Study on Recovering the Earth’s Potential Field Based on GOCE Gradiometry

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Abstract  Given the second radial derivative $V_\varphi(P)$ on the surface $\partial S$ corresponding to the satellite altitude, by using the fictitious comress recovery method, a fictitious regular harmonic field $r^2 V_\varphi^*(P)$ and a fictitious second radial gradient field $V_\varphi(P)$ in the domain outside an inner sphere $K_i$ can be determined, which coincides with the real field $V_\varphi(P)$ in the domain outside the Earth. $V_\varphi^*(P)$ could be further expressed as a uniformly convergent expansion series in the domain outside the inner sphere, because $r^2 V_\varphi^*(P)$ could be expressed as a uniformly convergent spherical harmonic expansion series due to its regularity and harmony in that domain. In another aspect, the fictitious field $V_\varphi^*(P)$ defined in the domain outside the inner sphere, which coincides with the real field $V_\varphi(P)$ in the domain outside the Earth, could be also expressed as a spherical harmonic expansion series. Then, the harmonic coefficients contained in the series expressing $V_\varphi^*(P)$ can be determined, and consequently the real field $V_\varphi(P)$ is recovered. Preliminary simulation calculations show that the second radial gradient field $V_\varphi(P)$ could be recovered based only on the second radial derivative $V_\varphi(P)|_{\text{satellite boundary}}$ given on the satellite boundary. Concerning the final recovery of the potential field $V_\varphi(P)$ based only on the boundary value $V_\varphi(P)|_{\text{boundary}}$, the simulation tests are still in process.

Keywords  GOCE gradiometry; second radial gradients; second radial gradient field recovery; potential field recovery

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Introduction

Several decades ago, geodesists concentrated on determining the Earth’s external gravity field based on the data given on the Earth’s surface $\partial\Omega$, where $\Omega$ denotes the domain occupied by the Earth. The problem is that one cannot obtain a complete or dense enough data coverage on the Earth’s surface, especially in the mountain areas and on the vast ocean surfaces.

Satellite missions provide an opportunity to supply complete or sufficient data coverage on a satellite surface that encloses the Earth. From satellite perturbation to satellite gradiometry, there are three development stages.

The first stage is based on the satellite orbit perturbation to determine the gravity field, referred to as the perturbation technique. With this technique, it is very difficult to get a gravity field model with high degree; because both the Earth’s gravitational field and the various non-Earth fields (such as radiation

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pressure, atmospheric drag, etc) have contributions to the orbit perturbations. It is almost impossible to efficiently separate one from another. Because of this, with perturbation technique, the gravity field model could be established with a not-so-high degree, e.g. only about \( N = 70 \).

The second stage is due to the on-board gravimetry, e.g. the launched CHAMP mission\(^1\) and GRACE mission\(^2\), referred to as the gravimetry technique, which can greatly improve the situation, because the non-conservative forces could be sensed by the on-board accelerometer. Based on the known models (such as the models for the Earth and ocean tides) as well as the precise GPS surveying, the purer Earth’s gravitational information could be provided. In this case, the gravity field model could be determined with quite a high degree, e.g. about \( N = 200 \) or even higher.

The third stage is due to the on-board gradiometry, e.g. the will-be-launched GOCE mission\(^3, 4\), which will further improve the situation, because this system could be used to separate gravitation from inertia almost completely. In this case, much purer gravitational information could be obtained. Consequently, it is expected that the gravity field could be more precisely determined with high accuracy and resolution; for instance, the degree \( N \) might be larger than 360. The actual results should be waited until the end of 2008 after GOCE is launched.

In this paper, we formulate a method for recovering the gravity field based only on the second radial derivatives \( V_{rr} \), which is supposed to be obtained on a “simply closed surface” defined by the GOCE satellite altitude, simply referred to as the satellite surface. Simulation calculation results are provided.

