A SCHUR-HORN THEOREM IN II$_1$ FACTORS

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Abstract. Given a II$_1$ factor $\mathcal{M}$ and a diffuse abelian von Neumann subalgebra $A \subset \mathcal{M}$, we prove a version of the Schur-Horn theorem, namely

$$E_A(U_{\mathcal{M}}(b))^{\sigma\text{-sot}} = \{a \in A^{sa} : a \prec b\}, \quad b \in \mathcal{M}^{sa},$$

where $\prec$ denotes spectral majorization, $E_A$ the unique trace-preserving conditional expectation onto $A$, and $U_{\mathcal{M}}(b)$ the unitary orbit of $b$ in $\mathcal{M}$. This result is inspired by a recent problem posed by Arveson and Kadison.

1. Introduction

In 1923, I. Schur [18] proved that if $A \in M_n(\mathbb{C})^{sa}$ (i.e., $A$ is selfadjoint) then

$$\sum_{j=1}^{k} \alpha_j^+ \leq \sum_{j=1}^{k} \beta_j^+, \quad k = 1, \ldots, n,$$

with equality when $k = n$ (denoted $\alpha \prec \beta$), where $\alpha = \text{diag}(A) \in \mathbb{R}^n$, $\beta = \lambda(A) \in \mathbb{R}^n$ the spectrum (counting multiplicity) of $A$, and $\alpha^+, \beta^+ \in \mathbb{R}^n$ are obtained from $\alpha, \beta$ by reordering their entries in decreasing order.

In 1954, A. Horn [12] proved the converse: given $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \prec \beta$, there exists a selfadjoint matrix $A \in M_n(\mathbb{C})$ such that $\text{diag}(A) = \alpha$, $\lambda(A) = \beta$. Since every selfadjoint matrix is diagonalizable, the results of Schur and Horn can be combined in the following assertion: if $\mathcal{D}$ denotes the diagonal masa in $M_n(\mathbb{C})$ and $E_{\mathcal{D}}$ is the compression onto $\mathcal{D}$, then

$$E_{\mathcal{D}}(\{U M_{\beta} U^* : U \in M_n(\mathcal{C}) \text{ unitary}\}) = \{M_{\alpha} \in \mathcal{D} : \alpha \prec \beta\},$$

where $M_\alpha$ is the diagonal matrix with the entries of $\alpha$ in the main diagonal.

This combination of the two results, commonly known as Schur-Horn theorem, has played a significant role in many contexts of matrix analysis: although simple, vector majorization expresses a natural and deep relation among vectors, and as such it has been a useful tool both in pure and applied mathematics. We refer to the books [5, 14] and the introductions of [6, 16] for more on this.

During the last 25 years, several extensions of majorization have been proposed by, among others, Ando [1] (to selfadjoint matrices), Kamei [13] (to selfadjoint operators in a II$_1$ factor), Hiai [10, 11] (to normal operators in a von Neumann algebra), and Neumann [16] (to vectors in $\ell^\infty(\mathbb{N})$). With these generalizations at hand, it is natural to ask about extensions of the Schur-Horn theorem.

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In [16], Neumann developed his extension of majorization with the goal of using it to prove a Schur-Horn type theorem in \( B(H) \) in the vein of previous works in convexity (see the introduction in [16] for details and bibliography). Other versions of the Schur-Horn theorem have been considered in [3] and [6]. It is interesting to note that the motivation in [16] comes from geometry, in [3] comes from the study of frames on Hilbert spaces, while in [6] it is of an operator theoretic nature.

In [6] Arveson and Kadison proposed the study of a Schur-Horn type theorem in the context of II_1 factors, which are for such purpose the most natural generalization of full matrix algebras. They proved a Schur type theorem for II_1 factors and they posed as a problem a converse of this result, i.e. a Horn type theorem. In this note we prove a Schur-Horn type theorem that is inspired by Arveson-Kadison’s conjecture (Theorem 3.4).

2. Preliminaries

Throughout the paper \( \mathcal{M} \) denotes a II_1 factor with normalized faithful normal trace \( \tau \). We denote by \( M^{sa}, M^{+}, U_M \), the sets of selfadjoint, positive, and unitary elements of \( M \), and by \( Z(\mathcal{M}) \) the center of \( M \). Given \( a \in M^{sa} \) we denote its spectral measure by \( p^a \). The characteristic function of the set \( \Delta \) is denoted by \( 1_\Delta \). For \( n \in \mathbb{N} \), the algebra of \( n \times n \) matrices over \( \mathbb{C} \) is denoted by \( M_n(\mathbb{C}) \), and its unitary group by \( U_n \). By \( dt \) we denote integration with respect to Lebesgue measure. To simplify terminology, we will refer to non-decreasing functions simply as “increasing”; similarly, “decreasing” will be used instead of “non-increasing”.

