Positive solutions of systems of perturbed Hammerstein integral equations with arbitrary order dependence

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Motivated by the study of systems of higher-order boundary value problems with functional boundary conditions, we discuss, by topological methods, the solvability of a fairly general class of systems of perturbed Hammerstein integral equations, where the nonlinearities and the functionals involved depend on some derivatives. We improve and complement earlier results in the literature. We also provide some examples in order to illustrate the applicability of the theoretical results.

This article is part of the theme issue ‘Topological degree and fixed point theories in differential and difference equations’.

1. Introduction

In this paper, we discuss the solvability of systems of perturbed Hammerstein integral equations of the form

\[ u_i(t) = \lambda_i \int_0^1 k_i(t, s) f_i(s, u_1(s), \ldots, u_n(s), \ldots, u_n^{(m_i)}(s)) \, ds + \sum_{j=1}^{p_i} \eta_{ij} \gamma_{ij}(t) h_{ij}[u] , \quad t \in [0, 1], \quad i = 1, 2, \ldots, n, \] (1.1)

where \( u = (u_1, \ldots, u_n) \), the kernels \( k_i \) are sufficiently regular, \( f_i \) are continuous, \( \gamma_{ij} \) are sufficiently smooth, \( h_{ij} \) are compact functionals that are allowed to take into account higher-order derivatives and \( \lambda_i, \eta_{ij} \) are parameters.

One motivation for studying the kind of equations that occur in (1.1) is that these often occur in applications; we refer the reader to the Introduction of [1] and
references therein. The case \( n = 1 \) has been studied recently by Goodrich [2,3], who complemented earlier works [1,4].

In particular, Goodrich studied the equation

\[
    u(t) = \lambda \int_0^1 k(t, s)f(s, u(s)) \, ds + \sum_{j=1}^2 \gamma_j(t)h_j[u],
\]

where the functionals \( h_j \) have the specific form

\[
    h_j[u] = \hat{h}_j(\alpha_j[u]). \tag{1.2}
\]

In (1.2), the functions \( \hat{h}_j \) are continuous and \( \alpha_j \) are linear functionals on the space \( C[0, 1] \) which can be represented as Stieltjes integrals, namely

\[
    \alpha_j[u] := \int_0^1 u(s) \, dA_j(s). \tag{1.3}
\]

The functional formulation (1.3) is well suited for handling, in a unified way, multi-point and integral boundary conditions (BCs). For an introduction to non-local BCs, we refer the reader to the reviews [5–11] and the articles [12–14].

The case \( n = 2 \) has been investigated in [1], where the authors studied the system

\[
    u_i(t) = \int_0^1 k_i(t, s)f_i(s, u_1(s), u_2(s)) \, ds + \sum_{j=1}^2 \gamma_{ij}(t)h_{ij}[u_1, u_2], \quad i = 1, 2,
\]

where the functionals \( h_{ij} \) act on the space \( C[0, 1] \times C[0, 1] \).

We stress that functionals involving higher-order derivatives play an important role in applications. In order to illustrate this fact in a simple situation, consider the boundary value problem (BVP)

\[
    u^{(4)}(t) = f(t, u(t)), \quad u(0) = h[u], \quad u''(0) = u(1) = u''(1) = 0. \tag{1.4}
\]

When \( h[u] \equiv 0 \) the BVP (1.4) can be used to describe the steady-state case of a simply supported beam of length 1. When the functional \( h \) is non-trivial the BVP (1.4) can be used to model a beam with a feedback control; for example, the case

\[
    h[u] = \hat{h}(u''(\xi)) \tag{1.5}
\]

models a beam with the right end simply supported and where the displacement in the left end is controlled (possibly in a nonlinear manner) by a sensor that measures the shear force in a point \( \xi \) placed along of the beam. The perturbed integral equation associated with (1.4) and (1.5) is

\[
    u(t) = \int_0^1 k(t, s)f(s, u_1(s)) \, ds + (1 - t)\hat{h}(u''(\xi)),
\]

a case that cannot be handled with the theory developed in [1–4] due to the third-order term occurring in (1.5).

