CRITERIA FOR COMPLETE INTERSECTIONS

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ABSTRACT. We obtain criteria for detecting complete intersections in projective varieties.

INTRODUCTION

Faltings [10] showed that a smooth subvariety of the projective space \( \mathbb{P}^n \), of codimension at most \( n/2 \), is a complete intersection as soon as its normal bundle decomposes into a direct sum of line bundles. Thus the property of being a complete intersection is reduced to the splitting of the normal bundle of the subvariety; the former is typically difficult to verify, while the latter condition is certainly necessary and it is, at least theoretically, much easier to check. His work was further elaborated under additional \textit{a priori} hypotheses in [2, 18].

The starting point of this article is the question whether a similar statement holds for subvarieties of other ambient varieties as well; indeed, complete intersections are the simplest examples of subvarieties and detecting them is an important task. By taking a glance at [10], one can see that at its base there is a Hauptlemma—main lemma—which is valid in great generality. However, its ‘implementation’ is strongly adapted to subvarieties of projective spaces. This may explain why, to our knowledge, the arguments have not been extended to a wider framework. It’s therefore natural to attempt bridging this difference. In this article we propose such a generalisation, which uses the author’s work [12] on partially ample subvarieties.

The following statement should convey the flavour of our results.

\textbf{Theorem} (cf. Section 2, 3) Let \( G \) be a simple, linear algebraic group of rank \( \ell \) and \( P \) a maximal parabolic subgroup. Let \( X \subset G/P \) be a smooth subvariety of codimension \( \delta \leq (\ell + 1)/3 \). Then \( X \) is a complete intersection in \( G/P \) as soon as either one of the following two conditions is satisfied:

\begin{itemize}
  \item Its normal bundle \( N_X \) splits into a direct sum of line bundles.
  \item There is a surjective homomorphism \( \bigoplus_{j=1}^{a} \mathcal{O}_{G/P}(d_j) \to N_X \), with \( a \leq (\ell + 1)/2 - \delta \).
\end{itemize}

So, if \( X \) is not a complete intersection and \( \delta \leq (\ell + 1)/4 \), then at least \( (\ell + 1)/2 - 2\delta \) equations in excess are necessary for defining \( X \).

Motivated by Hartshorne’s conjecture for projective spaces, we further elaborate the case of smooth two-codimensional subvarieties of rational homogeneous varieties. For projective spaces, a classical result of Ran [15] gives effective bounds for the Chern numbers of the normal bundle of the subvariety which imply the complete intersection property. What we show is that an analogous result holds in our case.

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**Theorem** (cf. Section 4) Let $G/P$ be as above with $\ell \geq 6$, and denote $m := \deg_L(\mathcal{I}_V) - 3$, where $L$ is the 1-dimensional Schubert line. Suppose $X$ is smooth, 2-codimensional, such that $[X] = d_X \cdot [O_{G/P}(1)^2]$, $\det(N_{X/V}) = O_X(n)$. If either

$$d_X \leq m(n - m) \quad \text{or} \quad \sqrt{d_X} \leq n/2 \leq m,$$

the subvariety $X \subset G/P$ is a complete intersection.

Several other criteria are obtained in the paper. We must say that our arguments are strongly influenced by Faltings’ and Ran’s methods. However, the interest in our viewpoint—we believe—consists in showing that the range of applications is substantially enlarged.

1. **Subvarieties with split normal bundle**

**Notation 1.1** We work over an algebraically closed field $k$ of characteristic zero. Throughout the article, $X$ is a smooth subvariety of an ambient smooth projective variety $V$, of codimension $2 \leq \delta \leq \dim V/2$, and defined by the sheaf of ideals $\mathcal{I}_X \subset O_V$; its co-normal bundle $N_{X/V} := \mathcal{I}_X/\mathcal{I}_X^2$ is locally free, of rank $\delta$. For a subvariety $Y$ of $V$ containing $X$, we denote $\mathcal{I}_{X \subset Y} := \mathcal{I}_X/\mathcal{I}_Y$.

This article deals with the following issue: assuming that $N_{X/V}$ decomposes into a direct sum of line bundles, can one deduce that $X$ a complete intersection of $\delta$ hypersurfaces in $V$? Faltings’ work [10] corresponds to the case where the ambient variety $V$ is a projective space.

Our investigations naturally lead us to consider flags of subvarieties, on the one hand, and partially ample subvarieties of a projective variety, on the other hand. Since our arguments require these definitions, for the reader’s comfort, we recall them here.

**Definition**

(i) (cf. [8]) A flag of (normal and irreducible) subvarieties of $V$ is a chain

$$V = Y_0 \supset Y_1 \supset \cdots \supset Y_d,$$

such that each $Y_j \subset Y_{j-1}$ is a Cartier divisor. Then $d$ is the length of the flag and $Y_d$ is its end.

(ii) (cf. [17]) Given a projective scheme $W$, $\mathcal{L} \in \text{Pic}(W)$, and an integer $0 \leq q \leq \dim W$, one says that $\mathcal{L}$ is $q$-ample if, for any coherent sheaf $\mathcal{G}$ on $W$, one has the following cohomology vanishing:

$$\exists m \in \mathbb{Z} \ \forall m \geq m_0 \ \forall t > q, \quad H^t(W, \mathcal{G} \otimes \mathcal{L}^m) = 0.$$

(For $q = 0$, $\mathcal{L}$ is ample in the usual sense, while for $q = \dim W$ the condition is vacuous.)

A line bundle which is both $q$-ample and semi-ample—that is, some tensor power is globally generated—is called $q$-positive; the name is justified by the terminology in complex geometry. A summary of these notions can be found in [12, Appendix].

