ONE SIDE CONTINUITY OF MEROMORPHIC MAPPINGS
BETWEEN REAL ANALYTIC HYPERSURFACES

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Abstract. We prove that a meromorphic mapping, which sends a piece of a real analytic strictly pseudoconvex hypersurface in $\mathbb{C}^2$ to a compact subset of $\mathbb{C}^N$ which does not contain germs of non-constant complex curves is continuous from the concave side of the hypersurface. This implies the analytic continuability along CR-paths of germs of holomorphic mappings from real analytic hypersurfaces with non-vanishing Levi form to the locally spherical ones in all dimensions.

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1. Introduction

1.1. Statement of the result. Let $0 \in M \subset U$ be a real analytic strictly pseudoconvex hypersurface in a neighborhood $U$ of zero of $\mathbb{C}^2$ defined as $M = \{z : Q(z, \bar{z}) = 0\}$, where $Q$ is a real analytic function in $U$ with the non-vanishing gradient. In an appropriate local coordinates we can suppose that $Q(z, \bar{z}) = y_2 - |z_1|^2 - r(z_1, \bar{z}_1, x_2)$, where $r$ is of order $\geq 3$ at zero, see (2.3) below. Set

$$U^+ = \{z \in U : y_2 > |z_1|^2 + r(z_1, \bar{z}_1, x_2)\}$$

and call this open set the concave side of $M$.

Let furthermore $K \subset U'$ be a compact in a complex manifold $U'$. Our goal in this paper is to prove the following

Theorem 1. Let $M$ and $K$ be as above and suppose, in addition, that $K$ does not contain germs of non-constant complex curves. Let $f : U \rightarrow U'$ be a meromorphic mapping such that $f|M[M] \subset K$. Then $f$ is continuous on $U^+$.

A meromorphic mapping $f : U \rightarrow U'$ between complex manifolds $U$ and $U'$ is defined by its graph $\Gamma_f$, which is a locally irreducible analytic subset of $U \times U'$ such that the natural projection $\pi : \Gamma_f \rightarrow U$ is proper and generically one to one. If $\pi : \Gamma_f \rightarrow U$ is
(proper and) generically d to 1 then f is called a d-valued meromorphic correspondence. In the case of \( U' = \mathbb{P}^N \) a meromorphic map f is defined by a couple f = \((f_1, \ldots, f_N)\) where \( f_j \) are meromorphic functions, see Section 2 for more details. f\(_{|M|}\) is by definition the closure of the set \( \{ f(z) : z \in M \setminus I_f \} \), where \( I_f \) denotes the set of points of indeterminacy of f. Therefore in our case condition f\(_{|M|}\) \( \subset K \) means that for every \( z \in M \setminus I_f \) one has that \( f(z) \in K \).

Notice that in the case \( K \subseteq \mathbb{C}^N \subseteq \mathbb{P}^N \) condition f\(_{|M|}\) \( \subset K \) easily (by maximum principle) implies that \( f(\overline{U^+}) \subset \hat{K} \), where \( \hat{K} \) is the polynomial hull of \( K \), i.e., that f is bounded from the concave side \( \overline{U^+} \) of \( M \). But we claim more: that f is continuous from this side up to \( M \).

**Remark 1.** A good example of a compact without germs of complex curves is a compact real analytic hypersurface in \( \mathbb{C}^d \), see [DE].

### 1.2. Applications

Let us explain the interest in such a theorem. Recall the following result of Pinchuk, see Theorem 6.2 of [P]. Every germ of a holomorphic mapping from a real analytic hypersurface \( M \subset \mathbb{C}^n \) to the unit sphere \( S^{2n-1} \subset \mathbb{C}^n \) analytically extends along any CR-path in \( M \). A CR-path in \( M \) is a path \( \gamma : [0,1] \to M \) such that \( \dot{\gamma}(t) \in T_{\gamma(t)}^* M \) for every \( t \in [0,1] \). The proof of this theorem in [P] consist of two steps. First, one proves that f extends *meromorphically* along \( \gamma \), see Lemma 6.7 there. Then one proves the holomorphicity of the extended map, see Lemma 6.6 in [P]. The proof of both lemmas in [P] crucially uses the assumption that \( M' = S^{2n-1} \) and does not hold already for a general locally spherical hypersurface on the place of \( S^{2n-1} \).

**Remark 2.** It is claimed in [SV] that a germ of a holomorphic mapping f : \((M,x) \to (M',x')\) of a strictly pseudoconvex hypersurface \( M \subset \mathbb{C}^n \) to a compact, real algebraic, strictly pseudoconvex hypersurface \( M' \subset \mathbb{C}^d \) analytically extends along any CR-path in \( M \). Unfortunately the proof of [SV] contains a serious gap in Lemma 3.4. Example of [IM] is actually a counterexample to this proof. However a careful inspection of [SV] yields a *meromorphic* extension of f. We attach an Appendix to our paper in order to help the interested reader to see that the proof of [SV] gives the following: *in the conditions as above f meromorphically extends along any CR-path in M starting at x′. See Theorem 6.2 in the Appendix.*

So, following the logic of [P] we are interested in proving that the extended meromorphic map is actually holomorphic, i.e., we want to have an analog of Lemma 6.6 from [P] in a possibly more general case. We can do that for locally spherical \( M' \)-s. Namely, combining our theorem with the result of Pinchuk we get the following

**Corollary 1.** Let \((M,x)\) be a germ of a real analytic s.p.c. hypersurface in \( \mathbb{C}^n \), \( n \geq 2 \) and \( M' \subset \mathbb{C}^d \) a compact locally spherical hypersurface. Let f : \((M,x) \to (M',x')\) be a germ of a meromorphic mapping. Then f is holomorphic.

Abbreviation s.p.c. stands for strictly pseudoconvex. Indeed, let \( F : \tilde{M}' \to S^{2n-1} \) be the development map constructed in [BS]. Here \( \pi : \tilde{M}' \to M' \) stands for the universal cover of \( M' \). Using our theorem (after the reduction to the dimension two), we can localize the problem from the concave side of \( M \) and then apply the result of Pinchuk to \( F \circ \pi^{-1} \circ f \). This gives us the holomorphicity of f.

**Remark** that we do not need algebraicity of \( M' \) here. Combining this result with already mentioned extract from [SV] we get the following
Corollary 2. Let \((M, x)\) be a germ of a real analytic hypersurface s.p.c. in \(\mathbb{C}^n\), \(n \geq 2\) and \(M' \subset \mathbb{C}^{n'}\) a compact real algebraic locally spherical hypersurface. Let \(f : (M, x) \to (M', x')\) be a germ of a holomorphic mapping. Then \(f\) analytically extends along every CR-path in \(M\).

Since the distribution of complex tangents on a strictly pseudoconvex hypersurface is contact one can draw a CR-path between any two points in \(M\), see [G]. Therefore we obtain one more corollary.

Corollary 3. Suppose that hypersurface \(M \subset \mathbb{C}^2\) is compact real analytic s.p.c. and simply connected. Let \(f : (M, x) \to (M', x')\) be a germ of a non-constant holomorphic map into a compact real algebraic s.p.c. hypersurface \(M' \subset \mathbb{C}^{n'}\). Then \(f\) extends to a proper holomorphic mapping \(F : D \to D'\) between domains bounded by \(M\) and \(M'\) respectively. Moreover \(F\) is continuous up to the boundary.

We do not know whether \(F\) is holomorphic in a neighborhood of \(\bar{D}\) in the non-spherical case, this is an open question. But \(F\) is meromorphic in a neighborhood of \(\bar{D}\).

Remark 3. 1) Algebraicity of \(M'\) in Corollaries 1, 2 is needed already for the meromorphic extension of \(f\), see example in [BS].

