Fast Dirichlet Optimal Parameterization of Disks and Sphere Sectors

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Abstract

We utilize symmetries of tori constructed from copies of given disk-type meshes in 3d, together with symmetries of corresponding tilings of fundamental domains of plane tori. We use these correspondences to prove optimality of the embedding of the mesh onto special types of triangles in the plane, and rectangles. The proof provides a certain framework for using symmetries of the image domain. The complexity is linear in the mesh size. We then use the method to prove a novel embedding of a 3-fold rotationally symmetric sphere-type mesh onto a set in the plane with 3-fold rotational symmetry. The only additional constraint on the set is that its translations tile the plane. The embedding is optimal under the symmetry and tiling constraint. This is done by a novel construction of a torus from 63 copies of the original sphere.

1 Introduction

In this work we start by describing a method for parametrizing a simplicial complex, which is a combinatorial disk, embedded in $\mathbb{R}^3$ onto three types of domains. By "parametrizing" we mean that we give an efficient way to compute its embedding. The three types of domains are a right-angled isosceles triangle, an equilateral triangle and a square. These parametrizations are optimal with respect to a Dirichlet energy functional defined on simplicial complexes. In practice one can write a suitable system of linear equations for the images in the plane of the vertices of the embedding. This method and proofs were described in [1]. However, by giving direct proofs which utilize in a novel way the symmetry of the constructed simplicial complex, we are able to address and prove the main contribution of the article, namely, parameterizing a 3-fold rotationally symmetric sphere-type mesh onto a tile in the plane, which has 3-fold rotational symmetry and free boundary under the constraint that the plane can be covered with translations of such tile.

For the matter of the proof, in each one of the cases we construct, out of copies of the given mesh, a suitable simplicial complex embedded in $\mathbb{R}^3$ which is
homeomorphic to a torus. We then embed this torus in a suitable plane torus, i.e. a quotient space of \( \mathbb{R}^2 \) a rank 2 lattice, with weights which are calculated according to the geometry of the given surface and guarantee minimization of the energy considered. The validity of the embedding is due to proofs by Lovasz and Gortler.

Parameterizations which minimize the Dirichlet energy are desired in the area of Geometry Processing as, at least in the case of triangles, they approximate conformal maps from Riemann surfaces for which the given complex is a good approximation. For example see [3]. We introduce a simple yet effective framework which allows to use symmetries of the image domain of an embedding map, to show symmetries of the embedding itself.

The framework consists of identifying a symmetry in the image domain, the plane torus in this case, and looking for a corresponding symmetric surface which can be constructed from copies of the given surface. For the different symmetries we define conjugate maps, one in the image domain and one in the source domain. These maps, when composed on the right side and on the left side respectively with the optimal embedding map, yield the same optimal embedding map. Since the surface constructed is symmetric with respect to the map in the image domain, we conclude that the embedding has a corresponding symmetry.

Applying this type of argument with different pairs of conjugate maps prove the desired symmetries of the embedding. Then we conclude optimality of the embedding map restricted to each one of the copies consisting of the constructed surface.

The most elaborate application of this method, and the most useful in our opinion, is the construction of a torus out of 63 copies of a 3-fold rotationally symmetric sphere-type mesh. We are able to parametrize the torus onto a 120 degrees rhombus as shown in Figure 1 by solving a full rank system of linear equations, whose size is approximately 63 times the number of vertices in the original sphere-type mesh. It is interesting to compare this to the embedding of a sphere described in [1] onto "Orbifold of type 2". If our symmetric sphere was constructed out of 3 spheres, we would expect to obtain the same result as they did. However, for any other 3-fold rotationally symmetric sphere, our method which provides the optimal parameterization will be better. We refer to the three symmetric parts of the sphere-type mesh as sphere sectors. Note that a 3-fold rotationally symmetric sphere, can be made from 3 copies of a disk-type surface, with a connecting scaffold. We think that such construction will be valuable in applications.

