What’s Decidable About Program Verification Modulo Axioms?

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Abstract. We consider the decidability of the verification problem of programs modulo axioms — automatically verifying whether programs satisfy their assertions, when the function and relation symbols are interpreted as arbitrary functions and relations that satisfy a set of first-order axioms. Though verification of uninterpreted programs (with no axioms) is already undecidable, a recent work introduced a subclass of coherent uninterpreted programs, and showed that they admit decidable verification [26]. We undertake a systematic study of various natural axioms for relations and functions, and study the decidability of the coherent verification problem. Axioms include relations being reflexive, symmetric, transitive, or total order relations, functions restricted to being associative, idempotent or commutative, and combinations of such axioms as well. Our comprehensive results unearth a rich landscape that shows that though several axiom classes admit decidability for coherent programs, coherence is not a panacea as several others continue to be undecidable.

1 Introduction

Programs are proved correct against safety specifications typically by induction—the induction hypothesis is specified using inductive invariants of the program, and one proves that the reachable states of the program stays within the region defined by the invariants, inductively. Though there has been tremendous progress in the field of decidable logics for proving that invariants are inductive, finding inductive invariants is almost never fully automatic. And completely automated verification of programs is almost always undecidable.

Programs can be viewed as working over a data-domain, with variables storing values over this domain and being updated using constants, functions and relations defined over that domain. Apart from the notable exception of finite data domains, program verification is typically undecidable when the data domain is infinite. In a recent paper, Mathur et. al. [26] establish new decidability results when the data domain is infinite. Two crucial restrictions are imposed — data domain functions and relations are assumed to be uninterpreted and programs are assumed to be coherent (the meaning of coherence is discussed later in this introduction). The theory of uninterpreted functions is an important theory in SMT solvers that is often used (in conjunction with other theories) to solve feasibility of loop-free program snippets, in bounded model-checking, and to validate verification conditions. The salient aspect of [26] is to show that entire
program verification is decidable for the class of coherent programs, without any user-provided inductive invariants (like loop invariants). While the results of [26] were mainly theoretical, there has been recent work on applying this theory to verifying memory-safety of heap-manipulating programs [27].

Data domain functions and relations used in a program usually satisfy special properties and are not, of course, entirely uninterpreted. The results of [26] can be seen as an approximate/abstraction-based verification method in practice — if the program verifies assuming functions and relations to be uninterpreted, then the program is correct for any data domain. However, properties of the data domain are often critical in establishing correctness. For example, in order to prove that a sorting program results in sorted arrays, it is important that the binary relation $<$ used to compare elements of the array is a total ordering on the underlying data sort. Consequently, constraining the data domain to satisfy certain axioms results in more accurate modeling for verification.

In this paper, we undertake a systematic study of the verification of uninterpreted programs when the data-domains are constrained using theories specified by (universally quantified) axioms. The choice of the axioms we study are guided by two principles. First, we study natural mathematical properties of functions and relations. Second, we choose to study axioms that have a decidable quantifier-free fragment of first order logic. The reason is that even single program executions can easily encode quantifier-free formulae (by computing the terms in variables, and assert Boolean combinations of atomic relations and equality on them). Since we are seeking decidable verification for programs with loops/iteration, it makes little sense to examine axioms where even verification of single executions is undecidable.

**Coherence modulo theories:** Mathur et. al. [26] define a subclass of programs, called coherent programs, for which program verification on uninterpreted domains is decidable; without the restriction of coherence, program verification on uninterpreted domains is undecidable. Since our framework is strictly more powerful, we adapt the notion of coherence to incorporate theories. A coherent program [26] is one where all executions satisfy two properties — memoizing and early-assumes. The memoizing property demands that the program computes any term, modulo congruence induced by the equality assumes in the execution, only once. More precisely, if an execution recomputes a term, the term should be stored in a current variable. The early-assumes restriction demands, intuitively, that whenever the program assumes two terms to be equal, it should do so early, before computing superterms of them.

We adapt the above notion to coherence modulo theories. The memoizing and early-assumes property are now required modulo the equalities that are entailed by the axioms. More precisely, if the theory is characterized by a set of axioms $\mathcal{A}$, the memoizing property demands that if a program computes a term
t and there was another term \( t' \) that it had computed earlier which is equivalent to \( t \) modulo the assumptions made thus far and the axioms \( A \), then \( t' \) must be currently stored in a variable. Similarly, the early-merges condition is also with respect to the axioms — if the program execution observes a new assumption of equality or a relation holding between terms, then we require that any equality entailed newly by it, the previous assumptions and the axioms \( A \) do not involve a dropped term. This is a smooth extension of the notion of coherence from \([20]\); when \( A = \emptyset \), we essentially retrieve the notion from \([20]\).

**Main Contributions**

Our first contribution is an extension of the notion of coherence in \([20]\) to handle the presence of axioms, as described above; this is technically nontrivial and we provide a natural extension.

Under the new notion of coherence, we first study axioms on relations. The EPR (effectively propositional reasoning) \([36]\) fragment of first order logic is one of the few fragments of first order logic that is decidable, and has been exploited for bounded model-checking and verification condition validation in the literature \([33,32,31]\). We study axioms written in EPR (i.e., universally quantified formulas involving only relations) and show that verification for even coherent programs, modulo EPR axioms, is undecidable.

Given the negative result on EPR, we look at particular natural axioms for relations, which are nevertheless expressible in EPR. In particular, we look at reflexivity, irreflexivity, and symmetry axioms, and show that verification of coherent programs is decidable when the interpretation of some relational symbols is constrained to satisfy these axioms. Our proof proceeds by instrumenting the program with auxiliary `assume` statements that preserve coherence and subtle arguments that show that verification can be reduced to the case without axioms; decidability then follows from results established in \([20]\).

We then show a much more nontrivial result that verification of coherent programs remains decidable when some relational symbols are constrained to be transitive. The proof relies on new automata constructions that compute streaming congruence closures while interpreting the relations to be transitive.

Furthermore, we show that combinations of reflexivity, irreflexivity, symmetry, and transitivity, admit a decidable verification problem for coherent program. Using this observation, we conclude decidability of verification when certain relations are required to be strict partial orders (irreflexive and transitive) or equivalence relations.

We then consider axioms that capture total orders and show that they too admit a decidable coherent verification problem. Total orders are also expressible in EPR and their formulation in EPR has been used in program verification, as they can be used in lieu of the ordering on integers when only ordering is important. For example, they can be used to model data in sorting algorithms, array indices in modeling distributed systems to model process ids and the states of processes, etc. \([33,32]\).
Our next set of results consider axioms on functions. Associativity and commutativity are natural and fundamental properties of functions (like $+$ and $\ast$) and are hence natural ways to capture/abstract using these axioms. (See [14] where such abstractions are used in program analysis.) We first show that verification of coherent programs is decidable when some functions are assumed to be commutative or idempotent. Our proof, similar to the case of reflexive and symmetric relations, relies on reducing verification to the case without axioms using program instrumentation that capture the commutativity and idempotence axioms. However, when a function is required to be associative, the verification problem for coherent programs becomes undecidable. This undecidability result was surprising to us.

The decidability results established for properties of individual relation or function symbols discussed above can be combined to yield decidable verification modulo a set of axioms. That is, the verification of coherent programs with respect to models where relational symbols satisfy some subset of reflexivity/ir-reflexivity/symmetry/transitivity axioms or none, and function symbols are either uninterpreted, commutative, or idempotent, is decidable.

Decidability results outlined above, apply to programs that are coherent modulo the axioms/theories. However, given a program, in order to verify it using our techniques, we would also like to decide whether the program is coherent modulo axioms. We prove that for all the decidable axioms above, checking whether programs are coherent modulo the axioms is a decidable problem. Consequently, under these axioms, we can both check whether programs are coherent modulo the axioms and if they are, verify them.

There are several other results that we mention only in passing. For instance, we show that even for single executions, verifying them modulo equational axioms is undecidable as it is closely related to the word problem for groups. And our positive results for program verification under axioms for functions (commutativity, idempotence), also shows that bounded model-checking under such axioms is decidable, which can have its own applications.

Due to the large number of results and technically involved proofs, we give only the main theorems and proof gists for some of these in the paper; details can be found in the Appendix.

2 Illustrative Example

Consider the problem of searching for an element $k$ in a sorted list. There are two simple algorithms for this problem. Algorithm 1 (Fig. 1, left) walks through the list from beginning to end, and if it finds $k$, it sets a Boolean variable exists to $T$. Notice this algorithm does not exploit the sortedness property of the list. Algorithm 2 (Fig. 1, right) also walks through the list, but it stops as soon as it either finds $k$ or reaches an element that is larger than $k$. If it finds the element it sets a Boolean variable found to $T$. If both algorithms are run on the same sorted list, then their answers (namely, exists and found) must be the same.
Fig. 1. Program $P_{\text{check-key}}$ (left) checks if the key $k$ exists in the list starting at $x$ and sets the variable $\text{exists}$ to $T$ if it does. Program $P_{\text{check-key-sorted}}$ (right) checks if the list starting at $x$ contains the key $k$ and works as expected on a sorted list. $<$ is interpreted as a strict total order. The condition $a \leq b$ is shorthand for $a < b \lor a = b$.

Fig. 2 (on the left) shows a program that weaves the above two algorithms together (treating Algorithm 1 as the specification for Algorithm 2). The variable $x$ walks down the list using the $\text{next}$ pointer. The variable $\text{stop}$ is set to $T$ when Algorithm 2 stops searching in the list. The precondition, namely that the input list is sorted, is captured by tracking another variable $\text{sorted}$ whose value is $T$ if consecutive elements are ordered as the list is traversed. The post condition demands that whenever the list is sorted, $\text{found}$ and $\text{exists}$ be equal when the list has been fully traversed. Note that the program’s correctness is specified using only quantifier-free assertions using the same vocabulary as the program.

The program works on a data domain that provides interpretations for the functions $\text{key}$, $\text{next}$, the initial values of the variables, and the relation $<$. When $<$ is interpreted to be a strict total order, the program is correct. However, if $<$ is not interpreted as a total order, then the program may be incorrectly deemed as buggy. To see this, consider the data model shown on the right in Fig. 2. The data domain has 9 elements in its universe, with the functions $\text{next}$ and $\text{key}$ interpreted as shown. Initially, $x, y$ have value $e_1$, NIL is $e_4$, $k$ is $e_7$, $T$ and $\text{sorted}$ are $e_8$, and $F, \text{found, exists, and stop}$ are $e_9$. The interpretation of $<$ is as follows — $e_5 < e_6$, $e_6 < e_7$, and $e_7 < e_5$. Clearly $<$ is not an order, but the program’s sortedness check “$\text{sorted} = T$” will pass. After the entire list is processed, $\text{exists}$ will be set to $T$ when $x = e_3$. On the other hand, $\text{stop}$ will be set to $T$ when $x = e_1$ because $k = e_7 < \text{key}(x)$. Therefore, at the end $\text{found} = F \neq \text{exists}$. The work presented in [26], where all functions and relations are uninterpreted, would therefore declare this program to be incorrect.

The goal of this paper is to explore several natural restrictions on data models and study the problem of verifying coherent programs for them. When $<$ is constrained to be a total order, the program in Fig. 2 is correct and coherent. Our results (see Section 5.5) show that verification of such programs when relations are constrained to be strict total orders is decidable, and hence we can build automatic decision procedures that will correctly verify such programs.
3 Preliminaries

We briefly recall the syntax and semantics of uninterpreted programs and the verification problem modulo axioms. Our presentation closely follows [26].

3.1 Program Syntax

We consider imperative programs with loops over a fixed finite set of variables $V$ and use constant ($C$), function ($F$), and predicate ($R$) symbols belonging to some first order signature $\Sigma = (C,F,R)$. Programs are then given by the syntax below:

$$
\langle \text{stmt} \rangle ::= x := c \mid x := y \mid x := f(z) \mid \text{assume}(\langle \text{cond} \rangle) \mid \text{skip} \mid \langle \text{stmt} \rangle ; \langle \text{stmt} \rangle \\
\text{while}(\langle \text{cond} \rangle) \langle \text{stmt} \rangle \mid \text{if}(\langle \text{cond} \rangle) \langle \text{stmt} \rangle \text{ else } \langle \text{stmt} \rangle
$$

$$
\langle \text{cond} \rangle ::= x = y \mid x = c \mid c = d \mid R(z) \mid \langle \text{cond} \rangle \lor \langle \text{cond} \rangle \mid \neg \langle \text{cond} \rangle
$$

Here, $f \in F$, $R \in R$, $c,d \in C$, $x,y \in V$, and $z$ is a tuple of variables in $V$ and constants in $C$. The syntax allows programs to have assignment statements, conditionals (if-then-else), looping constructs (while) and sequencing. Since constants can be modeled using variables that are never re-assigned, we will assume, without loss of generality, that the programs do not use constants. Further, arbitrary Boolean combinations of atomic predicates can be expressed using the if-then-else construct, and henceforth, we will also assume that all conditionals are atomic (i.e., of the form $x = y$, $x \neq y$, $R(z)$ or $\neg R(z)$).
3.2 Executions and Semantics of Uninterpreted Programs

Executions of programs over $\langle$stnt$\rangle$ are words over the following alphabet

$$\Pi = \{ "x := y", "x := f(z)", "\text{assume}(x = y)", "\text{assume}(x \neq y)", "\text{assume}(R(z))", "\text{assume}(\neg R(z))") \mid x, y, z \text{ are in } V \}$$

For a program $s \in \langle$stnt$\rangle$, the set of executions of $s$, denoted $\text{Exec}(s)$ is a regular language over the alphabet $\Pi$ and is given as follows (similar to [26]).

$$\begin{align*}
\text{Exec}(\text{skip}) &= \varepsilon \\
\text{Exec}(x := y) &= "x := y" \\
\text{Exec}(x := f(z)) &= "x := f(z)" \\
\text{Exec}(\text{assume}(c)) &= "\text{assume}(c)" \\
\text{Exec}(\text{if} c \text{ then } s_1 \text{ else } s_2) &= "\text{assume}(c)" \cdot \text{Exec}(s_1) + "\text{assume}(\neg c)" \cdot \text{Exec}(s_2) \\
\text{Exec}(\text{while } c \{ s \}) &= ["\text{assume}(c)" \cdot \text{Exec}(s_1)]^* \cdot "\text{assume}(\neg c)"
\end{align*}$$

The set of partial executions of $s$ is the set of prefixes of words in $\text{Exec}(s)$ and is also regular.

A data model $\mathcal{M} = (U_{\mathcal{M}}, \llbracket \cdot \rrbracket_{\mathcal{M}})$ for signature $\Sigma$ is a first order structure with a universe $U_{\mathcal{M}}$ of elements and interpretations for the constants ($\{ \llbracket c \rrbracket_{\mathcal{M}} \mid c \in C \}$), functions ($\{ \llbracket f \rrbracket_{\mathcal{M}} \mid f \in \mathcal{F} \}$) and relations ($\{ \llbracket R \rrbracket_{\mathcal{M}} \mid R \in \mathcal{R} \}$). Given a first order structure $\mathcal{M}$ over $\Sigma$ (also referred to as a data model in the rest of the presentation), and an execution $\rho \in \Pi^*$, the semantics of $\rho$ on $\mathcal{M}$ is given by $\text{eval}_{\mathcal{M}} : \Pi^* \times V \rightarrow U_{\mathcal{M}}$ that gives the the valuation of variables in $V$ at the end of an execution, and is defined as follows. Below, we assume that every variable $x \in V$ is associated with a designated constant $\hat{x} \in C$ which denotes its initial value.

$$\begin{align*}
\text{eval}_{\mathcal{M}}(\varepsilon, x) &= \llbracket \hat{x} \rrbracket_{\mathcal{M}} & \text{for every } x \in V \\
\text{eval}_{\mathcal{M}}(\rho \cdot "x := y", z) &= \text{eval}_{\mathcal{M}}(\rho, y) & \text{if } z \text{ is } x \\
\text{eval}_{\mathcal{M}}(\rho \cdot "x := f(z_1, \ldots, z_r)", y) &= \llbracket f \rrbracket_{\mathcal{M}}(\text{eval}_{\mathcal{M}}(\rho, z_1), \ldots, \text{eval}_{\mathcal{M}}(\rho, z_r)) & \text{if } y \text{ is } x \\
\text{eval}_{\mathcal{M}}(\rho \cdot a, x) &= \text{eval}_{\mathcal{M}}(\rho, x) & \text{otherwise}
\end{align*}$$

Example 1. Let us consider the program in Fig. 2. While the program does not strictly obey the syntax of $\langle$stnt$\rangle$, it can be easily transformed into one — all statements of the form if $(c)$ then $s$ else skip can be transformed to if $(c)$ then $s$ else skip. Further, complex assume statements like ‘assume($k = \text{key}(x)$)’ can be transformed using additional variables — in this case to ‘$kx := \text{key}(x)$; assume($k = kx$)’, where $kx$ is a new variable.

