Quantitative Néron theory for torsion bundles

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August 20, 2018

Let $R$ be a discrete valuation ring with algebraically closed residue field, and consider a smooth curve $C_K$ over the field of fractions $K$. For any positive integer $r$ prime to the residual characteristic, we consider the finite $K$-group scheme $\text{Pic}^r_{C_K}$ of $r$-torsion line bundles on $C_K$. We determine when there exists a finite $R$-group scheme, which is a model of $\text{Pic}^r_{C_K}$ over $R$; in other words, we establish when the Néron model of $\text{Pic}^r_{C_K}$ is finite. To this effect, one needs to analyse the points of the Néron model over $R$, which, in general, do not represent $r$-torsion line bundles on a semistable reduction of $C_K$. Instead, we recast the notion of models on a stack-theoretic base: there, we find finite Néron models, which represent $r$-torsion line bundles on a stack-theoretic semistable reduction of $C_K$. This allows us to quantify the lack of finiteness of the classical Néron models and finally to provide an efficient criterion for it.

1. Introduction

(1.1) The existence of a finite group $R$-scheme, which is an $R$-model of $\text{Pic}^r_{C_K}$. Let $R$ be a discrete valuation ring whose residue field is algebraically closed, we consider a smooth curve $C_K$ of genus $g \geq 2$ over the field of fractions $K$. For an integer $r > 2$ prime to the residual characteristic, we consider the étale and finite group $K$-scheme $\text{Pic}^r_{C_K}$ formed by the $r$-torsion line bundles on $C_K$.

We investigate the existence of a finite group $R$-scheme, which is an $R$-model of $\text{Pic}^r_{C_K}$. Using a result of Serre [Se60], Deschamps [De81, Lem. 5.17] shows that if such an $R$-model exists, then the minimal regular $R$-model of $C_K$ is a semistable $R$-curve (the argument is attributed to Raynaud and uses the fact that $r$ is greater than 2).

Clearly, this condition is not sufficient, see Example 5.6.1. On the other hand, a result of Lorenzini [Lo91] provides a sufficient condition: a finite $R$-model of $\text{Pic}^r_{C_K}$ equipped with a group structure exists if the dual graph of the special fibre of the minimal regular model of $C_K$ over $R$ is obtained from another graph by dividing all the edges in $r$ edges. This condition is not necessary, as we illustrate in Examples 5.6.9 and 5.6.10.

This paper places this issue in a different perspective: describing the interplay between the finiteness of the models of $\text{Pic}^r_{C_K}$ on a stack-theoretic base and the existence of stack-theoretic semistable reductions of $C_K$. As an application we finally find a necessary and sufficient condition on $C_K$ for the existence of a finite group scheme extending $\text{Pic}^r_{C_K}$ over $R$. Indeed, Corollary 5.5.1 shows that the following statements are equivalent.

* Financially supported by the Marie Curie Intra-European Fellowship within the 6th European Community Framework Programme, MEIF-CT-2003-501940.
(1) There exists a finite group $R$-scheme, which is an $R$-model of $\text{Pic}_{C_K}[r]$.

(2) The minimal regular model of $C_K$ over $R$ is semistable and the dual graph of its special fibre satisfies the following property: the (signed) number of edges common to any two circuits is always a multiple of $r$.  

(1.2) The problem in terms of Néron models. The question posed in (1.1) is about the Néron model of $\text{Pic}_{C_K}[r]$. Since $r$ is prime to the residual characteristic, by [BLR80, 7.1/Thm. 1], a finite group $R$-scheme, which is a model of $\text{Pic}_{C_K}[r]$ is necessarily the Néron model of $\text{Pic}_{C_K}[r]$, the universal $R$-model in the sense of the Néron mapping property. This model is a canonical $R$-model of $\text{Pic}_{C_K}[r]$ equipped with a group structure naturally induced by the universal property. In terms of the Néron model an equivalent reformulation of the question above is: when is $N(\text{Pic}_{C_K}[r])$ finite over $R$?

In order to explore Néron models of $\text{Pic}_{C_K}[r]$, it would be convenient to realise them in terms of the functor of $r$-torsion line bundles on a semistable reduction of $C_K$ over $R$. However, this is not possible in general. First, it may well happen that such a semistable reduction does not exist over $R$. Second, if the semistable reduction $C_K$ exists but is not of compact type, the functor of $r$-torsion line bundles is not finite, whereas the Néron model may well be finite, see Example 5.6.2.

(1.3) Quantitative theory of Néron models via twisted semistable reductions. We start by slightly recasting the notion of Néron model. We place it over a new stack-theoretic base $S[d]$: a proper modification of $S = \text{Spec } R$, containing the generic point $\text{Spec } K$, and having a stabiliser $\mu_d$ over the special point (we assume $d$ prime to the residual characteristic). On $S[d]$, there exists a Néron $d$-model $N_d(\text{Pic}_{C_K}[r])$: a universal $S[d]$-model in the sense of the Néron mapping property, see Definition 4.3.2. In this perspective, the problem is not only investigating geometric properties (such as the finiteness of Néron models, or the existence of semistable reductions) on a fixed base, but also measuring how large $\mu_d$ should be for such properties to hold on a stack-theoretic base. The question posed in (1.1) becomes: for which values of $d$ the Néron $d$-model of $\text{Pic}_{C_K}[r]$ is finite?

The advantage is that semistable reductions can be more efficiently used in this setting. Indeed, when $d$ is sufficiently large we can realise Néron models of $\text{Pic}_{C_K}[r]$ via stack-theoretic semistable reductions; then, by descent, we can more easily understand the picture for lower values of $d$. We use stack-theoretic curves whose nontrivial stabilisers occur only at the nodes (twisted curves in the sense of Abramovich and Vistoli). As shown in [Ch, Thm. 3.2.2], these curves carry as many $r$-torsion line bundles as smooth curves as soon as the size of the stabilisers on all nonseparating nodes divides $r$ (we recall the result in Theorem 3.2.1). Therefore, the problem of constructing finite Néron $d$-model is solved as soon as a twisted semistable reduction of this sort exists. The study of geometric invariants of $C_K$ allows us to determine when this is the case.

(1.4) The answer in terms of geometric invariants of $C_K$. The following natural invariants of $C_K$ play a crucial role. We assume that $\text{Pic}_{C_K}[r]$ is tamely ramified over $K$, otherwise, for any $d$, the Néron $d$-model is not finite. Then, we consider

- $m_1$, the least integer such that the minimal regular model of $C_K$ over $S[d]$ is semistable,
- $\Gamma$, the dual graph of the special fibre of the minimal regular model over $S[m_1]$.

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1 We always count the signed number of common edges as detailed in (2.2.3).
2 It contains nodes whose desingularisation is connected (nonseparating nodes).
Using these invariants, in Theorem 5.1.1 and Proposition 5.4.1, we provide explicit formulae for $m_2$, the least integer for which the Néron $d$-model is finite. On $S[m_1r]$, there exists a twisted semistable reduction $C$ of $C_K$ with stabilisers of order $r$ on all nonseparating nodes. As illustrated above, this yields a finite Néron $rm_1$-model of $Pic_{C_K}[r]$, which represents the $r$-torsion line bundles over a twisted curve $C$. For $d | r$, we can descend from $S[rm_1]$ to $S[dm_1]$ if the special fibre of $C$ and all its $r$-torsion line bundles are fixed by the natural action of $\xi_d$. In this way, the study of the action on the twisted curve finally yields (5.4.2)

$$m_2 = m_1r/\gcd\{r, c\},$$

where $c$ is the greatest common divisor of the number of edges shared by two circuits in the dual graph $\Gamma$. In particular, we find the answer to the question posed in (1.1), see Corollary 5.5.1.

A similar analysis allows us to determine $m_3$, the least integer for which the Néron $d$-model is finite and represents the $r$-torsion on a twisted semistable reduction of $C_K$. We have (equation (5.3.1))

$$m_3 = m_1r/\gcd\{r, t\},$$

where $t$ is the greatest common divisor of the thicknesses (see Notation 2.4.3) of the nonseparating nodes of the special fibre of the stable reduction of $C_K$ over $S[m_1]$. In (5.6), we compare the values of $m_3$ and $m_2$ in several examples.

(1.5) The group of connected components of the special fibre of $N(Pic_{C_K}^0)$. The fact that the Néron model of the $Pic_{C_K}[r]$ need not represent the functor of $r$-torsion line bundles is well known and holds in general for the entire functor $Pic_{C_K}^0$. In the literature, this issue motivated the introduction of a finite abelian group $\Phi$: the group of connected components of the special fibre of the Néron model of $Pic_{C_K}^0$. In [Ra70, §8.1.2, p. 64], Raynaud showed how, when the minimal regular model $C_R$ of $C_K$ is semistable, the group $\Phi$ can be defined in terms of the dual graph $\Gamma$ of the special fibre of $C_R$. The group $\Phi$ is studied from very different viewpoints (see §7.2), it is difficult to determine in general, and it is the subject of several open questions, see [Lo] for recent results.

Using $\Phi$, one can state an evident reformulation of our main question. The special fibre of the Néron model of $Pic_{C_K}[r]$ can be regarded as an extension of $\Phi[r]$ by the group of $r$-torsion line bundles on the special fibre of $C_R$. In this way, the finiteness of $N(Pic_{C_K}[r])$ is equivalent to

$$\Phi[r] \cong (\mathbb{Z}/r\mathbb{Z})^{\oplus b_1},$$

(1.5.1)

where $b_1$ is the first Betti number of the dual graph $\Gamma$. This is precisely the statement for which Lorenzini provides the sufficient condition [Lo91, Prop. 2] mentioned above. In this respect, our criterion stated in (1.1) can be also regarded as a statement on $\Phi$: it unravels the geometric condition on $C_K$ encoded in (1.5.1).

We find it interesting to give the proof of our claim both from the geometric point of view of twisted curves (Corollary 5.5.1) and from the abstract point of view of the group $\Phi$ (Propositions 7.4.1 and 7.5.1). Indeed, the approach via line bundles on twisted curves of the present paper is a new geometric interpretation of the $r$-torsion of the group $\Phi$. (Other interactions between the geometry of curves and the group $\Phi$ have been recently shown by Caporaso in [Cap].) In Section 7, we do not rely on the theory of twisted curves—we wrote the entire section so that the reader can start directly there if he wishes.
The torsor of \( r \)th roots of a line bundle. We also consider the case of \( r \)th roots of a given line bundle \( F_K \) on \( C_K \), whose degree is a multiple of \( r \). The \( r \)th tensor power \( L \mapsto L^{\otimes r} \) in the Picard group \( \text{Pic} C_K \) is étale and \( F_K \) is in the image; therefore the \( r \)th roots of \( F_K \) form an étale, finite \( K \)-torsor \( T_K \) over \( K \) under the kernel \( G_K = \text{Pic} C_K \lbrack r \rbrack \).

In Section 5 we generalise the results obtained for \( \text{Pic} C_K \lbrack r \rbrack \): we find Néron \( d \)-models of \( T_K \) which represent the \( r \)th roots of a line bundle extending \( F_K \) over a twisted semistable reduction of \( C_K \) over \( S \lbrack d \rbrack \). The criterion for the finiteness of \( N(T_K) \) over \( R \) is given in Corollary 6.2.1.

Structure of the paper. In Section 2 we fix the terminology. In Section 3 we recall some known results on twisted curves. In Section 4 we set up the notion of Néron \( d \)-models. In Section 5 we prove the results mentioned above, Theorem 5.1.1 and Corollary 5.5.1. In Section 6 we generalise the results to the torsor \( T_K \). Section 7 is self contained: we focus on the criterion stated above, and we prove it by means of the group \( \Phi \) and with no use of the theory of twisted curves. In Section 8 we show a technical lemma.

Acknowledgements. I would like to thank André Hirschowitz for encouraging me to investigate this problem. I'm indebted to Michel Raynaud for hours of explanations and suggestions which greatly influenced this work. I would like to thank Dino Lorenzini for attentively reading my early attempts to state Corollary 5.5.1 and for pointing out several results in the existing literature to me.