The paper starts with the case of embedding a complex onto a right-angled isosceles triangle. In Section 2 we construct the torus to be embedded for that case. In Section 3 we describe the energy functional to be minimized and explain what the weights are. In Section 4 we prove the desired symmetries of the embedding and conclude optimality of the embedding of a single copy of the given complex. In Section 5 we deal rigorously with the case of embedding onto an equilateral triangle. The case of embedding onto a rectangle is described very briefly in Section 6. In Section 7 a note about the complexity of implementing
the embedding. Then we deal with the most elaborate case - embedding of a sphere with 3-fold rotational symmetry.

Figure 1: Embedding of the torus constructed out of 63 sphere copies (using equal weights for simplicity).

2 Constructing the torus

We begin with the case of embedding a complex onto a right-angled isosceles triangle.

We start with two definitions.

**Definition 1.** A surface is a connected topological 2-manifold, that is, a connected Hausdorff space $S$ in which each point has a neighborhood that is home-
omorphic to an open subset of the plane. We require the transition maps to be continuous.

**Definition 2.** A disc-type mesh is is a combinatorial disc (that is, simplicial 2-complex which is simply connected, finite and with nonempty boundary) embedded in \( \mathbb{R}^3 \) (i.e. it is a surface, 1-simplices are embedded as straight lines).

We construct \( \widetilde{M} \), a simplicial 2-complex immersed in \( \mathbb{R}^3 \), with torus topology. By torus topology, we mean that it is homeomorphic to \( \mathbb{R}^2 / \mathbb{Z}^2 \).

Let \( M \) be a disc-type mesh. By definition \( M \) has a nonempty boundary and therefore its boundary contains at least 3 distinct points. We choose three distinct vertices on the boundary of \( M \): \( v_0, v_1, v_2 \in \mathbb{R}^3 \). We let \( \Gamma_0 \) be the path between \( v_0 \) and \( v_1 \) on the boundary of the disc. Let \( \Gamma_1 \) be the path on the disc’s boundary between \( v_1 \) and \( v_2 \), and let \( \Gamma_2 \) be the path on the disc’s boundary between \( v_2 \) and \( v_0 \). Simply connectedness implies that these paths are uniquely defined.

We now define eight duplicates of the disk-type mesh \( M \), this means eight identical combinatorial disks embedded identically in \( \mathbb{R}^3 \). Denote them by \( M_1, M_2, ..., M_8 \). For each copy \( M_i \) \((i = 1, ..., 8)\), let \( v'_0, v'_1, v'_2 \) be the vertices positioned at \( v_0, v_1, v_2 \) respectively. For each copy \( M_i \) \((i = 1, ..., 8)\), let \( \Gamma'_0 \) be the path between \( v'_0 \) and \( v'_1 \) on the boundary of \( M_i \). Let \( \Gamma'_1 \) be the path on the \( M_i \)’s boundary between \( v'_1 \) and \( v'_2 \), and let \( \Gamma'_2 \) be the path on \( M_i \)’s boundary between \( v'_2 \) and \( v'_0 \).

We now construct a simplicial 2-complex \( \widetilde{M}_\Delta \), immersed in \( \mathbb{R}^3 \), out of the copies \( M_i, i = 1, ..., 8 \). This is done by uniting the following pairs of paths: \( \Gamma_i'^0 \) is united with \( \Gamma_i'^{i+1} \) for \( i = 1, 3, 5, 7 \). We name the united paths \( \Gamma_0, \Gamma_0', \Gamma_0' \), \( \Gamma_1, \Gamma_1' \), \( \Gamma_2, \Gamma_2' \), \( \Gamma_4, \Gamma_4' \) respectively. Secondly, we unite the following pairs: \( \Gamma_1' \) with \( \Gamma_3' \), and \( \Gamma_1' \) with \( \Gamma_1'^{1+1} \) for \( i = 2, 4, 6 \). We name these united paths \( \Gamma_1, \Gamma_1' \), \( \Gamma_2' \), \( \Gamma_3' \), \( \Gamma_4' \) respectively. Lastly, we unite the following pairs: \( \Gamma_3' \) with \( \Gamma_3' \), \( \Gamma_2' \) with \( \Gamma_2' \), \( \Gamma_3' \) with \( \Gamma_3' \), and \( \Gamma_2' \) with \( \Gamma_2' \). We name these united paths \( \Gamma_3, \Gamma_3' \), \( \Gamma_2, \Gamma_2' \), \( \Gamma_3, \Gamma_3' \) respectively.

