On disjoint \((v, k, k - 1)\) difference families

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Abstract

A disjoint \((v, k, k - 1)\) difference family in an additive group \(G\) is a partition of \(G \setminus \{0\}\) into sets of size \(k\) whose lists of differences cover, altogether, every non-zero element of \(G\) exactly \(k - 1\) times. The main purpose of this paper is to get the literature on this topic in order, since some authors seem to be unaware of each other’s work. We show, for instance, that a couple of heavy constructions recently presented as new, had been given in several equivalent forms over the last forty years. We also show that they can be quickly derived from a general nearring theory result which probably passed unnoticed by design theorists and that we restate and reprove in terms of differences. We exploit this result to get an infinite class of disjoint \((v, k, k - 1)\) difference families coming from the Fibonacci sequence. Finally, we will prove that if all prime factors of \(v\) are congruent to 1 modulo \(k\), then there exists a disjoint \((v, k, k - 1)\) difference family in every group, even non-abelian, of order \(v\).

Keywords: disjoint difference family; zero difference balanced function; Frobenius group; Ferrero pair; Pisano period.

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1 Introduction

Throughout this paper all groups will be understood finite and written in additive notation but not necessarily abelian.

Given a subset $B$ of a group $G$, the list of differences of $B$ is the multiset $\Delta B$ of all possible differences between two distinct elements of $B$. A collection $\mathcal{F}$ of subsets of $G$ is a difference family (DF) of index $\lambda$ if the multiset sum $\Delta \mathcal{F} := \biguplus_{B \in \mathcal{F}} \Delta B$ covers every non-zero element of $G$ exactly $\lambda$ times. In particular, one says that $\mathcal{F}$ is a $(v, k, \lambda)$-DF if $G$ has order $v$ and its members (base blocks) have all size $k$. A difference family is said to be disjoint (DDF) if its blocks are pairwise disjoint. It is a partitioned difference family (PDF) if its blocks partition $G$.

Partitioned difference families, introduced by Ding and Yin [10] for the construction of optimal constant composition codes, are also important from the design theory perspective; for instance, in [7] it is shown a strict connection between PDFs having all blocks of the same size $k$ but one of size $k - 1$ and certain resolvable 2-designs (RBIBDs) with block size $k$. It seems that this paper passed almost completely unnoticed in spite of the fact that it contains many new RBIBDs a couple of which are particularly remarkable since their parameters are new; a $(45, 5, 2)$-RBIBD and a $(175, 7, 2)$-RBIBD. The importance of the former was pointed out in the paper itself, here we underline the importance of the latter considering that, according to Table 7.40 in [1], even the existence of a $(175, 7, 6)$-RBIBD was previously in doubt while it is now obvious that it can be obtained by simply tripling the obtained $(175, 7, 2)$-RBIBD.

After the above discussed paper the notion of a PDF apparently disappeared from the literature for a long time but, as a matter of fact, it has been considered under the name of a zero difference balanced function (ZDBF). A function $f$ from a group $G$ to a group $H$ is defined to be a $(v, \lambda)$-ZDBF if $\text{ord}(G) = v$ and the equation $f(g + x) = f(x)$ in the unknown $x$ has always $\lambda$ solutions whichever is $g \in G \setminus \{0\}$. It is an easy exercise to prove that this is completely equivalent to say that the set of non-empty fibers of $f$ form a PDF in $G$. It seems to be usual to assume that both $G$ and $H$ are abelian (see, e.g., [22] and [24]) but, in our opinion, there is no good reason to make this restriction.

Very recently PDFs have returned with their original name in a paper [19] where the authors develop the composition constructions of [7] making use of difference matrices.

It is clear that every DDF can be “completed” to a PDF by adding suitable blocks of size 1. We note that a $(v, k, \lambda)$-DDF necessarily has $1 \leq \lambda \leq k - 1$ apart the very trivial case of a $(k, k, k)$ difference set. The existence of $(v, k, 1)$-DDFs is in general a quite hard problem. Among the few results on this problem we recall that Dinitz and Rodney [13] found a $(v, 3, 1)$-DDF.
for any admissible \(v\) and that any radical \((v, k, 1)\)-DF (see [6]) is disjoint when \(k\) is odd. On the contrary, the literature on \((v, k, k-1)\)-DDFs is quite rich and our main purpose is to get this literature in order.

First of all it is worth mentioning that the \(G\)-orbit of any \((v, k, k-1)\)-DDF in \(G\) is the near resolution of a \((v, k, k-1)\) near resolvable design (NRB for short). We refer to [1] for general background on NRBs.

In the core of this paper we restate and reprove in terms of differences an old nearring theory result - probably passed almost unnoticed by other theorists - which, starting from Ferrero pairs, implicitly give a wide class of \((v, k, k-1)\)-DDFs in the kernel of a Frobenius group.

