Optimal mass normalizability for Gibbs measure associated with NLS on the 2D disc

Tianhao Xian

Abstract

We prove the normalizability of Gibbs measure associated with radial focusing nonlinear Schrödinger equation (NLS) on the 2-dimensional disc \( \mathbb{D} \), at critical mass threshold. The result completes the study of optimal mass normalizability on \( \mathbb{D} \) by Oh-Sosoe-Tolomeo (2021).

1 Introduction

In this paper, we complete Oh-Sosoe-Tolomeo’s study [15] of optimal normalizability threshold for the Gibbs measure for the focusing nonlinear Schrödinger equation (NLS), extending their result to the setting of \( L^2_{rad}(\mathbb{D}) \), the square integrable radial functions on unit disc.

The focusing nonlinear Schrödinger equation

\[
\begin{cases}
i\partial_t u(t, x) + \Delta u(t, x) + |u(t, x)|^{p-2}u(t, x) = 0, \\
u(0, \cdot) = u_0
\end{cases}
\]

(1)
is an evolution equation corresponding to the Hamiltonian

\[
H(u) = \frac{1}{2} \int_{\mathbb{D}} |\nabla u|^2 - \frac{1}{p} \int_{\mathbb{D}} |u|^p.
\]

The Gibbs measure is a probability on function space formally defined as a weighted Lebesgue measure

\[
d\rho = Z^{-1} e^{-H(u)} du,
\]

where \( Z \) is a normalization constant known as the partition function. The conservation of the Hamiltonian \( H \) (which holds for smooth data) suggests that \( \rho \) should be invariant under the flow of NLS. Since Lebowitz-Rose-Speer’s construction of the Gibbs measure [11], there have been numerous studies of the invariance of Gibbs measure with respect to the flow of NLS. (In particular, this yields almost surely global well-posedness of NLS.) McKean [12] proved invariance of the Gibbs measure for NLS on \( \mathbb{T} \). Meanwhile, Bourgain [3] proved the same result with a more analytic method, combining the deterministic local well-posedness with the invariant measure for the truncated NLS. Later, he constructed an invariant Gibbs
measure for a modified NLS equation on $\mathbb{T}^2$, where the local well-posedness used is probabilistic [2]. Bourgain’s method was then applied in other settings. Tzvetkov constructed an invariant measure for NLS on the unit disc in $\mathbb{R}^2$, with subcritical nonlinearity [18]. This result was improved by Bourgain-Bulut to include the critical nonlinearity with small mass initial data [4]. The restriction on the mass was imposed due to the lack of an optimal normalization result of the Gibbs measure for focusing NLS, which is the goal of this paper. For the defocusing NLS, the Gibbs measure can be constructed without restriction on the mass. Tzvetkov proved invariance of the defocusing sub-quintic NLS [17] and Bourgain-Bulut for defocusing cubic NLS on $3d$ unit ball [5].

Since a translation invariant measure on infinite dimensional space cannot be locally finite, the definition of the Gibbs measure in (2) is formal. To rigorously construct the Gibbs measure $\rho$, let $\{e_n\}$ be the orthonormal basis of $L^2_{rad}(\mathbb{D})$ consisting of eigenfunctions of $-\Delta$, with corresponding eigenvalues $\lambda_n$. Let $\{g_n\}_{n \geq 1}$ be a sequence of independent standard complex-valued Gaussian. Define the free Gaussian measure $\mu$ on $L^2_{rad}(\mathbb{D})$ as the law of random variable

$$u(\omega) = \sum_n g_n(\omega) \frac{1}{\sqrt{\lambda_n}} e_n.$$  \hspace{1cm} (3)

In other words, $\mu$ is the Gaussian measure on $L^2_{rad}(\mathbb{D})$ with mean 0 and covariance $(-\Delta)^{-\frac{1}{2}}$.

Writing $u = \sum_n u_n e_n$, then

$$d\mu(u) = \prod_n \frac{1}{2\pi \lambda_n} e^{-\frac{1}{4}|u_n|^2} d\Re u_n d\Im u_n = \mu_0^{-1} e^{-\frac{1}{2} \int \nabla u|^2} d\mu.$$ 

The Gibbs measure $\rho$ is then defined as a weighted $\mu$-measure with density $e^{\frac{1}{p+1} \int |u|^{p+1}}$. Unfortunately, this density is not $\mu$-integrable [11]. A remedy is to restrict the measure to a smaller set expected to be invariant under the equation. In [14], Oh-Quastel constructed a Gibbs measure on $\mathbb{T}$, conditioning on fixed mass $\int |u|^2$ and momentum $\int iu \bar{u}_x$. A more commonly used method is $L^2$-truncation: define

$$d\rho = d\rho_{K,p} = Z_{K,p}^{-1} 1_{\|u\|_{L^2} \leq K} e^{\frac{1}{p+1} \int |u|^{p+1}} d\mu.$$ \hspace{1cm} (4)

To guarantee $\rho_{K,p}$ is indeed a probability measure, we need to show the density is $\mu$-integrable, or equivalently, the partition function $Z_{K,p}$ is finite.

The study of the integrability of $\rho_{K,p}$ was initiated by Lebowitz-Rose-Speer [11], where they considered the Gibbs measure on $L^2(\mathbb{T})$. Previous results in the torus setting [11, 3, 15] are summarized as following

**Theorem 1.1** (Focusing Gibbs measure on torus). For $p \geq 2$, $K > 0$, the partition function of Gibbs measure for focusing NLS,

$$Z_{K,p} = E_{\mu} \left[ e^{\frac{1}{p+1} \int |u|^{p+1} dx}, \|u\|_{L^2(\mathbb{T})} \leq K \right],$$

1That is, $\Re g_n$ and $\Im g_n$ are independent with law $N(0,1/2)$.

2$(-\Delta)^{-\frac{1}{2}}$ is the solution map of the Poisson equation $-\Delta u = f$, $f \in L^2_{rad}(\mathbb{D})$, with Dirichlet boundary conditions.
satisfies

(a) (Subcritical) For \( p < 6 \), \( Z_{K,p} < \infty \).

(b) (Supercritical) For \( p > 6 \), \( Z_{K,p} = \infty \).

(c) (Critical) For \( p = 6 \), \( Z_{K,p} < \infty \) if and only if \( K \leq \|Q\|_{L^2(\mathbb{R})} \), where \( Q \) is the ground state solution (positive, radial, decreasing to 0 at infinity) of equation

\[-Q'' + Q - Q^5 = 0.\]

In [11], Lebowitz-Rose-Speer proved (a), (b). For the critical case \( p = 6 \), they showed that \( Z_{K,6} < \infty \) if \( K \) is small enough. Later Bourgain [3] reproved their results using the series expression (3). Part (c) was completed by Oh-Sosoe-Tolomeo [15].

Similar methods apply to the \( L^2_{rad}(\mathbb{D}) \) setting, where the critical nonlinearity is \( p = 4 \). In [18], Tzvetkov constructed the measure and proved invariance for the subcritical case \( p < 4 \). Later, Bourgain-Bulut [4] obtained a proof for the critical case with small \( L^2 \) cutoff \( K \). The optimal cutoff for \( K \) is \( \|Q\|_{L^2(\mathbb{R}^2)} \), where \( Q \) is the ground state solution of the equation

\[-\Delta Q + Q - Q^3 = 0.\]

Oh-Sosoe-Tolomeo [15] proved integrability for \( K < \|Q\|_{L^2} \) and non-integrability for \( K > \|Q\|_{L^2} \). The goal of this paper is to complete the remaining case \( K = \|Q\|_{L^2} \).

**Theorem 1.2** (Main result). The partition function for the Gibbs measure ([7]) at the critical threshold \( p = 4, K = \|Q\|_{L^2(\mathbb{R}^2)} \) is finite. More precisely,

\[ Z := \mathbb{E}_\mu \left[ e^{\frac{1}{2} \int_\mathbb{D} |u|^p dx}, \|u\|_{L^2(\mathbb{D})} \leq \|Q\|_{L^2(\mathbb{R}^2)} \right] < \infty. \]

### 1.1 Strategy of the proof

For now on, we discuss the partition function at critical nonlinearity and integrability threshold: \( p = 4, K = \|Q\|_{L^2(\mathbb{R}^2)} \). Looking at the formal density of \( \rho \) in (2), one sees that the key part of the proof is to bound the Hamiltonian \( H \) from below. The following strategy was introduced by Oh-Sosoe-Tolomeo [15].

The largest part of the density \( e^{-H} \) occurs at minimum points of \( H \), if they exist. Although \( \mu \) is a measure on \( L^2_{rad}(\mathbb{D}) \), by a density argument, we can restrict our discussion to a dense subset \( H^1_{rad,0}(\mathbb{D}) \), the set of radial \( H^1(\mathbb{D}) \) functions vanishing on \( \partial \mathbb{D} \). It turns out that, conditioning on \( \{ \|u\|_{L^2(\mathbb{D})} \leq \|Q\|_{L^2(\mathbb{R}^2)} \} \), the Hamiltonian \( H \) has no minimum point on \( H^1_{rad,0}(\mathbb{D}) \) (see remark 2.2 also [15]). On the other hand, \( H^1_{rad,0}(\mathbb{D}) \) embeds naturally into \( H^1_{rad}(\mathbb{R}^2) \) and \( H \) has the corresponding extension

\[ H_{\mathbb{R}^2}(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 - \frac{1}{4} \int_{\mathbb{R}^2} |u|^4. \]
The minima of the latter exist under the restriction \( \|u\|_{L^2(\mathbb{R}^2)} \leq \|Q\|_{L^2(\mathbb{R}^2)} \), and the minimal points consists of dilations and phase rotations of the ground state solution \( Q \), that is, \( e^{i\theta}Q_\delta \), for \( \theta \in \mathbb{T} \), \( \delta > 0 \) and \( Q_\delta = \delta^{-1}Q(\delta^{-1}) \). These minimizers form a 2-dimensional submanifold, the soliton manifold \( M \), \( \mathcal{M} \). It is natural to divide our domain \( H^1_{\text{rad}, 0}(\mathbb{D}) \) into two parts: the \( \epsilon \)-neighborhood in \( L^2 \) of the soliton manifold \( U_\epsilon \) and its complement.

The extremal problem for \( H_{\mathbb{R}^2} \) is closely related to the sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality (see Proposition 2.1). Away from the soliton manifold \( M \), the GNS inequality is not saturated. As a corollary, \( H(u) \geq c \int_D |\nabla u|^2 dx \), for some \( c > 0 \). Heuristically\(^4\)

\[
E_\mu \left[ e^{\frac{1}{4} \int_D |u|^p dx}, U_\epsilon \right] \leq \int e^{-c \int_D |\nabla u|^2} du < \infty.
\]

The integral on the right is bounded (formally) because its integrand can be regarded the density of the mean-zero Gaussian measure with covariance \( \Delta^{-1}/2c \).

On the neighborhood \( U_\epsilon \), the idea is to expand the Hamiltonian \( H \) along the soliton manifold \( M \). First, \( U_\epsilon \) can be treated as a normal bundle of \( M \) (with respect to the \( H^1 \) inner product). More precisely, for \( u \in U_\epsilon \), there is a unique decomposition \( u = e^{i\theta}Q_\delta + v \) with \( v \) in \( V_{\theta, \delta} \), the normal vector space at \( e^{i\theta}Q_\delta \). Formally,

\[
E_\mu \left[ e^{\frac{1}{4} \int_D |u|^p dx}, U_\epsilon \right] = \mathcal{I}_{U_\epsilon} e^{-H(u)} du.
\]

A change of variable formula allows us to rewrite it as a double integral

\[
\mathcal{I}_{\mathcal{M}} \int_{V_{\theta, \delta}} e^{-H(e^{i\theta}Q_\delta + v)} dv d\sigma,
\]

for some surface measure \( \sigma \ll d\theta d\delta \). Next, we expand the Hamiltonian as

\[
H(e^{i\theta}Q_\delta + v) = H(e^{i\theta}Q_\delta) + \frac{1}{2} \int_D |\nabla v|^2 + B_{\theta, \delta}(v) + \text{higher order terms.}
\]

By recognizing that \( e^{-\frac{1}{4} \int |\nabla v|^2} dv \) is the free Gaussian on \( V_{\theta, \delta} \), denoted as \( \mu_{V_{\theta, \delta}} \), the double integral \( (5) \) becomes

\[
\int_{\mathcal{M}} e^{-H(e^{i\theta}Q_\delta)} \left( \int_{V_{\theta, \delta}} e^{-B_{\theta, \delta}(v) + \text{h.o.t.}} dv \right) d\sigma.
\]

\(^3\)The name comes from the soliton solution of focusing nonlinear Schrödinger equation. See, e.g. [19, 9, 13].

\(^4\)Of course, the restriction \( \|u\|_{L^2(\mathbb{D})} \leq \|Q\|_{L^2(\mathbb{R}^2)} \) is implemented in the expectation. We omit here and below for simplicity.
$H(e^{i\theta}Q_\delta) \approx H_{\mathbb{R}^2}(Q)$ is uniformly bounded, and the high order terms can be tamed similarly as on $U^C$.

The term $B_{\theta,\delta}(v)$ in (6) is bounded from below by a quadratic form $\langle A_{\theta,\delta}v, v \rangle_{H^1}$ (on $H^1_{rad,0}(\mathbb{D})$). The analysis of this quadratic form is more involved than the in the one-dimensional case treated in [15]. We will show that $A_{\theta,\delta}$ is compact. Expressing the free Gaussian measure $\mu_{V_{\theta,\delta}}$ in the basis consisted of $A_{\theta,\delta}$’s eigenfunctions, the integral of $e^{-B_{\theta,\delta}(v)}$ term can be diagonalized and bounded by

$$\prod_n (1 + 2\lambda_n)^{-1/2} \lesssim \exp \left( -\sum_n \lambda_n \right),$$

with $\lambda_n$ be $A_{\theta,\delta}$’s eigenvalues. Therefore, we showed $e^{-B_{\theta,\delta}(v)}$ is integrable by proving an asymptotic lower bound for $\lambda_n$.

1.2 The outline of the paper

In Section 2, we introduce some notations and preliminaries. In Sections 3 and 4, we bound the integral defining the partition function away from the soliton manifold. Section 5 is devoted to the normal bundle decomposition of the neighborhood $U_\epsilon$ of the soliton, and a change of variables formula adapted to this decomposition. In Section 6, we conduct some spectral analysis to bound the integral of the quadratic part $e^{-B_{\theta,\delta}(v)}$. We finish the proof of our main theorem in Section 7.

2 Notation and preliminary

In this section, we summarize some of the notation we will use. $A \lesssim (\gtrsim) B$ means $A \leq (\geq) CB$ for some $C > 0$; $A \sim B$ means $A \lesssim B$ and $A \gtrsim B$. We denote $A \ll B$ as $A \leq cB$ for some small $c$.

In this paper, all function spaces consist of complex-valued functions. More precisely, for $\Omega = \mathbb{R}^2$ or $\mathbb{D}$, $L^2(\Omega)$ is equipped with inner product

$$\langle f, g \rangle := \Re \int_{\Omega} f(x)\overline{g(x)}dx.$$ 

The subscript $\text{rad}$ indicates the subspace of radial functions, e.g. $L^2_{\text{rad}}(\mathbb{D})$. We use $H^1_{\text{rad,0}}(\mathbb{D})$ to denote the subspace of radial functions in $H^1(\mathbb{D})$ which vanish on $\partial \mathbb{D}$.

