Abstract

There are a few results about the global stability of nontrivial solutions to quasilinear wave equations. In this paper we are concerned with the uniqueness and stability of traveling waves to the time-like extremal hypersurface in Minkowski space. Firstly, we can get the existence and uniqueness of traveling wave solutions to the time-like extremal hypersurface in $\mathbb{R}^{1+(n+1)}$, which can be considered as the generalized Bernstein theorem in Minkowski space. Furthermore, we also get the stability of traveling wave solutions with speed of light to time-like extremal hypersurface in $1+(2+1)$ dimensional Minkowski space.

Keywords: Quasilinear wave equations; Time-like extremal surface; Stability; Traveling wave solutions.

1. Introduction and main results

The extremal surface in Minkowski space is the $C^2$ surface with vanishing mean curvature. The time-like extremal surface is an interesting model which may be viewed as simple but nontrivial examples of membrane in field theory. The equation to time-like extremal hypersurface in $1+(n+1)$ dimensional Minkowski space are as follows

$$
\left(\frac{v_t}{\sqrt{1+|\nabla v|^2-v_t^2}}\right)_t - \nabla\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2-v_t^2}}\right) = 0. 
$$

(1.1)

where $\Delta = 1 + |\nabla v|^2 - v_t^2 > 0$, $v(t,x)$ is the scalar function, $t$ is the time variable and $x = (x_1, \cdots, x_n)$ is the space variable.

In this paper, we will give the uniqueness and stability of the traveling wave solution to the time-like extremal hypersurface in Minkowski space. There are two main parts. Firstly, we will give the existence and uniqueness of traveling wave solution to time-like extremal surface in Minkowski space, which is correspondent to the famous Bernstein theorem of minimal surface in $\mathbb{R}^n$. The classical Bernstein Theorem is solved by Bernstein in three dimensional Riemannian manifold [4]. It was proved in dimensions up to 8 by [33], [13], [12], [3], [38], [7]. For the space-like maximal surface in a $n$-dimensional Lorentzian manifold, there is the similar Calabi-Bernstein theorem, which was first proved by Calabi in [9], and extended to the general $n$-dimensional case by Cheng and Yau [10]. We can also refer to [23], [34], [15], [7], [30], [4]. Now we will consider the Bernstein type theorem of the system (1.1) and find out the representation of traveling wave solution. We assume that there exists a traveling wave solution of the form

$\text{...}$

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f(x − ct), where \( f \) is a scalar function. Without loss of generality, let the generalized velocity is \( \vec{c} = (0, \ldots, 0, c) \), \( c \neq 0 \).

Then, \( \Delta = 1 + \sum_{i=1}^{n-1} |\partial_i f|^2 + (1 - c^2) |\partial_n f|^2 \). Therefore, system (1.1) can be rewritten as

\[
c^2 \partial_n \left( \frac{\partial_n f}{\sqrt{\Delta}} \right) - \partial_1 \left( \frac{\partial_1 f}{\sqrt{\Delta}} \right) - \cdots - \partial_{n-1} \left( \frac{\partial_{n-1} f}{\sqrt{\Delta}} \right) - \partial_n \left( \frac{\partial_n f}{\sqrt{\Delta}} \right) = 0.
\] (1.2)

Thus

\[
\partial_1 \left( \frac{\partial_1 f}{\sqrt{\Delta}} \right) + \cdots + \partial_{n-1} \left( \frac{\partial_{n-1} f}{\sqrt{\Delta}} \right) + (1 - c^2) \partial_n \left( \frac{\partial_n f}{\sqrt{\Delta}} \right) = 0.
\] (1.3)

When \( |c| < 1 \), let \( x_n' = \frac{1}{\sqrt{1-c^2}}(x_n - ct) \), the above system (1.3) can be rewritten as

\[
\partial_1 \left( \frac{\partial_1 f}{\sqrt{\Delta}} \right) + \cdots + \partial_{n-1} \left( \frac{\partial_{n-1} f}{\sqrt{\Delta}} \right) + \tilde{\partial}_n \left( \frac{\partial_n f}{\sqrt{\Delta}} \right) = 0.
\] (1.4)

where \( \tilde{\partial}_n = \partial_n \). Then, system (1.4) can be considered as the equation to minimal surface in \( \mathbb{R}^n \). By Bernstein theorem of minimal surface in Euclidean space, \( f(x - ct) \) is the linear function of \( x_1, \ldots, (x_n - ct) \) for \( n \leq 8 \). Then, we can get the affine solutions of time like extremal surface

\[
f = a_1 x_1 + \cdots + a_n x_n' + b = a_1 x_1 + \cdots + \frac{1}{\sqrt{1-c^2}}(x_n - ct) + b.
\]

For the stability of this kind of flat plane solution for time-like extremal surface in Minkowski space, Allen et al. gave the positive answer about its stability in [1].

When \( c = 1 \), we can get \( \Delta = 1 + \sum_{i=1}^{n-1} |\partial_i f|^2 \). Therefore, the system (1.3) can be rewritten as

\[
\partial_1 \left( \frac{\partial_1 f}{\sqrt{\Delta}} \right) + \cdots + \partial_{n-1} \left( \frac{\partial_{n-1} f}{\sqrt{\Delta}} \right) = 0.
\] (1.5)

Then, the equation (1.5) can be considered as the minimal surface equation in \( \mathbb{R}^{n-1} \), which is independent of the \( n \)-th variable. Using Bernstein theorem in Euclidean space, we can get

\[
f(x_1, \ldots, x_n \pm t) = (a_1 x_1 + a_2 x_2 + \cdots + a_{n-1} x_{n-1} + b)F(x_n \pm t).
\] (1.6)

We can easily check the following form

\[
f(x_1, \ldots, x_n \pm t) = a_1 x_1 F_1(x_n \pm t) + a_2 x_2 F_2(x_n \pm t) + \cdots + a_{n-1} x_{n-1} F_{n-1}(x_n \pm t) + b F_n(x_n \pm t)
\] (1.7)

is also the exact solution of time-like extremal surface equation, where \( F_i \) (\( i = 1, \ldots, n \)) are \( C^2 \) functions.

**Remark 1.** In Minkowski space \( \mathbb{R}^{1+(1+n)} \), the authors [24] gave the coefficient and necessary condition of the global classical solution to time-like extremal surface in one dimensional space. Liu and Zhou gave the asymptotic behavior to global classical solutions, which tends to the combinations of traveling wave solutions [26] and got the exact solutions of the traveling wave solutions with the form \( \phi(x \pm t) \). The authors also got the stability of traveling wave solution to Cauchy problem to the equation of timelike extremal surface in Minkowski space \( \mathbb{R}^{1+(1+n)} \) [30]. The global existence of the initial boundary value problem of timelike extremal surface equation was studied in [28] and [29].

**Remark 2.** In this case, we get the exact solutions with the form as (1.6) or (1.7) for \( n \leq 9 \) for time-like extremal hypersurface in Minkowski space. It is different to the Bernstein theorem of minimal surface. In the second part in this paper, we will also consider the global stability of traveling wave solution having the special form with the speed of light.
When \( c > 1, \Delta = 1 + \sum_{i=1}^{n-1} |\partial_i f|^2 - (c^2 - 1)|\partial_{n} f|^2 \). Therefore,

\[
\partial_n \left( \frac{\partial_n f}{\sqrt{\Delta}} \right) - \frac{1}{c^2-1} \partial_1 \left( \frac{\partial_1 f}{\sqrt{\Delta}} \right) - \frac{1}{c^2-1} \partial_2 \left( \frac{\partial_2 f}{\sqrt{\Delta}} \right) - \cdots - \frac{1}{c^2-1} \partial_{n-1} \left( \frac{\partial_{n-1} f}{\sqrt{\Delta}} \right) = 0.
\]

Using the variable transformation \( x'_i = \frac{1}{\sqrt{c-1}}(x_i - ct) \), we have

\[
\tilde{\partial}_n \left( \frac{\tilde{\partial}_n f}{\sqrt{\Delta'}} \right) - \tilde{\partial}_1 \left( \frac{\tilde{\partial}_1 f}{\sqrt{\Delta'}} \right) - \tilde{\partial}_2 \left( \frac{\tilde{\partial}_2 f}{\sqrt{\Delta'}} \right) - \cdots - \tilde{\partial}_{n-1} \left( \frac{\tilde{\partial}_{n-1} f}{\sqrt{\Delta'}} \right) = 0. \tag{1.8}
\]

where \( \tilde{\partial}_n = \partial_{c'} \), \( \Delta' = 1 + \sum_{i=1}^{n-1} |\tilde{\partial}_i f|^2 - |\tilde{\partial}_{n'} f|^2 \). Then, we find that the system (1.8) is the equation of time-like extremal hypersurface in Minkowski space \( \mathbb{R}^{1,n-1} \).

**Remark 3.** The above results can be considered as the generalized Bernstein theorem of the time-like extremal surface in Minkowski space.

The equation (1.1) can be considered as the \( \Sigma \)-dimensional quasilinear wave equation. Most of the global results to nonlinear wave equations are concerned with Cauchy problem with small initial data, especially in high space dimension case. Recently, one kind of large solution called “short pulse solution” are considered in [11], [21]. For semilinear wave equations satisfying the null condition, global solution with large initial data is considered in [40], [41], [42]. Wang and Wei gave the global existence of short pulse solution to relativistic membrane equations [43]. For the stability of time-like extremal surface in Minkowski space, Brendle obtained the stability of one kind of Wenerstrass representation [31]. Recently, for the vanishing mean curvature equation, the existence of global smooth solutions for small initial data has been addressed successfully by Lindblad [25]. Allen, Andersson and Isenberg [11] proved the small data global existence for timelike extremal submanifold with codimension larger than one. Here, we first consider the stability of a class of traveling wave solution with the velocity 1. We denote \( v(t,x) \) as a small perturbation of the traveling wave solution with the speed of light. By rotational symmetry, we assume that the traveling wave is of the form \((ax_2 + b)F(x_1 + t)\). Let

\[
v(t,x) = (ax_2 + b)F(x_1 + t) + u(t,x) \tag{1.10}
\]

where \( x = (x_1, x_2) \). Then, we can get

\[
\frac{u_t + (ax_2 + b)F_t}{\sqrt{1 - Q_0(u,u) - 2((ax_2 + b)F(u_t - u_{x_1}) - au_{x_1}F')}} - \sum_{i=1}^{2} \frac{u_{x_i} + (ax_2 + b)F_{x_i}}{\sqrt{1 - Q_0(u,u) - 2((ax_2 + b)F'(u_t - u_{x_i}) - au_{x_i}F')}} \psi = 0. \tag{1.11}
\]

where \( Q_0(\phi, \psi) \equiv \phi_t \psi_t - \phi_{x_i} \psi_{x_i} - \phi_{x_j} \psi_{x_j} \) is the null form (see [20], [2]). Barbashov, Nesterenko and Chervyakov studied the nonlinear differential equations and obtained explicitly their general solutions to relativistic string in one dimensional case [3]. Milnor described all entire time-like minimal surfaces in the three-Minkowski space via a kind of Weierstrass representation [31]. Recently, for the vanishing mean curvature equation, the existence of global smooth solutions for small initial data has been addressed successfully by Lindblad [25]. Allen, Andersson and Isenberg [11] proved the small data global existence for timelike extremal submanifold with codimension larger than one. Here, we first consider the stability of a class of traveling wave solution with the velocity 1. We denote \( v(t,x) \) as a small perturbation of the traveling wave solution with the speed of light. By rotational symmetry, we assume that the traveling wave is of the form \((ax_2 + b)F(x_1 + t)\). Let

\[
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\]

where \( x = (x_1, x_2) \). Then, we can get

\[
\frac{u_t + (ax_2 + b)F_t}{\sqrt{1 - Q_0(u,u) - 2((ax_2 + b)F(u_t - u_{x_1}) - au_{x_1}F')}} - \sum_{i=1}^{2} \frac{u_{x_i} + (ax_2 + b)F_{x_i}}{\sqrt{1 - Q_0(u,u) - 2((ax_2 + b)F'(u_t - u_{x_i}) - au_{x_i}F')}} \psi = 0. \tag{1.11}
\]
We recast the system (1.11) as follows
\[
\Box u = \frac{1}{2} \frac{Q_0(u + (ax_2 + b)F, Q_0(u, u) + 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F)}{1 - Q_0(u, u) - 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F}
\]
\[
= \frac{1}{2}(1 - \hat{H})Q_0(u + (ax_2 + b)F, Q_0(u, u) + 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F)
\]
where \( \Box = \partial_{tt} - \partial_{x_11} - \partial_{x_22} \) and \( \hat{H} = \frac{1}{1 - Q_0(u, u) - 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F} + 1. \)

Then, because of the traveling wave solutions, there is one more linear term in above system than the original system. We can also rewrite the system as
\[
\Box u - Q_0(ax_2 + b)F, 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F)
\]
\[
= \frac{1}{2}(1 - \hat{H})Q_0(u, Q_0(u, u) + 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F) + Q_0((ax_2 + b)F, Q_0(u, u)))
\]
\[
= \frac{1}{2} \hat{H}Q_0((ax_2 + b)F, 2[(ax_2 + b)F'(u_1 - u_{x_1})] - au_{x_1}F)
\]

It is easily get the above system is also hyperbolic.

Under the assumptions
\[
\hat{H}_1 \quad ||(\xi, \frac{d}{d\xi})e^k(\xi)F(\xi)| \leq C_{k_1}(2 + \xi)^{-1}
\]
where \( k_1 \geq 0, k_2 \geq 0, k_1 + k_2 \leq s \) \( (s \geq 13) \) and \( t = x_1, \) we shall consider the following Cauchy problem
\[
 t = 0: \quad u = f(x), \quad u_t = g(x), \quad x \in \mathbb{R}^2
\]

with
\[
\text{Supp } \{f, g\} \subset \{x \mid |x| \leq 1\} \quad \text{and} \quad \|f\|_{H^{s-1}} + \|g\|_{H^s} < \varepsilon, \quad s \geq 13.
\]

By the finite propagation speed of waves, we can obtain
\[
\text{Supp } u(t, \cdot) \subset \{x \mid |x| \leq t + 1\}.
\]

Then, we can get the following result

**Theorem 1.1.** Under the assumption \( \hat{H}_1 \), there exists the global classical solutions to Cauchy problem (1.11) (1.13), provided that \( \varepsilon \) is sufficiently small.

**Remark 4.** The above main result establishes some kind of stability of the traveling wave solution \( F(x_1 + t) \) for the equation of time-like extremal hypersurface in Minkowski space. For the global solution \( u, u_t, \) and \( u_{x_1} \) are decaying in \( \xi \) with exponent \(-1/4\) and \( u_t, \) increasing in \( \xi \) with the rate \((2 + \xi)\delta\). The parameter \( \delta \) is the arbitrary small positive constant.

**Remark 5.** Using the above main result, we can also get the interesting result above the stability of certain kind of traveling wave solution with the speed larger than 1. For \( n = 2, \) system (1.8) can be considered as the time-like extremal hypersurface in Minkowski space \( \mathbb{R}^{1+1} \). Using the result in [28], there is an exact traveling solutions \( \Phi(x_1 \pm ct) \). The exact traveling wave solution of time-like extremal hypersurface in Minkowski space \( \mathbb{R}^{1+21} \) is \( \Phi(x_1 \pm \sqrt{\frac{c}{c-1}}(x_2 + c)) \), where the speed of traveling wave \( c \) is large than the speed of light. Using Lorentz transformation
\[
\tilde{x}_1 = \frac{\sqrt{c^2 - 1}}{c} x_1 + \frac{1}{c} x_2, \quad \tilde{x}_2 = \frac{1}{c} x_1 - \frac{\sqrt{c^2 - 1}}{c} x_2
\]

we can get the traveling wave solution \( \Phi(\sqrt{\frac{c}{c-1}}(\tilde{x}_1 - t)) = \tilde{\Phi}(\tilde{x}_1 - t) \). By the above main result of Theorem 1.1, we can get the stability of this kind of traveling wave solution.
Remark 6. The above result establishes the global existence of classical solutions for a class of large initial data of quasilinear wave equations.