(iii) (cf. [12]) Let $Y \subset W$ be a closed subscheme, $\tilde{W}$ the blow-up of the sheaf of ideals $\mathcal{I}_Y \subset O_W$, and $E_Y$ the exceptional divisor. One says that $Y$ is $q_Y$-ample, for $0 \leq q_Y \leq \dim Y$, if $O_{\tilde{W}}(E_Y)$ is $\delta_Y = (q_Y + \text{codim}_W(Y) - 1)$-ample: for all coherent sheaves $\mathcal{G}$ on $\tilde{W}$ there is an integer $m_\delta$, such that it holds

$$\forall m \geq m_\delta, \ \forall t \geq q_Y + \delta_Y, \quad H^t(\tilde{W}, \mathcal{G} \otimes O_{\tilde{W}}(mE_Y)) = 0.$$

These objects arise in numerous situations; in our context, the most relevant ones are the zero loci of globally generated vector bundles.
The partial amplitude consists of two properties, a local and a global one.

**Proposition** (cf. [12, Proposition 1.4]) Suppose $W$ is smooth and $Y$ is a local complete intersection (lci, for short). Then $Y$ is $q_Y$-ample if and only if the conditions below are satisfied:

$$N_Y/W \text{ is } q_Y\text{-ample; } \text{cd}(W \setminus Y) \leq q_Y + \delta_Y - 1. \quad (1.1)$$

Here $\text{cd}(\cdot)$ stands for ‘cohomological dimension’.

We assume henceforth that the conditions below are satisfied.

**Assumption 1.2**

(i) The conormal bundle of $X$ in $V$ splits into a direct sum of line bundles

$$I_X/I_X^2 = \bigoplus_{j=1}^{\delta} L_j^{-1} \otimes \mathcal{O}_X, \quad L_j \in \text{Pic}(V). \quad (1.2)$$

(Set $L_0 := \mathcal{O}_V$.) We denote by

$$x_j \in \Gamma(X, (I_X/I_X^2) \otimes L_j), \quad 1 \leq j \leq \delta, \quad (1.3)$$

the sections which respectively correspond to the direct summand $\mathcal{O}_X$.

(ii) For all $j = 1, \ldots, \delta$, we have: $L_j^{-1}L_j$ is semi-ample, hence $L_j$ is semi-ample too. (We say that $L_1, \ldots, L_\delta$ are ordered, for short.)

(iii) The subvariety $X \subset V$ is $q_X$-ample, with $q_X \leq \dim V - 2\delta = \dim X - \delta$.

**Remark 1.3** We imposed that the line bundles appearing in the splitting of the conormal bundle (over $X$) actually come from $V$. This is not very far from the full generality, where they would be defined only over $X$: the partial amplitude of $X$ implies that $\text{Pic}(V) \otimes \mathbb{Q} \to \text{Pic}(X) \otimes \mathbb{Q}$ is an isomorphism (for $\delta \geq 3$ and arbitrary $V$, cf. [12, Corollary 1.12]). However, in many situations and especially for subvarieties of rational homogeneous varieties and for zero loci of sections in vector bundles, the isomorphism holds with $\mathbb{Z}$-coefficients (cf. Barth-Larsen’s Lefschetz-type theorems [16]).

For $V = \mathbb{P}^n$, the line bundles $L_j$ are positive multiples of $\mathcal{O}_{\mathbb{P}^n}(1)$, so $q_j = 0$, for all $j$; thus our partial positivity assumption is a weakening of this situation. The semi-ampleness of the successive differences simply means that $L_1, \ldots, L_\delta$ are ordered ‘increasingly’; the condition is automatically satisfied for a suitable permutation, whenever $\text{Pic}(V) \cong \mathbb{Z}$.

The most important information given by the partial amplitude of $X \subset V$ is the upper bound (1.1) on the cohomological dimension of its complement. It substitutes the knowledge of the cohomology ring of $V$, which is necessary for applying the Hauptlemma [10] in order to ensure that certain intersections are non-trivial.

**Lemma 1.4** Let $V, X$ be as above. Suppose $Z \subset V$ is an arbitrary closed subscheme of codimension at most $\delta$. Then $Z$ intersects $X$ non-trivially.

**Proof.** The partial amplitude of $X \subset V$ bounds the cohomological dimension from above:

$$\text{cd}(V \setminus X) \leq q_X + \delta - 1 \leq \dim X - 1. \quad \text{Hence } Z, \text{ which is complete, can’t be contained in the complement of } X. \quad \square$$
Lemma 1.5 Let \( V, X \) be as above. Let \( Y \) be an irreducible, normal, lci subvariety of \( V \) which (strictly) contains \( X \) and is smooth along it, such that
\[
\mathcal{N}^\vee_{X/Y} = \mathcal{J}_{X \subset Y}/\mathcal{J}^2_{X \subset Y} = \bigoplus_{j=d}^\delta \mathcal{L}_j^{-1} \otimes \mathcal{O}_X.
\]
Then the following statements hold.

(i) There is a unique \( y \in \Gamma(Y, \mathcal{J}_{X \subset Y} \otimes \mathcal{L}_d) \) which induces \( x \in \Gamma(X, \mathcal{N}^\vee_{X/V} \otimes \mathcal{L}_d) \) under the natural homomorphism \( \Gamma(Y, \mathcal{L}_d) \to \Gamma(X, (\mathcal{O}_Y/\mathcal{J}^2_{X \subset Y}) \otimes \mathcal{L}_d) \).

(ii) Let \( Z \subset Y \) be the vanishing locus of the section \( y \). Then \( Z \) is still an irreducible, normal, lci variety containing \( X \), and is smooth along it; moreover, we have
\[
\mathcal{N}^\vee_{X/Z} = \mathcal{J}_{X \subset Z}/\mathcal{J}^2_{X \subset Z} = \bigoplus_{j=d+1}^\delta \mathcal{L}_j^{-1} \otimes \mathcal{O}_X.
\]

Proof. (i) Let \( \hat{Y}_X \) be the formal completion of \( Y \) along \( X \). The statement is deduced in two steps.

