The Scattering Theory of CP Violation

Decio Cocolicchio

*Dipartimento di Matematica, Univ. Basilicata, Potenza, Italy*

*Via N. Sauro 85, 85100 Potenza, Italy*

*Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy*

*Via G. Celoria 16, 20133 Milano, Italy*

ABSTRACT

The mixing effects and the $CP(T)$–violating formalism for Kaon, $B$–mesons and similar unstable oscillating systems, are recovered by means of a method based on the properties of the complex singularities in the $S$–matrix theory with unstable intermediate states. General $q^2$–dependent relations for $CP$–asymmetries are then introduced.
I. Introduction

Recently, there has been a renewed interest in the theoretical approach to neutral meson systems like $K^0 - \bar{K}^0$, $B^0 - \bar{B}^0$ or $D^0 - \bar{D}^0$, due mainly to the planned advances in the experimental methods in DAΦNE project at Frascati (Rome) and BABAR at SLAC (Stanford) to measure their small observables. Therefore, it becomes important to predict carefully these tiny effects. From the most general point of view of non relativistic Quantum Mechanics, the problem consists in defining correctly the position dependent probability which let us distinguish the factorized constituents ($K^0$ or $\bar{K}^0$ for example, but they can be also $B^0$ or $D^0$ systems) of the entangled mass eigenstates $K_{L,S}$ or $CP$-eigenstates $K_{1,2}$. In doing so, usually one restricts to a single pole (Lee-Oehme-Yang or LOY) approximation \cite{1} to describe the temporal evolution of the neutral kaon complex. The controversial point of the validity of this approximation consists in the fact that the characteristic time dependence of the survival probability of any metastable state $|\phi\rangle$

$$P(t) = |\alpha(t)|^2 = |\langle \phi|e^{-iHt}|\phi\rangle|^2 = \int e^{-iEt} \rho(E)dE$$  \hspace{1cm} (1)

can deviate from a pure exponential in the case that the decay energy spectrum

$$\rho(E) = |\langle \psi_E|\phi\rangle|^2$$  \hspace{1cm} (2)

is unbounded, being $|\psi_E\rangle$ the eigenstate of an effective Hamiltonian: $H|\psi_E\rangle = E|\psi_E\rangle$. In recent years, such a problem has been discussed extensively in the case of the neutral kaon system \cite{2}, with controversial results about the eventual physical effects of the regeneration of the physical kaons in the vacuum as due to the off-diagonal quenching of the resonant contributions. Nevertheless, since we are dealing with an approximate theory, it is not surprising that one expects departures from its predictions and an update of the LOY formalism seems needed \cite{3}. Deviations from the exponential decay law in the time evolution of any metastable system is expected at very short or very long times as compared to the lifetime of the unstable particle \cite{4} and in dependence of the structure of the prepared initial state \cite{5}. These basic features can be introduced
in the formalism of the density matrix with the generalization of the Liouville master equation which governs the decay and the evolution of the neutral kaon system and, in the same time, can distinguish the factorized constituents $K_{L,S}$ of the entangled state $K^0 - \bar{K}^0$ [6]. An alternative approach expresses the survival amplitude in terms of the relative propagator function and consider the non relativistic limit of the space-time evolution in terms of the coherent overlapping of two Wightmann expanded wave-packets which somehow mimics the interference effects of two kaons [7]. This treatment advocated the presence of two different energies and momenta in order to account for the separation between the $K_S$ and the $K_L$ states. Moreover, the breakdown of the exponential decay law can be noted also in the context of $S$-matrix scattering field theory [8]. In this context, the correct treatment of unstable oscillating systems, and our present understanding of the $CP$–violating effects, seems to be described by several formalisms: including the Wigner $R$–matrix approach (mainly adopted in nuclear theory [9]), the $K$-matrix technique [10], and methods based purely on the framework of the $S$-matrix perturbation theory [11]. The correct quantum field treatment of unstable particles within the framework of the $S$-matrix perturbation theory [4], is motivated mainly by the strength to generalize the Breit-Wigner propagator with the inclusion of the resummation of higher order quantum corrections, which often results in several pathologies in gauge field theories. Even though this approach may eventually furnishes gauge invariant results, nevertheless, the perturbation treatment introduces residual threshold terms which become more effective when $CP$–violating effects are considered [12]. The advanced features of today experimental methods suggest then to improve the phenomenological assumptions for those observables for which the $CP$–violating contributions are significant. One of the most sensible physical ground is the entangled $K^0 - \bar{K}^0$ interference amplitude. Although, its observables rest inextricably dominated by mixing (i.e. negligible “direct” $K \rightarrow (\pi\pi)_I$ contributions [13]), the assumption that, in general, the mixing of elementary particles is independent of the momentum squared of the underlying intermediate channels is more problematic, mainly in the case of the vector modes [14]. Width effects, for instance, have been suggested as a possible mechanism for generating resonant $CP$–violating contributions that could provide a window into new physics [15]. However, the inclusion of the precise $q^2$–dependence, usually thought to be small, could have unexpectedly large effects on the extraction
of those sensible observables which give significant contributions to \( CP \)-asymmetries. Nevertheless, the usual effective complex mass matrix methods of the narrow width LOY approximation neglects this effect, and also it rests completely inappropriate to implement the notion of the rest frame for an oscillating unstable composite system. All that, in turn, seems to require an explicit relativistic description of the kaon system, also in order to avoid the phase ambiguities that arise when the superposition of states with different mass and momentum are described in a Galilean invariant form [16]. Indeed, the essential content of the time dependent properties of the kaon complex can be introduced without recourse to the S-matrix formalism, by considering simply the subtleties related to the location of the complex singularities in the multiple sheets of the Riemann surface into which the Fourier transform of the propagator can be continued analytically. Such propagator’s method has the great advantage to appear natural and indeed independent of various production and decay mechanisms, although it is not immediate to have a model and to solve the ambiguities connected to its complex analytical structure. The propagator formalism for the \( K^0 - \bar{K}^0 \) system [17] results a rigorous relativistic treatment which arises naturally in the context of quantum field theory. In this paper, we propose to derive some general properties of kaon mixing in some detail, focusing on superweak CP-violating processes and combining the dynamics of the complex pole of the kaon field propagator with the results of the spectral formalism of the time dependent perturbation method.

