Emergent classical universes from initial quantum states in a tomographical description

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Abstract

Quantum and classical physical states are represented in a unified way when they are described by symplectic tomography. Therefore this representation allows us to study directly the necessary conditions for a classical universe to emerge from a quantum state.

In the previous study on the de Sitter universe this was done by comparing the classical limit of the quantum tomograms with the tomograms resulting from the classical cosmological equations.

In this paper we first review these results and extend them to all the de Sitter models. We show further that these tomograms can be obtained directly from transposing the Wheeler-De Witt equation to the tomographic variables. Subsequently, because the classic limits of the quantum tomograms are identified with their asymptotic expressions, we found the necessary conditions to extend the previous results by taking the tomograms of the WKB solutions of the Wheeler-DeWitt equation with a any potential.

Furthermore in the previous works we found that the de Sitter models undergo the quantum-to-classical transition when the cosmological constant decays to its present value, we discuss at the end how far we can extend this result to more general models.

In the conclusions, after discussing any improvements and developments of the results of this work, we sketch a phenomenological approach from which to extract information about the initial states of the universe.

1 Introduction

Understanding the universe we live in and its origins is one of the oldest questions of mankind. Only in the last century with the theory of general relativity
has it been possible to give a scientific description of the universe and its characteristics. However the study of general relativity has not completely resolved the knowledge of its origins.

The presence of initial singularity in the theory, identified with the Big Bang, highlights how the current universe must be the result of a transition from a previous initial state which is generally traced back to an initial quantum state, although, as indicated in [1][2][3], the universe may have emerged from a previous classical state described by a static model and from which the observed characteristics can be derived.

The various stages of the evolution of the universe can be described through the transitions between different physical states, which are not always described in the same way.

The state of a physical system is the set of all the informations necessary to determine its evolution, which in turn is described a sequence of states. For example in classical physics the state of a particle is well represented by a point on the phase space, or when the particle is in thermal bath by a distribution function of probabilities of its position and momentum on a given time. In classical cosmology, the state of the universe it should be sufficient to give the expansion factor \(a(t_0)\) at some time to determine because it is a constrained system, but we must consider also the various parameters present in the in the cosmological equations, like the spatial curvature \(k\), the present mass-energy density \(\rho_0\) and the cosmological constant \(\Lambda\) and so on, to define the state of the universe as their values determine its evolution.

In Schrödinger picture of quantum mechanics, the state of a particle is given by its wave function, which has not a direct physical interpretation, but which is an auxiliary function for defining the expectation values of the various observables quantities. The evolution of a quantum system is determined by the Schrödinger equation. Similarly in quantum cosmology the state of the quantum universe is described by the wave function of the universe solution of the Wheeler-DeWitt equation, which which unlike the previous case does not have an explicit dependence on time.

The difference of the classical and quantum description does not allow an immediate comparison between these different states and therefore to describe in a simple way the quantum-classical transition of the universe. We can overcome this problem by introducing the symplectic tomographic representation [4, 5, 6, 7, 8, 9].

In this representation both quantum and classical states are described with the same family of functions. So to understand if a quantum model evolves to a classic one, we must simply consider the classic limit obtained by letting \(\hbar \to 0\) [10, 11].

Another reason which justifies the use of tomograms is that they are observables and then allow to determine in laboratory the state of a quantum system (see for example [30, 31, 32]).

In our case the phenomenological determination of the classical tomogram may contain informations of the early quantum state. The possibility of realizing this proposal is also based on Hartle’s considerations in [12] where he points
out two important issues. The first one is that in the case of total absence of informations the density matrix would take the form $\rho = I/Tr(I)$ where $I$ is the unit matrix, but as $\rho \propto \exp(-H/kT)$, it would correspond to an infinite temperature in equilibrium, which would also correspond to an infinite temperature today, contrarily to the present observations. The second issue is that even if the entropy observed $\frac{S}{T} \sim 10^{80}$ is apparently very large, it is however very small compared to the maximum value possible $\frac{S}{T} \sim 10^{120}$ as showed by Penrose[13].

In [10] and [11] quantum and classical de Sitter models were analyzed in tomographic representation.

The de Sitter models present very simple properties so that a complete analytical description was available either in the quantum framework as in the classical. Therefore their study has allowed to understand the way to face more general models. In these papers only the Hartle-Hawking model was described in tomographical terms and it was evident that it does not have a classical limit, however it was possible to exhibit a tomogram which converged to the classical one derived from general relativity. Based on more careful considerations, in this work we proved that this tomogram can be derived from Vilenkin’s initial conditions.

Furthermore, a remarkable result was that in de Sitter’s models the quantum-to-classical transition is induced by the decay of the cosmological constant. To see if this result could be extended to more general models is the basis of the motivations for this work. However we have gone beyond this problem and we have determined systematic criteria to establish the compatibility between quantum and classical models starting from asymptotic solutions of the Wheeler-DeWitt equation with a generic potential to represent an extended class of sources.

The paper is organized in the following way. In the first two sections we resume the general properties for the classical de Sitter models in sect. 2 and the quantum models in sect. 3 known from literature.

In sect. 4 we summarize the definitions and the properties of classical and quantum tomograms. In sect. 5 we take up the results of [10] and complete them with the addition of the Vilenkin and Linde tomogram calculation.

In sect. 6 we complete the work on de Sitter models by introducing a tomographic version of the Wheeler-DeWitt equation and we show that the previous results are obtained as solutions of this equation.

In sect. 7 we show that taking the classical limit $\hbar \to 0$ of the de Sitter quantum tomograms is equivalent to take the asymptotic expressions of the tomograms and first we check if they converge to the classical tomogram and then we observe that the same asymptotic limits are obtained taking the limit $\lambda \to \lambda_{today}$ implying that the classical state of the universe may be the result of the decay of the cosmological constant.

We extend these results to a more general setting. In sect. 8 we look for a general definition of the classical limit of the quantum tomograms and by considering the WKB approximation in the Wheeler-DeWitt equation with a generic potential $V(q)$ and then we calculate their tomograms. This allows us to
determine which quantum tomograms converge to the classical ones and which do not. In sect\textsuperscript{9} we consider general models with the cosmological constant.

In sect\textsuperscript{10} we discuss the results, any improvement and the perspectives of this work.

2 The de Sitter models

Albert Einstein introduced the cosmological constant in order to have a static universe as it was expected at the beginning of the XXth century, while his original theory predicted a dynamical universe. But in 1917 Willem de Sitter showed that the static model was unstable and that the cosmological constant played instead an important dynamical role even in absence of material sources.

The dynamics of the de Sitter is very simple and describes an everlasting exponential expanding universe with an initial singularity in infinity past.

The de Sitter model has taken on an important role to introduce the inflationary paradigm needed to explain first of all in a generic way the flatness and homogeneity of the universe without recurring to a fine tuning of the initial conditions. The role of cosmological constant was attributed to a false vacuum state of the inflaton (a hypothetical particle responsible for this stage of the universe), represented by a stationary state (minimum or maximum) in the potential of this field.

