Relatively Weakly Open Convex Combinations of Slices and Scattered C*-Algebras

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Abstract. We prove that given a locally compact Hausdorff space $K$ and a compact C*-algebra $A$, the C*-algebra $C(K,A)$ satisfies (P1), namely that every convex combination of slices of the closed unit ball is a relatively weakly open subset of the closed unit ball, if and only if $K$ is scattered and $A$ is some $c_0$-sum of finite-dimensional C*-algebras. To obtain a similar characterization in the setting of general C*-algebras, we consider a weaker property ($\overline{P}_1$), namely for every convex combination of slices $C$ of the unit ball of a Banach space $X$ and $x \in C$, there exists a relatively weakly open set $W$ containing $x$, such that $W \subseteq C$. We prove that a C*-algebra has property ($\overline{P}_1$) if and only if it is scattered with finite-dimensional irreducible representations. We obtain some stability results for property ($\overline{P}_1$). For instance, this property passes down from Banach spaces to its closed ideals. As a consequence, we prove that an $L_1$-predual Banach space contains no isomorphic copy of $\ell_1$ if and only if it has property ($\overline{P}_1$).

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1. Introduction

In [14], Ghoussoub et al. exhibited a remarkable geometrical property of the face of the positive elements in the unit ball of the classical Banach space $L_1[0,1]$. Indeed, it is proved in [14, Remark IV.5, p. 48] that, for $\mathcal{F} := \{ f \in L_1[0,1] : f \geq 0 \|f\| = 1 \}$, any convex combination of a finite number of relatively weakly open subsets (in particular, slices) of $\mathcal{F}$ is a relatively weakly open subset of $\mathcal{F}$. It is well known that every relatively weakly open subset of the unit ball of a Banach space contains a convex combination of slices of the unit ball and that the reciprocal is not true (see [14, Lemma II.1]
and [14, Remark IV.5]). As in [18], we will say that a Banach space $X$ has property (P1) if:

*For every convex combination of slices $C$ of $B_X$ and $x \in C$, there exists $W$ relatively weakly open subset of the unit ball of $X$ containing $x$, such that $W \subseteq C$."

Quite recently, Abrahamsen and Lima obtained that given a scattered locally compact space $K$, the space of continuous functions on $K$ vanishing at infinity, $C_0(K, \mathbb{C})$, satisfies property (P1) [2, Theorem 2.3]. On the other hand, Haller, Kunusek and M. Põldvere showed that $K$ is scattered whenever $C_0(K, \mathbb{R})$ has property (P1) [18, Theorem 3.1]. In [1], the authors introduce another geometric property, named (co) (see Definition 2.2), and show that if a finite-dimensional normed space $X$ has this property, then for any scattered locally compact Hausdorff space $K$, the space $C_0(K, X)$ has property (P1).

In view of the results obtained in the aforementioned works, the study of property (P1) seems to be reduced to a very small family of Banach spaces that, except in the case of finite dimension, have in common that their dual has a structure similar to $\ell_1$. It is natural to wonder if those Banach spaces whose dual space has this certain similarity to $\ell_1$ have property (P1). In this sense, we introduce a property that we will be called (P1) and reads as follows:

*For every convex combination of slices $C$ of $B_X$ and $x \in C$, there exists $W$ relatively weakly open subset of the unit ball of $X$ containing $x$, such that $W \subseteq C$."

It is clear that (P1) implies (P1). As a consequence of Theorem 4.3, we have that (P1) and (P1) coincide in a wide class of Banach spaces, in particular in $C(K)$ spaces. However, we do not know whether (P1) and (P1) are in general the same property.

This paper is organized as follows. In Sect. 2, we present some stability results for properties (P1) and (P1). For instance, we prove that these properties are preserved by contractive projections (Proposition 2.3) and by $c_0$-sums (Proposition 2.5). We also show that every Banach space $X$, such that $X^* = \bigoplus_{\gamma \in \Gamma} X_{\gamma}$ for a family of finite-dimensional Banach spaces $\{X_{\gamma}\}_{\gamma \in \Gamma}$ with the (co) property, has property (P1) (Theorem 2.7). As a consequence, every isometric predual of $\ell_1$ has property (P1). This provides examples of Banach spaces with property (P1) that are not isomorphic to a complemented subspace of any $C(K)$-space (see [7, Corollary 1]). We conclude this section by showing that for Banach spaces, property (P1) is hereditary with respect to subspaces that are ideals (Proposition 2.12). As a consequence of the aforementioned results, we obtain that for a $L_1$-predual Banach space property (P1) is equivalent to containing no isomorphic copy of $\ell_1$ (Theorem 2.14).

Section 3 is mainly devoted to establish that finite dimensional C*-algebras have property (co) (see Definition 2.2, Theorem 3.8). One of the key ingredients in the proof of Theorem 3.8 is the perturbation theory in C*-algebras [8]. Concretely, in Theorem 3.6, we generalize some classical results of Davis in [11] for non-necessarily self-adjoint elements.
In Sect. 4, we will focus on the class of scattered C*-algebras introduced and characterized by Jensen in [21,22]. Since abelian C*-algebras are scattered if and only if they satisfy property (P1) [2,18], it is natural to study properties (P1) and ($\overline{\text{P1}}$) in the setting of general C*-algebras. We show that C*-algebras satisfying property (P1) (or ($\overline{\text{P1}}$)) turn out to be scattered (Proposition 4.2). We also prove that given a locally compact Hausdorff space $K$, and a compact C*-algebra $A$, the C*-algebra $C(K,A)$ has property (P1) if and only if $K$ is scattered and $A$ is some $c_0$-sum of finite-dimensional C*-algebras (Theorem 4.3). Finally, we provide a characterization of C*-algebras satisfying property ($\overline{\text{P1}}$) as those being scattered and having only finite dimensional representations (see Theorem 4.5).

2. Stability Results for Properties (P1) and ($\overline{\text{P1}}$)

We shall first introduce some notation. Let us consider $X$ a real or complex Banach space. We will denote by $B_X$ and $S_X$ the closed unit ball and the unit sphere of $X$. Given $C$ a norm-closed convex subset of $X$, for every $f \in B_X^*$ and $\varepsilon > 0$, we define a slice of $C$ by:

$$S(C,f,\varepsilon) := \{ x \in C : \Re f(x) > \sup_{c \in C} |f(c)| - \varepsilon \}.$$

Second, we will present the definitions object to the study in this paper.

**Definition 2.1.** Let us consider the following properties for a Banach space $X$:

(P1) For every convex combination of slices $C$ of $B_X$ and $x \in C$, there exists a relatively weakly open set $W$ containing $x$, such that $W \subseteq C$.

($\overline{\text{P1}}$) For every convex combination of slices $C$ of $B_X$ and $x \in C$, there exists a relatively weakly open set $W$ containing $x$, such that $W \subseteq C$.

As already remarked, (P1) was first defined in [18], while the existence of a Banach space having (P1) was first reveal in [2]. The next definition stems from [1].

**Definition 2.2.** A Banach space, $X$, is said to have property (co) if for every $n \in \mathbb{N}$, given $x_1, \ldots, x_n \in B_X$, $\lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $\varepsilon > 0$, there exist $\delta > 0$ and continuous functions $\Phi_i : B(x_0, \delta) \cap B_X \to B(x_i, \varepsilon) \cap B_X$, where $x_0 = \sum_{i=1}^n \lambda_i x_i$, satisfying $y = \sum_{i=1}^n \lambda_i \Phi_i(y)$ for every $y \in B(x_0, \delta)$.

Throughout this section, we will reveal new results in the setting of general Banach spaces concerning these three properties. We begin by showing that all these properties are preserved by contractive projections.

**Proposition 2.3.** Norm-one complemented subspaces of a Banach space inherit each of properties (P1), ($\overline{\text{P1}}$), and (co).

**Proof.** Let $X$ be a Banach space, let $P : X \to X$ be a contractive projection, and $Y = P(X)$. Let $\{S(B_Y,f_i,\varepsilon_i)\}_{i=1}^k$ be slices of $B_Y$, let $\lambda_i > 0$ with
\[
\sum_{i=1}^{k} \lambda_i = 1, \text{ and consider the convex combination of these slices:}
\]
\[
C = \sum_{i=1}^{k} \lambda_i S(B_Y, f_i, \varepsilon_i).
\]

Take \( y = \sum_{i=1}^{k} \lambda_i y_i \) in \( C \) with \( y_i \in S(B_Y, f_i, \varepsilon_i) \). We consider the slices of \( B_X \), \( \left\{ S(B_X, P^*(f_i), \varepsilon_i) \right\}_{i=1}^{k} \), and \( \tilde{C} := \sum_{i=1}^{k} \lambda_i S(B_X, P^*(f_i), \varepsilon_i) \). Assume that \( X \) has property (P1). Since \( y \in \tilde{C} \), there exists \( W \), a relatively weakly open subset of \( B_X \) containing \( y \), such that \( W \subseteq \tilde{C} \). Now, \( P(W) \) is a relatively weakly open subset of \( B_Y \) satisfying \( y \in P(\tilde{C}) \subseteq C \).

In the case that property (P1) is satisfied, the proof is similar and the case of the (co) property is trivial. \( \Box \)

The following is a frequently used technical result.

**Lemma 2.4.** Let \( X \) be a Banach space. Consider \( f \in B_{X^*}, \varepsilon \in \mathbb{R}^+ \) and \( x \in S(B_X, f, \varepsilon) \). Then, there exists \( \bar{\varepsilon}, \rho > 0 \), such that for all \( g \in B_{X^*} \) with \( \|f - g\| < \rho \), we have that:

\[
x \in S(B_X, g, \bar{\varepsilon}) \subseteq S(B_X, f, \varepsilon).
\]

**Proof.** Take \( \tilde{\varepsilon}, \rho \) any two positive numbers satisfying \( \tilde{\varepsilon} + 2\rho < \varepsilon \). For \( g \in B_{X^*} \) with \( \|f - g\| < \rho \) and \( x \in S(B_X, g, \tilde{\varepsilon}) \), we have that \( |\Re f(x) - \Re g(x)| < \rho \) and \( \Re f(x) > \Re g(x) - \rho > \|g\| - \tilde{\varepsilon} - \rho > \|f\| - 2\rho - \tilde{\varepsilon} > \|f\| - \varepsilon \); thus, \( x \) belongs to \( S(B_X, f, \varepsilon) \). \( \Box \)

Let \( \Gamma \) be a set, and let \( \{X_\gamma : \gamma \in \Gamma\} \) be a family of Banach spaces indexed by \( \Gamma \).

