Periodic orbits in evolutionary game dynamics: An information-theoretic perspective

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(Dated: October 4, 2022)

Even though existence of non-convergent evolution of the states of populations in ecological and evolutionary contexts is an undeniable fact, insightful game-theoretic interpretations of such outcomes are scarce in the literature of evolutionary game theory. Here we tap into the information-theoretic concept of relative entropy in order to construct a game-theoretic interpretation for periodic orbits in a wide class of evolutionary game dynamics. Effectively, we present a consistent generalization of the evolutionarily stable strategy—the cornerstone of the evolutionary game theory—and aptly term the generalized concept: information stable orbit. The information stable orbit captures the essence of the evolutionarily stable strategy in that it compares the total payoff obtained against an evolving mutant with the total payoff that the mutant gets while playing against itself. Furthermore, we discuss the connection of the information stable orbit with the dynamical stability of the corresponding periodic orbit.

Keywords: Evolutionary games, evolutionarily stable strategy, replicator map, escort-incentive dynamics, periodic orbits, relative entropy.

I. INTRODUCTION

Many ecological systems are known to exhibit cyclic evolution of the abundance of the constituent species—well-studied examples include the snowshoe-hare–lynx system [1], wolf-moose system [2] and lemming populations [3]. The simplest explanation for such a cyclic behaviour is mostly attributed to predation, which may be modelled by the corresponding Lotka–Volterra equation [4]. Since the Lotka–Volterra equation is mathematically mappable [5] onto the replicator equation [6, 7]—a paradigmatic evolutionary dynamic in the theory of evolutionary games [8]—the relevance of oscillatory dynamics in the evolutionary game theory is worth pondering.

The central idea in the evolutionary game theory is that of evolutionarily stable strategy (ESS) [9] which happens to be Nash equilibrium [10, 11] as well. Through the folk theorem (and other similar theorems) [12] of the evolutionary game theory, the convergent outcomes—represented by a fixed point in the phase space—of the replicator dynamics can be interpreted as Nash equilibrium and ESS whenever achievable. These game-theoretic interpretations of the fixed points can be further supplemented with interesting information-theoretic connection: A fixed point that is an ESS is locally asymptotically stable and an appropriately constructed Kullback–Leibler (KL) divergence (also called relative entropy) [13] is a Lyapunov function [14] for the fixed point, that is, the nonnegative function decreases with the passage of time [15] until the dynamics converges onto the fixed point eventually.

Nevertheless, there is a dearth of literature about the game-theoretic and the information-theoretic interpretations for non-convergent oscillatory outcomes. In this paper, we contribute in this area. For pragmatic reasons, it is rather convenient to work with discrete-time replicator equation (replicator map) as even with a population with two strategies (phenotypes), one can witness cyclic behaviour [16, 17]. In fact, using the replicator map, it was shown earlier [18] that a game-theoretic interpretation of periodic orbits is possible—an extension of ESS, called heterogeneity stable orbit (HSO), was proposed. However, unlike the ESS, the concept of the HSO does not consider fitness as the quantity to be optimized during natural selection; rather it concerns itself with the optimization of a weighted fitness—the weight being heterogeneity. The heterogeneity is the probability that two arbitrarily chosen members of the population belong to two different phenotypes. Although a justification for the HSO could be found in its connection with the stable periodic orbits just as ESS is connected with stable fixed points, going beyond the well-accepted notion of fitness to a weighted fitness appears unconventional. Moreover, it is not clear if HSO conforms with the principle of decreasing relative entropy [13].

A natural question emerges at this point: Using the notion of decreasing relative entropy during the evolution of the system, does there exist a natural extension of ESS for the case of periodic orbits such that fitness (sans any weight) is the object of optimization? We answer this question affirmatively in this paper by the introduction of the extension of ESS termed appropriately as Information Stable Orbit (ISO). We also find an encouraging connection of this game-theoretic concept—obtained through information-theoretic ideas—with the dynamical system theoretic concept of stability of periodic orbits. Our investigation is not restricted to only the replicator map, rather we expand our investigations to encompass a much wider class of evolutionary dynamics like incen-
tive [19] and escort-incentive dynamics [20] (which account for many well-studied dynamics like logit [21] and best-reply [22]).

Without further ado, in the immediately succeeding section, Sec. II, we elaborate on all the aforementioned evolutionary dynamics through the discussion of convergent outcomes. Subsequently, in Sec. III we come to the heart of the paper with the systematic exploration of the periodic orbits before further discussing and concluding the core results in Sec. IV.

II. INTERPRETING FIXED POINTS

With a view to keeping the presentation of the main ideas of the paper succinct, we confine ourselves to the simplest yet conceptually non-trivial set-up of two-person two-strategy games. Specifically, we consider an infinite unstructured population of two types whose frequencies change over time under some replication-selection rule. Between the individuals (or the players, in game-theoretic terminology), the interactions are assumed random and of two-player kind. The frequencies of the $i$-th type is denoted by $x_i$ and thus, $x = (x_1, x_2) = (x_1, 1 - x_1) = (x, 1 - x)$, in different equivalent notations, is the frequency (or state) vector of the population. Many kinds of dynamics for the evolution of the frequencies are studied in the literature.

