SYMPLECTIC RESOLUTIONS FOR CONICAL SYMPLECTIC VARIETIES

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Abstract. We introduce the notion of a conical symplectic variety, and show that any symplectic resolution of such a variety is isomorphic to the Springer resolution of a nilpotent orbit closure in a semisimple Lie algebra, composed with a linear projection.

1. Introduction

Definition 1. A conical symplectic variety is an affine variety $W \subset \mathbb{C}^N$, smooth in codimension 1, such that (i) $W$ is invariant under the dilation action of $\mathbb{C}^*$ on $\mathbb{C}^N$, (ii) There exists a holomorphic symplectic form $\omega$ on $W_{\text{reg}}$ such that $\lambda^*\omega = \lambda \omega$ for all $\lambda \in \mathbb{C}^*$, (iii) For any resolution $\pi : Z \to W$, the 2-form $\pi^*\omega$ (defined on $\pi^{-1}(W_{\text{reg}})$) extends to a regular 2-form $\Omega$ on $Z$. A resolution $\pi : Z \to W$ is called symplectic if $\Omega$ is a holomorphic symplectic form on the whole of $Z$.

For a conical symplectic variety $W \subset \mathbb{C}^N$, its normalization $\tilde{W}$ becomes a symplectic variety in the sense of Beauville [B2]. Typical examples of conical symplectic varieties are nilpotent orbit closures $\bar{O}$ in a semi-simple Lie algebra $\mathfrak{g}$ and their symplectic resolutions can be constructed as follows. For a flag variety $G/P$, its cotangent bundle $T^*G/P$ admits a natural Hamiltonian $G$-action, and the image of the moment map $T^*_G/P \to \mathfrak{g}^*$ is a nilpotent orbit closure $\bar{O}_P$ (the Richardson orbit of $P$). Hence we have a projective generically finite morphism $\mu : T^*_G/P \to \bar{O}_P$, which is called the Springer map associated to $G/P$. If $\mu$ is birational, it becomes a symplectic resolution of $\bar{O}_P$, which we call a Springer resolution. In [F], it is proven that symplectic resolutions of nilpotent orbit closures are exactly Springer resolutions.

Further examples of conical symplectic varieties can be constructed as follows: take a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ and a linear subspace $L \subset \mathfrak{g}$ such that the natural projection $\mathfrak{g} \to \mathfrak{g}/L$ maps $\mathcal{O}$ birationally to a subvariety $W$. Then $W$ is a conical symplectic variety provided it is smooth in codimension 1. This happens for example if $L$ is a line, not contained in the secant variety of $\mathcal{O}$. We suspect that this construction gives all conical symplectic varieties.

The main result of this note classifies symplectic resolutions for conical symplectic varieties. It generalizes the main theorem of [F], with a simpler proof.

Theorem 2. Let $\pi : Z \to W$ be a symplectic resolution for a conical symplectic variety. Then
(i) $Z \simeq T^*_G/P$ for some flag variety $G/P$.
(ii) The Springer map $\mu : T^*_G/P \to \bar{O}_P$ associated to $G/P$ is birational.
(iii) There exists a linear subspace $L \subset \mathfrak{g}$ such that $W \subset \mathbb{C}^N$ is the image of $\bar{O}_P \subset \mathfrak{g}$ under the projection $p : \mathfrak{g} \to \mathfrak{g}/L$. The induced map $\bar{O}_P \to W$ is birational.
(iv) The map $\pi$ is isomorphic to the composition $p \circ \mu$.
(v) If $W$ is normal, then $W = \mathcal{O}_P$.

Remark that our assumptions in Definition 1 are optimal for Theorem 2. This theorem is inspired by the alternative proof of [1] provided by Namikawa in Section 8 of [2]. As an application, we provide a simpler proof of the main theorem of [2], Section 7.

