On the sampling error for parametric curves

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ABSTRACT
Given a parametric polynomial curve \( \gamma : [a, b] \to \mathbb{R}^n \), how can we sample a point \( x \in \text{im}(\gamma) \) in such a way that it is distributed uniformly with respect to the arc-length? Unfortunately, we cannot sample exactly such a point—even assuming we can perform exact arithmetic operations. So we end up with the following question: how does the method we choose affect the quality of the approximate sample we obtain? In practice, there are many answers. However, in theory, there are still gaps in our understanding. In this paper, we address this question from the point of view of complexity theory, providing bounds in terms of the size of the desired error.

KEYWORDS
parametric curve, sampling, sampling error, Chebyshev, approximation

1 INTRODUCTION
Given a parametric polynomial curve \( \gamma : I := [a, b] \to \mathbb{R}^n \), we are interested in sampling a point \( x \in I \) so that \( \gamma(x) \) is uniformly distributed in the curve \( \gamma(I) \) (with respect the arc-length). To do this, we only need to sample \( t \in I \) with density proportional to the speed of the curve \( \|\gamma'(t)\|_2 \). Even if we perform exact arithmetic operations and operate with real numbers with infinite precision, the problem of sampling \( t \in I \) with density proportional to \( \|\gamma'(t)\|_2 \) cannot be solved exactly. The goal of this paper is to estimate how much the generated sample differs from the one that we want and how does the desired error affects complexity.

Sampling from a density is an important and well-studied problem in computational statistics. There are many methods to sample according to a certain distribution: Acceptance-Rejection (AR) method [10], Adaptive Rejection Sampling [7], Slice Sampling [15], etc. Currently, the state-of-the-art samplers are the so-called Markov Chain Monte Carlo (MCMC) algorithms have plenty of success stories, such as Hamiltonian Monte Carlo [16], Hit-and-Run [20] and Metropolis-Hastings [3]. However, most of them apply only for log-concave distributions [2, 5, 9, 11, 12]. The algorithm in [13] can be used to sample from (multivariate) non-convex density functions, but it does not handle the case where the density is restricted to an interval (or a bounded set in general). Moreover, for the univariate setting, the existing MCMC algorithms [8] either do not provide any guarantees or could lead to arbitrary high run-times.

The problem of sampling a parametric curve is empirically studied in [17]. They propose a practical method that samples from a (probability) density in one or two variables. Their method efficiently blends inverse transform sampling and Chebyshev approximations. Inverse transform sampling is based on solving the equation \( \Phi(x) = u \) for some uniformly distributed \( u \in [0, 1] \) and the cumulative distribution function \( \Phi \) of the considered distribution. If we could solve the equation exactly, then \( x \) would have the desired distribution. However, this is not true in general, so they employ Chebyshev approximations to be able to compute \( \Phi \) fast enough and then apply the bisection method. However, incorrect solutions lead to inexact sampling, and there does not seem to be any references regarding the error incurred in the sampling as a consequence of the considered approximations. We aim in filling this gap.

Sampling parameterized curves is a topic of interest. In [18], they propose a sampling method for a certain family of parameterized curves representation [19] based on a mixed parameterization between the arc-length and the curvature and they evaluate the reconstruction performance of their method. In [4], the author performs an adaptive method that samples with respect to the local curvature: he recursively performs bisections of the parameter domain and when a parameter interval corresponds to a parametric arc that is almost flat, then he considers the two endpoints as sample points.

Our contribution. We provide the first error analysis for sampling points of a polynomial parameterized curve using an algorithm that performs exact arithmetic operations and has a source of randomness. Although our algorithm is similar to that of [17], our novelty relies on the analysis that establishes a dependence between the desired error and the number of arithmetic operations needed.

We work in the BSS model (see Sec. 2.2) for convenience since the approximation of continuous distributions by discrete ones is highly dependent on the application. This model is suitable for developing a complexity theory over the real numbers, since a BSS machine is like a Turing machine but which can operate with real numbers and perform arithmetic operations and comparisons at cost one.

The main result of this paper is the following one.

Theorem 1.1. Fix \( \gamma : I \to \mathbb{R}^n \) be a polynomial parameterized curve. The algorithm CURVESAMPLER samples points from \( \gamma \) uniformly with respect to the arc-length by performing

\[ O\left( C^2+\log C_\gamma^3x\right) \]

arithmetic operations and with error \( 2^{-\ell} \) with respect to the total variation distance \( O \) has constants depending on the degree of \( \gamma \).

The techniques we employ seems to be quite general, so they open the door to extending our analysis to other continuous random variables in the future. Moreover, this paper is the first one studying reductions of randomness sources in the BSS model of computation.
We provide efficient computations for all the parameters involved where, by convention, we take $\ell_\infty$-norms. We also intervene in a review of Chebyshev approximations. We introduce the notion of an efficient sampler. In Sec. 3 we present our curve sampler, study its efficiency and analyze its complexity. In the end, we present our experimental results in Sec. 6.

## 2 HOW GOOD IS A SAMPLING METHOD?

### 2.1 Total Variation Distance

The total variation distance measures how much the probabilities of two arbitrary events differ; the smaller the total variation distance is, the harder it is to distinguish the sampled random variable from the target random variable.

