COMPUTING INTEGRAL POINTS ON HYPERELLIPTIC CURVES USING QUADRATIC CHABAUTY

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Abstract. We give a method for the computation of integral points on a hyperelliptic curve of odd degree over the rationals whose genus equals the Mordell-Weil rank of its Jacobian. Our approach consists of a combination of the $p$-adic approximation techniques introduced in previous work with the Mordell-Weil sieve.

1. Introduction

Let $f \in \mathbb{Z}[x]$ be a separable polynomial of degree at least 3. The problem of determining all integer solutions $(x, y)$ to the equation

$$y^2 = f(x)$$

is classical in nature and has been studied from several different viewpoints, yielding a toolbox of approaches. Techniques from Diophantine approximation, which make use of archimedean analysis, include linear forms in elliptic logarithms, $S$-unit equations, and Baker’s method. We discuss each of these techniques and their practical limitations – usually arising from astronomical height bounds or working with number fields of large degree – in Section 1.1 below.

In the realm of non-archimedean analysis, perhaps the most celebrated technique in the study of integral points is Coleman’s interpretation [19] of the method of Chabauty [18], allowing one to determine the rational points on a curve whose Jacobian has Mordell-Weil rank less than its genus. This is accomplished by computing a $p$-adic line integral, among whose zeros lie the rational points of the curve. The hypothesis that the rank is less than genus is essential, and in practice, it is perhaps more difficult to check this hypothesis than to carry out the actual construction of the $p$-adic integral.

Over the last decade, Kim has initiated an exciting program aimed at removing this restricting on rank, allowing the study of rational points on hyperbolic curves through the use of nonabelian geometric objects generalizing the role of the Jacobian in the Chabauty-Coleman method [28, 7]. Through delicate calculations in $p$-adic Hodge theory, this nonabelian Chabauty method can produce iterated $p$-adic integrals playing the role of the abelian integrals arising from the Jacobian of the curve [29, 8].

In the spirit of Kim’s program, we previously gave a method [5] based on $p$-adic height pairings to $p$-adically approximate the set of integral points in the case when the curve has Jacobian with Mordell-Weil rank equal to its genus. To state this more precisely, let us introduce some notation. Consider the genus $g$ hyperelliptic curve...
which has an affine model given by the equation \( y^2 = f(x) \) with \( \deg f = 2g + 1 \), and let \( J \) denote its Jacobian. Fix an odd prime \( p \) such that \( X \) has good reduction at \( p \). Define locally analytic functions \( f_i : X(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p \) by

\[
    f_i(z) = \int_{-\infty}^{z} \frac{x^i \, dx}{2y}, \quad 0 \leq i \leq g - 1
\]

and extend them linearly to functionals

\[
    f_i : J(\mathbb{Q}) \otimes \mathbb{Q} \to \mathbb{Q}_p.
\]

The method of Chabauty and Coleman can be interpreted as follows: If the Mordell-Weil rank of \( J(\mathbb{Q}) \) is less than \( g \), then one can construct a function given by a linear combination of the \( f_i \) on \( J(\mathbb{Q}_p) \) that vanishes on \( J(\mathbb{Q}) \). By restricting this function to \( X(\mathbb{Q}_p) \), one can approximate the points \( X(\mathbb{Q}) \).

In [5] we showed that under the hypotheses that \( J \) has ordinary reduction at \( p \), the Mordell-Weil rank of \( J(\mathbb{Q}) \) is equal to \( g \) and the \( f_i \) are linearly independent functionals on \( J(\mathbb{Q}) \otimes \mathbb{Q} \), one can construct a locally analytic function \( \rho \) that takes on a prescribed, finite set of constants \( T \) on integral points of \( X \). By setting \( \rho \) equal to each of these values, we find, among the set of \( p \)-adic points, the integral points on the working affine model of \( X \). The function \( \rho \) arises naturally from a local decomposition of the global \( p \)-adic height pairing, a natural quadratic form on the Jacobian; we refer to this method of finding integral points as quadratic Chabauty.

In this work, we show how quadratic Chabauty can be used in practice to provably find all integral points on \( X \) by combining it with the Mordell-Weil sieve. Since \( \rho \) can be written as a convergent power series on every residue disk, we can explicitly determine the finitely many solutions over \( \mathbb{Z}_p \) to the equations

\[
    \rho(z) = t,
\]

up to some finite precision \( p^N \), as \( t \) runs through \( T \). For this computation, we need to relate the global \( p \)-adic height to a natural basis of quadratic forms on \( J(\mathbb{Q}) \otimes \mathbb{Q} \), compute double Coleman integrals describing the local height contribution at \( p \), and calculate the arithmetic intersections describing the local height contributions away from \( p \). We briefly discussed how to compute these quantities in [5]; there we also presented two examples. In this paper, we give a full algorithm describing the necessary computations and provide an analysis of the \( p \)-adic precision which must be maintained throughout the computation. We furthermore detail how to combine quadratic Chabauty with the Mordell-Weil sieve to precisely find the set of integral points.

We note that the algorithms for computing integral points on elliptic curves over \( \mathbb{Q} \) are rather well-developed and perform particularly well in the case of rank 1 (see Section 1.1), so we have just a few remarks here, combining quadratic Chabauty with information about the group of \( \mathbb{F}_q \)-rational points for primes \( q \) of good reduction. This method is treated in Appendix A.

In this paper, we focus on a method for curves having genus at least 2. Here the existing approaches usually require either the rank \( r \) of \( J(\mathbb{Q}) \) to be strictly smaller than \( g \) or the availability of a set of Mordell-Weil generators, which currently is only possible for \( g \leq 3 \); see Section 1.1. Our approach here, combining quadratic Chabauty with the Mordell-Weil sieve, thus presents the first systematic and practical method to compute integral points for curves \( X \) as above with \( r = g \geq 4 \).
In broad terms, the idea is as follows: We first find the integral points of small height; our goal is to show that this set already contains all integral points. Then we apply quadratic Chabauty using several primes, producing an integer $M$ and a list of residue classes in $J(\mathbb{Q})/MJ(\mathbb{Q})$. We want to show that each of these residue classes cannot contain the image of a rational point on $X$. We accomplish this by applying the Mordell-Weil sieve. This method, pioneered by Scharaschkin [38], combines information modulo several primes $v$ of good reduction by finding the image of $X(\mathbb{F}_v)$ inside $J(\mathbb{F}_v)/MJ(\mathbb{F}_v)$.

The idea of using $p$-adic techniques, notably Chabauty’s method, to come up with congruence conditions and hence suitable lists of residue classes for the Mordell-Weil sieve is not new, see [13] and, in particular [37], which also uses congruence conditions modulo different primes. In the present work, however, we choose the primes $v$ in the Mordell-Weil sieve computation and the primes used for quadratic Chabauty at the same time, making it possible to keep the number of residue classes and/or the number (and size) of the primes $v$ comparatively small. This is necessary, because we want to consider examples of large genus (and, hence, large rank), and one typically needs to find about $v$ discrete logarithms in $J(\mathbb{F}_v)$, which becomes slow for large $v$ and $g$. See Example 9.3, where we use this approach to find the integral points on a curve of genus 4; here $v = 317$ is the largest prime we had to consider.

The structure of this paper is as follows: In Section 2, we review the method of quadratic Chabauty. In Section 3 we give a more detailed description of the algorithms needed to compute the quantities in quadratic Chabauty and to solve equation (1.1). In Section 4, we discuss some practical considerations arising from the method. We present the Mordell-Weil sieve and explain how to apply it in our situation in Sections 5-7. This allows us to describe a complete method for the computation of integral points when $X$ satisfies $r = g > 1$ in Section 8. Finally, we mention a few examples in Section 9, one of which has genus 4 and rank 4 and is not amenable to previous algorithms.

Most of our algorithms can be generalized to hyperelliptic curves defined over number fields. This is work in progress.

1.1. Other approaches. We discuss here how our method compares to other approaches in the literature. For completeness, we also give an overview of existing methods in the case of genus 1 as well.

The most efficient technique for the computation of integral points on elliptic curves relies on linear forms in elliptic logarithms. These can be used to compute an upper bound on the height of integral points [21, 27], or, equivalently, on the size of the coefficients of an integral point in terms of a given set of Mordell-Weil generators. This bound can then be reduced using the LLL-algorithm, see for instance [40, 34, 47]; frequently, the resulting bound is small enough to find all integral points simply by searching up to this bound. The computer algebra systems Sage [42] and Magma [11] contain an implementation of this algorithm.

Quadratic Chabauty can be combined with this approach, as it can be used to provide a lower bound on the height of all integral points that have not been found yet. However, here we are restricted to the rank 1 case, where usually a generator of the Mordell-Weil group modulo torsion is easily computed and the upper bounds after LLL-reduction are small enough so that only a few multiples of the generator have to be computed to provably find all integral points.
For genus greater than 1, no analogue of the algorithm based on elliptic logarithms is known. We briefly summarize the existing algorithms. More detail can be found in the introduction to [17]; see also [41].

The method of Chabauty-Coleman [19] is a \( p \)-adic method arising from Coleman’s reinterpretation of Chabauty’s theorem [18]; see [31] for an introduction to the method as well as some further variations. It can often be used to compute \( X(\mathbb{Q}) \) in practice, provided \( g \) is greater than the rank of \( J(\mathbb{Q}) \). It can also be combined with the Mordell-Weil sieve [16], which we discuss in detail in Section 5. On its own, the Mordell-Weil sieve is most useful to show that a given curve has no rational points, which is never the case for our curves.

Another class of approaches to the computation of \( X(\mathbb{Q}) \) is based on coverings of \( X \), see for instance [22, 15]. These require computations of the class group and unit group of number fields of potentially large degree. In practice, such methods often combine the use of coverings with elliptic curve Chabauty [12, 25], which requires the computation of Mordell-Weil groups of elliptic curves over number fields of large degree. Such computations can be prohibitively expensive in practice, but sometimes work rather well for reasonably small genus, rank and coefficients.

There are also methods which can be used to provably find all integral points. These are based on Baker’s method, which can be used to compute an upper bound on the height of integral points on \( X \), which provides a theoretical algorithm for the computation of the integral points. While Baker’s original bounds were too large to be of much practical use, a number of authors have produced substantial improvements; see [17, §1] for an overview. Some algorithms based on Baker’s method are given in [41].

In [17], a refinement of the Mordell-Weil sieve is discussed, which can be used to show that any rational points on \( X \) which have not been found yet must have extremely large height. It is possible to obtain lower bounds around \( 10^{2000} \) in this way. The authors of loc. cit. also produce explicit upper bounds on the height of integral points. These are often good enough to show that all integral points have been found when combined with the lower bounds. The main obstacle is that for the refinement of the Mordell-Weil sieve one needs an explicit set of generators of the Mordell-Weil group \( J(\mathbb{Q}) \). In practice, this is only possible when \( g \leq 3 \), see [24, 43, 45, 32, 46].

The method we present here in Section 8 is applicable whenever we can prove that the rank is equal to the genus and we have explicit generators of a subgroup of \( J(\mathbb{Q})/J(\mathbb{Q})_{\text{tors}} \) of finite index.

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2. Overview of quadratic Chabauty

In this section, we give an overview of the method of quadratic Chabauty [5] for hyperelliptic curves and describe the \( p \)-adic analytic functions that allow us to find
integral points. Let \( f \in \mathbb{Z}[x] \) be a separable polynomial of degree \( 2g + 1 \geq 3 \). The equation
\[
y^2 = f(x)
\]
defines a hyperelliptic curve \( X \) of genus \( g \) over \( \mathbb{Q} \) as follows: Let
\[
\mathcal{U} = \text{Spec}(\mathbb{Z}[x,y]/(y^2 - f(x))).
\]
Then \( X \) is the normalization of the projective closure of the generic fiber of \( \mathcal{U} \); it contains a unique point \( \infty \) at infinity, and this point is \( \mathbb{Q} \)-rational. We denote by \( \nu \) the embedding of \( X \) into its Jacobian \( J \) which maps a point \( P \) to the class of \( (P) - (\infty) \). Fix a branch of the \( p \)-adic logarithm, which we denote as \( \log_p \).

We assume that \( p \) is a prime of good reduction for \( X \). For a finite place \( v \) of \( \mathbb{Q} \) and divisors \( D_1 \) and \( D_2 \) on \( X(\mathbb{Q}_v) \) of degree zero with disjoint support, let
\[
h_v(D_1, D_2) \in \mathbb{Q}_p
\]
denote the local \( p \)-adic height pairing at \( v \) defined by Coleman and Gross [20] with respect to a direct sum decomposition of \( H^1_{\text{dR}}(X \times \mathbb{Q}_p) \):
\[
(2.1) \quad H^1_{\text{dR}}(X \times \mathbb{Q}_p) = W \oplus H^1_{\text{dR}}(X \times \mathbb{Q}_p, \Omega^1_{X \times \mathbb{Q}_p}).
\]
The local \( p \)-adic height pairing is symmetric if and only if \( W \) is isotropic with respect to the cup product pairing. When \( p \) is a prime of good ordinary reduction, we take \( W \) to be the unit root subspace for the action of Frobenius.

