PIXTON’S DOUBLE RAMIFICATION CYCLE RELATIONS

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Abstract. We prove a conjecture of Pixton, namely that his proposed formula for the double ramification cycle on $M_{g,n}$ vanishes in codimension beyond $g$. This yields a collection of tautological relations in the Chow ring of $M_{g,n}$. We describe, furthermore, how these relations can be obtained from Pixton’s 3-spin relations via localization on the moduli space of stable maps to an orbifold projective line.

1. Introduction

The double ramification cycle is a class $R_{g,A} \in A^g(M_{g,n})$ associated to any genus $g \geq 0$ and any collection of integers $A = (a_1, \ldots, a_n)$ whose sum is zero. Its restriction to the moduli space $M_{g,n} \subset M_{g,n}$ of smooth curves is the class of the locus of pointed curves $(C; x_1, \ldots, x_n)$ admitting a ramified cover $f : C \to \mathbb{P}^1$, for which the positive $a_i$ describe the ramification profile over 0 and the negative $a_i$ describe the ramification profile over $\infty$. This definition can be extended to all of $M_{g,n}$ via relative Gromov-Witten theory.

The question known as “Eliashberg’s problem” is, vaguely, whether one can give a more explicit description of the double ramification cycle. Toward this end, Faber and Pandharipande [14] proved that $R_{g,A}$ lies in the tautological ring, so Eliashberg’s problem can be refined by asking for a formula in terms of kappa and psi classes and their pushforwards from boundary strata.

In [20], Hain provided such a formula for the restriction of $R_{g,A}$ to the compact-type locus $\mathcal{M}^c_{g,n} \subset M_{g,n}$, which parameterizes curves whose dual graph is a tree. His proof relies on an alternative description of the double ramification cycle in terms of the universal Jacobian. Namely, on a smooth curve $C$, the existence of a ramified cover as prescribed by the definition of $R_{g,A}$ is equivalent to the requirement that

$$O_C(a_1[x_1] + \cdots + a_n[x_n]) \cong O_C.$$ 

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Thus, if
\[
\rho_A : \mathcal{M}_{g,n} \to \mathcal{X}_g
\]
is the map to the universal abelian variety defined by
\[
(C; x_1, \ldots, x_n) \mapsto \mathcal{O}_C(a_1[x_1] + \cdots + a_n[x_n]) \in \text{Jac}_C^0
\]
and \(\mathcal{Z}_g \subset \mathcal{X}_g\) is the zero section, then
\[
(1) \quad R_{g,A}|_{\mathcal{M}_{g,n}} = \rho_A^*[\mathcal{Z}_g].
\]
The map \(\rho_A\) extends without indeterminacy to \(\mathcal{M}_{g,n}^{ct}\), and Marcus and Wise [28], generalizing a previous result of Cavalieri-Marcus-Wise [3] for rational-tails curves, proved that the analogue of (1) still holds on the compact-type locus. On \(\mathcal{X}_g\), there is a theta divisor \(\Theta\) satisfying
\[
\Theta^g = g![\mathcal{Z}_g].
\]
Thus, we have
\[
R_{g,A}|_{\mathcal{M}_{g,n}^{ct}} = \frac{1}{g!}(\rho_A^*\Theta)^g,
\]
and Hain’s formula results from an explicit calculation of \(\rho_A^*\Theta\) in terms of tautological classes.

On the other hand, Grushevsky and Zakharov leveraged this same computation of \(\rho_A^*\Theta\) in a different way. Namely, they used the observation that
\[
\Theta^{g+1} = 0
\]
to derive tautological relations in \(A^d(\mathcal{M}_{g,n}^{ct})\) for any \(d > g\).

In recent work [37] (see also [2]), Pixton defined an extension of Hain’s class to the entire moduli space \(\overline{\mathcal{M}}_{g,n}\). More precisely, he extended the mixed-degree class \(e^\rho_A\Theta\) to a more general formula in terms of tautological classes, denoted \(\Omega_{g,A}\). To construct it, he first defined a family of classes \(\Omega^r_{g,A}\) depending on a positive integer parameter \(r\), which can in some sense be viewed as “mod \(r\)” versions of Hain’s expression for \(R_{g,A}|_{\mathcal{M}_{g,n}^{ct}}\). He then proved that \(\Omega^r_{g,A}\) is polynomial in \(r\) for \(r \gg 0\), and he defined \(\Omega_{g,A}\) as the constant term in this polynomial.

Simultaneously generalizing both Hain’s and Grushevsky-Zakharov’s arguments, Pixton conjectured the following:

**Conjecture 1.1 (Pixton).** Let \([\cdot]_d\) denote the codimension-\(d\) part of a class in \(A^*(\overline{\mathcal{M}}_{g,n})\). Then \(\Omega_{g,A}\) satisfies:

1. \([\Omega_{g,A}]_g = R_{g,A}\);
2. \([\Omega_{g,A}]_d = 0\) for all \(d > g\).
A proof of part (1) has been announced by Janda-Pandharipande-Pixton-Zvonkine [24], using localization on a moduli space of relative stable maps to an orbifold projective line. In particular, since $\Omega_{g,A}$ has an explicit expression in terms of the additive generators of the tautological ring, this yields a solution to Eliashberg’s problem.

The main result of the present paper is a proof of part (2):

**Theorem 1.2.** Let $\Omega_{g,A} \in A^*(\overline{M}_{g,n})$ be the mixed-degree class defined by (6), whose degree-$g$ component is equal to the double ramification cycle. Then the component of $\Omega_{g,A}$ in codimension $d$ vanishes for all $d > g$.

To prove the theorem, we make use of a geometric reformulation of $\Omega_{g,A}$ due to Zvonkine. Namely, we consider a moduli space $\overline{M}_{g,A}$ of pointed stable curves $(C; x_1, \ldots, x_n)$ equipped with a line bundle $L$ satisfying

$$L^{\otimes r} \cong O \left( -\sum_{i=1}^{n} a_i [x_i] \right).$$

There is a map

$$\phi : \overline{M}_{g,A} \to \overline{M}_{g,n}$$

forgetting the line bundle $L$, and if $\pi : C \to \overline{M}_{g,A}$ denotes the universal curve and $L_A$ the universal line bundle, set

$$\tilde{\Omega}^{r}_{g,A} := \frac{1}{r^{2g-1}} \phi_* \left( e^{r^2 c_1 (-R \pi_* L_A)} \right).$$

Like Pixton’s class, $\tilde{\Omega}^{r}_{g,A}$ is also polynomial in $r$ for $r \gg 0$, and the constant term in this polynomial is also equal to $\Omega_{g,A}$.

From here, the idea of the proof of Theorem 1.2 is to replace $A$ by a tuple $A'$ in such a way that $-R \pi_* L_{A'}$ becomes a vector bundle but the constant term in $r$ of (2) remains unchanged. Then, we replace the class $e^{r^2 c_1 (-R \pi_* L_A)}$ with the weighted total Chern class

$$c(r^2)(-R \pi_* L_{A'}) = 1 + r^2 c_1(-R \pi_* L_{A'}) + r^4 c_2(-R \pi_* L_{A'}) + \cdots.$$

Once again, this replacement only affects higher-order terms in $r$; the proof uses the fact that both $\Omega^{r}_{g,A}$ and the modification via (3) form Cohomological Field Theories (CohFTs), and the $R$-matrices can be explicitly calculated by Chiodo’s Grothendieck-Riemann-Roch formula [5]. The rank of $-R \pi_* L_{A'}$ is easy to compute, and for certain choices of $A$, the modification $A'$ can be chosen so that this rank equals precisely $g$. For such $A$, the fact that (3) manifestly vanishes in cohomological degrees greater than the rank proves the theorem. Then, using the fact
that $\Omega_{g,A}$ is polynomial in $A$ (as observed by Pixton [37]), we deduce
the theorem in general.

In fact, Pixton conjectured that the same vanishing result holds for
a more general class. This class can also be described by a Hain-type
formula, as we explain in Section 2.4, or, in the geometric reformulation,
it can be defined as

$$\Omega_{g,A,k} := \frac{1}{r^{2g-1}} \phi^*_s \left( e^{-c_1(-R_{\pi_*}L_A,k)} \right) \bigg|_{r=0},$$

where $L_{A,k}$ is the universal line bundle over the moduli space $\overline{M}_{g,A}$ of
pointed stable curves with a line bundle $L$ satisfying

$$L^{\otimes r} \cong \omega_{log}^k \left( -\sum_{i=1}^n a_i [x_i] \right).$$

(Evaluation at $r = 0$ occurs, as above, after taking $r$ sufficiently large
so that the class is polynomial.) The $k = 1$ case of the above, in
particular, is related to $r$-spin theory. We prove in Theorem 5.3 below
that this more general conjecture is also true, by essentially the exact
same proof as Theorem 1.2.

We remark that the tautological relations coming from vanishing of
the high-degree terms of (3) were previously observed in [8]. As was
explained in that paper, they can alternatively be derived from the
existence of the nonequivariant limit in the equivariant virtual cycle of
$\overline{M}_{g,n}(\mathbb{C}/\mathbb{Z}_r,0)$, a perspective that is useful in what follows.

It has been conjectured that the 3-spin relations constructed in [33]
generate all tautological relations on the moduli space of curves, so one
should expect the double ramification cycle relations to follow from
these. This is indeed the case. To prove it, we study the equivariant
Gromov-Witten theory of a projective line $\mathbb{P}[r,1]$ with a single orbifold
point of isotropy $\mathbb{Z}_r$. The associated CohFT is generically semisimple,
so, as explained in [22], tautological relations can be obtained by apply-
ing Givental-Teleman reconstruction to express the CohFT as a graph
sum and then using the existence of the limit as one moves toward a
nonsemisimple point. The relations thus obtained are equivalent to the
3-spin relations, via rather general machinery of the second author.

On the other hand, the same CohFT can be expressed as a graph
sum in a different way, via localization and Chiodo’s formula. A careful
matching reveals that the two graph sums agree, and the existence
of the nonsemisimple limit in the Givental-Teleman sum implies the
existence of the nonequivariant limit in the localization sum. Thus,
upon restriction to the substack of degree-zero maps to $\mathbb{P}[r,1]$, one
recovers the double ramification cycle relations in the form presented in [8].

1.1. Outline of the paper. We begin, in Section 2, by reviewing the definition of the double ramification cycle and Pixton’s conjectural formula in more detail. In Section 3, we recall Chiodo’s Grothendieck-Riemann-Roch formula for the Chern characters of the direct image of the universal line bundle on moduli spaces of rth roots and use it to make the formula for $\tilde{\Omega}^r_{g,A}$ more explicit. Section 4 reduces the proof of Theorem 1.2 to a comparison of $\tilde{\Omega}^r_{g,A}$ with the weighted total Chern class described in (3), and this comparison is accomplished in Section 5 by describing both classes in terms of the action of an explicit $R$-matrix on a Topological Field Theory, thus completing the proof of the main theorem and its generalization. Finally, in Section 6, we recast the double ramification cycle relations in terms of maps to an orbifold projective line, and use this perspective to show how they can be deduced from the 3-spin relations. Details of the localization on $\mathbb{P}[r,1]$, including a matching of the localization and reconstruction graph sums, are relegated to the appendix.

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2. Preliminaries on the double ramification cycle and Pixton’s conjectures

The exposition that follows is based on notes of Cavalieri [2] and Pixton [36].

2.1. The double ramification cycle. Fix a genus $g \geq 0$ and a collection of integers $A = (a_1, \ldots, a_n)$ whose sum is zero. Define a cycle on $\mathcal{M}_{g,n}$ as the class of the locus of pointed curves $(C; x_1, \ldots, x_n)$ for which there exists a ramified cover $f : C \to \mathbb{P}^1$ satisfying:

- $f^{-1}(0) = \{x_i \mid a_i > 0\},$
• the ramification profile over 0 is the partition \{a_i \mid a_i > 0\},
• \(f^{-1}(\infty) = \{x_i \mid a_i < 0\}\),
• the ramification profile over \(\infty\) is the partition \{|a_i| \mid a_i < 0\}.

We will denote by \(\mu\) the partition consisting of the positive \(a_i\) and by \(\nu\) the partition consisting of the absolute values of the negative \(a_i\); these are partitions of the same size since the sum of all \(a_i\)’s is zero. Further, denote \(n_0 = \#\{a_i = 0\}\).

To extend the class described above to the entire moduli space \(\overline{M}_{g,n}\), we compactify the space of such ramified covers by allowing degenerations of the target \(\mathbb{P}^1\). More specifically, there is a map

\[
\pi : \overline{M}_{g,n,0}(\mathbb{P}^1; \mu, \nu) \sim \to \overline{M}_{g,n}
\]

from the moduli space of rubber relative stable maps to \(\mathbb{P}^1\), and we set

\[
R_{g,A} := \pi_* \left[ \overline{M}_{g,n,0}(\mathbb{P}^1; \mu, \nu) \right]^{\text{vir}} \in A^g(\overline{M}_{g,n}).
\]

See [14] for a further discussion of rubber relative stable maps to the projective line.

