TENSOR COMPLETION WITH DCT BASED GRADIENT METHOD

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Abstract

Tensor Completion from a limited number of non-distorted observations, has enticed researchers interest. The color image has been considered as the three dimensional tensor. Low rank property in Optimization has been used to recover the tensors in the image. The Low rank prior alone not enough to tensor completion. The traditional tensor truncated nuclear norm approaches have been able to approximate the real rank of the tensor, but these are low rank prior approaches. Here a transformation-based optimization method has been proposed to complete the tensors of the image. The Discrete Cosine Transformation (DCT) has been used as transformation method. The tensor singular value decomposition (t-SVD) and accelerated proximal gradient line (APGL) approaches have been considered. The Full Reference metrics i.e., peak signal to noise ratio (PSNR) and structural similarity (SSIM) have been used to evaluate the proposed approach. The obtained results are superior to the existing algorithms. The PSNR and SSIM have been recorded as 27.30 dB and 0.8845 respectively.

Keywords: Tensor Completion, Tensor Singular Value Decomposition, Discrete Cosine Transform, Convex Optimization.

I. Introduction

An image lies in a low-dimensional space, estimating missing values in images has been considered as low-rank matrix approximation problem [I]. The Low rank matrix approximation approaches have been used to solve the corrupted input image of two-dimensional manner. The color images formed with three-dimensional space, to apply these approaches the image has to be separated as channels. These separated channels have been processed using the state-of-the art methods. Again, results have to be combined [II, III]. Here recent researchers the considered color image as the 3D data and formulated the problem as the Low Rank Tensor Completion [II, III, IV, V]. Tensor completion is the process of recovering the corrupted tensors of a color image. Tensor Completion has various applications such as machine learning, compressed
sensing and pattern recognition. The low rank minimization problem [II] to tensor completion has been formed as sum of metricized nuclear norm of a tensor.

$$\min \sum_{k=1}^{n} \| I_k \|_*, \text{subject to } I_\Omega = \mathcal{M}_\Omega$$  \hspace{1cm} (1)

where $I_k$ denotes the tensor unfolded along the $k^{th}$ mode tensor, $\mathcal{M}_\Omega$ represents the incomplete tensor. $\Omega$ indicates the known elements set. Kilmer et al. [VI] proposed a novel approach named tensor singular value decomposition (t-SVD) to perform the singular value decomposition on color images. Xue et al. [3] proposed a method incorporating the both truncated nuclear norm minimization [VII] and the t-SVD to an extension of matrix case. The objective function written as

$$\min_{J, W} \| J \|_* - \text{tr}(A_l * W * B_l^T)$$  \hspace{1cm} (2)

$$\text{subject to } J = W, W_\Omega = \mathcal{M}_\Omega$$

The truncated nuclear norm minimization alone not sufficient to recover the missing areas in two-dimensional case. By utilizing a transformation based truncated nuclear norm approach can recover the images with more structural and texture information [VIII, IX]. It has been extended to tensor case, [V] proposed an approach. The Discrete Cosine Transformation (DCT) has been incorporated with the Xue et al. proposed method. Here a DCT based Accelerated Proximal Gradient Line (APGL) approach has been introduced to tensor completion. It is providing the excellent recovery in the structure information of the image even the missing ratio is more than 80%.

The paper alignment has been done as the section 1 started with the introduction to tensors and literature review. Section 2 provided some basics of tensors and properties. Section 3 discussed the methodology to proposed approach. Section 4 has been accommodated with the results and discussion. Section 5 Carried out the conclusion.