Presently the cosmological constant is considered together with other dark matter candidates to explain the accelerated expansion of the universe as the observations on Type 1A supernovae indicates.

The physical interpretation of the cosmological constant is related by many authors to the quantum vacuum energy. But this interpretation poses a serious problem on why the cosmological constant is so small compared with the vacuum energy expected. In his seminal paper \cite{14} S. Weinberg estimates the value of the vacuum energy which should be larger than the actual value and eventually proportional to the Planck energy density where its estimated value of $\Lambda$ is $10^{122}$ orders of degree smaller.

In this paper we describe the de Sitter spacetime with the metric \cite{15}

$$ds^2 = -\frac{N^2}{q} d\tau^2 + q \frac{1}{1 - kR^2} d\tau^2 + qr^2 d\Omega^2 \quad (2.1)$$

where respect to the conventional metrics $q = a^2$ (where $a$ is the expansion factor) and the lapse is scaled by a factor $\sqrt{q}$. In the following only the case $k = 1$ will be considered.

From the Einstein equations with cosmological constant

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad (2.2)$$

and with $N = N(t)$, we have the following equations

$$\frac{1}{4} \frac{\dot{q}^2}{N^2} + \frac{1}{q} = \frac{8\pi G}{3} T_{00} + \frac{\Lambda}{3} \quad (2.3)$$
and
\[
\frac{\ddot{q}}{N^2} + \frac{1}{2} \frac{1}{N^2} \frac{\dot{q}^2}{q} - \frac{\dot{N}}{N} \dot{q} + \frac{2}{q} = -\frac{8\pi G}{3} T + \frac{4}{3} \Lambda.
\] (2.4)

The previous equations (putting for sake of simplicity \(N = 1\)) imply the conservation equation
\[
\dot{T}_{00} = \frac{1}{2} \frac{\dot{q}}{q} (T - 4T_{00}).
\] (2.5)

de Sitter’s model contemplates only the cosmological constant for which we will pose \(T_{ab} = 0\) in the rest of the section. We notice that in the gauge \(N = \sqrt{\frac{1}{q}}\) we obtain the classical solution
\[
q(t) = \Lambda^{-1} \cosh^2(\Lambda^{1/2} t).
\] (2.6)

For numerical reasons, in the following we multiply the metric by a factor \(\sigma^2 = 2G/3\pi\).

In order to quantize the gravitational system we need to move on to the Hamiltonian formalism. First we write the action, which after integrating the spatial part and eliminating a total derivative with respect to time, is
\[
S = \frac{1}{2} \int N \left[ -\frac{\dot{q}^2}{4N} + 1 - \lambda q \right] dt
\] (2.7)
where \(\lambda\) is now the cosmological constant in Planck units. The coordinates of the phase space are \((q, p)\) with the momentum \(p\) defined by
\[
p = \frac{\partial L}{\partial \dot{q}} = -\frac{\dot{q}}{4N}
\] (2.8)
using the Legendre transform, the action takes the form
\[
S = \int [p\dot{q} - N\mathcal{H}] dt
\] (2.9)
with
\[
\mathcal{H} = \frac{1}{2} (-4p^2 + \lambda q - 1)
\] (2.10)
The lapse function \(N\) is a Lagrange multiplier, the variation
\[
\frac{\delta S}{\delta \dot{N}} = 0
\] (2.11)
implies the Hamiltonian constraint,
\[
\frac{1}{2} (-4p^2 + \lambda q - 1) = 0
\] (2.12)
which is equivalent to equation (2.3).

The state of a classical system at a time \(t_0\) is given by the values of the positions and the momenta at that time. For a constrained system this values
are restricted to a submanifold of the phase space. Eq. (2.12) describes the phase space a curve of the states that the universe is allowed to take.

To express the state of the universe as a distribution on the phase space, let us assume that the universe is composed by $N$ local subsystems with $N$ is arbitrarily large, all with the same metric (2.1), but following a statistical behavior that can be described by Boltzmann equation

$$\frac{\partial f(q,p)}{\partial t} + [f, \mathcal{H}] = 0,$$

but with

$$\frac{\partial f(q,p)}{\partial t} = 0.$$ (2.14)

Then solutions to the Boltzmann equation are the functions $f(q,p)$ which commute with $\mathcal{H}$ the most simple are just of the form $f(\mathcal{H})$.

One choice can be

$$f(q,p) = \delta \left( -4p^2 + \lambda q - 1 \right).$$ (2.15)

which is equivalent to say that there is a strict homogeneity of all the subsystems composing the universe. Another choice could be

$$f_{\text{gauss}}(q,p) = \exp \left[ - \left( -4p^2 + \lambda q - 1 \right)^2 \right]$$ (2.16)

in which there is a gaussian fluctuation of the metric. Indeed it describes a universe where there are subsystems deviating from the condition (2.12). Such deviations are equivalent to local variations of the spatial curvature $k$. Because when $-4p^2 + \lambda q - 1 \neq 0$ it must be take a value $x_0$ and then

$$-4p^2 + \lambda q - k = 0$$

and where $k = 1 + x_0$. But the following we shall work only with (2.15).

3 Quantizing the de Sitter model and the initial conditions

In quantum cosmology the states of the universe are described by the wave functions $\Psi(q)$ defined on the minisuperspace, i.e. the space of the homogeneous metrics. They are solutions of the Wheeler-DeWitt equation, which is obtained by substituting the Hamiltonian constraint with the equation

$$\hat{\mathcal{H}} \Psi(q) = 0$$ (3.1)

where $\hat{\mathcal{H}}$ is the Hamiltonian operator obtained by substituting $q$ and $p$ in (2.12) with the respective operators

$$\hat{p} \Psi = -i\hbar \frac{d\Psi}{dq}$$ (3.2)
and
\[ \hat{q}\Psi(q) = q\Psi(q). \] 

(3.3)

In the de Sitter case the Wheeler-DeWitt equation takes the form
\[ (4\hbar^2 \frac{d^2}{dq^2} + \lambda q - 1) \psi(q) = 0. \] 

(3.4)

which can be reduced to the Airy equation
\[ \frac{d^2\psi(\xi)}{d\xi^2} - \xi\psi(\xi) = 0, \] 

(3.5)

with the change of variable
\[ \xi = \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}}. \] 

(3.6)

This equation has two independent solutions \( \text{Ai}(x) \) and \( \text{Bi}(x) \) whose integral representations are respectively
\[ \text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp \left[ i \left( \frac{z^3}{3} + xz \right) \right] dz \] 

(3.7)

and
\[ \text{Bi}(x) = \frac{1}{\pi} \int_{0}^{+\infty} \left[ \exp \left( \frac{z^3}{3} + xz \right) + \sin \left( \frac{z^3}{3} + xz \right) \right] dz. \] 

(3.8)

All the solutions of the Airy equation are linear combinations of \( \text{Ai}(x) \) and \( \text{Bi}(x) \).

