A KAM-Theorem for Persistence of Quasi-periodic Invariant Tori in Bifurcation Theory of Equilibrium Points *

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Abstract. In this paper, we establish a KAM-theorem for ordinary differential equations with finitely differentiable vector fields and multiple degeneracies. The theorem can be used to deal with the persistence of quasi-periodic invariant tori in multiple Hopf and zero-multiple Hopf bifurcations, as well as their subordinate bifurcations, of equilibrium points of continuous dynamical systems.

Keywords: Quasi-periodic invariant torus, Small frequency, Degeneracy, multiple Hopf bifurcation.

1. Introduction

To study the bifurcations of equilibria of a system of differential equations (ODEs, PDEs and functional differential equations), one usually reduces such a system to a lower-dimensional one on the center manifold by the Center Manifold Theorem. Possibly the reduced system is

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finitely differentiable even if the original system is analytic. When the equilibrium is partially elliptic and the normal form of the reduced subsystem has a normal form of Birkhoff type on the center manifold, then the truncated normal form may possess quasi-periodic invariant tori. In this case, a question arises naturally: does the original system (equivalently, the reduced system on the center manifold) have quasi-periodic invariant tori with the same dimension?

This problem can be discussed by KAM theory and a careful study leads us to consider the existence of quasi-periodic tori of the following system

\[
\begin{align*}
\dot{I}_1 &= \varepsilon^{q_1} [A_1(\xi, \varepsilon) I_1 + \varepsilon^{q_1} g_1(I, \varphi; \xi, \varepsilon)] \\
\dot{I}_2 &= \varepsilon^{q_2} [A_2(\xi, \varepsilon) I_2 + \varepsilon^{q_2} g_2(I, \varphi; \xi, \varepsilon)] \\
\dot{\varphi}_1 &= \varepsilon^{q_5} [\omega_1(\xi, \varepsilon) + \varepsilon^{q_5} g_3(I, \varphi; \xi, \varepsilon)] \\
\dot{\varphi}_2 &= \omega_2(\xi, \varepsilon) + \varepsilon^{q_7} g_4(I, \varphi; \xi, \varepsilon),
\end{align*}
\]

where \( I = \text{col}(I_1, I_2) \in \Omega \subset \mathbb{R}^{n_{11}} \times \mathbb{R}^{n_{12}} = \mathbb{R}^{n_1}, \varphi = \text{col}(\varphi_1, \varphi_2) \in T^{n_{21}} \times T^{n_{22}} = T^{n_2}, q_j \geq 0 (j = 1, \cdots, 7), \xi \in \Pi \subset \mathbb{R}^{n_3} \) is the bifurcation parameter, \( \varepsilon \) is a small perturbation parameter.

When \( g_j = 0, j = 1, \cdots, 4 \), \( I = 0 \) represents the quasi-periodic torus of the integrable part of (1.1) which corresponds to the invariant torus of truncated normal forms. The aim of the present paper is to examine the persistence of the quasi-periodic torus \( I = 0 \) under small perturbations (i.e., \( g_j \neq 0, j = 1, \cdots, 4 \)). We meet some difficulties: the perturbation terms \( g_j, j = 1, \cdots, 4 \) are only finitely differentiable, there exist small frequencies, small twist and higher-order degeneracy in (1.1) and the number of parameter variables is possibly less than the dimension of tori. We need to tackle these difficulties in constructing a new KAM theorem for (1.1).

In the context of finitely differentiable perturbations, the study on the persistence of quasi-periodic invariant tori has originated from the work of Moser [24] on area-preserving mappings of an annulus, which was extended to dissipative vector fields in [6] based on smoothing operator technique. Another important method, which can relax the requirement for regularity of perturbations, is to approximate a differentiable function by real analytic ones [25, 30, 39, 27, 35, 13, 3, 38]. Rüssmann proved an optimal estimate result on approximating a differentiable function by analytic ones. Following this approach Zehnder [39] established a generalized implicit function theorem and applied it to the existence of parameterized invariant tori of nearly integrable Hamiltonian systems in finitely differentiable case, Pöschel [27] showed that on a Cantor set, invariant tori of the perturbed Hamiltonian system form a differentiable family in the sense of Whitney. The results and ideas of Moser and Pöschel are extended to the case of symplectic mappings by Shang [35] and to the case of lower dimensional elliptic tori by Chierchia and Qian [13], respectively. Wagener [38] extended the modifying terms theorem of Moser [26] (i.e., introducing additional parameters) to finitely differentiable and Gevrey regular vector fields.

The results mentioned above, except for [13], were restricted to the case where the integrable part is analytic in coordinate variables as well as in parameters. The integrable part in [13] is
assumed to be Lipschitz with respect to parameters and the frequency map to be a Lipschitz homeomorphism. Of course, if the unperturbed (integrable) part and the perturbation are both of class $C^l$ with $l > 2n$ ($n$ is the number of degrees of freedom), it is reduced to the case where the integrable part is analytic and the perturbation is of class $C^l$ by regarding the initial values of action variables as parameters. The KAM theorems in [7, 38] can be applied to quasi-periodic bifurcations (i.e., bifurcations of quasi-periodic invariant tori). In this paper, we shall extend the result and method of Pöschel [27] to the dissipative system (1.1) with degeneracies, and provide a convenient tool to investigate the persistence of quasi-periodic invariant tori in bifurcation theory of equilibrium points.

The perturbation was assumed to be $C^{333}$ originally in the work of Moser [24] on area-preserving mappings of an annulus, and then was weakened to $C^5$ by Rüssmann [30] and to $C^l(l > 3)$ (meaning that the perturbation is of class $C^3$ and the derivatives of order 3 are Hölder continuous) by Rüssmann [32] and Herman [19], where a counterexample for $l < 3$ was given. For improvements on weakening the regularity of perturbations in the Hamiltonian case we refer to [3] and references therein.

The above mentioned results were proved under the so-called non-degeneracy conditions. In the context of degenerate KAM theory, i.e., if Kolmogorov’s non-degeneracy or Arnold’s isos-energetic non-degeneracy condition is violated, Arnol’d [2] established a properly degenerate KAM theorem (refined by [14, 12]) to deal with quasi-periodic motions in the planetary many body problem. In this case the integrable part does not depend on the full set of action variables, and the non-degeneracy conditions are imposed additionally on the averaged perturbation. The ideas of Arnol’d [2] were extended to the resonant torus case in [10, 23] and the normal zero-frequency case in [16, 17, 15] for Hamiltonian systems and in [5, 22, 20] for dissipative systems. Another method is to search for weaker non-degeneracy conditions concerning frequency maps, which have been studied in a series of papers, for example, by Bruno [8], Cheng and Sun [9], Rüssmann [33, 34], Han, Li and Yi [18] for finite dimensional Hamiltonian systems, and Bambusi, Berti and Magistrelli [4] for infinite dimensional case. The weaker non-degeneracy condition in [9] is that the image of the frequency map in an open set includes a curved $C^{m+2}$ one-dimensional submanifold. Rüssmann [33, 34] pointed out that the weaker non-degeneracy condition means that the image of the frequency map does not lie in an $(n - 1)$-dimensional linear subspace of $\mathbb{R}^n$ (this condition is also necessary in the analytic case). An interesting and real analytic Hamiltonian of the form

$$H(x, y, \varepsilon) = h_0(y^{(0)}) + \varepsilon^m h_1(y^{(1)}) + \cdots + \varepsilon^m h_m(y^{(m)}) + \varepsilon^{m+1} P(x, y, \varepsilon)$$

with the degeneracy involving several time scales was considered in [18]. The degeneracy in (1.1) is somewhat similar to the one in [18].

2. Statement of results

Let $\Omega_1$ and $\Omega_2$ be convex open neighbourhoods of the origin in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. $\Omega = \Omega_1 \times \Omega_2$, the parameter set $\Pi$ be a convex bounded open set of positive Lebesgue measure in $\mathbb{R}^n$. Let $|x|$ denote the maximum norm and $|x|_p$ the $p$-norm. In the following, $l$ and $\alpha$ represent the differentiability orders in the space variables $(I, \varphi)$ and the parameter variables $\xi$, respectively.
Definition 2.1 Let \( \alpha \) be a positive integer and \( l > 0 \), \( C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi) \) be the class of all functions \( f \) on \( \Omega \times \mathbb{T}^n \times \Pi \) whose partial derivatives \( \partial^\beta f \) with respect to the parameter variable \( \xi \in \Pi \) (which means the Whitney derivative if \( \Pi \) is a closed set) for all \( \beta, 0 \leq |\beta|_1 \leq \alpha \) are of class \( C^l \) in the space variable \( x = (I, \varphi) \in \Omega \times \mathbb{T}^n \), that is, there is some positive constant \( M \) such that the partial derivatives \( D^k \left( \partial^\beta f \right) \) of \( \partial^\beta f \) with respect to the space variable \( x = (I, \varphi) \in \Omega \times \mathbb{T}^n \) satisfy

\[
\left| D^k \left( \partial^\beta f(x, \xi) \right) \right| \leq M \tag{2.1}
\]

and

\[
\left| D^k \left( \partial^\beta f(x, \xi) \right) - D^k \left( \partial^\beta f(y, \xi) \right) \right| \leq M|x - y|^{l - |\beta|_1}, \quad |k|_1 = [l] \tag{2.2}
\]

for all \( x, y \in \Omega \times \mathbb{T}^n \) and all \( \beta, k \) with \( 0 \leq |\beta|_1 \leq \alpha, 0 \leq |k|_1 \leq [l] \), where \([l]\) is the integer part of \( l : l - [l] \in (0, 1) \), for nonnegative integer vectors \( k, \beta \), \( D^k = D_1^{k_1} \circ D_2^{k_2} \circ \cdots \circ D_{n_1+n_2}^{k_{n_1+n_2}}, \ D_j^{k_j} = \frac{\partial^{k_j}}{\partial x_j^{k_j}} \).

In addition, define a norm

\[
\|f\|_{C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi)} = \inf M
\]

is the smallest \( M \) for which the inequalities (2.1) and (2.2) hold. Then \( C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi) \) is a Banach space with respect to the norm \( \| \cdot \|_{C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi)} \), which is a generalization of the Hölder space to a parameter-depending case. The norms \( \| \cdot \|_{C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi)} \) and \( \| \cdot \|_{C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi)} \) are defined in a similar way, meaning the function depends on \( \varphi \in \mathbb{T}^n, \xi \in \Pi \) and \( \check{\xi} \in \Pi \), respectively.

When \( l \) is integer, we also introduce a generalization of the Zygmund space \( \hat{C}^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi) \) of all functions satisfying

\[
\left| D^k \left( \partial^\beta f(x, \xi) \right) \right| \leq M, \quad 0 \leq |k|_1 \leq l - 1 \tag{2.3}
\]

and

\[
\left| D^k \left( \partial^\beta f(x, \xi) \right) + D^k \left( \partial^\beta f(y, \xi) \right) - 2D^k \left( \partial^\beta f\left( \frac{1}{2}(x + y), \xi \right) \right) \right| \leq M|x - y|, \quad |k|_1 = l - 1, \tag{2.4}
\]

instead of (2.1) and (2.2), respectively, and the norm \( \|f\|_{\hat{C}^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi)} \) is the smallest \( M \) for which the inequalities (2.3) and (2.4) hold. For non-integer \( l > 0 \), \( \hat{C}^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi) = C^{l,\alpha}(\Omega \times \mathbb{T}^n, \Pi) \).

We shall sometimes drop parameters from functions whenever there is no confusion.

a) Assume

(H1) these non-negative constants \( q_1, \ldots, q_7 \) satisfy

\[
q_1 > q_3 \geq q_5, \quad q_7 \geq q_2 + q_5, \quad 0 < q_2 \leq \min\{q_4, q_6\};
\]

(H2) \( \omega_i, A_i \in C^\alpha(\Pi) \) with some positive integer \( \alpha \) and \( A_i \) is a diagonalizable matrix, \( A_i(\xi, \varepsilon) = B_i(\xi, \varepsilon)\Lambda_i(\xi, \varepsilon)B_i(\xi, \varepsilon)^{-1} \) for some diagonal matrix \( \Lambda_i, i = 1, 2 \). Denote \( \omega(\xi, \varepsilon) = \)
col(ε^νω_1, ω_2), Λ_1(ξ, ε) = diag(λ_1, · · · , λ_m), Λ_2(ξ, ε) = diag(λ_{m+1}, · · · , λ_n). Furthermore, assume that there are positive constants c_0, c_1 and ε* such that for all ε ∈ (0, ε*)

\[
\inf_{ξ ∈ Π} |λ_j(ξ, ε)| ≥ c_0, \quad \inf_{ξ ∈ Π} |λ_j(ξ, ε) - λ_i(ξ, ε)| ≥ c_0, \quad i ≠ j, 1 ≤ i, j ≤ n_1, \text{ or } n_1 + 1 ≤ i, j ≤ n,
\]

|B_i|_{α;Π}, |B_i^{-1}|_{α;Π}, |Λ_i|_{α;Π} ≤ c_1, \quad |ω_1|_{0;Π} = sup_{ξ ∈ Π} |ω_1| ≤ c_1, \quad i = 1, 2,

|∂^2 Ω|_{Π} = sup_{ξ ∈ Π} |∂^2 Ω(ξ, ε)| ≤ c_1ε^δ, \quad 1 ≤ |β| ≤ α;

\[
(H3) \quad g_j ∈ C^l(Ω × T^{n_2}, Π)(j = 1, · · · , 4) \text{ with } l > 2(α + 1)(l + 2) + α, \quad l > αn_2 - 1.
\]

**Remark 2.1** The requirement that Λ_1 does not have multiple eigenvalues is not necessary, only for the sake of simplification. The difficulty caused by multiple eigenvalues may be overcome by the technique of Rüssmann [34].

**Theorem 1** Suppose that the system (1.1) satisfies Assumptions (H1)-(H3). Then for any given 0 < γ ≪ 1, there is a sufficiently small 0 < ε_0 = o(γ^α) such that for 0 < ε ≤ ε_0, there exists a Canter set Π_γ ⊂ Π such for each ξ ∈ Π_γ, the system (1.1) admits a quasi-periodic invariant torus of the form I_1 = Φ_1(φ; η), I_2 = Φ_2(η; ξ), φ = col(φ_1, φ_2) ∈ T^{n_2} × T^{n_2} with frequencies ω^*(ξ) = (ε^νω_1, ω_2, ω_3(ξ)), which is of class C^α in Π ∈ Π, in the sense of Whitney and of class C^{l_1} in ξ ∈ Π_γ, together with derivatives up to order μ + 1 with respect to ξ for 0 < μ ≤ α (positive integer μ), the frequency map ω^*(ξ) is of class C^α in Π ∈ Π_γ in the sense of Whitney and satisfies

\[
|||Φ_1, Φ_2|||_{C^{l_1}(T^{n_2}; Π_γ)} ≤ Cε^{δ_2}γ^{−(μ+1)},
\]

\[
||ω^*_1 - ω_1||_{C^{l_1}(Π_γ)} ≤ Cε^{δ_6}, \quad ||ω^*_2 - ω_2||_{C^{l_1}(Π_γ)} ≤ Cε^{δ_7}.
\]

Moreover, there exist closed subsets Π_ν of Π, frequency vectors ω^ν(ξ) = col(ε^νω_1^ν(ξ), ω_2^ν(ξ)) and diagonal matrices Λ^ν(ξ) = diag(ε^νλ_1^ν(ξ), ε^νλ_2^ν(ξ), ε^νλ_3^ν(ξ)), for ν = 1, 2, · · · , satisfying

\[
||ω^ν_1 - ω_1||_{α;Π} ≤ Cε^{δ_6}, \quad ||ω^ν_2 - ω_2||_{α;Π} ≤ Cε^{δ_7}, \quad ||λ^ν_1 - λ_1||_{α;Π} ≤ Cε^{δ_2}, \quad ||λ^ν_2 - λ_2||_{α;Π} ≤ Cε^{δ_4}
\]

\[
(2.7)
\]

and

\[
Π_γ = Π_{γ−1} \setminus \bigcup_{k,m} \mathcal{R}_{km}^ν(γ)
\]

such that Π_γ = \bigcap_{ν=0}^{∞} Π_ν, where

\[
\mathcal{R}_{km}^ν(γ) = \{ξ ∈ Π_{γ−1} : |\sqrt{−1}(k, ω^{ν−1}) + (m, λ^{ν−1})| < γε^{δ_5}|k|^2 \}
\]

for m ∈ m, k ∈ Z^{n_2}, K_{ν−1} < |k|^2 ≤ K_ν, m = \{m ∈ Z^{n_2} : |m|^2 ≤ 2, Σ_j m_j = 0 or −1\}, ω_0 = ω, Λ_0 = Λ = diag(ε^θΛ_1, ε^θΛ_2), K_0 = 0 and K_1 = [K_1] + 1, K_1' = \frac{γ(ν + 1) - α}{\ln 3 + (n_2 + 1)(\ln r + \ln C)}, Π_0 is a closed subset of Π whose distance to the boundary of Π is at least equal to γ, and c and v are constants independent of ε, and γ, r is the radius of the neighbourhood Ω, ||·||_{C^{l_1}(Π_γ)} is the Whitney norm (see Appendix A.1), C = 24(n_2)!n_2^{α_1}e^{−n_2}, [K_1'] is the integer part of K_1'.

