LOWER BOUNDS FOR EIGENVALUES OF FINSLER MANIFOLDS

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ABSTRACT. In this paper, we study the spectrums induced by various dimension-like functions on a closed Finsler manifold and obtain a Gromov type and a Buser type lower bounds for general eigenvalues. In particular, for the Lusternik-Schnirelmann spectrum and Krasnoselskii spectrum, we not only obtain a better lower bound, but also estimate the multiplicity of an eigenvalue.

1. Introduction

Let \((M, F, dm)\) is a closed Finsler \(n\)-manifold and let \(H^{1,2}(M)\) be the standard Sobolev space. According to [13], the canonical energy functional (i.e., Rayleigh quotient) on \(H^{1,2}(M)\) is defined by

\[
E(u) := \frac{\int_M F^2(du)dm}{\int_M u^2dm}.
\]

Since \(E\) is positively 1-homogenous, it is convenient to consider the restriction of \(E\) on the "unit sphere"

\[
S := \left\{ u \in H^{1,2}(M) : \int_M u^2dm = 1 \right\}.
\]

In [18], the spectrum of \((M, F, dm)\) can be defined by a dimension-like function. More precisely, let \(\mathcal{F}\) be a certain collection of subsets of \(S\) and let \(\dim : \mathcal{F} \to \mathbb{N} \cup \{+\infty\}\) be a dimension-like function, i.e., \(\dim\) satisfies the following two conditions:

1. For any \(A \in \mathcal{F}\), \(\dim(A) \geq 0\) with equality if and only if \(A = \emptyset\);
2. \(\dim(A_1) \leq \dim(A_2)\), for any \(A_1, A_2 \in \mathcal{F}\) with \(A_1 \subset A_2\).

According to [11], the spectrum \(\{\lambda_k\}_{k=1}^{\infty}\) of \(E\) induced by \((\mathcal{F}, \dim)\) is defined as follows:

\[
\lambda_k := \sup \left\{ \lambda \in \mathbb{R}^+ \cup \{+\infty\} : \dim E^{-1}[0, \lambda] < k \right\}, \quad \forall k \in \mathbb{N}^+,
\]

where \(\dim E^{-1}[0, \lambda] := \sup(\dim(A) : A \in \mathcal{F}, A \subset E^{-1}[0, \lambda])\). It is remarkable that this definition is equivalent to the Min-Max theory (cf. [18] Proposition 5.2).

Usually various dimension-like functions can induce different spectrums. Before giving some interesting examples, we introduce some notations. Denote by \(\mathbb{P}(\mathcal{F}) := S/\mathbb{Z}_2\) the quotient space, i.e., the projective space, and use \(p : S \to \mathbb{P}(\mathcal{F})\) to

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denote the natural projection. Let \( \text{cat}_{\mathcal{P}(\mathcal{X})}(\cdot) \) and \( \text{ess}(\cdot) \) be the relatively Lusternik-Schnirelmann category (cf. [8, 19]) and the essential dimension (cf. [11]) on \( \mathcal{P}(\mathcal{X}) \), respectively. And \( \gamma(\cdot) \) denotes the Krasnoselskii genus (cf. [14, 19]).

Example 1. Set \( \mathcal{F}^{LS} := \{ A \subset \mathcal{S} : A \text{ is closed} \} \). Define the Lusternik-Schnirelmann dimension of \( A \) by

\[
\dim_{LS}(A) := \text{cat}_{\mathcal{P}(\mathcal{X})}(p(A)).
\]

The spectrum induced by \( (\mathcal{F}^{LS}, \dim_{LS}) \) is called the Lusternik-Schnirelmann spectrum, denoted by \( \{ \lambda_{k}^{LS} \}_{k=1}^{\infty} \).

Example 2. Set \( \mathcal{F}^{ES} := \{ A \subset \mathcal{S} : A \text{ is closed} \} \). Define the essential dimension of \( A \) by

\[
\dim_{ES}(A) := \text{ess}(p(A)) + 1.
\]

The spectrum induced by \( (\mathcal{F}^{ES}, \dim_{ES}) \) is called the essential spectrum, denoted by \( \{ \lambda_{k}^{ES} \}_{k=1}^{\infty} \).

Example 3. Set \( \mathcal{F}^{K} := \{ A \subset \mathcal{S} : A \text{ is closed and } A = -A \} \). For each \( A \in \mathcal{F}^{K} \), set

\[
\dim_{K}(A) := \gamma(A).
\]

The spectrum induced by \( (\mathcal{F}^{K}, \dim_{K}) \) is called the Krasnoselskii spectrum, denoted by \( \{ \lambda_{k}^{K} \}_{k=1}^{\infty} \).

According to [18, Theorem 5.15, Theorem 5.20, Theorem 5.30], these three spectrums share the following properties if \( F \) is reversible:

1. **Monotonicity.**
   \[
   0 = \lambda_{1} < \lambda_{2} \leq \ldots \leq \lambda_{k} \leq \ldots \nearrow +\infty.
   \]
   In particular, the multiplicity of each eigenvalue \( \lambda_{k} \) is always finite.

2. **First positive eigenvalue.**
   \[
   \lambda_{2} = \inf_{u \in \{ f \in H^{1,2}(M) : \int_{M} f \, dm = 0 \}} E(u).
   \]

3. **Existence of eigenfunction.**
   For each \( k \in \mathbb{N}^{+} \), there exists \( u = (m(M))^{-\frac{1}{2}} \) or \( u \in \{ f \in H^{1,2}(M) : \int_{M} f \, dm = 0 \} \) with
   \[
   \int_{M} (\Delta u + \lambda_{k} u) v \, dm = 0, \forall v \in H^{1,2}(M).
   \]

4. **Upper bound of eigenvalue.**
   If for some \( N \in [n, +\infty) \), the \( N \)-Ricci curvature satisfies \( \text{Ric}_{N} \geq (N - 1)K \), then
   \[
   \lambda_{k+1} \leq \frac{(N - 1)^{2}}{4} |K| + C(N) \left( \frac{k}{d} \right)^{2}, \forall k \in \mathbb{N},
   \]
   where \( C(N) \) is a constant only dependent on \( N \) and \( d := \text{diam}(M) \).

5. **Riemannian case.**
   The spectrum \( \{ \lambda_{k} \}_{k=1}^{\infty} \) is the standard spectrum when \( F \) is Riemannian.

In this paper, we continuously study the spectrums of a closed Finsler manifold. It is remarkable that Property (iv) (i.e., the Cheng type upper estimate) be can extended for nonreversible manifold and moreover, it remains valid for more general
dimension-like functions (cf. [13, Theorem 6.5, Remark 6.6]). Thus, a natural question is whether there is a uniform lower bound for the spectrum.

The purpose of this paper is to answer this question. Let us consider the spectrum \( \{ \lambda_k \}_{k=1}^{\infty} \) induced by \( (\mathcal{F}, \dim) \) with following Condition (I):

\[
(I) \begin{cases} 
(1) \text{All } \mathcal{F}_k \text{'s, } \forall k \in \mathbb{N}^+ \text{ are independent of the Finsler metric } F. \\
(2) \{ \lambda_k \}_{k=1}^{\infty} \text{ is the standard spectrum when } F \text{ is Riemannian.}
\end{cases}
\]

Obviously, these two conditions are quite natural. For instance, \( \{ \lambda_k^{LS} \} \), \( \{ \lambda_k^{ES} \} \) and \( \{ \lambda_k^K \} \) all satisfy Condition (I). Particularly, (1.1) always holds for a spectrum with Condition (I). Hence, \( \lambda_{k+1} \) is exactly the \( k \)-th positive eigenvalue and \( \lambda_2 \) is the first positive eigenvalue. On the other hand, another type of the first positive eigenvalue is defined in [13] by

\[
\overline{\lambda}_1 := \inf_{u \in \{ f \in H^1(M) : \int_M f \, \text{d}m = 0 \}} E(u).
\]

Some sharp lower bounds for \( \overline{\lambda}_1 \) have been obtained recently (cf. [20, 21, 22], etc.). However, it is usually hard to investigate whether \( \lambda_2 = \overline{\lambda}_1 \) unless the additional conditions are assumed. Nevertheless, \( \lambda_2 \geq \overline{\lambda}_1 \) holds if the eigenfunction corresponding to \( \lambda_2 \) exists, in which case the lower estimates on \( \overline{\lambda}_1 \) also work for \( \lambda_2 \).

More generally, we have the following results.

**Theorem 1.1.** Let \( (M, F, \text{d}m) \) be a closed Finsler \( n \)-manifold equipped with the Busemann-Hausdorff measure or the Holmes-Thompson measure. Suppose for some \( K \leq 0 \),

\[
\text{Ric} \geq (n - 1)K, \text{ diam}(M) = d.
\]

Thus, there is a positive constant \( C_1 = C_1(n) \) such that for any spectrum \( \{ \lambda_k \}_{k=1}^{\infty} \) satisfying Condition (I),

\[
\lambda_{k+1} \geq \frac{C_1}{\Lambda_F^{3n+6}} \cdot d^2 \sqrt{|K|}, \forall k \in \mathbb{N},
\]

where \( \Lambda_F \) is the uniformity constant of \( (M, F) \).

**Theorem 1.2.** Let \( (M, F, \text{d}m) \) be a closed Finsler \( n \)-manifold equipped with the Busemann-Hausdorff measure or the Holmes-Thompson measure. Thus, there are two positive constants \( C_2 = C_2(n) \) and \( C_3 = C_3(n) \) such that for any spectrum \( \{ \lambda_k \}_{k=1}^{\infty} \) satisfying Condition (I),

\[
\lambda_{k+1} \geq \frac{C_2}{\Lambda_F^{23n+16}} \left( \frac{k}{V} \right)^{\frac{n}{V}}, \forall k \geq C_3 \cdot \Lambda_F^{5n+5} \left( \frac{V}{V} \right),
\]

where \( V = \text{m}(M) \) and \( i_M \) is the injectivity radius of \( (M, F) \).

