Abstract. We deal with Bedford-Taylor type capacities on almost complex surfaces.

1. Introduction

Let \((M,J)\) be an almost complex surface (almost complex manifold of the real dimension 4). For \(u,v \in W^{1,2}_{loc}(\Omega)\), where \(\Omega \subset M\) is a domain, we can define a wedge product

\[ i\partial \bar{\partial} u \wedge i\partial \bar{\partial} v := -i\partial \bar{\partial} (i\partial u \wedge \bar{\partial} v) + \partial (\partial u \wedge \bar{\partial} v) + \bar{\partial} (\theta \partial u \wedge \bar{\partial} v) - \theta \partial u \wedge \bar{\partial} v \]

as a \((2,2)\) current.

If \(u,v\) are \(C^2\) functions then it is the standard wedge product of continuous forms. If \(u,v \in W^{1,2}_{loc}(\Omega)\) are plurisubharmonic then this is a regular Borel measure, see \([P2, P3]\) and \((i\partial \bar{\partial} u)^2\) is called the Monge-Ampère operator.

The goal of this article is to study plurisubharmonic functions, the Monge-Ampère operator and the relative capacity on almost complex surfaces.

All results proved in the paper, in the case of \(\mathbb{C}^n\), are proved in the classical papers \([B-T1, B-T2]\). The main difference between \(\mathbb{C}^n\) and almost complex manifold (with the not necessary integrable almost complex structure) is the fact that for plurisubharmonic function \(u\), the positive current \(i\partial \bar{\partial} u\) is not necessary closed. Thus the pluripotential theory on almost complex manifold is in some sense similar to pluripotential theory on hermitian manifold where the current \(i\partial \bar{\partial} u + \omega\) is not closed too. However, the theory in the non-integrable case is much

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more difficult. This is, among others, because in case of hermitian manifolds non-closed part of $i\partial \bar{\partial} u + \omega$ is just the hermitian form $\omega$ which is smooth and does not depend on $u$ but in our situation non-closed part of $i\partial \bar{\partial} u$ is only in $L^2$ (at least for bounded $u$) and strongly depends on $u$.

2. Preliminaries

2.1. Almost complex manifolds and plurisubharmonic functions. We say that $(M, J)$ is an almost complex manifold if $M$ is a manifold and $J$ is a $C^\infty$ smooth endomorphism of the tangent bundle $TM$, such that $J^2 = -\text{id}$. The real dimension of $M$ is even in that case. We will denote by $n$ the complex dimension of $M$: $n = \dim \mathbb{C} M = \frac{1}{2} \dim \mathbb{R} M$. All definitions below are exactly the same as in the case of complex manifolds.

As on complex manifolds we can define here $(p, q)$-forms and more generally $(p, q)$-currents. We have the decomposition of the exterior differential:

$$d = \partial + \bar{\partial} - \theta - \bar{\theta},$$

where operators $\partial$, $\bar{\partial}$, $-\theta$ and $-\bar{\theta}$ are respectively $(1, 0)$, $(0, 1)$, $(2, -1)$ and $(-1, 2)$ parts of $d$. On the level of functions we have

$$d = \partial + \bar{\partial}.$$

Let $\Omega \subset M$ be a domain and $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. In $\mathbb{C}$ we have the standard almost complex structure $J_{st}$. We say that a function $\lambda: \mathbb{D} \to D$ is $J$-holomorphic if $d\lambda J_{st} = J d\lambda$. We say that a function $u: \Omega \to [-\infty, +\infty)$ is plurisubharmonic iff

1. $u \neq -\infty$
2. $u$ is upper-semicontinuous and
3. $u \circ \lambda$ is subharmonic for any $J$-holomorphic function $\lambda: \mathbb{D} \to \Omega$.