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Here, we drop $\varepsilon$ from functions, the continuous differentiability of functions $\omega_i'$ and $\Lambda_i'(i = 1, 2)$ on the closed set $\Pi$, means that they are continuously differentiable in some neighbourhood of $\Pi$. Here and in the sequel, we also regard the $\Lambda$ as a column vector of its diagonal elements when $\Lambda$ is a diagonal matrix.

b) The Canter set $\Pi_\gamma$ is not empty and indeed the measure $\text{meas}(\Pi \setminus \Pi_\gamma) \to 0$ as $\gamma \to 0$ as long as we impose proper non-degeneracy conditions on frequencies. Since in applications the non-degeneracy conditions on frequencies are different, Theorem 1 does not involve the measure estimate of $\Pi_\gamma$ so that it can be used more widely. In the following theorem, we give some conditions to ensure that the Canter set $\Pi_\gamma$ is not empty.

By the assumption (H2), we can write $\omega_2$ as

$$\omega_2(\xi, \varepsilon) = \omega_{20} + \varepsilon^q \omega_{21}(\xi) + o(\varepsilon^q),$$

where $\omega_{20}$ is independent of $\xi$, $o(\varepsilon^q)$ represents infinitely small quantity of $\varepsilon^q$ up to $\alpha$-th derivatives. Denote $\Lambda_2(\xi) = \Lambda_2(\xi, \varepsilon)|_{\varepsilon = 0}$ and $\omega(\xi) = \text{col}(\omega_1(\xi, 0), \omega_{21}(\xi))$ in the case $n_{22} \neq 0$, $\omega(\xi) = \omega_1(\xi, 0)$ in the case $n_{22} = 0$.

**Theorem 2** Suppose that the system (1.1) satisfies the assumptions in Theorem 1, moreover assume that $n_3 + \cdots + n_3^\alpha \geq n_2$ and

(i) for all $\xi \in \Pi$

$$\text{rank}\left(\frac{\partial^{\beta_1}}{\partial \xi^{\beta_1}} \omega : 1 \leq |\beta_1| \leq \alpha\right) = n_2 \quad \text{in Case } n_{22} = 0,$$

$$\text{rank}\left(\frac{\partial^{\beta_1}}{\partial \xi^{\beta_1}} \omega : 1 \leq |\beta_1| \leq \alpha\right) = n_2 \quad \text{in Case } n_{22} \neq 0,$$

(ii) for all integer vectors $0 \neq k \in \mathbb{Z}^{n_2}, m = (m_1, \cdots, m_{n_{12}}) \in \mathbb{Z}^{n_{12}}$ with $1 \leq |m_1| \leq 2$ and $m_1 + \cdots + m_{n_{12}} = 0$ or $-1$

$$\text{meas}\left\{\xi \in \Pi : \sqrt{-1}\langle k, \omega_0 + \varepsilon^q \omega(\xi) \rangle + \varepsilon^q \langle m, \Lambda_2(\xi) \rangle = 0\right\} = 0,$$

where $\omega_0 = \text{col}(0, \omega_{20})$.

Then the Canter set $\Pi_\gamma$ defined in Theorem 1 is of positive Lebesgue measure

$$\text{meas} \Pi_\gamma = \text{meas} \Pi - O(\gamma^{1/\alpha})$$

for sufficiently small $\gamma$.

**Remark 2.2** If $q_3 > q_5 \geq 0$, then the assumption (ii) may be removed, see the proof of Theorem 2 and Remark 5.1 in Section 5.

More results on measure estimates of $\Pi_\gamma$ will be given in the forthcoming second part concerning on the persistence of quasi-periodic invariant tori in bifurcation theory.
c) we consider a specific form of (1.1) for the case \( n_{11} = n_{22} = 0, n_2 = n_3, q_3 = q_5 = 0 \) and \( q_4 = q_6 = 1 \), which means that the first and fourth equations in (1.1) are absent and the number of parameter variables equals the dimension of tori, the equation (1.1) reads

\[
\begin{align*}
\dot{I} &= A(\xi)I + \varepsilon g_1(I, \varphi; \xi, \varepsilon) \\
\dot{\varphi} &= \omega(\xi) + \varepsilon g_2(I, \varphi; \xi, \varepsilon).
\end{align*}
\] (2.11)

Denote \( \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_{n_1}) \) and \( \omega = \text{col}(\omega_1, \cdots, \omega_{n_1}) \), where \( \lambda_1, \cdots, \lambda_{n_1} \) are the eigenvalues of \( A, A(\xi) = B(\xi)\Lambda(\xi)B(\xi)^{-1} \). Assume

**H2’** \( \omega, A \in C^1(\Pi) \), the map \( \xi \to \omega(\xi) \) is a diffeomorphism between \( \Pi \) and its image, and there exist positive constants \( c_0, c_1 \) and \( c_2 \) such that \( \|B\|_{1;\Pi}, \|B^{-1}\|_{1;\Pi}, \|\Lambda\|_{1;\Pi}, \|\omega\|_{1;\Pi} \leq c_1 \).

\[
|\langle m, \Lambda(\xi) \rangle| \geq c_0, \quad \left\| \left( \frac{\partial \omega}{\partial \xi} \right)^{-1} \right\| \leq c_2 \text{ on } \Pi \] (2.12)

and

\[
\text{meas} \left\{ \xi \in \Pi : \sqrt{-1}\langle k, \omega(\xi) \rangle + \langle m, \Lambda(\xi) \rangle = 0 \right\} = 0 \] (2.13)

for all integer vectors \( 0 \neq k \in \mathbb{Z}^{n_2} \), \( m \in \mathbb{Z}^{n_1} \) with \( 1 \leq |m|_1 \leq 2 \) and \( m_1 + \cdots + m_{n_1} = 0 \) or \(-1\).

**H3’** \( g_j \in C^{l,1}(\Omega \times \mathbb{T}^{n_2}, \Pi)(j = 1, 2) \) with \( l > 5t + 8, \ t > n_2 - 1 \).

**Remark 2.3**

(i) When the real part \( \text{Re}\Lambda \) of \( \Lambda \) satisfies \( \langle m, \text{Re}\Lambda(\xi) \rangle \neq 0 \) on \( \Pi \), the condition (2.13) holds spontaneously. In particular, the condition (2.13) is satisfied if \( \Lambda \) is independent of \( \xi \).

(ii) The Assumption (H2’) implies that the condition (2.13) is satisfied if

\[
\left( \left( \frac{\partial \omega}{\partial \xi} \right)^{-1} \right)^T \frac{\partial}{\partial \xi} \langle m, \Lambda(\xi) \rangle = \sqrt{-1}k \quad \text{for} \ 0 \neq |k|_1 \leq 2n_2c_1c_2.
\]

Theorems 1 and 2 imply

**Corollary 2.1** Suppose that the system (2.11) satisfies Assumptions (H2’) and (H3’). Then for any given \( 0 < \gamma \ll 1 \), there is a sufficiently small \( \varepsilon^* > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon^* \), there exists a Cantor set \( \Pi_{\gamma} \subset \Pi \) with the Lebesgue measure

\[
\text{meas} \Pi_{\gamma} = \text{meas} \Pi - c\gamma
\]

and for each \( \xi \in \Pi_{\gamma} \), the system (2.11) possesses a quasi-periodic invariant torus \( I = \Phi(\varphi; \xi), \varphi \in \mathbb{T}^{n_2} \) consisting of quasi-periodic motions, which is of \( \tilde{C}^1 \) \( (l_1 = l - 4(t + 1) - 3) \) in \( \varphi \in \mathbb{T}^{n_2} \) and Lipschitz in \( \xi \in \Pi_{\gamma} \), where \( c \) is a constant independent of \( \gamma \) and \( \varepsilon \).
Usually the normal form (integrable part) of (1.1) related bifurcation problems of actual models is only finitely differentiable, not analytic in the parameter \( \xi \), and the frequency map is possibly degenerate so that we need the higher-order derivatives of the frequency map to estimate the Lebesgue measure of \( \Pi_y \) and obtain \( \Pi_x \) is the most part of \( \Pi \). Hence, we want to establish an approximation lemma and the corresponding inverse approximation lemma in which a finitely differentiable function is approximated by a sequence of functions being analytic in space variables, but finitely differentiable in parameter variables. These comprise Section 3. The proofs of Theorems 1 and 2 are given in Sections 4 and 5, respectively.

3 Approximation Lemmas

Zehnder [39] established the approximation and inverse approximation Lemmas on a finitely differentiable real function approximated by a sequence of real analytic functions, which was generalized to the anisotropic case by Pöschel [27], and was sharpened to covering the finitely differentiable and Gevrey regular cases by Wagener [38], respectively. Here, we give generalized versions of Zehnder’s approximation and inverse approximation Lemmas on a finitely differentiable function is approximated by a sequence of functions being analytic in \( \xi \) differentiable in parameter variables. These comprise Section 3.

a) We first introduce some notations. Let \( m, n \) and \( \alpha \) be positive integers, \( \mathcal{U} \subset \mathbb{C}^m \) and \( \Pi \subset \mathbb{R}^n \) be open sets, \( \mathcal{W}^a(\mathcal{U}, \Pi) \) be the class of all functions of \((z, \xi)\) on \( \mathcal{U} \times \Pi \) which are analytic in \( z \in \mathcal{U} \) and \( \alpha \)-times continuously differentiable in \( \xi \in \Pi \). For \( g \in \mathcal{W}^a(\mathcal{U}, \Pi) \), define

\[
|g|_{\mathcal{W}^a(\mathcal{U}, \Pi)} = \sup_{|\beta| \leq \alpha} \sup_{(z, \xi) \in \mathcal{U} \times \Pi} |\partial^\beta g(z, \xi)| .
\]

In particular, for \( \mathcal{U} = \{z \in \mathbb{C}^m : |\text{Im} z| \leq r, |\text{Re} z| \leq r, \text{Im} z| < r\} \), we denote \( |g|_{\mathcal{W}^a(\mathcal{U}, \Pi)} \) by \( |g|_{\mathcal{W}^a(\mathcal{U})} \).

Take an even function \( u_0 \in C_0^\infty(\mathbb{R}) \), vanishing outside the interval \([-1, 1]\) and identically equal to 1 in a neighbourhood of 0 (see [38] for the construction of such a function). For \( x \in \mathbb{R}^m \), let \( u(x) = u_0(\|x\|^2) \) and \( \tilde{u} \) be the inverse Fourier transform of \( u \)

\[
\tilde{u}(z) = (2\pi)^{-m} \int_{\mathbb{R}^m} u(x) e^{-i(z \cdot x)} dx .
\]

Let \( f \) be a real-valued function of class \( C^{l, \alpha}([0, 1], \mathbb{R}) \) (see Definition 2.1), \( f_r(0 < r \leq 1) \) be defined by the convolution

\[
f_r(x, \xi) := (S_r f_r)(x, \xi) = r^{-m} \int_{\mathbb{R}^m} \tilde{u}(r^{-1}(x - y)) f(y, \xi) dy \tag{3.1}
\]

for \( x \in \mathbb{C}^m \). We list some properties of the analytic smoothing operator \( S_r \) in Section A.3 of the appendix, which will be used in the proof of the next lemma.

**Lemma 1** Let \( f(x, \xi) \) be a real-valued function of class \( C^{l, \alpha}([0, 1], \mathbb{R}) \) for some real number \( l > 0 \) and \( \alpha \in \mathbb{N} \), where \( \Pi \subset \mathbb{R}^n \) is an open set. Then for every \( r \in (0, 1) \), the function \( f_r(x, \xi) \) is \( \alpha \)-times continuously differentiable in \( \xi \in \Pi \), entire real analytic in \( x \in \mathbb{C}^m \) together with derivatives up to order \( \alpha \) with respect to \( \xi \), and satisfies
Hence,
\[ q = \text{in the case without parameter-dependence. In the following, we will use } C \text{ constant depending } p \text{, } r \text{, and the dimension } m. \] Moreover, \( f \) is \( \omega \)-periodic in some variable if in which \( f \) is \( \omega \)-periodic.

\textbf{Proof} From (3.1) it is clear that \( f_t(x, \xi) \) is analytic in \( x \in \mathbb{C}^m \), and \( \alpha \)-times continuously differentiable in \( \xi \in \Pi \), taking real values on real variables \( x \), and if \( f \) is periodic in some variable, then so is \( f_t \). As differentiation may commute with integration in (3.1) for functions with bounded derivatives, we obtain \( \partial^\beta_t f_t = S_\beta (\partial^\beta_t f) \) for \( |\beta|_1 \leq \alpha \). Of course, we also have \( S_\beta (D^k f_t) = D^k (S_\beta(f)) \) for \( |k|_1 \leq l, k \in \mathbb{Z}^m \). Hence we only need to prove the estimates (i)-(iii) in the case without parameter-dependence. In the following, we will use \( C \) to denote some constant depending \( l, p \) and \( m \).

(i) The case where \( p \) is a integer, is proved by Chierchia [11], see Lemma [11] (f) in Appendix. Hence we only give the proof for the case \( p = q + \mu \leq l, \mu \in (0, 1), q \in \mathbb{Z}^m \). Denote \( g(x) = D^q f, |\beta|_1 = q \). Then by (a) and (b) in Lemma [11] we have for \( x, y \in \mathbb{R}^m \),

\[ \sup_{x \neq y} |x - y|^{-\mu} |(g - S_\beta g)(x) - (g - S_\beta g)(y)| \]

\[ = \sup_{x \neq y} |x - y|^{-\mu} \left| \int_{\mathbb{R}^m} \tilde{u}(z)[g(x) - g(x - rz) - g(y) + g(y - rz)] \, dz \right| \equiv (*) . \]

Case I: \( q = [l] \), the integer part of \( l \). For \( |x - y| \geq r \), by \( g \in C^{l-q} \) and Lemma [11] (d), we have

\[ (*) \leq \sup_{x \neq y} |x - y|^{-\mu} \left( \int_{\mathbb{R}^m} \tilde{u}(z)||g(x) - g(x - rz)|| + ||g(y) - g(y - rz)|| \, dz \right) \]

\[ \leq 2^{l-p} ||f||_{L^p} \int_{\mathbb{R}^m} \tilde{u}(z)||z||^{-q} \, dz \leq C r^{l-p} ||f||_{L^p} . \]

For \( |x - y| < r \), we also have

\[ (*) \leq \sup_{x \neq y} |x - y|^{-\mu} \left( ||f||_{L^p} |x - y|^{l-q} + \int_{\mathbb{R}^m} \tilde{u}(z)||g(x - rz) - g(y - rz)|| \, dz \right) \]

\[ \leq \left( 1 + \int_{\mathbb{R}^m} \tilde{u}(z) \, dz \right) |x - y|^{l-p} ||f||_{L^p} \leq C r^{l-p} ||f||_{L^p} . \]

Hence, \( ||g - S_\beta g||_{L^p} \leq C r^{l-p} ||f||_{L^p} \), which, combining with Lemma [11] (f) for the case of integers, implies (i) for the case \( q = [l] \).

Case II: \( q < [l] \). For \( |x - y| \geq r \), using the Taylor’s formula of \( h(rz) = g(x - rz) - g(y - rz) \) at \( z = 0 \) and Lemma [11] (c), we obtain

\[ (*) = \sup_{x \neq y} \frac{|x - y|^{-\mu}}{[l] - q)!} \left| \int_{\mathbb{R}^m} \tilde{u}(z)(-rz \cdot \nabla)^{|l| - q}(g(x - \theta rz) - g(y - \theta rz)) \, dz \right| \]

\[ \leq \sup_{x \neq y} |x - y|^{-\mu} r^{l-q} \sum_{|k|_1 = [l] - q} \frac{1}{k!} \left| \int_{\mathbb{R}^m} \tilde{u}(z)z^k(D^k g(x - \theta rz) - D^k g(y - \theta rz)) \, dz \right| , \quad (3.2) \]
where \( r_z \cdot \nabla = \sum_{j=1}^{m} r_j D_j \) and \( \theta \in (0, 1) \). Thus, equivalently, we need to estimate the following expression

\[
(**) \equiv \sup_{x \neq y} |x - y|^{-\mu} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (D^{k+\beta} f(x - \theta r_z) - D^{k+\beta} f(y - \theta r_z)) dz \right|, |k + \beta|_1 = |l|.
\]

By Lemma 11 (c), we get

\[
(**) = \sup_{x \neq y} |x - y|^{-\mu} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (D^{k+\beta} f(x - \theta r_z) - D^{k+\beta} f(x)) dz \right|
\]

\[
+ \int_{\mathbb{R}^n} \tilde{u}(z) z^k (D^{k+\beta} f(y) - D^{k+\beta} f(y - \theta r_z)) dz \leq C r^{l-|\mu|} ||f||_{L^\infty}, \tag{3.3}
\]

For \( |x - y| < r \), if \( |l| - q \geq 2 \), then similarly we have

\[
(*) \leq \sup_{x \neq y} |x - y|^{-\mu} r^{l-|\mu|} \sum_{|k|_1 = |l| - q - 1} \frac{1}{k!} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (D^k g(x - \theta r_z) - D^k g(y - \theta r_z)) dz \right|. \tag{3.4}
\]

The mean value theorem and Lemma 11 (c) deduce

\[
\sup_{x \neq y} |x - y|^{-\mu} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (D^k g(x - \theta r_z) - D^k g(y - \theta r_z)) dz \right|
\]

\[
\leq \sup_{x \neq y} |x - y|^{-\mu} \sum_{|k|_1 = 1} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (x - y)^k D^{k+k'} g(y - \theta r_z + \theta_{kk'}(x - y)) dz \right|
\]

\[
= \sup_{x \neq y} |x - y|^{-\mu} \sum_{|k|_1 = 1} \left| \int_{\mathbb{R}^n} \tilde{u}(z) z^k (x - y)^k (D^{k+k'} g(y - \theta r_z + \theta_{kk'}(x - y)) - D^{k+k'} g(x)) dz \right|
\]

\[
\leq \sup_{x \neq y} |x - y|^{-\mu} \sum_{|k|_1 = 1} |x - y||f||_{L^\infty} \int_{\mathbb{R}^n} |\tilde{u}(z)| z^k ||\theta r_z + (1 - \theta_{kk'}) (x - y)||^{l-|k|} dz
\]

\[
\leq C r^{l+1-|\mu|} ||f||_{L^\infty}, \tag{3.5}
\]

where \( \theta_{kk'} \in (0, 1) \). If \( |l| - q = 1 \), then by the mean value theorem,

\[
(*) \leq \sup_{x \neq y} |x - y|^{-\mu} \sum_{|k|_1 = 1} \left| \int_{\mathbb{R}^n} \tilde{u}(z) (x - y)^k (D^k g(y + \theta_{kk'}(x - y)) - D^k g(y - rz + \theta_{2k}(x - y))) dz \right|
\]

\[
\leq \sup_{x \neq y} |x - y|^{-\mu} \sum_{|k|_1 = 1} |x - y||f||_{L^\infty} \int_{\mathbb{R}^n} |\tilde{u}(z)| rz + (\theta_{1k'} - \theta_{2k'})(x - y)||^{l-|l|} dz
\]

\[
\leq C r^{l+1-|\mu|} ||f||_{L^\infty} = C r^{l-\mu} ||f||_{L^\infty}, \tag{3.6}
\]

where \( \theta_{1k'}, \theta_{2k'} \in (0, 1) \). Hence, (3.2)-(3.6) and Lemma 11 (f) imply (i) for the case \( q < |l| \).