Acknowledgements: We are grateful to Yoshinori Namikawa for pointing out an error in a previous version, and to the referees for their helpful comments. Baohua Fu is supported by NSFC (11031008 and 11225106).

2. Proof of Theorem 2

For a smooth variety $Y$, we denote by $T_Y$ the tangent bundle, by $T^*_Y$ the cotangent bundle, and by $K_Y$ the canonical bundle (the determinant of $T^*_Y$). A contact structure on $Y$ is a corank one subbundle $F \subset T_Y$ such that the bilinear form on $F$ with values in the quotient line bundle $L = T_Y/F$ deduced from the Lie bracket on $T_Y$ is everywhere non-degenerate. This implies that $\dim Y = 2n - 1$ is odd and $K_Y \simeq L^{2n-2}$. We call $L$ the contact line bundle of the contact structure on $Y$. A typical example of contact manifold is the projectivised cotangent bundle $\mathbb{P}(T^*_M) := (T^*_M \setminus \text{zero section})/\mathbb{C}^*$ of a smooth variety $M$, where the contact line bundle is $\mathcal{O}_{\mathbb{P}(T^*_M)}(1)$.

Proposition 3. Let $X \subset \mathbb{P}^{N-1}$ be a closed singular subvariety and $f : Y \to X$ a resolution. Assume that $Y$ has a contact structure with the contact line bundle $f^*\mathcal{O}_X(1)$. Let $\tilde{X} \to X$ be the normalization map. Then $Y \simeq \mathbb{P}(T^*_M)$ for some flag variety $G/P$ and the induced map $Y \to \tilde{X}$ yields the Stein factorization of the projectivised Springer map associated to $G/P$.

Proof. As $X$ is singular, we get $b_2(Y) \geq 2$. Note that $K_Y = f^*\mathcal{O}_X(-n)$ with $n = (\dim X + 1)/2$, hence $K_Y$ is not nef. By [1], we deduce that $Y \simeq \mathbb{P}(T^*_M)$ for some smooth projective variety $M$ and $\mathcal{O}_{\mathbb{P}(T^*_M)}(1) \simeq f^*\mathcal{O}_X(1)$. This implies that $\mathcal{O}_{\mathbb{P}(T^*_M)}(1)$, hence the tangent bundle $T_M$ is globally generated. We deduce that $M$ is a homogeneous variety, hence by [1] it is isomorphic to $G/P \times A$, where $G/P$ is a flag variety for some semisimple algebraic group $G$ and $A$ is an abelian variety.

Let $X \subset \mathbb{C}^N$ be the affine cone of $X$. As $f^*\mathcal{O}_X(-1) = \mathcal{O}_{\mathbb{P}(T^*_M)}(-1)$, the map $f$ pulls back the $\mathbb{C}^*$-bundle $\tilde{X} \setminus \{0\} \to X$ to the $\mathbb{C}^*$-bundle $T^*_M \setminus \text{zero section} \to \mathbb{P}(T^*_M)$, which gives a birational map $\tilde{f} : T^*_M \setminus \text{zero section} \to \tilde{X} \setminus \{0\}$. Note that $T^*_M \simeq T^*_{G/P} \times \mathbb{C}^g \times A$, where $g = \dim A$. As $\tilde{X}$ is affine, the map $\tilde{f}$ contracts $\{x\} \times A$ to one point for any $x \in (T^*_{G/P} \setminus \text{zero section}) \times (\mathbb{C}^g \setminus \{0\})$, hence $\tilde{f}$ factors through $T^*_M \setminus \text{zero section} \to (T^*_{G/P} \setminus \text{zero section}) \times (\mathbb{C}^g \setminus \{0\})$. As $\tilde{f}$ is birational, we get $g = 0$ and $M \simeq G/P$. If $\dim G/P = 1$, then $G/P \simeq \mathbb{P}^1$ and $X$ is smooth, which contradicts our assumption. Thus, $\dim G/P \geq 2$.

