Abstract

A Steiner structure $S = S_q[t,k,n]$ is a set of $k$-dimensional subspaces of $\mathbb{F}_q^n$ such that each $t$-dimensional subspace of $\mathbb{F}_q^n$ is contained in exactly one subspace of $S$. Steiner structures are the $q$-analogs of Steiner systems; they are presently known to exist only for $t = 1$, $t = k$, and for $k = n$. The existence of nontrivial $q$-analogs of Steiner systems has occupied mathematicians for over three decades. In fact, it was conjectured that they do not exist. In this paper, we show that nontrivial Steiner structures do exist. First, we describe a general method which may be used to produce Steiner structures. The method uses two mappings in a finite field: the Frobenius map and the cyclic shift map. These maps are applied to codes in the Grassmannian, in order to form an automorphism group of the Steiner structure. Using this method, assisted by an exact-cover computer search, we explicitly generate a Steiner structure $S_2[2,3,13]$. We conjecture that many other Steiner structures, with different parameters, exist.

Keywords: cyclic shifts, cyclotomic cosets, Frobenius map, Grassmannian scheme, $q$-analogs, Steiner structures.
1. Introduction

The Grassmannian $G_q(n, k)$ is the set of all $k$-dimensional subspaces of an $n$-dimensional subspace over the finite field $\mathbb{F}_q$. A code in $G_q(n, k)$ is a subset of $G_q(n, k)$. There has been lot of interest in these codes in the last five years due to their application in network coding [8]. Our motivation for this work also came from this application in network coding.

This paper is devoted to the existence of nontrivial Steiner structures, known also as $q$-analog of Steiner systems. A Steiner structure $S_q[t, k, n]$ is a set $S$ of $k$-dimensional subspaces of $\mathbb{F}_q^n$ such that each $t$-dimensional subspace of $\mathbb{F}_q^n$ is contained in exactly one subspace of $S$. Steiner structures were considered in many papers [1, 4, 10, 12, 13], where they have other names as well. An $S_q[t, k, n]$ can be readily constructed for $t = k$ and for $k = n$. If $t = 1$ these structures are called $k$-spreads and they are known to exist if and only if $k$ divides $n$. These structures are also considered to be trivial. The first nontrivial case is a Steiner structure $S_2[2, 3, 7]$. The possible existence of this structure was considered by several authors, and some conjectured [11] that it doesn’t exist and that generally nontrivial Steiner structures do not exist.

The main result of this paper is a description of a method to search for a structured Steiner structure $S_p[2, k, n]$. The search was successful in finding a Steiner structure $S_2[2, 3, 13]$.

The rest of this paper is organized as follows. In Section 2 we define two types of mappings and state some of their properties. In Section 3 we use the two types of mappings for an attempt to construct nontrivial Steiner structures $S_p[2, k, n]$, where $p$ and $n$ are prime integers. In Section 4 we construct a Steiner structure $S_2[2, 3, 13]$. In Section 5 we discuss the possible existence of more Steiner systems with the same parameters and with different parameters. In Section 6 we discuss some more combinatorial designs which are related to the constructed Steiner structures. Conclusion and problems for further research are given in Section 7.

2. Mappings in the Grassmannian

Let $\mathbb{F}_{p^n}$ be the finite field with $p^n$ elements, where $p$ and $n$ are primes, and let $\alpha$ be a primitive element of $\mathbb{F}_{p^n}$.

The Frobenius map $Y_\ell$, $0 \leq \ell \leq n - 1$, $Y_\ell : \mathbb{F}_{p^n} \setminus \{0\} \rightarrow \mathbb{F}_{p^n} \setminus \{0\}$ is defined by $Y_\ell(x) \triangleq x^{p^\ell}$ for each $x \in \mathbb{F}_{p^n} \setminus \{0\}$.

The cyclic shift map $\Phi_j$, $0 \leq j \leq p^n - 2$, $\Phi_j : \mathbb{F}_{p^n} \setminus \{0\} \rightarrow \mathbb{F}_{p^n} \setminus \{0\}$ is defined by $\Phi_j(\alpha^i) \triangleq \alpha^{i+j}$, for each $0 \leq i \leq p^n - 2$.

When we say the Frobenius map or the cyclic shift map, without specifying a particular map, we will mean $Y_1$ or $\Phi_1$, respectively.