Besides the usual operator norm in \( M \), we consider the 1-norm induced by the trace, \( \|x\|_1 = \tau(|x|) \). As we will be always dealing with bounded sets in a II_1 factor, we can profit from the fact that the topology induced by \( \| \cdot \| \) agrees with the \( \sigma \)-strong operator topology. Because of this we will express our results in terms of \( \sigma \)-strong closures although our computations are based on estimates for \( \| \cdot \|_1 \).

For \( X \subset M \), we shall denote by \( \overline{X} \) and \( \overline{X}^{\sigma\text{-sot}} \) the respective closures in the norm and in the \( \sigma \)-strong operator topology.

2.1. Spectral scale and spectral preorders. The spectral scale [17] of \( a \in M^{sa} \) is defined as

\[
\lambda_a(t) = \min\{ s \in \mathbb{R} : \tau(p^a(s, \infty)) \leq t \}, \quad t \in [0, 1).
\]

The function \( \lambda_a : [0, 1) \to [\|a\|, \infty) \) is decreasing and right-continuous. The map \( a \mapsto \lambda_a \) is continuous with respect to both \( \| \cdot \| \) and \( \| \cdot \|_1 \), since [17]

\[
\|\lambda_a - \lambda_b\|_\infty \leq \|a - b\|, \quad \|\lambda_a - \lambda_b\|_1 \leq \|a - b\|_1 \quad a, b \in M^{sa},
\]

where the norms on the left are those of \( L_\infty([0, 1], dt) \) and \( L^1([0, 1], dt) \) respectively.

We say that \( a \) is submajorized by \( b \), written \( a \prec_w b \), if

\[
\int_0^s \lambda_a(t) \, dt \leq \int_0^s \lambda_b(t) \, dt, \quad \text{for every } s \in [0, 1).
\]

If in addition \( \tau(a) = \tau(b) \) then we say that \( a \) is majorized by \( b \), written \( a \prec b \).

These preorders play an important role in many papers (among them we mention [8, 10, 11, 13]), and they arise naturally in several contexts in operator theory and operator algebras; some recent examples closely related to our work are the study of Young’s type [9] and Jensen’s type inequalities [2, 6, 7, 10].
Theorem 2.1 ([10]). Let $a, b \in \mathcal{M}^{sa}$. Then $a \prec b$ (resp. $a \prec_w b$) if and only if $\tau(f(a)) \leq \tau(f(b))$ for every convex (resp. increasing convex) function $f : J \to \mathbb{R}$, where $J$ is an open interval such that $\sigma(a), \sigma(b) \subseteq J$.

If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra and $b \in \mathcal{M}^{sa}$, we denote by $\Omega_{\mathcal{N}}(b)$ the set of elements in $\mathcal{N}^{sa}$ that are majorized by $b$, i.e.

$$\Omega_{\mathcal{N}}(b) = \{ a \in \mathcal{N}^{sa} : a \prec b \}.$$  

The unitary orbit of $a \in \mathcal{M}^{sa}$ is the set $\mathcal{U}_\mathcal{M}(a) = \{ u^*au : u \in \mathcal{U}_\mathcal{M} \}$.

Proposition 2.2. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_{\mathcal{N}}$ be the trace preserving conditional expectation onto $\mathcal{N}$. Then, for any $b \in \mathcal{M}^{sa},$

(i) $E_{\mathcal{N}}(b) \prec b$.

(ii) $\| E_{\mathcal{N}}(b) \|_1 \leq \| b \|_1$.

(iii) $E_{\mathcal{N}}(\mathcal{U}_\mathcal{M}(b))$ is a von Neumann subalgebra and $\mathcal{U}_\mathcal{M}(b) \prec b$.

Proof. (i) The map $E_{\mathcal{N}}$ is doubly stochastic (i.e. trace preserving, unital, and positive), so this follows from [10, Theorem 4.7] (see also [6, Theorem 7.2]).