Now, let us consider the following execution of this program.

$$\pi = \pi_0 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_1 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_2 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_3 \cdot \text{assume}(x = \text{NIL})$$

This execution corresponds to entering the loop body exactly three times. $\pi_0$ corresponds to the statements executed prior to entering the loop for the first
It is easy to see that infeasible modulo $M$ if for every prefix $\sigma$ the execution holds on the model. More precisely, an execution $\rho$ is said to be irreflexivity — if $\forall \sigma, \pi, z, R(x, y, z) \cdot x \neq y \Rightarrow x < y$ holds in every model $M$ that is feasible in $M$ if there is an $A$-model, denoted $M \models A$, if for every $\varphi \in A$, we have $M \models \varphi$. A formula $\varphi$ is $A$-valid, denoted $A \models \varphi$, if $\varphi$ holds in every model $M$ that satisfies $A$.

An execution $\rho$ is said to be feasible modulo $A$ if there is an $A$-model $M$ such that $\rho$ is feasible in $M$.

**Example 2.** Let us again consider the execution $\pi$ from Example 1. We first observe that $\pi$ is feasible on the model $M$ from Fig. 2 (right).

Now let us consider the set of axioms $\mathcal{A}_{STO}$ that states that the relation symbol $<$ used in the program in Fig. 2 (left) is interpreted to be a strict total order. That is

$$\mathcal{A}_{STO} = \{ \forall x, \neg(x < x), \forall x, y, z. x < y \land y < z \implies x < z, \forall x, y. x = y \lor x < y \lor y < x \}$$

Observe that the model $M$ is not a $\mathcal{A}_{STO}$-model because there is a cyclic dependency $-e_5 < e_6, e_6 < e_7$ and $e_7 < e_5$. Now consider the model $M'$ which differs from $M$ only in the interpretation of $<$ as: $\llbracket \mathcal{A}_{STO} \rrbracket_{M'} = \{ (e_5, e_6), (e_6, e_7), (e_5, e_7) \}$. It is easy to see that $M'$ is an $\mathcal{A}_{STO}$ model and the execution $\pi$ is not feasible on $M'$. In fact, there is no $\mathcal{A}_{STO}$-model on which $\pi$ is feasible, or, as we say, $\pi$ is infeasible modulo $\mathcal{A}_{STO}$.

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$^2$ A ground atomic predicate is of the form $t_1 \sim t_2$, or $R(t_1, \ldots, t_k)$ or $\neg R(t_1, \ldots, t_k)$, where $\sim \in \{ =, \neq \}$, $R$ is a relation symbol, and $t_i$s are ground terms.
3.4 Program Verification Modulo Axioms

We consider programs annotated with post-conditions that are over the following syntax below. Here, $x$, $y$ and $z$ belong to the set of program variables $V$ and $R \in \mathcal{R}$ is a relation symbol in $\Sigma$.

$$\mathcal{L} : \quad \varphi ::= x = y \mid R(z) \mid \varphi \lor \varphi \mid \neg \varphi$$

**Definition 1 (Program Verification Modulo Axioms).** For a program $s$ and a set of axioms $\mathcal{A}$, we say that $s$ satisfies a postcondition $\varphi$ over the syntax $\mathcal{L}$ modulo $\mathcal{A}$ if for every $\mathcal{A}$-model $M$ and for execution $\rho \in \text{Exec}(s)$ that is feasible in $M$, $M$ satisfies $\varphi[\text{eval}_M(\rho,V)/V]$ (i.e., where each variable $x \in V$ is replaced by $\text{eval}_M(\rho,V)$).

We remark that one can alternatively phrase the verification problem stated above in terms of feasibility. That is, a program $s$ satisfies a postcondition $\varphi$ modulo $\mathcal{A}$ iff every execution $\rho$ of $s'$ is infeasible modulo $\mathcal{A}$ (i.e., there is no $\mathcal{A}$-model $M$ such that $\rho$ is feasible in $M$), where $s' = s; \text{assume}(\neg \varphi)$.

4 Coherence Modulo Axioms

In this section we extend the notion of coherence from [26], adapting it to our current setting where we restrict data models using axioms $\mathcal{A}$. We will first recall the notion of terms computed by an execution, which will be used to define the notion of coherence.

4.1 Terms Computed and Assumptions Accumulated by Executions

We will associate a syntactic term with each variable after a partial execution $\rho$. This, intuitively, is the term computed by $\rho$ and stored in $x$. Let $\text{Terms}_\Sigma$ be the set of terms built using constants and functions in $\Sigma$. The term stored in $x$ after $\rho$ is defined inductively on $\rho$ as follows.

\[
\begin{align*}
\text{TEval}(\varepsilon, x) &= \hat{x} & \text{for every } x \in V \\
\text{TEval}(\rho \cdot "x := y", z) &= \text{TEval}(\rho, y) & \text{if } z \text{ is } x \\
\text{TEval}(\rho \cdot "x := f(z_1, \ldots, z_r)", y) &= f(\text{TEval}(\rho, z_1), \ldots, \text{TEval}(\rho, z_r)) & \text{if } y \text{ is } x \\
\text{TEval}(\rho \cdot a, x) &= \text{TEval}(\rho, x) & \text{otherwise}
\end{align*}
\]

The set of terms computed by an execution $\rho$ is $\text{Terms}(\rho) = \{ \text{TEval}(\rho', x) \mid \rho' \text{ is a prefix of } \rho, x \in V \}$.

As an execution proceeds, it accumulates assumptions over the terms it computes, and we will use $\kappa(\rho)$ to denote the assumptions made by the execution $\rho$. In [26], relations are modeled using functions (to Booleans) and hence relational assumes were avoided. In the current exposition, however, we will treat relations as first class objects and the set of assumptions will also include relational
predicates. Formally, $\kappa(\rho)$ is a set of ground predicates over $\Sigma \cup \{=\}$ defined as follows.

$$
\begin{align*}
\kappa(\varepsilon) &= \emptyset \\
\kappa(\sigma \cdot \text{"assume}(x = y)") &= \kappa(\sigma) \cup \{ \text{TEval}(\sigma, x) = \text{TEval}(\sigma, y) \} \\
\kappa(\sigma \cdot \text{"assume}(x \neq y)") &= \kappa(\sigma) \cup \{ \text{TEval}(\sigma, x) \neq \text{TEval}(\sigma, y) \} \\
\kappa(\sigma \cdot \text{"assume}(R(z_1, z_2, \ldots, z_k))") &= \kappa(\sigma) \cup \{ R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k)) \} \\
\kappa(\sigma \cdot \text{"assume}(\neg R(z_1, z_2, \ldots, z_k))") &= \kappa(\sigma) \cup \{ \neg R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k)) \} \\
\kappa(\sigma \cdot \mathbf{a}) &= \kappa(\sigma) \quad \text{otherwise}
\end{align*}
$$

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\kappa(\sigma \cdot \text{"assume}(R(z_1, z_2, \ldots, z_k))") &= \kappa(\sigma) \cup \{ R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k)) \} \\
\kappa(\sigma \cdot \text{"assume}(\neg R(z_1, z_2, \ldots, z_k))") &= \kappa(\sigma) \cup \{ \neg R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k)) \} \\
\kappa(\sigma \cdot \mathbf{a}) &= \kappa(\sigma) \quad \text{otherwise}
\end{align*}
$$

4.2 Coherence

Our definition of coherence modulo axioms is a smooth generalization of the definition of coherence in [26]. The notion of coherence consists of two properties — memoizing and early equality assumes. The memoizing property says, intuitively, when a term $t$ is computed after executing some prefix $\sigma$ of an execution, if $t$ is equivalent to some other term modulo the assumptions made in the execution so far, then $t$ must not have been dropped at the end of $\sigma$, i.e., a program variable must already hold this term. We replace the notion of equivalence of terms in this definition by equivalence modulo the axioms as well.

The notion of early assumes in [26] intuitively says that assumptions of equality (on terms $t_1$ and $t_2$) should be encountered early — earlier than dropping any superterm of $t_1$ or $t_2$. This notion of early assumes allows for effectively computing congruence closure on the set of terms computed by the execution, which in turn, is necessary to accurately maintain which terms are equivalent. However, we observe that the notion in [26] is too restrictive and not entirely necessary. In our paper, we generalize this notion in several ways, to a more semantic one as follows. Whenever an execution encounters an assumption of equality between two terms, we instead demand that only the equivalences that are additionally implied by this new assumption, can be inferred locally using the already known congruence between terms in the window, i.e., the set of terms pointed to by the
program variables when the equality assumption is encountered. Next, we incorporate axioms into this definition, by requiring that the notion of equivalence is also modulo the axioms, and further require that all assumptions (equality, disequality, relational) are required to be early (as against only restricting equality assumptions to be early like in [26]). We will elaborate on these differences using an example after presenting the formal definition next.

Given a set of first order sentences \( \Gamma \) and ground terms \( t_1 \) and \( t_2 \), we say that \( t_1 \equiv_\Gamma t_2 \) if \( \Gamma \vdash t_1 = t_2 \).

**Definition 2 (Coherence modulo axioms).** Let \( \mathcal{A} \) be a set of axioms and let \( \rho \) be a complete or partial execution over variables \( V \). Then, \( \rho \) is said to be coherent modulo \( \mathcal{A} \) if it satisfies the following two properties.

**Memoizing.** Let \( \pi = \sigma \cdot "x := f(x)" \) be a prefix of \( \rho \) and let \( t = TEval(\pi, x) \). If there is a term \( t' \in Terms(\sigma) \) such that \( t' \equiv_{A \cup \kappa(\sigma)} t \), then there must exist some variable \( y \in V \) such that \( TEval(\sigma, y) \equiv_{A \cup \kappa(\sigma)} t \).

**Early Assumes.** Let \( \pi = \sigma \cdot "\text{assume}(c)" \) be a prefix of \( \rho \), where \( c \) is any of \( x = y, x \neq y, R(z), \neg R(z) \). Let \( t \in Terms(\sigma) \) be a term computed in \( \sigma \) such that \( t \) has been dropped, i.e., for every \( x \in V \), we have \( TEval(\sigma, x) \not\equiv_{A \cup \kappa(\sigma)} t \). For any term \( t' \in Terms(\sigma) \), if \( t \equiv_{A \cup \kappa(\sigma)} t' \), then \( t \equiv_{A \cup \kappa(\sigma)} t' \).

**Remark.** We remark that every execution that is coherent as per the definition in [26], is also coherent modulo \( \mathcal{A} = \emptyset \) as in Definition 2. However, the converse is not true and we illustrate this difference below.

**Example 3.** Let us fix \( A = \emptyset \) for this example. Consider the execution \( \rho = \sigma \cdot \text{assume}(x = y) \) where,

\[
\sigma = \text{assume}(x = y) \cdot x' := f(x) \cdot y' := f(y) \cdot x' := f(x') \cdot y' := f(y')
\]

We first observe that the prefix \( \sigma \) is coherent both with respect to the definition in [26] and Definition 2. First there are no superterms of \( \tilde{x} = TEval(\epsilon, x) \) and \( \tilde{y} = TEval(\epsilon, y) \) when the first statement \( \text{assume}(x = y) \) is observed, and thus, this assume is early. Second, even though the statements “\( y' := f(y) \)” and “\( y' := f(y') \)” are computing a term that has been equivalently computed before (modulo the assumption \( \{\tilde{x} = \tilde{y}\} \)), a copy of these terms is available in some program variable (variable \( x' \) in both the cases) at the time of the execution, thus respecting the memoizing restriction.

Now let us discuss the execution \( \rho \). This execution is not coherent with respect to [26]. In particular, the last assume \( \text{assume}(x = y) \) is not early, as superterms \( f(\tilde{x}) \) and \( f(\tilde{y}) \) have been computed but dropped in the prefix \( \sigma \). However, observe that \( f(\tilde{x}) \equiv_{A \cup \kappa(\sigma)} f(\tilde{y}) \) (here, \( A \cup \kappa(\sigma) = \{\tilde{x} = \tilde{y}\} \)) and thus, \( \rho \) meets the early assumes restriction as per Definition 2 making \( \rho \) coherent.

Let us now consider an example which illustrates the notion of coherence in the presence of axioms.
Example 4. Let us now illustrate the notion of coherence in the presence of axioms using the execution \( \rho \) below.

\[
\rho = z_1 := f(x, y) \cdot z_2 := f(y, x) \cdot z_3 := g(z_1) \cdot z_4 := g(z_2) \cdot z_5 := z_6 := g(z_1)
\]

Let \( \rho_i \) denote the prefix of \( \rho \) of length \( i \). Here, \( \text{TEval}(\rho_3, z_3) = g(f(\bar{x}, \bar{y})) \), \( \text{TEval}(\rho_5, z_3) = z_6 \neq g(f(\bar{x}, \bar{y})) \) and \( \text{TEval}(\rho_6, z_6) = g(f(\bar{x}, \bar{y})) \).

When the set of axioms is \( \mathcal{A} = \emptyset \), this execution is not coherent modulo \( \mathcal{A} \) as it violates the memoizing requirement at the last statement \( z_6 := g(z_1) \) (no variable stores the term \( g(f(\bar{x}, \bar{y})) \) after \( \rho_5 \)).

Now, consider the axiom set denoting commutativity of \( f \), i.e., \( \mathcal{A}_{\text{comm}} = \{ \forall u, v. \bar{f}(u, v) = f(u, v) \} \). In this case, we observe that \( f(\bar{x}, \bar{y}) \cong_{\mathcal{A}_{\text{comm}}} f(\bar{y}, \bar{x}) \) and thus \( g(f(\bar{x}, \bar{y})) \cong_{\mathcal{A}_{\text{comm}}} g(f(\bar{y}, \bar{x})) \). Also, \( \text{TEval}(\rho_5, z_4) = g(f(\bar{y}, \bar{x})) \cong_{\mathcal{A}_{\text{comm}}} g(f(\bar{x}, \bar{y})) \). This ensures that \( \rho \) is indeed coherent modulo \( \mathcal{A}_{\text{comm}} \).

Let \( \text{CoherentExecs}(\Sigma, V, \mathcal{A}) \) denote the set of executions over the signature \( \Sigma \) and variables \( V \) that are coherent modulo the set of axioms \( \mathcal{A} \).

Definition 3. A program \( s \) over signature \( \Sigma \) and variables \( V \) is said to be coherent modulo \( \mathcal{A} \) if \( \text{Exec}(s) \subseteq \text{CoherentExecs}(\Sigma, V, \mathcal{A}) \).

In this paper, we explore several classes of axioms, studying when the verification problem for coherent programs modulo the axioms is decidable.

4.3 Recap of results from\([26]\)

We briefly state the main decidability results from\([26]\) about coherent programs, using the notation defined above, so the set of axioms \( \mathcal{A} \) is empty. The results hold even when the early assumes condition is generalized (Definition\([2]\) and relations are treated as first class objects, as we do in this paper.

Theorem 1 (Essentially\([26]\)). Let \( \Sigma \) be a first order signature and \( V \) a finite set of variables. The following observations hold when the set of axioms is empty.

1. There is a finite automaton \( \mathcal{F} \) (effectively constructable) of size \( O(2^{\text{poly}|V|}) \) such that for any coherent execution \( \rho \), \( \mathcal{F} \) accepts \( \rho \) iff \( \rho \) is feasible.
2. There is a finite automaton \( \mathcal{C} \) (effectively constructable) of size \( O(2^{\text{poly}|V|}) \) such that \( L(\mathcal{C}) = \text{CoherentExecs}(\Sigma, V, \emptyset) \).

As a consequence, the following problems are decidable in PSPACE.

- Given a coherent program \( P \), determine if \( P \) is correct.
- Given a program \( P \), determine if \( P \) is coherent.

The problems of verifying coherent programs and checking coherence, are also \( \text{PSPACE-hard} \).

Proof Sketch. These observations have been proved in\([26]\), but the proof is also sketched in Appendix\([A]\) for completeness and to account for the modified definitions. Intuitively, the automata to check feasibility and coherence of executions, track equivalences between program variables, functional and relational correspondences between them that hold based on the assumes seen. Crucial
to establishing the correctness of the automata constructions is the observation that, when the set of axioms is empty, equality of two terms does not depend on disequality and relational assumes seen in the execution. That is, if \( \kappa(\rho)_{eq} \) denotes the set of equality assumes in \( \rho \), then for any computed terms \( t_1, t_2 \),
\[
t_1 \equiv_{\kappa(\rho)} t_2 \iff t_1 \equiv_{\kappa(\rho)_{eq}} t_2.
\]

5 Axioms over Relations

In this section, we investigate the decidability of the verification problem for coherent programs modulo relational axioms, i.e., axioms which only involve relation symbols \( R \) in the signature \( \Sigma \).