The two types of mappings $Y_\ell$ and $\Phi_j$ can be applied on a subset or a subspace, by applying the map on each element of the subset or subspace, respectively. Formally, given two integers $0 \leq \ell \leq n - 1$ and $0 \leq j \leq p^n - 2$,

$$Y_\ell \{x_1, x_2, \ldots, x_r\} \triangleq \{Y_\ell(x_1), Y_\ell(x_2), \ldots, Y_\ell(x_r)\},$$

$$\Phi_j \{x_1, x_2, \ldots, x_r\} \triangleq \{\Phi_j(x_1), \Phi_j(x_2), \ldots, \Phi_j(x_r)\}.$$

**Lemma 1.** The mappings $Y_\ell$ and $\Phi_j$ are invertible.
Proof. Clearly, $Y_{\ell}^{-1} = Y_{n-\ell}$ and $\Phi_j^{-1} = \Phi_{2^{n-1}-j}$.

For a given integer $s \in \mathbb{Z}_{p^n-1}$, the cyclotomic coset $C_s$ is defined by

$$C_s \overset{\text{def}}{=} \{ s \cdot p^i : 0 \leq i \leq n-1 \}.$$ 

The smallest element in a cyclotomic coset is called the coset representative. Let $\rho(s)$ denote the coset representative of $C_s$, i.e. if $r$ is the coset representative for the coset of $s$, $C_s$, then $r = \rho(s)$. The following lemmas are well-known and can be easily verified.

**Lemma 2.** The size of a cyclotomic coset is either $n$ or one. There are exactly $\frac{p^n-p}{n}$ different cyclotomic cosets of size $n$.

**Lemma 3.** When applied on $\mathbb{F}_{p^n} \setminus \{0\}$ the Frobenius mappings forms an equivalence relation on $\mathbb{F}_{p^n} \setminus \{0\}$, where an equivalence class contains the powers of $\alpha$ which are exactly the elements of one cyclotomic cosets of $\mathbb{Z}_{p^n-1}$.

**Lemma 4.** The finite field $\mathbb{F}_{p^n}$ and the vector space $\mathbb{F}_p^n$ are isomorphic.

In view of Lemma 4 we can apply the Frobenius mappings and the cyclic shifts mappings on $\mathbb{F}_p^n$ exactly as they are applies on $\mathbb{F}_{p^n}$. If $h: \mathbb{F}_{p^n} \to \mathbb{F}_p$ is the isomorphism from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$ such that $y = h(x)$ for $y \in \mathbb{F}_p^n$ and $x \in \mathbb{F}_{p^n}$ then

$$Y_{\ell}(y) \overset{\text{def}}{=} h(Y_{\ell}(x)) \text{ and } \Phi_j(y) \overset{\text{def}}{=} h(\Phi_j(x)),$$

for every $0 \leq \ell \leq n-1$ and $0 \leq j \leq p^n-2$.

**Lemma 5.** Let $X$ and $Y$ be two $k$-dimensional subspaces of $\mathbb{F}_p^n$ such that there exist two integers, $\ell_1, 0 \leq \ell_1 \leq n-1$, and $j_1, 0 \leq j_1 \leq p^n-1$, such that $Y = \Phi_{j_1}(Y_{\ell_1}(X))$. Then there exist two integers, $\ell_2, 0 \leq \ell_2 \leq n-1$, and $j_2, 0 \leq j_2 \leq p^n-1$, such that $Y = Y_{\ell_2}(\Phi_{j_2}(X))$.

**Proof.** Let $X$ and $Y$ be two $k$-dimensional subspaces of $\mathbb{F}_p^n$ such that there exist two integers, $\ell_1, 0 \leq \ell_1 \leq n-1$, and $j_1, 0 \leq j_1 \leq p^n-1$, such that $Y = \Phi_{j_1}(Y_{\ell_1}(X))$. Then

$$Y = \alpha^{j_1}X^{p^{\ell_1}} \implies Y^{p^n-\ell_1} = (\alpha^{j_1}X^{p^{\ell_1}})^{p^n-1} \implies \alpha^{p^n-1-j_1}p^{n-\ell_1}Y^{p^n-\ell_1} = X,$$

and hence $X = \Phi_{p^n-1-j_1}p^{n-\ell_1}(Y^{p^n-\ell_1}(Y))$. Thus, $Y = Y_{\ell_1}(\Phi_{j_1}p^{n-\ell_1}(X))$. □

Similar manipulations as the ones in Lemma 5 lead to the following consequence.

