AN ESTIMATE FOR THE MULTIPLICITY OF BINARY RECURRENCES

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Abstract

We use a refined version of Roth’s Lemma, proved with the help of Faltings’ Product Theorem, in order to give an upper bound for the multiplicity of a binary linear recurrence.

Résumé

Nous utilisons une version du Lemme de Roth, provenant du Théorème du Produit de Faltings, pour majorer la multiplicité d’une récurrence linéaire binaire.

Version française abrégée

Dans l’article [7] H.-P. Schlickewei donne une majoration, indépendante du corps des nombres, de la multiplicité d’une récurrence linéaire binaire. La contribution principale à cette majoration est exprimée par une version, pour la droite projective, du théorème du sous-espace de Schmidt. Nous améliorons cette majoration en utilisant la même méthode, mais faisant appel à un Lemme de Roth plus puissant, démontré à partir d’idées liées au théorème du Produit de Faltings ([4], [3], [6]). Même si ce résultat n’est pas comparable à la majoration remarquable obtenue récemment par F. Beukers et H.-P. Schlickewei [2], elle montre jusqu’à quel point on peut arriver en utilisant les techniques du théorème du sous-espace de Schmidt dans ce contexte. Nous démontrons,

Théorème 1.1. Soient \( a, b, \alpha, \beta \) nombres complexes, tels que au moins un entre \( \alpha, \beta \) n’est pas une racine de l’unité. Alors il y a au moins \( 2^{57} \) entiers \( m \in \mathbb{Z} \) tels que

\[
ax^m + b\beta^m + 1 = 0.
\]

Pour démontrer ce théorème il nous faut la proposition suivante. Soit \( S \) un sous-ensemble fini de \( M_K \) contenant les places à l’infini. Pour tout \( v \in S \) nous donnons deux formes linéaires \( L_{1,v}(x), L_{2,v}(x) \in \{X_1, X_2, X_1 + X_2\} \) et deux nombres réels \( e_{1,v}, e_{2,v} \) tels que \( \sum_{v \in S}(e_{1,v} + e_{2,v}) = 0 \), et pour tout sous-ensembles \( S' \) de \( S \), et chaque \( (i(v))_{v \in S'} \) avec \( i(v) \in \{1, 2\} \), \( |\sum_{v \in S'} e_{i(v),v}| \leq 1 \). Ce sont les conditions (3.1) -- (3.3) de [7]. On obtient,

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Proposition 2.1. Supposons que $L_{v,i}, e_{v,i}$ soient comme en haut. Soit $0 < \delta < 1$ un nombre réel. Alors pour tout nombres réels $Q > 4^\delta$ les solutions $x \in K^2$ de
\[
\|L_{i,v}(x)\|_v < Q^{e_{v,i} - \delta[K_v:K]/[K_v:K]}, \quad v \in S, \quad v \not\mid \infty, \quad i \in \{1, 2\},
\]
\[
\|L_{i,v}(x)\|_v \leq Q^{e_{v,i}}, \quad v \in S, \quad v \not\mid \infty, \quad i \in \{1, 2\},
\]
\[
\|x\|_v \leq 1, \quad v \in S,
\]
sont contenues dans au plus $2^{227/10} \delta^{-3} \log \delta^{-1}$ droites de $K^2$.

Nous rappelons que l’indice $i_x, r(P)$ d’un polynôme $P$ en $2m$ variables, multihomogène de multidegré $r = (r_1, \cdots, r_m) \in \mathbb{Z}_m$ en un point $x \in K^{2m}$ est la multiplicité de $P$ en $x$ calculée avec le poids $1/r_i$ dans la $i$-ème direction. Nous avons utilisé ici la version suivante du Lemme de Roth.