**Definition 2.1.** [23] Let $x, \tilde{x} \in \mathbb{R}$ be random variables. The total variation distance (TV distance) of $x$ and $\tilde{x}$ is defined as

$$\text{dist}_{TV}(x, \tilde{x}) := \sup_{B \in \mathcal{B}} |\mathbb{P}(x \in B) - \mathbb{P}(\tilde{x} \in B)|,$$

where $\mathcal{B}$ is the set of Borel subsets of $\mathbb{R}$.

**Remark 2.2** (Discrete vs. Continuous). If the sampled random variable $\tilde{x}$ is discrete and the target random variable $x$ is continuous, then $\text{dist}_{TV}(x, \tilde{x}) = 1$. To see this, we only have to choose $B \in \mathcal{B}$ to be the support of the discrete random variable. Hence, the total variation distance does not allow us to evaluate how a discrete sampler approximates a continuous random variable.

### 2.2 What is an efficient sampler?

For a sampler to be efficient, we want it to run in time that is polynomial in the size of the input. We use the Blum-Shub-Smale (BSS) model of computation [1] to avoid problems arising from approximating continuous random variables with discrete ones. In the BSS model, real numbers can be stored exactly as a single unit during computation, and operations with real numbers are done at unit cost. We call BSS program, a program, i.e., a finite list of commands, that can be implemented in a BSS machine.

We introduce the notion of an efficient sampler, which will be useful in measuring the performance of our sampling method.

**Definition 2.3.** Given a random variable $x \in \mathbb{R}$, an efficient sampler for $x$ is a pair of BSS programs $S : N \times \mathbb{R}^k \times \{0, 1\}^l \to \mathbb{R}$ and $P : N \to \mathbb{R}^k$ such that: $S(1)$ on input $(t, u, x)$, the run-time of $S$ is at most $\exp(\ell)$, $S(2)$ on input $t$, the run-time of $P$ is at most $\exp(\ell)$, and $S(3)$ if $u \in \{0, 1\}^l$ is uniformly distributed, then $y_t := S(t, P(t), u)$ is a random variable such that

$$\text{dist}_{TV}(x_t, x) \leq 2^{-\ell}.$$

The program $P$ in Def. 2.3 represents the preprocessing that the sampler requires in order to produce the correct result and $u$ the source of randomness. The output of $P$ is used together with the

![Figure 1 A parametric polynomial curve of degree 3 in $\mathbb{R}^3$. From blue to yellow, from smaller to higher values on the $z$-axis. With red the 10 points we sampled with our implementation of Alg. 4.](https://github.com/TolisChal/sampling_curves)
source of randomness \( u \) as the inputs of the program \( S \), which produces the sample.

Remark 2.4 (On-line and off-line computations). In a more classical conception of an efficient sampler, the program \( P \) would be missing. However, we have to take into account that samplers are not intended to be a run-once program, but they are intended to run many times. Because of this, it is reasonable to allow off-line computations—precomputations—, even if these are expensive, as long as we don’t have to repeat them. In this way, our definition gives this possibility.

Remark 2.5 (Exact arithmetic operations). The total variation distance is only valid for continuous random variables (see Rem. 2.2). This means that if we want to use the TV distance as a measure of the efficiency of our sampler, the programs of the sampler should not operate under finite precision. The BSS model allows to assume that the arithmetic operations are performed exactly.

Developing programs that operate under finite precision arises when performing computations—precomputations—, even if these are expensive, as they are run many times. Because of this, it is reasonable to allow off-line computations—precomputations—, even if these are expensive, as long as we don’t have to repeat them. In this way, our definition gives this possibility.

The following proposition shows the main technique that we apply.

Assume, without loss of generality that for all \( i \), \( \mathbb{P}(x \in J_i) \) and \( \mathbb{P}(y \in J_i) \) are positive. Otherwise, the statement still holds, but the proof is slightly more convoluted. Fix \( B \in \mathcal{B} \). By the definition of the \( x_{J_i}, \mathbb{P}(x \in B \cap J_i | x \in J_i) = \mathbb{P}(x_{J_i} \in B) \).

\[
\mathbb{P}(x \in B) = \sum_{i=1}^{k} \mathbb{P}(x_{J_i} \in B) \mathbb{P}(x \in J_i),
\]

and so, after some elementary operations, we bound \( |\mathbb{P}(x \in B) - \mathbb{P}(y \in B)| \) by

\[
\sum_{i=1}^{k} |\mathbb{P}(x_{J_i} \in B) - \mathbb{P}(y_{J_i} \in B)| \mathbb{P}(x \in J_i) + \sum_{i=1}^{k} |\mathbb{P}(x \in J_i) - \mathbb{P}(y \in J_i)|.
\]

Now, by maximizing over \( B \in \mathcal{B} \), we conclude.

The above proposition suggests the strategy of partition sampling (Algorithm PartitionSampler). In other words, to sample \( x \), we only need to sample the \( x_{J_i} \) and to compute the probabilities \( \mathbb{P}(x \in J_i) \) with enough precision for some partition \( \{J_i\}_{i=1}^{k} \) of \( I \).