5.1 Verification modulo EPR axioms

A first-order formula is said to be an EPR formula [36] if it is of the form
\[
\exists x_1 \ldots x_k \forall y_1, \ldots, y_m \varphi
\]
where \( \varphi \) is quantifier-free and purely relational (uses no function symbols).

It is well known that satisfiability of EPR formulas is decidable, in fact by a reduction to Boolean satisfiability [24]. Consequently, the problem of checking whether a single execution is feasible under axioms written in EPR can be shown to be decidable, and has been exploited in bounded model-checking.

Consequently, we could reasonably ask whether verification of coherent programs under EPR axioms is decidable. Surprisingly, we show that they are not (proof details can be found in Appendix B.1).

**Theorem 2.** Verification of uninterpreted coherent programs modulo EPR axioms is undecidable.

Given the above result, we turn to several classes of quantified axioms, which are all expressible in EPR (and hence have a decidable bounded model checking problem) and examine their decidability for coherent program verification.

5.2 Reflexivity, Irreflexivity, and Symmetry

We consider program verification under the following axioms (individually):

\[
\begin{align*}
\varphi_{\text{refl}}^R & \triangleq \forall x \cdot R(x, x) \\
\varphi_{\text{irref}}^R & \triangleq \forall x \cdot \neg R(x, x) \\
\varphi_{\text{symm}}^R & \triangleq \forall x, y \cdot R(x, y) \implies R(y, x)
\end{align*}
\]

We show that verification is decidable modulo these axioms using a technique that we call *program instrumentation*. Let us fix a relation \( R \) and an axiom \( \varphi_p^R \), where \( p \in \{\text{refl}, \text{irref}, \text{symm}\} \). The idea is to find a function (in fact, a string homomorphism) \( h_p^R \) such that for any program \( P \), \( P \) is correct/coherent modulo
\{ \phi^p_R \} \text{ iff } h^R_p(\text{Exec}(P)) \text{ is correct/coherent modulo the empty axiom set. Decidability then follows by exploiting the results of [26]. The function } h^R_p \text{ will capture the properties of the axiom it is trying to eliminate, and so it will be different for different axioms. We first outline these function } h^R_p \text{, then state their property and prove the decidability result.}

For reflexivity, we transform an execution } \rho \text{ of } P \text{ to } \rho' \text{ where } \rho' \text{ is essentially } \rho, \text{ except that whenever we see the computation of a term, using an assignment of the form } "x := f(z)" , \text{ we immediately insert an assume statement that states that } R(x,x) \text{ holds. More precisely, the homomorphism is defined as,}

\[
h^R_{\text{refl}}(a) = \begin{cases} a \cdot \text{assume}(R(x,x)) & \text{if } a = "x := f(z)" \\ a & \text{otherwise} \end{cases}
\]

The homomorphisms used for irreflexivity and symmetry follow similar lines and are outlined in Appendix B.2.

**Theorem 3.** For any relation symbol } R \text{ and } p \in \{ \text{refl, irref, symm} \}, \text{ the problems of coherent verification modulo } \{ \phi^R_p \} \text{ and checking coherence modulo } \{ \phi^R_p \} \text{ are PSPACE-complete.}

### 5.3 Transitivity

We now consider the transitivity axiom for a relation } R \text{ which says

\[
\phi^{\text{trans}}_R = \forall x, y, z \cdot R(x, y) \land R(y, z) \implies R(x, z) \quad \text{(transitivity)} \tag{2}
\]

The proof for decidability modulo this axiom is different and more complex that the proofs for reflexivity, irreflexivity, and symmetry. Intuitively, the program instrumentation approach does not seem to work for transitivity. This is because transitivity effects can be global. For example, we may have that the execution asserts the sequence of relational assumes \(R(t_1, t_2), R(t_2, t_3), \ldots, R(t_{n-1}, t_n)\) (here, \(t_1, \ldots, t_n\) are terms computed by the execution), where some of the intermediate terms may have been dropped by the program (i.e., the variables holding these terms were reassigned). Consequently, relating \(t_1\) and (the possibly newly constructed term) \(t_n\) requires a principally new machinery. We modify the automaton construction from [26] so that it maintains the transitive closure of the assumptions the program makes. Our main observation is the following:

**Theorem 4.** Let } \Sigma \text{ be a first order signature and } V \text{ a finite set of program variables. Let } \mathcal{A} = \{ \phi^{\text{trans}}_R \mid R \in \mathcal{R}_{\text{trans}} \} \text{ for some set of relation symbol } \mathcal{R}_{\text{trans}} \text{ in } \Sigma. \text{ The following observation hold.}

1. There is a finite automaton } \mathcal{F}_{\text{trans}} \text{ (effectively constructable) of size } O(2^{\text{poly}(|V|)}) \text{ such that for any coherent execution } \rho \text{ that is coherent modulo } \mathcal{A}, \mathcal{F}_{\text{trans}} \text{ accepts } \rho \text{ iff } \rho \text{ is feasible.}

2. There is a finite automaton } \mathcal{C}_{\text{trans}} \text{ (effectively constructible) of size } O(2^{\text{poly}(|V|)}) \text{ such that } L(\mathcal{C}_{\text{trans}}) = \text{CoherentExecs}(\Sigma, V, \mathcal{A}).
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Fig. 3. Implied negative relational assumes for a transitive relation $R$. The dashed edges ($\cdots\cdots$) represent the inferred relationship implied from the relations marked by bold edges ($\rightarrow\rightarrow$).

Proof Sketch. These are in some sense a generalization of the automata constructions used to establish decidability in [26]. The automata $F_{\text{trans}}$ and $C_{\text{trans}}$ rely on tracking equivalence between values stored in variables, and functional and relational correspondences between these values. However, now since some relations maybe transitive, additional relational correspondences (or their absence) maybe implied for $R \in R_{\text{trans}}$. The basic idea is to maintain for transitive relations $R$ (a) the transitive closure of the positive relation assumes $\text{assume}(R(x,y))$, and (b) the negative relational assumes implied by the relational assumes seen in an execution. More precisely, if the execution sees assumes $\text{assume}(R(x,y))$ and $\text{assume}(R(y,z))$, then we also add the constraint $R(x,z)$ in the automaton’s state. Further, if the execution observes $\text{assume}(R(x,y))$ and $\text{assume}(\neg R(x,z))$, then one can infer the constraint $\neg R(y,z)$, and in this case, we accumulate this additional constraint in the state of the automaton. Similarly, if the execution observes $\text{assume}(R(y,z))$ and $\text{assume}(\neg R(x,z))$, then one can infer the constraint $\neg R(x,y)$, which is added in the automaton’s state. Both these scenarios are illustrated in Fig. 3. A detailed proof of this result is given in Appendix B.3.

As a consequence we have the following result.

Theorem 5. For $A = \{ R^R_{\text{trans}} \mid R \in R_{\text{trans}} \}$, the problems of coherent verification modulo $A$ and checking coherence modulo $A$ are PSPACE-complete.

5.4 Strict Partial Orders

We now turn our attention to axioms that dictate that certain relations be partial or total orders. The anti-symmetry axiom that holds for non-strict orders introduces subtle complications. Recall that $R$ is anti-symmetric if $\forall x, y. R(x,y) \land R(y,x) \Rightarrow x = y$; this axiom can imply equality between terms if $R$ holds between a pair of terms. Concretely, if $R$ is anti-symmetric, and the program makes assumptions in an execution that $R(t_1,t_2)$ and $R(t_2,t_1)$ hold, then any model in which such an execution is feasible must also ensure that $t_1 = t_2$. This implicit equality assumption interferes with the notions of coherence and the automata constructions (proofs of the results in [26] and Theorem 5) that compute a congruence closure on terms in a streaming fashion.

Hence, we only consider strict partial orders in this section. Recall that a relation $R$ is a strict partial order if it satisfies the irreflexivity axiom and the
transitivity axiom, together denoted $A_{SPO}^R$. We can prove decidability for problems modulo $A_{SPO}^R$ by using our algorithm for irreflexivity and transitivity.

**Theorem 6.** The following problems are PSPACE-complete.
1. Given a program $P$ that is coherent modulo $A_{SPO}^R$, determine if $P$ is correct.
2. Given a program $P$, determine if $P$ is coherent modulo $A_{SPO}^R$

### 5.5 Strict Total Orders

A relation $R$ is a strict total order if it is a strict partial order and satisfies:

$$\forall x, y : x \neq y \implies R(x, y) \lor R(y, x) \quad \text{(totality)} \quad (3)$$

Strict total orders are again tricky to handle as the axiom for totality can result in implicit equality between terms. For example, if $\neg R(x, y)$ and $\neg R(y, x)$ then it must be the case that $x = y$. However, if we restrict ourselves to executions that only have assumes of the form $\text{assume}(R(x, y))$ and do not have any assumes on $\neg R$, i.e., of the form $\text{assume}(\neg R(x, y))$ then there are no implicit equalities that are entailed.

Unfortunately, in general, program executions can contain negative assumes on $R$ (i.e., assumes of the form $\text{assume}(\neg R(x, y))$). In order to ensure that executions contain only positive assumptions on $R$, we must be careful when identifying executions of programs with conditionals — branches where the assumption $\neg R(x, y)$ holds must be translated to a branch that assumes $R(y, x)$ and a branch that assumes $x = y$.

That is, we modify the following rules defining executions of programs for branch statements; for all other statements, the rules are the same as in Section 3.2.

$$\text{Exec} \left( \text{assume}(\neg R(x, y)) \right) = \text{“assume}(R(y, x))" + \text{“assume}(x = y)"$$

$$\text{Exec} \left( \text{if } R(x, y) \text{ then } s_1 \text{ else } s_2 \right) = \text{“assume}(R(x, y))" \cdot \text{Exec}(s_1) + \text{Exec}(\text{assume}(\neg R(x, y))) \cdot \text{Exec}(s_2)$$

$$\text{Exec} \left( \text{while } R(x, y) \{ s \} \right) = \text{“assume}(R(x, y))" \cdot \text{Exec}(s_1)^* \cdot \text{Exec}(\text{assume}(\neg R(x, y)))$$

After such a translation, executions can now have additional equality assumes even if they did not appear in the program. When we refer to coherent programs, we mean that they are coherent according to the above modified notion of executions. This means for such programs to be coherent, all executions must ensure that the additional equality assumes are early. And when we talk about coherent verification of programs with total orders, we mean verification for programs that are coherent after this transformation.

We observe that in the absence of any assumes of the form $\neg R(x, y)$ the verification problem modulo strict total orders reduces that modulo strict partial orders, giving us the following ($A_{STO}^R$ denote the axioms of irreflexivity, transitivity and totality for the relation $R$).

**Theorem 7.** The problems of coherent verification, and checking coherence modulo $A_{STO}^R$ are PSPACE-complete.
6 Axioms Over Functions

We now discuss computational problems modulo axioms that involve function symbols. The treatment of axioms involving functions in the verification of coherent programs is inherently hard. This is because, like in the case of (nonstrict) partial orders and strict total orders, the axioms along with the assume-steps in the execution, can imply equalities between terms beyond those entailed by just the assume steps in the execution. For example, consider the axiom
\[ \forall x, y \cdot f(x, y) = f(y, x) \]
constraining \( f \) to be a commutative function. Then terms like \( f(f(x, y), z) \) are equal to terms like \( f(z, f(x, y)) \), and hence when building models we must make sure that functions/relations on such terms are defined in the same way. Terms made equivalent by the functional axioms can be syntactically very different, and keeping track of the equivalence on unbounded executions is hard using finite memory. We consider many natural classes of axioms, and proving both positive and negative results that help delineate the decidability/undecidability boundary.

6.1 Associativity

We now consider the associativity axiom for a function \( f \).
\[ \varphi^f_{\text{assoc}} \triangleq \forall x, y, z \cdot f(x, f(y, z)) = f(f(x, y), z) \]  
(associativity)  

We show, surprisingly to us, that coherent verification is undecidable modulo \( \{ \varphi^f_{\text{assoc}} \} \), i.e., even when we have only one axiom that requires only one function to be associative. In fact, the situation is a lot worse — checking the feasibility of even a single (even coherent) execution is undecidable, in the presence of a single associative function. The proof of the following result uses a reduction from the word problem for finitely generated semigroups [35].

**Theorem 8.** Given a a trace \( \rho \) that is coherent modulo \( \{ \varphi^f_{\text{assoc}} \} \), it is undecidable to determine if \( \rho \) is feasible. Therefore, the problem checking if a program \( P \) that is coherent modulo \( \{ \varphi^f_{\text{assoc}} \} \) is undecidable.

6.2 Commutativity

We now consider the commutativity axiom, which is the following
\[ \varphi^f_{\text{comm}} \triangleq \forall x, y \cdot f(x, y) = f(y, x) \]  
(commutativity)  

We augment executions with an auxiliary variable \( v^* \not\in V \) and transform executions using the following homomorphism that uses the auxiliary variable \( v^* \)
\[
h^f_{\text{comm}}(a) = \begin{cases} 
  a \cdot \text{"} v^* := f(y, x) \text{"} \cdot \text{"} \text{assume}(z = v^*) \text{"} & \text{if } a = \text{"} z := f(x, y) \text{"} \\
  a & \text{otherwise}
\end{cases}
\]

We show that the above transformation preserves feasibility and coherence, giving us the following result.

**Theorem 9.** Verification of coherent programs and checking coherence modulo commutativity axioms is decidable and is \text{PSPACE}-complete.
### 6.3 Idempotence

Next we consider the idempotence axiom for a unary function $f$:

$$\varphi_{\text{idem}} \triangleq \forall x \cdot f(x) = f(f(x)) \quad \text{(idempotence)} \quad (6)$$

Again, we show that there is a simple homomorphism $h_{\text{idem}}^f$ that preserves coherence and feasibility (see Appendix C.2) and reduces the verification question to that modulo $\mathcal{A} = \emptyset$, giving:

**Theorem 10.** Verification of coherent programs and checking coherence modulo idempotence axioms is $\text{PSPACE}$-complete.

### 7 Combining Axioms

We have thus far proved decidability results when a relation or functions satisfies certain properties like reflexivity/irreflexivity/symmetry/transitivity or commutativity/idempotence. We now show that all of these results can be combined. That is, we can consider a signature where relations and functions are assumed to satisfy some subset of these properties, and with some being uninterpreted, and the verification problem will remain decidable for coherent programs.

**Theorem 11.** Let $\mathcal{A}$ be a set of axioms where each relation symbol $R$ is either a total order or satisfies some (possibly empty) subset of properties out of reflexivity, irreflexivity, symmetry, transitivity, and each function symbol $f$ satisfies some (possibly empty) subset out of commutativity and idempotence. The verification problem for coherent programs modulo $\mathcal{A}$ is $\text{PSPACE}$-complete.

The proof of the above result proceeds by eliminating axioms one at a time. We first eliminate the relational axioms (reflexivity, irreflexivity, symmetry) in $\mathcal{A}$ using program instrumentation. We then eliminate the functional axioms in $\mathcal{A}$, again using program instrumentation. Our proof relies on this order of elimination of axioms. At this point, the only axioms remaining are those corresponding to transitivity of a subset of relational symbols, which is handled using the automata construction discussed in the proof of Theorem 1.

### 8 Related Work

The theory of equality with uninterpreted functions (EUF) is a widely used theory in many verification applications as it has decidable quantifier free fragment. EUF has been central to advances in verification of microprocessor control [64] and hardware verification [119] and property directed model checking [15]. EUF has been used as a popular abstraction in software verification [23]. Uninterpreted functions have also been studied for equivalence checking and translation validation [54]. Bueno et al [5] demonstrated the effectiveness of uninterpreted programs for verifying SVCOMP benchmarks against control flow properties.
Mathur et al. [26] introduced the class of coherent uninterpreted programs and showed that verification of coherent programs, with or without recursive function calls, is a decidable problem. This is one of the few subclasses of program verification over infinite domains that is known to be decidable. Previous works [13, 14, 30] have established decidability of verification of classes of uninterpreted programs with heavy syntactic restrictions such as disallowing conditionals inside loops or nested loops, etc. As noted in [26], the notion of coherence is close to the notion of a bounded pathwidth decomposition [37]. A term that is created in a coherent execution stays within some program variable (modulo congruence) until the first time all variables containing that term are over-written, and after this point, the execution never computes it again, and thus, the set of windows that contain a term form a contiguous segment of the program execution. Path decomposition and the related notion of tree decomposition have been exploited many times in the literature to give decidability in verification [25, 7, 8].