Lemme 3.1. (Lemme de Roth) Soit $m \geq 2$ un entier, $r = (r_1, \cdots, r_m) \in \mathbb{Z}_m$ des entiers positifs et $0 < \vartheta \leq m^2(m+1)$ un nombre réel. Supposons que pour tout entier $h$ avec $1 \leq h \leq m - 1$, $r_h/r_{h+1} \geq m^2(m+1)$. Soit $P$ un polynôme non-nul en $2m$ variables, multihomogène de multidegré $r$ et soient $L_1, \cdots, L_m$ des formes lineaires binaires avec coefficients en $\mathbb{Q}$ tels que, pour tout entier $h$ avec $1 \leq h \leq m$
\[
 r_h \log H(L_h) \geq \frac{7m(m!)^2m}{2 \vartheta} \sum_{i=1}^{m} r_i + \log H(P),
\]
Alors, pour chaque entier $h$ avec $1 \leq h \leq m$, il y a un $x_h \in V(L_h)(K)$ tel que pour $x = (x_1, \cdots, x_m)$ nous avons $i_{x,r}(P) < \vartheta$.

Le théorème principal decoule maintenant comme dans [7]. Nous utilisons le même principe des trous et les mêmes choix des paramètres.

1. Introduction

In his paper [7] H.-P. Schlickewei gives an upper bound, independent from the degree of the number field, for the multiplicity of a binary linear recurrence. The main contribution to his bound comes from a particular version of the Schmidt Subspace Theorem in the case of the projective line. We improve the upper bound for the multiplicity of a binary linear recurrence using the same method but applying a refined version of Roth’s Lemma, which combines the ideas of [3] and [6] arising from the Faltings’ Product Theorem ([4]). Although this result falls short of the remarkable bound recently obtained by F. Beukers and H.-P. Schlickewei [2], this method still gives better bounds compared to the result of E. Bombieri, J. Müller and M. Poe [1]. We obtain,

Theorem 1.1. Let $a, b, \alpha, \beta$ be complex numbers, such that one at least among $\alpha, \beta$ is not a root of unity. Then there are at most
\[
2^{57}
\]
integers $m \in \mathbb{Z}$ such that
\[
a \alpha^m + b \beta^m + 1 = 0.
\]
Corollary 1.2. 1. For \(0 \neq \nu_0, \nu_1\) complex numbers let
\[ u_{n+2} = \nu_1 u_{n+1} + \nu_0 u_n \]
be a binary linear recurrence sequence of complex numbers. Suppose that one at least of the roots \(\alpha_1, \alpha_2\) of the polynomial \(z^2 - \nu_1 z - \nu_0\) is not a root of unity, and if \(\alpha_1 \neq \alpha_2\) assume moreover that \(\alpha_1/\alpha_2\) is not a root of unity. Then \(\{u_n\}\) has multiplicity
\[ U \leq 2^{57}. \]
2. Let \(0 \neq \mu_0, \mu_1, \mu_2\) complex numbers, let us consider a ternary sequence
\[ v_{m+3} = \mu_2 v_{m+2} + \mu_1 v_{m+1} + \mu_0 \]
with \(|\nu_0| + |\nu_1| + |\nu_2| \neq 0\). If the polynomial \(z^3 - \mu_2 z^2 - \mu_1 z - \mu_0\) has three distinct roots \(\alpha_1, \alpha_2, \alpha_3\), assume that at least one of the quotients \(\alpha_1/\alpha_3, \alpha_2/\alpha_3\) is not a root of unity. Then \(\{v_n\}\) has zero-multiplicity
\[ U(0) \leq 2^{57}. \]

Proof. We prove this corollary as in [7] §1 using Theorem 1.1 instead of [7] Theorem 1. \(\square\)

Here for a complex number \(c\) we define the \(c\)-multiplicity \(U(c)\) of \(\{u_n\}\) as the number of solutions \(m \in \mathbb{Z}\) of the equation \(u_m = c\) and we write \(U = \sup_{c} U(c)\) and call \(U\) the multiplicity of the sequence.

Theorem 1 of [7] differs from Theorem 1.1 in that instead of \(2^{57}\) it has
\[ 2^{223} \]
as upper bound for the number of solutions of (1). We refer to [7] for a discussion concerning known results and conjectures on linear recurrences.