Algorithm 1: PartitionSampler

**Input**: Partition \( J_1, \ldots, J_k \) of \( I \)

\( \hat{p} \in \Delta^{k} = \{ p \in \mathbb{R}^k_+ | \|p\|_1 = 1 \} \)

Approximate samplers \( S_i \) for \( x_{J_i} \)

**Output**: Approximate sample of \( x \in I \)

1. Sample \( i \in \{1, \ldots, k\} \) with probability \( \hat{p}_i \)
2. \( x \leftarrow S_i \)
3. Output \( x \)

We omit the formal proof that this procedure gives an efficient sampler if the \( S_i \) are efficient samplers. To see this, we only have to note that this method’s run-time will be at most the run-time of the
4 INVERSE TRANSFORM SAMPLING

Let $I = [a, b] \subset \mathbb{R}$ and $\varphi : I \to (0, \infty)$ be a density function. Inverse transform sampling is based on the fact that the solution $x \in I$ of \( \int_a^x \varphi(s) \, ds = u \), for $u \in [0, 1]$ uniformly distributed, has density $\varphi$ (see Alg.2 for its pseudocode).

Algorithm 2: INVERSETransformSampler

| Input | $I = [a, b], \varphi : I \to [0, \infty]$ such that $\int_a^b \varphi(t) \, dt = 1$ |
|-------|------------------------------------------------------------------|
| Output| $x \sim \varphi$ |
| 1     | Sample $u \in [0, 1]$ uniformly |
| 2     | Find the solution $x$ of $\int_a^x \varphi(s) \, ds = u$ |
| 3     | Output $x$ |

However, any reader of this pseudocode will be suspicious about all the details swept under the rug in line 2. How do we solve $\int_a^x \varphi(s) \, ds = u$? And how fast can we do it? Even though this is an important question regarding the complexity of sampling, we feel that it is unaddressed by the literature, so we discuss it.

4.1 Inverse Transform by Bisection

Let the cumulative distribution function corresponding to the density $\varphi$ be

$$
\Phi(x) := \int_a^x \varphi(s) \, ds.
$$

We want to solve the equation

$$
\Phi(x) = u \in [0, 1].
$$

When solving this equation, the bisection method outputs an interval containing the root. This interval is found by repeatedly subdividing the initial interval, and selecting the one for whom $u - \Phi$ has different signs at the two endpoints. Let $\ell \in \mathbb{N}$; we stop subdividing after $\ell$ iterations. In the end, we choose a point at random in the final interval. We integrate the bisection method in inverse transform sampling in the algorithm BisectionSampler.

The following theorem shows that BisectionSampler produces a nice sampler.

Theorem 4.1. Let $\varphi : [a, b] \to (0, \infty)$, $x \sim \varphi$ and $x_\ell$ the output of BisectionSampler for $\ell \in \mathbb{N}$. Then

$$
\text{dist}_{TV}(x, x_\ell) \leq 2^{-\ell} |b - a| \max_{x \in [a, b]} |\varphi'(x)|. 
$$

Proof. Without loss of generality, assume that $a = 0$. For $k \in \{0, \ldots, 2^\ell - 1\}$, let $J_k := [bk/2^\ell, b(k + 1)/2^\ell]$. The $x_\ell$ produced by BisectionSampler can also be produced as follows:

1. Choose the interval $J_k$ at random with probability $P(x \in J_k)$.
2. Sample $x_k \in J_k$ uniformly.

Therefore, by Proposition 3.3, to compute $\text{dist}_{TV}(x, x_\ell)$, we only have compute the TV distance between $x_k$ and the uniformly distributed $x \in J_k$. Now, after an elementary computation and Proposition 3.1,

$$
\text{dist}_{TV}(x_k, x_m) \leq \mathbb{P}(x \in J_k)^{-1}||\varphi_k - 2^\ell \mathbb{P}(x \in J_k)||_1.
$$

Now, by the mean value theorem, $\mathbb{P}(x \in J_k) = b\varphi(x)/2^\ell$ for some $x_k \in J_k$, so we obtain

$$
\text{dist}_{TV}(x_k, x_m) \leq \mathbb{P}(x \in J_k)^{-1}||\varphi_k - \varphi(x)||_1.
$$

Now, $|\varphi(x) - \varphi(x_k)| \leq b2^{-\ell} \max_{x \in J_k} |\varphi'(x)|$, by the mean value theorem, and so $\text{dist}_{TV}(x_k, x_m) \leq \mathbb{P}(x \in J_k)^{-1} b2^{-2\ell} \max_{x \in J_k} |\varphi'(x)|$, concluding the proof.

Note that BisectionSampler needs to perform a minimum of

$$
\max \left\{ 0, \sup \{|\log |\varphi'(x)|| : x \in [a, b]\} \right\}
$$

iterations. Moreover, note that in the bisection method, we can interchange precomputation and computation with no effect to our notion of the efficient sampler.

4.2 Inverse Transform by other methods

Instead of the bisection method, suppose that we use Newton’s method. We will have a sequence of approximate solvers

$$
\Psi_\ell(u) = \Psi_{\ell-1}(u) + \frac{u - \Phi(\Psi_{\ell-1}(u))}{\Phi'(\Psi_{\ell-1}(u))}
$$

for $\ell \geq 1$, where $\Psi_0$ is an appropriately chosen continuous function to guarantee that $\{\Psi_n\}$ converges to the inverse of $\Phi$.