The different proposals given by Hartle and Hawking, Vilenkin and Linde’s are characterized by different combinations of these functions (see for example in [15], [18], [19]).

Indeed the Hartle and Hawking no boundary condition corresponds to the wave function [20]
\[ \psi_{\text{HH}} = a \text{Ai} \left( \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}} \right), \] 

(3.9)

Vilenkin’s tunneling from nothing [21] [22] is represented by the combination
\[ \psi_{V} \left( \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}} \right) = \frac{b}{2} \left( \text{Ai} \left( \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}} \right) + i \text{Bi} \left( \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}} \right) \right) \] 

(3.10)

and finally the initial condition discussed by Linde which is an extension of Vilenkin’s proposal is [23] is
\[ \psi_{L} = -ic \text{Bi} \left( \frac{1 - \lambda q}{(2\hbar\lambda)^{2/3}} \right) \] 

(3.11)

with \( a \), \( b \) and \( c \) are normalization constants.
The Airy functions oscillate for negative values of the argument (i.e. when \( q > 1/\lambda \)), while for \( q < 1/\lambda \) \( \text{Ai}(\frac{1-\lambda q}{(2\pi \lambda)^{2/3}}) \) decays rapidly and \( \text{Bi}(\frac{1-\lambda q}{(2\pi \lambda)^{2/3}}) \) grows to infinity.

These functions are better characterized by their asymptotic behavior of their oscillating part. In fact for \( q > 1/\lambda \) the Vilenkin wave function becomes

\[
\psi_V \approx b \frac{(2\pi \hbar)^{1/6}}{2[\pi^2(\lambda q - 1)]^{1/4}} \exp(-i\xi), \tag{3.12}
\]

similarly the Hartle and Hawking wave function takes the form

\[
\psi_{HH} \approx a \frac{\pi \hbar}{\pi^2(\lambda q - 1)^{1/4}} \cos \left( \xi - \frac{\pi}{4} \right), \tag{3.13}
\]

and Linde’s wave function takes

\[
\psi_L \approx ic \frac{\pi \hbar}{\pi^2(\lambda q - 1)^{1/4}} \sin \left( \xi - \frac{\pi}{4} \right), \tag{3.14}
\]

where \( \xi = \frac{2}{3}(\lambda q - 1)^{3/2} \). These three expressions are derived both formally from their analytic definitions \( (3.7) \) and \( (3.8) \) (see \cite{16} and \cite{17}), and by applying the WKB approximation to eq.(3.5) (see \cite{24}). They show that Vilenkin’s wave function \( (3.12) \) presents only expanding modes, whereas the other two wave functions present combinations of expanding and collapsing modes.

### 4 Classical and quantum states in tomographic representation

The aim of symplectic tomography is to reconstruct a physical state (classical or quantum) by using a set of marginal functions. A classical physical state is represented on the phase space by a distribution function \( f(q,p) \) solution of the Boltzmann equation. A quantum state is described by a wave function or by a Wigner function \( W(q,p) \) on the phase space. The Wigner function is called a quasi-distribution because it can take negative values, whereas a classical distribution function is always positive.

In order to represent quantum and classical state in a homogeneous way we can describe them in terms of symplectic tomography. For an introduction on this topic see for example \cite{26} and \cite{25}.

Let us consider a distribution on a one dimensional phase space where all the states are represented by a pair of coordinates \((q, p)\). We consider the projection of this distribution on the \( q \) axis i.e. we consider a function \( W(q) \). If we rotate the coordinate system by the transformations

\[
X = \mu q + \nu p \tag{4.1}
\]

and

\[
P = -\nu q + \mu p \tag{4.2}
\]
with $\mu = s\cos\theta$ and $\nu = s^{-1}\sin\theta$, where $s$ is a squeezing factor and $\theta$ is the rotation angle of the $(q,p)$ frame, we can take all the projections $\mathcal{W}(X,\mu,\nu)$ of the same distribution on each $X$ axis (where the initial $W(q) = \mathcal{W}(X,1,0)$). The set of all these projections is the tomogram. So we can reconstruct the distribution by appropriately treating the tomogram. This is done generally with a Radon transform as we will see in the following. Other reconstructing algorithms are used when a tomogram is obtained from a set of experimental data [32]. For this reason determining the tomogram of a system is equivalent to determine its state. It is important to remark that these transformations are linear canonical transformations [26].

Let us define the classical and quantum tomogram.

Given a classical probability distribution $f(q,p)$ on the phase space, we define the classical tomogram by

$$\mathcal{W}(X,\mu,\nu) = \int f(q,p)\delta(X - \mu q - \nu p)\,dq\,dp. \quad (4.3)$$

If $f(q,p)$ is normalized then also the tomogram is normalized and satisfies the following conditions,

$$\int \mathcal{W}(X,\mu,\nu) \,dX = 1 \quad (4.4)$$

$$\mathcal{W}(X,1,0) = \int f(q,p)\,dp \quad (4.5)$$

and

$$\mathcal{W}(X,0,1) = \int f(q,p)\,dq. \quad (4.6)$$

Similarly a quantum tomogram defined by

$$\mathcal{W}(X,\mu,\nu) = \int W(q,p)\delta(X - \mu q - \nu p)\,dq\,dp. \quad (4.7)$$

where $W(q,p)$ is the Wigner function. This is a general definition for pure and mixed states. For a pure state, given a wave function $\psi(q)$, an equivalent definition of tomogram is

$$\mathcal{W}(X,\mu,\nu) = \frac{1}{2\pi\hbar|\nu|} \left| \int \psi(y) \exp \left[ i \left( \frac{\mu}{\hbar\nu} y^2 - \frac{X}{\hbar\nu} y \right) \right] \,dy \right|^2. \quad (4.8)$$

In particular if $\nu = 0$ and $\mu = 1$ we have

$$\mathcal{W}(X,1,0) = \frac{1}{2\pi} |\psi(X)|^2 = \frac{1}{2\pi} |\psi(x)|^2.$$

and for $\mu = 0$ and $\nu = 1$ we have the Fourier transform with respect to the variable $p/\hbar$,

$$\mathcal{W}(X,0,1) = \frac{1}{2\pi\hbar} \left| \int \psi(y) \exp \left[ -i \frac{py}{\hbar} \right] \,dy \right|^2. \quad (4.9)$$
i.e. it is the square modulus of the wave function in the $p$ representation.

The symplectic tomograms $W(X, \mu, \nu)$ are probability functions and satisfy the following conditions:

1) **Nonnegativity**

$$W(X, \mu, \nu) \geq 0,$$

(4.10)

2) **Normalization**

$$\int W(X, \mu, \nu) dX = 1,$$

(4.11)

and 3) **Homogeneity**

$$W(\alpha X, \alpha \mu, \lambda \nu) = \frac{1}{|\lambda|} W(X, \mu, \nu).$$

(4.12)

The important thing is that all these relations can be inverted \[6\][10][26][25][28] and then tomograms represent the state of a classical or a quantum system. They are observables and have been used to reconstruct the Wigner function in quantum mechanics and quantum optics \[29\][28][30][31][32].

