Fidelity based unitary operation-induced quantum correlation for continuous-variable systems

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We propose a measure of nonclassical correlations $N_{GF}$ in terms of local Gaussian unitary operations based on fidelity for bipartite continuous-variable systems. This quantity is easier to be calculated or estimated and is a remedy for the local ancilla problem associated with the geometric measurement-induced nonlocality. A simple computation formula of $N_{GF}$ for any (1+1)-mode Gaussian states is presented and an estimation of $N_{GF}$ for any (n+m)-mode Gaussian states is given. For any (1 + 1)-mode Gaussian states, $N_{GF}$ does not increase after performing a local Gaussian channel on the unmeasured subsystem. Comparing $N_{GF}(\rho_{AB})$ with other quantum correlations such as Gaussian geometric discord for two-mode symmetric squeezed thermal states reveals that $N_{GF}$ is much better in detecting quantum correlations of Gaussian states.

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INTRODUCTION

The presence of correlations in bipartite quantum systems is one of the main features of quantum mechanics. The most important among such correlations is surely entanglement [1]. However, much attention has been devoted to the study and the characterization of quantum correlations that go beyond the paradigm of entanglement recently. Non-entangled quantum correlations also play important roles in various quantum communications and quantum computing tasks.

For the last two decades, various methods had been proposed to describe quantum correlations, such as quantum discord (QD) [2], geometric quantum discord [3–5], measurement-induced nonlocality (MIN) [6] and measurement-induced disturbance (MID) [7] for discrete-variable systems. For continuous-variable systems, Giorda, Paris [8] and Adesso, Datta [9] independently gave the definition of Gaussian QD for two-mode Gaussian states and discussed its properties. G. Adesso, D. Girolami in [10] proposed the concept of Gaussian geometric discord for Gaussian states. Measurement-induced disturbance of Gaussian states was studied in [11]. In [12], the MIN for Gaussian states was discussed. For other related results, see [13–19]...
and the references therein. Also, many efforts had been made to find simpler methods to quantify these correlations. However, it seems that this is a very difficult task, too. By now, for example, almost all known quantifications of various correlations for continuous-variable systems are difficult to evaluate and can only be calculated for \((1 + 1)\)-mode Gaussian states or some special states. Even for finite-dimensional cases, the authors in [20] proved that computing quantum discord is NP-hard. So it makes sense and is important to find more helpful quantifications of quantum correlations.

The purpose of this paper is to propose a correlation \(N_{\mathcal{F}} \) for bipartite Gaussian systems in terms of local Gaussian unitary operations based on fidelity. This correlation \(N_{\mathcal{F}} \) describes the same correlation as Gaussian geometric discord for Gaussian states but have some remarkable nice properties that the known quantifications are not possed: (1) \(N_{\mathcal{F}} \) is a quantum correlation without ancilla problem; (2) \(N_{\mathcal{F}} \) can be estimated easily for any \((n + m)\)-mode Gaussian states and calculated for any \((1 + 1)\)-mode Gaussian states; (3) \(N_{\mathcal{F}} \) is nonincreasing after performing local Gaussian operations on the unmeasured subsystem. Comparison \(N_{\mathcal{F}} \) with other quantum correlations for two-mode symmetric squeezed thermal states reveals that \(N_{\mathcal{F}} \) is better in detecting the nonclassicality in Gaussian states.

**GAUSSIAN STATES AND GAUSSIAN UNITARY OPERATIONS**

In this section we recall briefly some notions and notations concerning Gaussian states and Gaussian unitary operations. For arbitrary state \(\rho\) in a \(n\)-mode continuous-variable system with state space \(H\), its characteristic function \(\chi_{\rho}\) is defined as

\[
\chi_{\rho}(z) = \text{tr}(\rho W(z)),
\]

where \(z = (x_1, y_1, \cdots, x_n, y_n)^T \in \mathbb{R}^{2n}\), \(W(z) = \exp(iR^T z)\) is the Weyl displacement operator, \(R = (R_1, R_2, \cdots, R_{2n}) = (\hat{Q}_1, \hat{P}_1, \cdots, \hat{Q}_n, \hat{P}_n)\). As usual, \(\hat{Q}_i = (\hat{a}_i + \hat{a}_i^\dagger)/\sqrt{2}\) and \(\hat{P}_i = -i(\hat{a}_i - \hat{a}_i^\dagger)/\sqrt{2}\) \((i = 1, 2, \cdots, n)\) stand for respectively the position and momentum operators, where \(\hat{a}_i^\dagger\) and \(\hat{a}_i\) are the creation and annihilation operators in the \(i\)th mode satisfying the Canonical Commutation Relation (CCR)

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} I \quad \text{and} \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = [\hat{a}_i, \hat{a}_j] = 0, \quad i, j = 1, 2, \cdots, n.
\]

\(\rho\) is called a Gaussian state if \(\chi_{\rho}(z)\) is of the form

\[
\chi_{\rho}(z) = \exp[-\frac{1}{4}z^T \Gamma z + i\mathbf{d}^T z],
\]

where

\[
\mathbf{d} = (\langle \hat{R}_1 \rangle, \langle \hat{R}_2 \rangle, \cdots, \langle \hat{R}_{2n} \rangle)^T = (\text{tr}(\rho R_1), \text{tr}(\rho R_2), \cdots, \text{tr}(\rho R_{2n}))^T \in \mathbb{R}^{2n}
\]
is called the mean or the displacement vector of $\rho$ and $\Gamma = (\gamma_{kl}) \in M_{2n}(\mathbb{R})$ is the covariance matrix (CM) of $\rho$ defined by $\gamma_{kl} = \text{tr}[\rho(\Delta R_k \Delta R_l + \Delta R_l \Delta R_k)]$ with $\Delta R_k = \hat{R}_k - \langle \hat{R}_k \rangle$ ([21]). Note that $\Gamma$ is real symmetric and satisfies the condition $\Gamma + i\Delta \geq 0$, where $\Delta = \bigoplus_{i=1}^{n} \Delta_i$ with $\Delta_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for each $i$.

