ON THE LENGTHS OF COMPACT EXCEPTIONAL OBJECTS IN DERIVED MODULE CATEGORIES

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Abstract. Let $A$ be a finite dimensional algebra and $D^b(A)$ be the bounded derived category of finitely generated left $A$-modules. In this paper we consider lengths of compact exceptional objects in $D^b(A)$, and prove a sufficient condition such that these lengths are bounded by the number of isomorphism classes of simple $A$-modules. Moreover, we show that algebras satisfying this condition are bounded derived simple.

1. Introduction

In algebraic representation theory, one of the most interesting and complicated problems is to classify algebras up to derived equivalence. In other words, given an algebra $A$, we want to characterize or construct all algebras whose (bounded) derived module categories are triangulated equivalent to the (bounded) derived module category of $A$. This big project draws the attention of many people, and quite many results have been obtained in two different approaches. In one direction, people have classified certain types of algebras with special properties; see [1, 7, 9, 10, 11, 16]. In the other direction, a few properties have been shown to be invariant under derived equivalence, such as number of isomorphism classes of simple modules and finiteness of global dimensions ([14]), finiteness of finitistic dimensions ([20]), finiteness of strong global dimensions ([15]), self-injective property ([2]), etc. However, there are much more questions for which the answers are unknown. For instances, it is quite well known that a finite dimensional local algebra is only derived equivalent to itself up to Morita equivalence. But at this moment the classification of algebras having this property is not done yet.

According to a fundamental result of Rickard ([30, 31]), an algebra $\Gamma$ is derived equivalent to $A$ if and only if there is a tilting complex $T$ in the bounded derived module category $D^b(A)$ such that $\Gamma$ is isomorphic to the opposite algebra of $\text{End}_{D^b(A)}(T)$. Therefore, to classify algebras derived equivalent to $A$, it suffices to to construct all tilting complexes in $D^b(A)$ up to isomorphism. Recently, Al-Nofayee and Rickard pointed out in [2] that up to Morita equivalence, there are at most countably many basic algebras derived equivalent to a given algebra. This motivates us to investigate algebras derived equivalent to only finitely many basic algebras up to isomorphism. In particular, we are interested in algebras for which there are only finitely many basic tilting complexes $T$ in $D^b(A)$ up to degree shifts and isomorphisms, where by basic we mean $T$ has no repeated summands. Clearly, if this is true, then as a prerequisite, the lengths of tilting complexes (defined in Section 2) must be bounded by a fixed number. Example of algebras satisfying this property include finite dimensional local algebras, hereditary algebras, and piecewise hereditary algebras (by the main result of Happel and Zacharia in [15]).
For connected algebras, finiteness of lengths of indecomposable compact exceptional objects implies finiteness of lengths of tilting complexes (Proposition 2.1). Moreover, we show that finiteness of indecomposable compact exceptional objects is invariant under derived equivalences (Theorem 2.2). The following theorem and its corollary give us a sufficient condition such that the lengths of indecomposable compact exceptional objects are bounded.

**Theorem 1.1.** Let $A$ be a finite dimensional algebra and $P_S$ be a projective cover of a simple $A$-module $S$. Let $Q$ be the direct sum of all indecomposable projective $A$-modules (up to isomorphism) not isomorphic to $P_S$. Suppose that $P_S$ satisfies the following conditions:

1. The socle of $P_S$ contains a simple summand $S_0 \cong S$;
2. $S_0$ is not contained in the trace of $Q$ in $P_S$ as left $A$-modules;
3. dually, $DS_0$ is not contained in the trace of $DQ$ in $DP_S$ as right $A$-modules, where $D = \text{Hom}_A(\_, k)$.

Then the following conclusions are true:

1. For any minimal and exceptional object $P^\bullet \in K^b(A\mathcal{P})$, $P_S$ appears at no more than one degree of $P^\bullet$.
2. Let $X, Y \in K^b(A\mathcal{P})$ be minimal objects such that $\text{Hom}_{K^b(A\mathcal{P})}(X, Y[n]) = 0$ for all $n \in \mathbb{Z}$. Then $P_S$ cannot appear in both $X$ and $Y$.

**Corollary 1.2.** If all indecomposable projective $A$-modules satisfy the above conditions, then the length of every indecomposable exceptional object in $K^b(A\mathcal{P})$ is bounded by the number of isomorphism classes of simple $A$-modules. Moreover, every indecomposable projective $A$-module appears at precisely one degree for every minimal basic tilting complex $T \in K^b(A\mathcal{P})$, and the length of $T$ is bounded by the number of isomorphism classes of simple $A$-modules as well.

By [4] and [5], an algebra $A$ is called bounded derived simple if $D^b(A)$ has no nontrivial recollements by bounded derived module categories of algebras. Clearly, local algebras are bounded derived simple. But there are many other bounded derived simple algebras ([26, 27]). We show that algebras satisfying the conditions in the above theorem are bounded derived simple. That is:

**Theorem 1.3.** Let $A$ be a connected algebra satisfying the conditions in Theorem 1.1. Then every torsion pair $(\text{Tria} X, \text{Tria} Y)$ of $K^b(A\mathcal{P})$ induced by two objects $X, Y \in K^b(A\mathcal{P})$ coincides with a torsion pair $(\text{Tria} P, \text{Tria} Q)$ induced by two projective $A$-modules $P$ and $Q$. Moreover, $A$ is bounded derived simple.

The paper is organized as follow. In Section 2 we define lengths of compact objects in derived module categories, and prove Theorem 1.1 and its corollary. Stratification of bounded derived module categories is investigated in Section 3, where Theorem 1.3 is proved. In Sections 4 and 5 we describe some examples, and ask several questions for which the answers are not clear to us.

In this paper we only consider finite dimensional algebras over an algebraically closed field $k$, although many results are still true in a much more general framework. All modules, unless specified explicitly, are finitely generated left modules. Composition of maps and morphisms is from right to left. The zero module is regarded as a trivial projective or free module. For a fixed object $X$ in a module category or a derived category, $\text{add} X$ is the additive category constituted of direct summands of finite direct sums of $X$, and $\text{Tria} X$ is the smallest triangulated
category (closed under isomorphisms, degree shifts, direct summands, and finite coproducts) containing $X$. The degree shift functor [1] in derived categories is as usually defined.

2. Lengths of compact exceptional objects

Throughout this paper we assume that all finite dimensional $k$-algebras are basic and connected. That is, all indecomposable summands of $A$ are pairwise nonisomorphic, and $A$ is not a direct sum of two nontrivial algebras (or in other words, the ordinary quiver of $A$ is connected). Let $A$-mod be the category of finitely generated left $A$-modules, and let $D^b(A)$ be the bounded derived category of $A$-mod. We view an $A$-module as a stalk complex concentrated in degree 0.

Let $K^b(A_P)$ be the homotopy category of perfect complexes; that is, complexes of finite length each term of which is a finitely generated projective $A$-module. It is well known that $K^b(A_P)$ can be identified with a full subcategory of $D^b(A)$. We say an object $P^* \in K^b(A_P)$ is minimal if regarded as chain complex, $P^*$ has no summands of the following form:

$$\ldots \to 0 \to P \xrightarrow{id} P \to 0 \to \ldots,$$

where $P$ is a finitely generated projective $A$-module. It is easy to see that $P^*$ is minimal if and only if every differential map $d^i : P^i \to P^{i+1}$ sends $P^i$ into the radical of $P^{i+1}$. Moreover, an object $X \in D^b(A)$ is compact (that is, the functor $\text{Hom}_{D^b(A)}(X, -)$ commutes with small coproduct) if and only if it is quasi-isomorphic to a unique (up to isomorphism) minimal perfect complex in $K^b(A_P)$.

Let $X$ be a compact object in $D^b(A)$. To define its length, we first choose a minimal perfect complex $P^* \in K^b(A_P)$ quasi-isomorphic to $X$, and let $r$ and $s$ be the degrees of the first and the last nonzero terms in this complex. Then we define the length $\ell(X) = \ell(P^*)$ to be $s - r + 1$. Since every compact object in $D^b(A)$ is quasi-isomorphic to a unique minimal perfect complex, its length is well defined.

