Isochronous Oscillators

F. Calogero, F. Leyvraz

To cite this article: F. Calogero, F. Leyvraz (2010) Isochronous Oscillators, Journal of Nonlinear Mathematical Physics 17:1, 103–110, DOI: https://doi.org/10.1142/S1402925110000611

To link to this article: https://doi.org/10.1142/S1402925110000611

Published online: 04 January 2021
ISOCHRONOUS OSCILLATORS

F. CALOGERO

Dipartimento di Fisica, Università di Roma “La Sapienza”, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma
francesco.calogero@roma1.infn.it
francesco.calogero@uniroma1.it

F. LEYVRAZ

Centro Internacional de Ciencias, Cuernavaca, Mexico
Departamento de Física, Universidad de los Andes, Bogotá, Colombia
fa.leyvraz44@uniandes.edu.co

Received 12 May 2009
Accepted 11 August 2009

We exhibit the solution of the initial-value problem for the system of 2N + 2 oscillators characterized by the Hamiltonian

\[ H(\hat{p}_0, \hat{q}_0, \hat{p}_0, \hat{q}_0, \hat{p}_0, \hat{q}_0, \ldots, \hat{p}_0, \hat{q}_0) = \frac{1}{2}(\hat{p}_0^2 - \hat{q}_0^2 + \Omega^2(\hat{q}_0^2 - \hat{q}_0^2)) \]

\[ + \left( \frac{\hat{p}_0 - \Omega \hat{q}_0}{b} \right)^2 \sum_{n=1}^{N} \left( \hat{p}_n^2 - \hat{q}_n^2 \right) + \omega_n^2(\hat{q}_n^2 - \hat{q}_n^2) \]

where \( N \) is an arbitrary positive integer, \( \Omega \), \( b \) and \( \omega_n^2 \) are \( N + 2 \) arbitrary real constants, \( \hat{q}_m, \hat{p}_m \) with \( m = 0, 1, \ldots, N \) are the \( 2N + 2 \) canonical coordinates and \( \hat{p}_m, \hat{q}_m \) the corresponding \( 2N + 2 \) canonical momenta. In the classical context the solution is completely periodic with period \( T = 2\pi/|\Omega| \) (for arbitrary initial data). In the quantal context the (infinitely degenerate) spectrum is equispaced, with spacing \( \hbar |\Omega| \); all the corresponding eigenfunctions are also exhibited. This finding obtains as special case of a more general (new) class of isochronous Hamiltonians.

Keywords: Nonlinear oscillators; isochronous Hamiltonians; quantization; equispaced spectrum; infinite degeneracy.

1. Introduction

The investigation of periodic dynamical systems has been an important thread throughout the evolution of the mathematical formulation of the laws of physics and the development of the theory of differential equations. In the last century a major boost to this type of investigations, but mainly restricted to planar systems with 2 time-dependent variables, was motivated by the famous, still open, 16th Hilbert problem, see for instance the review papers [1–3].
Recently we introduced [4] a new technique, of rather general applicability (also to systems with an arbitrary number of degrees of freedom), allowing to modify a Hamiltonian so that, in the classical context, the solutions of the initial-value problem of the modified Hamiltonian are isochronous: completely periodic with a fixed period independent of the initial data — provided they are in an appropriate — open, fully dimensional — phase space region, which in the case of this new technique [4] actually coincides with the entire phase space (for a review of results on isochronous Hamiltonians, including this development, see [5]). A remarkable application of this technique has been discussed, in the context of the general many-body problem, in [6]. Another class of (classical) isochronous systems, not covered by the review [5], has been identified quite recently [7]. In the present paper we introduce yet another variant of the approach to generate isochronous Hamiltonians, and we report a simple application of this technique, yielding a dynamical system of coupled oscillators whose solutions are isochronous for arbitrary initial data. We moreover show that, in the quantal context, this Hamiltonian features an (infinitely degenerate) equispaced spectrum.

