Exponential Synchronization of Partially Coupled Heterogeneous Networks With Time-Delays and Heterogeneous Impulses

GUIZHEN FENG, JINDE CAO, (Fellow, IEEE), JIAN DING, AND YISHU WANG

1Department of Basic Courses, Nanjing Vocational University of Industry Technology, Nanjing 210023, China
2School of Mathematics, Southeast University, Nanjing 210096, China
3Yonsei Frontier Lab, Yonsei University, Seoul 03722, South Korea
4College of Math and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Corresponding author: Jian Ding (df2001101@126.com)

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ABSTRACT This work focuses on exponential synchronization for a class of partially coupled heterogeneous networks with time-delays and heterogeneous impulses. The synchronization targets are selected as the common equilibrium solution and the average trajectory, respectively. Some synchronization criteria are deduced by using Lyapunov function and comparison principle.

INDEX TERMS Exponential synchronization, heterogeneous networks, partially coupled, comparison principle, impulse.

I. INTRODUCTION

As an important mathematical model to describe the real world, complex dynamic networks have aroused strong research interests from scholars at home and abroad in recent years. Research on the cooperative collective behavior of complex networks is a mainstream direction of current complex network research. Cooperative collective behavior of networks mainly includes consistency [1], [2], stability [3]–[5] and synchronization [6], [7].

For a long time, research of complex networks has mainly focused on homogeneous networks that consist of a single type of objects and links, and the state of each network node follows the same evolutionary laws. Homogeneous networks cannot completely reveal the differences in individual state evolution and the corresponding changes in network couplings. Conversely, heterogeneous network models have advantages in revealing these differences. Therefore, the study of heterogeneous complex networks has great practical importance and application values.

When many individuals with different state evolution laws are coupled together to form a self-organizing and collaborative heterogeneous network, their cooperative collective behavior has great uncertainty. In the process of information transmission, a time-delay phenomenon inevitably occurs due to the influence of factors such as spatial distance, limited speed of information transmission, node competition, channel congestion and memory effect. The existence of time delay causes instability and oscillation of dynamic systems. Network nodes exchange information through multiple channels, which may be different for different nodes [8]. Therefore, studying the cooperative collective behavior of partially coupled heterogeneous networks with time-delay effects is important. Research on synchronization is widely investigated.

Synchronization refers to the process that all network nodes starting from a certain initial state reach the same state through the dynamic evolution of nodes own state, the coupling between the nodes, and the external control operations on the network. Only a few networks can achieve synchronization by adjusting system parameters, whereas most of the networks need to use control strategies to realize synchronization in accordance with the specific characteristics of the network [9]–[11]. For heterogeneous networks, the difference in node evolution makes it difficult to achieve full synchronization by virtue of static linear controllers. Some new control strategies have been proposed to realize synchronization. Reference [12] used a state-feedback control strategy to achieve synchronization by adding constraints or controllers to each node for compensating the differences between nodes. Some new synchronization concepts were introduced to replace the full synchronization.
References [13]–[15] proposed the concept of output synchronization. Cooperative control was added to achieve synchronization by using the output information of neighboring nodes. References [16] and [17] introduced the concept of quasi-synchronization that enables the network to achieve synchronization within a certain error bound.

Compared with continuous control strategies, impulsive control has obvious advantages because it only needs to design the controller in sparse time sequence, and has been successfully applied to many synchronization problems [12], [18]–[22]. In recent years, research on the synchronization of heterogeneous networks based on impulsive strategy has attracted considerable interests. He et al. analyzed the synchronization problem of master-slave heterogeneous network on the basis of distributed impulsive control [23], and presented a method to optimize the synchronization error bound and controller design. Reference [24] discussed the synchronization of heterogeneous networks with impulsive effects and coupling delays. This process compensates for the differences between nodes by adding additional constraints and control to each node.

This paper studies the synchronization of a class of heterogeneous networks. Different from the existing results that only consider time delays [12] or only consider partial coupling [8], [25], the network model in this paper simultaneously considers the effects of time delays and partial coupling on the cooperative collective behavior of heterogeneous networks. Under the action of a class of heterogeneous impulses that depend on time and node states, the heterogeneous network achieves exponential synchronization. The main results of this study are twofold. First, the exponential synchronization of the partially coupled time-delay network with heterogeneous impulses is proven by means of the comparison theorem and Lyapunov function if the nodes of the heterogeneous network have a common equilibrium state; Second, the partially coupled time-delay network with heterogeneous impulses exponentially synchronizes to the average state of the nodes if they do not have a common equilibrium state.

Notations: Throughout this paper, $\mathcal{N}$ is the set of positive integers, $I_n$ is the $n \times n$ identity matrix, $\mathbb{R}^n$ is the $n$-dimensional column vector space, $\mathbb{R}^{n \times n}$ is the set of all $n \times n$ matrices. For $A \in \mathbb{R}^{n \times n}$, let $\lambda_{\text{min}}(A)$ and $\lambda_{\text{max}}(A)$ be the minimum eigenvalue and maximum eigenvalue of $A$, respectively. The norm of $A$ is denoted by $\| A \|$, where $\| A \| = \sqrt{\lambda_{\text{max}}(A^T A)}$, and $A^T$ is the transpose of $A$. For $A \in \mathbb{R}^{n \times n}$, $A < 0$ implies $x^T A x < 0$ for any $x \in \mathbb{R}^n, x \neq 0$. For two matrices $A = (a_{ij})_{m \times n}$ and $B$, their Kronecker product $A \otimes B = (a_{ij} B)_{m \times n}$. For a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$, $Df$ denotes the Jacobian.