If $u$ is plurisubharmonic then it is locally integrable and $i\partial \bar{\partial} u \geq 0$, see [P]. The converse was proved by R. Harvey and B. Lawson in [H-L]. Namely they proved that, if $u \in L^1_{loc}$ and $i\partial \bar{\partial} u \geq 0$ then a function $\tilde{u}$, given by

$$\tilde{u}(z) = \text{ess lim sup} u(w),$$

is a plurisubharmonic function which is equal a.e. to the function $u$.

We say that a function $u$ on $\Omega$ is strictly plurisubharmonic iff for any $\varphi \in C^\infty_0(\Omega)$ there is $\varepsilon_0 > 0$ such that the function $u + \varepsilon \varphi$ is plurisubharmonic for $\varepsilon > 0$.

We say that a domain $\Omega \Subset M$ is strictly pseudoconvex (of class $C^\infty$), if there is a strictly plurisubharmonic function $\rho$ of class $C^\infty$ in
a neighborhood of $\Omega$, such that $\Omega = \{ \rho < 0 \}$ and $\nabla \rho \neq 0$ on $\partial \Omega$. We say that $M$ is almost Stain if there is exhausting smooth strictly plurisubharmonic function on $M$.

2.2. **Dirichlet problem for the Monge-Ampère equation.** Let $\Omega \subset M$ be strictly pseudoconvex domain. The following Theorem will be useful for us.

**Theorem 1.** There is a unique solution $u$ of the Dirichlet problem:

\[
\begin{cases}
  u \in \mathcal{PSH}(\Omega) \cap C^\infty(\bar{\Omega}) \\
  (i\partial \bar{\partial} u)^n = dV \text{ in } \Omega \\
  u = \varphi \text{ on } \partial \Omega
\end{cases},
\]

where $\varphi \in C^\infty(\bar{\Omega})$ and $dV$ is the volume form on a neighbourhood of $\bar{\Omega}$.

The proof of this theorem in [P1] has a gap (in the part about the second order estimate). The mistake is corrected in recent work of J. Chu, V. Tosatti and B. Weinkove [C-T-W].

3. **Estimates**

From here, we will assume that $M$ is an almost complex surface.

The following proposition was proved in [P3].

**Proposition 2** (proposition 4.2 in [P3]). Let $u \in \mathcal{PSH} \cap W^{1,2}_{\text{loc}}(\Omega)$ then:

i) If $v \in \mathcal{PSH}(\Omega)$ and $v \geq u$, then $v \in W^{1,2}_{\text{loc}}(\Omega)$;

ii) If a sequence $u_j$ of plurisubharmonic functions decreases to $u$, then it converges in $W^{1,2}_{\text{loc}}$.

Note here that i) imply that bounded plurisubharmonic functions are in $W^{1,2}$. From the proof of the above in [P1] we also get

**Proposition 3.** Let $D \subset \Omega$, $u, v \in \mathcal{PSH}(\Omega)$, $u \leq v \leq 0$ and $u \in W^{1,2}(\Omega)$. Then $v \in W^{1,2}(D)$ and $\|v\|_{W^{1,2}(D)} \leq C\|u\|_{W^{1,2}(\Omega)}$, where the constant $C$ depends only on $D$ and $\Omega$.

Note that Błocki in [B2] proved above estimate for subharmonic functions in $\mathbb{R}^n$.

As a Direct consequence we get the following

**Corollary 4.** If $K \subset \Omega$ and $u$ bounded plurisubharmonic function then

\[\|u\|_{W^{1,2}(K)} \leq C\|u\|_{\Omega}\]

**Proof:** By Stokes’ theorem we get

\[
\int_{\Omega} i\partial \bar{\partial} u \wedge i\partial \bar{\partial} v = \int_{\Omega} \theta \bar{\partial} u \wedge \bar{\partial} v - \theta \partial u \wedge \bar{\partial} v,
\]

and thus the statement follows. $\square$
Theorem 5 (Chern-Levine-Nirenberg inequalities). Let $K \subset \Omega$ and $u, v \in PSH \cap W^{1,2}(\Omega)$, then
\[
\int_K i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v \leq C\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)},
\]
and if in addition $u, v$ are bounded then
\[
\int_K i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v \leq C\|u\|_{\Omega}\|v\|_{\Omega}.
\]