Recall an algebra $A$ is derived equivalent to another algebra $\Gamma$ if the bounded derived category $D^b(A)$ is triangulated equivalent to $D^b(\Gamma)$. By Rickard’s theorem (see [30, 31], or [17] for left modules), $A$ is derived equivalent to $\Gamma$ if and only if there is some object $T \in K^b(A_P)$ which is exceptional (or rigid), i.e., $\text{Hom}_{K^b(A_P)}(T, T[n]) = 0$ for $n \neq 0$, and generates $K^b(A_P)$, such that $\Gamma$ is isomorphic to the opposite algebra of $\text{End}_{D^b(A_P)}(T)$. An object $X \in D^b(A)$ is called a tilting complex if it is quasi-isomorphic to some $T \in K^b(A_P)$ satisfying the above conditions.

Rickard’s theorem laid the foundation for classifying algebras up to derived equivalence. Indeed, to find all algebras derived equivalent to a given algebra $A$, it is enough to construct all basic tilting complexes $T \in K^b(A_P)$; that is, indecomposable summands of $T$ are pairwise non-isomorphic. It was pointed out in [2] that Corollary 9 of [18] implies that there are at most countably many tilting complexes in $D^b(A)$ up to isomorphism and degree shift. In [32] Rickard describes a way to construct infinitely many tilting complexes whose endomorphism algebras lie in

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1. We say $X$ classically generates $\text{Tri} X$ in this situation.

2. Note that the length we defined here is slightly different from that defined in [15]. The length defined here counts terms between the first and the last nonzero terms, whereas the authors of [15] defines the length of $X$ to be $s - r$, counting the number of differential maps between the first and the last nonzero terms.
different Morita equivalence classes. These results motivate us to consider algebras which has only finitely many tilting complexes up to isomorphism and degree shift. Of course, if this is true, the lengths of tilting complexes in $D^b(A)$ must be bounded.

The following preliminary observation tells us that finiteness of lengths of indecomposable compact exceptional objects implies finiteness of lengths of tilting complexes.

**Proposition 2.1.** Let $A$ be a connected algebra. If the lengths of all indecomposable compact exceptional objects are bounded by a fixed number, then the lengths of basic tilting complexes are also bounded.

**Proof.** Let $T = \oplus_{i=1}^n T_i$ be a basic tilting complex with $n$ indecomposable summands, where $n$ is the number of isomorphism classes of simple $A$-modules. Without loss of generality we assume that every indecomposable summand is minimal. If $r,s \in \mathbb{Z}$ satisfy $T^r \neq 0 \neq T^s$ and $T^i = 0$ for $i > s$ or $i < r$, the length of $T$ is $s - r + 1$. Since the length of each indecomposable summand is bounded by a fixed number $m$, $l(T) \leq nm$ is also bounded if we can show that there is no gap among these indecomposable summands. That is, for any $r \leq j \leq s$, we have $T^j \neq 0$.

But this is clear. Indeed, if it is not true, then we use this gap and decompose $T = T' \oplus T''$ such that $\text{Hom}_{K^b(A)}(T', T'') = 0 = \text{Hom}_{K^b(A)}(T'', T') = 0$. Consequently, $A$ is derived equivalent to $\text{End}_{K^b(A)}(T')^\text{op} \oplus \text{End}_{K^b(A)}(T'')^\text{op}$. This is impossible since connectedness is invariant under derived equivalences. □

Happel and Zacharia proved in [15] that finiteness of lengths of indecomposable compact objects is equivalent to piecewise hereditary property, which is invariant under derived equivalences. Similarly, finiteness of lengths of indecomposable compact exceptional objects is invariant under derived equivalences, too.

**Theorem 2.2.** Let $A$ and $B$ be two connected basic finite dimensional algebras. Suppose that $A$ and $B$ are derived equivalent. If lengths of indecomposable compact exceptional objects in $D^b(A)$ are bounded, then $B$ has the same property as well.

**Proof.** Without loss of generality we only consider minimal objects in homotopy categories of perfect complexes. Assume that $l(X) \leq m$ for all indecomposable exceptional objects $X \in K^b(A)$. Since $D^b(B)$ is derived equivalent to $D^b(A)$, by [30], there exists a tilting complex $T \in K^b(B)$ such that $\text{End}_{K^b(B)}(T)^\text{op} \cong A$. Moreover, $T$ induces a triangulated equivalence $F : K^b(A) \to K^b(B)$ such that $F(A) = T$. Let $G : K^b(B) \to K^b(A)$ be the quasi-inverse of $F$.

Now let $Y$ be a minimal, indecomposable exceptional object in $K^b(B)$. Then $G(Y)$ is an indecomposable exceptional object in $K^b(A)$, which has at most $m$ nonzero terms since by our assumption $G(Y) \in K^b(A)$ is minimal and by the given condition lengths of indecomposable exceptional objects in $K^b(A)$ are bounded by $m$.

Note that $FG(Y)$ is constructed as follows (Proposition 2.10 in [30]). First, for all $i \in \mathbb{Z}$, nonzero $G(Y)^i$ (which are projective $A$-modules) are replaced by objects in add $T$ to obtain a bigraded complex $Y^{**}$ over add $B$. Then we take the total complex $Y^*$ of $Y^{**}$ and define $FG(Y) = Y^*$. Clearly, $Y \cong FG(Y) = Y^*$, and $l(Y^*) \leq ml(T)$, where $l(T)$ is the length of $T$. That is, lengths of indecomposable exceptional objects in $K^b(B)$ are bounded by $ml(T)$. □

The main content of this section is to prove Theorem 1.1, giving a sufficient condition such that the lengths of indecomposable exceptional objects in derived
categories are bounded. We describe two conditions equivalent to (2) and (3) in Theorem 1.1 respectively, which are easier to check in the proof of the main result of this section.

**Lemma 2.3.** Let $A, S, P_S, S_o, Q,$ and $D$ be as in Theorem 1.1. Then

1. Condition (2) in Theorem 1.1 holds if and only if for any $P \in \text{add} Q$ and for any homomorphism $\alpha : P \to P_S$, $S_o$ is contained in the cokernel of $\alpha$.
2. Dually, condition (3) in Theorem 1.1 holds if for any $P \in \text{add} Q$ and for any homomorphism $\beta : P_S \to P$, $S_o$ is contained in the kernel of $\beta$.

**Proof.** Clearly, for and $P \in \text{add} Q$ and for any $\alpha : P \to P_S$, the image $\alpha(P)$ is a submodule of $\text{tr}_Q(P_S)$, the trace of $Q$ in $P_S$. Therefore, if condition (2) in Theorem 1.1 is true, then $S_o$ is not contained in $\alpha(P)$, and hence it must lie in the cokernel of $\alpha$. Conversely, suppose that for any $P \in \text{add} Q$ and for any homomorphism $\alpha : P \to P_S$, $S_o$ is contained in the cokernel of $\alpha$. Let $d = \dim_k \text{Hom}_A(Q, P_S)$. Then there is a map $\alpha : Q^d \to P_S$ such that $\text{tr}_Q(P_S)$ coincides with the image of $\alpha_o$. But $S_o$ is in the cokernel of $\alpha_o$, so it cannot be contained in $\text{tr}_Q(P_S)$. This proves the first statement.

By duality, for any $P \in \text{add} Q$ and for any homomorphism $\beta : P_S \to P$, $S_o$ is contained in the kernel of $\beta$ if and only if $DS_o$ is contained in the cokernel of $D\beta : DP \to DP_S$. Now the second statement follows from a similar argument as in the proof of the first one. \qed

Let $P^* \in K^b(A P)$. We say an indecomposable projective $A$-module $P$ *appears at degree* $i$ if $P^i$ has a summand isomorphic to $P$. The following proposition is the first part of Theorem 1.1.

**Proposition 2.4.** Let $A$ be a finite dimensional algebra and $P_S$ be a projective cover of a simple $A$-module $S$. Let $Q$ be the direct sum of all indecomposable projective $A$-modules (up to isomorphism) not isomorphic to $P_S$. Suppose that $P_S$ satisfies the following conditions:

1. The socle of $P_S$ contains a simple summand $S_o \cong S$;
2. $S_o$ is not contained in the trace of $Q$ in $P_S$ as left $A$-modules;
3. dually, $DS_o$ is not contained in the trace of $DQ$ in $DP_S$ as right $A$-modules, where $D = \text{Hom}_A(-, k)$.

Let $P^* \in K^b(A P)$ be minimal and exceptional. Then $P_S$ appears at no more than one degree of $P^*$.

