Keller-Osserman type conditions for
differential inequalities with gradient
terms on the Heisenberg group

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Abstract

The aim of this paper is to study the qualitative behaviour of non-negative
entire solutions of certain differential inequalities involving gradient terms
on the Heisenberg group. We focus our investigation on the two classes of
inequalities of the form $\Delta^\varphi u \geq f(u)(|\nabla u|)$ and $\Delta^\varphi u \geq f(u)-h(u)g(|\nabla u|)$,
where $f, l, h, g$ are non-negative continuous functions satisfying certain
monotonicity properties. The operator $\Delta^\varphi$, called the $\varphi$-Laplacian,
can be viewed as a natural generalization of the $p$-Laplace operator recently
considered by various authors in this setting. We prove some Liouville
theorems introducing two new Keller-Osserman type conditions, both ex-
tending the classical one which appeared long ago in the study of the
prototype differential inequality $\Delta u \geq f(u)$ in $\mathbb{R}^m$. Furthermore, we
show sharpness of our conditions when we specialize to the case of the $p$-
Laplacian. Needless to say, our results continue to hold, with the obvious
minor modifications, also in the Euclidean space.

1 Introduction and main results

To state our main results we first need to recall some preliminary facts and to
introduce the notations that we shall use in the sequel.

Let $H^m$ be the Heisenberg group of dimension $2m + 1$, that is, the Lie
group with underlying manifold $\mathbb{R}^{2m+1}$ and group structure defined as follows:
for all $q, q' \in H^m$, $q = (z, t) = (x_1, \ldots, x_m, y_1, \ldots, y_m, t)$, $q' = (z', t') = \ldots$
1 Introduction and main results

\[(x'_1, \ldots, x'_m, y'_1, \ldots, y'_m, t').\]

\[q \circ q' = \left( z + z', t + t' + 2 \sum_{i=1}^{m} (y_i x'_i - x_i y'_i) \right).\]

A basis for the Lie algebra of left-invariant vector fields on \(H^m\) is given by

\[X_j = \partial_{x_j} + 2 y_j \partial_{t}, \quad Y_j = \partial_{y_j} - 2 x_j \partial_{t}, \quad \partial_{t}\]

for \(j = 1, \ldots, m\). This basis satisfies Heisenberg’s canonical commutation relations for position and momentum,

\[[X_j, Y_k] = -4 \delta_{jk} \partial_{t},\]

all other commutators being zero. It follows that the vector fields \(X_j, Y_k\) satisfy Hörmander’s condition, and the Kohn-Spencer Laplacian, defined as

\[\Delta_{H^m} = \sum_{j=1}^{m} (X_j^2 + Y_j^2)\]

is hypoelliptic by Hörmander’s theorem (see [7]).

In \(H^m\) there are a “natural” origin \(o = (0,0)\) and a distinguished distance function from zero defined, for \(q = (z,t) \in H^m\), by

\[r(q) = r(z,t) = \left( |z|^4 + t^2 \right)^{1/4}\]

(where \(|\cdot|\) denotes the Euclidean norm in \(\mathbb{R}^{2m}\)), which is homogeneous of degree 1 with respect to the Heisenberg dilations \((z,t) \mapsto (\delta z, \delta^2 t), \ \delta > 0\). This gives rise to a distance on \(H^m\), called the Koranyi distance, and defined by

\[d(q, q') = r(q^{-1} \circ q'), \quad q, q' \in H^m.\]

We set

\[B_R(q_o) = \{q \in H^m : d(q, q_o) < R\}\]

to denote the (open) Koranyi ball of radius \(R\) centered at \(q_o\). We simply use \(B_R\) for balls centered at \(q_o = o\). The density function with respect to \(o\) is the function

\[\psi(q) = \psi(z, t) = \frac{|z|^2}{r^2(z, t)} \quad \text{for} \ q = (z, t) \neq o;\]

note that \(0 \leq \psi \leq 1\). For \(u \in C^1(H^m)\), the Heisenberg gradient \(\nabla_{H^m}u\) is given
by

\[ \nabla_{H^m} u = \sum_{j=1}^{m} (X_j u) X_j + (Y_j u) Y_j, \]  

(7)

so that, for \( f \in C^1(\mathbb{R}) \), \( \nabla_{H^m} f(u) = f'(u) \nabla_{H^m} u \), and a \( \cdot \) product on the span of \( X_j, Y_j \) is defined, for \( W = w^j X_j + \tilde{w}^j Y_j, Z = z^j X_j + \tilde{z}^j Y_j \) by the formula

\[ W \cdot Z = \sum_{j=1}^{m} w^j z^j + \tilde{w}^j \tilde{z}^j. \]  

(8)

By definition, \( |\nabla_{H^m} u|_{H^m}^2 = \nabla_{H^m} u \cdot \nabla_{H^m} u \), and we have the validity of the Cauchy-Schwarz inequality

\[ |\nabla_{H^m} u \cdot \nabla_{H^m} v|_{H^m} \leq |\nabla_{H^m} u|_{H^m} |\nabla_{H^m} v|_{H^m}. \]  

(9)

The distance function \( r \) satisfies the following fundamental relations involving \( \psi \):

\[ \Delta_{H^m} r = \frac{2m+1}{r} \psi \quad \text{in} \quad H^m \setminus \{o\}, \]  

(10)

\[ |\nabla_{H^m} r|_{H^m}^2 = \psi \quad \text{in} \quad H^m \setminus \{o\}. \]  

(11)

Recently, some authors (see, for example, [5], [4] and [1]) have studied a generalization of the Kohn Laplacian, defined, for \( p \in [2, +\infty) \), by

\[ \Delta^p_{H^m} u = \sum_{j=1}^{m} \left[ X_j \left( |\nabla_{H^m} u|_{H^m}^{p-2} X_j u \right) + Y_j \left( |\nabla_{H^m} u|_{H^m}^{p-2} Y_j u \right) \right], \quad u \in C^2(H^m), \]  

(12)

which can be considered as a natural \( p \)-Laplace operator in the setting of the Heisenberg group.

In this paper we consider a further generalization, which we shall call \( \varphi \)-Laplacian, \( \Delta^\varphi_{H^m} \), defined for \( u \in C^2(H^m) \) as follows:

\[ \Delta^\varphi_{H^m} u = \sum_{j=1}^{m} \left[ X_j \left( |\nabla_{H^m} u|_{H^m}^{-1} \varphi(|\nabla_{H^m} u|_{H^m}) X_j u \right) + Y_j \left( |\nabla_{H^m} u|_{H^m}^{-1} \varphi(|\nabla_{H^m} u|_{H^m}) Y_j u \right) \right], \]  

(13)

where \( \varphi \) satisfies the structural conditions

\[ \begin{cases} \varphi \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+), & \varphi(0) = 0, \\ \varphi' > 0 & \text{on} \mathbb{R}_0^+. \end{cases} \]  

(\( \Phi \))

This family of operators, containing the \( p \)-Laplacian (obtained with the choice \( \varphi(t) = t^{p-1}, p > 1 \)), has been recently studied in the context of Riemannian geometry (see, for example, [9] for motivations and further references). Although
we shall focus our attention on this generalization, the main example we keep in mind is the \( p \)-Laplacian itself, to which an entire section is devoted.

The aim of this paper is to study weak (in the sense of Subsection 2.2 below) non-negative entire solutions of differential inequalities of the form

\[
\Delta_{H^m}^\varphi u \geq f(u) l(|\nabla_{H^m} u|_{H^m}),
\]

(14)

where \( f \) and \( l \) satisfy respectively the following conditions:

\[
\begin{align*}
\{ f &\in C^0(\mathbb{R}^+_0), \quad f > 0 \text{ on } \mathbb{R}^+; \\
&f \text{ is increasing on } \mathbb{R}^+_0; \}
\end{align*}
\]

\( (F) \)

\[
\begin{align*}
\{ l &\in C^0(\mathbb{R}^+_0), \quad l > 0 \text{ on } \mathbb{R}^+; \\
&l \text{ is } C-\text{monotone non-decreasing on } \mathbb{R}^+_0; \}
\end{align*}
\]

\( (L) \)

We recall that \( l \) is said to be \( C-\text{monotone non decreasing} \) on \( \mathbb{R}^+_0 \) if, for some \( C \geq 1 \),

\[
\sup_{s \in [0, t]} l(s) \leq C l(t), \quad \forall t \in \mathbb{R}^+_0.
\]

Clearly, if \( l \) is monotone non decreasing on \( \mathbb{R}^+_0 \), then it is 1-monotone non-decreasing on the same set; in fact the above condition allows a controlled oscillatory behaviour of \( l \) on \( \mathbb{R}^+_0 \). To express our next requests, from now on we assume that

\[
\frac{t \varphi'(t)}{l(t)} \in L^1(0^+) \backslash L^1(+\infty), \quad \frac{\varphi(t)}{l(t)} = o(1) \quad \text{as} \quad t \to 0^+. \quad \text{(\( \Phi \& L \))}
\]

Note that often (e.g. in the case of the \( p \)-Laplacian) the latter condition directly assures integrability at \( 0^+ \) in the former. We define

\[
K(t) = \int_0^t \frac{s \varphi'(s)}{l(s)} \, ds;
\]

(15)

observe that \( K : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \) is a \( C^1 \)-diffeomorphism with

\[
K'(t) = \frac{t \varphi'(t)}{l(t)} > 0,
\]

thus the existence of the increasing inverse \( K^{-1} : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \). Finally we set

\[
F(t) = \int_0^t f(s) \, ds.
\]
Definition 1.1. The generalized Keller-Osserman condition for inequality
\[ \Delta_{H^m} \varphi u \geq f(u)l(|\nabla_{H^m} u|_{H^m}) \]
is the request:
\[ \frac{1}{K^{-1}(F(t))} \in L^1(+\infty). \] \hfill (KO)

Note that, in the case of the $p$-Laplace operator and $l \equiv 1$, (KO) coincides with the well known Keller-Osserman condition for the $p$-Laplacian, that is,
\[ \frac{1}{F(t)} \in L^1(+\infty). \]

