Small $x$ QCD

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Abstract — A review of some theoretical aspects of small $x$ QCD physics is given, with a particular emphasis to the relation between the BFKL and the colour dipole approaches. The nonlinear evolution equations one may construct, as a better approximation beyond the linear analysis, are discussed together with their relation to a possible saturation regime.

1 Introduction

Hadronic interactions at very high energies have been shown to be driven by states with very high partonic densities. The first phenomenological evidence for such states was obtained by the study of DIS experiments at HERA where, for very small $x$, very high gluon densities were found in the partonic description of the proton target.

The standard theoretical framework employed to study the parton densities and structure functions of the proton in DIS is based on the QCD collinear factorization, where the cross sections are decomposed in coefficient functions (hard, perturbative) and parton densities (soft, nonperturbative), the latter evolving according to the DGLAP equations [1]. These linear equations already provide the ingredients to a qualitative understanding of the raise of the gluon density at small $x$, but the domain of applicability is restricted by the assumptions adopted in their derivation.

Essentially they are based on the requirement of large enough $Q^2$ values in order to neglect any kind of higher twist corrections and the associated perturbative resummation is organized in powers $\alpha_s^n(\alpha_s \log Q^2)^n$, where the diagrams corresponding to $k$ fixed and all $n$ generate the $N^kL$ logarithmic approximation. At small $x$ one may take the DLA approximation of the DGLAP evolution to pick up the dominant contribution to the gluon density growth, corresponding to the contribution of the $(\alpha_s \log Q^2 \log 1/x)^n$ terms.

But to really extract the leading contribution at small $x$, if $Q^2$ is not extremely large, one may choose to resum all the $(\alpha_s \log 1/x)^n$ contributions. This has to be associated to the “high energy factorization” which decomposes the cross sections (and amplitudes) into impact factors integrated with Green’s functions in the transverse space. In particular the target impact factor integrated with the Green’s function corresponds to the unintegrated (in the transverse space) gluon density. In this framework also a set of higher twist contributions is taken into account.

This is known as the BFKL approach (LL) [2], which, as a matter of fact, predicts a very strong raise of the gluon density, stronger than what is shown by the experimental analysis in the $x$-window studied at HERA. There are also two theoretical problems in the approach: the appearance of an infrared instability related to a diffusion with the rapidity evolution and the lack of running coupling description within the LL approximation.

Successively an independent approach, equivalent to the LL BFKL, named “colour dipole picture”, has also been developed. It is useful to understand certain aspects of small $x$ physics in a more economical way.

Further studies in the BFKL approach have been devoted to the NLL case, where the resummation is extended to include contributions of order $\alpha_s(\alpha_s \log 1/x)^n$ [3]. The detailed analysis of these NLL contributions has shown that there is a sort of instability due to I.R. singularities which do not agree with the renormalization group flow. An established cure is to include a set of corrections (of higher order with respect to the NLL scheme) which are restoring the correct behavior. Within this context the growth of the gluon density and the cross sections $\sigma$ is characterized by an intercept $\lambda \simeq 0.2 (\sigma \approx x^{-\lambda})$. The other piece of information needed in order to compute the cross sections, the perturbative impact factors at NLL approximation, are partially known. The Jet Verteces [4] have been derived, while the virtual photon impact factor [5] is in a advanced status of computation.
There is also available a montecarlo approach (CASCADE) where the small $x$ regime is studied within the so called CCFM approach [6], which keeps ingredients of both the DGLAP and BFKL worlds in the LL approximation.

The main theoretical problem in common to the above described approaches is that only a linear evolution is taken into account with the resummation. This problem reflects the fact that unitarity is strongly violated, in particular the Froissart bound.

In the perturbative approach the set of diagrams associated to iterated effects of parton fusion or also of multiscattering (QM shadowing) are completely neglected.

From the experimental point of view a new information has been found, namely the observation that the DIS scattering data show, within a very wide range of small $x$ and $Q^2$, a scaling low [7]. This regime is characterized by the appearance of a hard scale (called saturation scale) and this fact is very interesting since it may provide a perturbative scale in the high density regime at small $s$. Moreover this scaling low, which exists also in the region where a linear evolution is considered a reasonable approximation, is implied if a saturation regime exists and provide a boundary condition to the linear evolution equations.

