The mixing time of switch Markov chains: a unified approach

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Abstract

Since 1997 a considerable effort has been spent to study the mixing time of switch Markov chains on the realizations of graphic degree sequences of simple graphs. Several results were proved on rapidly mixing Markov chains on unconstrained, bipartite, and directed sequences, using different mechanisms.

The aim of this paper is to unify these approaches. We will illustrate the strength of the unified method by showing that on any $P$-stable family of unconstrained/bipartite/directed degree sequences the switch Markov chain is rapidly mixing. This is a common generalization of every known result that shows the rapid mixing nature of the switch Markov chain on a region of degree sequences. Among the applications of this general result is an almost uniform sampler for power-law and heavy-tailed degree sequences. Another application shows that the switch Markov chain on the degree sequence of an Erdős-Rényi random graph $G(n, p)$ is asymptotically almost surely rapidly mixing if $p$ is bounded away from 0 and 1 by at least $\frac{5 \log n}{n - 1}$.

Keywords: degree sequences, realizations, switch Markov chain, rapidly mixing, MCMC, Sinclair’s multi-commodity flow method, $P$-stability, strong stability, power-law distribution

1. Introduction

An important problem in network science is to algorithmically construct typical instances of networks with predefined properties. In particular, special attention has been devoted to

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sampling simple graphs with a given degree sequence. In this paper only graphs without parallel edges and loops are considered and we restrict our study to degree sequences which have at least one realization (graphic). A realization of a degree sequence \( \mathbf{d} \in \mathbb{N}^n \) is a graph on the vertex set \([n] = \{1, \ldots, n\}\) whose degree sequence \( d(G) \) is equal to \( \mathbf{d} \) (i.e., the graphs are labeled). Sometimes we use a different labeling of the vertex set \( V = \{v_1, \ldots, v_n\} \), where \( V \ni v_i \leftrightarrow i \in [n] \). We will use the more general notation \([a, b]\) and \([a, b)\) for the closed-closed and closed-open intervals of integers between \( a \) and \( b \). We study the three most common degree sequence types: bipartite degree sequences, directed degree sequences and the usual degree sequences which we call unconstrained degree sequences (these are the degree sequences of simple graphs).

In 1997 Kannan, Tetali, and Vempala ([30]) proposed the use of the switch Markov chain (also known as the swap chain [33]) for uniformly sampling realizations of a degree sequence. For all three degree sequence types, the switch Markov chain can be thought of as the Markov chain of smallest possible modifications. To illustrate this, we give an informal description of the switch Markov chain on unconstrained degree sequences. If \( G_1, G_2 \) are two realizations of the same unconstrained degree sequence, it is easy to see that the minimum size of the symmetric difference \( E(G_1) \triangle E(G_2) \) is four. We say that \( G_1 \) and \( G_2 \) differ by a switch if this symmetric difference is exactly four. The states of the switch Markov chain are the realizations of the degree sequence and the probability of going from realization \( G_1 \) to \( G_2 \) is nonzero if and only if they differ by a switch (and this nonzero quantity is independent of \( G_1 \) and \( G_2 \)). For the precise definition of this chain, and for the definition of the chains for other degree sequence types, we refer the reader to Section 3 (unconstrained and bipartite) and Section 6 (directed).

Because the symmetric difference of graphs is ubiquitous in this paper, we often shorten the notation \( E(X) \triangle E(Y) \) to \( X \triangle Y \).

The following conjecture has been named after Kannan, Tetali, and Vempala, in recognition of their pioneering work.

**Conjecture 1.1** (the KTV conjecture). The switch Markov chain is rapidly mixing for any bipartite, directed, or unconstrained degree sequence.

To give some context to the Conjecture, we say that a Markov chain is rapidly mixing if the distribution on the state space is close in \( \ell_1 \) norm to the unique stationary distribution after \( \text{poly}(n) \) steps, where \( n \) is the size of the input. In our case, \( n \) is the number of vertices (or the number of entries in the degree sequence). This property means that sampling from the state space with the stationary distribution is a more or less tractable problem, even if the state space has exponential size.

It is not uncommon that uniformly randomly applied, small local modifications of combinatorial objects result in rapid mixing. This is the case for solutions of the 0–1 knapsack problem [34] (Morris and Sinclair, 2004), for the union of perfect and almost perfect matchings of a graph [24] (Jerrum and Sinclair, 1989), and two-rowed contingency tables [8] (Dyer and Greenhill, 2000), for example. In all of these cases, applying the smallest possible modifications of the respective combinatorial objects randomly, result in rapid mixing of the corresponding Markov chain.

Although Conjecture 1.1 is still open, there is a series of results that prove the rapid mixing of the switch Markov chain on various special degree sequence classes. We summarize
these results in a very compact way in Table 1 without presenting the sometimes lengthy definitions of the special classes, which can be found in the references provided. Some rapid mixing results on directed degree sequences work with directed graphs [20] (Greenhill, 2011), while some work in the bipartite representation with a forbidden perfect matching [12] (Erdős, Kiss, Miklós and Soukup, 2015). Since we use the bipartite representation in the present paper, we will not discuss directed degree sequences until Section 6.

| unconstrained deg. seq. | bipartite deg. seq. | directed deg. seq. |
|-------------------------|---------------------|--------------------|
| regular [6]             | (half-)regular [33] | regular [20]       |
| almost half regular [12]|                     |                    |
| \( \Delta \leq \frac{1}{3} \sqrt{2m} \) [22] | \( \Delta \leq \frac{1}{\sqrt{2}} \sqrt{m} \) [15] | \( \Delta < \frac{1}{\sqrt{2}} \sqrt{m - 4} \) [15] |
| power-law density-bound, \( \gamma > 2.5 \) [22] | | |
| \((\Delta - \delta + 1)^2 \leq \)  | \((\Delta - \delta)^2 \leq \)  | similar to bipartite case |
| \( \leq 4 \cdot \delta(n - \Delta - 1) \) | \( \leq \delta(\frac{n}{2} - \Delta) \) [14, 15] | [14, 15] |
| proof in [2, 3] | (or Corollary 18 in [2, 3]) | |
| bipartite Erdős-Rényi [14, 15] | similar to bipartite case | |
| p, \( 1 - p \geq 4 \sqrt{\frac{2 \log n}{n}} \) | | [14, 15] |
| strongly stable degree sequence classes [2, 3] | | |

Table 1: Some classes of degree sequences for which the rapid mixing of the switch Markov chain is was already known when the first draft of this paper was published on arXiv. Here \( \Delta \) and \( \delta \) denote the maximum and minimum degrees, respectively. Half of the sum of the degrees is \( m \), and \( n \) is the number of vertices. The notation is similar for bipartite and directed degree sequences. Some technical conditions are omitted.

Notice, that some, but not all of the results came in pairs for unconstrained and bipartite (directed) degree sequences. The reason for this discrepancy is the following: while both set of results are based on Sinclair’s multicommodity flow method, one of them has to deal with special circuits (one of the vertices may be visited at most twice) instead of just cycles in the decomposition of symmetric differences of two realizations of a degree sequence. The main goal of this paper to remedy this discrepancy between the machineries used for the bipartite and unconstrained degree sequences by decomposing into circuits where each vertex is visited at most twice. Along the way we also give new, much more transparent proofs for the main results in [33] (Miklós, Erdős and Soukup, 2013).

Greenhill and Sfragara suggested exploring the connection between the mixing rate of the switch Markov chain and stable degree sequences [22, Section 1.1] (Greenhill and Sfragara, 2018). The first notion of stability for unconstrained and bipartite degree sequences was \( P\text{-stability} \), introduced by Jerrum and Sinclair [25]. An unconstrained degree sequence on \( n \) vertices, usually denoted by \( d \), is an element of \([n - 1]^n\). The set of graphs with the degree
sequence \( \mathbf{d} \) is denoted by \( G(\mathbf{d}) \). To make the following and later stability definitions more readable, let \( G(\mathbf{d}) := \emptyset \) for a non-graphic degree sequence \( \mathbf{d} \).

**Definition 1.2.** Let \( \mathcal{D} \) be an infinite set of unconstrained degree sequences. We say that \( \mathcal{D} \) is \( P \)-stable, if there exists a polynomial \( p \in \mathbb{R}[x] \) such that for any \( n \in \mathbb{N} \) and any degree sequence \( \mathbf{d} \in \mathcal{D} \) on \( n \) vertices we have

\[
| G(\mathbf{d}) \cup \bigcup_{x,y \in [n], x \neq y} G(\mathbf{d} + 1_x + 1_y) | \leq p(n) \cdot |G(\mathbf{d})|,
\]

where \( 1_x \) is the \( x \)th unit vector.

Without proof we state, that the notion of \( P \)-stability does not change even if we require

\[
| \{ G \mid G \in G(\mathbf{d}'), \ d' \in \mathbb{N}^n, \ell_1(\mathbf{d}, \mathbf{d}') \leq 2 \} | \leq p(n) \cdot |G(\mathbf{d})|.
\]

With a bit of care, Definition 1.2 generalizes to bipartite and directed degree sequences: we require that the realizations of perturbed degree sequences are also bipartite or directed, respectively.

Informally, a class of unconstrained degree sequences is \( P \)-stable if making slight perturbations to the degree sequence from a \( P \)-stable class cannot increase the number of realizations too much. Thus \( P \)-stability is a property of the degree sequences directly, in the sense that it only cares about the number of their (perturbed) realizations.

**Theorem 1.3** (proved in Section 8). The switch Markov chain is rapidly mixing on \( P \)-stable unconstrained, bipartite, and directed degree sequence classes.

For the sake of transparency, we would like to mention that we were aware that a quirky proof of Theorem 1.3 for the bipartite and directed models can be read out already from [14, 15] (Erdős, Mezei, Miklós and Soltész, 2018). However, that paper used a complicated and opaque technique for describing the switch sequence (Section 5.5), which cannot be applied to unconstrained degree sequences. Instead of publishing an immature proof for the bipartite and directed cases, we opted to find a unified proof which naturally accommodates unconstrained degree sequences too.

**Comparison with previous results.** Amanatidis and Kleer [2, 3] introduced and studied the concept of strongly stable degree sequences. In contrast with \( P \)-stability, strong stability requires that for any realization of the perturbed degree sequence there exists a realization of the original degree sequence such that the two graphs are not too different. The precise definition of strong stability is given at the beginning of Section 9. In [2, 3], it is shown that strongly stable degree sequences are \( P \)-stable. It is not known whether these concepts of stability are equivalent.

**Theorem** ([2, 3], Amanatidis and Kleer, 2020). The switch Markov chain is rapidly mixing on strongly stable unconstrained and bipartite degree sequence classes.

Since [2] appeared, it has become a commonly held belief that strong stability and \( P \)-stability are equivalent. It is mentioned in [2] that the bipartite case does not immediately imply the directed one, because their proof still needs some work to accommodate forbidden
edge sets. Whether the conjecture on the equivalence of the two notions of stability holds or not, Theorem 1.3 is a strictly stronger theorem than the one above.

The main reason we could unify the proofs for the unconstrained, bipartite, and directed models is because of a technical novelty introduced in this paper: extending the $T$-operator of [33] (Miklós, Erdős and Soukup, 2013) from bipartite to unconstrained graphs. The atoms of a decomposition generated by the $T$-operator are so-called primitive circuits (see Definition 2.4), along which we recursively construct the multi-commodity flow required by the Sinclair method. The processing of the primitive circuits via Algorithm 2.1 is in turn an extension of the SWEEP algorithm of [33]. These refinements require a fairly extensive and detailed analysis of the structures of the realizations under study. In exchange, the unified framework in which we prove Theorem 1.3 allows us to treat the three models with minimal branching in the proof.

There are two interesting direct consequences of Theorem 1.3 concerning popular unconstrained random graph models. It turns out that asymptotically almost surely, the degree sequence of an Erdős-Rényi random graph $G(n, p)$ is $P$-stable if $p$ is bounded away from 0 and 1 by $\frac{5 \log n}{n - 1}$, see Corollary 9.6. Gao and Greenhill [17] show that power-law distribution-bounded degree sequences with parameter $\gamma > 2$ are $P$-stable. Consequently, Theorem 1.3 implies that the switch Markov chain is rapidly mixing on these degree sequences: see Section 9.2. This gives the first formal verification of the validity of generating random power-law (distribution-bounded) graphs via the switch Markov chain [28] (Jia, and Barabási, 2013) and [39] (Yan, Vétes, Towlson, Chew, Walker, Schafer and Barabási, 2017).

The proof of Theorem 1.3 relies on Sinclair’s multicommodity flow method (Section 3), which can be described informally in the case of the switch Markov chain as follows. Suppose that the chain has $N$ states, and let $G$ be the graph underlying the Markov chain: the vertex set of $G$ is the state space of the chain (in our applications, the states are realizations of a given degree sequence), and two states are adjacent in $G$ if the transition probability between them is non-zero in the chain. Sinclair’s multicommodity flow method ensures the rapid mixing of the chain if we can design a multicommodity flow on $G$ which transfers a unit amount of commodities between each pair of vertices (different commodities for different pairs), and no more than $N \cdot \text{poly}(\log(N))$ amount of commodities go through any vertex (no vertex is overloaded). Hence most of the present paper is devoted to the design of a flow and the proof that it does not overload any vertex.

1.1. Some related Markov chain approaches

In the literature there is at least one other Markov chain application where the applied operation is the switch. This studies binary contingency tables with fixed marginals. Such tables can be considered as bipartite graphs with fixed degree sequences.

Switch Markov chain on perfect matchings (or Diaconis chain). Diaconis, Graham and Holmes in 2001 ([7]) considered applications of (0,1)-permanents to problems in statistics. The permanent is equal to the number of the perfect matchings in the corresponding bipartite graph. They study the switch Markov chain approach on the perfect matchings: the state space is the set of perfect matchings of the fixed bipartite graphs and the transitions are generated by switches. It is easy to see that this Markov chain is not irreducible.
in general. For example, the hexagon has two distinct perfect matchings (every second edge from the cycle), but no switch can be applied to it. In [7] a structural constraint was imposed on the bipartite graph making the chain irreducible, and the authors conjectured that the switch Markov chain for perfect matchings is rapidly mixing on that graph class.

Dyer, Jerrum and Müller in 2017 ([9]) showed that the switch Markov chain for perfect matchings is irreducible if and only if the bipartite graph is chordal. However, as it turned out, the chain is not rapidly mixing on all chordal bipartite graphs. The largest hereditary subclass of the chordal balanced bipartite graphs (so called monotone balanced bipartite graphs) for which the switch Markov chain is rapidly mixing was also determined in [9]. The proof is based on Sinclair’s multicommodity flow method.

**Applications of the switch Markov chain in practice.** The switch Markov chain is often used in everyday practice. In social sciences or in ecology it is standard practice to generate an initial example graph with some required properties (for example, the degree sequence shall obey a power-law, see Section 9.2) and perturb the initial graph with a (not too long) series of switch operations. While there is no theoretical guarantee that the resulting graph is much “closer” to a random one than the initial graph, in practice, the resulting graph is sufficiently random.

**Sampling binary contingency tables with simulated annealing.** Returning to binary contingency tables with arbitrary but fixed marginals, the first problem is to count and randomly sample them for statistical purposes. As it can be known from Table 1, we cannot do that with confidence using the switch Markov chain. Jerrum, Sinclair and Vigoda [26] attacked the problem with simulated annealing in 2004. Their approach was improved by Bezáková, Bhatnagar and Vigoda in 2006 ([5]), who used a greedy starting point and subsequently perturbed it with a moderate number of switches to obtain a faster running time for bipartite graphs. However, Štefankovič, Vigoda and Wilmes [38] showed that there is no weighting scheme for which the Jerrum-Sinclair-Vigoda chain mixes rapidly in general. It is not known whether the switch chain rapidly mixes over the set of all graphic degree sequences.

**1.2. Structure of the paper**

The remainder of the paper is structured as follows. In Section 2 we give a slightly more detailed description of the flow and we present the necessary graph theoretic tools which we will use to construct paths that will form the flow. In Section 3 we give the formal definition of Sinclair’s multicommodity flow method and its simplified version that is tailored to our needs. In Section 4 we provide a high level description of the multicommodity flow. In Section 5 we introduce the $T$-operator and complete the description of the multicommodity flow. In Section 6 we define the switch Markov chain for directed degree sequences. Before turning into the home stretch, an auxiliary structure that tracks the defined flow is analyzed in Section 7. In Section 8 we finally prove Theorem 1.3 and we also provide the necessary modifications to deal with bipartite and directed degree sequences. Lastly, we describe the known $P$-stable regions of degree sequences in Section 9 and present the connections between Theorem 1.3 and the aforementioned popular graph models.
2. Definitions and preliminaries, the structure of the sets of realizations

Let us recall some well known notions and notations. Let \( d = (d(v_1), \ldots, d(v_n)) \) denote a unconstrained degree sequence on vertex set \( V = [n] \) and let

\[
D = ((d(u_1), \ldots, d(u_{n_1})), (d(v_1), \ldots, d(v_{n_2})))
\]

denote a bipartite degree sequence on the bipartition \((U, V) := ([n_1], [n_1 + 1, n_1 + n_2])\). (For convenience we assume that \( n_1 \geq n_2 \) and that \( D \) can also be considered as an \( n_1 + n_2 \) long vector.) We will use the notations \( G(d) \) and \( G(D) \) for the sets of all realizations of the corresponding degree sequences.

The switch operation exchanges two disjoint edges \( ac \) and \( bd \) in the realization \( G \) with \( ad \) and \( bc \) if the resulting configuration \( G' := G - ac - bd + ad + bc \) is again a simple graph (we denote the operation by \( ac, bd \Rightarrow ad, bc \)).

For an \( ac, bd \Rightarrow ad, bc \) switch operation to be valid, it is necessary but not always sufficient that both \( ac, bd \in E(G) \) and \( ad, bc \notin E(G) \) hold. For each setting (graphs, bipartite graphs, directed graphs) we will define a set of non-chords, which are pairs of vertices which are forbidden from forming edges. See Definition 2.1 below. A pair of vertices which is not a non-chord will be called a chord: such pairs are allowed to be edges, so they may be inserted or deleted. We emphasize that whether or not a pair of vertices forms a chord does not depend on the current realization.

The term chord/non-chord is motivated from Algorithm 2.1, which constructs a switch sequence that exchanges edges and non-edges of a circuit. To do so, it has to include some of the chords of the circuit in switches. Those chords that may not appear in a switch (because that would violate the graph model) are hence called non-chords.

Next, we reformulate the definition of the switch operation to avoid inserting non-chords: an \( ac, bd \Rightarrow ad, bc \) switch operation can be applied if \( ac, bd \in E(G) \), \( ad, bc \notin E(G) \), and \( ad, bc \) are both chords. We now define the set of chords and non-chords in the case of unconstrained and bipartite graph models.

**Definition 2.1.** For simple graphs, the non-chords are exactly the pairs of the form \((v, v)\), as loops are forbidden. Because no further constraints have to be set, we call their degree sequences unconstrained. In bipartite graphs, \((u, v)\) is a chord if and only if \( u \) and \( v \) are in different vertex classes.

In the case of directed graphs (Section 6), we further restrict the set of chords.

It is a well-known fact that the set of all possible realizations of a graphic unconstrained degree sequence is connected under the switch operation. See for example [23] (Havel, 1957) or [29] (Hakimi, 1962). It is interesting to know, however, that the first known proof is from 1891 [35] (Petersen, 1891). For bipartite graphs the equivalent results were proved in 1957 in [16] (Gale, 1957) and [36] (Ryser, 1957). The “classical” proofs work through so called “canonical” realizations. However, the paths between different realizations, created in this way, are very far from efficient for the purpose of applying them in Sinclair’s multicommodity-flow method. Therefore another way has to be designed to select the appropriate paths.

To that end, let us consider two realizations of the same (bipartite or unconstrained) degree sequence. Those edges, that are present or missing in both realizations, do not need
to be changed. The remaining pairs of vertices, each of which is an edge in exactly one of
the realizations, form the symmetric difference of the two realizations, usually denoted by $\nabla$.
To any alternating circuit decomposition of $\nabla$, we are going to assign a sequence of switches
that transform the first realization into the second (this is described right after the proof of
Lemma 2.6).

A graph $H$, with edges colored by either red or blue, will be called a **red-blue graph**.
For vertex $v$ let $d_r(v)$ and $d_b(v)$ be the degree of vertex $v$ in red and blue edges, respectively.
This red-blue graph is **balanced** if for each $v \in V(H)$ equality $d_r(v) = d_b(v)$ holds.

Let $X, Y$ both be realizations (on the same vertex set) of an unconstrained degree sequence
$d$ or a bipartite degree sequence $D$. Let the symmetric difference of the edges be

$$\nabla := E(X) \triangle E(Y).$$

Color the edges of $\nabla$ according to which graph they come from: the $E(X)$ edges are colored
red and the $E(Y)$ edges are colored blue. Equipped with this coloring, $\nabla$ is a balanced
red-blue graph.

A **circuit** in a graph $H$ is a closed trail (so any edge is traversed at most once). As
the graph is simple, a circuit is determined by the sequence of the vertices $v_0, \ldots, v_t$, where
$v_0 = v_t$. Note that there can also be other indices $i < j$ such that $v_i = v_j$. A circuit is called
a **cycle**, if its simple, i.e., for any $i < j$, $v_i = v_j$ only if $i = 0$ and $j = t$.

A circuit (or, in particular, a cycle) in a balanced red-blue graph is called **alternating**, if
the color of its edges alternates. In other words, the color of the edge from $v_i$ to $v_{i+1}$ differs
from the color of the edge from $v_{i+1}$ to $v_i$, and also edges $v_0v_1$ and $v_{t-1}v_t$ have different
colors. Consequently, alternating circuits have even length.

The meaning of “alternating” slightly varies with context. For a subgraph of $\nabla$, it is used
in the red/blue alternating sense; we may explicitly say $(X, Y)$-alternating in this case. For
realizations, “alternating” refers to the conventional edge/non-edge alternation. In practice,
this should not be confusing, since:

**Lemma 2.2.**

- A circuit $C$ which is alternating in the red-blue graph $\nabla$ is alternating between edges
  and non-edges of $X$ (and $Y$ as well).

- If $C$ is an alternating circuit in $X$ then $C$ is a red-blue alternating circuit with respect
to $X$ and $Y := X \triangle C$.

**Proof.** The blue edges are missing from $X$, while the red edges are contained in $X$. $\square$

The following observations are easy to see.

