We report on recent progress on the flavour non-singlet splitting functions in perturbative QCD. The exact four-loop ($N^3\text{LO}$) contribution to these functions has been obtained in the planar limit of a large number of colours. Phenomenologically sufficient approximate expressions have been obtained for the parts not exactly known so far. Both cases include results for the four-loop cusp and virtual anomalous dimensions which are relevant well beyond the evolution of non-singlet quark distributions, for which an accuracy of (well) below 1% has now been reached.
1. Introduction

Up to power corrections, observables in $ep$ and $pp$ hard scattering can be schematically expressed as

$$O^{ep} = f_i \otimes c_i^0, \quad O^{pp} = f_j \otimes f_k \otimes c_k^0$$

(1.1)

in terms of the respective partonic cross sections (coefficient functions) $c_i^0$ and the universal parton distribution functions (PDFs) $f_i(x, \mu^2)$ of the proton at a (renormalization and factorization) scale $\mu$ of the order of a physical hard scale, e.g., $M_H$ for the total cross section for the production of the Higgs boson. The dependence of the PDFs on the momentum fraction $x$ is not calculable in perturbative QCD; their scale dependence is governed by the renormalization-group evolution equations

$$\frac{\partial}{\partial \ln \mu^2} f_i(x, \mu^2) = \left[ P_{ik}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](x)$$

(1.2)

where $\otimes$ denotes the Mellin convolution. The splitting functions, which are closely related to the anomalous dimensions of twist-2 operators in the light-cone operator-product expansion (OPE), and the coefficient functions can be expanded in powers of the strong coupling $a_s \equiv \alpha_s(\mu^2)/(4\pi)$.

$$P = a_s P^{(0)} + a_s^2 P^{(1)} + a_s^3 P^{(2)} + a_s^4 P^{(3)} + \ldots,$$  

(1.3)

$$c_a^0 = a_s^{n_{\alpha}} \left[ c_0^{(0)} + a_s c_0^{(1)} + a_s^2 c_0^{(2)} + a_s^3 c_0^{(3)} + \ldots \right].$$

(1.4)

Together the first three terms in eqs. (1.3) and (1.4) provide the next-to-next-to-leading order (N$^3$LO) of perturbative QCD for the observables (1.1). This is now the standard approximation for many hard processes; see refs. [1–4] for the corresponding splitting functions.

Corrections beyond N$^3$LO are of phenomenological interest where high precision is required, such as in determinations of $\alpha_s$ from deep-inelastic scattering (DIS) (see refs. [5, 6] for the N$^3$LO corrections to the most important structure functions), and where the perturbation series shows a slow convergence, such as for Higgs production via gluon-gluon fusion calculated in ref. [7] at N$^3$LO. The size and structure of the corrections beyond N$^2$LO are also of theoretical interest.

Here we briefly report about considerable recent progress on the three four-loop (N$^3$LO) non-singlet splitting functions. We focus on the quantities $P_{ns}^{(3)}(x)$ for the evolution of flavour-differences $q_i \pm \bar{q}_j - (q_i \pm \bar{q}_j)$ of quark and antiquark distributions; for more details see ref. [8].

2. Diagram calculations of fixed-$N$ moments

Two methods have been applied for obtaining Mellin moments of the quantities $P^{(3)}$ in eq. (1.3). Depending on the function, both can be used to determine the same even-$N$ or the odd-$N$ moments.

In the first one calculates, via the optical theorem and a dispersion relation in $x$, the unfactorized structure functions in DIS, as done at two and three loops in refs. [9–12]. The construction of the FORCER program [13] has facilitated the extension of those computations (which also provide moments of the coefficient functions) to four loops. For the hardest diagrams, the complexity of these computations rises quickly with $N$, hence only $N \leq 6$ has been covered completely so far [14]. Much higher $N$ can be accessed for simpler cases, e.g., values up to $N > 40$ have been reached for high-$n_f$ parts. These were sufficient to determine the complete $n_f^2$ and $n_f^3$ parts of the non-singlet splitting functions $P_{ns}^{(3)}(x)$ and the $n_f^3$ parts of the corresponding flavour-singlet quantities [15].
The increase of the complexity of the Feynman integrals with $N$ is more benign for the second method based on the OPE which was applied to the present non-singlet cases at NLO in ref. [16], see also ref. [17]. FORCER calculations in this framework have reached $N = 16$ for all contributions to the functions $P_{ns}^{(3)}$, $N = 18$ for their $n_f$ parts and $N = 20$ for the complete limit of a large number of colours $n_c$ [8]. See refs. [18–21] for earlier calculations of $P_{ns}^{\pm(3)}$ at $N \leq 4$.