By the local existence result of nonlinear wave equations with small initial data, we can get the classical solutions in the time interval \([-2, 0]\). For getting the global existence result for nonlinear wave equations, the now classical method is to use Lorentz invariance and introduce the Klainerman’s vector fields. We will introduce operator \(Z\) which are infinitesimal generators of the Lorentz group as follows

\[
Z = \{\partial_t, \partial_{x_1}, \partial_{x_2}, L_0, L_1, L_2, \Omega\}
\]

(1.15)

where

\[
L_0 = t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2}, L_1 = x_1\partial_t + t\partial_{x_1}, L_2 = x_2\partial_t + t\partial_{x_2}, \Omega = x_1\partial_{x_2} - x_2\partial_{x_1}.
\]

(1.16)

These operators do not communicate with the linearized equation for \(u\). Compared to the \(Z\) operators, the \(\Gamma\) operator has just one operator less. However, this is the crucial point to prove our main result. Although, we can also get the decay in the global Klainerman Sobolev inequality in Klein-Gorden equations without the commutator \(L_0\) and wave equations with multiple speed without the commutator \(L_i\) by the Klainerman-Sideris inequality. Here we cannot get any decay of the classical solutions in \(t\) direction with only \(\Gamma\) operators. Because the effect of traveling wave solution, we will consider the system in Goursat coordinates. Fortunately, using the Goursat coordinates, we can get the decay in \(\xi\) direction, which is also more weak (only of exponent \(-1/4\)) than the usual Cauchy problem (which is \(-1/2\) in \(t\) direction). It is another main difficulty in our problem. Therefore, in the following we can recast our problem to study the generalized Goursat problem. Let \(\xi = t + x_1, \eta = t - x_1\), we consider the Goursat problem with the data as follows

\[
\xi = -1 : v = h_1(\eta, x_2); \quad \eta = -1 : v = h_2(\xi, x_2)
\]

and satisfy the compatibility condition of order \(s + 1\) at the line \((\xi, \eta) = (-1, -1)\). Moreover, by the local existence theorem of quasilinear wave equation, we have

\[
||h_1||_{H^{s+1}} \leq C_0\varepsilon, \quad ||h_2||_{H^{s+1}} \leq C_0\varepsilon
\]

where \(C_0\) is a positive constant independent of \(\varepsilon\). Moreover,

\[
\text{Supp } h_1 \subset \{-1 \leq \eta \leq 0, -1 \leq x_2 \leq 1\}
\]

\[
\text{Supp } h_2 \subset \{-1 \leq \xi \leq 0, -1 \leq x_2 \leq 1\}
\]

Noting (1.14), we can get

\[
\text{Supp } v \subset \{(\xi, \eta, x_2) | x_2| \leq \sqrt{(2+\xi)(2+\eta)}\].
\]

Therefore, we will consider the generalized Goursat problem in coordinates \((\xi, \eta, x_2)\) instead of the original system.
2. Preliminaries

For getting the stability result of the traveling wave solutions, we will give the key estimates in this section, which plays an important role in proving our main result. Noting (1.18), in coordinates \((\xi, \eta)\), we have

\[
\begin{aligned}
\Gamma_1 &= 2\partial_\xi, \quad \Gamma_2 = 2\partial_\eta, \quad \Gamma_3 = \partial_{x_2}, \\
\Gamma_4 &= 2\eta\partial_\eta + x_2\partial_{x_2}, \quad \Gamma_5 = 2\xi\partial_\xi + x_2\partial_{x_2}, \quad \Gamma_6 = \xi\partial_{x_2} + 2x_2\partial_\eta.
\end{aligned}
\] (2.1)

Firstly, the elementary facts about \(\Gamma\) operators are as follows

**Lemma 2.1.** \([27]\) Noting the relations of \(Z\) and \(\Gamma\), we can easily get

\[
\Gamma^4 Q_0(\phi, \psi) = \sum_{0 \leq k_1 + k_2 \leq k} A_{k_1 k_2} Q_0(\Gamma^{k_1} \phi, \Gamma^{k_2} \psi),
\] (2.2)

\[
\Box \Gamma^4 \psi = \Gamma^4 \Box \psi + \sum_{k < k_1 + k_2} A^{(1)}_{k_1 k_2} \Gamma^{k_1} \Box \Gamma^{k_2} \psi.
\] (2.3)

Through a simple computation, we can also get

\[
\Box = 4\partial_\xi \partial_\eta - \partial_{x_2}, \quad Q_0(\phi, \psi) = 2(\phi_2 \psi_\eta + \phi_\eta \psi_\xi) - \phi_{x_2} \psi_{x_2}.
\] (2.4)

In the following, we will consider the estimates of the commutators in coordinates \((\xi, \eta, x_2)\).

**Lemma 2.2.** \([17]\) In Goursat coordinates, for the null form \(Q_0\), there hold

\[
|Q_0(\phi, \psi)| \lesssim (2 + \xi)^{-1} (|\Gamma^4 \phi| |\nabla \psi| + |\nabla \phi| |\Gamma^4 \psi|),
\] (2.5)

\[
|Q_0(\phi, \psi)| \lesssim (2 + \eta)^{-1} (|\Gamma^4 \phi| (|\psi| + |\phi|) + (|\phi_2| + |\phi_{x_2}|) |\Gamma^4 \psi|),
\] (2.6)

where \(\nabla = \{\partial_\eta, \partial_{x_2}\}\).

**Proof.** Noting the null form of (2.4), we have

\[
(2 + \xi)Q_0(\phi, \psi) = 2Q_0(\phi, \psi) + \xi[2(\phi_2 \psi_\eta + \phi_\eta \psi_\xi) - \phi_{x_2} \psi_{x_2}]
\]

\[
= 2Q_0(\phi, \psi) + \Gamma_5 \phi_2 \psi_\eta + \Gamma_5 \phi_\eta \psi_\xi - \frac{1}{2} \Gamma_6 \phi \psi_{x_2} - \frac{1}{2} \Gamma_6 \phi \psi_{x_2}.
\]

Then, we can get the estimate of (2.5). By the similar way, we can easily obtain the estimate (2.6). \(\Box\)

**Lemma 2.3.** Let \(\phi\) having the compact support as \((1.17)\), we have

\[
|\phi(\xi, \eta, \cdot)|_{L^2(D)} \lesssim |\phi_2(\xi, \xi, \cdot)|_{L^2(\Gamma^2)}
\] (2.7)

\[
|\phi(\xi, \eta, x_2)|_{L^2(D)} \lesssim \sup_{x_2} |\phi_2(\xi, \xi, \cdot)|
\] (2.8)

where, the domain \(D = [(\eta, x_2)| -1 \leq \eta < +\infty, \ -\infty < x_2 < +\infty]\).

**Proof.** We first prove the estimate (2.7). It is only necessary to prove

\[
|f(x_2)|_{L^2(\Gamma^2)} \lesssim 2|f_2|_{L^2(\Gamma)}
\] (2.9)
provided that \( \text{Supp} f \subset \{ x_2 \mid |x_2| < a \} \). In fact, we can get the desired estimate by taking \( a = \sqrt{(3 + \xi)(3 + \eta)} \), \( f = \phi \) and taking \( L^2 \)-norm on the both side of (2.9) for \( \eta \).

\[
\frac{|f(x_2)|}{a - |x_2|^2} = \int_{(a-x_2)^2}^{|f(x_2)|^2} (a - |x_2|^2)dx_2
\]

\[
= \int_{-a}^{a} \frac{|f(x_2)|^2}{(a + x_2)^2}dx_2 + \int_{0}^{a} \frac{|f(x_2)|^2}{(a - x_2)^2}dx_2
\]

\[
= -\frac{f^2(0)}{a} + 2 \int_{-a}^{a} \frac{ff'}{a + x_2}dx_2 - 2 \int_{0}^{a} \frac{ff'}{a - x_2}dx_2
\]

\[
\leq 2\frac{|f(x_2)|}{a - |x_2|^2} |f(x_2)|^2.
\]

In a similar way, to prove (2.8), we only need to get

\[
\frac{|f(x_2)|}{a - |x_2|^2} \leq \sup_{x_2} |f(x_2)|. \tag{2.10}
\]

Without loss of generality, we may assume \( x_2 > 0 \) and \( f(a) = 0 \), then

\[
|f(x_2)| = | - \int_{x_2}^\infty f y dy | \leq \sup_{x_2} |f(x_2)|(a - x_2).
\]

\[\square\]

\subsection{2.1. Sobolev inequality}

For proving our main result, in this subsection we will give Sobolev inequalities as follows

\textbf{Proposition 1.} Let \( \phi \) having the compact support as (1.79), we have

\[
|\phi(\xi, \eta, x_2)| \lesssim (2 + \eta)^{-\frac{1}{2}}(2 + \xi)^{-\frac{1}{2}} \sum_{0 \leq k_1 \mid k_2 \leq 1} |\Gamma^{k_1} \nabla_{k_2} \phi(\xi, \eta, \cdot)|_{L^2(D)} \tag{2.11}
\]

where \( \nabla = \{ \partial_\eta, \partial_{x_2} \} \).

\textbf{Proof.} We first consider the case \( |x_2| \leq \frac{\sqrt{(2 + \xi)(2 + \eta)}}{4} \). Then, we have

\[
|\phi(\xi, \eta, \cdot)|_{L^\infty(L^2)} \leq C|\phi(\xi, \eta, \cdot)|_{L^2}(\int_{|x_2| \leq \lambda} |\partial_{x_2} \phi(\xi, \eta, \cdot)|_{L^2}^2 + \frac{1}{\lambda^2} |\phi(\xi, \eta, \cdot)|_{L^2}^2), \tag{2.12}
\]

When \( \lambda = 1 \), the above inequality follows from Nirenberg's inequality. Then the general case follows from the scaling. In our case, we take

\[
\lambda = \frac{\sqrt{(2 + \xi)(2 + \eta)}}{4}.
\]

Noting the definition of \( \Gamma \), we can get

\[
(2 + \eta)(\Gamma_6 \phi + 2 \phi_{x_2}) - x_2(\Gamma_4 \phi + 4 \phi_\eta) = [(2 + \xi)(2 + \eta) - x_2^2] \phi_{x_2},
\]

Then,

\[
|\phi_{x_2}| \leq [(2 + \eta)^{-\frac{1}{2}}(2 + \xi)^{-\frac{1}{2}} + (2 + \xi)^{-1}] |\Gamma \phi| . \tag{2.13}
\]

Therefore,

\[
|\phi(\xi, \eta, \cdot)|_{L^\infty(L^2)} \lesssim [(2 + \eta)^{-\frac{1}{2}}(2 + \xi)^{-\frac{1}{2}} + (2 + \xi)^{-1}] \sum_{|\xi| \leq 1} |\Gamma^k \phi(\xi, \eta, \cdot)|_{L^2(\mathbb{R})} \tag{2.14}
\]

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When $\xi \geq \eta$, we apply one dimensional Sobolev inequality for the $\eta$ variable to get
\[ |\phi(\xi, \eta, x_2)| \lesssim (2 + \eta)^{-\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} \sum_{0 \leq |k|, |l| \leq 1} |\Gamma^{k} \nabla^{l} \phi(\xi, \cdot)|_{L^{2}(\Omega)} \text{ for } |x_2| \leq \frac{(2 + \xi)(2 + \eta)}{4}. \tag{2.15} \]

When $\xi \leq \eta$, we have
\[ [(2 + \xi)(2 + \eta) - x_2^2] \partial_{r} = \frac{1}{2}((2 + \xi)(\Gamma_4 + 4 \partial_{\eta}) - x_2(\Gamma_6 + 2 \partial_{x_2})). \]
Therefore
\[ |\phi_{r}| \lesssim (2 + \eta)^{-\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} |\Gamma \phi|. \]
Noting that
\[ \phi(\xi, \eta, x_2)^2 = -\int_{\eta}^{\infty} \partial_{\eta} \phi^2 \, d\eta, \]
then apply one dimensional Sobolev inequality for the $x_2$ variable, we can get
\[ |\phi(\xi, \eta, x_2)| \lesssim (2 + \eta)^{-\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} \sum_{0 \leq |k|, |l| \leq 1} |\Gamma^{k} \nabla^{l} \phi(\xi, \cdot)|_{L^{2}(\Omega)} \text{ for } |x_2| \leq \frac{(2 + \xi)(2 + \eta)}{4}. \tag{2.16} \]

On the other hand, for the case of $|x_2| \geq \frac{(2 + \xi)(2 + \eta)}{4}$, we introduce the polar coordinates
\[ \sqrt{2 + \eta} = r \cos \theta, \quad x_2 = r \sin \theta. \]
Then, we can get
\[ r \partial_{r} = 2(2 + \eta) \partial_{r} + x_2 \partial_{x_2} = 4 \partial_{r} + \Gamma_4, \quad \partial_{\theta} = \sqrt{2 + \eta} \partial_{x_2} \frac{\sqrt{2 + \eta}(\partial_{x_1} - x_2 \partial_{\eta})}{2 \sqrt{2 + \eta} d\eta d\theta} = dr d\theta \tag{2.17} \]

By Nirenberg's inequality,
\[ \sup_{\omega \in \partial^{\Omega}} |\phi(\xi, r, \cdot)|^{2} \leq |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}^{2} (|\partial_{\rho} \phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} + |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}) \]
\[ = -\int_{\partial \Omega} \partial_{\rho} |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} (|\partial_{\rho} \phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} + |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}) \}
Noting,
\[ -\partial_{\rho} |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} = -\frac{1}{2 |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}} \int_{\partial \Omega} \phi_{\rho} \, d\theta \leq |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} \]
Similarly, we can get
\[ -\partial_{\rho} |\phi_{\rho}(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} \leq |\phi_{\rho}(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} \]
Then,
\[ \sup_{\omega \in \partial^{\Omega}} |\phi(\xi, r, \cdot)|^{2} \lesssim \frac{1}{\sqrt{(2 + \xi)(2 + \eta)}} \int_{\partial \Omega} r |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} (|\partial_{\rho} \phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} + |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}) \]
\[ + |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} (|\partial_{\rho} \phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} + |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)}) \} \int_{\partial \Omega} \phi_{\rho} \, d\theta \leq |\phi(\xi, r, \cdot)|_{L^{2}(\partial \Omega)} \tag{2.19} \]
Noting (2.17) (2.18), it is not difficulty to get
\[ |\phi(\xi, \eta, x_2)| \lesssim (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} \sum_{0 \leq |k|, |l| \leq 1} |\Gamma^{k} \nabla^{l} \phi(\xi, \cdot)|_{L^{2}(\Omega)} \text{ for } |x_2| \geq \frac{(2 + \xi)(2 + \eta)}{4}. \tag{2.20} \]
Therefore, the estimate (2.11) follows from (2.16) and (2.19).
Proposition 2. Let $\phi$ has the compact support as $\mathcal{B}(\xi, r)$, we can obtain
\[
|\phi_{\xi}(\xi, \eta, x_2)| \lesssim (2 + \eta)^{-\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} \sup_{0 \leq |k| + |\xi| + 1 \leq 2} |\Gamma^k \nabla^2_\xi \phi_{\xi}(\xi, \cdot)|_{L^2(D)} \tag{2.21}
\]
\[
|\phi_{\xi}(\xi, \eta, x_2)| \lesssim (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{-\frac{1}{2}} \sup_{0 \leq |k| + |\xi| + 1 \leq 2} |\Gamma^k \nabla^2_\xi \phi_{\xi}(\xi, \cdot)|_{L^2(D)} \tag{2.22}
\]

Proof. 

By Lemma 2.3 and Proposition 1.1, we get
\[
2(2 + \eta)\phi_{\eta} = (4\partial_{\eta} + \Gamma_4 - x_2 \partial_{x_2})\phi
\]

By Proposition 1.1, we have
\[
|x_2 \partial_{x_2} \phi| \leq (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} |\partial_{x_2} \phi|
\]
\[
\leq (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} \sup_{0 \leq |k| + |\xi| + 1 \leq 1} |\Gamma^k \nabla^2_\xi \phi_{\xi}|_{L^2(D)}
\]

By Lemma 2.2 and Proposition 1.1, we get
\[
(4\partial_{\eta} + \Gamma_4)\phi = (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} \sup_{x_2} \left| \frac{(4\partial_{\eta} + \Gamma_4)\phi}{(2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} - |x_2|} \right|
\]
\[
\leq (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} \sup_{x_2} \left| (4\partial_{\eta} + \Gamma_4)\phi_{x_2} \right|
\]
\[
\leq (2 + \eta)^{\frac{1}{2}}(2 + \xi)^{\frac{1}{2}} \sum_{0 \leq |k| + |\xi| + 1 \leq 2} |\Gamma^k \nabla^2_\xi \phi_{\xi}|_{L^2(D)}
\]

Combining the above estimates, we can get the proof of the estimate (2.21). Using the similar procedures, we can get the estimate (2.22).