In order to deal with the presence of the density function $\psi$ in the version of our inequalities that we shall describe below, we need to assume two “relaxed homogeneity” requests on $\varphi'$ and $l$:
\[ s\varphi'(st) \leq Ds^\tau \varphi'(t), \quad \forall s \in [0,1], \quad t \in \mathbb{R}_0^+, \] \hfill (\Phi 2)
\[ s^{1+\tau}l(t) \leq \Lambda l(st), \quad \forall s \in [0,1], \quad t \in \mathbb{R}_0^+, \] \hfill (L 2)
for some positive constants $D, \Lambda > 0$ and $\tau \geq 0$. We stress that (L 2) is a mild requirement: for example, it is satisfied by every $l(t)$ of the form
\[ l(t) = \sum_{k=0}^{N} C_k t^{\nu_k}, \quad N \in \mathbb{N}, \quad C_k \geq 0, \quad -\infty < \nu_k \leq 1 + \tau \quad \text{for every } k. \]
Indeed, since $s \leq 1$ we have
\[ l(st) = \sum_{k=0}^{N} C_k s^{\nu_k} t^{\nu_k} \geq \sum_{k=0}^{N} C_k s^{1+\tau} t^{\nu_k} = s^{1+\tau}l(t). \]

Note also that, if (L2) is true for some $\tau_o$, then it also holds for every $\tau \geq \tau_o$. This is interesting in the case of the $p$-Laplacian, which trivially satisfies (L 2) for every $0 \leq \tau \leq p - 1$. In this case the choice $\tau = p - 1$ is the least demanding on $l(t)$. We also observe that the coupling of (\Phi 2) and (L 2) does not automatically imply the integrability at $0^+$ in $[\Phi \& L]$. For instance if $\varphi(t) = t^\tau$ and $l(t) = t^{\tau+1}$, then (\Phi 2) and (L 2) are satisfied, but $\frac{\varphi'(t)}{l(t)} \notin L^1(0^+)$. We shall prove the following Liouville-type result:

**Theorem 1.1.** Let $\varphi$, $f$, $l$ satisfy (\Phi), (\Phi 2), (L 2) and $[\Phi \& L]$. Suppose also the validity of the relaxed homogeneity conditions (\Phi 2), (L 2). If the generalized Keller-Osserman condition (KO) holds, then every solution $0 \leq u \in C^1(H^m)$ of
\[ \Delta_{H^m} \varphi u \geq f(u)l(|\nabla_{H^m} u|_{H^m}) \quad \text{on } H^m \] holds,
\[ \frac{1}{K^{-1}(F(t))} \in L^1(+\infty). \]
is constant. Moreover, if \( l(0) > 0 \), then \( u \equiv 0 \).

The proof is achieved through the construction of a “radial” supersolution \( v \) of (16) (see the next section for the precise definition) on an annular region \( B_T \setminus B_t \), \( 0 < t_0 < T \), which is small near \( \partial B_t \) and blows up at \( \partial B_T \). A careful comparison between \( u \) and \( v \) allows us to conclude that \( u \) must necessarily be constant. As opposed to Osserman’s approach (see [8]), in order to construct the supersolution we have not tried to solve the radialization of (16), since the presence of the gradient term may cause different behaviours near the first singular time. Roughly speaking, even if we could prove the local existence of a radial solution in a neighborhood of zero (which is not immediate due to the singularity of \( 1/r \) and possibly of \( \varphi' \) in 0), we cannot be sure that, in case the interval of definition is \([0, T)\), \( T < +\infty \), the solution blows up at time \( T \): \( \textit{a priori} \), it may even happen that the solution remains bounded, but the first derivative blows up, giving rise to some sort of cusp. The necessity of excluding this case led us to a different approach: a blowing-up supersolution is explicitly constructed, exploiting directly the Keller-Osserman condition. Beside being elementary, this alternative method also reveals the reason why \((KO)\) is indeed natural as an optimal condition for the existence or non-existence of solutions.

As it will become apparent from the proof of Theorem 1.1 below, the result can be restated on the Euclidean space \( \mathbb{R}^m \) getting rid of request \((\Phi_2)\) and \((L_2)\), which are related to the density function \( \psi \). Indeed we have

**Theorem 1.2.** Let \( \varphi, f, l \) satisfy \((\Phi)\), \((\text{F})\), \((\text{L})\), \((\Phi \& \text{L})\) and the generalized Keller-Osserman condition \((KO)\). Let \( u \in C^1(\mathbb{R}^m) \) be a non-negative solution of

\[
\Delta_{\mathbb{R}^m} u = \text{div} \left( |\nabla u|^{-1} \varphi(|\nabla u|) \nabla u \right) \geq f(u)l(|\nabla u|) \quad \text{on} \quad \mathbb{R}^m. \tag{17}
\]

Then \( u \) is constant. Moreover, if \( l(0) > 0 \), then \( u \equiv 0 \).

To show the sharpness of \((KO)\), we produce a global unbounded subsolution of (14) when \((KO)\) is violated. For simplicity we only deal with the case of the \( p \)-Laplacian and we prove the following:

**Theorem 1.3.** Assume the validity of \((\text{F})\) and \((\text{L})\). Suppose that

\[
\frac{t^{p-1}}{l(t)} = o(1) \quad \text{as} \quad t \to 0^+ , \quad l(t) \leq B_1 + B_2 t^\mu \quad \forall \ t \in \mathbb{R}_0^+, \quad (p \& \text{L})
\]

where \( B_1, B_2 > 0 \) and \( 0 \leq \mu < 1 \). Assume also the relaxed homogeneity condition

\[
l(t)s^\mu \leq \Lambda l(st) \quad \forall \ s \in [0, 1], \ t \in \mathbb{R}_0^+ . \quad (L2_p)
\]
2 Preliminaries

Then the following conditions are equivalent:

(i) there exists a non-negative, non-constant solution $u \in C^1(H^m)$ of inequality $\Delta^p_{H^m} u \geq f(u)l(|\nabla_{H^m} u|_{H^m})$;

(ii) $\frac{1}{K^{-1}(F(t))} \notin L^1(+\infty)$.

As for Theorem 1.1, we can state the analogous result in Euclidean setting: in this latter case, assumption $(L^2_p)$ is unnecessary. We would like to stress that the subsolution constructed to prove the necessity part of the Keller-Osserman condition is unbounded. This fact is not accidental: indeed, in Section 5 we shall prove that, under all the assumptions of Theorem 1.1 but $(KO)$, bounded subsolutions still have to be constant.

In the last part of the paper we show how the techniques introduced can be implemented to study differential inequalities of the form

$$\Delta^p_{H^m} u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m}),$$

(18)

where the functions appearing in the RHS of the above are non-negative. The main results obtained are Theorem 6.3, that is, triviality of the solutions in the general setting under an appropriate Keller-Osserman condition, and Theorem 6.6 for the $p$-Laplace operator, where we show the sharpness of the condition in analogy with Theorem 1.3. Details appear in Section 6 below.

2 Preliminaries

The aim of this section is to introduce an explicit formula for the $\varphi$-Laplacian acting on radial functions and the appropriate notion of weak solution of differential inequalities of the type of (14) or, more generally, (18).

2.1 “Radialization” of the $\varphi$-Laplacian

Consider a radial function, that is, a function of the form

$$u(q) = \alpha(r(q)), \quad q \in H^m,$$

(19)

where $\alpha : R^+_0 \to R, \alpha \in C^2(R^+_0)$.

Now, a straightforward but somewhat lengthy computation yields the expression:

$$\Delta^p_{H^m} u = \sqrt{\psi} \left[ \sqrt{\psi} \varphi' \left( |\alpha'(r)| \sqrt{\psi} \right) \alpha''(r) + \frac{2m+1}{r} \operatorname{sgn} \alpha'(r) \varphi \left( |\alpha'(r)| \sqrt{\psi} \right) \right].$$

(20)
It is worth to stress the following property, which allows us to shift the origin for the Koranyi distance from $o$ to any other point $q_0$: if we denote with $\bar{r}(q) = d(q_0, q) = r(q_0^{-1} \circ q)$, a calculation shows that

$$[X_j(\bar{r})](q) = [X_j(r)](q_0^{-1} \circ q), \quad [Y_j(\bar{r})](q) = [Y_j(r)](q_0^{-1} \circ q),$$

hence we obtain the invariance with respect to the left multiplication

$$\Delta_H^\varphi(\alpha \circ \bar{r})(q) = \Delta_H^\varphi(\alpha \circ r)(q_0^{-1} \circ q). \quad (21)$$

The above relation will come in handy in what follows.

### 2.2 Weak formulation

In this section we derive a weak formulation for the differential inequality \((14)\). In order to simplify the notation, let us first introduce the function

$$A(t) = t^{-1} \varphi(t), \quad A(t) \in C^0(\mathbb{R}^+). \quad (22)$$

With the help of the matrix $B = B(q)$ (see [3], pg. 294), defined by

$$B(q) = B(z, t) = \begin{pmatrix} \quad I_{2m} & 2y_1 \\ \vdots & \ddots & \ddots \\ -2x_1 & \cdots & -2x_m & 4|z|^2 \end{pmatrix},$$

we can write the $\varphi$-Laplacian in divergence form. Indeed, indicating from now on with $\text{div}$, $\nabla$ and $\langle \ , \ \rangle$ respectively the ordinary Euclidean divergence, gradient and scalar product in $\mathbb{R}^{2m+1}$, given $u \in C^2(H^m)$ we have

$$\Delta_H^\varphi u = \sum_j [X_j(A(|\nabla_{H^m} u|_{H^m})X_j u) + Y_j(A(|\nabla_{H^m} u|_{H^m})Y_j u)] =$$

$$= \sum_j [A(|\nabla_{H^m} u|_{H^m})X_j(X_j u) + X_j(A(|\nabla_{H^m} u|_{H^m})X_j u +$$

$$+ A(|\nabla_{H^m} u|_{H^m})Y_j(Y_j u) + Y_j(A(|\nabla_{H^m} u|_{H^m})Y_j u] =$$

$$= A(|\nabla_{H^m} u|_{H^m}) \text{div}(B \nabla u) + \nabla_{H^m} A(|\nabla_{H^m} u|_{H^m}) \cdot \nabla_{H^m} u,$$
where with \( B \nabla v \) we mean the vector in \( \mathbb{R}^{2m+1} \) whose components in the standard basis \( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}, \frac{\partial}{\partial t} \) are given by the matrix multiplication of \( B \) with the components of \( \nabla v \) in the same basis. Having made this precise, it is easy to see that \( B \nabla v = \nabla H_m v \). Now, a standard check shows that, for \( u,v \in C^1(H^m) \),
\[
\langle \nabla u, B \nabla v \rangle = \nabla H_m u \cdot \nabla H_m v. \tag{23}
\]
Then, going back to the previous computation we have
\[
\Delta^\varphi_{H_m} u = A(\nabla H_m u) \text{div} (B \nabla u) + \langle \nabla A(\nabla H_m u), B \nabla u \rangle = \text{div} (A(\nabla H_m u)B \nabla u),
\]
which is the desired expression. Note that, when \( \varphi(t) = t \), the above becomes the well-known formula (see, e.g., [6] and [3]) for the Kohn-Spencer Laplacian, that is, \( \Delta^\varphi_{H_m} u = \text{div}(B \nabla u) \). It follows that (14) can be interpreted in the weak sense as follows: for all \( \zeta \in C_0^\infty(H^m) \), \( \zeta \geq 0 \), we have
\[
\int_{\mathbb{R}^{2m+1}} \zeta \Delta^\varphi_{H_m} u = \int_{\mathbb{R}^{2m+1}} \zeta \text{div} (A(\nabla H_m u)B \nabla u) = \int_{\mathbb{R}^{2m+1}} A(\nabla H_m u) \langle B \nabla u, \nabla \zeta \rangle = \int_{\mathbb{R}^{2m+1}} A(\nabla H_m u) \nabla H_m u \cdot \nabla H_m \zeta,
\]
and thus the weak form is
\[
- \int_{\mathbb{R}^{2m+1}} A(\nabla H_m u) \nabla H_m u \cdot \nabla H_m \zeta \geq \int_{\mathbb{R}^{2m+1}} f(u)(\nabla H_m u) \zeta \quad \text{(24)}
\]
as expected. Hence, an entire weak classical solution of (14) is a function \( u \in C^1(H^m) \) such that, for all \( \zeta \in C_0^\infty(H^m) \), \( \zeta \geq 0 \), (24) is satisfied. A similar definition of course holds for the differential inequality (18).

