Gromov-Hausdorff distances for Lorentzian length spaces

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Abstract

We construct three different analoga of Gromov-Hausdorff space for Lorentzian distances and show a Gromov precompactness result in this context.

Recently, attempts to 'synthesize' Lorentzian geometry in a similar fashion as this has been done in Riemannian geometry have attracted much attention. This article aims to show that many constructions of Gromov-Hausdorff spaces can be made in the (synthetic or not) Lorentzian situation. The author of this article wrote in a recent other paper about Lorentzian optimal transport and Lorentzian length spaces 'As the main structures of both frameworks satisfy the inverse triangle inequality instead of the triangle inequality, it seems quite unlikely that one could establish something comparable to the Gromov-Hausdorff distance with the Gromov compactness result, let alone an analogon to Perelman stability'. The present article serves mainly to partially revoke and correct this rather hasty judgement.

Let \( S \) be an arbitrary set fixed once and forever in this article. For all categories in this article we infer without further mention that all objects are of cardinality \( \leq \#S \). For a category \( \mathcal{C} \), let \( \mathcal{C}^I \) denote the set of isomorphism classes of objects of \( \mathcal{C} \). For a functor \( \mathcal{F} : X \to Y \) between two categories, let \( \mathcal{F}^I : X^I \to Y^I \) be the push-down of \( \mathcal{F} \). We denote the closure of \( A \subset X \) in a topological space \( X \) by \( \text{cl}(A, X) \) and the set-theoretic symmetric difference of two subsets \( A, B \) of a set \( C \) by \( A \Delta B \).

In synthetic Lorentzian geometry, there are two different functorial approaches. The first one uses the category \( \text{ALL} \) of (causal) almost Lorentzian pre-length spaces ([6], [11]), which are sets \( X \) with a function \( \sigma : X \times X \to \mathbb{R} \) that is antisymmetric and satisfies \( \sigma(x, z) \geq \sigma(x, y) + \sigma(y, z) \) whenever \( \sigma(x, y), \sigma(y, z) > 0 \). On Lorentzian manifolds, there is a natural such signed Lorentzian distance

\[ \sigma_\alpha(x, y) := \sqrt{\sigma(x, y)} \]

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1In the above references, \( \sigma \) is moreover supposed to be is supposed to be lower semi-continuous w.r.t. a topology given or defined from \( \sigma \) as the topology \( \tau_\alpha \) defined in [9]; here, however, we will add continuity to our assumptions explicitly if we need it.
function \( \sigma(p,q) := \pm \sup \{ \ell(c) | c : p \sim q \in J^\pm(p) \text{ causal} \} \) where \( \ell(c) \geq 0 \) is the length of a Lipschitz causal curve defined in analogy to metric length, replacing \( \sup \) with \( \inf \).

The other approach uses the category \textbf{POM} of (partially) ordered measure spaces. The two categories seem to be intimately connected (see below, see [11] for further details).

There are at least three approaches to induce generalized pseudometrics on reasonably large subsets of \textbf{ALL} or \textbf{POM}.

The first two approaches use functors from \textbf{ALL} resp. \textbf{POM} to the category of metric spaces via a map \( \Phi_f \) (depending on a locally bounded measurable function \( f : \mathbb{R} \to \mathbb{R} \)) from an object \( X \) to \( \mathbb{R}^X \) resp. to the space \( AE(X) \) of almost everywhere defined functions on \( X \), then defining (fixing \( p \in [1;\infty] \)) a map \( \Phi_{f,p} \) mapping the class of \( (X,\sigma) \in \text{Obj}(\text{ALL}) \) to the class of \( (X,d_{f,p}) \) where

\[
\Phi_f : X \ni x \mapsto f \circ \sigma_x, \quad d_{f,p}(x,y) := \Phi_{f,p}(\sigma) := ((\Phi_f)^*d_{L^p})(x,y) = |f \circ \sigma_x - f \circ \sigma_y|_{L^p(X)} \in [0;\infty]
\]

and \( \sigma_x := \sigma(x,\cdot) \). On the metric side we have the Gromov-Hausdorff distance \( d_{GH}^+ \) between isometry classes which we can pull back, inducing a map \( \Phi_{f,p}^*d_{GH}^+ : \text{ALL} \times \text{ALL} \rightarrow \mathbb{R} \cup \{\infty\} \) resp. \( \Phi_{f,p}^*d_{GH}^+ : \text{POM} \times \text{POM} \rightarrow \mathbb{R} \cup \{\infty\} \).

In either of the two categories, we want to restrict ourselves to the quantities given there, without borrowing structures from the other category, because ultimately we want to show continuity of the (push-down of the) functor w.r.t. the respective Gromov-Hausdorff topology, which means that, on \textbf{ALL}, we are only allowed to use the distance, limiting us to the choice \( p = \infty \), whereas on \textbf{POM}, we are only allowed to use the measure and the order, which implies \( f := g \circ \text{sgn} \) for some function \( g \). Let us pursue the case of \textbf{POM} first. The calculations should be compared to the proof of Th. 1 (ii) in [10].