- We claim that \( \Gamma(Y, \mathcal{L}_d) \to \Gamma(\hat{Y}_X, \mathcal{L}_d \otimes \mathcal{O}_{\hat{Y}_X}) \) is an isomorphism.
  It is sufficient to show that, for all sufficiently large integers \( r \), it holds
  \[
  H^0(Y, \mathcal{J}_{X \subset Y}^r \otimes \mathcal{L}_d) = H^1(Y, \mathcal{J}_{X \subset Y}^r \otimes \mathcal{L}_d) = 0.
  \]
  Note that \( X \subset Y \) is \((q_X + d - 1)\)-ample and \( q_X + d - 1 \leq \dim X - 1 \). Since \( Y \) is a Gorenstein variety (it is lci) and is smooth along \( X \), the blow-up \( \hat{Y} \) of \( X \) is still Gorenstein.
  Then we have, for \( t = 0, 1 \) and \( r \gg 0 \),
  \[
  H^1(Y, \mathcal{J}_{X \subset Y}^r \otimes \mathcal{L}_d) = H^{\dim Y - t}(\hat{Y}, \mathcal{O}_{\hat{Y}}(-rE_X) \otimes \mathcal{L}_d) = 0.
  \]
  The second claim is that the homomorphisms
  \[
  \Gamma(Y, (\mathcal{O}_Y/\mathcal{J}_{X \subset Y}^{r+1}) \otimes \mathcal{L}_d) \to \Gamma(Y, (\mathcal{O}_Y/\mathcal{J}_{X \subset Y}^{r}) \otimes \mathcal{L}_d), \quad r \geq 2,
  \]
  are isomorphisms. Indeed, we have the exact sequence
  \[
  0 \to \text{Sym}^r \left( \mathcal{J}_{X \subset Y}/\mathcal{J}^2_{X \subset Y} \right) \otimes \mathcal{L}_d \to \mathcal{O}_X/\mathcal{J}^{r+1}_{X \subset Y} \otimes \mathcal{L}_d \to \mathcal{O}_X/\mathcal{J}^r_{X \subset Y} \otimes \mathcal{L}_d \to 0.
  \]
  The assumption on the conormal bundle implies that the left-hand side is a direct sum of line bundles \( M^{-1} \) of the form
  \[
  M = (\mathcal{L}_{j_1} \otimes \cdots \otimes \mathcal{L}_{j_r} \otimes \mathcal{L}_d^{-1}) \otimes \mathcal{O}_X, \quad j_1, \ldots, j_r \geq d.
  \]
  Since \( \mathcal{L}_{j_1} \mathcal{L}_d^{-1} \) is semi-ample and \( r \geq 2 \), each such \( M \) is \( q_{j_2} \)-positive, with \( q_{j_2} \leq \dim X - 2 \) (cf. (1.1)).

Overall, \( \Gamma(Y, \mathcal{L}_d) \to \Gamma(X, (\mathcal{O}_Y/\mathcal{J}^2_{X \subset Y}) \otimes \mathcal{L}_d) \) is an isomorphism.

(ii) The scheme \( Z \) contains \( X \), by construction. The section \( x \) can be viewed as the differential of \( y \) along its vanishing locus, so the differential criterion implies that \( Z \) is smooth along \( X \). Thus there is a unique irreducible component \( Z_0 \subset Z \) containing \( X \).
Let $Z'$ be an arbitrary component of $Z$. Since it is the zero locus of a section in a line bundle on $Y$, the components of $Z$ are hypersurfaces in $Y$, $\dim Z' = \dim Y - 1$. Hence $Z$ is lci too. Furthermore, the previous Lemma implies that $Z' \cap X \neq \emptyset$, so $Z' \cap Z_0 \neq \emptyset$; the previous discussion yields $Z = Z_0$.

It remains to show that $Z$ is normal; it’s enough to prove that the singular locus $Z^{\text{sing}} \subset Z$ has codimension at least two. If $\dim Y = \dim X + 1$, then $Z_0 = X$ is smooth. Now suppose $\dim Y \geq \dim X + 2$. Then $Z^{\text{sing}}$ is disjoint of $X$, so we have

$$\dim Z^{\text{sing}} \leq \text{cd}(Y \setminus X) \leq \dim X - 1 \leq \dim Z - 2.$$  

The last statement follows from the exact sequence of conormal bundles. 

By inductively applying the previous lemma, we obtain the following statement which holds in great generality.

**Proposition 1.6** Suppose $V, X$ are as in 1.2. Then there is a flag of irreducible, normal, lci (hence Cohen-Macaulay) subvarieties

$$V = Y_0 \supset Y_1 \supset \cdots \supset Y_{\delta - 1} \supset Y_\delta = X.$$  

Each $Y_j \subset Y_{j-1}$ is the vanishing locus of a section $y_j \in \Gamma(Y_{j-1}, \mathcal{L}_j)$, for $j = 1, \ldots, \delta$, which extends $x_j \in \Gamma(X, \mathcal{N}_{Y/V} \otimes \mathcal{L}_j)$.

The result is not satisfactory because we are interested in complete intersections. The problem is with Lemma 1.5, which doesn’t extend the sections $\{x_j\}_j$ to the whole ambient space. Let us show that this is the only obstruction for concluding that $X$ is a complete intersection.

**Lemma 1.7** Suppose that $v_1 \in \Gamma(V, \mathcal{J}_X \otimes \mathcal{L}_1), \ldots, v_\delta \in \Gamma(V, \mathcal{J}_X \otimes \mathcal{L}_\delta)$ induce the sections $x_1, \ldots, x_\delta$, respectively; let $Y_d$ be their common (scheme-theoretic) vanishing locus. Then $Y_d$ is an irreducible, normal, $d$-codimensional complete intersection in $V$.

Let the ground field be $\mathbb{C}$. The same conclusion holds if, instead of 1.2, $X$ satisfies the following condition: the Poincaré dual of $[X]$ equals $d_X \cdot \chi^d$, where $d_X > 0$ and $\chi \in H^2(V; \mathbb{R})$ is an ample (Kähler) class.

**Proof.** By their very definition, the divisors $D_j := \{v_j = 0\}$ contain $X$, so $Y_d$ contains $X$ too. Moreover, the differential criterion shows that $D_1, \ldots, D_\delta$ intersect transversally along $X$, therefore $Y_d$ is smooth in a neighbourhood of $X$. We deduce that there is a unique irreducible component $Y \subset Y_d$ containing $X$ and this has codimension $d$ in $V$.

