The Phantom of the New Oscillatory Cosmological Phase

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Abstract. We study a recently proposed new cosmological phase where a scalar field moves periodically in an expanding spatially-flat Friedmann universe. This phase corresponds to a limiting cycle of the equations of motion and can be considered as a cosmological realization of a “time-crystal”. We show that this phase is only possible, provided the Null Energy Condition is violated and the so-called Phantom divide is crossed. We prove that in general k-essence models: i) this crossing causes infinite growth of quantum perturbations on short scales, and ii) exactly periodic solutions are only possible, provided the limiting cycle encircles a singularity in the phase plane. The configurations neighboring this singular curve in the phase space are linearly unstable on one side of the curve and superluminal on the other side. Moreover, the increment of the instability is infinitely growing for each mode by approaching the singularity, while for the configurations on the other side, the sound speed is growing without limit. We illustrate our general results by analytical and numerical studies of a particular class of such k-essence models.
1 Introduction

Usually the appearance of a periodic structure in space is attributed to formation of ordinary crystals. By analogy, the authors of [1] suggested to use the name “time-crystals” for the systems with vacuum solutions which are exactly periodic in time. In cosmology such vacuum solutions can be interesting in context of Dark Energy and early stages of Inflation. On the quantum level this idea was further elaborated in [2]. However, a cosmological expansion usually works as a friction and dissipates energy, so that an exactly periodic motion is impossible. Indeed, a usual system will finally approach an equilibrium static configuration with minimal energy density. However, in some non-canonical cases, the lowest possible available configuration can still be non-static and correspond to a motion. In particular, this is the case for the ghost condensate [3]. This is also the basis for the original k-inflation [4] attractor. Thus the solution with the lowest energy level can spontaneously break the time-translation invariance. Due to the shift-symmetry of the ghost condensate, this spontaneous symmetry breaking still results there in the time-independent and Lorentz-invariant energy momentum tensor (EMT). The latter corresponds to the normal de Sitter vacuum with some cosmological constant. It is rather interesting to consider whether one can spontaneously break time-translational invariance even on the level of the EMT so that the vacuum configuration is different from the de Sitter spacetime. Here we mean that this non-trivial solution should be valid for very long times up to the asymptotic future. In particular, if the motion in this vacuum state is periodic, with the period $T$, then the continuous time translation invariance, $t \rightarrow t + c$ with $c$ real, is only broken to the level of a discrete subgroup $t \rightarrow t + nT$, where $n$ is an integer. The recent work [5] proposed an interesting new phase of cosmological matter with these properties. In this phase a scalar field periodically moves in the constantly expanding universe. This oscillatory phase was realized by a non-canonical scalar field theory of the k-essence type [4, 6–8], see the next section for details. A similar idea of an oscillating dark energy was studied in e.g. [9, 10]. In this paper we discuss whether it is possible to realize a k-essence time-crystal in a cosmological setup.
2 NEC violation is needed for a limiting cycle

A presence of exactly periodic motion with the period $T$, in particular, requires that the energy density is periodic $\varepsilon (t + T) = \varepsilon (t)$. If at $t$ the energy density was decreasing, then somewhere between $t$ and $t + T$ the energy density should start to increase to compensate this reduction. And vice versa: for the originally increasing energy density there should be a moment between $t$ and $t + T$ where the energy density should start to decrease. Thus anyway during each cycle there are time intervals on which the energy density is growing and on which it is decreasing. Further, the conservation of energy in a Friedmann universe requires

$$\dot{\varepsilon} = -3H (\varepsilon + p), \quad (2.1)$$

where $H$ is the Hubble parameter and $p$ is the effective pressure of the oscillatory matter. Hence, in an expanding universe (with $H > 0$) the energy density can grow, only if the effective enthalpy density $\varepsilon + p$ is negative. The negative sign of $\varepsilon + p$ implies a violations of the Null Energy Condition (NEC) which states that for all null vectors $n^\mu$ the energy-momentum tensor should satisfy $T_{\mu\nu} n^\mu n^\nu \geq 0$. In turn, a violation of the NEC necessarily implies that the Hamiltonian density of the system is unbounded from below - for each constant there are such local values of the initial data that the energy density is more negative than this constant, see [11]. In other words a violation of NEC implies that the system has to possess configurations with arbitrary negative energy densities. For a recent discussion of NEC violation see e.g. [12]. Clearly, in a collapsing universe, a decrease in energy density would require an NEC violation as well. In cosmology, matter which violates NEC is often referred to as Phantom [13].

On the other hand, the time derivative of the energy density can change the sign, provided $H$ changes sign. In this case the spatially-flat Friedmann universe has to be able to evolve from expansion to contraction and then from contraction to expansion. The latter transition corresponds to a bounce. It is well known that a bounce of a spatially-flat Friedmann universe requires a violation of NEC. Indeed, the transition from contraction to expansion at time $t_b$ requires $H (t_b) = \dot{a}/a = 0$. Using the Friedmann equations

$$H^2 = \frac{1}{3} \varepsilon, \quad (2.2)$$

and

$$\dot{H} = -\frac{1}{2} (\varepsilon + p), \quad (2.3)$$

we obtain that at transition from contraction with $H < 0$ to expansion with $H > 0$ one has $\dot{H} > 0$ and therefore $\varepsilon + p < 0$ so that NEC is violated again. Hence we conclude that a violation of NEC is necessary to realize an exactly periodic oscillatory cosmological phase. Moreover, due to the periodicity the system should be able to evolve through the so-called Phantom divide - the border of the NEC-violating region. This border corresponds to the equation of state parameter $w = p/\varepsilon = -1$. The dynamical transition through $w = -1$ should cyclicly happen in both directions from above and from below.

It is well known that systems of k-essence type cannot dynamically violate NEC and cross the Phantom divide [14], see also [15–17] and for the review [18, 19]. Note that the pathologies associated with the crossing of the Phantom divide are more severe than those problems [20–23] arising just due to the violation of NEC. Below in the next section (3) we will refresh some basic facts about k-essence and provide simple arguments against a smooth transition of a k-essence field through the Phantom divide.
Finally it is useful to consider the average of \( w + 1 \) over some time interval \( T = t_f - t_i \). Using (2.1) and (2.2) under the assumption that \( \varepsilon > 0 \) between \( t_i \) and \( t_f \) we obtain

\[
\langle w(t) + 1 \rangle_T = \frac{1}{T} \int_{t_i}^{t_f} dt (1 + w(t)) = -\frac{1}{T} \int_{t_i}^{t_f} dt \frac{\dot{\varepsilon}}{\varepsilon \sqrt{3\varepsilon}} = \frac{1}{T} \sqrt{\frac{4}{3}} \left( \frac{1}{\sqrt{\varepsilon(t_f)}} - \frac{1}{\sqrt{\varepsilon(t_i)}} \right).
\]

If the time interval \( T \) is a multiple of the period of oscillations \( T \), then \( \langle w(t) + 1 \rangle_T = 0 \). Thus such an oscillatory stage can be interesting to model inflation and dark energy.