The work in [27] extends the work of [26] to updatable maps and identifies extensions of coherence that make verification decidable. It utilizes this to provide implementation of verification algorithms for memory safety for a class of heap manipulating programs, including traversal algorithms on data structures such as singly linked list, sorted lists, binary search trees etc. Combining the results of this paper with these results is an interesting future direction.

The class of EPR formulas that consist of universally quantified formulas over relational signatures is a well-known decidable class of first-order logic [36]. EPR-based reasoning has been proved powerful for verification of large-scale systems [32, 28, 38] and the Ivy [33, 29] system is one of the most notable framework that exploits EPR based reasoning for verifying program snippets without recursion. EPR encoding of order axioms such as reflexivity, symmetry, transitivity and total orders has been used in proving programs working over heaps [20].

The work in Kleene Algebra with Tests (KAT) [22] considers problems involving unbounded recursion and choice with abstractions of data, similar to our work. However, while we treat congruence axioms for equality faithfully in our work, it is unclear to us how to express these in KAT or its extensions [21, 23, 9]. Furthermore, the restrictions of coherence studied in [26] and the work here that are based on bounded path-width notions seem very different from studies of decidable problems in KAT. A study of whether our results can be adapted to yield decidable fragments for KAT is an interesting future direction.

A notable verification technique with an automata-theoretic foundation and that has been very effective in practice is that of trace abstraction due to Heizmann et al. [15, 16, 17, 11, 12]. In this technique, one constructs iteratively regular sets that (incompletely) capture the set of all infeasible executions, eventually striving to cover all failing executions of a program, but handling complex theories such as arithmetic. In contrast, our work here builds complete automata in one stroke that accept all infeasible traces over a vocabulary, but handles only simple theories with restricted sets of axioms, but yielding decidability. Combining these two lines of work for more efficient software verification is an interesting future direction.
9 Conclusions

By incorporating axioms on functions and relations, decidability results in this paper, enable a more faithfully automatic verification of programs. It is worth noting that the upper bound for all our decidability results is \( \text{PSPACE} \), which is the same as that for Boolean programs. Thus, though we consider programs over infinite domains with additional structure, our verification results have the same complexity as that for programs over Boolean domains.

One future direction is to adapt this technique for practical program verification. In this context, adapting our technique within the automata-theoretic technique of \[15,17,16,12,10\] seems most promising. Second, there are several program verification techniques that use EPR, and in several of these, EPR is used mainly to establish a linear order on the universe \[20\]. Automatically verifying such programs using our technique is worth exploring.

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A Handling Relations in Streaming Congruence Closure

The work in [26] omit relations and model them as functions. Specifically, all programs are assumed to have two fixed variables $T$ and $F$ (corresponding to Boolean constants true and false) that are never re-assigned. In the beginning of each program, there is an assume $\text{assume}(T \neq F)$. Further, for every $k$-ary relation $R$, there is a function $f_R$ and a variable $b_R$. Every assumption of the form “$\text{assume}(R(z))$” is translated to the sequence “$b_R := f_R(z)$ · “$\text{assume}(b_R = T)$”, and every assumption of the form “$\text{assume}(\neg R(z))$” is translated to the sequence “$b_R := f_R(z)$ · “$\text{assume}(b_R = F)$”.

This approach adds additional program variables and function symbols and further restricts the class of programs that are coherent because the memoizing restriction also applies to the newly introduced function symbols. In this paper, we show how to handle relations as first class symbols without modeling them using function symbols. For this, we will construct an automaton (similar to that in [26]) that accepts coherent executions (modulo the empty set of axioms $\emptyset$) iff they are feasible (modulo $\emptyset$).

Recall that executions are words over the alphabet $\Pi = \{ "x := y", "x := f(z)", "\text{assume}(x = y)", "\text{assume}(x \neq y)", "\text{assume}(R(z))", "\text{assume}(\neg R(z))" | x, y, z \text{ are in } V \}$. Let us denote by $A_{\text{SCC}}$ our automaton for streaming congruence closure. The states $Q_{\text{SCC}}$ are either the special reject state $q_{\text{reject}}$ or tuples of the form $(\equiv, d, P, \text{rel}^+, \text{rel}^-)$, where

- $\equiv$ is an equivalence relation over $V$,
- $d$ is a symmetric and irreflexive binary relation over $V/\equiv$ (equivalence classes of $\equiv$),
- $P$ is such that for every $k$-ary function $f \in \Sigma$, $P(f)$ is a partial mapping from $(V/\equiv)^k \rightarrow V/\equiv$, and
- $\text{rel}^+$ and $\text{rel}^-$ are such that for every $k$-ary relation $R$, $\text{rel}^+(R)$ and $\text{rel}^-(R)$ are sets of $k$-tuples of $V/\equiv$ such that $\text{rel}^+(R) \cap \text{rel}^-(R) = \emptyset$.

Notice that the first three components of the state are similar to [26]. The later two components intuitively accumulate the relational assumes (corresponding to $\gamma(\cdot)$ and $\delta(\cdot)$).

The transition relation $\delta_{\text{SCC}}$ of the automaton is defined as follows. Let $q = (\equiv, d, P, \text{rel}^+, \text{rel}^-)$. If $q = q_{\text{reject}}$, then $\delta_{\text{SCC}}(q, a) = q_{\text{reject}}$ for every $a \in \Pi$ (i.e., $q_{\text{reject}}$ is an absorbing state). Otherwise, we define the state $q' = \delta_{\text{SCC}}(q, a)$ as the tuple $(\equiv', d', P', \text{rel}^+', \text{rel}^-')$ below. In each of these cases, if $d'$ becomes irreflexive or there is a relation $R$ such that $\text{rel}^+(R) \cap \text{rel}^-(R) \neq \emptyset$, then we set $q'$ to be $q_{\text{reject}}$.

$a = "x := y"$.

Here, if $y = x$, $q' = q$. Otherwise, the variable $x$ gets updated to be in the equivalence class of $y$, and $d', P', \text{rel}^+'$ and $\text{rel}^−'$ are updated accordingly:

- $\equiv' := \equiv \cup \{(x, y), (y', x) \mid y' \equiv y\} \cup \{(x, x)\}$.
- $d' = \{(x_1 \equiv', x_2 \equiv) \mid x_1, x_2 \in V \setminus \{x\}, ([x_1] \equiv, [x_2] \equiv) \in d\}$
There are two cases to consider.

Case 2. \( \text{rel} \rightarrow \text{P} \rightarrow \text{rel} \)

Let \( P \) is such that for every \( r \)-ary function \( h \),

\[
P'(h)([x_1]_{=}^r, \ldots, [x_r]_{=}^r) = \begin{cases} 
[u]_{=}^r & x \notin \{u, x_1, \ldots, x_r\} \text{ and } [u]_{=} = P(h)([x_1]_{=}^r, \ldots, [x_r]_{=}^r) \\
\text{undef} & \text{otherwise}
\end{cases}
\]

\( \text{rel}^+ \) is such that for every \( k \)-ary relation \( R \),

\( \text{rel}^+(R) = \{([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \in \text{rel}^+(R)\} \)

\( \text{rel}^- \) is such that for every \( k \)-ary relation \( R \),

\( \text{rel}^-(R) = \{([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \in \text{rel}^-(R)\} \)

\( a = "x := f(z_1, \ldots, z_k)" \).

There are two cases to consider.

1. **Case** \( P(f)([z_1]_{=}^r, \ldots, [z_k]_{=}^r) \text{ is defined.} \)

   Let \( P(f)([z_1]_{=}^r, \ldots, [z_k]_{=}^r) = [v]_{=} \). This case is similar to the case when \( a \) is "\( x := y \)". That is, when \( x \in [v]_{=} \), then \( \equiv' = \equiv, d' = d \) and \( P' = P \).

   Otherwise, we have
   
   - \( \equiv' = \equiv \mid V \setminus \{x\} \cup \{(x, v), (v', x) \mid v' \equiv v\} \cup \{(x, x)\} \)
   - \( d' = \{([x_1]_{=}^r, [x_2]_{=}^r) \mid x_1, x_2 \in V \setminus \{x\}, ([x_1]_{=}^r, [x_2]_{=}^r) \in d\} \)

   \( P' \) is such that for every \( r \)-ary function \( h \),

   \[
P'(h)([x_1]_{=}^r, \ldots, [x_r]_{=}^r) = \begin{cases} 
[u]_{=}^r & x \notin \{u, x_1, \ldots, x_r\} \text{ and } [u]_{=} = P(h)([x_1]_{=}^r, \ldots, [x_r]_{=}^r) \\
\text{undef} & \text{otherwise}
\end{cases}
\]

\( \text{rel}^+ \) is such that for every \( k \)-ary relation \( R \),

\( \text{rel}^+(R) = \{([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \in \text{rel}^+(R)\} \)

\( \text{rel}^- \) is such that for every \( k \)-ary relation \( R \),

\( \text{rel}^-(R) = \{([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{=}^r, \ldots, [x_k]_{=}^r) \in \text{rel}^-(R)\} \)

2. **Case** \( P(f)([z_1]_{=}^r, \ldots, [z_k]_{=}^r) \text{ is undefined.} \)

   In this case, we remove \( x \) from its older equivalence class and make a new class that only contains the variable \( x \). We update \( P \) to \( P' \) so that the function \( f \) maps the tuple \( ([z_1]_{=}^r, \ldots, [z_k]_{=}^r) \) (if each of them is a valid/non-empty equivalence class) to the class \( [x]_{=}^r \). The set \( d' \) follows easily from the new \( \equiv' \) and the older set \( d \). Thus,

   - \( \equiv' = \equiv \mid V \setminus \{x\} \cup \{(x, x)\} \)
   - \( d' = \{([x_1]_{=}^r, [x_2]_{=}^r) \mid x_1, x_2 \in V \setminus \{x\}, ([x_1]_{=}^r, [x_2]_{=}^r) \in d\} \)
   - \( P' \) behaves similar to \( P \) for every function different from \( f \).
• For every $r$-ary function $h \neq f$,

$$P'(h)([x_1]_{\equiv'}, \ldots, [x_r]_{\equiv'}) = \begin{cases} [u]_{\equiv'} & \text{if } u \not\in \{u, x_1, \ldots x_k\} \\
undf & \text{otherwise} \end{cases}$$

• For the function $f$, we have the following.

$$P'(f)([x_1]_{\equiv'}, \ldots, [x_k]_{\equiv'}) = \begin{cases} [x]_{\equiv'} & \text{if } x_i = z_i \forall i \text{ and } x \not\in \{x_1, \ldots x_k\} \\
undf & \text{otherwise} \end{cases}$$

• $rel^+$ is such that for every $k$-ary relation $R$,

$rel^+(R) = \{([x_1]_{\equiv'}, \ldots [x_k]_{\equiv'}) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in rel^+(R))\}$

• $rel^-$ is such that for every $k$-ary relation $R$,

$rel^-(R) = \{([x_1]_{\equiv'}, \ldots [x_k]_{\equiv'}) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in rel^-(R))\}$

$a = \text{"assume}(x = y)"$.

Here, we essentially merge the equivalence classes in which $x$ and $y$ belong and perform the "local congruence closure". In addition, $d'$ and $P'$ are also updated as in [20].

- $\equiv'$ is the smallest equivalence relation on $V$ such that (a) $\equiv \cup \{(x, y)\} \subseteq \equiv'$,

and (b) for every $k$-ary function symbol $f$ and variables $x_1, x'_1, x_2, x'_2, \ldots, x_k, x'_k, z, z' \in V$ such that $[z]_{\equiv} = P(f)([x_1]_{\equiv}, \ldots, [x_k]_{\equiv}), [z']_{\equiv} = P(f)([x'_1]_{\equiv}, \ldots, [x'_k]_{\equiv})$, and $(x_i, x'_i) \in \equiv'$ for each $i$, we have $(z, z') \in \equiv'$.

- $d' = \{(x_1]_{\equiv}, [x_2]_{\equiv}) \mid ([x_1]_{\equiv}, [x_2]_{\equiv} \in d\}$

- $P'$ is such that for every $r$-ary function $h$,

$$P'(h)([x_1]_{\equiv'}, \ldots, [x_r]_{\equiv'}) = \begin{cases} [u]_{\equiv'} & \text{if } [u]_{\equiv} = P(h)([x_1]_{\equiv}, \ldots, [x_r]_{\equiv}) \\
undf & \text{otherwise} \end{cases}$$

- $rel^+$ is such that for every $k$-ary relation $R$,

$$rel^+(R) = \{([x_1]_{\equiv'}, \ldots [x_k]_{\equiv'}) \mid ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in rel^+(R))\}$$

- $rel^-$ is such that for every $k$-ary relation $R$,

$$rel^-(R) = \{([x_1]_{\equiv'}, \ldots [x_k]_{\equiv'}) \mid ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in rel^-(R))\}$$

$a = \text{"assume}(x \neq y)"$.

In this case, $\equiv' = \equiv, P' = P, rel^+ = rel^+$ and $rel^- = rel^-$. Further,

$$d' = d \cup \{([x]_{\equiv'}, [y]_{\equiv'}), ([y]_{\equiv'}, [x]_{\equiv'})\}$$
for the set of equality and positive relational atoms in $\kappa_M$ to define the minimal model. We will use the model.

**Lemma 1.**

If $a = \text{assume}(R(x_1, x_2, \ldots, x_k))$.

In this case, $\equiv' \equiv$, $P' = P$, $d' = d$ and $\text{rel}^\equiv = \text{rel}^\equiv$. Further, $\text{rel}^+(R') = \begin{cases} \text{rel}^+(R) \cup \{([x_1]_{\equiv'}, [x_2]_{\equiv'}, \ldots, [x_k]_{\equiv'})\} & \text{if } R' = R \\ \text{rel}^+(R') & \text{otherwise} \end{cases}$

$a = \text{assume}(\neg R(x_1, x_2, \ldots, x_k))$.

In this case, $\equiv = \equiv$, $P' = P$, $d' = d$ and $\text{rel}^\equiv = \text{rel}^\equiv$. Further, $\text{rel}^-(R') = \begin{cases} \text{rel}^-(R) \cup \{([x_1]_{\equiv'}, [x_2]_{\equiv'}, \ldots, [x_k]_{\equiv'})\} & \text{if } R' = R \\ \text{rel}^-(R') & \text{otherwise} \end{cases}$

We next give a proof gist for the correctness of the automaton construction.

The bulk of the proof is the same as that given in [26]. Here, we only discuss the details necessary to prove the correctness that relates to the relational assumes.

Let us define the notion of a minimal model. Intuitively, this model has the same algebraic structure (same interpretations for constants and functions) as the initial term model as defined in [26]. Further, we also add relations in the minimal model on top of the initial term model. For a set of ground equalities $A$, we will denote by $\mathcal{M}^\text{init}_A = (\mathcal{U}^\text{init}_A, P^\text{init}_A)$ the initial term model given by the congruence induced by $A$.

**Definition 4.** Let $\Gamma = \Gamma_\text{equalities} \cup \Gamma_\text{relations}$ be a set of atomic formulae of the form $(t_1 = t_2) \in \Gamma_\text{equalities}$ or $R(t_1, \ldots, t_k) \in \Gamma_\text{relations}$ where $t_1, \ldots, t_k$ are ground terms over our vocabulary $\Sigma$ and $R$ is a $k$-ary relation in our vocabulary $\Sigma$. The minimal model $\mathcal{M}^\text{min}_\Gamma = (\mathcal{U}^\text{min}_\Gamma, P^\text{min}_\Gamma)$ of $\Gamma$ is defined as follows.

- $\mathcal{U}^\text{min}_\Gamma = \mathcal{U}^\text{init}_\Gamma$.
- $\llbracket c \rrbracket^\text{min} = \llbracket c \rrbracket^\text{init}$ for $c \in C$.
- $\llbracket f \rrbracket^\text{min} = \llbracket f \rrbracket^\text{init}$ for $f \in F$, and
- $\llbracket R \rrbracket^\text{min} = \{\llbracket t_1 \rrbracket^\text{min}, \ldots, \llbracket t_k \rrbracket^\text{min} \mid R(t_1, \ldots, t_k) \in \Gamma_\text{relations}\}$.

For an execution $\rho$, the minimal model of $\rho$ is defined by the minimal model for the set of equality and positive relational atoms in $\kappa(\rho)$ (i.e., we do not include the dis-equality and the negative relational assumes accumulated by $\rho$) to define the minimal model. We will use $\mathcal{M}_\rho = (\mathcal{U}_\rho, P_\rho)$ to denote this minimal model.