First, let us recall the paradigmatic continuous-time replicator dynamic [6, 7] in the light of information theoretic ideas. Of particular use, is the relative entropy or the Kullback–Leibler (KL) divergence given by $D_{KL}(p||q) = \sum y p(y) \ln[p(y)/q(y)]$ where $y$ belongs to the support of the probability distributions $p$ and $q$. The KL-divergence helps to measure the distance between two probability distributions and herein, we use it as a measure of the distance of an evolving population state from a fixed (equilibrium) population state. If the fixed population state is chosen as the ESS state, the aforementioned KL-divergence decreases with time [15]. Given that the KL-divergence fits the criteria for being a valid Lyapunov function, one may also interpret this as a statement for dynamical stability. There exists, however, an important nuance to this fact: While the local asymptotic stability and evolutionary stability of a fixed point state are synonymous only for two-strategy games, for games with higher number of strategies, ESS implies local asymptotic stability of the corresponding fixed point but the converse is not necessarily true [6].

A. Incentive Dynamics

In this paper, we are motivated to carry out our investigations with discrete-time dynamics because they are a convenient test-bed for investigating non-convergent outcomes like periodic orbits and chaos. In particular, the following is a version of the discrete-time replicator dynamic [16–18, 22–28], henceforth referred to as the replicator map, that exhibits periodic orbits and chaos even in the simple two-strategy case:

$$\Delta x_i^{(k)} = x_i^{(k)} \left[ f_i(x^{(k)}) - \left< f(x^{(k)}) \right> \right].$$

Here $\Delta x_i^{(k)} \equiv x_i^{(k+1)} - x_i^{(k)}$ represents the difference between consecutive population states and $f$ is the fitness function for the types, with the average population fitness being given by $\left< f(x^{(k)}) \right> \equiv \sum_{i} x_i^{(k)} f_i(x^{(k)})$. The superscript denotes the time step. For this discrete map, it can be shown that a locally asymptotically stable fixed point implies ESS for two-strategy games [17, 29], without its converse being necessarily true.

In fact, a unification of quite a few evolutionary dynamics is mathematically possible through the incentive dynamic [19, 20, 30–34] whose discrete-time representation is given by

$$\Delta x_i^{(k)} = x_i^{(k)} (\delta \varphi_i^{(k)}) \equiv \varphi_i(x^{(k)}) - x_i^{(k)} \sum_j \varphi_j(x^{(k)}),$$

where $\varphi(x^{(k)})$ is the incentive vector. The notation $(\delta \varphi_i^{(k)})$ has been introduced for future convenience. Choice of different incentive functions, $\varphi$, correspond to different dynamics (such as replicator, projection [35], logit [21], and best response [36, 37]) as exhibited in Table I. A state of the population is an incentive stable state (ISS) [20], $\hat{x}$, if for any $x$ in the local deleted neighbourhood of $\hat{x}$, we have

$$\sum_i x_i \varphi_i(x) > \sum_i \varphi_i(x).$$

ISS reduces to ESS for the replicator equation.

For the two-strategy incentive map—as a generalization of the corresponding result of replicator map [17]—one can quite readily show using linear stability analysis that a locally asymptotically stable fixed point obeys the ISS condition. Moreover, it is an easy exercise to show that just like for the replicator dynamic, the ISS for the continuous incentive dynamic implies that the time derivative of the KL-divergence, $D_{KL}(\hat{x}||x)$, is negative. In what follows, we observe that negative discrete-time derivative of the KL-divergence, defined as

| Evolutionary Dynamic | Incentive ($\varphi_i(x)$) |
|----------------------|--------------------------|
| Projection           | $f_i(x)$                 |
| Replicator           | $x_i f_i(x) \exp(\beta f_i(x))$ $^a$ |
| Logit                | $\sum_i \exp(\beta f_i(x))$ $^a$ |
| Best Reply           | $x_i BR_i(x)^b$          |

$^a$ Here, $\beta$ is called the rationality factor, see [21].
$^b$ BR($x$) is the best response function, a detailed discussion is given in [22].

TABLE I. Incentive functions for various evolutionary dynamics.
where \( \Delta \text{D}_{\text{KL}}(\hat{x} | x^{(k)}) \equiv \text{D}_{\text{KL}}(\hat{x} | x^{(k+1)}) - \text{D}_{\text{KL}}(\hat{x} | x^{(k)}) \), implies the ISS state:

\[
\Delta \text{D}_{\text{KL}}(\hat{x} | x^{(k)}) = - \sum_{i=1}^{2} \hat{x}_i \ln \frac{x_i^{(k+1)}}{x_i^{(k)}} - 2 \sum_{i=1}^{2} \hat{x}_i [1 + (\delta_{\phi})_i^{(k)}] \geq - \ln \left[ 1 + \sum_{i=1}^{2} \hat{x}_i (\delta_{\phi})_i^{(k)} \right].
\]  

(4a)

The last inequality is obtained using Jensen’s inequality [13, 38] for convex functions. Now, for decreasing KL-divergence, we should have \( \Delta \text{D}_{\text{KL}}(\hat{x} | x^{(k)}) < 0 \), or, \( \sum_{i=1}^{2} \hat{x}_i (\delta_{\phi})_i^{(k)} > 0 \). This implies, \( \sum_{i=1}^{2} \varphi_i(x^{(k)}) - \sum_{i=1}^{2} \hat{x}_i \varphi_i(x^{(k)}) < 0 \) for all \( x^{(k)} \) in the deleted neighbourhood of \( \hat{x} \), which is the ISS condition. Note that if one could interpret the KL-divergence as the discrete-time Lyapunov function [39, 40] for this dynamic, this result shows that the stable fixed point implies ISS for two-strategy games, which is in line with the result known using linear stability analysis.