**Lemma 2.3** (adapted from [10], Erdős, Király and Miklós, 2013).

(i) If $H$ is a balanced red-blue graph then the edge set can be decomposed into alternating
circuits.
(ii) Let \( C = v_0, v_1, \ldots, v_{2t} = v_0 \) be an alternating circuit in a balanced red-blue graph \( H \), in which for some \( i < j < 2t \), \( j - i \) is even and \( v_i = v_j \). Then the circuit can be decomposed into two shorter alternating circuits:

\[
C = (v_i, v_{i+1}, \ldots, v_{j-1}, v_j) \uplus (v_j, v_{j+1}, \ldots, v_{2t-1}, v_0, v_1, \ldots, v_{i-1}, v_i).
\]

It is clearly possible that a vertex occurs twice in an alternating circuit without the possibility to divide it into two, smaller alternating circuits. The smallest example is a “bow-tie” circuit: \( v_1, v_2, v_3, v_1, v_4, v_5, v_1 \) with an alternating edge coloring. (The very first and very last occurrences of \( v_1 \) shows the closing of the alternating circuit.) Recalling our earlier discussion, these two copies of the vertex \( v_1 \) form a non-chord.

**Definition 2.4.** An alternating circuit is **primitive**, if it cannot be decomposed further in the way described in Lemma 2.3(ii).

From Lemma 2.1 it follows, that if a vertex appears twice in a primitive circuit, then the distance of two copies of the same vertex must be odd. Consequently, a vertex cannot be visited three times by a primitive circuit, because the three pairwise distances cannot be simultaneously odd. Therefore, in a bipartite graph, a primitive alternating circuit is an alternating cycle.

The definition of primitive alternating circuits is different from the definition of elementary alternating circuits introduced in [10] (Erdős, Király and Miklós, 2013). Without providing the definition here, we mention that, for example, the red-blue graph obtained from a red \( C_5 \) and its blue complement is a primitive alternating circuit, see Figure 1. Still, it can be decomposed into an alternating \( C_4 \) \((x_1, x_5, x_7, x_6)\) and an alternating bow-tie, so it is not elementary.

**Figure 1:** The cycle of length 5 has an alternating circuit traversing all of its edges and non-edges.
Lemma 2.5. Let $C$ be an alternating circuit of length 6 in $\nabla$. If the only non-chords are loops, then there is at most one vertex which is visited more than once by $C$. In other words, at most one of the three main diagonals of $C$ is a non-chord.

Proof. If $v$ is a vertex that is visited at least twice by $C$, it has at least four other neighbors that are pairwise distinct from each other and $v$. Since $v$ is counted twice in the length of $C$, every vertex of $C$ is accounted for, and the claim holds (and $C$ is a bow-tie). □

We will use Sinclair’s multicommodity flow method (Theorem 3.2) to bound the mixing time of the switch Markov chain. The multicommodity flow is given by a set of switch sequences between any two realizations of the degree sequence. The main idea behind the definition of the flow can be described roughly as follows:

For each pair of realizations $(X, Y)$ and every possible complete matching of the $X$-edges with $Y$-edges in $\nabla = X \Delta Y$ at every vertex, assign a switch sequence from $X$ to $Y$ as follows. The matchings decompose $\nabla$ into alternating circuits (Lemma 4.5 in Section 4.1). Refine each of the alternating circuits into primitive alternating circuits in a canonical way (Section 4). By concatenating the switch sequences given by Algorithm 2.1 for the primitive alternating circuits, a switch sequence from $X$ to $Y$ is obtained.

Here we reached a very important point: Algorithm 2.1 does not require an order for processing the primitive circuits; in principle it can be done arbitrarily. One of the novelties of this paper leading to the unified proof is constructing a very delicate order of processing the primitive circuits, which ultimately enables the unification of the proofs. We will return to this point in Section 4. We design the order of processing the primitive circuits in Section 5.

The Sweep procedure in Algorithm 2.1 will be used to construct the switch sequence between two realizations whose symmetric difference is a primitive alternating circuit $C = x_1x_2 \cdots x_{2t}$. Suppose that some of the alternating primitive circuits from $\nabla$ have already had their edges and non-edges exchanged compared to $X$ along the switch sequence; let the current realization be $G$. Let $C$ be the next alternating circuit in $X$ whose edges and non-edges we exchange using Sweep. We assume that $x_1x_2 \notin E(G)$. Most steps of the algorithm involve a single switch (line 26) which removes an edge $x_1x_{2t+2}$, fixes two edges $x_{2t}x_{2t+1}$, $x_{2t+1}x_{2t+2}$ of the symmetric difference, and inserts an edge $x_1x_{2t}$ (which has to be a chord). If $x_1x_{2t}$ is non-chord then a special switch avoiding $x_1x_{2t}$ is performed (lines 15 and 18).
Algorithm 2.1 Sweeping a primitive alternating circuit $C = (x_1, x_2, \ldots, x_{2\ell})$ in $G$. The algorithm assumes that $x_1x_2 \notin E(G)$.

1: procedure Sweep$(G, [x_1, x_2, \ldots, x_{2\ell}]) \rightarrow [Z_0, Z_1, Z_2, \ldots, Z_{\ell-2}, (Z_{\ell-1})]$
2: $Z_0 \leftarrow G$
3: $q \leftarrow 1$
4: end
5: if $\exists r \in \mathbb{N} x_1 = x_{2r}$ then
6: $L \leftarrow \{2i \in 2\mathbb{N}: 4 \leq 2i \leq 2\ell, x_1x_{2i} \in E(G) \text{ and } x_{2i} \neq x_{2r+1}\}$
7: else
8: $L \leftarrow \{2i \in 2\mathbb{N}: 4 \leq 2i \leq 2\ell, x_1x_{2i} \in E(G)\}$
9: end if
10: while $end < 2\ell$ do
11: $start \leftarrow \min\{2i \in L : 2i > end\}$
12: $2t \leftarrow start - 2$
13: while $2t \geq end$ do
14: if $x_1 = x_{2t}$ and $x_{2t+2} = x_{2t-1}$ then
15: $Z_q \leftarrow Z_{q-1} - \{x_{2t}x_{2t+1}, x_{2t-2}x_{2t-1}\} + \{x_{2t+1}x_{2t+2}, x_1x_{2t-2}\}$
16: $2t \leftarrow 2t - 4$
17: else if $x_1 = x_{2t}$ and $x_{2t+1} = x_{2t-2}$ then
18: $Z_q \leftarrow Z_{q-1} - \{x_1x_{2t+2}, x_{2t-2}x_{2t-1}\} + \{x_{2t+1}x_{2t+2}, x_{2t-1}x_{2t}\}$
19: $2t \leftarrow 2t - 4$
20: else if $x_1 = x_{2t}$ or $x_1x_{2t} \in E(Z_{q-1})$ then
21: $Z_q \leftarrow \text{DOUBLE STEP}(Z_{q-1}, x_1, [x_{2t-2}, \ldots, x_{2t+2}])$
22: $q \leftarrow q + 1$
23: $Z_q \leftarrow \text{DOUBLE STEP}(Z_{q-1}, x_1, [x_{2t-2}, \ldots, x_{2t+2}])$
24: $2t \leftarrow 2t - 4$
25: else
26: $Z_q \leftarrow Z_{q-1} - \{x_1x_{2t+2}, x_{2t}x_{2t+1}\} + \{x_1x_{2t}, x_{2t+1}x_{2t+2}\}$
27: $2t \leftarrow 2t - 2$
28: end if
29: $q \leftarrow q + 1$
30: end while
31: end
32: end procedure

Lemma 2.6. Suppose that $G$ contains a primitive alternating circuit $C$ of length $2\ell$. Let the vertex sequence of $C$ be $(x_1, x_2, \ldots, x_{2\ell})$, and suppose that $x_1x_2 \notin E(G)$.

Then Algorithm 2.1 provides a valid switch sequence between $G$ and $G \triangle C$ in the case of unconstrained and bipartite graph models. The length of the switch sequence is $\ell - 1$ in the bipartite case; in the unconstrained case, the length is either $\ell - 1$ or $\ell - 2$. 

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Proof. The processing done by Algorithm 2.1 is governed by two nested loops. The outer loop iterates the variable \( \text{start} \) through

\[
\mathcal{L} = \left\{ 2i \in 2\mathbb{N} : 4 \leq 2i \leq 2\ell, \; x_1x_{2i} \in E(G) \right\} \cap \left\{ 2i \in 2\mathbb{N} | x_{2i} \neq x_{2r+1} \text{ whenever } x_{2r} = x_1 \right\}
\]

in increasing order. The set \( \mathcal{L} \) is not empty since \( x_1x_{2\ell} \in E(G) \) and, if \( x_1 = x_{2r} \) for some \( r \) then \( x_{2\ell} \neq x_{2r+1} \). In the first iteration, \( \text{end} = 2 \), and in the successive iterations \( \text{end} \) takes the value taken by \( \text{start} \) in the previous iteration.

The inner loop performs a series of switches that changes the status of the edges and non-edges induced by consecutive vertices in the interval of vertices between \( x_{\text{start}}, \ldots, x_{\text{end}} \).

As a side effect, it also changes chords between \( x_1 \) and \( \{x_4, x_6, \ldots, x_{2\ell-4}, x_{2\ell-2}\} \), and rarely other chords induced by two vertices of \( C \).

We have to check that each time a new graph is obtained from \( Z_{q-1} \) (lines 15, 18, 21, 23, and 26 in Algorithm 2.1), the changes correspond to valid switches.

![Figure 2: An alternating cycle \( C \) in \( G \) where every \( x_1x_{2t} \) is a chord (an edge or a non-edge)](image)

Let us first check the case when \( G \) is bipartite; for an initial state, see Figure 2. As we remarked earlier, \( C \) must be a cycle: if \( x_i = x_j \) and \( i \neq j \), then Definition 2.4 implies that \( i \neq j \) (mod 2), but because \( C \) is alternating, \( x_i \) and \( x_j \) have to be in different vertex classes. Therefore lines 15 and 18 in Algorithm 2.1 are never reached. We will use induction to prove that whenever a new iteration of the inner loop (line 13) starts, we have

\[
Z_{q-1} = G \Delta \{x_i x_{i+1} : i \in [1, \text{end}) \cup [2t + 2, \text{start})\} \Delta \{x_1x_{\text{end}}, x_1x_{\text{start}}, x_1x_{2t+2}\}. \tag{2.1}
\]

If Equation (2.1) holds when line 13 is reached, then \( x_1x_{2t} \notin E(Z_{q-1}) \), which means that line 26 alone produces the switch sequence. For \( q = 1 \), equation (2.1) holds. Now assume that \( q \geq 2 \) and (2.1) holds. The set \( \{x_1, x_{2t}, x_{2t+1}, x_{2t+2}\} \) is guaranteed to be a set of four
vertices, and Equation (2.1) guarantees that \( x_1 x_{2t} \notin E(Z_{q-1}) \). It also follows that the switch on line 26 neither creates a multi-edge nor deletes an edge which is not present in \( Z_{q-1} \). Hence

\[
Z_q = Z_{q-1} \Delta \{x_1 x_{2t}, x_{2t} x_{2t+1}, x_{2t+1} x_{2t+2}, x_1 x_{2t+2}\}
\]

which implies that (2.1) holds for \( Z_q \) (as \( 2t + 2 \) becomes \( 2t \)). When \( 2t = \text{end} \), the graph assigned to \( Z_q \) on line 26 is \( G \Delta \{x_i x_{i+1} : i \in [1, \text{start})\} \Delta \{x_1 x_{\text{start}}\} \). In particular, \( Z_{\ell-1} = G \Delta C \), which is what we wanted.

Next, let us verify the algorithm for unconstrained degree sequences. Observe the following:

**Obs. (1)** If \( x_i = x_j \), then either \( i = j \) or \( i \not\equiv j \pmod{2} \), in accordance with Lemma 2.3(ii).

In particular, \( \{x_1, x_3, \ldots, x_{2\ell-1}\} \) and \( \{x_2, x_4, \ldots, x_{2\ell}\} \) are both sets of size \( \ell \), and

\[
\{x_{2i} x_{2j+1} : x_{2i} = x_{2j+1}\}
\]

is a (partial) pairing between the two sets.

**Obs. (2)** Suppose \( x_1 x_{2i} \) is a chord. (It may or may not be an edge in \( G \).) If \( x_1 x_{2i} \in C \), then \( \exists r \in \mathbb{N} \) such that \( x_1 = x_{2r} \) and \( x_{2i} \in \{x_{2r-1}, x_{2r+1}\} \).

Suppose first, that \( x_1 \) is visited exactly once by \( C \). Notice, that the arguments of the bipartite case go through seamlessly for the unconstrained case because of Obs. (1) and Obs. (2). The two observations guarantee that (2.1) holds by induction, since \( x_1 x_{2t} \) does not appear in \( \{x_i x_{i+1} : i \in [2, 2\ell - 1]\} \) for any \( 2 < 2t \leq 2\ell \). Figure 2 is still a faithful picture, although some of the vertices may be identical, edges and non-edges are not repeated.

![Figure 3](attachment:image.png)

Figure 3: A primitive circuit with 3 non-chords shown. Each non-chord joins two copies of a vertex, i.e., \( x_1 = x_8 \), etc. The end-points of non-chords are pairwise disjoint.

Suppose now, that \( x_1 \) is visited twice by \( C \), that is, \( x_1 = x_{2r} \) for some integer \( r \), where \( 2 < 2r < 2\ell \). See Figure 3 where \( x_1 = x_8 \). There are two ways the induction argument
may break when $x_1 = x_{2r}$. Since $x_1 x_{2r}$ is a non-chord, it cannot participate in a switch. Furthermore, it is possible that $x_1 x_{2r} \in \{x_i x_{i+1} : i \in [2, 2\ell - 1]\}$ for some $2t$. The only way for this to happen is if $x_1 = x_{2r}$, and $x_2 \in \{x_{2r+1}, x_{2r-1}\}$.

We will verify that Equation (2.1) holds whenever Algorithm 2.1 reaches line 13. Notice, that if $2t \in L$, then by definition $x_2 \neq x_{2r-1}$, and $x_2 \neq x_{2r-1}$, because $x_1 x_2 = x_{2r-1} x_{2r} \notin E(G) \ (C \text{ alternates in } G)$. When $x_2 \in \{x_{2r-1}, x_{2r}, x_{2r+1}\}$, the arguments of the previous case are still applicable, because $x_1 x_2, x_1 x_{2t+2}$ are chords and

$$x_1 x_{\text{start}}, x_1 x_{\text{end}}, x_1 x_2, x_1 x_{2t+2} \notin \{x_i x_{i+1} : i \in [2, 2\ell - 1]\}.$$

Let us check that Equation (2.1) is preserved by the iteration of the inner loop that starts with $2t = 2r$. Suppose $2t = 2r, x_{2r+1} \neq x_{2r-2}$ and $x_{2r-2} \neq x_{2r+1}$. We choose not to perform the standard switch on line 26, instead, we perform a DOUBLE STEP (Algorithm 2.2) to avoid using $x_1 x_{2r}$ in a switch. DOUBLE STEP is called on line 27 to apply a switch to $Z_{r-1}$. We have $\ell \leq 2r - 2, \text{ start } \geq 2r + 2$, and by induction, (2.1) holds for $Z_{r-1}$, thus $x_1 x_{2r-2} \notin E(Z_{r-1})$ and $x_1 x_{2r+2} \in E(Z_{r-1})$.

**Algorithm 2.2** The DOUBLE STEP avoids inserting or deleting $x_1 x_2$. The algorithm assumes that at most one of $x_2 - x_2 \in L_1, x_{2r-1} x_{2r+2}$ is a non-chord.

```plaintext
function DOUBLE STEP($Z, x_1, [x_2 \rightarrow x_{2t-1}, x_2, x_2t+1, x_{2t+2}]$)
    if $x_{2t-2} x_{2t+1}$ is a chord then
        if $x_{2t-2} x_{2t+1} \in E(G)$ then
            return $Z + \{x_{2t-2} x_{2t+1}, x_1 x_{2t+2}\}$
        else if $x_{2t-2} x_{2t+1} \notin E(G)$ then
            return $Z + \{x_{2t-2} x_{2t+1}, x_1 x_{2t+2}\}$
        end if
    else if $x_{2t-1} x_{2t+2}$ is a chord then
        if $x_{2t-1} x_{2t+2} \in E(G)$ then
            return $Z + \{x_{2t-1} x_{2t+2}, x_1 x_{2t+2}\}$
        else if $x_{2t-1} x_{2t+2} \notin E(G)$ then
            return $Z + \{x_{2t-1} x_{2t+2}, x_1 x_{2t+2}\}$
        end if
    end if
end function
```

All in all, if $2t = 2r$ and $x_{2r-1} x_{2r+2}, x_{2r-2} x_{2r+1}$ are chords, then the vertices

$$(x_1, x_{2r-2}, x_{2r-1}, x_2, x_{2t+1}, x_{2t+2}, x_1)$$

in this order trace out a bow-tie. Thus the first call to DOUBLE STEP on line 27 performs a valid switch. The next call to DOUBLE STEP on line 29 likewise performs a valid switch, and it restores alternation between edges along $(x_1, x_{2r-2}, x_{2r-1}, x_2, x_{2t+1}, x_{2t+2}, x_1)$; furthermore, $Z_{q+1}$ satisfies (2.1) (for $q + 2$).

Suppose $2t = 2r$. If $x_{2r-2} x_{2r+1}$ or $x_{2t-1} x_{2t+2}$ is a non-chord, we perform a special switch, either on line 15 or on line 18. It cannot happen that both $x_{2r-2} x_{2r+1}$ and $x_{2t-1} x_{2t+2}$ are
non-chords, because \( x_{2r+1} x_{2r+2} \notin E(G) \) and \( x_{2r-2} x_{2r-1} \in E(G) \). It is simple to check that in either case, a valid switch is performed, and (2.1) holds in the next iteration of the inner loop.

In any case, we will never have \( x_{2t+2} = x_{2r} \) when the algorithm reaches line 13.

If \( x_{2t} = x_{2r+1}, x_{2t+2} \neq x_{2r-1} \), and \( x_1 x_{2t} \notin E(Z_1) \), then line 26 performs a valid switch on \( Z_1 \), and (2.1) holds in the next iteration. Suppose, that \( x_{2t} = x_{2r+1}, x_{2t+2} \neq x_{2r-1} \), and \( x_1 x_{2t} \in E(G) \). In this iteration of the inner loop, two calls to DOUBLE STEP are made on lines 21 and 23. Note that \( 2t \notin \mathcal{L} \), which means that \( end < 2t < start \). By Equation (2.1), we have \( x_1 x_{2t+2} \in E(Z_1) \). By the assumption, \( x_2 x_{2r+1} \notin \{ x_i x_{i+1} : i \in [1, end] \cup [2t + 2, start] \} \), which implies that \( x_{2r-1} x_{2t+2} \) is also not contained in this set, so \( x_{2r-1} x_{2t+2} \notin E(Z_1) \). Therefore, even if \( x_{2t-2} = x_{2r-1} \), the vertices \( (x_1, x_{2t-2}, x_{2r-1}, x_{2t}, x_{2t+1}, x_{2t+2}, x_1) \) form an alternating cycle of 6 vertices if \( x_{2t-1} \neq x_{2t+2} \) and \( x_{2t-2} \neq x_{2t+1} \). The alternating circuit \((x_1, x_{2t-2}, x_{2r-1}, x_{2t}, x_{2t+1}, x_{2t+2}, x_1)\) is a bow-tie if \( x_{2t-1} = x_{2t+2} \) or \( x_{2t-2} = x_{2t+1} \). By Lemma 2.5, we cannot have \( x_{2t-1} = x_{2t+2} \) and \( x_{2t-2} = x_{2t+1} \) simultaneously, thus the two calls to DOUBLE STEP perform valid switches, and Equation (2.1) holds at the start of the next iteration.

The case of \( x_{2t} = x_{2r-1}, x_{2t+2} \neq x_{2r+1} \), and \( x_1 x_{2t} \notin E(Z_1) \) is similar to the previous case. We can deduce that

\[
x_{2r-1} x_{2r}, x_{2r} x_{2r+1} \in \{ x_i x_{i+1} : i \in [1, end] \cup [2t + 2, start] \},
\]
i.e., we have \( x_1 x_{2t-2} \notin E(Z_1) \), even if \( x_{2t-2} = x_{2r+1} \).

Lastly, suppose that \( x_{2t+2} \in \{ x_{2r-1}, x_{2r+1} \} \) and \( x_1 x_{2t} \notin \{ x_{2r-1}, x_{2r}, x_{2r+1} \} \). From the analysis of the previous cases it follows that \( x_1 x_{2t+2} \) is a chord and \( x_1 x_{2t+2} \in E(Z_1) \). We get \( x_1 x_{2t} \notin E(Z_1) \) from Equation (2.1). As before, line 26 performs a valid switch and (2.1) holds at the start of the next iteration.

We have thus shown that Algorithm 2.1 produces valid switches. If line 15 or 18 is reached during SWEEP, then \( Z_{t-2} = G \triangle C \), otherwise \( Z_{t-1} = G \triangle C \). This concludes the proof of Lemma 2.6.

We will demonstrate the algorithm on Figure 3. In the first iteration of the outer loop, start takes index 12 as its initial value. We call \( x_1 x_{12} \) the start-chord and \( x_1 x_2 \) the end-chord. The algorithm sweeps the alternating chords along the circuit between \( x_2 \) and \( x_{12} \), and vertex \( x_1 \) will be the cornerstone of this procedure.