2.1 Eigenfunctions and eigenvalues

On Hilbert space, consider $-\Delta$ as an operator with domain on $H^2_{\text{rad}}(\mathbb{D})$ with Dirichlet boundary conditions. Using the radial variable $r = |x|$, the eigenvalue equation can be written as

$$\begin{cases}
-(\partial_r^2 + \frac{1}{r}\partial_r)e = \lambda e, \\
\quad e'(0) = 0, e(1) = 0.
\end{cases}$$
The $L^2$-normalized eigenfunctions and eigenvalues are
\[ e_n(r) = \frac{J_0(z_n r)}{\|J_0(z_n r)\|_{L^2(D)}}, \quad \lambda_n = z_n^2, \quad (7) \]
where $J_0$ is the Bessel function of order 0. $z_n$ is its $n$th zero, with asymptotic behavior (see also Lemma 2.2 in [18])
\[ z_n = \pi(n - \frac{1}{4}) + O(\frac{1}{n}). \quad (8) \]
By Sturm–Liouville theory, $\{e_n\}$ (more precisely, $\{e_n, ie_n\}$) forms an orthonormal basis for $L^2_{rad}(\mathbb{D})$.

With the notation above, we obtain a series representation (3) of a random element $u$ distributed according to the free Gaussian measure $\mu$ is
\[ u(\omega) = \sum_{n \geq 1} \frac{g_n(\omega)}{z_n} e_n. \quad (9) \]

We can then define the dyadic\(^5\) projection $P_{\leq N}$ on the $2N$-dimensional (real) subspace $E_N := \text{span}\{e_n : n \leq N\}$,
and $P_N := P_{\leq N} - P_{\leq N/2}$.

2.2 Bessel functions of order 0

For the spectral analysis in Section 6.2 we need some asymptotic expansions for Bessel functions. The equation
\[ \partial_r^2 u + \frac{1}{r} \partial_r u + u = 0 \]
has a fundamental set of solution $J_0, Y_0$. $J_0$ is called Bessel function of the first kind while $Y_0$ is Bessel function of second kind.

When $r \to 0$, the Bessel functions have the following series expansions (see Section 10.8 in [7]):
\[ J_0(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left( \frac{r}{2} \right)^{2m}, \]
\[ Y_0(r) = \frac{2}{\pi} \ln(\frac{r}{2})J_0(r) + \frac{2}{\pi} p(r), \]
where
\[ p(r) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left( \gamma - \sum_{k=1}^{m} \frac{1}{k} \right) \left( \frac{r}{2} \right)^{2m}, \]
\(^5\)In this paper, the capital letter $N$ (and $M$) is always to denote some dyadic number $2^n$.  

and $\gamma$ is Euler’s constant. In particular, $J_0(0) = 1$ and $Y_0(r) \approx \ln(r)$, as $r \to 0$.

For $r \to \infty$, there are the following asymptotic expansions (see Section 10.7):

\[
J_0(r) = \sqrt{\frac{2}{\pi r}} \cos \left( r - \frac{1}{4} \pi \right) + o\left( \frac{1}{r^{1/2}} \right),
\]

\[
Y_0(r) = \sqrt{\frac{2}{\pi r}} \sin \left( r - \frac{1}{4} \pi \right) + o\left( \frac{1}{r^{1/2}} \right).
\] (10)

### 2.3 Gagliardo-Nirenberg-Sobolev inequality

The sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality in $\mathbb{R}^d$, $d \geq 2$ was proved by Weinstein [19]. We state the $d = 2$ case. See e.g. [9] for a proof.

**Proposition 2.1.** For any $p > 2$, $u \in H^1(\mathbb{R}^2)$ satisfies

\[
\|u\|_{L^p(\mathbb{R}^2)} \leq \frac{p}{2} \|Q\|_{L^2(\mathbb{R}^2)}^{2-p} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{2-p} \|u\|_{L^2(\mathbb{R}^2)}^2
\] (11)

where $Q$ is the unique positive, radial and exponentially decaying solution of

\[
(p - 2)\Delta Q + 2Q^{p-1} - 2Q = 0.
\] (12)

The equality holds if and only if $u(x) = aQ(b(x - c))$ for $a \in \mathbb{C} \setminus \{0\}$, $b > 0$ and $c \in \mathbb{R}$.

**Remark 2.2.** If we restrict the inequality to $H^1_{rad,0}(\mathbb{D})$, then there are no functions that saturate this inequality. Indeed, if $u \in H^1_{rad,0}(\mathbb{D})$ minimizes the GNS inequality on $\mathbb{D}$, since $u$ vanishes on $\partial \mathbb{D}$, it is also a minimizer of (11). This means $u$ equals to some $aQ(br)$, which is positive on the whole $\mathbb{R}^2$, a contradiction.

The function $Q$ is called the ground state solution. We state some simple bounds for $Q$.

**Lemma 2.3.** The radial ground state solution $Q = Q(r)$ satisfies

1. $Q(r)$ is decreasing for $r \in [0, \infty)$.
2. $Q(r), \nabla Q(r) = O(e^{-cr})$.
3. $Q(0) = \|Q\|_\infty \geq (p/2)^{1/(p-2)}$.

**Proof.** The proof of part (1), (2) can be found, for example, in [16, Appendix B].

For part (3), as a radial function, (12) can be written as

\[
Q_{rr} + \frac{1}{r} Q_r + \frac{2}{p-2} Q^{p-1} - \frac{2}{p-2} Q = 0.
\]

Define the energy

\[
E[Q(r)] = \frac{1}{2} Q^2_r(r) + \frac{2}{p(p-2)} Q^p(r) - \frac{1}{p-2} Q^2(r).
\]
A direct computation shows
\[ \frac{d}{dr} E[Q(r)] = -\frac{1}{r} Q_r^2(r) \leq 0. \]

Since \( Q \) is radial, \( Q_r(0) = 0 \). Moreover, \( Q(\infty) = Q_r(\infty) = 0 \) implies
\[ \frac{2}{p(p-2)} Q'(0) - \frac{1}{p-2} Q^2(0) = E[Q(0)] \geq E[Q(\infty)] = 0. \]

Since \( Q > 0 \), this implies \( Q(0) \geq \frac{2}{p} \left( p - \frac{2}{p-2} \right) \frac{1}{Q_r^2(0)} \). Since \( Q \) is decreasing, \( \|Q\|_{L^\infty(\mathbb{R}^2)} = Q(0) \).

### 3 Away from the soliton manifold

The argument for this part is in the the same spirit as the small truncation \( K \ll 1 \) case. We follow a modification of Bourgain’s argument, as in [15].

We begin with a large deviation result for \( \mu \).

**Lemma 3.1.** For any \( N \leq \infty \),
\[ \mathbb{E}_\mu \left[ \|P_{\geq N} u\|_{L^4(\mathbb{D})} \right] \lesssim (\log N)^{\frac{4}{p}} N^{-\frac{1}{4}}. \]

In particular, \( \int_\mathbb{D} |u|^4 dx \) is \( \mu \)-almost surely finite.

**Proof.** We use a bound for eigenfunctions (see [18]):
\[ \|e_n\|_{L^4(\mathbb{D})} \lesssim (\log(2 + n))^{\frac{1}{4}}. \]

Then,
\[ \mathbb{E}_\mu \left[ \|P_{\geq N} u\|_{L^4(\mathbb{D})}^4 \right] = \mathbb{E} \left[ \int_\mathbb{D} \frac{1}{2} \left( \sum_{n \geq N} g_n^2 e_n^2 \right)^2 - \sum_{n \geq N} g_n^4 e_n^4 \right] \lesssim \sum_{n \geq N} \int_\mathbb{D} \mathbb{E} \left[ g_n^4 g_m^4 \frac{1}{2} e_n e_m \right] \lesssim \left( \sum_{n \geq N} \frac{1}{n^4} \right)^2 \lesssim \log N \cdot N^{-2}. \]

The last inequality above used summation by parts:
\[ \sum_{n \geq N} \frac{(\log n)^{1/2}}{n^2} \leq \sum_{k \geq \log N} \sqrt{k} 2^{-k} = 2 \sqrt{\log N} N^{-1} + \sum_{k \geq \log N} \left( \sqrt{k} + 1 - \sqrt{k} \right) 2^{-k} \leq 2 \sqrt{\log N} N^{-1} + \sum_{k \geq \log N} \frac{1}{\sqrt{k}} 2^{-k} \leq 3 (\log N)^{1/2} N^{-1}. \]
Instead of directly working on the complement of some $\epsilon$-neighborhood of the soliton manifold $\mathcal{M}$, we start with an alternative characterization of the sharpness of GNS inequality. For small $\gamma > 0$, define subdomain

$$S_\gamma := \bigcap_{N} \left\{ u \in L^2_{rad}(\mathbb{D}) : \frac{1}{4} \int_{\mathbb{D}} |P_{\leq N} u|^4 \leq \frac{1 - \gamma}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right\}.$$ 

Our aim in this section is the following.

**Proposition 3.2.**

$$\mathbb{E}_\mu \left[ e^{\frac{1}{4} \int_{B} |u|^4}, S_\gamma \right] < \infty$$

**Proof.** We further slice $S_\gamma$ according to its $L^4$ size:

$$F_N := \left\{ u \in L^2_{rad}(\mathbb{D}) : \|P_{\geq M} u\|_4 > (\log M)^{3/4}, \ M < N; \ \|P_{\geq N} u\|_4 \leq (\log N)^{3/4} \right\}$$

As a corollary of Fernique’s theorem \[8\] (see also Lemma 4.2 in \[15\]), there exists some $c > 0$, such that

$$\mu \left( \|P_{\geq N} u\|_4 \geq t \mathbb{E} \left[ \|P_{\geq N} u\|_4 \right] \right) \leq e^{-ct^2}.$$ 

By Lemma 3.1, taking $t = (\log N)^{3/4} \cdot \mathbb{E} \left[ \|P_{\geq N} u\|_4 \right]^{-1} \geq (\log N)^{1/2} N^{1/2}$,

$$\mu(F_{2N}) \leq \mu(\|P_{\geq N} u\|_4 \geq (\log N)^{3/4}) \leq \exp \left( -c(\log N)N \right).$$ \hspace{1cm} (13)

By Young’s inequality,

$$\int_{\mathbb{D}} |u|^4 = \int_{\mathbb{D}} |P_{\leq N} u + P_{\geq 2N} u|^4 \leq (1 + \epsilon) \int_{\mathbb{D}} |P_{\leq N} u|^4 + C_\epsilon \int_{\mathbb{D}} |P_{\geq 2N} u|^4.$$ \hspace{1cm} (14)

On $S_\gamma \cap F_{2N}$, using (14), the definition of $S_\gamma$ and Hölder’s inequality,

$$\mathbb{E} \left[ \exp \left( \frac{1}{4} \int_{\mathbb{D}} |u|^4 \right), S_\gamma \cap F_{2N} \right] \leq e^{C_1(\log N)^3} \mathbb{E}_\mu \left[ \exp \left( \frac{1}{4} \int_{\mathbb{D}} (1 + \epsilon) |P_{\leq N} u|^4 \right), S_\gamma \right] \leq e^{C_1(\log N)^3} \mathbb{E}_\mu \left[ \exp \left( \frac{(1 - \gamma)(1 + \epsilon)}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right), F_{2N} \right] \leq e^{C_1(\log N)^3 - c_2(\log N)N} \mathbb{E}_\mu \left[ \exp \left( \frac{c_3}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right) \right] \leq e^{C_1(\log N)^3 - c_2(\log N)N} \mathbb{E}_\mu \left[ \exp \left( \frac{c_3}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right) \right] \leq e^{C_1(\log N)^3 - c_2(\log N)N} \mathbb{E}_\mu \left[ \exp \left( \frac{c_3}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right) \right].$$

Above $C_1 = C_1(\epsilon)$, $c_2 = c_2(\eta)$, and $c_3 = (1 - \gamma)(1 + \epsilon)(1 + \eta) < 1$ by choosing $\epsilon, \eta$ small enough. Since the expectation in the last line only involves $P_{\leq N} u \in E_N$,

$$\mathbb{E}_\mu \left[ \exp \left( \frac{c_3}{2} \int_{\mathbb{D}} |\nabla P_{\leq N} u|^2 \right) \right] = \prod_{n \leq N} \mathbb{E}_\mu \left[ e^{\frac{c_3}{2} g_n} \right] = \exp \left( \frac{N}{2} \log \left( \frac{1}{1 - c_3} \right) \right).$$ \hspace{1cm} (15)

Summing over $N$, $\mathbb{E}_\mu \left[ \exp \left( \frac{1}{4} \int_{B} |u|^4 \right), S_\gamma \right]$ is bounded. \hspace{1cm} $\square$
4 Reduce to the soliton neighborhood: a stability argument

The sharp GNS inequality (11), at critical $p = 4$ for $u \in H^1_{rad}(\mathbb{R}^2)$, reads as

$$\frac{1}{4} \| u \|_{L^4(\mathbb{R}^2)}^4 \leq \frac{1}{2} \| \nabla u \|_{L^2(\mathbb{R}^2)}^2 \left( \frac{\| u \|_{L^2(\mathbb{R}^2)}}{\| Q \|_{L^2(\mathbb{R}^2)}} \right)^2.$$ (16)

On $L^2$-cutoff $\| u \|_{L^2(\mathbb{R}^2)} \leq \| Q \|_{L^2(\mathbb{R}^2)}$, this implies

$$H_{\mathbb{R}^2}(u) \geq 0.$$

The minima of this functional coincide with the minimizers of GNS inequality with restriction $\| u \|_{L^2(\mathbb{R}^2)} = \| Q \|_{L^2(\mathbb{R}^2)}$. By Proposition 2.1, these minimum points form a 2-dimensional (real) submanifold, the soliton manifold

$$\mathcal{M}_{\mathbb{R}^2} := \{ e^{i\theta} Q_\delta : \theta \in \mathbb{T}, \delta > 0 \},$$

where

$$Q_\delta(x) = \delta^{-1}Q(\delta^{-1}x).$$

The scaling is chosen to keep the $L^2(\mathbb{R}^2)$ norm invariant. By (12),

$$\Delta Q_\delta + Q_\delta^3 - \delta^{-2}Q_\delta = 0.$$ (17)

Since $H^1_{rad,0}(\mathbb{D})$ embeds naturally into $H^1(\mathbb{R}^2)$, (16) holds true on $\mathbb{D}$. In particular, $H(u) > 0$ when $\| u \|_{L^2(\mathbb{D})} \leq \| Q \|_{L^2(\mathbb{R}^2)}$.

The $(\delta_*, \delta^*)$-segment of the $\epsilon$-neighborhood of $\mathcal{M}_{\mathbb{R}^2}$ in $L^2_{rad}(\mathbb{D})$ is defined as

$$U_\epsilon(\delta_*, \delta^*) := \left\{ u \in L^2_{rad}(\mathbb{D}) : \| u \|_{L^2(\mathbb{D})} \leq \| Q \|_{L^2(\mathbb{R}^2)}, \| u - e^{i\theta} Q_\delta \|_{L^2(\mathbb{D})} \leq \epsilon, \text{ for some } \theta \in \mathbb{T}, \delta \in (\delta_*, \delta^*) \right\}.$$

In Section 3, we proved integrability on $S_\gamma$. The following stability result shows that $S_\gamma$ contains $U^C_\epsilon$.

Lemma 4.1. Given $\epsilon$, $\delta^*$, there exists $\gamma = \gamma(\epsilon, \delta^*)$, such that

$$U_\epsilon(0, \delta^*)^C \subset S_\gamma,$$

where the complement of $U_\epsilon$ is taken in

$$\left\{ u \in L^2_{rad}(\mathbb{D}) : \| u \|_{L^2(\mathbb{D})} \leq \| Q \|_{L^2(\mathbb{R}^2)} \right\}.$$
Proof. The proof follows the idea of Lemma 6.3 in [15].