The case of higher-order dependence within the equation has been investigated recently, by means of the classical Krasnosel’skii’s theorem of cone compression-expansion, by de Sousa & Minhós [15]. In particular, de Sousa & Minhós [15] consider the existence of non-trivial solutions for the system of Hammerstein equations

\[
    u_i(t) = \int_0^1 k_i(t, s)f_i(s, u_1(s), \ldots, u_1^{(m_1)}(s), \ldots, u_n(s), \ldots, u_n^{(m_n)}(s)) \, ds, \quad i = 1, 2, \ldots, n.
\]
As an interesting application of their theory, de Sousa and Minhós apply their result to a system of BVPs of the form

\[
\begin{align*}
\begin{aligned}
u''(t) + f_1(t, u_1(t), u_1'(t), u_2(t), u_2'(t), u_2''(t), u_2'''(t)) &= 0, & t \in (0, 1), \\
u''(t) &= f_2(t, u_1(t), u_2(t), u_2'(t), u_2''(t)), & t \in (0, 1)
\end{aligned}
\end{align*}
\]  

(1.6)

and

\[
\begin{align*}
u_1(0) &= u_1(1) = u_2(0) = u_2(1) = u_2''(0) = u_2''(1) = 0.
\end{align*}
\]

The system (1.7) can be used as a model of the displacement of simply supported suspension bridge. In this model, the fourth-order equation describes the road bed and the second-order equation models the suspending cables, we refer to [15] for more details.

On the other hand, the case of equations of the form

\[
u(t) = \lambda \int_0^1 k(t, s)f(s, u(s), u'(s)) \, ds + \sum_{j=1}^{2} \eta_j \gamma_j(t) h_j[u],
\]

where the functionals \( h_j \) act on the space \( C^1[0,1] \), has been studied recently by Infante [16], by means of the classical fixed-point index. Here we develop further this approach and we extend the results of [16] to the case of systems and higher-order dependence in the nonlinearities and the functionals. We also improve the case \( n = 1 \) and \( m_1 = 1 \), by allowing more freedom in the growth of the nonlinearities near the origin; this is achieved by means of an eigenvalue comparison.

In order to illustrate the applicability of our theory, we discuss, merely as an example, the solvability of the system of the following model problem:

\[
\begin{align*}
u''(t) + \lambda_1 f_1(t, u_1(t), u_1'(t), u_2(t), u_2'(t), u_2''(t), u_2'''(t)) &= 0, & t \in (0, 1), \\
u''(t) &= \lambda_2 f_2(t, u_1(t), u_2(t), u_2'(t), u_2''(t)), & t \in (0, 1), \\
u_1(0) &= 0, \quad u_1(1) = \eta_{11} h_{11}(u_1, u_2), \\
u_2(0) &= \eta_{21} h_{21}(u_1, u_2), \quad u_2'(0) = u_2(1) = u_2''(1) = 0,
\end{align*}
\]

(1.7)

where \( h_{11}, h_{21} \) are non-negative, compact functionals defined on the space \( C^1[0,1] \times C^3[0,1] \). The interest in (1.7) arises in the fact that it presents a coupling in the nonlinearities near the origin; this is achieved by means of an eigenvalue comparison.

In order to illustrate the applicability of our theory, we discuss, merely as an example, the solvability of the system of the following model problem:

\[
\begin{align*}
u''(t) + \lambda_1 f_1(t, u_1(t), u_1'(t), u_2(t), u_2'(t), u_2''(t), u_2'''(t)) &= 0, & t \in (0, 1), \\
u''(t) &= \lambda_2 f_2(t, u_1(t), u_2(t), u_2'(t), u_2''(t)), & t \in (0, 1), \\
u_1(0) &= 0, \quad u_1(1) = \eta_{11} h_{11}(u_1, u_2), \\
u_2(0) &= \eta_{21} h_{21}(u_1, u_2), \quad u_2'(0) = u_2(1) = u_2''(1) = 0,
\end{align*}
\]

(2.1)

where \( u = (u_1, u_2) \). Throughout the paper, we make the following assumptions on the terms that occur in (2.1).