Let’s ignore the issue of choosing $\Psi_0$ and consider that we have a sequence $\{\Psi_\ell\}$ of easily computable functions converging to the inverse of $\Phi$. Under which conditions can we expect $\{\Psi_\ell(u)\}$, with
\[ u \in [0, 1] \text{ uniformly distributed, to converge to } x \text{ in the total variation distance?} \]

Now, by Proposition 3.1, we have that
\[
\text{dist}_{TV}(\Psi(u), x) \leq \|\delta_{\Psi(u)} - \varphi\|_1,
\]
where \(\delta_{\Psi(u)}\) is the density function of \(\Psi(u)\). Assuming that the \(\Psi\) is injective, we have, by the change of variables theorem, that
\[
\delta_{\Psi(u)}(x) = \frac{\chi_{\Psi[0,1]}(x)}{|\Psi'(\Psi^{-1}(x))|}
\]
where \(\chi_{\Psi[0,1]}\) is the characteristic function of \(\Psi[0,1]\). Hence, we need that
\[
\left\| \frac{\chi_{\Psi[0,1]}(x)}{|\Psi'(\Psi^{-1}(x))|} - \varphi \right\|_1 \to 0,
\]
or equivalently, using a change of variables, that
\[
\|1 - (\Phi \circ \Psi')\|_1 = \|1 - (\varphi \circ \Psi')\|_1 \to 0.
\]
Therefore, the problem lies in the fact that even if \(u \circ \Phi\) converges uniformly to zero, this convergence does not guarantee that we have \(L^1\)-convergence of the derivative. This motivates the following question:

**Question 4.2.** Under which conditions and for which method producing a sequence of approximations \(\Psi(t)\) of the inverse of \(\Phi: [a, b] \to [0, 1]\) can we guarantee that \(\Phi \circ \Psi\) converges to the identity not only uniformly but also at the level of the derivatives?

Answering the above question lies at the center of measuring the error of using Newton’s method for inverse transform sampling.

### 5 Sampling Points from a Curve

Let \(y : I := [-1, 1] \to \mathbb{R}^n\) be the parameterization of a real polynomial curve of degree \(d\)–after a linear change of coordinates we can always assume that \(I = [-1, 1]\). Since we want to sample a point \(x \in \gamma(I)\) uniformly with respect to the arc-length, we only need to sample a random parameter \(t \in I\) distributed according to the normalized speed
\[
\varphi(t) := \frac{\|y'(t)\|_2}{\int_{-1}^{1} \|y'(s)\|_2 ds}
\]
and then take the random variable \(y(t) \in \gamma(I)\) which will have the desired distribution.

When \(n \geq 2\), we have that \(\varphi\) is not a polynomial. Because of this, to perform the inverse transform sampling, even by bisection, we will approximate \(\varphi\) by a Chebyshev approximation \(\hat{\varphi}\) for which computing
\[
\hat{\Phi}(t) := \int_{-1}^{t} \hat{\varphi}(s) ds
\]
is a lot easier than computing
\[
\Phi(t) := \int_{-1}^{t} \varphi(s) ds.
\]
Now, to make the Chebyshev approximation faster, we will split the interval into subintervals.

First, we review the Chebyshev approximation; then, we apply it to the case of interest; finally, we show how splitting accelerates the Chebyshev approximation. The algorithm appears in Algorithm 4. Again, we observe that this algorithm is very similar to the one proposed in [17], but our main contribution is not the sampler itself but the error analysis using the total variation distance.

### 5.1 Chebyshev approximations

We follow mainly [14] and [22]. Recall that the \(k\)th Chebyshev polynomial is the polynomial given by
\[
\mathcal{U}_k(x) = \sum_{i=0}^{k} \binom{k}{i} (1 - x^2)^i x^{k-2i}, \quad (5.2)
\]
where \(\mathcal{U}\) is the first initial of Chebyshev in the Cyrillic script. Alternatively, note that \(\mathcal{U}_k\) satisfies
\[
\mathcal{U}_k(x) = \cos(k \arccos(x)), \quad (5.3)
\]
for \(x \in I\). As a consequence, the \(k\) zeros of \(\mathcal{U}_k\) are given by
\[
\zeta_{a,k} := \cos \left( \frac{(1 + 2a)\pi}{2k} \right) \quad (5.4)
\]
with \(a \in \{0, \ldots, k - 1\}\).

#### 5.1.1 Chebyshev interpolation

The \(k\)th Chebyshev interpolant of \(f : I \to \mathbb{R}\) is the unique degree \(k\) polynomial \(\mathcal{U}(f)\) satisfying for \(a \in \{0, \ldots, k - 1\}\),
\[
\mathcal{U}_k(f)(\zeta_{a,k+1}) = f(\zeta_{a,k+1}). \quad (5.5)
\]
To compute \(\mathcal{U}_k(f)\), there is no need to solve the system above thanks to the following proposition.

**Proposition 5.1.** [14, Thm. 6.7] Let
\[
\mathcal{U}_k(f) = \mathcal{U}_k(f_0) + \sum_{a=1}^{k} c_a \mathcal{U}_a,
\]
be the \(k\)th Chebyshev interpolant of \(f : I \to \mathbb{R}\). Then
\[
c_a = \frac{2}{k + 1} \sum_{i=0}^{k-1} f(\zeta_{i,k}) \mathcal{U}_a(\zeta_{i,k}). \quad \square
\]

Let us remind that \(\mathcal{U}_k(f)\) is not equivalent to truncating the Chebyshev series up to degree \(k\) [22, Ch. 4].

#### 5.1.2 Evaluation of Chebyshev interpolants

Given a Chebyshev interpolant \(\mathcal{U}(f)\), we can expand it in the monomial basis and then evaluate it using, for example, Ruffini-Horner’s method. However, we have a version of Ruffini-Horner’s method that works directly for Chebyshev expansions of a polynomial.

