EXCEPTIONAL SEQUENCES AND ROOTED LABELED FORESTS

KIYOSHI IGUSA AND EMRE SEN

Abstract. We give a representation-theoretic bijection between rooted labeled forests with \( n \) vertices and complete exceptional sequences for the quiver of type \( A_n \) with straight orientation. The ascending and descending vertices in the forest correspond to relatively injective and relatively projective objects in the exceptional sequence. We conclude that every object in an exceptional sequence for linearly oriented \( A_n \) is either relatively projective or relatively injective or both. We construct a natural action of the extended braid group on rooted labeled forests and show that it agrees with the known action of the braid group on complete exceptional sequences. We also describe the action of \( \Delta \), the Garside element of the braid group, on rooted labeled forests using representation theory and show how this relates to cluster theory.

Contents

Introduction 2
1. Rooted labeled forests 4
   1.1. The Hasse diagram of an exceptional sequence 4
   1.2. The exceptional sequence of a rooted labeled forests 7
   1.3. Relatively projective and injective objects 10
   1.4. Chord diagrams and rooted labeled trees 12
   1.5. Generating function for exceptional sequences 14
2. Braid group action on rooted labeled forests 15
   2.1. Braid group action 15
   2.2. Proof of Theorem 2.1 17
   2.3. Examples: \( A_2 \) and \( A_3 \) 18
3. Parking functions 19
   3.1. Parking functions and exceptional sequences 20
   3.2. Parking functions and rooted labeled forests 20
4. Action of the Garside element 22
   4.1. The fundamental braid \( \delta \) 22
   4.2. Garside element \( \Delta \) 25
   4.3. Clusters and signed exceptional sequences 28
   4.4. Action of extended braid group 30
   4.5. Comments 31
5. Acknowledgements 31

References 31

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Exceptional sequences of objects is a central subject in algebraic geometry, representation theory and combinatorics. For the Dynkin quiver of type $A_n$, there are $(n + 1)^{n-1}$ complete exceptional sequences [25]. From the combinatorial point of view, there is a vast list of enumeration problems in which we encounter sets of size $(n + 1)^{n-1}$ and indeed exceptional sequences of type $A_n$ can be interpreted as maximal chains in the poset (partially ordered set) of noncrossing partitions of the set $\{0, 1, 2, \cdots, n\}$, trees with $n$ labeled edges, factorizations of the cyclic permutation $(012\cdots n)$ as a product of $n$ transpositions [12], [2], [10], [28], [18], [21], [25], [17], [23]. Our aim is to give another combinatorial interpretation which has representation theoretical outcomes.

An exceptional sequence for a hereditary algebra $\Lambda$ over any field is a sequence of rigid indecomposable $\Lambda$-modules $(E_1, \cdots, E_k)$ so that $\text{Hom}_\Lambda(E_j, E_i) = \text{Ext}_\Lambda(E_j, E_i) = 0$ for all $1 \leq i < j \leq k$. The sequence is called complete if $k$ is equal to the number of simple $\Lambda$-modules which is the number of vertices in the quiver of $\Lambda$. There is lot of work on exceptional sequences and their generalizations from the representation theory point of view: [25], [22], [3], [4], [8], [14], [19], [20], [26], [27]. In this work, we consider a homological property of modules: which elements of an exceptional sequence can be either relatively projective or relatively injective. We recall that a term $E_j$ in an exceptional sequence is called relatively projective if it is a projective object of the perpendicular category which is the full subcategory of $\text{mod-}\Lambda$ of objects $X$ so that $\text{Hom}(E_k, X) = 0 = \text{Ext}(E_k, X)$ for all $k > j$. (See Definition [1.18]) In particular, $E_1, E_2, \cdots, E_j$ lie in this perpendicular category. Relatively injective objects are defined dually. (See section 1.3.) In the case of signed exceptional sequences introduced in [15], relatively projective terms are allowed to be “shifted”. This additional structure converts the poset of noncrossing partitions into a category [16] and maximal chains in the poset become maximal sequences of composable morphisms. By [15], signed exceptional sequences are in bijection with ordered clusters tilting sets. Instead of $(n + 1)^{n-1}$ exceptional sequences (in type $A_n$) we get $n!$ times the Catalan number $C_{n+1} = \frac{1}{n+1}(2n+2)^{n+1}$ signed exceptional sequences. (Theorem 4.15 below.)

In this paper we construct a very simple 1-1 correspondence between exceptional sequences for the linear $A_n$ quiver $Q : 1 \rightarrow 2 \rightarrow \cdots \rightarrow n$ and rooted labeled forests with $n$ vertices. This bijection has many nice applications including interpretation of relatively projective and injective modules. Each exceptional sequence is represented by a planar graph which makes many algebraic concepts clearly visible. The correspondence is given by the following theorem in which we represent modules by their supports.

An indecomposable representation of a linear quiver of type $A_n$ is given by its support which is a closed interval $[a, b]$ where $1 \leq a \leq b \leq n$. The representation with this support will be labeled $M_{ab}$. This module has top the simple module at $a$ and socle the simple module at $b$.

**Theorem A1.** (Theorem 1.13) For each complete exceptional sequences $(E_1, \cdots, E_n)$ for $A_n$ consider the partial ordering on the set $\{v_1, v_2, \cdots, v_n\}$ given by $v_i < v_j$ if the support of $E_i$ is contained in the support of $E_j$. The Hasse diagram on this poset is a
rooted labeled forest. Conversely, any rooted forest with $n$ vertices labeled $v_1, \ldots, v_n$ is the Hasse diagram for a unique exceptional sequence.

One key result of this paper is that the location of the relatively projective and relatively injective objects in an exceptional sequence are immediately visible in the labeled forest as descending and ascending vertices in the forest (plus the roots which are both relatively projective and relatively injective). More precisely:

**Theorem A2.** (Theorem 1.21) In a complete exceptional sequence for linear $A_n$, $E_i$ is relatively projective if and only if either the corresponding node $v_i$ is a root of a tree in the forest (i.e., the support of $E_i$ is maximal) or $i < j$ for $E_j$ the smallest object containing $E_i$ in its support. Dually, $E_i$ is relatively injective if and only if either $v_i$ is a root in the forest or $i > j$ for $E_j$ the smallest object containing $E_i$ in its support.

Theorems A1 and A2 are illustrated in Figure 1. The labeled forest shows that there are 5 relatively projective objects in this exceptional sequence: $E_2, E_6, E_1, E_4, E_5$. The first two correspond to the roots $v_2, v_6$ of the forest and the last three correspond to the vertices $v_1, v_4, v_5$ whose indices 1, 4, 5 are smaller than the indices 2, 6, 6 of their parents $v_2, v_6, v_6$ resp.

![Figure 1](image)

**Figure 1.** By Theorem A1 this figure indicates the rooted labeled forest corresponding to the complete exceptional sequence for the quiver $A_7$:

$$(E_1, E_2, E_3, E_4, E_5, E_6, E_7) = (M_{33}, M_{13}, M_{11}, M_{66}, M_{77}, M_{47}, M_{44})$$

$M_{ab}$ denotes the module with support $[a, b]$. E.g., $M_{13}$ contains $M_{11}$ and $M_{33}$ in its support and $M_{aa} = S_a$ is the simple module at $a$.

One consequence of Theorem A2 is the following.

**Corollary B.** (Corollary 1.22) Every object in a complete exceptional sequence for linear $A_n$ is either relatively projective or relatively injective and at least one object is both.

Also, there is a three variable generating function for ascending and descending vertices in rooted labeled forests which is directly applicable to the corresponding exceptional sequence, giving a generating function for exceptional sequences:

**Theorem C.** (Theorem 1.30) We have a three variable generating function

$$P_n(a, b, c) = \sum a^p b^q c^r = c(a + (n - 1)b + c)(2a + (n - 2)b + c) \cdots (n - 1)a + b + c)$$

where the sum is over all complete exceptional sequences $E_*$ for a linear quiver of type $A_n$ and the monomial $a^p b^q c^r$ is given by letting

$p = \text{number of relatively projective but not relatively injective objects in } E_*$,
\[ q = \text{number of relatively injective but not relatively projective objects in } E, \]
\[ r = \text{number of objects in } E \text{ which are both relatively projective and relatively injective}. \]

We have the added bonus of visualizing the action of the braid group on exceptional sequences. Crawley-Boevey showed [6] that the braid group acts transitively on the set of exceptional sequences for any acyclic quiver, and Ringel [23] extended this to all hereditary algebras. Here we show (in Figures 3, 4, 5, 6) how to visualize this action using rooted labeled forests. This visualization is good enough to see the action of the Garside element of the braid group (\( \Delta \)). This is important for several reasons.

For example, \( \Delta^2 \) generates the center of the braid group. For us, \( \Delta \) is important since it gives a formula for the correspondence (proved in [15]) between “support tilting sets” and “signed exceptional sequences”. (See Proposition 4.16). We give a detailed description of the action of \( \Delta \) on rooted labeled forests in section 4.2. Basically, \( \Delta \) takes the projective vertices of a forest (on the left side), moves them to the top (to form the new roots) and moves the roots to the right side, to form the new injective vertices. (Propositions 4.11, 4.20, Figure 13). Thus \( \Delta^2 \) converts projective vertices of a forest into injective vertices (Figure 11). Figure 11 above has two projective vertices \( v_5, v_6 \) on the left, two roots \( v_6, v_2 \) at the top and two injective vertices \( v_2, v_3 \) on the right.

It has been pointed out to us that the braid group action on exceptional sequences already has a visualization given by parking functions in [11]. In section 3 we review the bijection between exceptional sequences for linear \( A_n \) and parking functions and the corresponding rooted labeled forests using “Prüfer codes”. We show using an example that this correspondence between exceptional sequences and rooted labeled forests is very different from ours.

There is another well-known visualization of exceptional sequences of type \( A_n \) (with any orientation) given by “chord diagrams” which can be converted into labeled trees with \( n+1 \) vertices which are equivalent to rooted labeled forests with \( n \) vertices. That construction, from [12], is detailed in section 1.4. However, that construction does not keep track of relatively projective and injective objects which are the focus of this paper.

In conclusion, this paper gives an easy visualization of exceptional sequences using planar diagrams called “rooted labeled forests” (section 1) and these forests are used to visualize the action of the braid group on exceptional sequences (section 2), in particular the action of the important Garside element \( \Delta \) (section 4), and to show that exceptional sequences have a three variable generating function for its relatively projective and injective objects, the same as for rooted labeled forests.

1. Rooted labeled forests

We will construct a 1-1 correspondence between complete exceptional sequences for linear \( A_n \) and rooted labeled forest. Then, we derive some representation theoretic consequences.

1.1. The Hasse diagram of an exceptional sequence. We consider the Hasse diagram of an exceptional sequence and derive some of its properties. For example, we show that it is a rooted forest.
Definition 1.1. A rooted labeled forest is a forest with vertices numbered 1 through \(n\) and a root chosen for each component. By adding an extra vertex labeled 0 and attaching it to each of the roots, such a structure is equivalent to a tree with \(n + 1\) vertices labeled 0 through \(n\).

Remark 1.2. The nodes of a rooted tree are partially ordered by paths to the root. Conversely, the Hasse diagram of a finite poset is a rooted forest if it has the property that each node has at most one parent. In that case, the maximal elements are roots of disjoint trees.

Let \(E_* = (E_1, \ldots, E_n)\) be a complete exceptional sequence for \(A_n\). (Unless otherwise noted, we always take the orientation \(1 \rightarrow 2 \rightarrow \cdots \rightarrow n\).) We define a partial ordering on the objects \(E_i\) by \(E_i \leq E_j\) if the support of \(E_i\) is contained in the support of \(E_j\). In other words, \(a_j \leq a_i \leq b_i \leq b_j\) where \([a_i, b_i]\) denotes the support of \(E_i\). We claim that the Hasse diagram of this partial ordering is a rooted labeled forest. This follow from the following lemma.