(ii) Consider the convex function $f(x) = |x|$. Since $E_{\mathcal{N}}(b) \prec b$, using Theorem 2.1 we get

$$\| E_{\mathcal{N}}(b) \|_1 = \tau(f(E_{\mathcal{N}}(b))) \leq \tau(f(b)) = \| b \|_1.$$ 

(iii) By (i) and the fact that $u^*bu \prec b$ for every $u \in \mathcal{U}_\mathcal{M}$, we just have to prove that $\Omega_{\mathcal{N}}(b)$ is $\| \cdot \|_1$-closed. So, let $(a_n)_{n \in \mathbb{N}} \subset \Omega_{\mathcal{N}}(b)$ be such that $\lim_{n \to \infty} \| a_n - a \|_1 = 0$ for some $a \in \mathcal{N}$. Then, necessarily, $a \in \mathcal{N}^{sa}$. By (2),

$$\int_0^s \lambda_a(t) \, dt = \lim_{n \to \infty} \int_0^s \lambda_{a_n}(t) \, dt \leq \int_0^s \lambda_b(t) \, dt.$$ 

Also, $\tau(a) = \lim_n \tau(a_n) = \tau(b)$, so $a \prec b$. \hfill \Box

The following result seems to be well-known, but we have not been able to find a reference. Thus we give a sketch of a proof.

Proposition 2.3. Let $a \in \mathcal{A}^{sa}$, where $\mathcal{A} \subset \mathcal{M}$ is a diffuse von Neumann subalgebra. Then there exists a spectral resolution $\{ e(t) \}_{t \in [0,1]} \subset \mathcal{A}$ with $\tau(e(t)) = t$ for every $t \in [0,1]$, and such that

$$a = \int_0^1 \lambda_a(t) \, de(t).$$ 

Proof. Since $\tau(1) < \infty$, it is enough to show the result for $a \geq 0$. By [15, Theorem 3.2] there exists $a' \in \mathcal{A}$ with $g_{a'}(s) = \tau(p_a(-\infty,s])$ continuous in $\mathbb{R}$, and an increasing left-continuous function $f$ such that $p_a(-\infty,s] = p_{a'}(-\infty,f(s))$. Although the original statement in [15] involves a masa, only the fact that the algebra is diffuse is needed for its proof.

Let $g_a'(s) = \min\{ s : g_{a'}(s) \geq t \}$, and let $q(t) = p_{a'}(-\infty,g_a'(t])$. Since $\tau(q(t)) = g_{a'}(g_a'(t)) = t$ for every $t \in [0,1]$, it follows that $\{ q(t) \}_{t \in [0,1]}$ is a continuous spectral resolution. Moreover, $q(g_{a'}(t)) = p_{a'}(-\infty,t]$, so $p_a(-\infty,t] = q(g_{a'}(f(t)))$. As $g_{a'} \circ f$ is increasing and right-continuous, by [15, Theorem 4.4] there exists an increasing and left-continuous function $h_a$ such that $a = \int h_a(t) \, dq(t)$. Define $e(t) = 1 - q(1-t)$, and $h(t) = h_a(1-t)$. Then $\tau(e(t)) = t$, $a = \int h(t) \, de(t)$. As $h$ is decreasing and right-continuous, it can be seen that $h = \lambda_a$. \hfill \Box
A spectral resolution \( \{ e(t) \}_{t \in [0,1]} \) as in Proposition \( 2.3 \) is called a complete flag for \( a \).

3. A Schur-Horn Theorem for II \(_1\) factors

For each \( n \in \mathbb{N}, k \in 1, \ldots, 2^n \), let \( \{ I_k^{(n)} \}_{k=1}^{2^n} \) be the partition of \([0,1]\) associated to the points \( \{ h 2^{-n} : h = 0, \ldots, 2^n \} \).

**Definition 3.1.** For each \( n \in \mathbb{N} \) and every \( f \in L^1([0,1]) \), let

\[
E_n(f) = \sum_{i=1}^{2^n} \left( 2^n \int_{I_i^{(n)}} f \right) 1_{I_i^{(n)}}.
\]

It is clear that each operator \( E_n \) is a linear contraction for both \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \). Given the flag \( \{ e(t) \}_{t \in [0,1]} \), we write \( e([t_0, t_1]) \) for \( e(t_1) - e(t_0) \). Note that, since we consider \( e(t) \) diffuse, \( e([t_0, t_1]) = e([t_0, t_1]) = e([t_0, t_1]) \).

