Avalanches, Scaling and Coherent Noise

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Abstract

We present a simple model of a dynamical system driven by externally-imposed coherent noise. Although the system never becomes critical in the sense of possessing spatial correlations of arbitrarily long range, it does organize into a stationary state characterized by avalanches with a power-law size distribution. We explain the behavior of the model within a time-averaged approximation, and discuss its potential connection to the dynamics of earthquakes, the Gutenberg-Richter law, and to recent experiments on avalanches in rice piles.
I. INTRODUCTION

There has in the last few years been considerable interest in extended systems which self-organize into a state exhibiting large scale fluctuations and intermittent dynamics. One of the earliest attempts to model systems of this type was made in 1987 by Bak, Tang, and Wiesenfeld, who proposed a simple lattice model for the avalanches produced by depositing grains of sand on an ever-growing sand pile [1]. Despite having only short-range interactions and no tunable parameters, their model organizes itself into a state with long-range spatial correlations and avalanches of size not limited by any finite correlation length. It has been proposed that similar self-organized critical (SOC) behavior could lie behind a wide range of physical phenomena showing $1/f$ noise and scale-free fluctuation distributions. SOC models have been put forward to describe the dynamics of earthquakes [2], biological evolution [3] and extinction [4], forest fires [5], and many other systems [6]. The common features of these models are that (i) they are all driven very slowly, and (ii) they all have perfect memory, i.e., in the absence of the driving force the model would be entirely stationary.

A distinctive observable consequence of SOC dynamics is that the distribution of fluctuation (or avalanche) sizes takes a power-law form with characteristic exponent $\tau$:

$$p_{\text{aval}}(s) \propto s^{-\tau}. \quad (1)$$

The value of $\tau$ typically lies in the range $1 < \tau \leq \frac{3}{2}$, with the value $\frac{3}{2}$ corresponding to a critical branching process, appearing if one makes the “random neighbor approximation” in which each site interacts with a randomly-selected small number of other sites [7]. This approximation is equivalent to the limit of infinite dimension, and should give correct results for systems above their critical dimension. In reality however, a number of the systems which are modeled using SOC dynamics in fact display event size distributions with fairly large exponents. Terrestrial earthquakes, for example, appear to follow the Gutenberg–Richter law [9] with $\tau \approx 2.0$, and the 1D rice pile experiment of Frette and co-workers [10], which has been compared to the sandpile model of Bak, Tang and Wiesenfeld, gives $\tau = 2.1 \pm 0.1$. 

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A number of models have been proposed which offer explanations for these higher values of \( \tau \). The model discussed by Christensen and Olami [2] is one such, but it achieves its result only at the price of having an entirely deterministic dynamics; if one introduces randomness into the model, the simple scaling behavior is destroyed. The reason is that when the exponent describing the distribution of avalanches’ spatial extent becomes larger than 2, the mean avalanche size becomes finite and independent of system size, and the spatial overlap between subsequent avalanches becomes insignificant [3]. In the presence of randomness, this can prevent the system from building up any long-range correlations, and ultimately destroy the critical state. We conjecture that, in this regime, any randomness in the positions of the nucleation centers of the avalanches will destroy self-organization of long range spatial correlations.

In this paper, we present a different explanation to account for systems that have larger values of \( \tau \). We demonstrate that power-law event size distributions having \( \tau \) around 2 or greater, are typical of extended systems with quenched memory if they are driven by coherent noise, and that in such systems they are present even in the absence of any interaction between the different parts of the system. (This is different from the situation in the SOC models, where the system is driven by a local driving force, coupled with interactions between the components of the system.) The simplest model demonstrating the phenomenon is defined as follows. Consider a system of \( N \) agents, such as grains on the surface of a sand pile or points of contact in a subterranean fault. With each agent \( i \) we associate a threshold for movement \( x_i \) which can take values falling in some specified range and represents the amount of stress that the agent will withstand before it moves. For convenience, we choose to measure \( x_i \) on a scale on which \( 0 \leq x_i < 1 \). The dynamics of the model then consists of the repetition of two steps:

1. A fixed fraction \( f \) of the agents are selected at random, and the values of their threshold variables \( x_i \) are exchanged with new random numbers selected uniformly from the interval \([0, 1]\).
2. We select a random number or “stress level” $\eta$ from some distribution $p_{\text{stress}}(\eta)$. All $x_i$ below $\eta$ are also exchanged with new random numbers selected uniformly from the interval $[0,1)$. The number of agents whose thresholds are changed in this fashion is the size $s$ of the avalanche taking place in this time-step.