### 3 Refreshing k-essence and why cannot it cross the Phantom divide

Here we will collect and discuss basic mostly well known facts about k-essence which is a noncanonical, minimally coupled to gravity scalar field whose dynamics is described by the action

\[
S = \int d^4x \sqrt{-g} p(\varphi, X),
\]

where

\[
X = \frac{1}{2} g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}.
\]

In cosmological applications the field is only slightly inhomogeneous and anisotropic so that \( \varphi_{,\mu} \) is timelike, hence throughout the paper we assume that \( X > 0 \), as our signature convention \((+,−,−,−)\). This class of theories was introduced in [4] (see also [6–8]). The corresponding EMT is

\[
T_{\mu\nu} = p, X \varphi_{,\mu} \varphi_{,\nu} - pg_{\mu\nu},
\]

for every timelike \( \varphi_{,\mu} \) this EMT takes a form of a perfect fluid

\[
T_{\mu\nu} = (\varepsilon + p) u_\mu u_\nu - pg_{\mu\nu},
\]

with the four velocity

\[
u_\mu = \frac{\varphi_{,\mu}}{\sqrt{2X}},
\]

pressure \( p \) and the energy density

\[
\varepsilon = 2X p, X - p.
\]

Hence for all null vectors \( n^\mu \) we have

\[
T_{\mu\nu} n^\mu n^\nu = p, X (\varphi_{,\mu} n^\mu)^2 = \frac{\varepsilon + p}{2X} (\varphi_{,\mu} n^\mu)^2.
\]

The Null Energy Condition requires \( T_{\mu\nu} n^\mu n^\nu \geq 0 \) or \( p, X \geq 0 \). Thus to change the sign of \( T_{\mu\nu} n^\mu n^\nu \) the system has to change the sign of \( p, X \) or for time-like derivatives to change the sign of the enthalpy density \( \varepsilon + p \). In particular, if the transition is smooth, then at the Phantom divide \( p, X = 0 \).

Around any time-like, \( X > 0 \), background (in particular around cosmological backgrounds [24]) the perturbations propagate with the speed given by

\[
\varepsilon^2 = \left( \frac{\partial p}{\partial \varepsilon} \right)_\varphi = \frac{p, X}{\varepsilon, X}.
\]
The small perturbations of k-essence around any background propagate in an effective (contravariant) metric [25] which is conformally equivalent to

\[ G_{\mu\nu} = p_{,X} g_{\mu\nu} + p_{,XX} \varphi_{,\mu} \varphi_{,\nu}, \]  

(3.9)

see also [25–28], see also [29, 30] for the relativistic acoustic geometry in closely related irrotational perfect fluids. The equation of motion is

\[ G^{\mu\nu} \nabla_\mu \nabla_\nu \varphi + \varepsilon p_{,\varphi} = 0. \]  

(3.10)

The positivity of the right hand side of the expression (3.8) guaranties that this equation of motion is a hyperbolic quasilinear PDE. Note that hyperbolicity guaranties that Cauchy problem is well posed and evolution is predictable till some moment of time. However, it was demonstrated that breakdown of predictability due to the formation of caustics is a rather generic phenomenon, see recent discussion in [31, 32]. In cosmology the equation of motion reduces to

\[ \varepsilon_{,X} \ddot{\varphi} + 3H \dot{\varphi} p_{,X} + \varepsilon p_{,\varphi} = 0, \]  

(3.11)

where \( H \) is given by the first Friedmann equation (2.2) with the energy density (3.6). Instead of \( \varphi \) and \( \dot{\varphi} \) one can use \( \varepsilon \) and \( p \) as independent variables. In this case the Jacobian is \( J = p_{,\varphi} \varepsilon_{,\varphi} - \varepsilon_{,\varphi} p_{,\varphi}. \) The dynamical equation on energy density is (2.1) with the Hubble parameter given by the first Friedmann equation (2.2). While the dynamical equation for the pressure is

\[ \dot{p} = -\frac{1}{\varepsilon_{,X}} \left( 6X H p_{,X}^2 + p_{,\varphi} \varepsilon_{,\varphi} - \varepsilon_{,\varphi} p_{,\varphi} \right), \]  

(3.12)

where the Hubble parameter is expressed through the energy density by the first Friedmann equation (2.2) and all other quantities should be expressed through \( (\varepsilon, p) \) using implicit function theorem. Unfortunately this description is only valid outside of the \( \dot{\varphi} = 0 \) line. Indeed, the Jacobian can be written as

\[ J = p_{,\varphi} \varepsilon_{,\varphi} - \varepsilon_{,\varphi} p_{,\varphi} = \dot{\varphi} \left( p_{,\varphi} \varepsilon_{,X} - \varepsilon_{,\varphi} p_{,X} \right), \]  

(3.13)

where the right hand side is vanishing on the \( \dot{\varphi} = 0 \) line. However, if a limiting cycle exists, it has to cross this line. Thus the Jacobian is not sign-definite and it is not clear whether the expression in the brackets of (3.12) is sign-definite. For a shift-symmetric case (with the symmetry \( \varphi \to \varphi + c \)) the pressure evolves due to the (2.1). In this case a limiting cycle is not possible in an expanding universe, because \( \dot{\varphi} p_{,X} \propto a^{-3} \) so that \( \dot{\varphi} p_{,X} \) always decreasing.

Further it is useful to use the energy conservation (2.1) along with the (2.2) and the definition of energy density (3.6) to obtain

\[ \sqrt{\frac{4}{3} \varepsilon_f} - \sqrt{\frac{4}{3} \varepsilon_i} = \int_{\varepsilon_i}^{\varepsilon_f} \frac{d\varepsilon}{\sqrt{3\varepsilon}} = - \int_{t_i}^{t_f} dt \dot{\varphi} p_{,\varphi} = \int_{\varphi_i}^{\varphi_f} d\varphi p_{,\varphi}. \]  

(3.14)

3.1 Momentum invertibility and strong superluminality

Further it is illuminating to look at the canonical formulation of the dynamics. It is particularly useful as the authors of [5] are interested in degenerate Hamiltonians which are not smooth functions of canonical momenta. For smooth Lagrangians this happens when the field velocity cannot be uniquely expressed through the canonical momentum. We will
proceed using the ADM formalism, see [33] and [34]. In a spacetime foliation generated by a

time-like congruence with a tangent vector $t^\mu$ we can represent the metric as

$$ds^2 = N^2 dt^2 - \ell_{ik} \left( dx^i + N^i dt \right) \left( dx^k + N^k dt \right).$$  (3.15)

The unit normal to the hypersurface of constant $t$ is

$$U_\mu = N \partial_\mu t.$$  (3.16)

The relative three velocity $\nu$ of $U_\mu$ with respect to the k-essence fluid velocity $u^\mu$ (given by

\(3.5\)) can be found from

$$\frac{1}{\sqrt{1-\nu^2}} = U_\mu u^\mu = \frac{1}{N\sqrt{2X}} \left( \dot{\varphi} - N^i \varphi, i \right),$$  (3.17)

so that after some algebra

$$\nu^2 = \frac{N^2 \ell_{ik} \varphi, k \varphi, i}{\left( \varphi - N^i \varphi, i \right)^2},$$  (3.18)

where we have used the standard ADM results $g^{ik} = -\ell_{ik} + N^i N^k / N^2$, $g^{ti} = -N^i / N^2$ and $g^{tt} = 1 / N^2$, $\ell_{ik} \ell_{km} = \delta_{im}$.