### B. Escort-Incentive Dynamics

There is yet another way the replicator dynamic can be generalized. Specifically, consider the discrete \( q \)-deformed replicator dynamic [41] given by

\[
\Delta x_i^{(k)} = (x_i^{(k)})^q (f_i(x^{(k)}) - \langle f(x^{(k)}) \rangle_q)
\]  

(5)

where \( q \) is a positive real number and \( \langle f(x^{(k)}) \rangle_q := \sum_j x_j^{(k)} f_j(x^{(k)}) / \sum_j x_j^{(k)} \) is the \( q \)-generalized mean. Clearly, setting \( q \rightarrow 1 \) reduces this dynamic to the standard replicator dynamics. Eq. (5) is, in turn, a special case of a larger set of dynamics given by the following discrete-time escort-incentive dynamics [20]:

\[
\Delta x_i^{(k)} = \varphi_i(x^{(k)}) - \tilde{\sigma}_i(x^{(k)}) \sum_j \varphi_j(x^{(k)}),
\]  

(6)

where the escort distribution vector, \( \tilde{\sigma}(x^{(k)}) = (\tilde{\sigma}_1(x^{(k)}), \tilde{\sigma}_2(x^{(k)})) \) is defined as \( (\sum_{i=1}^{2} \sigma(x_i))^{-1} (\sigma(x_1), \sigma(x_2)) \). The escort function, \( \sigma \), is nondecreasing and positive on \([0,1]\). Different choices for different escort and incentive functions correspond to different dynamics, such as replicator, \( q \)-deformed replicator, projection, and exponential escort (see Table II). It is interesting to note that this generalization of the replicator dynamic is actually motivated through the information-theoretic framework [20].

It is natural that the concepts of ESS and ISS can be further extended for the escort-incentive dynamics: A state of the population is an escort incentive stable state (EISS) [20], \( \hat{x} \), if for any \( x \) in the local deleted neighbourhood of \( \hat{x} \), we have

\[
\sum_i \hat{x}_i \frac{\varphi_i(x)}{\sigma_i(x)} > \sum_i x_i \frac{\varphi_i(x)}{\sigma_i(x)}.
\]  

(7)

Again, for the two-strategy escort-incentive map, using linear stability analysis, it is an easy exercise to show—as a generalization of the corresponding result of replicator map [17]—that a locally asymptotically stable fixed point of the two-strategy escort-incentive map is an EISS.

### Table II. Escort and incentive function combinations for various evolutionary dynamics.

| Evolutionary dynamic | Incentive \((\varphi_i(x))\) | Escort \((\sigma_i(x))\) |
|----------------------|-------------------------------|--------------------------|
| Replicator           | \( x_i f_i(x) \)               | \( x_i \)                |
| Replicator with select | \( \beta x_i f_i(x) \)         | \( \beta x_i \)          |
| q-deformed replicator | \( x_i^q f_i(x) \)             | \( x_i^q \)             |
| Exponential escort   | \( e^{x_i} f_i(x) \)           | \( e^x \)               |

\( a \) Here, \( \beta \) is the intensity of selection in the dynamic.

\( b \) Here, \( q = 1 \) will give the replicator dynamic.

It stands proven in the literature [41] that the escort divergence is the Lyapunov function for the continuous escort-replicator equation (which is an alternate escort distribution motivated evolutionary dynamic). We recall that the escort divergence, \( D_\sigma(x | x') \)—a generalization of the KL-divergence—is given by \( D_\sigma(x | x') = \sum_i x_i (\log_\sigma(v) - \log_\sigma(x_i)) dv \), where \( \log_\sigma(x) = \int_x^{x'} (\log_\sigma(v) - \log_\sigma(x_i)) dv \), is called the escort logarithm. Motivated by this result, we now check what negative discrete-time derivative of the escort divergence, defined as \( \Delta D_\sigma(x | x^{(k)}) = D_\sigma(x | x^{(k+1)}) - D_\sigma(x | x^{(k)}) \), implies and how it is connected with a fixed point (\( \hat{x} \)) that is an EISS.

To this end, we need the following two inequalities obtained via usual bounds on Riemann integrals:

\[
\int_a^b \frac{1}{\sigma(u)} du \geq \frac{b - a}{\sigma(b)},
\]  

(8a)

\[
\int_a^b \log_\sigma(v) dv \geq (b - a) \log_\sigma(a).
\]  

(8b)
By making use of these, we find that
\[ \Delta D_\sigma(\hat{x}|x^{(k)}) = \sum_{i=1}^{2} \left( x_i - x_i^{(k+1)} \right) \log_\sigma(x_i^{(k+1)}) \]
\[ + \left( x_i - x_i^{(k)} \right) \log_\sigma(x_i^{(k)}) + \int_{x_i^{(k+1)}}^{x_i^{(k)}} \log_\sigma(v) dv \] \tag{9a}
\[ \geq \sum_{i=1}^{2} \left[ \left( x_i - x_i^{(k)} \right) \log_\sigma(x_i^{(k)}) - \log_\sigma(x_i^{(k+1)}) \right] \] \tag{9b}
\[ = \sum_{i=1}^{2} \left[ \left( x_i - x_i^{(k)} \right) \int_{x_i^{(k+1)}}^{x_i^{(k)}} \frac{1}{\sigma(u)} du \right] \] \tag{9c}
\[ \geq \sum_{i=1}^{2} \left[ \left( x_i - x_i^{(k)} \right) \frac{x_i^{(k)} - x_i^{(k+1)}}{\sigma(x_i^{(k)})} \right] + \left( x_2 - x_2^{(k)} \right) \left( x_2^{(k)} - x_2^{(k+1)} \right) \frac{1}{\sigma(x_2^{(k+1)})} \] \tag{9d}
\[ = - \sum_{i=1}^{2} \frac{\hat{x}_i \phi_i(x^{(k)})}{\sigma(x_i^{(k)})} + \sum_{i=1}^{2} \frac{\hat{x}_i \phi_i(x^{(k)})}{\sigma(x_i^{(k)})} - C_\sigma(\hat{x}, k). \] \tag{9e}