The random selection of different values for $\eta$ at each step may be thought of as imposing external stresses which coherently (in other words, simultaneously) influence all of the weaker agents—those having suitably low thresholds for stress—but leaves unchanged the stronger ones. It seems physically reasonable to assume that smaller stresses should be more common than larger ones, and in the following discussion we make the assumption that $p_{\text{stress}}(\eta)$ is largest at $\eta = 0$ and falls off to zero as $\eta$ becomes large. We denote the typical scale of the falloff by $\sigma$. The most interesting regime is when $\sigma \ll 1$ and $f \ll 1$.

II. RESULTS

We have examined the properties of this model both analytically and numerically. Instead of simulating the model directly, we have developed an algorithm which calculates the threshold distribution and avalanche sizes in a formally exact way for a system with $N = \infty$. Starting off with a uniform distribution of thresholds, the system evolves towards a statistically stationary state. In this state we record the mean threshold distribution and the frequency distribution of avalanches. The results are shown in Figures 1 and 2 for a simulation using exponentially distributed stresses:

$$p_{\text{stress}}(\eta) = \frac{1}{\sigma} \exp(-\eta/\sigma).$$

(2)

As Figure 1 shows, the distribution of avalanche sizes $s$ is flat up to a certain point (whose position varies with $\sigma$ and $f$) and then falls off as a power law according to Eq. (1) with $\tau \approx 2.0$. This power-law behavior appears to be robust in the regime of small $f$ and $\sigma$. If, for example, instead of Eq. (2) we employ a Gaussian stress distribution then, although the average distribution of thresholds (Figure 2) changes radically, the power-law form of the
avalanche distribution remains. Notice, however, that the exponent $\tau$ changes slightly as the applied stresses are varied. For the Gaussian distribution, for example, we find $\tau = 2.2 \pm 0.1$, as opposed to $\tau = 1.9 \pm 0.1$ for the exponential. And for steeper distributions of stresses ($p(\eta) \propto e^{-(\eta/\sigma)^q}$ with $q \geq 4$) we find $\tau = 2.4$ or greater.

In order to investigate possible connections with spatially-organized models, we have also implemented our model on a lattice and at each time-step eliminated not only those agents whose thresholds for stress fall below the selected level, but also their neighbors. In all cases we observe a power-law distribution of avalanches with exponent in the vicinity of $\tau = 2$.

In order to understand the appearance of this power law, let us consider the time-averaged behavior of the model. The statistically stationary state arises as a competition between the two processes comprising the dynamics: the stresses which tend to remove lower thresholds from the distribution and thus shift the weight of the distribution to higher values of $x$, and the aging, which tends to move weight back down again. The result is that the average threshold distribution $p_{\text{thresh}}(x)$ is a highly nonhomogeneous, monotonic increasing function of $x$ which, for small $\sigma$, tends to have a plateau as $x$ approaches unity (see Figure 2). By balancing the two competing processes, we can calculate $p_{\text{thresh}}(x)$ and hence the avalanche distribution. For concreteness, we perform the calculation here for the exponentially distributed stresses of Eq. (2).

