Equivalence of the Bergman and Teichmüller metrics on Teichmüller spaces

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1 Introduction

Let $R$ be a compact Riemann surface with genus $g > 1$. Denote by $\text{Teich}(R)$ the Teichmüller space of $R$. There are two canonical invariant metrics on $\text{Teich}(R)$, namely, the Teichmüller and Weil-Petersson metrics. By Bers embedding one can regard $\text{Teich}(R)$ as a bounded domain of holomorphy in $\mathbb{C}^{3g-3}$. Hence it carries four classical invariant metrics: the Carathéodory, Bergman, Kobayashi and Kähler-Einstein metrics. Royden’s theorem [12] states that the Teichmüller metric coincides with the Kobayashi metric. The Weil-Petersson metric is incomplete (cf. [6], [15]), while all the other metrics are complete since the Carathéodory metric is (cf. [3]). Recently, McMullen [7] introduced a new invariant metric $g_{1/l}$ equivalent to the Teichmüller metric in order to prove that the moduli space of Riemann surfaces is Kähler hyperbolic. By using the McMullen metric, Liu, Sun and Yau [5] proved the equivalence of the Teichmüller and Kähler-Einstein metrics. In this paper, we use McMullen’s $g_{1/l}$ metric and Takhtajan-Teo’s Kähler potential of the Weil-Petersson metric [13] together with the classical $L^2$—estimate to show the following

**Theorem 1.1.** The Bergman and Teichmüller metrics are equivalent on $\text{Teich}(R)$.

Throughout the paper, $A = O(B)$ means $A \leq CB$ and equivalence $A \asymp B$ means $\frac{1}{C}B \leq A \leq CB$ where $C > 0$ is a uniform constant on $\text{Teich}(R)$.

It is also interesting to invest the boundary behavior of the Bergman kernel and metric if we regard $\text{Teich}(R)$ as a bounded domain in $\mathbb{C}^{3g-3}$. Let $K_T$ denote the Bergman kernel function, let $\text{dist}_B$ be the Bergman distance and $\delta_T$ be the Euclidean boundary distance. We have
Theorem 1.2. (i) $K_T \geq C(\delta_T|\log \delta_T|)^{-2}$; (ii) Given $X_0 \in \text{Teich}(R)$, we have $\text{dist}_B(X_0, \cdot) \geq C|\log \delta_T|$. 

Remark. Ohsawa [10] has showed $K_T(X) \to \infty$ as $X \to \partial \text{Teich}(R)$. 

2 A review of Teichmüller theory

In this section, we review some basic definitions in the Teichmüller theory, for more detail, see [7].

A Riemann surface $R$ is called hyperbolic if its universal covering is the upper half plane $H$. The Poincaré metric $|dz|/\text{Im } z$ on $H$ descends to a complete metric on $R$ with constant curvature $-1$, which is called the hyperbolic metric.

Let $R$ be a hyperbolic Riemann surface. A Riemann surface $X$ is marked by $R$ if it is equipped with a quasiconformal homeomorphism $f : R \to X$. Two marked surfaces $(X_1, f_1), (X_2, f_2)$ are equivalent if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $X_1$ onto $X_2$. We call the set of all such equivalence classes $[X, f]$ the Teichmüller space of $R$ and denote it by $\text{Teich}(R)$. The Teichmüller distance between two points $p_i = [X_i, f_i], i = 1, 2$ in $\text{Teich}(R)$ is defined by

$$d_T(p_1, p_2) = \frac{1}{2} \inf K(h)$$

where $h$ is taken over all quasiconformal mappings of $X_1$ onto $X_2$ which are homotopic to $f_2 \circ f_1^{-1}$ and $K(h) \geq 1$ denotes the maximal dilatation of $h$. The Teichmüller space is topologically a cell.