**Proof.** We prove the conclusion by contradiction. Suppose that $P_S$ appears more than one degrees in $X$. Let $r$ and $s$ be the first degree and the last degree where $P_S$ appears. Therefore, we have the following decompositions: $P^r \cong P_S \oplus Q^r$, and $P^s \cong P_S \oplus Q^s$. Now we construct a chain map as follows:

\[
\cdots \xrightarrow{d_{s-2}} P^{s-2} \xrightarrow{d_{s-2}} P^{s-1} \xrightarrow{d_{s-1}} P_S \oplus Q^s \xrightarrow{d_s} P^{s+1} \xrightarrow{d_{s+1}} \cdots
\]

\[
\cdots \xrightarrow{d_{r-2}} P^{r-2} \xrightarrow{d_{r-2}} P^{r-1} \xrightarrow{d_{r-1}} P_S \oplus Q^r \xrightarrow{d_r} P^{r+1} \xrightarrow{d_{r+1}} \cdots
\]

where $\tilde{\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and $\alpha$ maps the top of $P_S$ to $S_o$ in the socle of $P_S$. \qed
We want to show that the chain map is well defined. It suffices to check that \( \alpha d_{s-1} = 0 = d_s \alpha \). Indeed, since \( P^* \) is minimal, \( d_{s-1}(P^{s-1}) \) is contained in the radical of \( P_S \oplus Q^s \). But \( \alpha \) maps the radical of \( P_S \oplus Q^s \) to 0, and we get \( \hat{\alpha} d_{s-1} = 0 \).

By definition, \( \hat{\alpha}(P_S \oplus Q^s) = S_0 \subseteq P_S \subseteq P_S \oplus Q^s \). Write \( P^{r+1} \cong P_S^\otimes n \oplus Q^{r+1} \) such that \( Q^{r+1} \) has no summand isomorphic to \( P_S \). Let \( p_1 : P^{r+1} \to P_S^\otimes n \) and \( p_2 : P^{r+1} \to Q^{r+1} \) be the projections. Therefore, to show \( d_s \alpha = 0 \), it suffices to check \( p_1 d_r(S_0) = 0 = p_2 d_r(S_0) \). Since \( p_2 d_r : P_S \to Q^{r+1} \), the second identity follows from the second statement of Lemma 2.3. Also, note that \( p_1 d_r : P_S \to P_S^\otimes n \).

But \( P^* \) is minimal, so \( p_1 d_r \) sends \( P_S \) into the radical of \( P_S^\otimes n \), and must maps \( S_0 \) to 0 since it is in the socle of \( P_S \). That is, the first identity is also true. This shows that the chain map is well defined.

We claim that this chain map is not null homotopic. Otherwise, there must exist maps \( h_s : P_S \oplus Q^s \to P^{r-1} \) and \( h_{s+1} : P^{s+1} \to P_S \oplus Q^r \) such that \( \alpha = h_s d_{r-1} + h_{s+1} d_r \). In particular, \( \alpha \) can be expressed as the composite of the following two maps:

\[
\begin{pmatrix} h_s \\ d_s \end{pmatrix} : P_S \to P^{r-1} \oplus P^{s+1} \to \begin{pmatrix} pd_{r-1} & ph_{s+1} \end{pmatrix} : P_S^\otimes n \oplus Q^{r+1} \to P_S,
\]

where \( i : P_S \to P_S \oplus Q^s \) and \( p : P_S \oplus Q^r \to P_S \) are the inclusion and projection respectively. But this is impossible. Indeed, since \( r \) and \( s \) are the first and the last degrees where \( P_S \) appears, we conclude that \( P^{r-1} \oplus P^{s+1} \) has no summand isomorphic to \( P_S \). Therefore, by the first statement of Lemma 2.3, \( S_0 \subseteq P_S \) must be in the cokernel of \( \begin{pmatrix} pd_{r-1} & ph_{s+1} \end{pmatrix} \), and hence must be in the cokernel of \( \alpha \) by the above factorization of \( \alpha \). This contradicts the definition \( \alpha(P_S) = S_0 \), and our claim is proved.

However, the above construction shows that \( \text{Hom}_{K^b(A \mathcal{P})}(P^*, P^*[s-r]) \neq 0 \), where \( s - r \neq 0 \). This contradicts the assumption that \( P^* \) is exceptional. The conclusion follows from contradiction.

The following result is actually implied implicitly by the proof of the previous proposition.

**Corollary 2.5.** Let \( P^* \in K^b(A \mathcal{P}) \) be a minimal object and suppose that \( P_S \) satisfies the conditions in the previous proposition. If \( i \) is the first degree where \( P_S \) appears in \( P^* \), then the socle of \( H^i(P^*) \) has a simple summand isomorphic to \( S \).

**Proof.** Consider the diagram

\[
\begin{array}{cccccc}
\ldots & P^{i-1} & \xrightarrow{d_{i-1}} & P_S & \oplus & Q & \xrightarrow{d_i} & P^{i+1} & \xrightarrow{d_{i+1}} & \ldots
\end{array}
\]

By Condition (2) in the previous proposition, \( S_0 \) is not contained in the image of \( d_{i-1} \). But \( S_0 \) also lies in the kernel of \( d_i \) since \( P_S \) is sent to the radical of \( P^{i+1} \); see the third paragraph of the proof of the previous proposition. Therefore, \( S_0 \) is contained in \( H^i(P^*) \). Clearly, it is in the socle of \( H^i(P^*) \).

Using the same technique, we can prove the second part of Theorem 1.1, which is the following proposition.

**Proposition 2.6.** Let \( A \) and \( P_S \) be as in proposition 2.4. If \( X,Y \in K^b(A \mathcal{P}) \) are two minimal objects satisfying \( \text{Hom}_{K^b(A \mathcal{P})}(X,Y[n]) = 0 \) for all \( n \in \mathbb{Z} \), then \( P_S \) cannot appear in both \( X \) and \( Y \).
Proof. We give a sketch. Suppose that $P_S$ appears in both $X$ and $Y$. Applying degree shifts if necessary, we can assume that both the last degree of $X$ where $P_S$ appears in $X$ and the first degree of $Y$ where $P_S$ appears in $Y$ are 0. Therefore, as did in the proof of the previous proposition, we get a chain map

$$\cdots \cdots \xrightarrow{d_{-3}} X^{-2} \xrightarrow{d_{-2}} X^{-1} \xrightarrow{d_{-1}} P_S \oplus Q^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} \cdots$$

$$\cdots \xrightarrow{d_{-3}} Y_{r-2} \xrightarrow{d_{-2}} Y_{r-1} \xrightarrow{d_{-1}} P_S \oplus R^0 \xrightarrow{d_0} Y^1 \xrightarrow{d_1} \cdots,$$

where $\alpha$ is defined as before. Now copy the previous proof, we can show that this chain map is well defined, and is not null homotopic. This contradicts the assumption that $\text{Hom}_{K^b(A_P)}(X,Y[n]) = 0$ for all $n \in \mathbb{Z}$, so the conclusion follows from contradiction.

We give a few direct corollaries, the first of which is already well known to people.

**Corollary 2.7.** If $A$ is a finite dimensional local algebra, then every indecomposable nonzero compact exceptional object $X \in D^b(A)$ is quasi-isomorphic to the stalk complex $\mathcal{A}A$ up to degree shift. In particular, $A$ is only derived equivalent to itself up to Morita equivalence.

**Proof.** Let $P^* \in K^b(A_P)$ be the unique (up to isomorphism) minimal exceptional object quasi-isomorphic to $X$. Then by proposition 2.4, there is exactly one $i \in \mathbb{Z}$ such that $P_i \neq 0$. Since $X$ is indecomposable, we know $P^i \cong _A\mathcal{A}$, and the conclusion follows.

**Corollary 2.8.** Let $e$ be a primitive idempotent in $A$ such that $eA(1-e) = 0$ and the socle of $Ae$ has a simple summand $S_e$ isomorphic to $Ae/\text{rad}Ae$. Let $P^* \in K^b(A_P)$ be a minimal exceptional object. Then $P = Ae$ appears at no more than one degree of $P^*$.

Since $eA(1-e) = 0$, the algebra $A$ is actually a triangular matrix algebra. That is, $A = \begin{bmatrix} eAe & 0 \\ (1-e)Ae & (1-e)A(1-e) \end{bmatrix}$.

**Proof.** We check the three conditions in Proposition 2.4. Condition (1) is given. It suffices to check (2) and (3). For $P \in \text{add}(A(1-e))$, we have $\text{Hom}_A(Ae,P) = 0$. By the second statement of Lemma 2.3, Condition (3) in Proposition 2.4 is true.