We will show that several constructions for \((v, k, k-1)\)-DDFs obtained over the years could be quickly obtained as a corollary of that result. In order to further appreciate its effectiveness we apply it to get some new DDFs, in particular the Pisano \((p^4, k, k-1)\)-DDFs in \(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}\) which arise from the Fibonacci sequence. In the last section we give an example of an infinite class of non-abelian DDFs obtainable via the Ferrero construction and, more importantly, we will prove that if all prime factors of \(v\) are congruent to 1 modulo \(k\), then there exists a \((v, k, k-1)\)-DDF in any group of order \(v\).

2 Some known results

A \((v, k, k-1)\)-DDF in \(G\) is known in each of the following cases.

(i) \(v\) is odd, \(k = 2\), and \(G\) is any group of order \(v\).

(ii) \(v \equiv 1 \pmod{4}\), \(k = 4\), \(G = \mathbb{Z}_v\) and there exists a \(\mathbb{Z}\)-cyclic whist tournament on \(v\) players (briefly \(Wh(v)\)).

(iii) \(v \equiv 1 \pmod{k}\) is a prime power and \(G\) is the additive group of \(\mathbb{F}_v\) (the field of order \(v\)).

(iv) The maximal prime power divisors of \(v\) are all congruent to 1 (mod \(k\)) and \(G\) is a direct product of elementary abelian groups.

(v) All prime factors of \(v\) are congruent to 1 (mod \(k\)) and \(G = \mathbb{Z}_v\).

A DDF as in (i) is nothing but a starter of \(G\) (see, e.g., [12]).

The reason of (ii) is that the initial round of a \(\mathbb{Z}\)-cyclic \(Wh(v)\) is a \((v, 4, 3)\)-DDF (but the converse is not generally true). For general background on \(\mathbb{Z}\)-cyclic whist tournaments we refer to [2].

For a DDF as in (iii) - which a special case of the DDFs in (iv) - one can simply take the set of all cosets of the \(k\)-th roots of unity in the multiplicative group of \(\mathbb{F}_v\). This DDF is usually attributed to Wilson [23] but we note that it was given earlier in an equivalent form by Ferrero [14]. We also note that this DDF can be presented as the \((v, k-1)\)-ZDBF mapping any \(x \in \mathbb{F}_v\) into \(x^{(v-1)/k} \in \mathbb{F}_v\).
A large class of DDFs as in (iii) and the additional condition that \( k \equiv 2 \) (mod 4) have been very recently given by Li [18]; each block of these DDFs is a suitable union of two cosets of the \( \frac{k}{2} \)-th roots of unity in \( \mathbb{F}_v \).

Results (iv) and (v) have been recently described using the language of ZDBFs and obtained with quite involved proofs in [9] and [11], respectively. We note, however, that a simple proof of (iv) was given by Furino in 1991 (see end of section 3 in [15]) and that the same proof was implicitly given by Boykett in 2001 (see the proof of Proposition 7 in [4]). There is another approach for getting (iv) very quickly from (ii) with the use of difference matrices; this has been very recently noted by Li, Wei and Ge [19] but traces of the same approach can be found in a very old paper by Jungrickel (see Corollary 4.5 in [17]).

We note that difference matrices would also allow to obtain (v) very quickly from the existence of a cyclic \((p^n, k, k - 1)\)-DDF for every prime \( p \equiv 1 \) (mod \( k \)). Such a difference family was obtained by Furino (see Lemma 4.3 in [14]) and it is also deducible from an even earlier result by Phelps [20] (see Theorem 4.6 in the paper [5] by the present author).

In the next section we will give a clean construction for \((v, k, k - 1)\)-DDFs in the kernel of a Frobenius group which can be deduced from an old nearring result. We will show how (iv) and (v) can be almost immediately obtained as special cases of this construction. In the last section we will prove that (v) can be generalized to any group \( G \) of order \( v \).

3 Ferrero \((v, k, k - 1)\) difference families

A Frobenius group \( F \) is a semidirect product \( G \rtimes A \) with \( A \) a non-trivial group of automorphisms of \( G \) acting semiregularly on \( G \setminus \{0\} \) (one also says that \( A \) is fixed point free). This means that for \( g \in G \) and \( \alpha \in A \) we have \( \alpha(g) = g \) if and only if either \( \alpha = id_G \) or \( g = 0 \). The groups \( G \) and \( A \) are said to be the kernel and the complement of \( F \), respectively. Any such pair \((G, A)\) is called a Ferrero pair by nearring theorists [4, 8]. Note that if \((G, A)\) is a Ferrero pair, then \((H, B)\) is a Ferrero pair as well for every non-trivial subgroup \( H \) of \( G \) and any non-trivial subgroup \( B \) of \( A \). For general background on Frobenius groups we refer to [16].