For \( v \neq p \), the symmetric bilinear pairing \( h_v(D_1, D_2) \) can be defined in terms of arithmetic intersection theory on a regular model of \( X \times \mathbb{Q}_v \) over \( \mathbb{Z}_v \), while for \( v = p \) it is given by a Coleman integral along \( D_2 \) of a certain differential of the third kind with residue divisor \( D_1 \).

If \( D_1 \) and \( D_2 \) are defined over \( \mathbb{Q} \), then the sum
\[
h(D_1, D_2) = \sum_v h_v(D_1, D_2)
\]
of local height pairings over the finite places \( v \) of \( \mathbb{Q} \) respects linear equivalence. It induces the global Coleman-Gross \( p \)-adic height pairing
\[
h : J(\mathbb{Q}) \times J(\mathbb{Q}) \to \mathbb{Q}_p.
\]
This is a symmetric bilinear pairing, which is conjectured to be nondegenerate.

In [4], it was shown that the restriction that \( D_1 \) and \( D_2 \) must have disjoint support can be removed by extending the local height pairing relative to a choice of tangent vectors. This theme was further explored in [5], where we used a certain consistent choice of tangent vectors to study local heights. The resulting local height pairing at \( p \) gives rise to a function
\[
\tau(z) = h_p((z) - (\infty), (z) - (\infty))
\]
defined on \( X(\mathbb{Q}_p) \); we can then extend \( \tau \) to a function on \( X(\bar{\mathbb{Q}}_p) \). This function turns out to be a Coleman function in the sense of [10] and, by [5, Theorem 2.2], a double Coleman integral. Hence it can be written as a convergent \( p \)-adic power series on every residue disk.

We define locally analytic functions \( f_i : X(\bar{\mathbb{Q}}_p) \to \bar{\mathbb{Q}}_p \) by
\[
f_i(z) = \int_{\infty}^{z} \omega_i, \quad 0 \leq i \leq g - 1
\]
where \( \omega_i = \frac{x_i}{2x^2} \). These \( f_i \) can be extended linearly to functionals
\[
f_i : J(\mathbb{Q}_p) \otimes \mathbb{Q} \to \mathbb{Q}_p.
\]
Because Coleman integrals are compatible with the action of the absolute Galois group of \( \mathbb{Q}_p \), the restrictions of these functionals to \( J(\mathbb{Q}) \otimes \mathbb{Q} \), also denoted by \( f_i \), are \( \mathbb{Q}_p \)-valued.

From now on, we make the assumption that \( f \) does not reduce to a square modulo \( q \) for any prime number \( q \). In [5] we proved the following result as Theorem 3.1:

**Theorem 2.1.** Suppose that the Mordell-Weil rank of \( J(\mathbb{Q}) \) is \( g \) and that the \( f_i \) are linearly independent as functionals on \( J(\mathbb{Q}) \otimes \mathbb{Q} \). Then there exist constants \( \alpha_{ij} \in \mathbb{Q}_p \) such that the locally analytic function
\[
\rho(z) = \tau(z) - \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} f_i(z)f_j(z)
\]
takes values on \( U(\mathbb{Z}[1/p]) \) in an effectively computable finite set \( T \subset \mathbb{Q}_p \).

**Remark 2.2.** In the statement of [5, Theorem 3.1] we incorrectly required only that no linear combination of the \( f_i \) vanished on \( U(\mathbb{Z}[1/p]) \), but what is actually needed is the stronger independence assumption stated here, which was indeed used in the proof of the theorem.

**Remark 2.3.** Note that we can apply a variant of the method of Chabauty-Coleman if a linear combination of the \( f_i \) vanishes on \( U(\mathbb{Z}[1/p]) \), so the “quadratic Chabauty” situation described in the theorem is really the novel case.

We give a sketch of the proof of Theorem 2.1. Because the rank is \( g \) and the \( f_i \) are linearly independent, the set
\[
\left\{ \frac{1}{2}(f_if_j + f_jf_i) \right\}_{0 \leq i \leq j \leq g-1}
\]
forms a basis of the space of \( \mathbb{Q}_p \)-valued bilinear forms on \( J(\mathbb{Q}) \otimes \mathbb{Q} \). The global \( p \)-adic height pairing \( h \) is also a \( \mathbb{Q}_p \)-valued bilinear form on \( J(\mathbb{Q}) \otimes \mathbb{Q} \), so there are constants \( \alpha_{ij} \in \mathbb{Q}_p \) such that
\[
h((P) - (\infty), (P) - (\infty)) = \sum_{i \leq j} \alpha_{ij} f_i(P)f_j(P)
\]
for all \( P \in X(\mathbb{Q}) \).

The right hand side of (2.3) extends to a locally analytic \( \mathbb{Q}_p \)-valued function on \( X(\mathbb{Q}_p) \). It then only remains to show that
\[
\rho(z) := \tau(z) - \sum_{i \leq j} \alpha_{ij} f_i(z)f_j(z)
\]
can only take values on \( U(\mathbb{Z}[1/p]) \subset X(\mathbb{Q}_p) \) in an effectively computable finite subset of \( \mathbb{Q}_p \). For this, we analyze the possible values of the local height pairings \( h_q((P) - (\infty), (P) - (\infty)) \) for \( q \neq p \) and \( P \in U(\mathbb{Z}[1/p]) \), and use that by construction
\[
\rho(P) = -\sum_{q \neq p} h_q((P) - (\infty), (P) - (\infty))
\]
for \( P \in X(\mathbb{Q}) \).
To ease notation, we will only deal with a single prime $q$ and omit it from the notation whenever possible. Following [5, §3], we let $X$ denote a strong desingularization of the Zariski closure $X'$ of $X \times \mathbb{Q}_q$ over $\text{Spec} \mathbb{Z}_q$. By the latter, we mean the scheme defined in the weighted projective plane $\mathbb{P}^2_{\mathbb{Z}_q}(1, g+1, 1)$ by the equation
\[ Y^2 = F(X, Z), \]
where $F(X, Z)$ is the degree $2g + 2$-homogenization of $f$. By abuse of notation, the section in $X(\mathbb{Q}_q)$ corresponding to a point $P \in X(\mathbb{Q}_q)$ will also be denoted by $P$.

Our assumptions on $f$ guarantee that $X'$ is normal with irreducible and reduced special fiber. Hence there is a unique component of the special fiber of $X'$ which dominates the special fiber of $X''$, and any $F \in X(\mathbb{Q}_q)$ whose reduction modulo $q$ is nonsingular has the property that the section $F$ intersects this component. Let $\Phi(\cdot)$ denote the rational-valued intersection multiplicity on $X$. For a divisor $Z$ on $X \times \mathbb{Q}_q$ of degree zero, we let $\Phi(Z)$ denote a vertical $\mathbb{Q}$-divisor on $X'$ such that $Z + \Phi(Z)$ has intersection multiplicity 0 with all vertical divisors on $X'$; such a $\mathbb{Q}$-divisor always exists by [30, Theorem III.3.6]. Then, by the proof of [5, Proposition 3.3], we have
\[ h_q((-\infty), (P) - (\infty)) = -(D_P \cdot D_P) \log_p(q), \]
where
\[ D_P = (P) - (\infty) + \Phi((P) - (\infty)) \in \text{Div}(X) \otimes \mathbb{Q}, \]
with the abuse of notation introduced above. Here the self-intersection is defined with respect to the choice of tangent vectors mentioned above; see the proof of [5, Proposition 3.3] and [26, §5] for further details.

Let $T(q) \subset \mathbb{Q}$ denote the set of all self-intersections $D_P^2 = (D_P \cdot D_P)$, where $P$ runs through $U(\mathbb{Z}_q)$.

**Lemma 2.4.** (i) The intersection multiplicity $D_P^2$ only depends on the fiber of $X_q$ which the section $P$ intersects.

(ii) If $T(q) \neq \{0\}$, then the special fiber $X_q$ is reducible or $q$ divides the leading coefficient of $f$. In particular, $T(q) = \{0\}$ if $q$ is a prime of good reduction.

(iii) If $q_1, \ldots, q_\ell$ are the primes $q$ for which $T(q) \neq \{0\}$, then
\[ \rho(U(\mathbb{Z}[1/p])) \subset \left\{ \sum_{i=1}^{\ell} d_i \log_p(q_i) : d_i \in T(q_i) \right\}. \]

**Proof.** For the first two statements see the proof of [5, Proposition 3.3]. The proof of (iii) follows from (2.4) and (2.5). \qed

In particular, the value of $\rho$ at a $p$-integral point $P$ on $X$ is completely determined by the intersection multiplicities $D_{P,q}^2 \in \mathbb{Q}$ for primes $q$ of bad reduction and these rational numbers do not depend on $p$! We encode this global data as follows:

**Definition 2.5.** We call a prime $q$ very bad for $X$ if $T(q) \neq \{0\}$. Suppose that $q_1 < q_2 < \ldots < q_\ell$ are the very bad primes for $X$. We call a tuple
\[ (d_1, \ldots, d_\ell) \in \mathbb{Q}^\ell \]
an intersection pattern if $d_i \in T(q_i)$ for all $1 \leq i \leq \ell$. Let $\mathcal{P}$ denote the set of intersection patterns; this set is finite by Lemma 2.4. We say a point $P \in U(\mathbb{Z}[1/p])$
has intersection pattern \( d = (d_1, \ldots, d_t) \in \mathcal{P} \), if \( D^{2}_{P,q_{i}} = d_{i} \) for \( i \in \{1, \ldots, t\} \). Moreover, we define a \( p \)-adic logarithm map \( \log_{p} : \mathcal{P} \rightarrow \mathbb{Q}_{p} \) by

\[
\log_{p}(d) = \sum_{i=1}^{n} d_{i} \log_{p}(q_{i}).
\]

If there are no very bad primes, then we set \( \mathcal{P} = \{1\} \) and accordingly \( \log_{p}(\mathcal{P}) = \{0\} \).

With this notation, the following result follows immediately from Lemma 2.4:

**Corollary 2.6.** Let \( T := \log_{p}(\mathcal{P}) \subset \mathbb{Q}_{p} \). Then we have

\[
\rho(\mathcal{U}(\mathbb{Z}[1/p])) \subset T.
\]

### 3. Algorithms for quadratic Chabauty

Building on the theory described in the previous section, in this section we discuss how to use Theorem 2.1 to \( p \)-adically approximate integral points in practice. In §§3.1 — 3.4 we give algorithms to compute the set \( T \) and each of the quantities in the equation giving the function \( \rho(z) \):

\[
(3.1) \quad \rho(z) := \tau(z) - \sum_{0 \leq i \leq g-1} \sum_{0 \leq j \leq g-1} \alpha_{ij} f_{i}(z) f_{j}(z) = t \in T.
\]

Finally, in §3.5, we discuss how to compute the solutions in \( \mathcal{U}(\mathbb{Z}_{p}) \) to (3.1). By Theorem 2.1, the set of solutions is finite and will contain the integral points on \( X \). We will henceforth assume that \( p \) is a prime of good ordinary reduction.

#### 3.1. Computing the \( \alpha_{ij} \)

The first step is to express the global \( p \)-adic height function as a \( \mathbb{Q}_{p} \)-linear combination of a natural basis of the space of \( \mathbb{Q}_{p} \)-valued quadratic forms on \( J(\mathbb{Q}) \otimes \mathbb{Q} \). To do this, we need independent non-torsion points \( P_{1}, \ldots, P_{g} \in J(\mathbb{Q}) \). In practice we usually have such points available if we know that \( r = g \), because the verification of this fact requires bounding \( r \) from below, typically by finding \( g \) independent non-torsion points in \( J(\mathbb{Q}) \).

We compute the global \( p \)-adic height pairings \( h(P_{1}, P_{1}), h(P_{1}, P_{2}), \ldots, h(P_{g}, P_{g}) \) using [9, Algorithm 3.8]; each \( p \)-adic height is a sum of Coleman-Gross local height integrals (a precision analysis of which is done in [3, §6]), as well as rational multiples of \( \log_{p}(q_{i}) \) for rational primes \( q_{i} \) (which can be computed to the necessary precision using standard methods).

Next, we check if the set \( \{f_{0}, \ldots, f_{g-1}\} \) gives a basis of the \( \mathbb{Q}_{p} \)-valued functionals on \( J(\mathbb{Q}) \otimes \mathbb{Q} \), or equivalently, if the set

\[
(3.2) \quad \left\{ \frac{1}{2}(f_{i} f_{j} + f_{j} f_{i}) \right\}_{0 \leq i \leq j \leq g-1}
\]

forms a basis for the space of \( \mathbb{Q}_{p} \)-valued bilinear forms on \( J(\mathbb{Q}) \otimes \mathbb{Q} \). We use the fact that we have, for each pair \((k, l)\), the relationship

\[
h(P_{k}, P_{l}) = \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} \cdot \left( \frac{1}{2} (f_{i}(P_{k}) f_{j}(P_{l}) + f_{j}(P_{k}) f_{i}(P_{l})) \right),
\]

if (3.2) gives a basis. In particular, by computing the determinant of the matrix whose entries are given by \( \left( \frac{1}{2} (f_{i}(P_{k}) f_{j}(P_{l}) + f_{j}(P_{k}) f_{i}(P_{l})) \right) \) and verifying that it is nonzero, we determine that the \( f_{i} \) are linearly independent \( \mathbb{Q}_{p} \)-valued functionals.
on $J(\mathbb{Q}) \otimes \mathbb{Q}$. If the $f_i$ aren’t linearly independent, then we instead use the classical Chabauty-Coleman method and compute $X(\mathbb{Q})$.