Suppose by contradiction that there exists $\epsilon > 0$ and $u_n \notin U_\epsilon$, $\|u_n\|_{L^2(\mathbb{D})} \leq \|Q\|_{L^2(\mathbb{R}^2)}$, such that, for some $N_n > 0$ and $\gamma_n \to 0$,

$$\frac{1}{4} \int_{\mathbb{D}} |P_{\leq N_n} u|^4 > \frac{1 - \gamma_n}{2} \int_{\mathbb{D}} |\nabla P_{\leq N_n} u|^2.$$  \hfill (18)

By GNS inequality (11), on the other hand,

$$\frac{1}{4} \int_{\mathbb{D}} |P_{\leq N_n} u|^4 \leq \frac{1}{2} \int_{\mathbb{D}} |\nabla P_{\leq N_n} u|^2 \frac{\|P_{\leq N_n} u\|^2_{L^2(\mathbb{D})}}{\|Q\|^2_{L^2(\mathbb{R}^2)}}.$$  \hfill (19)

By (18),

$$1 - \gamma_n \leq \frac{\|P_{\leq N_n} u\|^2_{L^2(\mathbb{D})}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \leq \frac{\|u_n\|^2_{L^2(\mathbb{D})}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \leq 1.$$  \hfill (20)

Thus

$$\|P_{\leq N_n} u_n\|_{L^2(\mathbb{R}^2)} \to \|Q\|^2_{L^2(\mathbb{R}^2)}, \quad \|u_n - P_{\leq N_n} u_n\|_{L^2(\mathbb{R}^2)} \to 0.$$  \hfill (21)

If $\|P_{\leq N_n} u_n\|_{H^1(\mathbb{D})}$ is uniformly bounded, then $P_{\leq N_n} u_n$ converges weakly to some $v \in H^1(\mathbb{D})$. Definition of the eigenfunctions (7) implies $P_{\leq N_n} u_n = 0$ on $\partial \mathbb{D}$, hence in $H^1(2\mathbb{D})$. By Rellich-Kondrachov theorem, $P_{\leq N_n} u_n \to v$ in $L^2(\mathbb{R})$ and $v = 0$ on $\mathbb{D}^c$, which in turn means $v \in H^1(\mathbb{R}^2)$. Using (18) again, we have

$$\frac{1}{4} \int_{\mathbb{D}} |v|^4 = \lim_{n \to \infty} \frac{1}{4} \int_{\mathbb{D}} |P_{\leq N_n} u|^4 \geq \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{D}} |\nabla P_{\leq N_n} u|^2 \geq \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2.$$  \hfill (22)

This shows $v \in H^1(\mathbb{R}^2)$ is an optimizer of GNS inequality supported on $\mathbb{D}$, a contradiction. Therefore, up to a subsequence,

$$d_n := \|P_{\leq N_n} u_n\|_{H^1(\mathbb{D})} \to \infty.$$  \hfill (23)

Set $w_n = d_n^{-1} P_{\leq N_n} u_n(d_n^{-1})$. By scaling,

$$\|w_n\|_{H^1(\mathbb{R}^2)} = 1, \quad \|w_n\|_{L^2(\mathbb{R}^2)} \to \|Q\|_{L^2(\mathbb{R}^2)},$$

and by (18),

$$\limsup_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^2} |w_n|^4 \geq \frac{1}{2}. \hfill (24)$$

Now, using the bubble decomposition (see [10], Proposition 3.1), there exist $J^* < \infty$ functions $\phi_j \in H^1(\mathbb{R}^2)$, and $x_j \in \mathbb{R}^2$ and for $J \leq J^*$, there exist $r_n^j \in H^1(\mathbb{R}^2)$, such that

$$w_n(x) = \sum_{j=1}^J \phi_j(x - x_n^j) + r_n^j(x).$$

11
with the following properties:

\[ \|w_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\phi_j\|_{L^2}^2 + \|r_n^J\|_{L^2}^2 + o(1), \]
\[ \|\nabla w_n\|_{L^2}^2 + \sum_{j=1}^{J} \|\nabla \phi_j\|_{L^2}^2 = \|\nabla r_n^J\|_{L^2}^2 + o(1), \]
\[ \limsup_{n \to \infty} \|w_n\|_{L^2}^4 = \sum_{j=1}^{J^*} \|\phi_j\|_{L^2}^4, \]
\[ \limsup_{J \to J^*} \limsup_{n \to \infty} \|r_n^J\|_{L^2}^4 = 0. \]

If \( J^* = 0 \), then \( \|w_n\|_{L^2} = \|r_n^0\|_{L^2} \to 0 \), which contradicts (20). For \( J^* \geq 1 \), combining with (20),

\[ \frac{1}{2} \leq \limsup_{n \to \infty} \frac{1}{4} \int_{\mathbb{R}^2} |w_n|^4 = \frac{1}{4} \sum_{j=1}^{J^*} \|\phi_j\|_{L^2}^4 \leq \frac{1}{2} \sum_{j=1}^{J^*} \|\nabla \phi_j\|_{L^2}^2 \|\phi_j\|_{L^2}^2 \leq \sup_{j} \|\phi_j\|_{L^2}^2 \frac{1}{2} \limsup_{n \to \infty} \|\nabla w_n\|_{L^2}^2 \frac{1}{\|Q\|_{L^2}^2} = \frac{1}{2} \sup_{j} \|\phi_j\|_{L^2}^2. \]

On the other hand,

\[ \frac{1}{2} \sup_{j} \|\phi_j\|_{L^2}^2 \leq \frac{1}{2} \sum_{j=1}^{J^*} \|\phi_j\|_{L^2}^2 \frac{1}{\|Q\|_{L^2}^2} = \frac{1}{2} \limsup_{n \to \infty} \|w_n\|_{L^2}^2 \frac{1}{\|Q\|_{L^2}^2} = \frac{1}{2}. \]

Thus \( J^* = 1 \), and \( w_n(x) = \phi_1(x - x_n^1) + r_n^1(x) \). This implies that \( \phi_1 \) is an optimizer of GNS inequality. Moreover, since \( \|r_n^1\|_{L^2} \to 0 \), up to subsequence \( w_n \to \phi_1(\cdot - x_n^1) \) a.e.; since \( \|w_n\|_{L^2} \to \|\phi_1\|_{L^2} \), \( w_n - \phi_1(\cdot - x_n^1) \to 0 \) in \( L^2 \). Since \( w_n \) is radial, \( x_n^1 \) has to be bounded. We may assume \( x_n^1 \to x_0 \). By uniqueness of the optimizer, \( \phi_1(\cdot - x_0) = e^{i\theta}Q\delta \), for some \( \theta, \delta \). By the definition of \( w_n \) and (19),

\[ \lim_{n \to \infty} \|u_n - e^{i\theta}Qd_n^{-1}\delta\|_{L^2} = \lim_{n \to \infty} \|P_{\leq N_n}u_n - e^{i\theta}Qd_n^{-1}\delta\|_{L^2} = 0, \]

contradicting to \( u_n \notin U_{\epsilon} \).

As a corollary of Proposition 3.2 and Lemma 4.1 we have

**Corollary 4.2.** For \( \epsilon, \delta^* > 0 \),

\[ \mathbb{E}_\mu \left[ e^{\frac{+}{\epsilon} \int_0^1 |w|^4} U_{\epsilon}(0, \delta^*)^C \right] < \infty. \]
5 Normal bundle decomposition

We now move on to the integrability on the soliton neighborhood $U_\varepsilon$. The goal for the remainder of the paper is to bound
\[ \mathbb{E}_\mu \left[ e^{\frac{1}{2} \int_{\mathbb{D}} |u|^4}, U_\varepsilon(0, \delta^*) \right]. \] (21)

Recall the Gibbs measure $\rho$ is formally written as $e^{-H} du$. As discussed in Section 4, the minima of the Hamiltonian $H^1_{\mathbb{R}^2}$ are achieved on the soliton manifold $M_{\mathbb{R}^2}$, the functions of form $e^{i\theta} Q_\delta$. As the scaling parameter $\delta$ tends to zero, $e^{i\theta} Q_\delta$ will essentially concentrate inside $\mathbb{D}$. Therefore, the restrictions of $e^{i\theta} Q_\delta$ to $\mathbb{D}$ almost minimize $H$, and form a “near soliton” manifold $\mathcal{M}$ in $U_\varepsilon$. The restriction of $e^{i\theta} Q_\delta$ does not lie in $H_{rad,0}(\mathbb{D})$. For the spectral analysis in Section 6, we define the restriction, written in boldface, as
\[ Q_\delta(r) := Q_\delta(r) - Q_\delta(1) = Q_\delta(r) + O(e^{-c\delta^{-1}}), \] (22)
and
\[ \mathcal{M} := \{ e^{i\theta} Q_\delta : \theta \in \mathbb{T}, \delta > 0 \}. \]

To estimate $H$ on $U_\varepsilon$, it is reasonable to expand it along normal directions to $\mathcal{M}$. By this we mean that at each point $e^{i\theta} Q_\delta \in \mathcal{M}$, we perturb $H$ along the normal vector space
\[ V_{\theta,\delta} := \{ v \in L^2_{rad}(\mathbb{D}) : \langle v, -\Delta(e^{i\theta} \partial_\delta Q_\delta) \rangle = \langle v, -\Delta(e^{i\theta} Q_\delta) \rangle = 0 \}. \] (23)

Note that
\[ \partial_\theta (e^{i\theta} Q_\delta) = ie^{i\theta} Q_\delta, \]
\[ \partial_\delta (ie^{i\theta} Q_\delta) = e^{i\theta} \partial_\delta Q_\delta = e^{i\theta} (\partial_\delta Q_\delta - \partial_\delta Q_\delta(1)). \]

$ie^{i\theta} Q_\delta$ and $e^{i\theta} \partial_\delta Q_\delta$ are two tangent vectors of $\mathcal{M}$ in $H^1_{rad,0}(\mathbb{D})$. Since our inner product is real-valued,
\[ \langle ie^{i\theta} Q_\delta, e^{i\theta} \partial_\delta Q_\delta \rangle = \langle ie^{i\theta} Q_\delta, (-\Delta) e^{i\theta} \partial_\delta Q_\delta \rangle = 0. \]

For simplicity, we only discuss the case when $\theta = 0$ (which is sufficient for our proof). Let $u = Q_\delta + v$ with $v \in V_{0,\delta},$
\[ H(u) = H(Q_\delta + v) = H(Q_\delta) + \langle -\Delta Q_\delta - Q_\delta^3, v \rangle + \frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2 \]
\[ - \langle Q_\delta^2, \frac{1}{2} v^2 + |v|^2 \rangle - \langle Q_\delta, |v|^2 v \rangle - \frac{1}{4} \int_{\mathbb{D}} |v|^4 \]
\[ \overset{6}{\text{Here we use radial variable } r = |x|.} \]
Constant part: By (22), exponential decay of the ground state $Q$ and $\nabla Q$ (see Lemma 2.3),
\[
H(Q_\delta) = \frac{1}{2} \int_D |\nabla Q_\delta|^2 - \frac{1}{4} \int_D |Q_\delta|^4
= \frac{1}{2} \int_D |\nabla Q_\delta|^2 - \frac{1}{4} \int_D |Q_\delta|^4 + O(e^{-\delta^{-1}})
= \delta^{-2} \left( \frac{1}{2} \int_{|x|>\delta^{-1}} |\nabla Q|^2 - \frac{1}{4} \int_{|x|>\delta^{-1}} |Q|^4 \right) + O(e^{-\delta^{-1}}) = O(e^{-\delta^{-1}}).
\]

Linear part: Assume $\|v\|_{L^2(\mathbb{D})} \leq 1$, which is indeed the case on $U_\epsilon$ (see Remark 5.11). By (17),
\[
\langle -\Delta Q_\delta - Q_\delta^3, v \rangle = \langle -\Delta Q_\delta - Q_\delta^3, v \rangle + O(e^{-\delta^{-1}}) = -\delta^{-2} \langle Q_\delta, v \rangle + O(e^{-\delta^{-1}}).
\]

Remark 5.1. The linear functional above is related to $dH$, the linearization of $H$ on $H^1(\mathbb{R}^2)$. Denote $M_{\mathbb{R}^2}(u) := \frac{1}{2} \|u\|_{L^2(\mathbb{R}^2)}^2$. Since $Q_\delta$ is a minimizer of $H_{\mathbb{R}^2}$ conditioned on $M_{\mathbb{R}^2} = \frac{1}{2} \|Q\|_{L^2(\mathbb{R}^2)}^2$, there exists a Lagrange multiplier $\lambda = \lambda(Q_\delta)$ such that
\[
dH_{\mathbb{R}^2}(Q_\delta) - \lambda dM_{\mathbb{R}^2}(Q_\delta) = 0, \text{ that is, } -\Delta Q_\delta - Q_\delta^3 - \lambda Q_\delta = 0.
\]
Using the equation of the ground state (17), we know $\lambda(Q_\delta) = -\delta^{-2}$.

Higher order terms: Applying Cauchy-Schwarz, for any small $\eta$
\[
\langle Q_\delta, |v|^2 v \rangle = \langle Q_\delta, |v|^2 v \rangle + O(e^{-\delta^{-1}}) \leq \eta \langle Q_\delta, |v|^2 \rangle + C_\eta \int_D |v|^4 + O(e^{-\delta^{-1}}).
\]

Collecting all the reductions above, we get
\[
H(u) \geq O(e^{-\delta^{-1}}) + \frac{1}{2} \int_D |\nabla v|^2 - B_\delta(v) - C_\eta \int_D |v|^4,
\]
where
\[
B_\delta(v) = \delta^{-2} \langle Q_\delta, v \rangle + \left\langle Q_\delta^2, \frac{1}{2} v^2 + (1 + \eta)|v|^2 \right\rangle.
\]

5.1 Change of variable formula
The decomposition $u = e^{i\theta}Q_\delta + v$ induces assign a normal bundle structure to the soliton neighborhood $U_\epsilon$.

Theorem 5.2 (Normal bundle decomposition). Given $\epsilon > 0$, and dyadic number $N \in [1, \infty]$, there exists $\delta_* = \delta_*(N, \epsilon)$, and $\delta^* = \delta^*(\epsilon)$, satisfying $N^{-1} \lesssim \delta_* < \delta^* \ll 1$. Such that at any point $e^{i\theta}Q_\delta \in \mathcal{M}$ with $\tilde{\theta} \in (\delta_*, \delta^*)$, there are a neighborhood $W$ of $(\tilde{\theta}, \tilde{\delta}, 0)$ in $T \times \mathbb{R} \times V_{\theta, \delta} \cap E_N$, and a diffeomorphism $G : W \to G(W) \subset L^2_{rad}(\mathbb{D}) \cap E_N$ defined as
\[
G(\theta, \delta, v) := P_{\leq N} \left( e^{i(\bar{\theta} + \theta)}Q_{\delta \tilde{\theta}} \right) + P_{V_{\tilde{\theta}, \delta} \cap E_N} v.
\]
Its image $G(W)$ contains \( \{ u \in L^2_{\text{rad}}(\mathbb{D}) \cap E_N : \| u - P_{\leq N} e^{i\theta} Q_\delta \|_{L^2(\mathbb{D})} \leq 2\epsilon \} \).

In particular, for any $u \in U_\epsilon(\delta_*, \delta^*)$, there exist $\theta, \delta$ and $v \in V_{\theta, \delta} \cap E_N$, such that

\[
P_{\leq N} u = P_{\leq N} (e^{i\theta} Q_\delta) + v.
\]

We postpone the proof to Section 5.3.