3. Towards all-$N$ expressions

If the anomalous dimensions $\gamma_{ns}(N) = -P_{ns}(N)$ at $N^{n+2}$LO are analogous to the lower orders, then they can be expressed in terms of harmonic sums $S_{\vec{w}}$ [22, 23] and denominators $D_a^k \equiv (N+a)^{-k}$ as

$$\gamma_{ns}^{(n)}(N) = \sum_{w=0}^{2n+1} c_{0w} S_{\vec{w}}(N) + \sum_{a} \sum_{k=1}^{2n+1} \sum_{w=0}^{2n+1-k} c_{aw} D_a^k S_{\vec{w}}(N) . \quad (3.1)$$

The denominators at the calculated values of $N$ indicate $a = 0, 1$ for $\gamma_{ns}^\pm$, with coefficients $c_{0w}, c_{aw}$ that are integer modulo low powers of $1/2$ and $1/3$. Sums up to weight $w = 2n + 1$ occur at $N^2$LO.

Based on a conformal symmetry of QCD at an unphysical number of space-time dimensions $D$, it has been conjectured that the $\overline{\text{MS}}$ functions $\gamma_{ns}(N)$ are constrained by ‘self-tuning’ [24, 25],

$$\gamma_{ns}(N) = \gamma_0 (N + \sigma \gamma_{ns}(N) - \beta(\alpha_s)/\alpha_s)) \quad (3.2)$$

where $\beta(\alpha_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 - \ldots$ is the beta function, for its present status see refs. [26, 27]. The initial-state (PDF) and final-state (fragmentation-function) anomalous dimensions are obtained for $\sigma = -1$ and $\sigma = 1$, respectively, and the universal kernel $\gamma_0$ is reciprocity respecting (RR), i.e., invariant under replacement $N \rightarrow (1-N)$. Eq. (3.2) implies that the non-RR parts and the spacelike/timelike difference are inherited from lower orders. Hence ‘only’ $\gamma_0$, which includes $2^{n-1}$ RR (combinations of) harmonic sums of weight $w$, needs to be determined at four loops.

Present information, given by the even-$N$ (odd-$N$) values $N \leq 16$ ($15$) of $\gamma_{ns}^{+(-)}(N)$ ($\gamma_{ns}^{-(+)}(N)$) and endpoint constraints (see below), is insufficient to determine the $n = 3$ coefficients in eq. (3.1). However, $\gamma_{ns}^{\pm} = \gamma_{ns}$ in the large-$n_c$ limit, hence the known even-$N$ and odd-$N$ values can be used. Moreover, alternating sums do not contribute to $\gamma_{ns}^\pm$ in this limit, leaving 1, 1, 2, 3, 5, 8, 13 = Fibonacci($w$) RR sums at weight $w = 1, \ldots, 7$ and a total of 87 basis functions for $n = 3$ in eq. (3.1).

Large-$N$ and small-$x$ limits provide more than 40 constraints on their coefficients. At large-$N$, the non-singlet anomalous dimensions have the form [33–35]

$$\gamma_{ns}^{(n-1)}(N) = A_n \ln \widetilde{N} - B_n + N^{-1} \{C_n \ln \widetilde{N} - \widetilde{D}_n + \frac{1}{2} A_n \} + O(N^{-2}) \quad (3.3)$$

with $\ln \widetilde{N} \equiv \ln N + \gamma_0$, where $\gamma_0$ denotes the Euler-Mascheroni constant. $C_n$ and $\widetilde{D}_n$ are given by

$$C(a_s) = (A(a_s))^2 \quad , \quad \widetilde{D}(a_s) = A(a_s) \cdot (B(a_s) - \beta(\alpha_s)/\alpha_s) \quad , \quad (3.4)$$

in terms of lower-order information on the cusp anomalous dimension $A(a_s) = A_1 a_s + A_2 a_s^2 + \ldots$ and the quantity $B(a_s) = B_1 a_s + B_2 a_s^2 + \ldots$ sometimes called the virtual anomalous dimension.