2) When \(M'\) is just a compact without the germs of complex curves as in our theorem one cannot hope to make \(f\) continuous also from the convex side of \(M\) (this would imply that \(f\) is actually holomorphic). A counterexample was given in [IM].

3) The result of Lemma 6.6 from [P] was later reproved in [C] in a slightly more general case of \(M'\) of the form

\[ M' = \{ z' \in \mathbb{C}^{n'} : \sum_{j=1}^{n'} |z_j|^{2m_j} < 1 \}, \]  

where each \(m_j\) is a positive integer. But let us remark that mapping \(\Phi : (z_1, \ldots, z_{n'}) \to (z_1^{m_1}, \ldots, z_{n'}^{m_{n'}})\) sends this \(M'\) to \(S^{2n'-1}\) and \(\Phi\) is proper. Therefore the result of [C] follows from that of [P].

1.3. Case of meromorphic correspondences. Now suppose that in the conditions of Theorem 1 our \(f\) is a \(d\)-valued meromorphic correspondence with values in a complex manifold \(U'\). When saying that \(f\) on \(M\) takes its values in \(K \subset U'\) we mean that for every \(m \in M \setminus I_f\) all values of \(f\) at \(m\) are contained in \(K\). By saying that \(f\) is continuous on \(U'\) we mean that the restriction \(f|_{\Gamma f \cap U'} : \Gamma_f \cap U' \to U'\) is finite to one everywhere.

Corollary 4. Let \(M\) and \(K\) be as in Theorem 1 and let \(f : (M, 0) \to U'\) be a germ of a meromorphic correspondence such that \(f|_{M[M]} \subset K\). Then \(f\) is continuous on \(U'\).

The proof uses the same ingredients as that of Theorem 1 plus a simple observation that \(d\)-valued meromorphic correspondence from \(U\) to \(U'\) can be viewed as a meromorphic mapping from \(U\) to \(\text{Sym}^d(U')\), here \(\text{Sym}^d(U')\) is the \(d\)-th symmetric power of \(U'\), and that \(\text{Sym}^d(K)\) does not contain germs of non-constant complex curves, see section 5 for more details.

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2. Intersections of analytic disks with a real analytic S.P.C. hypersurface

2.1. Generalities. Let \((M,0)\) be a germ of a strictly pseudoconvex hypersurface in \(\mathbb{C}^2\) defined in some sufficiently small neighborhood \(U\) of the origin as the zero set of a real analytic function \(M := \{z \in U : Q(z, \bar{z}) = 0\}\), \(Q(0) = 0\) and \(\text{grad}_z Q \neq 0\). In an appropriate coordinates \(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2\) of \(\mathbb{C}^2\) the defining function of \(M\) can be written as

\[
Q(z, \bar{z}) = y_2 - |z_1|^2 - r(z_1, \bar{z}_1, x_2). \tag{2.1}
\]

This represents our hypersurface \(M\) in the form

\[
M = \{(z_1, z_2) : y_2 = |z_1|^2 + r(z_1, \bar{z}_1, x_2)\}, \tag{2.2}
\]

where the rest \(r\) can be supposed to be put into the Chern-Moser normal form i.e., has the form

\[
r(z_1, \bar{z}_1, x_2) = \sum_{k,l \geq 2} r_{kl}(x_2) z_1^k \bar{z}_1^l. \tag{2.3}
\]

Here \(r_{kl}(x_2)\) are real analytic functions on \(x_2\) with \(r_{kl} = \bar{r}_{lk}\), see [CM]. In fact \(r_{kl}\) in the Chern-Moser normal form satisfy some additional properties, but we shall not need them.

Set \(U^\pm := \{z \in U : Q(z, \bar{z}) \geq 0\}\) and call \(U^\pm\) the concave/convex side of \(M\).

Meromorphic mapping \(f\) from a complex manifold \(U\) with values in a complex manifold (or complex space) \(U'\) is given by an irreducible and locally irreducible analytic subset \(\Gamma_f \subset U \times U'\) (the graph of \(f\)) such that the restriction \(\Gamma_f |_{\Gamma_f} : \Gamma_f \rightarrow U\) of the natural projection \(\Gamma_f : U \times U' \rightarrow U\) is generically one to one. Generically here means outside of a proper analytic set \(I_f\), which is called the indeterminacy set of \(f\). From the condition of the irreducibility of \(\Gamma_f\) and Remmert proper image theorem, it follows that \(\text{codim} I_f \geq 2\); in the case of \(n = 2\) this means a discrete set of points. When \(U' = \mathbb{P}^N\) then such \(f\) can be defined by an \(N\)-tuple of meromorphic functions \(f_1, \ldots, f_N\) and \(I_f\) is the union of the indeterminacy sets of all \(f_k\). Outside of divisors of poles of \(f_k\) we have a mapping to \(\mathbb{C}^N\) viewed as the standard affine part of \(\mathbb{P}^N\). With some abuse of notations we shall often write \(f : U \rightarrow \mathbb{C}^N\) instead of \(f : U \rightarrow \mathbb{P}^N\).

Our standing assumption about the meromorphic map \(f : U \rightarrow U'\) is that \(I_f \subset \{0\}\), i.e., either \(I_f = \{0\}\) or is empty, and that

\[
\text{f}|_M[M] \subset K, \tag{2.4}
\]

for some compact \(K\) in \(U'\). This will be written as \(f : (U, M) \rightarrow (U', K)\). Let us make precise that \(\text{f}|_M[M]\) is defined as

\[
\text{f}|_M[M] := \overline{f(M \setminus \{0\})}. \tag{2.5}
\]

This notation is coherent with that from algebraic geometry. In the latter case \(M\) is a complex subvariety of \(U\) and \(\text{f}|_M[M]\) is the so called proper image of \(M\) under \(f\).

2.2. Families of analytic disks. Recall that by an analytic disk in a complex manifold/space \(U\) one calls a holomorphic map \(\varphi : \Delta \rightarrow U\) of the unit disk \(\Delta\) to \(U\). The image \(\varphi(\Delta)\) of an analytic disk we shall denote as \(C\). For the purpose of this paper we shall need a precise information about intersections of certain holomorphic 1-parameter families of analytic disks \(\{C_\lambda\}\), all passing through the origin, with our hypersurface \(M\). Here \(\lambda\) is a complex parameter varying in a neighborhood of some \(\lambda_0 \in \mathbb{C}\). First we shall describe what families \(\{C_\lambda\}\) will occur below.
Let \( \pi : \mathcal{U} \to U \) be a tree of blowings-up over \( 0 \in U \subset \mathbb{C}^2 \). By saying this we mean that \( \pi \) is a composition of a finite number of \( \sigma \)-processes, i.e., \( \pi = \sigma_1 \circ \ldots \circ \sigma_N \). In more details denote by \( E_0 \) the line \( \{ z_2 = 0 \} \) in the initial neighborhood of zero \( U^{(0)} := U \subset \mathbb{C}^2 \) with “natural coordinates” \( z_1^{(0)} = z_1, z_2^{(0)} = z_2 \). In the natural affine charts \( (U_1^{(1)}, z_1^{(1)}, z_2^{(1)}) \) and \( (U_2^{(1)}, z_1^{(1)}, z_2^{(1)}) \) on the first blow-up \( U^{(1)} \) the blow-down map \( \pi_1 : U^{(1)} \to U^{(0)} \) can be written as

\[
\begin{align*}
&\begin{cases}
  z_1^{(1)} = z_1, \\
  z_2^{(1)} = z_2
\end{cases} & \text{in } U_1^{(1)} \quad \text{and} \quad \begin{cases}
  z_1^{(1)} = \tau, \\
  z_2^{(1)} = z_2
\end{cases} & \text{in } U_2^{(1)}. \quad (2.6)
\end{align*}
\]