**Proposition 5.2.** [6, pp. 55-56] Let \(p = \sum_{a=0}^{k} c_a \mathcal{U}_a\), then for every \(x \in \mathbb{R}\),
\[
p(x) = \frac{1}{2} (\Delta_0(x) - \Delta_2(x)) \quad (5.6)
\]
where \(\Delta_0(x)\) and \(\Delta_2(x)\) are computed through the following backwards-recursive relation
\[
\begin{cases}
\Delta_0(x) = 2x \Delta_1(x) - \Delta_3(x) + c_a \\
\Delta_{k+1}(x) = \Delta_{k+2}(x) - c_a
\end{cases} \quad \square
\]
5.1.3 Speed of convergence. To estimate the error of the Chebyshev approximation, we consider the so-called Bernstein ellipse given by

$$E_\rho := \{ z \in \mathbb{C} \mid |z + \sqrt{z^2 - 1}| = \rho \},$$

where $\rho > 1$. Note that $E_\rho$ is the ellipse with foci $-1$ and $1$ and focal distance $\rho + \rho^{-1}$. We have the following theorem.

**Theorem 5.3.** [22, Theorem 8.2]. If $f$ is analytic on the elliptic disc given by $E_\rho$, then

$$\| f - \Psi_{k}(f) \|_{\infty} \leq \frac{4\| f \|_{E_\rho} \rho^{-k}}{\rho - 1}, \quad (5.7)$$

where $\| f \|_{E_\rho} := \max_{z \in E_\rho} | f(z) |$.

To obtain theoretical bounds of $\| f \|_{E_\rho}$, the following inequality by Bernstein will make our job easier.

**Theorem 5.4.** [22, Exercise 8.6] Let $f$ be a polynomial of degree $d$ and $\rho > 1$. Then

$$\| f \|_{E_\rho} \leq \rho^d \| f \|_{\infty}. \quad \Box$$

Note that this inequality does not serve to approximate polynomials by a Chebyshev interpolant of a lower degree. However, we will use it to control the quantity of interest for $\varphi$, i.e., the normalized speed of $\gamma$ (5.1). The hard part will be estimating a sufficiently small $\rho$ so that $\varphi$ admits an analytic extension to the interior of $E_\rho$.

5.1.4 Integration formulas. Imagine we want to compute the integral (definite or indefinite) of $\Psi_{k}(f)$. To do this, we use the following proposition.

**Proposition 5.5.** [6, pp. 54-55][14, pp. 45-46, 59] Let $a = \frac{c_0}{2} + \sum_{a=1}^{k} c_a \Psi_{a}$.

Then

$$\rho = \sum_{a=1}^{k+1} c_a \Psi_{a}$$

with

$$pc_a = \begin{cases} \frac{c_{a+1} - c_{a}}{2a}, & \text{if } a = 1, \ldots, k - 1 \\ \frac{c_{a+1} - c_k}{2k}, & \text{if } a = k \\ \frac{c_{a+1} - c_{k+1}}{2(k+1)}, & \text{if } a = k + 1 \end{cases}$$

is a primitive function of $p$. Moreover,

$$\int_{-1}^{1} p(x) \, dx = c_0 - \sum_{a=2}^{n} \left( \frac{1 + (-1)^a}{a^2 - 1} \right) c_a. \quad \Box$$

The following fact will be useful later. Let $F(t) := \int_{-t}^{t} f(s) \, ds$ for $t \in [-1, 1]$, then

$$\| F \|_\infty \leq \| f \|_{1} \leq 2 \| f \|_{\infty}.$$ 

Hence, Theorem 5.3 allows us to also control the error of the integral approximation.

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5.2 Bernstein ellipse for the speed

The following theorem shows that the conditions of Theorem 5.3 are satisfied for the speed $\varphi$ and gives possible $\rho$s we can take.

**Theorem 5.6.** Let $\varphi$ be given as in (5.1), non-vanishing in $I$. Let $\rho > 1$ be such that

$$\rho < \rho^*(y) := \min_{z \in \mathbb{C}} \left\{ \frac{\| f \|_{E_\rho} \rho^{-k}}{\rho - 1} \right\}$$

then $\varphi$ admits an analytic extension $\varphi_{an}$ to the interior of $E_\rho$ and

$$\| \varphi_{an} \|_{E_\rho} \leq \rho^d \| \varphi \|_{\infty}.$$

**Proof.** Note that $\varphi$ is the square root of the polynomial $\| \varphi \|_{\infty}$. To analytically extend such a function to the interior of the ellipse $E_\rho$, we need that no complex root $z$ of $\| \varphi \|_{\infty}$ lies inside $E_\rho$. Now, $z$ lies inside the interior of $E_\rho$ if and only if

$$\rho \geq \frac{|z + 1| + |z - 1| + \sqrt{(|z + 1| + |z - 1|)^2 - 4}}{2}.$$

Recall that $E_\rho$ is the ellipse with foci $-1$ and $1$ and focal distance $\rho + \rho^{-1}$, so $\rho$ lies in its interior if and only if $|z + 1| + |z - 1| \leq \rho + \rho^{-1}$. The latter is equivalent to the above inequality for $\rho$.