1 Recovery of the second radial gradient field

 Suppose the GOCE satellite has a polar or near-polar orbit, then the second gradients \( V_{rr} \) (five dependent components) of the Earth’s gravitational potential are measured on the GOCE satellite surface, which encloses the Earth. That means the boundary values \( V_{rr} \mid_{\partial S} \) on the satellite surface \( \partial S \) are given. Then, one can obtain the second radial gradients \( V_{rr} \) on the boundary \( \partial S \), i.e. the boundary values \( V_{rr} \mid_{\partial S} \) are known.

Since the Earth’s external gravitational potential field \( V(P) \) is a regular harmonic function in the domain \( \overline{\Omega} \), the domain outside the Earth, it is easy to confirm that the following function

\[
u(P) = rrV_{rr}(P), \ P \in \overline{\Omega}
\]

is regular and harmonic, with the known boundary value

\[
u(P) \mid_{\partial S} = [rrV_{rr}(P)]_{\partial S}
\]

due to the fact that both the satellite boundary and the boundary value \( V_{rr} \mid_{\partial S} \) are known. Then, based on the fictitious compress recovery method\(^5, 6\), it could be determined a fictitious regular harmonic field \( u'(P)(P \in \overline{K}_i) \), where \( \overline{K}_i \) denotes the domain outside the previously chosen inner sphere \( K_i \) with the radius \( R_i \), and the fictitious field \( u'(P)(P \in \overline{K}_i) \) coincides with the real field \( u(P) \) in the domain \( \overline{\Omega} \), i.e. it holds that

\[
u'(P) = u(P) = [rrV_{rr}(P)], \ P \in \overline{\Omega}
\]

Hence, the second radial gradient field \( V_{rr}(P \in \overline{\Omega}) \) is determined by Eq.(4)

\[
V_{rr}(P) = \frac{1}{r^2}u'(P), \ P \in \overline{\Omega}
\]

2 Recovery of the potential field

Since \( u'(P)(P \in \overline{K}_i) \) is regular and harmonic, it could be expressed as a uniformly convergent spherical harmonic series\(^7\)

\[
u'(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\overline{R}_i}{r} (a_{mn} \cos \lambda + b_{mn} \sin \lambda) P_{nm}(\cos \theta), P \in \overline{K}_i
\]

where \( P_{nm}(\cos \theta) \) are the associated Legendre functions, the harmonic coefficients \( a_{mn} \) and \( b_{mn} \) can be determined by integrating on the surface of the inner sphere \( K_i \) based on the boundary value \( u_{\partial K_i} = u'(R_i, \theta, \lambda) \mid_{\partial K_i} \)\(^7, 8\).

In another aspect, the potential field \( V(P)(P \in \overline{\Omega}) \) could be also expressed as a uniformly convergent spherical harmonic series at least in the domain \( \overline{K}_g \), the domain outside Brillouin sphere \( K_g \) with radius \( R_g \)\(^9\), written as
$V(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{R}{r} \right)^n (a_{nm} \cos \theta + b_{nm} \sin \theta) P_{nm}(\cos \theta), P \in \mathbb{R}_s$  \hspace{1cm} (6)

In fact, based on the fictitious compress recovery method\[5\], Eq.(6) holds also in the whole domain outside the Earth\[6\]. Hence, Eq.(6) can be equivalently written as:

$V(P) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{a}{r} \right)^n (a_{nm} \cos \theta + b_{nm} \sin \theta) P_{nm}(\cos \theta), P \in \mathbb{R}_s$  \hspace{1cm} (7)

where $a$ is the Earth’s equatorial average radius, the harmonic coefficients $a_{nm}$ and $b_{nm}$ can be determined as follows.

