Deformations of $\mathcal{N}=2$ Dualities to $\mathcal{N}=1$ Dualities in $SU$, $SO$ and $USp$ Gauge Theories

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Abstract

We study deformations of dualities in finite $\mathcal{N}=2$ supersymmetric QCD. Adding mass terms for some quarks and the adjoint matter to the finite $\mathcal{N}=2$ theory, which is known to have dual descriptions, the correspondence of gauge invariant operators between the original and dual theory is deformed. As a result, we naturally obtain N.Seiberg’s $\mathcal{N}=1$ duality. Furthermore, we discuss the origin of the meson and superpotential in the dual theory. This approach can be applied to $SU(N)$, $SO(N)$, and $USp(2N)$ gauge theories, and we analyze all these cases.

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1 Introduction

In the last few years remarkable progress has been made in supersymmetric gauge theories in four dimensions. Using the methods developed by N. Seiberg, effective superpotential can often be determined exactly and lots of non-perturbative effects are found (see [1] for a review). One of the most interesting discoveries is duality in $\mathcal{N} = 1$ supersymmetric gauge theories. For example, in the pioneered work of N. Seiberg [2], it was found that $\mathcal{N} = 1$ supersymmetric $SU(N_c)$ QCD with $N_f$ flavors of quarks has a dual description, which is $\mathcal{N} = 1$ supersymmetric $SU(N_f - N_c)$ QCD with $N_f$ flavors of dual quarks and a gauge singlet meson field interacting with the dual quarks by the superpotential.

Up to now there is no rigorous proof of this duality, and so it is important to investigate non-trivial evidences for it. There are several arguments that the $\mathcal{N} = 1$ duality can be derived by analyzing $\mathcal{N} = 2$ supersymmetric QCD [3, 4, 5]. Since N. Seiberg and E. Witten’s two beautiful papers [6] appeared, it has become clear that the low energy effective theory of $\mathcal{N} = 2$ supersymmetric QCD can be analyzed exactly making use of hyperelliptic curves [7, 8]. In [3], it was shown that the low energy effective theory at the baryonic root of $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ QCD with $N_f$ flavors is $SU(N_f - N_c) \times U(1)^{2N_c - N_f}$ gauge theory, and adding $\mathcal{N} = 2$ breaking mass term for the adjoint chiral field, the theory flows to $\mathcal{N} = 1$ supersymmetric $SU(N_f - N_c)$ QCD which is consistent to the N. Seiberg’s duality in $\mathcal{N} = 1$ supersymmetric QCD. This argument can also be applied to $SO(N)$ and $USp(2N)$ gauge theory [9]. However, these works do not provide a complete derivation of the N. Seiberg’s duality since the origin of the meson field and the superpotential, needed in the dual theory, is not specified. An approach for this problem is given in [5], in which finite $\mathcal{N} = 2$ supersymmetric QCD is considered in detail. $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ QCD with $2N_c$ flavors is known to be finite and believed to have a dual description. Adding mass term for the adjoint chiral field to this theory and using Fierz transformation, the authors of [5] derived the dual $SU(N_c)$ theory with the meson field and the superpotential.

The purpose of this paper is to obtain deeper understanding of the relation between $\mathcal{N} = 1$ duality and $\mathcal{N} = 2$ duality, and to derive $\mathcal{N} = 1$ duality from $\mathcal{N} = 2$ duality. In particular, we generalize the argument in [5] for the case with $N_f(\leq 2N_c)$ massless flavors and propose the origin of the meson field and the superpotential.

We start with the finite $\mathcal{N} = 2$ theory with mass terms for hypermultiplets. The dual of this theory is determined so as to have the same hyperelliptic curve [7]. Then we add $\mathcal{N} = 2$ breaking mass term for the adjoint chiral field to obtain $\mathcal{N} = 1$ theory and see the change of the vacuum moduli space. This change of the vacuum moduli space is non-trivial because we must correctly consider non-perturbative effects. To have the same vacuum moduli space, we determine

*USp(2N_c) is the unitary symplectic group of rank N_c.*
corresponding deformation for the dual theory and the correspondence of gauge invariant operators between the original and dual theory. Now we can apply the transformation used in [5] and obtain N. Seiberg’s $N = 1$ dual theory including a meson field interacting with the dual quarks by the superpotential. This method can be applied to $SU(N_c)$, $SO(N_c)$ and $USp(2N_c)$ gauge theories. In the case of $SO(N_c)$ and $USp(2N_c)$ gauge theories, unlike the $SU(N_c)$ case, we find that the vacuum moduli space has several distinct branches, and the dual transformation maps each branch to the corresponding dual branch.

2 S-duality in $\mathcal{N} = 2$ $SU(N_c)$ Gauge Theory

2.1 a brief review of S-duality in $\mathcal{N} = 2$ theory

In this subsection we will make a brief review of S-dualities in $\mathcal{N} = 2$ supersymmetric QCD. Here we consider $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ QCD with $N_f$ hypermultiplets in the $N_c$ representation of the gauge group. The theory can be described in terms of $\mathcal{N} = 1$ superfields: $W_\alpha$ (a field strength chiral multiplet), $\Phi$ (a chiral multiplet in the adjoint representation of the gauge group), $Q^i$ and $\bar{Q}_i$ (chiral multiplets in the $N_c$ and $\overline{N_c}$ representation of the gauge group respectively), where $i = 1, \ldots, N_f$ are flavor indices. The superpotential is

$$W_{\text{ele}} = \sqrt{2} g Q^i \Phi \bar{Q}_i + \sqrt{2} m_{ij} Q^i \bar{Q}_j,$$

(2.1)

where $m = (m^2_{ij}) = \text{diag}(m_1, \ldots, m_{N_f})$ is a quark mass matrix.

When $N_f < 2N_c$, the theory is asymptotically free and when $N_f = 2N_c$, the theory is scale-invariant (for $m = 0$). Highly nontrivial evidences of S-duality have been found for $N_f = 2N_c$ and so we restrict our attentions to this case.

The vacuum moduli space of the theory was analyzed in detail in [7, 3]. They showed that the vacuum moduli space has various branches intersecting with each other, and is locally a product of a Coulomb branch and a Higgs branch [1]. The Coulomb branch can be analyzed exactly making use of the hyperelliptic curves derived in [3, 3]:

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + 4h(h+1) \prod_{i=1}^{N_f} (x - m_i - 2ms_h), \quad N_f = 2N_c$$

(2.2)

where $m_s \equiv (1/N_f) \sum m_i$ is the flavor-singlet mass, $h(\tau) \equiv \theta_4^1(\tau)/\theta_4^1(\tau) - \theta_4^1(\tau))$ is a specific modular function of the bare gauge coupling constant $\tau = \theta/\pi + i8\pi/g^2$. This curve is invariant under S-duality transformation $\tau \rightarrow -1/\tau$, $m_i \rightarrow \tilde{m}_i \equiv m_i - 2ms$. So it is strongly suggested that there is another description of the same physics. We want to call the original and dual description by the electric and

*Due to the non-renormalization theorem, the local product structure is retained quantum mechanically.
magnetic theory respectively. The magnetic theory is also $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ QCD with $N_f$ hypermultiplets in the $\mathbf{N}_c$ representation of the gauge group, but the bare masses and the couplings are different. The superpotential is

$$W_{\text{mag}} = \sqrt{2} \tilde{g} q_i \varphi \bar{q}^i + \sqrt{2} \tilde{m}_j^i q_j \bar{q}^i ,$$

(2.3)

where $\varphi$, $q_j$ and $\bar{q}^i$ are chiral multiplets in the adjoint, $\mathbf{N}_c$ and $\mathbf{\overline{N}_c}$ representation of the gauge group respectively, $\tilde{m}_j^i \equiv m_j^i - 2m_S \delta_j^i$ and $\tilde{g} \equiv 1/g$.

The Higgs branches do not receive quantum corrections and can be analyzed classically, due to the non-renormalization theorem [3]. They are parameterized by holomorphic gauge invariant operators with several constraints [9]. As a result of detailed analysis in [3], it was shown that there is a correspondence of gauge invariant operators between the electric and magnetic theory, which is compatible with all the constraints. Namely, the Higgs branches are also the same in the electric and magnetic theory.

These facts strongly suggest that the electric and magnetic theory describe the same physics. And more surprisingly, adding adjoint mass terms, this $\mathcal{N} = 2$ duality flows down to the duality of $\mathcal{N} = 1$ supersymmetric QCD found by N.Seiberg [2]. We examine this point in the next section.

All these analysis are applicable to $SO(N_c)$ and $USp(2N_c)$ gauge theories. We investigate these cases in section 4 and section 5.

### 2.2 the correspondence of the Higgs branches

Let us explain in some detail the electric-magnetic correspondence of gauge invariant operators in the Higgs branches of the $\mathcal{N} = 2$ supersymmetric $SU(N_c)$ QCD with $2N_c$ flavors. Using the same argument as in [3], we can determine the structure of the Higgs branches for the case there are mass terms for the hypermultiplets, and show that there is a electric-magnetic correspondence of gauge invariant operators.

In the electric theory, F-term equations are as follows.

$$Q_a^i \bar{Q}_b^b = \rho \delta_a^b, \quad (\rho \in \mathbb{C}),$$

(2.4)

$$g \Phi^a_b \bar{Q}_j^b + m^i_j \bar{Q}_a^i = 0,$$

(2.5)

$$g Q_a^i \Phi_b^a + m^i_j Q_b^j = 0.$$

(2.6)

We denote the vacuum expectation values of $\Phi$, $Q$ and $\bar{Q}$ by the same symbols. (2.3) and (2.6) imply that we do not need the gauge invariant operators which are the mixture of $\Phi$ and $(Q^i, \bar{Q}_i)$ when we describe the moduli by the gauge invariant operators. The moduli space, which consists of the Coulomb branch and the Higgs branches, is parameterized by $U_k \equiv \text{tr}(\Phi^k)$, $(k = 2, \cdots, N_c)$, the

*Note that we use the notation in [2] for $q_i$ and $\bar{q}^i$, which is different from that in [3].
meson and the baryons. Roughly speaking, the Coulomb branch is parameterized by $U_k$ and the Higgs branches are parameterized by the meson and the baryons.