The following theorem is the highlight of this section but we point out that the first part of the theorem is not really new since it is essentially the same as Theorem 5.5 in [8]. The crucial diversity is that our presentation and proof are given in terms of differences. In the original statement it is said that any Ferrero pair \((G, A)\) with \( \text{ord}(G) = v \) and \( \text{ord}(A) = k \) generates a \( 2-(v, k, k - 1) \) design and then in the proof it is shown that the blocks of this design are all the translates of the \( A \)-orbits of the non-zero elements of \( G \) under the natural action of \( G \). For the main subject of the present paper, the crucial fact is that the set of \( A \)-orbits on \( G \setminus \{0\} \) is a \((v, k, k - 1)\)-DDF.
Theorem 1. If \((G,A)\) is a Ferrero pair with \(\text{ord}(G) = v\) and \(\text{ord}(A) = k\), then the set of \(A\)-orbits on \(G \setminus \{0\}\) is a \((v,k,k-1)\)-DDF in \(G\). In the hypothesis that \(G\) is abelian and that \(vk\) is odd, this DDF is splittable into two \((v,k,\frac{k-1}{2})\)-DDFs.

Proof. By definition, \(A\) acts semiregularly on \(G \setminus \{0\}\). This easily implies that each \(A\)-orbit on \(G \setminus \{0\}\) has size \(k\) and that the map \(g \in G \rightarrow \alpha(g) - g \in G\) is a bijection for every \(\alpha \in A \setminus \{id\}\). Thus we can write:

\[
\{\alpha(g) - g \mid g \in G \setminus \{0\}\} = G \setminus \{0\} \quad \forall \alpha \in A \setminus \{id\} \tag{1}
\]

Let \(X\) be a complete set of representatives for the \(A\)-orbits on \(G \setminus \{0\}\) and for each \(x \in X\) let \(Orb(x)\) be the \(A\)-orbit of \(x\). We have to prove that \(\mathcal{O} := \{Orb(x) \mid x \in X\}\) is a \((v,k,k-1)\)-DDF. It is evident that the ordered pairs of distinct elements of \(Orb(x)\) with a fixed second coordinate \(y\) are exactly those of the form \((\alpha(y),y)\) with \(\alpha \in A \setminus \{id\}\). Thus we have \(\Delta Orb(x) = \bigcup_{y \in Orb(x)} \{\alpha(y) - y \mid \alpha \in A \setminus \{id\}\}\), hence

\[
\Delta \mathcal{O} = \bigcup_{x \in X} \bigcup_{y \in Orb(x)} \{\alpha(y) - y \mid \alpha \in A \setminus \{id\}\} = \bigcup_{g \in G \setminus \{0\}} \{\alpha(g) - g \mid \alpha \in A \setminus \{id\}\}.
\]

So we can write \(\Delta \mathcal{O} = \bigcup_{\alpha \in A \setminus \{id\}} \{\alpha(g) - g \mid g \in G \setminus \{0\}\}\) which, by \((1)\), is the union of \(k-1\) copies of \(G \setminus \{0\}\). The first part of the assertion follows.

From now we assume that \(kv\) is odd and that \(G\) is abelian. Suppose that two opposite elements \(g\) and \(-g\) are in the same \(A\)-orbit. In this case there is an \(\alpha \in A\) such that \(\alpha(g) = -g\). Then, by induction, we would have \(\alpha^i(g) = g\) or \(-g\) according to whether \(i\) is even or odd, respectively. Thus, in particular, we would have \(\alpha^k(g) = -g\). On the other hand \(k\) is the order of \(A\) so that \(\alpha^k = id\). It follows that \(g = -g\). So, considering that \(G\) does not have involutions since \(v\) is odd, we necessarily have \(g = 0\). We conclude that the set \(X\) considered in the first part of our proof can be chosen of the form \(X = Y \cup -Y\) with \(Y\) a suitable \(\frac{k-1}{2}\)-subset of \(G\). In this way we have that \(\mathcal{O}\) is splittable into the two parts \(\mathcal{O}_1 = \{Orb(y) \mid y \in Y\}\) and \(\mathcal{O}_2 = \{Orb(-y) \mid y \in Y\}\). Now note that \(Orb(-y) = -Orb(y)\). Hence we obviously have \(\Delta Orb(-y) = \Delta Orb(y)\) for each \(y \in Y\) since \(G\) is abelian. We conclude that \(\Delta \mathcal{O}_1 = \Delta \mathcal{O}_2\) and then, recalling that \(\mathcal{O}\) is a \((v,k,k-1)\)-DDF, we deduce that both \(\mathcal{O}_1\) and \(\mathcal{O}_2\) are \((v,k,\frac{k-1}{2})\)-DDFs. \qed

The \((v,k,k-1)\)-DDFs produced by the above theorem will be said Ferrero difference families.

Note that the patterned starter of a group \(G\) of odd order (namely the set of all possible pairs \(\{g,-g\}\) of opposite elements of \(G \setminus \{0\}\)) can be seen as the Ferrero DF determined by the Ferrero pair \((G,\{id,-id\})\) when \(G\) is abelian.
As a first immediate consequence of Theorem 1 we have the following result which was also stated in a weaker form by Furino ([15], Lemma 4.2).