The results of this paper are reported in the following Sec. 2; they are then proven in Sec. 3. A very terse Sec. 4, entitled "Outlook", concludes the paper.

2. Results

Proposition 1. The Hamiltonian

\[
H(p_0, \begin{pmatrix} q_0 \\ \hat{q}_0 \end{pmatrix}) = \frac{1}{2}(p_0^2 + \Omega^2 \hat{q}_0^2) + \frac{p_0 + \delta \Omega}{b} h(\begin{pmatrix} q_0 \\ \hat{q}_0 \end{pmatrix})
\]  

is isochronous: there is an open, fully dimensional region of its phase space where all its solutions are periodic with period

\[
T = \frac{2\pi}{|\Omega|} \quad (2a)
\]

\[
q_m(t+T) = q_m(t), \quad p_m(t+T) = p_m(t), \quad m = 0, 1, \ldots, N. \quad (2b)
\]

Notation and Related Remarks. Here (and in the following Propositions 2 and 3) the context is of course that of classical Hamiltonian dynamics: \( q_0 \) and the \( N \) components of the \( N \)-vector \( \hat{q} \equiv (q_1, \ldots, q_N) \) are the \( 1+N \) canonical coordinates, likewise \( p_0 \) and the \( N \) components of the \( N \)-vector \( \hat{p} \equiv (p_1, \ldots, p_N) \) are the corresponding \( 1+N \) canonical momenta; \( h(\begin{pmatrix} q_0 \\ \hat{q}_0 \end{pmatrix}) \) is a, largely arbitrary, Hamiltonian (independent of the canonical variables \( p_0 \) and \( q_0 \); it might of course depend on other, constant, quantities, besides the \( N \) canonical coordinates \( q_0 \) and the corresponding \( N \) canonical momenta \( p_0 \), with \( n = 1, \ldots, N \)); the two constants \( \Omega \) and \( b \) are real but otherwise arbitrary (both, of course, nonvanishing; indeed the constant \( b \) — with the dimensionality of a momentum — has been introduced merely to keep good track of dimensions and it only plays a quite marginal role, indeed the restriction that it be real could be lifted without significant consequences except for some notational complications in some of the following formulae); and \( i \) is the imaginary unit, \( i^2 = -1 \).

The independent variable \( t \) ("time") characterizing (see below) the evolution, due to this Hamiltonian, of the canonical variables \( q_m \equiv q_m(t), p_m \equiv p_m(t) \), is hereafter assumed to be real (indeed nonnegative, starting from the initial time \( t = 0 \)); differentiations with respect to it will be denoted below by superimposed dots. On the other hand, due to the complex
character of this Hamiltonian — caused by the explicit presence of the imaginary unit \(i\) in its definition (1) — the canonical variables \(q_m\) and \(p_m\) are complex numbers, and we will use superimposed hats respectively checks to identify their real respectively imaginary parts, according to the following definitions:

\[
q_m \equiv q_m + i\bar{q}_m, \quad p_m \equiv \hat{p}_m + i\check{p}_m, \quad m = 0, 1, \ldots, N.
\] (3)

However, as it is well known (see for instance [8]), the same time-evolution produced by the complex Hamiltonian (1) for the \(2N + 2\) complex canonical variables \(q_m, p_m\), and via these formulæ for the \(4N + 4\) real variables \(\hat{q}_m, \check{q}_m, \hat{p}_m, \check{p}_m\), is yielded, directly for these \(4N + 4\) variables, by the real Hamiltonian

\[
H(\hat{p}_0, \check{p}_0, \hat{q}_0, \check{q}_0, \hat{q}, \check{q}) = \text{Re}\{H(\hat{p}_0 - i\check{p}_0, \hat{q}_0 + i\check{q}_0, \hat{q} + i\check{q})\},
\] (4)

with the \(2N + 2\) real quantities \(\hat{q}_m, \check{q}_m\) playing now the role of canonical coordinates and the \(2N + 2\) real quantities \(\hat{p}_m, \check{p}_m\) being the corresponding canonical momenta (do note the negative signs appearing in the right-hand side of (4)). Here and hereafter whenever we use the indices \(n\) respectively \(m\) it is understood that they run from 1 to \(N\) respectively from 0 to \(N\).

