Sturm Theorem and a Refinement of Vietoris’ Inequality for Cosine Polynomials

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Abstract. In a recent work \cite{alzer2014}, the authors established the following refinement of the well-known 1958 result of Vietoris on nonnegative cosine polynomials.

\textbf{Proposition.} Let $$T_n(x) = \sum_{k=0}^{n} b_k \cos(kx)$$ with $$b_{2k} = b_{2k+1} = \frac{1}{4^k \binom{2k}{k}}$$ \textup{(}k \geq 0\textup{)}.

\textup{(}\ast\textup{)}

The inequalities $$T_n(x) \geq c_0 + c_1 x + c_2 x^2 > 0$$ \textup{(}c_k \in \mathbb{R}, k = 0, 1, 2\textup{)} hold for all $$n \geq 1$$ and $$x \in (0, \pi)$$ if and only if

$$c_0 = \pi^2 c_2, \quad c_1 = -2\pi c_2, \quad 0 < c_2 \leq \alpha,$$

where

$$\alpha = \min_{0 \leq t < \pi} \frac{T_0(t)}{(t - \pi)^2} = 0.12290\ldots$$

In four places of the proof, use was made of the classical Sturm Theorem on determining the number of real roots of an algebraic polynomial in a given interval. Although absolutely rigorous, the Sturm procedure involves lengthy technical computations carried out with the help of the software MAPLE 13. This article supplements \cite{alzer2014} by providing such details which were omitted in the latter.

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1. Introduction

Please refer to the Abstract for the motivation of writing this article.

Let
\[ X_0(y) = \sum_{k=0}^{n} c_k y^k \]  
(1.1)

be an AP (algebraic polynomial) with real coefficients \( c_k, k = 1, 2, \cdots, n \), and \((\alpha, \beta) \subset \mathbb{R}\) be a given subinterval of the real line. The celebrated Sturm Theorem furnishes a rigorous procedure to determine the number of real roots (multiple roots are counted once) of \( X_0(y) \) in \((\alpha, \beta)\). Details of the Sturm theorem are given in van der Waerden [3, p. 248]; see also Kwong [2].

The steps of the SP (Sturm procedure) are summarized as follows. Suppose that \( \alpha \) and \( \beta \) are not roots of \( X(y) \), which is true in most applications.

1. Compute \( X_1(y) = X_0'(y) \), the derivative of \( X_0(y) \).
2. Compute the Sturm sequence of polynomials \( X_2(y), X_3(y), \cdots \) using the Euclidean algorithm: Each \( X_i(y) \) is the negative of the remainder when \( X_{i-2} \) is divided by \( X_{i-1} \). In the algorithm used by MAPLE, each \( X_i \) is normalized (by dividing by a positive constant) so that the leading coefficient is \( \pm 1 \).
3. Count the number of sign changes in each of the two sequences \{\( X_i(\alpha) \)\} and \{\( X_i(\beta) \)\}.
4. The difference of these two numbers is the number of real roots of \( X(y) \) in \((\alpha, \beta)\).

The procedure can be easily modified if \( \alpha \) and/or \( \beta \) are roots of \( X(y) \). MAPLE provides a simple command to automate the procedure. It displays only the desirable output from step 4, and hides all the intermediate, less relevant data from steps 1 to 3.

Every CP (cosine polynomial) \( \sum b_k \cos(kx) \) can be rewritten as an AP of the variable \( y = \cos(x) \) while every SP (sine polynomial) \( \sum a_k \sin(kx) \) is the product of \( \sin(x) \) and an AP of \( y \). This simple observation has enabled the authors to successfully exploit the SP in helping (often only a relatively small portion of the entire proof is based on the SP) to obtain new results on NN (nonnegative) TP (trigonometric polynomials), some of which have been described in [2].

Intensive computations are involved in the practical implementation of the SP. In our study, we have used the software MAPLE 13 to carry out these computations. As we have adamantly pointed out in [2], proofs based on the SP are absolutely theoretically rigorous. However, some researchers consider the details of such computations to be too technical and uninteresting to be included in the presentation of the results in a theoretical research article. With this in mind when we wrote [1], we have omitted all such details, and referred interested readers to this article.

The authors do not intend to publish this article in a regular journal. It will be archived in arXiv and permanently available in the internet.

2. Use of SP in [1]

The following applications of the SP appear in the same order as in [1].

