial conditions to the travelling wave of minimal speed is demonstrated in \[4\] as well.

The purpose of this work is to establish a variational characterization for the speed \( c_\star(m, p) \). The exact value of the speed cannot be determined in general however upper and lower bounds on the speed for general values of \( m \) and \( p \) can be obtained. The main result of the present work (see Theorem 1 below) is the variational expression for the speed

\[ c_\star = \sup_g \left[ \frac{\left( \int_0^1 pu^{(m-1)(p-1)/p} \int_0^1 g(u) \int_0^1 h^{(p-1)/p} f^{(p-1)/p} g(du) \right)^{1/p}}{\int_0^1 g(u) du} \right], \]  

from where upper and lower bounds will be constructed. In \[4\], \( g \in C^1(0,1) \) is such that \( g(u) \geq 0 \), with \( h(u) \equiv -g'(u) > 0 \) in \( (0,1) \) and \( \int_0^1 g(u) du \) finite.
We find that for any $m > 0, p > 1$ (with $\gamma \geq 0$) the asymptotic speed is bounded by
\[
\left( \frac{m}{p-1} \int_0^1 u^{(m-1)} f(u) du \right)^{(p-1)/p} \leq c_*(m, p) \leq \frac{p}{\left( \frac{m}{p-1} \right)^{(p-1)/p}} \sup_u \left[ u^\gamma \left( \frac{f(u)}{u} \right)^{(p-1)} \right]^{1/p}.
\] (4)

The lower bound is a generalization of the Zeldovich-Frank-Kamenetskii (ZFK) bound (see, e.g., [10,9]). Effectively, for $m = 1, p = 2$ their classical bound $c_* \geq c_{ZFK} = \sqrt{2} \int_0^1 f(u) du$ is recovered. The upper bound, for $m = 1, p = 2$ reduces to the Aronson-Weinberger upper bound [1]
\[\frac{c}{\sup_u 2 \sqrt{f(u)/u}}.\]

An interesting case arises when $\gamma = 0$. As mentioned above the speed can be determined exactly [4] and it is given by $c_0(m, p)$ when $\gamma = 0$. Here we recover this result from the variational principle showing that when $\gamma = 0$ a local lower bound can be found choosing an adequate trial function $g(u)$. This lower bound is exactly $c_0(m, p)$. The upper bound given in (4) reduces to $c_0(m, p)$ when $\gamma = 0$ and $f(u)$ satisfies the KPP criterion $\sup_u \sqrt{f(u)/u} = f'(0)$. In the following sections we prove the statements made above. Our variational principle reduces to our standard variational principle (see [7], [8], [9]) when $p = 2$.

The rest of this manuscript is organized as follows: In Section 2 we derive the variational principle, in Section 3 the bounds for general values of $\gamma$ with $m > 0, p > 1$ are obtained, and in Section 4 we derive a lower bound of the Zeldovich-Frank-Kamenetskii type for any $\gamma \geq 0$.

2 Variational Principle

We consider left travelling wave solutions $u(\xi)$ with $\xi = x + ct$ so that the TW profile satisfies $u_\xi > 0$. The TW solution satisfies the ordinary differential equation (ODE)
\[cu_\xi = m^{p-1} \frac{d}{d\xi} \left( u^{(m-1)(p-1)}(u_\xi)^{p-1} \right) + f(u).\] (5)