Obviously, Lemma 11 (f) implies (ii), and the definition of \( f_r \) and Lemma 11 (b) and (d) imply (iii). \( \blacksquare \)

From Lemma 11 it follows the approximation lemma.
**Lemma 2** (Approximation Lemma) Let \( f(x, \xi) \) be a real-valued function of class \( C^{l,\alpha}(\mathbb{R}^m, \Pi) \) for some real number \( l > 0 \) and \( \alpha \in \mathbb{N} \), where \( \Pi \) is an open set, and let \( \{ r_j \}_{j=0}^{\infty} \) be a monotonically decreasing sequence of positive numbers with \( r_0 \leq 1 \) and tend to zero. Then there exists a sequence of functions \( \{ f_j(z, \xi) \}_{j=0}^{\infty} \), being of class \( C^\omega \) in \( \xi \in \Pi \), and entire, real analytic in \( z \in \mathbb{C}^n \) together with derivatives up to order \( \alpha \) with respect to \( \xi \), starting with \( f_0 \equiv 0 \), such that

\[
\lim_{j \to \infty} \| f_j - f \|_{p,\alpha;\mathbb{R}^m,\Pi} = 0 \quad \text{for all } 0 \leq p < l
\]

and

\[
\| f_j - f_{j-1} \|_{r_j,\alpha;\Pi} \leq C_0 r_j^j \| f \|_{l,\alpha;\mathbb{R}^m,\Pi} \quad \text{for } j \geq 1,
\]

where the constant \( C_0 \) depends on \( l \) and the dimension \( m \). Moreover, the \( f_j \) is \( \omega \)-periodic in each variable in which \( f \) is \( \omega \)-periodic.

b) Now, we want to apply the approximation lemma to the proof of Theorem 1 and obtain sequences of real analytic functions approximating \( g_i(i = 1, \ldots, 4) \) in the equation (1.1).

Without loss of generality, we take

\[
\Omega = \{ I = \text{col}(I_1, I_2) \in \mathbb{R}^{n_1} : |I| < 3 \tilde{r} \}
\]

for some constant \( 0 < \tilde{r} \leq 1 \). Let

\[
\Omega' = \{ I \in \mathbb{R}^{n_1} : |I| \leq 2 \tilde{r} \}, \quad r_j = \tilde{r} 3^{-j}, \quad j = 0, 1, 2, \ldots.
\]

Define complex neighbourhoods \( \mathcal{U}_j \) of \( \Omega' \times \mathbb{T}^{n_2} \) for \( j = 0, 1, 2, \ldots \) by

\[
\mathcal{U}_j = \{(I, \varphi) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} : \text{dist}(I, \Omega'_*) < 2 r_j, |\text{Im}\varphi| < 3 r_j \} := \Omega' \times \mathbb{T}^{n_2} + (r_j, 3 r_j).
\]

We first expand the definition domain \( \Omega' \times \mathbb{T}^{n_2} \times \Pi \) of \( g_i(i = 1, \ldots, 4) \) to \( \mathbb{R}^{n_1} \times \mathbb{T}^{n_2} \times \Pi \) in the following manner: we multiply \( g_i \) by a \( C^\infty \)-function on \( \mathbb{R}^{n_1} \) which identical 1 on \( \Omega'_* \) and vanishes outside \( \Omega \). The obtained function belongs to \( C^{l,\alpha}(\mathbb{R}^{n_1} \times \mathbb{T}^{n_2}, \Pi) \) and is equal to \( g_i \) on \( \Omega' \times \mathbb{T}^{n_2} \times \Pi \), its norm is bounded by \( c_l \| g_i \|_{l,\alpha;\Omega' \times \mathbb{T}^{n_2}, \Pi} \), where \( c_l \) is a constant depending \( l, n_1 \) and the chosen \( C^\infty \)-function. Then by the approximation lemma (Lemma 2) we have the following corollary.

**Corollary 3.2** If the system (1.1) satisfies Assumption (H3), then there exist sequences \( \{ g_i^j(I, \varphi, \xi) \}_{j=0}^{\infty} \) of real analytic functions, being of class \( C^\omega \) in \( \xi \in \Pi \), and entire, real analytic in \( (I, \varphi) \in \mathcal{U}_0 \), periodic in the variables \( \varphi \) with periodic \( 2\pi \) together with derivatives up to order \( \alpha \) with respect to \( \xi \), starting with \( g_0^i \equiv 0 \), such that

\[
\lim_{j \to \infty} \| g_i^j - g_i \|_{p,\alpha;\Omega' \times \mathbb{T}^{n_2}, \Pi} = 0 \quad \text{for all } 0 \leq p < l,
\]

\[
\| g_i^j - g_i^{j-1} \|_{\mathcal{U}_{j-1},\alpha;\Pi} \leq C_0 r_j^j \| g_i \|_{l,\alpha;\Omega' \times \mathbb{T}^{n_2}, \Pi} \quad \text{for } j \geq 1, i = 1, \ldots, 4,
\]

where \( C_0 \) is a constant depending only on \( l, n_1, \tilde{r} \) and \( c_l \).
c) Let $\Omega \subset \mathbb{R}^m$ be an open convex set, and $\Pi_0 \subset \mathbb{R}^n$ be a closed set,

$$\mathcal{W}_j = \Omega + r_j, \quad \Pi_j = \bigcup_{\xi \in \Pi_0} \{ \zeta \in \mathbb{R}^n : |\zeta - \xi| < s_j \}, \quad j = 0, 1, 2, \cdots,$$

where $r_j = r_0 \theta^j, 0 < \theta < 1$ and $\{s_j\}_{j=0}^\infty$ is a monotonically decreasing sequence of positive numbers with $s_0 \leq 1$ and tend to zero.

**Lemma 3** (Inverse Approximation Lemma) Let $\{f_j(x, \xi)\}_{j=0}^\infty$ be a sequence of functions such that $f_0 \equiv 0$, $f_j(x, \xi)$ is of class $C^l$ in $\xi \in \Pi_j$, real analytic in $x \in \mathcal{W}_j$ together with derivatives up to order $\alpha$ with respect to $\xi$, and

$$|f_j - f_{j-1}|_{\mathcal{W}_j, \alpha, \Pi_j} \leq M r_j^l$$

for every $j \geq 1$ and some constant $M$. If there exists a constant $c'_0 > 0$ such that $r'_j \leq c'_0 s_j^\delta$, $j = 1, 2, \cdots$, then there is a unique function $f(x, \xi)$ being of class $C^l$ in $\xi \in \Pi_0$ in the sense of Whitney (see Appendix A.1), and of class $\hat{C}^l$ in $x \in \Omega$ together with derivatives up to order $\alpha - 1$ with respect to $\xi$ such that

$$\|f\|_{C^l_{\alpha-1, \Omega, \Pi}} \leq C'_0 M \quad \text{and} \quad \lim_{j \to \infty} \|f - f_j\|_{\mathcal{W}_j, \alpha, \Pi} = 0 \quad \text{for all} \quad 0 \leq p < l.$$

Moreover, let $l = q + \mu, q \in \mathbb{Z}_+, \mu > 0$ and if $r'_j \leq c'_1 s_j^\delta$ for some constant $c'_1$ and $0 < \delta \leq 1$, then we may require the $(\alpha - 1)$-order derivatives $\partial^\beta f(x, \xi)$ with $|\beta|_1 = \alpha - 1$ to be uniformly $\delta$-Hölder continuous in $\xi \in \Pi_0$ in the space $C^q(\Omega)$, that is,

$$\|\partial^\beta f(\cdot, \xi) - \partial^\beta f(\cdot, \xi')\|_{C^q(\Omega)} \leq C'_1 M |\xi - \xi'|^\delta \quad \text{for} \quad \xi, \xi' \in \Pi_0, |\beta|_1 = \alpha - 1,$$

where the constant $C'_0$ and $C'_1$ depend on $l, m, n, \theta, c'_0$ and $c'_1$, $\hat{C}^l(\Omega)$ is the Zygmund space.

**Proof** By a similar proof to that of Lemma 2.2 (ii) in [39] (also see the proof of Lemma 4.3 in [21], Theorem A.3 in [38]), we can obtain that there exist functions $f^{(\beta)} \in \hat{C}^l(\Omega), |\beta|_1 \leq \alpha$ such that

$$\sup_{\xi \in \Pi} \|f^{(\beta)}(\cdot, \xi)\|_{\hat{C}^l(\Omega)} \leq C'_0 M \quad \text{and} \quad \lim_{j \to \infty} \|\partial^\beta f_j(\cdot, \xi) - f^{(\beta)}(\cdot, \xi)\|_{C^q(\Omega)} = 0$$

uniformly on $\Pi_0$ for all $0 \leq p < l$ and $|\beta|_1 \leq \alpha$. Set $f(x, \xi) = f^{(\beta)}(x, \xi)$ with $\beta = 0$. To prove the rest of the lemma we only need to verify (3.8) and the compatibility conditions in the definition of Whitney derivatives (see Appendix A.1)

$$f^{(\beta)}(x, \xi) = \sum_{|\beta + k|_1 = \alpha - 1} \frac{1}{k!} f^{(\beta+k)}(x, \xi)(\xi - \xi')^k + R^\beta(x, \xi, \zeta)$$

with

$$\sup_{x \in \Omega} |R^\beta(x, \xi, \zeta)| \leq CM|\xi - \xi'|^{\alpha - |\beta|_1}$$

for all $\xi, \zeta \in \Pi_0, |\beta|_1 \leq \alpha - 1$ and some finite constant $C$. 


Thus, we prove the compatibility conditions (3.9) and (3.10), and obtain
\[\Pi\] for \(j \geq 1, |\beta|_1 \leq \alpha - 1.\) Then
\[f^{(\beta)}(x, \xi) = \sum_{j=1}^{\infty} \partial_{\xi}^{j} h_j(x, \xi), \quad R^{(\beta)}(x, \xi, \zeta) = \sum_{j=1}^{\infty} R_{j}^{(\beta)}(x, \xi, \zeta), \quad |\beta|_1 \leq \alpha - 1. \quad (3.11)\]

If \(s_{j_0+1} \leq |\xi - \zeta| < s_{j_0}\) for some positive integer \(j_0\), then the line segment \(L\) connecting \(\xi\) to \(\zeta\) is contained in \(\Pi_j\) with \(1 \leq j \leq j_0\), and the Taylor expansion implies
\[
\sup_{x \in \Omega} |R_{j}^{(\beta)}(x, \xi, \zeta)| \leq C_1(\beta) M r_j^{j} |\xi - \zeta|^{|\beta|_1}, \quad 1 \leq j \leq j_0.
\]
And
\[
\sup_{x \in \Omega} |R_{j}^{(\beta)}(x, \xi, \zeta)| \leq C_2(\beta) M r_j^{j} s_{j_0+1}^{1(1-|\beta|_1)} |\xi - \zeta|^{|\beta|_1}, \quad j \geq j_0 + 1.
\]
Hence,
\[
\sup_{x \in \Omega} |R^{(\beta)}(x, \xi, \zeta)| \leq M |\xi - \zeta|^{|\beta|_1} \left( C_1 \sum_{j=1}^{j_0} r_j^{j} + C_2 \sum_{j=j_0+1}^{\infty} \left( \frac{r_j}{r_{j_0+1}} \right)^j \frac{r_j}{s_{j_0+1}^{1(1-|\beta|_1)}} \right) \leq C M |\xi - \zeta|^{|\beta|_1}
\]
If \(|\xi - \zeta| \geq s_1\), then we also have
\[
\sup_{x \in \Omega} |R^{(\beta)}(x, \xi, \zeta)| \leq C_2(\beta) M |\xi - \zeta|^{|\beta|_1} \left( \frac{r_j}{s_{j_0+1}^{1(1-|\beta|_1)}} \sum_{j=1}^{\infty} \left( \frac{r_j}{r_1} \right)^j \right) \leq C M |\xi - \zeta|^{|\beta|_1}
\]
Thus, we prove the compatibility conditions (3.9) and (3.10), and obtain \(\partial_{\xi}^{\beta} f(x, \xi) = f^{(\beta)}(x, \xi)\) for \(|\beta|_1 \leq \alpha - 1.\)

Now, we prove (3.8). Let
\[
u_j(x, \xi) = \partial_{\xi}^{j} h_j(x, \xi) \quad \text{and} \quad u_j(x, \xi) = \partial_{\xi}^{j} f(x, \xi), \quad |\beta|_1 = \alpha - 1.
\]
Then the (3.11) implies
\[
u(x, \xi) = \sum_{j=1}^{\infty} u_j(x, \xi) \quad \text{for} \quad (x, \xi) \in \Omega \times \Pi_0. \quad (3.12)
\]
By the Cauchy inequality and (3.7), we have
\[
|D^{k} u_{j}|_{\Omega, 1; \Pi_0} \leq C(k) M r_{j}^{(-|\beta|_1)} \quad \text{for} \quad |\beta|_1 \leq q. \quad (3.13)
\]
where \(C(k)\) is a constant depending only on \(k\). By a similar proof to one for the compatibility and replacing (3.7) with (3.13), (3.12) implies
\[
\sup_{x \in \Omega} |D^{k} u(x, \xi) - D^{k} u(x, \zeta)| \leq C_1 M |\xi - \zeta|^{|\beta|_1} \quad \text{for} \quad \xi, \zeta \in \Pi_0, |\beta|_1 \leq q.
\]
The proof of the lemma is complete.
4 Proof of Theorem 1

We first introduce some notation so that the system (1.1) is written in a compact form. Denote

\[ A^0 = \text{diag}(\varepsilon^{q_1}A_1, \varepsilon^{q_3}A_2), \quad B = \text{diag}(B_1, B_2), \]

\[ \Lambda^0 = \text{diag}(\varepsilon^{q_1}\Lambda_1, \varepsilon^{q_3}\Lambda_2), \quad \omega^0 = \text{col}(\varepsilon^{q_1}\omega_1, \omega_2), \]

\[ P_1 = \text{diag}(\varepsilon^{q_1}E_{n_{11}}, \varepsilon^{q_3+q_4-q_2}E_{n_{12}}), \quad P_2 = \text{diag}(\varepsilon^{q_3+q_6-q_2}E_{n_{21}}, \varepsilon^{q_3-q_1}E_{n_{22}}), \quad P = \text{diag}(P_1, P_2), \]

where \( E_n \) represents the \( n \times n \) identity matrix. Then the system (1.1) reads

\[
\begin{pmatrix} \dot{I} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} A^0(\xi, \varepsilon)I \\ \omega^0(\xi, \varepsilon) \end{pmatrix} + PG(I, \varphi, \xi, \varepsilon) \quad (4.1)
\]

with \( G = \varepsilon^{q_2}\text{col}(g_1, g_2, g_3, g_4) \).

