1 Introduction

In this paper, we focus on stratified pooling games as introduced by Schlicher et al. [9]. In these games, there are several players that are located geographically close together. They each keep spare parts in stock to protect for downtime of their high-tech machines and face (possibly) different downtime costs per stockout. The players can cooperate by forming a joint spare parts pool. A stratified pooling policy, which is optimal in terms of minimizing the long-run average costs, is then applied. It determines, depending on the real-time on-hand inventory, which players may take parts from the joint spare parts pool. This policy makes it hard to analyse the corresponding games. However, Schlicher et al. [9] were able to prove core non-emptiness of stratified pooling games, by formulating the underlying situation (of the game) in terms of a Markov decision problem (MDP) and the optimal pooling policy as a stationary policy in this MDP.

We consider a natural extension of stratified pooling games by allowing the players to optimize on the stock level of the joint spare parts pool as well (rather than assuming that this stock level is -simply- the sum of the stock levels of the individual players as is the case in the original game). We explicitely introduce the costs for holding spare
parts (which is excluded in the original game as well) such that players face a natural trade off between these holding costs and the downtime costs (to find the optimal stock level). For these games, which we call optimized pooling games, we are able to show core non-emptiness as well. We do so by using a proving technique that has similarities with the one in Schlicher et al. [9]. Besides the direct consequence of core non-emptiness, namely that there is an incentive for all players to fully cooperate, this result is of particular interest as extensions of spare parts pooling games, like (not) optimizing on stock levels, can (typically) break the core non-emptiness result (see e.g., Karsten et al. [2], who show that the core can be empty whenever spare parts levels are not optimized, and Schlicher et al. [8], who show that applying another type of pooling can make the difference between an empty and a non-empty core).

In the remainder of this paper, we first describe our new spare parts situation and the associated optimized pooling game. Then, we present our proof, showing that the core of any optimized pooling game is non-empty. All proofs of the lemmas are relegated to the appendix.

2 Model description

In this section, we define the spare parts situation and the associated optimized pooling game. Besides, we describe how the coalitional values of our games relate to the value function of a corresponding Markov decision process.

2.1 Spare parts situations

We consider an environment with a finite set \( N \subseteq \mathbb{N} \) of service providers that are located geographically close together and each keeps spare parts in stock to prevent costly downtime of their high-tech machines. We limit ourselves to one critical component, i.e., one stock-keeping unit, which is subject to failures. For each service provider \( i \in N \), it holds that a failure of a high-tech machine immediately leads to a demand for a spare part. This occurs according to a Poisson process with rate \( \lambda_i \in \mathbb{R}_+ \). We assume that each service provider \( i \in N \) keeps an initial amount of spare parts in stock, which is (possibly) restricted by the service provider’s maximal storage capacity of \( C_i \in \mathbb{N} \cup \{0\} \) spare parts. The cost for holding a single spare part is \( h \in \mathbb{R} \) per time unit per service provider. If a spare part is on hand when demand occurs, this demand is always satisfied and the failed part is sent to the repair shop of service
provider $i$, which repairs such parts one-by-one. Repair times of these parts are assumed to be independent and identically distributed according to an exponential distribution with mean $\mu_i^{-1} \in \mathbb{R}_+$. If no spare part is available when demand occurs, an emergency procedure is instigated, which means that a spare part is leased (from an external supplier with infinite supply) for the duration of the repair time of the failed component, which is sent to the repair shop (of service provider $i$). The expected costs associated with the extra idleness of the machine (due to the delivery time of a leased spare part), shipment of an emergency spare part, and so on, called downtime costs, are $d_i \in \mathbb{R}_+$ for service provider $i$. We assume that each service provider $i \in N$ wants to minimize its long-run average costs per time unit. This implies that each service provider faces a natural trade off between the holding costs and the downtime costs to determine the optimal amount of spare parts to stock (i.e., stock level), while taking into account the maximal storage capacity restriction.

To analyse this setting, we define a *spare parts situation* as a tuple $(N, C, d, \lambda, h, \mu)$ with $N, C = (C_i)_{i \in N}, d = (d_i)_{i \in N}, \lambda = (\lambda_i)_{i \in N}, h, \mu = (\mu_i)_{i \in N}$ as defined above. For short, we use $\theta$ to refer to a spare parts situation and $\Theta$ to refer to the set of spare parts situations.

### 2.2 Optimized pooling games

Consider spare parts situation $\theta = (N, C, d, \lambda, h, \mu)$ and coalition $S \subseteq N$ with $S \neq \emptyset$. The players in coalition $S$ can collaborate by pooling their storage capacities, demand streams and repair rates into a joint system with a, yet to be determined, base stock level, a (heterogeneous) demand rate $\lambda_S = \sum_{i \in S} \lambda_i$, and a single repair shop, in which components are repaired one-by-one with as repair rate $\mu_S = \sum_{i \in S} \mu_i$. In this joint system, the players face the problem of which (base) stock level to choose and which demand to accept or reject, such that the long-run average costs are minimized. Here, we assume that each failed component is sent to the repair shop immediately and that whenever a demand is rejected (and so the spare part is not satisfied from the joint spare parts pool), an emergency procedure is instigated (as discussed before), with downtime costs that depend on whose demand is rejected. We denote the minimal long-run average costs per time unit for coalition $S \subseteq N$ by $c^\theta(S)$ and set $c^\theta(\emptyset) = 0$. The associated game $(N, c^\theta)$ will be called an *optimized pooling game*. 
2.3 MDP formulation

In line with Schlicher et al. [9] the decision problem per coalition can be considered as a (discrete time) Markov decision process (MDP) as well. This is allowed since the decision problem per coalition can be recognized as a semi-Markov decision problem, which can be converted to an equivalent MDP by applying uniformization (see, e.g., Lippman [3]). For that, we add fictitious transitions of a state to itself to ensure that the total rate out of a state is equal for all states, the so-called uniformization rate. Then, we consider the embedded discrete-time MDP by looking at the system only at transition instants, which occur according to a Poisson process, with as rate the uniformization rate. This modelling technique turns out to be very useful. Let $\theta \in \Theta$ and $S \subseteq N$ with $S \neq \emptyset$. In what follows, we present this corresponding MDP.

2.3.1 State and action spaces

We define the state space to be $Y^S = \{0, 1, \ldots, C_S\}$ with $i \in Y^S$ representing the number of spare parts in stock of coalition $S$ and the action space to be $A^S(y) = \{a^S_{i-}, a^S_{i+}\}_{i \in S}$ with $a^S_{i-}(y) = \{0, 1\}$ for all $i \in S$ and all $y > 0$, $a^S_{i-}(0) = \{0\}$ for all $i \in S$, $a^S_{i+}(y) = \{0, 1\}$ for all $i \in S$ and all $y < C_S$ and $a^S_{i+}(C_S) = \{0\}$ for all $i \in S$. For sub action space $a^S_{i-}$, action 1 corresponds with the acceptance of a demand at a player, while action 0 corresponds with the rejection of such a demand. For sub action space $a^S_{i+}$, action 1 corresponds with the repair of a spare part, while action 0 corresponds with the decision to not repair a(nother) spare part. Note, in comparison with the original spare parts situation as introduced in Schlicher et al. [9], the players now have the possibility to decide on the amount of (repairable) spare parts to stock, which consequently results into an addition action space ($A^S_{i+}$).