Given $X \in \text{Teich}(R)$, let $Q(X)$ denote the Banach space of holomorphic quadratic differentials $\phi = \phi(z)dz^2$ on $X$ with

$$\|\phi\|_T = \int_X |\phi| < \infty.$$ 

Let $B(X)$ be the space of $L^\infty$ measurable Beltrami differentials $\mu = \mu(z)d\bar{z}/dz$ on $X$. A tangent vector $v \in T_X \text{Teich}(R)$ is represented by a $\mu \in B(X)$ and its Teichmüller norm is given by

$$\|\mu\|_T = \sup \left\{ \text{Re} \int_X \phi(z)\mu(z)dzd\bar{z} : \phi \in Q(X), \|\phi\|_T = 1 \right\}.$$
We have the isomorphism
\[ T_X \text{Teich}(R) \cong B(X)/Q(X)^\perp \]
and \( \|\mu\|_T \) gives infinitesimal form of the Teichmüller distance.

A \textit{projective structure} on \( X \) is a subatlas charts with Möbius transformations as transition functions. The space of all projective surfaces marked by \( R \) is a complex manifold fibering over \( \text{Teich}(R) \), which will be denoted by \( \text{Proj}(R) \). By Fuchsian Uniformization, there is a canonical section \( \sigma_F : \text{Teich}(R) \to \text{Proj}(R) \). Each fiber \( \text{Proj}_X(R) \) over \( X \in \text{Teich}(R) \) is an affine space modeled on the Banach space \( P(X) \) of all holomorphic quadratic differentials on \( X \) with
\[ \sup_X \rho^{-2}\|\phi\| < \infty. \]
Teich\((R)\) has a complexification defined by
\[ QF(R) = \text{Teich}(R) \times \text{Teich}(\overline{R}) \]
where \( \overline{R} \) is the complex conjugate of \( R \). The real-analytic map \( \sigma_F \) naturally induces a holomorphic map
\[ \sigma : QF(R) \to \text{Proj}(R) \times \text{Proj}(\overline{R}). \]
Denote by \( \sigma(X,Y) = (\sigma_{QF}(X,Y),\overline{\sigma_{QF}(X,Y)}) \). The \textit{Bers embedding} \( \beta_Y : \text{Teich}(R) \to P(Y) \) is given by
\[ \beta_Y(X) = \overline{\sigma_{QF}(X,Y)} - \sigma_F(Y). \]
One has the following well-known theorem

\textbf{Theorem 2.1.} The Bers embedding maps \( \text{Teich}(R) \) to a bounded domain in \( P(Y) \) which is contained in the ball with radius \( 3/2 \).

\section{Weil-Petersson metric}

Let \( R \) be a compact Riemann surface of genus \( g > 1 \). The \textit{Weil-Petersson norm} on the cotangent space \( Q(X) \cong T^*_{X} \text{Teich}(R) \) is defined by
\[ \|\phi\|^2_{WP} = \int_X \rho(z)^{-2}\|\phi(z)\|^2|dz|^2. \]
By duality, one gets a Riemann metric $g_{WP}$ on the tangent space of $\text{Teich}(R)$. Furthermore, it is a non-complete Kähler metric of negative sectional curvature (cf. [6], [14]–[16]). It follows from the Cauchy-Schwarz inequality that

$$\|v\|_{WP} \leq 2\sqrt{\pi(g-1)}\|v\|_T$$

holds for any tangent vector $v$ on $\text{Teich}(R)$. Recall that

$$\beta_X : \text{Teich}(R) \to Q(X) \cong T^*_X \text{Teich}(R).$$

It was shown by McMullen that for any fixed $Y \in \text{Teich}(R)$, the 1–form $\beta_X(Y)$ is bounded in the Teichmüller and Weil-Petersson metrics and satisfies $d\beta_X(Y) = i\omega_{WP}$ where $\omega_{WP}$ is the Kähler form of $g_{WP}$ (cf. Theorem 1.5 in [7]). Moreover, Takhtajan and Teo found a real-analytic function $S_Y$ on $\text{Teich}(R)$ coming from the Liouville action in string theory such that

$$-\beta_X(Y) = \sigma_F(X) - \sigma_{QF}(X,Y) = \frac{1}{2} \partial S_Y$$

(cf. Corollary 4.1 in [13]), which implies $-\frac{1}{2}S_Y$ is a Kähler potential for the Weil-Petersson metric with

$$\partial \bar{\partial}(-S_Y) \geq C \partial S_Y \bar{\partial} S_Y$$

for suitable constant $C > 0$.

