A Modest Revision of the Standard Model

G. J. Stephenson, Jr. and T. Goldman

Dept. of Physics and Astronomy,
University of New Mexico,
Albuquerque, NM 87501

and

Theoretical Division, MS-B283,
Los Alamos National Laboratory,
Los Alamos, NM 87545

Abstract

With a modest revision of the mass sector of the Standard Model, the systematics of the fermion masses and mixings can be fully described and interpreted as providing information on matrix elements of physics beyond the Standard Model. A by-product is a reduction of the largest Higgs Yukawa fine structure constant by an order of magnitude. The extension to leptons provides for insight on the difference between quark mixing and lepton mixing as evidenced in neutrino oscillations. The large difference between the scale for up-quark and down-quark masses is not addressed.
I. INTRODUCTION

For several decades, the quantum numbers and corresponding gauge interactions that distinguish the different families of fermions have been sought without overt success. The various efforts to understand the fermion masses have ranged from substructure to Grand Unification. The former approaches include rishons\cite{1} and technicolor\cite{2}, but always encounter a fundamental problem: For relativistic constituents, the limiting gaps between eigenstates tend to a constant value, as in the MIT bag model\cite{3} (surprisingly parallel to the eigen structure for the non-relativistic harmonic oscillator). With sufficiently strange potentials (e.g., the “dracula” potential\cite{4}), the lowest few states may be forced to match the increasing gaps found in the real world, but they ultimately tend to a constant gap and predict additional states within the range of experiment that remain unseen. The latter approach has found some interesting relations between quarks and leptons\cite{5, 6} and even some mixing angles\cite{7} but no convincing overall solution has been obtained. Efforts along these lines continue.\cite{8}

We consider a modest revision of the mass sector of the Standard Model (SM) by taking the point of view that all of the fermions with a given electric charge have nothing within the SM that makes their right-chiral, weak interaction singlet components distinguishable to the Higgs boson. That is, we take the Higgs coupling to be insensitive to “generation” and discard the Yukawa coefficients that have been (artificially) inserted in the SM to reproduce the observed masses.

Others have constructed similar mass matrices by starting from various symmetry assumptions. A few of them, of which we are aware, are referenced here. \cite{9, 10} They are all “top-down” approaches, making initial symmetry assumptions. Conversely, we take a “bottom-up” starting point focusing on the absence of known symmetry or quantum numbers. This may be viewed as an accidental $S_3$ symmetry but that is not the fundamental nature of our assumed starting point. Rather, we assume that the final form of the fermion mass matrix is determined by (loop) corrections to the SM from physics beyond the SM (BSM physics) and invert the relation to extract information on some matrix elements of BSM physics.

We apply this concept here to the quarks and comment on related implications for leptons, reserving a full discussion of the leptonic system including neutrinos and their additional complications to a later paper. This approach amounts to a small change in the mass sector of
the SM, (hence, a modest revision, albeit a “massive” one) which provides an interpretation of the deviations from the direct mass implications of our revised SM in terms of BSM physics.

Although this approach proposes a resolution of the differences in fermion masses between “generations”, it does not attempt to address the differences in mass scale between the fermions of different electric charge, generally described as within “families”.

A. Mass in the SM

We first briefly review the structure of SM mass terms that we will alter. They are set by (arbitrary) Yukawa couplings of the (one, or in supersymmetry, at least two) Higgs boson(s) to the weak interaction active left-chiral doublets and weak interaction “sterile” (except for their $U(1)_B$ quantum numbers) singlets. The form is

$$\mathcal{L}_m = \sum_{i,j} Y_{Uij} \left( \frac{1 + \gamma_5}{2} \right) \Psi_{Ui} < \phi^0, \phi^+ > \left( \frac{1 - \gamma_5}{2} \right) \delta_{ij} + \text{h.c.}$$

$$+ \sum_{i,j} Y_{Dij} \left( \frac{1 + \gamma_5}{2} \right) \Psi_{Di} < \phi^-, \phi^{0*} > \left( \frac{1 - \gamma_5}{2} \right) \delta_{ij} + \text{h.c.}$$

where the Higgs may be the same or different in the two sets of terms. Unlike the comparable terms for the interaction with the weak gauge bosons which is naturally in the current basis, this Dirac notation is in the mass eigenstate basis and suppresses the information that pairs of independent Weyl spinors are involved.

In a more general notation, a Dirac bispinor is composed from two Weyl spinors

$$\Psi = \begin{bmatrix} \xi \\ \chi \end{bmatrix}$$

where

$$\xi = \zeta_a + \zeta_b$$

$$\chi = \sigma_2(-\zeta_a^* + \zeta_b^*)$$

so that the Dirac mass term appears as

$$\Psi \Psi = -(\chi^\dagger \xi + \xi^\dagger \chi).$$
This makes it clear that the Dirac term is coupling two independent Weyl spinors (left- and right-chiral) one of which is weak interaction active and the other sterile as described above.

For completeness, we display the Majorana form for a Weyl spinor and its mass term:

\[
\Psi_M = \begin{bmatrix} \xi \\ -\sigma_2 \xi^* \end{bmatrix}, \quad \bar{\Psi}_M \Psi_M = -(\xi^\dagger \sigma_2 \xi^* + \xi^T \sigma_2 \xi) \tag{5}
\]

which will be relevant when we refer to neutrinos later.