II. The Kaon Complex Within and Beyond the Wigner–Weisskopf Narrow Width Approximation.

In this section, we investigate the shortcomings of the Lee-Oehme-Yang theory [1] of the decay and evolution of the neutral kaon system. It is characteristic of the Wigner–Weisskopf narrow width approximation that the effective Hamiltonian acts upon a Hilbert subspace in which the exponential decay law is assured in order to generate a dynamical semigroup evolution. The time evolution of the flavour states

\[
\begin{pmatrix}
|K^0(t)\rangle \\
|\bar{K}^0(t)\rangle
\end{pmatrix}
= \mathcal{U}(t)
\begin{pmatrix}
|K^0\rangle \\
|\bar{K}^0\rangle
\end{pmatrix},
\]

(3)
and similarly for the mass right–eigenstates

\[
\begin{pmatrix}
|K_S(t)\rangle \\
|K_L(t)\rangle
\end{pmatrix} = \mathcal{V}(t) \begin{pmatrix}
|K_S\rangle \\
|K_L\rangle
\end{pmatrix}
\]  

(4)

are governed by the matrix elements

\[
U_{ij} = \langle K_i | \exp \left[ -\frac{i}{\hbar} \mathcal{H} t \right] | K_j \rangle
\]

\[
V_{\alpha\beta} = \langle K'_\alpha | \exp \left[ -\frac{i}{\hbar} \mathcal{H} t \right] | K_\beta \rangle
\]

(5)

where the latin indices denote \(K^0, \bar{K}^0\) and the greek letters denote \(K_S, K_L\), respectively. The evolution matrices \(U\) and \(V\) are then related by the following similarity transformation

\[
U = \mathcal{R} \mathcal{V} \mathcal{R}^{-1}
\]

(6)

The effective Hamiltonian matrix is determined by eight real parameters, but only seven are physically meaningful because the absolute phase of \(H_{12}\) or \(H_{21}\) is meaningless, being the relative phase of \(|K^0\rangle\) and \(|\bar{K}^0\rangle\) arbitrary. They can be substituted by the two complex eigenvalues

\[
\lambda_S = \frac{1}{2} (C - D)
\]

\[
\lambda_L = \frac{1}{2} (C + D)
\]

(7, 8)

where, in a general theory, \(C = \lambda_L + \lambda_S = H_{11} + H_{22} = \text{tr} \mathcal{H}\) and \(D^2 = (\lambda_L - \lambda_S)^2 = (H_{11} - H_{22})^2 + 4H_{12}H_{21} = (\text{tr} \mathcal{H})^2 - 4(\text{det} \mathcal{H})\), and the two complex mixing parameters \(\epsilon_{S,L}\) are given by:

\[
\epsilon_s = \left( \frac{2H_{12} - D}{2H_{12} + D} \right) - \left( \frac{4H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} - H_{22} + D + 2H_{12}} \right) = \epsilon - \delta_S
\]

\[
\epsilon_L = \left( \frac{2H_{12} - D}{2H_{12} + D} \right) - \left( \frac{4H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} - H_{22} - D - 2H_{12}} \right) = \epsilon - \delta_L
\]

(9)

where

\[
\epsilon = \frac{2H_{12} - D}{2H_{12} + D} = \frac{\sqrt{H_{12}} - \sqrt{H_{21}}}{\sqrt{H_{12}} + \sqrt{H_{21}}} = \frac{1}{iD} (-\text{Im} M_{12} + i\text{Im} \Gamma_{12})
\]

\[
\delta_S = \left( \frac{2H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} + H_{12} - \lambda_S} \right)
\]

\[
\delta_L = \left( \frac{2H_{12}}{2H_{12} + D} \right) \left( \frac{H_{11} - H_{22}}{H_{11} - H_{12} - \lambda_L} \right)
\]