2.3.2 Costs and transition probabilities

Let $\gamma = \sum_{i \in N}[\lambda_i + \mu_i]$. We will use $\gamma$ as the uniformization rate, which is independent of $S$. In addition, let $\lambda_i^* = \lambda_i / \gamma$, $\mu_i^* = \mu_i / \gamma$ for all $i \in N$ and $h^* = h / \gamma$. Now, $C^S(y,a)$ denotes the expected costs collected over a single (uniformized) time epoch, given that the system begins the period in state $y \in Y^S$ and action $a = \{a^{-}_i, a^{+}_i\}_{i \in S} \in a^S(y)$ is taken. For our situation, we have

$$C^S(y,a) = \sum_{i \in S} \lambda_i^* \cdot (1 - a^{-}_i) \cdot d_i + h^* \cdot y \quad \text{for all } y \in Y^S \text{ and all } a \in a^S(y).$$

In addition, let $p^S(y'|y,a)$ denote the one-stage transition probability from state $y \in Y^S$.
to \( y' \in \mathcal{Y}^S \) under action \( a = \{ a_i^-, a_i^+ \}_{i \in S} \in \mathcal{A}^S(y) \). We have

\[
p^S(y'|y,a) = \begin{cases} 
\sum_{i \in S} \lambda_i^* \cdot a_i^- & \text{if } y' = y - 1, y > 0 \\
\sum_{i \in S} \mu_i^* \cdot a_i^+ & \text{if } y' = y + 1, y < C_S \\
1 - \sum_{i \in S} [\lambda_i^* \cdot a_i^- + \mu_i^* \cdot a_i^+] & \text{if } y' = y \\
0 & \text{otherwise,}
\end{cases}
\]

for all \( y \in \mathcal{Y}^S \) and all \( a \in \mathcal{A}^S(y) \).

### 2.3.3 Value function and equivalence

Now, we present the value function in a form suitable for this paper. This form can be obtained by some rewriting of the value function in standard form (see, e.g., Schlicher et al. [9, p.9] for the value function in standard form). The proof, showing how the value function in standard form can be rewritten into a form suitable for this paper, is straightforward and for this reason omitted (rather than relegated to the appendix).

**Lemma 1.** Let \( \theta \in \Theta \) and \( S \subseteq N \). For all \( y \in \mathcal{Y}^S \) and all \( t \in \mathbb{N} \cup \{0\} \), we have

\[
V_{t+1}^S(y) = \sum_{i \in S} \left[ \lambda_i^* \min_{l \in \{0, \min\{y,1\}\}} \left\{ V_t^S(y-l) + (1-l)d_i \right\} + \mu_i^* \min_{l \in \{0, \min\{1,C_S-y\}\}} V_t^S(y+l) \right] \\
+ h^* \cdot y + \left( 1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*] \right) \cdot V_t^S(y)
\]

and \( V_0^S(y) = 0 \) for all \( y \in \mathcal{Y}^S \).

Finally, we define \( g^S \) as the minimal long-run average costs per time epoch of the MDP. Similarly to Schlicher et al. [9], we show that there is a direct relation between \( g^S \) and the original minimal long-run average costs per time unit of coalition \( S \). Recall that proofs of lemmas are relegated to the appendix.

**Lemma 2.** Let \( \theta \in \Theta \) and \( S \subseteq N \) with \( S \neq \emptyset \). Then

\[
c^\theta(S) = \gamma \cdot g^S = \gamma \cdot \lim_{t \to \infty} \frac{V_t^S(y)}{t} \quad \text{for all } y \in \mathcal{Y}^S.
\]

### 3 Core non-emptiness of optimized pooling games

In this section, we will show that optimized pooling games have a non-empty core. In doing so, we need to introduce some definitions and well-known results. We closely follow the formulation as presented in Schlicher et al. [9].
A map $\kappa : 2^N \setminus \{\emptyset\} \to [0, 1]$ is called a balanced map for $N$ if
$$\sum_{S \in 2^N : i \in S} \kappa_S = 1 \quad \text{for all } i \in N.$$ A collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called balanced if there exists a balanced map $\kappa$ for which $\kappa_S > 0$ for all $S \in \mathcal{B}$ and $\kappa_S = 0$ otherwise. Moreover, a collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called minimal balanced if there exists no proper subcollection of $\mathcal{B}$ that is balanced as well. An advantage of minimal balanced collections is that for every minimal balanced collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ there exists exactly one associated balanced map $\kappa$ (Peleg and Sudhölter [6]). A game $(N, c)$ is called balanced if for every minimal balanced collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ with associated balanced map $\kappa$ it holds that
$$\sum_{S \in \mathcal{B}} \kappa_S \cdot c(S) \geq c(N).$$

Now, we are able to present a sufficient and necessary condition for core non-emptiness due to Bondareva [1] and Shapley [11].

**Theorem 1.** A game $(N, c)$ is balanced if and only if $\mathcal{C}(N, c) \neq \emptyset$.

Let $\theta \in \Theta$ and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. We define $\alpha \in \mathbb{N}$ as the smallest integer for which $\kappa_S \cdot \alpha \in \mathbb{N}$ for all $S \in \mathcal{B}$ and use $b_S = \kappa_S \cdot \alpha$ for all $S \in \mathcal{B}$ as a shorthand notation. Note that for these new definitions, we suppress the dependency on $\mathcal{B}$ of $\alpha$, $b_S$, and $\kappa_S$. So, in order to show balancedness for our optimized pooling game $(N, c^\theta)$, it suffices to check if for each $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ it holds that
$$\sum_{S \in \mathcal{B}} b_S \cdot c^\theta(S) \geq \alpha \cdot c^\theta(N). \quad (1)$$

In the remainder of this section, we prove balancedness for our game by showing that (1) holds for each minimal balanced collection. We use the 5-step proof technique as introduced by Schlicher et al. [9]. This proof technique consists of several steps, and to facilitate understanding of the steps we first informally summarize them:

0. **Definition** of an MDP for each coalition $S$ such that $c^\theta(S) = \gamma \cdot \lim_{t \to \infty} V^S_t(y)/t$, where $V^S$ is the value function corresponding to the MDP (see Section 2.3).

1. **Copy** of each coalition $S$ to obtain labeled coalitions $(S, k)$ for $k \in \{1, \ldots, b_S\}$ and associated value function $V^{S,k}$. Express the left-hand size of (1) in terms of $V^{S,k}$.
2. **Combination** of the value functions $V_{S,k}$ for all labeled coalitions $(S,k)$ into a single value function $V_{B}$. The combination of this new value function is semi-cartesian: each individual value function $V_{S,k}$ is retained in $V_{B}$ along with all its dynamics, while the transitions (due to demand arrivals or repair completions) are coupled across the individual value functions.

3. **Relaxation** of the possible transition actions in $V_{B}$ to obtain $\hat{V}_{B}$. This latter value function corresponds to a situation where demand of a labeled coalition can be satisfied using inventory of any labeled coalition and where a repair completion of a labeled coalition can be used to increase the inventory of any labeled coalition.

4. **Anonimization** of the state space belonging to $\hat{V}_{B}$ to obtain an MDP that only keeps track of the total inventory of all labeled coalitions together, with associated value function $V_{\alpha}$. In this MDP demands arrive in batches of size $\alpha$, and each repair completion simultaneously returns (at most) $\alpha$ parts to inventory.

5. **Uncopy** of value function $V_{\alpha}$ into $\alpha$-times the value function $V_{N}$, which is the value function of the grand coalition.

We next discuss steps 1-5 in detail and present a conclusion which proves (1). We want to stress that, although we use the proof technique as introduced in Schlicher et al. [9], the proof itself is new.

1. **Copy.** For each minimal balanced collection $B \subseteq 2^{N}\setminus\{\emptyset\}$, we introduce another set $\mathcal{L}$ that contains for each $S \in B$ exactly $b_{S}$ labeled copies of coalition $S$.

**Definition 1.** Let $\theta \in \Theta$ and $B \subseteq 2^{N}\setminus\{\emptyset\}$ be a minimal balanced collection. Then, we define

$$\mathcal{L} = \left\{ (S,k) \mid S \in B, k \in \{1,2,\ldots,b_{S}\} \right\}.$$  

The labeled copies are called labeled coalitions. For each labeled coalition $(S,k) \in \mathcal{L}$ we denote the value function by $V_{t}^{S,k}$ and the capacity by $C_{S,k}$.

**Lemma 3.** For every $\theta \in \Theta$ it holds for any minimal balanced collection $B \subseteq 2^{N}\setminus\{\emptyset\}$ that

$$\sum_{S \in B} b_{S} \cdot c_{\theta}(S) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \cdot \sum_{S \in B} \sum_{k=1}^{b_{S}} V_{t}^{S,k}(C_{S,k}).$$

2. **Combination.** We show that for any minimal balanced collection $B \subseteq 2^{N}\setminus\{\emptyset\}$ we can construct a combined value function (of some unspecified MDP) with a state space that keeps track of the inventory level of every labeled coalition $(S,k) \in \mathcal{L}$, an action
space that consists of all possible actions per labeled coalition \((S,k) \in \mathcal{L}\) given its inventory level, and for which the related costs coincide with \(\sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V^S_t(k_{C_S,k})\) for all \(t \in \mathbb{N} \cup \{0\}\). In order to do so, we first introduce a new state space.