**a) Outline of the proof** We are going to prove Theorem 1 by employing the KAM iteration process. By Corollary 3.2 (see Section 3), we obtain a sequence of real analytic functions

\[ G^0 = 0, \quad G^1 = \varepsilon^{q_2}\text{col}(g_1, g_2, g_3, g_4) \]

approximating \( G \) and

\[
\lim_{j \to \infty} ||G^j - G^i||_{p,0;\Omega^* \times \mathbb{T}^n, \Pi} = 0 \quad \text{for all } 0 \leq p < l, \quad (4.2)
\]

\[
||G^j - G^{j-1}||_{U_{j-1}; 0; \Pi} \leq C_0^j ||G^i||_{l; 0; \Omega^* \times \mathbb{T}^n, \Pi} \quad \text{for } j \geq 1. \quad (4.3)
\]

The definitions of \( \Omega^* \), \( U_j \) etc are seen above Corollary 3.2. Denote \( G_1^1 = \varepsilon^{q_2}\text{col}(g_1^1, g_2^1) \) and \( G_2^1 = \varepsilon^{q_2}\text{col}(g_3^1, g_4^1) \). We truncate \( G^1 \) to its lower-degree terms

\[
\Omega(G^1) := \begin{pmatrix} G_1^1(0, \varphi) + \partial_I G_1^1(0, \varphi)I \\ G_2^1(0, \varphi) \end{pmatrix} := \begin{pmatrix} u_0^1(\varphi) + u_1^0(\varphi)I \\ w_0^0(\varphi) \end{pmatrix}
\]

and write (4.1) as

\[
\begin{pmatrix} \dot{i} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} A^0I \\ \omega^0 \end{pmatrix} + P \begin{pmatrix} u_0^0(\varphi) + u_1^0(\varphi)I + H_1^0 \\ w_0^0(\varphi) + H_2^0 \end{pmatrix} + P(G - G^1), \quad (4.4)
\]

with \( \partial_I f(I, \varphi) \) represents the partial derivative (Jacobian matrix) of \( f \) with respect to the variable \( I \). Here, we drop parameters from functions and will do this also in the sequel whenever there is no confusion.
Moreover, the Cauchy inequality (see Lemma A.3 in [28]) implies
\[
\|u^0_0\|_{\nu,0;\Omega} \leq C_0 M^{\varepsilon_2} r_0^j, \\
\|u^0_1\|_{\nu,0;\Omega} \leq C_0 M^{\varepsilon_2} r_0^{-1}, \\
\|w^0\|_{\nu,0;\Omega} \leq C_0 M^{\varepsilon_2} r_0^j, \\
H_1^0 = O_{\nu,0;\Omega}(I^\nu), \\
H_2^0 = O_{\nu,0;\Omega}(I),
\] (4.5)
and
\[
H_1^0|_{I^\nu} \leq 2C_0 M^{\varepsilon_2} r_0^{-2}, \\
H_2^0|_{I^\nu} \leq C_0 M^{\varepsilon_2} r_0^{-1},
\] (4.6)
where $M^{\varepsilon_2} = \|G\|_{\nu,0;\Omega \times \mathbb{T}_2^*;\Omega}$. We want to look for a transformation $T_1$ to eliminate the lower-degree terms of $PG^1$ such that in new coordinates the lower-degree terms of analytic part in (4.4) are much smaller than the old ones. Assume that at the $\nu$-th step of the process, we have already found a coordinate transformation $T_\nu$, with $T_0 = \text{Id}$, the identity map, such that the system (4.1) is transformed into
\[
\begin{pmatrix}
i \\
\phi
\end{pmatrix} = \begin{pmatrix} A^\nu I \\
\omega^\nu
\end{pmatrix} + P \begin{pmatrix} u_0^\nu(\phi) + u_1^\nu(\phi) I + \tilde{H}_1^\nu \\
\tilde{w}^\nu(\phi) + \tilde{H}_2^\nu
\end{pmatrix} + P \mathcal{D}_\nu(G \circ T_\nu - G^\nu \circ T_\nu),
\] where $\tilde{H}_1^\nu = O(I^\nu)$, $\tilde{H}_2^\nu = O(I)$, $\mathcal{D}_\nu = P^{-1}(DT_\nu)^{-1} P$, the circle “$\circ$” indicates composition of functions and $DT_\nu$ the Jacobian matrix of $T$ with respect to coordinate variables. Then we replace $G^\nu$ with $G^{\nu+1}$ which is closer to $G$, and the above equation is rewritten as
\[
\begin{pmatrix}
i \\
\phi
\end{pmatrix} = \begin{pmatrix} A^\nu I \\
\omega^\nu
\end{pmatrix} + P \begin{pmatrix} u_0^\nu(\phi) + u_1^\nu(\phi) I + H_1^\nu \\
\tilde{w}^\nu(\phi) + H_2^\nu
\end{pmatrix} + P \mathcal{D}_\nu(G \circ T_\nu - G^{\nu+1} \circ T_\nu),
\] (4.8)
where
\[
\begin{pmatrix} u_0^\nu(\phi) + u_1^\nu(\phi) I \\
\tilde{w}^\nu(\phi) \\
\tilde{w}^\nu(\phi)
\end{pmatrix} = \begin{pmatrix} \tilde{u}_0^\nu(\phi) + \tilde{u}_1^\nu(\phi) I \\
\tilde{\omega}^\nu(\phi) \\
\tilde{\omega}^\nu(\phi)
\end{pmatrix} + \mathcal{L}(\mathcal{D}_\nu(G^{\nu+1} \circ T_\nu - G^\nu \circ T_\nu)),
\]
\[
H_1^\nu = O(I^\nu), \\
H_2^\nu = O(I).
\] We want to construct a coordinate change $T^{\nu+1}$ to eliminate the lower-degree terms in (4.8) such that the lower-degree terms of the next step are much smaller. Repetition of this process leads to a sequence of transformation $T_\nu = T_{\nu-1} \circ T_\nu$ with $T_0 = \text{Id}$, $\nu = 1, 2, \cdots$, the limit transformation of which, if converges, reduces (4.1) into a system without the lower-degree terms. Thus, we can obtain the quasi-periodic solution of (4.1). The proof of convergence is due to the following iteration lemma which describes quantitatively the KAM iteration process.

b) **Iteration Lemma** Before stating the iteration lemma we first introduce the iterative sequences and notations used at each iteration step. Set
\[
\varepsilon_0 = \varepsilon^{\nu_2}, \\
\|G\|_{\nu,0;\Omega \times \mathbb{T}_2^*;\Omega} = M\varepsilon_0,
\]
\[ \Omega = \{ I \in \mathbb{R}^{n_1} : |I| < \tilde{r} \}, \quad \Omega^* = \{ I \in \mathbb{R}^{n_1} : |I| \leq 2\tilde{r} \}, \quad \Omega_0 = \{ I \in \mathbb{R}^{n_1} : |I| < \tilde{r} \} \]

with some constant \( 0 < \tilde{r} \leq 1 \). For \( \nu \geq 1 \), let

(i) \( r_0 = \tilde{r}, \ r_v = \tilde{r}3^{-v} \),

\[ \mathcal{U}_v = \Omega^* \times \mathbb{R}^{n_2} + (3r_v, 3r_v), \quad \mathcal{V}_v = \Omega_0 \times \mathbb{R}^{n_2} + (r_v, r_v), \quad v \geq 0, \]

(ii) \( K_0 = 0, \quad K_v = [K'_v] + 1, \quad K'_v = 3^v r_0^{-1} (\ln \tilde{C} + (n_2 + 1) \ln r_0 + (l + (n_2 + 1)\nu - \alpha) \ln 3), \quad \tilde{C} = 24(n_2!)^{n_2} e^{n_2/2}, \ [K'_v] \) is the integer part of \( K'_v \);

(iii) \( s_0 = \gamma, \quad s_v = \gamma(16c_1 n_3 \sqrt{n_2} K'_v)^{-1}, \quad \Pi^+_v = \{ \xi \in \mathbb{R}^{n_3} : \text{dist}(\xi, \Pi_v) < s_v \}; \)

(iv) \( \chi_v = r_v^{-2} (\alpha + 1) (i + 1) - a - 3, \quad \chi_v = \sum_{j=1}^{\infty} \chi_j \), the assumption \( l > (\alpha + 1) (i + 2) + \alpha t \) implies \( \chi_v = \sum_{j=1}^{\infty} \chi_j < \frac{1}{2} \); 

(v) \( \delta_{\nu v} = \gamma^{-\nu - 1} r_v^{-l - (\alpha + \mu + 2)(i + 1) - a - 3} C_0 M e_0, \quad 0 \leq \mu \leq \alpha \); 

(vi) \( f(I, \varphi, \xi) = O_{\mathcal{U}, \alpha, \Pi^+_v}(I) \) denotes a map which is real analytic in coordinate variables \((I, \varphi) \in \mathcal{U} \), continuously differentiable up to order \( \alpha \) in parameter \( \xi \in \Pi^+_v \), and vanishes with \( I \)-derivatives up to order \( k - 1 \geq 0 \), and \( f \) and its \( \xi \)-derivatives up to order \( \alpha \) are bounded on \( \mathcal{U} \times \Pi^+_v \).

**Lemma 4 (Iteration Lemma)** Assume that for the equation \((4.8)\) with \( \nu \geq 0 \),

(v.1) (Frequency condition) let \( A' = \text{diag}(\varepsilon_1^{q_1} A'_1, \varepsilon_1^{q_2} A'_2), \quad A'' = \text{diag}(\varepsilon_0^{b_1} A''_1, \varepsilon_0^{b_2} A''_2), \quad \Lambda'_1 = \text{diag}(\lambda'_1, \cdots, \lambda'_n), \quad \Lambda''_1 = \text{diag}(\lambda''_1, \cdots, \lambda''_n), \quad A'_0 = B_i' A'_i B_i'^{-1} (i = 1, 2) \) and \( \omega = \text{col}(\varepsilon^q_0 \omega'_1, \omega''_2) \) satisfy, for \( \varepsilon \in (0, \varepsilon^*) \),

\[
\inf_{\xi \in \Pi^+_v} |\lambda_j| \geq c_0 (1 - X_i) \geq \frac{c_0}{2}, \quad \inf_{\xi \in \Pi^+_v} |\lambda_j - \lambda_i| \geq c_0 (1 - X_i) \geq \frac{c_0}{2}
\]

for \( i \neq j, \ 1 \leq i, j \leq n_1, \) or \( n_1 + 1 \leq i, j \leq n_1, \) and

\[
|\Lambda''_1 - \Lambda''_1|_{\| \cdot \|_{\Pi^+_v}} \leq c_0 C_0 M e_0 \varepsilon_0^{b_1} r_v^{-l - (\alpha + 1)(i + 2) - 1}, \quad |\omega_1 - \omega_1|_{\| \cdot \|_{\Pi^+_v}} \leq C_0 M e_0 \varepsilon_0^{b_1} r_v^{-l - (\alpha + 1)(i + 2)}, \quad \nu \geq 1
\]

for \( i = 1, 2, \) where \( c_0 \) is a positive constant, \( c_0 \) and \( c_0 \) are given in Assumption (H2), \( b_1 = 0, b_2 = q_4 - q_2, b_3 = q_6 - q_2, b_4 = q_7 - q_2 \); 

(v.2) (Small condition) the terms \( u''_0, u''_i \) and \( w \) satisfy the following estimates

\[
|u''_0|_{\| \cdot \|_{\Pi^+_v}} \leq 4 C_0 M e_0 r_v^l, \quad |u''_i|_{\| \cdot \|_{\Pi^+_v}} \leq C_0 M e_0 r_v^{l - (\alpha + 1)(i + 2) - 1}, \quad \| w \|_{\| \cdot \|_{\Pi^+_v}} \leq C_0 M e_0 r_v^{l - (\alpha + 1)(i + 2)},
\]

\( H'_1(I, \varphi, \xi) \) and \( H''_1(I, \varphi, \xi) \) fulfill

\[
H'_1 = O_{V_v, \alpha, \Pi^+_v}(I), \quad H''_1 = O_{V_v, \alpha, \Pi^+_v}(I), \quad |H'_1 - H''_1|_{\| \cdot \|_{\Pi^+_v}} \leq C_0 M e_0 \varepsilon_0
\]

for \( \nu \geq 1, \ i = 1, 2; \)

(v.3) (Transformation) the transformation \( T_v : V_v \times \Pi^+_v \rightarrow U_v \) is real analytic in coordinate variables \((I, \varphi) \in \mathcal{V}_v \) and continuously differentiable up to order \( \alpha \) in the parameter \( \xi \in \Pi^+_v \), satisfies

\[
|T_v - T_v|_{V_v, \alpha, \Pi^+_v} \leq (1 + X_v) C_0 M e_0 \gamma^{-1} r_v^{l - (\alpha + \mu + 2)(i + 1) - a - 2} < r_v \chi_v,
\]

\[(4.11)\]
\[|P^{-1}(DT_v - DT_{v-1})P|_{V_v, u, \Pi_v^\nu} \leq 2(1 + X_v)C_1C_0M\varepsilon_0\gamma^{-\mu-1}r_v^{(\alpha+\mu+2)((\nu+1) - \alpha - 3)} < X_v \quad (4.12)\]

with \( T_0 = \text{Id} \) and \( 0 \leq \mu \leq \alpha \), where \( C_1 \) is a constant independent of \( \nu \).

Then there exists a closed set \( \Pi_{v+1} \subset \Pi_v \)

\[\Pi_{v+1} = \{ \xi \in \Pi_v : |\sqrt{-1}(k, \omega^\nu) + (m, A^\nu)| \geq \varepsilon_0|k|^2, m \in m, k \in \mathbb{Z}^n, K_v < |k|_2 \leq K_{v+1} \}\]

(see Theorem 1 and (H3) for definitions of \( m \) and \( t \), respectively) and a coordinate transformation

\[T^{v+1} : V_{v+1} \times \Pi_{v+1}^{\nu+1} \rightarrow V_v \subset U \subset U_{v+1}\]

in the form

\[I = \rho + v_0^\nu(\phi, \xi) + v_1^\nu(\phi, \xi)\rho, \quad \varphi = \phi + \Phi^\nu(\phi, \xi), \quad (4.13)\]

where \( \rho \) and \( \phi \) are new coordinate variables, and all terms in the transformation are real analytic in \( \phi \) and continuously differentiable in \( \xi \) up to order \( \alpha \), satisfy the estimates

\[|\Phi^\nu|_{2r_v+1, \alpha; \Pi_{v+1}^{\nu+1}} \leq C_1C_0M\varepsilon_0\gamma^{-\alpha-1}r_v^{(\alpha+1)(2\alpha-3)}, \quad (4.14)\]

\[|v_0^\nu|_{2r_v+1, \alpha; \Pi_{v+1}^{\nu+1}} \leq C_1C_0M\varepsilon_0\gamma^{-\alpha-1}r_v^{(\alpha+1)(2\alpha-3)-\alpha}, \quad (4.15)\]

\[|v_1^\nu|_{2r_v+1, \alpha; \Pi_{v+1}^{\nu+1}} \leq C_1C_0M\varepsilon_0\gamma^{-\alpha-1}r_v^{(\alpha+1)(2\alpha-3)-1} \quad (4.16)\]

and

\[|P^{-1}(DT^{v+1})^{-1}P|_{V_{v+1}, 0; \Pi_{v+1}^{\nu+1}} < X_{v+1}, \quad \hat{\phi}_\xi \left( P^{-1}(DT^{v+1})^{-1}P \right)_{V_{v+1}, 0; \Pi_{v+1}^{\nu+1}} < X_{v+1} \quad (4.17)\]

for \( 1 \leq |\beta|_1 \leq \alpha \), such that the equation (4.8) is transformed into

\[
\begin{pmatrix}
\dot{\rho} \\
\phi
\end{pmatrix} = \begin{pmatrix}
A^{v+1} \rho \\
\omega^{v+1}
\end{pmatrix} + P \begin{pmatrix}
u_0^{v+1}(\phi) + u_1^{v+1}(\phi)\rho + H_1^{v+1} \\
w_1^{v+1}(\phi) + H_2^{v+1}
\end{pmatrix} + P\Sigma_{v+1}(G \circ T_{v+1} - G^{v+2} \circ T_{v+1})
\]

and the conditions (v.1)-(v.3) are satisfied by replacing \( \nu \) by \( \nu+1 \) and \( (I, \varphi) \) by \( (\rho, \phi) \), respectively, where \( T_{v+1} = T_v \circ T^{v+1}, \Sigma_{v+1} = P^{-1}(DT^{v+1})^{-1}P \).

c) **Proof of Theorem 1** Theorem 1 is easy to be proven by the Iteration Lemma and Inverse Approximation Lemma.

First the system (1.1) has been written in the form (4.4) just as (4.8) satisfying the conditions (v.1)-(v.3) with \( \nu = 0 \) in the Iteration Lemma by Assumptions (H2) and (H3), (4.5) and (4.6). We use the Iteration Lemma inductively to obtain a sequence of transformations \( T_v \) mapping \( V_v \times \Pi_v^\nu \) into \( V_0 \) and satisfying the estimate (4.11). Noting that \( V_v \) and \( \Pi_v^\nu \) are exactly regarded as those neighbourhoods of the open convex set \( \Omega_0 \times \mathbb{T}^n \subset \mathbb{R}^{m+1} \) and closed subset \( \Pi_v \subset \Pi \), respectively, and \( r_\nu^\nu / s_\nu^\nu \rightarrow 0 \) as \( \nu \rightarrow \infty \) \((l_1 = l - (\alpha + \mu + 2)(\nu + 1) - \alpha - 2 \) and the positive integer \( \mu \leq \alpha \) by the definition of \( s_\nu \), the Inverse Approximation Lemma and Condition (v.3) imply
that for every $\xi \in \Pi_\nu$, the limit map $T = \lim_{\nu \to \infty} T_\nu$ exists in $C^{\mu-1}(\Omega_0 \times \mathbb{T}^m, \Pi_\nu)$ for $0 \leq p < l_1$ and $T : \Omega_0 \times \mathbb{T}^m \times \Pi_\nu \to \Omega^* \times \mathbb{T}^m$ for sufficiently small $\varepsilon$, and is of the form

$$T : \quad I = \rho + V_0(\phi, \xi) + V_1(\phi, \xi)\rho, \quad \varphi = \phi + \Phi(\phi, \xi)$$

by (4.13), which is of class $C^\mu$ in $\xi \in \Pi_\nu$ in the sense of Whitney and of class $\hat{C}^{\mu}$ in $\varphi \in \mathbb{T}^m$ together with derivatives up to order $\mu - 1$ with respect to $\xi$ for $0 < \mu \leq \alpha$. Moreover, by (4.17), we obtain

$$\lim_{\nu \to \infty} \|G \circ T_\nu - G^{\nu+1} \circ T_\nu\|_{\rho, \mu-1, \Omega_0 \times \mathbb{T}^m, \Pi_\nu} = 0 \quad \text{for all } 0 \leq p < l_1, 0 < \mu \leq \alpha$$

(4.18)

and by Condition (v.3) and (4.17),

$$|\mathcal{D}_\nu - \mathcal{D}_{\nu-1}|_{\nu, \mu, \Pi_\nu} \leq C_2 C_0 M \varepsilon_0 \gamma^{-1} r_\nu^\mu - (\alpha + \mu + 2)(\alpha + 1 - \alpha - 3),$$

(4.19)

where $C_2$ is a constant independent of $\nu, \gamma$ and $\varepsilon_0$. It follows from (4.10), (4.18) and (4.19) that System (4.1) is transformed by $T$ into the system

$$\begin{cases}
\dot{\rho} = A^*(\xi)\rho + P_1 O(\rho^2) \\
\dot{\phi} = \omega^*(\xi) + P_2 O(\rho)
\end{cases}$$

(4.20)

for $(\rho, \phi) \in \Omega_0 \times \mathbb{T}^m, \xi \in \Pi_\nu$, where $A^*(\xi) = \text{diag}(\varepsilon^0 A_1^*(\xi), \varepsilon^0 A_2^*(\xi))$, $\omega^*(\xi) = \text{col}(\varepsilon^0 \omega_1^*(\xi), \omega_2^*(\xi))$, $A_i^* = \lim_{\nu \to \infty} A_i^\nu$ and $\omega_i^* = \lim_{\nu \to \infty} \omega_i^\nu$ ($i = 1, 2$) exist by Condition (v.1) and are of class $C^\nu$ in $\xi \in \Pi_\nu$ in the sense of Whitney by the Inverse Approximation Lemma since $s_\nu^{-(\alpha + 1)} s_\nu^{-\alpha} \to 0$ as $\nu \to \infty$. Thus, we obtain the quasi-periodic invariant torus of (4.1)

$$I = V_0(\phi, \xi), \quad \varphi = \phi + \Phi(\phi, \xi), \quad \phi = \omega^*(\xi)t + \phi_0$$

satisfying the estimates (2.5), (2.6) and (2.7) by Conditions (v.1) and (v.3). The rest of Theorem 1 can be derived immediately from the Iteration Lemma. 

