CONNECTION PRESERVING ACTIONS ARE TOPOLOGICALLY ENGAGING

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Abstract. Topologically and geometrically engaging actions have proved to be useful to obtain rigidity results for semisimple Lie group actions (see [7], [1]). We show that the action of a simple noncompact Lie group on a compact manifold preserving a unimodular rigid geometric structure of algebraic type (e.g. a connection together with a volume density) is topologically engaging on an open conull dense set.

1. Introduction

A fundamental problem in geometry is to determine the isometry group of a given manifold with a geometric structure. From a dynamical point of view, an even more interesting problem is to determine for a given Lie group $G$ the manifolds with geometric structures that admit an action by isometries from $G$. Particularly interesting problems arise when we assume $G$ to be a semisimple Lie group as it has been shown in the work of Adams, Feres, Katok, Spatzier and Zimmer, among others.

For semisimple Lie groups, the existence of actions preserving geometric structures impose strong restrictions on the manifolds that admit such actions. In a sense, this can be considered an extension of Margulis’ superrigidity theorem, since any action defines a representation of the group into the diffeomorphism group of the manifold. However, the techniques used to prove such restrictions are somehow more complicated.

When studying group actions it is very useful to distinguish those satisfying suitable conditions. In this work we want to focus on actions satisfying what is known as an engagement condition, with particular emphasis on topologically and geometrically engaging actions (see Definitions 2.3 and 2.7). For any such restriction to be useful we need it to have two important features: 1) The condition must allow to obtain interesting properties or apply known tools. 2) The condition must be satisfied by most actions under study or it must be a consequence of natural geometric/dynamic hypothesis. Theorems 2.6 and 2.15 are just two examples that show that topological and geometric engagement satisfy the first feature, and plenty of other results found in the references below provide more instances.

On the other hand, it turns out that topological engagement is satisfied for actions that preserve a connection and a finite volume, which is a pretty natural geometric condition. Even though this statement is claimed to be true in several

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of the references below there is no complete proof of this fact up to this date. The main goal of this work is to provide such a proof to completely settle this basic property of topologically engaging actions.

With respect to the organization of the article, in section 2 we define the engaging conditions discussed above and describe some of their applications. In section 3 we develop some of the basic lemmas needed in our proof that come from a result of Gromov known as the centralizer theorem. Section 4 contains the main result of this work, Theorem 4.8 where we prove that for simple noncompact Lie groups every analytic action preserving a unimodular rigid geometric structure of algebraic type (among which we have the structures consisting of a connection and a smooth measure) is topologically engaging on an open conull dense set. Finally, in section 5 we make some observations that are brought to light from the ideas in the proof of our main result.

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2. Actions and engagement conditions

In the rest of this article the (Lie group) actions of structure groups of fiber bundles are assumed to be on the right, except for the actions of fundamental groups on universal covers which we assume to be on the left. All other actions are assumed to be on the left as well.

A standard technique used to study actions of Lie groups is to require certain dynamical conditions to be fulfilled. For the case of semisimple Lie groups of noncompact type it has been found that it is particularly useful to consider restrictions on the actions lifted to coverings of the manifold being acted upon. In order to be more precise we state without proof the following standard result:

**Proposition 2.1.** Let $M$ be a connected manifold acted upon by a connected Lie group $G$, then for any covering $\pi: M' \to M$ there is an action of $\tilde{G}$, the universal covering of $G$, that commutes with the covering transformations and for which the covering maps are equivariant, in other words we have $\pi(gm) = \pi_0(g)\pi(m)$ for all $g \in \tilde{G}$ and $m \in M'$, where $\pi_0: \tilde{G} \to G$ is the canonical covering map.

If $G$ is a simple Lie group of noncompact type acting on a compact manifold $M$, then the orbits are typically very complicated, unless the action is trivial. However, it might be the case that after lifting the action, as in the previous proposition, the orbits have more manageable features. The following result (see [8]) provides a fundamental example to consider. We recall that a measure preserving action is ergodic if the only measurable invariant sets are either null or conull (see [8] for more details).

**Proposition 2.2.** Let $H$ be a simple noncompact Lie group and $\Gamma$ a lattice in $H$. If $G$ is noncompact closed subgroup of $H$, then the action of $G$ on $H/\Gamma$ is ergodic with respect to any bi-invariant measure on $H$.

The action of $G$ on $H/\Gamma$ is very complicated for $G$ a proper subgroup as above, but such action can be lifted so that $G$ (not just its universal cover $\tilde{G}$) acts on $H$ with closed orbits that define a quotient $G\backslash H$ that has an analytic manifold structure. We also remark that in the above result we can take $H$ to be a semisimple noncompact Lie group as long as we require $\Gamma$ to be an irreducible lattice (see [8]).
Trying to capture this special behavior, Zimmer has introduced in \[10\] the following notion:

**Definition 2.3.** A smooth action of a connected Lie group \(G\) on a manifold \(M\) is called topologically engaging if there is some \(\tilde{g} \in \tilde{G}\) that acts tamely on \(\tilde{M}\) (i.e., has locally closed orbits) and that projects to an element \(g \in G\) that does not lie in a compact subgroup.

**Remark 2.4.** Observe that the action considered in the previous proposition is topologically engaging. Also notice that for an action to be topologically engaging it is enough for the action of \(\tilde{G}\) on \(\tilde{M}\) to have locally closed orbits.