**Definition 2.** Let \(\theta \in \Theta\) and \(\mathcal{B} \subseteq 2^{\mathbb{N}} \setminus \{\emptyset\}\) be a minimal balanced collection. Then, we define

\[
\mathcal{Y}^\mathcal{B} = \left\{ (r_z)_{z \in \mathcal{L}} \mid r_z \in \{0, 1, \ldots, C_z\} \forall z \in \mathcal{L} \right\}.
\]

Secondly, we will introduce a new action space.

**Definition 3.** Let \(\theta \in \Theta\) and \(\mathcal{B} \subseteq 2^{\mathbb{N}} \setminus \{\emptyset\}\) be a minimal balanced collection. Then, for all \(r \in \mathcal{Y}^\mathcal{B}\) and all \(i \in \mathbb{N}\) we define

\[
\mathcal{A}^\mathcal{B}_{i,-}(r) = \left\{ (l_z)_{z \in \mathcal{L}} \mid l_z \in \{0, \min\{1, r_z\}\} \forall z \in \mathcal{L} : i \in S \right\}
\]

\[
\mathcal{A}^\mathcal{B}_{i,+}(r) = \left\{ (l_z)_{z \in \mathcal{L}} \mid l_z \in \{0, \min\{1, C_z - r_z\}\} \forall z \in \mathcal{L} : i \notin S \right\}.
\]

Subsequently, we introduce the new value function. We use that \(||\cdot||||\) is the \(L^1\) norm.

**Definition 4.** Let \(\theta \in \Theta\) and \(\mathcal{B} \subseteq 2^{\mathbb{N}} \setminus \{\emptyset\}\) be a minimal balanced collection. Then, for all \(r \in \mathcal{Y}^\mathcal{B}\) and all \(t \in \mathbb{N} \cup \{0\}\), we define the value function as

\[
V^{\mathcal{B}}_{t+1}(r) = \sum_{i \in \mathbb{N}} \left[ \lambda^*_i \min_{l \in \mathcal{A}^\mathcal{B}_{i,-}(r)} \left\{ (a - ||l||)d_i + V^{\mathcal{B}}_t(r - l) \right\} + \mu^*_i \min_{l \in \mathcal{A}^\mathcal{B}_{i,+}(r)} \left\{ V^{\mathcal{B}}_t(r + l) \right\} \right] + h^* \cdot ||r||_1.
\]

with \(V^{\mathcal{B}}_0(r) = 0\) for all \(r \in \mathcal{Y}^\mathcal{B}\).

Now, we are able to show for all time moments the equivalence between the costs of the new value function and \(\sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V^S_t(k_{C_S,k})\).

**Lemma 4.** Let \(\theta \in \Theta\) and \(\mathcal{B} \subseteq 2^{\mathbb{N}} \setminus \{\emptyset\}\) be a minimal balanced collection. Then, for all \(r \in \mathcal{Y}^\mathcal{B}\) and all \(t \in \mathbb{N} \cup \{0\}\) it holds that

\[
\sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V^S_t(k_{r_S,k}) = V^{\mathcal{B}}_t(r).
\]

3. Relaxation. We introduce a value function (related to some unspecified MDP), that coincides with the value function of Definition 4 except for a relaxed action space. In order to do so, we first need to introduce this relaxed action space.
Definition 5. Let $\theta \in \Theta$ and $B \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in Y^B$ and all $i \in N$ we define

$$\hat{A}_{i-}^B(r) = \left\{ (l_z)_{z \in L} \mid l_z \in \mathbb{N} \cup \{0\}, \forall z \in L, \sum_{z \in L} l_z \leq \alpha, \ r - l \in Y^B \right\}$$

$$\hat{A}_{i+}^B(r) = \left\{ (l_z)_{z \in L} \mid l_z \in \mathbb{N} \cup \{0\}, \forall z \in L, \sum_{z \in L} l_z \leq \alpha, \ r + l \in Y^B \right\}$$

The following result is a direct consequence of relaxing the action space. The proof is straightforward and for this reason omitted (rather than relegated to the appendix).

Lemma 5. Let $\theta \in \Theta$ and $B \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. Then, for all $r \in Y^B$ and all $i \in N$ it holds that $A_{i-}^B(r) \subseteq \hat{A}_{i-}^B(r)$ and $A_{i+}^B(r) \subseteq \hat{A}_{i+}^B(r)$.

Now, we present the value function with this relaxed action space.

Definition 6. Let $\theta \in \Theta$ and $B \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. For all $r \in Y^B$ and all $t \in \mathbb{N} \cup \{0\}$, we define

$$\hat{V}_{t+1}^B(r) = \sum_{i \in N} \left[ \lambda_i^* \min_{l \in \hat{A}_{i-}^B(r)} \left\{ (\alpha - ||l||_1) d_i + \hat{V}_{t}^B(r - l) \right\} + \mu_i^* \min_{l \in \hat{A}_{i+}^B(r)} \left\{ \hat{V}_{t}^B(r + l) \right\} \right] + h^* \cdot ||r||_1,$$

with $\hat{V}_0^B(r) = 0$ for all $r \in Y^B$.

Incorporating a relaxed action space in the new value function leads to related costs that are smaller than or equal to the original costs of the value function. The proof is straightforward (by induction on $t$) and therefore omitted (rather than relegated to the appendix).

Lemma 6. Let $\theta \in \Theta$ and $B \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. For all $r \in Y^B$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that $V_t^B(r) \geq \hat{V}_t^B(r)$.

4. Anonimization. We introduce a value function (which is related to some unspecified MDP for which the belongings and decisions of the labeled coalitions are anonimized) and show cost-equivalence with the one of Definition 6.
Definition 7. Let $\theta \in \Theta$ and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. Then, for all $j \in \{0, 1, \ldots, \alpha \cdot C_N\}$ and all $t \in \mathbb{N} \cup \{0\}$ we define

$$V_t^\alpha(i) = \sum_{i \in \mathbb{N}} \left[ \lambda_t^\alpha \min_{l \in \{0, \ldots, \min\{a, j\}\}} \left\{ (\alpha - l)d_i + V_t^\alpha(j - l) \right\} + \mu_t^\alpha \min_{l \in \{0, \ldots, \min\{a, \alpha \cdot C_N - j\}\}} V_t^\alpha(j + l) \right] + h^\alpha \cdot j$$

with $V_0^\alpha(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot C_N\}$.

Lemma 7. Let $\theta \in \Theta$ and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. For all $r \in \mathcal{Y}^\mathcal{B}$ it holds that

$$\hat{V}_t^\mathcal{B}(r) = V_t^\alpha(||r||_1) \quad \text{for all } t \in \mathbb{N} \cup \{0\}.$$

5. Uncopy. We identify some interesting properties of $V^\alpha$.

Lemma 8. Let $\theta \in \Theta$ and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. For all $t \in \mathbb{N} \cup \{0\}$ it holds that

(i) $V_t^\alpha(j) + V_t^\alpha(j + 2) \geq 2 \cdot V_t^\alpha(j + 1)$ for all $j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\}$;

(ii) $V_t^\alpha(k + j) + V_t^\alpha(k + j + 2) = 2 \cdot V_t^\alpha(k + j + 1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (C_N - 1)\alpha\}$.

The properties of Lemma 8 allow us to uncopy value function $V^\alpha$ into $\alpha$ - times value function $V^N$, i.e., the value function of coalition $N$.

Lemma 9. Let $\theta \in \Theta$ and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. For all $j \in \{0, \alpha, \ldots, C_N \cdot \alpha\}$ and all $t \in \mathbb{N} \cup \{0\}$ it holds that

$$V_t^\alpha(j) = \alpha \cdot V_t^N \left( \frac{j}{\alpha} \right)$$

Conclusion. Now, we integrate the previous steps to demonstrate validity of (1).

