Multi–Magnon Scattering in the Ferromagnetic 
XXX–Model with Inhomogeneities

by

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Abstract

We determine the transition amplitude for multi–magnon scattering induced through an inhomogeneous distribution of the coupling constant in the ferromagnetic XXX–model. The two and three particle amplitudes are explicitly calculated at small momenta. This suggests a rather plausible conjecture also for a formula of the general n–particle amplitude.
1 Introduction

We want to report in this article the calculation of transition amplitudes of multi-magnon scattering in the ferromagnetic Heisenberg XXX–chain with an inhomogeneous distribution of the coupling constant.

The Hamiltonian of the model under consideration is given by

\[ H = H_{\text{hom}} + H_{\text{inh}} \] (1)

\[ H_{\text{hom}} = \frac{J}{4} \sum_{n=-N}^{N} [\vec{\sigma}_n \vec{\sigma}_{n+1} - \mathbb{1}] \] (1a)

\[ H_{\text{inh}} = \frac{1}{4} \sum_{n=-N}^{N} z_n [\vec{\sigma}_n \vec{\sigma}_{n+1} - \mathbb{1}] \] (1b)

where \( \vec{\sigma} \) denotes the Pauli matrices operating in quantum spaces \( V_i; \ i = -N, \ldots, N \) attached to a one-dimensional lattice with \( 2N \) sites.

We choose in (1) the ferromagnetic sign of the coupling constant (\( J < 0 \)) and assume the inhomogeneous piece \( H_{\text{inh}} \) to be a small perturbation of the homogeneous part \( H_{\text{hom}} \), that is, we stipulate for the locally varying couplings \( z_i \)

\[ |z_i| \ll |J|. \]

The homogeneous XXX-chain is, as a prototype of an integrable model, one of the most thoroughly studied one-dimensional spin models.

A mathematically rigorous analysis of the model has been provided by Babbitt and Thomas [1]. For a treatment of the XXX–model in the framework of the algebraic Bethe Ansatz (ABA) cf. references [2]–[5]. It is found that the complete spectrum of the model is formed by quasi-particles, here called magnons, and bound states of magnons, the so-called string states. The integrability of the model implies that the interaction between magnons and strings is of a particularly simple structure. It is characterized by the following features [3]:

- multi-particle scattering factorizes into two-particle amplitudes
- the string states are absolutely stable bound states.

It follows from these properties that neither genuine multi-particle scattering takes place (with a non-trivial reshuffling of the particle momenta) nor does a break-up of the bound states occur.
We will make use of the technique of the ABA \[2\]–[3]. A first step in this direction is to embed the Heisenberg spin model (1a) into a family of vertex models. The latter models are defined through a monodromy matrix $T(\lambda)$ depending on a spectral parameter $\lambda$

$$T(\lambda) = L_N(\lambda) \cdots L_{-N}(\lambda)$$  \hspace{1cm} (2)

with the local “Lax-operators” $L_n$ given by

$$L_n(\lambda) = \frac{1}{2} [2i\lambda \mathbb{I}_0 \otimes \mathbb{I}_n + \vec{\sigma}_0 \otimes \vec{\sigma}_n]$$  \hspace{1cm} (3)

The unit operator $\mathbb{I}_0$ and the Pauli matrices $\sigma_0$ act in an auxiliary two-dimensional space, while $\mathbb{I}_n$ and $\sigma_n$ act in the quantum space $V_n$. The spin chain model (1a) emerges as the logarithmic derivative of the vertex model monodromy matrix

$$H = -\frac{J}{2} \frac{d}{d\lambda} \ln(tr_0 T(\lambda)) \bigg|_{\lambda=0} - J \frac{2N}{2} \mathbb{I} = \frac{J}{4} \sum_{n=-N}^{N} [\vec{\sigma}_n \vec{\sigma}_{n+1} - \mathbb{I}]$$  \hspace{1cm} (6)

with $tr_0$ denoting the trace with respect to the auxiliary space. The integrability of the vertex models and therewith also the integrability of the XXX-model is based on the fact that there is a c-number matrix $R = R(\lambda - \mu)$, s.t. the Yang-Baxter-Faddeev-Zamolodchikov (YBFZ) relation

$$R(\lambda - \mu) L_i(\lambda) \otimes L_i(\mu) = L_i(\mu) \otimes L_i(\lambda) R(\lambda - \mu)$$ \hspace{1cm} (4)

is satisfied. $R = R(\lambda - \mu)$ is in the case at hand given by

$$R(\lambda - \mu) = \begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix}$$ \hspace{1cm} (5)

with

$$f(\mu,\lambda) = 1 + \frac{ic}{\mu - \lambda}, \quad \text{and} \quad g(\mu,\lambda) = \frac{ic}{\mu - \lambda}.$$  

The parameter $c$ is set to unity in the following, meaning that the spectral parameter is taken as a dimensionless entity. The local relation (5) induces the global relation

$$R(\lambda - \mu) (T(\lambda) \otimes T(\mu)) = (T(\mu) \otimes T(\lambda)) R(\lambda - \mu)$$ \hspace{1cm} (6)

which might be considered here as hallmark of integrability.

The YBFZ-relations can be maintained in certain inhomogeneous generalizations of the above
models. One possibility is to choose different representations for different sites of the lattice. Such cases have been analyzed in references [7]–[9]. A conceptually simpler possibility, noticed in [4, 5], consists in attaching local parameters $z_i$ to the local Lax-operator, that is one substitutes $L_i(\lambda)$ by $L_i(\lambda - z_i)$ and obtains then also a modified monodromy matrix

$$T(\lambda, \{z_i\}) = T(\lambda; z_{-N} \ldots, z_N) = \prod_{i=N}^{-1} L_i(\lambda - z_i).$$  \hspace{1cm} (7)

One may easily see that the YBFZ relations remain intact,

$$R(\lambda - \mu) (T(\lambda, \{z_i\}) \otimes T(\mu, \{z_i\})) = (T(\mu, \{z_i\}) \otimes T(\lambda, \{z_i\})) R(\lambda - \mu).$$ \hspace{1cm} (8)

It should be noted that equation (8) only holds in general if one specifies for $T(\lambda, \{z_i\})$ and $T(\mu, \{z_i\})$ the same distribution of local parameters $\{z_{-N}, \ldots, z_N\}$. The physics of the model on the other hand appears to be invariant under permutations of the parameters. This is a consequence of the fact that the BA equations, to be mentioned shortly in the following section, which provide the spectrum of the eigenstates, are insensitive to these permutations. It is true (modulo some inessential caveats) that the order of the different representations along the lattice, as mentioned above, is for the same reason irrelevant. We therefore believe that genuine effects of inhomogeneties can only be realized outside the class of integrable models.

It is an easy undertaking to arrive from the inhomogeneous vertex model (7) at the inhomogeneous spin chain (1b). Let us make for this purpose the specifications

$$\lambda \rightarrow \epsilon \lambda \quad z_j \rightarrow \epsilon z_j.$$ \hspace{1cm} (9)

The inhomogeneous Heisenberg magnet (1b) is recovered as the logarithmic derivative of the vertex model

$$H = -J \frac{i}{2} \frac{d}{d\epsilon} \ln(tr_0 T(\epsilon \lambda, \{\epsilon z_i\}))\bigg|_{\epsilon=0} - \frac{J}{2} \sum_{n=1}^{N} (\lambda - z_n) \mathbb{1} = \frac{J}{4} \sum_{n=1}^{N} (\lambda - z_n) [\vec{\sigma}_n \vec{\sigma}_{n+1} - \mathbb{1}] =: \lambda H_0 - H_1.$$ \hspace{1cm} (10)

By taking the derivative with respect to a parameter $\epsilon$ which parametrizes different distributions, one expects to leave the realm of integrability. We will confirm this expectation by evaluating non-vanishing irreducible multi-particle scattering amplitudes which are supposed to vanish identically in integrable models.