The second radial derivative of Eq.(7) reads

$V_{rr}(P) = \frac{GM}{r^3} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \frac{a}{r} \right)^n \mu_n a_{nm}$  \hspace{1cm} (8)

where $\mu_n \equiv (n+1)(n+2)$  \hspace{1cm} (9)

Substituting Eqs.(5) and (8) into Eq.(3), we get immediately the following conclusion

$\left( \frac{R}{r} \right)^n a_{nm} = \left( \frac{a}{r} \right)^n \mu_n a_{nm}$  \hspace{1cm} (10)

$\left( \frac{R}{r} \right)^n b_{nm} = \left( \frac{a}{r} \right)^n \mu_n b_{nm}$

or, substituting Eq. (9) into Eq. (10), we get

$a_{nm} = \frac{1}{(n+1)(n+2)} \left( \frac{R}{a} \right)^n a_{nm}$  \hspace{1cm} (11)

$b_{nm} = \frac{1}{(n+1)(n+2)} \left( \frac{R}{a} \right)^n b_{nm}$

Hence, by substituting Eq.(11) into Eq.(7), the real field $V(P)(P \in \Omega)$ is recovered, based only on the given boundary values $V_{rr}(P)|_{S}$ on the satellite boundary $\partial S$.

3 Demonstration and experimental simulation

In order to confirm the validity and effectiveness of the formulations proposed in sections 2 and 3, two tests are presented. One is made by a uniform sphere model, and the other is made by a simplified EGM96 model with degree $N=12$. In both cases, the following conditions are assumed: (1) a circular polar orbit (with a revolution period of 1.5 hours) is fixed in a space-fixed system; (2) the gradiometry observations on the satellite orbit (satellite boundary) are created based on the uniform sphere model or the simplified EGM96 model; (3) the second radial gradient field $V_{rr}(P)$ and the gravitational potential field $V(P)$ are recovered by using only the observations $V_{rr}$ on the satellite boundary.

3.1 Demonstration with a uniform sphere model

The gravitational potential of a uniform sphere with radius $a$(e.g. $a = 6371$ km) is expressed as

$V(P) = \frac{GM}{r}$, $P \in \Omega$  \hspace{1cm} (12)

The second radial derivative of the potential field $V(P)$ in $\Omega$ reads

$V_{rr}(P) = \frac{2GM}{r^3}$, $P \in \Omega$  \hspace{1cm} (13)

which has the boundary value $V_{rr}(P)|_{S}$ on the satellite surface $\partial S$:

$V_{rr}(P) = \frac{2GM}{r^3}$, $P \in \partial S$  \hspace{1cm} (14)

and it is supposed that the boundary value $V_{rr}(P)|_{S}$ is measured by gradiometry technique.

Now, suppose we only know the boundary value $V_{rr}(P)|_{S}$, and the aim is to recovery both the second radial gradient field $V_{rr}(P)$ and the potential field $V(P)(P \in \Omega)$. For simplicity and without loss of the generality, we suppose the satellite surface $\partial S$ is a spherical surface. Since

$\nabla^2 V(P) = \frac{2GM}{r}$

is a regular harmonic function in $\Omega$, it is easy to demonstrate that the fictitious solution $V'(P)(P \in \Omega)$ coincides exactly with the real field $u(P)$ in the domain outside the “Earth”\[5\]. Consequently, in the present case, it has been demonstrated that the second radial gradient field $V_{rr}(P)$ can be recovered based only on the boundary value $V_{rr}(P)|_{S}$.

Further, since $V'(P)$ is a regular harmonic function in $\Omega$, it can be expanded into a uniformly convergent spherical harmonic series\[7\], which could be
expressed as:
\[
u^*(P) = \frac{GM}{r} a_{00} = 2 \frac{GM}{r}, \quad P \in \mathcal{K},
\]

because \( u^*(P) \) has a constant boundary value \( u^*(P)|_{\mathcal{K}} = \frac{2GM}{r} \) on \( \partial \mathcal{K} \), other coefficients \( a_{nm}^* \) and \( b_{nm}^* \) (except \( a_{00}^* = 2 \)) must be equal to zero, i.e.
\[
a_{00}^* = 2, \quad a_{nm}^* = 0 \quad (n \neq 0),
\]
\[
b_{nm}^* = 0 \quad (n \neq 0)
\]

Substituting the above results into Eq.(11), we have
\[
a_{00} = \frac{1}{2} a_{00}^* = 1, \quad a_{nm} = 0 \quad (n \neq 0),
\]
\[
b_{nm} = 0 \quad (n \neq 0)
\]

Consequently, by substituting Eq.(18) into Eq.(7), the recovered field is
\[
V(P) = \frac{GM}{r} a_{00} P_{00}(\cos \theta) = \frac{GM}{r}, \quad P \in \mathcal{O}
\]

which is exactly the previously provided real field. Hence, the potential field \( V(P) \) is recovered.

