BOUNDEDNESS OF STOCHASTIC SINGULAR INTEGRAL OPERATORS AND ITS APPLICATION TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

ILDOO KIM AND KYEONG-HUN KIM

ABSTRACT. In this article we present a stochastic counterpart of the Hörmander condition and Calderón-Zygmund theorem. Let $W_t$ be a Wiener process defined on a probability space $(\Omega, \mathbb{P})$ and $K(r, t, x, y)$ be a random kernel which is stochastically singular in the sense that

$$
\mathbb{E} \left[ \int_0^t \int_{|x-y|<\varepsilon} |K(s, t, x, y)| dydW_s \right]^p = \infty, \quad \forall t, p, \varepsilon > 0, \ x \in \mathbb{R}^d.
$$

We prove that the stochastic singular integral of the type

$$
Tg(t, x) := \int_0^t \int_{\mathbb{R}^d} K(t, s, x, y) g(s, y) dydW_s \quad (0.1)
$$

is a bounded operator on $L_p = L_p(\Omega \times (0, \infty); L_p(\mathbb{R}^d))$ for any $p \geq 2$ if it is bounded on $L_2$ and the following (which we call stochastic Hörmander condition) holds: there exists a quasi-metric $\rho$ on $(0, \infty) \times \mathbb{R}^d$ and a positive constant $C_0$ such that for $X = (t, x), Y = (s, y), Z = (r, z) \in (0, \infty) \times \mathbb{R}^d$,

$$
\sup_{\omega \in 0, X, Y} \int_0^\infty \left[ \int_{\rho(X, Z) \geq C_0\rho(X, Y)} |K(r, t, z, x) - K(r, s, z, y)| dz \right]^2 dr < \infty.
$$

As a consequence of our result on stochastic singular integral operators, we obtain the maximal regularity for a very wide class of stochastic partial differential equations.

1. INTRODUCTION

Since Calderón and Zygmund’s work, the singular integral theory has been one of most important fields in Mathematics and it has been developed considerably in various directions (see e.g. [2, 11]). In particular, due to Hörmander the singular integral

$$
Tf(x) := \int_{\mathbb{R}^d} K(x, y) f(y) dy \quad (1.1)
$$

2010 Mathematics Subject Classification. 60H15, 42B20, 35S10, 35K30, 35B45.

Key words and phrases. Stochastic Singular Integral Operator, $L_p$-estimate, Pseudo-differential operator, Stochastic Partial Differential Equation.

The first author was supported by the TJ Park Science Fellowship of POSCO TJ Park Foundation.

The second author was supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1401-02.
becomes a bounded operator on $L_p(\mathbb{R}^d)$ if the kernel $K$ satisfies the Hörmander condition (see [11, Theorem I.5.3])

$$\sup_{x,y \in \mathbb{R}^d} \int_{|x-z|>2|x-y|} |K(x,z) - K(y,z)| dz < \infty.$$  \hspace{1cm} (1.2)

Hörmander’s condition is considered as one of most general conditions in the theory of the singular integral, and there is a huge number of applications to partial differential equations. For instance, consider the heat equation

$$u_t = \Delta u + f, \quad (t,x) \in (0,T] \times \mathbb{R}^d, \quad u(0) = 0.$$  \hspace{1cm} (1.3)

As is well known, for the solution $u$ we have

$$u_{xx}(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla p_{t',x'}(s,t,x-y)f(s,y)dy ds,$$

where $p(t,x)$ is the heat kernel. One can prove that the kernel $K(s,t,x,y) = 1_{s<t}p_{t',x'}(t-s,x-y)$ is singular but satisfies (1.2) on $\mathbb{R}^{d+1}$. Consequently this leads to

$$\|u_{xx}\|_{L_p((0,T) \times \mathbb{R}^d)} \leq C\|f\|_{L_p((0,T) \times \mathbb{R}^d)}.$$  \hspace{1cm} (1.4)

In particular, he proved the $L_p$-boundedness of

$$\nabla u(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla p(t-s,x-y)g(s,y)dy dW_s.$$  \hspace{1cm} (1.5)

The right hand side of (1.5) becomes a stochastic singular integral in the sense that

$$\mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} |\nabla p(s,t,x-y)| dy dW_s \right]^p = \infty, \quad \forall t,p > 0.$$  \hspace{1cm} (1.6)

Lately, $L_p$-theory has been further developed for high-order stochastic PDEs, stochastic integro-differential equations and certain stochastic pseudo-differential equations. For related works, we refer to [4, 6, 7] (Krylov’s analytic approach) and [12, 13] ($H^\infty$-calculus). Krylov’s approach requires differentiability of the kernel, and $H^\infty$-calculus approach works only if the corresponding operator is a generator of bounded analytic semigroup and does not depend on the time variable.

Our primary goal is to introduce a theory with which one can investigate the maximal regularity for very large classes of stochastic partial differential equations. The stochastic singular integral of type (1.1) naturally appears if one tries to obtain the maximal $L_p$-regularity of solutions to stochastic partial differential equations. We prove that the stochastic Hörmander condition is sufficient for the $L_p$-boundedness of the stochastic integral and demonstrate that our result on stochastic singular integral (1.1) leads to the maximal $L_p$-regularity of large classes of stochastic partial differential equations.

Here is a brief comment on our approach. We noticed that some key techniques in Krylov’s approach, e.g. integration by parts, are not applicable for general kernels. Hence we combined Krylov’s idea with some tools used for the deterministic singular integral theory and Calderón-Zygmund theorem.
The article is organized as follows. The main theorem is given Section 2 and the related parabolic Littlewood-Paley inequality is introduced and proved in Section 3. In section 4, the main theorem is proved on the basis of the parabolic Littlewood-Paley inequality. Finally, the maximal $L_p$-regularity result for SPDEs is given in Section 5.

We finish the introduction with the notation used in the article. $\mathbb{N}$ and $\mathbb{Z}$ denote the natural number system and the integer number system, respectively. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$. For $i = 1, \ldots, d$, multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \ldots\}$, and functions $u(x)$ we set

$$u_{x_i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_{i_1}^{\alpha_1} \cdots D_{i_d}^{\alpha_d} u, \quad \nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_d}).$$

We also use the notation $D^m$ for a partial derivative of order $m$ with respect to $x$. For $p \in [1, \infty)$, a normed space $F$, and a measure space $(X, \mathcal{M}, \mu)$, $L_p(X, \mathcal{M}, \mu; F)$ denotes the space of all $F$-valued $\mathcal{M}$-measurable functions $u$ so that

$$\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|_F^p \mu(dx) \right)^{1/p} < \infty,$$

where $\mathcal{M}$ denotes the completion of $\mathcal{M}$ with respect to the measure $\mu$.

For $p = \infty$, we write $u \in L_\infty(X, \mathcal{M}, \mu; F)$ iff

$$\sup_x |u(x)| := \|u\|_{L_\infty(X, \mathcal{M}, \mu; F)} := \inf \{ \nu \geq 0 : \mu(\{x : \|u(x)\|_F > \nu\}) = 0\} < \infty.$$

If there is no confusion for the given measure and $\sigma$-algebra, we usually omit the measure and the $\sigma$-algebra. In particular, for a domain $\mathcal{O} \subset \mathbb{R}^d$ we denote $L_p(\mathcal{O}) = L_p(\mathcal{O}, \mathcal{F}, \ell; \mathbb{R})$ and $L_p(\mathbb{R}) = L_p(\mathcal{O}, \mathcal{L}, \ell; l_2)$, where $\mathcal{L}$ is the Lebesgue measurable sets, $\ell$ is the Lebesgue measure, and $l_2$ is the space of sequences $a = (a_n)$ so that

$$|a|_{l_2}^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

We use "=" to denote a definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $|a|$ is the biggest integer which is less than or equal to $a$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the d-dimensional Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x)dx$ and $\mathcal{F}^{-1}(f)(x) := 1/(2\pi)^d \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi)d\xi$. For a Lebesgue measurable set $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure. For a set $B, 1_B$ is the indicator of $B$, i.e. $1_B(b) = 1$ if $b \in B$ and $1_B(b) = 0$ otherwise. For a complex number $z$, $\overline{z}$ is the complex conjugate of $z$ and $\Re[z]$ is the real part of $z$. For functions depending on $\omega, t$, and $x$, the argument $\omega \in \Omega$ will be usually omitted. Usually $X_0, X, Y, Z$ denote the vectors in $(0, \infty) \times \mathbb{R}^d$ and are represented by

$$X_0 = (t_0, x_0), \quad X = (t, x), \quad Y = (s, y), \quad Z = (r, z),$$

where $t_0, t, s, r$ are positive numbers and $x_0, x, y, z$ are vectors in $\mathbb{R}^d$. Finally, $N$ denotes a generic constant which can differ from line to line and if we write $N = N(a, b, \ldots)$, then this means that the constant $N$ depends only on $a, b, \ldots$.

2. MAIN RESULT

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of $\sigma$-fields on $\Omega$ satisfying the usual condition, i.e. $\mathcal{F}_t \subset \mathcal{F}$ contains all $(\mathcal{F}, P)$-null sets and $\mathcal{F}_t = \bigcap_{s \geq t} \mathcal{F}_s$. By $\mathcal{P}$ we denote the predictable $\sigma$-algebra, that
is, \( \mathcal{P} \) is the smallest \( \sigma \)-algebra containing the collection of all sets \( A \times (s,t] \), where \( 0 \leq s \leq t < \infty \) and \( A \in \mathcal{F}_s \). Let \( W^1_t, W^2_t, \ldots \) be an infinite sequence of independent one-dimensional Wiener processes defined on \( \Omega \), each of which is a Wiener process relative to \( \{ \mathcal{F}_t, t \geq 0 \} \). For \( T \in (0, \infty] \) and a domain \( \mathcal{O} \subset \mathbb{R}^d \), we denote
\[
\mathcal{O}_T := (0, T) \times \mathcal{O}.
\]
Define
\[
\mathbb{L}_p(\mathcal{O}_T) = L_p(\Omega \times (0, T), \mathcal{P}; L_p(\mathcal{O})), \quad \mathbb{L}_p(\mathcal{O}_T, l_2) = L_p(\Omega \times (0, T), \mathcal{P}; L_p(\mathcal{O}; l_2)),
\]
and
\[
\|g\|_{\mathbb{L}_p(\mathcal{O}_T, l_2)} = \left( \mathbb{E} \int_0^T \int_\mathcal{O} |g|^p dx dt \right)^{1/p}.
\]
If \( \mathcal{O}_T = (0, \infty) \times \mathbb{R}^d \), we simply put \( \mathbb{L}_p(\mathcal{O}_T) = \mathbb{L}_p \) and \( \mathbb{L}_p(\mathcal{O}_T, l_2) = \mathbb{L}_p(l_2) \).

Denote \( \mathbb{R}_+ = (0, \infty) \) and let \( K(r, t, z, x) = K(\omega, r, t, z, x) \) be a \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{O}) \)-measurable function such that \( K(r, t, z, x) = 0 \) if \( r \geq t \). For \( g = (g^1, g^2, \ldots) \in \mathbb{L}_p(\mathcal{O}, l_2) \) and \((t, x) \in \mathcal{O}_T\), define
\[
\mathbb{T}_z g(t, x) := \int_0^{t-z} \int_\mathcal{O} K(r, t, z, x)g^k(r, z)dzdW^k_r
\]
and
\[
\mathbb{T}_g(t, x) := \int_0^t \int_\mathcal{O} K(r, t, z, x)g^k(r, z)dzdW^k_r
\]
\[
:= \lim_{\varepsilon \downarrow 0} \mathbb{T}_z g(t, x),
\]
where the sense of convergence will be specified in Assumption 2.3.