We will restrict our considerations to the scattering of elementary magnons (the incorporation of string states is technically definitely much more cumbersome). A simplifying aspect of the
problem can be found in that the inhomogeneous perturbation respects the same global \(SU(2)\)
invariance as the homogeneous term. This implies magnon number conservation.

The plan of the paper is as follows: in the subsequent section we recall some ingredients of the ABA, discuss the thermodynamic limit and introduce the so-called multisite formalism, taken from [10]. In section III we evaluate in first order perturbation theory multi-particle amplitudes at small momenta. Of crucial importance to achieve this goal will be the representation of form factors as deduced in [10]. The technical tools to be applied in our analysis are approximately the same as those used in [11] for perturbative calculations in antiferromagnetic environments (while of course our calculations are simpler and more simple-minded). The concluding section is devoted to a qualitative discussion of other kinematical regions of multi-particle scattering and a summary. In an appendix we report a perturbative calculation of the spectrum of low lying states which is conceptually not related to the theme of the bulk of the paper but on a technical level rather similar to the calculations in section III.

2 Basics of Bethe Ansatz

2.1 Algebraic Bethe Ansatz

We collect here for the sake of self-consistency of the paper some of the basic aspects of the ABA\(^1\).

Let the monodromy matrix (2) be parameterized as

\[
T(\lambda) = \begin{pmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{pmatrix}.
\] (11)

One deduces from the YBFZ relations (3) the commutators of the operators \(A(\lambda), \ldots, D(\lambda)\). Of these 16 relations we only list the following

\[
\begin{align*}
[B(\lambda), B(\mu)] &= 0 = [C(\lambda), C(\mu)] \\
[B(\lambda), C(\mu)] &= g(\lambda, \mu)(D(\lambda)A(\mu) - D(\mu)A(\lambda)) \\
A(\mu)B(\lambda) &= f(\mu, \lambda)B(\lambda)A(\mu) + g(\lambda, \mu)B(\mu)A(\lambda) \\
D(\mu)B(\lambda) &= f(\lambda, \mu)B(\lambda)D(\mu) + g(\mu, \lambda)B(\mu)D(\lambda) \\
C(\lambda)D(\mu) &= f(\lambda, \mu)D(\mu)C(\lambda) + g(\mu, \lambda)D(\lambda)C(\mu)
\end{align*}
\]

\(^1\)For a more thorough introduction see [2, 3].
\[ C(\lambda)A(\mu) = f(\mu, \lambda)A(\mu)C(\lambda) + g(\lambda, \mu)A(\lambda)C(\mu). \quad (12) \]

Let \(|0\rangle\) denote the state of highest weight with respect to the tensor product of \(SU(2)\) representations in the configuration space \(V = \prod V_i\). This state is annihilated by the operators \(C(\lambda)\) and is an eigenstate of the trace of the transfer matrix \(\tau(\lambda)\)

\[
\tau(\lambda) = tr_0 T(\lambda) = A(\lambda) + D(\lambda)
\]

\[ C(\lambda)|0\rangle = 0 \]

\[ (A(\lambda) + D(\lambda))|0\rangle = \left[ \left( i\lambda + \frac{1}{2} \right)^{2N} + \left( i\lambda - \frac{1}{2} \right)^{2N} \right] |0\rangle. \]

The ABA renders a representation of the eigenstates of the transfer matrix in terms of the operators \(B(\lambda)\) - being the hermitian conjugates of the operators \(C(\lambda)\) - which act as creation operators of quasi-particles (magnons) on the highest weight state \(|0\rangle\).

Introducing the notation

\[ |\Phi(\lambda_1, \ldots, \lambda_l)\rangle = \prod_{n=1}^{l} B(\lambda_n)|0\rangle \]

one arrives - exploiting the commutation relations (12) - at

\[
(A(\lambda) + D(\lambda)) \prod_{n=1}^{l} B(\lambda_n)|0\rangle = \Lambda(\lambda; \lambda_1, \ldots, \lambda_l) \prod_{n=1}^{l} B(\lambda_n)|0\rangle + \sum_{k=1}^{l} \tilde{\Lambda}_k(\lambda_1, \ldots, \lambda_l)B(\lambda) \prod_{j \neq k} B(\lambda_j)|0\rangle \]

with

\[
\Lambda(\lambda; \lambda_1, \ldots, \lambda_l) = \left( i\lambda + \frac{1}{2} \right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda - i}{\lambda_j - \lambda} + \left( i\lambda - \frac{1}{2} \right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda + i}{\lambda_j - \lambda}, \quad (15)\]

and

\[
\tilde{\Lambda}_k(\lambda_1, \ldots, \lambda_l) = g(\lambda_k, \lambda) \left[ \left( i\lambda_k + \frac{1}{2} \right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda_k - i}{\lambda_j - \lambda_k} - \left( i\lambda_k - \frac{1}{2} \right)^{2N} \prod_{j=1}^{l} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k} \right]. \quad (16)\]

The second bunch of terms on the right hand side comes from the exchange term in the commutation relations. The spectral parameters have to be specified s.t. these terms vanish. This gives rise to the Bethe Ansatz equations

\[
\left( i\lambda_k + \frac{1}{2} \right)^{2N} = \prod_{j=1}^{l} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}. \quad (17)\]

\[ |\Phi(\lambda_1, \ldots, \lambda_l)\rangle \] is an eigenstate of the transfermatrix if the last equations are satisfied. The eigenstates can be classified with respect to the eigenvalue of \( S^3 \):
\[
S^3|\Phi(\lambda_1, \ldots, \lambda_l)\rangle = \left( \frac{2N}{2} - l \right)|\Phi(\lambda_1, \ldots, \lambda_l)\rangle.
\] (18)

The dual wave functions are given by
\[
\langle \Phi(\{\lambda_j\})| = \langle 0| \prod_{j=1}^l B^\dagger(\lambda_j) = (-1)^l \langle 0| \prod_{j=1}^N C(\lambda_j) .
\] (19)

### 2.2 Thermodynamical limit

We introduce suitably normalized operators in order to deal with finite norms of states and finite eigenvalues in the thermodynamical (TD) limit \([12]\):
\[
\tilde{A}(\lambda) = a^{-1}(\lambda)A(\lambda) , \quad \tilde{D}(\lambda) = d^{-1}(\lambda)D(\lambda)
\]
\[
\tilde{B}(\lambda) = a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)B(\lambda) , \quad \tilde{C}(\lambda) = a^{-\frac{1}{2}}(\lambda)d^{-\frac{1}{2}}(\lambda)C(\lambda)
\] (20)

with \( a(\lambda) = \alpha(\lambda)^{2N} = (i\lambda + \frac{1}{2})^{2N} \) the eigenvalue of the operator \( A(\lambda) \) and \( d(\lambda) = \delta(\lambda)^{2N} = (i\lambda - \frac{1}{2})^{2N} \) the eigenvalue of \( D(\lambda) \) with respect to \( |0\rangle \). Thus the operators \( \tilde{A}(\lambda) \) and \( \tilde{D}(\lambda) \) have correspondingly the eigenvalue 1. Exploiting the relations \( ^3 \)
\[
\lim_{N \to \infty} g(\lambda - \mu) \exp \frac{i(p(\lambda) - p(\mu))N}{2} = -\pi \delta(\lambda - \mu)
\] (21)

and
\[
\frac{1}{x} = \frac{1}{x \pm i\epsilon} \pm i\pi \delta(x)
\]
which hold in the sense of generalized functions \([13]\) - not pointwise - we obtain in the TD limit the simplified relations
\[
\tilde{A}(\lambda)\tilde{B}(\mu) = f_-(\lambda, \mu)\tilde{B}(\mu)\tilde{A}(\lambda)
\]
\[
\tilde{C}(\lambda)\tilde{D}(\mu) = f_-(\lambda, \mu)\tilde{D}(\mu)\tilde{C}(\lambda)
\] (22)

with \( f_-(\lambda, \mu) = 1 + \frac{i}{\lambda - \mu - i\epsilon} \).