Proof: Take a nonnegative test function $\varphi$ which is equal 1 on $K$. By definition of the current $i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v$ and the integration by parts we can estimate:
\[
\int_K \varphi i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v
\leq \int_\Omega (i\partial\bar{\partial}\varphi i\partial\bar{\partial}u \wedge \bar{\partial}v - d\varphi(\partial u \wedge \bar{\partial}v + \theta \bar{\partial}u \wedge \bar{\partial}v))
\]
\[
+ \int_\Omega \varphi(\theta \bar{\partial}u \wedge \bar{\partial}v - \theta \bar{\partial}u \wedge \bar{\partial}v)
\leq C\|u\|_{W^{1,2}(\Omega)}\|v\|_{W^{1,2}(\Omega)},
\]
where $C$ depends on $\varphi$ and $J$. The second part follows from the first one and Corollary 4. $\square$

4. Convergence Theorem for Increasing Sequences

As in integrable case we define the relative capacity of the Borel subset $E$ of $\Omega$ as
\[
cap(E, \Omega) = \sup\{ \int_E (i\partial\bar{\partial}u)^2 : u \in PSH(\Omega), -1 \leq u \leq 0 \}.
\]
We shall also consider the following set function associated to the hermitian metric $\omega$:
\[
cap_\omega(E, \Omega) = \sup\{ \int_E (i\partial\bar{\partial}u) \wedge \omega : u \in PSH(\Omega), -1 \leq u \leq 0 \}.
\]
When $E \subset \Omega$ then by the Chern-Levine-Nirenberg inequalities we have $\cap(E, \Omega) < +\infty$ and thus if there is a bounded function $h \in PSH(\Omega)$ which satisfies $\omega \leq i\partial\bar{\partial}h$ we also have $\cap_\omega(E, \Omega) < +\infty$. We say that a sequence $u_k$ of plurisubharmonic functions defined on $\Omega$ converge with respect to capacity to a function $u$ if for any compact set $K \subset \Omega$ and $t > 0$
\[
\lim_{k \to \infty} \cap(K \cap \{|u - u_k| > t\}, \Omega) = 0.
\]
In the same way we define convergence with respect to $\cap_\omega$. 
**Proposition 6.** Let \( u_k \) be a sequence of plurisubharmonic functions which decreases to a bounded plurisubharmonic function \( u \). Then it converge with respect to \( \text{cap}_\omega \).

**Proof.** We can assume that all \( u_k \) are equal outside compact set \( E \subset \Omega \). We fix \( v \in \mathcal{P}SH(\Omega) \), \(-1 \leq v \leq 0\). Using integration by parts we can estimate

\[
0 \leq I_k = \int_{\Omega} (u_k - u)i\partial \bar{\partial}v \wedge \omega
= -\int_{\Omega} i\partial(u_k - u) \wedge \bar{\partial}v \wedge \omega + \int_{\Omega} i(u_k - u)\bar{\partial}v \wedge \partial \omega
\leq C\|u_k - u\|_{W^{1,1}(K)}\|v\|_{W^{1,1}(K)}.
\]

By Propositions 2 and 3 we get that \( I_k \to 0 \) as \( k \to \infty \) and (as in [K]) the Proposition follows. \(\square\)

**Proposition 7.** Let \( u \) be a bounded plurisubharmonic function on \( \Omega \) and \( \varepsilon > 0 \). Then, there exists an open set \( U \subset \Omega \) with \( \text{cap}_\omega(U, \Omega) < \varepsilon \) and such that \( u \) restricted to \( \Omega \setminus U \) is continuous.

Using previous result and regularization result from [P2] (see also [H-L-P]) we can prove it exactly like in the case of domains in \( \mathbb{C}^n \) (see for example proof of theorem 1.13 in [K]).