**Definition 2.1.** Let \( D \) be a subset of \( \mathbb{R}^{d+1} \). A function \( \rho(X, Y) \) defined on \( D \times D \) is called a quasi-metric if the following four properties hold:

(i) \( \rho(X, Y) \geq 0 \) for all \( X, Y \in D \)

(ii) \( \rho(X, Y) = 0 \) iff \( X = Y \)

(iii) \( \rho(X, Y) = \rho(Y, X) \) for all \( X, Y \in D \)

(iv) There exists a constant \( N_\rho \geq 1 \) such that \( \rho(X, Y) \leq N_\rho (\rho(X, Z) + \rho(Z, Y)) \) for all \( X, Y, Z \in D \).

Define balls related to the quasi-metric \( \rho \) as
\[
B_c(X) := \{ Z \in D : \rho(X, Z) < c \}, \quad X \in D, \ c > 0.
\]
Note that the center \( X \) of the ball \( B_c(X) \) is always in \( D \).

Throughout the article we assume that the quasi-metric \( \rho \) satisfies the doubling ball condition on \( D \), that is, for any \( \gamma > 0 \) there exists a constant \( N_\gamma \) so that
\[
|B_{\gamma c}(X)| \leq N_\gamma |B_c(X)| \quad \forall c > 0, \ X \in D. \tag{2.1}
\]

For a locally integrable function \( f \) on \( D \), define its sharp function as
\[
f^k(t, x) := \sup \int_{B_c(Y)} |f(r, z) - f_{B_c(Y)}| \ drdz
\]
\[
:= \sup \frac{1}{|B_c(Y)|} \int_{B_c(Y)} |f(r, z) - f_{B_c(Y)}| \ drdz
\]
\[
\approx \sup \frac{1}{|B_c(Y)|^2} \int_{B_c(Y)} \int_{B_c(Y)} |f(Z) - f(Z')| \ dZ dZ'.
\]
where the sup is taken over all $B_\varepsilon(Y)$ containing $X = (t, x)$ and

$$f_{B_\varepsilon(Y)} = \int_{B_\varepsilon(Y)} f(r, z) \, drdz.$$  

Similarly, the maximal function $Mf(t, x)$ is defined as

$$Mf(t, x) := \sup_{B_\varepsilon(Y)} \int_{B_\varepsilon(Y)} |f(r, z)| \, drdz,$$

where the sup is taken over all $B_\varepsilon(Y)$ containing $X = (t, x)$.

Below is a version of Hardy-Littlewood and Fefferman-Stein theorems.

**Theorem 2.2.** For any $p > 1$,

$$\|Mf\|_{L^p(D)} \leq N(p)\|f\|_{L^p(D)}, \quad \forall f \in L^p(D). \quad (2.2)$$

Furthermore, if $|D| = \infty$, then

$$\|f\|_{L^p(D)} \leq N_p\|f\|_{L^p(D)} \quad (2.3)$$

For the proof of this theorem, see e.g. [7, Theorem 2.2 and Theorem 2.4]. If \((2.3)\) holds, we say $\rho$ admits the Fefferman-Stein theorem (FS), which obviously holds if $|D| = \infty$.

**Assumption 2.3** ($L_2$-boundedness). For each $g \in L_2(O_T, l_2)$, $T\varepsilon g(t, x)$ converges in $L_2(O_T)$ as $\varepsilon \downarrow 0$. Moreover the operator $g \mapsto \mathbb{T}g$ is bounded from $L_2(O_T, l_2)$ to $L_2(O_T)$, i.e., there exists a constant $N_0$ such that for all $g \in L_2(O_T, l_2)$,

$$\|\mathbb{T}g\|_{L_2(O_T)} \leq N_0\|g\|_{L_2(O_T, l_2)}. \quad (2.4)$$

**Assumption 2.4** (A stochastic Hörmander condition). There exist positive constants $C_0$ and $N_1$ such that for all $X = (t, x), Y = (s, y), Z = (r, z) \in O_T$,

$$\sup_{\omega} \sup_{X \neq Y} \left( \int_0^T \int_{\rho(X, Z) \geq C_0\rho(X, Y)} |K(r, t, z, x) - K(r, s, z, y)| \, dz \right)^2 \, dr \leq N_1, \quad (2.5)$$

where $\rho$ is a quasi-metric admitting FS.

By $L(O_T, l_2)$ (simply $L(l_2)$ if $O_T = (0, \infty) \times \mathbb{R}^d$), we denote the space of the processes $g = (g^1, g^2, \ldots)$ such that $g^k = 0$ for all large $k$ and each $g^k$ is of the type

$$g^k(t, x) = \sum_{i=1}^{j(k)} 1(\tau_{i-1}, \tau_i)(t)g^{ik}(x),$$

where $g^{ik} \in C_0^\infty(O)$, and $\tau_i$ are stopping times so that $\tau_i \leq T$. It is known that $L(O_T, l_2)$ is dense in $L_p(O_T, l_2)$ for all $p \geq 1$ (for instance, see [7, Theorem 3.10]) if $O_T = (0, T) \times \mathbb{R}^d$. The idea of [7, Theorem 3.10] is easily applied even for general $O_T$.

Here is our main result.

**Theorem 2.5.** Suppose that Assumptions 2.3 and 2.4 hold. Then for any $p > 2$, the operator $\mathbb{T}$ can be continuously extended from $L(O_T, l_2)$ to $L_p(O_T, l_2)$. Moreover, for any $g \in L_p(O_T, l_2)$,

$$\|\mathbb{T}g\|_{L_p(O_T)} \leq N(d, p, C_0, N_0, N_1)\|g\|_{L_p(O_T, l_2)}.$$

The proof of this theorem will be given in Section 4.
3. Parabolic Littlewood-Paley inequality

For $l_2$-valued measurable functions $f = (f^1, f^2, \cdots)$ on $\mathcal{O}_T$, denote

$$ Gf(t, x) := \left( \int_0^t \int_\mathcal{O} K(r, t, z, x) f(r, z) dz \right)^{1/2} $$

$$ = \lim_{\varepsilon \downarrow 0} \left( \int_0^{t - \varepsilon} \int_\mathcal{O} K(r, t, z, x) f(r, z) dz \right)^{1/2}. \quad (3.1) $$

In this section we study the boundedness of operator $G$ in $L_p(\mathcal{O}_T; l_2)$. Since the integral above is deterministic one may assume that the kernel $K$ is nonrandom throughout this section.

Theorem 3.1 below is the main result of this section which we call “Parabolic Littlewood-Paley inequality”. This inequality was first proved by Krylov for $K(r, t, x, y) = \nabla_x p(t-r, x-y)$, where $p(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}$ is the heat kernel. If $f$ is independent of $t$ then parabolic Littlewood-Paley inequality with $K = \nabla_x p(t-r, x-y)$ leads to the classical (elliptic) Littlewood-Paley inequality (for instance, see [4, Section 1]).

**Theorem 3.1 (Parabolic Littlewood-Paley inequality).** Let $p \ge 2$. Suppose Assumptions 2.3 and 2.4 hold. Then for any $f \in L_2(\mathcal{O}_T; l_2) \cap L_\infty(\mathcal{O}_T; l_2)$,

$$ \|Gf\|_{L_p(\mathcal{O}_T)} \le N\|f\|_{L_p(\mathcal{O}_T; l_2)} $$

where $N$ depends only on $d, p, C_0, N_1, \text{ and } N_2$.

The proof of this theorem will be given at the end of this section.

**Lemma 3.2.** Suppose Assumption 2.3 holds. Then for each $f \in L_2(\mathcal{O}_T; l_2)$, $Gf(t, x)$ is finite almost everywhere, and moreover the operator $f \rightarrow Gf$ is a bounded operator from $L_2(\mathcal{O}_T; l_2)$ to $L_2(\mathcal{O}_T)$.

**Proof.** Obviously, since $f$ is nonrandom, $f \in L_2(l_2)$. By Itô's isometry and (2.4),

$$ \int_0^T \int_\mathcal{O} |Gf(t, x)|^2 dxdt = \mathbb{E} \int_0^T \int_\mathcal{O} |Tf(t, x)|^2 dxdt $$

$$ \le N_0 \mathbb{E} \int_0^T \int_\mathcal{O} |f(t, x)|^2 dxdt $$

$$ = N_0 \int_0^T \int_\mathcal{O} |f(t, x)|^2 dxdt. $$

Thus the lemma is proved. \qed

Denote

$$ Kf(r, t, x) = \int_\mathcal{O} K(r, t, z, x) f(r, z) dz $$

and

$$ Gf(t, s, x, y) = \left( \int_0^T |Kf(r, t, x) - Kf(r, s, y)|^2 l_2 dr \right)^{1/2}. $$
Observe
\[ \mathcal{G}f(t, x) = \left[ \int_0^t |Kf(r, t, x)|^2_{l_2} \, dr \right]^{1/2} \]
and
\[ Gf(t, s, x, y) = \left[ \int_0^T |1_{r<t}Kf(r, t, x) - 1_{r<s}Kf(r, s, y)|^2_{l_2} \, dr \right]^{1/2}, \]
where the last equality is due to the assumption that \( K(r, t, z, x) = 0 \) if \( t \leq r \).

**Lemma 3.3.** Let \((t_1, x_1) \in B_c(X_0)\) and suppose Assumption 2.3 holds. Then for any \( f_1, f_2 \in L_2(O_T; l_2)\),
\[
\int_{B_c(X_0)} \int_{B_c(X_0)} |\mathcal{G}(f_1 + f_2)(t, x) - \mathcal{G}(f_1 + f_2)(s, y)| \, dt \, dx \, ds \, dy \\
\leq 2M(\mathcal{G}f_1)(t_1, x_1) + \int_{B_c(X_0)} \int_{B_c(X_0)} Gf_2(t, s, x, y) \, dt \, dx \, ds \, dy. \tag{3.2}
\]

**Proof.** Set \( f = f_1 + f_2 \) and let \((t, x), (s, y) \in B_c(X_0)\). By Lemma 3.2 we may assume
\[ \mathcal{G}(f_1)(t, x) + \mathcal{G}(f_2)(t, x) + \mathcal{G}(f_1)(s, y) + \mathcal{G}(f_2)(s, y) < \infty. \]

Then by Minkowski’s inequality,
\[
|\mathcal{G}f(t, x) - \mathcal{G}f(s, y)| \\
= \left[ \int_0^T |1_{r<t}Kf(r, t, x)|^2_{l_2} \, dr \right]^{1/2} - \left[ \int_0^T |1_{r<s}Kf(r, s, y)|^2_{l_2} \, dr \right]^{1/2} \\
\leq \left[ \int_0^T |1_{r<t}Kf(r, t, x) - 1_{r<s}Kf(r, s, y)|^2_{l_2} \, dr \right]^{1/2} \\
\leq \left[ \int_0^T |1_{r<t}Kf_1(r, t, x)|^2_{l_2} \, dr \right]^{1/2} + \left[ \int_0^T |1_{r<s}Kf_2(r, s, y)|^2_{l_2} \, dr \right]^{1/2} \\
+ \left[ \int_0^T |1_{r<t}Kf_2(r, t, x) - 1_{r<s}Kf_2(r, s, y)|^2_{l_2} \, dr \right]^{1/2}.
\]
Taking mean average to the above inequality, we get (3.2). \( \square \)

Take constants \( N_\rho \) and \( C_0 \) from Definition 2.1 and Assumption 2.3 respectively, and denote
\[ \gamma_0 = \gamma_0(N_\rho, C_0) := (2C_0N_\rho + 1)N_\rho. \]

