SYZYGIES OF COHEN-MACAULAY MODULES AND
GROTHENDIECK GROUPS

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Abstract. We study the converse of a theorem of Butler and Auslander-Reiten. We show that a Cohen-Macaulay local ring with an isolated singularity has only finitely many isomorphism classes of indecomposable summands of syzygies of Cohen-Macaulay modules if the Auslander-Reiten sequences generate the relation of the Grothendieck group of finitely generated modules. This extends a recent result of Hiramatsu, which gives an affirmative answer in the Gorenstein case to a conjecture of Auslander.

1. Introduction

Throughout this note, let \((R, m, k)\) be a Cohen-Macaulay local ring with an isolated singularity. We denote by \(\text{CM}(R)\) (resp. \(\text{mod}(R)\)) the category of (maximal) Cohen-Macaulay \(R\)-modules (resp. finitely generated \(R\)-modules) with \(R\)-homomorphisms.

Let \(G(\text{CM}(R))\) be the quotient of the free abelian group \(\bigoplus \mathbb{Z}[X]\) generated by the isomorphism classes \([X]\) of modules \(X\) in \(\text{CM}(R)\) by the subgroup generated by

\[
\{[X] + [Z] - [Y] \mid Y \cong X \oplus Z\}.
\]

Thus \(G(\text{CM}(R))\) is isomorphic to the free abelian group generated by the isomorphism classes of indecomposable Cohen-Macaulay \(R\)-modules.

We denote by \(\text{Ex}(\text{CM}(R))\) the subgroup of \(G(\text{CM}(R))\) generated by

\[
\{[X] + [Z] - [Y] \mid \text{there exists an exact sequence } 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \text{CM}(R)\}\.
\]

Then the quotient group \(G(\text{CM}(R))/\text{Ex}(\text{CM}(R))\) is nothing but the Grothendieck group \(K_0(\text{CM}(R))\) of \(\text{CM}(R)\) and therefore coincides with the Grothendieck group of \(\text{mod}(R)\).

We also denote by \(\text{AR}(\text{CM}(R))\) the subgroup of \(G(\text{CM}(R))\) generated by

\[
\left\{[X] + [Z] - [Y] \mid \text{there exists an Auslander-Reiten sequence } 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \text{ in } \text{CM}(R)\right\}.
\]

Concerning the relationship between \(\text{Ex}(\text{CM}(R))\) and \(\text{AR}(\text{CM}(R))\), the following theorem holds; see [5], [3 Proposition 2.2] and [10 Theorem 13.7].

**Theorem 1.1** (Butler, Auslander-Reiten). *If \(R\) is of finite CM type, then \(\text{Ex}(\text{CM}(R)) = \text{AR}(\text{CM}(R))\).*

Here we say that \(R\) is of *finite CM type* if there are only finitely many isomorphism classes of indecomposable Cohen-Macaulay \(R\)-modules.

Auslander conjectured that the converse of Theorem 1.1 holds. Our main result is the following theorem, which yields a weaker version of the converse of Theorem 1.1.

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Theorem 1.2. If $\text{Ex}(\text{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes \mathbb{Z} \mathbb{Q}$, then there exist only finitely many isomorphism classes of indecomposable summands of (first) syzygies of Cohen-Macaulay $R$-modules.

If $R$ is Gorenstein, then every Cohen-Macaulay $R$-module is a first syzygy of some Cohen-Macaulay $R$-module. Hence Theorem 1.2 recovers the following result, which is proved by Hiramatsu [7] and gives an affirmative answer to Auslander’s conjecture in the case of Gorenstein local rings.

Corollary 1.3 (Hiramatsu). Assume that $R$ is Gorenstein. If $\text{Ex}(\text{CM}(R)) = \text{AR}(\text{CM}(R))$, then $R$ is of finite CM type.

When $R$ is a two dimensional complete local ring and $k$ is algebraically closed, it is shown in [6, Corollary 3.3] that $R$ has a finite number of isomorphism classes of indecomposable summands of syzygies of Cohen-Macaulay modules if and only if $R$ is a rational singularity. This fact provides the following corollary.

Corollary 1.4. Assume that $R$ is a complete local ring of dimension two with $k$ algebraically closed. If $\text{Ex}(\text{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} = \text{AR}(\text{CM}(R)) \otimes \mathbb{Z} \mathbb{Q}$, then $R$ is a rational singularity.