II. MODEL DESCRIPTIONS AND PRELIMINARIES

We consider the following dynamical networks

$$\dot{x}_i(t) = A_i x_i(t) + B_i f_i(x_i(t)) + C_i f_j(x_j(t - \tau(t))) + c \sum_{j=1,j \neq i}^{N} d_{ij} R H_j (x_j(t) - x_i(t)), \quad i = 1, \cdots, N,$$

where $x(t) = [x_1(t), \cdots, x_N(t)]^T \in \mathbb{R}^n$ is the state vector of the $i$th node at time $t$; $A_i, B_i, C_i \in \mathbb{R}^{n \times n}$ are constant matrices. $\tau(t)$ is the time-varying delay satisfying $0 \leq \tau(t) \leq \tau$; $f_i(x) = [f_{i1}(x_1), \cdots, f_{in}(x_n)]^T$ for $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n, i = 1, 2, \cdots, n$; $c > 0$ is the coupling strength; $R = \text{diag}(r_1, \cdots, r_n)(r_i > 0)$ is the inner coupling matrix; $D = (d_{ij})_{1 \leq i,j \leq N} \in \mathbb{R}^{N \times N}$ is a symmetric matrix which denotes connection weight of the dynamical networks: $d_{ij} = d_{ji} > 0$ if there is a connection from node $j$ to node $i$ ($i \neq j$), otherwise $d_{ij} = 0$; $H_j = \text{diag}(h_{ij})_j = 1, \cdots, n$. Suppose $\Gamma_{ii} = -\sum_{j=1,j \neq i}^{N} \Gamma_{ij}, 1 \leq i \leq N$. Then the network (1) can be rewritten as

$$\dot{x}_i(t) = A_i x_i(t) + B_i f_i(x_i(t)) + C_i f_j(x_j(t - \tau(t))) + c \sum_{j=1}^{N} \Gamma_{ij} x_j(t), \quad i = 1, \cdots, N.$$

We consider (1) or (2) with the following impulsive effects

$$x_i(t_k^+) = x_i(t_k^-) + \mu_i k x_i(t_k^-), \quad (3)$$

where $\mu_i k$ denotes the impulsive strength; $\{t_k\}$ is an impulsive sequence satisfying $0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{k \to \infty} t_k = \infty$; $x_i(t_k^+)$ and $x_i(t_k^-)$ denote the limit from the left and the right at time $t_k$, respectively. In this paper, we assume that the solution to (2) is right-hand continuous, then $x_i(t_k^+) = x_i(t_k), i = 1, 2, \cdots, N$ and $k \in \mathbb{N}$.

Remark 1: (2) is heterogeneous because $A_i, B_i, C_i$ vary with node state $x_i$. The impulse in (3) is also heterogeneous because $\mu_i k$ depends on impulsive time $t_k$ and node state $x_i$.

We make the following assumptions:

(A1) $f_i$ and $f_j$ satisfy the Lipschitz conditions: $\| f_i(x) - f_j(y) \| \leq l_i \| x - y \|, \quad i = 1, 2.$

(A2) The impulsive sequence $\{t_k\}$ satisfies that there exists $N_0 \in \mathbb{Z}^+$ and $T_a > 0$ such that

$$\frac{T-t}{T_a} - N_0 \leq N_0 (T, t) \leq \frac{T-t}{T_a} + N_0$$

for any $0 \leq t \leq T$, where $N_0(T, t)$ is the times of impulses in the interval $[t, T]$.

Definition 1: The upper-right Dini derivative $D^+ u(t)$ is defined as

$$D^+ u(t) = \lim_{h \to 0^+} (u(t + h) - u(t))/h.$$

Lemma 1 [9]: Suppose that $0 \leq \tau_i(t) \leq \tau, \quad i = 1, 2, \cdots, m, \quad F(t, u, u_1, u_2, \cdots, u_m) : \mathbb{R}^n \times \mathbb{R}^{m+1} \to \mathbb{R}$ is nondecreasing in $\bar{u}$ for each fixed $(t, u, u_1, \cdots, u_{i-1}, \bar{u}_{i+1}, \cdots, u_m), i = 1, 2, \cdots, m$, and $I_k(u) : \mathbb{R}^n \to \mathbb{R}$ is nondecreasing in $u, k \in \mathbb{N}^+$.

If

$$D^+ u(t) \leq F(t, u(t), u(t - \tau(t)), \cdots, u(t - \tau_m(t))), \quad t \neq t_k, \quad u(t_k^+) \leq I_k(u(t_k^-)), k \in \mathbb{N}^+,$$
and
\[
\begin{aligned}
D^+ v(t) &\geq F(t, v(t), v(t - \tau_1(t)), \cdots, v(t - \tau_m(t))), \quad t \neq t_k, \\
v(t^+_k) &\geq k (v(t^-_k)), \quad k \in \mathbb{N}^+,
\end{aligned}
\]
then \(u(t) \leq v(t)\) for \(-\tau \leq t \leq 0\) implies that \(u(t) \leq v(t)\) for \(t \geq 0\).