The canonical momentum is defined as

$$P = \frac{\partial}{\partial \dot{\varphi}} (\sqrt{-g} \rho) = \sqrt{-g} p_{,X} \frac{1}{N^2} \left( \dot{\varphi} - N^i \varphi, i \right).$$  (3.19)

Hence, the canonical momentum is always vanishing at the crossing of the Phantom divide $p_{,X} = 0$. The velocity $\dot{\varphi}$ can be found from the canonical momentum provided

$$\frac{\partial P}{\partial \dot{\varphi}} \neq 0.$$  (3.20)

In cosmology this momentum is $P = a^3 \rho_{,X}$ and cannot be locally expressed exclusively

through $\varphi$ and $\dot{\varphi}$ as $a(t) = \exp \left( \int^t dt' \sqrt{\varepsilon(\varphi(t'), \dot{\varphi}(t'))} / 3 \right)$. Using the definitions of the

relative velocity (3.18) and the sound speed (3.8) this invertibility condition (for a given scale

factor $a$, before solving the Hamiltonian constraint - the first Friedmann equation (2.2)) can

be written in the form

$$\frac{\partial P}{\partial \dot{\varphi}} = \sqrt{-g} \left( \left( \dot{u} \right)^2 2X p_{,XX} + p_{,X} g^{tt} \right) = \sqrt{-g} G^{tt} = \frac{\varepsilon X}{N} \sqrt{\ell} \left( \frac{1 - \nu^2 c^2 S}{1 - \nu^2} \right).$$  (3.21)

In particular, one can chose such a foliation that $\varphi$ is constant on the equal-time hypersurface,

so that $\varphi(t)$. In this foliation $\nu = 0$ and the invertibility of the momentum requires that

$\varepsilon_{X} \neq 0$. Note that cusp Hamiltonian and non-invertibility of momentum are among the

desirable features of the “time crystals” and one of the requirements imposed in [5] on a

general system with an oscillatory attractor as a ground state. Indeed, the ground state

implies a minimum of the Hamiltonian. But if all first derivatives of the Hamiltonian are

vanishing, there is no motion and an oscillatory configuration cannot be a ground state.

Clearly for systems with cusp Hamiltonians this argument does not work.

However, the homogeneous and isotropic cosmological dynamics of a k-essence scalar

field driving the expansion of the universe cannot be brought to this simple two dimensional
canonical form. Below, in section 4, using standard results from the theory of ODE, we prove that the changing of sign of \( \varepsilon, X \) is a necessary condition for the existence of cosmological limiting cycles. Hence one has to require that on some configurations \( \varepsilon, X = 0 \). But, for a generic k-essence Lagrangian, \( \varepsilon, X = 0 \) does not imply \( p, X = 0 \). Therefore, for \( X > 0 \) and a finite and non-vanishing \( p, X \), the singular surface where \( \varepsilon, X = 0 \) corresponds to the divergent speed of sound for the perturbations \( c_S^2 \to \infty \). Moreover, if \( \varepsilon, X \) changes sign (as required for the existence of a limiting cycle) it implies that on one side of the singular hypersurface there is a region of extreme superluminality, while the configurations on the other side suffer from infinitely strong gradient instabilities. On these latter configurations \( c_S^2 < 0 \) and the equation of motion (3.10) becomes an elliptic quasilinear PDE. Thus generically these “time crystals” k-essence systems require existence of configurations with an infinitely strong superluminality which are neighboring configurations with infinitely strong gradient instabilities.

As it follows from (3.12) the pressure has an infinite time derivative at the singularity. Thus the time derivative of the Ricci scalar \( R \) blows up as well. An effective action in gravity should contain the term \( (\partial R)^2 \) which blows up in this case. This is another way to see that EFT breaks down on the singular curve where \( \varepsilon, X = 0 \).

As it follows from (3.21), for superluminal speeds of sound the invertibility of the momentum-velocity relation can be also violated on a foliation with the relative velocity \( v^2 = 1/c_S^2 \). In this case the hypersurface of constant time coincides with the characteristic surface of the equation of motion. Clearly such a surface cannot be chosen for initial data and the foliation is not suitable for the Cauchy problem. For a detailed discussion see [25, 35]. Other works discussing the non-uniqueness of the Hamiltonian due to the multivalued relation between momentum and field velocity include [36–38].

### 3.2 Classical and quantum perturbations

Further the cosmological scalar perturbations are described by the action

\[
S = \frac{1}{2} \int d\eta d^3 x \ Z \left( (\mathcal{R}')^2 - c_S^2 (\partial_i \mathcal{R})^2 \right),
\]

where \( \eta \) is the conformal time and the curvature perturbation \( \mathcal{R} \) is constructed out of the perturbation of the k-essence field \( \delta \varphi \) and the Newtonian potential \( \Phi \)

\[
\mathcal{R} = \Phi + H \frac{\delta \varphi}{\dot{\varphi}},
\]

where both perturbations are written in terms of gauge-invariant variables which correspond to the Newtonian gage. Finally the normalization is

\[
Z = a^2 \left( \frac{\varepsilon + p}{c_S^2 H^2} \right) = \varepsilon, X \left( \frac{\dot{\varphi} a}{H} \right)^2.
\]

The action is ghosty provided \( Z < 0 \) which is equivalent to \( \varepsilon, X < 0 \). If NEC is violated, but there are no gradient instabilities so that the system is hyperbolic and \( c_S^2 > 0 \), then necessarily \( \varepsilon, X < 0 \) and the action is ghosty. Ghost instabilities are perturbative, but rely on interactions with other fields. In particular there is always an interaction through gravity. On the contrary the gradient instabilities are linear short scale instabilities.

If \( \varepsilon, X = 0 \) at the Phantom divide then the background solution has to go through a singularity of the equation of motion (3.11). For example it is the case in the example
provided in [10]. Even if we assume that $c_S$ remains nonzero and does not blow up one would need to guaranty that $\varepsilon, \varphi$ does not blow up. However, in that case $\varepsilon, \varphi$ should vanish at the same point where $\varepsilon, X$ is vanishing. This provides two equations in the phase space $(\varphi, \dot{\varphi})$ to satisfy. Therefore this can generically only happen on isolated points of the measure zero corresponding to extrema of the energy density. This degenerate case was in details discussed in [14]. Another crucial point is that at the Phantom divide with non-vanishing speed of sound the normalization factor $Z$ is vanishing. But $Z = 0$ corresponds to an infinitely strong coupling on all scales at for the QFT of perturbations. Indeed, the cubic terms in the action for perturbations contain higher order derivatives of the background quantities and are not vanishing at $Z = 0$. A QFT which only contains third order operators is strongly coupled on all scales.