From here on we denote $u' = u_\xi$. Following the usual procedure, we introduce the phase space coordinate
\[q(u) = u^{m-1} u'(u)\]
in terms of which the ODE for the travelling waves becomes, after dividing by $q$,
\[
\frac{c}{m^{p-1}} = \frac{d}{du}(q(u))^{p-1} + \frac{u^{m-1}f(u)}{m^{p-1}q(u)}. \tag{6}
\]
Here, it is convenient to define
\[
F(u) = \frac{u^{m-1}f(u)}{m^{p-1}}. \tag{7}
\]
In what follows, let us define the functional
\[
\mathcal{J}[g] \equiv \frac{pm^{p-1}}{(p-1)(p-1)^p} \int_0^1 h(u)^{1/p} F(u)^{(p-1)/p} g^{(p-1)/p} du \int_0^1 g(u) du, \tag{8}
\]
which acts on $D$, the space of functions $g \in C^1(0,1)$ such that $g(u) \geq 0$, with $h(u) = -g'(u) > 0$ in $(0,1)$ and $\int_0^1 g(u) du$ finite. Here the function $F(u)$ is given by (7) above. With this notation we state our main result, which is embodied in the following theorem.

**Theorem 1 (Variational characterization of $c_*$) Let $f \in C^1[0,1]$ with $f(0) = f(1) = 0$, $f(u) > 0$ in $(0,1)$, and $f(u)$ concave in $[0,1]$. Assume $\gamma = m(p-1) - 1 \geq 0$. Then,
\[
c_*(m,p) = J \equiv \sup\{\mathcal{J}[g] \mid g \in D\}. \tag{9}
\]
Moreover,

i) If $\gamma > 0$, there is a $g \in D$, $\tilde{g}$ say, such that $J = \mathcal{J}[\tilde{g}]$. This maximizing $\tilde{g}$ is unique up to a multiplicative constant, and

ii) If $\gamma = 0$ we construct and explicit maximizing sequence $g_\alpha \in D$ such that $\lim_{\alpha \to 0} \mathcal{J}[g_\alpha] = c_*(m,p)|_{\gamma=0}$, where $c_*(m,p)|_{\gamma=0}$ is given by (2) above.

**Proof** Let $g(u) \in D$. Multiplying (6) by $g(u)$ and integrating in $u$ between 0 and 1 we obtain after integrating by parts,
\[
\frac{c}{m^{p-1}} \int_0^1 g(u) du = \int_0^1 du \left( h(u)q(u)^{p-1} + \frac{g(u)F(u)}{q(u)} \right) = \int_0^1 \Phi(u) du. \tag{10}
\]
where $h(u) \equiv -g'(u) > 0$ and we assume that $g(u)$ is such that $\lim_{u \to 0} g(u)q(u)^{p-1} = 0$.

The integrand of the right side,
\[
\Phi = h(u)q(u)^{p-1} + \frac{g(u)F(u)}{q(u)}, \tag{11}
\]
at fixed $u$ can be considered as a function of $q$. It is clear from (11) that $\Phi(q)$ has a unique positive minimum at $\hat{q}$ so that $\Phi(q) \geq \Phi(\hat{q})$. A simple calculation yields
\[
\hat{q} = \left[ \frac{Fg}{(p-1)h} \right]^{1/p}, \tag{12}
\]
and
\[ \Phi(\hat{q}) = \frac{pgh^{1/p}F(p^{-1}/p)}{(p-1)(p^{-1}/p)}. \]

It follows from (10) that
\[ c^* \geq \frac{pm^{p-1}}{(p-1)(p^{-1}/p)} \int_0^1 h^{1/p} F(p^{-1}/p) g(p^{-1}/p) du \int_0^1 g(u) du, \tag{13} \]
for every \( g \in D \). To establish (9) we need only prove that the supremum of the right side of (13) over all \( g \in D \) is actually \( c^* \). We will do this separately in the cases \( \gamma > 0 \) and \( \gamma = 0 \).

