KÄHLERNESS OF MODULI SPACES OF STABLE SHEAVES OVER NON-PROJECTIVE K3 SURFACES

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Abstract. We show that a moduli space of slope-stable sheaves over a K3 surface is an irreducible hyperkähler manifold if and only if its second Betti number is the sum of its Hodge numbers $h^{2,0}$, $h^{1,1}$ and $h^{0,2}$.

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1. Introduction

Irreducible hyperkähler manifolds are compact, connected Kähler manifolds which are simply connected, holomorphically symplectic and have $h^{2,0} = 1$. Very few examples of them are known up to today, and all the known deformation classes arise from moduli spaces of semistable sheaves on a projective K3 surface or on an abelian surface.

In [12] we showed that if $S$ is any K3 surface, the moduli space $M_{\mu}^{\omega}(S, \omega)$ of $\mu_{\omega}$-stable sheaves on $S$ of Mukai vector $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$ is a compact, connected complex manifold, it carries a holomorphic symplectic form and it is of K3$^{[n]}$-type (i.e. it is deformation equivalent to a Hilbert scheme of points on a projective K3 surface). This holds under some hypothesis on $\omega$ and $v$ (the Kähler class $\omega$ has to be $v$-generic, and $r$ and $\xi$ have to be prime to each other: we refer the reader to [12] for the definition of $v$-genericity).

The main open question about these moduli spaces is if they carry a Kähler metric: if it is so, it follows that they are all irreducible hyperkähler manifolds of K3$^{[n]}$-type. The answer to this question is affirmative at least in three cases: when $S$ is projective; when $M_{\mu}^{\omega}(S, \omega)$ is a surface; when $M_{\mu}^{\omega}(S, \omega)$ parametrizes only locally free sheaves. This lead us to the following

**Conjecture 1.1.** The moduli spaces $M_{\mu}^{\omega}(S, \omega)$ are Kähler manifolds.

Evidences are provided by the previous examples where the moduli spaces are indeed Kähler, and by the fact that their geometry is somehow similar to that of an irreducible hyperkähler manifold: in [12] we show that on their second integral cohomology there is a non-degenerate quadratic form defined as the Beauville form of irreducible hyperkähler manifolds.

But still, this analogy is not sufficient to guarantee that the moduli spaces are Kähler: it is known since [5], [6] and [7] that there are examples of compact, simply connected, holomorphically symplectic manifolds having $h^{2,0} = 1$ which are not Kähler, but their second integral cohomology carries a non-degenerate quadratic form, and the Local Torelli Theorem holds.

The aim of this paper is to show that the previous conjecture holds true under some additional hypothesis on the second Betti number of $M_{\mu}^{\omega}(S, \omega)$.

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1.1. Main definitions and notations. In this section we collect all the definitions and notations we will use in the following.

Definition 1.2. A **holomorphically symplectic manifold** is a complex manifold which carries an everywhere non-degenerate holomorphic closed 2-form (called **holomorphic symplectic form**).

We notice that a compact holomorphically symplectic manifold is always of even complex dimension, and a holomorphic form $\sigma$ defines an isomorphism $\sigma : T_X \rightarrow \Omega^2_X$ of vector bundles, where $T_X$ is the tangent bundle of $X$, and $\Omega^2_X$ is the cotangent bundle of $X$ (i.e. the dual bundle of $T_X$).

Let $X$ be a compact, connected complex manifold of complex dimension $d$, and $k \in \{0, ..., 2d\}$ and $p, q \in \mathbb{N}$.

Definition 1.3. The $k$–**Betti number** of $X$ is

$$b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C}),$$

and the $(p, q)$–**Hodge number** of $X$ is

$$h^{p,q}(X) = \dim_{\mathbb{C}} H^q(X, \Omega^p_X).$$

The general relation between the Betti and the Hodge numbers of $X$ is that

$$b_k(X) \leq \sum_{p+q=k} h^{p,q}(X)$$

for every $k \in \{0, ..., 2d\}$, and the equality holds for every $k$ if and only if the Fr"ohlicher spectral sequence of $X$ degenerates at the $E_1$ level.

Definition 1.4. A complex manifold $X$ is a $b_2$–**manifold** if

$$b_2(X) = h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X).$$

Definition 1.5. A compact, connected complex manifold $X$ is in the **Fujiki class** $\mathcal{C}$ if $X$ is bimeromorphic to a compact K"ahler manifold.

Among all manifolds in the Fujiki class $\mathcal{C}$ we clearly have compact K"ahler manifolds. Moreover, a compact, connected complex manifold in the Fujiki class $\mathcal{C}$ verifies the $\partial \bar{\partial}$–lemma, and hence the Fr"ohlicher spectral sequence of $X$ degenerates at the $E_1$ level. In particular, it is a $b_2$–manifold.

If $f : \mathcal{X} \longrightarrow B$ is a holomorphic fibration, for every $b \in B$ we let $X_b := f^{-1}(b)$. Let $X$ be a compact, connected complex manifold and $B$ a connected complex manifold.

Definition 1.6. A **deformation of $X$ along** $B$ is a smooth and proper family $f : \mathcal{X} \longrightarrow B$ such that there is $0 \in B$ such that $X_0$ is biholomorphic to $X$. 
Let now \( \mathcal{P} \) be a property of complex manifolds.

**Definition 1.7.** We say that the property \( \mathcal{P} \) is **open in the analytic topology** (resp. **in the Zariski topology**) if for every deformation \( \mathcal{X} \) along a connected complex manifold \( B \), the set of those \( b \in B \) such that \( X_b \) verifies \( \mathcal{P} \) is an analytic (resp. Zariski) open subset of \( B \);

Kählerness is open in the analytic topology, but in general it is neither open in the Zariski topology, nor closed. Being a \( b_2 \)-manifold is open in the Zariski topology. Being in the Fujiki class \( \mathcal{C} \) is not open in general.

The last definitions we need are the following:

**Definition 1.8.** Let \( X \) be an compact, connected complex manifold.

1. The manifold \( X \) is **irreducible hyperkähler** if it is a Kähler manifold which is simply connected, holomorphically simplectic and \( h^{2,0}(X) = 1 \).
2. The manifold \( X \) is **deformation equivalent to an irreducible hyperkähler manifold** if there is a connected complex manifold \( B \) and a deformation \( \mathcal{X} \longrightarrow B \) of \( X \) along \( B \) for which there is \( b \in B \) such that \( X_b \) is an irreducible hyperkähler manifold.
3. The manifold \( X \) is **limit of irreducible hyperkähler manifolds** if there is a smooth and proper family \( \mathcal{X} \longrightarrow B \) along a smooth connected base \( B \) such that and a sequence \( \{ b_n \} \) of points of \( B \) converging to \( 0 \), such that \( X_{b_n} \) is an irreducible hyperkähler manifold.

1.2. **Main results and structure of the paper.** The main result of the paper is the following

**Theorem 1.9.** Let \( S \) be a K3 surface, \( \omega \) a Kähler class on \( S \). We let \( v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z}) \) be such that \( r > 0 \) and \( \xi \in NS(S) \). Suppose that \( r \) and \( \xi \) are prime to each other, and that \( \omega \) is \( v \)-generic. Then the moduli space \( M = M^v(S, \omega) \) of \( \mu_\omega \)-stable sheaves on \( S \) with Mukai vector \( v \) is Kähler if and only if it is a \( b_2 \)-manifold.

Clearly, this proves Conjecture 1.1 under the additional hypothesis of the moduli spaces being \( b_2 \)-manifolds. This has an immediate corollary:

**Corollary 1.10.** Let \( \mathcal{M} \longrightarrow B \) be any smooth and proper family of moduli spaces of sheaves verifying the conditions of Theorem 1.9. The set of \( b \in B \) such that \( M_b \) is Kähler is a Zariski open subset of \( B \).

In view of of Theorems 1.1 and 1.2 of [12], another immediate corollary is the following:
Corollary 1.11. Let $S$ be a $K3$ surface, $\omega$ a Kähler class on $S$ and $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$ be such that $r > 0$ and $\xi \in NS(S)$. Suppose that $r$ and $\xi$ are prime to each other, that $\omega$ is $v$—generic, and that $M^\mu_v(S, \omega)$ is a $b_2$—manifold.

1. The moduli space $M^\mu_v(S, \omega)$ is an irreducible hyperkähler manifold of $K3^{[n]}$—type, which is projective if and only if $S$ is projective.

2. If $v^2 \geq 2$, there is a Hodge isometry $\lambda_v : v^\perp \to H^2(M^\mu_v, \mathbb{Z})$.

The case $v^2 = 0$ was already treated in [12]: in this case there is a Hodge isometry

$$\lambda_v : v^\perp / \mathbb{Z}v \to H^2(M^\mu_v, \mathbb{Z}),$$

and there is no need to suppose that $M^\mu_v(S, \omega)$ is a $b_2$—manifold here. For as a consequence Theorem 1.1 of [12] we already know that $M^\mu_v(S, \omega)$ is a $K3$ surface.

The proof of Theorem 1.9 is an application of general results about compact, connected complex $b_2$—manifolds which are holomorphically symplectic and limit of irreducible hyperkähler manifolds. The starting point is the following:

Theorem 1.12. Let $X$ be a compact, connected holomorphically symplectic $b_2$—manifold which is deformation equivalent to an irreducible hyperkähler manifold. Then on $H^2(X, \mathbb{Z})$ there is a non-degenerate quadratic form $q_X$ of signature $(3, b_2(X) - 3)$, and the Local Torelli Theorem holds.

This result is due to Guan [7], and it is a generalization of the well-known analogue for irreducible hyperkähler manifolds proved by Beauville in [1]. By Local Torelli Theorem we mean that the period map is locally a biholomorphism (as in the case of irreducible hyperkähler manifolds). We will recall the definition of the Beauville form and the Local Torelli Theorem in Section 2.

Remark 1.13. In [12] we proved (see Theorem 1.1 there) that if $M$ is a moduli spaces of slope-stable sheaves over a non-projective $K3$ surface (verifying all the hypothesis of Theorem 1.9), then on $H^2(M, \mathbb{Z})$ there is a non-degenerate quadratic form of signature $(3, b_2(M) - 3)$. This is proved without any assumption on $b_2(M)$. Anyway, in [12] we have not proved the Local Torelli Theorem, and here we are able to prove it only by assuming $M$ to be a $b_2$—manifold.

In Section 3 we consider compact, connected holomorphically symplectic $b_2$—manifolds which are not only deformation equivalent to an
irreducible symplectic manifold, but which are moreover limit of irreducible hyperkähler manifolds. The main result of Section 3 is the following:

**Theorem 1.14.** Let $X$ be a compact, connected holomorphically symplectic $b_2$–manifold which is limit of irreducible hyperkähler manifolds. Then $X$ is bimeromorphic to an irreducible hyperkähler manifold (hence, it is in the Fujiki class $C$).

As we will see, a moduli space $M$ verifying the hypothesis of Theorem 1.9 is a compact, connected holomorphically symplectic $b_2$–manifold which is limit of irreducible hyperkähler manifolds. As a consequence, asking for these moduli spaces to be $b_2$–manifolds is enough to conclude that they are all bimeromorphic to a compact irreducible hyperkähler manifold.