The inner loop works from the start-chord \( x_1 x_{12} \) (edge) towards the end-chord \( x_1 x_2 \) (non-edge). The first value taken by \( 2t \) is 10. Since \( x_1 x_{10} \) is a chord, \( Z_1 \) is obtained by switching along \( x_1, x_{12}, x_{11}, x_{10} \). In the next step, \( 2t = 8 \). However, \( x_1 x_3 \) is a non-chord, therefore SWEEP branches into line 18 instead of line 26. \( Z_2 \) is obtained by switching along \( x_1, x_{10}, x_9 = x_6, x_7, x_8 = x_1 \). Subsequently \( Z_3 \) is obtained by switching along \( x_1, x_6, x_5, x_4 \); note that \( x_8 x_9 = x_1 x_6 \notin E(Z_3) \). The last iteration of the inner loop (for the first iteration of the outer loop) switches along \( x_1, x_4, x_3, x_2 \) and produces \( Z_4 \). Notice, that all of the chords on the circuit from \( x_2 \) to \( x_{10} \) changed their status and \( x_1 x_{12} \) is no longer an edge in \( Z_4 \), but the rest of the chords (except \( x_1 x_6 \) and \( x_1 x_{2t-2} \)) have the same status in \( Z_4 \) as they had in \( G \). Furthermore, the edges and non-edges of the circuit have all been exchanged on the segment from \( x_1 \) to \( x_{12} \).
For the second iteration of the outer loop, \( \text{end} = 12 \) and \( \text{start} \) is assigned a new value too. Eventually, \( \text{start} = 2\ell \) is set, which marks the last iteration of the outer loop. Until \( t = 2\ell - 2 \) (and \( q < \ell - 3 \)), the switches are produced by line 26. Note, that \( x_1 x_{2\ell-4} \notin E(Z_{\ell-4}) \), even when \( 2\ell - 4 \in \mathcal{L} \). However, \( x_1 x_{2\ell-2} \notin E(Z_{\ell-4}) \), thus \text{DOUBLE} \text{ step} \text{ is called on lines 21. The next call to \text{DOUBLE} \text{ step} \text{ on line 23 returns } G \triangle C \text{ and finishes the switch sequence.}

The demonstration shows that some chords that are not necessarily traversed by \( C \) change from being an edge to a non-edge and vice versa during this procedure. However, there are strict patterns that these irregularities must abide. We will collect in \( R \) the set of pairs of vertices \( xy \) of \( Z_q \) whose edge vs. non-edge status is only temporarily modified by the switch sequence from \( G \) to \( G \triangle C \). In other words, \( R \) contains those chords \( xy \) of \( Z_q \), that will be changed by a switch before the sequence reaches \( G \triangle C \) and \( xy \in E(G) \Leftrightarrow xy \notin E(Z_q) \).

**Lemma 2.7.** Suppose \( Z_q \) is an intermediate realization produced by Algorithm 2.1 on the switch sequence between \( G \) and \( G \triangle C \), where \( C = (x_1, x_2, \ldots, x_{2\ell}) \) is an alternating primitive circuit. Let

\[
R := (\{(Z_q \triangle G) \setminus E(C)\}) \cup Q,
\]

where \( Q := \emptyset \), except if \( Z_q \) is the return value of a call to \text{DOUBLE} \text{ step} \text{ on line 21 in which case}

\[
Q := \begin{cases}
\{x_{2t+1} x_{2t+2}\}, & \text{if } x_{2t-2} \neq x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1}), \\
\{x_{2t-2} x_{2t+1}\}, & \text{if } x_{2t-2} = x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1}), \\
\emptyset, & \text{otherwise.}
\end{cases}
\]

The following statements hold at the moment when \( Z_q \) is assigned a value in \text{SWEEP}.

(a) \( R \) is a set of chords with both end-vertices in \( \nabla \),

(b) \( R = \emptyset \), if \( Z_q \in \{G, G \triangle C\} \),

(c) \( R \subseteq \{x_1 x_{\text{start}}, x_1 x_{\text{end}}, x_1 x_{2\ell}\} \), if \( Z_q \) is produced on line 26

(d) \( R \subseteq \{x_1 x_{\text{start}}, x_1 x_{\text{end}}, x_1 x_{2\ell-2}\} \), if \( Z_q \) is produced on line 15, line 18, or line 23 (the second call to \text{DOUBLE} \text{ step}),

(e) \( R \subseteq \{x_1 x_{\text{start}}, x_1 x_{\text{end}}\} \cup H \), where \( H \) is defined in Equation (2.3), if \( Z_q \) is the return value of a call to \text{DOUBLE} \text{ step} \text{ on line 21.}

(f) \((Z_q \triangle R) \triangle G \) is a set of at most 2 edge-disjoint subtrails of \( C \), starting and ending at endpoints of chords in \( R \).

(g) The edges in \( R \) cover at most 5 vertices besides \( x_1 \). In the bipartite case, \( R \) covers at most 3 vertices other than \( x_1 \).

**Proof.** Most statements follow from the proof of Lemma 2.6. To verify (e) and (f) when \( Z_q \) is assigned a value on line 21 in Algorithm 2.1 we shall provide a formula for the \( Z_q \) returned on line 21. Let us define:

\[
F := \begin{cases}
[1, \text{end}] \cup [2t + 1, \text{start}], & \text{if } x_{2t-2} \neq x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1}) \\
[1, \text{end}] \cup [2t - 2, \text{start}], & \text{if } x_{2t-2} \neq x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1}) \\
[1, \text{end}] \cup [2t - 1, \text{start}], & \text{if } x_{2t-2} = x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1}) \\
[1, \text{end}] \cup [2t + 2, \text{start}], & \text{if } x_{2t-2} = x_{2t+1} \text{ and } x_{2t-2} x_{2t+1} \notin E(Z_{q-1})
\end{cases}
\]

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Moreover, set:

$$H := \begin{cases} x_1x_2t_{-2}, x_2t_{-2}x_2t_{+1} & \text{if } x_2t_{-2} \neq x_2t_{+1} \land x_2t_{-2}x_2t_{+1} \in E(Z_q-1) \\ x_1x_2t_{+2}, x_2t_{-2}x_2t_{+1}, x_2t_{+1}x_2t_{+2} & \text{if } x_2t_{-2} \neq x_2t_{+1} \land x_2t_{-2}x_2t_{+1} \notin E(Z_q-1) \\ x_1x_2t_{+2}, x_2t_{+2}x_2t_{-1} & \text{if } x_2t_{-2} = x_2t_{+1} \land x_2t_{+2}x_2t_{-1} \in E(Z_q-1) \\ x_1x_2t_{-2}, x_2t_{+2}x_2t_{-1}, x_2t_{-2}x_2t_{-1} & \text{if } x_2t_{-2} = x_2t_{+1} \land x_2t_{+2}x_2t_{-1} \notin E(Z_q-1) \end{cases}$$

(2.3)

The following equation holds when \text{DOUBLE step} is called on line 21:

$$Z_q = G \Delta F \Delta \{x_1x_{\text{end}}, x_1x_{\text{start}}\} \Delta H,$$

(2.4)

which verifies (e). If \(Q\) is non-empty, the single chord in \(Q\) is not involved in the \text{DOUBLE step} which created \(Z_q\), therefore \(Q \cap E(Z_q \Delta G) = \emptyset\), which completely verifies (f). \(\square\)

Let \(X\) and \(Y\) be two realizations. Assume that we can decompose the symmetric difference \(\nabla = X \Delta Y\) into \(k\) primitive circuits. Let \text{Sweep} process primitive circuits of the decomposition one by one. The concatenation of the switch sequences returned by \text{Sweep} is a switch sequence from \(X\) to \(Y\) with at most \(\frac{|\nabla|}{2} - k\) switch operations. The process only changes the statuses of chords induced by vertices of the circuits (which includes the edges of the circuit).

3. Sinclair’s multicommodity flow method

For unconstrained degree sequences we define our Markov chain \((G_d, P_d)\) as follows: in the Markov graph \(G_d(G(d), E_d)\) the pair \((G, G')\) is an edge if these two realizations differ in exactly one switch. To make a move, choose an unordered pair \(F\) of unordered pairs of distinct vertices, uniformly at random from \(G\), say \(F = \{(x, y), (z, w)\}\) and choose a perfect matching \(F'\) from the other two perfect matchings on the same four vertices. If \(F \subseteq E(G)\) and \(F' \cap E(G) = \emptyset\), then perform the switch (so \(E(G') = (E(G) \cup F') \setminus F\)). Assuming that \(P(G, G') \neq 0\) and \(G \neq G'\), we have

$$\text{Prob}(G \rightarrow G') = P(G, G') := \frac{1}{2\choose n_1 \choose -n_2}.$$  

(3.1)

Equation (3.1) immediately gives that the Markov chain is symmetric. Notice, that if \((F, F')\) corresponds to a feasible switch, then \((F', F)\) does not, therefore \(P(G, G) \geq \frac{1}{2}\) for any realization \(G\). A Markov chain possessing this property of staying in the current state with probability at least \(\frac{1}{2}\) is called lazy. Laziness implies that the eigenvalues of the transition matrix of the Markov chain are non-negative, and it also implies that the chain is aperiodic.

For bipartite degree sequences we define our Markov chain \((G_D, P_D)\) as follows: in the Markov graph \(G_D(V_D, E_D)\) the pair \((G, G')\) is an edge, if these two realizations differ in exactly one switch. The transition matrix \(P\) is defined as follows: we choose uniformly an unordered pair of distinct vertices from \(U\), and an unordered pair of distinct vertices from \(V\), then uniformly randomly choose one of the two matchings between these two pairs. If it preserves the degree sequence, we remove the chosen matching, and add the other. The switch moving from \(G\) to \(G'\) is unique, therefore the probability of this transformation (the jumping probability from \(G\) to \(G' \neq G\)) is:

$$\text{Prob}(G \rightarrow G') = P(G, G') := \frac{1}{2\choose n_1 \choose -n_2}.$$  

(3.2)
The transition probabilities are time- and edge-independent, and symmetric. The probability of staying in the same state is at least $\frac{1}{2}$.

To begin with, we recall some definitions and notations from the literature. Since the uniform distribution is the desired stationary distribution of the switch Markov chains, we conveniently present cited theorems for this specific case only. Let $P^t$ denote the $t^{th}$ power of the transition probability matrix and let $N := |V(G)|$ be the size of the state space of the Markov chain. For any element of the state space $X \in V(G)$, define

$$
\Delta_X(t) := \frac{1}{2} \sum_{Y \in V(G)} |P^t(X, Y) - 1/N|,
$$

We define the mixing time of a Markov chain $\mathcal{M}$ as

$$
\tau_\varepsilon(\mathcal{M}) := \max_{X \in V(G)} \min_t \{\Delta_X(t') \leq \varepsilon \text{ for all } t' \geq t\}.
$$

The Markov chain is said to be rapidly mixing if and only if

$$
\tau_\varepsilon(\mathcal{M}) \leq O\left(\text{poly}(\log(N/\varepsilon))\right).
$$

In this case the switch Markov chain method provides a fully polynomial almost uniform sampler (FPAUS) of the realizations of the given degree sequences. Using a different Markov chain, which changes at most 2 edges per step (but which may not preserve the degree sequence), Jerrum and Sinclair have shown that realizations of $\mathcal{P}$-stable degree sequences have a fully polynomial almost uniform sampler [25] (Jerrum and Sinclair, 1990).

Consider the different eigenvalues of the transition matrix of $\mathcal{M}$ in non-increasing order: if the Markov chain is lazy, we have

$$
1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N \geq 0.
$$

The relaxation time $\tau_{\text{rel}}(\mathcal{M})$ is defined as

$$
\tau_{\text{rel}}(\mathcal{M}) = \frac{1}{1 - \lambda^*}
$$

where $\lambda^*$ is the second largest eigenvalue modulus. So $\tau_{\text{rel}}(\mathcal{M}) = (1 - \lambda_2)^{-1}$ for lazy chains. The following result was proved implicitly by Diaconis and Strook in 1991, and explicitly stated by Sinclair [37, Proposition 1]:

**Theorem 3.1** (Sinclair). $\tau_\varepsilon(\mathcal{M}) \leq \tau_{\text{rel}}(\mathcal{M}) \cdot \log(N/\varepsilon)$.

Note, that the cardinality of the state space of the switch Markov chain trivially satisfies $\log |V(G)| \leq n^2$. By Theorem 3.1 it is sufficient to find a poly($\log N$) upper bound on $\tau_{\text{rel}}(\mathcal{M})$ to show that the Markov chain is rapidly mixing. Rapid convergence to the uniform distribution is highly desirable in any practical application.

There are different methods to prove rapid convergence, here we use – similarly to [30] (Kannan, Tetali and Vempala, 1997) – Sinclair’s multicommodity flow method [37, Theorem 5'] (Sinclair, 1992).
**Theorem 3.2** (Sinclair). Let \(\mathbb{H}\) be a graph whose vertices represent the possible states of a time reversible finite state Markov chain \(\mathcal{M}\), and where \((U, V) \in E(\mathbb{H})\) if and only if the transition probabilities of \(\mathcal{M}\) satisfy \(P(U, V)P(V, U) \neq 0\). For all \(X \neq Y \in V(\mathbb{H})\) let \(\Gamma_{X,Y}\) be a set of paths in \(\mathbb{H}\) connecting \(X\) and \(Y\) and let \(\pi_{X,Y}\) be a probability distribution on \(\Gamma_{X,Y}\). Furthermore let

\[
\Gamma := \bigcup_{X \neq Y \in V(\mathbb{H})} \Gamma_{X,Y}.
\]

We also assume that there is a stationary distribution \(\pi\) on the vertices \(V(\mathbb{H})\). We define the capacity of an edge \(e = (W, Z)\) as

\[
Q(e) := \pi(W)P(W, Z)
\]

and we denote the length of a path \(\gamma\) by \(|\gamma|\). Finally let

\[
\kappa_{\Gamma} := \max_{e \in E(\mathbb{H})} \frac{1}{Q(e)} \sum_{\gamma \in \Gamma_{X,Y} : e \in \gamma} \pi(X)\pi(Y)\pi_{X,Y}(\gamma)|\gamma|.
\] (3.3)

Then

\[
\tau_{\text{rel}}(\mathcal{M}) \leq \kappa_{\Gamma}
\] (3.4)

holds.

We are going to apply Theorem 3.2 for \((G, P)\), which is either the unconstrained \((G, P) = (G(d), P_d)\) or the bipartite \((G, P) = (G(D), P_D)\) switch Markov chain. Using the notation \(N := |V(G)|\), the (uniform) stationary distribution is given by \(\pi(X) = N^{-1}\) for all \(X \in V(G)\). So if we can design a multicommodity flow which is composed of paths shorter than an appropriate \(\text{poly}^f(n)\) function, then inequality (3.4) becomes

\[
\tau_{\text{rel}}(G, P) \leq \frac{\text{poly}^f(n)}{N} \left( \max_{e \in E(\mathbb{H})} \frac{1}{P(e)} \sum_{\gamma \in \Gamma_{X,Y} : e \in \gamma} \pi_{X,Y}(\gamma) \right).
\] (3.5)

If \(Z \in e\), then

\[
\sum_{\gamma \in \Gamma_{X,Y} : e \in \gamma} \pi_{X,Y}(\gamma) \leq \sum_{\gamma \in \Gamma_{X,Y} : Z \in \gamma} \pi_{X,Y}(\gamma),
\] (3.6)

so we have

\[
\tau_{\text{rel}}(G, P) \leq \left( \max_{e \in E(\mathbb{H})} \frac{1}{P(e)} \right) \cdot \frac{\text{poly}^f(n)}{N} \cdot \left( \max_{Z \in V(\mathbb{H})} \sum_{\gamma \in \Gamma_{X,Y} : Z \in \gamma} \pi_{X,Y}(\gamma) \right).
\] (3.7)

We make one more assumption. Namely, that for each pair of realizations \(X, Y \in V(G)\), there is a non-empty finite set \(S_{X,Y}\) (which draws its elements from a pool of symbols), and for each \(s \in S_{X,Y}\) there is a path \(\Upsilon(X, Y, s)\) from \(X\) to \(Y\). Let

\[
\Gamma_{X,Y} := \{\Upsilon(X, Y, s) : s \in S_{X,Y}\}.
\] (3.8)
It can happen that $\Upsilon(X, Y, s) = \Upsilon(X, Y, s')$ for $s \neq s'$, so we consider $\Gamma_{X,Y}$ a “multiset”. For any $\gamma \in \Gamma_{X,Y}$, we have

$$\pi_{X,Y}(\gamma) = \frac{|\{s \in S_{X,Y} : \gamma = \Upsilon(X, Y, s)\}|}{|S_{X,Y}|}.$$  

Putting together the observations and simplifications above we obtain a slightly weaker version of Theorem 3.2. Theorem 3.3 is simpler to use in our applications.

**Theorem 3.3.** Let us assume that there exists two polynomials $\text{poly}_D^\ell, \text{poly}_D^\rho \in \mathbb{R}[x]$ (which only depend on the degree sequence class $D$), a non-empty finite set $S_{X,Y}$ for each $X \neq Y \in V(G)$, and a path $\Upsilon(X, Y, s)$ from $X$ to $Y$ in $G$ for each $s \in S_{X,Y}$. If

- each path $\Upsilon(X, Y, s)$ is shorter than $\text{poly}_D^\ell(n)$, and
- for each $Z \in V(G)$

$$\sum_{X,Y \in V(G)} \frac{|\{s \in S_{X,Y} : Z \in \Upsilon(X, Y, s)\}|}{|S_{X,Y}|} \leq \text{poly}_D^\rho(n) \cdot |V(G)|,$$

then the Markov chain $(G, P)$ is rapidly mixing. Specifically, the mixing time is at most

$$\tau_\varepsilon(G, P) \leq \left( \min_{P(X,Y) \neq 0} P(X,Y) \right)^{-1} \cdot \text{poly}_D^\ell(n) \cdot \text{poly}_D^\rho(n) \cdot (n^2 - \log \varepsilon)$$

(3.10)

In the simple and bipartite cases, each transition $(X, Y) \in E(G)$ satisfies $P(X, Y) \geq n^{-4}$.

4. **Multicommodity flow - general considerations**

Typically, a successful application of Sinclair’s method requires decomposing $\nabla = X \Delta Y$ into alternating circuits in many ways, each decomposition yielding a different path in $G$. The decompositions are parameterized by $S_{X,Y}$ (see (3.8)). This parametrization (described in detail in Lemma 4.5) and its application to (Step 1) was introduced in [30] (Kannan, Tetali and Vempala, 1997).

Let $X$ and $Y$ be two realizations of the same (unconstrained or bipartite) degree sequence; they both belong to $G$. For each $s \in S_{X,Y}$ we will construct a path $\Upsilon(X, Y, s)$ in $G$ which connects $X$ to $Y$. A high level description of the construction of these paths follows.

**Step 1** Guided by $s \in S_{X,Y}$, decompose the symmetric difference $\nabla = E(X) \Delta E(Y)$ into $X, Y$-alternating circuits: $W_1^s, W_2^s, \ldots, W_p^s$.

**Step 2** Decompose every alternating circuit $W_k^s$ into primitive alternating circuits $C_1^k, C_2^k, \ldots, C_{\ell_k}^k$ (the parameter $s$ is omitted from the labels).

**Step 3** Process the primitive circuits iteratively via Algorithm 2.1. The returned switch sequences are concatenated. The resulting path in $G$ is labeled $\Upsilon(X, Y, s)$.  

20
We will describe how to perform [Step 1] in Section 4.1 and how to perform [Step 2] in Section 5.3. Now suppose that the decomposition processes in [Step 1] and [Step 2] are already complete.

**Definition 4.1.** For any \( 1 \leq k \leq p_s \) and \( 1 \leq r \leq \ell_k + 1 \) let

\[
G^k_r = X \Delta \left( \bigcup_{i=1}^{k-1} W^i_s \right) \Delta \left( \bigcup_{j=1}^{r-1} C^k_j \right).
\]

(4.1)

Each graph \( G^k_r \) is called a milestone of the path \( \Upsilon(X, Y, s) \).

Clearly, \( X = G^1_1 \) and \( Y = G^{p_s}_{\ell_{p_s} + 1} \). Also, \( G^k_{\ell_k + 1} = G^{k+1}_1 \) for \( 1 \leq k < p_s \). We are now equipped to provide a more technical description of [Step 3].

**Definition 4.2.** To any \( s \in S_{X,Y} \) we associate a path \( \Upsilon(X, Y, s) \) in the Markov graph \( G \) defined as the following sequence:

\[
\Upsilon(X, Y, s) := \left( \text{Sweep}(G^k_r, C^k_r) \right)_{r=1}^{\ell_k + 1} \bigg|_{k=1}^{p_s}.
\]

(4.2)

**Lemma 4.3.** The length of \( \Upsilon(X, Y, s) \) is at most \( \frac{1}{2} |E(X) \Delta E(Y)| \).

*Proof.* Follows from Lemma 2.6.

The most sensitive part of the construction is [Step 2]. We will need to ensure that the following property holds:

**Reconstructability**

Let \( Z \in G \) denote an arbitrary vertex along a path \( \Upsilon(X, Y, s) \). To apply Sinclair’s method we will need that \( s \in S_{X,Y} \) can be reconstructed from an element of \( S_{\nabla \cap E(Z'), \nabla \setminus E(Z')} \), \( \nabla \), and another small parameter set \( B \); here \( Z' \) is a slight perturbation of \( Z \) described by \( B \), such that \( |E(Z \Delta Z')| \) is at most a small constant.

In case of unconstrained degree sequences, Cooper, Dyer and Greenhill ([6], 2007) and Greenhill and Sfragara (2018 [22]) decompose \( W^s_k \) into “simple” circuits which have the following property: in each “simple” circuit \( C \) there is one predefined vertex (actually, the smallest vertex in a predefined vertex order), which occurs at most twice in \( C \). This made the reconstruction above relatively simple, but made processing such “simple” circuits relatively complicated.

As mentioned earlier, primitive circuits are cycles in bipartite graphs. For bipartite degree sequences such a cycle decomposition is available, which is provided by the \( T \)-operator defined in Section 5.2 of [33] (Miklós, Erdős and Soukup, 2013). To adapt this method to the
unconstrained degree sequences we cannot expect to be able to decompose $W_k$ into alternating cycles (recall the bow tie or Figures 11 and 3). In Section 5 we generalize the $T$-operator to simple graphs. For bipartite graphs, the generalized and the original $T$-operator in [33] produce the same decomposition of alternating cycles. The new proof described in Section 5.2 is simpler than that of [33] because it is described on a higher level of abstraction.

4.1. (Step 1) - parameterizing the circuit decomposition

Now we describe the parametrization process which was originally introduced by Kannan, Tetali and Vempala [30]. Let $W$ be the edges of the simple graph and assume that for each vertex $w \in W$ the $F$-degree and the $F'$-degree of $w$ are the same: $d_F(w) = d_{F'}(w)$ for all $w \in W$. An alternating circuit decomposition of $(F, F')$ is a circuit decomposition such that successive edges come alternately from $F$ and $F'$. By definition, that means that each circuit is of even length. To be more verbose, we may say that the circuit is $F, F'$-alternating.

The set of all edges in $F$ (in $F'$) which are incident to a vertex $w$ is denoted by $F(w)$ (by $F'(w)$, respectively). If $A$ and $B$ are sets, denote by $[A, B]$ the complete bipartite graph with classes $A$ and $B$.

Definition 4.4.

$$\mathcal{S}(F, F') = \{s : W \to 2^{E([F(w), F'(w)])} \text{ such that } s(w) \text{ is a maximum matching of } [F(w), F'(w)] \text{ for all } w \in W\}$$ (4.3)

Naturally, $s(w)$ is a perfect matching of $[F(w), F'(w)]$ if $d_F(w) = d_{F'}(w)$. (The definition is meaningful even when $d_F(w) = d_{F'}(w)$ does not hold for every $w$; this will be the case in Lemma 5.18)

Lemma 4.5. There is a natural one-to-one correspondence between the family of all alternating circuit decompositions of $(F, F')$ and the elements of $\mathcal{S}(F, F')$.