i) Case \( \gamma > 0 \). Below we show that when \( \hat{q} \) is the solution of (6) (with \( c = c^* \)), equality is attained in (13) for some \( g \in D \), so that we obtain the variational characterization for the speed
\[ c^* = \sup_{g} \frac{pm^{p-1}}{(p-1)(p^{-1}/p)} \int_0^1 h^{1/p} F(p^{-1}/p) g(p^{-1}/p) du \int_0^1 g(u) du. \tag{14} \]
We have already proven (see (13) above) that \( c^* \geq J[g] \) for every \( g \in D \). What we will actually show here is that when \( \gamma > 0 \), there exists a \( g \in D \), \( \tilde{g} \) say, such that \( c^* = J[\tilde{g}] \). Hence in the case \( \gamma > 0 \) the variational principle reads,
\[ c^* = \max_{g \in D} (J[g]). \tag{15} \]
In the case \( \gamma > 0 \), the existence of a travelling wave for any \( c \geq c^* \) was proven in Theorem 2.1 of Reference [4]. Moreover, in the case \( \gamma > 0 \), the solution of (6) satisfies,
\[ q(u)^{p-1} \approx \frac{c^*}{m^{p-1}} u, \tag{16} \]
in the neighborhood of \( u = 0 \). In order to show that the sup is actually attained in (9) we have to show that there exists \( \tilde{g} \in D \) satisfying (12) when \( \hat{q} \) is a solution of (6). To construct such a \( g \), let \( v \) be the solution of
\[ \frac{v'}{v} = \frac{c^*}{m^{p-1}} \frac{1}{q^{p-1}(u)}, \tag{17} \]
where \( q \) is a solution of (6). Notice that this \( v \) is unique up to a multiplicative constant. A simple calculation using (17), (6), and the definition (7) of \( F \), yields,
\[ \frac{v''}{v} = \frac{c^*}{m^{p-1}} \frac{F(u)}{q^{2p-1}(u)}. \tag{18} \]
Choosing
\[ \tilde{g}(u) = \frac{1}{(v'(u))^{1/(p-1)}}, \tag{19} \]
it follows from (17) and (18) that,
\[-\tilde{g}'(u) = \frac{1}{p-1} (v'(u))^{p/(p-1)} v'' = \frac{1}{p-1} \frac{1}{q'(u)} \frac{v''(u)}{v'/v} = \frac{1}{p-1} \tilde{g}(u) \frac{F'(u)}{q'(u)}.\] (20)

which is precisely (12). From (16) and (17) we have that
\[v(u) \approx A u \quad \text{and} \quad v'(u) \approx A,\] (21)

near \(u = 0\). Hence, it follows from (19) that
\[\tilde{g}(0) = A^{-1/(p-1)} < \infty.\] (22)

Integrating (17) and using (21) we can write explicitly,
\[v(u) = \exp \left( \int_{u_0}^u c_s m^{p-1} q^{p/(p-1)}(s) ds \right),\] (23)

for some \(0 < u_0 < 1\). Clearly, the value of \(A\) in (21) is determined by the value of \(u_0\). Finally, using (17), (19), and (23), we can write,
\[\tilde{g}(u) = \frac{m q(u)}{c_s^{1/(p-1)}} \exp \left( \frac{1}{p-1} \int_{u_0}^u c_s m^{p-1} q^{p/(p-1)}(s) ds \right).\] (24)

Since, the integrand in (24) is positive, \(u_0 < 1\), and \(q(1) = 0\), it follows from (24) that \(\tilde{g}(1) = 0\). From all the results above it follows that \(\tilde{g}\) given by (24) is in \(\mathcal{D}\), and that \(c_s = J[\tilde{g}]\).

It is clear from the construction above that \(\tilde{g}\) is unique up to a multiplicative constant. The uniqueness of the maximizing \(g \in \mathcal{D}\), however, can be seen directly from our variational principle (8). In fact, suppose that there are two different maximizers, say \(g_1, g_2 \in \mathcal{D}\), with \(\int_0^1 g_1(u) du = \int_0^1 g_2(u) du = 1\). Then, for any \(\alpha \in (0,1)\) consider now,
\[g_\alpha(u) = \alpha g_1(u) + (1-\alpha) g_2(u).\] (25)