Theorem 2. Optimized pooling games are balanced.

Proof: Let $\theta \in \Theta$ and $(N, c^\theta)$ be the associated optimized pooling game and $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. In addition, let $\tilde{r} = (C_{S,k})_{(S,k) \in \mathcal{L}^\mathcal{B}}$. Then, observe that

$$\sum_{S \in \mathcal{B}} b_S \cdot c^\theta(S) = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V_t^{S,k}(C_{S,k}) \geq \gamma \cdot \lim_{t \to \infty} \frac{\hat{V}_t^\mathcal{B}(\tilde{r})}{t} = \gamma \cdot \lim_{t \to \infty} \frac{V_t^\alpha(\alpha \cdot C_N)}{t}$$

$$= \gamma \cdot \lim_{t \to \infty} \frac{V_t^N(C_N)}{t} = \alpha \cdot c^\theta(N).$$
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4 Appendix

Proof of Lemma 2

The first equality holds by uniformization, which is allowed if transition rates are bounded and the MDP is multichain (see Puterman [7, p.568]). Notice that interarrival times of demands as well as repair times are exponentially distributed with rates that are bounded from above. In addition, for every stationary policy, there exist one or multiple recurrent classes. Hence, the MDP is multichain. With respect to the second equality, observe that state space $Y^S$ and action space $A^S$ of the MDP are finite. In addition, under stationary policy $f = (f_i(y))_{y \in Y^S, i \in S}$ with $f_i(y) = (1, 1)$ for all $i \in S$ and all $0 < y < C_S$ and $f_i(0) = (0, 1)$ for all $i \in S$ and $f_i(C_S) = (1, 0)$ for all $i \in S$, every state $y \in Y^S$ is accessible from any state $y' \in Y^S$ after (possibly) some arrivals and some (one-by-one) repair completions. Hence, the related Markov chain is irreducible. An irreducible Markov chain with finite state space is positive recurrent (see e.g., Modica and Poggiolini [4]). Finally, observe that the long-run average costs per time epoch under policy $f$ are bounded (naturally) by $\sum_{i \in S} \lambda^*_i \cdot d_i + h^* \cdot C_S$ and as a result of Sennott [10, Proposition 4.3], the second equality follows.

Proof of Lemma 3

Let $\theta \in \Theta$ and $B \subseteq 2^N \setminus \{\emptyset\}$ be a minimal balanced collection. It holds that

$$\sum_{S \in B} b_S \cdot c^\theta(S) = \gamma \cdot \sum_{S \in B} \sum_{k=1}^{b_S} \lim_{t \to \infty} \frac{V^{S,k}_i(C_{S,k})}{t} = \gamma \cdot \lim_{t \to \infty} \frac{1}{t} \cdot \sum_{S \in B} \sum_{k=1}^{b_S} V^{S,k}_i(C_{S,k}).$$

The first equality holds by exploiting all labeled coalitions, Lemma 2, and the fact that $C_{S,k} \in Y^{S,k}$ for all $(S,k) \in \mathcal{L}$. The last equality holds as all limits are well-defined and all sums are finite. This concludes the proof.

Proof of Lemma 4

Proof : This proof is by induction. By definition of the value functions $V^{S,k}_0(y) = 0$ for all $y \in Y^{S,k}$, and all $S \in \mathcal{B}$ and all $k \in \{1, 2, \ldots, b_S\}$. Similarly, $V^{A}_0(r) = 0$ for all $r \in A$ as well. Hence, $\sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V^{S,k}_0(r_{S,k}) = V^{A}_0(r)$ for all $r \in A$. Let $t \in \mathbb{N} \cup \{0\}$ and
assume that \( \sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V_{i+1}^{S,k}(r_{S,k}) = V_i^{\mathcal{B}}(r) \) for all \( r \in \mathcal{R} \). Let \( r \in \mathcal{R} \). Now, observe that

\[
\sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} V_{i+1}^{S,k}(r_{S,k}) = \sum_{S \in \mathcal{B}} \sum_{k=1}^{b_S} \left( \sum_{i \in S} \left( \lambda_i^* \min_{l \in \{0, \ldots, \min\{l_{S,k}\}\}} \left\{ V_i^{S,k}(r_{S,k} - l) + (1 - l)d_i \right\} + \mu_i^* \min_{l \in \{0, \ldots, \min\{l_{C-S-r}\}\}} V_i^{S,k}(r_{S,k} + l) \right) \right) + h^* \cdot \|r\|_1
\]

\[
= \sum_{i \in N} \left[ \lambda_i^* \left( \sum_{S \in \mathcal{B} : i \in S} \sum_{k=1}^{b_S} \min_{l \in \{0, \ldots, \min\{l_{S,k}\}\}} \left\{ V_i^{S,k}(r_{S,k} - l) + (1 - l)d_i \right\} + \sum_{S \in \mathcal{B} : i \notin S} \sum_{k=1}^{b_S} V_i^{S,k}(r_{S,k}) \right) \right] + \mu_i^* \left( \sum_{S \in \mathcal{B} : i \in S} \sum_{k=1}^{b_S} V_i^{S,k}(r_{S,k} + l) + \sum_{S \in \mathcal{B} : i \notin S} \sum_{k=1}^{b_S} V_i^{S,k}(r_{S,k}) \right) + h^* \cdot \|r\|_1
\]

\[
= \sum_{i \in N} \left[ \lambda_i^* \left( \min_{l \in \hat{\mathcal{A}}_i^{\mathcal{B}}(r)} \left\{ \sum_{S \in \mathcal{B} : i \in S} \sum_{k=1}^{b_S} V_i^{S,k}(r_{S,k} - l_{S,k}) + \sum_{S \in \mathcal{B} : i \notin S} \sum_{k=1}^{b_S} V_i^{S,k}(r_{S,k}) \right\} \right) \right] + \mu_i^* \left( \min_{l \in \hat{\mathcal{A}}_i^{\mathcal{B}}(r)} \left\{ V_i^{\mathcal{B}}(r - l) + (a - ||l||_1)d_i \right\} \right) + h^* \cdot \|r\|_1
\]

\[
= V_{i+1}^{\mathcal{B}}(r)
\]

The first equality holds by Lemma 1. The second equality holds as \( 1 - \sum_{i \in S} [\lambda_i^* + \mu_i^*] = \sum_{i \in N} [\lambda_i^* + \mu_i^*] - \sum_{i \in S} [\lambda_i^* + \mu_i^*] = \sum_{i \in N \setminus S} [\lambda_i^* + \mu_i^*] \) and the definition of \( L^1 \) norm. The third equality holds by conditioning on \( \lambda_i^* \) and \( \mu_i^* \) for all \( i \in N \). The fourth equality holds as the sum of minima can be rewritten as one minimum and \( \hat{\mathcal{A}}_i^{\mathcal{B}} \) and \( \hat{\mathcal{A}}_i^{\mathcal{B}} \) are defined such that the decisions made for all minima fit. Note that \( l_{S,k} = 0 \) if \( i \notin S \). The fifth equality holds by the induction hypothesis. The last equality holds by Definition 4. By the principle of mathematical induction, this completes the proof. \( \square \)