Lemma 1.3. Given two objects \(E_i = M_{a_i,b_i}\) and \(E_j = M_{a_j,b_j}\) in an exceptional sequence, the intervals \([a_i,b_i]\), \([a_j,b_j]\) are “noncrossing”, i.e. these closed interval are either disjoint or one is contained in the other.

Proof. Given two objects whose supports cross, say, \(M_{ab}, M_{cd}\) where \(a < c \leq b < d\), we have \(\text{Hom}(M_{cd}, M_{ab}) \neq 0\) since \(M_{cb}\) is a quotient module of \(M_{cd}\) and a submodule of \(M_{ab}\). Also, \(\text{Ext}(M_{ab}, M_{cd}) \neq 0\) since we have a nonsplit exact sequence

\[
0 \rightarrow M_{cd} \rightarrow M_{cb} \oplus M_{ad} \rightarrow M_{ab} \rightarrow 0.
\]

So, both pairs \((M_{ab}, M_{cd})\) and \((M_{cd}, M_{ab})\) are not exceptional. Therefore, they cannot occur in an exceptional sequence in either order. \(\square\)

Lemma 1.4. The set of elements above \(E_i\) form a linearly ordered set.

Proof. Suppose \(E_i \leq E_j, E_k\). Then the supports of \(E_j, E_k\) meet and therefore, by Lemma 1.3, one is contained in the other. So, they are linearly ordered. \(\square\)

Proposition 1.5. The Hasse diagram of the objects \(E_i\) in an exceptional sequence form a rooted labeled forest. Moreover, the roots are the maximal elements.

Proof. By Lemma 1.4 a node cannot have two parents. Therefore, each connected component of the Hasse diagram is a tree with root at the top. \(\square\)

Example 1.6. Consider the exceptional sequence for \(A_7\):

\((S_4, S_5, M_{15}, M_{67}, S_6, M_{12}, S_1)\)

The corresponding Hasse diagram is:

\[
\begin{aligned}
E_4 &= M_{67} \\
E_5 &= S_6 \\
E_2 &= S_5 \\
E_1 &= S_4 \\
E_6 &= M_{12} \\
E_7 &= S_1
\end{aligned}
\]
We note that the leaves are simple modules.

Remark 1.7. The labeled forest can be drawn more systematically by choosing the vertices to lie in the Auslander-Reiten quiver of \( \Lambda = KQ \). This is a planar diagram where the module \( M_{ab} \) for \( 1 \leq a \leq b \leq n \) is placed at the point \((-b-a, b-a+1)\). For Example 1.6 this is:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

\[
E_4 \quad E_5 \quad E_2 \quad E_1 \quad E_6 \\
\bullet \\
\bullet \\
\bullet
\]

Note that the \( y \)-coordinate of the point \( M_{ab} \) is its length \( b-a+1 \).

It is easy to see that, with these coordinates, the Hasse diagram is embedded. Indeed, if \( A \) is above \( X \) and \( B \) is above \( Y \) and if the edges \( AX, BY \) were to cross, then the supports of \( A, B \) intersect. So, one of these supports contains the other, say \( A \) is below \( B \). Then \( X \) and \( Y \) would both be below \( A \) and there would be no edge from \( Y \) to \( B \).

**Proposition 1.8.** The length of the object \( E_i = M_{a_i,b_i} \) (given by \( b_i - a_i + 1 \)) is the number of objects \( E_j \) which are \( \leq E_i \) in the partial ordering. In particular, the leaves (the minimal elements) have length 1, i.e., they are simple modules.

Given a rooted labeled tree, define the weight of any node \( v_i \) to be the number of nodes which are \( \leq v_i \). The statement of the proposition is that the weight of \( v_i \) as a node in the Hasse diagram is equal to the length of the corresponding module \( E_i \).

**Proof of Proposition 1.8.** We use the fact that the dimension vectors \( \dim E_i \) form a vector space basis for \( \mathbb{R}^n \). (See for example, [15 Proposition 2.7].) Suppose that \( E_i = M_{ab} \) with length \( k = b-a+1 \). Take the linear map \( \pi : \mathbb{R}^n \to \mathbb{R}^k \) given by projection onto coordinates \( a \) through \( b \). Then \( \dim E_i \) maps to the vector \((1,1,\cdots,1)\). For any \( E_j \) whose support is not contained in the interval \([a,b]\) we have by Lemma 1.3 that the support of \( E_j \) is either disjoint from the interval \([a,b]\) or it contains that interval. In either case, \( \pi(\dim E_j) \) is a scalar multiple of \( \dim E_i \). Therefore, the image of \( \pi \) is spanned by \( \pi(\dim E_j) \) for those \( E_j \) with support in \([a,b]\). So, there must be at least \( k \) of these. However, there are at most \( k \) linearly independent vectors with support in \([a,b]\). Therefore, the number of \( E_j \) with support in \([a,b]\) is exactly \( k \), the length of \( E_i \). So, the length of \( E_i \) is the weight of \( v_i \) as claimed. \( \square \)

Proposition 1.5 shows that every complete exceptional sequence gives rise to a rooted labeled forest, i.e., we obtain a mapping \( H \)

\[
\{ \text{complete exceptional sequences for } A_n \} \xrightarrow{H} \{ \text{rooted labeled forests on } n \}.
\]
We will show that this map is a bijection by constructing an inverse. Since the two sets have the same cardinality it will suffice to construct a right inverse to \( H \). So, for every rooted labeled forest on \( n \) we will construct the unique exceptional sequence having that forest as Hasse diagram. We will not prove uniqueness directly. Uniqueness follows from the counting argument.

1.2. The exceptional sequence of a rooted labeled forests. We give an explicit construction of the necessarily unique complete exceptional sequence for any rooted labeled forest. The construction will be recursive.

For \( 1 \leq i \leq n \), let \( v_i \) be the node labeled \( i \) in the forest \( F \). Let \( T_i \) be the tree having root \( v_i \), i.e., \( T_i \) consists of all nodes \( v_j \leq v_i \) in \( F \). Then \( v_j \leq v_i \) if and only if \( T_j \subset T_i \).

First, consider the connected case, i.e., \( F \) is a tree. Let \( v_r \) be the root of \( F \), i.e., \( F = T_r \). Let \( v_{i_1}, v_{i_2}, \ldots, v_{i_k} \) be the children of the root \( v_r \). Since the subscripts \( i_j \) are distinct integers different from \( r \) we may choose the labels so that

\[
i_1 < i_2 < \cdots < i_p < r < i_{p+1} < \cdots < i_k
\]

where \( 0 \leq p \leq k \). Recall that the weight \( w(v_i) \) is the size of \( T_i \) by definition. In particular, the weight of \( v_r \) is \( n \) and

\[
n = 1 + \sum w(v_{i_j})
\]

We assign the module \( E_r := M_{1n} \) to \( v_r \). Since \( M_{1n} \) is uniserial, it has a filtration with \( k+1 \) subquotients, one of length 1 and the others of length \( w(v_{i_j}) \). Denote the filtration:

\[
0 = A_{p+1} \subset A_p \subset A_{p-1} \subset \cdots \subset A_2 \subset A_1 \subset B_k \subset \cdots \subset B_{p+1} \subset B_p = M_{1n}
\]

We may choose these so that \( A_x/A_{x+1} \) has length \( w(v_{i_x}) \) when \( x \leq p \) and \( B_y/A_1 \) has length \( w(v_{i_y}) \) for \( y \geq p + 1 \). (This makes \( B_k/A_1 \) simple, of length 1.) Then we let

\[
E_{i_x} = A_x/A_{x+1}, \quad E_{i_y} = B_y/A_1.
\]

**Proposition 1.9.** \((E_{i_1}, \ldots, E_{i_p}, E_r, E_{i_{p+1}}, \ldots, E_{i_k})\) is an exceptional sequence.

**Proof.** It suffices to show that \((X, Y)\) is an exceptional pair for any subsequence \( X, Y \) of length 2.

Case 1: \( X = E_r \). Then \( Y = E_{i_y} \) for \( y > p \) which has no morphism to \( E_r \) since its support does not contain \( n \) by construction. \( \text{Ext}(Y, E_r) = 0 \) since \( E_r \) is injective.

Case 2: \( Y = E_r \). Then \( X = E_{i_x} \) for \( x \leq p \). These cannot be quotients of \( Y \). Hence there is no morphism \( Y \to X \) and \( \text{Ext}(E_r, X) = 0 \) since \( E_r \) is projective.

For the remaining pairs, there is no morphism since they have disjoint supports.

Case 3: \( X = E_{i_x} \) where \( x \leq p \) and \( Y = E_{i_y} \) where \( y \geq p + 1 \). Then the supports of \( X \) and \( Y \) are disjoint and not consecutive. Therefore there is no morphism and no extension.

In the remaining cases, \( X, Y \) are on the same side of \( E_r \). So, their supports are disjoint and \( \text{Ext}(Y, X) = 0 \) since, in any extension, \( Y \) would be the submodule. \( \Box \)

We come to the recursion step. Take any subtree of \( F \) with root \( v_s \). Suppose the corresponding module has been constructed, say \( E_s = M_{ab} \) with length \( b-a+1 = w(v_s) \). In particular \( E_s \) will be simple when \( v_s \) is a leaf. When \( v_s \) is not a leaf, we repeat the construction above for the children \( v_{j_s} \) of \( v_s \). This will assign modules \( E_{j_s} \) of length equal to the weight \( w(v_{j_s}) \) having supports which are closed disjoint subintervals of \([a, b]\) with
union \( [a, b] \) minus one point which we call the “gap”. (In the construction above, the gap is the support of the simple module \( B_k/A_1 \).) Continuing in this way we construct all modules \( E_i \) in the case when \( F \) is connected.

Now consider the case when \( F \) is a disjoint union of trees \( T_1, \cdots, T_k \) with roots \( v_{j_1}, \cdots, v_{j_k} \) where \( j_1 < j_2 < \cdots < j_k \). Then the weights \( w(v_{j_i}) \) add up to \( n \). Let \( E_{j_1}, \cdots, E_{j_k} \) be the modules with disjoint supports of length \( w(v_{j_i}) \) which are consecutive. For instance \( E_{j_1} = M_{1x} \) where \( x = w(v_{j_1}) \), \( E_{j_2} = M_{x+1,y} \) where \( y = x + w(v_{j_2}) \), etc.

**Proposition 1.10.** \( (E_{j_1}, \cdots, E_{j_k}) \) is an exceptional sequence.

**Proof.** There are no morphisms between these objects since their supports are disjoint. For any \( x < y \), \( \text{Ext}(E_{j_y}, E_{j_x}) = 0 \) since in any extension, \( E_{j_y} \) will be the submodule. \( \square \)

**Remark 1.11.** We say that a sequence of modules \( X_1, \cdots, X_k \) have consecutive supports if \( X_i = M_{a_i, b_i} \) where each \( a_i = b_{i-1} + 1 \). For example, the modules in Proposition 1.10 have consecutive supports. Modules with consecutive support form an exceptional sequence.

We list the basic properties of the modules \( E_i \) corresponding to the nodes \( v_i \) under the construction above.

**Proposition 1.12.** Given any rooted labeled forest \( F \), let \( E_i \) be the module corresponding to node \( v_i \) by the above recursive construction. Then

1. Each \( E_i \) has length equal to \( w(v_i) \).
2. The support of \( E_i \) is contained in the support of \( E_j \) if and only if \( v_i \leq v_j \).
3. The supports of \( E_i, E_j \) are disjoint if and only if \( v_i, v_j \) are not comparable.
4. If \( v_{i_1}, \cdots, v_{i_k} \) are the children of \( v_r \) and \( i_1 < i_2 < \cdots < i_p < r < i_{p+1} < \cdots < i_k \) then
   a. \( E_{i_p} \) is a submodule of \( E_r \) if and only if \( x = p \), i.e., if \( v_{i_x} \) is the child of \( v_r \) with the largest label less than \( r \). (None of the \( E_{i_x} \) are submodules of \( E_r \) when \( p = 0 \).)
   b. \( E_{i_y} \) is a quotient of \( E_r \) if and only if \( y = p + 1 \), i.e., if \( v_{i_y} \) is the child of \( v_r \) with the smallest label greater than \( r \). (None of the \( E_{i_y} \) are quotients of \( E_r \) when \( p = k \).)
   c. \( E_{i_1}, \cdots, E_{i_p} \) have consecutive supports.
   d. \( E_{i_{p+1}}, \cdots, E_{i_k} \) have consecutive supports.