Lemma 2: Let \(Q \in \mathbb{R}^{n \times n}\) be a positive semi-definite matrix, then the following inequality holds
\[
2x^T Q y \leq x^T Q x + y^T Q y
\]
for \(x, y \in \mathbb{R}^n\).

III. SYNCHRONIZATION CRITERIA
This section aims to supply some synchronization criteria for the network (1). Difference is found in achieving synchronization when different synchronization targets are considered, and the network synchronization to the anticipated states often fails. Therefore, the choice of target states is extremely crucial to achieve synchronization, especially for heterogeneous networks. In this paper, we focus on two types of target synchronization states of (1). The first type is the common equilibrium solution, and the second type is the average trajectory.

A. SYNCHRONIZATION TO THE COMMON EQUILIBRIUM SOLUTION
We assume that all the isolated nodes in the network (1) or (2) have a common equilibrium solution \(s(t)\), that is \(s(t)\) satisfies
\[
\dot{s}(t) = A_i s(t) + B_i f_1(s(t)) + C_i f_2(s(t - \tau(t)),
\]
and the network (1) (or (2)) is anticipated to synchronize to \(s(t)\).

Let \(e_i(t) = x_i(t) - s(t)\), then the error system is
\[
\dot{e}_i(t) = A_i e_i(t) + B_i g_1(e_i(t)) + C_i g_2(e_i(t - \tau(t)) + c \sum_{j=1}^{n} R_{ij} e_j(t), \quad i = 1, \cdots, N,
\]
where \(g_1(e_i(t)) = f_1(x_i(t)) - f_1(s(t)), g_2(e_i(t - \tau(t)) = f_2(x_i(t - \tau(t))) - f_2(s(t - \tau(t))).

Considering the effect of heterogeneous impulses (3), we obtain the following network:
\[
\begin{aligned}
\dot{e}_i(t) &= A_i e_i(t) + B_i g_1(e_i(t)) + C_i g_2(e_i(t - \tau(t)) + c \sum_{j=1}^{n} R_{ij} e_j(t), \\
&\quad t \neq t_k, k \in \mathbb{N}^+ \\
\end{aligned}
\]
and
\[
e_i(t^+_k) = e_i(t^-_k) + \mu_i e_i(t^-_k).
\]

The network (6) subject to the initial condition can be rewritten as
\[
\begin{aligned}
\dot{e}(t) &= A e(t) + B G_1(e(t)) + C G_2(e(t - \tau(t)) + H e(t), \\
&\quad t \neq t_k, k \in \mathbb{N}^+ \\
e(t^+_k) &= U_k e(t^-_k), \\
e(t) &= \Phi(t), \quad t \in [-\tau, 0],
\end{aligned}
\]
where \(e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t)), A = \text{diag}[A_1, A_2, \cdots, A_N], B = \text{diag}[B_1, B_2, \cdots, B_N], C = \text{diag}[C_1, C_2, \cdots, C_N], G_1(e(t)) = (g_1^T(e_1(t)), \cdots, g_N^T(e_N(t))), G_2(e(t - \tau(t)) = (g_1^T(e_1(t - \tau(t))), \cdots, g_N^T(e_N(t - \tau(t))))^T, \Gamma = (\Gamma_i)_{i \times N}, H = c(I_N \otimes R)\Gamma_k, U_k = \text{diag}[1 + \mu_1, 1 + \mu_2, \cdots, 1 + \mu_N] \otimes I_n, \Phi(t) \in \text{PRC}([-\tau, 0], \mathbb{R}^n) = \{\varphi: [-\tau, 0) \rightarrow \mathbb{R}^n | \varphi \text{ is piecewise right continuous}, \text{endowed with norm } \|\cdot\|_{\tau}: \|\varphi\|_{\tau} = \sup_{-\tau \leq t \leq 0} |\varphi(t)|\}

Theorem 1: Suppose that (A1) and (A2) hold, there exist a positive definite matrix-valued function \(P(t)\) and positive numbers \(\bar{\lambda}, \alpha, \beta\) and \(\mu \in (0, 1)\) such that
\[
\beta + \mu N_{0n} + \frac{\ln \left(\frac{\mu}{T_0}\right)}{T_0} < 0.
\]
For \(t \in [t_k, t_{k+1}], k \in \mathcal{N}\),
\[
P(t) \leq \bar{\lambda} I_{nN},
\]
\[
P(t)A + A^T P(t) + P(t) H + H^T P(t) + (2 - \alpha) P(t) + \bar{\lambda} I_2^T \|B\|^2 I_{nN} + \bar{P}(t) \leq 0,
\]
and
\[
\tilde{\lambda} I_2^T \|C\|^2 I_{nN} - \beta P(t - \tau(t)) \leq 0.
\]
For \(t = t_k, k \in \mathcal{N}\),
\[
U_k^T P(t_k) U_k < \mu P(t_{k-1}),
\]
then (1) exponentially synchronizes to \(s(t)\).