We are going to show that $Y_d = Y$ is irreducible. Take some other component $Y'$ of $Y_d$; its dimension is at least $\dim V - d$ and Lemma 1.4 shows that $Y' \cap X \neq \emptyset$. The smoothness of $Y_d$ along $X$ implies that $\text{codim}_V(Y') = d$ and $Y' \supseteq Y$. Thus $Y_d$ is an irreducible, complete intersection subvariety of $V$. The upper bound on the cohomological dimension of $V \setminus X$ forces the singular locus $Y'^{\text{sing}}_d$, which is disjoint of $X$, to have $\dim Y'^{\text{sing}}_d \leq \dim Y_d - 2$. It follows that $Y_d$ is a normal variety.

Now we turn to the second statement. The previous proof applies ad litteram if Lemma 1.4 is still valid: $X$ intersects any closed subscheme $Z$ of codimension at most $\delta$. We may assume that $Z$ is reduced, irreducible, $\delta$ codimensional, so its Poincaré dual belongs to $H^{2\delta}(V; \mathbb{R})$. Then $[Z] \cap (\chi^\delta \cup \chi^{\dim V - 2\delta})$ is the volume of $Z$ with respect to the Kähler form, a strictly positive number. This proves that $[Z] \cap [X] \neq 0$. 

\qed
Extending the sections \( y_j \in \Gamma(Y_{j-1}, \mathcal{L}_j) \) to \( V \) require stronger cohomology vanishing properties for the line bundles on \( V \); the assumptions which will be made in the sequel are imposed by the analysis of the general case. Recall that \( v_1 := y_1 \) is already defined over \( V \). Suppose that \( v_1, \ldots, v_{d-1}, \ d \leq \delta \), as above are already constructed, so we have the Koszul resolution
\[
0 \to \bigwedge^{d-1} N_{d-1}^{\vee} \to \bigwedge^{d-2} N_{d-1}^{\vee} \to \cdots \to N_0^{\vee} \to \mathcal{O}_X \to \mathcal{O}_{Y_{d-1}} \to 0, \quad N_{d-1} := \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_{d-1}.
\]

For lifting \( y_d \), it suffices to ensure that \( H^t \left( V; \bigwedge^d N_{d-1}^{\vee} \otimes \mathcal{L}_d \right) = 0 \), \( 1 \leq t \leq d-1 \). As \( d \) varies, within the brackets appear direct sums of the form \( \mathcal{L}_{j_1}^{-1} \cdots \mathcal{L}_{j_t}^{-1} \mathcal{L}_d \), so one should have
\[
\forall \ j_1 < \cdots < j_t < d \leq \delta, \quad H^{\dim_V - t} \left( V; (\omega_V^{-1} \otimes \mathcal{L}_{j_1}^{-1} \cdots \mathcal{L}_{j_t}^{-1} \mathcal{L}_d)^{-1} \right) = 0. \tag{1.4}
\]
Now we distinguish two alternatives to achieve this matter:
- either by imposing more conditions on the line bundles \( \mathcal{L}_j \);
- or by restricting the type of varieties \( V \) and keeping the line bundles \( \mathcal{L}_j \) arbitrary.

Faltings’ work concerns the projective space, so it enters into the latter category. To justify the following definition, observe that \( \mathbb{P}^n \) has the special feature that the line bundles on it have no intermediate cohomology.

**Definition 1.8** We say that the variety \( V \) is \( s \)-split, for an integer \( 0 < s < \dim V \), if it satisfies the following property:
\[
H^t(V, \mathcal{L}) = 0, \quad \forall 1 \leq t \leq s, \ \forall \mathcal{L} \in \text{Pic}(V).
\]

Any \( n \)-dimensional Fano variety \( V \) with cyclic Picard group is \( (n-1) \)-split. This property holds, in particular, for rational homogeneous varieties \( G/P \), where \( G \) is simple and \( P \) is a maximal parabolic subgroup. More generally, if \( V \) is \( s \)-split and \( X \) is a complete intersection of \( \delta \) ample divisors, \( \dim X \geq 3 \), then \( X \) is \( (s-\delta) \)-split.

**Theorem 1.9** Let \( V, X \) be as in 1.2. Suppose moreover that one of the following two properties is satisfied:
(i) The line bundle \( \omega_V^{-1} \mathcal{L}_1^{-1} \cdots \mathcal{L}_{\delta-1}^{-1} \) is ample; in particular, \( V \) is a Fano variety.
(Loosely speaking, \( X \) is a sufficiently low degree subvariety of \( V \).)
(ii) The ambient variety \( V \) is \( s \)-split and \( X \) has codimension \( \delta \leq s \) in \( V \).

Then \( X \) is a complete intersection in \( V \).

**Proof.** In both cases we claim that the cohomology vanishing (1.4) is satisfied, so the sections \( y_1, \ldots, y_\delta \) extend to the ambient space \( V \) and Lemma 1.7 applies. In the second case there is nothing to prove.

For the first case, we apply the Kodaira vanishing theorem. The successive differences \( \mathcal{L}_{j-1}^{-1} \mathcal{L}_j \) are semi-ample, so it’s enough to have the ampleness of \( \omega_V^{-1} \mathcal{L}_1^{-1} \cdots \mathcal{L}_{\delta-1}^{-1} \); this follows from the hypothesis, since \( \mathcal{L}_j, j \notin \{j_1, \ldots, j_{t-1}\} \), are semi-ample. \( \square \)

It is interesting to note that the first condition appears also in the recent article [3], where the authors study rigidity properties of complete intersections in rational homogeneous varieties (which are Fano). Our result shows that low-codimensional subvarieties are already complete intersections (see also Sections 2.1 and 4 below).
2. Examples

We illustrate our result in several situations.