Corollary 1. Combining Proposition 1 and Proposition 2, we can get
\[
|\phi_{\xi}(\xi, \eta, x_2)| \lesssim (2 + \eta)^{-\frac{1}{2}} \sum_{0 \leq |k| + |\xi| + 1 \leq 2} |\Gamma^k \nabla^2_\xi \phi(\xi, \cdot)|_{L^2(D)} \tag{2.23}
\]

3. Stability of the traveling wave solution with the form $F(x_1 + t)$ in $\mathbb{R}^{1+2+1}$

In the following we will prove the stability of the traveling wave solution with the form $F(x_1 + t)$, i.e. we take $a = 0, b = 1$ in (1.10). Then we can get the perturbed system of timelike extremal hypersurface in Minkowski space $\mathbb{R}^{1+2+1}$ as follows
\[
\left( \begin{array}{c}
\frac{u_1 + F'}{\sqrt{1 - Q_0(u, u) - 2F'(u_1 - u_1)}} \\
\frac{u_2 + F'}{\sqrt{1 - Q_0(u, u) - 2F'(u_2 - u_2)}}
\end{array} \right) = \left( \begin{array}{c}
\frac{u_1 + F'}{\sqrt{1 - Q_0(u, u) - 2F'(u_1 - u_1)}} \\
\frac{u_2 + F'}{\sqrt{1 - Q_0(u, u) - 2F'(u_2 - u_2)}}
\end{array} \right)_{x_2} = 0. \tag{3.1}
\]

We rewrite the system (3.1) as the following form
\[
\square u = -\frac{Q_0(u + F, Q_0(u, u) + 2F'(u_1 - u_1))}{2(1 - Q_0(u, u) - 2F'(u_1 - u_1))}. \tag{3.2}
\]

Under the weaker assumption than $\mathcal{H}_1$

\[
\mathcal{H}_1 \quad |(\xi \frac{d}{d \xi})^k (\frac{d}{d \xi})^l F'(\xi)| \lesssim C_{k, l}(2 + \xi)^{-1}
\]

we will prove the stability of traveling wave solution $F(x_1 + t)$ to system (3.1). The proof of the general traveling wave solution $\mathcal{H}_1$ is similar to that of above theorem. In the end of this paper, we will point out the key difference in the proof.
We can rewrite system (3.2) in coordinates \((\xi, \eta, \chi_2)\) as following
\[
\square u - 4F^2 \partial_{yy}u = 4\partial_{\xi\xi}u - \partial_{\xi\chi_2}u - 4F^2 \partial_{yy}u = \frac{1}{2}(1 - H)[Q_0(u, Q_0(u, u)) + 4F'Q_0(u, u) + 6u\xi F'\partial_{yy}u + 8F'u_0^2] - 4F^2 \partial_{yy}u
\]
where \(H(\xi) = 1 + \frac{1}{\xi^2}\). Taking the operator \(\Gamma^k\) to the above equation and denoting \(u_k = \Gamma^k u\), we have
\[
\square u_k - 4\Gamma^k(F^2 \partial_{yy}u) = 4\partial_{\xi\xi}u_k - \partial_{\xi\chi_2}u_k - 4\Gamma^k(F^2 \partial_{yy}u) = \frac{1}{2}(1 - H)[Q_0(u, 2Q_0(u, u_k)) + 6F'Q_0(u, u_{k2}) + J_k] - 4\sum_{k_1 + k_2 + k_3 \geq 1} \Gamma^{k_1}(F^2 \partial_{yy}u)\Gamma^{k_2}H
\]
with
\[
J_k = \sum_{k_1 + k_2 + k_3 \geq 1} \Gamma^{k_1}(1 - H)Q_0(\Gamma^{k_2}u, Q_0(\Gamma^{k_3}u, \Gamma^{k_4}u)) + \cdots + 4\Gamma^k(F' \partial_{yy}u).
\]

For proving main result, we will give the energy estimates. Define the higher order energy
\[
E_s = \sum_{|l| \leq s} \left( \int \int \frac{u_{l\eta}^2}{(2 + \xi)^{1+\frac{1}{m}}(2 + \eta)^{\frac{1}{m}}} d\xi d\eta d\chi_2 + \int \int \frac{u_{l\chi_2}^2}{(2 + \xi)^{1+\frac{1}{m}}(2 + \eta)^{\frac{1}{m}}} d\xi d\eta d\chi_2 \right) (3.3)
\]
and the lower order energies
\[
\bar{e}_s = \sup_{\xi} \sum_{|l| \leq s} \int \int (u_{l\eta}^2 + u_{l\chi_2}^2) d\eta d\chi_2. (3.4)
\]

For getting the low order energy estimate, we also introduce the weighted lower derivative \(L^\infty\) norm estimates
\[
\tilde{e}_s = (2 + \xi)^{-\delta} \sum_{|l| \leq s-7} \|\Gamma^l u\|_{L^\infty} (3.5)
\]

where the parameter \(\delta\) is an arbitrary positive small constant less than \(\frac{3}{10}\).

### 3.1. Higher order energy estimates

In the subsection we will give the energy estimates terms by terms. For get our main result, firstly we introduce the weight function \(B(\xi)\) satisfying
\[
B'(\xi) = F^2(\xi).
\]
Then, we have
\[
m \leq e^{-\ell B(\xi)} \leq M,
\]
where \(m, M\) are positive constants.
Firstly, multiplying \((2 + \xi)\) to the equation (3.4), we can get the left hand side term

\[
\iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20} e^{-B(E)}\,d\xi d\eta d\xi' d\eta' d\xi d\eta d\xi' d\eta'
\]

\[
= \iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20} e^{-B(E)} 4\partial_\xi u_{k} - \partial_{x_2} u_{k} = \iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20} e^{-B(E)}\,d\xi d\eta d\xi' d\eta' d\xi d\eta d\xi' d\eta'
\]

\[
= \iint \int 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20}] + \frac{1}{2} \frac{d}{d\eta}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} e^{-B(E)} u_{20}^2] - \frac{1}{2} \frac{d}{d\eta}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} e^{-B(E)} u_{20}^2] 
\]

\[
= \iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} e^{-B(E)} u_{20}^2\,d\xi d\eta d\xi' d\eta' d\xi d\eta d\xi' d\eta'
\]

The right hand side terms are as follows

\[
\iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20} e^{-B(E)} (1 - H) Q_{0}(u, 2Q_{0}(u, u)) + 4F' Q_{0}(u, u_0) + J_k
\]

By the bound of \(e^{-B(E)}\) and the decay property of \(B'(\xi)\), without loss of generality, we can estimate the right hand terms above without the weighted function \(e^{-B(E)}\). Then, the first term of the right hand side can be rewritten as

\[
\iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20}(1 - H) Q_{0}(u, 2Q_{0}(u, u_0)) d\xi d\eta d\xi' d\eta' d\xi d\eta d\xi' d\eta'
\]

\[
= \iint \int (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} u_{20}(1 - H) u_{20} u_{20} + 4u_{20} Q_{0} - 2u_{2} Q_{0} d\xi d\eta d\xi' d\eta' d\xi d\eta d\xi' d\eta'
\]

\[
= \iint \int 4 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] - (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}
\]

\[
= [(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] + 4 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]

\[
= (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0} - [(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]

\[
= 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] - (2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}
\]

\[
= [(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] + 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]

\[
= \iint \int 4 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] + 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]

\[
= 2 \iint \int [(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] + 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]

\[
= 2 \iint \int [(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}] + 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{d}{2}}(2 + \eta)^{-\frac{d}{2}} (1 - H) u_{20} u_{20} Q_{0}]
\]
The second term of the right hand side is

\[
2 \int \int (2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'Q_0(u, u^2_{\eta\xi})
\]

\[
= 2 \int \int (2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'[2u_\xi(u^2_{\eta\eta})_\eta + 2u_\eta(u^2_{\eta\eta})_\xi - u_\eta(u^2_{\eta\eta})_{\eta\xi}]d\xi d\eta dx_2
\]

\[
= \int \int \int \frac{d}{dy}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\xi u^2_{\eta\xi}] - 4 \frac{d}{d\xi}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\xi u^2_{\eta\xi}]
\]

\[
+ 4 \frac{d}{d\xi}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\xi u^2_{\eta\xi}]
\]

\[
- 2 \frac{d}{dx_2}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\eta u^2_{\eta\eta}] + 2 \frac{d}{dx_2}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\eta u^2_{\eta\eta}]
\]

\[
= \int \int \int \frac{d}{dy}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\xi u^2_{\eta\xi}] - 4 \frac{d}{d\xi}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\xi u^2_{\eta\xi}]
\]

\[
- 2 \frac{d}{dx_2}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\eta u^2_{\eta\eta}] + 2 \frac{d}{dx_2}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)F'u_\eta u^2_{\eta\eta}]
\]

\[
Q_0((2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}, u)Q_0(u, u)
\]

\[
= (2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)Q_0(u_{\xi\eta}, u)Q_0(u, u) + u_\eta Q_0((2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}, u)Q_0(u, u)
\]

\[
+ \frac{1}{2}u_\eta Q_0((2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}, u)Q_0(u, u)
\]

Then, the right hand side term as follows:

\[
\int \int \int (2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}u_{\eta\eta}(1 - H)\phi(2, Q_0(u, Q_0(u, u))) + 4F'Q_0(u, u_{\eta\xi}) + J_6 - 4 \sum_{k=1}^{K_2} \Gamma_i^k(F'Q_0)^k H d\xi d\eta dx_2
\]

\[
= \int \int \int \frac{d}{dy}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}Q_0] + 4 \frac{d}{d\xi}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_\eta u_{\eta\xi}Q_0]
\]

\[
- \frac{2}{d_{\xi\eta}}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}u_{\eta\xi}Q_0]d\xi d\eta dx_2
\]

\[
- \frac{2}{d_{\xi\eta}}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)Q_0(u_{\xi\eta}, u)] - [(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_\eta u_{\eta\xi}Q_0]
\]

\[
- 2(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)Q_0(u_{\xi\eta}, u)Q_0(u, u) + 2u_{\eta\xi}Q_0((2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H), u)Q_0(u, u)d\xi d\eta dx_2
\]

\[
- \frac{2}{d_{\xi\eta}}[(2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}(1 - H)u_{\xi\eta}u_{\eta\xi}Q_0]d\xi d\eta dx_2
\]

The same as follows:

\[
\int \int \int (2 + \xi)^{-\hat{\phi}}(2 + \eta)^{-\hat{\phi}}u_{\eta\eta}(1 - H)\phi(2, Q_0(u, Q_0(u, u))) + 4F'Q_0(u, u_{\eta\xi}) + J_6 - 4 \sum_{k=1}^{K_2} \Gamma_i^k(F'Q_0)^k H d\xi d\eta dx_2
\]
Finally

\[ -4[(2 + \xi)^{\tau}(2 + \eta)^{\tau}(1 - H)F'_{11}]u_{2}u_{2} - 4[(2 + \xi)^{\tau}(2 + \eta)^{\tau}(1 - H)F']_{22}u_{2}u_{2} \]
\[ + 2[(2 + \xi)^{\tau}(2 + \eta)^{\tau}(1 - H)F'_{22}]u_{2}u_{2}d\xi d\eta dx_{2} \]
\[ + \int \int \int (2 + \xi)^{\tau}(2 + \eta)^{\tau}u_{2}u_{2}(1 - H)J_{4}d\xi d\eta dx_{2} \]
\[ -4 \sum_{k_{1}+k_{2}=k>1} \int \int \int (2 + \xi)^{\tau}(2 + \eta)^{\tau}u_{2}u_{2}^k(F^2u_{2})^kHd\xi d\eta dx_{2}. \]

In the following we will estimate the above equation terms by terms. Firstly, we will deal with the term \( A_1 \). Noting Proposition 1 and Proposition 2,

\[ \int \int \int [(2 + \xi)^{\tau}(2 + \eta)^{\tau}(1 - H)Q_{0}^{2}(u_{2}, u)d\xi d\eta dx_{2} \]
\[ = \int \int \int \frac{1}{10}[(2 + \xi)^{\tau}(2 + \eta)^{\tau}(1 - H)Q_{0}^{2}(u_{2}, u) + (2 + \xi)^{\tau}(2 + \eta)^{\tau}H_{0}Q_{0}^{2}(u_{2}, u)d\xi d\eta dx_{2} \]
\[ \lesssim \int \int \int (2 + \xi)^{\tau}(2 + \eta)^{\tau}[H_{0}][u_{2}u_{2}^2 + u_{2}^2 + u_{2}^2]d\xi d\eta dx_{2} \]
\[ \lesssim \int \int \int (2 + \xi)^{\tau}(2 + \eta)^{\tau}[Q_{0} + 4F'(\xi)u_{2}] [u_{2}u_{2}^2 + u_{2}^2 + u_{2}^2]d\xi d\eta dx_{2} \]
\[ \lesssim (1 + \epsilon_{r}c_{r}F_{r}) + \int \int \int (2 + \xi)^{\tau}(2 + \eta)^{\tau}(Q_{0} + 4F'(\xi)u_{2}) [u_{2}u_{2}^2 + u_{2}^2 + u_{2}^2]d\xi d\eta dx_{2}. \]
By Lemma 3.2 and (3.6),
\[
\int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} Q_{ijl}(u_{ij}^2 u_{kl}^2 + u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2)d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} [(\Gamma u_0)||\nabla u_0|| + (\Gamma u_0)||\nabla u_0)||u_{ij}^2 u_{kl}^2 + u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2]d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} [(\Gamma u_0)||\nabla u_0|| + (\Gamma u_0)||\nabla u_0)||u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2]d\xi d\eta d\xi d\eta d\xi d\eta \\
+ \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} [(\Gamma u_0)||\nabla u_0|| + (\Gamma u_0)||\nabla u_0)||u_{ij}^2 u_{kl}^2 + u_{ij}^2 u_{kx}^2]d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} [(\Gamma u_0)||\nabla u_0|| + (\Gamma u_0)||\nabla u_0)||u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2 + e_\xi^2 E_x] \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} [(\Gamma u_0)||\nabla u_0|| + (\Gamma u_0)||\nabla u_0)||u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2 + e_\xi^2 E_x] \\
\leq (e_\xi^2 + \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq e_\xi^2 + \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta. \\
(3.10)
\]
Noting the assumption of $H_1$ and Proposition 2,
\[
4 \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} |F(\xi)u_{ij}||u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2 + u_{ij}^2 u_{kx}^2]d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{ij}^2 u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{ij}^2 u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta. \\
(3.11)
\]
By Proposition 1 and Proposition 2, we can get
\[
\int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{ij}^2 u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} e_\xi^2 u_{kx}^2 d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq e_\xi^2 E_x. \\
(3.12)
\]
Combining with Corollary 1, we also have
\[
\int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{ij}^2 u_{kx}^2 e_\xi^2 d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq e_\xi^2 \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{kx}^2 d\xi d\eta d\xi d\eta d\xi d\eta. \\
(3.13)
\]
Therefore, we can get the first term of $A_1$
\[
\int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} (1 - H) u_0 Q_{ij}(u_0, u) d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq (e_\xi + e_\xi^2 e_\xi + e_\xi^2) E_x. \\
(3.14)
\]
We will continue to estimate the second term of $A_1$. Using Proposition 1 and Proposition 2,
\[
\int \int \int 2(2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} (1 - H) u_0 Q_{ij}(u_0, u) Q_{0}(u_0, u) d\xi d\eta d\xi d\eta d\xi d\eta \\
\leq (1 + e_\xi) \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} (1 - H) u_0 Q_{ij}(u_0, u) Q_{ij}(u_0, u) d\xi d\eta d\xi d\eta d\xi d\eta \\
+ (1 + e_\xi) \int \int \int (2 + \xi)^{-\frac{\delta}{2}}(2 + \eta)^{-\frac{\delta}{2}} u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{lx}^2 + u_{ij}^2 u_{kx}^2 + u_{ij}^2 u_{kx}^2 d\xi d\eta d\xi d\eta d\xi d\eta. \\
(3.14)
\]
Noting Proposition 2 and Corollary 1, we have
\[ |u_{12}| |u_{22}| \lesssim (2 + \varepsilon)^{-1/2} (2 + \eta) \varepsilon (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1} e_s. \]

Therefore, we can get
\[ \iint\iint (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1} \tilde{u}_{\varepsilon}^2 \eta^2 \xi^2 d\xi d\eta d\tau \lesssim e_s E_s. \quad (3.15) \]

Noting Proposition 1, we have
\[ \iint\iint (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{u}_{\varepsilon x}^2 d\xi d\eta d\tau \lesssim e_s \iint\iint \eta^2 \xi^2 \tilde{u}_{\varepsilon x}^2 d\xi d\eta d\tau \lesssim e_s E_s. \quad (3.16) \]

