Quantum effective force in an expanding infinite square-well potential and Bohmian perspective

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Abstract

The Schrödinger equation is solved for the case of a particle confined to a small region of a box with infinite walls. If the walls of the well are moved, then, due to an effective quantum nonlocal interaction with the boundary, even though the particle is nowhere near the walls, it will be affected. It is shown that this force apart from a minus sign is equal to the expectation value of the gradient of the quantum potential for vanishing at the walls boundary condition. The variation of this force with time is studied. A selection of Bohmian trajectories of the confined particle is also computed.

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(Some figures may appear in color only in the online journal)

1. Introduction

In most of the problems of quantum mechanics, the Hamiltonian of the system is time dependent and so one needs to solve the time-dependent Schrödinger equation (TDSE). Problems with moving boundary conditions are an interesting class of such time-dependent problems. Such a system was first considered by Fermi [1] in connection with the study of cosmic radiation. After that, several authors have studied problems with moving boundaries ([2–4]; [5] and references therein).

Different aspects of the problem of a particle in a one-dimensional infinite square-well potential with one wall in uniform motion have been discussed by earlier authors. The exact solution of the TDSE for this problem was, as far as we know, first given by Doescher and Rice [2]. Schlitt and Stutz [6] considered the application of the sudden approximation to the rapid expansion of the well. Pinder [7] investigated the applicability of both adiabatic and sudden approximations for both expanding and contracting wells: it was shown that sudden approximation is appropriate for the expanding well provided the wall speed is sufficiently great, but this approximation may not be applied to the contracting well irrespective of the rate of contraction. Using the semiclassical approximation, Luz and Cheng [8] evaluated the exact propagator of the problem. The energy gain and the transition amplitudes and probabilities between initial and final energy eigenstates of the problem have been calculated [3]. A recent numerical study of a particle in a box with different laws for the movement of the wall shows that physical quantities such as probability density and expectation value of the position or mean value of the energy have a smooth behavior for small speeds of the moving wall. In contrast, if this speed becomes large, many irregularities appear as sharp bumps on the probability distribution or a chaotic shape on the averaged values of position and energy [9].

The aim of this paper is to probe some aspects of the time-dependent boundary condition for a particle confined in an infinite square well that have remained hitherto unnoticed.

Let us focus on the effect of the time-dependent boundary condition while one of the infinite boundary walls in a box is moved where we have a well-localized Gaussian wave packet which remains peaked at the center of the box, $x_c$, well away from the walls. Now, by calculating the effective quantum force [10], one can study the way this effective quantum
force changes with time. Due to such a force, the expectation value of the momentum with the direction perpendicular to the walls gradually changes with time. Then one can compare the curves of the quantum effective force for the static (when the wall is at rest) and the dynamic (when the wall moves) situations, and pinpoint the instant from which the dynamic curve deviates from the static one. Such an instant shows the time at which the confined particle begins to feel the motion of the wall.

Physically, one expects that when the width of the initial Gaussian packet is much smaller than the initial width of the box, $\sigma_0 \ll \ell_0$, one might consider a Gaussian wavepacket to be realizable in the box trap. But this approximation casts some doubts upon the computations (i.e. ‘is the effect an artifact of the tails of the Gaussian at the boundaries?’). So for this purpose, in addition, we consider a localized state with a finite support embedded in the support of the expanding box trap at time $t = 0$. It is natural to consider the particle-in-a-box eigenstates of a tiny box centered at $x = x_c$ and suddenly released at $t = 0$ to become the initial state without the necessary approximations to assume a Gaussian packet as an initial state. Such a state is called the ‘tiny-box state’ in the following.

The computed Bohmian trajectories [11–13] for the static and the dynamic situations are also instructive in revealing the conceptual ramifications of such an example. In Bohm’s model, each individual particle is assumed to have a definite position, irrespective of any measurement. The pre-measured value of position is revealed when an actual measurement is made. Over an ensemble of particles having the same value of position is revealed when an actual measurement. The pre-measured model, each individual particle is assumed to have a definite

The plan of this paper is as follows. Section 2 contains a very brief review of the relevant mathematical steps leading to the exact solution of the problem. In section 3, numerical computations related to the effect of the time-dependent boundary condition are presented. Finally, in section 4 we present the concluding remarks.