Quantum and classical tomograms differ in that the former must satisfy the uncertainty principle in the form stated by Robertson \[35\] and Schrödinger \[34\] (see also \[33\]-\[39\]), which is expressed in terms of the variance of the canonical variables $\sigma_{pp}$, $\sigma_{qq}$ and the covariance $\sigma_{qp}$,

$$\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \frac{1}{4} \hbar^2.$$  

(4.13)

Unlike the uncertainty principle enunciated by Heisenberg, it is invariant under the group of linear canonical transformations \[37\] so in a tomogram if it is verified for a pair $(\mu, \nu)$, it is true for every other pair.

However in many cases it is sufficient to write the uncertainty principle in its weakest form (with $\sigma_{qp} = 0$),

$$\left[ \int W(X, 1, 0) X^2 dX - \left\{ \int W(X, 1, 0) X dX \right\}^2 \right] \times \left[ \int W(X, 0, 1) X^2 dX - \left\{ \int W(X, 0, 1) X dX \right\}^2 \right] \geq \frac{1}{4} \hbar^2$$

(4.14)

If this last condition is violated certainly the tomogram is related to a classical system.

5 **Tomographic description of the de Sitter universe**

Let us now analyze the classical and quantum de Sitter models from the tomographic point of view.
If we apply (4.3) to the distribution (2.15)
\[ f(q,p) = \delta (-4p^2 + \lambda q - 1). \]  
we find the classical de Sitter tomogram
\[ \mathcal{W}(X,\mu,\nu) = \frac{1}{2|\mu|} \frac{1}{\sqrt{\frac{\lambda^2 \nu^2}{16\mu^2} + \frac{\lambda X}{\mu} - 1}}. \]  
(5.2)

It is clear that this function is not integrable on an infinite range, then its domain must be restricted to compact support such that the normalization condition (4.4) is satisfied. We choose a closed interval \( I \) such that the normalization condition is satisfied. For example if we choose the inferior value of the interval is such that
\[ \frac{\lambda^2 \nu^2}{16\mu^2} + \frac{\lambda X}{\mu} - 1 = 0 \]
we see that
\[ \int_{-\infty}^{\infty} \mathcal{W}(X,\mu,\nu) dX = \frac{1}{2|\mu|} \int_X \frac{dX}{\sqrt{\frac{\lambda^2 \nu^2}{16\mu^2} + \frac{\lambda X}{\mu} - 1}} = \frac{\mu}{|\mu|} = 1. \]  
(5.4)

if \( \mu > 0 \).

In conclusion the value of the cosmological constant of the order of \( 10^{-122} \lambda_{Planck} \), implies that the interval \( I \) becomes very narrow and the classical tomogram of a de Sitter universe is approximated by a delta function,
\[ \mathcal{W}(X,\mu,\nu) \approx \delta \left( \frac{\lambda X}{\mu} + \frac{\lambda^2 \nu^2}{16\mu^2} - 1 \right), \]  
(5.5)
which means that the extreme smallness of the cosmological constant implies a very high degree of homogeneity of the classical de Sitter universe. To loose the restrictions imposed by the cosmological constant we can define the tomogram on a larger interval \([X_1, X_2] \) provided to normalize the tomogram by
\[ \bar{\mathcal{W}}(X,\mu,\nu) = \frac{\mathcal{W}(X,\mu,\nu)}{\int_{X_1}^{X_2} \mathcal{W}(X,\mu,\nu)}. \]  
(5.6)

Now let’s determine the quantum tomograms from the wave functions (3.9)–(3.11) by applying the first fractional Fourier transform (4.8) and taking its square modulus of and divided by \( 2\pi \hbar |\nu| \). For (3.9) we have,
\[ \Psi_{HH}(X,\mu,\nu) = A \int \exp \left( i \left( \frac{z^3}{3} + \frac{1 - \lambda q}{(2\hbar \lambda)^{2/3}} z + \frac{\mu}{2\hbar \nu} q^2 - \frac{Xq}{\hbar \nu} \right) \right) dqdz \]  
(5.7)
and the Hartle-Hawking tomogram is
\[ \mathcal{W}_{HH}(X, \mu, \nu) = \frac{a^2}{2\pi \hbar |\mu|} \left| \text{Ai} \left( \frac{1}{(2\hbar \lambda)^{2/3}} \left( 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right) \right) \right|^2. \quad (5.8) \]

It is straightforward to extend this calculation to the Vilenkin and Linde wave functions by noticing that, the wave functions (3.10) and (3.11) can be written as [16] [17]

\[ \psi_V \left( 1 - \frac{\lambda q}{(2\hbar \lambda)^{2/3}} \right) = b e^{i\pi/3} \text{Ai} \left( e^{-2\pi i/3} \frac{1 - \lambda q}{(2\hbar \lambda)^{2/3}} \right) \quad (5.9) \]

and

\[ \psi_L \left( 1 - \frac{\lambda q}{(2\hbar \lambda)^{2/3}} \right) = c \left[ e^{4\pi i/3} \text{Ai} \left( e^{4\pi i/3} \frac{1 - \lambda q}{(2\hbar \lambda)^{2/3}} \right) + e^{2\pi i/3} \text{Ai} \left( e^{2\pi i/3} \frac{1 - \lambda q}{(2\hbar \lambda)^{2/3}} \right) \right]. \quad (5.10) \]

and finally we find that the Vilenkin and Linde tomograms are respectively

\[ \mathcal{W}_V(X, \mu, \nu) = \frac{b^2}{2\pi \hbar |\mu|} \left| \frac{1}{(2\hbar \lambda)^{2/3}} \left( 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right) \right|^2 \quad (5.11) \]

and

\[ \mathcal{W}_L(X, \mu, \nu) = \frac{c^2}{2\pi \hbar |\mu|} \left| \text{Bi} \left( \frac{1}{(2\hbar \lambda)^{2/3}} \left( 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right) \right) \right|^2. \quad (5.12) \]

### 6 The Wheeler-DeWitt tomographic equation

In this section we show that can find the tomograms (5.8), (5.11) and (5.12) as solutions of the Wheeler-DeWitt equation expressed with the tomographical variables \( X, \mu \) and \( \nu \).

This equation is obtained by introducing the correspondences between operators,

\[ q \rightarrow -\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{\nu}{2\hbar} \frac{\partial}{\partial X} \quad (6.1) \]

and

\[ \frac{d}{dq} \rightarrow \frac{\mu}{2} \frac{\partial}{\partial X} - i\hbar \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \nu} \quad (6.2) \]

which applied to (3.4) gives the tomographic Wheeler-DeWitt equation

\[ \left( \hbar^2 \mu^2 \frac{\partial^2}{\partial X^2} - 4i\hbar \mu \frac{\partial}{\partial \nu} - 4 \left( \frac{\partial}{\partial X} \right)^{-2} \frac{\partial^2}{\partial \nu^2} \right) \Psi(X, \mu, \nu) = 0 \quad (6.3) \]

\[ -\lambda \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{\hbar \lambda \nu}{2} \frac{\partial}{\partial X} - 1 \right) \Psi(X, \mu, \nu) = 0 \quad (6.4) \]
A tomogram is obtained by the square modulus of one of the solutions of this equation.