Remark 5.3. Although the decomposition happens in $L^2_{\text{rad}}(\mathbb{D})$, the normal vector space $V_{\theta, \delta}$ in (23) is actually with respect to $\dot{H}^1$ inner product. We make this choice because the free Gaussian measure $\mu$ is homogeneous on its Cameron-Martin space $\dot{H}^1_{\text{rad}, 0}$. For any vector space $\dot{H}^1_{\text{rad}, 0} = V_1 \oplus \dot{H}^1_{\text{rad}, 0}$, we have the splitting $\mu = \mu_{V_1} \otimes \mu_{V_2}$, where $\mu_{V_i}$ are the free Gaussians on $V_i$.

For integration on a normal bundle, we refer the following change of variable formula.

Lemma 5.4 (Change of variable, Lemma 6.11 in [15]). Let $M^d \subset \mathbb{R}^n$ be a closed submanifold and $N$ be its normal bundle. Suppose there is a decomposition (a diffeomorphism) mapping a neighborhood $U$ of $M$ to $N$ via

\[
u = x + v, \ u \in U, \ x \in M, \ v \in T_x^\perp M.
\]

Let $V = \{ (x, v) : x + v \in U \}$, then for any measurable function $f : \mathbb{R}^n \to \mathbb{R}$,

\[
\int_U f(u) du \lesssim \int_M \int_{T_x^\perp M} f(x + v) 1_V(x, v) dv d\sigma(x)
\]

the measure $\sigma(x)$ is defined as

\[
d\sigma(x) := \left(1 + \sup_{k=1, \ldots, d} |\nabla_x t_k(x)|^d \right) d\omega(x),
\]

where $d\omega$ is the surface measure on $M$ and $\{t_k(x)\}_{k=1}^d$ is an orthonormal frame of $M$.

Combining with the lower bound (24), Lemma 5.4 formally bounds the $\mu$-expectation (21) as

\[
\int_M \int_{V_{\theta, \delta}} e^{R_{\theta}(v) + C_n \int_{\mathbb{D}} |v|^4} e^{-\frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2} dv d\sigma(\theta, \delta).
\]

Note that “$e^{-\frac{1}{2} \int_{\mathbb{D}} |\nabla v|^2} dv = d\mu_{V_{\theta, \delta}}$.” $\mu_{V_{\theta, \delta}}$, the free Gaussian measure on $V_{\theta, \delta}$ can be defined as following: Let $\{h_n\}$ be a $\dot{H}^1$ orthonormal basis in the dense subspace $V_{\theta, \delta} \cap H^1_{\text{rad}, 0}(\mathbb{D})$, then $\mu_{V_{\theta, \delta}}$ is the law of random variable $v(\omega) = \sum_n g_n(\omega) h_n$, where $\{g_n\}$ is a sequence of independent standard complex-valued Gaussians. This construction coincides with $\mu$, where $\{\xi_n\}$ is the orthonormal basis in $H^1_{\text{rad}, 0}(\mathbb{D})$. With this argument in mind, we expect the following estimate.
Proposition 5.5. Given $\delta^* > 0$,

$$
\mathbb{E}_\mu \left[ e^{\frac{1}{\delta} \int_0^1 |u|^4} U_c(0, \delta^*) \right] \leq \int_0^{\delta^*} \left( \int_{V_{0,\delta}} e^{B_\delta(v) + C_n \int_0^{\delta^*} |v|^4} \, d\mu_{V_{0,\delta}}(v) \right) \delta^{-5} \, d\delta.
$$

where $\mu_{V_{0,\delta}}$ is the free Gaussian measure on $V_{0,\delta}$.

Proof. To simplify the notation, we denote $U_c$ for $U_c(0, \delta^*)$. Up to some mollification of the indicator function $1_{U_c}$, the Dominated Convergence Theorem gives

$$
\mathbb{E}_\mu \left[ e^{\frac{1}{\delta} \int_0^1 |u|^4} U_c(0, \delta^*) \right] = \lim_{N \to \infty} \int 1_{U_c}(P_{\leq N} u) e^{\frac{1}{\delta} \int_0^1 |P_{\leq N} u|^4} \, d\mu(u).
$$

Since $\mu$ is the law of (26),

$$
\int 1_{U_c}(P_{\leq N} u) e^{\frac{1}{\delta} \int_0^1 |P_{\leq N} u|^4} \, d\mu(u) = \int_{E_N \cap U_c} e^{\frac{1}{\delta} \int_0^1 |P_{\leq N} u|^4} \prod_{n=1}^N \frac{1}{2\pi} e^{-\frac{1}{2} |v|^2} \, dv.
$$

Using decomposition Theorem 5.2 and change of variable Lemma 5.4, the integral above is bounded by

$$
\int_{P_{\leq N} \mathcal{M}} \int_{V_{0,\delta} \cap E_N} (2\pi)^{-\frac{N}{2}} e^{-H(P_{\leq N} e^{i\theta} Q_\delta + v)} \, dvd\sigma(\theta, \delta).
$$

Note that on $V_{\theta,\delta} \cap E_N$, $d\mu_{V_{\theta,\delta}} = (2\pi)^{-\frac{N-2}{2}} e^{-\frac{1}{2} \int_0^{\delta^*} |P_{\leq N} u|^2} \, dv$. The integral above can be written as

$$
\int_{P_{\leq N} \mathcal{M}} \int_{V_{0,\delta}} f(P_{\leq N} e^{i\theta} Q_\delta, P_{\leq N} u) \, d\mu_{V_{0,\delta}}(v) \, d\sigma(\theta, \delta).
$$

where

$$
f(P_{\leq N} e^{i\theta} Q_\delta, P_{\leq N} u) = (2\pi)^{-1} \exp \left( -H \left( P_{\leq N} (e^{i\theta} Q_\delta + v) \right) + \frac{1}{2} \int_D |P_{\leq N} u|^2 \right)
$$

$$
= (2\pi)^{-1} \exp \left( \frac{1}{4} \int_D |P_{\leq N} (e^{i\theta} Q_\delta + v)|^4 - \frac{1}{2} \int_D |\nabla P_{\leq N} e^{i\theta} Q_\delta|^2 + \langle \Delta P_{\leq N} e^{i\theta} Q_\delta, v \rangle \right).
$$

By (25),

$$
d\sigma(\theta, \delta) = \left( 1 + \sup_{k=1,2} \| \partial_t t_k \|^2_{H^1} + \| \partial_t k \|^2_{H^1} \right) \, d\omega(\theta, \delta).
$$

Here $t_1$ is $\hat{H}^1$–normalized $P_{\leq N} e^{i\theta} Q_\delta$ whereas $t_2$ is $\hat{H}^1$–normalized $P_{\leq N} e^{i\theta} \partial_t Q_\delta$. By scaling argument, as $\delta \to 0$,

$$
\|Q_\delta\|_{H^k(\mathbb{R})} = O(\delta^{-k}), \quad \|\partial_t Q_\delta\|_{H^k(\mathbb{R})} = O(\delta^{-k-1}). \quad (26)
$$

To simplify the notation, we hide the restriction $\delta \geq \delta_* \gtrsim N^{-1}$ for $P_{\leq N} \mathcal{M}$, enforced by Theorem 5.2.
Therefore,
\[ \| \partial_t t_k \|_{L^2}^2 = O(1), \| \partial_\delta t_k \|_{L^2}^2 = O(\delta^{-2}). \]

For the surface measure, note that vectors \( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \delta} \) are orthogonal in \( \dot{H}^1 \). Hence
\[ d\omega(\theta, \delta) = \| P_{\leq N} e^{i\theta} Q_\delta \|_{H^1} \| P_{\leq N} e^{i\theta} \partial_\delta Q_\delta \|_{H^1} \lesssim \delta^{-3} d\delta d\theta. \]

Taking \( N \to \infty \), the Dominated Convergence Theorem yields
\[ \int_T \int_{\delta}^* \int_{V_{\theta,\delta}} f(P_{\leq N} e^{i\theta} Q_\delta, P_{\leq N} v) d\mu_{V_{\theta,\delta}}(v) \delta^{-3} d\delta d\theta \to \int_T \int_{\delta}^* \int_{V_{\theta,\delta}} f(e^{i\theta} Q_\delta, v) d\mu_{V_{\theta,\delta}}(v) \delta^{-3} d\delta d\theta. \]

Observe that \( f(e^{i\theta} Q_\delta, v) = f(Q_\delta, e^{-i\theta} v) \) and \( e^{-i\theta} V_{\theta,\delta} = V_{0,\delta} \). Therefore,
\[ \int_T \int_{\delta}^* \int_{V_{\theta,\delta}} f(e^{i\theta} Q_\delta, v) d\mu_{V_{\theta,\delta}}(v) \delta^{-3} d\delta d\theta = 2\pi \int_T \int_{\delta}^* \int_{V_{0,\delta}} f(Q_\delta, v) d\mu_{V_{0,\delta}}(v) \delta^{-3} d\delta. \]

Using \( 24 \),
\[ f(Q_\delta, v) = (2\pi)^{-1} e^{-H(Q_\delta + v) + \frac{1}{2} \int_0^1 |v|^2} \leq (2\pi)^{-1} e^{O(e^{-c_\delta} + B_\delta(v) + C_\eta f_\delta |v|^4}. \]

This gives the desired bound. \( \square \)

**Remark 5.6.** If we keep the constraint \( U_\varepsilon \) in the proof, the bound in Proposition 5.5 can be strengthened to
\[ \int_0^{\delta} \left( \int_{V_{0,\delta}} 1_{U_\varepsilon}(Q_\delta + v) e^{B_\delta(v) + C_\eta f_\delta |v|^4} d\mu_{V_{0,\delta}}(v) \right) \delta^{-5} d\delta. \]

In particular, by Theorem 5.2, this gives the bound
\[ \int_0^{\delta} \left( \int_{V_{0,\delta}, |v|_{L^2(B)}} e^{B_\delta(v) + C_\eta f_\delta |v|^4} d\mu_{V_{0,\delta}}(v) \right) \delta^{-5} d\delta. \]

### 5.2 Higher order term

Following \( 27 \) it is sufficient to estimate
\[ \int_{V_{0,\delta}, |v|_{L^2(B)} \leq \varepsilon} e^{B_\delta(v) + C_\eta f_\delta |v|^4} d\mu_{V_{0,\delta}}(v). \]

Applying Hölder’s inequality, we divide it into two part: quadratic term, \( B_\delta \) part and higher order term, \( \int_B |v|^4 \) part.

\[ \left( \int_{V_{0,\delta}} e^{(1+\eta)B_\delta(v)} d\mu_{V_{0,\delta}}(v) \right)^{\frac{1}{1+\eta}} \left( \int_{V_{0,\delta}, |v|_{L^2(B)} \leq \varepsilon} e^{C_\eta f_\delta |v|^4} d\mu_{V_{0,\delta}}(v) \right)^{\frac{\eta}{1+\eta}}. \]

In this subsection, we deal with the higher order term part and leave the quadratic part to Section \( 6 \).
Lemma 5.7.
\[
\int_{\|v\|_{L^2(\mathbb{D})} \leq \varepsilon} e^{C_n/\varepsilon} |v|^4 \, d\mu_{V_0, \delta}(v) < \infty
\]

Proof. Recall that
\[
V_{0, \delta} := \{ v \in L^2_{\text{rad}}(\mathbb{D}) : \langle v, -\Delta (e^{i\theta} Q_\delta) \rangle = \langle v, -\Delta (ie^{i\theta} Q_\delta) \rangle = 0 \}.
\]
Let \( t_1, t_2 \) be \( H^1 \)-normalized vectors of \( Q_\delta, iQ_\delta \), respectively, that is
\[
t_1 = \frac{\partial_\theta Q_\delta}{\|\partial_\theta Q_\delta\|_{H^1}}, \quad t_2 = \frac{iQ_\delta}{\|iQ_\delta\|_{H^1}}.
\]
Then the corresponding orthogonal projection \( P_V \) on \( V_{0, \delta} \) follows as
\[
P_{V_{0, \delta}}(u) := u - \sum_j \langle u, (-\Delta) t_j \rangle t_j.
\]
Since \( \mu_{V_{0, \delta}} \) is the free Gaussian measure on \( V_{0, \delta} \), we have
\[
\int_{\|v\|_{L^2} \leq \varepsilon} e^{C_n/\varepsilon} |v|^4 \, d\mu_{V_0, \delta}(v) = \int_{\|P_V u\|_{L^2} \leq \varepsilon} e^{C_n/\varepsilon} |P_V u|^4 \, d\mu(u).
\]
As in Proposition 3.2 we further slice the domain:
\[
F_{2N} := \{ u \in L^2_{\text{rad}}(\mathbb{D}) : \|P_{\geq M} P_V u\|_4 > \lambda_M, M < 2N; \|P_{\geq 2N} P_V u\|_4 \leq \lambda_{2N} \},
\]
where \( \lambda_N = (\log N)^{3/4} \). Therefore,
\[
E_\mu \left[ e^{C_n/\varepsilon} |P_V u|^4, F_{2N}, \|P_V u\|_2 \leq \varepsilon \right] \leq e^{C_1\lambda_{2N}} E_\mu \left[ e^{C_2/\varepsilon} |P_{\leq N} P_V u|^4, F_{2N}, \|P_V u\|_2 \leq \varepsilon \right].
\]
(28)
\( C_1, C_2 \) above depend only on \( \eta \). Applying GNS inequality and conditioning on \( \|P_V u\|_2 \leq \varepsilon \),
\[
\int |P_{\leq N} P_V u|^4 \lesssim \|P_{\leq N} P_V u\|_2^2 \|P_{\leq N} P_V u\|_{H^1}^2 \leq \varepsilon^2 \left\| P_{\leq N} u - \sum_j \langle u, (-\Delta) t_j \rangle P_{\leq N} t_j \right\|^2_{H^1}
\]
\[
\lesssim \varepsilon^2 \left( \|P_{\leq N} u\|_{H^1}^2 + \sum_j \langle u, (-\Delta) t_j \rangle^2 \|P_{\leq N} t_j\|_{H^1}^2 \right)
\]
\[
\leq \varepsilon^2 \left( \|P_{\leq N} u\|_{H^1}^2 + \sum_j \langle u, (-\Delta) t_j \rangle^2 \right).
\]
(28) is then bounded by
\[
e^{C_1\lambda_{2N}} E_\mu \left[ e^{C_2/\varepsilon} \|P_{\leq N} u\|_{H^1}^2 \right]^{1/4} \prod_j E_\mu \left[ e^{C_2/\varepsilon} \langle u, (-\Delta) t_j \rangle^2 \right]^{1/4} \mu(F_{2N})^{1/4}
\]
18
The first expectation has the same form as (15), therefore equals to \( \exp \left( \frac{N}{2} \log \left( \frac{1}{1 - 2C_3 \epsilon^2} \right) \right) \).

For the second expectation, note that \( \langle u, (\Delta) t_j \rangle \) is a mean-zero Gaussian. Using the series representation (9), its variance is bounded by \( 2 \| t_j \|_{H^1}^2 = 2 \). So, the expectation is bounded provided \( \epsilon \) is small enough.

Finally, \( \mu \{ F_{2N} \} \leq \mu \{ \| P_{\geq N} P V u \|_4 \geq \lambda_N \} \)
\( \leq \mu \{ \| P_{\geq N} u \|_4 \geq \lambda_N/3 \} + \sum_j \mu \{ \langle u, (\Delta) t_j \rangle \| P_{\geq N} t_j \|_4 \geq \lambda_N/3 \} \).

By (13) and \( \lambda_N = (\log N)^{3/4} \), \( \mu \{ \| P_{\geq N} u \|_4 \geq \lambda_N/3 \} \leq \exp (-c(\log N)N) \). Applying GNS inequality, \( \| P_{\geq N} t_j \|_4 \lesssim \| P_{\geq N} t_j \|_2^{1/2} \| P_{\geq N} t_j \|_{H^1}^{1/2} \leq \| P_{\geq N} t_j \|_2^{1/2} \).