### 3 Proof of Theorem 1.1

In order to prove Theorem 1.1 we shall need a comparison theorem and a maximum principle which are well-known for the Kohn-Spencer Laplacian (see [2]). Here we briefly prove the corresponding statements for the \( \varphi \)-Laplacian that we shall use below, basing on ideas taken from [10] and [11]. Throughout Subsections 3.1 and 3.2 we shall assume \([9]\) and \([12]\).
3 Proof of Theorem 1

3.1 Comparison principle

**Proposition 3.1.** Let \( \Omega \subset H^m \) be a relatively compact domain with \( C^1 \) boundary. Let \( u, v \in C^0(\overline{\Omega}) \cap C^1(\Omega) \) satisfy

\[
\begin{cases}
\Delta_{H^m}^\varphi u \geq \Delta_{H^m}^\varphi v & \text{on } \Omega \\
u \leq v & \text{on } \partial \Omega.
\end{cases}
\]  

(25)

Then \( u \leq v \) on \( \Omega \).

**Proof.** The proof basically follows the one in [9] pp.85–86. However, we reproduce the steps for the sake of completeness. Let \( w = v - u \). By contradiction assume that there exists \( \bar{q} \in \Omega \) such that \( w(\bar{q}) < 0 \), and let \( \varepsilon > 0 \) be such that \( w(\bar{q}) + \varepsilon < 0 \). The function \( w_\varepsilon = \min\{w + \varepsilon, 0\} \) has compact support in \( \Omega \), hence \( -w_\varepsilon \) is an admissible Lipschitz test function. The weak definition of (25), together with the divergence form of \( \Delta_{H^m}^\varphi \), reads:

\[
0 \geq \int_{\Omega} \left( |\nabla_{H^m} v|^{-1}_{H^m} \varphi(|\nabla_{H^m} v|_{H^m}) B \nabla v - |\nabla_{H^m} u|^{-1}_{H^m} \varphi(|\nabla_{H^m} u|_{H^m}) B \nabla u, \nabla w_\varepsilon \right).
\]

(26)

where \( E = \{q : w(q) < -\varepsilon\} \). We denote with \( h \) the integrand in (26). With the aid of the Cauchy-Schwarz inequality we have

\[
h \geq |\varphi(|\nabla_{H^m} v|_{H^m}) - \varphi(|\nabla_{H^m} u|_{H^m})| |\nabla_{H^m} v|_{H^m} - |\nabla_{H^m} u|_{H^m}| \geq 0,
\]

(27)

where the latter inequality is due to the monotonicity of \( \varphi \).

It follows from (26) and (27) that \( 0 \geq \int_{\Omega} h \geq 0 \), hence \( h = 0 \) a.e. on \( \Omega \).

This implies that \( |\nabla_{H^m} u|_{H^m} = |\nabla_{H^m} v|_{H^m} \) on \( E \), and therefore

\[
0 = h = |\nabla_{H^m} u|^{-1}_{H^m} \varphi(|\nabla_{H^m} u|_{H^m}) \langle B \nabla (v - u), \nabla (v - u) \rangle = |\nabla_{H^m} u|^{-1}_{H^m} \varphi(|\nabla_{H^m} u|_{H^m}) |\nabla (v - u)|^2_{H^m}.
\]

This shows that

\[
|\nabla (w_\varepsilon)|^2_{H^m} = 0,
\]

(28)

whence \( w_\varepsilon \) is constant. Indeed, from (28) we have \( X_j(w_\varepsilon) = Y_j(w_\varepsilon) = 0 \) for every \( j = 1 \ldots m \), and using the commutation law (2) we also have \( \partial w_\varepsilon / \partial t = 0 \); recalling the definition of \( X_j \) and \( Y_j \), all the components of the Euclidean gradient of \( w_\varepsilon \) vanish, proving the constancy of \( w_\varepsilon \). Since \( w_\varepsilon(\bar{q}) < 0 = w_\varepsilon|_{\partial \Omega} \) we reach the desired contradiction. \( \square \)
3 Proof of Theorem 1.1

3.2 Maximum principle

Proposition 3.2. Let \( \Omega \subset \mathbb{H}^m \) be a domain. Let \( u \in C^0(\overline{\Omega}) \cap C^1(\Omega) \) satisfy
\[
\Delta_{\mathbb{H}^m}^\varphi u \geq 0 \quad \text{in } \Omega \quad (29)
\]
and let \( u^* = \sup_{\Omega} u \). If \( u(q_M) = u^* \) for some \( q_M \in \Omega \), then \( u \equiv u^* \).

Proof. By contradiction, suppose the existence of a solution \( u \) of (29) and of \( q_M \in \Omega \) such that \( u(q_M) = u^* \), but \( u \not\equiv u^* \). Set \( \Gamma = \{ q \in \Omega : u(q) = u^* \} \). Let \( \delta > 0 \) and define
\[
\Omega^+ = \{ q \in \Omega : u^* - \delta < u(q) < u^* \}; \quad \Gamma_\delta = \{ q \in \Omega : u(q) = u^* - \delta \}; \quad (30)
\]
note that \( \partial \Omega^+ \cap \Omega = \Gamma \cup \Gamma_\delta \). Let \( q' \in \Omega^+ \) be such that \( d(q', \Gamma) < d(q', \Gamma_\delta) \), \( d(q', \Gamma) < d(q', \partial \Omega) \) (31) (this is possible up to choosing \( q' \) sufficiently close to \( q_M \)). Let \( B_R(q') \) be the largest Koranyi ball centered at \( q' \) and contained in \( \Omega^+ \). Then, by construction \( u < u^* \) in \( B_R(q') \) while \( u(q_0) = u^* \) for some \( q_0 \in \partial B_R(q') \). Since \( q_0 \) is an absolute maximum for \( u \) in \( \Omega \), we have \( \nabla u(q_0) = 0 \).

Now we construct an auxiliary function. Towards this aim, we consider the annular region
\[
E_R(q') = \overline{B_R(q')} \setminus B_{R/2}(q') \subset \Omega^+; \quad (32)
\]
we fix \( a \in (u^* - \delta, u^*) \) to be determined later and consider the following problem
\[
\begin{aligned}
\left[ \varphi(z') \right]' + \frac{2m+1}{4} \varphi(z') & \leq 0 \quad \text{in } (R/2, R) \\
z(R/2) = a, \quad z(R) = u^* \\
u^* - \delta \leq z \leq u^*, \quad z' > 0 \quad \text{in } [R/2, R].
\end{aligned} \quad (33)
\]
Notice that, for example, the function
\[
z(t) = \int_{R/2}^t \varphi^{-1}\left( \frac{c}{s^{2m+1}} \right) ds + a \quad (34)
\]
satisfies (33) for some suitable constant \( c \).

Using the invariance property (21), such a function gives rise to a \( C^2 \)-solution \( v(q) = z(\tilde{r}(q)) \), where \( \tilde{r}(q) = r(q'^{-1} \circ q) \), of
\[
\begin{aligned}
\Delta_{\mathbb{H}^m}^\varphi v & \leq 0 \quad \text{in } E_R(q') \\
v = a & \quad \text{on } \partial B_{R/2}(q'), \quad v = u^* \quad \text{on } \partial B_R(q') \\
u^* - \delta < v \leq u^*
\end{aligned} \quad (35)
\]
Indeed by hypothesis (Φ2) we have
\[
\Delta_H^m v \leq D \left( \sqrt{\psi} \right)^{r+1} \left\{ [\varphi(z')]' + \frac{2m+1}{t} \varphi(z') \right\} \leq 0.
\] (36)

It is important to point out that there exists a positive constant \(\lambda > 0\) such that
\[\langle \nabla v, \nabla \bar{r} \rangle = z'(\bar{r}) |\nabla \bar{r}|^2 \geq \lambda > 0 \text{ on } \partial E_R(q'); \] (37)

this follows since \(\bar{r}\) differs from \(r\) by a translation of the Heisenberg group (that is, a diffeomorphism), and \(|\nabla r|^2 = \frac{1}{4t} (|z|^6 + \frac{t^2}{4})\) only vanishes at the origin \(o\).

Next we choose \(a \in (u^* - \delta, u^*)\) close enough to \(u^*\) so that \(u \leq v\) on \(\partial B_{R/2}(q')\); this is possible since \(\partial B_{R/2}(q') \subset \subset \Omega^+\) and thus \(\max_{\partial B_{R/2}(q')} u < u^*\). Now \(u, v \in C^0(E_R(q')) \cap C^1(E_R(q'))\) and, since \(v \equiv u^*\) on \(\partial B_{R}(q')\), they satisfy
\[
\begin{cases}
\Delta_H^m u \geq \Delta_H^m v & \text{on } E_R(q') \\
u \leq v & \text{on } \partial E_R(q').
\end{cases}
\] (38)

Then by Proposition 3.1 we have \(u \leq v\) on \(E_R(q')\).