Let $\mu : T^*_{G/P} \to \tilde{\mathcal{O}}$ be the Springer map associated to $G/P$; this is a projective, generically finite morphism. By the Stein factorization, it follows that the algebra $A := H^0(T^*_{G/P}, \mathcal{O}_{T^*_{G/P}})$ is finitely generated, and $\mu$ factors as a birational morphism $\tilde{\mu} : T^*_{G/P} \to V := \text{Spec}(A)$ followed by a finite morphism $\eta : V \to \tilde{\mathcal{O}}$. Note that $\tilde{\mu}$ and $\eta$ are both $G \times \mathbb{C}^*$-equivariant; also, $V^{\mathbb{C}^*}$ is a single point, say $o$. 


Let $\mathbb{C}[\hat{X}]$ be the coordinate ring of the affine variety $\hat{X}$, then the birational morphism $\hat{f} : T^*_G/P \to \hat{X}\setminus\{0\} \subset \hat{X}$ induces a homomorphism of $\mathbb{C}$-algebras $\mathbb{C}[\hat{X}] \to H^0(T^*_G/P, 0, \mathcal{O}) = A$, where the latter equality follows from $\dim(G/P) \geq 2$. This gives a morphism $V \to \hat{X}$ and the map $\mu$ (restricted to $T^*_G/P, (0)$) yields the Stein factorization of $\hat{f}$. Hence we get that $V\setminus\{0\}$ is the normalization of $\hat{X}\setminus\{0\}$. By taking the projectivisation, we get our claim. \hfill $\square$

We now classify conical symplectic varieties with only isolated singularities.

**Lemma 4.** Let $W$ be a conical symplectic variety with only isolated singularities. Then $W = \mathcal{O}_{\min}$, where $\mathcal{O}_{\min}$ is the minimal nilpotent orbit in a simple Lie algebra.

**Proof.** As $W$ is invariant by the $\mathbb{C}^*$-action, it has a unique singular point, which is $\{0\}$. As $W\setminus\{0\} \to PW$ is a $\mathbb{C}^*$-bundle, we deduce that $PW$ is smooth. Note that by Lemma 1.4 [B1], the symplectic form on $W\setminus\{0\}$ induces a contact structure on $PW$ with contact line bundle $\mathcal{O}_{PW}(1)$. As $\mathcal{O}_{PW}(1)$ is very ample, we deduce that $PW \simeq \mathbb{P}\mathcal{O}_{\min}$ by Cor. 1.8 [B1]. This gives that $W = \mathcal{O}_{\min}$. \hfill $\square$

Now let us prove Theorem 2. First note that the $\mathbb{C}^*$-action on $W$ lifts to $Z$ (see for example Prop. A.7 of [N1]), which makes $\pi$ to be $\mathbb{C}^*$-equivariant. Let $\Omega$ be the symplectic form on $Z$ extending $\pi_\ast\omega$, then we have $\lambda^\ast\Omega = \lambda\Omega$. We denote by $\mathbb{P}Z$ the quotient $(Z\setminus\pi^{-1}(0))/\mathbb{C}^*$. Then we get a morphism $\tilde{\pi} : \mathbb{P}Z \to PW$. By Sections 3 and 4 of [N2], we have

**Lemma 5.** $\mathbb{P}Z$ is a smooth contact projective variety with the contact line bundle $L := \pi_\ast\mathcal{O}_{PW}(1)$.