The proof of Theorem 1.14 is based on a well-known strategy already used by Siu in [14] to show that all K3 surfaces are Kähler, and by Huybrechts in [8] to show that non-separated marked irreducible hyperkähler manifolds are in fact bimeromorphic. More precisely, if $\Lambda$ is a lattice, we say that a compact complex manifold $X$ carries a $\Lambda$–marking if on $H^2(X,\mathbb{Z})$ there is non-degenerate quadratic form, and there is an isometry $\phi : H^2(X,\mathbb{Z}) \to \Lambda$. The pair $(X,\phi)$ is called a $\Lambda$–marked manifold.

The set of (equivalence classes of) $\Lambda$–marked manifolds is denoted $M_\Lambda$: as a consequence of Theorem 1.12, it contains the subset $M_\Lambda^{\text{hk}}$ of $\Lambda$–marked manifolds $(X,\phi)$ where $X$ is a compact holomorphically symplectic $b_2$–manifold which is deformation equivalent to an irreducible hyperkähler manifold (and whose Beauville lattice is isometric to $\Lambda$).

By the Local Torelli Theorem we can give $M_\Lambda^{\text{hk}}$ the structure of complex space, in which we have a (non-empty) open subset $M_\Lambda^{\text{hk}}$ of irreducible hyperkähler manifolds. We let $\overline{M}_\Lambda^{\text{hk}}$ be its closure in $M_\Lambda^{\text{hk}}$.

Theorem 1.14 can be restated by saying that if $(X,\phi) \in \overline{M}_\Lambda^{\text{hk}}$, then $X$ is bimeromorphic to an irreducible hyperkähler manifold. This is the statement we prove: the idea of the proof is that if $(X,\phi) \in \overline{M}_\Lambda^{\text{hk}}$, then $(X,\phi)$ is non-separated from a point $(Y,\psi) \in M_\Lambda^{\text{hk}}$. A standard argument shows that $X$ and $Y$ have to be bimeromorphic.

Theorem 1.14 is just an intermediate result on the way to the Kähler-ness of the moduli spaces, and it is used in Section 4 to prove that on a compact, connected holomorphically symplectic $b_2$–manifold which is limit of irreducible hyperkähler manifolds, we can define an analogue of the positive cone of an irreducible hyperkähler manifold.
Recall that if \( X \) is irreducible hyperkähler and \( C_X \) is the cone of real \((1,1)\)-classes over which the Beauville form is strictly positive, the positive cone \( C^+_X \) is the connected component of \( C_X \) which contains the Kähler cone of \( X \). A result of Huybrechts shows that \( C^+_X \) is contained in (the interior of) the pseudo-effective cone of \( X \).

Theorem 1.14 is used to prove that on a compact, connected holomorphically symplectic \( b_2^- \)-manifold \( X \) which is limit of irreducible hyperkähler manifolds the intersection of the pseudo-effective cone of \( X \) and of \( C_X \) (which can be defined as for irreducible hyperkähler manifolds by Theorem 1.12) consists of exactly one of the two connected components of \( C_X \): such a component is the positive cone of \( X \), still denoted \( C^+_X \). We then prove:

**Theorem 1.15.** Let \( X \) be a compact, connected holomorphically symplectic \( b_2^- \)-manifold which is limit of irreducible hyperkähler manifolds. If there is \( \alpha \in C^+_X \) such that

1. \( \alpha \cdot C > 0 \) for every rational curve \( C \) on \( X \), and
2. for every \( \beta \in H^1(X) \cap H^2(X, \mathbb{Z}) \) we have \( q_X(\alpha, \beta) \neq 0 \),

then \( X \) is irreducible hyperkähler, and \( \alpha \) is a Kähler class on \( X \).

The proof of this result is based on Theorem 1.14, which gives a bimeromorphism \( f : Y \to X \) between \( X \) and an irreducible hyperkähler manifold \( Y \). Using twistor lines for real \((1,1)\)-classes on \( X \) (which can be defined similarly to the hyperkähler case thanks to Theorem 1.12) and a strategy used by Huybrechts for irreducible hyperkähler manifolds, we show that the conditions on \( \alpha \) imply that \( f^* \alpha \) is a Kähler class on \( Y \). An easy argument then shows that \( f \) is a biholomorphism, and that \( \alpha \) is a Kähler class on \( X \).

The last part of the paper is devoted to show that on \( M \) a class \( \alpha \) as in the statement of Theorem 1.15 exists. This is obtained by using the (Hodge) isometry

\[
\lambda_v : v^\perp \otimes \mathbb{R} \to H^2(M, \mathbb{R})
\]

(whose existence was proved in [12]) to produce classes in \( C_M \). By deforming to a moduli space of slope-stable sheaves on a projective K3 surface, and by using a classical construction of ample line bundles on \( M \) in this case (starting from an ample line bundle on \( S \)), we are able to conclude that a class as in Theorem 1.15 exists, concluding the proof of Theorem 1.9.

### 1.3. The Beauville form and the Local Torelli Theorem.

The starting point of the proof of Theorem 1.9 is Theorem 1.12 which is
due to Guan. We will not prove it here (the proof can be found in [7]), but we recall the definition of $q_X$ and the Local Torelli Theorem.

1.3.1. The Beauville form on $H^2(X, \mathbb{C})$. Let $X$ be a compact, connected holomorphically symplectic manifold of complex dimension $2n$.

The Beauville form of $X$ is a quadratic form on $H^2(X, \mathbb{C})$ defined as follows. First, choose a holomorphic symplectic form $\sigma$ on $X$, and assume for simplicity that $\int_X \sigma^n \wedge \overline{\sigma}^n = 1$. For every $\alpha \in H^2(X, \mathbb{C})$, we let

$$q_\sigma(\alpha) := \frac{n}{2} \int_X \alpha^2 \wedge \sigma^{n-1} \wedge \overline{\sigma}^{n-1} + (1-n) \int_X \alpha \wedge \sigma^n \wedge \overline{\sigma}^{n-1} \int_X \alpha \wedge \sigma^{n-1} \wedge \overline{\sigma}^n.$$

Note that $q_\sigma(\sigma + \overline{\sigma}) = (\int_X \sigma^n \wedge \overline{\sigma}^n)^2 = 1$ so $q_\sigma$ is non-trivial. Moreover, the quadratic form $q_\sigma$ depends a priori on the choice of $\sigma$.

1.3.2. The period map. Let $X$ be a compact, connected holomorphically symplectic $b_2-$manifold of complex dimension $2n$, and suppose that $h^{2,0}(X) = 1$. We let $f : \mathcal{X} \longrightarrow B$ be its Kuranishi family, and $0 \in B$ a point such that the fiber $X_0$ is isomorphic to $X$.

By Theorem 1 and the following Remark 1 of [7], it follows that $B$ is smooth, and that up to shrinking it we can even suppose that all the fibers of the Kuranishi family are holomorphically symplectic.

Up to shrinking $B$, for every $b \in B$ the fiber $X_b$ of $f$ is a compact, connected holomorphically symplectic $b_2-$manifold (since being a $b_2-$manifold is an open property). Moreover, again up to shrinking $B$, by the Ehresmann Fibration Theorem we can suppose that $\mathcal{X}$ is diffeomorphic to $X \times B$. In particular, this induces a diffeomorphism $u_b : X \longrightarrow X_b$ for every $b \in B$, and hence an isomorphism of complex vector spaces

$$u_b^* : H^2(X_b, \mathbb{C}) \longrightarrow H^2(X, \mathbb{C}).$$

We now let $\mathbb{P} := \mathbb{P}(H^2(X, \mathbb{C}))$, and

$$p : B \longrightarrow \mathbb{P}, \quad p(b) := [u_b^*(\sigma_b)],$$

where $\sigma_b$ is the holomorphic symplectic form on $X_b$ such that $\int_{X_b} \sigma_b^n \wedge \overline{\sigma}_b^n = 1$ (we notice that such a $\sigma_b$ is unique as $h^{2,0}(X) = 1$, and hence $h^{2,0}(X_b) = 1$). The map $p$ is holomorphic, and will be called period map of $X$.

We let $Q_\sigma$ be the quadric defined by the quadratic form $q_\sigma$ in $\mathbb{P}$, i.e.

$$Q_\sigma = \{ \alpha \in \mathbb{P} | q_\sigma(\alpha) = 0 \},$$

and $\Omega_\sigma$ be the open subset of $Q_\sigma$ defined as

$$\Omega_\sigma := \{ \alpha \in Q_\sigma | q_\sigma(\alpha + \overline{\alpha}) > 0 \}.$$
As showed in [7] we have the following, known as Local Torelli Theorem:

**Proposition 1.16.** Let $X$ be a compact, connected holomorphically symplectic $b_2 -$manifold, such that $h^{2,0}(X) = 1$.

1. The quadratic form $q_\sigma$ (and hence $Q_\sigma$ and $\Omega_\sigma$) is independent of $\sigma$, and will hence be denoted $q_X$ (and similarly $Q_X$ and $\Omega_X$).

2. Up to a positive rational multiple, the quadratic form $q_X$ is a non-degenerate quadratic form on $H^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$.

3. If $B$ is the base of the Kuranishi family of $X$, we have that $p(B) \subseteq \Omega_X$, and that $p : B \rightarrow \Omega_X$ is a local biholomorphism.

Using the non-generate quadratic form $q_X$ (of signature $(3, b_2(X) - 3)$) we let

$$C'_X := \{ \alpha \in H^2(X, \mathbb{R}) \mid q_X(\alpha) > 0 \},$$

which is an open cone in $H^2(X, \mathbb{R})$ having two connected components. Moreover, we let

$$\tilde{H}^{1,1}_\mathbb{R}(X) := \text{Im}(\{ \alpha \in H^{1,1}(X) \mid d\alpha = 0 \} \rightarrow H^2(X, \mathbb{C}) \cap H^2(X, \mathbb{R}),$$

and notice that this consists exactly of de Rham cohomology classes of real $d -$closed $(1, 1)$ -forms on $X$. We let

$$C_X := C'_X \cap \tilde{H}^{1,1}_\mathbb{R}(X),$$

which is then an open cone in $\tilde{H}^{1,1}_\mathbb{R}(X)$ having two connected components.

2. Limits of irreducible hyperkähler manifolds

This section is devoted to prove that every compact, connected holomorphically symplectic $b_2 -$manifold $X$ which is limit of irreducible hyperkähler manifold, is bimeromorphic to an irreducible hyperkähler manifold: in other words, we prove Theorem [1.14]

The proof is divided in several sections. First we construct a moduli space $\mathcal{M}_Z$ of marked manifolds, and thanks to the Local Torelli Theorem we may give it the structure of a (non-separated) complex space. It will carry a period map to some period domain, which is locally a biholomorphism.

Then we show that each point in the closure of the open subset of $\mathcal{M}_Z$ given by irreducible hyperkähler manifolds is non-separated from an irreducible hyperkähler manifold.