The meson and the baryons are defined as

\begin{align}
M^i_j & \equiv Q^i_a \tilde{Q}^a_j, \\
B^{i_1 \cdots i_{N_c}} & \equiv Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}, \\
\tilde{B}^{i_1 \cdots i_{N_c}} & \equiv \tilde{Q}^{a_1}_{i_1} \cdots \tilde{Q}^{a_{N_c}}_{i_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}.
\end{align}

By definition, the meson and the baryons are subjected to the constraints:

\begin{align}
(*)B \tilde{B} &= *(M^{N_c}), \\
(*)B \cdot M &= M \cdot \tilde{B} = 0, \\
(*)B \cdot B &= \tilde{B} \cdot \tilde{B} = 0.
\end{align}

Here we use the notations in [3] that the "\cdot" represents the contraction of a lower with an upper flavor index, and the "\ast" stands for contracting all flavor indices with the totally antisymmetric tensors $\epsilon^{i_1 \cdots i_{N_c}}$ or $\tilde{\epsilon}^{i_1 \cdots i_{N_c}}$. For example (2.10) is

\begin{equation}
\epsilon^{i_1 \cdots i_{N_c} k_1 \cdots k_{N_c}} B^{k_1 \cdots k_{N_c}} \tilde{B}^{j_1 \cdots j_{N_c}} = \epsilon^{i_1 \cdots i_{N_c} k_1 \cdots k_{N_c}} M^{k_1} \cdots M^{k_{N_c}}.
\end{equation}

There are further constraints for the meson and the baryons from the F-term equations (2.4)\textendash (2.6):

\begin{align}
M \cdot M' &= 0, \\
M' \cdot B &= B \cdot M' = 0, \\
*(m \cdot B) &= *(m \cdot \tilde{B}) = 0, \\
m \cdot M &= M \cdot m, \\
m^i_j M^j_i &= 0,
\end{align}

where $(M')^i_j \equiv M^i_j - \frac{1}{N_c} (\text{Tr } M) \delta^i_j$ \footnote{We use "Tr" for summing up flavor indices, while "tr" for color indices.}. Note that (2.10) can be rewritten in the more useful way $m \cdot B = m \cdot \tilde{B} = 0$ for generic choice of bare masses.

The constraints including $\Phi$ are as follows \footnote{We have assumed that the bare masses are chosen to be generic.}.

\begin{align}
S_k B &= S_k \tilde{B} = 0 \quad (k = 1, \cdots, N_c), \\
S_k M^{[i_1}_{j_1} \cdots M^{i_l]}_{j_l} &= 0 \quad (k + l > N_c),
\end{align}

where we have defined $S_k \equiv \epsilon^{a_1 \cdots a_{N_c}} \epsilon^{b_1 \cdots b_{k+1} \cdots a_{N_c}} \Phi^{a_1}_{b_1} \cdots \Phi^{a_k}_{b_k}$ instead of $U_k$. To show (2.20), we used the equation $m \cdot M = 0$ which will be deduced later from the explicit form of $M$, as well as the usual formula $\epsilon^{a_1 \cdots a_N} \epsilon^{b_1 \cdots b_N} = \delta^{[a_1}_{b_1} \cdots \delta^{a_N]}_{b_N}$. These constraints (2.10)\textendash(2.20) form a complete set of the classical constraints.
Owing to the non-renormalization theorem in $\mathcal{N} = 2$ supersymmetric gauge theory, we know that the Higgs branches do not receive quantum corrections \cite{3}, and so the classical constraints (2.10)$\sim$(2.18), which are the constraints for the Higgs branches, are correct quantum mechanically.

Let us solve these equations and determine the structure of the Higgs branches. We set $m = \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{2N_c})$ where $N_f$ is the number of massless flavors and $m_i$’s are chosen to be generic. Then (2.17) imply that $M$ can be put into the form

$$
\begin{pmatrix}
A & 0 \\
0 & d
\end{pmatrix},
$$

(2.21)

where $X$ is an $N_f \times N_f$ block and $d = \text{diag}(d_{N_f+1}, \cdots, d_{2N_c})$. Assuming that rank $X = r$ and rank $d = s$ \cite{4}, $M$ can be reduced to the following form up to $SU(N_f)$ massless flavor symmetry and massive flavor permutations.

$$
g^2 M = \begin{pmatrix}
X & Y \\
0 & 0 \\
& d_{N_f+1} \\
& & \ddots \\
& & & d_{N_f+s} \\
& & & & 0
\end{pmatrix},
$$

(2.22)

where $X$ is an $r \times r$ block and $Y$ is an $r \times (N_f - r)$ block with rank($XY$) = $r$, and $d_i \neq 0$ for $(i = N_f + 1, \cdots, N_f + s)$. Then (2.14) gives \(X - \frac{g^2}{N_c} \text{Tr} M \cdot (XY) = 0\) and \((d_i - \frac{g^2}{N_c} \text{Tr} M)d_i = 0\), implying

$$
X^i_j = \frac{1}{N_c}(\text{Tr} X + \sum_{k=1}^s d_{N_f+k})\delta^i_j,
$$

(2.23)

$$
d_i = \frac{1}{N_c}(\text{Tr} X + \sum_{k=1}^s d_{N_f+k}).
$$

(2.24)

The solutions of these equations exist for $r + s = N_c$ or $X = s = 0$.

First we consider the case $\text{Tr} M \neq 0$, which implies $r + s = N_c$. The constraint (2.18) gives

$$
\text{Tr} M \sum_{k=1}^s m_{N_f+k} = 0.
$$

(2.25)

For the generic choice of bare masses, this equation implies $s = 0$. $Y$ can be taken to be a diagonal matrix with real non-negative elements by an $SU(N_f)$ similarity

\footnote{\ref{2.10} and \ref{2.11} imply rank($M$) = $r + s \leq N_c$.}
transformation preserving the form (2.23). As a result, $M$ is of the form

$$g^2M = \begin{pmatrix} \rho & \kappa_1 & \cdots & \kappa_{N_f-N_c} \\ \kappa_1 & \rho & \cdots & \\ \vdots & \vdots & \ddots & \\
N_f-N_c & \kappa_{N_f-N_c} & \cdots & \rho \end{pmatrix},$$

(2.26)

where $\kappa_i \in \mathbb{R}_+$ and $\rho \in \mathbb{C}$.

For the case $\text{Tr} \: M = 0$, namely $A = s = 0$, we can diagonalize $Y$ by an $SU(N_f)$ similarity transformation:

$$g^2M = \begin{pmatrix} \kappa_1 & \cdots & \\ \vdots & \ddots & \\
2r & \kappa_r \end{pmatrix},$$

(2.27)

where $\kappa_i \in \mathbb{R}_+$ and $r \leq \lfloor N_f/2 \rfloor$.

Note that these solutions imply a useful equation $m \cdot M = 0$.

Let us consider about the baryons. For the case $\text{Tr} \: M = 0$, since rank $M \leq [N_f/2] < N_c$, (2.10) implies either $B = 0$ or $\tilde{B} = 0$. Without loss of generality, we can set $\tilde{B} = 0$ and the form of $M$ to be as in (2.27) with $\kappa_i \neq 0$. Then (2.13), (2.11) and $m \cdot B = 0$ imply that the only non-zero elements of $B^{i_1 \cdots i_{N_c}}$ are $B^{i_2 \cdots i_{r+1} \cdots i_{N_c}}$ with $2r < i_{r+1} < i_{r+2} \cdots < i_{N_c} \leq N_f$ (up to permutations of the flavor indices). So we find $B = 0$ for $N_f - N_c < r$. For $N_f - N_c \geq r$, $B$ can be non-zero. Using (2.12) and a flavor symmetry, which preserves the form of the meson (2.27), we can reduce the baryon to only one element, say $B^{1 \cdots r,2r+1 \cdots N_c+r}$. Using $U(1)_B$ symmetry, it can be chosen to be real. For the case $\text{Tr} \: M \neq 0$, the baryons are non-zero and expressed by the mesons using (2.10) up to the ratio of $B$ and $\tilde{B}$.

In summary, we have obtained two types of branches: the baryonic and the non-baryonic branches.

**< 1 > The Baryonic Branch**

$B \neq 0$ or $\tilde{B} \neq 0$, and the meson is as in (2.20). We also include the limit $B, \tilde{B} \to 0$. This branch exists for $N_c \leq N_f$.

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*This constraint is so called Plücker relation, and so we can apply Plücker embedding theorem.

†From (2.19), $B \neq 0$ or $\tilde{B} \neq 0$ implies $\Phi = 0$. 
< 2 > The non-Baryonic Branch

\[ B = \bar{B} = 0 \] and the meson is as in \((2.27)\).

As a check, we can determine the vacuum moduli space directly using the F-term and D-term equations for \(Q, \bar{Q}\) and \(\Phi\), and get the same result as above (see appendix B.1).

Next we investigate the magnetic theory and find a correspondence of the gauge invariant operators between the electric and magnetic theory.

The meson and the baryons in the magnetic theory are

\[
N^i_j \equiv q_{a_j} \tilde{q}^{a_i}, \\
b_{i_1 \cdots i_{N_c}} \equiv q_{a_1} \cdots q_{a_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}, \\
\tilde{b}^{i_1 \cdots i_{N_c}} \equiv \tilde{q}^{a_1} \cdots \tilde{q}^{a_{N_c}} \epsilon_{a_1 \cdots a_{N_c}}.
\]

(2.28) \hspace{3cm} (2.29) \hspace{3cm} (2.30)

The constraints for the meson and the baryons are

\[
(*b)\tilde{b} = *(N^{N_c}), \\
(*\tilde{b}) \cdot N = N \cdot *b = 0, \\
(*\tilde{b}) \cdot \tilde{b} = b \cdot *b = 0, \\
N \cdot N' = 0, \\
N' \cdot \tilde{b} = b \cdot N' = 0, \\
*(\tilde{m} \cdot b) = *(\tilde{m} \cdot \tilde{b}) = 0, \\
\tilde{m} \cdot N = N \cdot \tilde{m}, \\
\tilde{m}^j_i N^j_i = 0.
\]

We can see that the Higgs branches in the electric theory and those in the magnetic theory are the same under the correspondence of the gauge invariant operators:

\[
electric \leftrightarrow \text{magnetic} \\
g^2 M \leftrightarrow \tilde{g}^2 N' \\
g^{N_c} B \leftrightarrow (-\tilde{g})^{N_c} *b \\
g^{N_c} \bar{B} \leftrightarrow \tilde{g}^{N_c} *\tilde{b}.
\]

We must check that the correspondence is compatible with all the constraints. It is easy to show that \((2.11) \sim (2.13)\) imply \((2.32) \sim (2.38)\). To see that \((2.11)\) implies \((2.31)\), we use the solutions \((2.26)\) and \((2.27)\). On the baryonic branch,

\[
*\text{Note that our definition of the baryons } (b, \tilde{b}) \text{ are different from those in [3]. The baryons } (b, \tilde{b}) \text{ in [3] are defined in terms of the low energy effective theory at the root of the baryonic branch for } N_f < 2N_c.
\]
for example, we have

\[
\tilde{g}^2 N \leftrightarrow \tilde{g}^2 M' = \begin{pmatrix}
\kappa_1 & & & & \\
& \ddots & & & \\
& & \kappa_{N_f-N_c} & & \\
& & & -\rho & \\
& & & & \ddots \\
& & & & & -\rho
\end{pmatrix}.
\] (2.42)

Now it is easy to see that

\[
\tilde{g}^2 N_c \left[ i_1 \ldots N_c j_{N_c} \right] \leftrightarrow \tilde{g}^2 M'_1 \left[ i_1 \ldots M_{N_c} j_{N_c} \right] = (-g^2)^{N_c} \epsilon_{i_1 \ldots i_{2N_c}} \epsilon_{j_1 \ldots j_{2N_c}} M_{i_{N_c+1}}^{j_{N_c+1}} \ldots M_{i_{2N_c}}^{j_{2N_c}}
\] (2.43)

and that (2.10) implies (2.31). Thus we conclude that the Higgs branches in the electric and magnetic theory are the same. Combining with the fact that the Coulomb branches in the electric and magnetic theory are also the same, as mentioned in section 2.1, it means that the vacuum moduli spaces are exactly the same. This is one of the most non-trivial evidences for the existence of the S-duality.

