Dirac Structures, Moment Maps
and Quasi–Poisson Manifolds

Henrique Bursztyn
Marius Crainic

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Dirac structures, moment maps and quasi-Poisson manifolds

Henrique Bursztyn*
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3, Canada

Marius Crainic†
Department of Mathematics
Utrecht University, P.O. Box 80.010, 3508 TA
Utrecht, The Netherlands

Abstract

We extend the correspondence between Poisson maps and actions of symplectic groupoids, which generalizes the one between momentum maps and hamiltonian actions, to the realm of Dirac geometry. As an example, we show how hamiltonian quasi-Poisson manifolds fit into this framework by constructing an “inversion” procedure relating quasi-Poisson bivectors to twisted Dirac structures.

Dedicated to Alan Weinstein for his 60th birthday

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*henrique@math.toronto.edu
†crainic@math.uu.nl
1 Introduction

This paper builds on three ideas pursued by Alan Weinstein in some of his many fundamental contributions to Poisson geometry: First, Lie algebroids play a prominent role in the study of Poisson manifolds [8, 30]; second, Poisson maps can be regarded as generalized momentum maps for actions of symplectic groupoids [25, 31]; third, Poisson structures on manifolds are particular examples of more general objects, called Dirac structures [12, 13, 28]. The main objective of this paper is to combine these three ideas in order to extend the notion of “moment map” to the realm of Dirac geometry. As an application, we obtain an alternative approach to hamiltonian quasi-Poisson manifolds [2] which answers many of the questions posed in [28, 31], shedding light on the relationship between various notions of generalized Poisson structures, hamiltonian actions and reduced spaces.

Let \( g \) be a Lie algebra, and consider its dual \( g^* \), equipped with its Lie-Poisson structure. The central ingredients in the formulation of classical hamiltonian \( g \)-actions are a Poisson manifold \((Q, \pi_Q)\) and a Poisson map \( J : Q \to g^* \), which we use to define an action of \( g \) on \( Q \) by hamiltonian vector fields:

\[
J_v \mapsto \mathcal{X}(Q), \quad v \mapsto X_{J_v} := \iota_{dJ_v}(\pi_Q),
\]

where \( J_v \in C^\infty(Q) \) is given by \( J_v(x) = \langle J(x), v \rangle \). For the global picture, we assume that \( J \) is a complete Poisson map [8, Sec. 6.2], in which case the infinitesimal action (1.1) can be integrated to an action of the connected, simply connected Lie group \( G \) with Lie algebra \( g \), in such a way that \( J \) becomes \( G \)-equivariant with respect to the coadjoint action of \( G \) on \( g^* \). The map \( J \) is called a momentum map for the \( G \)-action on \( Q \), and we refer to the \( G \)-action as hamiltonian.

A key observation, described in [25, 31], is that this construction of a hamiltonian action out of a Poisson map holds in much more generality: one may replace \( g^* \) by any Poisson manifold, as long as Lie groups are replaced by symplectic groupoids [29]. In this sense, any Poisson map can be seen as a “Poisson-manifold valued moment map”.

In this paper, we show that the correspondence between Poisson maps and hamiltonian actions by symplectic groupoids can be further extended to the context of Dirac geometry: in this setting, Poisson maps must be replaced by special types of Dirac maps, called Dirac realizations (see Def. 3.11); for the associated global actions, twisted presymplectic groupoids [6] (alternatively called quasi-symplectic groupoids [33]) play the role of symplectic groupoids. Our main results show that various important notions of generalized hamiltonian actions, such as the “quasi” objects of [2, 3], fit nicely into the Dirac geometry framework.

We organize our results as follows:

In Section 2, we discuss important connections between Lie algebroids and bivector fields. Our main result is that, just as ordinary Poisson structures give rise to Lie algebroid structures on their cotangent bundles, a quasi-Poisson manifold [2, Def. 2.1] \((M, \pi)\) defines a Lie algebroid structure on \( T^*M \oplus g \), where \( g \) is the Lie algebra of the Lie group acting on \( M \). The leaves of this Lie algebroid coincide with the leaves of the “quasi-hamiltonian foliation” of [2, Sec. 9] in

4 Moment maps in Dirac geometry: the global picture

4.1 Integrating Lie algebroids and infinitesimal actions

4.2 Poisson maps as moment maps for symplectic groupoid actions

4.3 Dirac realizations as moment maps for presymplectic groupoid actions

4.4 Reduction in Dirac geometry

4.5 AMM-groupoids and hamiltonian quasi-Poisson \( G \)-manifolds
the hamiltonian case, though, in our framework, we make no assumption about the existence of group-valued moment maps.

In Section 3, we study hamiltonian actions in the context of Dirac geometry at the infinitesimal level. We observe that, just as Poisson maps, Dirac realizations are always associated with Lie algebroid actions. (This is, in fact, the guiding principle in our definition of Dirac realizations.) After discussing how classical notions of infinitesimal hamiltonian actions fit into this framework, we prove the main result of the section: Dirac realizations of Cartan-Dirac structures on Lie groups \([6, 28]\) are equivalent to quasi-Poisson \(g\)-manifolds carrying group-valued moment maps. This equivalence involves an “inversion” procedure relating twisted Dirac structures and quasi-Poisson bivectors, revealing that these two objects are in a certain sense “mirror” to one another. The main ingredients in this discussion are the Lie algebroids of Section 2 and the bundle maps which appear in \([6]\) as infinitesimal versions of multiplicative 2-forms. This result explains, in particular, the relationship between Cartan-Dirac and quasi-Poisson structures on Lie groups; on the other hand, it recovers the correspondence proven in \([2, \text{Thm. 10.3}]\) between “non-degenerate” hamiltonian quasi-Poisson manifolds (i.e., those for which the Lie algebroids of Section 2 are transitive) and quasi-hamiltonian spaces \([3]\).

In Section 4, we study moment maps in Dirac geometry from a global point of view. We show that complete Dirac realizations “integrate” to presymplectic groupoid actions, which are natural extensions of those studied in \([33]\). As our main example, we show that the “integration” of Dirac realizations of Cartan-Dirac structures on Lie groups results in hamiltonian quasi-Poisson \(G\)-manifolds. Finally, we show that the natural reduction procedure in the setting of Dirac geometry encompasses various classical reduction theorems \([21, 24, 25]\) as well as their “quasi” counterparts \([2, 3, 33]\).

We remark, following an observation of E. Meinrenken, that the results concerning quasi-Poisson manifolds in this paper only require the Lie algebras to be quadratic, in contrast with some of the constructions in \([2]\), in which the positivity of the bilinear forms plays a key role. (In particular, our results hold for quasi-Poisson \(G\)-manifolds when \(G\) is a noncompact semisimple Lie group.) Most of our constructions can be carried out in the more general setting of \([1]\), but this will be discussed in a separate paper.

A work which gave initial motivation and is closely related to the present paper is that of Xu \([33]\), in which a Morita theory of quasi-symplectic groupoids is developed in order to compare “moment map theories”. Our results show that twisted Dirac structures complement Xu’s picture in two ways: on one hand, by providing the infinitesimal framework for Morita equivalence; on the other hand, by leading to more general “modules” (i.e., hamiltonian spaces).

It is a pleasure to dedicate this paper to Alan Weinstein, whose work and insightful ideas have been an unlimited source of inspiration to us.

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Notation: We use the following conventions for bundle maps: if \(\pi\) is a bivector field on \(M\),
then $\pi^b : T^*M \to TM$, $\alpha \mapsto \pi(\alpha, \cdot)$; if $\omega$ is a 2-form, then $\omega^b : TM \to T^*M$, $X \mapsto \omega(X, \cdot)$.

The space of $\kappa$-multivector fields on $M$ is denoted by $\mathcal{X}^k(M)$.

On a Lie group $G$, with Lie algebra $\mathfrak{g}$, $(\cdot, \cdot)_G$ will denote a bi-invariant nondegenerate quadratic form; we write $\phi^G$ for the associated Cartan 3-form, and $\chi_G \in \Lambda^3 \mathfrak{g}$ for the dual trivector.

\section{Lie algebroids, bivector fields and Poisson geometry}

\subsection{Lie algebroids}

A Lie algebroid over a manifold $M$ is a vector bundle $A \to M$ together with a Lie algebra bracket $[\cdot, \cdot]$ on the space of sections $\Gamma(A)$, and a bundle map $\rho : A \to TM$, called the anchor, satisfying the Leibniz identity (2.1). Here, as usual, $[\cdot, \cdot] \in \mathfrak{g}$

\begin{equation}
[\xi, f\xi'] = f[\xi, \xi'] + \mathcal{L}_{\rho(\xi)}(f)\xi', \quad \text{for } \xi, \xi' \in \Gamma(A), \text{ and } f \in C^\infty(M). \tag{2.1}
\end{equation}

Whenever there is no risk of confusion, we will write $\mathcal{L}_{\rho(\xi)}$ simply as $\mathcal{L}_{\xi}$.

If $M$ is a point, then a Lie algebroid over $M$ is a Lie algebra in the usual sense. An important feature of Lie algebroids $A \to M$ is that the image of the anchor, $\rho(A) \subseteq TM$, defines a generalized integrable distribution, determining a singular foliation of $M$. The leaves of this foliation are the orbits of the Lie algebroid. The following example plays a key role in the study of hamiltonian actions and moment maps.

\textbf{Example 2.1} (Transformation Lie algebroids)

Consider an infinitesimal action of a Lie algebra $\mathfrak{g}$ on a manifold $M$, given by a Lie algebra homomorphism $\bar{\rho} : \mathfrak{g} \to \mathcal{X}(M)$. The transformation Lie algebroid associated with this action is the trivial vector bundle $M \times \mathfrak{g}$, with anchor $(x, v) \mapsto \rho(x, v) := \bar{\rho}(v)(x)$ and Lie bracket on $\Gamma(M \times \mathfrak{g}) = C^\infty(M, \mathfrak{g})$ defined by

\begin{equation}
[u, v](x) := [u(x), v(x)]_{\mathfrak{g}} + (\bar{\rho}(u(x)) \cdot v)(x) - (\bar{\rho}(v(x)) \cdot u)(x). \tag{2.2}
\end{equation}

We often denote a transformation Lie algebroid by $\mathfrak{g} \ltimes M$.

Note that $[\cdot, \cdot]$ is uniquely determined by the condition that it coincides with $[\cdot, \cdot]_{\mathfrak{g}}$ on constant functions and the Leibniz identity. The orbits of $\mathfrak{g} \ltimes M$ are the $\mathfrak{g}$-orbits on $M$.

The remaining of this section is devoted to examples of Lie algebroids closely related to Poisson manifolds.

\subsection{Bivector fields and Poisson structures}

If $(M, \pi)$ is a Poisson manifold, then $T^*M$ carries a Lie algebroid structure with anchor

\begin{equation}
\pi^b : T^*M \to TM, \quad \beta(\pi^b(\alpha)) = \pi(\alpha, \beta), \tag{2.3}
\end{equation}

and bracket

\begin{equation}
[\alpha, \beta] = \mathcal{L}_{\pi^b(\alpha)}(\beta) - \mathcal{L}_{\pi^b(\beta)}(\alpha) - d\pi(\alpha, \beta), \tag{2.4}
\end{equation}

uniquely characterized by $[df, dg] = d\{f, g\}$ and the Leibniz identity (2.1). Here, as usual, $\{f, g\} = \pi(df, dg)$ is the Poisson bracket on $C^\infty(M)$. In this case, the orbits of $T^*M$ are the symplectic leaves of $M$, i.e., the integral manifolds of the distribution defined by the hamiltonian vector fields $X_f = \pi^b(df)$.
**Example 2.2 (Lie-Poisson structures)**

Let \((\mathfrak{g}, [, ,])\) be a Lie algebra, and consider \(\mathfrak{g}^*\) equipped with the associated Lie-Poisson structure

\[
\{f, g\}(\mu) := \langle \mu, [df(\mu), dg(\mu)] \rangle, \quad \mu \in \mathfrak{g}^*. \tag{2.5}
\]

Under the identification \(T^*\mathfrak{g}^* \cong \mathfrak{g}^* \times \mathfrak{g}\), one can see that the Lie algebroid structure on \(T^*\mathfrak{g}^*\) induced by (2.5) is that of a transformation Lie algebroid \(\mathfrak{g}^* \ltimes \mathfrak{g}^*\), see Example 2.1, and a direct computation reveals that the action in question is the coadjoint action.

If \(\pi \in \mathcal{X}^2(M)\) is an arbitrary bivector field, let us consider \(\pi^\sharp, [, ,], \{ , , \}\) and \(X_f\) as defined by the previous formulas, and let \(\chi_\pi \in \mathcal{X}^3(M)\) be the trivector field defined by

\[
\chi_\pi := [\pi, \pi], \tag{2.6}
\]

i.e., \(\chi_\pi\) satisfies

\[
\frac{1}{2}\chi_\pi(df, dg, dh) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \{\{f, g\}, h\} + \text{c.p.},
\]

where we use c.p. to denote cyclic permutations.

**Lemma 2.3** For any bivector field \(\pi\) on \(M\), one has

\[
\pi^\sharp([\alpha, \beta]) = [\pi^\sharp(\alpha), \pi^\sharp(\beta)] + \frac{1}{2}\alpha \wedge \beta(\chi_\pi), \tag{2.7}
\]

\[
[[\alpha, \beta], \gamma] + \text{c.p.} = \frac{1}{2}(\mathcal{L}_{\alpha \wedge \beta}(\chi_\pi)(\gamma) + \text{c.p.}) - d(\chi_\pi(\alpha, \beta, \gamma)), \tag{2.8}
\]

for \(\alpha, \beta, \gamma \in \Omega^1(M)\). As a result, the following are equivalent:

(i) \(\pi\) is a Poisson tensor;

(ii) \(\pi^\sharp : \Omega^1(M) \to \mathcal{X}(M)\) preserves the brackets;

(iii) the bracket \([, ,]\) on \(\Omega^1(M)\) satisfies the Jacobi identity;

(iv) \((T^*M, \pi^\sharp, [, ,])\) is a Lie algebroid.

**Proof:** The key remark is that the difference between the left and right hand sides of each of (2.7) and (2.8) is \(C^\infty(M)\)-multilinear in \(\alpha, \beta\) and \(\gamma\). So it is enough to prove the identities on exact forms, which is immediate. \(\square\)

**Example 2.4 (Twisted Poisson manifolds)**

Consider a closed 3-form \(\phi \in \Omega^3(M)\). A \(\phi\)-twisted Poisson structure on \(M\) [19, 27] consists of a bivector field \(\pi \in \mathcal{X}^2(M)\) satisfying

\[
\frac{1}{2}[\pi, \pi] = \pi^\sharp(\phi).
\]

Here, we abuse notation and write \(\pi^\sharp\) to denote the map induced by (2.3) on exterior algebras. We know from Lemma 2.3 that the bracket (2.4) induced by \(\pi\) is not preserved by \(\pi^\sharp\) and does not satisfy the Jacobi identity. However,

\[
\pi^\sharp([\alpha, \beta] + i_{\pi^\sharp(\alpha)}\wedge \pi^\sharp(\beta))(\phi)) = [\pi^\sharp(\alpha), \pi^\sharp(\beta)].
\]
Hence, if we define a “twisted” version of the bracket (2.4),

$$\left[\alpha, \beta\right]_\phi := [\alpha, \beta] + i_{\pi^*\alpha} \wedge i_{\pi^*\beta}(\phi),$$

then \(\pi^*\) will preserve this new bracket, and \(\left[\cdot, \cdot\right]_\phi\) satisfies the Jacobi identity. As a result, \((T^*M, \pi^*, \left[\cdot, \cdot\right]_\phi)\) is a Lie algebroid. We leave it to the reader to prove a “twisted” version of Lemma 2.3.