2. Basic equations

Consider a narrow box inside a wide box with a particle inside the inner one. The walls of the outer box are at $x_L = 0$ and $x_R = \ell_0$ and the walls of the inner one are at $x_{L1} = (\ell_0 - \ell_1)/2$ and $x_{R1} = (\ell_0 + \ell_1)/2$ where $\ell_1 \ll \ell_0$. At time $t = 0$ the inner box is suddenly removed and the right wall of the outer box starts to move uniformly with velocity $u$. The infinite wall speed $u$ corresponds to a hard wall at $x = 0$. We discuss the solution of TDSE for two cases. The initial wavefunction has to be:

1. a Gaussian wave packet well localized in the center of the tiny box. In fact in this case we have a truncated Gaussian packet, because of the confinement of the

Gaussian packet with infinite tails in a narrow region, and hence the name ‘truncated Gaussian packet’ (TGP).

The problem concerning the tails of Gaussian packet that was mentioned in the introduction is now translated to the truncation.

2. the ground state of the narrow box with kick momentum $k$ (tiny-box state (TBS)).

For a picture, see figure 1.

Using the propagator of a rigid box with the left wall at $x_L = 0$ and the moving right wall in a constant velocity $u$ [8],

$$K(x, t; x', 0) = \frac{2}{\sqrt{\ell_0 \ell(t)}} e^{\frac{\ell}{\ell(t)^2} \left( x'^2 - x^2 \right)} \sum_{n=1}^{\infty} e^{\frac{\ell^2}{\ell(t)^2} \left( n^2 - \frac{x'^2 - x^2}{\ell(t)} \right)} \times \sin \left( \frac{n \pi x}{\ell(t)} \right) \sin \left( \frac{n \pi x'}{\ell(t)} \right),$$

and the relation

$$\psi(x, t) = \int dx' K(x, t; x', 0) \psi_0(x'),$$

one obtains the wavefunction at any time having $\psi_0(x)$ in hand. $\ell(t) = \ell_0 + ut$ shows the position of the moving wall at time $t$. At this point, it must be mentioned that due to the Galilean invariance of the Schrödinger equation [13], the case of both the moving walls is equivalent to the case of one wall in motion but with $u$ as the relative velocity of walls.

With the initial wavefunction to be a TGP well localized in the center of the box $x_c = \ell_0/2$,

$$\psi_0(x) = \frac{1}{(2\pi \sigma_0^2)^{1/4}} \exp \left[ -i k(x - x_c) - \frac{(x - x_c)^2}{4\sigma_0^2} \right] \times \Theta(x - x_c) \Theta(x_c - x),$$

where $\Theta(x)$ is the step function, one obtains

$$\psi(x, t) = \frac{2}{\sqrt{\ell_0 \ell(t)}} \sum_{n=1}^{\infty} e^{\frac{\ell^2}{\ell(t)^2} \left( n^2 \pi^2 \sigma_0^2 \right)} \sin \left( \frac{n \pi x}{\ell(t)} \right) \times f_n,$$

where $f_n$ is a function depending on $n$. For a picture, see figure 1.
where

\[ f_n = \frac{i}{2} \left( \frac{\pi}{2} \right)^{1/4} \sqrt{\frac{\sigma_0 \ell_0}{\hbar}} \exp \left[ -\frac{\left( \int \frac{\langle x | \hat{p} | x \rangle}{\langle x | x \rangle} + 8\pi^2 x^2 \sigma_0^2 \hbar + 16\pi \ell_0 \sigma_0^2 \hbar k \ell_0 + \ell_0^2 (4\pi \sigma_0^2 \hbar + 8\pi^2 \sigma_0^2 \hbar k (h \ell_0 + m \ell_0))}{8\ell_0 (h \ell_0 + 2m \ell_0 \sigma_0^2)} \right] \times \frac{\text{Erf}\left( -\frac{2i(2\pi \hbar - m \ell_0) x^2 + \ell_0 \hbar^2 (2h \mu - m \ell_0) \ell_0^2}{4 \sqrt{\ell_0 (h \ell_0 + 2m \ell_0 \sigma_0^2)} \right)}{4 \sigma_0 \hbar \ell_0 (h \ell_0 + 2m \ell_0 \sigma_0^2)}, \]

and Erf is the error function: \( \text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \). In the second case we take the initial wave function as

\[ \psi_0(x) = \sqrt{\frac{2}{\ell_1}} \sin \left[ \frac{\pi}{\ell_1} (x - x_1) \right] e^{i(k(x-x_1) + \Theta(x-\ell_1)(x_2-x))}. \]