Since $\varphi_{an}^2$ is a polynomial of degree $2d$ on $I$, it is a polynomial of degree $2d$ on the interior of $E_\rho$. Hence, by Bernstein’s inequality (Theorem 5.4),

$$\| \varphi_{an}^2 \|_{E_\rho} \leq \rho^d \| \varphi_{an}^2 \|_{\infty} \approx \rho^{2d} \| \varphi \|_{\infty}^2.$$

Since the square root of the maximum is the maximum of the square root, the desired bound follows. \Box

**Remark 5.7.** Note that Theorem 5.6 gives a conservative bound for $\| \varphi \|_{E_\rho}$, which we use for giving an upper bound for the complexity. Since we can precompute this quantity, we do this in the off-line part of CURVESAMPLER, so that we get better run times.

Observe that $\rho^*(y)$ is optimal. In this way, if we want a Chebyshev interpolant $\Psi_{k}(f)$ such that $\| \Psi_{k}(f) - \varphi \|_{\infty} \leq \varepsilon$, then the degree of this interpolant has to satisfy

$$k \geq \frac{1}{\log \rho^*(y)} \left( \frac{\ln \frac{1}{\varepsilon} + \log \| \varphi \|_{E_\rho},(y) + 2 - \log(\rho^*(y) - 1) \right). \quad (5.8)$$

by Theorem 5.3.

We now give theoretical bounds for $\rho^*$.

**Theorem 5.8.** Let $y : I \mapsto \mathbb{R}^n$ be the parameterization of a real polynomial curve of degree $d$ with non-vanishing speed and

$$C(y) := \| y' \|_{\infty} / \min_{t \in I} (\| y(t) \|_{2})$$

where $\| y' \|_{\infty} = \sum_{t=1}^{n} \sum_{j} \| y_{t,j} \|_{1}$ for $y_{t,j} = \sum_{j} y_{t,j} T_{j}$. Then

$$\rho^*(y) \geq 1 + \frac{1}{e \cdot d \cdot C(y)}.$$

**Proof.** Since $\varphi$ is non-vanishing, $C(y) < \infty$. Let $\varepsilon \in (0, 1/d)$ and consider $I_{\varepsilon} := \{ z \in C \mid \text{dist}(z, I) \leq \varepsilon \}$. Now, by [21, Proposition 3.6], for each $i$, the map

$$I_{\varepsilon} \ni z \mapsto \| y'(z) \|_{2} / \| y' \|_{\infty}$$

is $(e \cdot d)$-Lipschitz. Hence, the map

$$I_{\varepsilon} \ni z \mapsto \| y'(z) \|_{2} / \| y' \|_{\infty}$$

is \((e \cdot d)\)-Lipschitz. In this way, if \(\varepsilon = 1/(e \cdot d \cdot C(\gamma))\), we have that \(\|\gamma'(t)\|^2\) does not have zeros inside \(I_e\). Now, if for some \(\rho \geq 1, E_\rho \subseteq I_e\), then \(\rho \leq \rho^*(\gamma)\).

By the definition of \(E_\rho, E_\rho \subseteq I_e\) if and only if a) \((\rho + \rho^{-1})/2 - 1 \leq \varepsilon\) (major semiaxis bound) and b) \(\sqrt{\rho^2 + \rho^{-2}} - 2/2 \leq \varepsilon\) (minor semiaxis bound). There is such a \(\rho\) satisfying

\[ \rho = 1 + \varepsilon \]

when \(\varepsilon \leq 1\). Now, \(\varepsilon = 1/(e \cdot d \cdot C(\gamma)) \leq 1\). Hence the claim follows. \(\square\)

### 5.3 Acceleration through splitting the curve

Whenever we split the curve, we should expect the value of \(\rho^*(\gamma)\) to increase. The reason for this is that, after renormalization of a smaller interval, the zeros of \(\gamma'\) are further away and so the value of \(\rho\) should increase. However, we observe that this increase will depend on the value of \(\rho^*(\gamma)\), so there is not a uniform constant factor improvement independent of \(\rho^*(\gamma)\).

The above paragraph suggests that we can just perform a fixed number of binary subdivision steps to accelerate the algorithm. This strategy does indeed accelerate the sampler, as shown by experiments (see Figure 3).

To conclude, note that as we compute \(\rho^*(\gamma)\), we have to compute also the complex roots of \(\gamma'\). Therefore we can split the interval \(I\) along the points

\[ R_{z_1}, \ldots, R_{z_d} \]

where \(z_1, z_2, \ldots, z_d\) are the complex roots of \(\gamma'\). This subdivision accelerates the algorithm significantly as it forces the roots of \(\gamma'\) to lie on the endpoints of each interval or outside.

### 5.4 The sampler

We give the sampler for the curve \(\gamma : I \rightarrow \mathbb{R}^n\), CurveSampler, without indicating the subdivision procedure. We observe that excepting the last call, the sampler performs the majority of its operations off-line, so they don’t have to be repeated in each call.