**Remark 2.5.** For most applications it is enough for the above condition to hold on suitable open subsets, so we will say that the action is topologically engaging on a \(G\)–invariant open subset \(U \subset M\) if there is a \(\tilde{g} \in \tilde{G}\) whose orbits in \(\tilde{U}\) (the inverse image of \(U\) in \(\tilde{M}\)) are locally closed and that projects to an element \(g \in G\) that does not lie in a compact subgroup. Notice that since \(U\) is open the locally closed condition does not depend on whether it is being taken with respect to \(\tilde{U}\) or \(\tilde{M}\).

For a dynamical condition as topological engagement to be useful it has to hold for an important family of actions and also it has to have interesting consequences. As remarked before, the main goal of this work is to show that most “geometric” actions are topologically engaging. On the other hand, in \[7\] and \[10\] it has been shown that topological engagement ensures rigid behavior for actions of semisimple Lie groups of noncompact type. In particular, the following result is the main step in the proof of Theorem A in \[7\]:

**Theorem 2.6.** Let \(G\) be a connected noncompact simple Lie group with finite center, finite fundamental group and \(\mathbb{R}\)-rank\((G) \geq 2\). Let \(M\) be a compact manifold and suppose that there is a topologically engaging action of \(G\) on \(M\) preserving a finite measure. Then \(\pi_1(M)\) is not isomorphic to the fundamental group of any complete Riemannian manifold \(N\) with negative curvature bounded away from \(0\) and \(-\infty\).

In \[1\] we have introduced a dynamical condition for actions of semisimple Lie groups similar to Zimmer’s topological engagement. Such condition comes from a particular way of measuring how the orbits in the universal cover stretch out to infinity. The latter is more precisely stated in the following:

**Definition 2.7.** Let \(G\) be a connected semisimple Lie group of noncompact type, let \(X\) be the symmetric space of noncompact type associated to \(G\) and let \(M\) be a compact Riemannian manifold acted upon by \(G\). Choose a Cartan decomposition \(g = \mathfrak{k} \oplus \mathfrak{m}\) for the Lie algebra of \(G\) (with \(\mathfrak{k}\) a maximal compact subalgebra), and let \(\mathfrak{m}_1\) be the unit ball in \(\mathfrak{m}\) with respect to the Killing form of \(g\). For \(v \in \mathfrak{m}_1\) denote with \(g_v^t = \exp(tv)\) the one–parameter subgroup of \(\tilde{G}\) generated by \(v\). The pointwise stretch of the action of \(G\) on \(M\) is the function defined by:

\[
p-\text{stre}(G, M) : \mathfrak{m}_1 \times \tilde{M} \to \mathbb{R} \\
(v, x) \mapsto \liminf_{t \to \infty} \frac{d_{\tilde{M}}(g_v^t x, x)}{t}
\]
where \( \tilde{G} \) acts on \( \tilde{M} \) by an arbitrary but fixed lift of the action of \( G \) on \( M \). We say that the action of \( G \) on \( M \) has positive stretch if \( p\text{-stre}(G, M) \) is a positive function.

**Remark 2.8.** Notice that the pointwise stretch of an action does not depend on the choice of the lifted action to \( \tilde{M} \). Also notice that the pointwise stretch does depend on the choice of the Cartan decomposition of \( g \) and the Riemannian metric on \( M \). However, it is easily seen that the condition of having positive stretch does not depend on either of them.

**Remark 2.9.** In the definition of pointwise stretch, the distance \( d_{\tilde{M}}(g^t_x, x) \) measures the stretching of the orbit of \( x \) with respect to the one–parameter subgroup \( g^t \) being considered. On the other hand, from the basic theory of symmetric spaces (see [4]), if \( x_0 \in X \) is the point fixed by the subgroup generated by \( t \), then \( t \mapsto g^t_{x_0} \) is a unit speed geodesic, and so we have \( d_X(g^t_{x_0}, x_0) = t \). It follows that the limit that defines \( p\text{-stre}(G, M)(v, x) \) compares as \( t \to \infty \) the stretching out to infinity of the orbits of \( x \) with the stretching out to infinity of the geodesics in the space \( X \). In particular, when the pointwise stretch is positive, the orbits on \( \tilde{M} \) stretch out to infinity at least as fast as they do in the symmetric space \( X \).

**Remark 2.10.** It is easily seen that an action as above has positive pointwise stretch if and only if for every one–parameter subgroup \( g^t \) of \( \tilde{G} \) which does not map into a compact subgroup of \( \text{Ad}(G) \) we have:

\[
\liminf_{t \to \infty} \frac{d_{\tilde{M}}(g^t_x, x)}{t} > 0
\]

for every \( x \in \tilde{M} \). From this it is an easy matter to show that an action with pointwise positive stretch is topologically engaging.

The notion of pointwise positive stretch is a natural translation to actions of the notion of stretch considered in [3] and [5] for foliations. However, to obtain rigidity type results for actions the following slightly stronger notion is needed:

**Definition 2.11.** With the notation as in the previous definition, we say that the action of \( G \) on \( M \) is geometrically engaging if for every sequence \( (g^t_n) \) in \( G \) such that \( (g^t_n x_0) \) is a quasi–ray in \( X \) (the symmetric space associated to \( G \)) for some (and hence any) \( x_0 \in X \), the limit inferior:

\[
\liminf_{n \to \infty} \frac{d_{\tilde{M}}(g^t_n x, x)}{d_X(g^t_n x_0, x_0)} > 0
\]

for every \( x \in \tilde{M} \).