**Proof.** Property (1) is Proposition 1.8. Property (4) follow from Proposition 1.9. To prove (2) and (3) we note that, for any \( i, j \) there are only two possibilities: either one is \( \leq \) the other or they are not comparable. In the first case, suppose \( v_i < v_j \). Then there is a sequence \( v_i = v_{k_0} < v_{k_1} < v_{k_2} < \cdots < v_j \) where each \( v_{k_s} \) is a child of \( v_{k_{s+1}} \) in that case, \( \text{supp} E_{k_s} \subset \text{supp} E_{k_{s+1}} \) by construction. So, \( \text{supp} E_i \subset \text{supp} E_j \). In the second case, either \( v_i, v_j \) lie in disjoint trees with distinct roots \( v_k, v_{\ell} \) in which case \( E_k \) and \( E_{\ell} \) have disjoint supports by Proposition 1.10. Otherwise, \( v_i, v_j \) lie in the same tree. Let \( v_k \) be the smallest node \( \geq v_i, v_j \). Then \( v_i, v_j \) are \( \leq \) two children of \( v_k \) where the corresponding modules have disjoint supports. So, \( E_i, E_j \) have disjoint supports. \( \square \)
Theorem 1.13. (Theorem A1) Given any rooted labeled forest \( F \), let \( E_i \) be the module corresponding to node \( v_i \) by the above recursive construction. Then

\[
(E_1, E_2, \cdots, E_n)
\]

is a complete exceptional sequence.

To prove this theorem we consider the three cases when \( E_i \) is a submodule of \( E_j \), \( E_i \) is a quotient module of \( E_j \) and \( E_i, E_j \) extend each other.

Lemma 1.14. If \( E_i \) is a submodule of \( E_j \) and \( i \neq j \) then \( v_i < v_j \) and the nodes \( v_{k_1}, \cdots, v_{k_t} \) connecting \( v_i \) to \( v_j \) in \( F \) have increasing labels: \( i < k_1 < k_2 < \cdots < k_t < j \).

Proof. Since \( v_i < v_{k_1} < v_j \), \( \text{supp} \ E_i \subset \text{supp} \ E_{k_1} \subset \text{supp} \ E_j \). Since \( E_i \) contains the socle of \( E_j \), so does \( E_{k_1} \). So, \( E_{k_1} \) is a submodule of \( E_j \) and \( v_{k_1} \) is a child of \( v_j \). It follows that the label must be smaller by the construction: \( k_i < j \). By the same argument \( k_p < k_{p+1} \) for all \( p \) and \( i < k_1 \). \( \square \)

Lemma 1.15. If \( E_i \) is a quotient module of \( E_j \) and \( i \neq j \) then \( v_i < v_j \) and the nodes \( v_{k_1}, \cdots, v_{k_t} \) connecting \( v_i \) to \( v_j \) in \( F \) have decreasing labels: \( i > k_1 > k_2 > \cdots > k_t > j \).

Proof. This statement is dual to Lemma 1.14 and has an analogous proof. \( \square \)

Lemma 1.16. \( \text{Ext}(E_i, E_j) \neq 0 \) if and only if \( E_i, E_j \) have (disjoint) consecutive supports and, in that case, \( E_i \) is a submodule of some \( E_k \) (which may be \( E_j \)) and \( E_j \) is a quotient module of some \( E_\ell \) (which may be \( E_j \)) where \( v_k, v_\ell \) are sibling or they are both roots and \( i \leq k < \ell \leq j \).

Proof. Let \( \text{Ext}(E_i, E_j) \neq 0 \). Then either the supports are consecutive or they overlap. But the second case is excluded by Proposition 1.12 (2), (3). So, \( E_i, E_j \) have consecutive supports. The converse is clear.

Suppose the supports of \( E_i, E_j \) are consecutive, say \( \text{supp} \ E_i = [a, b] \), \( \text{supp} \ E_j = [b+1, c] \). Let \( v_k \) be maximal in \( F \) so that \( v_i \leq v_k \) and \( v_j \not\leq v_k \). Then \( E_k \) contains the support of \( E_i \) and is disjoint from the support of \( E_j \). So, \( b \in \text{supp} \ E_k \) and \( b+1 \not\in \text{supp} \ E_k \) making \( E_i \) a submodule of \( E_k \). By Lemma 1.14, \( i \leq k \).

Similarly, let \( v_\ell \) be maximal in \( F \) so that \( v_j \leq v_\ell \) and \( v_i \not\leq v_\ell \). Then \( \ell \leq j \) by Lemma 1.15. By maximality of \( v_k, v_\ell \), they must be sibling (or roots of disjoint trees). In either case, \( k < \ell \) since \( E_k, E_\ell \) have consecutive supports. (This follows from either Proposition 1.12 (4) or Proposition 1.10.) This proves the lemma. \( \square \)

Proof of Theorem 1.13. These lemmas imply that \( (E_1, \cdots, E_n) \) is a complete exceptional sequence. Indeed, if \( i < j \) we have \( \text{Hom}(E_j, E_i) = 0 \) since \( E_j \) is not a submodule of \( E_i \) by Lemma 1.14 and \( E_i \) is not a quotient module of \( E_j \) by Lemma 1.15. Also, \( \text{Ext}(E_j, E_i) = 0 \) by Lemma 1.16. \( \square \)

Example 1.17. It is easy to convert a rooted labeled forest into a complete exceptional sequence. Start with the following.
By Remark 1.11, the modules $E_3$, $E_5$ corresponding to the roots $v_3, v_5$ have consecutive supports. By Proposition 1.12 (1), these supports have lengths 3, 6. So, $E_3 = M_{13}$ and $E_5 = M_{49}$. Since $1 < 3$, $E_1$ is the length 2 submodule of $E_3$. So, $E_1 = M_{23}$. By Proposition 1.12(4), $E_6$ has support at 4 followed by a gap, then $E_2, E_4$ have consecutive supports in the interval $[6, 9]$. So, $E_6 = S_4$, $E_2 = S_6$ and $E_4 = M_{79}$. The children of $v_4$ are leaves with larger labels, so the corresponding modules are simples of injective type: $E_7 = S_7$, $E_9 = S_8$. Similarly, $E_8 = S_2$. The complete exceptional sequence is

$$E_* = (M_{23}, S_6, M_{13}, M_{79}, M_{49}, S_4, S_7, S_2, S_8)$$

This same example is done using chord diagrams following [12] in Figure 2.

1.3. Relatively projective and injective objects.

**Definition 1.18.** An object $E$ of a subcategory $\mathcal{C}$ of $\text{mod}\Lambda$ is called relatively projective, respectively relatively injective if it is a projective, respectively injective, object of the subcategory $\mathcal{C}$. The right perpendicular category $Z^\perp$ of any $\Lambda$-module $Z$ is defined to be the full subcategory of $\text{mod}\Lambda$ of all modules $Y$ so that

$$\text{Hom}_\Lambda(Z, Y) = 0 = \text{Ext}^1_\Lambda(Z, Y).$$

Similarly, the left perpendicular category $\perp Z$ of any $\Lambda$-module $Z$ is defined to be the full subcategory of $\text{mod}\Lambda$ of all modules $X$ so that

$$\text{Hom}_\Lambda(X, Z) = 0 = \text{Ext}^1_\Lambda(X, Z).$$

By abuse of terminology, we call an object $E_i$ in a complete exceptional sequence $(E_1, \cdots, E_n)$ relatively projective if it is a projective object of the right perpendicular category $Z^\perp$ of $Z = E_{i+1} \oplus \cdots \oplus E_n$. Similarly, we call an object $E_i$ in a complete exceptional sequence $(E_1, \cdots, E_n)$ relatively injective if it is an injective object of the left perpendicular category $\perp Z$ of $Z = E_1 \oplus \cdots \oplus E_{i-1}$.

**Proposition 1.19.** If any two of the objects in a short exact sequence $0 \to A \to B \to C \to 0$ lie in $Z^\perp$ (or $\perp Z$) then so does the third.

**Proof.** This follows from the 6-term exact sequence:

$$0 \to \text{Hom}(Z, A) \to \text{Hom}(Z, B) \to \text{Hom}(Z, C)$$

$$\to \text{Ext}(Z, A) \to \text{Ext}(Z, B) \to \text{Ext}(Z, C) \to 0.$$ 

An analogous argument works for $\perp Z$. \qed

Since $X \in Y^\perp$ if and only if $Y \in \perp X$, we get the following.

**Corollary 1.20.** For any short exact sequence $0 \to A \to B \to C \to 0$, $(A \oplus B)^\perp = A^\perp \cap B^\perp$ is contained in $C^\perp$ and similarly, $\perp (A \oplus B) \subseteq \perp C$. \qed
Theorem 1.21. (Theorem [A2]) Let $E_* = (E_1, \cdots, E_n)$ be a complete exceptional sequence for a linear quiver of type $A_n$. Let $F$ be the corresponding rooted labeled forest. For each $E_i$ in $E_*$, let $v_i$ be the corresponding node of $F$. Then,

1. $E_i$ is relatively projective and relatively injective if and only if $v_i$ is a root of $F$.
2. For $v_i$ not a root of $F$, let $v_j$ be the parent of $v_i$. Then, $E_i$ is relatively projective, resp. injective in $E_*$ if and only if $i < j$, resp $i > j$.

Corollary 1.22. (Corollary [B]) Every object in a complete exceptional sequence of type $A_n$ with straight orientation is either relatively projective or relatively injective and at least one object is both.

To prove Theorem 1.21 let $v_{i_1}, \cdots, v_{i_k}$ be the children of $v_j$ with

$$i_1 < i_2 < \cdots < i_p < j < i_{p+1} < \cdots < i_k.$$ 

As in Proposition 1.9 we have that $E_j = M_{ab}$ and $E_{i_1}, \cdots, E_{i_p}$ is a sequence of modules with consecutive support in the interval $[a+1, b]$ which is the support of $rM_{ab}$, the radical of $M_{ab}$. Also, $E_{i_p}$ is a submodule of $M_{ab}$. In more details, we have $E_{i_s} = M_{a_s,b_s}$ where $a \leq b_0 < b_1 < b_2 < \cdots < b_p = b$ and $a_s = b_{s-1} + 1$ for $1 \leq s \leq p$. In Example 1.6 $p = 2$ and $a, b_0, b_1, b_2 = 1, 3, 4, 5$ making $E_{i_1} = E_1 = M_{44}$ and $E_{i_2} = E_2 = M_{55}$.

Lemma 1.23. The objects $E_{i_y}$ for $y > p$ are not relatively projective in $E_*$. 

Proof. Let $W$ be the direct sum of the objects $E_t$ which come after $E_{i_p}$, i.e., $t > i_y$. We will show that $E_i$ is not a projective object of $W^\perp$.

Claim: For any $p < x \leq y$, $W^\perp$ contains a module $M_{cb}$ with socle $S_b$ which maps onto $E_{i_x}$.

Since $E_{i_x}$ is a proper quotient of $M_{cb}$, $E_{i_x}$ cannot be projective in $W^\perp$ for $p < x \leq y$. When $x = y$ this proves the lemma. So, it suffices to prove this claim.

The proof of the claim is by induction on $x$. For $x = p + 1$, $E_{i_{p+1}}$ is a quotient of $E_j = M_{ab}$ which lies in $W^\perp$ since it is to the left of $E_{i_y}$ in $E_*$. We take $c = a$ in that case. Suppose the claim holds for $x < y$. Then, by induction, there is a module $M_{cb} \in W^\perp$ which maps onto $E_{i_x}$ which also lies in $W^\perp$. Therefore the kernel $M_{ab}$ lies in the abelian category $W^\perp$. But $M_{ab}$ maps onto $E_{i_{x+1}}$ since $E_{i_x}, E_{i_{x+1}}$ have consecutive supports. So, the claim holds for $x + 1$. This proves the claim and the lemma.