Adapting an argument of Siu (for K3 surfaces) and Huybrechts (for higher dimensional irreducible hyperkähler manifolds), we conclude the proof of Theorem [1.14]
2.1. The moduli space of \( \Lambda \)-marked manifolds. In this section, we let \( Z \) be an irreducible hyperk"ahler manifold, and we write \((\Lambda, q) := (H^2(Z, \mathbb{Z}), q_Z)\) for the Beauville lattice of \( Z \). We let \( \mathbb{P}_\Lambda := \mathbb{P}(\Lambda \otimes \mathbb{C}) \), and inside of it we let

\[
Q_\Lambda := \{ \alpha \in \mathbb{P}_\Lambda \mid q(\alpha) = 0 \},
\]

which is the quadric defined by \( q \), and

\[
\Omega_\Lambda := \{ \alpha \in Q_\Lambda \mid q(\alpha + \overline{\alpha}) > 0 \}.
\]

The following is immediate, as \( X \) is a \( b_2 \)-manifold and the Hodge numbers are upper-semicontinuous.

**Proposition 2.1.** Let \( X \) be a compact, connected holomorphically symplectic \( b_2 \)-manifold which is deformation equivalent to an irreducible hyperk"ahler manifold \( Z \). Then for every \( p, q \geq 0 \) such that \( p + q = 2 \) we have \( h^{p,q}(X) = h^{p,q}(Z) \). In particular \( h^{2,0}(X) = 1 \).

If \( X \) is a compact, connected holomorphically symplectic \( b_2 \)-manifold which is deformation equivalent to \( Z \), by Proposition 2.1 and Theorem 1.12 we know that \( H^2(X, \mathbb{Z}) \) carries a non-degenerate quadratic form \( q_X \), and that there is an isometry \( \phi : H^2(X, \mathbb{Z}) \rightarrow \Lambda \). The isometry \( \phi \) is called \( \Lambda \)-marking on \( X \), and the pair \((X, \phi)\) is a \( \Lambda \)-marked manifold. The set of \( \Lambda \)-marked manifolds will be denoted \( \mathcal{M}'_Z \).

Moreover, we let \( \mathcal{M}_Z := \mathcal{M}'_Z / \sim \), where \((X, \phi) \sim (X', \phi')\) if and only if there is a biholomorphism \( f : X \rightarrow X' \) such that \( f^* \circ \phi' = \phi \). The set \( \mathcal{M}_Z \) will be referred to as the moduli space of \( \Lambda \)-marked manifolds. We let \( \mathcal{M}^{hk}_Z \) be the subset of \( \mathcal{M}_Z \) of pairs \((X, \phi)\) where \( X \) is an irreducible hyperk"ahler manifold: it will be called moduli space of \( \Lambda \)-marked hyperk"ahler manifolds.

We first show that \( \mathcal{M}_Z \) has the structure of complex space (hence justifying the name space we use for it): the following is a generalization of Proposition 4.3 of [10], and requires the same proof.

**Proposition 2.2.** Let \( Z \) be an irreducible hyperk"ahler manifold and \((\Lambda, q)\) its Beauville lattice.

1. For any \((X, \phi) \in \mathcal{M}_Z\) there is an inclusion \( i_X : B \rightarrow \mathcal{M}_Z \), where \( B \) is the base of the Kuranishi family of \( X \).
2. The set \( \mathcal{M}_Z \) has the structure of smooth complex space of dimension \( b_2(Z) - 2 \).
3. The subset \( \mathcal{M}^{hk}_Z \) is an open subset (in the analytic topology) of \( \mathcal{M}_Z \).
\textbf{Proof.} Let $X$ be a compact, connected holomorphically symplectic $b_2$–manifold which is deformation equivalent to $Z$, and $f : \mathcal{X} \rightarrow B$ its Kuranishi family.

Up to shrinking $B$ we can suppose that it is a complex disk of dimension $b_2(X) - 2 = b_2(Z) - 2$, and as we have seen before for every $b \in B$ we can suppose that $X_b$ is a compact, connected holomorphically symplectic $b_2$–manifold (which is clearly deformation equivalent to $Z$).

Moreover, we can suppose that $\mathcal{X}$ is diffeomorphic (over $B$) to the trivial family $X \times B$, and that we have a diffeomorphism $u_b : X \rightarrow X_b$ inducing an isometry $u_b^* : H^2(X_b, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$. We let $\phi_b := \phi \circ u_b^*$, which is then a $\Lambda$–marking on $X_b$, for every $b \in B$.

It follows that for every $b \in B$ we have $(X_b, \phi_b) \in \mathcal{M}_Z$, so that we have a map

$$i_X : B \rightarrow \mathcal{M}_Z, \quad i_X(b) := (X_b, \phi_b).$$

We show that $i_X$ is an inclusion. Let $b, b' \in B$ and suppose that $i_X(b) = i_X(b')$. This means that $(X_b, \phi_b) \sim (X_{b'}, \phi_{b'})$, i. e. there is a biholomorphism $f : X_b \rightarrow X_{b'}$ such that

$$f^* = \phi_{b'}^{-1} \circ \phi_b.$$

By definition of $\phi_b$ and $\phi_{b'}$, this means that

$$f^* = (\phi \circ u_b^*)^{-1} \circ (\phi \circ u_{b'}^*) = (u_b^*)^{-1} \circ u_{b'}^*.$$

Now, let $\sigma_b$ and $\sigma_{b'}$ be symplectic forms on $X_b$ and $X_{b'}$ respectively. As $f$ is a biholomorphism, the form $f^* \sigma_{b'}$ is holomorphic symplectic on $X_b$, and hence

$$[u_b^* \sigma_b] = [u_{b'}^* \sigma_{b'}].$$

But as $f^* = (u_b^*)^{-1} \circ u_{b'}^*$, this implies that $[u_b^* \sigma_b] = [u_{b'}^* \sigma_{b'}]$. By definition of the period map of $X$, this means that $p(b) = p(b')$.

But now recall that by point (3) of Proposition 1.16, the period map $p : B \rightarrow \Omega$ is a local biholomorphism: up to shrinking $B$, for $b \neq b' \in B$ we have $p(b) \neq p(b')$. It follows that up to shrinking $B$ the condition $i_X(b) = i_X(b')$ implies $b = b'$, and $i_X$ is an inclusion of $B$ in $\mathcal{M}_Z$. This proves point 1 of the statement.

To give $\mathcal{M}_Z$ the structure of a complex space, we just need to show that each point of $\mathcal{M}_Z$ has a neighborhood having the structure of a complex manifold, and that whenever two neighborhoods of this type intersect, the corresponding complex structures glue.

If $(X, \phi) \in \mathcal{M}_Z$, the previous part of the proof suggests to view $i_X(B)$ as a neighborhood $(X, \phi)$ in $\mathcal{M}_Z$. Now, let $(X, \phi), (X', \phi') \in \mathcal{M}_Z$, and we let $B$ and $B'$ be the bases of the Kuranishi families of $X$ and of $X'$, respectively. If $i_X(B) \cap i_X(B') \neq \emptyset$, then $B \cap B'$ is an open subset of $B$ and $B'$, over which the Kuranishi families coincide. This allows
us to glue the Kuranishi families along \( B \cap B' \), and hence the complex structures of \( i_X(B) \) and \( i_X(B') \) can be glued in \( \mathcal{M}_Z \). This shows that \( \mathcal{M}_Z \) has the structure of a complex space.

We notice that as each base \( B \) of a Kuranishi family of a compact, connected holomorphically symplectic \( b_2 \)-manifold is smooth (see section 2.2) of dimension \( b_2(Z) - 2 \), it follows that \( \mathcal{M}_Z \) is a smooth complex space, and its dimension is \( b_2(Z) - 2 \). This proves point 2 of the statement.

The fact that \( \mathcal{M}_{hk}^\Lambda \) is open in the analytic topology is a consequence of the fact that Kähleriness is an open property in the analytic topology.

The complex space \( \mathcal{M}_Z \) has two connected components, and one can pass from one to the other by sending \((X, \phi)\) to \((X, -\phi)\).

We now define the period map in this generality: we let \( \pi : \mathcal{M}_Z \rightarrow \mathbb{P}_\Lambda \), \( \pi(X, \phi) := [\phi(\sigma)] \), where \( \sigma \) is a holomorphic symplectic form on \( X \). Notice that the \( \Lambda \)-marking \( \phi \) induces an isomorphism \( \overline{\phi} : \mathbb{P} \rightarrow \mathbb{P}_\Lambda \), and as it is an isometry it induces an isomorphism \( \overline{\phi} : \Omega_X \rightarrow \Omega_\Lambda \).

If \( B \) is the base of the Kuranishi family of \( X \), we have \( \pi|_{i_X(B)} = \overline{\phi} \circ p \): if \( b \in B \) and \( \sigma_b \) is a symplectic form on \( X_b \), we have

\[
\overline{\phi}(p(b)) = \overline{\phi}(u^*_b \sigma_b) = [\phi(u^*_b \sigma_b)] = [\phi_b(\sigma_b)] = \pi(b).
\]

The first two points of the following Proposition are just a translation in this language of Theorem 1.12. For the last point: the surjectivity is Theorem 8.1 of [8]; the general injectivity is the Global Torelli Theorem of Verbitsky.

**Proposition 2.3.** We have the following properties:

1. the image of \( \pi \) is contained in \( \Omega_\Lambda \);
2. the map \( \pi \) is a local biholomorphism;
3. if \( \mathcal{M}_{Z,0}^{hk} \) is a connected component of \( \mathcal{M}_Z^{hk} \), the map \( \pi|_{\mathcal{M}_{Z,0}^{hk}} \) is surjective and generically injective.

Now, we let \( \mathcal{M}_Z^{hk} \) be the closure of \( \mathcal{M}_Z^{hk} \) in \( \mathcal{M}_Z \). Using this formalism, we can state Theorem 1.14 in an equivalent way:

**Proposition 2.4.** If \((X, \phi) \in \mathcal{M}_Z^{hk} \), then \( X \) is bimeromorphic to an irreducible hyperkähler manifold (hence, it is in the Fujiki class \( C \)).
This is the statement we will prove in the next sections.

2.2. Non-separatedness in \( \mathcal{M}_Z \). The first result we show is the following:

**Proposition 2.5.** Let \( (X, \phi) \in \mathcal{M}^{hk,0}_Z \). Then there is \( (Y, \psi) \in \mathcal{M}^{hk,0}_Z \) such that \( (X, \phi) \) and \( (Y, \psi) \) are non-separated in \( \mathcal{M}_Z \).

**Proof.** The statement is trivial if \( (X, \phi) \in \mathcal{M}^{hk,0}_Z \). We then suppose that \( (X, \phi) \in \mathcal{M}^{hk,0}_Z \setminus \mathcal{M}^{hk,0}_Z \).

We let \( p := \pi(X, \phi) \in \Omega_\Lambda \) be the period of \( (X, \phi) \). As \( \pi|_{\mathcal{M}^{hk,0}_Z} \) is surjective, there is \( (Y, \psi) \in \mathcal{M}^{hk,0}_Z \) such that \( \pi(Y, \psi) = p \). We show that \( (X, \phi) \) and \( (Y, \psi) \) are non-separated in \( \mathcal{M}_Z \).