For \( x \in X \), let \( \sigma_x^+ \) resp. \( \sigma_x^- \) denote the positive resp. negative part of \( \sigma_x \). For \( r \in [-1;1] \) we define

\[
F_r : \mathbb{R} \ni s \mapsto \frac{1}{2} + \frac{r}{2} \cdot \text{sgn}(s), \quad D_r := d_{F_r,2},
\]

then \( D_r \) interpolates between the past metric (taking into account only the past cones) \( D_{-1} \) with

\[
D_{-1}(x,y) := \| \chi_{(-\infty;0)} \circ \sigma_x - \chi_{(-\infty;0)} \circ \sigma_y \|_{L^2(X)} = \sqrt{\mu(J^-(x) \triangle J^-(y))}
\]

for \( F_{-1} = (1 - \theta_0) \) and the future metric \( D_1 \) (taking into account only the future cones) for \( F_1 = \theta_0 \), passing through \( D_0(x,y) = \frac{1}{2} \| \text{sgn} \sigma_x - \text{sgn} \sigma_y \|_{L^2} \). Whereas \( D_{\pm 1}^2 \) is a metric on \( X \setminus \partial^\pm X \), it vanishes identically on \( \partial^\pm X \times \partial^\pm X \). For fixed \( p,q \in X \), let \( V \) be the 2-dimensional linear subspace spanned by the vectors \( u^+ := \text{sgn} \sigma_p^+ - \text{sgn} \sigma_q^+ \) and \( u^- := \text{sgn} \sigma_p^- - \text{sgn} \sigma_q^- \). Then the \( L^2 \) scalar product on \( V \) is uniquely given by the corresponding quadratic form on three vectors any two of which are non-collinear, thus given by \( D_{-1/2}(p,q), D_0(p,q) \) and \( D_{1/2}(p,q) \). In other

\footnote{For brevity we denote by \( \| \cdot \|_{\infty} \) the norm induced by the supremum and not by the essential supremum, whenever no canonical measure is part of the data.}
words, the datum of those three recovers the whole family $D_r$. We are interested primarily in $D_{\pm 1}(p, q) = ||u^\pm||^2$. Elementary calculations reveal the explicit equation

$$||u^\pm||^2 = \frac{5}{8}||D_{\mp 1/2}(p, q)||^2 + \frac{21}{8}||D_{\pm 1/2}(p, q)||^2 - \frac{3}{8}||D_0(p, q)||^2. \tag{1}$$

Then we can identify future and past boundary as $\partial^\pm X = \{ x \in X \mid \exists y \in X \setminus \{ x \} : D_{\pm 1}(x, y) = 0 \}$. In the following we want to recover the causal structure. For $p, q \in X$ we define $\kappa_p^\pm := \text{sgn}(\sigma_p^\pm)$ and consider the following expressions only depending on the metrics $D_r$:

$$(D_{\pm 1}(p, q))^2 = \langle \kappa_p^+, \kappa_q^+ - \kappa_q^- \rangle_{L^2}$$

$$(D_0(p, q))^2 = \langle \kappa_p^+ - \kappa_p^-, \kappa_q^+ - \kappa_q^- \rangle_{L^2},$$

we calculate

$$(D_0(p, q))^2 - (D_{-1}(p, q))^2 - (D_1(p, q))^2 = 2(\langle \kappa_p^+, \kappa_q^- \rangle_{L^2} + \langle \kappa_p^-, \kappa_q^+ \rangle_{L^2} - \langle \kappa_p^+, \kappa_q^- \rangle_{L^2} - \langle \kappa_p^-, \kappa_q^+ \rangle_{L^2}), \tag{2}$$

where the last two terms vanish due to causality of $X$. This implies

$$(D_0(p, q))^2 - (D_{-1}(p, q))^2 - (D_1(p, q))^2 \neq 0 \iff p \ll q \lor q \ll p.$$

as $p \ll q \iff (\chi_{(0, \infty)} \circ \sigma_p, \chi_{(-\infty, 0)} \circ \sigma_q)_{L^2} \neq 0$. The distinction between the two relevant cases can be made by taking into account $p \ll q \iff \mu(J^+(p)) > \mu(J^+(q))$, so we only have to pick a point $r \in \partial^+ X$ (for which $D_1(p, r) = \sqrt{\mu(J^+(p))}$ and $D_1(q, r) = \sqrt{\mu(J^+(q))}$) and then

$$p \ll q \iff (D_0(p, q))^2 - (D_{-1}(p, q))^2 - (D_1(p, q))^2 \neq 0 \land D_1(p, r) > D_1(q, r).$$

We define

$$\Phi^X_r(X, \sigma) := (X, D_{-1/2}) \sqcup (X, D_0) \sqcup (X, D_{1/2})$$

and $d_{\text{GH}}^X$ as the associated Gromov-Hausdorff metric on $\text{POM}^1$, whose definition via distortion we now reconsider.

A relation $R \subset X \times Y$ is called a correspondence from $X$ to $Y$ if $\text{pr}_1 R = X$ and $\text{pr}_2 R = Y$, and the set of all correspondences between $X$ and $Y$ is denoted by $\text{Corr}(X, Y)$.

$$d_{\text{GH}}((X, d_X), (Y, d_Y)) := \frac{1}{2}(\text{dist}(R) | R \in \text{Corr}(X, Y)), \tag{3}$$
where for an arbitrary relation from \((X, d_X)\) to \((Y, d_Y)\) we define

\[
\text{dist}(R) := \sup \{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R\}.
\]

If we apply this to metric spaces \((X, d_X), (Y, d_Y)\), then \(d_{GH}\) is indeed the Gromov-Hausdorff metric.