Here $M_k(\mathbb{R})$ stands for the algebra of all $k \times k$ matrices over the real field $\mathbb{R}$.

Now assume that $\rho_{AB}$ is an $(n + m)$-mode Gaussian state with state space $H = H_A \otimes H_B$. Then the CM $\Gamma$ of $\rho_{AB}$ can be written as

$$\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

where $A \in M_{2n}(\mathbb{R})$, $B \in M_{2m}(\mathbb{R})$ and $C \in M_{2n \times 2m}(\mathbb{R})$. Particularly, if $n = m = 1$, by means of local Gaussian unitary (symplectic at the CM level) operations, $\Gamma$ has a standard form:

$$\Gamma_0 = \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix},$$

with $A_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $B_0 = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$, $C_0 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$, $a, b \geq 1$ and $ab - 1 \geq c^2(d^2)$.

For any unitary operator $U$ acting on $H$, the unitary operation $\rho \mapsto U \rho U^\dagger$ is said to be Gaussian if it maps Gaussian states into Gaussian states, and such $U$ is called a Gaussian unitary operator. It is well-known that a unitary operator $U$ is Gaussian if and only if $U^\dagger RU = SR + m$, for some vector $m$ in $\mathbb{R}^{2n}$ and some $S \in \text{Sp}(2n, \mathbb{R})$, the symplectic group of all $2n \times 2n$ real matrices $S$ that satisfy

$$S \in \text{Sp}(2n, \mathbb{R}) \Leftrightarrow S\Delta S^T = \Delta.$$

Thus, every Gaussian unitary operator $U$ is determined by some affine symplectic map $(S, m)$ acting on the phase space, and can be denoted by $U = U_{S,m}$ ([22, 23]).

We list some simple facts for Gaussian states and Gaussian unitary operations, and some useful results for matrix theory, which will be used frequently in the present paper.

**Lemma 1.** ([22]) *For any $(n + m)$-mode Gaussian state $\rho_{AB}$, write its CM $\Gamma$ as in Eq.(1). Then the CMs of the reduced states $\rho_A = \text{tr}_B \rho_{AB}$ and $\rho_B = \text{tr}_A \rho_{AB}$ are matrices $A$ and $B$, respectively.*

Denote by $S(H)$ the set of all quantum states of the system with state space $H$. 


Lemma 2. ([24]) Assume that $\rho_{AB} \in S(H_A \otimes H_B)$ is a $(n + m)$-mode Gaussian state. Then $\rho_{AB}$ is a product state, that is, $\rho_{AB} = \sigma_A \otimes \sigma_B$ for some $\sigma_A \in S(H_A)$ and $\sigma_B \in S(H_B)$, if and only if $\Gamma = \Gamma_A \oplus \Gamma_B$, where $\Gamma$, $\Gamma_A$ and $\Gamma_B$ are the CMs of $\rho_{AB}$, $\sigma_A$ and $\sigma_B$, respectively.

Lemma 3. ([22, 23]) Assume that $\rho$ is any $n$-mode Gaussian state with CM $\Gamma$ and displacement vector $d$, and assume that $U_{S,m}$ is a Gaussian unitary operator. Then the characteristic function of the Gaussian state $\sigma = U \rho U^\dagger$ is of the form $\exp(-\frac{1}{4}z^T \Gamma \sigma z + i d^T \sigma z)$, where $\Gamma = S \Gamma S^T$ and $d_\sigma = m + S d$.

Lemma 4. ([25]) For any quantum states $\rho$, $\sigma$ and any numbers $a > 1$, we have

$$\text{tr}(\rho \sigma) \leq (\text{tr} \rho^a)^{\frac{1}{a}} (\text{tr} \sigma^b)^{\frac{1}{b}},$$

where $b = \frac{a}{a-1}$.

Lemma 5. ([26]) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a square matrix.

(1) If $A$ is invertible, then its determinant $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (\det A)(\det(D - CA^{-1}B)).$

(2) If $D$ is invertible, then its determinant $\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (\det D)(\det(A - BD^{-1}C)).$

FIDELITY BASED NONCLASSICALITY OF GAUSSIAN STATES BY GAUSSIAN UNITARY OPERATIONS

Fidelity is a measure of closeness between two arbitrary states $\rho$ and $\sigma$, defined as $F(\rho, \sigma) = (\text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2$[27]. This measure has been explored in various context of quantum information processing such as cloning [28], teleportation [29], quantum states tomography [30], quantum chaos [31] and spotlighting phase transition in physical systems [32]. Though fidelity itself is not a metric, one can define a metric $D(\rho, \sigma) = g(F(\rho, \sigma))$, where $g$ is a monotonically decreasing function of distance measure. A few such fidelity induced metrics we mentioned here are Bures angle $A(\rho, \sigma) = \arccos \sqrt{F(\rho, \sigma)}$, Bures metric $B(\rho, \sigma) = (2 - 2\sqrt{F(\rho, \sigma)})^{\frac{1}{2}}$ and sine metric $C(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ [33].