Again exactly as in \( \mathbb{C}^n \), from above Proposition we get

**Corollary 8.** Let \( \mathcal{U} \) be a uniformly bounded family of plurisubharmonic functions in \( \Omega \). Suppose that \( u, v \in \mathcal{U} \) and \( (v_k) \subset \mathcal{U} \) and

\[i\partial \bar{\partial}v_k \to i\partial \bar{\partial}v.\]

Then

\[ui\partial \bar{\partial}v_k \to ui\partial \bar{\partial}v.\]

**Proposition 9.** Let \( \mathcal{U} \) be a uniformly bounded family of plurisubharmonic functions in \( \Omega \). Suppose that \( (u_k), (v_k) \subset \mathcal{U} \) are increase to \( u \) and \( v \) respectively. Then

\[i\partial u_k \wedge \bar{\partial}v_k \to i\partial u \wedge \bar{\partial}v.\]

**Proof.** First we will prove that

\[(1) \quad u_k \bar{\partial}v_k \to u \bar{\partial}v.\]

Let \( \varphi \in \mathcal{D}_{(2,1)} \). Using Stokes theorem we can calculate

\[
\int_{\Omega} u_k \bar{\partial}v_k \varphi - \int_{\Omega} u \bar{\partial}v \varphi
\]
\[
\int_\Omega (u_k - u) \bar{\partial} v_k \varphi + \int_\Omega u \bar{\partial}(v_k - v) \varphi \\
= \int_\Omega (u_k - u) \bar{\partial} v_k \varphi + \int_\Omega (v_k - v) \bar{\partial}(u \varphi).
\]

Since \(L^2\) norms of \(\bar{\partial} v_k \varphi\) and \(\bar{\partial}(u \varphi)\) depends only on \(\varphi\) and \(U\), using Helder inequality, we can choose constant \(C\) not depending on \(k\) such that
\[
\int_\Omega u_k \bar{\partial} v_k \varphi - \int_\Omega u \bar{\partial} v \varphi \leq C (\|u_k - u\|_{L^2(\Omega)} + \|v_k - v\|_{L^2(\Omega)}) \to 0.
\]

Thus (1) follows.

The second step is to obtain the following convergence
\[
(2) \quad u_k i \bar{\partial} v_k \to u i \bar{\partial} v.
\]

Let \(\varphi \in D_{(1,1)}\) be positive. By Corollary 8 we get
\[
\limsup_{k \to \infty} \int_\Omega u_k i \bar{\partial} v_k \wedge \varphi \leq \limsup_{k \to \infty} \int_\Omega u i \bar{\partial} v_k \wedge \varphi = \int_\Omega u i \bar{\partial} v \wedge \varphi.
\]

Set \(s \in \mathbb{N}\). Using Stokes' theorem we can estimate
\[
\liminf_{k \to \infty} \int_\Omega u_k i \bar{\partial} v_k \wedge \varphi \geq \liminf_{k \to \infty} \int_\Omega u_i \bar{\partial} v_k \wedge \varphi = \int_\Omega u_i \bar{\partial} v \wedge \varphi
\]
\[
= \int_\Omega v \bar{\partial} u_s \wedge \varphi + \int_\Omega u_s \bar{\partial} v \wedge \varphi + \int_\Omega v \bar{\partial} u \wedge \bar{\partial} \varphi.
\]

From (1) and again Corollary 8 the last line with \(s \to \infty\) converge to
\[
\int_\Omega v \bar{\partial} u \wedge \varphi + \int_\Omega u \bar{\partial} v \wedge \varphi + \int_\Omega v \bar{\partial} u \wedge \bar{\partial} \varphi = \int_\Omega u i \bar{\partial} v \wedge \varphi.
\]

This together with (3) gives us (2).