This class has an alternative description when restricted to the locus \(\mathcal{M}_{g,n}^{ct} \subset \overline{M}_{g,n}\) consisting of curves of compact type— that is, curves whose dual graph is a tree. As explained in the introduction, the Jacobian \(\text{Jac}_C^0\) of a compact-type curves is a compact abelian variety, and the map

\[
\rho_A : \mathcal{M}_{g,n} \to \mathcal{X}_g
\]

to the universal abelian variety defined by

\[
(C; x_1, \ldots, x_n) \mapsto \mathcal{O}_C(a_1[x_1] + \cdots + a_n[x_n]) \in \text{Jac}_C^0
\]
can be extended to \(\mathcal{M}_{g,n}^{ct}\). It is straightforward to see that, if \(\mathcal{Z}_g \subset \mathcal{X}_g\) denotes the zero section, then the class \(\rho^*_{A}[\mathcal{Z}_g]\) coincides with the double ramification cycle when one restricts to \(\mathcal{M}_{g,n}\). By the results of [3] and [28], this is also true for the extension to \(\mathcal{M}_{g,n}^{ct}\).

On the other hand, there is a theta divisor \(\Theta \in A^1(\mathcal{X}_g^0)\), which restricts in each fiber of the universal family to the prescribed polarization on the corresponding abelian variety, and which is trivial when restricted to the zero section. Using results of Deninger and Murre [11] (see [41] and [19] for further exposition), one can show that this divisor satisfies

\[
\Theta^g = g![\mathcal{Z}_g]
\]

and \(\Theta^{g+1} = 0\).

Hain [20] has computed \(\rho^*_{A}\Theta\) in terms of tautological classes on \(\mathcal{M}_{g,n}^{ct}\), which, via the above observations, implies a formula for the restriction
of the double ramification cycle. The result of his computation is:

\[ R_{g,A}^{ct} = \frac{1}{2^g g!} \left( -\frac{1}{2} \sum_{0 \leq l \leq g} a_l^2 \Delta_{l,l} \right)^g , \]

where

\[ a_I = \sum_{i \in I} a_i \]

and \( \Delta_{l,l} \) is defined as the class of the closure of the locus of curves with an irreducible component of genus \( l \) containing the marked points in \( S \) and an irreducible component of genus \( g - l \) containing the remaining marked points. (In the unstable cases where such curves do not exist, it is defined by convention: \( \Delta_{0,\{i\}} = \Delta_{g,n\setminus\{i\}} = -\psi_i \), and \( \Delta_{0,\emptyset} = 0 \).)

2.2. Pixton’s conjectural formula. The starting point for Pixton’s generalization of Hain’s formula (4) to all of \( \overline{M}_{g,n} \) is the observation that, by packaging the expressions for each power of \( \rho_1^*A_{\Theta} \) into the mixed-degree class \( e^{\rho_1^*A_{\Theta}} \), one obtains a “compact-type Cohomological Field Theory”. That is, if \( V \) is an infinite-dimensional vector space with generators \( e_a \) indexed by integers \( a \), then the association

\[ V \to H^*(\overline{M}_{g,n}^{ct}) \]

\[ e_{a_1} \otimes \cdots \otimes e_{a_n} \mapsto R_{g,A}^{ct} \]

satisfies all of the axioms of a CohFT except for the gluing axiom along nonseparating nodes, which do not occur in the compact-type moduli space. We refer the reader to [26] or [33] for a careful discussion of CohFTs and their axioms.

According to the results of Givental and Teleman [17, 39], a semisimple CohFT can be obtained via the action of an \( R \)-matrix on a Topological Field Theory; the result is an expression for the CohFT as a summation over graphs. A similar procedure works for \( R_{g,A}^{ct} \), and it can be used to write Hain’s formula as a graph sum. Namely, by expanding the exponential and using intersection theory on \( \overline{M}_{g,n} \), one finds that

\[ e^{\rho_1^*A_{\Theta}} = \sum_{\Gamma \in G_{g,n}^{ct}} \frac{\Gamma}{|\text{Aut}(\Gamma)|} \left( \prod_{i=1}^n e^{\frac{1}{2}a_i^2 \psi_i} \prod_{e=(h,h')} \frac{1 - e^{-\frac{1}{2}w(h)w(h')}(\psi_h + \psi_{h'})}{\psi_h + \psi_{h'}} \right) . \]

Here, \( G_{g,n}^{ct} \) denotes the set of decorated dual graphs of curves in \( \overline{M}_{g,n}^{ct} \). The set of edges of a graph \( \Gamma \) is denoted \( E(\Gamma) \), and each edge is written \( e = (h,h') \) for half-edges \( h \) and \( h' \). The classes \( \psi_h \) and \( \psi_{h'} \) are the first
Chern classes of the cotangent line bundles at the two branches of the node corresponding to \( e \), and \( \iota_\Gamma \) is the gluing map

\[
\iota_\Gamma : \prod_{\text{vertices } v} \mathcal{M}_{g(v), \text{val}(v)} \to \mathcal{M}_{g,n},
\]

in which \( g(v) \) is the genus of \( v \) and \( \text{val}(v) \) the valence (that is, the total number of half edges and legs incident to \( v \)).

Associated to each such graph \( \Gamma \) is a unique weight function

\[
w : H(\Gamma) \to \mathbb{Z}
\]
on the set \( H(\Gamma) \) of half-edges and legs, determined by:

(W1) \( w(h_i) = a_i \) for each leg \( h_i \) associated to a marked point \( x_i \);

(W2) if \( e = (h, h') \), then \( w(h) + w(h') = 0 \);

(W3) for each vertex \( v \), the sum of the weights of half-edges and legs incident to \( v \) equals zero.

The fact that these conditions uniquely determine \( w \) is a consequence of the fact that \( \Gamma \) is a tree.

Now, if one attempts to naively generalize the above formula to the full moduli space by allowing \( \Gamma \) to be any dual graph for a curve in \( \mathcal{M}_{g,n} \), then there will no longer be a unique choice of weight function \( w \) satisfying (W1) – (W3). Indeed, any loop in the dual graph permits infinitely many choices of weights, so the sum of the expressions in (5) over all possible weight functions will not converge.

To avoid such infinite sums, Pixton introduces an additional parameter \( r \) and restricts to weight functions

\[
w : H(\Gamma) \to \{0, 1, \ldots, r - 1\}
\]
satisfying the following three conditions:

(R1) \( w(h_i) \equiv a_i \mod r \) for each half-edge \( h_i \) associated to a marked point \( x_i \);

(R2) if \( e = (h, h') \), then \( w(h) + w(h') \equiv 0 \mod r \);

(R3) for each vertex \( v \), the sum of the weights of half-edges incident to \( v \) is zero modulo \( r \).

There are clearly only finitely many such weight functions associated to any dual graph \( \Gamma \). Set \( \Omega^r_{g,A} \) to be the class

\[
\sum_{\Gamma, w} \frac{1}{\text{Aut}(\Gamma)} \frac{1}{r^{h_1(\Gamma)}} \iota_\Gamma^* \left( \prod_{i=1}^n e^{\frac{1}{2}g_i^2 \psi_i} \prod_{e=(h, h')} \frac{1 - e^{-\frac{1}{2}w(h)w(h')(\psi_h + \psi_{h'})}}{\psi_h + \psi_{h'}} \right),
\]

where \( \Gamma \) ranges over all dual graphs of curves in \( \mathcal{M}_{g,n} \), and \( w \) ranges over weight functions satisfying (R1) – (R3).
As observed by Pixton, the class $\Omega_{g,A}^r$ satisfies a number of polynomiality properties:

**Lemma 2.1** (Pixton [37]). *For fixed $g$ and $A$, the class $\Omega_{g,A}^r$ is polynomial in $r$ for $r \gg 0$. Moreover, the constant term in this polynomial is itself polynomial in the arguments $A$.*

More generally, let $\Gamma$ be a dual graph with half-edges $h_1, \ldots, h_N$ and let $W$ be a polynomial in $N$ variables. Then the sum

$$\sum_w W(w(h_1), \ldots, w(h_N)),$$

where $w$ ranges over weight functions satisfying $(R1) – (R3)$, is a polynomial in $r$ for $r \gg 0$. This polynomial is divisible by $r^{h_1(\Gamma)}$ and its lowest degree term depends on $a_1, \ldots, a_n$ polynomially.

Given Lemma 2.1, Pixton’s conjectural formula for the double ramification cycle can now be defined:

$$\Omega_{g,A} := \Omega_{g,A}^r \bigg|_{r=0}. \tag{6}$$

(Throughout the paper, evaluation at $r = 0$ should always be understood as occurring after choosing $r$ large enough so that the class in question is polynomial.)

**2.3. Geometric reformulation.** A different perspective on $\Omega_{g,A}$, first suggested by Zvonkine, will be more useful for our proof of Theorem 1.2. Let $\overline{M}_{g,A}$ be the moduli space\(^1\) parameterizing pointed stable curves $(C; x_1, \ldots, x_n)$ equipped with a line bundle $L$ satisfying

$$L^\otimes r \cong O \left( -\sum_{i=1}^n a_i [x_i] \right). \tag{7}$$

There is a map

$$\phi : \overline{M}_{g,A} \to \overline{M}_{g,n}$$

forgetting the line bundle $L$ and the orbifold structure; this map has degree $r^{2g-1}$, as explained, for example, in [4]. If $\pi : C_A \to \overline{M}_{g,A}$ denotes the universal curve and $L_A$ denotes the universal line bundle on $C_A$, then the class

$$\tilde{\Omega}_{g,A}^r := \frac{1}{r^{2g-1}} \phi_* (e^{-r^2 \mathcal{C}_1 (R \pi, L_A)}) \tag{8}$$

\(^1\)Here, a compactification of the moduli space of such objects on smooth curves must be chosen. There are several ways to compactify, as summarized in Section 1.1.2 of [38]; for our purposes, we will allow orbifold structure at the nodes of $C$ and require only that $L$ is an orbifold line bundle.
is also polynomial in $r$ for $r \gg 0$, by Lemma 2.1, and

$$\Omega_{g,A} = \tilde{\Omega}^r_{g,A} \bigg|_{r=0}.$$

The fact that this definition of $\Omega_{g,A}$ agrees with the previous one can be proved by noting that $\tilde{\Omega}^r_{g,A}$ forms a semisimple CohFT on a vector space $V = \mathbb{C}\{e_0, e_1, \ldots, e_{r-1}\}$, expressing it as a dual graph sum using the Givental-Teleman reconstruction of semisimple CohFTs, and comparing the resulting dual graph sums using Lemma 2.1. We will return to this argument in Lemma 5.2 below.

### 2.4. Generalization to powers of the log canonical.

Both of these definitions of $\Omega_{g,A}$ are readily generalized to allow for powers of the log canonical. To do so, fix an integer $k$ and assume that $A = (a_1, \ldots, a_n)$ satisfies

$$\sum_{i=1}^n a_i = k(2g - 2 + n).$$

Let $\overline{M}^{k/r}_{g,A}$ be the moduli space parameterizing pointed stable curves $(C; x_1, \ldots, x_n)$ equipped with a line bundle $L$ satisfying

$$L^\otimes r \simeq \omega^\otimes \left( -\sum_{i=1}^n a_i[x_i] \right).$$

As above, there is a degree-$r^{2g-1}$ map

$$\phi : \overline{M}^{k/r}_{g,A} \to \overline{M}_{g,n}$$

forgetting $L$ and the orbifold structure on the curve. Set

$$\tilde{\Omega}^r_{g,A,k} \coloneqq \frac{1}{r^{2g-1}} \phi_* \left( e^{-r^2 c_1(R\pi_* L_{A,k})} \right),$$

where $\pi : C_{A,k} \to \overline{M}^{k/r}_{g,A}$ is the universal curve and $L_{A,k}$ the universal line bundle.