Since the computation of fidelity involves square root of density matrix, various forms of fidelity had been proposed to simplify the computation. In [34], the authors gave another form as

$$F(\rho, \sigma) = \frac{(\text{tr} \rho \sigma)^2}{\text{tr} \rho^2 \text{tr} \sigma^2}, \quad (3)$$

In [35], to capture global nonlocal effect of a quantum state due to locally invariant projective measurements, the authors used the fidelity in Eq.(3) to define a metric $C(\rho, \sigma) = \sqrt{1 - F(\rho, \sigma)}$ for any states $\rho$ and $\sigma$;
furthermore, for any finite-dimensional bipartite quantum state $\rho_{AB}$, a new kind of MIN in terms of this metric was defined as

$$N_F(\rho_{AB}) = \max_{\Pi^A} C^2(\rho_{AB}, \Pi^A(\rho_{AB})),\]$$

where the maximum is taken over all von Neumann measurements performing on subsystem A that are invariant at $\rho_A = \text{tr}_B(\rho_{AB})$, the reduced state of $\rho_{AB}$. They presented an analytic expression of this version of MIN for pure bipartite states and $2 \times n$ dimensional mixed states.

In the present paper, we define a quantum correlation $N^G_F$ for continuous-variable systems by local unitary operations for $(n+m)$-mode states using the same metric based on the fidelity Eq.(3).

Definition 1. For any $(n+m)$-mode state $\rho_{AB} \in \mathcal{S}(H_A \otimes H_B)$, the quantity $N^G_F(\rho_{AB})$ is defined by

$$N^G_F(\rho_{AB}) = \sup_U C^2(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I)) = \sup_U \{ 1 - \frac{\text{tr}(U^\dagger \otimes I)\rho_{AB}(U \otimes I)^2}{\text{tr}(\rho_{AB}^2)\text{tr}((U \otimes I)\rho_{AB}(U^\dagger \otimes I)^2)} \},$$

where the supremum is taken over all Gaussian unitary operators $U \in \mathcal{B}(H_A)$ satisfying $U \rho_A U^\dagger = \rho_A$.

Remark 1. For any Gaussian state $\rho_{AB}$, there are many nontrivial Gaussian unitary operators $U$ (other than the identity $I$) satisfying $U \rho_A U^\dagger = \rho_A$ [16], and hence Definition 1 makes sense. Different from [16], in which a quantum correlation $N$ is proposed by Gaussian unitary operations based on the Hilbert-Schmidt norm, the quantity $N^G_F(\rho_{AB})$ measures the global nonlocal effect of a quantum state due to locally invariant Gaussian unitary operations by the metric $C^2(\rho, \sigma) = 1 - \mathcal{F}(\rho, \sigma)$ with the fidelity $\mathcal{F}$ as in Eq.(3).

Recall that, the MIN [6] is defined as the square of Hilbert-Schmidt norm $\| \cdot \|_2 (\|A\|_2 = \sqrt{\text{tr}(A^\dagger A)})$ of difference of pre- and post-measurement states. i.e.,

$$N(\rho_{AB}) = \max_{\Pi^A} \| \rho_{AB} - (\Pi^A \otimes I)\rho_{AB}(\Pi^A \otimes I)^\dagger \|_2^2,$$

where the maximum is taken over all von Neumann measurements which maintain the reduced state $\rho_A$ invariant corresponding to part A. In [16], a kind of quantum correlation $N$ for $(n+m)$-mode continuous-variable systems is defined as the square of Hilbert-Schmidt norm of difference of pre- and post-transform states

$$N(\rho_{AB}) = \frac{1}{2} \sup_U \| \rho_{AB} - (U \otimes I)\rho_{AB}(U \otimes I)^\dagger \|_2^2,$$

where the supremum is taken over all unitary operators which maintain $\rho_A$ invariant corresponding to part A. There are other quantum correlations defined by Hilbert-Schmidt norm, for example, the quantum correlations proposed in [10, 13]. These kinds of quantity defined by Hilbert-Schmidt norm mentioned above
may change rather wildly through some trivial and uncorrelated actions on the unoperated party B. For example, if we append an uncorrelated ancilla C, and regarding the state $\rho_{ABC} = \rho_{AB} \otimes \rho_C$ as a bipartite state with the partition AB:C; after some straight calculations, one can easily get that

$$\mathcal{N}(\rho_{ABC}) = \mathcal{N}(\rho_{AB}) \text{tr} \rho_C^2,$$

which means that the quantity $\mathcal{N}$ differs arbitrarily due to local ancilla C as long as $\rho_C$ is mixed. While this problem can be avoided if one employs $\mathcal{N}^G_F$ as in Definition 1 since

$$\mathcal{F}(\rho_{ABC}, (U \otimes I \otimes I)\rho_{ABC}) = \mathcal{F}(\rho_{AB} \otimes \rho_C, (U \otimes I)\rho_{AB} \otimes I\rho_C) = \mathcal{F}(\rho_{AB}, (U \otimes I)\rho_{AB}) \cdot \mathcal{F}(\rho_C, \rho_C) = \mathcal{F}(\rho_{AB}, (U \otimes I)\rho_{AB}),$$

according to the multiplicativity of the fidelity [34]. Thus, we reach the following conclusion.

**Theorem 1.** $\mathcal{N}^G_F$ is a quantum correlation without ancilla problem.

We explore further the properties of $\mathcal{N}^G_F$ below.

**Theorem 2.** $\mathcal{N}^G_F$ is locally Gaussian unitary invariant, that is, for any $(n + m)$-mode Gaussian state $\rho_{AB} \in S(H_A \otimes H_B)$ and any Gaussian unitary operators $W \in B(H_A)$ and $V \in B(H_B)$, we have $\mathcal{N}^G_F((W \otimes V)\rho_{AB}(W^\dagger \otimes V^\dagger)) = \mathcal{N}^G_F(\rho_{AB})$.