2.3 The Lie algebroid of a quasi-Poisson manifold

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\), equipped with a bi-invariant nondegenerate quadratic form \((\cdot, \cdot)_\mathfrak{g}\). Let \(\phi^G\) be the bi-invariant Cartan 3-form on \(G\), and let \(\chi_G \in \Lambda^3\mathfrak{g}\) be its dual trivector. On Lie algebra elements \(u, v, w \in \mathfrak{g}\), we have

$$\phi^G(u, v, w) = \chi_G(u^\vee, v^\vee, w^\vee) = \frac{1}{2}(u, [v, w])_\mathfrak{g},$$

where \(u^\vee, v^\vee, w^\vee\) are dual to \(u, v, w\) via \((\cdot, \cdot)_\mathfrak{g}\); when \((\cdot, \cdot)_\mathfrak{g}\) is a metric and \(e_a\) is an orthonormal basis of \(\mathfrak{g}\), we can write\(^1\)

$$\chi_G = \frac{1}{12} \sum (e_a, [e_b, e_c])_\mathfrak{g} e_a \wedge e_b \wedge e_c.$$

A quasi-Poisson \(G\)-manifold [2] consists of a \(G\)-manifold \(M\) together with a \(G\)-invariant bivector field \(\pi\) satisfying

$$\chi_\pi = \rho_M(\chi_G), \tag{2.9}$$

where \(\rho_M : \mathfrak{g} \longrightarrow \mathcal{X}(M)\) is the associated infinitesimal action, and we keep the same notation for the induced maps of exterior algebras. When \(M\) is just a \(\mathfrak{g}\)-manifold, we call the corresponding object a quasi-Poisson \(\mathfrak{g}\)-manifold. The two notions are related by the standard procedure of integration of infinitesimal actions; in particular, they coincide if \(M\) is compact and \(G\) is simply connected.

In analogy with ordinary or twisted Poisson manifolds, are quasi-Poisson structures also associated with Lie algebroids? As we now discuss, the answer is yes. Let us consider a more general set-up: let \(M\) be a \(\mathfrak{g}\)-manifold and let \(\pi \in \mathcal{X}^2(M)\) be an arbitrary bivector field. Motivated by [2, Sec. 9], we consider on \(T^*M \oplus \mathfrak{g}\) the “anchor” map

$$r : T^*M \oplus \mathfrak{g} \longrightarrow TM, \quad r(\alpha, v) = \pi^* \alpha + \rho_M(v), \tag{2.10}$$

combining the bivector field and the action. On sections of \(T^*M \oplus \mathfrak{g}\), we consider the bracket defined by

$$[(\alpha, 0), (\beta, 0)] = ([\alpha, \beta], - \frac{1}{2} \rho_M^*(\alpha \wedge \beta)(\chi_G)), \tag{2.11}$$

$$[(0, v), (0, v')] = (0, [v, v']_\mathfrak{g}), \tag{2.12}$$

$$[(0, v), (\alpha, 0)] = (\mathcal{L}_{\rho_M(v)}(\alpha), 0), \tag{2.13}$$

for all 1-forms \(\alpha, \beta \in \Omega^1(M)\) and all \(v, v' \in \mathfrak{g}\) (thought of as constant sections in \(\mathcal{C}^\infty(M, \mathfrak{g})\)). As in Example 2.1, the definition of the bracket on general elements in \(\Gamma(T^*M \oplus \mathfrak{g}) = \Omega^1(M) \oplus \mathcal{C}^\infty(M, \mathfrak{g})\) is obtained from the Leibniz formula (2.1). With these definitions, we obtain a quasi-Poisson analogue of Lemma (2.3):

\(^1\)More generally, with no positivity assumptions on \((\cdot, \cdot)_\mathfrak{g}\), we can write \(\chi_G = \frac{1}{12} \sum (e_a, [e_b, e_c])_\mathfrak{g} f_a \wedge f_b \wedge f_c\), where \(f_a\) is a basis of \(\mathfrak{g}\) satisfying \((f_a, e_b)_\mathfrak{g} = \delta_{ab}\). A similar observation holds for (2.25).
Theorem 2.5 Let $M$ be a $\mathfrak{g}$-manifold equipped with a bivector field $\pi$. The following are equivalent:

(i) $(M, \pi)$ is a quasi-Poisson $\mathfrak{g}$-manifold;

(ii) $r : \Omega^1(M) \oplus C^\infty(M, \mathfrak{g}) \to \mathcal{X}(M)$ preserves brackets;

(iii) the bracket $[\cdot, \cdot]$ on $\Omega^1(M) \oplus C^\infty(M, \mathfrak{g})$ satisfies the Jacobi identity;

(iv) $(T^*M \oplus \mathfrak{g}, r, [\cdot, \cdot])$ is a Lie algebroid.

Proof: Note that $r$ preserves the bracket (2.12), since $\rho_M$ is an action. From the identity (2.7) in Lemma 2.3, it follows that $r$ preserves the bracket of type (2.11) if and only if $\chi_\pi = \rho_M(\chi_G)$. On the other hand, $r$ preserves the bracket of type (2.13) if and only if $\pi^2 \mathcal{L}_{\rho_M(v)}(\xi) = \mathcal{L}_{\rho_M(v)}\pi^2(\xi)$, which is equivalent to the $\mathfrak{g}$-invariance of $\pi$. This shows that (i) and (ii) are equivalent.

Let us prove that (i) implies (iii); from the proof, the converse will be clear. Assuming (i), we must show that $[\cdot, \cdot]$ on $\Omega^1(M) \oplus C^\infty(M, \mathfrak{g})$ satisfies the Jacobi identity. On elements of type $(0, v)$, this reduces to the Jacobi identity for $\mathfrak{g}$ (or, alternatively, for $\mathfrak{g} \ltimes M$). On elements $(0, v)$, $(0, w)$ and $(\alpha, 0)$, the Jacobi identity of $[\cdot, \cdot]$ reduces to the fact that $\rho_M$ is an action. Computing the "jacobiantor" for elements of type $(0, v), (\alpha, 0), (\beta, 0)$, we see that the first component is

$$[\mathcal{L}_{\rho_M(v)}(\alpha), \beta] + [\alpha, \mathcal{L}_{\rho_M(v)}(\beta)] - \mathcal{L}_{\rho_M(v)}([\alpha, \beta]).$$

Using the Leibniz identity, we see that the $C^\infty(M)$-linearity of (2.14) with respect to $\beta$ is equivalent to $\pi^2 \mathcal{L}_{\rho_M(v)}(\beta) = \mathcal{L}_{\rho_M(v)}\pi^2(\beta)$, i.e., to the $\mathfrak{g}$-invariance of $\pi$. Hence, if $\pi$ is invariant, (2.14) is $C^\infty(M)$-linear on $\alpha$ and $\beta$, and then one can check that it is zero by looking at the particular case when $\alpha$ and $\beta$ are exact. The second component of the jacobiantor of $(0, v), (\alpha, 0), (\beta, 0)$ can be computed similarly.

To complete the proof that (i) implies (iii), we must deal with the Jacobi identity for elements of type $(\alpha, 0), (\beta, 0), (\gamma, 0)$. To this end, we first need to find the expression for the bracket between elements of type $(0, \bar{v})$ and $(\alpha, 0)$, with $\bar{v} \in C^\infty(M, \mathfrak{g})$ not necessarily constant: pairing $d\bar{v} \in \Omega^1(M; \mathfrak{g})$ with an element $\mu \in C^\infty(M, \mathfrak{g}^*)$ gives us a 1-form on $M$, denoted by $A_{\bar{v}}(\mu)$, satisfying the following two properties:

$$A_{f\bar{v}}(\mu) = fA_{\bar{v}}(\mu) + \mu(\bar{v})df, \quad A_{\bar{v}}(f\mu) = fA_{\bar{v}}(\mu),$$

for $f \in C^\infty(M)$. We claim that

$$[(0, \bar{v}), (\alpha, 0)] = (\mathcal{L}_{\rho_M(\bar{v})}(\alpha) - A_{\bar{v}}(\rho_M(\alpha)), -\mathcal{L}_{\pi^1(\alpha)}(\bar{v})).$$

To see that, note that (2.15) holds when $\bar{v}$ is constant, and the difference between the left and right hand sides is $C^\infty(M)$-linear in $\bar{v}$. We remark that

$$A_{\mu \wedge \mu'}(\chi_G)(\mu'') + c.p. = 2d(\chi_G(\mu, \mu', \mu'')).$$

Again, it is easy to check this identity when $\mu$, $\mu'$ and $\mu''$ are constant, so (2.16) follows from $C^\infty(M)$-linearity. Also, denoting $\chi_M := \rho_M(\chi_G)$, a direct computation shows that

$$\rho_M(i_{\rho_M(\alpha \wedge \beta)}(\chi_G)) = i_{\alpha \wedge \beta}(\chi_M).$$
We now turn to the computation of the jacobian of the elements \((\alpha, 0), (\beta, 0)\) and \((\gamma, 0)\), that we denote by \(\text{Jac}(\alpha, \beta, \gamma)\). For the first component of \(\text{Jac}(\alpha, \beta, \gamma)\), we obtain

\[
([\alpha, \beta], \gamma) + \text{c.p.} - \frac{1}{2}(\mathcal{L}_{\alpha} \chi_M(\chi_M)(\gamma) + \text{c.p.}) + \frac{1}{2}A_{\rho_M^*(\alpha) \rho_M^*(\beta)}(\chi_M(\rho_M^*(\gamma))) + \text{c.p.}.
\] (2.17)

Combining the second identity of Lemma 2.3 with (2.16), we get that (2.17) equals

\[
d((\rho_M(\chi_G) - \chi_\pi)(\alpha, \beta, \gamma)),
\]

which vanishes by the condition \(\chi_\pi = \rho_M(\chi_G)\). So we are left with proving that the second component of \(\text{Jac}(\alpha, \beta, \gamma)\) vanishes, which amounts to showing that

\[
i_{\rho_M^*(\alpha, \beta)}(\chi_G) + \text{c.p.} = \mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)}(\chi_G) + \text{c.p.}.
\] (2.18)

In order to do that, consider the operators \(i_{\rho_M^*(\alpha, \beta)}\) and \(\mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)} - \mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)}\) acting on \(C^\infty(M, \Lambda g)\), for \(\alpha, \beta \in \Omega^1(M)\).

**Claim 2.6** On \(\Lambda g\), seen as constant functions in \(C^\infty(M, \Lambda g)\), we have

\[
i_{\rho_M^*(\alpha, \beta)} = \mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)} - \mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)}.
\] (2.19)

**Proof:** Both operators are derivations of degree -1 on \(\Lambda g\), hence it suffices to show (2.19) for elements \(v \in g\). As we now check, this follows from the definition of the bracket induced by \(\pi\) and the invariance of \(\pi\): on one hand,

\[
i_{\rho_M^*(\alpha, \beta)}(v) = [\alpha, \beta](\rho_M(v)) = i_{\rho_M(v)}\mathcal{L}_{\pi^*(\alpha)}(\beta) - i_{\rho_M(v)}\mathcal{L}_{\pi^*(\beta)}(\alpha) - i_{\rho_M(v)}d\pi(\alpha, \beta).
\] (2.20)

Using that \(i_XY = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X\) for vector fields \(X, Y\), we have

\[
i_{\rho_M(v)}\mathcal{L}_{\pi^*(\alpha)}(\beta) = \mathcal{L}_{\pi^*(\alpha)}(\beta(\rho_M(v))) - \beta(\pi^*(\alpha), \rho_M(v))
\]

\[= \mathcal{L}_{\pi^*(\alpha)}i_{\rho_M^*(\alpha)}(\beta(v)) - \pi(\mathcal{L}_{\rho_M(v)}(\alpha), \beta),
\] (2.21)

where the last equality follows from the \(g\)-invariance of \(\pi\). Using the identity (2.21) (and its analogue for \(\alpha\) and \(\beta\) interchanged) in (2.20), (2.19) follows. \(\square\)

Using the claim, we see that

\[
i_{\rho_M^*(\alpha, \beta)}(\chi_G) + \text{c.p.} = (i_{\rho_M^*(\gamma)}\mathcal{L}_{\pi^*(\beta)})(\alpha) - \mathcal{L}_{\pi^*(\beta)}i_{\rho_M^*(\alpha)} + \text{c.p.}.
\] (2.22)

when restricted to constant elements in \(C^\infty(M, \Lambda g)\). On the other hand, it follows from (2.19) that, on \(\Lambda g\), we can write

\[
i_{\rho_M^*(\alpha, \beta)} = [\mathcal{L}_{\pi^*(\alpha)}, i_{\rho_M^*(\beta)}] - [\mathcal{L}_{\pi^*(\beta)}, i_{\rho_M^*(\alpha)}]
\] (2.23)

since the Lie derivatives are zero on constant functions. But both sides of (2.23) are \(C^\infty(M)\)-linear, so this equality is valid for all \(C^\infty(M, g)\). So we can write

\[
i_{\rho_M^*(\alpha, \beta)}(\chi_G) + \text{c.p.} = -(i_{\rho_M^*(\alpha, \beta)}i_{\rho_M^*(\gamma)}) + \text{c.p.}
\]

\[= -([\mathcal{L}_{\pi^*(\alpha)}, i_{\rho_M^*(\beta)}] - [\mathcal{L}_{\pi^*(\beta)}, i_{\rho_M^*(\alpha)}])i_{\rho_M^*(\gamma)} + \text{c.p.},
\]
from where we deduce that
\[ i^{\ast}_{\rho_{M}}([\alpha,\beta \wedge \gamma]) + c.p. = 2(L_{\pi^{\ast}(\alpha)}i^{\ast}_{\rho_{M}}(\beta \wedge \gamma) + c.p.) - (i^{\ast}_{\rho_{M}}(\gamma)L_{\pi^{\ast}(\alpha)} i^{\ast}_{\rho_{M}}(\beta) - i^{\ast}_{\rho_{M}}(\beta)L_{\pi^{\ast}(\alpha)} i^{\ast}_{\rho_{M}}(\gamma) + c.p.). \]

On constant functions, we can use (2.22) to conclude that
\[ (i^{\ast}_{\rho_{M}}([\alpha,\beta \wedge \gamma]) + c.p.) = 2(L_{\pi^{\ast}(\alpha)}i^{\ast}_{\rho_{M}}(\beta \wedge \gamma) + c.p.) - (i^{\ast}_{\rho_{M}}([\alpha,\beta \wedge \gamma]) + c.p.), \]
i.e., \( i^{\ast}_{\rho_{M}}([\alpha,\beta \wedge \gamma]) + c.p. = L_{\pi^{\ast}(\alpha)}i^{\ast}_{\rho_{M}}(\beta \wedge \gamma). \) Evaluating this identity at \( \chi_{G} \) proves (2.18), showing that (i) implies (iii). Looking back into the proof, one can check that the same formulas show the converse, so that (i) and (iii) are equivalent.

Since (iii) and the Leibniz identity for \([\cdot, \cdot]\) are together equivalent to (iv), it follows that (i) – (iv) are equivalent to each other.

\[ \square \]

**Corollary 2.7** If \((M, \pi)\) is a quasi-Poisson \(g\)-manifold, then the generalized distribution
\[ \pi^{\ast}(\alpha) + \rho_{M}(v) \subseteq TM, \text{ for } \alpha \in T^{\ast}M, \ v \in g, \]
is integrable.

This result shows that the singular distribution discussed in [2, Thm. 9.2] in the context of hamiltonian quasi-Poisson manifolds is integrable even without the presence of a moment map (and without the positivity of \([\cdot, \cdot]_{g}\)). As in the case of ordinary Poisson manifolds, we call a quasi-Poisson manifold nondegenerate if its associated Lie algebroid is transitive (i.e., its anchor map is onto).

**Example 2.8** *(Quasi-Poisson structures on Lie groups)*

Let \( G \) be a Lie group with Lie algebra \( g \), which we assume to be equipped with an invariant nondegenerate quadratic form \((\cdot, \cdot)_{g}\). We consider \( G \) acting on itself by conjugation. As shown in [2, Sec. 3], the bivector field \( \pi_{G} \), defined on left invariant 1-forms by
\[ \pi_{G}(dl^{\ast}_{g^{-1}}(\mu), dl^{\ast}_{g^{-1}}(\nu)) := \frac{1}{2}((Ad_{g^{-1}} - Ad_{g})(\mu^{\vee}), \nu^{\vee})_{g}, \quad (2.24) \]
where \( l_{g} \) denotes left multiplication by \( g \in G, \mu, \nu \in g^{\ast}, \) and \( \mu^{\vee} \) is the element in \( g \) dual to \( \mu \) via \((\cdot, \cdot)_{g} \), makes \( G \) into a quasi-Poisson \( G \)-manifold. If \((\cdot, \cdot)_{g} \) is a metric, then we can write
\[ \pi_{G} = \frac{1}{2} \sum e^{r}_{a} \wedge e^{l}_{a}, \quad (2.25) \]
where \( e_{a} \) is an orthonormal basis of \( g \) and \( e^{r}_{a} \) (resp. \( e^{l}_{a} \)) are the corresponding right (resp. left) translations.