To set up the linear system relating the global height pairing $h$ to the elements in (3.2), we use the computed values of $h(P_k, P_l)$ as well as the values of the sums of products of Coleman integrals

$$\frac{1}{2}(f_i(P_k)f_j(P_l) + f_j(P_k)f_i(P_l)),$$

computed as in [6, Algorithm 11] to recover the $\alpha_{ij}$.

We postpone a discussion of all precision concerning Coleman integrals to §3.3. The precision of the integrals present in (3.3) is discussed in Proposition 3.3.

### 3.2. Computing $f_i(z)$

The Coleman integral

$$f_i(z) = \int_{\infty}^{z} \omega_i$$

of a regular 1-form $\omega_i$ is computed as a power series expansion for each residue disk. We compute the set of residue disks, then for each residue disk, do the following:

1. Take a lift of the residue disk to a point $P \in U(\mathbb{Z}_p)$. In particular, if working in a Weierstrass residue disk, take the lift to be the characteristic zero Weierstrass point; else, take an arbitrary lift.
2. Compute $f_i(P) = \int_{\infty}^{P} \omega_i = \frac{1}{2} \int_{-P}^{P} \omega_i$ using [6, Lemma 16], which sets the global constant of integration and reduces the computation of $f_i(z)$ to that of a tiny Coleman integral.
3. Take $z$ to be a local coordinate in the disk of $P$ and break up the path between $\infty$ and $z$ using

$$f_i(z) = \int_{\infty}^{P} \omega_i + \int_{P}^{z} \omega_i = f_i(P) + \int_{P}^{z} \omega_i,$$

which allows $\int_{P}^{z} \omega_i$ to be computed as a tiny Coleman integral, as in [6, Algorithm 8].

Computing $f_i(P)$ requires the precision given in Proposition 3.3.

### 3.3. Computing $\tau$

We begin with the observation that the local component of the $p$-adic height at $p$,

$$\tau(z) = h_p((z) - (\infty), (z) - (\infty)),$$

can be expressed as a sum of double Coleman integrals [5, Theorem 2.2]:

$$\tau(z) = -2 \int_{b_0}^{z} \sum_{i=0}^{q-1} \omega_i \tilde{\omega}_i,$$

where $b_0$ is a tangential basepoint at infinity, as in [5, §2]. Then computing the $p$-adic power series $\tau(z)$ amounts to

- computing the value $\tau(P)$ for an arbitrary point $P \in U(\mathbb{Z}_p)$ to set the global constant of integration and
• fixing a lift of Frobenius \( \phi \), computing the the action of \( \phi \) on the first Monsky-Washnitzer cohomology of an appropriate affine piece of the curve, which yields the analytic continuation of the double integral to each of the other residue disks not containing \( P \).

In practice, what is computationally easier is to compute the set of residue disks, then for each residue disk, do the following:

1. Take a lift to a point \( P \in \mathcal{U}(\mathbb{Z}_p) \) (if \( P \) is in a Weierstrass disk, take the lift to be the characteristic zero Weierstrass point; else, take an arbitrary lift) and compute \( \tau(P) \), setting the constant of integration.
2. Compute the dual basis \( \{\tilde{\omega}_i\} \subset W \) for \( \omega_0, \ldots, \omega_{g-1} \) with respect to the cup product pairing. This is done by
   
   (a) first computing, for \( j = g, \ldots, 2g-1 \), the differentials \( \tilde{\omega}_j \), which are the projection of the \( \omega_j \) on \( W \) along the space of holomorphic forms.
   
   Using the cup product matrix, this gives each \( \tilde{\omega}_0, \ldots, \tilde{\omega}_{g-1} \) as a \( \mathbb{Q} \)-linear combination of \( \tilde{\omega}_g, \ldots, \tilde{\omega}_{2g-1} \).

   (b) Then we compute the projection of \( \omega_g, \ldots, \omega_{2g-1} \) with respect to the basis \( \{\omega_0, \ldots, \omega_{g-1}, \tilde{\omega}_g, \ldots, \tilde{\omega}_{2g-1}\} \). The linear algebra allows us to write each \( \tilde{\omega}_g, \ldots, \tilde{\omega}_{2g-1} \), and consequently, each \( \tilde{\omega}_0, \ldots, \tilde{\omega}_{g-1} \) as a sum involving a \( \mathbb{Q}_p \)-linear combination of \( \{\tilde{\omega}_0, \ldots, \tilde{\omega}_{g-1}\} \), together with a \( \mathbb{Q} \)-linear combination of the \( \{\omega_g, \ldots, \omega_{2g-1}\} \).

3. Compute the double integral for \( \tau(z) \) as a power series in \( z \) using \( \tau(P) \); in particular, the relevant double integrals from \( P \) to \( z \) are computed as tiny integrals within that disk, using the fact that

\[
\tau(z) = \tau(P) - 2 \sum_{i=0}^{g-1} \left( \int_P^z \omega_i \tilde{\omega}_i + \frac{1}{2} \int_P^z \omega_i \int_{-P}^P \tilde{\omega}_i \right).
\]

The tiny double integrals are computed as in [2, Algorithm 5.1].

We now compute the necessary precision for each local height and Coleman integral discussed in this section. We begin by recalling the statement of Proposition 18 of [6], very slightly adapted to fit our purposes:

**Proposition 3.1** (Proposition 18, [6]). Let \( \int_P^Q \omega \) be a tiny integral in a residue disk not equal to the disk at infinity, with \( P, Q \in \mathcal{U}(\mathbb{Z}_p) \) accurate to \( n \) digits of precision. Let \( (x(t), y(t)) \) be the local interpolation between \( P \) and \( Q \) and \( \omega = g(x, y)dx \) a differential of the second kind such that \( h(t) = g(x(t), y(t)) \) belongs to \( \mathbb{Z}_p[[t]] \). If we truncate \( h(t) \) modulo \( t^m \), then the computed value of the integral \( \int_P^Q \omega \) will be correct to \( \min\{n, m + 1 - |\log(m + 1)|\} \) digits of precision.

The corresponding statement for tiny double integrals easily follows:

**Corollary 3.2.** Let \( \int_P^Q \omega \eta \) be a tiny double integral in a residue disk not equal to the disk at infinity, with \( P, Q \in \mathcal{U}(\mathbb{Z}_p) \) accurate to \( n \) digits of precision. Let \( (x(t), y(t)) \) be the local interpolation between \( P \) and \( Q \) and \( \omega \int \eta = g(x, y)dx \) a differential of the second kind such that \( h(t) = g(x(t), y(t)) \) belongs to \( \mathbb{Z}_p[[t]] \). If we truncate \( h(t) \) modulo \( t^m \), then the computed value of the integral \( \int_P^Q \omega \eta \) will be correct to \( \min\{n, m + 1 - |\log(m + 1)|\} \) digits of precision.

As a special case of [6, Proposition 19], we have
Proposition 3.3. Let \( \int_P^Q \omega \) be a Coleman integral of \( \omega \) a differential of the first kind or a differential of the second kind without poles in the disks of integration. Suppose \( P, Q \in \mathcal{U}(\mathbb{Z}_p) \) are in non-Weierstrass residue disks and given accurate to \( n \) digits of precision. Let \( \text{Frob} \) be the matrix of the action of Frobenius on the basis differentials. Set \( B = \text{Frob}^t - I_{2g \times 2g} \) and let \( m_1 = v_p(\det(B)) \). Then the computed value of the integral \( \int_P^Q \omega \) will be accurate to \( n - \max\{m_1, |\log n|\} \) digits of precision.

Proof. The Coleman integral is computed via a linear system, where the quantities involved are an inverted matrix \( B^{-1} \) times a vector of constants involving tiny integrals and integrals of exact forms evaluated at points. Suppose the entries of \( B \) are computed to precision \( n \). Then taking \( B^{-1} \), we must divide by \( \det(B) \), which lowers the precision by \( m_1 = v_p(\det(B)) \). By Proposition 3.1, computing tiny integrals with the series expansions truncated modulo \( t^n \) gives a result precise up to \( n - |\log n| \) digits. Thus the value of the integral \( \int_P^Q \omega \) will be correct to \( n - \max\{m_1, |\log n|\} \) digits of precision. \( \square \)

Finally, we discuss precision of local height integrals:

Proposition 3.4. Let \( P = (a, b) \in \mathcal{U}(\mathbb{Z}_p) \) be a point accurate to \( n \) digits of precision. Let \( \tau(P) \) be the local height pairing at \( p \) of \( (P) - (\infty) \) with itself. Let \( \text{Frob} \) be the matrix of the action of Frobenius on the basis differentials. Set \( B = \text{Frob}^t - I_{2g \times 2g} \) and let \( m_1 = v_p(\det(B)) \). There exists an algorithm to compute \( \tau(P) \) to \( n - \max\{m_1, |\log n|\} \) digits of precision.

Proof. Let \( a_{2g+1} \) denote the leading coefficient of the polynomial \( f \) defining \( X \). Suppose \( P \) is a Weierstrass point. Then by [5, Lemma 4.1], we have \( \tau(P) = \frac{1}{2}(\log_p(f'(a)) + \log_p(a_{2g+1})) \), and we are done.

Now suppose \( P \) is a non-Weierstrass point. We discuss the precision, mostly in terms of local coordinates, necessary to compute \( \tau(P) \) to \( n - \max\{m_1, |\log n|\} \) digits of precision. Throughout, we recall formulas given in [5, §4] and [3, §6].

We begin with the following equation for \( \tau(P) \) from [5, §4]:

\[
\tau(P) = \frac{1}{4}(\log_p(4b^2) + 2\log_p(a_{2g+1})) + \frac{1}{4}\left( \int_{w(P)}^P \frac{b \, dx}{y(x-a)} - \int_{w(P)}^P \eta \right),
\]

where \( \eta \) is the holomorphic projection of the differential \( \frac{b \, dx}{y(x-a)} \), computed from the splitting (2.1). Computing Coleman integrals as in Proposition 3.3 gives the integral of \( \eta \) correct to \( n - \max\{m_1, |\log n|\} \) digits of precision.

The integral \( \int_{w(P)}^P \frac{b \, dx}{y(x-a)} \) is somewhat more delicate to analyze, since it is the Coleman integral between two different residue disks of a differential form with poles at both of its endpoints, \( P \) and \( w(P) \). To compute it, the strategy is to break up the path to produce two slightly simpler integrals: the sum of

- a tiny integral of a differential form with precisely one pole at an endpoint, which can be computed by normalizing the resulting integral with respect to a choice of cotangent vector and
- a Coleman integral between two different residue disks with poles within the disks of integration (but away from the endpoints).
We begin by breaking up the path from $w(P)$ to $P$ using an arbitrary point $Q$ (distinct from $P$) in the disk of $P$:

\[
\int_{w(P)}^{P} \frac{b \, dx}{y(x-a)} = \int_{w(P)}^{Q} \frac{b \, dx}{y(x-a)} + \int_{Q}^{P} \frac{b \, dx}{y(x-a)} \\
= \int_{w(Q)}^{P} \frac{b \, dx}{y(x-a)} + \int_{Q}^{P} \frac{b \, dx}{y(x-a)} \\
= \int_{w(Q)}^{Q} \frac{b \, dx}{y(x-a)} + \int_{Q}^{P} \frac{b \, dx}{y(x-a)} + \int_{Q}^{P} \frac{b \, dx}{y(x-a)},
\]

which gives

\[
(3.5) \quad \int_{w(P)}^{P} \frac{b \, dx}{y(x-a)} = -2 \int_{P}^{Q} \frac{b \, dx}{y(x-a)} + \int_{w(Q)}^{Q} \frac{b \, dx}{y(x-a)}.
\]

The integral (3.6) is now a tiny integral, but it still has a pole at one of its endpoints, $P$. Let $z = \frac{x-a}{2b}$ be the normalized parameter at $P$, used to normalize the integral, as in [5, §4]. Writing $\frac{b \, dx}{y(x-a)}$ in terms of this parameter yields a Laurent series expansion of the form $(z^{-1} + a_0 + a_1 z + \cdots)dz$, because $\frac{b \, dx}{y(x-a)}$ has a simple pole with residue 1 at $P$. The normalized integral $\int_{P}^{Q} \frac{b \, dx}{y(x-a)}$ is the normalized integral of $\frac{b \, dx}{y(x-a)}$ evaluated at $Q$; in particular, the normalization gives that the constant term of the integral with respect to $z$ is 0, which lets us write it as

\[
\int_{P}^{Q} \frac{b \, dx}{y(x-a)} = \log_p(z) + a_0 z + \frac{a_1}{2} z^2 + \cdots |_{z=z(Q)}.
\]

Changing to the parameter $t = (x-a) = 2bz$, we obtain

\[
(3.6) \quad \int_{P}^{Q} \frac{b \, dx}{y(x-a)} = \log_p \left( \frac{t}{2b} \right) + a_0 \left( \frac{t}{2b} \right) + \frac{a_1}{2} \left( \frac{t}{2b} \right)^2 + \cdots |_{t=t(Q)}
\]

\[
= \log_p(t(Q)) - \log_p(2b) + \int_{0}^{t(Q)} \left( \frac{b \, dx(t)}{y(t)(t-a)} - t^{-1} \right) dt.
\]

Since the precision of this last integral is found by computing a truncation of the tiny integral modulo $tm' - \log(m') \geq n$, the value of (3.6) will be correct to $n$ digits of precision.