**Corollary 1.** The combination of the Frobenius map and the cyclic shift mappings induces an equivalence relation of the set of all $k$-dimensional subspaces of $\mathbb{F}_p^n$.

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3. Steiner Structures $S_p[2,k,n]$

We suggest to construct a set $S$ of $k$-dimensional subspaces of $\mathbb{F}_p^n$, $n$ prime, which has the cyclic shift map and the Frobenius map as generators for its automorphism groups. For simplicity we will consider the case where $p = 2$, but the method is easily generalized for larger primes. The nonzero elements of the field will be represented as one cycle for this construction. Given a $k$-dimensional subspace

$$\{0, \alpha^{i_1}, \alpha^{i_2}, \ldots, \alpha^{i_{2^k-1}}\}$$

in $S$, we require that for each $0 \leq \ell \leq n - 1$ and $0 \leq j \leq 2^n - 2$,

$$\{0, \alpha^{i_12^\ell + j}, \alpha^{i_22^\ell + j}, \ldots, \alpha^{i_{2^k-1}2^\ell + j}\}$$

will be also a $k$-dimensional subspace of $S$. In other words, $X \in S$ if and only if $\Phi_j(Y_\ell(X)) \in S$, for every $0 \leq \ell \leq n - 1$ and $0 \leq j \leq p^n - 2$.

For a given $k$-dimensional subspace

$$X = \{0, \alpha^{i_1}, \alpha^{i_2}, \ldots, \alpha^{i_{2^k-1}}\}$$

of $\mathbb{F}_2^n$, let the difference set of $X$, $\Delta(X)$, be the set of integers defined by

$$\Delta(X) = \{i_r - i_s : 1 \leq r, s \leq 2^k - 1, r \neq s\}.$$

**Lemma 6.** If $X$ is a $k$-dimensional subspace of $\mathbb{F}_2^n$, where $\gcd(2^k - 1, 2^n - 1) = 1$, then the cyclic shifts of $X$ form $2^n - 1$ distinct $k$-dimensional subspaces.

A $k$-dimensional subspace $X$ of $\mathbb{F}_2^n$ will be called complete if $|\Delta(X)| = (2^k - 1)(2^k - 2)$. Two complete $k$-dimensional subspaces $X$, $Y$ of $\mathbb{F}_2^n$ will be called disjoint complete if $\Delta(X) \cap \Delta(Y) = \emptyset$. Each $k$-dimensional subspace $X$ of $\mathbb{F}_2^n$ contains $\frac{(2^k-1)(2^{k-1}-1)}{3}$ two-dimensional subspaces of $\mathbb{F}_2^n$. If $|\Delta(X)| = (2^k - 1)(2^k - 2)$ then no two of these $\frac{(2^k-1)(2^{k-1}-1)}{3}$ two-dimensional subspaces are cyclic shifts of each other. Hence, we have

**Lemma 7.** If $X$ is a $k$-dimensional subspace of $\mathbb{F}_2^n$ and $|\Delta(X)| = \frac{(2^k-1)(2^{k-1}-1)}{3}$ then the cyclic shifts of $X$ form $2^n - 1$ distinct $k$-dimensional subspaces. The $\frac{(2^k-1)(2^{k-1}-1)}{3}$ two-dimensional subspaces of these $2^n - 1$ distinct $k$-dimensional subspaces are all distinct.

The first consequence of the theory is

**Theorem 1.** If there exist $\frac{2^n-2}{(2^k-1)(2^n-2)}$ pairwise disjoint complete $k$-dimensional subspaces then there exists a Steiner structure $S_2[2,k,n]$.

A Steiner structure derived via Theorem 1 has an automorphism which consists of a cycle of length $2^n - 1$. Such a Steiner system will be called cyclic. Related codes in the Grassmannian which have automorphism of size $2^n - 1$ are called cyclic codes. Such codes were considered in [6, 9]. The search for such large codes become time consuming and less efficient and hence another tool to simplify the search (on the expense of a more structured code) should be added.
Therefore, we try to add one more automorphism group into the structure by adding the Frobenius mappings into the equation.

