MAPPING PROPERTIES OF THE ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present paper, the order of convexity of \( \, _2F_1(a, b; c; z) \) is first given under some conditions on the positive real parameters \( a, b \) and \( c \). Then we show that the image domains of the unit disc \( \mathbb{D} \) under some shifted zero-balanced hypergeometric functions \( \, _2F_1(a, b; a + b; z) \) are convex and bounded by two horizontal lines which solves the problem raised by Ponnusamy and Vuorinen in [9].

1. Introduction and main results

The Gaussian hypergeometric function is defined by the power series

\[
\, _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n
\]

for \( z \in \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \), where \( (a)_n \) is the Pochhammer symbol; namely, \( (a)_0 = 1 \) and \( (a)_{n+1} = (a)_n(a + n) = a(a + 1) \cdots (a + n) \) for all \( n \in \mathbb{N} := \{0, 1, 2, \ldots \} \). Here \( a, b \) and \( c \) are complex constants with \( -c \not\in \mathbb{N} \). Hypergeometric functions can be analytically continued along any path in the complex plane that avoids the branch points 1 and \( \infty \).

If \( c = a + b \), it is called the zero-balanced one and \( \, _2F_1(a, b; c; z) \) is usually called a shifted hypergeometric function. For instance the function \( \, _2F_1(1, 1; 2; z) = -\log(1 - z) \) is a shifted zero-balanced hypergeometric function. In the present paper, we only restrict to the real parameters \( a, b \) and \( c \). For the basic properties of hypergeometric functions we refer to [1], [8] and [17].

For a function \( f \) analytic in \( \mathbb{D} \) and normalized by \( f(0) = f'(0) - 1 = 0 \), the order of convexity and the order of starlikeness (with respect to the origin) of \( f \) are defined by

\[
\kappa = \kappa(f) := 1 + \inf_{z \in \mathbb{D}} \frac{\Re z f''(z)}{f'(z)} \in [-\infty, 1]
\]

and

\[
\sigma = \sigma(f) := \inf_{z \in \mathbb{D}} \frac{\Re z f'(z)}{f(z)} \in [-\infty, 1]
\]

respectively. We follow the convention in [3] due to Küstner, that is \( \kappa(f) = -\infty \) only if \( f' \) is zero-free in \( \mathbb{D} \) and \( \Re [zf''(z)/f'(z)] \) is not bounded from below in \( \mathbb{D} \), whereas \( \kappa(f) \) is regarded to be not defined if \( f' \) has zeros in \( \mathbb{D} \). The corresponding convention can also be made for \( \sigma(f) = -\infty \). It is known that \( f \) is convex, i.e. \( \kappa(f) \geq 0 \) if and only if \( f \) is

2010 Mathematics Subject Classification. Primary 30C45; Secondary 33C05.

Key words and phrases. Gaussian hypergeometric function, order of convexity, zero-balanced one, Ramanujan’s formula.

This research is supported by National Natural Science Foundation of China (No. 11901086) " and the Fundamental Research Funds for the Central Universities” in UIBE (No. 18YB02).
univalent in $D$ and $f(D)$ is a convex domain; and $f$ is starlike, i.e. $\sigma(f) \geq 0$ if and only if $f$ is univalent in $D$ and $f(D)$ is a starlike domain with respect to the origin. It is also true that if $\kappa(f) \geq -1/2$, then $f$ is univalent in $D$ and $f(D)$ is convex in (at least) one direction, see [10] and [11] p.17, Thm.2.24; p.73]. If $f$ is convex in $D$, then the order of starlikeness of $f$ is at least $1/2$ (see [10] p. 49, Theorem 2.1]).

The study on the order of starlikeness of the shifted hypergeometric functions is dated from 1961, see [7] by Merkes and Scott. Since then, several authors have researched this project by using different methods, for instance Ruscheweyh and Singh [12], Ponnusamy and Vuorinen [9], K"ustner [4] and so on. With regard to the order of convexity, there are few papers in the literature comparing with the order of starlikeness. Such kind of research can be found in [13] Theorem 4 due to Silverman, [5] Corollary 8, Corollary 9] by K"ustner in 2007 and our paper [15, Theorem 1.3, Theorem 1.5]. Our first aim in this paper is to provide explicit orders of convexity in terms of $a$, $b$ and $c$ for some shifted hypergeometric functions. As an application, we solve the following problem on the zero-balanced hypergeometric functions which is raised by Ponnusamy and Vuorinen in [9], see also [3].