We recall that the \( c_0 \)-sum of the family \( \{X_\gamma : \gamma \in \Gamma\} \), denoted by \( \bigoplus_{\gamma \in \Gamma} X_\gamma \), is the Banach space:

\[
\bigoplus_{\gamma \in \Gamma} X_\gamma := \{(x_\gamma) : x_\gamma \in X_\gamma, \lim_{\gamma \in \Gamma} \|x_\gamma\|_\gamma = 0\}
\]

and \( \|(x_\gamma)\| = \sup\{\|x_\gamma\|_\gamma : \gamma \in \Gamma\} \) for each \( (x_\gamma) \in \bigoplus_{\gamma \in \Gamma} X_\gamma \).

The \( \ell_1 \)-sum of the family \( \{X_\gamma : \gamma \in \Gamma\} \), denoted by \( \bigoplus_{\gamma \in \Gamma} \ell_1 X_\gamma \), is the Banach space:

\[
\bigoplus_{\gamma \in \Gamma} \ell_1 X_\gamma := \{(x_\gamma) : x_\gamma \in X_\gamma, \gamma \in \Gamma, \sum_{\gamma \in \Gamma} \|x_\gamma\|_\gamma < \infty\}
\]

and \( \|(x_\gamma)\| = \sum_{\gamma \in \Gamma} \|x_\gamma\|_\gamma \).

Finally, the \( \ell_\infty \)-sum of the family \( \{X_\gamma : \gamma \in \Gamma\} \), \( \bigoplus_{\gamma \in \Gamma} \ell_\infty X_\gamma \), is the Banach space:

\[
\bigoplus_{\gamma \in \Gamma} \ell_\infty X_\gamma := \{(z_\gamma) : z_\gamma \in X_\gamma, \sup \|x_\gamma\|_\gamma < \infty\}
\]

and \( \|(z_\gamma)\| = \sup\{\|z_\gamma\|_\gamma : \gamma \in \Gamma\} \) for each \( (z_\gamma) \in \bigoplus_{\gamma \in \Gamma} \ell_\infty X_\gamma \).
It is also well known that \((\bigoplus_{\gamma \in \Gamma} X_{\gamma})^* = \bigoplus_{\gamma \in \Gamma} X_{\gamma}^*\) and \((\bigoplus_{\gamma \in \Gamma} X_{\gamma})^* = \bigoplus_{\gamma \in \Gamma} X_{\gamma}^*\). For each \(\mathcal{R} \subseteq \Gamma\), we denote by \(P^0_{\mathcal{R}}\) (respectively, \(P^1_{\mathcal{R}}\), \(P^\infty_{\mathcal{R}}\)) the canonical projection of \(\bigoplus_{\gamma \in \Gamma} X_{\gamma}\) (respectively, \(\bigoplus_{\gamma \in \Gamma} X_{\gamma}^*\), \(\bigoplus_{\gamma \in \Gamma} X_{\gamma}^{**}\)) onto \(\bigoplus_{\gamma \in \mathcal{R}} X_{\gamma}\) (respectively, \(\bigoplus_{\gamma \in \mathcal{R}} X_{\gamma}^*\), \(\bigoplus_{\gamma \in \mathcal{R}} X_{\gamma}^{**}\)).

The following result generalizes Theorem 5.2 in [1], since every finite-dimensional Banach space with the (co) property has (P1) [1, Proposition 2.2].

**Proposition 2.5.** Let \(\{X_{\gamma} : \gamma \in \Gamma\}\) be a family of Banach spaces indexed by \(\Gamma\). Then, \(Z := \bigoplus_{\gamma \in \Gamma} X_{\gamma}\) has property (P1) (respectively, \((P1)\)) if and only if \(X_{\gamma}\) has property (P1) (respectively, \((P1)\)) for every \(\gamma \in \Gamma\).

**Proof.** The only if part is given by Proposition 2.3. Now, let \(\{S(B_Z, f_i, \varepsilon_i)\}_{i=1}^n\) be slices of \(B_Z, f_1, \ldots, f_n \in S_{Z^*}\), let \(\lambda_i > 0\) with \(\sum_{i=1}^n \lambda_i = 1\), and consider the convex combination of these slices:

\[ C = \sum_{i=1}^n \lambda_i S(B_Z, f_i, \varepsilon_i). \]

Let \(z = \sum_{i=1}^n \lambda_iz_i \in C\) with \(z_i \in S(B_Z, f_i, \varepsilon_i)\). Our goal is to find a non-empty relatively weakly open subset of \(B_Z\) containing \(z\) that is contained in \(C\). By Lemma 2.4, given \(i \in \{1, \ldots, n\}\) and \(\varepsilon_i\), there exists \(\tilde{\varepsilon}_i, \rho_i > 0\), such that for all \(g \in B_{Z^*}\) with \(\|f_i - g\| < \rho_i\), we have that:

\[ z_i \in S(B_Z, g, \tilde{\varepsilon}_i) \subseteq S(B_Z, f_i, \varepsilon_i). \]

Fix \(\mathcal{R} \subseteq \Gamma\) a finite subset with \(N \in \mathbb{N}\) elements, such that \(||f_i - P^1_{\mathcal{R}}(f_i)|| < \rho_i\) for all \(i \in \{1, \ldots, n\}\). Then, we have that:

\[ z \in \sum_{i=1}^n \lambda_i S(B_Z, P^1_{\mathcal{R}}(f_i), \tilde{\varepsilon}_i) \subseteq C. \]

Since for \(i \in \{1, \ldots, n\}\), \(\Re P^1_{\mathcal{R}}(f_i)(z_i) > ||P^1_{\mathcal{R}}(f_i)|| - \tilde{\varepsilon}_i\), we set \(\varepsilon > 0\), such that:

\[ \Re P^1_{\mathcal{R}}(f_i)(z_i) - N\varepsilon > ||P^1_{\mathcal{R}}(f_i)|| - \tilde{\varepsilon}_i \]

for all \(i \in \{1, \ldots, n\}\).

Given \(i \in \{1, \ldots, n\}\) and \(j \in \mathcal{R}\), we consider the following slices in the closed unit ball of the Banach space \(X_j\):

\[ S^j_i := S(B_{X_j}, P^1_{\mathcal{R}}(f_i)(j), ||P^1_{\mathcal{R}}(f_i)(j)|| - \Re P^1_{\mathcal{R}}(f_i)(z_i(j)) + \varepsilon). \]

Now, for every \(j \in \mathcal{R}\), we have that:

\[ z(j) = \sum_{i=1}^n \lambda_iz_i(j) \in \sum_{i=1}^n \lambda_i S^j_i, \]

and since \(X_j\) has property (P1), there exists a relatively weakly open subset \(W_j\) of \(B_{X_j}\) containing \(z(j)\), such that \(W_j \subseteq \sum_{i=1}^n \lambda_i S^j_i\). We define the set:

\[ W := \left( \prod_{j \in \mathcal{R}} W_j \right) \times B_{(I-P^0_{\mathcal{R}})(Z)}, \]
which is clearly a relatively weakly open subset $B_Z$ containing $z$.

We will finish the proof by showing that $W \subseteq C$. Indeed, given $y \in W$, we have that $y(j) \in W_j$ for every $j \in R$ and, thus, $y(j) = \sum_{i=1}^{n} \lambda_i y(j)_i$ where each $y(j)_i$ belongs to $S^j_i$. Therefore, we can define, for each $i \in \{1, \ldots, n\}$, the element $y_i \in Z$ given by:

$$y_i(\gamma) := \begin{cases} y(\gamma)_i, & \gamma \in R \\ y(\gamma), & \gamma \in \Gamma \setminus R \end{cases}$$

satisfying $y = \sum_{i=1}^{n} \lambda_i y_i$ and:

$$\text{Re} P^1_\Gamma(f_\gamma) y_i = \sum_{j \in R} \text{Re} P^1_\Gamma(f_j) y(j)_i > \sum_{j \in R} (\text{Re} P^1_\Gamma(f_j)(z(j)) - \varepsilon)$$

$$= \text{Re} P^1_\Gamma(f_i)(z_i) - N \varepsilon > \|P^1_\Gamma(f_i)\| - \varepsilon,$$

so thus $y \in \sum_{i=1}^{n} \lambda_i S(B_Z, P^1_\Gamma(f_i), \varepsilon_i) \subseteq C$.

In case we are dealing with property $(P1)$, the proof is analogous. \hfill $\Box$

The following result provides new examples of Banach spaces of type $C(K, X)$ with property $(P1)$, where $X$ is an infinite-dimensional Banach space.

**Corollary 2.6.** Let $K$ be a scattered compact Hausdorff space and let $\{X_\gamma\}_{\gamma \in \Gamma}$ be a family of finite-dimensional Banach spaces with $(co)$ property. Then, $C(K, \bigoplus_{\gamma \in \Gamma}^c X_\gamma)$ has property $(P1)$.

**Proof.** For every family of Banach spaces $\{X_\gamma\}_{\gamma \in \Gamma}$, the equality $C(K, \bigoplus_{\gamma \in \Gamma}^c X_\gamma) = \bigoplus_{\gamma \in \Gamma}^c C(K, X_\gamma)$ holds. Since $X_\gamma$ has the $(co)$ property, by [1, Theorem 2.5], $C(K, X_\gamma)$ has property $(P1)$ for every $\gamma \in \Gamma$. Finally, Proposition 2.5 applies. \hfill $\Box$

**Theorem 2.7.** Let $\{X_\gamma\}_{\gamma \in \Gamma}$ be a family of finite-dimensional Banach spaces with the $(co)$ property. Let $Z$ be a Banach space, such that $Z^* = \bigoplus_{\gamma \in \Gamma} X_\gamma$. Then, $Z$ has property $(P1)$.

**Proof.** As before, let $C$ be a convex combination of slices of the unit ball of $Z$:

$$C = \sum_{i=1}^{n} \lambda_i S(B_Z, f_i, \varepsilon_i).$$

Fix $z_0 = \sum_{i=1}^{k} \lambda_i z_i \in C$ with $z_i \in S(B_Z, f_i, \varepsilon_i)$. Our goal is to find a non-empty relatively weakly open subset of $B_Z$ containing $z_0$, that is contained in $C$.

Let $\eta$ be a positive number satisfying $\text{Re} f_i(z_i) - 3\eta > \|f_i\| - \varepsilon_i$ for all $i \in \{1, \ldots, n\}$. Associated to $\eta$, we can find a finite set $R \subseteq \Gamma$, such that $\|f_i - P^1_\Gamma(f_i)\| < \eta$ for all $i \in \{1, \ldots, n\}$. Let $N \in \mathbb{N}$ denote the cardinality of $R$.

By hypothesis, $Z^* = \bigoplus_{\gamma \in \Gamma}^c X_\gamma$, and hence, $Z^{**} = \bigoplus_{\gamma \in \Gamma} X_\gamma$. We will represent the elements of $Z$ as elements in $Z^{**}$.