**Lemma 3.4.** Suppose Assumption 2.3 holds, \( f \) belongs to \( L_2(O_T; l_2) \cap L_\infty(O_T; l_2) \) and vanishes outside of \( Q_{\tau_0}(t_0, x_0) \). Then
\[
\int_{B_c(X_0)} \int_{B_c(X_0)} Gf(t, s, x, y) \, dt \, dx \, ds \, dy \leq N\|f\|_{L_\infty(O_T; l_2)}, \tag{3.3}
\]
where \( N \) depends only on \( d, \gamma_0, \) and \( N_0 \).
Moreover by Hölder’s inequality and Lemma 3.2, that Assumption 2.3 and Assumption 2.4 hold. Then
\[ \int_{B_c(X_0)} Gf(t, x) dtdx \]
where Minkowski’s inequality, 
\[ Gf(t, s, x, y) \leq Gf(t, s) + Gf(s, y) \]
Thus recalling the definition of \( Gf \),
\[ Gf(t, s, x, y) \leq Gf(t, s) + Gf(s, y) \]
Therefore the left side of (3.3) is less than or equal to
\[ 2 \int_{B_c(X_0)} Gf(t, x) dtdx. \]
Moreover by Hölder’s inequality and Lemma 3.2
\[ \int_{B_c(X_0)} Gf(t, x) dtdx \leq \frac{1}{|B_c(X_0)|^{1/2}} \left[ \int_{B_c(X_0)} |Gf(t, x)|^2 dtdx \right]^{1/2} \]
\[ \leq \frac{1}{|B_c(X_0)|^{1/2}} \left[ \int_{Q_T} |Gf(t, x)|^2 dtdx \right]^{1/2} \]
\[ \leq \frac{N_0}{|B_c(X_0)|^{1/2}} \left[ \int_{Q_T} |f(t, x)|^2 dtdx \right]^{1/2} \]
\[ \leq N(d, \gamma_0, N_0) \|f\|_{L_\infty(Q_T; l_2)}, \]
where the last inequality is due to the assumption that \( f = 0 \) outside of \( B_{\gamma_0c}(X_0) \) and (2.1). Thus the lemma is proved. \( \square \)

**Lemma 3.5.** Let \( f \in L_2(Q_T; l_2) \cap L_\infty(Q_T; l_2) \) and \( f = 0 \) on \( Q_{\gamma_0c}(t_0, x_0) \). Suppose that Assumption 2.3 and Assumption 2.4 hold. Then
\[ \int_{B_c(X_0)} \int_{B_c(X_0)} Gf(t, s, x, y) dtdxdsdy \leq N\|f\|_{L_\infty(Q_T; l_2)}, \]
where \( N \) depends only on \( N_1 \).

Proof. If \( X = (t, x), Y = (s, y) \in B_c(X_0) \), and \( Z = (r, z) \in \mathcal{O}_T \setminus Q_{\gamma_0c}(t_0, x_0) \), then
\[ \rho(Z, X) \geq \frac{\rho(Z, X_0)}{N_\rho} - \rho(X, X_0) \geq 2C_0N_\rho c \geq C_0\rho(X, Y). \] (3.4)

Thus recalling the definition of \( Gf \) and the assumptions on \( f \), we have
\[ |Gf(t, s, x, y)|^2 \]
\[ \leq \int_0^T \left[ \int_{\mathcal{O}} |K(r, t, z, x) - K(r, s, z, y)| |f(r, z)| dz \right]^2 dr \]
\[ \leq \|f\|_{L_\infty(Q_T; l_2)}^2 \int_0^T \left[ \int_{A(t, r, s, x, y)} |K(r, t, z, x) - K(r, s, z, y)| dz \right]^2 dr, \]
where \( A(t, r, s, x, y) \) is the set of all \( z \in \mathbb{R}^d \) for which inequality (3.3) holds. Therefore by (2.1),
\[ |Gf(t, s, x, y)| \leq N_1^{1/2}\|f\|_{L_\infty(Q_T; l_2)} \]
and
\[ \int_{B_c(X_0)} \int_{B_c(X_0)} Gf(t, s, x, y) dtdxdsdy \leq N_1^{1/2}\|f\|_{L_\infty(Q_T; l_2)}. \]
The lemma is proved. \( \square \)
Lemma 3.6. Let \( f_1 \in L_2(\mathcal{O}_T; l_2), \ f_2 \in L_2(\mathcal{O}_T; l_2) \cap L_\infty(\mathcal{O}_T; l_2) \), and suppose Assumptions 2.3 and 2.4 hold. Then for each \((t_1, x_1) \in \mathcal{O}_T, \)
\[
|\mathcal{G}(f_1 + f_2)|^2(t_1, x_1) \leq 2M(\mathcal{G}f_1)(t_1, x_1) + N\|f_2\|_{L_\infty(\mathcal{O}_T; l_2)},
\]
where \( N \) depends only on \( d, \gamma, N_0, C_0, \) and \( N_1. \)

Proof. Let \((t_1, x_1) \in B_{\epsilon}(X_0). \) Then by Lemma 3.3
\[
\int_{B_{\epsilon}(X_0)} \int_{B_{\epsilon}(X_0)} |\mathcal{G}(f_1 + f_2)(t, x) - \mathcal{G}(f_1 + f_2)(s, y)| \ dt \ dx \ dy
\]
\[
\leq 2M(\mathcal{G}f_1)(t_1, x_1) + \int_{B_{\epsilon}(X_0)} \int_{B_{\epsilon}(X_0)} Gf_2(t, s, x, y) \ d\tau \ dx \ dy.
\]

Moreover, defining \( f_{2,1}(t, x) := f_2(t, x)1_{Q_{\infty}(t_0, x_0)}(t, x) \) and \( f_{2,2}(t, x) := f_2(t, x) - f_{2,1}(t, x), \) we have
\[
\int_{B_{\epsilon}(X_0)} \int_{B_{\epsilon}(X_0)} Gf_2 \ d\tau \ dx \ dy
\]
\[
\leq \int_{B_{\epsilon}(X_0)} \int_{B_{\epsilon}(X_0)} Gf_{2,1} \ d\tau \ dx \ dy + \int_{B_{\epsilon}(X_0)} \int_{B_{\epsilon}(X_0)} Gf_{2,2} \ d\tau \ dx \ dy.
\]
Therefore we obtain (3.5) by applying Lemma 3.4 and Lemma 3.3. The lemma is proved.

Proof of Theorem 3.1.

Since the case \( p = 2 \) is already proved in Lemma 3.2, we assume \( p > 2. \) Let \( f \in L_2(\mathcal{O}_T; l_2) \cap L_\infty(\mathcal{O}_T; l_2). \) For \( \lambda > 0, \) we put
\[
f_{1,\lambda}(t, x) = f(t, x)1_{|f| \leq \delta \lambda}(t, x) \quad \text{and} \quad f_{2,\lambda}(t, x) = f(t, x)1_{|f| > \delta \lambda}(t, x),
\]
where \( \delta \) is a positive constant which will be specified later. Obviously,
\[
f = f_{1,\lambda} + f_{2,\lambda}.
\]
Assume
\[
\lambda \leq |\mathcal{G}(f)|^2(t, x).
\]

Then by Lemma 3.6
\[
\lambda \leq |\mathcal{G}(f)|^2(t, x) \leq 2M(\mathcal{G}f_{1,\lambda})(t, x) + N\|f_{2,\lambda}\|_{L_\infty(\mathcal{O}_T; l_2)}
\]
\[
\leq 2M(\mathcal{G}f_{1,\lambda})(t, x) + N\delta \lambda,
\]
where \( N \) is independent of \( \lambda \) and \( \delta. \) Take \( \delta > 0 \) so that \( N\delta < 1/2. \) Then the above inequality implies that
\[
\lambda \leq 4M(\mathcal{G}f_{1,\lambda})(t, x).
\]
Thus
\[
|\{(t, x) \in \mathcal{O}_T : \lambda \leq |\mathcal{G}(f)|^2(t, x)\}| \leq |\{(t, x) \in \mathcal{O}_T : \lambda \leq 4M(\mathcal{G}f_{1,\lambda})(t, x)\}|
\]
(3.6)
By (2.2)
\[
\|\mathcal{G}f\|_{L_p(\mathcal{O}_T)} \leq N_p\|\mathcal{G}f\|_{L_p(\mathcal{O}_T)}^p
\]
Observe that
\[
\|\mathcal{G}f\|_{L_p(\mathcal{O}_T)}^p = p \int_0^\infty \lambda^{p-1} |\{(t, x) \in \mathcal{O}_T : \lambda \leq |\mathcal{G}(f)|^2(t, x)\}| \ d\lambda.
\]
Therefore by (3.6), Chebyshev's inequality, (2.2), and Lemma 3.2,

\[
\|Gf^\#\|_{L^p(O_T)} \leq \int_0^\infty \lambda^{p-3} \int_{O_T} |\mathcal{M}(gf_1,\lambda)(t,x)|^2 dtdx d\lambda
\]

\[
\leq N \int_0^\infty \lambda^{p-3} \int_{O_T} |f_1,\lambda(t,x)|^2 dtdx d\lambda
\]

\[
\leq N \int_0^\infty \lambda^{p-3} \int_{O_T \cap \{|f| > \delta\}} |f(t,x)|^2 dtdx d\lambda
\]

\[
=N \int_{O_T} \left( \int_0^\infty \lambda^{p-3} d\lambda \right) |f(t,x)|^2 dtdx
\]

\[
\leq N \|f\|_{L^p(O_T;L^2)}.
\]

The last inequality is due to \(p > 2\). The theorem is proved. \(\square\)

### 4. Proof of Theorem 2.5

Let \(g \in L^2(O_T)\). Then for each \(\omega\), \(g \in L^2(O_T;l^2) \cap L^\infty(O_T;l^2)\). Therefore by Fubini’s Theorem, Burkholder-Davis-Gundy’s inequality, and Theorem 3.1,

\[
\mathbb{E} \int_0^T \|Tg(t,\cdot)\|^p_{L^p} dt
\]

\[
= \int_0^T \int_{\mathbb{R}^d} \mathbb{E} \left| \int_0^t \int_{\mathbb{R}^d} K(r,t,z,x)g^k(r,z)dzdW^k_r \right|^p dx dt
\]

\[
\leq N(p) \int_0^T \int_{\mathbb{R}^d} \left( \int_0^t \left| \int_{\mathbb{R}^d} K(r,t,z,x)g(r,z)dz \right|_{l^2}^2 dr \right)^{p/2} dx dt
\]

\[
\leq N \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |Gg|^p dx dt \leq N \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |g|^p dx dt.
\]

The theorem is proved. \(\square\)

### 5. Application to SPDE: Maximal \(L_p\)-regularity

We study the maximal \(L_p\)-regularity of SPDEs of the type

\[
du(t,x) = A(t)u(t,x)dt + \sum_{k=1}^\infty g^k(t,x)dW^k_t, \quad (t,x) \in (0,\infty) \times \mathbb{R}^d; \quad u(0) = 0,
\]

(5.1)

where \(W^k_t\) are independent one-dimensional Wiener process defined on \(\Omega\).
5.1. **Time measurable pseudo-differential operator.** Assume that \( A(t) \) is a pseudo differential operator with the symbol \( \psi(t, \xi) \), that is,

\[
A(t)f(x) := \mathcal{F}^{-1}[\psi(t, \xi)\mathcal{F}(f)(\xi)](x).
\]

We set

\[
d_0 := \left\lfloor \frac{d}{2} \right\rfloor + 1,
\]

and assume there exists a constant \( \nu > 0 \) such that

\[
\Re[-\psi(t, \xi)] \geq \nu|\xi|^{\gamma},
\]

and

\[
\int_{R \leq |\xi| < 2R} \prod_{i=1}^{d_0} |D_{\xi}^{\alpha_i} \psi(t, \xi)|^{k_i} d\xi 
\leq \nu^{-1} N(d, c) R^{k_1(\gamma - |\alpha_1|) + k_2(\gamma - |\alpha_2|) + \cdots + k_{d_0}(\gamma - |\alpha_{d_0}|)}
\]

for any \( R > 0 \), multi-indexes \( \alpha_i \in (\mathbb{Z}^+)^d \) and \( k_i \in \mathbb{Z}^+ \) \( (i = 1, 2, \ldots, d_0) \) such that

\[
\sum_{i=1}^{d_0} |\alpha_i| + \sum_{i=1}^{d} k_i \leq d_0 + \sum_{i=1}^{d_0} 1_{k_i > 0}.
\]

**Remark 5.1.** Here is a sufficient condition for (5.3): \( \exists c > 0 \) such that

\[
|D_{\xi}^2 \psi(t, \xi)| \leq c|\xi|^{-|\alpha|}, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \forall |\alpha| \leq d_0.
\]

Indeed, if this holds then, for \( R \leq |\xi| < 2R \),

\[
\prod_{i=1}^{d_0} |D_{\xi}^{\alpha_i} \psi(t, \xi)|^{k_i} \leq N(c, d) R^{k_1(\gamma - |\alpha_1|) + k_2(\gamma - |\alpha_2|) + \cdots + k_{d_0}(\gamma - |\alpha_{d_0}|)}
\]

Thus by integrating on \( \{ \xi \in \mathbb{R}^d : R \leq |\xi| < 2R \} \) we certainly get (5.3).