Since \( t_j \) are smooth functions, we have
\[
\| P_{\geq N} t_j \|_2^2 = \sum_{n \geq N} \langle t_j, e_n \rangle^2 \leq \| (\Delta)^k t_j \|_2^2 \sum_{n \geq N} \frac{\| e_n \|_2^2}{\gamma_n^{2k}} \lesssim_k \frac{1}{N^{4k-1}}.
\]

Thus, \( \mu \{ \langle u, (\Delta) t_j \rangle \| P_{\geq N} t_j \|_4 \geq \lambda_N/3 \} \leq \mu \{ \langle u, (\Delta) t_j \rangle \geq N \} \leq \exp (-cN^2) \).

Now summing the factors above,
\[
\int_{\| v \|_2 \leq \epsilon} e^{C_0 \int_0^1 |v|^4} d\nu_{\nu, \delta} (v) \lesssim \sum_N \exp \left( C_1 (\log 2N)^3 + \frac{N}{8} \log \left( \frac{1}{1 - 2C_3 \epsilon^2} \right) - cN^2 \right) < \infty.
\]

\( \square \)

### 5.3 Proof of Theorem 5.2

Fix \( \tilde{\theta}, \tilde{\delta} \), let \( G \) be the diffeomorphism defined in the statement. The differentiability of \( G \) is straightforward. To prove that \( G \) is invertible, we use the following version of Inverse Function Theorem.

**Theorem 5.8.** Let \( X, Y \) be Banach spaces, and \( f : X \to Y \) be a \( C^1 \) maps. Suppose \( df(x_0) \) is invertible and for some \( \kappa < 1 \), \( R > 0 \),
\[
\| df(x_0)^{-1} \circ df(x) - Id_X \| \leq \kappa, \quad x \in \bar{B}^X (x_0, R).
\]

Then \( f \) is invertible near \( x_0 \). Moreover, for \( r := \frac{1 - \kappa}{\| df(x_0)^{-1} \|} R \),
\[
B^Y (f(x_0), r) \subset f(B^X (x_0, R)).
\]

19
Proof. Assume \( x_0 = 0 \) and \( f(0) = 0 \). Denote \( A = df(0) \). For any \( y \in \hat{B}^X(0, R) \), define map \( H_y : \hat{B}^X(0, R) \to X \) as
\[
H_y(x) := A^{-1}y - (A^{-1}f(x) - x).
\]
By (29), for \( x_1, x_2 \in B^X(0, R) \),
\[
\|H(x_1) - H(x_2)\| = \|(A^{-1}f - Id)(x_1) - (A^{-1}f - Id)(x_2)\| \leq \kappa \|x_1 - x_2\|.
\]
Moreover,
\[
\|H_y(x)\|_X \leq \|A^{-1}y\|_X + \|A^{-1}f(x) - x\|_X \leq \|A^{-1}\| r + \kappa R \leq R.
\]
So \( H \) is a contraction map on \( \hat{B}^X(0, R) \). Banach’s Fixed-Point Theorem implies the existence a unique fixed point \( x \) such that \( H_y(x) = x \), that is \( f(x) = y \).

To apply the Inverse Function Theorem, we need operator bounds for \( dG^{-1}(0, 1, 0) \circ dG(\theta, \delta, w) - Id \).

In the rest of this subsection, we set, with out loss of generality, \( \tilde{\theta} = 0 \).

Lemma 5.9. For \( \delta \gtrsim N^{-1} \), the derivative
\[
dG(0, 1, 0) : \mathbb{R} \times \mathbb{R} \times V_{0, \delta} \cap E_N \to L^2_{rad}(\mathbb{D}) \cap E_N
\]
is invertible and the inverse \( dG^{-1}(0, 1, 0) \) is bounded uniformly in \( \delta \).

Proof. Given \( v = (\alpha, \beta, v) \in \mathbb{R} \times \mathbb{R} \times V_{0, \delta} \cap E_N \), denote \( u := dG(0, 1, 0)v \). Direct computation gives,
\[
u = \alpha P_{\leq N}iQ_{\delta} + \beta \delta P_{\leq N}\partial_\delta Q_{\delta} + v.
\]
In coordinate \((P_{\leq N}iQ_{\delta}, P_{\leq N}\partial_\delta Q_{\delta}, V_{0, \delta} \cap E_N)\) of \( L^2_{rad}(\mathbb{D}) \cap E_N \), the map \( dG(0, 1, 0) \) has matrix representation
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \tilde{\delta} & 0 \\
0 & 0 & Id
\end{pmatrix},
\]
which is invertible. By orthogonality, (30) yields
\[
\alpha = \frac{\langle u, (-\Delta)iQ_{\delta} \rangle}{\langle P_{\leq N}Q_{\delta}, (-\Delta)Q_{\delta} \rangle},
\]
\[
\beta = \delta^{-1} \frac{\langle u, (-\Delta)\partial_\delta Q_{\delta} \rangle}{\langle P_{\leq N}\partial_\delta Q_{\delta}, (-\Delta)\partial_\delta Q_{\delta} \rangle},
\]
\[
v = u - \alpha P_{\leq N}iQ_{\delta} - \beta \delta P_{\leq N}\partial_\delta Q_{\delta}.
\]
Next we will obtain lower bounds for the denominators. Since $Q_\delta$ differs $Q_\delta$ only by $O(e^{-c\delta^{-1}})$ (also true for its derivatives), it suffices to estimate the corresponding expression with direct truncation $Q_\delta$. Recall $e_n$ are $L^2$-normalized eigenfunctions. We have

$$\langle Q_\delta, e_n \rangle = \int_{B} \delta^{-1} Q(\delta^{-1}x) J_0(z_n x) \|J_0(z_n \cdot)\|_{L^2(B)}^{-1} \, dx$$

$$= \|J_0(z_n \cdot)\|_{L^2(B)}^{-1} \int_{\delta^{-1}B} Q(y) J_0(z_n \delta y) \, dy$$

$$= \|J_0(z_n \cdot)\|_{L^2(B)}^{-1} \delta \left( \hat{Q}(z_n \delta) + O(\exp(-c\delta^{-1})) \right).$$

Here, using that $J_0$ is bounded near 0 and asymptotic approximation \(^1\)0, \(\hat{Q}(\xi) := \int_{\mathbb{R}^2} Q(x) J_0(\xi x) \, dx\)

$$\lesssim \int_{0}^{\frac{1}{\xi}} Q(r) \, dr + \int_{\frac{1}{\xi}}^{\infty} Q(r) \cos(\xi r - \frac{\pi}{4}) (\xi r)^{-\frac{1}{2}} \, dr \lesssim \langle \xi \rangle^{-2}.$$

$$\|J_0(z_n \cdot)\|_{L^2(B)}^2 = \frac{1}{z_n^2} \int_{z_n B} J_0(y)^2 \, dy = \frac{1}{z_n^2} \int_{0}^{z_n} J_0(r)^2 \, r \, dr$$

$$= \frac{1}{z_n^2} \left( \int_{0}^{1} O(1) \, dr + \int_{1}^{z_n} \cos(\pi r) \, dr \right) \sim \frac{1}{z_n}.$$

Combining with \(^3\)1

$$\langle P_{\leq N}(-\Delta) Q_\delta, Q_\delta \rangle = \sum_{n \leq N} z_n^2 \langle Q_\delta, e_n \rangle^2 \sim \sum_{n \leq N} z_n^3 \delta^2 \left( \hat{Q}(z_n \delta)^2 + O(\exp(-c\delta^{-1})) \right)$$

$$\sim \delta^{-2} \pi \sum_{n \leq N} (z_n \delta)^3 \hat{Q}(z_n \delta)^2 \cdot \delta(z_n - z_{n-1})$$

$$\sim \delta^{-2} \pi \int_{0}^{\delta z_N} y^3 \hat{Q}(y)^2 \, dy$$

\((\text{for } \delta \gtrsim \frac{1}{z_N} \sim \frac{1}{N}) \geq c_1 \delta^{-2}.\)

Similar computation yields, for $\delta \gtrsim N^{-1}$,

$$\langle P_{\leq N}(-\Delta) \partial_\delta Q_\delta, \partial_\delta Q_\delta \rangle \gtrsim \delta^{-4}$$

(32)

Using \(^2\)6, \(^3\)1, \(^5\)2 and Cauchy-Schwarz, \(\delta^{-2} \|u\|_{L^2(B)} \|Q_\delta\|_{H^2(B)} \lesssim \|u\|_{L^2(B)} \cdot \)

$$\alpha \lesssim \delta^2 \|u\|_{L^2(B)} \|Q_\delta\|_{H^2(B)} \lesssim \|u\|_{L^2(B)} \cdot$$

$$\beta \lesssim \delta^3 \|u\|_{L^2(B)} \|\partial_\delta Q_\delta\|_{H^2(B)} \lesssim \|u\|_{L^2(B)} \cdot$$

$$\|v\|_{L^2(B)} \lesssim \|u\|_{L^2(B)} + |\alpha| + |\beta| \lesssim \|u\|_{L^2(B)}.\)
Lemma 5.10. For $\tilde{\delta} \gtrsim N^{-1}$, we have

$$\|dG^{-1}(0,1,0) \circ dG(\theta, \delta, w) - Id\|_{op} \lesssim |\theta| + |\delta - 1|(|\delta| + 1) + \|w\|_{L^2(\mathbb{D})}.$$ 

Proof. Given $v = (\alpha, \beta, v) \in \mathbb{R} \times \mathbb{R} \times V_{0,\delta} \cap E_N$; denote the image by $\tilde{v} = (\tilde{\alpha}, \tilde{\beta}, \tilde{v})$ and the intermediate image by $u$. That is,

$$dG(\theta, \delta, w)v = u = dG(0,1,0)\tilde{v}.$$ 

Direct computation yields,

$$dG(0,1,0)\tilde{v} = \tilde{\alpha}P_{\leq N}\delta_\delta \beta + \tilde{\beta} \tilde{\delta} P_{\leq N}\partial_\delta Q_\delta + \tilde{v},$$

$$dG(\theta, \delta, w)v = \alpha P_{\leq N}ie^{i\theta}Q_\delta + \beta \tilde{\delta} P_{\leq N}(\partial_\delta Q_\delta) + P_{\Delta N}(\partial_\delta \cap E_N) v + d\left(P_{\Delta N}(\partial_\delta \cap E_N) \right)(\alpha, \beta).$$

Here, $(\partial_\delta Q_\delta)$ means $\frac{\partial}{\partial s}Q_\delta|_{s=\delta}$, and

$$d\left(P_{\Delta N}(\partial_\delta \cap E_N) \right)(\alpha, \beta) = (\alpha \partial_\theta + \beta \partial_\delta) \left( -\frac{\langle w, P_{\leq N}(\Delta)ie^{i\theta}Q_\delta \rangle}{\langle P_{\leq N}(\Delta)ie^{i\theta}Q_\delta \rangle} + \frac{\langle w, P_{\leq N}(\Delta)\delta e^{i\theta}(\partial_\delta Q_\delta) \rangle}{\langle P_{\leq N}(\Delta)\delta e^{i\theta}(\partial_\delta Q_\delta) \rangle} \right).$$

Similar to the proof of Lemma 5.9 applying (31), (32) and (26) we get

$$\|d\left(P_{\Delta N}(\partial_\delta \cap E_N) \right)(\alpha, \beta)\|_{L^2(\mathbb{D})} \lesssim (|\alpha| + |\beta|) \|w\|_{L^2(\mathbb{D})}.$$ (33)

By orthogonality,

$$\tilde{\alpha} = \frac{\langle u, (\Delta)Q_\delta \rangle}{\langle P_{\leq N}Q_\delta, (\Delta)Q_\delta \rangle} = \alpha \frac{\langle P_{\leq N}ie^{i\theta}Q_\delta, (\Delta)Q_\delta \rangle}{\langle P_{\leq N}Q_\delta, (\Delta)Q_\delta \rangle} + \beta \frac{\langle P_{\leq N}e^{i\theta}(\partial_\delta Q_\delta), (\Delta)Q_\delta \rangle}{\langle P_{\leq N}Q_\delta, (\Delta)Q_\delta \rangle} + \frac{\langle P_{\Delta N}(\partial_\delta \cap E_N) v, (\Delta)Q_\delta \rangle}{\langle P_{\Delta N}(\partial_\delta \cap E_N) v, (\Delta)Q_\delta \rangle} + \frac{\langle d\left(P_{\Delta N}(\partial_\delta \cap E_N) \right)(\alpha, \beta), (\Delta)Q_\delta \rangle}{\langle P_{\Delta N}(\partial_\delta \cap E_N) \rangle(\alpha, \beta), (\Delta)Q_\delta \rangle}.$$ 

The first term is the dominant part. Applying Cauchy-Schwartz, (31), (32) and (26),

$$\frac{\langle P_{\leq N}ie^{i\theta}Q_\delta, (\Delta)Q_\delta \rangle}{\langle P_{\leq N}Q_\delta, (\Delta)Q_\delta \rangle} = 1 + \frac{\langle P_{\leq N}(ie^{i\theta}Q_\delta - iQ_\delta), (\Delta)Q_\delta \rangle}{\langle P_{\leq N}Q_\delta, (\Delta)Q_\delta \rangle} = 1 + O\left(\|e^{i\theta}Q_\delta - Q_\delta\|_{L^2(\mathbb{D})} \right) + O(e^{-c\delta^{-1}}).$$

<sup>9</sup>Heuristically, each $\Delta$ gives $\delta^{-2}\delta^{-2}$, while each $\partial_\delta$ gives $\delta^{-1}$. 