The probability of an agent possessing a threshold $x$ lying below the stress level $\eta$ at any given time-step (and hence of it moving during this time-step) is

$$P_{\text{move}}(x) = \int_x^{\infty} p_{\text{stress}}(\eta) \, d\eta = e^{-x/\sigma}.$$  \hfill (3)

The total time-averaged rate at which agents move in the interval between $x$ and $x + dx$ is then

$$P_{\text{move}}(x) p_{\text{thresh}}(x) \, dx + f p_{\text{thresh}}(x) \, dx = W \, dx$$  \hfill (4)

where the $x$-independent constant $W$ on the right-hand side is the time-averaged rate at which probability is added to $p_{\text{thresh}}$. Rearranging we have
\[ p_{\text{thresh}}(x) = \frac{W}{f + e^{-x/\sigma}}. \]  

The constant is easily fixed by requiring that \( p_{\text{thresh}}(x) \) integrate to unity, giving
\[ W = \frac{f}{\sigma} \left[ \log \frac{f e^{1/\sigma} + 1}{f + 1} \right]^{-1}. \]

For small \( f \) and \( \sigma \), \( p_{\text{thresh}}(x) \) rises exponentially from zero and then levels off in a plateau around \( x = -\sigma \log f \). Physically, this arises because agents possessing thresholds above this point are affected only by the aging process, which treats them all equally. Below this level, the stress process is important too, and it preferentially moves those with lower thresholds.

The avalanche size distribution is given by
\[ p_{\text{aval}}(s) = \int_0^\infty p(s|\eta) p_{\text{stress}}(\eta) \, d\eta. \]

The probability \( p(s|\eta) \) of getting an avalanche of a certain size given a certain stress level, depends on the distribution of thresholds, which will in general vary from one time-step to another. However, if we make the “time averaged approximation” (TAA) whereby one assumes that at each time-step the threshold distribution can be approximated by its time-averaged value, then \( p(s|\eta) = \delta(s(\eta) - s) \) where \( s(\eta) \) is just
\[ s(\eta) = \int_0^\eta p_{\text{thresh}}(x) \, dx. \]

The avalanche size distribution then becomes
\[ p_{\text{aval}}(s) = \int_0^\infty \delta(s(\eta) - s) p_{\text{stress}}(\eta) \, d\eta = \frac{p_{\text{stress}}(\eta(s))}{p_{\text{thresh}}(\eta(s))} \left( e^{-\eta(s)/\sigma} (f + e^{-\eta(s)/\sigma}) \right) \]

where we have used Eqs. (5) and (8). We can calculate the stress level \( \eta(s) \) corresponding to an avalanche of size \( s \) from the same two equations, which give
\[ s = \left[ \log \frac{1 + f e^{\eta/\sigma}}{1 + f} \right] / \left[ \log \frac{1 + f e^{1/\sigma}}{1 + f} \right] \approx \sigma \log (1 + f e^{\eta/\sigma}) - \sigma f \]
for $e^{-1/\sigma} \ll f \ll 1$ and $\sigma \ll 1$. We can now distinguish a number of different regimes. For small avalanches, such that $s \ll \sigma$, the logarithm on the right-hand side can be expanded giving $s + \sigma f \approx \sigma f e^{\eta/\sigma}$. Substituting into Eq. (9)

$$p_{\text{aval}}(s) \propto [s + \sigma f]^{-2} \quad \text{for } s \ll \sigma.$$  

(11)

This gives a flat avalanche distribution for small $s$ up to about $s = \sigma f$, and then a power-law distribution for larger $s$ with exponent $\tau = 2$. The approximation breaks down when $s \approx \sigma$, giving way to a regime in which $e^{\eta/\sigma} \sim e^s$, and hence the avalanche distribution falls off exponentially with $s$. The various regimes can clearly be seen in the numerical results presented in Figure 1, and the predicted cross-over points between them agree well with the theory.

When $f$ decreases below $e^{-1/\sigma}$, the approximations in Eq. (10) break down and instead it becomes valid to write $e^{\eta/\sigma} \approx 1 + se^{1/\sigma}$. In this regime the theory predicts a breakdown in the scaling, a phenomenon which is also seen in the simulations. Thus the reloading process, whose scale is set by $f$, must be small but necessarily non-zero if we are to see power-law behavior in the avalanche distribution. Notice however that at precisely $f = 0$ the theory predicts a return to $\tau = 2$ scaling, which is not seen in the simulations, implying that the TAA breaks down in this regime because the distribution $p(s|\eta)$ becomes too broad to be well approximated by a $\delta$-function.