The fact that the validity (proven in the following section) of Proposition 1 requires hardly any restriction on the Hamiltonian \(h(\hat{p}, \check{q})\) might be considered remarkable or trivial, depending on the level of understanding of the mechanism that underpins this fact (as is indeed the case for any valid mathematical finding). This mechanism is analogous to — yet different from — that reported in previous papers [4–6]: indeed the isochronous Hamiltonian, (1), introduced here is to some extent simpler than that introduced previously, in particular it involves \(h(\hat{p}, \check{q})\) linearly, see (1), rather than quadratically, see [4] (another Hamiltonian involving \(h(\hat{p}, \check{q})\) linearly was introduced in [6], see Eq. (42) there, but the property of periodicity of its generic solution is incorrectly attributed there to this Hamiltonian). On the other hand the Hamiltonian (1) has the disadvantage of being complex; yielding a complex dynamics; as indicated above this can be remedied, allowing to return to a context of real Hamiltonian dynamics, but at the cost of doubling the number of canonical variables and possibly of causing the resulting Hamiltonian (involving the real and imaginary parts of \(h(\hat{p} - i\check{p}, \hat{q} + i\check{q})\); see (4), (3) and (1)) to be much uglier — although the example on which we focus in this paper shows that this need not happen, see below.

**Proposition 2.** The solution of the initial-value problem for the complex Hamiltonian

\[
H(\hat{p}_0, \check{p}_0, \hat{q}_0, \check{q}_0) = \frac{1}{2} (\hat{p}_0^2 + \Omega^2 \check{q}_0^2) + \frac{p_0 + d\Omega}{6} \sum_{n=1}^{N} h_n(p_n, q_n),
\] (5a)

\[
h_n(p_n, q_n) = \frac{1}{2} (q_n^2 + \Omega^2 \check{q}_n^2),
\] (5b)

is provided by the following formulæ:

\[
q_n(t) = q_n(0) + \frac{E}{\Omega^2} \cos(\Omega t) + \left[p_n(0) + \frac{E}{\Omega} \right] \frac{\sin(\Omega t)}{\Omega} - \frac{E}{\Omega^2},
\] (6a)

\[
p_n(t) = p_n(0) + \frac{E}{\Omega} \cos(\Omega t) - \left[q_n(0) + i\frac{E}{\Omega} \right] \frac{\sin(\Omega t)}{\Omega} + \frac{E}{\Omega},
\] (6b)
The solution of the initial-value problem for the real Hamiltonian

\begin{align}
H(p_0, \hat{p}_0, \hat{p}_b; \hat{q}_0, \hat{\dot{q}}_0, \hat{\ddot{q}}_0) &= \frac{1}{2} [\hat{\dot{p}}_0^2 - \hat{\dot{p}}_b^2 + \Omega^2 (\hat{q}_0^2 - \hat{\dot{q}}_0^2)] \\
+ \sum_{n=1}^{N} \hat{h}_n(p_n, \hat{p}_n, \hat{q}_n, \hat{\dot{q}}_n) + \sum_{n=1}^{N} \hat{h}_n(p_n, \hat{p}_n, \hat{q}_n, \hat{\ddot{q}}_n),
\end{align}

(8a)

is provided by the following formulas:

\begin{align}
\hat{q}_m(t) &= \text{Re}[q_m(t)], \quad \hat{\dot{q}}_m(t) = \text{Im}[q_m(t)], \\
\hat{p}_m(t) &= \text{Re}[p_m(t)], \quad \hat{\dot{p}}_m(t) = \text{Im}[p_m(t)],
\end{align}