Since $X_\gamma$ has the $(co)$ property for every $\gamma \in \Gamma$, given $\frac{n}{3N} > 0$ (and $z_0(\gamma) \in X_\gamma$), there exist $\delta(i, \gamma) > 0$ and continuous functions $\Phi_{(i, \gamma)} : B(z(\gamma),$
\[ \delta_{(i, \gamma)}(B(z_i(\gamma)), \frac{\eta}{3N}) \cap B_{X_{\gamma}}, \] such that for every \( y \in B(z_0(\gamma), \delta_{(i, \gamma)}) \cap B_{X_{\gamma}}, \) we have \( y = \sum_{i=1}^{k} \lambda_i \phi_{(i, \gamma)}(y), \) where \( i \in \{1, \ldots, n\}. \)

We take \( \delta := \min\{\delta_{(i, \gamma)} : 1 \leq i \leq k, \gamma \in \mathcal{R}\}, \frac{\eta}{3N}. \)

For every \( \gamma \in \mathcal{R}, \) we can choose a \( \delta/9 \)-net of the unit sphere of \( X_{\gamma}, \)

\[ (\varphi_j(\gamma))^{M_\gamma}_{j=1}, \] where \( M_\gamma \) is the (finite) cardinal of the net. We can extend this functional to \( S_{Z^*} \) in a natural way, defining \( \varphi_{(i, \gamma)}(\xi) = \delta_{\gamma, \xi} \varphi_j(\gamma) \) for each \( \xi \in \mathcal{R} \) (and zero elsewhere), where, in this case, \( \delta_{\gamma, \xi} \) is the Kronecker delta.

We now define the relatively weakly open subset of \( B_{Z^*} \):

\[ U = \{ z \in B_{Z^*} : \|g(z - z_0)\| < \delta/9, \gamma \in \mathcal{R} \}, \]

where \( H := \{ \varphi_{(j, \gamma)} : \gamma \in \mathcal{R}, 1 \leq j \leq M_\gamma\} \cup \{ P_{R}^1(f_i) : 1 \leq i \leq k\}. \)

We consider \( \tilde{C} := \sum_{i=1}^{n} \lambda_i S(B_{Z^*}, f_i, \varepsilon_i) \) and:

\[ \tilde{U} = \{ z \in B_{Z^*} : \|g(z - z_0)\| < \delta/9, \gamma \in \mathcal{R} \}. \]

We claim that \( \tilde{U} \subseteq \tilde{C}. \)

Let \( z \) be in \( \tilde{U}. \) We will define \( z_i \in S(B_{Z^*}, f_i, \varepsilon_i), \) \( i = 1, 2, \ldots, k, \) and show that \( z \) can be written as \( z = \sum_{i=1}^{k} \lambda_i z_i \in \tilde{C}. \)

Since \( z \in \tilde{U}, \) for \( \gamma \in \mathcal{R} \) and \( 1 \leq j \leq M_\gamma: \)

\[ |\varphi_{(j, \gamma)}(z_0 - z)| < \delta/9, \]

and hence, \( \|z_0(\gamma) - z(\gamma)\|_{X_{\gamma}} < \frac{\delta}{3}. \)

Having in mind that \( z_0(\gamma) = \sum_{i=1}^{n} \lambda_i z_i(\gamma) \) for \( \gamma \in \mathcal{R}, \) that \( X_{\gamma} \) has the (co) property, and that \( \|z_0(\gamma) - z(\gamma)\|_{X_{\gamma}} < \frac{\delta}{3}, \) we have that:

\[ z(\gamma) = \sum_{i=1}^{n} \lambda_i \Phi_{(i, \gamma)}(z_i(\gamma)) \quad \text{with} \quad \|z_i(\gamma) - \Phi_{(i, \gamma)}(z_i(\gamma))\|_{X_{\gamma}} < \frac{\eta}{3N}. \]

We define \( \tilde{z}_i \in B_{Z^*} \) by \( \tilde{z}_i(\gamma) = \Phi_{(i, \gamma)}(z_i(\gamma)) \) whenever \( \gamma \in \mathcal{R} \) and \( \tilde{z}_i(\gamma) = z(\gamma) \) otherwise.

It is clear that \( z = \sum_{i=1}^{n} \lambda_i \tilde{z}_i \) and it only remains to show that \( \tilde{z}_i \in S(B_{Z^*}, f_i, \varepsilon_i). \)

We have:

\[ |P_{R}^{1}(f_{i})(\tilde{z}_{i} - z_{i})| \leq \sum_{\gamma_{\in \mathcal{R}}} |P_{R}^{1}(f_{i})(\gamma)(\tilde{z}_{i}(\gamma) - z_{i}(\gamma))| \]

\[ = \sum_{\gamma_{\in \mathcal{R}}} |P_{R}^{1}(f_{i})(\gamma)(\Phi_{(i, \gamma)}(z_{i}(\gamma)) - z_{i}(\gamma))| \]

\[ \leq N \|z_{i}(\gamma) - \Phi_{(i, \gamma)}(z_{i}(\gamma))\|_{X_{\gamma}} < \frac{\eta}{3N} < \frac{\eta}{3}. \]

Since \( \|f_{i} - P_{R}^{1}(f_{i})\| < \eta, \) we have \( \|f_{i}(z_{i}) - P_{R}^{1}(f_{i})(z_{i})\| < \eta \) and \( |f_{i}(\tilde{z}_{i}) - P_{R}^{1}(f_{i})(\tilde{z}_{i})| < \eta. \) Hence, \( |f_{i}(\tilde{z}_{i} - z_{i})| < 3\eta, \) so that:

\[ \Re f_{i}(\tilde{z}_{i}) \geq \Re f_{i}(z_{i}) - 3\eta > 1 - \varepsilon_{i}, \]

and we are done.

Finally, for each \( y \in U, \) we have that \( y = \sum_{i=1}^{n} \lambda_i \tilde{z}_i \) with \( \tilde{z}_i \in S(B_{Z^*}, f_i, \varepsilon_i). \) Since \( B_{Z^*} \) is \( w^* \)-dense in \( B_{Z^*}, \) we have that \( y \) is the \( w^* \)-limit in \( Z^{**} \) of the net \( (\sum_{i=1}^{n} \lambda_i x_{(i, \alpha)}) \) with \( x_{(i, \alpha)} \in S(B_{Z^*}, f_i, \varepsilon_i). \) Therefore, it follows that
y is the \( w \)-limit of \((\sum_{i=1}^{n} \lambda_i x_{i,\alpha})\), and since \( C \) is a convex subset of \( B_Z \), we conclude that \( y \in \overline{C} \).

\[ \square \]

**Remark 2.8.** Looking at the proof of Theorem 2.7, we realize that if \( Z \) is a Banach space, such that \( Z^* = \bigoplus_{\gamma \in \Gamma} X_\gamma \) for a family of finite-dimensional Banach spaces \( \{X_\gamma\}_{\gamma \in \Gamma} \) with property (co), then every convex combination of \( w^* \)-slices of the unit ball of \( Z^* \) is a relatively \( w^* \)-open subset of \( B_{Z^{**}} \). In the particular case that \( X_\gamma = \mathbb{R} \) for every \( \gamma \in \Gamma \), this result is obtained in [15, Proposition 4.5].

The latter theorem will be decisive to characterize \( C^* \)-algebras with property \((\overline{P})\) (see Theorem 4.5). Having in mind that \( \mathbb{C} \) and \( \mathbb{R} \) have the (co) property (Proposition 2.3 in [1]), we obtain the following result.

**Corollary 2.9.** Every isometric predual of \( \ell_1 \) has property \((\overline{P})\).

As a consequence of the above corollary and [7, Corollary 1], there exist Banach spaces with property \((\overline{P})\) that are not isomorphic to a complemented subspace of any \( C(K) \)-space.

The following reformulation of property \((\overline{P})\) will be useful in succeeding results.

**Lemma 2.10.** Let \( X \) be a Banach space. Then, \( X \) has property \((\overline{P})\) if and only if for every convex combination of \( w^* \)-slices \( C \) of \( B_{X^{**}} \) and \( x \in C \cap X \), there exists a relatively weakly open set \( W \) containing \( x \), such that \( W \subseteq \overline{C}^{w^*} \).

**Proof.** Assume first that \( X \) has property \((\overline{P})\). We consider a convex combination \( C_{X^{**}} \) of \( w^* \)-slices of \( B_{X^{**}} \) and \( x_0 \in C_{X^{**}} \cap X \). Now, \( C_X = C_{X^{**}} \cap B_X \) is a convex combination of slices of \( B_X \) and \( x_0 \in C_X \). Since \( X \) has property \((\overline{P})\), there is a relatively weakly open subset \( U \) of \( B_X \) with \( x_0 \in U \subseteq \overline{C}_X \). Then, \( U \) contains a set \( V \) of the form:

\[
\{ x \in B_X : |g_i(x - x_0)| < \delta, \ i = 1, \ldots, n \},
\]

for suitable \( \delta > 0 \), \( n \in \mathbb{N} \), and \( g_1, \ldots, g_n \in X^* \). Set

\[
V^{**} := \{ z \in B_{X^{**}} : |g_i(z - x_0)| < 1, \ i = 1, \ldots, n \}.
\]

Since \( V^{**} \) is relatively \( w^* \)-open in \( B_{X^{**}} \), and \( B_X \) is \( w^* \)-dense in \( B_{X^{**}} \), the set \( v := V^{**} \cap B_X \) is \( w^* \)-dense in \( V^{**} \). Having in mind that \( x_0 \in V \subseteq \overline{C}_X \), it follows that:

\[
x_0 \in V^{**} = \overline{V}^{w^*} \subseteq \overline{C}_X^{w^*} \subseteq \overline{C}_{X^{**}}^{w^*}.
\]

We assume now that \( X^{**} \) satisfies the assumptions, and claim that \( X \) has property \((\overline{P})\).