The physical principle behind the appearance of a power-law distribution here is the interdependence of the avalanche and threshold distributions; the avalanche distribution is a function of the particular distribution of thresholds at any time, but the threshold distribution is itself produced by the action of the avalanches.

**III. CONNECTION WITH OTHER MODELS**

There are clear similarities between our model and the sand pile model, in which sites also possess a certain threshold stress that they will withstand without adjusting. Furthermore
we have a source term, the reloading or aging fraction $f$ of agents which at each time-step lose memory of their previously assigned thresholds. This source term is similar in effect to the addition of the single grains of sand in the sand pile models. There are however some important differences between our model and the SOC models. First, the stresses in our model are coherent, rather than localized as they are in the sandpile. Second, the agents are, at least in the simplest versions of the model, entirely non-interacting. In SOC models, it is the interactions which give rise to avalanches. In our model on the other hand the avalanches of simultaneously moving agents arise because all the agents feel the same externally imposed stresses. There is no causal connection between the events which comprise an avalanche; each agent moves independently of the others.

Unlike other model systems for large scale fluctuations, such as the Burridge-Knopoff (BK) model [11] and the recycled version of the Democratic Fiber Bundle Model (DFBM) [12], the model presented here does not make a clear distinction between small, finite-sized events, and large ones whose size scales like the size of the system. In the BK model, for instance, the spectrum of event sizes contains two separate parts, one composed of small events which scales as $s^{-2}$, and another composed of the big events, which occur quasiperiodically. The BK and DFBM models are not statistically stationary, by contrast with our model whose dynamics rapidly reaches a statistically stationary state. Models such as BK and DFBM also show “foreshock” events in which large avalanches are preceded by smaller ones. Our dynamics does not have foreshocks but does display aftershock events, a phenomenon which we discuss in greater detail in next section.

**IV. DISCUSSION**

Next, we would like to examine the potential relationship of our model to processes occurring in real physical systems. First we consider earthquakes. To begin with, we ignore spatial correlations and consider the variables $x_i$ to be thresholds for movement at various points along a fault. The coherent stress $\eta$ is provided by long-wavelength background noise
from some external source, such as other distant tremors, or movements in the deeper regions of the earth, and the reloading $f$ is due to slow plastic deformation from tectonic movements of the crust. As we have seen, these elements alone lead directly to a power-law distribution of earthquake sizes very close to the observed Gutenberg-Richter law, without the need to invoke interactions between neighboring parts of the fault. That is not to say that such interactions do not exist, only that they are not necessary to produce the observed power law. (Kagan [13] has presented evidence of a fractal pattern in the spatial distribution of earthquake activity, which is an indication that interactions are a feature of the dynamics. This however need not lead us to conclude that these local interactions are necessary for producing the observed size distribution of events.)

Another interesting feature of our model is that it shows clear aftershocks. The mechanism for these is straightforward. When a large avalanche takes place, a significant fraction of the thresholds in the system are replaced with new, uniformly distributed ones. Because of the monotonic increasing form of the threshold distribution, this has the effect of shifting the weight of the distribution downwards, increasing the fraction of agents with low threshold for movement. The result is that subsequent stresses on the system have a larger-than-normal effect, and we see an amplification of the usual level of “background” avalanches in the aftermath of a particularly large event. In Figure 3, we show a section of a time series of avalanches from one of our simulations, which clearly displays this aftershock effect. Notice that if we apply the argument iteratively, we would also expect to see sequences of “after-aftershocks” following each of the aftershocks, a behavior which is indeed evident in Figure 3. We have also measured the average probability of getting an event of significant size in the aftermath of another large one, and found that for small times its distribution goes approximately as $t^{-1}$ (Figure 4). A similar result is seen in the data from real earthquakes, and is commonly referred to as Omori’s law [14].