**Problem 1.1.** Do there exist positive numbers $\delta_1$, $\delta_2$ such that for $a \in (0, \delta_1)$ and $b \in (0, \delta_2)$ the normalized function $z_2F_1(a, b; a + b; z)$ ($z_2F_1(a, b; a + b; z^2)$ resp.) satisfies the property that maps the unit disc $D$ into a strip domain?

We restrict our attention only on the mapping $z \mapsto z_2F_1(a, b; a + b; z)$ in the present paper. In view of Ramanujan’s formula:

\[
_2F_1(a, b; a + b; z) = \frac{1}{B(a, b)} \left( R(a, b) - \log(1 - z) \right) + O \left( |1 - z| \log \frac{1}{|1 - z|} \right),
\]

as $z \to 1$ in $D$, where $R(a, b) = 2\psi(1) - \psi(a) - \psi(b)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $B(a, b)$ denote the digamma function and the Beta function respectively, the zero-balanced hypergeometric function behaves like the function $-\frac{1}{B(a, b)} \log(1 - z)$. Thus it is easy to see that it maps the unit disc $D$ into a strip domain. It seems that it is trivial to consider only Problem 1.1 along this direction. In fact, our aim is to find conditions on the pair $(a, b)$ such that the image domain of $D$ under the mapping $z \mapsto z_2F_1(a, b; a + b; z)$ is convex and lies in a parallel strip. Along this line, the author in [15, Corollary 1.2] (cf. [5] Corollary 8, Corollary 9]) got the following result:

**Theorem A.** If $a$ and $b$ are real constants satisfying $-a \notin \mathbb{N}$, $-b \notin \mathbb{N}$ and $ab < 1$, then the shifted zero-balanced hypergeometric function $z_2F_1(a, b; a + b; z)$ is not convex in $D$. 

If furthermore, we suppose $a$ and $b$ are both positive. According to Theorem A, in order to consider the convexity of the shifted zero-balanced hypergeometric functions the condition $ab \geq 1$ is required, which further implies $a \geq 1$ or $b \geq 1$. Furthermore if $a + b = 2$, it follows from $2 = a + b \geq 2\sqrt{ab}$ that $ab \leq 1$ which together with $ab \geq 1$ implies $ab = 1$. Thus when investigating the convexity of the shifted zero-balanced hypergeometric functions $z_2F_1(a, 2 - a; 2; z)$ with $a > 0$, it is sufficient to consider only the function $z_2F_1(1, 1; 2; z) = -\log(1 - z)$.

For $0 < a \leq b < 1$, even though $z_2F_1(a, b; a + b; z)$ is not convex, it is natural to ask the question: whether the function $z_2F_1(a, b; a + b; z)$ possesses the other geometric properties,
for example starlikeness and univalence and so on. In fact, K"ustner in [5, Theorem 1] (cf. [4, Theorem 1.1, Remark 2.3]) got the following result on the order of starlikenss of the shifted hypergeometric functions.

**Theorem B.** If \( 0 < a \leq b \leq c \), then

\[
\sigma(z_2 F_1(a, b; c; z)) = 1 - \frac{2 F_1'(a, b; c; -1)}{2 F_1(a, b; c; -1)} \geq 1 - \frac{ab}{b + c}
\]

As a consequence of Theorem B, the function \( z_2 F_1(a, b; a + b; z) \) is starlike thus univalent in \( \mathbb{D} \) if \( 0 < a \leq b < 1 \).

From now on, we focus our attention on the case \( a \geq 1 \). First if \( a = 1 \), we proved the next result in [15] on the convexity of \( z_2 F_1(1, b; c; z) \).

**Theorem C.** ([15, Corollary 4.3]) Assume \( b \) and \( c \) are real constants satisfying \( 0 < b \leq c \), the function \( z_2 F_1(1, b; c; z) \) is convex in \( \mathbb{D} \) if one of the following conditions holds:

1. \( 0 \leq b \leq 4 \) and \( c \geq 2 \);
2. \( 1 \leq c < \min\{2, 1 + b\} \) and \( c \geq 3 - b \).

For general \( a > 1 \), by making use of Riemman-Stieltjes type integral of the ratio of hypergeometric functions, we first obtain the order of convexity of the shifted hypergeometric functions as follows.