Let \( \{S(B_X, f_i, \varepsilon_i)\}_{i=1}^k \) be slices of \( B_X \), let \( \lambda_i > 0 \) with \( \sum_{i=1}^k \lambda_i = 1 \), and consider the convex combination of these slices, \( C_X = \sum_{i=1}^k \lambda_i S(B_X, f_i, \varepsilon_i) \), and \( x_0 \in C_X \). Put \( C_{X^{**}} = \sum_{i=1}^k \lambda_i S(B_{X^{**}}, f_i, \varepsilon_i) \), a convex combination of \( w^* \)-slices of \( B_{X^{**}} \). Since \( x_0 \in C_{X^{**}} \), by assumption, there exists \( W \), a relatively \( w^* \)-open neighborhood of \( x \), such that \( x \in W \subseteq \overline{C}_X^{w^{**}} \). Since \( S(B_{X^{**}}, f_i, \varepsilon_i) \) is relatively \( w^* \)-open in \( B_{X^{**}} \) and \( B_X \) is \( w^* \)-dense in \( B_{X^{**}} \), the
set \( S(B_X, f_i, \varepsilon_i) \) is \( w^* \)-dense in \( S(B_{X**}, f_i, \varepsilon_i) \) for every \( i = 1, \ldots, n \). Therefore, \( C_{X^{**}}^{w^*} = C_X^{w^*} \) and \( x \in W \subseteq C_X^{w^*} \). Now, \( x \in V \), where \( V := W \cap B_X \) is a relatively weakly open subset of \( B_X \), and for \( y \in V \), we have that \( y = \sum_{i=1}^{k} \lambda_i z_i \) with \( z_i \in S(B_{X**}, f_i, \varepsilon_i) \). Since \( B_X \) is \( w^* \)-dense in \( B_{Z**} \), we have that \( y \) is the \( w^* \)-limit of \( \{ \sum_{i=1}^{k} \lambda_i x(i,\alpha) \} \) with \( x(i,\alpha) \in S(B_X, f_i, \varepsilon_i) \) in \( X** \). Therefore, it follows that \( y \) is the \( w \)-limit of \( \{ \sum_{i=1}^{k} \lambda_i x(i,\alpha) \} \), and since \( C_X \) is a convex subset of \( B_X \), we conclude that \( y \in \overline{C_X} \). 

Let \( X \) be a Banach space and \( Y \) a subspace of \( X \). According to the terminology introduced by Godefroy et al. (see [16]) we recall that \( Y \) is an ideal in \( X \) if \( Y^\perp \), the annihilator of \( Y \) in \( X^* \), is the kernel of a norm one projection on \( X^* \). We also recall that a closed subspace \( Y \) of a Banach space \( X \) is said to be locally 1-complemented in \( X \) if, for every finite-dimensional subspace \( E \) of \( X \) and every \( \varepsilon > 0 \), there exists a linear operator \( P_E : E \to Y \) with \( P_E x = x \) for all \( x \in E \cap Y \) and \( \|P_E\| \leq 1 + \varepsilon \). A linear operator \( \Phi : Y^* \to X^* \) is called a Hahn–Banach extension operator, if \( (\Phi y^*)(y) = y^*(y) \) and \( \|\Phi y^*\| = \|y^*\| \) for all \( y \in Y \) and \( y^* \in Y^* \).

The equivalence of statements (1)–(3) in the following Lemma is due to Fakhoury [13] in the real case. Fakhoury, also in that paper, characterized real \( L_1 \)-preduals as those spaces that are ideals in every super-space. Kalton in [23], independently proved this equivalence in case of quasi-Banach spaces (see also [26] and [29, Corollary 3.3]). The equivalence with item (4) in Lemma 2.11 can be obtained by taking the adjoint of the norm one projection.

**Lemma 2.11.** [23,26] Let \( Y \) be a subspace of a Banach space \( X \). The following statements are equivalent:

1. \( Y \) is an ideal in \( X \).
2. There exists a Hahn–Banach extension operator \( \Phi : Y^* \to X^* \).
3. \( Y \) is locally 1-complemented in \( X \).
4. \( Y^\perp^\perp \) is the range of a norm one projection in \( X^{**} \).

The next result represents an improvement of Proposition 2.3, since every norm-one complemented subspace of a Banach space is an ideal.

**Proposition 2.12.** Let \( X \) be a Banach space with property \((\overline{P_1})\). Then, every ideal \( Y \) in \( X \) has property \((\overline{P_1})\).

**Proof.** Let \( Y \) be an ideal in \( X \). By Lemma 2.11, \( Y \) is locally 1-complemented in \( X \) and there exists an extension operator \( \Phi : Y^* \to X^* \) with \( \|\Phi\| \leq 1 \). There exists a projection \( P : X^* \to X^* \) onto \( Z := \Phi(Y^*) \) with \( \|P\| = 1 \) and \( P^*(X^{**}) = Z^* \). Indeed, denote by \( R_Y : X^* \to Y^* \) the natural restriction operator, \( R_Y(x^*) = x^*|Y \) for \( x^* \in X^* \). Then, \( P = \Phi R_Y : X^* \to X^* \) is a contractive projection on \( X^* \) with range \( Z \) and \( \ker P = Y^\perp \). We also have that \( P^* : X^{**} \to Y^\perp^\perp = Z^* \) is a contractive projection.

Let \( \{ S(B_Y, f_i, \varepsilon_i) \}_{i=1}^{k} \) be slices of \( B_Y \), let \( \lambda_i > 0 \) with \( \sum_{i=1}^{k} \lambda_i = 1 \), and consider the convex combination of these slices, \( C_Y = \sum_{i=1}^{k} \lambda_i S(B_Y, f_i, \varepsilon_i) \).
Having in mind that \( \Phi(f_i)(y) = f_i(y) \) for every \( y \in Y \), we have that
\[
C_Y = \sum_{i=1}^{k} \lambda_i S(B_Y, \Phi(f_i), \varepsilon_i).
\]
We consider the slices:
\[
C_{Z^*} = \sum_{i=1}^{k} \lambda_i S(B_{Z^*}, \Phi(f_i), \varepsilon_i) \quad \text{and} \quad C_{X^{**}} = \sum_{i=1}^{k} \lambda_i S(B_{X^{**}}, \Phi(f_i), \varepsilon_i).
\]

It is clear that \( S(B_{Z^*}, \Phi(f_i), \varepsilon_i) \subset S(B_{X^{**}}, \Phi(f_i), \varepsilon_i) \).

Given \( z \in S(B_{X^{**}}, \Phi(f_i), \varepsilon_i) \), we have that \( \Phi(f_i)(P^*(z)) = P(\Phi(f_i))(z) = \Phi(f_i)(z) \), and hence, \( P^*(z) \in S(B_{Z^*}, \Phi(f_i), \varepsilon_i) \). Thus, \( P^*(S(B_{X^{**}}, \Phi(f_i), \varepsilon_i)) = S(B_{Z^*}, \Phi(f_i), \varepsilon_i) \) and, therefore, \( P^*(C_{X^{**}}) = C_{Z^*} \).

Let \( y_0 \in C_Y \). Since \( X \) has property (PT), by Lemma 2.10, for \( C_{X^{**}} \) and \( y_0 \in C_{Z^*} \cap Y \subseteq C_{X^{**}} \cap X \), there exists \( W \), a relatively \( w^* \)-open neighborhood of \( y_0 \) in \( X^{**} \), such that \( y_0 \in W \subseteq \overline{C_{X^{**}}} \). Then, \( W \) contains a set of the form:
\[
V := \{ z \in B_{X^{**}} : |g_i(z - y_0)| \leq \delta \quad \forall i = 1, \ldots, n \},
\]
for suitable \( \delta > 0 \), \( n \in \mathbb{N} \), and \( g_1, \ldots, g_n \in B_{X^{**}} \). Since \( P \) is a norm one projection on \( X^* \) with range \( Z \), we put \( y_i^* = P(g_i) \) for every \( i = 1, \ldots, n \). We define a relatively weakly open neighborhood of \( y_0 \) in \( Y \):
\[
U := \{ y \in B_Y : |y_i^*(y - y_0)| < \delta \quad \forall i = 1, \ldots, n \}.
\]
For \( y \in U \), we have that:
\[
\delta > |y_i^*(y - y_0)| = |\Phi(y_i^*)(y - y_0)| = |P(g_i)(y - y_0)| = |g_i(P^*(y - y_0))|,
\]
and since \( y, y_0 \in Y^{**} \), it follows that \( y \in V \). Put:
\[
U_1 := \{ y \in B_Y : |\Phi(y_i^*)(y - y_0)| < \delta \quad \forall i = 1, \ldots, n \},
\]
and
\[
U_1^{**} := \{ z \in B_{Z^*} : |\Phi(y_i^*)(z - y_0)| < \delta \quad \forall i = 1, \ldots, n \}.
\]

We consider the topology \( w^*_Z := \sigma(Z^*, Z) \) in \( Z^* \). Since \( B_Y \) is a norming subset of \( B_{Z^*} \) for \( Z \), we have that \( B_Y \) is \( w^*_Z \)-dense in \( B_{Z^*} \), and hence, the set \( U_1 \) is \( w^*_Z \)-dense in \( U_1^{**} \). It follows that \( U_1^{**} = \overline{U_1 w^*_Z} \subseteq V \subseteq \overline{C_{X^{**}}} \). This implies that:
\[
U_1^{**} = P^*(U_1^{**}) \subseteq P^*(\overline{C_{X^{**}}} w^*_Z) = \overline{P^*(C_{X^{**}}) w^*_Z} = \overline{C_{Z^*} w^*_Z}.
\]
We have obtained that for \( y_0 \in C_Y \), there exists a relatively weakly open subset \( U \) of \( B_Y \) with \( y_0 \in U \) and \( U = U_1 \subseteq \overline{C_{Z^*} w^*_Z} \). Given \( y \in U \), since \( \overline{C_Y w^*_Z} = \overline{C_{Z^*} w^*_Z} \), we have that \( y \) is the \( w^*_Z \)-limit in \( Z^* \) of the net \( \left( \sum_{i=1}^{n} \lambda_i y_{(i, \alpha)} \right) \) with \( y_{(i, \alpha)} \in S(B_Y, \Phi(f_i), \varepsilon_i) \). We recall that \( S(B_Y, f_i, \varepsilon_i) = S(B_Y, \Phi(f_i), \varepsilon_i) \) and that \( \Phi(y_i^*)(y) = y_i^*(y) \) for all \( y \in Y \) and \( y^* \in Y^* \). It follows that \( y \) is the \( w \)-limit of \( \left( \sum_{i=1}^{n} \lambda_i y_{(i, \alpha)} \right) \), and since \( C_Y \) is a convex subset of \( B_Y \), we have that \( y \in \overline{C}_Y \). We conclude that \( Y \) has property (PT) \( \square \).

We recall the classical result of Pelczynski and Semadeni [31], for a compact Hausdorff topological space \( K \), \( C(K) \) contains no isomorphic copy of \( \ell_1 \) if and only if \( K \) is scattered. Recently, Abrahamsen and Lima and Haller et al. show that \( C(K) \) has property (P1) if and only if \( K \) is scattered (see
The properties (P1) and (P1) are equivalent in $C(K)$ (see Theorem 4.3). Therefore, the following result can be stated.

**Corollary 2.13.** Let $K$ be a locally compact Hausdorff topological space. The following assertions are equivalent:

i) $C(K)$ contains no isomorphic copy of $\ell_1$.

ii) $C(K)$ has property $(\text{P1})$.

The next theorem is a generalization of the above corollary in the setting of $L_1$-predual Banach spaces.