We will show that \( \widetilde{M}_\Delta \) is homeomorphic to the 2-torus. We will refer to the 8 different sub-simplices of \( \widetilde{M}_\Delta \) corresponding to \( M_i, i = 1, ..., 8 \), as \( M'_i, i = 1, ..., 8 \) in accordance. Note that the \( M'_i \)'s are not disjoint, as we united edges and vertices to construct \( \widetilde{M}_\Delta \). We use Tutte embedding (see [11]), to map each \( M'_i \) to \( M'_i \) in the \( \mathbb{R}^2 / \mathbb{Z}^2 \) according to the following diagram. Each \( M'_i, i = 1, ..., 8 \), is mapped to \( M'_i \) in the diagram, where each bordering path \( \Gamma_j^k, j \in \{0, 1, 2\}, k \in \{1, 2, 3, 4\} \) is mapped onto \( \gamma_j^k \) up to a translation. We demand that the vertices on each path are mapped to equally spaced vertices on the line it is mapped onto, and of course that the extreme vertices of each path are mapped to the extreme points of that line. If we consider the well defined total map from \( \widetilde{M}_\Delta \) to \( \mathbb{R}^2 / \mathbb{Z}^2 \), we have a homeomorphism based on the properties of Tutte embedding.

From now on, we refer to \( \widetilde{M}_\Delta \) as \( \widetilde{M} \). We regard \( \widetilde{M} \) as a surface, but also regard the immersed (or embedded) subsets of the underlying simplicial complex structure of \( \widetilde{M} \).
3 Optimal embedding of the torus

Given two vectors \( v_1, v_2 \in \mathbb{R}^2 \), let \( \Lambda \) be the lattice generated by \( v_1, v_2 \). We wish to find an embedding of \( \mathcal{M} \) onto \( \mathbb{R}^2/\Lambda \), which minimizes the Dirichlet energy functional. The proof for the validity of the embedding itself is due to results by Gortler or Lovasz [6][11].

In the general setting the Dirichlet energy of a map \( U \) is defined as follows. Let \( \mathcal{X} = \mathcal{X}^n \) and \( \mathcal{Y} = \mathcal{Y}^m \) be two smooth compact Riemannian manifolds of dimension \( n \) and \( m \), respectively. We assume \( \mathcal{X} \) and \( \mathcal{Y} \) are equipped with metric tensor \((g_{\alpha\beta})\) and \((\gamma_{ij})\), respectively, in some local coordinate charts \((x_1, ..., x_n)\) at \( x \), and \((U^1, ..., U^m)\) of \( U(x) \) on \( \mathcal{X} \) and \( \mathcal{Y} \), respectively. The Dirichlet energy of a smooth map \( U : \mathcal{X} \to \mathcal{Y} \) is defined as the integral of the square of the derivatives \( dU \). More precisely, the energy density of \( U \) is

\[
e(x, U) := \frac{1}{2} |dU_x|^2 = \frac{1}{2} \text{tr}[(dU_x)^* dU_x]
\]

and the Dirichlet energy of \( U \) is

\[
\mathcal{E}(U, \mathcal{X}) := \int_{\mathcal{X}} e(x, U) \text{dvol}_\mathcal{X} = \frac{1}{2} \int_{\mathcal{X}} |dU_x|^2 \text{dvol}_\mathcal{X}.
\]