22
Fundamental theorem of calculus gives
\[
\| e^{i\theta} Q_{\delta\delta} - Q_{\delta} \|_{L^2(D)} \leq |e^{i\theta} - 1| \| Q_{\delta\delta} \|_{L^2(D)} + \| Q_{\delta\delta} - Q_{\delta} \|_{L^2(D)} \\
\lesssim |\theta| + \int_{\delta}^{\delta\delta} \| (\partial_\delta Q)_{\delta} \|_{L^2} \; ds \\
\lesssim |\theta| + |\ln \delta| \lesssim |\theta| + |\delta - 1|.
\]

Then use orthogonality to estimate the 2nd and 3rd terms:
\[
\frac{\langle P_{\leq N} e^{i\theta}(\partial_\delta Q)_{\delta\delta}, (-\Delta) i Q_{\delta} \rangle}{\langle P_{\leq N} Q_{\delta}, (-\Delta) Q_{\delta} \rangle} = \frac{\langle P_{\leq N} (e^{i\theta}(\partial_\delta Q)_{\delta\delta} - \partial_\delta Q_{\delta}) \cdot (-\Delta) i Q_{\delta} \rangle}{\langle P_{\leq N} Q_{\delta}, (-\Delta) Q_{\delta} \rangle} \\
\lesssim \| e^{i\theta}(\partial_\delta Q)_{\delta\delta} - \partial_\delta Q_{\delta} \|_{L^2(D)} \\
\lesssim |\theta| + \int_{\delta}^{\delta\delta} \| (\partial_\delta Q)_{\delta} \|_{L^2} \; ds \\
\lesssim |\theta| + |\delta - 1|^{\delta\delta} \delta^{-1},
\]

and
\[
\frac{\langle P_{V_0,\delta\delta \cap E_N} v, (-\Delta) i Q_{\delta} \rangle}{\langle P_{\leq N} Q_{\delta}, (-\Delta) Q_{\delta} \rangle} = \frac{\langle P_{V_0,\delta\delta \cap E_N} v, (-\Delta) (i Q_{\delta} - i e^{i\theta} Q_{\delta\delta}) \rangle}{\langle P_{\leq N} Q_{\delta}, (-\Delta) Q_{\delta} \rangle} \\
\lesssim \| v \|_{L^2} \cdot (|\theta| + |\delta - 1|).
\]

For the last term, applying Cauchy-Schwartz,
\[
\frac{\langle d \left( P_{V_0,\delta\delta \cap E_N} w \right) (\alpha, \beta), (-\Delta) i Q_{\delta} \rangle}{\langle P_{\leq N} Q_{\delta}, (-\Delta) Q_{\delta} \rangle} \lesssim \| d \left( P_{V_0,\delta\delta \cap E_N} w \right) (\alpha, \beta) \|_{L^2(D)} \\
\lesssim (|\alpha| + |\beta|) \| w \|_{L^2(D)}.
\]

Thus
\[
\tilde{\alpha} = \alpha + O \left( (|\theta| + |\delta - 1| \delta^{-1} + 1) \cdot (\| v \|_{L^2} + |\alpha|) + (|\alpha| + |\beta|) \| w \|_{L^2(D)} \right).
\]

Similarly,
\[
\tilde{\beta} = \beta + O \left( (|\theta| + |\delta - 1| \delta^{-1} + 1) \cdot (\| v \|_{L^2} + |\beta|) + (|\alpha| + |\beta|) \| w \|_{L^2(D)} \right).
\]

Finally,
\[
\tilde{v} - v = dG(\theta, \delta, w)v - \tilde{\alpha} P_{\leq N} Q_{\delta} - \tilde{\beta} P_{\leq N} \partial_\delta Q_{\delta} - v \\
= \alpha P_{\leq N} i e^{i\theta} Q_{\delta\delta} + \tilde{\beta} P_{\leq N} (-\Delta) Q_{\delta} + d \left( P_{V_0,\delta\delta \cap E_N} w \right) (\alpha, \beta) \\
+ \left( P_{V_0,\delta\delta \cap E_N} v - v \right) - \tilde{\alpha} P_{\leq N} Q_{\delta} - \tilde{\beta} P_{\leq N} \partial_\delta Q_{\delta} \\
= (\alpha - \tilde{\alpha}) P_{\leq N} i e^{i\theta} Q_{\delta\delta} + \alpha P_{\leq N} i e^{i\theta} Q_{\delta\delta} - i Q_{\delta}) + (\beta - \tilde{\beta}) \delta P_{\leq N} \partial_\delta Q_{\delta} \\
+ \beta \tilde{\delta} P_{\leq N} ((\partial_\delta Q)_{\delta\delta} - \partial_\delta Q_{\delta}) + d \left( P_{V_0,\delta\delta \cap E_N} w \right) (\alpha, \beta) + \left( P_{V_0,\delta\delta \cap E_N} v - v \right).
\]
Using orthogonality, rewrite the last term as

\[
P_{V_0,\delta} \cap E_N v - v = - \frac{\langle v, P_{\leq N} (-\Delta) e^{i\theta} Q_{\delta\delta} \rangle}{\langle P_{\leq N} (-\Delta) e^{i\theta} Q_{\delta\delta}, e^{i\theta} Q_{\delta\delta} \rangle} e^{i\theta} Q_{\delta\delta} \\
- \frac{\langle v, P_{\leq N} (-\Delta) \overline{\delta e^{i\theta} (\overline{\partial_\delta Q}_{\delta\delta})} \rangle}{\langle P_{\leq N} (-\Delta) \overline{\delta e^{i\theta} (\overline{\partial_\delta Q}_{\delta\delta})}, \overline{\delta e^{i\theta} (\overline{\partial_\delta Q}_{\delta\delta})} \rangle} \overline{\delta e^{i\theta} (\overline{\partial_\delta Q}_{\delta\delta})} \\
- \frac{\langle v, P_{\leq N} (-\Delta) (\delta e^{i\theta} (\partial_\delta Q)_{\delta\delta} - i Q_{\delta\delta}) \rangle}{\langle P_{\leq N} (-\Delta) (\delta e^{i\theta} (\partial_\delta Q)_{\delta\delta} - i Q_{\delta\delta}), \delta e^{i\theta} (\partial_\delta Q)_{\delta\delta} \rangle} \delta e^{i\theta} (\partial_\delta Q)_{\delta\delta}.
\]

Bound all these terms as before, using that \( \tilde{\delta} \ll 1 \) and the estimates for \( \tilde{\alpha}, \tilde{\beta} \),

\[
\| \tilde{v} - v \|_{L^2(\mathbb{D})} \lesssim |\alpha - \tilde{\alpha}| + |\alpha| (|\theta| + |\delta - 1|) + |\beta - \tilde{\beta}| + |\beta| |\theta| + |\delta - 1| |\delta^{-1} - 1| \\
+ (|\alpha| + |\beta|) \| w \|_{L^2} + \| v \|_{L^2} (|\theta| |\delta\delta| + |\delta - 1|) \\
\lesssim (|\alpha| + |\beta| + \| v \|_{L^2}) (|\theta| + |\delta - 1| |\delta^{-1} - 1| + \| w \|_{L^2}) \\
\lesssim (|\alpha| + |\beta| + \| v \|_{L^2}) (|\theta| + |\delta - 1| |\delta^{-1} - 1| + \| w \|_{L^2}).
\]

To sum up,

\[
\| dG^{-1}(0,1,0) \circ dG(\theta, \delta, 0)v - v \|_{L^2(\mathbb{D})} \leq |\tilde{\alpha} - \alpha| + |\tilde{\beta} - \beta| + \| \tilde{v} - v \|_{L^2} \\
\lesssim \| v \|_{L^2} (|\theta| + |\delta - 1| |\delta^{-1} - 1| + \| w \|_{L^2}).
\]

\[\square\]

**Proof of Theorem 5.12.** Recall that we assume, without loss of generality, \( \tilde{\theta} = 0 \). We apply the Inverse Function Theorem 5.8 to \( G \). By Lemma 5.10,

\[
\kappa := \| dG^{-1}(0,1,0) \circ dG(\theta, \delta, w) - Id \|_{op} \leq C_0 (|\theta| + |\delta - 1| + \| w \|_{L^2}).
\]

Define

\[
W = \left\{ (\theta, \delta, w) : |\theta| + |\delta - 1| + \| w \|_{L^2} < \min \left\{ 8\varepsilon \| dG^{-1}(0,1,0) \|, \frac{C_0^{-1}}{2} \right\} \right\}.
\]

Then, for \( (\theta, \delta, w) \in W \), \( \kappa \leq \frac{1}{2} \). Since \( \frac{1}{\| dG^{-1}(0,1,0) \|} \cdot 4\varepsilon \| dG^{-1}(0,1,0) \| \leq 2\varepsilon \),

\[
\left\{ u \in L^2_{rad}(\mathbb{D}) \cap E_N : \| u - P_{\leq N} Q_{\delta} \|_{L^2} < \varepsilon \right\} \subset G(W).
\]

\[\square\]

**Remark 5.11.** If we consider general \( \tilde{\theta} \), in the definition of \( W \), \( |\theta| \) has to be replaced by \( |\theta - \tilde{\theta}| \). Moreover, from the definition of \( W \), for any decomposition \( P_{\leq N} u = P_{\leq N} \left( e^{i\theta} Q_{\delta} \right) + v \), we have \( \| v \|_{L^2} \leq C\varepsilon \). The constant \( C \), by Lemma 5.9, is independent of \( \delta \).
6 Quadratic part: a spectral analysis

6.1 Reduction to quadratic form

In this section, we estimate the quadratic part:

\[ \int_{V_{0,\delta}} e^{(1+\eta)B_\delta(v)} d\mu_{V_{0,\delta}}(v), \]

where

\[ B_\delta(v) = \delta^{-2} \langle Q_\delta, v \rangle + \left\langle Q_\delta^2, \frac{1}{2} v^2 + (1 + \eta)|v|^2 \right\rangle. \]

The strategy is to compare \( B_\delta(v) \) with a simpler quadratic form. We illustrate our intuition on \( H^1(\mathbb{R}^2) \). Using the notation in Remark 5.1, recall that \( Q_\delta \) is a minimizer of \( H_{\mathbb{R}^2} \) with constraint \( M_{\mathbb{R}^2}(u) = \frac{1}{2} \|Q\|^2_{L^2(\mathbb{R}^2)} \). A second derivative test yields

**Lemma 6.1.** For any \( w \in H^1(\mathbb{R}^2) \) with \( \langle w, Q_\delta \rangle = 0 \),

\[ \frac{1}{2} \langle -\Delta w, w \rangle - \left\langle Q_\delta^2, \frac{1}{2} w^2 + |w|^2 \right\rangle + \frac{\delta^{-2}}{2} \langle w, w \rangle \geq 0. \]  \hspace{1cm} (34)

**Proof.** For any \( w \) with \( \langle w, Q_\delta \rangle = 0 \), define a path \( u(t) = \frac{\|Q\|_2}{\|Q_\delta + tw\|_2} (Q_\delta + tw) \). Then \( u(t) \) lies in the constraint set \( \{ M_{\mathbb{R}^2}(u) = \frac{1}{2} \|Q\|^2_{L^2(\mathbb{R}^2)} \} \), and \( u(0) = Q_\delta, \ u_t(0) = w \). Thus, \( H_{\mathbb{R}^2}(u(t)) \) reaches its minimum at \( t = 0 \). The second derivative test yields

\[ 0 \leq \left. \frac{d^2}{dt^2} H_{\mathbb{R}^2}(u(t)) \right|_{t=0} = \langle d^2 H_{\mathbb{R}^2}(Q_\delta) w, w \rangle + \langle dH_{\mathbb{R}^2}(Q_\delta), u_{tt}(0) \rangle. \]  \hspace{1cm} (35)

Since \( M(u(t)) \) is constant, we have

\[ 0 = \left. \frac{d^2}{dt^2} M_{\mathbb{R}^2}(u(t)) \right|_{t=0} = \langle d^2 M_{\mathbb{R}^2}(Q_\delta) w, w \rangle + \langle dM_{\mathbb{R}^2}(Q_\delta), u_{tt}(0) \rangle. \]

Recall that the Lagrange multiplier method gives \( dH_{\mathbb{R}^2}(Q_\delta) - \lambda dM_{\mathbb{R}^2}(Q_\delta) = 0 \). Thus,

\[ \langle dH_{\mathbb{R}^2}(Q_\delta), u_{tt}(0) \rangle = \lambda \langle dM_{\mathbb{R}^2}(Q_\delta), u_{tt}(0) \rangle = -\lambda \langle d^2 M_{\mathbb{R}^2}(Q_\delta) w, w \rangle. \]

The second derivative test (35) now reads as

\[ d^2 H_{\mathbb{R}^2}(Q_\delta) - \lambda d^2 M_{\mathbb{R}^2}(Q_\delta) \geq 0. \]

We get the desired inequality by noting

\[ \langle d^2 H_{\mathbb{R}^2}(Q_\delta) w, w \rangle = \langle -\Delta w, w \rangle - \langle Q_\delta^2, w^2 + 2|w|^2 \rangle, \]

\[ \langle d^2 M_{\mathbb{R}^2}(Q_\delta) w, w \rangle = \langle w, w \rangle, \]

and \( \lambda = -\delta^{-2} \) (ref. Remark 5.1).
The first term \( \frac{1}{2} \langle -\Delta w, w \rangle = \frac{1}{2} \int |\nabla w|^2 \) is contained implicitly in the formal density of \( \mu_{V_0, \delta} \), \( e^{-\frac{1}{2} \int |\nabla w|^2} dw \) (despite the additional constraint \( \langle w, Q_\delta \rangle = 0 \)). The second derivative test suggests us to compare \( B_\delta(v) \) with

\[
\left\langle Q^2_\delta, \frac{1}{2} w^2 + |w|^2 \right\rangle - \frac{\delta^{-2}}{2} \langle w, w \rangle. \tag{36}
\]

There are two difficulties. First, the linear term \(-\frac{\delta^{-2}}{2} \langle Q_\delta, v \rangle\) cannot be bounded separately. Indeed, \( \langle Q_\delta, v \rangle \) is a real valued Gaussian with mean 0 and variance \( \langle -\Delta^{-1} Q_\delta, Q_\delta \rangle \sim \delta^2 \). Therefore

\[
\mathbb{E}_{\mu_{V_0, \delta}} \left[ e^{-\delta^{-2} \langle Q_\delta, v \rangle} \right] \gtrsim e^{c\delta^{-2}},
\]

which is too large. This issue will be addressed in Lemma 6.2. Second, the derivative test (34) holds with the additional constraint \( \langle w, Q_\delta \rangle = 0 \), to overcome this, we need to single out the \( Q_\delta \) direction. Define

\[
e := \frac{P_{V_0, \delta} (-\Delta^{-1} Q_\delta)}{\|P_{V_0, \delta} (-\Delta^{-1} Q_\delta)\|_{H^1(\mathbb{D})}} = \frac{-\Delta^{-1} Q_\delta}{\|\Delta^{-1} Q_\delta\|_{H^1(\mathbb{D})}} + O(e^{-c\delta^{-1}}).
\]

Recall \( \Delta^{-1} \) is the solution map of the Poisson equation \( \Delta u = f \), \( f \in L^2_{rad}(\mathbb{D}) \), with Dirichlet boundary condition. Accordingly, let \( W_\delta \) be the subspace \( H^1 \)-orthogonal to \( \text{span}\{e\} \). We have

\[
V_{0, \delta} = W_\delta \oplus H^1, \text{ span}\{e\}.
\]

Write \( v = ge + w \) for some \( g \in \mathbb{R} \), \( w \in W_\delta \). We obtain

\[
B_\delta(v) = \delta^{-2} \langle Q_\delta, ge \rangle + \left\langle Q^2_\delta, \left( \frac{3}{2} + \eta \right)(ge + \Re w)^2 + \left( \frac{1}{2} + \eta \right)(\Im w)^2 \right\rangle
\leq (1 + 3\eta) \left\langle Q^2_\delta, \left( \frac{3}{2}(\Re w)^2 + \left( \frac{1}{2}(\Im w)^2 \right) + \delta^{-2} \langle Q_\delta, ge \rangle + C_\eta \langle Q^2_\delta, (ge)^2 \rangle. \right.
\]

Compare with the reference form (36), we expect the following.

**Lemma 6.2.** Given \( \eta > 0 \), there exists \( \epsilon^* = \epsilon^*(\eta) \) small enough, such that for all \( \epsilon \leq \epsilon^* \),

\[
\delta^{-2} \langle Q_\delta, ge \rangle + C_\eta \langle Q^2_\delta, (ge)^2 \rangle \leq -\left( 1 - \eta \right) \frac{\delta^{-2}}{2} \langle w, w \rangle + O(e^{-c\delta^{-1}}).
\]

**Proof.** The proof is almost identical to Lemma 6.13 in [15], we rephrase it here to be self-contained.

Since \( \|Q_\delta\|_{L^\infty} \sim \delta^{-1} \), it is equivalent to show

\[
\langle Q_\delta, ge \rangle + C_\eta \|ge\|_2^2 + \left( \frac{1 - \eta}{2} \right) \|w\|_2^2 \lesssim e^{-c\delta^{-1}}. \tag{37}
\]
By the normal bundle decomposition\(^{10}\)
\[
\|Q\|_{L^2(\mathbb{R}^2)}^2 \geq \|Q_\delta + v\|_2^2 = \|Q_\delta\|_2^2 + 2 \langle Q_\delta, v \rangle + \|v\|_2^2
\]
\[
= \|Q_\delta\|_2^2 + 2 \langle Q_\delta, ge \rangle + \|v\|_2^2
\]
\[
= \|Q_\delta\|_2^2 + 2 \langle Q_\delta, ge \rangle + \|w\|_2^2 + 2 \langle ge, w \rangle + \|ge\|_2^2.
\]
Since \(\|Q\|_{L^2(\mathbb{R}^2)}^2 - \|Q_\delta\|_2^2 = O(e^{-\delta^{-1}})\),
\[
2 \langle Q_\delta, ge \rangle + \|v\|_2^2 = O(e^{-\delta^{-1}}).
\] (38)

By the definition of \(e\), we have
\[
\|e\|_2 \sim \|Q_\delta\|_{H^{-1}} \sim \delta \sim \langle Q_\delta, e \rangle.
\] (39)

When \(g \geq 0\), by (38),
\[
\|ge\|_2 \sim \langle Q_\delta, ge \rangle = O(e^{-\delta^{-1}}),
\]
\[
\|w\|_2 \leq \|v\|_2 + \|ge\|_2 \lesssim e^{-\delta^{-1}},
\]
which implies (37).