Proof: Consider the following Lyapunov function
\[
V(t) = e^T(t) P(t) e(t).
\]
Taking the derivative of \(V(t)\) along the trajectories of (7), we obtain
\[
D^+ V(t) = e^T(t) [A^T P(t) + P(t) A + H^T P(t) + P(t) H^T] e(t) + 2 \tilde{\lambda} I_2^T \|B\|^2 I_{nN} e(t) + e^T(t) P(t) G_1(e(t))
\]
\[
+ e^T(t) P(t) G_2(e(t - \tau(t))) \leq e^T(t) P(t) e(t) + e^T(t) P(t) B G_1(e(t))
\]
\[
+ e^T(t) P(t) B G_2(e(t - \tau(t))) \leq e^T(t) P(t) e(t) + \tilde{\lambda} I_2^T \|B\|^2 I_{nN} e(t) + e^T(t) P(t) e(t) + \tilde{\lambda} I_2^T \|C\|^2 e^T(t - \tau(t)) e(t).
\]
By Lemma 2 and (A1),
\[
G_1^T(e(t)) B^T P(t) e(t) + e^T(t) P(t) B G_1(e(t)) \leq e^T(t) P(t) e(t) + \tilde{\lambda} I_2^T \|B\|^2 I_{nN} e(t) + e^T(t) P(t) e(t)
\]
\[
G_2^T(e(t - \tau(t))) C^T P(t) e(t) + e^T(t) P(t) C G_2(e(t - \tau(t))) \leq e^T(t) P(t) e(t) + \tilde{\lambda} I_2^T \|C\|^2 e^T(t - \tau(t)) e(t).
\]
Then, we obtain
\[
D^+ V(t) = e^T(t) [P(t) A + A^T P(t) + P(t) H + H^T P(t) + (2 - \alpha) P(t) + \tilde{\lambda} I_2^T \|B\|^2 I_{nN} + \bar{P}(t)] e(t)
\]
from solving the linear system (14) has a solution of the form

\[
    \begin{align*}
    \phi(t) &= \alpha t + \beta v(t), \\
    \psi(t) &= \mu t + \gamma \mu (t), \\
    \theta(t) &= \eta t + \zeta \eta \tau(t), \\
    \end{align*}
\]

Therefore, for any \(m\)

\[
    \begin{align*}
    \mu_s^{m+k} &= \mu_s^{m-k} = \mu_s^m \mu_s^{-k}, \\
    \end{align*}
\]

where \(\mu_s^{m-k} \leq \mu_s^{m} \mu_s^{-k} \leq \mu_s^m\).

By (A2), for \(t \in [t_k, t_{k+1}]\) and \(s \in [t_k, t_{k+1}], 0 \leq k \leq m, \)

\[
    t - s - N_0 \leq m - k + 1,
\]

and then

\[
    \begin{align*}
    \mu_s^{m-k} &\leq \mu_s^{m} \mu_s^{-k} - N_0 - 1 = \frac{1}{\mu_0^{N_0+1}} e^{\frac{ln \mu}{\mu_0} (t-s)}, \\
    \mu_s^m &\leq \frac{1}{\mu_0^{N_0+1}} e^{\frac{ln \mu}{\mu_0} t}. \\
    \end{align*}
\]

Let \(\eta = - (\alpha + \ln \mu_{+a}), \xi = \frac{\lambda_{-a}}{\mu_0^{N_0+1}} \sup_{-a} t, \Phi(t)^T \Phi(t), \mu_0 = \mu_0^{N_0+1} \). Substituting (17) and (18) into (17), we obtain

\[
    \begin{align*}
    (t) < \xi [\xi e^{\eta t} + \int_0^t e^{\eta (t-s)} \Phi(t)(s) + e]ds. \\
    \end{align*}
\]

Therefore, there exists a unique \(\lambda\) such that \(\Phi(\lambda) = 0\), which derives

\[
    \begin{align*}
    \beta e^{\lambda t} = \mu_0 (\eta - \lambda). \\
    \end{align*}
\]

Since, \(\mu_0 \eta - \beta > 0\), we have

\[
    \begin{align*}
    \phi(t) \leq \xi [\xi e^{\eta t} + \frac{\epsilon}{\mu_0 \eta - \beta}], \quad t \in [-\tau, 0]. \\
    \end{align*}
\]

We claim that (21) holds for all \(t \geq 0\).

If not, then there exists a \(t^* > 0\) such that

\[
    \begin{align*}
    \phi(t^*) > \xi [\xi e^{\eta t^*} + \frac{\epsilon}{\mu_0 \eta - \beta}], \\
    \end{align*}
\]

and

\[
    \begin{align*}
    \phi(t) < \xi [\xi e^{\eta t} + \frac{\epsilon}{\mu_0 \eta - \beta}], \quad t < t^*. \\
    \end{align*}
\]

By (19) and (23), we obtain

\[
    \begin{align*}
    \phi(t) < \xi [\xi e^{\eta t} + \int_0^{t^*} e^{\eta (t-s)} \phi(s) + e]ds \\
    < e^{\eta t^*} [\xi + \frac{\epsilon}{\mu_0 \eta - \beta} + \frac{\beta e^{\lambda t^*}}{\mu_0} \int_0^{t^*} e^{\lambda (t-s)}ds \\
    + \frac{\eta \epsilon}{\mu_0 \eta - \beta} \int_0^{t^*} e^{\lambda ds}] \\
    = e^{\eta t^*} [\xi + \frac{\epsilon}{\mu_0 \eta - \beta} + \frac{\beta e^{\lambda t^*}}{\mu_0 (\eta - \lambda)} (e^{\lambda t^*} - 1) \\
    + \frac{\eta \epsilon}{\mu_0 \eta - \beta} (e^{\lambda t^*} - 1)] \\
    = e^{\eta t^*} [\xi e^{\eta t^*} e^{-\lambda t^*} + \frac{\epsilon}{\mu_0 \eta - \beta} e^{\eta t^*}]. \\
    \end{align*}
\]

This contradicts (22), and then (21) holds for all \(t \geq 0\).