II. Notations and Preliminaries

The italic capital letters indicates the tensor, normal capital letters are indicates the matrix, italic small letter indicates the vector and the normal small letter indicates the scalar. The image tensor will be $J \in \mathbb{C}^{(n_1 \times n_2 \times n_3)}$. The tensor contains the horizontal, lateral and frontal slice. The frontal slice $J(:, :, k)$ can be represented as the $J^k$. The block circulant matrix and unfolded tensor have been written in eq.3 and eq.4.
\[b\text{cir}c(\mathcal{I}) = \begin{bmatrix}
I^{(1)} & I^{(n3)} & \cdots & I^{(2)} \\
I^{(2)} & I^{(1)} & \cdots & I^{(3)} \\
\vdots & \vdots & \ddots & \vdots \\
I^{(n3)} & I^{(n3-1)} & \cdots & I^{(1)}
\end{bmatrix}\]

(3)

\[\text{un}f\text{o}ld(\mathcal{I}) = \begin{bmatrix}
I^{(1)} \\
I^{(2)} \\
\vdots \\
I^{(n3)}
\end{bmatrix}\]

(4)

**Definition: Tensor Product** [X] Let \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) and \(Y \in \mathbb{R}^{n_2 \times n_4 \times n_3}\). The tensor product between \(X\) and \(Y\) can be written as

\[X \ast Y = \text{fold}(b\text{cir}c(X), \text{un}f\text{o}ld(Y))\]

(5)

where \(\cdot\) indicates the matrix product.

**Definition: Tensor Transpose** [X] The transpose of a tensor \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) becomes as \(X^T\) with order \(n_2 \times n_1 \times n_3\). The frontal slice need to be transposed and then reversing the order of the frontal slices from 2 to \(n_3\).

\[ (X^T)^{(1)} = (X^1)^{(T)} \]

\[ (X^T)^{(i)} = (X^{n_3-i+2})^{(T)}, i = 2,3,\ldots,n_3.\]

(6)

**Definition: Identity Tensor** [X] The identity tensor \(I \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) contains the first frontal slice \(n \times n\) identity matrix, remaining frontal slices contains all zeros.

**Definition: Orthogonal Tensor** [X] A tensor \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) orthogonal if it satisfies

\[X^T \ast X = X \ast X^T = I\]

(7)

**Theorem: T-SVD** [X] Let \(X \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) by applying TSVD the \(X\) decomposed as the \(U \in \mathbb{R}^{n_1 \times n_1 \times n_3}, S \in \mathbb{R}^{n_1 \times n_2 \times n_3}\) and \(V \in \mathbb{R}^{n_2 \times n_2 \times n_3}\). The Fig 1 provides the basic multiplication order of T-SVD.

\[X = U \ast S \ast V^T\]

(8)
Definition: Tensor Singular Value Thresholding\[III\] Let T-SVD of tensor $X \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ will be $U^*S^*V^T$. The Singular Value Thresholding (SVT) has been applied to each frontal slice of the tensor $S$.

$$\mathcal{D}_T(X) = U \ast \mathcal{D}_T(S) \ast V^T$$ (9)

III. Tensor Completion With DCT Based Optimization

The DCT has been used in image compression, image de-noising and designing the transform coders etc., wang et al. [VIII] extended the DCT with gradient based optimization towards the image completion. Ping-Ping Wang et al. The n-dimensional DCT has been calculated by utilizing the separability property n times the one-dimensional DCT will be calculated. Here the image contains the three spaces, first images has been divided in to $8 \times 8$ blocks and applied the two-dimensional DCT, then applied one dimensional DCT to the same block. The one-dimensional DCT and two dimensional DCT can be written as

$$\mathbb{C}(j) = \alpha(j) \sum_{i=0}^{n-1} u(i) \cos \left( \frac{\pi(2i+1)}{2n} j \right)$$ (10)

where $\alpha_j = \frac{1}{n} \sqrt{\frac{2}{n}}$ if $j=0$ else $\frac{2}{n} \sqrt{\frac{2}{n}}$

$$\mathbb{C}_{j_1,j_2} = \alpha_{j_1} \alpha_{j_2} \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} c(i_1,i_2,j_1,j_2) I_{i_1,i_2} = G_{j_1,j_2} \ast 1$$ (11)

where $n_1,n_2$ and $n_3$ are the size of the image block. $\alpha_{j_1} = \frac{1}{n_1} \sqrt{\frac{2}{n_1}}$ if $j_1 = 0$ else $\frac{2}{n_1} \sqrt{\frac{2}{n_1}}$, $\alpha_{j_2} = \frac{1}{n_2} \sqrt{\frac{2}{n_2}}$ if $j_2 = 0$ else $\frac{2}{n_2} \sqrt{\frac{2}{n_2}}$. The cosine basis function written in eqn.11

$$c(i_1,i_2,j_1,j_2) = \cos \left( \frac{\pi(2i_1+1)}{2n_1} j_1 \right) \cos \left( \frac{\pi(2i_2+1)}{2n_2} j_2 \right)$$ (12)