In this example, the image of \( \pi^{\ast}_{G} \) is tangent to the \( G \)-orbits, so the leaves of the corresponding foliation are the conjugacy classes. The formula for the Lie algebroid bracket on \( T^{\ast}G \oplus g \) bears close resemblance with the one for the bracket in the “double” of the Lie quasi-bialgebra of \( G \), as in [4]. We will discuss this connection in a separate work.

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3 Moment maps in Dirac geometry: the infinitesimal picture

3.1 Dirac manifolds

Let $\phi$ be a closed 3-form on a manifold $M$. A $\phi$-twisted Dirac structure on $M$ [28] is a subbundle $L \subset E = TM \oplus T^*M$ satisfying the following two conditions:

1. $L$ is maximal isotropic with respect to the symmetric pairing $\langle \cdot, \cdot \rangle : \Gamma(E) \times \Gamma(E) \to C^\infty(M)$,
   
   \[ \langle (X, \alpha), (Y, \beta) \rangle := \beta(X) + \alpha(Y); \quad (3.1) \]

2. The space of sections $\Gamma(L)$ is closed under the bracket $[\cdot, \cdot]_\phi : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$,
   
   \[ [(X, \alpha), (Y, \beta)]_\phi := ([X, Y], \mathcal{L}_X \beta - i_Y d\alpha + i_X \wedge \phi). \quad (3.2) \]

Since the pairing (3.1) has zero signature, condition 1. is equivalent to requiring that $L$ has rank equal to $\dim(M)$ and that $\langle \cdot, \cdot \rangle|_L = 0$. The bracket (3.2) is the $\phi$-twisted Courant bracket considered in [28]. When $\phi = 0$, this bracket is a non-skew-symmetric version of Courant’s original bracket introduced in [12].

Twisted Dirac structures are always associated with Lie algebroids. Indeed, the restriction of the Courant bracket $[\cdot, \cdot]_\phi$ to a Dirac subbundle $L \subset TM \oplus T^*M$ defines a Lie algebra bracket on the space of sections $\Gamma(L)$, making $L \to M$ into a Lie algebroid with anchor

\[ \rho = \text{pr}_1|_L : L \to TM, \]

where $\text{pr}_1$ is the first projection. The orbits of this algebroid are also called the leaves of $L$.

**Example 3.1 (Twisted Poisson structures)**

If $\pi$ is a bivector field on $M$, then

\[ L_\pi := \text{graph}(\pi^\sharp) \subset TM \oplus T^*M \]

satisfies condition 1., and $L_\pi$ is a $\phi$-twisted Dirac structure if and only if $\pi$ is a $\phi$-twisted Poisson structure in the sense of Example 2.4. In this case, the second projection

\[ \text{pr}_2|_L : L \to T^*M \]

establishes an isomorphism of Lie algebroids, where $T^*M$ is equipped with the Lie algebroid structure described in Example 2.4. Setting $\phi = 0$, we obtain a one-to-one correspondence between ordinary Poisson structures on $M$ and Dirac structures satisfying the extra condition $L \cap TM = \{0\}$.

**Example 3.2 (Twisted presymplectic forms)**

Similarly, the graph associated with a 2-form $\omega \in \Omega^2(M)$, $L_\omega = \text{graph}(\omega^\sharp)$, is a $\phi$-twisted Dirac structure if and only if

\[ d\omega + \phi = 0, \]

and we refer to $\omega$ as a $\phi$-twisted presymplectic form. In this case, setting $\phi = 0$, we have an identification of closed 2-forms on $M$ with Dirac structures satisfying $L \cap T^*M = \{0\}$. 

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In general, the leaves of a $\phi$-twisted Dirac structure $L$ carry twisted presymplectic forms defined as follows: at each $x \in M$, we define a skew symmetric bilinear form $\theta_x$ on $\rho(L)_x = \text{pr}_1(L)_x$ by

$$\theta_x(X_1, X_2) = \alpha(X_2), \quad (3.3)$$

where $\alpha$ is any element in $T_x^*M$ satisfying $(X_1, \alpha) \in L_x$. The fact that $L$ is maximal isotropic with respect to (3.1) guarantees that (3.3) is independent of the choice of $\alpha$, and these forms fit together into a smooth leafwise 2-form $\theta$. Using that $\Gamma(L)$ is closed with respect to (3.2), one can show that, on each leaf $\iota : \mathcal{O} \hookrightarrow M$, the 2-form $\theta$ satisfies

$$d\theta + \iota^* \phi = 0.$$ 

At each point $x \in M$, the kernel of $\theta$ coincides with $L_x \cap T_xM$, which shows that the leafwise presymplectic forms are nondegenerate if and only if $L$ comes from a $\phi$-twisted Poisson structure.

We will denote the distribution $L \cap TM$ on $M$ by $\ker(L)$.

Since Dirac structures are always associated with Lie algebroids, it is natural to consider how to obtain Dirac structures from them. The following is a useful construction, see [6]: for a Lie algebroid $A$ over $M$ with anchor $\rho : A \rightarrow TM$, we define a $\phi$-IM form of $A$ to be any bundle map $\sigma : A \rightarrow \mathcal{T} \subset \mathcal{A}$ satisfying the following properties:

$$\langle \sigma(\xi), \rho(\xi') \rangle = -\langle \sigma(\xi'), \rho(\xi) \rangle;$$

$$\sigma([\xi, \xi']) = L_{\xi} (\sigma(\xi')) - L_{\xi'} (\sigma(\xi)) + d(\langle \sigma(\xi), \rho(\xi') \rangle + i_{\rho(\xi)} \wedge \rho(\xi') (\phi),$$

for $\xi, \xi' \in \Gamma(A)$ (here $\langle \cdot, \cdot \rangle$ denotes the usual pairing between a vector space and its dual). Let $L_\sigma \subset TM \oplus T^*M$ be the image of the map $(\rho, \sigma) : A \rightarrow TM \oplus T^*M$. Then the following is immediate.

**Lemma 3.3** If $\sigma$ is a $\phi$-IM form of $A$ and $\text{rank}(L_\sigma) = \text{dim}(M)$, then $L_\sigma$ is a $\phi$-twisted Dirac structure on $M$.

Of course, any Dirac structure can be realized as the image of an IM form by taking $A = L$, viewed as an algebroid with $\rho = \text{pr}_1|L$, and $\sigma = \text{pr}_2|L$.

The following is a key example.

**Example 3.4** (*Cartan-Dirac structures on Lie groups*)

Cartan-Dirac structures on Lie groups play a role in Dirac geometry analogous to the one played by Lie-Poisson structures (Example 2.2) in Poisson geometry. Just as Lie-Poisson structures on the dual of Lie algebras are completely determined by the Kostant-Kirillov-Souriau (KKS) symplectic forms along coadjoint orbits, Cartan-Dirac structures on Lie groups “assemble” certain 2-forms defined on conjugacy classes defined as follows.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, and let $\langle \cdot, \cdot \rangle_\mathfrak{g}$ be a bi-invariant nondegenerate quadratic form, which we use to identify $TG$ and $T^*G$. For $v \in \mathfrak{g}$, let $v_G = v_r - v_l$ be the

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2These bundle maps are infinitesimal versions of multiplicative 2-forms on groupoids, see [6]; the terminology “IM” stands for “infinitesimal multiplicative”. 

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infinitesimal generator of the action of $G$ on itself by conjugation. We define, on each conjugacy class $\iota : \mathcal{C} \hookrightarrow G$, a 2-form $\theta$ by

$$\theta_g(u_G, v_G) := \left( \frac{1}{2}(\text{Ad}^{-1} - \text{Ad}) u, v \right)_g, \ g \in \mathcal{C}. \quad (3.6)$$

Direct computations show that $d\theta + \iota^* \phi^G = 0$, where $\phi^G$ is the bi-invariant Cartan 3-form on $G$, and that $\theta_g$ is nondegenerate at a point $g$ if and only if $(\text{Ad} + 1)$ is invertible. The 2-forms (3.6) appear in [17] in the study of symplectic structures of moduli spaces, and we will refer to them as GHJW 2-forms.

Since GHJW 2-forms are not symplectic, but twisted presymplectic, they should correspond to a $\phi^G$-twisted Dirac structure $L_G$ on $G$ rather than a Poisson structure. A simple computation shows that

$$L_G = \left\{ (v_r - v_l, \frac{1}{2}(v_r + v_l)) \mid v \in \mathfrak{g} \right\} \subseteq TG \oplus TG. \quad (3.7)$$

(Recall that we are identifying $TG$ with $T^*G$ via $(\cdot, \cdot)_g$.) We call $L_G$ the Cartan-Dirac structure on $G$ associated with $(\cdot, \cdot)_g$ [28, 6].

Note that $\varpi(v) = v_r - v_l$ is the anchor of the action Lie algebroid (Example 2.1) $\mathfrak{g} \ltimes G$ with respect to the action by conjugation, and the map

$$\sigma : G \times \mathfrak{g} \to TG, \quad \sigma(v) = \frac{1}{2}(v_r + v_l) \quad (3.8)$$

satisfies the conditions of Lemma 3.3. So $\sigma$ is a $\phi^G$-IM form of $\mathfrak{g} \ltimes G$, and the Cartan-Dirac structure $L_G$ arises as the image of $(\rho, \sigma)$. In this case, $(\rho, \sigma)$ actually establishes an isomorphism between $\mathfrak{g} \ltimes G$ and $L_G$. (Note the analogy with Example 2.2, which shows that Lie algebroids of Lie-Poisson structures are isomorphic to action Lie algebroids for the coadjoint action!)

Let us finally recall an important operation involving Dirac structures: if $L$ is a $\phi$-twisted Dirac structure on $M$ and $B \in \Omega^2(M)$, then

$$\tau_B(L) := \left\{ (X, \alpha + B^\phi(X)) \mid (X, \alpha) \in L \right\} \quad (3.9)$$

defines a $(\phi - dB)$-twisted Dirac structure on $M$ [28]. The operation $\tau_B$ is called a gauge transformation associated with $B$, and it has the effect of modifying $L$ by adding the pull-back of $B$ to the presymplectic form on each leaf.

### 3.2 Dirac maps

Since Dirac structures generalize both Poisson and presymplectic structures, we have two possible definitions of Dirac maps, see [7].

Let $(M, L_M)$ and $(N, L_N)$ be twisted Dirac manifolds. A smooth map $f : N \to M$ is a forward Dirac map, or $f$-Dirac in short, if $L_N$ and $L_M$ are related as follows:

$$L_M = \left\{ (df(Y), \alpha) \mid Y \in TN, \alpha \in T^*M \text{ and } (Y, df^*(\alpha)) \in L_N \right\}. \quad (3.10)$$

If $L_M$ and $L_N$ are associated with twisted Poisson structures, then an f-Dirac map is equivalent to a Poisson map. The terminology “forward” is due to the fact that, at each point, (3.10) extends the usual notion of “push-forward” of a linear bivector. For this reason, we may write

$$L_M = f_LN$$
instead of (3.10), in analogy with the notation for “f-related” bivector fields on a manifold.

Similarly, \( f : N \rightarrow M \) is a **backward Dirac map**, or simply **b-Dirac**, if

\[
L_N = \{(Y, df^*\alpha) \mid Y \in TN, \alpha \in T^*M \text{ and } (df(Y), \alpha) \in L_M\}. \tag{3.11}
\]

If \( L_M \) and \( L_N \) are associated with twisted presymplectic structures \( \omega_M \) and \( \omega_N \), then a b-Dirac map is just a map satisfying \( f^*\omega_M = \omega_N \). As before, we will write

\[
L_N = f^*L_M
\]

to denote that \( f \) is a b-Dirac map. Note that \( f^*L_M \) is always a well-defined, though not necessarily smooth, subbundle of \( TN \), in contrast with \( f_\ast L_N \), which may not be well-defined at all. In fact, \( f^*L_M \) defines a Dirac structure on \( N \) provided it is smooth, which is the case, e.g., when \( f \) is a submersion. However, as illustrated in the next example, \( f^*L_M \) may define a Dirac structure even when \( f \) is not a submersion.

**Example 3.5 (Inclusion of presymplectic leaves)**

Let \( L \) be a twisted Dirac structure on \( M \), and consider a presymplectic leaf \( O \), equipped with Dirac structure \( L_\theta \) associated with the twisted presymplectic form \( \theta \). Denoting by \( \iota : O \hookrightarrow M \) the inclusion map, it follows from the definition of \( \theta \) that

\[
L_\theta = \{(X, i_X\theta) \mid X \in T_O = \{(X, (di)^*\alpha) \mid (d\iota(X), \alpha) \in L\} = \iota^*L. \tag{3.12}
\]

So \( \iota : (O, L_\theta) \hookrightarrow (M, L) \) is a b-Dirac map. On the other hand, at each point of \( M \), we have

\[
\iota_\ast L_\theta = \{(d\iota(X), \alpha) \mid (X, (di)^*\alpha) \in L_\theta\}.
\]

By the second equality in (3.12), it follows that \( \iota_\ast L_\theta \subseteq L \), but since they have the same dimension, we get

\[
\iota_\ast L_\theta = L, \tag{3.13}
\]

so \( \iota \) is f-Dirac as well.

Note that the fact that the inclusion of presymplectic leaves into a Dirac manifold is an f-Dirac map is a direct generalization of the fact that the inclusion of symplectic leaves into a Poisson manifold is a Poisson map. As a simple consequence, we have

**Corollary 3.6** Let \((N, L_N)\) and \((M, L_M)\) be twisted Dirac manifolds. A map \( J : N \rightarrow M \) is f-Dirac if and only if its restriction to each presymplectic leaf of \( N \) is f-Dirac.

We remark that Example 3.5 is very special in that the inclusion map of presymplectic leaves is both forward and backward Dirac (see also Remark 4.12). In general, f-Dirac maps need not be b-Dirac, nor the other way around.

### 3.3 Poisson maps as infinitesimal Hamiltonian actions

The usual notion of Lie algebra action can be extended to the realm of Lie algebroids, the main difference being that algebroids, rather than acting on manifolds, act on maps from manifolds into their base [18]: An **action of a Lie algebroid** \( A \rightarrow M \) on a map \( J : N \rightarrow M \) consists of a Lie algebra homomorphism \( \rho_N : \Gamma(A) \rightarrow \mathcal{X}(N) \) satisfying

\[
dJ \circ \rho_N(\xi) = \rho(\xi), \quad \text{for all } \xi \in \Gamma(A), \tag{3.14}
\]
and such that, for \( f \in C^\infty(M) \) and \( \xi \in \Gamma(A) \), \( \rho_N(f \xi) = J^* f \rho_N(\xi) \) (i.e., the induced map \( \Gamma(J^*A) \to \mathcal{X}(N) \) comes from a vector bundle morphism \( J^*A \to TN \), where \( J^*A = A \times_M N \) is the pull-back of the vector bundle \( A \) by \( J \).

**Example 3.7 (Actions of transformations Lie algebroids)**

Consider an infinitesimal action \( \rho \) of \( \mathfrak{g} \) on a manifold \( M \). Then an action \( \rho_N \) of the transformation Lie algebroid \( A = \mathfrak{g} \times M \) on a map \( J : N \to M \) is equivalent to an infinitesimal action \( \bar{\rho}_N \) of \( \mathfrak{g} \) on \( N \) for which \( J \) is \( \mathfrak{g} \)-equivariant. Indeed, \( \rho_N \) and \( \bar{\rho}_N \) are related by the formula

\[
\rho_N(v)_y = \bar{\rho}_N(v(J(y)))_y, \quad \text{where} \ v \in C^\infty(M, \mathfrak{g}), \ y \in N,
\]

and the \( \mathfrak{g} \)-equivariance of \( J \) corresponds to (3.14).

In Poisson geometry, Poisson maps are always associated with Lie algebroid actions: If \( (Q, \pi_Q) \) and \( (P, \pi_P) \) are Poisson manifolds, then any Poisson map \( J : Q \to P \) induces a Lie algebroid action of \( T^*P \) on \( Q \) by

\[
\Omega^1(P) \to \mathcal{X}(Q), \quad \alpha \mapsto \pi_Q^\sharp(J^*\alpha).
\]

When the target \( P \) is the dual of a Lie algebra, we recover a familiar example:

**Example 3.8 (Infinitesimal hamiltonian actions)**

Consider \( \mathfrak{g}^* \) equipped with its Lie-Poisson structure. As remarked in Example 2.2, the Lie algebroid structure on \( T^*\mathfrak{g}^* \) induced by (2.5) is that of a transformation Lie algebroid \( \mathfrak{g} \times \mathfrak{g}^* \) with respect to the coadjoint action. If \( J : Q \to \mathfrak{g}^* \) is a Poisson map, then it induces an action of \( T^*\mathfrak{g}^* \) via (3.16), which, by Example 3.7, is equivalent to an ordinary \( \mathfrak{g} \)-action on \( Q \) for which \( J \) is equivariant. A simple computation shows that the \( \mathfrak{g} \)-action arising in this way is just a hamiltonian action in the usual sense, making \( Q \) into a hamiltonian Poisson \( \mathfrak{g} \)-manifold having \( J \) as a momentum map.

