THE MESON MODEL ON THE BASIS OF THE STRING SOLUTION OF THE HEISENBERG EQUATION

V.D.Dzhunushaliev

Theoretical physics department, the Kyrgyz State National University, 720024, Bishkek, Kyrgyzstan

Abstract

The axially symmetric non-local solution in the Heisenberg equation is found. It is regular in the whole space and has the finite energy on the unit of length according to this we may consider the solution as a string. Taking the non-local spherically symmetric solution, which was found by Finkelstein et. al., and our solution in account we suggest to consider the Heisenberg equation as a quantum equation for non-local objects (strings, flux tubes, membranes and so on). The received solution is used for the obtaining the meson model as a rotating string with the quark on its ends.

PACS number: 03.65.Pm; 11.17.-w

In the 50-th years W.Heisenberg introduce the nonlinear term in the Dirac equation [1]-[3]. The given equation (Heisenberg equation(HE)) has the discrete spectrum of the spherically symmetric solution with the finite energy even in the classical region [4], [3]. This gave the hope that in Unified field theory based on HE all the fundamental characteristic of the elementary particles would be derive. The further development of the theory straight this direction showed that this hope could not be realized.

Notice that the main peculiarity of HE is the fact that it has the non-local solution: for example, in [1],[3] the spherically symmetric particlable
solutions are received. It can be suppose that also the axially symmetric non-local solutions exist in HE which can be described strings or flux tubes. Recall that at present the strings and the flux tubes are actively discussed in the superstring theory [6] and in quantum chromodynamics (see for example [7]-[8]). This paper is devoted to the string solutions in HE. Remark that the axially symmetric string solution has been already found in the nonlinear Schrödinger equation [9].

Let us write down the HE by the next view:

\[
\left[ i \gamma^\mu D_\mu - m + \lambda \langle \bar{\psi} \psi \rangle \right] \psi = 0, \tag{1}
\]

where \( \bar{\psi} \) is the Dirac adjoint spinor; \( D_\mu = \partial/\partial x^\mu + iG_\mu \) is the covariant derivative; \( G_\mu \) is the Yang-Mills field; \( \lambda \) is the some coefficient; \( \hbar = c = 1; \mu = 0, 1, 2, 3 \). The nonlinear term in Eq.(1) can take various forms: \( \gamma^\mu (\bar{\psi}\gamma_\mu \psi) \), \( \gamma^\mu \gamma^5 (\bar{\psi}\gamma_\mu \gamma^5 \psi) \), but we choose the simplest case represented in Eq.(1). We will search for the axially symmetric solution of the system (1). For that we write down \( \psi(t, \rho, \varphi) \) as follows:

\[
\psi(t, \rho, \varphi) = \begin{pmatrix}
A(\rho) \\
B(\rho) \\
C(\rho) \\
D(\rho)
\end{pmatrix} e^{i\varphi} \exp(-iEt), \tag{2}
\]

where \( \rho, z \) and \( \varphi \) are the coordinates in the cylindrical coordinate system; \( G_\mu = 0, E \) - some constant. The substitution of (2) into Eq.(1) leads to the next set of the equations:

\[
\begin{align*}
(E - m)A - D' - \frac{D}{\rho} + \lambda A(A^2 + B^2 - C^2 - D^2) &= 0, \\
(E - m)B - C' + \lambda B(A^2 + B^2 - C^2 - D^2) &= 0, \\
-(E + m)C - B' - \frac{B}{\rho} + \lambda C(A^2 + B^2 - C^2 - D^2) &= 0, \\
-(E + m)D - A' + \lambda D(A^2 + B^2 - C^2 - D^2) &= 0.
\end{align*}
\]

We consider the simplest case \( B = C = 0 \). We introduce the dimensionless variable \( x = \rho \beta; \beta = E/m \) and the functions \( A(\rho) = a(x)(m/\lambda)^{1/2}, D(\rho) = d(x)(m/\lambda)^{1/2} \). Thus we get the next set of equations:

\[
\begin{align*}
a' + d(1 + \beta - a^2 + d^2) &= 0, \\
d' + \frac{d}{x} + a(1 - \beta - a^2 + d^2) &= 0.
\end{align*}
\]
As it easy to note, the given set has the following trivial solutions:

\[ a = d = 0, \quad \text{(9)} \]
\[ d = 0; \quad a = \pm \sqrt{1 - \beta}. \quad \text{(10)} \]

The solution (9) is the saddle point in the phase space \((ad)\) and the solutions (10) are two stables focuses.

The set (7,8) coincides (up to the factor 2 at the second summand in Eq.(8)) with that investigated in [4],[5]. On those papers the qualitative investigation of the set of equations for spherically symmetric case was carried out, and as a result of it was shown that there exists the finite discrete spectrum of the solutions making a physical sense. These solutions asymptotically drop to zero by an exponential law and are regular in the whole space, so they have a finite energy. It is logical to suppose that in the axially symmetric case this features will take place too, that is solutions making a physically sense with the finite energy on the unit of length will take place. Such the solutions would be named as string ones and it would allow to get a new approach to the HE.