### 3.2 Simulation test with a simplified EGM96 model

As a preliminary experimental test, based on EGM96 model with degree \( N = 360 \) \[^{[10]}\], we truncate it into a simplified model with degree \( N = 12 \), expressed as
\[
V'(P) = \frac{GM}{r} \sum_{n=0}^{12} \sum_{m=0}^{n} \frac{(-1)^n}{r^n} (a_{nm} \cos \lambda + b_{nm} \sin \lambda) P_m(\cos \theta), \quad P \in \mathcal{O}
\]

where the coefficients \( a_{nm} \) and \( b_{nm} \) are given\(^{[10]}\). Setting a “normal” potential field \( U(P) \):
\[
U(P) = \frac{GM}{r} \sum_{n=0}^{12} \sum_{m=0}^{n} \frac{(-1)^n}{r^n} (a_{nm} \cos \lambda + b_{nm} \sin \lambda) P_m(\cos \theta), \quad P \in \mathcal{O}
\]

we have a disturbing potential field
\[
V(P) = \frac{GM}{r} \sum_{n=4}^{12} \sum_{m=0}^{n} \frac{(-1)^n}{r^n} (a_{nm} \cos \lambda + b_{nm} \sin \lambda) P_m(\cos \theta), \quad P \in \mathcal{O}
\]

It is noted that using \( V(P) \) instead of \( T(P) \) to denote the disturbing potential is only for the convenience in the present case.

Based on Eq.(22), the second radial gradient field \( V_{rr}(P) \) could be expressed as
\[
V_{rr}(P) = \frac{GM}{r^3} \sum_{n=4}^{12} \sum_{m=0}^{n} \frac{(-1)^n}{r^n} \mu_n (a_{nm} \cos \lambda + b_{nm} \sin \lambda) P_m(\cos \theta), \quad P \in \mathcal{O}
\]

where the factor \( \mu_n \) is given by Eq.(9).

Now, it is granted that the real potential field \( V(P) \) and the second radial gradient field \( V_{rr}(P) \) are given by Eq.(22) and Eq.(23), respectively. The aim is to recover them based only on the boundary value \( V_{rr}(P)|_{\mathcal{K}} \) on the spherical satellite surface \( \partial \mathcal{K} \) with radius \( R_s = 6600 \) km. The boundary value \( V_{rr}(P)|_{\mathcal{K}} \) is calculated from Eq.(23). In the sequel, the experimental calculations will show that the second radial gradient field \( V_{rr}(P) \) could be recovered based only on the “observations” \( V_{rr}|_{\mathcal{K}} \).

By using discrete approach and dividing the spherical surface into \( 1^\circ \times 1^\circ \) grids, there are \( 64800 \) point values \( V_{rr}|_{\mathcal{K}} \) on the boundary \( \mathcal{K} \), which are given by Eq.(23), and consequently the boundary values \( u(P)|_{\mathcal{K}} = rrV_{rr}|_{\mathcal{K}} \) are known. Then, based on the fictitious compress recovery method, the fictitious boundary values \( [rrV_{rr}(P)]_{\mathcal{K}} \) are determined, on the basis of which the fictitious field \( [rrV_{rr}(P)] \) is determined, i.e. the value \( [rrV_{rr}(P)] \) at any point in \( \mathcal{K} \) can be determined. In theory, the fictitious field \( [rrV_{rr}(P)] \) coincides with the real field \( rrV_{rr}(P) \) in the domain outside the Earth. Hence, the second fictitious radial gradient field \( V_{rr}(P) \) is determined, which in theory coincides with the real field \( V_{rr}(P) \) in the domain outside the Earth. In our calculations, the radius of the inner sphere is \( 6000 \) km, and the iterative times is 35.