Define

\[
p(s, t, x) := 1_{0 < s < t} \mathcal{F}^{-1}\left[\exp\left(\int_s^t \psi(r, \xi) dr\right)\right](x).
\]

and

\[
(-\Delta)^{\gamma/4} p(s, t, x) := 1_{0 < s < t} \mathcal{F}^{-1}\left[|\xi|^{\gamma/2} \exp\left(\int_s^t \psi(r, \xi) dr\right)\right](x).
\]

Then for any \( g \in L_2(\mathbb{R}^d) \), the (weak) solution to (5.4) is given by

\[
u(t, x) = \int_0^t \int_{\mathbb{R}^d} p(s, t, x - y) g(y) dy dW^k_s.
\]

(5.5)

See e.g. [7, Theorem 4.2] for details. Actually in [7] the representation formula of the weak solution is derived only for \( \psi(t, \xi) = -|\xi|^2 \), but one can easily check the argument there works for the general case.

Due to (5.2) we may say \( A(t) \) is a linear operator of order \( \gamma \). Applying the Ito’s formula to \( |u(t, x)|^2 \), taking the expectation, and then integrating over \( \mathbb{R}^d \), we get for any \( t > 0 \),

\[
\mathbb{E}\|u(t)\|_{L_2(\mathbb{R}^d)}^2 - 2\Re \left[\mathbb{E}\int_0^t \int_{\mathbb{R}^d} u A(x, s) ds \right] = \mathbb{E}\int_0^t \|g\|_{L_2(\mathbb{R}^d \times J_t)}^2 ds.
\]
By Plancherel’s theorem and (5.2),
\[-\Re \left[ \int_{\mathbb{R}^d} u \overline{A} u \, dx \right] = -\Re \left[ \int_{\mathbb{R}^d} \psi(s, \xi) \mathcal{F}(u)(\xi)^2 \, d\xi \right],\]
\[= \int_{\mathbb{R}^d} \Re[\psi(s, \xi)] |\mathcal{F}(u)(\xi)|^2 \, d\xi \geq \nu \int_{\mathbb{R}^d} |\xi|^\gamma |\mathcal{F}(u)(\xi)|^2 \, d\xi = \nu \|(-\Delta)^{\gamma/4} u\|^2_{L^2(\mathbb{R}^d)}.
\]
It follows that
\[\mathbb{E} \|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\nu \mathbb{E} \int_0^t \|(-\Delta)^{\gamma/4} u\|^2_{L^2(\mathbb{R}^d)} \, ds \leq \mathbb{E} \int_0^t \|g(t, \cdot\|_{L^p_{\nu}(\mathbb{R}^d)}^p \, dt.\]

The following theorem extends the above $L_2$-estimate to $L_p$-estimate.

**Theorem 5.2.** Let $p \geq 2$ and assume (5.2) and (5.3) hold. Then for any $g \in \mathbb{L}(l_2)$ and $u$ defined as in (5.5), we have
\[\mathbb{E} \|u(t)\|^p_{L^p(\mathbb{R}^d)} \leq N(d, p, \gamma, \nu) \mathbb{E} \int_0^t \|g(t, \cdot\|_{L^p_{\nu}(\mathbb{R}^d)}^p \, dt.\] (5.6)

**Remark 5.3.** A proof of (5.6) is given in [4] with a stronger condition than (5.4), that is
\[|D_\xi^\alpha \psi(t, \xi)| \leq \nu^{-1} |\xi|^{-|\alpha|}, \quad \forall |\alpha| \leq d_0 + 1.
\] The proof of [4] highly depends on the integration by parts, which requires the stronger assumption on $\psi(t, \xi)$.

**Example 5.4.** Let $m \in \mathbb{N}$ and $A(t) = (-1)^{m-1} \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(t) D^{\alpha+\beta}$ be a $2m$-order differential operator. Assume that $a^{\alpha\beta}(t)$ are bounded complex-valued measurable functions and satisfy an ellipticity condition, i.e.,
\[\nu |\xi|^{2m} \leq \sum_{|\alpha|=|\beta|=m} \xi^\alpha \xi^\beta \Re[a^{\alpha\beta}(t)] \quad \forall \xi \in \mathbb{R}^d.
\] Then $A(t)$ is the pseudo-differential operator whose symbol is given by $\psi(t, \xi) = (-1)^{m} \sum_{|\alpha|=|\beta|=m} a^{\alpha\beta}(t) \xi^\alpha \xi^\beta$. Obviously $\psi(t, \xi)$ satisfies (5.3) and (5.4) with $\gamma = 2m$.

**Example 5.5.** The class of pseudo-differential operators we are considering in this article covers a certain class of non-local operators. Let $\gamma \in (0, 2)$ and denote
\[A(t)u = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(t, x + y) - u(t, x) - \chi(y)(\nabla u(t, x), y) \right) \frac{m(t, y)}{|y|^{d+\gamma}} \, dy\]
where $\chi(y) = I_{\gamma > 1} + I_{|y| \leq 1} I_{\gamma = 1}$ and $m(t, y) \geq 0$ is a measurable function satisfying the following conditions (i)-(iv):
(i) If $\gamma = 1$ then
\[\int_{\partial B_1} m(t, w) S_1(dw) = 0, \quad \forall t > 0,\] (5.7)
where $\partial B_1$ is the unit sphere in $\mathbb{R}^d$ and $S_1(dw)$ is the surface measure on it.
(ii) The function $m = m(t, y)$ is zero-order homogeneous and differentiable in $y$ up to $d_0 = [\frac{d}{2}] + 1$. 
(iii) There is a constant $K$ such that for each $t \in \mathbb{R}$

$$\sup_{|\alpha| \leq d_0, |y| = 1} |D^\alpha_y m^{(\alpha)}(t, y)| \leq K.$$  

(iv) There exists a constant $c > 0$ so that $m(t, y) > c$ on a set $E \subset \partial B_1$ of positive $S_1(dw)$-measure.

Using (i)-(iv) one can check that $A(t)$ is a pseudo differential operator with the symbol $\psi(t, \xi)$ satisfying (5.2) and (5.4), where

$$\psi(t, \xi) = -c_1 \int_{\partial B_1} |(w, \xi)|^{\gamma} [1 - i\varphi^{(\gamma)}(w, \xi)] m(t, w) S_1(dw),$$

$$\varphi^{(\gamma)}(w, \xi) = c_2 \frac{(w, \xi)}{|(w, \xi)|} \ln |(w, \xi)| I_{\gamma \neq 1} - \frac{2}{\pi |(w, \xi)|} \ln |(w, \xi)| I_{\gamma = 1},$$

and $c_1(\gamma, d), c_2(\gamma, d)$ are certain positive constants (see [9] for the detail).

To apply Theorem 2.5 we set $T = \infty, \mathcal{O} = \mathbb{R}^d$, and

$$\rho(X, Y) = |t - s|^{1/\gamma} + |x - y|,$$

where $X = (t, x)$ and $Y = (s, y)$. Since $\rho$ is a quasi-metric with the doubling ball condition and $|(0, \infty) \times \mathbb{R}^d| = \infty$, $\rho$ admits the Fefferman-Stein theorem. Define

$$T_{\varphi, \varepsilon} g := \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^d} (-\Delta)^{\gamma/4} p(r, t, x - y) g^k(r, y) dy dW^k_r$$

and

$$T_{\varphi} g := \lim_{\varepsilon \downarrow 0} T_{\varphi, \varepsilon} g,$$

where the limit is in the sense of $L_2$-norm.

In the next lemma, we first show that $T_{\varphi, \varepsilon} g$ converges with respect to the norm in $L_2$ and $T_{\varphi}$ is a bounded operator from $L_2(l_2)$ to $L_2$.

**Lemma 5.6.** For each $g \in \mathbb{L}_2(l_2)$, $T_{\varphi} g(t, x)$ converges in $\mathbb{L}_2$ as $\varepsilon \downarrow 0$. Moreover the operator $g \mapsto T_{\varphi} g$ is bounded from $\mathbb{L}_2(l_2)$ to $\mathbb{L}_2$, i.e., there exists a constant $N_0$ such that for all $g \in \mathbb{L}_2(l_2)$,

$$E \int_0^\infty \|T_{\varphi} g(t, \cdot)\|_{L_2}^2 dt \leq N_0 E \int_0^\infty \|g(t, \cdot)\|_{L_2(l_2)}^2 dt.$$  (5.8)
Proof. Let $\varepsilon_1 > \varepsilon_2 > 0$. Then by Fubini's theorem, Itô's isometry, and Plancherel’s theorem,

\[
E \int_0^{\infty} \left\| T_{\psi, \varepsilon_1} g(t, \cdot) - T_{\psi, \varepsilon_2} g(t, \cdot) \right\|^2_{L^2} dt \\
= \int_0^{\infty} E \int_{t - \varepsilon_1}^{t - \varepsilon_2} \int_0^t \left( -\Delta \right)^{\gamma/4} p(r, t, x - y) g^k(r, y) dW^k_r dy dt dx \\
= \int_0^{\infty} E \int_{t - \varepsilon_1}^{t - \varepsilon_2} \int_0^t \left( -\Delta \right)^{\gamma/4} p(r, t, x - y) g^k(r, y) dy \left\| dW^k_r \right\|_{L^2} dx dt dr \\
= N(d) \int_0^{\infty} E \int_{t - \varepsilon_1}^{t - \varepsilon_2} \left\| |\xi|^\gamma \exp \left( 2 \int_r^t \Re|\psi(\rho, \xi)| dp \right) |\mathcal{F}(g)(r, \cdot)|_{L^2}^2 d\xi dr dt \\
\leq N(d) \int_0^{\infty} \int_{t - \varepsilon_1}^{t - \varepsilon_2} \left\| \exp \left( -2\mu|\xi|\right) \right\| |\mathcal{F}(g)(r, \cdot)|_{L^2}^2 d\xi dr \\
\leq N(d) \int_0^{\infty} \int_{t - \varepsilon_1}^{t - \varepsilon_2} \left\| \exp \left( -2\nu_1|\xi|\right) - \exp \left( -2\nu_2|\xi|\right) \right\| |\mathcal{F}(g)(r, \cdot)|_{L^2}^2 d\xi dr.
\]

The last term goes to zero as $\varepsilon_1, \varepsilon_2 \to 0$ by the Lebesgue dominated convergence theorem. Therefore $T_0 g$ is well-defined and using Fubini’s theorem, Itô’s isometry, and Plancherel’s theorem again, we get (5.8). The lemma is proved. \qed

Due to Lemma 5.6, to prove (5.6) it suffices to show that Assumption 2.1 holds with

\[
K(r, t, z, x) = 1_{0 < r < t} (-\Delta)^{\gamma/4} p(r, t, x - z).
\]

For $0 < s < t$ and $x \in \mathbb{R}^d$, denote

\[
q_1(s, t, x) = \mathcal{F}^{-1} \left[ \exp \left( \int_s^t \psi(r, (t - s)^{-1/\gamma} \xi) dr \right) \right] (x),
\]

and

\[
q_2(s, t, x) = (t - s)^{d/\gamma} p(s, t, (t - s)^{1/\gamma} x) = \mathcal{F}^{-1} \left[ \psi(t, (t - s)^{-1/\gamma} \xi) |\xi|^{\gamma/2} \exp \left( \int_s^t \psi(r, (t - s)^{-1/\gamma} \xi) dr \right) \right] (x).
\]

