ON THE INDUCTIVE GALOIS–MCKAY CONDITION FOR FINITE GROUPS OF LIESTYPE IN THEIR DEFINING CHARACTERISTIC

BIRTE JOHANSSON

Abstract. We verify the inductive Galois–McKay condition in the defining characteristic for the finite groups of Lie type with exceptional graph automorphisms, $B_n(2)$ $(n \geq 2)$, and the groups of Lie type with non-generic Schur multiplier.

1. Introduction

Let $G$ be a finite group, $p$ a prime and $P$ a Sylow-$p$-subgroup of $G$. The McKay conjecture claims that there exists a bijection between the sets of characters of $G$ and of the normalizer $N_G(P)$ with degree not divisible by $p$. Navarro refined this conjecture and proposed that there is a bijection between these sets such that the same number of characters are fixed under the action of certain Galois automorphisms [Nav04]. This is called the Galois–McKay or Navarro–McKay conjecture.

Navarro, Späth and Vallejo reduced the Galois–McKay conjecture to a problem about simple groups in [NSV20]. If the inductive Galois–McKay condition [NSV20, Definition 3.1] is satisfied for all simple groups, the Galois–McKay conjecture itself holds for all groups. In [Ruh20], Ruhstorfer showed that the inductive Galois–McKay condition is true for many groups of Lie type in their defining characteristic. We verify the inductive Galois–McKay condition for some of the remaining groups of Lie type in their defining characteristic.

Theorem 1.1. The inductive Galois–McKay condition [NSV20, Definition 3.1] is satisfied in their defining characteristic for the groups $B_2(2^i)$, $G_2(3^i)$, $F_4(2^i)$, $B_n(2)$ for integers $i \geq 1$, $n \geq 2$ as well as for $G_2(2)$ and the simple groups of Lie type with non-generic Schur multiplier.

As an outlook, for the groups of Lie type in their defining characteristic it remains to verify the inductive Galois–McKay condition for the Suzuki and Ree groups. We plan to address this in the future.

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2. The inductive Galois–McKay condition

2.1. Notation. First we introduce some notation, mostly following the notation used in [NSV20].

Let $p$ be a prime and $G$ a finite group. We call a character of $G$ with degree not divisible by $p$ a $p'$-character and denote the set of irreducible $p'$-characters of $G$ by $\text{Irr}_{p'}(G)$.

In the following, let $H \subseteq G := \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ be the subgroup consisting of those $\sigma \in G$ such that there exists an $n \in \mathbb{Z}$ with $\sigma(\xi) = \xi^{p^n}$ for every root of unity $\xi$ of order not divisible by $p$. We say that $\sigma$ is associated to the integer $n$. The group $H$ acts naturally on $\text{Irr}(G)$ for a finite group $G$ by applying the elements of $H$ to the character values. This action obviously preserves the character degrees. For a character $\psi$ the orbit of $\psi$ under the action of $H$ is denoted by $\psi^H$. In the same way, $\sigma \in H$ acts on...
a (projective) representation $\mathcal{P}$ over $\mathbb{Q}^{ab}$ via $\mathcal{P}^\sigma(g) := \mathcal{P}(g)^\sigma$ for all $g \in G$. Here, $\mathcal{P}(g)^\sigma$ is obtained by applying $\sigma$ to all matrix entries of $\mathcal{P}(g)$.

Let $f$ be an automorphism of $G$ and $\psi$ a character of a normal subgroup $N \trianglelefteq G$. Then we denote by $\psi^f$ the character defined by $\psi^f(x) = \psi(x^{f^{-1}})$ for all $x \in N$, using exponential notation for the image of $x$ under the automorphism $f$. For $g \in G$, we often use the element itself to denote the conjugation by $g$. Thus we have $\psi^g(x) = \psi(x^{g^{-1}}) = \psi(gxg^{-1})$ for all $x \in N$. The stabilizer of $\psi$ in $G$ under this conjugation action is denoted by $G_\psi$. In the same way, automorphisms of a group act on its (projective) representations.

The group of inner automorphisms of a group $G$ consists of the automorphisms induced by conjugation with the elements of $G$ and will be denoted by $\text{Inn}(G)$. The conjugation automorphisms of two elements are the same if and only if they are in the same coset modulo the centre $Z(G)$. We will denote the subset of inner automorphisms of $G$ that are induced by elements of a subgroup $H \leq G$ with $Z(G) \leq H$ by $\text{Inn}(G | H)$. If $\Gamma$ is a group of automorphisms of $G$, we denote by $\Gamma_{\text{out}} \subseteq \Gamma$ a subgroup isomorphic to the outer automorphism group of $G$ that is a complement to $\text{Inn}(G) \cap \Gamma$. Note that this subgroup does not exist for all groups.

For $N \trianglelefteq G$ and $\psi$ a $G$-invariant character of $N$ (i.e. $\psi^g = \psi$ for all $g \in G$), let $\mathcal{P}$ be a projective representation of $G$ such that

- its restriction to $N$ affords $\psi$,
- $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ and $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ for all $n \in N$ and $g \in G$.

Then we say that $\mathcal{P}$ is associated to $\psi$. We know by [NSV20, Corollary 1.2] that we can always find such a projective representation with matrix entries in $\mathbb{Q}^{ab}$.

For an arbitrary $\psi \in \text{Irr}(N)$, let $\mathcal{P}$ be a projective representation of $G_\psi$ over $\mathbb{Q}^{ab}$ associated to $\psi$. Then for $(g, \sigma) \in G \times \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ with $\psi^{-\sigma} := (\psi^g)^\sigma = \psi$, let $\mu_{g\sigma} : G_{\psi} \to \mathbb{C}^\times$ be the unique function constant on cosets of $N$ with $\mu_{g\sigma}(1) = 1$ such that

$$\mathcal{P}^{\sigma} := (\mathcal{P}^g)^\sigma \sim \mu_{g\sigma} \mathcal{P}$$

where $\sim$ denotes similarity between the projective representations [NSV20, Lemma 1.4].

### 2.2. Inductive Galois–McKay condition.

The inductive Galois–McKay condition from [NSV20, Definition 3.1] is the following.