Letting \(\epsilon \to 0\), we obtain from (15) and (21) that

\[
    V(t) \leq \phi(t^*) \leq \xi e^{-\lambda t^*}.
\]

Therefore, (1) exponentially synchronizes to the common equilibrium solution.
Remark 2: The positive definite matrix $P(t)$ is used to design the Lyapunov function $V(t)$. In [24], a linear matrix-valued function $P(t)$ connecting two fixed matrices in each impulsive interval is designed to construct the Lyapunov function. Compared with the synchronization criteria in [24], Theorem 1 in this work has a wider application.

In Theorem 1, if $P(t)$ is constant $\bar{P}$, then we can easily obtain the following result.

Corollary 1: Suppose that $(A_1)$ and $(A_2)$ hold, there exist a positive definite matrix $P$ and positive numbers $\hat{\lambda}, \alpha, \beta$ and $\mu \in (0, 1)$ such that $(9)$ and

$$\frac{\hat{\lambda}}{\beta} \nu^2 \|C\|^2 \leq \lambda_{\text{min}}(P) \leq \lambda_{\text{max}}(P) \leq \hat{\lambda},$$

(24)

and

$$U_k^T P U_k < \mu P,$$

(26)

then (1) exponentially synchronizes to $s(t)$.

Remark 3: In Corollary 1, inequality $U_k^T P U_k < \mu P$ implies that the impulses in $(1)$ are synchronizing up to $|1 + \mu_k| < 1$. Theorem 1 can be applied to investigate the synchronization problem of dynamical networks when the impulses are synchronizing, desynchronizing ($|1 + \mu_k| > 1$) or inactive ($|1 + \mu_k| = 1$). If the impulses are inactive or desynchronizing, then $P(t)$ is used to offset the negative impulsive effects, as shown in (12).

Remark 4: The impulsive intervals are usually assumed to be bounded when the synchronization problem with impulsive effects is considered, especially with inactive or desynchronizing impulsive effects. This assumption limits the frequency of these negative impulses in a fixed time period. This work uses the average impulsive interval to limit the impulsive interval bounds. Thus, the results in this work are less conservative.

**B. SYNCHRONIZATION TO THE AVERAGE TRAJECTORY**

In this subsection, we consider the following average state

$$s(t) = \frac{1}{N} \sum_{k=1}^{N} x_k(t).$$

Let $e_i(t) = x_i(t) - s(t)$. Obviously, $\sum_{i=1}^{N} e_i(t) = 0$.

Since

$$\dot{s}(t) = \frac{1}{N} \sum_{k=1}^{N} [A_k x_k(t) + B_k f_1(x_k(t)) + C_k f_2(x_k(t) - s(t))] + c \sum_{j=1}^{N} RT_{ij} e_j(t),$$

we obtain

$$\dot{e}(t) = A(s(t) + e(t)) + B f_1(s(t) + e(t))$$

$$+ C f_2(s(t) - s(t)) + e(t) + (t - \tau(t))) + c \sum_{j=1}^{N} RT_{ij} e_j(t)$$

$$- \frac{1}{N} \sum_{k=1}^{N} [A_k (s(t) + e_k(t)) + B_k f_1(s(t) + e_k(t))$$

$$+ C_k f_2(s(t) - s(t)) + e_k(t) - \tau(t)))]$$

$$= A(s(t) + e(t))$$

$$+ B (\int_{0}^{1} df_1(s(t) + e_k(t))dt + e(t))$$

$$+ C (\int_{0}^{1} df_2(s(t) - s(t)) + e(t))$$

$$+ c \sum_{j=1}^{N} R_{ij} e_j(t) - \frac{1}{N} \sum_{k=1}^{N} A_k (s(t) + e_k(t))$$

$$- \frac{1}{N} \sum_{k=1}^{N} B_k (\int_{0}^{1} df_1(s(t) + e_k(t))dt + e(t))$$

$$- \frac{1}{N} \sum_{k=1}^{N} C_k (\int_{0}^{1} df_2(s(t) - s(t)) + e(t))$$

Then

$$\dot{e}(t) = He(t) + \text{diag}(A_1 + B_1 \int_{0}^{1} df_1(s(t) + e(t))dt,$$

$$\cdots, A_N + B_N \int_{0}^{1} df_1(s(t) + e_N(t))dt)e(t)$$

$$+ \text{diag}(C_1 \int_{0}^{1} df_2(s(t) - s(t)) + e(t))dt,$$

$$\cdots, C_N \int_{0}^{1} df_2(s(t) - s(t)) + e(t))dt)e(t - \tau(t))$$

$$- \frac{1}{N} \left[ \begin{array}{cccc} \wedge_1, & \cdots, & \wedge_N \\ \wedge_1, & \cdots, & \wedge_N \\ \wedge_1, & \cdots, & \wedge_N \end{array} \right] e(t)$$