When \(g < 0\), according to Remark 5.11, \(\|v\|_2 \leq \epsilon. \) (38) implies
\[
\|ge\|_2 \sim |\langle Q_\delta, ge \rangle| = \frac{1}{2} \|v\|_2^2 + O(e^{-\delta^{-1}}) \leq \epsilon,
\]
and
\[
\|v\|_2^2 \leq 2 |\langle Q_\delta, ge \rangle| + O(e^{-\delta^{-1}}).
\]
On the other hand,
\[
\|v\|_2^2 = \|w\|_2^2 + 2 \langle ge, w \rangle + \|ge\|_2^2 \geq (1 - \eta/2) \|w\|_2^2 - K_\eta \|ge\|_2^2.
\]
Thus,
\[
2 |\langle Q_\delta, ge \rangle| \geq \|v\|_2^2 - O(e^{-\delta^{-1}}) \geq (1 - \eta/2) \|w\|_2^2 - K_\eta \|ge\|_2^2 - O(e^{-\delta^{-1}}),
\]
that is
\[
(1 - \eta/2) \|w\|_2^2 + 3C_\eta \|ge\|_2^2 \leq 2 |\langle Q_\delta, ge \rangle| + (K_\eta + 3C_\eta) \|ge\|_2^2 + O(e^{-\delta^{-1}})
\]
\[
\leq (2 + O(\epsilon)) |\langle Q_\delta, ge \rangle| + O(e^{-\delta^{-1}}).
\]
Choose \(\epsilon = \epsilon(\eta)\) small enough, we can multiply the inequality by a suitable constant less than 1, such that
\[
(1 - \eta) \|w\|_2^2 + 2C_\eta \|ge\|_2^2 \leq 2 |\langle Q_\delta, ge \rangle| + O(e^{-\delta^{-1}}).
\]
which gives (37). \(\square\)

\(^{10}\)In this proof, \(\|\cdot\|_{2} \) is short for \(\|\cdot\|_{L^2(\mathbb{D})} \).
6.2 Spectral analysis

So far, we have shown:

\[ B_\delta(v) \leq (1 + 3\eta) \left( \frac{Q_\delta^2}{2} + \frac{1}{2}(\Im w)^2 \right) - (1 - \eta) \frac{\delta^{-2}}{2} \langle w, w \rangle + O(e^{-c\delta^{-1}}). \]

In this section, \( H^1_{rad,0}(\mathbb{D}) \) is a Hilbert space equipped with inner product \( \langle \cdot, \cdot \rangle_{H^1} = \langle (-\Delta)^{1/2}, \cdot \rangle \). Denote \( \Re H^1_{rad,0} \) (resp. \( \Im H^1_{rad,0} \)) the subspace of real (resp. pure-imaginary) valued functions in \( H^1_{rad,0} \). Accordingly, define the real and imaginary part of \( W_\delta \cap H^1_{rad,0}(\mathbb{D}) \) as

\[
W_R := W_\delta(\delta) = \left\{ w \in \Re H^1_{rad,0}(\mathbb{D}) : \langle w, (-\Delta)\partial_\delta Q_\delta \rangle = \langle w, Q_\delta \rangle = 0 \right\},
\]

\[
W_I := W_\delta(\delta) = \left\{ w \in \Im H^1_{rad,0}(\mathbb{D}) : \langle w, (-\Delta)Q_\delta \rangle = 0 \right\}.
\]

Let \( P_{W_R} \) (resp. \( P_{W_I} \)) be the \( \dot{H}^1 \) orthogonal projection on \( W_R \) (resp. \( W_I \)). Then define operators on \( H^1_{rad,0}(\mathbb{D}) \) as

\[
A_1 = P_{W_R}(-\Delta)^{-1} \left( - (1 + 5\eta) \frac{3}{2} Q_\delta^2 + \frac{\delta^{-2}}{2} \right) P_{W_R},
\]

\[
A_2 = P_{W_I}(-\Delta)^{-1} \left( - (1 + 5\eta) \frac{1}{2} Q_\delta^2 + \frac{\delta^{-2}}{2} \right) P_{W_I}.
\]

Clearly,

\[
B_\delta(v) \leq - (1 - \eta) \langle A_1 \Re w, \Re w \rangle_{H^1} - (1 - \eta) \langle A_2 \Im w, \Im w \rangle_{H^1} + O(e^{-c\delta^{-1}}).
\]

At the end of this section, we will show

**Proposition 6.3.** For \( \eta > 0 \) small enough,

\[
\int_{W_\delta} \exp \left( - (1 - \eta) \langle A_1 \Re w, \Re w \rangle_{H^1} - (1 - \eta) \langle A_2 \Im w, \Im w \rangle_{H^1} \right) d\mu_{W_\delta}(w) \lesssim e^{-c\delta^{-1}}.
\]

The implicit constant is independent of \( \delta \).

By definition, \( A_1, A_2 \) are symmetric. Since \( \|Q_\delta\|_\infty \sim \delta^{-1}, \langle A_1 w, w \rangle_{H^1} \lesssim \delta^{-2} \|w\|_{L^2(\mathbb{D})} \). The Rellich–Kondrachov theorem then implies that \( A_i \) are compact on \( H^1_{rad,0}(\mathbb{D}) \). Since the range of \( A_1 \) (resp. \( A_2 \)) is in \( W_R \) (resp. \( W_I \)), it is also a compact operator on \( W_R \) (resp. \( W_I \)). In particular, its spectrum consists of eigenvalues, with the only possible essential spectrum at 0. Let \( h_n \in W_R \) be a normalized eigenfunction of \( A_1 \) corresponding to eigenvalue \( \lambda_n \). Then \( \{h_n\} \) forms an orthonormal basis of \( \ker A_1 \). By construction of free Gibbs measure,

\[
\int e^{-(1-\eta)(A_1 w, w)_{H^1}} d\mu_{W_R}(w) = \prod_n \mathbb{E} \left[ e^{-(1-\eta)\lambda_n g^2} \right] = \prod_n (1 + 2(1 - \eta)\lambda_n)^{-1/2},
\]

where \( g \) is a real-valued standard Gaussian. In order to get a finite product, we first need to check \( 1 + 2(1 - \eta)\lambda_n > 0 \).
Proposition 6.4. The smallest eigenvalue of $A_1$ (resp. $A_2$) is greater than $-\frac{1}{2} + \epsilon_0$, where $\epsilon_0 \to 0$ as $\delta \to 0$.

Proof. Assume by contradiction that there exists a fixed $\epsilon_0 > 0$, and eigenfunctions $w_n \in W_R(\delta_n)$, such that $\delta_n \to 0$ but

$$\langle A_1 w_n, w_n \rangle_{H^1} \leq \left(-\frac{1}{2} + \epsilon_0\right) \langle w_n, w_n \rangle_{H^1}.$$ 

Then

$$\left\langle \left(-(1 + 5\eta)\frac{3}{2}Q_{\delta_n}^2 + \frac{\delta_n^{-2}}{2} + \left(\frac{1}{2} - \epsilon_0\right)(-\Delta)\right) w_n, w_n \right\rangle \leq 0.$$ 

Set $\tilde{w}_n = \delta_n w_n(\delta_n \cdot)$ and extend by 0 outside $\delta^{-1}D$, then the inequality above implies

$$\left\langle \left(-(1 + 5\eta)\frac{3}{2} \left(Q - Q(\delta_n^{-1})\right)^2 + \frac{1}{2} + \left(\frac{1}{2} - \epsilon_0\right)(-\Delta)\right) \tilde{w}_n, \tilde{w}_n \right\rangle_{L^2(\mathbb{R}^2)} \leq 0. \quad (41)$$

Define

$$W_{\mathbb{R}^2,R}(\delta) := \left\{ u \in \mathfrak{R}L^2_{rad} \cap H^1(\mathbb{R}^2) : \left\langle 1_{\delta^{-1}D} \cdot u, Q - Q(\delta_n^{-1}) \right\rangle_{L^2(\mathbb{R}^2)} = \left\langle 1_{\delta^{-1}D} \cdot u, (-\Delta)\partial_\delta Q \right\rangle_{L^2(\mathbb{R}^2)} = 0 \right\},$$

and

$$B_n := P_{W_{\mathbb{R}^2,R}(\delta_n)} \left(-(1 + 5\eta)\frac{3}{2} \left(Q - Q(\delta_n^{-1})\right)^2 1_{\delta^{-1}D} + \frac{1}{2} + \left(\frac{1}{2} - \epsilon_0\right)(-\Delta)\right) P_{W_{\mathbb{R}^2,R}(\delta_n)},$$

as an operator on $L^2_{rad}(\mathbb{R}^2)$. Since $(Q - Q(\delta_n^{-1}))^2 1_{\delta^{-1}D}$ is bounded, the Schrödinger operator inside the parenthesis has essential spectrum on $[\frac{1}{2}, \infty)$. Note that $B_n$ is a restriction of this Schrödinger operator on a subspace, therefore the essential spectrum of $B_n$ is contained in $[\frac{1}{2}, \infty)$. Moreover,

$$\langle B_n w, w \rangle_{L^2(\mathbb{R}^2)} \geq -(1 + 5\eta)\frac{3}{2} \left\| Q - Q(\delta_n^{-1}) \right\|_{L^\infty}^2 \langle w, w \rangle_{L^2(\mathbb{R}^2)}.$$ 

$B_n$ is semi-bounded. Thus, we can let $\lambda_n$ be infimum of the spectrum. $\lambda_n$ is bounded from below. Using (11) and $\tilde{w}_n \in W_{\mathbb{R}^2,R}(\delta_n)$, we see that $\lambda_n \leq 0$, therefore it is an eigenvalue. Up to a subsequence, we may assume $\lambda_n \to \lambda \leq 0$. Define the limit operator

$$B_R(\epsilon_0, \eta) := P_{W_{\mathbb{R}^2,R}} \left(-(1 + 5\eta)\frac{3}{2}Q^2 + \frac{1}{2} + \left(\frac{1}{2} - \epsilon_0\right)(-\Delta)\right) P_{W_{\mathbb{R}^2,R}},$$

with

$$W_{\mathbb{R}^2,R} := \left\{ u \in \mathfrak{R}L^2_{rad} \cap H^1(\mathbb{R}^2) : \langle u, Q \rangle_{L^2(\mathbb{R}^2)} = \langle u, (-\Delta)\partial_\delta Q \rangle_{L^2(\mathbb{R}^2)} = 0 \right\}.$$ 

We are going to derive a contradiction by showing that $B_R(\epsilon_0, \eta) \geq \theta > 0$. By continuity, it is sufficient to prove this for $\epsilon_0 = \eta = 0$. 

29
The same argument for $A_2$ would require us to show
\[ B_I(0, 0) := P_{W_{\mathbb{R}^2,I}} \left( -\frac{1}{2}Q^2 + \frac{1}{2} + \frac{1}{2}(-\Delta) \right) P_{W_{\mathbb{R}^2,I}} \geq \theta > 0, \]
where
\[ W_{\mathbb{R}^2,I} := \left\{ u \in \mathcal{L}^2_{\text{rad}}(\mathbb{R}^2) : \langle u, (-\Delta)Q \rangle_{L^2(\mathbb{R}^2)} = 0 \right\}. \]
Using Lemma 6.1 with $\delta = 1$, for any real valued $w$, we have
\[ \left\langle \left( \frac{1}{2}(1 - \Delta) - \frac{3}{2}Q^2 \right) w, w \right\rangle_{L^2(\mathbb{R}^2)} \geq 0, \quad \text{for } \langle w, Q \rangle_{L^2(\mathbb{R}^2)} = 0 \]
\[ \left\langle \left( \frac{1}{2}(1 - \Delta) - \frac{1}{2}Q^2 \right) w, w \right\rangle_{L^2(\mathbb{R}^2)} \geq 0. \]
Then $B_R(0, 0)$ (resp. $B_I(0, 0)$) is the restriction of $T_R$ (resp. $T_I$) on $W_{\mathbb{R}^2,R}$ (resp. $W_{\mathbb{R}^2,I}$). We need to show $T_R$ (resp. $T_I$) is strictly positive on $W_{\mathbb{R}^2,R}$ (resp. $W_{\mathbb{R}^2,I}$). By the min-max principle, the spectrum of $T_R$ (resp. $T_I$) is contained in $[0, \infty)$.

Since $Q$ is a Schwartz function, $T_I$, as an operator on $L^2_{\text{rad}}(\mathbb{R}^2)$, has the same essential spectrum as $1 - \Delta$, that is, $[1, \infty)$.

The ground state equation $-\Delta Q - Q^3 + Q = 0$ implies $T_I Q = 0$. Using the radial variable $r$, the linear ODE
\[ T_I u = \frac{1}{2}(-\partial_r^2 - \frac{1}{r}\partial_r + 1 - Q^2)u = 0 \]
has Wronskian $W = \frac{1}{r}$. Thus, if there is any other linearly independent solution $u$, we must have, as $r \to 0$,
\[ u'(r) = \frac{1}{rQ(r)} + Q'(r)Q(r)u(r) \sim \frac{1}{r} + o(1) u(r) \]
which implies
\[ \|u\|_{H^1(\mathbb{R}^2)}^2 \gtrsim \int_0^1 \frac{1}{r} - o(1) \|u\|_{L^2(\mathbb{R}^2)}^2 = \infty. \]
So the eigenspace for the minimal eigenvalue $\lambda_1 = 0$ is $\text{span}\{Q\}$. We conclude that, on $W_{1,I}$, $T_I \geq \theta > 0$. Indeed, the normal component of $Q$ to $(-\Delta)Q$, denoted as $Q_\perp$, has strictly smaller norm: $\|Q_\perp\|_2 < \|Q\|_2$. Thus, for any $u \in W_{\mathbb{R}^2,I}$, the component on $Q$ direction has size
\[ \frac{\langle u, Q \rangle_{L^2(\mathbb{R}^2)}}{\|Q\|_2} = \frac{\langle u, Q_\perp \rangle_{L^2(\mathbb{R}^2)}}{\|Q\|_2} \leq \|u\|_2 \frac{\|Q_\perp\|_2}{\|Q\|_2} = c_0 \|u\|_2, \quad c_0 \in (0, 1). \]
Hence, write $u = P_Q u + w$, where $\langle w, Q \rangle_{L^2(\mathbb{R}^2)} = 0$, and let $\lambda_2$ be the second smallest point of the spectrum. We have
\[ \langle T_I u, u \rangle_{L^2(\mathbb{R}^2)} = \langle T_I w, w \rangle_{L^2(\mathbb{R}^2)} \geq \lambda_2 \|w\|_2^2 \geq \lambda_2(1 - c_0^2) \|u\|_2^2. \]
By the similar argument, it suffices to show that $P_Q T_R P_Q u = 0$ for $u \perp_{L^2(\mathbb{R}^2)} W_{\mathbb{R}^2,R}$.  

Lemma 6.5. \( T_R \), as an operator on \( L^2_{rad}(\mathbb{R}^2) \) has trivial kernel: \( \ker T_R = \{0\} \).