Lemma 1.24. $rM_{ab}$ is a projective object in the perpendicular category $M_{ab}^\perp$.

Proof. When $b = n$, $rM_{ab}$ is projective and lies in $M_{ab}^\perp$.

For $b < n$, $rM_{ab}$ is a quotient of $\tau M_{ab} = M_{a+1,b+1}$. So, $\text{Hom}(rM_{ab}, \tau M_{ab}) = 0$ and $\text{Hom}(M_{ab}, rM_{ab}) = 0$. So, $rM_{ab} \in M_{ab}^\perp$.

To show that $rM_{ab}$ is projective in $M_{ab}^\perp$, suppose not. Then there is another object $X \in M_{ab}^\perp$ and an epimorphism $X \twoheadrightarrow rM_{ab}$. But then $X = M_{a+1,c}$ for some $c > b$. Then we have an epimorphism $X \twoheadrightarrow \tau M_{ab}$ contradicting the assumption that $X \in M_{ab}^\perp$. So, $rM_{ab}$ is projective in $M_{ab}^\perp$.

Lemma 1.24 is the case $s = p$ of the following lemma:

Lemma 1.25. $M_{a+1,b_s}$ is a projective object in the perpendicular category $W^\perp$ where $W = M_{ab} \oplus E_{i_{x+1}} \oplus \cdots \oplus E_{i_p}$.
Proof. For \( b_s = b, M_{a+1,b_s} = r M_{ab} \) and the previous lemma applies.

For \( b_s < b \), \( M_{a+1,b_s} \) is a quotient of \( r M_{ab} = M_{a+1,b+1} \). So, \( M_{a+1,b_s} \in M_{ab}^\perp \). Also, \( M_{a+1,b_s} \in E_{i_t}^\perp \) for \( s < t \leq p \) since the support of \( E_{i_t} \) lies in \((b_s,n]\). So, \( M_{a+1,b_s} \in W^\perp \).

The module \( M_{ab} \) is an iterated quotient of components of \( W \) since \( M_{ab} = M_{ab}/E_{i_p} \) and \( M_{ab} = M_{a,b_s}/E_{i_{s+1}} \). Therefore, \( W^\perp \subset M_{a,b_s}^\perp \) by Corollary 1.20. \( M_{a+1,b_s} \) is projective in \( M_{a,b_s}^\perp \) by the previous lemma and therefore it is projective in \( W^\perp \). \( \square \)

**Theorem 1.26.** Let \( v_i \) be a child of \( v_j \) in the rooted labeled forest \( F \). Then the object \( E_j \) in the corresponding complete exceptional sequence \( E_s = (E_1, \ldots, E_n) \) is relatively projective if and only if \( i < j \).

Proof. If \( i > j \) then, by Lemma 1.23, \( E_i \) is not relatively projective. So, suppose \( i < j \). Then, in the notation above, \( i = i_s \) for some \( s \leq p \).

Let \( W \) be the direct sum of all objects \( E_k \) in the exceptional sequence with \( k > i \). This includes the summands of \( W_0 = E_j \oplus E_{i_{s+1}} \oplus \cdots \oplus E_{i_p} \). Thus \( W^\perp \subset W_0^\perp \). By Lemma 1.25, \( W_0^\perp \) contains as a projective object the module \( M_{a+1,b_s} \) which has \( E_{i_s} = M_{a,b_s} \) as a submodule. Since \( W_0^\perp \) is a hereditary abelian category, this makes \( E_{i_s} \) a projective object of \( W_0^\perp \). So, \( E_{i_s} \) is a projective object of \( W^\perp \subset W_0^\perp \). \( \square \)

We have the following dual statement with an analogous argument.

**Theorem 1.27.** Let \( v_i \) be a child of \( v_j \) in the rooted labeled forest \( F \). Then the object \( E_j \) in the corresponding complete exceptional sequence \( E_s = (E_1, \ldots, E_n) \) is relatively injective if and only if \( i > j \). \( \square \)

By the discussion above, the only \( E_j \) which have a chance to be both relatively projective and relatively injective are ones corresponding to the roots of the components of \( F \). Let \( v_{j_1}, v_{j_2}, \ldots, v_{j_k} \) be the roots of \( F \) with \( j_1 < j_2 < \cdots < j_k \). Then the modules \( E_{j_1}, \ldots, E_{j_k} \) have consecutive supports with union \([1,n]\).

**Theorem 1.28.** The objects \( E_{j_1}, \ldots, E_{j_k} \) corresponding to the roots of the forest \( F \) are relatively projective and relatively injective in the complete exceptional sequence \( E_s \).

Proof. First, \( E_{j_k} \) is projective since it is a submodule of the projective module \( M_{1n} \). Then, as in the proof of Lemma 1.23, the quotient \( M_{1n}/E_{j_k} \) is a projective object in the category \( E_{j_k}^\perp \). Since \( E_{j_{k-1}} \) is a submodule of \( M_{1n}/E_{j_k} \), it also becomes projective in \( E_{j_k}^\perp \) and therefore projective in \( W^\perp \) where \( W \) is the direct sum of all \( E_t \) for \( t > j_{k-1} \). Repeating this process, we see that each \( E_{j_t} \) is relatively projective in \( E_s \).

The dual arguments show that each \( E_{j_t} \) is relatively injective in \( E_s \). \( \square \)

Theorem 1.21 is given by combining Theorems 1.26, 1.27, and 1.28.

### 1.4. Chord diagrams and rooted labeled trees.

First, we note that a rooted forest with \( n \) labeled vertices is equivalent to a rooted tree with \( n+1 \) vertices and \( n \) labeled edges. (Just move the edge labels to the endpoint furthest from the root, delete the root of the tree, then declare the adjacent vertices to be roots of the resulting forest.)

A bijection between exceptional sequences on quivers of type \( A_n \) (with any orientation) and rooted trees with \( n \) labeled edges is given as follows.
Figure 2. The exceptional sequence \((M_{23}, S_6, M_{13}, M_{79}, M_{49}, S_4, S_7, S_2, S_8)\) from Example 1.17 is drawn as the sequence of chords \((24, 67, 14, 70, 40, 45, 78, 23, 89)\) in red. The corresponding tree with root \((\ast)\) in the region containing the arc 0 to 1 is indicated in blue. Deleting the root gives the rooted labeled forest in Example 1.17.

Goulden and Yong [12] (and earlier [21]) showed that rooted trees with \(n\) labeled edges are in bijection with factorizations of the \(n + 1\) cycle \((012 \cdots n)\) as a product of \(n\) transpositions \(\tau_i = (a_i, b_i)\). The bijection is given by drawing chords (labeled with \(i\)) between points \(a_i\) and \(b_i\) in a circle as indicated in Figure 2. The dual graph of these chords is the tree. The root is the region adjacent to the arc from 0 to 1. Brady and Watt [2] showed that such factorizations of the \(n + 1\) cycle are in bijection with exceptional sequences where \((\tau_1, \cdots, \tau_n) = ((a_1, b_1), \cdots, (a_n, b_n))\) corresponds to the exceptional sequence \((M_{a_1, b_1-1}, \cdots, M_{a_n, b_n-1})\) where the indices are taken modulo \(n + 1\).

It is easy to see why the bijection between exceptional sequences and rooted labeled forests (or trees), as illustrated in Figure 2 is the same as the bijection given in Theorem 1.13. Any chords in the construction, say 14, separates the circle in two parts. One part contains the integers in the interval \([1, 4]\). The other part contains the arc 01 and therefore the root of the tree. So, any other chord \((a, b)\) will be further away from the root if and only if \([a, b] \subsetneq [1, 4]\), or, equivalently, \([a, b] \subsetneq [1, 4]\). Therefore, the rooted forest is the Hasse diagram of the set of corresponding intervals \([a_i, b_i]\) which are the supports of the objects in the exceptional sequence.

The correspondence between exceptional sequences and factorizations of the Coxeter element of the Weyl group (the \(n + 1\) cycle in the symmetric group for \(A_n\)) has been generalized to any quiver in [14]. The correspondence with chord diagrams is also known for any orientation of \(A_n\) [8]. See also [17] for the relation to noncrossing partitions.
1.5. **Generating function for exceptional sequences.** The following theorem and its proof come from [10]. (See also [28].) This is a special case of a result from [9].

**Theorem 1.29.** For each \( n \geq 1 \), let \( P_n(a, b, c) \) be the three variable polynomial for rooted labeled forest given by

\[
P_n(a, b, c) = \sum a^p b^q c^r \text{ where the sum is over all rooted labeled forests, } r \text{ is the number of roots of the forest, } p \text{ is the number of nodes } v_i \text{ whose parent } v_j \text{ has label } j > i, \text{ q is the number of nodes } v_i \text{ with parent } v_j \text{ so that } j < i. \]

Then

\[
P_n(a, b, c) = c(a + (n - 1)b + c)(2a + (n - 2)b + c) \cdots ((n - 1)a + b + c).
\]

**Proof.** We associate a weight to each node of a forest. The weight will be \( a, b, c \) depending on which type of node it is. Then the monomial \( a^p b^q c^r \) for a forest is the product of the weights of its nodes.

The proof is by induction on \( n \). For \( n = 1 \), \( P_1(a, b, c) = c \) since there is only one forest with one node which has weight \( c \).

For \( n \geq 2 \), we write \( P_n(a, b, c) \) as the sum of two generating function

\[
P_n(a, b, c) = P_n(a, b, c)_1 + P_n(a, b, c)_2
\]

the first is for forest in which \( v_1 \) is a root. We remove the root \( v_1 \), decrease the labels of the other nodes by 1 and keep the weights of the children of \( v_1 \) to be \( b \). This gives a rooted labeled forest with \( n - 1 \) nodes in which the roots have weight either \( b \) or \( c \). Since a node with weight \( c \) was removed we have:

\[
P_n(a, b, c)_1 = cP_{n-1}(a, b, b + c).
\]

For the remaining forests, we remove the node \( v_1 \) and make the children of \( v_1 \) roots, but keeping the weight \( b \). We obtain a forest with roots of weight either \( b \) or \( c \). To reassemble the original forest, we need to choose one node to be the parent of \( v_1 \), we need to make the root of the tree in which this node lives to have weight \( c \). The other \( b \) weighted roots become the children of \( v_1 \). So,

\[
P_n(a, b, c)_2 = a(n - 1) \frac{c}{b + c} P_{n-1}(a, b, b + c).
\]

A straightforward calculation proves the result. \( \square \)

Let \( P_{A_n}(a, b, c) = \sum \lambda_{pqr} a^p b^q c^r \) where \( \lambda_{pqr} \) is the number of complete exceptional sequences for linear \( A_n \) having

1. \( p \) objects \( E_i \) which are relatively projective but not relatively injective,
2. \( q \) objects which are relatively injective but not relatively projective
3. \( r \) objects which are both relatively projective and relatively injective.

**Theorem 1.30.** (Theorem [C])

\[
P_{A_n}(a, b, c) = P_n(a, b, c) = c \prod_{j=1}^{n-1} (ja + (n - j)b + c).
\]

**Proof.** This follows from Theorem [1.29] and the properties of the bijection between complete exceptional sequences and rooted labeled forests given by Theorem [1.21]. \( \square \)
For example, when \( n = 3 \) this is:

\[
P_{A_3}(a, b, c) = P_3(a, b, c) = c(a + 2b + c)(2a + b + c) = 2a^2c + 2b^2c + 5abc + 3ac^2 + 3bc^2 + c^3.
\]

**Remark 1.31.** Theorem 1.30 implies that there is only one exceptional sequence in which all objects are both relatively projective and relatively injective. (The \( c^3 \) term in \( P_{A_3} \) above.) This is never true for other orientations of \( A_n \) since such an exceptional sequence is given by a sequence of simple modules in which the support of each object is a source in the complement of the supports of the ones on its left and there are at least two such sequences for any nonlinear \( A_n \). For example, for \( A_3 : 1 \to 2 \leftarrow 3 \), there are two exceptional sequences with all terms relatively projective and relatively injective. This gives the term \( 2c^3 \) in its three variable generating function:

\[
P_{1\to 2\leftarrow 3}(a, b, c) = 2a^2b + 2b^2c + 4abc + 4ac^2 + 2bc^2 + 2c^3
\]

So, Theorem 1.30 hold only for \( A_n \) with straight orientation.