$$- \frac{1}{N} \left[ \begin{array}{cccc} \vee_1, & \cdots, & \vee_N \\ \vee_1, & \cdots, & \vee_N \\ \vee_1, & \cdots, & \vee_N \end{array} \right] e(t - \tau(t))$$

$$+ \left[ \begin{array}{c} \Delta_1 \\ \vdots \\ \Delta_N \end{array} \right]$$

$$+ \left[ \begin{array}{c} \vee_1 \\ \vdots \\ \vee_N \end{array} \right]$$
where
\[
\begin{align*}
\Delta_t &= (A_i - \frac{1}{N} \sum_{k=1}^{N} A_k) s(t) + (B_i - \frac{1}{N} \sum_{k=1}^{N} B_k) f_i(s(t)), \\
\nabla_i &= (C_i - \frac{1}{N} \sum_{k=1}^{N} C_k) f_i(s(t)),
\end{align*}
\]
i = 1, 2, \ldots, N.

Let
\[
\begin{align*}
\Theta(t) &= \text{diag}[A_1 + B_1 \int_0^t D f_1(s(t) + \omega e_1(t)) d\omega, \\
&\quad \ldots, A_N + B_N \int_0^t D f_1(s(t) + \omega e_N(t)) d\omega], \\
\Xi(t - \tau(t)) &= \text{diag}[C_1 \int_0^t D f_2(s(t) - \tau(t)) + \omega e_1(t - \tau(t))) d\omega, \\
&\quad \ldots, C_N \int_0^t D f_2(s(t) - \tau(t)) + \omega e_N(t - \tau(t))) d\omega], \\
\Pi(t) &= \frac{1}{N} \begin{bmatrix}
\Lambda_1 & \cdots & \Lambda_N \\
\Lambda_1 & \cdots & \Lambda_N \\
\cdots & \cdots & \cdots \\
\Lambda_1 & \cdots & \Lambda_N
\end{bmatrix}, \\
\Upsilon(t - \tau(t)) &= \frac{1}{N} \begin{bmatrix}
\nabla_1 & \cdots & \nabla_N \\
\nabla_1 & \cdots & \nabla_N \\
\cdots & \cdots & \cdots \\
\nabla_1 & \cdots & \nabla_N
\end{bmatrix}, \\
\Psi(t) &= \begin{bmatrix}
\Delta_1 \\
\cdots \\
\Delta_N
\end{bmatrix}, \\
\Omega(t - \tau(t)) &= \begin{bmatrix}
\nabla_1 \\
\cdots \\
\nabla_N
\end{bmatrix}.
\end{align*}
\]

then the error system with initial condition reads as
\[
\begin{cases}
ed_i(t) = H e_i(t) + \Theta_i(t) e_i(t) + \Xi_i(t - \tau_i(t)) e_i(t - \tau_i(t)) - \Pi_i(t) e_i(t) - \Upsilon_i(t - \tau_i(t)) e_i(t - \tau_i(t)) + \Psi_i(t) + \Omega_i(t - \tau_i(t)), \\
ed_i^+(t) = U_i e_i(t_i).
\end{cases}
\]

**Theorem 2:** Suppose that (A_1) and (A_2) hold, there exists a positive definite matrix \( P(t) \), positive numbers \( \lambda, \alpha, \beta \) and \( \mu \in (0, 1) \) such that
\[
2\beta + \mu N_0 \lambda + \frac{\ln \mu}{T_a} < 0,
\]
and for \( t \in [t_k, t_{k+1}) \), \( k \in \mathcal{N} \),
\[
P(t) \leq \lambda I_{nN},
\]
\[
H^T P(t) + P(t) H + \Theta^T(t) P(t) + P(t) \Theta(t)
\]
\[
-\Pi^T(t) P(t) - P(t) \Pi(t) + (4 - \alpha) P(t) + \dot{P}(t) \leq 0,
\]
\[
\Xi^T(t - \tau(t)) P(t) \Xi(t - \tau(t)) + \Upsilon^T(t - \tau(t)) P(t) \Upsilon(t - \tau(t))
\]
\[
\leq \beta \tilde{P}(t - \tau(t)).
\]
Then, we obtain
\[ D^+ V(t) \leq \alpha V(t) + \beta V(t - \tau(t)) + \beta \xi e^{\lambda(t)} . \] (31)

Obviously,
\[ V(t_k) < \mu V(t_k^-). \] (32)

Similar to the proof of Theorem 1, we consider the following modified comparing system
\[
\begin{cases}
\dot{v}(t) = \alpha v(t) + \beta v(t - \tau(t)) + \beta \xi e^{\lambda(t)} + \epsilon, \\
 v(t_k^+) = \mu v(t_k^-), \\
 v(t) = V(t), \\
t \neq t_k, \quad k \in N^+, \\
t \in [-\tau, 0],
\end{cases}
\] (33)

By Lemma 1,
\[ V(t) \leq v(t) \quad \text{for} \quad t \geq 0. \] (34)

We use the method of variation of parameters to estimate the solution of (33), and suppose that the solution of (33) has the form of
\[ v(t) = C(t)\mu^m e^{at}, \quad t \in [t_m, t_{m+1}], m \in N. \]