Proof. It is known (see for example [9]) that the corresponding \( L^2(\mathbb{R}^2) \) operator \( L^+ := -\Delta + 1 - 3Q^2 \) has \( \ker L^+ = \text{span}\{\partial_{x_1} Q, \partial_{x_2} Q\} \). Clearly \( \ker T_R \subset \ker L^+ \). If there is some \( u \in \ker T_R \setminus \{0\} \), then by normalizing, we may assume \( u = \partial_{b} Q \), for some unit vector \( v \in \mathbb{R}^2 \). Moreover, since \( Q \) is radial, we may further rotate \( u \) to get \( u = \partial_{x_1} Q \). Since \( u \) is radial, \( u(x, \cdot) \) would be even. But \( Q(x, \cdot) \) is also even and therefore its derivative, \( u(x, \cdot) = \partial_{x_1} Q(x, \cdot) \), has to be odd. This forces \( u \equiv 0 \), a contradiction.  

Thus \( T_R u = 0 \), if and only if \( u = 0 \). By differentiating (17) with respect to \( \delta \), we obtain \( T_R(\partial_{b} Q) = Q \). Combining with Lemma 6.5 yields that \( P_Q T_R P_Q u = 0 \) if and only if \( u \in \text{span}\{Q, \partial_{b} Q\} \), which is orthogonal to \( W_{R^2, R} \).

Next, we will show the eigenvalues in (40) is summable.

Proposition 6.6. List the non-zero eigenvalues of \( A_1 \) (or \( A_2 \)) as \( \lambda_-^1 \leq \ldots \leq 0 < \ldots \leq \lambda_-^n \leq \lambda_+^n \). Then

\[
\lambda_n^+ \gtrsim \frac{\delta^{-2}}{n^2}, \quad \lambda_n^- \gtrsim \frac{1}{n^2}.
\]

Proof. We only prove this result for \( A_1 \), same argument applies for \( A_2 \) case.

Define \( \tilde{T}_R := (-\Delta)^{-1} \left[ -(1 + 5\eta)\frac{3}{2} Q^2 + \frac{\delta^{-2}}{2} \right] \) as an operator on \( H^1_{rad,0}(\mathbb{D}) \). Then \( A_1 = P_{W_R} \tilde{T}_R P_{W_R} \) is a finite rank perturbation of \( \tilde{T}_R \), thus the eigenvalues of \( A_1 \) are interlaced with eigenvalues of \( \tilde{T}_R \). It is sufficient to get the same estimates (42) for \( B_R \), whose non-zero eigenvalues we list as \( \tilde{\lambda}_1^- \leq \ldots < 0 < \ldots \leq \tilde{\lambda}_1^+ \).

The strategy of the proof is to compare \( B_R \) simpler Schrödinger operators. We investigate the positive and negative eigenvalues separately.

Positive eigenvalues: By Lemma 2.3, we can choose some constant \( a > 0 \) large enough such that \( Q(r) \leq 1/4 \) for \( r \geq a \). Let

\[
S_+(r) = \begin{cases} 
-\infty, & r \leq a\delta \\
\frac{1}{4}\delta^{-2}, & r > a\delta
\end{cases}
\]

Then \( S_+ \leq -(1 + 5\eta)\frac{3}{2} Q^2 + \frac{\delta^{-2}}{2} \). Define \( (-\Delta)^{-1} S_+ \) as an operator on the subspace \( X_+ := \{u \in H^1_{rad,0}(\mathbb{D}) : u(r) = 0 \text{ for } r \in [0, a\delta]\} \).
Clearly, $(-\Delta)^{-1}S_+$ is a compact operator, and we list its positive eigenvalues as $0 < \cdots \leq \mu_2 \leq \mu_1$. By the min-max principle,

$$-\bar{\lambda}^+_n = \min_{\dim L=n} \max_{w \in L} \left\{ \frac{\langle -T_RW, w \rangle_{H^1}}{\|w\|_{H^1}^2} \right\}$$

$$\leq \min_{\dim L=n} \max_{w \in L} \left\{ \frac{\langle -S_+w, w \rangle}{\|w\|_{H^1}^2} : w \in X_+ \right\} \cup \{\infty : w \notin X_+\}$$

$$= \min_{\dim L=n, L \subseteq X_+} \max_{w \in L} \left\{ \frac{\langle -S_+w, w \rangle}{\|w\|_{H^1}^2} : w \in X_+ \right\} = -\mu_n.$$

The eigenvalue equation $(-\Delta)^{-1}S_+f = \mu f$ can be written as

$$\left( \partial_r^2 + \frac{1}{r} \partial_r \right)f = -\frac{1}{4} \frac{\delta^{-2}}{\mu} f =: r \in [a\delta, 1],$$

with boundary value condition $f(a\delta) = f(1) = 0$.

As discussed in Section 2.2, equation $\left( \partial_r^2 + \frac{1}{r} \partial_r \right)u = -w^2u$ has a general solution of the form $c_1J_0(wr) + c_2Y_0(wr)$. Denote $B = \sqrt{\left| \frac{1}{4} \frac{\delta^{-2}}{\mu} \right|}$. Since $\mu > 0$, $B \in (0, \infty)$. The boundary value conditions yield

$$0 = f(1) = c_1J_0(B) + c_2Y_0(B),$$

$$0 = f(a\delta) = c_1J_0(Ba\delta) + c_2Y_0(Ba\delta).$$

The equations above yields the cross-product relation

$$J_0(B)Y_0(Ba\delta) - Y_0(B)J_0(Ba\delta) = 0. \tag{43}$$

Let $B_n$ denotes the $n$th root of (43). It is known (see [6]) that $B_n \sim \frac{n\pi}{1-a\delta}$.

Since $B \sim \frac{\delta^{-1}}{\sqrt{\mu}}$, we have $\mu_n \sim \frac{\delta^{-2}(1-a\delta)^2}{n^2}$. Hence

$$\bar{\lambda}^+_n \gtrsim \frac{\delta^{-2}}{n^2}.$$

**Negative eigenvalues:** Using the same constant $a$ as in positive eigenvalue part, let

$$S_-(r) = \begin{cases} -3\delta^{-2}, & r \leq 5\delta \\ -\infty, & r > 5\delta \end{cases}.$$ 

By Lemma 2.3, $\|Q\|_{L^\infty} \leq \sqrt{2}$. Thu, $S_- \leq -(1 + 5\eta)\frac{3}{2}Q^2_0 + \frac{\delta^2}{2}$. Then the operator $(-\Delta)^{-1}S_-$ is defined on subspace

$$X_- := \left\{ u \in H^1_{rad}(B) : u(r) = 0 \text{ for } r \in [a\delta, 1] \right\}.$$ 

\footnote{For more results on zeros of cross-product of Bessel function, see Section 10.2 in [7] and references wherein.}
List the negative eigenvalues of \((-\Delta)^{-1} S_\nu\) as \(\nu_1 \leq \nu_2 \leq \cdots < 0\). The min-max principle yields \(\lambda_n^- \geq \nu_n\).

The eigenvalue equation \((-\Delta)^{-1} S_\nu f = \nu f\) can be written as
\[
\left(\partial^2_r + \frac{1}{r} \partial_r\right) f = \frac{3\delta^{-2}}{\nu} f, \quad r \leq a\delta,
\]
with boundary condition \(f(a\delta) = 0\). Since \(f \in H^1_{\text{rad},0}(\mathbb{D})\), in radial variable \(r\), we impose initial conditions \(f'(0) = 0\) and normalized with \(f(0) = 1\).

Denote \(A = \sqrt{|\frac{3\delta^{-2}}{\nu}|}\). Since \(\nu < 0\), \(A \in (0, \infty)\), the initial condition \(f'(0) = 0\), \(f(0) = 1\) implies
\[
f(r) = J_0(Ar), \quad r \leq a\delta.
\]
Using \(f(a\delta) = 0\), we obtain
\[
f(a\delta) = J_0(Aa\delta) = 0.
\]
Let \(A_n\) be the \(n\)th root of (44). By Theorem 7.2.1 in [1], \(A_n \sim \frac{\pi n}{a\delta}\). Since \(A \sim \frac{\delta^{-1}\sqrt{|\nu|}}{a\delta}\), we have \(\nu \sim -\frac{1}{n^2}\). Hence:
\[
\lambda_n^- \lesssim -\frac{1}{n^2}.
\]

**Proof of Proposition 6.3.** For \(j = 1, 2\), list the eigenvalues of \(A_j\) as \(\lambda_{j,1}^- \leq \lambda_{j,2}^- \leq \cdots < 0 \leq \lambda_{j,1}^+ \leq \lambda_{j,2}^+ \cdots\), and denote \(h_{j,n}^\pm\) as the \(\dot{H}\)-normalized eigenfunction corresponding to eigenvalue \(\lambda_{j,n}^\pm\). Since \(W_\delta \cap H^1_{\text{rad},0}(\mathbb{D}) = W_R \oplus W_I\) is a dense subspace, \(\{h_{1,n}^\pm, ih_{2,n}^\pm\}\) forms an orthogonal basis of \(W_\delta\). The free Gaussian measure \(\mu_{W_\delta}\) is then the law of random function
\[
w = \sum_{\nu \in \{+,-\}} \sum_{j \in \{1,2\}} \sum_n g_{j,n}^\nu h_{j,n}^\nu,
\]
where \(g_{j,n}^\nu\) are independent real-valued standard Gaussian. Therefore,
\[
\int_{W_\delta} \exp \left(- (1-\eta) \langle A_1 \Re w, \Re w \rangle_{H^1_R} - (1-\eta) \langle A_2 \Im w, \Im w \rangle_{H^1_I} \right) d\mu_{W_\delta}(w)
\]
\[
= \prod_{\nu \in \{+,-\}} \prod_{j \in \{1,2\}} \prod_n \mathbb{E} \left\{ e^{-(1-\eta)\lambda_{j,n}^\nu (g_{j,n}^\nu)^2} \right\}
\]
\[
= \prod_{\nu \in \{+,-\}} \prod_{j \in \{1,2\}} \prod_n (1 + 2(1-\eta)\lambda_{j,n}^\nu)^{-\frac{1}{2}}.
\]

33
Using Proposition 6.4 and Proposition 6.6 we get
\[
\ln \left( \prod_n (1 + 2(1 - \eta)\lambda_{j,n}^+) \right)^{-\frac{1}{2}} = \sum_n -\frac{1}{2} \ln \left( 1 + 2(1 - \eta)\lambda_{j,n}^+ \right)
\lesssim -(1 - \eta) \sum_n \lambda_{j,n}^+
\lesssim -(1 - \eta) \sum_{n \geq \delta^{-1}} \frac{\delta^{-2}}{n^2} \leq -c_\eta \delta^{-1},
\]
whilst
\[
\ln \left( \prod_n (1 + 2(1 - \eta)\lambda_{j,n}^-) \right)^{-\frac{1}{2}} \lesssim (1 - \eta) \sum_n \frac{1}{n^2} \leq C_\eta.
\]
Hence (45) is \( O_\eta(\epsilon^{-\delta^{-1}}) \).

7 Proof of the main result

We now summarize what has been proved to conclude Theorem 1.2. Corollary 4.2 shows
\[
\mathbb{E}_\mu \left[ e^{\frac{1}{4} \int_{\mathbb{R}} |u|^4, U_\epsilon(0, \delta^*) C} \right] < \infty.
\]
Applying Lemma 5.7 and Proposition 6.3 Proposition 5.5 yields
\[
\mathbb{E}_\mu \left[ e^{\frac{1}{4} \int_{\mathbb{R}} |u|^4, U_\epsilon(0, \delta^*)} \right] \lesssim \int_0^{\delta^*} e^{-c_\delta^{-1} \delta^{-5}} d\delta < \infty.
\]
In summary, we have proved normalizability of Gibbs measure \( \rho \).

References

[1] R. Beals and R. Wong. Special Functions: A Graduate Text. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010. isbn: 9781139490436.

[2] Jean Bourgain. “Invariant measures for the 2D-defocusing nonlinear Schrödinger equation”. In: Communications in Mathematical Physics 176.2 (1996), pp. 421–445.

[3] Jean Bourgain. “Periodic nonlinear Schrödinger equation and invariant measures”. In: Communications in Mathematical Physics 166.1 (1994), pp. 1–26.

[4] Jean Bourgain and Aynur Bulut. “Almost sure global well posedness for the radial nonlinear Schrödinger equation on the unit ball I: The 2D case”. In: Annales de l'Institut Henri Poincare Section (C) Non Linear Analysis 31.6 (2014), pp. 1267–1288.
[5] Jean Bourgain and Aynur Bulut. “Almost sure global well-posedness for the radial nonlinear Schrödinger equation on the unit ball II: the 3d case”. In: *Journal of the European Mathematical Society* 16.6 (2014), pp. 1289–1325.

[6] James Alan Cochran. “Remarks on the Zeros of Cross-Product Bessel Functions”. In: *Journal of the Society for Industrial and Applied Mathematics* 12.3 (1964), pp. 580–587. ISSN: 03684245.

[7] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.1.3 of 2021-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. URL: http://dlmf.nist.gov/.

[8] X. Fernique. “Regularité des trajectoires des fonctions aléatoires gaussiennes”. In: *Ecole d’Eté de Probabilités de Saint-Flour IV—1974*. Ed. by P. L. Hennequin. Berlin, Heidelberg: Springer Berlin Heidelberg, 1975, pp. 1–96. ISBN: 978-3-540-37600-2.

[9] R. Frank. Ground states of semi-linear PDEs. 2014. URL: http://www.math.caltech.edu/~rlfrank/ (visited on 04/10/2022).

[10] Taoufik Hmidi and Sahbi Keraani. “Blowup theory for the critical nonlinear Schrödinger equations revisited”. In: *International Mathematics Research Notices* 2005.46 (Jan. 2005), pp. 2815–2828. ISSN: 1073-7928.

[11] Joel L Lebowitz, Harvey A Rose, and Eugene R Speer. “Statistical mechanics of the nonlinear Schrödinger equation”. In: *Journal of statistical physics* 50.3 (1988), pp. 657–687.

[12] H. P. McKean. “Statistical mechanics of nonlinear wave equations (4): Cubic Schrödinger”. English (US). In: *Communications in Mathematical Physics* 168.3 (Apr. 1995), pp. 479–491. ISSN: 0010-3616.

[13] K. Nakanishi and W. Schlag. *Invariant Manifolds and Dispersive Hamiltonian Evolution Equations*. Zurich lectures in advanced mathematics. European Mathematical Society, 2011. ISBN: 9783037190951.

[14] Tadahiro Oh and Jeremy Quastel. “On invariant Gibbs measures conditioned on mass and momentum”. In: *J. Math. Soc. Japan* 65 (Dec. 2010).

[15] Tadahiro Oh, Philippe Sosoe, and Leonardo Tolomeo. “Optimal integrability threshold for Gibbs measures associated with focusing NLS on the torus”. In: *Inventiones mathematicae* (2021).

[16] Terence Tao. *Nonlinear dispersive equations*. Vol. 106. CBMS Regional Conference Series in Mathematics. Local and global analysis. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006, pp. xvi+373. ISBN: 0-8218-4143-2.

[17] Nikolay Tzvetkov. “Invariant measures for the defocusing nonlinear Schrödinger equation”. In: *Annales de l’Institut Fourier*. Vol. 58. 7. 2008, pp. 2543–2604.
[18] Nikolay Tzvetkov. “Invariant measures for the nonlinear Schrödinger equation on the disc”. In: Dynamics of Partial Differential Equations 3.2 (2006), pp. 111–160.

[19] Michael I Weinstein. “Nonlinear Schrödinger equations and sharp interpolation estimates”. In: Communications in Mathematical Physics 87.4 (1982), pp. 567–576.