Proof. If $C = \{C_1, C_2, \ldots, C_n\}$ is an alternating circuit decomposition of $(F, F')$, then define $s_C \in \mathcal{S}(F, F')$ as follows:

$$s_C(w) := \{(w, u), (w, u') \in [F(w), F'(w)] : (w, u) \text{ and } (w, u') \text{ are successive edges in some } C_i \in C\}.$$ (4.4)

In turn, to each $s \in \mathcal{S}(F, F')$ assign an alternating circuit decomposition

$$C_s = \{W_1^s, W_2^s \ldots, W_p^s\}$$

of $(F, F')$ as follows: Consider the bipartite graph $\mathcal{F}_s$, whose vertex classes are $F$ and $F'$, which are the edges of the simple graph $K = (W, F \cup F')$. The edge set of $\mathcal{F}_s$ is

$$E(\mathcal{F}_s) = \{(w, u), (u', w') : w \in W \text{ and } ((u, w), (u', w)) \in s(w)\}.$$ 

In other words, $E(\mathcal{F}_s)$ is the union of $s(w)$ for $w \in W$. $\mathcal{F}_s$ is a 2-regular graph, because for each edge $(u, v) \in F \cup F'$ there is exactly one $(u, w) \in F \cup F'$ with $((u, v), (u, w)) \in s(u)$,
there is exactly one \((t, v) \in F \cup F'\) with \(((u, v), (t, v)) \in s(v)\), therefore the \(F_s\)-neighbors of \((u, v)\) are \((u, w)\) and \((t, v)\).

Since \(F_s\) is 2-regular, it is the union of vertex disjoint cycles \(\{W^s_i : i \in I\}\). Now \(W^s_i\) can also be viewed as a sequence of edges in \(F \cup F'\), which is a circuit in the graph \(K\) that alternates between \(F\) and \(F'\). In conclusion, \(\{W^s_i : i \in I\}\) is an alternating circuit decomposition of \((F, F')\). Since

\[
s_{C_s} = s,
\]

the proof is complete.

If the degree sequence of both \(F\) and \(F'\) is \((d_1, \ldots, d_k)\), then write

\[
t_{F,F'} = \prod_{i=1}^{k} (d_i!).
\]  

(4.5)

Clearly,

\[
|\mathcal{S}(F, F')| = t_{F,F'}.
\]  

(4.6)

We are ready to describe \(S_{X,Y}\) of Theorem 3.3 (\(S_{X,Y}\) first appears after Equation 3.7).

**Definition 4.6.** For any two graphs \(X\) and \(Y\) whose degree sequences are identical, let

\[
S_{X,Y} := S(E(X) - E(Y), E(Y) - E(X)).
\]

By Lemma 4.5, every \(s \in S_{X,Y}\) corresponds to an \((X, Y)\)-alternating circuit decomposition of \(\nabla = X \triangle Y\):

\[
\nabla = W^s_1 \uplus W^s_2 \uplus \cdots \uplus W^s_p,
\]  

(4.7)

where \(\uplus\) is the disjoint union. Since the vertex set of a realization is \([n]\), the natural ordering on \([n]\) induces a lexicographical order on the edges, which are unordered pairs of \([n]\). We order the circuits of the decomposition (4.7) in the order of their lexicographically first edges: for \(W^s_i\) and \(W^s_j\), we have \(i < j\) if and only if the lexicographically first edge of \(W^s_i\) is lexicographically smaller than the lexicographically first edge of \(W^s_j\).

In each \(W^s_i\), the matching-system \(s\) induces an \((X, Y)\)-alternating Eulerian circuit. For readability, we omit \(s\) from the superscript of \(W^s_i\) in Section 5. The detailed description of [[Step 1]] is now complete.

5. Multicommodity flow - designing and counting paths in \(G(d)\).

5.1. Preparatory considerations

So far, we have described Sinclair’s multicommodity flow method in general and we have learnt how to shape a realization into another one if their difference is exactly one primitive circuit. We defined our irreducible Markov chain on the state space \(G\). Finally we described a set of parametrizations of the difference of any two realizations \(X\) and \(Y\), and decided that for each such parametrization we will design one switch sequence which transform \(X\) into \(Y\). We have reached the technically most challenging part of any such proof: we have to design one particular path (switch sequence) for every possible parameter set \((X, Y, s)\) in such a way
that the ensemble does not overload any edge in the Markov graph. Typically this is not an easy problem. To quote [9] Dyer, Jerrum and Müller, “Achieving low congestion [...] is a delicate matter.”

In our case the problems come from two sources. The first one is connected to the alternating circuit decomposition of the symmetric difference of the two realizations $X$ and $Y$. It will be relatively simple to keep track of both $E(X) \cap E(Y)$ and $E(X) \triangle E(Y)$, but extra care is needed to decide whether an edge of $E(X) \triangle E(Y)$ originates from $X$ or $Y$. At the beginning of the switch sequence, it is clear which edges come from $X$, and which come from $Y$, but after processing several primitive circuits, we lose this information (without the help of further parameters).

One can try the following trivial strategy: for each primitive circuit, we define a Boolean variable, initiated to zero. Whenever we process a primitive circuit, we change the value of the associated Boolean variable to one. This is useful when we want to list the edges of $W_k$ in their original order defined by $s$.

When a primitive circuit $C$ is processed by SWEEP, we reverse the order of its edges compared to the original order $s$. If the original alternating enumeration is $(a, b, c, d)$, then after SWEEP processes $C$, the order is $(d, c, b, a)$. The resulting order is an Eulerian circuit of $W_k$ which alternates in the next milestone. If $X \triangle Y$ is known, then the Boolean variables tell us which which primitive circuits have already been processed by SWEEP. If the Boolean switch is zero for $C$, then the current realization restricted to $E(C)$ is identical to $X \cap E(C)$; if the Boolean switch is one, then the current realization restricted to $E(C)$ is identical to $Y$. This allows us to restore the original order $s$. Thus the current values of the variables determine the origin of the edges.

This is a plausible solution if we have a constant number of primitive circuits. However, if this number is not bounded by a constant, say, it is linear in $n$, then the cardinality of all possible configurations is exponential in $n$, therefore the cardinality of possible values taken by the auxiliary parameter set $B$ is also exponential. This is simply not sufficient to prove rapid mixing.

To produce a successful proof (of the polynomial mixing time of the switch Markov chain) we shall keep track of both the current alternating edge sequence and the origin of the edges (whether an edge of $X \triangle Y$ comes from $X$ or $Y$). Our solution relies on an enumeration algorithm, which, after processing a certain primitive circuit, will change the trailing order of the edges not only in the current circuit, but also in a well-defined neighborhood of that circuit. This is achieved via the $T$-operator (see Section 5.2), which is the tool that provides the (Reconstructability) property of the multicommodity-flow we are building. Ultimately, it enables us to reconstruct the realizations $X$ and $Y$ when at milestones (Definition 4.1).

The second problem crops up while SWEEP processes a primitive circuit. Algorithms 2.1 and 2.2 change the status of some chords which do not belong to the symmetric difference of $X$ and $Y$. We will use the auxiliary matrix $\hat{M}$ introduced in Definition 5.23 to track $X \cup Y$ and $X \triangle Y$ using the current $Z$. Very often, the row- and column-sums of $\hat{M}$ correspond to $d$, i.e., $\hat{M}$ is the adjacency matrix of a realization of $d$. However, when we switch an edge from outside the symmetric difference, then $\hat{M}$ is not a 0-1 matrix anymore. This does not impact the reconstructability of $X$ and $Y$ from our auxiliary data, but we have to ensure that the same $\hat{M}$ does not appear too often. Papers [11] and [12] used the same coding method.
as this paper does, but others, like [6] or [22], used different parameters. All known proofs control the number of “invalid” non-0-1 occurrences, by an appropriate “critical lemma” (a term coined in [21]). In this paper, this quantity is controlled directly by $P$-stability.

In the next subsection we will introduce and study the $T$-operator to tackle the first problem mentioned at the beginning of this section.

5.2. Preparing for (Step 2) - the $T$-operator

The $T$-operator is an abstract description of the algorithm we use to decompose $W_k$ into primitive $X,Y$-alternating circuits, as foretold in Section 4. Suppose the edges of $W_k$ are colored green and $\pi_0$ is a permutation of the edges in which $s \in S_{XY}$ traverses $W_k$, started from a fixed predefined vertex. The $T$-operator will be called repeatedly, where each iteration outputs a primitive circuit. As its input, the $T$-operator takes an Eulerian circuit on $W_k$ and a red-green coloring of the edges $W_k$. To find a primitive circuit, $W_k$ is traversed along the Eulerian circuit until the set of visited green edges contains an $X,Y$-alternating primitive circuit $C$. The output of the $T$-operator modifies the input coloring by recoloring the edges of $C$ red, moreover, reverses the order of the edges of $C$ and the red-neighborhood of its edges in the Eulerian circuit.

This procedure tracks the path $\Upsilon(X,Y,s)$: when the primitive circuit $C$ is processed by Sweep, the portion of $s$ corresponding to $W_k$ is modified to encode the Eulerian circuit produced by the $T$-operator. The modified trail is alternating between edges and non-edges in the current realization, because the order of the edges of $C$ is reversed by the $T$-operator.

We will show that iteratively applying the $T$-operator to $\pi_0$ and the identically green coloring stabilizes at the reverse (not inverse!) of $\pi_0$ and the identically red coloring. This will show, that the $T$-operator is indeed capable of producing a primitive circuit decomposition of $W_k$.

Let us list a number of abstract objects. After the definitions, we will roughly give the correspondence between the abstractions and their application to $W_k$.

Let $[\mu] = \{1,2,\ldots,\mu\}$ be a base set, denote by $S_{[\mu]}$ the symmetric group on $[\mu]$ and let $Pos$ denote the set of positions, where $Pos = \{(1)^+,(2)^+,(\mu-1)^+\}$. For convenience, we consider $(i)^+ = (i+1)^-$ and allow the alternating naming $Pos = \{(2)^-,(\mu)^-\}$. Let $f$ be a two-coloring on $Pos$ with $f \in \{\text{green},\text{red}\}^{Pos}$. We will describe the state of our system with the pair

$$(\pi,f) : \pi \in S_{[\mu]}, f \in \{\text{green},\text{red}\}^{Pos}.$$ 

Let $E \subset \binom{[\mu]}{2}$ be a fixed subset which we call the set of eligible reversals. Assume that the connected components of the simple graph $([\mu],E)$ are cliques. (5.1)

It is important to recognize that each eligible reversal consists of a pair of elements of the base set, and they do not depend on the image of those elements under $\pi$. Accordingly, to make the definitions more readable, let us define

$$\pi^{-1}(E) = \left\{ \{\pi^{-1}(x),\pi^{-1}(y)\} : \{x,y\} \in E \right\}.$$ 

Let us emphasize that $E,\pi,f$ are abstractions, they will gain their meaning and actual contents later on in Section 5.3.
We define max $\emptyset$ which occur at identical vertices with the same parity. Stated simply, in applications, $E$ describes every possible primitive circuit which is contained as a subsequence in the Eulerian circuit. As mentioned in Section 5.2, $\pi$ describes an Eulerian circuit on $W_k$, and $f^{-1}(\text{red})$ is the union of the edges of primitive circuits already processed by SWEEP.

We now define an operator $T_\pi$, or $T$ for short, as $E$ is fixed anyway. This $T$ is a function mapping $S_{[\mu]} \times \{\text{green, red}\}^{Pos}$ into itself. To determine the image of $(\pi, f)$ under $T$, an interval will be selected first. For that end let

$$j(\pi, f) := \min \left\{ j' \in [\mu] \mid \exists i' < j' : f ((i')^+) = f ((j')^-) = \text{green}, \ {i', j'} \in \pi^{-1}(E) \right\}$$

then let

$$i(\pi, f) := \max \left\{ i' < j(\pi, f) \mid f ((i')^+) = \text{green}, \ {i', j(\pi, f)} \in \pi^{-1}(E) \right\}.$$ 

We define $\max \emptyset = -\infty$ and $\min \emptyset = +\infty$. For any integer $k : 1 \leq k \leq \mu$ we select two positions from $Pos$. Let $a(\pi, f)(k) := k$ if $f((k)^-) = \text{green}$, and let

$$a(\pi, f)(k) := \min \left\{ i' \leq k \mid \forall i'' \text{ s.t. } i' \leq i'' < k : f ((i'')^+) = \text{red} \right\}$$

otherwise. Furthermore, let $b(\pi, f)(k) := k$ if $f((k)^+) = \text{green}$, and let

$$b(\pi, f)(k) := \max \left\{ j' \geq k \mid \forall j'' \text{ s.t. } k < j'' \leq j' : f ((j'')^-) = \text{red} \right\}$$

otherwise. By definition, $a(\pi, f)(k) \leq k \leq b(\pi, f)(k)$ for all $k \in [\mu]$. For a visualization, see the top half of Figure 4.

For any $k \in [\mu]$, let $\overline{k}(\pi, f) := a(\pi, f)(k) + b(\pi, f)(k) - k$. We omit the index $(\pi, f)$ in the following. Informally, $\overline{k}$ takes the largest red segment $[a(\pi, f)(k), b(\pi, f)(k)]$ around $k$, and flips $k$ around the center of the segment; see the bottom half of Figure 4.

**Definition 5.1 (T-operator).** We define the function

$$T : S_{[\mu]} \times \{\text{green, red}\}^{Pos} \rightarrow S_{[\mu]} \times \{\text{green, red}\}^{Pos}.$$ 

Let $(\pi, f) \in S_{[\mu]} \times \{\text{green, red}\}^{Pos}$. If $j(\pi, f) = +\infty$, then let $T(\pi, f) := (\pi, f)$ be a fixed point. If $j(\pi, f)$ is finite, define $T : (\pi, f) \mapsto (\pi', f')$ as follows:

$$\pi'(k) = \begin{cases} \pi(a(i) + b(j) - k) & \text{if } k \in [a(i), b(j)], \\ \pi(k) & \text{if } k \notin [a(i), b(j)], \end{cases}$$

$$f'((k)^+) = \begin{cases} f((k)^+) & \text{if } 1 \leq k < a(i) \text{ or } b(j) \leq k < \mu, \\ \text{red} & \text{if } a(i) \leq k < b(j). \end{cases}$$

Observe Figure 5. The $T$-operator reverses the order of the green sections in $[a(i), b(j)]$, such that the order is not reversed on the already red sections. Then the region $[a(i), b(j)]$ is colored red.
Figure 4: An demonstrative example where $\pi = \text{id}_{11}$. The curved arcs represent the pairs in $\mathcal{E}$. The encircled numbers are $\pi(x)$ and $\pi(x^{-})$, respectively. Do note, that the displayed $\mathcal{E}$ cannot occur in applications, as it connects vertices of different parity.

Let us give another, more verbose description of $T(\pi, f)$. Let

$$\{(a_{(\pi,f)}(i_{(\pi,f)}))+, \ldots, (b(\pi,f)(j_{(\pi,f)}))-\}$$

be the maximal interval (with integer endpoints) containing $\{i_{(\pi,f)}+, \ldots, j_{(\pi,f)}-\}$ such that the $f$-image of the new positions of the extended interval are red. Take a look at Figure 5. To construct $\pi'$ from $\pi$, every maximal red interval $\{x^+, \ldots, y^-\}$ in $\{a^+, \ldots, b^-\}$ is shifted to $\{(a + b - y)^+, \ldots, (a + b - x)^-\}$, and the green positions within $\{a^+, \ldots, b^-\}$ are taken in reverse order in the remaining positions between the shifted red intervals.

Now that $T$ is defined, we will use $T$ iteratively. For $r \in \mathbb{N}$, let

$$T^r = T \circ T \circ \ldots \circ T.$$ 

Given any permutation $\pi_0$ on $[\mu]$, let $(\pi_r, f_r) := T^r(\pi_0, \text{green})$, where green is the identically green function. In subscripts, we shorten $(\pi_r, f_r)$ by writing $r$ instead. For example, $j_r = j(\pi_r, f_r)$, etc.

**Lemma 5.2.** For any $r \geq 0$, the pair of endpoints of a maximal path formed by elements of $f_r^{-1}(\text{red})$ is an element of $\pi_r^{-1}(\mathcal{E})$. 

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Figure 5: An example for $T(\pi, f) = (\pi', f')$. The curved arcs represent the pairs in $\mathcal{E}$. The encircled numbers are $\pi(x)$ and $\pi'(x)$, respectively, where $x = 1, \ldots, 11$ from left to right ($\pi$ is identity).

Proof. The statement is vacuously true when $r = 0$. Use property (5.1) and the fact that $f^{-1}_r((i_r)^+) = f^{-1}_r((j_r)^-) = \text{green}$. By induction, either $a_r(i_r) = i_r$ or $\{a_r(i_r), i_r\} \in \pi^{-1}_r(\mathcal{E})$. By definition, $\{i_r, j_r\} \in \pi^{-1}_r(\mathcal{E})$, thus we also have $\{a_r(i_r), j_r\} \in \pi^{-1}_r(\mathcal{E})$. The same argument goes through for $j_r$ and $b_r(j_r)$.

Lemma 5.3. The following statements hold for $r \geq 0$.

(i) If $(\pi_r, f_r)$ is not a fixed point of the $T$ operator then $j_r < j_{r+1}$;

(ii) $b_r(j_r) = j_r$;

(iii) $f^{-1}_{r+1}(k^+) = \text{green}$ for $j_r \leq k \leq \mu$;

Proof. We proceed by induction on $r$. Statements (ii), (iii) are true when $r = 0$. Now suppose that $r \geq 0$ is such that statements (ii), (iii) are true for $r$. If $(\pi_r, f_r)$ is a fixed point of $T$ then we are done. Otherwise, since

$$f^{-1}_{r+1}((\text{green})) \subsetneq f^{-1}_r((\text{green})),$$

we have $f_r((i_{r+1})^+) = f_r((j_{r+1})^-) = \text{green}$. Clearly, $\{\pi_{r+1}(i_{r+1}), \pi_{r+1}(j_{r+1})\} \in \mathcal{E}$, so

$$\{\pi^{-1}_r(\pi_{r+1}(i_{r+1})), \pi^{-1}_r(\pi_{r+1}(j_{r+1}))\} \in \pi^{-1}_r(\mathcal{E}).$$

(5.2)
Since \( f_{r+1}((j_{r+1})^-) = \text{green} \), we must have \( j_{r+1} > b_r(j_r) = j_r \) or \( j_{r+1} \leq a_r(i_r) \). The first case gives \( j_r < j_{r+1} \) immediately.

Next, suppose that \( j_{r+1} < a_r(i_r) \). Then \( \pi_{r+1}(j_{r+1}) = \pi_r(j_{r+1}) \) and \( \pi_{r+1}(i_{r+1}) = \pi_r(i_{r+1}) \). Plugged into Equation (5.2), the definition of \( j_r \) implies that \( j_{r+1} \geq j_r > a_r(i_r) \), a contradiction.

Finally, suppose that \( j_{r+1} = a_r(i_r) \). Recalling that \( \pi_{r+1}(a_r(i_r)) = \pi_r(j_r) \), we have

\[
\{ \pi^{-1}_r(\pi_{r+1}(i_{r+1})), \pi^{-1}_r(\pi_{r+1}(j_{r+1})) \} = \{ i_{r+1}, j_r \}.
\]

Now Equation (5.2) gives

\[
\{ i_{r+1}, j_r \} \in \pi^{-1}_r(\mathcal{E}),
\]

and property (5.1) implies that \( \{ i_{r+1}, i_r \} \in \pi^{-1}_r(\mathcal{E}) \). If \( f_r((i_r)^-) = \text{green} \) then this contradicts the definition of \( j_r \). Otherwise, \( \{ a_r(i_r), \ldots, i_r \} \) is a maximal red interval in \( f_r \) and hence, by Lemma 5.2 and property (5.1) we conclude that \( \{ i_{r+1}, a_r(i_r) \} \in \pi^{-1}_r(\mathcal{E}) \). Again, this contradicts the choice of \( j_r \).

Hence in all cases we conclude that \( j_r < j_{r+1} \), which implies that \( b_{r+1}(j_{r+1}) = j_{r+1} \) and \( f_{r+2}((k)^+) = \text{green} \) for \( j_{r+1} \leq k \leq \mu \). This completes the proof. \( \square \)

The next lemma is very important for understanding how the \( T \)-operator works. In Definition 5.1 we could have taken \( \pi(i + j - k) \) for \( k \in [i,j] \), and so far, every stated lemma would still hold. The reason we take \( a(i) \) instead of \( i \) is to ensure that the order \( \pi_r \) on maximal red regions in \( f_r \) are just the reverse of their original order in \( \pi_0 \).

**Lemma 5.4.** For arbitrary \( \pi_0 \), \( r \geq 0 \), and \( k \in [\mu] \), we have

\[
\pi_r(k) = \pi_0[a_r(k) + b_r(k) - k].
\]

**Proof.** If \( r = 1 \), the statement immediately follows from the definition. Suppose the statement holds for \( r - 1 \). If \( k \notin [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})] \) then \( a_{r-1}(k) = a_r(k) \) and \( b_{r-1}(k) = b_r(k) \), so

\[
\pi_r(k) = \pi_{r-1}(k) = \pi_0(a_{r-1}(k) + b_{r-1}(k) - k) = \pi_0(a_r(k) + b_r(k) - k),
\]

as we wished.

Suppose from now on that \( k \in [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})] \). Let

\[
\ell = a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - k.
\]

Since the edges in \([a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})]\) are all \( \text{red} \) in \( f_r \), we have

\[
a_r(k) = a_{r-1}(i_{r-1}), \quad b_r(k) = b_{r-1}(j_{r-1}).
\]

Writing \( \ell(r-1) \) for \( \ell(\pi_{r-1}j_{r-1}) \), by induction we have

\[
\pi_r(k) = \pi_{r-1}\left(\ell(r-1)\right) = \pi_0\left(a_{r-1}\left(\ell(r-1)\right) + b_{r-1}\left(\ell(r-1)\right) - \ell(r-1)\right),
\]

Since \( a_{r-1}\left(\ell(r-1)\right) = a_{r-1}(\ell) \) and \( b_{r-1}\left(\ell(r-1)\right) = b_{r-1}(\ell) \), the right hand side is equal to \( \pi_0(\ell) \). Expanding it, we get

\[
\pi_0(\ell) = \pi_0(a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - k) = \pi_0(a_r(k) + b_r(k) - k),
\]

which is what we intended to prove. \( \square \)
Repeated application of the $T$-operator turns every position red eventually, if the trivial necessary condition is satisfied:

**Lemma 5.5.** If $\{1, \mu\} \in \mathcal{E}$, then $\exists z \in \mathbb{N}$ such that $f_z^{-1}(\text{red}) = \text{Pos}$.