2. Terminology

Context. We work with schemes locally of finite type over an algebraically closed field \( k \). By \( R \) we denote a complete discrete valuation ring with residue field \( k \) and fraction field \( K \). We write \( r \) for a positive integer prime to \( \text{char}(k) \).

Graphs. In this paper we say graphs for 2-graphs possibly having loops and multiple edges such as in

In fact, all graphs considered in this paper arise in the following way.

Notation 2.2.1 (dual graph). The dual graph \( \Gamma_C \) of a semistable curve \( C \) over \( k \) is the graph whose set of vertices \( V \) is the set of irreducible components of \( C \) and whose set of edges \( E \) is the set of nodes of \( C \). An edge \( e \) associated to a node in \( C \) joins the vertices \( v' \) and \( v'' \) if the following condition is satisfied: each irreducible component corresponding to \( v' \) and \( v'' \) contains a branch of the node.

We say that an edge \( e \) is nonseparating if the graph remains connected after taking off \( e \) from \( E \). In this way the nodes of \( C \) are either separating or nonseparating according to the corresponding edge in the dual graph.

Notation 2.2.2 (paths and circuits). Each edge in a graph is oriented if it is equipped with an ordering for its vertices: we refer to the first vertex and the last vertex as the tail and the tip. A path in a graph is a sequence of oriented edges \( e_0, \ldots, e_{n-1} \) such that the tip of \( e_i \) is also the tail of \( e_{i+1} \). In this way, a path determines a sequence of vertices \( v_0, \ldots, v_n \) such that from the vertex \( v_i \) there is an edge \( e_i \) to the vertex \( v_{i+1} \) for \( 0 \leq i < n \). A simple path is a path whose
vertices $v_0, \ldots, v_n$ are distinct. A circuit, is a path whose vertices $v_0$ and $v_n$ coincide and with no repeated vertices aside from the first and the last. Note that the edges of paths and circuits are always oriented: these paths are sometimes called directed; here we omit the adjective directed, because no ambiguity may arise.

**Convention 2.2.3** (signed number of edges). When counting the edges common to two circuits in a graph, we always count the signed number of edges. More precisely, the edges lying on both circuits are counted with a positive or negative sign according to whether their orientations in the two circuits agree or disagree.

**Notation 2.2.4** (chain and cochain complexes). Let $\Gamma$ be a graph with an arbitrary fixed orientation. We have a chain complex with differential

$$\partial: C_1(\Gamma, \mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}),$$

where the edge $e$ is sent to $[v_+] - [v_-]$ if $v_+$ is the tip and $v_-$ is the tail of $e$. Since $C_0(\Gamma, \mathbb{Z})$ and $C_1(\Gamma, \mathbb{Z})$ are canonically isomorphic to the group of cochains $C^0(\Gamma, \mathbb{Z})$ and $C^1(\Gamma, \mathbb{Z})$ we can regard the differential of the cochain complex $C^\bullet(\Gamma, \mathbb{Z})$ as

$$\delta: C_0(\Gamma, \mathbb{Z}) \to C_1(\Gamma, \mathbb{Z}),$$

where the vertex $v$ is sent to the sum of the edges ending at $v$ minus the sum of the edges starting at $v$.

For any positive integer $q$, after tensoring with $\mathbb{Z}/q\mathbb{Z}$, we get differentials of $\mathbb{Z}/q\mathbb{Z}$-valued chain and cochain complexes

$$\partial_q: C_1(\Gamma, \mathbb{Z}/q\mathbb{Z}) \to C_0(\Gamma, \mathbb{Z}/q\mathbb{Z}), \quad \delta_q: C_0(\Gamma, \mathbb{Z}/q\mathbb{Z}) \to C_1(\Gamma, \mathbb{Z}/q\mathbb{Z}).$$

(2.3) **Stacks.** We always work with algebraic stacks locally of finite type over a locally noetherian scheme (we refer to [LM00] for the main definitions). When working with algebraic stacks with finite diagonal, we use Keel and Mori’s Theorem [KM97]: there exists an algebraic space $|X|$ (the coarse space) associated to the stack $X$ and a morphism $\pi_X: X \to |X|$ (or simply $\pi$) which is universal with respect to morphisms from $X$ to algebraic spaces.

A geometric point $p \in X$ is an object $\text{Spec} \ k \to X$. We denote by $\text{Aut}(p)$ the stabiliser of $p$: the automorphism group of $p$ regarded as an object of the fibred category $X_k$. In order to identify certain stacks and morphisms between stacks locally at a geometric point $p$, we adopt the standard convention (see [ACV03, §1.5]) of exhibiting the strict henselisation of the stack and of the morphisms involved; we call this process “local picture at $p$” (this avoids repeated mention of strict henselisation of the morphisms and of the stacks).

(2.4) **Semistable curves.** All curves appearing in this paper are semistable: for sake of clarity we recall the definition.

**Definition 2.4.1.** A semistable curve of genus $g \geq 2$ on a scheme $X$ is a proper and flat morphism $C \to X$ whose fibres $C_x$ over geometric points $x \in X$ are reduced, connected, 1-dimensional, and satisfy the following conditions:

1. $C_x$ has only ordinary double points (the nodes),
2. if $E$ is a nonsingular rational component of $C_x$, then $E$ meets the other components of $C_x$ in at least two points.
(3) we have \( \dim_{k(x)} H^1(C_x, O_{C_x}) = g. \)

The curve is stable if, in (2), we require that \( E \) meets the other components of \( C_x \) in at least three points.

**Remark 2.4.2** (line bundles on semistable curves). If \( C \) is a semistable curve over \( k \), an essentially complete description of \( \text{Pic}^C = H^1(C, \mathbb{G}_m) \) and of its \( r \)-torsion subgroup \( \text{Pic}^C[r] = H^1(C, \mu_r) \) is given as follows. Consider the short exact sequences \( 1 \to \mathbb{G}_m \to \nu_v \mathbb{G}_m \to \nu_v \mathbb{G}_m/\mathbb{G}_m \to 1 \) and \( 1 \to \mu_r \to \nu_v \mu_r \to \nu_v \mu_r/\mu_r \to 1 \). Let us choose an orientation for the dual graph of \( C \). We denote by \( \nu : C^\nu \to C \) the normalisation of \( C \). Then, the long exact sequences of cohomology induced by the above sequences are

\[
1 \to \mathbb{G}_m \to (\mathbb{G}_m)^V \to (\mathbb{G}_m)^E \to \text{Pic}^C \xrightarrow{\nu^*} \text{Pic}^C[\nu] \to 1,
\]

and

\[
1 \to \mu_r \to (\mu_r)^V \to (\mu_r)^E \to \text{Pic}^C[\nu] \xrightarrow{\nu^*} \text{Pic}^C[\nu] \to 1,
\]

where \( V \) and \( E \) are the vertices and the edges of the dual graph and the morphisms \( (\mathbb{G}_m)^V \to (\mathbb{G}_m)^E \) and \( (\mu_r)^V \to (\mu_r)^E \) can be regarded as coboundary homomorphisms \( \delta \) with the assigned orientation as in Notation 2.2.4. In this way, the number of \( r \)-torsion line bundles on a semistable curve coincides with \( r^{2g-b_1} \), where \( b_1 \) is the first Betti number of the dual graph.

**Notation 2.4.3** (thickness). Let \( C_R \) be a semistable curve over the complete discrete valuation ring \( R \). When the fibre over the fraction field \( K \) is smooth, the local picture at a node \( e \in C \) is given by \( \text{Spec} R[z,w]/(zw - \pi^e) \), for \( \pi \) a uniformiser of \( R \) and \( \eta(e) \) a positive integer, which we call the thickness of the node \( e \).

**Notation 2.4.4** (stable model and semistable minimal regular model). Let us assume that a smooth curve \( C_K \) of genus \( g \geq 2 \) over \( K \) is given and that there exists a semistable reduction \( C_R \) over \( R \), i.e. \( C_R \) is a semistable curve over \( R \) and its generic fibre is isomorphic to \( C_K \). In this situation, there may be several semistable reductions, but among them there are two special choices. Indeed, there exists a unique stable curve \( C^\text{st}_R \to \text{Spec} R \) whose generic fibre is isomorphic to \( C_K \); we refer to it as the stable model of \( C_K \to \text{Spec} K \). Furthermore, there exists a unique semistable curve \( C^\text{reg}_R \to \text{Spec} R \) for which \( C^\text{reg}_R \) is regular and we refer to it as the semistable minimal regular model of \( C_K \to \text{Spec} K \) (indeed \( C^\text{reg}_R \) is minimal with respect to regular models of \( C_K \)).

There is a natural \( R \)-morphism \( C^\text{reg}_R \to C^\text{st}_R \) obtained by contraction of all rational lines of selfintersection \(-2\) in \( C^\text{reg}_R \). Each node of thickness \( \eta(e) \) in \( C^\text{st}_R \) is the contraction of a chain of \( \eta(e) - 1 \) rational curves in \( C^\text{reg}_R \). The statements above are well known from the theory of semistable reduction [SGA7II], [DM69].

**2.5** Line bundles on semistable reductions: multidegrees and dual graph. Consider the semistable minimal regular model \( C^\text{reg}_R \) of \( C_K \). We study line bundles \( F \) on \( C^\text{reg}_R \) whose restriction on \( C_K \) is trivial: \( F|_{C_K} \cong O_{C_K} \).

They can be written as \( O(\sum_{v \in V} a_v X_v) \), where \( V \) is the set of irreducible components \( X_v \) of the special fibre and \( \bar{a} = (a_v)_{V} \) is a multiindex with entries in \( \mathbb{Z} \). Note that the multiindex \( \bar{d}(F) = (d_v)_{V} \) given by the degree of \( F = O(\sum_{v \in V} a_v X_v) \) on all irreducible components satisfies

\[
\bar{d}(F) = M \bar{a},
\]
where $M$ is the intersection matrix $(X_{v_1} \cdot X_{v_2})$.

This shows that the set of multiindices determined by the degrees on all irreducible components of all line bundles $F$ extending $\mathcal{O}_{C_K}$ on $C^{\text{reg}}_R$ equals $\text{im}(M) \subset C_0(\Gamma, \mathbb{Z})$, where $\Gamma$ is the dual graph of the special fibre of $C^{\text{reg}}_R$ and $M$ is regarded as an endomorphism of $C_0(\Gamma, \mathbb{Z})$.

Remark 2.5.1. It is easy to see that we have $-M = \partial \circ \delta$.

Remark 2.5.2. For any line bundle $F_K$, whose relative degree is a multiple of $r$, it is easy to see that the following conditions are equivalent.

1. There exists a line bundle $F$ on $C^{\text{reg}}_R$ satisfying $F|_{C_K} \cong F_K$, whose degree is a multiple of $r$ on each irreducible component of the special fibre.

2. There exists a line bundle $F$ on $C^{\text{reg}}_R$ satisfying $F|_{C_K} \cong F_K$ and such that the multiindex $\vec{d}(F) \mod r$ belongs to $\text{im}(\partial_r \circ \delta_r)$.

(2.6) Twisted curves. Consider a proper and flat morphism from a stack $C$ to a scheme $X$, for which the fibres are purely one-dimensional with at most nodal singularities, the order of all stabilisers is prime to char($k$), the coarse space is a semistable curve $|C| \rightarrow X$ of genus $g$, and the smooth locus is a scheme.

These stack-theoretic curves $C \rightarrow X$ are called twisted curves if the following condition introduced by Abramovich and Vistoli [AV02] is satisfied. The local picture at a node is given by $[U/\mu_l] \rightarrow T$, where

- $T = \text{Spec} \ A$ for $A$ a ring,

- $U = \text{Spec} \ A[z, w]/(zw - t)$ for some $t \in A$,

- the action of $\mu_l$ is given by $(z, w) \mapsto (\xi_l z, \xi_l^{-1} w)$ for $l$ a positive integer and $\xi_l$ a primitive $l$th root of unity.