In the rest of this note, we give a proof of Theorem 1.2.

2. Proof of our theorem

As in the introduction, we always assume that $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring with an isolated singularity. All $R$-modules are assumed to be finitely generated.

We denote by $\text{mod}(R)$ (resp. $\text{CM}(R)$) the stable category of $\text{mod}(R)$ (resp. $\text{CM}(R)$). These categories are defined in such a way that the objects are the same as those of $\text{mod}(R)$ (resp. $\text{CM}(R)$), and for objects $M, N$, the set of morphisms from $M$ to $N$ is $\text{Hom}_R(M, N)$, defined to be the quotient of $\text{Hom}_R(M, N)$ by the $R$-submodule consisting of homomorphisms factoring through free $R$-modules.

To give a proof of Theorem 1.2 we prepare several lemmas. The first one is given in [7, Lemma 2.1].

Lemma 2.1. There exists a Cohen-Macaulay $R$-module $X$ such that for any non-free Cohen-Macaulay $R$-module $M$ one has $\text{Hom}_R(M, X) \neq 0$.

We denote by $\Omega M$ the first syzygy of $R$-module $M$, and by $\text{Tr} M$ the (Auslander) transpose of $M$; see [10, Definition (3.5)]. The modules $\Omega M$ and $\text{Tr}$ are uniquely determined by $M$ up to free summands.

We have the lemma below; see [9, Proposition 2.7] for instance.

Lemma 2.2. Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an exact sequence in $\text{mod}(R)$ and $U$ an $R$-module. Then there exists a long exact sequence

$$\text{Hom}_R(U, \Omega A) \rightarrow \text{Hom}_R(U, \Omega B) \rightarrow \text{Hom}_R(U, \Omega C) \rightarrow \text{Hom}_R(U, A) \rightarrow \text{Hom}_R(U, B) \rightarrow \text{Hom}_R(U, C).$$

Here, we define $\Omega^{-1} M$ to be the module $\text{Tr} \Omega \text{Tr} M$ for an $R$-module $M$. The assignments $M \mapsto \Omega M$, $M \mapsto \text{Tr} M$ and $M \mapsto \Omega^{-1} M$ define additive endofunctors of $\text{mod}(R)$. Moreover, the following lemma holds; see [4, Corollary 3.3].
Lemma 2.3. Let \( X, Y \) be \( R \)-modules. Then there is an isomorphism
\[
\text{Hom}_R(X, \Omega Y) \to \text{Hom}_R(\Omega^{-1}X, Y),
\]
which is natural in \( X, Y \) (i.e. one has an adjoint pair \((\Omega^{-1}, \Omega) : \text{mod}(R) \to \text{mod}(R))\).

We denote by \( \Omega \text{CM}(R) \) the full subcategory of \( \text{mod}(R) \) consisting of first syzygies of Cohen-Macaulay \( R \)-modules.

Remark 2.4. It is easy to see that \( \Omega \text{CM}(R) \) is closed under direct summands. In particular, the following are equivalent.

1. There are only finitely many non-isomorphic indecomposable modules in \( \Omega \text{CM}(R) \).
2. There are only finitely many non-isomorphic indecomposable summands of modules in \( \Omega \text{CM}(R) \).

If one/both of these conditions is/are satisfied, we say that \( R \) is of finite \( \Omega \text{CM} \) type.

Now we give some properties of modules in \( \Omega \text{CM}(R) \).

Lemma 2.5. If \( M \in \Omega \text{CM}(R) \), then \( \Omega^{-1}M \) is in \( \text{CM}(R) \) and \( M \cong \Omega \Omega^{-1}M \) up to free summands.

Proof. Since \( M \) is in \( \Omega \text{CM}(R) \), there is a Cohen-Macaulay module \( N \) such that \( M \) is the first syzygy of \( N \). By the proof of [2, Proposition 2.21], there is an exact sequence \( 0 \to R^{\oplus a} \to \Omega^{-1}M \oplus R^{\oplus b} \to N \to 0 \) with some integers \( a, b \). This implies that \( \Omega^{-1}M \) is a Cohen-Macaulay module. As \( M \) is a syzygy, \( M \) is isomorphic to \( \Omega \Omega^{-1}M = \Omega \text{Tr} \Omega \text{Tr} M \) up to free summands by [2, Theorem 2.17]. \( \square \)

Next we investigate the non-free part of a given module.