**Proof of Lemma 7**

**Proof:** This proof is by induction. By definition of the value functions \( \hat{V}_0^{\mathcal{B}}(r) = 0 \) for all \( r \in \mathcal{R} \) and \( V_0^{\mathcal{B}}(||r||_1) = 0 \) for all \( r \in \mathcal{R} \). Hence, \( \hat{V}_0^{\mathcal{B}}(r) = V_0^{\mathcal{B}}(||r||_1) \) for all \( r \in \mathcal{R} \).
Let $t \in \mathbb{N} \cup \{0\}$ and assume that $\hat{V}_t^\beta(r) = V_t^\alpha(||r||_1)$ for all $r \in \mathcal{R}$. Let $r \in \mathcal{R}$. Now, observe that

$$\hat{V}_{t+1}^\beta(r) = \sum_{i \in N} \lambda_i^* \min_{l \in \hat{a}_i^\beta(r)} \left\{ (\alpha - ||l||_1)d_i + \hat{V}_t^\beta(r - l) \right\} + \sum_{i \in N} \mu_i^* \min_{l \in \hat{a}_i^\beta(r)} \left\{ \hat{V}_t^\beta(r + l) \right\} + h^* \cdot ||r||_1$$

$$= \sum_{i \in N} \lambda_i^* \min_{z \in \{0,1,\ldots,\min(\alpha,||r||_1)\}} \left\{ \min_{l \in \hat{a}_i^\beta(r); ||l||_1 = z} \left\{ (\alpha - z)d_i + \hat{V}_t^\beta(r - l) \right\} \right\}$$

$$+ \sum_{i \in N} \mu_i^* \min_{z \in \{0,1,\ldots,\min(\alpha,\alpha \cdot C_N - ||r||_1)\}} \left\{ \min_{l \in \hat{a}_i^\beta(r); ||l||_1 = z} \hat{V}_t^\beta(r + l) \right\} + h^* \cdot ||r||_1$$

$$= \sum_{i \in N} \lambda_i^* \min_{z \in \{0,1,\ldots,\min(\alpha,||r||_1)\}} \left\{ (\alpha - z)d_i + V_t^\alpha(||r||_1 - z) \right\}$$

$$+ \sum_{i \in N} \mu_i^* \min_{z \in \{0,1,\ldots,\min(\alpha,\alpha \cdot C_N - ||r||_1)\}} \left\{ V_t^\alpha(||r||_1 + z) \right\} + h^* \cdot ||r||_1$$

$$= V_{t+1}^\alpha(||r||_1).$$

The first equality holds by Definition 6. The second equality holds by rewriting the minimum as a two-step minimization. The third equality holds by the induction hypothesis. The last equality holds by Definition 7. By the principle of mathematical induction, this completes the proof. \qed
Proof of Lemma 8

Proof: First, the value function will be rewritten. For all \( j \in \{0,1,\ldots,\alpha \cdot C_N\} \) and all \( t \in \mathbb{N} \cup \{0\} \), we have

\[
V^\alpha_{t+1}(j) = \sum_{i \in N} \lambda^*_i \min_{l \in \{0,1,\ldots,\min\{a_i\}\}} \left\{ (\alpha - l)d_i + V^\alpha_t(j - l) \right\} \\
+ \sum_{i \in N} \mu^*_i \min_{l \in \{0,\ldots,\min\{\alpha,\beta - j\}\}} \left\{ V^\alpha_t(j + l) + h^* \cdot j \right\}
\]

\[
= \sum_{i \in N} \lambda^*_i \left[ \min_{l \in \{0,1,\ldots,\min\{a_i\}\}} \left\{ (j - l)d_i + V^\alpha_t(j - l) \right\} + (\alpha - j)d_i \right] \\
+ \sum_{i \in N} \mu^*_i \min_{l \in \{0,\ldots,\min\{\alpha,\beta - j\}\}} \left\{ V^\alpha_t(j + l) + h^* \cdot j \right\}
\]

\[
= \sum_{i \in N} \lambda^*_i \left[ \min_{l \in \{\max\{0,j - \alpha\},\ldots,j\}} \left\{ ld_i + V^\alpha_t(l) \right\} + (\alpha - j)d_i \right] \\
+ \sum_{i \in N} \mu^*_i \min_{l \in \{j,\ldots,\min\{j + \alpha,\beta - C_N\}\}} \left\{ V^\alpha_t(l) + h^* \cdot j \right\},
\]

where the second equality holds as \((\alpha - l)d_i = (\alpha - j)d_i + (j - l)d_i\) and \((\alpha - j)d_i\) is a constant. The last equality holds by substituting \( j - l \) into a new variable and by substituting \( j + l \) into a new variable.

In addition, we define for all \( j \in \{0,1,\ldots,\alpha \cdot C_N\} \) and all \( t \in \mathbb{N} \cup \{0\} \)

\[
V^{\alpha_1}_{t+1}(j) = \sum_{i \in N} \lambda^*_i \min_{l \in \{\max\{0,j - \alpha\},\ldots,j\}} \left\{ ld_i + V^\alpha_t(l) \right\}
\]

\[
V^{\alpha_2}_{t+1}(j) = \sum_{i \in N} \mu^*_i \min_{l \in \{j,\ldots,\min\{j + \alpha,\beta - C_N\}\}} \left\{ V^\alpha_t(l) \right\}
\]

\[
V^{\alpha_3}_{t+1}(j) = h^* \cdot j + \sum_{i \in N} \lambda^*_i (\alpha - j)d_i.
\]

Note that \( V^\alpha_t(j) = V^{\alpha_1}_{t+1}(j) + V^{\alpha_2}_{t+1}(j) + V^{\alpha_3}_{t+1}(j) \) for all \( j \in \{0,1,\ldots,\alpha \cdot C_N\} \) and all \( t \in \mathbb{N} \cup \{0\} \).

(i) Now, we will prove by induction that \( V^\alpha_t(j) + V^\alpha_t(j + 2) \geq 2 \cdot V^\alpha_t(j + 1) \) for all \( j \in \{0,1,\ldots,\alpha \cdot C_N - 2\} \) and all \( t \in \mathbb{N} \cup \{0\} \). By definition of the value functions \( V^\alpha_0(j) = 0 \) for all \( j \in \{0,1,\ldots,\alpha \cdot C_N\} \). Hence, \( V^\alpha_0(j) + V^\alpha_0(j + 2) \geq 2 \cdot V^\alpha_0(j + 1) \) for all \( j \in \{0,1,\ldots,\alpha \cdot C_N - 2\} \). Let \( t \in \mathbb{N} \cup \{0\} \) and assume that \( V^\alpha_t(j) + V^\alpha_t(j + 2) \geq 2 \cdot V^\alpha_t(j + 1) \) for all \( j \in \{0,1,\ldots,\alpha \cdot C_N - 2\} \).

We first focus on \( V^{\alpha_2}_{t+1}(j) \), thereafter we focus on \( V^{\alpha_1}_{t+1}(j) \), and finally we focus on \( V^{\alpha_3}_{t+1}(j) \).

15
Let \( j \in \{0, 1, \ldots, \alpha \cdot (C_N - 1) - 2\} \). Now, observe that

\[
V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j + 2) = \sum_{i \in N} \mu_i^* \min_{l \in \{j, \ldots, \min\{j + \alpha, C_N\}\}} V_t^\alpha(l) + \sum_{i \in N} \mu_i^* \min_{l \in \{j + 2, \ldots, \min\{j + 2 + \alpha, C_N\}\}} V_t^\alpha(l)
\]

\[
= \sum_{i \in N} \mu_i^* \min_{l_1 \in \{j, \ldots, j + \alpha\}} V_t^\alpha(l_1) + \sum_{i \in N} \mu_i^* \min_{l_2 \in \{j + 2, \ldots, j + \alpha + 2\}} V_t^\alpha(l_2)
\]

\[
\geq 2 \sum_{i \in N} \mu_i^* \min_{l_3 \in \{j + 1, \ldots, j + \alpha + 1\}} V_t^\alpha(l_3)
\]

\[
= 2 \sum_{i \in N} \mu_i^* \min_{l_3 \in \{j + 1, \ldots, \min\{j + \alpha + 1 + \alpha, C_N\}\}} V_t^\alpha(l_3)
\]

\[
= 2 \cdot V_{t+1}^{\alpha_2}(j + 1)
\]

The inequality holds by Lemma \[10\] (see page 22 and page 23) with \( f(x) = V_t^\alpha(x) \), \( a = j \), \( b = j + \alpha \), \( c = d = 2 \). The last but one equality holds as \( j \leq \alpha \cdot (C_N - 1) - 2 \).