### 2.3 the baryonic root

Before closing this section, we want to comment on the unbroken gauge group at the baryonic root \[.\] We now consider the baryonic branch with \( \text{Tr} \, M \neq 0 \) in the electric theory. From the F-term equations (2.4)–(2.6), we have

\[
\left( \frac{1}{N_c} \text{Tr} \, M \right) \text{tr}(\Phi^k) = \sum_{i=1}^{2N_c} \left( -\frac{m_i}{g} \right)^k M_i^i.
\] (2.44)

Recall that we have set \( m = \text{diag}(0, \ldots, 0, m_{N_f+1}, \ldots, m_{2N_c}) \), and \( M \) can be written as in (2.26). So, the right hand side of (2.44) vanishes, because \( N_c \leq N_f \), and (2.44) implies \( \text{tr}(\Phi^k) = 0 \). Taking the limit \( M, B, \tilde{B} \to 0 \) along this branch, we expect \( SU(N_c) \) gauge symmetry unbroken.

On the other hand, similar equation in the magnetic theory is

\[
\left( \frac{1}{N_c} \text{Tr} \, N \right) \text{tr}(\varphi^k) = \sum_{i=1}^{2N_c} \left( -\frac{\tilde{m}_i}{g} \right)^k N_i^i.
\] (2.45)

Substituting the form of \( N \) on the baryonic branch (2.42), we have

\[
\rho \text{tr}(\varphi^k) = \rho \sum_{i=N_c+1}^{2N_c} \left( -\frac{\tilde{m}_i}{g} \right)^k.
\] (2.46)

* The baryonic root is a point where \( M = B = \tilde{B} = 0 \) on the baryonic branch.
For $\rho \neq 0$, we find $\tilde{g}^k \text{tr}(\varphi^k) = \sum_{i=N_c+1}^{2N_c+1} (-\tilde{m}_i)^k$, which implies the form
\[
\tilde{g} \varphi = \text{diag}(2m_s, \ldots, 2m_s, 2m_s - m_{N_f+1}, \ldots, 2m_s - m_{2N_c})
\] (2.47)
up to permutations. Now the unbroken gauge group $SU(N_f - N_c) \times U(1)^{2N_c - N_f}$ is expected at the limit $b, \tilde{b}, N \to 0$. This result is consistent with [3].

3 Deformations of $\mathcal{N} = 2$ Theory to $\mathcal{N} = 1$ Theory

In this section, we will see how the electric and magnetic theory are deformed when we break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ adding the adjoint mass term and how the duality changes.

3.1 $\mathcal{N} = 1$ deformed electric theory

We now add the adjoint mass term $\mu \text{tr} \Phi^2$, which breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ explicitly, to the superpotential (2.1)
\[
W_{\text{ele}} = \sqrt{2} g Q^i \tilde{\Phi} Q_i + \sqrt{2} m_i Q^i \tilde{Q}_i + \frac{\mu}{\sqrt{2}} \text{tr} \Phi^2.
\] (3.1)
The F-term equation (2.4) is modified
\[
g \left( Q^i_a \tilde{Q}^b_i - \frac{1}{N_c} (Q^i \tilde{Q}_i) \delta^b_a \right) + \mu \Phi^b = 0.
\] (3.2)
We can eliminate $\Phi$ using this equation. The F-term equations (2.3) and (2.4) become
\[
\tilde{Q}^a_i (\tilde{M})^i_j = (\tilde{M})^i_j Q^a_i = 0,
\] (3.3)
where we have defined $g^2 \tilde{M} \equiv g^2 M' - \mu m$. The constraints (2.14) and (2.13) are modified to be
\[
M \cdot \tilde{M} = 0,
\] (3.4)
\[
\tilde{M} \cdot B = \tilde{B} \cdot \tilde{M} = 0.
\] (3.5)
Other constraints (2.10), (2.11), (2.12), (2.16) and (2.17) are not modified, while (2.18) is modified to a redundant constraint.

Since we have broken $\mathcal{N} = 2$ supersymmetry, the Higgs branches will receive quantum corrections. But we analyze classically for the time being and later we consider the quantum effects.

* This form is exactly the same as that in [3] which is deduced by the requirement of IR-freedom and the existence of a purely hypermultiplet Higgs branch.
Now we study the classical moduli space. As (2.10), (2.11) and (2.17) are not modified, $M$ can be put into the form (2.22)

$$g^2 M = \begin{pmatrix} X & Y \\ 0 & 0 \\ d_{N_f+1} & & \ddots \\ \vdots & & & \ddots \\ d_{N_f+s} & & & & 0 \end{pmatrix}.$$  \tag{3.6}

Then (3.4) implies that

$$X_j = \frac{1}{N_c} (\text{Tr} X + \sum_{k=1}^s d_{N_f+k}) \delta^i_j,$$  \tag{3.7}

$$d_{N_f+i} = \frac{1}{N_c} (\text{Tr} X + \sum_{k=1}^s d_{N_f+k}) + \mu m_{N_f+i}.$$  \tag{3.8}

These equations imply

$$(N_c - r - s) g^2 \text{Tr} M = N_c \mu (\sum_{i=1}^s m_{N_f+i}),$$  \tag{3.9}

which has three classes of solutions.

< 1 > The Baryonic Branch ($r = N_c, \ s = 0$)

For the case $r + s = N_c$, as we have chosen bare masses to be generic, (3.4) implies that $s = 0$ and $M$ is of the form

$$g^2 M = \begin{pmatrix} \rho & & & & \kappa_1 \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \rho & \kappa_{N_f-N_c} \\ \kappa_1 & \vdots & \vdots & \kappa_{N_f-N_c} & \rho \end{pmatrix},$$  \tag{3.10}

where $\rho \in \mathbb{C}$ and $\kappa_i \in \mathbb{R}_+.$

< 2 > The non-Baryonic Branch ($r < N_c, \ s = 0$)
For the case \( r + s < N_c \) and \( s = 0 \), we get

\[
g^2 M = \begin{pmatrix}
\kappa_1 \\
\vdots \\
\kappa_r \\
\end{pmatrix},
\]  

(3.11)

where \( \kappa_i \in \mathbb{R}_+ \) and \( r \leq \lfloor N_f/2 \rfloor \).

**< 3 > The Exceptional Branch \((s \neq 0)\)**

For the case \( r + s < N_c \) and \( s \neq 0 \), the value of \( \text{Tr} M \) is fixed:

\[
c \equiv g^2 \frac{N_c}{N_c - r - s} \mu \sum_{i=1}^{s} m_{N_f+i},
\]  

(3.12)

and \( M \) becomes

\[
g^2 M = \begin{pmatrix}
c & \kappa_1 \\
\vdots & \vdots \\
c & \kappa_r' \\
\end{pmatrix},
\]  

(3.13)

where \( \kappa_i \in \mathbb{R}_+ \), \( d_{N_f+i} = c + \mu m_{N_f+i} \) and \( r' \leq \min \{ N_f - r, r \} \).

The baryons take the same form as in the \( \mathcal{N} = 2 \) case on the baryonic and non-baryonic branches, while these are all zero on the exceptional branch. We can show that these solutions are the same as the solutions derived from the F-term and D-term equations (see appendix B.2), and that the modified constraints, which we have considered, form a complete set of constraints.

We then consider non-perturbative effects. On the baryonic branch the gauge group is broken completely and the theory is weakly coupled. So we expect that this branch will not be lifted quantum mechanically. On the non-baryonic
branch of rank \( M = r \) the gauge group is broken to \( SU(N_c - r) \) and the number of the massless quarks are \( N_f - 2r \). For \( r > N_f - N_c \), Affleck-Dine-Seiberg superpotential is generated \([10]\), and the classical vacua are lifted. The non-baryonic branch of \( r \leq N_f - N_c \) is a submanifold of the baryonic branch, i.e. \( \rho = 0 \) in \([11]\). On the exceptional branch the theory become \( \mathcal{N} = 1 \) \( SU(N_c - r - s) \) Super Yang-Mills theory with several massless singlets. Then owing to the gaugino condensation, the dynamical superpotential is generated and we expect that this branch will disappear quantum mechanically.

As a result, we conclude that only the baryonic branch remains as the vacuum moduli space.

### 3.2 \( \mathcal{N} = 1 \) deformed magnetic theory and duality

Before presenting the answer, it would be instructive to show how we find the duality. Notice that the equations \([3.4]\), \([2.11]\) and \([3.5]\) are symmetric under Hodge dual transformations for the baryons \( (B \rightarrow \ast B, \tilde{B} \rightarrow \ast \tilde{B}) \) and interchanging \( M \) and \( \tilde{M} \). This fact suggests that there is a dual (magnetic) theory in which the meson \( \tilde{g}^2 \tilde{N} \) corresponds to \( g^2 \tilde{M} = g^2 M' - \mu m \). Then we find that \([3.4]\) implies

\[
\tilde{N} \cdot N = 0, \quad (3.14)
\]

where \( \tilde{g}^2 \tilde{N} \equiv \tilde{g}^2 N' + \mu \tilde{m} \). The superpotential, whose F-term equation implies \([3.14]\), is

\[
W_{\text{mag}} = \sqrt{2} \tilde{g} q_i \varphi \tilde{q}^i + \sqrt{2} \tilde{m} q_i \tilde{q}^i - \frac{\mu}{\sqrt{2}} \text{tr} \varphi^2. \quad (3.15)
\]

Note that the sign of the adjoint mass term is different from that in the electric theory.

We claim that this theory is dual to the electric theory \([3.1]\). The correspondence of the meson and the baryons is

\[
electric \leftrightarrow \text{magnetic} \quad g^2 M \leftrightarrow \tilde{g}^2 \tilde{N} \quad (3.16)
\]

\[
g^{N_c} B \leftrightarrow (-\tilde{g})^{N_c} \ast b \quad (3.17)
\]

\[
g^{N_c} \tilde{B} \leftrightarrow \tilde{g}^{N_c} \ast \tilde{b}. \quad (3.18)
\]

This correspondence smoothly flows to that in \( \mathcal{N} = 2 \) theory in the limit \( \mu \rightarrow 0 \).

The inverse map is

\[
electric \leftrightarrow \text{magnetic} \quad g^2 \tilde{M} \leftrightarrow \tilde{g}^2 N \quad (3.19)
\]

---

* In order to show these facts, it may be useful to see the explicit form of \( Q \) and \( \tilde{Q} \), which is listed in appendix \([3.2]\).
\[ g^{Nc} \ast B \Leftrightarrow \tilde{g}^{Nc}b \]  
\[ (-g)^{Nc} \ast \tilde{B} \Leftrightarrow \tilde{g}^{Nc}\tilde{b}. \]  
\[ (3.20) \]
\[ (3.21) \]

The following correspondence of the constraints is almost trivial.

\[ M \cdot \tilde{M} = 0 \Leftrightarrow \tilde{N} \cdot N = 0 \]  
\[ M \cdot (*B) = M \cdot (\ast \tilde{B}) = 0 \Leftrightarrow \tilde{N} \cdot b = \tilde{N} \cdot \tilde{b} = 0 \]  
\[ \tilde{M} \cdot B = \tilde{M} \cdot \tilde{B} = 0 \Leftrightarrow N \cdot (\ast \tilde{b}) = N \cdot (\ast \tilde{b}) = 0 \]  
\[ m \cdot M = M \cdot m \Leftrightarrow \tilde{m} \cdot N = N \cdot \tilde{m}. \]  
\[ (3.22) \]
\[ (3.23) \]
\[ (3.24) \]
\[ (3.25) \]

Now, we must check

\[ (*B)\tilde{B} = *(M^{Nc}) \Leftrightarrow (\ast b)\tilde{b} = *(N^{Nc}). \]  
\[ (3.26) \]

\( M \) is of the form \((3.10)\) and then \( N \) is

\[ \tilde{g}^2N \leftrightarrow g^2\tilde{M} = \begin{pmatrix} \kappa_1 \\ \vdots \\ -\rho \\ \kappa_{N_f-N_c} \\ \vdots \\ -\rho \\ -\sigma_{N_f+1} \\ \vdots \\ -\sigma_{2N_c} \end{pmatrix}, \]  
\[ (3.27) \]

where \( \sigma_{N_f+i} = \rho + \mu m_{N_f+i} \).