One notes, comparing with equation \([12]\), that the exchange terms have dropped out. The

\( ^2 \)Our prescription differs from the one given in \([12]\) in an inessential way.

\( ^3 \)Terms of the form \( \frac{1}{x} \) have to be evaluated according to the principle–value prescription.
BAE can therefore be disregarded in the TD limit (as long as one restricts the attention to the sector of elementary magnons). Normalized asymptotic scattering states are generated by acting with creation operators $Z(\lambda) = \tilde{B}(\lambda)\tilde{A}^{-1}(\lambda)$ on the vacuum (the highest weight state) and are annihilated by operators $Z^\dagger(\lambda) = -\tilde{D}^{-1}(\lambda)\tilde{C}(\lambda)$. The action of the operators $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$ are easily deduced from the relations (22) and from the fact that the vacuum is an eigenstate with unit eigenvalue of $\tilde{A}(\lambda)$ and $\tilde{D}(\lambda)$ and therefore also of $\tilde{A}^{-1}$ and $\tilde{D}^{-1}$.

An incoming scattering state is given by

$$Z(\lambda_1)\ldots Z(\lambda_n)|0\rangle$$

if the rapidities are ordered in such a way that $\lambda_1 < \ldots < \lambda_n$, and represents an outgoing state for $\lambda_1 > \ldots > \lambda_n$.

To relate the incoming to the outgoing states, use has to be made of the relation

$$Z(\lambda)Z(\mu) = S(\lambda, \mu)Z(\mu)Z(\lambda)$$

with $S(\lambda, \mu) = \frac{f(\lambda, \mu)}{f(\mu, \lambda)}$ the two-body S-matrix. It is easily seen that the n-magnon S-matrix is given as a product of 2-magnon S-matrices.

The wavefunctions (23) are normalized to delta functions with a unit prefactor.

2.3 Multisite formalism

To evaluate scattering amplitudes in the Born approximation, we have to determine formfactors of the type

$$\langle 0|\prod_i^l Z^\dagger(\lambda_i)\mathcal{O}\prod_j^k Z(\lambda_j)|0\rangle$$

where the operator $\mathcal{O}$ is given by $\mathcal{O} = \sum_{n=-N}^N z_n \sigma_n \sigma_{n+1} \equiv \sum_n \mathcal{O}_{n,n+1}$. So we are lead to consider matrixelements of the form

$$\langle 0|\prod_i^l Z^\dagger(\lambda_i)\mathcal{O}_{n,n+1}\prod_j^k Z(\lambda_j)|0\rangle = \prod_{i>j} f^{-1}(\lambda_i^C, \lambda_j^C) f^{-1}(\lambda_i^B, \lambda_j^B)\langle 0|\prod_i^l \tilde{C}(\lambda_i^C)\mathcal{O}_{n,n+1}\prod_j^k \tilde{B}(\lambda_j^B)|0\rangle$$

where the latter identity is a straightforward consequence of the definition of $Z$ and the commutation relation (22).

The basic strategy for the determination of the r.h.s. of (26) will consist in decomposing the monodromy matrix into parts as follows:

$$T(\lambda) = T(3|\lambda)T(2|\lambda)T(1|\lambda)$$

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\[ T(1|\lambda) = L_{n-1}(\lambda) \cdots L_{-N}(\lambda) \]
\[ T(2|\lambda) = L_{n+1}(\lambda)L_i(\lambda) \]
\[ T(3|\lambda) = L_N(\lambda) \cdots L_{n+2}(\lambda). \]

The sub–monodromy matrices may be parametrized as \( T(\lambda) \) above

\[ T(j|\lambda) = \begin{pmatrix} A_j(\lambda) & B_j(\lambda) \\ C_j(\lambda) & D_j(\lambda) \end{pmatrix} \]

with \( j=1,2,3 \). The product in (27) is meant to be ordinary matrix multiplication of 2 \times 2 matrices. The \( T(j|\lambda) \) fulfill the global YBFZ commutation relation separately, acting on the vector space with highest weight \( |0\rangle_j \). The highest weight state of the total space is given as a tensor product

\[ |0\rangle = |0\rangle_3 \otimes |0\rangle_2 \otimes |0\rangle_1. \]  

(28)

Using the commutation relations, the operators \( A_j, D_j \), which appear if the \( B(\lambda) \) are expressed through operators of the subspaces, can be commuted through to the vacuum. This yields the so–called multisite formula [5]

\[ \prod_{j=1}^l B(\lambda_j^B)|0\rangle = \sum_{\{\lambda^{BI}\}_{j_B} \cup \{\lambda^{BII}\}_{j_B} \cup \{\lambda^{III}\}_{j_B}} \prod_{j_B}^{l_B} \prod_{k_B}^{l_B} \prod_{n_B}^{l_B} K_B 
\]

\[ B_3(\lambda_j^III)|0\rangle_3 \otimes B_2(\lambda_j^{BII})|0\rangle_2 \otimes B_1(\lambda_j^{BI})|0\rangle_1 \]

(29)

where

\[ K_B = a_2(\lambda_j^B)d_1(\lambda_j^{BI})a_3(\lambda_j^{BI})d_1(\lambda_j^{III})a_3(\lambda_j^{III})d_2(\lambda_j^{III}) \]

\[ \times f(\lambda_j^B, \lambda_j^{BI})f(\lambda_j^B, \lambda_j^{BII})f(\lambda_j^{BII}, \lambda_j^{BII}). \]

The summation in (23) is with respect to the partition of the set of all Bethe parameters \( \{\lambda_j\} \) in 3 disjunct subsets \( \{\lambda^{BI}\}, \{\lambda^{BII}\} \) and \( \{\lambda^{III}\} \) with

\[ \text{card}\{\lambda^{BI}\} = l_1^B, \quad \text{card}\{\lambda^{BII}\} = l_2^B, \quad \text{card}\{\lambda^{III}\} = l_3^B. \]

A similar representation can be derived for the dual vector \( \langle 0|\prod_{j=1}^r C(\lambda_j^C) \)

\[ \langle 0|\prod_{j=1}^r C(\lambda_j^C) = \sum_{\{\lambda^C\}_{j_C} \cup \{\lambda^{CIII}\}_{j_C} \cup \{\lambda^{CII}\}_{j_C}} \prod_{j_C}^{l_C} \prod_{k_C}^{l_C} \prod_{l_C}^{l_C} K_C 
\]

\[ \langle 0|_3 C_3(\lambda_j^{BIII}) \otimes \langle 0|_2 C_2(\lambda_j^{BII}) \otimes \langle 0|_1 C_1(\lambda_j^{BII}) \]