Another way to understand the pathology is to introduce the canonical variable (Mukhanov-Sasaki variable) which for for k-essence is [24, 39]

$$v = zR,$$  \hspace{1cm} (3.25)

with

$$z = a\frac{\dot{\varphi}}{H} \left[ \frac{\varepsilon + p}{2Xc_S^2} \right]^{1/2} = a\frac{\dot{\varphi}}{H} \sqrt{\varepsilon, X}. \hspace{1cm} (3.26)$$

The canonical variable can be written as

$$v = a\sqrt{\varepsilon, X} \left( \delta \varphi + \Phi \frac{\dot{\varphi}}{H} \right). \hspace{1cm} (3.27)$$

The dynamics of this variable is described by the action

$$S = \frac{1}{2} \sigma \int d\eta d^3x \left( (v')^2 + c_S^2 v \Delta v + \frac{z''}{z} v^2 \right), \hspace{1cm} (3.28)$$

$\sigma = \text{sign}(\varepsilon, X)$. The corresponding dispersion relation is

$$\omega_k^2 = c_S^2 k^2 - \frac{z''}{z}, \hspace{1cm} (3.29)$$

where $-z''/z$ plays the role of the square of effective time-dependent mass. Positive $z''/z$ indicates the Jeans instability operating on large scales where one cannot neglect expansion of spacetime. This mass is infinitely growing (in the positive or negative direction) at the Phantom divide with $\varepsilon, X = 0$. Clearly this does not allow to separate scales and treat the perturbations as an EFT.

Another problem is rooted in the speed of sound. Let’s now assume that $\varepsilon, X > 0$ at the Phantom divide. Then we have $c_S^2(t_{pd}) = 0$ and by continuity $c_S^2$ changes sign at this point. This change of sign of $c_S^2$ implies the transition of the equation of motion (3.10) from the hyperbolic to the elliptic type of PDE or vice versa. It is well known that the Cauchy problem is ill-posed for elliptical PDE’s. The problem is in the short wavelength instabilities behaving as

$$\delta \varphi_k \sim \exp(|c_S k| t), \hspace{1cm} (3.30)$$

so that the increment of instability in mode $k$ is $|c_S k|$ and grows for large wave numbers without any bound. In reality the bound is provided by a lattice size or by the strong coupling scale. In both these cases the instability has a characteristic scale corresponding to
the cut off scale completely destroying the predictability of the theory. If a system enters the elliptic regime on a classical solution which is in the formal region of validity of the EFT, than the system stays there much longer than the inverse frequency cut off scale. In that case all physical modes get a tremendous exponential amplification due to (3.30).

It is worth looking at the quantum fluctuations to understand the problem. If we start from a healthy background \( c_S^2 \) > 0 and \( \varepsilon_{,X} \) > 0, then in a Hadamard state (short scales vacuum state) the mode functions \( v_k \) for the canonical variable (3.27) are normalized inside the horizon (for scales \( c_S k \ll |z''/z| \)) as

\[
|v_k| = \sqrt{\frac{\hbar}{\omega_k}} \simeq \sqrt{\frac{\hbar}{c_S k}}. \tag{3.31}
\]

Where we have explicitly written \( \hbar \) to stress the quantum origin of this quantity. Then the characteristic quantum fluctuation of the canonical variable on scale \( k \) is

\[
\delta v_k \sim \sqrt{\hbar} |v_k| k^{3/2} \simeq \sqrt{\frac{\hbar}{c_S}}. \tag{3.32}
\]

On these ultrashort physical length scales \( \ell = a/k \) we can neglect the fluctuations of the Newtonian potential (see below) so that

\[
\delta \varphi_{,\ell} \sim \sqrt{\hbar} \left( \frac{k}{a} \right) = \left( \frac{\hbar^2}{\varepsilon_{,X} c_S} \right)^{1/4} \cdot \frac{1}{\ell} = \left( \frac{2X\hbar^2}{\varepsilon_{,X} (\varepsilon + p)} \right)^{1/4} \frac{1}{\ell}. \tag{3.33}
\]

Here we assumed that \( \ell \gg L_{UV} \), where the latter is the UV-cutoff length scale which is typically present in such derivatively coupled theories. Thus if either of the quantities \( \varepsilon_{,X} \) or \( p_{,X} \) is vanishing, the quantum fluctuations \( \delta \varphi_{,\ell} \) blow up on short scales. In particular, for \( c_S \to 0 \) quantum fluctuations at every given scale \( \ell \) inside the horizon blow up and completely invalidate the applicability of the whole theory of cosmological perturbations.

On the other hand the fluctuations of the Newtonian potential on short scales are scale-independent

\[
\Phi_{,\ell} \sim \sqrt{\hbar} \left( \frac{\varepsilon + p}{c_S} \right)^{1/2} \sim \sqrt{\hbar} (X\varepsilon_{,X} (\varepsilon + p))^{1/4}, \tag{3.34}
\]

see page 345 [39]. These fluctuations are always small provided \( c_S \gtrsim \varepsilon + p \). The latter condition can also be written in standard units as

\[
c_S \gtrsim \left( \frac{\varepsilon}{\varepsilon_{Pl}} \right) (1 + w), \tag{3.35}
\]

where \( \varepsilon_{Pl} \) is the Planckian energy density\(^1\). It is worthwhile to compare the right hand side of this expression with the Ricci curvature \( R = -\varepsilon (1 - 3w) \).

The magnitude of the quantum fluctuations \( \delta \varphi_{,\ell} \) on a short scale \( \ell \) can be also obtained from the following uncertainty relation

\[
\delta \varphi_{,\ell} \cdot \delta P_{,\ell} \gtrsim \hbar \ell^{-3}, \tag{3.36}
\]

where \( \delta P_{,\ell} \) is the fluctuation of the canonical momentum on this scale, for a detailed discussion see [40]. Further, the canonical momentum of fluctuations is

\[
\delta P_{,\ell} = G^{tt} \delta \dot{\varphi}_{,\ell} = \varepsilon_{,X} \delta \varphi_{,\ell}. \tag{3.37}
\]

\(^1\)Clearly this is is a rather weak lower bound on the sound speed for a dust-like k-essence.
The fluctuation of the field velocity on short-scale $\ell$ can be estimated as
\[
\delta \dot{\varphi}_\ell \simeq \omega_\ell \delta \varphi_\ell \simeq (c_\ell / \ell) \delta \varphi_\ell. \quad (3.38)
\]

For an oscillator the vacuum saturates the uncertainty relation therefore for a collection of oscillators
\[
\delta \varphi^2_\ell \simeq \frac{\hbar}{\varepsilon, \varepsilon, c_\ell} \cdot \frac{1}{\ell^2},
\]
which again gives (3.33). This estimation is not applicable, if the system is strongly coupled for scales $c_\ell k \lesssim |z''/z|$. However, in that case the predictive power of such theory is rather limited.