(9a)

where \(q_m(t), p_m(t)\) are given by the explicit expressions (6) (for \(m = 0\)) and (7) (for \(m = n\), of course with \(q_m(0) = \hat{q}_m(0) + \hat{\dot{q}}_m(0), p_m(0) = \hat{p}_m(0) + i \hat{\dot{p}}_m(0)\), see (3). \(\Box\)

Note that the assumed property of the constant \(\Omega\) to be real and nonvanishing guarantees that, for arbitrary initial data, the quantities \(q_m(t), p_m(t)\) given by these formulae are all periodic with period \(T\), see (2a); but this property does not require that the \(N + 1\) constants \(\omega_n^2\) and \(b\) be real — although this requirement is indeed essential to guarantee that the Hamiltonian (8) be real. Of course the specific evolution of these solutions within the period \(T\) depends quite significantly on the sign of the real quantities \(\omega_n^2\) and as well on the magnitude of these quantities and of \(\Omega\).

In the following, to treat the quantal case, we assume for simplicity \(\Omega, b\) and all the circular frequencies \(\omega_n\) to be real.

Proposition 4. The stationary Schrödinger equation obtained, in the quantal context, by applying the standard quantization prescription

\begin{align}
\hat{q}_m \Rightarrow x_m, \quad \hat{\dot{q}}_m \Rightarrow y_m, \quad \hat{p}_m \Rightarrow -i \hbar \frac{\partial}{\partial x_m}, \quad \hat{p}_m \Rightarrow -i \hbar \frac{\partial}{\partial y_m}, \quad m = 0, \ldots, N,
\end{align}

(10)
to the real Hamiltonian (8), features an (infinitely degenerate) equispaced spectrum with spacing $\hbar$: 

$$
\left\{ \frac{1}{2} \hbar^2 \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) + \Omega^2 (x_n^2 - y_n^2) \right\} 
- \hbar^{-1} \left( i\hbar \frac{\partial}{\partial x_n} + \Omega y_n \right) \frac{1}{2} \sum_{n=1}^{N} \left[ \hbar^2 \left( \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial y_n^2} \right) + \omega_n^2 (x_n^2 - y_n^2) \right] 
- \hbar^{-1} \left( i\hbar \frac{\partial}{\partial y_n} + \Omega x_n \right) \frac{1}{2} \sum_{n=1}^{N} \left[ \hbar^2 \frac{\partial^2}{\partial y_n^2} + \omega_n^2 x_n y_n \right] \right\} \Psi 
= k \hbar \Omega \Psi, \quad k = 0, \pm 1, \pm 2, \ldots \square
$$

Explicit expressions of the eigenfunctions $\Psi \equiv \Psi(x_n, y_n, x, y, k)$ are easily obtained from the proof of this proposition, as provided in the following section; they allow a precise accounting of the infinite degeneracy of the energy level corresponding to each (arbitrary integer) value of the quantum number $k$.

### 3. Proofs

The point of departure to prove the two Propositions 1 and 2 are the equations of motion implied by the Hamiltonian (1), reading

\begin{align}
\dot{q}_0 &= p_0 + \frac{E}{b}, \\
\dot{p}_0 &= -\Omega \left( \Omega q_0 + \frac{E}{b} \right), \quad (12a) \\
\dot{q}_n &= p_n + \Omega \partial h(p, q) \frac{\partial h(p, q)}{\partial p_n}, \\
\dot{p}_n &= -p_n + \Omega \partial h(p, q) \frac{\partial h(p, q)}{\partial q_n}. \quad (12b)
\end{align}

To write these formulae, and below, we use the fact — obviously implied by the definition of the Hamiltonian (1) — that $h(p, q)$ is a constant of motion, here denoted by $E$, which can of course be identified with the initial value of $h(p, q)$,

$$
E = h(p(0), q(0)).
$$

It is then clear (from (12a)) that the time evolution of the two canonical variables $q_0(t)$ and $p_0(t)$ is given by (6), implying

$$
\frac{p_0(t) + \Omega q_0(t)}{b} = \frac{p_0(0) + \Omega q_0(0)}{b} \exp(i\Omega t) = \hat{v}(t).
$$