**Theorem 2.14.** Let $X$ be a $L_1$-predual Banach space. Then, $X$ contains no isomorphic copy of $\ell_1$ if and only if $X$ has property $(\text{P1})$.

**Proof.** We assume first that $X$ contains no isomorphic copy of $\ell_1$. Then, by [17], $B_{X^*}$ is the closed convex hull of the extreme points of $B_{X^*}$, and hence, $X^*$ is purely atomic. This implies that $X^* = \ell_1(\Gamma)$ for some set $\Gamma$. By Theorem 2.7, $X$ has property $(\text{P1})$.

Now, suppose that $X$ is not separable, has property $(\text{P1})$, and that $X$ contains an isomorphic copy of $\ell_1$. Then, there exists a separable subspace $Y$ of $X$, such that $Y$ is isomorphic to $\ell_1$. Combining [32, Theorem] with Lemma 2.11, there exists a separable ideal $Z$ in $X$, such that $Y \subset Z \subset X$. In a recent work, Bandyopadhyay et al. prove that if $X$ is a non-separable $L_1$-predual Banach space, then every separable ideal in $X$ is an $L_1$-predual Banach space (see [6, Theorem 2.8]). By Proposition 2.12, we have that $Z$ is a separable $L_1$-predual Banach space with property $(\text{P1})$. Since $Z$ contains an isomorphic copy of $\ell_1$, we have that $Z^*$ is not separable. By a result of Lazar and Lindenstrauss [25, Theorem 2.3], $Z$ contains a 1-complemented subspace isometric to $C(\Delta)$, the Banach space of continuous functions on the Cantor discontinuum $\Delta$. Since property $(\text{P1})$ is inherited by 1-complemented subspaces (Proposition 2.3), we obtain that $C(\Delta)$ has property $(\text{P1})$. This is a contradiction, since $\Delta$ is a non-scattered compact topological space (see [18, Theorem 3.1]).

In the case that $X$ is separable, has property $(\text{P1})$ and that $X$ contains an isomorphic copy of $\ell_1$ we argue similarly to the final part of the previous reasoning.

In any case, we conclude that $X$ contains no isomorphic copy of $\ell_1$. □

### 3. Finite-Dimensional C*-Algebras have Property (co)

The following result is a generalization of Lemma 2.2 in [2] to the setting of $C^*$-algebras.

**Lemma 3.1.** Let $A$ be a $C^*$-algebra. Let $x$, $y$ be two elements in $B_A$ and $d = \|x + y\|$. Then, for every $\lambda \in [0, \frac{1}{2}]$, we have:

$$\|\lambda x + (1 - \lambda)y\| \leq \sqrt{1 - (4 - d^2)(\lambda - \lambda^2)} \leq 1 - \frac{(4 - d^2)\lambda}{4}.$$
Proof. We have that $4 \geq d^2 = \|x + y\|^2 = \|(x + y)^*(x + y)\| = \sup\{\phi((x + y)^*(x + y)) : \phi \in S(A)\}$, where the supremum is taken over the states of $A$, $S(A)$, and hence, $\phi(x^*y + y^*x) \leq d^2 - \phi(x^*x) - \phi(y^*y)$ for every $\phi \in S(A)$.

Now, it is straightforward to verify that:

$$
\|\lambda x + (1 - \lambda)y\|^2 = (\lambda x + (1 - \lambda)y)^*(\lambda x + (1 - \lambda)y)
= \sup\{\lambda^2\phi(x^*x) + (1 - \lambda)^2\phi(y^*y) + \lambda(1 - \lambda)\phi(x^*y + y^*x) : \phi \in S(A)\}
\leq \sup\{\lambda^2\phi(x^*x) + (1 - \lambda)^2\phi(y^*y) + \lambda(1 - \lambda)(d^2 - \phi(x^*x) - \phi(y^*y)) : \phi \in S(A)\}
= \sup\{(\lambda^2 - \lambda(1 - \lambda))\phi(x^*x) + ((1 - \lambda)^2 - \lambda(1 - \lambda))\phi(y^*y) + \lambda(1 - \lambda)d^2 : \phi \in S(A)\}
\leq (\lambda^2 - \lambda(1 - \lambda)) + ((1 - \lambda)^2 - \lambda(1 - \lambda)) + \lambda(1 - \lambda)d^2 = 1 - (4 - d^2)(\lambda - \lambda^2).
$$

The last inequality follows from the facts $\sqrt{1 + t} \leq 1 + \frac{t}{2}$ for $t \geq -1$ and $\lambda - \lambda^2 \geq \frac{1}{2}$.

Given two partial isometries $e, f$ in a $C^*$-algebra $A$ we say that $e \leq f$ whenever $ee^*f = e$ and $fe^*e = e$ (equivalently $f = e + (1 - ee^*)f(1 - e^*e)$). Given a positive element $a$ in the closed unit ball of $A$, the sequence $(a^n)$ converges to a projection in $A^*$ (in the weak$^*$-topology) called the support projection of $a$. This projection can also be defined as the biggest projection $p$ in $A^*$ satisfying $p \leq a$. Given $x \in B_A$ with polar decomposition $x = v|x|$, we define $s(x)$, the support partial isometry of $x$ (in $A^*$) as $vp$, where $p$ is the support projection of $|x|$. It is clear that $x = s(x) + (1 - s(x)s(x)^*s(x))$ and $s(x)$ is the biggest partial isometry in $A^*$ satisfying this property.

Given a $C^*$-algebra $A$, every partial isometry $e$ in $A$ induces a decomposition of $A$ as the direct sum $A_2(e) \oplus A_1(e) \oplus A_0(e)$, where $A_2(e) = ee^*Ae^*e$, $A_1(e) = ee^*A(1 - e^*e) \oplus (1 - ee^*)Ae^*e$ and $A_0(e) = (1 - ee^*)A(1 - e^*e)$.

The corresponding (natural) projections onto these subspaces, $P_i(e)$ ($i \in \{0, 1, 2\}$), called Peirce projections, are known to be contractive and every element in $A_2(e)$ is orthogonal to any element in $A_0(e)$.

**Lemma 3.2.** Let $A$ be a $C^*$-algebra and let $x, y$ be two elements in $B_A$. Then, there exists a partial isometry $e$ in $A^*$ satisfying:

$$
\lambda x + (1 - \lambda)y = e + P_0(e)(\lambda x + (1 - \lambda)y) \text{ for all } \lambda \in [0, 1],
$$

and being maximal for this property.

In particular, when $A$ is a finite-dimensional $C^*$-algebra, we have that $e$ is a finite rank partial isometry in $A$ and $\|P_0(e)(\lambda x + (1 - \lambda)y)\| < 1$ for all $\lambda \in ]0, 1[$.

**Proof.** Fix $\lambda_0 \in ]0, 1[$ and set $e = s(\lambda_0x + (1 - \lambda_0)y) \in A^*$ whenever $\|\lambda_0x + (1 - \lambda_0)y\| = 1$ and $e = 0$ otherwise. In case $e = 0$, the desired equality is trivially satisfied for every $\lambda \in [0, 1]$, and thus, we can assume that $\|\lambda_0x + (1 - \lambda_0)y\| = 1$ and $e = s(\lambda_0x + (1 - \lambda_0)y)$.

Since $F_e = (e + A_0^*(e)) \cap B_A$ is a norm-closed face in the closed unit ball of $A$ [4, Theorem 4.10], we have that $\lambda x + (1 - \lambda)y \in F_e$ for every $\lambda \in [0, 1]$. Therefore, $s(\lambda x + (1 - \lambda)y) \geq e = s(\lambda_0x + (1 - \lambda_0)y)$ for every $\lambda \in [0, 1]$. The
arbitrarily of \( \lambda_0 \) gives \( s(\lambda x + (1 - \lambda)y) = e \) for every \( \lambda \in [0, 1] \), showing the maximality of \( e \) among the partial isometries satisfying this property.

The final comments should be clear from properties of the support partial isometry of a norm-one element in a finite-dimensional C*-algebra (see, for example, [19, Theorem 3.1] or [8, page 19]). \( \square \)

The following result is probably part of the folklore. We include here a proof for the sake of completeness.

**Lemma 3.3.** Let \( \mathcal{A} \) be a C*-algebra and let \( e, f \) be two partial isometries in \( \mathcal{A} \), such that \( \|e - f\| \leq \varepsilon \), where \( \varepsilon \) is positive. Then, the following inequalities hold:

(a) \( \|P_2(e) - P_2(f)\| \leq 4 \varepsilon \), \( \|P_1(e) - P_1(f)\| \leq 8 \varepsilon \), \( \|P_0(e) - P_0(f)\| \leq 4 \varepsilon \).

(b) \( \|P_k(u)P_j(v)\| \leq 4 \varepsilon \) where \( u, v \in \{e, f\} \) distinct and also \( k, j \in \{0, 1, 2\} \) are distinct.

In particular, given a norm one element \( x \in \mathcal{A} \), satisfying \( x = e + P_0(e)x \), we have that \( \|P_1(f)x\|, \|P_2(f)x - f\|, \|x - (f + P_0(f)x)\| \leq 5 \varepsilon \).

**Proof.** (a) Given \( x \in \mathcal{A} \) with \( \|x\| = 1 \), we have that \( \|ee^*xe^*e - f^*xf^*f\| \leq \|(e - f)e^*xe^*e\| + \|f^*xf^*f\| \leq \varepsilon + \|f(e^* - f^*)xe^*e\| + \|f^*xf^*f\| \leq 3 \varepsilon + \|f^*xf^*(e - f)\| \leq 4 \varepsilon \). Since \( P_0(e)x = (1 - ee^*)x(1 - e^*e) \) and \( P_1(e)x = (1 - ee^*)xe^*e + e^*e(1 - e^*e) \), the remaining inequalities follow in the same manner.

(b) \( \|P_2(e)P_1(f)\| = \|(P_2(f) - P_2(e))P_1(f)\| \leq 4 \varepsilon \) by (a). The rest of cases can be obtained analogously.

Finally, given \( x \in \mathcal{A} \), a norm one element with \( x = e + P_0(e)x \), we have that \( P_1(f)x = P_1(f)(e - f) + P_1(f)P_2(f)x, P_2(f)x - f = P_2(f)(e - f) + P_2(f)P_0(e)x \) and \( x - (f + P_0(f)x) = (e - f) + (P_0(e) - P_0(f))x \); thus, (a) and (b) apply. \( \square \)

The following two remarks contain information concerning perturbation theory in C*-algebras.