We denote coordinates both in \( U_1^{(1)} \) and \( U_2^{(1)} \) with the same letters and call them the “natural coordinates” in \( U^{(1)} \), this will not lead us to a confusion. The exceptional curve \( E_1 \) is given by \( \{ z_1^{(1)} = 0 \} \) in \( U_1^{(1)} \) and by \( \{ z_2^{(1)} = 0 \} \) in \( U_2^{(1)} \). After that one blows-up some point on \( E_1 \) and denotes this as \( \pi_2 : U^{(2)} \to U^{(1)} \), and so on. Each time in the similar way one obtains “natural coordinates” \( z_1^{(k)}, z_2^{(k)} \) on affine charts \( U_1^{(k)} \) and \( U_2^{(k)} \) of \( U^{(k)} \). The exceptional curve \( E_k \) of \( \pi_2 : U^{(k)} \to U^{(k-1)} \) writes in these coordinates as either \( \{ z_1^{(k)} = 0 \} \) or as \( \{ z_2^{(k)} = 0 \} \). We shall not distinguish between \( E_k \) and its strict transforms under the further \( \sigma \)-processes, i.e., \( \pi_{k+1} E_k \) will be denoted still as \( E_k \) and so on. Therefore the exceptional divisor \( E \) of the resulting \( \pi : U^{(N)} \to U \) represents as the union \( E = E_1 \cup \ldots \cup E_N \).

For a fixed \( k \) between 1 and \( N \) consider the following families:

\[
\Delta_\lambda = \{ z_2^{(k)} = \lambda, |z_1^{(k)}| < 1 \}_{|\lambda-\lambda_0|<\varepsilon} \quad \text{if the equation of } E_k \quad \text{is } \{ z_1^{(k)} = 0 \}, \quad (2.7)
\]

or

\[
\Delta_\lambda = \{ z_1^{(k)} = \lambda, |z_2^{(k)}| < 1 \}_{|\lambda-\lambda_0|<\varepsilon} \quad \text{if the equation of } E_k \quad \text{is } \{ z_2^{(k)} = 0 \}. \quad (2.8)
\]

Here \( \lambda_0 \in \mathbb{C} \) and \( \varepsilon > 0 \) are chosen arbitrarily. In other words these are the families of disks which intersect \( E_k \) orthogonally in “natural coordinates” on \( U^{(k)} \). The holomorphic 1-parameter families we shall be interested in are the push-downs of \( \Delta_\lambda \)'s under the blow-down map \( \pi_1 \circ \ldots \circ \pi_k : U^{(k)} \to U^{(0)} \). And they will be denoted actually as \( C_\lambda \), i.e., \( C_\lambda := \pi_1 \circ \ldots \circ \pi_k(\Delta_\lambda) \). The number \( 1 \leq k \leq N \) will be clear from the context.

Families \( \{ C_\lambda \} \) are quite special. Their equations are polynomial in \( z_1, z_2 \) and \( \lambda \), this can be easily proved by the induction on the number \( N \) of \( \sigma \)-processes in \( \pi \). In our applications we shall see below that without loss of generality we can assume that the center \( \lambda_0 \) can be taken generic (e.g., avoiding points of intersection of \( E_j \) with \( E_k \)) and \( \varepsilon > 0 \) as small as we wish. Generically here means outside of a finite set. Therefore we can assume that

- \( C_\lambda \) do not degenerate to a point for any \( \lambda \);
- the tangent cone to any \( C_\lambda \) at zero does not contain either the line \( \{ z_1 = 0 \} \) or the line \( \{ z_1 = 0 \} \).

The first assertion is obvious since the proper transform of \( C_\lambda \) under \( \pi \) is \( \Delta_\lambda \), and the latter is a non-constant analytic disk. The second is a bit more subtle. Would the tangent cone to \( C_\lambda \) contain more than one line then \( \pi_1^* C_\lambda \) would intersect \( E_1 \) by more than one point. But this contradicts to the construction of the family \( \{ C_\lambda \} \).

In what follows we suppose that the line \( \{ z_1 = 0 \} \) is not in the tangent cone of \( C_\lambda \) at zero, the case with \( \{ z_2 = 0 \} \) can be treated analogously. By genericity we can assume that this
holds for all $|\lambda - \lambda_0| < \varepsilon$. Therefore there exists a bidisk $\Delta^2(\delta) = \Delta(\delta_1) \times \Delta(\delta_2)$ independent of $\lambda$ such that $C_\lambda \cap \Delta^2(\delta)$ has as its defining function the Weierstrass polynomial

$$W_\lambda(z) = z_2^d + a_1^\lambda(z_1)z_2^{d-1} + \ldots + a_d^\lambda(z_1)$$

for some $d \geq 1$, (2.9)

where coefficients $a_d^\lambda(z_1)$ are holomorphic both in $z_1$ and in $\lambda$, and $a_d^\lambda(0) \equiv 0$ again by genericity of $\lambda_0$. Moreover the degree $d$ is independent of $\lambda$. Since $C_\lambda$ are obviously irreducible at the origin the polynomials $W_\lambda$ should be irreducible too. By $D_\lambda(z_1)$ we denote the discriminant of $W_\lambda$. $D_\lambda(z_1)$ is holomorphic in both variables and $D_\lambda(0) \equiv 0$ if $d > 1$ because in this case $W_\lambda(0, z_2) = z_2^d$ has zero as a root of higher order. Write

$$D_\lambda(z_1) = b_k(\lambda)z_1^k + b_{k+1}(\lambda)z_1^{k+1} + \ldots,$$

(2.10)

where $k \geq 1$ and $b_k(\lambda) \neq 0$. Perturbing $\lambda_0$, i.e., taking it generically, and taking $\varepsilon$ smaller we can assume that the discriminants $D_\lambda(z_1)$ of the equations $\{W_\lambda = 0\}$ of $C_\lambda$ do not vanish for $z_1 \in \Delta(\delta_1) \setminus 0$ for some $\delta_1 > 0$ independent of $\lambda$. Therefore in the bidisk $\Delta^2(\delta)$ our curves $C_\lambda$ are given by the equations

$$C_\lambda = \left\{ z_2 = h_\lambda(z_1^{\frac{1}{d}}) \right\}, h_\lambda(0) = 0,$$

(2.11)

see [F]. Dependence of $h_\lambda$ on $\lambda$ stays to be holomorphic. If $d = 1$ one readily gets the same form (2.11) for $C_\lambda$.

2.3. Intersection of real hypersurfaces with families of analytic disks. Let $R$ be the intersection of a non-constant analytic disk $C = \varphi(\Delta) \ni 0$ with the strictly pseudoconvex real analytic hypersurface $0 \in M \subset \mathbb{C}^2$ as above, i.e., $R = \varphi^{-1}(C \cap M)$. Then $R$ is a one dimensional real analytic set, and it has a non-empty (one dimensional) interior provided $C \cap U^+ \neq \emptyset$. More precisely $R = S \cup \Gamma$, where $S$ is a discrete in $\Delta$ set of points $\{s_k\}$ and $\Gamma$ is a locally finite union of smooth arcs $\{\gamma_l\}$ with ends on $\{s_k\}$.

Now consider a holomorphic 1-parameter family $\{\varphi_\lambda : |\lambda - \lambda_0| < \varepsilon\}$ of analytic disks $\varphi_\lambda : (\Delta, 0) \to (\mathbb{C}^2, 0)$ such that $C_\lambda = \varphi_\lambda(\Delta) \cap \Delta^2(\delta) = \{(z_1, h_\lambda(z_1^{\frac{1}{d}})) : z_1 \in \Delta \delta_1\}$ are as above with $R_\lambda$, $\Gamma_\lambda$ and $S_\lambda$ having an obvious meaning in the parameter case.