**d) Proof of Iteration Lemma** To simplify the notation, we denote quantities referring to $\nu + 1$ with + such as $u^{\nu+1}$ by $u^+, r_{\nu+1}$ by $r_+$, and those referring to $\nu$ without the $\nu$ such as $u^\nu$ by $u$, $r_\nu$ by $r$. Substituting the transformation $T^+$ into (4.8), the transformation $T^+$ will be obtained by solving the homological equations

$$\begin{align*}
\partial_\phi v_0 \cdot \omega - A v_0 &= P_1 \Gamma_{K_\nu} u_0(\phi), \\
\partial_\phi v_1 \cdot \omega + v_1 A - A v_1 &= P_1 (\Gamma_{K_\nu} u_1(\phi) - B \text{diag}(B^{-1} \tilde{u}_1(0)B)B^{-1}), \\
\partial_\phi \Phi \cdot \omega &= P_2 (\Gamma_{K_\nu} w(\phi) - \tilde{w}(0)),
\end{align*}$$

(4.21) (4.22) (4.23)

where $B = \text{diag}(B_1, B_2)$, $\text{diag}(B^{-1} \tilde{u}_1(0)B)$ denotes a diagonal matrix whose elements are the diagonal elements of $B^{-1} \tilde{u}_1(0)B$, $\tilde{u}_1(0)$ and $\tilde{w}(0)$ denote the mean values (that is, the zero-order coefficients of the Fourier series expansions) of $u_1$ and $w$ over $\mathbb{T}^m$, respectively, $\Gamma_{K_\nu}$ is the
truncation operator of the Fourier series expansions defined in Lemma 12 and the notation
\( \partial_t f \cdot \omega = \sum_{j=1}^{n_2} \frac{\partial f}{\partial \theta_j} \) for \( \omega = \text{col}(t_1, \cdots, t_{n_2}) \). Here, the homological equations are approximated by truncating the Fourier series expansions of \( u_0, u_1 \) and \( w \) so that the solutions are defined on an open set of parameters. This idea is due to Arnol’d [11] and Pöschel [27].

d1) Solutions of (4.21)–(4.23) and estimates. Set

\[ \Pi_+ = \{ \xi \in \Pi_v : |\sqrt{-1}(k, \omega) + \langle m, \Lambda \rangle| \geq \gamma \varepsilon_0 |k|_{2}^{-i}, m \in \mathbb{m}, K < |k|_2 \leq K_+ \} \]

and

\[ \Pi_+^{s_+} = \{ \xi \in \mathbb{R}^{n_3} : \text{dist}(\xi, \Pi_+) < s_+ \} \subset \Pi_+^{s_+} \]

Lemma 5 For every \( \xi \in \Pi_+^{s_+} \), we have

\[ |\sqrt{-1}(k, \omega(\xi)) + \langle m, \Lambda(\xi) \rangle| \geq \frac{1}{4} \gamma \varepsilon_0 |k|_{2}^{-i}, \quad 0 < |k|_2 \leq K_+, m \in \mathbb{m}. \tag{4.24} \]

Proof We first prove

\[ |\sqrt{-1}(k, \omega(\xi)) + \langle m, \Lambda(\xi) \rangle| \geq \frac{1}{2} \gamma \varepsilon_0 |k|_{2}^{-i}, \quad 0 < |k|_2 \leq K_+, m \in \mathbb{m} \tag{4.25} \]

for every \( \xi \in \Pi_+ \). Noting the fact that \( K_j+1 \sqrt{j^{-1}(a+1)(a+2)} \to 0 \) as \( j \to \infty \), the (4.9) implies that for \( 0 < |k|_2 \leq K_j, 1 \leq j \leq \nu \),

\[ |\sqrt{-1}(k, \omega^j(\xi) - \omega^j-1(\xi)) + \langle m, \Lambda^j(\xi) - \Lambda^j-1(\xi) \rangle| \leq \varepsilon_0 (\sqrt{n_2 K_j + 2 \varepsilon_0}) C_0 M \varepsilon_0 K^{-j-1}_j \]

for sufficiently small \( \varepsilon_0 \). As for \( K_{j+1} \sqrt{j^{-1}(a+1)(a+2)} \to 0 \) as \( j \to \infty \),

\[ |\sqrt{-1}(k, \omega^j(\xi)) + \langle m, \Lambda^j(\xi) \rangle| \]

\[ \geq |\sqrt{-1}(k, \omega^j-1(\xi)) + \langle m, \Lambda^j-1(\xi) \rangle| - \sum_{i=j}^{\nu} |\sqrt{-1}(k, \omega^j(\xi) - \omega^j-1(\xi)) + \langle m, \Lambda^j(\xi) - \Lambda^j-1(\xi) \rangle| \]

\[ \geq \gamma \varepsilon_0 |k|_{2}^{-i} - \sum_{i=j}^{\nu} r_i K^{-i}_i \geq \frac{1}{2} \gamma \varepsilon_0 |k|_{2}^{-i}, \]

which implies (4.25).

For every \( \xi \in \Pi_+^{s_+} \subset \Pi_+^{s_+} \), there is \( \xi_0 \in \Pi_+ \) such that \( |\xi - \xi_0| < s_+ \). The condition (v.1) and (4.25) imply

\[ |\sqrt{-1}(k, \omega(\xi)) + \langle m, \Lambda(\xi) \rangle| \]

\[ \geq |\sqrt{-1}(k, \omega(\xi_0)) + \langle m, \Lambda(\xi_0) \rangle| - |\sqrt{-1}(k, \omega(\xi) - \omega(\xi_0)) + \langle m, \Lambda(\xi) - \Lambda(\xi_0) \rangle| \]

\[ \geq \frac{1}{2} \gamma \varepsilon_0 |k|_{2}^{-i} - 2 c_1 r_3 \varepsilon_0 (\sqrt{n_2} |k|_2 + 2)s_+ \]

\[ \geq \frac{1}{4} \gamma \varepsilon_0 |k|_{2}^{-i}. \]

\[ \blacksquare \]
The procedure of solving (4.21)-(4.23) is standard in KAM theory. Expanding the functions into the Fourier series in \( \phi \), and substituting in (4.21)-(4.23) and comparing coefficients of the term \( e^\sqrt{I(k,\phi)} \), one obtain the solutions

\[
\begin{align*}
\nu_0(\phi) &= P_1 \sum_{|k| \leq K_+} B(\sqrt{-1}(k, \omega) - \Lambda)^{-1} B^{-1} \tilde{u}_0(k) e^{\sqrt{-1}(k, \phi)}, \\
\nu_1(\phi) &= P_1 \sum_{|k| \leq K_+} BV_1(k) B^{-1} e^{\sqrt{-1}(k, \phi)}, \\
\Phi(\phi) &= P_2 \sum_{0 < |k| \leq K_+} (\sqrt{-1}(k, \omega))^{-1} \tilde{w}(k) e^{\sqrt{-1}(k, \phi)},
\end{align*}
\]

where

\[
(V_1(k))_{ij} = \begin{cases} 
(\sqrt{-1}(k, \omega) + \varepsilon^{\alpha_i} \lambda_j - \varepsilon^{\alpha_i} \lambda_i)^{-1}(\tilde{U}_1(k))_{ij}, & |k| + |i - j| \neq 0 \\
0, & |k| + |i - j| = 0,
\end{cases}
\]

\( U_1(\phi) = B^{-1} u_1(\phi) B \), \( a_i = q_i \) if \( 1 \leq i \leq n_{11} \), \( = q_3 \) if \( n_{11} + 1 \leq i \leq n_1 \), \( (V_1(k))_{ij} \) and \( (\tilde{U}_1(k))_{ij} \) represent elements of the matrices \( V_1(k) \) and \( \tilde{U}_1(k) \), respectively, and \( \tilde{u}(k) \) is the \( k \)-order coefficients of the Fourier series expansions of \( u \). Hence, Lemma 13, (4.26)-(4.28) and Conditions (v.1)-(v.2) imply that \( \nu_0, \nu_1 \) and \( \Phi \) are real analytic in \( \phi \in \mathcal{W} := \mathbb{T}^{q_2} + 2r_\epsilon \), continuously differentiable up to order \( \alpha \) in \( \xi \in \Pi^x_{\epsilon} \). Meanwhile using Lemma 14, one easily gets the estimates (4.14)-(4.16) (we denote \( |·|_{r\mu;\Pi^{x}_{\epsilon}} \) by \( |·|_{r\mu;\xi} \) for simplicity) and

\[
\begin{align*}
|\nu_0|_{2r_\epsilon, \mu;\xi} &\leq C_1 \delta_{+\mu} r_{+}^{(\alpha+1)(\alpha+2)+3}, & |P_1^{-1} \partial_\phi \nu_0|_{2r_\epsilon, \mu;\xi} &\leq C_1 \varepsilon^{-q_5} \delta_{+\mu} r_{+}^{(\alpha+1)(\alpha+2)+2}, \\
|\nu_1|_{2r_\epsilon, \mu;\xi} &\leq C_1 \delta_{+\mu} r_{+}, & |P_1^{-1} \partial_\phi \nu_1|_{2r_\epsilon, \mu;\xi} &\leq C_1 \delta_{+\mu} r_{+}, & |P_1^{-1} \partial_\phi \nu_1|_{2r_\epsilon, \mu;\xi} &\leq C_1 \varepsilon^{-q_5} \delta_{+\mu} r_{+} \\
|\Phi|_{2r_\epsilon, \mu;\xi} &\leq C_1 \delta_{+\mu} r_{+}^2, & |P_2^{-1} \partial_\phi \Phi|_{2r_\epsilon, \mu;\xi} &\leq C_1 \varepsilon^{-q_5} \delta_{+\mu} r_{+}
\end{align*}
\]

for \( 0 \leq \mu \leq \alpha \) and an appropriate choice of the constant \( C_1 \) independent of \( \nu \).

It is easy to see that when the \( \varepsilon_0 \) is sufficiently small, the transformation \( T^+ \) maps \( \mathcal{V}_+ \subset \mathcal{V} \) and \( \mathcal{V}_+ \) into \( \mathcal{V}_+ \), respectively, and

\[
|T^+ - \text{Id}|_{\mathcal{V}_+^{\ast}, \mathcal{V}_+^{\ast}} \leq C_1 r_\epsilon \delta_{+0}, \quad |\partial_\xi T^+|_{\mathcal{V}_+^{\ast}, \mathcal{V}_+^{\ast}} \leq C_1 r_\epsilon \delta_{+\mu}, \quad 1 \leq |\beta| = \mu \leq \alpha.
\]

\[\text{d2) Proof of (4.17).} \] Corresponding to the transformation \( T^+ \), we have its Jacobian matrix

\[
DT^+ = \begin{pmatrix}
E_{n_1} + \nu_1 & \partial_\phi \nu_0 + \partial_\phi \nu_1 \\
0 & E_{n_2} + \partial_\phi \Phi
\end{pmatrix}
\]
and the inverse

\[
( DT^+ )^{-1} = \begin{pmatrix}
(E_{n_1} + v_1)^{-1} & -(E_{n_1} + v_1)^{-1}(\partial_0 v_0 + \partial_0 v_1 \rho)(E_{n_2} + \partial_0 \Phi)^{-1} \\
0 & (E_{n_2} + \partial_0 \Phi)^{-1}
\end{pmatrix}.
\]

(4.34)

Thus, (4.29)–(4.31) imply

\[
|P^{-1} DT^+ P|_{\mathcal{V}_{+,0,s}^+} \leq 1 + \chi_{+} r_{+}^{\alpha(l+1)}, \quad |P^{-1} (DT^+ - E) P|_{\mathcal{V}_{+,0,s}^+} \leq C_1 \delta_{+\mu} \quad \text{for } 1 \leq \mu \leq \alpha.
\]

(4.35)

Noting that the derivatives of $DT^+$ with respect to the parameter $\xi$ is sufficiently small and that for a matrix $M(\xi)$ with a small norm, differentiating the left- and right-hand sides of $(E + M(\xi))^{-1}(E + M(\xi)) = E$ and using the Leibniz formula, we find

\[
\delta^\beta_\xi (E + M(\xi))^{-1} = - \sum_{k<\beta} \begin{pmatrix} \beta \\ k \end{pmatrix} \delta^k_\xi (E + M(\xi))^{-1} \cdot \delta^{\beta-k}_\xi (E + M(\xi)) \cdot (E + M(\xi))^{-1},
\]

where $k, \beta \in \mathbb{Z}_{+}^n$, \( \begin{pmatrix} \beta \\ k \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n_3} \\ k_1 \\ \vdots \\ k_{n_3} \end{pmatrix}, \quad E \text{ is the identity matrix, the estimates (4.30) and (4.31) imply}

\[
|P^{-1}_{1}(E_{n_1} + v_1)^{-1} P_{1}|_{2r_{+},0,s_{+}} \leq 1 + \chi_{+} r_{+}^{\alpha(l+1)+1},
\]

(4.36)

\[
|P^{-1}_{2}(E_{n_2} + \partial_0 \Phi)^{-1} P_{2}|_{2r_{+},0,s_{+}} \leq 1 + \chi_{+} r_{+}^{\alpha(l+1)+1},
\]

(4.37)

\[
|\partial^\beta_\xi (P^{-1}_{1}(E_{n_1} + v_1)^{-1} P_{1})|_{2r_{+},0,s_{+}} \leq 2(1 + r_{+} \chi_{+})^2 |P^{-1}_{1} v_1 P_{1}|_{2r_{+},0,s_{+}},
\]

(4.38)

\[
|\partial^\beta_\xi (P^{-1}_{2}(E_{n_2} + \partial_0 \Phi)^{-1} P_{2})|_{2r_{+},0,s_{+}} \leq 2(1 + r_{+} \chi_{+})^2 |P^{-1}_{2} \partial_0 \Phi P_{2}|_{2r_{+},0,s_{+}}.
\]

(4.39)

for $1 \leq |\beta|_1 \leq \alpha$ and sufficiently small $\varepsilon_0$. Hence, (4.17) follows from (4.34), (4.29), (4.30) and (4.36)–(4.39). Moreover, we have

\[
|P^{-1}_{1}(E_{n_1} + v_1)^{-1} P_{1} F|_{\mathcal{V}_{+,0,s}^+} \
|P^{-1}_{2}(E_{n_2} + \partial_0 \Phi)^{-1} P_{2} F|_{\mathcal{V}_{+,0,s}^+} < (1 + r_{+} \chi_{+}) |F|_{\mathcal{V}_{+,0,s}^+}
\]

(4.40)

for suitable $F$ which is real analytic in $\mathcal{V}_{+}$ and continuously differentiable up to order $\alpha$ in $\xi \in \Pi_{+}^s$.  

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d3) We proceed to verify (v.3) with \( \nu + 1 \) replacing \( \nu \). As the transformation \( T^+ : \mathcal{V}_+ \times \Pi_+ \to \mathcal{V}_+ \) (or \( \mathcal{V}_+^\ast \times \Pi_+^\ast \to \mathcal{V} \)) is real analytic in coordinate variables and continuously differentiable up to order \( \alpha \) in parameter \( \xi \), so is \( T^+_+ = T \circ T^+ \).