**Theorem 1.2.** Let \( a, b, c \) be real constants satisfying \( 1 < a \leq b \leq 4 \), \( c > 2 \) and \( b \leq c \leq 2b \leq 2(1 + a) \). The order of convexity of the function \( z_2 F_1(a, b; c; z) \) is

\[
\kappa(z_2 F_1(a, b; c; z)) = \frac{5 - c - a - b}{2} + \frac{c - 2 - (1 - a)(1 - b)}{2 \left( 1 - a + a z_2 F_1(a + 1, b; c; -1) / z_2 F_1(a, b; c; -1) \right)}.
\]

In particular, \( z_2 F_1'(a, b; c; z) \neq 0 \) for all \( z \in \mathbb{D} \).

We remark that in [15], the order of convexity of \( z_2 F_1(a, b; c; z) \) has already been dealt with, but the condition \( 0 < a \leq 1 \) is required there. Thus Theorem 1.2 extends the discussion to \( a > 1 \). By virtue of Theorems C and 1.2 we obtain the next theorem which not only gives the convexity of some shifted zero-balanced hypergeometric functions but also demonstrates the explicit bounds of the strip domains of the image domains. Thus we provide a strong version of solution to Problem 1.1.

**Theorem 1.3.** Let \( a \) and \( b \) be real constants. If one of the following conditions holds:

1. \( a = 1 \) and \( 1 \leq b \leq 4 \);
2. \( 1 < a \leq b \leq \min\{3, 1 + a, \frac{4a}{5a - 4}\} \),

then the shifted zero-balanced hypergeometric function \( z_2 F_1(a, b; a + b; z) \) is convex and maps the unit disc \( \mathbb{D} \) into a strip domain which is bounded by two horizontal lines \( \text{Im } w = \frac{\pi}{2 \pi B(a, b)} \). The boundary lines are optimal.
2. Some lemmas

In this section, we list some auxiliary lemmas which play important roles in the proofs of the main results. The first lemma is due to Ruscheweyh, Salinas and Sugawa [11]. Later, Liu and Pego in [6] pointed out that one condition can be deduced from the others. So the lemma can be finally characterized in the following form.

Lemma 2.1. Let \( F(z) \) be analytic in the slit domain \( \mathbb{C} \setminus [1, +\infty) \). Then

\[
F(z) = \int_0^1 \frac{d\mu(t)}{1-tz}
\]

for some probability measure \( \mu \) on \([0, 1]\), if and only if the following conditions are fulfilled:

1. \( F(0) = 1 \);
2. \( F(x) \in \mathbb{R} \) for \( x \in (-\infty, 1) \);
3. \( \text{Im} \ F(z) \geq 0 \) for \( \text{Im} \ z > 0 \);
4. \( \limsup_{x \to +\infty} F(-x) \geq 0 \).

The measure \( \mu \) and the functions \( F \) are in one-to-one correspondence.

The forthcoming four lemmas describe the properties of the ratio to two hypergeometric functions in different aspects.

Lemma 2.2. ([4, Thm. 1.5], [17, p.337-339 and Thm.69.2]) If \( -1 \leq a \leq c \) and \( 0 \leq b \leq c \neq 0 \), the ratio of two hypergeometric functions can be written in integral as

\[
\frac{2F_1(a + 1, b + 1; c + 1; z)}{2F_1(a + 1, b; c; z)} = \int_0^1 \frac{d\mu(t)}{1-tz}, \quad z \in \mathbb{C} \setminus [1, +\infty)
\]

where \( \mu : [0, 1] \to [0, 1] \) is nondecreasing with \( \mu(1) - \mu(0) = 1 \).

Lemma 2.3. Let \( a, b \) and \( c \) be real constants with \( 0 < a \leq b \leq 1 + a, \ c - a \not\in -\mathbb{N} \) and \( c - b \not\in -\mathbb{N} \). Then

\[
\lim_{x \to +\infty} \frac{x_2F_1(a + 1, b + 1; c + 1; -x)}{2F_1(a + 1, b; c; -x)} = \infty.
\]

Proof. Denote \( H(z) = 2F_1(a+1, b+1; c+1; z) \) and \( G(z) = 2F_1(a+1, b; c; z) \) for convenience. In order to show the claimed equation, the next two linear transforms (see [1] p.559, 15.3.3, 15.3.5, 15.3.7) are required:

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}2F_1(a, 1-c+a; 1-b+a; 1/z)
\]

(2.1)

\[+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}2F_1(b, 1-c+b; 1-a+b; 1/z), \quad (|\arg(-z)| < \pi)\]

and

\[
2F_1(a, b; c; z) = (1-z)^{-b}2F_1(b, c-a; c; \frac{z}{z-1}), \quad (|\arg(1-z)| < \pi).
\]
Then the proof can be separated into three cases according to the relationship between $a$ and $b$.