*Proof.* Lemma 5.2 implies that $\{1, \min\{t : f_r((t)^+) = \text{green}\}\} \in \pi_r^{-1}(\mathcal{E})$, therefore

\[
\{\min\{t : f_r((t)^+) = \text{green}\}, \mu\} \in \pi_r^{-1}(\mathcal{E}),
\]

except if $f_r^{-1}(\text{red}) = \text{Pos}$ already. □

The following lemma shows that those segments whose endpoints form an eligible reversal cannot simultaneously start on a green position and end on a red position.

**Lemma 5.6.** Given $r \geq 0$ and any $\{x, y\} \in \pi_r^{-1}(\mathcal{E})$ such that $x < y$, either $\{x, y\} = \{i_r, j_r\}$, or $f((x)^+) = \text{red}$, or $y \geq j_r + 1$.

*Proof.* The lemma trivially holds for $r = 0$. Suppose now, that $r \geq 1$.

Suppose first, that $f_r((x)^+) = f_r((y)^-) = \text{green}$. By definition, $y \geq j_r$. If $y \geq j_r + 1$, the lemma holds. If $y = j_r$, then definition of $i_r$ implies that $x \leq i_r$. If $y = j_r$ and $x < i_r$, then $x < a_r(i_r)$. By property (5.1) and Lemma 5.2, $\{x, a_r(i_r)\} \in \pi_r^{-1}(\mathcal{E})$ holds. Since $f_r((a_r(i_r))^-) = \text{green}$, we have a contradiction with the definition of $j_r$.

Suppose, that $f_r((x)^+) = \text{green}$, $f_r((y)^-) = \text{red}$, and the lemma does not hold. By Lemma 5.3 we have $y \leq j_{r-1}$. Then we must also have $x < a_{r-1}(i_{r-1})$ (otherwise $f_r((x)^+) = \text{red}$, a contradiction). Thus

\[
\pi_{r-1}^{-1}(\pi_r(x)) = x \text{ and } x \leq \pi_{r-1}^{-1}(\pi_r(y)) \leq j_{r-1}
\]

so $\{x, \pi_{r-1}^{-1}(\pi_r(y))\} \in \pi_r^{-1}(\mathcal{E})$. By induction, we should have $f_{r-1}((x)^+) = \text{red}$, which implies $f_r((x)^+) = \text{red}$, a contradiction.

We have checked and eliminated every possible case where the statement of the lemma is not satisfied. □

The next lemma provides our desired **(Reconstructability)** property. By including a natural number $w$ between 0 and $n^2$ in the parameter set $B$ in Equation (5.18), we need not know the current coloring $f_r$ to reconstruct $\pi_0$ from $\pi_r$:

**Theorem 5.7.** $\forall r \in \mathbb{N} \exists w \in \mathbb{N} \text{ and } \exists g \in \{\text{green, red}\}^{\text{Pos}}$ such that

\[
T^w(\pi_r, \text{green}) = (\pi_0, g).
\]

*Proof.* If $f_r(\text{Pos}) \equiv \text{red}$, then Lemma 5.2 implies that $\{1, \mu\} \in \pi_r^{-1}(\mathcal{E})$. By Lemma 5.4, $\pi_r(k) = \pi_0(1 + \mu - k)$. By Lemma 5.5 there exists a $\pi \in S[\mu]$ and a $z \in \mathbb{N}$ for which $T^z(\pi, \text{green}) = (\pi, \text{red})$. By Lemma 5.4 we have

\[
\pi(k) = \pi_r(1 + \mu - k) = \pi_0(k),
\]

which is what we wanted.
If \( f^{-1}_r(\text{red}) \) is composed of multiple components, then a repeated application of the \( T \)-operator works successively in these components. Lemma 5.3 says that the order of elements in each of these components have been reversed in \( \pi_r \) compared to \( \pi_0 \). Outside these intervals, however, \( \pi_r \) is identical to \( \pi_0 \).

Because of Lemma 5.2, we see that Lemma 5.3 implies that the maximal \text{red} intervals will be completely processed after a certain number of steps. Lemmas 5.3 and 5.6 together imply that if the \( T \)-operator starts working inside a component of \( f^{-1}_r(\text{red}) \) then the next selected interval \([i, j]\) will also be inside until the whole component becomes \text{red} again. At this point, by Lemma 5.4, the order of the elements inside each \text{red} component have been reversed for a second time, so the final permutation equals \( \pi_0 \), as claimed.

**Definition 5.8.** The restriction of a permutation \( \pi \) to an interval \([\alpha, \beta]\), denoted by \( \pi|_{[\alpha, \beta]} \), means that original domain \([\mu]\) of \( \pi \) is replaced with \([\alpha, \beta]\).

The \( T \)-operator naturally generalizes to injective maps from \([\alpha, \beta]\) to an arbitrary set.

**Lemma 5.9.** Suppose that \( r \in \mathbb{N} \) and \([\alpha, \beta] \subset [i_r, j_r] \) (where \( \alpha < \beta \)) is a proper subinterval such that any component of \( f^{-1}_r(\text{red}) \) is either disjoint from \( \text{Pos}[\alpha, \beta] := \{(\alpha)^+, \ldots, (\beta)^-\} \) or entirely contained by it. If \( f_r((\alpha)^+) = f_r((\beta)^-) = \text{green} \), then

\[
(\pi|_{[\alpha, \beta]}, f_r|_{\text{Pos}[\alpha, \beta]}) \text{ is a fixed point of the } T \text{-operator and } \quad (\pi|_{[\alpha, \beta]}, f_r|_{\text{Pos}[\alpha, \beta]}) = T^r(\vartheta, \text{green}),
\]

where \( \vartheta = \pi_0|_{[\alpha, \beta]} \cup \{\alpha \mapsto \pi_r(\alpha), \beta \mapsto \pi_r(\beta)\} \).

**Proof.** Both \( \pi_0|_{[\alpha, \beta]}, \pi_r|_{[\alpha, \beta]} \) map \([\alpha, \beta]\) to the same set, because of the assumption on the components of \( f^{-1}_r(\text{red}) \). Lemma 5.6 implies that \( \{\alpha, \beta\} \notin \mathcal{E} \) and that \( (\pi_r|_{[\alpha, \beta]}, f_r|_{[\alpha, \beta]}) \) is a fixed point of the \( T \)-operator. Lemma 5.2 guarantees that for \( k \leq r \), either \( \alpha < i(\pi_k, f_k) < j(\pi_k, f_k) < \beta \), or \([i(\pi_k, f_k), j(\pi_k, f_k)] \cap (\alpha, \beta) = \emptyset \). In the latter case, it is possible that \( j(\pi_k, f_k) = \alpha \) or \( i(\pi_k, f_k) = \beta \), which is why we defined \( \vartheta(\alpha) \) and \( \vartheta(\beta) \) separately. In the former case, the \( T \)-operator and the restriction operation trivially commute:

\[
T(\pi_k|_{[\alpha, \beta]}, f_k|_{[\alpha, \beta]}) = T(\pi_k, f_k)|_{[\alpha, \beta]}.
\]

This implies that

\[
(\pi_r|_{[\alpha, \beta]}, f_r|_{\text{Pos}[\alpha, \beta]}) = T^q(\vartheta, \text{green})
\]

for some \( q \leq r \). Since the left hand side is a fixed point of the \( T \)-operator, applying \( T^{r-q} \) acts identically on it. \( \square \)

**5.3. (Step 2) - decomposing a circuit into primitive circuits**

Given \( X, Y \in \mathcal{G} \) (we do not specify whether the degree sequence is unconstrained or bipartite), and \( s \in \mathcal{S}_{X,Y} \), we construct a path between \( X \) and \( Y \) in \( \mathcal{G} \) as follows. Recall Equation (4.7): the matching-system \( s \) decomposes \( \nabla = X \Delta Y \) into alternating circuits

\[
\nabla = W_1 \uplus \ldots \uplus W_p.
\]

The number \( p \) and the circuits \( W_k \) depend on \( s \); we omit \( s \) from super- and subscripts, as usual. The goal in this section is to apply the \( T \)-operator to decompose each circuit \( W_k \) into primitive circuits, and show that **(Reconstructability)** holds.
Let us fix an arbitrary circuit $W_k$ from the collection (5.3). The vertices of $W_k$ are a subset of $\mathbb{N}$, so there is a natural ordering on them. Let $v_1v_2$ be the lexicographically first edge of $W_k$, where $v_1$ precedes $v_2$. As stated below Equation (5.7), the matching system $s$ induces an $(X,Y)$-alternating Eulerian circuit on $W_k$. Let us list this Eulerian trail starting with the edge $v_1v_2$:

$$W_k : (v_1, v_2, v_3, v_4, \ldots, v_{|E(W_k)|}, v_{|E(W_k)|+1}), \quad \text{where } v_{|E(W_k)|+1} = v_1. \quad (5.4)$$

Let $\mu = |E(W_k)| + 1$, $\pi_0 = \text{id}_{[\mu]}$, $f_0 = \text{green}$, and

$$\mathcal{E} = \left\{ (x, y) \in \left( \left[ \frac{\mu}{2} \right] \right) : v_x = v_y \text{ and } x \equiv y \pmod{2} \right\}. \quad (5.5)$$

By transitivity, this set possesses property (5.4), so we can apply the $T$-operator on $\pi_0$ with $\mathcal{E}$ as the set of eligible reversals. Let $(\pi_r, f_r) = T^r(\pi_0, f_0)$ for $r \in \mathbb{N}$.

**Lemma 5.10.** Given $r \in \mathbb{N}$, visiting the vertices $v_{\pi_r(1)}v_{\pi_r(2)} \cdots v_{\pi_r(|E(W_k)|+1)}$ is an Eulerian circuit of $W_k$.

**Proof.** Easily seen by induction on $r$. Lemma 5.2 and the definition of the $T$-operator implies that we get $\pi_r$ by reversing some intervals of the trail defined by $\pi_{r-1}$ whose first and last vertices are identical. Consequently, every edge is visited by the new trail too. $\square$

It is clear by definition, that $(x)^+$ in the set $\text{Pos}$ coincides with $v_{\pi_r(x)}v_{\pi_r(x+1)}$ on the Eulerian circuit determined by $\pi_r$ in Lemma 5.10. Subsequently, $f_r$ defines a corresponding coloring: the edge $v_{\pi_r(x)}v_{\pi_r(x+1)}$ has color $f_r((x)^+)$. By Lemma 5.3

$$v_{\pi_r(x)}v_{\pi_r(x+1)} = \begin{cases} v_xv_{x+1} & \text{if } f_r((x)^+) = \text{green}, \\ v_{a_r(x) + b_r(x) - x - 1}v_{a_r(x) + b_r(x) - x} & \text{if } f_r((x)^+) = \text{red}. \end{cases} \quad (5.6)$$

By definition,

$$\{v_{a_r(x) + b_r(x) - x - 1}v_{a_r(x) + b_r(x) - x} : f_r((x)^+) = \text{red}\} = \{v_xv_{x+1} : f_r((x)^+) = \text{red}\}. \quad (5.7)$$

To complete Definition (5.11) we shall determine the primitive circuits $C^k_r$.

**Definition 5.11.** Let $\ell_k$ be the maximum $\ell$ for which $\pi_{\ell-1} \neq \pi_\ell$. For any $1 \leq r \leq \ell_k$, let

$$E(C^k_r) := \{v_xv_{x+1} : x \in [i_{r-1}, j_{r-1} - 1] \text{ and } (x)^+ \in f_{r-1}^{-1}(\text{green})\} = \{v_xv_{x+1} : (x)^+ \in f_{r-1}^{-1}(\text{red}) \setminus f_{r-1}^{-1}(\text{red})\}. \quad (5.8)$$

The equality of the two right hand sides in (5.8) follows from (5.7). Take the list of edges of $W_k$ starting with $v_1v_2$ in the order defined by the Eulerian circuit $\pi_0 = \text{id}_{[\mu]}$. This order can be restricted to the edges of $C^k_r$, so there is a natural Eulerian circuit on $C^k_r$, too.

Equations (5.6) and (5.8) show that $C^k_r$ is a subsequence of edges of the Eulerian circuit defined by $\pi_r$ on $W_k$:

$$E(C^k_r) = \{v_{\pi_r(x)}v_{\pi_r(x+1)} : x \in [i_{r-1}, j_{r-1} - 1] \text{ and } (x)^+ \in f_{r-1}^{-1}(\text{green})\} \quad (5.9)$$

We want to show that $C^k_r$ is $(X,Y)$-alternating. We need the following lemma.
Lemma 5.12. For any \( r \in \mathbb{N} \), we can describe \( \pi_r^{-1}(E) \) as the set of endpoints of even circuits formed by subintervals of the Eulerian circuit defined by \( \pi_r \) in \( W_k \):

\[
\pi_r^{-1}(E) = \left\{ \{x, y\} \in \binom{[\mu]}{2} : v_{\pi_r(x)} = v_{\pi_r(y)} \text{ and } x \equiv y \pmod{2} \right\}.
\]

Proof. From (5.3), we have

\[
\pi_r^{-1}(E) = \left\{ \{\pi_r^{-1}(x), \pi_r^{-1}(y)\} : \{x, y\} \in \binom{[\mu]}{2}, v_x = v_y \text{ and } x \equiv y \pmod{2} \right\}
\]

\[
= \left\{ \{x, y\} \in \binom{[\mu]}{2} : v_{\pi_r(x)} = v_{\pi_r(y)} \text{ and } \pi_r(x) \equiv \pi_r(y) \pmod{2} \right\},
\]

because \( \pi_r \) is a permutation. It is enough to show that \( \pi_r \) preserves parity; i.e., \( \pi_r(x) \equiv x \pmod{2} \) for any \( x \). For \( r = 0 \) this is trivial. Suppose \( \pi_{r-1} \) preserves parity. For \( x \notin [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})] \), we have \( \pi_r(x) = \pi_{r-1}(x) \pmod{2} \), so parity is preserved.

Now suppose that \( x \in [a_{r-1}(i_{r-1}), b_{r-1}(j_{r-1})] \). First observe that Lemma 5.2 implies that \( a_{r-1}(y) \equiv b_{r-1}(y) \pmod{2} \) for arbitrary \( y \). Hence \( a_{r-1}(i_{r-1}) \equiv i_{r-1} \pmod{2} \), and since \( \{i_{r-1}, j_{r-1}\} \in \pi_{r-1}(E) \) we know that \( i_{r-1} \equiv j_{r-1} \). As \( j_{r-1} = b_{r-1}(j_{r-1}) \) by Lemma 5.3, it follows that \( a_{r-1}(i_{r-1}) \equiv b_{r-1}(j_{r-1}) \pmod{2} \). Now

\[
\pi_r(x) = \pi_{r-1}(a_{r-1}(z) + b_{r-1}(z) - z)
\]

where

\[
z = a_{r-1}(i_{r-1}) + b_{r-1}(j_{r-1}) - x \equiv x \pmod{2}.
\]

Hence, by induction,

\[
\pi_r(x) \equiv a_{r-1}(z) + b_{r-1}(z) - z \equiv z \equiv x \pmod{2},
\]

completing the proof. \( \square \)

Lemma 5.13. \( C_r^k \) is a primitive \((X, Y)\)-alternating circuit for every \( 1 \leq r \leq \ell_k \). Moreover,

\[
W_k = C_1^k \uplus C_2^k \uplus \ldots \uplus C_{\ell_k}^k.
\]

Proof. Since \( W_k \) is an alternating circuit, \(|E(W_k)|\) is divisible by two, so \( \{1, |E(W_k)| + 1\} \in E \). According to Lemma 5.5 there exists a smallest \( \ell_k \in \mathbb{N} \) such that \( f_{\ell_k}(v_x v_{x+1}) = \text{red} \) for every \( x = 1, \ldots, |E(W_k)| \). As \( r \) increases, \( \text{red} \) edges stay \( \text{red} \), so it follows from Equation 5.8 that \( \bigcup_{r=1}^{\ell_k} C_r^k \) is an edge-disjoint partition of \( W_k \).

Recall (5.9). By Lemma 5.2 and Lemma 5.12 \( C_r^k \) is an even circuit. The proof of Lemma 5.12 also shows that \( \pi_r \) preserves parity. By (5.8), it is such a subsequence of the Eulerian circuit \( \pi_0 \), which starts with the odd indexed \( v_{i_{r-1}} \) and ends with the odd indexed \( v_{j_{r-1}} \); moreover, by (5.7) and Lemma 5.12 the \( \text{red} \) edges between \( v_{i_{r-1}} \) and \( v_{j_{r-1}} \) start and end on vertices of the same parity. This shows that \( C_r^k \) is \((X, Y)\)-alternating.

Suppose \( C_r^k \) visits some vertex three times, that is

\[
\exists x < y < z \text{ such that } i_{r-1} \leq x, y, z < j_{r-1} \text{ and } v_{\pi_r(x)} = v_{\pi_r(y)} = v_{\pi_r(z)}.
\]
If $x \equiv y \pmod{2}$ then $y \geq j_{r-1}$, a contradiction. Similarly, we must have $y \not\equiv z \pmod{2}$ and $x \not\equiv z \pmod{2}$, which is a contradiction.

If an even number of steps lead from one copy of a vertex to another copy of it on $C^{k}$, then $j_{r-1}$ is not minimal, a contradiction. This proves that $C^{k}$ is a primitive circuit. □

Recall Definition 4.1. Let us repeat Equation (4.1):

$$G^{k}_{r} = X \Delta \left( \bigcup_{i=1}^{k-1} W_{i} \right) \Delta \left( \bigcup_{j=1}^{r-1} C^{k}_{j} \right).$$

In words, we obtain the milestone $G^{k}_{r}$ from $X$ by exchanging edges with non-edges (and vice versa) in the following subsets: each $W_{i}$ for $1 \leq i \leq k-1$ and each edge in $W_{k}$ which is red in $f_{r}$. Milestones are special realizations: both $G^{k}_{r} \Delta X$ and $G^{k}_{r} \Delta Y$ are subgraphs of $X \Delta Y$. Milestones are uniquely determined by $(X, Y, s)$ and the fixed lexicographical order.

**Lemma 5.14.** For any $0 \leq r \leq \ell_{k}$,

$$\bigcup_{j=1}^{r} C^{k}_{j} = \left\{ v_{x}v_{x+1} : (x)^{+} \in f^{-1}_{r}(red) \right\}.$$

Furthermore, the Eulerian circuit described by $\pi_{r}$ in $W_{k}$ is alternating in $G^{k}_{r}$. For any $1 \leq r \leq \ell_{k}$, the circuit $C^{k}_{r}$ is alternating in $G^{k}_{r}$.

**Proof.** Equation (5.10) follows from (5.7) and (5.8). The Eulerian circuit determined by $\pi_{0}$ is by definition $(X, Y)$-alternating in $W_{k}$, and thus it is alternating in $G^{k}_{0}$. Consequently, by (5.8), $C^{k}_{r}$ is alternating in $W_{k}$. As described by (5.6), $\pi_{r}$ reverses the order of maximal red segments (with respect to the coloring $f_{r}$). From the definition of $G^{k}_{r}$, it follows that the Eulerian circuit determined by $\pi_{r}$ in $W_{k}$ is alternating in $G^{k}_{r}$. □

**5.4. (Step 3) - Describing the switch sequence along a primitive circuit**

Recall Equation 4.2 which describes the switch sequence from $X$ to $Y$ determined by $s$.

$$\Upsilon(X, Y, s) := \left( \left( \text{Sweep}(G^{k}_{r}, C^{k}_{r}) \right)_{r=1}^{k+1} \right)_{s=1}^{k}.$$

In this section we provide one last missing detail of the construction, and check that $\Upsilon(X, Y, s)$ is indeed a path in $\mathbb{G}$. By Equation 4.1

$$G^{k}_{r+1} = G^{k}_{r} \Delta C^{k}_{r}.$$ 

By Lemma 5.14 the primitive circuit $C^{k}_{r}$ alternates in $G^{k}_{r}$. By Lemma 2.6, Sweep returns a valid switch sequence between $G^{k}_{r}$ and $G^{k}_{r+1}$. We have one degree of freedom left in (4.2): the order in which Sweep enumerates the edges (and vertices) of $C^{k}_{r}$ is not specified yet.

Let $x_{1} := v_{y} \in V(C^{k}_{r})$, where $y$ minimizes

$$\deg_{X[V(C^{k}_{r})]}(v_{y}) + \deg_{Y[V(C^{k}_{r})]}(v_{y}) - \deg_{G^{k}_{r}[V(C^{k}_{r})]}(v_{y}),$$

and $y$ is minimal with respect to this condition.

Label the vertices of $C^{k}_{r}$ by $x_{1}, x_{2}, \ldots$ following the natural Eulerian circuit (5.4) on $W^{k}$ (either forwards or in reverse), starting with $x_{1}x_{2}$, where $x_{1}x_{2} \notin G^{k}_{r}$. By Lemma 2.6, $\text{Sweep}(G^{k}_{r}, C^{k}_{r})$ returns a switch sequence between $G^{k}_{r}$ and $G^{k}_{r+1}$. 34
5.5. Reconstructing the endpoints $X, Y$ of the switch sequence

Suppose that $Z \in \text{SWEEP}(G_r^k, C_r^k)$ is a realization which lies on the path $Y(X,Y,s)$ from $X$ to $Y$ with respect to some $s \in S_{X,Y}$. Recall Lemma 2.7 and Equation (2.2), which in our setting translates to

$$R = (\{Z \cap G_r^k\} \setminus E(C_r^k)) \cup Q,$$

(5.12)

where either $Q = \emptyset$ or $Q$ contains exactly one edge of $C_r^k$. Unfortunately, $Z$ may differ from $X$ on $W_i$ even when $i \neq k$. These are exactly the differences that are tracked by $R$, therefore it is much more comfortable to continue working in this section with

$$Z' = Z \Delta R.$$

(5.13)

Simply stated, the edges which are temporarily modified by $\text{SWEEP}(G_r^k, C_r^k)$ in $Z$ are returned to their appropriate state in $Z'$. Generally $Z'$ is not a realization of $d$, but it is well-behaved with respect to $W_i$ for $i \neq k$. From Definition 4.1 and Equation (5.12) it follows that

$$Z' \cap X, Z' \cap Y \subseteq E(C_r^k) \subseteq X \cap Y,$$

$$E(Z') \cap E(W_i) = E(X) \cap E(W_i) \text{ for } i > k,$$

$$E(Z') \cap E(W_i) = E(Y) \cap E(W_i) \text{ for } i < k,$$

$$E(Z') \cap E(C_r^k) = E(X) \cap E(C_r^k) \text{ for } j > r,$$

$$E(Z') \cap E(C_r^k) = E(Y) \cap E(C_r^k) \text{ for } j < r.$$

(5.14)

By Lemma 2.7(b), if $Z$ is a milestone (that is, $Z = G_r^k$ for some $k, r$), then $R = \emptyset$ and we have $Z' \cap C_r^k = Y \cap C_r^k$. The Eulerian circuit associated to $\pi_r$ is alternating in $Z = G_r^k$.