After obtaining the coefficients the selection has been considered. Based on the selection mask the values have been considered in the inverse DCT. here the selection mask $S_{pq}$ considered based on $p$ and $q$ values. $p=1,2,3$ and $q=1,2,3$
\[ S = \begin{cases} S_{pq} = 1 & \text{if } p \geq 1,2,3 \quad q \geq 1,2,3 \\ S_{pq} = 0 & \text{Otherwise} \end{cases} \]  

The inverse DCT has been applied to the block to retain image \( I_k \). The optimization process goes as in eqn. 15, 16 and 17.

**Update \( J_{k+1} \):**

\[ J_{k+1} = \arg \min \| \| I \| + \frac{1}{2t_k} \| J_k - (\mathcal{M}_k - (t_k \nabla f(J_k))) \|_F^2 \]  

The eqn. 14 has been solved by utilizing the T-SVT. i.e.,

\[ J_k = U_l D_t(S_l) V_l^T \]  

where \( D_t(S_l) = \text{diag}(\max(\sigma_l - \tau), 0)_{1 \times \tau^{st}} \), \( l = 1,2, \ldots, n_3 \)

**Update \( t_{k+1} \):**

\[ t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2} \]  

**Update \( \mathcal{M}_{k+1} \):**

\[ \mathcal{M}_{k+1} = J_{k+1} + \frac{t_k - 1}{t_{k+1}} (J_{k+1} - J_k) \]  

**Algorithm:** DCT with t-SVD and APGL for Tensor Completion

**Observation:** \( J, \Omega, \Omega_c, \gamma, R ; \)

**Initialization:** \( t = 1, \mathcal{M} = J, k = 1 \) ;

Calculate the gradient of \( f(I) \) to the Equation (22)

\[ [U, S, V] \leftarrow T - \text{SVD}(\mathcal{M}_k - t_k \nabla f(J_k)) \]

update \( J_{k+1}, t_{k+1}, \mathcal{M}_{k+1} ; \)

k = k+1;

Until \( \| J_{k+1} - J_k \| \leq \epsilon \)

**Output:** The Recovered Image \( \hat{J} = J_{k+1} \)

**IV. Results and Discussion**

**Parameter Settings:** The DCT block size considered as 8’8, the minimal rank is 1 and the maximum rank is 20. The lambda and tolerance values have been considered as 0.01 and 0.0001 respectively. In the image randomly 80% of tensors have been corrupted. These corrupted images have been recovered with the proposed approach.
and the results have been evaluated. The Full-Reference Image Quality Assessment Measures (FR-IQA) i.e., PSNR and SSIM have been used to evaluate.

Figure 2: Recovered images from 80% missing ratio

Fig2. consists of two images evaluated with PSNR and SSIM. Fig 2 a) and c) have been plotted with PSNR vs rank. The proposed method has high values in comparison with the existing LRTNN method. The maximum PSNR value obtained at Rank 8 i.e. 27.30 dB to parrot image. The Fig 2 b) and d) shows the SSIM vs Rank, the SSIM value attain the maximum at rank = 8 as 0.8845. To the existing LRTNN method got PSNR as 22.61 and SSIM as 0.7895. The DCT transformation has the feature of energy compaction and smoothing gives the scope in recovering of structure information.

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V. Conclusions

The proposed approach DCT transformation based APGL Optimization has been recovering the tensors missing in the image. The T-SVD and the truncated nuclear norm have been utilized instead of other existing methods. The recovery has been evaluated using the FR-IQA measures named PSNR and SSIM. The improvement values PSNR and SSIM are 5.92 dB and 0.1433 respectively. The DCT based TSVD optimization method has been outperformed.

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