We note
\[ Q_0((2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H), u) + Q_0(u, u) \]
\[ = 2[(2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H), u] u_{\xi} + 2[(2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H)]_\nu u_{\xi} \]
\[ - [(2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H)]_\xi u_{\eta} + 2u_{\xi} u_{\eta} - u_{\xi x} u_{\eta}, \]
\[ = - \left( \frac{1}{2} \right) \left( 2 + \varepsilon \right)^{-1/2} (2 + \eta)^{-1/2} (1 - H) + 2(2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} \]
\[ + 2 \left( \frac{1}{2} \right) (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} + (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\eta u_{\xi} \]
\[ - (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\eta u_{\xi} u_{\eta}, \]
\[ \text{Noting proposition 1 and corollary 1, the last term of } A_1 \]
\[ \iint\iint 2u_{\xi} Q_0((2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H), u) Q_0(u, u) d\xi d\eta d\tau \]
\[ = \iint\iint 2u_{\xi} [2u_{\xi} u_{\eta} + 2u_{\xi} u_{\eta} - u_{\xi x} u_{\eta}] \left( \frac{1}{2} \right) (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H) + 2(2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} \]
\[ + 2 \left( \frac{1}{2} \right) (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} + (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\eta u_{\xi} \]
\[ \lesssim \iint\iint [u_{\xi}^2 |u_{\xi}| + |u_{\xi} u_{\eta}||u_{\eta}| + |u_{\xi} u_{\xi x}||u_{\eta}||((2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} + H_{\xi} u_{\xi} + H_{\xi} u_{\eta}) \]
\[ + (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\xi u_{\eta} + (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \tilde{H}_\eta u_{\xi} (1 - H)) ] \]
\[ \lesssim e_s E_s + \iint\iint (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} \eta^2 \xi^2 d\xi d\eta d\tau \]
\[ + \iint\iint (2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H)(u_{\xi} + u_{\eta} + u_{\xi}) u_{\xi}^2 d\xi d\eta d\tau \]
\[ \lesssim e_s E_s. \quad (3.17) \]

Using the similar method as (3.5)–(3.15) in the above inequality, we can get
\[ \iint\iint 2u_{\xi} Q_0((2 + \varepsilon)^{-1/2} (2 + \eta)^{-1/2} (1 - H), u) Q_0(u, u) d\xi d\eta d\tau \]
\[ \lesssim e_s + e_s E_s + e_s^2 E_s. \quad (3.18) \]
In the following we will estimate the term $A_2$.

\[
\int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} (1 - H)u_{\infty}(\square u + 4F^2 u_{\eta}) Q_0(u, u) d\xi d\eta dx_2
\]
\[
\leq \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} |u_{\eta}| |Q_0(u, Q_0(u, u)) + 4FQ_0(u, u_{\eta}) + 4F' u_{\eta}^2| [u_{\infty}u_\xi] + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] d\xi d\eta dx_2
\]
\[
+ \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} [u_{\eta}] |Q_0||u_{\infty}u_\xi] + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] d\xi d\eta dx_2.
\]

By the assumption of $F$ and Proposition 2, we first give the estimate

\[
\int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} (1 - H)^2 |u_{\infty}| |4FQ_0(u, u_{\eta}) + 4F' u_{\eta}^2| [u_{\infty}u_\xi] + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] d\xi d\eta dx_2
\]
\[
\leq \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} |u_{\eta}| |Q_0(u, u_{\eta})| [u_{\infty}u_\xi] + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] d\xi d\eta dx_2
\]
\[
+ \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} |u_{\eta}| u_\xi^2 [u_{\infty}u_\xi] + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] d\xi d\eta dx_2
\]
\[
\leq (1 + e_s)^2 e_s^2 \int \int \int (2 + \xi)^{-1} (2 + \eta)^{-1} |u_{\eta}| |u_{\xi}| + (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} u_{\eta}^2 + (2 + \xi)^{-1} (2 + \eta)^{-1} |u_{k_3}u_{k_3}] |d\xi d\eta dx_2
\]
\[
\leq (1 + e_s)^2 e_s^2 E_s.
\]

Using the similar method, we can also get the estimate of last term. Then, noting (3.3), we can get

\[
\int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} (1 - H)u_{\infty}(\square u + 4F^2 u_{\eta}) Q_0(u, u) d\xi d\eta dx_2
\]
\[
\leq \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} |u_{\eta}| Q_0(u, Q_0(u, u)) |u_{\infty}u_\xi| + |u_{\xi}u_\eta| + |u_{k_3}u_{k_3}] |d\xi d\eta dx_2 + (1 + e_s)^2 e_s^2 E_s
\]
\[
\leq \int \int \int (2 + \xi)^{-\frac{5}{4}} (2 + \eta)^{-\frac{5}{4}} |Q_0(u, Q_0(u, u))| u_{\xi}^2 [u_\xi] + u_{\eta}^2 [u_\eta] + u_{k_3}^2 [u_{k_3}] + u_{k_3}^2 [u_{k_3}] d\xi d\eta dx_2
\]
\[
+(1 + e_s)^2 e_s^2 E_s. \quad (3.19)
\]

Furthermore,

\[
Q_0(u, Q_0(u, u)) |u_{\xi}^2 [u_\xi] + u_{\eta}^2 [u_\eta] + u_{k_3}^2 [u_{k_3}] + u_{k_3}^2 [u_{k_3}]|
\]
\[
= 2u_{\eta}Q_0 + 2u_{\eta}Q_0 - u_{\xi}Q_0 |u_{\xi}^2 [u_\xi] + u_{\eta}^2 [u_\eta] + u_{k_3}^2 [u_{k_3}] + u_{k_3}^2 [u_{k_3}]|
\]

Here we only estimate the first term and the other terms can be obtained using the similar way. By Proposition 1 and
Proposition 2, we have
\[
\iint (2 + \xi) \cdot (2 + \eta) \cdot (1 - H) \cdot \eta \cdot (1 - H) \cdot u_{\eta} \cdot u_{\xi} + u_{\eta} \cdot u_{\xi} + u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
\lesssim \iint (2 + \xi) \cdot (2 + \eta) \cdot (2 + \eta) \cdot \eta \cdot (1 - H) \cdot \eta \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
+ \iint (2 + \xi) \cdot (2 + \eta) \cdot (2 + \eta) \cdot \eta \cdot (1 - H) \cdot \eta \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
+ C \iint (2 + \xi) \cdot (2 + \eta) \cdot (2 + \eta) \cdot \eta \cdot (1 - H) \cdot \eta \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
\lesssim \iint (2 + \xi) \cdot (2 + \eta) \cdot (2 + \eta) \cdot \eta \cdot (1 - H) \cdot \eta \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
+ \iint (2 + \xi) \cdot (2 + \eta) \cdot (2 + \eta) \cdot \eta \cdot (1 - H) \cdot \eta \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
\lesssim (\xi e_1^2 + e_1^2) E_x. \tag{3.20}
\]

Then we can get
\[
A_2 \lesssim e_1^2 E_x + \xi e_1^2 E_x + e_1^2 E_x + e_1^2 E_x. \tag{3.21}
\]

Noting the decay of \( F' \) and the above process, we will estimate the term \( A_3 \)
\[
A_3 = \iint (2 + \xi) \cdot (2 + \eta) \cdot (1 - H) \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} + (2 + \xi) \cdot (2 + \eta) \cdot (1 - H) \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
+ \iint (2 + \xi) \cdot (2 + \eta) \cdot (1 - H) \cdot F' \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} + (2 + \xi) \cdot (2 + \eta) \cdot (1 - H) \cdot F' \cdot u_{\eta} \cdot u_{\xi} \cdot u_{\xi} \cdot d\xi d\eta dx
\]
\[
\lesssim \xi e_1^2 E_x + e_1^2 E_x + e_1^2 E_x + \xi e_1^2 E_x + e_1^2 E_x. \tag{3.22}
\]

For getting the estimation of the term \( A_4 \), we will first estimate \( J_k \). Denote
\[
J_{k1} = \sum_{k_1 + k_2 + k_3 + k_4 = k} \Gamma^{k_1}(1 - H) Q_0(\Gamma^{k_2} u, Q_0(\Gamma^{k_3} u, \Gamma^{k_4} u))
\]
\[
J_{k2} = \sum_{0 \leq k_1 \leq k} \Gamma^{k_1}(1 - H) \Gamma^{k-k_1} [F' Q_0(u, u_\eta)]
\]
\[
J_{k3} = \sum_{0 \leq k_1 \leq k} \Gamma^{k_1}(1 - H) \Gamma^{k-k_1} [F' u_\eta^2].
\]

For \( J_{k1} \), when \( k_1 \leq \frac{4k}{5} \), we have
\[
|J_{k1}| = \left| \sum_{k_1 + k_2 + k_3 + k_4 = k} \Gamma^{k_1}(1 - H) Q_0(\Gamma^{k_2} u, Q_0(\Gamma^{k_3} u, \Gamma^{k_4} u)) \right|
\]
\[
\lesssim (1 + e_1) |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u) + |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u) + |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u). \tag{3.23}
\]

Without loss of generality, we assume \( k_1, k_4 \leq \frac{4k}{5} \) and noting \( 2.21 \), then
\[
J_{k1} \lesssim (1 + e_1) |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u) + |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u) + |(\Gamma^{k_1} u)| Q_0(\Gamma^{k_2} u, \Gamma^{k_4} u) \]
\[
\approx J_{k11} + J_{k12} + J_{k13}. \tag{3.24}
\]

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Noting Lemma 3.2, Lemma 3.3 and Proposition 1, \[ J_{k1} = C(1 + e,)(\Gamma^{k}u)_{\eta}||Q_{0}(\Gamma^{k}u, \Gamma^{k}u)|| \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(\Gamma^{k}u)_{\eta}]||\nabla \Gamma^{k}u|| (2 + \xi)(2 + \eta)^{\frac{1}{2}}[(\nabla \Gamma^{k}u)_{\eta}] \frac{\Gamma^{k}u}{\sqrt{(2 + \xi)(2 + \eta)}} \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(\Gamma^{k}u)_{\eta}]||\nabla \Gamma^{k}u|| (2 + \xi)(2 + \eta)^{\frac{1}{2}}[(2\nabla \Gamma^{k}u_{\eta} + \Gamma_{4}\nabla \Gamma^{k}u - x_{2}\nabla \Gamma^{k}u_{\xi})||\Gamma^{k}u|| \eta] \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(\Gamma^{k}u)_{\eta}]||\nabla \Gamma^{k}u|| (2 + \xi)(2 + \eta)^{\frac{1}{2}}(\nabla \Gamma^{k}u_{\eta}) \]
\[ \lesssim (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \lesssim (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \] (3.25)

Meanwhile,
\[ J_{k1} = C(1 + e,)(\Gamma^{k}u)_{\eta}||Q_{0}(\Gamma^{k}u, \Gamma^{k}u)|| \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(\Gamma^{k}u)_{\eta}]||\nabla \Gamma^{k}u|| (2 + \xi)(2 + \eta)^{\frac{1}{2}}[(\nabla \Gamma^{k}u)_{\eta}] \frac{\Gamma^{k}u}{\sqrt{(2 + \xi)(2 + \eta)}} \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}(2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}e_{j} + (2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}e_{j}^{T}[(\nabla \Gamma^{k}u)_{\eta}] \]
\[ \lesssim (2 + \xi)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \lesssim (2 + \xi)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \] (3.26)

and
\[ J_{k1} = C(1 + e,)(\Gamma^{k}u)_{\eta}||Q_{0}(\Gamma^{k}u, \Gamma^{k}u)|| \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(\Gamma^{k}u)_{\eta}]||\nabla \Gamma^{k}u|| (2 + \xi)(2 + \eta)^{\frac{1}{2}}[(\nabla \Gamma^{k}u)_{\eta}] \frac{\Gamma^{k}u}{\sqrt{(2 + \xi)(2 + \eta)}} \]
\[ \lesssim (1 + e,)(\Gamma^{k}u)_{\xi}(2 + \xi)^{-1}[(2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}e_{j} + (2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}e_{j}^{T}[(\nabla \Gamma^{k}u)_{\eta}] \]
\[ \lesssim (2 + \xi)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \lesssim (2 + \xi)^{-\frac{1}{2}}(1 + e,)(\Gamma^{k}u)_{\xi} \] (3.27)

For \( J_{k1} \), when \( k_{1} \geq \left\lceil \frac{1+2}{2} \right\rceil \), we have
\[ |J_{k1}| \lesssim \sum_{k_{1}^{'},k_{2}+k_{3}+k_{4}=k_{1},k_{4}<k_{3},k_{3}<k} \Gamma^{k_{1}}(Q_{0} + F' u_{\eta})|Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]
\[ \leq \sum_{k_{1}^{'},k_{2}+k_{3}+k_{4}=k_{1},k_{4}<k_{3},k_{3}<k} \Gamma^{k_{1}}(Q_{0}(u, u)) + |\nabla \Gamma^{k_{1}}(F' u_{\eta})||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]

In the following we will estimate the above two parts separately.
\[ \sum_{k_{1}^{'},k_{2}+k_{3}+k_{4}=k_{1},k_{4}<k_{3},k_{3}<k} \Gamma^{k_{1}}(Q_{0}(u, u)||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)) \]
\[ = \sum_{k_{2}+k_{3}+k_{4}=k} |\Gamma^{k_{1}}u_{\eta} + \Gamma^{k_{1}}u_{\xi}||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]
\[ \leq \sum_{k_{2}+k_{3}+k_{4}=k} |\Gamma^{k_{1}}u_{\eta} + \Gamma^{k_{1}}u_{\xi}||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]
\[ + \sum_{k_{2}+k_{3}+k_{4}=k} |\Gamma^{k_{1}}u_{\eta} + \Gamma^{k_{1}}u_{\xi}||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]
\[ \leq B_{1} + B_{2} \] (3.28)

where
\[ B_{1} \leq (2 + \xi)^{-\frac{1}{2}}(\Gamma^{k_{1}}u_{\eta} + \Gamma^{k_{1}}u_{\xi}||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)| \]
\[ + (2 + \xi)^{-\frac{1}{2}}(\Gamma^{k_{1}}u_{\eta}||Q_{0}(\Gamma^{k_{1}}u, \Gamma^{k_{1}}u)) \]
\[ \leq B_{1} + B_{12} + B_{13} \] (3.29)
For the case $|k_5| \geq |k_6|$, by Corollary 1 and Lemma 3, we have

$$B_{11} = (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9 + \Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6 + \Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim |\Gamma^{\delta_k} u_9|(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}\frac{\Gamma^{\delta_k} u_9}{\sqrt{(2 + \eta)(2 + \varepsilon)}}(2 + \eta)^{-\frac{5}{2}}e^2 \\
\lesssim |\Gamma^{\delta_k} u_9|(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9|||2 + \eta)^{-\frac{5}{2}}e^2 \\
\lesssim (2 + \eta)^{-\frac{5}{2}}|\Gamma^{\delta_k} u_9| |\nabla^{\delta_k} u_9| + (2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim e^2(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| + (2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2.$$

For the case $|k_5| \leq |k_6|$, by Proposition 2 and Lemma 3, we have

$$B_{11} = (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9 + \Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6 + \Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}\frac{\Gamma^{\delta_k} u_9}{\sqrt{(2 + \eta)(2 + \varepsilon)}}(2 + \eta)^{-\frac{5}{2}}e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim e^2(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| + (2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2.$$

The estimation of $B_{12}$ can be gotten using the similar way to $B_{11}$. Next we will give the estimation of $B_{13}$. For the case $|k_5| \geq |k_6|$, by Corollary 1 and Lemma 3, we have

$$B_{13} = (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9 + \Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6 + \Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}\frac{\Gamma^{\delta_k} u_9}{\sqrt{(2 + \eta)(2 + \varepsilon)}}(2 + \eta)^{-\frac{5}{2}}e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim e^2(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| + (2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2.$$

For the case $|k_5| \leq |k_6|$, by Proposition 2 and Lemma 3, we have

$$B_{13} = (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9 + \Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9 \Gamma^{\delta_k} u_6 + \Gamma^{\delta_k} u_6 \Gamma^{\delta_k} u_9)|\|\nabla^{\delta_k} u_9\|\|\nabla^{\delta_k} u_6\| \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}\frac{\Gamma^{\delta_k} u_9}{\sqrt{(2 + \eta)(2 + \varepsilon)}}(2 + \eta)^{-\frac{5}{2}}e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim (2 + \varepsilon)^{-1}|(\Gamma^{\delta_k} u_9)(2 + \eta)^{-\frac{5}{2}} + |\Gamma^{\delta_k} u_6|(2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2 \\
\lesssim e^2(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| + (2 + \eta)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}|\nabla^{\delta_k} u_9| e^2.$$

Using the similar procedures, we can get the estimate of $B_{2}$. Then, we can get the estimate of $J_{k_1}$. 