Case I: Assume that $a < b < 1 + a$. By virtue of (2.1), we obtain
\[
H(-x) = \frac{\Gamma(c+1)\Gamma(b-a)}{\Gamma(b+1)\Gamma(c-a)} x^{-a-1} + O(x^{-b-1})
\]
and
\[
G(-x) = \frac{\Gamma(c)\Gamma(a+1-b)}{\Gamma(a+1)\Gamma(c-b)} x^{-b} + O(x^{-a-1})
\]
as $x \to +\infty$. It follows from the above asymptotic behaviors that $\lim_{x \to +\infty} xH(-x)/G(-x) = \infty$ since $a < b < 1 + a$.

Case II: Assume $b = a$. We deduce from the identity (2.2) and Ramanujan’s formula (1.1) that
\[
H(-x) = \binom{a+1, a+1; c+1; -x}{1+x}^{-a-1} 2F_1 \left( a+1, c-a-1; \frac{x}{x+1} \right)
\]
as $x \to +\infty$. Similarly, applying the identity (2.2) and Gauss summation formula:
\[
\lim_{x \to +\infty} 2F_1(a, b; c; x) = 2F_1(a, b; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},
\]
if $c-a-b > 0$ (see [1, p.556, 15.1.20]), we have
\[
\lim_{x \to +\infty} (1+x)^aG(-x) = \lim_{x \to +\infty} 2F_1(a, c-a-1; \frac{x}{x+1}) = \frac{\Gamma(c)}{\Gamma(c-a)\Gamma(c+1)}
\]
Thus by substituting the proceeding descriptions of $H(-x)$ and $G(-x)$ as $x \to +\infty$, we deduce that
\[
\lim_{x \to +\infty} \frac{xH(-x)}{G(-x)} = \lim_{x \to +\infty} x(1+x)^{-a-1} \left( \frac{\log(1+x)}{B(a+1, c-a-1)} + O(1) \right) \frac{G(-x)}{G(-x)} = \infty
\]
in this case.

Case III: Assume that $b = 1 + a$. Applying the same techniques in Case II, we find that
\[
\lim_{x \to +\infty} (1+x)^{a+1}H(-x) = \frac{\Gamma(c+1)}{\Gamma(c-a)\Gamma(a+2)},
\]
and
\[
G(-x) = (1+x)^{-a-1} \left( \frac{\log(1+x)}{B(a+1, c-a-2)} + O(1) \right), \quad x \to +\infty,
\]
which yields the required claim again.

We complete all these three cases, thus the proof is done.
Lemma 2.4. Let $a$, $b$ and $c$ be real constants with $a, b, c \not\in -\mathbb{N}$, $c-a \not\in -\mathbb{N}$ and $c-b \not\in -\mathbb{N}$.

1. If $c < a + b$, then
   \[ \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} = \frac{a + b - c}{a(1 - z)} + O\left(|1 - z|^{a+b-c}\right). \]

2. If $c = a + b$, then
   \[ \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} = \frac{1}{-a(1 - z)\log(1 - z)} + O\left(\log\frac{1}{|1 - z|}\right). \]

3. If $a + b < c < a + b + 1$, then
   \[ \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} = \frac{A}{(1 - z)^{1-a}} + O(|1 - z|^{c-1}) \]
   where
   \[ A = \frac{\Gamma(a + b + 1 - c)\Gamma(c - a)\Gamma(c - b)}{\Gamma(a + 1)\Gamma(b)\Gamma(c - a - b)}. \]
   \[ \alpha = c - a - b \in (0, 1) \text{ and } \varepsilon = \min\{2\alpha, 1\}. \]