Substituting \( v(t) \) into (33), we obtain, for any \( m \in N, \quad t \in [t_m, t_{m+1}] \)
\[ C(t) = C(0) + \int_0^t [\beta v(s - \tau(s)) + \beta \xi e^{\lambda(t-s)} + \epsilon] e^{-as} \mu_s^{-k} ds, \]
where \( \mu_s^{-k} = \mu^{-k} \) when \( s \in [t_k, t_{k+1}], 0 \leq k \leq m. \)

Then, using the same estimation in the proof of Theorem 1, we have, for any \( m \in N \) and \( t \in [t_m, t_{m+1}], \)
\[ v(t) = C(0)\mu^m e^{at} + \int_0^t [\beta v(s - \tau(s)) + \beta \xi e^{\lambda(t-s)} + \epsilon] e^{at} \mu_s^{-k} ds, \]
\[ < \xi e^{-\eta t} \]
\[ + \int_0^t \frac{e^{-\eta(t-s)}}{\mu_0} [\beta v(s - \tau(s)) + \beta \xi e^{\lambda(t-s)} + \epsilon] ds, \]
where \( \mu_s^{-k} = \mu^{-k} \) when \( s \in [t_k, t_{k+1}], 0 \leq k \leq m, \mu_0 = \mu_{N^0+1} \) and \( \xi, \eta \) and \( \lambda \) are defined as in Theorem 2.

We show that
\[ v(t) < \xi e^{-\lambda t} + \frac{\epsilon}{\mu_0 \eta - 2\beta}, \] (35)
holds for all \( t \geq 0. \)

If not, there exists a \( t^* > 0 \) such that
\[ v(t^*) \geq \xi e^{-\lambda t^*} + \frac{\epsilon}{\mu_0 \eta - 2\beta}, \] (36)
and
\[ v(t) < \xi e^{-\lambda t} + \frac{\epsilon}{\mu_0 \eta - 2\beta}, \quad t < t^*. \] (37)

By (35) and (37), we obtain
\[ v(t^*) < \xi e^{-\eta t^*} + \int_0^{t^*} \frac{e^{-\eta(t^*-s)}}{\mu_0} [\beta v(s - \tau(s)) + \beta \xi e^{\lambda(t-s)} + \epsilon] ds \]
which is contradictory to (1), and the proof is completed.

**Remark 5:** An error system is usually nonstandard when a heterogeneous network is considered to synchronize a non-common equilibrium point. Thus, investigating the full synchronization for heterogeneous network is difficult. The quasi-synchronization is raised, thereby allowing all node states tend to a manifold rather than a fixed point [23]. This work transforms the error system into a standard form when the average trajectory is selected as the target state and gives a full synchronization criterion.

**Remark 6:** Linear matrix inequality (LMI) is unsuitable to solve (32) and (33) due to the presence of \( \int_0^t D_j(s + \omega_e) d\omega \) and \( \int_0^t D_f(s(t - \tau(t)) + \omega_e(t - \tau(t))) d\omega \). However, finding solutions to (32) and (33) is possible in some special cases. The details can be found in [26].

**Remark 7:** The time-dependent Lyapunov function plays an important role when heterogeneous impulses are considered. Reference [24] investigated the exponential synchronization of time-delay homogeneous networks with heterogeneous impulses by virtue of Lyapunov function and the comparison principle. Different from [24], here we consider time-delay heterogeneous networks with heterogeneous impulses and deduce a synchronization criterion such that all nodes synchronize to the average trajectory. Reference [26] used matrix decomposition techniques to synchronize a heterogeneous network to the average trajectory. However, this method is unsuitable for time-delay heterogeneous networks.

In this work, we use the Lyapunov function combined with the comparison principle to investigate the exponential synchronization for a class of time-delay heterogeneous networks with heterogeneous impulses.

**Remark 8:** We need carefully analyze the decay rate of Lyapunov function over time when we consider exponential synchronization of networks. Estimating the Lyapunov function is difficult when time-delays and impulses are simultaneously considered. The reason lies in that time-delays and impulsive intervals are usually mixed up. The comparison theorem supplies us a method of applying ODE theory to investigate synchronization of networks. After obtaining some preliminary estimation to the Lyapunov function, comparison theorem allows us to consider an ODE. Then we can
use the method of variation of parameters to investigate the
effects of time-delays and impulses, and obtain an accurate
estimation of the Lyapunov function. In recent years, Lyapunov
function combined with the comparison principle has been widely applied in impulsive systems [9]–[11], [21], [27].

IV. NUMERICAL EXAMPLES
In this section, a numerical example is presented to illustrate
the effectiveness of the theoretical results.