**Remark.** The datum of \(D_1\) alone is sufficient to reconstruct the (undirected) causal structure, as it is easy to see that under the mild assumptions that \(X\) is distinguishing and has open time cones and that any open set is of positive measure, a curve is \(D_1\)-geodesic if and only if it is causal. The author was not able to either construct an example of noninjectivity of \(D_1\) or else proving injectivity. For our purposes, the construction with \(d_{GH}^\times\) will prove sufficient.

Let \(\text{POM}^I_{fv}\) be the subset of \(\text{POM}^I\) consisting of all the classes s.t. the volumes of future and past cones is finite.

**Theorem 1** \(d_{GH}^\times\) is a generalized pseudometric on \(\text{POM}^I\) and a metric on \(\text{POM}^I_{fv}\). Moreover, the volume of causal diamonds depends uniformly continuously on the \(d_{GH}^\times\)-isometry class in the following sense: For each \(\varepsilon > 0\) there is a \(\delta > 0\) such that for every correspondence \(C : X \to Y\) between objects of \(\text{POM}\) with \(\text{dist}^\times(C) < \delta\) we have

\[
\forall (x_1, y_1), (x_2, y_2) \in C : |\mu_X(J_X(x_1, x_2)) - \mu_Y(J_Y(y_1, y_2))| < \varepsilon.
\]

**Proof.** The first statement is immediate, as the pull-back of a metric is always a pseudometric. For the second statement, we only have to show injectivity of \(\Phi_r^\times\) for \(N = 3\), which has been shown in the preceding paragraph. The last statement follows directly from Eqs 1 and 2. 

How to use the freedom of post-composition with \(f\) reasonably? In [10] it has been shown that the metric defined by Noldus in [12], which in our terminology corresponds to \(d_{g,|\cdot|,\infty}\), cannot be made a length metric by applying \(\lambda\) to it. It turns out that for \(f_n := x^n\) for \(n\) odd and \(f_n(x) := \text{sgn}(x) \cdot x^n\) for \(n\) even, we get that for \(d_{GH}^\times := d_{g, f_n, \infty}\), the generalized length metric \(\lambda(d_{GH}^\times)\) is a length metric (i.e., finite) for \((X, g)\) a Cauchy slab. Some properties are obvious: If \((X, \sigma)\) is distinguishing, i.e., if for all \(x, y \in X\), \(x \neq y\) implies \(I^\pm(x) \neq I^\pm(y)\), then \(d^+\) is a metric an \((X, \sigma)\) w.r.t. which \(\sigma\) is continuous, more precisely: \(|\sigma(a, b) - \sigma(c, d)| \leq d^+(a, c) + d^+(b, d)\). Moreover, \(|\sigma(x, y)| \leq d^+(x, y)\). However, the topology generated by \(d^+\) differs in general from the topology \(\tau_+\) defined in [9] (of course, if \(\sigma\) is continuous w.r.t. \(\tau_+\), as it is the case for \((X, \sigma)\) being globally hyperbolic, both topologies coincide). Moreover, the functor \((X, \sigma) \mapsto (X, d^+)\) does not allow for a reconstruction of the Lorentzian data from the data on the metric side. Namely, denoting by \(\text{ALL}_{fd}^I\) the subset of those isomorphism classes of compact almost Lorentzian pre-length spaces whose Lorentzian distance is finite, we obtain:
Moreover, \( \Phi \) is a generalized pseudometric space, and \((\text{ALL}_{\text{id}}^1, \Phi_{\text{f, } \infty}^+ d_{\text{GH}}^+)\) is a pseudometric space but in general not a metric space, as in general, even in the case that \( f : \mathbb{R} \to \mathbb{R} \) is a diffeomorphism, \( \Phi_f \) is not injective.

**Proof.** Let \((X, \sigma)\) be defined by \( X := [-1; 1] \times \mathbb{R} \) (for \( b \geq 1 \)) and \( \sigma \) being the signed Lorentzian distance of the Lorentzian metric \( k^2 \cdot (-dx_0^2 + dx_1^2) \), where \( k \in C^\infty(X, (0; 1]) \) in such a way that there are no lines through 0. For example, for \( a \in (0; 1/4) \) we can choose \( k_a(x) = 1 \) for all \( x \in X \setminus (J(\{0\} \times (-a; a)) \cap x_0^{-1}((-1/4; 1/4)) \) and \( k_a(x) = a \) for all \( x \in J((0, 0)) \cap x_0^{-1}((-1/4 + a; 1/4 - a)) \). Then, indeed, as each maximally extended line begins at \( \partial^- X = \mathbb{R} \times \{-1\} \) and ends at \( \partial^+ X = \mathbb{R} \times \{1\} \), the image of no one can contain \((0, 0)\): Any curve the image of which does has length \( < 3 + 5a \), but for each \( p \in \partial^- X \) and each \( q \in \partial^+ X \) there is a future curve from \( p \) to \( q \) of length greater or equal to the piecewise affine future curve \( c \) from \((-1, 0)\) to \((1, 0)\) via \((-1/4, -1/4 - a) \) and \((1/4, -1/4 + a) \), which has length tending, with \( a \to 0 \), to \( 1/2 + \sqrt{8} > 3 \).