4 General phase space analysis and the Bendixson–Dulac theorem

The cosmological equation of motion (3.11) can be written in the first order form
\[
\frac{d\dot{\varphi}}{dt} = -3Hc_\ell^2 - \frac{\varepsilon, \varepsilon, X}{\varepsilon, \varepsilon},
\quad (4.1)
\]
\[
\frac{d\varphi}{dt} = \dot{\varphi}.
\]

The first equation of this system above is singular for configurations with $\varepsilon, X = 0$. The integral curves for the vector field $(\dot{\varphi}, \ddot{\varphi})$ of on the phase space $(\varphi, \dot{\varphi})$ are given by
\[
\frac{d\dot{\varphi}}{d\varphi} = -3He_\ell^2 - \frac{\varepsilon, \varepsilon, \dot{\varphi}}{\varepsilon, \varepsilon}. \quad (4.2)
\]

For every k-essence with the Lagrangian $p(\varphi, \dot{\varphi})$ it is convenient to introduce an auxiliary system of ODE [14]
\[
\frac{du}{dt} = \alpha(u, v) = -p_u \sqrt{3\varepsilon} - \varepsilon, v,
\quad (4.3)
\]
\[
\frac{dv}{dt} = \beta(u, v) = \varepsilon, u,
\]
with $\varepsilon = up_u - p$. Without the friction term $p_u \sqrt{3\varepsilon}$ these ODEs are the Hamilton equations of motion with canonical momentum $u$, coordinate $v$ and Hamiltonian $\varepsilon$. Without this friction term the motion happens on the curves of constant $\varepsilon$. As $p_u = up_X$, the effective friction coefficient is $p_X \sqrt{3\varepsilon}$. If the sign of the friction coefficient is always positive the dissipative motion cannot be periodic. Hence $p_X$ has to change the sign for the existence of a limiting cycle. This is a reformulation of the same statement presented in at the beginning of the paper. This system (4.3) is different from (3.11) but is constructed out of the same function $p(\varphi, \dot{\varphi})$ where instead of $\varphi$ one plugs in $v$ and instead of $\dot{\varphi}$ one plugs in $u$. The solutions $u(t)$ and $v(t)$ are different from $\varphi(t)$ and $\dot{\varphi}(t)$. In particular, $\dot{v} \neq u$. The main point is that this system (4.3) has the same integral curves on $(v, u)$ as the equation of motion (3.11) on $(\varphi, \dot{\varphi})$, but it is clearly less singular. This integral curves are locally given by (4.2). In particular a limiting cycle corresponds to a closed integral curve on $(\varphi, \dot{\varphi})$ plane and on $(v, u)$ plane. The time flow in $(v, u)$ goes in the opposite direction to the time flow in $(\varphi, \dot{\varphi})$ for the regions with $\varepsilon, X < 0$ as both first order equations (4.3) for $v$ and $u$ have an opposite sign to the corresponding equations (4.1).
Then by the Bendixson–Dulac theorem, if there exist a $C^1$ function $f(u,v)$ (called the Dulac function) such that
\[
\frac{\partial (f\alpha)}{\partial u} + \frac{\partial (f\beta)}{\partial v} > 0,
\] almost everywhere in a simply connected region of the plane, then there are no periodic solutions lying entirely within the region.

Further we will assume that in the region where of phase space where the limiting cycle is located the energy density is strictly positive, $\varepsilon > 0$. This is needed for the differentiability of the right hand side of the system (4.3). Moreover, this excludes a possibility of a bounce happening in this region of phase space.

In that case one can chose the Dulac function $f(\varepsilon) = 1/\sqrt{3\varepsilon}$ for which we have\(^2\)
\[
\frac{\partial (f\alpha)}{\partial u} + \frac{\partial (f\beta)}{\partial v} = \left(-f(\varepsilon)p_{,u}\sqrt{3\varepsilon}\right)_{,u} = -\varepsilon_{,X},
\] where we have used that $p_{,uu} = (up_{,X})_{,u} = p_{,X} + 2Xp_{,XX} = \varepsilon_{,X}$. Hence the system should necessary possess a singularity where $\varepsilon_{,X}$ changes sign or at least vanishing. Otherwise the limiting cycle cannot exist. This goes beyond the statement that the limiting cycle should have a fixed point of the system (4.3) inside. Indeed, for a fixed point $\varepsilon_{,u} = \varepsilon_{,X} u$ should vanish. The latter is always realized at the origin where $u = 0$. Note that the change of sign of $\varepsilon_{,X}$ is a strong requirement for the existence of a limiting cycle. It is not clear whether this condition can be obtained just form the requirement that the sign of the time-derivative of the pressure (3.12) should change.

It is important to note that $u$ does not correspond to a naive flat space canonical momentum $\pi = p_{,\phi} = P/a^3$ for the field $\phi$. Indeed, for this momentum $\pi$ we obtain using (3.11)
\[
\dot{\pi} = -3H\pi + \left(\frac{\partial p}{\partial \dot{\phi}}\right)_{\phi},
\] instead of the first equation of the auxiliary system (4.3). From this equation above it follows that on the limiting cycle the $p_{,\phi}$ (taken by constant $\dot{\phi}$) should change the sign. Thus a limiting cycle crosses the curve on phase space $(\phi, \dot{\phi})$ where $p_{,\phi} = 0$. Indeed, otherwise it is impossible to overcome the Hubble friction and the flat space canonical momentum $\pi$ would always decrease in an expanding universe. Note that the transition to $\pi$ is not that useful in the region with a singularity $\varepsilon_{,X} = 0$, as there the relation between momentum and field velocity becomes not invertible, see (3.21). Thus this description would not be useful to prove the statement that this singularity is required for a limiting cycle. Now we can look at the flat space Hamiltonian
\[
\mathcal{H} = \dot{\phi}\pi - p.
\] Clearly this Hamiltonian (4.7) is equal to $\varepsilon$ by value, but $\varepsilon(u,v)$ and $\mathcal{H}(\phi, \pi)$ are two different functions. Further using
\[
\left(\frac{\partial p}{\partial \dot{\phi}}\right)_\pi = \left(\frac{\partial p}{\partial \dot{\phi}}\right)_{\phi} + \left(\frac{\partial p}{\partial \phi}\right)_\varphi \left(\frac{\partial \phi}{\partial \phi}\right)_\pi,
\]\(^2\)It is important to stress the power of the Dulac generalization of the original Bendixson theorem which only allowed for $f = 1$. In that case one would obtain that $\varepsilon_{,X} (3\varepsilon + p)$ should change the sign for the existence of the limiting cycle. This is clearly a weaker requirement.
we get that
\[
\frac{\partial p}{\partial \phi} = -\left(\frac{\partial H}{\partial \phi}\right)_\pi,
\] (4.9)
so that (4.6) differs from the canonical equation by a friction term with \(3H = \sqrt{3}H\) friction coefficient. The first order equation for \(\phi\) does not have the friction term and remains canonical
\[
\dot{\phi} = \left(\frac{\partial H}{\partial \pi}\right)_\phi.
\] (4.10)
Contrary to the auxiliary system (4.3) the trajectories in \((\phi, \pi)\) are not identical to the integral curves of (3.11).