\( (C_1) \) For every \( i = 1, \ldots, n \), \( k_i : [0,1] \times [0,1] \to [0, +\infty) \) is measurable in \( s \) for every \( t \) and continuous in \( t \) for almost every (a.e.) \( s \), that is, for every \( \tau \in [0,1] \) we have

\[
\lim_{t \to \tau} |k_i(t,s) - k_i(\tau,s)| = 0 \text{ for a.e. } s \in [0,1];
\]
furthermore, there exists a function $\Phi_{i0} \in L^1(0, 1)$ such that $0 \leq k_i(t, s) \leq \Phi_{i0}(s)$ for $t \in [0, 1]$ and a.e. $s \in [0, 1]$.

(C2) For every $i = 1, \ldots, n$ and for every $l_i \in \mathbb{N}$, with $l_i < m_i$, the partial derivative $\frac{\partial^{l_i} k_i}{\partial t^{l_i}}$ is measurable in $s$ for every $t$, continuous in $t$ for a.e. $s$, and there exists $\Phi_{i}(s) \in L^1(0, 1)$ such that $|\frac{\partial^{l_i} k_i}{\partial t^{l_i}}(t, s)| \leq \Phi_{i0}(s)$ for $t \in [0, 1]$ and a.e. $s \in [0, 1]$.

(C3) For every $i = 1, \ldots, n$, $\frac{\partial^{m_i} k_i}{\partial t^{m_i}}$ is measurable in $s$ for every $t$, continuous in $t$ except possibly at the point $t = s$ where there can be a jump discontinuity, that is right and left limits both exist, and there exists $\Phi_{im}(s) \in L^1(0, 1)$ such that $|\frac{\partial^{m_i} k_i}{\partial t^{m_i}}(t, s)| \leq \Phi_{im}(s)$ for $t \in [0, 1]$ and a.e. $s \in [0, 1]$.

(C4) For every $i = 1, \ldots, n$, $\gamma_i(t) : [0, 1] \times [0, +\infty) \times \mathbb{R}^{m_i} \to [0, +\infty)$ is continuous.

(C5) For every $i = 1, \ldots, n$ and $j = 1, \ldots, p_i$, we have $\gamma_{ij} \in C^m_\infty[0, 1]$ and $\gamma_{ij}(t) \geq 0$ for every $t \in [0, 1]$.

(C6) For every $i = 1, \ldots, n$ and $j = 1, \ldots, p_i$, we have $\lambda_i, \eta_{ij}, \in [0, +\infty)$.

Owing to the assumptions above, for every $i = 1, \ldots, n$, the linear Hammerstein integral operator

$$L_i w(t) := \int_0^1 k_i(t, s)w(s) \, ds$$

is well defined and compact in the space $C[0, 1]$, where we adopt the standard norm $\|w\|_\infty := \max_{t \in [0, 1]} |w(t)|$. We recall that a cone $K$ in a real Banach space $X$ is a closed convex set such that $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geq 0$ and satisfying $K \cap (-K) = \{0\}$. It is clear that the operator $L_i$ leaves invariant the cone

$$\hat{P} := \{w \in C[0, 1] : w \geq 0 \text{ for every } t \in [0, 1]\}.$$ 

We denote by $r(L_i)$ the spectral radius of $L_i$ and assume

(C7) For every $i = 1, \ldots, n$, we have $r(L_i) > 0$.

Note that, since $\hat{P}$ is a reproducing cone in $C[0, 1]$, the assumption (C7) allows us to apply the well-known Krein–Rutman theorem and therefore $r(L_i)$ is an eigenvalue of $L_i$ with a corresponding eigenfunction $\varphi_i \in \hat{P} \setminus \{0\}$, that is

$$L_i \varphi_i(t) = \int_0^1 k_i(t, s) \varphi_i(s) \, ds = r(L_i) \varphi_i(t). \quad (2.2)$$

In what follows we shall make use of the eigenfunction $\varphi_i$ and the corresponding characteristic value

$$\mu_i := \frac{1}{r(L_i)}.$$ 

Note that the non-negative eigenfunction $\varphi_i$ inherits, from the kernel $k_i$, further regularity properties: indeed, since we have

$$\varphi_i(t) = \mu_i \int_0^1 k_i(t, s) \varphi_i(s) \, ds, \quad (2.3)$$

and, due to the assumptions (C1)–(C3), the r.h.s. of (2.3) is, as a function of the variable $t$, in $C^m_\infty[0, 1]$ we obtain

$$\varphi_i \in (\hat{P} \setminus \{0\}) \cap C^m_\infty[0, 1].$$

**Remark 2.1.** The assumption (C7) is frequently satisfied in applications. A sufficient condition, for details see [22], is given by