The resummation of small-$x$ double logarithms [28–31] provides the four-loop coefficients of $x^n \ln^b x$ at $4 \leq b \leq 6$ and all $a$ in the large-$n_c$ limit (in full QCD, this holds only at even $a$ for $P_{ns}^+(x)$ and odd $a$ for $P_{ns}^-(x)$). Moreover, a relation leading to a single-logarithmic resummation at $a = 0$,

$$\gamma_{ns}^+(N) \cdot (\gamma_{ns}^-(N) + N - \beta(\alpha_s)/\alpha_s) = O(1) \quad , \quad (3.5)$$

2
has been conjectured in ref. [32]. As far as it can be checked so far, this relation is found to be correct except for terms with \( \zeta_2 = \pi^2/6 \) that vanish in the large-\( n_c \) limit.

Taking into account all the above information, it is possible to set up systems of Diophantine equations for the coefficients \( c_{006}, c_{a0\ell} \) of \( \gamma_{\text{ins}}^{\pm(3)}(N) \) in the large-\( n_c \) limit that can be solved using the moments \( 1 \leq N \leq 18 \), leaving the results of the diagram calculation at \( N = 19, 20 \) as checks.

4. All-\( N \) anomalous dimension in the large-\( n_c \) limit

The exact expressions for the new \( n_f^0 \) and \( n_f^1 \) parts cannot be shown here due to their length, they can be found in eq. (3.6) and (3.7) of ref. [8]. For the \( n_f^2 \) and \( n_f^3 \) terms see ref. [15]. The resulting large-\( N \) coefficients \( A_{L,A} \) and \( B_{L,A} \) – the subscript \( L \) indicates the large-\( n_c \) limit – are found to be

\[
A_{L,A} = C_F n_c^3 \left( \frac{84278}{81} \xi_2 + \frac{20992}{27} \xi_3 + 1804 \xi_4 - \frac{352}{3} \xi_2 \xi_3 - 352 \xi_4 - \frac{32}{3} \xi_2 - 876 \xi_5 \right)
- C_F n_c^3 n_f \left( \frac{39883}{81} - \frac{26692}{81} \xi_2 + \frac{16252}{27} \xi_3 + \frac{440}{3} \xi_4 - \frac{256}{3} \xi_2 \xi_3 - 224 \xi_5 \right)
- C_F n_c^2 n_f^2 \left( \frac{2219}{81} - \frac{32}{81} \xi_2 + \frac{1280}{27} \xi_3 - \frac{64}{3} \xi_4 \right) - C_F n_f^3 \left( \frac{32}{81} - \frac{64}{27} \xi_5 \right) \tag{4.1}
\]

and

\[
B_{L,A} = C_F n_c^3 \left( -\frac{1379569}{5184} + \frac{24211}{27} \xi_2 - \frac{9803}{162} \xi_3 - \frac{9382}{9} \xi_4 + \frac{838}{9} \xi_2 \xi_3 + 1002 \xi_3 + \frac{16}{3} \xi_3^2 
+ 135 \xi_4 - 80 \xi_2 \xi_5 + 32 \xi_3 \xi_4 - 560 \xi_5 \right)
+ C_F n_c^2 n_f \left( -\frac{353}{3} - \frac{85175}{162} \xi_2 - \frac{137}{9} \xi_3 + \frac{16186}{27} \xi_4 - \frac{584}{9} \xi_2 \xi_3 - \frac{248}{3} \xi_5 - \frac{16}{3} \xi_3^2 - 144 \xi_5 \right)
- C_F n_c^2 n_f^2 \left( \frac{127}{18} - \frac{5036}{81} \xi_2 + \frac{932}{27} \xi_3 + \frac{1292}{27} \xi_4 - \frac{160}{9} \xi_2 \xi_3 - \frac{32}{3} \xi_5 \right)
- C_F n_f^3 \left( \frac{131}{81} - \frac{32}{81} \xi_2 - \frac{304}{81} \xi_3 + \frac{32}{27} \xi_4 \right) \tag{4.2}
\]