4. If $c = 1 + a + b$, then
   \[ \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} = -b\log(1 - z) + O(1). \]

Proof. The first three assertions can be found in [15, Lemma 2.3]. We need only to prove the last one. First recall the linear transform
\begin{equation}
\genfrac{2}{1}{2}{a}{b}{c} = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\genfrac{2}{1}{2}{a}{b}{c + 1}2F_1(a, b; a + b - c + 1; 1 - z) + (1 - z)^{c - a - b}\frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}\genfrac{2}{1}{2}{a - c, c - b, c - a - b + 1}{1}{1}{1}2F_1(c - a, c - b; c - a - b + 1; 1 - z).
\end{equation}
Applying the identity above and Ramanujan’s formula [11, 1], we have
\[ \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} = \frac{1}{\text{B}(1+a,b)(R(1+a,b)-\log(1-z)) + O\left(|1 - z|\log\frac{1}{|1 - z|}\right)} \]
\[ = \frac{\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\Gamma(a)\Gamma(b)}{\frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} + O(|1 - z|)} \]
\[ = -b\log(1 - z) + O(1) \]
as $z \to 1$ in $\mathbb{D}$, since $c = 1 + a + b$. \qed

Lemma 2.5. ([15], see also [4, Ramark 2.3]) If $-1 \leq a \leq c$ and $0 \leq b \leq c \neq 0$, then
\[ \frac{c}{b+c} \leq \frac{\genfrac{2}{1}{2}{a+1}{b}{c}}{\genfrac{2}{1}{2}{a}{b}{c}} \leq \frac{2c - b}{2c}. \]
Lemma 2.6. (see [13 Lemma 3.2]) Let \( \Omega \) be an unbounded convex domain in \( \mathbb{C} \) whose boundary is parametrized positively by a Jordan curve \( w(t) = u(t) + iv(t), \) \( 0 < t < 1, \) with \( w(0^+) = w(1^-) = \infty. \) Suppose that \( u(0^+) = +\infty \) and that \( v(t) \) has a finite limit as \( t \to 0^+. \) Then \( v(t) \leq v(0^+) \) for \( 0 < t < 1. \)

In the end of this section, the behavior of a special zero-balanced hypergeometric function \( {}_2F_1(1, 1, 2; z) \) around \( z = 1 \) is given.

Lemma 2.7. For \( \theta \in (0, 2\pi) \), we have

\[
\lim_{\theta \to 0^+} \Re \left[ -e^{i\theta} \log(1 - e^{i\theta}) \right] = +\infty \quad \text{and} \quad \lim_{\theta \to 0^+} \Im \left[ -e^{i\theta} \log(1 - e^{i\theta}) \right] = \frac{\pi}{2}.
\]

Proof. For \( z = e^{i\theta} \) with \( \theta \in (0, 2\pi) \), we do the following computations:

\[
-e^{i\theta} \log(1 - e^{i\theta}) = -e^{i\theta} \left[ 2 \log \left( \frac{\sin \frac{\theta}{2}}{\theta} \right) + i \frac{\theta - \pi}{2} \right]
= -2 \cos \theta \log \left( \frac{\sin \frac{\theta}{2}}{\theta} \right) + \frac{\theta - \pi}{2} \sin \theta - i \left[ 2 \sin \theta \log \left( \frac{\sin \frac{\theta}{2}}{\theta} \right) + \frac{\theta - \pi}{2} \cos \theta \right].
\]

We deduce the required identities by letting \( \theta \to 0^+ \) in the above form. \( \square \)

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.2. Let \( F(z) = {}_2F_1(a, b; c; z), \) \( G(z) = {}_2F_1(a + 1, b; c; z) \) and \( H(z) = {}_2F_1(a + 1, b + 1; c + 1; z) \) for simplicity. In order to obtain the order of convexity of \( zF(z), \) we should first compute its second logarithmic derivative and then evaluate the real part of this derivative in the unit disc \( \mathbb{D}. \) According to the equation (3.2) in [15], we get

\[
(3.1) \quad W(z) := 1 + \frac{z (zF(z))^\prime\prime}{(zF(z))'} = \frac{3 - c + (a + b - 2)z}{1 - z} + \frac{c - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + aG/F)}. \]

We will show that the infimum real part of \( W(z) \) in \( \mathbb{D} \) is attained at \( z = -1 \) under the assumptions of this theorem.