Let us consider the function \(v - u\): it satisfies \(v - u \geq 0\) on \(E_R(q')\) and \(v(q_0) - u(q_0) = u^* - u^* = 0\), so that \(\langle \nabla (v - u), \nabla \bar{r} \rangle(q_0) \leq 0\). Therefore
\[
0 = \langle \nabla u, \nabla \bar{r} \rangle(q_0) \geq \langle \nabla v, \nabla \bar{r} \rangle(q_0) > 0,
\] (39)
a contradiction. \(\square\)

**Remark 3.3.** Obviously, one can state an analogous minimum principle using the substitution \(v(q) = -u(q)\); however, a direct proof of the minimum principle following the above steps reveals some further difficulties due to the density function, which is not bounded from below away from zero.

### 3.3 Construction of the supersolution

In order to construct the radial supersolution for (14) we point out the validity of the next technical Lemma. We refer to the Introduction for notations and properties.

**Lemma 3.4.** Let \(\sigma \in (0, 1]\); then the generalized Keller-Osserman condition \([KO]\) implies
\[
\frac{1}{K^{-1}(\sigma F(t))} \in L^1(+\infty).
\] (40)
Proof. We perform the change of variables $t = s\sigma$ to have
\[
\int_{-\infty}^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))} = \sigma^{-1} \int_{-\infty}^{+\infty} \frac{dt}{K^{-1}(\sigma F(\sigma^{-1}t))}.
\]
Since $f$ and $K^{-1}$ are increasing by assumption, we get
\[
F(\sigma^{-1}t) = \int_{0}^{\sigma^{-1}t} f(z) \, dz = \sigma^{-1} \int_{0}^{t} f(\sigma^{-1}\xi) \, d\xi \geq \sigma^{-1} \int_{0}^{t} f(\xi) \, d\xi = F(t)
\]
and
\[
K^{-1}(\sigma F(\sigma^{-1}t)) \geq K^{-1}(F(t)),
\]
thus
\[
\int_{-\infty}^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))} \leq \sigma^{-1} \int_{-\infty}^{+\infty} \frac{dt}{K^{-1}(F(t))} < +\infty. \tag{41}
\]

Here is the construction of the supersolution.

Lemma 3.5. Suppose the validity of (Φ), (F), (L), (Φ & L) and of the Keller-Osserman (KO). Fix $0 < \varepsilon < \eta$ and $0 < t_0 < t_1$. Then, for every $\tilde{B} > 0$ there exist $T > t_1$ and a strictly increasing, convex function
\[
\alpha : [t_0, T) \to [\varepsilon, +\infty)
\]
satisfying
\[
\begin{cases}
(\varphi(\alpha'))' + \frac{2m+1}{2} \varphi(\alpha') \leq \tilde{B} f(\alpha) l(\alpha'); \\
\alpha(t_0) = \varepsilon, \quad \alpha(t_1) \leq \eta; \\
\alpha(t) \uparrow +\infty \text{ as } t \to T^-.
\end{cases} \tag{42}
\]

Proof. Consider $\sigma \in (0, 1]$ to be determined later and choose $T_\sigma > t_0$ such that
\[
T_\sigma - t_0 = \int_{\varepsilon}^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))}.
\]
Note that the RHS is well defined by Lemma 3.4 and, since it diverges as $\sigma \to 0^+$, up to choosing $\sigma$ sufficiently small we can shift $T_\sigma$ in such a way that $T_\sigma > t_1$. We implicitly define the $C^2$-function $\alpha(t)$ by requiring
\[
T_\sigma - t = \int_{\alpha(t)}^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))} \quad \text{on } [t_0, T_\sigma).
\]
We observe that, by construction, $\alpha(t_0) = \varepsilon$ and, since $K > 0$, $\alpha(t) \uparrow +\infty$ as
We deduce that is, the trivial identity

Using \( \Phi \) and \( \mathcal{L} \), we deduce the following chain of inequalities:

\[
\left[ \varphi(\alpha') \right]' + \frac{2m+1}{t} \varphi(\alpha') = \\
\sigma f(\alpha)l(\alpha') + \frac{2m+1}{t} \varphi(\alpha'(t_0)) + \frac{2m+1}{t} \int_{t_0}^{t} f(\alpha(s))l(\alpha'(s)) \, ds = \\
\left[ \sigma + \frac{2m+1}{t} \frac{\varphi(\alpha'(t_0))}{f(\alpha(t))l(\alpha'(t))} + \frac{2m+1}{t} \int_{t_0}^{t} \frac{f(\alpha(s))l(\alpha'(s))}{f(\alpha(t))l(\alpha'(t))} \, ds \right] f(\alpha(t))l(\alpha'(t)) \leq \\
\left[ \sigma + \frac{2m+1}{t} \frac{\varphi(\alpha'(t_0))}{f(\alpha(t))l(\alpha'(t))} + \frac{2m+1}{t} \frac{\sigma f(\alpha(t))l(\alpha'(t))}{f(\alpha(t))l(\alpha'(t))} \right] f(\alpha(t))l(\alpha'(t)),
\]

that is,

\[
\left[ \varphi(\alpha') \right]' + \frac{2m+1}{t} \varphi(\alpha') \leq \left[ \frac{2m+1}{t} \frac{\varphi(\alpha'(t_0))}{f(\alpha(t))l(\alpha'(t_0))} + 2(m+1)\sigma \right] f(\alpha(t))l(\alpha'(t)). \tag{43}
\]

Since \( K(0) = 0 \), \( \alpha'(t_0) = K^{-1}(\sigma F(\varepsilon)) \to 0 \) as \( \sigma \to 0 \), and using \( \Phi \land \mathcal{L} \), choosing \( \sigma \) small enough we can estimate the whole square bracket with \( \tilde{B} \) to show the validity of the first of \( \Phi \).

It remains to prove that, possibly with a further reduction of \( \sigma \), \( \alpha(t_1) \leq \eta \). From the trivial identity

\[
\int_{t_1}^{\infty} \frac{ds}{K^{-1}(\sigma F(s))} = T_\sigma - t_1 = (T_\sigma - t_0) + (t_0 - t_1) = \int_{t}^{\infty} \frac{ds}{K^{-1}(\sigma F(s))} + (t_0 - t_1),
\]
we deduce
\[ \int_{\varepsilon}^{\alpha(t_1)} \frac{ds}{K^{-1}(\sigma F(s))} = t_1 - t_0. \]
It suffices to choose \( \sigma \) such that \( \int_{\varepsilon}^{\eta} \frac{ds}{K^{-1}(\sigma F(s))} > t_1 - t_0 \); then obviously \( \alpha(t_1) \leq \eta \).
This completes the proof of the Lemma.

### 3.4 Last step of the proof

We denote with \( u^* = \sup u \) and we first suppose that \( u^* < +\infty \). We reason by contradiction and assume \( u \neq u^* \); by Proposition 3.2 \( u < u^* \) on \( H^m \). Choose \( r_0 > 0 \) and define
\[ u_0^* = \sup_{B_{r_0}} u < u^*. \]
Fix \( \eta > 0 \) sufficiently small such that \( u^* - u_0^* > 2\eta \), and choose \( \tilde{q} \in H^m \setminus \overline{B}_{r_0} \) such that \( u(\tilde{q}) > u^* - \eta \).
We then define \( \tilde{r} = r(\tilde{q}) \) and we construct the radial function \( v(q) = \alpha(r(q)) \) on \( B_{\tilde{r}} \setminus B_{r_0} \), with \( \alpha \) and \( T > \tilde{r} \) as in Lemma 3.5, \( \tilde{B} = 1/(\Lambda D) \), and satisfying the further requirement:
\[ \varepsilon \leq v \leq \eta \quad \text{on} \quad B_{\tilde{r}} \setminus B_{r_0}. \]
We observe that \( v \) is a supersolution for (14). Towards this aim, first we note that by integration, (39) and \( s \in [0,1] \), (40) implies the inequality
\[ \varphi(st) \leq Ds^\tau \varphi(t), \quad t \in R_0^+, \quad s \in [0,1]. \quad (44) \]
Next, considering the radial expression (20), using (11), (42), (43) and Lemma 3.5, we have
\[ \Delta_{H^m}^\varphi, \alpha(r(q)) = \sqrt{\psi} \left[ \sqrt{\psi} \varphi' \left( \alpha'(r) \sqrt{\psi} \right) \alpha''(r) + \frac{2m + 1}{r} \varphi \left( \alpha'(r) \sqrt{\psi} \right) \right] \leq \]
\[ \leq \left( \sqrt{\psi} \right)^{1+\tau} D \left[ \varphi'(\alpha'(r)) \alpha''(r) + \frac{2m + 1}{r} \varphi(\alpha'(r)) \right] \leq \]
\[ \leq \left( \sqrt{\psi} \right)^{1+\tau} D \left[ \frac{1}{\Lambda D} f(\alpha(r)) l(\alpha'(r)) \right] \leq \]
\[ \leq f(\alpha(r)) l(\alpha'(r) \sqrt{\psi}) = f(\alpha(r)) l(|\nabla H = \alpha(r)|_{H^m}). \]
Moreover
\[ u(\tilde{q}) - v(\tilde{q}) > u^* - \eta - \eta = u^* - 2\eta, \]
and, on \( \partial B_{r_0} \),
\[ u(q) - v(q) \leq u_0^* - \varepsilon < u^* - 2\eta - \varepsilon. \]
Thus, considering the difference \( u - v \) on the annular region \( B_T \setminus B_{r_0} \), since by construction
\[
u(q) - v(q) \to -\infty \quad \text{as } r(q) \to T,
\]

it follows that \( u - v \) attains a positive maximum \( \mu \) in \( B_T \setminus B_{r_0} \). Let \( \Gamma_\mu \) be a connected component of
\[
\{ q \in B_T \setminus B_{r_0} : u(q) - v(q) = \mu \}.
\]
Let \( \xi \in \Gamma_\mu \) and note that \( u(\xi) > v(\xi) \) and \(|\nabla_{H^m} u(\xi)|_{H^m} = |\nabla_{H^m} v(\xi)|_{H^m}|.\) As a consequence, since \( f \) is strictly increasing,
\[
\Delta_{H^m}^u(\xi) \geq f(u(\xi))l(|\nabla_{H^m} u(\xi)|_{H^m}) > f(v(\xi))l(|\nabla_{H^m} v(\xi)|_{H^m}) \geq \Delta_{H^m}^v(\xi).
\]