For a given $k$-dimensional subspace

$$X = \{0, a^{i_1}, a^{i_2}, \ldots, a^{i_{2^k-1}}\}$$

of $\mathbb{F}_2^n$ let the coset difference set of $X$, $\rho(\Delta(X))$, be the set of integers defined by

$$\rho(\Delta(X)) = \{\rho(i_r - i_s) : 1 \leq r, s \leq 2^k - 1, r \neq s\}.$$

A $k$-dimensional subspace $X$ of $\mathbb{F}_2^n$ will be called coset complete if $|\rho(\Delta(X))| = (2^k - 1)(2^k - 2)$. Two coset complete $k$-dimensional subspaces $X$, $Y$ of $\mathbb{F}_2^n$ will be called disjoint coset complete if $\rho(\Delta(X)) \cap \rho(\Delta(Y)) = \emptyset$.

**Theorem 2.** If $n$ is a prime and there exist $\frac{2^n - 2}{(2^k - 1)(2^k - 2)n}$ pairwise disjoint coset complete $k$-dimensional subspaces then there exists a Steiner structure $S_2[2, k, n]$.

A search for pairwise disjoint coset complete $k$-dimensional subspaces of $\mathbb{F}_2^n$ is done as follows. $\{0, a^{i_1}, a^{i_2}, a^{i_3}\}$ is a two-dimensional subspace if and only if $a^{i_1} + a^{i_2} + a^{i_3} = 0$. Also, $\{0, a^{i_1}, a^{i_2}, a^{i_3}\}$ is a two-dimensional subspace if and only if $\{0, a^{i_1+i}, a^{i_2+j}, a^{i_3+j}\}$ is a two-dimensional subspace for every integer $j$. Therefore, $\rho(i_2 - i_1), \rho(i_1 - i_2), \rho(i_3 - i_1), \rho(i_1 - i_3), \rho(i_3 - i_2), \rho(i_2 - i_3)$, always appear together in a coset difference set. It follows that we can partition the $\frac{2^n - 2}{n}$ cyclotomic cosets of size $n$, into $\frac{2^n - 2}{6n}$ groups and instead of $(2^k - 1)(2^k - 2)$ integers in a coset difference set we should consider only $\frac{(2^k - 1)(2^k - 1 - 1)}{3}$ such integers representing such a $k$-dimensional subspace. We form a graph $G(V, E)$ as follows. The set $V$ of vertices for $G$ are represented by $\frac{(2^k - 1)(2^k - 1 - 1)}{3}$-subsets of the set of $\frac{2^n - 2}{6n}$ elements which represents the $\frac{2^n - 2}{6n}$ groups of the cosets. Such a $\frac{(2^k - 1)(2^k - 1 - 1)}{3}$-subset $v$ represents a vertex if and only if there exists a $k$-dimensional subspace $X$ of $\mathbb{F}_2^n$ whose coset difference set $\rho(\Delta(X))$ is represented by $v$. For two vertices $v_1, v_2 \in V$ represented by $\frac{(2^k - 1)(2^k - 1 - 1)}{3}$-subsets, there is an edge $\{v_1, v_2\} \in E$ if and only if $v_1 \cap v_2 = \emptyset$. A clique with $m$ vertices in $G$ represents $m$ pairwise disjoint coset complete $k$-dimensional subspaces. A clique with $\frac{2^n - 2}{(2^k - 1)(2^k - 1)n}$ vertices represents a Steiner structure $S_2[2, k, n]$.

**4. Steiner Structure $S_2[2, 3, 13]$**

A program which generates this graph for $n = 13$ and $k = 3$ was written. For $n = 13$ we have $\frac{2^{13} - 2}{13} = 630$ cyclotomic cosets of size 13, and 105 groups. A clique of size $\frac{2^{13} - 2}{42n} = 15$ in this graph represents a Steiner structure $S_2[2, 3, 13]$. Unfortunately, it is not feasible to check even a small fraction of the subsets with 15 vertices of this graph and we found numerous number of cliques of size 14. In the conference on "Trends in Coding Theory" (Ascona, October 28, 2012 - November 2, 2012) Patric Östergård suggested to use the exact cover problem (to find 15 vertices with disjoint integers for a total of 105 integers) to find a solution. He pointed on a program in

http://www.cs.helsinki.fi/u/pkaski/libexact/ based on the backtrack algorithm and the dancing links.
data structure of Knuth [7]. We added this algorithm into our search. We formed GF\(2^{13}\) with the primitive element \(\alpha\) which is a root of the primitive polynomial \(x^{13} + x^4 + x^3 + x + 1 = 0\). The following fifteen pairwise disjoint coset complete 3-dimensional subspaces given in Table 4 form by Theorem 2 the required \(S_2[2, 3, 13]\).