Note that \( F \) is Riemannian if and only if \( \Lambda_F = 1 \) and hence, Theorem 1.1 (resp., Theorem 1.2) is exactly Gromov’s estimate [12] (resp., Buser’s estimate [4]). And the following example shows that Condition (I) is necessary for both Theorem 1.1 and Theorem 1.2.

**Example 4.** Let \( \mathcal{C} := \{ A \subset S : A \text{ is closed} \} \) and set

\[
\dim_{MC}(A) := \dim_{\mathcal{C}}(A) + 1,
\]
where $\dim C$ is the Lebesgue covering dimension. Thus, the spectrum $\{\lambda_{k}^{MC}\}$ induced by $(\mathcal{F}, \dim C)$ does not satisfy Condition (I). Moreover, Proposition 5.35 and the proof of Theorem 5.6 yields $\lambda_{k}^{MC} = 0$ for all $k \in \mathbb{N}^+$. On the other hand, although Theorem 1.1 and Theorem 1.2 are valid for $\{\lambda_{k}^{F}\}_{k=1}^{\infty}$ and $\{\lambda_{k}^{LS}\}_{k=1}^{\infty}$, we have a better estimate which holds for all measures.

Theorem 1.3. Let $(M, F, dm)$ be a closed reversible Finsler $n$-manifold. Suppose for some $K \leq 0$,

$$\text{Ric}_{N} \geq (N - 1)K, \quad \text{diam}(M) = d,$$

where $N \in [n, +\infty)$. Then there exists a positive constant $C_1 = C_1(N)$ such that

$$\lambda_{k+1}^{K} = \lambda_{k+1}^{LS} \geq \frac{C_1}{d^2} k^\frac{2}{N}, \quad \forall \; k \in \mathbb{N}.$$

In particular, there is a positive constant $C_2 = C_2(N)$ such that

$$m_{K}(k) = m_{LS}(k) \leq C_2 \left(1 + d\sqrt{K}\right) \left(d\sqrt{K}\right)^N + k^N, \quad \forall \; k \in \mathbb{N}^+,$$

where $m_{K}(k)$ (resp., $m_{LS}(k)$) is the multiplicity of the $k$-th eigenvalue with respect to the spectrum $\{\lambda_{k}^{K}\}_{k=1}^{\infty}$ (resp., $\{\lambda_{k}^{LS}\}_{k=1}^{\infty}$).

2. Preliminaries

In this section, we recall some definitions and properties about Finsler manifolds. See Lichnerowicz, etc., for more details.

A Finsler $n$-manifold $(M, F)$ is an $n$-dimensional differential manifold $M$ equipped with a Finsler metric $F$ which is a nonnegative function on $TM$ satisfying the following two conditions:

1. $F$ is positively homogeneous, i.e., $F(\lambda y) = \lambda F(y)$, for any $\lambda > 0$ and $y \in TM$;
2. $F$ is smooth on $TM \setminus \{0\}$ and the Hessian $\frac{1}{2}[F^2]_{y,y}$ is positive definite, where $F(x, y) := F(y^i\pi^i_x|_x)$.

Let $\pi : PM \to M$ and $\pi^*TM$ be the projective sphere bundle and the pullback bundle, respectively. Then a Finsler metric $F$ induces a natural Riemannian metric $g = g_{ij}(x, [y]) dx^i \otimes dx^j$, which is the so-called fundamental tensor, on $\pi^*TM$, where

$$g_{ij}(x, [y]) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad dx^i = \pi^* dx^i.$$

The Euler theorem yields that $F^2(x, y) = g_{ij}(x, [y]) y^i y^j$, for $(x, y) \in TM \setminus \{0\}$. It also should be remarked that $g_{ij}$ can be viewed as a local function on $TM \setminus \{0\}$, but it cannot be defined on $y = 0$ unless $F$ is Riemannian.

Note that $F$ may be irreversible, that is, $F(x, y) \neq F(x, -y)$. Hence, Rademacher and Egloff introduced the reversibility $\lambda_F$ and the uniformity constant $\Lambda_F$ to describe its asymmetry. More precisely, set

$$\lambda_F := \sup_{y \in SM} F(-y), \quad \Lambda_F := \sup_{X, Y, Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)},$$

where $S_x M := \{y \in T_x M : F(x, y) = 1\}$ and $SM := \cup_{x \in \mathbb{O}} S_x M$. Clearly, $\Lambda_F \geq \lambda_F^2 \geq 1$. It is easy to see that $\lambda_F = 1$ iff $F$ is reversible, while $\Lambda_F = 1$ iff $F$ is Riemannian.
On the other hand, $F$ also induces the average Riemannian metric $\hat{g}$ on $M$, which is defined by

$$\hat{g}(X,Y) := \frac{1}{\nu(S_x M)} \int_{S_x M} g_y(X,Y) d\nu_x(y), \ \forall \ X, Y \in T_x M,$$

where $\nu(S_x M) = \int_{S_x M} d\nu_x(y)$, and $d\nu_x$ is the Riemannian volume form of $S_x M$ induced by $F$. It is noticeable that

$$\Lambda_F^{-1} \cdot F^2(X) \leq \hat{g}(X,X) \leq \Lambda_F \cdot F^2(X),$$

with equality iff $F$ is Riemannian.

The dual Finsler metric $F^*$ on $M$ is defined by

$$F^*(\eta) := \sup_{X \in T_x M \setminus \{0\}} \frac{\eta(X)}{F(X)}, \ \forall \eta \in T^*_x M,$$

which is also a Finsler metric on $T^* M$. The Legendre transformation $\mathfrak{L} : TM \to T^* M$ is defined by

$$\mathfrak{L}(X) := \begin{cases} g_X(X, \cdot) & X \neq 0, \\ 0 & X = 0. \end{cases}$$

In particular, $F^*(\mathfrak{L}(X)) = F(X)$. Now let $f : M \to \mathbb{R}$ be a smooth function on $M$. The gradient of $f$ is defined as $\nabla f = \mathfrak{L}^{-1}(df)$. Thus, $df(x) = g_{\nabla f}(\nabla f, x)$.

The geodesic coefficient is defined by

$$G^i(y) := \frac{1}{4} g^{ij}(y) \left\{ 2 \frac{\partial g_{ij}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^i}(y) \right\} y^i y^k.$$

The Riemannian curvature $R_y$ of $F$ is a family of linear transformations on tangent spaces. More precisely, set $R_y := R^i_k(y) \frac{\partial}{\partial x^i} \otimes dx^k$, where

$$R^i_k(y) := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^k \partial y^j} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.$$

The Ricci curvature of $y$ is defined by

$$\text{Ric}(y) := \sum_i K(y, e_i) = \frac{R^i_k(y)}{F^2(y)},$$

where $e_1, \ldots, e_n$ is a $g_y$-orthonormal base on $(x, y) \in TM \setminus \{0\}$.

Let $\gamma : [0,1] \to M$ be a Lipschitz continuity path from $p, q$. The length of $\gamma$ is defined by

$$L_F(\gamma) := \int_0^1 F(\dot{\gamma}(t)) dt.$$

Define the distance function $d_F : M \times M \to [0, +\infty)$ by $d_F(p,q) := \inf L_F(\gamma)$, where the infimum is taken over all Lipschitz continuous paths $\gamma : [a, b] \to M$ with $\gamma(a) = p$ and $\gamma(b) = q$. Generally speaking, $d_F$ satisfies all axioms for a metric except symmetry, i.e., $d_F(p,q) \neq d_F(q,p)$, unless $F$ is reversible.

Given $R > 0$, the forward and backward metric balls $B^+_p(R)$ and $B^-_p(R)$ are defined by

$$B^+_p(R) := \{ x \in M : d_F(p,x) < R \}, \ B^-_p(R) := \{ x \in M : d_F(x,p) < R \}.$$

If $F$ is reversible, forward metric balls coincide with backward ones.

There is only one reasonable notion of the measure for Riemannian manifolds. However, the situation is different in Finsler geometry, because the determinant of the fundamental tensor depends on the direction of $y$. Thus, measures on a Finsler
manifold can be defined in various ways and essentially different results may be obtained.

Let \( dm \) be a smooth measure on \( M \). In a local coordinate system \((x^i)\), express \( dm = \sigma(x)dx^1 \wedge \cdots \wedge dx^n \). There are two measures used frequently in Finsler geometry, which are the so-called Busemann-Hausdorff measure \( dm_{BH} \) and Holmes-Thompson measure \( dm_{HT} \). They are defined by

\[
dm_{BH} := \frac{\text{vol}(B^n)}{\text{vol}(B_x M)} dx^1 \wedge \cdots \wedge dx^n,
\]

\[
dm_{HT} := \left( \frac{1}{\text{vol}(B^n)} \right) \int_{B_x M} |\det g_{ij}(x,y)dy^1 \wedge \cdots \wedge dy^n| dx^1 \wedge \cdots \wedge dx^n,
\]

where \( B_x M := \{ y \in T_x M : F(x,y) < 1 \} \). Each of them becomes the canonical Riemannian measure if \( F \) is Riemannian.