In the last step we will finish the proof. By (1) and (2) we can conclude
\[
i \partial u_k \wedge \bar{\partial} v_k = i \partial (u_k \bar{\partial} v_k) - u_k i \bar{\partial} v_k \to i \partial (u \bar{\partial} v) - u i \bar{\partial} v = i \partial u \wedge \bar{\partial} v.
\]

\[\square\]

**Corollary 10.** Suppose that \(u_k\) is a sequence of locally bounded plurisubharmonic functions which increase to plurisubharmonic function \(u\) a. e.. Then \(u_k\) converge to \(u\) in \(W^{1,2}_{\text{loc}}(\Omega)\).

**Theorem 11** (Convergence Theorem for increasing sequences). Suppose that \(u_k\) and \(v_k\) are sequences of locally bounded plurisubharmonic functions which increase to plurisubharmonic functions \(u\) and \(v\) respectively a. e. Then
\[
i \partial u_k \wedge i \bar{\partial} v_k \to i \partial u \wedge i \bar{\partial} v.
\]
5. Pluripolarity

Proposition 12. Assume that $E \subset \Omega \subset M$. Then

$$E \text{plp} \Rightarrow cap(E, \Omega) = 0.$$  

Proof. We can assume that there is compactly supported nonnegative function $\varphi$ which is equal 1 on $E$. Let $U \in \mathcal{PSH}$ is such that $U \neq -\infty$ and $U|_{E} = -\infty$. Set $v \in \mathcal{PSH}$ such that $-1 \leq v \leq 0$. Since $U \in L^1_{loc}$ (see [H-L]) the sequence $U/k$ increase to 0 a.e. on the open set $\Omega^\prime = \{ U < 0 \}$. Thus the sequence $v_k = \max\{U/k, v\}$ increase to 0 a.e. too. From convergence theorem $(i\partial\bar{\partial}v_k)^2 \rightarrow 0$. On the other hand, on the open set $\Omega_k = \{ U < k \}$ we have $v_k = v$. Thus

$$\int_{E} (i\partial\bar{\partial}v)^2 \leq \int_{\Omega^\prime} \varphi (i\partial\bar{\partial}v_k)^2 \rightarrow 0,$$

and we can conclude that $\int_{E} (i\partial\bar{\partial}v)^2 = 0$. Because it is for all $v$ from the definition of the capacity we get that $cap(E) = 0$. □

There is a constant $c_0$ (which depend on $\Omega$) such that

$$\theta\bar{\partial}\varphi \wedge \bar{\theta}\partial\varphi \leq c_0 i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega$$

for any smooth function $\varphi$ defined on $\Omega$. To prove the next result about pluripolarity we need the following Lemma:

Lemma 13. Let $h \in C(\bar{\Omega})$ be such that

$$i\partial\bar{\partial}h \geq 9c_0\omega \text{ on } \Omega \text{ and } \liminf_{z \rightarrow \partial\Omega} h(z) \geq 0.$$  

Let $D = \{ h < 0 \}$. For $u \in \mathcal{PSH} \cap L^\infty(D)$ satisfying

$$\inf u = -1, \liminf_{z \rightarrow \partial D} u(z) \geq 0 \text{ and } (i\partial\bar{\partial}u)^2 = 0 \text{ on } D$$

we have $u \geq h$.

Proof. Set $\varepsilon > 0$. Since every connected component of $D$ is almost Stain manifold there exists a sequence $u_k$ of smooth plurisubharmonic functions on $D$, which decreases to $u + \varepsilon$. Assume that there is $k_1 \in \mathbb{N}$ such that the set $\{ u_k, < \rho \}$ is not empty. By convergence theorem for decreasing sequences there is $k_2 \in \mathbb{N}$ such that for $k \geq k_2$ we have

$$\int_{\{ u < -\varepsilon \}} (i\partial\bar{\partial}u_k)^2 < \frac{1}{9} \int_{\{ u_{k_1} < h \}} (i\partial\bar{\partial}h)^2 \leq \frac{1}{9} \int_{\{ u_k < h \}} (i\partial\bar{\partial}h)^2.$$  