**Lemma 1.** Let $R$ be a ring of order $v$ with unity, and let $U(R)$ be the group of units of $R$. If $U$ is a subgroup of order $k$ of $U(R)$ with $u - 1 \in U(R)$ for each $u \in U \setminus \{1\}$, then there exists a Ferrero $(v, k, k - 1)$-DDF in the additive group of $R$.

**Proof.** Any subgroup $U$ of $U(R)$ can be seen as an automorphism group of the additive group $G$ of $R$. Indeed any $u \in U(R)$ can be identified with the automorphism of $G$ mapping $x$ into $ux$. It is also clear that $u - 1 \in U(R)$ for each $u \in U \setminus \{1\}$ implies that $U$ acts semiregularly on $G \setminus \{0\}$. The assertion then follows from Theorem 1. □

Now we show how the previous lemma allows to obtain result (iv) very quickly. We essentially give the same old easy proof given by Furino [15], not comparable to the recent tortuous proof in [11].

**Corollary 1.** Let $v$ be a product of prime powers $q_1, \ldots, q_n$ all congruent to $1 \pmod{k}$. Then there exists a Ferrero $(v, k, k - 1)$-DDF in the additive group of $\mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_n}$.

**Proof.** For $1 \leq i \leq n$, take a $k$-th primitive root $u_i$ of $\mathbb{F}_{q_i}$. It is immediate that $u := (u_1, \ldots, u_n)$ is a unit of order $k$ of the ring $R = \mathbb{F}_{q_1} \times \cdots \times \mathbb{F}_{q_n}$ and that $u^i - 1$ is a unit of $R$ for $1 \leq i \leq k - 1$. The assertion then follows from Lemma 1. □

Let us say that a Ferrero pair $(G, A)$ has parameters $(v, k)$ if $v$ and $k$ are the orders of $G$ and $A$, respectively. A trivial necessary condition for $(v, k)$ to be the parameters of a suitable Ferrero pair is that $k$ divides $v - 1$. From the above corollary one deduces that a sufficient condition is that $q \equiv 1 \pmod{k}$ for every maximal prime power factor $q$ of $v$. This condition has been proved to be also necessary by Boykett ([4], Corollary 6) as a consequence of other results using, in particular, Thompson’s theorem on Frobenius groups. We reprove this below in a simpler and more direct way.

**Proposition 1.** There exists a suitable Ferrero pair of parameters $(v, k)$ - or equivalently a Ferrero $(v, k, k - 1)$-DDF in a suitable group - if and only if $q \equiv 1 \pmod{k}$ for every maximal prime power factor $q$ of $v$.

**Proof.** ($\implies$) Let $(G, A)$ be a Ferrero pair of parameters $(v, k)$ and let $p^e$, $p$ prime, be a maximal prime power factor of $v$. The group $A$ acts as a permutation group on the set $Syl_p(G)$ of all Sylow $p$-subgroups of $G$. If $A$ does not fix any member of $Syl_p(G)$, then each $A$-orbit on $Syl_p(G)$ would have size $|A| = k$ so that $k$ divides $|Syl_p(G)|$. In its turn $|Syl_p(G)|$ divides $\frac{v}{p^e}$ by the third Sylow theorem. We conclude that $k$ divides both $v - 1$ and $\frac{v}{p^e}$, hence $k = 1$ which is absurd. So $A$ fixes a suitable $S \in Syl_p(G)$. It follows
that \((S, A)\) is a Ferrero pair so that \(ord(A) = k\) divides \(ord(S) - 1 = p^r - 1\) which is the assertion.

\((\iff)\) See Corollary 1

Now we show how result \((v)\) can be generalized to any abelian group and that this can be also quickly obtainable as a corollary of Lemma 1.

**Corollary 2.** If all prime factors of \(v\) are congruent to 1 \((\mod k)\), then there exists a Ferrero \((v, k, k - 1)\)-DDF in any abelian group \(G\) of order \(v\).

**Proof.** First recall that if \(q = p^e\) with \(p\) prime, then \(U(\mathbb{Z}_q)\) is cyclic of order \(\phi(q) = p^{e-1}(p - 1)\) (see, e.g., [3]). Also, the subgroup \(S_q\) of \(U(\mathbb{Z}_q)\) of order \(p^{e-1}\) consists of all elements \(s \in \mathbb{Z}_q\) with \(s \equiv 1 \pmod{p}\).

Let \(G\) be an abelian group of order \(v\). By the Fundamental Theorem of Finite Abelian Groups, there are suitable prime powers \(q_1, \ldots, q_n\) dividing \(v\) such that, up to isomorphism, \(G\) is the additive group of the ring \(R = \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_n}\). Of course we have \(U(R) = U(\mathbb{Z}_{q_1}) \times \cdots \times U(\mathbb{Z}_{q_n})\).