From the imaginary part we obtain the equation

$$-4\mu \frac{\partial \Psi}{\partial \nu} + \frac{\lambda \nu}{2} \frac{\partial \Psi}{\partial X} = 0$$  \hspace{1cm} (6.5)$$

that gives us the expression

$$\frac{\partial^2 \Psi}{\partial \nu^2} = \lambda \frac{\partial \Psi}{8\mu \partial X} + \left(\frac{\lambda \nu}{8\mu}\right)^2 \frac{\partial^2 \Psi}{\partial X^2}$$ \hspace{1cm} (6.6)$$

that can be inserted in the real part to obtain finally

$$\hbar^2 \mu^2 \frac{\partial^2 \Psi}{\partial X^2} = -\frac{\lambda}{2\mu} \left(\frac{\partial}{\partial X}\right)^{-1} \Psi - \left(\frac{\lambda \nu^2}{4\mu}\right)^2 \Psi - \lambda \left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial \Psi}{\partial \mu} - \Psi = 0$$ \hspace{1cm} (6.7)$$

To solve this equation we first take the Fourier transform of $\Psi$, $\hat{\Psi}(k, \mu, \nu)$ so that equation (6.7) becomes

$$\frac{\partial \hat{\Psi}}{\partial \mu} = \left(-i \left(\frac{\hbar^2 \mu^2 k^3}{\lambda} - \lambda \left(\frac{\nu^2}{4\mu}\right)^2 k - \frac{k}{\lambda}\right) - \frac{1}{2\mu}\right) \hat{\Psi}$$ \hspace{1cm} (6.8)$$

which integrated gives

$$\hat{\Psi}(k, \mu, \nu) = \frac{A}{\sqrt{\mu}} \exp\left[-i \left(\frac{\hbar^2 \mu^3 k^3}{3\lambda} + \left(\frac{\mu}{\lambda} - \frac{\lambda \nu^2}{16\mu}\right) k\right)\right]$$ \hspace{1cm} (6.9)$$

and we finally obtain

$$\Psi(X, \mu, \nu) = \frac{A}{\sqrt{\mu}} \int_{-\infty}^{\infty} \exp\left[-i \left(\frac{\hbar^2 \mu^3}{\lambda} - i \left(\frac{\mu}{\lambda} - \frac{\lambda \nu^2}{16\mu}\right) X\right) k\right] dk$$ \hspace{1cm} (6.10)$$

using the relation

$$Ai(ax) = \frac{1}{2\pi a} \int_{-\infty}^{\infty} e^{i\frac{1}{3}z^3 + iax} dz$$ \hspace{1cm} (6.11)$$

we obtain the following solutions of the Airy equation

$$\Psi(X, \mu, \nu) = A \frac{\mu^{1/2} \nu^{1/2} \hbar^{2/3}}{\lambda^{1/3}} Ai \left(\frac{e^{i(1 - n)\pi}}{\hbar\lambda^{2/3}} \left(1 - \frac{\lambda X}{\mu} - \frac{1}{16 \mu^2 \lambda^2}\right)\right).$$ \hspace{1cm} (6.12)$$

where $A$ is an arbitrary integration coefficient which to be determined by requiring that the asymptotic approximations have the same coefficient of the classical tomogram.

Finally the tomogram is given by

$$W(X, \mu, \nu) = |\Psi(X, \mu, \nu)|^2$$ \hspace{1cm} (6.13)$$
Which becomes, after applying the scaling property of the tomograms,

\[ W(\alpha X, \alpha \mu, \alpha \nu) = \frac{1}{|\alpha|} W(X, \mu, \nu) \]  

(6.14)

\[ W(X, \mu, \nu) = A^2 \frac{\hbar^{4/3}}{|\mu| \lambda^{2/3}} \left| \text{Ai} \left( \frac{e^{i(1-\xi)} \pi}{(\hbar \lambda)^{2/3}} \left( 1 - \frac{\lambda X}{\mu} - \frac{1}{16} \frac{\nu^2}{\mu^2} \lambda^2 \right) \right) \right|^2 \]  

(6.15)

The values \( n = 3 \) and \( n = 1 \) correspond respectively to the Hartle-Hawking and to the Vilenkin tomograms. The Linde solution is a linear combination of two independent solutions of eq. (6.3).

We notice that there is a slight difference with the functions obtained with the fractional Fourier transform by a factor \( 2^{2/3} \) in the denominator of the argument, this difference can be remedied by changing the definition of relations between operators (6.1) and (6.2).

### 7 Classical limits of the de Sitter tomograms and the classical tomogram

In this section discuss the relation between the quantum and classical tomograms. To this aim we look at the classical limit of a quantum tomogram by taking the limit \( \hbar \to 0 \), but for the tomograms (5.8), (5.11) and (5.12) we can take the limit \( (2\hbar \lambda)^{2/3} \to 0 \), which leads to their asymptotic expansions. They coincide with the asymptotic expansions of the Airy functions which change depending on whether argument is positive or negative. In the first case we have an exponential growth, for the Bi function and an exponential decrease for the Ai function. In the second case both functions oscillate.

Let us call the argument of the de Sitter tomograms

\[ S = \frac{1}{3 \hbar \lambda} \left( 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right)^{3/2} - \frac{\pi}{4}. \]

For \( S < 0 \) then the Hartle and Hawking tomogram is asymptotically

\[ \mathcal{W}_{HH}(X, \mu, \nu) \approx \frac{a^2}{8 \pi^2 \hbar |\mu|} \frac{(2\hbar \lambda)^{1/3}}{1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2}} \times |\cos(S)|^2, \]  

(7.1)

(7.2)

when \( S > 0 \)

\[ \mathcal{W}_{HH}(X, \mu, \nu) \approx \frac{a^2}{16 \pi^2 \hbar |\mu|} \frac{(2\hbar \lambda)^{1/3}}{1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2}}^{1/2} e^{-2(S+\xi)}, \]  

(7.3)
Similarly the asymptotic expressions for Linde’s tomogram are for \( S < 0 \)
\[
\mathcal{W}_L(X, \mu, \nu) \approx \frac{c^2}{8\pi^2 h|\mu|} \frac{(2h\lambda)^{1/3}}{\left| 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right|^{1/2}} \times |\sin (S)|^2, \tag{7.4}
\]
and
\[
\mathcal{W}_L(X, \mu, \nu) \approx \frac{c^2}{8\pi^2 h|\mu|} \frac{(2h\lambda)^{1/3}}{\left| 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right|^{1/2}} e^{2(S+\zeta)}. \tag{7.5}
\]
for \( S > 0 \)