We can start from the system
\[
\dot{\pi} = -\sqrt{3}H \pi - H_{,\phi},
\]
\[
\dot{\phi} = H_{,\pi},
\] (4.11)
and use the Bendixson–Dulac theorem with the Dulac function \(f = 1/\sqrt{3H}\) to obtain that \(\partial f(\alpha) + \partial f(\beta) = -1\). Which implies that the limiting cycle is not possible, provided the conditions of the Bendixson–Dulac theorem are satisfied. In particular, the relevant assumption there was that \(\alpha(\phi, \pi)\) and \(\beta(\phi, \pi)\) are smooth - i.e. that the flat space Hamiltonian \(H\) is smooth. Hence a limiting cycle is not possible for k-essence with a smooth Hamiltonian. Moreover, one can expect that the cosmocanonical system (4.11) is universal – holds for any generic cosmological scalar field beyond k-essence i.e. Galileons, Horndeski theories etc \(^3\). Therefore we expect that the limiting cycle in an expanding universe is only possible for systems with a not smooth flat space Hamiltonian.

5 Phase space analysis of the particular model

Here we provide a phase space analysis of the class of k-essence systems discussed in [5]. The expansion of the universe will be assumed to be driven by the scalar field itself through the first Friedmann equation (2.2). This analysis will mostly serve for the illustrative purposes of our generic statements made above. The Lagrangian studied in [5] is
\[
p(\phi, X) = \left(3b \phi^2 - 1\right)X + X^2 - V(\phi),
\] (5.1)
where the double-well potential is given by
\[
V(\phi) = \Lambda + \frac{1}{12a} - \frac{1}{2} \phi^2 + \frac{3a}{4} \phi^4.
\] (5.2)
Without potential and with \(b = 0\) this system corresponds to simple k-inflation [4] and ghost condensate [3]. The authors only considered \(a > 0\) and \(b > 0\), which we will assume in this paper as well. The extrema of the potential are given by
\[
\phi_0 = 0, \phi_\pm = \pm \frac{1}{\sqrt{3a}}.
\] (5.3)
\(^3\)We assume here that the system (4.11) is valid for a cosmological scalar field with a second order covariant equation of motion. A detailed discussion will be provided somewhere else.
Thus on in the phase space of the homogeneous solutions \((\varphi, \dot{\varphi})\) there are always three equilibrium points - fixed points \((0, 0), (\pm 1/\sqrt{3}a, 0)\). Clearly the first trivial point \((0, 0)\) without symmetry breaking is unstable. There \(V(\varphi_0) = \Lambda + 1/(12a)\). While the fixed points with symmetry breaking correspond to de Sitter solutions with

\[ V(\varphi_{\pm}) = \Lambda. \tag{5.4} \]

The NEC is violated when \(p, X < 0\) which occurs inside of the phantom divide which is given by the ellipse

\[ \dot{\varphi}^2 + 3b \varphi^2 = 1. \tag{5.5} \]

For values of \(\varphi\) and \(\dot{\varphi}\) outside of the ellipse the system does not violate the NEC. On the ellipse the equation of state \(w = -1\). From (3.33) it follows that by approaching the phantom divide ellipse from outside the quantum perturbations on short scales diverge.

The energy density is given by (3.6)

\[ \varepsilon = \left(3b \varphi^2 - 1\right) X + 3X^2 + V(\varphi). \tag{5.6} \]

The singularity in the equation of motion (3.11) occurs when \(\varepsilon, X = 0\) or on the ellipse

\[ 3\dot{\varphi}^2 + 3b \varphi^2 = 1. \tag{5.7} \]

From (3.33) it follows that by approaching this singularity ellipse with \(\varepsilon, X = 0\) from inside the quantum perturbations on short scales diverge. Note that both ellipses (5.5) and (5.7) share the same \(\varphi\) axis, but the axis in \(\dot{\varphi}\) for the NEC violation is in \(\sqrt{3}\) times larger. Hence the singularity ellipse (5.7) is always inside of the NEC-violation ellipse (5.5).

For the sound speed (3.8) as a function on phase space we have

\[ c^2_S(\varphi, \dot{\varphi}) = \frac{\dot{\varphi}^2 + 3b \varphi^2 - 1}{3\dot{\varphi}^2 + 3b \varphi^2 - 1} = 1 - \frac{2\dot{\varphi}^2}{3\dot{\varphi}^2 + 3b \varphi^2 - 1}. \tag{5.8} \]

Thus for all \(\varepsilon, X = 3\dot{\varphi}^2 + 3b \varphi^2 - 1 < 0\) i.e. inside of the ellipse (5.7) the sound speed is always superluminal except of points \(\dot{\varphi} = 0\). For \(\varepsilon, X > 0\) the speed of sound is never superluminal. By approaching this singular ellipse (5.7) from inside the sound speed grows without any limit. It is important that the superluminality is separated from a limiting cycle by the singularity ellipse (5.7). Indeed, no trajectory can cross this border, while we know that the limiting cycle (if exists) has to cross the Phantom divide ellipse (5.5) which is located outside of the singularity ellipse. In the region of the phase space between the NEC-violation ellipse (5.5) and the singularity ellipse (5.7) the sound speed is imaginary and the the system is elliptic. The increment of this linear instability grows for each mode without any bound by approaching the singularity ellipse (5.7). The linear instability for high \(k\) modes was mentioned in [5], albeit the authors used a different formula for the sound speed. Namely it was assumed that one can treat the system as a fluid so that the sound speed can be inferred from \(c^2_s = \partial \langle p \rangle / \partial \langle \varepsilon \rangle\) where the averaging is done for many oscillations. Clearly this formula is different from (3.8).

On the other hand for the derivative of the energy density we have

\[ \varepsilon, \varphi = \varphi \left(3b \dot{\varphi}^2 + 3a \varphi^2 - 1\right). \tag{5.9} \]
Thus there is a ellipse of the vanishing $\varepsilon,\varphi$

$$3b\dot{\varphi}^2 + 3a\varphi^2 = 1. \quad (5.10)$$

As we have showed in the section (4) the limiting cycle has to go through the curve $p,\varphi = 0$. This curve is given by

$$p,\varphi = \varphi \left(3b\dot{\varphi}^2 + 1 - 3a\varphi^2\right). \quad (5.11)$$

Thus it is enough to evolve through the $\varphi = 0$.