### 2. Braid group action on rooted labeled forests

It is well-known that the braid group acts transitively on the set of complete exceptional sequences for any Dynkin quiver, in particular, for type \( A_n \) [6], [23]. In this section we construct a natural action of the braid group on rooted labeled forests and we show, in Theorem 2.1, that this action corresponds to the action on complete exceptional sequences under the bijection given in the previous section. We also show that this action of the braid group is different from the one given in [11] since we are using a different bijection between the two sets.

The braid group \( B_n \) on \( n \) strands is given by generators and relations as follows. The generators are \( \sigma_i \) for \( 1 \leq i < n \) with two relations:

1. \( \sigma_i, \sigma_j \) commute if \( |j - i| \geq 2 \)
2. \( \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \).

Given a rooted labeled forest \( F \), two node \( v_i, v_j \) are said to be close if one of the following holds.

1. \( v_i, v_j \) are roots of distinct components of \( F \).
2. \( v_i, v_j \) are sibling, i.e., they have the same parent.
3. One of \( v_i, v_j \) is the parent of the other.

All remaining cases are not close.

To simplify notation, we add a root \( v_0 \), which we call the master root, at the top of the forest giving a rooted tree \( F_+ \). Then Case 1 is a special case of Case 2: Roots in \( F_+ \) are sibling in \( F \).

#### 2.1. Braid group action

The braid group \( B_n \) acts on the set of rooted labeled forests on \( n \) as follows. We give the action of \( \sigma_i \) for \( 1 \leq i < n \). The partial ordering on the set \( [n] = \{1, 2, \ldots, n\} \) with Hasse diagram \( F \) and the one with Hasse diagram \( \sigma_i(F) \) are the same for nodes other than \( v_i, v_{i+1} \). We keep the same notation for these other nodes in \( \sigma_i(F) \). But, we denote by \( v_i', v_{i+1}' \) the new nodes of \( \sigma_i(F) \) with labels \( i, i + 1 \). These are given as follows. See Figure 4.
Case 0: If nodes $v_i, v_j$ are not close then $\sigma_i(F)$ is given by switching the labels $i, i+1$, i.e., $\sigma_i(F) = F$ with nodes relabeled: $v_i' = v_{i+1}, v_{i+1}' = v_i$.

Case 1: Suppose $v_i$ is the parent of $v_{i+1}$, $v_k$ is the parent of $v_i$ in $F_+$, $X$ is the set of other children of $v_i$ and $Y$ is the set of children of $v_{i+1}$.

Then $\sigma_i(F)_+$ is given by removing node $v_{i+1}$, relabeling $v_i$ to $v_{i+1}'$, then adding a new node $v_i'$ and making it a child of $v_{i+1}'$ and a parent of all elements of $X$. Thus $Y \cup \{v_i'\}$ is the set of children of the new node $v_{i+1}'$ and $v_k$ is the parent of $v_{i+1}'$ in $\sigma_i(F)_+$.

Case 2: Suppose $v_i$ is a child of $v_{i+1}$, $v_k$ is the parent of $v_{i+1}$ in $F_+$, $Y$ is the set of other children of $v_{i+1}$ and $X$ is the set of children of $v_i$. Then $\sigma_i(F)_+$ is given by removing node $v_{i+1}$, relabeling $v_i$ to $v_{i+1}'$, then adding a new node $v_i'$ with parent $v_k$ and children $v_{i+1}'$ and the elements of $X$.

Note that $\sigma_i$ takes examples from Cases 1,2,3 to examples from Cases 2,3,1 respectively. Furthermore, $\sigma_i^3$ is the identity map on these three cases. See Figure 3.

Figure 3. Cases 1,2,3 of the braid move $\sigma_i$ are indicated. $v_k$ denotes the smallest node in $F_+$ which is above both $v_i$ and $v_{i+1}$ and $Z$ denotes the set of other children of $v_k$. $\sigma_i^3$ is the identity in all three cases.

Theorem 2.1. The action of $\sigma_i$ on the set of rooted labeled forests described above corresponds to the action of $\sigma_i$ on exceptional sequences under the bijection given in the previous section.

Corollary 2.2. The action of $\sigma_i$ on the set of rooted labeled forests satisfies the braid relations and therefore gives an action of the braid group.

The following property of $\sigma_i$, which follows easily from its description, will be needed in the next section.

Proposition 2.3. The vertex $v_{i+1}$ is a root of the forest $F$ if and only if $v_i'$ is a root of $\sigma_i F$. 
2.2. Proof of Theorem 2.1

Lemma 2.4. Let $F$ be a rooted labeled forest with associated exceptional sequence $E_\ast = (E_1, \cdots, E_n)$. Then for any $i < n$ we have the following.

1. $E_i$ is a submodule of $E_{i+1}$ if and only if $v_i$ is a child of $v_{i+1}$ in $F$.
2. $E_{i+1}$ is a quotient of $E_i$ if and only if $v_{i+1}$ is a child of $v_i$.
3. $E_i, E_{i+1}$ have consecutive supports if and only if $v_i, v_{i+1}$ are sibling in $F_+$.
4. $E_i, E_{i+1}$ are Hom and Ext orthogonal (in both directions) if and only if none of the above hold, i.e., if and only if $v_i, v_{i+1}$ are not close.

Proof. If $E_i \subset E_{i+1}$ then, by Lemma 1.14, $v_i$ must be a child of $v_{i+1}$ since there are no $k$ with $i < k < i+1$. If $E_{i+1}$ is a quotient of $E_i$ then, by Lemma 1.15, $v_{i+1}$ is a child of $v_i$ since there is no $k$ between $i$ and $i+1$. The converses are given by Proposition 1.12 (a) and (b) which follow from Proposition 1.9. This proves (1) and (2).

If $E_i, E_{i+1}$ have consecutive supports, Lemma 1.16 implies $v_i, v_{i+1}$ must be sibling since we must have $i = k < \ell = j = i+1$. The converse is given by Proposition 1.12 (c) and (d) and by Proposition 1.10 when $v_i, v_{i+1}$ are both roots. This proves (3). All other cases are in (4). □

Proof of Theorem 2.1. The braid move $\sigma_i$ acts on $E_\ast$ in all cases by moving $E_i$ to position $i+1$ and inserting a uniquely determined new object $E_i'$ in position $i$ to give a new exceptional sequence

$$\sigma_i(E_\ast) = (E_1, \cdots, E_{i-1}, E_i', E_i, E_{i+2}, \cdots, E_n).$$

Case a: If $E_i, E_{i+1}$ are Hom and Ext orthogonal then they commute and $E_i' = E_{i+1}$.

Case b: Suppose that $E_i$ is a submodule of $E_{i+1}$. Then we know that $E_i'$ is the quotient module $E_{i+1}/E_i$. By Lemma 2.4(1), $v_i$ is a child of $v_{i+1}$. Let $Y$ denote the other children of $v_{i+1}$. These have disjoint supports whose union is the support of $E_{i+1}/E_i$ minus one point. Therefore, $Y$ is the set of children of $v_i'$ the node in $\sigma_i(F)$ corresponding to $E_i'$. $E_i$ has not changed except for its label which is now $i+1$. So, it has the same set of children. This is Case 2 in subsection 2.3 and $\sigma_i(F)$ is the Hasse diagram of $\sigma_i(E_\ast)$.

Case c: Suppose that $E_{i+1}$ is a quotient of $E_i$, say $E_{i+1} = M_{ab}$ and $E_i = M_{ac}$. Then $E_i' = M_{b+1,c}$ is the kernel of the epimorphism $E_i \to E_{i+1}$. By Lemma 2.4(2), $v_{i+1}$ is a child of $v_i$. If $X$ is the set of the other children of $v_i$ in $F$ then the corresponding modules must have support in $[a, c]$ but disjoint from $[a, b]$. So, their supports are in $\text{supp } E_i' = [b+1, c]$. So, $X$ is the set of children of the new $v_i'$ in $\sigma_i(F)$. Since $v_{i+1}$ in $F$ has been removed, its children $Y$ become children of $v_i$, the new $v_i'+1 \in \sigma_i(F)$.

Case d: Suppose that $v_i, v_{i+1}$ are sibling in $F_+$ with sets of children $Y, X$, resp. Then $E_i, E_{i+1}$ must have disjoint consecutive supports by Lemma 2.4(3), say $\text{supp } E_i = [a, b]$, $\text{supp } E_{i+1} = [b+1, c]$. Since $E_i$ extends $E_{i+1}$, the new exceptional pair $(E_i', E_i)$ must be $(M_{ac}, M_{ab})$. The children of the new $v_i'$ with module $M_{ac}$ are the new $v_i'+1$ with module $M_{ab}$ and $X$. The children of $v_i'+1$ with module $M_{ab}$ are $Y$.

This concludes the proof of Theorem 2.1. See Figure 4 for an example. □
Figure 4 illustrates the sequence of five rooted labeled forests corresponding to the following five complete exceptional sequences for $A_{10}$:

$\sigma_3 \mapsto (M_{99}, M_{1,10}, M_{46}, M_{12}, M_{79}, M_{77}, M_{11}, M_{44}, M_{55})$
$\sigma_2 \mapsto (M_{99}, M_{4,10}, M_{1,10}, M_{46}, M_{12}, M_{79}, M_{77}, M_{11}, M_{44}, M_{55})$
$\sigma_5 \mapsto (M_{99}, M_{4,10}, M_{1,10}, M_{46}, M_{12}, M_{79}, M_{77}, M_{11}, M_{44}, M_{55})$
$\sigma_4 \mapsto (M_{99}, M_{4,10}, M_{1,10}, M_{49}, M_{46}, M_{12}, M_{77}, M_{11}, M_{44}, M_{55})$

2.3. Examples: $A_2$ and $A_3$. For $n = 2$, there are 3 rooted labeled forests and $\sigma_1$ permutes them in a 3-cycle:

Figure 5. The three rooted labeled forests on $n = 2$ are cyclically permuted by $\sigma_1$. 
Exceptional Sequences and Rooted Labeled Forests

For $n = 3$ there are 16 rooted labeled forests. Figure 6 illustrates 8 of them. Figure 7 illustrates how the rooted labeled forest can be visualized when the exceptional sequence is embedded in the Auslander-Reiten quiver of the path algebra.

3. Parking Functions

There is another action of the braid group on rooted labeled forests coming from the action on “parking functions” given in [11]. This action is difficult to describe, but we give an example to demonstrate that it is not the same as our action which is given in Figure 3. First, we review the definitions and known results about parking functions.

A parking function on $n$ is a function $f : [n] \to [n]$, where $[n] = \{1, 2, \cdots, n\}$ so that $f^{-1}[k]$ has at least $k$ elements for all positive $k \leq n$. This is equivalent to the condition that there are at most $p$ elements $i \in [n]$ with $f(i) \geq n - p + 1$.

The name comes from an interpretation using $n$ cars parking in $n$ spaces numbered 1 through $n$. Assume that Car $i$ has preferred parking space $a_i$. If that space is not available, Car $i$ will park in the next available space. If $f(i) = a_i$ is a parking function, each car $i$ will find a space $b_i \geq a_i$ and $g(i) = b_i$ will give a bijection $g : [n] \to [n]$.

As an example, suppose $n = 4$ and consider the parking function $(1, 1, 2, 2)$. Assume the cars come in reverse order. Then Car 4 will park in its preferred spot $b_4 = 2$, Car 3 will go to the next spot $b_3 = 3$, Car 2 will park in its favorite spot $b_2 = 1$. Finally, Car 1 will park in space $b_1 = 4$. The result is an exceptional sequence:

$$(M_{14}, M_{11}, M_{23}, M_{22}).$$
3.1. Parking functions and exceptional sequences. We review the well-known bijection between parking functions and exceptional sequences for linear $A_n$.