Here, \( C_\sigma(\hat{x}, k) \equiv \frac{\hat{x}_1 - x_1^{(k)}}{\sigma(x_1^{(k)})} \frac{x_1^{(k)} - x_1^{(k+1)}}{\sigma(x_1^{(k+1)})} - \frac{x_2^{(k)} - x_2^{(k+1)}}{\sigma(x_2^{(k)})} \]. Using the second inequality (Eq. (8b)) in Eq. (9a) yields Eq. (9b) and using the first inequality (Eq. (8a)) in Eq. (9c) yields Eq. (9d). We note that Eq. (9c), which is sum of two integrals, involve two coefficients of the integrals and those coefficients are related through the relation, \( \hat{x}_2 - x_2^{(k)} = - (\hat{x}_1 - x_1^{(k)}) \). Hence, without any loss of generality, one can fix \( \hat{x}_1 - x_1^{(k)} > 0 \) and consequently, Eq. (9e) is implied by Eq. (9d). Now, since the escort function in nondecreasing, we have \( \frac{x_2^{(k)} - x_2^{(k+1)}}{\sigma(x_2^{(k)})} \geq 0 \). This implies that the \( C_\sigma(\hat{x}, k) \) is nonnegative; and thus, for the stability of fixed point, requiring \( \Delta D_\sigma(\hat{x}|x^{(k)}) < 0 \), gives us the condition: \( \sum_{i=1}^{2} \frac{\hat{x}_i \phi_i(x^{(k)})}{\sigma(x_i^{(k)})} - \sum_{i=1}^{2} \frac{x_i \phi_i(x^{(k)})}{\sigma(x_i^{(k)})} + C_\sigma(\hat{x}, k) \) which is automatically satisfied by the EISS condition, Eq. (7).

In summary, we have understood how the decrease in an information-theoretic divergence, namely, the escort divergence (of which KL-divergence is a special case) correspond to the fixed points that are ESS, ISS, and EISS of wide class of discrete-time evolutionary dynamics.

### III. INTERPRETING PERIODIC ORBITS

Non-convergent outcomes, like periodic orbits, are quite common in evolutionary dynamics. In the discrete-time dynamics discussed in the preceding section, such outcomes [16–18, 42] are possible even with only two-strategy games, thus rendering the investigations about them analytical tractable. If we represent a finite sequence of \( m \) states representing iterates of a map by \( \{x^{(k)}\}_{k=m}^{k=m} \) (where \( k \) denotes the time step), then a periodic orbit of orbit \( m \) (that is, an \( m \)-period orbit) can be denoted by an infinite sequence \( \{x^{(k)}\}_{k=1}^{\infty} \) such that \( x^{(k)} = x^{(k+m)} \), for any \( k \geq 1 \). Since a periodic orbit only has \( m \) unique states, it may be compactly denoted by a finite sequence given by \( \{x^{(k)}\}_{k=m}^{k=m} \). A time-ordered collection of \( m \)-period points. Now we intend to show that the information-theoretic concept of KL-divergence (or more generally the escort divergence) presents a unique viewpoint that helps to construct a game-theoretic interpretation for the non-convergent dynamical equilibrium, specifically periodic orbit.

It is imperative to bring to the readers’ attention that an exercise of imparting game-theoretic meaning to the periodic orbits was performed [18]. We first critically revisit that work and scrutinize the concepts related to the periodic orbits within the game-theoretic framework. We confine ourselves only to the standard replicator map in this discussion.

### A. Scrutiny of Heterogeneity Stable Orbit

Two definitions were introduced [18]: Firstly, a sequence of states \( \{\hat{x}^{(k)} : x^{(k)} \in (0,1), k = 1, 2, \ldots, m\} \) where \( x^{(i)} \neq \hat{x}^{(j)} \) for \( i \neq j \), is a heterogeneity orbit (HO) of order \( m \) if \( \forall j \in \{1, 2, \ldots, m\} \).

\[ \sum_{k=1}^{m} H_{\hat{x}^{(k)}} \left[ x^{(j)} \cdot f(x^{(k)}) \right] = \sum_{k=1}^{m} H_{\hat{x}^{(k)}} \left[ x \cdot f(\hat{x}^{(k)}) \right]. \] \tag{10}