The agreement of the four-loop cusp anomalous dimension \( \gamma_{\text{ins}}^{\pm(3)} \) with the result obtained from the large-\( n_c \) photon-quark form factor [36, 37] provides a further non-trivial check of the determination of the all-\( N \) expressions from the moments at \( N \leq 18 \), and hence also of the relations (3.1) – (3.5).

The maximum-weight \( \xi_3^2 \) and \( \xi_6 \) parts of eq. (4.1) also agree with the result obtained in planar \( \mathcal{N} = 4 \) maximally supersymmetric Yang-Mills theory (MSYM) obtained before in ref. [38]. There is no such direct connection between the four-loop virtual anomalous dimension (4.2) and its counterparts in planar \( \mathcal{N} = 4 \) MSYM; see ref. [39] where the maximum-weight part of eq. (4.2) has been employed to derive the four-loop collinear anomalous dimension in planar \( \mathcal{N} = 4 \) MSYM.

The all-\( N \) large-\( n_c \) limit of \( \gamma_{\text{ins}}^{\pm(3)}(N) \) is compared in fig. 1 with the integer-\( N \) QCD results at \( N \leq 16 \). As illustrated in the left panel, the former are a decent approximation to the latter for the individual \( n_f^3 \) contributions. However, as shown in the right panel, there are considerable cancellations between the these contributions. These cancellations are most pronounced for the physically relevant number of \( n_f = 5 \) light quark flavours outside the large-\( N \)/large-\( x \) region. Hence the large-\( n_c \) suppressed contributions – indicated by the subscript \( N \) below – need to be taken into account in phenomenological N\(^3\)LO analyses.
Figure 1: The large-$n_c$ limit of the four-loop anomalous dimensions $\gamma_{\text{ns}}^{(3)}(N)$ (lines) compared to the QCD results for $\gamma_{\text{ns}}^{(3)}(N)$ at even $N$ and $\gamma_{\text{ns}}^{(3)}(N)$ at odd $N$ (points). Left: the $n_f$-independent contributions. Right: the results for physically relevant values of $n_f$. The values have been converted to an expansion in $\alpha_s$.

5. $x$-space approximations of the large-$n_c$ suppressed parts

With eight integer-$N$ moments known for both $P_{\text{ns}}^{+(3)}(x)$ and $P_{\text{ns}}^{-(3)}(x)$ and the large-$x$ and small-$x$ knowledge discussed in section 2, it is possible to construct approximate $x$-space expressions which are analogous to (but more accurate than) those used before 2004 at $N^2$LO, see refs. [40–43]. For this purpose an ansatz consisting of

- the two large-$x$ parameters $A_4$ and $B_4$ in eq. (3.3),
- two of three suppressed large-$x$ logs $(1-x)\ln^k(1-x), k=1,2,3$,
- one of ten two-parameter polynomials in $x$ that vanish for $x \to 1$,
- two of the three unknown small-$x$ logarithms $\ln^k x, k=1,2,3$

is built for the large-$n_c$ suppressed $n_f^0$ and $n_f^1$ parts $P_{\text{N,0/1}}^{+(3)}$ of $P_{\text{ns}}^{+(3)}(x)$. This results in 90 trial functions, the parameters of which can be fixed from the eight available moments. Of these functions, two representatives $A$ and $B$ are then chosen that indicate the remaining uncertainty, see fig. 2.