By the change of variables,

\[
(t - s)^{d/\gamma} p(s, t, (t - s)^{1/\gamma} x) = q_1(s, t, x),
\]

and

\[
(t - s)^{d/\gamma} (t - s)^{1/2} (-\Delta)^{\gamma/4} p(s, t, (t - s)^{1/\gamma} x) = (-\Delta)^{\gamma/4} q_1(s, t, x), \quad \text{(5.9)}
\]

and

\[
\frac{\partial}{\partial t} (-\Delta)^{\gamma/4} p(s, t, x) = (t - s)^{-d/\gamma} (t - s)^{-1} q_2(s, t, (t - s)^{-1/\gamma} x). \quad \text{(5.10)}
\]

Lemma 5.7. There exists a constant $N = N(d, \nu, \gamma)$ so that for any multi-index $\alpha$ with $|\alpha| \leq d_0$, $0 < s < t$, and $i = 1, \ldots, d$,

\[
\int_{\mathbb{R}^d} \left| D^\alpha_{\xi} \left( |\xi|^{\gamma/2} \mathcal{F}(q_1(t, s, \cdot)(\xi)) \right) \right| d\xi + \int_{\mathbb{R}^d} \left| D^\alpha_{\xi} \left( |\xi|^{\gamma/2} \mathcal{F}(q_1(t, s, \cdot)(\xi)) \right) \right| d\xi + \int_{\mathbb{R}^d} \left| D^\alpha_{\xi} \left( \mathcal{F}(q_2(t, s, \cdot)(\xi)) \right) \right| d\xi \leq N.
\]
Proof. Because of the similarity, we only show
\[
\int_{\mathbb{R}^d} \left| D_\xi^q \left( |\xi|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \right| d\xi \leq N.
\]
This is an easy consequence of (5.2) and (5.3). Indeed,
\[
\int_{\mathbb{R}^d} \left| D_\xi^q \left( |\xi|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \right| d\xi
\leq \sum_{n \in \mathbb{Z}} \int_{2^n \leq |\xi| < 2^{n+1}} \left| D_\xi^q \left( |\xi|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \right| d\xi
= \sum_{n \in \mathbb{Z}} \int_{2^n \leq |\xi| < 2^{n+1}} \left| D_\xi^q \left( |\xi|^{\gamma/2} \exp \left( \int_s^t \psi(r, (t-s)^{-1/\gamma}) dr \right) \right) \right| d\xi
\leq N \sum_{n \in \mathbb{Z}} \sum_{k=1}^{2^n} 2^{n(d+\frac{3\gamma}{2}-k)} e^{-\nu 2^n} \leq N(d, \nu, \gamma).
\]
The lemma is proved. \qed

Note that for any \( f \in L_1(\mathbb{R}^d) \),
\[
\sup_{x \in \mathbb{R}^d} |F^{-1}(f)(x)| \leq N(d)\|f\|_{L_1(\mathbb{R}^d)}.
\]
Thus by Lemma 5.7 there exists a constant \( N = N(d, \nu, \gamma) \) so that for any \( t > s \) and \( x \in \mathbb{R}^d \)
\[
\left| (-\Delta)^{\gamma/4} q_1(s, t, x) \right| + \left| \partial_{x^i} (-\Delta)^{\gamma/4} q_1(s, t, x) \right| + |q_2(s, t, x)| \leq N. \quad (5.11)
\]

Lemma 5.8. Let \( \varepsilon \in \left[ 0, \frac{d+3\gamma-2(d_0-1)}{2} \right] \). Then, there exists a constant \( N = N(d, \nu, \gamma, \varepsilon) \) so that for any multi-index \( \alpha \) with \( |\alpha| \leq d_0 - 1, \) \( 0 < s < t, \) and \( i = 1, \ldots, d, \)
\[
\int_{\mathbb{R}^d} \left| \xi \right|^{-\varepsilon} D_\xi^q \left( \left| \xi \right|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \left| \xi \right|^{-\varepsilon} D_\xi^q \left( \left| \xi \right|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \left| \xi \right|^{-\varepsilon} D_\xi^q \left( \left| \xi \right|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \right| d\xi
\leq N.
\]

Proof. Because of the similarity, we only show
\[
\int_{\mathbb{R}^d} \left| \xi \right|^{-\varepsilon} D_\xi^q \left( \left| \xi \right|^{\gamma/2} F_q(t, s, \cdot)(\xi) \right) \right| d\xi \leq N.
\]
Since \( d + 3\gamma - 2\varepsilon - 2(d_0 - 1) > 0, \)
\[
\sum_{n=-\infty}^{1} 2^{n(d+3\gamma-2\varepsilon-2(d_0-1))} < \infty.
\]
Therefore by (5.2) and (5.3),
\[
\int_{\mathbb{R}^d} \left| \xi \right|^{-\varepsilon} D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right)^2 \, d\xi \\
\leq \sum_{n \in \mathbb{Z}} \int_{2^n \leq \left| \xi \right| < 2^{n+1}} \left| \xi \right|^{-\varepsilon} D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right)^2 \, d\xi \\
= \sum_{n \in \mathbb{Z}} \int_{2^n \leq \left| \xi \right| < 2^{n+1}} \left| \xi \right|^{-\varepsilon} D_\xi^2 \left( \left| \xi \right|^{\gamma/2} \exp \left( \int_s^t \psi((t-s)^{-1/\gamma}) \, dr \right) \right)^2 \, d\xi \\
\leq N \sum_{n \in \mathbb{Z}} \sum_{k=1}^{2^n} \sum_{|\alpha|} \left( 2^{n(d+3\gamma-2k)} \right) e^{-\nu2^n} \leq N(d, \nu, \gamma, \varepsilon).
\]

The lemma is proved. \qed

Lemma 5.9. There exists a constant \( N = N(d, \nu, \gamma) \) so that for all \( c > 0 \), multi-index \( |\alpha| \leq d_0, 0 < s < t, \) and \( i = 1, \ldots, d, \)
\[
\int_{|\xi| \geq c} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right) \right|^2 \, d\xi \\
+ \sum_{n \in \mathbb{Z}} \int_{2^n \leq \left| \xi \right| < 2^{n+1}} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right) \right|^2 \, d\xi \\
\leq N \left( 1 + 1_{c<1} d^{d+3\gamma-2d_0} \right).
\]

Proof. As in the proofs of the previous lemmas, we only show
\[
\int_{|\xi| \geq c} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right) \right|^2 \, d\xi \leq N.
\]

By (5.2) and (5.3),
\[
\int_{|\xi| \geq c} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right) \right|^2 \, d\xi \\
\leq \sum_{n \in \mathbb{Z}} \int_{2^n \leq \left| \xi \right| < 2^{n+1}} 1_{|\xi| \geq c} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} F(q_1(t, s, \cdot)(\xi)) \right) \right|^2 \, d\xi \\
= \sum_{n \geq c/2} \sum_{2^n \leq \left| \xi \right| < 2^{n+1}} 1_{|\xi| \geq c} \left| D_\xi^2 \left( \left| \xi \right|^{\gamma/2} \exp \left( \int_s^t \psi((t-s)^{-1/\gamma}) \, dr \right) \right) \right|^2 \, d\xi \\
\leq N \sum_{n \geq c/2} \sum_{k=1} \left( 2^{n(d+3\gamma-2k)} \right) e^{-\nu2^n} \leq N(d, \nu, \gamma) \left( 1 + 1_{c<1} d^{d+3\gamma-2d_0} \right).
\]

The lemma is proved. \qed

Lemma 5.10. Let \( 0 < \delta < \left( \frac{\gamma}{2} \wedge \frac{1}{4} \right) \). Then there exists a constant \( N = N(d, \nu, \gamma, \delta) \) so that for any \( 0 < s < t \)
\[
\int_{\mathbb{R}^d} \left| x \right|^{\frac{d}{4} + \delta} (-\Delta)^{\gamma/4} q_1(s, t, x) \right|^2 \, dx \leq N, \tag{5.12}
\]
\[
\int_{\mathbb{R}^d} \left| x \right|^{\frac{d}{4} + \delta} \frac{\partial}{\partial x^i} (-\Delta)^{\gamma/4} q_1(s, t, x) \right|^2 \, dx \leq N, \tag{5.13}
\]

Therefore by (5.2) and (5.3),
and
\[
\int_{\mathbb{R}^d} \left| x \right|^{\frac{d}{2} + \delta} q_2(s, t, x) \right|^2 dx \leq N. \tag{5.14}
\]

Proof. We only prove (5.12). The proofs of (5.13) and (5.14) are similar. Note that it suffices to show that for each \( j = 1, \ldots, d, \)
\[
\int_{\mathbb{R}^d} \left| (ix)^{d-1} f(\xi) \right| dx = \left( -\Delta \right)^{\gamma}/4 q_1(s, t, x)(\xi).
\]

The left hand side of (5.15) is equal to
\[
\int_{\mathbb{R}^d} \left| (ix)^{d-1} f(\xi) \right| dx = \left( -\Delta \right)^{\gamma}/4 q_1(s, t, x)(\xi).
\]

Moreover by Plancherel’s theorem, the last term above equals to
\[
N(d) \int_{\mathbb{R}^d} \left| (-\Delta)^{\gamma/2} D_{\xi}^{d-1} \hat{q}(s, t, \xi) \right|^2 d\xi. \tag{5.17}
\]

Obviously, \( \varepsilon \in \left( 0, 1 \wedge \frac{d+\gamma-2(d-1)}{2} \right) \). Using the integral representation of the Fractional Laplacian operator \((-\Delta)^{\gamma/2}\) we get
\[
(-\Delta)^{\gamma/2}(D_{\xi}^{d-1} \hat{q}(s, t, \xi)) = N \int_{\mathbb{R}^d} \frac{D_{\xi}^{d-1} \hat{q}(s, t, \xi + \eta) - D_{\xi}^{d-1} \hat{q}(s, t, \xi)}{\left| \eta \right|^{d+\varepsilon}} d\eta.
\]

We divide \((-\Delta)^{\gamma/2}(D_{\xi}^{d-1} \hat{q}(s, t, \xi))\) into two terms:
\[
N \int_{\left| \eta \right| \geq 1} \frac{D_{\xi}^{d-1} \hat{q}(s, t, \xi + \eta) - D_{\xi}^{d-1} \hat{q}(s, t, \xi)}{\left| \eta \right|^{d+\varepsilon}} d\eta
\]
\[
+ N \int_{\left| \eta \right| < 1} \frac{D_{\xi}^{d-1} \hat{q}(s, t, \xi + \eta) - D_{\xi}^{d-1} \hat{q}(s, t, \xi)}{\left| \eta \right|^{d+\varepsilon}} d\eta =: I_1(s, t, \xi) + I_2(s, t, \xi).
\]

By Minkowski’s inequality and Lemma 5.8
\[
\left[ \int_{\mathbb{R}^d} I_1(s, t, \xi) \right]^{1/2} \leq 2 \left\| D_{\xi}^{d-1} \hat{q}(s, t, \cdot) \right\|_{L_2(\mathbb{R}^d)} \int_{\left| \eta \right| \geq 1} \frac{1}{\left| \eta \right|^{d+\varepsilon}} d\eta \leq N(d, \nu, \gamma).
\]

We split \( I_2 \) into \( I_{2,1}, I_{2,2}, \) and \( I_{2,3}, \) where
\[
I_{2,1}(s, t, \xi) := \int_{\left| \eta \right| < 1} \frac{D_{\xi}^{d-1} \hat{q}(s, t, \xi + \eta) - D_{\xi}^{d-1} \hat{q}(s, t, \xi)}{\left| \eta \right|^{d+\varepsilon}} d\eta
\]
\[ \mathcal{I}_{2.3}(s, t, \xi) := - \int_{|\eta| < 1} 1_{|\eta| \geq \frac{1}{\sqrt{d}}} \frac{D^{d_0-1}_\xi \tilde{q}(s, t, \xi + \eta)}{|\eta|^{d+\varepsilon}} d\eta. \]