2. Subspace Theorem

Let \(K\) be an algebraic number field. Denote its ring of integers by \(O_K\) and its collection of places (equivalence classes of absolute values) by \(M_K\). For \(v \in M_K\), \(x \in K\) we define the absolute value \(|x|_v\) by
1. \(|x|_v = |\sigma(x)|^{1/[K:\mathbb{Q}]}\) if \(v\) corresponds to a real embedding \(\sigma : K \hookrightarrow \mathbb{C}\),
2. \(|x|_v = |\sigma(x)|^{2/[K:\mathbb{Q}]}\) if \(v\) corresponds to a pair of conjugate complex embeddings \(\sigma, \overline{\sigma} : K \hookrightarrow \mathbb{C}\),
3. \(|x|_v = (N\mathcal{P})^{-ord_P(x)}/[K:\mathbb{Q}]\) if \(v\) corresponds to the prime ideal \(\mathcal{P}\) of \(O_K\).

Here \(N\mathcal{P} = #(O_K/\mathcal{P})\) is the norm of \(\mathcal{P}\) and \(ord_P(x)\) is the exponent of \(\mathcal{P}\) in the prime ideal decomposition of the principal ideal generated by \(x\), the order of 0 is \(\infty\). We denote by \(K_v\) the algebraic closure of the \(v\)-adic completion of \(K\). In the first two cases we call \(v\) infinite and write \(v\nmid \infty\), in case 3 we call \(v\) finite and write \(v \nmid \infty\). These absolute values satisfy the product formula \(\prod_v |x|_v \in M_K = 1\) for \(x \in K^*\). If \(x = (a_1, \cdots, a_m) \in K^m \setminus \{0\}\) we put
\[
||x||_v = \left( \sum_{i=1}^m |a_i|^{2[K:\mathbb{Q}]} \right)^{1/[K:\mathbb{Q}]}, \quad \text{if } v \text{ is real,}
\]
\[
||x||_v = \left( \sum_{i=1}^m |a_i|^{[K:\mathbb{Q}]} \right)^{1/[K:\mathbb{Q}]}, \quad \text{if } v \text{ is complex,}
\]
\[
||x||_v = \max\{|a_1|, \cdots, |a_m|\}, \quad \text{if } v \nmid \infty.
\]
Now define the *height* of \( x \) as

\[ H(x) = \prod_{v \in M_K} \|x\|_v. \]

By the product formula this defines a function on the projective space \( \mathbb{P}^{m-1}(K) \). Further it depends only on the point \( x \) and not on the choice of the number field \( K \) containing the coordinates of \( x \). For a linear form \( L(x) = a_1x_1 + \cdots + a_nx_n \) with algebraic coefficients, we define the height \( H(L) \) as the height of the point \( (a_1, \cdots, a_m) \). Moreover we define the height \( H(V(L)) \) of the \( (n-1) \)-dimensional linear subspace of \( K^n \)

\[ V(L) = \{ x \in K^n : L(x) = 0 \} \]

as the height of the linear form \( L \). Similarly the height of a polynomial is the height of the sequence of its coefficients.

Let us consider for \( n = 2 \) the set of linear forms given by

\[ L = \{ L_1(x) = x_1, \ L_2(x) = x_2, \ L_3(x) = x_1 + x_2 \}. \]

Let \( S \) be a finite subset of \( M_K \) containing all infinite places. We suppose that for each \( v \in S \) we are given a pair of different linear forms \( L_{1,v}(x), L_{2,v}(x) \) out of \( L \) and a pair of real numbers \( e_{1,v}, e_{2,v} \) such that

\[ \sum_{v \in S} (e_{1,v} + e_{2,v}) = 0, \tag{2} \]

and for each subset \( S' \) of \( S \), and any tuple \( (i(v))_{v \in S'} \) with \( i(v) \in \{1, 2\} \),

\[ |\sum_{v \in S'} e_{i(v),v}| \leq 1. \tag{3} \]

These are the conditions (3.1) – (3.3) of [7]. Consider, for a real number \( 0 < \delta < 1 \), the simultaneous inequalities

\[ \|L_{i,v}(x)\|_v < Q^{e_{i,v} - \delta|K_v:K|/|K:Q|}, \quad v \in S, \quad v|\infty, \quad i \in \{1, 2\}, \tag{4} \]

\[ \|L_{i,v}(x)\|_v \leq Q^{e_{i,v}}, \quad v \in S, \quad v \nmid \infty, \quad i \in \{1, 2\}, \]

\[ \|x\|_v \leq 1, \quad v \in S, \]

which correspond to condition (3.4) of [7]. We are now ready to state our improvement of Schlickewei’s Lemma 3.1 [7].