(1) Lemma 1.
\[ \min_{0 \leq t < \pi} \frac{T_6(x)}{(x - \pi)^2} = 0.1229... \]  
(2.1)

attained at \( x = 0.726656896 \ldots \).
The proof makes use of the following functions

$$\eta(x) = 10x^6 + 6x^5 - 12x^4 - \frac{11}{2}x^3 + \frac{29}{8}x^2 + \frac{11}{8}x + \frac{9}{16} \tag{2.2}$$

$$\mu(x) = \eta(x)\eta''(x) - \frac{1}{2}\eta'(x)^2. \tag{2.3}$$

and

$$\nu(x) = (1 - x^2)^3\mu(x)^2 - 4(0.1229)x^2\eta(x)^3 \tag{2.4}$$

and states the claim that $\mu$ and $\nu$ have no zeros in $[0.65, 0.95]$.

Using MAPLE, we obtain the explicit expressions

$$\mu(x) = 1200x^{10} + 1200x^9 - 1890x^8 - 1854x^7 + 938x^6 + 987x^5 + \frac{615}{8}x^4 - \frac{835}{8}x^3 - \frac{1659}{16}x^2 - \frac{297}{16}x + \frac{401}{128}$$

and

$$\nu(x) = -1440000x^{26} - 2880000x^{25} + 7416000x^{24} + 17625600x^{23} - 14981700x^{22} -$$

$$47224920x^{21} + \frac{62018162}{5}x^{20} + \frac{1799880978}{25}x^{19} + \frac{922239083}{250}x^{18} -$$

$$\frac{8419950417}{1250}x^{17} - \frac{92661304227}{5000}x^{16} + \frac{190171138621}{5000}x^{15} +$$

$$\frac{76293005877}{40000}x^{14} - \frac{107540100801}{10000}x^{13} - \frac{810190932293}{8000}x^{12} -$$

$$\frac{3311349049}{8000}x^{11} + \frac{880096473123}{32000}x^{10} + \frac{10330939713}{8000}x^9 -$$

$$\frac{236121121411}{1280000}x^8 - \frac{425398321107}{1280000}x^7 - \frac{840336154901}{10240000}x^6 +$$

$$\frac{19241962691}{1280000}x^5 - \frac{164343549777}{10240000}x^4 + \frac{18143739883}{5120000}x^3 -$$

$$\frac{428328477}{1280000}x^2 - \frac{119097}{1024}x + \frac{160801}{16384}.$$ 

Suppose that in a MAPLE session, the variables “mu” and “nu” have been assigned the polynomials $\mu(x)$ and $\nu(x)$, respectively. Applying the SP to the two polynomials is accomplished simply by the commands

```
sturm(mu, x, 65/100, 95/100); sturm(nu, x, 65/100, 95/100);
```

The output are both 0, meaning that $\mu(x)$ and $\nu(x)$ have no zeros in $[0.65, 0.95]$. That is all we are interested to know.

Note that in the above commands, we have to use 65/100 and 95/100 in place of the decimal forms 0.65 and 0.95, respectively. If the decimal forms are used, MAPLE will perform the SP using floating point arithmetic instead of exact arithmetic. In the floating point mode, rounding errors can and very often do lead to erroneous results.

MAPLE actually makes use of the Sturm sequence in its internal computation, without displaying them explicitly. If one is curious, one can see them using the command

```
sturmseq(mu, x);
```
Just for the sake of illustration, we list below the first few members of the Sturm sequence of \( \mu(x) \), starting with \( \mu_1(x) = \mu'(x) \).

\[
\begin{align*}
\mu_1 &= x^9 + \frac{9}{10} x^8 - \frac{63}{50} x^7 + \frac{213}{2000} x^6 + \frac{469}{800} x^5 + \frac{329}{41600} x^4 + \frac{41}{1600} x^3 - \frac{167}{6400} x^2 - \frac{553}{32000} x - \frac{99}{64000} \\
\mu_2 &= x^8 + \frac{5/6}{x^7} - \frac{25249}{24300} x^6 + \frac{2429}{2700} x^5 + \frac{43}{6280} x^4 + \frac{6091}{38880} x^3 + \frac{473}{2880} x^2 + \frac{1951}{64800} x + \frac{10619}{155200} \\
\mu_3 &= x^7 + \frac{164171}{403140} x^6 - \frac{253857}{134380} x^5 + \frac{24703}{26876} x^4 + \frac{86929}{161256} x^3 + \frac{104433}{430016} x^2 + \frac{24223}{537520} x + \frac{50933}{12900480} \\
\mu_4 &= -x^6 + \frac{361783625529}{2968641792581} x^5 + \frac{617789163720}{2968641792581} x^4 + \frac{5544754939585}{11874567170324} x^3 - \frac{1101526123005}{47498266861296} x^2 - \frac{1058471}{94996537362592} x + \frac{94000000}{94996537362592}
\end{align*}
\]