Lemma 3.1. Let $(a_1, a_2, \ldots, a_n)$ be a nondecreasing parking function on $n$. Let $b_1, \ldots, b_n$ be the permutation of $n$ given by the locations of the cars with preferred positions $a_i$ assuming they park in reverse order as outlined above. Then $M_{a_1, b_1}, \ldots, M_{a_n, b_n}$ is an exceptional sequence for linear $A_n$.

Proof. For $i < j$ we have $\text{Ext}(M_{a_j, b_j}, M_{a_i, b_i}) = 0$ since $a_i \leq a_j$. It remains to show

$$\text{Hom}(M_{a_j, b_j}, M_{a_i, b_i}) = 0$$

for $i < j$. When $a_i = a_j$, we have $b_i > b_j$ since Car $j$ has parked in space $b_j$ after finding spaces $a_j$ to $b_j - 1$ occupied. Therefore, Car $i$ will find that spaces $a_i$ to $b_j$ are taken and must park in $b_i > b_j$. Then $M_{a_j, b_j}$ is a quotient of $M_{a_i, b_i}$ and $\text{Hom}(M_{a_j, b_j}, M_{a_i, b_i}) = 0$.

When $a_i < a_j$, there is no morphism $M_{a_j, b_j} \to M_{a_i, b_i}$ since $b_i$ is not in the closed interval $[a_j, b_j)$ since those spaces are occupied by the time Car $i$ parks. Thus either

1. $b_i < a_j$, making the supports of the two modules disjoint or
2. $b_i > b_j$ in which case $M_{a_j, b_j}$ is a subquotient of $M_{a_i, b_i}$ and there is no morphism.

Therefore $M_{a_1, b_1}, \ldots, M_{a_n, b_n}$ is an exceptional sequence. □

Lemma 3.2. Let $(E_1, \ldots, E_n)$ be an exceptional sequence with sequence of tops $(a_1, \ldots, a_n)$. Then, for any $i$, there is another exceptional sequence with $a_i, a_{i+1}$ switched in the sequence of tops.

Proof. If $a_i = a_{i+1}$ there is nothing to do. Otherwise, suppose $a_i < a_{i+1}$. Then there is a unique object $E'_{i+1}$ so that $(E_1, \ldots, E'_{i+1}, E_i, E_{i+2}, \ldots, E_n)$ is an exceptional sequence. But, the dimension vector of $E'_{i+1}$ is congruent to $\dim E_{i+1}$ modulo $\dim E_i$. So, the top of $E'_{i+1}$ is equal to that of $E_{i+1}$, i.e., the tops $a_i, a_{i+1}$ have switched. The case $a_i > a_{i+1}$ is similar. □

Theorem 3.3. [11] The construction above gives a unique exceptional sequence having any given parking function as sequence of tops.

Proof. By Lemma 3.1 the construction works for any nondecreasing parking function. By Lemma 3.2 we can permute the parking function. Since the two sets have the same cardinality $(n + 1)^n - 1$, this construction gives a bijection. □

3.2. Parking functions and rooted labeled forests. There is another well-known bijection between parking functions and rooted labeled forests using Prüfer codes which we will review briefly here. We will use this bijection to compare the braid group action on parking functions given in [11] (using Theorem 3.3) to the one we give in the previous section using rooted labeled forests.

We take as an example, the parking function $(1, 1, 1, 1)$ on $A_4$ and the action of $\sigma_1$ then $\sigma_2$ then $\sigma_3$ using the top function (Theorem 3.3):

$$(1, 1, 1, 1) \xrightarrow{\sigma_1} (4, 1, 1, 1) \xrightarrow{\sigma_2} (4, 3, 1, 1) \xrightarrow{\sigma_3} (4, 3, 2, 1).$$

The Prüfer code corresponding to a parking function $(a_1, a_2, \ldots, a_n)$ is given by

$$(a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1})$$
where these numbers are taken modulo $n + 1$ which is $4 + 1 = 5$ in our case. Thus, the Prüfer codes corresponding to our sequence of parking functions are:

$$(0, 0, 0) \xrightarrow{\sigma_1} (2, 0, 0) \xrightarrow{\sigma_2} (4, 3, 0) \xrightarrow{\sigma_3} (4, 4, 4).$$

For example, the parking function $(4, 3, 1, 1)$ has Prüfer code $(-1, -2, 0) = (4, 3, 0)$.

The bijection between Prüfer codes and rooted labeled forests, given below, gives the corresponding sequence of forests:

$$\begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_1} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_2} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_3} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{align*}$$

(3.1)

The Prüfer code corresponding to a rooted labeled forest is given by $(p_1, \ldots, p_{n-1})$ where $p_1$ is the label of the unique node adjacent to the leaf of the forest with the largest label ($p_1 = 0$ if this maximal leaf is a root). When that leaf is removed from the forest, the label of the unique node adjacent to the leaf with the largest label is $p_2$ and so on. For example, in the third forest, the largest root is 2. Above 2 is $p_1 = 4$. After removing 2, the largest root is 4 and above 4 is $p_2 = 3$. What remains are two roots. So, $p_3 = 0$. The resulting Prüfer code is $(4, 3, 0)$.

We now examine the same sequence of complete exceptional sequences and corresponding sequence of rooted labeled forests using our new direct bijection between these two structures. The parking function $(1, 1, 1, 1)$ corresponds to the sequence of injective modules $(M_{14}, M_{13}, M_{12}, M_{11})$. The sequence of braid moves $\sigma_1, \sigma_2, \sigma_3$ from above with corresponding rooted labeled forests, Prüfer codes and parking functions are given by:

| Exceptional sequence | Forest | Prüfer code | Parking function |
|----------------------|--------|-------------|------------------|
| $(M_{14}, M_{13}, M_{12}, M_{11})$ | \begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_1} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_2} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_3} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{align*} | $(3, 2, 1)$ | $(1, 4, 1, 2)$ |
| $\xrightarrow{\sigma_1} (M_{44}, M_{14}, M_{12}, M_{11})$ | \begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_2} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_3} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{align*} | $(3, 2, 2)$ | $(1, 4, 1, 3)$ |
| $\xrightarrow{\sigma_2} (M_{44}, M_{34}, M_{14}, M_{11})$ | \begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} & \xrightarrow{\sigma_3} & \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{align*} | $(3, 2, 3)$ | $(3, 1, 3, 1)$ |
| $\xrightarrow{\sigma_3} (M_{44}, M_{34}, M_{24}, M_{14})$ | \begin{align*}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\end{align*} | $(2, 3, 4)$ | $(2, 4, 2, 1)$ |

This example uses Figure 3 to perform the braid group action on our exceptional sequence. The action is easy to perform on the rooted labeled forests. We saw that the
braid group action on parking functions from [11] was equivalent to a different and more complicated action of the braid group on these forests as shown in (3.1).

4. Action of the Garside element

We recall the definition of the fundamental braid \( \delta \) and recall the well-known action of \( \delta \) on complete exceptional sequences over any hereditary algebra. The corresponding action of \( \delta \) on forests has an easy description. We also describe the action of the Garside element \( \Delta \) on rooted labeled forests. \( \Delta \) is an important element of the braid group for many reasons [7]. For example, \( \Delta^2 = \delta^n \) generates the center of the braid group [5] and the action of \( \Delta \) on complete exceptional sequences over any hereditary algebra converts support tilting objects to \( c \)-vectors. See section 4.3 below for a full explanation. (This is a comment made without proof in [4].) In section 4.4 we use \( \Delta \) to describe an action of the extended braid group \( \tilde{B}_n \rtimes \mathbb{Z}_2 \) on the set of rooted labeled forests.

To do these detailed computations, we need more notation about exceptional sequences. First, the Euler pairing for \( \Lambda \)-modules \( X, Y \), also called the Euler characteristic of \( \Lambda \) [1] is the biadditive integer pairing on \( \mathbb{Z}^n \) characterized by the property that

\[
\langle \dim X, \dim Y \rangle_\Lambda := \dim \text{Hom}_\Lambda(X, Y) - \dim \text{Ext}_\Lambda(X, Y).
\]

We will use the shorthand notation:

\[
\chi_\Lambda(X, Y) := \langle \dim X, \dim Y \rangle_\Lambda.
\]

4.1. The fundamental braid \( \delta \). This is the braid group element

\[
\delta = \delta_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1}.
\]

This pushed the last (nth) strand underneath the first \( n-1 \) strands to put it on the far left. (See Figure 8.) The action of \( \delta \) on an exceptional sequence is

\[
\delta(E_1, E_2, \cdots, E_n) = (E'_n, E_1, E_2, \cdots, E_{n-1}).
\]

Remark 4.1. Each object in a complete exceptional sequence is uniquely determined up to isomorphism by its position and the other objects. In this case we must have

1. \( E'_n = \tau E_n \), the Auslander-Reiten translation of \( E_n \), if \( E_n \) is not projective since, in that case,

\[
\chi_\Lambda(E_n, X) = -\chi_\Lambda(X, \tau E_n).
\]

2. When \( E_n = P_i \) is projective with simple top \( S_i \) then

\[
\chi_\Lambda(P_i, X) = \dim X_i = \chi_\Lambda(X, I_i)
\]

where \( I_i \) is the injective envelope of \( S_i \). So, \( E'_n = I_i \) in this case.

(2) can also be seen combinatorially using Proposition 2.3 which implies that \( v_n \) is a root of \( F \) if and only if \( v'_1 \) is a root of \( F' = \delta F \). But this is equivalent to \( E'_n \), the first object of \( \delta E_n \), being injective. We use the shorthand notation

\[
\tau_\Lambda X := \begin{cases} 
\tau X & \text{if } X \text{ is not projective} \\
I_i & \text{if } X = P_i
\end{cases}
\]
Since the formula for $\delta$ depends on whether $E_n$ is projective, we need a combinatorial characterization of vertices corresponding to the projective (and injective) objects.

**Lemma 4.2.** The projective objects $P_1 \subset P_2 \subset \cdots \subset P_p$ of a complete exceptional sequence correspond to vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_p}$ of the forest $F$ given as follows.

1. $v_{j_p}$ is the last root of $F$, i.e., the other roots have smaller labels.
2. $v_{j_i}$ is the last relatively projective child of $v_{j_{i+1}}$, i.e., $j_i$ is maximal among labels of children of $v_{j_{i+1}}$ so that $j_i < j_{i+1}$.

**Lemma 4.3.** The injective objects $I_1 \rightarrow I_2 \rightarrow \cdots \rightarrow I_q$ of a complete exceptional sequence correspond to vertices $v_{k_1}, v_{k_2}, \ldots, v_{k_q}$ of the forest $F$ given as follows.

1. $v_{k_1}$ is the first root of $F$, i.e., the other roots have larger labels.
2. $v_{k_{i+1}}$ is the first relatively injective child of $v_{k_i}$, i.e., $k_{i+1}$ is minimal among labels of children of $v_{k_i}$ with $k_{i+1} > k_i$.

We call the vertices $v_j$ satisfying Lemma 4.2 projective vertices and those satisfying Lemma 4.3 the injective vertices of the rooted labeled forest $F$. These lemmas imply the following.

**Lemma 4.4.** In a complete exceptional sequence for linear $A_n$, $E_n$ is a projective module if and only if $v_n$ is a root of $F$. Similarly, $E_1$ is injective if and only if $v_1$ is a root.

**Theorem 4.5.** Let $F, \delta F$ be the rooted labeled forests of $E_\ast = (E_1, \ldots, E_n)$ and $\delta E_\ast = (\tau E_n, E_1, \ldots, E_{n-1})$ with labeled vertices $v_i$ and $v'_i$ respectively.

1. If $v_n$ is not a root of $F$, then $\delta F$ is equal to $F$ with the labels cyclically permuted so that $v'_i = v_n$ and $v'_i = v_{i-1}$ for $1 < i \leq n$.
2. If $v_n$ is a root of $F$, then $\delta F$ is obtained by cyclically permuting the labels of $F$, then exchanging $v'_1 = v_n$ with the master root $v_0$.