We infer from the linear transformation (see [2 eq. 27] or [4 eq. (2.4)])

\[
\frac{{}_2F_1(a + 1, b; c ; z)}{{}_2F_1(a, b; c ; z)} = \frac{1}{1 - \frac{bz_2F_1(a + 1, b + 1; c + 1; z)}{c} {}_2F_1(a, b; c; z)}, \quad z \in \mathbb{C} \setminus [1, +\infty)
\]

that the denominator of the second term in \( W(z) \) can be transformed into

\[
1 - a + aG/F = \frac{1}{1 - a + a/[1 - b/czH(z)/G(z)]} = \frac{1}{1 - a + a - 1 + \frac{(a-1)b}{c} zH/F.g}.
\]
Substituting the equation above into equation (3.1), we have

\[
W(z) = \frac{3 - c + (a + b - 2)z}{1 - z} + \frac{c - 2 + (1 - a)(1 - b)z}{(1 - a)(1 - z)} + \frac{a}{(a - 1)(1 - z)} \frac{c - 2 + (1 - a)(1 - b)z}{1 + \left(\frac{a-1}{c}H \frac{1}{G}\right) z^1 - z} + \left(\frac{a}{a-1}\right) \frac{1}{1 - z} + \frac{a}{(a - 1)(1 - z)} \frac{c - 2 + (1 - a)(1 - b)z}{1 + \left(\frac{a-1}{c}H \frac{1}{G}\right) z^1 - z}.
\]

Denote

\[
M_1(z) = \frac{c - 2 + (1 - a)(1 - b)z}{1 - z}, \quad M_2(z) = 1 + \frac{\tau H}{G},
\]

and \(M(z) = \frac{M_1(z)}{M_2(z)}\) for simplicity where \(\tau = \frac{(a-1)b}{c}\).

Next we show that under the conditions of this theorem, there exists a probability measure \(\nu\) on \([0, 1]\) such that

\[
(3.2) \quad M(z) = (c - 2) \int_0^1 \frac{d\nu(t)}{1 - tz}.
\]

By virtue of Lemma 2.1, it is sufficient to verify the four conditions there.

The first and second ones are easy to check since \(a, b\) and \(c\) are real constants with \(c > 2\). We proceed to verify the third one. It follows from Lemma 2.2 that there exists a probability measures \(\mu\) on \([0, 1]\) such that

\[
\frac{H(z)}{G(z)} = \int_0^1 \frac{d\mu(t)}{1 - tz},
\]

which implies that

\[
M_2(z) = \int_0^1 \frac{1 + (\tau - t)z}{1 - tz} d\mu(t).
\]

Thus for \(z = x + iy\) with \(y > 0\), an elementary computation generates that

\[
\begin{align*}
\text{Im } M(z)|M_2(z)|^2 &= - \text{Re } M_1 \text{ Im } M_2 + \text{ Im } M_1 \text{ Re } M_2 \\
&= - \int_0^1 \frac{\tau y}{|1 - tz|^2} \frac{c - 2 + (1 - a)(1 - b)|z|^2 - [c - 2 + (1 - a)(1 - b)]x}{|1 - z|^2} \, d\mu(t) \\
&\quad + \int_0^1 \frac{1 + (t^2 - \tau t)|z|^2 + (\tau - 2t)x [c - 2 + (1 - a)(1 - b)]y}{|1 - tz|^2} \, d\mu(t) \\
&\quad := y \int_0^1 \frac{I(t, x, y)}{|1 - z|^2 |1 - tz|^2} \, d\mu(t).
\end{align*}
\]
Letting \( p = c - 2 + (1 - a)(1 - b) \) and \( q = c - 2 - (1 - a)(1 - b) \) for simplicity, the numerator of the integrand can be simplified into

\[
I(t, x, y) = p(t^2 - \tau t)|z|^2 + \tau(1 - a)(1 - b)|z|^2 + p(\tau - 2t)x + \tau qx + p - \tau(c - 2)
\]

\[
= \tau(1 - a)(1 - b) + p(t^2 - \tau t)|z|^2 + p(\tau - 2t) + \tau x + p - \tau(c - 2)
\]

\[
:= Q(t)y^2 + Q(t) \left( x + \frac{\tau(c - 2) - pt}{Q(t)} \right)^2 + \frac{[p - \tau(c - 2)]Q(t) - [\tau(c - 2) - pt]^2}{Q(t)}
\]

\[
:= Q(t)y^2 + Q(t) \left( x + \frac{\tau(c - 2) - pt}{Q(t)} \right)^2 + \frac{S(t)}{Q(t)}
\]

since \( Q(t) \neq 0 \) as \( 1 < a \leq b \). Thus for \( y = \text{Im} z > 0 \), \( \text{Im} M(z) \geq 0 \) is valid if

\[
\begin{cases}
Q(t) = \tau(1 - a)(1 - b) + p(t^2 - \tau t) \geq 0; \\
S(t) = [p - \tau(c - 2)]Q(t) - [\tau(c - 2) - pt]^2 \geq 0,
\end{cases}
\]

hold for any \( t \in [0, 1] \).