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In the following, we will give the estimation of $J_{k2}$. When $k_1 \leq \lfloor \frac{1}{2} \rfloor$, the other case can be get easily.

$$J_{k2} \leq \sum_{0 \leq k_1 + k_2 = k} \Gamma^{k_1}(1 - H) \Gamma^{k_2}[F'Q_0(u, u_\eta)]$$

$$\lesssim (1 + e_s) \sum_{0 \leq k_1 + k_2 = k} \Gamma^{k_1} F \Gamma^{k_2} Q_0(u, u_\eta)$$

$$\lesssim (1 + e_s)(1 + \xi)^{-2} \sum_{k_1 + k_2 = k} Q_0(\Gamma^{k_1} u, \Gamma^{k_2} u_\eta)$$

$$\lesssim (1 + e_s)(1 + \xi)^{-2} \sum_{k_1 + k_2 = k} [\Gamma^{k_1} u_\xi [\Gamma^{k_2} u_{\eta}] + \Gamma^{k_1} u_\eta [\Gamma^{k_2} u_\xi] + \Gamma^{k_1} u_\xi |\Gamma^{k_2} u_\eta|].$$

For the case of $|k_1| \geq |k_2|$, noting Corollary 1, we can get

$$J_{k2} \lesssim (1 + e_s)(2 + \xi)^{-1} (2 + \eta)^{-\frac{1}{2}} e_s^{\frac{1}{2}} [\Gamma^{k_1} u_\xi | + \Gamma^{k_1} u_\eta | + \Gamma^{k_2} u_{\eta} |].$$

(3.30)

For the case of $|k_1| \leq |k_2|$, we can get

$$J_{k2} \lesssim (1 + e_s)(2 + \xi)^{-1} e_s^{\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} [2 + \eta)^{-\frac{1}{2}} [\Gamma^{k_1} u_{\eta}] + (2 + \eta)^{-\frac{1}{2}} [\Gamma^{k_1} u_\eta] + (2 + \xi)^{-\frac{1}{2}} [\Gamma^{k_2} u_{\eta} |].$$

(3.31)

Furthermore, we will estimate the last term $J_{k3}$. When $k_1 < k_2$, it is easily to get

$$J_{k3} = \sum_{0 \leq k_1 + k_2 = k} \Gamma^{k_1}(1 - H) \Gamma^{k_2}[F'' u_{\eta}^2]$$

$$\lesssim (1 + e_s)e_s^{\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} \sum_{0 \leq k_1 \leq k_2} [\Gamma^{k_1} u_{\eta}] \lesssim (1 + e_s)e_s^{\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} [\Gamma^{k_1} u_{\eta} |].$$

(3.32)

When $k_1 > k_2$, we have

$$J_{k3} \lesssim \sum_{0 \leq k_1 + k_2 = k} [\Gamma^{k_1} (Q_0 + 8 F' u_{\eta}) || \Gamma^{k_1} F' u_{\eta}^2 ||]$$

$$\lesssim (2 + \xi)^{-2} (2 + \eta)^{-1} e_s \sum_{0 \leq k_1 \leq k_2} [\Gamma^{k_1} (Q_0 u, u) | + \Gamma^{k_1} u_{\eta}]$$

$$\lesssim (2 + \xi)^{-2} (2 + \eta)^{-1} e_s \sum_{0 \leq k_1 \leq k_2} [(2 + \xi)^{\frac{1}{2}} (2 + \eta)^{\frac{1}{2}} |e_s^{\frac{1}{2}} u_{\eta} | + u_{\eta} |u_{\xi} | + u_{\xi} |u_{\eta} |]$$

$$\lesssim (2 + \xi)^{-2} (2 + \eta)^{-1} e_s \sum_{0 \leq k_1 \leq k_2} [(2 + \xi)^{\frac{1}{2}} (2 + \eta)^{\frac{1}{2}} e_s^{\frac{1}{2}} |u_{\eta} | + |u_{\xi} | + u_{\eta} |u_{\xi} | + |u_{\xi} |$$

$$\lesssim (2 + \xi)^{-2} (2 + \eta)^{-1} e_s e_s^{\frac{1}{2}} (2 + \xi)^{\frac{1}{2}} (2 + \eta)^{\frac{1}{2}} e_s^{\frac{1}{2}} |u_{\xi} | + (2 + \xi)^{-2} (2 + \eta)^{-1} e_s^{\frac{1}{2}} |u_{\eta} |. \quad (3.33)$$

Then, the estimation of $A_4$ can be got

$$A_4 = \iint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{\xi}(1 - H) d\xi d\eta d\xi d\eta$$

$$\lesssim e_s^{\frac{1}{2}} [1 + e_s^{\frac{1}{2}} + e_s^{\frac{1}{2}} + e_s^{\frac{1}{2}} + e_s^{\frac{1}{2}}] E_s. \quad (3.34)$$
There is only last term $A_3$ to be estimated.

$$A_3 = -4 \sum_{k_1 + k_2 = k} \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\eta\eta} \Gamma^2 \alpha_{\eta\eta}(F^2 \eta_{\eta\eta}) \eta_{\eta\eta} d\xi d\eta dx_2$$

\begin{align*}
&\leq \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left[ u_{\eta\eta} H u_{\xi\xi} + u_{\eta\eta} \Gamma^2 (Q_0 + 4 F' \eta_{\eta\eta}) u_{\xi\eta} \right] d\xi d\eta dx_2 \\
&\leq \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left[ |H| \frac{d}{d\eta} u_{\xi\xi}^2 + |u_{\eta\eta}| \left( |Q_0(u, u_x)| + |u_\xi||u_{\xi\xi}| \right) \right] d\xi d\eta dx_2 \\
&\leq \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left[ |H| u_{\xi\xi}^2 + |u_{\eta\eta}| \left( |Q_0(u, u_x)| + |u_\xi||u_{\xi\xi}| \right) \right] d\xi d\eta dx_2 \\
&\quad + \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} |H| |u_{\xi\xi}|^2 d\eta dx_2 \\
&\leq \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left[ |Q_0(u, u_{\eta\eta})| + F' \eta_{\eta\eta} u_{\xi\xi}^2 + |u_{\eta\eta}| |u_\xi u_{\xi\xi} + u_{\eta\eta} u_{\xi\xi} + u_x u_{\xi\xi}| \right] d\xi d\eta dx_2 + \varepsilon_5 E_5 + e_5^1 E_5 \\
&\leq \iiint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left[ |u_{\eta\eta} u_{\xi\xi} + u_\xi u_{\eta\eta} + u_x u_{\xi\eta}| \right] d\xi d\eta dx_2 + \varepsilon_5 E_5 + e_5^1 E_5 + e_5^1 E_5 \\
&\leq \varepsilon_5 E_5 + e_5 E_5 + e_5^1 E_5.
\end{align*}

Then, we get the all estimates of \((3.35)\).

Multiplying \((2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi} e^{H(t)}\) to the system \((3.4)\) and integrating it about \(\xi, x_2\) and \(\eta\), we can get the left hand side parts of system

\begin{align*}
4 \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \partial_{\xi\xi} u_{\xi\xi} d\eta dx_2 - \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \partial_{\xi\xi} u_{\xi\xi} d\eta dx_2 \\
+ 4 \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \Gamma^2 (F^2 u_{\eta\eta}) u_{\xi\xi} d\eta dx_2 \\
= 2 \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \frac{d}{d\eta} u_{\xi\xi}^2 d\eta dx_2 + \iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} \left( \frac{d}{d\eta} (u_{\xi\xi} u_{\xi\xi}) - \frac{1}{2} \frac{d}{d\xi} u_{\xi\xi}^2 \right) d\eta dx_2 \\
= -\frac{1}{2} \iint \frac{d}{d\xi} \left( (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi}^2 \right) d\eta dx_2 + \frac{1}{20} \iint \left[(2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi}^2 \right] d\eta dx_2 \\
+ \frac{1}{5} \iint \frac{d}{d\xi} \left( (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi}^2 \right) d\eta dx_2 + \iint \frac{d}{d\eta} \left[ (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi}^2 \right] d\eta dx_2 \\
+ \iint \frac{d}{d\xi} \left( (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi}^2 \right) d\eta dx_2. \tag{3.35}
\end{align*}

The right hand side parts can be obtained as follows

\begin{align*}
\iint (2 + \xi)^{-\frac{3}{2}} (2 + \eta)^{-\frac{3}{2}} u_{\xi\xi} (1 - H)[Q_0(u, 2 Q_0(u, u_x)) + 4 F' Q_0(u, u_x)] d\xi d\eta dx_2 \\
\leq H_1 + H_2 + H_3 + H_4.
\end{align*}
Then, we can get
\[ H_1 = \int \int \int (2 + \xi)^\frac{\partial}{\partial (2 + \eta)} \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)[4u_{\xi}Q_{0}(u, u_{\xi}) + 4u_{\eta}Q_{\xi}(u, u_{\xi}) - 2u_{\xi}Q_{0}(u, u_{\xi})]d\xi d\eta dx_{2} \]
\[ = \int \int \int 4\left(\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} - (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}ight)\]
\[ = \int \int \int 4\left(2 + \xi \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + \frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}ight)\]
\[ = \int \int \int 4\left(2 + \xi \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 4\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}ight)\]
\[ = \int \int \int 2(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 2\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}\]
\[ = \int \int \int 2(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 2\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}\]
\[ = \int \int \int 2(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 2\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}\]
\[ = \int \int \int 2(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 2\frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)u_{\xi}Q_{\xi}\]
and
\[ H_2 = 4 \int \int \int (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} + 2u_{\eta}u_{\xi}Q_{0} - u_{\xi}u_{\xi}Q_{0}]d\xi d\eta dx_{2} \]
\[ = 8 \int \int \int (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} - \frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)F^2 u_{\xi}Q_{0}\]
\[ = 8 \int \int \int (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} - \frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)F^2 u_{\xi}Q_{0}\]
\[ = 8 \int \int \int (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} - \frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)F^2 u_{\xi}Q_{0}\]
\[ = 8 \int \int \int (2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}(1 - H)F^2 u_{\xi}Q_{0}(1 - H)u_{\xi}u_{\xi}Q_{0} - \frac{d}{d\xi}(2 + \xi) \frac{\partial}{\partial (2 + \eta)} u_{\xi}Q_{0}(1 - H)F^2 u_{\xi}Q_{0}\]

Noting \(4u_{\xi}Q_{0} = \Box u_{\xi} + u_{\xi}u_{\xi}Q_{0} + 4\Gamma^2(F^2 u_{\eta}).\)
Then, the last term of the above equation can be rewritten as

\[
\begin{align*}
4 & \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H)F' u_{x_2} u_{k_5} d\xi d\eta d\phi \\
= & \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H)F' u_{x_2} u_{k_5} d\xi d\eta d\phi + \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H)F' u_{x_2} u_{k_5} u_{k_5} d\xi d\eta d\phi \\
+ & 4 \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H)F' u_{x_2} [u_{k_5} + 4 F'^2 u_{q_5}] u_{k_5} d\xi d\eta d\phi \\
+ & \frac{1}{2} \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} \left[ \frac{d}{dx_2} [(1 - H)F' u_{x_2} u_{k_5}^2] - [(1 - H)F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{1}{20} & \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5}^2 d\xi d\eta d\phi + \frac{1}{5} \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5}^2 d\xi d\eta d\phi \\
= & \frac{1}{2} \int \int \int \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5}^2] d\xi d\eta d\phi - 2 \frac{d}{d\eta} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5}^2] d\xi d\eta d\phi \\
- & \int \int \int \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5}^2] d\xi d\eta d\phi \\
+ & \int \int \int 4 \frac{d}{d\eta} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) u_{x_2} u_{k_5} Q_0] + 4 \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) u_{x_2} u_{k_5} Q_0] \\
- & 2 \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) u_{x_2} u_{k_5} Q_0] d\xi d\eta d\phi \\
- & 2 \int \int \int Q_0 [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) u_{x_2} u_{k_5} Q_0] d\xi d\eta d\phi \\
- & 2 \int \int \int (2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) u_{x_2} [u_{k_5} + 4 F'^2 u_{q_5}] Q_0(u_{x_2}, u) d\xi d\eta d\phi \\
+ & \int \int \int 8 \frac{d}{d\eta} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5} u_{k_5}] - 4 \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
- & 8 \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5} u_{k_5}] - 4 \frac{d}{d\xi} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
+ & 4 \int \int \int \frac{d}{d\eta} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi - 4 \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
- & 4 \int \int \int \frac{d}{d\eta} [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
+ & 4 \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
+ & \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}}(1 - H) F' u_{x_2} [u_{k_5} + 4 F'^2 u_{q_5}] u_{k_5}] d\xi d\eta d\phi \\
+ & \frac{1}{7} \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} \frac{d}{dx_2} [(1 - H) F' u_{x_2} u_{k_5}^2] - [(1 - H) F' u_{x_2} u_{k_5}^2] d\xi d\eta d\phi \\
+ & \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5} (1 - H) J_4 d\xi d\eta d\phi \\
+ & \int \int \int [(2 + \xi)^{-\frac{5}{2}}(2 + \eta)^{-\frac{5}{2}} u_{k_5} (1 - H) 4 F'^2 u_{q_5} H] d\xi d\eta d\phi \quad (3.36)
\end{align*}
\]
Noting
\[ Q_0(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)u_{\xi \eta}, u) \]
\[ = (2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)Q_0(u_{\xi \eta}, u) + u_{\xi \eta}Q_0((2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H), u), \]
then, we have
\[ 2Q_0((2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)u_{\xi \eta}, u)Q_0(u, u_\xi) \]
\[ = (2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)\frac{d}{d\xi}Q_0(u, u_\xi) - 2(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)Q_0(u, u)Q_0(u_{\xi \eta}, u_\xi) \]
\[ + 2u_{\xi \eta}Q_0((2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H), u)Q(u, u) \]
Furthermore, noting
\[ Q_0(u, u_{\xi \eta}) = 2u_{\xi \eta}u_{\xi \eta} + u_{\xi \eta}u_{\xi \eta} - u_{\xi \eta}u_{\xi \eta}, \]
then we can rewrite \( \Box u_{\xi \eta} \) as follows
\[ \Box u_{\xi \eta} = (1 - H)[Q_0(u, 2Q_0(u, u_{\xi \eta})) + 4F^\prime u_{\xi \eta}Q_0(u, u_{\eta}) + q - 4\Gamma^2(F^2 u_{\eta} u_{\xi})] \quad (3.37) \]
Substituting the above equations into (3.36), we can get
\[ \frac{1}{20} \int \int \int \int (2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}u_{\xi \eta}^2 d\xi d\eta dx + \frac{1}{2} \int \int \int (2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}u_{\xi \eta}^2 d\xi d\eta dx \]
\[ = \int \int \int \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}u_{\xi \eta}^2] d\xi d\eta dx \]
\[ - \frac{1}{2} \int \int \int [2(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}u_{\xi \eta}^2] d\xi d\eta dx \]
\[ + \int \int \int \frac{4d}{d\eta}[(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)u_{\eta}u_{\xi \eta}Q_0] \]
\[ - 2 \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)u_{\xi \eta}^2] d\xi d\eta dx \]
\[ - 2 \int \int \int (2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)\frac{d}{d\xi}Q_0(u, u_{\xi}) - 2(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)Q_0(u, u)Q_0(u_{\xi}, u_{\xi}) \]
\[ + 2u_{\xi \eta}Q_0((2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H), u)Q(u, u) d\xi d\eta dx \]
\[ - 8 \int \int \int \frac{8d}{d\eta}[(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)F^\prime u_{\xi \eta}^2] u_{\xi \eta}^2 d\xi d\eta dx \]
\[ - 8 \int \int \int [(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)F^\prime u_{\xi \eta}^2] d\xi d\eta dx \]
\[ + 8 \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{n}}(2 + \eta)^{-\frac{1}{n}}(1 - H)F^\prime u_{\xi \eta}^2] d\xi d\eta dx \]
\[ \begin{align*}
- \frac{4}{3} \iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) F^* u_{x_1} u_{x_2} u_{x_3} dx d\eta d\xi + \frac{4}{3} \iiint [2 + \xi]^2 (1 - H) F^* u_{x_1} u_{x_2} u_{x_3} dx d\eta d\xi \\
+ \iiint P(Q_0(u, 2Q_0(u, u_0)) + 8F' u_{x_2} u_{x_3} + 2F' u_{x_2} u_{x_3} - 4F' u_{x_2} u_{x_3} + J_k + 8F' u_{y_1} F_0^{2}(F_0^{2} u_{y_0}) - 41 \Gamma_4(F_0^{2} u_{y_0}) u_{x_1} d\xi d\eta d\xi d\eta \\
+ \frac{1}{3} \iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} [1 - (H) F^* u_{x_1} u_{x_2} u_{x_3}] - [(1 - H) F^* u_{x_1} u_{x_2} u_{x_3}] d\xi d\eta d\xi \\
+ \iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2} (1 - H) J d\eta d\xi d\eta \\
+ \iiint (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2} (1 - H) 41 \Gamma_4 (F_0^{2} u_{y_0}) d\eta d\xi d\eta \\
\text{where}
\begin{align*}
P(\xi, \eta, x_2) &= (2 + \xi)^{-2} (2 + \eta)^{-2} \frac{(1 - H) F^* u_{x_1}}{1 - 2(1 - H) F^* u_{y_0}}. \tag{3.38}
\end{align*}
\end{align*} \]