In both cases $\delta F_+ \cong F_+$ as unlabeled trees.

**Proof.** If $v_n$ is not a root of $F$ then, by Lemma 4.4, $E_n$ is not projective. So, $E'_n = \tau^* E_n$ which has the same length as $E_n$. Since the other objects are the same as before, all objects have the same length as before. So, the support containment relations cannot change. So, $\delta F$ must be the same forest as $F$ with labels cyclically permuted.
If \( v_n \) is a root of \( F \), then \( E_n \) is projective. Every other object in the sequence has support contained either in the support of \( E_n \) or the support of the corresponding injective object \( E'_n = \tau^*_A E_n \). Therefore, the new root \( v'_1 \) of \( \delta F \) covers exactly those vertices of \( F \) which were not descendants of \( v_n \). This is equivalent to saying that \( v_n \) has been exchanged with the master root and relabeled \( v'_1 \). □

\[
\begin{array}{c}
  2 \\
  1 \quad 3 \\
\end{array} \xrightarrow{\delta} \begin{array}{c}
  3 \\
  2 \quad 1 \\
\end{array} \xrightarrow{\delta} \begin{array}{c}
  1 \\
  2 \quad 3 \\
\end{array} \xrightarrow{\delta} \begin{array}{c}
  1 \\
  2 \quad 3 \\
\end{array}
\]

**Figure 9.** Theorem 4.5 in the case \( n = 3 \) is illustrated (from Figure 6).

**Corollary 4.6.** Let \( F, \delta^n F \) be the rooted labeled forests of \( E_* = (E_1, \cdots, E_n) \) and \( \delta^n E_* = (\tau^* E_1, \cdots, \tau^* E_n) \). Let \( v_{j_1}, \cdots, v_{j_p} \) be the projective vertices of \( F \). Then \( \delta^n F_+ \cong F_+ \) with projective vertices together with the master root \( v_0 \) cyclically permuted and all other vertices remaining the same. Thus \( v'_{j_p} = v_0, v'_0 = v_{j_1}, v'_i = v_{j_{i+1}} \) for \( 1 \leq i < p \) and \( v'_k = v_k \) for all other values of \( k \). In particular, \( v'_{j_1}, \cdots, v'_{j_p} \) are the injective vertices of \( \delta^n F \) and the roots of \( \delta^n F \) consist of \( v'_{j_1} \) and the children of \( v_{j_1} \) in \( F \). (See Figure 10.)

**Proof.** The first part follows from Theorem 4.5. The last part follows from Lemma 4.3 since \( j_1 \) must be smaller than the labels of the other roots of \( \delta^n F \). These other roots in \( \delta^n F \) were the children of \( v_{j_1} \) in \( F \) which must have larger index by Lemma 4.2.2. □

\[
\begin{array}{c}
  v_{j_p} \\
  \cdots \\
  v_{j_2} \\
  v_{j_1} \\
\end{array} \xrightarrow{\delta^n = \Delta^2} \begin{array}{c}
  X_{p+1} \\
  X_p \\
  X_2 \\
  X_1 \\
\end{array} \xrightarrow{\delta^n F :} \begin{array}{c}
  v'_{j_1} \\
  v'_{j_2} \\
  v'_{j_p} \\
  v'_{j_2} \\
\end{array} \xrightarrow{\delta^n F :} \begin{array}{c}
  X_3 \\
  \cdots \\
  X_{p+1} \\
  X_j \\
\end{array}
\]

**Figure 10.** Illustrating Corollary 4.6, the projective vertices of \( F \) become the injective vertices of \( \delta^n F \). The children of the smallest projective vertex \( v_{j_1} \) in \( F \) have larger indices and become the other roots of \( \delta^n F \) with the injective vertex \( v'_{j_1} \) being the first root of \( \delta^n F = \Delta^2 F \).
4.2. Garside element $\Delta$. The Garside element of the braid group $B_n$ is given by

$$\Delta = \delta_{n-1}\delta_{n-2}\cdots\delta_2\delta_1$$

where, for each $k < n$, $\delta_k$ is given by

$$\delta_k = \sigma_1\sigma_2\cdots\sigma_k.$$

In particular $\delta_{n-1} = \delta$ is the fundamental braid. The commutator relation:

$$\Delta\sigma_i\Delta^{-1} = \sigma_{n-i}$$

implies that $\Delta^2$ lies in the center of the braid group $B_n$. In fact, $\Delta^2$ generates the center of $B_n$ by [5]. It is also an easy observation that $\Delta^2 = \delta^n$. So, the action of the central element $\Delta^2$ is given by Corollary 4.6 and Figure 10.

![Figure 11](image1)

Figure 11. The Garside braid $\Delta$ for $n = 5$. Each strand goes over the ones on its right and under the ones on its left.

![Figure 12](image2)

Figure 12. The action of $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ is shown in the case $n = 3$ using Figure 6. Proposition 4.11 explains the combinatorics of $\Delta$.

The Garside braid is shown in Figure 11. Examples of the action of $\Delta$ on exceptional sequences for $A_3$ were computed in Figure 6 and are summarized in Figure 12 below. The use of the Garside element in cluster theory is explained in the next subsection. Here we discuss the action of $\Delta$ on forests.

We use the notation

$$\Delta(E_1, \cdots, E_n) = (E'_1, \cdots, E'_n).$$
The first thing to notice is that $E'_n = E_1$ since the first strand goes over the other strands in each step of the braid $\Delta$. Similarly, every strand goes over the ones on its right and under the ones on its left.

Let $C_k = Z^\perp$, the right perpendicular category of $Z = E_{k+1} \oplus \cdots \oplus E_n$. This is the category of all $\Lambda$-modules $X$ so that $\chi_{\Lambda}(E_j, X) = 0$ for all $j > k$.

**Lemma 4.7.** The sequence $(E_1, \cdots, E_k)$ is a complete exceptional sequence for the category $C_k$.

**Proof.** The objects $E_1, \cdots, E_k$ lie in $C_k$ by definition of an exceptional sequence. To show it is complete, suppose we could add an object of $C_k$ to this sequence. Then, that object could also be added to the complete exceptional sequence $(E_1, \cdots, E_n)$ to the left of $E_{k+1}$ contradicting its maximality. \hfill \Box

We recall our main theorem (Theorem 1.21) which says that $v_k$ is descending in $F$ (either a root or the child of some $v_j$ for $j > k$) if and only if $E_k$ is relatively projective which means $E_k$ is a projective object of the category $C_k$.

**Lemma 4.8.** The object $E'_{n-k+1}$ in $\Delta E_* = (E'_1, \cdots, E'_n)$ is the unique indecomposable object of $C_k$ having the property that $(E'_{n-k+1}, E_1, \cdots, E_{k-1})$ is a complete exceptional sequence for $C_k$. Consequently,

1. If $E_k$ is not relatively projective, then $E'_{n-k+1} = \tau_k E_k$ where $\tau_k$ is Auslander-Reiten translation in $C_k$. In that case, $\chi_{\Lambda}(E_k, E'_{n-k+1}) = -1$.
2. If $E_k$ is relatively projective, then $E'_{n-k+1}$ is the injective envelope in $C_k$ of the relatively simple $C_k$-top of $E_k$. In this case, $\chi_{\Lambda}(E_k, E'_{n-k+1}) = +1$.

In particular, $E'_1 = \tau^* E_n$ (just as in $\delta E_*$).

**Proof.** The braid move $\delta_{k-2} \cdots \delta_1$ on $E_*$ produces the exceptional sequence

$$(E'_{n-k+2}, \cdots, E'_n, E_k, E_{k+1}, \cdots, E_n)$$

which implies that $(E'_{n-k+2}, \cdots, E'_n)$ is a complete exceptional sequence for $C_{k-1}$. Applying the braid move $\delta_{k-1}$ produces the exceptional sequence

$$(E'_{n-k+1}, E'_{n-k+2}, \cdots, E'_n, E_{k+1}, \cdots, E_n)$$

which implies that $E'_{n-k+1}$ is the unique object of $C_k$ in the right perpendicular category of $C_{k-1}$. Equivalently, $(E'_{n-k+1}, E_1, \cdots, E_{k-1})$ is a complete exceptional sequence for $C_k$ as claimed. The rest follows by analogy with Theorem 4.5. \hfill \Box

This lemma immediately implies the following.

**Lemma 4.9.** Let $\Delta E_* = (E'_1, \cdots, E'_n)$. Then $E_k$ is relatively projective in $E_*$ if and only if $E'_{n-k+1}$ is relatively injective in $\Delta E_*$. 

In the proof of Lemma 4.8 we saw that

$$\delta_{k-1}^{-1}(E'_{n-k+1}, E'_{n-k+2}, \cdots, E'_n) = (E'_{n-k+2}, \cdots, E'_n, E_k).$$

If we apply this braid move to the last $k$ objects in $\Delta E_*$ we obtain the following.
Lemma 4.10. With the notation $\Delta(E_1, \cdots, E_n) = (E'_1, \cdots, E'_n)$ we have that 
$$(E'_1, \cdots, E'_{n-k}, E'_{n-k+2}, \cdots, E'_n, E_k)$$
is a complete exceptional sequence. In particular, 
$$\chi_{\Lambda}(E_k, E'_j) = 0$$
for $j \neq n - k + 1$.

Proposition 4.11. Let $F' = \Delta F$ for a rooted labeled forest $F$ with $n$ vertices. Then

1. $v_{j_1}, \cdots, v_{j_p}$ are the projective vertices of $F$ if and only if $v'_{n-j_1+1}, \cdots, v'_{n-j_p+1}$ are the roots of $\Delta F$.

2. $v_{k_1}, \cdots, v_{k_r}$ are the roots of $F$ if and only if $v'_{n-k_1+1}, \cdots, v'_{n-k_r+1}$ are the injective vertices of $\Delta F$.

Proof. Since $\Delta^2 = \delta^r$, it follows from Corollary 4.6 that (1) and (2) are equivalent. So, we prove only (1).

Statement (1) says that $v_k$ is a projective vertex of $F$ if and only if $v'_{n-k+1}$ is a root of $F' = \Delta F$. We prove this by induction on $n - k$. For $k = n$, $E_n$ is projective if and only if $v_n$ is a root of $F$ if and only if the first vertex $v'_1$ of $\Delta F$ is a root (See Remark 4.1). So, the statement holds for $k = n$.

Now suppose the statement holds for $k + 1$. Then, we observe that the braid move $\sigma_k$ moves $E_k$ to the $k + 1$st position in $\sigma_k E$:
$$\sigma_k(E_1, \cdots, E_k, \cdots, E_n) = (E_1, \cdots, E_{k-1}, E'_{k+1}, E_k, E_{k+2}, \cdots, E_n).$$
Therefore, by induction on $n - k$ we have that $E_k$ is a projective module if and only if the $(n - k)$th vertex of $\Delta \sigma_k F = \sigma_{n-k} \Delta F$ is a root. By Proposition 2.3, this is equivalent to the $(n - k + 1)$st vertex of $\Delta F$ being a root. Thus the statement holds for $k$.

This proves (1) in the proposition. □

Corollary 4.12. $\Delta E_*$ has a simple injective object if and only if $E_*$ has only one injective object. Similarly, $E_*$ has a simple projective object if and only if $\Delta E_*$ has only one projective object.

Figure 13. Illustrating Proposition 4.11 The projective vertices $v_1, v_2, v_3$ of $F$ become the roots $v_7, v_6, v_5$ resp. of $\Delta F$. The only root $v_3$ of $F$ becomes the only injective vertex $v_5$ of $\Delta F$. The injective vertices $v_3, v_7$ of $F$ correspond to the last injective $v_5$ in $\Delta F$ and its child $v_1$. The first projective in $F$ and its children become the projective vertices of $\Delta F$. For the placement of $v_2$ in $\Delta F$ we go to $\Delta^{-1} F$ then apply $\delta^7 = \Delta^2$. 
4.3. Clusters and signed exceptional sequences. The Garside element $\Delta$ takes a “support tilting set” for any hereditary algebra to the corresponding “signed exceptional sequence” [15]. This is a comment made in [4] using different terminology (before the introduction of signed exceptional sequences in [15]).