Remark 2.6.1 (unbalanced twisted curves). In the existing literature twisted curves satisfying the above local condition are often called balanced, [AV02]. We drop the adjective balanced, because we never consider unbalanced twisted curves.

All the notions introduced above for semistable curves generalise to twisted curves.

- Clearly, the notion of dual graph extends word for word from semistable curves over $k$ to twisted curves over $k$.

- Furthermore, the exactness of the sequences $1 \rightarrow G_m \rightarrow \nu^* G_m \rightarrow \nu^* G_m / G_m \rightarrow 1$ and $1 \rightarrow \mu_r \rightarrow \nu^* \mu_r \rightarrow \nu^* \mu_r / \mu_r \rightarrow 1$ holds for twisted curves as well as for semistable curves. However note that, in general, the long exact sequences written in Remark 2.4.2 are exact only up to the last homomorphism in the sequence, the pullback $\nu^*$, which in general is not surjective. This happens because the higher cohomology groups do not vanish, see [Ch, §3] for more details.

- For a twisted curve $C_R$ over $R$ with smooth generic fibre, the notion of thickness of a node $e$ also extends. The thickness $\eta(e)$ is a positive integer such that the local picture of $C_R$ at $e$ is given by $[U/\mu_l]$ where $U$ is the scheme $\text{Spec} R[z, w]/(zw - \pi^{\eta(e)})$, for $\pi$ a uniformiser of $R$. 

7
Given a smooth curve $C_K$ over the fraction field $K$ of a discrete valuation ring $R$, we say that $C_R$ is a twisted semistable reduction of $C_K$ on $R$ if $C_R \to \text{Spec } R$ is a twisted curve over $R$ whose generic fibre is isomorphic to $C_K$.

**Remark 2.6.2** (adding cyclic stabilisers on thick nodes). Let $C_K$ be a smooth curve and let $C_R$ be a semistable reduction. By definition, $C_R$ is a twisted semistable reduction, because a semistable reduction is just a representable twisted semistable reduction. Let $e$ be a node of thickness $\eta$ in the special fibre of $C_R$. Assume that $\eta$ factors as $\eta = dh$ where $d$ is a prime to $\text{char}(k)$. Then, we can construct a twisted semistable reduction

$$C(d) \to \text{Spec } R$$

with a stabiliser of order $d$ overlying $e$.

Indeed, take an étale neighbourhood of $e \in C_R$ of the form $\overline{U} = \text{Spec } R[w,z]/(zw = \pi^n)$. Consider the quotient stack $[U/\mu_d] = [(zw = \pi^{\eta/d}]/\mu_d]$ with $\mu_d$ acting as $(z,w) \mapsto (\xi_d z, \xi_d^{-1} w)$. The action is free outside the origin $p = (z = w = 0)$, and $[U/\mu_d] \to \overline{U}$ is invertible away from $p$. We define a twisted reduction $C_R(d)$ by gluing $[U/\mu_d]$ to $C \setminus \{e\}$ along the isomorphism $[U/\mu_d] \setminus \{p\} \to \overline{U} \setminus \{e\}$. Note that $C(d)$ has thickness $\eta/d$ at $p \in [U/\mu_d]$.

**Remark 2.6.3**. We point out that this shows that—in characteristic $0$—any curve $C_K$ admitting a semistable reduction also admits a twisted semistable reduction over $R$ which, as a stack, is regular. This fact explains why extending line bundles is easier if we allow twisted semistable reductions.

### 3. Taking $r$th roots on twisted curves

**3.1 Line bundles on twisted curves.** Consider the following situation: $L$ is a line bundle on a twisted curve over $k$. Locally at a node whose stabiliser has order $l$, the line bundle $L$ can be regarded as the trivial line bundle over $\{xy = 0\}$ with $\mu_l$-linearisation

$$((x,y), \lambda) \mapsto ((\xi_l x, \xi_l^{-1} y), \xi_l^q \lambda).$$

We point out that the multiplicity index $q$ is uniquely determined in $\{0, \ldots, l-1\}$ whenever one of the branches of the node is specified. Indeed, $q \in \{0, \ldots, l-1\}$ is determined by the line bundle $L$, the node, and the choice of the branch as follows. The tangent line along the chosen branch can be regarded as the $\mu_l$-representation $T$: $z \mapsto \xi_l z$ for a suitable primitive $l$th root of unity. Similarly, the restriction of $L$ to the chosen branch is a $\mu_l$-representation and is a power of order $q \in \{0, \ldots, l-1\}$ of $T$.

**3.2 The functor of $r$th roots.** For any twisted curve $C$ on $S = \text{Spec } R$ and for any line bundle $F$ on $C$ whose relative degree is a multiple of $r$, the $r$th roots of $F$ on $C$ are represented by an étale $S$-scheme

$$F^{1/r} \to S.$$

More precisely, following [Ch], §3, we can consider the category $\text{Roots}$, formed by the objects $(T, M_T, j_T)$, where $T$ is an $S$-scheme, $M_T$ is a line bundle on the semistable curve $C \times_S T \to T$, and $j_T$ is an isomorphism of line bundles $M_T^{\otimes r} \cong F \times_S T$. The category $\text{Roots}$ is a stack of Deligne–Mumford type, étale on $S$, [Ch], Prop. 3.1.3. 

8
In order to obtain the scheme $F^{1/r}$, let us point out that every object $(T, M_T, j_T)$ has an automorphism group given by multiplication by an $r$th root of unity on $T$ along the fibre of $M_T$. Then, $F^{1/r} \rightarrow S$ can be regarded as the “rigidification along $\mu_r$ of $\text{Roots}_r$” in the sense of Abramovich, Corti, and Vistoli, [ACV03, §5] (denoted by “$\sslash$”, see [Ro] for a careful treatment of this issue)

$$F^{1/r} = \text{Roots}_r \sslash \mu_r.$$ 

This process of rigidification provides a natural framework to a standard procedure that occurs systematically in the construction of Picard functors. Indeed, for any morphism of schemes $f: Y \rightarrow S$, the natural functor $T \mapsto \text{Pic}(Y_T)$ from $X$-schemes to sets is in general a presheaf and is not represented by a scheme. The actual “relative Picard functor” $\text{Pic}_{Y/S}$ is defined by the passage to the associated sheaf (see [BLR80, Ch. 8]). In this way, when $C$ is a semistable curve on $S$, the construction of $F^{1/r}$ yields the same scheme as the subscheme of $r$th roots of $F$ in the relative Picard functor $\text{Pic}_{C/S}$. In particular, the $K$-group $\text{Pic}_{C_K}[r]$ equals $\mathcal{O}^{1/r}$ and the $K$-torsor of $r$th roots of $F_K$ in $\text{Pic}_{C_K}$ equals $F^{1/r}_K$.

The scheme $F^{1/r}$ is étale on $S$. If we assume that $C_K$ is a smooth curve, then the generic fibre of the scheme $F^{1/r}$ contains $r^{2g}$ points. In this way, $\mathcal{O}^{1/r}$ is a finite group scheme and $F^{1/r}$ is a finite torsor under $\mathcal{O}^{1/r}$ if and only if the restriction $F|_{C_k}$ to the special fibre $C_k$ has exactly $r^{2g}$ $r$th roots. This motivates the following result.

**Theorem 3.2.1 ([Ch Thm. 3.2.2]).** Let $\pi: C \rightarrow |C|$ be a twisted curve of genus $g$ over $k$. Let $F$ be a line bundle on $|C|$, whose total degree is a multiple of $r$. There are exactly $r^{2g}$ roots of $\pi^* F$ on $C$ if and only if

$$\#(\text{Aut}(e)) \in r\mathbb{Z} \quad \text{for each nonseparating node } e,$$

$$\#(\text{Aut}(e))d(e) \in r\mathbb{Z} \quad \text{for each separating node } e,$$

where, for each separating node, $d(e)$ stands for the degree of $\pi^* F$ on one of the connected components of the partial normalisation of $C$ at $e$. \hfill \Box

### (3.3) An exact sequence relating $\text{Pic}_{C[r]}$ and $\text{Pic}_{C[|C|][r]}$

In fact if $\pi: C \rightarrow |C|$ is a twisted curve over $k$, for which all nodes have stabilisers of order $r$, once an orientation of the corresponding dual graph $\Gamma$ is chosen, we have the following exact sequence from [Ch 3.0.8]

$$1 \rightarrow \text{Pic}_{C[|C|][r]} \xrightarrow{\pi^*} \text{Pic}_{C[r]} \xrightarrow{j^*} C_1(\Gamma, \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\partial} C_0(\Gamma, \mathbb{Z}/r\mathbb{Z}) \xrightarrow{\varepsilon} \mathbb{Z}/r\mathbb{Z} \rightarrow 1 \quad (3.3.1)$$

where the homomorphisms are defined as follows. The differential $\partial$ is the boundary homomorphism with respect to the orientation of $\Gamma$, and $\varepsilon$ denotes the augmentation homomorphism $(h_v)_v \mapsto \sum h_v \in \mathbb{Z}/r\mathbb{Z}$. Finally $j^*$ can be regarded as the pullback to the singular locus via $j: \text{Sing}_C \rightarrow C$ or, more explicitly, as the homomorphism mapping $L \in \text{Pic}_{C[r]}$ to $(q_e)_{e \in E} \in C_1(\Gamma, \mathbb{Z}/r\mathbb{Z})$, where $q_e \in \{0, \ldots, r-1\}$ is defined as in §3.1.1.

### (3.4) The automorphism group of a twisted curve

Let $\pi: C \rightarrow |C|$ be a twisted curve over $k$ with all stabilisers of order $r$. In [ACV03 Prop. 7.1.1] the group $\text{Aut}(C, |C|)$ of automorphisms of $C$ fixing the coarse space $|C|$ is explicitly calculated: for any twisted curve $C$ over $k$, we have an isomorphism

$$\text{Aut}(C, |C|) \cong (\langle \mu_r \rangle)^E.$$
In fact [ACV03, Thm. 7.1.1] shows that we can choose a set of independent generators as follows. For $e \in C$, we have an automorphism $g \in \text{Aut}(C, |C|)$ such that the restriction of $g$ to $C \setminus \{e\}$ is the identity, and the local picture at $e$ is given by

$$(z, w) \mapsto (z, \xi_r w).$$

(3.4.1)

operating on the scheme $\text{Spec}(k[z, w]/(zw))$. (Note that morphisms of stacks are given up to natural transformations. In this way no branch has been privileged. The $1$-automorphism in the local picture above is in fact, $2$-isomorphic to the morphism $(z, w) \mapsto (\xi_r^h z, \xi_r^{1-h} w)$ for any $h = 0, \ldots, r - 1$.)

**Remark 3.5.2.** In fact $g$ acts by pullback on the entire Picard group. By [Ch, Prop. 2.5.4] the action on $\text{Pic} C$ is again the tensor product of the identity and a morphism $L \mapsto T_L$ given by composing $j^*: \text{Pic} C \to C_1(\Gamma, \mu_r)$ with $C_1(\Gamma, \mu_r) \to \text{Pic} C[r]$.

**Proposition 3.5.1.** The pullback via the automorphism $g \in \text{Aut}(C, |C|)$ satisfying (3.4.1) can be written as

$$g^* : \text{Pic} C[r] \to \text{Pic} C[r],$$

where $L \mapsto T_L$ is the composite homomorphism of $\text{Pic} C[r] \to C_1(\Gamma, \mu_r)$ and $C_1(\Gamma, \mu_r) \to \text{Pic} C[r]$ fitting in the exact sequences recalled above and in (3.3.1).