It is clear from (25) that \(g_\alpha \in \mathcal{D}\) and that \(\int_0^1 g_\alpha(u) du = 1\). Using H"{o}lder’s inequality with exponents \(p\) and \(p' = p/(p-1)\), it follows from (8) that
\[J[g_\alpha] > \alpha J[g_1] + (1-\alpha) J[g_2] = c_s,\] (26)

which is a contradiction with the fact that \(g_1\) and \(g_2\) are the maximizers. Notice that the inequality in (26) is strict if \(g_1 \neq g_2\).

ii) Case \(\gamma = 0\). For later purposes it is convenient to denote
\[J_g[f] = \int_0^1 [u^{m-1} h(u)^m f(u) q(u)]^{1/(m+1)} du.\] (27)
It then follows from (7) and (8) that

$$\mathcal{J}[g] = p m^{2/(m+1)} J_g[f]$$  \hspace{1cm} (28)$$

in the case $\gamma = 0$, when we conveniently normalize $g$ so that $\int_0^1 g(u) \, du = 1$.

Now, choose as a trial function the sequence

$$g_\alpha(u) = \frac{\alpha}{1 - \alpha} (u^{\alpha - 1} - 1), \quad 0 < \alpha < 1, \quad \text{with} \quad \alpha \to 0. \hspace{1cm} (29)$$

Notice that for each $\alpha \in (0,1)$, $g_\alpha(u) > 0$, $g_\alpha'(u) < 0$, $g_\alpha(1) = 0$, and $\lim_{\alpha \to 0} [u g_\alpha(u)] = 0$, so these are appropriate trial functions. Moreover, we have normalized the $g_\alpha$'s so that $\int_0^1 g_\alpha(u) \, du = 1$.

With this choice we will show that $\lim_{\alpha \to 0} J_{g_\alpha}[f] = f'(0)^{1/(m+1)}$ so that

$$\mathcal{J}[g_\alpha] \to (m^2 p^{m+1} f'(0))^{1/(m+1)} = c_0(m,p) \text{ as } \alpha \to 0. \hspace{1cm} (30)$$

To do so we write

$$J[f] = J[u f'(0)] + J[f] - J[u f'(0)]$$

and show that

$$J_{g_\alpha}[u f'(0)] \to f'(0)^{1/(m+1)}, \quad J_{g_\alpha}[f] - J_{g_\alpha}[u f'(0)] \to 0, \quad \text{as} \quad \alpha \to 0. \hspace{1cm} (30)$$

While the proof of the second limit is given in the Appendix, the proof of the first is as follows. Using (27) with $g = g_\alpha$ we have,

$$J_{g_\alpha}[u f'(0)] = f'(0)^{1/(m+1)} \alpha (1 - \alpha)^{-1/(m+1)} \int_0^1 (u^{m(\alpha - 1)}(u^{\alpha - 1} - 1))^{1/(m+1)} \, du = f'(0)^{1/(m+1)} \alpha (1 - \alpha)^{-(m+2)/(m+1)} B \left( \frac{m+2}{m}, \frac{\alpha}{1-\alpha} \right),$$

where $B(x, y)$ denotes the Euler Beta function. Now, $B(t, s) = \Gamma(t) \Gamma(s)/\Gamma(t+s)$, hence

$$J_{g_\alpha}[u f'(0)] = f'(0)^{1/(m+1)} \alpha (1 - \alpha)^{-(m+2)/(m+1)} \frac{\Gamma(\frac{m+2}{m}) \Gamma(\frac{\alpha}{1-\alpha})}{\Gamma(\frac{m+2}{m} + \frac{\alpha}{1-\alpha})} \hspace{1cm} (32)$$

Using $\lim_{x \to 0} x \Gamma'(x) = 1$ to evaluate the limit of the right side of (32) when $\alpha \to 0$, we finally conclude, $J_{g_\alpha}[u f'(0)] \to f'(0)^{1/(m+1)}$ as $\alpha \to 0$ from above. As indicated before, (30) then implies that $\mathcal{J}[g_\alpha] \to (m^2 p^{m+1} f'(0))^{1/(m+1)} = c_0(m,p)$ as $\alpha \to 0$, which concludes the proof of the Theorem.