\( \forall x \in (0, 1) \). Here, heterogeneity factor, \( H_x \equiv 2x(1-x) \), is the probability that two arbitrarily chosen members of the population belong to two different types. The condition of HO is equivalent to the condition of periodicity given by \( \sum_{j=k}^{k+m-1} H_{\hat{x}^{(k)}} \left[ f_1(\hat{x}^{(j)}) - f_2(\hat{x}^{(j)}) \right] = 0 \), where \( \hat{x}^{(k)} \) with \( 1 \leq k \leq m \), is any one of the \( m \) states of an \( m \)-periodic orbit. For \( m = 1 \), it reduces to the condition of the Nash equilibrium—\( x \cdot f(\hat{x}) = x \cdot f(\hat{x}) \) for any \( x \) in the interior of the simplex. Secondly, heterogeneity stable orbit (HSO) of order \( m \) is a sequence of states, \( \{\hat{x}^{(k)} : x^{(k)} \in (0,1), k = 1, 2, \ldots, m; x^{(i)} \neq \hat{x}^{(j)} \forall i \neq j \} \) such that

\[ \sum_{k=1}^{m} H_{x^{(k)}} \left[ x^{(1)} \cdot f(x^{(k)}) \right] > \sum_{k=1}^{m} H_{\hat{x}^{(k)}} \left[ x^{(1)} \cdot f(x^{(k)}) \right]. \] \tag{11}

for any sequence of states \( \{x^{(k)} : x^{(k)} \in (0,1), k = 1, 2, \ldots, m\} \). The HSO reduces to ESS for \( m = 1 \) and extends the concept of ESS to periodic orbits.

Using linear stability, one can show that all locally asymptotically stable periodic orbits obey the HSO criterion, although the converse is not necessarily true [18]. Moreover, HSO implies HO, just as ESS implies Nash equilibrium. To truly interpret the condition game-theoretically, however, a connection to underlying strategy space is necessary; and in this context, it can be
shown that just as the ESS condition is associated with the idea of strong stability [5], HSO may also be associated with the extension of strong stability for periodic orbits [18].

Two criticisms of the definition of the HSO are in order. First, the introduction of the heterogeneity factor is somewhat ad hoc. Second, a close inspection reveals that the HSO condition is a bit odd in the following sense: The total payoff on the right hand side of Eq. (11) is not the sum of the evolving mutant state playing with itself, but instead is the sum of payoffs of the mutant playing against a temporarily fixed mutant state chosen [note the fixed (1) superscript on the right hand side of Eq. (11)]. In this paper, we circumvent the aforementioned two drawbacks by introducing a new generalization of the ESS.

### B. Information Stable Orbit

Let us generalize the concept of ESS to define information stable orbit (ISO) as follows:

**Definition of ISO:** A sequence of $m$ states, $\{\hat{x}(k) : \hat{x}(k) \in (0, 1); k = 1, 2, \cdots , m; \hat{x}(i) \neq \hat{x}(j) \forall i \neq j\}$, is ISO of order $m$ of the replicator map if

$$\sum_{k=1}^{m} \hat{x}(k) \cdot f(\hat{x}(k)) > \sum_{k=1}^{m} x(k) \cdot f(x(k)) , \quad (12)$$

for any sequence of $m$ states $\{x(k) : x(k) \in (0, 1); k = 1, 2, \cdots , m\}$ of the map starting in some infinitesimal deleted neighbourhood of $\hat{x}(1)$.

The definition of the ISO of order $m$ qualifies any given finite sequence of $m$ states of a map, $\{\hat{x}(k)\}_{k=1}^{m}$, by comparing it—using Eq. (12)—with a finite sequence of $m$ mutant states, $\{x(k)\}_{k=1}^{m}$, also governed by the map. In doing so, only the first element of the given sequence, $\hat{x}(1)$, is explicitly used in the inequality, with the implicit knowledge that the remaining elements of this sequence are specified by the map exactly. The sequence for mutant states starts from the deleted neighbourhood of the first element of the ISO, and the remaining elements of this sequence are also determined exactly by the map. This formulation of a stability criterion for periodic orbits is inspired from that of the previously defined HSO [18].

Now, in order to qualify a periodic orbit by ISO, we must ensure that each of the $m$ sequences of length $m$ starting from each of the $m$ distinct states of an $m$-periodic orbit should obey the ISO condition. Here, all the $m$ sequences $\{x(j)\}_{j=k}^{j=k+m-1}$ equivalently denote the same $m$-periodic orbit and corresponding nearby sequences $\{\hat{x}(j)\}_{j=k}^{j=k+m-1}$ start in the infinitesimal deleted neighbourhood of $\hat{x}(k)$. Hence, we can recast the ISO condition (Eq. (12)) for periodic orbits as

$$\sum_{j=k}^{k+m-1} \hat{x}(k) \cdot f(\hat{x}(j)) > \sum_{j=k}^{k+m-1} x(j) \cdot f(x(j)) , \quad (13)$$

$\forall k \in \{1, 2, \cdots , m\}$. Note that the ISO reduces to the ESS for $m = 1$, which is consistent with the fact that a fixed point can be understood as a 1-period orbit.

Since any state of an $m$-period orbit of a nonlinear map is a fixed point of the $m$-th iterate of the map, it is very natural to ask if the condition of decrease in the KL-divergence leads to an extension of ESS, either HSO or ISO; after all, we have seen earlier that $\Delta D_{KL}(\hat{x}|x(k)) < 0$ [see Eqs. (4) a-c] leads to ESS. To this end, we introduce the notation, $\Delta_m D_{KL}(\hat{x}(k))|x(k)) = D_{KL}(\hat{x}(k)|x(k+m)) - D_{KL}(\hat{x}(k)|x(k))$ which measures the change in the KL-divergence over $m$ time steps. Now,

$$\Delta_m D_{KL}(\hat{x}(k)|x(k)) = -\sum_{i=1}^{2} \hat{x}(k) \ln \frac{\hat{x}(k+m)}{\hat{x}(k)} , \quad (14a)$$

$$= -m \sum_{i=1}^{2} \hat{x}(k) \sum_{j=k}^{k+m-1} \frac{1}{m} \ln \left[ 1 + \left( \frac{\delta_x f_i(j) }{f_i(j)} \right) \right] , \quad (14b)$$