Define the distortion of \((M,F,\text{d}m)\) as

\[
\tau(y) := \log \frac{\text{det} g_{ij}(x,y)}{\sigma(x)}, \quad \text{for } y \in T_x M \setminus \{0\}.
\]

And the \( S\)-curvature \( S \) is defined by

\[
S(y) := \left. \frac{d}{dt} \right|_{t=0} \tau(\gamma_y(t)),
\]

where \( \gamma_y(t) \) is the geodesic with \( \dot{\gamma}_y(0) = y \). It is easy to see both the distortion and the \( S\)-curvature vanish in the Riemannian case. Given \( N \in (-\infty, 0) \cup (n, +\infty) \), the \textit{weighted Ricci curvature} in \([15]\) is defined by

\[
\text{Ric}_N(y) := \text{Ric}(y) + \frac{d}{dt} \bigg|_{t=0} S(\gamma_y(t)) - \frac{S^2(y)}{N-n},
\]

\[
\text{Ric}_n(y) := \lim_{N \uparrow n} \text{Ric}_N(y).
\]

Let \( i : \Gamma \rightarrow M \) be a smooth hypersurface embedded in \( M \). For each \( x \in \Gamma \), there exist two 1-forms \( \omega_\pm(x) \in \Lambda^1 \) satisfying \( i^*(\omega_\pm(x)) = 0 \) and \( F^*(\omega_\pm(x)) = 1 \). Then \( n_\pm(x) := \mathcal{L}^{-1} i^*(\omega_\pm(x)) \) are two unit normal vectors on \( \Gamma \). Let \( dA_\pm \) denote the (area) measures induced by \( n_\pm \), i.e., \( dA_\pm = i^*(n_\pm)dm \) (cf. \( [17, \text{p.27-33}] \)). In general, \( n_- \neq -n_+ \) and \( A_+ (\Gamma) \neq A_- (\Gamma) \). See \([2]\) for some interesting examples.

The definitions and basic properties of the spectrums needed in this paper are presented in Section 1. Also refer to \([13, 15]\) for more details.

3. The proofs of Theorem \( \text{(1.1)} \) and Theorem \( \text{(1.2)} \)

In this section, we assume that \((M,F,\text{d}m)\) be a closed Finsler \( n\)-manifold, where \( \text{d}m \) is either the Busemann-Hausdorff measure or the Thompson-Holmes measure. Let \( \{ \lambda_k \}_{k=1}^\infty \) be the spectrum satisfying Condition (I).

**Lemma 3.1.** Let \((M,F,\text{d}m)\) be a closed Finsler \( n\)-manifold. Then we have

\[
\lambda_k \geq \frac{1}{A_F^{\frac{2n}{2n+2}}} \sup_{\{u_i\}_{i=1}^{k-1} \in L^2(M)} \inf_{\{f \in H^1(M) : \langle f, u_i \rangle_{L^2} = 0 \}} E(f),
\]

where supremum over any set of \( k-1 \) functions \( \{u_i\}_{i=1}^{k-1} \in L^2(M) \) and infimum over all function \( f \) which are perpendicular to \( u_i \) with respect to \( (\cdot, \cdot)_{L^2} \).
where $d\text{vol}_{\hat{g}}$ is the Riemannian measure induced by $\hat{g}$. Then a direct calculation together with Condition (I) yields
\[
\frac{\lambda_k^{\Lambda}}{\Lambda_F^{6n+1}} = \inf_{A \in \mathscr{A}} \sup_{u \in A} \frac{\int_M ||u||_F^2 \text{vol}_{\hat{g}}}{\int_M u^2 \text{vol}_{\hat{g}}} \leq \inf_{A \in \mathscr{A}} \sup_{u \in A} E(u) = \lambda_k < +\infty,
\]
where $\{\lambda_k^{\Lambda}\}_{k=1}^{\infty}$ is the standard spectrum of $\hat{g}$. Now we are done by using the max-min principle of $\lambda_k^{\Lambda}$ (cf. [6, p.17]).}

**Remark 3.2.** Let $(M,F,\text{dm})$ be a closed Finsler manifold equipped with any measure and let $\{\lambda_k\}_{k=1}^{\infty}$ be a spectrum with Condition (I). Then a argument similar to the above one yields a constant $C = C(\text{dm}, \lambda_F) \geq 1$ only dependent on $\text{dm}$ and $\lambda_F$ such that $C^{-1} \cdot \lambda_k^{\Lambda} \leq \lambda_k \leq C \cdot \lambda_k^{\Lambda}$, which implies
\[
0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \nearrow +\infty.
\]

The lemma above implies that one can obtain the lower bound of eigenvalue by selecting suitable $k - 1$ functions $\{u_i\}_{i=1}^{k-1} \in L^2(M)$ and considering the energy on $\{f \in H^{1,2}(M) : (f,u_i)_{L^2} = 0\}$. In the following, we use Buser’s method [3] to construct these $u_i$’s.

Let $\{B_{p_i}^+(r)\}_{i=1}^m$ be a maximal family of disjoint forward $r$-balls in $(M,F)$. For convenience, we call $\{p_i\}_{i=1}^m$ a complete $r$-package. Clearly, $\{B_{p_i}^+(1+\lambda_F r)\}_{i=1}^m$ covers $M$ and hence,
\[
(3.1) \quad m = \text{Cap}_M(r) \leq \text{Cov}_M(r) \leq \text{Cap}_M\left(\frac{r}{1+\lambda_F}\right),
\]
where
\[
\text{Cap}_M(r) := \text{maximum number of disjoint forward } r\text{-balls in } M,
\]
\[
\text{Cov}_M(r) := \text{minimum number of forward } r\text{-balls it takes to cover } M.
\]

For each $i$, define the **Dirichlet region**
\[
D_i := \{q \in M : d(p_i, q) \leq d(p_j, q), \text{ for all } j = 1, \ldots, m\}.
\]

Thus, for any $q \in D_i$, we claim $d(p_i, q) < (1+\lambda_F)r$. Otherwise, the covering implies that there exists $p_j \neq p_i$ such that $q \in B_{p_i}^+((1+\lambda_F)r)$ and hence, $d(p_j, q) < d(p_i, q)$, which is a contradiction! Therefore,
\[
(3.2) \quad B_{p_i}^+(r) \subset D_i \subset B_{p_i}^+((1+\lambda_F)r).
\]

**Proposition 3.3.** $\{D_i\}_{i=1}^m$ is a covering of $M$ with
\[
\text{m}(D_i \cap D_j) = 0, \quad \forall i \neq j.
\]

**Proof.** Since $m < \infty$, the definition of Dirichlet region implies that $\{D_i\}_{i=1}^m$ is a covering of $M$. Now we show $\text{m}(D_i \cap D_j) = 0$, if $i \neq j$. First, we set
\[
f_j(x) := d(p_j, x) - d(p_i, x), \quad A := f_j^{-1}(0) \cap (\text{Cut}_{p_i} \cup \text{Cut}_{p_j}), \quad B := f_j^{-1}(0) - A.
\]

Equip $f_j^{-1}(0)$ with the induced topology from $M$. According to [3, Lemma 8.5.4], $A$ is a closed subset of $f_j^{-1}(0)$ with zero measure. Thus, $B$ is an open set of $f_j^{-1}(0)$ and hence, there exists an open subset $N$ of $M$ such that $B = N \cap f_j^{-1}(0)$. In the
following, we show that $B$ is a $(n-1)$-dimensional submanifold of $N$ and hence, $m(B) = 0$.

Note that $f_j|_N$ is smooth. We claim $df_j(q) \neq 0$ for any $q \in B$. Otherwise, $df_j(q) = 0$ implies $\nabla d(p_i, q) = \nabla d(p_j, q)$, which yields $p_i = p_j$ due to $q \notin (\text{Cut}_{p_i} \cup \text{Cut}_{p_j})$ and $d(p_i, q) = d(p_j, q)$. Thus, $df_j|_B \neq 0$ and hence, a standard argument yields $B$ is a $(n-1)$-dimensional submanifold (of $N$).

Since both $A$ and $B$ are zero-measurable, $m(D_i \cap D_j) \leq m(f_j^{-1}(0)) = m(A) + m(B) = 0$. \hfill $\Box$

**Proposition 3.4.** $m(\text{int}(D_i)) = m(D_i)$.

**Proof.** Consider the open subset

$$U_i := \{ q \in M : d(p_i, q) < d(p_j, q), \forall j \neq i \} \subset \text{int}(D_i).$$

and

$$D_i - U_i = \bigcup_{j \neq i} (D_i \cap f_j^{-1}(0)).$$

The proposition above implies that

$$m(D_i - U_i) \leq \sum_{j \neq i} m(f_j^{-1}(0)) = 0 \Rightarrow m(D_i) = m(U_i) \leq m(\text{int}(D_i)) \leq m(D_i).$$

$\hfill \Box$

Let $D$ be a domain and let $p \in M$. $D$ is called *starlike with respect to* $p$ if each minimizing geodesic from $p$ to an arbitrary $q \in D$ is always contained in $D$.

**Proposition 3.5.** $\text{int}(D_i)$ is starlike with respect to $p_i$.

**Proof.** First, for each $j \neq i$, set

$$D_{ij} := \{ q \in M : d(p_i, q) \leq d(p_j, q) \}.$$

Then $D_i = \bigcap_{j \neq i} D_{ij}$ and hence,

$$(3.3) \quad \text{int}(D_i) = \bigcap_{j \neq i} \text{int}(D_{ij}) = \bigcap_{j \neq i} \{ q \in M : d(p_i, q) < d(p_j, q) \}.$$

Given any $q \in \text{int}(D_i)$, let $\gamma(t)$ be the normal minimal geodesic from $p_i$ to $q$. Now consider the function

$$f_j(t) = d(p_j, \gamma(t)) - d(p_i, \gamma(t)) = d(p_j, \gamma(t)) - t =: \rho_j(\gamma(t)) - t.$$

If $q$ is not a cut point of $p$ along $\gamma(t)$, we have

$$\frac{d}{dt}f_j(t) = g_{\nu \rho_j}(\nabla \rho_j, \gamma(t)) - 1 \leq F(\nabla \rho_j)F'(\gamma(t)) - 1 \leq 0, t \in [0, d(p_i, q)].$$

Since $q \in \text{int}(D_{ij})$ for all $j \neq i$ (see (3.3)), we have $f_j(d(p_i, q)) = d(p_j, q) - d(p_i, q) > 0$ and hence, $f_j(t) > 0$, for $t \in [0, d(p_i, q)]$, which implies $\gamma(t) \subset \text{int}(D_{ij})$. Then (3.3) implies $\gamma(t) \subset \text{int}(D_i)$. If $q$ is a cut point, then for any small $\epsilon > 0$, the above proof yields that $f_j(t) \geq f_j(d(p_i, q) - \epsilon), t \in [0, d(p_i, q) - \epsilon)$ and the same statement follows from the continuousness of $f_j$. \hfill $\Box$

**Definition 3.6.** Given an open subset $D \subset M$, its *Cheeger’s constant* $\mathcal{h}(D)$ is defined by

$$\mathcal{h}(D) = \inf_{\Gamma} \min \{ \frac{A_+(D \cap \Gamma)}{\min \{m(D_1), m(D_2)\}} \},$$

where $\Gamma$ varies over compact $(n-1)$-dimensional submanifolds of $M$ which divide $D$ into disjoint open subsets $D_1, D_2$ of $D$ with common boundary $\partial D_1 \cap \partial D_2 = \Gamma$. 
In particular, we have the following Cheeger type inequality. The proof is given in Appendix A.