And so,

\[
\mathcal{E}(U, \mathcal{X}) = \frac{1}{2} \int_{\mathcal{X}} g^{\alpha\beta} \gamma_{ij} \frac{\partial U^i}{\partial x^\alpha} \frac{\partial U^j}{\partial x^\beta} \text{dvol}_\mathcal{X}.
\]
According to Theorem 3.3 in [6], we have a function \( \tilde{U} \) from the set of all faces, \( \tilde{M} \), onto \( \mathbb{R}^2 / \Lambda \). For \( \Delta \in \tilde{M} \) and \( \tilde{U}(\Delta) \) the tensor metrics are the ones induced from the Euclidean Riemannian metric on \( \mathbb{R}^3 \). The simplicial Dirichlet energy is then defined to be

\[
    E(\tilde{U}, \tilde{M}) := \sum_{\Delta \in \tilde{M}} E(\tilde{U}|_{\Delta}, \Delta).
\]

We use Pinkall and Polthier classic work for the calculation of the "cotangent weights" which yield the optimal embedding, \( \tilde{U} \), with respect to the simplicial Dirichlet energy functional. The critical point for the Dirichlet energy is a harmonic function, and it’s a minimum point. The image of each vertex in \( \tilde{M} \) is mapped to a weighted average of the images of its neighbours according to the following equation

\[
    \sum_{i=1}^{n} (\cot \alpha_i + \cot \beta_i) (x_i - x_0) = 0, x_i \in \mathbb{R}^2,
\]

where \( x_i = 1, ..., n \), is an ordering of the 1-ring of \( x_0 \), and \((\alpha_i)_{i=1}^{n}, (\beta_i)_{i=1}^{n} \) are defined as follows. Set \( x_{n+1} = x_1 \). For \( i = 1, ..., n \), let \( \alpha_i \) be the angle between the lines \( x_0x_i \) and \( x_ix_{i+1} \), and let \( \beta_i \) be the angle between the lines \( x_0x_{i+1} \) and \( x_ix_{i+1} \). In the calculation of the weights, the ordering of the 1-rings should be in the direction which keeps a consistent orientation of the faces of the simplicial complex.

Throughout this work we will assume that the weights are positive. This is the case if no obtuse angles occur.

4 Properties of the mapped torus

We now show the symmetry properties of the one-to-one Dirichlet energy minimizing mappings of \( \tilde{M} \) onto \( \mathbb{R}^2 / \Lambda \), where \( \Lambda \) is a lattice generated by two unit vectors, i.e. \( \Lambda = \langle v_1, v_2 \rangle \) and \( \|v_1\| = \|v_2\| = 1 \). (Norm is the 2-norm.)

We first examine the one-to-one Dirichlet energy minimizing mapping of \( \tilde{M} \) onto \( \mathbb{R}^2 / \Lambda \), where \( \Lambda = \langle e_1, e_2 \rangle \), \( e_1 = (1 0)^T, e_2 = (0 1)^T \), i.e. the case where \( \Lambda = \mathbb{Z}^2 \).

Let \( l_H = \mathbb{R} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( l_V = \mathbb{R} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), two infinite lines in the plane. Let \( R_H, R_V \) be the reflections along \( l_H, l_V \) respectively. In addition, let \( l_{DP} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( l_{DS} = \mathbb{R} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \), corresponding to the primary and second diagonal of the square with corners \( (0 0)^T, (1 1)^T \), a fundamental domain in \( \mathbb{R}^2 \) for \( \mathbb{R}^2 / \Lambda \). Let \( R_{DP}, R_{DS} \) be reflections along \( l_{DP}, l_{DS} \) respectively.