In this case the relation of \( \psi(x, t) \) is cumbersome; there are eight modified error functions, \( \text{Erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} \, dt \) in its summand.

In the original Bohm approach, the Schrödinger equation is decomposed into two real equations by expressing the wavefunction in polar form \( \psi = R e^{iS/\hbar} \). Then, the vector field \( \mathbf{v} = \mathbf{p}/m \) is constructed from the vector field \( \mathbf{p} = \nabla S \) and assuming that \( \mathbf{v} \) defines at each space–time point the tangent to a possible particle trajectory passing through that point. In this interpretation of quantum mechanics, one obtains

\[ \frac{d\mathbf{p}}{dt} = -\nabla (V + Q), \]

where \( Q = -(\hbar^2/2m) \nabla^2 R/R \) is known as quantum potential. Analogous to classical physics, in Bohm’s model of quantum theory one has

\[ \frac{d\langle \mathbf{p} \rangle}{dt} = \langle \frac{d\mathbf{p}}{dt} \rangle, \]

where the mean value is defined for an ensemble of density \( R^2 \) and momentum \( \mathbf{p} = \nabla S \). In the standard approach to quantum mechanics the right-hand side of equation (8) is meaningless [13, pp 111–13]. Using equation (8) and taking the expectation value of equation (7), one obtains

\[ \frac{d\langle \mathbf{p} \rangle}{dt} = -(\nabla (V + Q)). \]

For the case of a particle within a box with one wall moving, using the integration by part one can find that

\[ -\frac{2m}{\hbar^2} \langle \nabla \psi \rangle = \int_0^{\ell(t)} \frac{d\langle \mathbf{p} \rangle}{dt} = -\frac{1}{\hbar^2} \int_0^{\ell(t)} \hbar^2 \frac{\partial}{\partial x} \left( \frac{\partial^2 \mathbf{R}}{\partial x^2} \right) dx - \frac{\hbar}{\hbar^2} \int_0^{\ell(t)} \mathbf{R} \frac{\partial}{\partial x} \left( \frac{\partial \mathbf{R}}{\partial x} \right) dx \]

\[ = -\left( \frac{\partial \mathbf{R}}{\partial x} \right)^2 \bigg|_0^{\ell(t)}, \]

where we have used the fact that the wavefunction is zero on both walls. The general case of boundary condition will be considered in the appendix. One obtains,

\[ \frac{\partial \psi}{\partial x} = \frac{\partial \mathbf{R}}{\partial x} + \frac{i}{\hbar} \mathbf{R} \frac{\partial}{\partial x} = \frac{\partial \mathbf{R}}{\partial x}, \]

where the second equality holds at the boundaries only. As a consequence, we have

\[ \frac{d\langle \mathbf{p} \rangle}{dt} = -\frac{\hbar^2}{2m} \left( \left| \frac{\partial \psi}{\partial x} (x = \ell(t), t) \right|^2 - \left| \frac{\partial \psi}{\partial x} (x = 0, t) \right|^2 \right) = \langle \mathbf{p} \rangle_{\text{qm}}(t), \]

where \( \langle \mathbf{p} \rangle_{\text{qm}}(t) \) was called the quantum effective force by Dodonov and Andreata [10]. It must be mentioned that in the context of the standard approach to quantum mechanics, one can obtain equation (10) by the simultaneous application of the Schrödinger equation

\[ \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = i\hbar \frac{\partial \psi}{\partial t}, \]

and the time-derivative of the expectation value of the momentum operator

\[ \frac{d\langle \mathbf{p} \rangle}{dt} = \frac{d}{dt} \int_0^{\ell(t)} \psi^*(x, t) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi(x, t), \]

as was first done by Dodonov and Andreata [10] for the case of an impenetrable wall at \( x = 0 \).