**Proof.** The claim follows from [Ch, Prop. 2.5.4] which identifies the line bundle $T_L$ with the composite morphism described above. Indeed $T_L$ is the sheaf of regular functions $f$ on $C^*$ whose values on the preimages $p_+$ and $p_-$ of a node $e$ are related by $f(p_+) = \xi_r^e f(p_-)$. This is precisely the image via $C_1(\Gamma, \mu_r) \to \text{Pic} C[r]$ of $(\xi_r^e)_{e \in E} \in C_1(\Gamma, \mu_r)$, with $q_e$ defined as in §(3.1) for each node $e$.

**4. Néron $d$-models**

**The definition of Néron model.** For any $K$-scheme $X_K$, a scheme $Y$ over $S$ is an $S$-model of $X_K$ if the generic fibre is $X_K$. There is an abundance of $S$-models; on the other hand, the Néron model is a canonical $S$-model smooth, separated, and of finite type, and satisfying a universal property, the Néron mapping property, which determines it uniquely, up to a canonical isomorphism:

For each smooth $S$-scheme $Y \to S$ and each $K$-morphism $u_K : Y_K \to X_K$, there is a unique $S$-morphism $u : Y \to X$ extending $u_K$.

The Néron model commutes with étale base changes, with the passage to henselisation [Ra70].
(4.2) Existence of Néron models. The existence of the Néron models of group schemes and torsors is proven in [BLR80] 4.3/Thm. 6, [BLR80] 4.4/Cor. 4] under a boundedness assumption [BLR80] 1.1/Def. 2. In this paper we only consider Néron models of proper $K$-group schemes and proper $K$-torsors, so that the boundedness assumption is automatically satisfied. We have the following statements.

Let $G_K \to \text{Spec} K$ be a group scheme. Assume that it is smooth, of finite type, and proper over $K$. Then, a Néron model $G$ on $S = \text{Spec} R$ exists and is unique, up to a canonical isomorphism. The structure of $G_K$ as a group $K$-scheme extends uniquely to a structure of $G$ as a group $S$-scheme.

Furthermore, assume that $T_K$ is also smooth, of finite type, and proper over $K$ and is a torsor on $\text{Spec} K$ under $G_K$. Then, a Néron model $T$ on $S = \text{Spec} R$ exists and is unique, up to a canonical isomorphism. If $T \to S$ is surjective, the structure of $T_K$ as a torsor on $\text{Spec} K$ under $G_K$ extends uniquely to a structure of $T$ as a torsor on $S$ under $G$, see [BLR80] 6.5/Cor. 3.4.

(4.3) The definition of Néron $d$-model. We place the notion of Néron model on a stack-theoretic base. Instead of $S$, for a suitable positive integer $d$ prime to $\text{char}(k)$, we take as a base the quotient stack $S[d] = \{\text{Spec} R_d/\mu_d\}$, where $R_d$ is equal to $R[\pi]/(\pi^d - \pi)$ for a uniformiser $\pi$ of $R$ and $t \in \mu_d$ acts on $R_d$ by $t(\pi)t = t\pi$ and fixes $R$. In this way, we have

$$\text{Spec} K \xrightarrow{i} S[d] \xrightarrow{p} \text{Spec} R,$$

where $i$ is an open and dense immersion and $p$ is the (proper) morphism to the coarse space.

Definition 4.3.1 ($S[d]$-model). For any $K$-scheme $X_K$, a representable morphism of stacks $Y \to S[d]$ is an $S[d]$-model of $X_K$ if its generic fibre $Y \times_{S[d]} \text{Spec} K$ is $X_K$.

We recall that, since we are working inside the 2-category of algebraic stacks, an $S[d]$-morphism from $f: \mathcal{X} \to S[d]$ to $f': \mathcal{X}' \to S[d]$ is a morphism $g: \mathcal{X} \to \mathcal{X}'$ alongside with a 2-isomorphism $g \circ f' \Rightarrow f$. Note, however, that the 2-isomorphism is uniquely determined, because any automorphism of a representable smooth morphism $f: \mathcal{X} \to S[d]$ is trivial (Lemma 4.2.3 of [AV02] applies since $f$ maps the open dense subscheme $X_K$ of $\mathcal{X}$ into the open dense subscheme $\text{Spec} K$ of $S[d]$).

Definition 4.3.2 (Néron $d$-model). Let $X_K$ be a smooth and separated $K$-scheme of finite type. For $d$ prime to $\text{char}(k)$, a Néron $d$-model is an $S[d]$-model $X$ of $X_K$, which is smooth, separated, and of finite type, and which satisfies the following universal property analogue to the Néron mapping property:

For each representable and smooth morphism of stacks $Y \to S[d]$, and each $K$-morphism $u_K: Y_K \to X_K$, there is an $S[d]$-morphism $u: Y \to X$, which extends $u_K$ and is unique, up to a unique natural transformation

$$\begin{array}{ccc}
Y_K & \xrightarrow{u_K} & Y \\
\downarrow & & \downarrow u \\
X_K & \xrightarrow{u} & X \\
\downarrow & & \downarrow \\
\text{Spec} K & \xrightarrow{} & S[d].
\end{array}$$

Remark 4.3.3. It follows from the Definition 4.3.2 that any two Néron $d$-models of $X_K$ are isomorphic and the isomorphism is unique up to a unique natural transformation.
Proposition 4.3.4. Let \( d \) be invertible in the residue field. For \( \pi \) a uniformiser of \( R \), write \( \tilde{R} \) for \( R[\pi]/(\pi^d - \pi) \), \( \tilde{K} \) for the corresponding field of fractions, and \( \tilde{S} \) for \( \text{Spec} \, \tilde{R} \), with the natural \( \mu_d \)-action. Let \( X_K \) be a smooth and separated \( K \)-scheme of finite type, and let \( \tilde{X}_K = X_K \otimes_K \tilde{K} \) be the corresponding \( \tilde{K} \)-scheme.

I. If \( N \) is the Néron \( d \)-model of \( X_K \), then the \( \tilde{S} \)-scheme \( \tilde{N} \) fitting in

\[
\begin{array}{ccc}
\tilde{N} & \longrightarrow & N \\
\downarrow & & \downarrow \\
\tilde{S} & \longrightarrow & S[d]
\end{array}
\]

is the Néron model of \( \tilde{X}_K \).

II. Conversely, assume that the scheme \( \tilde{X}_K \) has a Néron model \( N(\tilde{X}_K) \) on \( \tilde{S} \). Then, there is a natural \( \mu_d \)-action on \( N(\tilde{X}_K) \) together with a \( \mu_d \)-equivariant morphism \( N(\tilde{X}_K) \to \tilde{S} \), and the corresponding morphism of stacks

\[
[N(X_K)/\mu_d] \to S[d]
\]

is the Néron \( d \)-model of \( X_K \).

Proof. Since \( \tilde{S} \to S[d] \) is étale, point (I) is a straightforward consequence of the compatibility of the formation of Néron models with finite étale base change, \([Ra70]\).

For (II), note that \( \mu_d \) acts on \( N(X_K) \), because it acts on \( X_K \) and, by the Néron mapping property, the action extends to \( N(X_K) \). Therefore, \( [N(X_K)/\mu_d] \) is an \( S[d] \)-model of \( X_K = [X_K/\mu_d] \).

In order to check the universal property of Definition 4.3.2, consider a smooth and representable morphism \( Y \to S[d] \) and a morphism \( u_K : Y_K \to X_K \). Note that \( Y \to S[d] \) can be regarded as a \( \mu_d \)-equivariant smooth \( \tilde{S} \)-scheme \( \tilde{Y} \); indeed, \( \tilde{Y} \) is defined as \( Y \times_{S[d]} \tilde{S} \), which is a scheme by the representability assumption, and the \( \mu_d \)-action is defined by pullback of \( \mu_d \times \tilde{S} \to \tilde{S} \). In this way, \( u_K \) lifts to a \( \mu_d \)-equivariant morphism \( \tilde{Y}_K \to X_K \). By the Néron mapping property for \( N(X_K) \), we have a \( \mu_d \)-equivariant \( \tilde{S} \)-morphism \( \tilde{u} : \tilde{Y} \to N(X_K) \) extending \( \tilde{Y}_K \to X_K \). So, \( u : Y = [\tilde{Y}/\mu_d] \to [N(X_K)/\mu_d] \) extends \( u_K : Y_K \to X_K \).

Finally, take a morphism \( u' : Y \to [N(X_K)/\mu_d] \) such that \( u' \otimes K \) coincides with \( u_K \) on \( Y_K \). In fact, \( u' \otimes K \) and \( u_K \) are lifted by morphisms from \( \tilde{Y}_K \to X_K \) and the two liftings coincide after composition with the action of an element of \( \mu_d \). By the Néron mapping property, the extension to \( \tilde{Y} \) also coincide up to the action of an element of \( \mu_d \); this means that the morphisms of stacks \( u \) and \( u' \) are isomorphic up to a unique natural transformation. \( \square \)

Remark 4.3.5. By Proposition 4.3.4, the existence of Néron \( d \)-models is guaranteed under the properness assumptions of §4.2. Furthermore, the Néron \( d \)-model of a group \( K \)-scheme is equipped with a unique structure of group stack on \( S[d] \), and the Néron \( d \)-model of a \( K \)-torsor is equipped with a unique structure of torsor on \( S[d] \) if it surjects on \( S[d] \).

Notation 4.3.6. Whenever \( d \) is invertible in the residue field \( k \) and \( X_K \) satisfies the hypothesis of §4.2, the Néron \( d \)-model of \( X_K \) exists and we denote it by

\[
N_d(X_K) \to S[d].
\]
5. Finite Néron $d$-models of the $r$-torsion of the Picard group

(5.1) The finiteness of the Néron $d$-model with respect to $d$. In the following theorem we identify the finite Néron $d$-models and those who are represented by $r$-torsion line bundles on a twisted semistable reduction.

**Theorem 5.1.1.** Let $r > 2$ be an integer prime to $\text{char}(k)$, and let $C_K$ be a smooth curve of genus $g \geq 2$. Assume that the group $K$-scheme $G_K = \text{Pic}_{C_K}[r]$ is tamely ramified on $K$, i.e. all its points correspond to tame extensions of $K$.

Then, there exist three integer and positive invariants $m_1, m_2, m_3$ of $C_K$ satisfying

$$m_2 \in m_1 \mathbb{Z},$$
$$m_3 \in m_2 \mathbb{Z},$$
$$rm_1 \in m_3 \mathbb{Z},$$

and the following conditions.

1. There is a semistable reduction of $C_K$ on $S[d]$ if and only if $d$ is a multiple of $m_1$.
2. The Néron $d$-model $N_d(G_K)$ is a finite group scheme if and only if $d$ is a multiple of $m_2$.
3. The Néron $d$-model $N_d(G_K)$ is a finite group scheme and represents the $r$th roots of $\mathcal{O}$ on a twisted semistable reduction $C$ of $C_K$ on $S[d]$ if and only if $d$ is a multiple of $m_3$.

If $C_K$ admits a semistable reduction over $R$, then the condition $r > 2$ is superfluous and the condition that $G_K$ is tamely ramified is always true. In this case, for any $r$ prime to $\text{char}(k)$, we have $m_1 = 1$, $m_2 \mid m_3$, and $m_3 \mid r$.

**Remark 5.1.2.** Point (1) follows easily from a version of the theorem of semistable reduction (the argument is given in [De81, §5] by Deschamps and is attributed to Raynaud). We also point out that $m_2$ is a multiple of $m_1$ is merely a reformulation of a criterion due to Serre (see [De 81, §5]). These facts are reviewed in the course of the proof because they are needed in the rest of the paper.

**Proof of Theorem 5.1.1.** We prove (2), from which we deduce (1). Finally, we show point (3).

Point (2) affirms the existence of $m_2$ such that $N_d(\text{Pic}_{C_K}[r])$ is finite if and only if $m_2$ divides $d$. This is a simple fact in Galois theory, which we need later; we recall it in the following lemma.

**Lemma 5.1.3.** Let $G_K$ be a tamely ramified finite $K$-scheme. Then there is an integer and positive invariant $m(G_K)$ such that the Néron $d$-model of $G_K$ is finite if and only if $d$ is a multiple of $m(G_K)$.