**Claim 1.** The one-to-one Dirichlet energy minimizing mapping of \( \tilde{M} \) onto \( \mathbb{R}^2 / \Lambda \) is symmetric with respect to reflections along \( l_H, l_V, l_{DP}, l_{DS} \).
Proof. Let $\tilde{X}$ be the simplicial complex, which is an embedding of $\tilde{M}$ by the one-to-one Dirchlet energy minimizing map of $\tilde{M}$ onto $\mathbb{R}^2/\Lambda$. We denote this map by $\Phi$ ($\Phi : \tilde{M} \rightarrow \mathbb{R}^2/\Lambda$). It carries 2-simplices to 2-simplices, 1-simplices to 1-simplices, and so on. We extend in a natural way $R_H, R_V, R_{DP}, R_{DS}$ to be maps on simplicial complexes in the plane, and regard $\tilde{X}$ as a simplicial complex in the fundamental domain. Note that the image of an embedding of a simplicial complex on the fundamental domain under the reflection operations is still an embedding of the complex onto $\mathbb{R}^2/\Lambda$.

We define the maps $S_H, S_V, S_{DP}, S_{DS}$ which are all automorphisms of the surface $\tilde{M}$, in the sense that they are one-to-one, onto, and continuous. We define the map $S_H$ as the "horizontal reflection" (formal explanation in the next lines) of the surface $\tilde{M}$. $S_H$ reflects $\tilde{M}$ along $\Gamma^1$ concatenated by $\Gamma^3$, meaning it maps $M_1$ to $M_8, M_2$ to $M_7$ and so on. (In general for $i = 1, 2, 3, 4$ it maps $M_i$ to $M_{9-i}$.) All these maps, each a restriction of $S_H$, are the natural maps from one disk-type mesh to another copy of the same disk-type mesh. $S_H$, defined this way, is an automorphism of $\tilde{M}$. The maps $S_V, S_{DP}, S_{DS}$ are defined in a similar manner. They are the "reflections" of $\tilde{M}$ along: (Case $V$) $\Gamma^2_1$ concatenated by $\Gamma^3_1$, (Case $DP$) $\Gamma^3_0$ concatenated by $\Gamma^3_0$, (Case $DS$) $\Gamma^2_0$ concatenated by $\Gamma^4_0$; respectively.

We have the following: $\Phi = R_H \circ \Phi \circ S_H$, but also $S_H \tilde{M} \equiv \tilde{M}$ (the two sides are equal up to renaming), and so $\Phi(\tilde{M}) = R_H \circ \Phi(\tilde{M})$. (The first equation can be seen to hold from observing that both sides satisfy the fully constrained system of linear equations as in Gortler’s embedding of a torus in the plane.) The same argument applies for $R_V, S_V, R_{DP}$ and $S_{DP}, R_{DS}$ and $S_{DS}$. This finishes the proof of the claim.

\[\square\]

Corollary 1. $\Phi(M_1)$ is precisely one of the octants of the fundamental domain of $\mathbb{R}^2/\Lambda = \mathbb{R}^2/\mathbb{Z}^2$, and is a right angle isosceles triangle with leg length $\frac{1}{2}$ (see Figure 2).

Proof. Denote by $O$ the octant of the fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$ which has non-empty intersection with $\Phi(M_1)$. Then if it does not hold that $O \subset \Phi(M_1)$, by the symmetries of $\Phi(\tilde{M})$ we get a contradiction for the fact that $\Phi$ is an embedding onto $\mathbb{R}^2/\mathbb{Z}^2$. On the other hand, if it is strictly that $O \subset \Phi(M_1)$, then we get a contradiction for the assumption that $\Phi$ is one-to-one on its range, and so we conclude the statement.

\[\square\]

Corollary 2. $\mathcal{E}(\Phi|_{M_1}, M_1) = \min_{\phi \in A} \mathcal{E}(\phi, M_1)$, where $A$ is the set of embeddings of $\tilde{M}$ onto the triangle with corners $(0,0), (0,5), (5,0)$ such that $v_0,v_1,v_2$ are mapped to these corner points respectively.