B. Basic concept

If the Higgs boson does not recognize any quantum number information associated with the Weyl spinor fermions that are singlets under the weak interaction (right-chiral projections in the SM), then the entries to the mass matrix for all of the fermion fields of a given charge should be identical. The corresponding mass matrix appears to have been first described as “democratic” by Jarlskog [10], although, for larger dimensional forms, it is known in nuclear and condensed matter physics as the origin of the “pairing gap”. The eigenstates for such a system consist of one (non-zero energy or) massive state with all other (two in the case of interest here) eigenstates being (at zero energy or) massless. This provides for a natural starting point consistent with the large gap between the heaviest and the lighter known fundamental fermions (of each charged type).

Experimentally, however, the lighter fermions in each grouping are not massless. This deviation from zero mass for the lighter fermions can be accommodated by assuming small deviations from “democracy”, consistent with perturbative corrections from BSM physics, which quite naturally allows for small mass eigenstates in the final result. A particular constraint on the parameters of the BSM physics is required for one of these to be much smaller than the other. Under our assumption, these effects afford a glimpse into the nature and structure of the BSM physics by requiring specific relationships between some matrix elements (that would be calculable from any given BSM theory).

The constraints can be discerned by examining the matrix, \(U\), that relates the BSM-corrected mass eigenstates to the weak interaction eigenstates, for the up-quarks, and the corresponding matrix, \(V\), for the down-quarks, combined into the form \(UV^\dagger\) that produces the Cabibbo-
Kobayashi-Maskawa \cite{11, 12} (CKM) matrix in the form as described by the Particle Data Group \cite{13} (PDG). That is, our approach recognizes that the Higgs coupling to the “active”, left-chiral, weak iso-doublet fermions is aligned with the weak gauge boson coupling, and so is diagonal in that (Weyl spinor) basis. As stated above, however, the right-chiral, weak iso-singlet Weyl spinors all present themselves indistinguishably to the weak iso-singlet object formed by combining the Higgs doublet with the active fermion doublet. Thus the CKM matrix is produced by the misalignment between the mass and weak eigenstates, here determined from the two separate components, \( U \) and \( V^\dagger \).

Another feature of the mass eigenstates of the \( 3 \times 3 \) “democratic” matrix is that the eigenvectors for the mass eigenstates can take the form of a tri-bi-maximal (TBM) mixture of the original (current) eigenstates. One state must be maximally mixed; for the other two, one has a degeneracy choice. However, there is no impediment to an overall TBM choice. While useful to simplify calculations, this is not particularly significant for the quarks, as the TBM structure cancels out in construction of the CKM matrix. (See Eq. (36) below.) However, it does have significant implications for the difference between the CKM and the corresponding Pontecorvo-Maki-Nakagawa-Sakata \cite{14} (PMNS) matrix for mixing in the lepton sector. We will comment upon this difference in our conclusions.

II. STARTING POINT AND POSITIVE INDICATIONS

The tri-bi-maximal (TBM) matrix

\[
TBM = \begin{pmatrix}
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
-\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}}
\end{pmatrix}
\]

(6)

diagonalizes the “democratic” matrix

\[
M_{\text{dem}} = \frac{1}{3} \times \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

(7)

to

\[
M_m = TBM^\dagger \times M_{\text{dem}} \times TBM
\]
where we have chosen the overall scale so the nonzero eigenvalue is unity.

The efficacy of our conjecture can be tested by inverting the TBM transformation on the known quark masses (taken from the Particle Data Group (PDG)[13] and) placed into diagonal mass matrices, viz.,

\[
m_u = \begin{bmatrix} 2.3 & 0 & 0 \\ 0 & 1275 & 0 \\ 0 & 0 & 173500 \end{bmatrix}
\]

and

\[
m_d = \begin{bmatrix} 3.8 & 0 & 0 \\ 0 & 95 & 0 \\ 0 & 0 & 4150 \end{bmatrix}
\]

where all values are expressed in MeV/c^2. (We will ignore the significant uncertainties and variation with scale of these masses [13] as the ratios vary less dramatically, although the values of even the ratios are not known to very high accuracy.)

We now transform these inversely using the TBM matrix given in Eq.(6) and find that the resulting mass matrices are indeed quite “democratic”:

\[
m_{u-reduced} = TBM \times m_u \times TBM^\dagger
= (173500) \times \begin{bmatrix} 0.33701 & 0.32966 & 0.33333 \\ 0.32966 & 0.33701 & 0.33333 \\ 0.33333 & 0.33333 & 0.33334 \end{bmatrix}
\]

and similarly

\[
m_{d-reduced} = (4150) \times \begin{bmatrix} 0.34493 & 0.32204 & 0.33303 \\ 0.32204 & 0.34493 & 0.33303 \\ 0.33303 & 0.33303 & 0.33394 \end{bmatrix}
\]

where we have scaled out the overall factor of the largest mass in each case. Although the true accuracy is, of course, far less, we keep the extra digits to to display which matrix
elements are not identical after the (inverse) TBM transformation and so convey the patterns
that will survive even substantial (within experimental uncertainties) changes in the ratios
of the diagonal values.

This demonstrates that only perturbatively small corrections to a democratic starting point
are needed. (We have ignored $CP$-violation considerations here, but will return to them
below.) The deviations from “democracy” are exceptionally small, barely exceeding 1% in
the up-quark sector and less than 4% in the down-quark sector. It is clear from this that
something close in structure to $M_{dem}$ (times an overall scale, $m$) is a reasonable ansatz to
consider for an initial mass matrix. (A similar result holds for the charged leptons.)

This result confirms that the wide range of quark masses is well described by an almost
“democratic” mass matrix for each charge set of quarks, leaving only the overall scale differ-
ence between up-quarks and down-quarks (and also leptons) to be understood. We do not
address that difference here.