If $PW$ is smooth, then by Lemma 4 $W = \mathcal{O}_{\min} \subset g$. As $W$ admits a symplectic resolution, this implies that $g$ is of type $A$ and $Z \simeq T^*_p$ (cf. [E] or Proposition 6 below). Assume now that $PW$ is singular, then we can apply Lemma 5 and Proposition 3 to conclude that $(\mathbb{P}Z, L) \simeq (\mathbb{P}(T^*_G/P), \mathcal{O}_{\mathbb{P}(T^*_G/P)}(1))$. Note that we may take $G = \text{Aut}^0(G/P)$, up to changing $G$ and $P$. By the proof of Proposition 3 we have the following diagram:

$$
\begin{array}{cccccc}
Z & \leftarrow & Z\setminus\pi^{-1}(0) & \xrightarrow{\sim} & T^*_G/P \setminus(\text{zero section}) & \rightarrow & T^*_G/P \\
\downarrow & & \downarrow & & \downarrow & \downarrow \\
W & \leftarrow & W\setminus\{0\} & \xrightarrow{\text{normalization}} & V\setminus\{0\} & \rightarrow & V \simeq \hat{W}
\end{array}
$$

As $PW$ is not smooth, we may assume $G/P \neq \mathbb{P}^n$. Now we can apply Lemma 5 [N2] and the argument in loc. cit. (p. 183) to deduce that $Z \simeq T^*_G/P$ and this identifies $\tilde{\pi} : Z \to \hat{W}$ with the map $\tilde{\mu}$ from the Stein factorization of the Springer map $\mu : T^*_G/P \to \mathcal{O}$. As seen in the proof of Proposition 3 it follows that $W$ admits a $G \times \mathbb{C}^*$-action such that $\tilde{\pi}$ is equivariant.

As $W \subset \mathbb{C}^N$ and $\mathbb{C}^*$ acts on $\mathbb{C}^N$ by dilations, the coordinate ring $\mathbb{C}[W]$ is a subalgebra of $\mathbb{C}[\hat{W}]$ generated by elements of degree 1. On the other hand, we have $\mathbb{C}[\hat{W}] = H^0(T^*_G/P, \mathcal{O}_{T^*_G/P})$. The space of degree 1 elements in $H^0(T^*_G/P, \mathcal{O}_{T^*_G/P})$ is $H^0(G/P, T^*_G/P)$, which is $\text{aut}(G/P) = g$. Thus, the algebra $\mathbb{C}[W]$ is generated by a linear subspace of $g$, that we view as the orthogonal of a linear subspace $L \subset g$. So
the normalization map $\tilde{W} \to W$ factors as the map $\tilde{W} \to \overline{O} \subset \mathfrak{g} \cong \mathfrak{g}^*$ corresponding to the inclusion $\mathfrak{g} \subset H^0(T_{G/P}^*, \mathcal{O}_{T_{G/P}})$, followed by a map $p : \overline{O} \to W$, the restriction of the quotient map $\mathfrak{g} \to \mathfrak{g}/L$. Thus $p$ is birational. If $W$ is normal, then $W = \tilde{W}$, hence we get $W = \overline{O}$. This finishes the proof of Theorem 2.

In the proof of Theorem 2, we used [F] to deduce that among minimal nilpotent orbit closures, only that of type $A$ admits a symplectic resolution. One can in fact prove a stronger result.

**Proposition 6.** Let $W$ be a normal variety of dimension $2n \geq 4$ with an isolated singularity $0$ and a contracting $\mathbb{C}^*$-action. Assume that $W \setminus \{0\}$ admits a symplectic form $\omega$ satisfying $\lambda^*\omega = \lambda\omega, \forall \lambda \in \mathbb{C}^*$. If $W$ admits a symplectic resolution $\pi : Z \to W$, then $Z \simeq T^*_n$ and $W \simeq \overline{O}_{\min} \subset \mathfrak{s}l_{n+1}$.

**Proof.** Recall that the action of $\mathbb{C}^*$ on $W$ lifts to an action on $Z$ (see Prop. A.7 of [N1]). Let $Z^{\mathbb{C}^*}$ be the subvariety of $Z$ consisting of $\mathbb{C}^*$-fixed points, which is a disjoint union of smooth subvarieties since $Z$ is smooth. As $W^{\mathbb{C}^*} = \{0\}$, we have that $Z^{\mathbb{C}^*}$ is contained in the fiber $\pi^{-1}(0)$, hence it is a union of projective manifolds.