Noting
\[ \begin{align*}
\int \int \int (2 + \xi)^{-2} (2 + \eta)^{-2} [1 - (H) F^* u_{x_1} u_{x_2} u_{x_3}] - [(1 - H) F^* u_{x_1} u_{x_2} u_{x_3}] d\xi d\eta d\xi \\
= \int \int \int \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} [1 - (H) Q_0(u, u_0)] d\xi d\eta d\xi \\
- \int \int \int [2 + \xi]^2 (1 - H) u_{x_1} u_{x_2} Q_0 d\xi d\eta d\xi \\
\text{Then}
\begin{align*}
&\frac{1}{20} \iiint (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2}^2 d\xi d\eta d\xi + \frac{1}{5} \iiint (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2}^2 d\xi d\eta d\xi \\
= &\frac{1}{2} \iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2}^2 d\xi d\eta d\xi - 2 \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2}^2 d\xi d\eta d\xi \\
- &\iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} u_{x_2}^2 d\xi d\eta d\xi \\
+ &\iiint 4 \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) u_{x_1} u_{x_2} Q_0 d\xi d\eta d\xi \\
- &2 \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) u_{x_1} u_{x_2} Q_0 d\xi d\eta d\xi \\
+ &2 \iiint \frac{d}{dn} [2 + \xi]^2 (1 - H) Q_0(u, u_0) d\xi d\eta d\xi \\
- &2 \iiint \frac{d}{dn} [2 + \xi]^2 (1 - H) Q_0(u, u_0) d\xi d\eta d\xi \\
+ &\iiint 4(2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) Q_0(u, u_0) Q_0(u, u_0) - 4 u_{x_1} u_{x_2} Q_0(2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) u_{x_1} Q_0(u, u_0) d\xi d\eta d\xi \\
- &2 \iiint \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) u_{x_1} u_{x_2} Q_0(u, u_0) d\xi d\eta d\xi \\
+ &\iiint 8 \frac{d}{dn} (2 + \xi)^{-2} (2 + \eta)^{-2} (1 - H) F^* u_{x_1} u_{x_2} u_{x_3} d\xi d\eta d\xi \\
- &8 \iiint [2 + \xi]^2 (1 - H) F^* u_{x_1} u_{x_2} u_{x_3} d\xi d\eta d\xi + \iiint 4 \frac{d}{dn} [2 + \xi]^2 (1 - H) d\xi d\eta d\xi \\
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\end{align*} \]
In the following we also deal with the term

$$+ 4 \int \int \int \frac{d}{d\eta}(2 + \xi)^{-\tilde{\nu}}(2 + \eta)^{-\tilde{\mu}}(1 - H)F'\eta \mu \xi \gamma \delta d\xi d\eta dx_2 - 4 \int \int \int [2 + \xi)^{-\tilde{\nu}}(2 + \eta)^{-\tilde{\mu}}(1 - H)F'\eta \mu \xi \gamma \delta d\xi d\eta dx_2$$

$$- 4 \int \int \int \frac{d}{d\eta}(2 + \xi)^{-\tilde{\nu}}(2 + \eta)^{-\tilde{\mu}}(1 - H)F'\eta \mu \xi \gamma \delta d\xi d\eta dx_2$$

In the following we also deal with the term

$$\int \int \int P(Q_0(u,2Q_0(u,u)) + 8F'\eta \mu \xi \delta d\xi d\eta dx_2 + 4F'\eta \mu \xi \delta d\xi d\eta dx_2 + J_k + 8F'\eta \mu \eta \xi \delta d\eta dx_2$$

$$+ \frac{1}{2} \int \int \int (2 + \xi)^{-\tilde{\nu}}(2 + \eta)^{-\tilde{\mu}}u_{\xi \eta}(1 - H)J_k \xi d\eta dx_2$$

$$+ \int \int \int (2 + \xi)^{-\tilde{\nu}}(2 + \eta)^{-\tilde{\mu}}u_{\xi \eta}(1 - H)4\tilde{\nu}(F' - \eta \mu \xi \gamma \delta d\eta dx_2$$

In the following we also deal with the term

$$\int \int \int P(Q_0(u,2Q_0(u,u)) + 8F'\eta \mu \xi \delta d\xi d\eta dx_2 + 4F'\eta \mu \xi \delta d\xi d\eta dx_2 + J_k + 8F'\eta \mu \gamma \delta d\eta dx_2 + 4\tilde{\nu}(F' \eta \mu \xi \gamma \delta d\eta dx_2$$

$$\Leftrightarrow III_1 + \cdots + III_7 \quad (3.40)$$

$$III_1 = \int \int \int P(Q_0(u,2Q_0(u,u))u_{\xi \eta} \xi d\eta dx_2 - \int \int \int P[4\eta \xi \eta - 2u_{\xi \eta}, u_{\xi \eta}, u_{\xi \eta} \xi d\eta dx_2$$

$$= \int \int \int \frac{d}{d\eta}(4P_{\xi \eta}Q_{0 \xi \eta}) - 4P_{\xi \eta}Q_{0 \xi \eta} - 4P_{\xi \eta}Q_{0 \xi \eta} \xi d\eta dx_2$$

$$+ \int \int \int \frac{d}{d\xi} (2P_{\xi \eta}Q_{0 \xi \eta}) - 2P_{\xi \eta}Q_{0 \xi \eta} \xi d\eta dx_2$$

$$\Leftrightarrow 2 \int \int \int P(Q_0(u,u_{\xi \eta}))Q_0 \xi d\eta dx_2$$

$$\Leftrightarrow 2 \int \int \int [4P_{\xi \eta} + 4P(x)\eta] - 2[\xi \eta]Q_{0 \xi \eta} \xi d\eta dx_2$$

Using the integration by parts, we have

$$2 \int \int \int P(Q_0(u,u_{\xi \eta}))Q_0 \xi d\eta dx_2 = \int \int \int P \frac{d}{dx_2} Q^2_0(u,u_{\xi \eta}) d\xi d\eta dx_2$$

$$= \int \int \int \frac{d}{dx_2} [PQ_0^2(u,u_{\xi \eta})] d\xi d\eta dx_2 - \int \int \int P \xi Q_0(u,u_{\xi \eta}) d\xi d\eta dx_2$$

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Then,

\[ H_1 = \int \int \int \frac{dQ_0}{\partial \xi} [4Pu_\eta Q_0 u_{k_\xi}] + \frac{d}{d\xi} [4Pu_\eta Q_0 u_{k_\xi}] - \frac{d}{d\xi} [2Pu_\eta Q_0 u_{k_\xi}] d\xi d\eta dx_2 \]

\[ - \int \int \int \frac{d}{d\xi} [PQ_0(u, u_\xi)] d\xi d\eta dx_2 + \int \int \int P_\xi Q_0(u, u_\xi) d\xi d\eta dx_2 \]

\[ = \int \int \int [4Pu_\eta] + [4Pu_\eta]_x = 2[Pu_\eta]_x Q_0 u_{k_\xi} d\xi d\eta dx_2. \]

\[ H_2 = \int \int \int P8F' u_\eta u_{k_\xi} d\xi d\eta dx_2 \]

\[ = 8 \int \int \int \frac{d}{d\eta} [PF' u_\eta u_{k_\xi}] - [PF' u_\eta u_{k_\xi}] - [PF' u_\eta]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ + 8 \int \int \int \frac{d}{d\eta} [PF' u_\eta u_{k_\xi}] - 4 \frac{d}{d\xi} [PF' u_\eta u_{k_\xi}] + 4 [PF' u_\eta]_x u_{k_\xi} - 8 [PF' u_\eta]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ H_3 = \int \int \int P2F' u_\eta u_{k_\xi} d\xi d\eta dx_2 = \int \int \int PF' u_\eta \frac{d^2}{dx_2} u_{k_\xi} d\xi d\eta dx_2 \]

\[ = \int \int \int \frac{d}{d\eta} [PF' u_\eta u_{k_\xi}^2] - [PF' u_\eta]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ H_4 = \int \int \int P4F' u_\eta u_{k_\xi} u_{k_\xi} d\xi d\eta dx_2 = \int \int \int 2PF' u_{k_\xi} \frac{d}{dx_2} u_{k_\xi} d\xi d\eta dx_2 \]

Then

\[ \int \int \int P[Q_0(u, 2Q_0(u, u_\xi))] + 8F' u_\eta u_{k_\xi} + 2F' u_\eta u_{k_\xi} + 4F' u_{k_\xi} u_{k_\xi} + J + 8F' u_\eta (F' u_\eta) - 4F' (F' u_\eta) u_{k_\xi} d\xi d\eta dx_2 \]

\[ = \int \int \int \frac{d}{d\eta} [4Pu_\eta Q_0 u_{k_\xi}] + \frac{d}{d\xi} [4Pu_\eta Q_0 u_{k_\xi}] - \frac{d}{d\xi} [2Pu_\eta Q_0 u_{k_\xi}] d\xi d\eta dx_2 \]

\[ - \int \int \int \frac{d}{d\xi} [PQ_0(u, u_\xi)] d\xi d\eta dx_2 + \int \int \int P_\xi Q_0(u, u_\xi) d\xi d\eta dx_2 \]

\[ + \int \int \int 8 \frac{d}{d\eta} [PF' u_\eta u_{k_\xi}] - 4 \frac{d}{d\xi} [PF' u_\eta u_{k_\xi}] + 4 [PF' u_\eta]_x u_{k_\xi} - 8 [PF' u_\eta]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ + \int \int \int \frac{d}{d\eta} [PF' u_\eta u_{k_\xi}^2] - [PF' u_\eta]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ + 2 \int \int \int \frac{d}{d\eta} [PF' u_{k_\xi} u_{k_\xi}^2] - [PF' u_{k_\xi}]_x u_{k_\xi} d\xi d\eta dx_2 \]

\[ - 4 \int \int \int \frac{d}{d\eta} [PF' u_{k_\xi} u_{k_\xi}^2] - P_\eta F' u_{k_\xi} u_{k_\xi} - \frac{1}{2} \frac{d}{dx_2} [PF' u_{k_\xi}]^2 + \frac{1}{2} P_\xi F' u_{k_\xi}^2 d\xi d\eta dx_2 \]

\[ + 8 \int \int \int \frac{d}{d\eta} [PF' u_{k_\xi} u_{k_\xi}^2] - P_\eta F' u_{k_\xi} u_{k_\xi} - \frac{1}{2} \frac{d}{dx_2} [PF' u_{k_\xi}]^2 + \frac{1}{2} P_\xi F' u_{k_\xi}^2 d\xi d\eta dx_2 \]

\[ + \int \int \int P, u_{k_\xi} d\xi d\eta dx_2 \]
Finally, we can get

\[
\frac{1}{20} \int \int \int (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2}^2 \, d\xi d\eta dx_2 + \frac{1}{5} \int \int \int (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2}^2 \, d\xi d\eta dx_2
\]  
\[
= \frac{1}{2} \int \int \int \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2}^2] \, d\xi d\eta dx_2 - \frac{2}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2}^2] \, d\xi d\eta dx_2
\]  
\[
- \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2}^2] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{4}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2] + 4 \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2]
\]  
\[
- \frac{2}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
= 2 \int \int \int \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{4}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) Q(u, u) u_2(u, u)] - 4 u_{x_2} Q_0 u_2(u, u) (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2 \, d\xi d\eta dx_2
\]  
\[
- 2 \int \int \int \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{8}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] - 4 \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
- 8 \int \int \int \frac{d}{d\xi}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
+ 4 \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
- 4 \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
+ 4 \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} (1 - H) F' u_{x_2} u_2] \, d\xi d\eta dx_2
\]  
\[
+ \frac{1}{2} \int \int \int (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} \frac{d}{d\xi}[(1 - H) F' u_{x_2} u_2^2] - [(1 - H) F' u_{x_2} u_2^2] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{d}{d\eta}[(2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} u_{x_2} (1 - H) J_2] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{d}{d\eta}[(4 P_{x_2} Q_0 u_{x_2}) + d_{x_2}[(4 P_{x_2} Q_0 u_{x_2}) - d_{x_2}[(2 P_{x_2} Q_0 u_{x_2})] \, d\xi d\eta dx_2
\]  
\[
- \int \int \int d_{x_2}[(P Q_0^2 u_2) \, d\xi d\eta dx_2 + \int \int \int P_{x_2}(x) Q_0^2 u_2 \, d\xi d\eta dx_2
\]  
\[
- \int \int \int [4 P_{x_2} Q_0 u_{x_2}] \, d\xi d\eta dx_2
\]  
\[
+ \int \int \int \frac{d}{d\eta}[(PP' u_{x_2} u_{x_2})] - \frac{d}{d\eta}[(PP' u_{x_2} u_{x_2})] \, d\xi d\eta dx_2
\]  
\[
- \int \int \int \frac{d}{d\eta}[(PP' u_{x_2} u_{x_2})] \, d\xi d\eta dx_2
\]  
\[
+ 2 \int \int \int d_{x_2}[(PP' u_{x_2} u_{x_2})] - [PP' u_{x_2} u_{x_2}] \, d\xi d\eta dx_2
\]  
\[
+ 2 \int \int \int d_{x_2}[(PP' u_{x_2} u_{x_2})] - [PP' u_{x_2} u_{x_2}] \, d\xi d\eta dx_2
\]
\[ + \iint P J_k u_k , d\xi d\eta dx + \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} u_k d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] L_0^2 (u, u_k) d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] Q_0 (u_k, u) Q_0 (u_k, u) - u_k Q_0 ((2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H), u) Q(u_k, u) d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] F' u_k d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] J_0 u_k d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] J_0 u_k d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] J_0 u_k d\xi d\eta dx \]
\[ + \iint \left[ (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) \right] J_0 u_k d\xi d\eta dx \]
\[ = \mathcal{A}_1 + \cdots + \mathcal{A}_{14}. \]

In the following, we will estimate \( \mathcal{A}_i \), \( i = 1, \cdots, 14 \) respectively. Using the similar procedures to (3.8) and (3.14), (3.18), we can get the estimations of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \)

\[ \mathcal{A}_1 = \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) L_0^2 (u, u_k) d\xi d\eta dx \]
\[ \leq \varepsilon + (e_x + e_y + e_z + e^2) E_x, \quad (3.41) \]

\[ \mathcal{A}_2 = \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) J_0 (u_k, u) Q_0 (u_k, u_k) - u_k J_0 ((2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H), u) Q(u_k, u_k) d\xi d\eta dx \]
\[ \leq (\varepsilon + e^2) E_x, \quad (3.42) \]

Noting the estimate of \( \mathcal{A}_2 \), we can get

\[ \mathcal{A}_3 = \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) u_k d\xi d\eta dx \]
\[ \leq e^2 E_x + e_x E_x + e_y E_x + e_z E_x, \quad (3.43) \]

Noting,

\[ \mathcal{A}_4 = \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) F' u_k d\xi d\eta dx \]
\[ \mathcal{A}_5 = \iint (2 + \xi)^{- \frac{1}{2}} (2 +\eta)^{- \frac{1}{2}} (1 - H) F' u_k d\xi d\eta dx \]
we have

\[ \mathcal{A}_{41} \lesssim (1 + e_s) \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} u_{\xi \xi}] u_{\xi \eta} \xi d \eta d \xi \] 

\[ + \int \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} |H_\eta F'u_{\xi}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim (1 + e_s) \epsilon_s \{ E_s + \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(Q_{\eta \eta} + 4 F'(\xi) u_{\eta \eta}] u_{\eta \eta] d \xi d \eta d \xi] \} \] 

\[ \lesssim (1 + e_s) \epsilon_s \{ E_s + \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(e_s \frac{1}{2} + e_s)](2 + \xi)^\frac{\theta}{2} |e_{\eta}] u_{\xi \eta} d \xi d \eta d \xi \] 

Furthermore, noting Proposition 3 and Corollary 1,

\[ \mathcal{A}_{42} = \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} (1 - H) F' u_{\xi}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim \int \int (2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(1 - H) F' u_{\xi}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ + \int \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} |H_\xi F'u_{\xi}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim \int \int (2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(1 - H) u_{\xi \eta}] [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} e_s + (2 + \xi)^\frac{\theta}{2} e_s + e_s] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ + \int \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(Q_{\eta \eta} + 4 F'(\xi) u_{\eta \eta}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim (1 + e_s) \epsilon_s \{ E_s + (2 + \xi)^\frac{\theta}{2} \eta_{\xi \eta} d \xi d \eta d \xi \] 

\[ + \int \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(1 + (2 + \xi)^\frac{\theta}{2} e_s + (2 + \xi)^\frac{\theta}{2} e_s + e_s)](2 + \xi)^\frac{\theta}{2} |e_{\eta}] u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim [(1 + e_s) e_s + e_s + e_s + e_s] E_s. \]  

Denote

\[ \mathcal{A}_5 = \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} (1 - H) F' u_{\eta \eta} \eta u_{\xi \eta} d \xi d \eta d \xi. \] 