**Condition 2.1.** For a finite non-abelian simple group $S$ and $p$ a prime dividing $|S|$, let $G$ be a universal covering group of $S$, $R \in \text{Syl}_p(G)$ and $\Gamma := \text{Aut}(G)_R := \{ f \in \text{Aut}(G) | f(R) = R \}$. Then $S$ satisfies the inductive Galois–McKay condition for $p$ if the following holds for all $\psi \in \text{Irr}_p(G)$:

1. There exists a $\Gamma$-stable subgroup $N_G(R) \subseteq N \trianglelefteq G$ and a $\Gamma \times H$-equivariant bijection

$$\Omega : \text{Irr}_p(G) \to \text{Irr}_p(N).$$

2. There exist projective representations $\mathcal{P}$ of $G \rtimes \Gamma_\psi$ and $\mathcal{P}'$ of $N \rtimes \Gamma_{\text{Irr}(\psi)}$ with entries in $\mathbb{Q}^{ab}$ associated to $\psi$ respectively $\Omega(\psi)$ such that the associated factor sets $\alpha, \alpha'$ take roots of unity values, coincide on $(N \rtimes \Gamma_\psi) \times (N \rtimes \Gamma_\psi)$ and $\mathcal{P}(c), \mathcal{P}'(c)$ are associated with the same scalar for all $c \in C_{G \rtimes \Gamma_\psi}(N)$.

3. $\mu_{\alpha}, \mu_{\alpha'}$ agree on $N \rtimes \Gamma_\psi$ for all $a \in ((N \rtimes \Gamma_\psi) \times H)_\psi$.

We will see now that this is the same as in [NSV20, Definition 1.5], i.e. that the conditions given by [NSV20, Definition 1.5 (i), (ii)] always hold in this case.

**Lemma 2.2.** Consider the setting of Condition 2.1

(a) **Assuming Condition 2.1 (1) holds, the equations**

- $G \rtimes \Gamma_\psi = G(N \rtimes \Gamma_{\text{Irr}(\psi)})$,
- $G \cap (N \rtimes \Gamma_{\text{Irr}(\psi)}) = N$,
- $C_{G \rtimes \Gamma_\psi}(G) \subseteq N \rtimes \Gamma_{\text{Irr}(\psi)}$,
- $((N \rtimes \Gamma_{\text{Irr}(\psi)}) \times H)_\psi = ((N \rtimes \Gamma_{\text{Irr}(\psi)}) \times H)_{\text{Irr}(\psi)} = ((N \rtimes \Gamma) \times H)_\psi = ((N \rtimes \Gamma) \times H)_{\text{Irr}(\psi)}$

are always satisfied.
(b) We have

\[ C_{G \rtimes \Gamma_{\psi \Gamma}}(G) = \{ (g, \gamma_g^{-1}) \mid g \in N \} \]

with \( \gamma_g \) denoting the automorphism by conjugation with \( g \).

**Proof.** The first two equalities of (a) are obvious since \( N \leq G \). Note that (b) implies the third equation. For \( g \in G \), \( \tau \in \Gamma \) and with \( id \in \Gamma \) denoting the identity automorphism, we have

\[
\begin{align*}
(g, \tau) \in C_{G \rtimes \Gamma}(G) & \iff (g, \tau)(h, id) = (h, id)(g, \tau) \quad \text{for all } h \in G \\
& \iff (g \tau(h), \tau) = (h g, \tau) \quad \text{for all } h \in G \\
& \iff g \tau(h) = h g \quad \text{for all } h \in G \\
& \iff \tau(h) = g^{-1} h g \quad \text{for all } h \in G.
\end{align*}
\]

Since \( \text{Inn}(G \mid N) \) is the subgroup of inner automorphisms in \( \Gamma \) and acts trivially on the characters of \( G \) and \( N \), the claim

\[ C_{G \rtimes \Gamma_{\psi \Gamma}}(G) = \{ (g, \gamma_g^{-1}) \mid g \in N \} \subseteq N \rtimes \Gamma_{\psi \Gamma} \]

follows. For a \( \Gamma \times \mathcal{H} \)-equivariant bijection \( \Omega \), the fourth equation follows by equivariance and since \( N \) acts trivial on all characters of \( G \) and \( N \).

For groups \( N \leq G \), \( \Gamma \leq \text{Aut}(G)_N \), \( \psi \in \text{Irr}(G) \) and a map \( \Omega : \text{Irr}_{p'}(G) \to \text{Irr}_{p'}(N) \) satisfying the equations from Lemma 2.3(a) and Condition 2.1(2) and (3), we also write

\[ (G \rtimes \Gamma_{\psi \Gamma}, G, \psi) \preccurlyeq \{ (N \rtimes \Gamma_{\psi \Gamma}, N, \Omega(\psi)) \} \]

For groups \( N \leq G \), \( \Gamma \leq \text{Aut}(G)_N \), \( \psi \in \text{Irr}(G) \) and a map \( \Omega : \text{Irr}_{p'}(G) \to \text{Irr}_{p'}(N) \) satisfying the equations from Lemma 2.3(a) and Condition 2.1(2) and (3), we also write

\[ (G \rtimes \Gamma_{\psi \Gamma}, G, \psi) \preccurlyeq \{ (N \rtimes \Gamma_{\psi \Gamma}, N, \Omega(\psi)) \} \]

**Remark 2.3.** (a) Note that we already know that the inductive McKay condition from [IMN07, Section 10] is satisfied for groups of Lie type in their defining characteristic [Spä12, Theorem 1.1] (for a different phrasing of the inductive McKay condition more similar to Condition 2.1 see [Nav18, Definition 10.23]). Thus, if \( \mathcal{H} \) acts trivially on all occurring characters of \( p' \)-degree, we already know that there exists a \( \Gamma \)-invariant bijection from \( \text{Irr}_{p'}(G) \) to \( \text{Irr}_{p'}(N) \) satisfying (2) of Condition 2.1

(b) Assume that the projective representation \( P \) associated to some \( \psi \in \text{Irr}_{p'}(G) \) from Condition 2.1(2) is even an ordinary representation. Then every such \( \mu_a \), for \( a \in (N \rtimes \Gamma_{\psi \Gamma}) \times \mathcal{H}_\psi \) is a character of \( (G \rtimes \Gamma_{\psi \Gamma})/G \) by Gallagher’s lemma (see for example [Isa76, Corollary 6.17]). This follows from the fact that both \( P \) and \( P^a \) afford characters of \( G \rtimes \Gamma_{\psi} \) extending \( \psi \).

Let \( X \) be a representation of \( G \) affording \( \psi \). This representation can be canonically extended to a representation \( \tilde{X} \) of \( G \rtimes \text{Inn}(G \mid N) \) by setting

\[
\tilde{X}((g, \gamma_h)) = X(g)X(h)
\]

with \( \gamma_h \) the inner automorphism associated to \( h \in N \). Let \( a \in (N \rtimes \Gamma_{\psi \Gamma}) \times \mathcal{H}_\psi \), then we have \( \psi = \psi^a \). Consequently there exists some invertible matrix \( M \) such that

\[
X(g)^a = M^{-1}X(g)M
\]

for all \( g \in G \). Then

\[
\tilde{X}((g, \gamma_h))^a = X(g)^aX(h)^a = M^{-1}X(g)MM^{-1}X(h)M = M^{-1}X(g)X(h)M \sim \tilde{X}((g, \gamma_h))
\]

for all \( (g, \gamma_h) \in G \rtimes \text{Inn}(G \mid N) \). Thus, if \( P \) is an ordinary representation that extends \( \tilde{X} \), both \( P \) and \( P^a \) afford characters extending the same character of \( G \rtimes \text{Inn}(G \mid N) \). Therefore \( \mu_a \) is even a character of \( (G \rtimes \Gamma_{\psi})/(G \rtimes \text{Inn}(G \mid N)) \) in this case.