**Remark 2.12.** Recall that a sequence \( x_n \) in \( X \) is called a quasi–ray if there exist constants \( A > 0 \) and \( B \geq 0 \) such that:

\[
A^{-1}|m - n| - B \leq d_X(x_n, x_m) \leq A|m - n| + B
\]

for all \( m, n \geq 0 \).

**Remark 2.13.** Observe that the condition of geometric engagement does not depend on the choice of \( x_0 \in X \) or the Riemannian metric on \( M \). However, the actual value of the limit inferior may depend on such choices.

**Remark 2.14.** Since a geodesic in a symmetric space can always be seen as an orbit by a one–parameter subgroup, it is easily shown that a geometrically engaging action has positive pointwise stretch and so it is topologically engaging as well.
Since geometric engagement is a stronger condition than topological engagement it is expected to provide stronger rigidity type results, which is the case as the following theorem shows (see [1]):

**Theorem 2.15.** Let $G$ be a connected noncompact simple Lie group whose associated symmetric space $X$ either has rank at least 2 or is a quaternionic or Cayley hyperbolic space and let $N$ be a compact Riemannian manifold with nonpositive sectional curvature when $X$ has rank $\geq 2$ and with nonpositive complexified sectional curvature otherwise. Suppose $G$ has a geometrically engaging action on a compact manifold $M$. If $\pi_1(M) \cong \pi_1(N)$, then there is an isometric totally geodesic immersion $X \rightarrow N$. In particular, for rank($X) \geq 2$, the space $M$ cannot have the fundamental group of a compact manifold with strictly negative sectional curvature.

3. Gromov’s centralizer theorem

From a dynamical point of view, for an action to be more manageable it is desirable to have some sort of invariant structure on the manifold being acted upon. As it is shown by Gromov in [2], for superrigidity results, the most natural kind of structures that one can consider are the rigid geometric structures. We will start this section by reviewing some of the basic definitions and notation used to describe these geometric structures and we refer to [2] and [5] for further details.

Let $Gl^{(k)}(n)$ be the Lie group of $k$-jets of diffeomorphisms of $\mathbb{R}^n$ fixing the origin; this group is easily seen to be an algebraic group. Recall that a $k$-frame of an $n$-dimensional manifold $M$ at a point $m \in M$ is the $k$-jet of a local diffeomorphism $(\mathbb{R}^n,0) \rightarrow (M,m)$. Then for such manifold $M$ there is a $Gl^{(k)}(n)$-principal fiber bundle $L^{(k)}(M)$, called the $k$-th order frame bundle, which consists of the $k$-frames of $M$. In particular, for $k=1$ the group $Gl^{(1)}(n)$ is the usual general linear group and $L^{(1)}(M)$ is the usual linear frame bundle of $M$. Let $Q$ be a smooth manifold that admits a smooth action of $Gl^{(k)}(n)$ and denote with $E^Q$ the fiber bundle associated to $L^{(k)}(M)$ with standard fiber $Q$; then a geometric structure on $M$ of order $k$ and type $Q$ is a smooth section of $E^Q$, and such structure is called of algebraic type if $Q$ is a real algebraic variety and the $Gl^{(k)}(n)$-action is algebraic as well. Every diffeomorphism $\phi$ of $M$ induces corresponding bundle diffeomorphisms for both $L^{(k)}(M)$ and $E^Q$, and for a geometric structure of order $k$ and type $Q$ we will say that $\phi$ is an isometry or automorphism if the corresponding section of $E^Q$ is equivariant with respect to such diffeomorphisms induced by $\phi$. A smooth vector field on $M$ is called a Killing vector field for a given geometric structure if its local flow acts by local automorphisms for the structure.

A geometric structure $\omega$ of order $k$ and type $Q$ on $M$ is called unimodular if for each $m \in M$ the $Gl^{(k)}(n)$-orbit in $Q$ of $\omega(m)$ has stabilizers whose images in $Gl^{(1)}(n) = Gl(n)$ under the natural jet projection $Gl^{(k)}(n) \rightarrow Gl^{(1)}(n)$ are contained in the group of matrices with determinant $\pm 1$. It is easily seen that such structure defines a reduction of the linear frame bundle $L^{(1)}(M)$ to the group of matrices with determinant $\pm 1$, and so induce a volume density on $M$.

For a structure $\omega$ of order $k$ and type $Q$, $l \geq k$ and $x,y \in M$ we define $Aut^{(l)}(\omega,x,y)$ to be the set of $l$-jets of diffeomorphisms of $M$ taking $x$ to $y$ and $\omega(x)$ to $\omega(y)$ up to order $l$; any such jet is called an infinitesimal automorphism of $\omega$. We also denote $Aut^{(l)}(\omega,m) = Aut^{(l)}(\omega,m,m)$, which is clearly a group.
Notice that whenever a manifold has a geometric structure there is a corresponding geometric structure on its universal cover for which the covering map is a local isometry and the fundamental group acts by isometries as well. For such setup we will denote both geometric structures with the same symbol.