**Algorithm 4: CurveSampler**

**Input:** \(\gamma : I \rightarrow \mathbb{R}^n\) of degree \(d\)

**Output:** Approximate sample \(t\) of \(\varphi := \|\gamma'(t)\|^2 / \int_I \|\gamma'\|^2(s)ds\)

1. \(\varphi \leftarrow \|\gamma'(t)\|^2 / \int_I \|\gamma'\|^2(s)ds\) /* off-line */
2. \(Z \leftarrow \{z \in \mathbb{C} | \gamma'(z) = 0\}\) /* off-line */
3. \(\rho^* \leftarrow \min_{z \in Z} |z| (|z| + 1) + \sqrt{(|z| + 1)^2 - 4} / 2\) /* off-line */
4. \(M \leftarrow \|\varphi\|_{E^\rho}\) /* off-line */
5. \(K \leftarrow 5 + \ell + \left\lfloor (\log M - \log(\rho^* - 1))/\log \rho^* \right\rfloor\) /* off-line */
6. \(\Phi \leftarrow \Phi_k(\varphi) / \int \Phi_k(\varphi)ds\) /* off-line */
7. \(\Phi \leftarrow \text{Integral of } \Phi_k(\varphi)\) /* off-line */
8. \(\ell_B \leftarrow 1 + \ell + \max(0, \max \|\Phi'\|_{\infty})\) /* off-line */
9. \(t \leftarrow \text{BisectionSampler}(\Phi, \ell_B)\)
10. Output \(t\)

**5.5 Complexity of CurveSampler**

Recall that we are working in the BSS model with square roots, so we assume that we can evaluate \(\|\gamma'(t)\|^2\) exactly. Our main theorem is the following one.

**Theorem 5.9.** Let \(\gamma : I \rightarrow \mathbb{R}^n\) be a polynomial parameterized curve. CurveSampler is an efficient sampler for \(t \in I\) uniformly distributed with respect to the normalized speed of \(\gamma\). Moreover, it performs

\[ O(\ell^2(1 + \log C(\gamma))^2C(\gamma)^2)\]

off-line arithmetic operations, and

\[ O(\ell^2(1 + \log C(\gamma))^3C(\gamma)^2)\]

on-line arithmetic operations to achieve an error of \(2^{-\ell}\) in the TV distance. In both cases, \(O\) has constants depending on the degree of \(\gamma\).

**Remark 5.10.** Even though, we are ignoring the complexity of the offline part—many of those parts can be done in \(\text{poly}(d)\) arithmetic operations up to the desired degree of precision. This is why we focus on the dependence on the error.

**Remark 5.11.** Note that the complexity depends on the parameter \(C(\gamma)\) introduced in Theorem 5.8. Note that \(C(\gamma) \geq 1\) and that it is finite as long as \(\varphi\) is non-vanishing in \(I\). We can consider this as a condition number for sampling \(\gamma\), although further research on it is needed.

**Proof of Theorem 5.9.** On the one hand, by Theorem 5.3,

\[ \|\varphi - \varphi\| \leq 2\|\Phi_k(\varphi) - \varphi\| \leq \frac{16M}{\rho^* - 1} \left|\rho^* - 1\right|^k \leq 2^{-k\ell} \]

(5.10)

where the first inequality follows from \(\|\varphi\| = 1\), and so if \(s \sim \varphi\), then, by Proposition 3.1,

\[ \text{dist}_{TV}(s, t) \leq 2^{-\ell} \]

(5.11)

where \(t \sim \varphi\). On the other hand, by Theorem 4.1,

\[ \text{dist}_{TV}(t, I) \leq 2^{1-t} \leq 2^{-1+t} \]

Hence,

\[ \text{dist}_{TV}(s, t) \leq 2^{-\ell} \]

and so to show that CurveSampler is an efficient sampler for \(\varphi\), we only need to bound the complexity as desired.

For the off-line part, we need to bound the number of arithmetic operations in terms of \(\ell\). For line 2, we use some solver; for line 3, we just minimize over the roots found in line 2; for line 4, we only have to parameterize the boundary of \(E_\rho\) and find the minimum, due to the maximum modulus principle; for line 5, we do the assignment; for line 6, we use Proposition 5.1 and the second part of Proposition 5.5; for line 7, we use the first part of Proposition 5.5. By observing these, we see that the number of arithmetic operations is at most \(O(k^2)\), which by the definition of \(k\) in line 5 and Theorem 5.6 transforms to

\[ O(\ell^2(1 + \log C(\gamma))^2C(\gamma)^2)\]

where we use that \(1/\log(1 + x) \leq 2/x\).

For the on-line part, we perform \(\ell_B\) evaluations of \(\Phi\), taking each evaluation, by Proposition 5.2, \(O(k)\) operation. Thus we perform \(O(\ell_kk)\) arithmetic operation. We have to bound \(\ell_B\) now. However, this is equivalent to bounding

\[ \|\varphi'\|_{\infty}. \]
Using the theory of Chebyshev polynomials [22], we have that
\[ \| \mathcal{U}_k(\phi) - \phi \|_\infty \leq 16Mk^2(\rho^*)^{3-k}/((\rho^*) - 1). \]

Thus,
\[ \| \phi' \|_\infty \leq \| \phi'' \|_\infty + \frac{16Mk^2\rho^{3-k}}{\rho - 1} \leq \| \phi'' \|_\infty + 4k^2(\rho^*)^{2-\varepsilon}. \]

Now, on the one hand,
\[ 4k^2(\rho^*)^{2-\varepsilon} \leq O(t^2(1 + \log C(y))C(y)); \]
and on the other hand,
\[ |\phi'\prime | = \langle y \prime, y \prime \prime \rangle \leq \| y \prime \|_2 \| y \prime \prime \|_2 \]
where \( \| y \prime \|_2 \leq d\| y \prime \|_0 \) by [21, Proposition 3.6]. Thus
\[ \| \phi' \|_\infty \leq C(y). \quad (5.12) \]

Putting this together, we obtain the bound \( t_B \leq O(t + \log C(y) + O(k^2)/C(y)) = O(f + \log C(y) + f(1 + \log C(y))C(y)). \) Hence we are done. \( \square \)