Proof. This is clear - as if this is not the case, there exists $\phi \in A$ such that $\mathcal{E}(\Phi|_{M_1}, M_1) > \mathcal{E}(\phi, M_1)$, and we can construct a map $\Phi'$ for which $(\Phi', \tilde{M}) < (\Phi, \tilde{M})$. Contradiction.

\[\square\]
5 Mapping to an equilateral triangle

We show how to use the torus embedding to embed a disk-type mesh onto an equilateral triangle in the plane. Again, assume that we have \( v_0, v_1, v_2, 3 \) marked vertices on the boundary of the mesh, \( \mathcal{M} \). As shown in Figure 4, the plane can be tiled by regular hexagons, and one can be convinced that exactly seven regular hexagons can be arranged on a flat torus whose fundamental domain is a rhombus with angles of 60 degrees and 120 degrees, such that their interiors are disjoint. We define \( w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} \Re(e^{i\pi/3}) \\ \Im(e^{i\pi/3}) \end{pmatrix}, \) and \( \Lambda_R = \langle w_1, w_2 \rangle \). Each hexagon in the tiling of the plane, can be divided into six regular triangles with disjoint interiors having a common corner at the center of the hexagon. We define an equivalence relation on the triangles consisting of the triangles with disjoint interiors having a common corner at the center of the colored hexagon of the triangle attached to \( v_i \). For each \( i \), such that \( 1 \leq i \leq 42 \), let \( v_j^i, v_k^i \) correspond to the two other corners of the triangle attached to \( M_i \), such that \( v_j^i \) corresponds to the vertex marked by '1', and \( v_k^i \) corresponds to the vertex marked by '2'.

We now unite the \( M_i, i = 1, ..., 42 \), into one connected mesh according to Figure 4. This is done by looking at a bordering edge between two triangles in the figure. If the triangles correspond to \( M_{i_1} \) and \( M_{i_2} \), the bordering edge corresponds to a path of edges between \( v_j^{i_1} \) and \( v_j^{i_2} \) in \( M_{i_1} \), and between \( v_k^{i_1} \) and \( v_k^{i_2} \) in \( M_{i_2} \). Looking at all the edges in Figure 4 bordering two distinct triangles, we unite all the paths in \( M_i, i = 1, ..., 42 \) corresponding to these edges, and we obtain one connected simplicial complex denoted by \( \mathcal{M}' \).

We look at the boundary of \( \mathcal{M}' \). The boundary consists of 21 paths corresponding to the 21 boundary edges of the seven hexagons tiling in Figure 4. A boundary edge in Figure 4 corresponds to a path between vertices \( v_j^{i_0} \) and \( v_k^{i_0} \) on \( M_{i_0} \subset \mathcal{M}' \) for some \( i_0 \in \{1, 2, ..., 42\} \). If we were to extend the tiling of the regular triangle along with the marking of the corners, the edge in the figure would be bordering the original triangle and another triangle in the extended tiling. This other triangle is equivalent to one of the tiles in Figure 4, say the triangle that corresponds to \( M_{i_0} \). We then unite the path between \( v_j^{i_0} \) and \( v_k^{i_0} \) in \( M_{i_0} \subset \mathcal{M}' \), with the path between \( v_j^{i_0} \) and \( v_k^{i_0} \) \( M_{i_0} \subset \mathcal{M}' \). We do this for all the 21 paths corresponding to boundary edges in Figure 4. We obtain a simplicial complex \( \mathcal{M} \).

As a surface \( \mathcal{M} \) is homeomorphic to \( \mathbb{R}^2/\Lambda_R \), as one be convinced from con-
considering Figure 4 and employing Tutte embedding as before.