By the fundamental theorem of calculus,

\[ |\mathcal{I}_{2.1}(s, t, \xi)| \leq \int_0^1 \int_{|\eta| < 1} 1_{|\eta| < |\xi|} \frac{\nabla D^{d_0}_\xi \tilde{q}(s, t, \xi + \theta \eta)}{|\eta|^{d+\varepsilon-1}} d\eta d\theta. \]

Hence by Minkowski’s inequality and Lemma 5.9

\[ \|\mathcal{I}_{2.1}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \leq \left[ \int_{|\eta| < 1} \left( \int_{|\eta| < |\xi|} \frac{\nabla D^{d_0}_\xi \tilde{q}(s, t, \xi + \eta)}{|\eta|^{d+\varepsilon-1}} d\eta \right)^2 \right]^{1/2} \leq N \left[ \int_{|\eta| < 1} \frac{1 + |\eta|^{(d+3\gamma-2d_0)/2}}{|\eta|^{d+\varepsilon-1}} d\eta \right] \leq N(d, \nu, \gamma) \]

since

\[ (d + 3\gamma - 2d_0)/2 - d - \varepsilon + 1 > -d. \]

On the other hand, if $|\xi| \geq 2$, then $\mathcal{I}_{2.2}(s, t, \xi) = \mathcal{I}_{2.3}(s, t, \xi) = 0$ and thus we may assume $|\xi| \leq 2$. Recalling the range of $\varepsilon$, we have

\[ \varepsilon + \gamma < \frac{d + 3\gamma - 2(d_0 - 1)}{2}. \]

Hence by Hölder’s inequality and Lemma 5.8

\[ |\mathcal{I}_{2.2}(s, t, \xi)| \]

\[ \leq \left[ \int_{|\eta| < 1} 1_{|\eta| \geq \frac{1}{\sqrt{d}}} \frac{|\xi + \eta|^{2\varepsilon + 2\gamma}}{|\eta|^{2d + 2\varepsilon}} d\eta \right]^{1/2} \left[ \int_{\mathbb{R}^d} \frac{1}{|\eta|^{d+\varepsilon-1}} d\eta \right]^{1/2} \]

\[ \leq N \left[ \int_{|\eta| < 1} 1_{|\eta| \geq \frac{1}{\sqrt{d}}} \frac{|\xi + \eta|^{2\varepsilon + 2\gamma}}{|\eta|^{2d + 2\varepsilon}} d\eta \right]^{1/2} \left[ \int_{\mathbb{R}^d} \frac{1}{|\eta|^{d+\varepsilon-1}} d\eta \right]^{1/2} \]

\[ \leq N \left( 1 + |\xi|^{-d+2\varepsilon} \right). \]

Therefore we have

\[ \|\mathcal{I}_{2.2}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \leq N \int_{|\xi| < 2} \left( 1 + |\xi|^{-d + 2\varepsilon} \right) d\xi \leq N(d, \nu, \gamma). \]

Finally by Lemma 5.8 again,

\[ \|\mathcal{I}_{2.3}(s, t, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \leq N. \]

Due to (5.16) and (5.17), combining all estimates for $\mathcal{I}_1, \mathcal{I}_{2.1}, \mathcal{I}_{2.2}, \mathcal{I}_{2.3}$, we have (5.15). The lemma is proved. \qed
Remark 5.11. If $\gamma$ is not small, Lemma [5,10] is easily obtained from properties of the Fourier transform. Indeed,

$$
\int_{\mathbb{R}^d} \left| \frac{x}{|x|} D_{\xi}^{d_1-1} \left( D_{\xi}^d \hat{q}(s, t, \xi) \right) (x) \right|^2 \, dx
$$

$$
\leq N \sum_{j=1}^d \int_{\mathbb{R}^d} \left| (ix)^j D_{\xi}^{d_1-1} \left( \hat{q}(s, t, \xi) \right) (x) \right|^2 \, dx
$$

$$
= N \sum_{j=1}^d \int_{\mathbb{R}^d} \left| D_{\xi}^d \hat{q}(s, t, \xi) \right|^2 \, d\xi.
$$

Due to (5.2) and (5.3), the above term is finite if $3\gamma + d > 2d_0$.

Lemma 5.12. Let $\delta \in (0, \frac{1}{2} \wedge \frac{2}{7})$. Then there exists a constant $N(d, \nu, \gamma, \delta)$ such that for all $0 < s < t, c > 0$, $a \in \mathbb{R}$,

$$
\int_0^t \left( \int_{|z| \geq c} |(-\Delta)^{\gamma/4} p(r, t, z) | \, dz \right)^2 \, dr \leq N \left( (t-s)^{1/\gamma c^{-1}} \right)^{2\delta},
$$

(5.18)

$$
\int_0^a \left[ \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/4} p(r, t, z + h) - (-\Delta)^{\gamma/4} p(r, t, z) | \, dz \right]^2 \, dr \leq N \left( |h|(t-a)^{-1/\gamma} \right)^2,
$$

(5.19)

and

$$
\int_0^a \left[ \int_{\mathbb{R}^d} |(-\Delta)^{\gamma/4} p(r, t, z) - (-\Delta)^{\gamma/4} p(r, s, z) | \, dz \right]^2 \, dr \leq N \left( (t-s)(s-a)^{-1} \right)^2.
$$

(5.20)

Proof. First we prove (5.18). By (5.9), H"older’s inequality, and (5.12),

$$
\left| \int_{|z| \geq c} |(-\Delta)^{\gamma/4} p(r, t, z) | \, dz \right|^2
$$

$$
= (t-r)^{-1} \left| \int_{(t-r)^{1/\gamma} |z| \geq c} |(-\Delta)^{\gamma/4} q_1(r, t, z) | \, dz \right|^2
$$

$$
\leq (t-r)^{-1} \int_{(t-r)^{1/\gamma} |z| \geq c} |z|^{-d+2\delta} \, dz \int_{(t-r)^{1/\gamma} |z| \geq c} \left| z^{\frac{d}{2} + \delta} (-\Delta)^{\gamma/4} q_1(r, t, z) \right|^2 \, dz
$$

$$
\leq (t-r)^{-1+2\delta/\gamma c^{-2\delta}}.
$$

Hence we have

$$
\int_0^t \left( \int_{|z| \geq c} |(-\Delta)^{\gamma/4} p(r, t, z) | \, dz \right)^2 \, dr \leq N \left( (t-s)^{1/\gamma c^{-1}} \right)^{2\delta}.
$$

Next we prove (5.19). From (5.9),

$$
\frac{\partial}{\partial x^i} (-\Delta)^{\gamma/4} p(r, t, z) = (t-r)^{-d/\gamma} (t-r)^{-1/2-1/\gamma} \frac{\partial}{\partial x^i} (-\Delta)^{\gamma/4} q_1(r, t, (t-r)^{-1/\gamma} z),
$$

(5.21)
and by Hölder’s inequality, (5.11), and (5.13),
\[
\left(\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(r,t,z) \right| dz \right)^2 \leq N + \int_{|z| \geq 1} |z|^{-d-2\delta} dz \int_{\mathbb{R}^d} \left| z^{\frac{d}{2}+\delta} \frac{\partial}{\partial x^i} \Delta^{\gamma/2} q_1(r,t,z) \right|^2 dz \leq N,
\]
where \(N\) is independent of \(t\) and \(r\). Therefore, by the fundamental theorem of calculus,
\[
\int_0^a \left( \int_{\mathbb{R}^d} \left| (-\Delta)^{\gamma/4} p(r,t,z+h) - (-\Delta)^{\gamma/4} p(r,t,z) \right| dz \right)^2 dr \\
\leq |h|^2 \int_0^a \left( \int_{\mathbb{R}^d} \left| \nabla \Delta^{\gamma/2} p(r,t,z) \right| dz \right)^2 dr \\
\leq |h|^2 \int_0^a (t-r)^{-1-2/\gamma} \left( \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x^i} (-\Delta)^{\gamma/4} q_1(r,t,z) \right| dz \right)^2 dr \\
\leq N|h|^2 \int_0^a (t-r)^{-1-2/\gamma} dr \leq N|h| \int_{t-a}^{\infty} r^{-1-2/\gamma} dr = N \left( |h|(t-a)^{-1/\gamma} \right)^2.
\]
It only remains to prove (5.20). By the mean-value theorem and (5.10),
\[
\left| (-\Delta)^{\gamma/4} p(r,t,z) - (-\Delta)^{\gamma/4} p(r,s,z) \right| \\
\leq |t-s| [\theta t + (1-\theta)s - r]^{-d/\gamma-3/2} \left| q_2(r,\theta t + (1-\theta)s, \theta t + (1-\theta)s - r)^{-1/\gamma} \right|,
\]
where \(\theta \in [0,1]\). Moreover by Hölder’s inequality, (5.11), and (5.14),
\[
\int_{\mathbb{R}^d} \left| q_2(r,\theta t + (1-\theta)s, z) \right| dz < N,
\]
where \(N\) is independent of \(t, s, r, \) and \(\theta\). Therefore,
\[
\int_0^a \left( \int_{\mathbb{R}^d} \left| (-\Delta)^{\gamma/4} p(r,t,z) - (-\Delta)^{\gamma/4} p(r,s,z) \right| dz \right)^2 dr \\
\leq \int_0^a \frac{|t-s|^2}{(\theta t + (1-\theta)s - r)^{3/2}} dr \leq |t-s|^2 (s-a)^{-2}.
\]
The lemma is proved.

In the following corollary, we finally prove that the kerene
\[
K(r,t,z,x) := 1_{0\leq r < t}(-\Delta)^{\gamma/4} p(r,t,x-z)
\]
satisfies Assumption 2.4. Recall
\[
\rho(X,Y) = |t-s|^{1/\gamma} + |x-y|.
\]
For \(r > 0\) and \(X = (t,x), Y = (s,y) \in (0, \infty) \times \mathbb{R}^d\), set
\[
A(r,X,Y) := \left\{ z \in \mathbb{R}^d : \rho(X,Z) \geq 4 \cdot 2^{1/\gamma} \rho(X,Y) \right\} \\
= \left\{ z \in \mathbb{R}^d : |t-r|^{1/\gamma} + |x-z| \geq 4 \cdot 2^{1/\gamma} (|t-s|^{1/\gamma} + |x-y|) \right\},
\]
where \(Z = (r,z)\).
Corollary 5.13. There is a constant $N = N(d, \nu, \gamma)$ so that for any $X = (t, x), Y = (s, y) \in (0, \infty) \times \mathbb{R}^d$ and $r > 0$,

$$
\int_0^\infty \left( \int_{A(r, X, Y)} \left| 1_{0 < r < t}(-\Delta)^{\gamma/4}p(r, t, x - z) - 1_{0 < r < s}(-\Delta)^{\gamma/4}p(r, s, y - z) \right| \, dz \right)^2 \, dr 
\leq N.
$$

Proof. We use the notation $(-\Delta)^{\gamma/4}p(r, t, x)$ instead of $1_{0 < r < t}(-\Delta)^{\gamma/4}p(r, t, x)$. In other words, we assume that $(-\Delta)^{\gamma/4}p(r, t, x)$ is $t$-homogeneous. To this end, we may assume $t \geq s$ without loss of generality. Since the proof of the case $t = s$ is simpler, we only prove the case $t > s$.