Recall that a Poisson map \( J : Q \to P \) is called a symplectic realization if \( Q \) is symplectic. The following is an immediate consequence:

**Proposition 3.9** There is a one-to-one correspondence between Poisson maps into \( \mathfrak{g}^* \) and hamiltonian Poisson \( \mathfrak{g} \)-manifolds, and this correspondence restricts to a one-to-one correspondence between symplectic realizations of \( \mathfrak{g}^* \) and hamiltonian symplectic \( \mathfrak{g} \)-manifolds.

**Remark 3.10** An analogue of Prop. 3.9 holds more generally in the context of Poisson-Lie groups [20, 22]. Let \((G, \pi)\) be a simply-connected Poisson-Lie group, and let \( G^* \) be its dual. The Lie algebroid structure on \( T^*G^* \cong G^* \times \mathfrak{g} \) induced from the dual Poisson structure is a transformation Lie algebroid, now associated with the infinitesimal dressing action of \( \mathfrak{g} \) on \( G^* \). For a Poisson map \( J : Q \to G^* \), the general Lie algebroid action described by (3.16) reduces to a Poisson \( \mathfrak{g} \)-action for which \( J \) is an equivariant momentum map in the sense of Lu [21].

In order to extend this discussion to Dirac geometry, let us consider \( L_{\pi_P} = \text{graph}(\pi_P^\sharp) \), the associated Dirac structure on \((P, \pi_P)\). Using the Lie algebroid isomorphism \( T^*P \cong L_{\pi_P} \), we can rewrite the infinitesimal action (3.16) as

\[
\Gamma(L_{\pi_P}) \to \mathcal{X}(Q), \quad (X, \alpha) \mapsto Y,
\]

where \( Y \in \mathcal{X}(Q) \) is uniquely determined by the condition \( (Y, J^*\alpha) \in L_{\pi_Q} \). Also note that \( Y \) is related to \( X \) by \( dJ(Y) = X \), since \( J \) is a Poisson map. The question of whether this procedure can be carried out for f-Dirac maps leads us to the notion of Dirac realization.
3.4 Dirac realizations

If \((N, L_N)\) and \((M, L_M)\) are twisted Dirac manifolds, then, by definition, a smooth \(J : N \to M\) is an \(f\)-Dirac map if and only if, given \((X, \alpha) \in (L_M)_y\), there exists a \(Y \in T_yN\) with the property that
\[
(Y, dJ^*\alpha) \in (L_N)_y \quad \text{and} \quad X = (dJ)_y(Y).
\] (3.18)

It is natural to try to define an action of \(L_M\) on \(N\) just as in the case of Poisson maps, see (3.17), except that (3.18) does not determine \(Y\) uniquely in general. In fact, this is the case if and only if the following extra “nondegeneracy” condition holds:
\[
\ker(dJ) \cap \ker(L_N) = \{0\}.
\] (3.19)

A similar argument as in [6, Section 7.1] shows that (3.19) is equivalent to \(J : \ker(L_N) \to \ker(L_M)\) being an isomorphism.

**Definition 3.11** A Dirac realization of a \(\phi\)-twisted Dirac manifold \((M, L_M)\) is an \(f\)-Dirac map \(J : (N, L_N) \to (M, L_M)\), where \(L_N\) is a \(J\phi\)-twisted Dirac structure on \(N\), satisfying (3.19).

As a consequence of Definition 3.11, we have

**Corollary 3.12** Let \(J : N \to M\) be a Dirac realization. Then the map \(\Gamma(L_M) \to \mathcal{X}(N)\), \((X, \alpha) \mapsto Y\), where \(Y\) is determined by the conditions in (3.18), is a Lie algebroid action.

Dirac realizations \(J : N \to M\) for which \(N\) is presymplectic were studied in [6, Sec. 7.1] under the name of presymplectic realizations. As a result of Corollary 3.6, we have

**Corollary 3.13** A map \(J : N \to M\) is a Dirac realization if and only if its restriction to each presymplectic leaf of \(N\) is a presymplectic realization.

Similarly to Poisson geometry, the connection between Dirac realizations and “hamiltonian actions” is established by a suitable choice of “target” \(M\). Following the analogy between Lie-Poisson structures and Cartan-Dirac structures, it is natural to study the “hamiltonian spaces” associated with Dirac realizations of Cartan-Dirac structures. The particular case of presymplectic realizations is discussed in [6, Sec. 7.2]:

**Example 3.14** (Presymplectic realizations of Cartan-Dirac structures)

Let \(G\) be a Lie group with Lie algebra \(\mathfrak{g}\), equipped with a bi-invariant nondegenerate quadratic form \((\cdot, \cdot)_g\). Let \(L_G\) be the associated Cartan-Dirac structure on \(G\). If \((M, \omega_M)\) is a twisted presymplectic manifold, then the conditions for \(J : M \to G\) being a presymplectic realization can be expressed as follows:

1. \(\omega_M\) is \(\mathfrak{g}\)-invariant and satisfies \(d\omega_M = -J^*\phi^G\);
2. at each \(x \in M\), \(\text{Ker}(\omega_M)_x = \{ (\rho_M)_x(v) : v \in \text{Ker}(\text{Ad}_J(p) + 1) \}\);
3. the map \(J\) satisfies the moment map condition
\[
\omega^g J^* \rho_M = J^* \sigma.
\] (3.20)
The invariance of $\omega_M$ in 1. is with respect to the $\mathfrak{g}$-action $\rho_M$ induced by $J$ (recall that $L_G \cong \mathfrak{g} \ltimes G$, see Example 3.4, so an $L_G$-action defines an ordinary $\mathfrak{g}$-action), for which $J$ is equivariant; in 3., $\sigma$ is the IM-form of the Cartan-Dirac structure (3.8),

$$\sigma : \mathfrak{g} \longrightarrow T^*G, \quad \sigma(v) = \frac{1}{2}(v_r + v_l)^\vee,$$

where $v \longrightarrow v^\vee$ denotes the isomorphism $TG \longrightarrow T^*G$ induced by the quadratic form. The “relative closedness” of $\omega_M$ in 1. expresses that the associated Dirac structure is $J^*\phi_G$-twisted, while condition 2. is the “non-degeneracy” condition (3.19) applied to this particular case; finally, condition 3. follows from $J$ being an $f$-Dirac map.

Conditions 1., 2. and 3. are exactly the defining axioms of a **quasi-hamiltonian** $\mathfrak{g}$-space, in the sense of [3], for which $J$ is the group-valued moment map. Conversely, any group-valued moment map of a quasi-hamiltonian $\mathfrak{g}$-space is a presymplectic realization of $(G, L_G)$.

We summarize Example 3.14 in the next result, analogous to Proposition 3.9, see [6, Thm. 7.6].

**Theorem 3.15** There is a one-to-one correspondence between presymplectic realizations of $G$ endowed with the Cartan-Dirac structure, and quasi-hamiltonian $\mathfrak{g}$-manifolds.

Combining Corollary 3.13 with Theorem 3.15, we conclude that general Dirac realizations of Cartan-Dirac structures must be “foliated” by quasi-hamiltonian $\mathfrak{g}$-manifolds. Since hamiltonian quasi-Poisson manifolds, in the sense of [2], also have this property [2, Sec. 10], we are led to investigate the relationship between these objects.

### 3.5 Dirac realizations and hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds

#### 3.5.1 The equivalence theorem

For a quasi-Poisson $\mathfrak{g}$-manifold $(M, \pi)$, a **moment map** is a $\mathfrak{g}$-equivariant map $J : M \longrightarrow G$ (with respect to the infinitesimal action by conjugation on $G$) satisfying the condition [2, Lem. 2.3]

$$\pi^*J^* = \rho_M \sigma^\vee,$$

where

$$\sigma^\vee : T^*G \longrightarrow \mathfrak{g}, \quad \sigma^\vee(\xi_g) = \frac{1}{2}(dr_{g^{-1}}(\xi_g^\vee) + dl_{g^{-1}}(\xi_g^\vee))$$

is the adjoint of $\sigma$ (3.21) with respect to the form $(\cdot, \cdot)_g$, and $\rho_M : \mathfrak{g} \rightarrow TM$ is the $\mathfrak{g}$-action. Here $l_g$ and $r_g$ denote the left and right translations by $g$, respectively, and, as in Example 3.14, the symbol $\vee$ on elements of $T^*G$ is used to denote the corresponding element in $TG$ via the identification induced by $(\cdot, \cdot)_g$ (and vice-versa).

The following is our main result in this section.

**Theorem 3.16** There is a one-to-one correspondence between Dirac realizations of $G$, endowed with the Cartan-Dirac structure, and hamiltonian quasi-Poisson $\mathfrak{g}$-manifolds.

Before proving Theorem 3.16, let us collect some useful formulas relating the maps $\sigma$, $\sigma^\vee$, $\rho : \mathfrak{g} \rightarrow TG$, $\rho(v) = v_r - v_l$, and, for symmetry, the dual of $\rho$ with respect to $(\cdot, \cdot)_g$.

$$\rho^\vee : TG \longrightarrow \mathfrak{g}, \quad \rho^\vee(V_g) = dr_{g^{-1}}(V_g) - dl_{g^{-1}}(V_g).$$

The following lemma follows from a straightforward computation.
Lemma 3.17 The following formulas hold true:

\[
\begin{align*}
4\sigma^\vee \rho + \rho^\vee \rho &= 4\text{Id}_\mathfrak{g}, \\
4\sigma^\vee + (\rho^\vee)^* &= 4\text{Id}_{T^*G}, \\
\sigma^\vee \rho &= \rho^\vee \sigma, \\
\sigma \rho^\vee &= -(\rho^\vee)^* \sigma^*, \\
\rho^\vee (\sigma^\vee)^* &= -\sigma^\vee (\rho^\vee)^*.
\end{align*}
\]

Motivated by the equivalence between quasi-hamiltonian manifolds and nondegenerate hamiltonian quasi-Poisson manifolds [2, Thm. 10.3], which will also follow from Theorem 3.16, it is natural to combine the two moment-map conditions (3.20) and (3.22). By applying \(\pi^\sharp\) to (3.20) and using (3.22), we obtain, in particular, the relation

\[
\rho_M \sigma^\vee \sigma = \pi^\sharp \omega^\mathfrak{g}_M.
\]

This suggests the importance of writing \(\rho_M \sigma^\vee \sigma\) as the composition of some operator \(C : TM \to TM\) with \(\rho_M\) in general. Using (3.25) and the equivariance of \(J\), written as \(\rho = dJ \circ \rho_M\), it is easy to find an expression for \(C\) (which already appears in [2, Lem. 10.2]):

Lemma 3.18 For any manifold \(M\) equipped with an infinitesimal action \(\rho_M : \mathfrak{g} \to TM\), and any \(\mathfrak{g}\)-equivariant map \(J : M \to G\), the operator

\[
C = 1 - \frac{1}{4} \rho_M \rho^\vee (dJ) : TM \to TM,
\]

and its dual \(C^* : T^*M \to T^*M\), satisfy the formulas

\[
\rho_M \sigma^\vee \sigma = C \rho_M, \quad \text{and} \quad J^* \sigma^\vee \sigma = C^* J^*.
\]

Theorem 3.16 follows from the next two propositions, each one of them describing explicitly one direction of the asserted one-to-one correspondence.

Proposition 3.19 Let \(M\) be a quasi-Poisson \(\mathfrak{g}\)-manifold, and let \(A = T^*M \oplus \mathfrak{g}\) be its associated Lie algebroid, with anchor \(r\). Then any moment map \(J : M \to G\) induces a \(\phi\)-IM form of \(A\) by

\[
s : A \to T^*M, \quad s(\alpha, v) = C^*(\alpha) + J^* \sigma(v),
\]

so that the image \(L\) of the map \((r, s) : A \to TM \oplus T^*M\) is a \(J^* \phi^G\)-twisted Dirac structure on \(M\), and \(J : (M, L) \to (G, L_G)\) is a Dirac realization of the Cartan-Dirac structure on \(G\).

This proposition also suggests the converse construction.

Proposition 3.20 Let \(J : (M, L) \to (G, L_G)\) be a Dirac realization of the Cartan-Dirac structure on \(G\). Then

\[(i)\quad \text{for any } v \in \mathfrak{g} \text{ there is an unique vector } V \in TM \text{ satisfying}
\]

\[
dJ(V) = v, \quad \text{and} \quad (V, J^* \sigma(v)) \in L.
\]
(ii) for any \( \alpha \in T^*M \), there is a unique vector \( X \in TM \) satisfying
\[ dJ(X) = - (\rho_M(\sigma^*)^* \alpha), \]
\[ (X, C^*(\alpha)) \in L. \]

Moreover, \( v \mapsto \rho_M(v) := V \) defines a \( g \)-action on \( M \), and \( \alpha \mapsto \pi^g(\alpha) := X \) defines a quasi-Poisson tensor \( \pi \) on \( M \) so that \( (M, \pi) \) is a hamiltonian quasi-Poisson \( g \)-manifold with moment map \( J \).

Note that the \( g \)-action defined by (3.32) is just the one induced by the infinitesimal \( L_G = g \times G \)-action with “moment” \( J : M \to G \).

It is simple to check that the constructions in Propositions 3.19 and 3.20 are inverses to one another. For example, if \( L \) is obtained from \( \pi \) as in Prop. 3.19, then
\[ L = \{ (\pi^g(\alpha) + \rho_M(v), C^*(\alpha) + J^*(\sigma(v))) \mid \alpha \in T^*M, v \in g \}, \]
and it is clear that, given \( \alpha \in T^*M \), \( X = \pi^g(\alpha) \) satisfies conditions (3.33) (which is just the dual of the moment map condition (3.22)) and (3.34), so Prop. 3.20 constructs \( \pi \) back.

Since the proofs of Propositions 3.19 and 3.20 involve long computations, we will postpone them to the next subsection; we will discuss some examples and implications of the results first.

**Example 3.21 (Nondegenerate quasi-Poisson and quasi-hamiltonian \( g \)-manifolds)**

It is clear from the correspondence constructed in Prop. 3.19 that the singular foliation associated with \( \pi \), tangent to \( \text{Im}(\pi^g) + \text{Im}(\rho_M) \subseteq TM \), coincides with the singular foliation of the Dirac structure \( L \), tangent to \( \text{pr}_1(L) \subseteq TM \). In other words, the Lie algebroids associated with \( \pi \) and \( L \) have the same leaves, so one is transitive if and only if the other one is. Note that, for Dirac structures, \( \text{pr}_1(L) = TM \) means exactly that \( L \) is defined by a 2-form. As a result, it follows that the correspondence established by Theorem 3.16 restricts to a one-to-one correspondence between nondegenerate hamiltonian quasi-Poisson manifolds and presymplectic realizations of Cartan-Dirac structures on Lie groups.

Combining Example 3.21 with Theorem 3.15, we obtain

**Corollary 3.22** There is a one-to-one correspondence between

(i) non-degenerate quasi-Poisson hamiltonian \( g \)-manifolds.

(ii) quasi-hamiltonian \( g \)-manifolds.

(iii) presymplectic realizations of \( G \) endowed with the Cartan-Dirac structure.

Of course, in general, the leaves of a hamiltonian quasi-Poisson \( g \)-manifolds are quasi-hamiltonian \( g \)-manifolds, which can now be seen as a particular case of Dirac structures having presymplectic foliations (see also Corollary 3.13). The equivalence of (i) and (ii) can be found in [2].

The next example answers a question posed in [28, Ex. 5.2].

**Example 3.23 (Cartan-Dirac and quasi-Poisson structures on Lie groups)**

If \( G \) is a Lie group with Lie algebra \( g \) equipped with a bi-invariant nondegenerate quadratic form \( \langle \cdot, \cdot \rangle_g \), then one can regard it as a \( g^G \)-twisted Dirac manifold with respect to the Cartan-Dirac structure \( L_G \), or as a hamiltonian quasi-Poisson \( g \)-manifold. In the latter case, we consider
Since the moment map have the same dimension. Indeed, starting with hamiltonian quasi-Poisson \( f \)-Dirac, we have to compare, at each point of 3.16 preserves maps.