It is in this direction that recently the small $x$ scientific community has drawn his attention. In particular the resummation of particular kind of parton fusion effects has led to the formulation of non linear evolution equations. Some ideas, previously pioneered and developed in the DLA approximation [8], have been developed in the colour dipole picture (BK) [9] and in the LL BFKL framework [10][11].

The BK equation, which is the nonlinear equation for the colour dipole evolution determined in the large $N_c$ leading approximation, has been extensively studied both numerically [10] and, more recently, analytically [12], showing both the features of scaling and saturation as a universal phenomenon. The extension of this equation in the next-to-leading $1/N_c$ approximation has been given in [11] and one expects the same qualitative behavior of the solution.

Still these approaches are only partially addressing the requirements of unitarity and are therefore theoretically not very satisfactory, since, beyond the fan structure which are typically generating the nonlinear behavior, also some loop contributions are important.

The interest in the high density regime of QCD at small $x$ has been raising in the last years also in connection with the heavy-ion physics. A saturated regime, also called Color Glass Condensate [13], is seen as a possible prelude to a thermalized phase, the Quark Gluon Plasma, which may exist in specific cases for very short periods and is currently actively searched in the experiments at RHIC.

2 Linear Evolution: BFKL and Colour Dipole Physics

Let us consider an high energy scattering process in the Regge limit and within the perturbative regime ($s \gg -t \gg \Lambda_{QCD}$ in terms of Mandelstam variables) in order to very shortly illustrate some features of the high energy factorization and define a link between the LL BFKL and the colour dipole approaches [11].

The main result of the BFKL approach is that the leading contribution to the cross section can be written as an integral in the transverse space

$$
\sigma \simeq \int d\mu_T \Phi_1(y) \Phi_2, \quad \frac{\partial}{\partial y} G = \delta + \frac{\alpha_s}{2} K G, \quad K = \omega_1 + \omega_2 + V,
$$

where $y$ is the rapidity which plays the role of an evolution parameter and $d\mu_T$ is the measure in the transverse space and $K$ is the BFKL kernel. The perturbative kernel $K$ is getting contributions from virtual ($\omega_i$) and real ($V$) (is related to the square of the Lipatov vertex for gluon production in the LL approximation) with no I.R. singularities resulting from the sum. The kernel may be seen as defining a quantum mechanical Hamiltonian for two reggeized gluons, with the $\omega_i$ playing the role of the kinetic term and $V$ the interaction potential between them. The Green’s function $G$ defined above may be written when a spectral basis of the operator $K$ is known. Usually in the literature the BFKL equation is the eigenvalue equation for $K$. The interacting two reggeized gluon system in a colour singlet state is known as the perturbative BFKL Pomeron and the eigenvalues of the kernel are related to its intercept. Let us note that the reggeized gluons are defined in a selfconsistent way looking at the BFKL kernel properties in the colour octet state in the $t$-channel.

The impact factors of colourless external particles possess the nice property of being zero when the transverse momentum of one of the two exchanged gluons is zero. This property is related to the gauge invariance and let us have the freedom to look for different possible representations for the space of functions of the impact factors.
and for the domain of the operator $K$. In particular one may consider the M"obius representation when restricts to a space of functions $f(\rho_1, \rho_2)$ such that $f(\rho, \rho) = 0$. In such a case the BFKL equation is invariant under the M"obius, conformal, transformations.

The properties of the BFKL kernel in the M"obius representations are very interesting. The most important is the holomorphic separability, which means that in the coordinate space, when defining a complex $\rho = \rho_x + i\rho_y$, the kernel can be decomposed in a sum $K = h + h^*$, where $h = h(\rho_1)$.

Starting from the Feynman diagrams derivation, in momentum space, one obtains the following form for the BFKL kernel in terms of pseudodifferential operators acting on functions with complex variables:

$$-K = H_{12} = \ln |p_1|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_1|^2 p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_2|^2 p_1^* p_2 - 4\Psi(1).$$  (2)

There are different possible ways to write this operator and the alternative forms may differ depending on the space of functions considered. Among these forms there is one, for function related to the M"obius representation, defined by a purely integral operator, which coincides with the kernel of the evolution of the colour dipoles in the large $N_c$ limit [15]:

$$KN = \int \frac{d^2 \rho_3}{\pi} \frac{|\rho_1|^2}{|\rho_3|^2 |\rho_2|^2} \left( N(\rho_1, \rho_3) + N(\rho_3, \rho_2) - N(\rho_1, \rho_2) \right).$$  (3)