(10)
The complex scaling matrix \( \mathcal{R} \), which diagonalizes the effective Hamiltonian, is then given by
\[
\mathcal{R} = \left( \begin{array}{cc} \frac{a_S}{2H_{12}}(H_{11} - H_{22} + D) & -\frac{a_L}{2H_{12}}(H_{11} - H_{22} - D) \\ N_S(1 + \epsilon_S) & N_L(1 + \epsilon_L) \\ -N_S(1 - \epsilon_S) & N_L(1 - \epsilon_L) \end{array} \right) = \left( \begin{array}{cc} p & p' \\ -q & q' \end{array} \right),
\]
where \( a_S \) and \( a_L \) are fixed only once the eigenvectors normalization is realized and
\[
\eta_S = \frac{-q}{p} = \frac{(1 - \epsilon_S)}{(1 + \epsilon_S)} = \frac{-(H_{11} - H_{22} + D)}{2H_{12}} = \frac{2H_{21}}{(H_{11} - H_{22} - D)}
\]
\[
\eta_L = \frac{q'}{p'} = \frac{(1 - \epsilon_L)}{(1 + \epsilon_L)} = \frac{-(H_{11} - H_{22} - D)}{2H_{12}} = \frac{2H_{21}}{(H_{11} - H_{22} + D)}
\]
being \( N_{S,L}^2 = 2(1 + |\epsilon_{S,L}|^2) \). In any CPT–invariant theory, \( H_{11} = H_{22} \) and there is only one mixing parameter \( \epsilon = \epsilon_S = \epsilon_L \) and therefore \( \eta_S = -\eta_L, N_S^{-2} = N_L^{-2} = N^{-2} = 2(1 + |\epsilon|^2), p = p', q = q' \). In this case, we have that
\[
\lambda_S = H_{11} - \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2}\Gamma_{11} - \frac{D}{2} = m_S - \frac{i}{2}\gamma_S
\]
\[
\lambda_L = H_{11} + \sqrt{H_{12}H_{21}} = M_{11} - \frac{i}{2}\Gamma_{11} + \frac{D}{2} = m_L - \frac{i}{2}\gamma_L
\]
with
\[
D = 2\sqrt{H_{12}H_{21}} = 2\sqrt{(M_{12} - \frac{i}{2}\Gamma_{12})(M_{12}^* - \frac{i}{2}\Gamma_{12}^*)} = \left( \Delta m - \frac{i}{2}\Delta \gamma \right).
\]
These real \((m_{S,L})\) and imaginary \((\gamma_{S,L})\) components will define the masses and the decay widths of the \( \mathcal{H} \) eigenstates \( K_S \) and \( K_L \) in the narrow width approximation. These short- and long-lived particles result then a linear combination of the flavour \( K^0 \) and \( \bar{K}^0 \) states:
\[
\left( \begin{array}{c} |K_S\rangle \\ |K_L\rangle \end{array} \right) = \mathcal{R}^t \left( \begin{array}{c} |K^0\rangle \\ |\bar{K}^0\rangle \end{array} \right),
\]
where usually \( \mathcal{R}^t \) is preferably parameterized according to the following relations
\[
\mathcal{R}^t = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left( \begin{array}{cc} (1 + \epsilon) & -(1 - \epsilon) \\ (1 + \epsilon) & (1 - \epsilon) \end{array} \right) = \frac{1}{\sqrt{1 + |\eta|^2}} \left( \begin{array}{cc} 1 & \eta \\ 1 & -\eta \end{array} \right) = \left( \begin{array}{cc} p & -q \\ p & q \end{array} \right).
\]
After corresponding normalization of the eigenvectors
\[
\langle K_L|K_L\rangle = \langle K_S|K_S\rangle = |p|^2 + |q|^2 = 1 ,
\]
(18)
the impurity parameters can be connected by the simple relations

$$\epsilon = \frac{p - q}{p + q} = \frac{(\sqrt{H_{12}} - \sqrt{H_{21}})}{(\sqrt{H_{12}} + \sqrt{H_{21}})} = i \frac{\text{Im} M_{12} - \frac{i}{2} \text{Im} \Gamma_{12}}{\text{Re} M_{12} - \frac{i}{2} \text{Re} \Gamma_{12} + \frac{D}{2}}.$$  \hspace{1cm} (19)

We remember that the phases of $p$, $q$ may be altered by redefining the phases of the $K$ states. To the extent that $M_{12}$ and $\Gamma_{12}$ share the same phase: $M_{12} = |M_{12}|e^{i\phi}$, $\Gamma_{12} = |\Gamma_{12}|e^{i\phi}$, there is no CP-violation, since $\epsilon = i \frac{\sin \phi}{1 + \cos \phi}$. In general, by means of a suitable choice of the relative phase between $K_0$ and $\bar{K}_0$ one can make $\Gamma_{12}$ real negative. Anyway, both $p$ and $q$ are not measurable quantities, whereas the overlap between $K_S$ and $K_L$

$$\langle K_S | K_L \rangle = \frac{2 \text{Re} \epsilon}{1 + |\epsilon|^2} = \frac{1 - |\eta|^2}{1 + |\eta|^2} \approx \frac{2z}{4z^2 + 1} \arg(M_{12}) \hspace{1cm} (20)$$

is independent of any phase convention, with $z = \frac{|M_{12}|}{|\Gamma_{12}|} = (0.477 \pm 0.003)$. Nevertheless, the magnitude of

$$\eta = \frac{-q}{p} = \frac{1 - \epsilon}{1 + \epsilon} = -\sqrt{\frac{M_{12}^* - \frac{i}{2} \Gamma_{12}^*}{M_{12} - \frac{i}{2} \Gamma_{12}}}$$

is certainly independent of any phase convention and, assuming $\Delta S = \Delta Q$ rule conserved, it is directly connected to the amount of the kaon semileptonic charge rate

$$A_{SL} = \frac{\Gamma(K_L \to \ell^+\nu X) - \Gamma(K_L \to \ell^-\nu X)}{\Gamma(K_L \to \ell^+\nu X) + \Gamma(K_L \to \ell^-\nu X)} = \frac{1 - |\eta|^2}{1 + |\eta|^2}. \hspace{1cm} (22)$$

Its experimental value is $A_{SL} = (3.27 \pm 0.12) \cdot 10^{-3}$ and then the relative phase between $M_{12}$ and $\Gamma_{12}$ results $(6.53 \pm 0.24) \cdot 10^{-3}$. In the Wu-Yang convention $\text{Im} \Gamma_{12} = 0$, we obtain that

$$\arg(\epsilon) \approx \begin{cases} \pi - \Phi_{SW} & \text{for } \text{Im}M_{12} > 0 \\ \Phi_{SW} & \text{for } \text{Im}M_{12} < 0 \end{cases} \hspace{1cm} (23)$$

being the superweak phase $\Phi_{SW} = \tan^{-1}(2z)$. Another powerful experimental observable, which can determine the CP violation mainly in the $B_d$-meson mixing [18], in pair decays

$$e^+e^- \longrightarrow \Upsilon \longrightarrow (B^0\bar{B}^0)_L \longrightarrow (\ell^-\nu X^+)(\ell^+\nu Y^-). \hspace{1cm} (24)$$
is the dilepton charge asymmetry

\[
A_{CP} = \left| \frac{\langle \ell^+ \ell^+ | \mathcal{H} | \bar{B} B \rangle^2 - \langle \ell^+ \ell^- | \mathcal{H} | \bar{B} B \rangle^2}{\langle \ell^+ \ell^+ | \mathcal{H} | \bar{B} B \rangle^2 + \langle \ell^+ \ell^- | \mathcal{H} | \bar{B} B \rangle^2} \right| = \frac{|\eta|^4 - 1}{|\eta|^4 + 1} = -\frac{4 \text{Re} \epsilon (1 + |\epsilon|^2)}{(1 + |\epsilon|^2)^2 + 4(\text{Re} \epsilon)^2} = \left( \frac{\text{Im} \Gamma_{12}}{\Gamma_{12}} \right) \frac{1}{1 + \frac{1}{4} \frac{\Gamma_{12}}{|\Gamma_{12}|}} \tag{25}
\]