(c) It actually occurs that the characters \( \mu_a \) as in the constructions in Remark 2.3(b) are not trivial. An example for this is \( G_2(3) \) in its defining characteristic. Since the outer automorphism group of the Schur cover is cyclic, we know that all irreducible characters \( \psi \) of \( G \) extend as in (b) canonically to \( G \rtimes \text{Inn}(G \mid N) \) and also to \( G \rtimes \Gamma_{\psi} \). Computing the \( \mu_a \) for all of these extensions, we see that there is a \( \psi \in \text{Irr}_{p'}(G) \) of degree 14 that corresponds to a non-trivial \( \mu_a \) for some \( a \in (N \rtimes \Gamma_{\psi \Gamma}) \times \mathcal{H}_\psi \). For more details see the proof of Proposition 2.3.
3. $B_2(2^i)$, $G_2(3^i)$ and $F_4(2^i)$ in their defining characteristic

In this section we verify the inductive Galois–McKay condition \([24]\) for the groups $B_2(2^i)$, $G_2(3^i)$ and $F_4(2^i)$ in their defining characteristic for $i \geq 2$. The case $i = 1$ will be studied in Section \([5]\). We follow \([Mas10]\) and \([Ruh20]\) and extend the results from there. Note that the constructions there are valid for our groups, even though the exceptional graph automorphisms were not considered.

3.1. Notation. We now fix the notation we will use throughout this section. Let $G \in \{B_2(2^i), F_4(2^i), G_2(3^i)\}$ and $p$ be the defining characteristic 2 or 3, respectively. The group $G$ is simple, non-abelian, has trivial Schur multiplier and trivial centre \([MT11, Table 24.2, Remark 24.19]\).

Let $G$ be the corresponding algebraic group of type $B_2$, $F_4$, or $G_2$, respectively, defined over an algebraic closure $k$ of $\mathbb{F}_p$. Let $F$ be a Frobenius endomorphism such that $G^F = G$. We fix an $F$-stable maximal torus $T \subset G$ and an $F$-stable Borel subgroup $B \supset T$. Let $U$ be the unipotent radical of $B$. By \([MT11, Corollary 24.11]\), $U = U^F$ is a Sylow-$p$-subgroup of $G$ with normalizer $B = B^F$. We denote by $\Phi$ the root system of $G$ with respect to $T$ and by $\Phi^\vee$ the set of coroots. Let $n$ be the rank of $\Phi$ and denote the set of simple roots with respect to $B$ by $\Delta = \{\alpha_i \mid 1 \leq i \leq n\}$. Let $X(T)$ be the character group of $T$ and $Y(T)$ the group of cocharacters of $T$. For $\alpha \in \Phi$, let $U_\alpha$ be the root subgroup associated to $\alpha$ and fix an isomorphism $x_\alpha : (k, +) \to U_\alpha$.

Following \([Mas10]\), we can now define dual fundamental weights $\omega_j^\vee \in Y(T)$ such that $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$ for $1 \leq i, j \leq n$. For $t \in k^\times$ set

$$h_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t) x_\alpha(-1) x_{-\alpha}(1) x_\alpha(-1).$$

3.2. Automorphisms of $G^F$. Let $\gamma$ be the bijective morphism of $G$ induced by the exceptional automorphism $\rho$ of the Dynkin diagram. We denote by $F_\rho$ the field automorphism of $G$ such that $F_\rho(x_\alpha(\alpha)) = x_\alpha(\alpha^p)$ for all $\alpha \in \Phi$. Then we have $\gamma^2 = F_\rho$ and

$$\gamma(x_\alpha(t)) = \begin{cases} x_{\rho(\alpha)}(t) & \text{if } \alpha \text{ is long,} \\ x_{\rho(\alpha)}(t^p) & \text{if } \alpha \text{ is short,} \end{cases}$$

see e.g. \([GLS98, Theorem 1.15.4]\). Up to an inner automorphism of $G$, $F$ is of the form $F_\rho^i$. By taking a suitable conjugate $F_\rho^i$-stable torus $T$ and Borel subgroup $B$ we can assume from now on $F = F_\rho^i$. We follow the ideas in \([Ruh20]\) and imitate the considerations that were already made there for groups without exceptional graph automorphisms.

**Lemma 3.1.** The action of $\gamma$ on the dual weights is given by

$$\gamma(\omega_j^\vee(\mu)) = \begin{cases} \omega_{\rho(j)}^\vee(\mu) & \text{if } \alpha \text{ is long,} \\ \omega_{\rho(j)}^\vee(\mu^p) & \text{if } \alpha \text{ is short,} \end{cases}$$

where we use $\rho$ also to denote the permutation of the indices of the $\alpha_i$ induced by $\rho$.

**Proof.** Since the centre of $G$ is trivial, we can write

$$\omega_j^\vee : k^\times \to T, \quad \omega_j^\vee(\mu) = \prod_{k=1}^n h_{\alpha_k}(\mu^{a_{ij}})$$

with $(a_{ij})_{i,j} \in \mathbb{Q}^{n \times n}$ the inverse of the Cartan matrix of $G$ and $l$ the exponent of $X(T)/\mathbb{Z}\Phi$, i.e. $l = 1$ for $G_2$, $F_4$ and $l = 2$ for $B_2$. \([Mas10, Chapter 6]\). Thus this is just a simple computation using the inverses of the Cartan matrices as in \([Mas10, Appendix]\).

As already pointed out in Remark \([23]\), we know that the inductive McKay condition holds in defining characteristic. More precisely, we know by \([Bru09, Theorem 5]\) that there exists a $\Gamma$-invariant bijection

$$\Omega : \text{Irr}_{p'}(G) \to \text{Irr}_{p'}(B)$$
such that Condition 2.1 (2) holds. Let $\sigma \in \mathcal{H}$ be associated to the integer $f$. Then we know by [Ruh20, Remark 5.2]

$$\chi^\sigma = \chi^{F_\sigma t}$$

for all $\chi \in \text{Irr}_{F_\sigma}(G)$. The same holds for the characters in $\text{Irr}_{F_\sigma}(B)$.