First we estimate $I_1(X, Y)$. By (5.18),

$$
I_1(X, Y) = \left[ \int_{A(r, X, Y)} \left| (-\Delta)^{\gamma/4}p(r, t, x - z) - (-\Delta)^{\gamma/4}p(r, s, y - z) \right| \, dz \right]^2.
$$

We split $I_2$. Observe

$$
I_2 \leq I_{2,1} + I_{2,2} = \int_0^{2s-t} \left[ \int_{A(r, X, Y)} \left| (-\Delta)^{\gamma/4}p(r, t, x - z) - (-\Delta)^{\gamma/4}p(r, t, y - z) \right| \, dz \right]^2 \, dr 
+ \int_0^{2s-t} \left[ \int_{A(r, X, Y)} \left| (-\Delta)^{\gamma/4}p(r, t, y - z) - (-\Delta)^{\gamma/4}p(r, s, y - z) \right| \, dz \right]^2 \, dr.
$$

If $|x - y| \leq (t - s)^{1/\gamma}$ then by (5.19),

$$
I_{2,1} \leq N \left( |x - y|(t - s)^{-1/\gamma} \right)^2 \leq N.
$$

On the other hand, if $|x - y| > (t - s)^{1/\gamma}$, then

$$
I_{2,1} \leq 2I_{2,1,1} + I_{2,1,2},
$$

where

$$
I_{2,1,1} := \int_{s - |x - y|}^t \left[ \int_{|z| \geq |t - s|^{1/\gamma} + |x - y|} \left| (-\Delta)^{\gamma/4}p(r, t, z) \right| \, dz \right]^2 \, dr,
$$
and
\[ I_{2.1,2} = \int_0^{s-|x-y|}\left[ \int_{\mathbb{R}^d} \left| (-\Delta)^{\gamma/4} p(r,t,x-z) - (-\Delta)^{\gamma/4} p(r,t,y-z) \right| \, dz \right]^2 \, dr. \]

By (5.18) again,
\[ I_{2.1,1} \leq N \left( (t-s+|x-y|)^{-1/\gamma} \left( (t-s)^{1/\gamma} + |x-y| \right) \right)^2 \leq N \]
and by (5.19)
\[ I_{2.1,2} \left( |x-y| (t-s+|x-y|)^{-1/\gamma} \right)^2 \leq N. \]

It only remains to estimate \( I_{2.2} \). However, this is an easy consequence of (5.20) since \( 2s-t < t \). Indeed,
\[ I_{2.2} \leq N \left( (t-s|t-s|^{-1})^2 \leq N. \right. \]
The corollary is proved. □

Finally, applying Theorem 2.5 with \( T = T_{\psi,\epsilon} \) and \( T = T_\psi \), we obtain (5.6).

5.2. Infinitesimal generators of subordinate Brownian motions. In this subsection we consider the infinitesimal generators of subordinate Brownian motions. In general the symbols of such operators do not satisfy (5.2) which is assumed in the previous subsection.

Let \( S_t \) be a subordinator, that is, an increasing Lévy process taking values in \([0, \infty)\) with \( S_0 = 0 \). A subordinator \( S \) is completely characterized by its Laplace exponent \( \phi \), i.e.
\[ E e^{-\lambda S_t} = e^{-t\phi(\lambda)} \quad \text{for} \quad \lambda > 0. \]
Actually a function function \( \phi : (0, \infty) \to (0, \infty) \) with \( \phi(0+) = 0 \) is a Laplace exponent of a subordinator if and only if it is a Bernstein function (i.e. \((-1)^n D^n \phi \leq 0, \forall n\))
Also it is of the form
\[ \phi(\lambda) = b\lambda + \int_{(0, \infty)} \left( 1 - e^{-\lambda t} \right) \mu(dt), \quad \lambda > 0, \]
where \( b \geq 0 \) and \( \mu \) is a measure on \((0, \infty)\) satisfying \( \int_{(0, \infty)} (1 \wedge t) \mu(dt) < \infty \), called the Lévy measure. Let \( B_t \) be a \( d \)-dimensional Brownian motion independent of \( S_t \). Then \( \phi(\Delta) \) can be defined as the infinitesimal generator of the subordinate Brownian motion \( B_{S_t} \):
\[ \phi(\Delta)f(x) = \lim_{t \to 0} \frac{\mathbb{E} f(x+B_{S_t}) - f(x)}{t}, \quad f \in C^2_b(\mathbb{R}^d), \]
and its integral version is
\[ b\Delta f(x) + \int_{\mathbb{R}^d} \left( f(x+y) - f(x) - \nabla f(x) \cdot y \mathbbm{1}_{|y| \leq 1} \right) J(y) \, dy, \quad (5.21) \]
where \( J(x) = j(|x|) \) with \( j : (0, \infty) \to (0, \infty) \) given by
\[ j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt). \]
Let \( \phi \) be a sufficient condition to \((5.22)-(5.25)\) can be found e.g. in [10]. Consider the operator \( A(t) = \phi(\Delta) \). Then \((5.1)\) has a solution \( u \) given by

\[
    u(t, x) = \int_0^t \int_{\mathbb{R}^d} p(t - s, x - y) g(s, y) dydW_s^k,
\]

where

\[
    p(t, x) = \mathcal{F}^{-1} \left[ \exp \left( -t\phi(\xi^2) \right) \right](x).
\]

Let \( \phi^{-1} \) denote the generalized inverse of \( \phi \), i.e.

\[
    \phi^{-1}(t) := \inf \{ s > 0 : \phi(s) \geq t \}.
\]

**Assumption 5.14.** (i) There exists a constant \( N \) such that for all \( t \leq T \in (0, \infty) \) and \( x \in \mathbb{R}^d \)

\[
    \left| \phi(\Delta)^{1/2} p(t, \cdot)(x) \right| \leq N \left( t^{-1/2}(\phi^{-1}(t^{-1}))^{d/2} \wedge \frac{\phi(|x|^{-2})^{1/2}}{|x|^d} \right), \tag{5.22}
\]

\[
    \left| \phi(\Delta)^{1/2} \nabla p(t, \cdot)(x) \right| \leq N \left( t^{-1/2}(\phi^{-1}(t^{-1}))^{(d+1)/2} \wedge \frac{\phi(|x|^{-2})^{1/2}}{|x|^d} \right), \tag{5.23}
\]

and

\[
    \left| \phi(\Delta)^{3/2} p(t, \cdot)(x) \right| \leq N \left( t^{-3/2}(\phi^{-1}(t^{-1}))^{d/2} \wedge \frac{\phi(|x|^{-2})^{1/2}}{|x|^d} \right). \tag{5.24}
\]

(ii) \( \phi \) satisfies the following scaling property: there exist positive constants \( N_1 \), \( N_2 \), \( \delta_1 \), and \( \delta_2 \) so that

\[
    N_1 \left( \frac{b}{a} \right)^{\delta_1} \leq \frac{\phi(b)}{\phi(a)} \leq N_2 \left( \frac{b}{a} \right)^{\delta_2}, \quad \forall \ 0 < a \leq b. \tag{5.25}
\]

Using (5.24), one can find constants \( \delta_1 \), \( \delta_2 \), \( \delta_1 \), and \( \delta_2 \) depending only on \( \delta_1 \), \( \delta_2 \), \( \delta_1 \), and \( \delta_2 \) so that

\[
    N_1 \left( \frac{b}{a} \right)^{\delta_1} \leq \frac{\phi^{-1}(b)}{\phi^{-1}(a)} \leq N_2 \left( \frac{b}{a} \right)^{\delta_2}, \quad 0 < a \leq b. \tag{5.26}
\]

Furthermore,

\[
    \lim_{t \uparrow \infty} \phi(t) = \lim_{t \downarrow 0} \phi^{-1}(t) = \infty, \quad \lim_{t \uparrow 0} \phi(t) = \lim_{t \downarrow 0} \phi^{-1}(t) = 0,
\]

and

\[
    \phi(\phi^{-1}(t)) = t. \tag{5.27}
\]

**Example 5.15.** A sufficient condition to \((5.22)-(5.24)\) can be found e.g. in [3]:

(H1): \( \exists \) constants \( 0 < \delta_1 \leq \delta_2 < 1 \) and \( c_1, c_2 > 0 \) such that

\[
    c_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq c_2 \lambda^{\delta_2} \phi(t), \quad \lambda \geq 1, t \geq 1;
\]

(H2): \( \exists \) constants \( 0 < \delta_3 \leq 1 \) and \( c_3 > 0 \) such that

\[
    \phi(\lambda t) \leq c_3 \lambda^{\delta_3} \phi(t), \quad \lambda \leq 1, t \leq 1.
\]
Actually using (H1) and (H2) one can prove (see [4])
\[
\left| \phi(\Delta)^{n/2}D^\beta p(t,\cdot)(x) \right| \leq N \left( t^{-n/2}(\phi^{-1}(t^{-1}))^{(d+|\beta|)/2} \wedge t^{-(n-1)/2} \frac{\phi(|x|^{-2})^{1/2}}{|x|^{d+|\beta|}} \right)
\]
for any \( n \leq 3 \) and multi-index \( \beta \) with \( |\beta| \leq 2 \).

Here are some examples of Bernstein functions satisfying (H1) and (H2):

1. \( \phi(\lambda) = \lambda^\alpha + \lambda^\beta, \quad 0 < \alpha < \beta < 1; \)
2. \( \phi(\lambda) = (\lambda + \lambda^\gamma)^\delta, \alpha, \beta \in (0, 1); \)
3. \( \phi(\lambda) = \lambda^\alpha \log(1 + \lambda))^\beta, \alpha \in (0, 1), \beta \in (0, 1 - \alpha); \)
4. \( \phi(\lambda) = \lambda^\alpha \log(1 + \lambda))^{-\beta}, \alpha \in (0, 1), \beta \in (0, \alpha); \)
5. \( \phi(\lambda) = \log(\cosh(\sqrt{\lambda}))^\alpha, \alpha \in (0, 1); \)
6. \( \phi(\lambda) = \log(\sinh(\sqrt{\lambda})) - \log \sqrt{\lambda})^\alpha, \alpha \in (0, 1). \)

For example, the subordinate Brownian motion corresponding to the example (1) \( \phi(\lambda) = \lambda^\alpha + \lambda^\beta \) is the sum of two independent symmetric \( \alpha \) and \( \beta \) stable processes, and its infinitesimal generator is \(-(-\Delta)^{\beta/2} - (-\Delta)^{\alpha/2} \).

Define
\[
\mathbb{T}_{\phi,\epsilon} g := \int_{0}^{t-\epsilon} \int_{\mathbb{R}^d} \phi(\Delta)^{1/2} p(t-r, x-y)g^k(r, y)dydW^k_r
\]
and
\[
\mathbb{T}_\phi g := \lim_{\epsilon \downarrow 0} \int_{0}^{t-\epsilon} \int_{\mathbb{R}^d} \phi(\Delta)^{1/2} p(t-r, x-y)g^k(r, y)dydW^k_r,
\]
where the limit is in the sense of \( L_2 \)-norm.

Here is the main result of this subsection.

Theorem 5.16. Let \( p \in [2, \infty) \) and Assumption 5.14 hold. Then
\[
\| \mathbb{T}_\phi g \|_{L_p(O_T)} \leq N \| g \|_{L_p(O_T, l_2)} \quad \forall g \in L_p(O_T, l_2),
\]
where \( N \) depends only on \( d \) and the constants appearing in Assumption 5.14.

To apply Theorem 2.26, we set \( O = \mathbb{R}^d \) and
\[
\rho(X, Y) = \left( \phi^{-1}(|t-s|^{-1}) \right)^{-1/2} + |x-y|,
\]
where \( X = (t, x), \ Y = (s, y), \) and \( (\phi^{-1}(0^{-1}))^{-1/2} := 0. \) Due to (5.20), one can easily check that \( \rho \) is a quasi-metric and satisfies the doubling ball condition. Thus, we only need to check that
\[
K(t-r, z, x) := 1_{0 < r < t < T} \phi(\Delta)^{1/2} p(t-r, x-z)
\]
satisfies Assumption 2.24 since the proof of \( L_2 \)-boundedness of \( \mathbb{T}_\phi g \) can be easily proved as the proof of Lemma 5.6.

Lemma 5.17. There exists a constant \( N \) such that for all \( 0 < a < s < t < T, \) \( c > 0, \)
\[
\int_{s}^{t} \left( \int_{|z| \geq c} |\phi(\Delta)^{1/2} p(t-r, z)| \ dz \right)^2 dr \leq N(t-s)\phi(c^{-2}), \quad (5.28)
\]
\[
\int_0^a \left( \int_{\mathbb{R}^d} |\phi(\Delta)^{1/2}p(t-r, z + h) - \phi(\Delta)^{1/2}p(t-r, z)| \, dz \right)^2 \, dr \\
\leq N|h|^2 \phi^{-1} ((t-a)^{-1}),
\]
(5.29)

and
\[
\int_0^a \left( \int_{\mathbb{R}^d} |\phi(\Delta)^{1/2}p(t-r, z) - \phi(\Delta)^{1/2}p(s-r, z)| \, dz \right)^2 \, dr \leq N(t-s)^2(s-a)^{-2},
\]
(5.30)

where \( N \) depends only on \( d \) and the constants appearing in Assumption 5.14.