Note that the trace of $Q = A(1-e)$ is $A(1-e)Ae$, which coincides with $(1-e)Ae$ by the triangular matrix algebra structure. But $S_e \subseteq eAe$ is not contained in $(1-e)Ae$, so Condition (2) in Proposition 2.4 is also true.

**Corollary 2.9.** If all indecomposable projective $A$-modules satisfy the above conditions, then the length of every indecomposable exceptional object in $K^b(A_P)$ is bounded by the number of isomorphism classes of simple $A$-modules. Moreover, every indecomposable projective $A$-module appears at precisely one degree for every minimal basic tilting complex $T \in K^b(A_P)$, and the length of $T$ is bounded by the number of isomorphism classes of simple $A$-modules as well.

**Proof.** Take an indecomposable exceptional object $P^* \in K^b(A_P)$. Without loss of generality we can assume that $P^*$ is minimal. Let $r, s \in \mathbb{Z}$ such that $P^r$ and $P^s$ are the first and last nonzero terms in $P^*$. Then $l(P^*) = s - r + 1$ by our
definition. Since $P^\bullet$ is indecomposable, for $r \leq t \leq s$, $P^t \neq 0$. However, by Proposition 2.4, every indecomposable projective $A$-module (up to isomorphism) appears at no more than one degree in $P^\bullet$. Therefore, $P^\bullet$ can have at most $n$ nonzero terms, where $n$ is the number of isomorphism classes of simple $A$-modules. That is, $l(P^\bullet) = s - r + 1 \leq n$. This proves the first statement.

Note that every tilting complex $T$ is exceptional. Therefore, by Proposition 2.4, every indecomposable projective $A$-module appears at no more than one degree of $T$. But since $\text{Tria}T = K^b(A)P$, every indecomposable projective $A$-module must appear at some degree of $T$. Otherwise, suppose that $P^i \sim A\rho^i$ does not appear, where $\rho \in A$ is a primitive idempotent. Then $\text{Tria}T \subseteq \text{Tria}A(-\rho) \neq K^b(A)P$. This is impossible. The second statement follows from this observation.

To prove the third statement, we observe that every basic tilting complex $T$ must be connected as in the proof of Proposition 2.1. That is, if $r,s \in \mathbb{Z}$ satisfy $T^r \neq 0 \neq T^s$ and $T^i = 0$ for $i > s$ or $i < r$, then $T^i \neq 0$ for $r \leq j \leq s$. Therefore, the length of $T$ must be bounded by the number of isomorphism classes of simple $A$-modules by the second statement.

In the following example we use the above results to classify the derived equivalence classes of a particular algebra $A$. A similar example has been considered in [6]; see Example 5.13.

**Example 2.10.** Let $A$ be the path algebra of the following quiver with relations $\delta^2 = \rho^2 = \rho\alpha = \alpha\delta = 0$.

$$
\begin{array}{ccc}
  x & \xrightarrow{\alpha} & y \\
  \Downarrow & & \Downarrow \\
  y & \xrightarrow{\beta} & x
\end{array}
$$

Up to isomorphism, this algebra has two indecomposable projective modules as follows:

$$
P_x = \begin{array}{c} x \\ y \end{array}, \quad P_y = \begin{array}{c} y \\ y \end{array},
$$

both of which satisfy the conditions in Proposition 2.4.

According to Proposition 2.4, minimal indecomposable exceptional objects in $K^b(A)P$ have lengths at most 2. Moreover, if $l(P^\bullet) = 2$, then there exists a certain $i \in \mathbb{Z}$ such that $P^i \cong P^y_i$ and $P^{i+1} \cong P^x_i$ with $a,b \geq 1$. We can check that $P^\bullet$ is indecomposable if and only if $a = b = 1$. In conclusion, up to degree shift and quasi-isomorphisms, $K^b(A)P$ only has three indecomposable exceptional objects: stalk complexes $P_x$ and $P_y$, and

$$
X := P_y \xrightarrow{d} P_x,
$$

where $d$ maps the top of $P_y$ onto the simple summand $S_y$ in the socle of $P_x$.

Using Corollary 2.9, the reader can check that up to degree shift and quasi-isomorphism $K^b(A)P$ has three basic tilting complexes: $T_1 = P_x \oplus P_y$, $T_2 = P_x[-1] \oplus X$, and $T_3 = P_y \oplus X$. Clearly, $\text{End}_{K^b(A)P}(T_1)^{op} \cong A$.

By computation, $B^{op} = \text{End}_{K^b(A)P}(T_2)$ is isomorphic to the path algebra of the following quiver with relations $\beta\alpha\beta = 0 = \delta^2$, $\delta\alpha = \beta\delta = 0$:

$$
\begin{array}{ccc}
  x & \xrightarrow{\alpha} & y \\
  \Downarrow & & \Downarrow \\
  y & \xrightarrow{\beta} & x
\end{array}
$$
Similarly, by computation, \( C^{\text{op}} = \text{End}_{K^b(A \mathcal{P})}(T_3) \) is isomorphic to the path algebra of the following quiver with relations \( \alpha \beta \alpha = 0 = \delta^2, \delta \alpha = \beta \delta = 0 \):

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha} & y \\
\beta & & \end{array}
\]

We conclude that up to Morita equivalence, \( A \) is derived to three algebras: \( A, B, \) and \( C \) as they lie in different Morita equivalence classes.

A careful observation tells us that \( B \cong C^{\text{op}} \). This is reasonable since \( A \cong A^{\text{op}} \). Moreover, \( B \) has a tilting module \( T = B_1 \oplus M \) where \( M \cong B_1y/B_1x \). It is easy to check that \( \text{End}_B(T^{\text{op}}) \cong A \).

3. Stratification of bounded derived module categories

In this section we consider stratification of bounded derived module categories of algebras satisfying the conditions in Theorem 1.1, showing that they are bounded derived simple. First we introduce the definition of torsion pairs for triangulated categories, which is taken from Section 2, Chapter I of [8].

**Definition 3.1.** A pair of strict full subcategories \((T, F)\) of a triangulated category \( C \) is called a torsion pair if the following conditions hold:

1. \( C(T, F) = 0 \) for any \( T \in T \) and \( F \in F \).
2. \( T \in T \) implies \( T[1] \in T \); and \( F \in F \) implies \( F[-1] \in F \).
3. For any \( X \in C \), there is a triangle

\[
T_X \rightarrow X \rightarrow F_X \rightarrow T_X[1].
\]

Note that \( T \) is the left perpendicular category of \( F \), and \( F \) is the right perpendicular category of \( T \) (see [13]).

The following lemmas are crucial to prove the main result of this section.

**Lemma 3.2.** Let \( A \) be a finite dimensional algebra. Let \( e \) and \( f \) be two orthogonal idempotents such that \( e + f = 1 \). If there exist \( X \in \text{Tria} A e \) and \( Y \in \text{Tria} A f \) such that \((\text{Tria} X, \text{Tria} Y)\) is a torsion pair of \( K^b(A \mathcal{P}) \), then \( \text{Tria} X = \text{Tria} A e \) and \( \text{Tria} Y = \text{Tria} A f \).

**Proof.** It is clear that \( \text{Tria} X \subseteq \text{Tria} A e \) and \( \text{Tria} Y \subseteq \text{Tria} A f \). Therefore, we only need to show the other inclusions.

Take an arbitrary object \( U \in \text{Tria} A f \). Since \((\text{Tria} X, \text{Tria} Y)\) is a torsion pair of \( K^b(A \mathcal{P}) \), there is a canonical triangle

\[
V \rightarrow U \rightarrow W \rightarrow V[1]
\]

with \( V \in \text{Tria} X \subseteq \text{Tria} A e \) and \( W \in \text{Tria} Y \subseteq \text{Tria} A f \). However, since both \( U \) and \( W \) are contained in \( \text{Tria} A f \), so is \( V \). Therefore, \( V \in \text{Tria} A e \cap \text{Tria} A f \). We claim that \( V \) is quasi-isomorphic to 0. If this holds, then \( U \cong W \in \text{Tria} Y \), so \( \text{Tria} A f \subseteq \text{Tria} Y \).