Lemma 2.6. Let \( M \) be a finitely generated \( R \)-module, and \( \hat{R} \) the completion of \( R \).

1. \( M \) has an \( R \)-free summand if and only if \( M \otimes_R \hat{R} \) has a \( \hat{R} \)-free summand.
2. There is a unique decomposition \( M \cong M' \oplus F \) of \( M \) up to isomorphism with \( F \) free such that \( M' \) has no free summands. We call this module \( M' \) the non-free part of \( M \).
3. Let \( A, B \) be finitely generated \( R \)-modules. If \( A \) is a direct summand of \( B \), then \( A \) is a direct summand of \( B \).

Proof. (1) The assertion follows from [3, Corollary 1.15 (i)].

(2) We can take a maximal free summand \( R^{\oplus a} \) of \( M \) to have a decomposition \( M \cong M' \oplus R^{\oplus b} \) where \( M' \) has no free summands. Suppose that there is another decomposition \( M \cong M'' \oplus R^{\oplus b} \) where \( M'' \) has no free summands. Taking the completion, we have \( M \otimes_R \hat{R} \cong (M' \otimes_R \hat{R}) \oplus \hat{R}^{\oplus b} \cong (M'' \otimes_R \hat{R}) \oplus \hat{R}^{\oplus b} \). By (1), \( M' \otimes_R \hat{R} \) and \( M'' \otimes_R \hat{R} \) have no free summands. Since the Krull-Schmidt property holds over \( \hat{R} \), we have \( M' \otimes_R \hat{R} \cong M'' \otimes_R \hat{R} \) and \( a = b \). Using [3, Corollary 1.15 (ii)], we have \( M' \cong M'' \).

(3) Suppose that \( A \) is a direct summand of \( B \). Then \( A \) is also a direct summand of \( B \). Hence we have a decomposition \( B \cong A \oplus C \). It follows from (2) that the non-free part \( B \) of \( B \) is isomorphic to the module \( A \oplus C \). In particular, \( B \) has \( A \) as a direct summand. \( \square \)

Since there is an isomorphism \( \text{Hom}_R(M, N) \cong \text{Tor}^R_1(\text{Tr}(M), N) \) for finitely generated \( R \)-modules \( M, N \) (see [10, Lemma (3.9)]) and since we assume that \( R \) is an isolated singularity, we can show that the length of the \( R \)-module \( \text{Hom}_R(M, N) \) is finite for any
Let \( M, N \) in \( \text{CM}(R) \). We denote by \([M, N]\) the integer \( \text{length}_R(\text{Hom}_R(M, N)) \). The following proposition plays a key role in the proof of our theorem. For the definition and basic properties of an Auslander-Reiten sequence, we refer the reader to [8, 10].

**Proposition 2.7.** Let \( 0 \to A \to B \to C \to 0 \) be an Auslander-Reiten sequence, and \( U \) be a non-free Cohen-Macaulay \( R \)-module. Then the following hold.

1. The induced sequence \( \text{Hom}_R(U, B) \to \text{Hom}_R(U, C) \to 0 \) is exact if and only if \( C \) is not a direct summand of \( U \).
2. Suppose that \( U \) is an indecomposable module in \( \Omega \text{CM}(R) \).
   a. If \( U \) is not isomorphic to the non-free part of \( \Omega C \), then the induced sequence \( 0 \to \text{Hom}_R(U, A) \to \text{Hom}_R(U, B) \) is exact.
   b. If \([U, C] + [U, A] - [U, B] \neq 0\), then \( U \) is isomorphic to either \( C \) or the non-free part of \( \Omega C \).