A generalization of Pixton’s class can be defined by

$$\Omega_{g,A,k} = \tilde{\Omega}^r_{g,A,k} \bigg|_{r=0},$$

in the language of Section 2.3. Alternatively, in Pixton’s original formulation, the generalized class is defined by replacing condition (R3) above by

$$(R3')$$ for each vertex $v$, the sum of the weights of half-edges incident to $v$ is $k(2g(v) - 2 + \text{val}(v))$ modulo $r$. 
and setting $\Omega^r_{g,A,k}$ to be the class

$$\sum_{\Gamma,w} \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{r^{h_1(\Gamma)}} \cdot \iota_{\Gamma*} \left( \prod_v e^{-\frac{1}{2}k^2e_1} \prod_{i=1}^n e^{\frac{1}{2}a_i^2\psi_i} \prod_{e=(h,h')} 1 - e^{-\frac{1}{2}w(h)w(h')(\psi_h + \psi_{h'})} \psi_h + \psi_{h'} \right),$$

where $\Gamma$ ranges over all dual graphs of curves in $\overline{M}_{g,n}$, $v$ ranges over vertices of $\Gamma$, and $w$ ranges over weight functions satisfying (R1), (R2), and (R3'). Pixton has also proved an analogue of Lemma 2.1 for $\Omega^r_{g,A,k}$:

**Lemma 2.2** (Pixton [37]). For fixed $g$ and $A$, the class $\Omega^r_{g,A,k}$ is polynomial in $r$ for $r \gg 0$. Moreover, the constant term in this polynomial is itself polynomial in $k$ and the arguments $A$.

More generally, let $\Gamma$ be a dual graph with half-edges $h_1, \ldots, h_N$ and let $W$ be a polynomial in $N$ variables. Then the sum

$$\sum_w W(w(h_1), \ldots, w(h_N)),$$

where $w$ ranges over weight functions satisfying (R1), (R2), and (R3'), is a polynomial in $r$ for $r \gg 0$. This polynomial is divisible by $r^{h_1(\Gamma)}$ and its lowest degree term depends on $k$ and $a_1, \ldots, a_n$ polynomially.

We can therefore define $\Omega_{g,A,k}$ as the constant term of the polynomial in $r$ corresponding to $\Omega^r_{g,A,k}$. When $k = 0$, we recover the previous definitions of $\Omega_{g,A}$. Until otherwise stated, we will always assume that $k = 0$ in what follows.

### 3. Chiodo’s Grothendieck–Riemann–Roch formula

In this section, we recall Chiodo’s formula for the Chern characters of the direct image $R\pi_* \mathcal{L}_A$, which, in particular, can be used to write (8) explicitly in terms of tautological classes when $r$ is sufficiently large.

Fix a tuple of integers $A = (a_1, \ldots, a_n)$. In fact, one need not assume that the sum of the $a_i$ is zero, as was the case above, but only that

$$\sum_{i=1}^n a_i \equiv 0 \mod r;$$

this more general version will be important later. Let $\pi$ and $\mathcal{L}_A$ be as above. Then Chiodo’s formula states:

$$\text{ch}_d(R\pi_* \mathcal{L}_A) = \frac{B_{d+1}(0)}{(d+1)!} \kappa_d - \sum_{i=1}^n \frac{B_{d+1}(\frac{a_i}{r})}{(d+1)!} \psi_i^d +$$

...
\[ \frac{r}{2} \sum_{0 \leq i \leq g} B_{d+1} \left( \frac{g-l}{r} \right) \frac{1}{(d+1)!} i_{(l,I)}(\gamma_{d-1}) + \frac{r-1}{2} \sum_{q=0}^{r-1} B_{d+1} \left( \frac{q}{r} \right) \frac{1}{(d+1)!} j_{(\text{irr},q)}(\gamma_{d-1}), \]

using the presentation given in Corollary 3.1.8 of [5].

Let us summarize the notation appearing in this formula. First, \(B_{d+1}(x)\) are the Bernoulli polynomials, defined by the generating function
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\]
The \(\kappa\) and \(\psi\) classes are defined as usual, using the cotangent line to the coarse underlying curve.

Let \(Z_{(l,I)}\) be the substack of \(\mathcal{C}_A\) consisting of nodes separating the curve \(C\) into a component of genus \(l\) containing the marked points in \(I\) and a component of genus \(g - l\) containing the other marked points, subject to the requirement that stable curves of this type exist. Let \(Z'_{(l,I)}\) be the two-fold cover of \(Z_{(l,I)}\) given by a choice of branch at each such node. Then
\[
i_{(l,I)} : Z'_{(l,I)} \to \overline{\mathcal{M}}_{g,A}
\]
is the composition of this two-fold cover with the inclusion into \(\mathcal{C}_A\) and projection. The index \(q_{l,I} \in \{0, 1, \ldots, r - 1\}\) is the multiplicity of \(L\) at the chosen branch, which is defined by
\[
q_{l,I} + \sum_{i \in I} a_i \equiv 0 \mod r.
\]
If \(\psi\) is the first Chern class of the line bundle over \(Z'_{(l,I)}\) whose fiber is the cotangent line to the coarse curve at the chosen branch of the node, and \(\hat{\psi}\) is the first Chern class of the bundle whose fiber is the cotangent line to the coarse curve at the opposite branch, then \(\gamma_d\) is defined by
\[
\gamma_d = \frac{\psi^{d+1} + (-1)^d \hat{\psi}^{d+1}}{\psi + \hat{\psi}} = \sum_{i+j=d} (-\psi)^i \hat{\psi}^j.
\]
Finally, let \(Z'_{(\text{irr},q)}\) be given by nonseparating nodes in \(\mathcal{C}_A\) together with a choice of branch, such that the multiplicity of the line bundle \(L\) at the chosen branch is equal to \(q\). We have morphisms
\[
j_{(\text{irr},q)} : Z'_{(\text{irr},q)} \to \overline{\mathcal{M}}_{g,A}
\]
given, as before, by the two-fold cover, inclusion into the universal curve, and projection. The class \(\gamma_d\) is again defined by (10).
Fix a collection of integers $A = (a_1, \ldots, a_n)$ whose sum is zero. Suppose that $n > 0$ and exactly one $a_i$ is negative; without loss of generality, we may assume that $a_1 < 0$ and $a_i \geq 0$ for all $i \geq 2$. Choose any $r > \max\{|a_i|\}$, and set

$$A' = (a'_1, \ldots, a'_n) = (a_1 + r, a_2, \ldots, a_n),$$

which is now a collection whose sum is $r$ and for which every element is nonnegative.

The definitions of the moduli space $\overline{M}_{g,A}$ and the class $\tilde{\Omega}^r_{g,A}$ extend verbatim to tuples of integers whose sum is not necessarily zero but merely zero modulo $r$. In particular, $\tilde{\Omega}^r_{g,A'}$ is defined, and in fact, its constant term in $r$ is the same as that of $\tilde{\Omega}^r_{g,A}$:

**Lemma 4.1.** If $A$ and $A'$ are as above, then

$$\tilde{\Omega}^r_{g,A} \bigg|_{r=0} = \tilde{\Omega}^r_{g,A'} \bigg|_{r=0}.$$

**Proof.** Via Chiodo’s formula, $\tilde{\Omega}^r_{g,A}$ can be written as

$$\frac{1}{r^{2g-1}} \Phi \left( \exp \left( -r^2 \frac{B_2(0)}{2} \kappa_1 + r^2 \sum_{i=1}^{n} \frac{B_2(\frac{a_i}{r})}{2} \psi_i - r^3 \sum_{\Gamma} \frac{B_2(\frac{q\gamma}{r})}{2} [\Gamma] \right) \right),$$

where the sum is over one-noded graphs $\Gamma$ decorated with a multiplicity $q_\gamma$ at the node, and $[\Gamma]$ is the corresponding boundary divisor. (Note that since $B_2(x) = B_2(1-x)$, we need not distinguish between the two choices of branch.)

Because $B_2$ is a degree-two polynomial, the replacement $A \mapsto A'$ only affects the higher-order terms in $r$ in the argument of $\Phi$. Some care is required to ensure that the same is true after applying $\Phi$, since the degree of $\Phi$ on a codimension-$d$ boundary stratum is in general equal to $r^{2g-1-d}$ due to the presence of “ghost” automorphisms. This is indeed the case, though, because the replacement of $A$ by $A'$ does not change the boundary term. \qed

We have thus re-expressed Pixton’s conjectural formula (for $A$ satisfying the above conditions) as

$$\Omega_{g,A} = \tilde{\Omega}^r_{g,A'} \bigg|_{r=0}.$$

The advantage of having replaced $A$ by $A'$ is that $R^0 \pi_* \mathcal{L}_{A'}$ vanishes. This was proved in Section 2.3 of [8]; on smooth curves, it follows immediately from the fact that an $r$th root of $\mathcal{O}(-\sum a'_i [x_i])$ has negative degree, while in general, one proves that a section must be identically
zero by induction on the irreducible components, beginning with one on which the degree is negative. Since $R^0\pi_*L_{A'}$ vanishes, it follows that $-R\pi_*L_{A'}$ is a vector bundle.

We consider the class

$$c_{(r^2)}(-R\pi_*L_{A'}) := 1 + r^2c_1(-R\pi_*L_{A'}) + r^4c_2(-R\pi_*L_{A'}) + \cdots,$$

a weighted total Chern class. This can be expressed in terms of Chern characters as

$$c_{(r^2)}(-R\pi_*L_{A'}) = \exp\left(\sum_{d \geq 1} (-r^2)^d (d - 1)!\text{ch}_d(R\pi_*L_{A'})\right),$$

and hence, it also admits an explicit description via Chiodo’s formula. Let

$$C_{g,A'}^r := \frac{1}{r^{2g-1}}\phi_*(c_{(r^2)}(-R\pi_*L_{A'})).$$

**Lemma 4.2.** One has

$$\tilde{Q}_{g,A'}^r|_{r=0} = C_{g,A'}^r|_{r=0}.$$ 

A proof of this lemma will imply Theorem 1.2 for the tuples $A$ under consideration, since $C_{g,A'}^r$ clearly vanishes past the rank of the bundle $-R\pi_*L_{A'}$ and a straightforward Riemann–Roch computation shows that

$$\text{rank}(-R\pi_*L_{A'}) = g - 1 + \frac{1}{r} \sum_{i=1}^n a'_i = g.$$ 

The proof of Lemma 4.2 follows the same lines as that of Lemma 4.1. However, to make the argument carefully, one must be vigilant about the boundary terms appearing in both classes. The most streamlined way to handle these is to realize that both $\tilde{Q}_{g,A'}^r$ and $C_{g,A'}^r$ can be encoded as semisimple CohFTs, and hence can be expressed as the result of an $R$-matrix action on a Topological Field Theory (TFT). The two classes are then compared by explicitly computing both the $R$-matrix and the TFT in each case. This is the content of the following section.

**5. The CohFTs and their $R$-matrices**

The results of this section are well-known to experts— in particular, closely-related computations appear in [7], [38], and [6]— but we recall them here for clarity.
5.1. The CohFTs. Recall that a CohFT, as originally defined by Kontsevich and Manin [26], consists of a finite-dimensional \( \mathbb{C} \)-vector space \( V \) equipped with a nondegenerate pairing \( \eta \), a distinguished element \( 1 \in V \), and a system of homomorphisms 
\[
\Omega_{g,n} : V^\otimes n \to H^*(\overline{M}_{g,n}; \mathbb{C})
\]
satisfying a number of compatibility axioms. Any CohFT yields a quantum product \( \ast \) on \( V \), defined by
\[
\eta(v_1 \ast v_2, v_3) = \Omega_{0,3}(v_1 \otimes v_2 \otimes v_3),
\]
and we say that the CohFT is semisimple if \( \ast \) makes \( V \) into a semisimple \( \mathbb{C} \)-algebra— that is, if there exists a basis \( \epsilon_1, \ldots, \epsilon_r \) for \( V \) for which
\[
\epsilon_i \ast \epsilon_j = \delta_{ij} \epsilon_i.
\]
The work of Givental and Teleman [16, 39] implies that a semisimple CohFT can be expressed as
\[
\Omega = R \cdot \omega,
\]
where
\[
R = R(z) \in \text{End}(V)[z]
\]
is an \( R \)-matrix and \( \omega \) is the Topological Field Theory obtained by projecting \( \Omega \) to \( H^0(\overline{M}_{g,n}; \mathbb{C}) \).