**Proof.** Assume that $\rho_{AB} \in S(H_A \otimes H_B)$ is any $(n + m)$-mode Gaussian state. For given Gaussian unitary operators $W \in B(H_A)$ and $V \in B(H_B)$, let $\sigma_{AB} = (W \otimes V)\rho_{AB}(W^\dagger \otimes V^\dagger)$. Let $\mathcal{U}^G(H_A)$ be the set of all Gaussian unitary operators acting on $H_A$. Since

$$\mathcal{N}^G_F(\rho_{AB}) = \sup_{U \in \mathcal{U}^G(H_A), U\rho_A U^\dagger = \rho_A} \mathcal{C}^2(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I)) = \sup_{U \in \mathcal{U}^G(H_A), U\rho_A U^\dagger = \rho_A} \{1 - \mathcal{F}(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I))\} = 1 - \inf_{U \in \mathcal{U}^G(H_A), U\rho_A U^\dagger = \rho_A} \mathcal{F}(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I)),$$

to demonstrate that $\mathcal{N}^G_F$ is locally Gaussian unitary invariant, it suffices to prove

$$\inf_{U \in \mathcal{U}^G(H_A), U\rho_A U^\dagger = \rho_A} \mathcal{F}(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I)) = \inf_{U' \in \mathcal{U}^G(H_A), U'\sigma_{AB} U'^\dagger = \sigma_{AB}} \mathcal{F}(\sigma_{AB}, (U' \otimes I)\sigma_{AB}(U'^\dagger \otimes I)),$$

where $\sigma_{AB} = (W \otimes V)\rho_{AB}(W^\dagger \otimes V^\dagger)$, $W$ and $V$ are given Gaussian unitary operators acting on Hilbert spaces $H_A$ and $H_B$, respectively.
Note that \( \sigma_A = W \rho_A W^\dagger \). For any Gaussian unitary operator \( U \in \mathcal{B}(H_A) \) satisfying \( U \rho_A U^\dagger = \rho_A \), let \( U' = WUW^\dagger \). Then \( U' \) is also a Gaussian unitary operator and satisfies \( U' \sigma_A U'^\dagger = WUW^\dagger W \rho_A W^\dagger W U'^\dagger W^\dagger = \sigma_A \). Conversely, if \( U' \sigma_A U'^\dagger = \sigma_A \), \( U = WU'W \) will satisfy \( U \rho_A U^\dagger = \rho_A \).

By Eq. (3), we have
\[
\mathcal{F}(\rho_{AB},(U \otimes I)\rho_{AB}(U^\dagger \otimes I)) = \frac{(\text{tr} \rho_{AB}(U \otimes I)\rho_{AB}(U^\dagger \otimes I))^2}{\text{tr} \rho_{AB}^2 \text{tr}((U \otimes I)\rho_{AB}(U^\dagger \otimes I))^2}
\]
\[
= \frac{(\text{tr} (W \otimes V^\dagger)\sigma_{AB}(W \otimes V)(U \otimes I)(W \otimes V^\dagger)\sigma_{AB}(W \otimes V)(U^\dagger \otimes I))^2}{\text{tr}((W \otimes V^\dagger)\sigma_{AB}(W \otimes V)(U \otimes I)(W \otimes V^\dagger)\sigma_{AB}(W \otimes V)(U^\dagger \otimes I))^2}
\]
\[
= \frac{(\text{tr} \sigma_{AB}(U' \otimes I)\sigma_{AB}(U'^\dagger \otimes I))^2}{\text{tr} \sigma_{AB}^2 \text{tr}((U' \otimes I)\sigma_{AB}(U'^\dagger \otimes I))^2} = \mathcal{F}(\sigma_{AB},(U' \otimes I)\sigma_{AB}(U'^\dagger \otimes I)).
\]

Therefore, Eq. (5) holds, as desired. The proof is completed. \(\square\)

Notice that, for any \((n + m)\)-mode product quantum state \(\rho_{AB}\), one must have \(N_{\mathcal{F}}^G(\rho_{AB}) = 0\) by the definition. But for Gaussian states, the converse is also true. Hence, when restricted to Gaussian states, the correlation \(N_{\mathcal{F}}^G\) describes the same nonclassicality as that described by Gaussian QD (two-mode) \([8, 9]\), Gaussian geometric discord \([10]\), the correlations \(Q, Q_T\) discussed in \([13]\) and the correlation \(N\) discussed in \([16]\).

**Theorem 3.** For any \((n + m)\)-mode Gaussian state \(\rho_{AB} \in \mathcal{S}(H_A \otimes H_B)\), \(N_{\mathcal{F}}^G(\rho_{AB}) = 0 \) if and only if \( \rho_{AB} \) is a product state.

**Proof.** By Definition 1, the “if” part is apparent. Let us check the “only if” part. Since the mean of any Gaussian state can be transformed to zero under some local Gaussian unitary operation, by Theorem 2, it is sufficient to consider the Gaussian states whose mean are zero.

Assume that \(\rho_{AB}\) is a \((n + m)\)-mode Gaussian state with CM \( \Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \) as in Eq. (1) and zero mean such that \(N_{\mathcal{F}}^G(\rho_{AB}) = 0\). By Lemma 1, the CM of \(\rho_A\) is \(A\). According to the Williamson Theorem, there exists a symplectic matrix \(S_0\) such that \(S_0 A S_0^T = \bigoplus_{i=1}^n v_i I\) and \(U_0 \rho_A U_0^\dagger = \bigoplus_{i=1}^n \rho_i\), where \(U_0 = U_{S_0,0}\) and \(\rho_i\)’s are some thermal states. Write \(\sigma_{AB} = (U_0 \otimes I)\rho_{AB}(U_0^\dagger \otimes I)\). It follows from Theorem 2 that \(N_{\mathcal{F}}^G(\sigma_{AB}) = N_{\mathcal{F}}^G(\rho_{AB}) = 0\). Obviously, \(\sigma_{AB}\) has the CM
\[
\Gamma' = \begin{pmatrix} \bigoplus_{i=1}^n v_i I & C' \\ C'^T & B' \end{pmatrix}
\]
and the mean 0.