**Proof.** Denote by $\overline{K}$ a separable algebraic closure of $K$; we write $\overline{G}$ for $G_K \otimes \overline{K}$. By Proposition 4.3.1, we only need to determine the integers $d$ for which $R_d = R[\pi']/(\pi'^d - \pi)$ satisfies the following condition: the Néron model of the pullback $G_d$ of $G_K$ on the corresponding valuation field $K_d \supset R_d$ is finite on $R_d$. This happens if and only if $G_d$ is not ramified.

By descent theory, there is a natural morphism

$$\text{Gal}(\overline{K}/K) \to \text{Aut}(\overline{G}).$$  \hfill (5.1.4)
Note that the $K_d$-scheme $G_d$ is not ramified if and only if $\text{Gal}(\overline{K}/K_d)$ is contained in the kernel of the above morphism. Since $G_{K_d}$ is tamely ramified, the image of $\overline{\text{Gal}(K'/K_d)}$ is a finite cyclic group whose order is prime to $\text{char}(k)$. Let $m(G_{K_d})$ be such order. Then, $G_d$ is not ramified on $R_d$ if and only if $d$ is a multiple of $m(G_{K_d})$.

Point (1) follows from the following statement and from the application of Lemma 5.1.3.

Lemma 5.1.5 ([De81, §5]). There exists a group $K$-scheme $E_K$ associated to $C_K$ and to $r > 2$ satisfying the following property. The curve $C_K$ has a semistable reduction on $S[d]$ if and only if the Néron $d$-model of $E_K$ is finite.

Proof. In order to define $E_K$, one needs to introduce a finite Galois extension $K \subset K'$ for which $C_K \otimes K'$ has semistable reduction over $R'$, the integral closure of $R$ in $K'$ (such an extension exists by the theorem of semistable reduction). Then, there is a unique group $E_K$ satisfying the following conditions.

Let $U'$ be the group $R'$-scheme representing the functor of line bundles of degree zero on all irreducible components of the fibres of the semistable minimal regular model of $C_K$ on $\tilde{R}$. Let $P$ be the $r$-torsion subgroup of $U'$. The group $R'$-scheme $P$ is the disjoint union $P = P_{\text{gen}} \sqcup P_{\text{fin}}$, where $P_{\text{gen}}$ is a component contained in the generic fibre and a $P_{\text{fin}}$ is a finite group scheme over the base ring $R'$. Then, $E_K$ is determined by descent: the group $K$-scheme $E_K$ is the group satisfying $E_K \otimes K' = P_{\text{fin}} \otimes K'$. The fact that $P_{\text{fin}} \otimes K'$ descends and that this definition does not depend on the Galois extension $K'$ is shown in [De81, §5]. It remains to show that the equivalence holds.

By construction of $E_K$, if $C_K$ has stable reduction on $S[d]$, then $N_d(E_K)$ is finite. Indeed, since the definition of $E_K$ does not depend on the extension, we can define $E_K$ by descent from $K_d = K[p^r]/(p^r - 1)$ to $K$. The group $K_d$-scheme $E_K \otimes K_d$ is the generic fibre of a finite group scheme $P_{\text{fin}}$ over the integral closure $R_d$ of $R$ in $K_d$. Then, the Néron model of $E_K \otimes K_d$ over $R_d$ is finite. By Proposition 1.3.4 this is equivalent to say that the Néron $d$-model of $E_K$ is finite.

Conversely we prove that, if the Néron model of $E_K \otimes K_d$ is finite over $R_d$, then the minimal regular model of $C_K \otimes K_d$ is semistable over $R_d$. Recall that we can choose a finite Galois extension $K'$ containing $K_d$ such that the minimal regular model of $C_K \otimes K'$ over the integral closure $R'$ of $R$ in $K'$ is semistable. The claim is equivalent to showing that the jacobian variety of $C_K \otimes K_d$ has semistable reduction over $R_d$. This is the case if there is a semistable reduction of the jacobian variety of $C_K \otimes K'$ which descends from $R'$ to $R_d$.

Now, note that such a semistable reduction is provided by the group $R'$-scheme $U'$ defined above. Indeed, it descends from $R'$ to $R_d$ because the action of $\text{Gal}(K'/K_d)$ on its special fibre is trivial. In order to see that, via Serre’s lemma [Se60], it is enough to show that $\text{Gal}(K'/K_d)$ acts trivially on the special fibre of the kernel of the multiplication by $r$: the scheme $P$ introduced above. In fact, the special fibre of $P$ coincides with the special fibre of $P_{\text{fin}}$ and the action of $\text{Gal}(K'/K_d)$ on the special fibre of $P_{\text{fin}}$ is trivial as soon as $P_{\text{fin}}$ descends to $R_d$. This is indeed a consequence of the fact that the Néron model of $E_K \otimes K_d$ is finite over $R_d$.

Note that, since $E_K$ is a subgroup of $\text{Pic}_{C_K}[r]$ by definition, the above statement also shows that $m_1$ divides $m_2$.

Point (3) determines all indices $d$ satisfying the property $P(d)$: the Néron $d$-model $N_d(G_{K_d})$ is a finite group scheme and represents the $r$th roots of $0$ on a twisted semistable reduction $C$ of $C_K$ on $S[d]$. This is a consequence of Theorem 5.2.21 which states that a twisted curve
over $k$ of genus $g$ has $r^{2g}$ $r$-torsion line bundles if and only if the order of the stabiliser of each nonseparating node is a multiple of $r$. Applying this fact, we can immediately show that for any $d \in rm_1\mathbb{Z}$ the condition $P(d)$ is satisfied. Indeed, let $C_{m_1}$ be the stable reduction of $C_K$ over $S[m_1]$. Then, the pullback to $S[d]$ is a stable reduction $C_d$ of $C_K$. Note that all nodes have thicknesses in $r\mathbb{Z}$ because the base change of $\{zw = s^l\}$ via $s \mapsto s^h$ yields $\{zw = s^{hl}\}$ for all positive integers $h$ and $l$. Then, by Remark 2.6.2, on $S[d]$ there is a twisted curve $C_d$ whose nonseparating nodes have stabilisers whose order lies in $r\mathbb{Z}$. Finally, by Theorem 3.2.1, the group scheme formed by $r$th roots of $\mathcal{O}$ on $\tilde{C}$ is finite and is the Néron $d$-model of its generic fibre.

This shows that $rm_1\mathbb{Z}$ is included in the set of indices for which $P(d)$ is satisfied. Assume that we modify the statement $P(d)$: we require that $P(d)$ is true and also that the coarse space $|C|$ of $C$ is stable. Let us call this property $P(d)'$. Then the argument given above shows without change that the indices $d$ satisfy $P(d)'$ if and only if they are multiple of $m_1r/gcd\{r, t\}$, where $t$ is the greatest common divisor of the thicknesses of nonseparating nodes of the stable model $C_{m_1}$ over $S[m_1]$. In fact, this proves point (3) entirely, because we have $P(d) \iff P(d)'$. Indeed, $P(d)' \Rightarrow P(d)$ is obvious. Conversely, if $P(d)$ is true, then there is a twisted semistable reduction $C$ of $C_K$ on $S[d]$ such that $r$ divides the height of all nonseparating nodes in the coarse space $|C|$. After contraction of all projective lines with only two nodes in the special fibre of $|C|$, we obtain the stable model for which the thicknesses of the nonseparating nodes still belong to $r\mathbb{Z}$. This elementary fact easily follows from the fact that, in the semistable minimal regular model obtained by desingularising $|C|$, the number of nonseparating nodes in each chain of $-2$-curves is a multiple of $r$.  

The proof of Theorem 5.1.1 provides some characterisations of the invariants $m_1$, $m_2$, and $m_3$, which we state explicitly.

(5.2) **The indices $m_1$ and $m_2$.** By Lemma 5.1.3 we have

$$m_1 = \#(\text{im}(d_{E_K})), $$

where $E_K$ is the subgroup of $G_K$ satisfying Lemma 5.1.5 and $d_{E_K}$ is the morphism $\text{Gal}(\overline{K}/K) \to \text{Aut}(E_K \otimes \overline{K})$ for a separable closure $\overline{K}$ of $K$. Similarly, we have

$$m_2 = \#(\text{im}(d_{G_K})).$$

(5.3) **The ratio $m_3/m_1$.** The ratio $m_3/m_1$ can be expressed in terms of the geometry of the stable reduction of $C_K$ on $S[m_1]$. Indeed, we have

$$m_3/m_1 = r/gcd\{r, t\}, $$

where $t$ is the greatest common divisor of the thicknesses of the nonseparating nodes appearing in the stable reduction of $C_K$ on $S[m_1]$. As a consequence we have the following statement.

**Corollary 5.3.2.** The Néron model of $\text{Pic}_{C_K}[r]$ over $R$ is finite and represents the functor of $r$-torsion line bundles on a twisted semistable reduction of $C_K$ over $R$ if and only if $C_K$ admits a stable model in which the thickness of each nonseparating nodes in the special fibre is a multiple of $r$.

(5.4) **The ratio $m_2/m_1$.** The argument of the proof of Theorem 5.1.1 allows us to express the ratio $m_2/m_1$ in a similar way.
Proposition 5.4.1. We have

\[ m_2/m_1 = r/\gcd\{r,c\}, \]  

(5.4.2)

where \( c \) is the greatest common divisor of the number of edges common to two circuits in the dual graph of the special fibre of the semistable minimal regular model of \( C_K \) on \( S[m_1] \).

Proof. We already showed the relations \( m_1 \mathbb{Z} \subseteq m_2 \mathbb{Z} \subseteq m_3 \mathbb{Z} \subseteq rm_1 \mathbb{Z} \). It remains to show that the divisor \( q \) of \( r \) satisfying \( rm_1 = qm_2 \) equals \( \gcd\{r,c\} \). The claim can be equivalently restated as follows. Assume \( m_1 = 1 \); then \( \gcd\{r,c\} \) is the highest among the divisors \( q \) of \( r \) for which the finite Néron \( r \)-model descends to \( S[r/q] \). In other terms, \( \gcd\{r,c\} \) is the highest among the divisors \( q \) of \( r \) for which \( \mu_q \subseteq \mu_r \) acts trivially on the special fibre of \( N_r(G_K) \).

We can realise \( N_r(G_K) \) as the group of \( r \)-torsion line bundles on a twisted semistable reduction of \( C_K \) on \( S[r] \) whose fibers only contain stabilisers of order \( r \) on each node. We construct this twisted semistable reduction as we did in the proof of Theorem 5.1.1 by iterating the construction illustrated in Remark 2.6.2. In this way, we can assume that we have a twisted semistable reduction of \( C_K \) on \( S[r] \) such that the coarse space descends on \( S \) and is the semistable minimal regular model of \( C_K \). We regard this twisted curve on \( S[r] \) as a \( \mu_r \)-equivariant twisted curve \( \tilde{C} \) over the discrete valuation ring \( \tilde{R} = R[\tilde{\pi}]/(\tilde{\pi}^r - \pi) \).