**Remark 3.4.** Let \( \mathcal{A} \) be a C*-algebra. It is well known that the absolute value is a continuous function on \( \mathcal{A} \), where the absolute value of an element \( x \) is the square root of \( x^*x \) and denoted by \( |x| \). Concretely, given \( x, y \in \mathcal{A} \), we have that \( \|x^2 - y^2\| = \|x^*x - y^*y\| = \|x^*(x - y) + (x^* - y^*)y\| \leq \|x - y\| (\|x\| + \|y\|) \) and by [28, Theorem]

\[
\|x - y\| \leq \sqrt{\|x - y\|^2 (\|x\| + \|y\|)}.
\]

Therefore, we have that small perturbations of an element in a C*-algebra give rise to small perturbations of its absolute value.

**Remark 3.5.** Let \( \mathcal{A} \) be a finite-dimensional C*-algebra. Given \( x \in \mathcal{A} \), there exist a unitary \( u \in \mathcal{A} \), such that \( x = u|x| \). The eigenvalues of \( |x| \) are called the *singular values* of \( x \) and we may consider them as a vector, \( \text{Sing}(x) = \{\sigma_1(x), \ldots, \sigma_n(x)\} \in \mathbb{R}^n \), where \( n \) is the rank of \( \mathcal{A} \), counting the eigenvalues with multiplicity and in decreasing order.
Given \( x, y \in \mathcal{A} \), there exists a connection between the distance of the corresponding singular values of \( x \) and \( y \) and the distance between \( x \) and \( y \). Concretely, \( \max\{ |\sigma_i(x) - \sigma_i(y)| : i \in \{1, \ldots, n\} \} \leq \|x - y\| \) (see [8, Theorem 9.8]). There is no such a relation, in general, between the distance of the corresponding eigenvectors (see for example [9, page 46]). However, for some particular small perturbations of positive elements, we get small changes in the corresponding spectral resolutions.

More concretely, take a positive element \( h \) in a finite-dimensional C*-algebra \( \mathcal{A} \). Let \( \beta, \gamma > 0 \) and assume \( p \) is the spectral resolution of \( h \) associated with the set \([\nu, \mu]\), where \( \mu - \nu \leq 2\beta \) and the sets \([\nu - \gamma, \nu[, [\mu, \mu + \gamma[\) contain no eigenvalues of \( h \). Given a positive \( b \in \mathcal{A} \) with \( \|b - h\| \leq \delta < \frac{\gamma}{2} \) and \( q \) the projection of \( b \) associated with \( p \) (i.e., \( q \) is the spectral resolution of \([\nu - \delta, \mu + \delta]\)), Davis obtained in [11, Theorem 2.1] that:

\[
\|q(1 - p)\| = \|(1 - p)q\| \leq \frac{\beta + \delta}{\beta + \gamma}.
\]

Whenever \( \delta < \frac{\gamma}{4} \), applying [11, Theorem 2.1] to \( q \) and \( p \) with the new parameters \( \beta' = \beta + \delta \) and \( \gamma' = \frac{\gamma}{2} \), we have that:

\[
\|p(1 - q)\| = \|(1 - q)p\| \leq \frac{\beta + 2\delta}{\beta + \frac{\gamma}{2}}.
\]

In the particular case of \( \beta = 0 \) (\( p \) is the spectral resolution of a single eigenvalue) and \( \delta < \frac{\gamma}{4} \), we have that:

\[
\|p - q\| = \|p(1 - q) - (1 - p)q\| \leq \|p(1 - q)\| + \|(1 - p)q\|
\leq \frac{\delta}{\gamma} + \frac{4\delta}{\gamma} \leq \frac{16}{3} \frac{\delta}{\gamma},
\]

(see [33, Theorem 2] where a different bound is given).

For our purposes, given an element \( x \) in a finite-dimensional C*-algebra \( \mathcal{A} \) of rank \( n \), it will be more convenient to express \( x \) as the sum \( \sum_{i=1}^{n_0} \sigma_i(x)e_i \) where \( \{\sigma_i(x) : i \in \{1, \ldots, n_0\}\} \) are the singular values of \( x \) (eigenvalues of \( |x| \)) taken in decreasing order not counting multiplicity. Moreover, \( x = u|x| \) with \( u \) unitary in \( \mathcal{A} \), \( e_i = up_i \) is a finite rank partial isometry in \( \mathcal{A} \) with \( p_i \) the spectral resolution of \( |x| \) associated with the singular value \( \sigma_i(x) \) and \( \sum_{i=1}^{n_0} \text{rank}(e_i) = n \) (see for example [19, Theorem 3.1]).

Having in mind Remark 3.5 above, we have that, once fixed a positive \( \delta \) smaller than one half the distance between the elements of the union of the singular values of \( x \) and \( 0 \), for every element \( y \) satisfying \( \|y - x\| \leq \delta \), associated with each \( p_i \), we have a projection \( q_i \), the spectral resolution of \( |y| \) with respect to the set \([\sigma_i(x) - \delta, \sigma_i(x) + \delta]\). Notice that \( |y| \) does not coincide in general with a linear combination of these projections. Therefore, for each \( i \in \{1, \ldots, n_0\} \), we have associated a partial isometry \( f_i = vq_i \), where \( v \) is the unitary, such that \( y = v|y| \).

The following result exhibits the continuity at some fixed point of the perturbed spectral resolutions.
Theorem 3.6. Let $A$ be a finite-dimensional $C^*$-algebra and let $x$ be an element in $A$. Then, for every positive $\varepsilon$, there exists $\delta > 0$, such that for every $y$ in the closed ball centered at $x$ of radius $\delta$, $\|e_i - f_i\| \leq \varepsilon$ for each $i \in \{1, \ldots, n\}$, such that $\sigma_i(x) > 0$.

Proof. We define $\gamma = \min\{\sigma_i(x) - \sigma_{i+1}(x) : \sigma_i(x) > 0\}$ where $\sigma_i(x)$ are the singular values of $x$ and $\sigma_j(x) = 0$ in case $j$ is greater that the dimension of $\mathcal{A}$. Let $\delta$ be a positive number satisfying:

$$\max \left\{ \frac{16}{3} \sqrt{\left(2\|x\| + \delta\right) \delta}, \frac{1}{\sigma_i(x)} \left(\sqrt{\left(2\|x\| + \delta\right) \delta} + \delta\right) : \sigma_i(x) > 0 \right\} \leq \varepsilon.$$

Take $y \in A$ with $\|x - y\| \leq \delta$. Let $n_0, e_i, f_i, p_i, q_i$ be defined as in the comments preceding this theorem. By Remark 3.4, we have that $\|x - y\| = \|y\| = \sqrt{(2\|x\| + \delta) \delta}$. Now, Remark 3.5 gives $\|p_i - q_i\| \leq \frac{16}{3} \sqrt{(2\|x\| + \delta) \delta} \gamma$ for each $i \in \{1, \ldots, n\}$. By polar decomposition, we have that $x = u|x|$ and $y = v|y|$, where $u$ and $v$ are unitaries in $A$, with $e_i = up_i$ and $f_i = vq_i$.

Clearly, $|x|p_i = \sigma_i(x)p_i$, and it is straightforward to check that $\|x - v^*x\| = \|x\| - |y| + |y| - v^*x\| \leq \|x\| - |y| + \|y\| - v^*x\| \leq \|x\| - |y| + \|y - x\| \leq \sqrt{(2\|x\| + \delta) \delta} + \delta$.

Moreover, for each $\sigma_i(x) > 0$, we have:

$$\|f_i - e_i\| = \|q_i - v^*up_i\| = \|q_i - p_i + p_i - v^*up_i\| \leq \|q_i - p_i\| + \|p_i - v^*up_i\| \leq \|q_i - p_i\| + \|(|x| - v^*u|x|) \frac{p_i}{\sigma_i(x)}\| \leq \|q_i - p_i\| + \frac{1}{\sigma_i(x)} \| |x| - v^*x\| \leq \frac{16}{3} \sqrt{(2\|x\| + \delta) \delta} \gamma + \frac{1}{\sigma_i(x)} \left(\sqrt{(2\|x\| + \delta) \delta} + \delta\right) \leq \varepsilon.$$

It will be convenient to highlight and isolate a particular case of Theorem 3.6.

Remark 3.7. Let $A$ be a finite-dimensional $C^*$-algebra. Let $x, y$ be elements in the closed unit ball of $A$. Assuming that $\|x\| = 1$, we denote by $e = s(x)$ the support partial isometry of $x$ (i.e., $\gamma_1(x) = 1$ and $e = e_1$), and $\gamma = 1 - \|x - s(x)\|$. Let $\delta$ be a positive number with $\delta < \sqrt{2\delta} < \frac{\gamma}{4}$ and suppose that $\|x - y\| \leq \delta$. Denoting by $f$ the spectral resolution of $y$ corresponding to the set $[1 - \delta, 1]$, we have that:

$$\|e - f\| \leq \delta + \sqrt{2\delta} + \frac{16}{3} \frac{\sqrt{2\delta}}{\gamma}.$$

We are now in a position to prove the main result of this section. The proof is highly influenced by the proof of Theorem 2.3 in [2].

Theorem 3.8. Every finite-dimensional $C^*$-algebra has property (co).
Proof. Let $\mathcal{A}$ be a finite-dimensional C*-algebra, $n \in \mathbb{N}$. Let $x_1, \ldots, x_n$ be elements in the closed unit ball of $\mathcal{A}$ and let $\lambda_1, \ldots, \lambda_n$ be positive numbers with $\sum_{i=1}^n \lambda_i = 1$. We claim that for every positive $\varepsilon$, there exist a positive $\delta$, such that given $y \in B_\mathcal{A}$ with $\|y - \sum_{i=1}^n \lambda_i x_i\| \leq \delta$, there exist $\tilde{x}_1, \ldots, \tilde{x}_n$ in $B_\mathcal{A}$ satisfying $y = \sum_{i=1}^n \lambda_i \tilde{x}_i$ and $\|x_i - \tilde{x}_i\| \leq \varepsilon$, for each $i \in \{1, \ldots, n\}$.

Assume first that $\|\sum_{i=1}^n \lambda_i x_i\| = 1$.

In this case, by Lemma 3.2, denoting by $e$ the support partial isometry of $\sum_{i=1}^n \lambda_i x_i$, we have that $e \neq 0$ and $e$ is also the support partial isometry of any other (strict) convex combination of the elements $\{x_1, \ldots, x_n\}$.

We set, for each $j \in \{1, \ldots, n\}$, $a_j = \sum_{i=1, i \neq j}^n \frac{x_i}{n-1} \in B_\mathcal{A}$. We define $d = \max\{\|P_0(e)(a_j + x_j)\| : j \in \{1, \ldots, n\}\}$. It should be clear from Lemma 3.2 that $d < 2$. It is also direct to verify that $\sum_{j=1}^n (a_j - x_j) = 0$ and $\lambda a_j + (1 - \lambda)x_j = e + P_0(e)(\lambda a_j + (1 - \lambda)x_j)$ for every $\lambda \in [0, 1]$.