That is, for sufficiently small \( a \) there is no line through \((0, 0)\). We choose \( f := \chi_{[0; \infty)} \). Now, for each two \( p, q \in X \), the supremum of \( |f \circ \sigma_p - f \circ \sigma_q| \) is attained on the boundary \( \partial X \). We construct an almost Lorentzian length \( \delta \) on \( X \) such that for \( A := X \times \partial^+ X \cup \partial^- X \times X \) we have \( \delta|_A = \sigma|_A \) (so \( \Phi(\delta) = \Phi(\sigma) \)). It is easily seen via the inverse triangle inequality (which is sharp along lines) that the unique such distance \( \delta \) satisfies \( \delta(x, y) := \inf \{ \sigma(x, z) - \sigma(y, z) | z \in \partial^+ X \cap J^+(y) \} \) for all \( x \leq y \). We define \( \delta(x, y) := \inf \{ \sigma(x, z) - \sigma(y, z) | z \in \partial^+ X \cap J^+(y) \} \) for all \( x \leq y \) and 0 otherwise.

Let \( A(X) := (X, \delta) \), then \( d_{\text{GH}}(\Phi_f(X), \Phi_f(A(X))) = 0 \). Indeed, \( A(X) \) is a pre-length space: For \( x \ll y \ll z \),

\[
\delta(x, y) + \delta(y, z) = \inf \{ \sigma(x, z) - \sigma(y, z) | z \in J^+(y) \cap \partial^+(y) \} + \inf \{ \sigma(x, z) - \sigma(y, z) | z \in J^+(y) \cap \partial^+(y) \} \\
\leq \inf \{ \sigma(x, z) - \sigma(u, z) | z \in J^+(u) \cup \partial^+(y) \} = \delta(x, u).
\]

We define the **central set** of a globally hyperbolic compact Lorentzian pre-length space \( X \) as the minimizing locus of the **central distance** \( D : X \ni p \mapsto \sup \{ \sigma(p, x) | x \in \partial^+ X \} + \sup \{ \sigma(y, p) | y \in \partial^- X \} \). Isomorphisms of \( \text{ALL}^1 \) preserve central sets and central distances. In \( X \) and \( A(X) \) each, the central set contains the single point \( \{0\} \), but the respective central distances do not coincide, thus \( X \) and \( A(X) \) are not isomorphic. \( \blacksquare \)

As a third approach, we can mimic the definition of \( d_{\text{GH}} \) via correspondences and apply them directly to Lorentzian distance functions instead of metrics. Indeed, the definition of Gromov-Hausdorff distance via distortion by Eqs 3 and 4 equally well applies to \( d_X, d_Y \) being signed Lorentzian distance functions, where we denote the assignment by \( \text{dist}^- \) and \( d_{\text{GH}}^- \) for better distinction. So we equip the set \( \text{ALL}^1 \) (resp. \( \text{ALL}^1_c \subset \text{ALL}^1_{\text{id}} \)) of isomorphism classes of objects (resp. compact objects) of \( \text{ALL} \) with the assignment \( d_{\text{GH}}^- \) and obtain:

**Theorem 3** \((\text{ALL}^1, d_{\text{GH}}^-)\) is a generalized pseudometric space, and \((\text{ALL}^1_c, d_{\text{GH}}^-)\) is a metric space. Moreover, \( \Phi_{\text{id}}^* d_{\text{GH}}^- \leq \frac{1}{2} d_{\text{GH}}^- \).

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\textbf{Proof.} Nonnegativity and symmetry are obvious, and each isomorphism $I : (X, d_X) \mapsto (Y, d_Y)$ is a correlation with $\text{dist}^{-}(I) = 0$. For the triangle inequality consider for two correspondences $R_1 \in \text{Corr}(X, Y)$ and $R_2(Y, Z) \in \text{Corr}(Y, Z)$ their composition $R_2 \circ R_1 \in \text{Corr}(X, Z)$. Then

$$
\text{dist}^{-}(R_2 \circ R_1) = \text{sup}\{|d_X(x, x') - d_Z(z, z')| : (x, z), (x', z') \in R_2 \circ R_1\}
$$

$$
= \text{sup}\{|d_X(x, x') - d_Y(y, y') + d_Y(y, y') - d_Z(z, z')| : (x, y), (x', y') \in R_1 \land (y, z), (y', z') \in R_2\}
$$

$$
\leq \text{sup}\{|d_X(x, x') - d_Y(y, y')| + |d_Y(y, y') - d_Z(z, z')| : (x, y), (x', y') \in R_1 \land (y, z), (y', z') \in R_2\}
$$

$$
\leq \text{sup}\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in R_1\}
$$

$$
+ \text{sup}\{|d_Y(y, y') - d_Z(z, z')| : (y, z), (y', z') \in R_2\} = \text{dist}^{-}(R_1) + \text{dist}^{-}(R_2)
$$