III. A MODESTLY REVISED MASS SECTOR

With these comments and results in mind, we propose the mrSM (modestly revised Standard
Model) which differs from the SM only in the mass sector. In terms parallel to those of the
SM used above, we have

$$\mathcal{L}_{mr} = Y_U \Sigma_{i,j} \left( \frac{1 + \gamma_5}{2} \right) \Psi_{U_i} < \phi^0, \phi^+ > \left( \frac{1 - \gamma_5}{2} \right) \begin{bmatrix} \Psi_{U_j} \\ \Psi_{D_j} \end{bmatrix} + \text{h.c.}$$

$$+ Y_D \Sigma_{i,j} \left( \frac{1 + \gamma_5}{2} \right) \Psi_{D_i} < \phi^-, \phi^{0*} > \left( \frac{1 - \gamma_5}{2} \right) \begin{bmatrix} \Psi_{U_j} \\ \Psi_{D_j} \end{bmatrix} + \text{h.c.} \quad \text{(13)}$$

There is now only one value of $Y$ each for all of the up- and down-quarks and these two
values are approximately one third of the value of the two largest $Y_{ij}$ in the SM.

We note, as lagniappe, that this factor of 3 reduction in these remaining two Yukawa cou-
plings to the Higgs field from the largest value in each case in the SM improves the formal
perturbative character of this part of the SM by reducing the Higgs fine structure constant
by an order of magnitude. Of course, the total size of the effects of quark loops are unaltered,
as the sum over all channels produces the same significant net effect as in the diagonal mass
basis. However, the size of the total contribution becomes due to the number of diagrams contributing, not to any individual large one.

We take the full Higgs plus BSM-loop-corrected mass matrix to have the form

$$M_{mrSM} = m \times [M_{dem} + \epsilon M_{BSM}]$$

(14)

now in the current quark basis consistently defined by the Higgs and weak vector boson couplings. That is, we define the mass matrix for each set of 3 quarks of a given electric charge as $m$ (which is approximately one-third of the mass of the most massive of each triple of the fermions of a given non-zero electric charge) times the matrix $\mathcal{M}$, given by

$$\mathcal{M} = \frac{(1 + \epsilon \xi)}{3} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(15)

$$+ \epsilon \times \begin{bmatrix} \sqrt{\frac{2}{3}} y_0 + y_3 + \frac{1}{\sqrt{3}} y_8 & y_1 - I y_2 & y_4 - I y_5 \\ y_1 + I y_2 & \sqrt{\frac{2}{3}} y_0 - y_3 + \frac{1}{\sqrt{3}} y_8 & y_6 - I y_7 \\ y_4 + I y_5 & y_6 + I y_7 & \sqrt{\frac{2}{3}} y_0 - \frac{2}{\sqrt{3}} y_8 \end{bmatrix}$$

This allows for the most general set of perturbative (for small $\epsilon$) deviations possible for a Hermitean $3 \times 3$ matrix from the democratic mass matrix produced by uniform Higgs’ coupling in each quark charge sector. The coefficients are chosen to match the normalization of the standard Gell-Mann $SU(3)$ ($U(3)$) basis matrices.

The BSM corrections are all taken to be proportional to the small quantity, $\epsilon$, defined by the diagonal matrix of known mass eigenvalues, (again, with the overall scale factored out)

$$m \times \begin{bmatrix} \epsilon \delta & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(16)

Here, $\delta$ is the ratio of the lightest mass of the three quarks (with the same electric charge) to the mass of the intermediate mass quark, and $\epsilon$ is the ratio of that quark to the most massive of the three. In particular, for the quark mass values referred to above,

$$\epsilon_u = 7.35 \times 10^{-4}, \quad \delta_u = 1.8 \times 10^{-3}$$

$$\epsilon_d = 2.29 \times 10^{-2}, \quad \delta_d = 4.0 \times 10^{-2}$$

(17)
Even the largest of these values easily qualifies as a small expansion parameter.

The overall factor \((1 + \epsilon \xi)\) in Eq.(15) is introduced to rescale the largest eigenvalue of \(\mathcal{M}\) to unity, needed to account for the effect of the \(\epsilon y_0\) correction from the BSM physics. We will see below how a correlation between some of these \(y_i\) constants (which we view as matrix elements of BSM physics loop corrections, see below) reduces the smallest eigenvalue from \(\epsilon\) to \(\epsilon \delta\).

The quantities \(y_0\), \(y_3\) and \(y_8\) describe the possibilities for dynamical symmetry breaking from the BSM physics: While \((1 + \epsilon \xi)\) is only an overall scale revision, \(y_3\) and \(y_8\) parameterize the usual Cartan sub-algebra of \(SU(3)\) symmetry breaking allowed for 3 complex degrees of freedom, and \(y_0\) parameterizes the correction allowed by the \(U(1)\) factor of the overall \(U(3)\). These are all multiplied by a factor of \(\epsilon\) in the expectation that the BSM corrections are small, as is suggested by the data referred to above. It is apparent from the numerics above that this assumption is self-consistent.

We also allow for off-diagonal corrections, reflecting a possible more complete breaking of the (accidental) \(U(3)\) symmetry. We have included this element of necessity, having found that, with only \(y_3\) and \(y_8\), the desired result of the CKM matrix (in the form of the combination \(UV^\dagger\)) cannot match all of the (moduli of the) elements of that matrix as reported by the PDG [13]. However, as emphasized by Leviatan [16], some elements of a partially broken symmetry may survive when additional components outside the Cartan subalgebra, or even outside the parent spectrum generating algebra, appear.