Given a hyperbolic geodesic $\gamma$ on $R$, let $l_\gamma(X)$ denote the hyperbolic length of the corresponding geodesic on $X \in \text{Teich}(R)$. The length function is very useful in the Teichmüller theory. For instance, it relates the Teichmüller metric as follows

$$\|\partial \log l_\gamma\|_T \leq 2$$

(cf. Theorem 4.2 in [7]). Let $\text{Log} : \mathbb{R}_+ \to [0,\infty)$ be a smooth function such that

$$\text{Log}(t) = \begin{cases} \log t & \text{if } t \geq 2 \\ 0 & \text{if } t \leq 1. \end{cases}$$

McMullen [7] defined a new invariant Kähler metric by

$$g_{1/\ell} = g_{WP} - \delta \sum_{l_\gamma(X) < \epsilon} \partial \bar{\partial} \text{Log} \frac{\epsilon}{l_\gamma}$$
where the sum is over primitive short geodesics $\gamma$ on $X$; at most $3g - 3$ terms occur in the sum.

**Theorem.** (cf. Theorem 5.1 in [7]) For all $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that $g_{1/\mu}$ is equivalent to the Teichmüller metric.

Set
\[
\psi = -\frac{S_Y}{2} - \delta \sum_{l_\gamma(X) < \epsilon} \log \frac{\epsilon}{l_\gamma}.
\]

**Proposition 3.1.** There is a constant $C > 0$ such that
\[
g_{1/\mu} = \partial \bar{\partial} \psi \geq C \partial \bar{\partial} \psi.
\] (3)

**Proof.** It suffices to show the inequality in (3). For any $\epsilon/2 < l_\gamma(X) < \epsilon$, one has
\[
\| \partial \log (\epsilon/l_\gamma) \|_T \leq \sup_{t \in [\epsilon/2, \epsilon]} |\log'(t)| \cdot \frac{\epsilon}{l_\gamma} \| \partial \log l_\gamma \|_T = O(1)
\]
by (2). By (1), (2) and the above theorem, the desired inequality follows immediately from the Cauchy-Schwarz inequality.

### 4 Proofs of Theorems 1.1 and 1.2

Let $M$ be a complex manifold of dimension $m$. Let $\mathcal{H}_1$ denote the space of holomorphic $m$–forms $s$ on $M$ such that
\[
\|s\|_2^2 = \left| \int_M s \wedge \bar{s} \right| \leq 1.
\]
The Bergman kernel on $M$ is defined by
\[
K_M(z) = \sup \{ s \wedge \bar{s}(z) : s \in \mathcal{H}_1 \}
\] (4)
where $s_1 \wedge \bar{s}_1(z) \leq s_2 \wedge \bar{s}_2(z)$ means the ratio of the left and right sides is bounded by 1. If $K_M$ is nowhere vanishing on $M$, one can define the Bergman metric by $ds_M^2 := \partial \bar{\partial} \log K_M^*$ where
\[
K_M = K_M^* dz_1 \wedge \cdots \wedge d\bar{z}_m \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_m
\]
in local coordinates (note that the definition of $ds_M^2$ does not depend on the choice of coordinates hence is globally defined). It has the following extreme property:

$$ds_M^2(z; v) = \frac{1}{K_M^*(z)} \sup \left\{ |\partial s^*(v)|^2(z) : s \in \mathcal{H}_1, s(z) = 0, s = s^* dz_1 \wedge \cdots \wedge dz_n \right\}$$

(5)

for all $v \in T_z M$.