Now, fix $c > 0$ satisfying $0 < c \leq \frac{\varepsilon}{4} \min\{\lambda_i : i \in \{1, \ldots, n\}\}$ and define $\mu_j = \frac{c}{\lambda_j}$. It is clear that:

$$\max\{\mu_j : j \in \{1, \ldots, n\}\} = \frac{c}{\min\{\lambda_j : j \in \{1, \ldots, n\}\}} \leq \frac{\varepsilon}{4}. \quad (1)$$

We set $\gamma = 1 - \|P_0(e)(\sum_{i=1}^n \lambda_i x_i)\|$, which is positive by Lemma 3.2.

We can associate with every positive $\delta$ the following positive number, $\varepsilon_1 = \varepsilon_1(\delta) = \delta + \sqrt{2\delta + \frac{16}{3} \frac{\varepsilon}{\gamma}}$, which satisfies $\lim_{\delta \to 0} \varepsilon_1(\delta) = 0$.

Take $\delta > 0$ satisfying $\delta < \sqrt{2\delta} < \frac{\varepsilon}{4}$, then:

$$4\varepsilon_1 + \delta < \min\{\lambda_j : j \in \{1, \ldots, n\}\}(\frac{4 - d^2}{4}) \quad (2)$$

and

$$10\varepsilon_1 + 2\delta < \frac{\varepsilon}{2}. \quad (3)$$

Applying Remark 3.7 to any $y$ in the closed unit ball of $\mathcal{A}$ with $\|\sum_{i=1}^n \lambda_i x_i - y\| < \delta$ and denoting by $f$ the spectral resolution of $y$ associated with the set $[1 - \delta, 1]$, we have that:

$$\|f - e\| \leq \varepsilon_1. \quad (4)$$

We next define the elements $\tilde{x}_j$ and check the desired statements.

For each $j \in \{1, \ldots, n\}$, we define:

$$\tilde{x}_j = P_2(f)y + P_0(f)[x_j + \mu_j(a_j - x_j)] + y - \sum_{i=1}^n \lambda_i x_i.$$
It follows straightforwardly that:

\[
\sum_{j=1}^{n} \lambda_j \tilde{x}_j = \sum_{j=1}^{n} \lambda_j P_2(f) y + P_0(f) \left[ \sum_{j=1}^{n} \lambda_j x_j + \sum_{j=1}^{n} \lambda_j \mu_j (a_j - x_j) + \sum_{j=1}^{n} \lambda_j y \right] \\
- \left[ \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i x_i \right] \\
= P_2(f) y + P_0(f) \left[ \sum_{j=1}^{n} \lambda_j x_j + c \sum_{j=1}^{n} (a_j - x_j) + y - \sum_{i=1}^{n} \lambda_i x_i \right] \\
= P_2(f) y + P_0(f) y = y.
\]

It is also satisfied that \( \| x_j - \tilde{x}_j \| \leq \varepsilon \) for every \( j \in \{1, \ldots, n\} \). Indeed, remembering that \( \| \sum_{i=1}^{n} \lambda_i x_i - y \| < \delta \), we have:

\[
\| x_j - \tilde{x}_j \| = \| x_j - P_2(f) y - P_0(f) [x_j + \mu_j (a_j - x_j) + y - \sum_{i=1}^{n} \lambda_i x_i] \| \\
= \| P_2(f) x_j + P_1(f) x_j + P_0(f) x_j - P_2(f) y + f - P_0(f) x_j \| \\
- \mu_j P_0(f) a_j + \mu_j P_0(f) x_j + P_0(f) (y - \sum_{i=1}^{n} \lambda_i x_i) \| \\
\leq \| P_2(f) x_j - f \| + \| f - P_2(f) y \| + \| P_1(f) x_j \| + \| \mu_j P_0(f) (x_j - a_j) \| \\
+ \| P_0(f) (y - \sum_{i=1}^{n} \lambda_i x_i) \| \\
(by (4), Lemma 3.3 and the definition of f ) \\
\leq 5 \varepsilon_1 + \delta + 5 \varepsilon_1 + 2 \mu_j + \delta \leq (by (1) and (3) ) \leq \varepsilon.
\]

Finally, we will show that \( \| \tilde{x}_j \| \leq 1 \) for every \( j \in \{1, \ldots, n\} \). Since \( \| \tilde{x}_j \| = \max \{ \| P_2(f) y \|, \| P_0(f) [x_j + \mu_j (a_j - x_j) + y - \sum_{i=1}^{n} \lambda_i x_i] \| \} \), we only have to check that the second item is less than or equal to 1. Now:

\[
\| P_0(f) [x_j + \mu_j (a_j - x_j) + y - \sum_{i=1}^{n} \lambda_i x_i] \| \leq \| P_0(f) [(1 - \mu_j) x_j + \mu_j a_j] \| + \| y - \sum_{i=1}^{n} \lambda_i x_i \| \\
\leq \| P_0(e) [(1 - \mu_j) x_j + \mu_j a_j] \| + \| (P_0(e) - P_0(f)) [(1 - \mu_j) x_j + \mu_j a_j] \| + \| y - \sum_{i=1}^{n} \lambda_i x_i \| \\
(by Lemma 3.1 and Lemma 3.3 a) \leq 1 - \frac{4 - d^2}{4} \mu_j + 4 \varepsilon_1 + \delta \leq (by (2)) \leq 1.
\]

The case \( \| \sum_{i=1}^{n} \lambda_i x_i \| < 1 \) is even simpler. Notice that in this case \( e = 0 \), so that \( P_2(e) = P_1(e) = 0 \) and \( P_0(e) = \text{Id}_{|A|} \), and if \( \delta < \frac{\gamma}{4} \), the spectral resolution of \( y \) corresponding to the set \( \{1 - \delta, 1\} \), \( f \), is also zero. Defining \( \tilde{x}_j \) in the same manner, with the less restrictive assumption \( \delta \leq \min \{ \frac{\varepsilon_1}{2}, \min \{ \frac{4 - d^2}{4} \mu_j : j \in \{1, \ldots, n\} \} \} \), we arrive at the desired conclusion.

To prove that \( A \) has property \( (co) \), and once we have fixed \( \delta > 0 \), we only have to check that the functions \( \phi_j : B(x, \delta) \cap B_A \rightarrow B(x_j, \varepsilon) \cap B_A \) defined by \( \phi_j(y) = \tilde{x}_j \) for every \( j \in \{1, \ldots, n\} \) are continuous.
Given $y, z$ in $B(x, \delta) \cap B_A$, we have that:
\[
\|\phi_j(y) - \phi_j(z)\| = \|P_2(f_y)y + P_0(f_y)[x_j + \mu_j(a_j - x_j) + y - \sum_{i=1}^{n} \lambda_i x_i] - P_2(f_z)z - P_0(f_z)[x_j + \mu_j(a_j - x_j) + z - \sum_{i=1}^{n} \lambda_i x_i]\|
\]
\[
= \|y - z + (P_0(f_y) - P_0(f_z))[x_j + \mu_j(a_j - x_j) - \sum_{i=1}^{n} \lambda_i x_i]\| \leq \|y - z\| + 8\|f_y - f_z\|
\]

where, in the last inequality, we have used Lemma 3.3 and $\|x_j + \mu_j(a_j - x_j) - \sum_{i=1}^{n} \lambda_i x_i\| \leq 2$. Theorem 3.6 assures that the functions $\phi_j$ are continuous.

\[\square\]

4. Scattered C*-Algebras and Properties (P1) and (P1)

A topological space $K$ is called scattered if each closed subset of $K$ has an isolated point. It is well known that a locally compact Hausdorff topological space is scattered if and only if there exists no non-zero atomless regular Borel measure on $K$ (see [27,31]).

An AC*-algebra $A$ is said to be a scattered C*-algebra if every positive functional on $A$ is the sum of a sequence of pure functionals [21]. Clearly, an abelian C*-algebra is scattered if and only if the space of characters (equipped with the weak* topology) is scattered. There are several characterizations of scattered C*-algebras by different authors. For example, they are Type I with scattered spectrum [22, Corollary 3], its bidual coincides with an $\ell_\infty$-sum of factors of type I [21, Theorem 2.2], it admits a composition series, such that every quotient algebra is elementary [22, Theorem 2], its dual has the Radon–Nikodým property [10, Theorem 3] and every C*-subalgebra has real rank-zero [24, Theorem 2.3]. However, we will take advantage of the characterization obtained by Huruya [20, Theorem] which assures that a C*-algebra is scattered if and only if the spectrum of every hermitian element is scattered (i.e., countable).

Recently, Abrahamsen and Lima and Haller et al. have shown that given $K$ a compact Hausdorff topological space, the space of continuous $\mathbb{K}$-valued functions on $K$, $C(K, \mathbb{K})$, has property (P1) if and only if $K$ is scattered (see [2,18]). The main goal of this section is to present connections between properties (P1) and (P1) and being scattered in the setting of general C*-algebras.

Remark 4.1. In [18], the authors show that for every non-scattered locally compact Hausdorff space $K$, there exist a convex combination of slices in the unit ball of $A = C_0(K, \mathbb{R})$ with empty interior. This statement is stronger than $C_0(K, \mathbb{R})$ fails property (P1). If we only want to show the latter result, the (same) arguments given in the proof of Theorem 3.1 in [18] can be shortened. Concretely, since $K$ is not scattered, there exists an atomless measure
\( \mu \) with \( \mu(K) = 1 \). Let \( \varepsilon \in [0, 1] \) and let us define \( S_1 = S(B_{A_2}, \mu, \varepsilon) = \{ x \in C_0(K, \mathbb{R}) : \| x \| \leq 1, \mu(x) > 1 - \varepsilon \} \), \( S_2 = -S_1 \) and \( C = \frac{S_1 + S_2}{2} \). Clearly, 0 belongs to \( C \) and for each weakly open neighborhood of 0, \( U \), there exist disjoint compact sets \( K^1, K^2 \) contained in \( K \), a positive \( \delta \), such that \( 3\delta + 2\sqrt{\delta} < 1 - \varepsilon \), \( |\mu(K^1) - \mu(K^2)| < \delta \), \( \mu(K^1 \cup K^2) < \delta \), and a continuous function \( y_\delta \) in the unit sphere of \( C_0(K, \mathbb{R}) \) satisfying \( y_\delta \in U \), \( y_\delta(t) = 1 \quad \forall t \in K^1 \), \( y_\delta(t) = -1 \quad \forall t \in K^2 \) and \( y_\delta \notin C \).

**Proposition 4.2.** Every \( C^* \)-algebra with property (PT) is scattered.

*Proof.* Let \( A \) be a \( C^* \)-algebra satisfying property (PT). \( A \) will be scattered the moment we show that the spectrum of every self-adjoint element is scattered.