(C’7) There exist a subinterval $[a_i, b_i] \subseteq [0, 1]$ and a constant $c_i = c(a_i, b_i) \in (0, 1)$ such that

$$k_i(t, s) \geq c_i \Phi_{i0}(s) \text{ for } t \in [a_i, b_i] \text{ and a.e. } s \in [0, 1].$$
Owing to the hypotheses above, we work in the product space $\prod_{i=1}^{n} C^{m_i}[0, 1]$ endowed with the norm

$$\|u\| := \max_{i=1,\ldots,n} \{\|u_i\|_{C^{m_i}}\},$$

where $\|u_i\|_{C^{m_i}} := \max_{j=0,\ldots,m_i} \{\|u_i^{(j)}\|_{\infty}\}$. We use the cone

$$P := \left\{ u \in \prod_{i=1}^{n} C^{m_i}[0, 1] : u_i \geq 0 \text{ for every } t \in [0, 1], \ i = 1, \ldots, n \right\},$$

and we require the nonlinear functionals $h_{ij}$ to act positively on the cone $P$ and to be compact, that is:

$$(C_8) \text{ For every } i = 1, \ldots, n \text{ and } j = 1, \ldots, p_i, h_{ij} : P \rightarrow [0, +\infty) \text{ is continuous and maps bounded sets into bounded sets.}$$

We define the operator $T : P \rightarrow P$ as

$$Tu := (T_iu)_{i=1,\ldots,n}.$$  \hfill (2.4)

We make use of the following basic properties of the fixed-point index (we refer the reader to [23,24] for more details).

**Proposition 2.2 ([23,24])**. Let $K$ be a cone in a real Banach space $X$ and let $D$ be an open bounded set of $X$ with $0 \in D_K$ and $\overline{D}_K \neq K$, where $D_K = D \cap K$. Assume that $\overline{T} : D_K \rightarrow K$ is a compact map such that $x \neq \overline{T}x$ for $x \in \partial D_K$. Then the fixed-point index $i_K(\overline{T}, D_K)$ has the following properties:

1. If there exists $e \in K \setminus \{0\}$ such that $x \neq \overline{T}x + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(\overline{T}, D_K) = 0$.
2. If $\overline{T}x \neq \lambda x$ for all $x \in \partial D_K$ and all $\lambda > 1$, then $i_K(\overline{T}, D_K) = 1$.
3. Let $D^1$ be open in $X$ such that $\overline{D}^1_K \subset D_K$. If $i_K(T, D_K) = 1$ and $i_K(\overline{T}, D^1_K) = 0$, then $\overline{T}$ has a fixed point in $D_K \setminus \overline{D}^1_K$. The same holds if $i_K(T, D_K) = 0$ and $i_K(\overline{T}, D^1_K) = 1$.

For $\rho \in (0, \infty)$, we define the sets

$$P_\rho := \{u \in P : \|u\| < \rho\}, \quad I_\rho := [0, 1] \times \prod_{i=1}^{n}([0, \rho] \times [-\rho, \rho]^{m_i})$$

and the quantities

$$\tilde{f}_{i,\rho} := \max_{I_\rho} f_i(t, x_{1i1}, \ldots, x_{ini}, x_{n0}, \ldots, x_{nm_n}), \quad \tilde{H}_{ij,\rho} := \sup_{u \in \partial P_\rho} h_{ij}[u],$$

$$K_{i\ell} := \left\{ \begin{array}{ll}
\sup_{t \in [0, 1]} \int_{0}^{t} k_{i}(t, s) \, ds, & l = 0, \\
\sup_{t \in [0, 1]} \int_{0}^{t} \left| \frac{\partial \tilde{K}_{i\ell}}{\partial t} \right| (t, s) \, ds, & l = 1, \ldots, m_i. \end{array} \right.$$  \hfill (2.5)

With these ingredients, we can state the following existence and localization result.