### 3.3. Characters of $U^F/[U, U]^F$.

Following [Mas10, Section 8], we fix a non-trivial character $\phi_0 \in \text{Irr}(\mathbb{F}_q^*)$. We can define a character $\phi_\sigma \in \text{Irr}(U^F_{\alpha_i})$ by $\phi_\sigma(x_{\alpha_i}(a)) = \phi_0(a)$ for all $a \in \mathbb{F}_q$. By [DLM92] we know

$$U^F/[U, U]^F \cong \prod_{i=1}^n U^F_{\alpha_i}.$$  

Consequently, for $S \subseteq \{1, \ldots, n\}$ we can define

$$\phi_S := \prod_{i \in S} \phi_i,$$

where the value on the root subgroups not appearing in this product is given by the trivial character. The next lemma is [Ruh20, Lemma 6.3] but for groups of Lie type with exceptional graph automorphisms.

**Lemma 3.2.** For every $\sigma \in \mathcal{G}$ there exists a $t \in T^{(\gamma)}$ such that $\phi_S^t = \phi_S$ for all $S \subseteq \{1, \ldots, n\}$.

**Proof.** As in the proof of [Ruh20, Lemma 6.3(a)], let $b \in \mathbb{F}_p^*$ such that $\phi_S^t(a) = \phi_S(ba)$ for all $a \in \mathbb{F}_q$. It is shown there that for $s_i := \omega_{\rho(j)}(b) \in T^{F_\rho}$ with $1 \leq i \leq n$ and $t := \prod_{i=1}^n s_i \in T^{F_\rho}$ we have $\phi_S^t = \phi_S^S$. Since $b^p = b$, it follows with Lemma 3.1 that

$$\gamma(t) = \prod_{i=1}^n \gamma(s_i) = \prod_{i=1,\alpha_i \text{ long}}^{n} \omega_{\rho(j)}(b) \prod_{i=1,\alpha_i \text{ short}}^{n} \omega_{\rho(j)}(b^p) = \prod_{i=1}^{n} \omega_{\gamma}(b) = t,$$

thus $t$ is $\gamma$-invariant. \hfill \Box

### 3.4. Verification of the inductive condition.

Now we can continue in the same way as in [Ruh20, Section 6]. We do not repeat the constructions related to quasi-central automorphisms and Gelfand–Graev characters given there. For every $\sigma \in \mathcal{H}$ associated to the integer $e$, we set $x_{\sigma} := F_p^e \sigma^{-1} t$ with $t \in T^{(\gamma)}$ as in Lemma 3.2.

**Proposition 3.3.** Let $\psi \in \text{Irr}_{F_\rho}(G)$. Then there exists an extension $\tilde{\psi} \in \text{Irr}(G \rtimes (\gamma)_{\psi})$ such that $\tilde{\psi}^{x_\sigma} = \tilde{\psi}$ for every $\sigma \in \mathcal{H}$. The same is true in the local case for $\psi \in \text{Irr}_{F}(B)$ and $\tilde{\psi} \in \text{Irr}(B \rtimes (\gamma)_{\psi})$.

**Proof.** Let $F_0$ be a generator of $(\gamma)_{\psi}$. Then this follows from [Ruh20, Proposition 6.7] for $D = (\gamma)$ and trivial $\mu \in \text{Irr}(D_{\psi})$ since $t$ is $D$-invariant. In the local case the claim can be shown exactly as in [Ruh20, Proposition 6.10]. \hfill \Box

**Theorem 3.4.** Condition 2.1 is satisfied for $S \subseteq \{B_2(2^e), G_2(3^e), F_4(2^e)\}$ with $i \geq 2$ and $p$ the defining characteristic $p = 2$ or $p = 3$, respectively.

**Proof.** We use the notation of Condition 2.1. As already mentioned, the group $S$ is simple, non-abelian and has trivial Schur multiplier, thus we can consider $G = S$. We want to verify the condition for $N = B$.

The outer automorphism group of $G$ is cyclic with generator $\gamma$ [GLS98, Theorem 2.5.12]. Since $U = \prod_{\alpha \in \Phi^+} U_\alpha$ with $\Phi^+$ the set of positive roots with respect to $\Delta$ [GLS98, Theorem 1.12.7], $\gamma$ fixes $U$ by the description of its action in Section 3.2. Thus we have $\Gamma = (\gamma, \text{Inn}(G \rtimes N))$. Let $\Omega$ be a $\Gamma$-equivariant bijection from $\text{Irr}_{F_\rho}(G)$ to $\text{Irr}_{F}(B)$ as described above. Every element of $\mathcal{H}$ acts on both $\psi \in \text{Irr}_{F_\rho}(G)$ and $\Omega(\psi) \in \text{Irr}_{F}(B)$ in the same way as some element of $\Gamma$ acts on $\psi$ and $\Omega(\psi)$. Thus, $\Omega$ is also $\Gamma \times \mathcal{H}$-equivariant. Let $D = (\gamma)$. We know by Proposition 3.3 that for all $\psi \in \text{Irr}_{F_\rho}(G)$ we find an extension $\tilde{\psi} \in \text{Irr}(G \rtimes D_{\psi})$ such that $\tilde{\psi}^{x_\sigma} = \tilde{\psi}$ for all $\sigma \in \mathcal{H}$. Now every $a \in (D \times \mathcal{H})_\psi$ is of
the form \((F^e_p, \sigma^{-1})\) for some \(\sigma \in \mathcal{H}\) associated to the integer \(e\). As an element of \(T^D \subseteq T^F \subseteq B \subseteq G\), \(t\) acts trivially on \(\tilde{\psi}\) and we have
\[
\tilde{\psi}^x = \tilde{\psi}F_p^x \sigma^{-1} = \tilde{\psi}F_p^x \sigma^t = \tilde{\psi}.
\]
We can now extend \(\tilde{\psi}\) to the inner automorphisms in \(\Gamma\) as described in Remark 2.3 [1]. Then a representation affording \(\tilde{\psi}\) leads to trivial \(\mu_\psi\) for all \(a \in ((B \times \Gamma) \times \mathcal{H})_\psi\). The same can be done for the local character \(\Omega(\psi)\), thus (2) and (3) of Condition 2.1 also hold.

\[\square\]

4. \(B_n(2)\) in defining characteristic

We verify Condition 2.1 for \(S = B_n(2)\) with an integer \(n \geq 4\) and \(p = 2\). For \(n = 2\) and \(n = 3\) this will be treated separately in Proposition 5.3.

4.1. Action of Galois automorphisms in the global case. The group \(S\) is simple, non-abelian and has trivial Schur multiplier [MT11, Remark 24.19], hence we can take \(G = S\) in Condition 2.1. We want to show that the Galois automorphisms in \(\mathcal{H}\) act trivially on the characters in \(\text{Irr}_{2'}(G)\).