The cross section in the colour dipole picture is written as

$$\sigma \simeq \int d^2 p_1 d^2 p_2 \int_0^1 dx |\psi(\rho_1, \rho_2; x)|^2 N(\rho_1, \rho_2; y),$$  (4)

and the relation with the form in Eq. [1] can be understood on observing how to relate the impact factor $\Phi_1$ to the particle wave functions, for example for a virtual photon [11]. One has $\Phi_1 = \int dx|\psi|^2 \theta_{1R}$, where $\theta_{1R}$ are the phase factors which describes the four ways the two gluons attach to the $q\bar{q}$ pair. The factor $\theta_{1R} \to 0$ when one of the two gluons has zero 2-d transverse momentum. Nothing changes therefore adding terms which are distributions having zero-support on that region. Using such a gauge freedom one may write, in terms of an operator which projects onto the M"obius space of functions, $\theta^{UV}$, such that $\theta^{UV} f(\rho_1, \rho_2) = f(\rho_1, \rho_2) - 1/2 f(\rho_1, \rho_1) - 1/2 f(\rho_2, \rho_2)$:

$$\sigma \simeq \Phi_1 \otimes G \otimes \phi_2 = \int dx|\psi|^2 \theta_{1R} \otimes G \otimes \Phi_2 = \int dx|\psi|^2 \otimes \theta^{UV} G \otimes \Phi_2 = \int dx|\psi|^2 \otimes N.$$  (5)

We have already reminded that the linear approximations such as BFKL are violating unitarity. One may study in the LL approximation the linear evolution of more complicated systems, with many reggeized gluons (more than 2) in the $t$-channel as a first step towards the recovery of unitarity [16] [17].

The homogeneous equation (BKP) for $n$-gluon colour singlet states are governed by a kernel $K_n$ which is a sum of 2-gluon kernels. Again the Green’s function will be deigned by

$$\frac{\partial}{\partial y} G_n = \delta + \frac{\alpha_s}{2} K_n G_n.$$  (6)

Even if the n-gluon impact factors have the property of vanishing for a zero gluon momentum, the gauge freedom does not allow to restrict all the solution to a space of function which are null when two coordinates coincide. And, infact, there exist the leading intercept Odderon solution [13] which does not posseses this property.

Anyway there exist interesting states which belong to a (generalized) M"obius representation which are dynamically compatible with the BFKL evolution and which will be considered in the following step towards the unitarization, in section 4. But before proceeding in the illustration of some other interesting phenomena appearing in the small $x$ QCD dynamics, we shall remind some recent facts related to the strong interation, observed in the DIS data measured at HERA.

3 Scaling in DIS Data

Some interesting features have been recently found in the HERA DIS data for the total cross section $\gamma^* p$ [7]. Infact it has been observed that

$$\sigma_{\gamma^* p}(Q^2, x) \approx \sigma \left( \frac{Q^2}{Q^2_s(x)} \right).$$  (7)
in the region \( x \leq 10^{-2} \) and \( 0.045 \leq Q^2 \leq 450 \). The specific condition for this scaling to be realized is given by the saturation scale dependence on \( x \): \( Q_s^2 \sim Q_0^2 \left( \frac{x}{x_0} \right)^\lambda \).

Therefore there exist a region where the general dependence on the two kinematical variables reduces into a dependence on one new variable, resulting from a combination of the two. Analysing the \( \sigma_{\gamma p} \) data with respect to the \( \tau = \frac{Q^2}{Q_s^2(x)} \) variable has indeed led to the plot of Fig. 1.

Figure 1: The \( \sigma_{\gamma p} \) HERA data in the kinematic range \( x \leq 10^{-2} \) and \( 0.045 \leq Q^2/\text{GeV}^2 \leq 450 \) are plotted with respect to \( \tau = \frac{Q^2}{Q_s^2(x)} \).

A second interesting observation is that in the small \( x \) region \( Q_s \) constitutes an hard scale, since typically the parameters are such that \( Q_0 = 1\text{GeV} \) for \( x_0 \sim 3 \times 10^{-4} \) and \( \lambda \sim 0.3 \).