By Royden’s theorem [12], given $X_0 \in \text{Teich}(R)$, there is an embedded polydisk

$$\iota : (\Delta^{g-3}, 0) \rightarrow (\text{Teich}(R), X_0)$$

such that the Teichmüller(=Kobayashi) and Euclidean metrics are equivalent on $\Delta^{g-3}$. For any $s \in \mathcal{H}_1$ with $s = s^* dz_1 \wedge \cdots \wedge dz_{3g-3}$ in $\Delta^{g-3}$, we obtain from Cauchy’s estimate that

$$|\partial^\alpha s^*/\partial z^\alpha(0)| = O(1)$$

(6)

holds for any multi-indices $\alpha$. Let $\psi$ be as in section 3. By McMullen’s theorem and Proposition 3.1, one has

$$|\psi(X) - \psi(X_0)| \leq \left( \sup_{\text{Teich}(R)} \|d\psi\|_T \right) d_T(X_0, X) \leq O(d_T(X_0, X)).$$

Thus there exists a constant $c_0 > 0$ independent of $X_0$ such that

$$\iota(\Delta^{g-3}) \subset \{ X \in \text{Teich}(R) : |\psi(X) - \psi(X_0)| < c_0 \}. \quad (7)$$

Set

$$\lambda(X) = -e^{-\frac{C}{4}(\psi(X) - \psi(X_0))}.$$

By (3), one has

$$\partial \bar{\partial} \lambda = -\frac{C \lambda}{4} (2\partial \bar{\partial} \psi - C \partial \psi \bar{\psi}) \geq -\frac{C \lambda}{4} \partial \bar{\partial} \psi = -\frac{C \lambda}{4} g_1/1.$$
Hence by (7), we find a constant $C' > 0$ independent of $X_0$ such that
\[ \partial \bar{\partial} \lambda \geq C' \partial \bar{\partial} |z|^2, \quad \text{on } \Delta^{3g-3}. \tag{8} \]

Let us recall the following well-known $L^2$-estimate:

**Theorem.** (cf. [2], [9]) Let $M$ be a complete Kähler manifold of dimension $m$ and let $\varphi$ be a $C^\infty$ strictly psh function on $M$. Then for any $\bar{\partial}$-closed $(m,1)$ form $w$ with $\int_M |w|^2_{\bar{\partial}\varphi} e^{-\varphi} dV_\varphi < \infty$, there is an $m$-form $u$ on $M$ such that $\bar{\partial} u = w$ and
\[ \left| \int_M u \wedge \bar{u} e^{-\varphi} \right| \leq \int_M |w|^2_{\bar{\partial}\varphi} e^{-\varphi} dV_\varphi \]
where $dV_\varphi$ denotes the volume with respect to $\bar{\partial}\varphi$.

Let $\chi : \mathbb{R} \rightarrow [0,1]$ be a smooth function such that $\chi|_{(-\infty,1/2]} = 1$ and $\chi|_{[1,\infty)} = 0$. Set $w = z^\alpha \bar{\partial}\chi(|z|) \wedge dz_1 \wedge \cdots \wedge dz_{3g-3}$. Applying the above theorem for $M = \text{Teich}(R)$ with respect to the regularization of the following psh function $\varphi$ from above (cf. [11])
\[ \varphi = N\lambda + 2(3g - 3 + |\alpha|)\chi(|z|) \log |z| \]
for sufficiently large constant $N$, we obtain a form $u$ on $\text{Teich}(R)$ satisfying $\bar{\partial} u = w$ and
\[ \left| \int_{\text{Teich}(R)} u \wedge \bar{u} e^{-\varphi} \right| = O(1) \]
because of (7), (8). Then we obtain a holomorphic $3g-3$ form $s$ on $\text{Teich}(R)$ by setting $s = z^\alpha \chi(|z|)dz_1 \wedge \cdots \wedge dz_{3g-3} - u$ such that
\[ \frac{\partial^\alpha s^*}{\partial z^\alpha}(0) = 1, \quad \frac{\partial^\beta s^*}{\partial z^\beta}(0) = 0, \quad \forall |\beta| < |\alpha|, \quad \text{and} \quad \|s\|_2 = O(1) \tag{9} \]
since $\varphi < 0$ and $\varphi \sim 2(3g - 3 + |\alpha|) \log |z|$ near 0. By (4), (5), (6) and (9), the proof of Theorem 1.1 is complete.