**Definition 3.1.** A geometric structure $\omega$ is called $k$–rigid if for each $m \in M$ and $l \geq k$, the natural jet projection map $\text{Aut}^{(l)}(\omega, m) \to \text{Aut}^{(k)}(\omega, m)$ is injective. A geometric structure is called rigid if it is $k$–rigid for some $k$.

**Remark 3.2.** For $k$–rigid geometric structures the infinitesimal automorphisms at a point are completely determined by its $k$–jet at that point.

**Remark 3.3.** It is easy to show that pseudoRiemannian metrics are rigid structures of order 1 and that affine connections are rigid structures of order 2. Also both are structures of algebraic type. Moreover, any finite type structure in the sense of Cartan (see [5]) is rigid.

**Remark 3.4.** Notice that all definitions and constructions above can be performed replacing the smooth maps and manifolds by analytic ones, and that the corresponding remarks and properties mentioned above still hold true.

Gromov has extensively studied in [2] the properties of rigid structures that relate to the superrigid behavior of actions of semisimple Lie groups. A very important result of such study is the abundance of Killing vector fields for suitable rigid geometric structures that commute with the action of a simple Lie group that preserve any such structure. More precisely, we have the following result that appears as Corollary 4.3 in [11]:

**Theorem 3.5.** Suppose $G$ is a noncompact simple Lie group with finite center acting analytically and non trivially on a compact manifold $M$ preserving an analytic unimodular, rigid, structure $\omega$ of algebraic type. Identify the Lie algebra $\mathfrak{g}$ of $G$ with a Lie algebra of globally defined Killing fields on $\tilde{M}$ via the action of $M$. Let $\mathfrak{z}$ be the centralizer of $\mathfrak{g}$ in the Lie algebra of globally defined Killing vector fields on $\tilde{M}$. For $x \in \tilde{M}$, let $\mathfrak{z}(x)$ and $\mathfrak{g}(x)$ be the images of $\mathfrak{z}$ and $\mathfrak{g}$ respectively under the evaluation map at $x$. Then for a.e. $x \in \tilde{M}$ we have $\mathfrak{z}(x) \supset \mathfrak{g}(x)$.

As an immediate corollary of this we have the following result. From now on, given a manifold $M$ and a point $x \in M$ we will denote with $\text{ev}_x$ the evaluation map at $x$ for vector fields on $M$, and we define $\tilde{\text{ev}}_x$ similarly for $\tilde{M}$.

**Theorem 3.6** (Gromov’s centralizer theorem). Let $G$ be a simply connected, simple, noncompact Lie group with finite center acting on a compact manifold $M$ via analytic diffeomorphisms, and preserving a unimodular, rigid, analytic structure $\omega$ of algebraic type. Denote with $\mathcal{G}$ the Lie algebra of Killing vector fields on $\tilde{M}$ induced by the $G$–action.

Let $\mathcal{V}$ denote the collection of all analytic vector fields $X \in \mathfrak{X}(\tilde{M})$ such that

- $X$ centralizes $\mathcal{G}$, and
- $X$ is a Killing field for $\omega$.

Then:

1. $\mathcal{V}$ is $\pi_1(M)$-invariant.
2. $\mathcal{V}$ is finite dimensional.
3. $\mathcal{V}$ centralizes $\mathcal{G}$. 

Proof. Conclusion (1) follows from the fact that the actions of $\pi_1(M)$ and $G$ lifted to $\widetilde{M}$ commute, and also uses that $\omega$ in $\widetilde{M}$ is $\pi_1(M)$–invariant. Conclusion (2) follows from the fact that a rigid structure has a finite dimensional Lie algebra of Killing vector fields (see [5] for a proof of this claim in the case of finite type structures). Conclusion (3) is immediate from the definition of $\mathcal{V}$.

The only nontrivial claim is given by (4), which follows from Theorem 3.5 since $\mathfrak{g}(x) = T_xGx$. □

This remarkable result essentially states that, with the given hypotheses, whenever the Lie algebra of Killing fields for the geometric structure contains $\mathfrak{g}$ (the Lie algebra of $G$), through the action of $G$, then there a whole new vector space of Killing fields that in virtue of conclusion (4) somehow still contains $\mathfrak{g}$ and commutes with the original space of Killing fields. A simple but important example to have in mind is the action of $G$ as above on $G/\Gamma$, where $\Gamma$ is a cocompact lattice. In this case we can take the geometric structure to be the bi–invariant pseudoRiemannian metric on $G$ coming from the Killing form and observe that the $G$–action lifts to the left action of $G$ on itself so that $G$ is the Lie algebra of right invariant vector fields and $\mathcal{V}$ is the Lie algebra of left invariant vector fields.

Remark 3.7. Observe that the hypotheses of Gromov’s centralizer theorem are satisfied if we assume that the action leaves invariant an affine connection and a volume density as long as we assume both to be analytic.

In the rest of the article we will assume that $M$ is a compact manifold acted upon by a simple Lie group $G$ so that the hypotheses of Gromov’s centralizer theorem are satisfied. We will also denote with $\Gamma$ the fundamental group of $M$.

Remark 3.8. Since $\mathcal{V}$ as above is $\Gamma$–invariant, Gromov’s centralizer theorem provides a representation $\rho: \Gamma \rightarrow \text{Gl}($ $\mathcal{V}$ $)$ that can be used to understand some of the properties of $\Gamma$. As an example, in the following theorem due to Gromov (see [2] and [11]) the representation of $\Gamma$ is the one provided by Gromov’s centralizer theorem.