For the reader’s convenience, we briefly recall the definition of the action of an \( R \)-matrix on a CohFT; more detailed information can be found in [33]. We have:
\[
R \cdot \omega := \sum_{\Gamma \in G_{g,n}} \frac{1}{|\text{Aut}(\Gamma)|} \text{Cont}_\Gamma,
\]
where \( G_{g,n} \) is the set of decorated dual graphs of curves in \( \overline{M}_{g,n} \), and \( \text{Cont}_\Gamma \in H^*(\overline{M}_{g,n}) \otimes (V^*)^\otimes n \) is defined via contraction of tensors as follows:
\begin{itemize}
  \item at each vertex of \( \Gamma \), place the tensor 
  \[
  (T\omega)_{g(v),\text{val}(v)} \in H^*(\overline{M}_{g(v),\text{val}(v)}) \otimes (V^*)^\otimes \text{val}(v)
  \]
described below;
  \item at each leg \( l \) of \( \Gamma \) attached to a vertex \( v \), place 
  \[
  R^{-1}(\psi_l) \in H^*(\overline{M}_{g(v),\text{val}(v)}) \otimes \text{End}(V);
  \]
  \item at each edge \( e = (h, h') \) of \( \Gamma \) joining vertices \( v \) and \( v' \), place 
  \[
  \frac{\eta^{-1} - R^{-1}(\psi_h)\eta^{-1}R^{-1}(\psi_{h'}^t)}{\psi_h + \psi_{h'}} \in H^*(\overline{M}_{g(v),\text{val}(v)}) \otimes H^*(\overline{M}_{g(v'),\text{val}(v')}) \otimes V^\otimes 2.
  \]
\end{itemize}
In the vertex contribution, the translation operator $T$ is defined by
\[ T(z) := z1 - zR^{-1}(z)1 \in z^2V[[z]], \]
and $(T\omega)_{g,n}(v_1 \otimes \cdots \otimes v_n)$ is
\[ \sum_{m \geq 0} \frac{1}{m!} p_m(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+m})), \]
where $p_m : \overline{M}_{g,n+m} \to \overline{M}_{g,n}$ is the forgetful map.

In our case, the underlying vector space is $V = \mathbb{C}\{\zeta_0, \zeta_1, \ldots, \zeta_{r-1}\}$
with the pairing
\[ \eta(\zeta_i, \zeta_j) = \begin{cases} 1 & \text{if } i + j \equiv 0 \pmod{r} \\ 0 & \text{otherwise.} \end{cases} \]

We define two CohFTs on this vector space.

The first CohFT is
\[ \tilde{\Theta}_{g,n}^r(\zeta_1 \otimes \cdots \otimes \zeta_n) = r^g \cdot \tilde{\Theta}_{g,A}^r = \frac{1}{r^{g-1}} \phi_* \left( e^{r\tau c_1(\mathcal{L}_A)} \right), \]
where $A = (a_1, \ldots, a_n)$. The second is
\[ C_{g,n}^r(\zeta_1 \otimes \cdots \otimes \zeta_n) = r^g \cdot C_{g,A}^r = \frac{1}{r^{g-1}} \phi_* \left( c_{(r^2)}(\mathcal{L}_A) \right). \]

In both cases, the class is set to zero when the moduli space $\overline{M}_{g,A}$ does not exist—that is, whenever the condition
\[ \sum_{i=1}^n a_i \equiv 0 \pmod{r} \]
is not satisfied.

We remark that $C_{g,n}^r$ has another interpretation, as discussed in [8]. Namely, we consider the orbifold $[\mathbb{C}/\mathbb{Z}_r]$, on which $\mathbb{C}^*$ acts by multiplication. Then, if $\lambda$ denotes the equivariant parameter, one has
\[ \phi_* \left( \left[ \overline{M}_{g,a}([\mathbb{C}/\mathbb{Z}_r], 0) \right]_{\mathbb{C}^*}^{\text{vir}} \right) = \sum_{i=0}^{\infty} \left( \frac{\lambda}{r} \right)^{g-1+\frac{i}{r}} \sum_{a_i \equiv 0 \pmod{r}} \phi_* \left( c_i(\mathcal{L}_A) \right), \]
where $\overline{M}_{g,a}([\mathbb{C}/\mathbb{Z}_r], 0)$ denotes the substack of the moduli space of stable maps to $[\mathbb{C}/\mathbb{Z}_r]$ where the monodromy at the $i$th marked point is given by $a_i$. This follows, for example, from the localization computations in Appendix A.3 for $\mathbb{P}[r, 1]$, in the case where the degree $d$ is zero.
It follows that
\[(11) \quad C^r_{g,n}(\zeta_{a_1} \otimes \cdots \otimes \zeta_{a_n}) = r^{g-1+\frac{2}{r} \sum a_i} \phi_\ast \left( \left[ \overline{\mathcal{M}}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0) \right]^{\text{vir}} \right)_{|\lambda = \frac{1}{r}}. \]

Note that one must be careful in the situation where \(a_1 = \cdots = a_n = 0\), since in this case, \(\overline{\mathcal{M}}_{g,a}(\mathbb{C}/\mathbb{Z}_r, 0)\) is noncompact, and the virtual cycle should be understood as defined via the localization formula.

**Lemma 5.1.** Both \(\tilde{\Omega}^r_{g,n}\) and \(C^r_{g,n}\) form semisimple Cohomological Field Theories with unit \(\zeta_0\).

**Proof.** The fact that \(\tilde{\Omega}^r_{g,n}\) forms a CohFT is well-known; in particular, it is a “twisted” theory in the sense of [7]. For \(C^r_{g,n}\), the CohFT property follows from the interpretation (11). Indeed, the equivariant Gromov-Witten theory of \([\mathbb{C}/\mathbb{Z}_r]\) forms a CohFT under the pairing
\[\eta_{[\mathbb{C}/\mathbb{Z}_r]}(\zeta_i, \zeta_j) = \begin{cases} \frac{1}{r} & \text{if } i = j = 0 \\ \frac{1}{r} & \text{if } 0 \neq i + j \equiv 0 \mod r \\ 0 & \text{otherwise} \end{cases}, \]

and the pre-factor \(r^{g-1+\frac{2}{r} \sum a_i}\) can easily be shown to respect the decomposition properties.

The quantum product in either case can be computed explicitly, since the only contribution to the genus-zero three-point invariants comes in cohomological degree zero. Thus,
\[\tilde{\Omega}^r_{0,3}(\zeta_{a_1} \otimes \zeta_{a_2} \otimes \zeta_{a_3}) = C^r_{0,3}(\zeta_{a_1} \otimes \zeta_{a_2} \otimes \zeta_{a_3}) = \begin{cases} 1 & \text{if } \sum a_i \equiv 0 \mod r, \\ 0 & \text{otherwise.} \end{cases} \]

It follows that the quantum products are both
\[\zeta_i \ast \zeta_j = \zeta_{i+j} \mod r. \]

This shows that the unit is \(\zeta_0\), and moreover, that the ring structure on \(V\) is
\[\mathbb{C}[\zeta_1] / (\zeta_1^r = 1). \]

It is easy to see that this ring is semisimple, with idempotents given by
\[\epsilon_i := \frac{1}{r} \sum_{j=0}^{r-1} \xi^{ij} \zeta_1^j \]
for \(i \in \{0, \ldots, r-1\}\), where \(\xi\) is a primitive \(r\)th root of unity. \(\square\)
It follows from Lemma 5.1 that both $\tilde{\Omega}_r^{g,n}$ and $C_r^{g,n}$ can be computed in terms of an $R$-matrix action on a TFT. The TFTs are easy to calculate, since they arise from projecting the CohFT to cohomological degree zero; the result, in either case, is

$$\omega_{g,n}(\zeta_{a_1} \otimes \cdots \otimes \zeta_{a_n}) = \begin{cases} r^g & \text{if } \sum_{i=1}^n a_i \equiv 0 \mod r, \\ 0 & \text{otherwise.} \end{cases}$$

5.2. Computation of $R$-matrices. Fix the basis $\{\zeta_0, \ldots, \zeta_{r-1}\}$ for $V$. We claim that, in this basis, the $R$-matrix associated to the CohFT $\tilde{\Omega}_r^{g,n}$ is equal to

$$\tilde{R}_r^g(z) = \exp \begin{pmatrix} -\frac{r^2B_2(0)}{2}z \\ & \ddots \\ & & -\frac{r^2B_2(r-1)}{2}z \end{pmatrix},$$

and that the $R$-matrix associated to the CohFT $C_r^{g,n}$ is

$$R_C^r(z) = \exp \begin{pmatrix} \sum_{d=1}^\infty \frac{B_{d+1}(0)}{d(d+1)}(-r^2z)^d \\ & \ddots \\ & & \sum_{d=1}^\infty \frac{B_{d+1}(r-1)}{d(d+1)}(-r^2z)^d \end{pmatrix},$$

where in both cases the matrix inside the exponential is diagonal.\(^2\)

The fact that these matrices satisfy the symplectic condition $R(z) \cdot R^*(-z) = 1$, where $*$ denotes the adjoint with respect to the pairing, is a straightforward consequence of the identity $B_n(1-x) = (-1)^n B_n(x)$.

\(^2\)The fact that these matrices satisfy the symplectic condition $R(z) \cdot R^*(-z) = 1$, where $*$ denotes the adjoint with respect to the pairing, is a straightforward consequence of the identity $B_n(1-x) = (-1)^n B_n(x)$. 
Here,
\[
T(z) = z1 - z(R_C^r)^{-1}(z)1 \\
= z \left(1 - \exp \left(-\sum_{d=1}^{\infty} \frac{B_{d+1}(0)}{d(d+1)}(-r^2z)^d\right)\right) \zeta_0.
\]

Using the definition of \(\omega_{g,n}\) and applying Lemma 2.3 of [35] to the power series
\[
X(t) = 1 - \exp \left(-\sum_{d=1}^{\infty} \frac{B_{d+1}(0)}{d(d+1)}(-r^2t)^d\right),
\]
we can re-write (14) as
\[
r^g \exp \left(\sum_{d=1}^{\infty} (-1)^d \left(\frac{r^{2d}B_{d+1}(0)}{d(d+1)}\kappa_d - \sum_{j=1}^{n} \frac{r^{2d}B_{d+1}(a_j)}{d(d+1)}\psi_j^d\right)\right).
\]
The classes \(\kappa_d\) and \(\psi_j\) are pulled back under the degree-\(r^{2g-1}\) map \(\phi: M_{g,A} \rightarrow M_{g,n}\). Thus, the above is equal to
\[
\frac{1}{r^{g-1}}\phi_* \exp \left(\sum_{d=1}^{\infty} (-1)^d \left(\frac{r^{2d}B_{d+1}(0)}{d(d+1)}\kappa_d - \sum_{j=1}^{n} \frac{r^{2d}B_{d+1}(a_j)}{d(d+1)}\psi_j^d\right)\right),
\]
where we use the same notation for the \(\kappa\) and \(\psi\) classes on \(M_{g,n}\) as for their pullbacks to \(M_{g,A}\). Now, by Chiodo’s formula, the above coincides precisely with the restriction of \(C_{g,n}^r(\zeta_{a_1} \otimes \cdots \otimes \zeta_{a_n})\) to \(M_{g,n}\).

5.3. Proof of Theorem 1.2. We can now conclude the proof of the main theorem.

Proof of Lemma 4.2 and Theorem 1.2. When \(A\) has exactly one negative entry, we have reduced the claim to proving Lemma 4.2, or in other words that
\[
\frac{1}{r^g} (\bar{R}_r \cdot \omega)_{g,n}(\zeta_{a_1'} \otimes \cdots \otimes \zeta_{a_n'}) \bigg|_{r=0} = \frac{1}{r^g} (R_C^r \cdot \omega)_{g,n}(\zeta_{a_1} \otimes \cdots \otimes \zeta_{a_n}) \bigg|_{r=0}.
\]
This follows from Lemma 2.1, using the fact that the lowest-order terms in \(r\) of the two \(R\)-matrices agree.

Thus, the theorem is proved in the case where exactly one \(a_i\) is negative. Since, \(\Omega_{g,A}\) is polynomial in \(A\) by Lemma 2.1, this implies the result in general as long as \(n > 0\).

If \(n = 0\), the initial step of replacing \(A\) by \(A'\) is no longer valid, but the above nevertheless implies that
\[
\Omega_{g,0} = \frac{1}{r^{2g-1}}\phi_* \left(c_{r^2}( -R_{\pi_*}L_{\emptyset})\right) \bigg|_{r=0}.
\]
In this case, $R^0\pi_*L_{\emptyset}$ is a trivial line bundle, while $R^1\pi_*L_{\emptyset}$ is the pullback under $\phi$ of the Hodge bundle $E$ on $\overline{M}_g$. Thus, the vanishing of $\Omega_{g,\emptyset}$ in degrees past $g$ follows from the fact that $E$ is a rank-$g$ vector bundle. □

The computation of $R$-matrices also reveals why the geometric reformulation of $\Omega_{g,A}$ as the constant term of $\tilde{\Omega}_{g,A}^r$ matches Pixton’s original presentation as the constant term of $\Omega_{g,A}^r$.