(30)
with
\[ K_C = d_2(\lambda_{IC}^C)a_1(\lambda_{KC}^{CH})d_3(\lambda_{IC}^{CH}) \]
\[ \times f(\lambda_{IC}^{CH}, \lambda_{KC}^{CH})f(\lambda_{IC}^{CH}, \lambda_{KC}^{CH})f(\lambda_{KC}^{CH}, \lambda_{IC}^{CH}) . \]

Inserting (25) and (30) into (26) we obtain
\[ \langle 0 | \prod_{j=1}^l C(\lambda_j^C)O_{n,n+1} \prod_{k=1}^l B(\lambda_k^B)|0 \rangle = \sum_{I,III,II,II,III} \prod_{I \leq j < K \leq III} a_j(\lambda_j^C)d_k(\lambda_j^C)a_k(\lambda_j^B)d_j(\lambda_k^B) \]
\[ \times f(\lambda_j^C, \lambda_k^C) f(\lambda_k^B, \lambda_j^B) S_{i_1}^{(I)}(\{\lambda_j^C\}, \{\lambda_j^B\}) \]
\[ \times S_{i_3}^{(III)}(\{\lambda_{II}^C\}, \{\lambda_{II}^B\}) \langle 0 | C_2(\lambda_{II}^C)O_{n,n+1} B(\lambda_{II}^B)|0 \rangle \]
(31)

with \( S_{i_1}^{(I)}(\{\lambda_j^C\}, \{\lambda_k^B\}) = \langle 0 | \prod_i C(\lambda_i^C) \prod_j B(\lambda_j^B)|0 \rangle \) being the scalar product in the j-th space.

The cardinality of the partition sets \( \{\lambda_j^C\} \) and \( \{\lambda_j^B\} \) are equal to each other. Matrixelements with \( card(\lambda_j^C) \neq card(\lambda_j^B) \) vanish.

Taking into account the normalization of the operators \( \tilde{B} \) and \( \tilde{C} \) relative to \( B \) and \( C \) we arrive at the following expression
\[ \langle 0 | \prod_{j=1}^l \tilde{C}(\lambda_j^C)O_{n,n+1} \prod_{k=1}^l \tilde{B}(\lambda_k^B)|0 \rangle = \sum_{I,III,II,II,III} \prod_{I \leq j < K \leq III} f(\lambda_j^C, \lambda_k^C) f(\lambda_k^B, \lambda_j^B) \]
\[ \times \alpha \delta(\lambda_j^C)\frac{-N_1}{2} \alpha \delta(\lambda_j^{CII})\frac{-N_2}{2} \alpha \delta(\lambda_j^{CIII})\frac{-N_3}{2} \alpha \delta(\lambda_j^{BII})\frac{-N^2}{2} \alpha \delta(\lambda_j^{IB}) \]
\[ \times \left[ \frac{r(\lambda_j^B)}{r(\lambda_j^C)} \right]^{\frac{N_1}{2}} \left[ \frac{r(\lambda_{II}^C)}{r(\lambda_j^B)} \right]^{\frac{N_2}{2}} \left[ \frac{r(\lambda_{III}^C)}{r(\lambda_j^B)} \right]^{\frac{N_3}{2}} S_{i_1}^{(I)}(\{\lambda_j^C\}, \{\lambda_j^B\}) S_{i_3}^{(III)}(\{\lambda_{II}^C\}, \{\lambda_{II}^B\}) \]
(32)

where \( \alpha \delta(\lambda) \equiv \alpha(\lambda)\delta(\lambda) \) and \( r(\lambda) \equiv \frac{\alpha(\lambda)}{\delta(\lambda)} \).

One has the following recursion relation for the scalarproducts [10]:

\[ S_l(\{\lambda_j^C\}, \{\lambda_k^B\}) = a(\lambda_j^C) \sum_{n=1}^l g(\lambda_n^B, \lambda_j^C) \prod_{j \neq 1} \prod_{k \neq n} g(\lambda_k^B, \lambda_n^B) S_{l-1}(\hat{a}_1(\lambda), \hat{d}_1(\lambda)) + \]
\[ d(\lambda_j^C) \sum_{n=1}^l a(\lambda_n^B) g(\lambda_n^B, \lambda_j^C) \prod_{j \neq 1} \prod_{k \neq n} g(\lambda_k^B, \lambda_n^B) S_{l-1}(\hat{a}_2(\lambda), \hat{d}_2(\lambda)) \]

\[ \text{for } l \geq 2, \text{with } \hat{a}_1(\lambda) = a(\lambda) \hat{d}_1(\lambda) = d(\lambda) \hat{a}_2(\lambda) = a(\lambda) \hat{d}_2(\lambda) = d(\lambda) \]

\[ \text{and } \lambda \in \{\lambda_j^C\} \text{ or } \lambda_j^C \text{ and } \lambda \in \{\lambda_j^B\} \text{ and } \lambda_j^B \text{ if } l \geq 2. \]
with \( \hat{a}_1(\lambda) = a(\lambda) h(\lambda, \lambda^B_1) \), \( \hat{a}_2(\lambda) = a(\lambda) h(\lambda, \lambda^C_1) \) and \( \hat{d}_1(\lambda) = d(\lambda) h(\lambda^C_1, \lambda) \), 
\( \hat{d}_2(\lambda) = d(\lambda) h(\lambda^B_1, \lambda) \), while \( h(\lambda, \mu) = \frac{g(\lambda \mu)}{\lambda \mu} = 1 + \frac{\mu}{\lambda} \). We have quoted here on the r.h.s. the functional dependence of the scalar products on the vacuum eigenvalues which have changed going from the l.h.s. to the r.h.s. from \( a(\lambda) \) to \( a(\lambda) \) and \( d(\lambda) \) to \( d(\lambda) \) respectively, which makes the solution of the recursion relation difficult in general. The two-term recursion relation simplifies in the TD limit, if we concentrate on the irreducible part of the amplitude.

We get for the normalized scalar product in the limit \( N \to \infty \):

\[
\lim_{N \to \infty} \prod_{CI} \prod_{BI} \left[ \frac{r(\lambda^B_I)}{r(\lambda^C_I)} \right]^{N^*_2 + 1} \alpha \delta(\lambda^C_I) - \frac{N}{2} \alpha \delta(\lambda^B_I) - \frac{N}{2} S_{St}(\{\lambda^C_I\}, \{\lambda^B_I\}) = \\
\left\{ \prod_{n_1=1}^{l_1} \left[ \frac{r(\lambda^B_{n_1})}{r(\lambda^C_{n_1})} \right]^{N_{n_1} - N_{n_1}^* + 1} g(\lambda^C_{n_1}, \lambda^B_{n_1}) \prod_{j_1 \neq 1}^{l_1} g(\lambda^C_{j_1}, \lambda^C_{j_1}) \prod_{k_1 \neq n_1}^{l_1} g(\lambda^B_{k_1}, \lambda^B_{k_1}) \right\} \\
- \pi \sum_{n_1=1}^{l_1} \delta(\lambda^C_{n_1} - \lambda^B_{n_1}) \prod_{j_1 \neq 1}^{l_1} g(\lambda^C_{j_1}, \lambda^C_{j_1}) \prod_{k_1 \neq n_1}^{l_1} g(\lambda^B_{k_1}, \lambda^B_{k_1})
\] 