**Proposition 2.1.** Suppose that \( \delta, L_{v,i}, e_{v,i} \) are as above and that (2), (3) hold. Then for all real numbers

\[ Q > 4^\delta \]

the solutions \( x \in K^2 \) of (4) are contained in the union of at most

\[ 2^{227/10} \delta^{-3} \log \delta^{-1} \]

one-dimensional linear subspaces of \( K^2 \).
3. Roth’s Lemma

Let \( P(x_{11}, x_{12}, \ldots; x_{m1}, x_{m2}) \) be a polynomial with rational coefficients, multihomogeneous of multidegree \( \mathbf{r} = (r_1, \ldots, r_m) \). Given a point \( x \in K^m \) we define the index \( i_{x, \mathbf{r}}(P) \) of \( P \) with respect to \( (x, \mathbf{r}) \) as the weighted multiplicity of \( P \) at \( x \) with weights \( 1/d_j \). This turns out to be the same as defining the index with respect to binary linear forms as in [7] §5.

**Lemma 3.1. (Roth’s Lemma)** Let \( m \geq 2 \) be an integer, \( \mathbf{r} = (r_1, \ldots, r_m) \) a \( m \)-tuple of positive integers and \( 0 < \vartheta \leq m^2(m+1) \) a real number. Suppose that for all integers \( h \) with \( 1 \leq h \leq m-1 \)

\[
\frac{r_h}{r_{h+1}} \geq \frac{m^2(m+1)}{\vartheta}.
\]

Furthermore, let \( P \) be a non-zero polynomial in \( 2m \) variables, multihomogeneous of multidegree \( \mathbf{r} \) and let \( L_1, \ldots, L_m \) be binary linear forms with coefficients in \( \mathbb{Q} \) such that, for all integers \( h \) with \( 1 \leq h \leq m \)

\[
\frac{r_h}{r_{h+1}^m} \geq \frac{7m!(m!)^2}{2\vartheta^m} \sum_{i=1}^{m} r_i + \log H(P),
\]

Then, for all integers \( h \) with \( 1 \leq h \leq m \), there is a \( x_h \in V(L_h)(K) \) such that for \( x = (x_1, \ldots, x_m) \) we have

\[
i_{x, \mathbf{r}}(P) < \vartheta.
\]

**Proof.** We follow the proof of [3] Theorem 3 avoiding [7] Lemma 9 but using [6] Proposition 5.3 instead. Let \( h \) be an integer with \( 1 \leq h \leq m-1 \) then

\[
r_h \log H(V(L_h)) \leq \frac{m!(m+1)^m}{\vartheta^m} \left( \frac{m!}{2} \sum_{i=1}^{m} r_i + m!(\log H(P) + \log(2^{-1+\sum_{i=1}^{m} r_i}(\sum_{i=1}^{m} r_i)^{m-1})) \right)
\]

The difference comes out in [3] (5.3) where we obtain the inequality

\[
r_h \log H(V(L_h)) \leq \frac{(m!)^2m^m}{\vartheta^m} \left( \frac{1}{2} + \log 2 \right) \sum_{i=1}^{m} r_i + (m-1) \log(\sum_{i=1}^{m} r_i) + \log H(P) \leq \frac{7m!(m!)^2}{2\vartheta^m} \sum_{i=1}^{m} r_i + \log H(P).
\]

Since \( H(L_h) = H(V(L_h)) \) this completes the proof of Lemma 3.1. \( \square \)

4. Proof of the Subspace Theorem

Let \( g_1(Q), g_2(Q), V_h(Q) \), for \( h \) integer with \( 1 \leq h \leq m \), be as in [7] §5. Then we have the following version of [7] Lemma 6.1,

**Lemma 4.1.** Suppose \( 0 < \delta < 1 \) is real and that

\[
m > 28800\delta^{-2}.
\]