Indeed, since \( V \in \text{Tria} A e \), we get a minimal representation of \( V \)

\[
P^*: \ldots \rightarrow 0 \rightarrow P^r \rightarrow \ldots \rightarrow P^s \rightarrow 0 \rightarrow \ldots
\]

such that all \( P^i \) are contained in \( \text{add} A e \). Similarly, since \( V \in \text{Tria} A f \), we get another minimal representation of \( V \)

\[
Q^*: \ldots \rightarrow 0 \rightarrow Q^l \rightarrow \ldots \rightarrow Q^s \rightarrow 0 \rightarrow \ldots
\]
such that all $Q^i$ are contained in $\text{add} Af$. Note that $P^\bullet$, $V$ and $Q^\bullet$ are quasi-isomorphic in $K^b(\mathcal{A} P)$. Therefore, there exist quasi-isomorphisms $\varphi_i : P^\bullet \to Q^\bullet$ and $\psi_i : Q^\bullet \to P^\bullet$. Let $\delta_i = \psi_i \varphi_i$ for all $i \in \mathbb{Z}$. Then $\delta_i : P^\bullet \to P^\bullet$ is again a quasi-isomorphism. On the other hand, for each $i \in \mathbb{Z}$, the component $\varphi_i$ sends $P^i$ into the radical of $Q^i$ since they do not have isomorphic summands. By the same reason, $\psi_i$ sends $Q^i$ into the radical of $P^i$. Consequently, $\delta_i$ maps $P^i$ into the radical of $P^i$, and hence is nilpotent as $P^i$ has finite length. That is, for each $i \in \mathbb{Z}$, there is some $m_i$ such that $\delta_i^{m_i} = 0$. As there are only finitely many nonzero terms in $P^\bullet$, we can find some $m$ such that $\delta_i^{m} = 0$ for all $i \in \mathbb{Z}$; i.e., $\delta^{m} = 0$. But $\delta^{m}$ is a quasi-isomorphism (since $\delta$ is). This happens if and only if $P^\bullet$ and hence $V$ are quasi-isomorphic to 0. Our claim is proved.

We have shown $\text{Tria} Y \supseteq \text{Tria} Af$. The fact that $\text{Tria} X \supseteq \text{Tria} Ae$ can be shown by using the same argument. This finishes the proof.

A immediate corollary is:

**Corollary 3.3.** Let $X,Y,e,f$ be as in the previous proposition. Then $A$ is isomorphic to the triangular matrix algebra (see [25])

$$
\begin{bmatrix}
  eAe & 0 \\
  fAe & fAf
\end{bmatrix}.
$$

**Proof.** It is clear since $Ae \in \text{Tria} X$, $Af \in \text{Tria} Y$, and $(\text{Tria} X, \text{Tria} Y)$ is a torsion pair of $K^b(\mathcal{A} P)$.

**Lemma 3.4.** If $A$ satisfies the conditions in Proposition 2.4, then the finitistic dimension of $A$ is 0.

Recall that the finitistic dimension of $A$ is the supremum of projective dimensions of finitely generated $A$-modules with finite projective dimension. The finitistic dimension of $A$ equals 0 if and only if all finitely generated $A$-modules having finite projective dimension are projective.

**Proof.** Suppose that the conclusion is wrong. Then we can find some $M \in \mathcal{A}$-mod such that the projective dimension of $M$ is finite and is strictly greater than 0. Take a minimal projective resolution of $M$ as follows

$$
0 \to P^n \to P^{n-1} \to \ldots \to P^0 \to 0.
$$

By our assumption, $P^n \neq 0$, $n \geq 1$, and the map $d_n : P^n \to P^{n-1}$ must be injective.

Take a nonzero indecomposable summand $P_S$ of $P^n$, which is a projective cover of a certain simple $A$-module $S$. Denote by $\iota$ the inclusion $P_S \to P^n$. Decompose $P^{n-1} \cong P_S^{\oplus m} \oplus Q$ where $Q$ has no summand isomorphic to $P_S$, and denote by $p_1, p_2$ the projection $P^{n-1} \to P_S^{\oplus m}$ and the projection $P^{n-1} \to Q$ respectively. Now consider the map $d_{n,t} : P_S \to P^{n-1}$. It must be injective as well since both $\iota$ and $d_n$ are injective.

By the decomposition, $d_{n,t} = \begin{bmatrix} p_1 d_{n,t} \\ p_2 d_{n,t} \end{bmatrix}$. By Condition (1) in Proposition 2.4, the socle of $P_S$ contains a simple summand $S_0$ isomorphic to $S$, and $p_2 d_{n,t}(S_0) = 0$ by Condition (2) of Lemma 2.3. Moreover, since the projective resolution is minimal, $d_n$ maps $P^n$ into the radical of $P^{n-1}$, so $p_1 d_{n,t}$ maps $P_S$ into the radical of $P_S^{\oplus m}$. Consequently, $p_1 d_{n,t}(S_0) = 0$ as well. Therefore, $S_0$ is in the kernel of $d_{n,t}$, contradicting the assertion that $d_{n,t}$ is injective. The conclusion follows from contradiction.

$\square$
Now we begin to prove the main result of this section. First we define recollements of triangulated categories; for more details, see [3, 4, 5, 6, 12, 19, 28].

**Definition 3.5.** A recollement of a triangulated category $\mathcal{C}$ by triangulated categories $\mathcal{D}$ and $\mathcal{E}$ is expressed diagrammatically as follows

\[
\begin{array}{c}
\mathcal{D} \\
\text{i}^* \downarrow \\
\mathcal{C} \\
\text{j}^* \downarrow \\
\mathcal{E}
\end{array}
\]

with six additive functors $i^*, i_*, i_!,$ and $j^*, j_!, j_*$ satisfying the following conditions:

1. $(i^*, i_*, i_!)$ and $(j^*, j_!, j_*)$ both are adjoint triples;
2. $i_!, j_!$ and $j_*$ are fully faithful;
3. $i_! j_* = 0$;
4. For each $X \in \mathcal{C}$, there are triangles

\[
i_* i_!(X) \rightarrow X \rightarrow j_* j^* (X) \rightarrow i_* i_!(X)[1],
\]

\[
j_* j^* (X) \rightarrow X \rightarrow i_* i^* (X) \rightarrow j_* j^* (X)[1].
\]

The following theorem, proved in [4], gives a relation between recollements of derived module categories and exceptional compact objects.

**Theorem 3.6.** (Corollary 2.5 in [4]) Let $\Gamma$ be a finite dimensional algebra. If the bounded derived category $D^b(\Gamma)$ is a recollement of $D^b(B)$ and $D^b(C)$, then there are objects $T_1, T_2 \in D^b(\Gamma)$ satisfying:

1. Both $T_1$ and $T_2$ are compact and exceptional;
2. $\text{Hom}_{D^b(\Gamma)}(T_1, T_2[n]) = 0$ for all $n \in \mathbb{Z}$;
3. $T_1 \oplus T_2$ generates $D^b(\Gamma)$ as triangulated category.

Moreover, we have $\text{End}_{D^b(\Gamma)}(T_1)^{\text{op}} \cong C$ and $\text{End}_{D^b(\Gamma)}(T_2)^{\text{op}} \cong B$.

According to [26], an algebra $\Gamma$ is said to be bounded derived simple if $D^b(\Gamma)$ has no nontrivial recollements by bounded derived module categories of algebras.

**Theorem 3.7.** Algebras $A$ satisfying conditions in Proposition 2.4 are bounded derived simple.

**Proof.** This is clearly true if $A$ is a local algebra. Suppose that $A$ is not local and is not bounded derived simple. Therefore, by Theorem 5.12 in [6], $A$ is not $K^b(A)$-simple as well. That is, the homotopy category $K^b(A)$ has a nontrivial recollement as follows

\[
\begin{array}{c}
K^b(S P) \\
\text{i}^* \downarrow \\
K^b(A P) \\
\text{j}^* \downarrow \\
K^b(R P)
\end{array}
\]

where $R$ and $S$ are finite dimensional algebras. It is clear that $i_* S$ and $j_! R$ are compact. Without loss of generality we can assume that they both are minimal. Since $K^b(S P) = \text{Tria} S$ and $i_*$ is a full embedding, we have $\text{Im} i_* = \text{Tria} i_* S$.

---

[3] Here we say $T_1 \oplus T_2$ generates $D^b(\Gamma)$ if for any nonzero object $X \in D^b(\Gamma)$, there is a certain $n \in \mathbb{Z}$ such that $\text{Hom}_{D^b(\Gamma)}(T_1 \oplus T_2, X[n]) \neq 0$. This holds if $\text{Tria}(T_1 \oplus T_2) = K^b(A P)$; i.e., $T_1 \oplus T_2$ classically generates $K^b(A P)$.
Similarly, \( \text{Im} j_i = \text{Tria} j_i R \). Moreover, by Lemma 2.6 in \([12]\), \((\text{Tria} j_i R, \text{Tria} i_s S)\) gives rise to a torsion pair of \( K^b(A_P) \).