$$\geq -m \sum_{i=1}^{2} \hat{x}(k) \ln \left[ 1 + \sum_{j=k}^{k+m-1} \frac{m}{1} \left( \frac{\delta_x f_i(j) }{f_i(j)} \right) \right] , \quad (14c)$$

$$\geq -m \ln \left[ 1 + \sum_{i=1}^{2} \hat{x}(k) \sum_{j=k}^{k+m-1} \frac{m}{1} \left( \frac{\delta_x f_i(j) }{f_i(j)} \right) \right] , \quad (14d)$$

where the last two inequalities are obtained using Jensen’s inequality. One finds that demanding $\Delta_m D_{KL}(\hat{x}(k)|x(k)) < 0$ implies the condition of ISO as given in Eq. (12) and not the condition of HSO. Thus, this information-theoretic fact justifies the aptness of the name ISO. One should observe the unidirectionality in the arguments presented in Eqs (14)a-d. In particular, just as there exist some unstable fixed points that are ESS in the discrete replicator map, we may also find some unstable periodic orbits that obey the ISO condition. This unidirectionality is also true for the HSO condition and appears to be an artefact of the temporal discretization of the evolutionary dynamics.

As can be guessed, the generalization of the ISO for the incentive dynamics is straightforward:

**Definition of ISO for incentive map:** A sequence of $m$ states, $\{\hat{x}(k) : \hat{x}(k) \in (0, 1); k = 1, 2, \cdots , m; \hat{x}(i) \neq \hat{x}(j) \forall i \neq j\}$, is ISO of order $m$ of the incentive map if

$$\sum_{k=1}^{m} \sum_{i=1}^{m} \hat{x}(i) f(\hat{x}(k)) > \sum_{k=1}^{m} \sum_{i=1}^{m} \phi_i(\hat{x}(k)) , \quad (15)$$

for any sequence of states $\{x(k) : x(k) \in (0, 1); k = 1, 2, \cdots , m\}$ of the map starting in some infinitesimal deleted neighbourhood of $\hat{x}(1)$.

We observe that this condition reduces to ISO on setting $\phi(x) = x f(x)$ as expected for the replicator map. Also, $m = 1$ reproduces the ISS condition (Eq. (3)) as required. Just like for the case of the ISO, here one can show that the decreasing KL-divergence constructed using a periodic orbit and its neighbouring orbit implies the
C. Generalization to Escort-Incentive Dynamics

ISO’s further generalization for the case of the escort-incentive dynamics is also possible.

Definition of ISO for escort-incentive map: A sequence of m states, \( \{ \hat{x}^{(k)} : \hat{x}^{(k)} \in (0,1) \}, k=1,2,\ldots,m \), is ISO of order m of the escort-incentive map if

$$\sum_{k=1}^{m} \sum_{i} \hat{x}^{(1)} \frac{\varphi_i(\hat{x}^{(k)})}{\sigma_i(\hat{x}^{(k)})} > \sum_{k=1}^{m} \sum_{i} \hat{x}^{(1)} \frac{\varphi_i(x^{(k)})}{\sigma_i(x^{(k)})}$$

(16)

for any sequence of states \( \{ x^{(k)} : x^{(k)} \in (0,1) \} \) of the map starting in some infinitesimal deleted neighborhood of \( \hat{x}^{(1)} \).

Here, in the context of information theoretic interpretation of this condition that is an m-periodic orbit, the appropriate information-theoretic concept is the escort divergence. Hence, we introduce the notation, \( \Delta_m D_\sigma(\hat{x}^{(k)})|x^{(j)}| = D_\sigma(\hat{x}^{(k)})|x^{(j+1)}| - D_\sigma(\hat{x}^{(k)})|x^{(j)}| \)

which measures the change in the escort divergence over m time steps. Subsequently, notice that

$$\Delta_m D_\sigma(\hat{x}^{(k)})|x^{(j)}|$$

$$= \sum_{k=j}^{k+m-1} \left[ D_\sigma(\hat{x}^{(k)})|x^{(j+1)}| - D_\sigma(\hat{x}^{(k)})|x^{(j)}| \right]$$

(17a)

and hence

$$\sum_{j=k}^{k+m-1} \Delta D_\sigma(\hat{x}^{(k)})|x^{(j)}|$$

(17b)

$$\geq - \sum_{j=k}^{k+m-1} \sum_{i=1}^{2} \hat{x}_i^{(k)} \frac{\varphi_i(x^{(j)})}{\sigma(x^{(j)})} + \sum_{j=k}^{k+m-1} \sum_{i=1}^{2} x_i^{(j)} \frac{\varphi_i(x^{(j)})}{\sigma(x^{(j)})} - \sum_{j=k}^{k+m-1} C_\sigma(\hat{x}^{(k)}, j).$$

(17c)

The last inequality is obtained by using Eq. (9e) and hence \( \sum_{j=k}^{k+m-1} C_\sigma(\hat{x}^{(k)}, j) \) —a sum of nonnegative quantities—must be a nonnegative quantity. This means that if escort divergence is decreasing, that is, \( \Delta_m D_\sigma(\hat{x}^{(k)})|x^{(j)}| < 0 \), then following is implied:

$$\sum_{j=k}^{k+m-1} \sum_{i=1}^{2} \hat{x}_i^{(k)} \frac{\varphi_i(x^{(j)})}{\sigma(x^{(j)})}$$

$$> \sum_{j=k}^{k+m-1} \sum_{i=1}^{2} x_i^{(j)} \frac{\varphi_i(x^{(j)})}{\sigma(x^{(j)})} - \sum_{j=k}^{k+m-1} C_\sigma(\hat{x}^{(k)}, j).$$

(18)

We observe that if a periodic orbit satisfies the definition of ISO for escort-incentive map (Eq. (16)), then it automatically satisfies the immediately preceding inequality (18). Hence, the relation of the ISO for escort-incentive map, that is a periodic orbit, with the decreasing escort divergence is transparent.