Let \( j \in \{\alpha \cdot (C_N - 1) - 1, \alpha \cdot (C_N - 1), \ldots, \alpha \cdot C_N - 2\} \). Now, observe that

\[
V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j + 2) = \sum_{i \in N} \mu_i^* \min_{l \in \{j, \ldots, \min\{j + \alpha, C_N\}\}} V_t^\alpha(l) + \sum_{i \in N} \mu_i^* \min_{l \in \{j + 2, \ldots, \min\{j + 2 + \alpha, C_N\}\}} V_t^\alpha(l)
\]

\[
\geq \sum_{i \in N} \mu_i^* \min_{l_1 \in \{j, \ldots, \alpha C_N\}} \{V_t^\alpha(l_1) + V_t^\alpha(l_2)\}
\]

\[
\geq 2 \sum_{i \in N} \mu_i^* \min_{l_3 \in \{j + 1, \ldots, \alpha C_N\}} V_t^\alpha(l_3)
\]

\[
= 2 \sum_{i \in N} \mu_i^* \min_{l_3 \in \{j + 1, \ldots, \min\{j + \alpha + 1 + \alpha, C_N\}\}} V_t^\alpha(l_3)
\]

\[
= 2 \cdot V_{t+1}^{\alpha_2}(j + 1)
\]

The first inequality holds as adding a possible term to a set from which its minimum is selected will not increase the minimum. The second inequality holds by Lemma \[10\] with \( f(x) = V_t^\alpha(x) \), \( a = j \), \( b = \alpha \cdot C_N \), \( c = 2 \), and \( d = 0 \). The last but one equality holds as \( j + 1 + \alpha \geq \alpha \cdot C_N \).

So, for all \( j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\} \) it holds that \( V_{t+1}^{\alpha_2}(j) + V_{t+1}^{\alpha_2}(j + 2) \geq 2V_{t+1}^{\alpha_2}(j + 1) \).
Let \( j \in \{0, 1, \ldots, \alpha - 1\} \). Now, observe that

\[
V_{t+1}^{\alpha_1}(j) + V_{t+1}^{\alpha_1}(j + 2)
= \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j - \alpha\}, \ldots, j\}} \left\{ I d_i + V_i^a(l) \right\} + \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j + 2 - \alpha\}, \ldots, j + 2\}} \left\{ I d_i + V_i^a(l) \right\}
\geq \sum_{i \in N} \lambda_i^* \min_{\substack{l_1 \in \{0, \ldots, j\} \atop l_2 \in \{0, \ldots, j + 2\}}} \left\{ (l_1 + l_2) d_i + V_i^a(l_1) + V_i^a(l_2) \right\}
\geq 2 \sum_{i \in N} \lambda_i^* \min_{l_3 \in \{0, 1, \ldots, j + 1\}} \left\{ V_i^a(l_3) + l_3 d_i \right\}
= 2V_{t+1}^{\alpha_1}(j + 1).
\]

The first inequality holds as adding a (possible) term to a set from which its minimum is selected will not increase the minimum. The second inequality holds by Lemma 10 with \( f(x) = V_i^a(x) + x \cdot d_i \) (which is convex as the sum of a convex function \( V_i^a(x) \)) and a linear function \( x \cdot d_i \) is still convex), \( a = 0, \ b = j, \ c = 0, \) and \( d = 2 \). The last but one equality holds as \( j + 1 - \alpha \leq 0 \).

Let \( j \in \{\alpha, \alpha + 1, \ldots, \alpha + C_N - 2\} \). Now, observe that

\[
V_{t+1}^{\alpha_1}(j) + V_{t+1}^{\alpha_1}(j + 2)
= \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j - \alpha\}, \ldots, j\}} \left\{ I d_i + V_i^a(l) \right\} + \sum_{i \in N} \lambda_i^* \min_{l \in \{\max\{0, j + 2 - \alpha\}, \ldots, j + 2\}} \left\{ I d_i + V_i^a(l) \right\}
= \sum_{i \in N} \lambda_i^* \min_{\substack{l_1 \in \{0, j - \alpha\}, \ldots, j\} \atop l_2 \in \{0, j + 2 - \alpha, \ldots, j + 2\}} \left\{ (l_1 + l_2) d_i + V_i^a(l_1) + V_i^a(l_2) \right\}
\geq 2 \sum_{i \in N} \lambda_i^* \min_{l_3 \in \{j + 1 - \alpha, \ldots, j + 1\}} \left\{ V_i^a(l_3) + l_3 d_i \right\}
= 2V_{t+1}^{\alpha_1}(j + 1).
\]

The first inequality holds by Lemma 10 with \( f(x) = V_i^a(x) + x \cdot d_i \) (which is convex as the sum of a convex function \( V_i^a(x) \)) and a linear function \( x \cdot d_i \) is still convex), \( a = j - \alpha, \ b = j, \ c = 2, \) and \( d = 2 \). The last but one equality holds as \( j + 1 - \alpha \leq 0 \).

So, for all \( j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\} \) it holds that \( V_{t+1}^{\alpha_1}(j) + V_{t+1}^{\alpha_1}(j + 2) \geq 2V_{t+1}^{\alpha_1}(j + 1) \).
Let $j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\}$. Now, observe that

$$V_{t+1}^{a_3}(j) + V_{t+1}^{a_3}(j+2) = h^* \cdot j + \sum_{i \in N} \lambda_i^* (\alpha - j) d_i + h^* \cdot (j + 2) + \sum_{i \in N} \lambda_i^* (\alpha - (j + 2)) d_i$$

$$= 2 \cdot h^* \cdot (j + 1) + 2 \sum_{i \in N} \lambda_i^* (\alpha - (j + 1)) d_i$$

$$= 2 \cdot V_{t+1}^{a_3}(j + 1).$$

So, for all $j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\}$ it holds that $V_{t+1}^{a_3}(j) + V_{t+1}^{a_3}(j + 2) \geq 2V_{t+1}^{a_3}(j + 1)$.

We conclude, for all $j \in \{0, 1, \ldots, \alpha \cdot C_N - 2\}$, it holds that

$$V_{t+1}^{a_3}(j) + V_{t+1}^{a_3}(j + 2) = V_{t+1}^{a_1}(j) + V_{t+1}^{a_2}(j) + V_{t+1}^{a_3}(j) + V_{t+1}^{a_2}(j + 2) + V_{t+1}^{a_3}(j + 2)$$

$$\geq 2V_{t+1}^{a_1}(j + 1) + 2V_{t+1}^{a_2}(j + 1) + 2V_{t+1}^{a_3}(j + 1)$$

$$= 2V_{t+1}^{a_3}(j + 1).$$

(ii) By definition of the value functions $V_0^a(j) = 0$ for all $j \in \{0, 1, \ldots, \alpha \cdot C_N\}$. Hence, $V_0^a(k + j) + V_0^a(k + j + 2) = 2 \cdot V_0^a(k + j + 1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (C_N - 1)\alpha\}$. Let $t \in \mathbb{N} \cup \{0\}$ and assume that $V_t^a(k + j) + V_t^a(k + j + 2) = 2 \cdot V_t^a(k + j + 1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (C_N - 1)\alpha\}$.

First, observe that function $V_t^a(j) + j \cdot d_i$ is convex in $j$ for all $i \in N$ as $V_t^a(\cdot)$ is convex by (i) and $j \cdot d_i$ is linear. By our induction hypothesis, it holds that $V_t^a(k + j) + V_t^a(k + j + 2) = 2 \cdot V_t^a(k + j + 1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (C_N - 1)\alpha\}$, which implies that $V_t^a(\cdot) + j \cdot d_i$ is also piecewise linear. So, there exists a $p \in \{0, \alpha, 2\alpha, \ldots, C_N\alpha\}$ for which it holds that $V_t^a(p) + p \cdot d_i \leq V_t^a(j) + j \cdot d_i$ for all $j \in \{0, 1, \ldots, \alpha \cdot C_N\}$ and all $i \in N$. We fix such a $p$ and denote it by $p^* \in \{0, \alpha, \ldots, C_N \cdot \alpha\}$.

Then, for all $k \in \{0, \alpha, \ldots, p^* - \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$V_{t+1}^{a_1}(k + j) = \sum_{i \in N} \lambda_i^* \left[ \min_{l \in \{\max(0, k + j - \alpha), \ldots, k + j\}} \left\{ ld_i + V_t^a(l) \right\} \right]$$

$$= \sum_{i \in N} \lambda_i^* \left[ (k + j) d_i + V_t^a(k + j) \right],$$

where the second equality holds, because the minimum of convex function $V_t^a(j) + j \cdot d_i$ is attained at $j = p^*$ and so the minimum in the first equality is attained at $k + j$.