To simplify the notation, we will denote the in\( \text{\-}\)finitesimal actions of \( g \) ¡ \( f \), and we obtain

\[
\text{df} = \varphi (\beta) = f (\varphi (\beta)) \quad \text{for} \quad f (\varphi (\beta)) = \varphi (f (\beta))
\]

for \( L_1 \) corresponds to \( \pi_1 \), it follows that, at each point, \( L_2 \subseteq f_L L_1 \). But since they have equal dimension, we conclude that \( L_2 = f_L L_1 \), so \( f \) is forward Dirac.

For the converse, suppose that \( f_L L_1 = L_2 \) and \( J_2 \circ f = J_1 \). It is easy to check that, in this case, \( f \) is automatically \( g \)-equivariant with respect to (3.32). From that, it follows that \( df L_1 = C_2 df \), and we obtain

\[
\text{df} (C_2(\beta) + J_2^* \sigma (v)) = (\text{df} (\beta)) + \sigma (v).
\]

Since

\[
(\pi_1 (\beta), \pi_2 (\beta)) + J_1^* \sigma (v)) \in L_1,
\]

for \( L_1 \) corresponds to \( \pi_1 \), it follows that, at each point, \( L_2 \subseteq f_L L_1 \). But since they have equal dimension, we conclude that \( L_2 = f_L L_1 \), so \( f \) is forward Dirac.

To complete the “duality” picture between Dirac realizations of Cartan-Dirac structures and hamiltonian quasi-Poisson \( g \)-manifolds, we note that the correspondence established in Theorem 3.16 preserves maps.

If \((M_i, J_i)\) is a hamiltonian quasi-Poisson \( g \)-manifold with moment map \( J_i : M_i \to G, i = 1, 2\), then a map \( f : M_1 \to M_2 \) is a hamiltonian quasi-Poisson map if \( f \circ J_1 = J_2 \circ f \), \( f \) is \( g \)-equivariant, and \( J_2 \circ f = J_1 \). Suppose that \( L_i \) is the Dirac structure on \( M_i \) corresponding to \( \pi_i, i = 1, 2 \), via Theorem 3.16.

**Proposition 3.24** A map \( f : (M_1, J_1) \to (M_2, J_2) \) is a hamiltonian quasi-Poisson map if and only if \( f : (M_1, L_1) \to (M_2, L_2) \) is \( f \)-Dirac and commutes with the realization maps, \( J_2 \circ f = J_1 \).

**Proof:** Suppose \( f : M_1 \to M_2 \) is a hamiltonian quasi-Poisson map. In order to check that \( f \) is \( f \)-Dirac, we have to compare, at each point of \( M_2, L_2 \) with

\[
f_L L_1 = \{((\text{df} (X), \beta) \mid (X, \text{df} (\beta)) \in L_1 \}.
\]

To simplify the notation, we will denote the in\( \text{\-}\)finitesimal actions of \( g \) on \( M_i \) by \( \rho_i \), and \( C_i = 1 - (1/4) \rho_i \rho^d J_i \), \( i = 1, 2 \). Since \( L_2 \) corresponds to \( \pi_2 \), we have

\[
L_2 = \{(\alpha \rho_2 (v), C_2 (\beta) + J_2^* \sigma (v)) \mid \beta \in \text{TM}_2, v \in g \}.
\]

Using that \( \text{df} \pi_1^* \text{df}^* = \pi_2^* \) (which is another way of writing \( f \circ \pi_1 = \pi_2 \)) and \( \text{df} \rho_1 = \rho_2 \) (which is \( f \)-\( g \)-equivariance), we deduce that

\[
\pi_2^* (\beta) + \rho_2 (v) = \text{df} (\pi_1 (\text{df}^* (\beta)) + \rho_1 (v)).
\]

On the other hand, using the \( g \)-equivariance of \( f \) and \( J_2 \circ f = J_1 \), it follows that \( df C_1 = C_2 df \), and we obtain

\[
\text{df} (C_2^* (\beta) + J_2^* \sigma (v)) = C_1^* (\text{df}^* (\beta)) + J_1^* \sigma (v).
\]

Since

\[
(\pi_1 (\text{df}^* (\beta)) + \rho_1 (v), C_1^* (\text{df}^* (\beta)) + J_1^* \sigma (v)) \in L_1,
\]

for \( L_1 \) corresponds to \( \pi_1 \), it follows that, at each point, \( L_2 \subseteq f_L L_1 \). But since they have equal dimension, we conclude that \( L_2 = f_L L_1 \), so \( f \) is forward Dirac.

For the converse, suppose that \( f_L L_1 = L_2 \) and \( J_2 \circ f = J_1 \). It is easy to check that, in this case, \( f \) is automatically \( g \)-equivariant with respect to (3.32). From that, it follows that \( df C_1 = C_2 df \).

In order to prove that \( f \) is a hamiltonian quasi-Poisson map, we must still check that \( f \circ \pi_1 = \pi_2 \), or, equivalently, that \( \text{df} \pi_1 \text{df}^* = \pi_2^* \). By Prop. 3.20, it suffices to prove that, for \( \beta \in \text{T}^* M_2 \),

\[
(Y, C_2^* (\beta)) \in L_2 \quad \text{and} \quad \text{d} J_2 (Y) = -p_2^* \sigma (Y^*) \beta.
\]
Since \( f \) is an \( f \)-Dirac map, the first condition in (3.38) holds since

\[
(\pi^1_1^*df^*(\beta), df^*(C_2^*(\beta))) = (\pi^1_1^*df^*(\beta), C_1^*(df^*(\beta))) \in L_1,
\]

for \( L_1 \) corresponds to \( \pi_1 \). The second condition holds since

\[
dJ_2(df\pi^1_1^*df^*(\beta)) = dJ_1(df\pi^1_1^*df^*(\beta)) = -(\rho_1\sigma^\vee)^*df^*(\beta) = -(\rho_2\sigma^\vee)^*(\beta).
\]

We now proceed to the proofs of Propositions 3.19 and 3.20.

3.5.2 The proofs

Proof of Proposition 3.19:
To simplify our formulas, we set

\[
T = \rho_M\rho^\vee dJ,
\]

and we denote by \( \langle \cdot, \cdot \rangle \) the pairing between vector spaces and their duals.

First, we have to show that \( \langle s(\xi), s(\xi') \rangle \) is antisymmetric in \( \xi, \xi' \in A = T^*M \oplus g \). We check this on elements of type \((\alpha,0), (\beta,0)\), and we leave the other cases to the reader. Since \( \pi \) is antisymmetric, we only have to show that \( \langle T^*\beta, \pi^I(\alpha) \rangle \) is antisymmetric in \( \alpha \) and \( \beta \). Using that \( dJ\pi^J = -(\sigma^\vee)^* (\rho_M)^* \), which is the adjoint of the moment map condition (3.22), we see that

\[
\pi(\alpha, T^*\beta) = \langle T^*\beta, \pi^I(\alpha) \rangle = -\langle \rho_M(\beta), \rho^\vee(\sigma^\vee)^* \rho^\vee_M(\alpha) \rangle,
\]

which is antisymmetric by (3.29).

We now turn to proving that \( s \) satifies (3.5). For sections of \( A \) of type \( \xi = (0, u), \xi' = (0, v) \), with \( u, v \in g \), (3.5) follows from the next lemma.

Lemma 3.25 Given a \( g \)-manifold \( M \), and an equivariant map \( J : M \rightarrow G \), then, for any \( u, v \in g \),

\[
J^*\sigma([u, v]) = L_{\rho_M(u)}(J^*(\sigma(v))) - L_{\rho_M(v)}(J^*(\sigma(u))) + d(J^*(\sigma(v), \rho_M(u)) + i_{\rho_M(u)}(\rho_M(v)) (J^*\phi^G).
\]

Proof: Using the equivariance of \( J \), \( dJ\rho_M = \rho \), we immediately see that this equation is the pull-back by \( J \) of (3.5) for \( \sigma \) (for instance, the last term in the equation equals to \( J^*\phi^G \)). \( \square \)

Next we consider the case \( \xi = (0, u) \) and \( \xi' = (\alpha, 0) \), which is handled by the next result.

Lemma 3.26 Given a \( g \)-manifold \( M \), an equivariant map \( J : M \rightarrow G \), and a bivector \( \pi \) on \( M \) satisfying the moment map condition (3.22), then, for all \( v \in g \) and \( \alpha \in T^*M \),

\[
C^*L_{\rho_M(v)}(\alpha) = L_{\rho_M(v)}(C^*(\alpha)) - L_{\pi^I(v)}(J^*(\sigma(v))) - \rho_M(\pi^I(v)) + i_{\rho_M(v)}(\pi^I(v)) + i_{\rho_M(v)}(\pi^I(v))(J^*\phi^G).
\]

Although Lemma 3.26 is more difficult than Lemma 3.25, it is still simpler than the next case, treated in Lemma 3.27 below. Since a formula that holds under the same assumptions and is proven by the same method is proven in detail in Claim 3.37 below, we will omit its proof.

Let us now consider \( \xi = (\alpha, 0) \) and \( \xi' = (\beta, 0) \). Comparing with Lemmas 3.25 and 3.26, the greater technical difficulty of this case comes from the fact that now the formulas involve both \( \chi_G \) and \( \phi^G \), the bracket \([,\] \) induced by \( \pi \) on 1-forms, and require more than just the moment map condition.
Lemma 3.27 Given a \( g \)-manifold \( M \), an equivariant map \( J : M \rightarrow G \), and an invariant bivector \( \pi \) on \( M \) satisfying the moment map condition (3.22), then, for all \( \alpha, \beta \in T^* M \),

\[
C^*(\alpha, \beta) - \frac{1}{2} J^* \sigma(i(\rho_M)^*(\alpha \wedge \beta)) = L_{\pi^*(\alpha)}(\beta) - L_{\pi^*(\beta)}(\alpha) - d\left( C^*(\beta), \pi^*(\alpha) \right) + i_{\pi^*(\alpha) \wedge \pi^*(\beta)}(J^* \phi^G).
\]

**Proof:** Since \( C = 1 - \frac{1}{4} T \), using the definition of \( s \) and of the bracket in \( \Gamma(A) \), we see that we can rewrite the equation in the lemma as

\[
T^*([\alpha, \beta]) = -L_{\pi^*(\alpha)}(T^*(\beta)) + L_{\pi^*(\beta)}(T^*(\alpha)) + d\pi(\alpha, T^*(\beta)) = -2J^* \sigma i_{\rho_M^*(\alpha) \rho_M^*(\beta)} \chi G - 4i_{\pi^*(\alpha) \wedge \pi^*(\beta)} J^* \phi^G. \quad (3.40)
\]

Let us evaluate all the terms of (3.40) on an arbitrary vector field \( X \in \mathcal{X}(M) \). To simplify the formulas, we set

\[
a = \rho_M^*(\alpha), \quad b = \rho_M^*(\beta), \quad V = J(X) \quad (3.41)
\]

and we consider the Hom\((g^*, g)\)-valued function on \( G \) given by

\[
D = \rho^*(\sigma^*)^*.
\]

Claim 3.28 The following formula holds:

\[
\langle d\pi(\alpha, T^*(\beta)), X \rangle = \langle a, L_V(Db) \rangle - \langle b, L_V(Da) \rangle - \langle a, L_V(D)b \rangle. \quad (3.42)
\]

**Proof:** The left hand side of (3.42) is \( L_X \pi(\alpha, T^*(\beta)) \). Hence, using (3.39), it equals

\[
-\langle L_{J(X)} \rho_M^*(\beta), \rho^*(\sigma^*)^* \rho_M^*(\alpha) \rangle - \langle \rho_M^*(\beta), L_{dJ(X)}(\rho^*(\sigma^*)^*) \rangle \rho_M^*(\alpha). \quad (3.43)
\]

With the notation of (3.41), and using \( D^* = -D \) (i.e. \( 3.29 \)) to rewrite the first term, we see that (3.43) equals

\[
\langle DL_V(b), a \rangle - \langle b, L_V(Da) \rangle.
\]

To obtain (3.42), we write \( DL_V(b) = L_V(Db) - L_V(D)(b) \).

From the definition of \([\alpha, \beta] \) (2.4), we have

\[
T^*([\alpha, \beta]) = T^* L_{\pi^*(\alpha)}(\beta) - T^* L_{\pi^*(\beta)}(\alpha) - T^* d\pi(\alpha, \beta). \quad (3.44)
\]

Claim 3.29 The following formula holds:

\[
\langle T^* d\pi(\alpha, \beta), X \rangle = \pi(L_{T(X)}(\alpha), \beta) + \pi(\alpha, L_{T(X)}(\beta)). \quad (3.45)
\]

**Proof:** This follows from the invariance of \( \pi \) and the fact that the image of \( T \) sits inside that of \( \rho_M \).

Using (3.44) and (3.45), we can split the left hand side of (3.40) as a difference of two terms which are symmetric to each other. The next claim deals with such a term.

Claim 3.30 The following formula holds:

\[
\left\langle T^* L_{\pi^*(\alpha)}(\beta) - L_{\pi^*(\alpha)}(T^*(\beta)), X \right\rangle = \pi(L_{T(X)}(\alpha), \beta) + \langle b, L_V(Da) \rangle + \langle b, d\rho^*((\sigma^*)^*(a), V) \rangle \quad (3.46)
\]
Proof: The left hand side of (3.46) equals
\[ -\left\langle \beta, [\pi^\sharp(\alpha), T(X)] + T([\pi^\sharp(\alpha), X]) \right\rangle. \] (3.47)
To rewrite \([\pi^\sharp(\alpha), T(X)]\), we note that
\[ [\pi^\sharp(\alpha), \rho_M(\bar{v})] = -\pi^\sharp \mathcal{L}_{\rho_M(\bar{v})}(\alpha) + \rho_M \mathcal{L}_{\pi^\sharp(\alpha)}(\bar{v}), \] (3.48)
for all \(\bar{v} \in C^\infty(M, \mathfrak{g})\): Indeed, due to \(C^\infty(M)-\)linearity with respect to \(\bar{v}\), it suffices to check (3.48) for \(\bar{v}\) constant; in this case, the equation is just the invariance of \(\pi\). We now use (3.48) for \(\bar{v} = \rho^\vee J(X)\) to get
\[ [\pi^\sharp(\alpha), T(X)] = -\pi^\sharp \mathcal{L}_{T(X)}(\alpha) + \rho_M \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee J(X)). \]
We deduce that (3.47) equals to
\[ \left\langle \beta, \pi^\sharp \mathcal{L}_{T(X)}(\alpha) \right\rangle - \left\langle \rho^\ast_M(\beta), \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee J(X)) \right\rangle - \left\langle \rho^\ast_M(\beta), \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee J(X)) \right\rangle = \pi(\mathcal{L}_{T(X)}(\alpha), \beta) - \left\langle \rho^\ast_M(\beta), \mathcal{L}_{\pi^\sharp(\alpha)}(\rho^\vee J(X)) \right\rangle. \] (3.49)
On the other hand, for all vector fields \(Y\) on \(M\) and \(\mathfrak{g}\)-valued 1-forms \(\nu\) on \(G\), we have
\[ \mathcal{L}_Y(J^*\nu) = \mathcal{L}_{dJ(Y)}(\theta(dJ(Y))) + (d\theta)(dJ(Y), dJ(X)). \]
Using this identity for \(Y = \pi^\sharp\alpha\) and \(\nu = \rho^\vee \) in (3.49), together with the dual of the moment map condition, \(dJ\pi^\sharp = -((\sigma)^\vee)^*(\rho_M)^\ast\), we obtain (3.46) and prove the claim. □

Combining the formulas of claims 3.28, 3.29 and 3.30, we conclude that the left hand side of (3.40) evaluated at a vector field \(X\) is
\[ \left\langle b, d\rho^\vee((\sigma)^\vee)^*a, V \right\rangle - \left\langle a, d\rho^\vee((\sigma)^\vee)^*b, V \right\rangle - \left\langle a, \mathcal{L}_V(D)b \right\rangle. \] (3.50)
On the other hand, the right hand side of (3.40) applied to \(X\) equals to
\[ -2(\chi_G(\rho_M^\ast(\alpha), \rho_M^\ast(\beta), \sigma^\ast dJ(X)) + 2\phi^G(dJ\pi^\sharp(\alpha), dJ\pi^\sharp(\beta), dJ(X))) = -2(\chi_G(a, b,\sigma^\ast V) + 2\phi^G((\sigma)^\vee)^*a, (\sigma)^\vee)^*b, V \right\rangle, \] (3.51)
where we have used again that \(dJ\pi^\sharp = -((\sigma)^\vee)^*(\rho_M)^\ast\).
To conclude the proof of the lemma, it suffices to show that \(\chi_G\) and \(\phi^G\) are related as follows.