Now, taking the integral of both sides of equation (10) leads to

\[ \langle \mathbf{p} \rangle(t) = \langle \mathbf{p} \rangle(0) + \int_0^t \langle \mathbf{p} \rangle_{\text{qm}}(t) \, dt, \]

where in the case of TGP, \( \langle \mathbf{p} \rangle(0) = \hbar \text{Erfi} \left( \frac{x}{2\sqrt{2m}} \right) \), whereas for the case of TBS, \( \langle \mathbf{p} \rangle(0) = \hbar k \).

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Figure 2. A selection of Bohm trajectories, \( x(t) \), for \( u = 100 \pi \) and an initially motionless wavefunction: (a) TBS, the static case; (b) TBS, the dynamic case; (c) TGP, the static case; and (d) TGP, the dynamic case. In each figure, the black curve starts at \( x_0 = x_c - 2\sigma_0 \), the red one at \( x_0 = x_c \) and the green one at \( x_0 = x_c + 2\sigma_0 \).

3. Numerical calculations

In this section, we work in a unit system where \( \hbar = 1 \) and \( m = 0.5 \). The other parameters are chosen as \( \ell_1 = \ell_0/20 \), \( \sigma_0 = \ell_1/10 \) and \( \ell_0 = 1 \). Conservation of the probability, given by 
\[
\int_0^\ell \psi^* \psi \, dx = 1,
\]
can be used as a parameter that gives us a test on the precision of the results (it must be noted that in the case of TGP because of truncation the total probability is not equal to unity; instead it is equal to \( \text{Erf}\left[5/\sqrt{2}\right] = 0.9999994267 \)). We use Simpson’s rule for taking the integral of equation (2). Using the conservation of the probability as the control variable, we got a good numerical stability by just taking the first 400 terms of the infinite sum appearing in the relation of \( \psi(x, t) \) in both the TGP and TBS cases. Using the Runge–Kutta method for solving the guidance differential equation, a selection of Bohmian trajectories is presented.

Figure 1 shows the initial wave function for both TGP and TBS and the walls of the well to give a feeling about how narrow the initial wavefunction is.

In figure 2(a), selection of Bohm paths is presented for an initially motionless wavefunction, i.e. \( k = 0 \) in equations (3) and (6). From parts (a) and (c), one can see that a Bohm particle in the static case remains at rest in the center of the box. We have checked the long-time behavior of Bohm trajectories that starts at the tail of the leading half in the dynamic situation and saw that for TGP, the particle eventually moves on a path approximately parallel (approximately, because it has very small oscillation around the parallel path) to the path of the moving wall, i.e. with the velocity of the wall, but for TBS it moves with a velocity less than the velocity of the moving wall.

Figure 3 show the expectation value of the position operator for an initially motionless wavefunction. From this figure and figure 2 one finds that in the static case, i.e. the fixed wall, the Bohm path, which was initially placed at the center of the motionless packet, \( x_0 = \langle x \rangle(0) \) (in the case of TBS, \( x_0 = 0.5 \), whereas \( x_0 = (1/2)\text{Erf}[5/\sqrt{2}] = 0.4999997 \) in the case of a truncated Gaussian), moves with the center point subsequently, \( x(t) = \langle x \rangle(t) \), at least up to time 0.003 that we have considered. But this is not true in the dynamic case. The reason is the quantum effective force which has been shown in figure 4: in the static case \( f_{qm}(t) \) is zero for both the tiny-box and truncated Gaussian states. Deviation of \( \langle x \rangle(t) \) from its static value \( \langle x \rangle(0) \) takes place sooner in the TBS. This shows that the particle begins to feel the motion of the wall sooner for the case of TBS compared to the case of TGP. In figures 5 and 6, we have plotted the time dependence of the quantum effective force for a fixed value of the speed of the moving wall, \( u = 100 \pi \), but different values of kick momentum \( k \).

Figure 4. Quantum effective force versus time, \( f_{qm}(t) \), for an initially motionless wavefunction: (a) TBS and (b) TGP. In each figure, the black curve shows \( f_{qm}(t) \) for the dynamic case and the red one shows \( f_{qm}(t) \) for the static case.