**Lemma 5.2.** The two definitions of $\Omega_{g,A}$ described in Sections 2.2 and 2.3 agree:

$$\Omega_{g,A}^r\bigg|_{r=0} = \tilde{\Omega}_{g,A}^r\bigg|_{r=0}.$$

**Proof.** The formula for $r^{-g}\Omega_{g,A}^r$ via the $R$-matrix action exactly agrees with $\Omega_{g,A}^r$, except that the modifications

$$e^{-\frac{1}{2}r^2B_2(0)\kappa_1} \hookrightarrow 1,$$

$$e^{\frac{1}{2}r^2B_2(\frac{1}{r})\psi_i} \hookrightarrow e^{\frac{1}{2}r^2\psi_i},$$

$$\frac{1 - e^{-\frac{1}{2}r^2(B_2(\frac{w(h)}{r})\psi_h + B_2(\frac{w(h')}{r})\psi_{h'})}}{\psi_h + \psi_{h'}} \hookrightarrow \frac{1 - e^{-\frac{1}{2}w(h)w(h')\psi_h + \psi_{h'}}}{\psi_h + \psi_{h'}},$$

need to be done for the vertex, leg and edge factors, respectively. Now note that

$$B_2(x) = x^2 - x + \frac{1}{6}.$$

Hence, the first two modifications amount to a multiplication of

$$e^{\frac{r^2}{2}\kappa_1} \prod_{i=1}^n e^{\frac{1}{2}(r a_i - \frac{1}{2}r^2)\psi_i},$$

which leaves constant terms in $r$ invariant. The third modification also does not affect constant-in-$r$-terms, since

$$r^2B_2\left(\frac{w(h)}{r}\right) = r^2B_2\left(\frac{w(h')}{r}\right) = (w(h))^2 + rw(h) + \frac{r^2}{6} \equiv (w(h))^2 \equiv -w(h)w(h') \pmod{r}. □$$

### 5.4. Relations with powers of the log canonical.

Fix an integer $k$ and a tuple of integers $A = (a_1, \ldots, a_n)$ for which

$$\sum_{i=1}^n a_i \equiv k(2g - 2 + n) \pmod{r}.$$
As above, let $\overline{M}_{g,A}^{k/r}$ denote the moduli space of pointed stable curves with a line bundle $L$ satisfying (9).

Chiodo’s formula extends to these more general moduli spaces with only a small modification. It reads:

$$(15)\quad \text{ch}_d(R\pi_* L_{A,k}) = B_{d+1}(\frac{k}{r}) \kappa_d - \sum_{i=1}^{n} B_{d+1}(\frac{a_i}{r}) \psi_i^d +$$

$$+ \frac{r}{2} \sum_{0 \leq l \leq g} B_{d+1} \left( \frac{a_l}{r} \right) \left( \frac{d-1}{d} \right) p^* \phi_{i(l)} \left( \gamma_{d-1} \right) + \frac{r}{2} \sum_{q=0}^{r-1} B_{d+1}(\frac{2}{r}) \left( \frac{d-1}{d+1} \right) j_{(\text{irr},q)} \left( \gamma_{d-1} \right),$$

and the multiplicities $q_{l,I}$ are now determined by the condition

$$q_{l,I} + \sum_{i \in I} a_i \equiv k(2g - 2 + n) \mod r.$$

Using this, the proof of Theorem 1.2 is readily generalized.

**Theorem 5.3.** For any $k$ and any tuple $A$ of integers satisfying $\sum a_i = k(2g - 2 + n)$, the component of $\Omega_{g,A,k}$ in degree $d$ vanishes for all $d > g$.

**Proof.** Recall from Lemma 2.2 that the class $\Omega_{g,A,k}$ is polynomial in $k$. Therefore, it suffices to prove the theorem only for $k < 0$. In this case, the argument in the proof of Theorem 1.2 extends straightforwardly.

Specifically, Lemma 5.2 again shows that the two definitions of $\Omega_{g,A,k}$ agree, so it suffices to prove the vanishing for the geometrically formulated class. When exactly one $a_i$ is negative, one can also replace $A$ by $A' = (a_1 + r, a_2, \ldots, a_n)$, which again makes $-R\pi_*(L_{A',k})$ a vector bundle of rank $q$ but does not affect the lowest-order term in $r$ of $\phi_*(e^{r^2c_1(-R\pi_* L_{A,k})})$. From here, one proves again that the constant-in-$r$ term of

$$(16)\quad \frac{1}{r^{2g-1}} \phi_*(e^{r^2c_1(-R\pi_* L_{A',k})})$$

agrees with that of

$$(17)\quad \frac{1}{r^{2g-1}} \phi_*(c_{(r^2)}(-R\pi_* L_{A',k})),$$

assuming $r$ is first chosen sufficiently large. The proof is the same as previously; indeed, after multiplying by $r^q$, both (16) and (17) form CohFTs on the same vector space $V$ with the same pairing $\eta$ as considered previously. The TFT, on the other hand, is now nonzero only when $\sum_{i=1}^{n} a_i \equiv k(2g - 2 + n) \mod r$, and the unit is not $\zeta_0$ but $\zeta_k$. 
This shifted unit, which appears in the definition of the translation operator $T$, precisely accounts for the modification to Chiodo’s formula. A comparison of the $R$-matrices again completes the proof. \[\square\]

6. Connection to the 3-spin relations

Theorem 1.2 can be viewed as a collection of tautological relations in $A^*(\overline{\mathcal{M}}_{g,n})$, which we refer to as the double ramification cycle relations. Given that Pixton’s 3-spin relations, described in [34] and proved in [33], are conjectured to generate all relations in the tautological ring, one would expect the double ramification cycle relations to follow from these. This is indeed the case, as we explain in this section.

More precisely, what we prove is that the double ramification cycle relations $[\Omega_{g,A}]_d = 0$ in which exactly one of the arguments $a_i$ is negative can be derived from the 3-spin relations. By polynomiality of $\Omega_{g,A}$ in $A$, this is sufficient to derive all of the double ramification cycle relations.

It should be noted that the arguments of this section do not apply to the relations of Theorem 5.3. To address this more general situation, one would need to construct a new variant of moduli spaces of stable maps and study its intersection theory.

6.1. Equivariant orbifold projective line. Let $X = \mathbb{P}[r,1]$ denote an orbifold projective line, with one orbifold point of isotropy $\mathbb{Z}_r$ located at 0. More explicitly, $X$ can be expressed as a weighted projective space

$$X = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^*$$

in which $\mathbb{C}^*$ acts by $\sigma \cdot (x,y) = (\sigma^r x, \sigma y)$.

Consider the action of $\mathbb{C}^*$ on $X$ by $t \cdot [x,y] = [tx, ty]$, and let $\lambda$ denote the equivariant parameter. The equivariant Chen-Ruan cohomology of $X$ is described in Appendix A.1.

One can encode the equivariant orbifold Gromov-Witten theory of $X$ in a CohFT on the vector space $H^*_{CR}(X)$ depending on $\lambda$, a Novikov variable $q$ and a formal coordinate $t \in H^*_{CR}(X)$ as follows. For any $v_1, \ldots, v_n \in H^*_{CR}(X)$ and any $g, n$ such that $2g - 2 + n > 0$, define

$$\Omega^t_{g,n}(v_1 \otimes \ldots \otimes v_n) := \sum_{d,m \geq 0} \frac{q^d}{m!} (p_{d,m})^* \left( \prod_{i=1}^n ev^*_i(v_i) \cap \prod_{i=n+1}^{n+m} ev^*_i(t) \cap [\overline{\mathcal{M}}_{g,n+m}(X,d)]_{\text{vir}} \right),$$
where \( p_{d,m} : \overline{\mathcal{M}}_{g,n+m}(X,d) \to \overline{\mathcal{M}}_{g,n} \) is the forgetful map. It is well-known that \( \Omega_{g,n}^t \) forms a CohFT, and that it is semisimple for generic choices of the parameters (see Appendix A.2).

The localization formula expresses \( \Omega_{g,n}^t(v_1 \otimes \ldots \otimes v_n) \) as a sum over graphs. More specifically, just as in the case of ordinary \( \mathbb{P}^1 \), a graph \( \Gamma \) corresponds to a fixed locus in \( \overline{\mathcal{M}}_{g,n+m}(X,d) \) parameterizing maps \( f : C \to X \), where vertices of \( \Gamma \) indicate components of \( C \) contracted by \( f \) and edges indicate noncontracted components. The contributions to the localization formula from each such graph have been explicitly calculated by Johnson [25]; we review his calculation in Appendix A.3.

### 6.2. Nonequivariant limit and the double ramification cycle relations.

Throughout this subsection, we work at the basepoint \( t = 0 \), and we set \( q = 0 \), so in the localization formula for \( \Omega_{g,n}^0 \), only graphs with a single vertex contribute.

Fix insertions
\[
\zeta_0^{a_1}, \ldots, \zeta_0^{a_n} \in H^*_{CR}(X),
\]
where, in the notation of Appendix A.1, \( \zeta_0 \) is the generator of the twisted sector of age \( 1/r \) supported at \( 0 \in X \). Here, we assume that \( 0 \leq a_i < r \) for each \( i \) and that \( \sum a_i = r \). In the case \( a_i = 0 \), we mean by \( \zeta_0^{a_i} \) the identity of \( H^*_{CR}(X) \).

Since at least one \( a_i \) is nonzero, the single vertex in the localization graph must map to \( 0 \in X \). The fixed locus associated to this graph is
\[
\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0) \subset \overline{\mathcal{M}}_{g,n}(X, 0).
\]
Recall that \( \overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0) \) can be viewed as the moduli space of tuples \( (C; x_1, \ldots, x_n; L) \), in which \( (C; x_1, \ldots, x_n) \) is a stable orbifold curve and \( L \) is an orbifold line bundle on \( C \) satisfying \( L^{\otimes r} \cong O_C \). The integrand \( \prod_{i=1}^n \text{ev}_i^*(c_0^{a_i}) \) is supported on the open and closed substack of \( \overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_r, 0) \) on which the isotropy group of \( C \) at \( x_i \) acts on the fiber of \( L \) with weight \( a_i \). This is nothing but the moduli space \( \overline{\mathcal{M}}_{g,A} \) considered above.

Thus, by the contribution computed in Appendix A.3, the localization expression for \( \Omega_{g,n}^0(\zeta_{a_1} \otimes \ldots \otimes \zeta_{a_n})|_{q=0} \) is
\[
\sum_{i=0}^{\infty} \lambda^{g-i} r^{i-g} \phi_* \left( c_i (-R\pi_* \mathcal{L}_A) \right).
\]

The fact that (18) admits a nonequivariant limit is equivalent to the vanishing of \( \phi_*(c_i (-R\pi_* \mathcal{L}_A)) \) for \( i > g \).

On the other hand, the proof of Theorem 1.2 shows that the double ramification cycle relations (with exactly one negative argument)
follow from the vanishing of $C_{r,A}^g$ in degrees past $g$. Up to a factor of a power of $r$, the degree-$i$ part of $C_{r,A}^g$ is $\phi_*(-R\pi_*\mathcal{L}_A)$. Thus, the double ramification cycle relations follow from the existence of the nonequivariant limit in the localization expression for $\Omega_{g,n}^0|_{q=0}$.

6.3. Nonsemisimple limit and the 3-spin relations. An a priori entirely different way to express $\Omega_{g,n}^t$ as a graph sum is through Givental-Teleman reconstruction. Namely, for any $t$ for which $\Omega_{g,n}^t$ is semisimple, it can be written as the action of an $R$-matrix on a TFT. The resulting expression, when viewed as a formal function of $t$, appears to have poles at values of $t$ for which $\Omega_{g,n}^t$ is not semisimple, reflecting the fact that reconstruction fails at these basepoints. Yet the original CohFT $\Omega_{g,n}^t$ is defined for any choice of $t$. Thus, the non-semisimple limit of the reconstruction graph sum exists, and so the coefficient of any apparent pole in $t$ must vanish. This yields a family of relations in $H^*(\overline{M}_{g,n})$.

The same reasoning produces relations associated to any CohFT for which generic shifts are semisimple; in particular, Pixton’s 3-spin relations arise in this way from a CohFT defined via Witten’s 3-spin class. Although the resulting relations appear to depend crucially on the particular generically semisimple CohFT at hand, they are in fact independent of the chosen CohFT.

**Theorem 6.1** ([23], Theorem 3.3.6). Let $\Omega$ be a CohFT, for which generic shifts $\Omega_{g,n}^t$ exist and are semisimple but there exists at least one nonsemisimple shift. Then the tautological relations arising from the existence of a nonsemisimple limit in the Givental-Teleman reconstruction formula for $\Omega$ are equivalent to Pixton’s 3-spin relations.

We briefly digress to recall the definition of the formal strata algebra, following [18] and [35]. Let $\Gamma$ be a stable graph of genus $g$ with $n$ legs, and let

$$t_\Gamma : \overline{M}_\Gamma := \prod_{\text{vertices } v} \overline{M}_{g(v),\text{val}(v)} \to \overline{M}_{g,n}$$

be the gluing morphism whose image is the boundary stratum dictated by $\Gamma$. A basic class on $\overline{M}_\Gamma$ is defined as a class of the form

$$\gamma := \prod_{\text{vertices } v} \theta_v,$$

where $\theta_v$ is a monomial in the $\kappa$ and $\psi$ classes on the vertex moduli space $\overline{M}_{g(v),\text{val}(v)}$.