Theorem 3.9. Let $M$ be a compact manifold acted upon by a simple Lie group $G$ satisfying the hypotheses of Gromov’s centralizer theorem. Then there is a representation $\rho: \Gamma \rightarrow \text{Gl}(q)$ for some $q$ such that the Zariski closure of $\rho(\Gamma)$ contains a group locally isomorphic to $G$.

Such result imposes strong restrictions on the fundamental group of $M$. For example, using a theorem of Moore (see [8]) it follows that the fundamental group of $M$ cannot be amenable. In the basic theory of Lie groups it is a well known fact that the semisimple Lie groups of noncompact type provide a family of groups which is completely disjoint from the family of amenable groups, and every Lie group is built out of both families; more precisely, every connected semisimple amenable Lie group is compact and every connected Lie group is (up to a covering) the semidirect product of an amenable group and a semisimple Lie group of noncompact type. Then the previous result states that (under suitable restrictions) the fundamental group of a manifold acted upon by a simple noncompact Lie group $G$ lies in the same sort of family that contains $G$. 
4. Topological engagement

In this section we will prove that actions satisfying the hypotheses of Gromov’s centralizer theorem are topologically engaging. Our main tool will be the representation considered in Theorem 3.9. In this section $\mathcal{M}$ will denote a manifold acted upon by $G$ satisfying the hypotheses of Gromov’s centralizer theorem, $\Gamma$ will denote the fundamental group of $\mathcal{M}$ and $\rho$ the representation defined in Theorem 3.9.

Consider the product left action of $\Gamma$ on $\mathcal{V} \times \tilde{\mathcal{M}}$ which defines a manifold $\Gamma \backslash (\mathcal{V} \times \tilde{\mathcal{M}})$ that fibers as a vector bundle over $\mathcal{M}$. We will denote by $E V$ the total space of this vector bundle. It is easily seen that the principal frame bundle $L(E V)$ associated to $E V$ is canonically isomorphic to $\Gamma \backslash (\text{Gl}(V) \times \tilde{\mathcal{M}})$. Notice that by conclusion (2) from Gromov’s centralizer theorem, the space $V$ is finite dimensional and so $\text{Gl}(V)$ is a finite dimensional Lie group. Also observe that the evaluation map defines an analytic vector bundle map $ev: E V \rightarrow TM$. Since $G$ is simply connected its left action on $\mathcal{M}$ lifts to an action on the principal bundle $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ that commutes with the (left) action of $\Gamma$. Being $E V$ a fiber bundle associated to $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ there is an induced left action of $G$ on the bundles $E V \rightarrow \mathcal{M}$ and $L(E V) \rightarrow \mathcal{M}$ by bundle automorphisms, and it is easily checked that such actions are given by $g[X, m] = [gX, gm]$, where (for the case of $E V$) $X \in \mathcal{V}$, $m \in \tilde{\mathcal{M}}$ and $g \in G$, with a similar expression for $L(E V)$.

Lemma 4.1. The vector bundle map $ev: E V \rightarrow TM$ is $G$–equivariant with respect to the actions induced on $E V$ and $TM$ as bundles associated to the principal bundle $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$.

Proof. Given a diffeomorphism $\phi$ of $\mathcal{M}$ and a vector field $X$ on $\mathcal{M}$ we clearly have $d\phi(X_m) = d\phi(X)_{\phi(m)}$ for every $m \in \mathcal{M}$. From this it follows that the evaluation map $ev$ is $G$–equivariant with respect to the natural action of $G$ on $TM$ and the action of $G$ on $E V$ given by $g[X, m] = [gX, gm]$, where $X \in \mathcal{V}$ and $gX$ denotes the vector field obtained by letting the diffeomorphism defined by $g$ act on $X$. By conclusion (3) of Gromov’s centralizer theorem, the action of $G$ on $\mathcal{V}$ is trivial, i.e. $gX = X$ for all $X \in \mathcal{V}$ and $g \in G$. Then ev is $G$–equivariant for the action of $G$ on $E V$ as associated bundle of $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$. □

A very useful property of simple Lie groups is that their actions are essentially locally free. More precisely we have the following result that appears as Corollary 3.6 from [11] (see also [9]).

Theorem 4.2. Suppose $G$ is a simple noncompact Lie group acting non trivially on a manifold $\mathcal{M}$ with a finite invariant smooth measure. Then there is an open dense conull $G$–invariant set for which the stabilizers are discrete.

Remark 4.3. We recall that an action with discrete stabilizers is called locally free and it is called essentially locally free if it is locally free on a conull set.

A direct application of Frobenius theorem provides the following:

Corollary 4.4. Suppose $G$ is a simple noncompact Lie group acting non trivially on a manifold $\mathcal{M}$ with a finite invariant smooth measure. Then on an open conull dense $G$–invariant subset $U_0$ of $\mathcal{M}$ the $G$–orbits define a smooth foliation and the tangent spaces to the orbits define a smooth subbundle $TO$ of the tangent bundle $TU_0$. Moreover, if the action is analytic, the foliation and $TO$ are analytic as well.
This result allows us to improve the conclusions of Gromov’s centralizer theorem.