**Proof.** First we prove (5.28). This is an easy consequence of assumptions on \( \phi \). Indeed, by (5.22) and (5.25),
\[
\int_{|z| \geq c} |\phi(\Delta)^{1/2}p(t-r, z)| \, dz \leq \int_{|z| \geq c} \frac{\phi(|z|^{-2})^{1/2}}{|z|^d} \, dz \\
\leq \int_{|z| \geq 1} \frac{\phi(|cz|^{-2})^{1/2}}{|z|^d} \, dz \\
\leq N\phi(c^{-2})^{1/2}.
\]

Next we prove (5.29). By the fundamental theorem of calculus,
\[
\int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2}p(t-r, z + h) - \phi(\Delta)^{1/2}p(t-r, z) \right| \, dz \\
\leq |h| \int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2}\nabla p(t-r, z) \right| \, dz.
\]

Denote
\[
a_t := (\phi^{-1}(t^{-1}))^{1/2}.
\]

Then by (5.22), (5.24), and (5.26),
\[
\int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2}\nabla p(t-r, z) \right| \, dz \\
= \int_{|z| < 1/a_t} \left| \phi(\Delta)^{1/2}\nabla p(t-r, z) \right| \, dz + \int_{|z| \geq 1/a_t} \left| \phi(\Delta)^{1/2}\nabla p(t-r, z) \right| \, dz \\
\leq N \left( \left( 1/a_t \right)^d (t-r)^{-1/2}(\phi^{-1}((t-r)^{-1}))^{(d+1)/2} + \int_{|z| \geq 1/a_t} \frac{\phi(|z|^{-2})^{1/2}}{|z|^{d+1}} \, dx \right) \\
\leq N \left( (t-r)^{-1/2}(\phi^{-1}((t-r)^{-1}))^{1/2} \right).
\]

Hence by (5.26),
\[
\int_0^a \left( \int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2}p(t-r, z + h) - \phi(\Delta)^{1/2}p(t-r, z) \right| \, dz \right)^2 \, dr \\
\leq N|h|^2 \left( \int_0^a (t-r)^{-1}\phi^{-1} ((t-r)^{-1}) \, dr \right) \\
\leq N|h|^2 \phi^{-1} ((t-a)^{-1}).
\]
Finally we prove (5.30). By the mean-value theorem, (5.24), and (5.25),
\[
\int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2} p(t-r,z) - \phi(\Delta)^{1/2} p(s-r,z) \right| \, dz \\
= (t-s) \int_{\mathbb{R}^d} \frac{d}{dr} \phi(\Delta)^{1/2} p(\theta t + (1-\theta)s - r, z) \, dz \\
= (t-s) \int_{\mathbb{R}^d} \phi(\Delta)^{3/2} p(\theta t + (1-\theta)s - r, z) \, dz \\
\leq N(t-s)(\theta t + (1-\theta)s - r)^{-3/2} \\
+ N(t-s) \int_{|z| \geq 1/\alpha_{r,t-s-r}} (\theta t + (1-\theta)s - r)^{-1} \frac{\phi(|z|^{-2})^{1/2}}{|z|^d} \, dz \\
\leq N(t-s)(\theta t + (1-\theta)s - r)^{-3/2},
\]
where \( \theta \in [0,1] \). Therefore,
\[
\int_0^a \left[ \int_{\mathbb{R}^d} \left| \phi(\Delta)^{1/2} p(t-r,z) - \phi(\Delta)^{1/2} p(s-r,z) \right| \, dz \right]^2 \, dr \\
\leq N(t-s)^2 \int_0^a (\theta t + (1-\theta)s - r)^{-3} \, dr \leq (t-s)^2(s-a)^{-2}.
\]
The lemma is proved. \( \square \)

In the following corollary, we finally prove that the kernel
\[
K(t-r, z, x) := 1_{0 < r < t < T} \phi(\Delta)^{1/2} p(t-r, x-z)
\]
satisfies Assumption 2.4. Recall
\[
\rho(X, Y) = (\phi^{-1}(|t-s|^{-1}))^{-1/2} + |x-y|.
\]
Due to (5.20), there exists a constant \( N_\phi \geq 1 \) so that
\[
\phi^{-1}(a) \leq N_\phi \phi^{-1}(2^{-1}a) \quad \forall a > 0.
\]
For \( r > 0 \), \( X = (t,x), Y = (s,y) \in (0,\infty) \times \mathbb{R}^d \) set
\[
A(r, X, Y) := \{ z \in \mathbb{R}^d : \rho(X, Z) \geq 4N_\phi \rho(X, Y) \},
\]
where \( Z = (r,z) \). For the notational convenience, we use
\[
\phi(\Delta)^{1/2} p(t-r, x-z)
\]
to denote
\[
1_{0 < r < t < T} \phi(\Delta)^{1/2} p(t-r, x-z),
\]
that is, we assume
\[
\phi(\Delta)^{1/2} p(t-r, x-z) = 0
\]
unless \( 0 < r < t < T \).

**Corollary 5.18.** There exists a constant \( N \) so that for all \( X = (t,x), Y = (s,y) \in (0,T) \times \mathbb{R}^d \),
\[
\int_0^T \left[ \int_{A(r, X, Y)} \left| \phi(\Delta)^{1/2} p(t-r, x-z) - \phi(\Delta)^{1/2} p(s-r, y-z) \right| \, dz \right]^2 \, dr \leq N,
\]
where \( N \) depends only on \( d \) and the constants appearing in Assumption 5.14.
Proof. Without loss of generality, we assume \( t > s \). We only focus on proving the case \( t > s \) since the proof of the case \( t = s \) is simpler. Denote

\[
I(r, X, Y) = \left[ \int_{A(r, X, Y)} \phi(\Delta)^{1/2} p(t - r, x - z) - \phi(\Delta)^{1/2} p(s - r, y - z) \, dz \right]^2.
\]

If \( r \geq t \), then \( I(r, X, Y) = 0 \). Thus

\[
\int_0^T I(r, X, Y)dr = \int_{2s-t}^t I(r, X, Y)dr + \int_0^{2s-t} I(r, X, Y)dr =: I_1(X, Y) + I_2(X, Y).
\]

First we estimate \( I_1(X, Y) \). By (5.28),

\[
I_1(X, Y) \leq \int_{2s-t}^t \left[ \int_{|z| \geq (\phi^{-1}(|t-s|^{-1}))^{-1/2}} \phi(\Delta)^{1/2} p(t - r, z) \, dz \right]^2 dr
\]

\[
+ \int_0^{2s-t} \left[ \int_{|z| \geq (\phi^{-1}(|t-s|^{-1}))^{-1/2}} \phi(\Delta)^{1/2} p(s - r, z) \, dz \right]^2 dr \leq N.
\]

We split \( I_2 \). Observe

\[
I_2 \leq I_{2,1} + I_{2,2}
\]

\[
:= \int_{2s-t}^t \left[ \int_{A(r, X, Y)} \phi(\Delta)^{1/2} p(t - r, x - z) - \phi(\Delta)^{1/2} p(t - r, y - z) \, dz \right]^2 dr
\]

\[
+ \int_0^{2s-t} \left[ \int_{A(r, X, Y)} \phi(\Delta)^{1/2} p(t - r, y - z) - \phi(\Delta)^{1/2} p(s - r, y - z) \, dz \right]^2 dr.
\]

If \(|x - y| \leq (\phi^{-1}(|t-s|^{-1}))^{-1/2}\) then by (5.29),

\[
I_{2,1} \leq N|x - y|^2 (\phi^{-1}(|t-s|^{-1})) \leq N.
\]

On the other hand, if

\[
|x - y| > (\phi^{-1}(|t-s|^{-1}))^{-1/2},
\]

then

\[
I_{2,1} \leq 2I_{2,1,1} + I_{2,1,2},
\]

where

\[
I_{2,1,1} := \int_{s - (\phi(|x - y|^{-2})^{-1}}^t \left[ \int_{|z| \geq (\phi^{-1}(|t-s|^{-1}))^{-1/2} + |x - y|} \phi(\Delta)^{1/2} p(t - r, z) \, dz \right]^2 dr
\]

\[
I_{2,1,2}
\]

\[
:= \int_0^{s - (\phi(|x - y|^{-2})^{-1}} \left[ \int_{\mathbb{R}^d} \phi(\Delta)^{1/2} p(t - r, x - z) - \phi(\Delta)^{1/2} p(t - r, y - z) \, dz \right]^2 dr.
\]
By (5.28),
\[ \mathcal{I}_{2,1,1} \leq N \left[ (t - s + \phi \left( |x - y|^2 \right))^{-1} \phi \left( \left( \phi^{-1}((t - s)^{-1} + |x - y|^2) \right)^{-1/2} \right) \right] \]
and by (5.29),
\[ \mathcal{I}_{2,1,2} \leq N \left| x - y \right|^2 \phi^{-1} \left( (t - s + \phi \left( |x - y|^2 \right))^{-1} \right) \leq N. \]

It only remains to estimate \( \mathcal{I}_{2,2} \). However, this is an easy consequence of (5.30). Indeed,
\[ \mathcal{I}_{2,2} \leq N \left( (t - s)^{-1} \right)^2 \leq N. \]

The corollary is proved. \( \square \)

Consequently, to prove Theorem 5.16 it is enough to apply Theorem 2.5 with 
\[ T_\varepsilon = T_{\phi, \varepsilon} \quad \text{and} \quad T = T_\phi. \]

REFERENCES

[1] H. Dong and D. Kim. On \( L_p \)-estimates for elliptic and parabolic equations with \( A_p \) weights. *arXiv preprint arXiv:1603.07844*, 2016.
[2] L. Grafakos. Modern Fourier Analysis. Springer, 2008.
[3] I. Kim, K.-H. Kim, and P. Kim. Parabolic Littlewood-Paley inequality for \( \phi(\Delta) \)-type operators and applications to stochastic integro-differential equations. *Advances in Mathematics*, 249:161–203, 2013.
[4] I. Kim, K.-H. Kim, and S. Lim. Parabolic Littlewood-Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order, and applications to high-order stochastic PDE. *Journal of Mathematical Analysis and Applications*, 436(2):1023–1047, 2016.
[5] I. Kim, K.-H. Kim, and S. Lim. An \( L_q(L_p) \)-Theory for Parabolic Pseudo-Differential Equations: Calderón-Zygmund Approach. *Potential Analysis*, pages 1–21, 2016.
[6] N. V. Krylov. A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations. *Ulam Quart*, 2(4):16, 1994.
[7] N. V. Krylov. An analytic approach to SPDEs. *Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs*, 64:185–242, 1999.
[8] N. V. Krylov. *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, volume 96. American Mathematical Society Providence, RI, 2008.
[9] R. Mikulevičius and H. Pragarauskas. On \( L_p \)-estimates of Some Singular Integrals Related to Jump Processes. *SIAM Journal on Mathematical Analysis*, 44(4):2305–2328, 2012.
[10] R. L. Schilling, R. Song, and Z. Vondracek. *Bernstein functions: theory and applications*, volume 37. Walter de Gruyter, 2012.
[11] E. M. Stein and T. S. Murphy. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 3. Princeton University Press, 1993.
[12] J. Van Neerven, M. Veraar, and L. Weis. Maximal \( L^p \)-Regularity for Stochastic Evolution Equations. *SIAM Journal on Mathematical Analysis*, 44(3):1372–1414, 2012.
[13] J. Van Neerven, M. Veraar, and L. Weis. Stochastic maximal \( L_p \)-regularity. *The Annals of Probability*, 40(2):788–812, 2012.