Here the second equality is clearly implied by the definition (7c) of $\tau$: and it implies that the equations of motion (12b) can now be reformulated as follows:

$$
\zeta_n = \frac{\partial h(p, q)}{\partial q_n}, \quad \eta_n = -\frac{\partial h(p, q)}{\partial p_n}, \quad (15)
$$

via the formal change of dependent variables

$$
\xi(t) = q(t), \quad \eta(t) = p(t). \quad (16a)
$$
with (since \( \tau(0) = 0 \), see (7c))

\[
g(0) = \xi(0), \quad p(0) = \eta(0).
\]  

(16b)

Note that these equations of motion, (15), are just the standard equations of motion yielded by the Hamiltonian \( h(\eta, \xi) \): of course the appended primes in the left-hand sides of (15) denote differentiations with respect to the (complex) independent variable \( \tau \), and we are assuming that this makes good sense, namely that \( h(\eta, \xi) \) is an analytic (but not necessarily holomorphic) function of its 2N arguments, namely of the 2N components of the two \( N \)-vectors \( \eta \) and \( \xi \).

We therefore conclude that the solution of the initial-value problem for the equations of motion (12b) is given, via the change of independent variable (16) with (7c), by the solution of the initial-value problem — with the same initial data — for the equations of motion (15). Hence — see (7c) — the solution \( g(t), p(t) \) of (12b) is completely periodic in the real independent variable \( t \) — see (2b) — if the corresponding solution \( \xi(\tau), \eta(\tau) \) of (15) is holomorphic, as a function of the complex variable \( \tau \), in the disk \( D \) enclosed, in the complex \( \tau \)-plane, by the circle \( C \) of radius \( \tau \), centered at the origin of the complex \( \tau \)-plane, by the circle \( C \) and having radius \( \tau \) — see (2a) — by the variable \( \tau = \tau(t) \) when the real variable \( t \) progresses continuously from its initial value \( t = 0 \). This circle \( C \) — see (7c) — has, in the complex \( \tau \)-plane, a diameter one end of which is at the point

\[
d = 2|p_0(0) - \Omega q_0(0)|/(\Omega);
\]  

(17) hence it is enclosed inside the circular disk \( \hat{D} \) centered at the origin of the complex \( \tau \)-plane and having radius \( |d| \). Since every solution \( \xi(\tau), \eta(\tau) \) of the equations of motion (15) is holomorphic — as a function of the complex variable \( \tau \) — in the neighborhood of the origin \( \tau = 0 \) where the initial data \( \xi(0) = g(0), \eta(0) = p(0) \) (see (16b)) are assigned (provided of course these initial data are not assigned where the right-hand sides of the equations of motion (15) are singular, see for instance [9, Sec. 12.21]) and, since the extent of the region around \( \tau = 0 \) where the solution \( \xi(\tau), \eta(\tau) \) is holomorphic depends on the initial data \( \xi(0) = g(0), \eta(0) = p(0) \) of the equations of motion (12) (see for instance [9, Sec. 12.21]) while the radius \( |d| \) of the disk \( \hat{D} \) can be made arbitrarily small by an appropriate choice of the initial data \( p_0(0) \) and \( q_0(0) \), see (17), it is clear that there is an open, fully dimensional set of these initial data \( \xi(0) = g(0), \eta(0) = p(0) \) yielding isochronous solutions, see (2b), for any arbitrary (nonvanishing) assignment of the real constant \( \Omega \). Proposition 1 is thus proven.