Lemma 4.5. Consider a group $G$ acting on a manifold $M$ as in Gromov’s centralizer theorem. Then there is an open conull dense subset $\tilde{U}$ of $\tilde{M}$, which is both $G$ and $\Gamma$–invariant, such that conclusion (4) from Gromov’s centralizer theorem holds for every $x \in \tilde{U}$. Moreover, $G$ acts locally freely on $\tilde{U}$.

Proof. Let $\tilde{U}_0$ be an open $G$–invariant conull subset of $\tilde{M}$ on which the $G$–action is locally free as provided by Corollary 4.3. Note that $\tilde{U}_0$ may be assumed to be $\Gamma$–invariant. Consider the natural evaluation map $\tilde{ev}: \tilde{U}_0 \times \mathcal{V} \to TM$. Since $\tilde{ev}$ is analytic it is easily seen that there is an open conull $\Gamma$ and $G$–invariant open subset $\tilde{U} \subset \tilde{U}_0$ (the complement of an analytic set) on which $\text{rank}(\tilde{ev}_x)$ is maximal. From this it follows that $\bigcup_{x \in \tilde{U}} \tilde{ev}_x(\mathcal{V})$ defines an analytic vector subbundle of $TM$ on $\tilde{U}$ and its continuity, together with that of the tangent bundle to the orbits, implies that the set $A$ consisting of those points $x \in \tilde{U}$ such that $T_xGx$ is not contained in $\tilde{ev}_x(\mathcal{V})$ is open in $\tilde{U}$. But Gromov’s centralizer theorem implies that $A$ is null in $\tilde{U}_0$ and since the measure on $\tilde{M}$ is smooth it must be that $A$ is empty. Hence, $\tilde{U}$ satisfies conclusion (4) and has the required properties. \hfill \square

The following will also prove to be a useful property to consider.

Lemma 4.6. Let $G$ and $M$ be as in Corollary 4.3. Then on an open conull dense $G$–invariant subset $U_0$ of $M$ both $TO$ and its frame bundle $L(TO)$ are trivial. Moreover, there is a trivialization $L(TO) \cong \text{Gl}(\mathfrak{g}) \times U_0$, where $\mathfrak{g}$ denotes the Lie algebra of $G$, so that the $G$–action is given by $g(A, m) = (\text{Ad}_G(g) \circ A, gm)$ for every $A \in \text{Gl}(\mathfrak{g})$, $g \in G$ and $m \in U_0$.

Proof. Let $U_0$ be an open subset of $M$ as in Corollary 4.3 and for every $X \in \mathfrak{g}$ let $X^*$ be the vector field on $U_0$ induced by $X$, i.e. $X^*$ is given at a point $m \in U_0$ by

$$X^*_m = \frac{d}{dt} \bigg|_{t=0} (\exp(tX)m)$$

Consider the bundle map defined by

$$\alpha: \mathfrak{g} \times U_0 \to TO$$

$$(X, m) \mapsto X^*_m$$

Since the action is locally free every basis $(X_i)_i$ of $\mathfrak{g}$ induces at every point $m \in U_0$ a family of vectors $\{(X_i)_m^*\}_i$ that also defines a base for the tangent space to the orbit at $M$, i.e. to the fiber of $TO$ at $m$. In particular, $\alpha$ trivializes $TO$ and $L(TO)$ is trivial as well.

On the other hand, for $X \in \mathfrak{g}$, $m \in U_0$ and $g \in G$ we have:

$$gX^*_m = g\frac{d}{dt} \bigg|_{t=0} (\exp(tX)m)$$

$$= \frac{d}{dt} \bigg|_{t=0} (g \exp(tX)m)$$

$$= \frac{d}{dt} \bigg|_{t=0} (g \exp(tX)g^{-1}gm)$$

$$= \frac{d}{dt} \bigg|_{t=0} (\exp(t\text{Ad}_G(g)(X))gm)$$

$$= \text{Ad}_G(g)(X)^*_m$$
In particular, the trivialization $\alpha$ is $G$-equivariant for the action on $g \times U_0$ given by $g(X, m) = (\text{Ad}_G(g)(X), gm)$. Now observe that if $\alpha_m$ denotes the linear map at the fibers over $m$ induced by $\alpha$, then the trivialization of $L(TO)$ is given by:

$$\beta: \text{Gl}(g) \times U_0 \rightarrow L(TO)$$

$$(A, m) \mapsto \alpha_m \circ A$$

where we have taken $\frak{g}$ as the standard fiber of $TO$ when considering $L(TO)$ as a frame bundle for $TO$. We then have that for $A \in \text{Gl}(g)$, $m \in U_0$ and $g \in G$:

$$(g(\alpha_m \circ A))(v) = g(\alpha_m \circ A)(v)$$

$$= gA(v)^*_m$$

$$= (\text{Ad}_G(g)(A(v)))^*_m$$

$$= (\alpha_m \circ \text{Ad}_G(g) \circ A)(v)$$

for all $v \in \frak{g}$. It follows that $\beta$ is $G$-equivariant for the $G$-action on $\text{Gl}(g) \times U_0$ given by $g(A, m) = (\text{Ad}_G(g) \circ A, gm)$ for every $A \in \text{Gl}(g)$, $g \in G$ and $m \in U_0$. □

For every $x \in M$, denote with $ev_x$ the linear map given by the bundle map $ev: E^V \rightarrow TM$ at the fibers over $x$, and define the subspaces of $E^V_x$ (the fiber of $E^V$ at $x$) given by $T_x = ev^{-1}_x(T_xGx)$ and $K_x = ev^{-1}_x(0)$. The following result states that over a suitable open set the spaces $T_x$ and $K_x$ define analytic vector bundles.