Finally asymptotic form of Vilenkin’s tomogram for \( S < 0 \) is
\[
\mathcal{W}_V(X, \mu, \nu) \approx \frac{b^2}{8\pi^2 h|\mu|} \frac{(2h\lambda)^{1/3}}{\left| 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right|^{1/2}} |\sin (S)|^2. \tag{7.6}
\]
and for \( S > 0 \)
\[
\mathcal{W}_V(X, \mu, \nu) \approx \frac{c^2}{8\pi^2 h|\mu|} \frac{(2h\lambda)^{1/3}}{\left| 1 - \frac{\lambda X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} \right|^{1/2}}
\times \left( e^{2(S+\zeta)} + e^{-2(S+\zeta)} \right) \tag{7.7}
\]

We fix the normalization coefficients \( a, b \) and \( c \) imposing that the quantum and classical tomograms have the same coefficient,
\[
a = \frac{b}{2} = c = \frac{2^{5/6} \pi h^{1/3}}{\lambda^{1/6}}. \tag{7.10}
\]

Therefore we draw the following conclusions. First we notice that in the limit \((2h\lambda)^{2/3} \to 0\) for \( S > 0 \) the tomograms go either to zero or to infinity and they do not match the classical model \([5,2]\). For \( S < 0 \), both the Hartle-Hawking and Linde tomograms do not have a limit due to the presence of the the cosine square and sine square factors which go to infinite oscillations. On the other side Vilenkin tomogram converges to \([5,2]\) due to the fact that it contains only the expanding model.

However when we take the limit \( \lambda \to \lambda_{today} \) with \( \hbar \) constant, we see that the extremely small value of \( \lambda_{today} \) which is of the order of \( 10^{-122}\lambda_{Planck} \) justifies taking the expressions \([7.3]\), \([7.4]\) and \([7.7]\) of the tomograms. While Vilenkin’s tomogram becomes again the classical tomogram, the Hartle-Hawking and Linde tomograms have now a limit and become good candidates to represent a (pseudo-)classical universe. In other word the decay of the cosmological constant to the present value is responsible for the smooth transition of the universe to a the classical regime. The Hartle-Hawking and Linde tomograms present an interference pattern which should be observable, at least in principle, with a large scale survey of the universe.
8 General relations between the asymptotic solutions and the classical models

The results found in the previous section suggest that to study the relation between quantum and classical cosmological states with any potential \( V(q) \), it is sufficient to construct the quantum tomograms from the asymptotic solutions of the Wheeler-DeWitt equation, whereas the classical tomograms are derived by introducing a distribution function according to the criterion introduced in sect.5.

To determine the form of the classical tomograms we consider the Hamiltonian

\[
\mathcal{H} = \frac{1}{2} (-4p^2 + V(q))
\]  

(8.1)

and associate to it the phase space distribution,

\[
g(q,p) = \delta (-4p^2 + V(q))
\]  

(8.2)

where \( V(q) \) is a generic potential. We split \( g(q,p) \) in the following way

\[
g(q,p) = \frac{\delta \left( p - \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|} + \frac{\delta \left( p + \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|}
\]  

(8.3)

where we used the rule that for any \( F(x) \),

\[
\delta(F(x)) = \frac{\delta(x - x_0)}{|F'(x_0)|}
\]  

(8.4)

if \( x_0 \) is the only solution of equation \( F(x) = 0 \) and

\[
\delta(F(x)) = \sum_i \frac{\delta(x - x_i)}{|F'(x_i)|}
\]  

(8.5)

if \( F(x) = 0 \) has more solutions \( x_i \). The tomogram is then given by

\[
\mathcal{W}(X, \mu, \nu) = \int \left[ \frac{\delta \left( p - \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|} + \frac{\delta \left( p + \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|} \right] \delta(X - \mu q - \nu p) dq dp
\]

\[
= \frac{1}{|\nu|} \int \left[ \frac{\delta \left( \frac{X - \mu q}{\nu} - \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|} + \frac{\delta \left( \frac{X - \mu q}{\nu} + \frac{\sqrt{V(q)}}{2} \right)}{|\sqrt{V(q)}|} \right] dq
\]

\[
= \frac{2}{|\nu|} \left( \frac{1}{2\nu \sqrt{V(q(1))} - V'(q(1))} + \frac{1}{2\nu \sqrt{V(q(2))} + V'(q(2))} \right)
\]

(8.6)
where \( q(1) = q(1)(X, \mu, \nu) \) and \( q(2) = q(2)(X, \mu, \nu) \) are the solutions of the equations

\[
\frac{X - \mu q(1)}{\nu} - \sqrt{V(q(1))} = 0 \tag{8.7}
\]

and

\[
\frac{X - \mu q(2)}{\nu} + \sqrt{V(q(2))} = 0 \tag{8.8}
\]

If eq. (8.7) has \( m \) solutions \( q_1^1 \ldots q_m^1 \) and eq. (8.8) has \( n \) solutions \( q_1^2 \ldots q_n^2 \) the tomogram becomes

\[
W(X, \mu, \nu) = \frac{2}{|\nu|} \left( \sum_{i=1}^{m} \frac{1}{-2\mu \sqrt{V(q_i^1)} - V'(q_i^1)} + \sum_{j=1}^{n} \frac{1}{2\mu \sqrt{V(q_j^2)} + V'(q_j^2)} \right) \tag{8.9}
\]

Now let us look for the necessary steps to construct the quantum tomograms from the asymptotic solutions of the Wheeler-DeWitt equation,

\[
4\hbar^2 \frac{d^2 \psi(q)}{dq^2} - V(q)\psi(q) = 0. \tag{8.10}
\]

In the limit \( \hbar \to 0 \) by applying the WKB method we find the solutions of eq. (8.10), whose leading terms

- for \( V(x) > 0 \) the solution is

\[
\psi(q) \approx \frac{A}{|V(q)|^{1/4}} e^{\pm \frac{1}{2} \int q \sqrt{V(y)} dy}, \tag{8.11}
\]

and for \( V(q) < 0 \)

\[
\psi(q) \approx \frac{A}{|V(q)|^{1/4}} e^{\pm i \int \sqrt{V(y)} dy}, \tag{8.12}
\]

where \( A \) is a normalization constant.