For $k = p^*$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$V_{t+1}^{a_1}(k + j) = \sum_{i \in N} \lambda_i^* \left[ \min_{l \in \{\max(0, k + j - \alpha), \ldots, k + j\}} \left\{ ld_i + V_t^a(l) \right\} \right]$$

$$= \sum_{i \in N} \lambda_i^* \left[ k \cdot d_i + V_t^a(k) \right],$$

18
where the second equality results from the fact that the minimum of \( V_1^a(j) + j \cdot d_i \) is attained at \( j = p^* \) and so the minimum in the first equality is attained at \( k (= p^*) \).

For all \( k \in \{ p^* + a, p^* + 2 \cdot a, \ldots, (C_N - 1) \cdot a \} \) and all \( j \in \{ 0, 1, \ldots, \alpha - 2 \} \) it holds that

\[
V_{t+1}^{a_1}(k + j) = \sum_{i \in N} \lambda_i^* \left[ \min_{l \in \{ \max(0, k+j - \alpha), \ldots, k+j \}} \left\{ ld_i + V_t^a(l) \right\} \right] = \sum_{i \in N} \lambda_i^* \left[ (k + j - \alpha) d_i + V_t^a(k + j - \alpha) \right],
\]

(4)

where the second equality results from the fact that the minimum of \( V_1^a(j) + j \cdot d_i \) is attained at \( j = p^* \) and so the minimum in the first equality is attained at \( k + j - \alpha \).

So, for all \( k \in \{ 0, 1, \ldots, p^* - a \} \) and all \( j \in \{ 0, 1, \ldots, \alpha - 2 \} \), we have

\[
V_{t+1}^{a_1}(k + j) + V_{t+1}^{a_1}(k + j + 2) = 2 \sum_{i \in N} \lambda_i^* \left[ V_t^a(k + j + 1) + (k + j + 1) \cdot d_i \right]
\]

where the first equality holds by (2) and the second one by the induction hypothesis.

For \( k = p^* \) and all \( j \in \{ 0, 1, \ldots, \alpha - 2 \} \) it holds that

\[
V_{t+1}^{a_1}(k + j) + V_{t+1}^{a_1}(k + j + 2) = 2 \sum_{i \in N} \lambda_i^* \left[ V_t^a(k) + k \cdot d_i \right]
\]

where the first equality holds by (3). The second equality holds by the induction hypothesis.

For all \( k \in \{ p^* + a, p^* + 2 \cdot a, \ldots, (C_N - 1) \cdot a \} \) and all \( j \in \{ 0, 1, \ldots, \alpha - 2 \} \) it holds that

\[
V_{t+1}^{a_1}(k + j) + V_{t+1}^{a_1}(k + j + 2) = 2 \sum_{i \in N} \lambda_i^* \left[ V_t^a(k + j + 1 - \alpha) + (k + j + 1 - \alpha) d_i \right]
\]

where the first equality holds by (3). The second equality holds by the induction hypothesis.
where the first equality holds by (4) and the second equality by the induction hypothesis.

We conclude that for all $k \in \{0, \alpha, \ldots, \alpha \cdot (C_N - 1)\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ we have

$$V_i^{\alpha_1}(k+j) + V_i^{\alpha_1}(k+j+2) = 2 \cdot V_i^{\alpha_1}(k+j+1). \quad (5)$$

Recall that $V_i^{\alpha}(\cdot)$ is convex by (i). By our induction hypothesis, it holds that $V_i^{\alpha}(k+j) + V_i^{\alpha}(k+j+2) = 2 \cdot V_i^{\alpha}(k+j+1)$ for all $j \in \{0, 1, \ldots, \alpha - 2\}$ and all $k \in \{0, \alpha, 2\alpha, \ldots, (C_N - 1)\alpha\}$. So, there exists a $p \in \{0, \alpha, 2\alpha, \ldots, C_N\alpha\}$ for which it holds that $V_i^{\alpha}(p) \leq V_i^{\alpha}(j)$ for all $j \in \{0, 1, \ldots, \alpha \cdot C_N\}$ and all $i \in N$. We fix such a $p$ and denote it by $\bar{p} \in \{0, \alpha, \ldots, C_N \cdot \alpha\}$. Note that $p^*$ and $\bar{p}$ do not coincide necessarily.

Now, for all $k \in \{0, \alpha, \ldots, p^* - 2\alpha\}$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$V_i^{\alpha_2}(k+j) = \sum_{i \in N} \mu_i^* \min_{l \in \{k+j, \ldots, \min\{k+j+\alpha \cdot C_N\}\}} V_i^{\alpha}(l) = \sum_{i \in N} \mu_i^* V_i^{\alpha}(k+j+\alpha), \quad (6)$$

where the second equality holds as the minimum of convex function $V_i^{\alpha}(l)$ is attained at $l = \bar{p}$ and so the minimum in the first equality is attained at $k+j+\alpha$.

For $k = \bar{p} - \alpha$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$V_i^{\alpha_2}(k+j) = \sum_{i \in N} \mu_i^* \min_{l \in \{k+j, \ldots, \min\{k+j+\alpha \cdot C_N\}\}} V_i^{\alpha}(l) = \sum_{i \in N} \mu_i^* V_i^{\alpha}(\bar{p}), \quad (7)$$

where the second equality results from the fact that the minimum of $V_i^{\alpha}(l)$ is attained at $l = \bar{p}$ and so the minimum in the first equality is attained at $\bar{p}$.

For all $k \in \{\bar{p}, \bar{p} + \alpha, \ldots, (C_N - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha\}$ it holds that

$$V_i^{\alpha_2}(k+j) = \sum_{i \in N} \mu_i^* \min_{l \in \{k+j, \ldots, \min\{k+j+\alpha \cdot C_N\}\}} V_i^{\alpha}(l) = \sum_{i \in N} \mu_i^* V_i^{\alpha}(k+j), \quad (8)$$

where the second equality results from the fact that the minimum of $V_i^{\alpha}(l)$ is attained at $l = \bar{p}$ and so the minimum in the first equality is attained at $k+j$. 

20
So, for all $k \in \{0, 1, \ldots, \tilde{p} - 2\alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$, we have

$$V_{t+1}^{a_2}(k + j) + V_{t+1}^{a_2}(k + j + 2)$$

$$= \sum_{i \in N} \mu_i^* V_i^a(k + j + \alpha) + \sum_{i \in N} \mu_i^* V_i^a(k + j + 2 + \alpha)$$

$$= 2 \sum_{i \in N} \mu_i^* V_i^a(k + j + 1 + \alpha)$$

$$= 2 V_{t+1}^{a_2}(k + j + 1),$$

where the first equality holds by (6) and the second equality holds by the induction hypothesis.

For $k = \tilde{p} - \alpha$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$V_{t+1}^{a_2}(k + j) + V_{t+1}^{a_2}(k + j + 2)$$

$$= \sum_{i \in N} \mu_i^* V_i^a(h) + \sum_{i \in N} \mu_i^* V_i^a(h)$$

$$= 2 \sum_{i \in N} \mu_i^* V_i^a(h)$$

$$= 2 V_{t+1}^{a_2}(k + j + 1),$$

where the first equality holds by (7) and the second equality holds by the induction hypothesis.

For all $k \in \{\tilde{p}, \tilde{p} + \alpha, \ldots, (CN - 1) \cdot \alpha\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ it holds that

$$V_{t+1}^{a_2}(k + j) + V_{t+1}^{a_2}(k + j + 2)$$

$$= \sum_{i \in N} \mu_i^* V_i^a(k + j) + \sum_{i \in N} \mu_i^* V_i^a(k + j)$$

$$= 2 \sum_{i \in N} \mu_i^* V_i^a(k + j)$$

$$= 2 V_{t+1}^{a_2}(k + j + 1),$$

where the first equality holds by (8) and the second equality holds by the induction hypothesis.