Let \( S_R : \mathcal{M} \to \mathcal{M} \) be the map which "rotates" \( \mathcal{M} \) one step counter-clockwise around the vertex corresponding to the center of the brown hexagon in Figure 3, meaning it maps \( \mathcal{M}_k \subset \mathcal{M} \) for some \( k \in \{1, \ldots, 42\} \) to \( \mathcal{M}_{k'} \), where the triangle corresponding to \( \mathcal{M}_{k'} \) is the triangle corresponding \( \mathcal{M}_k \) rotated by 60 degrees counter-clockwise around the center of the brown hexagon. Let \( \Phi : \mathcal{M} \to \mathbb{R}^2/\Lambda_R \) be the Dirichlet optimal embedding map. Denote by \( p_0 \) the image under \( \Phi \) of the vertex of \( \mathcal{M} \) corresponding to the center of the brown hexagon. Let \( H_{RF} : \mathbb{R}^2/\Lambda_R \to \mathbb{R}^2/\Lambda_R \) be the automorphism of \( \mathbb{R}^2/\Lambda_R \), which rotates the torus 60 degrees clockwise around pivot point \( p_0 \), and reflects each regular triangle along its height emanating from the corner marked by 0 (in the tiling of the torus consisting of 42 regular triangles). We call it the "rotate and flip symmetry", for a 60 degrees counter-clockwise rotation.

We define a "translate map" \( S_T : \mathcal{M} \to \mathcal{M} \). The translate map \( S_T \) translates the torus cyclically in the following way: it maps the disk \( \mathcal{M} \) corresponding to the brown hexagon, to the disk corresponding to the light blue hexagon lying to the north of it; it maps the disk corresponding to the purple hexagon to the disk corresponding to the brown hexagon lying north of it; the disk corresponding to the red hexagon to the disk corresponding to the yellow hexagon; and so on. One can refer to 4 and the marked fundamental domain, to see how the mapping translates the hexagons (yellow goes to dark blue, light blue goes to red and so on). The mapping by \( S_T \) of the disks in \( \tilde{\mathcal{M}} \) corresponding to the partition of the hexagon into triangles, is such that the correspondences of the images of the disks to triangles preserves the orientation of the original correspondences. By looking at Figure 4 one can see that the order of \( S_T \) is 7.

We now define the "conjugate" map of \( S_T \) on the torus, which we will denote by \( H_T \). Let \( q \in \mathbb{R}^2/\Lambda_R \) be the vector equal to the difference between the center of the brown hexagon and the center of the purple hexagon, if one has a seven hexagons tiling of the torus as in Figure 3 where the center of the brown hexagon is at \( p_0 \) (previously defined). Define \( H_T : \mathbb{R}^2/\Lambda_R \to \mathbb{R}^2/\Lambda_R \) to be the mapping \( x \mapsto x + q \) for \( x \in \mathbb{R}^2/\Lambda_R \). We have that for \( k = 1, 2, 3, 4, 5, 6, \) \( \tilde{\Phi} = H_T^k \circ \Phi \circ S_R^k \), and so we conclude that if \( M^H \subset \tilde{\mathcal{M}} \) is the disk corresponding to the brown hexagon in the figure, then \( \Phi(M^H) \) is exactly a hexagon. This is because \( \tilde{\Phi} \) is a union of translates of \( \Phi(M^H) \), where the interiors are disjoint, and if it weren’t the case, then a contradiction for the injectivity or the surjectivity of the embedding would follow. Going back to the "rotate and flip map", we conclude that for \( i = 1, \ldots, 42 \), each copy of the original mesh, \( \mathcal{M}_i \), is embedded by \( \tilde{\Phi} \) unto an equilateral triangle. These embeddings are identical up to translation of the range, and thus each has equal minimal Dirichlet energy.
6 The rectangular case

Given a disk-type mesh $\mathcal{M}$, and four marked vertices on its boundary, we can make four copies of $\mathcal{M}$ and glue them according to Figure 5, making a torus $\tilde{\mathcal{M}}$. The embedding of $\tilde{\mathcal{M}}$ (with "cotan weights") onto $\mathbb{R}^2/\mathbb{Z}^2$ will have horizontal and vertical symmetries, and each copy of $\mathcal{M}$ will be embedded optimally onto a square. The proof is similar to the previous proofs. One can also replace $\mathbb{Z}^2$ by a different lattice generated by two orthogonal vectors.