Figure 6 reveals that in the case of TGP, the quantum effective force is the same for both the static and dynamic cases for \( k < 0 \) and is zero for \( k > 0 \) in the dynamic case, at least for our parameters and time domain \( t \in [0, 0.0005] \).
The quantum effective force $f_{qm}$ deviates from zero in the positive direction for $k < 0$ but in the negative direction for $k > 0$, i.e., the particle accelerates for $k < 0$ but decelerates for $k > 0$.

Noting figure 7, which displays $f_{qm}(t)$ for the motionless TBS and TGP but for different wall speeds, one finds: (i) in the presented region of time, the quantum effective force for TGP is negligible compared to the TBS (one must note that in the longer time limit the opposite behavior takes place according to figure 4), (ii) the direction of deviation from zero changes with $u$ and (iii) the deviation time increases with $u$.

4. Summary and discussion

In this study, we obtained the solution of TDSE for a particle in (i) TBS and (ii) TGP of an infinite square well with one wall in uniform motion. We showed that due to a quantum effective force, which apart from a minus sign is the expectation value of the gradient of the quantum potential in the context of Bohmian mechanics, the expectation value of the momentum operator changes gradually with time. We studied the variation of this quantum effective force with time for different values of the speed of the moving wall in the case of a motionless packet and different values of the kick momentum but fixed value of the speed of the moving wall. Some Bohm trajectories for the motionless packet were also plotted. We have learned from the numerical calculations that the particle in TBS begins to feel the motion of the wall sooner in comparison to TGP. This may be understood by computing the speed of propagation [13] for both TBS and TGP. Other ramifications of this study, such as contracting box, the dependence of quantum effective force on related parameters such as the mass of the confined particle, width of the initial packet, other initial packets such as excited particle-in-a-box eigenstates and other types of boundary conditions, such as periodic ones, call for further consideration.

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Appendix. The general form of the quantum effective force for a particle in a box

A.1. Bohmian mechanics

In the context of Bohmian mechanics, the actual momentum of the particle is given by $p = \hbar S/\partial x$, where $S/h$ is the phase of the wave function. Thus, for the time-derivative of the expectation value of the actual momentum of the confined particle inside the one-dimensional box, one has [13]

$$f_{qm}(t) \equiv \frac{\partial \langle p \rangle}{\partial t} = \frac{\partial}{\partial t} \int_0^{(t)} dx \, x^2 \frac{\partial S}{\partial x} = \dot{\ell}(t) \left( R^2 \frac{\partial S}{\partial x} \right)_{x = \ell(t)} + \int_0^{(t)} dx \, \frac{\partial R^2}{\partial t} \frac{\partial S}{\partial x} + \int_0^{(t)} dx \, R^2 \frac{\partial}{\partial t} \frac{\partial S}{\partial x}. \quad \text{(A.1)}$$
where we have used Leibnitz’s formula and \( \dot{\psi}(t) = d\psi(t)/dt \).

Now, using the continuity equation

\[
\frac{\partial R^2}{\partial t} + \frac{\partial}{\partial x} \left( R^2 \frac{1}{m} \frac{\partial S}{\partial x} \right) = 0 \tag{A.2}
\]

and the generalized Hamilton–Jacobi equation

\[
- \frac{\partial S}{\partial t} = \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + Q, \tag{A.3}
\]

which can be obtained by putting the polar form \( \psi(x, t) = R(x, t) e^{iS(x,t)/\hbar} \) in the Schrödinger equation, one obtains

\[
f_{qn}(t) = \dot{\psi}(t) \left( R^2 \frac{\partial S}{\partial x} \right) \bigg|_{x(t)} - \int_0^{\ell(t)} dx \frac{\partial}{\partial x} \left( R^2 \frac{1}{m} \frac{\partial S}{\partial x} \right) \
- \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + V + Q \bigg). \tag{A.4}
\]