Now let $\varepsilon > 0$ be given, let $d_{\text{GH}}(X, Y) =: r$ and $d_{\text{GH}}(Y, Z) =: s$, then there are correspondences $R_1 : X \rightarrow Y$ and $R_2 : Y \rightarrow Z$ with $\text{dist}^{-}(R_1) < r + \varepsilon/2$ and $\text{dist}^{-}(R_2) < s + \varepsilon/2$ and thus $\text{dist}^{-}(R_2 \circ R_1) < r + s + \varepsilon$, therefore $d_{\text{GH}}(X, Z) \leq d_{\text{GH}}(X, Y) + d_{\text{GH}}(Y, Z)$. To see $d_{\text{GH}}(X, Y) = 0 \Rightarrow X = Y$ and completeness, we first note $\text{dist}^{-}(R) = \text{dist}^{-}(\text{cl}(R, X \times Y))$. Obviously, every correspondence $f : X \rightarrow Y$ between distinguishing ordered measure spaces $X$ and $Y$ with $\text{dist}^{-}(f) = 0$ is a bijective map and an isomorphism of \textbf{ALL}. Now assume $d_{\text{GH}}(X, Y) = 0$. Then there is a sequence $a$ in $\text{Corr}(X, Y)$ with $\text{dist}^{-}(a(n)) \rightarrow_{n \rightarrow \infty} 0$. Let $\Sigma$ be a dense countable set in $X$, which exists due to compactness and metrizability \textbf{(by $d^+$)} of $X$. Compactness of $Y$ and Cantor’s diagonal procedure ensures existence of a subsequence $b$ of $a$ such that for every $s \in \Sigma$ some points in $b_n(s)$ converge to some $\tilde{b}_\infty(s) \in Y$, so $\tilde{b}_\infty : \Sigma \rightarrow Y$. Let $b_\infty := \text{cl}(\tilde{b}_\infty, X \times Y)$, then $b_\infty$ is left-total, as $\Sigma$ is dense in $X$, and it is right-total, as by $\text{dist}^+ < 2\text{dist}^{-}$, the sets $b_n(\Sigma)$ are dense in $Y$. Thus $b_\infty$ is a correlation. Then for all $(p, r), (q, s) \in b_n$, we get $|\sigma_Y(r, s) - \sigma_Y(p, q)| \leq \text{dist}^{-}(b_n) \rightarrow_{n \rightarrow \infty} 0$ \textbf{(the mapping sending a compact subset to its image under a continuous function is continuous in the Hausdorff topology)}, thus $\sigma(b_\infty(p), b_\infty(q)) = \sigma(p, q)$ and $b_\infty$ is an \textbf{ALL}-isomorphism.

Now we show the estimate between the metrics: For every correspondence $R : X \rightarrow Y$ we have

$$
\text{dist}^+(R) = \text{sup}_{(x_1, y_1), (x_2, y_2) \in R}\{|\sup_{x_3 \in X}|\sigma(x_1, x_3) - \sigma(x_2, x_3)| - \sup_{y_3 \in Y}|\sigma(y_1, y_3) - \sigma(y_2, y_3)|\}
$$

Supposing w.l.o.g. that $\sup_{x_3 \in X}|\sigma(x_1, x_3) - \sigma(x_2, x_3)| \geq \sup_{y_3 \in Y}|\sigma(y_1, y_3) - \sigma(y_2, y_3)|$, we get

$$
\text{dist}^+(R) \leq \text{sup}_{(x_1, y_1), (x_2, y_2) \in R}\{|\sup_{x_3 \in X}|\sigma(x_1, x_3) - \sigma(x_2, x_3)| - \sup_{y_3 \in Y}|\sigma(y_1, y_3) - \sigma(y_2, y_3)|\}
$$

$$
\leq \text{sup}_{(x_1, y_1), (x_2, y_2) \in R}\{|\sigma(x_1, x_3) - \sigma(y_1, y_3)| + |\sigma(x_2, x_3) - \sigma(y_2, y_3)|\}
$$

$$
= 2\text{dist}^{-}(R)
$$

(where the inequality in the second last line is an application of the general rule $||A-B|-|C-D|| \leq |(A-B)-(C-D)| \leq |A-C| + |B-D|$ for all real numbers $A, B, C, D$). Thus $d_{\text{GH}}^+ \leq 2d_{\text{GH}}^-$. ■
As an example, we let \( r \in (0; \infty] \) and consider \( L_{n,r} := \frac{1}{n} \mathbb{Z}^{1,1} \cap x_0^{-1}(-r; r) \subset \mathbb{R}^{1,1} \cap x_0^{-1}(-r; r) =: L_{\infty,r} \). Then we can consider \( R_n \in \text{Corr}(L_{n,r}, L_{\infty,r}) \) defined by \( xR_n y \iff y \in B_{\mathbb{R}^{2}}(x, 2/n) \), so indeed we have \( L_n \to_{n \to \infty} L_{\infty} \) for \( r < \infty \) but not for \( r = \infty \). In the (non-Hausdorff) topology generated by the pseudometric \( \Phi_{f,\infty}^* d_{\text{GH}} \) for every continuous bijective function \( f \). For further example, consider the last paragraph of this article.

McCann and Sämann [2] construct a natural measure on Lorentzian pre-length spaces (and so a functor \( \text{MCS} : \text{ALL} \to \text{POM} \)), essentially equivalent (see [11]) to the following: For \( A \subset X \), \( N > 0 \) we define (keeping in mind that \( I(x,y) \) is open with compact causally convex closure for all \( x, y \in X \) if \( X \) is globally hyperbolic):

\[
CC_\delta(A) := \{(p, q) \in (X^N)^2 | A \subset \bigcup_{k=1}^\infty J(p(k), q(k)) \land \text{diam}(I(p(k), q(k))) < \delta \forall k \in \mathbb{N}\},
\]