Claim 3.31 For all \(a, b \in \mathfrak{g}^\ast\) and all vector fields \(V\) on \(G\), one has
\[ \frac{1}{2}\chi_G(a, b, \sigma^\ast V) + \phi^G((\sigma)^\vee)^*a, (\sigma)^\vee)^*b, V \right\rangle = \frac{1}{4}\left(\left\langle a, d\rho^\vee((\sigma)^\vee)^*b, V \right\rangle - \left\langle b, d\rho^\vee((\sigma)^\vee)^*a, V \right\rangle \right) + \left\langle a, \mathcal{L}_V(D)b \right\rangle. \] (3.52)
Proof: It suffices to prove (3.52) on elements of type
\[ a = u^\vee, b = v^\vee, V = w_r, \]
where \(u, v, w \in \mathfrak{g}\), and we recall that \(u^\vee \in \mathfrak{g}^\ast\) denotes the dual of \(u\) with respect to the quadratic form, and \(w_r\) is the vector field on \(G\) obtained from \(w\) by right translations. We will also denote
by Ad(u) ∈ C∞(G, g) the function g → Adg(u), and we define Ad−1(u) similarly. We will need the explicit formulas for σ* and (σ*)*:

\[ \sigma^*(w_r) = \frac{1}{2} (w + Ad^{-1}(w))_g, \quad (\sigma^*)^*(u^\vee) = \frac{1}{2} (u_r + u_l). \]

Using these formulas, combined with the invariance of χG and φG, the formula ut = Ad(ut), and the explicit formulas for χG and φG on elements of g, one can check that the left hand side of (3.52) is

\[ \frac{1}{8} \left( ([u, v], w + Ad^{-1}(w))_g + ([u + Ad(u), v + Ad(v)], w)_g \right). \quad (3.53) \]

Since ρ\vee is the difference between the right and left Maurer-Cartan forms on G, we have

\[ (d\rho^\vee)(u_r, v_r) = [u, v] + Ad^{-1}[u, v]. \]

By the invariance of (·, ·)g with respect to Ad, we get that

\[ \langle a, d\rho^\vee((\sigma^*)^*b, V) \rangle = \frac{1}{2} \left( ([u, v, w] + Ad^{-1}([w, v]))_g + (u, [Ad(v), w] + [v, Ad^{-1}(w)])_g \right) \]

\[ = \frac{1}{2} \left( [u + Ad(u), v + Ad(v)], w \right)_g. \quad (3.54) \]

Since D = \frac{1}{2}(Ad − Ad−1), and L_{w_r}(D)(v) = [w, Ad(v)] + Ad−1([w, v]), it follows that

\[ \langle a, L_V(D)b \rangle = -\frac{1}{2} \left( ([u, Ad(v)], w)_g + ([Ad(u), v], w)_g \right). \]

Hence the right hand side of (3.52) equals to

\[ \frac{1}{8} \left( 2([u + Ad(u), v + Ad(v)], w)_g - ([u, Ad(v)], w)_g - ([Ad(u), v], w)_g \right), \]

which is easily seen to coincide with (3.53). This concludes the proof of the claim. □

Using Claim 3.31, we conclude that (3.50) and (3.51) coincide, and this proves Lemma 3.27 □

From Lemmas 3.25, 3.26 and 3.27, it follows that s is a φG-IM form for A. To conclude that L = Im(r, s) is a Dirac structure, we must still prove that L has rank n = dim(M). This follows from the next lemma (which also serves as inspiration for the proof of Prop. 3.20).

**Lemma 3.32** The sequence

\[ 0 → T^*G → j → A \overset{(r,s)}{→} L → 0 \]

is exact, where \( j(a) = (-J^*a, \sigma^\vee(a)) \), \( a ∈ T^*G \).

**Proof:** The fact that \((r, s) ∘ j = 0\) is equivalent to (3.22) and the second formula in (3.30). We define the maps

\[ U : A → T^*G, \quad U(α, v) = -\frac{1}{4} (ρ^\vee)^*ρ_M^*(α) + σ(v), \quad (3.55) \]

23
\[
i : L \rightarrow A, \quad i(X, \alpha) = (\alpha, \frac{1}{4} \rho^\vee d J(X)). \tag{3.56}
\]

We claim that
\[U \circ j = \text{Id}, \quad (r, s) \circ i = \text{Id}, \quad \text{and} \quad j \circ U + i \circ (r, s) = \text{Id}, \tag{3.57}
\]
and these identities imply that the sequence is exact. For the first identity in (3.57), write
\[U(j(a)) = \frac{1}{4} \rho^\vee \rho_M^* a + \sigma \sigma^\vee a,
\]
and then, using that \(dJ \rho_M = \rho\) and (3.26), we see that \(U \circ j = \text{Id}\). The second identity is immediate from the first and the last ones. To prove the last identity, we evaluate \(j(U(\alpha, v))\):
The first component gives
\[-J^*( -\frac{1}{4} (\rho^\vee)^* \rho_M^* (\alpha) + \sigma(v)) = -J^* \sigma(v) + \alpha - (1 - \frac{1}{4}(\rho_M \rho^\vee J)^*) \alpha = -(J^* \sigma(v) + C^*(\alpha)) + \alpha \tag{3.58}
\]
The second component is
\[\sigma^\vee(-\frac{1}{4}(\rho^\vee)^* \rho_M^* \alpha + \sigma(v)). \tag{3.59}
\]
But \(\sigma^\vee(\rho^\vee)^* \rho_M^* = -\rho^\vee(\sigma^\vee)^* \rho_M^* = \rho^\vee dJ \pi^\sharp\), where we have used (3.29), and the moment map condition (3.22). Expressing \(\sigma^\vee\sigma\) using (3.25), we see that (3.59) is
\[v - \frac{1}{4} \rho^\vee dJ(\rho_M(v) + \pi^\sharp(\alpha)). \]
Hence
\[j(U(\alpha, v)) = (\alpha, v) - (s(\alpha, v), \frac{1}{4} \rho^\vee d J r(\alpha, v)), \]
i.e. \(j \circ U + i \circ (r, s) = \text{Id}\).

This concludes the proof of Proposition 3.19.

**Proof of Proposition 3.20:**
Let \(J : (M, L) \rightarrow (G, L_G)\) be a Dirac realization. Identifying \(L_G\) with \(\mathfrak{g} \ltimes G\), we know that there is an induced action of \(\mathfrak{g}\) on \(M\), denoted by \(\rho_M\). Spelling out the definition, for \(v \in \mathfrak{g}\), \(\rho_M(v)\) is the unique vector field satisfying the equations in (3.32).

From the second condition in (3.32) and the fact that \(L\) is isotropic, we immediately deduce

**Lemma 3.33** For all \((X, \alpha) \in L\),
\[\rho_M^*(\alpha) + \sigma^* d J(X) = 0. \tag{3.60}
\]

Inspired by Lemma 3.32, we prove:

**Lemma 3.34** There is an exact sequence
\[
0 \longrightarrow L \overset{i}{\longrightarrow} T^* M \oplus \mathfrak{g} \overset{U}{\longrightarrow} T^* G \longrightarrow 0,
\]
where \(U\) and \(i\) are given by (3.55) and (3.56), respectively.
Proof: First of all, $U$ is surjective since, as in Lemma 3.32 (and keeping the same notation), $U \circ j = \mathrm{Id}$. Next, $U \circ i = 0$ is an immediate consequence of (3.28) and (3.60). Finally, using the nondegeneracy condition (3.19) for a Dirac realization, it follows that $i$ is injective. By a dimension argument, it follows that the sequence is exact. \hfill \Box

We now concentrate on constructing the quasi-Poisson bivector field $\pi$. Following (ii) of Prop. 3.20, we have

Claim 3.35 $\pi^\sharp$ is well defined. 

Proof: We first show that (3.33) and (3.34) have a solution $X$, for any given $\alpha$: the point is that the element

$$(-C^*(\alpha), \frac{1}{4} \rho^\vee(\sigma^\vee)^* \rho_M^*(\alpha))$$

is in the kernel of $U$: this is a simple computation using (3.26) and (3.29). Hence it must be in the image of $i$. More explicitly, we find that there exists an $X$ such that

$$(X, C^*(\alpha)) \in L, \quad \rho^\vee(dJ(X) + (\sigma^\vee)^* \rho_M^*(\alpha)) = 0. \quad (3.61)$$

On the other hand, applying Lemma 3.33 to $(X, C^*(\alpha))$, and then using the first equation in (3.30) to replace $C \rho_M$, we find that

$$\sigma^*(dJ(X) + (\sigma^\vee)^* \rho_M^*(\alpha)) = 0. \quad (3.62)$$

Since $\mathrm{Ker}(\sigma^*) \cap \mathrm{Ker}(\rho^\vee) = 0$, equations (3.61) and (3.62) imply (3.33). The uniqueness of $X$ follows from the nondegeneracy condition (3.19). \hfill \Box

Claim 3.36 $\pi^\sharp$ defines a bivector field $\pi$ which satisfies the moment map condition (3.22).

Proof: We have to show that

$$\alpha(\pi^\sharp(\beta)) + \beta(\pi^\sharp(\alpha)) = 0$$

for all 1-forms $\alpha$ and $\beta$. Let $X = \pi^\sharp(\alpha)$ and $Y = \pi^\sharp(\beta)$. Using (3.34) for $(\alpha, X)$ and $(\beta, Y)$, the fact that $L$ is isotropic, and the definition of $C$, we find that

$$4(\alpha(Y) + \beta(X)) = \alpha(\rho_M \rho^\vee dJ(Y)) + \beta(\rho_M \rho^\vee dJ(X)). \quad (3.63)$$

Let us show that the right hand side of (3.63) is zero: using (3.33), (3.63) becomes:

$$\alpha(\rho_M \rho^\vee(\rho_M \sigma^\vee)^*(\beta)) + \beta(\rho_M \rho^\vee(\rho_M \sigma^\vee)^*(\alpha)),$$

and this is zero due to (3.29). On the other hand, (3.33) shows that $dJ \pi^\sharp = -\rho_M \sigma^\vee$; dualizing it (and using $(\pi^\sharp)^* = -\pi^\sharp$, which holds by the first part of the lemma), we obtain the moment map condition. \hfill \Box

Claim 3.37 The bivector field $\pi$ is $\mathfrak{g}$-invariant.
Proof: We have to show that \( \mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)) = \pi^\sharp(\mathcal{L}_{\rho_M(v)}(\alpha)) \) for \( v \in \mathfrak{g} \), and 1-forms \( \alpha \). For that, it suffices to show that \( \mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)) \) satisfies (3.33) and (3.34).

\[
\begin{align*}
&dJ(\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha))) = -(\rho_M \sigma^\vee) \mathcal{L}_{\rho_M(v)}(\alpha), \\
&(\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)), C^* \mathcal{L}_{\rho_M(v)}(\alpha)) \in L
\end{align*}
\]

These conditions are related to Lemma 3.26. Let us first prove (3.65). Using (3.34), (3.32), and the fact that \( L \) is isotropic, we conclude that

\[
([\rho_M(v), \pi^\sharp(\alpha)], \mathcal{L}_{\rho_M(v)}(C^* \alpha) - \mathcal{L}_{\pi^\sharp(\alpha)}(J^* \sigma(v)) + d \langle J^* \sigma(v), \pi^\sharp(\alpha) \rangle + i_{\rho_M(v) \wedge \pi^\sharp(\alpha)}(J^* \phi_G) \) \in L.
\]

Using Lemma 3.26, we see that this expression is precisely \( (\mathcal{L}_{\rho_M(v)}(\pi^\sharp(\alpha)), C^* \mathcal{L}_{\rho_M(v)}(\alpha)) \).

Formula (3.64) is closely related to the one in Lemma (3.26): the proofs are similar and hold under the same hypothesis (which might be a bit surprising since (3.64) says that, although the invariance condition on \( \pi \) is not assumed, it must be satisfied modulo the kernel of \( J \)). Since we have omitted the proof of Lemma 3.26, we will give the details for (3.64). We evaluate both sides of (3.64) on an arbitrary 1-form \( \mu \in \Omega^1(G) \). The left hand side gives

\[
\begin{align*}
\langle J^* \mu, [\rho_M(v), \pi^\sharp(\alpha)] \rangle &= d \langle J^* \mu(\rho_M(v), \pi^\sharp(\alpha)) \rangle + \mathcal{L}_{\rho_M(v)} \langle J^* \mu, \pi^\sharp(\alpha) \rangle - \mathcal{L}_{\pi^\sharp(\alpha)} \langle J^* \mu, \rho_M(v) \rangle \\
&= - \langle d \mu(\rho(v), (\sigma^\vee)^* \rho^\sharp_M(\alpha)) - \mathcal{L}_{\rho_M(v)} \langle \mu, (\sigma^\vee)^* \rho^\sharp_M(\alpha) \rangle \rangle + L_{(\sigma^\vee)^* \rho^\sharp_M(\alpha)} \langle \mu, \rho(v) \rangle
\end{align*}
\]

Evaluating \( \mu \) on the right hand side, we get

\[-\langle \mathcal{L}_{\rho_M(v)}(\alpha), \rho_M \sigma^\vee \mu \rangle = - \langle \mathcal{L}_{\rho_M(v)} \langle \alpha, \rho_M \sigma^\vee \mu \rangle \rangle + \langle \alpha, [\rho_M(v), \rho_M \sigma^\vee \mu] \rangle.
\]

Now, using \([\rho_M(v), \rho_M(\tilde{v})] = \rho_M([v, \tilde{v}]) + \rho_M \mathcal{L}_{\rho_M(v)}(\tilde{v}) \) for \( \tilde{v} = \sigma^\vee \mu \in C^\infty(M, \mathfrak{g}) \), we get

\[-\mathcal{L}_{\rho_M(v)} \langle \rho^\sharp_M(\alpha), \sigma^\vee \mu \rangle + \langle \rho^\sharp_M(\alpha), [v, \sigma^\vee \mu] \rangle + \langle \rho^\sharp_M(\alpha), \mathcal{L}_{\rho_M(v)}(\sigma^\vee \mu) \rangle.
\]

We have to show that this coincides with r.h.s. of (3.66). Comparing the two formulas, we see that the resulting equation makes sense for \( \rho^\sharp_M(\alpha) \) replaced by any element in \( C^\infty(M, \mathfrak{g}^*) \). On the other hand, since the equation is \( C^\infty(M) \)-linear with respect to this element, we may assume that the element is a constant \( a \in \mathfrak{g}^* \) (and the remaining appearances of \( \rho_M \) become \( \rho \)). The identity to be proven, relating the r.h.s of (3.66) and (3.67), becomes

\[-(d \mu)(\rho(v), (\sigma^\vee)^* a) - \mathcal{L}_{\rho(v)} \langle \mu, (\sigma^\vee)^* a \rangle + L_{(\sigma^\vee)^* a} \langle \mu, \rho(v) \rangle = \langle a, [v, \sigma^\vee \mu] \rangle,
\]

or, equivalently,

\[-\mu([\rho(v), (\sigma^\vee)^* a]) = \langle a, [v, \sigma^\vee \mu] \rangle.
\]

We may assume that \( \mu \) is the dual (with respect to the quadratic form) of the vector field \( w_r \) for some \( w \in \mathfrak{g} \), and that \( a \) is the dual of an element \( u \in \mathfrak{g} \). Equation (3.68) becomes (after multiplying by 2):

\[-(w_r, [v_r - v_l, u_r + u_l]) = (u, [v, w + \text{Ad}^{-1}(w)]),
\]

and this can be proven to hold from the invariance of the quadratic form and the identities \([v_l, u_l] = -[v, u_l] \) (see (4.1) for the convention), \([v_r, u_l] = [v_l, u_r] \).

\[\square\]
Claim 3.38 The bivector field $\pi$ is a quasi-Poisson tensor.