By the Bialynicki-Birula decomposition (see [BB] which applies in this non-projective setting, since $Z$ is proper over $W$ which is contracted to $0$ by the $\mathbb{C}^*$-action), there exists an irreducible component $M$ of $Z^{\mathbb{C}^*}$ such that the set $U := \{z \in Z | \lim_{\lambda \to 0} \lambda \cdot z \in M\}$ is open in $Z$. Let $\Omega$ be the symplectic form on $Z$, which is the extension of $\pi^*\omega$. As $\lambda^*\Omega = \lambda\Omega$ for all $\lambda \in \mathbb{C}^*$, we obtain that $(U, \Omega) \simeq (T^*_M, \omega_{can})$ as $\mathbb{C}^*$-varieties (cf. Lemma 3.7 [F]), where $\omega_{can}$ is the canonical symplectic form on $T^*_M$. On the other hand, we have $\dim \pi^{-1}(0) \leq \frac{1}{2} \dim W = \dim M$ (Cor. 8.5 [CMSB]). This implies that $M$ is an irreducible component of $\pi^{-1}(0)$. As $M$ is smooth, by Cor. 8.7 [CMSB], we get that $M \simeq \mathbb{P}^n$. It follows (for example by the arguments above) that the map $U \to W$ is the Springer map $\mu : T^*_n \to \overline{O}_{\min} \subset \mathfrak{s}l_{n+1}$. As $\mu$ is projective, we get $U = Z$. \hfill \Box

### 3. An application

A **symplectic variety** is a normal variety $W$ with a symplectic form $\omega$ on its smooth locus such that for any resolution $\phi : Z \to W$, the 2-form $\phi^*\omega$ defined on $\phi^{-1}(W_{\text{reg}})$ extends to a regular 2-form on $Z$.

As an application of Theorem 2, we provide an alternative proof of the following main result of [N2].

**Theorem 7** (Namikawa). Let $(W, \omega)$ be a singular symplectic variety embedded in $\mathbb{C}^N$ as a complete intersection of hypersurfaces defined by homogeneous polynomials. Assume that the symplectic form $\omega$ satisfies $\lambda^*\omega = \lambda^k\omega, \forall \lambda \in \mathbb{C}^*$ for some $k$. Then $(W, \omega)$ is isomorphic to the nilpotent cone $(\mathcal{N}, \omega_{KK})$ of a semisimple complex Lie algebra $\mathfrak{g}$ together with the Kirillov- Kostant form $\omega_{KK}$.

**Proof.** By Section 2 of [N2], $W$ is a conical symplectic variety with a symplectic resolution, hence by Theorem 2 we get that $(W, \omega)$ is a nilpotent orbit closure in a semi-simple Lie algebra $\mathfrak{g}$. But every nilpotent orbit closure $\overline{O} \subset \mathfrak{g}$ which is a complete intersection in $\mathfrak{g}$ must be the full nilpotent cone, by the main result of
Section 7 of \([N2]\). We now provide a proof of that result, which is somehow shorter and more uniform than the original one.

Let \(d_1, \ldots, d_r \geq 2\) be the degrees of defining equations of \(\bar{\mathcal{O}}\) in \(\mathfrak{g}\). Then \(r = \text{codim}_\mathfrak{g}(\mathcal{O})\) and by \([N2]\) (Section 2, p. 160), we have

\[
(3.1) \quad \sum_{i=1}^{r} d_i = \frac{1}{2} \dim \mathcal{O} + \text{codim}_\mathfrak{g}(\mathcal{O}), \quad \text{and} \quad \text{codim}_\mathfrak{g}(\mathcal{O}) \leq \frac{1}{3} \dim \mathfrak{g}.
\]