However, in the \( B \) sector, the situation is still unclear, being \( z_B = \frac{|M_B|}{|\Gamma_{12}|} \) very large, \( |\eta| \approx 1 \) and \( A_{CP} \) under investigation \[19\]. From the known experimental inputs of \( x_d \approx \Delta m_B \tau_B = (0.67 \pm 0.10) \) and the fact that \( \Delta m_B \approx 2|M_{12}^B| = (4.0 \pm 0.8) \cdot 10^{-13} \) we find that \( \arg(M_{12}^B) \approx \left( \frac{\text{Im} M_{12}^B}{|M_{12}^B|} \right) \) is in a wide range between 0.03 and 0.73. It can be easily shown that \( |\eta| = 1 \) is a necessary and sufficient condition for the \( CP \) conservation in the mixing.

The method, we outlined, appears more transparent from a phenomenological point of view, just because its components are expressed directly in terms of observables. Coming back to the specimen situation of the kaon complex, in order to find the structure of the time evolution matrix elements \( U_{ij} \), we need to generalize the relevant analytic properties of the usual quantum theory of the reduced resolvent of a linear operator to the case of a non Hermitian Hamiltonian. Moreover, here, we prefer do not consider a specific dynamical model \[2\] and we suppose only to decompose the original non Hermitian Hamiltonian into two non commutating part

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \tag{26}
\]

where \( \mathcal{H}_0 \) contains the strong interactions under which the kaons appear as stable particles and where \( \mathcal{H}_I \) is supposed to induce their decay modes into the continuum spectrum of \( \mathcal{H}_0 \). If the interaction is small, the time evolution matrix elements can then be derived extracting the poles \( \lambda_\alpha \) on the second Riemann sheet and studying the expansion criteria for these poles dominance in the spectral decomposition

\[
U_{ij} = \int_{\text{Spec}(H)} d\lambda e^{-i\lambda t} \rho_{ij}(\lambda) \tag{27}
\]

where the integration extends over the whole spectrum of the Hamiltonian and \( \rho_{ij}(\lambda) \) is given by

\[
\rho_{ij} = \langle K_i | \mathcal{G} | K_j \rangle \tag{28}
\]
where $G(z) = [zI - \mathcal{H}]^{-1}$ rigorously must represent the complete reduced resolvent. Beyond the pole dominance, the details of the result depend of course upon the spectral matrix $\rho_{ij}(\lambda)$. Nevertheless, a sensible spectrum is expected to be bounded from below and it is possible to normalize the ground state to have the zero energy eigenvalue. The integration range of the spectrum then extends from 0 to $\infty$. Despite this cut-off, it is evident that the complete time evolution of the system is determined in principle by the positivity requirement of the spectral content of the initial state. The Wigner Weisskopf approximation amounts to assume that the spectrum density $\rho_{ij}$ has simple poles. In this narrow width approximation, the time evolution matrix is then

$$U(t) = \begin{pmatrix} f_+(t) & \frac{1}{\eta} f_- (t) \\ \eta f_- (t) & f_+(t) \end{pmatrix}$$

(29)

where $f_{\pm} (t) = \frac{1}{2} (V_{ss} \pm V_{ll}) = \frac{1}{2} (e^{-i \lambda \tau} \pm e^{-i \lambda L \tau})$. Nevertheless, since it is an approximate theory, it is not surprising that, for example, this procedure cannot satisfy exactly the unitarity requirement [20], which is essential for the basic interpretation of any theory. The assumption of constant decay rates, as they arise by the unitarity sum rules [21] connecting the kaon system with the space of all the decaying final states, then, cannot be justified in an exact sense. In fact, for instance, the modulus of the ratio of the off-diagonal elements

$$r(t) = \frac{U_{12}(t)}{U_{21}(t)} = \left( \frac{p}{q} \right)^2$$

(30)

differs from unity and could vary with time [2] if the time-reversal $T$–invariance is not a symmetry of the underlying Hamiltonian. In fact, in this case, the condition of reciprocity is not properly satisfied [20]. Indeed, the inclusion of off-diagonal terms $V_{sl} = -V_{ls}$ in the evolution matrix $\mathcal{V}$, induces a modification both of the time evolution matrix and also of the previous ratio. Assuming a global $CPT$–invariant propagation, in fact, it becomes

$$r(t) = \left( \frac{p}{q} \right)^2 \left( \frac{1 - A}{1 + A} \right)$$

(31)

On more general assumptions like causality and analycity, the time dependence of the vacuum regeneration term $A = (V_{sl} - V_{ls})/(V_{ss} - V_{ll})$ can be obtained to study the expansion criteria for pole dominance. The details of the results depend, of course, upon the spectral function $\rho_{ij}$ defined in Eq. (28). If the interacting sector $\mathcal{H}_I$ of the Hamiltonian is small, there will be two (non degenerate) poles $\lambda_\alpha$ on the second Riemann
sheet. At this level of generalization [22] the time evolution matrix elements are given according to the following expression

\[ U_{ij} = \sum_{r=0}^{\infty} \frac{1}{2\pi i} \int_{C} e^{-izt} \langle K_i|((G_0 H_I)^r G_0|K_j)dz \]  

(32)

where \( C \) is a closed curve encircling all the complex eigenvalues of the effective Hamiltonian \( H \) and

\[ G_0(z) = \frac{1}{z\mathcal{I} - \mathcal{H}_0} \]  

(33)

represents the analytically continued Green’s function of the non interacting theory.