If $Z$ is not a milestone, then it is an intermediate realization strictly between $G_r^k$ and $G_{r+1}^k$ on the switch sequence. There generally does not exist an Eulerian circuit on $W_k$ which is alternating in $Z' = Z \Delta R$. For $W_i$ ($i \neq k$), the trail induced by $s \in S_{X,Y}$ on $W_i$ is alternating in $Z'$ (but generally it is not alternating in $Z$). Our next goal is to slightly modify $\pi_{r-1}$, such that it induces an Eulerian circuit in $W_k$ which is alternating in $Z'$, except at a constant number of vertices.

According to Lemma 2.7(d), the symmetric difference of $Z'$ and $G_r^k$ are two subtrails of $C_r^k$. Without loss of generality,

$$Z' \Delta G_r^k = \{v_x v_{x+1} : x \in \mathcal{I} \text{ and } f_{r-1}(x^+) = \text{green}\}$$

(5.15)

where $\mathcal{I} = [\alpha, \beta] \cup [\gamma, \delta]$ or $\mathcal{I} = [i_{r-1}, \alpha] \cup [\beta, \gamma] \cup [\delta, j_{r-1}]$, such that $i_{r-1} \leq \alpha \leq \beta \leq \gamma \leq \delta \leq j_{r-1}$ and $\alpha, \beta, \gamma, \delta$ are chosen in such a way that $\mathcal{I}$ is minimal. Let us define $\pi_{Z'}$ as follows ($\pi_{Z'}$ depends on $X, Y, s$, too). If $\mathcal{I} = [\alpha, \beta] \cup [\gamma, \delta]$, then let

$$\pi_{Z'}(x) := \begin{cases} 
\pi_{r-1}(\frac{x}{x^+(r-1)}), & \text{if } x \in [\alpha, \beta] \cup [\gamma, \delta] \\
\pi_{r-1}(x), & \text{otherwise.}
\end{cases}$$

(5.16)

In other words, $\pi_{r-1}$ is reversed on the maximal $f_{r-1}$-red intervals of $[\alpha, \beta] \cup [\gamma, \delta]$.

If $\mathcal{I}$ does not match the previous form, then we have $\mathcal{I} = [i_{r-1}, \alpha] \cup [\beta, \gamma] \cup [\delta, j_{r-1}]$, such that $i_{r-1} < \alpha$ and $\delta < j_{r-1}$. In this case, let

$$\pi_{Z'}(x) := \begin{cases} 
\pi_{r-1}(\frac{(i_{r-1}+j_{r-1}-x)}{x^+(r-1)}), & \text{if } x \in [i_{r-1}, \alpha] \cup [\beta, \gamma] \cup [\delta, j_{r-1}] \\
\pi_{r-1}(i_{r-1}+j_{r-1}-x), & \text{otherwise.}
\end{cases}$$

(5.17)
Lemma 5.15. The Eulerian circuit defined by \( \pi_{Z'} \) on \( W_k \) alternates in \( Z' \) with the exception of at most 4 pairs of chords.

Proof. Since \( \pi_{r-1} \) alternates on \( G_k \), it follows that \( \pi_{r-1} \) also alternates at \( v_{\pi_{r-1}(x)} \) in \( Z' \) for \( x \notin \mathcal{I} \). For any \( x \in \mathcal{I} \), \( \{ \alpha, \beta, \gamma, \delta \} \), \( \pi_{r-1} \) alternates at \( v_{\pi_{r-1}(x)} \) in \( Z' \) if \( f^{-1}(x^-) = f^{-1}(x^+) \).

To ensure alternation at \( v_{\pi_{r-1}(x)} \) when \( f^{-1}(x^-) \neq f^{-1}(x^+) \), \( \alpha < \delta < \beta < \gamma \) and \( x \notin \mathcal{I} \). The two subtrails are reversed on the edges which also belong to \( f^{-1}(\text{red}) \). In addition, if \( \mathcal{I} = [i_{r-1}, \alpha) \cup [\beta, \gamma) \cup [\delta, j_{r-1}) \) such that \( i_{r-1} < \alpha < \delta < j_{r-1} \), then we reverse the trail on the whole circuit \( C_k \), so that the trail defined by \( \pi_{Z'} \) alternates at \( v_{\pi_{Z'}(i_{r-1})} \) and \( \pi_{Z'}(j_{r-1}) \) in \( Z' \). Overall there are at most 4 non-alternations of the trail associated to \( \pi_{Z'} \) in \( Z' \), which occur at a subset of \( \{v_{\pi_{Z'}(\alpha)}, v_{\pi_{Z'}(\beta)}, v_{\pi_{Z'}(\gamma)}, v_{\pi_{Z'}(\delta)} \} \). □

Definition 5.16. Denote by \( \sigma \) the assembly structure which describes how to put together the Eulerian circuit defined by \( \pi_{Z'} \) on \( W_k \) from the at most three alternating subtrails of it that run between the at most four sites \( \{v_{\pi_{Z'}(\alpha)}, v_{\pi_{Z'}(\beta)}, v_{\pi_{Z'}(\gamma)}, v_{\pi_{Z'}(\delta)} \} \) of possible non-alternation.

Ideally we want to choose a matching-system from \( S_{\nabla \cap E(Z')}, \nabla \setminus E(Z') \), but unfortunately this set is not well-defined: the degree sequences of \( \nabla \cap E(Z') \) and \( \nabla \setminus E(Z') \) can be slightly different. Recall, that Definition 4.4 can be applied to graphs with different degree sequences. Instead of \( \nabla \cap E(Z') \) and \( \nabla \setminus E(Z') \), we will often write \( \nabla \cap Z' \) and \( \nabla \setminus Z' \).

Definition 5.17. Let \( s(X, Y, Z) \in S(\nabla \cap Z', \nabla \setminus Z') \) be the system of matchings we obtain by modifying the original \( s \in S_{X,Y} \) such that it traces \( \pi_{Z'} \) on \( W_k \) in \( Z' \) (see Lemma 5.10). The non-alternating pairs are not stored in \( s(X, Y, Z) \).

Lemma 5.18. \( |S(\nabla \cap Z', \nabla \setminus Z')| \leq n^4 \cdot |S_{X,Y}| \).

Proof. According to Lemma 5.15 the sites of non-alternation of \( \pi_{Z'} \) in \( Z' \) are vertices of certain edges in \( R \) (\( x_1 \) and the other ends of the start-, end- and current-chord). At each of these sites, \( \text{deg}_{\nabla \cap Z'}(v) = \frac{1}{2} \text{deg}_{\nabla}(v) + 1 \) and \( \text{deg}_{\nabla \setminus Z'}(v) = \frac{1}{2} \text{deg}_{\nabla}(v) - 1 \), or the other way around. Recall Equations 4.5 and 4.6. The number of maximum matchings between edges incident to \( v \) in \( \nabla \cap Z' \) and in \( \nabla \setminus Z' \) is \( [\frac{1}{2} \text{deg}_{\nabla}(v) + 1]! \). Thus there is an extra factor of \( \frac{1}{2} \text{deg}_{\nabla}(v) + 1 \leq n \) compared to the respective factor in the enumeration of \( S_{X,Y} = S(X \setminus Y, Y \setminus X) \). □

Lemma 5.17 proves that an appropriate \( w \) exists in the next definition.

Definition 5.19. Let us define the list of additional parameters:

\[
B(X, Y, Z, s) := (x_1, \sigma, R, w),
\]

where \( w \in \mathbb{N} \) satisfies \( T^w(\pi_r, \text{green}) = (\pi_0, g) \) for some \( g \), and \( w \) is minimal.

Definition 5.20. Let \( \mathbb{B} \) be the set which contains all possible tuples of additional parameters, defined by

\[
\mathbb{B} := \left\{ B(X, Y, Z, s) : Z \in \Upsilon(X, Y, s), \ X, Y \in \mathbb{G}, \ s \in S_{X,Y} \right\}.
\]

Lemma 5.21. The cardinality of the parameter set \( \mathbb{B} \) is \( O(n^8) \) in the unconstrained case, and \( |\mathbb{B}| = O(n^6) \) in the bipartite case.
Proof. Clearly $0 \leq w \leq n^2$, because the generalized $T$-operator decreases the number of green positions in each iteration. Lemma 2.7(g) claims that given $x_1$, $R$ has only $O(n^5)$ possible values ($O(n^3)$ in the bipartite case). Since $\sigma$ has a constant size description and there are at most $n$ choices for $x_1$, we have $|B| \leq O(n^8)$ (and $O(n^6)$ in the bipartite case).

Lemma 5.22. The quadruplet composed of the graphs $Z$, $\nabla$, the matching-system $s(X, Y, Z)$ and $B(X, Y, Z, s)$ uniquely determines the triplet $(X, Y, s)$.

Proof. The matching $s(X, Y, Z)$ assembles $W_i$ and the trails given by $s$ on them for $i \neq k$ (because the lexicographical order is used to number the $W_i$). The edges $E(W_k)$ are assembled into at most three trails.

If $E(W_k)$ is assembled into one trail by $s(X, Y, Z)$, then $Z' = Z \triangle R = G^k_\ell$ and thus $\pi^r_\ell = \pi_{r-1}$. A single bit in $\sigma$ is dedicated to indicate which of the two directions the trail of $\pi_{r-1}$ starts from $v_1$. We can reconstruct $\pi_0$ from $\pi_{r-1}$ and $w$ via Theorem 5.7. The $T$-operator reproduces the primitive alternating circuits $C^k_j$ for $1 \leq j \leq \ell_k$ (Lemma 5.13).

Clearly, $X = G^k_\ell \triangle \cup_{i=1}^{\ell-1} W_i \triangle \cup_{j=1}^{\ell-1} C^k_j$ and $Y = X \triangle \nabla$.

If $E(W_k)$ is not assembled into one trail by $s(X, Y, Z)$, the reconstruction is trickier. Let $\sigma$ contain the list of assembly structures of the at most three trails (the original trail $s$ is a closed trail). The non-alternations in $s(X, Y, Z)$ occur precisely at the boundaries of intervals of positions corresponding to $Z$. A flag in $\sigma$ is dedicated to signaling whether Equation (5.16) or (5.17) holds for $\pi^r_\ell$.

Suppose first, that Equation (5.16) holds. Then $Z = [\alpha, \beta) \cup \gamma, \delta)$, and the values of $\alpha, \beta, \gamma, \delta$ are known (these are the sites of non-alternation). Apply the $T$-operator repeatedly to the positions in the interval $[\alpha, \beta)$ (i.e., restrict the $T$-operator to this interval), starting with an identically green coloring until a fixed point of the $T$-operator is reached. According to Lemma 5.9 this transforms $\pi^r_\ell|_{[\alpha, \beta)}$ back to $\pi_{r-1}|_{[\alpha, \beta]}$. Repeat the procedure for the $[\gamma, \delta)$ interval, too. As in the case when $Z$ is a milestone, $\pi_{r-1}$ and $w$ determine $\pi_0$, which in turn determines $C^k_j$ and then $X$ and $Y$.

Lastly, if Equation (5.17) holds, then $Z = [i_{r-1}, \alpha) \cup [\beta, \gamma) \cup [\delta, j_{r-1})$, such that $i_{r-1} < \alpha$ and $\delta < j_{r-1}$. Compared to the previous case, the extra complexity is determining $i_{r-1}$ and $j_{r-1}$ before we could determine $\pi_{r-1}$. Because $f_{r-1}(\alpha^+) = f_{r-1}(\delta^-) = green$, we have

$$
\begin{align*}
\hat{j}_{r-1} &= \min \{ x > \delta : \exists y < \alpha \text{ such that } (v_x = v_y) \land (x \equiv y \mod 2) \}, \\
\hat{i}_{r-1} &= \max \{ y < \alpha : (v_{j_{r-1}} = v_y) \land (j_{r-1} \equiv y \mod 2) \}.
\end{align*}
$$

Definition 5.23. Let us define the auxiliary structure:

$$\begin{align*}
\hat{M}(X, Y, Z) &:= A_X + A_Y - A_Z, \\
M &:= \left\{ \hat{M}(X, Y, Z) : Z \in \Upsilon(X, Y, s), X, Y \in \mathbb{G}, s \in S_{X,Y} \right\}. 
\end{align*}
$$

where $A_X, A_Y, A_Z$ are the adjacency matrices of $X, Y, Z$, respectively.

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The rows and columns of adjacency matrices are enumerated in such a way that the matrix is symmetric and it has an identically zero main diagonal. In this section we only focus on the role of \( \hat{M} \) in the reconstruction process. We will further discuss its properties in Section 7. We have arrived at the finale of the reconstruction method.

**Lemma 5.24.** The function

\[
\Phi_Z(X, Y, s) := (B(X, Y, Z, s), \hat{M}(X, Y, Z), s(X, Y, Z))
\]

is injective on the tuples satisfying \( Z \in \mathcal{Y}(X, Y, s) \). In other words, \( \Phi_Z \) has an inverse function

\[
\Psi_Z : \mathcal{M} \times \mathcal{B} \times \bigcup_{Z' \in \nabla} \mathcal{S}(\nabla \cap Z', \nabla \setminus Z') \to \{(X, Y, s) : Z \in \mathcal{Y}(X, Y, s), X, Y \in \mathcal{G}, s \in \mathcal{S}_{X,Y}\}
\]

(naturally, \( \Psi_Z \) is only defined on the image set of \( \Phi_Z \)).

**Proof.** The graph \( \nabla = X \triangle Y \) is determined by \( Z \) and \( \hat{M} \), but this information alone does not separate the \( X \)-edges and \( Y \)-edges. The graph \( Z' \) is determined by \( B(X, Y, Z, s) \) and \( Z \). Lemma 5.22 claims that \((X, Y, s)\) can be reconstructed from these objects. \( \square \)

6. **Directed degree sequences**

Now it is time to extend our management for directed degree sequences. This short description goes more or less in parallel with [15] (Erdős, Mezei, Miklós and Soltész, 2018).

Let \( \vec{G} \) be a simple directed graph (parallel edges and loops are forbidden, but oppositely directed edges between two vertices are allowed) with vertex set \( \vec{G}(\vec{G}) = \{x_1, x_2, \ldots, x_n\} \) and edge set \( E(\vec{G}) \). For every vertex \( x_i \in X \) we associate two numbers: the out-degree and the in-degree of \( x_i \). These numbers form the directed degree bi-sequence \( \vec{d} = (\vec{d}_{\text{out}}(x_1), \ldots, \vec{d}_{\text{out}}(x_n)), (\vec{d}_{\text{in}}(x_1), \ldots, \vec{d}_{\text{in}}(x_n))\).

We introduce the following bipartite representation of \( \vec{G} \): let \( B(\vec{G}) = (U, V; E) \) be a bipartite graph where each class consists of one copy of every vertex from \( \vec{G}(\vec{G}) \). The edges adjacent to a vertex \( u_x \) in class \( U \) represent the out-edges from \( x \), while the edges adjacent to a vertex \( v_x \) in class \( V \) represent the in-edges to \( x \) (so a directed edge \( xy \) corresponds to the edge \( u_xv_y \)). If a vertex has zero in- (respectively out-) degree in \( \vec{G} \), then we delete the corresponding vertex from \( B(\vec{G}) \). (This representation was used by Gale [16], but one can find it already in [35] (Petersen, 1981).) The directed degree bi-sequence \( \vec{d} \) gives rise to a bipartite degree sequence \( \vec{D} \):

\[
\vec{D} = ((\vec{d}_{\text{out}}(x_1), \ldots, \vec{d}_{\text{out}}(x_n)), (\vec{d}_{\text{in}}(x_1), \ldots, \vec{d}_{\text{in}}(x_n))).
\]

Since there are no loops in our directed graph, there cannot be any \((u_x, v_x)\) edge in its bipartite representation: these vertex pairs are **non-chords**. It is easy to see that these forbidden edges form a forbidden (partial) matching \( \mathcal{F} \) in the bipartite graph \( B(\vec{G}) \), or in more general terms, in \( B(\vec{D}) \), and we call this a **restricted** bipartite degree sequence.

**Definition 6.1.** For restricted bipartite degree sequences, the set of chords is the vertex pairs of the form \( u_xv_y \) where \( x \neq y \).
By definition, $\mathcal{G}(\widetilde{D})$ is the set of all bipartite realizations of $\widetilde{D}$ which avoid the non-chords from $\mathcal{F}$. Now it is easy to see that the bipartite graphs in $\mathcal{G}(\widetilde{D})$ are in one-to-one correspondence with the possible realizations of the directed degree bi-sequence.

Consider two oppositely oriented triangles, $\overrightarrow{C_3}$ and $\overleftarrow{C_3}$. In order to move between these two realizations of the same degree sequence, Kleitman and Wang \cite{Kleitman:1990:NPB} observed that a new operation is needed, and they proved that with this extra “switching” operation the space of realizations becomes irreducible. Take the symmetric difference $\nabla$ of the bipartite representations $B(\overrightarrow{C_3})$ and $B(\overleftarrow{C_3})$. It contains exactly one alternating cycle (the edges come alternately from $B(\overrightarrow{C_3})$ and $B(\overleftarrow{C_3})$), s.t. each vertex pair of distance 3 along the cycle in $\nabla$ is a non-chord. In this alternating cycle no “classical” switch can be performed. To address this issue, we need an extra switch operation, which is the bipartite analogue of the operation introduced by Kleitman and Wang: we exchange all edges coming from $B(\overrightarrow{C_3})$ with all edges coming from $B(\overleftarrow{C_3})$ in one operation.

In general, a **triple-switch** is defined as follows: take a length-6 alternating cycle $C$ in $\nabla$, and if one of the three vertex pairs of distance 3 in $C$ forms a non-chord, we exchange all edges of $C$ to non-edges and vice versa. It is a well-known fact \cite{Erdos:2013:NPB} \cite{Erdos:2015:NPB} that the set $\mathcal{G}(B(\widetilde{D}))$ of all realizations is irreducible under switches and triple-switches that avoid the $\mathcal{F}$-edges.

The example of $\overrightarrow{C_3}$ and $\overleftarrow{C_3}$ demonstrates why the triple-switch operation is necessary. However, as long as some steps of the Markov-chain require choosing 6 vertices, it seems wasteful to not perform the triple-switch simply because some of the vertex pairs of distance 3 are chords.

In this paper, we relax the restrictions on triple-switches: given a length-6 alternating cycle $C$ in $\nabla$, a triple switch is valid if and only if at least one of the three vertex pairs of distance 3 in $C$ is a non-chord. This relaxation allows us to shave off a factor of $n^4$ from the mixing time of the Markov chain. To see this, compare the proofs of Theorem \ref{thm:8.3} and Theorem \ref{thm:8.8}.

The inner loop of Algorithm \ref{alg:2.1} has to be modified, because the conclusion of Lemma \ref{lem:2.3} does not necessarily hold in the directed case. The adaptation of SWEEP in Algorithm \ref{alg:6.1} works on the bipartite representation $B(\overrightarrow{G})$ instead of the directed graph $\overrightarrow{G}$. If $Z_q$ gets its value from **TRIPLE-SWITCH**, then Lemma \ref{lem:2.4}[d] applies to it, otherwise Lemma \ref{lem:2.4}[f] holds for $Z_q$. Because of this, the statements of Lemma \ref{lem:2.4}[f] and \ref{lem:2.4}[g] and Lemma \ref{lem:5.24} about the bipartite case apply to the directed case as well.

We are ready to define our switch Markov chain on $(\mathcal{G}(\widetilde{D}), P)$ for the restricted bipartite degree sequence $\widetilde{D}$. The transition (probability) matrix $P$ of the Markov chain is defined as follows: let the current realization be $G$. Then

(a) with probability $1/2$ we uniformly choose a set of two vertices $\{u,u'\}$ from $U$ and a set of two vertices $\{v,v'\}$ from $V$. There are two matchings, $\{uv, u'v'\}$ and $\{uv', u'v\}$, between these sets. Let $F$ be one of these matchings, chosen randomly, and let $F'$ be the other matching. If both $F$ and $F'$ consist of chords only and $F \subseteq E(\overrightarrow{G})$ and $F' \cap E(\overrightarrow{G}) = \emptyset$, then perform the switch (so $E(\overrightarrow{G}') = (E(\overrightarrow{G}) \cup F') \setminus F$), otherwise $G' = G$.

(b) With probability $1/2$ we choose a set of three vertices from $U$ and a set of three vertices from $V$. Let $F$ and $F'$ be a uniformly randomly selected pair of disjoint perfect
Algorithm 6.1 Sweeping a primitive circuit in the bipartite representation. The DIRECTED SWEEP assumes that $x_1 x_2 \notin E(G)$ and that $C = (x_1, x_2, \ldots, x_{2\ell})$ is a primitive alternating circuit.

function TRIPLE-SWITCH($G, x_1, [x_{2t-2}, x_{2t-1}, x_{2t}, x_{2t+1}, x_{2t+2}]$)
    return $G + \{x_{2t-2} x_{2t-1} x_{2t} x_{2t+1} x_{2t+2}\} - \{x_{2t-2} x_{2t-1} x_{2t} x_{2t+1} x_{2t+2}\}$
end function

procedure DIRECTED SWEEP($G, [x_1, x_2, \ldots, x_{2\ell}] \rightarrow [Z_1, Z_2, \ldots, Z_{\ell-2}, (Z_{\ell-1})]$)
    $Z_0 \leftarrow G$
    $q \leftarrow 1$
end

$L \leftarrow \{2i \in 2\mathbb{N} : 4 \leq 2i \leq 2\ell, x_1 x_{2i} \text{ is a chord and } x_1 x_{2i} \in E(G)\}$

while $\text{end} < 2\ell$ do
    $\text{start} \leftarrow \min \{2i \in L : 2i > \text{end}\}$
    $2t \leftarrow \text{start} - 2$

    while $2t \geq \text{end}$ do
        if $x_1 x_{2t}$ is a non-chord then
            $Z_q \leftarrow \text{TRIPLE SWITCH}(Z_{q-1}, x_1, [x_{2t-2}, \ldots, x_{2t+2}])$
            $2t \leftarrow 2t - 2$
        else if $x_1 x_{2t}$ is a chord then
            $Z_q \leftarrow Z_{q-1} - \{x_1 x_{2t+2}, x_{2t} x_{2t+1}\} + \{x_1 x_{2t}, x_{2t+1} x_{2t+2}\}$
        end if
        $q \leftarrow q + 1$
        $2t \leftarrow 2t - 2$
    end while

    $\text{end} \leftarrow \text{start}$
end while

end procedure
matchings between these sets. If both $F$ and $F'$ consist of chords only, and the remaining matching between the two chosen sets contains a non-chord, and $F \subseteq E(G)$ and $F' \cap E(G) = \emptyset$, then perform the triple-switch (so $E(G') = E(G) \cup F' \setminus F$), otherwise $G' = G$.