**Proposition 4.7.** Let $G$ be a Lie group acting on a manifold $M$ satisfying the hypotheses of Gromov’s centralizer theorem. Then there is a conull dense open $G$-invariant subset $U$ of $M$ so that the following subsets of $E^V$ define analytic subbundles of $E^V$ over $U$:

$$T|_U = \bigcup_{x \in U} T_x$$

$$K|_U = \bigcup_{x \in U} K_x$$

**Proof.** Let $U$ be an open subset of $M$ whose inverse image under the natural projection $\tilde{M} \rightarrow M$ satisfies the conclusion of Lemma 4.5. Then we have for every $x \in U$ that $ev_x(E^V_x) \supset T_xO$, where the latter denotes the fiber of $TO$ at $x$. Since $ev$ is an analytic bundle map and since

$$\dim T_x = \dim V + \dim G - \text{rank}(ev_x)$$

$$\dim K_x = \dim V - \text{rank}(ev_x)$$

it is enough to observe that, by the proof Lemma 4.5, the map $x \mapsto \text{rank}(ev_x)$ is constant on $U$ to obtain the conclusion. □

Finally, we prove the main result of this article:

**Theorem 4.8.** Let $M$ be a compact analytic manifold acted upon on the left by a simply connected simple non-compact Lie group $G$ with finite center preserving a rigid unimodular analytic geometric structure of algebraic type (e.g. a connection and a volume form, both analytic). Then there is an open conull dense $G$-invariant open subset $U$ of $M$ whose lift to $\tilde{M}$ has locally closed $G$-orbits. In particular, the $G$-action is topologically engaging on $U$. 


Proof: By Corollary 4.3 and Proposition 4.7 there is an open conull dense $G$--

invariant subset $U$ of $M$ so that the action on $U$ is locally free and the sets $T|_U$

and $K|_U$ define analytic subbundles of $E^V$ over $U$. From the proof of the previous

results it follows easily that $U$ is $\Gamma$--invariant.

Let us denote with $N$, $m$ and $n$ the ranks of the bundles $E^V$, $T|_U$ and $K|_U$, respectively. Hence, it is clear that the subset of $L(E^V)$ given by $L(T,K)|_U = \{ u \in L(E^V)|_U \mid u(\mathbb{R}^m) = T_{p(u)}$, $u(\mathbb{R}^n) = K_{p(u)} \} \}$ (each $u \in L(E^V)$ is considered as a frame of $E^V$) is an analytic principal subbundle of $L(E^V)$ over $U$, where $L(E^V)|_U$ is the restriction of $L(E^V)$ to $U$ and $p: L(E^V) \to M$ is the canonical projection. Also observe that the bundles $T|_U$, $K|_U$ and $L(T,K)|_U$ are all $G$--invariant.

Let $\tilde{U} = \pi^{-1}(U)$ be the open subset of $\tilde{M}$ where $\pi$ is the projection of $\tilde{M}$ onto $M$. Given $m_0 \in \tilde{U}$ we will prove that the $G$--orbit of $m_0$ is closed in $\tilde{U}$.

For $A_0 \in \text{Gl}(V)$ consider the map $\lambda: \tilde{M} \to L(E^V) = \Gamma \backslash (\text{Gl}(V) \times \tilde{M})$ given by $\lambda(m) = [A_0,m]$. Observe that as a subset of $L(E^V) = \Gamma \backslash (\text{Gl}(V) \times \tilde{M})$ we can identify $L(T,K)|_U = \{ [A,m] \mid m \in \tilde{U}, A \in \text{Gl}(V), A(T_0) = T_m, A(K_0) = K_m \}$, where $T_m = \{ X \in V \mid X_m \in T \mathcal{O} \}$, $K_m = \{ X \in V \mid X_m = 0 \}$ and $T_0, K_0$ are the corresponding spaces at a fixed base point of $\tilde{U}$.

Since $G$ acts trivially on $V$ it is straightforward to check that $T_{gm} = gT_m = T_m$

and $K_{gm} = gK_m = K_m$ for every $m \in \tilde{U}$ and $g \in G$. Since the fibers of $L(E^V)$ are

acted upon by the structure group transitively, we can choose $A_0$ so that $\lambda(m_0) \in L(T,K)|_U$, and we then have in particular that

$$
A_0(T_0) = T_{m_0}
$$

$$
A_0(K_0) = K_{m_0}
$$

But since $\lambda$ is a $G$--equivariant map and $L(T,K)|_U$ is $G$--invariant we conclude that

$\lambda(Gm_0) \subset L(T,K)|_U$.

We claim that we further have $\lambda(\text{cl}_{\tilde{U}}(Gm_0)) \subset L(T,K)|_U$, where $\text{cl}_{\tilde{U}}(Gm_0)$ is the closing of $Gm_0$ in $\tilde{U}$. To see this we need to show that for every sequence $(g_n m_0)_n$ in $\tilde{U}$ that converges to $m_1 \in \tilde{U}$ we have $\lambda(m_1) \in L(T,K)|_U$. In other words, we need to show that $A_0(T_0) = T_{m_1}$ and $A_0(K_0) = K_{m_1}$, and equation (11) makes this equivalent to showing that $T_{m_0} = T_{m_1}$ and $K_{m_0} = K_{m_1}$. Choose $X \in T_{m_0}$ and observe that $X \in T_{m_0}$ for every $n$. In other words, $X_{g_n m_0} \in T \mathcal{O}$ and since $g_n m_0 \to m_1$ in $\tilde{U}$ and $T \mathcal{O}$ is the tangent bundle to a foliation in $U$ we conclude that $X_{m_1} \in T \mathcal{O}$, i.e. $X \in T_{m_1}$. Since both spaces have the same dimension it follows that $T_{m_0} = T_{m_1}$, and a similar argument proves the claim for $K$.