III. The Relativistic Treatment of the Neutral Kaon System.

In this section, we reconsider the propagator formalism of the Quantum Field Theory for the scalar mesons mixing, at the basis for the correct approach of the \( K^0 - \bar{K}^0 \) mixing. In the absence of weak interactions, \( K^0 \) and \( \bar{K}^0 \) are eigenstates of the strong interactions and form a degenerate particle–antiparticle pair in flavour state, with a common mass \( m_\circ \) (whatever we assume CPT invariance). When higher order weak interactions are introduced, transitions are induced between \( K^0 \) and \( \bar{K}^0 \). Thus, mixing, due to quantum corrections, prohibits the \( K^0, \bar{K}^0 \) scalar mesons from propagating independently of each other. Consequently, the propagator of the kaon system has to be considered as a \( 2 \times 2 \) matrix. In other words, since strangeness is not conserved in the kaon decays, we must consider two propagators, one for each sense of strangeness. Therefore, to describe the \( K^0 - \bar{K}^0 \) transitions induced by higher order weak interactions, the essential tools consist in introducing four full dressed kaon propagators:

\[ \Delta'_{ij}(k^2) : \Delta'_{\circ\circ}(k^2) , \ \Delta'_{\circ\bar{\sigma}}(k^2) , \ \Delta'_{\bar{\sigma}\circ}(k^2) , \ \Delta'_{\bar{\sigma}\bar{\sigma}}(k^2) , \]  

(34)

which can be computed perturbatively using one-particle irreducible self-energy parts

\[ \Pi_{ij}(k^2) : \Pi_{\circ\circ}(k^2) , \ \Pi_{\circ\bar{\sigma}}(k^2) , \ \Pi_{\bar{\sigma}\circ}(k^2) , \ \Pi_{\bar{\sigma}\bar{\sigma}}(k^2) . \]  

(35)

In these expressions, the subscripts \( \circ, \bar{\sigma} \) correspond to incoming and outgoing eigenstates of the strong interactions \( K^0 \) and \( \bar{K}^0 \). The regularized propagators connecting the \( K^0 \)
and $K^0$ states to themselves and to each other can be obtained according to the matrix Dyson equation

$$\Delta' = \Delta + \Delta \Pi \Delta'$$  \hspace{1cm} (36)$$

where the following matrix notation has been used

$$\Delta = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}, \quad \Delta' = \begin{pmatrix} \Delta'_{\sigma\sigma} & \Delta'_{\sigma\pi} \\ \Delta'_{\pi\sigma} & \Delta'_{\pi\pi} \end{pmatrix}, \quad \Pi = \begin{pmatrix} \Pi_{\sigma\sigma} & \Pi_{\sigma\pi} \\ \Pi_{\pi\sigma} & \Pi_{\pi\pi} \end{pmatrix}.$$  \hspace{1cm} (37)$$

If we turned off weak interactions, the bare components of the tree level propagator are given by

$$\Delta_{ij}(k^2) = \frac{\delta_{ij}}{k^2 - m_0^2}$$  \hspace{1cm} (38)$$

In presence of interactions, the full expression for the dressed kaon propagator matrix $\Delta'$ can be determined in principle by the inversion of

$$[\Delta']^{-1} \approx \left( \Delta^{-1} - \Pi \right) = \begin{pmatrix} [k^2 - m_0^2 + \Pi_{\sigma\sigma}] & \Pi_{\sigma\pi} \\ \Pi_{\pi\sigma} & [k^2 - m_0^2 + \Pi_{\pi\pi}] \end{pmatrix}$$  \hspace{1cm} (39)$$

Keeping the leading terms, we find

$$\Delta'_{\sigma\sigma} = \left( k^2 - m_0^2 + \Pi_{\sigma\sigma} \right) - \frac{\Pi_{\sigma\pi} \Pi_{\pi\pi}}{k^2 - m_0^2 + \Pi_{\pi\pi}}^{-1},$$
$$\Delta'_{\pi\pi} = - \Pi_{\pi\sigma} \left( k^2 - m_0^2 + \Pi_{\pi\sigma} \right) \left( k^2 - m_0^2 + \Pi_{\sigma\sigma} \right) - \Pi_{\sigma\pi} \Pi_{\pi\pi}^{-1},$$
$$\Delta'_{\sigma\pi} = \left( k^2 - m_0^2 + \Pi_{\sigma\pi} \right) - \frac{\Pi_{\sigma\sigma} \Pi_{\pi\sigma}}{k^2 - m_0^2 + \Pi_{\pi\pi}}^{-1},$$
$$\Delta'_{\pi\sigma} = \left( k^2 - m_0^2 + \Pi_{\pi\sigma} \right) - \frac{\Pi_{\sigma\pi} \Pi_{\pi\sigma}}{k^2 - m_0^2 + \Pi_{\pi\pi}}^{-1}$$  \hspace{1cm} (40)$$

where the dependence of the self-energies on $k^2$ is implied. The renormalized self-energies $\Pi_{ij}(k^2)$ vanish by definition as $k^2 \to m_0^2$ at least as fast as $(k^2 - m_0^2)^2$. The subtleties of the higher order terms in the elements of the propagators are relevant only close the $(K^0\bar{K}^0)_L$ resonance. We assume that the pure fields $K^0$ and $\bar{K}^0$ have already been renormalized in the sense that the relevant counterterms have been absorbed into the mass and wavefunction renormalized factors. As a consequence of $CPT$ invariance, the diagonal matrix elements are equal:

$$\langle \bar{K}^0 | \Delta' | K^0 \rangle = \langle K^0 | \Delta' | K^0 \rangle$$  \hspace{1cm} (41)$$
whereas the off-diagonal elements $\Delta'_{ij}$ are equal only in the case the interactions are all $CP$ invariant. In order to describe the kaon system in terms of uncoupled channels, we need to diagonalize $\Delta'$. In principle, the matrix propagator