Then we have

\[ \tilde{g}^{2Nc}J_{J_1}^{i_1} \cdots J_{J_{Nc}}^{i_{Nc}} \leftrightarrow g^{2Nc}\tilde{M}_{J_1}^{i_1} \cdots \tilde{M}_{J_{Nc}}^{i_{Nc}} \]
\[ = (-g^2)^{Nc} \left( \prod_{i=N_f+1}^{2N_c} \frac{\sigma_i}{\rho} \right)^{1/2} \epsilon_{i_1 \cdots i_{2Nc}}^{a_1 \cdots a_{Nc}} q_{i_1}^{a_1} \cdots q_{i_{Nc}}^{a_{Nc}} M_{i_{Nc+1}}^{i_1} \cdots M_{i_{2Nc}}^{i_{2Nc}} \]  
\[ (3.28) \]

So, if we choose the normalization factor for baryons as

\[ b_{i_1 \cdots i_{Nc}} = \left( \prod_{i=N_f+1}^{2N_c} \frac{\rho}{\sigma_i} \right)^{1/2} \epsilon_{a_1 \cdots a_{Nc}} q_{i_1}^{a_1} \cdots q_{i_{Nc}}^{a_{Nc}}, \]  
\[ (3.29) \]
\[ \tilde{b}^{i_1 \cdots i_{Nc}} = \left( \prod_{i=N_f+1}^{2N_c} \frac{\rho}{\sigma_i} \right)^{1/2} \epsilon_{a_1 \cdots a_{Nc}} \widehat{q}_{i_1}^{a_1} \cdots \widehat{q}_{i_{Nc}}^{a_{Nc}}, \]  
\[ (3.30) \]
where
\[ \rho = \frac{g^2}{N_c} \text{Tr} N - 2\mu m_S \leftrightarrow \frac{g^2}{N_c} \text{Tr} M, \quad (3.31) \]
\[ \sigma_i = \frac{g^2}{N_c} \text{Tr} N + \mu \bar{m}_i \leftrightarrow \frac{g^2}{N_c} \text{Tr} M + \mu m_i, \quad (3.32) \]
we get the correspondence (3.26). Note that the normalization factor can be rewritten in the flavor singlet form:
\[ \prod_{i=N_f+1}^{2N_c} \frac{\sigma_i}{\rho} = \prod_{i=1}^{2N_c} \frac{\sigma_i}{\rho} = \frac{\det(\sigma^i_j)}{\rho^{2N_c}}, \quad (3.33) \]
where we have defined
\[ \sigma^i_j \equiv -\frac{g^2}{N_c} \text{Tr} N \delta^i_j + \mu \bar{m}^i_j. \quad (3.34) \]

If there were the non-baryonic branches of \( r \geq N_f - N_c \) or the exceptional branches in the electric theory, the rank of the corresponding \( N \) would become larger than \( N_c \) contradicting with (2.31) and (2.32).

### 3.3 the gauge group in the magnetic theory

In the last subsection, we have seen that the vacuum moduli space in the electric and magnetic theory are the same, and claimed that there is a duality in the \( \mathcal{N} = 1 \) deformed theories. Let us show that the magnetic theory is \( \mathcal{N} = 1 \) supersymmetric \( SU(N_f - N_c) \) QCD as expected from the \( \mathcal{N} = 1 \) duality of N.Seiberg.

The F-term equations in the \( \mathcal{N} = 1 \) deformed magnetic theory are
\[ \tilde{g} \left( q_{ai} \tilde{q}^{bi} - \frac{1}{N_c} (q_i \tilde{q}^i) \sigma_a^b \right) = \mu \varphi_a^b, \quad (3.35) \]
\[ \tilde{g} \varphi^a_b \tilde{q}^{b_j} + \bar{m}^i_j \tilde{q}^{ai} = 0, \quad (3.36) \]
\[ \tilde{g} q_{ai} \varphi^a_b + \bar{m}^i_j q_{bj} = 0. \quad (3.37) \]

From these equations we have recursive relations of \( u_k \equiv \text{tr} \varphi^k \):
\[ u_k = \frac{\tilde{g}^2}{\mu} \left( -\frac{1}{N_c} (\text{Tr} N) u_{k-1} + \sum_i (-\bar{m}^i_j) (k-1) N_i^j \right) \quad (3.38) \]
with the initial condition \( u_1 = 0 \). We are interested in the baryonic root where \( SU(N_c) \) gauge symmetry is expected to be unbroken in the electric theory. The

\[ ^* \text{There is a subtlety for the case } \rho = 0 \text{ or } \sigma_i = 0, \text{ but this arises only for submanifolds of the baryonic branch.} \]
point \( M = 0 \) corresponds to the point \( g^2 N = -\mu m \) (see (3.19)) in the magnetic theory. At this point, the solution of (3.38) is again
\[
\begin{align*}
  u_k &= \text{tr} \varphi^k = \frac{2 N_c}{\sum_{i=N_c+1}^{2 N_c}} \left( -\frac{m_i}{g} \right)^k, \\
  \varphi &= \text{the same as in (2.47)}.
\end{align*}
\]
and \( \varphi \) is the same as in (2.47). So, the gauge group is broken by \( \langle \varphi \rangle \) to \( SU(N_f - N_c) \times U(1)^{2 N_c - N_f} \), and the condensations of dual quarks
\[
\langle q_i \bar{q}_i \rangle = -\frac{\mu m_i}{g^2}, \quad (i = N_f + 1, \cdots, 2 N_c)
\]
break the \( U(1)^{2 N_c - N_f} \) factor. As a result, the magnetic theory is \( \mathcal{N} = 1 \) supersymmetric \( SU(N_f - N_c) \) QCD with \( N_f \) flavors.

### 3.4 Leigh-Strassler transformation

The magnetic theory of N.Seiberg’s duality in \( \mathcal{N} = 1 \) supersymmetric QCD is \( \mathcal{N} = 1 \) supersymmetric \( SU(N_f - N_c) \) theory with \( N_f \) flavors of dual quarks \( q_i, \bar{q}_i \) \((i = 1, \cdots, N_f)\) and a gauge singlet meson field \( M^i_j \), interacting with the dual quarks by the superpotential:
\[
W_{\text{mag}}^{\mathcal{N}=1} = q_i M^i_j \bar{q}_j.
\]

So far, we have needed neither the meson field \( M^i_j \) nor the superpotential \( W_{\text{mag}}^{\mathcal{N}=1} \). So our derivation of \( \mathcal{N} = 1 \) duality may seem to be inconsistent. But, it is not the case. There was an argument given by R.G.Leigh and M.J.Strassler [5] which showed the origin of the meson field and the superpotential for \( N_f = 2 N_c \) case. We will now show that applying their argument to the case with quark mass terms, there appears the superpotential (3.41) with the meson field \( M \).

The electric theory discussed in section 3.1 has the superpotential (3.4):
\[
W_{\text{ele}} = \sqrt{2} g Q^i \Phi Q_i + \sqrt{2} m_i Q^i \bar{Q}_i + \frac{\mu}{\sqrt{2}} \text{tr} \Phi^2.
\]
Integrating out \( \Phi \), (i.e. inserting (3.2)), we get
\[
W_{\text{ele}} = -\frac{g^2}{\sqrt{2} \mu} \left( (Q^i \bar{Q}_i)(Q^j \bar{Q}_j) - \frac{1}{N_c} (Q^i \bar{Q}_i)^2 \right) + \sqrt{2} m_i Q^i \bar{Q}_i.
\]
We consider the theory around \( Q^i = \bar{Q}_i = 0 \). In the limit \( g \to 0 \), we can neglect the first term:
\[
W_{\text{ele}} \sim \sqrt{2} m_i Q^i \bar{Q}_i.
\]
Integrating out the massive quarks, the theory turns out to be \( \mathcal{N} = 1 \) supersymmetric QCD with \( N_f \) massless flavors.
On the other hand, the magnetic theory discussed in section 3.2 has the superpotential (3.13):

\[
W_{\text{mag}} = \sqrt{2} \tilde{g} q_i \varphi \tilde{q}^i + \sqrt{2} \tilde{m}_i q_i \tilde{q}^i - \frac{\mu}{\sqrt{2}} \text{tr} \varphi^2.
\] (3.45)

Similarly, we integrate out \( \varphi \), (i.e. inserting (3.35)):

\[
W_{\text{mag}} = \frac{\tilde{g}^2}{\sqrt{2} \mu} \left( (q_i \tilde{q}^i)(q_j \tilde{q}^j) - \frac{1}{N_c} (q_i \tilde{q}^i)^2 \right) + \sqrt{2} \tilde{m}_i q_i \tilde{q}^i
\] (3.46)

\[
= q_i \mathcal{N}^\prime_{ij} \tilde{q}^j + \sqrt{2} \tilde{m}_i (q_i \tilde{q}^i) - \frac{\mu}{2\sqrt{2} \tilde{g}^2} \left( \text{Tr} \mathcal{N}^\prime \mathcal{N}^\prime - \frac{1}{N_c} (\text{Tr} \mathcal{N}^\prime)^2 \right),
\] (3.47)

where we have introduced an auxiliary field \( \mathcal{N}^\prime\) which can be eliminated by the equation of motion (F-term equation)

\[
\mathcal{N}^\prime_{ij} = \frac{\sqrt{2} \tilde{g}^2}{\mu} \left( q_j \tilde{q}^j - \frac{1}{N_c} \text{Tr}(q \tilde{q}) \delta_{ij} \right) = \frac{\sqrt{2} \tilde{g}^2}{\mu} (N^\prime)_{ij}.
\] (3.48)

From the correspondence of the meson (3.16), it is clear that we should rewrite the superpotential using

\[
\mathcal{M}^i_j \equiv \mathcal{N}^\prime_{ij} + \sqrt{2} \tilde{m}_j
\] (3.49)

\[
= \frac{\sqrt{2}}{\mu} \left( \tilde{g}^2 (N^\prime)_{ij} + \mu \tilde{m}_j \right) \quad \text{(on shell)}
\] (3.50)

\[
\leftrightarrow \frac{\sqrt{2}}{\mu} \tilde{g}^2 M^i_j.
\] (3.51)

Then the superpotential becomes

\[
W_{\text{mag}} = q_i \mathcal{M}^i_j \tilde{q}^j + \frac{\mu m_i}{g^2} \mathcal{M}^i_i - \frac{\mu}{2\sqrt{2} g^2} \left( \text{Tr} \mathcal{M}^2 - \frac{1}{N_c} (\text{Tr} \mathcal{M})^2 \right) + \text{const.}
\] (3.52)

We take the same limit as above \( \tilde{g} \sim 1/g \to \infty \), fixing (3.39) and (3.40). Then we can neglect the third term:

\[
W_{\text{mag}} \sim q_i \mathcal{M}^i_j \tilde{q}^j + \frac{\mu m_i}{g^2} \mathcal{M}^i_i.
\] (3.53)

The equations of motion,

\[
q_i \tilde{q}^i = -\frac{\mu m_i}{g^2} \neq 0 \quad (i = N_f + 1, \cdots, 2N_c)
\] (3.54)

imply that the gauge group is broken to \( SU(N_f - N_c) \), and the first term in (3.53) gives masses to \( q_i, \tilde{q}^i \), and \( \mathcal{M}^i_i \) for \( i, j = N_f + 1, \cdots, 2N_c \). Thus the massless fields are \( N_f \) dual-quarks \( (q_i, \tilde{q}^i) \) and the meson field \( (\mathcal{M}^i_j, (i, j = 1, \cdots, N_f)) \) which are
interacting with each other by the superpotential \((3.41)\). So, the theory is exactly the magnetic theory of N.Seiberg’s duality. Note that although the meson field \(M_{ij}\) appeared as an auxiliary field, we expect that this field becomes dynamical as a result of quantum effects, otherwise 't Hooft anomaly matching conditions are not satisfied \([2]\).