The $t^{-1}$ distribution can be understood as follows. A large avalanche will redistribute the thresholds of a large fraction of the agents in the system uniformly across the interval of allowed values ($0 \leq x < 1$ in this case). A subsequent stress of magnitude $\eta_1$ will remove
all those agents with $x < \eta_1$, and produce an aftershock extinction of a certain magnitude. In order to get another significant aftershock we now need a stress $\eta_2 > \eta_1$ in order to reach those agents which were not affected by the first aftershock. In general, if it took a time $t$ to get the first stress, then it will on average, take as long again to get another of the same magnitude, or an aggregate time of $2t$ until a stress of size $\eta_2$ comes along. Repeating the argument, it will take as long again, or a total time of $4t$ to get the third aftershock, and so forth. Given this exponential increase in the time intervals between these events, it is not hard to show that the histogram of aftershock events should have a $t^{-1}$ power-law form, regardless of the precise distribution of stresses applied to the system.

Note that our mechanism is by no means the only way to obtain aftershocks. An alternative mechanism has been proposed by Nakanishi [15] using a Burridge-Knopoff-like model in which relaxation processes are introduced by considering the geometry of stress redistribution following large quakes. As with the BK model, Nakanishi’s model has a quasiperiodic dynamics.

Second, let us compare our model with the results of recent studies of one-dimensional rice piles by Frette et al. [10]. In these studies the experimenters found a frequency distribution of avalanche sizes $s$ which was flat up to a certain fraction of the total size of the pile, and then fell off as a power of $s$ for larger avalanches according to Eq. (1), with a measured exponent of $\tau = 2.1 \pm 0.1$. A similar behavior is seen in the simulation results from our model (Figure 1) which also display a flat distribution of avalanches up to a certain fraction $\sim \sigma / f$ of the total system size, and then a power-law fall in avalanche frequency for larger sizes with exponent close to two. A possible interpretation of the experimental data then is that the dynamics of the rice pile is one of avalanches produced by the interplay of reloading with coherent stress. The reloading $f$ could arise as a result of newly added grains of rice, which tend to randomize the thresholds for grains on the surface, and the stresses might come from the tumbling of new grains as they are added to the pile. The plateau in the avalanche distribution for small sizes $s$ is then caused by rice grains which tumble past a number of sites before coming to rest, but have only enough energy to disturb the most
unstable of those sites, and the larger events which form the $\tau = 2$ power law are the result of occasional larger stresses in the tail of the distribution. The one-dimensional nature of the system ensures that all input disturbances propagate through a large portion of the system, and thus may be treated as coherent. In a two-dimensional system this would not be the case, and the pile might well show entirely different dynamics, either possessing a shallower power-law distribution $\tau < 2$, indicating perhaps that a true SOC dynamics is at work, or not possessing a power-law distribution at all, indicating that coherent driving forces are the only mechanism responsible for power laws in this system.

Our model also makes quantitative predictions about the scaling of the line between the two regimes in the avalanche distribution: the position of the line should go like $N\sigma f$, the factor of the system size $N$ appearing when we shift from measuring avalanches as fractions of the system size to measuring the total energy they release. Scaling of precisely this form with $N$ is indeed seen in the experiments. Frette et al. also mention that simple scaling disappears when the experiment is repeated with “rounder” rice. We can explain this result in terms of the narrower distribution of thresholds that round rice can support, which corresponds to larger values of both $\sigma$ and $f$.

The results of these experiments have also been modeled by Christensen using a SOC model with interacting elements. Clearly, there are aspects of the dynamics captured by their model which are missing from ours, particularly geometrical effects concerned with the spatial distributions of avalanches and the corresponding transport properties of rice in the pile. However, because the exponent $\tau$ is greater than 2, making $\langle s \rangle$ independent of system size, we can expect these properties to be independent of the largest avalanche events (though on the other hand, they should now depend strongly on the position and nature of the crossover between the two regimes of the avalanche distribution). We suggest that the reverse is also true, i.e., that the observed large avalanches could appear even in the absence of long range spatial correlations.