2.1. Subvarieties of homogeneous varieties. Take $V = G/P$, where $G$ is a semi-simple rational algebraic group and $P$ is a parabolic subgroup. A crucial and highly technical ingredient are Faltings’ upper bounds [9, Satz 5, Satz 7] for the cohomological dimension of the complement $V \setminus X$ and of the amplitude of the tangent bundle of $V$:

$$\text{cd}(V \setminus X) \leq \dim V - \ell + 2\delta - 2, \quad \mathfrak{T}_V \text{ is } (\dim X - \ell)\text{-ample.} \quad (2.1)$$

They involve the quantity $\ell := \min \{ \text{rank}(H) \mid H \text{ is a simple factor of } G \}$. Hence the line bundles $\mathcal{L}_j \otimes \mathcal{O}_X$ in (1.2) are $(\dim X - \ell)$-ample and the equivalence (1.1) shows that any smooth subvariety of $G/P$ is $q_X$-ample, with

$$q_X = \dim X - (\ell - 2\delta + 1). \quad (2.2)$$

By imposing the conditions 1.2, we obtain the upper bound on the codimension of $X$ for which Theorem 1.9 applies:

$$q_X \leq \dim X - \delta \quad \Rightarrow \quad \delta \leq (\ell + 1)/3.$$

Thus the only assumption which remains is that the differences $\mathcal{L}_{j-1}^{-1} \mathcal{L}_j$ should be semi-ample; since $V$ is rational homogeneous, this is the same as being effective.

**Theorem 2.1** Let $V = G/P$ be a rational homogeneous variety. Suppose $X$ is a smooth subvariety of codimension at most $(\ell + 1)/3$ and whose normal bundle splits into a direct sum of line bundles which are ‘ordered’. Then $X$ is a complete intersection in $G/P$.

In the particular case when $G$ is simple and $P$ is maximal, the Picard group of $G/P$ is cyclic and the ordering condition is automatically satisfied. Hence we obtain the result stated in the Introduction. The table below summarizes the situation for the various simple linear algebraic groups.

| group $G$ | $\text{SL}(\ell + 1)$ | $\text{SO}(2\ell + 1)$ | $\text{Sp}(\ell)$ | $\text{SO}(2\ell)$ | $E_6$, $E_7$ | $E_8$ |
|-----------|----------------------|----------------------|------------------|------------------|-------------|-------|
| $\delta \leq \ldots$ | $(\ell + 1)/3$ | $(\ell + 1)/3$ | $(\ell + 1)/3$ | $(\ell + 1)/3$ | 2 | 3 |

**Remark 2.2** We mentioned in the Introduction that, by considering partially ample subvarieties, the range of applications of Faltings’ methods are substantially extended. Now we can justify this claim.

The embedding dimension of the Grassmannian $\text{Gr}(k; \ell + 1)$ is $(\ell + 1)/k$, so the codimension exceeds very much the dimension and [10, Satz 5] doesn’t apply. However, the same criterion for complete intersections holds for subvarieties of codimension up to $(\ell + 1)/3$. Similar statements hold for the groups of type $B_\ell, C_\ell, D_\ell$, which yield the isotropic (orthogonal and symplectic) Grassmannians. Note also that we obtain ‘exotic’ criteria, e.g. for 2-codimensional subvarieties of the Cayley plane and Freudenthal’s variety.

On the lower side, we remark that for $\mathbb{P}^n$ we get the weaker bound $\delta \leq (n + 1)/3$ rather than $n/2$, as Faltings does. This is due to the fact that his proof uses a technical $G3$-criterion (in the terminology of Hironaka-Matsumura) for subvarieties of projective spaces, still due to him. In here, this issue is hidden at the first step of the proof of Lemma 1.5, since the partial amplitude assumption yields the $G3$-property (see [12, Section 2.2] for details).
2.2. **two-codimensional subvarieties.** The interest in them stems from Hartshorne’s conjecture saying that the complete intersections are the only such subvarieties of $\mathbb{P}^n$, $n \geq 6$. However, the same issue can be raised more generally, so one may wonder what information can be extracted from the present work.

Theorem 1.9 yields a flag $V \supset Y_1 \supset Y_2 = X$ of length two. The obstruction for obtaining a complete intersection is the lifting of the section $y_2 \in \Gamma(Y_1, \mathcal{L}_2)$ to the ambient space $V$, so it is sufficient to ensure the vanishing of $H^1(V, \mathcal{L}_1^{-1}\mathcal{L}_2)$ whenever $\mathcal{L}_1^{-1}\mathcal{L}_2$ is semi-ample. Thus one obtains further criteria, specific to this case.

**Corollary 2.3** Suppose $V$ satisfies either one of the following conditions:
- it is 1-split;
- it is a Fano variety;
- it is Frobenius split, compatible with an ample divisor (e.g. toric, spherical variety, cf. [7]).

Let $X \subset V$ be a smooth, 2-codimensional, $(\dim X - 2)$-ample subvariety, such that
$$N_{X/V} = (\mathcal{L}_1 \oplus \mathcal{L}_2) \otimes \mathcal{O}_X, \quad \mathcal{L}_1, \mathcal{L}_2 \in \text{Pic}(V),$$
with $\mathcal{L}_1^{-1}\mathcal{L}_2$ semi-ample. Then $X$ is a complete intersection.

In particular, the statement holds for rational homogeneous varieties $V = G/P$, with $\ell \geq 5$.

3. **On the number of equations defining subvarieties**

Vector bundles on varieties $X \subset \mathbb{P}^n$ with ‘few’ generators are split (cf. [10, Satz 2, 3]); the proof of this fact is based on a general Hauptlemma (recalled below) and then uses the cohomological properties of the projective space. The case of vector bundles on hypersurfaces in $\mathbb{P}^n$ is elaborated in [14]; the knowledge of the cohomology ring, given by the Lefschetz hyperplane theorem, plays a crucial role.

Below we obtain a splitting criterion for vector bundles on subvarieties of homogeneous varieties. The bounds on the codimension and the number of generators determined by our approach are in the same vein as in loc. cit. Although they are most likely not optimal (for $\mathbb{P}^n$ they are not), we did not find references dealing with the topic, in this generality.

**Hauptlemma** Let $0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathcal{F} \to 0$ be an exact sequence of vector bundles on a projective variety $X$, with $\mathcal{A} = \bigoplus_{j=1}^{a} \mathcal{A}_j$, $\mathcal{A}_j \in \text{Pic}(X)$. (We say that $\{\mathcal{A}_j\}$ generate $\mathcal{F}$, for short.)