Assume that there exists \( h \in A_{sa} \) with non-scattered spectrum. Let us denote by \( C_h \) the abelian \( C^* \)-subalgebra generated by \( h \), which is isomorphic to \( C_0(\text{Sp}(h), \mathbb{C}) \) ([30, Proposition 1.1.8]). Since the spectrum of \( h \) is non-scattered there exists an atomless regular Borel measure on \( \text{Sp}(h) \), \( \mu \), with \( \mu(\text{Sp}(h)) = 1 \). Fix \( 0 < \varepsilon < 1 \) and define \( S_1 = S(B_{A_2}, \mu, \varepsilon) = \{ z \in B_{A_2} : \Re(\mu(z)) > 1 - \varepsilon \} \), \( S_2 = -S_1 \) and \( S = \frac{S_1 + S_2}{2} \). Clearly, 0 belongs to \( S \) and we will show that every relatively weakly open subset of the closed unit ball of \( A \) containing 0 has an element not contained in \( S \).

Let \( W = \{ z \in B_{A_2} : |\varphi_i(z)| < \gamma, i = 1, \ldots, n \} \), where \( \varphi_i \in A^* \) (\( i = 1, \ldots, n \)) and \( \gamma > 0 \). Let us denote by \( f_i \) the restriction of \( \varphi_i \) to \( C_h \). Clearly, \( \mathcal{U} = \{ z \in B_{C_h} : |f_i(z)| < \gamma, i = 1, \ldots, n \} \subseteq W \) is a neighborhood of 0 in the weak topology of \( C_h \) restricted to its unit ball. Let \( K^1, K^2, \delta \) and \( y_\delta \) be as in Remark 4.1.

We will consider \( \mu \) both as a measure on \( \text{Sp}(h) \) and as a functional on \( A^{**} \) via the identification of the sets contained in \( \text{Sp}(h) \) and their associated characteristic functions in \( A^* \). Under this considerations, we denote by \( p_1 \) (respectively, \( p_2 \)) the projection in \( A^{**} \) given by the characteristic function associated to the set \( K^1 \) (respectively, \( K^2 \)) and we set \( p = p_1 + p_2 \). Clearly, \( p_1 \) and \( p_2 \) are orthogonal, \( \mu(p_j) = \mu(K^j) \) (\( j = 1, 2 \)) and \( |\mu(p_1) - \mu(p_2)| < \delta \).

Having in mind that \( \mu(\text{Sp}(h)) = 1 \), we have that:

\[
1 = \mu(\text{Sp}(h)) = \mu(K^1) + \mu(K^2) + \mu(\text{Sp}(h) \setminus (K^1 \cup K^2)) < \mu(K^1) + \mu(K^2) + \delta,
\]

and consequently, \( \mu(p) = \mu(p_1) + \mu(p_2) > 1 - \delta \).

The M-orthogonality between \( pA^{**}p \) and \((1 - p)A^{**}(1 - p)\) together with \( \mu(p) > 1 - \delta \) assures that \( \| \mu|_{(1 - p)A^{**}(1 - p)} \| < \delta \). Moreover, \( \mu(1 - p) < \delta \) and by the Cauchy–Schwarz inequality [30, Theorem 3.1.3], we have that \( |\mu((1 - p)z)|, |\mu(z(1 - p))| < \sqrt{\delta} \) for every \( z \in B_{A_2} \).

Now, suppose that \( y_\delta \) belongs to \( S \), so that \( y_\delta = \lim x_n + y_n \) with \( x_n \in S_1, y_n \in S_2 \). Denoting by \( e = p_1 - p_2 \) (a symmetry in the \( C^* \)-algebra \( pA^{**}p \)), simple arguments with respect to the order in \( pA^{**}p \) give \( e = \lim px_n p \) and \( e = \lim py_n p \). Therefore, for every natural \( n \) with \( \| e - px_n p \| \leq \delta \), we have that:

\[
|\mu(x_n)| \leq |\mu(px_n p - e)| + |\mu(e)| + |\mu((1 - p)x_n p)| + |\mu(px_n (1 - p))| + |\mu((1 - p)x_n (1 - p))| \leq 3\delta + 2\sqrt{\delta} < 1 - \varepsilon,
\]

which gets in contradiction with \( x_n \in S_1 \). \( \square \)
Notice that there exist scattered C*-algebras not satisfying property (P1). For example, it is straightforward to check that the (scattered) C*-algebra of compact operators on an infinite-dimensional Hilbert space does not satisfy property (P1). We recall that a compact C*-algebra is some c₀-sum of algebras of compact operators on a Hilbert space (see [5, Theorem 8.2]). Having in mind Proposition 2.5 we have that a compact C*-algebra fails property (P1) whenever one of its summands is infinite dimensional. The following result is a generalization of [2, Theorem 2.3] and [18, Theorem 3.1].

**Theorem 4.3.** Let K be a locally compact Hausdorff topological space and let A be a compact C*-algebra. The following assertions are equivalent:

i) C₀(K, A) has property (P1).

ii) C₀(K, A) has property (P̅I).

iii) K is scattered and A is the c₀-sum of finite-dimensional C*-algebras.

**Proof.** Clearly, i) implies ii).

To show that ii) implies iii), let us assume that C₀(K, A) has property (P̅I). It is well known that C₀(K) is isometrically isomorphic to a norm-closed one-complemented subspace of C₀(K, A). By Proposition 2.3, we have that C₀(K) has property (P̅I) and Proposition 4.2 assures that K is scattered. Since K has isolated points, we deduce that A is also a 1-complemented subspace of C₀(K, A) and, hence, has property (P̅I), which gives that A is a c₀-sum of finite-dimensional C*-algebras.

Finally, if K is scattered and A is a c₀-sum of finite-dimensional C*-algebras, C₀(K, A) has property (P1) by Corollary 2.6 and Theorem 3.8.

□

Given a C*-algebra A, there exists a factorization of its bidual as the sum of its atomic and non-atomic ideals. More concretely, the atomic representation of A is an ℓ∞-sum of B(Hᵦ) where Hᵦ are Hilbert spaces associated to (unitarily equivalent) irreducible representations and the sum is taken over the spectrum of A (see [30, 4.3.7 and 4.3.8]). We say that every factor in the atomic decomposition of A is finite-dimensional whenever dim(Hᵦ) < ∞ for each irreducible representation.

**Proposition 4.4.** Let A be a C*-algebra satisfying property (P̅I). Then, every factor appearing in the atomic decomposition of A is finite-dimensional.

**Proof.** Assume that there exists an irreducible representation of A over B(H), with H infinite-dimensional Hilbert space. Fix a norm one vector η₀ ∈ H and an orthonormal system {ηᵢ : i ∈ I}, such that {η₀} ∪ {ηᵢ : i ∈ I} is a basis of H. We define a minimal projection p = η₀ ⊗ η₀ and rank one partial isometries uᵢ = ηᵢ ⊗ η₀ in A**. Clearly, puᵢ = uᵢ, uᵢp = 0, and uᵢuᵢ* p = uᵢuᵢ* = p for every i ∈ I. Moreover, the closure of the linear span of {p} ∪ {uᵢ : i ∈ I} is isometric to the Hilbert space H (i.e., pA** = pB(H) △ H) and, hence, for every φ ∈ A* and δ > 0 the set {i ∈ I : |φ(uᵢ)| ≥ δ} is finite.

We denote by φ₀ the support functional associated to the minimal projection p (the extreme point in the unit ball of A* satisfying φ₀(p) = 1). More
concretely, \( \varphi_0(z) = \varphi_0(pzp) \) for every \( z \in A^{**} \) and hence \( \|x\| = |\varphi_0(x)| = \varphi_0(x^*)^{1/2} \) for all \( x \in pA^{**}p \).

Given \( 0 < \varepsilon < 1 \), we define \( S_1 = S(B_A, \varphi_0, \varepsilon) = \{ x \in B_A : \Re \varphi_0(x) > 1 - \varepsilon \} \), \( S_2 = -S_1 \) and \( S = S_1 + S_2 \).

For each \( x \in S_1 \), we have that \( \|px\|^2 = \|pxp\|^2 + \|px(1-p)\|^2 \), and thus, \( \|px(1-p)\| \leq (1 - \|x\|^2)^{1/2} = (1 - |\varphi_0(x)|^2)^{1/2} \leq (1 - (1 - \varepsilon)^2)^{1/2} \) and the same happens with elements \( y \in S_2 \), and thus:

\[
\|pz(1-p)\| \leq (1 - (1 - \varepsilon)^2)^{1/2} < 1 \quad \text{for all } z \in \tilde{S}.
\]

Given a relatively weakly open subset of \( B_A \) containing \( 0 \), \( U = \{ x \in B_A : |\varphi_j(x)| < \delta, j = 1, \ldots, n \} \), where \( \varphi_j \in A^* \) for every \( j = 1, \ldots, n \) and \( \delta \) is a positive number, we will find a norm-one element \( a \in U \) satisfying \( \|pa(1-p)\| = 1 \), so that \( a \notin \tilde{S} \) by (5).

As claimed at the beginning of the proof, there exists \( u \in \{ u_i : i \in I \} \) satisfying \( |\varphi_j(u)| < \frac{\delta}{2} \) for every \( j = 1, \ldots, n \). We recall that, by Kadison’s transitivity theorem, rank one partial isometries in \( A^{**} \) are compact (belong locally to \( A \) in the terminology of [4]), and thus, there exists a net \( (a_\lambda) \) in \( B_A \) converging in the weak*-topology to \( u \) and satisfying \( (a_\lambda) = u + (1 - uu^*)(a_\lambda)(1 - u^*u) \) for every \( \lambda \) (see [4] or [12, Theorem 5.1]).

The weak*-convergence of \( (a_\lambda) \) to \( u \) assures the existence of \( \lambda_0 \), such that \( |\varphi_j(a_{\lambda_0})| < \delta \) for every \( j = 1, \ldots, n \). Therefore, writing \( a = a_{\lambda_0} \), we have that \( a \in U \) and \( a = u + (1 - uu^*)a(1 - u^*u) \). It is straightforward to verify that:

\[
\|pa(1-p)\| = \|p(u + (1 - uu^*)a(1 - u^*u))(1-p)\| = \|u + p(1 - uu^*)a(1 - u^*u)(1-p)\| = \|u + (1 - uu^*)pa(1-p)(1-u^*u)\| = 1,
\]

which gives a contradiction with (5).

Our final result will be a complete characterization of \( C^* \)-algebras satisfying property (P).

**Theorem 4.5.** A \( C^* \)-algebra satisfies property \((P)\) if and only if it is scattered with finite dimensional irreducible representations.

**Proof.** The only if part was given in Proposition 4.2 and Proposition 4.4, while the if part is a consequence of Theorems 3.8 and 2.7. \( \square \)

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