To prove Proposition 2 we note that the Hamiltonian (3) corresponds to the Hamiltonian (1) with

\[
h(\eta, \xi) = \sum_{n=1}^{N} h_n(p_n, q_n),
\]  

(18) see (5b). Note that, via (13), this entails (6c), while the equations of motion (15) now read, via (18) and (16), as follows:

\[
\xi_n' = q_n, \quad \eta_n' = -\omega_n^2 \xi_n.
\]  

(19)
entailing
\[ \xi_n(\tau) = \xi_n(0) \cos(\omega_n \tau) + \eta_n(0) \frac{\sin(\omega_n \tau)}{\omega_n}, \quad (20a) \]
\[ \eta_n(\tau) = \eta_n(0) \cos(\omega_n \tau) - \xi_n(0) \omega_n \sin(\omega_n \tau). \quad (20b) \]

These formulae yield, via (16) and (7e), the three formul\ae{} (7), while the three formul\ae{} (6) are implied by our previous treatment, see (12a) with (13) and (18). This completes the proof of Proposition 2.

The proof of Proposition 3 is an immediate consequence of Proposition 2, see the remarks made in the last part of the paragraph entitled Notation and related remarks reported in Sec. 2 after Proposition 1: note that indeed \( h_n(p - i\dot{p}, \dot{q} + i\dot{q}) = \hat{h}_n(p; \hat{p}; \hat{q}; \hat{q}) + i\hat{h}_n(p; \hat{p}; \hat{q}; \hat{q}) \), see (5b), (8b) and (8c).

The first step to prove Proposition 4 is to observe that the \( 1+2N \) operators obtained from the \( 1+2N \) Hamiltonians \( H(p_0, \hat{p}_0; \hat{p}; \hat{q}) \), \( \hat{h}_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n) \) and \( h_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n) \) via the standard quantization prescription (10) commute among themselves: note that neither here nor below there is any ordering problem in the transition from the classical to the quantal contexts. Next we note that the following formula holds:
\[ H(p_0, \hat{p}_0; \hat{p}; \hat{q}) = \frac{1}{2}(\hat{P}^2 + \Omega^2 \hat{Q}^2) - \frac{1}{2} \left( \hat{P}^2 + \Omega^2 \hat{Q}^2 \right), \quad (21a) \]
\[ \hat{P} = \hat{p}_0 - \frac{1}{2} \sum_{n=1}^{N} h_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n), \quad \hat{Q} = \hat{q}_0 - \frac{1}{2} \sum_{n=1}^{N} h_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n). \quad (21b) \]
\[ \hat{P} = \hat{p}_0 - \frac{1}{2} \sum_{n=1}^{N} \hat{h}_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n), \quad \hat{Q} = \hat{q}_0 + \frac{1}{2} \sum_{n=1}^{N} \hat{h}_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n). \quad (21c) \]

and that these definitions entail that the quantities \( \hat{Q} \) and \( \hat{Q} \) can be considered — in the classical context — as two independent canonical variables, with the two quantities \( \hat{P} \) and \( \hat{P} \) the corresponding two canonical momenta. Hence in the quantal context these quantities satisfy the standard canonical commutation rules. It is then clear that the Hamiltonian \( H(p_0, \hat{p}_0; \hat{p}; \hat{q}) \) has a discrete spectrum with eigenvalues \( (k - \hat{k}) \hbar \Omega = k \hbar \Omega \), with the two quantum numbers \( k \) and \( \hat{k} \) required to be nonnegative integers, hence the quantum number \( k = \hat{k} - \hat{k} \) required to be an arbitrary integer. This essentially completes the proof of Proposition 4. The explicit construction of the eigenfunctions \( \Psi \) — hence the corresponding accounting of the infinite degeneracy of the energy level \( k \hbar \Omega \), see (11) — can be left as an exercise for the diligent reader, who shall take advantage of the property of the quantities \( \hat{h}_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n) \) and \( \hat{h}_n(p_n; \hat{p}_n; \hat{q}_n; \hat{q}_n) \) to commute among themselves and with \( \hat{Q}, \hat{Q}, \hat{P}, \hat{P} \) and \( H(p_0, \hat{p}_0; \hat{p}; \hat{q}) \), as well as the fact that the \( N \) quantities \( h_n(p; \hat{p}; \hat{q}) \) — being just the difference among two standard harmonic oscillator Hamiltonians — have a discrete spectrum, while the \( N \) quantities \( h_n(p; \hat{p}; \hat{q}) \) have a continuous spectrum. This latter fact is conveniently seen — and the corresponding eigenfunctions obtained — by performing the following simple (linear) canonical transformation,
\[ p_k = \frac{\hat{p} \pm \hat{p}}{\sqrt{2}}, \quad q_k = \frac{\hat{q} \pm \hat{q}}{\sqrt{2}}, \quad (22a) \]
yielding
\[ \hat{h}_n(\hat{p}, \hat{q}; \hat{\rho}, \hat{\sigma}) = \frac{1}{2}(p^2_\rho - \omega^2_n q^2_\rho) - \frac{1}{2}(p^2_\sigma - \omega^2_n q^2_\sigma). \] (22b)