Finally, the integral (3.7) can be computed as discussed in [5, §4.2]. We recall the formula [5, (4.4)] used to evaluate it:

\[
(3.7) \quad \int_{w(Q)}^{Q} \frac{b \, dx}{y(x-a)} = \frac{1}{1-p} \left( \Psi(A) \cup \Psi(\beta) + \sum_{A \in S} \text{Res}_A \left( \alpha \int \beta \right) - 2 \int_{Q}^{\phi(Q)} \frac{b \, dx}{y(x-a)} \right),
\]

where $\alpha = \phi^* \left( \frac{b \, dx}{y(x-a)} \right) - p \left( \frac{b \, dx}{y(x-a)} \right)$, $\beta$ is a differential form with residue divisor $(Q - w(Q))$; $S$ is the set of closed points on $X \times \mathbb{Q}_p$, and $\Psi(\xi)$ is the logarithm of a differential $\xi$ as in [3, Prop 2.3], where the map $\Psi$ takes a differential of the third kind to a differential of the second kind modulo exact differentials and is computed using definite Coleman integrals of a basis of $H^1_{dR}(X)$. 


Note that the differential $\alpha$ is chosen so that it is “essentially” of the second kind, so that within each residue disk, the residue along a sufficiently narrow annulus at the boundary of the disk is 0; for further details, see [3, Remark 4.9]. We have that $\Psi(\alpha)$ and $\Psi(\beta)$ have precision prescribed by definite Coleman integrals of a basis of $H^1_{dR}(X)$; that is, computing the Coleman integrals as in Proposition 3.3, this is \( n - \max\{m_1, \lceil \log n \rceil \} \).

For \( \sum_{\alpha \in \mathcal{S}} \text{Res}(\alpha \int \beta) \), we have three cases:

1. A finite Weierstrass,
2. A non-Weierstrass and not in residue disks containing the support of the residue divisor of $\beta$,
3. A non-Weierstrass and in disks containing the support of the residue divisor of $\beta$.

Case (1) is the most straightforward. To compute $\text{Res}_A(\alpha \int \beta)$ to $n$ digits of $p$-adic precision, by [3, Proposition 6.5], we need to compute the local coordinate at $A$, $(x(t), y(t))$ to $O(t^{2m-p-3})$. The dependence on $p$ in the exponent arises from analyzing the contributing terms in a $t$-adic expansion of $\alpha$, which amounts to a binomial expansion of $\phi^* \left( \frac{b \, dx}{y(x-a)} \right)$ where $\phi(x) = x^p$. For case (2), suppose $A$ is defined over a degree $d$ extension $K/Q_p$ and we have a working precision of $n$ digits, so that $A$ has precision $dn$ in terms of a uniformizer $\pi$ of $K$. As in [3, Corollary 6.4], let $m''$ be such that $dm'' + 1 - \lceil \log(dm'' + 1) \rceil \geq n$. Then computing $\beta$ in terms of a local coordinate $(x(t), y(t))$ at a point $B$ in the disk of $A$ defined over $Q_p$, we have that $\beta$ can be truncated modulo $t^{dm''}$ for $\sum_{\alpha \in \mathcal{S}} \text{Res}(\alpha \int \beta)$ to be correct to $n$ digits, where $U_A$ is the residue disk containing $A$. For case (3), we use [5, Lemma 4.2] to first rewrite $\int \beta$, and then use the truncation in case (2).

Lastly, we analyze the precision of \( \int_{Q}^{d(Q)} \frac{b \, dx}{y(x-a)} \). We take this opportunity to make a small correction to the discussion above [5, Lemma 4.2]; indeed, the same technique described in [5, Lemma 4.2] is also used to compute this integral, since the residue divisor of $\frac{b \, dx}{y(x-a)}$ is $(P) - (w(P))$, and by construction, $Q, \phi(Q)$, and $P$ are all in the same residue disk. So by computing the integrand in terms of a local coordinate at $Q$ and truncating modulo $t^{m''}$ as above, we have that the integral will be correct to $n$ digits of precision.

\[ \square \]

3.4. Computing the intersection patterns. We now discuss the computation of a set $T \subset \mathbb{Q}_p$ which contains the values of $\rho$ on $p$-integral points. By Lemma 2.4, it suffices to compute the set $\mathcal{P}$ of intersection patterns, since we can take $T = \log_{q}(\mathcal{P})$. Hence we compute the sets $T(q)$ of possible values of $D_{P,q}$ (see (2.6)), as $P$ runs through $U(\mathbb{Z}_q)$, where $q$ is a prime of bad reduction. In fact, it follows from Lemma 2.4 that it suffices to do this for $q = 2$ and for primes $q$ which divide the leading coefficient of $f$ or satisfy $v_q(\text{disc}(f)) \geq 2$, since all other primes are not very bad.

So let $q$ be such a prime; we will omit it from the notation in the following. We first compute a strong desingularization $\mathcal{X}$ of the Zariski closure of $X \times \mathbb{Q}_q$ over $\text{Spec}(\mathbb{Z}_q)$. Suppose that, as a divisor, $\mathcal{X}_q$ is given by $\sum_{i=1}^{n} a_i \Gamma_i$, where $\Gamma_1, \ldots, \Gamma_n$ are the irreducible components of $\mathcal{X}_q$ and $a_1, \ldots, a_n$ are positive integers. For $i, j \in \{1, \ldots, n\}$ let $m_{ij} = (a_i \Gamma_i \cap a_j \Gamma_j)$, so that $M = (m_{ij})_{i,j}$ is the intersection matrix of $\mathcal{X}_q$. Let $M^+$ denote the Moore-Penrose pseudoinverse [33] of $M$. 

Recall the adjunction formulas on $\mathcal{X}_{q}$: If $\mathcal{K} \in \text{Div}(\mathcal{X}) \otimes \mathbb{Q}$ is a canonical $\mathbb{Q}$-divisor on $\mathcal{X}$ (meaning $\mathcal{O}(\mathcal{K})$ is isomorphic to the relative dualizing sheaf of $\mathcal{X}$ over $\text{Spec}(\mathbb{Z}_q)$), then we have
\begin{equation}
(K \cdot \Gamma_i) = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2
\end{equation}
for all $i$, where $p_a(\Gamma_i)$ denotes the arithmetic genus of $\Gamma_i$.

For $P \in \mathcal{U}(\mathbb{Z}_q)$, we now give simple formulas for $D^2_P$ in terms of linear algebra on the special fiber $\mathcal{X}_q$. Recall from [5, Lemma 3.4] and the proof of [5, Proposition 3.3] that if $P \in X(\mathbb{Q}_q)$, then $D^2_P$ is given by
\begin{equation}
D^2_P = \Phi(((P) - (\infty))^2 - (P \cdot \text{div}_X(\omega_0)) - (\infty \cdot \text{div}_X(\omega_{g-1})).
\end{equation}
We consider each term on the right hand side of (3.9) individually, drawing on §5.1]. As these terms only depend on the component that the section $P$ intersects, we can compute $T(q)$ by computing them for each component of multiplicity 1.

3.4.1. Computing $\Phi((P) - (\infty))^2$. Let $u(P)$ denote the column vector whose $i$th entry is $((P) - (\infty) \cdot a_i \Gamma_i)$. Then we have
\begin{equation}
\Phi((P) - (\infty))^2 = u(P)^T M^+ u(P).
\end{equation}
by [5, (5.3)].

3.4.2. Computing $(P \cdot \text{div}_X(\omega_0))$. Recall from [5, §5.1] that
\begin{equation}
\text{div}_X(\omega_0) = (2g - 2)\infty + V,
\end{equation}
where $V$ is vertical. As $P \in \mathcal{U}(\mathbb{Z}_q)$, the section associated to it does not intersect the section $\infty$, so it follows from [5, §5.1] that
\begin{equation}
(P \cdot \text{div}_X(\omega_0)) = (P \cdot V) = (P \cdot V'),
\end{equation}
for such a point $P$. Here $V'$ is the unique vertical $\mathbb{Q}$-divisor such that $(2g - 2)\infty + V'$ is a canonical $\mathbb{Q}$-divisor and such that the support of $V'$ is disjoint from the component $\Gamma_0$. Writing $V' = \sum_{i=1}^{n} v_i' \Gamma_i$, $v_i' \in \mathbb{Q}$, it therefore suffices to compute the column vector $v' = (v'_1, \ldots, v'_n)^T$.

Suppose that $\Gamma_{i_0}$ dominates the Zariski closure of $\mathcal{X}$ and that the section $\infty$ intersects $\Gamma_{i_\infty}$, where $i_0, i_\infty \in \{1, \ldots, n\}$. Then $v'$ can be computed as follows: Let $b$ denote $(b_1, \ldots, b_n)^T$, where
\begin{equation}
b_i = -\Gamma_i^2 + 2p_a(\Gamma_i) - 2 - (2g - 2)(\infty \cdot \Gamma_i).
\end{equation}
By (3.8) we have
\begin{equation}
v' = M' \cdot b,
\end{equation}
where $M'$ is the matrix obtained from $M$ by setting the $i_\infty$-th row and column equal to 0.

3.4.3. Computing $(\infty \cdot \text{div}_X(\omega_{g-1}))$. The divisor of the function $x^{g-1} \in \mathbb{Q}(\mathcal{X})$ is equal to $(g - 1)D_0 - (2g - 2)(\infty)$, where $D_0 = (P_0) + (w(P_0)) \in \text{Div}(\mathcal{X} \otimes \mathbb{Q}_q)$ for a point $P_0 \in \mathcal{X}(\mathbb{Q}_q)$ with $x$-coordinate 0. Thus we have
\begin{equation}
\text{div}_X(\omega_{g-1}) = (g - 1)D_0 + W
\end{equation}
for some vertical divisor $W$. In analogy with (3.11), we have
\begin{equation}
(\infty \cdot \text{div}_X(\omega_{g-1})) = (\infty \cdot W) = (\infty \cdot W'),
\end{equation}

\vspace{1cm}
where

\[ W' = \sum_{i=1}^{n} w'_i \Gamma_i, \quad w'_1, \ldots, w'_n \in \mathbb{Q}, \]

is the unique vertical \( \mathbb{Q} \)-divisor such that \((2g-2)D_0 + W'\) is a canonical \( \mathbb{Q} \)-divisor and such that the support of \( W' \) is disjoint from the component \( \Gamma_0 \). If \( c = (c_1, \ldots, c_n)^t \) is defined by

\[ c_i = -\Gamma_i^2 + 2p_i(\Gamma_i) - 2 - (g-1)D_0.\Gamma_i, \]

then it follows from (3.8) that

\[(w'_1, \ldots, w'_n)^t = M' \cdot c.\]

### 3.5. Computing the solutions to \( \rho(z) \in T \)

Here we describe how to find all \( p \)-adic solutions to \( \rho(z) \in T \) in \( U(\mathbb{Z}_p) \) to precision \( p^N \).

1. We begin by enumerating the list of residue disks not including the disk at infinity: that is, the set \( X(F_p) \setminus \infty \).
2. Compute the dual basis \( \{\bar{\omega}_i\} \).
3. Initialize sets \( Z = \emptyset \) and \( \mathcal{Z} = \emptyset \).
4. For each disk, we do the following:
   a. If the disk is Weierstrass, we take \( P \) to be the characteristic \( 0 \) Weierstrass point in the disk. If the disk is not Weierstrass, we take \( P \) to be an arbitrary lift to \( \mathbb{Z}_p \) of the \( F_p \)-point.
   b. Compute \( \tau(P) \), a local coordinate \( (x(z), y(z)) \) at \( P \), and \( f_i(z), \tau(z) \) as in \( \S \S 3.2 - 3.3 \).
   c. Use the values of \( \alpha_{ij} \) as computed in \( \S 3.1 \) to produce the power series expansion

\[(3.12) \quad \rho(z) = \tau(P) - 2 \sum_{i=0}^{q-1} \left( \int_0^z \omega_i \bar{\omega}_i + \frac{1}{2} \int_0^z \omega_i \int_{-p}^p \bar{\omega}_i \right) - \sum_{0 \leq i \leq g-1} \alpha_{ij} f_i(z) f_j(z) \]

   d. For each value of \( t \in T \), solve the equation \( \rho(z) \equiv t \mod p^N \). For each \( z_i \) such that \( \rho(z_i) \equiv t \mod p^N \), check whether \( \rho'(z_i) \equiv 0 \mod p^N \). If \( \rho'(z_i) \not\equiv 0 \mod p^N \), use the local coordinate at \( P \) to find the corresponding \( y \)-coordinate of the point on \( X \) in the disk with \( x \)-coordinate \( x(z_i) \) and append \( ((x(z_i), y(z_i)), 1) \) to the set \( \mathcal{Z} \). If \( \rho'(z_i) \equiv 0 \mod p^N \), approximate \( m = \text{ord}_{z = z_i} \rho(z) \) using our working precision and append \( ((x(z_i), y(z_i)), m) \) to the set \( \mathcal{Z} \).