5. Are there Other Steiner Structures?

The existence of the Steiner structure \(S_2[2, 3, 13]\) provides some evidence that more Steiner structures exist. But, we certainly believe that for most parameters nontrivial Steiner structures do not exist. We believe that more Steiner structures \(S_q[2, 3, n]\) exist.

**Conjecture 1.** If \(n \equiv 1 \pmod{6}\), \(n \geq 13\), is a prime then there exists a Steiner structure \(S_2[2, 3, n]\) generated by a set of \(\frac{2^n - 2}{42n}\) pairwise disjoint coset complete 3-dimensional subspaces.

For \(q = 3\) we have found that there are no Steiner structure \(S_3[2, 3, 7]\) which has the cyclic shift map and the Frobenius map as generators of its automorphism groups. We also found that a Steiner structure \(S_2[2, 4, 13]\) with these automorphisms does not exist.

The Steiner structure \(S_2[2, 3, 13]\) that we found is not the only one with these parameters. By using an invertible linear transformation, defined by a binary \(13 \times 13\) invertible matrix, on the set of 3-dimensional subspaces of \(S_2[2, 3, 13]\) another Steiner structure with the same parameters will be obtained. Changing the primitive element or substituting in a solution another primitive element instead of \(\alpha\) will also produce new Steiner structures with the same parameters. Some of these structures will have the cyclic shift map and the Frobenius map as generators for their automorphism groups. Some of them won’t.

We believe that most Steiner structures do not have a nice mathematical structure and hence it is still probable that a Steiner structure \(S_2[2, 3, 7]\) exists. But, it is almost improbable that it will be found in the near future.

6. Other Related Designs

The existence of a Steiner structure implies the existence of other combinatorial designs. Moreover, if the steiner structure is cyclic then more designs are derived.

A Steiner system \(S(t, k, n)\) is a set \(S\) of \(k\)-subsets of an \(n\)-set, say \(Z_n\), such that each \(t\)-subset of \(Z_n\) is contained in exactly one subset of \(S\). The following result was obtained in [4].

**Theorem 3.** The existence of a Steiner structure \(S_2[2, k, n]\) implies the existence of a Steiner system \(S(3, 2^k, 2^n)\).

**Corollary 2.** There exists a Steiner system \(S(3, 8, 8192)\).

An \((n, w, \lambda)\) difference family is a set of \(w\)-subsets with elements taken from an additive group \(G, |G| = n\), such that each element of \(G \setminus \{0\}\) appears exactly \(\lambda\) times as a difference between elements \(w\)-subsets. There is an extensive literature on difference families, e.g. [2] [14]. The following theorem is easily verified.

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Theorem 4. If there exists a Steiner structure $S_2[2, k, n]$ formed by \( \frac{2^n - 2}{(2^k - 1)(2^k - 2)} \) pairwise disjoint coset complete $k$-dimensional subspaces, then there exist a $(2^n - 1, 2^k - 1, 1)$ difference family, where the elements are taken from the group $\mathbb{Z}_{2^n - 1}$.

Finally, Steiner structures are also diameter perfect codes in the Grassmann scheme \([1]\). Hence, the new Steiner structure $S_2[2, k, n]$ is a diameter perfect code in the Grassmann scheme.

7. Conclusion and Future Work

We have presented a framework for a possible structure of Steiner structures. This framework involves two mappings, the Frobenius map and the cyclic shift map. These mappings form automorphism groups in the system. A Steiner structure $S_2[2, 3, 13]$ was found by computer search based on this frame. This is the first known nontrivial Steiner structure. We believe that more Steiner structures exist and for future work we propose to find more by computer search and to provide a construction for an infinite family of such structures. One intriguing question is whether such a structure can be formed from a known construction of a difference family.

Note added: This is a preliminary version; an extended version will be submitted soon. Part of this work appears in an early version \([5]\), which covers all the material presented on November 1, 2012, in the conference “Trends in Coding Theory”, Ascona, Switzerland, October 28, 2012 - November 2, 2012.