For \(1 \leq i \leq n\), set \(q_i = p_i^{e_i}\) with \(p_i\) prime. By assumption, \(k\) divides \(p_i - 1\), hence \(U(\mathbb{Z}_{q_i})\) has an element \(u_i\) of order \(k\). Obviously \(\langle u_i \rangle\) has trivial intersection with \(S_{q_i}\), the subgroup of \(U(\mathbb{Z}_{q_i})\) of order \(p_i^{e_i-1}\). Hence, for \(1 \leq i \leq n\) and \(1 \leq j \leq k - 1\), we have \(u_i^j \not\equiv 1 \pmod{p_i}\).

We conclude that \(u := (u_1, \ldots, u_n)\) is a unit of \(R\) of order \(k\) and that \(u^j - 1\) is also a unit of \(R\) for \(1 \leq j \leq k - 1\). The assertion then follows from Lemma 1.

There are infinite classes of Ferrero DDFs which neither Corollary 1 nor Corollary 2 are able to capture. One of these classes will be given in the next section.

Here we only give an easy example of a Ferrero \((q^4, 3, 2)\)-DDF in \(\mathbb{Z}_{q^2} \times \mathbb{Z}_{q^2}\) for any prime power \(q\) not divisible by 3. Such a DDF cannot be obtained from Corollary 2 when \(q\) is a power of 2 or the power of a prime \(p \equiv 5 \pmod{6}\). Consider the automorphism \(\alpha\) of \(\mathbb{Z}_{q^2} \times \mathbb{Z}_{q^2}\) defined by \(\alpha(x, y) = (y - x, -x)\). One can see that the group \(A\) generated by \(\alpha\) has order 3 and acts semiregularly on \(G \setminus \{(0, 0)\}\). So we obtain the required \((q^4, 3, 2)\)-DDF by applying Theorem 1.

For instance, for \(q = 2\), we get the following Ferrero \((16, 3, 2)\)-DDF in \(\mathbb{Z}_4 \times \mathbb{Z}_4\) (in order to save space, any pair \((x, y)\) is denoted by \(xy\)):

\[
\{01, 10, 33\}, \ \{02, 20, 22\}, \ \{03, 30, 11\}, \ \{12, 13, 23\}, \ \{21, 32, 31\}.
\]

4 **Ferrero difference families from the Fibonacci sequence**

Here we present an infinite class of Ferrero PDFs obtainable via the Fibonacci sequence which, in many cases, cannot be deduced from Corollary 1 or Corollary 2. For this, we need to recall some number theoretic arguments.
The *Pisano period modulo* \( n \), denoted \( \pi(n) \), is the period of the Fibonacci sequence modulo \( n \) which is also equal to the period of the Fibonacci matrix
\[
\mathbf{F} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\]
in the group \( GL_2(n) \) of all \( 2 \times 2 \) invertible matrices of the ring \( \mathbb{Z}_n \). There is no known formula for \( \pi(p) \) with \( p \) a prime but the following properties have been established so far (see, e.g., [21]):

\( P_1 \) \( \pi(p^2) \) is equal either to \( p\pi(p) \) or \( \pi(p) \);

\( P_2 \) \( \pi(p) \) is equal to the least common multiple of the periods of the two eigenvalues of \( \mathbf{F} \) in the multiplicative group of \( \mathbb{F}_{p^2} \):

\[
(P_3) \quad \pi(p) = \begin{cases} 
3 & \text{if } p = 2 \\
20 & \text{if } p = 5 \\
an \text{ even divisor of } p - 1 & \text{if } p \equiv \pm 1 \pmod{10} \\
\frac{2(p+1)}{d} & \text{with } d \text{ an odd divisor of } p + 1 \text{ if } p \equiv \pm 3 \pmod{10}
\end{cases}
\]

Regarding property \( P_1 \), it should be noted that there is no known prime \( p \) for which \( \pi(p^2) = \pi(p) \) holds.

**Proposition 2.** There exists a Ferrero \((p^1, k, k - 1)\)-DDF in \( \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \) for any prime \( p \neq 5 \) and any divisor \( k \) of the Pisano period \( \pi(p) \).

**Proof.** An example of a \((16, 3, 2)\)-DDF in \( \mathbb{Z}_4 \times \mathbb{Z}_4 \) has been given at the end of the previous section, therefore the assertion is true for \( p = 2 \). For \( p \equiv \pm 1 \pmod{10} \) the assertion is an immediate consequence of Corollary [2] and property \( P_3 \). So, in the following, we will assume that \( p \equiv \pm 3 \pmod{10} \). This implies that 5 is not a square in \( \mathbb{F}_p \), hence the two eigenvalues \( \lambda_1, \lambda_2 \) of \( \mathbf{F} \) are “conjugates” in \( \mathbb{F}_{p^2} \). Indeed we have \( \{\lambda_1, \lambda_2\} = \{\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2}\} \). So the \( i \)-th powers of \( \lambda_1 \) and \( \lambda_2 \) are conjugates as well. It follows that \( \lambda_1^i = 1 \) if and only if \( \lambda_2^i = 1 \) which clearly implies that \( \lambda_1 \) and \( \lambda_2 \) have the same period in the multiplicative group of \( \mathbb{F}_{p^2} \).