**Proof.** (1) Assume that \( C \) is not a direct summand of \( U \). Using the lifting property of an Auslander-Reiten sequence, every homomorphism from \( U \) to \( C \) factors through the map \( B \to C \). This means that \( \text{Hom}_R(U, B) \to \text{Hom}_R(U, C) \) is surjective. Conversely, suppose that \( C \) is a direct summand of \( U \). Then there is a split epimorphism \( f : U \to C \). Let \( g : C \to U \) be the right-inverse of \( f \). If \( \text{Hom}_R(U, B) \to \text{Hom}_R(U, C) \) is surjective, then so is the morphism \( \text{Hom}_R(U, B) \to \text{Hom}_R(U, C) \) because of the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_R(U, B) & \longrightarrow & \text{Hom}_R(U, C) \\
\downarrow & & \downarrow \\
0 & & \text{Ext}_R^1(U, A)
\end{array}
\]

with exact rows and columns; see the proof of [9, Proposition 2.7 (2)]. Hence there is a lift \( h : U \to B \) of \( f \). The morphism \( hg : C \to B \) is the right-inverse of the morphism \( B \to C \). This contradicts the definition of the Auslander-Reiten sequences.

(2a) Let \( K \) be the kernel of the map \( \text{Hom}_R(U, A) \to \text{Hom}_R(U, B) \). By Lemma 2.2, the sequence \( \text{Hom}_R(U, \Omega B) \to \text{Hom}_R(U, \Omega C) \to K \to 0 \) is exact. Then the equality \( K = 0 \) is equivalent to the exactness of \( \text{Hom}_R(\Omega^{-1}U, B) \to \text{Hom}_R(\Omega^{-1}U, C) \to 0 \), by using Lemma 2.3. Lemma 2.5 implies that \( \Omega^{-1}U \) is Cohen-Macaulay. By (1), \( K \) is zero if and only if \( C \) is not a direct summand of \( \Omega^{-1}U \).

Suppose that \( C \) is a direct summand of \( \Omega^{-1}U \). By Lemma 2.6, there is a free \( R \)-module \( F \) such that \( \Omega C \) is a direct summand of \( U \oplus F \). Lemma 2.6 (3) implies that the non-free part of \( \Omega C \) is a direct summand of \( U \), and is isomorphic to \( U \) as \( U \) is indecomposable.

(2b) Let \( L \) be the cokernel of the map \( \text{Hom}_R(U, B) \to \text{Hom}_R(U, C) \). Then we have an equality \([U, C] + [U, A] - [U, B] = \text{length}_R(L) + \text{length}_R(K)\). By (1), \( \text{length}_R(L) \neq 0 \) implies that \( U \) is isomorphic to \( C \). By (2a), \( \text{length}_R(K) \neq 0 \) implies that \( U \) is isomorphic to the non-free part of \( \Omega C \).

Now we can give a proof of our theorem.
Proof of Theorem 1.2. Let \( X \) be the module that satisfies the conditions in Lemma 2.1. Then there is an exact sequence with a free module \( P \):

\[
0 \to \Omega X \to P \to X \to 0.
\]

Since \( \text{Ex}(\mathcal{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} = \mathcal{AR}(\mathcal{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} \), there are a finite number of indecomposable Cohen-Macaulay \( R \)-modules \( C_1, \ldots, C_n \) and an equality in \( \mathcal{G}(\mathcal{CM}(R)) \):

\[
a([X] + \Omega X - [P]) = \sum_{i=1}^{n} b_i([A_i] + [C_i] - [B_i]),
\]

where \( a \) is a positive integer, \( b_i \) are integers and \([A_i] + [C_i] - [B_i]\) come from Auslander-Reiten sequences \( 0 \to A_i \to B_i \to C_i \to 0 \). We have an equality in \( \mathbb{Z} \):

\[(2.7.1)\quad a[U, X \oplus \Omega X] = \sum_{i=1}^{n} b_i([U, A_i] + [U, C_i] - [U, B_i])\]

for each non-free indecomposable module \( U \) in \( \Omega \mathcal{CM}(R) \), since in general the equality

\[
[U, \bigoplus_{s=1}^{t} L_s] = \sum_{s=1}^{t} [U, L_s]
\]

holds for any \( R \)-modules \( L_1, \ldots, L_t \). The left-hand side of (2.7.1) is nonzero by the choice of \( X \) in Lemma 2.1 and hence so is the right-hand side. By Proposition 2.7 (2b), this can occur only when \( U \) is isomorphic to either \( C_i \) or the non-free part of \( \Omega C_i \) for some \( i \), and we conclude that the number of isomorphism classes of such modules \( U \) is finite.