A. TBM reduction

With what we identify as BSM corrections, after the TBM transformation as in Eq.(8), the (intermediate) form of the mass matrix becomes:

\[
\mathcal{M}_{int} = 
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 + \epsilon \xi
\end{bmatrix} + \epsilon \times  
\]

(18)
We have fixed the value of $\xi$ by the condition that the trace of $M_{\text{int}}$ must be equal to the trace of the (scale removed) diagonal matrix of eigenvalues, from Eq.(16), so that,

$$\xi = 1 + \delta - \sqrt{6} y_0.$$  \hfill (19)

We have also introduced a $p$ and $m$ notation to indicate sums and differences of the numerically labelled $y_i$'s combining them into shorter labelled forms to reduce the complexity of the appearance of the form of $M_{\text{int}}$.

In addition, we have set $y_2 = y_5 - y_7$ above, eliminating any imaginary terms from the (1, 2) and (2, 1) elements of $M_{\text{int}}$, to avoid a problem later with $CP$-violation. (The notation is also even more cumbersome without doing this now.) However, if $y_2$ is kept through the block diagonalization below, it becomes clear that this constraint is required to eliminate imaginary-valued terms of order $\epsilon$ from the (1, 2) and (2, 1) entries of $M_{2 \times 2}$ below. If such terms of order $\epsilon$ appear in those positions, they produce a $CP$-violation of order one (due to a phase angle proportional to $y_2 - y_{5m7}$), rather than of order $\epsilon^2$, as discussed below.

B. Block diagonalization

We now first block-diagonalize the $3 \times 3$ matrix in Eq.(18). With the expectation that the third component will dominate the eigenvector for the large (unit) eigenvalue of the full system, we choose an eigenvector of the form

$$vec3 = \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix}$$  \hfill (20)

where we expect $\alpha$ and $\beta$ to be of order $\epsilon$, and solve the first two of the equations in

$$M_{\text{int}} vec3 = 1 \cdot vec3$$  \hfill (21)

for $\alpha$ and $\beta$. Their values to leading order in $\epsilon$ are

$$\alpha = \frac{1}{3\sqrt{2}} \epsilon \left(2\sqrt{3}y_8 + 2y_1 - y_{4p6} - 3Iy_{5p7}\right)$$  \hfill (22)
\[ \beta = -\frac{1}{\sqrt{6}} \epsilon (2y_3 + y_{4m6} - 3Iy_{5m7}) \] (23)

The value of \( y_0 \) can be found from the third component relation in Eq.(21). To the leading order approximation in \( \epsilon \) used above, it is

\[ y_0 = \frac{1}{2\sqrt{6}}(3(1 + \delta) + 2(y_1 + y_{4p6})) \] (24)

It is now straightforward to construct two vectors orthogonal to \( vec3 \). With these, and after normalizing all three vectors, we can block diagonalize \( M_{int} \) as an intermediate step to the full diagonalization. Since we are only working, at this point, to order \( \epsilon \), we do this in a way that minimally affects the \( 2 \times 2 \) subspace, choosing

\[
\begin{align*}
vec1 &= \begin{bmatrix} 1 \\ 0 \\ -\alpha^* \end{bmatrix} \\
vec2 &= \begin{bmatrix} 0 \\ 1 \\ -\beta^* \end{bmatrix}
\end{align*}
\] (25-26)

where the minus signs appear because the full matrix of eigenvectors is necessarily of the form of a complex rotation. (Although what we show here is only normalized through order \( \epsilon \), we have carried out the full calculation through order \( \epsilon^2 \). The cumbersome formulae so produced provide no additional insight beyond those provided by the order \( \epsilon \) results presented here.)

This allows us to produce \( X_{3 \rightarrow 2} \)

\[
X_{3 \rightarrow 2} = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ -\alpha^* & -\beta^* & 1 \end{bmatrix}
\] (27)

which block diagonalizes \( M_{int} \) to the unit eigenvalue and a 2-by-2 block

\[
M_{2 \times 2} = \epsilon \times \begin{bmatrix} \frac{1}{3}(1 + \delta) - \frac{1}{\sqrt{3}}y_8 + \frac{1}{3}(2y_1 - y_{4p6}) & -\frac{1}{\sqrt{3}}(y_3 - y_{4m6}) \\ -\frac{1}{\sqrt{3}}(y_3 - y_{4m6}) & \frac{1}{3}(1 + \delta) + \frac{1}{\sqrt{3}}y_8 - \frac{1}{3}(2y_1 - y_{4p6}) \end{bmatrix}
\] (28)

Note that as discussed above, there are no order \( \epsilon \) imaginary terms in \( M_{2 \times 2} \). It is here that the requirement that there be no order \( \epsilon \) imaginary contribution in \( M_{2 \times 2} \) has become...
apparent: Once the factor of $\epsilon$ is removed from $M_{2x2}$, it is clear that $X_{2x2}$, the matrix that diagonalizes $M_{2x2}$, would include an eigenvector with an order one imaginary phase which would lead to a large $CP$ violation, inconsistent with observation. Our notation, $X$, is intended to reflect the fact that the form of this matrix will be applied to both the up-quarks, where the diagonalization matrix is conventionally labelled as $U$, and the down-quarks, conventionally labelled as $V$.