To do so, let \( U_X \) and \( U_Y \) be two open neighborhoods of \( (X, \phi) \) and \( (Y, \psi) \) respectively in \( \mathcal{M}_Z \). Up to shrinking \( U_X \) and \( U_Y \), we can suppose that \( \pi(U_X) = \pi(U_Y) =: V \).

Moreover, by point (3) of Proposition 1.16, up to shrinking \( U_X \) and \( U_Y \) we can suppose that \( \pi|_{U_Y} : U_Y \to V \) and \( \pi|_{U_X} : U_X \to V \) are biholomorphisms. Finally, as Kählerness is an open property in the analytic topology, up to shrinking \( U_X \) and \( U_Y \) we can suppose that \( U_Y \subseteq \mathcal{M}^{hk,0}_Z \).

Now, as \( (X, \phi) \in \mathcal{M}^{hk,0}_Z \), there is a hyperkähler manifold \( X' \) and a marking \( \phi' \) on \( X' \) such that \( (X', \phi') \in U_X \cap \mathcal{M}^{hk,0}_Z \). We can choose \( (X', \phi') \) to be generic. Let \( p' := \pi(X', \phi') \in V \); as \( \pi|_{U_Y} : U_Y \to V \) is surjective, there is \( (Y', \psi') \in U_Y \) such that \( \pi(Y', \psi') = p' \), and as \( U_Y \subseteq \mathcal{M}^{hk,0}_Z \), we have that \( Y' \) is an irreducible hyperkähler manifold.

Hence \( (X', \phi') \) and \( (Y', \psi') \) are two generic points in \( \mathcal{M}^{hk,0}_Z \); by point (3) of Proposition 2.3, we then have \( (X', \phi') = (Y', \psi') \) in \( \mathcal{M}_Z \), so that \( U_X \cap U_Y \neq \emptyset \), and we are done. \( \square \)

This result will be the starting point of the proof of Proposition 2.4.

2.3. The proof of Theorem 1.14. We now prove a key result in the proof of Proposition 2.4.

**Lemma 2.6.** Let \( B \) be a connected complex manifold and \( \mathcal{X} \to B \) and \( \mathcal{Y} \to B \) be two smooth, proper families verifying the following properties:

1. for every \( b \in B \) the fiber \( Y_b \) is an irreducible hyperkähler manifold with a \( \Lambda \)-marking \( \psi_b \);
2. for every \( b \in B \) the fiber \( X_b \) is a compact, connected holomorphically symplectic \( b_2 \)-manifold deformation equivalent to an irreducible hyperkähler manifolds, which has a \( \Lambda \)-marking \( \phi_b \);
(3) there is a non-empty open subset $V$ of $B$ such that for each $b \in V$ there is an isomorphism $f_b : Y_b \to X_b$ such that $f_b^* = \psi_b^{-1} \circ \phi_b$;

(4) for generic $b \in V$ we have $H^{1,1}(X_b) \cap H^2(X_b, \mathbb{Z}) = 0$.

Then $V$ is dense in $B$.

**Proof.** We choose a point $0 \in B$ and let $X := X_0$ and $Y := Y_0$. We show that $\partial V := \overline{V} \setminus V$ is contained in a countable union of analytic subvarieties of $B$. It has then real codimension at least 2 in $B$, hence it cannot separate the disjoint open subsets $V$ and $B \setminus \overline{V}$. As $V \neq \emptyset$, it follows that $B = \overline{V}$.

In order to show that $\partial V$ is contained in a countable union of analytic subvarieties of $B$, we show that if $s \in \partial V$, then $Y_s$ has either effective divisors or curves. This implies that

$$s \in \bigcup_{\alpha} S_{\alpha},$$

where $\alpha \in H^2(Y, \mathbb{Z})$ and $S_{\alpha}$ is the analytic subvariety of $B$ given by those $b \in B$ such that $\alpha \in NS(Y_b)$. We proceed by contradiction: we let $s \in \partial V$, and we suppose that $Y_s$ has no effective divisors and no curves.

As $s \in \partial V$, it follows that $(X_s, \phi_s)$ and $(Y_s, \psi_s)$ are non-separated points in $\mathcal{M}_Z$, were $Z$ is an irreducible hyperkähler manifold among all the $Y_b$. We first show that $X_s$ and $Y_s$ are bimeromorphic. To show this, let $\beta_s$ be a Kähler form on $Y_s$ and $\alpha_s$ a closed $(1,1)$−form on $X_s$ whose cohomology class is in $C_{X_s}$.

Consider a continuous family $\{\beta_t\}_{t \in B}$, where $\beta_t$ is a closed $(1,1)$−form on $Y_t$. As Kählerness is an open property in the analytic topology, there is an analytic open neighborhood $U$ of $s$ in $B$ such that for every $t \in U$ the form $\beta_t$ is Kähler on $Y_t$.

Moreover, consider a continuous family $\{\alpha_t\}_{t \in B}$ where $\alpha_t$ is a closed $(1,1)$−form on $X_t$. Up to shrinking $U$, and as the positivity of $q_X$ is an open property, we can suppose that for every $t \in U$ the cohomology class of $\alpha_t$ is in $C_{X_t}$.

Notice that as $s \in \partial V$, the intersection of $U$ with $V$ is not empty, and the generic point $t \in U \cap V$ is such that $f_t : Y_t \to X_t$ is a biholomorphism such that $f_t^* = \psi_t^{-1} \circ \phi_t$. By hypothesis, we have that $X_t$ is an irreducible hyperkähler manifold with $NS(X_t) = 0$. By Corollary 5.7 of [8], this implies that the Kähler cone of $X_t$ is one of the two connected components of $C_{X_t}$.

As the cohomology class $[\alpha_t]$ of $\alpha_t$ is in $C_{X_t}$, it follows that either $[\alpha_t]$ or $-[\alpha_t]$ is in the Kähler cone of $X_t$. Up to changing the sign of $\alpha_t$,
we can then suppose that for the generic $t \in U \cap V$ the class $[\alpha_t]$ is a Kähler class, and that $\alpha_t$ is a Kähler form.

In conclusion, there is a sequence $\{t_m\}_{m \in \mathbb{N}}$ of point of $U \cap V$ which converges to $s$, and such that for every $m \in \mathbb{N}$ we have a Kähler form $\alpha_m := \alpha_{t_m}$ on $X_m := X_{t_m}$ and a Kähler form $\beta_m := \beta_{t_m}$ on $Y_m := Y_{t_m}$, such that $\alpha_m$ converges to $\alpha_s$ and $\beta_m$ converges to $\beta_s$.

As $t_m \in V$, we have an isomorphism $f_m : Y_m \to X_m$ such that $f_m^* = \psi_m^{-1} \circ \phi_m$ (where $\psi_m := \psi_{t_m}$ and $\phi_m = \phi_{t_m}$). We let $\Gamma_m$ be the graph of $f_m$, and we compute its volume in $X_m \times Y_m$ with respect to the the form $p_1^* \alpha_m + p_2^* \beta_m$, where $p_1$ and $p_2$ are the projections of $X_m \times Y_m$ to $X_m$ and $Y_m$, respectively.

We then have

$$\text{vol}(\Gamma_m) = \int_{Y_m} (\beta_m + f_m^* \alpha_m)^{2n} = \int_{Y_m} ([\beta_m] + f_m^*[\alpha_m])^{2n} = \int_{Y_m} ([\beta_m] + \psi_m^{-1} \circ \phi_m([\alpha_m]))^{2n}.$$ 

Taking the limit for $m$ going to infinity we get

$$\lim_{m \to +\infty} \text{vol}(\Gamma_m) = \int_{Y_s} ([\beta_s] + \psi_s^{-1} \circ \phi_s([\alpha_s]))^{2n} < +\infty.$$

Hence, the volumes of the $\Gamma_m$ are bounded, so that by the Bishop Theorem the cycles $\Gamma_m$ converge to a cycle $\Gamma$ of $X_s \times Y_s$ with the same cohomological properties of the $\Gamma_m$'s: namely, we have $[\Gamma] \in H^{4n}(X_s \times Y_s, \mathbb{Z})$, and if $p_1$ and $p_2$ are the two projections from $X_s \times Y_s$ to $X_s$ and $Y_s$ respectively, we have $p_1^*[\Gamma] = [X_s]$, $p_2^*[\Gamma] = [Y_s]$, and

$$[\Gamma] \cdot \gamma := p_2^*[\Gamma] \cdot p_1^*[\gamma] = \psi_s^{-1}(\phi_s(\gamma)),$$

for every $\gamma \in H^2(X_s, \mathbb{Z})$.

Now, let us split $\Gamma$ in its irreducible components: by the previous properties we then have that either $\Gamma = Z + \sum_i D_i$ where $p_1 : Z \to X_s$ and $p_2 : Z \to Y_s$ are both generically one-to-one, or $\Gamma = Z_1 + Z_2 + \sum_i D_i$ where $p_1 : Z_1 \to X_s$ and $p_2 : Z_2 \to Y_s$ are generically one-to-one, but neither $p_1 : Z_2 \to Y_s$ nor $p_2 : Z_1 \to X_s$ are generically finite. In both cases we have $p_1^*[D_i] = p_2^*[D_i] = 0$. As shown in [10] (proof of Theorem 4.3), the second case cannot happen: it follows that $Z$ is a bimeromorphism between $X_s$ and $Y_s$.

Now, recall that $Y_s$ is supposed to have no effective divisors nor non-trivial curves. It follows from this that the bimeromorphism $Z$ is indeed an isomorphism: hence there is an isomorphism $f : Y_s \to X_s$ whose graph is $Z$, and $X_s$ is Kähler (as $Y_s$ is). Moreover, by Corollary 5.7 of [8], the Kähler cone of $Y_s$, and hence that of $X_s$, is one of the
components of \( C_Y \), and the morphisms \( [D_i]_*: H^2(X_s, \mathbb{Z}) \to H^2(Y_s, \mathbb{Z}) \) are all trivial (see Lemma 5.5 in [8]). In particular, we get that \( f^* = [Z]_* = [\Gamma]_* \).

We now prove that \( \Gamma = Z \). To do this, recall that the class of \( \alpha_s \) is Kähler, and let \( \gamma_s := f^*\alpha_s \), which is Kähler again on \( Y_s \). We compute the volumes on \( X_s \times Y_s \) with respect to the Kähler class \( p_1^*\alpha_s + p_2^*\gamma_s \).

We have
\[
\text{vol}(\Gamma) = \text{vol}(Z) + \sum_i \text{vol}(D_i) = \int_Z (p_1^*\alpha_s + p_2^*\gamma_s)^{2n} + \sum_i \text{vol}(D_i) =
\]
\[
= \int_{Y_s} ([Z]_*\alpha_s + \gamma_s)^{2n} + \sum_i \text{vol}(D_i) = \int_{Y_s} (f^*\alpha_s + \gamma_s)^{2n} + \sum_i \text{vol}(D_i) =
\]
\[
= \int_{Y_s} (2\gamma_s)^{2n} + \sum_i \text{vol}(D_i),
\]
where we used that \( f^* = [Z]_* = [\Gamma]_* \).