Notice that an execution $\rho$ only defines a relation on the set of computed terms and thus, the minimal model never relates elements outside of the set of computed terms using relation symbols. This is formalized below.

**Lemma 1.** Let $\rho$ be an execution and let $\mathcal{M}_\rho$ be the minimal model of $\rho$. Let $(e_1, \ldots, e_k) \in (\mathcal{U}_\rho)^k$ be a tuple of elements in the minimal model such that one of $e_1, \ldots, e_k$ is not computed by the execution (i.e., there is an $1 \leq i \leq k$ such that for every $t \in \text{Terms}(\rho)$, $[t]_\rho = e_i$). Then, we have $(e_1, \ldots, e_k) \not\in [R]_\rho$ for every $k$-ary relation $R$. 

An important property about the minimal model defined above is that there is a relation preserving homomorphism from this model to any other model that satisfies the assumptions in the execution. Formally,

**Lemma 2.** Let \( M = (U_M, \mathbb{I}_M) \) be a first order structure and let \( \rho \) be an execution that is feasible in \( M \). Then, there is a morphism \( h : U_\rho \to U_M \) such that

- \( h([c]_\rho) = [c]_M \) for every constant \( c \),
- \( h([f]_\rho(e_1, \ldots, e_k)) = [f]_M(h(e_1), \ldots, h(e_k)) \) for every \( k \)-ary function \( f \), and
- for every \( e_1, \ldots, e_k \in U_\rho \) and for every \( k \)-ary function, we have
  \[
  (e_1, \ldots, e_k) \in [R]_\rho \iff (h(e_1), \ldots, h(e_k)) \in [R]_M
  \]

Finally, we have that the minimal model is a sufficient to check for feasibility of an execution in some model (of course it is also necessary but that is evident). That is,

**Lemma 3.** Let \( \rho \) be an execution. If there is model \( M \) such that \( \rho \) is feasible in \( M \), then \( \rho \) is feasible in the minimal model \( M_\rho \).

Below, we present necessary inductive hypotheses to prove the correctness of the automaton construction. The full proof of correctness can be re-constructed using the following lemma and those used by Mathur et al in [20].

**Lemma 4.** Let \( \rho \) be an execution that is coherent modulo \( \varnothing \). Let \( q = (\equiv, d, P, \text{ref}^+, \text{ref}^-) \) be the state reached after reading \( \rho \) in the automaton, i.e., \( q = \delta_{\text{SCC}}(q_0, \rho) \). If \( q \neq \text{reject} \), then we have

- for every \( x_1, x_2, \ldots, x_k \in V \) and for every \( k \)-ary relation \( R \), such that \( ([x_1]\equiv, [x_2]\equiv, \ldots, [x_k]\equiv) \notin \text{ref}^+(R) \), we have \( (e_1, e_2, \ldots, e_k) \notin [R]_\rho \), in the minimal model of \( \rho \), where \( e_i = \llbracket T_{\text{Eval}}(\rho, x_i) \rrbracket^\text{min} \).
- for every \( x_1, \ldots, x_k \in V \), and for every \( k \)-ary relation \( R \), we have \( ([x_1]\equiv, [x_2]\equiv, \ldots, [x_k]\equiv) \notin \text{ref}^-(R) \) iff for every model \( M = (U_M, \mathbb{I}_M) \) for which \( \rho \) is feasible in \( M \), we have
  \[
  ([T_{\text{Eval}}(\rho, x_1)]_M, \ldots, [T_{\text{Eval}}(\rho, x_k)]_M) \notin [R]_M
  \]

### B Proofs from Section 5

**B.1 Proof of Theorem 2**

**Proof.** The undecidability is proved through a reduction from Post’s Correspondence Problem (PCP). Recall that PCP is the following problem.

**PCP.** Let \( \Delta = \{a_1, a_2, \ldots, a_k\} \) be a finite alphabet (\(|\Delta| > 2\)). Given strings \( \alpha_1, \alpha_2 \ldots \alpha_N, \beta_1, \beta_2, \ldots, \beta_N \in \Delta^* \) (with \( N > 0 \)), determine if there is a sequence \( i_1, i_2, \ldots, i_M \) such that \( 1 \leq i_j \leq N \) for every \( 1 \leq j \leq M \) and

\[
\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_M} = \beta_{i_1} \cdot \beta_{i_2} \cdots \beta_{i_M}
\]
What's Decidable About Program Verification Modulo Axioms?

(* For $1 \leq p \neq q \leq N - 1$ *)

\begin{verbatim}
assume (z_p \neq z_q);

assume (R(x, y));
assume (z_0 \neq M);
j := z_0;
while \((j \neq M)\) {
    assume (g(j));
    if \((i_j = z_1)\)
        
        \begin{verbatim}
          x' := f(x);
          assume (S(x, x'));
          assume (Q_{a_1}(x'));
          x := x';
        \end{verbatim}
      
        \begin{verbatim}
          y' := f(y);
          assume (S(y, y'));
          assume (Q_{b_1}(y'));
          y := y';
        \end{verbatim}
      
      \begin{verbatim}
        \end{verbatim}
  
    } else {
        \begin{verbatim}
          x' := f(x);
          assume (S(x, x'));
          assume (Q_{a_{N-1}}(x'));
          x := x';
        \end{verbatim}
      
        \begin{verbatim}
          y' := f(y);
          assume (S(y, y'));
          assume (Q_{b_{N-1}}(y'));
          y := y';
        \end{verbatim}
      
      \begin{verbatim}
        \end{verbatim}
  
    } \end{verbatim}
\end{verbatim}

\textbf{Fig. 4.} Program $P_{\text{EPR}}$ for showing verification is undecidable when there are relations and obey EPR axioms in Fig. 5.

It is well known that the PCP problem is undecidable. We will prove that given a PCP instance $I = (\Delta, \alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N)$, we can construct a set of EPR axioms $A$, a program $P_{\text{EPR}}$ that is coherent with respect to $A$, and a postcondition $\phi$ such that $I$ is a YES instance of PCP iff $P_{\text{EPR}}$ does not satisfy $\phi$.

Let us fix a PCP instance $I$. The desired program $P_{\text{EPR}}$ (with post condition $\phi$) is shown in Fig. 4 and the set of EPR axioms $A$ is shown in Fig. 5.

The signature $\Sigma$ consists of unary functions $f, g$ and $s$. The set of relations in $\Sigma$ is

$$\mathcal{R} = \{ R, S \} \cup \{ Q_a \mid a \in \Delta \}.$$

$$\forall x, y_1, y_2 \cdot R(x, y_1) \land R(x, y_2) \implies y_1 = y_2$$  \hspace{1cm} (7)
$$\forall x_1, x_2, y \cdot R(x_1, y) \land R(x_2, y) \implies x_1 = x_2$$  \hspace{1cm} (8)
$$\forall x, y_1, y_2 \cdot S(x, y_1) \land S(x, y_2) \implies y_1 = y_2$$  \hspace{1cm} (9)
$$\forall x_1, y_1, x_2, y_2 \cdot R(x_1, y_1) \land S(x_1, x_2) \land S(y_1, y_2) \implies R(x_2, y_2)$$  \hspace{1cm} (10)
For every $a \in \Delta$, we have $\forall x, y \cdot R(x, y) \land Q_a(x) \implies Q_a(y)$  \hspace{1cm} (11)
For every $a \in \Delta$, we have $\forall x, y \cdot R(x, y) \land Q_a(y) \implies Q_a(x)$  \hspace{1cm} (12)
For every $a \neq b \in \Delta$, we have $\forall x, y \cdot Q_a(x) \implies \neg Q_b(x)$  \hspace{1cm} (13)

Fig. 5. Axioms for the relations used in $P_{EPR}$.

The relations $R$ and $S$ are binary, while the rest are unary relations. The set of variables in the program are

$$V = \{z_1, \ldots, z_{N-1}\} \cup \{x, x', y, y', y, z_0, j, j_1, j, M\}$$

Intuitively, the program constructs two strings that prove that $I$ is a YES instance of PCP — the positions on one string are indexed by the variable $x$ and positions on the second string are indexed by the variable $y$. Variable $M$ intuitively stores the number of $a_i$’s that need to be concatenated to get a solution. The value of $M$ is fixed by the data model; this way of exploiting data models to get “nondeterminism” is key in this reduction. The variable $z_0$ stores “0”, and the variables $z_i$ ($i > 0$) store indices of strings in the input instance $I$. In each iteration of the while-loop, the index of the next pair of strings in the solution is “picked” by applying the (uninterpreted) function $g$; here again the data model that interprets $g$ resolves the non-determinism. Once the index is picked, the appropriate strings are “concatenated”. This happens step-by-step by generating the next index by applying function $f$, and fixing the symbol at that position. Here the relation $Q_a$ plays a role; if $Q_a(x)$ holds then intuitively it means that symbol $a$ appears in position $x$ of the string. Finally, after the next pair is concatenated, the index of the number of strings in the solution (a.k.a. $j$) is “incremented” (by using $s$).

The relations $R$ and $S$ play an important role. $S$ is the successor relation on string positions, and so appropriate assumes on $S$ are inserted whenever $f$ is used. The relation $R$ relates positions of the two constructed strings if the prefix up to that position is identical in the two strings — we start with requiring that the first positions are related by $R$ and our post condition demands that the last two positions are not $R$-related to say that the constructed strings are not a solution to the PCP instance.

The axioms in $A$ ensure that the relations $R, S$, and $Q_a$ are interpreted consistently with the above intuition. Axioms (1) and (2) require that a position in the first/second string is $R$ related to at most one position in the second/first string. Axioms (5) and (6) say that $R$-related positions have the same symbol, while axiom (4) says that if two positions are $R$ related then so are their “successors” (i.e., $S$-related elements). Axiom (3) requires $S$ to behave like a successor
relation — any position as at most one $S$-related position. Finally axiom (7)
intuitively says that there is at most one symbol at any position.

We will now prove the correctness of the reduction outlined in Fig. 4 and
Fig. 5.

Let us first argue why $P_{EPR}$ is coherent modulo the axioms $A$ in Fig. 5.

We first argue that in any execution $\rho$ of $P_{EPR}$, there are no equalities implied
by the relational assumes. The only candidate axioms that might imply equalities
are (1), (2) and (3). In any execution $\rho$, the only relational assumes of the form
$R(a, b)$ that are implied are of the form $R(f^n(\bar{x}), f^n(\bar{y}))$ ($n \geq 0$) and thus for a
given $t_1$, there is a syntactically unique $t_2$ for which $R(t_1, t_2)$ is implied on the
computed set of terms, and thus there is no implied equality using (1) or (2).
Next, the only assumptions of the form $S(f^n(2), f^{n+1}(2))$ ($n \geq 0$ and $z \in \{x, y\}$). Thus, no equality assumes are implied by (3).

Now, the only equality-\textbf{assume} in $\rho$ is the one at the end of the \textbf{while} loop — \textbf{assume}(\textbf{j} = \textbf{M}). At the point where this assume is seen, neither \textbf{j} nor \textbf{M} have
any superterms and thus there are no implied equalities due to this assume.

Let us now see why $\rho$ is memoizing. The terms in $\textbf{j}$ are always growing: $\textbf{s}^n(\textbf{j})$
in the $n^{th}$ iteration. So both the assignments "$\textbf{i} = \textbf{j}$" and "$\textbf{j} := \textbf{s}(\textbf{j})$" are
memoizing as they never recompute terms. The same reasoning also applies
to the terms in $\textbf{x}$ and $\textbf{y}$.

Let us now argue the correctness of the reduction.

$(\Rightarrow)$. Let us assume that the given PCP instance is a YES instance. Then, there
is a sequence $i_1, i_2, \ldots, i_M$ such that $\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_M} = \beta_{i_1} \cdot \beta_{i_2} \cdots \beta_{i_M}$. We can
now construct a model that satisfies the EPR axioms in Fig. 5 and violates the
post condition. In this model, $s$ is the successor function over $\mathbb{N}$, $z_0$ is the
number 0 and $g$ maps $i$ to $i_1$ based on the witness sequence above. Further, $z_r$
is interpreted as the number $r$. The variables $\textbf{x}$ and $\textbf{y}$ map to $\bar{x}$ and $\bar{y}$ respectively,
which are distinct elements. The function $f$ is such that $f^n(\bar{x}) \neq f^n(\bar{y})$ and $f^i(\bar{x}) \neq f^j(\bar{y})$ for every $i \neq j \in \mathbb{N}$ and further $f^i(\bar{x}) \neq f^j(\bar{y})$ for every
$i, j \in \mathbb{N}$. The relations $Q_a$ are interpreted as follows: $Q_a(f^n(\bar{x}))$ holds iff $a$ is the
$n^{th}$ character in the sequence $\alpha_{i_1} \cdots \alpha_{i_M}$ Similarly, $Q_a(f^n(\bar{y}))$ holds iff $a$ is the
$n^{th}$ character in the sequence $\beta_{i_1} \cdots \beta_{i_M}$. Then, since $\alpha_{i_1} \cdots \alpha_{i_M} = \beta_{i_1} \cdots \beta_{i_M}$,
we must have $R(\textbf{x}, \textbf{y})$ at the end of the computation.

$(\Leftarrow)$. In this case we have a feasible execution $\rho$ with the statement \textbf{assume} $R(\textbf{x}, \textbf{y})$ at the end.

Consider the initial term model $T$ for the vocabulary $\Sigma$ (without the relations)
and the starting constants $V = \{\hat{x} \mid x \in V\}$. We show that it is possible
to extend the term model $T$ with interpretations of relations such that the
resulting model $T_{\text{rels}}$ is such that $\rho$ is feasible on $T_{\text{rels}}$. In fact, the extension is the
following model: each binary and unary relation is interpreted to be the smallest
relation that satisfies the \textbf{assume}'s in $\rho$ as well as the EPR axioms. This is well
defined because the assumes on relations in $\rho$ are all positive assumes and all
EPR axioms are monotonic, except possibly the last one, which can be handled easily: \( Q_a(t) \) holds if \( \rho \) explicitly demands it. As can be seen, \( T_{\text{rels}} \) does not violate any negative assume on the relations since there are none. Further, all equality and disequality assumes are unaffected as in \( T_{\text{rels}} \), there are no terms \( t_x, t_y, t_{x1}, t_{x2}, t_{y1}, t_{y2} \) that can be instantiated for variables in the axioms (1), (2) and (3), as these relations are smallest. Thus, \( \rho \) is feasible on \( T_{\text{rels}} \).

Now from this model, we will construct the sequence \( i_1, \ldots, i_M \). The length of this sequence \( M \) will be the number of times the \textbf{while} loop is executed. Clearly, the loop is executed at least once and thus \( M > 0 \). Let \( t_x = f^{n1}(\hat{x}) \) and \( t_y = f^{n1}(\hat{y}) \) be the values of the variables \( x \) and \( y \) (in the term model \( T_{\text{rels}} \)).

We first argue that \( n_1 = n_2 \). Assume on the contrary that \( n_1 < n_2 \) (w.l.o.g.). Then, one can inductively show that \( R(f^{n1}(\hat{x}), f^{n1}(\hat{y})) \); this is because for every \( i < n_1 \), we have \( S(f^i(\hat{x}), f^{i+1}(\hat{x})) \) and \( S(f^i(\hat{y}), f^{i+1}(\hat{y})) \) and \( R(\hat{x}, \hat{y}) \). But then, in the term model we have \( f^{n1}(\hat{y}) \neq f^{n2}(\hat{y}) \) and this violates the assumption at the end of \( \rho \) (because of axiom (1)). Hence, we have \( n_1 = n_2 \).