Observe that on $U$ the bundle map ev induces an isomorphism between the 

quotient bundle $T|_U/K|_U$ and $T \mathcal{O}$ ($T \mathcal{O}$ is defined on $U$ only) which is essentially a consequence of the definitions of $T|_U$ and $K|_U$. If we denote with $L(T \mathcal{O})$ the 

principal fiber bundle over $U$ associated to $T \mathcal{O}$, then it is easy to check that the map:

$$
\mu: L(T,K)|_U \to L(T \mathcal{O})
$$

$$
u \to \tilde{u}
$$

where $\tilde{u}$ denotes the isomorphism $\mathbb{R}^{m-n} \to T_{p(u)}p(u)G$ induced by $u$ and $ev_p(u)$ ($p(u) \in M$ is the base point of $u$), defines a homomorphism of principal bundles which is easily seen to be $G$--equivariant.
By Lemma 4.6, the bundle $L(TO)$ over $U$ is $G$–equivariantly analytically equivalent to $\text{Gl}(g) \times U$ where the $G$–action on the latter is given by $g(A, m) = (\text{Ad}_G(g) \circ A, gm)$.

Now let $m_1 \in \text{cl}\tilde{U}(m_0G)$, so there is a sequence $(g_n m_0)_n$ that converges to $m_1$. From the above it follows that $g_n \mu \circ \lambda(m_0) \to \mu \circ \lambda(m_1)$, i.e. we have $g_n(A_1, m_0) \to (A_2, m_1)$ in $L(TO)$ for some $A_1, A_2 \in \text{Gl}(g)$, so that given the above action on $L(TO) \cong \text{Gl}(g) \times U$ it follows that $(\text{Ad}_G(g_n))_n$ converges in $\text{Ad}_G(G)$ and since $G$ has finite center we can replace $(g_n)_n$ by a subsequence to assume that $(g_n)_n$ converges to some $g \in G$. From this it follows that $m_1 = gm_0$ and so the orbit $Gm_0$ is closed in $\tilde{U}$. □

A straightforward consequence is given by the following result.

**Corollary 4.9.** Let $G$ be a group acting on a manifold $M$ as in Theorem 4.8. If the $G$–action on $M$ is minimal, i.e. all orbits are dense, then the $G$–action on $M$ is topologically engaging.

5. **Further developments**

As it is observed in the previous sections, geometric engagement is a condition stronger to but closely related to topological engagement. A natural problem is to determine whether or not connection preserving actions as those studied here are geometrically engaging. In [1] it has been proved that essentially all known actions of that sort are geometrically engaging, but the problem still remains open.

Nevertheless, its the authors belief that connection preserving actions are geometrically engaging. Moreover, we expect that some (nontrivial) extensions of the arguments in this work might allow to prove this fact. Notice that topological engagement is a purely topological condition while geometric engagement requires the choice of a Riemannian metric and some distance estimates, so the proof of geometric engagement should be considerably more complicated. On the other hand, the applications that would arise from such fact would be stronger than some of those obtained from topological engagement, as it has been remarked above and in [1].

**References**

[1] A. Candel and R. Quiroga–Barranco, *Fundamental groups of compact manifolds and symmetric geometry of noncompact type*, Comment. Math. Helv. 74 1 (1999) 63–83.

[2] M. Gromov, *Rigid transformations groups*, Géométrie différentielle, Colloque Géométrie et Physique de 1986 en l'honneur de André Lichnerowicz (D. Bernard and Y. Choquet-Bruhat, eds.), Hermann, 1988, pp. 65–139.

[3] __________, *Foliated Plateau problem, part II: Harmonic maps of foliations*, Geom. Funct. Anal. 1 (1991), 253–320.

[4] S. Helgason, Differential geometry Lie groups and symmetric spaces, Pure and Appl. Math., vol. 80, Academic Press, New York-San Francisco-London, 1978.

[5] S. Kobayashi, *Transformation groups in differential geometry*, Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[6] R. Quiroga-Barranco, *The stretch of a foliation and geometric superrigidity*, Trans. Amer. Math. Soc. 349 (1997), 2391–2426.

[7] R. J. Spatzier and R. J. Zimmer, *Fundamental groups of negatively curved manifolds and actions of semisimple groups*, Topology 30 (1991), 591–601.

[8] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Boston, 1984.

[9] __________, *On the automorphism group of a compact Lorentz manifold and other geometric manifolds*, Invent. Math. 83 (1986), 411–424.
[10] __________, Representations of fundamental groups of manifolds with a semisimple transformation group, J. Amer. Math. Soc. 2 (1989), 201–213.

[11] __________, Automorphism groups and fundamental groups of geometric manifolds, Proc. Symp. Pure Math., 54 (1993), Part 3, 693–710.

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