Now let \( E \) be a complete set of primitive orthogonal idempotents of \( A \), and define
\[
E_1 = \{ \epsilon \in E | Ae \text{ appears in } j_i R \}, \quad E_2 = \{ \epsilon \in E | Ae \text{ appears in } i_s S \}.
\]
Define \( e = \sum_{\epsilon \in E_1} \epsilon \) and \( f = \sum_{\epsilon \in E_2} \epsilon \). Since \((\text{Tria} j_i R, \text{Tria} i_s S)\) is a torsion pair of \( K^b(A_P) \), on one hand \( \text{Hom}_{K^b(A_P)}(j_i R, i_s S[n]) = 0 \) for all \( n \in \mathbb{Z} \), so \( E_1 \cap E_2 = \emptyset \) by Proposition 2.6; on the other hand, \( \text{Tria}(j_i R \oplus i_s S) = K^b(A_P) \), so \( E_1 \cup E_2 = E \).

Consequently, \( e + f = 1 \). Therefore, by Lemma 3.2, we know \( \text{Tria} j_i R = \text{Tria} Ae \) and \( \text{Tria} i_s S = \text{Tria} Af \). Moreover,
\[
A \cong \begin{bmatrix} eAe & 0 \\ fAe & fAf \end{bmatrix}.
\]

by Corollary 3.3.

It is well known that \((\text{Im} i_s, \text{Im} j_i)\) also gives a torsion pair of \( K^b(A_P) \); see Lemma 2.6 in \([12]\). Moreover, since \( \text{Im} i_s = \text{Tria} Af \), for any \( Y \in \text{Im} j_i \) and \( n \in \mathbb{Z} \), we should have \( \text{Hom}_{K^b(A_P)}(Af, Y[n]) = 0 \). That is, \( H^n(Y) \) has no composition factors isomorphic to summands of \( Af/\text{rad} Af \).

Now consider the canonical triangle
\[
i_s i^! A \longrightarrow A \longrightarrow j_s j^* A \longrightarrow i_s i^! A[1],
\]
which is unique up to isomorphism. Note that \( A \cong (eAe, fAf, fAf) \) is a triangular matrix algebra. Moreover, the first term lies in \( \text{Tria} Af \), so every homology \( H^n(i_s i^! A) \) has no composition factors isomorphic to summands of \( Ae/\text{rad} Ae \). In other words, every composition factor of \( H^n(i_s i^! A) \) is a summand of \( Af/\text{rad} Af \).

We also know that every homology of the third term has no composition factors isomorphic to summands of \( Af/\text{rad} Af \). In other words, every composition factor of \( H^n(j_s j^* A) \) is a summand of \( Ae/\text{rad} Ae \). Consequently, for \( m, n \in \mathbb{Z} \), \( H^m(i_s i^! A) \) and \( H^n(j_s j^* A) \) have no common composition factors, so there is no nonzero module homomorphisms between them.

Take homologies of the above triangle we obtain the long exact sequence with finitely many nonzero terms:
\[
\ldots \rightarrow 0 \rightarrow H^{-2}(j_s j^* A) \rightarrow H^{-1}(i_s i^! A) \rightarrow 0 \rightarrow H^{-1}(j_s j^* A) \\
\rightarrow H^0(i_s i^! A) \rightarrow A \rightarrow H^0(j_s j^* A) \rightarrow H^1(i_s i^! A) \rightarrow 0 \rightarrow \ldots
\]

Since the maps \( H^n(j_s j^* A) \rightarrow H^{n+1}(i_s i^! A) \) are 0 for \( n \in \mathbb{Z} \), this forces \( H^n(j_s j^* A) = 0 = H^n(i_s i^! A) \) for all \( n \neq 0 \). Therefore, \( j_s j^* A \) is a minimal projective resolution of \( H^0(j_s j^* A) \) and \( i_s i^! A \) is a minimal projective resolution of \( H^0(i_s i^! A) \). Moreover, we also have \( H^0(j_s j^* A) \cong eAe \) and \( H^0(i_s i^! A) \cong fAf \oplus fAf \) from the above description of composition factors of these homologies. But this implies that \( fAf \oplus fAf \) has finite projective dimension since \( i_s i^! A \in K^b(A_P) \), which is impossible by the next lemma. The conclusion follows from this contradiction. \( \square \)

Lemma 3.8. Let \( E_1, E_2, e, f \) be as in the previous theorem. Then \( fAf \) as a left \( A \)-module has infinite projective dimension.

Proof. Since \( A \) is isomorphic to the following triangular matrix algebra
\[
\begin{bmatrix} eAe & 0 \\ fAe & fAf \end{bmatrix},
\]
we have the short exact sequence $0 \to fAe \to Ae \to eAe \to 0$ of $A$-modules. Therefore, $fAe$ has finite projective dimension if and only if $eAe$ has finite projective dimension. However, by Lemma 3.4, this happens if and only if $eAe$ is a projective $A$-module. But $eAe$ and $Ae$ have the same top. Therefore, this is true if and only if $Ae = eAe$. In other words, $fAe = 0$. But then $A = eAe \oplus fAf$ is not connected, which is impossible. □

However, the algebras $A$ in the above theorem in general are not derived simple. Indeed, the algebra $A$ in Example 2.11 is bounded derived simple by the above theorem, but it is not derived simple. Actually, for every triangular matrix algebra $A = (eAe, fAf, fAe)$, as we pointed out in [25], $D(A)$ always has the following nontrivial recollement

\[
D(fAf) \xrightarrow{i^*} D(A) \xrightarrow{j^*} D(eAe),
\]

where the functors are specified as in Proposition 3.6 of [25].

4. **Weakly directed algebras**

In this section we apply the results in the previous sections to a special class of algebras. As before, we only consider connected basic finite dimensional algebras over an algebraically closed field $k$.

**Definition 4.1.** A finite dimensional algebra $A$ is said to be weakly directed if there is a complete set of primitive orthogonal idempotents $E = \{e_i\}_{i=1}^n$ and a partial order $\leq$ on $E$ such that $\text{Hom}_A(Ae_j, Ae_i) \cong e_jAe_i \neq 0$ only if $e_j \geq e_i$.

Quotient algebras of finite dimensional hereditary algebras are weakly directed. Other examples include local algebras, category algebras of skeletal finite EI categories ([21]), extension algebras of standard modules of standardly stratified algebras ([22]).

Sometimes it is useful to study weakly directed algebras from a categorical viewpoint. For a weakly directed algebra $A$, define $\mathcal{A}$ to be a $k$-linear category with objects elements in $E$, and $\mathcal{A}(e_i, e_j) = e_jAe_i$. The composition of morphisms is precisely product of elements in $A$. Then $\mathcal{A}$ is a finite directed category as defined in [23, 24]. We call $\mathcal{A}$ the associated category of $A$. Define a representation of $\mathcal{A}$ to be a covariant $k$-linear functor from $\mathcal{A}$ to the category of finite dimensional vector spaces. Then the reader can check that representations of $\mathcal{A}$ coincide with finitely generated left $A$-modules. For a fixed representation $M$ of $\mathcal{A}$, its value on a object $e_i$ is defined to be $e_iM \cong \text{Hom}_A(Ae_i, M)$, which is a left $e_iAe_i$-module. If $e_iM \neq 0$, we say $M$ is supported on $e_i$. We will often refer to these definitions later. For more details, see [23, 24].

The following lemma is described in [24].

**Lemma 4.2.** Let $A$ be a weakly directed algebra. Then every simple $A$-module can be identified with a simple $eAe$-module for some $e \in E$. Moreover, for any $M \in A\text{-mod}$ and $f \in E$, the left $fAf$-module $fM$ is also an $A$-module.

**Proof.** The first statement is contained in Proposition 2.2 in [24]. To show the second one, note that by the weakly directed structure of $A$, all non-endomorphisms
in the associated category $\mathcal{A}$ form a two-sided ideal $J$ of $A$. Therefore, $\oplus_{i=1}^{n} e_i A e_i \cong A/J$ is a quotient algebra of $A$, so $fAf$ is quotient algebra of $A$ as well. The second statement follows from this observation. \hfill $\square$

By this lemma, the $A$-module $fM$ and the $fAf$-module $fM$ have the same composition factors, and hence the same Lowey length.

The proposition below gives us a practical method to check conditions in Proposition 2.4 for weakly directed algebras.