### 4. Complementing quadratic Chabauty

Using the algorithms described in Section 3, we can compute a finite set of \( p \)-adic points on \( X \) which contains the set \( U(\mathbb{Z}) \) of integral points on \( X \), provided the conditions of Theorem 2.1 are satisfied. More precisely, we can compute the set \( \mathcal{P} \) of intersection patterns, and we can find all solutions \( z \in U(\mathbb{Z}_p) \) to the equations

\[(4.1) \quad \rho(z) = \log_p(d), \quad d \in \mathcal{P} \]

to \( N \) digits of precision. Theorem 2.1 implies that the set of solutions will contain the set of integral points on \( X \).

For every solution \( z \) to (4.1), there are three possible scenarios:

a. \( z \) lifts to a point in \( U(\mathbb{Z}) \),

b. \( z \) lifts to a point in \( X(\mathbb{Q}) \setminus U(\mathbb{Z}) \), or
(c) $z$ does not lift to a point in $X(\mathbb{Q})$.
In cases (a) and (b), it is usually trivial to find the corresponding rational point.
The reason is that because of our assumption $g > 1$, we do not expect $X(\mathbb{Q})$ to
contain any points of large height. So quadratic Chabauty leaves us with a list of
integral points $A \subset U(\mathbb{Z})$, and we want to prove that this list is complete.

There are essentially two problems we have to solve. The first one is caused by
the fact that we only compute our solutions to our finite working precision $p^N$.

**Problem 1.** Given a solution $z_0$ lifting to a point $P_0 \in X(\mathbb{Q})$, we have to show
that there are no integral points $\neq P_0$ congruent to $z_0$ modulo $p^N$.

If the power series expansion of $\rho - t$ in the residue disk containing $z_0$ only has
a simple zero at $z_0$, then we are obviously done.

So suppose that $\text{ord}_{z=z_0} \rho(z) - t = m > 1$. In practice, our approach is to simply
increase precision until $z_0$ separates into $m$ different points. If this does not work
for reasonably small precision $p^N$, then we are in one of the following two cases:

(i) $z_0$ really occurs as a multiple solution of $\rho(z) - t$.
(ii) $z_0$ is $p$-adically closer than $p^N$ to another point which satisfies $\rho(z) - t = 0$.

In case (ii), we will of course detect this by increasing precision, though we have
no a priori bound on when to stop (if $z_0$ is integral, we could derive an enormous
bound using linear forms in logarithms, but this would not be practical). However,
neither (i) nor (ii) occurred in our examples, and we do not expect this to occur in
practice.

Here is a heuristic explanation for this: If the original expansion of $\rho$ in the disk of
$z_0$ did not use $z_0$ as a basepoint, recompute $\rho(z) = t$ at $z_0$. Call $F(z) = \rho(z) - t$.
To state the simplest case of the problem, suppose we are trying to resolve a potential
double root; that is, our expansion of $F$ yields $F(z) = a_2 z^2 + \cdots$. Either we actually
have a double root at $z_0$ or $a_1$ is $p$-adically too small to show up as a coefficient of
the power series, given our working precision. Note that each element of the dual
basis $\omega_i$ is a $\mathbb{Q}_p$-linear combination of the $g$ regular 1-forms (from the projection
along the unit root subspace, using Frobenius) and a $\mathbb{Q}$-linear combination of the
$g$ meromorphic 1-forms $\{\omega_g, \ldots, \omega_{2g-1}\}$ (from the cup product matrix). So, using a
chosen certain choice of cotangent vector $b_0$ where necessary, as in [5, §§2–4], we have $F(z) := \rho(z) - t$

$$F(z) = -2\int_{b_0}\sum_{i=0}^{g-1}\omega_i\tilde{\omega}_i - \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} f_i(z) f_j(z) - t$$

$$= -2\int_{b_0}\sum_{i=0}^{g-1} \sum_{j=g}^{2g-1} b_{ij} \omega_{ij} - 2\int_{\infty}^{z} \sum_{i=0}^{g-1} \omega_i \sum_{j=0}^{g-1} b_{ij} \omega_j - \sum_{0 \leq i \leq j \leq g-1} \alpha_{ij} \int_{\infty}^{z} \omega_i \int_{\infty}^{z} \omega_j - t$$

$$= (q_1 z + q_2 z^2 + \cdots) + (s_1 z + s_2 z^2 + \cdots),$$

where the first power series has rational coefficients $q_i$ (the sums of $\mathbb{Q}$-linear combi-
nations of double integrals which are computed using a local coordinate expansion
at the integral point $z_0$) and the second power series has $p$-adic coefficients $s_i$. This
means that $F(z)$ has linear coefficient 0 or very close to 0, that is, either $q_1 = s_1$
(and that $s_1$ actually a rational number) or $q_1 - s_1 \approx 0$, or that $s_1 \approx q_1$, that is,
$s_1$ is very close to a rational number, both of which are scenarios that seem highly
unlikely.
So Problem 1 should not lead to any practical difficulties. However, there is a much more serious obstacle we have to overcome.

**Definition 4.1.** Let \( \mathcal{F}_p \) denote the set of solutions to the equations (4.1) which do not lift to a \( \mathbb{Q} \)-rational point to our working precision.

As we do not expect any \( \mathbb{Q} \)-rational points of large height, we have to solve the following problem.

**Problem 2.** Show that no element of \( \mathcal{F}_p \) lifts to an integral point.

By the discussion above, it suffices to solve Problem 2 in order to provably find all integral points on \( X \). So how can we attack Problem 2? This question will be answered in Section 6, after we establish the basics of the Mordell-Weil sieve in the next section.

### 5. The Mordell-Weil sieve

From now on, we suppose that the genus \( g \) of \( X \) is at least 2. We recall the Mordell-Weil sieve, a method to obtain results on rational points of \( X(\mathbb{Q}) \) using information about \( J(\mathbb{Q}) \) and information over finite fields. It was developed by Scharaschkin in his PhD thesis [38] and adapted and applied to great success by a number of authors, for instance Flynn [23], Poonen-Schaefer-Stoll [37] and Bruin-Stoll [14, 16]. The Mordell-Weil sieve is particularly useful for proving that \( X(\mathbb{Q}) \) is empty, but it has many other applications as well, see [16, §4.2]. In our discussion we follow [16].

The general idea is as follows: Let \( M \) be a positive integer and let

\[
\pi : J(\mathbb{Q}) \to J(\mathbb{Q})/MJ(\mathbb{Q})
\]

be the canonical epimorphism. Suppose that \( C_M \subset J(\mathbb{Q})/MJ(\mathbb{Q}) \) is a set of residue classes for which we want to show that no rational point \( P \in X(\mathbb{Q}) \) maps to \( C_M \) under \( \pi \circ \iota \). In the simplest case we pick a prime \( v \) of good reduction and consider the following commutative diagram:

\[
\begin{array}{ccc}
X(\mathbb{Q}) & \xrightarrow{\pi \circ \iota} & J(\mathbb{Q})/MJ(\mathbb{Q}) \\
\downarrow & & \downarrow \\
X(\mathbb{F}_v) & \xrightarrow{\beta_v} & J(\mathbb{F}_v)/MJ(\mathbb{F}_v) \\
\end{array}
\]

Here \( \alpha_v \) is induced by the reduction map \( J(\mathbb{Q}) \to J(\mathbb{F}_v) \) and \( \beta_v \) is induced by \( \iota : X(\mathbb{F}_v) \to J(\mathbb{F}_v) \). If \( \alpha_v(C_M) \cap \beta_v(X(\mathbb{F}_v)) = \emptyset \), then we are obviously done.

More generally, we can apply this argument to several primes of good reduction at once: Let \( S \) be a finite set of such primes and consider the commutative diagram

\[
\begin{array}{ccc}
X(\mathbb{Q}) & \xrightarrow{\pi \circ \iota} & J(\mathbb{Q})/MJ(\mathbb{Q}) \\
\downarrow & & \downarrow \\
\prod_{v \in S} X(\mathbb{F}_v) & \xrightarrow{\beta_S} & \prod_{v \in S} J(\mathbb{F}_v)/MJ(\mathbb{F}_v) \\
\end{array}
\]

Then it suffices to show that

\[
\alpha_S(C_M) \cap \beta_S \left( \prod_{v \in S} X(\mathbb{F}_v) \right) = \emptyset.
\]
In other words, we want to find $S$ such that

$$A(S, C_M) = \left\{ c \in C_M : \alpha_S(c) \in \beta_S \left( \prod_{v \in S} X(F_v) \right) \right\}$$

is empty. Heuristically, we can estimate the size of $A(S, C_M)$ as follows: For a prime $v$ of good reduction, we denote by $X_{M,v} = \beta_v(X(F_v))$ the image of $X(F_v)$ in $J(F_v)/M^* J(F_v)$ and we set

$$\gamma(v, M) = \frac{\# X_{M,v}}{\#J(F_v)/M^* J(F_v)}.$$

Note that $v$ can only be useful for our purposes if $\gamma(v, M) < 1$. The expected size of $A(S, C_M)$ is

$$n(S, C_M) = \# C_M \prod_{v \in S} \gamma(v, M),$$

so we want $n(S, C_M)$ to be small.

**Remark 5.1.** In addition to choosing $S$, we can of course also increase $M$ by taking some multiple $M'$ of $M$ and constructing the set $C_{M'} \subset J(Q)/M' J(Q)$ of residue classes which map to $C_M$ via the Chinese remainder theorem. It seems very difficult to prove that there are choices of $S$ and $M'$ so that this approach will show $\pi(\iota(X(Q))) \cap C_M = \emptyset$ as desired, so that at present the Mordell-Weil sieve is not known to be effective. See Poonen [36] for related heuristics, which predict, in particular, that such a choice should always exists for curves that have $X(Q) = \emptyset$.

We refer to Bruin and Stoll [16] for a thorough discussion of practical aspects of the Mordell-Weil sieve. We finally mention that it is also possible to use primes of bad reduction and “deep” information for the Mordell-Weil sieve, but we have not tried to use this for our purposes. See [16] on a description of these ideas for genus 2.

The practicality of the Mordell-Weil sieve is discussed in [16, §§7–8]. Essentially, two problems can arise. First, the set $A(S, C_M)$ can become too large. Secondly, the computation of the images of the sets $X(F_v)$ in $J(F_v)$ can become infeasible. The latter can occur because that computation requires approximately $v$ discrete logarithms in the group $J(F_v)$. We can avoid this issue by only considering primes such that $\# J(F_v)$ is sufficiently smooth, meaning that it has no large prime divisors, as in this case we can apply Pohlig-Hellman reduction [35], so that only a small number of relatively easy discrete logarithms have to be computed.

### 6. Combining quadratic Chabauty and the Mordell-Weil sieve

Now we show how quadratic Chabauty can be combined with the Mordell-Weil sieve to provably compute all integral points on $X$, where we assume $X$ to have genus at least 2. See Appendix A for a brief discussion of the genus 1 case.

Recall that in Section 3 we discussed algorithms to compute a finite subset of $U(Z_p)$ to $N$ digits of precision which contains the set $U(Z)$ of integral points, provided the conditions of Theorem 2.1 are satisfied. Having identified some of these $p$-adic points as actual rational points, we showed in Section 4 that in order to provably compute all integral points, it suffices to solve Problem 2: We need
to prove that the set \( F_p \) of remaining \( p \)-adic points do not lift to integral points. In fact, we will discuss how to prove that they do not lift to rational points by translating this into a problem that we can solve using the Mordell-Weil sieve.

For simplicity, we first assume that the torsion subgroup of \( J(\mathbb{Q}) \) is trivial and that we have generators \( P_1, \ldots, P_g \) of \( J(\mathbb{Q}) \) available. See Remarks \( \ref{remark:trivial-torsion} \) and \( \ref{remark:available-generators} \) on how to remove these assumptions.