Acknowledgment

We started our computer search for a Steiner structure $S_2[2, 3, 13]$ by trying to find a clique of size 15 in the graph on the 105 cosets (cf. Section 4). However, using a backtracking algorithm, we were able to search only a small fraction of this graph. On October 30, 2012, during the conference “Trends in Coding Theory” held in Ascona, Switzerland, Patric Östergård suggested that it would be better to search for an exact cover instead, and pointed out existing algorithms for the exact cover problem. Following this suggestion, we have changed our programs and immediately found a $S_2[2, 3, 13]$ Steiner structure. We are deeply indebted to Patric Östergård; without his advice, it is not clear whether we would have been able to complete this work.

Based upon the method presented in Sections 2 and 3 of this paper, along with Östergård’s suggestion to look for an exact cover, Braun and Wasserman \([3]\) ran a computer search and independently found numerous instances of $S_2[2, 3, 13]$.

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|   | representative                                                                 |
|---|-----------------------------------------------------------------------------|
| 1 | \( \{ 0, \alpha^0, \alpha^1, \alpha^{1249}, \alpha^{5040}, \alpha^{7258}, \alpha^{7978}, \alpha^{8105} \} \) |
| 2 | \( \{ 0, \alpha^0, \alpha^7, \alpha^{1857}, \alpha^{6681}, \alpha^{7259}, \alpha^{7381}, \alpha^{7908} \} \) |
| 3 | \( \{ 0, \alpha^0, \alpha^9, \alpha^{1144}, \alpha^{1945}, \alpha^{6771}, \alpha^{7714}, \alpha^{8102} \} \) |
| 4 | \( \{ 0, \alpha^0, \alpha^{11}, \alpha^{209}, \alpha^{1941}, \alpha^{2926}, \alpha^{3565}, \alpha^{6579} \} \) |
| 5 | \( \{ 0, \alpha^0, \alpha^{12}, \alpha^{2181}, \alpha^{2519}, \alpha^{3696}, \alpha^{6673}, \alpha^{6965} \} \) |
| 6 | \( \{ 0, \alpha^0, \alpha^{13}, \alpha^{4821}, \alpha^{5178}, \alpha^{7823}, \alpha^{8052}, \alpha^{8110} \} \) |
| 7 | \( \{ 0, \alpha^0, \alpha^{17}, \alpha^{291}, \alpha^{1199}, \alpha^{5132}, \alpha^{6266}, \alpha^{8057} \} \) |
| 8 | \( \{ 0, \alpha^0, \alpha^{20}, \alpha^{1075}, \alpha^{3939}, \alpha^{3996}, \alpha^{4776}, \alpha^{7313} \} \) |
| 9 | \( \{ 0, \alpha^0, \alpha^{21}, \alpha^{2900}, \alpha^{4226}, \alpha^{4915}, \alpha^{6087}, \alpha^{8008} \} \) |
| 10 | \( \{ 0, \alpha^0, \alpha^{27}, \alpha^{1190}, \alpha^{3572}, \alpha^{4989}, \alpha^{5199}, \alpha^{6710} \} \) |
| 11 | \( \{ 0, \alpha^0, \alpha^{30}, \alpha^{141}, \alpha^{682}, \alpha^{2024}, \alpha^{6256}, \alpha^{6406} \} \) |
| 12 | \( \{ 0, \alpha^0, \alpha^{31}, \alpha^{814}, \alpha^{1161}, \alpha^{1243}, \alpha^{4434}, \alpha^{6254} \} \) |
| 13 | \( \{ 0, \alpha^0, \alpha^{37}, \alpha^{258}, \alpha^{2093}, \alpha^{4703}, \alpha^{5396}, \alpha^{6469} \} \) |
| 14 | \( \{ 0, \alpha^0, \alpha^{115}, \alpha^{949}, \alpha^{1272}, \alpha^{1580}, \alpha^{4539}, \alpha^{4873} \} \) |
| 15 | \( \{ 0, \alpha^0, \alpha^{119}, \alpha^{490}, \alpha^{5941}, \alpha^{6670}, \alpha^{6812}, \alpha^{7312} \} \) |

Table 1: The fifteen pairwise disjoint coset complete 3-dimensional subspaces from which a Steiner structure \( S_2[2, 3, 13] \) is obtained