Now, the sequence \( i_1, \ldots, i_M \) can be deduced by the conditional branches in the while loop: the index \( i_j \) is the index of the branch taken in the \( j \)th iteration. Let \( \alpha = \alpha_{i1} \cdot \alpha_{i2} \cdots \alpha_{iM} \) and \( \beta = \beta_{i1} \cdot \beta_{i2} \cdots \beta_{iM} \). First we note that \( n_1 = |\alpha| \) and \( n_2 = |\beta| \) and thus \( |\alpha| = |\beta| \). Let \( \alpha_n \) and \( \beta_n \) be the \( n \)th characters of \( \alpha \) and \( \beta \) respectively. Then, one can see that \( Q_{\alpha_n}(f^n(\hat{x})) \) and \( Q_{\beta_n}(f^n(\hat{y})) \) hold in the term model. Now, axioms (5), (6) and (7) ensure that \( \alpha_n = \beta_n \). Thus, \( \alpha = \beta \).

\[ \square \]

**B.2 Proof of Theorem 3**

**Homomorphisms for Irreflexivity and Symmetry.** For irreflexivity, whenever we see the computation of a term using an assignment of the form “\( x := f(z) \)”, we insert an assume statement that demands that \( \neg R(x, x) \) holds. That is, we instrument executions using the following homomorphism.

\[ h^R_{\text{irref}}(a) = \begin{cases} 
  a \cdot \text{assume}(\neg R(x, x)) & \text{if } a = "x := f(z)" \\
  a & \text{otherwise}
\end{cases} \]

For the symmetry axiom on a relation \( R \), whenever we see an assumption of the form “\( \text{assume}(R(x, y)) \)”, we insert an assumption that \( R(y, x) \) holds. In other words, we use the following homomorphism.

\[ h^R_{\text{symm}}(a) = \begin{cases} 
  a \cdot \text{assume}(R(y, x)) & \text{if } a = "\text{assume}(R(x, y))" \\
  a \cdot \text{assume}(\neg R(y, x)) & \text{if } a = "\text{assume}(\neg R(x, y))" \\
  a & \text{otherwise}
\end{cases} \]

Proof of Theorem 3 follows from the more general result Theorem 11.

**B.3 Proof of Theorem 5 (Transitivity Axioms)**

In this section, we prove coherence modulo transitivity is decidable. More specifically, let \( R_{\text{trans}} \) be the set of binary relations that are transitive and let \( A_{\text{trans}} = \)
\{ \varphi^R \mid R \in \mathcal{R}_{trans} \}. We will show that the set \text{CoherentExecs}(\Sigma,V,\mathcal{A}_{\text{trans}}) is a regular language:

**Theorem 4.** Let \Sigma be a first order signature and V a finite set of program variables. Let \mathcal{A} = \{ \varphi^R \mid R \in \mathcal{R}_{trans} \} for some set of relation symbol \mathcal{R}_{trans} in \Sigma. The following observation hold.

1. There is a finite automaton \mathcal{F}_{trans} (effectively constructable) of size \(O(2^{\text{poly}(|V|)})\)
   such that for any coherent execution \(\rho\) that is coherent modulo \(\mathcal{A}\), \(\mathcal{F}_{trans}\) accepts \(\rho\) iff \(\rho\) is feasible.
2. There is a finite automaton \(\mathcal{C}_{trans}\) (effectively constructable) of size \(O(2^{\text{poly}(|V|)})\)
   such that \(L(\mathcal{C}_{trans}) = \text{CoherentExecs}(\Sigma,V,\mathcal{A})\).

For this, we modify the automaton construction in Appendix A to accommodate transitive relations.

The states of the automaton are still the same as that described in Appendix A. Further, the transition function \(\delta_{\text{SCC}}\) is such that for a state \(q \neq q_{\text{reject}}\), \(\delta_{\text{SCC}}(q,a)\) is the same as before when \(a \notin \{\text{"assume}(R(x,y))", \text{"assume}(\neg R(x,y))" \mid R \in \mathcal{R}_{trans}\}\). Below we give the modified transitions for these cases.

The intuitive idea behind the modification is as follows. For \(R \in \mathcal{R}_{trans}\) component \(\text{rel}^+(R)\) stores the pairs of equivalence classes which are implied by the transitive closure of the observed assume statements “\text{assume}(R(x,y))”. For example, if the execution observes “\text{assume}(R(x,y))” and “\text{assume}(R(y,z))”, then the component \(\text{rel}^-(R)\) stores the pair \(([x]_\equiv,[z]_\equiv)\) in addition to the pairs \(([x]_\equiv,[y]_\equiv)\) and \(([y]_\equiv,[z]_\equiv)\). Next, for every \(R \in \mathcal{R}_{trans}\), the component \(\text{rel}^-(R)\) also adds additional pairs \((c_1,c_2)\) of equivalence classes for which \(\neg R(c_1,c_2)\) is implied by the positive and negative assumes in the execution. More precisely, if the execution observes \(\text{assume}(R(x,y))\) and \(\text{assume}(\neg R(x,z))\), then one can infer the constraint \(\neg R(y,z)\), and in this case, we also add \(([y]_\equiv,[z]_\equiv)\) in \(\text{rel}^-(R)\) in addition to \(([x]_\equiv,[z]_\equiv)\). Similarly, if the execution observes \(\text{assume}(R(y,z))\) and \(\text{assume}(\neg R(x,z))\), then one can infer the constraint \(\neg R(x,y)\), and in this case, we also add \(([y]_\equiv,[z]_\equiv)\) in \(\text{rel}^-(R)\) in addition to \(([x]_\equiv,[z]_\equiv)\).

Let us now give the formal definition of \(\delta_{\text{SCC}}(q,a)\) when \(q \neq q_{\text{reject}}\) and when \(a \notin \{\text{"assume}(R(x,y))", \text{"assume}(\neg R(x,y))" \mid R \in \mathcal{R}_{trans}\}\). As before, if \(\text{rel}^+(R) \cap \text{rel}^-(R) \neq \emptyset\), we go to the state \(q_{\text{reject}}\).

\[ a = \text{"assume}(R(x,y))" \] .

In this case, \(\equiv' \equiv \equiv, P' = P, d' = d\). Further, \(\text{rel}^+(R') = \text{rel}^+(R')\) and \(\text{rel}^-(R') = \text{rel}^-(R')\) for every \(R' \neq R\). Further,

- \(\text{rel}^+(R)\) is the smallest set such that
  
  (a) \(\text{rel}^+(R) \subseteq \text{rel}^+(R)\), and
  
  (b) is transitively closed, i.e., for all \(x,y,z \in V\) if \(([x]_\equiv',[y]_\equiv') \in \text{rel}^+(R)\) and \(([y]_\equiv',[z]_\equiv') \in \text{rel}^+(R)\) then \(([x]_\equiv',[z]_\equiv') \in \text{rel}^+(R)\).

- \(\text{rel}^-(R)\) is the smallest set such that
  
  (a) \(\text{rel}^-(R) \subseteq \text{rel}^-(R)\), and
  
  (b) for all \(x,y,z \in V\) if \(([x]_\equiv',[y]_\equiv') \in \text{rel}^+(R)\) and \(([x]_\equiv',[z]_\equiv') \in \text{rel}^-(R)\) then \(([y]_\equiv',[z]_\equiv') \in \text{rel}^+(R)\), and
In order to argue correctness, we extend the notion of minimal model to transitivity.

**Definition 5.** Let $\mathcal{A}_{\text{trans}}$ be the set of transitivity axioms on some finite set of binary relations $\mathcal{R}_{\text{trans}}$. Let $\rho$ be an execution and let $A$ be the set of equalities in $\kappa(\rho)$. Let $\mathcal{M}_\rho$ be the minimal model for $\rho$. Define the minimal transitive model (with respect to $\mathcal{R}_{\text{trans}}$) of $\rho$ to be the model $\mathcal{M}_\rho^{\text{trans}} = (U^{\text{trans}}_\rho, \llbracket \cdot \rrbracket^{\text{trans}}_\rho)$ such that $U^{\text{trans}}_\rho = U_\rho$, $\llbracket e \rrbracket^{\text{trans}}_\rho = \llbracket e \rrbracket_\rho$ for every $e \in C$, $\llbracket f \rrbracket^{\text{trans}}_\rho = \llbracket f \rrbracket_\rho$ for every $f \in \mathcal{F}$ and $\llbracket R \rrbracket^{\text{trans}}_\rho = \llbracket R \rrbracket_\rho$ for every $R \in \mathcal{R} \setminus \mathcal{R}_{\text{trans}}$. Further, for every $R \in \mathcal{R}_{\text{trans}}$, define $\llbracket R \rrbracket^{\text{trans}}_\rho$ to be the smallest transitive set containing $\llbracket R \rrbracket_\rho$.

Notice that the execution $\rho$ only defines a relation on the set of computed terms, and thus the transitive closure of the observed assumes also stays with the set of computed terms. This is formalized below.

**Lemma 5.** Let $\rho$ be an execution and let $\mathcal{M}_\rho^{\text{trans}}$ be the minimal transitive model as defined above. Let $e_1, e_2 \in U^{\text{trans}}_\rho$ be elements in the minimal model such that either $e_1$ or $e_2$ is not computed by the execution (i.e., there is an $i \in \{1, 2\}$ such that for every $t \in \text{Terms}(\rho)$, $\llbracket t \rrbracket^{\text{trans}}_\rho \neq e_i$). Then, we have $(e_1, e_2) \notin \llbracket R \rrbracket^{\text{trans}}_\rho$.

An important property about the minimal transitive model defined above is that there is a relation preserving homomorphism from this model to any other model that satisfies the assumptions in the execution and the transitivity axioms. Formally,

**Lemma 6.** Let $\mathcal{M} = (U_\mathcal{M}, \llbracket \cdot \rrbracket_\mathcal{M})$ be a first order model and let $\rho$ be an execution that is feasible in $\mathcal{M}$, modulo $\mathcal{A}_{\text{trans}}$. Then, there is a morphism $h : U^{\text{trans}}_\rho \to U_\mathcal{M}$ such that

- $h(\llbracket f \rrbracket^{\text{trans}}_\rho(e_1, \ldots, e_k)) = \llbracket f \rrbracket_\mathcal{M}(h(e_1), \ldots, h(e_k))$ for every $k$-ary function $f$, and
- for every $e_1, \ldots, e_k \in U^{\text{trans}}_\rho$ and for every $k$-ary function, we have $(e_1, \ldots, e_k) \in \llbracket R \rrbracket^{\text{trans}}_\rho \implies (h(e_1), \ldots, h(e_k)) \in \llbracket R \rrbracket_\mathcal{M}$

Finally, we have that the minimal model is enough to check for feasibility of an execution in some model. That is,
Lemma 7. Let ρ be an execution that is feasible in M. Let A_{trans} be the set of
transitivity axioms for relations in R_{trans}. If there is model M such that ρ is
feasible in M, then ρ is feasible in the minimal model M^\rho_{trans}.

We prove the correctness of the automaton construction by inducting on the
length of the word. For this, we will be using the following inductive invariants.

Lemma 8. Let A_{trans} be the set of transitivity axioms on some finite set of
binary relations R_{trans}. Let ρ be an execution that is coherent modulo A_{trans}. Let
q = (≡, d, P, ref^+, ref^−) be the state reached after reading ρ in the automaton,
i.e., q = δ_{SCC}(q_0, ρ). If q ≠ q_{reject}, then we have (here R ∈ R_{trans})
– for every x, y ∈ V such that ([x]≡, [y]≡) ∉ ref^+ (R), we have (e_x, e_y) ∉
[R]_{trans}^\rho in the minimal model of ρ, where e_x = [TEval(ρ, x)]_{trans}^\rho and e_y =
[R]_{trans}^\rho
– for every x, y ∈ V, ([x]≡, [y]≡) ∈ ref^− (R) iff for every model M = (U_M, []_M)
for which ρ is feasible in M, we have ([TEval(ρ, x)]_M, [TEval(ρ, y)]_M) ∉
[R]_M.

B.4 Proof of Theorem 6

Follows from the more general result Theorem 11

B.5 Proof of Theorem 7

We first observe that when executions have only positive R assumes, checking
properties modulo A^R_{STO} is equivalent to checking properties modulo A^R_{SPO}. This
will allow us to reduce the case of strict total orders to the case of strict partial
orders.

Lemma 9. Let A be a set of first order sentences that do not mention R. Let ρ
be an execution that does not have any symbols of the form “assume(¬R(x, y))”.
Then the following two observations hold.

1. ρ is feasible modulo A ∪ A^R_{STO} iff ρ is feasible modulo A ∪ A^R_{SPO}; note that ρ
   may or may not be coherent.
2. ρ is coherent modulo A ∪ A^R_{STO} iff ρ is coherent modulo A ∪ A^R_{SPO}.

Proof. We first argue about feasibility. One direction is obvious: if ρ is feasible
modulo A ∪ A^R_{STO}, then ρ is feasible modulo A ∪ A^R_{SPO}. Let us consider the other
direction. Let M = (U_M, []_M) be a model in which ρ is feasible, such that
M ⊨ A ∪ A^R_{SPO}. Notice that [R]_M is a partial order on U_M. Let S be any linear
extension of [R]_M. Consider the model M’ = (U_M’, []_{M’}), where U_M’ = U_M
and for every constant symbol c, function symbol f and relation symbol Q
different from R, we have [c]_{M’} = [c]_M, [f]_{M’} = [f]_M and [Q]_{M’} = [Q]_M.
Finally, let [R]_{M’} = S. First, observe that M’ ⊨ A because no sentence in A
mentions R. Second, M’ ⊨ A^R_{STO} by construction. Finally, [R]_{M’} ≤ [R]_{M’}
and thus M’ ⊨ κ(ρ).
Let us now note that for any two computed terms $t_1, t_2 \in \text{Terms}(\rho)$, we have $t_1 \equiv_{A \cup A_{\text{STO}}^R} t_2$ iff $t_1 \equiv_{A \cup A_{\text{SPO}}^R} t_2$. The proof of this observation is similar to the proof of Lemma 16 and is skipped.

Now, based on the above observations, one can easily conclude that $\rho$ is coherent modulo $A \cup A_{\text{STO}}^R$ iff $\rho$ is coherent modulo $A \cup A_{\text{SPO}}^R$.

The proof of Theorem 7 follows from Theorem 6 and Lemma 9.

C Proofs from Section 6

C.1 Proof of Theorem 8

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$P_{\text{assoc}}$:

(* generate $u_1$ *)

(* $u_{1,i}$ is the $i^{th}$ letter in $u_1$ *)

$x_{1,2} := f(u_{1,1}, u_{1,2})$;
$x_{1,3} := f(x_{1,2}, u_{1,3})$;

(* generate $u_N$ *)

$x_1 := x_{1,|u_1|}$;

(* generate $v_1$ *)

$y_{1,2} := f(v_{1,1}, v_{1,2})$;
$y_{1,3} := f(y_{1,2}, v_{1,3})$;

(* generate $v_N$ *)

$y_1 := y_{1,|v_1|}$;

(* assume $u_1 = v_1$ *)

assume($x_1 = y_1$);

(* generate $u_2$ *)

(* generate $v_2$ *)

(* assume $u_2 = v_2$ *)

assume($x_2 = y_2$);

(* generate $u_0$ *)

$x_{0,2} := f(u_{0,1}, u_{0,2})$;
$x_{0,3} := f(x_{0,2}, u_{0,3})$;

(* assume $v_N = v_{v_N}$ *)

assume($x_N = y_N$);

(* generate $v_0$ *)

$y_{0,2} := f(v_{0,1}, v_{0,2})$;
$y_{0,3} := f(y_{0,2}, v_{0,3})$;

(* assume $x_0 \neq y_0$ *)

$y_0 := y_{0,|v_0|}$;

Fig. 6. Execution $P_{\text{assoc}}$ for showing checking feasibility of a single coherent execution with one associative function is undecidable.
To prove Theorem 8, we recall a classical computational problem called the word problem for a semi-group. Recall that a semi-group is an algebra consisting of a universe on which a single associative binary operation (often denoted \( \circ \)) is defined. A semi-group \( S \) is generated from a finite set \( \Delta \), every element in the universe of \( S \) can be constructed starting from \( \Delta \) using the operation \( \circ \). The word problem over semi-groups is the following.

**Word Problem over Semi-Groups.** Let \( \Delta \) be a finite set and \( \circ \) be the concatenation operation. Given word identities \( u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n \), and an additional identity \( u_0 = v_0 \), determine if for any semi-group \( S \) generated from \( \Delta \) in which the identities \( u_i = v_i \), for \( 1 \leq i \leq n \) hold, whether \( u_0 = v_0 \) holds.

This problem is known to be undecidable.

**Theorem 12 (Post’47 [35]).** Word problem for finitely generated semigroups is undecidable.

Using Post’s result, we prove undecidability to check the feasibility of a single coherent execution.

We show the following reduction. Given an instance \( I = (\Delta, \circ, u_1, v_1, \ldots, u_n, v_n, u_0, v_0) \) there is an execution \( \rho \) that is coherent modulo \( \{ \varphi^f_{\text{assoc}} \} \) such that \( I \) is a YES instance of the work problem iff \( \rho \) is infeasible modulo \( \{ \varphi^f_{\text{assoc}} \} \).