C. Subspace diagonalization

On removing the common factor of $\epsilon$, and defining

$$y_a = \frac{2}{3}(2y_1 - y_{4p6} - \sqrt{3}y_8)$$
$$= (1 - \delta) \cos(2\omega)$$

$$y_b = \frac{2}{\sqrt{3}}(y_3 - y_{4m6})$$
$$= -(1 - \delta) \sin(2\omega)$$

we find the eigenvalues of

$$M_{\text{reduced}} = \begin{pmatrix}
\frac{1}{2}(1 + \delta + y_a) & -\frac{1}{2}y_b \\
-\frac{1}{2}y_b & \frac{1}{2}(1 + \delta - y_a)
\end{pmatrix}$$

(31)

to be $\frac{1}{2}(1 + \delta \pm \sqrt{y_a^2 + y_b^2})$. These must equal $\delta$ and 1 to complete the mass spectrum, and indeed they do as we have

$$\sqrt{y_a^2 + y_b^2} = (1 - \delta)$$

(32)

Defining

$$X_{2x2} = \begin{pmatrix}
\cos(\omega) & \sin(\omega) \\
-\sin(\omega) & \cos(\omega)
\end{pmatrix}$$

(33)

as the transformation matrix that accomplishes the diagonalization, we obtain the rotation angle $\omega$ as

$$\omega = \frac{1}{2} \arctan\left(\frac{y_b}{y_a}\right)$$

(34)
Combining this diagonalization with the block diagonalization, the total unitary transformation that diagonalizes the mass matrix in Eq.(15) is given by the product, $X_{\text{tot}}$, where

$$X_{\text{tot}} = TBM \times X_{3\to2} \times \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (35)$$

To compute the CKM matrix, we need this evaluated for both the up-quarks, and for the down-quarks. Again, the separate matrices for these are conventionally labelled $U$ and $V$ respectively [13], so that

$$\text{CKM} = U \times V^\dagger.$$  \hspace{1cm} (36)$$

Note that the PDG description is one in which these matrices transform from mass eigenstates to current eigenstates, but our derivation above is for the transformation of current eigenstates to mass eigenstates. Hence, the Hermitian conjugates are interchanged and the $V^\dagger$ of the PDG is our $X_d$ for the down-quarks and similarly, their $U$ is the Hermitian conjugate, $X_u^\dagger$, of our $X_u$ for the up-quarks. Thus, as $TBM$ is common to both factors, it cancels out in the product. Hence, it is sufficient to define

$$X_{\text{net}} = X_{3\to2} \times \begin{bmatrix} \cos(\omega) & \sin(\omega) & 0 \\ -\sin(\omega) & \cos(\omega) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (37)$$

and substitute up-quark and down-quark parameter values to produce $U = X_u^\dagger$ and $V = X_d$ (where, again, the subscripts indicate that the appropriate parameters have been substituted for the generic parameters in $X_{\text{net}}$).

On doing so, we can identify, in the product, the Cabibbo angle as $\Theta_C$, where

$$\Theta_C = \omega_d - \omega_u$$  \hspace{1cm} (38)$$

$$= 0.2273$$

It is here that the $CP$-violation problem that would arise without the constraint that $y_2 = y_{5m7}$ applying separately for both the up- and down-quarks becomes explicit. With the
constraint, the Cabibbo angle appears in (1,2) matrix element of the $CKM$ matrix in the form

$$\sin(\omega_d)\cos(\omega_u) - \cos(\omega_d)\sin(\omega_u)$$  \hspace{1cm} (39)

but if $y_2 \neq y_5 m_7$, we obtain

$$\sin(\omega_d)\cos(\omega_u)e^{i\phi_d} - \cos(\omega_d)\sin(\omega_u)e^{i\phi_u}$$  \hspace{1cm} (40)

where

$$\tan(\phi) = \frac{y_2 - y_5 m_7}{y_3 - y_4 m_6}$$  \hspace{1cm} (41)

for both up- and down-quarks. As the $y_i$ parameters are “naturally” of order one, these phases will not, in general, be resolvable to reproduce $\sin(\Theta_C)$ in the (1,2) matrix element of the $CKM$ matrix. Even if they were equal so that the phase could be factored out to reproduce the Cabibbo angle, an order one $CP$-violation would remain.

We thus obtain the following leading order in $\epsilon$ approximation for the $CKM$ matrix:

$$CKM_{prx} = \begin{bmatrix} \cos(\Theta_C) & \sin(\Theta_C) & X_{13} \\ -\sin(\Theta_C) & \cos(\Theta_C) & X_{23} \\ X_{31} & X_{32} & 1 \end{bmatrix}$$  \hspace{1cm} (42)

where

$$X_{13} = \cos(\omega_u)(A_{du} + IF_{du}) - \sin(\omega_u)(B_{du} + IG_{du})$$  \hspace{1cm} (43)

$$X_{23} = \sin(\omega_u)(A_{du} + IF_{du}) + \cos(\omega_u)(B_{du} + IG_{du})$$  \hspace{1cm} (44)

$$X_{31} = -\cos(\omega_u + \Theta_c)(A_{du} - IF_{du}) + \sin(\omega_u + \Theta_c)(B_{du} - IG_{du})$$  \hspace{1cm} (45)

$$X_{32} = -\sin(\omega_u + \Theta_c)(A_{du} - IF_{du}) - \cos(\omega_u + \Theta_c)(B_{du} - IG_{du})$$  \hspace{1cm} (46)

and

$$A_{du} = Re(\alpha_d) - Re(\alpha_u)$$  \hspace{1cm} (47)

$$B_{du} = Re(\beta_d) - Re(\beta_u)$$  \hspace{1cm} (48)

$$F_{du} = Im(\alpha_d) - Im(\alpha_u)$$  \hspace{1cm} (49)

$$G_{du} = Im(\beta_d) - Im(\beta_u)$$  \hspace{1cm} (50)
with $\alpha$ and $\beta$ as given in Eqs. (22, 23) for $u$ and $d$ parameter values, e.g.,