The action of \( \mu_r \) on \( \tilde{C} \) is trivial on the special fibre of the coarse space \( \tilde{C} \), because the coarse space descends to \( S \). In this way, the action of \( \mu_r \) on \( \tilde{C} \) is characterised by its local picture at the nodes. We recall that the local picture of \( \tilde{C} \) at a node is \( [U/\mu_r] \), where \( U \) is \( zw = \tilde{\pi} \), and the quotient stack is defined by the \( \mu_r \)-action \( (z,w,\pi) \mapsto (\xi_r z, \xi_r^{-1} w, \pi) \). Therefore, we can write \( \{xy = \tilde{\pi}^r\} \) for the local picture of \( \tilde{C} \) (with \( x = z^r \) and \( y = z^r \)), and we notice that \( \{xy = \tilde{\pi}^r\} \) is the pullback via \( \tilde{\pi} \mapsto \tilde{\pi}^r \) of \( \{xy = \pi\} \), the local picture of the minimal regular model. Now, \( \xi_r \in \mu_r \) operates nontrivially on \( [U/\mu_r] \) but trivially on the special fibre of the coarse scheme. Therefore, \( \xi_r \in \mu_r \) fixes \( zw = \tilde{\pi}, x = z^r, y = q^r \), and acts by multiplication of \( \tilde{\pi} \). This implies that \( \xi_r \) operates on \( [U/\mu_r] \) as

\[ (z,w,\pi) \mapsto (\xi_r^a z, \xi_r^b w, \xi_r \tilde{\pi}) \quad \text{with} \quad a + b \equiv 1 \mod r. \]

Note that, up to natural transformation, this local picture identifies a single morphism \( g \) whose local picture at all nodes is given by

\[ (z,w,\pi) \mapsto (z, \xi_r w, \xi_r \tilde{\pi}). \]  

(5.4.3)

In this way, the action of \( \mu_r \) on the special fibre \( \tilde{C}_k \) is generated by the automorphism \( g \) studied in Proposition 3.5.1.

Using Proposition 3.5.1 we conclude that the highest divisor \( q \) of \( r \) for which \( \mu_q \in \mu_r \) acts trivially on \( \Pic_{\tilde{C}_k}[r] \) is the highest index \( q \) for which the endomorphism of \( \Pic_{\tilde{C}_k}[r] \) given by \( L \mapsto T^{\oplus r/q}_L \) vanishes. Proposition 3.5.1 claims that \( L \mapsto T_L \) fits in the following diagram where

\[ \text{The natural transformations have equation} \quad (z,w,\pi) \mapsto (\xi_r^h z, \xi_r^{-h} w, \tilde{\pi}) \]
the vertical and the horizontal sequences are exact.

\[
\xymatrix{
\text{Pic}_C[r] 
\ar[r]^{j^*} & C_1(\Gamma, \mu_r) 
\ar[r]^{\partial_r} 
\ar[d]_{\delta_r} & C_0(\Gamma, \mu_r) 
\ar[d] & C_0(\Gamma, \mu_r). 
}
\tag{5.4.4}
\]

Therefore, \( L \mapsto T_L \) vanishes if and only if the dual graph \( \Gamma \) of \( \tilde{C}_k \) satisfies \( \ker(\partial_r) \subseteq \text{im}(\delta_r) \). More generally, it is easy to see that \( L \mapsto T_L^{\otimes r/q} \) vanishes if and only if \( \Gamma \) satisfies \( \ker(\partial_q) \subseteq \text{im}(\delta_q) \). This completes the proof, because imposing \( \ker(\partial_q) \subseteq \text{im}(\delta_q) \) is equivalent to imposing the condition that the number of edges common to any two circuits is always a multiple of \( q \). See Section 8 for a proof.

\section{The finiteness criterion for \( N(\text{Pic}_{C_K}[r]) \).}

As a consequence of the previous results, we can determine whether the Néron model of \( \text{Pic}_{C_K}[r] \) is finite over \( R \) simply by looking at the special fibre of the minimal regular model of \( C_K \).

\begin{corollary}
Let \( r > 2 \) be prime to the residual characteristic, and let \( C_K \) be a smooth curve of genus \( g \geq 2 \).

The Néron model of \( \text{Pic}_{C_K}[r] \) is finite if and only if \( C_K \) admits a semistable minimal regular model for which the dual graph of the special fibre satisfies the following property: the (signed) number of edges common to any two circuits is always a multiple of \( r \).
\end{corollary}

\begin{proof}
The Néron model \( N(\text{Pic}_{C_K}[r]) \) is finite if and only if \( m_2 = 1 \). By (5.4.2), \( m_2 = 1 \) if and only if \( m_1 = 1 \) and \( c \in r\mathbb{Z} \). This is precisely the statement.
\end{proof}

\begin{remark}
Note that the two conditions related in the statement of Corollary 5.5.1 hold only if the group \( K \)-scheme \( \text{Pic}_{C_K}[r] \) is tamely ramified. Therefore, do not need to assume this condition in the statement.
\end{remark}

\begin{remark}
The criterion given above holds with the same proof for \( r = 2 \) if we make the assumption that a semistable reduction of \( C_K \) over \( R \) exists. Otherwise, in the statement, “if and only if” should be replaced by “if”.
\end{remark}

\section{Some examples.}

We compare the values of \( m_1, m_2, \) and \( m_3 \) in some concrete examples. We always consider a stable curve \( C \) over \( R \) satisfying the following conditions:

1. the generic fibre is a smooth curve \( C_K \) over \( K \),
2. \( C \) is regular (in other words, \( C \) is the minimal regular model of \( C_K \)).

The dual graph of the special fibre is denoted by \( \Gamma \). We list the values of \( m_1, m_2, \) and \( m_3 \) in the following table, where we assume for simplicity that \( r \) is a positive multiple of 4. Note that \( m_1 \) always equals 1 because the curve \( C_K \) admits a semistable reduction. Furthermore, \( m_3 \) equals \( r \) because all nodes of the stable model have thickness 1. On the other hand \( m_2 \) is determined by the (signed) numbers of edges shared by two circuits in the graph. The greatest common divisor of these numbers is 1, 2, 4, 2, 4, and 2 in the four cases considered; so \( m_2 \) equals \( r, r/2, r/4, r/2, r/4, \) and \( r/2 \), respectively.
Example 5.6.1 (the special fibre is irreducible and has a single node). This case is well known. The curve $C_K$ is smooth and its minimal regular model is a stable curve whose special fibre contains a single node, which is nonseparating. The group $\text{Pic}_{C_K}[r]$ is ramified over $K$ and, therefore, its Néron model over $R$ is not finite.

After base change to $K_d = K[\tilde{\pi}]/(\tilde{\pi}^d - \pi)$ we obtain a group with finite Néron model if and only if $d$ is a multiple of $r$. This is consistent with Theorem 5.1.1 and equation (5.4.2), which yields $m_1 = 1$, $m_2 = r$, and $m_3 = r$.

Example 5.6.2 (the special fibre has two irreducible components with two nodes). In this case the minimal regular model of $C_K$ is a stable curve whose special fibre has two irreducible components and two (nonseparating) nodes. On the one hand, this is a case where the classical Néron model of the group of 2-torsion line bundles is finite as we illustrate hereafter. On the other hand, the finite Néron model does not represent the group of 2-torsion line bundles of any semistable reduction of $C_K$. Indeed this is a consequence of Remark 2.4.2: as soon as $C_K$ admits a semistable reduction containing nonseparating nodes, the functor of $r$-torsion bundles on any semistable reduction is not finite.

Let us focus on the finiteness of the Néron model or, equivalently, on the proof of $m_2 = 1$ for the group $\text{Pic}_{C_K}[2]$. The dual graph is the second graph of the table above. There is only one nontrivial circuit in the graph and it has two edges; we have $c = 2$. By (5.4.2), we have $m_2 = 1$; i.e. $N(\text{Pic}_{C_K}[2])$ is finite. In general, for $\text{Pic}_{C_K}[r]$, we have $m_1 = 1$, $m_2 = r/2$, and $m_3 = r$.

Remark 5.6.3. The previous example shows that $m_2$ and $m_3$ are not equal in general. This means that a finite Néron $d$-model need not represent the $r$-torsion line bundles on a twisted semistable reduction on $S[d]$.

Example 5.6.4 (the dual graph of the special fibre is a circuit). The above example can be generalised without change to the case where the dual graph of the special fibre is a circuit.
having $h$ nodes and $h$ edges. In the table we consider the case $h = 4$. If $r$ is a multiple of $h$, then we get $m_1 = 1$, $m_2 = r/h$, and $m_3 = r$. For $h = r$ this provides a class of examples satisfying the necessary and sufficient condition of Corollary [5.5.1] for the finiteness of the Néron model of $\text{Pic}_{C_K}[r]$.

**Example 5.6.5.** The fourth graph provides an example where, for $r = 4$, we have $m_1 = 1$, $m_2 = 2$, and $m_3 = 4$. Since $m_2 \neq 1$, the Néron (1-)model is not finite. Indeed, the condition of Corollary [5.5.1] is not satisfied for $r = 4$: there are circuits sharing only two edges.

**Remark 5.6.6.** For $r = 4$, the two previous examples exhibit a situation where $C_K$ admits a semistable minimal regular model for which the dual graph of the special fibre satisfies the following property. The number of edges of the circuits contained in the dual graph of the special fibre is always a multiple of $r$. Example [5.6.2] may lead us to believe that this property implies the finiteness of the Néron model of $\text{Pic}_{C_K}[r]$. Instead, Example [5.6.5] shows that the Néron model of $\text{Pic}_{C_K}[r]$ may not be finite.

The condition that the number of edges of all circuits contained in the dual graph is always a multiple of $r$ is necessary, but not sufficient. The right sufficient and necessary condition on the graph involves the (signed) number of edges shared by two circuits and is the one stated in Corollary [5.5.1].

**Remark 5.6.7.** We also point out that a sufficient condition on the dual graph $\Gamma$ implying that the Néron model of $\text{Pic}_{C_K}[r]$ is finite is given by the following definition.

**Definition 5.6.8.** A graph is $r$-divided if it is obtained by taking another graph $\tilde{\Gamma}$ and by dividing all its edges in $r$ edges.

Indeed, Proposition 2 of [Lo91] implies immediately that the Néron model of $\text{Pic}_{C_K}[r]$ is finite as soon as $\Gamma$ is $r$-divided. We illustrate with some examples that this condition is not necessary.

**Example 5.6.9** (the dual graph of the special fibre need not be $r$-divided). A simple example illustrating this fact is given by taking a graph with $2r + 1$ edges, where the first $2r$ edges form two disjoint circuits with $r$ edges and the last edge joins two vertices lying on the first and the second circuit. This graph is not $r$-divided (the fifth diagram in the table provides a picture for $r = 4$).

In fact, even if we restrict to graphs which do not contain separating edges, “$r$-divided” is stronger than the condition stated in Corollary [5.5.1]. We show it in the following example.

**Example 5.6.10.** We consider the last graph of the table above. All the circuits share an even number of edges, but the graph is not 2-divided. Indeed, it contains two chains (the horizontal edges in the diagram) whose number of edges is not a multiple of 2.

### 6. Finite Néron d-models of the torsor of $r$th roots

We generalise the above results to the functor of $r$th roots $F_{K}^{1/r}$ of a line bundle $F_K$ over the curve $C_K$ smooth over $K$. We assume that the degree of $F_K$ is a multiple of $r$. The $K$-scheme $F_{K}^{1/r}$ is the preimage of $F_K$ with respect to the $r$th tensor power in $\text{Pic}_{C_K}$. It has a natural structure of étale $K$-torsor under the kernel of the $r$th tensor power: the $K$-group $\text{Pic}_{C_K}[r]$.

**The finiteness of the Néron d-model of the torsor of $r$th roots.** The following proposition exhibits a finite Néron $d$-models of $F_{K}^{1/r}$ that represents the functor of $r$th roots over a twisted semistable reduction of $C_K$. 

19
Proposition 6.1.1. Under the conditions of Theorem 5.1.1, we assume that $F_K$ is a line bundle on $C_K$, whose degree is a multiple of $r$. Recall that $C_K$ has semistable reduction on $S[d]$ if and only if $d$ is a multiple of $m_1$.

If $d$ is a multiple of $rm_1$, then the Néron $d$-model of $F_K^{1/r}$ is finite and there is a line bundle $F$ on a twisted semistable reduction $C$ of $C_K$ on $S[d]$ extending $F_K \rightarrow C_K$ and satisfying

$$N_d(F_K^{1/r}) = F^{1/r},$$

where $F^{1/r}$ represents the functor of $r$th roots of $F$ on $C \rightarrow S[d]$. In that case, the torsor structure of $N_d(F_K^{1/r})$ under $N_d(\text{Pic}_{C_K}[r])$ is the natural torsor structure of $F^{1/r}$ under the group of $r$-torsion line bundles on $C$.