The constructed execution \( \rho \) is shown in Fig. 6. The signature \( \Sigma \) consists of a binary function \( f \) which obeys the associativity axiom \( \varphi^f_{\text{assoc}} \). The set of variables in the program are

\[
\mathcal{V} = \{a_1, \ldots, a_k\} \cup \bigcup_{i=0}^{N} \{x_i, x_{i,2}, \ldots, x_{i,|u_i|}\} \cup \bigcup_{i=1}^{N} \{y_i, y_{i,2}, \ldots, y_{i,|v_i|}\}
\]

The post-condition \( \phi \) is \( x_0 = y_0 \).

Our reduction uses the associative function \( f \) to model concatenation. A word \( w = a_1, \ldots, a_m \) is modeled as the term \( t_w = f(a_1, f(a_2, \ldots, f(a_{m-1}, a_m) \ldots)) \).

Intuitively, the execution first creates the words \( u_1, v_1, u_2, v_2, \ldots, u_N, v_N \) and assumes \( u_1 = v_1, u_2 = v_2, \ldots, u_N = v_N \). It then creates the words \( u_0, v_0 \) and checks \( u_0 = v_0 \) in the postcondition. Proof that \( \rho \) is coherent and the reduction is correct is straightforward, but for completeness, the proof can be found in Appendix C.1.

We prove that the execution \( \rho \) shown in Fig. 6 is coherent and the reduction is correct.

Let us first argue why \( \rho_{\text{assoc}} \) is coherent modulo associativity of \( f \). This follows because all created terms are being retained in some program variables.

Now, we argue the correctness of the reduction.

\( (\Leftarrow) \). Assume that the given instance of the word problem is a NO instance. Then, there is a semi group \( (A, \circ) \) and a homomorphism \( h : S \rightarrow A \) such that for each \( 1 \leq i \leq N \), \( h(u_i) = h(v_i) \) and \( h(u_0) \neq h(v_0) \). Then, the model \( M = (U_M, [f]_M) \) with \( U_M = A \) and \( [f]_M = \circ \) is the model on which \( \rho_{\text{assoc}} \) is feasible. Further, \([f]_M\) is associative and all the assumptions in \( \rho_{\text{assoc}} \) hold in the model.
(...). Assume the execution $\rho_{\text{assoc}}$ is feasible modulo associativity. That is, there is a model $M = (U_M, \llbracket M \rrbracket)$ such that $\llbracket f \rrbracket_M$ is associative and all the assumes in the execution are true in the model. Then, clearly $U_M$ with $\llbracket f \rrbracket_M$ as concatenation is a semigroup. Further, there is a homomorphism $h$ from $S$ to $A = (U_M, \llbracket f \rrbracket_M)$ given by $h(a_i) = \llbracket a_i \rrbracket_M$ for every $a_i \in \Delta$. Since the string $u_0$ and $v_0$ are not equal in $A$, the equality $u_0 = v_0$, the given instance is a NO instance of the word problem.

C.2 Proofs of Theorem 9 and Theorem 10

Homomorphism for Idempotence. We use an auxiliary variable $v^* \not\in V$ and use the following homomorphism.

$$h^\ast_{\text{idem}}(a) = \begin{cases} a \cdot "v^* := f(y)" \cdot "\text{assume}(y = v^*)" & \text{if } a = "y := f(x)" \\ a & \text{otherwise} \end{cases}$$

Proof of Theorem 9 and Theorem 10 Follows from the more general result Theorem 11.

D Combinations

Definition 6 (Closure of binary relations). Let $R \subseteq S \times S$ be a binary relation on a set $S$. Let $p \in \{\text{refl, irref, symm, trans}\}$ and let $\varphi^R_p$ be the axiom of reflexivity, irreflexivity, symmetry or transitivity of $R$ (depending upon what $p$ is). Then, the $p$-closure of $R$, denoted $p(R)$ is defined as

- the smallest binary relation $R' \subseteq S \times S$ such that $R \subseteq R'$ and $R'$ satisfies $\varphi^R_p$, if $p \in \{\text{refl, symm, trans}\}$, or
- the largest binary relation $R' \subseteq S \times S$ such that $R \supseteq R'$ and $R'$ satisfies $\varphi^R_p$, if $p = \text{irref}$

Definition 7 (Relational Closure Extension). Let $M = (U_M, \llbracket M \rrbracket)$ be a first order model over the signature $\Sigma = (C, F, R)$. Let $R \in \mathcal{R}$ be a binary relation and let $p \in \{\text{refl, symm, irref, trans}\}$. The $(p, R)$-closure of $M$, denoted $\text{ClosureExt}^R_p(M)$ is the model $M' = (U_{M'}, \llbracket M' \rrbracket)$, where $U_{M'} = U_M$,

- for every $c \in C$, we have $\llbracket c \rrbracket_{M'} = \llbracket c \rrbracket_M$,
- for every $f \in F$, we have $\llbracket f \rrbracket_{M'} = \llbracket f \rrbracket_M$, and
- for every $Q \in \mathcal{R} \setminus \{R\}$, we have $\llbracket Q \rrbracket_{M'} = \llbracket Q \rrbracket_M$, and
- $\llbracket R \rrbracket_{M'} = p(\llbracket R \rrbracket_M)$.

We define the $(p, R)$-transitive closure of $M$, denoted $\text{TransClosureExt}^R_p(M)$ to be the model $M' = \text{ClosureExt}^R_{\text{trans}}(\text{ClosureExt}^R_p(M))$.

Proposition 1. Let $\Sigma$ be a FO signature, $R \in \mathcal{R}$, $p \in \{\text{refl, symm, irref, trans}\}$ and let $M$ be a $\Sigma$-structure. Let $t$ be a term over the signature $\Sigma$. Then, $\llbracket t \rrbracket_M = \llbracket t \rrbracket_{\text{ClosureExt}^R_p(M)} = \llbracket t \rrbracket_{\text{TransClosureExt}^R_p(M)} \in U_M$. 
Definition 8 (Invariance Under Relational Closure Extension). Let $A$ be a set of first order sentences over $\Sigma$. Let $R \in \mathcal{R}$ be a binary relation and let $p \in \{\text{refl, irref, symm, trans}\}$. $A$ is said to be invariant under $(p, R)$-closure extension if for every first order structure $M$, we have

$$M \models A \implies \text{ClosureExt}^R_p(M) \models A$$

Similarly, $A$ is said to be invariant under $(p, R)$-transitive closure extension if for every first order structure $M$, we have

$$M \models A \implies \text{TransClosureExt}^R_p(M) \models A$$

Lemma 10 (Preservation of invariance under unions). Let $A_1, A_2$ be two sets of first order sentences over $\Sigma$. Let $R \in \mathcal{R}$ be a binary relation and let $p \in \{\text{refl, irref, symm, trans}\}$. If both $A_1$ and $A_2$ are invariant under $(p, R)$-closure extension, then so is $A_1 \cup A_2$.

Proof. Follows easily from definitions. \hfill \square

Lemma 11 (Invariance Under Relational Closure Extensions). Let $\Sigma = (\mathcal{C}, F, \mathcal{R})$ be a FO signature, let $R \in \mathcal{R}$ be a binary relation and let $p \in \{\text{refl, irref, symm}\}$. Then we have the following.

1. The empty set of axioms $A = \emptyset$ is invariant under $(p, R)$-closure extension.
2. The singleton set $A = \{\phi\}$ ($\phi$ is a FO sentence over $\Sigma$) is invariant under $(p, R)$-closure extension if one of the following holds:
   a. $\phi$ does not syntactically mention the symbol $R$.
   b. $p = \text{refl}$ and $\phi \in \{R^{\text{refl}}(\phi)^R, R^{\text{symm}}(\phi)^R, R^{\text{trans}}(\phi)^R\}$
   c. $p = \text{irref}$ and $\phi \in \{R^{\text{irref}}(\phi)^R, R^{\text{symm}}(\phi)^R, R^{\text{trans}}(\phi)^R\}$
   d. $p = \text{symm}$ and $\phi \in \{R^{\text{symm}}(\phi)^R, R^{\text{trans}}(\phi)^R\}$
3. The singleton set $A = \{\phi\}$ is invariant under $(p, R)$-transitive-closure extension if $p = \text{symm}$ and $\phi = \phi^{R^{\text{trans}}}$.

Definition 9 (1-element extensions). Let $M = (U_M, \llbracket M \rrbracket)$ be a first order model over the signature $\Sigma = (\mathcal{C}, F, \mathcal{R})$. The one variable extension $1\text{Ext}(M)$ of $M$ is another model $M' = (U_{M'}, \llbracket M' \rrbracket)$, where $U_{M'} = U_M \cup \{e_*\}$, where $e_* \notin U_M$ is a fresh element, and

- for every $c \in \mathcal{C}$, we have $[c]_{M'} = [c]_M \in U_{M'}$,
- for every $R \in \mathcal{R}$ of arity $r$, we have $[R]_{M'} = [R]_M$,
- for every $f \in F$ of arity $r$, we have

$$\llbracket f \rrbracket_{M'}(e_1, \ldots, e_r) = \begin{cases} \llbracket f \rrbracket_M(e_1, \ldots, e_r) & \text{if } e_* \notin \{e_1, \ldots, e_r\} \\ e_* & \text{otherwise} \end{cases}$$

The above is well defined, in that for every model $M$, there is a unique (upto first order isomorphism\footnote{More precisely, for every first order structure $M$ over $\Sigma$, for two 1 element extensions $M_1$ and $M_2$ of $M$, and for every first order formula $\phi$ over $\Sigma$, $M_1 \models \phi$ iff $M_2 \models \phi$.}) one element extension $1\text{Ext}(M)$. 
Proposition 2. Let \( \Sigma \) be a FO signature and let \( M \) be a \( \Sigma \) structure. Let \( t \) be a term over the signature \( \Sigma \). Then, 
\[ \llbracket t \rrbracket_M = \llbracket t \rrbracket_{1\text{Ext}(M)} \in U_M. \]

Definition 10 (Invariance Under 1-element Extension). Let \( A \) be a set of first order sentences over \( \Sigma \). \( A \) is said to be invariant under 1-element extension if for every first order structure \( M \), we have 
\[ M \models A \implies 1\text{Ext}(M) \models A. \]

Lemma 12 (Preservation of invariance under unions). Let \( A_1, A_2 \) be two sets of first order sentences over \( \Sigma \). If both \( A_1 \) and \( A_2 \) are invariant under 1-element extensions, then so is \( A_1 \cup A_2 \).

Proof. Follows easily from definitions.

Lemma 13 (Invariance Under One Element Extension). Let \( \Sigma = (C, F, R) \) be a FO signature. Then,

1. The empty set of axioms \( A = \emptyset \) is invariant under 1-element extensions.
2. Let \( \varphi \) be a FO sentence and let \( A = \{ \varphi \} \). Then \( A \) is invariant under 1-element extensions if one of the following holds:
   a. \( \varphi \) is quantifier free. That is \( \varphi \) is a boolean combination of ground equality atoms or predicate atoms.
   b. \( \varphi \in \{ \varphi_{\text{comm}}, \varphi_{\text{idem}} \} \), where \( f \in F \) is either a unary or binary function.
   c. \( \varphi = \varphi_{\text{trans}}^R \), where \( R \in R \) is a binary relation.

Lemma 14. Let \( R \in R \) be a binary relation, \( f \in F \) be a unary function, \( g \in F \) be a binary function and let \( \rho \) be an execution. Then for every variable \( x \),

1. \( TEval(\rho, x) = TEval(h^R_p(\rho), x) \) for \( p \in \{ \text{refl}, \text{irref}, \text{symm} \} \),
2. \( TEval(\rho, x) = TEval(h^f_{\text{idem}}(\rho), x) \), and
3. \( TEval(\rho, x) = TEval(h^g_{\text{comm}}(\rho), x) \)

Proof. The only difference between \( \rho \) and \( h^R_p(\rho) \) is the fact that \( h^R_p(\rho) \) has additional assumes. The observation therefore, follows.

Corollary 1. For \( p \in \{ \text{refl}, \text{irref}, \text{symm} \} \) and any execution \( \rho \), \( \kappa(\rho) \subseteq \kappa(h^R_p(\rho)) \).

Definition 11 (Execution-restriction). Let \( \rho \) be an execution, \( R \in R \) be a relation and \( M \) be a model over which \( \rho \) is feasible. The \((R, \rho)\)-restriction of \( M \), denoted \( \text{ExecRest}_R(M, \rho) \) is the model \( M' = (U_{M'}, \llbracket \cdot \rrbracket_{M'}) \), where \( U_{M'} = U_M \),

- for every \( c \in C \), we have \( \llbracket c \rrbracket_{M'} = \llbracket c \rrbracket_M \),
- for every \( f \in F \), we have \( \llbracket f \rrbracket_{M'} = \llbracket f \rrbracket_M \), and
- for every \( Q \in R \setminus \{ R \} \), we have \( \llbracket Q \rrbracket_{M'} = \llbracket Q \rrbracket_M \), and
- \( \llbracket R \rrbracket_{M'} = \{ \llbracket t_1 \rrbracket_M, \ldots, \llbracket t_k \rrbracket_M \ \mid \ R(t_1, \ldots, t_k) \in \kappa(\rho) \} \).
Lemma 15 (Preservation of Feasibility). Let \( \rho \) be an execution, \( R \in \mathcal{R} \) and \( p \in \{ \text{refl}, \text{irref}, \text{symm} \} \). Let \( \mathcal{A} \) be a set of first order sentences over \( \Sigma \) such that if \( p = \text{symm} \) and \( \varphi^R_{\text{trans}} \in \mathcal{A} \), then the only other sentences in \( \mathcal{A} \) that mention \( R \) are \( \varphi^R_{\text{refl}}, \varphi^R_{\text{irref}} \) or \( \varphi^R_{\text{trans}} \). Further, assume that \( \mathcal{A} \) is invariant under \((p,R)\)-closure extension (or \((p,R)\)-transitive closure extension if \( \varphi^R_{\text{trans}} \in \mathcal{A} \)). \( \rho \) is feasible modulo \( \mathcal{A} \) if \( h^R_\mathcal{A} \) is feasible modulo \( \mathcal{A} \).

Proof. (\( \Rightarrow \)). Let \( \rho \) be feasible modulo \( \mathcal{A} \cup \{ \varphi^R_p \} \). Then, there is a model \( \mathcal{M} \) such that \( \mathcal{M} \models \mathcal{A} \cup \{ \varphi^R_p \} \) and \( \mathcal{M} \models \kappa(\rho) \). Now, we first note that all the ground predicates \( \phi \in \kappa(h^R_\mathcal{A}(\rho)) \setminus \kappa(\rho) \), we have that \( \mathcal{M} \models \phi \) because \( \mathcal{M} \models \{ \varphi^R_p \} \). This means, that \( h^R_\mathcal{M}(\rho) \) is feasible in \( \mathcal{M} \), thus implying that \( h^R_\mathcal{A}(\rho) \) is feasible modulo \( \mathcal{A} \).

(\( \Leftarrow \)). Let \( \rho' = h^R_\mathcal{A}(\rho) \) be feasible modulo \( \mathcal{A} \). Then, there is a model \( \mathcal{M} \) such that \( \mathcal{M} \models \mathcal{A} \) and \( \mathcal{M} \models \kappa(h^R_\mathcal{A}(\rho)) \).

Now we define a model \( \mathcal{M}' \) as follows.

1. If \( p \neq \text{symm} \) or \( \varphi^R_{\text{trans}} \notin \mathcal{A} \), then \( \mathcal{M}' = \text{ClosureExt}^R_{\mathcal{A}}(\mathcal{M}) \).
2. If \( p = \text{symm} \), \( \varphi^R_{\text{trans}} \in \mathcal{A} \) and \( \varphi^R_{\text{refl}} \notin \mathcal{A} \), then \( \mathcal{M}' = \text{TransClosureExt}^R_{\text{symm}}(\mathcal{M}'') \), where \( \mathcal{M}' = \text{ClosureExt}^R_{\text{symm}}(\mathcal{M}, h^R_\text{symm}(\rho))) \).
3. If \( p = \text{symm} \), \( \varphi^R_{\text{trans}} \in \mathcal{A} \) and \( \varphi^R_{\text{refl}} \in \mathcal{A} \), then \( \mathcal{M}' = \text{TransClosureExt}^R_{\text{irref}}(\mathcal{M}'') \), where \( \mathcal{M}' = \text{TransClosureExt}^R_{\text{irref}}(\text{ExecRestr}^R(\mathcal{M}, h^R_\text{symm}(\rho))) \).

Observations about \( \mathcal{M}'' \). We first make some simple observations about the intermediate model \( \mathcal{M}'' \) defined in the last two cases above.