Put

\[
E = \frac{m^2(m+1)}{240}, \quad F = \frac{7}{2} m(m!)^2 \left( \frac{m}{480} \right)^m.
\]
Then the numbers $Q$ with

\begin{align}
(7) & \quad L_1(g_1(Q))L_2(g_1(Q))L_3(g_1(Q)) \neq 0, \\
(8) & \quad \lambda_1(Q) < Q^{-\delta}, \\
(9) & \quad Q^{\delta^2} > 2^{600mF},
\end{align}

are contained in the union of at most $m - 1$ intervals of the type

$$Q_h < Q \leq Q_h^E.$$

**Proof.** The proof goes along the same lines as the proof of [7] Lemma 6.1. We construct as in loc. cit. the numbers $Q_h$ for $1 \leq h \leq m$ with $Q_{h+1} \geq Q_h^E$, we choose as there $\varepsilon = \delta/60$ and put $r_h = \lceil r_1 \log Q_1/\log Q_k \rceil + 1$. Then the polynomial $P$ of [7] Lemma 5.1 has index $\geq m\varepsilon$ with respect to $(V_1(Q_1), \ldots, V_m(Q_m); r)$. Put $\vartheta = 480$. By construction, for all integers $h$ with $1 \leq h \leq m - 1$, we have $r_h/r_{h+1} \geq E/2 = m^2(m+1)/r^{-1}$, so (5) is satisfied. If $\Gamma = \delta/10$, then we obtain as in [7] Lemma 6.1 that

$$r_h \log H(V_h) > r_1 \Gamma \log H(V_1).$$

Moreover by [7] Lemma 4.2 and (9),

$$\log H(V_h) \geq \Gamma \log H(V_h) > \Gamma^2 \log Q_h = \frac{\delta^2}{100} \log Q_h > 6mF \log 2.$$

By [7] Lemma 5.1 $P$ has log $H(P) < 4mr_1 \log 2$. Finally by combining this with (10) we get

$$\frac{7m(m!)^2m^m}{2^d^m} \left( \sum_{j=1}^{m} r_j + \log H(P) \right) < F(mr_1 + 4mr_1 \log 2) < 6mFr_1 \log 2 < \frac{\delta^2 r_1}{100} \log Q_h < \Gamma r_1 \log H(V_1) \leq r_h \log H(V_h).$$

The conclusion of Lemma 3.1 is that there is a point $x = (x_1, \ldots, x_m) \in K^m$ such that for all integers $h$ with $1 \leq h \leq m$, $x_h \in V_h$ and

$$\text{ind}_{x,r}P < \vartheta = 480 < \frac{8.60}{\delta} = \frac{8}{\varepsilon} < m\varepsilon.$$

This yields the desired proof. \(\square\)

The proof of Proposition 2.1 now follows easily. It is clear from [7] §6 that there are no more than

$$m(1 + \frac{4}{\delta} \log E) + (1 + \frac{4}{\delta} \log(300\delta^{-1}F))$$

subspaces. Since $0 < \delta < 1$ we can choose $m \leq 28801.\delta^{-2}$, then we get

$$\log E < 26 + 6 \log \delta^{-1},$$

and by Stirling formula

$$\log(300\delta^{-1}F) < 767865\delta^{-2} \log \delta^{-1}.$$  

The bound (11) does not exceed $2^{227/10}\delta^{-3} \log \delta^{-1}$, and this confirms Proposition 2.1.
5. Proof of the Main Theorem

We follow here the same proof as [7] §10. We use $\delta = 1/9$. By Proposition 2.1 for values of $Q > 4^9$, given any pair $(i(v), (e_{i,v}))$ of [7] Lemma 8.1, the solutions of (1) are contained in the union of not more than

$$2^{227/10} \cdot 3^6 \log 3$$

one-dimensional linear subspaces. The other constants that are relevant for our estimate remain the same as in [7] §10, thus the number of solutions of (1) is bounded above by

$$2(2^7 \cdot 4800.36 \log 4 + 1 + 2^{227/10} \cdot 6^{12} \log 3) < 2^{57}.$$

This proves Theorem 1.1.

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