This non-rigorous procedure can be checked by comparing the same treatment for the large-$n_c$ parts to our exact results. Moreover, the trial functions lead to very similar values for the next moment, e.g., $N = 18$ for $P_{\text{ns}}^{+(3)}$. The residual uncertainty at this $N$-value is a consequence of the width of the band at large $x$, which in turn is correlated with the uncertainties at smaller $x$. If the spread of the result $A$ and $B$ would underestimate the true remaining uncertainties, then a comparison with an additional analytic result at this next value of $N$ should reveal a discrepancy.
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Figure 2: About 90 trial functions for the $n_f$-independent contribution to the large-$n_c$ suppressed part of splitting function $P_{N,0}^{(3)+}(x)$, multiplied by $x^{0.4}(1-x)$. The two functions chosen to represent the remaining uncertainty are denoted by $A$ and $B$ and shown by solid (blue) lines. Due to the factor $(1-x)$ the contribution $A_{N,4}$ to the four-loop cusp anomalous dimension can be read off at $x = 1$.

We were able to extend the diagram computations of the $n_f$ parts of $P_{N,0}^{(3)+}(x)$ to $N = 18$ and find

$$P_{N,0}^{(3)+}(N = 18) = 195.8888792 B < 195.8888857 \text{exact} < 195.8888968 A.$$  

A similar check for $P_{N,0}^{(3)-}$ has been carried out by deriving a less accurate approximation using only seven moments and comparing the results to the now unused value at $N = 16$.

The case of $P_{N,0}^{(3)-}(x)$ has been treated in the same manner, but taking into account that only its leading small-$x$ logarithm is known up to now [29]. See ref. [8] for the (large-$N$ suppressed) additional $d^{abc} d_{abc}$ contribution $P_{N,0}^{s(3)}(x)$ to the splitting function for the total valence quark PDF.

6. Numerical results for the cusp and virtual anomalous dimensions

Combining the exact large-$n_c$ results, the approximations for the remaining $n_f^0$ and $n_f^1$ contributions and the complete high-$n_f$ contributions of ref. [15], the four-loop cusp anomalous dimension for QCD with $n_f$ quark flavours are given by

$$A_4 = 20702(2) - 5171.9(2) n_f + 195.5772 n_f^2 + 3.272344 n_f^3,$$

where the numbers in brackets represent a conservative estimate of the remaining uncertainty. The conversion of this result to an expansion in powers of $\alpha_s$ leads to

$$A_q(\alpha_s, n_f = 3) = 0.42441 \alpha_s (1 + 0.72657 \alpha_s + 0.73405 a_s^2 + 0.6647(2) a_s^3 + \ldots),$$

$$A_q(\alpha_s, n_f = 4) = 0.42441 \alpha_s (1 + 0.63815 \alpha_s + 0.50998 a_s^2 + 0.3168(2) a_s^3 + \ldots),$$

$$A_q(\alpha_s, n_f = 5) = 0.42441 \alpha_s (1 + 0.54973 \alpha_s + 0.28403 a_s^2 + 0.0133(2) a_s^3 + \ldots).$$  

(6.2)
The corresponding results for the virtual anomalous dimension, i.e., the coefficient of $\delta(1-x)$ show a similarly benign expansion with

$$B_4 = 23393(10) - 5551(1)n_f + 193.8554n_f^2 + 3.014982n_f^3$$

and

$$B_q(\alpha_s, n_f=3) = 0.31831\alpha_s (1 + 0.99712\alpha_s + 1.24116a_s^2 + 1.0791(13)a_s^3 + \ldots),$$
$$B_q(\alpha_s, n_f=4) = 0.31831\alpha_s (1 + 0.87192\alpha_s + 0.97833a_s^2 + 0.5649(13)a_s^3 + \ldots),$$
$$B_q(\alpha_s, n_f=5) = 0.31831\alpha_s (1 + 1.74672\alpha_s + 0.71907a_s^2 + 0.1085(13)a_s^3 + \ldots).$$

Due to constraints by large-$N$ moments, the errors of $A_4$ and $B_4$ are fully correlated. The accuracy in eqs. (6.2) and (6.4) should be amply sufficient for phenomenological applications.