D. Numerical Verification

![FIG. 1. Stable periodic orbits are ISO: The figure presents the bifurcation diagram for the replicator map for a two-player two-strategy game. The payoff matrix, \( \Pi \), is chosen such that \( T = 1.5 + S, R = 1 + P = 0 \). Upper inset: Blue curves depict positivity of \( \delta^2 \equiv \delta_{\varphi,\sigma}(\hat{x}^{(k)}) \) for the two states of the stable 2-period orbit points, \( (a,b) \approx (0.73,0.26) \) corresponding to \( S = 4.5 \). Lower inset: Red curves depict positivity of \( \delta^2 \equiv \delta_{\varphi,\sigma}(\hat{x}^{(k)}) \) for the four states of the 4-period orbit \( (c,d,e,f) \approx (0.74,0.63,0.20,0.14) \) existing at \( S = 5.7 \).](image)
as to achieve forward-invariance of the corresponding maps.

For each of the dynamics, we first demonstrate the existence of periodic orbits by generating their bifurcation diagrams, and subsequently choose parameter values to obtain examples of 2-period and 4-period orbits. We then verify the positivity of $\delta^4 \equiv \delta^4_{\phi,\sigma}(\hat{x}^{(k)})$ for each of these examples using small values of $\epsilon$. The stability of the periodic orbit can be further verified using linear stability analysis and by observing the asymptotic behaviour of the corresponding time series. We systematically exhibit examples of stable 2-period orbit ($\{a, b\}$) and 4-period orbit ($\{c, d, e, f\}$) of replicator, logit and $q$-replicator dynamics (see Figs. 1, 2 and 3 respectively) to illustrate that they follow the ISO criterion.

### E. ISO and Dynamic Stability

In the preceding discussion, we have observed numerically that stable periodic orbits correspond to ISO. It is worth recalling [18] that a locally asymptotically stable periodic orbit of (two-player two-strategy) replicator map is an HSO. Does such a fact exist between periodic orbit and ISO as well? We find that the answer is in affirmative but the mathematical proof is elusive for periodic orbit of general period. However, we can prove the following for the two-period orbit:

**Proposition:** A locally asymptotically stable 2-period orbit of a replicator map for two-player two-strategy game is an ISO.

This proposition can be proven as follows: Since we know that a locally asymptotically stable 2-periodic orbit must be HSO [18], we write the HSO condition (Eq. (11)) but in a slight different form given by

$$
\sum_{j=1}^{2} \left[ \hat{x}^{(1)} \cdot f(x^{(j)}) - x^{(j)} \cdot f(\hat{x}^{(1)}) \right] >
$$

$$
\frac{H_{x^{(2)}}}{H_{x^{(1)}}} \left( (x^{(1)} - \hat{x}^{(1)}) \cdot f(x^{(2)}) - (x^{(2)} - \hat{x}^{(2)}) \cdot f(x^{(1)}) \right).
$$

(21)

This way of writing makes the terms, required to define the ISO condition, conveniently appear together in the left hand side; hence, to show that the ISO condition holds, one has to simply show that the right hand side is nonnegative. Now, the right hand side may be recast as

$$
[f_1(x^{(2)}) - f_2(x^{(2)})] \left[ (\hat{x}^{(1)} - x^{(2)}) - \frac{H_{x^{(2)}}}{H_{x^{(1)}}} (\hat{x}^{(1)} - x^{(1)}) \right]
$$

$$
= 2(H_{x^{(2)}})^{-1} \Delta x^{(2)} \left[ (\hat{x}^{(1)} - x^{(2)}) - \frac{H_{x^{(2)}}}{H_{x^{(1)}}} (\hat{x}^{(1)} - x^{(1)}) \right],
$$

(22)

where the meaning of $\Delta$ is as in Eq. (1). We are free to choose small enough neighbourhood of $\hat{x}^{(1)}$ so that $x^{(1)}$ is close enough such that the term containing $\hat{x}^{(1)} - x^{(1)}$
in the right hand side of Eq. (22) can be made arbitrarily small (note that its prefactor composed of heterogeneity factors is finite). Furthermore, for stable periodic orbits and the aforementioned small neighbourhood, $\Delta x^{(2)}$ and $\dot{x}^{(1)} - x^{(2)}$ are of same sign. In conclusion, the right hand side of Eq. (21) is a positive quantity for some small neighbourhood of $\dot{x}^{(1)}$. Therefore, the right hand side of Eq. (21) is greater than zero, and the aforementioned proposition stands proven.