**Lemma 4.1.** The Galois automorphisms in \(\mathcal{H}\) act trivially on \(\text{Irr}_{2'}(G)\).

**Proof.** Let \(m\) be the exponent of \(G\) and \(\sigma \in \mathcal{H}\) with \(\sigma(\xi_m) = \xi_m^r\) for an \(m\)-th root of unity \(\xi_m\) and \(r \in \mathbb{Z}\) coprime to \(m\). We want to show that for such an \(r\) every semisimple element \(s \in G\) is conjugate to its \(r\)-th power \(s^r\). We can consider every element of \(G = B_n(2) = \text{Sp}_{2n}(2)\) as a \(2n \times 2n\)-matrix.

As in the proof of [Cab11, Proposition 2], we obtain a bijection between the classes of semisimple elements of \(G\) and the set of self-dual polynomials \(f \in \mathbb{F}_2[X]\) of degree \(2n\) by taking the characteristic polynomial. Let \(l\) be the smallest integer such that every self-dual \(f \in \mathbb{F}_2[X]\) splits into linear factors over \(\mathbb{F}_2\). Then every \(f\) is uniquely described by its set of roots over \(\mathbb{F}_2\). Since \(f\) is the characteristic polynomial of an element in a semisimple conjugacy class of \(G\), the roots of \(f\) are the eigenvalues of every element in this class. It follows that every conjugacy class of semisimple elements can be uniquely described by the eigenvalues of its elements.

Let \(s \in G\) be semisimple with eigenvalues \(\lambda_1, \ldots, \lambda_{2n} \in \mathbb{F}_2^\times\). With \(\alpha\) a generator of \(\mathbb{F}_2^\times\) and \(\xi \in \mathbb{Q}^{ab}\) a fixed primitive \((2^l - 1)\)-th root of unity, we can define a bijective group homomorphism
\[
*: \mathbb{F}_2^\times \rightarrow \langle \xi \rangle, \quad \alpha \mapsto \xi.
\]
Now \(\sigma \in \mathcal{H}\) acts on \(\langle \xi \rangle\) via \(\sigma(\xi) = \xi^{2^l}\) for some \(f \in \mathbb{Z}\). Because all matrix entries of \(s\) lie in \(\mathbb{F}_2\), we have \(\{\lambda_1, \ldots, \lambda_{2n}\} = \{(\lambda_1^*)^*, \ldots, (\lambda_{2n}^*)^*\}\). Taking the inverse under \(*\), this implies
\[
\{\lambda_1, \ldots, \lambda_{2n}\} = \{\lambda_1^*, \ldots, \lambda_{2n}^*\}.
\]
Thus, the set of eigenvalues of \(s\) coincides with the set of eigenvalues of \(s^r\) and it follows that \(s\) and \(s^r\) are conjugate.

By [Lus77, p. 164], there exists a bijection between the rational semisimple elements of \(\text{Sp}_{2n}(2)\) and of its dual \(\text{SO}_{2n+1}(2)\) with an isomorphism of centralizers. Thus, the Jordan decomposition of \(\chi \in \text{Irr}(G)\) can be written as \((s, \nu)\) with \(s \in G\) semisimple and \(\nu\) a unipotent character of \(C_G(s)\) [GM20, Section 2.6]. Now \(B_n\) defined over \(\mathbb{F}_2\) is connected reductive with connected centre. By [SV20], \(\psi^\sigma\) has Jordan decomposition \((s^\sigma, \nu^\sigma)\) for \(\sigma \in \mathcal{H}\) as described above. As in the proof of [Cab11, Proposition 2], we have \(C_G(s) \cong \text{Sp}_{2k}(2) \times C\) for some \(0 \leq k \leq n\) and \(C\) a product of finite linear groups for every semisimple \(s \in G\). By [Lus72, Corollary 1.12], every unipotent character of a group of type \(A\) or \(B\) is rational-valued. Thus, it follows \(\nu^\sigma = \nu\) for every unipotent character \(\nu\) of \(C_G(s)\). For all \(\sigma \in \mathcal{H}\) we also know that \(s^\sigma\) is conjugate to \(s\), thus the Jordan decompositions of \(\chi\) and \(\chi^\sigma\) are the same and it follows \(\chi = \chi^\sigma\). \(\square\)
4.2. Action of Galois automorphisms in the local case. Let $\text{Sym}_n(2)$ be the additive group of symmetric $n \times n$ -matrices over $\mathbb{F}_2$, $U_n(2) \leq \text{GL}_n(2)$ the group of upper triangular unipotent matrices over $\mathbb{F}_2$ and

$$J := \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \in \text{GL}_n(\mathbb{F}_2), \quad R := \left\{ \begin{pmatrix} x & xsJ \\ 0 & J(x^{-1})TJ \end{pmatrix} \mid x \in U_n(2), \ s \in \text{Sym}_n(2) \right\}.$$

By [Cab11] Proposition 3, $R$ is a self-normalizing Sylow-$2$-subgroup of $G$. We want to show that Condition 2.4 is satisfied for $N = R$.

**Lemma 4.2.** The Galois automorphisms in $H$ act trivially on $\text{Irr}_{2'}(R)$.

**Proof.** We want to show that all linear characters of $R$ have integer character values. Since all linear characters of $R$ can be obtained as inflated characters from its abelianization $R/R'$, it suffices to show that all linear characters of $R/R'$ have integer character values.

As in [Cab11], we see $R \cong \text{Sym}_n(2) \times U_n(2)$ with $U_n(2)$ acting on $\text{Sym}_n(2)$ by $x.s = xsx^T$ for $x \in U_n(2)$ and $s \in \text{Sym}_n(2)$. With the same considerations as in [Cab11] proof of Proposition 3, it follows

$$R/R' \cong (\text{Sym}_n(2)/[\text{Sym}_n(2), U_n(2)]) \times (U_n(2)/U_n(2)') \cong (C_2)^{n+1}.$$

Thus, every value of a linear character of $R/R'$ is either $1$ or $-1$ and thereby an integer. Now the claim follows. \qed

**Proposition 4.3.** Condition 2.4 is satisfied for $S = B_n(2)$ with $n \geq 4$ in defining characteristic $p = 2$.

**Proof.** We use the notation that was introduced in the beginning of the section. The group $G = S$ has trivial outer automorphism group [GLS98] Section 2.5], thus $\Gamma = \text{Inn}(G \mid R)$ acts trivial on all characters in $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(R)$.