Set $k \geq k_2$ and put $v = u_k$, $\bar{v} = (v + 1)^2 - 1$, $E = \{ v < 0 \}$, $\bar{E} = E \cap \{ \frac{k+1}{3} > v \}$, $F = \{ v < h \}$. Since $2v \geq \bar{v}$ on $E$ and $\bar{v} = v$ on $\partial E$ we have

$$F \subset \bar{E} \subset E.$$
For enough small $\delta > 0$ we still have $E_\delta = E \cap \{ \frac{h+\bar{v}}{3} > v - \delta \} \in E$. Let choose $\delta$ such that the set $\partial E_\delta$ has Lebesgue measure equal 0. Let $\varphi = v - \delta$ and $\psi = \max\{\varphi, \frac{h+\bar{v}}{3}\}$

Using (4), the assumption about $i\partial \bar{\partial} h$ and inequality

$$i\partial \bar{\partial} \bar{v} \geq 2i\partial v \wedge \bar{\partial} v$$

we can estimate

$$\int_E (dd^c \varphi)^2 < \frac{1}{9} \int_E (i\partial \bar{\partial} h)^2 + 2 \int_{E_{\delta}} \bar{\partial} v \wedge \bar{\partial} v + \int_{E \setminus E_{\delta}} (dd^c \psi)^2$$

$$\leq \frac{1}{9} \int_{E_{\delta}} (i\partial \bar{\partial} h)^2 + 2c_0 \int_{E_{\delta}} i\partial v \wedge \bar{\partial} v \wedge \omega + \int_{E \setminus E_{\delta}} (dd^c \psi)^2.$$ 

$$\leq \frac{1}{9} \int_{E_{\delta}} (i\partial \bar{\partial} h)^2 + \frac{2}{9} \int_{E_{\delta}} i\partial v \wedge \bar{\partial} v \wedge i\partial \bar{\partial} h + \int_{E \setminus E_{\delta}} (dd^c \psi)^2 \leq \int_{E} (dd^c \psi)^2.$$ 

But this inequality contradicts with Stokes theorem which gives us that $F$ is empty for any choose of $k \in \mathbb{N}$ and $\varepsilon > 0$. We thus get $u \geq h$. \hfill \Box

**Lemma 14.** Let $\Omega$ be strictly pseudoconvex domain. Let $u \in L^\infty \cap \mathcal{PSH}(\Omega)$ is such that $\lim_{z \to \partial \Omega} u(z) = 0$ and $(i\partial \bar{\partial} u)^2 = 0$. Then $u = 0$ in $\Omega$.

**Proof.** To prove the Lemma by contradiction let us assume that $u \neq 0$. Put $u_1 = \frac{u}{\|u\|_{L^\infty(\Omega)}}$ and $u_{k+1} = 2u_k + 1$ for $k \geq 1$. We can choose the defining function $h_1$ for $\Omega$ such that $i\partial \bar{\partial} h_1 \geq 9c_0 \omega$. Let $h_{k+1} = k_{k+1} + \frac{1}{2} = h_1 + \frac{k}{2}$ and $D_k = \{ z \in \Omega : h_k < 0 \}$. By Lemma 13 and induction we easily get that $h_k \leq u_k$. On the other hand $\inf u_k = -1$ and $\inf h_k \to \infty$. Contradiction! \hfill \Box

For an open set $V \subset M$ and a subset $E \subset V$ we put

$$u_E = u_{E,V} = \sup \{ v \in \mathcal{PSH}(V) : v \leq 0 \text{ and } v|_{E} \leq -1 \}.$$ 

**Lemma 15.** A function $u_E^*$ is plurisubharmonic and $\sup (i\partial \bar{\partial} u_E^*)^2 \subset \partial E$.

**Proof.** By the Choquet theorem there is an increasing sequence of plurisubharmonic functions $u_j \geq -1$ with $(\lim u_j)^* = u_E^*$. Using characterization of plurisubharmonic functions from [H-L] (see Preliminaries) we get that $(\lim u_j)^*$ is plurisubharmonic and the Lebesgue measure of the set $\{ \lim u_j \neq (\lim u_j)^* \}$ is equal 0.