**Proof.** Consider the semistable minimal regular model $C^{\text{reg}}$ of $C_K$ on $S[m_1]$. Since $C^{\text{reg}}$ is regular, there exists a line bundle $F^{\text{reg}}$ extending $F_K$ on $C^{\text{reg}}$. For $d \in rm_1 \mathbb{Z}$, over $S[d]$ there is a twisted curve $C$ whose nodes have stabiliser of order $r$ and whose coarse space fits in

$$(\text{we construct } C \text{ by iterating the construction of Remark 2.6.2 with } d = r \text{ at all nodes}).$$

Consider the pullback $F$ of $F^{\text{reg}}$ on $C$ via the projection to $C^{\text{reg}}$ in the diagram above and via $\pi : C \rightarrow |C|$. Notice that $F^{1/r}$ is a finite torsor on $S[d]$ by Theorem 3.2.1. Then, for $\tilde{R} = R[\pi]/(\pi^d = \pi)$, the pullback of $F^{1/r}$ on $\text{Spec} \tilde{R}$ is the Néron model of $F^{1/r}_K$ by [BLR80, 7.1/Thm. 1]. We conclude that $F^{1/r}$ is the Néron $d$-model of $F_K^{1/r}$ by Proposition 4.3.4. □

Remark 6.1.2. Note that the argument above shows that the line bundle $F$ satisfying $N_d(F_K^{1/r}) = F^{1/r}$ can be chosen in such a way that it descends to a line bundle over the semistable minimal regular model of $C_K$ on $S[m_1]$.

(6.2) The finiteness criterion for $N(F_K^{1/r})$. We apply the above result to determine when the usual Néron model of $F_K^{1/r}$ is finite over $R$.

**Corollary 6.2.1.** Let $r > 2$ be prime to $\text{char}(k)$, and let $C_K$ be a smooth curve of genus $g \geq 2$. The Néron model of $F_K^{1/r}$ is finite if and only if $C_K$ admits a semistable minimal regular model $C_R$ satisfying the following conditions:

(1) The dual graph of the special fibre of $C_R$ verifies the condition of Corollary 5.5.1

(2) There exists a line bundle $F_R$ on $C_R$ isomorphic to $F_K$ on the generic fibre and of degree $d \in r \mathbb{Z}$ on each irreducible component of the special fibre.

**Proof.** We note first that the two conditions in the statement hold only if $m_1$ equals 1. Indeed, if the Néron model of the torsor is finite, then the Néron model of $\text{Pic}_{C_K}[r]$ is finite, hence $m_2 = m_1 = 1$.

Using Proposition 6.1.1, we regard the Néron $r$-model of $F_K^{1/r}$ as the torsor of rth roots of a line bundle $F$ extending $F_K$ on a twisted reduction $C$ with stabilisers of order $r$ at all nodes. Note that $C$ can be chosen in such a way that the coarse space descends to the semistable minimal regular model $C_{\text{reg}}^{\text{reg}}$ of $C_K$ on $S$. In Remark 6.1.2 we point out that $F$ descends to $C_{\text{reg}}^{\text{reg}}$.  

20
Claiming that the Néron model of the torsor is finite is equivalent to saying that $N_r(F_{1/r}^1)$ also descends to $S$. Denote by $C_k$ the special fibre of $C$, whose coarse space is the special fibre of $C_{\reg}$, then, the descent of $N_r(F_{1/r}^1)$ to $S$ is equivalent to the claim that the automorphism $g$ defined in (5.5.3) acts trivially on the $r$th roots of $F|_{C_k}$.

Recall that the pullback $j^*$ via the embedding $j$ of the singular locus into $C_k$, can be regarded as a homomorphism from Pic to $(\mu_r)^F = C_1(\Gamma, \mu_r)$. By Proposition 3.5.1 and Remark 3.5.2 $g$ acts trivially on $F_{1/r}^1$ if and only if $j^*(F|_{C_k}^1)$ lies in the image of $\delta_r : C_0(\Gamma, \mu_r) \to C_1(\Gamma, \mu_r)$. This is equivalent to two inclusions: first we require that $\im \delta_r$ includes $j^*(\Pic_{C_k}|[r])$, second we require that there exists one root of $F|_{C_k}$ whose image via $j^*$ is contained in $\im \delta_r$. The first inclusion $\im \delta_r \supset j^*(\Pic_{C_k}|[r])$ amounts to require that $C_R$ satisfies the condition (1) of the statement above. The second inclusion simply means that there exists an $r$th root $L$ of $F$ on $C_k$ such that $\im(\partial_r \circ \delta_r)$ contains $\delta_r(j^*L)$. A direct calculation shows that $\partial_r(j^*L)$ is the multiindex of degrees modulo $r$ of $L^{\otimes r} \cong F$ on each irreducible component of $C_k$. Therefore such a multiindex lies in $\im(\partial_r \circ \delta_r)$ modulo $r$, which is, by Remark 2.5.2, what needed to be shown. 

**Remark 6.2.2.** For $r = 2$, Remark 5.5.3 extends word for word. The criterion holds in the case $r = 2$ if we assume that the minimal regular model $C_R$ of $C_K$ is semistable. More explicitly, for any line bundle $F_K$ of even degree on $C_K$, we have $N(F_{1/2}^1)$ if and only if

1. the circuits in the dual graph of the special fibre of $C_R$ always share an even number of edges.
2. there exists a line bundle $F_R$ on $C_R$ isomorphic to $F_K$ on the generic fibre of $C_R$ and of even degree on each irreducible component of the special fibre of $C_R$.

**Example 6.2.3** (square roots of a line bundle of multidegree $(1, -1)$). We fix $r = 2$, and we assume that the minimal regular model $C_K$ is a stable curve $C_R$, whose special fibre $C_k$ has two connected components $C_{1,k}$ and $C_{2,k}$ and two nodes.

Let us consider a line bundle $F_K$ of even degree over $C_K$. Since $C_R$ is regular, it is the minimal regular model of $C_K$. Furthermore, the study of the dual graph of the special fibre already carried out in Example 5.6.2 implies that condition (1) is satisfied. The finiteness of $N(F_{1/2}^1)$ only depends on the existence of a line bundle $F_R$ extending $F_K$ with even degrees $d_1$ and $d_2$ on $C_{1,k}$ and $C_{2,k}$.

In fact, for any line bundle $F_R$ extending $F_K$ on $C_R$, the degrees $d_1$ and $d_2$ have the same parity: $d_1 + d_2 \in 2\Z$. If both degrees are even, then the condition (2) of Corollary 6.2.1 is satisfied and $N(F_{1/2}^1)$ is finite. If both degrees are odd, then the analysis carried out in §2.5.1 shows immediately that there is no line bundle $F_R$ on $C_R$ extending $F_K$ with even degrees $d_1$ and $d_2$ on the special fibre. Then, $N(F_{1/2}^1)$ is not finite.

On the other hand, even in the case $d_1, d_2 \in 1 + 2\Z$, Proposition 6.1.1 says that the Néron 2-model is finite. Indeed its special fibre represents $2^{2g}$ square roots on a twisted curve $C_k$ with stabiliser of order 2 on each node. The degree of each square root on the two irreducible components of $C_k$ belongs to $1/2 + \Z$.

**7. The Néron model of Pic_{C_K}|[r]** without the theory of twisted curves

As stated in the introduction, this section is self contained and can be read straight after §1 assuming some well known facts on semistable curves recalled in §2.1-5. We consider the question of the finiteness of Pic_{C_K}|[r] under the same conditions of §1.1.
The group of connected components of the special fibre of the Néron model. We consider the subfunctor $\text{Pic}^0_{C_K}$ of line bundles of degree 0. It is represented by a proper, geometrically connected scheme of finite type over $K$: the jacobian variety of $C_K$. Let $N(\text{Pic}^0_{C_K})$ be its Néron model over $R$. In general, its special fibre is not geometrically connected and the connected components form a finite abelian group.

If $C_K$ admits a semistable minimal regular model $C_R$, then the group of components of the special fibre of $N(\text{Pic}^0_{C_K})$ admits a simple description, which is due to Raynaud [Ra70]. Indeed, we can realise this group by applying to the dual graph $\Gamma$ of the special fibre of $C_R$ a construction which associates to any connected graph a finite group.

**Definition 7.1.1.** Let $\Gamma$ be an oriented connected graph. The group $\Phi_\Gamma$ is an invariant of the graph $\Gamma$ given by

$$\Phi_\Gamma = \frac{\partial(C_1(\Gamma, \mathbb{Z}))}{\partial(\delta(C_0(\Gamma, \mathbb{Z})))}.$$  

**Remark 7.1.3.** Since $\Gamma$ is connected, the homomorphism $\partial: C_1 \rightarrow C_0$ and the augmentation homomorphism $\varepsilon: C_0 \rightarrow \mathbb{Z}$ form an exact sequence $C_1 \rightarrow C_0 \rightarrow \mathbb{Z}$. We have

$$\Phi_\Gamma = \frac{\ker \varepsilon}{\operatorname{im} M},$$

where $M$ is the endomorphism of $C_0$ defined by $-\partial \circ \delta$. (This endomorphism has a geometric interpretation, see (2.5) and in particular Remark 2.5.1.)

In fact $\Phi_\Gamma$ is a finite abelian group, [BLR80, 9/10]. Furthermore, write $h$ for the number of vertices of $V$, the Smith normal form of $M$ is given by Diag($n_1, \ldots, n_{h-1}, 0$), with $n_i \neq 0$ all $i$. In this way $\Phi_\Gamma$ is isomorphic to $\bigoplus_i \mathbb{Z}/n_i\mathbb{Z}$.

**Definitions of $\Phi_\Gamma$ in the literature.** This group appeared first in 1970, with the definition (7.1.4), in Raynaud’s paper [Ra70, §8.1.2, p. 64]. There, we consider a curve $C_K$ whose minimal regular model $C_R$ is semistable. The graph $\Gamma$ is the dual graph of the special fibre of $C_R$, and $\Phi_\Gamma$ is identified with the group of connected components of the special fibre of the Néron model of $\text{Pic}^0_{C_K}$.

It is worth mentioning that, in this perspective, the elements of this group can be regarded as classes of multidegrees of line bundles over a semistable curve; therefore Caporaso refers to $\Phi_\Gamma$ as the “degree class group”, [Cap].

The same group was studied from different viewpoints and with different names: Berman [Be86], Lorenzini [Lo89], Dahr [Da90] (sandpile group), Bacher–de la Harpe–Nagnibeda [BDN97] (Picard group), Biggs [Bi99] (critical group).

**Reformulating the definition of $\Phi_\Gamma$ and describing its $r$-torsion.** In view of the proof of our criterion for the finiteness of $N(\text{Pic}_{C_K}[r])$ we give a new characterisation of $\Phi_\Gamma$.
Lemma 7.3.1. The following diagram is commutative and all sequences are exact

\[
\begin{array}{cccccccc}
0 & \to & H_1/ (H_1 \cap \delta(C_0)) & \to & H^1 & \to & \Phi_\Gamma & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_1 & \to & C_1 & \to & \partial(C_1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H_1 \cap \delta(C_0) & \to & \delta(C_0) & \to & \partial(\delta(C_0)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

(7.3.2)

where \( H_1 \) and \( H^1 \) denote \( H_1(\Gamma, \mathbb{Z}) \) and \( H^1(\Gamma, \mathbb{Z}) \), and \( C_0 \) and \( C_1 \) denote \( C_0(\Gamma, \mathbb{Z}) \) and \( C_1(\Gamma, \mathbb{Z}) \).