Noting Corollary 1, we can get

\[ \mathcal{A}_5 = \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} (1 - H) F' u_{\eta \eta} \eta u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim (1 + e_s) \int \int (2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} [(1 + \chi)^\frac{\theta}{2} (1 - H) F' u_{\eta \eta}] \eta u_{\xi \eta} d \xi d \eta d \xi \] 

\[ + \int \int [(2 + \xi)^\frac{\theta}{2} (2 + \eta)^\frac{\theta}{2} |H_\eta F'u_{\eta \eta}] \eta u_{\xi \eta} d \xi d \eta d \xi \] 

\[ \lesssim (1 + e_s) \epsilon_s \{ E_s + 4 F'(\xi) u_{\eta \eta} \eta u_{\xi \eta} d \xi d \eta d \xi \] 

\[ + \int \int [Q_{\eta \eta} + 4 F'(\xi) u_{\eta \eta}] \eta u_{\xi \eta} d \xi d \eta d \xi. \]
\[
\begin{align*}
\lesssim (1 + e_s) & \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} e_t^2 u_{k}^2 d\xi d\eta dx_2 \\
& + \int \int \int (2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (|u_{k}||u_{\xi}| + |u_{\eta}||u_{\eta}| + |u_{x_2}||u_{x_2}| + (2 + \eta)^{-\frac{1}{n}} e_t^2 [(2 + \eta)^{-\frac{1}{n}} e_t^2 u_{k}^2 d\xi d\eta dx_2 \\
\lesssim (1 + e_s) e_t^\frac{1}{2} E_t + \int \int \int (2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} ((2 + \xi)^{-\frac{1}{m}} e_t^2 [(2 + \eta)^{-\frac{1}{n}} e_t^2 + (2 + \eta)^{-\frac{1}{n}} e_t^2 u_{k}^2 d\xi d\eta dx_2 \\
\lesssim (1 + e_s) e_t^\frac{1}{2} E_t + (e_s + e_t^\frac{1}{2}) E_t. \quad (3.45)
\end{align*}
\]

Denote
\[
\begin{align*}
\tilde{\Lambda}_6 &= \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (1 - H) F' u_{k_1} | u_{\xi} u_{x_2} | d\xi d\eta dx_2 \\
& + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (1 - H) F' u_{k_1} u_{x_2}^2 d\xi d\eta dx_2 \doteq \tilde{\Lambda}_{61} + \tilde{\Lambda}_{62}. \quad (3.46)
\end{align*}
\]

Then, noting the estimate of \( \tilde{\Lambda}_5 \),
\[
\begin{align*}
\tilde{\Lambda}_{61} \lesssim & \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (1 - H) |u_{k_1}||u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
& + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (1 - H) |u_{k_1}||u_{x_2}| d\xi d\eta dx_2 + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |H_0 u_{k_1}| |u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} e_t^2 |u_{\xi} u_{x_2}| d\xi d\eta dx_2 + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |H_0| e_t^2 |u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (Q_{0_1} + 4 F' (\xi)|u_{\eta}|)|e_t^\frac{1}{2} |u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} e_t^2 (2 + \eta)^{-\frac{1}{n}} e_t^2 + (2 + \eta)^{-\frac{1}{n}} e_t^2 |u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + (e_s + e_t^\frac{1}{2}) \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |u_{\xi} u_{x_2}| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + (e_s + e_t^\frac{1}{2}) E_t. \quad (3.47)
\end{align*}
\]

and
\[
\begin{align*}
\tilde{\Lambda}_{62} \lesssim & \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (1 - H) |u_{k_1}| u_{x_2}^2 d\xi d\eta dx_2 + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |H_0 u_{k_1}| u_{x_2}^2 d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} e_t^2 u_{x_2}^2 d\xi d\eta dx_2 + \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |H_0| e_t^2 u_{x_2}^2 d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + (1 + e_s) \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} (Q_{0_2} + 4 F' (\xi)|u_{\eta}|)|e_t^\frac{1}{2} u_{x_2}^2 d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + (1 + e_s) (e_s + e_t^\frac{1}{2}) \int \int \int \int [(2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} |u_{x_2}^2| d\xi d\eta dx_2 \\
\lesssim & (1 + e_s) e_t^\frac{1}{2} E_t + (1 + e_s) (e_s + e_t^\frac{1}{2}) E_t. \quad (3.48)
\end{align*}
\]

Denote
\[
\tilde{\Lambda}_7 = \int \int \int \int (2 + \xi)^{-\frac{1}{m}}(2 + \eta)^{-\frac{1}{n}} u_{k}^2 (1 - H) J_0 d\xi d\eta dx_2.
\]

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Similar to the estimate of the term $A_k$, we will first estimate $J_k$. Let

\[ J_{k1} = \sum_{k_1+k_2+k_3+k_4+k_5=k} \Gamma^k(1-H)Q_0(\Gamma^{k_1}u, Q_0(\Gamma^{k_2}u, \Gamma^{k_3}u)) \]

\[ J_{k2} = \sum_{0 \leq k_2 \leq k} \Gamma^{k_1}(1-H)\Gamma^{k-1}k[F'Q_0(u, u_0)] \]

\[ J_{k3} = \sum_{0 \leq k_2 \leq k} \Gamma^{k_1}(1-H)\Gamma^{k-1}k[F'\eta' u_0]. \]

For $J_{k1}$, when $k_1 \leq \lfloor \frac{k+2}{2} \rfloor$, we have

\[ |J_{k1}| = | \sum_{k_1+k_2+k_3+k_4+k_5=k} \Gamma^k(1-H)Q_0(\Gamma^{k_1}u, Q_0(\Gamma^{k_2}u, \Gamma^{k_3}u)) | \]

\[ \leq (1 + e_1)(|Q_0(\Gamma^{k_1}u, \Gamma^{k_2}u)| + |(\Gamma^{k_1}u_0)Q_0(\Gamma^{k_2}u, \Gamma^{k_3}u)| + |(\Gamma^{k_1}u_0)Q_0(\Gamma^{k_2}u, \Gamma^{k_3}u)|) \]

\[ \leq (1 + e_1)^2(1 + \eta)^{-1}(e_s + 2 \partial_{x_5}e_s)(\Gamma^{k}u_0)(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)(\Gamma^{k_3}u_0). \]

Without loss of generality, we assume $|k_3|, |k_4| \leq \lfloor \frac{k+4}{2} \rfloor$ and noting (2.21), then

\[ J_{k1} \lesssim (1 + e_1)(1 + \eta)^{-1}(e_s + 2 \partial_{x_5}e_s)(\Gamma^{k}u_0)(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)(\Gamma^{k_3}u_0). \]

Noting Lemma 3.2, Lemma 3.3 and Proposition 1,

\[ J_{k11} \lesssim (1 + e_1)(|Q_0(\Gamma^{k_1}u, \Gamma^{k_2}u)|) \]

\[ \lesssim (1 + e_1)(1 + \eta)^{-1}(e_s + 2 \partial_{x_5}e_s)(\Gamma^{k}u_0)(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)(\Gamma^{k_3}u_0). \]

By the definition of $\Gamma$ operator, we have

\[ 2x_5 \partial_{x_5}(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0) = (\Gamma_0 - \xi \partial_{x_5})(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0) \]

\[ = (\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0) + (\Gamma^{k_2}u_0)(\Gamma^{k_1}u_0) - \xi (\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0). \]

Then, noting Lemma 3, Proposition 1 and Proposition 2,

\[ |2x_5 \partial_{x_5}(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)| \lesssim |(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)| + |(\Gamma^{k_1}u_0)(\Gamma^{k_2}u_0)| + |(\Gamma^{k_2}u_0)(\Gamma^{k_1}u_0)| \]

\[ \lesssim e_s + (2 + \eta)^{1/2}(2 + \xi)^{1/2}(2 + \eta)^{1/2}e_s. \]

Therefore, we can obtain

\[ J_{k11} \lesssim (1 + e_1)e_s(2 + \eta)^{-1}(|\Gamma^{k_1}u_0|). \]

Meanwhile, noting Proposition 2, we can get

\[ J_{k12} = C(1 + e_1)(|\Gamma^{k_1}u_0|)Q_0(\Gamma^{k_1}u, \Gamma^{k_1}u_0) \]

\[ \lesssim (1 + e_1)(|\Gamma^{k_1}u_0|)(|\Gamma^{k_1}u_0|)(\Gamma^{k_1}u_0) + (\Gamma^{k_1}u_0)(\Gamma^{k_1}u_0)). \]

\[ \lesssim (1 + e_1)(|\Gamma^{k_1}u_0|)(2 + \eta)^{1/2}(2 + \xi)^{1/2}(2 + \eta)^{1/2}e_s + (2 + \xi)^{1/2}(2 + \eta)^{1/2}e_s. \]

\[ \lesssim (1 + e_1)e_s(2 + \eta)^{1/2}(2 + \xi)^{1/2}(2 + \eta)^{1/2}e_s. \]
and noting the estimate of $k_{\ell_11}$

$$J_{k_{\ell_11}} \lesssim (1 + e_\xi)|[T^{k_2}u]_{k_3}||Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)|$$

$$\lesssim (1 + e_\xi)(2 + \eta)^{-1}|[T^{k_2}u]_{k_3}||T^{k_2}u_{k_3}||T^{k_2}u|$$

$$\lesssim (1 + e_\xi)|[T^{k_2}u]_{k_3}|(2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}e^\frac{1}{2}_r(2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}e^\frac{1}{2}_r$$

$$\lesssim (1 + e_\xi)e_\xi(2 + \eta)^{-1}|(T^{k_2}u)_{k_3}|.$$  

(3.53)

When $k_1 \geq \frac{1}{12}k$, we have

$$|J_{k_1}| \lesssim \sum_{k_1' + k_2 + k_3 = k_1' + k_2 + k_3 < k} \Gamma^{k_1}(Q_0 + F''u_\eta)Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)$$

$$\lesssim \sum_{k_1' + k_2 + k_3 < k} |[\Gamma^{k_1}(Q_0(u, u)| + |[\Gamma^{k_1}(F''u_\eta)|]|Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)].$$

In the following we will estimate the above two parts separately.

$$\sum_{k_1' + k_2 + k_3 + k_4 < k} |[\Gamma^{k_1}(Q_0(u, u)|][Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)]$$

$$= \sum_{k_2 + k_3 + k_4 + k_5 < k} |[\Gamma^{k_1}u_k T^{k_2}u_{k_3} + \Gamma^{k_1}u_{k_3} T^{k_2}u_k]|Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)].$$

When $k_5 \leq k_6$, noting Lemma 2, we have

$$|[\Gamma^{k_1}u_k T^{k_2}u_{k_3} + \Gamma^{k_1}u_{k_3} T^{k_2}u_k]|Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)$$

$$\lesssim |[\Gamma^{k_1}u_k(2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}(2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}e^\frac{1}{2}_r + (2 + \xi)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}}e^\frac{1}{2}_r]|$$

$$\lesssim |[\Gamma^{k_1}u_k(2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}}[e^\frac{1}{2}_r e^\frac{1}{2}_r]|. (3.54)$$

For the case $|k_5| \geq |k_6|$, by Corollary 1 and Lemma 3, we have

$$|[\Gamma^{k_1}u_k T^{k_2}u_{k_3} + \Gamma^{k_1}u_{k_3} T^{k_2}u_k]|Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)$$

$$\lesssim |[\Gamma^{k_1}u_k(2 + \eta)^{-\frac{1}{2}}e^\frac{1}{2}_r + (2 + \eta)^{\frac{1}{2}}e^\frac{1}{2}_r]|$$

$$\lesssim |[\Gamma^{k_1}u_k(2 + \eta)^{-\frac{1}{2}}e^\frac{1}{2}_r + (2 + \eta)^{\frac{1}{2}}e^\frac{1}{2}_r]|.$$  

(3.55)

It is easy to get the estimate of second part $|[\Gamma^{k_1}(F''u_\eta)|]|Q_0(T^{k_2}u_{k_3}, \Gamma^{k_2}u)$). Then, we can get the estimate of $J_{k_1}$.

In the following, we will give the estimation of $J_{k_2}$. When $k_1 \leq \frac{1}{12}k$, the other case can be get easily.

$$J_{k_2} \lesssim \sum_{0 \leq k_1 + k_3 < k} |[\Gamma^{k_1}(1 - H)||[\Gamma^{k_1}]F''Q_0(u, u)|]$$

$$\lesssim (1 + e_\xi) \sum_{0 \leq k_1 + k_3 + k_2} |[\Gamma^{k_1}]F''Q_0(u, u)|$$

$$\lesssim (1 + e_\xi)(2 + \eta)^{-1} \sum_{k_1 + k_3 + k_2} |Q_0(T^{k_1}u_{k_3}, \Gamma^{k_1}u)|$$

$$\lesssim (1 + e_\xi)(2 + \eta)^{-1} \sum_{k_1 + k_3 + k_2} |[\Gamma^{k_1}u_k T^{k_2}u_{k_3} + \Gamma^{k_1}u_{k_3} T^{k_2}u_k]|.$$
For the case of $|k_3| \geq |k_6|$, noting Corollary 1,
\[
|\Gamma^3 u_{q6}| \lesssim (2 + \eta)^{-1}|(\partial_\eta + \Gamma_4 - x_2 \partial_{x_2})(\Gamma^3 u)_q| \\
\lesssim (2 + \eta)^{-1}|(\Gamma^3 u)_{q6} + \Gamma_3 (\Gamma^3 u)_q| + (2 + \eta)^{-1}(2 + \xi)^{-1}(\Gamma^3 u)_{x_6}
\lesssim (2 + \eta)^{-1} e^\frac{1}{\eta} + (2 + \eta)^{-1}(2 + \xi)^{-1} e^\frac{1}{\xi}.
\]
Then, we can get
\[
J_{12} \lesssim (1 + e_\eta)(2 + \xi)^{-1} e^\frac{1}{\xi} |\Gamma^3 u| + |\Gamma^3 u_q| + |\Gamma^3 u_{x_6}|. \tag{3.56}
\]
For the case of $|k_3| \leq |k_6|$, noting Lemma 2.2 and Lemma 2.3, we can have
\[
J_{12} \lesssim (1 + e_\eta)(1 + \xi)^{-1} \sum_{k_3 + k_6 = k} |Q_0(\Gamma^3 u, \Gamma^3 u_q)| \\
\lesssim (1 + e_\eta)(2 + \xi)^{-1} (2 + \eta)^{-1} |\Gamma^3 u| |\Gamma^3 u_q| + |\Gamma^3 u| |\Gamma^3 u_q| \\
\lesssim (2 + \xi)^{-1} (2 + \eta)^{-1} (1 + e_\eta) e^\frac{1}{\xi} |\Gamma^3 u| + |\Gamma^3 u_q|.
\]
Moreover, it is easily to get the estimate
\[
J_{13} = \sum_{0 \leq k_3 + k_6 \leq k} \Gamma^3 (1 - H) \Gamma^3 [F'' u^2] \lesssim (1 + e_\eta) (2 + \xi)^{-1} (2 + \eta)^{-1} \sum_{0 \leq k_3 + k_6 \leq k} |\Gamma^3 u_q| \tag{3.57}
\]
Then, the estimation of $A_4$ can be get
\[
\tilde{A}_7 \lesssim (e^\frac{1}{\eta} + e^\frac{1}{\xi}) E_\eta. \tag{3.58}
\]
Denote
\[
\tilde{A}_8 = \iiint P_{x_3}(x) Q_0^3(u, u_3) d\xi d\eta dx_2 + \iiint [4 [P_{x_1}]_q + [P_{x_2}]_q - 2 [P_{x_3}]_q] Q_0 u_{x_3} d\xi d\eta dx_2 \approx \tilde{A}_{81} + \tilde{A}_{82}
\]
where,
\[
\tilde{A}_{81} = \iiint P_{x_3}(x) Q_0^3(u, u_3) d\xi d\eta dx_2 \\
\lesssim \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} \frac{1}{1 - 2(1 - H)F'' u_q} |\Gamma^3 u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \\
\lesssim (1 + e_\eta) \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |\Gamma^3 u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \\
+ (1 + e_\eta)^2 \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |\Gamma^3 u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \\
+ (1 + e_\eta)^3 \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |\Gamma^3 u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \\
\lesssim (1 + e_\eta) \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |Q_{0x_3} + 4F''(\xi) u_{x_3} u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \tag{3.59}
\]
\[
+ (1 + e_\eta)^2 \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |Q_{0x_3} + 4F''(\xi) u_{x_3} u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2 \\
+ (1 + e_\eta)^3 \iiint (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} |Q_{0x_3} + 4F''(\xi) u_{x_3} u_{x_3}| |u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2 + u_{x_3}^2 u_{x_3}^2| d\xi d\eta dx_2.
\]
By Proposition 2, Proposition 3 and Corollary 1, we have

\[ \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |Q_{0\xi}, u_{\xi}| u_{\xi}^2 + u_{\eta}^2 + u_{\xi}^2 u_{\xi 1} d\xi d\eta dx_2 + (1 + \epsilon_s) e_{\epsilon_s}^2 E_s \]

\[ (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |Q_{0\xi}, u_{\xi}| u_{\xi}^2 + u_{\eta}^2 + u_{\xi}^2 u_{\xi 1} d\xi d\eta dx_2 + (1 + \epsilon_s)^2 e_{\epsilon_s}^2 E_s \]

\[ (1 + \epsilon_s)^3 e_{\epsilon_s}^3 E_s + (1 + \epsilon_s)^2 e_{\epsilon_s}^2 E_s \]

\[ (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |Q_{0\xi}, u_{\xi}| u_{\xi}^2 + u_{\eta}^2 + u_{\xi}^2 u_{\xi 1} d\xi d\eta dx_2 \]