**Lemma 2.1.** For $\varepsilon > 0$ small enough the following holds:

i) either for all $|\lambda - \lambda_0| < \varepsilon$ the 1-dimensional part $\Gamma_\lambda$ of $R_\lambda := \varphi_\lambda^{-1}(M \cap C_\lambda)$ contains a component $\gamma_\lambda$ with an end point at zero;

ii) or $R_\lambda = \{0\}$ for all $\lambda$ and then $C_\lambda \setminus 0 \subset U^-;

iii) or there exist sequences $\lambda_n \to \lambda_0$ and $\varepsilon_n \to 0$ such that (i) holds for all $|\lambda - \lambda_n| < \varepsilon_n$ and for all $n$.

**Proof.** Equations of analytic disks $C_\lambda$ and hypersurface $M$ can be written as

a) $C_\lambda = \{z_2 = h_\lambda(z_1^{\frac{1}{d}})\}$, $h_\lambda(0) = 0$ as in (2.11).

b) $M = \{y_2 = |z_1|^2 + \sum_{k,l \geq 2} a_{kl}(x_2)z_1^kz_1^l\}$ as in (2.2).

Making the substitution

$$\begin{cases}
  z_1 \to z_1^d \\
  z_2 \to z_2
\end{cases}$$

(2.12)

we rewrite (a) and (b) in the form
\[ C_\lambda = \{ z_2 = z_2^q h_\lambda(z_1) \} \text{ with some } q \geq 1, \]
\[ M = \{ y_2 = |z_1|^{2d} + \sum_{k,l \geq 2} a_{kl} x_2 z_1^{dk} z_2^{dl} \}, \]
for some (other) holomorphic \( h_\lambda \) such that \( h_\lambda(0) \neq 0 \) as a function of \( \lambda \).

**Case 1.** \( q > 2d \). In that case we obviously and directly from (2.13) get that \( C_\lambda \cap \bar{U}^+ = \{0\} \) for all \( \lambda \), i.e., we get the option \((\text{ii})\) of the lemma.

**Case 2.** \( h_{\lambda_0}(0) \neq 0 \). Taking \( \varepsilon > 0 \) smaller, if necessary, we can suppose that \( h_{\lambda}(0) \neq 0 \) for all \( |\lambda - \lambda_0| < \varepsilon \). Write \( h_\lambda(0) = a(\lambda) e^{\theta(\lambda)} \). Using polar coordinates \( z_1 = r_1 e^{i\varphi_1} \) and \( z_2 = r_2 e^{i\varphi_2} \) we get that on \( C_\lambda \cap M \) one has

\[
\begin{align*}
\{ r_2 \sin \varphi_2 &= r_1^{2d} + O(r_1^{4d}), \\
r_2 \sin \varphi_2 &= a(\lambda) r_1^{q\varphi} + O(r_1^{q+1}), \\
r_2 \cos \varphi_2 &= a(\lambda) r_1^{q\cos} (q\varphi_1 + \theta(\lambda)) + O(r_1^{q+1}).
\end{align*}
\]

Now we have the following two subcases.

**Subcase 2a.** \( q < 2d \). From (2.14) we get that

\[ r_1^{2d} + O(r_1^{4d}) = a(\lambda) r_1^{q\varphi_1 + \theta(\lambda)} + O(r_1^{q+1}), \]

which implies

\[ a(\lambda) \sin(q\varphi_1 + \theta(\lambda)) = r_1^{2d-q} + O(r_1^{4d-q}) + O(r_1) = O(r_1). \]

The last equation has \( 2q \) curves of solutions as \( r_1 \) tends to zero:

\[ \varphi_1 = -\frac{1}{q} \theta(\lambda) + \frac{1}{q} \arcsin \left( \frac{1}{a(\lambda)} O(r_1) \right) + \frac{2\pi k}{q}, \quad -q \leq k \leq q - 1, \]

and all these curves end at the origin.

After we had determined \( \varphi_1 \) as a function of \( r_1 \) from equations one and two of (2.14) we can easily find \( z_2 = r_2 (\cos \varphi_2 + i \sin \varphi_2) \) as a function of \( r_1 \) from the equations one and three of (2.14). Since \( q \geq 1 \) we shall have that \( z_2(r_1) \to 0 \) as \( r_1 \to 0 \). I.e., the option (i) of our lemma realizes.

**Subcase 2b.** \( q = 2d \). Again, from (2.14) we get that

\[ r_1^{2d} + O(r_1^{4d}) = a(\lambda) r_1^{2d\varphi_1 + \theta(\lambda)} + O(r_1^{2d+1}), \]

which implies

\[ a(\lambda) \sin(q\varphi_1 + \theta(\lambda)) = 1 + O(r_1). \]

If \( a(\lambda_0) < 1 \) then equation (2.15) has no solutions for small \( r_1 \) and we fall to the option (\( \text{ii} \)) of our lemma. If \( a(\lambda_0) > 1 \) then we are again in (i) with the curves of solutions

\[ \varphi_1 = -\frac{1}{q} \theta(\lambda) + \frac{1}{q} \arcsin \left( \frac{1}{a(\lambda)} (1 + O(r_1)) \right) + \frac{2\pi k}{q}, \quad -q \leq k \leq q - 1. \]

Finally, if \( a = 1 \) then by open mapping theorem one finds \( \lambda \) arbitrarily close to \( \lambda_0 \) such that \( a(\lambda) < 1 \) and the one gets the option (\( \text{iii} \)) of our lemma.

**Case 3.** If \( h_{\lambda_0}(0) = 0 \) we can take \( \lambda_n \) arbitrarily close to \( \lambda_0 \) such that \( h_{\lambda_n}(0) \neq 0 \) but \( a(\lambda_n) := |h(\lambda_n)| \) small. We fall into the assumptions of Case 2 with an additional condition that \( a(\lambda) < 1 \). This gives us the option (\( \text{iii} \)) of our lemma.

\[ \square \]
3. **Strict transform of real analytic hypersurfaces under a modification**

3.1. **Strict transform.** Let $M \ni 0$ be a real analytic strictly pseudoconvex hypersurface near the origin in $\mathbb{C}^2$. Let $\pi : \hat{U} \to U$ be a tree of blowings-up over the origin, $U$ stands here for some neighborhood of zero in $\mathbb{C}^2$. The exceptional divisor of $\pi$ is denoted as $E$. Denote by

$$\pi^* M := \overline{\pi^{-1}(M \setminus 0)}$$  \hspace{1cm} (3.1)

the proper preimage (or strict transform) of $M$ under $\pi$. Set $M^*_0 := \pi^* M \cap E$. Note that $M^*_0$ is connected. This follows from the fact that this set is the intersection of connected sets

$$M^*_0 = \bigcap_{\varepsilon > 0} \overline{\pi^{-1}((B^4_\varepsilon \setminus 0) \cap M)},$$  \hspace{1cm} (3.2)

here $B^4_\varepsilon$ stands for the ball of radius $\varepsilon$ centered at the origin of $\mathbb{C}^2$. We are going to prove that $M^*_0$ is a **massive set** in the topology of $E$.

**Lemma 3.1.** Let $\hat{p} \in M^*_0$ be a point on the strict transform of $M$ over the origin which is not a singular point of $E$ and let $E_k$ be an irreducible component of $E$ containing $\hat{p}$. Then $\hat{p}$ is an accumulation point of $\operatorname{int}(M^*_0 \cap E_k)$. In particular $M^*_0 \cap E_k$ has the non-empty interior in the topology of $E_k$ for every $E_k$ such that $M^*_0 \cap E_k \not\subset \operatorname{Sing} E$.