By Lemma 3 and \([16]\), for any Gaussian unitary operator \(U_{S,m} \in \mathcal{B}(H_A)\) so that \(m = 0\) and \(S = \bigoplus_{i=1}^n S_{\theta_i}\) with
\[
S_{\theta_i} = \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}
\]
for some $\theta_i \in [0, \frac{\pi}{2}]$, we have $U_{S,m}\sigma_AU_{S,m}^\dagger = \sigma_A = \text{tr}_B(\sigma_{AB})$. Then, by the definition Eq.(4), $N_F^G(\sigma_{AB}) = 0$ entails

$$(\text{tr}_A\sigma_{AB}(U_{S,m} \otimes I)\sigma_{AB}(U_{S,m}^\dagger \otimes I))^2 = \text{tr}_A^2\text{tr}_B((U_{S,m} \otimes I)\sigma_{AB}(U_{S,m}^\dagger \otimes I))^2.$$ 

Since the Holder’s inequality (Lemma 4) asserts that $\text{tr}(\rho \sigma)^2 \leq \text{tr}\rho^2 \text{tr}\sigma^2$ and clearly, the equality holds if and only if $\sigma = \rho$, we must have

$$\sigma_{AB} = (U_{S,m} \otimes I)\sigma_{AB}(U_{S,m}^\dagger \otimes I).$$

Hence $\sigma_{AB}$ and $(U_{S,m} \otimes I)\sigma_{AB}(U_{S,m}^\dagger \otimes I)$ have the same CMs, that is,

$$
\begin{pmatrix}
\oplus_{i=1}^n v_i I & C' \\
C'^T & B'
\end{pmatrix} = 
\begin{pmatrix}
\oplus_{i=1}^n v_i I & SC'
\end{pmatrix} = 
\begin{pmatrix}
\oplus_{i=1}^n v_i I & SC'^T
\end{pmatrix}.
$$

If we take $\theta_i \in (0, \frac{\pi}{2})$ for each $i$, then $I - S$ is an invertible matrix, which forces $C' = 0$. So $\sigma_{AB}$ is a product state by Lemma 2. It follows that $\rho_{AB} = (U_0^\dagger \otimes I)\sigma_{AB}(U_0 \otimes I)$ is also a product state. \qed

In the rest of this paper, we mainly consider the case when the states $\rho_{AB}$ are Gaussian.

Now we turn to the question how to calculate $N_F^G$. For any two-mode Gaussian state $\rho_{AB}$, we can give an analytic computation formula.

**Theorem 4.** For any $(1 + 1)$-mode Gaussian state $\rho_{AB}$ whose CM has the standard form

$$\Gamma_0 = \begin{pmatrix}
A_0 & C_0 \\
C_0^T & B_0
\end{pmatrix},$$

we have

$$N_F^G(\rho_{AB}) = 1 - \frac{(ab - c^2)(ab - d^2)}{(ab - c^2/2)(ab - d^2/2)}.$$ 

Particularly, the value of $N_F^G(\rho_{AB})$ is independent of the mean of states.

**Proof.** For any $(1 + 1)$-mode Gaussian state $\rho_{AB}$ with CM $\Gamma'$ and mean $(d'_A, d'_B)$, we can always find two Gaussian operators $U$ and $V$ so that the CM $\Gamma_0$ of $\sigma_{AB} = (U \otimes V)\rho_{AB}(U^\dagger \otimes V^\dagger)$ is of the standard form

$$\Gamma_0 = \begin{pmatrix}
A_0 & C_0 \\
C_0^T & B_0
\end{pmatrix} = \begin{pmatrix}
a & 0 & c & 0 \\
0 & a & 0 & d \\
c & 0 & b & 0 \\
0 & d & 0 & b
\end{pmatrix}. $$

Denote the mean of $\sigma_{AB}$ by $(d_A, d_B)$. Since $N^G_F$ is locally Gaussian unitary invariant, one has $N^G_F(\rho_{AB}) = N^G_F(\sigma_{AB})$. Hence, we may assume that the CM of $\rho_{AB}$ is $\Gamma_0$ and the mean of $\rho_{AB}$ is $(d_A, d_B)$.

For any Gaussian unitary operator $U_{S,m}$ such that $U_{S,m}\rho_{A}U_{S,m}^\dagger = \rho_A$, we see that $S$ and $m$ meet the conditions $SA_0S^T = A_0$ and $Sd_A + m = d_A$. As $A_0 = aI_2$, we have $SS^T = I_2$. It follows from $S\Delta S^T = \Delta$ that there exists some $\theta \in [0, \frac{\pi}{2}]$ such that $S = S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. So the CM of Gaussian state $(U_{s,m} \otimes I)\rho_{AB}(U_{s,m}^\dagger \otimes I)$ is

$$\Gamma_\theta = \begin{pmatrix} a & 0 & c\cos\theta & d\sin\theta \\ 0 & a & -c\sin\theta & d\cos\theta \\ c\cos\theta & -c\sin\theta & b & 0 \\ d\sin\theta & d\cos\theta & 0 & b \end{pmatrix},$$

and the mean of $(U_{s,m} \otimes I)\rho_{AB}(U_{s,m}^\dagger \otimes I)$ is

$$(S \oplus I)(d_A \oplus d_B) + m \oplus 0 = (Sd_A + m) \oplus d_B = d_A \oplus d_B = (d_A, d_B)$$
as $Sd_A + m = d_A$. Conversely, for any $S_\theta$, taking $m = d_A - S_\theta d_A$, we have $U_{S_\theta,m}$ satisfies the condition $U_{S_\theta,m}\rho_{A}U_{S_\theta,m}^\dagger = \rho_A$.