**Theorem 2.3**. Assume there exist $r, R, \delta \in (0, +\infty)$, with $r < R$, and $i_0 \in \{1, 2, \ldots, n\}$ such that the following three inequalities are satisfied:

$$\lambda_{i_0} \geq \frac{\mu_{i_0}}{\delta}, \quad f_{i_0}(t, x_{1i_0}, \ldots, x_{1ni_0}, \ldots, x_{n0}, \ldots, x_{nm_n}) \geq \delta x_{i_0} \text{ on } I_r.$$  \hfill (2.6)

Then the system (2.1) has a solution $u \in P$ such that

$$r \leq \|u\| \leq R.$$
Note that which contradicts the fact that from (2.7) we obtain, for $t \in [0, 1]$,

$$
\sigma u^{(l_0)}_{i_0}(t) = \lambda_i \int_0^1 \frac{\partial k_{i_0}(t, s)}{\partial t_{l_0}} f_{i_0}(s, u_1(s), \ldots, u_{l_1}^{(m_1)}(s), \ldots, u_n(s), \ldots, u_{l_n}^{(m_n)}(s)) \, ds + \sum_{j=1}^{p_i} \eta_{i_0 j} y_{i_0 j}^{(l_0)}(s) h_{i_0 j}[u].
$$

From (2.7) we obtain, for $t \in [0, 1]$,

$$
\sigma |u^{(l_0)}_{i_0}(t)| \leq \lambda_i \int_0^1 \frac{\partial k_{i_0}(t, s)}{\partial t_{l_0}} |f_{i_0}(s, u_1(s), \ldots, u_{l_1}^{(m_1)}(s), \ldots, u_n(s), \ldots, u_{l_n}^{(m_n)}(s))| \, ds
$$

$$
+ \sum_{j=1}^{p_i} \eta_{i_0 j} |y_{i_0 j}^{(l_0)}(s)| h_{i_0 j}[u] \leq \lambda_i \int_0^1 k_{i_0}(t, s) \delta u_{i_0}(s) \, ds + \sum_{j=1}^{p_i} \eta_{i_0 j} \|y_{i_0 j}^{(l_0)}\|_\infty h_{i_0 j} \leq R.
$$

Taking in (2.8) the supremum for $t \in [0, 1]$ yields $\sigma \leq 1$, a contradiction. Therefore, we have $i_p(T, P_R) = 1$.

We now consider the function $\varphi(t) := (\varphi_1(t), \ldots, \varphi_n(t))$, where $t \in [0, 1]$ and $\varphi_i$ is given by (2.2). Note that $\varphi \in \mathcal{P} \setminus \{0\}$. We show that

$$
u \neq Tu + \sigma \varphi$$

for every $u \in \partial P$ and every $\sigma > 0$.

If not, there exist $u \in \partial P$ and $\sigma > 0$ such that $u = Tu + \sigma \varphi$. In particular, we have $u_{i_0}(t) = T_{i_0} u(t) + \sigma \varphi_{i_0}(t)$ for every $t \in [0, 1]$ and therefore $u_{i_0}(t) \geq \sigma \varphi_{i_0}(t)$ in $[0, 1]$. Observe that we have $r \geq \|u_{i_0}\|_\infty \geq \sigma \|\varphi_{i_0}\|_\infty > 0$.

For every $t \in [0, 1]$, we have

$$u_{i_0}(t) = k_{i_0}(t, s) f_{i_0}(s, u_1(s), \ldots, u_{l_1}^{(m_1)}(s), \ldots, u_n(s), \ldots, u_{l_n}^{(m_n)}(s)) \, ds
$$

$$+ \sum_{j=1}^{p_i} \eta_{i_0 j} y_{i_0 j}^{(l_0)}(s) h_{i_0 j}[u] + \sigma \varphi_{i_0}(t) \geq \lambda_i \int_0^1 k_{i_0}(t, s) \delta u_{i_0}(s) \, ds + \sigma \varphi_{i_0}(t)
$$

$$\geq \lambda_i \int_0^1 k_{i_0}(t, s) \delta \sigma \varphi_{i_0}(s) \, ds + \sigma \varphi_{i_0}(t)
$$

$$= \sigma \lambda_i \frac{\delta}{\mu_{i_0}} \varphi_{i_0}(t) + \sigma \varphi_{i_0}(t) \geq 2 \sigma \varphi_{i_0}(t).$$

By iteration we obtain, for $t \in [0, 1]$,

$$u_{i_0}(t) \geq n \sigma \varphi_{i_0}(t)$$

for every $n \in \mathbb{N}$, which contradicts the fact that $\|u_{i_0}\|_\infty \leq r$.