Take $s \in \text{Hom}(\mathcal{A}_j, \mathcal{A})$ whose $j$th component is the identity, and denote by $Z, W$ the zero loci of $\beta \circ s$ and $\text{pr}_j \circ \alpha$, respectively ($\text{pr}_j$ stands for the projection onto $j$th component). Then the intersection $Z \cap W$ is empty.

Now we are going to specialize this result to $X \subset V = G/P$. For shorthand, let
$$a := \text{rank}(\mathcal{A}), \quad f := \text{rank}(\mathcal{F}), \quad e := \text{rank}(\mathcal{E}).$$

We assume that the line bundles $\mathcal{A}_j$ are ordered:
$$\mathcal{A}_{j-1} \mathcal{A}_j, \ j \geq 2, \text{ are effective.}$$
(Over homogeneous varieties, this is the same as globally generated.)

**Lemma 3.1** Let the situation as above. Suppose that either one of the following inequalities hold: $3\delta + f + a \leq \ell + 1$ or $3\delta + e + a \leq \ell + 1$. 


Then \( F \) is isomorphic, through \( \beta \), to the direct sum of \( f \) line bundles appearing in \( \mathbb{A} \). (Similar statement holds for \( E \).)

**Proof.** Let us assume that the first inequality holds. We argue by induction on the pair \((e, f)\), ordered lexicographically; if \( e = 0 \), there is nothing to prove. The ordering assumption implies that, for \( s \in \text{Hom}(A_1, \mathbb{A}) \) general, the induced section \( \beta \circ s \) has one of the following properties:

- \( Z = \emptyset \).

Then \( \mathcal{F} = \mathcal{F}/(\beta \circ s)A_1 \) is locally free and we have \( 0 \rightarrow \mathcal{E} \rightarrow \bigoplus_{j=2}^{a} A_j \rightarrow \mathcal{F}' \rightarrow 0 \). The induction hypothesis implies (\( f \) decreases) that \( \mathcal{F}' \) splits and so does \( \mathcal{F} \).

- \( Z \subset X \) is smooth, non-empty, and \( Z \cap W = \emptyset \).

In this case, we are going to show that \( W = \emptyset \); that is, \( \mathcal{E} \rightarrow A_1 \) is surjective. Hence the kernel \( \mathcal{E}' \) fits into \( 0 \rightarrow \mathcal{E}' \rightarrow \bigoplus_{j=2}^{a} A_j \rightarrow \mathcal{F} \rightarrow 0 \) (\( e \) decreases), so \( \mathcal{F} \) splits.

So let us prove that \( W \) is empty. Faltings’ bound applied to \( Z \subset X \) yields

\[
\text{cd}(X \setminus Z) \leq \text{cd}(V \setminus Z) \leq \dim V - \ell + 2\delta_Z - 2, \quad \delta_Z = \text{codim}_V(Z) = \delta + f.
\]

(During the inductive process, this is decreasing with \( f \).)

If \( W \neq \emptyset \), then it is a projective subscheme of \( V \) of dimension at least \( \dim X - e \). (In the inductive process, when \( e \) decreases, this quantity increases.) Now observe that the hypothesis yields \( \dim W \geq \text{cd}(V \setminus Z) + 1 \), a contradiction.

If the second inequality of the proposition is satisfied, we dualize the sequence and repeat the previous argument. (Here one starts with a general homomorphism \( A_n^{-1} \rightarrow A^\vee \).)

By applying the lemma to the (co)normal bundle of \( X \) in \( V \), thus \( f = \delta \), we deduce the following result.

**Theorem 3.2** Let \( X \subset V = G/P \) be a \( \delta \)-codimensional subvariety, such that either \( N_{X/V} \) or \( N_{X/V}^\vee \) is generated by \( a \geq \delta \) line bundles (defined over \( V \)) which can be ordered; so \( a - \delta \) is the 'number of relations' between the generators. If either

\[
(a - \delta) \leq \ell + 1 - 5\delta \quad \text{or} \quad (a - \delta) \leq \frac{\ell + 1}{2} - 2\delta,
\]

the normal bundle splits; it is the direct sum of \( \delta \) of these line bundles.

In particular, when \( P \) is maximal parabolic, we obtain the result stated in the Introduction.

4. **ON A RESULT OF Ran**

In this part we further investigate the two-codimensional case. Although it logically belongs to Subsection 2.2, we deferred it here because the methods differ from the previous ones. Keeping in mind Hartshorne’s conjecture, one would like to remove the assumption on the splitting of the normal bundle. In this sense, the references [6, 15, 4, 13] prove that subvarieties \( X \subset \mathbb{P}^n \) of sufficiently low degree are complete intersections; that is, the splitting of the normal bundle is automatically satisfied. Ran’s article, based on the analysis of the secant lines of \( X \), yields effective bounds for the degree. What we are going to show is that his arguments carry over to homogeneous varieties.

Suppose \( V = G/P \), where \( G \) is simple of rank \( \ell \) and \( P \) is a maximal parabolic subgroup; we work over the ground field \( \mathbb{k} = \mathbb{C} \). A basis of the (co)homology group of \( V \) is given by
the Schubert subvarieties; the various incidences between them are encoded in the so-called Hasse-diagram. Note that \( \text{Pic}(V) = H^2(V; \mathbb{Z}) = \mathbb{Z} \cdot \mathcal{O}_V(1) \) is determined by the simple root defining \( P \). The generator \( l \) of \( H_2(V; \mathbb{Z}) \) is represented by lines \( L \) of degree one (with respect to \( \mathcal{O}_V(1) \)), embedded in \( V \); we call them straight lines, in analogy with \( \mathbb{P}^n \). We denote

\[
m := c_1(V) \cdot l - 3.
\]

(The notation is chosen to coincide with the number \( m \) appearing in Ran’s article.)