The common equilibrium solution \( s(t) \) satisfies the
following system described by

\[
\dot{s}(t) = A_1 s(t) + B_1 f_1(s(t)) + C_1 f_2(s(t - \tau(t)))
\]

where \( s(t) = (s_1(t), s_2(t), s_3(t))^T \), \( A_1 = -0.1 I_3 \), \( B_1 = \\
\begin{bmatrix}
0.09 & 0 & -0.1 \\
0 & 0.02 & 0 \\
-0.06 & 0 & -0.1
\end{bmatrix} \), \( C_1 = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.1 & 0 & -0.2 \\ -0.5 & 0 & -0.1 \end{bmatrix} \), \( A_2 = \begin{bmatrix} 0.6A_1 & B_2 & C_2 \\ 0.8B_1 & C_1 & A_3 \end{bmatrix} \), \( A_3 = 0.9B_1 \), \( C_3 = 0.8C_1 \), \( A_4 = 0.5A_1 \), \( B_4 = 0.7B_1 \), \( C_4 = 0.7C_1 \), \( A_5 = 0.4A_1 \), \( B_5 = 0.4B_1 \), \( C_5 = 0.6C_1 \), \( A_6 = 0.3A_1 \), \( B_6 = 0.5B_1 \), \( C_6 = 0.5C_1 \), \( f_1(s(t)) = (0.5(|s_1(t)| - |s_1(t)| - 1), 0, 0)^T \), and

\( f_2(s(t - \tau(t))) = (0.5(|s_1(t - \tau(t))| + 1 - |s_1(t - \tau(t)) - 1|), 0, 0)^T \). Then, Lipschitz constants can be got \( l_1 = l_2 = 1 \).

Referring to the topology of the heterogeneous partial
couple network in the work of Lu et al. [28], the network is
described by

\[
\dot{x}_i(t) = A_i x_i(t) + B_i f_1(x_i(t)) + C_i f_2(x_i(t - \tau(t)))
\]

where \( x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T \), \( c = 1 \), \( R = 0.02 I_3 \),

\[
D = \begin{bmatrix}
-11.3 & 7 & 0 & 0.2 & 0 & 4.1 \\
4.2 & -11.2 & 7 & 0 & 0 & 0 \\
0 & 4.1 & -11.1 & 7 & 0 & 0 \\
0 & 0.1 & 4.1 & -11.2 & 7 & 0 \\
0 & 0 & 0 & 3.9 & -10.9 & 7 \\
7 & 0 & 0 & 0 & 4 & -11
\end{bmatrix}
\]

The channel matrix of \( H_{ij} \) can be seen in the work [28].

Let \( N_0 = 2, 0 < \tau(t) < \tau, \tau = 0.25 \), according to Corollary 1, by solving the linear matrix inequalities (9), (27), and (28), we can find a feasible solution with \( T_1 = 0.01 \), \( \alpha = 11.2, \beta = 1.55, \mu = 0.9, P = 0.11I_{18} \).

Consequently, by Corollary 1, we can obtain exponential
synchronization of the heterogeneous systems. Fig.1, Fig.2 and Fig.3 depict the trajectory of the error state

\( e_i(t) = x_i(t) - s(t), i = 1, 2, 3, 4, 5, 6 \).

V. CONCLUSION
In this paper, we investigate the synchronization for a class of
partially coupled heterogeneous impulsive networks with
time-delays, and obtain some sufficient conditions to realize
exponential synchronization. The main results are divided
into two parts, where the networks are anticipated to syn-
chronize to the common equilibrium solution and the average
trajectory. The method is based on Lyapunov function and the
comparison principle. The results serve as a useful supple-
ment to the full synchronization of heterogeneous impulsive
networks with time-delays. The heterogeneity of networks
makes the form of the error system unstandard if the nodes
have no common equilibrium point. Fortunately, we obtain some standard error system by linearization when we select the average trajectory as the synchronization target. The conditions in Theorem 2 are difficult to solve by using LMI. Thus, we will modify or simplify these conditions in future works.

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GUIZHEN FENG received the M.S. degree in applied mathematics from Southeast University, Nanjing, China, in 2002, and the Ph.D. degree from the University of Southern California, in 2015. Her research interests include stability theory, synchronization of complex dynamical networks, and consensus in multi-agent systems.

JINDE CAO (Fellow, IEEE) received the B.S. degree from Anhui Normal University, Wuhu, China, in 1986, the M.S. degree from Yunnan University, Kunming, China, in 1989, and the Ph.D. degree from Sichuan University, Chengdu, China, in 1998, all in mathematics/applied mathematics. He is currently an Endowed Chair Professor and the Dean of the School of Mathematics, and the Director of the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence of China and the Research Center for Complex Systems and Network Sciences, Southeast University.

Dr. Cao is elected as a member of the Academy of Europe and the European Academy of Sciences and Arts, a Foreign Member of the Russian Academy of Engineering, the Russian Academy of Natural Sciences, and the Lithuanian Academy of Sciences, a Fellow of the Pakistan Academy of Sciences, a Fellow of the African Academy of Sciences, an IASCYS academican, and a Full Member of the Sigma Xi. He was a recipient of the National Innovation Award of China, Obada Prize, and the Highly Cited Researcher Award in Engineering, Computer Science, and Mathematics by Thomson Reuters/Clarivate Analytics.
JIAN DING received the B.S. degree in mathematics from Qufu Normal University, Qufu, China, in 1998, and the M.S. and Ph.D. degrees in mathematics from Southeast University, Nanjing, China, in 2002 and 2010, respectively. His research interests include stability theory, Hamiltonian systems, and variational method.

YISHU WANG received the B.S. degree in mathematics from Xiangtan University, Xiangtan, China, in 2014, and the M.S. and Ph.D. degrees from the Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China, in 2016 and 2019, respectively. His research interests include numerical solution of partial differential equations, finite element methods, and domain decomposition methods.

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