Now, choose a sequence \( \{t_m\}_{m \in \mathbb{N}} \) of points in \( V \cap U \) converging to \( s \), and consider the graphs \( \Gamma_m \) of the isomorphisms \( f_m : Y_m \to Y_m \). Letting \( \gamma_m := f_m^*\alpha_m \), the sequence \( \gamma_m \) converges to \( \gamma_s \), and the sequence \( \text{vol}(\Gamma_m) \) converges to \( \text{vol}(\Gamma) \). But
\[
\text{vol}(\Gamma_m) = \int_{Y_m} ([\Gamma_m]_*\alpha_m + \gamma_m)^{2n} = \int_{Y_m} (2\gamma_m)^{2n},
\]
hence this sequence converges to
\[
\int_{Y_s} (2\gamma_s)^{2n},
\]
so that
\[
\text{vol}(\Gamma) = \int_{Y_s} (2\gamma_s)^{2n}.
\]
It turns then out that \( \text{vol}(D_i) = 0 \), hence \( D_i = 0 \) and \( \Gamma = Z \).

It follows that
\[
f^* = [Z]_* = [\Gamma]_* = \psi_s^{-1} \circ \phi_s.
\]
But this implies that \( s \in V \), contradicting that \( s \in \partial V \). In conclusion if \( s \in \partial V \) on \( Y_s \) there are either effective divisors or curves, and we are done.

We are now able to conclude the proof of Proposition 2.4.

**Proof.** By Proposition 2.5 we know that there is an irreducible symplectic manifold \( Y \), together with a marking \( \psi \), such that \( (X, \phi) \) is non-separated from \( (Y, \psi) \) in \( \mathcal{M}_Z \).
Consider $Def(X)$ and $Def(Y)$, the bases of the Kuranishi families of $X$ and $Y$ respectively. Up to shrinking them, as the points $(X, \phi)$ and $(Y, \psi)$ are non-separated, the Local Torelli Theorem allows us to identify them. Hence the Kuranishi families of $X$ and $Y$ are over the same base $B$, and we suppose that $X$ and $Y$ are over the same point $0 \in B$.

The non-separatedness implies that there is $t \in B$ such that $X_t$ and $Y_t$ are isomorphic under an isomorphism $f_t$ such that $f_t^* = \psi_t^{-1} \circ \phi_t$. Let $V$ be the biggest open subset of $B$ given by all $t \in B$ verifying this same property. As $V$ is open in $B$, and as $B$ is open in $M$, the generic point $t$ of $V$ is such that $NS(X_t) = 0$.

We can then apply Lemma 2.6 to conclude that $V = B$. Now, notice that if $0 \in V$, then $X$ and $Y$ are isomorphic, and we are done. Hence we can suppose that $0 \notin V$, so that $0 \in \partial V$. We can then apply the same argument in the proof of Lemma 2.6 to conclude that $X$ and $Y$ are bimeromorphic. □

3. Criterion for Kählerness

We now want to prove a Kählerness criterion for a compact, connected holomorphic symplectic $b_2$–manifold $X$ which is limit of irreducible hyperkähler manifolds.

Let us first recall a notation. In this section, $X$ is a compact, connected holomorphic symplectic $b_2$–manifold $X$ which is limit of irreducible hyperkähler manifolds. By Theorem 1.14, we know that $X$ is in the Fujiki class $C$, hence $H^2(X, \mathbb{C})$ has a Hodge decomposition. In particular, we have

$$\tilde{H}^{1,1}_R(X) = H^2(X, \mathbb{R}) \cap H^{1,1}(X) =: H^{1,1}(X, \mathbb{R}).$$

3.1. Twistor lines. If $\sigma$ is a holomorphic symplectic form on $X$, the cohomology class of $\sigma$ allows us to define a real plane

$$P(X) := (\mathbb{C} \cdot \sigma \oplus \mathbb{C} \cdot \overline{\sigma}) \cap H^2(X, \mathbb{R})$$

in $H^2(X, \mathbb{R})$, which is independent of $\sigma$ (as $h^{2,0}(X) = 1$). If $\alpha \in H^{1,1}(X, \mathbb{R})$ we let

$$F(\alpha) := P(X) \oplus \mathbb{R} \cdot \alpha,$$

which is a 3–dimensional real subspace of $H^2(X, \mathbb{R})$, and we let $F(\alpha)_\mathbb{C} := F(\alpha) \otimes \mathbb{C}$.

Now, let $Z$ be an irreducible hyperkähler manifold which is deformation equivalent to $X$, and let $(\Lambda, q)$ its Beauville lattice. If $\phi : H^2(X, \mathbb{Z}) \to \Lambda$ is a $\Lambda$–marking on $X$ (which exists by Theorem
consider the point \((X, \phi)\) of \(M_Z\). As \(X\) is limit of irreducible hyperkähler manifolds, we have \((X, \phi) \in \overline{M}_Z^{hk}\).

Notice that \(F(\alpha)_C\) is a 3–dimensional linear subspace of \(H^2(X, \mathbb{C})\), hence \(\phi(F(\alpha)_C)\) is a 3–dimensional subspace of \(\Lambda \otimes \mathbb{C}\), and \(\mathbb{P}(\phi(F(\alpha)_C))\) is a plane in \(\mathbb{P}_A\). Hence \(\mathbb{P}(\phi(F(\alpha)_C)) \cap \Omega_A\) is a curve in \(\Omega_A\) passing through \(\pi(X, \phi)\).

If \(B\) is the base of the Kuranishi family of \(X\), the inverse image

\[
T(\alpha) := \pi^{-1}(\mathbb{P}(\phi(F(\alpha)_C)) \cap \Omega_A) \cap B
\]

is then a curve in \(B\), which will be called twistor line of \(\alpha\). The restriction of the Kuranishi family of \(X\) to \(T(\alpha)\) will be denoted

\[
\kappa_\alpha : X(\alpha) \to T(\alpha).
\]

For every \(t \in T(\alpha)\) there is real \((1, 1)\)–class \(\alpha_t\) on the fiber \(X_t\) of the Kuranishi family of \(X\) over \(t\), and the sequence \(\{\alpha_t\}\) converges to \(\alpha\). If \(\alpha\) is Kähler, then \(T(\alpha) \simeq \mathbb{P}^1\), and all \(\alpha_t\) are Kähler on \(X_t\).

### 3.2. Cones in \(H^{1,1}(X, \mathbb{R})\).

Using the notation introduced in the previous section, we have

\[
C_X = \{ \alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0 \},
\]

which is an open cone in \(H^{1,1}(X, \mathbb{R})\) having two connected components.

If \(X\) is Kähler (and hence irreducible hyperkähler), the Kähler cone \(\mathcal{K}_X\) of \(X\) (i. e. the open convex cone of Kähler classes on \(X\)) is contained in one of them: such a component is usually called positive cone of \(X\), and denoted \(C^+_X\). The other component will be denoted \(C^-_X\). If \(NS(X) = 0\), Corollary 5.7 of [8] gives us that \(\mathcal{K}_X = C^+_X\), a fact that has already been used in the previous sections.

Theorem 1.1 of [2] tells us that if \(X\) is irreducible hyperkähler, then \(\alpha \in C^+_X\) is in the Kähler cone of \(X\) if and only if

\[
\int_C \alpha > 0
\]

for every rational curve \(C\) of \(X\). Our aim is to show a similar result for a compact, connected holomorphic symplectic \(b_2\)–manifold \(X\) which is limit of irreducible hyperkähler manifolds.

As on such a manifold the Kähler cone could be empty, we cannot use it to define the positive cone of \(X\). Anyway, we can use the pseudo-effective cone \(\mathcal{E}_X\) of \(X\), i. e. the closed connected cone of classes of positive closed real \((1, 1)\)–currents on \(X\). If \(X\) is irreducible hyperkähler, by point i) of Theorem 4.3 of [3] we have \(C^+_X \subseteq \mathcal{E}_X\).
Popovici and Ugarte (see Theorem 5.9 of [13]) showed that if $X \to B$ is a smooth and proper family of sGG manifolds and $\{b_n\}$ is a sequence of points of $B$ converging to a point $b \in B$, then the limit of the pseudo-effective cones of $X_{b_n}$ is contained in $E_{X_b}$, i.e. the pseudo-effective cone varies upper-semicontinuously along $B$.

As all manifolds in the Fujiki class $\mathcal{C}$ are sGG manifolds (see [13]), we conclude that the pseudo-effective cone varies upper-semicontinuously in families of class $\mathcal{C}$ manifolds.

We now prove the following general fact about convex cones in a real finitely dimensional vector space.

**Lemma 3.1.** Let $V$ a real vector space of finite dimension $n$, and let $A, B \subseteq V$ two cones in $V$ such that:

1. the cone $A$ is strictly convex (i.e. it does not contain any linear subspace of $V$) and closed;
2. the cone $B$ is open and has two connected components, each of which is convex;
3. for every $a \in B$, we have either $a \in A$ or $-a \in A$.

Then $A \cap B$ is one of the connected components of $B$.

**Proof.** We first notice that if $B^+$ and $B^-$ are the two connected components of $B$, if $B^+ \subseteq A$ we have $B^+ = B \cap A$. Indeed, if $b' \in B^- \cap A$, then $-b' \in B^+ \subseteq A$. It follows that $b', -b' \in A$, which is not possible as $A$ is a strictly convex cone.

We are left to prove that there is a connected component of $B$ which is contained in $A$. To do so, let $b_0 \in B \cap A$, and let $B^+$ be the connected component of $B$ which contains $b_0$. We show that if $b_1 \in B^+$, then $b_1 \in A$.

Consider the segment $[b_0, b_1] := \{b_t := (1 - t)b_0 + tb_1 \mid t \in [0, 1]\}$.

Suppose that $b_1 \notin A$, we have to find a contradiction. First, notice that as $b_1 \notin A$, there is $t \in [0, 1)$ such that $b_t \notin A$: indeed, if for every $t \in [0, 1)$ we have $b_t \in A$, as $A$ is closed we would have $b_1 \in A$.

As $b_t$ and $b_1$ are not in $A$, for every $s \in [t, 1)$ we have $b_s \notin A$: indeed, as $b_t, b_1 \notin A$ but $b_t, b_1 \in B$, we have $-b_t, -b_1 \in A$. As $A$ is convex, the segment $[-b_t, -b_1]$ (whose elements are the $-b_s$ for $s \in [t, 1]$) is contained in $A$. But this means that $b_s \notin A$ as $A$ is a strictly convex cone.