3 An upper bound on the speed for $\gamma \geq 0$

In this section we derive from our variational principle (i.e., from Theorem 1 above) an explicit upper bound on the speed of fronts. In order to do this we rewrite (14) as

$$c_* = \sup_g \left[ \frac{p m^{p-1}}{(p-1)(p^1/p)} \left[ \int_0^1 [h F(p-1)/g]^{1/p} g(u) \, du \right] \right]. \hspace{1cm} (33)$$
Since the mapping \( x \rightarrow x^{1/p} \) is concave for \( p > 1 \), defining the probability measure \( dv = g(u)du/\int_0^1 g(u)du \), and using Jensen’s inequality we get,

\[
c_* \leq \sup_g \left( \frac{p m^{p-1}}{(p-1)^{p-1}p} \right) \left[ \frac{\int_0^1 hF(u)u \, du}{\int_0^1 g(u)du} \right]^{1/p} \leq \frac{p m^{p-1}}{(p-1)^{p-1}p} \sup_u \left( \frac{F^{p-1}}{u} \right) \sup_g \left[ \frac{\int_0^1 h(u)u \, du}{\int_0^1 g(u)du} \right]^{1/p}.
\]  \quad (34)

Integrating \( \int_0^1 h(u)u \, du \) by parts, using \( g(1) = 0 \), and \( \lim_{u \to 0} u g(u) = 0 \) it follows from (34) that

\[
c_* \leq \frac{p m^{p-1}}{(p-1)^{p-1}p} \sup_u \left( \frac{F^{p-1}}{u} \right)^{1/p}.
\]

Replacing the expression for \( F(u) \) in terms of \( f(u) \) we finally obtain the upper bound

\[
c_* \leq p \left( \frac{m}{p-1} \right)^{(p-1)/p} \sup_u \left( \frac{f(u)}{u} \right)^{(p-1)} \right]^{1/p}.
\]  \quad (35)

with \( \gamma = m(p - 1) - 1 \) as defined before. When \( \gamma = 0 \) the expression above reduces to

\[
c_*|_{\gamma=0} \leq p(m^2)^{1/(m+1)} \sup_u \left( \frac{f(u)}{u} \right)^{1/(m+1)}.
\]  \quad (36)

In particular, when \( m = 1 \) (i.e., \( p = 2 \) since \( \gamma = 0 \)), (36) is the classical upper bound of Aronson and Weinberger [1]. Notice that for the reaction profiles considered here (i.e., \( f(u) \) positive and concave in \([0, 1]\), \( f(0) = f(1) = 0 \) and \( f \in C^1[0, 1] \), we clearly have that \( \sup_{u \in [0,1]} f(u)/u = f'(0) \), and in fact we have equality in (36).

4 Integral lower bound: a Zeldovich–Frank–Kamenetskii type bound

From the variational characterization lower bounds can be constructed choosing specific values for the trial function \( g(u) \). In this section we construct a lower bound which involves the integrals of the reaction term as the Zeldovich–Frank–Kamenetskii classical bound [15,10,9]. Our ZFK type bound is embodied in the following lemma.