Definition 4.13. We define a support tilting set for a hereditary algebra $\Lambda$ with $n$ simple objects to be a set of $n$ nonisomorphic modules and “shifted projective modules” $T_1, \cdots, T_n$ (each $T_i$ is either an indecomposable module $M_i$ or a shifted indecomposable projective module $P_k[1]$) having the property that $\text{Hom}_{\mathcal{D}^b}(T_i, T_j[1]) = 0$ in the bounded derived category $\mathcal{D}^b = \mathcal{D}^b(\text{mod-}\Lambda)$ for all $i, j$. In other words, $\text{Ext}^1_\Lambda(M_i, M_j) = 0$ for all $i, j$ and $\text{Hom}_\Lambda(P_k, M_i) = 0$ for all $k, i$.

For example, in $A_3$, $\{S_1, S_3, P_2[1]\}$ is a support tilting set since $S_1, S_3$ do not extend each other ($\text{Ext}^1_\Lambda(S_1, S_3) = \text{Ext}^1_\Lambda(S_3, S_1) = 0$) and $\text{Hom}_\Lambda(P_2, S_1 \oplus S_3) = 0$.

Definition 4.14. By a signed exceptional sequence we mean an exceptional sequence $E_* = (E_1, \cdots, E_n)$ together with a sequence $\varepsilon_*$ of signs $\varepsilon_i \in \{-1, +1\}$ so that $\varepsilon_i = -1$ implies $E_i$ is relatively projective. We write the sequence as $(X_1, \cdots, X_n)$ where

$$X_i = \begin{cases} E_i[1] & \text{if } \varepsilon_i = -1 \\ E_i & \text{otherwise.} \end{cases}$$

For example, in $A_3$, $(S_1[1], S_3, P_2)$ is a signed exceptional sequence since $S_1$ is relatively projective. (The first object in an exceptional sequence is always relatively projective.)

In [15], an explicit correspondence between signed exceptional sequences and support tilting objects is shown.

Theorem 4.15. [15] Over any hereditary algebra, there is a 1-1 correspondence between signed exceptional sequences and ordered support tilting sets.

Thus, for type $A_n$, the number of signed exceptional sequences is $n!$ times the Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+2}{n+1}$. For linear $A_n$, this can also be seen from our generating function $p_n(a, b, c)$ in Theorem 1.30. Since the relative projective objects can have any sign, the number of signed exceptional sequences is

$$p_n(2, 1, 2) = 2(n-1 + 2(2))(n-2 + 2(4)) \cdots (1 + 2(n)) = 2 \frac{(2n+1)!}{(n+2)!} = n! C_{n+1}.$$  

For linear $A_n$, there is an explicit correspondence using the Garside element $\Delta$.

Proposition 4.16. Any support tilting set can be arranged into a signed exceptional sequence $E_* = (E_1, \cdots, E_n)$ with signs $\varepsilon_1, \cdots, \varepsilon_n$. Furthermore,

$$\Delta E_* = (E'_n, \cdots, E'_1)$$  

with signs $\varepsilon'_n, \cdots, \varepsilon'_1$ is a signed exceptional sequence with the property that

$$\varepsilon_k \varepsilon'_j \chi_\Lambda(E_k, E'_j) = \delta_{kj}.$$  

In addition, the signed exceptional sequence $E_*$ corresponds to the ordered support tilting set $(\varepsilon_n E_n, \cdots, \varepsilon_1 E_1)$, which is $E_*$ with signs $\varepsilon_i$ in reverse order.
Remark 4.17. It is stated in [4] that this holds for any hereditary algebra. Here we prove it in the case of linear $A_n$ using rooted labeled forests. Also, Equation (4.2) implies that $-\varepsilon'_j \dim E'_j$ are the “c-vectors” of the “cluster” $\{\varepsilon_k \dim E_k\}$ if we take the “initial seed” to be the shifted projectives $E_{s_i}, \varepsilon_{s_i}$.

Proof. It is an observation originally due to Schofield [21] that a support tilting set can be arranged in an exceptional sequence. In the case of finite type, this is very easy to see: take the left-to-right order of the objects as they appear in the Auslander-Reiten quiver. The shifted projective objects can be placed after these in increasing order of size. Since all negative (shifted) objects are projective, the result is a signed exceptional sequence.

To see that the exceptional sequence $\Delta E_*$ with signs $\varepsilon_*$ is a signed exceptional sequence we examine which objects have negative signs.

1. If $E_k$ is a projective module then $E_k$ and $E'_k$ have the same sign: $\varepsilon_k = \varepsilon'_k$. Since $v_k$ is a root of the forest $\Delta F$, $E'_k$ is relatively projective and either sign is allowed.
2. If $E_k$ is any relative projective object which is not projective then it is positive. So, $\varepsilon'_k = \varepsilon_k = +1$.
3. In all other cases, $E_k$ is relatively injective but not relatively projective. So, $E_k$ is positive and $\varepsilon'_k = -\varepsilon_k = -1$. This is OK since $E'_k$ is relatively projective (but not relatively injective) in $\Delta E_*$.

The conclusion is that all objects in $\Delta E_*$ which are relatively projective but not relatively injective have a negative sign and those which are relatively injective but not relatively projective have positive sign. The roots might have either sign. So, $\Delta E_*$ with signs $\varepsilon'_*$ is a signed exceptional sequence.

By Lemma 4.8 we have $\chi_\Lambda(E_k, E'_k) = \varepsilon_k \varepsilon'_k$. By Lemma 4.10 $\chi_\Lambda(E_k, E'_j) = 0$ when $j \neq k$. Equation (4.2) follows.

To prove the last statement we recall some results from [15]. By [15 Corollary 2.17], the dimension vectors $\varepsilon'_i \dim E'_i$ of the signed exceptional sequence $(E'_n, \cdots, E'_1)$, with signs $\varepsilon'_i$, corresponds to the ordered support tilting set $(T_n, \cdots, T_1)$, with signs $\varepsilon_i$, are equal to the negative c-vector: $-\beta_i$ corresponding to the ordered support tilting set:

$$\varepsilon'_i \dim E'_i = -\beta_i$$

assuming that $(T_1, \cdots, T_n)$, which are $T_i$ in reverse order, form an exceptional sequence. This is the case here by construction if we take $T_i = E_i$.

By [15 Theorem 2.14], the c-vectors $\beta_i$ corresponding to the support tilting set $\{\varepsilon_i T_i\}$ are the unique solutions of the equation:

$$\varepsilon_k \chi_\Lambda(T_k, \beta_j) = -\delta_{kj}.$$  

Combining (4.4), (4.3) and (4.2) we see that the signed exceptional sequence (4.1) corresponds to the ordered cluster tilting set $(T_n, \cdots, T_1)$, with signs $\varepsilon_i$ as claimed. □

Example 4.18. For $A_2$, there are five support tilting objects. These are listed below, ordered as signed exceptional sequences. Applying $\Delta$ we get $\Delta(E_1, E_2) = (E'_2, E'_1)$ with $E'_1 = E_1$ and $E'_2 = -\tau E_2$ so that $\chi(E_i, E'_j) = \varepsilon_i \varepsilon'_j \delta_{ij}$. We interpret $X[1]$ as negative $X$.

1. $(P_2, P_1) \overset{\Delta}{\rightarrow} (S_1, P_2)$
shifted: \((P_2[1], P_1)\)

(3) \((S_1, P_2[1]) \xrightarrow{\Delta} (P_1[1], S_1)\)

(4) \((P_2[1], P_1[1]) \xrightarrow{\Delta} (S_1[1], P_2[1])\)

(5) \((P_2, P_1[1]) \xrightarrow{\Delta} (S_1[1], P_2)\)

For example, in (2), \(\tau S_1 = P_2\). So, \(\Delta(P_1, S_1) = (-\tau S_1, P_1) = (P_2[1], P_1)\) with \(\chi(S_1, P_2) = \dim \text{Hom}_A(S_1, P_2) - \dim \text{Ext}_A(S_1, P_2) = -1\), the sign of \(P_2[1]\).

**Example 4.19.** None of the forests in Figure 13 are examples of support tilting sets. In the first and second forest, there is a vertex with two relatively injective children. These extend each other. The last \(\Delta F\) has three roots. The smaller two extend each other. (We could shift the projective module \(E_7\).) In Figure 12, the second and last forests are not support tilting for the same reasons. The second has two relatively injective children of the root and the last has three roots which is not allowed. The first forest is support tilting if we shift the projective module \(E_3\). This gives \((S_2, I_2, S_3[1])\). The corresponding signed exceptional sequence is given by the second forest with its root shifted: \((P_1[1], S_1, S_2)\). To illustrate Equation (12) note that \(\text{Hom}(I_2, S_1) = K\), but \(\text{Hom}(I_2, S_2) = 0 = \text{Ext}(I_2, S_2)\) and \(\text{Hom}(I_2, P_1) = 0 = \text{Ext}(I_2, P_1)\).

4.4. **Action of extended braid group.** The *extended braid group* is

\[
\tilde{B}_n := B_n \rtimes_C \mathbb{Z}/2
\]

using the outer involution \(C\) given by \(\sigma_i \mapsto \sigma_i^{-1}\). We refer to \(C\) as complex conjugation. Equivalently, we use \(D = \Delta C\) in \(\tilde{B}_n\) conjugation by which gives another involution:

\[
D\sigma_i D = \sigma_{n-i}^{-1}.
\]

We call this *duality* since it corresponds to \(D : \text{mod-}\Lambda \rightarrow \text{mod-}\Lambda^{op}\) given by

\[
DM = \text{Hom}_K(M, K).
\]

This reverses the order of an exceptional sequence:

\[
D(E_1, \ldots, E_n) = (DE_n, \ldots, DE_1)
\]

since \(\chi_A(X, Y) = \chi_{A^{op}}(DY, DX)\). The operation \(D\) gives an involution on the set of complete exceptional sequences of the linear \(A_n\) quiver since this quiver is isomorphic to its opposite quiver. The modules \(DE_i\) have the same support inclusion relations as the \(E_i\). Therefore, duality \(D\) acts on the corresponding rooted labeled forests by reversing the order of vertices: \(DF \cong F\) with vertex labels \(v' \in \nu_{n+1}\).

Complex conjugation \(C = D\Delta\) has a cleaner description than the Garside element:

**Proposition 4.20.** Let \(F' = CF = D\Delta F\). Then \(F = D\Delta F'\), i.e., \(C = D\Delta\) is an involution. Furthermore:

1. \(v_{i_1}, \ldots, v_{i_p}\) are the projective vertices of \(F\) if and only if \(v'_{j_1}, \ldots, v'_{j_p}\) are the roots of \(F'\).
2. \(v_{k_1}, \ldots, v_{k_q}\) are the injective vertices of \(F\) with \(k_1 < k_2 < \cdots < k_q\) if and only if \(v'_{k_1}\) is the first projective vertex of \(F'\) and \(v'_{k_2}, \ldots, v'_{k_q}\) are its children.
3. Dually, \(v_{\ell_q}\) is the last injective vertex of \(F\) and \(v_{\ell_1}, \ldots, v_{\ell_{q-1}}\) are its children if and only if \(v''_{\ell_1}, \ldots, v''_{\ell_{q-1}}\) are the projective vertices of \(F'' = D\Delta^{-1} F\).
4.5. **Comments.** We note that the results of this paper partially answer a question in [26] raised by the second author. The question was: “Is it possible to interpret complete exceptional sequences of linear Nakayama algebras of rank \( n \) by certain rooted labeled forests with \( n \) labels?” But another question is raised: In [26] a bijection is constructed between rooted labeled trees of height at most 1 and complete exceptional sequences for \( A_n \) with radical square zero. How is this related to the bijection constructed in the present paper? Can this bijection be extended to rooted labeled forests of larger bounded heights and exceptional sequences of other algebras?

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*Email address: igusa@brandeis.edu*

*Email address: emresen641@gmail.com*