We prove the first inequality. By an elementary calculation, we find that

\[
(3.3) \quad \begin{cases}
Q(0) = \tau(1 - a)(1 - b) > 0; \\
Q(1) = c - 2 + \frac{(a - 1)(2b - c)}{c} \geq 0.
\end{cases}
\]

Note also that the conditions \( 1 < a \leq b \leq 4 \) and \( b \leq c \) imply \( 0 < \tau \leq 3 \). Thus if \( 2 \leq \tau \leq 3 \), the equations in \((3.3)\) yield that the quadratic function \( Q(t) \) is nonnegative in \([0, 1]\). Hence we remain to consider the case \( 0 < \tau < 2 \), and in this case the inequality

\[
Q(t) \geq Q \left( \frac{\tau}{2} \right) = \frac{\tau^2}{4b}[4(b - 1)c - pb],
\]

always holds for \( t \in [0, 1] \). Therefore in order to show the non-negativity of \( Q(t) \) on \([0, 1]\), it suffices to prove \( 4(b - 1)c - pb \geq 0 \). As \( 1 < a \leq b \) implies

\[
4(b - 1)c - pb = (3b - 4)c + 2b - (a - 1)(b - 1)b \geq (3b - 4)c + 2b - (b - 1)^2b := h(b, c),
\]

we divide the proof into two cases according to the sign of \( 3b - 4 \).

Case I: Assume that \( 1 < b \leq 4/3 \). Then \( h(b, c) \geq 2b(3b - 4) + 2b - (b - 1)^2b = -b(b - 1)(b - 7) > 0 \) as \( c \leq 2b \).

Case II: Assume that \( b > 4/3 \). Then \( h(b, c) \geq (3b - 4)b + 2b - (b - 1)^2b = -b(b^2 - 5b + 3) \geq 0 \) as \( b \leq c \) and \( 4/3 < b \leq 4 \).

Thus we obtain \( Q(t) \geq 0 \) for \( t \in [0, 1] \) under the assumptions of this theorem.

We next verify the property of \( S(t) \). After computing, we get

\[
S(t) = -p\tau[(c - 2)t + p - (c - 2)(\tau + 2)t] + [p - \tau(c - 2)]\tau(1 - a)(1 - b) - (c - 2)^2\tau^2
\]

and \( S(1) = 0. \) Furthermore an elementary computation yields that \( S(0) = pec^2(2b - c) \), which is nonnegative as \( c \leq 2b \). Therefore the quadratic function \( S(t) \geq 0 \) for all \( t \in [0, 1] \).

In the end, we turn to show the final condition \( \limsup M(-x) \geq 0 \) in Lemma 2.5.

Lemma 2.3 shows that \( \lim_{x \to +\infty} M_2(-x) = \infty \) which in conjunction with \( \lim_{x \to +\infty} M_1(-x) = \infty \)
\[-(a - 1)(b - 1) \neq 0\] generates that
\[
\lim_{x \to +\infty} M(-x) = 0
\]
under the condition \(a \leq b \leq 1 + a\).

We thus obtain the integral form (3.2) which immediately yields that
\[
W(z) = 1 - a + \frac{a(a + 1 - c)}{a - 1} \frac{1}{1 - z} + \frac{a(c - 2)}{a - 1} \int_0^1 \frac{d\nu(t)}{1 - tz}.
\]
where \(\nu\) is a probability measure on \([0, 1]\). Therefore \(W(z)\) is analytic on \(D\) and since \(a > 1\) and \(c > 2\), then (3.4)
\[
\Re W(z) \geq W(-1), \quad \text{for } |z| = 1.
\]
Next we will prove the above inequality for all \(z \in D\) which is exactly the assertion of this theorem. If \(c \leq a + 1\) in addition, it is obvious that inequality (3.4) holds for \(z \in D\) since \(\Re 1/(1 - z) > 1/2\) is valid for \(z \in D\). We need only to deal with the case \(c > a + 1\). First we observe that \(c \leq 1 + a + b\) as \(c \leq 2b\) and \(b \leq 1 + a\). As a direct consequence of Lemma 2.4, we conclude that
\[
\lim_{z \to 1} \frac{G(z)}{F(z)} = +\infty,
\]
for \(1 + a < c \leq 1 + a + b\). Thus it follows from the above equation that
\[
W(z) = \frac{3 - c + (a + b - 2)z}{1 - z} + \frac{e - 2 + (1 - a)(1 - b)z}{(1 - z)(1 - a + aG/F)}
\]
\[
= \frac{3 - c + (a + b - 2)z}{1 - z} + o(1) + \frac{1}{1 - z}
\]
\[
= \frac{1 + a + b - c + o(1)}{1 - z}
\]
as \(z \to 1\) along the real axis in \(D\). Since \(W(z)\) is analytic in \(D\), we obtain that (3.4) holds for all \(z \in D\) if \(c < 1 + a + b\). As for the case \(c = 1 + a + b\), by virtue of equation (2.3) in Lemma 2.4, we conclude that
\[
W(z) = 2 - a - b + \frac{e - 2 + (1 - a)(1 - b)z}{1 - z} \frac{1}{1 - a - ab \log(1 - z)} + O(1)
\]
\[
\to +\infty,
\]
as \(z \to 1\) along the real axis in \(D\). Thus the inequality (3.4) holds for all \(z \in D\) if \(c = 1 + a + b\), since \(W(z)\) is analytic in \(D\).