For our further analysis of the phase curves it is convenient to use the auxiliary system (4.3)

$$\frac{du}{dt} = -p,u\sqrt{3\varepsilon} - \varepsilon,u = -u \left(3bv^2 + u^2 - 1\right)\sqrt{3}\varepsilon - v \left(3bu^2 + 3av^2 - 1\right), \quad (5.12)$$

$$\frac{dv}{dt} = \varepsilon,u = u \left(3bv^2 + 3u^2 - 1\right),$$

where the energy density

$$\varepsilon (v,u) = \frac{1}{2} \left(3b v^2 - 1\right) u^2 + \frac{3}{4} u^4 + \Lambda + \frac{1}{12a} - \frac{1}{2} v^2 + \frac{3a}{4} v^4. \quad (5.13)$$

For this system the singular ellipse (5.7) does not correspond to any singularity. In generic case when $a \neq b \neq 1$ the flow $\vec{F} = (\dot{v}, \dot{u})$ on the singular ellipse is parallel to $u$-axis, see Fig. (2) for a degenerate case. For small $v$ or close to the $u$-axis the flow is always pointing out outside of the singular ellipse, as the cosmological friction dominates. However, by approaching the $v$-axis the second term in the equation (5.12) starts to dominate. And the direction of flow changes inwards. The point of equilibrium is between these two forces is a nontrivial fixed point.

Without the friction term the motion occurs in $(v,u)$ along the contours of the constant energy density. If we find an energy level completely enclosing the phantom divide from the outside where $p,X > 0$ (so that in this model $\varepsilon,X > 0$ too) the flow of the time evolution in an expanding universe will be directed to the interior phase space of this level of energy density. Indeed, the normal to the $\varepsilon (v,u) = \varepsilon_0$ contour is $\vec{N} = (\varepsilon,u,\varepsilon,u)$, it points out outside of the closed contour, while the scalar product of a $\vec{F}$ with flow $\vec{F} = (\dot{v}, \dot{u})$ is

$$\vec{F} \cdot \vec{N} = -\varepsilon,u p,u\sqrt{3}\varepsilon = -2X\varepsilon,X p,X\sqrt{3}\varepsilon. \quad (5.14)$$

It is easy to check that the curve of constant energy density

$$\varepsilon (v,u) = \varepsilon_{pd} = \frac{1}{4} + \frac{1}{12a} + \Lambda, \quad (5.15)$$

where $\varepsilon (v,u)$ is given by (5.13) has the smallest energy density among the levels encircling the Phantom divide ellipse (5.5), see Fig. (1). Thus any infinitesimally small change of the contour $\varepsilon (v,u) = \varepsilon_{pd} (1 + \epsilon)$ where $\epsilon \ll 1$ encloses the Phantom divide and has the flow of trajectories pointing inside. No trajectory can escape this contour (5.15). As a limiting cycle has to cross the Phantom divide, there cannot be limiting cycles outside of the contour of no return. This contour of no return crosses the $v$-axes at

$$\varphi_{nr \pm} = \pm \sqrt{\frac{1 + \sqrt{1 + 3a}}{3a}}. \quad (5.16)$$
The limiting cycle cannot cross the singularity ellipse (5.7). Hence we have the following bound on the amplitude of the oscillations of the field

\[
\frac{1}{\sqrt{3b}} < \varphi_{\text{max}} < \sqrt{\frac{1 + \sqrt{1 + 3a}}{3a}}.
\] (5.17)

Let us study the fixed points in details. We start from the trivial fixed point \((u, v) = 0\) in that case

\[
\varepsilon = \Lambda + \frac{1}{12a} + \mathcal{O}\left(u^2, v^2\right) = V_0 + \mathcal{O}\left(u^2, v^2\right),
\] (5.18)

so that the linearized system takes the form

\[
\begin{align*}
\frac{du}{dt} &= u\sqrt{3V_0} + v, \\
\frac{dv}{dt} &= -u,
\end{align*}
\] (5.19)

with the corresponding eigenvalues

\[
\lambda^0_\pm = \sqrt{\frac{3V_0}{4}} \pm \sqrt{\frac{3V_0}{4} - 1}.
\] (5.20)

For \(V_0 \geq 4/3\) both eigenvalues are real and positive so that the fixed point is an unstable node. While for \(V_0 < 4/3\) both root are complex with a positive real part so that the fixed point is an unstable focus. In both cases all trajectories leave the small neighborhood of the trivial fixed point.

Now let’s consider other fixed points \(\left(0, \pm \left(3a\right)^{-1/2}\right)\). Because of (5.16) these fixed points are always inside of the no return contour (5.15). If these points are stable and are outside of the Phantom divide \(b > a\), they allow to trajectories to end up without creating the limiting cycle. There we have for the energy

\[
\varepsilon = \Lambda + \mathcal{O}\left(\delta u^2, \delta v^2\right),
\] (5.21)

so that the linearized system is

\[
\begin{align*}
\frac{du}{dt} &= -u \left(\frac{b}{a} - 1\right) \sqrt{3\Lambda} - 2\delta v, \\
\frac{d\delta v}{dt} &= u \left(\frac{b}{a} - 1\right).
\end{align*}
\] (5.22)

The corresponding eigenvalues are

\[
\lambda^c_\pm = \frac{1}{2} \left[ \left(1 - \frac{b}{a}\right) \sqrt{3\Lambda} \pm \sqrt{3\Lambda \left(1 - \frac{b}{a}\right)^2 + 8 \left(1 - \frac{b}{a}\right)} \right].
\] (5.23)

If \(b > a\) both roots have negative real part. In that case \(\varphi_\pm\) is outside of the phantom divide and both fixed points are stable focuses, see (3). In that case trajectories starting crossing the no return contour (5.15) from outside can end on these stable focuses. If \(b < a\) then both roots are real and \(\lambda_+^c > 0\) while \(\lambda_-^c < 0\). Hence the fixed points are saddle points i.e unstable for \(b < a\). Thus for a limiting cycle we need that \(b < a\).

Other nontrivial fixed points are located at the singular ellipse.
6 Conclusions and Discussion

The new oscillatory state of cosmological matter corresponds to a limiting cycle of the classical equations of motion. This state can also be considered as a cosmological realization of a time-crystal. This regime can be of interest to cosmology, because on average the universe undergoes the de Sitter expansion with $\langle w \rangle = -1$. Hence, a time-crystal can be used to model early stages of Inflation or late stages of Dark Energy. We have shown that any realization of the new oscillatory state of cosmological matter requires not only a violation of the NEC but also a crossing of the Phantom divide. The systems studied in [5] are known to be incapable to achieve this crossing [14], see also [15–17] and for a review [18, 19]. In particular, the crossing generically implies gradient instabilities where the sound speed is imaginary, $c_s^2 < 0$.

We have showed in section 3 that the quantum perturbations of k-essence on short scales grow without any bound in an attempt to evolve across $w = -1$.