Remark 2.8. The converse of Theorem 1.1 has been proved by Auslander for artin algebras and by Auslander-Reiten for one dimensional complete local domains. We shall give examples of finite dimensional local algebras and one dimensional complete local domains which are of finite \( \Omega \mathcal{CM} \) type but not of finite \( \mathcal{CM} \) type. Thus finite \( \Omega \mathcal{CM} \) type is not sufficient to hold the equality \( \text{Ex}(\mathcal{CM}(R)) = \mathcal{AR}(\mathcal{CM}(R)) \), and the converse of Theorem 1.2 is not true in general if we replace the condition \( \text{Ex}(\mathcal{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} = \mathcal{AR}(\mathcal{CM}(R)) \otimes \mathbb{Z} \mathbb{Q} \) with the condition \( \text{Ex}(\mathcal{CM}(R)) = \mathcal{AR}(\mathcal{CM}(R)) \).

Example 2.9. Let \( R = k[X, Y]/(X, Y)^2 \) with \( k \) a field. Then \( R \) is a finite dimensional local \( k \)-algebra and not of finite \( \mathcal{CM} \) type. Since the first syzygy \( M \) of a non-free \( R \)-module is a submodule of a direct sum of copies of the maximal ideal \( \mathfrak{m} \), the module \( M \) is annihilated by \( \text{ann}(\mathfrak{m}) = \mathfrak{m} \). So \( M \) is a module over \( R/\mathfrak{m} \). In particular, every non-free indecomposable module in \( \Omega \mathcal{CM}(R) \) is isomorphic to \( R/\mathfrak{m} \), and \( R \) is of finite \( \Omega \mathcal{CM} \) type.

Example 2.10. Let \( S = k[[t]] \) with \( k \) a field, \( n \geq 1 \) be an integer and \( R = k[[t^n, t^{n+1}, \ldots, t^{2n-1}]] \) be the subring of \( S \). Then \( R \) is a one dimensional complete local domain and the maximal ideal \( \mathfrak{m} = t^nS \) of \( R \) is isomorphic to \( S \) as an \( R \)-module. Let \( M \) be a non-free indecomposable module in \( \Omega \mathcal{CM}(R) \). We show that \( M \) can be regarded as an \( S \)-submodule of some free \( S \)-module. In fact, there exist a Cohen-Macaulay \( R \)-module \( N \) and a short exact sequence \( 0 \to M \to R^{\oplus a} \to N \to 0 \) coming from a minimal free resolution of \( N \). By the minimality, we have a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathfrak{m}^{\oplus a} & \longrightarrow & L & \longrightarrow & 0 \\
| & | & | & | & | & | & | & | \\
0 & \longrightarrow & M & \longrightarrow & R^{\oplus a} & \longrightarrow & N & \longrightarrow & 0.
\end{array}
\]
By the snake lemma, \( L \) is viewed as a submodule of \( N \) and thus a Cohen-Macaulay \( R \)-module. Replacing \( m \) with \( S \) and multiplying \( t^n \), we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & M & \rightarrow & S^{a} & \rightarrow & L & \rightarrow & 0 \\
& & \downarrow{t^n} & & \downarrow{t^n} & & \downarrow{t^n} & & \\
0 & \rightarrow & M & \rightarrow & S^{a} & \rightarrow & L & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
M/t^nM & \rightarrow & S^{a}/t^nS^{a} & \rightarrow & 0 & & 0 & & \\
\end{array}
\]

where the rows and columns are both exact. Applying the snake lemma again, we see that the morphism \( M/t^nM \rightarrow S^{a}/t^nS^{a} \) in the diagram above is injective, as \( L \) is a Cohen-Macaulay \( R \)-module. Since \( t^{n+1} \) annihilates \( S^{a}/t^nS^{a} \), it also annihilates \( M/t^nM \). Hence \( t^{n+1}M \subset t^nM \). Identifying \( M \) as an \( R \)-submodule of \( S^{a} \), we observe \( tM \subset M \), which makes \( M \) be an \( S \)-submodule of \( S^{a} \). Since \( S \) is a discrete valuation ring, the submodule \( M \) of the free \( S \)-module \( S^{a} \) is free. This shows that the nonisomorphic indecomposable \( R \)-modules in \( \Omega CM(R) \) are \( R \) and \( S(\cong m) \), which especially says that \( R \) is of finite \( \Omega CM \) type. On the other hand, \( R \) is not of finite \( CM \) type when \( n \geq 4 \) (see [8, Theorem 4.10]).

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