By Lemma 2.1 and Lemma 2.2 the group $H$ acts trivially on these characters. We know $|\text{Irr}_{2'}(G)| = |\text{Irr}_{2'}(R)| = 2^{n+1}$ by [Cab11] Section 2. Thus, there obviously exists a $\Gamma \times H$-equivariant bijection between $\text{Irr}_{2'}(G)$ and $\text{Irr}_{2'}(R)$. Since $\Gamma = \text{Inn}(G \mid R)$, we can extend a representation $\chi$ of $G$ affording $\psi \in \text{Irr}_{2'}(G)$ to $G \times \Gamma$ canonically as described in Remark 2.3 [3].

Again by Remark 2.3 [3], any $\mu_a$ for $a \in ((N \times \text{Inn}(G \mid N)) \times H$ has to be a character of $(G \rtimes \Gamma)/(G \rtimes \text{Inn}(G \mid R))$ and thereby trivial. The same is true for $\Omega(\psi)$, thus (2) and (3) of Condition 2.4 hold. \qed

5. Groups with non-generic Schur multiplier and $G_2(2)$

In [Ruh20], the group $G_2(2)$ and all groups that have non-generic Schur multiplier are excluded from the considerations. Here, we show that the inductive Galois–McKay condition is also satisfied for $G_2(2)$ and the finite groups of Lie type with non-generic Schur multiplier (see e.g. [MT11] Table 24.3) in their defining characteristic.

5.1. About $p$-extensions. As we see in the following lemma, we often do not have to consider the full Schur cover. Given a group $S$ and some covering group $G$ of it, the $p'$-part of the covering group is some covering group $H$ with $G \to H \to S$ such that $|G|/|H|$ is a $p$-power and $|H|/|S|$ is prime to $p$.

**Lemma 5.1.** Let $S$ be a simple non-abelian group, $p$ a prime, $G$ the Schur cover of $S$ and $H$ the $p'$-part of the Schur cover. Let $M$ be the normalizer of a Sylow-$p$-subgroup $P$ of $H$ and set $\Gamma = \text{Aut}(H)_p$. Assume that there exists a $\Gamma \times H$-equivariant bijection $\Omega': \text{Irr}_{p'}(H) \to \text{Irr}_{p'}(M)$ such that for all $\psi \in \text{Irr}_{p'}(H)$ we have

$$(H \rtimes \Gamma_\psi, H, \psi) \geq_c (M \rtimes \Gamma_\psi, M, \Omega'(\psi)).$$

Then the inductive Galois–McKay condition holds for $S$. 
Proof. We have \( G/Z(G) \cong S \) and \( G/Z(G)_p \cong H \), thus \( G \) is a central \( p \)-extension of \( H \). Consider an irreducible character of \( G \) such that its restriction to \( Z(G)_p \) is not trivial. Let \( d \) be the degree of the character. Then we find a \( z \in Z(G)_p \) such that an affording representation \( \chi \) is of the form

\[ \chi(z) = \varepsilon I_d \]

for some \( \varepsilon \in \mathbb{C}^\times \) with \( \varepsilon |Z(G)_p| = 1 \) and \( \varepsilon \neq 1 \). Since \( S \) is perfect, \( G \) is also perfect \cite[Theorem B.4]{Nav18} and its only linear character is the trivial one. Thus, the determinant of \( \chi(g) \) has to be 1 for all elements \( g \in G \) and it follows

\[ \det(\chi(z)) = \varepsilon^d = 1. \]

Since \( |Z(G)_p| \) is a \( p \)-power and \( \varepsilon \neq 1 \), \( d \) and \( p \) cannot be coprime and it follows \( p \mid d \). Therefore, all irreducible \( p' \)-characters of \( G \) have \( Z(G)_p \) in their kernel and can be obtained by inflating irreducible \( p' \)-characters of \( H \). Thus, deflation yields a bijection \( \text{Def}^G_H \) between \( p' \)-characters of \( G \) and \( H \).

Let \( R \leq G \) be the preimage of \( P \) and \( N \) its normalizer in \( G \). Since \( Z(G)_p \leq N \) by definition and Sylow theory, \( N \) is a central \( p \)-extension of \( M \). Thus, inflation yields a bijection \( \text{Inf}^M_N \) between the \( p' \)-characters of \( M \) and \( N \). It follows that \( \Omega \) induces a bijection

\[ \Omega := \text{Inf}^M_N \circ \Omega' \circ \text{Def}^G_H : \text{Irr}_{p'}(G) \to \text{Irr}_{p'}(N). \]

By \cite[Corollary B.8]{Nav18}, the automorphism groups of \( G \) and \( H \) are both isomorphic to the automorphism group of \( S \). Consequently, every automorphism of \( H \) stabilizing \( P \) uniquely determines an automorphism of \( G \) stabilizing \( R \). Under this identification we will denote the group of automorphisms of \( G \) by \( \Gamma \) as well. Since a \( \chi \in \text{Irr}_{p'}(G) \) is uniquely determined by the values of \( \chi \) on \( H, \Gamma \) and \( \mathcal{H} \) act on \( \text{Irr}_{p'}(G) \) in the same way as on \( \text{Irr}_{p'}(H) \), i.e. \( \text{Def}^G_H \) is \( \Gamma \times \mathcal{H} \)-equivariant. The same is true for \( \text{Inf}^N_M \), thus \( \Omega \) is \( \Gamma \times \mathcal{H} \)-equivariant.

It remains to show that (2) and (3) in Condition 2.1 hold. For a \( \psi \in \text{Irr}_{p'}(G) \), we can extend every projective representation of \( H \times \Gamma \text{Def}^G_H(\psi) \) to \( G \times \Gamma \psi \) canonically such that it is trivial on \( Z(G)_p \). The same can be done for projective representations of \( M \times \Gamma \text{Def}^G_H(\psi) \). By assumption, we have projective representations of \( H \times \Gamma \text{Def}^G_H(\psi) \) and \( M \times \Gamma \text{Def}^G_H(\psi) \) satisfying Condition 2.1 (2), (3) for \( \text{Def}^G_H(\psi) \). It is easy to see that the trivially extended representations of \( G \times \Gamma \psi \) and \( N \times \Gamma \Omega(\psi) \) satisfy these conditions for \( \psi \) as well. Therefore, Condition 2.1 holds for \( S \). \( \square \)

Thus, it suffices to consider the \( p' \)-part of the Schur multiplier in Condition 2.1.

5.2. Explicit computations. We first prove a simple lemma stating some facts that will be used frequently to simplify the explicit computations for many of the mentioned groups.

**Lemma 5.2.** Let \( \psi \in \text{Irr}_{p'}(G) \) with the notation of Condition 2.1.

(a) If \( \Gamma \psi/\text{Inn}(G \mid N) \) is cyclic, \( \psi \) extends to \( G \rtimes \Gamma \psi \).