Fig.1 shows the theoretical values of the second radial gradient \( V_{rr}(P) \) on the satellite boundary \( \partial \mathcal{K} \), calculated from Eq.(23). On the basis of these initial values and by using the fictitious compress recovery method, the fictitious field \( V_{rr}(P) \) is determined. It is noted that, in practical calculations, at the beginning the fictitious field \( [rrV_{rr}(P)] \) is determined based on the boundary value \( rrV_{rr}(P) \) on the satellite boundary \( \partial \mathcal{K} \), and then \( V_{rr}(P) \) is determined, because \( rrV_{rr}(P) \) but \( V_{rr}(P) \) is a regular harmonic function in \( \mathcal{O} \).

The differences (residuals) between the real field \( V_{rr}(P) \) and the fictitious field \( V_{rr}(P) \) on the satellite boundary are calculated, shown in
The statistical analysis about the residuals on the satellite surface $\partial K_S$ is summarized in Table 1. From Table 1 it can be concluded that the recovered field $V'_r(P)(P \in K)$ coincides with the real field $V'_r(P)(P \in \Omega)$ on the satellite boundary $\partial K_S$ under the (RMS) accuracy level $0.1 \times 10^{-14}$ $\text{s}^{-2}$, which is equivalent to the height variation 4 cm.

**Table 1  Analysis of the residuals on $\partial S$**

| Stat. Anal. | Max  | Min  | Mean | RMS  |
|-------------|------|------|------|------|
| Results     | 0.96 | -0.78| 0.001| 0.10 |

Fig.3 shows the theoretical values of the second radial gradient $V'_r(P)$ on the Earth’s surface $\partial \Omega$, calculated from Eq.(23). The differences (residuals) between the real field $V'_r(P)(P \in \Omega)$ and the fictitious field $V'_r(P)(P \in K)$ on the Earth’s surface $\partial \Omega$ are calculated, which is shown by Fig.4.

The statistical analysis about the residuals on the Earth’s surface $\partial \Omega$ is summarized in Table 2. From Table 2 it can be concluded that the recovered field $V'_r(P)(P \in K)$ coincides with the real field $V'_r(P)(P \in \Omega)$ on the Earth’s surface $\partial \Omega$ under the (RMS) accuracy level $0.5 \times 10^{-14}$ $\text{s}^{-2}$, which is equivalent to the height variation of 20 cm. This accuracy is not good enough due to the following two reasons: (1) the grid is not small enough; (2) the iterative times is not sufficient enough.

**Table 2  Analysis of the residuals on $\partial \Omega$**

| Stat. Anal. | Max  | Min  | Mean | RMS  |
|-------------|------|------|------|------|
| Results     | 0.96 | -0.78| 0.001| 0.10 |

Concerning the recovery of the potential field $V(P)$, the simulation calculations are still in process.

**4 Conclusion**

In theory, given only the second-order radial derivatives of the gravitational potential on the satellite surface, based on the fictitious compress recovery method the Earth’s external second radial gradient field $V'_r(P)$ as well as the potential field $V(P)$ can be recovered. This also solves the “downward continuation” problem\[11\]. With uniform spherical model this conclusion is confirmed. By using a simplified EGM96 model with degree $N=12$, preliminary
simulation experiments show that the second radial gradient field $V_r(P)$ can be recovered based only on the boundary value $V_r(P)|_{ss}$ given on the satellite surface. The simulation tests about the recovery of the potential field $V(P)$ based only on the boundary value $V_r(P)|_{ss}$ are still in process. In another aspect, a more complicated model has been taken, and at the same time a more dedicated grid program, e.g. $15'\times15'$, will be made.

Similar to recovering both the second radial gradient field $V_r(P)$ and the potential field $V(P)$ based only on the second-order radial derivatives of the potential given on the satellite surface, the conclusion can be also made that both the first radial gradient field $V_r(P)$ and the potential field $V(P)$ can be recovered based only on the first-order radial derivative of the potential provided on the satellite surface.

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