We first prove (4.11) and (4.12) inductively. For \( \nu = 1 \), by (4.29)-(4.33), it implies
\[
|T_1 - \text{Id}|_{\mathcal{V}_1,\mu;1} \leq C_1 r_1 \delta_1 \mu < r_1 \chi_1, \quad |DT_1 - E|_{\mathcal{V}_1,\mu;1} \leq C_1 \delta_1 \mu < \chi_1
\]
and
\[
|P^{-1}(DT_1 - E)P|_{\mathcal{V}_1,\mu;1} \leq C_1 \delta_1 \mu < \chi_1
\]
for \( 0 \leq \mu \leq \alpha \). Assume that at the \( \nu \)-th step we have
\[
|T - T_{\nu-1}|_{\mathcal{V}_\nu,\mu;\nu} \leq (1 + X)C_1 r_\delta \delta_\mu < r_\chi, \quad |DT - DT_{\nu-1}|_{\mathcal{V}_\nu,\mu;\nu} \leq (1 + X)C_1 \delta_\nu \mu < \chi
\]
and
\[
|P^{-1}(DT - DT_{\nu-1})P|_{\mathcal{V}_\nu,\mu;\nu} \leq 2(1 + X)C_1 \delta_\nu \mu < \chi
\]
for \( 0 \leq \mu \leq \alpha \), here we have omitted the subscript \( \nu \) from the quantities referring to \( \nu \). Then in view of the induction assumptions we obtain
\[
|DT|_{\mathcal{V}_0,\nu} \leq 1 + X, \quad |\partial^\nu DT|_{\mathcal{V}_0,\nu} \leq X \quad \text{for} \ 1 \leq |\beta|_1 \leq \alpha.
\]
(4.41)

Combining (4.32), (4.41) and Lemma 13(i) we get
\[
|T_+ - T|_{\mathcal{V}_1,\mu;1} = |T \circ T^+ - T|_{\mathcal{V}_1,\mu;1} \leq (1 + X_+)C_1 r_+ \delta_+ \mu < r_+ \chi_+ \quad \text{for} \ 0 \leq \mu \leq \alpha,
\]
(4.42)
which, together with the Cauchy inequality, implies
\[
|DT_+ - DT|_{\mathcal{V}_1,\mu;1} \leq (1 + X_+)C_1 \delta_+ \mu < \chi_+ \quad \text{for} \ 0 \leq \mu \leq \alpha.
\]
(4.43)

Similarly, we have
\[
|P^{-1}DTP|_{\mathcal{V}_0,\nu} \leq 1 + X, \quad |\partial^\nu (P^{-1}DTP)|_{\mathcal{V}_0,\nu} \leq X \quad \text{for} \ 1 \leq |\beta|_1 \leq \alpha
\]
(4.44)

and
\[
|(P^{-1}DTP) \circ T^+ - P^{-1}DTP|_{\mathcal{V}_1,\mu;1} \leq (1 + \chi_+)(1 + X)C_1 \delta_+ \mu \quad \text{for} \ 0 \leq \mu \leq \alpha
\]
(4.45)
by (4.32), (4.44) and Lemma 13(i). Based on the observation
\[
P^{-1}(DT_+ - DT)P = ((P^{-1}DTP) \circ T^+ - P^{-1}DTP)(P^{-1}DT^+P) + P^{-1}DTP(P^{-1}(DT^+ - E)P),
\]
from (4.45), (4.35), (4.44) and the Leibniz formula, it follows
\[
|P^{-1}(DT_+ - DT)P|_{\mathcal{V}_1,\mu;1} \leq 2(1 + X_+)C_1 \delta_+ \mu < \chi_+
\]
for \( 0 \leq \mu \leq \alpha \) and sufficiently small \( \varepsilon_0 \). Thus, we have proved (4.11) and (4.12) with \( \nu + 1 \).
Now, we show that $T_+$ maps $\mathcal{V}_+$ into $\mathcal{U}_+$. Noting the expression of $T_+$ in angle variable direction is independent of $\rho$, we set $T_+(\rho, \phi) = \text{col}(v_+(\rho, \phi), \Phi_+(\phi))$. In view of the induction hypotheses, (4.42) implies

$$|T_+ - \text{Id}|_{V_+,0;x_s} \leq \sum_{j=1}^{v+1} |T_j - T_{j-1}|_{V_+,0;x_s} \leq \sum_{j=1}^{v+1} r_j < r_0,$$

and (4.43) implies

$$|DT_+|_{V_+,0;x_s} < 1 + X_+ < 2.$$  

Hence, the first component of $T_+$ is mapped into $\Omega^* + r_+$. For $\phi$ with $|\text{Im} \phi| < r_+$, there exists a $\phi_0 \in \mathbb{T}^{n_1}$ such that $|\phi - \phi_0| < r_+$. Therefore, the $\Phi_+$ being real analytic and (4.46) imply

$$|\text{Im} \Phi_+(\phi)| = |\text{Im} (\Phi_+(\phi) - \Phi_+(\phi_0))| \leq |DT_+|_{V_+,0;x_s} |\phi - \phi_0| < 2r_+.$$ 

Thus, for $\xi \in \Pi^*_r$, $T_+$ maps $\mathcal{V}_+$ into $\Omega^* \times \mathbb{T}^{n_1} + (2r_+, 2r_+) \subset \mathcal{U}_+$, as claimed. Furthermore, let $6(\rho, \phi) = \mathcal{D}_1(G^{v+2} \circ T_+ - G^{v+1} \circ T_+)$, then (4.3), (4.11), (4.12), Lemma 13 (ii) and the Cauchy inequality imply

$$|6|_{V_+,\alpha:x_s} \leq 2r_+^{l-\alpha} C_0 M \varepsilon_0$$  

and

$$|\partial_\rho 6|_{V_+,\alpha:x_s} \leq 2r_+^{l-\alpha-1} C_0 M \varepsilon_0$$

for sufficiently small $\varepsilon_0$.

d4) Estimates of remainder terms. Denote

$$W(I, \varphi) = \text{col}(u_0(\varphi) + u_1(\varphi)I, w(\varphi)), \quad A^+ = A + P_1 \bar{A}, \quad \Lambda^+ = \Lambda + P_1 \bar{\Lambda}, \quad \omega^+ = \omega + P_2 \bar{\omega},$$

where

$$\bar{A} = B(\text{diag}(B^{-1} \bar{u}_1(0)B))B^{-1}, \quad \bar{\Lambda} = \text{diag}(B^{-1} \bar{u}_1(0)B), \quad \bar{\omega} = \bar{w}(0).$$

Then the assumption (H2) implies that there is a constant $\tilde{c}_0 \geq 1$ such that

$$|\text{diag}(B^{-1} \bar{u}_1(0)B)|_{\alpha,x_s} \leq \tilde{c}_0 |u_1|_{\alpha,x_s}, \quad |\bar{A}|_{\alpha,x_s} \leq \tilde{c}_0 |u_1|_{\alpha,x_s}.\quad (4.49)$$

By Assumption (H1) and Condition (v.2), it is easy to see $A^+, \Lambda^+$ and $\omega^+$ satisfy (v.1) with $\nu$ replaced by $\nu + 1$. We have found the transformation $T^+$ which transforms the equation (4.8), by using (4.21)-(4.23), into the following one in the new variables

$$\begin{pmatrix} \dot{\rho} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} A^+ \rho \\ \omega^+ \end{pmatrix} + P \mathcal{D}^+ \\ P^{-1}(E - DT^+)P \begin{pmatrix} \bar{A} \rho \\ \bar{\omega} \end{pmatrix} + (\text{Id} - \Gamma_{\mathcal{K}})W(\rho, \phi)$$

$$+ W \circ T^+(\rho, \phi) - W(\rho, \phi) + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \circ T^+(\rho, \phi) + P(6(\rho, \phi) + P \mathcal{D}_1(G - G^{v+2}) \circ T_+(\rho, \phi),$$

where

$$H_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
where \( \mathcal{D}^+ = (P^{-1}DT^+P)^{-1} \), \( \mathcal{D}_+ = (P^{-1}DT_+P)^{-1} \). We use the notation \( Lf(\rho, \phi) \) to denote the linear part of a function \( f \) in \( \rho \), that is

\[
Lf(\rho, \phi) = f(0, \phi) + \partial_\rho f(0, \phi) \rho
\]

and denote \( \delta(\rho, \phi) = \text{col}(\delta_1(\rho, \phi), \delta_2(\rho, \phi)) \), rewrite the above equation in the form of (4.3),

\[
\begin{pmatrix}
\dot{\rho} \\
\dot{\phi}
\end{pmatrix}
= A^+ \rho + P \begin{pmatrix}
\begin{pmatrix}
0 & \partial_\rho \Phi \\
\partial_\rho \Phi & 0
\end{pmatrix}
\end{pmatrix} u_0^+ + u_1^+ + \begin{pmatrix}
0 & \partial_\rho \Phi \\
\partial_\rho \Phi & 0
\end{pmatrix} \rho + H_1^+ + P \mathcal{D}_+(G - G^{\nu+2}) \circ T_+(\rho, \phi),
\]

where

\[
w^+ = P_2^{-1}(E_n + \partial_\rho \Phi) - P_2 \left[-P_2^{-1} \partial_\rho \Phi P_2 \hat{\omega} + (\text{Id} - \Gamma_{K_2})w + w(\Phi + \Phi) - w(\Phi) \right] + H_2(v_0, \phi + \Phi) + \delta_2(0, \phi),
\]

\[
u_0^+ = P_1^{-1}(E_n + v_1 - 1)P_1 \left[-P_1^{-1} \partial_\rho v_0 P_1 \hat{\omega} + w^+(\phi) + (\text{Id} - \Gamma_{K_1})u_0 + u_0(\phi + \Phi) - u_0(\phi) \right] + u_1(\phi + \Phi) + \delta_2(0, \phi),
\]

\[
u_1^+ = P_1^{-1}(E_n + v_1 - 1)P_1 \left[-P_1^{-1} \partial_\rho v_1 P_1 \hat{\omega} + w^+(\phi) + (\text{Id} - \Gamma_{K_1})u_1 + u_1(\phi + \Phi) - u_1(\phi) + u_1(\phi + \Phi) + v_1 \right] + \delta_2(0, \phi),
\]

\[
Q = \partial_\rho H_2(v_0, \phi + \Phi)(E_n + v_1) + \partial_\rho \delta_2(0, \phi),
\]

or in another form,

\[
H_1^+ = P_2^{-1}(E_n + \partial_\rho \Phi) - P_2 \left[H_2 \circ T^+(\rho, \phi) + \delta_2(\rho, \phi) - H_2 \circ T^+(0, \phi) - \delta_2(0, \phi) \right],
\]

\[
H_2^+ = P_1^{-1}(E_n + v_1 - 1)P_1 \left[(\text{Id} - L)(H_1 \circ T^+(\rho, \phi) + \delta_2(\rho, \phi)) \right] + P_1^{-1} \partial_\rho v_0 (E_n + \partial_\rho \Phi)^{-1} P_2 Q \rho - P_1^{-1} (\partial_\rho v_0 + \partial_\rho v_1 \rho) P_2 H_2^+.
\]

By (4.7) and (4.10), we have

\[
|H_i|_{V, \alpha, \beta} \leq \bar{C}_0 |M\varepsilon_0| + |H^0_{i\xi}|_{U, \alpha, \beta} < \bar{C}_0 C_0 M\varepsilon_0, \quad i = 1, 2,
\]

where \( \bar{C}_0 = 2 + \sum_{j=1}^\infty j^\beta < \infty \).

Now, we proceed to prove (v.2) for \( \nu + 1 \) and first estimate the three terms \( u_0^+, u_1^+ \) and \( w^+ \).

We will use \( C_\alpha \) to denote a constant only depending on \( \alpha \). By using the Taylor expansions of \( H_1 \) and \( H_2 \), Lemma (13) (ii), (4.29)-(4.31), (4.10), (4.56), (4.47) and the Cauchy inequality we find

\[
|H_1(v_0, \phi + \Phi)|_{r_0, \alpha, \beta} \leq C_\alpha \bar{C}_0 C_0 M\varepsilon_0 r_0^{-2} |v_0|_{r_0, \alpha, \beta} \ll C_0 M\varepsilon_0 r_0^{-\alpha},
\]

\[
|H_2(v_0, \phi + \Phi)|_{r_0, \alpha, \beta} \leq C_\alpha \bar{C}_0 C_0 M\varepsilon_0 r_0^{-1} |v_0|_{r_0, \alpha, \beta} \ll C_0 M\varepsilon_0 r_0^{-\alpha+1}(1+2),
\]

\[
|Q|_{r_0, \alpha, \beta} \leq C_\alpha \bar{C}_0 C_0 M\varepsilon_0 r_0^{-1}
\]
and
\[ |\partial_1 H_1(v_0, \phi + \Phi)(E_{n_1} + v_1)|_{r_s, \alpha; s} \leq C_a \widetilde{C} C_0 M \varepsilon_0 r_{+}^{-2} |v_0|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1} \] (4.60)

for sufficiently small \( \varepsilon_0 \). From the Cauchy inequality, Condition (v.2), Lemma 13(i) and (4.31), it follows
\[ |u_0(\phi + \Phi) - u_0(\phi)|_{r_s, \alpha; s} \leq C_a r_{+}^{-1} |u_0|_{r_s, \alpha; s} |\Phi|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-\alpha}, \] (4.61)
\[ |u_1(\phi + \Phi) - u_1(\phi)|_{r_s, \alpha; s} \leq C_a r_{+}^{-1} |u_1|_{r_s, \alpha; s} |\Phi|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}, \] (4.62)
\[ |w(\phi + \Phi) - w(\phi)|_{r_s, \alpha; s} \leq C_a r_{+}^{-1} |w|_{r_s, \alpha; s} |\Phi|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)}. \] (4.63)

Lemma 12: Condition (v.2) and the definition of \( K_+ \) imply
\[ |(\text{Id} - \Gamma_{K_+})u_0|_{r_s, \alpha; s} \leq r_s C_0 M \varepsilon_0 r_{+}^{l-\alpha}, \quad |(\text{Id} - \Gamma_{K_+})u_1|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}, \] (4.64)
\[ |(\text{Id} - \Gamma_{K_+})w|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)}, \] (4.65)
and (4.30), (4.31), Condition (v.2) and (4.49),
\[ |P_1^{-1} \partial_v v_1 P_1 \tilde{A}|_{r_s, \alpha; s} \leq C_a |P_1^{-1} \partial_v v_1 P_1 \tilde{A}|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}, \] (4.66)
\[ |P_2^{-1} \partial_\Phi P_2 \tilde{\omega}|_{r_s, \alpha; s} \leq C_a |P_2^{-1} \partial_\Phi P_2 \tilde{\omega}|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)}. \] (4.67)

Combining the estimates (4.58), (4.63), (4.65), (4.67), (4.47) and (4.40) for \( w^+ \), we have
\[ |w^+|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)} \] (4.68)
by (4.51) and choosing small \( \varepsilon_0^* \). By the estimates (4.29), (4.30), (4.59), (4.40), (4.68) and Condition (v.2), we also have
\[ |P_1^{-1} \partial_\Phi v_0 P_2 \tilde{\omega} + w^+|_{r_s, \alpha; s} \ll C_a |P_1^{-1} \partial_\Phi v_0 P_2 \tilde{\omega} + w^+|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}, \] (4.69)
\[ |P_1^{-1} \partial_\Phi v_1 P_2 \tilde{\omega} + w^+|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1} \] (4.70)
and
\[ |P_1^{-1} \partial_\Phi v_0 (E_{n_2} + \partial_\Phi \Phi)^{-1} P_2 Q|_{r_s, \alpha; s} \leq C_a |P_1^{-1} \partial_\Phi v_0 (E_{n_2} + \partial_\Phi \Phi)^{-1} P_2 Q|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}. \] (4.71)

From Lemma 13(ii), Condition (v.2), (4.29) and (4.30), it follows
\[ |u_1(\phi + \Phi)v_0|_{r_s, \alpha; s} \leq C_a |u_1(\phi + \Phi)|_{r_s, \alpha; s} |v_0|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-\alpha}, \] (4.72)
\[ |u_1(\phi + \Phi)v_1|_{r_s, \alpha; s} \leq C_a |u_1(\phi + \Phi)|_{r_s, \alpha; s} |v_1|_{r_s, \alpha; s} \ll C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}. \] (4.73)

On account of the estimates (4.57), (4.60), (4.61), (4.62), (4.64), (4.69)-(4.73) for \( u_n^+ \) and \( u_1^+ \), (4.40) and (4.47)-(4.48), and the expressions (4.52) and (4.53), we can choose \( \varepsilon_0^* \) so small that our estimates yield
\[ |u_0^+|_{r_s, \alpha; s} \leq 4 C_0 M \varepsilon_0 r_{+}^{l-\alpha}, \quad |u_1^+|_{r_s, \alpha; s} \leq C_0 M \varepsilon_0 r_{+}^{l-(a+1)(\alpha+2)-1}. \]
To turn to the estimates of $H^+_1$ and $H^+_2$, by (4.54) and (4.55) we have

\[ H^+_1 - H_1 = P^{-1}_1(E_n + v_1)P_1 [H_1 \circ T^+(\rho, \phi) - H_1(\rho, \phi) - \partial_1 H_1(v_0, \phi + \Phi)(E_n + v_1)\rho - H_1(v_0, \phi + \Phi) + \delta_1(\rho, \phi) - \delta_1(0, \phi) - \partial_\rho \delta_1(0, \phi)\rho + P^{-1}_1 \partial_\phi v_0(E_n + \partial_\phi \Phi)^{-1}P_2 Q\rho - P^{-1}_1(\partial_\phi v_0 + \partial_\phi v_1\rho)P_2 H^+_2 - P^{-1}_1 v_1 P_1 H_1] \quad (4.74) \]

and

\[ H^+_2 - H_2 = P^{-1}_2(E_n + \partial_\phi \Phi)^{-1}P_2 [H_2 \circ T^+(\rho, \phi) - H_2(\rho, \phi) - H_2(v_0, \phi + \Phi) + \delta_2(\rho, \phi) - \delta_2(0, \phi) - P^{-1}_2 \partial_\phi \Phi P_2 H_2(\rho, \phi)] \quad (4.75) \]