(\( \text{diam} \) is w.r.t. \( L \)) if \( \text{diam}_\text{w.r.t. } \Phi \ast \ell \) and defined the Lorentzian length \( \ell \) and \( \text{L} \) for all \( \text{r} \) \( \text{L} \)Recovers the dimension \( \text{L} \) at \( \text{b} \) (which recovers the dimension \( n \) at every \( b \in X \) if \( X \) is a spacetime) by

\[
Dm(b) := \limsup_{I^+(b) \ni x \to b^- \in I^-} (-\log_2(\Phi(a, c)) + 1) \in [1; \infty],
\]

where \( (\text{with Vol}(J(a, c)) \neq 0) \) for \( a, c \in (X,g) \),

\[
\Phi(a, c) := \sup \left\{ \frac{\text{Vol}(J(a, b)) + \text{Vol}(J(b, c))}{\text{Vol}(J(a, c))} \mid b \in J(a, c) \land \text{Vol}(J(a, b)) = \text{Vol}(J(b, c)) \right\} \in [1; \infty],
\]

and defined the Lorentzian length \( \ell(c) \in [0; \infty] \) of a timelike curve \( c : I \to X \) by

\[
\ell(c) = \inf \left\{ \sum_{k=1}^{N} \frac{\text{Dm}(p_k) \Gamma\left(\frac{\text{Dm}(p_k) - 1}{2}\right)}{\pi} \cdot \text{Vol}(J(p_k, p_{k+1})) \mid \{t_0, ..., t_N\} \text{partition of } I \text{ with } p(n) := c(t_n) \right\},
\]

and finally put \( \mathcal{F}(\leq, \text{Vol}) := K^{-}(\ell) \) where \( K^{-}(\ell)(x,y) := \sup \{ \ell(c) \mid c : x \sim y \text{ causal} \} \). Let us give a statement of equivalence of the topologies generated by the two Gromov-Hausdorff metrics. It turns out to be crucial to stay away from the boundary, which we now formalize.
Let GOM\(^1\)_n resp. GOM\(^1\)\(_{\leq n}\) be the space of globally hyperbolic elements of POM\(^1\)_v of constant dimension \(n\) resp. of local dimension \(\leq n\) and let ALL\(^1\)_n be the space of elements of ALL\(^1\) of constant dimension \(n\) resp. of local dimension \(\leq n\). Let tdiam\((X) := \sup\{\sigma(x,y)| x,y \in X\}, then

\[
\text{tdiam}(X) = \inf\{\sup\{\sigma(x,y)| x \in \partial^-X\}| y \in \partial^+X\} = \inf\{\sup\{\sigma(x,y)| y \in \partial^+X\}| x \in \partial^-X\}.
\]

Moreover, let, for \(\varepsilon > 0\),

\[K_{\varepsilon}(X) := \{p \in X| \sup\{\sigma(p,y)| y \in \partial^+X\} \geq \varepsilon \wedge \sup\{\sigma(x,p)| x \in \partial^-X\} \geq \varepsilon\}.
\]

For \(\varepsilon < \text{tdiam}(X)\), \(X \mapsto K_{\varepsilon}(X)\) preserves compactness, connectedness, and global hyperbolicity.

**Theorem 4** Let \(\varepsilon > 0\) and \(n \in \mathbb{N}\). Then \(\mathcal{MC}_S \circ K_{\varepsilon} : (\text{ALL}_1^1, d_{\text{GH}}^-) \rightarrow (\text{POM}_1, d_{\text{GH}}^\times)\) and \(\mathcal{F} : (\text{GOM}_1^1, d_{\text{GH}}^\times) \rightarrow (\text{ALL}_1^1, d_{\text{GH}}^-)\) are continuous.

**Proof.** For the first assertion, let \(X_\infty \in \text{ALL}_1^1\) be given, we want to show continuity of \(\mathcal{MC}_S \circ K_{\varepsilon}\) at \(X_\infty\). Let \(n \mapsto (C_n : X_n \rightarrow X_\infty)\) be a sequence of correspondences between objects of ALL with \(\text{dist}^{-}(C_n) \rightarrow_{n \rightarrow \infty} 0\). It is sufficient to prove that \(\text{dist}^\times(\mathcal{MC}_S(K_{\varepsilon}(C_n))) \rightarrow_{n \rightarrow \infty} 0\). We want to show that the distortion w.r.t. the outer measure \(\mu_N\) converges to 0. Assume the opposite, then there is \(\varepsilon > 0\) and points \(p_n, q_n \in X_n\) as well as \(p_\infty, q_\infty \in X_\infty\) with \((p_n, q_\infty), (q_n, p_\infty) \in C_n\) and \(d^\times(p_n, q_n) - d^\times(p_\infty, q_\infty) > \varepsilon\). On the other hand, we have \(d^\times(p_n, q_n) = |\sigma_{p_n} - \sigma_{q_n}|_{L^2(X_n)}\) and \(d^\times(p_\infty, q_\infty) = |\sigma_{p_\infty} - \sigma_{q_\infty}|_{L^2(X_\infty)}\). As we know from the proof of Th. that \(\text{dist}^+(C_n) \leq 2\text{dist}^-(C_n)\), the respective integrands (in the definition of the \(L^2\) norm) become arbitrarily close in the supremum norm, thus the only thing that remains to be shown to complete the proof by contradiction is that the measures of the \(X_n\) are uniformly bounded. To this aim, let \(\delta > 0\) be fixed. Let, for \(\varepsilon > 0\), \(I_{\varepsilon}(x,y) := \{p \in X| \sigma(x,p), \sigma(p,y) > \varepsilon\}\),

\[
CC^\times_{\delta}(A) := \{(p,q) \in (X^N)^2| A \subset \bigcup_{k=1}^\infty I_{\varepsilon}(p(k), q(k)) \wedge \text{diam}(I_{\varepsilon}(p(k), q(k))) < \delta \forall k \in \mathbb{N}\}.
\]