\[ \times \lim_{N \to \infty} \prod_{CI \neq BI \neq n_1} \prod_{n_1=1}^{l_1} \left[ \frac{r(\lambda^B_I)}{r(\lambda^C_I)} \right]^{N^*_2 + 1} \alpha \delta(\lambda^C_I) - \frac{N}{2} \alpha \delta(\lambda^B_I) - \frac{N}{2} S_{St}(\{\lambda^C_I\}, \{\lambda^B_I\}) = \\
\sum_{n_1=1}^{l_1} \sum_{n_2 \neq n_1}^{l_2} \cdots \sum_{n_{l_1-1} \neq n_1 \cdots n_{l_1-1}}^{l_{l_1-1}} \prod_{j_{i+1} \neq 1}^{l_1} g(\lambda^C_{n_i}, \lambda^C_{n_j}) \prod_{k_1 \neq n_1}^{l_1} g(\lambda^B_{k_1}, \lambda^B_{n_1}) \cdots \prod_{k_{l_1-1} \neq n_1 \cdots n_{l_1}}^{l_1} g(\lambda^B_{k_{l_1}}, \lambda^B_{n_{l_1}}) \\
\times \prod_{k \geq 1}^{l_1} \frac{r(\lambda^B_k)}{r(\lambda^C_k)} = \\
\sum_{n_1=1}^{l_1} \sum_{n_2 \neq n_1}^{l_2} \cdots \sum_{n_{l_1-1} \neq n_1 \cdots n_{l_1-1}}^{l_{l_1-1}} \prod_{j_{i+1} \neq 1}^{l_1} g(\lambda^C_{n_i}, \lambda^C_{n_j}) \prod_{k_1 \neq n_1}^{l_1} g(\lambda^B_{k_1}, \lambda^B_{n_1}) \cdots \prod_{k_{l_1-1} \neq n_1 \cdots n_{l_1}}^{l_1} g(\lambda^B_{k_{l_1}}, \lambda^B_{n_{l_1}}). \tag{33}
\]
with the subscript "irr" indicating the restriction to that part of the scalar product that contributes finally to the irreducible amplitude. The extension of the r.h.s. of equation (34) by factors \(1 = g(\lambda, \mu) g(\lambda, \nu)\) enables us to represent it as a determinant multiplied by some overall factor:

\[
\lim_{N \to \infty} \prod_{CI, BI} l_i \prod_{j, i > j} \left[ \frac{r(\lambda_i^C)}{r(\lambda_j^C)} \right] \frac{N_j}{2} \quad \text{with } M^{(I)} = \frac{g(\lambda_i^C, \lambda_j^B)}{h(\lambda_i^{CI}, \lambda_j^{BI})}.
\]

A similar relation holds for the part denoted with III (here only the second term in the recursion relation of the scalar product contributes):

\[
\lim_{N \to \infty} \prod_{III, BI} l_i \prod_{j, i > j} \left[ \frac{r(\lambda_i^{CIII})}{r(\lambda_j^{BIII})} \right] \frac{N_j}{2} \quad \text{with } M^{(III)} = \frac{g(\lambda_i^{CIII}, \lambda_j^{BIII})}{h(\lambda_i^{CIII}, \lambda_j^{BIII})}.
\]

Inserting (35) and (34) into (31) one obtains

\[
\langle 0 | \prod_{j=1}^l \tilde{C}(\lambda_j^C) O_{n,n+1} \prod_{k=1}^l \tilde{B}(\lambda_k^B) | 0 \rangle_{\text{irr}} = \prod_{i} \left[ \frac{r(\lambda_i^C)}{r(\lambda_i^B)} \right]^{N_i - N_j} \prod_{j,i > j} g(\lambda_j^C, \lambda_i^C) g(\lambda_j^B, \lambda_i^B) \sum_{I, II, III} (-1)^{|P_C| + |P_B|} \prod_{II} \tilde{C}_2(\lambda_i^C) O_{n,n+1} \prod_{II} \tilde{B}_2(\lambda_i^B) | 0 \rangle
\]

\[
\times \prod_{I \leq J < K \leq III} h(\lambda_K^C, \lambda_J^C) h(\lambda_J^B, \lambda_K^B) \prod_{i,j} h(\lambda_i^{CII}, \lambda_j^{BII}) \prod_{i,j} h(\lambda_i^{BIII}, \lambda_j^{CIII})
\]

\[
\times \det \left( R^{-1}(\lambda_i^C) M^{(I)}(\lambda_i^C, \lambda_j^B) R(\lambda_j^B) \right) \det \left( R(\lambda_i^{CIII}) M^{(III)}(\lambda_i^{CIII}, \lambda_j^{BIII}) R^{-1}(\lambda_j^{BIII}) \right)
\]

(37)

with \(R(\lambda)_{ij} = r(\lambda_i) \delta_{ij}\). While deriving this result we used \(f(\lambda, \mu) = g(\lambda, \mu) h(\lambda, \mu)\) and the antisymmetry of the \(g\)'s.

\([P_B]\) stands for the parity of the permutation

\[
P_B : \{\lambda_I^B\} \cup \{\lambda_I^{BIII}\} \cup \{\lambda_I^{CIII}\} \to \{\lambda_B\}
\]
while \([P_C]\) stands for the parity of the permutation
\[
P_C : \{\lambda^C_{II}\} \cup \{\lambda^C_I\} \cup \{\lambda^C_{III}\} \rightarrow \{\lambda^C\}
\]
with the enumeration in each subset according to the original one.

It is possible in principle to write the result in a more compact way, namely as the determinant
of the sum of three matrices [14]. As it is not useful for our purpose we will not pursue this
line of reasoning.

### 3 Low-energy limit

To start with let us make the simplifications which are due to the special form of the pertur-
bation. The matrixelement
\[
2\langle 0| \prod_i \tilde{C}_2(\lambda^C_i) \sigma_n \sigma_{n+1} \prod_i \tilde{B}_2(\lambda^B_i)|0\rangle_2
\]
appearing in equation (37) is to be evaluated with respect to the two–site highest weight state
\(|0\rangle_2\). There can therefore at most two operators \(B\) and \(C\) show up in (39) (applying two
operators \(B_2\) to \(|0\rangle_2\) one reaches the state of lowest weight of the two-site vector space). Since
we are anyhow restricting our attention to scattering events in non-forward direction we may
evaluate instead of (39) the matrixelement
\[
2\langle 0| \prod_i \tilde{C}_2(\lambda^C_i) (\sigma_n \sigma_{n+1} - \mathds{1}_n \cdot \mathds{1}_{n+1}) \prod_i \tilde{B}_2(\lambda^B_i)|0\rangle_2
\]
with \(\mathds{1}_n\) the identity in \(V_n\) (the addition of \(\mathds{1}_n \cdot \mathds{1}_{n+1}\) gives only a contribution to the amplitude
in forward direction). But (40) vanishes on the state of highest weight (no operators \(B_2\)
and \(C_2\)) and on the state of lowest weight (two operators \(B_2\) and \(C_2\)). We are left with the
matrixelement with one operator \(B_2\) and \(C_2\), which is straightforwardly calculated
\[
2\langle 0| \tilde{C}_2(\lambda^C_{II}) (\sigma_n \sigma_{n+1} - \mathds{1}_n \cdot \mathds{1}_{n+1}) \tilde{B}_2(\lambda^B_{II})|0\rangle_2 = 2 \frac{1}{\alpha \delta(\lambda^B_{II})} \frac{1}{\alpha \delta(\lambda^C_{II})} .
\]
Taking the normalization and the last result into account we obtain for the transition amplitude
the representation
\[
\sum_n \langle 0| \prod_{j=1}^l Z^\dagger(\lambda^C_j) \mathcal{O}_{n,n+1} \prod_{k=1}^l Z(\lambda^B_k)|0\rangle =
\]
For the next order of the expansion in powers of momenta we obtain in a straightforward manner the following result:

\[
\sum_n z_n 2^n \prod_i \frac{l}{r(\lambda_i^C)} \prod_{j>i} l \frac{g(\lambda_j^C, \lambda_i^C)}{f(\lambda_j^C, \lambda_i^C)} \frac{g(\lambda_j^B, \lambda_i^B)}{f(\lambda_j^B, \lambda_i^B)} \sum'(-1)^{|P_C|+|P_B|} \frac{1}{\alpha \delta(\lambda^B) \alpha \delta(\lambda^C)} \prod_{l \leq j < K \leq III} h(\lambda^K_C, \lambda^K_J) h(\lambda^B_J, \lambda^B_K) \prod_{i} h(\lambda_J^{CI}, \lambda_J^{BI}) \prod_{j>i} h(\lambda_J^{BIII}, \lambda_J^{CIII})
\]

\[
\times \det \left( R^{-1}(\lambda^C_J) M^{(I)}(\lambda_I^C, \lambda_J^C) R(\lambda_J^B) \right) \det \left( R(\lambda_{III}^C) M^{(III)}(\lambda_{III}^C, \lambda_{III}^B) R^{-1}(\lambda_{III}^B) \right)
\]

(42)

The slash on the sum over the partitions is supposed to indicate that only partitions with exactly one representative present in the subset labeled by \( II \) are to be taken.

We are now prepared to examine the behaviour of irreducible scattering amplitudes at low momenta \( \lambda_i \approx p_i \) for small momentum) with two or more than two magnons involved (the one particle amplitude will be quoted below for the sake of completeness). An obvious method to get a handle on formula (12) consists in a systematic expansion in powers of momenta, as far as they appear in functions \( h \) and keeping at the same time the functions \( g \) unexpanded. The leading term is obtained by putting \( h \) consistently to one at all places where it appears in (12). This yields for \( \lambda_\alpha \in \{ \lambda^C \} \)

\[
\sum_n \langle 0 | \prod_{j=1}^l Z_j^\dagger (\lambda_j^C) O_{n,n+1} \prod_{k=1}^l Z_k (\lambda_k^B) | 0 \rangle \approx
\]

\[
\sum_n 32 z_n \prod_{j>i} \frac{g(\lambda_j^C, \lambda_i^C)}{f(\lambda_j^C, \lambda_i^C)} \frac{g(\lambda_j^B, \lambda_i^B)}{f(\lambda_j^B, \lambda_i^B)} \sum'(-1)^{|P_C|+|P_B|} \det g(\lambda_J^C, \lambda_I^C) \det g(\lambda_I^B, \lambda_J^B) \]

(43)

The prefactors \( \prod_{j>i} \frac{g(\lambda_j^C, \lambda_i^C)}{f(\lambda_j^C, \lambda_i^C)} \) may also be put equal to one in leading order by noting that \( \frac{g(\lambda \mu)}{f(\lambda \mu)} \approx 1 + O (\lambda - \mu) \). The sum over the partitions I and III in (13) renders a vanishing result as one infers from the Laplace formula for the determinant of a sum of matrices [3]:

\[
det (A + B) = \sum_{P_L, P_C} (-1)^{|P_L|+|P_C|} \det A_{P_L, P_C} \det B_{P_L, P_C}
\]

(44)

where \( P_L \) is the partition of rows in subsets of rows of \( A \) and \( B \), while \( P_C \) is analogous the partition of columns, and the fact that \( g \) is odd

\[
g(\lambda_j^C, \lambda_j^B) + g(\lambda_j^C, \lambda_j^B) = 0
\]

For the next order of the expansion in powers of momenta we obtain in a straightforward manner the following result:

\[
\sum_n \langle 0 | \prod_{j=1}^l Z_j^\dagger (\lambda_j^C) \sum_{i=1}^l \frac{1}{r(\lambda_i^C)} \prod_{k=1}^l Z_k (\lambda_k^B) | 0 \rangle \approx 2^l \prod_{j>i} (\lambda_j^C, \lambda_i^C) h^{-1}(\lambda_j^B, \lambda_i^B) \prod_i \left[ \frac{r(\lambda_i^C)}{r(\lambda_i^B)} \right]^{N_1-N_3} \]

\[
\times 16 z_1 \sum_C \lambda_\alpha \sum_{I,II,III} (-1)^{|P_C|+|P_B|} (l_3 - l_1 + 6 \epsilon_\alpha_{I,II,III}) \det g(\lambda_I^C, \lambda_I^B) \det g(\lambda_{III}^B, \lambda_{III}^C)
\]

(45)
with

\[ \epsilon^{I,III}_\alpha = \begin{cases} 
+1; & \lambda_\alpha \in I \\
-1; & \lambda_\alpha \in III \\
0; & \lambda_\alpha \in II 
\end{cases} \]

If \( \lambda_\alpha \in \{\lambda^B\} \) we get the same result up to an overall minus sign.

The result can be simplified further, using the following chain of identities:

\[
\sum_{I,III} (-1)^{[P_B]+[P_C]} (l_3 - l_1) \det g(\lambda^C_I, \lambda^B_I) \det g(\lambda^B_{III}, \lambda^C_{III}) \\
= \frac{\partial}{\partial x} \sum_{I,III} (-1)^{[P_B]+[P_C]} \det g(x\lambda^C_I, x\lambda^B_I) \det g(x^{-1}\lambda^B_{III}, x^{-1}\lambda^C_{III})|_{x=1} \\
= \frac{\partial}{\partial x} \left( \sum_1^{l_1} - \sum_1^{l_3} \right) \det g(\lambda^C, \lambda^B)|_{x=1} \\
= 0 \quad \text{for } l_1 + l_3 > 1
\]

where we used again the Laplace formula and the antisymmetry of the \( g \)'s.

The remaining term of the first-order Taylor expansion is

\[
\langle 0 | \prod_{j=1}^l Z^I(\lambda^C_j) z_i \sigma_i \sigma_{i+1} \prod_{k=1}^l Z(\lambda^B_k) | 0 \rangle = \prod_{j>i} h^{-1}(\lambda^C_j, \lambda^C_i) h^{-1}(\lambda^B_j, \lambda^B_i) \prod_{i} \left[ \frac{r(\lambda^C_i)}{r(\lambda^B_i)} \right]^{N_i-N_{i+1}} \\
\times 16 \cdot 6i \sum_{C} \lambda_i \sum_{l,III} (-1)^{[P_B]+[P_C]} (\epsilon^{I,III}_\alpha) \det g(\lambda^C_I, \lambda^B_I) \det g(\lambda^B_{III}, \lambda^C_{III}).
\]

The sum over the first and the third partition can be combined to a determinant of the sum of two matrices

\[
\sum_{l,III} (-1)^{[P_B]+[P_C]} (\epsilon^{I,III}_\alpha) \det g(\lambda^C_I, \lambda^B_I) \det g(\lambda^B_{III}, \lambda^C_{III}) \\
= \det \left( g(\lambda^C, \lambda^B) + \hat{g}_\alpha(\lambda^B, \lambda^C) \right)
\]

The matrix \( \hat{g}_\alpha(\lambda^B, \lambda^C) \) differs from \( g(\lambda^B, \lambda^C) \) in that the \( \alpha \)-th row is multiplied by \(( -1 )\). The sum of the two matrices is thus a matrix with only one row of non-vanishing entries. The determinant is again zero except for \( l_1 + l_3 = 1 \). We keep as net result that the first–order term of the Taylor expansion gives a non–vanishing contribution only for the one– and two–particle amplitude.