**Lemma 3.2.** Let \( \{ e(t) \}_{t \in [0,1]} \), \( \{ I_i^{(n)} \}_{i=1}^{2^n}, n \in \mathbb{N} \) and \( \{ E_n \}_{n \in \mathbb{N}} \) as above. Then, for each \( a \in \mathcal{M}^a \),

\[
\lim_{n \to \infty} \| a - \int_0^1 E_n(\lambda_n)(t) de(t) \|_1 = 0.
\]

**Proof.** By continuity of the trace, we only need to check that

\[
\text{lim}_{n \to \infty} \| \lambda_n - E_n(\lambda_n) \|_1 = 0
\]

in \( L^1([0,1]) \). Consider first a continuous function \( g \). By uniform continuity, \( \| g - E_n(g) \|_1 \to 0 \). Since continuous functions are dense in \( L^1([0,1]) \) and because the operators \( E_n \) are \( \| \cdot \|_1 \)-contractive for every \( n \in \mathbb{N} \), a standard \( \varepsilon/3 \) argument proves (4) for any integrable function. \( \square \)

Recall that \( D \) denotes the diagonal masa in \( M_\alpha(\mathbb{C}) \), and that for \( \alpha \in \mathbb{R}^n \) we denote by \( M_\alpha \) the matrix with the entries of \( \alpha \) in the diagonal and zero off-diagonal. The projection \( E_D \) of \( M_\alpha(\mathbb{C}) \) onto \( D \) is then given by \( E_D(A) = M_{\text{diag}(A)} \), where \( \text{diag}(A) \in \mathbb{R}^n \) is the main diagonal of \( A \). We use \( \{ e_{ij} \} \) to denote the canonical system of matrix units in \( M_\alpha(\mathbb{C}) \).

**Lemma 3.3.** Let \( \mathcal{N} \subset \mathcal{M} \) be a von Neumann subalgebra, and assume \( E_{\mathcal{N}} \) denotes the unique trace preserving conditional expectation onto \( \mathcal{N} \). Let \( \{ p_i \}_{i=1}^n \subset Z(\mathcal{N}) \) be a set of mutually orthogonal equivalent projections such that \( \sum_{i=1}^n p_i = I \). Then there exists a unital \( * \)-monomorphism \( \pi : M_\alpha(\mathbb{C}) \to \mathcal{M} \) satisfying

\[
\pi(e_{ii}) = p_i, \quad 1 \leq i \leq n,
\]

\[
E_{\mathcal{N}}(\pi(A)) = \pi(E_D(A)), \quad A \in M_\alpha(\mathbb{C}).
\]

**Proof.** Since the projections \( p_i \) are equivalent in \( \mathcal{M} \), for each \( i \) there exists a partial isometry \( v_{1i} \) such that \( v_{1i}v_{1i}^* = p_i \) and \( v_{1i}^*v_{1i} = p_i \). Let \( v_{i1} = p_1, v_{ii} = v_{1i}^* \) for \( 2 \leq i \leq n \) and \( v_{ij} = v_{1i}v_{1j} \) for \( 1 \leq i, j \leq n \). In this way we get the standard associated system of matrix units \( \{ v_{ij} \}, 1 \leq i, j \leq n \) in \( \mathcal{M} \). Define \( \pi : M_\alpha(\mathbb{C}) \to \mathcal{M} \) by \( \pi(A) = \sum_{i,j=1}^n a_{ij} v_{ij} \).

The matrix unit relations imply that \( \pi \) is a \( * \)-homomorphism and it is clear that (5) is also satisfied. Moreover,

\[
E_{\mathcal{N}}(v_{ij}) = E_{\mathcal{N}}(p_i v_{ij} p_j) = p_i E_{\mathcal{N}}(v_{ij}) p_j = \delta_{ij} p_i,
\]
since \( p_ip_j = \delta_{ij}p_i, \) \( E_N(v_{ii}) = E_N(p_i) = p_i, \) and \( p_i \in Z(N). \) Finally, we check (6):
\[
E_N(\pi(A)) = \sum_{i,j} a_{ij} E_N(v_{ij}) = \sum_i a_{ii} p_i = \pi(E_D(A)).
\]
\[
\square
\]

Next we state and prove our version of the Schur-Horn theorem for \( \Pi_1 \) factors. Note the formal analogy with (1).