Before proving Theorem 1.2, let us recall the following

**Definition.** (cf. [1]) Suppose that $(M, \omega)$ is a complete Kähler manifold of dimension $m$. We say that $(M, \omega)$ has bounded geometry if and only if for each $x_0 \in M$ there exists an embedded polydisk $\iota : (\Delta^m,0) \rightarrow (M,x_0)$.
such that the Euclidean metric and $\omega$ are equivalent on $\Delta^m$ and for any integer $l$, there is a constant $C_l > 0$ such that for any multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \leq l$ we have

$$\left| \frac{\partial^{(|\alpha|+|\beta|)}}{\partial z^{\alpha} \partial \bar{z}^{\beta}} g_{ij} \right| \leq C_l$$

on $\Delta^m$ where $\omega = \sum g_{ij} dz_i dz_j$.

By extreme properties of the derivatives of the Bergman metric similar as (4), (5), it is not difficult to verify that the Teichmüller space equipped with the Bergman metric has bounded geometry. According to the Schwarz lemma of Yau (cf. Theorem 3 in [17]), one has

$$\frac{dV_{KE}}{dV_B} = O(1)$$

where $dV_{KE}$ and $dV_B$ denote the volume forms of the Kähler-Einstein and Bergman metrics respectively. Now we view Teich($R$) as a bounded domain in $\mathbb{C}^{3g-3}$ equipped with the canonical coordinate $\zeta$. If we write

$$dV_{KE} = V_{KE}(\zeta)(\partial \bar{\partial} |\zeta|^2)^{3g-3},$$

then

$$\frac{dV_{KE}}{dV_B} = O(1)$$

(cf. [8]). Note that

$$K_{\text{Teich}(R)} = K_{\text{Teich}(R)}^* dz_1 \wedge \cdots \wedge dz_{3g-3} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{3g-3} = K_T d\zeta_1 \wedge \cdots \wedge d\zeta_{3g-3} \wedge d\bar{\zeta}_1 \wedge \cdots \wedge d\bar{\zeta}_{3g-3},$$

which implies

$$K_T = K_{\text{Teich}(R)}^* \cdot |\det(\partial z_j/\partial \zeta_k)|^2.$$ 

Since the Bergman and Teichmüller metrics are equivalent, one has

$$\frac{dV_B}{(\partial \bar{\partial} |\zeta|^2)^{3g-3}} \asymp |\det(\partial z_j/\partial \zeta_k)|^2 \asymp K_T.$$

Hence $V_{KE} = O(K_T)$, verifying (i) of Theorem 1.2.

Since $\det(\partial z_j/\partial \zeta_k)$ is nowhere vanishing on $\Delta^{3g-3}$, we can take a single-valued branch of $f$ of $\log\det(\partial z_j/\partial \zeta_k)$. Applying the Schwarz-Pick lemma to the holomorphic map $f : \Delta^{3g-3} \rightarrow \{ w \in \mathbb{C} : |\text{Im } w| < \pi \}$, we obtain

$$\| \partial f \|_{\partial \bar{\partial} |z|^2}(X_0) = O(1),$$

8
which implies
\[
\left\| \frac{\partial}{\partial \log K_T} \right\|_{\partial \bar{\partial} \log K_T} (X_0) = O \left( \| \partial \log K_{T}^{*} \|_{\partial \bar{\partial} |z|^2} (X_0) + \| \partial f \|_{\partial \bar{\partial} |z|^2} (X_0) \right) = O(1). \tag{10}
\]

The assertion (ii) then follows from (10) and (i).

**Remark.** By (10), the function
\[
r = -e^{-\tau \log K_T} = -K_T^{-\tau}
\]
is a bounded strictly psh exhaustion function on Teich(R) for sufficiently small \( \tau > 0 \).

Clearly
\[
\frac{1}{C} \delta_{T}^{c_1} \leq -r \leq C \delta_{T}^{c_2}
\]
holds for suitable \( C, c_1, c_2 > 0 \), since trivially one has \( K_T = O(\delta_T^{-6g+6}) \). Some bounded psh exhaustion functions without estimate were given in [4], [18].

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