By continuity, there exists an open set \( V \supset \Gamma_\mu \) such that
\[
\Delta_{H^m}^u \geq \Delta_{H^m}^v \quad \text{on } V. \tag{45}
\]
Fix now \( \xi \in \Gamma_\mu \) and a parameter \( 0 < \rho < \mu \); let \( \Omega_{\xi, \rho} \) be the connected component containing \( \xi \) of the set
\[
\{ q \in B_T \setminus B_{r_0} : u(q) > v(q) + \rho \}.
\]
We observe that \( \xi \in \Omega_{\xi, \rho} \) for every \( \rho \) and that \( \Omega_{\xi, \rho} \) is a nested sequence as \( \rho \) converges to \( \mu \). We claim that if \( \rho \) is close to \( \mu \), then \( \overline{\Omega_{\xi, \rho}} \subset V \). This can be shown by a compactness argument such as the following: since \( \Gamma_\mu \) is closed and bounded, there exists \( \varepsilon > 0 \) such that \( d(V^c, \Gamma_\mu) \geq \varepsilon \). Suppose, by contradiction, that there exist sequences \( \rho_n \uparrow \mu \) and \( \{q_n\} \) such that \( q_n \in \Omega_{\xi, \rho_n} \) and \( d(q_n, \Gamma_\mu) > \varepsilon \). Then, we can assume that the sequence is contained in \( \Omega_{\xi, \rho_0} \), which, by construction, has compact closure; passing to a subsequence converging to some \( \overline{\eta} \), we have by continuity
\[
d(\overline{\eta}, \Gamma_\mu) \geq \varepsilon, \tag{46}
\]
but, on the other hand, \( (u - v)(\overline{\eta}) = \lim_n (u - v)(q_n) \geq \lim_n \rho_n = \mu \), hence \( \overline{\eta} \in \Gamma_\mu \) and this contradicts \( \Gamma_\mu \). Therefore, \( d(\partial \Omega_{\xi, \rho}, \Gamma_\mu) \to 0 \) as \( \rho \to \mu \), and the claim is proved.

On \( \partial \Omega_{\xi, \rho} \) we have \( u(q) = v(q) + \rho \); since \( v(q) + \rho \) solves
\[
\Delta_{H^m}^v(\rho + v) = \Delta_{H^m}^v \leq f(v(l(|\nabla_{H^m} v||H^m)) \leq f(v + \rho)(l(|\nabla_{H^m} (v + \rho)||H^m)).
\]
by Proposition 3.1
\[ u(q) \leq v(q) + \rho. \]
But \( u(\xi) = v(\xi) + \mu \) and \( \xi \in \Omega_{\xi, \rho} \), a contradiction. The case \( u^* = +\infty \) is easier and can be treated analogously. This shows that \( u \equiv c \), where \( c \) is a non-negative constant; in case \( l(0) > 0 \) we have \( 0 = \Delta_{H^m}^c \geq f(c)l(0) \). This implies \( f(c) = 0 \), hence \( c = 0 \).

## 4 Proof of Theorem 1.3

This section is devoted to proving the result stated in Theorem 1.3; first of all we observe that the sufficiency of the Keller-Osserman condition, i.e. implication \( ii) \Rightarrow i) \), follows from Theorem 1.1. In particular, it is easy to see that \( L_{p, L} \) implies \( L_{2, L} \) and that \( L_{2, p} \) implies \( L_{2} \). This latter follows since \( \Delta_{H^m}^c \) satisfies \( (\Phi) \) for every \( 0 \leq \tau \leq p - 1 \) (as we have already pointed out), and \( \tau = p - 1 \) is the best choice. Our aim is therefore to provide existence of unbounded \( C^1 \)-solutions of inequality (16) under the assumption that \( (KO) \) is not satisfied; this will be achieved through a careful pasting of two subsolutions defined on complementary sets. First, we deal with “radial stationary functions”, that is, functions of the form
\[ v(q) = w(|z|), \quad q = (z, t) \in H^m, \]
where \( w : \mathbb{R}_0^+ \to \mathbb{R}, w \in C^2(\mathbb{R}_0^+) \). Performing computations very similar to those in Subsection 2.1 we obtain the following identities:
\[ |\nabla_{H^m}|z||_{H^m} \equiv 1, \quad \Delta_{H^m}|z| = \frac{2m - 1}{|z|}, \]
and thus the expression of the \( \varphi \)-Laplacian for a radial stationary function is
\[ \Delta_{H^m}^c v = \varphi'(|w'(|z|)|)w''(|z|) + \frac{2m - 1}{|z|} \text{sgn} (w'(|z|))\varphi(|w'(|z|)|). \quad (47) \]
This shows that radial stationary functions in the Heisenberg group behave as Euclidean radial ones, and this fact allows us to avoid dealing with the density function.

Now let \( \epsilon > 0 \) and \( \sigma \geq 1 \) to be determined later and define \( w_\sigma(t) \) implicitly by
\[ t = \int_{\epsilon}^{w_\sigma(t)} \frac{ds}{K^{-1}(\sigma F(s))}, \quad (48) \]
The existence of $w_\sigma$ on all $\mathbb{R}^+_0$ is ensured by the negation of the Keller-Osserman condition, through the reversing of Lemma 3.4. Observe that $w_\sigma(0) = \varepsilon$ and $w'_\sigma(t) = K^{-1}(\sigma F(w_\sigma(t))) \geq K^{-1}(F(\varepsilon)) > 0$ on $\mathbb{R}^+_0$. Define

$$t_\sigma = \int_{\varepsilon}^{2\varepsilon} \frac{ds}{K^{-1}(\sigma F(s))},$$

so that $w_\sigma(t_\sigma) = 2\varepsilon$.

The function $u_2(z, t) = w_\sigma(|z|)$ is $C^1$ for $|z| \geq t_\sigma$ and satisfies

$$\Delta_{H^m}^\sigma u_2 \geq \sigma f(u_2)|\nabla_{H^m} u_2|_{H^m} \geq f(u_2)|\nabla_{H^m} u_2|_{H^m}$$

since $\varphi'(w(|z|))w''(|z|) = \sigma f(w(|z|))l(w'(|z|))$ and $\varphi$ is non-negative. Unfortunately, $u_2$ is only Lipschitz on the line $|z| = 0$. One might get rid of this problem modifying the base point of the integral (48), that is, substituting $\varepsilon$ with 0, but then one should require $1/K^{-1}(\sigma F(s)) \in L^1(0^+)$, an assumption which we want to avoid. Therefore we solve the problem by using a gluing technique and pasting together a subsolution defined on $|z| \leq t_\sigma$ and a modification of $u_2$ on $|z| \geq t_\sigma$.

First of all we consider the Cauchy problem

$$\begin{cases}
[\varphi(\alpha')]' = \Theta & \text{on } [0, +\infty) \\
\alpha(0) = \alpha'(0) = 0,
\end{cases}$$

with $\Theta$ a constant to be determined later. This problem has the solution $\alpha \in C^1(\mathbb{R}^+_0) \cap C^2(\mathbb{R}^+_0)$

$$\alpha(t) = \int_0^t \varphi^{-1}(\Theta s) \, ds;$$

note that $\alpha'(t) > 0$ when $t > 0$. Choosing $\Theta = \varphi(1)/t_\sigma$, we have

$$\alpha'(t_\sigma) = 1 \quad \text{and} \quad \alpha(t_\sigma) = \int_0^{t_\sigma} \varphi^{-1}(\Theta s) \, ds \leq t_\sigma,$$

and if we fix an $\varepsilon > 0$ so that $K^{-1}(F(\varepsilon)) > 1$, we also have that

$$\frac{\alpha'(t_\sigma)}{w'_\sigma(t_\sigma)} = \frac{1}{K^{-1}(\sigma F(\varepsilon))} \leq \frac{1}{K^{-1}(F(\varepsilon))} < 1. \quad (49)$$

Furthermore, noting that $t_\sigma \to 0$ as $\sigma \to +\infty$, up to choosing $\sigma$ sufficiently large, we have

$$\alpha(t_\sigma) < \varepsilon, \quad (50)$$
and since $\Theta = \frac{2\epsilon}{t_0} \to +\infty$ as $\sigma \to +\infty$, we can choose $\sigma$ large enough so that

$$f(\alpha(t))l(\alpha'(t)) \leq \Theta \quad \forall t \in [0, t_\sigma].$$

This last condition implies that the composition $u_1(z, t) = \alpha(|z|)$, which is $C^1$ even at $|z| = 0$, satisfies

$$\Delta_H^\varphi u_1 \geq f(u_1)l(|\nabla_H u_1|) \quad \text{on } \overline{B_{T_\sigma}}.$$

Now we need to glue the solutions $u_1$ and $u_2$ together, and to this end we define a real $C^2$-function $\gamma_\sigma : [w_\sigma(t_\sigma), +\infty) \to [\alpha(t_\sigma), +\infty)$ such that

$$\gamma_\sigma(w_\sigma(t_\sigma)) = \alpha(t_\sigma), \quad 0 < \gamma_\sigma' \leq 1, \quad \gamma_\sigma'(w_\sigma(t_\sigma)) = \frac{\alpha'(t_\sigma)}{w_\sigma'(t_\sigma)}, \quad \gamma_\sigma'' \geq 0 \quad (53)$$

Using (49) and (50), it is not hard to see that the above conditions are not contradictory: in particular from $\alpha(t_\sigma) < \epsilon = w_\sigma(0) < w_\sigma(t_\sigma)$ and $\alpha'(t_\sigma) < w_\sigma'(t_\sigma)$, we see that the requests involving $\gamma_\sigma'(t)$ are indeed compatible, and it also holds

$$\gamma_\sigma(t) \leq t \quad \text{on } [w_\sigma(t_\sigma), +\infty).$$

Next, we consider the following function, depending on the parameter $\sigma$:

$$u(z, t) = \begin{cases} 
    u_1(z, t) = \alpha(|z|) & \text{if } |z| \in [0, t_\sigma] \\
    (\gamma_\sigma \circ u_2)(z, t) = (\gamma_\sigma \circ w)(|z|) & \text{if } |z| \in [t_\sigma, +\infty) 
\end{cases} \quad (55)$$