Let us identify any matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_{p^2}) \) with the automorphism of \( \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2} \) mapping \((x, y)\) into \((ax + by, cx + dy)\).

The period of the Fibonacci matrix \( \mathbf{F} \) in \( GL_2(\mathbb{Z}_{p^2}) \) is equal to \( \pi(p^2) \) which, by property \( P_1 \), is equal either to \( p\pi(p) \) or \( \pi(p) \). Let \( A \) be the subgroup of order \( \pi(p) \) of the group generated by \( \mathbf{F} \). Thus \( A = \langle \Phi \rangle \) with \( \Phi = \mathbf{F} \) or \( \Phi = \mathbf{F}^p \) according to whether \( \pi(p^2) = \pi(p) \) or \( \pi(p^2) = p\pi(p) \), respectively.

Assume that 1 is an eigenvalue of \( \Phi \). Then, considering that for any matrix \( M \) and any positive integer \( j \) we have \( Spec(M^j) = \{\lambda^j \mid \lambda \in Spec(M)\} \), we have

\[
\begin{align*}
\lambda_1^j = \lambda_2^j &= 1 & \text{if } \pi(p^2) = \pi(p) \\
\lambda_1^{pi} = \lambda_2^{pi} &= 1 & \text{if } \pi(p^2) = p\pi(p)
\end{align*}
\]
Thus, by property \((P_3)\), we have that \(\pi(p)\) divides \(i\) in the former case while \(\pi(p)\) divides \(pi\) in the latter. Anyway \(\pi(p)\) and \(p\) are coprimes by property \((P_3)\) so that, in both cases, \(i\) should be divisible by \(\pi(p)\). Recalling that \(\pi(p)\) is the order of \(\Phi\), we conclude that \(\Phi^i\) is the identity matrix.

Now note that a matrix \(M \in GL_2(\mathbb{Z}/p^2\mathbb{Z})\) is fixed point free if and only if 1 does not belong to \(\text{Spec}(M)\). So we have proved that the group \(A\) acts semiregularly on \(\mathbb{Z}/p^2\mathbb{Z}\) \(\{0,0\}\), i.e., \((\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p^2, A)\) is a Ferrero pair. Of course \((\mathbb{Z}/p^2 \times \mathbb{Z}/p^2, \langle \Phi^{\pi(p)/k} \rangle)\) is a Ferrero pair as well for each divisor \(k\) of \(\pi(p)\) and then the assertion follows from Theorem [1] □

The Pisano DDFs, namely the \((p^4, k, k - 1)\)-DDFs which are built as in the proof of the above proposition, allow to largely enrich the set of known values of \(k\) for which there exists a \((p^4, k, k - 1)\)-DDF in \(\mathbb{Z}/p^2 \times \mathbb{Z}/p^2\) in the case of \(p \equiv \pm 3 \pmod{10}\). In particular, for a Mersenne prime \(p = 2^{4n+3} - 1\) we have \(p \equiv -3 \pmod{10}\) and then we necessarily have \(\pi(p) = 2^{4n+4}\) by property \((P_4)\). Thus there exists a Pisano \((p^4, 2^i, 2^i - 1)\)-DDF in \(\mathbb{Z}/p^2 \times \mathbb{Z}/p^2\) for \(1 \leq i \leq 4n + 4\).

As an example, let us apply Proposition [2] with \(p = 3\). We have \(\pi(3) = 6\) and \(\pi(3^2) = 3 \cdot \pi(3) = 18\). Then the matrix \(\Phi\) considered in the proof of the above proposition is \(F^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}\) and the group \(A\) generated by \(\Phi\) is the following:

\[
\left\{ \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 8 \\ 8 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 7 \\ 7 & 3 \end{pmatrix}, \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}, \begin{pmatrix} 6 & 7 \\ 7 & 8 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}
\]

The ten orbits of \(A\) on \(\mathbb{Z}_9 \times \mathbb{Z}_9 \setminus \{(0,0)\}\) are listed below where, again, any pair \((x, y)\) will be simply denoted by \(xy\):

\[
B_0 = \{21, 85, 73, 08, 78, 14, 26, 01\}; \quad B_1 = \{42, 71, 56, 07, 57, 28, 43, 02\};
\]

\[
B_2 = \{63, 66, 30, 06, 36, 33, 60, 03\}; \quad B_3 = \{64, 52, 13, 05, 15, 47, 86, 04\};
\]

\[
B_4 = \{32, 48, 17, 80, 67, 51, 82, 10\}; \quad B_5 = \{53, 34, 81, 88, 46, 65, 18, 11\};
\]

\[
B_6 = \{74, 20, 64, 87, 25, 70, 35, 12\}; \quad B_7 = \{68, 72, 77, 83, 31, 27, 22, 16\};
\]

\[
B_8 = \{37, 54, 55, 76, 62, 45, 44, 23\}; \quad B_9 = \{58, 40, 38, 75, 41, 50, 61, 24\}.
\]

Thus the \(B_i\)'s are the blocks of a Pisano \((81, 8, 7)\)-DDF in \(\mathbb{Z}_9 \times \mathbb{Z}_9\).