The formal strata algebra $S_{g,n}$ is generated as a $\mathbb{Q}$-vector space by pairs $[\Gamma, \gamma]$, where $\Gamma$ is a stable graph and $\gamma$ is a basic class on $\overline{M}_\Gamma$. A
multiplication rule can be defined using intersection theory with respect to the morphisms $\xi_\Gamma$, so that the association
\[
q : S_{g,n} \to A^*(\bar{M}_{g,n})
\]
\[
[\Gamma, \gamma] \mapsto \iota_{\Gamma, *}(\gamma)
\]
is a homomorphism of rings. It was proved by Graber-Pandharipande [18] that the classes $q([\Gamma, \gamma])$ are additive generators of the tautological ring; thus, tautological relations can be understood explicitly as elements of the kernel of $q$.

From here, the derivation of the double ramification cycle relations from the 3-spin relations proceeds in two steps.

**Lemma 6.2.** Pixton’s 3-spin relations imply that the reconstruction of the CohFT $\Omega^t$ (as an element of the formal strata algebra) is regular in the variables $\lambda, q$, and $t$.

**Proof.** By results of Iritani [21], under any specialization of the parameters $\lambda$ and $q$, $\Omega^t$ becomes a CohFT regular in a neighborhood of $t = 0$. Thus, by Theorem 6.1, Pixton’s 3-spin relations are equivalent to the regularity in $t$ of the reconstruction of any such specialization that is generically semisimple.

We claim that any choice of $(\lambda, q) \neq (0, 0)$ yields a generically semisimple CohFT. To prove this, it suffices to find a single value of $t$ for which $\Omega^t$ is semisimple, since semisimplicity is an open condition on $t$. In particular, set all coordinates of $t$ except for the hyperplane coordinate $t_1$ equal to zero, which essentially corresponds to considering the small quantum cohomology. On this line on the Frobenius manifold the CohFT is semisimple away from the vanishing locus of the discriminant
\[
d_{\lambda, q}(t_1) = (-1)^{r+1} \left( \frac{(r+1)^{r+1}}{r^r} (qe^{t_1})^r - \frac{1}{r} \lambda^{r+1} \right)
\]
of the defining polynomial (see Appendix A.2). As long as $(\lambda, q) \neq (0, 0)$, there exists a choice of $t_1$ for which $d_{\lambda, q}(t_1) \neq 0$, and hence this choice makes $\Omega^{0, \ldots, 0, t_1}$ semisimple.

Thus, Pixton’s 3-spin relations imply that the reconstruction of $\Omega^t$ is regular in the variables $\lambda, q$, and $t$, at least away from the locus $\{\lambda = q = 0\}$. However, since this locus has codimension two, it follows that the reconstruction of $\Omega^t$ is regular everywhere. \qed

**Lemma 6.3.** The regularity in $\lambda, q$, and $t$ of the reconstruction graph sum expression for $\Omega^t$ implies the double ramification cycle relations.
Proof. The key point is that the graph sum expression for $\Omega^t$ obtained via reconstruction agrees term-by-term with the graph sum expression obtained via localization. More precisely, after an application of Chiodo’s Grothendieck-Riemann-Roch formula at each vertex of a localization graph, the virtual localization formula presents $\Omega^t$ as an element of the formal strata algebra, which equals the element of the formal strata algebra given by expressing $\Omega^t$ through reconstruction. The proof of this assertion is technical, and is relegated to Appendices A.4 – A.7.

Assuming this, regularity of the reconstruction graph sum implies regularity of the localization graph sum. In particular, it implies that we can set $q = t = 0$ in the localization graph sum, and the result will be regular in the remaining parameter $\lambda$. As observed in Section 6.2, this regularity yields the double ramification cycle relations. □

Combining the results of this section, we have shown:

**Theorem 6.4.** The double ramification cycle relations are a consequence of Pixton’s 3-spin relations.

**Appendix A. Localization on an orbifold projective line**

Let $X = \mathbb{P}[r_0, r_\infty]$ be a projective line with an orbifold point of order $r_0$ at 0 and an orbifold point of order $r_\infty$ at $\infty$. (The case needed above is $r_0 = r$ and $r_\infty = 1$.)

Let $\mathbb{C}^*$ act on $X$ with weights $(0, 1)$. The goal of this appendix is to compare the computation of the CohFT corresponding to the equivariant Gromov-Witten theory of $X$ via localization and via the Givental-Teleman classification. We will show that they give the same result not only on the level of cohomology, but also on the level of Chow and of the formal strata algebra. In particular, this gives a proof of the Givental-Teleman classification in Chow in this case.

Sections A.1 and A.2 set up notation and recall some basic facts about the classical and quantum cohomology of $X$. In Section A.3, we present the localization formula for the equivariant virtual fundamental class of $\overline{M}_{g,n}(X, d)$, following [25]. From this formula, we can define a CohFT $\Omega$ encoding the equivariant Gromov-Witten theory of $X$, taking values not in $H^*(\overline{M}_{g,n})$ but in the formal strata algebra; this is made explicit in Section A.4, and is also generalized to a shifted CohFT $\Omega^t$.

Given that the morphism from $\overline{M}_{g,n+m}(X, d)$ to $\overline{M}_{g,n}$ involves contraction of unstable components, $\Omega^t$ has many localization contributions to each dual graph $\Gamma$ in $\overline{M}_{g,n}$. These come from trees of rational curves connecting a marking to a vertex in $\Gamma$, unmarked trees emanating from a vertex, and trees along an edge of $\Gamma$. In particular, these
contributions can be encoded in certain generating series of genus-zero
Gromov-Witten invariants, and in Section A.5 we set up the requisite
generating series and prove the formal properties we will need. Section
A.6 contains the heart of the computation, in which the properties from
the previous section are used to write \( \Omega^t \) as \( R_t \cdot \Omega_{\text{Ch},u} \), where \( \Omega_{\text{Ch}} \) is
a CohFT defined in terms of a weighted total Chern class of \(-R\pi_*\mathcal{L}\)
on \( \overline{\mathcal{M}}_{g,n}(B\mathbb{Z}, 0) \) (which is also strata-algebra-valued, by Chiodo’s for-
mula) and \( \Omega_{\text{Ch},u} \) is a certain shift. Finally, in Section A.7, we express
\( \Omega_{\text{Ch},u} \) as the action of an \( R \)-matrix on a TFT, using the fact that
Chiodo’s formula does the same for the unshifted CohFT \( \Omega_{\text{Ch}} \). From
here, the equality of the localization and reconstruction expressions for
\( \Omega^t \) is readily concluded.

This strategy is closely modeled on [10]. The manipulations of \( \Omega^t \)
that are required are almost entirely formal, relying only on the com-
parison of \( \psi \) classes under the forgetful maps \( \pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \).

A.1. **Classical cohomology.** The torus action on \( X \) has two fixed
points, 0 and \( \infty \). We let \( \lambda \) denote the equivariant parameter. The
equivariant Chen-Ruan cohomology ring of \( X \) is isomorphic to \( H_0 \oplus H_\infty \),
where

\[
H_0 = \mathbb{C}[[\zeta_0]]/(\zeta_0^{r_0} - \lambda/r_0), \quad H_\infty = \mathbb{C}[[\zeta_\infty]]/(\zeta_\infty^{r_\infty} - \lambda/r_\infty).
\]

Here, \( \zeta_0, \ldots, \zeta_0^{r_0-1} \) for \( i \in \{0, \infty\} \) are the generators of the twisted
sectors, and the untwisted sector is generated by the classes \( \phi_0 := [0]/\lambda \) and
\( \phi_\infty := [-\infty]/\lambda \), where \([i]\) are the equivariant classes of the fixed
points. The classes \( \phi_0 \) and \( \phi_\infty \) also act as the identities in \( H_0 \) and
\( H_\infty \). The identity in cohomology is \( 1 = \phi_0 + \phi_\infty \). Also, let \( h \) be the
equivariant lift \( h = [0] = \lambda \phi_0 \) of the hyperplane class.

The equivariant Poincaré pairing \( \eta \) is

\[
\eta(\phi_0, \phi_0) = \frac{1}{\lambda}, \quad \eta(\phi_\infty, \phi_\infty) = -\frac{1}{\lambda}, \quad \eta(\zeta_i, \zeta_i^{r_i-1}) = \frac{1}{r_i}
\]

for \( j \in \{1, \ldots, r_i - 1\} \), and all other pairings between basis vectors
vanish.

The equivariant Chen-Ruan cohomology ring is semisimple. If

\[
\zeta_0^{r_0} - \frac{\lambda}{r_0} = \prod_{i=1}^{r_0} (\zeta_0 - a_i), \quad \zeta_\infty^{r_\infty} - \frac{\lambda}{r_\infty} = \prod_{i=1}^{r_\infty} (\zeta_\infty - b_i)
\]
for formal roots $a_i$ and $b_i$, then the $r_0 + r_\infty$ idempotents of the equivariant Chen-Ruan cohomology are given by the Lagrange basis polynomials

$$
\prod_{i \neq j} (\zeta_0 - a_i \phi_0), \quad j \in \{1, \ldots, r_0\}, \quad \prod_{i \neq k} (\zeta_\infty - b_i \phi_\infty), \quad k \in \{1, \ldots, r_\infty\}.
$$

A.2. Quantum cohomology. The equivariant quantum cohomology ring is a deformation of the above-described classical equivariant Chen-Ruan cohomology ring. The deformation is parametrized by a formal point

$$
t = \sum_{i=1}^{r_0-1} t_i/r_0 \zeta_0^i + \sum_{i=1}^{r_\infty-1} t_i/r_\infty \zeta_\infty^i + t_0 1 + t_1 h
$$
on $H$ and a formal parameter $q$. The quantum cohomology is also semisimple, with each idempotent being given by a deformation of a classical idempotent.

In the case where $t_i = 0$ for $i \neq 1$, we obtain the small quantum cohomology of $X$, for which an explicit description was given by Milanov-Tseng [30]. Specifically, let $f(x)$ be the mirror polynomial:

$$
f(x) = e^{r_0 x} + q^{r_\infty} e^{r_\infty(t_1 - x)} + \lambda(t_1 - x).
$$

Then the small equivariant quantum cohomology ring of $X$ is isomorphic to the ring generated by $e^{\pm x}$ modulo the $x$-derivative $f'$ of $f$, under the identification:

$$
\zeta_0^i \mapsto e^{ix}, \quad \zeta_\infty^i \mapsto q^i e^{i(t_1 - x)}, \quad 1 \mapsto 1, \quad h \mapsto r_\infty q^{r_\infty} e^{r_\infty(t_1 - x)} + \lambda.
$$

A.3. Localization. For any choice of $\phi_1, \ldots, \phi_n \in H^*_CR(X)$, the equivariant class

$$
(19) \quad \prod_{i=1}^n \text{ev}_i^*(\phi_i) \cap [\overline{M}_{g,n}(X, d)]^{vir}_{C^*}
$$
can be computed via localization. We recall this computation, following Johnson [25].

First, recall that the fixed loci of the $C^*$ action on $\overline{M}_{g,n}(X, d)$ are indexed by certain graphs $\Gamma$, decorated as follows:

- Each edge $e$ is labeled with a positive integer $d(e)$.
- Each vertex $v$ is labeled with a nonnegative integer $g(v)$ and an element $j(v) \in \{0, \infty\}$. 


• Each leg \( l \) is labeled with an element of \( \{1, \ldots, n\} \) and an element
\[
\rho(l) \in \{0, 1, \ldots, r_{j(v)} - 1\}.
\]

Then there is a fixed locus in \( \overline{\mathcal{M}}_{g,n}(X, d) \) parameterizing maps \( f : C \rightarrow X \), for which:

• Edges of \( \Gamma \) correspond to components \( C_e \) of \( C \) not contracted by \( f \). Such components must be genus-zero Galois covers of \( X \) ramified only over 0 and \( \infty \), and \( d(e) \) denotes the degree of the restriction of \( f \) to \( C_e \).

• Vertices of \( \Gamma \) correspond to components of \( C \) contracted by \( f \), and \( g(v) \) denotes the genus of the component. Such a component must map to one of the fixed points of \( X \), which is specified by \( j(v) \).

• Legs of \( \Gamma \) correspond to marked points, and \( \rho(l) \) denotes the twisted sector in \( X \) to which \( f \) maps the marked point.

Let \( h(v) \) denote the number of half-edges incident to a vertex \( v \), and let \( n(v) \) denote the number of legs. There are three exceptional cases, in which a vertex corresponds not to a contracted component but to a single point of \( C \). Namely:

1. If \( (g(v), h(v), n(v)) = (0, 1, 0) \), then \( v \) corresponds to an unmarked ramification point of the map \( C_e \rightarrow X \), where \( e \) is the unique edge incident to \( v \);
2. If \( (g(v), h(v), n(v)) = (0, 1, 1) \), then \( v \) corresponds to a marked ramification point of the map \( C_e \rightarrow X \);
3. If \( (g(v), h(v), n(v)) = (0, 2, 0) \), then \( v \) corresponds to a node at which two noncontracted components meet.