We calculate the partial Fourier transforms of the oscillating functions (8.12) because it is where we can apply the Hartle criterion according to which the correlation between the variables can be found only where the wave function is strongly peaked, while the functions (8.11) either grow to infinite or go to zero rapidly. This criterion gives a necessary condition to find a classical solution. Indeed we calculate the tomogram by applying the Laplace method to the integral

\[
\psi(X, \mu, \nu) = \int \frac{A}{|V(q)|^{1/4}} e^{\pm \frac{1}{2} \int q y \sqrt{V(y)} dy + i \frac{m^2}{2m^2} - i \frac{Xy}{2m^2} dq}, \tag{8.13}
\]
we develop the exponent to the second order to find

\[
\psi(X, \mu, \nu) = \int \frac{A}{|V(q_0)|^2} \exp \left[ \pm \frac{1}{2\hbar} \left( i \int_{q_0}^{q_0} \sqrt{V(y)} dy + \frac{i \mu q_0^2}{2\hbar \nu} - \frac{i X q_0}{\hbar \nu} \right) \right] dq
\]

\[
+ \frac{i}{\hbar} \left( \pm \sqrt{V(q_0)} + \frac{\mu}{\nu} q_0 - \frac{X}{\nu} \right) (q - q_0) + \left( \pm \frac{1}{2} \frac{V'(q_0)}{\sqrt{V(q_0)}} + \frac{\mu}{\nu} \right) \frac{(q - q_0)}{2} \right] dq
\]

(8.14)

where \( q_0 \) is a stationary point so that the first derivative must be zero. This corresponds precisely to solve eqs. (8.7) and (8.8).

The condition on the first derivative is the correlation condition between momenta and coordinates because from (4.1)

\[
p = \frac{X - \mu q}{\nu}
\]

(8.15)

and \( \sqrt{V(q)} \) is the derivative of the phase term of the wave function (8.12),

\[
p = \pm \sqrt{V(q)}.
\]

(8.16)

which is equivalent to find the peaks of the function \( \psi(q) \exp(i \frac{\mu q^2}{2\nu} - i \frac{X q}{\nu}) \).

Furthermore (8.16) makes us recognize that the negative and positive exponents in (8.12) represent respectively the expanding and contracting modes of the universe due to the relation (2.8).

After calculating the Gaussian integral we obtain,

\[
\psi_1(X, \mu, \nu) = \left| \frac{4\pi \hbar}{\sqrt{V'(q(1)) + \frac{2 \mu}{\nu} \sqrt{V(q(1))}}} \right| e^{\left( \mp \int^{q(1)} V(y) dy + \frac{i \mu q^2}{2\nu} - \frac{i X q}{\nu} \right)}
\]

(8.17)

and

\[
\psi_2(X, \mu, \nu) = \left| \frac{4\pi \hbar}{\sqrt{-V'(q(2)) + \frac{2 \mu}{\nu} \sqrt{V(q(2))}}} \right| e^{\left( -\mp \int^{q(2)} V(y) dy + \frac{i \mu q^2}{2\nu} - \frac{i X q}{\nu} \right)}.
\]

(8.18)

The general solution is

\[
\psi(X, \mu, \nu) = c_1 \psi_1(X, \mu, \nu) + c_2 \psi_2(X, \mu, \nu)
\]

(8.19)

and the tomogram is given by

\[
\mathcal{W}(X, \mu, \nu) = \frac{1}{2\pi \hbar |\nu|} \left| \psi \right|^2 = \frac{2}{|\nu|} \left[ c_1^2 |\psi_1|^2 + c_2^2 |\psi_2|^2 + 2c_1 c_2 \psi_1 \psi_2 \right]
\]

(8.20)

Among all the possible choices of \( c_1 \) and \( c_2 \), we find the tomograms related to the Hartle-Hawking, Vilenkin and Linde models respectively when \( c_1 = c_2 \), \( c_2 \neq 0 \) and \( c_1 = 0 \), and \( c_1 = -c_2 \).
In general all these tomograms, which are now classical, as the left hand side of the uncertainty principle (4.14) goes to zero, present interference terms as for the Hartle-Hawking and Linde tomograms in the de Sitter universes.

Only the set of tomograms obtained from the combination

$$\psi(X,\mu,\nu) = \psi_1(X,\mu,\nu) + i\psi_2(X,\mu,\nu),$$

(8.21)

coincides with (8.6). But when eqs. (8.7) and (8.8) have multiple solutions, we have that

$$\psi_1(X,\mu,\nu) = \sum_{i=1}^{m} \frac{4\pi\hbar}{V'(q^{(1)}_i) + 2\mu \sqrt{V(q^{(1)}_i)}} \left( e^{\frac{\imath}{\hbar} \int_{q^{(1)}_i} \psi(q_{(1)})} \right)$$

(8.22)

and

$$\psi_2(X,\mu,\nu) = \sum_{j=1}^{n} \frac{4\pi\hbar}{-V'(q^{(2)}_j) + 2\mu \sqrt{V(q^{(2)}_j)}} \left( e^{-\frac{\imath}{\hbar} \int_{q^{(2)}_j} \psi(q_{(2)})} \right).$$

(8.23)

and the tomograms (8.20) obtained from (8.22) and (8.23) never converge to (8.9) due to the unavoidable presence of the mixed products.

9 Models with the cosmological constant

So far we have considered general models with any potential and we found the conditions of compatibility between quantum and classical tomograms. However, no obvious criteria appear for the transition to a classical universe. In this section we want to understand if the decay of the cosmological constant can responsible for this transition as in the case of de Sitter universes. Therefore we will consider the potentials that contain the cosmological constant explicitly

$$4\hbar^2 \frac{d^2 \psi(q)}{dq^2} + (\lambda q - 1 - \varepsilon f(q)) \psi(q) = 0,$$

(9.1)

and we can write the Wheeler-DeWitt equation as a modification of eq.(3.4).

We can face this problem in three different ways. If $f(q)$ is a perturbative term we can solve the equation eq. (9.1) by a regular perturbation method[24], the second is to solve it by taking an uniform asymptotic expansion of solutions [16], the third one by applying the method illustrated in the previous section.

In the first case, with the change of variables (3.6), we have

$$\frac{d^2 \psi(\xi)}{d\xi^2} - \left( \xi + \varepsilon f \left( \frac{1 - (2\hbar \lambda)^{2/3}}{\lambda} \right) \right) \psi(\xi) = 0,$$

(9.2)

We consider the expansion

$$\psi(\xi) = \psi_0(\xi) + \varepsilon \psi_1(\xi) + \ldots$$

(9.3)
and we obtain the equations
\[
\frac{d^2 \psi_0(\xi)}{d\xi^2} - \xi \psi_0(\xi) = 0, 
\]
(9.4)
\[
\frac{d^2 \psi_1(\xi)}{d\xi^2} - \xi \psi_1 = f \left( 1 - \frac{(2\hbar \lambda)^{2/3} \xi}{\lambda} \right) \psi_0(\xi), 
\]
(9.5)
this second equation is an inhomogeneous Airy equation. Let’s take for sake of brevity \( \psi_0(\xi) = \text{Ai}(\xi) \), the perturbed solution is
\[
\psi(\xi) = \left( 1 + \varepsilon \left( c_1 + \pi \int \text{Ai}(\xi) \text{Bi}(\xi) f \left( \frac{1 - \xi}{\lambda} \right) d\xi \right) \right) \text{Ai}(\xi) 
\]
\[+ \varepsilon \left( c_2 - \pi \int \text{Ai}^2(\xi) f \left( \frac{1 - (2\hbar \lambda)^{2/3} \xi}{\lambda} \right) d\xi \right) \text{Bi}(\xi), \]
(9.6)
(9.7)
Where \( c_1 \) and \( c_2 \) are arbitrary integration constants. The solutions are proportional to the unperturbed de Sitter solutions, so from eq. (4.8) we derive the perturbed de Sitter tomograms provided the integrals of the inhomogeneous solution are well-behaved.