We may assume that \(\mathfrak{g}\) is simple. We denote by \(I\) the ideal of \(\bar{\mathcal{O}}\) in the coordinate ring \(\mathbb{C}[\mathfrak{g}]\), and by \(\mathfrak{m}\) the maximal ideal of 0 in \(\mathbb{C}[\mathfrak{g}]\). Let \(G\) be the adjoint group of \(\mathfrak{g}\); then \(G \times \mathbb{C}^*\) acts on \(\mathfrak{g}\) (where \(\mathbb{C}^*\) acts by dilations) and stabilizes \(\bar{\mathcal{O}}\) and 0. Hence \(G \times \mathbb{C}^*\) acts on \(\mathbb{C}[\mathfrak{g}]\) and stabilizes \(I \subset \mathfrak{m}\). Since \(G \times \mathbb{C}^*\) is reductive, we may find a submodule \(M \subset \mathfrak{m}\) which is mapped isomorphically to \(I/\mathfrak{m}I\) under the quotient map \(I \to I/\mathfrak{m}I\). By the graded Nakayama lemma, a homogeneous basis of \(M\) yields a minimal generating system of the ideal \(I\), and hence a regular sequence in \(\mathbb{C}[\mathfrak{g}]\) since \(\bar{\mathcal{O}}\) is a complete intersection. In geometric terms, the morphism

\[
f: \mathfrak{g} \longrightarrow M^*: = V
\]

corresponding to the inclusion \(M \subset \mathbb{C}[\mathfrak{g}]\) is flat and its (scheme-theoretic) fiber \(f^{-1}(0)\) equals \(\bar{\mathcal{O}}\) (this is a slightly more precise version of Lemma 3 in \([N2]\)). Thus, \(f\) is open, and hence surjective by \(\mathbb{C}^*\)-equivariance.

Choose a maximal torus \(T \subset G\) and denote by \(\mathfrak{t}\) its Lie algebra; this is a Cartan subalgebra of \(\mathfrak{g}\). Since \(G\) is dense in \(\mathfrak{g}\), there exists \(x \in \mathfrak{t}\) such that the differential \(df_x: \mathfrak{g} \to V\) is surjective. As \(df_x\) is linear and \(T\)-equivariant, it follows that each weight of the \(T\)-module \(V\) is also a weight of \(\mathfrak{g}\). In particular, the highest weight of any simple summand of the \(G\)-module \(V\) is either the highest root, or the highest short root \(\lambda\) (if \(G\) is not simply laced), or 0. Moreover, the highest root cannot occur, since the corresponding simple module is just \(\mathfrak{g}\), and \(\dim(V) = \dim(\mathfrak{g}) - \dim(\bar{\mathcal{O}}) < \dim(\mathfrak{g})\).

If \(G\) is simply laced, then \(V\) must be the trivial \(G\)-module, and hence \(\bar{\mathcal{O}}\) contains the nilpotent cone \(\mathcal{N}\); so we conclude that \(\bar{\mathcal{O}} = \mathcal{N}\). Thus, we may assume that \(G\) is not simply laced; then we have an isomorphism of \(G\)-modules

\[
V \cong pV(0) \oplus qV(\lambda),
\]

where \(p, q\) are non-negative integers, \(V(0)\) denotes the trivial \(G\)-module \(\mathbb{C}\), and \(V(\lambda)\), the simple \(G\)-module with highest weight \(\lambda\). (The latter module is called the "little adjoint module" in \([P]\), where its invariant theoretical properties are investigated.)

If \(q = 0\) then we conclude as above that \(\bar{\mathcal{O}} = \mathcal{N}\); thus, we may further assume that \(q \geq 1\).

If \(G\) of type \(C_n\) (resp. \(F_4, G_2\)), then \(V(\lambda)\) has dimension \(2n^2 - n - 1\) (resp. 26, 7), which contradicts the inequality \(\text{codim}_\mathfrak{g}(\mathcal{O}) = \dim V \leq \frac{1}{3} \dim(\mathfrak{g})\) in (3.1). Thus, we may assume that \(G = \text{SO}(2n + 1)\); then \(V(\lambda)\) is the natural \(G\)-module \(\mathbb{C}^{2n+1}\).