$$\alpha_u = \frac{1}{3\sqrt{2}} \epsilon_u \left(2\sqrt{3}y_{8u} + 2y_{1u} - y_{4p6u} - 3Iy_{5y7u}\right)$$
$$= \frac{1}{\sqrt{2}} \epsilon_u \left(\sqrt{3}y_{8u} + \frac{1}{2}(1 - \delta_u)\sin\left(\omega_u\right) - Iy_{5y7u}\right)$$  \hspace{1cm} (51)

$$\beta_d = -\frac{1}{\sqrt{6}} \epsilon_d \left(2y_{3d} + y_{4m6d} - 3Iy_{5m7d}\right)$$
$$= -\frac{1}{\sqrt{6}} \epsilon_d \left(3y_{3d} - \frac{\sqrt{3}}{2}(1 - \delta_d)\sin\left(\omega_d\right) - 3Iy_{5m7d}\right)$$  \hspace{1cm} (52)

using Eqs. (29, 30) and similarly for the other combinations.

We note here that Eq. (52) demonstrates why we were not able to match a CKM matrix element (in particular, the value of $X_{23}$, see below) without including (real) off-diagonal BSM mass corrections. Without $y_{4m6d}$, the real part of $\beta_d$ is proportional to a trigonometric factor times a fraction of $\epsilon_d$ which has a value of only about half of that of $X_{23}$. Of course, the complex nature of the CKM matrix requires some (imaginary) off-diagonal BSM mass corrections to provide for $CP$-violation but this demonstrates that inclusion of at least some of these (real) terms is independently required.

IV. FITS

Taking values from the PDG [13], and using their parametrization, we obtain central values for the real and imaginary parts of these quantities in the relations:

$$X_{13} = 0.00126 - 0.00328I$$  \hspace{1cm} (53)
$$X_{23} = 0.04120 + 0.0I$$  \hspace{1cm} (54)
$$X_{31} = 0.00806 - 0.00319I$$  \hspace{1cm} (55)
$$X_{32} = -0.04042 - 0.000737I$$  \hspace{1cm} (56)

where the rhs in each case is the corresponding entry of the PDG CKM matrix when the particular set of signs is chosen for the three angles corresponding to them all being in the first quadrant. Other choices should be investigated as well, but again, these demonstrate that at least one solution exists. (We have investigated this and find that the largest differences,
apart from signs, are in the real part of $X_{31}$ and the imaginary part of $X_{23}$, but the changes are not large, e.g., $\sim 20\%$ in the real part of $X_{31}$.)

![Variation of combined BSM parameters for real and imaginary contributions to CKM matrix as functions of $\omega_u$.](image)

This would appear to afford 8 relations for 5 unknowns, but as is evident from the construction, there are trigonometric relations that reduce this to only four independent relations. We can, however, solve Eqs. (47, 48, 49, 50) for those quantities as a function of $\omega_u$, which provides a number of constraints on values of the BSM parameters. The results, divided by $\epsilon_d$, are plotted in Fig. (1); analytically in $\omega_u$, with numerical evaluation of the coefficients, they are:

$$A_{du} = 0.001262 \cos(\omega_u) + 0.04120 \sin(\omega_u)$$  \hspace{1cm} (57)

$$B_{du} = 0.04120 \cos(\omega_u) - 0.001262 \sin(\omega_u)$$  \hspace{1cm} (58)
\[ F_{du} = -0.003275 \cos(\omega_u) \]  
\[ G_{du} = +0.003275 \sin(\omega_u) \]

In the Figure, we divided by $\epsilon_d$ since both $\epsilon$ factors are included in the definitions of $A_{du}$, $B_{du}$, $F_{du}$, $G_{du}$. The scale of the result in the Figure shows that the parameter values of the $y_i$ are “natural”, that is, of order one rather than excessively large (or small, except possibly for the imaginary components related to $CP$-violation).

**A. Parameter relations**

Using these results in the previous relations allows for determination of the parameter values for $d$-quarks in terms of those for $u$-quarks:

\[ y_{5p7d} = -0.20227 \cos(\omega_u) + 0.32096 y_{5p7u} \]  
\[ y_{5m7d} = -0.11678 \sin(\omega_u) + 0.32096 y_{5m7u} \]
\[ y_{8d} = +0.32096 y_{8u} + 0.045001 \cos(\omega_u) + 1.4690 \sin(\omega_u) \]
\[ +0.092487 \cos(2\omega_u) - 0.27713 \cos(2(\omega_u + \Theta_C)) \]
\[ y_{3d} = +0.32096 y_{3u} - 1.4690 \cos(\omega_u) + 0.045001 \sin(\omega_u) \]
\[ -0.092487 \sin(2\omega_u) + 0.27713 \sin(2(\omega_u + \Theta_C)) \]

where $\Theta_C = 0.22729$. Note that the coefficient 0.32096 is just the ratio $\epsilon_u/\epsilon_d$. The first two equations here can, of course, be used to determine $y_{5d}$ and $y_{7d}$ separately in terms of $y_{5u}$ and $y_{7u}$. Of course, they may all be inverted to determine $u$-quark BSM parameters in terms of those for $d$-quarks if so preferred.