**Lemma 4.3.** Let $A$ be a weakly directed algebra. Then conditions in Proposition 2.4 hold if and only if for any $e \in E$, the socle of $eAe$ contains a simple summand $S_e$ isomorphic to $eAe/\text{rad}eAe$ such that for any $x \in eA(1-e)$ we have $S_e x = 0$.

By Lemma 4.2, the simple $A$-module $S_e$ is a simple $eAe$-module as well, so $S_e \subseteq eAe$. Therefore, the right action of $x \in eA(1-e)$ on $S_e$ is induced from the multiplication of $eAe$ by $eA(1-e)$ from the right side.

**Proof.** We first prove the if part by checking conditions in Proposition 2.4. Condition (1) is given, so it suffices to check (2) and (3). Note that the trace of $Q = A(1-e)$ is $A(1-e)Ae$, which is not supported on $e$ since $eA(1-e)Ae = 0$ by the weakly directed structure. But $S_e \subseteq eAe$ is not contained in $(1-e)Ae$, so Condition (2) in Proposition 2.4 is also true.

Let $\beta : A \rightarrow P$ be an $A$-module homomorphism, where $P \in \text{add} A(1-e)$. Since $\text{Hom}_A(Ae, A(1-e)) \cong eA(1-e)$, $\beta$ corresponds to an element $x \in \text{add} eA(1-e)$, and $\beta(eA) = Aex$. By the assumption, $S_e x = 0$. Therefore, $\beta(S_e) = 0$. In other words, $S_e$ is contained in the kernel of $\beta$, and hence Condition (2) in Proposition 2.4 follows from the second statement of Lemma 2.3.

Conversely, suppose that conditions in 2.4 hold. Then by the previous paragraph, for any $x \in eA(1-e)$, we can find a module homomorphism $\beta \in \text{Hom}_A(eAe, A(1-e))$ such that $\beta(eA) = Aex$. In particular, $S_e x = \beta(S_e) = 0$ since by (2) of Lemma 2.3 $S_e$ is in the kernel of $\beta$. This finishes the proof. \hfill $\square$

**Remark 4.4.** Since for any $e \in E$, $eA(1-e)$ is a finite dimensional vector space, we can take a finite basis $\{x_i\}_{i=1}^{n}$ of $eA(1-e)$. Moreover, we can choose a minimal set of elements $\{v_j\}_{j=1}^{m} \subseteq eAe$ which generates the socle of $eAe$. Therefore, in practice we only need to verify that $v_j x_i = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Consider the following example:

**Example 4.5.** Let $A$ be the path algebra of the following quiver with relations $\delta^2 = \theta^2 = \rho^2 = 0$, $\theta \rho = \rho \theta = 0$, $\alpha \delta = \beta \delta = 0$, and $\theta \alpha = \rho \beta = 0$.

\[ \xymatrix{ x \ar[rr]^\alpha \ar[rd]_\beta & & y \ar[ld] \ar[dl] \ar[ld] } \]

Indecomposable projective modules are:

\[ P_x = x \quad y \quad y, \quad P_y = y \quad y \]

It is routine to check that $P_x$ satisfies all conditions in Proposition 2.4. The indecomposable projective module $P_y$ satisfies the first two conditions, but fails the
third one. Indeed, there is a short exact sequence

$$0 \longrightarrow P_y \longrightarrow P_x \longrightarrow M \longrightarrow 0,$$

where $M = \frac{x}{y}$. This is because the socle of $P_y$ is spanned by $\rho$ and $\theta$, but $\theta \beta \neq 0$ and $\rho \alpha \neq 0$.

The following proposition tells us that conditions in Proposition 2.4 are preserved by ideals or coideals of the poset $(E, \leq)$.

**Proposition 4.6.** Let $e_0 \in E$ be a maximal (or a minimal) idempotent, i.e., for any $e_0 \neq e \in E$ we have $eAe_0 = 0$ (or $e_0Ae = 0$). Then the algebra $B = (1-e_0)A(1-e_0)$ also satisfies the conditions in Proposition 2.4.

**Proof.** Assume that $e_0$ is maximal. By the weakly directed structure of $A$, we know that $A$ is actually the following triangular matrix algebra

$$A = \begin{bmatrix} B & 0 \\ e_0A(1-e_0) & e_0Ae_0 \end{bmatrix}.$$

In particular, for every $e_0 \neq e \in E$, the socle of $Be = Ae/e_0Ae$ still has a simple summand isomorphic to $eBe/\text{rad} eBe$ because $eBe = eAe$. Moreover, since $1_B = 1 - e_0$, then

$$eB(1_B - e) = e((1 - e_0)A(1 - e_0)(1 - e - e_0)) = eA(1 - e - e_0) \subseteq eA(1 - e).$$

Now it is easy to see that the conditions in Proposition 2.4 also hold for $B$. The situation that $e_0$ is minimal can be proved similarly by using the facts that

$$A = \begin{bmatrix} e_0Ae_0 & 0 \\ (1 - e_2)e_0 & B \end{bmatrix},$$

and $Be = Ae$ for $e \neq e_0$. \qed

We end this section by introducing a way to construct weakly directed algebras satisfying conditions in Proposition 2.4. Let $Q = (Q_0, Q_1)$ be a finite, connected quiver without loops and oriented cycles, where $Q_0$ and $Q_1$ are the vertex set and the arrow set respectively. Now we define a $k$-linear category $\mathcal{R}$ as follows. Objects in $\mathcal{R}$ are precisely vertices in $Q_0$. To each vertex $v \in Q_0$, we assign a finite dimensional algebra $R_v$; and to each arrow $\alpha : v \to w$, we assign a finitely generated nonzero $(R_w, R_v)$-bimodule $B_\alpha$. Then we define $\mathcal{R}(v, w) = R_v$.

To define $\mathcal{R}(v, w)$ for $v \neq w$, we let

$$R_1 = \bigoplus_{\alpha \in Q_1} B_\alpha,$$

Then $R_1$ is an $(R_0, R_0)$-bimodule, so we can define $R = R_0[R_1]$, the tensor algebra generated by $R_0$ and $R_1$. Finally, we define $\mathcal{R}(v, w) = 1_vR1_w$. Composition of morphisms in $\mathcal{R}$ is precisely product in $R$. This construction has been described and applied in [21]. Note that $R$ is a finite dimensional algebra since the tensor power eventually becomes 0 by the directed structure of the quiver. We call $Q$ the underlying quiver of $\mathcal{R}$.

Now we make the following assumptions. Firstly, we assume that each $R_v$ is isomorphic to $k[t]/(t^{n_v})$ for some $n_v \geq 2$. Moreover, for each arrow $\alpha : v \to w$, $B_\alpha$ as a left $R_w$ module (resp., a right $R_v$-module) is indecomposable and not projective. This means $B_\alpha \cong k[t]/(t^l)$ as vector spaces and $0 < l < n_w$ (resp.,
Define $J$ to be the space constituted of all non-endomorphism in $R$, or equivalently,

$$J = R_1 \oplus R_1 \otimes_{R_0} R_1 \oplus \ldots = \bigoplus_{v, w \in Q_0; v \neq w} \mathcal{R}(v, w),$$

which is a two-sided ideal of $R$. Finally, we define $A = R/I$, where $I \subseteq J^2$ is a two-sided ideal of $R$. The reader can check that algebras $A$ constructed in this way satisfy conditions in Proposition 2.4, and hence is bounded derived simple.

Here is an example explaining the above construction.

**Example 4.7.** Let $A$ be the path algebra of the following quiver with relations

1. $\delta^3 = \rho^3 = \theta^3 = 0$;
2. $\alpha \delta = \rho \alpha$, $\alpha \delta^2 = 0$;
3. $\beta \rho = \theta \beta$, $\beta \rho^2 = 0$;
4. $\beta \alpha = 0$.

This algebra can be constructed as follows. Take the quiver $x \rightarrow y \rightarrow z$ and let $A_x \cong A_y \cong A_z \cong k[t]/(t^3)$. Define $B_\alpha \cong B_\beta \cong k[t]/(t^2)$, where the bimodule action is induced by the left and right multiplication. Finally we let the composite of any two non-endomorphisms in this category be 0.