7 Complexity of the mappings

In each of the three embeddings described so far - embedding onto a right-angled isosceles triangle, onto an equilateral triangle and onto a rectangle, we start with disk-type mesh $\mathcal{M}$ with, say, $n$ vertices. For the matter of the proof, we construct a torus out of copies of $\mathcal{M}$. However, we stress that if one wishes to implement such an embedding, this could be done with solving two systems of $n$ linear equations with $n$ variables. The variables in one system of equations correspond to the $x$ coordinates of the image of the vertices, and in the other system of equations they correspond to the $y$ coordinates of the image of the vertices.

Figure 3: Partial gluing of all copies of disk mesh consisting of a torus.
vertices. In each of the cases, we embed the torus, which has more than \( n \) vertices, but the proofs show that the embeddings of the different copies are related by affine maps, and so one can actually write an \( n \) linear equations systems for an embedding of one copy onto each of the domains.

8 Embedding of a 3-fold symmetric sphere

In this section we let our simplicial sphere \( \tilde{S} \) be a simplicial complex which is homeomorphic to \( S^2 \subset \mathbb{R}^3 \). We assume to have three symmetric simple paths, mutually disjoint, on \( \tilde{S} \) emanating from \( p_0 \) - one of the points which lie on the sphere and on the axis of symmetry. We now make 63 copies of the sphere \( \tilde{S} \).

On each copy we have the same identical paths as on the original simplicial sphere. For each copy denote the paths on it by \( \gamma^i_0, \gamma^i_1, \gamma^i_2 \), where \( i \) is the index of the copy (\( i = 1, \ldots, 63 \)).

We create a torus from the copies, by cutting and stitching along paths
\( \gamma_i^0, \gamma_i^1, \gamma_i^2, i = 1, \ldots, 63 \). The gluing instructions can be found in Figure 7. Each hexagon not lying on the two gray rectangles, represents one of the simplicial sphere. In Figure 7, '11' and '12' denote the two sides created by "cutting" along path \( \gamma_i^0 \) (considering sphere \( i \)). '21' and '22' denote the two sides created by "cutting" along path \( \gamma_i^1 \), and '31' and '32' denote the two sides created by "cutting" along path \( \gamma_i^2 \).

Looking at the hexagons copied to the top gray rectangle from the bottom part, and the hexagons copied to the right gray rectangle from the left, one can be convinced that the stitching actually yields a torus.

We then embed the torus \( \mathcal{T} \) onto \( \mathbb{R}^2/\Lambda R \) \((3 \times 3 \times 7 = 63)\). By analogous arguments \( \mathcal{S} \subset \mathcal{T} \) (that is, any sphere in the constructed torus) is mapped to a tile which has 3-fold rotational symmetry. The map has optimal Dirichlet energy under the symmetry and tiling constraint.

The conjugate maps which prove the symmetric nature of the embedding are the rotations of all spheres by \( 2\pi/3 \) around their axis of symmetry and the rotations of the plane by \( 2\pi/3 \) around the image of the antipodal point of \( p_0 \). Then we consider conjugate pairs of maps - a translation map on the torus and a map that maps one sphere to another accordingly. The proof is similar to the proof in the embedding of a disk onto an equilateral triangle.

In Figure 7 the embedding of the whole torus can be seen. In Figure 8 the original sphere that we took can be seen. The torus was constructed out of 63 copies of the depicted sphere. In Figure 8 a zoom-in of the parameterization can be seen.
Figure 6: The original spheres - on the left with the special points from which the cuts run, on the right with three cuts in different colors.

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Figure 7: Gluing instructions for stitching the 63 spheres.
Figure 8: Zoomed-in picture of the parameterization of the 63-spheres torus.