Note that, by construction, $u$ has global $C^1$-regularity even on the cylinder $|z| = t_\sigma$. It remains to prove that, up to choosing $\sigma$ large enough, it is a subsolution of (16) on the whole $H^m$. By (51), we only need to check this for $|z| \geq t_\sigma$, but unfortunately, in order to treat this case, we need to assume some homogeneity conditions which would give $\varphi$ a structure very similar to the one of the $p$-Laplacian. Therefore, it is more enlightening to treat directly the $p$-Laplacian case, where things get simpler. A computation that uses (53), (54).
the $C$-monotonicity of $l$ and the monotonicity of $f$ shows that
\[
\Delta_{H^m}^p u = \gamma'_\sigma [(\gamma'_\sigma)^{p-2}] \nabla_{H^m} u_2 |_{H^m}^{p-2} \Delta_{H^m} u_2 + \\
+ (\gamma'_\sigma)^{p-2} \nabla_{H^m} u_2 |_{H^m}^{p-2} \gamma''_\sigma \nabla_{H^m} u_2 |_{H^m}^{p} + \\
+ (p-2)(\gamma'_\sigma)^2 (\gamma'_\sigma)^{p-3} \nabla_{H^m} u_2 |_{H^m}^{p-3} \nabla_{H^m} u_2 |_{H^m} \cdot \nabla_{H^m} |\nabla_{H^m} u_2|_{H^m} + \\
+ (p-2)\gamma'_\sigma \gamma'_\sigma \nabla_{H^m} u_2 |_{H^m}^{3} (\gamma'_\sigma)^{p-3} \nabla_{H^m} u_2 |_{H^m} \geq \\
\geq (\gamma'_\sigma)^{p-1} (|\nabla_{H^m} u_2|_{H^m}^{p-2} \Delta_{H^m} u_2 + (p-2)|\nabla_{H^m} u_2|_{H^m}^{p-3} \nabla_{H^m} u_2 |_{H^m} \cdot \nabla_{H^m} |\nabla_{H^m} u_2|_{H^m}) = \\
= (\gamma'_\sigma)^{p-1} \Delta_{H^m}^p u_2 \geq (\gamma'_\sigma)^{p-1} \sigma f(u_2) |\nabla_{H^m} u_2|_{H^m} \geq \\
\geq (\gamma'_\sigma (w(t_\sigma)))^{p-1} f(\gamma'_\sigma \sigma u_2) \frac{\sigma}{C} \gamma'_\sigma |\nabla_{H^m} u_2|_{H^m} = \\
\geq \left( \frac{1}{K^{-1}(\sigma F(2\epsilon))}\right)^{p-1} \frac{\sigma}{C} f(u) |\nabla_{H^m} u|_{H^m}. \\
\tag{56}
\]

The proof is now complete provided we show that
\[
\frac{\sigma}{[K^{-1}(\sigma F(2\epsilon))]^{p-1}} \rightarrow +\infty \quad \text{as} \quad \sigma \rightarrow +\infty.
\]

Using the definition of $K$ and the growth condition $L^2_p$, we deduce
\[
K(t) = (p-1) \int_0^t \frac{s^{p-1}}{l(s)} ds \geq (p-1) \int_0^t \frac{s^{p-1}}{B_1 + B_2 s^p} ds \simeq t^{p-\mu} \quad \text{as} \quad t \rightarrow +\infty.
\]

Hence, for some positive constant $\tilde{C}$ we get
\[
K^{-1}(t) \leq \tilde{C} t^{1-p-\frac{1}{\mu}}.
\]

It follows that, since $\mu < 1$,
\[
\frac{\sigma}{[K^{-1}(\sigma F(2\epsilon))]^{p-1}} \geq \frac{\sigma}{\tilde{C}(\epsilon) \sigma^{p-1}} \rightarrow +\infty \quad \text{as} \quad \sigma \rightarrow +\infty.
\]

Up to choosing $\sigma$ sufficiently large we can deduce from (56)
\[
\Delta_{H^m}^p u \geq f(u) |\nabla_{H^m} u|_{H^m} \quad \text{on} \quad |z| \geq t_\sigma
\]
and we have the desired conclusion. To end the proof of the theorem, we note that the $C^1$ regularity of $u$ on the cylinder $|z| = t_\sigma$ and at the origin $o$ makes it necessary to proceed with the weak formulation. Nevertheless, this is a standard matter because of the continuity of $\nabla_{H^m} u$: however, for the sake of complete-
ness, let $\xi \in C^\infty_0(H^m)$ and define

$$V = \{ q = (z,t) \in H^m : |z| < t_\sigma \} \cap \text{supp}(\xi),$$

$$W = \{ q = (z,t) \in H^m : |z| > t_\sigma \} \cap \text{supp}(\xi),$$

$$\Gamma = \{ q = (z,t) \in H^m : |z| = t_\sigma \} \cap \text{supp}(\xi).$$

Through a suitable partition of unity, we can find $\xi_1, \xi_2 \in C^\infty_0(H^m)$ such that $\xi = \xi_1 + \xi_2$ and

$$\text{supp}(\xi_1) \subset \{ (z,t) \in H^m : |z| < t_\sigma \}, \quad \text{supp}(\xi_2) \subset \{ (z,t) \in H^m : |z| > \frac{t_\sigma}{2} \}.$$

Because of linearity, it is sufficient to show inequality (10) for $\xi_1$ and $\xi_2$. For $\xi_1$ the weak formulation of (10) is immediate: indeed, on $\text{supp}(\xi_1)$, $u \equiv u_1$ which solves (52) weakly. Hence, we only need to consider $\xi = \xi_2$. Using the weak formulation (24), the definition of $u$, $V$, $W$, and remembering that

(i) $u_1, u_2$ are pointwise subsolutions on $V\setminus \{|z| = 0\}$, $W$ respectively, with non-vanishing gradient,

(ii) $\xi = 0$ in a neighborhood of $\{|z| = 0\}$,

we deduce, denoting with $\nu_V$ and $\nu_W$ the (Euclidean) normals to $\partial V$ and $\partial W$:

$$\int_{H^m} \nabla_{H^m} u_1 p^{-2}_{H^m}(B \nabla u_1, \nabla \xi) = \int_V \nabla_{H^m} u_1 p^{-2}_{H^m}(B \nabla u_1, \nabla \xi) +$$

$$+ \int_W (\gamma_{\sigma}') p^{-1} \nabla_{H^m} u_2 p^{-2}_{H^m}(B \nabla u_2, \nabla \xi) = \int_{\partial V} \nabla_{H^m} u_1 p^{-2}_{H^m}(B \nabla u_1, \nu_V) \xi - \int_V \xi \Delta_{H^m} u_1 +$$

$$+ \int_W \nabla_{H^m} u_2 p^{-2}_{H^m}(B \nabla u_2, \nabla ((\gamma_{\sigma}') p^{-1} \xi)) - \int_W (p-1) (\gamma_{\sigma}') p^{-2}_{H^m} \nabla_{H^m} u_2 p^{-1}_{H^m} \cdot$$

Using $\gamma_{\sigma}' \geq 0$ and the divergence theorem for the third addendum, we obtain

$$\int_{H^m} \nabla_{H^m} u_1 p^{-2}_{H^m}(B \nabla u_1, \nabla \xi) \leq \int_{\partial V} \nabla_{H^m} u_1 p^{-2}_{H^m}(B \nabla u_1, \nu_V) \xi - \int_V \xi \Delta_{H^m} u_1 +$$

$$+ \int_{\partial W} \nabla_{H^m} (\gamma_{\sigma} \circ u_2) p^{-2}_{H^m}(B \nabla (\gamma_{\sigma} \circ u_2), \nu_W) \xi - \int_W (\gamma_{\sigma}') p^{-1} \Delta_{H^m} u_2 \xi.$$

Note that the only possibly non-null part of the boundary integrals is along $\Gamma$, for which $\nu_V = -\nu_W$. Since $u$ is $C^1$ on $\Gamma$, the boundary terms cancel and, by
5 Non-existence of bounded solutions

(i), (ii) together with the final estimates of (56) we get

\[
\int_{H^m} |\nabla_{H^m} u|^p \langle B \nabla, \nabla \xi \rangle \leq - \int_V f(u_1) l(|\nabla_{H^m} u_1|_{H^m}) \xi + \int_{\Gamma} \xi f(\gamma \sigma \circ u_2) l(|\nabla_{H^m} (\gamma \sigma \circ u_2)|_{H^m}) \xi = - \int_{H^m} f(u) l(|\nabla_{H^m} u|_{H^m}) \xi.
\]

Therefore \( u \) is a weak subsolution, and the proof is complete.

5 Non-existence of bounded solutions

The aim of this section is to show that the differential inequality (16) admits no non-constant, non-negative bounded solutions in general, that is, even if the Keller-Osserman condition is not satisfied.

Theorem 5.1. Let \( \varphi, f, l \) satisfy (Φ), (F), (L), (Φ & L), (Φ2) and (L2). Then every non-negative bounded \( C^1 \)-solution \( u \) of

\[
\Delta_{H^m} \varphi \geq f(u) l(|\nabla_{H^m} u|_{H^m}) \quad \text{on } H^m
\]

is constant; moreover, if \( l(0) > 0 \), then \( u \equiv 0 \).

Proof. Let \( u \) be a non-negative bounded solution of (57) and let \( u^* = \sup_{H^m} u \). We follow the same steps of the proofs of Lemma 3.5 and Theorem 1.1 and define a radial supersolution \( v(q) = \alpha(r(q)) \), where \( \alpha : [r_0, T_\sigma] \to \mathbb{R}_+ \) is defined by

\[
T_\sigma - t = \int_{\alpha(t)}^A \frac{ds}{K^{-1}(\sigma F(s))},
\]

with \( A \) any constant greater than \( u^* \). Note that, as before, \( \alpha(r_0) = \varepsilon, \alpha(\tilde{r}) < \eta \) and \( \alpha'(t) > 0 \) on \([r_0, T_\sigma]\), while \( \alpha(T_\sigma) = A \).

Now choose \( r_0 \) as in Section 3.4 and consider the difference \( u - v \) in the annular region \( B_{T_\sigma} \setminus B_{r_0} \); note that, on \( \partial B_{r_0} \), \( u - v < u^* - 2\eta - \varepsilon \), there exists \( \tilde{q} \) such that \( u(\tilde{q}) - v(\tilde{q}) > u^* - 2\eta \), and, on \( \partial B_{T_\sigma} \), \( u - v < u^* - A \). Thus \( u - v \) attains a positive maximum \( \mu \) at some point of \( B_{T_\sigma} \setminus \overline{B_{r_0}} \).

Hereafter, the proof proceeds exactly as that of Theorem 1.1 so we omit the details.

6 More differential inequalities

The aim of this section is to show that the method used so far allows us to treat some other cases; in particular, we focus our attention on the differential inequality (18), that is,
\[ \Delta_{H^m}^\varphi u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m}). \]

As a matter of fact, the most interesting case arises when \( h \geq 0 \) and \( g \geq 0 \), that is, when we have the action of two opposite terms and when the standard comparison arguments do not apply. Indeed, as we shall see, in the generalized Keller-Osserman condition the terms \( h \) and \( f \) play very different roles.