Conversely, we can begin our argument with N.Seiberg’s \(\mathcal{N}=1\) duality. Consider \(\mathcal{N}=1\) supersymmetric \(SU(N_c)\) QCD with \(2N_c\) flavors and its dual. The meson field \(M\) in the magnetic theory corresponds to \(\tilde{Q}\tilde{Q}\) in the electric theory. When we deform the superpotential in the electric theory as \((3.43)\), the corresponding deformation of the superpotential in the magnetic theory is as \((3.52)\). Introducing the adjoint auxiliary field, we can rewrite the superpotential as \((3.42)\) or \((3.45)\).

\section{Duality in \(SO(N_c)\) Gauge Theory}

\subsection{S-duality in \(\mathcal{N}=2\) theory}

We consider in this subsection \(\mathcal{N}=2\) supersymmetric \(SO(N_c)\) QCD with \(N_c-2\) hypermultiplets in the vector representation of the gauge group. We have chosen the number of the hypermultiplets so that the theory is scale invariant, and has a dual description. As in the \(SU(N_c)\) case, we describe the theory in terms of \(\mathcal{N}=1\) superfields \([\star]\) by a field strength chiral multiplet \(W_{ab}^\alpha\) and a chiral multiplet \(\Phi_{ab}\), both in the adjoint representation of the gauge group, and chiral multiplets \(Q^i_a\) in the vector representation of the gauge group, where \(a, b = 1, \cdots, N_c\) are color indices, and \(i = 1, \cdots, 2(N_c-2)\) are flavor indices. The superpotential is

\[ W_{\text{ele}} = \sqrt{2} g Q^i_a \Phi_{ab} Q^j_b \mathbb{J}_{ij} + \sqrt{2} m_{ij} Q^i_a Q^j_a, \]

where \(\mathbb{J} \equiv (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \otimes \mathbb{I}\) is the symplectic metric and \(\mathbb{I}\) is the \((N_c-2) \times (N_c-2)\) identity matrix. We raise and lower the flavor indices by contracting with \(\mathbb{J}\). See appendix \([\star]\) for our conventions.

Since the pairs \((Q^i, Q^{N_c-2+i})\) make up \(\mathcal{N}=2\) hypermultiplets, the bare mass matrix \(m_{ij}\) is \(m = (m_{ij}) \equiv (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) \otimes \text{diag}(m_1, \cdots, m_{N_c-2})\).

As commented in section \([\star]1\), S-dualities in \(\mathcal{N}=2\) supersymmetric \(SO(N_c)\) theories are also known \([\star]4, \star]8\). The hyperelliptic curve which describes the coulomb phase of the theory is

\[ y^2 = x \prod_{a=1}^{[N_c/2]} (x - \phi_a^0)^2 + 4fx^{3-\epsilon} \prod_{j=1}^{N_c-2} (x - m_j^2), \]

where \(f(\tau) = \theta_2^4 \theta_4^4 / (\theta_2^4 - \theta_4^4)^2\) and \(\epsilon = (N_c \mod 2) \[\star]1, \star]8\). This curve is invariant under \(\tau \to -1/\tau\), and so we expect that the magnetic theory is also \(\mathcal{N}=2\) supersymmetric \(SO(N_c)\) QCD with \(N_c-2\) hypermultiplets in the vector representation

\footnote{We use the notations in \([\star]4, \star]7\).}
of the gauge group, and has the superpotential
\[ W_{\text{mag}} = \sqrt{2} \tilde{g} q_i^a \phi^{ab} q_j^b \bar{q}^{ij} + \sqrt{2} m^{ij} q_i^a q_j^a, \] (4.3)
where \( \tilde{g} = 1/g. \)

### 4.2 \( \mathcal{N}=1 \) deformed electric theory

As in the \( SU \) case, we add the adjoint mass term and break \( \mathcal{N} = 2 \) supersymmetry to \( \mathcal{N} = 1 \) explicitly:
\[ W_{\text{ele}} = \sqrt{2} g Q_a^i \Phi_{ab} Q^j_b \bar{q}^{ij} + \sqrt{2} m_{ij} Q_a^i Q_j^a + \frac{\mu}{\sqrt{2}} \text{tr} \Phi^2. \] (4.4)

The F-term equations are
\[
\begin{align*}
g Q_a^i \bar{q}^{ij} Q_j^b - \mu \Phi_{ab} &= 0, \\
g \Phi_{ab} Q^j_b \bar{q}^{ij} + m_{ij} Q_a^i &= 0.
\end{align*}
\]

From (4.5), we can eliminate \( \Phi \), and so the vacuum moduli space is parameterized by the meson and the baryon which are defined by
\[
\begin{align*}
M^{ij} &= Q_a^i Q_j^a, \\
B^{i_1 \cdots i_{N_c}} &= Q_{a_1}^{i_1} \cdots Q_{a_{N_c}}^{i_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}.
\end{align*}
\]

The constraints of these operators following from the definitions are
\[
\begin{align*}
(*B)B &= *(M^N), \\
M \cdot *B &= 0, \\
(*B) \cdot B &= 0.
\end{align*}
\]

Here the meaning of “*” and “:” are almost the same as that defined in section 2.2, but for \( *(M^N) \) we mean \( \epsilon_{i_1 \cdots i_2 N_c-4} M^{i_1 i_2} \cdots M^{i_{N_c} j_{N_c}} \).

Inserting (4.5) into (4.4), we get
\[ Q_a^i \bar{q}^{ij} \tilde{M}^{jk} = 0, \] (4.12)
where we have defined \( g^2 \tilde{M}^{jk} \equiv g^2 M^{jk} - \mu m^{jk} \) and \( m^{jk} \equiv \bar{q}^{ij} m_{j'k'} q^{k'j} (= m_{jk}). \)

The constraints following from this equation are
\[
\begin{align*}
M^{ij} \bar{q}^{jk} \tilde{M}^{kl} &= 0, \\
\tilde{M} \cdot \bar{q} \cdot B &= 0, \\
*(m \cdot \bar{q} \cdot B) &= 0.
\end{align*}
\]

(4.9) and (4.10) imply rank \( M \leq N_c \). From (4.9) it follows that \( B \) is zero for rank \( M < N_c \) and \( B \) can be expressed by \( M \) for rank \( M = N_c \). It is easy to
check that the other constraints including $B$ are redundant. Thus the vacuum moduli space is parameterized by $M$ only, and the constraints are (4.13) and rank $M \leq N_c$.

We can determine the vacuum moduli space. (4.13) implies

$$M^{ij}J_{jk}m^{kl} = m^{ij}J_{jk}M^{kl}. \quad (4.16)$$

We set $m^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{N_c-2})$, where $m_i$s are chosen to be generic, and then (4.16) implies

$$g^2 M = \begin{pmatrix} X & 0 & Y & 0 \\ 0 & 0 & 0 & d \\ Y^T & 0 & Z & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \quad (4.17)$$

where $X$ and $Z$ are $N_f \times N_f$ symmetric matrices, $Y$ is an $N_f \times N_f$ matrix and $d = \text{diag}(d_{N_f+1}, \cdots, d_{N_c-2})$. From (4.13) we find

$$\begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix} = 0, \quad (4.18)$$

$$d_i(d_i - \mu m_i) = 0, \quad i = N_f + 1, \cdots, N_c - 2. \quad (4.19)$$

It was shown in [4] that the solutions of (4.18) can be reduced to $X = \text{diag}(a_1, \cdots, a_{N_f})$, $(a_i \in \mathbb{R}_+)$, $Y = Z = 0$, using similarity transformations of massless flavor symmetry group $USp(2N_f)$. So we get

$$g^2 M = \begin{pmatrix} X & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix}, \quad (4.20)$$

where $X = \text{diag}(a_1, \cdots, a_r, 0, \cdots, 0), a_i \in \mathbb{R}_+, d = \text{diag}(\mu m_{N_f+1}, \cdots, \mu m_{N_f+s}, 0, \cdots, 0)$ and $r + 2s \leq N_c$.

This form is the same as that derived from the F-term and D-term equations for $Q$ and $\Phi$ (see appendix [B.3]), and so the constraints form a complete set of the classical constraints.

Let us consider the quantum effects. When the meson is as (4.20), the theory is $\mathcal{N} = 1$ supersymmetric $SO(N_c-r-2s)$ QCD with $2(N_f-r)$ massless quarks in the vector representation. If $2(N_f-r) \leq (N_c-r-2s)-5$ (i.e. $r-2s > 2N_f-N_c+4$), Affleck-Dine-Seiberg type dynamical superpotential is generated and the classical vacua are lifted [11]. Therefore the vacuum moduli space consists of the branches of $r-2s \leq 2N_f-N_c+4$.

The structure of the vacuum moduli space is rather different from that in the $SU(N_c)$ theory. We have seen in section 3.1 that there is essentially one branch
(baryonic branch) in the $SU(N_c)$ theory. In the $SO(N_c)$ theory, however, we have found several distinct branches for $d$ in (4.20) is a fixed matrix. Each branch has a point with unbroken $SO(N_c - 2s)$ gauge symmetry ($X = 0$ in (4.20)). So we have a series of $SO$ theories (i.e. $SO(N_c)$, $SO(N_c - 2)$, $\cdots$, $SO(2N_f - N_c + 4)$) all at once.