By repeating the approximation procedure in section 5 for individual colour factors, it is possible to obtain corresponding approximate coefficients for $A_4$ and $B_4$ which can be summarized as (for a table of the relevant group invariants see, e.g., appendix C of ref. [44])

| \(A_4\) | \(B_4\) |
| --- | --- |
| \(C_F^4\) | 0 | 197. ± 3. |
| \(C_F^2C_A\) | 0 | −687. ± 10. |
| \(C_F^2C_A^2\) | 0 | 1219. ± 12. |
| \(C_F^2C_A^3\) | 610.3 ± 0.3 | 295.6 ± 2.4 |
| \(n_fC_F^2\) | −31.00 ± 0.4 | 81.4 ± 2.2 |
| \(n_fC_F^2C_A\) | 38.75 ± 0.2 | −455.7 ± 1.1 |
| \(n_fC_F^2C_A^2\) | −440.65 ± 0.2 | −274.4 ± 1.1 |
| \(n_fC_F^2C_A^3\) | −123.90 ± 0.2 | −143.5 ± 1.2 |
| \(n_fC_F^2\) | −21.31439 | −5.775288 |
| \(n_f^2C_F\) | 58.36737 | 51.03056 |
| \(n_f^2C_F\) | 2.454258 | 2.261237 |

where the exactly known \(n_f^2\) and \(n_f^3\) coefficients have been included for completeness. Due to the constraint provided by the exact large-\(n_c\) limit, the errors in this table are highly correlated; for numerical applications in QCD eqs. (6.2) and (6.4) should be used instead. The above results show that both quartic group invariants definitely contribute to the four-loop cusp anomalous dimension, for this issue see also refs. [45–48] and references therein. This implies that the so-called Casimir scaling between the quark and gluon cases, \(A_q = C_F/C_AA_0\), does not hold beyond three loops.

7. \(N^3\)LO corrections to the evolution of non-singlet PDFs

The effect of the fourth-order contributions on the evolution of the non-singlet PDFs can be illustrated by considering the logarithmic derivatives of the respective combinations of quark PDFs with respect to the factorization scale, \(q_{\text{in}} = d\ln q_{\text{in}}/d\ln \mu_f\), at a suitably chosen reference point.
As in ref. [1], we choose the schematic, order-independent initial conditions

\[ xq_{+}^{\pm \nu}(x, \mu_0^2) = x^{0.5}(1-x)^3 \quad \text{and} \quad \alpha_s(\mu_0^2) = 0.2. \]  

(7.1)

For \( \alpha_s(M_Z^2) = 0.114 \ldots 0.120 \) this value for \( \alpha_s \) corresponds to \( \mu_0^2 \simeq 25 \ldots 50 \text{ GeV}^2 \) beyond the leading order, a scale range typical for DIS at fixed-target experiments and at the \( e\bar{p} \) collider HERA.

The new \( N^3\text{LO} \) corrections to \( \dot{q}_{ns} \) are generally small, hence they are illustrated in fig. 3 by comparing their relative effect to that of the \( N^2\text{LO} \) contributions for the standard identification \( \mu_r = \mu_f = \mu \) of the renormalization scale with the factorization scale. Except close to the sign change of the scaling violations at \( x \simeq 0.07 \), the relative \( N^3\text{LO} \) effects are (well) below 1% for the flavour-differences \( q_{+}^{\nu} \) and \( q_{-}^{\nu} \) (left and middle panel). The \( N^2\text{LO} \) and \( N^3\text{LO} \) corrections are larger for the valence distribution \( q_{v}^{\nu} \) at \( x < 0.07 \) due to the effect of the \( d^{abc}d_{abc} \) ‘sea’ contribution \( P_{ns}^{s}(x) \), note the different scale of the right panel in fig. 3. Also in this case the \( N^3\text{LO} \) evolution represents a clear improvement, and the relative four-loop corrections are below 2%.