The (triple-)switch moving from $G$ to $G'$ is unique, therefore the probability of this transformation (the jumping probability from $G$ to $G' \neq G$) is:

$$\text{Prob}(G \to (a) G') := P(G, G') = \frac{1}{4} \cdot \frac{1}{\binom{|U|}{2} \binom{|V|}{2}}.$$  (6.1)

and

$$\text{Prob}(G \to (b) G') := P(G, G') = \frac{1}{24} \cdot \frac{1}{\binom{|U|}{3} \binom{|V|}{3}}.$$  (6.2)

The probability of transforming $G$ to $G'$ (or vice versa) is time-independent and symmetric. Therefore, $P$ is a symmetric matrix, where the entries in the main diagonal are non-zero, but (possibly) distinct values. Again, $P(G, G) \geq \frac{1}{4}$, because if $(F, F')$ corresponds to a feasible (triple-)switch, then $(F', F)$ does not. Therefore the chain is aperiodic and the eigenvalues of its transition matrix are non-negative. Each transition $(X, Y) \in E(G(\tilde{D}))$ satisfies $P(X, Y) \geq n^{-4}$.

However it is important to recognize that in papers [20] (Greenhill, 2011) and [22] (Greenhill and Sfragara, 2018) a slightly different Markov chain is studied, where it is assumed that the degree sequences under study are irreducible using switches only. This is the case, for example, for regular directed degree sequence. Papers [4] (Berger and Müller-Hannemann, 2010) and [32] (LaMar, 2011) provide a full characterization of directed degree sequences with this property.

7. The auxiliary matrix $\hat{M}$

The auxiliary matrix $\hat{M} = A_X + A_Y - A_Z$ defined in (5.20) is a linear combination of three adjacency matrices. The row and columns sums are equal to the corresponding degrees prescribed by $\mathbf{d}$. If $Z = G^k_{r_1}$, then $G^k_{r_1} \Delta X \subseteq X \Delta Y$ implies that $\hat{M}$ is a 0–1 matrix. If $Z$ is an intermediate realization, $\hat{M}$ is still a 0–1 matrix except on the entries associated to edges in $R$, since $(Z \Delta R) \Delta X \subseteq X \Delta Y$. These +2 and −1 entries will be called bad entries, and the chords to which they correspond are called type-(2) and type-(−1) chords, respectively.

Lemma 7.1. If $R$ falls under case [c] or [d] of Lemma 2.7, then $R$ contains at most two type-(2) and at most one type-(−1) chords.

Proof. Lemma 2.7[c] or [d] claims that $R$ has at most three elements. Of these, $x_1x_{\text{start}}$ and $x_1x_{\text{end}}$ are edges in $X$, so the entries associated to them in $\hat{M}$ are symmetric pairs of +2 or +1 entries. In case [c] if $R$ contains the third chord, $x_1x_{2t}$, and it is an edge in $X$, then we must have $\text{end} = 2t$, so $R$ actually does not contain $x_1x_{2t}$. Thus $x_1x_{2t} \in R \implies x_1x_{2t} \notin E(X)$, so the entries associated to $x_1x_{2t}$ in $\hat{M}$ are −1’s or 0’s. Case [d] is similar to case [c].
The switch operation is extended to symmetric matrices as follows. Suppose $M \in \mathbb{Z}^{[k] \times [k]}$. For any $x, y \in [k]$ we define the one-edge graph $G_{x,y} = ([k]; \{xy\})$ with the adjacency matrix $A_{xy}$. Clearly, $A_{xy}$ is a symmetric matrix with two 1’s. Let $(x, y; z, w)$ be a list of four pairwise distinct elements of $[k]$. Switching along these four vertices produces the symmetric matrix

$$M' = M + A_{xz} - A_{zy} + A_{yw} - A_{wx}.$$  \hspace{1cm} (7.1)

Clearly, the row and column sums of $M'$ are identical to that of $M$. Notice, that a switch in $Z$ translates into a switch on $\hat{M}$.

Notice, that for bipartite degree sequences, the “top-right” submatrix of this $\hat{M}$ is equal to the auxiliary matrix used in [33] (Miklós, Erdős and Soukup, 2013) (the bipartite adjacency matrix).

**Lemma 7.2.** Let $M \in \mathbb{Z}^{[k] \times [k]}$ be a symmetric matrix with 0’s in the diagonal, such that each row and column sum is in the interval $[1, k - 2]$. Also, suppose that the row sum of the first row is minimal. If the entries of $M$ are 0 and 1, except for at most two symmetric pairs of entries of $+2$ in the first row and in the first column, and at most one symmetric pair or $-1$ entries anywhere in the matrix, then there exist at most 2 switches that transform $M$ into a 0–1 matrix except for at most one pair of symmetric $-1$ entries.

**Proof.** Suppose $M_{1,j} = 2$. We must have $j \neq 1$, which means that the maximum of an entry in the rest of the column of $j$ is 1. Because the column sum is at most $k - 2$ and there is at most one $-1$ in the column, there exist $i \in [k] \setminus \{1, j\}$ such that $M_{i,j} \in \{-1, 0\}$. We have two cases.

\begin{enumerate}
    
    
    \item Suppose that there exists $\ell \in [k] \setminus \{1, i\}$ such that $M_{i,\ell} > M_{1,\ell}$. Since $1 \neq i, \ell$, we assume that $M_{i,\ell} < 2$, therefore $M_{1,\ell} \in \{0, -1\}$. Switch along $(1; i, \ell, j)$ in $M$. The operation decreases $M_{1,j}$ to 1. If $M_{i,\ell} = 0$ then $M_{1,\ell} = -1$, so when the switch creates a symmetric pair of $-1$’s, it also eliminates another pair. The matrix resulting from the switch operation satisfies the assumptions of this lemma and contains two fewer $+2$ entries.

    \item Otherwise, for all $\ell \in [k] \setminus \{1, i\}$ we have $M_{i,\ell} \leq M_{1,\ell}$. Since the row sum of the first row is minimal, we have

    $$\sum_{\ell=1}^{k} M_{1,\ell} \leq \sum_{\ell=1}^{k} M_{i,\ell}$$

    and hence

    $$0 \leq \sum_{\ell \in \{1, i, j\}} (M_{i,\ell} - M_{1,\ell}) = M_{i,1} - M_{1,i} + M_{i,j} - M_{1,j} = M_{i,j} - 2,$$

    because $M$ is symmetric with 0 diagonal. The inequality implies that $M_{i,j} = 2$, so either $i = 1$ or $j = 1$, which contradicts our choice of $i, j$.

\end{enumerate}

By recursion a second pair of entries which equal $+2$ can also be eliminated. \hfill $\square$
8. Applications of the unified method

In this Section we harvest some fruits of our unified machinery, proving a rather general result for all typical degree sequence types.

In 1990 Jerrum and Sinclair published a very influential paper \[25\] (Jerrum and Sinclair, 1990) about fast uniform generation of regular graphs and about realizations of degree sequences where no degree exceeds \(\sqrt{n/2}\). To achieve this goal, they applied the Markov chain they have developed in \[24\] (Jerrum and Sinclair, 1989). Informally it is known as JS chain, and it is sampling the perfect and near-perfect 1-factors on the corresponding Tutte gadget. The rapid mixing nature of the JS chain depends on the ratio of the number of perfect and the number near-perfect 1-factors. As they proved it is applicable if and only if the degree sequence \(d\) belongs to a \(P\)-stable class.

Recall the definition of \(P\)-stability (introduced in Definition 1.2). Careful examination of the known results about rapidly mixing switch Markov chains revealed the fact that all known “good” degree sequence classes (for unconstrained degree bipartite or directed degree sequences) are \(P\)-stable. It raises the conjecture that the switch Markov chains on \(P\)-stable degree classes are rapidly mixing. We resolve this conjecture affirmatively in this section.

For a fixed \(B \in \mathbb{B}\), let the set of compatible auxiliary structures be

\[
\mathcal{M}_B = \left\{ \hat{M} : \exists X, Y, Z \in \mathcal{G}(d), s \in S_{X,Y} \right. \\
\left. \text{s.t. } \hat{M} = A_X + A_Y - A_Z, \ B = B(X, Y, Z, s) \right\}.
\]

To apply the simplified Sinclair’s method, it is sufficient to estimate from above the value of

\[
\sum_{X,Y \in V(G)} \frac{|\{ s \in S_{X,Y} : Z \in \mathcal{T}(X, Y, s) \}|}{|S_{X,Y}|}
\]

for any realization \(Z\).

**Lemma 8.1.** The following bound holds for any unconstrained, bipartite, and directed degree sequence:

\[
\sum_{X,Y \in V(G)} \frac{|\{ s \in S_{X,Y} : Z \in \mathcal{T}(X, Y, s) \}|}{|S_{X,Y}|} \leq n^4 \cdot \sum_{B \in \mathbb{B}} |\mathcal{M}_B|.
\]

**Proof.** According to Lemma 5.24 expression (8.1) can be rewritten as follows:

\[
\sum_{X,Y \in V(G)} \frac{\left| \{ \Psi_Z(\hat{M}(X, Y, Z), B(X, Y, Z, s), s(X, Y, Z)) : Z \in \mathcal{T}(X, Y, s) \} \right|}{|S_{X,Y}|}.
\]

Observe, that \(|S_{X,Y}|\) is already determined by \(\nabla = X \triangle Y\), which in turn is determined by \(Z\) and \(\hat{M}\). Let \(t_{\nabla} := |S_{X,Y}|\). Furthermore, \(Z'\) is determined by \(B\) and \(Z\) (see Equations (5.13) and (5.18)). Let

\[
\mathbb{B}_{\hat{M}} = \left\{ B(X, Y, Z, s) : \exists X, Y \text{ such that } \hat{M} = A_X + A_Y - A_Z, \ s \in S_{X,Y} \right\}.
\]
Continue writing (8.2) as follows and apply Lemma 5.18

\[
\sum_{\hat{M} \in M} \prod_{\{\Psi_Z(\hat{M}, B, s^*) : \exists B \in B, s^* \in S(\nabla \cap Z', \nabla \backslash Z')\}} \leq t_{\nabla} \sum_{\hat{M} \in M} |B_{\hat{M}}| \cdot n^4 \cdot t_{\nabla} \leq n^4 \cdot \sum_{\hat{M} \in M} |B_{\hat{M}}| \leq n^4 \cdot \sum_{B \in B} |M_B|.
\]

□

At this point, the proofs for unconstrained, bipartite, and directed degree sequences slightly diverge. The most general of these is the case of unconstrained degree sequences. First, we discuss this case. Having understood the argument, it is relatively simple to fit it to the cases of the bipartite and directed degree sequence cases. Moreover, the tools required for proving our results on the latter two classes have already been published in [33] (Miklós, Erdős and Soukup, 2013), so their proofs will be less verbose than the next section on unconstrained degree sequences.

8.1. Unconstrained degree sequences

First, let us bound the number of auxiliary structures compatible with a given parameter set. Recall Definition 1.2. The $x^{th}$ unit vector is denoted by $1_x$.

Lemma 8.2. If the stability of an unconstrained degree sequence $d$ is bounded by the polynomial $p(n)$, then

\[|M_B| \leq n^6 \cdot p(n) \cdot |G(d)|\]

holds for any $B \in \mathcal{B}$.

Proof. Equation (5.8) defines $B = (x_1, \sigma, R, w)$. Recall, that $(Z \triangle R) \triangle X \subseteq X \triangle Y$, so the bad entries (+2 and -1 values) in $\hat{M}$ correspond to positions assigned to chords in $R$. Let $M$ be the symmetric submatrix of $\hat{M}$ induced by the vertices of $C_k$ as rows and columns. We have two cases.

Case 1: $\nexists f \in R$ where $x_1 \notin f$ and $f \notin E(C_k)$

All of the non 0–1 entries of $\hat{M}$ are contained in $M$, in the rows and columns associated to $x_1$. Since $Z$ is an intermediate realization of the switch sequence from $G_k^{C_{k+1}}$ to $G_k^{C_{k+1}}$, the degree sequence of $Z[V(C_k)]$ is equal to the degree sequence of $G_k^{C_{k+1}}[V(C_k)]$. Therefore, Lemma 7.1 and Assumption (5.11) implies that we can use Lemma 7.2 to remove the +2’s from $M$ with at most two switches. Hence the same applies to $\hat{M}$. For each switch, the type-(2) chord determines two vertices of the switch, thus there are $n^4$ ways to choose the at most two switches that eliminate the +2 entries.

Let $\hat{M}'$ be the matrix we get after applying the switches defined by Lemma 7.2. Either $\hat{M}'$ is an adjacency matrix of a realization of $d$, or $\hat{M}'$ contains -1 entries at positions associated to the chord $xy$. In the former case $\hat{M}' \in G(d)$, and in the latter $\hat{M}' + A_{xy} + A_{yx} \in G(d + 1_x + 1_y)$.  

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Case 2: \( \exists! f \in R \text{ such that } x_1 \notin f \text{ and } f \notin E(C^k_r) \)

This is only possible if \( R \) falls under case \([e]\) of Lemma 2.7. The auxiliary structure belonging to the intermediate realization before or after \( Z \) on the switch sequence is one switch away from \( \hat{M} \), moreover this switch touches \( f \). There are at most \( n^2 \) switches satisfying these conditions, because \( f \) already determines two vertices. After performing the appropriate switch on \( \hat{M} \), the enumeration of the previous case applies. \( \square \)

We are ready to prove one of the main results of this paper.

**Theorem 8.3.** The switch Markov chain is rapidly mixing on \( P \)-stable unconstrained degree sequence classes.

**Proof.** From Lemma 8.2 and Lemma 5.21 we get:

\[
n^4 \cdot \sum_{B \in \mathcal{B}} |M_B| \leq n^6 \cdot |\mathcal{B}| \cdot p(n) \cdot |G(d)| \leq O(n^{18}) \cdot p(n) \cdot |G(d)|.
\]

Equation (3.9) now follows from Lemma 8.1. Every condition of the simplified Sinclair’s method is satisfied, so the switch Markov chain on \( G(d) \) is rapidly mixing. Theorem 3.3 states that the mixing time is

\[
\tau_\varepsilon(G(d), P) \leq n^4 \cdot m \cdot O(n^{18}) \cdot p(n) \cdot (n^2 - \log \varepsilon) \leq O(n^{22}) \cdot p(n) \cdot m \cdot (n^2 - \log \varepsilon).
\]

where the length of \( T(X, Y, s) \) is bounded by Lemma 4.3. \( \square \)

**8.2. Bipartite degree sequences**

Let \( D \) denote a bipartite degree sequence on \( n = n_1 + n_2 \) vertices.

**Definition 8.4.** Let \( \mathcal{D} \) be an infinite set of bipartite degree sequences. We say that \( \mathcal{D} \) is \( P \)-stable, if there exists a polynomial \( p \in \mathbb{R}[x] \) such that for any \( n_1, n_2 \in \mathbb{N}, n_1 \geq n_2 \), and any degree sequence \( D \in \mathcal{D} \) on \( n_1 \) and \( n_2 \) vertices we have

\[
|G(D) \cup \bigcup_{x \in [n_1], y \in [n_2]} G(D + 1_x + 1_{(n_1+y)})| \leq p(n) \cdot |G(D)|,
\]

where \( 1_x \) is the \( x \)th unit vector.

Recall that a primitive alternating circuit on a bipartite graph is a cycle. Also, for any \( X, Y, Z \in G(D) \), the auxiliary structure \( \hat{M} = A_X + A_Y - A_Z \) is determined by the submatrix spanned by \( U \times V \subset (U \cup V)^2 \). This is the “top-right” submatrix, often called the bipartite adjacency matrix. The “top-left” and the “bottom-right” submatrices are zero.

**Lemma 8.5.** If the stability of a bipartite degree sequence \( D \) is bounded by the polynomial \( p(n) \), then

\[
|M_B| \leq n^4 \cdot p(n) \cdot |G(D)|
\]

holds for any \( B \in \mathcal{B} \).
Proof. The proof is simpler and slightly different than that of Lemma 8.2. Double step is never called in the bipartite case (Lemma 2.6), so either $R$ is empty or Lemma 7.1 applies to it. Hence, there are \( \binom{n_1 n_2}{2} \) possibilities to choose the remaining vertices of the (at most two) switches that eliminate the type-(2) bad chords.

Secondly, we have to make sure that the switches produced by Lemma 7.2 respect the bipartition. As before, let $M$ be the submatrix of $\hat{M}$ induced by the vertices of $C_k r$. Let $H = K_{U(C^r_k)} \uplus K_{V(C^r_k)}$ be the disjoint union of the two cliques within the classes. Instead of applying Lemma 7.2 on $M$, apply it on $M + A_H$. Each row and column sum increased by the same number, therefore assumptions of the lemma are still satisfied. Any switch which eliminates a $+2$ from this matrix which is valid in the unconstrained sense also respects the bipartition. □

Theorem 8.6. The switch Markov chain is rapidly mixing on $P$-stable bipartite degree sequence classes.

Proof. Instead of Lemma 8.2 we use Lemma 8.5. The bound on the size of $\mathcal{B}$ is $O(n^6)$ according to Lemma 8.24. Other than these differences, the proof is identical to that of Theorem 8.3

\[
\sum_{B \in \mathcal{B}} |M_B| \leq n^4 \cdot p(n) \cdot |G(\vec{D})| \leq O(n^{14}) \cdot p(n) \cdot |G(\vec{D})|
\]

Theorem 8.3 states that the mixing time is

\[
\tau_\varepsilon(G(\vec{D}), P) \leq n^4 \cdot m \cdot O(n^{14}) \cdot p(n) \cdot (n^2 - \log \varepsilon) \leq O(n^{18}) \cdot p(n) \cdot m \cdot (n^2 - \log \varepsilon)
\]

where the length of $\Upsilon(X, Y, s)$ is bounded by Lemma 4.3. □

8.3. Directed degree sequences

Recall from Section 6 that instead of directly manipulating directed graphs, we work on their bipartite representations. Formally, the degree sequence of the directed graph is identical to that of its bipartite representation. Through the bipartite representation, directed degree sequence classes inherit a definition of $P$-stability (thus the realizations of the representing [perturbed] bipartite degree sequences also avoid the non-chords). Let $\vec{D}$ denote the bipartite representation of a directed degree sequence $\vec{d}$ on $n$ vertices (so the length of the vector $\vec{D}$ is $2n$).

Lemma 8.7. If the stability of a directed degree sequence $\vec{d}$ is bounded by the polynomial $p(n)$, then

\[
|M_B| \leq n^4 \cdot p(n) \cdot |G(\vec{D})| = n^4 \cdot p(n) \cdot |G(\vec{D})|
\]

holds for any $B \in \mathcal{B}$.

Proof. The proof of Lemma 8.5 applies to the bipartite representation, but we have to check that applying Lemma 7.2 on $M + A_H$ produces switches that avoid the non-chords. Indeed, this is the case, because the non-chords of the form $u_x v_x$ correspond to the main diagonal in $M$, which the switches chosen by the lemma avoid. □
Theorem 8.8. The switch Markov chain is rapidly mixing on $P$-stable directed degree sequence classes.

Proof. By Lemma 5.24, the bound on the size of $B$ is $O(n^6)$, as in the proof of Theorem 8.6. From the previous lemma, we get:

$$n^4 \cdot \sum_{B \in \mathbb{B}} \left| M_B \right| \leq n^4 \cdot |\mathbb{B}| \cdot n^4 \cdot p(n) \cdot |\mathcal{G}(\hat{D})| \leq O(n^{14}) \cdot p(n) \cdot |\mathcal{G}(\hat{D})|.$$ 

Theorem 3.3 states that the mixing time is

$$\tau_\varepsilon(G(D), P) \leq n^6 \cdot m \cdot O(n^{14}) \cdot p(n) \cdot (n^2 - \log \varepsilon) \leq O(n^{20}) \cdot p(n) \cdot m \cdot (n^2 - \log \varepsilon).$$

where the length of $\Upsilon(X, Y, s)$ is bounded by Lemma 4.3. □

9. $P$-stable degree sequence classes

In the proof of almost every previous result on rapid mixing of the switch Markov chain, it turns out there is a short hidden proof that the degree sequences under study are $P$-stable. The unified proof contains most of the technical difficulty of proving rapid mixing of the switch Markov chain.

There have already been successful attempts at unifying some of the proofs, most notably by Amanatidis and Kleer [2, 3], who study the notion of strong stability:

Definition 9.1 (adapted from [2]). Let $D$ be a set of degree sequences. Let $G'(d) = \bigcup_{x,y \in [n]} G(d - 1_x - 1_y)$.

We say that $D$ is strongly stable if there exists a constant $\ell$ such that for any $d \in D$ and any $G' \in G'(d)$ there exists $G \in G(d)$ (which depends on $G'$) such that $|E(G') \Delta E(G)| \leq 2\ell$.

For bipartite graphs, the definition is analogous. The above definition is easily seen to be equivalent with the one given in [2], and it has the advantage that it does not rely on the definition of the Jerrum-Sinclair chain. Having formally defined strong stability, we restate the relevant theorem of Amanatidis and Kleer.

Theorem 9.2 ([2, 3]). The switch Markov chain is rapidly mixing on strongly stable unconstrained and bipartite degree sequence classes.

In the following subsections of this section we discuss all known $P$-stable degree sequence regions. It is an intriguing problem to discover other $P$-stable regions.

9.1. Unconstrained degree sequences

For the sake of having more readable and compact formulas, let $\Delta = \max d$, $\delta = \min d$, and $m = \frac{1}{2} \sum_{v \in V} d(v)$ be functions of $d$.