Let $p \in V \setminus E$. There is a domain $D \subset V \setminus E$ which is a smooth strictly pseudoconvex neighborhood of $p$. For $j \in \mathbb{N}$ let $\varphi^{(j)}_k$ be a sequence of
smooth functions which decrease to \( u_j \) on \( \partial D \). By Theorem 1 we can solve Dirichlet problem:

\[
\begin{align*}
  w_k^{(j)} &\in C^\infty(\bar{D}) \cap \text{PSH}(D), \\
  (i\partial \bar{\partial} w_k^{(j)})^2 &= k^{-1} \omega^2, \\
  w_k^{(j)} |_{\partial D} &= \varphi_k^{(j)}.
\end{align*}
\]

Put

\[
w_j = \begin{cases} 
  u_j & \text{on } V \setminus D, \\
  \lim_{k \to \infty} w_k^{(j)} & \text{on } D.
\end{cases}
\]

Then \( w_j \) is a sequence of plurisubharmonic functions increasing a.e. to \( u_E \). Moreover by convergent theorem for decreasing sequences \((i\partial \bar{\partial} w_j)^2 = 0\) on \( D \) and thus by convergent theorem for increasing sequences \((i\partial \bar{\partial} u_E^*)^2 = 0\) on \( D \). But we can choose \( D \) as a neighborhood of any point in \( V \setminus E \) which gives us that \( \text{supp} (i\partial \bar{\partial} u_E^*)^2 \subset \partial E \).

**Proposition 16.** Let \( \Omega \) be a strictly pseudoconvex domain. Assume that \( E \) is \( F_\sigma \) subsets of \( \Omega \) and \( \text{cap}(E, \Omega) = 0 \). Then \( E \) is pluripolar. Moreover there is a plurisubharmonic function \( u \) on \( \Omega \) such that \( u|_E = -\infty \).

**Proof.** Let \( E_i \) be increasing sequence of compact subsets such that

\[
\sum E_i = E.
\]

Put \( w_i = u_{E_i}^* \). By Lemma 15 we get that \((i\partial \bar{\partial} w_i)^2 = 0\). Because \( \Omega \) is strictly pseudoconvex \( \lim_{z \to \partial \Omega} w_i(z) = 0 \) and by Lemma 14 \( w_i = 0 \).

Similar as in Lemma 15 by the Choquet lemma, for any \( i \), there is an increasing sequence of plurisubharmonic functions \( v_k^{(i)} \) such that \( \lim_{k \to \infty} v_k^{(i)} = 0 \) a.e. and \( v_k^{(i)} \leq -1 \) on \( E_i \). By the Lebesgue’s Monotone Convergence Theorem we can choose for any \( i \) a number \( k \) such that for \( h_i = v_k^{(i)} \) we have \( \|h_i\|_{L^1(\Omega)} \leq \frac{1}{2^i} \). We can conclude that a function

\[
u = \sum_{i=1}^{\infty} h_i
\]

is plurisubharmonic and \( u|_E = -\infty \).

**Corollary 17.** For any \( J \)-holomorphic function \( u : \mathbb{D} \to M \), a set \( u(\mathbb{D}) \) is pluripolar.

**Proof.** Let \( p \in u(\mathbb{D}) \). We can choose a strictly pseudoconvex neighbourhood \( U \) of \( p \). Let \( E = u(\mathbb{D}) \cap U \). The function \( u \) have at most countable many singular points (see for example lemma 2.7 in [M]). Thus using Rosay theorem we get that \( E \) is a sum of countable many compact pluripolar sets. This implies that \( \text{cap}(E, U) = 0 \) and by Proposition 16 \( E \) is pluripolar. Thus Corollary follows.
Proposition 18. Let $M$ be an almost stein manifold and let $E \subset M$ be pluripolar $F_\sigma$ set. Then there is a plurisubharmonic function $u$ on $\Omega$ such that $u|_E = -\infty$.