In this way, we have

\[
\Phi_\Gamma \cong \frac{H^1}{H_1/(H_1 \cap \delta(C_0))}.
\]

(7.3.3)

Since \( \Phi_\Gamma \) is a finite abelian group its \( r \)-torsion \( \Phi_\Gamma[r] \) is given by \( \Phi_\Gamma \otimes \mathbb{Z}/r\mathbb{Z} \). By tensoring both sides of (7.3.2), we get

\[
\Phi_\Gamma[r] \cong \frac{\partial_r(C_1(\Gamma, \mathbb{Z}/r\mathbb{Z}))}{\partial_r(\delta_r(C_0(\Gamma, \mathbb{Z}/r\mathbb{Z})))}.
\]

Write the analogue of diagram (7.3.2) with \( C_0(\Gamma, \mathbb{Z}/r\mathbb{Z}), C_1(\Gamma, \mathbb{Z}/r\mathbb{Z}), \partial_r, \) and \( \delta_r \) instead of \( C_0(\Gamma, \mathbb{Z}), C_1(\Gamma, \mathbb{Z}), \partial, \) and \( \delta \). The diagram is still exact, provided that we replace \( H_1 \) and \( H^1 \) by \( H_1(\Gamma, \mathbb{Z}/r\mathbb{Z}) = \ker(\delta_r) \) and \( H^1(\Gamma, \mathbb{Z}/r\mathbb{Z}) = \coker(\delta_r) \). In this way, we get the following exact sequence.

Lemma 7.3.4. We have

\[
0 \to H_1(\Gamma, \mathbb{Z}/r\mathbb{Z})/(H_1(\Gamma, \mathbb{Z}/r\mathbb{Z}) \cap \delta_r(C_0(\Gamma, \mathbb{Z}/r\mathbb{Z}))) \to H^1(\Gamma, \mathbb{Z}/r\mathbb{Z}) \to \Phi_\Gamma[r] \to 0
\]

(7.3.5)

Recall that \( H^1(\Gamma, \mathbb{Z}/r\mathbb{Z}) \) is isomorphic to \( (\mathbb{Z}/r\mathbb{Z})^{\oplus b_1(\Gamma)} \). Therefore, \( \Phi_\Gamma[r] \) is a quotient of a group of order \( r^{2g} \).

(7.4) The group \( \Phi_\Gamma \) and the finiteness of the Néron model of \( \text{Pic}_{C_K}^r \). Applying Raynaud’s description of the Néron model of the jacobian variety, the condition that \( N(\text{Pic}_{C_K}^r) \) is finite can be characterised in terms of the group \( \Phi_\Gamma \).

Proposition 7.4.1. Let \( r > 2 \) be prime to \( \text{char}(k) \). The Néron model of \( \text{Pic}_{C_K}^r \) is finite if and only if the minimal regular model of \( C_K \) over \( R \) is semistable and the dual graph \( \Gamma \) of its special fibre satisfies

\[
\Phi_\Gamma[r] \cong (\mathbb{Z}/r\mathbb{Z})^{\oplus b_1(\Gamma)}.
\]

(7.4.2)

Proof. Since \( r > 2 \), the fact that the Néron model \( N(\text{Pic}_{C_K}^r) \) is finite implies that the minimal regular model \( C_R \) of \( C_K \) over \( R \) is semistable (see [De81, 5.17]). Therefore, we can take for granted the existence of a semistable regular reduction \( C_R \).

Then the \( r \)-torsion of the special fibre of \( N(\text{Pic}^0_{C_K}) \) is the special fibre of the \( r \)-torsion of \( N(\text{Pic}^0_{C_K}) \), which is the Néron model of \( \text{Pic}^0_{C_K}[r] = \text{Pic}_{C_K}[r] \). In this way, the Néron model
$N(Pic_{C_k}[r])$ is finite, if and only if the special fibre of the Néron model $N(Pic_{C_k}[r])$ consists of $r^{2g}$ points. By [Ra70] and [BLR80, 9], the special fibre of $N(Pic_{C_k}[r])$ is an extension of $\Phi_{r}[r]$ by the $r$-torsion of the Picard group of the special fibre $C_k$ of $C_R$. Now, Pic$C_k[r]$ consists of $r^{2g-h_1(r)}$ points. Therefore, the number of points of the special fibre of $N(Pic_{C_k}[r])$ is $r^{2g}$ if and only if the group $\Phi_{r}[r]$ has order $r^{h_1}$. This fact, by Lemma $7.3.4$, is equivalent to $\Phi_{r}[r] \cong (\mathbb{Z}/r\mathbb{Z})^\oplus h_1(r)$.

(7.5) When is the $r$-torsion subgroup of $\Phi_{r}$ isomorphic to $(\mathbb{Z}/r\mathbb{Z})^\oplus h_1$? We obtain again the answer of Corollary $5.5.1$.

**Proposition 7.5.1.** For any positive $r$, the $r$-torsion subgroup of the group $\Phi_{r}$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^\oplus h_1(r)$ if and only if the (signed) number of edges common to any two circuits is always a multiple of $r$.

**Proof.** Recall the exact sequence of Lemma $7.3.4$ and the fact that $H^1(\Gamma, \mathbb{Z}/r\mathbb{Z})$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^\oplus h_1(r)$. Then, $\Phi_{r}[r]$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^\oplus h_1(r)$ if and only if $H^1(\Gamma, \mathbb{Z}/r\mathbb{Z}) = H^1(\Gamma, \mathbb{Z}/r\mathbb{Z}) \cap \delta(C_0(\Gamma, \mathbb{Z}/r\mathbb{Z}))$. This yields $\ker(\partial_r) \subseteq \text{im}(\delta_r)$. Lemma $8.0.2$ proven in Section $8$ implies the statement of the theorem.

8. The proof of the combinatorial lemma on circuits

We give a proof of a combinatorial lemma used in Proposition $5.4.1$ and in Proposition $7.5.1$.

**Lemma 8.0.2.** For any positive integer $r$ and for any connected graph $\Gamma$ the following conditions are equivalent:

1. $\ker(\partial_r) \subseteq \text{im}(\delta_r)$;

2. the number of edges of any two circuits is always a multiple of $r$.

**Proof of Lemma $8.0.2$.** Using (2), we show that $\ker(\partial_r)$ is included in $\text{im}(\delta_r)$. It is enough to prove this claim for any circuit $C = \sum_{i=0}^{n-1} e_i$ passing through $n$ vertices $v_0, v_1, \ldots, v_{n-1}$: for $0 \leq i < n - 1$, $v_i$ to $v_{i+1}$ are the tail and the tip of $e_i$, whereas $v_{n-1}$ and $v_0$ are the tail and the tip of $e_{n-1}$. We have $C \in \ker(\partial_r)$, and we want to show that there exists $A = \sum_{v \in V} a(v)[v]$ such that $\delta_r(A) = C$.

For an arbitrary vertex $v$, we define $a(v)$ as follows: if there exists a simple path joining $v$ to $v_i$ with all vertices outside $C$ apart from $v_i$, then we set $a(v) \equiv i \mod r$. Note that it may happen that a vertex $v$ off $C$ can be connected in the way described above to both $v_i$ and $v_j$ with $i < j$ via two simple paths $S_i$ and $S_j$. Then, we can choose the last vertex $l$ common to $S_i$ and $S_j$, and define a circuit $\tilde{C}$ joining $l$ to $v_i$ via $S_i$, $v_i$ to $v_j$ via $C$, and, finally, $v_j$ to $l$ via $-S_j$. The condition (2) applied to $\tilde{C}$ and $C$ yields $j - i \equiv 0 \mod r$. It is easy to check that, for $A = \sum_{v \in V} a(v)[v]$, we have $\delta_r(A) = C$ by construction.

Conversely, we consider two circuits $C$ and $\tilde{C}$ and we count the common edges (with sign). We write $\tilde{C}$ as $\sum_{j=1}^{m-1} [f_j]$, the circuit whose vertices are $w_0, \ldots, w_{m-1}$, whose $j$th edge $f_j$ is oriented from $w_j$ to $w_{j+1}$ for $0 \leq j < m - 1$, and whose edge $f_{m-1}$ is oriented from $w_{m-1}$ to $w_0$. By $\ker(\delta_r) \subseteq \text{im}(\delta_r)$ there exists $A = \sum_{v \in V} a(v)[v] \in C_0(\Gamma, \mathbb{Z}/r\mathbb{Z})$ such that $\delta_r(A) = \tilde{C}$. Using $A$, for any $j \in \{0, \ldots, m - 1\}$, we associate to each vertex $w_j$ an index $a(w_j) \in \mathbb{Z}/r\mathbb{Z}$, which we denote by $a_j \in \mathbb{Z}/r\mathbb{Z}$. The condition $\partial_r(A) = \tilde{C}$ implies

$$m \in r\mathbb{Z} \quad \text{and} \quad ax - ay = x - y \quad \text{in} \ \mathbb{Z}/r\mathbb{Z},$$

(8.0.3)
if $x$ and $y$ are values in $\{0, \ldots, m - 1\}$. We also observe that, by definition,
\begin{equation}
a(v_+) = a(v_-) \quad \text{in } \mathbb{Z}/r\mathbb{Z} \tag{8.0.4}
\end{equation}
if $v_+$ and $v_-$ are the tip and the tail of an edge not included in $\tilde{C}$.

We assume that $C$ and $\tilde{C}$ have some edges in common, but not all (otherwise the claim follows from $m \in r\mathbb{Z}$ in (8.0.3)). We write $C = \sum_{i=0}^{n} [e_i]$ for the circuit $C$ passing through the vertices $v_0, \ldots, v_{n-1}$. We choose the notation so that, for $0 \leq i < n - 1$, the edge $e_i$ is oriented from $v_i$ to $v_{i+1}$, whereas the edge $e_{n-1}$ is oriented from $v_{n-1}$ to $v_0$ and does not lie on $\tilde{C}$. By (8.0.4), this implies
\begin{equation}
a(v_{n-1}) = a(v_0) \quad \text{in } \mathbb{Z}/r\mathbb{Z}. \tag{8.0.5}
\end{equation}

Consider the indices $0, 1, \ldots, n - 1$ of the vertices of $C$. Define a non decreasing sequence of integers $0 \leq t_1 \leq \cdots \leq t_{2h} < n$ with $h \in \mathbb{N}$ so that $i$ and $i + 1$ are contained in one of the intervals $[t_1, t_2], [t_3, t_4], \ldots, [t_{2h-1}, t_{2h}]$ if and only if there exists an edge of $\tilde{C}$ joining $v_i$ and $v_{i+1}$ (in any direction). Note that we can insert in the sequence two values $t_{2j-1}$ and $t_{2j}$ with $t_{2j-1} = t_{2j}$ for any $j \in \mathbb{N}$, as long as we reparametrise and we respect the monotony. In this way, we can adjust the choice of the values $\{t_i\}_{0 < i \leq 2h}$ so that $h$ is even and for each even (resp. odd) $k$ satisfying $0 < k \leq h$, the simple path $\sum_{t_{2k-1} < i < t_{2k}} e_i$ joining $v_{t_{2k-1}}$ to $v_{t_{2k}}$ lies on $\tilde{C}$ and $C$ with the same (resp. the opposite) orientation. Now, (8.0.3) implies
\begin{equation}
a(v_{t_{2k}}) - a(v_{t_{2k-1}}) = (-1)^k (t_{2k} - t_{2k-1}) \quad \text{in } \mathbb{Z}/r\mathbb{Z}; \tag{8.0.6}
\end{equation}
Clearly, the following relation hold in $\mathbb{Z}/r\mathbb{Z}$
\begin{equation*}
0 = (a(v_0) - a(v_{n-1})) + \sum_{0 \leq j < n-1} (a(v_{j+1}) - a(v_j)).
\end{equation*}
By (8.0.3), the first summand on the right hand side vanishes. In fact, using (8.0.4), we get
\begin{equation*}
0 = \sum_{0 < k \leq h} (a(v_{t_{2k}}) - a(v_{t_{2k-1}})).
\end{equation*}
By (8.0.6), we have
\begin{equation*}
0 = \sum_{0 < k \leq h} (-1)^k (t_{2k} - t_{2k-1}).
\end{equation*}
Note that the last equation in $\mathbb{Z}/r\mathbb{Z}$ is the claim (2).

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