The two sequences of numbers mentioned in step 3 of the SP are

\[
\{\mu_1(0.65)\} = \{0.006, \ 0.01, \ 0.01, \ -0.01, \ -0.05, \ 0.01, \ 0.01, \ -0.4, \ -0.6, \ -0.9, \ -1\} \\
\{\mu_1(0.95)\} = \{0.2, \ 0.2, \ 0.09, \ -0.4, \ -1.1, \ 0.9, \ -0.1, \ -1.3, \ -1.3, \ -1.2, \ -1\}
\]

There are 3 changes of sign in the first sequence, after the third, fifth, and seventh terms, respectively. There are also 3 changes of sign in the second sequence, after the third, fifth, and sixth terms, respectively. Hence, the outcome of step 4 is 0.

We are lucky that MAPLE hides all such gory details from us.

\( \Box \)

\textbf{(2) Lemma 7.} Let

\[
C_n(x) = \sum_{k=0}^{n} (-1)^k b_k \cos(kx).
\]

(where \( b_k \) are defined by \( \ast \) in the Abstract.) If \( 2 \leq n \leq 21 \ (n \neq 6) \) and \( x \in (5\pi/8, \pi) \), then

\[
\frac{820}{33} \left(1 - \cos \frac{x}{10}\right) \leq C_n(x).
\]

\textbf{Proof.} We set \( y = x/10 \) and

\[
P_n(y) = C_n(10y) - \frac{820}{33} \left(1 - \cos(y)\right).
\]

Letting \( Y = \cos(y) \). Then \( P_n(y) \) is an algebraic polynomial in \( Y \). We denote this polynomial by \( P_n(Y) \), where \( Y \in [\cos(\pi/10), \cos(\pi/16)] = [0.951..., 0.980...] \). Applying Sturm’s theorem gives that \( P_n(Y) \) has no zero on \([0.951, 0.981]\) and satisfies \( P_n(0.97) > 0 \). It follows that \( P_n \) is positive on \([\pi/16, \pi/10]\). This implies that (2.5) holds.

Let us use the case \( n = 2 \) as an illustration. We have

\[
P_2(y) = 1 - \cos(10y) + \frac{1}{2} \cos(20y) - \frac{-820}{33} \left(1 - \cos(y)\right)\]

Assume that the MAPLE variable “\( P \)” has already been assigned this cosine polynomial. The command

\[
X := \text{subs}(\cos(y)=Y, \text{expand}(P));
\]

produces the AP \( P_2(Y) \) mentioned in the proof and assigns it to the variable “\( X \).

\[
X = 262144 Y^{20} - 1310720 Y^{18} + 2785280 Y^{16} - 3276800 Y^{14} + 2329600 Y^{12} - 1025536 Y^{10} + 275840 Y^8 - 43360 Y^6 + 3700 Y^4 - 150 Y^2 + \frac{820}{33} Y - \frac{-1475}{66}
\]

Our initial goal is to affirm that \( X \) is NN in the interval \( Y \in [\cos(\pi/10), \cos(\pi/16)] \). However, the MAPLE command “\text{sturm}” can only handle intervals where the endpoints \( \alpha \) and \( \beta \) are
rational numbers. To overcome that obstacle, we show that \( X \) is NN in the slightly larger interval \([0.951, 0.981]\) using the command
\[
\text{sturm}(X, Y, 951/1000, 981/1000);
\]
The same process has to be repeated for the other cases \( 2 < n \leq 21 \) \((n \neq 6)\). The following MAPLE snippet can be used for that purpose.

```maple
b := proc (n) local n1;
    if n = 1 or n = 0 then 1
    else n1 := floor((1/2)*n)+1;
        factorial(2*n1-3)/(2^(2*n1-3)*factorial(n1-1)*factorial(n1-2))
    end if
end proc;

C := n -> 1 + add((-1)^k*b(k)*cos(k*x), k = 1 .. n);

for n from 2 to 21 do
    P := C(n) - 820/33 * (1-cos((1/10)*x));
    X := subs(cos(y) = Y, expand(subs(x = 10*y, expand(P))));
    print((n,sturm(X, Y, 951/1000, 981/1000)));
end do:
```

The first six lines define a procedure to generate the coefficients of the Vietoris polynomial: the command “\( b(n) \)” produces \( b_n \) as defined in \((\ast)\). The next line defines a procedure to generate \( C_n(x) \) defined in Lemma 7. The remaining lines constitute a do loop to apply the SP to verify \((2.5)\), for \( n \) from 2 to 21.