$$\Delta' = \left[Ik^2 - \Lambda(k^2)\right]^{-1},$$

where the effective square-mass matrix is given by

$$\Lambda(k^2) = \left[m_0^2 I + \Pi(k^2)\right],$$

can be brought into a diagonal form

$$\Delta'_{\alpha\beta}(k^2) = \begin{pmatrix} \Delta'_{S}(k^2) & 0 \\ 0 & \Delta'_{L}(k^2) \end{pmatrix}$$

through a $k^2$-dependent transformation. We may suppose that the dynamics of the system is governed by the poles $\lambda_{S,L}$ in the propagator that are the complex solutions deriving from the vanishing of the determinant of the inverse propagator

$$\det(\lambda^2 I - \Lambda) = 0$$

This dispersion relation is just the equation which locates the position of poles and let us write the diagonal propagator according to the following expression

$$\Delta'_{\alpha\beta} = \begin{pmatrix} \frac{1}{k^2 - \lambda^2_S} & 0 \\ 0 & \frac{1}{k^2 - \lambda^2_L} \end{pmatrix}.$$ 

Nevertheless, since in our case the coupling is weak, it is sufficient to approximate

$$\Pi_{ij}(k^2) \simeq \Pi_{ij}(m_0^2).$$

In this case,

$$\Lambda = \left[m_0^2 I + \Pi(m_0^2)\right]$$

represents the square of an effective complex mass matrix. It can be diagonalized by a similarity complex transformation $R:

$$[RAR^{-1}]_{\beta\alpha} = \lambda^2 \delta_{\beta\alpha}$$
where, in brief we obtain
\[ \lambda^2_S = \frac{1}{2} [(\Lambda_{11} + \Lambda_{22}) - Q] \]
\[ \lambda^2_L = \frac{1}{2} [(\Lambda_{11} + \Lambda_{22}) + Q] \]
(50)

with
\[ Q = \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}\Lambda_{21}} \]
(51)

The complex scaling matrix \( \mathbf{R} \) exists unless both \( Q = 0 \) and the off–diagonal elements \( \Lambda_{12}, \Lambda_{21} \) are different from zero. The physical fields \( K_L \) and \( K_S \) corresponding to the eigenvalues
\[ \lambda^2_{S,L} = \left( m_{S,L} - \frac{i}{2} \gamma_{S,L} \right)^2 \approx m^2_{S,L} - im_{S,L} \gamma_{S,L} \]
(52)

are combinations of the \( K^0 \) and \( \overline{K}^0 \) for which only the diagonal elements of the propagator matrix contain poles in the \( K_L, K_S \) basis. We define the transformation between the physical and flavour pure bases as usual with relations analogous to Eq. (16). Any invariance of the theory will reflect itself in an invariance of the propagator and then also of the square mass matrix \( \Lambda \).

The fact that \( \Lambda \) is, in general, momentum dependent does not introduce any additional complications, in practice, since \( k^2 \) is always fixed by the on-shell condition of the initial particles. Anyway, the resulting eigen-physical fields are those with a definite propagation behaviour. However, disregarding, at the moment, the complications regarding the higher order differences among the possible schemes of renormalization, for a given \( s = k^2 \), we can write the regularized inverse propagator as

\[ \Delta'^{-1}_{ij} = [sI - \Lambda] = \mathbf{R}_{i\alpha} \Delta'^{-1}_{\alpha\beta} \mathbf{R}^{-1}_{\beta j} \]
(53)

where
\[ \Delta'^{-1}_{\alpha\beta} = [s\delta_{\alpha\beta} - N_{\alpha\beta}] \]
(54)

and
\[ N_{\alpha\beta} = [\mathbf{R}^{-1}\Lambda\mathbf{R}]_{\alpha\beta} . \]
(55)

In general, \( \Pi \) (and hence \( \Lambda \)), is momentum dependent. Expanding with respect to the \( s_p \) pole of the scattering amplitude where \( \Delta'^{-1}_{ij}(s_p) = 0 \), we obtain
\[ \Pi_{ij} \approx \Pi(s_p) + (s - s_p) \Pi'(s_p) + \ldots \]
(56)
Extracting the leading terms of this Laurent expansion about \(s_p\), we can write \(\Delta^{-1}_{\alpha\beta}\) according to

\[
\Delta^{-1}_{\alpha\beta} \simeq \frac{\delta_{\alpha\beta}}{(s - s_\alpha)}
\]

where

\[
s_\alpha \simeq \lambda^2_\alpha [1 - \Pi'_\alpha(s_p)]
\]

It is worth noting that \(\Lambda(k^2)\) shares all the properties of the effective Hamiltonian in the description of the kaon system. In particular, \(CPT\) invariance requires that \(\Lambda_{11} = \Lambda_{22}\) and \(CP\) invariance prescribes the equality of the off-diagonal elements \(\Lambda_{12} = \Lambda_{21}\). Thus, the \(K^0 - \bar{K}^0\) mixing gives rise to \(CP\) violation through the effective mass-squared matrix \(\Lambda(k^2)\). The basic parameter which characterizes the indirect \(CP\) violation induced by the mixing in the kaon system is given by

\[
\eta = \frac{-q}{p} = \frac{-(1 - \epsilon)}{(1 + \epsilon)} = \sqrt{\frac{\Lambda_{21}}{\Lambda_{12}}}
\]

which is a rephasing invariant quantity and hence physically meaningful.