(b) Assume either

- \( \Gamma \psi = \text{Inn}(G \mid N) \) or
- \( \Gamma \psi = \Gamma \psi^a \) and there exists a subgroup \( \Gamma \psi^a \text{out} \leq \Gamma \psi \) isomorphic to the corresponding outer automorphism group and some extension of \( \psi \) to \( G \rtimes (\Gamma \psi^a \text{out}) \) with character values fixed by \( \mathcal{H} \).

Then there exists a representation of \( G \rtimes \Gamma \psi \) associated to \( \psi \) such that \( \mu_a \) is trivial for all \( a \in ((G \rtimes \Gamma) \times \mathcal{H}) \psi \).

(c) Assume \( \psi \) is linear, \( \Gamma \psi/\text{Inn}(G \mid N) \) is cyclic and \((\Gamma \psi^a \text{out}) \text{out}\) exists and is abelian. Then we find an extension of \( \psi \) to \( G \rtimes \Gamma \psi \) such that \( \mu_a \) is trivial for all \( a \in ((G \rtimes \Gamma) \times \mathcal{H}) \psi \).

The same holds for \( \psi \in \text{Irr}_{p'}(N) \) if we replace \( G \) by \( N \) in the respective semidirect products.

**Proof.** We can extend any representation affording \( \psi \) canonically to a representation of \( G \rtimes \text{Inn}(G \mid N) \) as in Remark 2.3(a). Now \( \Gamma \psi/\text{Inn}(G \mid N) \) is cyclic and there exists a representation of \( G \rtimes \Gamma \psi \) associated to \( \psi \) by \cite[Theorem 5.1]{Nav18}. This shows (a).
The canonical extension of a representation affording \( \psi \) to \( G \times \text{Inn}(G \mid N) \) corresponds to trivial \( \mu_a \) by Remark 5.2. Thus (b) holds if \( \Gamma = \text{Inn}(G \mid N) \).

To show (b) in the second case, let \( X \) be a representation of \( G \times (\Gamma_{\psi_0}^*) \) affording the described \( \mathcal{H}_\psi \)-invariant character \( \psi_0 \). Then we can extend \( X \) canonically to a representation \( \mathcal{P} \) of \( G \times \Gamma_{\psi_0}^* \).

Let \( \psi_1 \) be the character afforded by this representation. Since \( \psi_0 \) only has \( \mathcal{H}_\psi \)-invariant values and \( \psi_1((1, \gamma_g)) = \psi_0((g, \text{id})) \) for \( \gamma_g \) the inner automorphism associated to \( g \in G \), \( \psi_1 \) also has \( \mathcal{H}_\psi \)-invariant character values. We have

\[
\mathcal{P}(xyy^{-1}) = \mathcal{P}(y)\mathcal{P}(x)\mathcal{P}(y^{-1}) \sim \mathcal{P}(x)
\]

for all \( x, y \in G \times \Gamma_{\psi_0}^* \). Thus, for all \( a \in ((N \times \Gamma_{\psi_0}^*) \times \mathcal{H})_{\psi} = (N \times \Gamma_{\psi_0}^*) \times \mathcal{H}_\psi \), only the \( \mathcal{H} \)-component determines \( \mu_a \). Let \( \sigma \) be the \( \mathcal{H} \)-component of \( a \). Then we have

\[
\psi_1^a = \mu_a \psi_1 = \psi_1^0 = \psi_1
\]

and the claim follows.

For (c), we see as in the proof of [Nav18, Theorem 5.1] that we find an extension \( \psi_0 \in \text{Irr}(G \times \Gamma_{\psi}) \) by setting

\[
\psi_0(tu^k) = \sigma^t \psi(t)
\]

for any \( t \in G \), \( u \in \Gamma_{\psi} \) such that \( \Gamma_{\psi} = \{u, \text{Inn}(G \mid N)\} \) and some \( z \in \mathbb{C} \) with \( z^{\text{ord}(u)} = 1 \). We have \( (tu)^\gamma = \gamma(t)u \) for all \( \gamma \in (\Gamma_{\psi}^*) \). Setting \( z = 1 \), we obtain

\[
\psi_0(tu^k) = \psi(t) = \psi_0(tu^k)
\]

for all \( (\gamma, \sigma) \in ((\Gamma_{\psi}^*) \times \mathcal{H})_\psi \). The claim holds since we can extend the character canonically to the inner automorphisms in \( \Gamma \).

In the following proof, all explicit computations were made with GAP [GAP19].

**Proposition 5.3.** The inductive Galois–McKay condition holds for \( G_2(2)' \) and the simple groups of Lie type with non-generic Schur multiplier in their defining characteristic.

**Proof.** Let \( S \) be a simple group of Lie type, \( p \) its defining characteristic and \( G \) the \( p' \)-part of the Schur cover of \( S \). Note that the proof of [Rub20, Theorem 7.3] does not use the fact that \( G^F = G \) is the universal covering group of \( S \). Except for \( B_2(2)' \), the exceptional part of the Schur multiplier is always a \( p \)-group [MTT1 Table 24.3]. Therefore, by Lemma 5.1 and [Rub20, Theorem 7.3], the inductive Galois–McKay condition is satisfied in defining characteristic for \( S \) being one of the groups \( \text{PSL}_2(4), \text{PSL}_3(4), \text{PSL}_4(2), \text{PSL}_2(2), \text{PSL}_2(2), \text{PSU}_3(2), 2^E\text{PSL}_2(2), 2^B\text{PSU}_2(2), 3^2\text{PSU}_2(2), 2^B\text{PSU}_2(2), 3^2\text{PSU}_2(2), 2^B\text{PSU}_2(2) \).

The remaining simple groups of Lie type with exceptional Schur multiplier are \( \text{Sp}_6(2), \text{F}_4(2), \text{G}_2(3), 2^B\text{PSL}_3(2), 2^B\text{PSL}_3(2) \).

Note that \( G_2(2)' \) has trivial Schur multiplier. For \( S = \{\text{Sp}_6(2), \text{G}_2(3), 2^B\text{PSL}_3(2), 2^B\text{PSL}_3(2) \} \), the outer automorphism group of \( S \) is cyclic by [CCN+88, 5]. By Lemma 5.1, it suffices to let \( G \) be the \( p' \)-part of the Schur cover of \( S \) with \( p \) being the defining characteristic of \( S \). Then the outer automorphism group of \( G \) is also cyclic [Nav18, Corollary B.8]. One can show computationally that with \( R \in \text{Sym}_p(\Gamma) \) and \( \Gamma = \text{Aut}(G)\) the actions of \( \Gamma \times H \) on \( \text{Irr}_p(G) \) and on \( \text{Irr}_p(N_G(R)) \) are permutation isomorphic. Thus, we can easily find a \( \Gamma \times H \)-equivariant bijection between \( \text{Irr}_p(G) \) and \( \text{Irr}_p(N_G(R)) \). By constructing the factor map \( \pi : \text{Aut}(G) \to \text{Aut}(G) / \text{Inn}(G) \), we find a subgroup \( \Gamma_\text{out} \leq \Gamma \) such that the restriction of \( \pi \) to \( \Gamma_\text{out} \) is surjective. Thus, we can compute the character tables of \( G \times \Gamma_\text{out} \) and \( N_G(R) \times \Gamma_\text{out} \). We see that we can apply Lemma 5.2 for the characters in all groups except \( G_2(3) \). It follows that the inductive Galois–McKay condition is true for these groups.