**Proposition 3.7.** Let $D \subset M$ be an open subset of $M$. Then

$$
\inf_{\{f \in H^{1,2}(M) \setminus \{0\}: \int_D f^2 dm = 0\}} E(u) \geq \frac{\mathcal{h}^2(D)}{4\lambda_F},
$$

where $\lambda_F$ is the reversibility of $(M, F)$.

We also need the following result. Although it has been proven by the second and third authors in [24], we give the proof in Appendix 3.8 for the completeness.

**Lemma 3.8.** Let $D$ be a starlike open set (with respect to $p$) in $M$ with $B_p^+(r) \subset D \subset B_p^+(R)$. If for some $K \leq n \leq 0$, $\text{Ric} \geq (n-1)K$, then

$$
\mathcal{h}(D) \geq 4\max_{0 < \beta < \sqrt{\frac{R}{2 n K}}} \left\{ \Lambda_{n,K}(\beta) \left[ V_n(K) \left( \sqrt{\frac{2}{\Lambda_F}} \right) - V_n,K(\beta) \right] \right\} \geq \frac{C_1 + \sqrt{|K|}}{\Lambda_F} r^{n-1},
$$

where $C_1 = C_1(n) < 1$ is a positive constant only dependent on $n$.

From above, we can show Theorem 1.1.

**The proof of Theorem 1.1.** Suppose $K < 0$. Without loss of generality, we can assume $K = -1$. Given any $\varepsilon > 0$, let $\{p_i\}_{i=1}^m$ be a complete $\varepsilon$-package and let $\{D_i\}_{i=1}^m$ be the Dirichlet regions. Let $\varphi_i$ be the characteristic function of $D_i$. For each $f \in H^{1,2}(M)$ with $(f, \varphi_i)_{L^2} = 0$, $i = 1, \ldots, m$, Proposition 3.3 together with Proposition 3.4 and Proposition 3.7 yields

$$
\int_M F^{*2}(df)dm = \sum_{i=1}^m \int_{D_i} F^{*2}(df)dm = \sum_{i=1}^m \int_{\text{int}(D_i)} F^{*2}(df)dm
$$

$$
\geq \frac{\mathcal{h}^2(D_i)}{4\Lambda_F} \int_{\text{int}(D_i)} f^2 dm \geq \frac{\min\{\mathcal{h}^2(D_i), i = 1, \ldots, m\}}{4\Lambda_F} \sum_{i=1}^m \int_{\text{int}(D_i)} f^2 dm
$$

$$
= \frac{\min\{\mathcal{h}^2(D_i), i = 1, \ldots, m\}}{4\Lambda_F} \int_M f^2 dm,
$$

where $\mathcal{h}(D_i) := \mathcal{h}(\text{int}(D_i))$. Now it follows from Lemma 3.1 that

$$
\lambda_{m+1} \geq \frac{\min\{\mathcal{h}^2(D_i), i = 1, \ldots, m\}}{4\Lambda_F^{2m+3}}.
$$

Now (3.1) implies $m = \text{Cap}_M(\varepsilon) \leq \text{Cov}_M(\varepsilon) =: k$. And (3.2) implies

$$
B_{p_i}^+ (\varepsilon) \subset D_i \subset B_{p_i}^+ \left( 2\sqrt{\frac{\Lambda_F}{\varepsilon}} \right), \text{ if } 0 < \varepsilon \leq \frac{d}{2\sqrt{\Lambda_F}};
$$

$$
B_{p_i}^+ (\varepsilon) \subset D_i \subset B_{p_i}^+ (d), \text{ if } \varepsilon \geq \frac{d}{2\sqrt{\Lambda_F}}.
$$

which together with (3.2) and Lemma 3.8 yields a positive constant $C_2 = C_2(n) < 1$ such that

$$
(3.3) \quad \lambda_{k+1} \geq \lambda_{m+1} \geq \frac{C_2^2 + d}{\Lambda_F^{2m+3} \varepsilon^2}, \forall \varepsilon > 0.
$$
On the other hand, Lemma 3.2 (4) yields that
\[
\left\{ \begin{array}{ll}
\varepsilon \leq 2\Lambda_F^\frac{n}{2} \cdot n^\frac{n}{2} \cdot \left[ \frac{d^n_{\text{vol}}(n-1)}{k} \right] \cdot \frac{1}{k}, & \text{if } 0 < \varepsilon \leq d; \\
k = 1, & \text{if } \varepsilon \geq d,
\end{array} \right.
\]
which together with (3.3) yields
\[
\lambda_{k+1} \geq \frac{\varepsilon_0^{1+d}}{\Lambda_F^{2n+6} \cdot d^2 k^\frac{n}{2}}, \forall k \in \mathbb{N}.
\]
It is easy to see that same argument also works in the case of \( K = 0 \). \( \square \)

In order to show Theorem 1.2, we need the following Croke type inequality. Also see [26, Theorem 4.6] for the one of Berger type.

**Lemma 3.9.** (23, 26). Let \((M, F, dm)\) be a closed Finsler n-manifold. Then for any \( x \in M \) and \( 0 < r < i_M/(1 + \sqrt{\Lambda_F}) \), we have
\[
m(B_x^+(r)) \geq C_3 \cdot \Lambda_F^{-(4n+9/2)} \cdot r^n,
\]
where \( C_3 = \frac{\text{vol}(\mathbb{S}^{n-1})}{(\text{vol}(\mathbb{S}^n)/2)^{n+1-n}}. \)

Now the following result implies Theorem 1.2 directly.

**Theorem 3.10.** Let \((M, F, dm)\) be a closed Finsler n-manifold. If for some \( K \leq 0 \) and \( V > 0 \),
\[
\text{Ric} \geq (n-1)K, \ m(M) = V,
\]
then there exists positive constant \( \mathcal{E} = \mathcal{E}(n) \) such that
\[
\lambda_{k+1} \geq \left\{ \begin{array}{ll}
\frac{\exp \left[ \varepsilon \left( 1 + \sqrt{\Lambda_F} |K| + \frac{9(n+1)}{2} \right) \right]}{\Lambda_F^{2n+11 + \frac{n}{2}}}, & \text{if } k \leq \frac{2n+2\Lambda_F^{2n/2+5} \cdot V}{\mathcal{E} \cdot i_M}, \\
\frac{\exp \left[ \varepsilon \left( 1 + \sqrt{\Lambda_F} |K| \left( \frac{V}{2} \Lambda_F^{4n+9/2} \right)^\frac{n}{2} \right) \right]}{\Lambda_F^{2n+11 + \frac{n}{2}}}, & \text{if } k \geq \frac{2n+2\Lambda_F^{2n/2+5} \cdot V}{\mathcal{E} \cdot i_M}.
\end{array} \right.
\]

**Proof.** Suppose \( K < 0 \). For convenience, we can assume that \( K = -1 \). Thus, for any \( \varepsilon > 0 \), let \( \{p_i\}_{i=1}^m \) be a complete \( \varepsilon \)-package, where \( m = \text{Cap}_M(\varepsilon) \).

**Case 1.** If \( \varepsilon \leq \frac{1}{2\sqrt{\Lambda_F}} \), then Lemma 3.9 yields
\[
(3.4) \quad m(B_{p_i}^+(\varepsilon)) \geq C_3 \cdot \Lambda_F^{-(4n+9/2)} \cdot \varepsilon^n \Rightarrow m = m(\varepsilon) \leq \frac{V}{C_3 \cdot \varepsilon^n \Lambda_F^{4n+9/2}}.
\]

**Case 2.** Suppose that \( \varepsilon > \frac{1}{2\sqrt{\Lambda_F}} \). Now we claim that each \( B_{p_i}^+(\varepsilon) \) contains at least \( \left[ \frac{\varepsilon_{i_M} + \frac{1}{2}}{2\sqrt{\Lambda_F}} \right] \) disjoint forward \( \frac{1}{2\sqrt{\Lambda_F}} \)-balls. In fact, choose any \( x \in \partial B_{p_i}^+(\varepsilon) \), let \( \gamma(t), 0 \leq t \leq 1 \) be the normal minimal geodesic from \( p_i = \gamma(0) \) to \( x = \gamma(1) \). Choose a sequence \( \{q_i := \gamma(t_i)\}_{i=1}^\epsilon \) on \( \gamma \) such that
By the definition, it is easy to see that
\[ s = 1 + \left[ \frac{\varepsilon - \frac{i_M}{2\sqrt{\Lambda_F}}}{1 + \sqrt{\Lambda_F} \frac{i_M}{2\sqrt{\Lambda_F}}} \right] \geq \left[ \frac{\varepsilon}{i_M} + 1 \right], \]
which implies \( K = 0. \)

It is easy to check that the inequality and (3.6). It is easy to check that the argument above still works if \( K = 0. \)
4. The proof of Theorem 1.3

According to [11], we introduce the definition of a counting function.

**Definition 4.1.** Let $\mathcal{F}$ be a certain collection of subsets of $\mathcal{S}$ and let $\dim : \mathcal{F} \to \mathbb{N}$ be a dimension-like function. Given $\lambda > 0$, the counting function corresponding to $(\mathcal{F}, \dim)$ is defined by

$$N(\lambda) := \sup \left\{ \dim(A) : A \in \mathcal{F}, \sup_{u \in A} E(u) < \lambda \right\}.$$  

The following result implies the relationship between the bounds of a counting function and the bounds of the spectrum.