4.3 $\mathcal{N}=1$ deformed magnetic theory and duality

The superpotential of the $\mathcal{N}=1$ deformed magnetic theory is

$$W_{\text{mag}} = \sqrt{2} \tilde{g}q^a_i \varphi^{ab} q^b_j \mathbb{J}^{ij} + \sqrt{2} m^{ij} q^a_i q^a_j - \frac{\mu}{\sqrt{2}} \text{tr} \varphi^2,$$

where $\tilde{g} \equiv 1/g$. We define the meson and the baryon as follows.

$$N_{ij} \equiv q^a_i q^a_j,$$  \hspace{1cm} (4.22)

$$b_{i_1 \cdots i_{N_c}} \equiv q^a_{i_1} \cdots q^a_{i_{N_c}} \epsilon^{a_1 \cdots a_{N_c}}.$$  \hspace{1cm} (4.23)

As in the electric theory, the baryon is redundant. We claim that the magnetic theory is dual to the electric theory under the correspondence:

$$\text{electric} \leftrightarrow \text{magnetic}$$

$$g^2 M^{ij} \leftrightarrow \tilde{g}^2 \tilde{N}^{ij} \equiv \tilde{g}^2 \mathbb{J}^{ij} \tilde{N}^{ij}$$

$$\left( g^2 \tilde{M}^{ij} \leftrightarrow \tilde{g}^2 N^{ij} \right),$$  \hspace{1cm} (4.24)

where $\tilde{g}^2 \tilde{N}^{ij} \equiv \tilde{g}^2 N^{ij} + \mu m^{ij}$.

It is trivial to check the correspondence of the constraint:

$$M^{ij} \mathbb{J}^{jl} \tilde{M}^{kl} = 0 \leftrightarrow \tilde{N}^{ij} \mathbb{J}^{jk} N_{kl} = 0.$$  \hspace{1cm} (4.25)

When the meson is as (4.20), the rank of $N$ is $r + 2(N_c - 2 - N_f - s)$. As explained in the last subsection, we know that $r - 2s \leq 2N_f - N_c + 4$, which implies rank $N \leq N_c$. As a result, we conclude that the vacuum moduli space in the electric and magnetic theory are the same.

4.4 the correspondence of the gauge group

We can determine the gauge groups in the electric and magnetic theory, and show that the duality in the last subsection is consistent with N.Seiberg’s duality.

The F-term equation (4.13) implies

$$\text{tr} \Phi^k = \left( \frac{g}{\mu} \right)^k \text{Tr}(\mathbb{J} \cdot M)^k.$$  \hspace{1cm} (4.26)

*Here we call a connected component of the vacuum moduli space by a branch.
If we choose the branch in (4.20), and take \( X = 0 \) then
\[
g^k \text{tr} \Phi^k = \begin{cases} 
0 & (k : \text{odd}) \\
2 \sum_{i=1}^s (m_{N_f+i})^k & (k : \text{even}).
\end{cases}
\] (4.27)

From this, we have
\[
g\Phi = \begin{pmatrix} 
0 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & im_{N_f+1} & \cdots \\
-\im_{N_f+1} & \cdots & -\im_{N_f+s} & \cdots \\
\vdots & \cdots & \ddots & \ddots \\
-\im_{N_f+s+1} & \cdots & \im_{N_c-2} & \cdots \\
0 & \cdots & \ddots & \ddots \\
\vdots & \cdots & \ddots & \ddots \\
\end{pmatrix}.
\] (4.28)

So we expect \( SO(N_c-2s) \) gauge symmetry unbroken ¹, and there are \( 2N_f \) massless quarks in the vector representation of the unbroken gauge group.

On the other hand, on the corresponding branch in the magnetic theory,
\[
\tilde{g}^k \text{tr} \varphi^k = \left( \frac{-g^2}{\mu} \right)^k \text{Tr}(\mathbb{J}^{-1} \cdot N)^k = \begin{cases} 
0 & (k : \text{odd}) \\
2 \sum_{i=N_f+s+1}^{N_c-2} (m_i)^k & (k : \text{even}),
\end{cases}
\] (4.29)
implying
\[
\tilde{g}\varphi = \begin{pmatrix} 
0 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & im_{N_f+s+1} & \cdots \\
-\im_{N_f+s+1} & \cdots & -\im_{N_c-2} & \cdots \\
\vdots & \cdots & \ddots & \ddots \\
\end{pmatrix}.
\] (4.30)

Thus the magnetic theory is expected to be \( SO(2N_f - (N_c-2s) + 4) \) gauge theory with \( 2N_f \) massless dual quarks in the vector representation. This result supports N.Seiberg’s duality in \( \mathcal{N} = 1 \) supersymmetric \( SO \) QCD ² ³ ⁴ ⁵.

Note that unlike the \( SU \) case, in which we have obtained a duality for \( SU(N_c) \leftrightarrow SU(N_f-N_c) \), we have obtained a series of the duality for \( SO(N_c-2s) \leftrightarrow SO(2N_f - (N_c-2s) + 4) \), \( s = 0, 1, \cdots, N_c-2-N_f \).

¹\( U(1) \) factors are broken by \( M \neq 0 \) (see section 3.3).

²Here \( N_f \) is the number of the massless hypermultiplets, and so the number of \( \mathcal{N} = 1 \) chiral multiplets in the vector representation of the gauge group is \( 2N_f \).
4.5 Leigh-Strassler transformation

Similar arguments as in section 3.4 can be applied to the $SO(N_c)$ theory, and we can show that the meson field and the superpotential are needed in the magnetic theory. For simplicity, we investigate the duality on the branch of $s = 0$ in (4.20).

Integrating out $\Phi$ in the electric theory (4.4), we have

$$W_{\text{ele}} = \frac{g^2}{\sqrt{2\mu}}(Q^iQ^j)J_{ik}J_{jl}(Q^kQ^l) + \sqrt{2}m_{ij}Q^iQ^j. \quad (4.31)$$

If we take the limit $g \to 0$, we can neglect the first term:

$$W_{\text{ele}} \sim \sqrt{2}m_{ij}Q^iQ^j. \quad (4.32)$$

Integrating out massive flavors, the theory turns out to be $N = 1$ supersymmetric $SO(N_c)$ QCD with $2N_f$ massless chiral multiplets in the vector representation of the gauge group.

In the magnetic theory, on the other hand, integrating out $\phi$, we get

$$W_{\text{mag}} = -\tilde{g}^2\frac{\mu}{\sqrt{2\mu}}(q_iq_j)\tilde{J}^{ij}(q_kq_l) + \sqrt{2}m^{ij}q_iq_j. \quad (4.33)$$

We introduce an auxiliary field $\mathcal{M}^{ij}$ and rewrite this superpotential:

$$W_{\text{mag}} = \mathcal{M}^{ij}q_iq_j + \frac{\mu}{\tilde{g}^2}m_{ij}\mathcal{M}^{ij} - \frac{\mu}{2\sqrt{2}\tilde{g}^2}\mathcal{M}_{ij}\mathcal{M}^{ij}. \quad (4.34)$$

The equation of motion implies $\mu\mathcal{M}^{ij} = \sqrt{2}(\tilde{g}^2N^{ij} + \mu m^{ij}) \leftrightarrow \sqrt{2}\tilde{g}^2\mathcal{M}^{ij}$. In the limit $\tilde{g} \sim 1/g \to \infty$, fixing $\langle qq \rangle \sim \mu m/\tilde{g}^2$ and $\langle \varphi \rangle \sim m/\tilde{g}$, we get

$$W_{\text{mag}} \sim \mathcal{M}^{ij}q_iq_j + \frac{\mu}{\tilde{g}^2}m_{ij}\mathcal{M}^{ij}. \quad (4.35)$$

The theory flows down to $N = 1$ supersymmetric $SO(2N_f - N_c + 4)$ QCD with $2N_f$ massless flavors interacting with a meson field $\mathcal{M}^{ij}$, which is the magnetic theory in the N.Seiberg’s duality [2, 11].

5 Duality in $USp(2N_c)$ Gauge Theory

5.1 S-duality in $N = 2$ theory

In this subsection, we consider $N = 2$ supersymmetric $USp(2N_c)$ QCD with $2N_c + 2$ hypermultiplets in the $2N_c$ representation of the gauge group. The superpotential is

$$W_{\text{ele}} = \sqrt{2}gQ^a\Phi_b\Phi_c\Phi_b^{bc}Q^i_c + \sqrt{2}m_{ij}Q^i_aQ^j_b. \quad (5.1)$$
where \(a, b = 1, \ldots, 2N_c\) are color indices, \(i, j = 1, \ldots, 4N_c + 4\) are flavor indices, and \(m = (m_{ij}) \equiv (0_{-1}^{1}) \otimes \text{diag}(m_1, \ldots, m_{2N_c+2})\) is an anti-symmetric matrix.

The hyperelliptic curve for the theory was determined in [7] and was found to be invariant under \(T: \tau \rightarrow \tau + 1, \prod m_j \rightarrow -\prod m_j\), and \(ST^2S: \tau \rightarrow \tau/(1-2\tau)\). Although it is not a simple strong-weak duality, we expect that there is a magnetic theory, which is also an \(N = 2\) supersymmetric \(USp(2N_c)\) QCD with \(2N_c + 2\) hypermultiplets in the \(2N_c\) representation of the gauge group, whose superpotential is

\[
W_{\text{mag}} = \sqrt{2} \tilde{g} q_i^a \phi_a b q^c_i + \sqrt{2} m_{ij} q_i^a \phi_{ab} q^b_j, \tag{5.2}
\]

where \(\tilde{g}\) is the dual gauge coupling in \(\tilde{\tau} = \tau/(1-2\tau)\).

### 5.2 \(\mathcal{N}=1\) deformed electric theory

As we have investigated in \(SU\) and \(SO\) case, we break \(\mathcal{N} = 2\) supersymmetry to \(\mathcal{N} = 1\) supersymmetry by adding the adjoint mass term:

\[
W_{\text{ele}} = \sqrt{2} g Q^i_a \Phi^b_c q^c_i + \sqrt{2} m_{ij} q^i_a \phi_{ab} q^j_b + \frac{\mu}{\sqrt{2}} \text{tr} \Phi^2. \tag{5.3}
\]

The F-term equations are

\[
g Q^i_a Q^j_c + \mu \phi^{bc}_i q^c_i = 0, \tag{5.4}
\]

\[
g Q^i_a \phi^b_c - m_{ij} Q^j_b = 0. \tag{5.5}
\]

In the \(USp\) theory, there is no baryon and the vacuum moduli space is parameterized by the meson \(M^{ij} \equiv Q^i_a \phi^{ab} Q^j_b\).

The constraints for the meson are

\[
M \cdot \tilde{M} = 0, \tag{5.6}
\]

\[
\epsilon_{i_1 \cdots i_{4N_c+4}} M^{i_1 i_2} \cdots M^{i_2 N_c+1 i_2 N_c+2} = 0, \tag{5.7}
\]

where we have defined \(g^2 \tilde{M} \equiv g^2 M + \mu m\).