**Proof:** We must show that $\pi^2(\alpha, \beta) = [\pi^2(\alpha), \pi^2(\beta)] + \frac{1}{2}i_{\alpha \wedge \beta}(\rho_M(\chi G))$. Using the definition of $\pi^2$ (ii) of Prop. 3.20) evaluated at $[\alpha, \beta]$, we have to show that

$$J([(\pi^2(\alpha), \pi^2(\beta)] + \frac{1}{2}i_{\alpha \wedge \beta}(\rho_M(\chi G))) = -(\rho_M\sigma^\gamma)[\alpha, \beta], \quad (3.69)$$

$$((\pi^2(\alpha), \pi^2(\beta)] + \frac{1}{2}i_{\alpha \wedge \beta}(\rho_M(\chi G)), C^*(\alpha, \beta)) \in L. \quad (3.70)$$

Similarly to the discussion in the previous claim, these conditions are related to Lemma 3.27. The first equation holds under the same assumptions, and it is proven by the same method, so it will be left to the reader (similar to the discussion in the previous proof, the equation tells us that, although the quasi-Poisson condition is not assumed, it must be satisfied modulo the kernel of $J$).

We now prove (3.70). First, we use that $(\pi^2(\alpha), C^*(\alpha)) \in L$, $(\pi^2(\beta), C^*(\beta)) \in L$, the fact that $L$ is isotropic, and then apply the formula in Lemma 3.27, to conclude that

$$([[\pi^2(\alpha), \pi^2(\beta)], C^*(\alpha, \beta)] - \frac{1}{2}J^*\sigma_i^\gamma\rho_M(\alpha \wedge \beta)(\chi G)) \in L. \quad (3.71)$$

On the other hand, applying the action in (3.32) to $v = i_{\rho_M(\alpha \wedge \beta)(\chi G)}$ and observing that $\rho_M(v) = i_{\alpha \wedge \beta}(\rho_M(\chi G))$, we find that

$$(i_{\alpha \wedge \beta}(\rho_M(\chi G)), J^*\sigma_i^\gamma\rho_M(\alpha \wedge \beta)(\chi G)) \in L. \quad (3.72)$$

Since $L$, at each point, is a vector space, (3.71) and (3.72) imply (3.70). \qed

4 Moment maps in Dirac geometry: the global picture

4.1 Integrating Lie algebroids and infinitesimal actions

Lie groupoids are the global counterparts of Lie algebroids. In order to fix our notation, we recall that a Lie groupoid over a manifold $M$ consists of a manifold $\mathcal{G}$ together with surjective submersions $t, s : \mathcal{G} \rightarrow M$, called **target** and **source**, a partially defined multiplication $m : \mathcal{G}^2 \rightarrow \mathcal{G}$, where $\mathcal{G}^2 := \{(g, h) \in \mathcal{G} \times \mathcal{G} | s(g) = t(h)\}$, a **unit section** $\varepsilon : M \rightarrow \mathcal{G}$ and an **inversion** $\mathcal{G} \rightarrow \mathcal{G}$, all related by the appropriate axioms, see e.g. [8]. To simplify our notation, we will often identify an element $x \in M$ with its image $\varepsilon(x) \in \mathcal{G}$.

For a Lie groupoid $\mathcal{G}$, the associated Lie algebroid $A(\mathcal{G})$ consists of the vector bundle

$$\ker(ds)|_M \rightarrow M, \quad (4.1)$$

with anchor $\rho = dt : \ker(ds)|_M \rightarrow TM$ and bracket induced from the Lie bracket on $\mathcal{X}(\mathcal{G})$ via the identification of sections $\Gamma(\ker(ds)|_M)$ with right-invariant vector fields on $\mathcal{G}$ tangent to the $s$-fibers.

An **integration** of a Lie algebroid $A$ is a Lie groupoid $\mathcal{G}$ together with an isomorphism $A \cong A(\mathcal{G})$. Unlike Lie algebras, not every Lie algebroid admits an integration, see [14] for a description of the obstructions. On the other hand, if a Lie algebroid is integrable, then there exists a canonical source-simply-connected integration $\mathcal{G}(A)$, see [14].

If $M$ is a point, then a Lie groupoid over $M$ is a Lie group, and the associated Lie algebroid is its Lie algebra.
Example 4.1 (Transformation Lie groupoids)

Let $G$ be a Lie group acting from the left on a manifold $M$. The associated **transformation Lie groupoid**, denoted by $G \ltimes M$, is a Lie groupoid over $M$ with underlying manifold $G \ltimes M$, source map $s(g, x) = x$, target map $t(g, x) = g \cdot x$, and multiplication

$$(g, x) \cdot (g', x') = (gg', x').$$

In this case, $A(G \ltimes M) = g \ltimes M$, the transformation Lie algebroid associated with the infinitesimal action of $g$ on $M$ corresponding to the given $G$-action. (However, even if a $g$-action does not come from a global action of a Lie group, one can always find a Lie groupoid integrating the transformation Lie algebroid $g \ltimes M$, see [16, 26].)

Similarly to infinitesimal actions, Lie groupoids act on maps into their identity sections: if $G$ is a Lie groupoid over $M$, then a (left) action of $G$ on a map $J : N \to M$ is a map $m_N : G \times M N \to N$, $(g, y) \mapsto g \cdot y$, satisfying

1. $J(g \cdot y) = t(g)$,
2. $(gg')y = g(g'y)$,
3. $J(y) \cdot y = y$.

Here $G \times_M N := \{(g, y) \in G \times N | s(g) = J(y)\}$. For reasons that will be clear in the next two subsections, the map $J : N \to M$ is often referred to as the **moment map** of the action $m_N$ [25].

Example 4.2 (Actions of transformation Lie groupoids)

Analogously to Example 3.7, an action $m_N$ of a transformation Lie groupoid $G = G \ltimes M$ on a map $J : N \to M$ is equivalent to an ordinary action $\overline{m}_N$ of the Lie group $G$ on $N$ for which $J$ is $G$-equivariant. Indeed, $m_N$ and $\overline{m}_N$ are related by

$$m_N((g, J(y)), y) = \overline{m}_N(g, y), \quad \text{where } g \in G \text{ and } y \in N. \quad (4.2)$$

The link between infinitesimal and global actions is based on the following notion: An infinitesimal action $\rho_N$ of a Lie algebroid $A$ is called **complete** if $\rho_N(\xi) \in \mathfrak{X}(N)$ is a complete vector field whenever $\xi \in \Gamma(A)$ has compact support. As in the case of Lie algebras, a complete action of a Lie algebroid $A$ can be integrated to an action of its canonical source-simply-connected integration $G(A)$, see e.g. [26].

### 4.2 Poisson maps as moment maps for symplectic groupoid actions

A 2-form $\omega$ on a Lie groupoid $\mathcal{G}$ is called **multiplicative** if the graph of the groupoid multiplication $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is an isotropic submanifold of $(\mathcal{G}, \omega) \times (\mathcal{G}, \omega) \times (\mathcal{G}, -\omega)$. Equivalently, the multiplicativity condition for $\omega$ can be written as

$$m^* \omega = \text{pr}_1^* \omega + \text{pr}_2^* \omega, \quad (4.3)$$

where $\text{pr}_i : \mathcal{G}^{(2)} \to \mathcal{G}$, $i = 1, 2$, are the canonical projections. A **symplectic groupoid** [29] is a Lie groupoid together with a multiplicative symplectic form.

Symplectic groupoids are the global counterparts of Poisson manifolds in the following sense: If $\pi$ is a Poisson structure on a manifold $P$ inducing an integrable Lie algebroid structure on
A = T^*P (as in Section 2.2), then the associated source-simply-connected groupoid \( \mathcal{G}(P) := \mathcal{G}(A) \) carries a natural multiplicative symplectic structure [9, 15, 23]; on the other hand, on any symplectic groupoid \((\mathcal{G}, \omega)\) over a manifold \(P\), condition (4.3) automatically implies that \(P\) has an induced Poisson structure uniquely determined by the condition that the target map \(t : \mathcal{G} \to P\) (resp. source map \(s : \mathcal{G} \to P\)) is a Poisson map (resp. anti-Poisson map) [11].

An integration of a Poisson manifold \((P, \pi)\) is a symplectic groupoid \((\mathcal{G}, \omega)\) over \(P\) for which the induced Poisson structure coincides with \(\pi\). Note that the symplectic form \(\omega\) defines a vector bundle map
\[
\ker(d\omega|_P) \longrightarrow T^*P, \quad \xi \mapsto i_\xi \omega|_{TP}
\]
(4.4) inducing an isomorphism of Lie algebroids \(A(\mathcal{G}) \cong T^*P\) [11]. This immediately implies that \(\dim(\mathcal{G}) = 2 \dim(P)\).

Example 4.3 (Integrating Lie-Poisson structures)

Let us consider \(g^*\), equipped with its Lie-Poisson structure. If \(G\) is a Lie group with Lie algebra \(g\), then the transformation groupoid \(\mathcal{G} = G \times g^*,\) with respect to the coadjoint action, integrates \(T^*g^* = g \times g^*\). The identification \(G \times g^* \cong T^*G\) by right translations induces a multiplicative symplectic form \(\omega\) on \(\mathcal{G}\), in such a way that \((\mathcal{G}, \omega)\) is a symplectic groupoid integrating \(g^*\).

Remark 4.4 The construction of the symplectic groupoid in the previous example can be extended to the context of Poisson-Lie groups, see Remark 3.10: If \((G, \pi)\) is a simply-connected Poisson-Lie group and \(G^*\) is its dual, then, assuming that the dressing action is complete, the transformation groupoid \(G \times G^*\) carries a symplectic structure making it into a symplectic groupoid integrating \(G^*\). (This symplectic structure is basically the one associated with the semi-direct product Poisson structure on \(G \times G^*\) induced from the action of \(G\) on itself by right multiplication.) For a more general construction when the actions are not complete, see [22].

Let us assume that \(P\) is an integrable Poisson manifold. We have seen that any Poisson map \(J : Q \to P\) induces a Lie algebroid action of \(T^*P\) on \(Q\). Analogous to the case of Lie algebras, when this action is complete, it can be “integrated” to an action of \(\mathcal{G}(P)\), the canonical source-simply-connected symplectic groupoid of \(P\). We remark that the completeness of the \(T^*P\) action in the Lie algebroid sense coincides with the notion of \(J : Q \to P\) being complete as a Poisson map, i.e., if \(J \in C^\infty(P)\) has compact support (or if \(x_J\) is complete), then \(x_J \cdot (J)\) is complete.

The global action \(m_N : \mathcal{G}(P) \times_P Q \to Q\) arising in this way is compatible with the Poisson structure on \(Q\) in the sense that \(\text{graph}(m_N)\) is a lagrangian submanifold of \((\mathcal{G}(P), \pi) \times (Q, \pi_Q) \times (Q, -\pi_Q)\), where \(\pi\) is the Poisson structure associated with the symplectic form \(\omega\) on \(\mathcal{G}(P)\). Since inclusions of symplectic leaves of Poisson manifolds are Poisson maps, an equivalent way to express this compatibility is that the restricted action \(m_N : \mathcal{G}(P) \times_P S \to S\) to each symplectic leaf \((S, \omega_S) \hookrightarrow (Q, \pi_Q)\) satisfies
\[
m_N^*\omega_S = \text{pr}_P^*\omega + \text{pr}_S^*\omega_S,
\]
where \(\text{pr}_Q : \mathcal{G}(P) \times_P S \to \mathcal{G}(P)\) and \(\text{pr}_S : \mathcal{G}(P) \times_P S \to S\) are the natural projections, see [25, 32]. On the other hand, if \((Q, \pi_Q)\) is a Poisson manifold and \(m_N\) is an action of a symplectic groupoid \(\mathcal{G}\) on \(J : Q \to P\) compatible with \(\pi_Q\) in the sense just described, then \(J\) is automatically a Poisson map (this is just a leafwise version of [25, Thm. 3.8]).

The next example is the global version of Example 3.8.

\[\text{A submanifold } C \text{ of a Poisson manifold } (P, \pi) \text{ is lagrangian if, at each } x \in P, \text{ the intersection of } T_xC \text{ with } \pi(T^*_xP), \text{ the tangent space to the symplectic leaf at } x, \text{ is a lagrangian subspace of } \pi(T^*_xP).\]
Example 4.5 (Global hamiltonian actions)

Consider $\mathfrak{g}^*$ with its Lie-Poisson structure, and let $G$ be the simply-connected Lie group with Lie algebra $\mathfrak{g}$. As in Example 3.8, the starting point is a Poisson map $J : Q \to \mathfrak{g}^*$. Note that $J$ is complete as a Poisson map if and only if the associated infinitesimal $\mathfrak{g}$-action is by complete vector fields. In this case, the global action of the symplectic groupoid $T^*G \cong G \ltimes \mathfrak{g}^*$ is equivalent, in the sense of Example 3.7, to the hamiltonian $G$-action obtained by integrating the infinitesimal hamiltonian $\mathfrak{g}$-action on $Q$.

So, in the previous example, the “moment” $J : Q \to \mathfrak{g}^*$ for the symplectic groupoid action of $T^*G$ is just a momentum map for a hamiltonian $G$-action in the ordinary sense.

Remark 4.6 Analogously to the previous example and following Remarks 3.10 and 4.4, a Poisson map $J : Q \to G^*$, where $G^*$ is the dual group to a complete simply-connected Poisson Lie group, can be “integrated” to an action of the symplectic groupoid $G \ltimes G^*$, which is equivalent to a $G$-action on $Q$ for which $J$ is equivariant (with respect to the dressing action on $G^*$). The “moment” $J$ in this case coincides with Lu’s momentum map [21] for a Poisson action of a Poisson-Lie group on a Poisson manifold.

4.3 Dirac realizations as moment maps for presymplectic groupoid actions

In order to describe the global actions “integrating” Dirac realizations, we should first identify the global objects integrating Dirac manifolds, generalizing symplectic groupoids. This was done in [6]: if $\phi$ is a closed 3-form on $M$, then a $\phi$-twisted presymplectic groupoid over $M$ is a Lie groupoid $G$ over $M$ equipped with a multiplicative 2-form $\omega$ such that

1. $d\omega = s^*\phi - t^*\phi$,
2. dim$(G) = 2 \dim(M)$,
3. ker$(\omega_x) \cap \ker(d_x s) \cap \ker(d_x t) = \{0\}$, for all $x \in M$.

(Twisted presymplectic groupoids are called quasi-symplectic groupoids in [33].) The multiplicativity of $\omega$ and condition 1. in this definition guarantee that the map

$$\sigma_\omega : A \to T^*M, \quad \xi \mapsto i_{\xi}\omega|_{TM} \quad (4.6)$$

is a $\phi$-IM form for $A$, while 2. and 3. are the extra-conditions needed in Lemma 3.3 to insure that the image $L$ of $(\rho, \sigma_\omega)$ is a $\phi$-twisted Dirac structure. When $(G, \omega)$ is a symplectic groupoid, such $L$ is precisely the Dirac structure associated with the induced Poisson structure on $M$. As proven in [6], $L$ is uniquely determined by the condition that $t$ is an $f$-Dirac map (resp., $s$ is an anti-$f$-Dirac map). Conversely, the canonical groupoid $G(L)$ integrating the Lie algebroid associated with a $\phi$-twisted Dirac structure (assuming it is integrable) is naturally a $\phi$-twisted presymplectic groupoid [6, Sec. 5]. This correspondence generalizes the one between Poisson manifolds and symplectic groupoids [9, 15, 23] (see also [10] for the integration of twisted Poisson structures).

We now have all the ingredients to generalize the “integration” procedure of Poisson maps to symplectic groupoid actions, explained in Section 4.2, to the context of Dirac geometry. Let $L_M$ be a $\phi$-twisted Dirac structure on $M$ associated with an integrable Lie algebroid. We call a Dirac realization $J : N \to M$ complete if the induced Lie algebroid action of $L_M$ on $N$ is complete, in which case it integrates to an action $m_N : G(L_M) \times_M N \to N$, where $(G(L_M), \omega)$
is the canonical twisted presymplectic groupoid associated with \( L_M \). In this situation, we will simply say that the action \( m_N \) integrates the realization \( J \).

**Theorem 4.7** Let \((M, L_M)\) be a \( \phi \)-twisted Dirac manifold and assume that \( L_M \) is integrable. A complete Dirac realization \( J : N \to M \) integrates to an action \( m_N : \mathcal{G}(L_M) \times_M N \to N \) satisfying

\[
m_N^* L_N = \tau_{pr_N^* \omega}(pr_N^* L_N),
\]

where \( pr_\boldsymbol{\varepsilon} \) and \( pr_N \) are the projections from \( \mathcal{G}(L_M) \times_M N \) onto \( \mathcal{G}(L_M) \) and \( N \), respectively, and \( \tau_{pr_N^* \omega} \) denotes a gauge transformation.