Let \(A := K_{\varepsilon}(X_\infty)\) and let us consider an element \(D_\infty\) of \(CC^\times_{\delta}(A)\) with \(\lambda_n(D_\infty) > \mu_{n,\delta}(A) - \theta\). By compactness of \(A\), there is a finite subcover \(D_\infty^0\). Then we choose, for every \((p,q) \in D_\infty^0\), an element \((p^{(m)}, q^{(m)})\) with \((p^{(m)}, p), (q^{(m)}, q) \in C_m\) and obtain a finite cover \(D_m \in CC_{\delta + 2\varepsilon}(X_m)\) of cardinality \(N\). Indeed, \(\text{dist}^+(C_m) < 2\varepsilon\) ensures that each \(x \in X_n\) is contained in some \(I(p_k, q_k)\). If \(\sigma(p,q) \geq \varepsilon\), then \(\sigma(p^{(m)}, q^{(m)}) < 2\sigma(p,q)\), whereas if \(\sigma(p,q) \leq \varepsilon\), then \(\sigma(p^{(m)}, q^{(m)}) \leq 3\varepsilon\). Thus \(\lambda_n(D_m) < 2^n\lambda_n(D_\infty) + N(3\varepsilon)^n < 2^n(\mu_{n,\delta}(A) - \theta) + N(3\varepsilon)^n\).

For the second assertion, let \(n \mapsto (K_n : Y_n \rightarrow Y_\infty)\) be a sequence of correspondences between objects of POM with \(\text{dist}^\times(K_n) \rightarrow_{n \rightarrow \infty} 0\). We want to show \(\text{dist}^-(\mathcal{F}(K_n)) \rightarrow_{n \rightarrow \infty} 0\). By compactness, we find \(x_\infty, y_\infty \in X_\infty\) and \(x_n, y_n \in X_n\) for each \(n \in \mathbb{N}\) with \((x_n, x_\infty) \in C_n\) and \((y_n, y_\infty) \in C_n\) and \(\sigma(x_n, y_n) \neq_{n \rightarrow \infty} \sigma(x_\infty, y_\infty)\). Let \(c\) be a causal curve from \(x_\infty\) to \(y_\infty\) with \(\ell(c) > \sigma(x_\infty, y_\infty) - \varepsilon\), thus
there is a finite causal chain \( p \) from \( x_\infty \) to \( y_\infty \) with \( \ell(p) > \sigma(x_\infty, y_\infty) - \varepsilon \). Let \( M := \#p \). We want to pull back the causal chain to a causal chain in \( X_n \). To this aim let \( \delta := \min\{d^X(p_i, X \setminus J^-(p_{i+1})) | i \in \mathbb{N}_{M-2} \} \). Let \( p^{(n)}_i \in C^{-1}_n(p_i) \). There is \( n_0 \in \mathbb{N} \) such that \( \text{dist}^X(C_n) < \{\frac{\varepsilon}{2M}, \delta\} \) for all \( n \geq n_0 \). Then \( p^{(n)} \) is a causal chain from \( x_n \) to \( y_n \) for all \( n \geq n_0 \) and \( \sigma(x_\infty, y_\infty) - 2\varepsilon < \ell(p(n)) \leq \sigma(p_n, q_n) \) (the last inequality holds as \( p^{(n)} \) is the length of a curve by geodesicity). This shows the lower semicontinuity part \( \sigma(x_\infty, y_\infty) \leq \liminf_{n \to \infty} \sigma(x_n, y_n) \). For upper semicontinuity, choose a subsequence \( n \mapsto \gamma_n \) of \( n \mapsto c_n \) with \( \gamma_n \to d^\infty_n \gamma_\infty \) and \( \tilde{\gamma}_n \) by dyadic convergence resp. convergence (can be chosen Lipschitz). Then \( \tilde{\gamma}_n \to n \to \gamma_\infty \), and \( \ell \) is upper semicontinuous (Prop. 3.17 in [6]).

**Remark.** Due to the different behaviour of the two metrics under scaling for \( n \neq 1 \), they cannot be strongly (Lipschitz) equivalent.

**Theorem 5** The functors \( \mathcal{F} \) and \( \mathcal{MC}S \) are inverse to each other on the closure of the set of classes of g.h. spacetimes in \( \text{ALL}^1_n \) resp. \( \text{POM}^1_n \).

**Proof.** Assume they are not inverse to each other on an element \( X \) of the closure, then there is some \( \varepsilon > 0 \) such that \( \mathcal{F} \circ \mathcal{MC}S \circ K_\varepsilon(X) \neq K_\varepsilon(X) \). But the last equality holds for spacetimes, the maps are continuous, so the equality holds for the closure, contradiction.

Interestingly, despite the fact that \( d^-_{GH} \) is not a metric between metric spaces, there is an adapted Gromov precompactness result in this context. For a full and g.h. almost Lorentzian length space \( X \) we could define an \( \varepsilon \)-net to be a subset \( N \subset X \) such that for each \( x \in X \) there are \( p, q, y \in N \) with \( x, y \in J(p, q) \) and \( ||\sigma_x - \sigma_y||_{L^\infty(J(p, q))} < \varepsilon \), and then mimick BBI 7.4.9, 7.4.12, 7.4.15 and 7.5.1 and show completeness. However, in the following we will use another way. Let \( \text{ALL}^1_\text{c} \) be the category of compact almost Lorentzian length spaces and \( \text{MET} \) the category of metric spaces. Then for each \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) and each \( A \subset \text{MET}^1 \) such that \( \text{diam}(X) \leq T \) for all \( X \in A \) we get

For every \( X \in A \), there is an \( \varepsilon \)-net of cardinality \( N \) in \( X \)

\[ \iff \text{there is a metric d on } N^*_N \text{ such that } d_{GH}(X, (N, d)) < \varepsilon/2 \]
\[ \iff A \subset B_{d_{GH}}(A_{N, T+\varepsilon}, \varepsilon/2) \text{ for the (compact) set } A(N, T + \varepsilon) \text{ of isometry classes of metric spaces of cardinality } \leq N \text{ and diameter } \leq T + \varepsilon. \]

Thus one could define: \( X \) uniformly totally bounded :\( \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \exists T > 0 : A \subset B_{d_{GH}}(A_{N, T}, \varepsilon). \)

This is the definition most useful for our aims as it persists for non-metric Gromov-Hausdorff spaces.