Suppose that \( z \in U(\mathbb{Z}_p) \) is accurate to \( N \) digits of precision. If \( z \) lifts to rational point \( P \in X(\mathbb{Q}) \), then there are integers \( a_1, \ldots, a_g \) such that

\[
(6.1) \quad \iota(P) = a_1 P_1 + \ldots + a_g P_g.
\]

By properties of the Coleman integral, this implies

\[
f_i(z) = \int_{\infty}^{z} \omega_i = a_1 \int_{P_1} \omega_i + \ldots + a_g \int_{P_g} \omega_i \quad \text{for all } i \in \{0, \ldots, g-1\}.
\]

Let \( (\tilde{a}_1, \ldots, \tilde{a}_g) \in \mathbb{Z}/p^N\mathbb{Z} \) be defined by

\[
(6.2) \quad \begin{pmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_g \end{pmatrix} = \begin{pmatrix} \int_{P_1} \omega_0 & \cdots & \int_{P_g} \omega_0 \\ \vdots & \ddots & \vdots \\ \int_{P_1} \omega_{g-1} & \cdots & \int_{P_g} \omega_{g-1} \end{pmatrix}^{-1} \begin{pmatrix} \int_{\infty}^{z} \omega_0 \\ \vdots \\ \int_{\infty}^{z} \omega_{g-1} \end{pmatrix}.
\]

Note that if \( (6.1) \) holds, then we must have \( a_i \equiv \tilde{a}_i \pmod{p^N} \) for all \( i \in \{1, \ldots, g\} \).

So, if we are given \( z \) as above, and we want to prove that \( z \) does not lift to a rational point, we compute \( (\tilde{a}_1, \ldots, \tilde{a}_g) \in \mathbb{Z}/p^N\mathbb{Z} \), noting that if \( d \) is the valuation of the determinant of the matrix we are inverting, we need its entries to precision \( p^{N+d} \). Then it suffices to show that there is no tuple \( (a_1, \ldots, a_g) \in \mathbb{Z}^g \) such that

\[
(i) \quad a_i \equiv \tilde{a}_i \pmod{p^N} \quad \text{for all } i \in \{1, \ldots, g\},
\]

\[
(ii) \quad a_1 P_1 + \ldots + a_g P_g \quad \text{is in the image of } X(\mathbb{Q}) \quad \text{in } J(\mathbb{Q}) \quad \text{under the embedding } \iota.
\]

The tuple \( (\tilde{a}_1, \ldots, \tilde{a}_g) \) induces a residue class \( c \in J(\mathbb{Q})/p^N J(\mathbb{Q}) \) via

\[
c = \tilde{a}_1 P_1 + \ldots + \tilde{a}_g P_g;
\]

we say that \( c \) corresponds to \( z \).

Then, denoting by \( \pi: J(\mathbb{Q}) \to J(\mathbb{Q})/p^N J(\mathbb{Q}) \) the canonical epimorphism, we want to show that there is no point \( P \in X(\mathbb{Q}) \) whose image under \( \pi \circ \iota \) maps to the residue class \( c \) in \( J(\mathbb{Q})/p^N J(\mathbb{Q}) \) corresponding to \( z \) — which is precisely the type of problem the Mordell-Weil sieve is likely to solve! We set \( M = p^N \), let \( C_M \subset J(\mathbb{Q})/MJ(\mathbb{Q}) \) denote the set of residue classes corresponding to elements of \( F_p \) and look for a set \( S \) of primes of good reduction such that \( A(S, C_M) = \emptyset \), where \( A(S, C_M) \) is defined in \( (5.1) \).

Unfortunately, as stated, this approach is not likely to succeed in practice. The reason is that for \( M \) a prime power the set \( A(S, C_M) \) is unlikely to be empty, as we cannot expect \( #J(\mathbb{F}_v) \) to be divisible by \( p \) for many small primes \( v \). Of course, we can simply take a multiple \( M' \) of \( p^N \) as discussed in Remark \( 5.1 \). However, we would like to apply our method to curves of genus (and hence rank) at least 4, so this strategy would likely lead to large sets \( A(S', C_{M'}) \). Moreover, the discrete logarithms needed to compute the images of \( X(\mathbb{F}_v) \) in \( J(\mathbb{F}_v) \) would become infeasible, since the shape of \( M \) would force us to consider primes \( v \) such that \( #J(\mathbb{F}_v) \) has large prime divisors and the computation of discrete logarithms (in fact explicit computations in \( J(\mathbb{F}_v) \) in general) becomes slow as \( g \) and \( v \) become large.
Fortunately, it turns out that we can actually do much better. Namely, we use the following strategy: We combine the information coming from quadratic Chabauty for several primes of good ordinary reduction \( p_1, \ldots, p_s \) to \( N_1, \ldots, N_s \) digits of precision, respectively. Let \( m \) be a small integer coprime to the \( p_k \) and set
\[
M = m \cdot p_1^{N_1} \cdots p_s^{N_s}.
\]
If \( P \in U(\mathbb{Z}) \), then there is an intersection pattern \( d \in \mathcal{P} \) such that we have
\[
\rho_{p_k}(P) = \log_{p_k}(d) \quad \text{for all} \quad k \in \{1, \ldots, s\}.
\]
We can partition \( \mathcal{F}_{p_k} \) into subsets
\[
\mathcal{F}_{p_k,d} = \mathcal{F}_{p_k} \cap \rho_{p_k}^{-1}(\log_{p_k}(d)),
\]
where \( d \) runs through \( \mathcal{P} \). For an intersection pattern \( d \), we define \( C_d \) to be the set of residue classes \( c \in J(\mathbb{Q})/MJ(\mathbb{Q}) \) such that \( c \) corresponds to an element of \( \mathcal{F}_{p_k,d} \) for every \( k \in \{1, \ldots, s\} \), without any condition modulo \( m \). Finally, we set
\[
C_M := \bigcup_{d \in \mathcal{P}} C_d
\]
and apply the Mordell-Weil sieve with these choices of \( M \) and \( C_M \) for some set \( S \) of primes of good reduction so that \( A(S, C_M) \) is likely to be empty.

**Remark 6.1.** If there is nontrivial rational torsion on \( J \), then we have to replace \( C_M \) by the set of all classes \( c + Q \in J(\mathbb{Q})/MJ(\mathbb{Q}) \), where \( c \in C_M \) and \( Q \) runs through \( J(\mathbb{Q})_{\text{tors}} \setminus \{O\} \). This increases the size of \( C_M \), but otherwise does not cause any problems.

**Remark 6.2.** If the points \( P_i \) only generate a subgroup of finite, but unknown index, then it suffices to show that this index is coprime to all \( v \in S \) used in the Mordell-Weil sieve computation a posteriori, because then \( \langle P_1, \ldots, P_g \rangle \) and \( J(\mathbb{Q})/J(\mathbb{Q})_{\text{tors}} \) have the same image in \( J(\mathbb{F}_v) \).

### 7. Choosing the parameters for quadratic Chabauty and the Mordell-Weil sieve

The previous section described a general approach to combining quadratic Chabauty with the Mordell-Weil sieve in order to provably compute all integral points on \( X \). Now we discuss how to make this practical. To this end, we need to establish how to choose the various parameters for the Mordell-Weil sieve and quadratic Chabauty computations. More precisely, we need to choose
- the primes \( p_1, \ldots, p_s \) of good ordinary reduction,
- the numbers \( N_1, \ldots, N_s \) of digits of precision,
- the integer \( m \),
- the set \( S \) of primes of good reduction,
so that the set \( A(S, C_M) \) has a good chance to be empty, where \( M = m \cdot p_1^{N_1} \cdots p_s^{N_s} \) and \( C_M \) is defined in (6.4). If we can then show that \( A(S, C_M) \) is indeed empty, we have proved that our list of known integral points is complete. To this end, we will make all the choices listed above at the same time, as opposed to choosing the \( p_k \), the \( N_k \) and \( m \) a priori and then looking for a suitable set \( S \).

With notation as in Section 5, the expected size of the set \( A(S, C_M) \) is
\[
n(S, C_M) = \#C_M \prod_{v \in S} \gamma(v, M).
\]
We would like to find $S$ and $M$, so that $n(S, C_M)$ is small, where $C_M$ is as in (6.4), before we actually start the quadratic Chabauty computations.

So suppose we have candidate $S$ and $M$. It is straightforward to compute $\gamma(v, M)$ for given $v$ and $M$, see [16, §3]. However, the size of $C_M$ is given by

$$
\#C_M = \#J(\mathbb{Q})_{\text{tors}}/MJ(\mathbb{Q})_{\text{tors}} \cdot \#J(\mathbb{Q})/mJ(\mathbb{Q}) \cdot \sum_{d \in \mathcal{P}} \prod_{k=1}^{s} \#\mathcal{F}_{p_k, d}.
$$

Hence we cannot compute $\#C_M$ without already knowing the sizes of the sets $\mathcal{F}_{p_k, d}$ for the various $d \in \mathcal{P}$. But these sets come out of the quadratic Chabauty computation for $p_k$! So instead, we settle for an integer $\theta(C_M)$ such that we expect

$$
\#C_M \leq \theta(C_M)
$$

to hold, using the following strategy:

We first compute the very bad primes and the set $\mathcal{P}$ of intersection patterns. Then we find all small integral points on $X$, say of logarithmic height at most $B_2$. For every such point $P$, we use (2.4) to find the element $d \in \mathcal{P}$ such that $\rho_{p_k}(P) = \log_{p_k}(d)$ for all primes $p_k$. Let $\sigma_d$ denote the number of known integral points corresponding to $d$; this can be computed easily once we have computed the regular models $X$ as in Section 3.4. Experimental data suggests that we should expect at most two solutions to the equation

$$
\rho_{p_k}(z) = \log_{p_k}(d)
$$
in every affine residue disk. So we expect

$$
\#\mathcal{F}_{p_k, d} \leq \sum_{d \in \mathcal{P}} (2 \cdot \#X(\mathbb{F}_{p_k}) - 2 - \sigma_d)
$$

and hence $\#C_M \leq \theta(C_M)$, where

$$
\theta(C_M) = \#J(\mathbb{Q})_{\text{tors}}/MJ(\mathbb{Q})_{\text{tors}} \cdot \#J(\mathbb{Q})/mJ(\mathbb{Q}) \cdot \sum_{d \in \mathcal{P}} \prod_{k=1}^{s} (2 \cdot \#X(\mathbb{F}_{p_k}) - 2 - \sigma_d)
$$

Thus we expect the size of $A(S, C_M)$ to be at most

$$(7.1) \quad n'(S, C_M) = \theta(C_M) \cdot \prod_{v \in S} \gamma(v, M).$$

So our goal is to find $S$ and $M$ as above such that $n'(S, C_M)$ is small. Our approach to this is as follows:

(a) Fix positive constants $\varepsilon$ and $B_1$, let $p_1, \ldots, p_u$ denote the primes $\leq B_1$ of good ordinary reduction, and set $S = \emptyset$.

(b) Append primes $v$ of good reduction to $S$ such that $\#J(\mathbb{F}_v)$ is divisible by at least one prime $p_k$ and only has prime divisors below some chosen bound, and such that we have $n'(S', C_{M'}) < \varepsilon$, where

- $M' = M_{l-1} \cdot M_{l-1}^{-1} = \prod_{v \not\in S} J(\mathbb{F}_v) \cong \mathbb{Z}/M_1 \mathbb{Z} \times \cdots \times \mathbb{Z}/M_l \mathbb{Z}$, and
- $M_j \mid M_{j+1}$ for $j = 1, \ldots, l - 1$.

(c) Find $M = m \cdot p_1^{N_1} \cdots p_u^{N_u} \mid M'$ and $S' \subset S$ such that $n'(S, C_M) < \varepsilon$ and such that $A(S, C_M)$ can be computed efficiently.
This is essentially analogous to the techniques proposed by Bruin and Stoll in [16]. In particular, see [16, §3.1] for an argument why \( M' = M_{l-g-1-j} \) is likely to yield small values of \( n(S', C_{M'}) \) and [16, §3.2] for criteria which guarantee that \( A(S, C_M) \) can be computed efficiently. Their arguments are detailed for the case \( C_M = J(Q)/MJ(Q) \), but can be adapted easily to remain valid in our situation. In particular, we restrict to primes \( v \) such that the prime divisors of \( \#J(F_v) \) are not too large to keep the discrete logarithms needed in the Mordell-Weil sieve computation feasible. In practice we usually choose \( B_1 \approx 50 \), so that the quadratic Chabauty computations are reasonably fast. Note that usually only small powers of the \( p_k \) will divide some \( \#J(F_v) \), so we need only a few digits of \( p \)-adic precision. Using the strategy outlined above, we can keep a good balance between the size of the sets \( C_M \), and a suitable flexibility in choosing \( S \).

8. A METHOD FOR COMPUTING THE INTEGRAL POINTS WHEN \( r = g > 1 \)

We collect the techniques described in this paper into a complete method for computing the set of integral points on a hyperelliptic curve with Mordell-Weil rank equal to its genus.

Algorithm 8.1 (Computing the set of all integral points on \( X \)).

**Input:** A separable polynomial \( f \in \mathbb{Z}[x] \) of degree \( 2g + 1 \geq 5 \)
- that does not reduce to a square modulo any prime number and
- such that the Jacobian \( J \) of the hyperelliptic curve \( X \) defined by \( y^2 = f(x) \) has rank \( J(Q) \) equal to \( g \).

We assume that an explicit set of independent non-torsion points \( P_1, \ldots, P_g \in J(Q) \) is readily available.

**Output:** This procedure either computes the set \( U(\mathbb{Z}) \) of all integral points on \( X \) or terminates with an error.