IV. DISCUSSION AND CONCLUSIONS

Our results in this paper lie in the exciting overlapping area of evolutionary game theory, dynamical systems theory, and information theory. Specifically, we have shown that an extension of the idea of ESS is possible for periodic orbits in a large class of time-discrete evolutionary dynamics. We have termed the extended concept as ISO (and its generalizations) because it can be motivated through the information-theoretic idea of decreasing KL-divergence—a rather general principle that extends the idea of entropy maximization [13] in natural world. Thus, the concept of the ISO—in line with similar recent developments [15, 43, 44]—highlights the tightly knit connections between the fields of evolutionary dynamics and information theory. These connections between evolutionary game dynamics and information theory is a promising avenue of research, and we suspect that a lot of information-theoretic concepts (such as Rényi entropy, Fisher information, and information geometry) may be transported to this context to improve our understanding of evolutionary systems. In particular, the possibility of an information-metric under which the time-discrete replicator map is a gradient-like system is worth pondering, and it could lead to tighter conditions of the stability for periodic orbits in game-theoretic terms.

From the game-theoretic viewpoint, the ISO is a more satisfying extension of the ESS as compared to HSO [18] because it does not require the use of the ad hoc heterogeneity factor in its definition. Furthermore, recall that ESS condition is tied to the concept of strong stability [5] and it is not surprising that HSO is connected to the concept of the strongly stable strategy set (SSSS) [18]—the extension of strong stability for periodic orbits. This is an important qualification, for the strong stability provides perhaps the ‘best validation for the concept of evolutionary stability’ [5]. Since locally asymptotically stable 2-period orbit—being HSO—is ISO, when the orbit is SSSS, it is ISO as well. Next we observe how the game-theoretic idea of stability in terms of comparison between payoffs beautifully translates to the stability criterion for periodic orbits as well. An ESS state is a state such that its payoff, when played against any other state in its neighbourhood, is greater than the payoff gained when the other state plays with itself. Analogously, for the ISO, we see that the relevant quantity for comparison is the sum of payoffs for a particular state $\dot{x}$ of the $m$-period orbit. Here we must look at a sequence of $m$ states starting from the neighbourhood of the state $\dot{x}$ and compare the sum of the payoffs—as gained by the state $\dot{x}$ while in play with each of the states in the sequence— with the sum of the payoff that the states in the sequence gain while playing against themselves. In other words, we may say that the $m$-period orbit is evolutionarily stable when a state in the orbit outperforms an evolving mutant state as compared to when the mutant state keeps playing with itself.

From a dynamical systems theory point of view, we have illustrated a connection between the dynamical stability of periodic outcomes and the ISO equilibrium. We conjecture that such results may also be derived for higher period orbits although it appears to be mathematically more involved and require the specific properties of the dynamic in question. Nevertheless, before we end, we would like to delve into some related subtlety that deserves thorough future investigation. We wonder how the unstable periodic orbits connect with the ISO. We know that some unstable orbits are HSO and some are not [18]. Although we are presently unable to provide any rigorous proof but we conjecture that a similar scenario is true for the case of ISO: A trivial example is that of the replicator map for two-payer two-strategy stag-hunt game where there is an unstable interior fixed point (1-period orbit) that is not an ESS (or ISO of order 1) [17] (cf. Appendix A).

To end with an optimistic note, we envisage useful extensions of the concepts developed in this paper for asymmetric games and extensive games, as well as for games in the finite populations. Probably, the idea of ISO can be useful for the time-continuous evolutionary dynamics as well through the construction of Poincaré maps—an avenue worth pursuing in future.

ACKNOWLEDGMENTS

SC acknowledges the support from SERB (DST, govt. of India) through Project No. MTR/2021/000119.

Appendix A: A note on 2-period orbit and ISO

Here we demonstrate a curious result for the analytically tractable case of 2-period orbits in the replicator map. Consider a periodic orbit $\{\dot{x}^{(1)}, \dot{x}^{(2)}\}$ and note that, by definition, $\Delta \dot{x}^{(1)} \Delta \dot{x}^{(2)} < 0$. If we consider the two points of the periodic orbit to be well-separated and a state $x^{(1)}$ in the infinitesimal small neighbourhood of $\dot{x}^{(1)}$, then we expect $\Delta x^{(1)} \Delta x^{(2)} < 0$—even if the periodic orbit is unstable, as long as $x^{(2)}$ (the state to which $x^{(1)}$ is mapped) is in small neighbourhood of $\dot{x}^{(2)}$, far
away from $x^{(1)}$. We should always be able to write
\[ \Delta x^{(1)} \Delta x^{(2)} + \sum_{j=1}^{2} \left( \frac{H_{x^{(j)}}}{H_{x^{(j)}}} \right) \Delta x^{(j)} \left( x^{(1)} - \hat{x}^{(1)} \right) < 0 \quad (A1) \]

because the second term in the left hand side can be made infinitesimally small by taking $x^{(1)} \to \hat{x}^{(1)}$ as the heterogeneity factor is a positive quantity. Rewriting this equation, we arrive at
\[ \sum_{j=1}^{2} \left( H_{x^{(j)}} \right)^{-1} \Delta x^{(j)} \left( x^{(j)} - \hat{x}^{(1)} \right) < 0, \quad (A2) \]

which may be further rewritten using the replicator map to yield
\[ \sum_{j=1}^{2} (x^{(j)} - \hat{x}^{(1)}) \left[ f_1(x^{(j)}) - f_2(x^{(j)}) \right] < 0. \quad (A3) \]

This equation is equivalent to the ISO condition for 2-period orbits. Thus, it appears that a 2-period orbit is ISO irrespective of its stability property. However, the fraction of mutants required to invade ISO has to be comparative larger for the case of stable 2-period orbits. In must be kept in mind that the analytically tractable case of the 2-period orbits of the replicator map appears to be very special in this respect: It remains to be proven that how much of this discussion goes over to the higher period orbits in arbitrary escort-incentive dynamics.

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