**Theorem 6** (Gromov precompactness for \( d^-_{GH} \)) Let \( A \subset \text{ALL}^1_\text{c} \) is precompact w.r.t. \( d^-_{GH} \) iff

1. There is \( T > 0 \) with \( \text{diam}(X) < T \forall X \in A; \)
2. For all \( \varepsilon > 0 \) there is \( N(\varepsilon) \in \mathbb{N} \) such that for all \( X \in A \) there is \( \gamma_\varepsilon \in \text{ALL}^1 \) with \( \#\gamma_\varepsilon \leq N(\varepsilon) \) and \( d^-_{GH}(X, \gamma_\varepsilon) < \varepsilon. \)
Conjecture: The set of $n$-dimensional Riemannian manifolds is dense in the set of all metric spaces whose local Hausdorff dimension around every point is less than or equal to $n$. The functor assigning to a (class of a) metric space the (class of the) measured metric space whose measure is the $n$-dimensional Hausdorff measure is continuous. Analogously, the set of classes of g.h. $n$-dimensional Lorentzian manifolds is dense w.r.t. $d_{\text{GH}}^\times$ in $\text{ALL}^I_{\leq n}$ and w.r.t. $d_{\text{GH}}^\times$ in $\text{POM}^I_{\leq n}$.
By Theorem 5, the conjecture would mean that the functors $F$ and $MCS$ are inverse to each other on $\text{ALL}^{1} \leq n$ resp. $\text{POM}^{1} \leq n$.

As it becomes more and more transparent that ordered measure spaces and almost Lorentian length spaces are intimately related such that, under mild hypotheses, both structures can be used at a time, one can try to set up a version of analysis on metric measure spaces as initiated by Ambrosio, Gigli and many others (for an overview see [3], [4]), in this new context. The theory of Lorentzian optimal transport as in [13], [8] naturally defines curvature bounds.

To define a pointed version for the noncompact case, there are two possible lines of attack:

1. Begin with Gromov’s pointed metric as laid down in Herron [5] and try to reformulate it via correspondences.

2. Use the Busemann metric instead of the Hausdorff metric and try to reformulate the corresponding Gromov-Busemann distance in terms of correspondences.

We will need a replacement for balls, or, more generally, compact subsets exhausting the spaces (recall that the pointed Hausdorff distance is a true metric only on the set of isometry classes of proper metric spaces). A small consideration shows that there is no $SO(1, n)$-covariant map from $\mathbb{R}^{1,n}$ to the set of open precompact subsets of $\mathbb{R}^{1,n}$. So we would at least need a double puncture, which corresponds to the well-known observer moduli space where the admissible diffeomorphisms for the quotient space are required to fix a tangent vector.

For a Lorentzian length space $(X, \sigma)$ and a complete length space $(Y, d)$ we define their product $(X \times Y, p(\sigma, d))$ by $p(\sigma, d)((x_1, y_1), (x_2, y_2)) := \pm \sqrt{\sigma^2(x_1, x_2) - d^2(y_1, y_2)}$ for $\pm(\sigma(x_1, x_2) > d(y_1, y_2))$ and 0 otherwise. It is easy to see that $(X \times Y, p(\sigma, d))$ is a Lorentzian length space, and that we have $d^{-}_{\text{GH}}(X \times Y, A \times B) = \sqrt{d^{-}_{\text{GH}}(X,Y)^2 + d^{-}_{\text{GH}}(A,B)^2}$ (as the natural product of correspondences satisfies the corresponding equation). Furthermore, we obtain $d^{-}_{\text{GH}}((X, r \cdot \sigma), (X, \sigma)) = r \cdot \text{tdiam}(X, \sigma)$ for all $r > 0$ if $\text{tdiam}(X, \sigma) < \infty$ in exactly the same manner as in the metric case, i.e. by calculating the distortion of the identity. This implies e.g. that the curve $t \mapsto ((\mathbb{R}, -dt^2) \times \mathbb{S}^n(\frac{r}{2}))$ is geodesic, where $\mathbb{S}^n(r) = r \cdot \mathbb{S}^n(1)$ is the standard sphere of radius $r$ in Euclidean $\mathbb{R}^n$.

Only two days after the prepublication of the first version of this article on arXiv.org (which differed from this one only by some shortenings and this last paragraph), Ettore Minguzzi and Stefan Suhr prepublished on arXiv.org the results of their independent work [7] on the same subject in a very recommendable article which contains other very interesting aspects, like stability under $d^{-}_{\text{GH}}$-convergence in the set of Lorentzian pre-length spaces (a) of the property of being a so-called bounded Lorentzian length space with $i_0$ ([7], Th. 5.18) and (b) of timelike curvature bounds ([7], Th. 6.7).
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