Let \( X \) be a smooth, 2-codimensional subvariety. Since \( H^4(V; \mathbb{Z}) \) is not necessarily cyclic, we impose that the Poincaré-dual class to \( X \) is a multiple of \( [\mathcal{O}_V(1)]^2 \):

\[
[X] = d_X \cdot \chi^2, \quad \text{with } \chi := [\mathcal{O}_V(1)] \in H^2(V; \mathbb{Z}).
\]

This condition is certainly necessary for concluding that \( X \) is a complete intersection. If \( \text{Pic}(V) \to \text{Pic}(X) \) is an isomorphism, the Hartshorne-Serre construction [1] shows that \( X \) is the vanishing locus of a rank-two vector bundle on \( V \). The table below gives the range of values of \( \ell \) for which the isomorphism between the Picard groups is satisfied (cf. Barth-Lefschetz-type theorems [16]):

| group \( G \) | SL(\( \ell + 1 \)) | SO(2\( \ell + 1 \)) | Sp(\( \ell \)) | SO(2\( \ell \)) | \( E_6, E_7, E_8 \) |
|-----------------|------------------|-------------------|------------|------------------|------------------|
| Pic(\( V \)) \( \xrightarrow{\sim} \) Pic(\( X \)) | \( \ell \geq 6 \) | \( \ell \geq 4 \) | \( \ell \geq 6 \) | \( \ell \geq 5 \) | yes \( \checkmark \) |

So, in this case, we have a locally free resolution

\[
0 \to \mathcal{O}_V(-n) \to N^\vee \to \mathcal{I}_X \to 0,
\]

where \( N \) is a rank-two vector bundle on \( V \), with the following Chern classes:

\[
c_2(N) = [X] = d_X \cdot \chi^2, \quad \det N = \mathcal{O}_V(n),
\]

\[
N \otimes \mathcal{O}_X \simeq N^\vee_X, \quad c_1(N^\vee_X) = \mathcal{O}_X(n).
\]  

Observe that \( n > 0 \), since \( N^\vee_X \) is globally generated and partially ample (cf. (2.1)), being a quotient of \( \mathcal{I}_V \). For \( k \in \mathbb{Z} \), we denote

\[
e(k) := d_X - nk + k^2, \quad \text{so } c_2(N^\vee(k)) = e(k) \cdot \chi^2.
\]

\[
\Delta(N) := c_2(\mathcal{End}(N)) = (4d_X - n^2) \cdot \chi^2, \quad \text{the discriminant}.
\]

**Theorem 4.1** Let the situation be as above and assume that either one of the following conditions is satisfied:

(i) \( d_X \leq m(n - m) \);

(ii) \( \sqrt{d_X} \leq n/2 \leq m \).

Then \( X \subset V \) is a complete intersection.

We are going to show that the argument in [15] generalizes straightforwardly. The only change is that we find more comfortable working with stable maps (in the sense of Kontsevich-Manin, see e.g. [11]) instead of secants, in order to understand the various strata which appear.

Let \( \tilde{V} \xrightarrow{\sigma} V \) be the blow-up along \( X \). The exceptional divisor \( E_X \) is a \( \mathbb{P}^1 \)-bundle over \( X \), in fact \( E_X = \mathbb{P}(N^\vee_X) \xrightarrow{\sigma_X} X \); the fibres of \( \sigma_X \) are called vertical lines and their homology class is denoted by \( \phi \). Note that

\[
H_2(\tilde{V}; \mathbb{Z}) = H_2(V; \mathbb{Z}) \oplus \mathbb{Z} \cdot \phi = \mathbb{Z} \cdot l \oplus \mathbb{Z} \cdot \phi.
\]
The lifting of $l \in H_2(V; \mathbb{Z})$ to $H_2(\bar{V}; \mathbb{Z})$ is given by the pre-image of a straight line which avoids $X$. For $k \geq 0$, denote $b_k = l - k\phi \in H_2(\bar{V}; \mathbb{Z})$; it is determined by the conditions $\sigma_\varphi(b_k) = l$ and $b_k \cdot E_X = k$.

We consider the moduli space $\overline{M}_{0,p}(\bar{V}; b_k)$ of $p$-pointed, genus zero stable maps, which represent the homology class $b_k$; the latter are denoted $(T, t) \Rightarrow \bar{V}$. So $T$ is a quasi-stable curve (tree) $T$ whose components are isomorphic to $\mathbb{P}^1$ and $t = (t_1, \ldots, t_p)$ stands for the markings. The reason for introducing $b_k$ is clear: if a straight line in $\bar{V}$ meets $X$ at $k$ points with multiplicity one, its proper transform in $\bar{V}$ represents $b_k$. Conversely, any $T \Rightarrow \bar{V}$ representing $b_k$ has a unique ‘main’ component $\mathbb{P}^1_{\text{main}} \subset T$ which is mapped by $\sigma_\varphi$ to a straight line in $V$. The other components are sent to vertical lines, thus $u_*[\mathbb{P}^1_{\text{main}}] = b_j$, $j \geq k$.

Let us fix a general point $pt \in V$, in a sense to be precised. For $k \geq 1$, we define

$$\Sigma_k := \overline{M}_{0,1+k}(\bar{V}; b_k) \times \bar{V}^{1+k} (pt \times E^X_k) \subset \overline{M}_{0,1+k}(\bar{V}; b_k).$$

It is analogous to the space of $k$-secants to $X$, passing through $pt$; by abuse of language, we call it the same. Now observe that $\overline{M}_{0,1}(\bar{V}; \phi) = E_X \to X \cong \overline{M}_{0,0}(\bar{V}; \phi)$. Thus, as $k$ increases, we obtain a sequence of morphisms

$$\Sigma_0 \leftarrow \Sigma_1 \leftarrow \ldots \leftarrow \Sigma_k \leftarrow \Sigma_{k+1} \leftarrow \ldots$$

induced by the gluing map

$$\overline{M}_{0,1+(k+1)}(\bar{V}; b_{k+1}) \times \bar{V} \to \overline{M}_{0,k}(\bar{V}; b_k)$$

which inserts a vertical line at the last marked point. It is known that $\overline{M}_{0,0}(\bar{V}; l) = \overline{M}_{0,0}(\bar{V}; b_0)$ is irreducible, normal, of dimension $c_1(V) \cdot l + \dim(V) - 3$, so we have

$$\dim(\Sigma_0) = c_1(V) \cdot l - 2 = m + 1.$$

(We agree henceforth that $\dim \emptyset = -1$.) Clearly, $\Sigma_1 \neq \emptyset$ and, as $k$ increases, at each step one expects the dimension of $\Sigma_k$ to drop by one, since the main component must intersect $X$ at one more point, so

$$\dim(\Sigma_k) \geq m + 1 - k, \; \forall \; k \geq 1.$$

Hence the last non-empty variety should be $\Sigma_m$, a finite set of points. However, this is not necessarily true. We are interested in the case when the process terminates sooner; then, necessarily, at some moment the dimension must drop by two, at least.