The output consists of pairs of numbers, “\( n \)” and the number of roots of \( X \) in the interval in question. The latter is 0 except for \( n = 6 \). That is why this case is excluded in Lemma 7.

Remark 1. The statement (fourth line from the bottom) used to generate \( X \) from \( P \) is different from that given earlier. This is because the MAPLE command “\text{expand}” only knows how to expand \( \cos(kx) \) when \( k < 100 \). The earlier statement will fail if \( n \geq 10 \). The modified statement expands the expression before making the substitution \( y = x/10 \). Then after the substitution another expansion is performed to obtain \( X \).

Remark 2. As \( n \) increases, the degrees of the corresponding \( P \) and \( X \) increase and the time needed by the SP increases significantly. The SP is therefore not recommended for dealing with TP of high degrees.

In fact, the SP is seldom adequate when establishing general results. At best, it plays an assistant role used to take care of a few exceptional cases involving lower degree polynomials.

Remark 3. The symbol “;” is used to end the do loop in the last line to suppress printing of the intermediate computation results, namely, the various \( P \) and \( X \) in the iterations. If one is curious to see what these results are, one can use the regular statement terminator “;,” instead. When \( n \) is large, the expression \( X \) is rather lengthy.

(3) Lemma 8. Let
\[
\Delta(x) = \sum_{k=0}^{21} (-1)^k (b_k - b_{22}) \cos(kx) - \frac{820}{33} \left( 1 - \cos \frac{x}{10} \right).
\]
If $5\pi/8 \leq x \leq 2.68$, then $\Delta(x) > 0.29$;
if $2.68 \leq x \leq 2.83$, then $\Delta(x) > 0.46$;
if $2.83 \leq x \leq 2.98$, then $\Delta(x) > 0.64$;
if $2.98 \leq x \leq 2.970$, then $\Delta(x) > 0.90$;
if $2.970 \leq x \leq 3.021$, then $\Delta(x) > 1.32$;
if $3.021 \leq x \leq 3.051$, then $\Delta(x) > 1.78$.

Proof. Let $5\pi/8 \leq x \leq 2.68$. We have $\cos(\pi/16) = 0.980...$ and $\cos(0.268) = 0.964...$. The function $\Delta - 0.29$ is an AP in $Y = \cos(x/10)$. An application of Sturm’s theorem shows that this function is positive on $[0.964, 0.981]$. This leads to $\Delta(x) > 0.29$ for $x \in [5\pi/8, 2.68]$. Using the same method of proof we obtain the other estimates for $\Delta(x)$. $\square$

Essentially this Lemma reveals good lower bounds of $\Delta(x)$ in progressively smaller subintervals of $[5\pi/8, \pi]$. Numerical optimization methods can be used to estimate $\min \{\Delta(x)\}$ in any given interval, but that cannot be taken as a rigorous proof. Especially when the objective function is oscillatory in the interval, as is the case of $\Delta(x)$ on $[5\pi/8, 2.68]$, a numerical algorithm may erroneously yield one of the larger local minimum instead of the global one. The SP provides a theoretically rigorous lower bound.

(4) As usual, we adopt the notation

$$(a)_0 = 1, \quad (a)_n = \prod_{k=0}^{n-1} (a + k) = \frac{\Gamma(a + n)}{\Gamma(a)} \quad (n \geq 1),$$

and define

$$d_{2k} = d_{2k+1} = \frac{(69/100)k}{k!} \quad (k = 0, 1, 2, ...). \quad (2.7)$$

**Lemma 11.** Let

$$I(x) = \sum_{k=0}^{21} \left( b_k - \frac{b_{22}d_k}{d_{22}} \right) \cos(kx). \quad (2.8)$$

If $0 < x \leq 0.1$, then $I(x) > 1.5$.

Proof. With $Y = \cos(x) \in [\cos(0.1), 1] \subset [0.995, 1]$, $(I(x) - 1.5)$ is an AP in $Y$. The SP reveals that it is positive in the interval. It follows that $I(x) > 1.5$ for $x \in [0, 0.1]$. $\square$

Here we are dealing with a single CP, $(I(x) - 15)$. Again to avoid an irrational endpoint, we use the slightly larger interval $[995/1000, 1]$ for the corresponding AP. Suppose that $I(x)$ has been assigned to the variable “$Ix”$, the MAPLE command to use is

```
sturm(subs(cos(x)=Y, expand(Ix)), Y, 995/1000, 1);```

**References**

[1] H. Alzer, M.K. Kwong, A Refinement of Vietoris’ Inequality for Cosine Polynomials (2014 preprint).
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