The remaining uncertainty due to the approximate character of the four-loop splitting functions beyond the large-\( n_c \) limit is indicated by the difference between the solid and dotted (red) curves in fig. 3 and fig. 4 below. Due to the small size of the four-loop contributions and the ‘\( x \)-averaging’ effect of the Mellin convolution,

\[
[P_{ns} \otimes q_{ns}](x) = \int_{1/x}^{x} \frac{dy}{y} P_{ns}(y) q_{ns}\left(\frac{x}{y}\right),
\]

(7.2)

the results of section 4 are safely applicable to lower values of \( x \) than one might expect from fig. 2.

The stability of the NLO, \( N^2\text{LO} \) and \( N^3\text{LO} \) results under variation of the renormalization scale over the range \( \frac{1}{2} \mu_f^2 \leq \mu_r^2 \leq 8 \mu_f^2 \) is illustrated in fig. 4 at typical values of \( x \). Except close to the sign change of \( \dot{q}_{ns}^+ \), the variation is well below 1% for the conventional interval \( \frac{1}{2} \mu_f \leq \mu_r \leq 2 \mu_f \).

![Figure 3: The relative \( N^2\text{LO} \) and \( N^3\text{LO} \) corrections to the logarithmic scale derivative of the non-singlet combinations \( q_{ns}^\mu \) of quark PDFs for the schematic order-independent input (7.1) for \( n_f = 4 \) at \( \mu_r = \mu_f \).](image-url)
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The dependence of the NLO, N$^2$LO and N$^3$LO results for $\Delta q^{+}_{\text{ns}} \equiv d \ln q^{+}_{\text{ns}} / d \ln \mu^2$ on the renormalization scale $\mu$, at six typical values of $x$ for the initial conditions (7.1) and $n_f = 4$ flavours. The remaining uncertainty of the four-loop splitting function $P^{+}_{\text{ns}}(x)$ leads to the difference of the solid and dotted curves.

8. Summary and Outlook

The splitting functions for the non-singlet combinations of quark PDFs have been addressed at the fourth-order (N$^3$LO) of perturbative QCD. The quantities $P^{+}_{\text{ns}}(x)$ are now known exactly in the limit of a large number of colours $n_c$. Present results for the large-$n_c$ suppressed contributions with $n_f^0$ and $n_f^1$ are still approximate, but sufficiently accurate for phenomenological applications in deep-inelastic scattering and collider physics. FORM and FORTRAN files of these results can be obtained by downloading the source of ref. [8] from arXiv.org.

It would be desirable, mostly for theoretical purposes, to obtain also the analytic forms $n_f^0$ and $n_f^1$ parts of $P_{\text{ns}}^{+}(3)$. So far, only their contributions proportional to the values $\zeta_4$ and $\zeta_5$ of the Riemann $\zeta$-function have been completely determined, together with the (unpublished) $\zeta_3$ part of the $n_f^1$ contributions. The $\zeta_4$ parts are particularly simple; in fact, it turns out that they (and other $\pi^2$ terms) can be predicted via physical evolution kernels from lower-order quantities, see refs. [49,50].

The $\zeta_5$ part of $P_{\text{ns}}^{+}(3)$, presented in appendix D of ref. [8], includes a (non large-$n_c$) contribution

$$- \frac{128}{3} \left\{ 3 C_F^2 C_A^2 - 2 C_F C_A^3 + 12 d^{abcd}_i d^{abcd}_i / N_R \right\} 5 \zeta_5 |S_1(N)|^2. \quad (8.1)$$

The resulting $\ln^2 N$ large-$N$ behaviour needs to be compensated by non-$\zeta_5$ terms. Eq. (8.1) looks exactly like the $\zeta_5$-‘tail’ of the so-called wrapping correction in the anomalous dimensions in $\mathcal{N} = 4$ maximally supersymmetric Yang-Mills theory, see refs. [51,52].
Phenomenologically, of course, one rather needs corresponding results for the flavour-singlet splitting functions $H^{(3)}_{ij}(x)$, $i, j = q, g$. At present, it appears computationally too hard to obtain moments of all four functions beyond $N = 6$ using the method of refs. [9–12]. Therefore one will need to resort to the OPE, which offers additional theoretical challenges in the massless flavour-singlet case, see refs. [53–55]. We hope to address this issue in a future publication.

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