As a further remark, we can stress that these states of definite mass and lifetime are never more states with a definite \(CP\) character. The experimental evidence [23] in 1964, that both the short-lived \(K_S\) and long-lived \(K_L\) states decayed to \(\pi\pi\), suggests that it is important to consider the \(CP\) eigenstates \(K_1\) and \(K_2\), with the conventional choice of phase. The transformation between the physical and this new basis can be parameterized by means of the impurity complex parameter \(\epsilon\) which encodes the indirect mixing effect of \(CP\) violation in the following form

\[
K_S = \frac{1}{\sqrt{1 + |\epsilon|^2}} (K_1 - \epsilon K_2) = \frac{1}{\sqrt{2}} [(p + q)K_1 - (p - q)K_2]
\]

\[
K_L = \frac{1}{\sqrt{1 + |\epsilon|^2}} (K_2 + \epsilon K_1) = \frac{1}{\sqrt{2}} [(p - q)K_1 + (p + q)K_2]
\]

Moreover, the propagation of this different linear \(CP\)-invariant combination of \(K^0\), \(\bar{K}^0\) cannot be regarded as physical, in the sense they cannot be directly produced or detected.

A last remark regards the fact that the most general (non \(CPT\) invariant or generally \(k^2\)-dependent) transformation between the physical and the bare states involves two
impurity $\epsilon_S \neq \epsilon_L$ factors

\[ K_S = \frac{1}{\sqrt{1 + |\epsilon_S|^2}} (K_1 - \epsilon_S K_2) \]

\[ K_L = \frac{1}{\sqrt{1 + |\epsilon_L|^2}} (K_2 + \epsilon_L K_1) \]  

(61)

In fact, the condition that the off–diagonal components of the propagator matrix $\Delta'_{\alpha\beta}$ contain no poles, will fix $\epsilon_{S,L}$. When the theory violates the CPT invariance and the $\Pi_{ij}(k^2)$ is $k^2$ dependent, we see explicitly that $\epsilon_S \neq \epsilon_L$ and the transformation from the bare to the physical kaons is not a simply rotation. Furthermore, one can note that the physical masses which arise from locating the poles in the diagonalised propagator matrix no longer correspond to exact eigenstates. To lowest order in the quantum corrections the physical mass is given by

\[ M_{\text{phys}}^2 = m^2 + \text{Re} \Pi_{ii}((M_{\text{phys}}^2)^2) \right)^{1/2} \]

that is the real part of the pole in $\Delta'_{oo}$. Such kind of generalization of the pole mass renormalization scheme has been already outlined in Ref. [4] in the absence of particle mixing. In case of mass matrices, these conditions have to be fulfilled by the corresponding eigenvalues, resulting in complicated expressions. These relations can be considerably simplified by requiring simultaneously the on–shell conditions for the renormalization matrices. So that, we can state that the renormalized one-particle irreducible two-point functions are diagonal if the external lines are on their mass–shell. The diagonal elements are then fixed such that the renormalized fields are properly normalized, i.e. the residues of their renormalized propagators are equal to one. This choice of field renormalization implies that the renormalization conditions for the mass parameters involve only the corresponding diagonal self–energies. Assuming that $\text{Re} \Pi_{ii}(k^2)$ vanish as $k^2 \to m_0^2$, at least as $(k^2 - m_0^2)^2$, the following prescriptions

\[ \text{Re} \Pi_{ij}(m_0^2) = \text{Re} \Pi_{ji}(m_0^2) = 0, \]

\[ \lim_{k^2 \to m_0^2} \frac{1}{k^2 - m_0^2} \text{Re} \Pi_{ii}(k^2) = 0. \]  

(62)

lead to a standard Breit-Wigner form for the propagators. Note that any deviation of the Breit-Wigner form and/or any non linearity in the $k^2$–dependence will produce a non–zero off–diagonal element of the propagator matrix even in the physical basis. However, we may simply choose that the self-energies are regularized by requiring that the physical complex pole positions of the matrix elements are not shifted rather to impose
the usual on-shell renormalization conditions. Moreover, the inclusion of this formalism in a gauge theory requires the possibility to choose a pole $s_p$ which can regularize the self-energies to produce gauge invariant results. In fact, the question of the correct treatment of unstable particles in any underlying theory faced with the problem to select gauge independent observables. Until the dynamics of unstable particles has been described in terms of initial and final asymptotic states, it results unitary and causal. Nevertheless, this use of on–shell particle configurations becomes misleading if the resummation of the unstable particle self-energy graphs takes into account higher–order corrections. Nevertheless, unitarity relations, like those in Ref. [21], which no longer relate real quantities, become unclear with unstable states as interacting particles. Evidently, the problem of gauge dependence with unstable particles needs a consistent computational scheme which could avoid the artifacts of the resummation method. Actually, we neglect the non-resonant parts of the transition amplitudes that is equivalent to consider isolated narrow particles. The resulting Breit-Wigner form of the propagator although, in general, is not enough to preserve gauge invariance, but it is expected to contain the biggest contribution of the absorptive part. Namely, far from the resonant region, the decay rate of the kaon is so small that it is legitimate to use kaons $K^0, \bar{K}^0$ as two asymptotic states of the $S$-matrix. Within the spirit of the mass–mixing formalism we take the initial widths to be constant, with no explicit functional dependence on mass. Inclusion of such mass dependence, or working with mass rather mass-squared matrix, results in amplitudes changes we expected negligible in the limited energy range one usually works. Of course, this view is not justified for a very short time interval (much shorter than the mean life of $K_S$). In fact, in this time interval, decay processes cannot be described by the simple complex pole dynamics. Finally, the question that the off diagonal elements of the self energy matrix are momentum-dependent implies that the conventional assumption of constant mixing ratio $r(q^2)$ remains questionable [22].

**ACKNOWLEDGEMENTS**

I wish to thank F. Botella, N. Paver and A. Pugliese for useful discussions and B. Chiu, H. Y. Cheng and L. Kalfin for correspondence.
References

[1] T. D. Lee, R. Oehme and C. N. Yang, Phys. Rev. 106 (1957) 340; P. K. Kabir, The CP Puzzle (Academic Press, New York, 1968), Appendix A.

[2] L. A. Khalfin, “New Results on the CP–violation Problem”, Univ. Texas Report DOE-ER40200-211 (February 1990); C.B. Chiu and E.C.G. Sudarshan, Phys. Rev. D42 (1990) 3712; Ya. I. Azimov, JETP Lett. 58 (1993) 159; K. Urbanowsky, Int. J. Mod. Phys. A10 (1995) 1151.