This fact has been immediately considered associated to a possible saturation in the gluon density for the kinematical region wherein \( Q^2 \leq Q_s^2(x) \) but it has also been recognized that the scaling behavior extends beyond the saturation region, up to \( Q^2 \ll Q_s^1(x)/\Lambda_{QCD}^2 \). In this extra domain of scaling the linear evolution equations can be used to describe the gluon density, and it has been shown that just a saturated boundary condition has to be supplied to the linear equations to recover the scaling behavior.

The picture one may get for the different kinematical regions is therefore described in Fig. 2 where the \( \log Q^2 - \log \frac{1}{x} \) plane is presented.

Figure 2: Dynamical regimes in different kinematical regions of the \( \log Q^2 - \log \frac{1}{x} \) plane.

4 Non Linear Evolution: Leading Large \( N_c \) and a Step Beyond

Let us now consider the linear evolution of systems with different number of gluons in the \( t \)-channel. Starting from the elementary vertices which define the transition between states of different number of gluons in the \( t \)-channel in
the LL approximation \[^{16}\] , it is possible to organize a hierarchy of an infinite number of coupled linear equations.

In particular these set of equations has been written and analyzed explicitly in a systematic way for the systems of up to 6 gluons \[^{19}\ 20\] . In the 4 gluon system one may extract an effective vertex \(V_{2 \rightarrow 4}\) which defines the transition between 2 and 4 reggeized gluons. Any solution can be decomposed \[^{19}\] in the sum of two terms, \(D_R^4\) and \(D_I^4\), the latter being related to such a transition, followed by an evolution governed by \(G_4\). The former term is governed by the gluon reggeization and different choices can be made in the large \(N_c\) limit \[^{21}\] . In the large \(N_c\) limit, when only colour planar diagrams dominates, the effective vertex \(V_{2 \rightarrow 4}\) becomes a simpler vertex which describes the splitting of a BFKL Pomeron into two.

In the 6 gluon system the solution of the coupled equations can be decomposed in a sum of terms, one of them, \(D_R^6\) is again related to gluon reggeization, another contains a new effective vertex \(V_{2 \rightarrow 6}\), related to a transition where the colour structure of two Odderon is present, and another may be seen as an iteration of two splitting of the kind \(V_{2 \rightarrow 4}\) in sequence in rapidity, each followed by the BFKL evolution.

The diagrams with successive splitting in rapidity, denoted fan diagrams, have been defined in the past in a different context \[^{8}\] . Therefore one is tempted to consider all the diagrams, where splitting in sequence are present, and to define an object which describes a full resummation. This approach has led to derive in a special case \[^{10}\] the BK equation, previously obtained by the resummation of colour dipole splitting in nuclear targets \[^{9}\] in the limit \(N_c \rightarrow \infty\). More recently, the fan resummation has been reconsidered in order to investigate a general relation between the BFKL and the dipole approaches and to define an extension of the non linear evolution equation beyond the leading large \(N_c\) approximation \[^{11}\].

As usual, the non linear evolution appears when one is insisting in defining an approximation of the full system, which is governed by linear equations, in terms of a single smaller object and neglecting all kind of higher correlations.

As a first step the leading fan structure is extracted on considering the substitution \(G_4 \rightarrow G_2 \otimes G_2\) so that one may write

\[
\frac{\partial}{\partial y} \Psi = \frac{\bar{\alpha}_s}{2} K \Psi - \bar{\alpha}_s^2 V \otimes \Psi \Psi ,
\]

(8)

which resums the fan diagrams of Fig. 3. With \(V\) we have denoted the effective transition vertex.

![Figure 3](image-url)

**Figure 3:** Fan diagrams which are resummed, with the coupling of the gluons to the quark lines understood in all possible ways.