In these situations, \( v \) is referred to as unstable; otherwise, \( v \) is stable.

Given a stable vertex \( v \), the decorations on \( \Gamma \) determine the monodromy of the map at all marked points on the corresponding contracted curve \( C_v \). Moreover, the monodromy at nodes of \( C_v \) is determined. Namely, by Lemma II.12 of [25], the monodromy of \( C_v \) at a node where \( C_v \) meets a noncontracted component \( C_e \) is equal to \(-d(e) \mod r_{j(v)} \). Thus, the decorations on \( \Gamma \) yield a tuple

\[
\rho(v) \in \{0, 1, \ldots, r_{j(v)} - 1\}^{h(v)+n(v)}
\]

recording the monodromy at all special points of \( C_v \). We denote

\[
\iota(\rho(v)) = \sum_{a \in \rho(v)} \frac{a}{r_{j(v)}}.
\]
Let $V(\Gamma)$ denote the vertex set of $\Gamma$ and let $E(\Gamma)$ denote the edge set. Denote

$$\overline{M}_\Gamma := \prod_{v \in V(\Gamma)} \overline{M}_{g(v), \rho(v)}(B\mathbb{Z}_{r_{j(v)}}, 0).$$

Then there is as canonical family of $\mathbb{C}^*$-fixed stable maps to $X$ over $\overline{M}_\Gamma$, yielding a morphism

$$\iota_\Gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}(X, d).$$

This is not exactly the inclusion of the fixed locus associated to $\Gamma$, because elements of $\overline{M}_{g,n}(X, d)$ have automorphisms not accounted for by the contracted components individually. Specifically, there are additional automorphisms from permuting vertex components via an automorphism of the graph $\Gamma$, multiplying noncontracted components $C_e$ by a $d(e)$th root of unity, and acting around a node by a “ghost” automorphism. Nevertheless, there is a finite map from $\overline{M}_\Gamma$ to the associated fixed locus, and one can explicitly compute that the degree of this map is

$$(20) \quad |\text{Aut}(\Gamma)| \cdot \prod_{e \in E(\Gamma)} \frac{d(e)}{\text{gcd}(r_0, d(e))\text{gcd}(r_\infty, d(e))}.$$ 

Thus, applying the virtual localization formula, $[\overline{M}_{g,n}(X, d)]_{\mathbb{C}^*}^{\text{vir}}$ can be expressed as a sum over decorated graphs $\Gamma$ of contributions pushed forward from the moduli spaces $\overline{M}_\Gamma$.

For each such decorated graph, the contribution is given by the inverse equivariant Euler class of a virtual normal bundle, which is computed by Johnson in [25]. After some simplification and accounting for the degree $(20)$, his calculations show:

$$(21) \quad \left[ \overline{M}_{g,n}(X, d) \right]_{\mathbb{C}^*}^{\text{vir}} =$$

$$\sum_{\Gamma} \frac{(\iota_\Gamma)_*}{|\text{Aut}(\Gamma)|} \left( \prod_{e \in E(\Gamma)} C(e) \prod_{v \in V(\Gamma)} C(v) \prod_{v \in V(\Gamma)} (-\psi_v) \prod_{\text{nodes}} \frac{\eta^{-1}_{e,v}}{-\psi - \psi'} \right).$$

Here, setting $\lambda_0 := \lambda$ and $\lambda_\infty := -\lambda$, we have

$$C(e) := \lambda_0^{\left \lfloor \frac{d(e)}{r_0} \right \rfloor} \lambda_\infty^{\left \lfloor \frac{d(e)}{r_\infty} \right \rfloor} \cdot \frac{d(e)}{r_0} \left \lfloor \frac{d(e)}{r_0} \right \rfloor + \left \lfloor \frac{d(e)}{r_\infty} \right \rfloor - 1 \cdot \frac{d(e)}{r_\infty} \left \lfloor \frac{d(e)}{r_\infty} \right \rfloor!.$$
and

\[
C(v) := \sum_{i=0}^{\infty} \left( \frac{\lambda_j(v)}{r_j(v)} \right)^{g(v) - 1 + \varepsilon(r_j(v)) - i} c_i(-R\pi_* L),
\]

where \(\pi\) is the universal curve of \(\mathcal{M}_{g(v)}, \rho(v)(\mathbb{BZ}_{r_j(v)})\) and \(L\) is the universal \(r_j(v)\)th root. In the third product, \(\psi_v\) denotes the equivariant cotangent line class of the coarse underlying curve— i.e., \(-\psi_v = \lambda_j(v)/d(e)\), where \(e\) is the unique adjacent edge. The last product is over nodes forced on the source curve by \(\Gamma\); these are either pairs of a stable vertex \(v\) together with an adjacent edge \(e\), or unstable vertices \(v\) with \((g(v), h(v), n(v)) = (0, 2, 0)\). Here, \(\psi\) and \(\psi'\) stand for the equivariant cotangent line classes of the coarse underlying curves joined by the node and

\[
\eta_{i,v}^{-1} := \begin{cases} 
\lambda_j(v) & \text{if } d(e) \equiv 0 \mod r_j(v) \\
r_j(v) & \text{if } d(e) \not\equiv 0 \mod r_j(v).
\end{cases}
\]

**Remark A.1.** In the following, only the exact expressions for the factors in (21) corresponding to stable vertices and nodes at stable vertices will play a role. The other factors will only be part of certain genus zero generating series, which we will determine indirectly.

**A.4. CohFTs on the level of the strata algebra.** We want to define the CohFT \(\Omega\) corresponding to the equivariant Gromov-Witten theory as an element of the strata algebra using the localization computation of the preceding section.

First, for a vertex \(v\) mapped to the fixed point \(j\), the pushforward of the contribution \(C(v)\) under the map \(\phi : \mathcal{M}_{g,\rho}(\mathbb{BZ}_{r_j}) \to \mathcal{M}_{g,n}\) forgetting the line bundle and orbifold structure motivates the definition

\[
\Omega_{g,n}^{\text{Ch},j}(\zeta_j^0, \ldots, \zeta_j^n) = \sum_{i=0}^{\infty} \left( \frac{\lambda_j}{r_j} \right)^{g-1 + \sum_{k=1}^{n} a_k} \phi_* (c_i(-R\pi_* L)).
\]

Extending multilinearly, this defines a CohFT \(\Omega_{g,n}^{\text{Ch}}\) on the vector space \(H_j\). Let \(\Omega_{\text{Ch}}\) be the CohFT on \(H\) defined as the direct sum of \(\Omega_{\text{Ch},0}\) and \(\Omega_{\text{Ch},\infty}\). Using Chiodo’s formula, we can view the image of \(\Omega_{\text{Ch}}\) not merely as \(H^* (\mathcal{M}_{g,n})\) but as the strata algebra.

Now set for \(v_1, \ldots, v_n \in H\)

\[
\Omega_{g,n}(v_1, \ldots, v_n) = \sum_{d=0}^{\infty} q^d p_*( \prod_{i=1}^{n} \ev_i^* (v_i) \cap [\mathcal{M}_{g,n}(X, d)]^{\text{vir}} ) ,
\]

where \(p : \mathcal{M}_{g,n}(X, d) \to \mathcal{M}_{g,n}\) forgets the map and orbifold structure and stabilizes the source curve. By the virtual localization formula
described in the previous section, $\Omega$ can be defined in terms of $\Omega_{\text{Ch}}$, and hence it, too, can be defined on the level of the strata algebra.

More generally, for a formal point $t$ on $H$, we will also need to consider the shifted CohFT $\Omega^t$ defined by

$$\Omega^t_{g,n}(v_1, \ldots, v_n) = \sum_{m=0}^{\infty} \frac{1}{m!} \pi_* \Omega_{g,n+m}(v_1, \ldots, v_n, t, \ldots, t),$$

where $\pi$ forgets the last $m$ markings. We use the analogous definition for $\Omega^t_{\text{Ch}}$.

A.5. Genus-zero Gromov-Witten theory. In this section, we define various generating series of genus-zero Gromov-Witten invariants of $X$ and derive identities necessary for the next section.

For $f_i \in H[[z]]$, we define genus-zero correlators by

$$\langle f_1(\psi), \ldots, f_n(\psi) \rangle_n := \sum_{d=0}^{\infty} q^d \int_{[\overline{M}_{0,n}(X,d)]^{vir}} \prod_{i=1}^{n} \text{ev}_i^*(f_i),$$

where $\text{ev}_i^*$ pulls back elements in $H$ and replaces $z$’s by $\psi_i$’s. The multilinear form $\langle \cdot, \ldots, \cdot \rangle^\text{Ch}_n$ is defined similarly.

In the following we will often work with power series in the variables $w^{-1}, z^{-1}, w/\lambda, z/\lambda$. While there can be arbitrary positive and negative powers in $w$ and $z$, the multiplication of these series is always well-defined when it appears.

We define an endomorphism-valued power series $S$ by

$$S_t(z) = v + \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{v}{z - \psi}, \eta^{-1}, t, \ldots, t \right\rangle_{2+n},$$

where the argument $\eta^{-1}$ stands for raising an index in order to obtain a vector from a linear form. The series $S^\text{Ch}_t$ is defined similarly, using the correlators $\langle \cdot, \ldots, \cdot \rangle^\text{Ch}_n$. We use the convention here and everywhere else that integrals over non-existent moduli spaces are defined to be zero.

By the WDVV and string equations, the bivector

$$V_t(z, w) := \frac{\eta^{-1}}{z + w} + \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{z - \psi}, \frac{\eta^{-1}}{w - \psi}, t, \ldots, t \right\rangle_{2+n},$$

can be computed in terms of $S$ via

$$V_t(z, w) = \frac{(S^*_t(z) \otimes S^*_t(w)) \eta^{-1}}{z + w},$$

(22)
where $S^*$ is the adjoint of $S$ with respect to $\eta$. In particular, $S$ satisfies the symplectic condition

$$S_\ell(z)S^*_\ell(-z) = \text{Id}.$$  

Equivalently to (22), for any $f \in H[z]$ we have

$$f(z) + \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{z-\psi}, f(-\psi), t, \ldots, t \right\rangle_{2+n} = S^*_\ell(z)[S_\ell(-z)f(z)]_+,$$

where the bracket $[-]_+$ picks out the regular part in a power series in $z$.

The localization computations will use the following lemma repeatedly. In all cases it will be applied to the series $\epsilon(z)$ defined as $t$ plus the sum of localization contributions to computing

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{z-\psi}, t, \ldots, t \right\rangle_{1+n}$$

such that the first marking lies on a noncontracted Galois cover.

**Lemma A.2.** Let $\epsilon \in H[z]$ be arbitrary. Then there exists

$$u(\epsilon) := \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle \eta^{-1}, 1, \epsilon(\psi), \ldots, \epsilon(\psi) \right\rangle_{2+n}^{\text{Ch}} \in H$$

such that for any $f_1, f_2 \in H[z]$,

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left\langle f_1(\psi), f_2(\psi), \epsilon(\psi), \ldots, \epsilon(\psi) \right\rangle_{2+n}^{\text{Ch}} = \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle f_1(\psi), f_2(\psi), u(\epsilon), \ldots, u(\epsilon) \right\rangle_{2+n}^{\text{Ch}}.$$

**Proof.** This is a general result in Gromov-Witten theory attributed in [10] to Dijkgraaf-Witten [12]. The vanishing of

$$\sum_{n=1}^{\infty} \frac{1}{n!} \left\langle f_1(\psi), f_2(\psi), \epsilon(\psi), \ldots, \epsilon(\psi), u(\epsilon) - \epsilon(\psi), \ldots, u(\epsilon) - \epsilon(\psi) \right\rangle_{2+n+k}^{\text{Ch}}$$

for any $k \geq 1$ can be proven inductively using the string equation and genus-zero topological recursion relations. This vanishing immediately implies the lemma. \qed

We want to apply localization to the computation of $S_\ell(z)v$ for some $v \in H$. For each fixed locus, the first marking will lie either on a Galois cover or on a contracted component connected to several trees...
of rational curves. Let $T$ be the generating series of contributions of the (possibly empty) tree connecting to the second marking. More explicitly, $T(z)$ is an endomorphism-valued power series, and $T_t(z)v$ is defined as $v$ plus the localization contributions to

$$
\sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{v}{z-\psi}, \eta^{-1}, t, \ldots, t \right\rangle_{2+n}
$$

such that the first marking lies on a noncontracted Galois cover. Since $\epsilon$ records the contribution of any one of the other trees at the contracted component containing marking one, the result of the localization is

$$
S_t(z)v = T_t^*(z)v + \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle \frac{v}{z-\psi}, T_t(-\psi)\eta^{-1}, \epsilon(\psi), \ldots, \epsilon(\psi) \right\rangle_{2+n}^{\text{Ch}}
$$

$$
= T_t^*(z)v + \sum_{n=1}^{\infty} \frac{1}{n!} \left\langle \frac{v}{z-\psi}, T_t(-\psi)\eta^{-1}, u(\epsilon), \ldots, u(\epsilon) \right\rangle_{2+n}^{\text{Ch}}.
$$

In the second equality we have used Lemma A.2. Using the adjoint of (24), we can rewrite this as

$$
S_t(z) = R_t(z)S_u^{\text{Ch}}(z),
$$

where $u = u(\epsilon)$ and

$$
R_t(z) = [T_t^*(z)(S_u^{\text{Ch}}(z))^{-1}]_+.
$$

By (23) (which applies also to $S_u^{\text{Ch}}$), $R$ also satisfies the symplectic condition

$$
R_t(z)R_t^*(-z) = \text{Id}.
$$

However, in contrast to usual $R$-matrices, $R_t(0)$ equals the identity only modulo $q$.