Proof. Let $\rho$ be an exhaustion smooth strictly plurisubharmonic function on $M$. By the Sard’s theorem there is a sequence $(a_k) \subset \mathbb{R}$ for which $a_{k+1} \geq a_k + 1$ and all connected components of $\Omega_k = \{ z \in M : \rho(z) < a_k \}$ are strictly pseudoconvex. Like in the proof of Proposition 16 we can choose a function $u_k \in PSH(\Omega_k)$ such that $-1 \leq u_k \leq 0$, $u_k|_{E \cap \Omega_k} = -1$ and $\|u_k\|_{L^1(\Omega_k)} < \frac{1}{2^k}$. Put

$$v_k = \begin{cases} \max\{\rho - a_{k+1}, u_{k+2}\} & \text{on } \Omega_{k+2}, \\ \rho - a_{k+1} & \text{on } M \setminus \Omega_{k+2}, \end{cases}$$

and $u = \sum v_k$. Since $v_k = u_{k+2}$ on $\Omega_k$ it is clear that $u$ has required properties. □

REFERENCES

[B-T1] E. Bedford, B. A. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math. 37(1976), 1-44,

[B-T2] E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math. 149(1982), 1-41,

[B1] Z. Błocki, The complex Monge-Ampère operator in pluripotential theory, unfinished lecture notes based on graduate course at Jagiellonian University, 1997, (see http://gamma.im.uj.edu.pl/~blocki/publ/ln/wykl.pdf),

[B2] Z. Blocki, On the definition of the Monge-Ampère operator in $\mathbb{C}^2$, Math. Ann. 328 (2004), 415-423,

[C-T-W] J. Chu, V. Tosatti, B. Weinkove, The Monge-Ampère equation for non-integrable almost complex structures, arXiv:1603.00706 to appear in J. Eur. Math. Soc. (JEMS) 2018.

[E] F. Elkhadra, J-pluripolar subsets and currents on almost complex manifolds, Math. Z. 264 (2010), no. 2, 399-422.

[H-L] F. R. Harvey, H. B. Lawson, Jr., Potential Theory on Almost Complex Manifolds, Ann. Inst. Fourier (Grenoble) 65 (2015), no. 1, 171-210,

[H-L-P] F. R. Harvey, H. B. Lawson, Jr., S. Pliš, Smooth Approximation of Plurisubharmonic Functions on Almost Complex Manifolds, Math. Ann. 366 (2016), no. 3-4, 929-940,

[K] S. Kołodziej, The complex Monge-Ampère equation and pluripotential theory, Mem. Amer. Math. Soc. 178 (2005), no. 840,

[L] M. Lejmi, Strictly nearly Kähler 6-manifolds are not compatible with symplectic forms, C. R. Math. Acad. Sci. Paris 343 (2006), no. 11-12, 759-762,

[M] D. McDuff, The local behaviour of holomorphic curves in almost complex 4-manifolds, J. Differential Geom. 34 (1991), no. 1, 143-164,
[P] N. Pali, *Fonctions plurisousharmoniques et courants positifs de type (1,1) sur une variété presque complexe*, Manuscripta Math. 118 (2005), no. 3, 311-337.

[P1] S. Pliś, *The Monge-Ampère equation on almost complex manifolds*, Math. Z. 276 (2014), no. 3-4, 969-983.

[P2] S. Pliś *On the regularization of $J$-plurisubharmonic functions*, C. R. Math. Acad. Sci. Paris 353 (2015), no. 1, 17-19.

[P3] S. Pliś, *Monge-Ampère operator on four dimensional almost complex manifolds*, J. Geom. Anal. 26 (2016), no. 4, 2503-2518.

[R] J.-P. Rosay *J-holomorphic submanifolds are pluripolar*, Math. Z. 253 (2006), no. 4, 659-665.

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