Conversely, if \( m_N \) is an action of \( \mathcal{G}(L_M) \) on \( J : N \to M \) satisfying (4.7), then \( J \) is f-Dirac; if \( J \) also satisfies (3.19), then it is a Dirac realization whose integration is \( m_N \).

In order to prove the theorem, we need the following result.

**Lemma 4.8** Let \((M, L_M)\) be a \( \phi \)-twisted Dirac manifold and assume that \( L_M \) is integrable. Let \( m_N : \mathcal{G}(L_M) \times_M N \to N \) be an action of \( \mathcal{G}(L_M) \) on \( J : N \to M \), and assume that \( N \) is equipped with a \( J^* \phi \)-twisted presymplectic form \( \omega_N \). Then \( J \) is an f-Dirac map if and only if

\[
m_N^* \omega_N = \text{pr}_N^* \omega + \text{pr}_s^* \omega.
\]

**Proof:** To simplify the notation, let \( \mathcal{G} = \mathcal{G}(L_M) \), and let us denote by \( A \) the corresponding Lie algebroid (which is just \( L_M \)). The source and target maps in \( \mathcal{G} \) are denoted by \( s \) and \( t \). Also, let \( \omega_1 = m_N^* \omega_N - \text{pr}_N^* \omega_N \) and \( \omega_2 = \text{pr}_s^* \omega \). With these definitions, our goal is to show that \( J \) is f-Dirac if and only if \( \omega_1 = \omega_2 \).

The key observation is that if we regard \( \mathcal{G} \times_M N \) as a transformation Lie groupoid over \( N \), with source \( \text{pr}_N \) and target \( m_N \), a direct computation shows that both \( \omega_1 \) and \( \omega_2 \) are multiplicative. Hence, by [6, Thm. 2.5], \( \omega_1 = \omega_2 \) if and only if the corresponding bundle maps

\[
\sigma_{\omega_i} : A \times_M N \to T^* N, \quad \xi_y \mapsto \sigma_{\omega_i} = (i_{\xi_y} \omega_i)|_N
\]

for \( i = 1, 2 \), see (4.6), coincide. For \( \xi_y \in A \times_M N \) (\( \xi \in A_x \) and \( x = J(y) \)) and \( Y \in TN \) (as usual, we identify \( TN \) with \( T\mathcal{G}(N) \), where \( \varepsilon : N \to \mathcal{G} \times_M N \) is the identity section), we have

\[
\sigma_{\omega_1}(\xi_y, Y) = \omega_N(dm_N(\xi_y), Y) \quad \text{and} \quad \sigma_{\omega_2}(\xi_y, Y) = \omega(\xi, dJ(Y)).
\]

For the first identity in (4.9), we used that \( i_{\xi_y} \text{pr}_N^* \omega_N = 0 \) for \( \xi_y \in A \times_M N \), since \( \text{pr}_N \) is the source map in \( \mathcal{G} \times_M N \); and \( A \times_M N \) is its Lie algebroid, which is tangent to the source fibres along the identity section.

Since \( L_M = \{(dx/\xi) \mid \xi \in A \}, J : N \to M \) being f-Dirac means that

\[
\{(dx/\xi) \mid \xi \in A \} = \{(dJ(\xi), \alpha) \mid \xi \in A \} = \{(dJ(Y), \alpha) \mid i_Y \omega_N = J^* \alpha \}.
\]

But, for \( \xi_y \in A \times M N \), we have \( dJ(dm_N(\xi_y)) = dt(\xi) \). It then follows from (4.10) that

\[
\omega(\xi, dJ(Y)) = \omega_N(dm_N(\xi_y), Y)
\]

for all \( Y \in TN \), which implies that \( \sigma_{\omega_1} = \sigma_{\omega_2} \), i.e., \( \omega_1 = \omega_2 \).

The converse follows from the same arguments, reversing the steps. 

We can now prove Theorem 4.7:
Proof: We keep writing \( G \) for \( G(L_M) \). Suppose that \( m_N \) integrates a Dirac realization \( J : N \to M \). The bundles \( m^*_NL_N \) and \( pr^*_NL_N \), seen as subbundles of \( T(G \times_M N) \oplus T^*(G \times_M N) \), have the same projection onto the first factor: at a point \((g, y)\), they both coincide with \( T_gG \times_T yO \), where \( O \) is the leaf of \( L_N \) through \( y \). Note that, since \( pr_N \) is a submersion, \( pr^*_NL_N \) is a smooth subbundle, so it is an honest Dirac structure.

By Corollary 3.13, since \( J \) is a Dirac realization of \( M \), its restriction to any leaf of \( L_N \), \((O, \theta)\), is a presymplectic realization, and \( m_N \) is tangent to the leaves. By Lemma 4.8,

\[
m^*_N\theta = pr^*_N\theta + pr^*_N\omega,
\]

which implies the compatibility (4.7).

Conversely, (4.7) implies that \( J \) is tangent to the leaves of \( L_N \). Restricting \( m_N \) to these leaves, (4.7) amounts to (4.8). So, by Lemma 4.8, \( J \) is an f-Dirac map when restricted to each leaf, which implies that \( J \) is f-Dirac by Corollary 3.6. The last statement follows from Corollary 3.13 and a direct check.

Remark 4.9 The presymplectic groupoid actions resulting from presymplectic realizations are exactly the “modules” considered in the Morita theory developed in [33] to compare various notions of moment maps. More general Dirac realizations give rise to more general “hamiltonian spaces” which still fit with the constructions in [33].

We will discuss examples of the “integration” in Theorem 4.7 related to “quasi” hamiltonian actions in Section 4.5.

4.4 Reduction in Dirac geometry

Just as in Poisson geometry, one can also carry out reduction in the context of Dirac manifolds. The general construction described in this section recovers reduction procedures in various settings, including [2, 3, 25, 33].

The set-up is as follows. Let \( J : N \to M \) be a Dirac realization of a \( \phi \)-twisted Dirac manifold \((M, L_M)\). Let \( x \in M \) be a regular value of \( J \), and consider the submanifold \( \iota : C = J^{-1}(x) \hookrightarrow N \). Following [25, 33], let \( \mathfrak{g}_x = \ker(\rho)_x \) be the isotropy Lie algebra of \( L_M \) at \( x \). Since the anchor \( \rho \) is the projection \( pr_1|_{L_M} \), it follows that

\[
\mathfrak{g}_x = (L_M \cap T^*M)_x.
\]

The induced Lie algebroid action of \( L_M \) on \( J : N \to M \) defines a vector bundle morphism \( L_M \times_M N \to TN \), and a simple computation shows that this morphism gives rise to an action of the Lie algebra \( \mathfrak{g}_x \) on \( C \). Our object of interest is the orbit space \( C/\mathfrak{g}_x \).

Lemma 4.10 If the stabilizer algebras of the \( \iota_x \)-action on \( C \) have constant dimension (on each component), then \( \iota^*L_N \) is a (untwisted) Dirac structure on \( C \).

Proof: As mentioned in Section 3.2, the conclusion in the lemma holds as long as we show that \( \iota^*L_N \) is a smooth subbundle of \( TC \oplus T^*C \).

As a vector bundle, \( \iota^*L_N \) is naturally identified with

\[
\frac{L_N \cap (TC \oplus T^*N)}{(L_N \cap TC^c)},
\]

(4.12)
see [12], and \( \iota^*L_M \) will be smooth if we show that both bundles in (4.12) are smooth. For that, it suffices to show that each one has constant dimension. But since their quotient \( \iota^*L_N \) has constant dimension, it suffices to show that either \( L_N \cap (TC \oplus T^*N) \) or \( L_N \cap TC^o \) has constant dimension. We will prove that for \( L_N \cap TC^o \).

On one hand,

\[
L_N \cap TC^o = \{(0, \beta) \in L_N \mid \iota^*\beta = 0 \} = \{(0, dJ^*\alpha) \in L_N \mid \alpha \in T^*M \}.
\]

It follows from \( J \) being an f-Dirac map that if \( (0, dJ^*\alpha) \in L_N \), then \( (0, \alpha) \in L_M \). So, if \( \rho_N \) is the infinitesimal action of \( L_M \) on \( J \), we can write

\[
L_N \cap TC^o = \{(0, dJ^*\alpha) \in L_N \mid \alpha \in T^*M \} \equiv \frac{\ker(\rho_N) \cap L_M \cap T^*M}{\ker(dJ^*)}.
\]

But \( \ker(\rho_N) \cap L_M \cap T^*M \) is the stabilizer of the \( l_x \)-action on \( C \), which is assumed to have constant dimension. Since \( x \) is a regular value, \( dJ \) has maximal rank on \( C \), so \( \ker(dJ^*) \) also has constant dimension. As a result, the dimension of (4.12) is constant, and \( \iota^*L_N \) is a smooth bundle.

Finally, note that \( \iota^*L_N \) is a \((\iota^*J^*\phi)\)-Dirac structure on \( C \), but \( \iota^*J^*\phi = 0 \). So \( \iota^*L_N \) is an ordinary Dirac structure.

We now show that the quotient \( C/l_x \) carries a natural Poisson structure.

**Theorem 4.11** Suppose that the orbit space \( C/l_x \) is a smooth manifold so that projection \( C \rightarrow C/l_x \) is a submersion. Then there is a unique Poisson structure \( \pi_{red} \) on \( C/l_x \) for which the projection \((C, \iota^*L_N) \rightarrow (C/l_x, \pi_{red})\) is an f-Dirac map.

**Remark 4.12** The projection \((C, \iota^*L_N) \rightarrow (C/l_x, \pi_{red})\) is also a b-Dirac map, and this property characterizes \( \pi_{red} \) uniquely as well.

**Proof:** It follows from our assumptions that the \( l_x \)-orbits on \( C \) have constant dimension, so the same holds for the stabilizer algebras. By Lemma 4.10, \((C, \iota^*L_N)\) is a Dirac manifold.

The *admissible functions* on \((C, \iota^*L_N)\), i.e., the set of functions on \( C \) whose differential vanish on \( \ker(\iota^*L_N) = \iota^*L_N \cap TC \) form a Poisson algebra, see [12, Sec. 2.5], under the bracket

\[
\{f; g\} := \mathcal{L}_fg,
\]

where \( X_f \) is a local vector field such that \((X_f, df) \in \iota^*L_N \). We will show that this Poisson algebra induces a Poisson structure on \( C/l_x \) by showing that the kernel of \( \iota^*L_N \) coincides with the \( l_x \)-orbits, i.e.,

\[
\ker(\iota^*L_N) = \rho_N(l_x).
\]

On one hand,

\[
\iota^*L_N \cap TQ = \{Y \in TQ \mid \exists \beta \in T^*N \text{ with } (Y, \beta) \in L_N, \iota^*\beta = 0 \} = \{Y \in TQ \mid \exists \alpha \in T^*M \text{ with } (Y, dJ^*\alpha) \in L_N \}.
\]

But since \( J \) is f-Dirac and \( dJ(Y) = 0 \), we can write

\[
\iota^*L_N \cap TQ = \{Y \in TQ \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N \}.
\]

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On the other hand,

\[ \rho_N(l_x) = \{ Y \in TN \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N, \ dJ(Y) = 0 \} \]

\[ = \{ Y \in TQ \mid \exists \alpha \in L_M \cap T^*M \text{ with } (Y, dJ^*\alpha) \in L_N \}. \]

So (4.13) follows.

The fact that the projection \((\mathcal{C}, \iota^*L_N) \to (\mathcal{C}/l_x, \pi_{red})\) is an f-Dirac map and the claim in Remark 4.12 follow from a direct computation, see e.g. [7].

Of course, if the Dirac realization \(J : N \to M\) is complete, one can state Theorem 4.11 in terms of the action of the isotropy group of \(G(L_M)\) at \(x \in M\) on \(C = J^{-1}(x)\). Versions of Theorem 4.11 can also be derived when this action is locally free and the quotient is an orbifold, as well as for more general “intertwiner spaces” [33].

**Remark 4.13 (Other reductions)**

The following are important particular cases of the reduction in Theorem 4.11:

- If \(M\) is Poisson and \(J : N \to M\) is a symplectic realization, we recover [25, Thm.3.12]; in particular, when \(M = g^*\), this reduces to Marsden-Weinstein classical theorem [24], and when \(M = G^*\), the dual of a Poisson-Lie group, we get Lu’s reduction [21]. If \(J : N \to M\) is a Poisson map, we get the “Poisson-version” of these results.

- If \(M\) is \(\phi\)-twisted Dirac and \(J : N \to M\) is a presymplectic realization, then we obtain Xu’s reduction [33, Thm. 3.17]; in particular, when \(M\) is a Lie group equipped with Cartan-Dirac structure, one recovers the quasi-hamiltonian reduction of [3].

- If \(J : N \to G\) is a general Dirac realization of a Lie group with Cartan-Dirac structure, then we recover the reduction of quasi-Poisson manifolds of [2] via the identification established in Theorem 3.16, see Remark 4.16 below.

### 4.5 AMM-groupoids and hamiltonian quasi-Poisson G-manifolds

We now discuss global actions, in the sense of Theorem 4.7, associated with complete Dirac realizations of Cartan-Dirac structures.

Let \(G\) be a Lie groups equipped with a Cartan-Dirac structure \(L_G\) with respect to a bi-invariant nondegenerate quadratic form \((\cdot, \cdot)_g\). The first step is to identify \(G(L_G)\), the canonical presymplectic groupoid integrating \(L_G\).

As shown in [6, Sec. 7], \(G(L_G)\) is closely related to the AMM-groupoids of [5]: if \(G = G \times G\) is the transformation groupoid with respect to the conjugation action, then the 2-form [3]

\[ \omega(g,x) = \frac{1}{2} ((\text{Ad}_x p_g^*\lambda, p_g^*\lambda)_g + (p_g^*\lambda, p_g^*\lambda + \bar{\lambda})_g), \]

where \(p_g, p_x : G \times G \to G\) are the first and second projections, and \(\lambda\) and \(\bar{\lambda}\) are the left and right Maurer-Cartan forms, makes \(G\) into a \(\phi^G\)-twisted presymplectic groupoid. If \(G\) is simply-connected, then \((G, \omega)\) is isomorphic to \(G(L_G)\), the canonical source-simply-connected integration of \(L_G\). In general, \(G(L_G)\) is obtained from the AMM groupoid by pulling back \(\omega\) to \(\tilde{G} \times G\), where \(\tilde{G}\) is the universal cover of \(G\) [6, Thm. 7.6]. As a result, just as Lie-Poisson structures “integrate” to cotangent bundles of Lie groups, see Example 4.3, Cartan-Dirac structures “integrate” to the “double” \((G \times G, \omega)\) in the sense of [3].
For simplicity, let $G$ be simply connected. A complete Dirac realization $J : M \to G$ induces a presymplectic groupoid action of $(G, \omega)$, as in Theorem 4.7, which is equivalent to a $G$-action on $M$ for which $J$ is $G$-equivariant, see Example 4.2: this $G$-action is just an integration of the infinitesimal $\mathfrak{g}$-action which makes $M$ into a quasi-Poisson $\mathfrak{g}$-manifold, as constructed in Proposition 3.20. So $M$ becomes a hamiltonian quasi-Poisson $G$-manifold for which $J : M \to G$ is the group valued moment map [2]. This construction yields the following global version of Theorem 3.16.

**Theorem 4.14** There is a one-to-one correspondence between complete Dirac realizations of $(G, L_G)$ and hamiltonian quasi-Poisson $G$-manifolds.

**Corollary 4.15** There is a one-to-one correspondence between compact Dirac realizations of $(G, L_G)$ and compact hamiltonian quasi-Poisson $G$-manifolds.

Of course, a global version of Prop. 3.24 also holds.

**Remark 4.16** (Reduction)

Given a Dirac realization of $(G, L_G)$, $J : M \to G$, the Dirac reduction of Theorem 4.11 produces Poisson spaces $J^{-1}(g)/G_g$, where $G_g$ is the centralizer of $g \in G$. Using Remark 4.13 and [2, Prop. 10.6], one can check that these are the same Poisson spaces obtained by quasi-Poisson reduction [2, Thm.6.1] if we regard $M$ as a hamiltonian quasi-Poisson $G$-manifold instead.

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