1. First, \( [R]_{\mathcal{M}''} \) is a symmetric relation.
2. Second, \( \mathcal{M}'' \models \mathcal{A} \). To see this, observe that \( \mathcal{M}' \models \mathcal{A}' \), where \( \mathcal{A}' \) is the subset of \( \mathcal{A} \) that do not contain sentences involving \( R \). Further, \( \mathcal{M}' \) is transitively (and also reflexively, as appropriate) closed and the only sentences that mention \( R \) are reflexivity, irreflexivity or transitivity axioms.
3. Third, \( [R]_{\mathcal{M}''} \subseteq [R]_{\mathcal{M}} \).
4. Fourth, \( \mathcal{M}'' \models \kappa(h^R_\text{symm}(\rho)) \). This is because all terms evaluate to the same elements in \( \mathcal{M} \) and \( \mathcal{M}'' \) so all equality and disequality assumes in \( \kappa(h^R_\text{symm}(\rho)) \) hold in \( \mathcal{M}'' \). Similarly, for every relation \( Q \) different from \( R \), all the assumptions involving \( Q \) is in \( \kappa(h^R_\text{symm}(\rho)) \) hold. Let us consider the assumptions involving \( R \). All positive assumptions hold because of the way \( \mathcal{M}'' \) is defined. Let us consider a negative assume \( \neg R(t_1,t_2) \in \kappa(h^R_\text{symm}(\rho)) \). Let \( e_1 = [t_1]_{\mathcal{M}} \) and \( e_2 = [t_2] \). We know that \( \mathcal{M} \models \kappa(h^R_\text{symm}(\rho)) \) and thus \( (e_1,e_2) \notin [R]_{\mathcal{M}} \) (also since \( [R]_{\mathcal{M}} \) is symmetric, we have \( (e_2,e_1) \notin [R]_{\mathcal{M}} \)). Assume on the contrary that \( (e_1,e_2) \in [R]_{\mathcal{M}''} \). Then, there are two possible cases. The first case is \( e_1 = e_2 \) (and thus \( e_1,e_2 \) is in \( [R]_{\mathcal{M}} \) because of reflexive closure). But then \( \mathcal{A} \) contains the reflexivity axiom and thus \( (e_1,e_1) \in [R]_{\mathcal{M}} \) giving us a contradiction. The second case is that \( e_1 \neq e_2 \). In this case, there must be elements \( f_1,\ldots,f_k \) such that \( (f_i,f_{i+1}) \in [R]_{\mathcal{M}} \), \( e_1 = f_1 \) and \( e_2 = f_k \). But then, since \( \mathcal{M} \models \kappa(h^R_\text{symm}(\rho)) \), it must be that \( (f_i,f_{i+1}) \in [R]_{\mathcal{M}} \). Also, since \( \mathcal{M} \models \varphi^R_{\text{trans}} \), we must have \( (e_1,e_2) \in [R]_{\mathcal{M}} \) because of transitivity, giving us a contradiction.
5. $M' = M''$.

We will now argue that $M' \models A \cup \{ \varphi^R_p \}$ and is also feasible in $\rho$, which will imply $\rho$ is feasible modulo $A \cup \{ \varphi^R_p \}$. Since $A$ is invariant under $(p, R)$-transitive closure extension (or $(p, R)$-transitive closure extension, as appropriate), we have that $M' \models A$ (as $M'$ is either the closure extension of $M$ or $M''$, both of which satisfy $A$). Further, $M' \models \varphi^R_p$ by definition. Thus, $M' \models A \cup \{ \varphi^R_p \}$. Now, we argue that $M' \models \kappa(\rho)$. Let $t_1 \lor t_2 \in \kappa(\rho)$ be an equality or disequality atom in $\kappa(\rho)$. First, observe that $t_1 \lor t_2 \in \kappa(h^R_p(\rho))$ as $\kappa(\rho) \subseteq \kappa(h^R_p(\rho))$, and thus $M \models t_1 \lor t_2$ (and when $M''$ is defined, $M'' \models t_1 \lor t_2$). Further, observe that for every term $t$, $[t]_{M'} = [t]_M$ (or $[t]_{M'} = [t]_{M''}$ as appropriate) (see Proposition 1). This means that $M' \models t_1 \lor t_2$ A similar argument ensures that for every predicate $\psi$ of the form $Q(t_1, \ldots, t_k)$ or $\neg Q(t_1, \ldots, t_k)$ (where $Q$ is different from $R$ in $\kappa(\rho)$), $M' \models \psi$. Finally, we argue about predicate atoms involving $R$. We do a case-by-case analysis depending upon what $p$ is.

$p \in \{ \text{symm, refl} \}$ For every positive predicate $R(t_1, t_2) \in \kappa(\rho)$, we have that $R(t_1, t_2) \in \kappa(h^R_p(\rho))$ and thus $M \models R(t_1, t_2)$. Now, notice that $[R]_M \subseteq [R]_{M'}$ (because of the way symmetric or reflexive closure is defined), we have that $M' \models R(t_1, t_2)$. Let us now consider a negative predicate $\neg R(t_1, t_2) \in \kappa(h^R_p(\rho))$. Let $e_1 = [t_1]_{M'}$ and $e_2 = [t_2]_{M'}$. Suppose on the contrary that $(e_1, e_2) \in [R]_{M'}$ but $(e_1, e_2) \notin [R]_M$.

Here we have the following subcases -

- **Case $p = \text{refl}$**. By definition of $[R]_{M'}$, it must be that $e_1 = e_2$. However, by definition of $h^R_p(\rho)$, it must be that $R(t_1, t_1) \in \kappa(h^R_p(\rho))$ since $t_1$ is a computed term. Now since $M \models \kappa(h^R_p(\rho))$, it must be that $(e_1, e_1) \in [R]_M$ giving us a contradiction.

- **Case $p = \text{symm}$ and $\varphi^R_{\text{trans}} \notin A$**. By definition of symmetric closure, we must have $(e_1, e_2) \in [R]_M$. Now, $\psi = \neg R(t_1, t_2) \in \kappa(\rho)$. We therefore have, by definition of $h^R_p$, $\psi' = \neg R(t_2, t_1) \in \kappa(h^R_p(\rho))$ and thus $(e_2, e_1) \notin [R]_M$, giving us a contradiction.

- **Case $p = \text{symm}$, $\varphi^R_{\text{trans}} \in A$**. Argued in the paragraph titled ‘Observations about $M''$’ above.

$p = \text{irref}$ For a negative predicate $\psi = \neg R(t_1, t_2) \in \kappa(\rho)$, it is easy to see that $M' \models \psi$. Let us consider a positive predicate $\psi = R(t_1, t_2) \in \kappa(\rho)$. Let $e_1 = [t_1]_{M'}$ and $e_2 = [t_2]_{M'}$. Since $\kappa(\rho) \subseteq \kappa(h^R_p(\rho))$, we have that $(e_1, e_2) \in [R]_{M'}$. Suppose on the contrary that $M' \models \psi$ and thus $(e_1, e_2) \notin [R]_{M'}$. By definition of irreflexivity closure, we must have $e_1 = e_2$. However note that by definition of $h^R_{\text{irref}}$, $\neg R(t_1, t_1) \in \kappa(h^R_p(\rho))$ and thus $(e_1, e_2) \notin [R]_M$ giving us a contradiction.

Thus, $\rho$ is feasible in $M'$ which is a $A$ model. \hfill \Box

Lemma 16 (Preservation of Term Equalities). Let $\Sigma = (C, F, R)$ be an FO signature, $R \in C$ a binary relation and $p \in \{ \text{refl, irref, symm} \}$. Let $A$ be a set of first order sentences over $\Sigma$ such that if $p = \text{symm}$ and $\varphi^R_{\text{trans}} \in A$, then the only other sentences in $A$ that mention $R$ are $\varphi^R$ or $\varphi^R_{\text{irref}}$. Further,
assume that $A$ is invariant under $(p, R)$-closure extension (or $(p, R)$-transitive closure extension if $\varphi^R_{\text{trans}} \in A$). For any execution $\rho$, and any two computed terms $t_1, t_2 \in \text{Terms}(h^R_p(\rho))$,

$$t_1 \equiv_{A \cup \{ \varphi^R_p \}} t_2 \iff t_1 \equiv_{A \cup \{ \varphi^R_p \} \cup \kappa(\rho)} t_2.$$  

Proof. First observe that for every $\psi \in \kappa(h^R_p(\rho)) \setminus \kappa(\rho)$, we have $A \cup \{ \varphi^R_p \} \models \psi$. Therefore, every $A \cup \{ \varphi^R_p \} \kappa(\rho)$-model is also a $A \cup \kappa(h^R_p(\rho))$-model. Hence, if $t_1 \equiv_{A \cup \kappa_1(\rho)} t_2$ then $t_1 \equiv_{A \cup \kappa_2(\rho)} t_2$.

For the other direction, suppose $t_1 \not\equiv_{A \cup \kappa_1(\rho)} t_2$. Then by definition, there is a $A \cup \kappa(h^R_1(\rho))$ model $\mathcal{M}$ such that $[t_1]_\mathcal{M} \neq [t_2]_\mathcal{M}$. Consider the execution $\rho_1 = \rho \cdot \text{assume}(t_1 \neq t_2)$. Technically $\rho_1$ is not an execution by our definition. What we mean is to copy the terms $t_1$ an $t_2$ in fresh variables when they are computed, and assume that those variables are not equal; we skip doing this precisely. Observe that $h^R_p(\rho_1) = h^R_p(\rho) \cdot \text{assume}(t_1 \neq t_2)$. Based on our assumptions, $h^R_p(\rho_1)$ is feasible in $\mathcal{M}$. By Lemma 13 we have $\rho_1$ is feasible in some $A \cup \{ \varphi^R_p \}$-model $\mathcal{M}'$. Thus, $[t_1]_{\mathcal{M}'} \neq [t_2]_{\mathcal{M}'}$, and so $t_1 \not\equiv_{A \cup \{ \varphi^R_p \} \cup \kappa(\rho)} t_2$. □

Lemma 17 (Preservation of Coherence). Let $\rho$ be an execution, $R \in \mathcal{R}$ and $p \in \{ \text{refl, irref, symm} \}$. Let $A$ be a set of first order sentences over $\Sigma$ such that if $p = \text{symm}$ and $\varphi^R_{\text{trans}} \in A$, then the only other sentences in $A$ that mention $R$ are $\varphi^R_{\text{refl}}, \varphi^R_{\text{irref}}$ or $\varphi^R_{\text{trans}}$. Further, assume that $A$ is invariant under $(p, R)$-closure extension (or $(p, R)$-transitive closure extension if $\varphi^R_{\text{trans}} \in A$). $\rho$ is coherent modulo $A \cup \{ \varphi^R_p \}$ if and only if $h^R_p(\rho)$ is coherent modulo $A$.

Proof. Follows from Lemma 14 and Lemma 17 □

Lemma 18 (Preservation of Feasibility). Let $\Sigma = (\mathcal{C}, \mathcal{F}, \mathcal{R})$ be a FO signature, $f \in \mathcal{F}$ be a unary or binary function and let $p \in \{ \text{comm, idem} \}$. Let $A$ be a set of first order sentences over $\Sigma$ invariant under 1-element extension. Also assume that $A$ has no sentence that mentions $f$. $\rho$ is an execution.

1. $\rho$ is feasible modulo $A \cup \{ \varphi^f_p \}$ if $h^f_p(\rho)$ is feasible modulo $A$.
2. $\rho$ is coherent modulo $A \cup \{ \varphi^f_p \}$ if $h^f_p(\rho)$ is coherent modulo $A$.

Proof. Let us first argue preservation of feasibility. ($\Rightarrow$). Let $\rho$ be feasible modulo $A \cup \{ \varphi^f_p \}$. Then, there is a model $\mathcal{M}$ such that $M \models A \cup \{ \varphi^f_p \}$. Now, we first note that all the ground predicates $\phi \in \kappa(h^f_p(\rho)) \setminus \kappa(\rho)$, we have that $M \models \phi$ because $M \models \{ \varphi^f_p \}$. This means, that $h^f_p(\rho)$ is feasible in $M$, thus implying that $h^f_p(\rho)$ is feasible modulo $A$.

($\Leftarrow$). Let $h^f_p(\rho)$ be feasible modulo $A$. Then, there is a model $\mathcal{M}$ such that $M \models A$ and $M \models \kappa(h^f_p(\rho))$. Let $M' = \text{1Ext}(M)$ and let $e_*$ be the extra element added in the construction of $M'$. By Lemma 13 we have $M' \models A$ and $M' \models \kappa(h^f_p(\rho))$. Now, consider the model $M'' = (\mathcal{U}_{M''}, \llbracket M'' \rrbracket)$ with $\mathcal{U}_{M''} = \mathcal{U}_{M'}$.
Theorem 11. Let $\emptyset$ be a set of axioms where each relation symbol $R$ is either a total order or satisfies some (possibly empty) subset of properties out of reflexivity, irreflexivity, symmetry, transitivity, and each function symbol $f$ satisfies some (possibly empty) subset of commutativity and idempotence. The verification problem for coherent programs modulo $\emptyset$ is PSPACE-complete.

Proof. PSPACE-hardness follows from the PSPACE-hardness of verification modulo $\emptyset$ as proved in [29]. We focus on the PSPACE upper bound, for which we will show that the set of executions that are feasible and coherent modulo $\emptyset$ is regular and accepted by an automaton of size $O(2^{\text{poly}(|V|)})$.

Let $L = \text{Exec}(s)$ be the set of executions of the given coherent program $s$; $L$ is regular. Since total orders are reducible to partial orders (Lemma 9) (under appropriate assumptions on the trace), we will assume we only have combinations of the other axioms we consider (and not total orders). Let $A_{\text{rel}} = \{\varphi^R_{p_i} \}_{i=1}^{k_{\text{rel}}}$ be some arbitrary ordering on the set of relational axioms in $A_{\text{rel}}$. We define a sequence of languages $L_0, \ldots, L_{k_{\text{rel}}}$ as: $L_0 = L$, $L_{i+1} = h_{p_{i+1}}(L_i)$. Let $L_{\text{rel}} = L_{k_{\text{rel}}}$. We can inductively argue that -

1. $L_{\text{rel}}$ is regular (since regular languages are closed under homomorphism),
2. \( L_{rel} \) is feasible modulo \( A \setminus A_{rel} = A_{fun} \uplus A_{trans} \) iff \( L \) is feasible modulo \( A \) (using Lemma 13), and
3. \( L_{rel} \) is coherent modulo \( A_{fun} \uplus A_{trans} \) iff \( L \) is coherent modulo \( A \) (using Lemma 17).

Since the given program \( s \) is assumed to be coherent modulo \( A \), we have \( L_{rel} \) is indeed coherent modulo \( A_{fun} \uplus A_{trans} \).

Here, by feasibility (resp. coherence) of a language, we mean feasibility (resp. coherence) of each of the strings in the language.

We now analogously get rid of axioms in \( A_{fun} \) one at a time. Let \( A_{fun} = \{ \phi_{q_i} \}_{i=1}^{k_{fun}} \) be some arbitrary ordering on the set of functional axioms in \( A_{fun} \). We define a sequence of languages \( K_0, \ldots, K_{k_{fun}} \) as: \( K_0 = L_{rel}, K_{i+1} = h_{q_{i+1}}(K_i) \). Let \( L_{fun} = K_{k_{fun}} \). We can inductively argue that -

1. \( L_{fun} \) is regular (since regular languages are closed under homomorphism).
2. \( L_{fun} \) is feasible modulo \( A_{trans} \) iff \( L_{rel} \) is feasible modulo \( A_{fun} \uplus A_{trans} \) (using Lemma 18). This implies that \( L_{fun} \) is feasible modulo \( A_{trans} \) iff \( L \) is feasible modulo \( A \).
3. \( L_{fun} \) is coherent modulo \( A_{trans} \) iff \( L_{rel} \) is coherent modulo \( A_{fun} \uplus A_{trans} \) (using Lemma 18). Together with the previous observations, we have \( L_{fun} \) is indeed coherent modulo \( A_{trans} \).

Thus, the verification problem reduces to checking if \( L_{fun} \) is feasible modulo \( A_{trans} \). This problem is decidable as a consequence of Theorem 4 and the fact that \( L_{fun} \) is coherent modulo \( A_{trans} \). In other words, we need to check for containment of two regular languages - \( L_{fun} \subseteq L(F_{trans}) \), where \( F_{trans} \) is the automaton in Theorem 4 which is decidable.

The complexity arguments follows from the observation that \( L \) (and thus \( L_{fun} \)) are recognizable by NFAs of size linear in \(|s|, |V| \) and \(|A_{fun} \uplus A_{rel}| \) and are also effectively constructible and that the containment check can be done in \( \text{PSPACE} \).