(1) Choose positive integers \( B_1, B_2 \) and a positive real number \( \varepsilon \).
(2) Compute the rational points on \( X \) of height \( \leq B_2 \) and let \( A \) denote the set of integral points among these.
(3) Compute the very bad primes and the set \( \mathcal{P} \) of intersection patterns as in §3.4.
(4) Compute the torsion subgroup \( J(Q)_{\text{tors}} \).
(5) Using the strategy discussed in §7, find a set \( S \) of primes of good reduction and a positive integer \( M = m \cdot p_1^{N_1} \cdots p_s^{N_s} \) such that
   - every \( p_k \) is a good ordinary prime \( \leq B_1 \);
   - \( \gcd(p_1 \cdots p_s, m) = 1 \);
   - we have \( n'(S, C_M) < \varepsilon \), where \( C_M \) is as in (6.4).
(6) For every \( p \in \{ p_1, \ldots, p_s \} \), do the following:
   - Using the estimates of §3, find the working precision \( p^{N'} \) needed for the results of 7aiii) to be correct to \( N \) digits.
   - Compute the constants \( \alpha_{ij} \) as in §3.1. If we find that the functions \( f_0, \ldots, f_{g-1}: J(Q) \otimes \mathbb{Q} \to \mathbb{Q}_p \) are dependent, use Chabauty’s method to compute \( X(Q) \) and exit.
(c) Compute a power series expansion of $f_i(z)$ and $\tau(z)$ in every residue disk as in §§3.2–3.3.

(7) For every $d \in \mathcal{P}$ do the following:
(a) For every $p \in \{p_1, \ldots, p_s\}$, do the following:
   (i) Compute the set $Z$ of solutions (with multiplicities) to $\rho(z) = \log_p(d)$ as in §3.5.
   (ii) If there is a solution $z_0 \in Z$ with multiplicity $> 1$ which lifts to a rational point, increase the working precision $p^{N'}$. If $N'$ exceeds a pre-determined bound, exit with an error; else go to Step (6b).
   (iii) For every solution $z$ lifting to a point $P \in \mathcal{U}(Z)$, set $A = A \cup \{P\}$.
   (iv) Compute the residue classes $c \in J(Q)/p^N J(Q)$ corresponding to the points $z \in \mathcal{F}_{p_k,d}$.

(b) Apply the Chinese remainder theorem to find the set $C_d$ containing all residue classes $c \in J(Q)/M J(Q)$ such that $c$ corresponds to an element of $\mathcal{F}_{p_k,d}$ modulo $p^N$ for all $k$.

(c) For every $c' \in J(Q)_{\text{tors}}/M J(Q)_{\text{tors}} \setminus \{O\}$ and every $c \in C_d$, append $c + c' \in J(Q)/M J(Q)$ to $C_d$.

(8) Set $C_M = \bigcup_{d \in \mathcal{P}} C_d$.

(9) Compute $A(S, C_M)$ using the Mordell-Weil sieve.

(10) If $A(S, C_M) = \emptyset$, then:
(a) Check if the index $(J(Q)/J(Q)_{\text{tors}} : \langle P_1, \ldots, P_g \rangle)$ is prime to $v$ for every $v \in S$ which was used in the Mordell-Weil sieve.
(b) If yes, return $A$.
(c) Otherwise, saturate $\langle P_1, \ldots, P_g \rangle$ at $v$ for every relevant $v \in S$ and go to Step (6) with the new points $P_1, \ldots, P_g$ resulting from the saturation process.

Regarding Step (10), it often happens that some primes $v \in S$ are superfluous in the Mordell-Weil sieve computation, so we do not have to check whether they divide the index. See [37, §13.7] or [24] where it is explained how to show that a certain prime does not divide the index. In practice, we usually expect the index to be 1, so we expect Step (10c) to never occur. See [24] for a method that can be used to saturate at $v$ when the genus is 2. Alternatively, if we find that some prime $v$ that was used in the Mordell-Weil sieve computation divides the index, we can try to rerun Step (9) without $v$. If this fails to show $A(S \setminus \{v\}, C_M) = \emptyset$, we can go back to Step (5), keeping the same constants $B_i$ and $\varepsilon$, but a priori excluding all elements known to divide the index from $S$. 

Our implementation of Step (3) crucially relies on Magma’s RegularModels package, which computes a strong desingularization of the Zariski closure of $X$. Our implementation of the choice of $S$ and $M$ (Step (5)) and of the actual Mordell-Weil sieve computation (Step (9)) is based on Magma code of Michael Stoll. See [16, §3.3] for a description of the ideas underlying that implementation. The $p$-adic analytic steps in the algorithm were implemented in Sage.
9. Examples

Here we give some examples illustrating the techniques described in this paper. We start by finishing the computation of the integral points on a genus 2 curve, to which we applied quadratic Chabauty in [5]. Then we compute the integral points on a modular curve of genus 3. Finally, we do the same for a curve of genus 4, which does not seem to be amenable to any techniques previously available.

Example 9.1. Let $X$ be the genus 2 curve
\[ y^2 = x^3(x - 1)^2 + 1. \]
In [5, §7.2] we showed the following:
(i) The Jacobian $J$ of $X$ has rank 2 over $\mathbb{Q}$.
(ii) $J(\mathbb{Q})_{\text{tors}} = \{O\}$.
(iii) The points $P_1 = [(P) - (\infty)]$ and $P_2 = [(Q) - (R)]$ generate $J(\mathbb{Q})$, where $P = (2, -3)$, $Q = (1, -1)$, $R = (0, 1)$.
(iv) The only very bad prime is 2.
(v) We have $P = \{0, \frac{2}{3}, \frac{4}{3}\}$.

We then computed the solutions to $\rho(z) \in T$ at $p = 11$. Among these solutions, the only ones that appeared to be actual integral points were $P, Q, R$ and their images under $w$.

Let us now prove that these are indeed the only integral points on $X$. Note that this has been proved already by Michael Stoll using the methods of [17]. We apply quadratic Chabauty for the primes $p_1 = 5$ and $p_2 = 11$, to respective precision $N_1 = 4$ and $N_2 = 6$ and run the Mordell-Weil sieve with $M = 5^4 \cdot 11^6$ and $S = \{17, 863, 7193\}$. After taking out residue classes containing integral points, we are left with 209 residue classes in $J(\mathbb{Q})/M J(\mathbb{Q})$; the Mordell-Weil sieve computation then shows that none of the 209 residue classes contain the image of a rational point on the curve. This proves that our list of integral points is complete.

Example 9.2. Let $X$ be the genus 3 hyperelliptic curve
\[ y^2 = (x^3 + x + 1)(x^4 + 2x^3 - 3x^2 + 4x + 4). \]
This is the curve $C_{496}^{\text{new}}$, a new modular curve of level 496 with $\mathbb{Q}$-simple Jacobian in the language of [1]. Applying 2-descent as described in [44] and implemented in Magma, we find that the Jacobian $J$ of $X$ has rank 3 over $\mathbb{Q}$. The torsion subgroup has order 2, the nontrivial element coming from the factorisation $(x^3 + x + 1)(x^4 + 2x^3 - 3x^2 + 4x + 4)$. Let $P = (-1, 2)$, $Q = (0, 2)$, $R = (-2, 12)$, $S = (3, 62)$. We want to prove that, up to the hyperelliptic involution, these are the only integral points on $X$.

The points on the Jacobian represented by differences of known rational points on $X$ generate a subgroup of $J(\mathbb{Q})/J(\mathbb{Q})_{\text{tors}}$ of finite index; an explicit set of generators of this subgroup is given by
\[ \{ P_1 = [(P) - (\infty)], P_2 = [(S) - (w(Q))], P_3 = [(w(S)) - (R)] \}. \]
As we do not need it for our method, we did not check whether these points generate the Mordell-Weil group modulo torsion; however, this should be possible using the techniques described in [46].

We apply the method from Section 8; leading to the following choices:
(i) $M = 3 \cdot 7^3 \cdot 17^3 \cdot 37^2$,
(ii) $S = \{5, 41, 607, 617, 1861, 11313, 17209\}$.

The very bad primes are 2 and 31 and the set of intersection patterns is

$$P = \{(a, b) : a \in \left\{0, 1, \frac{5}{4}, \frac{7}{3}\right\}, b \in \left\{0, \frac{1}{2}\right\}\}.$$

Now we give some detail on the quadratic Chabauty computation for $p_1 = 7$.

We find the following $\mathbb{Z}_7$-points having $\rho$-values in the set $T = \log_7(P)$.

| disk  | $x(z)$                                    | $\rho(z)$          |
|-------|-------------------------------------------|--------------------|
| $(3, \pm 1)$ | $3 + 3 \cdot 7 + O(7^3)$ | 0                  |
|       | $3 + 3 \cdot 7 + 3 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ |
|       | $3 + 2 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ | $\log_7(2)$        |
|       | $3 + 2 \cdot 7 + 7^2 + O(7^3)$         | $\log_7(2) + \frac{1}{2} \log_7(31)$ |
|       | $3 + 4 \cdot 7^2 + O(7^3)$             | $\frac{5}{2} \log_7(2)$ |
|       | $3 + O(7^3)$                           | $\frac{5}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|       | $3 + 3 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ | $\frac{7}{2} \log_7(2)$ |
|       | $3 + 3 \cdot 7 + 7^2 + O(7^3)$         | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |

| $(4, \pm 1)$ | $4 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ | 0                  |
|             | $5 \cdot 7 + 6 \cdot 7^2 + O(7^3)$ | 0                  |
|             | $4 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ |
|             | $5 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ |
|             | $O(7^3)$                               | $\log_7(2)$        |
|             | $2 \cdot 7 + 2 \cdot 7^2 + O(7^3)$   | $\log_7(2)$        |
|             | $2 \cdot 7^2 + O(7^3)$                | $\log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $2 \cdot 7 + O(7^3)$                  | $\log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $4 \cdot 7 + O(7^3)$                  | $\frac{7}{2} \log_7(2)$ |
|             | $5 \cdot 7 + 4 \cdot 7^2 + O(7^3)$   | $\frac{7}{3} \log_7(2)$ |
|             | $4 \cdot 7 + 4 \cdot 7^2 + O(7^3)$   | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $5 \cdot 7 + O(7^3)$                  | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |

| $(5, \pm 2)$ | $5 + 6 \cdot 7 + 7^2 + O(7^3)$ | 0                  |
|             | $5 + 4 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ | 0                  |
|             | $5 + 6 \cdot 7 + 4 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ |
|             | $5 + 4 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ |
|             | $5 + 2 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{5}{2} \log_7(2)$ |
|             | $5 + 7 + 2 \cdot 7^2 + O(7^3)$       | $\frac{7}{2} \log_7(2)$ |
|             | $5 + 2 \cdot 7 + 7^2 + O(7^3)$       | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $5 + 7 + 3 \cdot 7^2 + O(7^3)$       | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $5 + 6 \cdot 7 + 6 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2)$ |
|             | $5 + 6 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|             | $5 + 6 \cdot 7 + 4 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
After carrying out the quadratic Chabauty computations for $7, 17$ and $37$, we are left with $31488$ residue classes modulo $7^3 \cdot 17^3 \cdot 37^2$ for which we have to show that they do not contain the image of a rational point. We also add all residue classes modulo $3$, leading to $850176$ residue classes modulo $M = 3 \cdot 7^3 \cdot 17^3 \cdot 37^2$. Applying the Mordell-Weil sieve, we find that none of these residue classes can contain the image of a rational point. Since the index of the subgroup generated by differences of the known rational points on $X$ in $J(\mathbb{Q})/J(\mathbb{Q})_{\text{tors}}$ is easily shown not to be divisible by any prime in the set $S$, this proves that our list of integral points is indeed complete.

**Example 9.3.** Let $X$ be the hyperelliptic curve

$$y^2 = x^4(x - 2)^2(x - 1)(x + 1)(x + 2) + 4$$

of genus $4$. Let $P = (0, 2)$, $Q = (1, 2)$, $R = (2, -2)$, $S = (-1, -2)$, $U = (-2, 2)$. We want to show that these points, together with their images under $w$, form the complete list of integral points.