Also, notice that, for any $n$-mode Gaussian states $\rho, \sigma$ with CMs $V_\rho, V_\sigma$ and means $d_\rho, d_\sigma$, respectively, it is shown in [36] that

$$\text{Tr}(\rho\sigma) = \frac{1}{\sqrt{\det[(V_\rho + V_\sigma)/2]}} \exp[-\frac{1}{2} \delta(d)^T \det[(V_\rho + V_\sigma)/2]^{-1} \delta(d)],$$

(6)

where $\delta(d) = d_\rho - d_\sigma$.

Hence, by Eq.(4) and Eq.(6) as well as the fact that $\text{det} \Gamma_\theta = \text{det} \Gamma_0 = (ab - c^2)(ab - d^2)$, one obtains

$$N^G_F(\rho_{AB}) = \sup_{U \in \mathcal{U}(H_A), \ U\rho_{A}U^\dagger = \rho_{A}} \text{C}^2(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I))$$

$$= \sup_{U \in \mathcal{U}(H_A), \ U\rho_{A}U^\dagger = \rho_{A}} \left\{ 1 - \frac{(\text{tr} \rho_{AB}(U \otimes I)\rho_{AB}(U^\dagger \otimes I))^2}{\text{tr}(\rho_{AB}^2)\text{tr}((U \otimes I)\rho_{AB}(U^\dagger \otimes I))^2} \right\}$$

$$= \sup_{\theta \in [0, \frac{\pi}{2}]} \left\{ 1 - \frac{\sqrt{\det \Gamma_0 \det \Gamma_\theta}}{\det((\Gamma_0 + \Gamma_\theta)/2)} \right\}$$

$$= \max_{\theta \in [0, \frac{\pi}{2}]} \left\{ 1 - \frac{(ab - c^2)(ab - d^2)}{[ab - c^2(1 + \cos \theta)/2][ab - d^2(1 + \cos \theta)/2]} \right\}$$

$$= 1 - \frac{(ab - c^2)(ab - d^2)}{(ab - c^2/2)(ab - d^2/2)}$$

and, this quantity is independent of the mean of $\rho_{AB}$, completing the proof.

Next, we are going to give an estimate of $N^G_F$ for any $(n + m)$-mode Gaussian state $\rho_{AB}$.
Theorem 5. For any \((n + m)\)-mode Gaussian state \(\rho_{AB}\) with CM \(\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}\), \(N^G_{\mathcal{F}}(\rho_{AB})\) is independent of the mean of \(\rho_{AB}\) and

\[
0 \leq N^G_{\mathcal{F}}(\rho_{AB}) \leq 1 - \frac{\text{det}(B - C^T A^{-1} C)}{\text{det} B} < 1.
\]

Furthermore, the upper bound 1 is tight.

Proof. Let \(\rho_{AB}\) be any \((n + m)\)-mode Gaussian state with CM \(\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}\) and mean \(d = (d_A, d_B)\). Note that, by Lemma 1, the CM of \(\rho_A\) is \(A\). Write \(\sigma_{AB} = (U_{S,m} \otimes I)\rho_{AB}(U^\dagger_{S,m} \otimes I)\), where \(U_{S,m}\) is any Gaussian unitary operator of the subsystem \(A\). Clearly, \(U_{S,m}\rho_A U^\dagger_{S,m} = \rho_A\) if and only if the symplectic matrix \(S\) satisfies \(S\sigma_{AB}^{\dagger} = A\) and the vector \(m = d_A - Sd_A\). In this case \(\sigma_{AB}\) has the CM

\[
\Gamma_{S} = \begin{pmatrix} A & SC \\ C^{T}S^{T} & B \end{pmatrix}
\]

and the mean \(d_S = (S \otimes I)(d_A \otimes d_B) + m \oplus 0 = (Sd_A + m) \oplus d_B = d_A \oplus d_B = (d_A, d_B) = d\).

Denote by \(S(2n) = \text{Sp}(2n, \mathbb{R})\), the set of all \(2n \times 2n\) symplectic matrices. Then, by Eq.(6),

\[
N^G_{\mathcal{F}}(\rho_{AB}) = \sup_{U \in U^d(H_A), U \rho_A U^\dagger = \rho_A} C^2(\rho_{AB}, (U \otimes I)\rho_{AB}(U^\dagger \otimes I))
\]

\[
= \sup_{U \in U^d(H_A), U \rho_A U^\dagger = \rho_A} \left\{ 1 - \frac{(\text{tr}(\rho_{AB}(U \otimes I)\rho_{AB}(U^\dagger \otimes I))^2}{\text{tr}(\rho_{AB}^2)} \right\}
\]

\[
= \sup_{S \in S(2n), S\sigma_{AB}^{\dagger} = A} \left\{ 1 - \frac{1}{\sqrt{\text{det} \Gamma} \sqrt{\text{det} \Gamma_S}} \right\}
\]

\[
= \sup_{S \in S(2n), S\sigma_{AB}^{\dagger} = A} \left\{ 1 - \frac{\sqrt{\text{det} \Gamma} \sqrt{\text{det} \Gamma_S}}{\sqrt{(\Gamma + \Gamma_S)/2}} \right\}
\]

That is,

\[
N^G_{\mathcal{F}}(\rho_{AB}) = \sup_{S \in S(2n), S\sigma_{AB}^{\dagger} = A} \left\{ 1 - \frac{\sqrt{\text{det} \Gamma} \sqrt{\text{det} \Gamma_S}}{\sqrt{(\Gamma + \Gamma_S)/2}} \right\}.
\] (7)

Obviously, \(N^G_{\mathcal{F}}(\rho_{AB})\) is independent of the mean \(d\).