Since \(f\) is surjective and \(G\mathfrak{g}^T\) is dense in \(\mathfrak{g}\), it follows that \(GV^T\) is dense in \(V\). But this does not hold for \(V = 2V(\lambda)\) (as \(V(\lambda)^{T}\) is a line), and hence \(q \leq 1\). So we may take \(V = pV(0) \oplus V(\lambda)\). Since the algebra of invariant functions \(\mathbb{C}[V(\lambda)]^G\) is generated by the quadratic form defining \(G\), it follows that the categorical quotient \(V/\!/G := \text{Spec} \mathbb{C}[V]^G\) is an affine space, and the quotient morphism

\[
q_v: V \longrightarrow V/\!/G
\]
is flat with reduced fibers. Since $q_V$ sits in a commutative square

$$
\begin{array}{c}
g \\
\downarrow q_0 \\
g//G
\end{array} \begin{array}{c}
\rightarrow \\
V \\
\downarrow q_V \\
V//G
\end{array},
$$

where $f, q_0$ are flat with reduced fibers, we see that $f//G$ is flat with reduced fibers, too.

We claim that $f//G$ is also finite. Consider indeed the restriction $f^T : g^T = t \rightarrow V^T$. Then the fiber of $f^T$ at 0 equals $\mathcal{O} \cap t = \{0\}$ (as a set). Since $f^T$ is equivariant for the natural actions of $\mathbb{C}^*$, it follows that $f^T$ is finite. Hence so is $f^T/W : g^T/W \rightarrow V^T/W$, where $W$ denotes the Weyl group of $(G, T)$. But the natural map $g^T/W \rightarrow g//G$ is an isomorphism by the Chevalley restriction theorem; moreover, the analogous map $V^T/W \rightarrow V//G$ is finite, and dominant since $GV^T$ is dense in $V$. The finiteness of $f//G$ follows from this in view of the commutative square

$$
\begin{array}{c}
g^T/W \\
\downarrow \\
g//G
\end{array} \begin{array}{c}
\rightarrow \\
V^T/W \\
\downarrow \\
V//G
\end{array},
$$

Since $f//G$ is also flat with reduced fibers, it is a finite étale cover, and hence an isomorphism as $V//G$ is an affine space. In other words, the algebra $\mathbb{C}[g]^G$ is freely generated by the pull-backs under $f$ of the $p$ projections $V \rightarrow V(0)$ and of the basic invariant of $V(\lambda)$. The degrees of these invariants are $a_1, \ldots, a_p, 2a_{p+1}$, where $a_1, \ldots, a_p$ denote the weights of the action of $\mathbb{C}^*$ on $pV(0)$, and $a_{p+1}$ the weight of that action on $V(\lambda)$. On the other hand, the degrees of the basic invariants of $\mathbb{C}[g]^G$ are $2, 4, \ldots, 2n$. In particular, we have $n = p + 1$ and

$$a_1 + \cdots + a_{n-1} + 2a_n = n^2 + n.$$

This implies that $r = \dim V = 3n$ and $\dim \mathcal{O} = \dim g - 3n = 2n^2 - 2n$. Moreover, we may assume that the degrees of defining equations of $\mathcal{O}$ satisfy $d_i = a_i$ for $i = 1, \ldots, n-1$, and $d_n = \cdots = d_{3n} = a_n$. By (3.1), we have $d_1 + \cdots + d_{3n} = (n^2 - n) + 3n$, which gives that

$$a_1 + \cdots + a_{n-1} + (2n + 1)a_n = n^2 + 2n.$$

Combining the two displayed equalities yields $(2n - 1)a_n = n$ which is not possible since $a_n = d_n \geq 2$.

\[\blacksquare\]

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