**Proposition 4.2.** Given $\lambda \in (0, \infty)$, the following results hold:

1. If $N(\lambda) \leq k$ for some $k \in \mathbb{N}$, then
   $$\lambda N(\lambda) < \lambda \leq \lambda N(\lambda) + 1 \leq \lambda^{k+1}.$$  

2. Suppose $\lambda_k \geq f(k)$ for any $k \in \mathbb{N}^+$, where $f$ is a strictly increasing nonnegative function. Thus,
   $$N(\lambda) < \lfloor f^{-1}(\lambda) \rfloor + 1,$$

where $[\cdot]$ is the Gauss function.

**Proof.** (1) It suffices to show $\lambda_k < \lambda \leq \lambda_{k+1}$ if $N(\lambda) = k$. Definition 4.1 yields for any $\epsilon \in (0, 1)$, there exists $B \subseteq \mathcal{F}$ such that $\sup_{u \in B} E(u) < \lambda$ and $k = \dim(B)$. Then [18] Proposition 5.2 implies

$$\lambda_k = \inf_{A \subseteq \mathcal{F}_k} \sup_{u \in A} E(u) \leq \sup_{u \in B} E(u) < \lambda.$$  

On the other hand, if $\lambda_{k+1} < \lambda$, then $N(\lambda) \geq k + 1$ which is a contradiction, and hence the first statement follows.

(2) We first claim that $N(f(k)) < k$. In fact, if $N(f(k)) \geq k$, then there exists $A \subseteq \mathcal{F}_k$ such that $\sup_{u \in A} E(u) < f(k)$, which implies $\lambda_k \leq \lambda_{k+1} < \lambda$. This is a contradiction and hence, the claim is true. For any $\lambda > 0$, since $\lambda < f([f^{-1}(\lambda)] + 1)$, the claim implies

$$N(\lambda) \leq N(f([f^{-1}(\lambda)] + 1)) < \lfloor f^{-1}(\lambda) \rfloor + 1.$$  

$\square$

**Proposition 4.3.** For a reversible compact Finsler manifold, we have

$$\begin{cases}
m_K(k) = m_{LS}(k), \\
N_K(\lambda) = N_{LS}(\lambda), \quad \forall \lambda \in (0, \infty), \\
m_K(k) \leq N_K(\lambda_k^K + 0),
\end{cases}$$

where $N_K$ and $N_{LS}$ denote the counting functions corresponding to $(\mathcal{F}_K, \dim_K)$ and $(\mathcal{F}_{LS}, \dim_{LS})$, respectively.

**Proof.** According to [18] Theorem 5.20, $\lambda_k^K = \lambda_{k+1}^{LS}$ for all $k \in \mathbb{N}^+$ and $\dim_K(A) = \dim_{LS}(A \cup -A)$ for any closed set $A \subseteq \mathcal{S}$, which imply $m_K(\lambda_k^K) = m_{LS}(\lambda_{k+1}^{LS})$ and $N_K(\lambda) = N_{LS}(\lambda)$ directly. On the other hand, suppose that the multiplicity of the eigenvalue $\lambda_k$ is $l$, that is, $\lambda_{k-1}^K = \cdots = \lambda_{k-l}^K = \cdots = \lambda_{k+l}^K = \lambda$. According to [18] The proof of Prop. 5.13, there is a compact set $A \subseteq \mathcal{S}$ with $\dim_K(A) \geq l$ and $\sup_{u \in A} E(u) = \lambda$. This implies $m_K(k) = l \leq \dim_K(A) \leq N_K(\lambda + 0)$.  

$\square$
By a same argument as [7] Theorem 2.11 and [15] Theorem 5.2, one can easily show the following result. Also refer to [27] Theorem 5) for the nonreversible case.

**Theorem 4.4.** Let \((M, F, dm)\) be a complete reversible Finsler \(n\)-manifold. If for some \(N \in [n, +\infty)\), \(\text{Ric}_N \geq (N - 1)K \leq 0\). Let \(A_i, i = 1, 2\) be two bounded open subsets and let \(W\) be an open subset such that for each two \(x_i \in A_i\), the normal minimal geodesic \(\gamma_{x_1,x_2}\) from \(x_1\) to \(x_2\) is contained in \(W\). Thus, for any non-negative integrable function \(f\) on \(W\), we have

\[
\int_{A_1 \times A_2} \left( \int_0^{d(x_1,x_2)} f(\gamma_{x_1,x_2}(s)) ds \right) dm_x \leq C(N, K, d) \left[ m(A_1) \text{diam}(A_2) + m(A_2) \text{diam}(A_1) \right] \int_W f dm,
\]

where \(dm_x\) is the product measure induced by \(dm\), \(d := \sup_{x \in A_1, x_2 \in A_2} d(x_1, x_2)\) and

\[
C(N, K, d) = \sup_{0 < \frac{1}{2} r \leq s \leq r} \left( \frac{g_K(r)}{g_K(s)} \right)^{N-1} \leq 2^{N-1} e^{(N-1)\sqrt{Kd}}.
\]

The above theorem yields the following result.

**Lemma 4.5.** Let \((M, F, dm)\) be a closed reversible Finsler manifold with \(\text{Ric}_N \geq (N - 1)K\), where \(N \in [0, +\infty)\) and \(K \leq 0\). Then there exists a constant \(C = C(N)\) such that for any \(R\)-ball \(B_x(R)\), we have

\[
\int_{B_x(R)} |u - u_{x,R}|^2 dm \leq 2^{N+2} R^2 e^{(N-1)\sqrt{K}|R|} \int_{B_x(2R)} F^{*2}(du) dm, \quad \forall u \in H^{1,2}(M),
\]

where \(u_{x,R}\) is the mean value of \(u\) on \(B_x(R)\), i.e.,

\[
u_{x,R} := \frac{1}{m(B_x(R))} \int_{B_x(R)} u dm.
\]

**Proof.** Without loss of generality, we can assume \(u \in C^\infty(M)\). First, a direct calculation yields

\[
|u - u_{x,R}|(z) \leq \frac{1}{m(B_x(R))} \int_{B_x(R)} |u(z) - u(y)| dm(y),
\]

which together with the Hölder inequality implies

\[
|u - u_{x,R}|^2(z) \leq \frac{1}{m(B_x(R))} \int_{B_x(R)} |u(z) - u(y)|^2 dm(y).
\]

Integrating the above inequality on \(B_x(R)\), we obtain

\[
\int_{B_x(R)} |u - u_{x,R}|^2(z) dm(z) \leq \frac{1}{m(B_x(R))} \int_{B_x(R)} dm(z) \int_{B_x(R)} |u(z) - u(y)|^2 dm(y).
\]

(4.1)

According to [13], we have

\[
|u(z) - u(y)| \leq \int_0^{d(z,y)} F^*(du \circ \gamma_{z,y}(s)) ds \leq \left( \int_0^{d(z,y)} F^{*2}(du \circ \gamma_{z,y}(s)) ds \right)^{\frac{1}{2}} (2R)^\frac{1}{2},
\]
where $\gamma_{z,y}$ is a normal minimal geodesic from $z$ to $y$. This inequality together with (4.1) yields
\[
\int_{B_x(R)} |u - u_{x,R}|^2(z) \, dm(z) \leq \frac{2R}{m(B_x(R))} \int_{(B_x(R)) \times (B_x(R))} \left( \int_0^{d(z,y)} F^{*2}(du \circ \gamma_{z,y}(s)) \, ds \right) \, dm_x.
\]

Now set $A_1 = A_2 := B_x(R), W := B_x(2R)$. Then Theorem 4.4 furnishes
\[
\int_{B_x(R) \times B_x(R)} \left( \int_0^{d(z,y)} F^{*2}(du \circ \gamma_{z,y}(s)) \, ds \right) \, dm_x \leq 2^{N+1} R e^{(N-1) \sqrt{|K|} R} m(B_x(R)) \int_{B_x(2R)} F^{*2}(du) \, dm,
\]
which together with (4.2) yields the lemma. □

Inspired by [10], we obtain the following estimate, which together with Proposition 4.2, Proposition 4.3 and (1.2) (i.e., [18 Therem 1.4]) yields Theorem 1.3 directly.

**Theorem 4.6.** Let $(M, F, dm)$ be a closed reversible Finsler $n$-manifold with $\text{diam}(M) = d$. If for some $N \in [n, +\infty)$ and $K \leq 0$, $\text{Ric}_N \geq (N - 1)K$. Then for any $\lambda > 0$, there exists a constant $\mathcal{C} = \mathcal{C}(N)$ such that
\[
N_K(\lambda) \leq \max \left\{ \mathcal{C}^{1+\sqrt{|K|} dN \lambda^2}, 1 \right\}.
\]

**Proof.** Given $\lambda > 0$, for any $u \in E_\lambda := \{u \in \mathcal{S} : E(u) < \lambda\}$, we have
\[
\int_M F^{*2}(du) \, dm < \lambda \int_M u^2 \, dm = \lambda.
\]

For any $r > 0$, let $\{p_i\}_{i=1}^m$ be a complete $r$-package. Thus, $\{B_i := B_{p_i}(2r)\}$ is a covering of $M$. Define a linear map
\[
\Phi_{\lambda,r} : H^{1,2}(M) \to \mathbb{R}^m,
\]
\[
u \mapsto \left( \frac{1}{m(B_1)} \int_{B_1} u \, dm, \ldots, \frac{1}{m(B_m)} \int_{B_m} u \, dm \right).
\]

It should be remarked that $\Phi_{\lambda,r}$ is a continuous and odd map.