It is easy to verify that \(\text{det} \Gamma = \text{det} \Gamma_S\). Since \(\Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}\) > 0, by Lemma 4, we have

\[
0 < \text{det} \Gamma = \text{det} A \text{det}(B - C^T A^{-1} C) = \text{det} \Gamma_S,
\]

which implies that \(\text{det}(B - C^T A^{-1} C) > 0\). In addition, as \(\frac{\Gamma + \Gamma_S}{2} = \begin{pmatrix} A & C + SC \\ C^T + C^T S^T & B \end{pmatrix} \) and \(A, B\) are positive-definite, by Fischer’s inequality ([26, pp.506]), we have \(\text{det} \frac{\Gamma + \Gamma_S}{2} \leq \text{det} A \text{det} B\). Hence, by
Eq.(7), we have
\[ 0 \leq N^G_F(\rho_{AB}) \leq 1 - \frac{\det A \det (B - C^T A^{-1} C)}{\det A \det B} = 1 - \frac{\det (B - C^T A^{-1} C)}{\det B} < 1. \]

We claim that the upper bound 1 is tight, that is, we have
\[ \sup_{\rho_{AB}} N^G_F(\rho_{AB}) = 1. \tag{8} \]

To see this, consider a two-mode squeezed vacuum state \( \rho(r) = S(r)|00\rangle\langle 00|S^\dagger(r) \), where \( S(r) = \exp(-r \hat{a}_1 \hat{a}_2 + r \hat{a}_1^\dagger \hat{a}_2^\dagger) \) is a two-mode squeezing operator with squeezed number \( r \geq 0 \) and \( |00\rangle \) is the vacuum state \((37)\). The CM of \( \rho(r) \) is \( \frac{1}{2} \begin{pmatrix} A_r & C_r \\ C^T_r & B_r \end{pmatrix} \), where \( A_r = B_r = \begin{pmatrix} \exp(-2r) + \exp(2r) & 0 \\ 0 & \exp(-2r) + \exp(2r) \end{pmatrix} \) and \( C_r = C^T_r = \begin{pmatrix} -\exp(-2r) + \exp(2r) & 0 \\ 0 & \exp(-2r) - \exp(2r) \end{pmatrix} \). By Theorem 4, it is easily checked that
\[ N^G_F(\rho(r)) = 1 - \frac{1}{\left(\frac{e^{-4r} + e^{4r}}{2} + 3\right)^2}. \]

Clearly, \( N^G_F(\rho(r)) \to 1 \) as \( r \to \infty \). So \( \sup_{\rho_{AB}} N^G_F(\rho_{AB}) = 1 \) and Eq.(8) is true.

Suppose that \( \rho_{AB} \) is a \((n + m)\)-mode Gaussian state with CM \( \Gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \) as in Eq.(1). One can always perform a local Gaussian unitary operation on the state \( \rho_{AB} \), say \( \sigma_{AB} = (U_{S_A} \otimes V_{S_B}) \rho_{AB} (U_{S_A}^\dagger \otimes V_{S_B}^\dagger) \), such that the corresponding CM of \( \sigma_{AB} \) is of the form \( \Gamma' = \begin{pmatrix} \oplus_i^m v_i I_2 & C' \\ C'^T & \oplus_i^{m'} s_i I_2 \end{pmatrix} \), where \( v_i \) and \( s_i \) are the symplectic roots of \( \rho_A \) and \( \rho_B \) respectively, \( C' = S_A C S_B^T \). By Theorem 2, \( N^G_F(\sigma_{AB}) = N^G_F(\rho_{AB}) \).

This gives an estimation of \( N^G_F(\rho_{AB}) \) for \((n + m)\)-mode Gaussian state \( \rho_{AB} \) in terms of symplectic roots of the CMs for \( \rho_A \) and \( \rho_B \):
\[ 0 \leq N^G_F(\rho_{AB}) \leq 1 - \frac{\det(\oplus_i^m s_i I_2 - S_B C'^T S_A^T(\oplus_i^n 1/v_i I_2) S_A C S_B^T)}{\prod_{i=1}^{m} s_i^2} < 1. \]

**NONLOCALITY CONNECTED TO GAUSSIAN CHANNELS**

In this section we intend to investigate the fidelity based nonlocality connected to a Gaussian quantum channel. Here we mainly consider the \((1 + 1)\)-mode Gaussian states whose CM are of the standard form.

Since a Gaussian state \( \rho \) is described by its CM \( \Gamma \) and displacement vector \( d \), we can denote it as \( \rho = \rho(\Gamma, d) \). Recall that a Gaussian channel is a quantum channel that transforms Gaussian states into
Gaussian states. Assume that $\Phi$ is a Gaussian channel of $n$-mode Gaussian systems. Then, there exist real matrices $M, K \in M_{2n}(\mathbb{R})$ satisfying $M = M^T \geq 0$ and $\det M \geq (\det K - 1)^2$, and a vector $\overline{d} \in \mathbb{R}^{2n}$, such that, for any $n$-mode Gaussian state $\rho = \rho(\Gamma, d)$, we have $\Phi(\rho(\Gamma, d)) = \rho(\Gamma', d')$ with

$$d' = Kd + \overline{d} \text{ and } \Gamma' = K\Gamma K^T + M.$$ 

So we can parameterize the Gaussian channel $\Phi$ as $\Phi = \Phi(K, M, \overline{d})$.