**Theorem 3.4.** Let \( A \subset M \) be a diffuse abelian von Neumann subalgebra and let \( b \in M^{sa}. \) Then
\[
E_A(U_M(b))^{\sigma-sot} = \Omega_A(b).
\]

**Proof.** By Proposition (2.2) we only need to prove \( E_A(U_M(b))^{\sigma-sot} \supseteq \Omega_A(b). \)

So let \( a \in A^{sa} \) with \( a < b. \) By Proposition (2.3)
\[
a = \int_0^1 \lambda_a(t) dt = \int_0^1 \lambda_b(t) df(t)
\]
where \( \{e(t)\} \) and \( \{f(t)\} \) are complete flags with \( \tau(e(t)) = \tau(f(t)) = t, e(t) \in A, \)
\( t \in [0,1]. \)

Let \( \{I_i^{(n)}\}_{i=1}^{2^n}, n \in \mathbb{N}, \) be the family of partitions considered before and let \( \epsilon > 0. \) By Lemma (3.2) there exists \( n \in \mathbb{N} \) such that
\[
\left\| a - \sum_{i=1}^{2^n} \alpha_i p_i \right\|_1 < \epsilon, \quad \left\| b - \sum_{i=1}^{2^n} \beta_i q_i \right\|_1 < \epsilon,
\]
where \( \alpha_i = 2^n \int_{I_i^{(n)}} \lambda_a(t) dt, \beta_i = 2^n \int_{I_i^{(n)}} \lambda_b(t) dt, p_i = e(I_i^{(n)}), q_i = f(I_i^{(n)}), 1 \leq i \leq 2^n. \) Note that \( \tau(p_i) = \tau(q_i) = 2^{-n}. \) Let \( \alpha = (\alpha_1, \ldots, \alpha_{2^n}), \beta = (\beta_1, \ldots, \beta_{2^n}) \in \mathbb{R}^{2^n}. \) From the fact that \( \lambda_a \) and \( \lambda_b \) are decreasing, the entries of \( \alpha \) and \( \beta \) are already in decreasing order. Using that \( a < b \) IN \( M, \) we conclude that \( a < \beta \) in \( \mathbb{R}^n. \)

By the classical Schur-Horn theorem (1), there exists \( U \in \mathcal{U}_n(\mathbb{C}) \) such that
\[
E_D(U M_3 U^*) = M_\alpha
\]
Consider the *-monomorphism \( \pi \) of Lemma (3.3) associated with the orthogonal family of projections \( (p_i)_{i=1}^{2^n} \subset A. \)
Let \( w \in U_M \) such that \( wq_iw^* = p_i, i = 1, \ldots, 2^n, \)
and put \( u := \pi(U) w \in U_M. \) By (6) and (9),
\[
E_A\left( u \left( \sum_{i=1}^{2^n} \beta_i q_i \right) u^* \right) = E_A(\pi(U M_3 U^*)) = \pi(E_D(U M_3 U^*)) = \pi(M_\alpha) = \sum_{i=1}^{2^n} \alpha_i p_i.
\]
Using (8), we conclude that
\[
\left\| E_A(ubu^*) - a \right\|_1 \leq \left\| E_A(u(b - \sum_{i=1}^{2^n} \beta_i q_i) u^*) \right\|_1 + \left\| a - \sum_{i=1}^{2^n} \alpha_i p_i \right\|_1 < 2\epsilon.
\]
As \( \epsilon \) was arbitrary, we obtain \( a \in E_A(U_M(b))^{\sigma-sot}. \)
\[
\square
\]

**Corollary 3.5.** For each \( b \in M^+, \) the set \( E_A(U_M(b))^{\sigma-sot} \) is convex and \( \sigma \)-weakly compact.
In \[6\], Arveson and Kadison posed the problem whether for \(b \in \mathcal{M}^{sa}\), with the notations of Theorem 3.4,
\[
E_A \left( \overline{U_M(b)} \right) = \Omega_A(b).
\]
Since (13)
\[
U_M(b) = U_M(b)^{\sigma-sot} = \{ a \in \mathcal{M}^{sa} : \lambda_a = \lambda_b \},
\]
an affirmative answer to the Arveson-Kadison problem is equivalent to
(10)
\[
E_A \left( \overline{U_M(b)}^{\sigma-sot} \right) = \Omega_A(b).
\]
As a description of the set \(\Omega_A(b)\), (7) is weaker than (10), since in general
(11)
\[
E_A(\overline{U_M(b)}^{\sigma-sot}) \subset E_A(\overline{U_M(b)})^{\sigma-sot}.
\]
An affirmative answer to the Arveson-Kadison problem would imply equality in (11). We think it is indeed the case, although a proof of this does not emerge from our present methods.

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