6 IMPLEMENTATION AND EXPERIMENTS

We provide an open-source software in Matlab\textsuperscript{\textregistered} to sample uniformly from a given parametric polynomial curve. It is an original implementation of CurveSampler\footnote{https://github.com/TolisChal/sampling_curves} using BisectionSampler for the computation of the roots in Inverse Transform sampling. Our implementation also uses a few standard routines from Matlab\textsuperscript{\textregistered}’s toolbox. In particular, we use the functions (i) chebyshevT() to evaluate the \( k \)-th degree Chebyshev polynomial \( \mathcal{U}_k(x) \), (ii) roots() to compute the zero set of the speed \( \| y' \prime \|_2 \) for the computation of \( \rho^* \) in Theorem 5.3, and (iii) fmincon() to solve the optimization problem required to compute \( \| f \|_{E, \rho} \) again in Theorem 5.3. All computations were performed on a PC with Intel\textsuperscript{\textregistered} Pentium(R) CPU G4400 \( @ \) 3.30GHz \( \times \) 2 CPU and 16GB RAM.

To set \( k \), the order of Chebyshev approximation in an interval \([-1, 1]\), we employ Prop. 5.3. We compute both \( \rho^* \) and \( \| f \|_{E, \rho} \). Then, we set the degree of Chebyshev approximation equal to \( k \) according to Eq. 5.8, for several small values of the Chebyshev approximation error \( \varepsilon \).

We empirically capture the dependency between the error \( \varepsilon \) and the degree \( k \). The plot in Fig. 2 illustrates \( k \) as \( 1/\varepsilon \) increases for a fixed curve of degree \( d = 10 \) in \( \mathbb{R}^{50} \). The plot confirms that \( k \) increases proportionally to \( \log(1/\varepsilon) \). We further empirically examine how the splitting of the interval \([a, b]\) could improve the efficiency of our implementation. In Fig. 3 we sample from the same curve as in Fig. 2. We additionally, split \([a, b]\) to 4 equilength subintervals, and we report with red dots the maximum Chebyshev degree \( k_{\text{max}} \) among all subintervals. We notice that the splitting accelerates our implementation, while the average ratio of Chebyshev degrees is 2.5.

We sample from parametric polynomial curves up to 100 dimensions and degree 20 to further illustrate the efficiency of our algorithm. Table 1 illustrates the degree of Chebyshev approximation \( k \) (without splitting), the run-time of the preprocessing, and the time per generated sample after preprocessing. The degree \( k \) increases with the degree of \( d \) of the polynomial curve, however, it grows sublinear clearly. The run-time of the preprocessing takes a few seconds, while the time per sample after preprocessing is smaller than 1 second for every instance.
On the sampling error for parametric curves

| $d$ | $n$ | $k$ | $\Pr. T$ | $T$/s | $\epsilon = 0.1$ | $T$/s | $\epsilon = 0.01$ |
|-----|-----|-----|----------|-------|----------------|-------|----------------|
| 20  | 20  | 0.82| 0.53     | 0.47  | 0.50           | 0.47  |
| 40  | 32  | 1.08| 0.44     | 0.56  | 0.53           | 0.53  |
| 50  | 60  | 0.92| 0.44     | 0.59  | 0.59           | 0.59  |
| 80  | 29  | 1.04| 0.43     | 0.63  | 0.63           | 0.63  |
| 100 | 25  | 1.24| 0.54     | 0.82  | 0.82           | 0.82  |
| 20  | 34  | 1.34| 0.57     | 0.66  | 0.66           | 0.66  |
| 40  | 21  | 0.76| 0.40     | 0.59  | 0.59           | 0.59  |
| 10  | 60  | 1.17| 0.44     | 0.63  | 0.63           | 0.63  |
| 80  | 38  | 1.31| 0.53     | 0.66  | 0.66           | 0.66  |
| 100 | 37  | 1.64| 0.64     | 0.69  | 0.69           | 0.69  |
| 20  | 35  | 1.29| 0.59     | 1.00  | 1.00           | 1.00  |
| 40  | 46  | 1.53| 0.58     | 1.00  | 1.00           | 1.00  |
| 15  | 60  | 3.40| 0.49     | 1.00  | 1.00           | 1.00  |
| 80  | 43  | 1.65| 0.49     | 1.00  | 1.00           | 1.00  |
| 100 | 49  | 2.36| 0.63     | 1.00  | 1.00           | 1.00  |
| 20  | 67  | 2.82| 0.70     | 0.66  | 0.66           | 0.66  |
| 40  | 63  | 2.07| 0.53     | 0.69  | 0.69           | 0.69  |
| 20  | 60  | 2.38| 0.66     | 0.76  | 0.76           | 0.76  |
| 80  | 70  | 2.60| 0.59     | 0.82  | 0.82           | 0.82  |
| 100 | 56  | 2.55| 0.62     | 0.82  | 0.82           | 0.82  |

Table 1 Sampling from random parametric polynomial curves with our implementation of Alg. 4. All polynomials’ coefficients are generated from the standard Gaussian $N(0, 1)$, $\delta$ denotes the degree of each univariate polynomial; $n$ is the dimension; $\epsilon$ is the approximation error in Prp. 5.3; $\kappa$ is the degree of Chebyshev approximation; $\Pr. T$ is the pre-process time in seconds; $T$/s is the time in seconds per generated sample after preprocessing.

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