We complete the proof. \(\square\)

Proof of Theorem 1.3. We first prove that \(z_2 F_1(a, b; a + b; z)\) is convex in \(D\) under the assumptions. If \(a = 1\) and \(1 \leq b \leq 4\), the convexity of \(z_2 F_1(a, b; a + b; z)\) is a consequence of the case (2) in Theorem C. While if \(1 < a \leq b \leq 3\), the order of convexity
of \( z_2F_1(a, b; a + b; z) \) is

\[
\kappa = \frac{5 - 2a - 2b}{2} + \frac{2(a + b) - 3 - ab}{2 \left( 1 - a + a \frac{\sum_{i=0}^{\infty} \binom{a+b}{i+1} \left( -\frac{a+b}{2} \right)^i}{2F_1(a, b; a + b; 1)} \right)}
\]

according to Theorem 1.2. It is easy to see that \( 2(a + b) - 3 - ab \geq 0 \) since \( 1 < a \leq b \leq 3 \). Thus in view of Lemma 2.5, we get

\[
\kappa \geq \frac{5 - 2a - 2b}{2} + \frac{2(a + b) - 3 - ab}{2 \left( 1 - a + a \frac{2a+b}{2(a+b)} \right)} = \frac{4 - 5a}{b} + \frac{4a}{2(2a + 2b - ab)}.
\]

By observing the numerator of the above lower bound, we conclude that if \( 1 < a \leq b \leq \frac{4a}{5a - 4} \), then the order of convexity \( \kappa \) is non-negative, which means \( z_2F_1(a, b; a + b; z) \) is convex in \( \mathbb{D} \).

On the other hand, we consider the image domain \( \Omega \) of \( \mathbb{D} \) under the mapping \( z \mapsto z_2F_1(a, b; a + b; z) \). For \( 0 < t < 1 \), let \( w(t) = u(t) + iv(t) \) stand for the boundary \( e^{2\pi it}z_2F_1(a, b; a + b; e^{2\pi it}) \) of the domain \( G \), which is a Jordan convex curve. In view of Lemma 2.7 and Ramanujan’s formula (2.2), we have

\[
\lim_{t \to 0^+} w(t) = \lim_{t \to 1^-} w(t) = \infty, \quad \lim_{t \to 0^+} u(t) = +\infty \quad \text{and} \quad \lim_{t \to 0^+} v(t) = \frac{\pi}{2B(a, b)}.
\]

Therefore it follows from Lemma 2.6 that \( v(t) \leq v(0^+) \) holds for \( 0 < t < 1 \). Since the image domain \( G \) is symmetric with respect to the real axis, we finally have

\[
|v(t)| \leq \frac{\pi}{2B(a, b)}, \quad \text{for} \ 0 < t < 1,
\]

which means that the image domain of \( \mathbb{D} \) under the function \( z_2F_1(a, b; a + b; z) \) is bounded by two horizontal lines \( \operatorname{Im} w = \pm \frac{\pi}{2B(a, b)} \).

The proof is finished. \( \square \)

Acknowledgements. The author would like to appreciate Professor Toshiyuki Sugawa for discussions and guidance, without whom I could’t finish this work. The author is also indebted to Professor Matti Vuorinen for drawing my attention to this problem.

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