One characteristic which does seem to distinguish our model from the SOC alternatives is the existence of aftershock events. It might therefore might be profitable to investigate the
existence of aftershock avalanches in the experimental data, in order to make a quantitative
distinction between the two classes of dynamics.

Finally we would like to point out that models of the type introduced here do not
constrain $\tau$ to values close to 2. Although the values found with the simple version of the
model outlined in Section II all lie approximately in the range $1.8 < \tau < 2.4$, we have
investigated other variants on the model which produce values outside this range. One
particularly interesting version is one in which we allow for the possibility of there being
many different kinds of stress on an agent. We suppose that agent $i$ is subject to $M$
independent types of stress, and that it has a separate threshold for yielding to each one,
making $x_i$ an $M$-dimensional vector quantity. One then assumes that all $M$
components of $x_i$ are to be replaced with new values every time any one of the types of stress exceeds
the corresponding threshold value. In the limit $M = 1$, this model is clearly just the same
as the version discussed above, and for higher values of $M$ we continue to see a power-law
distribution of avalanche sizes, regardless of the nature of the applied stresses. However,
the exponent of the power law becomes steeper as the value of $M$ increases, and appears
to approach 3 as $M$ becomes large. (We have investigated the model numerically up to
$M = 50$.) It is interesting to note that stock market fluctuations show power-law fluctuation
distributions with exponents close to $\tau = 3$ \[18\]. One may speculate whether these so-called
“fat tails” in the distribution are the natural response to the action of external stresses on
the market (of which there are indeed many).

V. CONCLUSION

To summarize, we have demonstrated that coherent noise in large systems typically gives
rise to intermittent behavior with an “avalanche” type dynamics characterized by a power-
law distribution of avalanche sizes with exponent in the vicinity of $\tau = 2$. This value is similar
to that seen in a number of real systems, including rice piles and earthquakes, suggesting
that these systems may in fact be driven by external noise, rather than self-organizing under
the influence of short-range internal interactions. If one allows more elaborate types of stress on the system one can obtain power laws with exponents as high as $\tau = 3$.

We believe that the study of systems driven in this fashion by coherent external noise may offer new interpretations of intermittent dynamics in a variety of extended non-equilibrium systems in terms of a direct interplay between small scale structures and long wavelength fluctuations in the system. Such systems might include not only the ricepiles and earthquakes considered here, but possibly also extended chaotic systems such as economics [18] and turbulence [19].
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J. de Boer, B. Derrida, H. Flyvbjerg, A. D. Jackson and T. Wettig, Phys. Rev. Lett. 73, 906 (1994).

[8] In fact it could be argued that $\tau = 2$ is the upper “critical” value for 1D SOC models because the spatial organization on a length scale $L$ is due to avalanches of size larger than $L$. The probability for these to appear is of order $1/L^{\tau-1}$, and, since they affect $L$ sites their weighted contribution to ordering on length scale $L$ is $L/L^{\tau-1}$. This average organization on length scale $L$ should be compared with the disorganization caused by non-overlapping avalanches appearing with frequency $\propto 1-1/L^{\tau-1}$. Criticality demands that organization should be larger than disorganization for $L \to \infty$, implying that $\tau < 2$.

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FIG. 1. Simulation results for the frequency distribution of avalanches with exponentially distributed stresses (solid line) and Gaussian ones (dashed line), with $f = 10^{-3}$ and $\sigma = \frac{1}{20}$ in each case.

FIG. 2. Simulation results for the time-averaged distribution of thresholds $x$ with exponentially distributed stresses (solid line) and Gaussian ones (dashed line). As in Figure 1, $f = 10^{-3}$ and $\sigma = \frac{1}{20}$ in each case.
FIG. 3. Time series plot of avalanche sizes during a portion of a simulation showing clear aftershocks.

FIG. 4. Histogram of the time distribution of aftershocks following a major avalanche. The histogram follows a power law with an exponent close to one (Omori’s law).