Thus we obtain $i_p(T, P_R) = 0$.

Therefore, we have

$$i_p(T, P_R \setminus \overline{P_R}) = i_p(T, P_R) - i_p(T, P_R) = 1,$$

which proves the result.

We now illustrate the applicability of theorem 2.3.
Example 2.4. We focus on the system
\[
\begin{align*}
&u''_1(t) + \lambda_1 f_1(t, u_1(t), u'_1(t), u_2(t), u'_2(t), u''_2(t), u''''_2(t)) = 0, \quad t \in (0, 1), \\
&u''_2(t) = \lambda_2 f_2(t, u_1(t), u'_1(t), u_2(t), u'_2(t), u''_2(t), u''''_2(t)), \quad t \in (0, 1), \\
&u_1(0) = 0, \quad u_1(1) = \eta_{11} h_{11}[(u_1, u_2)] \\
\text{and} \quad &u_2(0) = \eta_{21} h_{21}[(u_1, u_2)], \quad u''_2(0) = u_2(1) = u''_2(1) = 0,
\end{align*}
\]
where \( h_{11}, h_{21} \) are non-negative, compact functionals acting on the cone
\[
P = \{ (u_1, u_2) \in C^1[0, 1] \times C^3[0, 1] : u_1, u_2 \geq 0 \text{ for every } t \in [0, 1] \}.
\]

With our methodology, we could study a more complicated version of this BVP, by adding more functional terms in the BCs, but we refrain from doing so for the sake of clarity.

It is routine to show that the solutions of (2.9) can be written in the form
\[
\begin{align*}
u_1(t) &= \eta_{11} t h_{11}[(u_1, u_2)] + \lambda_1 \left[ \int_0^1 k_1(t, s) f_1(s, u_1(s), u'_1(s), u_2(s), u'_2(s), u''_2(s), u''''_2(s)) \, ds \right] \\
u_2(t) &= \eta_{21} (1 - t) h_{21}[(u_1, u_2)] + \lambda_2 \left[ \int_0^1 k_2(t, s) f_2(s, u_1(s), u'_1(s), u_2(s), u'_2(s), u''_2(s), u''''_2(s)) \, ds \right]
\end{align*}
\]
where
\[
k_1(t, s) = \begin{cases} s(1 - t), & s \leq t, \\ (1 - s), & s > t, \end{cases} \quad \text{and} \quad k_2(t, s) = \begin{cases} \frac{1}{6} s(1 - t)(2t - s^2 - t^2), & s \leq t, \\ \frac{1}{6} (1 - s)(2s - t^2 - s^2), & s > t. \end{cases}
\]

It is known that the kernels \( k_1 \) and \( k_2 \) that occur in (2.11) are continuous, non-negative, satisfy condition \( (C_7) \) and (e.g. [22,25,26])
\[
K_{10} = \frac{1}{8}, \quad \mu_1 = \pi^2, \quad K_{20} = \frac{5}{384}, \quad \mu_2 = \pi^4.
\]

By direct calculation, we obtain
\[
\begin{align*}
\frac{\partial k_1}{\partial t}(t, s) &= \begin{cases} -s, & s < t, \\ (1 - s), & s > t, \end{cases} \quad \text{and} \quad \frac{\partial k_2}{\partial t}(t, s) = \begin{cases} \frac{1}{6} s(-6t + s^2 + 3t^2 + 2), & s \leq t, \\ (1 - s)(-s^2 + 2s - 3t^2), & s > t, \end{cases} \\
\frac{\partial^2 k_2}{\partial t^2}(t, s) &= \begin{cases} s(t - 1), & s \leq t, \\ t(s - 1), & s > t, \end{cases} \quad \text{and} \quad \frac{\partial^3 k_2}{\partial t^3}(t, s) = \begin{cases} s, & s < t, \\ (s - 1), & s > t. \end{cases}
\end{align*}
\]