**Proof.** Let $\hat{p} \in M^*_0 \cap E_k$ be as in this lemma. From (3.2) it is clear that we can find a sequence $\hat{p}_n \to \hat{p}$ such that $p_n := \pi(\hat{p}_n) \in U^+$ and then necessarily $p_n \to 0$. Moreover, since $U^+$ is open we can choose these $\hat{p}_n$ generically. For any fixed $n > 1$ find natural coordinates $z_1^{(k)}, z_2^{(k)}$ in an affine neighborhood of $\hat{p}_n$ and set $\lambda_0 := z_1^{(k)}(\hat{p}_n)$ or $\lambda_0 := z_2^{(k)}(\hat{p}_n)$ depending on what is the equation of $E_k$ in these coordinates, see (2.7) and (2.8). Let $\{\Delta_\lambda\}_{|\lambda - \lambda_0| < \varepsilon}$ and $\{C_\lambda\}_{|\lambda - \lambda_0| < \varepsilon}$ be the holomorphic 1-parameter families constructed as there. By the genericity of the choice of $\hat{p}_n$ these families can be chosen generically as well. Since these disks cut $U^+$ the case (ii) of Lemma 2.1 does not occur and we conclude that either $\lambda_0$ is an interior point of $M^*_0$ or it can be approximated by interior points. This gives in its turn the approximation of $\hat{p}$ by the interior points of $M^*_0$.

Indeed, suppose we are under the case (i) of that lemma. Then $C_\lambda$ intersects $M$ by an 1-dimensional real analytic set which accumulates to zero. Let $\gamma_\lambda$ be some 1-dimensional local component of this set which accumulates to 0. Its strict transform $\tilde{\gamma}_\lambda := \pi^{-1}(\gamma_\lambda \setminus 0)$ accumulates $\lambda \in E_k$, i.e., $\lambda$ is viewed as the point of intersection of $\Delta_\lambda$ with $E_k$. Therefore $\lambda$ belongs to $M^*_0 \cap E_k$ for all $\lambda$ close to $\lambda_0$. The case (iii) is clear.

**Remark 3.1.** Observe that $E_k \cap \operatorname{Sing} E$ consists from a finite set of points. From connectivity of $M^*_0 \cap E_k$ it follows that if $M^*_0 \cap E_k$ is non-empty and is contained in $\operatorname{Sing} E$ then $M^*_0 \cap E_k = \{\text{point}\}$.

3.2. **Strict transform to the first two blowings-up: example.** It is not necessary for the proofs of this paper but is very instructive to compute the strict transform of a real analytic hypersurface onto first few blowings-up.

1) Write the equation (2.2) in the form

$$M = \{ \text{Im} z_2 = |z_1|^2 + r(z, \bar{z}) \},$$  \hspace{1cm} (3.3)

If we denote by $M_1$ the proper transform of $M$ under $\pi_1$ and use notations of subsection 2.2 we see that $M_1 \setminus E_1$ has equations

\[ \text{...} \]

...
where $r_1(z, z) = r(z, z, z, z, z, z)$, and

$$M_1 \setminus E_1 = \{ \text{Im}(z_1, z_2) = |z_1|^2 + r_1(z, z) \} \quad \text{in} \quad U_1^{(1)},$$

(3.4)

and

$$M_1 \setminus E_1 = \{ \text{Im}(z_2) = |z_2|^2 + r_2(z) \} \quad \text{in} \quad U_2^{(1)},$$

(3.5)

where $r_2(z, z) = r(z, z, z, z, z, z)$. We see that the rest $r_1$ and $r_2$ have order of vanishing at zero not less than the original $r$.

The closure of $M_1 \setminus E_1$ in $U_2^{(1)}$, that is actually $M_1 \setminus U_2^{(1)}$, is a smooth hypersurface (one easily checks that the gradient never vanishes) with the same equation as (3.3) and $M_1 \setminus U_2^{(1)} \supset (E_1 \cap U_2^{(1)})$. Now remark that $M_1 \setminus E_1$ in $U_1^{(1)} \setminus E_1$ is defined by

$$M_1 \setminus E_1 = \left\{ \text{Im} \left( \frac{z_2}{z_1} \right) = 1 + \frac{r_1(z_1, z)}{z_1 z_1} \right\},$$

(3.6)

where $z_1(z_1, z) = O(z_1)$.

We see that $M_1 \setminus U_1^{(1)}$ is a real cone with vertex at the origin. On the diagram of moduli $(r_1, r_2) = (|z_1|^2, |z_2|^2)$ it is tangent to the cone $r_2 \geq r_1$, see Fig. 1(a). At this stage the proper transform $M_1$ of $M$ contains the entire exceptional curve $E_1$.

2) Now let us blow-up the point $0^{(1)}$, by which we denote the origin in $U_1^{(1)}$. In what follows coordinates $(z_1, z_2)$ will be redefined simply as $(z_1, z_2)$ in order to simplify the notations, the same for $z^{(2)}$ below. We get the following equations in charts $U_1^{(2)}$ and $U_2^{(2)}$:

$$\text{Im}(z_1^2 z_2) = |z_1|^2 + r_1(z, \bar{z}) \quad \text{in} \quad U_1^{(2)},$$

(3.7)

and

$$\text{Im}(z_1 z_2^2) = |z_1|^2 |z_2|^2 + r_2(z, \bar{z}) \quad \text{in} \quad U_2^{(2)},$$

(3.8)

with an appropriate $r_1$ and $r_2$. If we denote by $M_2$ the proper transform of $M_1$ under $\pi_2 : U^{(2)} \to U^{(1)}$ and by $E_2$ the corresponding exceptional curve then for $M_2 \setminus E_2$ in $U_1^{(2)} \setminus E_2$ and for $M_2 \setminus E_2$ in $U_2^{(2)} \setminus E_2$ we get correspondingly the equations

$$\text{Im} \left( \frac{z_1}{z_2} \right) = 1 + O(z) \quad \text{and} \quad \text{Im} \left( \frac{z_2}{z_2} \cdot \frac{1}{z_1} \right) = 1 + O(z).$$

(3.9)

Now we see that $M_2$ intersects the second exceptional curve $E_2$ by the closed disk $D$, which in coordinates of $U_1^{(2)}$, corr. of $U_2^{(2)}$, is given as

$$D = \{ z_1 = 0, |z_2| \geq 1 \} \quad \text{cor. as} \quad D = \{ z_2 = 0, |z_1| \leq 1 \},$$

(3.10)

see Fig. 1(b).

4. Continuity of the mapping

4.1. The limit set of $f$ from the concave side of $M$. Denote by

$$\lim_{z \to z_0} f(z) \quad \text{correspondingly} \quad \lim_{z \to z_0} f(z),$$

the sets of cluster points of all sequences $f(z_n)$ when $z_n \in U^+, z_n \to 0$ (corr. when $z_n \in M \setminus 0, z_n \to 0$).
Figure 1. The first proper transform $M_1$ of $M$, see picture (a), contains the first exceptional curve $E_1$ entirely. The second $M_2$, see picture (b), contains $E_1$ and only a part of $E_2$.