Further in section (4) we used the Bendixson–Dulac theorem to prove that for k-essence i) to realize a cosmological limiting cycle classically the system has to have a flat-space Hamiltonian with cusps and that ii) the existence of a limiting cycle implies that the system has a singularity where $\varepsilon_{,X}$ changes sign and where the canonical momentum does not define the velocity uniquely. The appearance of this singularity implies the presence of configurations with strong superluminal propagation of the small perturbations and existence of other configurations with strong gradient instability. On this singularity the small quantum perturbations are infinitely strongly coupled. The Bendixson–Dulac theorem dictates that the singularity curve on which $\varepsilon_{,X}$ is vanishing, should be located inside of the limiting cycle, as the latter cannot cross the singularity. For some systems, including the one studied in [5], the configurations with superluminality are separated from the limiting cycle by the singularity curve with infinitely strongly coupled perturbations. Thus one can consider that the oscillatory solution and the superluminal configurations are described by two separate EFT’s. There are arguments [35] that the presence of superluminality implies that the system cannot be UV-completed in the usual local and Lorentz-invariant way. If the superluminal configurations should be described by a disconnected EFT different from the one which describes the limiting cycle and neighboring configurations these arguments would not apply.

Then in section (5) we analyzed in details the dynamics of the particular class of k-essence models from [5]. These theories have three free parameters. Following the spirit of the Poincare-Bendixson theorem we analyzed for which values of the parameters the limiting cycle is possible. This section serves for the illustrative purposes of our analytical results from the previous sections. We provide numerically obtained phase plots for a better visualization. In this particular class of theories the configurations with the superluminality are always screened by the singularity from the time-crystal or limiting cycle. However, the latter is also disconnected from the trivial Lorentz-invariant vacuum. Hence the EFT describing the time-crystal does not have any Lorentz-invariant solutions. It is interesting to understand under which conditions this feature allows for the theory to have a Lorentz-invariant and local UV-completion.

The models which can violate NEC and can cross the Phantom divide without immediate pathologies are Generalized Galileons / Horndeski theories [41–43]. Recently it has been argued [44] that for the subclass of these theories introduced in [45] it is still rather hard to have a singularity free cosmological evolution without any pathology and with an NEC-violating phase. Further these results were generalized to whole class of Horndeski theories in [46]. Both these works only consider external matter which does not have any direct coupling
to the considered Galileon scalar field. May be a presence of an additional degree of freedom particularly coupled to the Galileon can ameliorate the problem.

To conclude, we find the idea of cosmological time-crystals rather interesting, but it seems to be very hard to realize this idea in a physically plausible way.

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Figure 1. Phase plot with a limiting cycle for the system (5.1) with the parameters $a = 1$, $b = 1/2$ and $\Lambda = 1$. The values of parameters are chosen for illustrative purposes only. The limiting cycle is the yellow less-symmetric curve. The red ellipse is the singularity where $\varepsilon, \chi = 0$, which is given by (5.7). Inside of this singular curve the perturbations are ghosts. On the singular curve the perturbations are infinitely strongly coupled. The green ellipse is the Phantom divide on which $p, \chi = 0$ and $w = -1$. This green ellipse is given by (5.5). Inside this green ellipse the NEC is violated. Between these two ellipses the sound speed is imaginary and the system is linearly unstable. By approaching the red ellipse from outside this linear instability becomes infinitely strong for any given mode $k$. The blue contour is the lowest level of constant energy density enclosing the Phantom divide, given by (5.15). One can clearly see five fixed points: two focuses on the singular ellipse, one node in the origin and two saddles on the $\varphi$-axis.

Clearly the system possesses a trivial equilibrium / fix point at the origin $(\varphi, \dot{\varphi}) = (0, 0)$ where the system is linearly stable but is ghosty and suffers therefore from the ghosts instabilities due to interactions (e.g. unavoidable interactions through gravity). This trivial vacuum is isolated by the red ellipse of the infinitely strong coupling from the yellow oscillatory attractor which can be considered as a nontrivial ground state. However, this nontrivial ground state is plagued by linear instabilities. Indeed, the yellow cyclic trajectory clearly crosses the green ellipse four times - enters twice the region where the sound speed is imaginary. Moreover, the classical evolution breaks down by approaching the green ellipse as the quantum perturbations on short scales grow without any bound, see (3.33).
Figure 2. Phase plot with a limiting cycle for the system (5.1) with the degenerate values of the parameters: $a = 1$, $b = 1$ and $\Lambda = 0$ on the right and $a = 1$, $b = 1$ and $\Lambda = 1$ on the left. The values of parameters are chosen for illustrative purposes only. The limiting cycle is the yellow less-symmetric curve. The red ellipse is the singularity where $\varepsilon_X = 0$, which is given by (5.7). Inside of this singular curve the perturbations are ghosts. On the singular curve the perturbations are infinitely strongly coupled. The green ellipse is the curve on which $p_X = 0$ and $w = -1$. This green ellipse is given by (5.5). Inside this green ellipse the NEC is violated. Between these two ellipses the sound speed in imaginary and the system is linearly unstable. By approaching the red ellipse from outside this linear instability becomes infinitely strong for any given mode $k$.

Clearly both systems possess a trivial equilibrium / fix point at the origin $(\varphi, \dot{\varphi}) = (0, 0)$ where the system is linearly stable but is ghosty and suffers therefore from the ghosts instabilities due to interactions (e.g. unavoidable interactions through gravity). This trivial vacuum is isolated by the red ellipse of the infinitely strong coupling from the oscillatory attractor which can be considered as a nontrivial ground state. However, this is nontrivial ground state is plagued by linear instabilities. Indeed, the yellow cyclic trajectory clearly crosses the green ellipse four times - enters twice the region where the sound speed is imaginary. Moreover, the classical evolution breaks down by approaching the green ellipse as the quantum perturbations on short scales grow without any bound, see (3.33).
Figure 3. Phase plot without the limiting cycle for the system (5.1) with the values of the parameters $a = 1$, $b = 2$ and $\Lambda = 1$ on the left. The values of parameters are chosen for illustrative purposes only. The red ellipse is the singularity where $\varepsilon_X = 0$, which is given by (5.7). Inside of this singular curve the perturbations are ghosts. On the singular curve the perturbations are infinitely strongly coupled. The green ellipse is the curve on which $p_X = 0$ and $w = -1$. This green ellipse is given by (5.5). Inside this green ellipse the NEC is violated. Between these two ellipses the sound speed in imaginary and the system is linearly unstable. By approaching the red ellipse from outside this linear instability becomes infinitely strong for any given mode $k$. The blue contour is the lowest level of constant energy density enclosing the Phantom divide, given by (5.15). One can clearly see five fixed points: two saddles on the singular ellipse, one node in the origin and two stable fixed points on the $\varphi$-axis. These fixed points are stable focuses and attract trajectories crossing the blue contour of no return. Outside of the blue contour there cannot be limiting cycles as the latter has to cross through the green ellipse - the Phantom divide.