(5.6) implies

\[
M \cdot m = m \cdot M. \tag{5.8}
\]

We set \(m_{ij} = (0_{-1}^{1}) \otimes \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{N_c-2})\), where \(m_i\)'s are chosen to be generic, and then (5.8) implies

\[
g^2 M = \begin{pmatrix}
X & 0 & Y & 0 \\
0 & 0 & 0 & d \\
-Y^T & 0 & Z & 0 \\
0 & -d & 0 & 0
\end{pmatrix}, \tag{5.9}
\]
where $X$ and $Z$ are $N_f \times N_f$ anti-symmetric matrices, $Y$ is an $N_f \times N_f$ matrix and $d = \text{diag}(d_{N_f+1}, \cdots, d_{2N_c+2})$. From (5.6) we find,

$$
\begin{pmatrix}
X & Y \\
-Y^T & Z
\end{pmatrix}
\begin{pmatrix}
X & Y \\
-Y^T & Z
\end{pmatrix} = 0,
\quad (5.10)
$$

$$
d_i(d_i - \mu m_i) = 0, \quad i = N_f + 1, \ldots, 2N_c + 2 \quad (5.11)
$$

Using similarity transformation of the massless flavor symmetry group $O(2N_f)$, the solutions can be reduced to $X = Z = 0$ and

$$
Y = \begin{pmatrix}
-\hat{q}^2 & -i\hat{q}^2 \\
-i\hat{q}^2 & \hat{q}^2 \\
0
\end{pmatrix},
\quad \hat{q} = \text{diag}(q_1, \cdots, q_r), \quad q_i \in \mathbb{R}_+,
\quad d = \text{diag}(\mu m_{N_f+1}, \cdots, \mu m_{N_f+s}, 0, \cdots, 0),
\quad s \leq 2N_c + 2 - N_f. \quad (5.12)
$$

(5.7) implies $r + s \leq N_c$. We can check that this form is exactly the same as that derived from the D-term and F-term equations for $Q$ and $\Phi$ (see appendix B.4), and so the constraints (5.6) and (5.7) form a complete set.

Next we consider quantum effects. When the meson is as above, the theory turns out to be $\mathcal{N} = 1$ supersymmetric $USp(2(N_c - r - s))$ QCD with $(N_f - 2r)$ massless flavors in the defining representation. For $N_f - 2r \leq N_c - r - s$ Affleck-Dine-Seiberg type superpotential is generated non-perturbatively, and the classical vacua are lifted [12]. For $N_f - 2r = N_c - r - s + 1$ the theory is in the confining phase and the classical moduli space is deformed quantum mechanically [12]. We will consider the branches which have a point $Y = 0$ in (5.12), and discuss the duality on these branches. Thus we find a constraint

$$
N_c - N_f + r - s + 1 < 0.
$$

5.3 $\mathcal{N}=1$ deformed magnetic theory and duality

The superpotential of $\mathcal{N} = 1$ deformed magnetic theory is

$$
W_{\text{mag}} = \sqrt{2} \tilde{g} q_i^a \varphi_a^b \mathbb{J}_{bc} q_i^c + \sqrt{2} m^{ij} q_i^a \mathbb{J}_{ab} q_j^b - \frac{\mu}{\sqrt{2}} \text{tr} \varphi^2.
\quad (5.14)
$$

The meson in the magnetic theory is defined as $N^{ij} = N_{ij} \equiv q_i^a \mathbb{J}_{ab} q_j^b$ which is constrained by the similar equation $N \cdot \tilde{N} = 0$, where $\tilde{g}^2 \tilde{N} \equiv \tilde{g}^2 N - \mu m$, and rank $N \leq 2N_c$.

The correspondence between the electric and magnetic theory is

$$
\begin{align*}
\text{electric} & \leftrightarrow \text{magnetic} \\
g^2 M & \leftrightarrow \tilde{g}^2 \tilde{N} \\
(g^2 \tilde{M} & \leftrightarrow \tilde{g}^2 N). \quad (5.15)
\end{align*}
$$
The correspondence of the constraint (5.3) is trivial:

$$M \cdot \tilde{M} = 0 \leftrightarrow N \cdot \tilde{N} = 0. \tag{5.16}$$

When $M$ takes the form as in (5.12) and (5.13), we find rank $N = 2(2N_c - N_f + r - s + 2)$. As explained above, the vacuum moduli space is constrained with $N_c - N_f + r - s + 1 < 0$ and so we find rank $N \leq 2N_c$. These facts imply that the vacuum moduli spaces in the electric and magnetic theory are the same.

We can determine the unbroken gauge groups in the same way as in section 4.4. (5.4) and the similar equation in the magnetic theory imply that

$$\text{tr } \Phi^k = \left(\frac{-g}{\mu}\right)^k \text{Tr } M^k, \quad \text{tr } \varphi^k = \left(\frac{\bar{g}}{\mu}\right)^k \text{Tr } N^k. \tag{5.17}$$

The point $M = 0$ in the electric theory at which $USp(2N_c)$ gauge symmetry is expected to be unbroken, corresponds to the point $\bar{g}^2 N = \mu m$. At this point, (5.17) implies that

$$\bar{g} \varphi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \ddots \\ 0 \\ m_{N_f+1} \\ \ddots \\ m_{2N_c+2} \end{pmatrix}. \tag{5.18}$$

So we expect that the magnetic theory at this point is a $USp(2(N_f - N_c - 2))$ gauge theory. The result is consistent to N-Seiberg’s duality in $\mathcal{N} = 1$ $USp$ QCD \[2, 12\]. Similarly, when we consider the branch of rank $d = s$ in (5.13) in the electric theory, we get the duality for $USp(2(N_c - s)) \leftrightarrow USp(2(N_f - N_c + s - 2))$. Thus we have found a series of $USp$ duality as in the case of the $SO$ theory.

We can also carry out Leigh-Strassler transformation in this case. Integrating out $\Phi$ in (5.3),

$$W_{\text{ele}} = -\frac{g^2}{\sqrt{2} \mu} (Q^j \bar{Q}^i)(Q^i \bar{Q}^j) + \sqrt{2} m_{ij} (Q^i \bar{Q}^j). \tag{5.19}$$

In the limit $g \to 0$, we have

$$W_{\text{ele}} \sim \sqrt{2} m_{ij} (Q^i \bar{Q}^j). \tag{5.20}$$

The theory is $\mathcal{N} = 1$ supersymmetric $USp(2N_c)$ QCD with $N_f$ massless flavors.

On the other hand, introducing suitable auxiliary meson field $\mathcal{M}^{ij}$, the superpotential for the magnetic theory becomes

$$W_{\text{mag}} = -\mathcal{M}^{ij} (g_{ij} \bar{q}_j) + \frac{\mu}{g^2} m^{ij} \mathcal{M}^{ij} + \frac{\mu}{2\sqrt{2} g^2} \mathcal{M}^{ij} \mathcal{M}^{ij}. \tag{5.21}$$
The equation of motion implies $\mu \mathcal{M}^{ij} = \sqrt{2} \left( g^2 N^{ij} - \mu \mathcal{M}^{ij} \right) \leftrightarrow \sqrt{2} g^2 \mathcal{M}^{ij}$. In the same limit $\tilde{g} \sim 1/g \to \infty$, fixing $\langle q^\dagger q \rangle \sim \mu m/\tilde{g}^2$ and $\langle \varphi \rangle \sim m/\tilde{g}$, we get

$$W_{\text{mag}} \sim -\mathcal{M}^{ij}(q^\dagger q) + \frac{\mu}{g^2} m^{ij} \mathcal{M}^{ij}. \quad (5.22)$$

The theory again flows down to the magnetic theory of N.Seiberg’s duality, namely $\mathcal{N} = 1$ supersymmetric $USp(2(N_f - N_c - 2))$ QCD with $N_f$ massless flavors interacting with a meson field $\mathcal{M}^{ij}$.

6 Summary and Comments

We have studied the deformations of $\mathcal{N} = 2$ supersymmetric QCD, adding the adjoint mass term, and seen that the S-dualities in finite $\mathcal{N} = 2$ theories naturally flow to N.Seiberg’s $\mathcal{N} = 1$ dualities. Generalizing the argument in [5], we obtained the gauge singlet meson field and the superpotential needed in the magnetic theory.

We have shown that much information on the duality can be obtained using the classical equations of motion. This fact would be the remnant of the non-renormalization theorem of the Higgs branches in $\mathcal{N} = 2$ theories. Of course, it is important to take the non-perturbative effects into account, when we break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. In particular, in order to show that the vacuum moduli spaces in the electric and magnetic theory are the same, the non-perturbative effects are essential. However, we have not revealed more fruitful structures in the $\mathcal{N} = 1$ theories, such as the confining phases, quantum deformations of the moduli spaces, or electric-magnetic-dyonic triality in the $SO$ theory [1, 11].

Our methods are quite simple and seem to have other applications. For example, instead of adding the adjoint mass term, we can add various $\mathcal{N} = 2$ breaking terms to the superpotential and investigate deformations of the S-dualities. We hope that this approach will give a useful guide for searching new dualities.

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Appendices
A  convention

We write down the conventions used in the $SO(N_c)$ gauge theory.

- $J = (J_{ij}) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I$
- $J^{-1} = (J^{ij}) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I$
- $J_{ij} J^{jk} = \delta^i_j$
- $m_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(m_1, \ldots, m_N)$
- $m_{ij} \equiv J^k_{ik} m_{kj} = (\mu_{ij}) = m_{ij}$
- $m_{ij} J_{jk} = J_{ij} m_{jk}$
- $N_{ij} = q^a_i q^a_j$
- $N^{ij} \equiv J^k_{ik} N_{kj} = (\mu N)$

B  the form of Q

We list the D-term and F-term equations and the classical moduli spaces in terms of $Q, \tilde{Q}$ and $\Phi$.

B.1  $\mathcal{N} = 2$ $SU$ theory

The D-term and F-term equations are

\[
\begin{align*}
[\Phi, \Phi^\dagger] &= 0, \quad (\text{B.1}) \\
Q^i_{\alpha} Q^i_{\overline{\beta}} - \overline{Q}^i_{\alpha} \overline{Q}^i_{\overline{\beta}} &= \nu \delta^i_{\alpha}, \quad (\text{B.2}) \\
Q^i_{\alpha} \overline{Q}^i_{\overline{\beta}} &= \rho \delta^i_{\alpha}, \quad (\text{B.3}) \\
g \Phi^a \overline{Q}^b_j + m^i_j \overline{Q}^a_i &= 0, \quad (\text{B.4}) \\
g \Phi^a \Phi^b_{\alpha} + m^i_j \Phi^a_i &= 0, \quad (\text{B.5})
\end{align*}
\]

where $\nu \in \mathbb{R}$, $\rho \in \mathbb{C}$ and $m^i_j = \text{diag}(0, \ldots, 0, m_{N_f+1}, \ldots, m_{2N_c})$ [1]. The vacuum moduli space consists of the Coulomb, baryonic and non-baryonic branches.

* Using the F-term equations (B.4) and (B.5), we find

\[
D^2 = g^2 \left( |Q^\dagger T^\alpha Q - \overline{Q} T^\alpha \overline{Q}^\dagger|^2 + 2 \text{tr}([\Phi^\dagger, \Phi]^2) + 4(|Q^\dagger m + Q^\dagger \Phi|^2 + |m^\dagger \overline{Q} + \Phi^\dagger \overline{Q}^\dagger|^2) \right),
\]

implying (B.1) and (B.2).
The Coulomb branch \( Q \) and \( \tilde{Q} \) are zero.

\[
\Phi = \begin{pmatrix}
\phi_1 \\
\vdots \\
\phi_{N_c}
\end{pmatrix},
\]

(B.6)

where \( \phi_i \in \mathbb{C} \) and \( \sum \phi_i = 0 \).