[3] C. Bernardini, L. Maiani and L. Testa, Phys. Rev. Lett. 71 (1993) 2687.

[4] D. Cocolicchio, “The Characterization of Unstable Particles”, preprint IFUM 514/FT-96, Istituto Nazionale di Fisica Nucleare, Sezione di Milano, submitted to Physical Review D.

[5] J. Schwinger, Ann. Phys. 9 (1960) 169.

[6] J. Ellis, J.S. Hagelin, D.V. Nanopoulos and M. Srednicki, Nucl Phys. B241 (1984) 381; P. Huet and M.E. Peskin, Nucl. Phys. B434 (1995) 3; J. Ellis, J.L. Lopez, N.E. Mavromatos and D.V. Nanopoulos, Phys. Rev. D53 (1996) 3846; F. Benatti and R. Floreanini, Phys. Lett. B389 (1996) 100.

[7] E. Sassaroli, Y.N. Srivastava and A. Widom, Phys. Lett. 344B (1995) 436; H. J. Lipkin, Phys. Lett. B348 (1995) 604.

[8] P.T. Matthews and A. Salam, Phys. Rev. 115 (1959) 1079; M. Levy, Nuovo Cimento 13 (1959) 115; 14 (1959) 612; R. Jacob and R.G. Sachs, Phys. Rev. 121 (1961) 350; M. Veltman, Physica 29 (1963) 186; H. P. Stapp, Nuovo Cimento, 32 (1964) 103; J. Gunson, J. Math. Phys. 6 (1965) 827, 845, 852; R. Eden, P. Landshoff, D. Olive and J. Polkinghorne, “The Analytic S-matrix”, (Cambridge Univ. Press, Cambridge, 1966).

[9] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. 30 (1958) 257; Y. Yamaguchi, J. Phys. Soc. Japan 58 (1989) 4375; 60 (1991) 1541; V.V. Sokolov and V. G. Zelevinsky, Nucl. Phys. A504 (1989) 562.
[10] R. L. Warnock, Ann. Phys. (NY) 65 (1971) 386; R. G. Newton, “Scattering Theory of Waves and Particles”, (McGraw-Hill, New York, 1966); A.N. Kamal, N. R. Sinha, Z. Phys. C41 (1988) 207; M. Wanninger, L.M. Sehgal, Z. Phys. C50 (1991) 47.

[11] G. C. Wick, Phys. Lett. 30B (1969) 126; L. Wolfenstein, Phys. Rev. 188 (1969) 2536; Y. Dothan and D. Horn, Phys. Rev. D1 (1970) 916; L. Stodolsky, Phys. Rev. D1 (1970) 2683.

[12] A. Pilaftsis, Z. Phys. C47 (1990) 95; J. Liu and G. Segre, Phys. Rev. D49 (1994) 1342; J. Papavassiliou and A. Pilaftsis, Phys. Rev. D53 (1996) 2128.

[13] A. J. Buras, “CP Violation: Present and Future”, Proceedings 1st Int. Conf. on Phenomenology of Unification, Rome, 1994.

[14] S. Coleman and H. Schnitzer, Phys. Rev. 134 (1964) B863; R. G. Sachs and J. F. Willemsen, Phys. Rev. D2 (1970) 133; F. M. Renard, Springer Tracts Mod. Phys. 63 (1972) 98.

[15] R. Cruz, B. Grzakowski and J. F. Gunion, Phys. Lett. B289 (1992) 440; D. Atwood, G. Eilam, A. Soni, R. R. Mendel and R. Migneron, Phys. Rev. Lett. 70 (1993) 1364; J. Liu, Phys. Rev. D47 (1993) R1741; M. Nowakowski and A. Pilaftsis, Z. Phys. C60 (1993) 121; U. Baur and D. Zeppenfeld, Phys. Rev. lett. 75 (1995) 1002; T. Arens and L. M. Sehgal, Phys. Rev. D51 (1995) 3525.

[16] V. Bargmann, Ann. Phys. 54 (1954) 1; A. W. Wightman, Rev. Mod. Phys. 34 (1962) 845; J. M. Levy-Leblond, J. Math. Phys. 4 (1963) 776 and Comm. Math. Phys. 4 (1967) 157; E. Henley and W. Thirring, Elementary Quantum Field Theory, McGraw Hill, New York (1963).

[17] R. G. Sachs, Ann. Phys. 22 (1963) 239; J. Harte and R. G. Sachs, Phys. Rev. 135 (1964) B459; R. G. Sachs, The Physics of Time Reversal Invariance (University of Chicago Press, Chicago, 1988); O. Nachtmann, Acta Phys. Austr. Suppl. 6 (1969) 485; D. Sudarsky, E. Fishbach, C. Talmadge, S. H. Aronson and H. Y. Cheng, Ann. Phys. 207 (1991) 103.
[18] D. Cocolicchio and L. Maiani, Phys. Lett. B291 (1992) 155.

[19] A. Acuto and D. Cocolicchio, Phys. Rev. D47 (1993) 3945.

[20] B. G. Kenny and R. G. Sachs, Phys. Rev. D8 (1973) 1605; P. K. Kabir and A. Pilaftsis, Phys. Rev. A53 (1996) 66.

[21] J. S. Bell and J. Steinberger, Proceedings Oxford Int. Conf. on Elementary Particles 1965, ed. R. G. Moorehouse et al. (Rutherford HEP Lab., Chilton, Didcot, Berkshire, England, 1966) p. 195.

[22] D. Cocolicchio and M. Viggiano, “The Quantum Theory of the Kaon Oscillations”, preprint IFUM FT-97, Istituto Nazionale di Fisica Nucleare, Sezione di Milano.

[23] J. H. Christenson, J. W. Cronin, V. L. Fitch, and R. Turlay, Phys. Rev. Lett. 13 (1964) 138.