Lemma 4.1. Let a meromorphic mapping $f$ satisfies (2.4), where $M$ is as above and $M' \subseteq U'$ is an arbitrary compact. Then

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} f(z).$$

(4.2)

Proof. Let $\pi : \hat{U} \to U$ be a resolution of indeterminacies of $f$, i.e., a tree of blowings-up over the origin, such that the pull-back $\hat{f} := f \circ \pi$ of $f$ to $\hat{U}$ is holomorphic. Furthermore, let $z_n \in U^+$ be a sequence of points such that $z_n \to 0$ and $f(z_n) \to w_\infty$. The latter is some point of $M'$. Denote by $\hat{z}_n \in \hat{U}^+ := \pi^{-1}(U^+)$ the preimages of $z_n$ under $\pi$. Taking a subsequence we can suppose that $\hat{z}_n$ converge to some $\hat{z}_\infty \in E := \pi^{-1}(0)$, the latter is the exceptional divisor of the modification $\pi$. We have that $\hat{f}(\hat{z}_n) \to w_\infty = \hat{f}(\hat{z}_\infty)$. Take the family $\{\Delta_\lambda\}_{|\lambda - \lambda_0| < \epsilon}$ of disks as in (2.7) and (2.8) in such a way that $\Delta_\lambda \cap E = \hat{z}_\infty$.

Before applying Lemma 2.1 to the corresponding family $C_\lambda$ we perturb $\lambda_0$ and take $\epsilon > 0$ in order to fulfill the usual assumptions we impose on our family. Remark also that we can find $\lambda$-s arbitrarily close to $\lambda_0$ in such a way that in addition $C_{\lambda} \setminus 0$ cannot be contained in $U^-$. For example take $\lambda$ such that $\Delta_\lambda$ contains some $\hat{z}_n$, this means that $C_{\lambda} \ni z_n \in U^+$ (or any $\lambda$ close to this). Therefore the case (ii) of Lemma 2.1 will not occur. Remark that we perturbed also ours $\hat{z}_\infty$ and $w_\infty$.

Case 1. If we are in the case (i) of Lemma 2.1 then

$$\lim_{z \to 0} f(z) = \lim_{z \to \hat{z}_\infty} \hat{f}(z) = w_\infty.$$  

(4.3)

But $C_{\lambda_0} \cap M$ contains a real analytic curve $\gamma$ accumulating to zero in this case. The limit in (4.3) is the same as the limit along this $\gamma$ which is a subset of $M$. This proves the inclusion

$$\lim_{z \to 0} f(z) \subseteq \lim_{z \to \hat{z}_\infty} \hat{f}(z)$$

(4.4)

in this case.
Lemma 4.2. In the conditions of Lemma 4.1 suppose, in addition, that the compact set \( \Delta_{\lambda_n} \) with \( E \). Remark that \( \hat{\lambda}_n \rightarrow \tilde{\lambda}_\infty \). Let \( \gamma_n \) be a component of \( C_{\lambda_n} \cap M \) accumulating to zero. Then

\[
\hat{f}(\hat{z}_n') = \lim_{\hat{z} \in \Delta_{\lambda_n}} \hat{f}(\hat{z}) = \lim_{\hat{z} \in \Delta_{\lambda_n}} \hat{f}(\hat{z}) = \lim_{\hat{z} \in \gamma_n} \hat{f}(\hat{z}) \subset \lim_{\hat{z} \in M} \hat{f}(\hat{z}). \tag{4.5}
\]

By holomorphicity of \( \hat{f} \) we have that \( \hat{f}(\hat{z}_n') \rightarrow \hat{f}(\hat{\lambda}_\infty) = w_\infty \). I.e., the inclusion (4.4) is proved also in this case.

In fact up to now proved that for some \( \hat{z}_n'' \in E, \hat{z}_n'' \rightarrow \hat{\lambda}_\infty \) we have that \( \hat{f}(\hat{z}_n'') \in f|_M[E] \). But the considerations applied in the Case 2 show that this implies that \( \hat{f}(\hat{\lambda}_\infty) \in f|_M[E] \) as well. I.e., the inclusion (4.4) is proved.

The inverse inclusion is obvious. For if

\[
w_\infty = \lim_{\hat{z}_n \rightarrow 0} f(\hat{z}_n),
\]

then we find \( \hat{z}_n \in E^+ \) sufficiently close to \( z_n \) and the limit \( \hat{f}(\hat{z}_n) \) will be the same.

\[\Box\]

Remark 4.1. The set in the right hand side of (4.2) we shall denote as \( f|_M[0] \). Observe the obvious inclusion

\[
\lim_{\hat{z} \in \hat{M} \setminus 0} \hat{f}(\hat{z}) = f|_M[0] \subset f|_M[E]. \tag{4.6}
\]

4.2. Continuity of the mapping from the concave side of \( M \). Now we can proof the announced continuity of \( f \) from the side \( U^+ \) of \( M \).

Lemma 4.2. In the conditions of Lemma 4.1 suppose, in addition, that the compact \( M' \) does not contain any germ of a complex curve. Then the restriction \( f|_{D^+ \setminus 0} \) is continuous up to \( M \).

Proof. We need to prove the continuity at zero only. Moreover, due to Lemma 4.1 the only thing to prove is that

\[
f|_M[0] = \{ \text{point} \}. \tag{4.7}
\]

Let \( \pi : \hat{U} \rightarrow U \) be, as above, the resolution of indeterminacies of \( f \) and \( E \) its exceptional divisor. It is clear that

\[
f|_M[0] = \hat{f}(M_0^*). \tag{4.8}
\]

By \( E^* \) denote the union of components of \( E \) on which the lift \( \hat{f} \) is constant. Therefore if \( M_0^* \subset E^* \) then the lemma is proved. Suppose that this is not the case. This means that there exists an irreducible component \( E_k \) of \( E \) such that \( M_0^* \cap E_k \neq \emptyset \) and \( \hat{f}|_{E_k} \neq \text{const} \). If \( M_0^* \cap E_k \subset \text{Sing } E \), i.e., is just one point \( \hat{p} \), see Remark 3.1 then there should be another component \( E_l \) of \( E \) intersecting \( E_k \) at this point. \( \hat{p} \) cannot be an isolated point of \( E_l \) too and we can find a point \( \hat{q} \in M_0^* \) close to \( \hat{p} \) which is not singular for \( E \) and then Lemma 2.1 applies to \( \hat{q} \) on the place of \( \hat{p} \).

According to this lemma there exists \( \hat{q} \in E_0 \) close to \( \hat{p} \) and a neighborhood \( \hat{V} \) of \( \hat{q} \) on \( E_k \) (or on \( E_l \)) such that \( \hat{V} \subset M_0^* \). But this means that \( \hat{f}|_{E_k}(\hat{V}) \subset M' \) (or \( \hat{f}|_{E_l}(\hat{V}) \subset M' \)). Since \( M' \) does not contain germs of complex curves this means that \( \hat{f} \) is constant on \( E_k \) (or on \( E_l \)). In the former case this is a contradiction. In the latter \( E_k \cap M_0^* = \{ \hat{p} \} \in E^*_f \).
Therefore \( M^*_0 \cap E_k \subset E^*_f \) for every component \( E_k \) of \( E \) which intersects \( M^*_0 \) and the lemma is proved.

**Remark 4.2.** If no further assumption are imposed on the image set \( K \) mapping \( f \) (being continuous from the concave side \( U^+ \) of \( M \)) can be still **meromorphic** in general, i.e., it can happen that 0 is really an indeterminacy point of \( f \). Such an example was constructed in [IM].

5. **APPLICATIONS AND GENERALIZATIONS**

5.1. **Proof of Corollaries 1, 2 and 3.** Let a germ \( f : (M,x) \to (M',x') \) of a holomorphic mapping from a real analytic hypersurface \( M \subset \mathbb{C}^n \) to a compact locally spherical \( M' \subset \mathbb{C}'^n \) be given. By the result of [SN] \( f \) meromorphically extends along any given CR-path \( \gamma \subset M \) starting at \( x \). All we need to prove is that this extension is holomorphic. Denote by 0 \( \in M \) the point in which we shall prove the holomorphicity of \( f \).