If \( S = G_2(3) \), we see that for both \( G \) and \( N = N_G(R) \) we can apply Lemma 5.2 for all \( 3' \)-characters but \( \chi_2 \in \text{Irr}_p(G) \) of degree 14 as in [CCN+88, p.60] and a character \( \varphi \in \text{Irr}_p(N) \) of degree 2 (it can be characterized as the only character of degree 2 fixed by \( \Gamma \) that has character value 1 for some group elements). The actions of both \( \mathcal{H} \) and \( \Gamma \) are trivial on these characters, thus we can assume that \( \Omega \) maps the characters onto another and we have

\[
((G \times \Gamma_\chi) \times \mathcal{H})_{\chi_2} = (G \times \Gamma) \times \mathcal{H}.
\]
We can now compute the character tables of \( G \times \Gamma_{\text{out}} \) and \( N \times \Gamma_{\text{out}} \) and determine the \( \mu_a \) corresponding to \( \chi_2 \) and \( \varphi \) for \( a \in (G \times \Gamma) \times \mathcal{H} \). With Remark [2.3][b] it follows for both \( \chi_2 \) and \( \varphi \)

\[
\mu_{(y, \sigma)} = \begin{cases} 
\chi & \text{if } \xi_3^\sigma = \xi_3^2, \\
1 & \text{if } \xi_3^\sigma = \xi_3,
\end{cases}
\]

for \( \xi_3 \) a primitive third root of unity and \( (y, \sigma) \in (N \times \Gamma) \times \mathcal{H} \). Here, \( \chi \) is the character of \( G \times \Gamma_{\text{out}} \) (resp. \( N \times \Gamma_{\text{out}} \)) given by inflation from the non-trivial character of \( G \times \Gamma_{\text{out}} / G \cong C_2 \) (resp. \( N \times \Gamma_{\text{out}} / N \cong C_2 \)), i.e.

\[
\chi(g, \tau) = \begin{cases} 
1 & \text{if } \tau = \text{id}, \\
-1 & \text{else},
\end{cases}
\]

for all \( (g, \tau) \in N \times \Gamma_{\text{out}} \). Since we can extend the characters canonically to the inner automorphisms in \( \Gamma \), Condition [2.3] is satisfied.

For \( S = F_4(2) \), we know that the Schur multiplier has order 2 and can therefore consider \( G = S \). We see that \( R \in \text{Syl}_2(G) \) is self-normalizing and that the character values of the linear characters of \( R \) are integers. The values of the characters in \( \text{Irr}_2'(G) \) are given in [CCN+85] and we see that \( \mathcal{H} \) acts trivially on both \( \text{Irr}_2(G) \) and \( \text{Irr}_2(R) \). The outer automorphism group of \( G \) is generated by an outer automorphism \( \gamma \) stabilizing \( R \) with \( \gamma^2 = \text{id} \), induced by the exceptional graph automorphism of \( G \) [GLS98].

We can compute the actions of \( \gamma \) on the conjugacy classes of \( G \) and \( N \) and we see that the actions of \( \gamma \) on \( \text{Irr}_2(G) \) and \( \text{Irr}_2(R) \) are permutation isomorphic. Thus, for \( \psi \in \text{Irr}_2'(G) \), we now have either \( \Gamma_{\psi} = \text{Inn}(G | R) \) or we can read off the character values of \( G \times \langle \gamma \rangle \) from the character table of the split extension \( F_4(2) \times 2 \) given in [CCN+85]. There we see that \( \mathcal{H} \) acts trivial on the extensions, thus by Lemma [6.2][b] we find an extension of \( \psi \) to \( G \times \Gamma_{\psi} \) such that all corresponding \( \mu_a \) for \( a \in (G \times \langle \gamma \rangle) \times \mathcal{H} \) are trivial. The same holds for the characters in \( \text{Irr}_2'(R) \) by Lemma [5.2][c].

We now consider \( S = B_2(2)' \cong \text{PSL}_2(9) \) which has cyclic Schur multiplier of order 6 and thereby a non-trivial exceptional Schur multiplier in both defining characteristics \( p = 2 \) and \( p = 3 \). For \( p = 3 \), this was already treated above, thus let \( p = 2 \). Let \( G \) be the 3-cover of \( S \), \( R \) a Sylow-2-subgroup of \( G \) and \( N = \text{N}_G(R) \). Note first \( \Gamma = \langle \text{Inn}(G | N), \gamma_1, \gamma_2 \rangle \) with \( \gamma_1, \gamma_2 \in \text{Aut}(G) \) of order 2 and that \( N \) is indeed \( \Gamma \)-stable. We have the additional difficulty that there is no subgroup of \( \Gamma \) that is isomorphic to the outer automorphism group. Thus, we choose a subgroup \( \Gamma' \subseteq \Gamma \) containing representatives of all outer automorphisms of \( G \). We see that all considered characters extend to \( G \times \Gamma' \) resp. \( N \times \Gamma' \) but we cannot assume that these character extensions agree with the described canonical extensions to the semidirect product with the inner automorphisms. In particular, we do not know anymore that the centres of \( G \times \Gamma'_{\psi} \) and \( N \times \Gamma'_{\psi} \) are in the kernels of the extended characters.

We can explicitly compute the actions of \( \Gamma' \) and \( \mathcal{H} \) on \( \text{Irr}_2'(G) \) and \( \text{Irr}_2'(N) \). Since these are permutation isomorphic, we can construct a \( \Gamma \times \mathcal{H} \)-equivariant bijection between \( \text{Irr}_2'(G) \) and \( \text{Irr}_2'(N) \). For all characters in \( \text{Irr}_2'(G) \) we can show that there is at least one character extension such that all occurring \( \mu_a \) for \( a \in (N \times \Gamma'_{\psi}) \times \mathcal{H} \) are trivial, either by applying Lemma [5.2] or by individually determining the action of \( (G \times \Gamma'_{\psi}) \times \mathcal{H} \) on the relevant characters of \( G \times \Gamma'_{\psi} \). By computing the values of these extended characters on the centre of \( G \times \Gamma'_{\psi} \), we see that all corresponding central characters are trivial. The same can be done for \( \text{Irr}_2'(N) \), thus Condition [2.3] holds.

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