We may use
\[
\Phi_{20}(s) = \begin{cases} \frac{\sqrt{3}}{27} s(1 - s^2)^{3/2}, & \text{for } 0 \leq s \leq \frac{1}{2}, \\ \frac{\sqrt{3}}{27} (1 - s)s^2 (2 - s)^{3/2}, & \text{for } \frac{1}{2} < s \leq 1, \end{cases}
\]
\[
\Phi_{21}(s) = \frac{1}{6} s(2 + s^2), \quad [15],
\]
and, by direct calculation, we take
\[
\Phi_{10}(s) = \Phi_{22}(s) = s(1 - s), \quad \Phi_{11}(s) = \Phi_{23}(s) = |s - \frac{1}{2}| + \frac{1}{2}.
\]

Therefore the assumptions \( (C_1) - (C_3) \) are satisfied. By direct computation, we obtain
\[
K_{22} = \frac{1}{8}, \quad K_{11} = K_{23} = \frac{1}{2}, \quad K_{21} \leq \frac{5}{24}.
\]

Note that we have
\[
\gamma_{11}(t) = t, \quad \gamma_{11}'(t) = 1, \quad \gamma_{21}(t) = (1 - t), \quad \gamma_{21}'(t) = -1.
\]
and therefore we get
\[ \| \gamma_1 \|_\infty = \| \gamma_1' \|_\infty = \| \gamma_2 \|_\infty = \| \gamma_2' \|_\infty = 1, \| \gamma_2'' \|_\infty = \| \gamma_2''' \|_\infty = 0, \]

Thus the condition (2.5) is satisfied if
\[
\max \left\{ \frac{1}{2} \lambda \bar{f}_{1R} + \eta_1 H_{11R}, \frac{5}{24} \lambda \bar{f}_{2R} + \eta_2 H_{21R}, \frac{1}{2} \lambda \bar{f}_{2R} \right\} \leq R. \tag{2.12}
\]

Let us now fix the nonlinearities \( f_i \) and the functionals \( h_{1i} \), say
\[
f_1(t, u_1(t), u_1'(t), u_2(t), u_2'(t), u_2''(t)) = u_1^2(t)(2 - t \sin(u_1'(t) + u_2'(t)) - t^2(u_1(t) + u_2(t))),
\]
\[
f_2(t, u_1(t), u_1'(t), u_2(t), u_2'(t), u_2''(t)) = \sqrt{u_2(t)} e^{\sigma(u_1(t) + u_2'(t))},
\]
\[
h_{11}[(u_1, u_2)] = \int_0^1 (u_1'(t) + u_2'(t))^2 \, dt
\]
and
\[
h_{21}[(u_1, u_2)] = (u_1'(1/4))^4 + (u_2'(3/4))^4,
\]
and prove the existence of solutions in \( P \) with \( \| u \| \leq 1 \). Thus we fix \( R = 1 \). Since \( \bar{f}_{1i} \leq 3, \bar{f}_{2i} \leq e^2 \), \( H_{11} \leq 4, H_{21} \leq 2 \), the condition (2.12) is satisfied if the inequality
\[
\max \left\{ \frac{3}{2} \lambda \bar{f}_{1i} + 4 \eta_1, \frac{5e^2}{24} \lambda \bar{f}_{2i} + 2 \eta_2, \frac{e^2}{2} \lambda \bar{f}_{2i} \right\} \leq 1 \tag{2.13}
\]
holds. Note that \( f_2 \) satisfies condition (2.6) for every fixed \( \lambda_2 > 0 \), by choosing \( r \) sufficiently small. Therefore, for the range of parameters that satisfy the inequality (2.13) with \( \lambda_2 > 0 \), theorem 2.3 provides the existence of a solution of the system (2.10) in \( P \), with \( 0 < \| u \| \leq 1 \); this occurs, for example, for \( \lambda_1 = 1/10, \lambda_2 = 1/5, \eta_1 = 1/5, \eta_2 = 1/3 \).

We now use an elementary argument to prove a non-existence result.