Using *Magma*, we show that the rank of the Jacobian over $\mathbb{Q}$ is $4$ and that the torsion subgroup is trivial. A set of generators of the finite-index subgroup $G$ of the Jacobian is given by: 

| $a$ | $b$ | $c$ | $d$ |
|-----|-----|-----|-----|
| (6, $\pm 2$) & $6 + 3 \cdot 7 + O(7^3)$ & 0 & 0 |
|  & $6 + 7 + 3 \cdot 7^2 + O(7^3)$ & $\frac{1}{2} \log_7(31)$ & $\frac{1}{2} \log_7(31)$ |
|  & $6 + 3 \cdot 7 + 3 \cdot 7^2 + O(7^3)$ & $\frac{5}{4} \log_7(2)$ & $\frac{5}{4} \log_7(2)$ |
|  & $6 + 6 \cdot 7 + 6 \cdot 7^2 + O(7^3)$ & $\frac{5}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{5}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|  & $6 + 5 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ & $\frac{7}{4} \log_7(2)$ & $\frac{7}{4} \log_7(2)$ |
|  & $6 + 6 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|  & $6 + 5 \cdot 7 + 6 \cdot 7^2 + O(7^3)$ & $\frac{7}{4} \log_7(2)$ & $\frac{7}{4} \log_7(2)$ |
|  & $6 + 3 \cdot 7 + 5 \cdot 7^2 + O(7^3)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|  & $6 + 7 + 2 \cdot 7^2 + O(7^3)$ & $\frac{7}{4} \log_7(2)$ & $\frac{7}{4} \log_7(2)$ |

| (2, $\pm 3$) | $2 + 7^2 + O(7^3)$ | 0 & 0 |
|  & $2 + 5 \cdot 7 + 2 \cdot 7^2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ & $\frac{1}{2} \log_7(31)$ |
|  & $2 + O(7^3)$ | $\frac{1}{2} \log_7(31)$ & $\frac{1}{2} \log_7(31)$ |
|  & $2 + 7 + 3 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2)$ | $\frac{7}{4} \log_7(2)$ |
|  & $2 + 7 + 4 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|  & $2 + 4 \cdot 7 + 3 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2)$ | $\frac{7}{4} \log_7(2)$ |
|  & $2 + 5 \cdot 7 + 6 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
|  & $2 + 3 \cdot 7^2 + O(7^3)$ | $\frac{7}{4} \log_7(2)$ | $\frac{7}{4} \log_7(2)$ |
|  & $2 + 5 \cdot 7 + O(7^3)$ | $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ & $\frac{7}{4} \log_7(2) + \frac{1}{2} \log_7(31)$ |
$J(\mathbb{Q})$ generated by differences of the known rational points on $X$ is given by
\[ \{P_1 = [(P) - (Q)], P_2 = [(R) - (S)], P_3 = [(U) - (w(P))], P_4 = [(w(Q)) - (w(S))] \} . \]

Because we cannot compute generators for the full Mordell-Weil group in genus 4, the techniques of [17] are not applicable. Moreover, the rank is equal to the genus, so the method of Chabauty-Coleman is not applicable. The $S$-unit techniques mentioned in Section 1.1 fail because in order to apply them in this case, we would need to compute class groups and fundamental units of number fields of degree 144. A similar complication arises when trying a combination of covering techniques with elliptic curve Chabauty. Hence none of the previously known methods are practical for this example.

First we compute that 2 is the only very bad prime and that $P = \{0, \frac{1}{2}, \frac{12}{5}\}$. We then apply quadratic Chabauty for the good ordinary primes 5, 7, 11, 13 and 17, to 6, 4, 4, and 4 digits of precision, respectively. For instance, we find the following $\mathbb{Z}_5$-points having $\rho$-values in the set $T = \log_5(P)$:

| disk | $x(z)$ | $\rho(z)$ |
|------|--------|------------|
| $(0, \pm 2)$ | $0 + O(5)$ | $\frac{12}{5} \log_5(2)$ |
| | $3 \cdot 5 + 5^2 + 2 \cdot 5^4 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| $(1, \pm 2)$ | $1 + O(5)$ | $\frac{12}{5} \log_5(2)$ |
| | $1 + 5 + 2 \cdot 5^2 + 5^4 + 4 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| | $1 + 4 \cdot 5 + 2 \cdot 5^3 + 2 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| | $1 + 2 \cdot 5 + 2 \cdot 5^3 + 3 \cdot 5^4 + 2 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| $(2, \pm 2)$ | $2 + O(5)$ | $\frac{12}{5} \log_5(2)$ |
| | $2 + 5 + 2 \cdot 5^3 + 4 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| $(3, \pm 2)$ | $3 + 5 + 4 \cdot 5^2 + 2 \cdot 5^3 + 5^4 + O(5^6)$ | $0$ |
| | $3 + 5^2 + 3 \cdot 5^3 + 2 \cdot 5^4 + 3 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| | $3 + 4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| $(4, \pm 2)$ | $4 + 5^2 + 3^5 + 5^4 + 3 \cdot 5^5 + O(5^6)$ | $0$ |
| | $4 + 5 + 3 \cdot 5^4 + O(5^6)$ | $0$ |
| | $4 + 4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |
| | $4 + 2 \cdot 5 + 2 \cdot 5^3 + 5^4 + 2 \cdot 5^4 + 5^5 + O(5^6)$ | $\frac{12}{5} \log_5(2)$ |

This leads to 9660096 residue classes modulo $M = (5 \cdot 7 \cdot 11 \cdot 13 \cdot 17)^3$. Using the set $S = \{13, 19, 83, 103, 167, 727, 971, 2909\}$, all of whose elements are easily shown to be coprime to the index $(J(\mathbb{Q}) : G)$, the Mordell-Weil sieve proves that none of these residue classes contain the image of a rational point on $X$. 
Appendix A. Computing all integral points on elliptic curves using quadratic Chabauty

Let $X$ be an elliptic curve of rank 1 over $\mathbb{Q}$. For simplicity, we also assume that $X(\mathbb{Q})$ is torsion-free. As described in Section 1.1, there exist quite efficient algorithms for the computation of the integral points on $X$; the most practical ones are based on elliptic logarithms. Nevertheless, we can use quadratic Chabauty, for this purpose as well, as sketched below.

Suppose we have already found the set $\mathcal{A} \subset U(\mathbb{Z})$ of suspected integral points on $X$, and our goal is to prove the equality $\mathcal{A} = U(\mathbb{Z})$.

Let $P$ be the generator of $X(\mathbb{Q})$. The first helpful observation is that we only have to consider residue disks not corresponding to affine multiples of the reduction $\bar{P}$ of $P$ in the quadratic Chabauty computation.

We define the function $\log_P$ on $X(\mathbb{Q}_p)$ by

$$\log_P(z) = \log_{P,P}(z) = \frac{\int_{\infty}^{z} \omega_0}{\int_{\infty}^{1} \omega_0},$$

noting that this is indeed the discrete logarithm with respect to $P$ for points in $X(\mathbb{Q})$.

Let $\mathcal{P}$ be the set of intersection patterns. We try to prove that $\mathcal{A} = U(\mathbb{Z})$ by showing, for each $d \in \mathcal{P}$, that there are no points in $U(\mathbb{Z}) - \mathcal{A}$ with intersection pattern $d$.

We use quadratic Chabauty for several primes. Let us first discuss the case of 2 primes. Following Silverman-Stange [39], we call a pair of distinct primes $(p_1,p_2)$ amicable if

$$p_2 = \#X(\mathbb{F}_{p_1}) \quad \text{and} \quad p_1 = \#X(\mathbb{F}_{p_2}).$$

More generally, we call a pair of distinct primes $(p_1,p_2)$ pseudo-amicable if

$$p_2 \mid \#X(\mathbb{F}_{p_1}) \quad \text{and} \quad p_1 \mid \#X(\mathbb{F}_{p_2}).$$

Suppose that $p_1$ and $p_2$ are primes of good ordinary reduction such that $(p_1,p_2)$ is pseudo-amicable. For simplicity, assume that $p_1 = \#X(\mathbb{F}_{p_2})$. We first apply quadratic Chabauty to $X$ for $p_1$ and $p_2$, keeping in mind that only residue classes corresponding to affine multiples of $\bar{P}$ have to be considered. Suppose that $d \in \mathcal{P}$ and that $z \in X(\mathbb{Q}_{p_1})$ is a solution to $\rho_{p_1}(z) = \log_{p_1}(d)$ which lifts to a rational point on $X$. Then we have $z = aP$ for some $a \in \mathbb{Z}$. The discrete logarithm $a_1$ of $z$ mod $p_1$ with respect to $P$ is then congruent to $a$ modulo $p_2$. From $a = \log_{P,p_2}(z) \mod p_1$ we know the residue disk $U_{z,p_2}$ of the reduction of $z$ modulo $p_2$ (more generally, if we had $p_1 \mid \#X(\mathbb{F}_{p_2})$, then we would know that the reduction of $z$ modulo $p_2$ lies in one of $\#X(\mathbb{F}_{p_2})/p_1$ explicitly given residue disks).

We run through all solutions $w \in X(\mathbb{Q}_{p_2})$ to $\rho_{p_2}(w) = \log_{p_2}(d)$ in the residue disk $U_{z,p_2}$ and compute

$$a_2 = \log_{P,p_2}(z) \mod p_2.$$

Finally, we check whether $a_1 = a_2$; if not, then we know that $w$ doesn’t correspond to an integral point. If this happens for all such $w$, then we can discard $z$.

Example A.1. Let $X$ denote the elliptic curve

$$y^2 = x^3 - 4.$$

Then $X$ has rank 1 and trivial torsion over $\mathbb{Q}$ and $P = (2,2)$ is a generator of $X(\mathbb{Q})$. Using an implementation of the algorithm based on elliptic logarithms mentioned in
Section (1.1) (for instance in Magma or Sage), we easily find that the only \( \mathbb{Z} \)-integral points on \( X \) are \( \pm P \) and \( \pm Q \), where \( Q = (5, 11) \). We want to recover this result using the techniques discussed above. First we compute that 2 is the only very bad prime and that \( \mathcal{P} = \{0, 1\} \) and \( \# X(\mathbb{F}_v) \) for all primes \( v < 100 \) of good and ordinary reduction.

We find that the pair \((13, 7)\) is pseudo-amicable, since \( \# X(\mathbb{F}_{13}) = 21 \) and \( \# X(\mathbb{F}_{7}) = 13 \). Using the algorithms in Section 3 we compute all solutions to \( \rho_{13}(z) \in \{0, \log_{13}(2)\} \). Since the order of the reduction \( \tilde{P} \) of \( P \) modulo 13 is 7, it suffices to consider the 6 residue disks corresponding to nontrivial multiples of \( \tilde{P} \). We then proceed as follows: For every solution \( z \notin \{\pm P, \pm Q\} \) to \( \rho_{13}(z) = \log_{13}(d) \), where \( d \in \mathcal{P} \), we assume that \( z \) does correspond to a rational point \( a \cdot P \in X(\mathbb{Q}) \) and we compute

- the discrete logarithm \( a' \in \mathbb{F}_7 \) of \( z \) mod 13 with respect to \( \tilde{P} \);
- the residue disk \( U_{z, 7} \) of \( z \) mod 7, by computing \( \int_{\infty}^{z} \omega \cdot (\int_{\infty}^{t} \omega)^{-1} \).

So we only have to check for all solutions \( w \in U_{z, 7} \) to \( \rho_{7}(w) = \log_{7}(t) \), whether

\[
a'' = \int_{\infty}^{w} \omega \cdot (\int_{\infty}^{t} \omega)^{-1} \mod 7
\]

coincides with \( a' \); if not, then we know that \( z \notin X(\mathbb{Q}) \).

Using this approach, we succeeded in showing that indeed \( \pm P, \pm Q \) are the only integral points on \( X \).

More generally, we recall [39, Sec. 2, Definition] that an aliquot cycle of length \( l \) for \( X \) is a finite sequence of primes \( \overline{\mathcal{P}} = (p_1, p_2, \ldots, p_l) \) with the property that \( \# X(\mathbb{F}_{p_i}) = p_{i+1} \), where \( i \) is taken modulo \( l \). We can similarly define pseudo-aliquot cycles by replacing the condition with the weaker one asserting only that \( p_{i+1} \) divides \( \# X(\mathbb{F}_{p_i}) \).

**Definition A.2.** Let \( \overline{\mathcal{P}} \) be a pseudo-aliquot cycle of length \( l \) for \( X \). A lift of \( \overline{\mathcal{P}} \) is a collection of points \( z_i \in X(\mathbb{Q}_{p_i}) \), for each \( i = 1, \ldots, l \), such that \( \rho_{p_i}(z_i) = \log_{p_i}(d) \), and such that the entire collection satisfies the following conditions with \( n_i = \log_{p_i}(z_i) \in \mathbb{Q}_{p_i} \):

- We have \( n_i \in \mathbb{Z}_{p_i} \).
- As \( p_{i+1} \) divides \( \# X(\mathbb{F}_{p_i}) \), \( n_{i+1} \) is restricted to certain residue classes in \( X(\mathbb{Q}_{p_i}) \) and we assume that \( z_i \) sits in one such residue class.

Clearly an integral point \( Q \in U(\mathbb{Z}) \) with intersection pattern \( d \in \mathcal{P} \) gives rise to the lift of \( \overline{\mathcal{P}} \) where \( z_i \) is the image of \( Q \) in \( X(\mathbb{Q}_{p_i}) \). We will say that \( \overline{\mathcal{P}} \) disqualifies \( d \) if there are no lifts of \( \overline{\mathcal{P}} \) other than the ones corresponding to points in \( \mathcal{A} \). This then proves that there are no points in \( U(\mathbb{Z}) - \mathcal{A} \) with intersection pattern \( d \). If we manage to disqualify all intersection patterns in \( \mathcal{P} \) we prove that \( \mathcal{A} = U(\mathbb{Z}) \).

We have not tried to carry this out for cycles of length \( > 2 \) in practice.
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