### 6.1 Basic assumptions and a new adapted Keller-Osserman condition

We collect the following further set of hypotheses:

\[ h \in C^0(\mathbb{R}^+), \ h(t) \geq 0 \text{ on } \mathbb{R}^+, \ h \in L^1(0^+), \ h \text{ monotone non-increasing}; \quad (H) \]

\[ t\varphi'(t) \in L^1(0^+); \quad (\Phi 0) \]

\[ \exists B > 0, \ \theta \in (-\infty, 2) : \varphi'(ts) \geq B\varphi'(t)s^{-\theta} \ \forall t \in \mathbb{R}^+, \forall s \in [1, +\infty). \quad (\Phi 3) \]

Integrating, it is easy to deduce that the following condition is implied by (\( \Phi 3 \)):

\[ \varphi(ts) \geq B\varphi(t)s^{1-\theta} \ \forall t \in \mathbb{R}^+, \forall s \in [1, +\infty), \quad (58) \]

Note that \( \varphi(t) = t^{p-1}, p > 1 \) satisfies (\( \Phi 3 \)) with \( B = 1, 2 - p \leq \theta < 2 \). Again, by way of example, if

\[ \varphi(t) = \int_0^t \frac{ds}{P(s)}, \]

where \( P(s) \) is a polynomial with degree at most \( \theta \), non-negative coefficients and such that \( P'(0) > 0 \), then \( \varphi \) satisfies (\( \Phi 3 \)). We would also like to stress that conditions (\( \Phi 3 \)) and (\( \Phi 2 \)) are compatible, as it is apparent, for instance, for the \( p \)-Laplacian.

As in the previous theorems, the necessity of dealing with the density function leads us to require a relaxed homogeneity also on \( g \), as expressed by the following inequality:

\[ g(st) \leq \tilde{D}s^{\tau+1}t^2\varphi'(t) \ \forall s \in [0, 1], \ t \in \mathbb{R}^+ \quad (G) \]

where \( \tau \) is as in (\( \Phi 2 \)) and \( \tilde{D} \) is a positive constant; this bound on \( g \) is also due to a structural constraint which comes from the construction of the supersolution. Unfortunately, for the \( p \)-Laplacian this turns out to be quite restrictive. For example, if \( g(t) = Dt^\nu \), for some \( 0 \leq \nu \) and some constant \( D > 0 \), it is not hard to see that (\( G \)) holds if and only if \( \nu = p \). However, since (\( \Phi 5 \)) is an inequality, solving for this \( g \) will solve for any other smaller \( g \).

We now examine the steps leading to the definition of the Keller-Osserman
condition adapted to inequality (15). Setting \( t = 1 \) in (15) we have

\[
\varphi'(s) \geq B\varphi'(1)s^{-\theta},
\]

and since \( \varphi'(1) > 0 \) we deduce, integrating and using \( \theta < 2 \),

\[
t\varphi'(t) \notin L^1(\mathbb{R}^+) .
\]

In the present case, \( l \equiv 1 \) and the definition of \( K \) given in (15) becomes

\[
K(t) = \int_0^t s\varphi'(s)\,ds.
\]

It follows that (15) with \( \theta \leq 2 \) implies that \( K \) is a \( C^1 \)-diffeomorphism from \( \mathbb{R}^+ \) onto itself. From (15) we also have, for \( s \in \mathbb{R}^+, y \in [1, \infty) \),

\[
\int_0^t sy\varphi'(sy)\,ds \geq By^{1-\theta} \int_0^t s\varphi'(s)\,ds,
\]

so that

\[
K(ty) \geq By^{2-\theta}K(t) \quad \forall t \in \mathbb{R}^+, \forall y \in [1, \infty).
\] (59)

Next, we define

\[
\hat{F}(t) = \int_0^t f(s)e^{(2-\theta)\int_0^s h(x)\,dx}\,ds.
\]

For \( s \in \mathbb{R}^+ \) we let

\[
t = K^{-1}\left( \sigma\hat{F}(s) \right).
\]

Since \( K^{-1} \) is non-decreasing we get

\[
y = \frac{K^{-1}(\hat{F}(s))}{K^{-1}(\sigma\hat{F}(s))} \geq 1,
\]

and applying inequality (59) we deduce

\[
K\left( K^{-1}(\hat{F}(s)) \right) \geq BK\left( K^{-1}(\sigma\hat{F}(s)) \right) \left[ \frac{K^{-1}(\hat{F}(s))}{K^{-1}(\sigma\hat{F}(s))} \right]^{2-\theta} .
\]

Hence we obtain

\[
\left[ \frac{K^{-1}(\hat{F}(s))}{K^{-1}(\sigma\hat{F}(s))} \right]^{2-\theta} \leq \frac{1}{B\sigma} .
\] (60)
Since $\theta < 2$ this can be written as

$$
\frac{\sigma \mu}{K^{-1}(\sigma \hat{F}(s))} \leq \frac{B - \mu}{K^{-1}(\hat{F}(s))}, \quad s \in \mathbb{R}^+.
$$

(61)

In conclusion, the following inequality holds:

$$
\int_{-\infty}^{+\infty} \frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\sigma \hat{F}(s))} \, ds \leq \left( \frac{1}{B\sigma} \right)^{\frac{1}{\theta}} \int_{-\infty}^{+\infty} \frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\hat{F}(s))} \, ds.
$$

(62)

We are now ready to introduce the further generalized Keller-Osserman condition in the form

**Definition 6.1.** The **generalized Keller-Osserman condition** for inequality

$$
\Delta_{H^\infty}^\varphi u \geq f(u) - h(u)g(|\nabla_{H^\infty} u|_{H^\infty})
$$

is the request:

$$
\frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\sigma \hat{F}(t))} \in L^1(+\infty).
$$

\((\bar{KO})\)

As we have already mentioned, the roles of $f$ and $h$ in the above condition are far from being specular. In particular, $h$ has two opposite effects: on the one hand the explicit term $e^{\int_0^t h(x) \, dx}$ supports the non-integrability, hence the existence, on the other hand its presence in the expression for $\hat{F}(t)$ favours integrability.

We observe that, under assumptions \((\Phi)\) and \((\Phi_3)\), inequality \((\bar{KO})\) implies that, if \((\bar{KO})\) holds, then for every $\sigma \in (0, 1]$

$$
\frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\sigma \hat{F}(t))} \in L^1(+\infty).
$$

(63)

A particular case arises when $h \in L^1(+\infty)$. We are going to see that, independently of the sign of $h$, condition \((\bar{KO})\) and \((\bar{CO})\) are indeed equivalent:

**Proposition 6.1.** Assume \((\Phi)\), \((\bar{F})\), \((\Phi_3)\) and suppose that $h : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is a continuous function such that $h \in L^1(+\infty)$. Then

$$
\frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\hat{F}(t))} \in L^1(+\infty) \quad \text{if and only if} \quad \frac{1}{K^{-1}(\hat{F}(t))} \in L^1(+\infty).
$$
Proof. First of all we observe that, since $\theta < 2$,

$$\hat{F}(t) = \int_0^t f(s) e^{(2-\theta) \int_0^s h(x) \, dx} \, ds \leq e^{(2-\theta) \|h\|_{L^1}} \int_0^t f(s) \, ds = \Lambda_1 F(t)$$

with $\Lambda_1 \geq 1$. Similarly $F(t) \leq \Lambda_2 \hat{F}(t)$ with $\Lambda_2 \geq 1$.

Thus, since $K^{-1}$ is non-decreasing

$$\int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(F(s))} \leq \int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(\Lambda_1^{-1} \hat{F}(s))}. \quad (64)$$

We now perform the change of variables $t = s\Lambda_1^{-1}$. Thus

$$\int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(\Lambda_1^{-1} \hat{F}(s))} \leq \Lambda_1 \int_{+\infty}^{+\infty} \frac{dt}{K^{-1}(\Lambda_1^{-1} \hat{F}(\Lambda_1 t))}. \quad (65)$$

Since $\Lambda_1 \geq 1$, denoting with $a(s) = f(s) e^{(2-\theta) \int_0^s h(x) \, dx}$ we have

$$\hat{F}(\Lambda_1 t) = \int_0^{\Lambda_1 t} a(y) \, dy = a(\Lambda_1 x) \, dx \geq \Lambda_1 e^{-(2-\theta) \|h\|_{L^1}} \int_0^t a(z) \, dz = \Lambda \hat{F}(t)$$

for some constant $0 < \Lambda \leq \Lambda_1$. Hence $\Lambda_1^{-1} \hat{F}(\Lambda_1 t) \geq \sigma \hat{F}(t)$, where $\sigma = \Lambda \Lambda_1^{-1} \leq 1$. Using (64), (65), the monotonicity of $K^{-1}$ and Lemma 3.4 (in particular inequality (41)) we show that

$$\int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(\hat{F}(s))} \leq \int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(\Lambda_1^{-1} \hat{F}(\Lambda_1 t))} \leq \Lambda_1 \int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(\sigma F(s))} \leq \Lambda_1 \sigma \int_{+\infty}^{+\infty} \frac{ds}{K^{-1}(F(s))}. \quad (66)$$

Therefore, $h \in L^1(\mathbb{R}^+)$ and (66) immediately imply that

$$\frac{e^{\int_0^t h(x) \, dx}}{K^{-1}(\hat{F}(t))} \in L^1(+\infty) \quad \text{if and only if} \quad \frac{1}{K^{-1}(F(t))} \in L^1(+\infty).$$

\[\square\]

### 6.2 Construction of the supersolution and final steps

Now we proceed with the construction of the supersolution; the idea follows the lines of Lemma 6.2 but we briefly reproduce the main steps.