We shall solve the set (7-8) by numerical way. The solution has the following form near the axis \(z = 0\):

\[ a = a_0 + a_2 \frac{x^2}{2} + \cdots, \quad \text{(11)} \]
\[ d = d_1 x + \cdots, \quad \text{(12)} \]
\[ a_2 = d_1 (a_0^2 - 1 - \beta), \quad \text{(13)} \]
\[ d_1 = \frac{1}{2} a_0 (a_0^2 - 1 + \beta). \quad \text{(14)} \]

From (11-14) we see that all the solutions depend upon two parameters: \(a_0 = a(x = 0)\) and \(\beta\). It is easy to note that there is a continuous region of values \(a_0\) for which \(a(x) \xrightarrow{x\to\infty} +1-\beta)\text{1/2},\) it is the stable focus (7-8) in the solution of the set (7-8). Moreover, there is a continuous region of values \(a_0\) for which \(a(x) \xrightarrow{x\to\infty} -(1-\beta)\text{1/2},\) it is the stable focus (7-8). It is evident that the exceptional solution take place on the boundary of these regions (for which \(a(x) \xrightarrow{x\to\infty} 0,\) \(d(x) \xrightarrow{x\to\infty} 0\)). It is the separatrix passing through the saddle point (8). Denote the boundary value of the exceptional solution by the following: \(a(x = 0) = a^*_n\), where \(n\) the number of the points in which \(a(x) = 0\).
These exceptional solutions \( a_n(x) \) and \( d_n(x) \) may be found by a sequential approximation method. Two solutions \( a_0(x) \), \( d_0(x) \) and \( a_1(x) \), \( d_1(x) \) are displayed on the Fig.1 (\( \beta = 0.5 \)). By the numerical method the following boundary values are evaluated: \( a_0^* = 1.298569 \ldots \); \( a_1^* = 1.491163 \ldots \); \( a_2^* = 1.576012 \ldots \). By further numerical analyze was find out that solutions are singular by \( x > 2.0 \).

Apparently, the asymptotic behavior of the given exceptional solutions is:

\[
a \approx d \approx \exp \left\{ -2x \sqrt{\beta (1 - \beta)} \right\} \sqrt{x}. \quad (15)
\]

It is clearly seen that such an asymptotic behavior of the functions \( a_n(x) \) and \( d_n(x) \) tends to the fact that the linear density of the energy is finite for the given string.

Now we are proceeding to the physical interpretation of the received results. We calculate the linear energy density of the string. According to the general Hamiltonian definition, the linear energy density is equal:

\[
h = \int \bar{\psi} \left[ -i\gamma^j \partial_j + m - \frac{\lambda}{2} (\bar{\psi}\psi) \right] \psi d\rho d\varphi, \quad (16)
\]

where \( j = 1, 2, 3 \). Then we receive the following expression for the energy density:

\[
h = \frac{2\pi}{\lambda} I_n, \quad (17)
\]

where

\[
I_n = \int_0^\infty \left[ \beta \left( a^2 + b^2 \right) + \frac{1}{2} \left( a^2 - b^2 \right)^2 \right] x dx. \quad (18)
\]

The numerical calculations give us the following results: \( I_0 \approx 3.5524 \), \( I_1 \approx 21.4293 \), \( I_2 \approx 51.7859 \).

Despite the boundary effect we assume that the quark and antiquark rotating around the common center are placed on the ends of the string having the linear energy density (16). In according with well-known opinion this construction is a meson. Take another frame of reference connected with the rotating quark, after that its potential energy \( V \) is equal:

\[
V(r) = \frac{2\pi}{\lambda} I_n r + \frac{L^2}{2Mr^2}, \quad (19)
\]
where $r$ is the distance from quark to the mass center; $M$ is the quark mass; $L$ is the angular momentum of the quark. This potential will have the minimum if only $I_n > 0$. In this case the quark will be in the potential hole and its distance $r_0$ to the rotation center will equal:

$$r_0 \approx \left( \frac{\lambda}{2\pi I_n M} \right)^{1/3},$$  \hspace{1cm} (20)

where we take into account that the characteristic angular momentum for meson is $L \approx \hbar = 1$. As the suggested string model is a phenomenological model, we cannot define the parameter $\lambda$. The parameters $\lambda, E$ and $m$ can be define in the scope of Yang-Mills nonperturbative quantum Yang-Mills field theory, by analogy with that occure in superconductivity theory during the definition of the parameters of Ginzburg-Landau equation from the microscopic theory. It is interesting is to estimate the value of the parameter $\lambda$ defining the quantum self-action of the string wave function; i.e.

$$\lambda \approx 2\pi I_n M r_0^3$$  \hspace{1cm} (21)

The characteristic quark mass is $M \approx 0.5 GeV$, characteristic meson length (string length) is $r_0 \approx 1 Fm$, hence then the parameter $\lambda \approx 4.8 Fm^2$ with $n = 0$. It is easy to show that according to dimensional consideration we can write down this parameter as follows $\lambda = \hbar l^2 = l^2$, where $l \approx 2.2 Fm$ is a certain length scale constant.

It is obvious that this consideration is an approximative model of meson as a string stretched between rotating quark-antiquark pair. As the next approximation we have to take into account the chromodynamical field created by the quark.

Finally, we see that the HE has the non-local solutions and in particular it has the string solution with the finite energy on the unit of length. By analogy with the Dirac equation describing the quantum pointlike particle we can assume that the Heisenberg equation describe the quantum non-local objects, for example: axially symmetric objects (strings, flux tubes), spherically symmetric objects (membranes) and so on. The received solution can use to the obtaining the string model of meson.

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Fig. 1.

1 is $a_0(x)$ function; 2 is $d_0(x)$ function; 3 is $a_1(x)$ function; 4 is $d_1(x)$ function.