Then,

\[ \tilde{A}_{81} \lesssim (1 + \epsilon_s)^2 e_{\epsilon_s}^2 E_s + (1 + \epsilon_s)^3 e_{\epsilon_s}^3 E_s. \] (3.60)

\[ \tilde{A}_{82} = \int \int \left( 4P_{u\xi} + [Pu_{\xi}]_{\eta} - 2P_{u_{\xi 1}} \right) Q_{0}(u_{\xi}, u) d\xi d\eta dx_2 \]

\[ \lesssim \int \int \left( P_{u\xi} + P_{u_{\xi 1}} + P_{u_{\xi 1}} \right) Q_{0}(u_{\xi}, u) d\xi d\eta dx_2 \]

\[ \lesssim \tilde{A}_{821} + \tilde{A}_{822}. \]

Noting Proposition 2,

\[ \tilde{A}_{822} \lesssim (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |F^r| u_{\xi} (u_{\eta} | u_{\xi 2} | u_{\xi 2} | d\xi d\eta dx_2 \]

\[ + (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |F^r| u_{\xi} (u_{\eta} | u_{\xi 2} | u_{\xi 2} | d\xi d\eta dx_2 \]

\[ + (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |F^r| u_{\xi} (u_{\eta} | u_{\xi 2} | u_{\xi 2} | d\xi d\eta dx_2 \]

\[ \lesssim (1 + \epsilon_s)^2 \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} e_{\epsilon_s}^2 u_{\xi 1} u_{\xi 1} d\xi d\eta dx_2 \]

\[ + \int \int \int (1 + \epsilon_s)^2 (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} (2 + \xi)^{-\frac{1}{2}} (2 + \eta)^{-\frac{1}{2}} e_{\epsilon_s} |u_{\xi 2} u_{\xi 2}| d\xi d\eta dx_2 \]

\[ \lesssim (1 + \epsilon_s)^2 e_{\epsilon_s}^2 E_s + (1 + \epsilon_s)^2 e_{\epsilon_s} \int \int \int (2 + \xi)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} |u_{\xi 1} u_{\xi 1} u_{\xi 1} | d\xi d\eta dx_2 \]

\[ \lesssim (1 + \epsilon_s)^2 e_{\epsilon_s}^2 E_s. \] (3.61)
In the following, we will estimate $\tilde{A}_{821}$.

\[
\tilde{A}_{821} \lesssim \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} \frac{(1-H)^2 u_{\xi z}}{1-2(1-H) u_{\eta z}} + u_{\xi z} + u_{\eta z}) d\xi d\eta d\eta_2
\]

\[
+ \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} \frac{(1-H)^2 u_{\xi z}}{1-2(1-H) u_{\eta z}} + u_{\xi z} + u_{\eta z}) d\xi d\eta d\eta_2
\]

\[
+ \iint (2+\xi)^{-\beta}(2+\eta)^{-\beta} \frac{(1-H)^2 u_{\xi z}}{1-2(1-H) u_{\eta z}} + u_{\xi z} d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon)^2 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^2 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^2 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon)^2 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon + (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^3 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon + (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon)^2 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon + (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
+ (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon)^2 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon + (1+e_\varepsilon)^3 \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon)^2 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon. \tag{3.62}
\]

Therefore, we can get

\[
\tilde{A}_8 \lesssim (1+e_\varepsilon)^2 (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon. \tag{3.63}
\]

Denote

\[
\tilde{A}_0 \doteq \iint [PF' u_{\xi}]_2 u_{\xi z}^2 d\xi d\eta d\eta_2 = \hat{A}_1 + \tilde{A}_2.
\]

By Proposition 2, it is easily to get

\[
\hat{A}_9 = \iint [PF' u_{\xi}]_2 u_{\xi z}^2 d\xi d\eta d\eta_2
\]

\[
\lesssim \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim \iint ((2+\xi)^{-\beta}(2+\eta)^{-\beta} |u_\xi||Q_0(u_\xi, u)| |u_\xi z| d\xi d\eta d\eta_2
\]

\[
\lesssim (1+e_\varepsilon) (e_\varepsilon + e_\varepsilon^2)^2 E_\varepsilon. \tag{3.64}
\]
Therefore, we can obtain the estimate
\[
\bar{A}_{9} \lesssim (1 + \varepsilon_{s})^{2} e_{s} E_{s} + (1 + \varepsilon_{s})^{3}(e_{s} + e_{s}^{2} + e_{s}^{3} + e_{s}^{4}) E_{s}.
\] (3.66)

Using the similar method to estimate \( \bar{A}_{9} \), we can get
\[
\bar{A}_{10} \doteq \iint \iint \frac{[PF'u_{\xi}]_{x_{1}}u_{x_{1}}u_{x_{1}}^{2}e_{s}d\xi d\eta d\xi d\eta d\xi d\eta} \lesssim (1 + \varepsilon_{s})^{2} e_{s} E_{s} + (1 + \varepsilon_{s})^{3}(e_{s} + e_{s}^{2} + e_{s}^{3} + e_{s}^{4}) E_{s}.
\] (3.67)

We denote
\[
\bar{A}_{11} = \iint \iint (2 + \varepsilon)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} \frac{(1 - H)^{\gamma} u_{x_{1}}u_{x_{1}}^{2}e_{s}d\xi d\eta d\xi d\eta d\xi d\eta}{1 - 2(1 - H)F'u_{\eta}} J_{x_{1}}u_{x_{1}}^{2}d\xi d\eta d\xi d\eta d\xi d\eta.
\] (3.68)

Using the similar method to estimate \( \bar{A}_{11} \), we can get the estimate of \( \bar{A}_{11} \). Similarly, we can get the estimate
\[
\bar{A}_{12} = \iint \iint (2 + \varepsilon)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} u_{x_{1}}^{2}(1 - H)F'u_{\eta}Hd\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta.
\]
\[
\lesssim \iint \iint (2 + \varepsilon)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} u_{x_{1}}^{2}Hd\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta + \iint \iint (2 + \varepsilon)^{-\frac{1}{2}}(2 + \eta)^{-\frac{1}{2}} u_{x_{1}}^{2}(\Gamma H)d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta.
\]
\[
\lesssim \varepsilon + (e_{s} + \tilde{e}_{s}^{2} + e_{s}^{3}) E_{s}.
\]

It is easily to get the estimate
\[
\bar{A}_{13} = -\iint \iint P[4\Gamma^{4}(F'u_{\eta})^{2}]u_{x_{1}}^{2}d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta.
\] (3.69)
\[
\lesssim \varepsilon + (e_{s} + \tilde{e}_{s}^{2} + e_{s}^{3}) E_{s}
\]
and
\[
\bar{A}_{14} = \iint \iint P[8F'u_{\eta}^{2}F'_{\xi}(F'u_{\eta})]u_{x_{1}}^{2}d\xi d\eta d\xi d\eta d\xi d\eta d\xi d\eta.
\]
\[
\lesssim \varepsilon + (\tilde{e}_{s} + e_{s} + e_{s}^{3}) E_{s}.
\] (3.70)
3.2. Low order energy estimates

In this subsection, we will give the low order energy estimates. Firstly, we can get the following important lower order $L^\infty$ estimate.

**Proposition 3.** If we suppose that $\epsilon_s \ll \epsilon$, we can get

$$\epsilon_s \ll \epsilon.$$  \hspace{1cm} (3.71)

**Proof.** Using the foundational solution of system (3.3), we have

$$\Gamma^4 u_0 = \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(u, u) + 2F(u_1, u_1)}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt'$$

$$\Gamma^4 u_0 = \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(u, u) + 2F(u_1, u_1)}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt' + \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(F, F(u_1, u_1))}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt'$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(F, F(u_1, u_1))}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt'$$

$$\lesssim \epsilon + \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(u, u) + 2F(u_1, u_1)}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt'$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(F, F(u_1, u_1))}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] |x - x'|^2 \, dx' \, dt'.$$

By integrating in part, we have

$$\Gamma^4 u \lesssim \epsilon + \int_0^\tau \int_{\mathbb{R}^3} \partial \Gamma^4 \left[ \frac{Q_0(u, u) + 2F(u_1, u_1)}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] \, dx \, dt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \partial \Gamma^4 \left[ \frac{Q_0(F, F(u_1, u_1))}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] \, dx \, dt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \Gamma^4 \left[ \frac{Q_0(F, F(u_1, u_1))}{1 - Q_0(u, u) + 2F(u_1, u_1)} \right] \, dx \, dt.$$

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Then

\[
|\Gamma^s u| \lesssim \varepsilon + \iint \iint |\Gamma^{s'}|(1 - H)(Q_0(u, Q_0(u, u)) + 6F'Q_0(u, u) + 8F''u_0^2) + 4F^2u_0^2Hd\xi d\eta dx_2 |
\]

\[
+ \iint \iint |\Gamma^{s'}u_0|^2 (\xi - \xi')(\eta - \eta') - (x_2 - x_2')^2 d\xi d\eta dx_2 
\]

\[
\leq \varepsilon + \iint \iint |\Gamma^{s'}|(Q_0(u, Q_0(u, u)) + 6F'Q_0(u, u) + 8F''u_0^2) + 4F^2u_0^2Hd\xi d\eta dx_2 
\]

\[
+ \iint \iint (2 + \varepsilon)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}} \xi^{\frac{1}{2}} + (2 + \varepsilon)^{\frac{1}{2}}(2 + \eta)^{\frac{1}{2}} e_k d\xi d\eta dx_2 
\]

\[
\leq \varepsilon + \iint \iint (2 + \varepsilon)^{-1}(2 + \eta)^{-1} |[\nabla^s u_0]| [\nabla^s u_0]| + [\nabla^s u_0]| [\nabla^s u_0]|d\xi d\eta dx_2 
\]

\[
+ \iint \iint (2 + \varepsilon)^{-1}(2 + \eta)^{-1} (2 + \eta)^{\frac{1}{2}} |\Gamma^{s'}u_0|^2 |\Gamma^{s'}u_0|d\xi d\eta dx_2 
\]

\[
+ \iint \iint (2 + \varepsilon)^{-1} |\Gamma^{s'}u_0|^2 |\Gamma^{s'}u_0|d\xi d\eta dx_2 
\]

\[
\leq \varepsilon + \iint \iint (2 + \varepsilon)^{\frac{1}{2}}(2 + \eta)^{-1} (2 + \eta)^{\frac{1}{2}} (2 + \eta)^{\frac{1}{2}} \xi e_k d\xi d\eta + e_k + e_k 
\]

\[
\leq \varepsilon + \iint (2 + \varepsilon)^{-1} d\xi d\eta + e_k + e_k 
\]

\[
\leq \varepsilon + (2 + \varepsilon)^{\frac{1}{2}} e_k + e_k 
\]

\[
(3.72)
\]

where we use the bootstrap step in the last two steps. Then, we can get the conclusion. \(\square\)

Taking the operator \(\Gamma^{s'}\) to (3.3), we have

\[
\Box \Gamma^{s'}u + 4\Gamma^{s'}F\cdot u_0 = \Gamma^{s'}\left[\frac{1}{2}(1 - H)(Q_0(u, Q_0(u, u)) + 6F'Q_0(u, u) + 8F''u_0^2) - 4F^2u_0^2H\right],
\]

where \(\Box = \Gamma^{s'} + \sum_{\ell < l} A_{\ell \ell} \Gamma^{s'}\). Multiplying \(u_0 e^{-B(\ell)}\) into the above equation and integrating it for the variables \(x_2\) and \(\eta\), we have

\[
2\frac{d}{d\xi} \iint u_0^2 e^{-B(\ell)} dx_2 d\eta = \iint e^{-B(\ell)} u_0 \Gamma^{s'}\left[\frac{1}{2}(1 - H)(Q_0(u, Q_0(u, u)) + 6F'Q_0(u, u) + 8F''u_0^2) - 4F^2u_0^2H\right] dx_2 d\eta.
\]

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Without loss of generality, we assume $|l| \geq 7$. Noting the bound of $B(\xi)$, we can get

$$
\frac{d}{d\xi} \int U_{\eta}^\alpha dx_2 d\eta \\
\lesssim \sum_{|l| \leq 5} \int \int |u_{\eta}| |Q_0| \left((1 - H)|Q_0(u, Q_0(u, u)) + 6F^*Q_0(u, u) + 4F^* \partial \eta H| \right) dx_2 d\eta \\
\lesssim \sum_{|l| \leq 5} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int |u_{\eta}| |Q_0| (1 - H) dx_2 d\eta \\
+ \sum_{|l| \leq 5} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int |u_{\eta}| |Q_0| (1 - H) dx_2 d\eta.
$$

Then, integrating $\xi$ from the Goursat boundary to $\xi$ and noting Lemma 2.2, we obtain

$$
\int \int U_{\eta}^\alpha dx_2 d\eta \\
\lesssim \xi + (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
\lesssim \xi + (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ (1 + e_\xi) E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi \\
+ E_{\xi} \sum_{|l| + |\eta| + l + |\eta| \leq l = 14} \int \int \int (2 + \xi)^{\frac{3}{2}} (2 + \eta)^{\frac{3}{2}} |Q_0| \Gamma^4 u, Q_0(\Gamma^4 u, \Gamma^4 u)|^2 dx_2 d\eta d\xi.
$$

(73)
Using the similar procedure to the proof of Proposition 3, we have

\[
\int \int \int u_{\eta}^{2}dx_{2}d\eta \lesssim \varepsilon + (1 + e_{s})E_{s}^{+} (1 + e_{s}) \int \int \int (2 + \xi)^{-2} (2 + \eta)^{-2} |\nabla T^{\xi}u|^{2} |\nabla T^{\xi}u|^{2} dx_{2} d\eta d\xi d\eta
\]

Similarly, multiplying \( u_{\eta} e^{-iB(\xi)} \) into the above equation and integrating it for the variables \( x_{2} \) and \( \eta \), by the bound of \( B(\xi) \) we have

\[
\int \int \int u_{\eta} \Gamma^{4}(1 - H) [Q_{0}(u_{\eta} + Q_{0}(u_{\eta})) + 4 F'Q_{0}(u_{\eta}) + 4 F''u_{\eta}^{2}] - 4 F^{2}u_{\eta}^{2} dx_{2} d\eta \]

Similarly, multiplying \( u_{\eta} e^{-iB(\xi)} \) into the above equation and integrating it for the variables \( x_{2} \) and \( \eta \), by the bound of \( B(\xi) \) we have

\[
\int \int \int u_{\eta}^{2} e^{-iB(\xi)} dx_{2} d\eta \lesssim \varepsilon + (1 + e_{s})E_{s}^{+} (1 + e_{s}) \int \int \int (2 + \xi)^{-2} (2 + \eta)^{-2} |\nabla T^{\xi}u|^{2} |\nabla T^{\xi}u|^{2} dx_{2} d\eta d\xi d\eta
\]

(3.74)
Therefore, Combining all the above estimates and by the bootstrap method, we can get the energy estimates
\[ E_t \lesssim \varepsilon, \quad e_s \lesssim \varepsilon. \] (3.77)

Then, we can get the stability of the traveling wave solution \( F(x_1 + t) \) to the time-like extremal hypersurface in Minkowski space \( \mathbb{R}^{1+2+1} \).

**Remark 7.** Here we will give the difference of the proof of stability result to the general traveling wave solutions \((a + bx_2)F(x_1 + t)\). There is one more term \( Q_0(a, Fu_{x_2}) \) in (1.12) than the terms in (3.2). In the proof of stability result, the main step is to get the decay of the variables \( \xi \) and \( \eta \). We can get the following two decay estimates
\[
\begin{align*}
|Q_0(a, Fu_{x_2})| &\lesssim (2 + \xi)^{-1}(2 + \eta)^{-1}|\nabla u_{x_2}| + |uv_2|u_{x_2}|| |
\end{align*}
\]
There is at least one good derivative in the right hand side. Using the similar procedures, we can get the main stability result for the traveling wave solutions with the general form. We omit the details.

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