**Lemma 6.2.** Assume the validity of \([\Phi], \, \hat{F}, \, (H), \, \Phi_3\) and of the Keller-Osserman assumption \([\hat{K}O]\). Fix $0 < \varepsilon < \eta$, $0 < t_0 < t_1$. Then there exists
6 More differential inequalities

\( \sigma \in (0,1], \ T_\sigma > t_1 \) and \( \alpha : [t_0, T_\sigma) \to [\varepsilon, +\infty) \) satisfying

\[
\begin{align*}
(\varphi(\alpha'))' + \frac{2m+1}{2} \varphi(\alpha') &\leq f(\alpha) - h(\alpha)(\alpha')^2 \varphi(\alpha); \\
\alpha' &> 0, \ \alpha(t) \uparrow +\infty \ \text{as} \ t \to T_\sigma^-, \\
\alpha(t_0) &\equiv \varepsilon \ \text{and} \ \alpha(t) \leq \eta \ \text{on} \ [t_0, t_1].
\end{align*}
\] (67)

**Proof.** First of all we observe that, using (K) and (63) we have that

\[
\int_{-\infty}^{+\infty} e^{\int_0^\sigma h(x) \, dx} \, ds \uparrow +\infty \ \text{as} \ \sigma \downarrow 0^+.
\] (68)

We thus fix \( \sigma_0 \in (0,1] \) so that, for every \( \sigma \in (0, \sigma_0] \)

\[
T_\sigma = t_0 + \int_{-\infty}^{+\infty} e^{\int_0^\sigma h(x) \, dx} \, ds > t_1.
\] (69)

Implicitly define the \( C^2 \)-function \( \alpha : [t_0, T_\sigma) \to [\varepsilon, +\infty) \) by setting

\[
T_\sigma - t = \int_{\alpha(t)}^{+\infty} e^{\int_0^\sigma h(x) \, dx} \, ds.
\] (70)

By construction, \( \alpha(t_0) = \varepsilon \) and \( \alpha(t) \to +\infty \) as \( t \to T_\sigma^- \). We differentiate (69) a first time to obtain

\[
K^{-1}(\sigma \hat{F}(s)) = \alpha' e^{\int_0^\sigma h(x) \, dx}
\] (71)

so that \( \alpha' > 0 \). Transforming the above into \( \sigma \hat{F}(\alpha) = K\left(\alpha' e^{\int_0^\sigma h(x) \, dx}\right) \), differentiating once more and using the definition of \( \hat{F} \) and \( K \) we arrive at

\[
\sigma f(\alpha)e^{(2-\theta)\int_0^\sigma h(x) \, dx} - \alpha' e^{\int_0^\sigma h(x) \, dx} \varphi'(\alpha') e^{\int_0^\sigma h(x) \, dx} \left[ \alpha'' + (\alpha')^2 h(\alpha) \right].
\]

We use (62) and \( \alpha' > 0 \) to deduce

\[
\sigma f(\alpha) \geq B \varphi'(\alpha') \left[ \alpha'' + (\alpha')^2 h(\alpha) \right]
\]

and thus

\[
\varphi'(\alpha') \alpha'' \leq \frac{\sigma}{B} f(\alpha) - (\alpha')^2 \varphi'(\alpha') h(\alpha).
\] (72)

Integrating (71) on \([t_0, t]\) and using \( \alpha' > 0, \ \varphi' \geq 0, \ \hat{F} \) and (H) we obtain

\[
\varphi(\alpha'(t)) \leq \varphi(\alpha'(t_0)) + \frac{\sigma}{B} t f(\alpha(t)).
\] (73)
Putting together (71) and (72) and using (F)
\[
\varphi'(\alpha')\alpha'' + \frac{2m+1}{t}\varphi(\alpha') \leq f(\alpha)\left[\frac{\sigma}{B}2(m+1) + \frac{2m+1}{t_0} \frac{\varphi(\alpha'(t_0))}{f(\alpha(t_0))}\right] - (\alpha')^2 h(\alpha)\varphi'(\alpha').
\]
(73)
From (70)
\[
\alpha'(t_0) = K^{-1}\left(\sigma\hat{F}(\varepsilon)\right)e^{-\int_0^t h(x)dx}.
\]
Therefore, since \(\varphi(t) \to 0\) as \(t \to 0^+\), choosing \(\sigma \in (0,\sigma_0]\) sufficiently small, (63) yields
\[
\varphi'(\alpha')\alpha'' + \frac{2m+1}{t}\varphi(\alpha') \leq \frac{1}{D}f(\alpha) - h(\alpha)(\alpha')^2 \varphi'(\alpha')
\]
on \([t_0,T_\sigma]\). To prove that \(\alpha(t) \leq \eta\) on \([t_0,t_1]\) we observe that
\[
t_1 - t_0 = T_\sigma - t_0 + t_1 - T_\sigma = \int_{t_0}^{t_1} e^{\int_0^t h(x)dx} ds.
\]
Hence, since the integrand goes monotonically to \(+\infty\) as \(\sigma \to 0^+\), we need to have \(\alpha(t_1) \to \varepsilon\) as \(\sigma \to 0^+\). Since \(\alpha' > 0\) this proves the desired property.

We are now ready to state the non-existence result for inequality (18). The proof is a minor modification of the one given for Theorem 1.1; therefore we only sketch the main points referring to Section 3.4 for definitions and notations.

**Theorem 6.3.** Let \(\varphi, f, h, g\) satisfy \((P), (E), (H), (G), (FP), (F2), (F3), and (KO)\). Let \(u\) be a non-negative \(C^1\)-solution of
\[
\Delta_{H^m}^u \geq f(u) - h(u)g(|\nabla H^m u|_{H^m}) \quad \text{on } H^m.
\]
(74)
Then \(u \equiv 0\).

**Proof.** First of all, note that it is sufficient to prove that \(u\) is equal to a constant \(c\); indeed, by assumption \((Q)\), \(0 = \Delta_{H^m}^u c \geq f(c) - h(c)g(0) = f(c)\) and the conclusion follows from (E). Now we prove that a maximum principle holds for equation (18) on a domain \(\Omega\); indeed, if we assume \(u(q) = u^*\) for some \(q \in \Omega\), then there exists a neighbourhood \(U_q \subseteq \Omega\) such that, for every \(\varepsilon > 0\), \(g(|\nabla H^m u|_{H^m}) < \varepsilon\) on \(U_q\). This implies, up to choosing \(\varepsilon\) sufficiently small, \(\Delta_{H^m}^u u \geq f(u) - h(u^*)\varepsilon \geq 0\) on \(U_q\). Then, by Theorem 5.2 \(u \equiv u^*\) on such neighbourhood, and thus the set \(\{q \in \Omega : u(q) = u^*\}\) is non-empty, open and closed in \(\Omega\); therefore, \(u \equiv u^*\) in \(\Omega\).

Eventually, in order to prove the constancy of \(u\), assume, by contradiction, that there exists \(q_0 \in H^m\) such that \(u(q_0) < u^*\); then, by the maximum principle,
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$u < u^*$ on $H^m$. We now proceed as in the proof of Theorem 1.1 and define $r_0$, $\eta$, $\tilde{q}$, $\tilde{r}$ in the same way. Then, we construct the function $v(q) = \alpha(r(q))$, with $\alpha$ as in Lemma 6.2. A calculation shows that

$$
\Delta_{R^m} \varphi_H v = \sqrt{\psi} \varphi'(\alpha'(r)\sqrt{\psi}) \alpha''(r) + \frac{2m+1}{r} \varphi(\alpha'(r)\sqrt{\psi}) \leq \\
\leq \left(\sqrt{\psi}\right)^{1+\tau} D \left[ \varphi'(\alpha'(r)) \alpha''(r) + \frac{2m+1}{r} \varphi(\alpha'(r)) \right] \leq \\
\leq \left(\sqrt{\psi}\right)^{1+\tau} D \left[ \frac{1}{D} f(\alpha) - h(\alpha)\alpha' \varphi' \right] \leq \\
\leq f(\alpha(r)) - \frac{D}{D} h(\alpha(r)) g(\alpha'(r)\sqrt{\psi}) \leq f(v) - h(v) g(|\nabla_{H^m} v|_{H^m}),
$$

where in the last inequality we have used (G) and we have chosen $D$ big enough to ensure $D \geq \bar{D}$.

If $\xi$ lies in the connected component $\Gamma_\mu$, using (F), (H) and $|\nabla_{H^m} u(\xi)|_{H^m} = |\nabla_{H^m} v(\xi)|_{H^m}$ we obtain

$$
\Delta_{H^m} u(\xi) \geq f(u(\xi)) - h(u(\xi)) g(|\nabla_{H^m} u(\xi)|_{H^m}) > \\
> f(v(\xi)) - h(v(\xi)) g(|\nabla_{H^m} v(\xi)|_{H^m}) \geq \Delta_{H^m} v(\xi). \tag{76}
$$

The rest of the proof is much the same. \hfill \Box

**Remark 6.4.** We note that the maximum principle is indeed unnecessary for the proof of the final steps in Theorems 1.1 and 6.3. If we assume that $u$ is not constant, we can consider a point $q_0$ such that $u(q_0) < u^*$ and, by continuity, a small radius $r_0$ such that $u|_{\partial B_{r_0}}(q_0) < u^*$. Using the invariance property, we can consider $q_0$ as the origin for the Koranyi distance, and proceed analogously to the end.

As for Theorem 1.1, we can state the Euclidean counterpart of Theorem 6.3 substituting assumption (G) with the request

$$
g(t) \leq Dt^2 \varphi'(t) \quad \text{on} \ (0, +\infty). \quad (\tilde{G})
$$

We have:

**Theorem 6.5.** Let $\varphi, f, h, g$ satisfy (D), (F), (H), (G), $\Phi_1$, $\Phi_2$, $\Phi_3$, and $\tilde{K}G$. Let $u \in C^1(R^m)$ be a non-negative solution of

$$
\Delta_{R^m} u \geq f(u) - h(u) g(|\nabla u|) \quad \text{on} \ R^m. \tag{77}
$$

Then $u \equiv 0$. 
6.3 Another existence result for the $p$-Laplacian

As a quick application of Lemma 6.1 and Theorem 1.3, we can deduce that the modified Keller-Osserman condition (KO) is optimal in the case of the $p$-Laplacian.

**Theorem 6.6.** Let $f, h, g$ satisfy (F), (H), (G), (F2) and (F3) with $\tau = 0$. Furthermore suppose that $h \in L^1(\mathbb{R}^+)$. Then, the following conditions are equivalent:

i) there exists a non-negative, non-constant solution $u \in C^1(H^m)$ of inequality $\Delta_{H^m}^p u \geq f(u) - h(u)g(|\nabla_{H^m} u|_{H^m});$

ii) $\frac{1}{K^{-1}(F(t))} \not\in L^1(+\infty)$.

**Proof.** First, we deduce from the assumptions and from Lemma 6.1 the equivalence between (KO) and (KO). We have already pointed out that the $p$-Laplacian satisfies (F2) for every $0 \leq \tau \leq p - 1$: as it can be checked, the choice of $\tau = 0$ is the least stringent on (G). Furthermore, (F0) is automatic. This shows that implication $i) \Rightarrow ii)$ is an immediate application of Theorem 6.3. Regarding the other one, set $l(t) \equiv 1$ and apply the existence part of Theorem 1.3 (note that all the assumptions are satisfied), to get a solution of

$$\Delta_{H^m}^p u \geq f(u).$$

Since the RHS is trivially greater than $f(u) - h(u)g(|\nabla_{H^m} u|_{H^m})$ we have the desired conclusion.

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