Take a vector \( v \in T_0M \) such that \( L_M(0)[v] \neq 0 \). Taking such \( v \) generically we can preserve this condition and, additionally, taking a transverse to \( M \) vector \( n \) we can achieve that \( L \cap I_f \) is discrete for a subspace \( L \) of \( \mathbb{C}^n \) spanned by \( v \) and \( n \). Then \( L \cap I_f \) will be discrete as well. All we need is to prove our theorem for the restriction of \( f \) to this subspace. Indeed, after a coordinate change we can suppose that \( L = \{z_3 = ... = z_n = 0\} \) and that \( f \) is meromorphic in the unit polydisk \( U = \Delta^n \). Shrinking \( U \) if necessary and assuming that our theorem is proved when \( n = 2 \) we get that \( f|_{L \cap U} \) is holomorphic and therefore the graph \( \Gamma_f|_{L \cap U} \) is Stein. Take a Stein neighborhood \( V \) of \( \Gamma_f|_{L \cap U} \) in \( U \times \mathbb{P}^N \). Then for every \( z'' = (0,0,z_3,...,z_n) \) close to zero we have that \( f(z) := (z,f(z)) \) is holomorphic in a neighborhood of \( \partial \Delta^2 \times \{z''\} \) with values in \( V \). The holomorphicity of \( f \) follows now from the Hartogs extension theorem for holomorphic functions. Therefore from now on we shall assume that \( n = 2 \), \( M \) strictly pseudoconvex at 0 and zero is the only eventual indeterminacy point of \( f \).

Let \( M' \) be our compact locally spherical hypersurface in \( \mathbb{C}'^n \). This means that for every point \( x' \in M' \) there exists a neighborhood \( U' \supset x' \) in \( \mathbb{C}'^n \) and a biholomorphism \( \Phi \) of \( U' \) with values in \( \mathbb{C}^n \) such that \( \Phi(U' \cap M') \subset S^{2n'-1} \). In [BS] it was proved that the universal cover \( \tilde{M}' \) of \( M' \) admits a biholomorphic **development** mapping \( F : \tilde{M}' \to S^{2n'-1} \). Moreover, construction in [BS] obviously gives a complex neighborhood \( V' \supset \tilde{M}' \) such that the covering map \( \pi : \tilde{M}' \to M' \) extends as a locally biholomorphic map to \( V' \) and this \( f \) is an embedding of \( \tilde{V}' \) to \( \mathbb{C}^n \).

Shrinking \( (U,M) \), if necessary, we can suppose that \( f(\overline{U'}) \subset U' \), where this time \( U' \) is a neighborhood of \( 0' = f(0) \subset M' \) such that \( \pi : \tilde{U'} \to U' \) is a biholomorphism. Here \( \tilde{U}' \) is an appropriate neighborhood of some preimage \( \tilde{0}' \) of \( 0' \) by \( \pi \) in \( \tilde{V}' \). As a result the composition \( \tilde{H} := F \circ \pi^{-1} \circ f : \overline{\tilde{U}'} \to \mathbb{C}^n \) is well defined and maps \( M \) to \( S^{2n'-1} \). Moreover, it extends holomorphically to a neighborhood of any point \( x \in M \setminus \{0\} \). Applying the quoted theorem of Pinchuk we extend \( \tilde{H} \) holomorphically to a neighborhood of \( M \), denote this neighborhood by \( \tilde{U} \) again. This gives us the desired holomorphicity of \( f \) and proves Corollaries 1 and 2.

**Proof of Corollary 3.** As it was said in Introduction the distribution of complex tangents on a strictly pseudoconvex hypersurface is contact and by Theorem of Gromov, see [G], one can draw a CR-path between any two points of \( M \). Since, in addition \( M \) is supposed to be
simply connected our \( f \) extends meromorphically to a neighborhood of \( M \) and therefore on \( D \). If \( n = 2 \) we can directly apply our Theorem 1 and continuity of \( f \) up to the boundary. This gives us the statement of Corollary 3.

**Remark 5.1.** Corollary 3 most probably stays to be true for all \( n \geq 2 \) in the source and not only for \( n = 2 \). But the proof requires the study of multidimensional blowings-up of real hypersurfaces and is out of the range of this paper.

**5.2. Proof for meromorphic correspondences.** Let \( U \) and \( U' \) be complex manifolds. Recall that a meromorphic correspondence \( f : U \to U' \) is an analytic subset \( \Gamma_f \subset U \times U' \) such that the restriction \( \text{pr}_1|_{\Gamma_f} : \Gamma_f \to U \) of the natural projection \( \text{pr}_1 : U \times U' \to U \) onto \( \Gamma_f \) is proper and generically d to 1. Here \( d \geq 1 \) is called the order of \( f \) and \( \Gamma_f \) is its graph. Meromorphic correspondence of order 1 is just a meromorphic mapping. In general \( f \) is called a d-ordered meromorphic correspondence. If \( \Gamma_f = \emptyset \) the correspondence is called holomorphic. Here \( I_f \) stands for the set of indeterminacy points of \( f \), i.e., \( x \in I_f \) if \( \dim \text{pr}_1|_{\Gamma_f}^{-1}(x) \geq 1 \).

Let us prove now Corollary 3. The symmetric power \( \text{Sym}^d(U') \) of \( U' \) of order \( d \) is a normal complex space and \( f \) naturally defines a meromorphic mapping \( f^d : U \to \text{Sym}^d(U') \). Condition that \( f(m) \subset K \) for \( m \in M \setminus \{0\} \) implies (is equivalent to) that \( f^d(m) \in \text{Sym}^d(K) \). Would \( \text{Sym}^d(K) \) contain a germ of a non-constant complex curve \( \varphi^d(\lambda) = \text{Sym}(\varphi_1(\lambda),...,\varphi_d(\lambda)) \) then \( K \) would contain all \( \varphi_k \). At least one of them should be non-constant. Contradiction. Therefore \( \text{Sym}^d(K) \) does not contain such a germ. Now one can literally repeat the proof of Theorem 1 for the meromorphic mapping \( f^d : U \to \text{Sym}^d(U') \) to get the conclusion of Corollary 3.

**6. Appendix: Result of Shafikov-Verma**

In this appendix we explain that the paper [SV] contains the following statement.

**Theorem 6.1.** Let \( M \) be a smooth real analytic minimal hypersurface in \( \mathbb{C}^n \) and let \( M' \) be a smooth compact real algebraic hypersurface in \( \mathbb{C}^{n'}, 1 < n \leq n' \). Then every germ \( f : (M,x) \to (M',x') \) of a holomorphic map from \( M \) to \( M' \) extends as a meromorphic correspondence \( F \) along any CR-path on \( M \) and \( F|_M[M] \subset M' \). If, moreover, \( M' \) is strictly pseudoconvex then \( F \) is a meromorphic map.

Recall that a real hypersurface \( M \) is called \textit{minimal} if it does not contain a non-constant germ of a complex hypersurface. The statement of this theorem is implicit in [SV] and is hidden inside of the proof of a stronger statement about \textit{holomorphicity} of \( F \). Unfortunately the proof of the holomorphicity of extension \( F \) in [SV] contains a gap. To make this point clear we give an outline of the proof of Theorem 6.1 referring step by step to [SV].

**Proof.** The proof of the theorem breaks into several steps. Recall that a real submanifold \( \Sigma \) of \( \mathbb{C}^n \) is called \textit{generic} if its tangent space at any point contains a complex subspace of minimal possible dimension. If \( \dim_{\mathbb{R}}\Sigma = 2n - 2 \) this means simply that \( T_{b} \Sigma \) is not a complex subspace of \( \mathbb{C}^n \) for all \( b \in \Sigma \). Genericity is obviously an open condition.