Recently, Greenhill and Sfragara [22] published a breakthrough result on the rapid mixing of the switch Markov chain.
Theorem 9.3 (22). The switch Markov chain is rapidly mixing on the following family of unconstrained degree sequences:

\[ \mathcal{D}_{GS} := \left\{ \mathbf{d} \in \mathbb{Z}^+ : \delta \geq 1, \ 3 \leq \max \mathbf{d} \leq \frac{1}{3} \sqrt{2m} \right\} \tag{9.1} \]

It turns out that the authors implicitly prove on page 10 of 22 that \( \mathcal{D}_{GS} \) is a \( P \)-stable class. However, this implicit result is actually not new: Jerrum, McKay, and Sinclair extensively studied the notion of \( P \)-stability in their seminal work 27.

Theorem 9.4 (Jerrum, McKay, Sinclair – Theorem 8.1 in 27). The family of unconstrained degree sequences

\[ \mathcal{D}_{JMS} := \left\{ \mathbf{d} \in \mathbb{N}^n : (\max \mathbf{d} - \min \mathbf{d} + 1)^2 \leq 4 \cdot \min \mathbf{d} \cdot (n - \max \mathbf{d} - 1) \right\} \]

is \( P \)-stable.

Theorem 9.5 (Jerrum, McKay, Sinclair – Theorem 8.3 in 27). The family of unconstrained degree sequences

\[ \mathcal{D}_{JMS^+} := \left\{ \mathbf{d} \in \mathbb{N}^n : (2m - n\delta)(n\Delta - 2m) \leq (\Delta - \delta)((2m - n\delta)(n - \Delta - 1) + (n\Delta - 2m)\delta) \right\} , \]

is \( P \)-stable.

Theorem 9.3 implies that the switch Markov chain is rapidly mixing on elements of \( \mathcal{D}_{JMS} \) and \( \mathcal{D}_{JMS^+} \). Moreover, it is easy to see that \( \mathcal{D}_{GS} \subset \mathcal{D}_{JMS^+} \). However, the proofs of Theorems 9.4 and 9.5 actually prove a bit more than just \( P \)-stability. In 27 it is also shown that \( \mathcal{D}_{JMS} \) and \( \mathcal{D}_{JMS^+} \) are strongly stable regions with \( \ell \leq 10 \), so Theorem 9.2 already applies to them.

The following corollary is a consequence of the fact that the degrees in an Erdős-Rényi random graph are tightly concentrated around their expected value.

Corollary 9.6. Let \( G(n,p) \) be an Erdős-Rényi random graph of order \( n \geq 100 \) with edge probability \( p \), where \( p \) is bounded away from 0 and 1 by at least \( \frac{5 \log n}{n-1} \). Then \( \Pr(\mathbf{d}(G(n,p)) \in \mathcal{D}_{JMS^+}) \geq 1 - \frac{3}{n} \).

Proof. We may suppose that \( p \leq \frac{1}{2} \) by taking the complement of \( G \) if necessary. Let \( p = p(n), \ v_1 = \sqrt{\frac{5 \log n}{n-1}} \) and \( m = \frac{1}{2} \sum_{v \in V} \mathbf{d}(v) = \binom{n}{2}(p + \varepsilon_2) \). By Hoeffding’s inequality, we have

\[ \Pr \left( \Delta(G) > (p + \varepsilon_1) \cdot (n - 1) \right) \leq \sum_{v \in V(G)} \Pr \left( \mathbf{d}(v) < (p + \varepsilon_1) \cdot (n - 1) \right) \leq n \cdot e^{-2\varepsilon_1^2(n-1)} \leq \frac{1}{n} , \]

and similarly

\[ \Pr \left( \delta(G) < (p - \min(\varepsilon_1, p)) \cdot (n - 1) \right) \leq \frac{1}{n} . \]
The degree sequence \( d(G(n,p)) \) is in \( D_{\text{JMS+}} \) if it satisfies
\[
(2m - n\delta)(n\Delta - 2m) \leq (\Delta - \delta)((2m - n\delta)(n - \Delta - 1) + (n\Delta - 2m)\delta). \quad (9.2)
\]

First suppose that \( p \geq \varepsilon_1 \). Because increasing \( \Delta \) or decreasing \( \delta \) makes the inequality stricter, without loss of generality, we may substitute \( \Delta = (p + \varepsilon_1) \cdot (n - 1) \) and \( \delta = (p - \varepsilon_1) \cdot (n - 1) \) into (9.2). We calculate
\[
(2m - n\delta) = \left(2 \binom{n}{2}(p + \varepsilon_2) - n(p + \varepsilon_1)(n - 1)\right) = n(n - 1)(\varepsilon_1 + \varepsilon_2),
\]
\[
(n\Delta - 2m) = n(n - 1)(\varepsilon_1 + \varepsilon_2),
\]
\[
(2m - n\delta)(n - \Delta - 1) = n(n - 1)(\varepsilon_1 + \varepsilon_2) \cdot (1 - p - \varepsilon_1)(n - 1),
\]
\[
(n\Delta - 2m)\delta = n(n - 1)(\varepsilon_1 + \varepsilon_2) \cdot (p - \varepsilon_1)(n - 1).
\]

Therefore (9.2) holds if
\[
(n(n - 1)(\varepsilon_1 + \varepsilon_2))^2 \leq 2\varepsilon_1(n - 1) \cdot n(n - 1)(\varepsilon_1 + \varepsilon_2) \cdot (1 - 2\varepsilon_1)(n - 1),
\]
or after simplification,
\[
n\varepsilon_2 \leq (n - 2)\varepsilon_1 - 4\varepsilon_1^2(n - 1).
\]

If \( \varepsilon_2 \leq \frac{\sqrt{\log n}}{n-1} \) then the last inequality is satisfied whenever
\[
n \frac{\sqrt{\log n}}{n-1} \leq (n - 2)\sqrt{\frac{\log n}{n-1}} - 4\log n.
\]

Clearly, the right hand side grows \( \Theta(n^{3/4}) \) faster than the left hand side as \( n \to \infty \), and the inequality already holds for \( n = 100 \). Now for any \( c > 0 \),
\[
\Pr\left(\frac{1}{2} \sum_{v \in V} d(v) > (p + c)\binom{n}{2}\right) \leq e^{-2c^2\binom{n}{2}},
\]
and substituting \( c = \frac{\sqrt{\log n}}{n-1} \) shows that \( \varepsilon_2 \leq \frac{\sqrt{\log n}}{n-1} \) with probability at least \( 1 - 1/n \). Overall, \( \Pr(d(G(n,p)) \notin D_{\text{JMS+}}) \leq \frac{1}{n} + \frac{1}{n} + \frac{1}{n} \) if \( p \geq \varepsilon_1 \).

Now suppose that \( \frac{5\log n}{n-1} \leq p < \varepsilon_1 \). Substituting \( \Delta = (p + \varepsilon_1) \cdot (n - 1) \) and \( \delta = 0 \) into (9.2), we find that \( d(G(n,p)) \in D_{\text{JMS+}} \) if
\[
2m(n\Delta - 2m) \leq \Delta \cdot 2m(n - \Delta - 1).
\]
Rearranging, this inequality holds if \( \Delta (\Delta + 1) \leq 2m \), and substituting for \( \Delta \) and simplifying gives the sufficient condition \( 4\varepsilon_1^2 \leq p - \varepsilon_2 \).

The last inequality is satisfied if \( \varepsilon_2 = \frac{\sqrt{\log n}}{n} \). Therefore, if \( \frac{5\log n}{n-1} \leq p < \varepsilon_1 \), then \( \Pr(d(G(n,p)) \notin D_{\text{JMS+}}) \leq \frac{2}{n} \).

Similar results have already been proved for bipartite Erdős-Rényi graphs \( [14,15] \) (Erdős, Mezei, Miklós and Soltész, 2018), with the requirement that \( p \) is bounded away from 0 and 1 by at least \( 4\sqrt{\frac{2\log n}{n}} \).
9.2. Unconstrained power-law bounded degree sequences

Let us quote two definitions introduced by Gao and Wormald (2016).

Definition 9.7 ([18]). Suppose \( d \in \mathbb{N}^n \) is a degree sequence. If \( \exists C > 0 \), then \( d \) is

- **power-law density-bounded** with parameter \( \gamma \), if for all \( i \in [1, n] \),
  \[
  \left| d^{-1}(i) \right| \leq C n i^{-\gamma}
  \]

- **power-law distribution-bounded** with parameter \( \gamma \), if for all \( i \in [1, n] \)
  \[
  \sum_{j=i}^{n} \left| d^{-1}(j) \right| \leq \sum_{j=1}^{\infty} C n j^{-\gamma}.
  \]

Barabási and Albert (1999) [1] recognized in their seminal paper that a lot of real world networks grow via some form of preferential attachment, and these networks have power-law like degree distributions. The preferential attachment model and its relatives produce graphs with power-law density-bounded degree sequences (see Definition 9.7). However, the degree sequences of most real world networks deviate somewhat from the degree sequences of such synthetic networks. Instead, as noted in [18], the degree sequence of a real world network is much more likely to obey the less restrictive power-law distribution-bound. Compared to the former bound, the latter allows relatively high maximum degrees and longer tails in the degree distribution.

The switch Markov chain is not the only way to exactly sample the uniform distribution on the realizations of a degree sequence. Recently, Gao and Wormald (2018) presented in [19] the first “provably practical” sampler for power-law distribution-bounded degree sequence where \( \gamma \) is allowed to be less than 3; in fact they can go as low as 2.8811. For such degree sequences, they provide a linear time approximate sampler and a polynomial time exact sampler.

In degree distributions of empirical networks following a power-law, the parameter \( \gamma \) is usually between 2 and 3.

Gao and Wormald [18] compute the number of realizations for several types of heavy-tailed degree sequences, and in-turn, those formulas imply \( P \)-stability of the respective classes. They conjectured that the degree sequences obeying a power-law distribution-bound with \( \gamma > 2 \) are \( P \)-stable, which was shown by Gao and Greenhill (2020) in [17]. This is a corroborative example for the applicability of Theorem 8.3, because rapid mixing of such degree sequences is immediately verified, independently from the rapid mixing result of [17].

9.3. Bipartite degree sequences

Let \( D \) be a bipartite degree sequence on \( U \) and \( V \) as vertex classes. We use the following shorthands in this sub-section:

\[
\delta_U = \min_{u \in U} D(u), \quad \delta_V = \min_{v \in V} D(v),
\]

\[
\Delta_U = \max_{u \in U} D(u), \quad \Delta_V = \min_{v \in V} D(v),
\]

and \( m = \sum_{u \in U} D(u) = \sum_{v \in V} D(v) \).
Theorem 9.8 (implicitly proved in Theorem 2 in [15] Erdős, Mezei, Miklós, and Soltész, 2018). The set of bipartite degree sequences $D$ that satisfy

$$2 \leq \Delta \leq \sqrt{\frac{m}{2}},$$

(9.3)

is $P$-stable.

Clearly, Theorems 9.3 and 9.8 are closely related, the difference in constants is caused by the different structural constraints only.

Theorem 9.9 (implicitly proved in Theorem 3 in [15]). The set of bipartite degree sequences $D$ that satisfy

$$(\Delta_U - \delta_U - 1)(\Delta_V - \delta_V - 1) \leq \max \left( \delta_U(|U| - \Delta_V + 1), \delta_V(|V| - \Delta_U + 1) \right)$$

(9.4)

is $P$-stable.

Amanatidis and Kleer presented a bipartite analogue of Theorem 9.4.

Theorem 9.10 (Corollary 18 in [2]). The set of bipartite degree sequences that satisfy both

$$(\Delta_U - \delta_U)^2 \leq 4\delta_V \cdot (|V| - \Delta_U)$$

$$(\Delta_V - \delta_V)^2 \leq 4\delta_U \cdot (|U| - \Delta_V)$$

(9.5)

is $P$-stable (because it is strongly stable).

The following Theorem 9.11 is a bipartite analogue of Theorem 9.4. In some sense, it is stronger than either Theorem 9.9 or 9.10. If one side is regular $(\Delta_U = \delta_U)$, then Inequality (9.4) and (9.6) are automatically satisfied. Inequality (9.4) trivially holds even for almost half-regular bipartite degree sequences $(\Delta_U \leq \delta_U + 1)$. Assuming $|U| = |V|$, $\Delta_U = \Delta_V$, $\delta_U = \delta_V$ are all satisfied, (9.5) and (9.6) are equivalent, and are loosely speaking 4 times better than (9.4).

Theorem 9.11. The set of bipartite degree sequences that satisfy

$$(\Delta_U - \delta_U) \cdot (\Delta_V - \delta_V) \leq 4 \cdot \min \left( \delta_U(|U| - \Delta_V), \delta_V(|V| - \Delta_U) \right)$$

(9.6)

is $P$-stable (because it is strongly stable).

Proof. If the degrees of two vertices in the same class are increased by one in a graphic bipartite degree sequence, then the resulting degree sequence is not graphic. Let $G$ be a realization of $D + 1_u + 1_v$ on $U$ and $V$ as vertex classes. The degree sequence of $G$ is

$$D(G) = \begin{cases} 
    d(x) + 1 & \text{if } x = u_1 \text{ or } v_1, \\
    d(x) & \text{otherwise}.
\end{cases}$$

We claim that there exists an alternating path $P$ of length at most 7 between $u_1$ and $v_1$ in $G$, such that the first and last edges of $P$ are edges of $G$. Assume that no such path exists. Let $U_1, V_2, U_3$ be the set of vertices that are reachable from $v_1$ via an alternating path
Without loss of generality, we may assume that \( k \) is a good candidate for \( P \). Similarly, \( G[U_2, V_2] \) is an empty graph (no alternating paths of length 5), and \( G[U_3, V_3] \) is a complete bipartite graph (no alternating paths of length 7). These observation also imply that \( \{u_1\} \uplus U_1 \uplus U_2 \uplus U_3 \) and \( \{v_1\} \uplus V_1 \uplus V_2 \uplus V_3 \) are subpartitions of \( U \) and \( V \), respectively. Let \( U_4 \) and \( V_4 \) be the remaining vertices of \( U \) and \( V \), respectively.

By definition and the fact that \( G[U_3, V_3] \) is a complete bipartite graph, \( G[U_1 \cup U_3, V_1 \cup V_3] \) is also a complete bipartite graph. The vertices in \( U_2 \) are only adjacent to elements of \( V_1 \cup V_3 \), therefore

\[
|E(U_2, V_1 \cup V_3)| = |E(U_2, V)| \geq \delta_U|U_2|.
\]

Every vertex which is joined by a non-edge to a vertex of \( V_1 \) is contained in \( U_2 \). Therefore \( G[U_4, V_1] \) is a complete bipartite graph and \( |V_1| = d(u_1) \geq \delta_U + 1 \), thus

\[
|E(U_4, V_1 \cup V_3)| \geq |E(U_4, V_1)| > \delta_U|U_4|.
\]

Since \( G[U_1 \cup U_3, V_1 \cup V_3] \) is a complete bipartite graph, we have

\[
|E(U_2 \cup U_4, V_1 \cup V_3)| \leq |V_1 \cup V_3| \cdot (\Delta_V - |U_1 \cup U_3|).
\]

Combining the previous inequalities, we get

\[
\delta_U(|U| - |U_1 \cup U_3| - 1) = \delta_U \cdot |U_2 \cup U_4| < |V_1 \cup V_3| \cdot (\Delta_V - |U_1 \cup U_3|).
\]

Let us substitute \( k_1 = |U_1 \cup U_3| \) and \( k_2 = |V_1 \cup V_3| \), leading to

\[
\begin{align*}
\delta_U(|U| - \Delta_V - 1) &< (\Delta_V - k_1) \cdot (k_2 - \delta_U), \\
\delta_V(|V| - \Delta_U - 1) &< (\Delta_U - k_2) \cdot (k_1 - \delta_V).
\end{align*}
\]

The second inequality is obtained by symmetry. Solving for \( k_1 \), we get

\[
\frac{\delta_V(|V| - \Delta_U - 1)}{\Delta_U - k_2} + \delta_V < k_1 < \Delta_V - \frac{\delta_U(|U| - \Delta_V - 1)}{k_2 - \delta_U}.
\]

Without loss of generality, we may assume that \( \delta_V(|V| - \Delta_U - 1) \leq \delta_U(|U| - \Delta_V - 1) \). Omitting \( k_1 \) from the middle of the above inequality, we have

\[
\delta_V(|V| - \Delta_U - 1)(\Delta_U - \delta_U) < (\Delta_V - \delta_V)(\Delta_U - k_2)(k_2 - \delta_U).
\]

The left hand side in the last inequality is maximal if \( k_2 = \frac{1}{2}(\Delta_U + \delta_U) \). We get

\[
4\delta_V(|V| - \Delta_U - 1) < (\Delta_V - \delta_V)(\Delta_U - \delta_U),
\]

which contradicts the assumptions of this theorem. Thus there exists a suitable alternating path of length at most 7 starting on an edge of \( u_1 \) and ending on an edge of \( v_1 \).

Switching the edges along the alternating path \( P \) transforms \( G \) into a realization of \( D \). The procedure consists of at most 8 non-deterministic choices (vertices of the alternating path), so \( p(|U| + |V|) = (|U| + |V|)^8 \) is a good witness to stability. □
9.4. Directed degree sequences

Let \( \vec{d} \) be a directed degree sequence on \( X \) as vertices. Let \( \vec{d}_{\text{out}} \) be the out-degree sequence and \( \vec{d}_{\text{in}} \) be the in-degree sequence. We use the following abbreviations in this sub-section:

\[
\delta_{\text{out}} = \min_{x \in X} \vec{d}_{\text{out}}(x), \quad \delta_{\text{in}} = \min_{x \in X} \vec{d}_{\text{in}}(x),
\]

\[
\Delta_{\text{out}} = \min_{x \in X} \vec{d}_{\text{out}}(x), \quad \Delta_{\text{in}} = \min_{x \in X} \vec{d}_{\text{in}}(x),
\]

and \( m = \sum_{x \in X} \vec{d}_{\text{out}}(x) = \sum_{x \in X} \vec{d}_{\text{in}}(x) \).

**Theorem 9.12** (implicitly proved in [22] Greenhill and Sfragara, 2018). The set of bipartite degree sequences \( \vec{d} \) that satisfy

\[
2 \leq \max(\Delta_{\text{out}}, \Delta_{\text{in}}) \leq \frac{1}{4} \sqrt{m}, \tag{9.7}
\]

is \( P \)-stable.

**Theorem 9.13** (implicitly proved in Theorem 4 in [15] Erdős, Mezei, Miklós and Soltész, 2018). The set of directed degree sequences \( \vec{d} \) satisfying

\[
2 \leq \max(\Delta_{\text{out}}, \Delta_{\text{in}}) < \frac{1}{\sqrt{2}} \sqrt{m - 4},
\]

is \( P \)-stable.

**Theorem 9.14** (implicitly proved in Theorem 5 in [15]). The set of directed degree sequences \( \vec{d} \) satisfying

\[
(\Delta_{\text{out}} - \delta_{\text{out}}) \cdot (\Delta_{\text{in}} - \delta_{\text{in}}) \leq 2 - n + \\
\max \left( \delta_{\text{out}}(n - \Delta_{\text{in}} - 1) + \delta_{\text{in}} + \Delta_{\text{out}}, \delta_{\text{in}}(n - \Delta_{\text{out}} - 1) + \delta_{\text{out}} + \Delta_{\text{in}} \right)
\]

is \( P \)-stable.

10. Summary

To summarize the new results of the paper we present Table 2, an updated version of Table 1 which contains both entirely new and improved results. A strongly stable class is, as the name suggests, naturally \( P \)-stable, see the papers of Amanatidis and Kleer [2, 3]. Their results already provide a unified framework for proving all previously known bipartite and unconstrained degree sequence results.

The flexibility of our unified method allowed us to extend the rapid mixing results of the switch Markov chain in two directions (in the table):

- vertically (power of machinery) to \( P \)-stable degree sequence classes, and
- horizontally (applicability of machinery) to directed degree sequences.
| unconstrained degree sequences | bipartite deg. seq. | directed deg. seq. |
|-------------------------------|---------------------|-------------------|
| regular [6]                   | (half-)regular [33] | regular [20]      |
| almost half regular [12]      |                     |                   |
| \( \Delta \leq \frac{1}{3} \sqrt{2m} \) [22] | \( \Delta \leq \frac{1}{\sqrt{2}} \sqrt{m} \) [15] | \( \Delta < \frac{1}{\sqrt{2}} \sqrt{m} - 4 \) [15] |
| Power-law distribution-bound, \( \gamma > 2 \) \text{\textsuperscript{†}} | \( (\Delta_U - \delta_U) \cdot (\Delta_V - \delta_V) \leq 4 \delta_U(|U| - \Delta_V), 4 \delta_V(|V| - \Delta_U) \) | similar to bipartite case |
| \( (\Delta - \delta + 1)^2 \leq 4 \cdot \delta (n - \Delta - 1) \) proof in [2, 3] | Theorems 9.9 and 9.10 |                   |
| Erdős-Rényi \( G(n, p) \) | Bipartite Erdős-Rényi [14, 15] | similar to bipartite case |
| \( p, 1 - p \geq \frac{5 \log n}{n - 1} \) | \( p, 1 - p \geq 4 \frac{2 \log n}{n} \) [14, 15] |                   |
| strongly stable degree sequence classes [2, 3] | \( P \)-stable degree sequence classes |

Table 2: Updated version of Table 1 with the new results in this paper. Here \( \Delta \) and \( \delta \) denote the maximum and minimum degrees, respectively. Half of the sum of the degrees is \( m \), and \( n \) is the number of vertices. The notation is similar for bipartite and directed degree sequences. Some technical conditions have been omitted. Gray text is used for previously known results.

\( \text{\textsuperscript{†}} \) Gao and Greenhill [17] list two sufficient conditions for some kind of stability, both of which apply to power-law distribution-bound degree sequences with \( \gamma > 2 \). Rapid mixing is shown through Condition 1 in [17]. Condition 2 of [17] has slightly better constants and implies rapid mixing in combination with Theorem 8.3 or Theorem 9.2.

We have also shown that the degree sequence of the Erdős-Rényi random graph \( G(n, p) \) is rapidly mixing with high probability as \( n \to \infty \), for any edge probability \( p \) satisfying \( p, 1 - p \geq \frac{5 \log n}{n - 1} \).

The notion of \( P \)-stability arises naturally when studying the rapid mixing of the switch Markov chain [25, 27] (Jerrum and Sinclair, 1990, Jerrum, McKay and Sinclair, 1992). It would be really intriguing to find even a small rapidly mixing degree sequence class which is not \( P \)-stable. Finding the bipartite and directed analogues of Theorem 9.5 seems to be a relatively easy and moderately rewarding open problem.

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