After a short calculation, we find

\[ |H_1 \circ T^+ - H_1|_{V_{\alpha,0};s} \leq C_o r^{-1}_+ |H_1|_{V_{\alpha,0};s}|T^+ - 1d|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

and

\[ |H_2 \circ T^+ - H_2|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

by Lemma 13 (i), the Cauchy inequality, (4.32) and (4.56). It implies

\[ |\delta_1(\rho, \phi) - \delta_1(0, \phi) - \partial_\rho \delta_1(0, \phi)\rho|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

and

\[ |\delta_2(\rho, \phi) - \delta_2(0, \phi)|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

by (4.47),

\[ |P^{-1}_1 v_1 P_1 H_1|_{V_{\alpha,0};s} \leq C_o |P^{-1}_1 v_1 P_1|_{r_{\alpha,0};s}|H_1|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

and

\[ |P^{-1}_2 \partial_\phi \Phi P_2 H_2|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

by (4.30), (4.31) and (4.56), and

\[ |P^{-1}_1 \partial_\phi v_0(E_n + \partial_\phi \Phi)^{-1}P_2 Q|{\alpha,0;\alpha,0;} \leq C_o |P^{-1}_1 \partial_\phi v_0 P_2|{r_{\alpha,0};s} |Q|{r_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0 \]

by (4.40), (4.29) and (4.59). The above estimates, (4.40), (4.58) and the expression (4.75) yield

\[ |H^+_2 - H^+_2|_{V_{\alpha,0};s} \leq \chi_+ C_0 M \varepsilon_0, \quad |H^+_2|_{V_{\alpha,0};s} \leq (X_+ + r^{-1}_0)C_0 M \varepsilon_0 < \tilde{C}_0 C_0 M \varepsilon_0, \]

which, together with (4.29) and (4.30), implies

\[ |P^{-1}_1(\partial_\phi v_0 + \partial_\phi v_1\rho)P_2 H^+_2|_{V_{\alpha,0};s} \ll \chi_+ C_0 M \varepsilon_0. \]

Thus, the above estimates, (4.40), (4.57)-(4.60) and the expression (4.74) also yield

\[ |H^+_1 - H_1|_{V_{\alpha,0};s} \leq \chi_+ C_0 M \varepsilon_0, \quad |H^+_1|_{V_{\alpha,0};s} \leq (X_+ + 2r^{-2}_0)C_0 M \varepsilon_0 < \tilde{C}_0 C_0 M \varepsilon_0. \]

Obviously,

\[ H^+_1 = \varepsilon_0 O_{V_{\alpha,0};s}(\rho^2), \quad H^+_2 = \varepsilon_0 O_{V_{\alpha,0};s}(\rho). \]

This completes the proof of the Iteration Lemma. ■
5 Proof of Theorem 2

The conditions (2.8), (2.9), Lemma 10 and Remark A.2 imply that there is a constant $c_3 > 0$ such that

$$\max_{0 \leq \mu \leq \alpha} \| D^\mu \langle b, \omega(\xi) \rangle \| \geq c_3$$

in Case $n_{22} = 0$

and

$$\max_{1 \leq \mu \leq \alpha} \| D^\mu \langle b, \omega(\xi) \rangle \| \geq c_3$$

in Case $n_{22} \neq 0$

for all $\xi \in \Pi, b \in S_{n_{21}} := \{ b \in \mathbb{R}^{n_2} : |b_2| = 1 \}$. By (2.7), (H1), (H2) and the Whitney extension theorem (see Lemma 8), still using $\omega^\nu$ and $\Lambda^\nu$ to denote their extensions, we can find a constant $c_4$ such that

$$\| \Lambda^\nu \|_{0,\Pi} \leq c_4 \epsilon^{q_1} \leq c_4 \epsilon^{q_5}, \quad \| \omega^{\nu} - \omega \|_{0,\Pi} \leq c_4 \epsilon^{q_2+q_5}, \quad \nu = 0, 1, 2, \ldots .$$

Hence, for sufficiently small $\epsilon$, all $\xi \in \Pi, b \in S_{n_{21}}, \nu = 0, 1, 2, \ldots$

$$\max_{0 \leq \mu \leq \alpha} \| D^\mu \langle b, \omega^{\nu}(\xi, \epsilon) \rangle \| \geq \frac{c_3}{2} \epsilon^{q_5}$$

in Case $n_{22} = 0$

and

$$\max_{1 \leq \mu \leq \alpha} \| D^\mu \langle b, \omega^{\nu}(\xi, \epsilon) \rangle \| \geq \frac{c_3}{2} \epsilon^{q_5}$$

in Case $n_{22} \neq 0$. (5.1)

Set

$$f_{km}^\nu(\xi) = \langle k, \omega^\nu(\xi, \epsilon) \rangle + \langle m, \text{Im} \Lambda^\nu(\xi, \epsilon) \rangle$$

for $0 \neq k \in \mathbb{Z}^{n_2}, K_{\nu} < |k_2| \leq K_{\nu+1}; m = \text{col}(m_1, \cdots, m_{n_1}) \in \mathbb{Z}^{n_1}, |m_1| \leq 2$ and $m_{1} + \cdots + m_{n_1} = 0$ or $-1, \nu = 0, 1, 2, \cdots$. Here $\text{Im} \Lambda^\nu$ is the imaginary part of $\Lambda^\nu$. Then

$$3_{km}^\nu(\gamma) \subset \{ \xi \in \Pi_0 : | f_{km}^{\nu-1}(\xi) | < \gamma \epsilon^{q_5} |k_2| \}, \quad \nu = 1, 2, \cdots ,$$

where $\Pi_0$ is the closed subset of $\Pi$ defined in Theorem 1.

**Lemma 6** If $|k_2| \geq \frac{8}{c_5} c_4 |m_1| n_3^{\alpha/2}, 0 \neq k \in \mathbb{Z}^{n_2}$. Then

$$\text{meas} 3_{km}^\nu(\gamma) \leq c_5 (\text{diam} \Pi_0)^{\nu_{3a-1}} (\gamma |k_2|^{\nu-1})^{\frac{\alpha}{2}}$$

(5.2)

for some positive constant $c_5$, where $\text{diam} \Pi_0$ represents the diameter of $\Pi_0$.

**Proof** We only give the proof for the case $n_{22} \neq 0$, the proof of the case $n_{22} = 0$ is analogous and is omitted.

Due to the continuity of the derivatives and the compactness of $\Pi_0$ and $S_{n_{21}}$, the non-degenerate condition (5.1) implies that there exist finite covers $\{ \Pi_{i} \}_{i=1}^{j_0}$ and $\{ S_{i} \}_{i=1}^{j_0}$ of $\Pi_0$ and $S_{n_{21}}$, respectively, and $\mu_{ij} : 1 \leq \mu_{ij} \leq \alpha, i = 1, \cdots, i_0; j = 1, \cdots, j_0; \Pi'$ is chosen to be convex, such that

$$\| D^{\mu_{ij}} \langle b, \omega^{\nu}(\xi, \epsilon) \rangle \| \geq \frac{c_3}{4} \epsilon^{q_5}$$

for all $\xi \in \Pi_{i}, b \in S_{i}$.
Hence, for $0 \neq k \in \mathbb{Z}^n$, $\frac{k}{|k|} \in S^i$, we have

$$\| D^{\mu_{i,k}} f_{km}(\xi) \| \geq |k|_2 \| D^{\mu_{i,k}} (\frac{k}{|k|}, \omega^i(\xi, \varepsilon)) \| \geq c_4 m |\varepsilon^{\delta_3 n_3^i}| \geq \frac{c_3}{8} \varepsilon^{\delta_3} |k|_2$$

(5.3)

for all $\xi \in \Pi^i, i = 1, \cdots, i_0$, admitted $m$ and $\nu$ if $|k|_2 \geq \frac{8}{c_3} c_4 m |n_3^{\alpha/2}|$.

Now we estimate the measure of $\mathcal{R}^y_{km}(\gamma) \cap \Pi^i$. It follows by (5.3) and the definition of the norm (see Lemma [10]) that there is a vector $a \in \mathcal{S}_{n_3,1}$ such that

$$| D^{\mu_{i,k}} f_{km}(\xi) a^{\mu_{i,k}} | \geq \frac{c_3}{8} \varepsilon^{\delta_3} |k|_2 \quad \text{for all } \xi \in \Pi^i. \quad (5.4)$$

Write $\xi = at + \zeta$ with $t \in \mathbb{R}, \zeta \in a^\perp$ and let $f(t) = f_{km}(at + \zeta)$, $I_\zeta = \{ t \in \mathbb{R} : at + \zeta \in \Pi^i \}$. The inequality (5.4) means

$$\frac{\partial^{\mu_{i,k}} f(t)}{\partial t} \geq \frac{c_3}{8} \varepsilon^{\delta_3} |k|_2 \quad \text{for all } t \in I_\zeta.$$

By Fubini’s theorem and Lemma [9] it implies

$$\text{meas}(\mathcal{R}^y_{km}(\gamma) \cap \Pi^i) \leq c_6 (\text{diam} \Pi_0)^{n_3 - 1} (\varepsilon |k|_2^{t_1 - 1})^{\frac{1}{2}},$$

where $c_6 = 4 \max\{1, (4\alpha! / c_3)^{1/\alpha}\}$. Therefore

$$\text{meas}(\mathcal{R}^y_{km}(\gamma) \cap \Pi^i) \leq \sum_{i=1}^{i_0} \text{meas}(\mathcal{R}^y_{km}(\gamma) \cap \Pi^i) \leq i_0 c_6 (\text{diam} \Pi_0)^{n_3 - 1} (\varepsilon |k|_2^{t_1 - 1})^{\frac{1}{2}}.$$ 

The estimate (5.2) is proved by setting $c_5 = i_0 c_6$.  

Now, Let $K_1 = 1 + \frac{8}{c_3} c_4 n_3^{\alpha/2}$, $K_2 = 1 + \frac{16}{c_3} c_4 n_3^{\alpha/2}$ and

$$\mathcal{S}_1 = \{ (k, m) \in \mathbb{Z}^m \times \mathbb{Z}^{n_3} : 0 < |k|_2 < K_1, |m|_1 = 1, m_1 + \cdots + m_{n_3} = -1 \},$$

$$\mathcal{S}_2 = \{ (k, m) \in \mathbb{Z}^m \times \mathbb{Z}^{n_3} : 0 < |k|_2 < K_2, |m|_1 = 2, m_1 + \cdots + m_{n_3} = 0 \}.$$

Lemma 7 If $(k, m) \in \mathcal{S}_1 \cup \mathcal{S}_2$, then

$$\text{meas} \mathcal{R}^y_{km}(\gamma) \leq \gamma. \quad (5.5)$$

Proof Case 1: $q_5 > 0$. Denote $\mathcal{S} = \{ (k, m) \in \mathcal{S}_1 \cup \mathcal{S}_2 : \langle k, \omega_0 \rangle \neq 0 \}$. On one hand, noting that $\omega_0$ is independent of $\xi$, we can find a positive constant $c_7$ such that

$$|\langle k, \omega_0 \rangle| \geq c_7 \quad \text{for all } (k, m) \in \mathcal{S},$$

thus $\mathcal{R}^y_{km}(\gamma) = \emptyset$ for sufficiently small $\gamma$ if $(k, m) \in \mathcal{S}$. On the other hand, the condition (ii) and the assumption $q_1 > q_3 \geq q_5$ imply that there exists a positive constant $c_8$ such that

$$\text{meas} \{ \xi \in \Pi : |\sqrt{-1} \varepsilon^{\delta_3} (k, \omega(\xi)) + \varepsilon^{\delta_3} (m^2, \Lambda_2(\xi)) | < c_8 \varepsilon^{\delta_3} \} \leq \gamma$$

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for \((k, m) \in \mathcal{R}_1 \cup \mathcal{R}_2 \setminus \mathcal{R}\) and \(m = \text{col}(m^1, m^2)\) with \(m^1 \in \mathbb{Z}^{n_1}, m^2 \in \mathbb{Z}^{n_2}\), which implies the inequality (5.5) for sufficiently small \(\gamma\).

**Case 2:** \(q_5 = 0\). The condition (ii) implies

\[
\text{meas}[\xi \in \Pi : |\sqrt{-1}\langle k, \tilde{\omega}_0 + \tilde{\omega}(\xi)\rangle + \epsilon q_3\langle m^2, \Lambda_2(\xi)\rangle| < c_0] \leq \gamma
\]

with a constant \(c_0 > 0\) and \((k, m) \in \mathcal{R}_1 \cup \mathcal{R}_2\) for sufficiently small \(\gamma\), which also implies the inequality (5.5).

**Remark 5.1** If \(q_3 > q_5 \geq 0\), then without the condition (ii), we can obtain

\[
\text{meas}\mathcal{R}_{km}^\gamma(\chi) \leq c_5(\text{diam}\Pi_0)^{n_3-1}(2\gamma|k|_2^{i-1})^{\frac{n}{2}} \quad \text{for all} \quad (k, m) \in \mathcal{R}_1 \cup \mathcal{R}_2.
\]

In fact, for sufficiently small \(\gamma\) (equivalently, sufficiently small \(\epsilon\)), we have

\[
\mathcal{R}_{km}^\gamma(\chi) \subset \{\xi \in \Pi_0 : |(k, \omega^\gamma(\xi, \epsilon))| < 2\epsilon q_5|k|_2^i\}
\]

for all \((k, m) \in \mathcal{R}_1 \cup \mathcal{R}_2\). From Lemma 6 with \(|m|_1 = 0\), it follows

\[
\text{meas}\mathcal{R}_{km}^\gamma(\chi) \leq \text{meas}\{\xi \in \Pi_0 : |(k, \omega^\gamma(\xi, \epsilon))| < 2\epsilon q_5|k|_2^i\}
\]

\[
\leq c_5(\text{diam}\Pi_0)^{n_3-1}(2\gamma|k|_2^{i-1})^{\frac{n}{2}}
\]

By Lemmas 6 and 7 we obtain

\[
\text{meas}(\Pi_0 \setminus \Pi_\gamma) \leq \sum_{y=1}^{\infty} \sum_{K_{y-1} < |k|_2 \leq K_y} \left(\text{meas}\mathcal{R}_{k0}^\gamma + \sum_{|m|_1 = 1} \text{meas}\mathcal{R}_{km}^\gamma + \sum_{|m|_2 = 2} \text{meas}\mathcal{R}_{km}^\gamma\right)
\]

\[
\leq \sum_{(k, m) \in \mathcal{R}_1 \cup \mathcal{R}_2} \text{meas}\mathcal{R}_{km}^\gamma + c_5(\text{diam}\Pi_0)^{n_3-1}\gamma^{\frac{n}{2}} \left|\sum_{0 \leq k_1 \leq 2^{|m|_2}} |k|_2^{\frac{n}{2}}\right|
\]

\[
+ n_1 \sum_{|k|_2 \geq K_1} |k|_2^{\frac{n}{2}} + n_1(n_1 - 1) \sum_{|k|_2 \geq K_2} |k|_2^{\frac{n}{2}}
\]

\[
\leq c_{10} \gamma^{\frac{n}{2}}
\]

where \(c_{10}\) is a positive constant depending on \(n_1, n_2, \text{diam}\Pi, \omega_i\) and \(\Lambda_i (i = 1, 2)\), and \(\text{meas}(\Pi \setminus \Pi_0) = O(\gamma)\). The proof of Theorem 2 is complete.

**Appendix**

**A.1. Whitney extension theorem**

Let \(\overline{\Omega} \subset \mathbb{R}^n\) be a closed set, \(p\) be a non-negative integer, \(p < l \leq p + 1\). \(C^l_w(\overline{\Omega})\) is the class of all collections \(f = \{f^{(k)} \}_{|k|_1 \leq p}\) of functions defined on \(\overline{\Omega}\) which satisfy, for some finite \(M\),

\[
|f^{(k)}(x)| \leq M, \quad |f^{(k)}(x) - P_k(x, y)| \leq M|x - y|^{l-|k|_1}\quad (6.1)
\]
for all \( x, y \in \Omega \) and \(|k|_1 \leq p\), where
\[
P_k(x, y) = \sum_{|k+j|_1 \leq p} \frac{1}{j!} f^{(k+j)}(y)(x-y)^j
\]
is the analogue of the \( k \)-th Taylor polynomial. \( f \) is called \( C^l \) Whitney in \( \Omega \) with Whitney derivatives \( D^k f = f^{(k)} \) for \(|k|_1 \leq p\). Define a norm
\[
\|f\|_{C^l_w(\Omega)} = \inf M
\]
is the smallest \( M \) for which both inequalities in (6.1) hold. Then \( C^l_w(\Omega) \) with the norm is a Banach space.

The following extension theorem indicates that a Whitney differentiable function has an extension to \( \mathbb{R}^n \) which is differentiable in the standard sense.

**Lemma 8** (Whitney extension theorem, [40, 37, 27]) Let \( \Omega \) be a closed set in \( \mathbb{R}^n \), \( p \in \mathbb{Z}_+ \) and \( p < l \leq p + 1 \). Then there exists a linear extension operator
\[
\mathcal{E} : C^l_w(\Omega) \to C^l(\mathbb{R}^n), \quad f = \{f^{(k)}\}_{|k|_1 \leq p} \to F = \mathcal{E}f
\]
such that
\[
D^k F|_{\Omega} = f^{(k)}, \quad |k|_1 \leq p
\]
and
\[
\|F\|_{C^l(\mathbb{R}^n)} \leq C\|f\|_{C^l_w(\Omega)},
\]
where the constant \( C \) depends only on \( l \) and the dimension \( n \), but not on \( \Omega \). Moreover, if \( \Omega = \Omega_1 \times \mathbb{T}^{n_2} \subset \mathbb{R}^{n_1} \times \mathbb{T}^{n_2} \), then the extension can be chosen to be defined on \( \mathbb{R}^{n_1} \times \mathbb{T}^{n_2} \), so that the periodicity is preserved.