4.1 Non-abelian disjoint \((v, k, k - 1)\) difference families

Here we give some constructions for DDFs in non-abelian groups. Let us start by giving a class of non-abelian Ferrero DDFs.

**Proposition 3.** If \(R = (V, +, \cdot)\) is a ring with unity admitting a group \(U\) of units such that \(u^2 - 1 \in U(R)\) for each \(u \in U \setminus \{1\}\), then there exists a non-abelian Ferrero \((v^3, k, k - 1)\)-DDF with \(v = |V|\) and \(k = |U|\).
Proof. Let us equip the set $V^3$ with the operation $\oplus$ defined by the rule

$$(x_1, y_1, z_1) \oplus (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 \cdot y_2).$$

It is an easy exercise to check that $G = (V^3, \oplus)$ is a group. Also note that $G$ is non-abelian since we have, for instance, $(0, 1, 0) \oplus (1, 0, 0) = (1, 1, 0)$ while $(1, 0, 0) \oplus (0, 1, 0) = (1, 1, 1)$. Now note that for each $u \in U$, the map $\alpha_u : (x, y, z) \in V^3 \rightarrow (u \cdot x, u \cdot y, u^2 \cdot z) \in V^3$ is an automorphism of $G$ and that $(G, A := \{\alpha_u \mid u \in U\})$ is a Ferrero pair. The assertion then follows from Theorem 1.

We remark that a group $U$ as in the statement of the above proposition is necessarily of odd order. Indeed, in the opposite case, $U$ would have at least one involution, say $u$, and then $u^2 - 1 = 0 \notin U(R)$ against the assumption.

Applying Proposition 3 with $R = \mathbb{F}_q$ we obtain the following.

**Corollary 3.** There exists a non-abelian Ferrero $(q^3, k, k - 1)$-DDF for any pair $(q, k)$ with $q$ a prime power and $k$ any odd divisor of $q - 1$.

Let $F$ be the $(v^3, k, k - 1)$-DDF in $(V^3, \oplus)$ obtainable by Proposition 3. We remark that $F$ actually coincides with the $(v^3, k, k - 1)$-DDF in the abelian group of the ring $R \times R \times R$ obtainable using Lemma 1. On the other hand to consider $F$ as a DDF in $(V^3, +)$ is not the same as to consider $F$ as a DDF in $(V^3, \oplus)$. Indeed the $(v^3, k, k - 1)$-NRB whose near resolution is the orbit of $F$ under $(V^3, \oplus)$ does not coincide with the $(v^3, k, k - 1)$-NRB whose near resolution is the orbit of $F$ under $(V^3, +)$.

By Corollary 2 there exists a $(v, k, k - 1)$-DDF in any abelian group $G$ of order $v$ provided that all primes in $v$ are congruent to 1 (mod $k$). We are going to see that this result remains true if one removes the hypothesis of commutativity of $G$. The present author proved that the existence of a $(p, k, \lambda)$-DF for any prime $p$ in $v$ implies the existence of a $(v, k, \lambda)$-DF in any group $G$ of order $v$ (see Corollary 5.5 in [5]). Now we reprove this theorem showing that if all component $(p, k, \lambda)$-DFs are disjoint, then the resultant $(v, k, \lambda)$-DF in $G$ is disjoint as well.

**Theorem 2.** If $G$ is a group of order $v$ and there exists a $(p, k, \lambda)$-DF (resp. DDF) for every prime factor $p$ of $v$, then there exists a $(v, k, \lambda)$-DF (resp. DDF) in $G$.

**Proof.** The cases $k = 1$ and $k = 2$ are trivial. So, in the following, we assume $k > 2$. We prove the theorem by induction on $v$. The assertion is trivially true for $v = 1$; in this case the required DF is the empty family. Let $G$ be a group of order $v > 1$ as in the statement and assume that the assertion is true for all groups of order less than $v$. First observe that $v$ is necessarily odd since the existence of a $(p, k, \lambda)$-DF for any prime factor $p$ of $v$ implies that $p \geq k > 2$ for any such prime $p$. It follows, by the Feit-Thompson theorem,
that \( v \) is solvable. Thus, in particular, \( G \) has a normal subgroup \( N \) of prime index, say \( p \). By hypothesis there exists a \((p, k, \lambda)\)-DF (resp. DDF), say \( \mathcal{F}_1 \), in \( G/N \). By induction, there also exists a \((\frac{v}{p}, k, \lambda)\)-DF (resp. DDF), say \( \mathcal{F}_2 \), in \( N \). For each block \( B = \{g_1 + N, \ldots, g_k + N\} \in \mathcal{F}_1 \) and any \( n \in N \) consider the \( k \)-subset \( B(n) \) of \( G \) defined by \( B(n) = \{g_i + in \mid 1 \leq i \leq k\} \). We claim that

\[
\mathcal{F} := \{B(n) \mid B \in \mathcal{F}_1, n \in N\} \cup \mathcal{F}_2
\]

is a \((v, k, \lambda)\)-DF (resp. DDF) in \( G \).