Under assumptions from particular models of the BSM physics, such as $y_1 = 0$ or $y_4 + y_6 = 0$ and $y_5 - y_7 = 0$, for example, the remaining individual parameters may be determined and checked for self-consistency. Such relations would follow from invoking a specific form of the BSM physics and calculating the induced corrections, and amounts to finding a specific direction for the non-Cartan components in what used to be called $U$-spin and $V$-spin when triuplicity of flavor referred to the $u$, $d$ and $s$ quarks of $SU(3)_{\text{flavor}}$. One constraint immediately apparent from these equations is that one cannot have either of $y_5 = \pm y_7$ for both up- and
down-quarks simultaneously.

\[ \begin{align*}
K^0 & \quad 0.00003 \\
K & \quad 0.00002 \\
K & \quad 0.00001 \\
K & \quad 0.00001
\end{align*} \]

FIG. 2. Variation of coefficients of BSM parameters with parameter $\omega_u$ for contributions to the Jarlskog invariant for $CP$-violation.

B. CP Violation

As noted by Kobayashi and Maskawa (KM) [12], there is a phase freedom in the creation and annihilation operators for the quark (and other fermion) fields of the SM in the case of three (but not two) “generations” (or “families”) that allows for the absorption of all but one phase in the $UV^\dagger$ construction of the CKM matrix. Our mrSM does not change this but since we have already introduced a complex component, it behooves us to examine what $CP$-violating implications are so introduced under the assumption that this is the only
source of $CP$-violation.

The invariant characterization of $CP$-violation was described by Jarlskog [17]. The Jarlskog invariant quantity, which we label $J$, appears only at order $\epsilon^2$, and is given by [13]

$$J = \text{Im}CKM_{i,j}CKM_{k,l}CKM_{i,l}CKM_{k,j}$$

$$= \cos(\theta_{12})\sin(\theta_{12})\cos(\theta_{23})\sin(\theta_{23})\sin(\theta_{13})\cos^2(\theta_{13})\sin(\delta)$$

$$= 2.96 \times 10^{-5}$$

(65)

up to an overall sign ambiguity, in the standard PDG representation of the CKM matrix.

In terms of the parametrization we have developed here, using $i = 2 = j$ and $k = 3 = l$, we obtain

$$J_{2233} = \pm \cos(\theta_C)\sin(\theta_C)[(F_{du}B_{du} - A_{du}G_{du})]$$

(66)

which necessarily agrees with the PDG value since we have fitted all of these parameters to the corresponding PDG values.

At this first order in $\epsilon$ level of approximation, only this combination of matrix elements reproduces the correct result for $J$. We have checked, however, that completing the unitary structure of $X_{\text{tot}}$ to second order in $\epsilon$ produces the correct result from any $i,j,k,l$ combination. (We note one last time the requirement that $y_2 = y_{5m7}$, as without it, the phases referred to above in the $CKM\ (1,2)$ matrix element would produce an order one $CP$-violation in the calculation of $J_{1122}$ unless $y_2 - y_{5m7}$ or each of the separate $y_i$ that are involved are “unnaturally” small.)

In Fig.(2), we display the coefficients of the contributions to $J$ after employing all of the relations available above.

V. DISCUSSION

The mass ratios of the quarks are scale dependent, and one could examine the effects of that scale dependence on the CKM matrix and our fit. However, the ratios are generally poorly known and do not vary significantly with scale over the range from 2 GeV where the lightest quark masses are generally defined and determined to the scale of the $b$-quark, nor from there to the weak scale which is also very close to the top quark mass. Refinements responding to
these issues are certainly warranted, but we do not expect them to produce large corrections to our BSM parameter constraints determined here. In fact, since the effects considered here are dominated by the values of $\epsilon$, only the uncertainties associated with the masses of the strange and charmed quarks should be significant, as the $b$- and $t$-quark masses are relatively accurately known. Fortunately, the very large uncertainties associated with the ratios of the two lightest quarks do not play a significant role in establishing the configuration, although they will be important for precision analyses.

We have carried out the straightforward extension of our results to the next higher order in $\epsilon$ which could, in principle be used to further constrain the values of the unknown parameters. Unfortunately, utilization requires knowledge of the relevant experimental values to order $\epsilon^2$, i.e., to of order parts in $10^4$, which is an accuracy generally not presently available. More accurate measurements would certainly change this conclusion.

A. BSM contributions

In general terms, Fig.(3) shows the nature of expected BSM corrections that do distinguish the different fermions and could lead to the small corrections that we find in our fits. The Lagrangian structure that we have in mind uses Weyl spinors for the separate left-chiral ($d_x$, for member of a weak interaction doublet) and right-chiral ($s_x$, for a weak interaction singlet) parts of the fermion Dirac bispinors, but nonetheless produces Dirac mass terms which may be simply represented as above.

Interestingly, we expect these loop calculations to be finite as they involve only differences within the triples of fermions. This effect was observed in Ref.[18], where a symmetric, overall divergence appears but the differences in mass corrections are finite, although the model is for a quite different application of symmetry-breaking mass corrections. Also, the calculation there was done only for Cartan sub-algebra corrections, but since off-diagonal corrections simply refer to a different Cartan sub-algebra, those corrections should also be finite.

The Figure is drawn for a BSM gauge vector boson interaction, but the BSM vector could in principle also couple the Weyl spinor $d_x$ to a ($CP$-conjugate of the) Weyl spinor $s_x$, which simply requires interchanging the labels on either side of the Higgs’ coupling to complete the
FIG. 3. A BSM loop correction to the fermion-Higgs-boson vertex that alters the charged fermion mass matrix from “democratic” to that shown in Eq. (15). Left-chiral fermions with weak interactions are labelled $d_x$ for “doublet” and right-chiral fermions with no weak interactions are labelled $s_x$ for “singlet” or “sterile”.

In that latter configuration of labels (without the $CP$-conjugation), the figure could also apply to a BSM scalar boson in the loop as well.