We turn now to the second-order contribution. The computation is tedious, but it proceeds otherwise along the same lines as the first-order calculation. We thus only quote the result
\[
\langle 0 \| \prod_{j=1}^{l} Z_j^\dagger (\lambda_j^C) \; z_i \sigma_i \sigma_{i+1} \prod_{k=1}^{l} Z_k (\lambda_k^B) \| 0 \rangle \sim \\
16 \frac{1}{2} \sum_{\alpha \beta} \lambda_\alpha \lambda_\beta \; \sum'_{I,II,III} \left( -1 \right)^{[Pa]+[Pc]} \left[ i \epsilon_{\alpha}^{BC} (l_1 - l_3 - 6 \epsilon_{\alpha}^{I,III}) i \epsilon_{\beta}^{BC} (l_1 - l_3 - 6 \epsilon_{\beta}^{I,III}) \right] \right. \\
+ \left. \delta_{\alpha \beta} (34 \check{\epsilon}^{I,III} + l_1 + l_3) \right] \det g(\lambda_i^C, \lambda_i^B) \det g(\lambda_{II}^B, \lambda_{I}^C)
\]

\[ (46) \]

with

\[
\epsilon_{\alpha}^{BC} = \begin{cases} 
+1; & \lambda_\alpha \in B \\
-1; & \lambda_\alpha \in C
\end{cases}
\]

and

\[
\check{\epsilon}^{I,III} = \begin{cases} 
+1; & \lambda \text{ in } I,III \\
0; & \text{otherwise}
\end{cases}
\]

When \( l_1 + l_3 > 2 \) this term vanishes as can be shown by generalizing the considerations used with the first-order calculation:

- terms proportional to \((l_1 + l_3 + \text{const})\) vanish by the same argument as used in the first-order calculation (even for \( l_1 + l_3 = 2 \))
- terms proportional to \((l_1 - l_3)^2\) vanish when regarded as the second derivative with respect to \( x \) at \( x = 1 \) (this term renders for \( l_1 + l_3 = 2 \) the only nonvanishing contribution)
- terms proportional to \(\epsilon_{\alpha}^{I,III} (l_1 - l_3)\) give a matrix with a prefactor \( \left( \frac{1}{x} - x \right)^{l_1 + l_3 - 1} \left( \frac{1}{x} + x \right) \) of which the derivative with respect to \( x \) at \( x = 1 \) vanishes
- terms proportional to \(\epsilon_{\alpha}^{I,III} \epsilon_{\beta}^{I,III}\) give a matrix with at most 2 columns or rows or one column and one row not zero after applying the Laplace formula

We finish this section by quoting the leading terms of the transition amplitudes at small momenta with the explicit expressions for \( l < 3 \) in lowest order:

- \( l = 1 \)

\[
\sum_n z_n \langle 0 \| Z^\dagger (\lambda) \; \sigma_n \sigma_{n+1} Z (\mu) \| 0 \rangle = -2 \sum_n z_n \left[ \frac{r(\lambda)}{r(\mu)} \right] \frac{N_1 - N_2}{2} \frac{1}{\lambda^2 + \frac{1}{4} \mu^2 + \frac{1}{4}}
\]

\[ (47) \]
• \( l = 2 \), cf. equation (43)

\[
\sum_n z_n \langle 0 | Z_i^C Z_j^C Z_i^B Z_j^B | 0 \rangle = -64 \sum_n z_n \left( \sum_C \lambda^C - \sum_B \lambda^B \right)^2 \det_2 \left( \frac{1}{\lambda^C - \lambda^B} \right)
\]

(48)

• \( l = 3 \), cf. equation (46)

\[
\sum_n z_n \langle 0 | \prod_{i=1}^3 Z_i^C \sigma_n^+ \sigma_{n+1}^+ \prod_{i=1}^3 Z_i^B | 0 \rangle = -128 \sum_n z_n \left( \sum_C \lambda^C - \sum_B \lambda^B \right)^3 \det_3 \left( \frac{1}{\lambda^C - \lambda^B} \right)
\]

(49)

with \( \det_l \left( \frac{1}{\lambda^C - \lambda^B} \right) \) denoting the Cauchy determinant of a \( l \times l \) matrix

\[
det_l \left( \frac{1}{\lambda^C - \lambda^B} \right) = (-1)^{m+n} \frac{\prod_{i<j}^l (\lambda_i - \lambda_j) \prod_{i<j}^l (\mu_i - \mu_j)}{\prod_{i,j}^l (\lambda_i - \mu_j)}.
\]

A couple of remarks may be in order:

1. The one-particle amplitude quoted above is in fact the full Born term (not the leading piece at small momentum).

2. We have left out in (48) and (49) factors \( \prod_i \left[ \frac{r(\lambda_i)}{r(\mu_i)} \right]^{N_\lambda - N_\mu} \). The latter give rise to a Fourier transform of the distribution of coupling constants. We have effectively evaluated the distribution at zero momentum.

3. To apply the above expressions to physical processes of magnon scattering one has to restrict the respective expressions to the energy shell, given by \( \sum_i \lambda_i^B^2 = \sum_i \lambda_i^C^2 \).

4 Conclusion

The main result of this paper are the formulas (48) and (49) for two- and three-magnon scattering at small momenta. An obvious generalization to \( n \)-particle scattering may be con-
jectured:

$$\langle 0 | \prod_{i=1}^{l} Z^\dagger (\lambda_i^C) \bar{\sigma}_i \sigma_{i+1} \prod_{i=1}^{l} Z (\lambda_i^B) |0 \rangle \approx -16 2^l \left( \sum_C \lambda^C - \sum_B \lambda^B \right)^l \det \left( \frac{1}{\lambda^C - \lambda^B} \right).$$  \hspace{1cm} (50)

We are not able to prove this conjecture so far.

There are other kinematical regions besides the one of low momenta for which simple and reliable estimates can be made. If all momenta and all differences of momenta become large, the $n$–particle transition amplitude decreases with $\rho^{-(n+3)}$ – $\rho$ denoting a common scale of all momenta – as can be inferred from an inspection of equation (50). An interesting kinematical region – also accessible to a rather detailed analytical description – is given by the setting

$$|\lambda_i^B - \lambda_j^B| \ll 1, \quad |\lambda_i^C - \lambda_j^C| \ll 1, \quad |\lambda_i^B - \lambda_j^C| \gg 1, \quad \forall \, i, j, \quad |\lambda_i^B| \sim |\lambda_i^C| \sim \rho \gg 1.$$

This situation is realized if a bunch of particles travelling approximately with the same velocity is collectively scattered backwards at the inhomogeneity.

The piece of (57) supplying the $\rho$ dependence in this case is given by

$$2 \langle 0 | \tilde{C}_2 (\lambda_i^C) \mathcal{O}_{n,n+1} \tilde{B}_2 (\lambda_i^B) |0 \rangle 2 \prod_{i,j}^{l} h(\lambda_i^C, \lambda_j^B) \prod_{i,j}^{l} h(\lambda_i^{BIII}, \lambda_j^{CIII}) \times \det \left( \mathcal{M}^{(I)} (\lambda_i^C, \lambda_i^B) \right) \det \left( \mathcal{M}^{(III)} (\lambda_i^{CIII}, \lambda_i^{BIII}) \right)$$

for which one easily calculates the scaling behaviour $\rho^{-(n+3)}$.