The set of those $t \in [0, 1]$ for which $b_t \notin A$ surely has an infimum $t_0 \in [0, 1]$. Hence, for every $t < t_0$ we have $b_t \in A$, and for every $t > t_0$ we have $b_t \notin A$. As $b_t \in B$, this implies that $-b_t \notin A$ for every $t > t_0$. But as $A$ is closed, these conditions give $b_{t_0} \in A$ (as $b_t \in A$ for every
<t < t_0) and -b_{t_0} \in A \text{ (as } -b_t \in A \text{ for every } t > t_0). \text{ But as } A \text{ is a strictly convex cone, we get a contradiction.} \qquad \square

This fact will be used in the proof of the following:

**Lemma 3.2.** Let $X$ be a compact, connected holomorphic symplectic $b_2-$manifold which is limit of irreducible hyperkähler manifolds. Then $C_X \cap E_X$ consists of exactly one connected component of $C_X$.

**Proof.** The pseudo-effective cone $E_X$ is strictly convex and closed in $H^{1,1}(X, \mathbb{R})$. The cone $C_X$ is open and has two connected components, each of which is convex. We show that if $\alpha \in C_X$, then either $\alpha \in E_X$ or $-\alpha \in E_X$ (which in particular implies that $C_X \cap E_X \neq \emptyset$). Once this is done, the statement follows from Lemma 3.1.

Fix an irreducible hyperkähler manifold $Z$ which is deformation equivalent to $X$, and we let $(\Lambda, q)$ be its Beauville lattice. Moreover, let $\alpha \in C_X$, and consider a $\Lambda-$marking $\phi$ on $X$ (whose existence comes from Theorem 1.12). As $X$ is limit of irreducible hyperkähler manifolds, we have $(X, \phi) \in \mathcal{M}_Z^{hk}$.

Let $\mathcal{X} \to B$ be the Kuranishi family of $X$, and let $0$ be the point of $B$ over which the fiber of $X_0$ is $X$. By Theorem 1.14 $X$ is in the Fujiki class $C$, hence it is a sGG-manifold. This being an open condition (see [13]), up to shrinking $B$ we can suppose that for every $b \in B$ the manifold $X_b$ is sGG.

Moreover, as $(X, \phi) \in \mathcal{M}_Z^{hk}$, there is a sequence $\{b_n\}$ of points of $B$ converging to $0$ over which the fiber $X_b$ is irreducible hyperkähler, and we can even suppose that $NS(X_n) = 0$.

Consider a sequence $\{\alpha_n\}$ given by $\alpha_n \in C_{X_n}$ converging to $\alpha$, hence either $\alpha_n \in C_{X_n}^+$ (for all $n$), or $-\alpha_n \in C_{X_n}^+$ (for all $n$). As recalled before, we have $C_{X_n}^+ \subseteq E_{X_n}$: it follows that either $\alpha_n \in E_{X_n}$ (for all $n$) or $-\alpha_n \in E_{X_n}$ (for all $n$). By Theorem 5.9 of [13] we then conclude that either $\alpha \in E_X$ or $-\alpha \in E_X$. \qquad \square

The connected component of $C_X$ contained in $E_X$ will be denoted $C_X^+$ and called positive cone of $X$, in analogy with the hyperkähler case. The other connected component of $C_X$ will be denoted $C_X^-$.  

3.3. **Deformations and Kähler classes.** The first result we prove is the following:

**Proposition 3.3.** Let $X$ be a compact, connected holomorphic symplectic manifold in the Fujiki class $C$, which is limit of irreducible hyperkähler manifolds. Let $\alpha \in C_X$. 


(1) If for every $\beta \in H^2(X, \mathbb{Z})$ we have $q(\alpha, \beta) \neq 0$, then there is $t \in T(\alpha)$ such that $X_t$ is Kähler and either $\alpha_t$ or $-\alpha_t$ is a Kähler class on $X_t$.

(2) If moreover $\alpha \in C_X^+$, then $\alpha_t$ is Kähler.

Proof. Let $Z$ be an irreducible hyperkähler manifold which is deformation equivalent to $X$, and let $(\Lambda, q)$ be its Beauville lattice. Fix a marking $\phi$ on $X$ (which exists by Theorem 1.12), and consider the point $(X, \phi) \in M_Z$. As $X$ is limit of irreducible hyperkähler manifolds, we have $(X, \phi) \in \overline{M}_Z^{hk}$.

We first show that for the generic $t \in T(\alpha)$ the fiber $X_t$ is in $\overline{M}_Z^{hk}$. Let $X \to B$ be the Kuranishi family of $X$, and let $0$ be the point of $B$ over which the fiber $X_0$ is $X$.

As $(X, \phi) \in \overline{M}_Z^{hk}$, there is a sequence $\{b_n\}$ of points of $B$ verifying the following properties:

1. The sequence $b_n$ converges to $0$ in $B$;
2. For each $n$ the fiber $X_n$ of $X$ over $b_n$ is an irreducible hyperkähler manifold such that $H^{1,1}(X_n) \cap H^2(X_n, \mathbb{Z}) = 0$;
3. For each $n$ there is $\alpha_n \in C_{X_n}$ such that the sequence $\alpha_n$ converges to $\alpha$.

As $H^{1,1}(X_n) \cap H^2(X_n, \mathbb{Z}) = 0$, up to changing the sign of $\alpha$, and hence of $\alpha_n$, we can suppose that $\alpha_n \in K_{X_n}$ for every $n$. We let $T_n$ be the twistor line of $\alpha_n$, which is a rational curve in $B$ passing through the point $(X_n, \phi_n)$.

As $(X_n, \phi_n)$ converges to $(X, \phi)$, and as $\alpha_n$ converges to $\alpha$, we see that the twistor lines $T_n$ converge to $T(\alpha)$. This means that if $t \in T(\alpha)$, there is a sequence $\{s_{t,n}\}$ of points of $B$ such that

1. The sequence $\{s_{t,n}\}$ converges to $t$;
2. For each $n$ we have $s_{t,n} \in T_n$.

As $s_{t,n} \in T_n$, and as $T_n$ is the twistor line of the Kähler class $\alpha_n$, we see that the fiber $X_{s_{t,n}}$ of the twistor family of $\alpha_n$ over $s_{t,n}$ is an irreducible hyperkähler manifold. As $s_{t,n}$ converges to $t$, we then see that $X_t$ is limit of irreducible hyperkähler manifolds. This means that $(X_t, \phi_t) \in \overline{M}_Z^{hk}$.

In particular, by Proposition 2.4 this implies that $X_t$ is bimeromorphic to an irreducible hyperkähler manifold $Y_t$. Now, by hypothesis we have $q_X(\alpha, \beta) \neq 0$ for each $\beta \in H^2(X, \mathbb{Z})$. This implies that $T(\alpha)$ does not intersect in $0$ (and hence generically) any of the hypersurfaces $S_\beta$, i.e. for a generic $t \in T(\alpha)$ the period of $(X_t, \phi_t)$ is generic in $\Omega_\Lambda$. 
As the periods of \((X_t, \phi_t)\) and \((Y_t, \psi_t)\) are equal, it follows that for a generic \(t \in T(\alpha)\) the irreducible hyperkähler manifold \(Y_t\) is such that \(H^{1,1}(Y_t) \cap H^2(Y_t, \mathbb{Z}) = 0\), so that \(X_t\) and \(Y_t\) are biholomorphic.

It follows that \(X_t\) is irreducible hyperkähler, and that \(\mathcal{K}_{X_t}\) is one of the components of \(\mathcal{C}_{X_t}\). As \(\alpha_t \in \mathcal{C}_{X_t}\), it follows that either \(\alpha_t\) or \(-\alpha_t\) is a Kähler class on \(X_t\).

Let us now suppose moreover that the class \(\alpha\) is even pseudo-effective, and by contradiction that \(\alpha\) is not a Kähler class. By what we just proved, it follows that \(-\alpha_t\) is Kähler for generic \(t \in T(\alpha)\). As \(\mathcal{K}_{X_t}\) is contained in \(E_{X_t}\), we then have a family \(-\alpha_t\) of pseudo-effective classes converging to \(-\alpha\).

Now, by Theorem 5.9 of [13] (which we can apply as by the previous part of the proof the family \(\mathcal{X}(\alpha) \rightarrow T(\alpha)\) is a family of class \(\mathcal{C}\) manifolds, and hence of sGG manifolds) we know that a limit of pseudo-effective classes along the family \(\mathcal{X}(\alpha)\) is a pseudo-effective class on \(X\).

This means that \(-\alpha\) is a pseudo-effective class on \(X\). As by hypothesis \(\alpha\) is pseudo-effective too, it follows that \(\alpha = 0\), which is not possible as \(q_X(\alpha) > 0\). This shows that \(\alpha\) is a positive pseudo-effective class such that \(q_X(\alpha, \beta) \neq 0\) for every \(\beta \in H^2(X, \mathbb{Z})\), then for a generic \(t \in T(\alpha)\) the class \(\alpha_t\) is Kähler. \(\square\)

We now use the previous Proposition to show the following, which is an improved version of Proposition 2.4.

**Proposition 3.4.** Let \(X\) be a compact holomorphic symplectic manifold in the Fujiki class \(\mathcal{C}\), which is limit of irreducible hyperkähler manifolds, and let \(\alpha \in \mathcal{C}_X^+\) be such that \(q_X(\alpha, \beta) \neq 0\) for every \(\beta \in H^2(X, \mathbb{Z})\). Then there is an irreducible hyperkähler manifold \(Y\) and a cycle \(\Gamma = Z + \sum_i D_i\) in \(X \times Y\) such that the following properties are verified:

1. the cycle \(Z\) defines a bimeromophic map between \(X\) and \(Y\);
2. the projections \(D_i \rightarrow X\) and \(D_i \rightarrow Y\) have positive dimensional fibers;
3. the cycle \(\Gamma\) defines a Hodge isometry \(\Gamma\) between \(H^2(X, \mathbb{Z})\) and \(H^2(Y, \mathbb{Z})\);
4. the class \(\Gamma, \alpha\) is Kähler.

**Proof.** Consider the family \(\kappa_\alpha : \mathcal{X}(\alpha) \rightarrow T(\alpha)\). By Proposition 3.3 we know that for a generic \(t \in T(\alpha)\) the fiber \(X_t\) of \(\kappa_\alpha\) over \(t\) is an irreducible hyperkähler manifold, and that \(\alpha_t\) is a Kähler class on it.

Let \(\mathcal{X}' \rightarrow T(\alpha_t)\) be the twistor family of \((X_t, \alpha_t)\), and notice that \(\pi(T(\alpha))\) is identified with an open subset of \(\pi(T(\alpha_t))\), and that for every \(s \in T(\alpha_t)\) the fiber \(X'_s\) of \(\mathcal{X}'\) over \(s\) is Kähler.
Restricting the twistor family $\mathcal{X}'$ to such an open subset, we then find two families $\mathcal{X}(\alpha) \to C$ and $\mathcal{X}' \to C$ over the same base curve, which have isomorphic fibers over $t$, and the fibers of $\mathcal{X}'$ are all Kähler. We let $0 \in C$ be the point over which the fiber of $\mathcal{X}$ is $X$, and we let $X'$ be the fiber of $\mathcal{X}'$ over $0$.