Now we claim that if $r > 0$ satisfies
\[
\lambda \leq 2^{N+4} \cdot 12^N \cdot r^2 e^{14(N-1) \sqrt{|K|} r} \left[ 1 \right]^{-1},
\]
then $0 \notin \Phi_{\lambda,r}(E_\lambda)$. Suppose not, that is, there exists $u \in E_\lambda$ with $\Phi_{\lambda,r}(u) = 0$. Thus, for all $1 \leq i \leq m$,
\[
u_{B_i} := \frac{1}{m(B_i)} \int_{B_i} u \, dm = 0 \Rightarrow |u - u_{B_i}|^2 = u^2,
\]
which together with Lemma 4.3 and Lemma B.3 yields

\[
\int_M u^2 \, dm \leq \sum_{i=1}^m \int_{B_i} u^2 \, dm \\
\leq 2^{N+2} \cdot (2r)^2 e^{2(N-1)\sqrt{|K|}} \sum_{i=1}^m \int_{B_{\lambda,r}(4r)} F^{*2}(du) \, dm \\
\leq 2^{N+4} \cdot 12^N \cdot r^2 e^{14(N-1)\sqrt{|K|}} \int_M F^{*2}(du) \, dm \\
< 2^{N+4} \cdot 12^N \cdot r^2 e^{14(N-1)\sqrt{|K|}} = \lambda \int_M u^2 \, dm.
\]

Thus, we have

\[
\lambda > \left[ 2^{N+4} \cdot 12^N \cdot r^2 e^{14(N-1)\sqrt{|K|}} \right]^{-1},
\]

which contradicts (4.3). Since the claim is true, for sufficiently small \( r > 0 \), (4.3) holds and hence, \( \Phi_{\lambda,r} : E_\lambda \rightarrow \mathbb{R}^m \setminus \{0\} \) is continuous and odd.

Thus, for any \( A \subset E_\lambda \cap \mathcal{F}_K \), \( \Phi_{\lambda,r} \mid A : A \rightarrow \mathbb{R}^m \setminus \{0\} \) is a continuous and odd map. Then the definition of Krasnosekii Genus (cf. [19, Definition 5.1, p.86]) yields

\[
(4.4) \quad \dim_K(A) \leq m \implies N_K(\lambda) \leq m.
\]

Now choose

\[
(4.5) \quad r_0 := \left( \lambda \cdot 2^{N+4} \cdot 12^N \cdot e^{14(N-1)\sqrt{|K|}d} \right)^{-\frac{1}{2}}.
\]

**Case 1.** If \( r_0 \leq d \),

\[
\lambda = \left[ 2^{N+4} \cdot 12^N \cdot r_0^2 e^{14(N-1)\sqrt{|K|}d} \right]^{-1} \leq \left[ 2^{N+4} \cdot 12^N \cdot r_0^2 e^{14(N-1)\sqrt{|K|}r_0} \right]^{-1},
\]

which together with (4.3) implies \( 0 \notin \Phi_{\lambda,r_0}(E_\lambda) \). Note that \( \Phi_{\lambda,r_0} \) is constructed by the covering \( \{ B_i := B_i(r_0) \}_{i=1}^m \). Now (4.4) together with Lemma B.3 and (4.5) furnishes

\[
(4.6) \quad N_K(\lambda) \leq m \leq e^{(N-1)d\sqrt{|K|}} \left( \frac{d}{r_0} \right)^N \leq C^{1+d\sqrt{|K|}dN} \lambda^\frac{N}{2},
\]

where

\[
C := \max \left\{ 2^{\frac{N(N+1)}{2}} \cdot 12 \cdot r_0^2, e^{(7N+1)(N-1)} \right\}.
\]

**Case 2.** If \( r_0 > d =: r_* \), then

\[
B_1(r_0) = M = B_1(r_*), \quad 1 = m = \text{Cap}_M(r_0) = \text{Cap}_M(r_*).
\]

Now it follows from (4.5) that

\[
\lambda < \left[ 2^{N+4} \cdot 12^N \cdot r_*^2 e^{14(N-1)\sqrt{|K|}r_*} \right]^{-1}.
\]

Now we consider \( r_* \) instead of \( r_0 \) since the coverings \( \{ B_1 \} \) induced by \( r_0 \) and \( r_* \) are same. From above, we can see that \( 0 \notin \Phi_{\lambda,r_*}(E_{\lambda}) \) and hence,

\[
N_K(\lambda) \leq m = 1,
\]

which together with (4.6) yields the theorem. \( \square \)
Appendix A. The proof of Proposition 3.7

Lemma A.1. Given a smooth nonnegative function \( f \in C^\infty(M) \). Then for any open set \( D \subset M \), we have

\[
\begin{align*}
(1) \int_0^\infty \min \{ m(\Omega(t)), m(D) - m(\Omega(t)) \} dt &\leq \int_D F^*(df) dm, \\
(2) \int_0^\infty \| m(\Omega(t)) \| dm &\leq \int_D m(\Omega(t)) dt,
\end{align*}
\]

where \( \Omega(t) := \{ x \in D : f(x) \geq t \} \).

Proof. Since \( D \) is open and \( f \in C^\infty(M) \), \( f \big|_D \in C^\infty(D) \). For convenience, we abuse the notation and use \( f \) to denote \( f \big|_D \). Thus,

(1). The layer cake representation yields that

\[
\int_D f dm = \int_D dm \int_0^f dt = \int_0^\infty dt \int_{\Omega(t)} dm = \int_0^\infty m(\Omega(t)) dt.
\]

(2). The co-area formula ([17 Theorem 3.3.1]) together with Definition 3.6 yields

\[
\int_D F^*(df) dm = \int_0^\infty A_m(\partial\Omega(t)) dt \geq \int_0^\infty \min \{ m(\Omega(t)), m(D) - m(\Omega(t)) \} dt,
\]

where \( A_m = \frac{\nabla f \cdot dm}{\nabla f} \) for the regular value of \( f \). \( \square \)

Before showing Proposition 3.7, we recall some facts concerned with \( H^{1,2}(M) \). For any \( f \in C^\infty(M) \), set

\[
\| f \|_H := \| f \|_{L^2} + \frac{1}{2} \| F^*(df) \|_{L^2} + \| F^*(-df) \|_{L^2},
\]

where

\[
\| f \|_{L^2} := \left( \int_M f^2 dm \right)^{\frac{1}{2}}, \quad \| F^*(df) \|_{L^2} := \left( \int_M F^{*2}(df) dm \right)^{\frac{1}{2}}.
\]

It is easy to check \( \| f \|_H \) is a norm. Let \( H^{1,2}(M) \) be the completeness of \( C^\infty(M) \) under the norm \( \| \cdot \|_H \). Since \( A_F < \infty \), one can show \( H^{1,2}(M) \) is the standard Sobolev space equipped with the standard topology (cf. [15][18]).

The proof of Proposition 3.7. Given a smooth function \( f \in C^\infty(M) \) with \( \int_D f dm = 0 \). Since \( f \big|_D \in C^\infty(D) \), there exists a median \( \alpha \) of \( f \big|_D \), i.e.,

\[
m(\{ x \in D : f(x) \geq \alpha \}) \geq \frac{1}{2} m(D), \quad m(\{ x \in D : f(x) \leq \alpha \}) \geq \frac{1}{2} m(D).
\]

It is easy to see that such \( \alpha \) always exists. Set \( f_+ := \max\{ f \big|_D - \alpha, 0 \} \) and \( f_- := \min\{ f \big|_D - \alpha, 0 \} \). By the definition of median, one can check that for any \( t > 0 \),

\[
m(\{ x \in D : f_+^2(x) \geq t \}) \leq \frac{1}{2} m(D), \quad m(\{ x \in D : f_-^2(x) \geq t \}) \leq \frac{1}{2} m(D).
\]
Hence, the above inequalities together with Lemma A.1 yield
\[
\mathfrak{h}(D) \int_D |f - \alpha|^2 \, dm = \mathfrak{h}(D) \int_D (f^2_+ + f^2_-) \, dm
\]
\[
= \mathfrak{h}(D) \left( \int_0^\infty m(\{x \in D : f^2_+(x) \geq t\}) \, dt + \int_0^\infty m(\{x \in D : f^2_-(x) \geq t\}) \, dt \right)
\]
\[
\leq \int_D F^*(df^2_+) \, dm + \int_D F^*(df^2_-) \, dm = 2 \int_D f_+ F^*(df_+) + (-f_-)F^*(-df_-) \, dm
\]
\[
\leq 2\lambda_F \int_D |f - \alpha| F^*(df) \, dm \leq 2\lambda_F \left( \int_D |f - \alpha|^2 \, dm \right)^{\frac{1}{2}} \left( \int_D F^{*2}(df) \, dm \right)^{\frac{1}{2}}.
\]
Hence,
\[
\int_D F^{*2}(df) \, dm \geq \frac{\mathfrak{h}^2(D)}{4\lambda_F^2} \int_D |f - \alpha|^2 \, dm.
\]
Since \( \int_D f \, dm = 0 \),
\[
\inf_{\alpha \in \mathbb{R}} \int_D |f - \alpha|^2 \, dm \geq \int_D f^2 \, dm.
\]
Now the result follows from the density mentioned above. 