**Lemma 4.2** Let $1 \leq j \leq m$ be such that $\text{codim}_{\Sigma_j}(\Sigma_{j+1}) \geq 2$. Then we have $e(j) = 0$.

(The codimension is the maximal codimension the irreducible components.)

**Proof.** In this case, $\dim \Sigma_j \geq 1$ and there is a complete one-dimensional family of $j$-secants to $X$. The same proof as [15, Proposition] yields the conclusion. $\square$

**Lemma 4.3** Suppose $\Delta(N) \leq 0$. Then there is an integer $1 \leq k \leq n/2$, such that $e(k) \geq 0$. Moreover, $\Gamma(V, \mathcal{I}_X(k)) \neq 0$ and therefore we have $\Sigma_{k+1} = \emptyset$.

**Proof.** We distinguish two possibilities.

Case $\Delta(N) = 0$. Then $4d_X = n^2$, so we have $n = 2k$, $c_1(N^\vee(k)) = 0$, $c_2(N^\vee(k)) = 0$. If $N^\vee(k)$ is semi-stable, it is associated to a representation of the fundamental group of $V$, which is trivial, hence $N$ splits into two copies of $O_V(k)$. Otherwise, $N$ is not semi-stable and we proceed as below.
Case $\Delta(N) < 0$  Bogomolov’s theorem implies that $N^\vee$ is not semi-stable, so there is a non-trivial injective homomorphism $\mathcal{O}_V(-k) \to N^\vee$, $k \in \mathbb{Z}$; that is, a section $\nu$ of $N^\vee(k)$. The minimal such integer $k_{\text{min}}$ has the following properties:

- $k_{\text{min}} < n/2$  It is the unstability condition. More precisely, $\mathcal{O}_V(-k_{\text{min}})$ is the first term of the Harder-Narasimhan filtration of $N^\vee$; this uniquely characterises $k_{\text{min}}$.
- $c(k_{\text{min}}) \geq 0$  The minimality of $k_{\text{min}}$ implies that the vanishing locus of $\nu$ is a two-codimensional, possibly empty, subscheme which represents $c_2(N^\vee(k_{\text{min}}))$.
- $k_{\text{min}} \geq 1$  Since $n \geq 1$, the left-hand side of the exact sequence below vanishes

$$0 \to \Gamma(\mathcal{O}_V(k_{\text{min}} - n)) \to \Gamma(N^\vee(k_{\text{min}})) \to \Gamma(J_X(k_{\text{min}})) \to 0,$$

hence we obtain a non-vanishing section $J_X(k_{\text{min}}) \subset \mathcal{O}_V(k_{\text{min}})$.

In both cases, the variety $X$ is contained in the hypersurface $Y$ of degree $k_{\text{min}}$, corresponding to the induced section in $\mathcal{O}_V(k_{\text{min}})$. We claim that $\Sigma_j = \emptyset$, $\forall j \geq k_{\text{min}} + 1$. (The general point $pt \in V$, necessary for defining $\Sigma_0$, is chosen outside of the hypersurface $Y$.) Indeed, let $\tilde{Y} = \sigma^*Y - \epsilon E_X$, $\epsilon \geq 1$, be the proper transform of $Y$ and take $u : \mathbb{P}^1 \to \tilde{V}$ representing the class $\beta_j$. Note that $u_*(\mathbb{P}^1)$ is not contained in $Y$ because $pt \notin Y$, so we have

$$0 \leq u_*[\mathbb{P}^1] \cdot \tilde{Y} = (l - j\phi) \cdot (k_{\text{min}} \chi - \epsilon E_X) = k_{\text{min}} - \epsilon j. \tag*{$\Box$}$$

Proof. (Theorem 4.1) We remark that the hypotheses are made in such a way to ensure that $k_{\text{min}} \leq m$: for (i), this follows from $c(k_{\text{min}}) \geq 0 \geq c(m)$, combined with the monotonicity of the function $j \mapsto c(j)$; for (ii), this is Lemma 4.3. Since $\Sigma_{k_{\text{min}} + 1} = \emptyset$, the discussion preceding Lemma 4.2 implies that $\Sigma_j \subseteq \emptyset$ for $j \leq k_{\text{min}} + 1$; let $j_{\text{min}}$ be minimal with this property, so $1 \leq j_{\text{min}} \leq k_{\text{min}} \leq n/2$. The monotonicity of the function $j \mapsto c(j)$ yields

$$0 = c(j_{\text{min}}) \geq c(k_{\text{min}}) \geq 0 \quad \Rightarrow \quad j_{\text{min}} = k_{\text{min}}, \ c(k_{\text{min}}) = 0.$$

We conclude that $c_2(N^\vee(k_{\text{min}})) = 0$, so the zero locus of the section $\nu$ is empty. One gets an exact sequence of vector bundles

$$0 \to \mathcal{O}_V(-k_{\text{min}}) \to N^\vee \to \mathcal{O}_V(k_{\text{min}} - n) \to 0,$$

which splits, since the relevant $\text{Ext}^1$-group vanishes.

In both cases—$\Delta(N) = 0$ and $\Delta(N) < 0$—the bundle $N$ splits, so $N_{X/V}$ too, and the sections $x_1, x_2$ in (1.3) extend to $V$. By Lemma 1.7, $X$ is a complete intersection. \hfill $\Box$

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