**Lemma 1** For any \( m > 0 \), \( p > 1 \), \( \gamma \geq 0 \) and \( f \) satisfying the hypothesis of Theorem 1, we have that

\[
c_* \geq \left( \frac{m p}{p-1} \right)^{(p-1)/p} \left[ \int_0^1 u^{m-1} f(u) \, du \right]^{(p-1)/p}.
\]  \quad (37)
Proof Choose as a trial function of our variational principle (9) the function
\[ g(u) = \left( \int_u^1 F(u') \, du' \right)^{1/p}. \]

It is simple to verify that \( g \in C^1(0, 1), h = -g' > 0, g(1) = 0, \) and \( g(u) \geq 0 \) in \([0, 1]\). Moreover, since \( g \) is decreasing, \( \int_0^1 g(u) \, du \leq g(0) = \left( \int_0^1 F(u') \, du' \right)^{1/p} < \infty \). Hence, \( g \in \mathcal{D} \). A simple calculation yields
\[ h(u) = \frac{F(u)}{p} \left( \int_u^1 F(u') \, du' \right)^{1/p - 1} \]
and \( hg^{p-1} = F(u)/p \). It follows then from (13) that
\[ c_\alpha \geq \frac{pm^{p-1}}{(p-1)(p-1)/p} \left( \frac{1}{p} \right)^{1/p} \int_0^1 \frac{F(u) \, du}{\int_0^1 g(u) \, du}. \]

Now, since \( g(u) \) is a decreasing positive function, \( \int_0^1 g(u) \, du \leq g(0) \). Hence,
\[ c_\alpha \geq \frac{pm^{p-1}}{(p-1)(p-1)/p} \left( \frac{1}{p} \right)^{1/p} \left( \int_0^1 F(u) \, du \right)^{1/(p-1)}. \quad (38) \]

If we express the right side of (38) in terms of the original reaction term \( f(u) \) we get (37) which proves the lemma.

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Appendix

In this appendix we show that
\[ J_{g_\alpha}[f] = J_{g_\alpha}[uf'(0)] \to 0 \quad \text{when} \quad \alpha \to 0, \quad (39) \]
where \( g_\alpha \) is given by (29). We defined
\[ J_{g_\alpha}[f] = \int_0^1 [u^{m-1} h_\alpha^m f(u) g_\alpha(u)]^{1/(m+1)} \, du, \]
so that
\[ |J_{g_\alpha}[f] - J_{g_\alpha}[uf'(0)]| \leq \int_0^1 (u^{m-1} h_\alpha^m g_\alpha(u))^{1/(m+1)} |f(u)^{1/(m+1)} - (uf'(0))^{1/(m+1)}| \, du. \quad (40) \]
Since for $m \geq 0$, $1/(m + 1) \leq 1$, it is not difficult to verify the inequality $\left| a^{1/(m+1)} - b^{1/(m+1)} \right| \leq \left| a - b \right|^{1/(m+1)}$ for all $a \geq 0, b \geq 0, m \geq 0$. In the present case, we have

$$|f(u)^{1/(m+1)} - (uf'(0))^{1/(m+1)}| \leq |f(u) - uf'(0)|^{1/(m+1)}. \quad (41)$$

If $f(u)$ and its derivative are continuous in $[0, 1]$, there exist $d > 0, k > 0$ such that

$$\frac{|f(u) - uf'(0)|}{u} < d \frac{u^k}. \quad (42)$$

Using (41) and (42) in (40), together with the explicit form of $g_\alpha$ we have that

$$|J_{g_\alpha} [f] - J_{g_\alpha} [uf'(0)]| \leq \frac{\alpha}{(1 - \alpha)^{1/(m+1)}} \int_0^1 d^{1/(m+1)} u^{N(\alpha)} \, du,$$

where

$$N(\alpha) = \alpha - 1 + \frac{k}{(m + 1)}.$$

Since $\alpha > 0$ and $k > 0$, $N(\alpha) > -1$ and $u^{N(\alpha)}$ is integrable. Performing the integral we finally find

$$|J_{g_\alpha} [f] - J_{g_\alpha} [uf'(0)]| \leq \frac{m + 1}{\alpha(m + 1) + k} \frac{\alpha d^{1/(m+1)}}{(1 - \alpha)^{1/(m+1)}} \to 0 \quad \text{when} \quad \alpha \to 0.$$

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