Both families are endowed with natural markings $\phi_s$ and $\phi'_s$ for each $s$, such that $(\phi'_s)^{-1} \circ \phi_t$ is induced by the isomorphism $X_t \simeq X'_t$. The class $\alpha'_s := (\phi'_s)^{-1} \circ \phi_s(\alpha_s)$ is a Kähler class on $X'_s$ for every $s \in C$. In particular the class $\alpha' := (\phi'_0)^{-1} \circ \phi_0(\alpha)$ is Kähler on $X'$.

By Lemma 2.6 the points $(X, \phi_0)$ and $(X', \phi'_0)$ are non-separated points in $\mathcal{M}_Z$. As $(X, \phi_0) \in \mathcal{M}^{hk}_Z$ and $(X', \phi'_0) \in \mathcal{M}^{hk}_Z$, we can apply Proposition 2.4 to show that there is a cycle $\Gamma = Z + \sum_i Y_i$ on $X \times X'$ such that

1. the cycle $Z$ defines a bimeromorphic map between $X$ and $X'$;
2. the projections $Y_i \to X$ and $Y_i \to X'$ have positive dimensional fibers;
3. the cycle $\Gamma$ defines a Hodge isometry $[\Gamma]_s$ between $H^2(X, \mathbb{Z})$ and $H^2(X', \mathbb{Z})$;

Notice that $[\Gamma]_s \alpha = \alpha'$, which is Kähler.

3.4. The proof of Theorem 1.15. We are now ready to prove Theorem 1.15, namely that if $X$ is a compact, connected holomorphically symplectic $b_2$-manifold which is limit of irreducible hyperkähler manifolds, any very general class $\alpha \in C^+_X$ (i.e. $q_X(\alpha, \beta) \neq 0$ for every $\beta \in H^2(X, \mathbb{Z})$) such that $\alpha \cdot C > 0$ for every rational curve $C$ on $X$ is a Kähler class on $X$, and in particular $X$ is Kähler.

Proof. By Proposition 3.4 as $\alpha \in C^+_X$ is such that $q_X(\alpha, \beta) \neq 0$ for every $\beta \in H^2(X, \mathbb{Z})$, then there is an irreducible hyperkähler manifold $Y$ and a cycle $\Gamma = Z + \sum_i D_i$ in $X \times Y$ such that the following properties are verified:

1. the cycle $Z$ defines a bimeromorphic map between $X$ and $Y$;
2. the projections $D_i \to X$ and $D_i \to Y$ have positive dimensional fibers;
3. the cycle $\Gamma$ defines a Hodge isometry $[\Gamma]_s$ between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$;
4. the class $\alpha' := [\Gamma]_s \alpha$ is Kähler on $Y$.

The argument used in the proof of Theorem 2.5 of [9] shows that since $[\Gamma]_s \alpha$ is a Kähler class on $Y$ and $\alpha \cdot C > 0$ for every rational curve $C$ on $X$, then all the irreducible components $D_i$ of $\Gamma$ which are contracted by the projection $p_X$ of $X \times Y$ to $X$ are such that the codimension in $X$ of $p_X(D_i)$ is at least 2. By Lemma 2.2 of [8] it then follows that
the morphisms $[D_i]_* : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ are all trivial. As a consequence, we have $\alpha = [\Gamma]_* \alpha' = [Z]_* \alpha'$.

We let $f : Y \to X$ be a bimeromorphism whose graph is $Z$. As $\alpha'$ is Kähler, then for every rational curve $C'$ in $Y$ we have

$$\int_{C'} \alpha' > 0.$$ 

Notice that $\alpha' = f^* \alpha$, so that we have

$$\int_C \alpha > 0, \quad \int_{C'} f^* \alpha > 0$$

for every rational curve $C$ in $X$ and every rational curve $C'$ in $Y$. By Proposition 2.1 of [9] it follows that $f$ extends to an isomorphism, and $\alpha$ is then a Kähler class. \[\Box\]

4. Kählerness of moduli spaces of sheaves

This last section is devoted to the proof of Theorem 1.9. Hence, we let $S$ be a K3 surface, $v \in H^2(S, \mathbb{Z})$ be of the form $v = (r, \xi, a)$ where $r > 0$ and $\xi \in NS(S)$ are prime to each other. Moreover, we let $\omega$ be a Kähler class on $S$, which we suppose to be $v$–generic.

We want to show that if the moduli space $M := M^v(S, \omega)$ is a $b_2$–manifold, then it is Kähler. To do so, we apply Theorem 1.15 to $M$: we then need to prove that $M$ is a compact, connected holomorphically symplectic $b_2$–manifold which is limit of irreducible hyperkähler manifolds, and we need to provide a very generic class $\alpha \in C^+_M$ such that $\alpha \cdot C > 0$ for every rational curve $C$ in $M$.

We will always assume that $v^2 \geq 2$, as the cases $v^2 \leq 0$ are already known: if $v^2 < -2$ we have $M = \emptyset$; if $v^2 = -2$ then $M$ is a point; if $v^2 = 0$, then $M$ is a K3 surface by Corollary 5.3 of [12].

A. The moduli space $M$ is a compact, connected holomorphically symplectic $b_2$–manifold which is limit of irreducible hyperkähler manifolds. The fact that $M$ is a compact, holomorphically symplectic manifold is due to Toma (see Remark 4.5 of [15]). The connectedness is given by Proposition 4.24 of [12]. The fact that $M$ is a $b_2$–manifold is supposed to hold true.

We are then left to prove that $M$ is a limit of irreducible hyperkähler manifolds. To do so, let $S \to B$ be the Kuranishi family of the K3 surface $S$, where $B$ is a complex manifold of dimension 20. Let $B_0 \subseteq B$ be the subvariety of $B$ given by those $b \in B$ such that $\xi \in NS(S_b)$. Similarly, let $B_\omega \subseteq B$ be the subvariety of $B$ given by those $b \in B$ such that the class $\omega \in H^{1,1}(S_b, \mathbb{R})$. Moreover, we let $B_{\xi, \omega} := B_{\xi} \cap B_{\omega}$. 
Recall that $B_\xi$ and $B_\omega$ are smooth hypersurfaces of $B$, and as $\xi$ and $\omega$ are linearly independent, then $B_\xi$ and $B_\omega$ intersect transversally, so that $B_{\xi,\omega}$ is a smooth analytic subset of $B$ (of positive dimension). By Theorem 3.5 of [9], the subset $B_\mu^{\xi,\omega}$ of $B_{\xi,\omega}$ given by those $b$ such that $S_b$ is projective is dense in $B_{\xi,\omega}$.

We now consider the restriction $S' := S_{|B_{\xi,\omega}}$, together with a morphism $S' \rightarrow B_{\xi,\omega}$. We suppose $0 \in B_{\xi,\omega}$ be such that $S_0 \sim S$.

Recall that $\omega$ is a Kähler class on $S$: as Kählerness is an open property in the analytic topology, there is an analytic open subset $D' \subset B_{\xi,\omega}$ containing $0$ and such that for every $b \in D'$ the class $\omega$ is Kähler on $S_b$. We then can consider the relative moduli space $M \rightarrow D'$, whose fiber over $b$ is the moduli space $M_b = M_{\mu}(S_b, \omega)$ of $\mu$-stable sheaves on $S_b$ whose Mukai vector is $v$.

As the $v$-genericity is an open property in the analytic topology, there is an open subset $D$ of $D'$ such that for every $b \in D$ the class $\omega$ is $v$-generic. We then consider the restriction $S_D$ of $S$ to $D$, together with a morphism $S_D \rightarrow D$. For every $d \in D$ the K3 surface $S_d$ comes equipped with a Mukai vector $v = (r, \xi, a)$ and a $v$-generic polarization $\omega$.

As a consequence, the restriction of $M$ to $D$, denoted $M_D$ is such that for every $d \in D$ the fiber $M_d$ is a compact, connected complex manifold (see again Proposition 4.24 of [12]). The morphism $M_D \rightarrow D$ being submersive, it follows that the family $M_D \rightarrow D$ is a smooth and proper family, whose fiber over $0$ is $M_{\mu}(S, \omega)$.

Now, as $B_{\xi,\omega}^\mu$ is dense in $B_{\xi,\omega}$, it follows that $B_{\xi,\omega}^\mu \cap D$ is dense in $D$. Hence, for the generic point $d \in D$ the fiber $S_d$ is a projective K3 surface, and the fiber $M_d$ is an irreducible hyperkähler manifold (see Theorem 3.4 of [12]). Hence $M_{\mu}(S, \omega)$ is limit of irreducible hyperkähler manifolds.

Remark 4.1. As a consequence of what we just proved, by Theorem 1.14 we see that if $M_{\mu}(S, \omega)$ verifies the assumptions of Theorem 1.9 then it is in the Fujiki class $C$.

B. A very generic class $\alpha \in C_M^+$ such that $\alpha \cdot C > 0$ for every rational curve $C$ of $M$. In order to prove Theorem 1.9 by the previous part of this section we just need to find such a class $\alpha$ on $M$. To do so, recall that by [12] there is a morphism

$$\lambda_v : v^\perp \rightarrow H^2(M, \mathbb{Z})$$

which is an isometry (since $v^2 \geq 2$) with respect to the Mukai pairing on $v^\perp$ and the Beauville form of $M$.  

Moreover, by the previous paragraph the moduli space \( M \) is in the Fujiki class \( C \), hence \( H^2(M, \mathbb{Z}) \) has a Hodge decomposition, and \( \lambda_v \) is a Hodge morphism. In particular, \( \lambda_v \) is a Hodge isometry. This remains true if we tensor with \( \mathbb{R} \), and we get a Hodge isometry
\[
\lambda_v : v^\perp \otimes \mathbb{R} \longrightarrow H^2(M, \mathbb{R}).
\]
We will then construct the desired class \( \alpha \) by taking an appropriate element of \( v^\perp \otimes \mathbb{R} \).

The choice we make is the following: let \( m \in \mathbb{N} \) and
\[
\alpha_{m, \omega} := (-r, -mr\omega, a + m\omega \cdot \xi).
\]
First of all, we remark that \( \alpha_{m, \omega} \in v^\perp \otimes \mathbb{R} \), as
\[
(v, \alpha_{m, \omega})_S = mr\omega \cdot \xi - r(a + m\omega \cdot \xi) + ra = 0.
\]
Moreover, as \( \omega \) is a real \((1, 1)-\)class on \( S \), the class \( \alpha_{m, \omega} \) is a real \((1, 1)-\)class orthogonal to \( v \).

It then follows that
\[
\alpha := \lambda_v(\alpha_{m, \omega}) \in H^{1,1}(M, \mathbb{R}).
\]
We prove that \( \alpha \) is a very general class in \( C_M^+ \) such that \( \alpha \cdot C > 0 \) for every rational curve \( C \) on \( M \).

We start by showing that \( \alpha \) is very general in \( H^{1,1}(M, \mathbb{R}) \)

**Lemma 4.2.** If \( \omega \) is sufficiently generic, then for every \( \beta \in H^2(M, \mathbb{Z}) \) we have \( q_M(\alpha, \beta) \neq 0 \).