\section*{Appendix B. Volume comparison}

Let \((M, F, dm)\) be a forward complete Finsler \(n\)-manifold. Given \(p \in M\), denote by \((r, y) = (r, \theta^n)\), \(1 \leq \alpha \leq n\), the polar coordinates about \(p\). Express
\[
dm = \hat{\sigma}_p(r, y) dr \wedge dv_p(y),
\]
where \(dv_p\) is the measure on \(S^pM\) induced by \(F\). Note that \(d\mathfrak{A}_i|_{(r, y)} := \hat{\sigma}_p(r, y) dv_p(y)\) is the measure on \(S^p(r)\) induced by \(\nabla r\). Then we have the following result.

\begin{lemma}
Let \(i: \Gamma \hookrightarrow M\) be a smooth hypersurface and let \(d\mathfrak{A}_\pm\) be the measure of \(\Gamma\) induced by the unit normal fields \(n_\pm\). Then
\[
d\mathfrak{A}_\pm|_{(r, y)} \geq \lambda_F^{-1} \cdot d\mathfrak{A}_i|_{(r, y)}
\]
for any point \((r, y) = x \in \Gamma \ (r > 0)\).
\end{lemma}

\begin{proof}
Let \(n\) denote a unit normal vector field on \(\Gamma\). Thus, the coarea formula (\cite[Theorem 3.3.1]{17}) yields
\[
d\mathfrak{A}_+ = |i^*(\nabla r) \, dm| = |g_n(n, \nabla r)| dA_n \leq \lambda_F \, dA_n.
\]
\end{proof}

It is remarkable that \(e^r \in \mathfrak{A}^n, \mathfrak{A}^n\) if \(dm\) is either the Busemann-Hausdorff measure or the Holmes-Thompson measure (cf. \cite[Proposition A.1]{27}). Then \cite[Remark 3.5]{26} together with \cite[(3.1)]{31} yields the following. Also refer to \cite[Lemma 4.2]{23}.

\begin{lemma}
Let \((M, F, dm)\) be a closed Finsler \(n\)-manifold, where \(dm\) is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. If for some \(K \leq 0\) and \(d > 0\),
\[
\text{Ric} \geq (n - 1)K, \text{ diam}(M) = d,
\]
then for any \(\alpha \in \mathbb{R}\),
\[
\int_D |f - \alpha|^2 \, dm \geq \frac{\mathfrak{h}^2(D)}{4\lambda_F^2} \int_D |f - \alpha|^2 \, dm.
\]
\end{lemma}
then

(1) \( \hat{\sigma}_p(\min\{i_y, r\}, y) \geq \Lambda F_{n,K}^{2n} \frac{A_{n,K}(r)}{V_{n,K}(R)} - \frac{V_{n,K}(r_1) - V_{n,K}(r_0)}{R - r_1} \int_r^R \hat{\sigma}_p(\min\{i_y, t\}, y) dt, \ \forall 0 < r \leq R; \)

(2) \( \int_{\min\{i_y, r_0\}}^r \hat{\sigma}_p(\min\{i_y, t\}, y) dt \geq \Lambda F_{n,K}^{2n} \frac{V_{n,K}(r_1) - V_{n,K}(r_0)}{R - r_1} \int_r^R \hat{\sigma}_p(\min\{i_y, t\}, y) dt, \ \forall 0 < r_0 < r_1 < r_2; \)

(3) \( \int_0^r \hat{\sigma}_p(\min\{i_y, t\}, y) dt \geq \Lambda F_{n,K}^{2n} \frac{V_{n,K}(r_1) - V_{n,K}(r_0)}{R - r_1} \int_{\min\{i_y, r_0\}}^r \hat{\sigma}_p(\min\{i_y, t\}, y) dt, \ \forall 0 < r \leq R; \)

(4) \( \text{Cov}_M(r) \leq \Lambda F_{n,K}^{2n} \frac{V_{n,K}(d)}{V_{n,K}(\frac{r}{1+\Lambda})}, \ \forall 0 < r \leq d. \)

Here, \( V_{n,K}(r) \) (resp. \( A_{n,K}(r) \)) is the volume (resp. area) of ball (resp. sphere) with radius \( r \) in the Riemannian space form of constant curvature \( k \), that is,

\[
A_{n,K}(r) = \text{vol}(S^{n-1}) s_{K}^{n-1}(r), \quad V_{n,K}(r) = \text{vol}(S^{n-1}) \int_0^r s_{K}^{n-1}(t) dt.
\]

Lemma B.3. Let \((M, F, dm)\) be a closed reversible Finsler \( n \)-manifold. Suppose for some \( K \leq 0 \) and \( d > 0 \),

\[
\text{Ric}_N \geq (N - 1)K \leq 0, \quad \text{diam}(M) = d,
\]

where \( N \in [n, \infty) \). Let \( \{p_i\}_{i=1}^m \) be a complete \( r \)-package. Then we have

(1) For any \( 0 < r \leq d \),

\[
m = \text{Cap}_M(r) \leq e^{(N-1)d\sqrt{|K|}} \left( \frac{d}{r} \right)^N.
\]

(2) Given any \( x \in M \), the number of \( B_{p_i}(4r) \)'s containing \( x \) is not greater than

\[
12^N e^{12(N-1)r\sqrt{|K|}}.
\]

Proof. For any \( x \in M \) and \( 0 < r \leq R \), [15] Theorem 5.2 yields that

\[
\frac{m(B_x(R))}{m(B_x(r))} \leq \frac{\int_0^R s_{K}^{n-1}(t) dt}{\int_0^r s_{K}^{n-1}(t) dt} \leq e^{(N-1)R\sqrt{|K|}} \left( \frac{R}{r} \right)^N.
\]

which implies (1) directly. We now show (2). Suppose that \( m(B_x(r)) = \inf_i m(B_{p_i}(r)) \), where \( \inf \) is defined on all \( i \) with \( x \in B_{p_i}(4r) \). It is easy to check that

\[
B_{p_i}(4r) \subset B_{p_i}(12r).
\]

Hence, the number of \( B_{p_i}(4r) \) containing \( x \) is less than

\[
\frac{m(B_{p_i}(12r))}{m(B_{p_i}(r))} \leq 12^N e^{12(N-1)r\sqrt{|K|}}.
\]
The proof of Lemma 3.8. Let $\Gamma$ be a smooth hypersurface embedded in $D$ which divides $D$ into disjoint open sets $D_1, D_2$ in $D$ with common boundary $\partial D_1 = \partial D_2 = \Gamma$. Without loss of generality, we assume that $m(D_1 \cap B^+_p(r/(2\sqrt{\Lambda F}))) \leq \frac{1}{2} m(B^+_p(r/(2\sqrt{\Lambda F})))$. Hence, $m(D_1 \cap B^+_p(r/(2\sqrt{\Lambda F}))) \leq m(D_2 \cap B^+_p(r/(2\sqrt{\Lambda F})))$.

For convenience, set $B(\eta) := B^+_p(\eta)$, for any $\eta > 0$. Let $\alpha \in (0, 1)$ be a constant which will be chosen later.

Step 1: Suppose $m(D_1 \cap B(r/(2\sqrt{\Lambda F}))) \leq \alpha m(D_1)$.

For each $q \in D_1 - \text{Cut}_p$, set $q^*$ is the last point on the minimal geodesic segment $\gamma_{pq}$ from $p$ to $q$, where this ray intersects $\Gamma$. If the whole segment $\gamma_{pq}$ is contained in $D_1$, set $q^* := p$.

Fix a positive number $\beta \in (0, r/(2\sqrt{\Lambda F}))$. Let $(t, y)$ denote the polar coordinate system about $p$. Given a point $q = (\rho, y) \in D_1 - \text{Cut}_p - B(r/(2\sqrt{\Lambda F}))$, set $\text{rod}(q) := \{(t, y) : \beta \leq t \leq \rho\}$.

Define

\[ D_1^1 := \{q \in D_1 - \text{Cut}_p - B(r/(2\sqrt{\Lambda F})) : q^* \notin B(\beta)\}; \]
\[ D_1^2 := \{q \in D_1 - \text{Cut}_p - B(r/(2\sqrt{\Lambda F})) : \text{rod}(q) \subset D_1\}; \]
\[ D_1^3 := \{q \in B(r/(2\sqrt{\Lambda F})) - B(\beta) : \exists x \in D_1^2 \text{ such that } q \in \text{rod}(x)\}. \]

By Lemma B.2, we obtain that

\[ \frac{m(D_1^3)}{m(D_1^1)} \geq \Lambda_F^{-2n} V_{n,K}(r/(2\sqrt{\Lambda F})) - V_{n,K}(\beta) =: \gamma^{-1}. \]

It follows from the assumption that

\[ (1-\alpha)m(D_1) \leq m(D_1 - B(r/(2\sqrt{\Lambda F}))), \quad m(D_1^3) \leq m(D_1 \cap B(r/(2\sqrt{\Lambda F}))) \leq \alpha m(D_1). \]

Note that $D_1 - B(r/(2\sqrt{\Lambda F})) \subset D_1^1 \cup D_1^2$. From above, we have

\[ (1-\alpha)m(D_1) \leq m(D_1^3) + m(D_1^1) \leq \gamma \alpha m(D_1) + m(D_1^1). \]
Thus, \( \beta \leq c_{j_y} \) and \( \exp_p(c_{j_y}, y) \in \Gamma \). Lemma B.2 then yields
\[
\sum_{j_y} \int_{a_{j_y}}^{b_{j_y}} \sigma_p(t, y) dt \leq \sum_{j_y} \int_{c_{j_y}}^{b_{j_y}} \sigma_p(t, y) dt \leq \Lambda_F^{2n} \sum_{j_y} V_{n,K}(b_{j_y}) - V_{n,K}(c_{j_y}) \sigma_p(c_{j_y}, y) \leq \Lambda_F^{2n} \frac{\sum_{j_y} \sigma_p(c_{j_y}, y)}{A_{n,K}(\beta)}.
\]
The inequality above together with Lemma B.1 yields
\[
\mathbf{m}(D_1) \leq \Lambda_F^{2n} \frac{V_{n,K}(R) - V_{n,K}(\beta)}{A_{n,K}(\beta)} \int_{\mathcal{D}_1} \sum_{j_y} \sigma_p(c_{j_y}, y) d\nu_p(y) \leq \Lambda_F^{2n+\frac{1}{2}} \frac{V_{n,K}(R) - V_{n,K}(\beta)}{A_{n,K}(\beta)} A_\pm(\Gamma).
\]
Combining (C.1) and (C.2), we obtain
\[
\frac{A_\pm(\Gamma)}{\mathbf{m}(D_1)} = \frac{A_\pm(\Gamma)}{\mathbf{m}(D_1)} \frac{\mathbf{m}(D_1)}{\mathbf{m}(D_1^* \cap B(r/(2\sqrt{\Lambda_F})))} \geq \Lambda_F^{-(2n+\frac{1}{2})}(1 - \alpha(1 + \gamma)) \frac{A_{n,K}(\beta)}{V_{n,K}(R) - V_{n,K}(\beta)}.
\]

Step 2: Suppose \( \mathbf{m}(D_1 \cap B(r/(2\sqrt{\Lambda_F}))) \geq \mathbf{om}(D_1) \).