It is thus seen that the hamiltonian \( \hat{h}_n(\hat{p}, \hat{q}; \hat{\rho}, \hat{\sigma}) \) is again the difference among two simple Hamiltonians, each however representing now an antiharmonic oscillator (note the minus signs in front of \( \omega^2_n \)), hence characterized by a continuous spectrum. But let us reemphasize that these feature, while relevant to identify the degeneracy of the spectrum of the Hamiltonian \( H(\hat{p}_0, \hat{q}_0; \hat{p}, \hat{q}; \hat{\rho}_0, \hat{\sigma}_0) \) and correspondingly its eigenfunctions (which can be identified with a certain freedom, due to this degeneracy), do not affect the conclusion of Proposition 4 concerning its spectrum.

Finally let us note that the approach, via appropriate canonical transformations, employed here to prove Proposition 4, could obviously have been as well used to prove the three previous propositions.

4. Outlook

The explicit solvability of the dynamical system characterized by the Hamiltonian (8) in both the classical and the first quantized contexts raises the challenge of its possible solvability in a second quantized context.

Acknowledgments

We wish to acknowledge with thanks the hospitality, extended in more than one occasion, to one of us (FC) by the Centro Internacional de Ciencias in Cuernavaca and to the other one of us (FL) by the Physics Department of the University of Roma “La Sapienza”. FL also wishes to thank the Centro de Investigación en Complejidad Básica y Aplicada at the Universidad Nacional in Bogotá, Colombia for the opportunity of spending a sabbatical year there while this paper was written and acknowledges the financial support of the following projects: CONACyT 44020 and DGAPA IN112307.

References

[1] J. Chavarriga and M. Sabatini, A survey of isochronous systems, Qual. Theory Dyn. Syst. 1 (1999) 1–70.
[2] A. Cima, F. Mañosas and J. Villadelprat, Isochronicity for several classes of Hamiltonian systems, J. Diff. Eq. 157 (1999) 373–413.
[3] J. Chavarriga and M. Grau, Some open problems related to 16th Hilbert problem, Scientia (Series A: Math Sciences: Universidad Técnica Federico Santa María, Valparaíso, Chile, ISSN 0716-8446) 9 (2003) 1–26.
[4] F. Calogero and F. Leyvraz, General technique to produce isochronous Hamiltonians, J. Phys. A: Math. Theor. 40 (2007) 12931–12944.
[5] F. Calogero, Isochronous systems (Oxford University Press, Oxford, 2008).
[6] F. Calogero and F. Leyvraz, Spontaneous reversal of irreversible processes in a many-body Hamiltonian evolution, New J. Phys. 10 (2008) 023042.
[7] F. Calogero and F. Leyvraz, A new class of isochronous dynamical system, J. Phys. A: Math. Theor. 41 (2008) 295101.
[8] F. Calogero, Classical many-body problems amenable to exact treatments, Lecture Notes in Physics Monograph m 66 (Springer, Berlin, 2001).
[9] E. L. Ince, Ordinary Differential Equations (Dover, New York, 1956).