**Proof.** As \( \beta \in H^2(M, \mathbb{Z}) \), there are \( s, b \in \mathbb{Z} \) and \( D \in H^2(S, \mathbb{Z}) \) such that \( D \cdot \xi = sa + rb \) (i.e. \( \gamma := (s, D, b) \in v^\perp \)) and \( \beta = \lambda_v(\gamma) \).

It follows that
\[
q_M(\alpha, \beta) = q_M(\lambda_v(\alpha_{m, \omega}), \lambda_v(\gamma)) = (\alpha_{m, \omega}, \gamma)_S = -m\omega \cdot (rD + s\xi) + rb - sa.
\]
Suppose that \( q(\alpha, \beta) = 0 \): this is then equivalent to
\[
\omega \cdot (rD + s\xi) = \frac{D \cdot \xi - 2sa}{m},
\]
which means that \( \omega \) is on some hyperplane in \( H^2(S, \mathbb{R}) \) associated to \( D \). As the family of these hyperplanes is countable (since the family of \( D \in H^2(S, \mathbb{Z}) \) is countable), and as \( \omega \) is sufficiently generic, we see that \( q(\alpha, \beta) \neq 0 \) for every \( \beta \in NS(M) \). \( \Box \)

We notice that we can move \( \omega \) in the \( v \)-chamber of the Kähler cone of \( S \) where it lies without changing \( M \) (see Proposition 3.2 of \([12]\)), hence we can always suppose that \( \omega \) is sufficiently generic, and hence that \( \alpha \) is very generic.
We now show that \( q_M(\alpha) > 0 \) and that it is a pseudo-effective class on \( M \).

**Lemma 4.3.** If \( m \gg 0 \) we have \( \alpha \in \mathcal{C}_M^+ \).

**Proof.** We first prove that \( \alpha \in \mathcal{C}_M \), and then that \( \alpha \in \mathcal{E}_M \).

We have

\[
q_M(\alpha) = q_M(\lambda_v(\alpha_{m,\omega})) = (\alpha_{m,\omega}, \alpha_{m,\omega})_S = m^2 r^2 \omega^2 + 2ra - 2mr_\omega \cdot \xi.
\]

As \( m \gg 0 \) and \( \omega^2 > 0 \) (since \( \omega \) is Kähler on \( S \)), we then see that \( q_M(\alpha) > 0 \), i.e. \( \alpha \in \mathcal{C}_M \).

We now have to show that \( \alpha \in \mathcal{E}_M \). To show this, consider the deformation

\[
\mathcal{M} \longrightarrow B_{\xi, \omega}
\]

we introduced in the previous paragraph. We let \( 0 \in B_{\xi, \omega} \) be the point over which the fiber is \( M^\mu_v(S, \omega) \). For a generic \( b \in B_{\xi, \omega} \) the fiber is \( M_b = M^\mu_v(S_b, \omega) \) where \( S_b \) is a projective K3 surface, so that the fiber is a projective irreducible hyperkähler manifold.

Notice that \( \omega \) is still a \( v \)-generic Kähler class on \( S_b \), and the class \( \alpha \) is still in \( \mathcal{C}_{M_b} \), and this for every \( b \in B_{\xi, \omega} \). We write \( \alpha_0 := \alpha \).

Now, as shown in Remark 3.5 of [12], in the same \( v \)-chamber where \( \omega \) lies there is a class of the form \( \omega' = c_1(H) \) for some ample line bundle \( H \) on \( S_b \). We let \( \alpha_1 := \lambda_v(\alpha_{m,\omega}) \). Moreover, for every \( t \in [0,1] \) we let \( \omega_t := (1-t)\omega + tw' \), which is a segment contained in the \( v \)-chamber where \( \omega \) and \( \omega' \) are, and we let \( \alpha_t := \lambda_v(\alpha_{m,\omega_t}) \). By linearity of \( \lambda_v \), we have

\[
\alpha_t = (1-t)\alpha_0 + t\alpha_1,
\]

and the image of the map \( \alpha : [0,1] \longrightarrow H^{1,1}(M_b, \mathbb{R}) \) defined by letting \( \alpha(t) := \alpha_t \) is a segment in \( \mathcal{C}_{M_b} \).

Our aim is to show that \( \alpha \in \mathcal{E}_M \). As the family \( \mathcal{M} \longrightarrow B \) is a family of manifolds in the Fujiki class \( \mathcal{C} \) by the previous paragraph, by Theorem 5.9 of [13] it is sufficient to show that \( \alpha_0 \in \mathcal{E}_{M_b} \) for a generic \( b \) around 0. As for the generic \( b \) around 0 we have that \( M_b \) is irreducible hyperkähler, this is equivalent to show that \( \alpha_0 \in \mathcal{C}_{M_b}^+ \).

As \( \mathcal{C}_{M_b}^+ \) is a convex cone and the segment \([\alpha_0, \alpha_1]\) is contained in \( \mathcal{C}_{M_b}^+ \), to show that \( \alpha_0 \in \mathcal{C}_{M_b}^+ \) it is sufficient to show that \( \alpha_1 \in \mathcal{C}_{M_b}^+ \).

But now, as \( S_b \) is projective we can use a general construction presented in [11]: if \( H \) is a \( v \)-generic ample line bundle on \( S_b \), we can construct an ample line bundle \( L(H) \) on \( M^\mu_v(S_b, H) \), and we have \( c_1(L(H)) = \lambda_v(\alpha_{m,\omega(H)}) \). As \( M^\mu_v(S_b, \omega) = M^\mu_v(S_b, H) \) (as \( \omega \) and \( c_1(H) \) are in the same \( v \)-chamber, by [12]), the class \( \lambda_v(\alpha_{m,\omega'}) \) is an ample...
class on $M^n_b(S_b, \omega)$. It then lies in the Kähler cone of $M_b$, and hence in $C^+_M$.

In conclusion, we have shown that up to choosing $m \gg 0$ and $\omega$ sufficiently generic, the class $\alpha$ is a very generic class in $C^+_M$. We are left to show that $\alpha \cdot C > 0$ for every rational curve $C$ in $M$.

**Lemma 4.4.** If $m \gg 0$ and $\omega$ is sufficiently generic, we have $\alpha \cdot C > 0$ for every rational curve $C$ on $M$.

**Proof.** Let $[C] \in H^{2n-1,2n-1}(M, \mathbb{Z})$, and let $\beta_C \in NS(M)$ be the dual of $[C]$, so that
\[
\alpha \cdot C = q_M(\alpha, \beta_C).
\]
We then just need to prove that $q_M(\alpha, \beta_C) > 0$ for every rational curve $C$ on $M$.

Let $S \to B$ be the Kuranishi family of $S$, and let $0 \in B$ be such that $S_0 = S$. We let $B_C$ be the subset of $B$ of those $b \in B$ such that $\beta_C \in NS(S_b)$, i.e. $C$ is a rational curve on $S_b$. Consider the intersection $D_C := D \cap B_C$, which is an analytic subset of $D$, whose generic point $d$ is such that $S_d$ is a projective K3 surface.

We let $\mathcal{M}_C$ be the restriction of the relative moduli space $\mathcal{M} \to D$ to $D_C$. Notice that for every $d \in D_C$ we have the class $\alpha \in C_{M_d}$ and the rational curve $C$ on $M_d$. As the intersection product of $\alpha$ with $C$ is constant along $D_C$, it is sufficient to show that
\[
q_{M_d}(\alpha, \beta_C) > 0
\]
for some $d \in D_C$.

As $\beta_C \in NS(M_d)$, there are $s, b \in \mathbb{Z}$ and $\zeta \in NS(S_d)$ such that $\gamma := (s, \zeta, b) \in v^\perp$ and $\beta_C = \lambda_v(\gamma)$. As $\lambda_v$ is an isometry, we have
\[
q_{M_d}(\alpha, \beta_C) = q_{M_d}(\lambda_v(\alpha_{m, \omega}), \lambda_v(\gamma)) = (\alpha_{m, \omega}, \gamma)_{S_d}.
\]
It is then sufficient to show that $(\alpha_{m, \omega}, \gamma)_{S_d} > 0$.

Now, by Lemma 3.3 of [12] there is an ample class $\omega'$ on $S_d$ which is in the same $v$-chamber of $\omega$ and such that for every $\eta \in NS(S_d)$ we have $\omega \cdot \eta = \omega' \cdot \eta$. Then we have $\alpha_{m, \omega'} \in v^\perp$, and
\[
(\alpha_{m, \omega}, \gamma)_{S_d} = (\alpha_{m, \omega'}, \gamma)_{S_d}.
\]
It is then sufficient to show that $(\alpha_{m, \omega'}, \gamma)_{S_d} > 0$.

To do so, consider a rational $\omega''$ in a neighborhood of $\omega'$ in the ample cone of $S_d$, and let $p \in \mathbb{N}$ and $H$ an ample line bundle on $S_d$ such that $p\omega'' = c_1(H)$. As we can choose $m \gg 0$, we can suppose that $m = m'p$ for some very big $m' \in \mathbb{N}$. As $H$ is $v$-generic we have that $\lambda_v(\alpha_{m', c_1(H)})$ is the first Chern class of an ample line bundle, so that
\[
\lambda_v(\alpha_{m', c_1(H)}) \cdot C > 0.
\]
It follows that
\[(\alpha_{m',c_1(H)}, \gamma)_{S_d} > 0.\]

Now, notice that
\[\alpha_{m',c_1(H)} = \langle -r, m'r c_1(H), a + m'c_1(H) \cdot \xi \rangle = \langle -r, m'pr c_1(H)/p, a + m' p \xi \cdot c_1(H)/p \rangle = \langle -r, m r \omega'', a + m \omega'' \cdot \xi \rangle = \alpha_{m,\omega''}.\]

Hence we get
\[(\alpha_{m,\omega'', \gamma})_{S_d} > 0\]

As this is true for all rational classes \(\omega''\) in a neighborhood of \(\omega'\), this implies that
\[(\alpha_{m,\omega, \gamma})_{S_d} \geq 0.\]

As we saw before, this implies that \(\alpha \cdot C \geq 0\) for every rational curve \(C\) in \(M\). But as \(\beta_C \in NS(M)\) and \(\alpha \cdot C = q_M(\alpha, \beta_C)\), and as we know that \(\alpha\) is very generic by Lemma 4.2, it follows that \(\alpha \cdot C \neq 0\). In conclusion we have \(\alpha \cdot C > 0\), and we are done. \(\square\)

Now, by paragraph A we know that if \(M^\mu(S, \omega)\) verifies the assumptions of Theorem 1.9 then it is a compact, connected holomorphically symplectic \(b_2\)–manifold which is limit of irreducible hyperkähler manifolds. Moreover, Lemmas 4.2, 4.3 and 4.4 we know that the class \(\alpha\) is a very generic class in \(C^+_M\) such that \(\alpha \cdot C > 0\) for every rational curve \(C\) of \(M^\mu(S, \omega)\).

We can then apply Theorem 1.15 to conclude that \(M^\mu(S, \omega)\) is a Kähler manifold (actually irreducible hyperkähler), and that \(\alpha\) is a Kähler class on it. This concludes the proof of Theorem 1.9.

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