Note that without the intermediation of the Higgs scalar vacuum expectation value, both the $d_x$ and the $s_x$ pass through the loop unchanged and not coupled to mass, so the BSM correction affects only vertex renormalization.

B. Leptons

Of course, application to the leptons also comes to mind. For the charged leptons, Fig. (3) still applies, as it also does for Dirac mass terms for neutrinos. However, producing these requires the existence of uncharged Weyl (right-chiral) fermions with no SM interactions at all, i.e., the so-called “sterile” neutrinos, as opposed to the known ones which are “active”
with respect to the weak interactions. Under the long honored assumption of Zweig-Glashow (quark-lepton) symmetry, the existence of these states has been widely presumed since the early days of Grand Unified Theories\[19\], starting with $SO(10)$. Aside from the prediction of the charm quark, this symmetry (or regularity) successfully predicted the existence of the $t$- and $b$-quarks as soon as the $\tau$-lepton was discovered. \[20\] There may even be some recent experimental evidence for “sterile” neutrinos. \[21\]

A $6 \times 6$ structure in terms of Weyl spinors is now required, rather than the simple $3 \times 3$ Dirac mass structure that the charged fermions can be reduced to by the standard construction of Dirac bispinors. The Dirac mass terms of the neutrinos, $m_D$, have the same form as that for the charged fermions as described above, but they now appear in $3 \times 3$ off-diagonal blocks in this $6 \times 6$ matrix. The upper left $3 \times 3$ block remains zero, as the SM does not produce Majorana masses for the active neutrinos, nor is it necessary for BSM physics to do so directly: The “see-saw” mechanism\[6\] produced by a sufficiently massive lower right $3 \times 3$ block of “sterile” neutrinos leads to light, Majorana mass eigenstates that are dominated by active neutrino amplitudes.

That lower right $3 \times 3$ block of Majorana masses (with a structure as given in Eq.(5) above) for the “sterile” Weyl spinors (corresponding to what would have been the right-chiral component of a normal Dirac neutrino wavefunction) is unconstrained. As we pointed out many years ago\[22\], neutrino mass mixing of the almost purely “active” eigenstates can be expected to be similar to that for the quarks, unless there is some particular structure to this $3 \times 3$ block of Majorana masses for the “sterile” Weyl spinors. That is, the PMNS matrix should be similar to the CKM matrix. At the time, the concern was to determine whether or not neutrinos should be expected to have mass and whether or not their mixing should be expected to be large enough to measure. As we know now, the masses are small but the mixing is even larger and very close to the particular TBM form that we showed above applies separately to the up- and down-type quarks, but cancels in the CKM matrix.

We have identified a procedure that provides for a determination of the relations between the sterile neutrino Majorana mass matrix that produces a fully diagonalized $6 \times 6$ mass matrix with no mixing of the active parts of the mostly active Majorana mass eigenstates relative to their initial structure. Thus for the weak interaction currents, the factor of the leptonic analog of the $CKM$ matrix (i.e., the $PMNS$-matrix \[14\]) contributed by neutrinos
will be the identity (or close to it). It follows that the $PMNS$-matrix for the weak lepton currents, will be almost of the $TBM$ form with corrections coming from the diagonalization of the charged leptons, and possibly only those corrections:

$$PMNS \approx TBM,$$

i.e., the $PMNS$-matrix will be approximately the same as the $TBM$-matrix, but not exactly the same. In this happy instance, our “democratic” plus BSM hypothesis for the fermion mass matrices also provides for a unified understanding of all of the weak current mixing structures simultaneously, subject to additional constraints on BSM physics. We will report on a detailed analysis of this conjecture in a future paper.

VI. CONCLUSION

We have assumed that within the SM, for the fermions with a given electric charge, the Higgs doublet is sensitive only to the quantum numbers of the left-chiral Weyl spinor parts of the Dirac wave functions. With this assumption, the iso-singlet terms formed between them and the Higgs doublet couple equally to all of the right-chiral Weyl spinor parts. This in turn implies that the SM mass matrix should have a “democratic” form, with one massive eigenstate and two massless ones. Upon adding perturbative corrections of an almost completely general form, presumed to arise from BSM physics, we find that a consistent set of parameters may be extracted that conforms to the known quark mass spectra and CKM mixing matrix, including $CP$-violation. These parameter value constraints provide information on matrix elements of the manner in which BSM physics couples to SM degrees of freedom. The question of why the overall mass scale for the up-quarks is significantly larger than that for the down-quarks remains unresolved, but may well be related to the supersymmetric conjecture of a pair of Higgs bosons, one providing mass for the up-quarks and the other providing for the down-quarks.

It is clear that, in this approach, extracting more detailed information on the nature of BSM physics and the value of BSM matrix elements requires more accurate determination of the quark mass ratios and their mixing amplitudes in the weak interaction. Experimental determinations of the separate real and imaginary parts of the CKM matrix elements will also be of great value.
Under the “see-saw” assumption regarding the existence and nature of sterile neutrino components, the extension of these ideas to leptons can produce the beginning of an understanding of both why the PMNS matrix is approximately of tri-bimaximal form and also why it is not exactly so. Along with the information from the violations of “democracy”, the potential information on the sterile neutrino mass matrix may open the door to learning about the physics in the dark matter sector, with the sterile neutrinos as the first component known from other than gravitational interactions.

Finally, we note that the intermediate propagation of sterile as well as active neutrinos in variations of the graphs above applied to weak decay box graphs may be relevant to recent observations of violations of lepton universality, such as in Ref.([23]).

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