For simplicity, set \( W_i := D_i \cap B(r/(2\sqrt{\Lambda_F})) \), \( i = 1, 2 \). Now we consider the product space \( W_1 \times W_2 \) with the product measure \( dm_x := dm \times dm \). Let
\[
N := \{(q, w) \in W_1 \times W_2 : q \in \text{Cut}_w \text{ or } w \in \text{Cut}_q \}.
\]
Then Fubini’s theorem together with [3] Lemma 8.5.4] yields \( \mathbf{m}_x(N) = 0 \). For each \( (q, w) \in (W_1 \times W_2) \setminus N \), there exists a unique minimal geodesic \( \gamma_{wq} \) from \( w \) to \( q \) with the length \( L_F(\gamma_{wq}) \leq r \). The triangle inequality implies \( \gamma_{wq} \subset B(r) \). Denote by \( q^* \) the last point on \( \gamma_{wq} \) where \( \gamma_{wq} \) intersects \( \Gamma \). Now define
\[
V_1 := \{(q, w) \in W_1 \times W_2 - N : d(w, q^*) \geq d(q^*, q)\},
\]
\[
V_2 := \{(q, w) \in W_1 \times W_2 - N : d(w, q^*) \leq d(q^*, q)\}.
\]
Since \( \mathbf{m}_x(V_1 \cup V_2) = \mathbf{m}_x(W_1 \times W_2) \), we have
\[
\mathbf{m}_x(V_1) \geq \frac{1}{2} \mathbf{m}_x(W_1 \times W_2) \text{ or } \mathbf{m}_x(V_2) \geq \frac{1}{2} \mathbf{m}_x(W_1 \times W_2).
\]

Case I: Suppose that \( \mathbf{m}_x(V_1) \geq \frac{1}{2} \mathbf{m}_x(W_1 \times W_2) \).
Note that
\[ m_\times(V_1) = \int_{w \in W_2} dm \int_{q \in W_1 : d(w, q^*) \geq d(q^*, q) \geq q} dm. \]
Thus, there exist a point \( w_2 \in W_2 \) and a measurable set \( U_1 \subset W_1 \) such that

1. For each \( q \in U_1 \), \( d(w_2, q^*) \geq d(q^*, q) \) and \( (q, w_2) \notin N \).
2. \( \text{m}(U_1) \geq \frac{1}{2} \text{m}(W_1) \).

Let \((t, y)\) denote the polar coordinate system about \( w_2 \). For \( q = (\rho, y) \in U_1 \), set \( q^* = (\rho^*, y) \). Since \( \rho^* = d(w, q^*) \geq d(q^*, q) = \rho - \rho^* \), \( \rho^* \geq \rho/2 \). Set \( \rho^{**} := \sup\{s : \exp_{w_2}(ty), \, t \in [\rho', s)\} \), which is contained in \( U_1 \setminus \text{Cut}_{w_2} \). Then \( \hat{q} := (\rho^{**}, y) \in B(r/(2\sqrt{\Lambda_F})) \), which implies
\[ \rho^{**} = d(w_2, \hat{q}) \leq d(w_2, p) + d(p, \hat{q}) < \frac{r}{2} + \frac{r}{2\sqrt{\Lambda_F}} \leq r. \]

Since \((\hat{q})^* = q^*, \, \rho^* \geq \rho^{**}/2 \), Lemma B.2 then yields
\[ \int_{\rho^*}^{\rho^{**}} \hat{\sigma}_w(t, y) dt \geq \Lambda_F^{-2n} \frac{A_{n,K}(\rho^*)}{V_{n,K}(r/2)} \geq \Lambda_F^{-2n} \frac{A_{n,K}(\rho^{**}/2)}{V_{n,K}(\rho^{**}/2)} \]
\[ \geq \Lambda_F^{-2n} \frac{A_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)}. \]

Lemma B.1 now yields that
\[ d A_\perp(\rho^*, y) \geq \Lambda_F^{-(2n + \frac{1}{2})} \frac{A_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)} \left( \int_{\rho^*}^{\rho^{**}} \hat{\sigma}_p(t, y) dt \right) d\nu_{w_2}(y). \]

Hence,
\[ A_\perp(\Gamma) \geq A_\perp(\Gamma \cap B(r/(2\sqrt{\Lambda_F}))) \geq \Lambda_F^{-(2n + \frac{1}{2})} \frac{A_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)} \text{m}(U_1). \]

By assumption, we have
\[ \alpha \text{m}(D_1) \leq \text{m}(D_1 \cap B(r/(2\sqrt{\Lambda_F}))) = \text{m}(W_1) \leq 2 \text{m}(U_1), \]
which implies
\[ \frac{A_\perp(\Gamma)}{\text{m}(D_1)} \geq \frac{\alpha A_{n,K}(r/2)}{2 \Lambda_F^{2n + \frac{1}{2}} V_{n,K}(r) - V_{n,K}(r/2)}. \]

**Case II:** Suppose that \( m_\times(V_2) \geq \frac{1}{2} m_\times(W_1 \times W_2) \).

Then Fubini’s theorem yields that there exist a point \( q_1 \in W_1 \) and a measurable set \( U_2 \subset W_2 \) such that

1. For each \( w \in U_2 \), \( d(w, q_1^*) \leq d(q_1^*, q_1) \) and \( (q_1, w) \notin N \).
2. \( \text{m}(U_2) \geq \frac{1}{2} \text{m}(W_2) \).

It is noticeable that \( q_1^* \) is dependent on the choice of \( w \). Let \( w^\perp \) denote the first point on \( \gamma_{wq_1} \), where the segment intersects \( \Gamma \). Thus, for each \( w \in U_2 \),
\[ d(w, w^\perp) \leq d(w, q_1^*) \leq d(q_1^*, q_1) \leq d(w^\perp, q_1). \]

Let \( \tilde{F} \) denote the reverse of \( F \). It follows from [25] Lemma 3.1] that the reverse of the geodesic \( \gamma_{wq_1} \) is a minimal geodesic \( \tilde{\gamma}_{q_1w} \) from \( q_1 \) to \( w \) in \((M, \tilde{F})\). Note that \( w^\perp \) is the last point on \( \tilde{\gamma}_{q_1w} \) where \( \tilde{\gamma}_{q_1w} \) intersects \( \Gamma \). Let \( \tilde{N} \) be defined as \( N \) in \((M, \tilde{F})\). Denote by \( \tilde{d} \) the metric induced by \( \tilde{F} \). Thus, \( U_2 \subset W_2 \) satisfies

1. For each \( w \in U_2 \), \( \tilde{d}(q_1, w^\perp) \geq \tilde{d}(w^\perp, w) \) and \( (q_1, w) \notin \tilde{N} \).
2. \( \tilde{m}(U_2) \geq \frac{1}{2} \tilde{m}(W_2) \);
To obtain the best possible bound, we set a similar argument to the one in Case I (with respect to $(M, \text{the same Busemann-Hausdorff measures and Holmes-Thompson measures.})\) Thus, an easy calculation then yields

$$A_{\mp}(\Gamma) \geq A_{\mp}^{(2n+\frac{1}{2})} \frac{A_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)} m(U_2).$$

By assumption, we have

$$\alpha m(D_1) \leq m(D_1 \cap B(r/(2\sqrt{A_F}))) \leq m(D_2 \cap B(r/(2\sqrt{A_F}))) = m(W_2) \leq 2m(U_2).$$

Hence,

$$\frac{A_{\pm}(\Gamma)}{m(D_1)} \geq \frac{\alpha}{2\Lambda^{2n+\frac{1}{2}} F} A_{n,K}(r/2) \frac{V_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)}.$$

**Step 3:** From above, we obtain

$$A_{\pm}(\Gamma) \geq \frac{1-\alpha(1+\Lambda_{2n+\frac{1}{2}}F)}{\Lambda_{2n+\frac{1}{2}} F} \mathcal{A}, \quad \frac{\alpha}{2\Lambda^{2n+\frac{1}{2}}} \mathcal{B}, \quad \frac{\alpha}{2\Lambda^{2n+\frac{1}{2}}} \mathcal{F}.$$

where

$$\mathcal{A} := \frac{A_{n,K}(\beta)}{V_{n,K}(r) - V_{n,K}(r/2)}, \quad \mathcal{B} := \frac{A_{n,K}(r/2)}{V_{n,K}(r) - V_{n,K}(r/2)}, \quad \mathcal{F} := \frac{V_{n,K}(r) - V_{n,K}(r/(2\sqrt{A_F}))}{V_{n,K}(r/(2\sqrt{A_F})) - V_{n,K}(r)}.$$

To obtain the best possible bound, we set

$$\frac{1 - \alpha(1+\Lambda_{2n+\frac{1}{2}}F)}{\Lambda_{2n+\frac{1}{2}} F} \mathcal{A} = \frac{\alpha}{2\Lambda^{2n+\frac{1}{2}}} \mathcal{B}.$$

Thus,

$$\alpha = \frac{2\mathcal{A}}{\mathcal{B} + 2\mathcal{A}(1+\Lambda_{2n+\frac{1}{2}}F)}.$$

An easy calculation then yields

$$A_{\pm}(\Gamma) \geq \frac{A_{n,K}(\beta)}{2\Lambda_{2n+\frac{1}{2}} F} \frac{V_{n,K}(r) - V_{n,K}(\beta)}{V_{n,K}(r/(2\sqrt{A_F})) - V_{n,K}(\beta)}.$$

We are done by

$$\frac{A_{\pm}(\Gamma)}{\min\{m(D_1), m(D_2)\}} \geq \frac{A_{\pm}(\Gamma)}{m(D_1)}. \quad \Box$$

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