Very similar results come from the application of the uniform asymptotic expansion of solutions[16], where if \( \xi \) varies on a bounded interval \( a \leq \xi \leq b \) and where for each fixed \( (2\hbar \lambda)^{-2/3} \), \( f \left( \frac{1 - (2\hbar \lambda)^{2/3} \xi}{\lambda} \right) \) is continuous in \( \xi \) for \( a \leq \xi \leq b \) there are solutions \( \psi_a(\xi) \) and \( \psi_b(\xi) \) such that uniformly on \( \xi \) for \( a \leq \xi \leq b \)
\[
\psi_a(\xi) = \text{Ai}(\xi)(1 - O((2\hbar \lambda)^{-2/3})) \quad (2\hbar \lambda)^{-2/3} \to 0 
\]
\[
\psi_b(\xi) = \text{Bi}(\xi)(1 - O((2\hbar \lambda)^{-2/3})) 
\]
(9.8)
and uniformly on \( \xi \) for \( 0 \leq \xi \leq b \)
\[
\psi_a(\xi) = \text{Ai}(\xi)(1 - O((2\hbar \lambda)^{-2/3})) + \text{Bi}(\xi)O((2\hbar \lambda)^{-2/3}) \quad (2\hbar \lambda)^{-2/3} \to 0 
\]
\[
\psi_b(\xi) = \text{Bi}(\xi)(1 - O((2\hbar \lambda)^{-2/3})) + \text{Ai}(\xi)O(2\hbar \lambda)^{-2/3} 
\]
(9.9)
Finally to apply the WKB method, we take the change of variables \( y = 1 - \lambda q \) and the Wheeler-DeWitt becomes
\[
4\hbar^2 \lambda^2 \frac{d^2 \psi(y)}{dy^2} - \left( \xi - f \left( \frac{1 - y}{\lambda} \right) \right) \psi(y) = 0. 
\]
(9.10)
And we consider the oscillating asymptotic solutions,
\[
\psi(y) = \frac{1}{\sqrt{|\xi + g(\frac{1 - y}{\lambda})|}} e^{\pm \pi \hbar \sqrt{y + g(\frac{1 - y}{\lambda})}} dy 
\]
(9.11)
we then proceed calculating the fractional Fourier transform and we recover all the results of sect. [8].

We conclude that also in this case the decay of the cosmological constant plays a crucial role in the transition to a classical universe.
10 Conclusions

Symplectic tomography offers us the possibility to express quantum and classical cosmological states in an unified formalism giving us the possibility to compare them and to determine when a quantum to classical transition is possible. This approach allowed us to study in general the de Sitter models. The results of [10] and in sect. 5 have been recalled and the relation of the various proposals of initial conditions of the universe with the classical tomogram (2.13) were examined. We found that only the Vilenkin tomogram converges to a classical universe, while the Hartle and Hawking as well as the Linde’s tomograms do not have a limit. However when we consider the decay of the cosmological constant $\lambda \to \lambda_{\text{today}}$ we can take as well the tomograms in their asymptotic form and the universe enters to a classical state. Because this limit is to a finite value we are led to consider also the Hartle and Hawking and Linde as ”classical” models with interference terms which are peculiar and unexpected properties if observed.

These interference terms are a general characteristic in most of the quantum tomograms and their classical limits as it came out in sect. 5. The quantum tomograms converge to the classical tomogram (8.6) only , when they satisfy the condition (8.21) and in some particular cases.

Here by classical we mean tomograms obtained from the classical Einstein equations, but we could also consider as ”classical” all the tomograms obtained asymptotically as they violate the uncertainty principle. They may equally describe the present universe correctly. Finally in sect. 9 we established the criteria by which the cosmological constant can induce the transition to a classical universe.

Although we have found general results in this paper in the tomographic analysis of the quantum-to-classical transition, we want to highlight that there are some points that still need to be explored in the future work.

For example, the fractional Fourier transforms we have used so far are all defined in an interval ranging from $-\infty$ to $+\infty$ even if variable $q$ is explicitly positive. Therefore it may seem incorrect to use it in general. There are different way around this problem, for example we may consider only wave functions with a compact support , or we can an infinite barrier at $q = 0$ in the potential and finally instead of using the fractional Fourier transform we can introduce the fractional Hankel transform (see for example[44] and [45]) where the integration range runs from 0 to $+\infty$.

As we have seen, the decay of the cosmological constant can be a mechanism which drives the quantum universe to a classical state. This could explain two crucial points, the transition itself and the smallness of the cosmological constant [14]. This phenomenon in cosmology is not new at it explains the exit from inflation in most models. But in this case this problem has to be considered in a quantum universe. Perhaps address this problem taking models with scalars fields or with other approaches like in [46] [47] [48].

Further considerations lead to broaden the cosmological constant problem to more general forms of dark energy. These can be treated in terms of extended
theories $F(R)$. In this case we believe that it might be interesting to check in 
these theories if there is self-consistent way to describe the quantum to classical transition.

So far we have discussed the mathematical implications of the tomographic representation. We recall here that the important property of tomograms is that they observable and therefore it is presumable that we can build a model of the universe in tomographic terms based on the observable quantities such as such as the Hubble constant $H_0$, the deceleration parameter $q_0$, and the density parameters $\Omega_k$, $\Omega_\Lambda$ and $\Omega_m$.

In the de Sitter models the argument of the tomogram can be interpreted by writing explicitly $X = \mu q + \nu p$ in the argument of the tomogram and using the equations we find

\[
1 - \frac{X}{\mu} - \frac{\lambda^2 \nu^2}{16 \mu^2} = 1 - \lambda q - \frac{\lambda \nu}{\mu} + \frac{\lambda^2 \nu^2}{16 \mu^2} = 1 - \lambda q \left( 1 - \frac{1}{4} \frac{\nu}{\mu} + \frac{1}{24} \frac{\nu^2}{\mu^2} \right) \quad (10.1)
\]

where we have used $p = -\frac{\dot{q}}{4}$ and $\dot{q} = \frac{2}{3} \lambda$ according and to (2.4) so we see that these terms are related to the Hubble constant and the deceleration parameter and suggest that $\nu/\mu \propto \Delta \tau$. In the other tomograms we expect more complicated relations between the variables $(X, \mu, \nu)$ and the cosmological observables obtained after solving the equations (8.7) and (8.8).

Tomograms are probability functions, so we adopt the point of view that they describe the statistics of a universe made by a patchwork of regions with the same homogeneous metric, but where the parameters that characterize them can vary. This happens, for example, when cosmological perturbations are involved. Then the statistical description of the observations will give us more complete information on the nature of the potential $V(q)$ and therefore a more precise knowledge of the initial state of the universe.

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