Given \( g \in G \setminus N \), let \( g + N = (g_i + N) - (g_j + N) \) be a representation of \( g + N \) as a difference from a block \( B = \{g_1 + N, \ldots, g_k + N\} \) of \( \mathcal{F}_1 \). Consider the element \( n := -g_i + g + g_j \), necessarily belonging to \( N \), and let \( \text{ord}(n) \) be its order. We have \( 1 \leq |i - j| < k \), hence \( i - j \) is coprime with \( \text{ord}(n) \) since, by assumption, every divisor of \( \text{ord}(G) \) distinct by 1 is clearly greater than \( k \). Thus there exists the inverse, say \( x \), of \( i - j \) modulo \( \text{ord}(n) \). Now check that \( g \) is the difference between the \( i \)-th element and the \( j \)-th element of the block \( B(xn) \) of \( \mathcal{F}_1 \):

\[
(g_i + ixn) - (g_j + jxn) = g_i + (i - j)xn - g_j = g_i - g_j + g + j - g_j = g.
\]

In this way we have proved that each of the \( \lambda \) representations of \( g + N \) as a difference from \( \mathcal{F}_1 \) leads to a representation of \( g \) as a difference from \( \mathcal{F} \).

Thus every element of \( G \setminus N \) is covered at least \( \lambda \) times by \( \Delta \mathcal{F} \). The same is true for all elements of \( N \setminus \{0\} \) since \( \Delta \mathcal{F}_2 \) is \( \lambda \) times \( N \setminus \{0\} \). Now note that the number of blocks of \( \mathcal{F} \) is given by

\[
|\mathcal{F}| = |\mathcal{F}_1| \cdot |N| + |\mathcal{F}_2| = \frac{\lambda(v-1)}{k(k-1)} \cdot \frac{v}{p} + \frac{\lambda}{k(k-1)}(\frac{v}{p} - 1) = \frac{\lambda(v-1)}{k(k-1)}
\]

and then \( |\Delta \mathcal{F}| = k(k - 1)|\mathcal{F}| = \lambda(v - 1) \). It follows, by the pigeon hole principle, that every non-zero element of \( G \) is covered by \( \Delta \mathcal{F} \) exactly \( \lambda \) times, i.e., \( \mathcal{F} \) is a \((v, k, \lambda)\)-DF.

It remains to prove that \( \mathcal{F} \) is disjoint in the hypothesis that both \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are disjoint.

Every block of the form \( B(n) \) with \( B = \{g_1 + N, \ldots, g_k + N\} \in \mathcal{F}_1 \) and \( n \in N \) has no element in \( N \) otherwise we would have \( g_i + in \in N \) for some \( i \), hence \( g_i + N = N \) contradicting the fact that the blocks of \( \mathcal{F}_1 \) partition \( G/N \setminus \{N\} \). Thus \( B(n) \) is disjoint with every block of \( \mathcal{F}_2 \).

Now assume that \( B(n) \) and \( B'(n') \) have an element in common for some blocks \( B = \{g_1 + N, \ldots, g_k + N\} \) and \( B' = \{g'_1 + N, \ldots, g'_k + N\} \) of \( \mathcal{F}_1 \) and some elements \( n, n' \) of \( N \). Thus we have \( g_i + in = g'_j + jn' \) for suitable \( i, j \in \{1, \ldots, k\} \). This implies that \( g_i + N = g'_j + N \), hence \( B = B' \) and \( i = j \) since \( \mathcal{F}_1 \) is disjoint. It follows that \( i(n - n') = 0 \), hence \( \text{ord}(n - n') \) is a divisor of \( i \) which implies \( \text{ord}(n - n') = 1 \) since every divisor of \( \text{ord}(G) \) distinct by 1 is greater than \( k \). We conclude that \( B = B' \) and \( n = n' \), i.e., \( B(n) = B'(n') \).
Finally any two distinct blocks of $F_2$ are disjoint by assumption. The assertion follows.

We are now finally able to prove the main result of this section.

**Corollary 4.** If all prime divisors of $v$ are congruent to $1 \pmod{k}$, then there exists a $(v,k,k-1)$-DDF and a $(v,k,k-1)^2$-DDF in $G$ for any group $G$ of order $v$.

**Proof.** We know that there exists a $(p,k,k-1)$-DDF and a $(p,k,k-1)^2$ for any prime $p \equiv 1 \pmod{k}$. Then the assertion immediately follows from Theorem 2.

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