To arrive at this conclusion it is essential to view the determinants in the above formula as derivatives of Cauchy determinants:

$$\det \mathcal{M}_{ij} \approx \det_\lambda \left( \frac{1}{(\lambda_i - \lambda_j)^2} \right) \sim \frac{\partial}{\partial \lambda_i^C} \cdots \frac{\partial}{\partial \lambda_n^C} \det_\lambda \left( \frac{1}{(\lambda_i - \lambda_j)} \right)$$

A completely open problem within our approach is the treatment of string states. The determination of break–up amplitudes for string states seems to us a particularly challenging problem.

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In this appendix we use the result (31) to determine the shift of energy eigenvalues caused by
\[ H_{\text{inh}} = \frac{1}{4} \sum_{n=1}^{N} z_n [\vec{\sigma}_n \vec{\sigma}_{n+1} - \mathbb{1}] \]
The lowest excitation is generated by flipping one spin \( l = 1 \). The solution of the Bethe Ansatz equation is in this case
\[ \lambda = \frac{1}{2} \cot \frac{p_0}{2} , \quad p_0 = \frac{2\pi k}{N} ; \quad k = 1, ..., N . \] (51)
Taking into account parity degeneracy the first-order correction to the energy \( E^{(0)}(\lambda) = \frac{1}{2} \lambda^2 - \frac{1}{4} \) is found to be
\[
E^{(1)}(\lambda) = E^{(0)}(\lambda) \left[ \left( \frac{1}{N} \sum_{j=1}^{N} z_j \right) \pm \sqrt{\frac{1}{N} \sum_{j,k=1}^{N} z_j z_k \exp(-2ip_0(\lambda)(j - k))} \right].
\] (52)
This shows that the energy correction depends in first-order both on the mean–value of the couplings \( \bar{z} = \frac{1}{N} \sum_{j=1}^{N} z_j \) and on the Fouriertransform of the distribution \( \frac{1}{N} \sum_{j=1}^{N} z_j \exp(\pm 2ip_0j) \)
\( (p_0 = \frac{1}{i} \ln \frac{1 + \lambda}{1 - \lambda}) \).
The second-order corrections can be obtained from the secular equation
\[
E^{(2)}_n(\lambda) = \sum_{m \neq n} \frac{\tilde{V}_{nm} \tilde{V}_{mn}}{E^{(0)}_n - E^{(0)}_m}
\]
where the matrixelements are taken with respect to the corrected wave-function in the zeroth approximation
\[
\tilde{V}_{nm} = \langle c_1^{(0)} \Phi(\lambda) \pm c_2^{(0)} \Phi(-\lambda) | H_1 | \Phi(\mu) \rangle
= c_1^{(0)} \mathcal{V}(\lambda, \mu) \pm c_2^{(0)} \mathcal{V}(-\lambda, \mu)
\]
with \( \mathcal{V}(\lambda, \mu) \) defined as in (52) and \( c^{(0)} \) being the following expressions
\[
c_1^{(0)} = \sqrt{\frac{\mathcal{V}(\lambda, -\lambda)}{2|\mathcal{V}(\lambda, -\lambda)|}}
\]
\[
c_2^{(0)} = \sqrt{\frac{\mathcal{V}(-\lambda, \lambda)}{2|\mathcal{V}(-\lambda, \lambda)|}}.
\]
Inserting the explicit formulas yields
\[
E^{(2)}_n(\lambda) = \frac{1}{4} \sum_{\mu \neq \pm \lambda} \frac{1}{\lambda^2 - \mu^2} \frac{1}{N^2} \sum_{j,k} z_j z_k \left\{ \left( \frac{i\mu - \frac{1}{2}}{\mu + \frac{1}{2}} \right)^{j-k} \left[ \left( \frac{i\lambda + \frac{1}{2}}{i\lambda - \frac{1}{2}} \right)^{j-k} \left( \frac{i\lambda - \frac{1}{2}}{i\lambda + \frac{1}{2}} \right)^{j-k} \right] \right\}.
\]

\(K_{\pm}\) is a quotient of Fourier transforms
\[
K_{\pm} = \left[ \sqrt{\sum_{j=1}^{N_j} z_j^2 \left( \frac{(i\lambda + \frac{1}{2})}{(i\lambda + \frac{1}{2})} \right)^{2j}} \right]^{1/2}.
\]

The sum over \(\mu\) can be transformed for \(N \to \infty\) into a principle-value integral
\[
E^{(2)}_n(\lambda) = \frac{1}{8\pi} \sum_{j,k} \frac{z_j z_k}{N} \text{vp} \int_{-\infty}^{+\infty} \frac{1}{\lambda^2 - \mu^2} \left\{ \left( \mu + \frac{i}{2} \right)^{(j-k-1)} \left( \mu - \frac{i}{2} \right)^{(k-j-1)} 2 \cos [p(\lambda)(j-k)] \right\} \left\{ \left( \frac{i\lambda + \frac{1}{2}}{i\lambda - \frac{1}{2}} \right)^{2j} \right\} \text{vp} \int_{-\infty}^{+\infty} \frac{1}{\lambda^2 - \mu^2} \right\} d\mu.
\]

This principle-value integral can be evaluated by deforming the integration contour into the complex plane, closing it at infinity, which is possible as the integrand vanishes as \(r^{-4}\) at infinity. Thus only the pole structure of the integral matters.

There are poles at \(\lambda = \pm \mu\) for all values of \(j, k\), at \(\mu = \pm \frac{i}{2}\) for \(j \geq k\) in the first term and for \(j \leq k\) in the second term and at \(\mu = -\frac{i}{2}\) with \(j, k\) dependence of the first and second term interchanged. It is convenient to split the sum over \(j, k\) into three parts:
\[
\sum_{j,k} = \sum_{j=k} + \sum_{j \geq k} + \sum_{j \leq k}.
\]

For each sum the contour can be deformed in such a way that the integrand only contains poles of first order, for which the residues are easily calculated.

The result of the integration is
\[
E^{(2)}_n(\lambda) = -E^0(\lambda) \left\{ \frac{1}{2} \sum_j \frac{z_j^2}{N} [2 \pm f(\lambda, j = k)]
- \frac{2}{\lambda} \sum_{j > k} \frac{z_j z_k}{N} \sin [p(\lambda)(j-k)] [2 \cos [p(\lambda)(j-k)] \pm f(\lambda, j > k)] \right\}
\]

with \(f(\lambda, j, k) = [K_+ e^{-ip(\lambda)(j+k)} + K_- e^{ip(\lambda)(j+k)}] \)
For the second lowest excitation (two magnons) the computation is more involved, but still elementary, so we only give the result for the first-order correction to the energy of the two-magnon state:

\[ E^{(1)}(\mu_1, \mu_2) = E^{(0)}(\mu_1, \mu_2) \pm 2 \left[ \sum_{j,k=1}^N z_j z_k \exp \left[ -2ip_0(\mu_1)(j - k) \right] \exp \left[ -2ip_0(\mu_2)(j - k) \right] \right] \]

Furthermore there exist complex solutions of the Bethe Ansatz equations. They describe bound states \[ [17] \] with momentum

\[ e^{ip(x)} = \left( \frac{x + i}{x - i} \right) \]

and energy

\[ E^0_{\text{String}}(x) = \frac{1}{x^2 + 1} \]

where \( x \) denotes the center of the bound state.

The first-order correction for the two magnon bound state is

\[ E^{(1)}_{\text{String}}(x) = E^{(0)}_{\text{String}}(x) \left[ \left( \frac{1}{N} \sum_{j=1}^N z_j \right) \pm E^0_{CM} \sqrt{\sum_{j,k=1}^N \frac{z_j z_k}{N^2} \exp(-2ip(x)(j - k))} \right] \]

with \( E^0_{CM} \) the energy of the center of the bound state.

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