For later use, we compute the localization series $\epsilon$. For this we use the $J$-function

$$
J_t(z) = z1 + t + \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{z-\psi}, t, \ldots, t \right\rangle_{1+n} = zS_t^*(z)1,
$$

where the last equality follows from the string equation. Evaluating $J_t(z)$ via virtual localization, we find

$$
J_t(z) = z1 + \epsilon(-z) + \sum_{n=2}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{z-\psi}, \epsilon(\psi), \ldots, \epsilon(\psi) \right\rangle_{1+n}^{\text{Ch}}.
$$

Here, the second summand corresponds to the case that the first marking lies on a Galois cover (and also includes the summand $t$) and the third summand corresponds to the case that the first marking lies on
a contracted component. The third summand is a power series in $z^{-1}$ with no constant part. Therefore, we can compute $\epsilon$ via

$$\epsilon(z) = [z(\text{Id} - S_t^*(-z))]_+.$$  

Note that, in contrast to $S_u^\text{Ch}(z)$, the endomorphism $S_t(z)$ has both positive and negative powers in $z$.

A.6. **Main computation.** We are now ready to consider the computation of

$$\Omega^t_{g,n}(v_1, \ldots, v_n)$$

via virtual localization. For each localization graph, there exists a dual graph $\Gamma$ recording the topological type of the stabilization of a generic source curve inside the corresponding fixed locus, and we compute the sum of localization contributions for a fixed dual graph $\Gamma$.

At each vertex of $\Gamma$, we need to sum over an arbitrary number of possible trees not containing any of the $n$ markings. As in Section A.5, the contribution of any such tree is given by the vector $\epsilon$. Therefore, we will need to compute at any vertex of $\Gamma$ sums of the form

$$\sum_{m=0}^{\infty} \frac{1}{m!} \Omega^\text{Ch}_{g(v),\text{val}(v)+m}(f_1(\psi), \ldots, f_n(\psi), \epsilon(\psi), \ldots, \epsilon(\psi)),$$

where $f_1, \ldots, f_n$ are contributions from trees containing one of the $n$ markings or connecting $v$ to another stable vertex. We rewrite

$$\sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \Omega^\text{Ch}_{g(v),\text{val}+m+l}(f_1(\psi), \ldots, f_n(\psi), (\epsilon(\psi) - u)^{\otimes m}, u^{\otimes l})$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \Omega^\text{Ch,\text{u}}_{g(v),\text{val}+m+l}(\tilde{f}_1(\psi), \ldots, \tilde{f}_n(\psi), ([S_u^\text{Ch}(\psi)(\epsilon(\psi) - u)]_+)^{\otimes m}),$$

where we have used short-hand notation to denote multiple identical arguments and $\tilde{f}_i(z) := [S_u^\text{Ch}(z)f_i(z)]_+$. In the second step, we have used the splitting property of $\Omega^\text{Ch}$ and the comparison

$$\psi_i^j = \pi^* \psi_i^j + \sum_{a=1}^j \delta_i \psi_i^a \pi^* \psi_i^{j-a}$$

for any $i \in \{1, \ldots, n(v) + m\}$ and $j \geq 0$, where $\pi$ is the forgetful map $\pi: \mathcal{M}_{g(v),n(v)+m+l} \to \mathcal{M}_{g(v),n(v)+m}$ and $\delta_i$ is the boundary divisor where, generically, marking $i$ lies on a genus-zero component containing none of the first $\text{val}(v) + m$ markings.
We compute
\[
[S^\text{Ch}_u(z)(\epsilon(z) - u)]_+ = [S^\text{Ch}_u(z)\{z - [zS^\text{Ch}_u(-z)1]_+\}]_+ - u
= (z1 + u) - [zS^\text{Ch}_u(z)S^\text{Ch}_u(z)]_+ - u = z(1 - R^{-1}_t(z))1.
\]
In the first step we have used (27). The second step uses the fact that, since \(S^\text{Ch}_u(z)\) is a power series in \(z^{-1}\) for any \(A\), we have \([S^\text{Ch}_u(z)[A]_+)_+ = [S^\text{Ch}_u(z)A]_+\). The third step uses (25) and the symplectic condition (23).

Similarly, we study the contributions from the trees containing markings or connecting stable components. If \(f_i(z)\) corresponds to the contribution of trees carrying marking \(j\), we have \(f_i(z) = T_t(z) - v_j\) and therefore
\[
\tilde{f}_i(z) = R^{-1}_t(z)v_j,
\]
by (25) and (26).

The contribution of the trees connecting two stable vertices naturally forms a bivector-valued series \(E(z, w)\). The same series already appears in genus-zero localization computations, and we can precisely define \(E_t(z, w)\) to be the localization contribution to the integral
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{-z - \psi}, \frac{\eta^{-1}}{-w - \psi}, t, \ldots, t \right\rangle_{2+n}
\]
such that the first and second marking both lie on Galois covers. In the higher-genus localization, the variables \(z\) and \(w\) in \(E_t(z, w)\) will be replaced by \(\psi\)-classes at the two sides of the node and inserted in the corresponding arguments of \(\Omega^\text{Ch}\). As we will see soon,

\[
[S^\text{Ch}_u(z) \otimes S^\text{Ch}_u(w)]_+ = (1 - R^{-1}_t(z) \otimes R^{-1}_t(w)) \frac{\eta^{-1}}{w + z}.
\]

In total, we have shown that
\[
\Omega^t = R_t \cdot \Omega^\text{Ch,u}
\]
as elements of the strata algebra, using the definition of the action of \(R_t\) recalled in Section 5.1.

To see why (28) is true, we compute \(V_t(-z, -w)\) via virtual localization. Using Lemma A.2 for contributions where the first two markings lie at the same vertex, this gives
\[
V_t(-z, -w) = V^\text{Ch}_u(-z, -w) + (F_t^z \otimes F_t^w)E_t(z, w),
\]
where \(V^\text{Ch}_u\) is defined analogously to \(V_t\), the endomorphism \(F_t^z\) of \(H[z]\) is defined by
\[
F_t^z v(z) = v(z) + \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \frac{\eta^{-1}}{-z - \psi}, v(\psi), \epsilon(\psi), \ldots, \epsilon(\psi) \right\rangle_{2+n}^\text{Ch},
\]
and the endomorphism $F^w_t$ of $H[w]$ is defined similarly. By Lemma A.2 and (24), we can write
\[ F^w_t v(z) = (S^\text{Ch}_u(z))^{-1}[S^\text{Ch}_u(z)v(z)]_+. \]
After applications of (22) and (25), we arrive at (28).

A.7. Shifting semisimple CohFTs. To conclude our arguments, we need to compare $\Omega^\text{Ch,u}$ to $\Omega^\text{Ch}$. For this we can use the following more general lemma.

**Lemma A.3.** Let $\Omega$ be a semisimple CohFT (on the level of the strata algebra) on a vector space $H$ defined by the action of an $R$-matrix $R$ on a TFT and let $t$ be a formal point on $H$. Then the shifted CohFT $\Omega^t$ is given by the action of a certain modified $R$-matrix on a modified TFT.

Together with (29), this lemma shows that
\[ \Omega^t = R^t_t \cdot (R^\text{Ch}_u \cdot \omega) \]
for an $R$-matrix $R^\text{Ch}_u$ and a TFT $\omega$. Since the $R$-matrix action lifts to the strata algebra, it follows that $\Omega^t = \tilde{R}^t_t \cdot \tilde{\omega}$ where
\[ \tilde{R}^t_t(z) = R^t_t(z)R^\text{Ch}_u(z)R^{-1}_t(0) \]
and $\tilde{\omega} = R^t_t(0) \cdot \omega$. Now, the power series $\tilde{R}$ has constant term $\text{Id}$. By the dilaton equation, $\tilde{\omega}$ is a TFT and by restricting to genus zero and three markings, we see that it has to coincide with the TFT from the equivariant Chen-Ruan cohomology of $X$. Starting from the localization formula and using Chiodo’s formula, we have arrived at the formula for $\Omega^t$ in terms of the action of an $R$-matrix on the TFT.

**Proof of Lemma A.3.** In the reconstruction formula, we need to consider for each dual graph $\Gamma$ an expression of the form
\[ \sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \omega_{g(v),\text{val}(v)+m}(f_1(\psi), \ldots, f_{\text{val}(v)}(\psi), t, \ldots, t), \]
where $\omega$ is the TFT corresponding to $\Omega$ and the $f_i$ record contributions from legs or edges. We use the comparison of $\psi$ classes between $\mathcal{M}_{g(v),\text{val}(v)}$ and $\mathcal{M}_{g(v),\text{val}(v)+m}$ to rewrite this as
\[ \sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \omega_{g(v),\text{val}(v)+m}(\left[S_t(\bar{\psi})f_1(\bar{\psi})\right]_+, \ldots, \left[S_t(\bar{\psi})f_{\text{val}(v)}(\bar{\psi})\right]_+, t, \ldots, t) \]
\[ = \omega_{g(v),\text{val}(v)}(\left[S_t(\psi)f_1(\psi)\right]_+, \ldots, \left[S_t(\psi)f_{\text{val}(v)}(\psi)\right]_+), \]
where \( S_t(z) \) is the operator of quantum multiplication by \( e^{t/z} \), the \( \tilde{\psi} \) are \( \psi \)-classes pulled back from \( \overline{\mathcal{M}}_{g(v), \text{val}(v)} \), and the equality uses the string equation. Let us define new matrices \( \tilde{R}_t \), \( \tilde{R}_t^0 \) and \( R_t \) by

\[
\tilde{R}_t^{-1}(z) = \left[ S_t(z) R_t^{-1}(z) \right]_+,
\]

\[
\tilde{R}_t^0 = \tilde{R}_t(0), \quad R_t^{-1}(z) = \tilde{R}_t^0 \tilde{R}_t^{-1}(z).
\]

So far we have shown that \( \Omega^t_{g,n}(v_1, \ldots, v_n) \) is given by a dual graph sum

\[
\sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} t^{\Gamma_s} \left( \prod_{\gamma \in \Gamma} \sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \omega^t_{g,n}(v_1, \ldots, v_n, \gamma, \gamma, \ldots) \right),
\]

where

\[
\omega^t_{g,n}(v_1, \ldots, v_n) := \sum_{m=0}^{\infty} \frac{1}{m!} \pi^* \omega_{g,n,m} \left( \tilde{R}_t^0 v_1, \ldots, \tilde{R}_t^0 v_n, \psi(\text{Id} - \tilde{R}_t^0)1, \ldots, \psi(\text{Id} - \tilde{R}_t^0)1 \right)
\]

and the arguments of \( \omega^t \) corresponding to legs, edges, and extra legs should be filled with

\[
R_t^{-1}(\psi_i) v_i,
\]

\[
((\tilde{R}_t^0)^{-1} \otimes (\tilde{R}_t^0)^{-1}) \eta^{-1} - (R_t^{-1}(\psi) \otimes R_t^{-1}(\psi')) \eta^{-1},
\]

\[
\psi(\text{Id} - R_t^{-1}(\psi))1,
\]

respectively. In order for the edge term to be well-defined, the symplectic conditions

\[
\tilde{R}_t^0(\tilde{R}_t^0)^* = \text{Id}, \quad R_t(z) R_t^*(-z) = \text{Id}
\]

must hold. It is now straightforward to check that \( \omega^t \) is a TFT with respect to \( \eta \) and with the unit \( 1 \). Therefore, \( \Omega^t \) is given by the action of \( R_t \) on \( \omega^t \). \( \square \)

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