We conclude that for all $k \in \{0, \alpha, \ldots, \alpha \cdot (CN - 1)\}$ and all $j \in \{0, 1, \ldots, \alpha - 2\}$ we have

$$V_{t+1}^{a_2}(k + j) + V_{t+1}^{a_2}(k + j + 2) = 2 \cdot V_{t+1}^{a_2}(k + j + 1). \quad (9)$$
Moreover, for all \( k \in \{0,1,\ldots,(C_N-1)\alpha\} \) and all \( j \in \{0,1,\ldots,\alpha-2\} \) we have

\[
V_{t+1}^{a_3}(k+j) + V_{t+1}^{a_3}(k+j+2) = h^*(k+j) + \sum_{i \in N} \lambda_i^*(\alpha-j) d_i \\
+ h^*(k+j+2) + \sum_{i \in N} \lambda_i^*(\alpha-(j+2)) d_i \\
= 2h^*(k+j+1) + 2 \sum_{i \in N} \lambda_i^*(\alpha-(j+1)) d_i \\
= 2V_{t+1}^{a_3}(k+j+1).
\]

(10)

By combining (8), (9) and (10), we have for all \( k \in \{0,1,\ldots,(C_N-1)\alpha\} \) and all \( j \in \{0,1,\ldots,\alpha-2\} \)

\[
V_{t+1}^a(k+j) + V_{t+1}^a(k+j+2) = V_{t+1}^{a_1}(k+j) + V_{t+1}^{a_2}(k+j) + V_{t+1}^{a_3}(k+j) \\
+ V_{t+1}^{a_1}(k+j+2) + V_{t+1}^{a_2}(k+j+2) + V_{t+1}^{a_3}(k+j+2) \\
= 2 \cdot (V_{t+1}^{a_1}(k+j+1) + V_{t+1}^{a_2}(k+j+1) + V_{t+1}^{a_3}(k+j+1)) \\
= 2 \cdot V_{t+1}^a(k+j+1),
\]

where the second equality follows from (5), (9), and (10). By the principle of mathematical induction, this completes the proof. \( \square \)

Proof of Lemma 9

**Proof**: Based on (i) and (ii) of Lemma 8, it follows directly that for all \( j \in \{0,\alpha,\ldots,\alpha \cdot C_N\} \) and all \( t \in \mathbb{N} \cup \{0\} \) it holds that

\[
V_{t+1}^a(j) = \sum_{i \in N} \left[ \lambda_i^* \min_{l \in \{0,\min\{j,\alpha\}\}} \{V_t^a(j-l) + (\alpha-l)d_i\} \right] \\
+ \sum_{i \in N} \left[ \mu_i^* \min_{l \in \{0,\min\{\alpha,C_N-j\}\}} \{V_t^a(j+l)\} \right] + h^* \cdot j
\]

By definition of the value functions \( V_0^a(j) = 0 \) for all \( j \in \{0,\alpha,\ldots,C_N \cdot \alpha\} \) and \( V_0^N(j) = 0 \) for all \( j \in \{0,1,\ldots,C_N\} \). Hence, \( V_0^a(j) = \alpha \cdot V_0^N \left( \frac{j}{\alpha} \right) \) for all \( j \in \{0,\alpha,\ldots,C_N \cdot \alpha\} \). Let \( t \in \mathbb{N} \cup \{0\} \) and assume that \( V_t^a(j) = \alpha \cdot V_t^N \left( \frac{j}{\alpha} \right) \) for all \( j \in \{0,\alpha,\ldots,C_N \cdot \alpha\} \).
Let \( j \in \{0, \alpha, \ldots, C_N \cdot \alpha\} \). Then, observe that

\[
V^\alpha_{i+1}(j) = \sum_{i \in N} \left[ \lambda^*_i \min_{l \in \{0, \min(j, \alpha)\}} \{V^\alpha_i(j - l) + (\alpha - l)d_i\} \right] \\
+ \sum_{i \in N} \left[ \mu^*_i \min_{l \in \{0, \min(\alpha, \alpha C_N - j)\}} \{V^\alpha_i(j + l)\} \right] + h^* \cdot j \\
= \sum_{i \in N} \left[ \lambda^*_i \min_{l \in \{0, \min(j, \alpha)\}} \left\{ \alpha \cdot V^N_i \left( \frac{j - l}{\alpha} \right) + (\alpha - l)d_i \right\} \right] \\
+ \sum_{i \in N} \left[ \mu^*_i \min_{l \in \{0, \min(\alpha, \alpha C_N - j)\}} \left\{ \alpha \cdot V^N_i \left( \frac{j + l}{\alpha} \right) \right\} \right] + h^* \cdot j \\
= \sum_{i \in N} \left[ \lambda^*_i \min_{z \in \{0, \min(\frac{j}{\alpha}, 1)\}} \left\{ \alpha \cdot V^N_i \left( \frac{j}{\alpha} - z \right) + \alpha \cdot (1 - z)d_i \right\} \right] \\
+ \sum_{i \in N} \left[ \mu^*_i \min_{z \in \{0, \min(\frac{j}{\alpha}, 1)\}} \left\{ \alpha \cdot V^N_i \left( \frac{j}{\alpha} + z \right) \right\} \right] + \alpha \cdot h^* \cdot \frac{j}{\alpha} \\
= \alpha \cdot \left( \sum_{i \in N} \left[ \lambda^*_i \min_{z \in \{0, \min(\frac{j}{\alpha}, 1)\}} \left\{ V^N_i \left( \frac{j}{\alpha} - z \right) + (1 - z)d_i \right\} \right] \\
+ \sum_{i \in N} \left[ \mu^*_i \min_{z \in \{0, \min(\frac{j}{\alpha}, 1)\}} \left\{ V^N_i \left( \frac{j}{\alpha} + z \right) \right\} \right] + h^* \cdot \frac{j}{\alpha} \right) \\
= \alpha \cdot V^N_{i+1} \left( \frac{j}{\alpha} \right).
\]

The first equality holds by definition. The second equality holds by the induction hypothesis. The third equality holds by introducing a new variable \( z = l/\alpha \). The fourth equality holds by the induction hypothesis (again). The fifth equality holds as \( \alpha \) can be taken outside the summations. The last equality holds by Lemma 1. By the principle of mathematical induction, this completes the proof. \( \square \)
Lemma 10. Let $f : \mathbb{N} \to \mathbb{N}$ with $f(x) + f(x + 2) \geq 2 \cdot f(x + 1)$ for all $x \in \mathbb{N}$ and $a, b \in \mathbb{N}$ with $a \leq b$, $c, d \in \{0, 2\}$ and $a + c \leq b + d$. Then, it holds that
\[
\min_{x \in \{a, a+1, \ldots, b\}} \min_{y \in \{a+c, a+c+1, \ldots, b+d\}} \{f(x) + f(y)\} \geq 2 \min_{z \in \{a+\frac{c}{2}, a+\frac{c}{2}+1, \ldots, b+\frac{d}{2}\}} \{f(z)\}.
\]

Proof of Lemma 10

Let $x, y \in \mathbb{N}$. Observe that
\[
\min_{x \in \{a, a+1, \ldots, b\}} \min_{y \in \{a+c, a+c+1, \ldots, b+d\}} \{f(x) + f(y)\} \\
\geq \min \left[ \left\{ f(z) + f(z + 1) | z = a + \frac{c}{2}, \ldots, b + \frac{d}{2} - 1 \right\} \cup \left\{ 2f(z) | z = a + \frac{c}{2}, \ldots, b + \frac{d}{2} \right\} \right] \\
\geq \min \left\{ 2f(z) \mid z = a + \frac{c}{2}, \ldots, b + \frac{d}{2} \right\} \\
= 2 \min_{z \in \{a+\frac{c}{2}, a+\frac{c}{2}+1, \ldots, b+\frac{d}{2}\}} \{f(z)\}.
\]

The first inequality holds by midpoint convexity, i.e., the fact that $f(x) + f(y) \geq f \left( \left\lfloor \frac{x+y}{2} \right\rfloor \right) + f \left( \left\lfloor \frac{x+y}{2} \right\rfloor \right)$, which holds true as long as $f$ is convex (see e.g. [Murato, 2006, Thm. 1.2]). The second inequality holds as $\min\{2a, a+b, 2b\} = \min\{2a, 2b\}$ for any $a, b \in \mathbb{R}$. The last equality follows by taking (constant) 2 outside the minimum.