**Step 1.** The proof of Theorem 6.1 can be reduced to the following statement: Let \( \Omega \) be a domain in \( M \) such that \( f \) is meromorphic on \( \Omega \) and let \( b \in \partial \Omega \) be a boundary point such that \( \Sigma := \partial \Omega \) is a generic submanifold in a neighborhood of \( p \). Then \( f \) meromorphically extends to a neighborhood of \( b \).
The proof is given in the section 4.1 of [SV] and relies on the construction of special ellipsoids from [MP]. It is true that in section 4.1 of [SV] $f$ is already supposed to be holomorphic on $\Omega$, but the proof goes through for any analytic objects, e.g., for meromorphic correspondences.

Let $Q_b$ be the Segre variety of $M$ through $b$. By Proposition 5.1 from [S] there exists a dense open subset $\omega$ of $Q_b$ such that for every $a \in \omega$ the intersection $Q_a \cap \Omega$ is non-empty. Moreover, since $I_f$ is of codimension $\geq 2$ for a generic $a$ this intersection will be not contained in $I_f$. I.e., we can find $\xi \in \Omega \cap Q_a$ such that $f$ is holomorphic in a neighborhood $V_\xi$ of $\xi$. Let $V$ be a neighborhood of $Q_\xi$.

**Step 2.** For an appropriate choice of $V_\xi$ and $V$ the set

$$A := \{(z,z') \in V \times \mathbb{C}^n : f(Q_z \cap V_\xi) \subset Q_{z'}'\}$$

is analytic in $V \times \mathbb{C}^n$, extends to an analytic set in $V \times \mathbb{P}^n$, and this extension contains the graph of $f$ over $V_\xi$.

Indeed, if $Q'$ is the defining polynomial of $M'$ the condition $f(Q_z \cap U_\xi) \subset Q_{z'}'$ can be expressed as

$$Q'(f'(t,h(t,\bar{z})),\bar{z}') = 0.$$  

(6.2)

Here $t = (t_1,...,t_{n-1})$ is a coordinate on the tangent plane $T_zM$ and $t_n = h(t,\bar{z})$ is the equation of the Segre variety $Q_z$. After conjugation it is clear that this equation is holomorphic in $(z,z')$, see [S] [SV] for more details. Moreover, since (6.2) is polynomial in $z'$ our set $A$ closes to an analytic set in $V \times \mathbb{P}^n$. This closure will be still denoted as $A$. The fact that $A$ contains the graph of $f$ over $V_\xi$ immediately follows from the invariance of Segre varieties: $f(Q_z) \subset Q_{f(z)}'$ whenever everything is well defined (i.e., sufficiently localized).

Remark that dimension of $A$ might be bigger than $n$ simply because the set of $z'$ such that $Q_{z'} \supset f(Q_z)$ can have a positive dimension for a fixed $z$. In order to reduce the size of $A$ proceed as follows. Consider the natural projections $\pi : A \rightarrow V$ and $\pi' : A \rightarrow \mathbb{P}^n$. Remark that $\pi$ is proper simply because $\mathbb{P}^n$ is compact. For an appropriate neighborhoods $V^*$ of $Q_a$ and $V_a$ of a set

$$A^* := \{(z,z') \in V^* \times \mathbb{P}^n : \pi^{-1}(Q_z \cap V_a) \subset \pi'^{-1}(Q_{z'}')\}.$$  

(6.3)

**Step 3.** The set $A^*$ is analytic in $V^* \times \mathbb{P}^n$ of dimension $n$ and contains the graph of $f$ over (a shrinked, if necessary) $V_\xi$.

The rough reason for $A^*$ to be of dimension $n$ is that both $\pi^{-1}(Q_z \cap V_a)$ and $\pi'^{-1}(Q_{z}')$ are hypersurfaces in $A$ and therefore the set of $z' \in \mathbb{P}^n$ such that $\pi^{-1}(Q_z \cap V_a) \subset \pi'^{-1}(Q_{z}')$ for a given $z \in V^*$ is generically finite. We refer to §3.2 of [SV] for more details.

As the graph of the correspondence $F$ which extends $f$ we take the irreducible component of $A^*$ which contains the germ of the graph of $f$ over $\xi$. At points $z \in M \setminus I_F$ all values of $F$ are contained in $M'$ by unique continuation property of analytic functions and by the connectivity of the graph of $F$.

**Step 4.** If $M'$ is strictly pseudoconvex then $F$ is generically singlevalued, i.e., is a meromorphic map.

Suppose $z' \in F(z)$ for some $z \in M \setminus (I_F \cup R_F)$, where $R_F$ is the divisor of ramification of $F$. By the invariance of Segre varieties $F(Q_z) \subset Q_{z'}'$. This means that for all branches $F_1,...,F_d$ of $F$ near $z$ one has $F_j(Q_z) \subset Q_{z'}'$. Let $z' = F_1(z)$ for simplicity and suppose that there is $w' = F_2(z)$ different from $z'$. Then $F_1(Q_z) \subset Q_{w'}'$ as well. But due to the
strict pseudoconvexity of $M'$ we have that $Q'_{z'} \cap M' = \{z'\}$ and $Q'_{w'} \cap M' = \{w'\}$. I.e., the germs of $Q'_{z'}$ and $Q'_{w'}$ are disjoint. Contradiction, i.e., $z' = w'$ and $F$ is singlevalued on $M \setminus (I_F \cup R_F)$. This implies that $R_F = \emptyset$ and $F$ is singlevalued on $M \setminus I_F$.

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References

[BS] Burns D., Shnider S.: Spherical Hypersurfaces in Complex Manifolds. Invent. math. 33, 223-246 (1976).

[CM] Chern S., Moser J.: Real hypersurfaces in complex manifolds. Acta Math. 133, 219-271 (1974).

[C] Chiappari S.: Holomorphic Extension of Proper Meromorphic mappings. Michigan Math. J. 38, 167-174 (1991).

[DF] Diederich K., Fornaess J.-E.: Pseudoconvex domains with real-analytic boundary. Ann. Math. 107, no.: 2, 371-384 (1978).

[F] Fischer G.: Plane Algebraic Curves. AMS, Stud. Math. Libr. V. 16 (2001).

[G] Gromov M.: CarnotCarathéodory Spaces Seen From Within. Progress in Mathematics, 144, 79-318, Birkhäuser, Basel (1996).

[GR] Gunning R., Rossi H.: Analytic Functions of Several Complex Variables. Prentice-Hall (1965).

[IM] Ivashkovich S., Meylan F.: An Example Concerning Holomorphicity of Meromorphic Mappings Along Real Hypersurfaces. Mich. Math. J. 64, 487-491 (2015).

[MP] Merker J., Porten E.: On wedge extendability of CR-meromorphic functions. Math. Z. 241, no: 3, 485-512 (2002).

[Na] Narasimhan R.: Introduction to the Theory of Analytic spaces. Lecture Notes Mathematics, 25, Springer, Berlin (1966).

[P] Pinchuk S.: Analytic continuation of holomorphic mappings and the problem of holomorphic classification of multidimensional domains. Doctor Nauk Thesis (Habilitation), Chelyabinsk (1979).

[S] Shafikov R.: Analytic continuation of germs of holomorphic mappings between real hypersurfaces in $\mathbb{C}^n$. Mich. Math. J. 47, no: 1,133-149 (2000).

[SV] Shafikov R., Verma K.: Extension of holomorphic maps between real hypersurfaces in different dimension. Ann. Inst. Fourier 57, 2063-2080 (2007).