Integrating by part leads to

\[
\int_0^{\ell(t)} dx \frac{\partial}{\partial x} \left( R^2 \frac{1}{m} \frac{\partial S}{\partial x} \right) \bigg|_{x(t)} = \frac{1}{m} \int_0^{\ell(t)} dx \frac{\partial}{\partial x} \left( R^2 \left( \frac{\partial S}{\partial x} \right)^2 \right) \bigg|_{x(t)} \n- \frac{1}{m} \int_0^{\ell(t)} dx \frac{\partial^2}{\partial x^2} R \frac{\partial S}{\partial x} \frac{\partial S}{\partial x} \bigg|_{x(t)} \n= \frac{1}{m} \left( \frac{\partial S}{\partial x} \right)^2 \bigg|_{x(t)} - \frac{1}{2m} \int_0^{\ell(t)} dx \frac{\partial^2}{\partial x^2} R \frac{\partial S}{\partial x} \bigg|_{x(t)}^2. \tag{A.5}
\]

Thus, using the fact that inside the box the classical potential is zero, one has

\[
f_{qn}(t) = \dot{\psi}(t) \left( R^2 \frac{\partial S}{\partial x} \right) \bigg|_{x(t)} - \frac{1}{m} \left( \frac{\partial S}{\partial x} \right)^2 \bigg|_{x(t)} \n+ \int_0^{\ell(t)} dx \frac{\partial}{\partial x} \left( - \frac{\partial Q}{\partial x} \right). \tag{A.5}
\]

Now, using the definition of quantum potential \( Q = -(\hbar^2/2m)\nabla^2 R / R \),

\[
\int_0^{\ell(t)} dx \left( \frac{\partial^2}{\partial x^2} R \right) \bigg|_{x(t)} = -R^2 Q \bigg|_{x=\ell(t)} + \int_0^{\ell(t)} dx Q \left( \frac{\partial^2}{\partial x^2} R \right) \bigg|_{x=\ell(t)} \n= \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} R \right) \bigg|_{x=\ell(t)} - \frac{\hbar^2}{2m} \int_0^{\ell(t)} dx \left( \frac{\partial^2}{\partial x^2} R \right) \bigg|_{x=\ell(t)} \n\bigg( \frac{\partial^2}{\partial x^2} \bigg) \bigg|_{x=\ell(t)} \tag{A.5}
\]

The last integral can be evaluated as follows:

\[
\int_0^{\ell(t)} dx \left( \frac{\partial^2}{\partial x^2} R \right) \bigg|_{x=\ell(t)} = \int_0^{\ell(t)} dx \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial x} \right) \bigg|_{x=\ell(t)} = \left( \frac{\partial}{\partial x} \right) \bigg|_{x=\ell(t)} \tag{A.5}
\]

Finally, one obtains

\[
f_{qn}(t) = \dot{\psi}(t) \left( R^2 \frac{\partial S}{\partial x} \right) \bigg|_{x=\ell(t)} + \frac{\hbar^2}{2m} \left( \frac{\partial^2 R}{\partial x^2} - \left( \frac{\partial R}{\partial x} \right)^2 \right) \bigg|_{x=\ell(t)} \n- \frac{1}{m} \left( \frac{\partial S}{\partial x} \right)^2 \bigg|_{x=0} \tag{A.6}
\]

for the quantum effective force. Now, one can use the general form (A.6) for other types of boundary conditions such as periodic ones, \( \psi(0, t) = \psi(\ell(t), t) \) and \( \psi'(0, t) = \psi'(\ell(t), t) \), for which the momentum operator does have eigen-states obeying this periodic boundary condition.

A.2. Standard quantum mechanics

The simultaneous application of equations (11) and (12) with Leibnitz’s formula leads to

\[
f_{qn}(t) = \frac{d(p)}{dt} = \frac{\hbar}{i} \dot{\psi}(t) \frac{\partial \psi}{\partial x} \bigg|_{x=\ell(t)} \n+ \frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} - \left( \frac{\partial \psi}{\partial x} \right)^2 \right) \bigg|_{x=\ell(t)} \tag{A.7}
\]

where \( \Re(z) \) and \( \Im(z) \) show the real and imaginary parts of \( z \), respectively. Equation (A.8) is converted to equation (A.6) by using the polar form of the wave function. It is easily seen for ‘vanishing at the walls’ that boundary condition equation (A.8) yields the simple form (10).

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