**Theorem 2.5.** Assume that there exist \( \tau_i, \xi_{ij} \in (0, +\infty) \) such that
\[
0 \leq f_i(t, x_{1i}, \ldots, x_{ni}, x_{ni0}, \ldots, x_{ni0}) \leq \tau_i x_{0i}, \text{ on } [0, 1] \times \prod_{i=1}^n ([0, +\infty) \times \mathbb{R}^{m_i}), \]
\[
h_{ij}[u] \leq \xi_{ij} \| u \|_\infty, \text{ for every } u \in P, \quad i = 1 \ldots n, \quad j = 1 \ldots p_i,
\]
\[
\max_{i=1, \ldots, n} \left\{ \lambda_i \tau_i K_{\rho} + \sum_{j=1}^{p_i} \eta_{ij} \xi_{ij} \| y_{ij} \|_\infty \right\} < 1. \tag{2.14}
\]

Then the system (2.1) has at most the zero solution in \( P \).

**Proof.** Assume that there exist \( u \in P \setminus \{0\} \) such that \( Tu = u \). Then there exists \( i_0 \in \{1, \ldots, n\} \) such that \( \| u_{i_0} \|_\infty = \rho \), for some \( \rho > 0 \). Then, for every \( t \in [0, 1] \), we have
\[
u_{i_0}(t) = \lambda_{i_0} \int_0^1 k_{i_0}(t, s) f_{i_0}(s, u_1(s), \ldots, u_{(m_i)}(s), \ldots, u_n(s), \ldots, u_{(m_n)}(s)) \, ds + \sum_{j=1}^{p_{i_0}} \eta_{i_0 j} Y_{i_0 j}(t) h_{i_0 j}[u]
\]
\[
\leq \lambda_{i_0} \int_0^1 k_{i_0}(t, s) \tau_{i_0} u_{i_0} \, ds + \sum_{j=1}^{p_{i_0}} \eta_{i_0 j} Y_{i_0 j}(t) h_{i_0 j}[u]
\]
\[
\leq \lambda_{i_0} \int_0^1 k_{i_0}(t, s) \tau_{i_0} \rho \, ds + \sum_{j=1}^{p_{i_0}} \eta_{i_0 j} Y_{i_0 j}(t) \xi_{i_0 j} \rho
\]
\[
\leq \lambda_{i_0} \tau_{i_0} K_{\rho} \rho + \sum_{j=1}^{p_{i_0}} \eta_{i_0 j} \| Y_{i_0 j} \|_\infty \xi_{i_0 j} \rho. \tag{2.15}
\]

Taking the supremum for \( t \in [0, 1] \) in (2.15) gives \( \rho < \rho \), a contradiction. \( \blacksquare \)
We conclude by illustrating the applicability of theorem 2.5.

**Example 2.6.** Let us now consider the system

\[
\begin{align*}
    u_1''(t) + \lambda_1 u_1(t)(2 - t \sin(u_2(t)u_1'(t))) &= 0, \quad t \in (0, 1), \\
    u_2^{(4)}(t) &= \lambda_2 u_2(t)(2 - t \cos(u_1(t)u_2''(t))), \quad t \in (0, 1), \\
    u_1(0) = 0, \quad u_1(1) &= \eta_{11} 1/4 \cos^2(u_1'(1/4)u_2''(1/4)) \\
    u_2(0) &= \eta_{21} 1/3 \sin^2(u_1'(1/4)u_2''(3/4)), \quad u_2'(0) = u_2(1) = u_2''(1) = 0.
\end{align*}
\]

In this case, we may take \( \tau_1 = \tau_2 = 3, \xi_{11} = \xi_{21} = 1 \). Then the condition (2.14) reads

\[
    \max \left\{ \frac{3}{8} \lambda_1 + \eta_{11}, \frac{5}{128} \lambda_2 + \eta_{21} \right\} < 1. \tag{2.17}
\]

Since \((0, 0)\) is a solution of the system (2.16), for the range of parameters that satisfy the inequality (2.17), theorem 2.5 guarantees that the only possible solution in \( P \) of the BVP (2.16) is the trivial one; this occurs, for example, for \( \lambda_1 = 1, \lambda_2 = 5, \eta_{11} = 1/2, \eta_{21} = 1/3 \).

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