5. **Questions and remarks**

In this section let $A$ be an arbitrary finite dimensional algebra. If there are only finitely many basic tilting complexes in $D^b(A)$ up to isomorphism and degree shift, then $A$ is only derived equivalent to finitely many basic algebras up to isomorphism. Unfortunately, the converse statement is not true. For instance, let $A$ be a hereditary algebra of infinite representation type. Then there exist infinitely many pairwise nonisomorphic tilting modules. Indeed, since $A$ is of infinite representation type, there are infinitely many indecomposable modules in a preprojective component of the Auslander-Reiten quiver of $A$. Take an arbitrary indecomposable $A$-module $M$ from this component. It is well known that $M$ is a partial tilting module. By Bongartz’s lemma, we can always complete $M$ to a basic tilting module. Since each basic tilting module only has finitely many indecomposable summands, we conclude that there are infinitely many pairwise nonisomorphic tilting modules. However, it is not guaranteed that these basic tilting modules will produce infinitely many pairwise nonisomorphic basic algebras, as shown in the following example.

**Example 5.1.** Let $A$ be the path algebra of the Kroneck quiver with two vertices $x$ and $y$, and two arrows from $x$ to $y$. Let $\Gamma$ be a basic algebra derived equivalent to $A$. Since the number of isomorphism classes of simple modules is invariant under derived equivalence, we know that $\Gamma$ has two simple modules up to isomorphism. Moreover, $\Gamma$ is piecewise hereditary, so it is directed. Therefore, $\Gamma$ is isomorphic to the path algebra of a directed quiver with two vertices $u$ and $v$, and $n$ arrows from $u$ to $v$. Note that the characteristic polynomial of the Coxeter transformation of $A$ and $\Gamma$ must be the same; see [20]. This happens if and only if $n = 2$ by computation. Therefore, $A$ is only derived equivalent to itself up to Morita equivalence. But $A$ is hereditary and is of tame representation type, so it has infinitely many pairwise nonisomorphic basic tilting modules.
The following result would not be surprising to the reader at all, though we do not find an explicit statement in literature.

**Proposition 5.2.** Any connected hereditary algebra of finite representation type is only derived equivalent to finitely many basic algebras up to isomorphism.

**Proof.** Let $T$ be a basic tilting complex in $D^b(A)$ where $A$ is connected and hereditary and has finite representation type. Note that the length of every indecomposable exceptional object in $K^b(_A P)$ is bounded by 2. Therefore, the length of $T$ is bounded by $2n$, where $n$ is the number of isomorphism classes of simple $A$-modules. But it is clear that $T$ corresponds to sequence

$$T \cong M_1[r_1] \oplus M_2[r_2] \oplus \ldots \oplus M_n[r_n], \quad r_1 \leq r_2 \leq \ldots \leq r_n,$$

where all $M_i$ are indecomposable $A$-modules. Therefore, $r_n - r_1 \leq 2n$. Since $A$ is of finite representation type, the number of such sequences of length at most $2n$ is finite. That is, there are only finitely many basic tilting complexes up to degree shift and isomorphism. The conclusion follows clearly. □

In Section 2 we proved that the lengths of all basic tilting complexes must be bounded if the lengths of all indecomposable compact exceptional objects are bounded. We wonder whether the converse statement is true. This is not obvious since Bongartz’s lemma is not true in derived categories; see [30]. Therefore, given an indecomposable compact exceptional object $T$ in $D^b(A)$, people do not know whether there must exist another compact exceptional object $T' \in D^b(A)$ such that $T \oplus T'$ is a basic tilting complex. It is also not clear to the author whether there exists an algebra $A$ for which the lengths of compact exceptional objects are bounded, but there are infinitely many basic tilting objects in $D^b(A)$ up to isomorphism and degree shifts, or even more stronger, $A$ is derived equivalent to infinitely many basic algebras up to isomorphism. It would be interesting to describe some concrete examples, or show that it is not possible.

**References**

[1] H. Abe and M. Hoshino, *On derived equivalences for selfinjective algebras*, Comm. Algebra 34 (2006), 4441-4452.

[2] S. Al-Nofayee and J. Rickard, *Rigidity of tilting complexes and derived equivalence for self-injective algebras*, preprint, available at arXiv:1311.0504.

[3] L. Angeleri Hügel, S. Koenig, and Q. Liu, *Recollements and tilting objects*, J. Pure App. Algebra 215 (2011), 420-438.

[4] L. Angeleri Hügel, S. Koenig, and Q. Liu, *Jordan-Hölder theorems for derived module categories of piecewise hereditary algebras*, J. Algebra 352 (2012), 361-381.

[5] L. Angeleri Hügel, S. Koenig, and Q. Liu, *On the uniqueness of stratifications of derived module categories*, J. Algebra 359 (2012), 120-137.

[6] L. Angeleri Hügel, S. Koenig, Q. Liu, and Y. Dong, *Derived simple algebras and restrictions of recollements of derived module categories*, preprint, available at arXiv:1310.3479.

[7] H. Asashiba, *The derived equivalence classification of representation-finite selfinjective algebras*, J. Algebra 214 (1999), 182-221.

[8] A. Beligiannis and I. Reiten, *Homological and homotopic aspects of torsion theories*, Mem. Amer. Math. Soc. 188 (2007), 1-207.

[9] J. Białkowski, T. Holm, and A. Skowroński, *Derived equivalences for tame weakly symmetric algebras having only periodic modules*, J. Algebra 269 (2003), 652-668.

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4Actually, $r_{i+1} - r_i \leq 1$ for $1 \leq i \leq n - 1$ since $\text{Ext}_A^2(\cdot, \cdot) = 0$. Otherwise, as did in the proof of the previous proposition, we can show that $A$ is derived equivalent to an algebra which is not connected. This is impossible.
[10] R. Bocian, T. Holm, and A. Skowroński, Derived equivalence classification of one-parametric self-injective algebras, J. Pure Appl. Algebra 207 (2006), 491-536.

[11] T. Brüstle, Derived-tame tree algebras, Compositio Math. 129 (2001), 301-323.

[12] H. Chen and C. Xi, Good tilting modules and recollements of derived module categories, Proc. London Math. Soc. 104 (2012), 959-996.

[13] W. Geigle and H Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991), 273-343.

[14] D. Happel, Triangulated categories in the representation theory of finite-dimensional algebras, Lond. Math. Soc. Lecture Note Series 119, Cambridge University Press, Cambridge, (1988).

[15] D. Happel and D. Zacharia, A homological characterization of piecewise hereditary algebras, Math. Z. 260 (2008), 177-185.

[16] T. Holm, Derived equivalence classification of algebras of dihedral, semidihedral, and quaternion type, J. Algebra 211 (1999), 159-205.

[17] T. Holm, Derived categories, derived equivalences and representation theory, Proceedings of the Summer School on Representation Theory of Algebras, Finite and Reductive Groups (Cluj-Napoca, 1997), "Babes-Bolyai" Univ., Cluj-Napoca (1998), 33-66.

[18] B. Huisgen-Zimmermann and M. Saorín, Geometry of chain complexes and outer automorphisms under derived equivalence, Trans. Amer. Math. Soc. 353 (2001), 4757-4777.

[19] S. König, Tilting complexes, perpendicular categories and recollements of derived module categories of rings, J. Pure App. Algebra 73 (1991), 211-232.

[20] H. Lenzing, Coxeter transformations associated with finite-dimensional algebras, Computational methods for representations of groups and algebras (Essen, 1997), 287308, Progr. Math., 173, Birkhäuser, Basel, 1999.

[21] L. Li, Algebras stratified for all linear orders, Alg. Rep. Theory 16 (2013), 1085-1108.

[22] L. Li, Extention algebras of standard modules, Comm. Algebra 41 (2013), 3445-3464.

[23] L. Li, A generalized Koszul theory and its application, Trans. Amer. Math. Soc. 366 (2014), 931-977.

[24] L. Li, Stratifications of finite directed categories and generalized APR tilting modules, to appear in Comm. Algebra, available at arXiv:1212.0896.

[25] L. Li, Triangular matrix algebras: recollements, torsion theories, and derived equivalences, preprint, available at arXiv:1311.1258.

[26] Q. Liu and D. Yang, Blocks of group algebras are derived simple, Math. Z. 272 (2012), 913-920.

[27] Q. Liu and D. Yang, Stratification of algebras with two simple modules, preprint, available at arXiv:1310.3480.

[28] P. Nicolás and M. Saorín, Parametrizing recollement data for triangulated categories, J. Algebra 322 (2009), 1220-1250.

[29] S. Pan and C. Xi, Finiteness of finitistic dimension is invariant under derived equivalences, J. Algebra 322 (2009), 21-24.

[30] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) 39 (1989), 436-456.

[31] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. (2) 43 (1991), 37-48.

[32] J. Rickard, Infinitely many algebras derived equivalent to a block, preprint, available at arXiv:1310.2403.

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