### A.2. Measure estimate lemmas

**Lemma 9** ([34]) Let \( f : [a, b] \to \mathbb{R} \) with \( a < b \) be an \( \alpha \)-times continuously differentiable function satisfying
\[
\left| \frac{d^\alpha f(x)}{dx^\alpha} \right| \geq c, \quad x \in [a, b]
\]
for some \( \alpha \in \mathbb{N} \) and a constant \( c > 0 \). Then we have the measure estimate
\[
\text{meas}\{x \in [a, b] : |f(x)| \leq \varepsilon\} \leq 4 \left( \frac{\alpha!}{2c} \frac{\varepsilon}{\alpha} \right) \quad \text{for all } \varepsilon > 0.
\]

**Lemma 10** Let \( \Pi \subset \mathbb{R}^p \) be a bounded closed set, \( f_j : \Pi \to \mathbb{R} \) be of \( C^\alpha \) on \( \Pi \) with a positive integer \( \alpha \), \( j = 1, \cdots, q \). Denote \( f(\xi) = \text{col}(f_1(\xi), \cdots, f_q(\xi)) \). Assume for \( \xi \in \Pi \),
\[
\text{rank}\left(f(\xi), \frac{\partial^\beta f(\xi)}{\partial \xi^\beta} : 1 \leq |eta|_1 \leq \alpha\right) = q \quad \text{and} \quad 1 + p + p^2 + \cdots + p^\alpha \geq q. \quad (6.2)
\]
Then there is a constant $c > 0$ such that

$$\max_{0 \leq \mu \leq \alpha} \|D^\mu(b, f(\xi))\| \geq c \quad \text{for all } b \in S_{q,1}, \xi \in \Pi.$$  

Here $D$ represents the differential operator with respect to the variable $\xi$,

$$S_{q,1} = \{b \in \mathbb{R}^q : |b|_2 = 1\}, \quad \|D^\mu(b, f(\xi))\| = \max_{a \in S_{p,1}} |D^\mu(b, f(\xi))a^{\otimes \mu}|,$$

$a^{\otimes \mu} = (a_1, a_2, \cdots, a_\mu)$ with $a_i = a$, $i = 1, 2, \cdots, \mu$.

**Remark A.1** Here, by the Whitney extension theorem we assume the continuous differentiability of a function $f$ with respect to the parameter variable $\xi$ on a closed set $\Pi$ means that $f$ is continuously differentiable in some neighbourhood of $\Pi$.

**Proof** Suppose such a constant $c$ does not exist. Then for any positive integer $n$, we can find $\xi_n \in \Pi$ and $b_n \in S_{q,1}$ satisfying

$$\max_{0 \leq \mu \leq \alpha} \|D^\mu(b_n, f(\xi_n))\| < \frac{1}{n}, \quad n = 1, 2, \cdots.$$  

Based on the compactness of $\Pi$ and $S_{q,1}$ there are convergent subsequences of $\{b_n\}$ and $\{\xi_n\}$, respectively, still denoting by $\{b_n\}$ and $\{\xi_n\}$, such that $b_n \to b_0 \in S_{q,1}, \xi_n \to \xi_0 \in \Pi$ as $n \to \infty$. Thus, the continuity of the derivatives implies

$$\|D^\mu(b_0, f(\xi_0))\| = 0 \quad \text{for all } 0 \leq \mu \leq \alpha.$$  

Noting that

$$|D^\mu(b_0, f(\xi_0))(a_1, \cdots, a_\mu)| \leq \frac{\mu^\mu}{\mu!} |D^\mu(b_0, f(\xi_0))|$$

for all $a_i \in S_{p,1}, i = 1, \cdots, \mu$, we have

$$b_0^T \left(f(\xi_0), \frac{\partial^\beta f(\xi_0)}{\partial \xi^\beta} : 1 \leq |\beta|_1 \leq \alpha\right) = 0,$$

which implies

$$\text{rank} \left(f(\xi_0), \frac{\partial^\beta f(\xi_0)}{\partial \xi^\beta} : 1 \leq |\beta|_1 \leq \alpha\right) < q$$

being in contradiction with the condition (6.2). The lemma is proved.  

**Remark A.2** From the proof of Lemma 10 it is easy to see that if the condition (6.2) is replaced by

$$\text{rank} \left(\frac{\partial^\beta f(\xi)}{\partial \xi^\beta} : 1 \leq |\beta|_1 \leq \alpha\right) = q \quad \text{and} \quad p + p^2 + \cdots + p^\alpha \geq q,$$

then we also have

$$\max_{1 \leq \mu \leq \alpha} \|D^\mu(b, f(\xi))\| \geq c \quad \text{for all } b \in S_{q,1}, \xi \in \Pi.$$  

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$$\text{rank} \left(\frac{\partial^\beta f(\xi)}{\partial \xi^\beta} : 1 \leq |\beta|_1 \leq \alpha\right) = q \quad \text{and} \quad p + p^2 + \cdots + p^\alpha \geq q,$$

then we also have

$$\max_{1 \leq \mu \leq \alpha} \|D^\mu(b, f(\xi))\| \geq c \quad \text{for all } b \in S_{q,1}, \xi \in \Pi.$$
A.3. Properties of analytic smoothing operator

Let $l > 0, m \in \mathbb{N}$ and $C^l(\mathbb{R}^m)$ be the Hölder space defined in Definition 2.1 without parameter variables, $u_0 \in C^l_0(\mathbb{R}^m)$ be an even function, vanishing outside the interval $[-1, 1]$ and identically equal to 1 in a neighbourhood of 0, $u(x) = u_0(|x|/2)$ for $x \in \mathbb{R}^m$ and

$$\tilde{u}(z) = \int_{\mathbb{R}^m} u(x)e^{\sqrt{-1}(x,z)}dx \quad \text{for} \quad z \in \mathbb{C}^m,$$

$$f_r(x) := (S_r f)(x) := r^{-m} \int_{\mathbb{R}^m} \tilde{u}((x-y)/r)f(y)dy$$

for $x \in \mathbb{C}^m$ and $r \in (0, 1]$.

**Lemma 11** The following assertions are valid

(a) $\int_{\mathbb{R}^m} \tilde{u}(x)dx = u(0) = 1$;

(b) $(S_r f)(x) = \int_{\mathbb{R}^m} \tilde{u}(y)f(x-ry)dy$ for $x \in \mathbb{R}^m$;

(c) $\int_{\mathbb{R}^m} x^k \tilde{u}(x)dx = 0$ for $0 \neq k \in \mathbb{Z}^m_+$;

(d) for any $p \in \mathbb{N}$, there is a constant $C_p > 0$ such that

$$|D^k \tilde{u}(z)| \leq \frac{C_p}{(1 + |z|_2)^p}e^{\text{Im}z}$$

for all $|k|_1 \leq p, k \in \mathbb{Z}^m_+$,

where $D^k = D_1^{k_1} \circ D_2^{k_2} \circ \cdots \circ D_m^{k_m}$, and $D_j^{k_j} = \frac{\partial^{k_j}}{\partial x_j}$;

(e) if $P$ is a polynomial, then $(S_r P)(x) = P(x)$;

(f) there exists a constant $C_1 > 0$ such that

$$\left| D^k f_r(x) - \sum_{|\beta|_1 \leq l-|k|_1} D^{k+\beta} f(\text{Re}x) \frac{(\sqrt{-1}\text{Im}x)^\beta}{\beta!} \right| \leq C_1 r^{l-|k|_1} \|f\|_{L^1;\mathbb{R}^m}, \quad |\text{Im}x| \leq r \leq 1$$

for all $k \in \mathbb{Z}^m_+$ with $|k|_1 \leq l$. In particular, for $x \in \mathbb{R}^m$ and $p \in \mathbb{Z}_+$,

$$\|f_r - f\|_{p;\mathbb{R}^m} \leq C_{lp} r^{l-p} \|f\|_{L^p;\mathbb{R}^m}, \quad p \leq l$$

for a suitable constant $C_{lp}$ depending on $l, p$ and $m$.

**Proof** The definitions of $\tilde{u}$ and $S_r$ imply (a) and (b), respectively. Noting that the $\tilde{u}$ is a Schwartz function (see (d)), and the Fourier transformation and differentiation can be exchanged, we have

$$\int_{\mathbb{R}^m} x^k \tilde{u}(x)dx = (\sqrt{-1})^{|k|_1} D^k \int_{\mathbb{R}^m} \tilde{u}(x)e^{-\sqrt{-1}(y,x)}dx \bigg|_{y=0} = (\sqrt{-1})^{|k|_1} D^k u(y) \bigg|_{y=0} = 0,$$

which verifies (c). See Lemma 9, Proposition 8 and Remark 15 (i) in [11] for (d), (e) and (f), respectively, also see the proof of Lemma 2.1 in Part I of [39] for (e).
Lemma 12 Let $K$ be a positive integer and $f$ be a bounded and analytic function in the strip $\{x : |\text{Im} x| < r\}$ of $\mathbb{T}^n$, $f(x) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k)e^{\sqrt{-1}(k,x)}$. Define the truncation operator $\Gamma_K$ as follows

$$\Gamma_K f = \sum_{|k| \geq K} \hat{f}(k)e^{\sqrt{-1}(k,x)}.$$  

If $K > (2\rho)^{-1}$, then we have

$$|\Gamma_K^r|_{2\rho} < C(n)|f|_{2\rho} e^{-\rho K},$$  

where $C(n) = 6(n!)n^n e^{-n}$.

Proof: Set $\sigma = 2\rho$. Based on the fact that the number of all $k$ with $|k|_1 = m$ is bounded by $2nm^{n-1}$, we have

$$|\Gamma_K^r|_{2\rho} \leq \sum_{|k|_1 > m} |f|_{2\rho} e^{-\sigma |k|_1} \leq \sum_{|k|_1 > m} |f|_{2\rho} e^{-\sigma |k|_1} \leq |f|_{2\rho} \sum_{m > k} 2nm^{n-1} e^{-\sigma m}. \quad (6.3)$$

Here we use Lemma A.1 in [29]. Since the function $y^n e^{-\sigma y}$ is monotonically decreasing in the interval $[\frac{3}{\sigma}, +\infty)$ and $K > \sigma^{-1}$, therefore,

$$\sum_{m > K} m^{n-1} e^{-\sigma m} < \int_K^{+\infty} y^n e^{-\sigma y} dy = \left( \frac{1}{\sigma}K^{n-1} + \frac{n - 1}{\sigma^2}K^{n-2} + \cdots + \frac{(n-1)!}{\sigma^n} \right) e^{-\sigma K} < 3(n-1)!K^n e^{-\sigma K}.$$  

Hence, by (6.3) we obtain

$$|\Gamma_K^r|_{2\rho} \leq 6n!|f|_{2\rho} K^n e^{-\sigma K}. \quad (6.4)$$

Noting that the maximum of the function $y^n e^{-y}$ on the interval $(0, +\infty)$ is $n^n e^{-n}$, (6.4) implies

$$|\Gamma_K^r|_{2\rho} \leq 6(n!)n^n e^{-n}|f|_{2\rho} e^{-\rho K}. \quad \Box$$

Let $\Omega_1$ and $\Omega_2$ be domains in $\mathbb{C}^n$, $\Pi$ be an open set in $\mathbb{R}^m$, $f(x, \xi)$ and $g(x, \xi)$ be analytic in $x \in (\Omega_1 + r)$ and in $x \in \Omega_2$ respectively, and continuously differential up to order $\alpha$ in $\xi \in \Pi$, $g : \Omega_2 \times \Pi \to \Omega_1$, where $r > 0$, $\Omega_1 + r = \{x \in \mathbb{C}^n : \text{dist}(x, \Omega_1) < r\}$.

We introduce the notation for $1 \leq \mu \leq \alpha$,

$$|Df|_\mu := \max_{\mu \leq |\beta|_1 \leq \alpha} \left| \partial_\beta^\mu Df \right|_{\Omega_1 + r, 0, \Pi}, \quad |g|_\mu := \max_{\mu \leq |\beta|_1 \leq \alpha} \left| \partial_\beta^\mu g \right|_{\Omega_2, 0, \Pi},$$

where $Df$ represents the differential operator with respect to the coordinate variable $x$. Using the Chain Rule on differentiation of a composition of mappings and Cauchy inequality, we easily prove the following lemma.

Lemma 13 Let $\beta \in \mathbb{Z}^m_+$ and $|\beta|_1 = \mu, 1 \leq \mu \leq \alpha$. Then
and let \( K(m, j) \) denote the number of points in \( K(m, j) \). Then we have

\[
K(m, j)^\# \leq (2n)^n (2m)^{n-1} 2^{-\frac{j}{2(n-1)(j-1)}}. \tag{6.8}
\]
In fact, if \( k, k' \in K(m, j) \) are different points, then
\[
\gamma |k - k'|^r \leq |\langle k - k', \omega \rangle| \leq |\langle k, \omega \rangle| + |\langle k', \omega \rangle| + |\lambda| < \gamma 2^{1 - j},
\]
which implies
\[
|k - k'| \geq n^{-1} |k - k'|^r > n^{-1} 2^{\frac{j - 1}{r}} := 2\rho_j.
\]
Noting \( |k - k'| \leq 2m \) we get \( \rho_j \leq m \). If we encircle every point \( k \in K(m, j) \) by a cube \( C_k : |x - k| \leq \rho_j \), then these cubes are mutually disjoint. The intersections of these cubes \( C_k \) with the curved surface \( |x| = m \) are disjoint \( n - 1 \) dimensional sets with \( n - 1 \) dimensional volume \( \geq \rho_j^{n - 1} \).

As the \( n - 1 \) dimensional volume of the curved surface \( |x| = m \) is \( 2n(2m)^{n-1} \), we obtain
\[
K(m, j)^\# \leq \frac{2n(2m)^{n-1}}{\rho_j^{n-1}},
\]
which verifies the inequality (6.8). Thus we have
\[
\sum_{K(m, j)} |\langle k, \omega \rangle + \lambda|^{-b} \leq \gamma^{-b} 2^{h(j+1)} K(m, j)^\# \leq 2^{2(n+b-1)n^m m^\tau} 2^{|\langle k, \omega \rangle + \lambda|^{-b} C_1 2^{(b - \frac{n-1}{r})(j-1)}} := C_1 2^{(b - \frac{n-1}{r})(j-1)}.
\]

Let \( j^* \) be the greatest occurring \( j \) for which \( K(m, j) \neq \emptyset \). Then the facts that
\[
\gamma^{-1} 2^{j^*} < |\langle k, \omega \rangle| \leq \gamma^{-1} |k|^{r} \leq \gamma^{-1} (nm)^r
\]
and
\[
\{ k \in \mathbb{Z}^n : |k| = m \}^\# = (2m + 1)^n - (2m - 1)^n < 2n(4m)^{n-1}
\]
imply
\[
\sum_{|k|=m} |\langle k, \omega \rangle + \lambda|^{-b} \leq \sum_{|k|=m, |\langle k, \omega \rangle + \lambda|^{-b} \leq 2\gamma^{-1}} |\langle k, \omega \rangle + \lambda|^{-b} + \sum_{j=1}^{j^*} \sum_{K(m, j)} |\langle k, \omega \rangle + \lambda|^{-b}
\]
\[
\leq n^2 2^{n+b-1} m^{n-1} \gamma^{-b} + C_1 \sum_{j=1}^{j^*} 2^{(b - \frac{n-1}{r})(j-1)}
\]
\[
\leq \tau^2 2^{(n+b-1) n^{\tau} + 1} (\tau b - n + 1)^{-1} \gamma^{-b} m^{\tau} := C_2 m^{\tau}.
\]

Therefore,
\[
\sum_{0 \neq k \in \mathbb{Z}^n} |k||\langle k, \omega \rangle + \lambda|^{-b} e^{-\epsilon |k|} \leq \sum_{m=1}^{\infty} \sum_{|k|=m} (nm)^\# |\langle k, \omega \rangle + \lambda|^{-b} e^{-\epsilon |k|}
\]
\[
\leq C_2 n^{\tau} \sum_{m=1}^{\infty} m^{\tau + \nu} e^{-\epsilon m}.
\]
\[
(6.9)
\]
Noting that the function \( g(x) = x^{\tau+b} e^{-\epsilon x} \) on the interval \([1, \infty)\) gets its maximum at \( x_0 = \frac{\tau b + \nu}{\epsilon} \), moreover is strictly increasing and decreasing on \([1, x_0)\) and \((x_0, \infty)\), respectively. Denote the
integer part of $\frac{\tau b + v}{\sigma}$ by $m_0$. Then $m_0 \geq 1$ and
\[
\sum_{m=1}^{\infty} m^{\tau b + v} e^{-\sigma m} \leq \int_1^{m_0} x^{\tau b + v} e^{-\sigma x} dx + g\left(\frac{\tau b + v}{\sigma}\right) + \int_{m_0}^{\infty} x^{\tau b + v} e^{-\sigma x} dx
\]
\[
\leq g\left(\frac{\tau b + v}{\sigma}\right) + \sigma^{-\left(\tau b + v + 1\right)} \int_0^{\infty} y^{\tau b + v} e^{-\sigma y} dy
\]
\[
= \left(\frac{\tau b + v}{e\sigma}\right)^{\tau b + v} + \sigma^{-\left(\tau b + v + 1\right)} \Gamma(\tau b + v + 1). \quad (6.10)
\]
By the Stirling formula of the gamma function, we have
\[
\Gamma(\tau b + v + 1) < \frac{11}{4} \sqrt{\tau b + v} \left(\frac{\tau b + v}{e}\right)^{\tau b + v}. \quad (6.11)
\]
Combining (6.9)-(6.11), we obtain the estimate (6.7). The proof of the lemma is complete. ■

**Remark A.3** From the proof of Lemma 14 it is easy to see that if the norm $|k|_2$ in the condition (6.5) is replaced by the norm $|k|_1$, then the estimates (6.6) and (6.7) are still valid.

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