The Baryonic Branch \( \Phi \) is zero.

\[
Q = \begin{pmatrix}
\kappa_1 \\
\vdots \\
\kappa_{N_c}
\end{pmatrix},
\]

(B.7)

\[
\tilde{Q} = \begin{pmatrix}
\tilde{\kappa}_1 & \lambda_1 \\
\vdots & \vdots \\
\tilde{\kappa}_{N_c} & \lambda_{N_f-N_c}
\end{pmatrix},
\]

(B.8)

where \( \kappa_i, \lambda_i \in \mathbb{R}_+ \),

\[
\kappa_i \tilde{\kappa}_i = \rho, \quad \text{for all } i,
\]

\[
\kappa_i^2 - (|\tilde{\kappa}_i|^2 + \lambda_i^2) = \nu, \quad i \leq N_f - N_c,
\]

\[
\kappa_i^2 - |\tilde{\kappa}_i|^2 = \nu, \quad i \geq N_f - N_c + 1.
\]

The non-Baryonic Branch

\[
\Phi = \text{diag}(0, \ldots, 0, \phi_{r+1}, \ldots, \phi_{N_c}),
\]

(B.9)

\[
Q = \begin{pmatrix}
\kappa_1 \\
\vdots \\
\kappa_r
\end{pmatrix},
\]

(B.10)

\[
\tilde{Q} = \begin{pmatrix}
0 & \kappa_1 \\
\vdots & \vdots \\
0 & \kappa_r
\end{pmatrix},
\]

(B.11)

where \( \phi_i \in \mathbb{C} \), \( \sum \phi_i = 0 \), \( \kappa_i \in \mathbb{R}_+ \) and \( r \leq [N_f/2] \).

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The D-term equations (B.1) and (B.2) are not changed when we add the adjoint mass term $\mu \tr \Phi^2/\sqrt{2}$.

The D-term and F-term equations are

$$[\Phi, \Phi^\dagger] = 0, \quad (B.12)$$

$$Q_i^a Q_b^i - Q_b^a Q_i^i = \nu, \quad (B.13)$$

$$g \left( Q_i^a \tilde{Q}_i^b - \frac{1}{N_c} (Q_i^a \tilde{Q}_i^i) \delta_a^b \right) + \mu \Phi_a^b = 0, \quad (B.14)$$

$$g \Phi_b^a \tilde{Q}_j^b + m_j^i \tilde{Q}_i^a = 0, \quad (B.15)$$

$$g \Phi_a^b \tilde{Q}_b^i + m_j^i Q_j^a = 0, \quad (B.16)$$

where $\nu \in \mathbb{R}_+$ and $m_j^i = \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{2N_c})$.

The classical moduli space consists of the baryonic, non-baryonic and exceptional branches. The baryonic and non-baryonic branches are the same as those in $\mathcal{N} = 2$ theory ($\Phi$ is fixed to be zero).

**The Exceptional Branch**

$$g \Phi = \text{diag}(0, \cdots, 0, c, \cdots, c, -m_{2N_c-s+1}, \cdots, -m_{2N_c}), \quad (B.17)$$

$$Q = \begin{pmatrix} \kappa_1 \\ \vdots \\ \kappa_r \end{pmatrix}, \quad (B.18)$$

$$\tilde{Q} = \begin{pmatrix} \tilde{\kappa}_1 \\ \vdots \\ \tilde{\kappa}_r \\ \lambda_1 \\ \vdots \\ \lambda_{r'} \end{pmatrix}, \quad (B.19)$$

where $(\kappa_i, d_i, \lambda_i) \in \mathbb{R}_+, \ (\tilde{\kappa}_i, \tilde{d}_i) \in \mathbb{C}$.
\[ r + s \leq N_c - 1, \quad s \leq 2N_c - N_f, \quad r' = \min\{r, N_f - r\}, \]
\[ c = \frac{1}{N_c - r - s} \sum_{2N_c - s + 1}^{2N_c} m_i, \]
\[ g^2 \kappa_i \bar{\kappa}_i = \mu c, \quad \kappa_i^2 - (|\bar{\kappa}_i|^2 + \lambda_i^2) = 0, \]
\[ d_i = |\bar{d}_i|, \quad g^2 d_i \bar{d}_i = \mu c + \mu m_i \quad (N_f + 1 \leq i \leq N_f + s). \]

### B.3 \( \mathcal{N} = 1 \) deformed SO theory

As in the \( SU \) theory, the D-term equations are not changed from \( \mathcal{N} = 2 \) theory. The D-term and F-term equations are

\[ [\Phi, \Phi^\dagger] = 0, \]
\[ \text{Im}(Q_i^a Q_j^b) = 0, \]
\[ gQ_i^{a\bar{a}ij}Q_j^b - \mu \Phi_{ab} = 0, \]
\[ g\Phi_{ab}Q_i^{a\bar{a}ij} + m_{ij} Q_{a}^{j} = 0, \]

where \( m = (m_{ij}) \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{N_c-2}) \).

The classical moduli space is as follows.

\[ g\Phi = \begin{pmatrix} 0 & \cdots & 0 & \ldots & \text{im} N_f + 1 \\ \text{im} N_f + 1 & \cdots & 0 & \ldots & \text{im} N_f + s \\ \text{im} N_f + s & \cdots & \text{im} N_f + s & \cdots & 0 \\
\end{pmatrix}, \]

\[ Q = \begin{pmatrix} q_1 & \cdots & q_r \\ \cdots & \cdots & \cdots \\ d_1 & -i d_1 & \cdots & \cdots & d_1 \\ -i d_1 & \cdots & \cdots & i d_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_s & -i d_s & \cdots & \cdots & d_s \\ -i d_s & \cdots & \cdots & i d_s & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}. \]

where \( q_i \in \mathbb{R}_+ \), \( r + 2s \leq N_c \), \( r \leq N_f \), \( s \leq N_c - 2 - N_f \) and \( d_i = \sqrt{\frac{\mu m_{N_f+1}}{2g^2}} \).
B.4 \( \mathcal{N} = 1 \) deformed \( USp \) theory

The D-term and F-term equations are

\[
[\Phi, \Phi^\dagger] = 0,
\]

\[
Q_a^i Q_b^i + Q_b^i Q_a^i = 0,
\]

\[
gQ_a^i Q_c^i + \mu \Phi_b^b \Phi_c^c = 0,
\]

\[
gQ_a^i \Phi^\dagger_b - m_{ij} Q_b^j = 0,
\]

where \( Q_a^i \equiv Q_{ib}^i \mathbb{J}_{ba} \) and \( m = (m_{ij}) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \text{diag}(0, \cdots, 0, m_{N_f+1}, \cdots, m_{2N_c+2}) \).

The classical moduli space is as follows.

\[
g\Phi = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \text{diag}(0, \cdots, 0, im_{N_f+1}, \cdots, im_{N_f+s}, 0, \cdots, 0), \tag{B.30}
\]

\[
Q = \begin{pmatrix} Q' & 0 & 0 & 0 \\ 0 & iD & 0 & -D \\ 0 & 0 & Q' & 0 \\ 0 & -D & 0 & iD \end{pmatrix}, \tag{B.31}
\]

where

\[
Q' = \begin{pmatrix} q_1 & \cdots & iq_1 \\ \vdots & \ddots & \vdots \\ q_r & \cdots & iq_r \end{pmatrix}, \tag{B.32}
\]

\( Q' \) is an \((N_c-s) \times (N_f)\) matrix,

\[ r \leq \min\{N_c-s, [N_f/2]\}, \quad q_i \in \mathbb{R}_+, \]

\[
gD = \begin{pmatrix} \sqrt{\mu m_{N_f+1}/2} \\ \vdots \\ \sqrt{\mu m_{N_f+s}/2} \end{pmatrix}, \tag{B.33}
\]

where \( s \leq 2N_c + 2 - N_f, \quad r + s \leq N_c \).

References

[1] K.Intriligator and N.Seiberg, “Lectures on supersymmetric gauge theories and electric-magnetic duality,” hep-th/9509066.

[2] N.Seiberg, “Electro-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories,” Nucl. Phys. B435 (1995) 129.

[3] P.C.Argyres, M.R.Plesser and N.Seiberg, “The Moduli Space of Vacua of \( \mathcal{N} = 2 \) SUSY QCD and Duality in \( \mathcal{N} = 1 \) SUSY QCD,” hep-th/9603042, Nucl. Phys. B471 (1996) 159.
[4] P.C.Argyres, M.R.Plesser and A.D.Shapere, “$\mathcal{N} = 2$ Moduli Spaces and $\mathcal{N} = 1$ Dualities for $SO(N_c)$ and $USp(2N_c)$ Super-QCD,” hep-th/9608129, Nucl. Phys. B483 (1997) 172.

[5] R.G.Leigh and M.J.Strassler, “Exactly Marginal Operators and Duality in Four Dimensional $\mathcal{N} = 1$ Supersymmetric Gauge Theory,” Nucl. Phys. B447 (1995) 95.
M.J.Strassler, “Manifolds of Fixed Points and Duality in Supersymmetric Gauge Theories,” hep-th/9602021, Prog. Theo. Phys. Suppl. No.123 (1996) 373.

[6] N.Seiberg and E.Witten, “Electro-Magnetic Duality, Monopole Condensation and Confinement in $\mathcal{N} = 2$ Supersymmetric Yang-Mills Theory,” Nucl. Phys. B426 (1994) 19.
N.Seiberg and E.Witten, “Monopoles, Duality and Chiral Symmetry Breaking in $\mathcal{N} = 2$ Supersymmetric QCD,” Nucl. Phys. B431 (1994) 484.

[7] P.C.Argyres, M.R.Plesser and A.D.Shapere, “The Coulomb Phase of $\mathcal{N} = 2$ Supersymmetric QCD,” Phys. Rev. Lett. 75 (1995) 1699.
P.C.Argyres and A.D.Shapere, “The Vacuum Structure of $\mathcal{N} = 2$ Super-QCD with Classical Gauge Groups,” Nucl. Phys. B461 (1996) 437-459.

[8] A.Hanany and Y.Oz, “On The Quantum Moduli Space of Vacua $\mathcal{N} = 2$ Supersymmetric $SU(N_c)$ Gauge Theories,” Nucl. Phys. B452 (1995) 283.
A.Hanany, “On The Quantum Moduli Space of Vacua $\mathcal{N} = 2$ Supersymmetric Gauge Theories,” Nucl. Phys. B466 (1996) 85.

[9] M.Luty and W.Taylor IV, “Varieties of Vacua in Classical Supersymmetric Gauge Theories,” Phys. Rev. D53 (1996) 3399.

[10] I.Affleck, M.Dine and N.Seiberg, “Dynamical Supersymmetry Breaking in Supersymmetric QCD,” Nucl. Phys. B241 (1984) 493. I.Affleck, M.Dine and N.Seiberg, “Dynamical Supersymmetry Breaking in Four-Dimensions and Its Phenomenological Implications,” Nucl. Phys. B256 (1985) 557.

[11] K.Intriligator and N.Seiberg, “Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric $SO(N_c)$ Gauge Theories,” Nucl. Phys. B444 (1996) 125.

[12] K.Intriligator and P.Pouliot, “Exact Superpotentials, Quantum Vacua and Duality in Supersymmetric $SP(N_c)$ Gauge Theories,” Phys.Lett. B353 (1995) 471.