Since the effective vertex correctly possesses the property, as the impact factors, of being zero when any gluon line carries zero momentum, one may perform a subtraction allowed by the gauge freedom and so greatly simplify the vertex. To this end we can rewrite Eq. (5) in the Möbius space of functions and choose the convenient rescaling

\[
N_{\rho_1,\rho_2 \to \rho_3} = 8\pi \alpha_s \left( \Psi(\rho_1, \rho_2) - \frac{1}{2} \Psi(\rho_1, \rho_1) - \frac{1}{2} \Psi(\rho_2, \rho_2) \right),
\]

(9)

in order to obtain the BK equation

\[
\frac{d}{dy} N_{x,y} = \bar{\alpha}_s \int \frac{d^2 z}{2\pi} \frac{|x - y|^2}{|x - z|^2 |y - z|^2} \left( N_{x,z} + N_{y,z} - N_{x,y} - N_{x,z}N_{z,y} \right).
\]

(10)

This equation has been derived in the large \(N_c\) limit approximation. To go one step beyond and take into account the next-to-leading \(1/N_c\) corrections, one has to consider two kind of contributions: the one coming from the
nonplanar term, subleading in 1/N_c, in the effective vertex and the one coming from the lowest 1/N_c correction to the 4-gluon Green’s function, which at leading order is just the product of 2-gluon Green’s functions, so that one introduces and studies \( N_4(\rho_1, \rho_2; \rho_3, \rho_4) = N_{p_1,p_2}N_{p_3,p_4} + \Delta N_4(\rho_1, \rho_2; \rho_3, \rho_4) \), which evolves according to \( \frac{d}{dy} H_4 = \frac{\alpha_s}{N_c} K_4 N_4 \).

Therefore it is possible to write a system of two coupled equations [11]:

\[
\frac{d}{dy} N_{x,y} = \alpha_s \int \frac{d^2 z}{2\pi} \frac{|x-y|^2}{|x-z|^2 |y-z|^2} \left[ N_{x,z} + N_{z,y} - N_{x,y} - N_{x,z}N_{z,y} \right] \\
- \Delta N_4(x, z; y, z) - \frac{1}{2} \frac{1}{N_c - 1} (N_{x,z} + N_{z,y} - N_{x,y})^2 \right]
\]

\[
\frac{d}{dy} \Delta N_4(\rho_1, \rho_2; \rho_3, \rho_4) = \frac{\alpha_s}{2(N_c^2 - 1)} (K_{12} + K_{34}) (N_{\rho_1,\rho_3}N_{\rho_2,\rho_4} + N_{\rho_1,\rho_4}N_{\rho_2,\rho_3}) \\
+ \frac{\alpha_s}{2} (K_{12} + K_{34}) \Delta N_4(\rho_1, \rho_2; \rho_3, \rho_4). \tag{11}
\]

The next-to-leading 1/N_c corrections may provide a quantitative difference from the leading description, but it is believed that the qualitative picture will not change. Of great importance would also be to investigate the NLL corrections (in log 1/x) to the above equations.

The simpler BK equation (see Eq. (10)) has been already analyzed numerically showing the saturation phenomenon [10]. For an analytical analysis it may be written, in a simbolic notation, in momentum space as \( \frac{d}{dy} \hat{N} = \alpha_s(\partial_{in}) \hat{N} - \alpha_s \hat{N}^2 \) and recently it has been studied in the quadratic approximation of the complicated \( \chi(\partial_{in}) \) pseudodifferential operator.

Infact in this way [12] it reduces to the Fisher and Kolmogorov-Petrovsky-Piscounov (KPP) equation which, for reasonable boundary condition (in agreement with colour transparency) admits solutions which are traveling waves, i.e. functions which depends only on a combination of the two kinematical variables \( Q^2 \) and \( x \) (or \( y \sim \log(1/x) \)), for large enough rapidities. They are characterized by a universal behavior described by Eq. (7) and by a general pattern of corrections.

5 Conclusions

Small x QCD is showing, in certain kinematical regions, the formation of a dynamical regime which can be partially studied with perturbative field theory tools and was not expected a priory. It is characterized by high parton densities and it has been shown to lead to dynamical interactions which may be described by non linear equations if one insists to keep simple objects (gluon densities, perturbative Pomerons, or colour dipoles) to describe the data. The picture seems to be favoured by the DIS data which presents an interesting scaling behavior in the small x region.

As a general comment it is worth to note that most of the theoretical and analytical work in the small x QCD has to be seen as an effort to better understand the QCD and the strong interactions. A large fraction of the QCD community is tempted to use what is already known as a tool in the challenge to understand the Standard Model (SM), the mass generation mechanism (Higgs?) and the physics beyond the SM. Nonetheless QCD still constitutes a challenge by itself since many dynamical aspects still wait to be understood and therefore deserves further studies.

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