**Theorem 6.** Consider the $(1+1)$-mode continuous-variable system $AB$. Let $\Phi = \Phi(K, M, \overline{d})$ be a Gaussian channel performed on the subsystem $B$ with $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ and $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$. Assume that $\rho_{AB} \in \mathcal{S}(H_A \otimes H_B)$ is any $(1+1)$-mode Gaussian state with CM $\Gamma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix}$.

Then

$$N^G_{\overline{d}}((I \otimes \Phi)\rho_{AB}) = 1 - \frac{(ab - c^2)(ab - d^2)n_1 + a(ab - c^2)n_2 + a(ab - d^2)n_3 + a^2 n_4}{(ab - c^2/2)(ab - d^2/2)n_1 + a(ab - c^2/2)n_2 + a(ab - d^2/2)n_3 + a^2 n_4},$$

where $n_0 = (1 + \cos \theta)/2$, $n_1 = k_{11}^2 k_{22}^2 + k_{12}^2 k_{21}^2 - 2k_{11}k_{12}k_{21}k_{22}$, $n_2 = m_{22}k_{11}^2 + m_{11}k_{21}^2 - m_{12}k_{11}k_{21}$, $n_3 = m_{22}k_{12}^2 + m_{11}k_{22}^2 - 2m_{12}k_{12}k_{22}$ and $n_4 = m_{11}m_{22} - m_{12}^2$.

**Proof.** Suppose that the $(1 + 1)$-mode Gaussian state $\rho_{AB}$ has CM $\Gamma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix}$ and the mean $(d_A, d_B)$. Then the CM $\Gamma'$ and the mean $d'$ of $\sigma'_{AB} = (I \otimes \Phi)\rho_{AB}$ respectively are

$$\Gamma' = \begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & K^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} A_0 & C_0 K^T \\ K C_0^T & K B_0 K^T + M \end{pmatrix},$$

and

$$d' = (I \oplus K)(d_A \oplus d_B) + 0 \oplus \overline{d} = d_A \oplus (Kd_B + \overline{d}).$$

After a local invariant Gaussian unitary operation on the subsystem $A$, one has $(U \otimes I)\sigma'_{AB}(U^\dagger \otimes I) = \sigma'_{AB}$. Remind that $U \rho_A U^\dagger = \rho_A$, which forces that, in the symplectic transformation level, $U = U_{S, m}$ with $m = 0$ and $S = S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for some $\theta \in [0, \frac{\pi}{2}]$. Hence the CM and the mean of $\sigma'_{AB}$ are
respectively
\[
\Gamma'_S = \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_0 & C_0K^T \\ KC_0^T & KB_0K^T + M \end{pmatrix} \begin{pmatrix} S^T & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_0 & SC_0K^T \\ KC_0^T S^T & KB_0K^T + M \end{pmatrix}
\]
and
\[
d'_S = (S \oplus I)(d_A \oplus (Kd_B + \mathfrak{d})) + m \oplus 0 = (Sd_A + m) \oplus (Kd_B + \mathfrak{d}) = d_A \oplus (Kd_B + \mathfrak{d}).
\]
After some straightforward calculations, one can immediately get
\[
N^G_F((I \otimes \Phi)\rho_{AB}) = N^G_F((I \otimes \Phi)\rho_{AB}) = \sup_{U \in U^G(H_A), U \sigma_A U^\dagger = \sigma_A} \left\{ 1 - \frac{\sqrt{\det \Gamma' \det \Gamma'_S}}{\det((\Gamma' + \Gamma'_S)/2)} \right\}
\]
By the fact that \(\det \Gamma' = \det \Gamma'_S = \det A_0 \det(KB_0K^T + M - KC_0^T A_0^{-1}C_0K^T)\), the above formula can rewritten as the following
\[
N^G_F((I \otimes \Phi)\rho_{AB}) = \sup_{\theta \in [0, \frac{\pi}{2}]} \left\{ 1 - \frac{\det \begin{pmatrix} A_0 & C_0K^T \\ KC_0^T & KB_0K^T + M \end{pmatrix}}{\det \begin{pmatrix} A_0 & \frac{(I+S_\theta)C_0K^T}{2} \\ \frac{KC_0^T(I+S_\theta^T)}{2} & KB_0K^T + M \end{pmatrix}} \right\}
\]
Clearly, the quantity \(N^G_F((I \otimes \Phi)\rho_{AB})\) is independent of the parameter \(\mathfrak{d}\), and \(K, M\) can not be zero simultaneously. Substituting \(S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}\) into the above equation, after tedious calculations, one gets
\[
N^G_F((I \otimes \Phi)\rho_{AB}) = \sup_{\theta \in [0, \frac{\pi}{2}]} \left\{ 1 - \frac{\det(K(B_0 - C_0^T A_0^{-1}C_0)K^T + M)}{\det(K(B_0 - \frac{C_0^T(I+S_\theta^T)}{2} A_0^{-1}(I+S_\theta)C_0K^T)K^T + M)} \right\}
\]
where
\[
\begin{align*}
n_0 &= (1 + \cos \theta)/2, \\
n_1 &= k_{11}^2 k_{22}^2 + k_{12}^2 k_{21}^2 - 2k_{11}k_{12}k_{21}k_{22}, \\
n_3 &= m_{22}k_{12}^2 + m_{11}k_{12}^2 - 2m_{12}k_{11}k_{22}, \\
n_4 &= m_{11}m_{22} - m_{12}^2.
\end{align*}
\]

The proof is completed.

**Remark 2.** If \( K = 0 \), then \( \det M \geq 1 \), and we have
\[
N^G_F((I \otimes \Phi)\rho_{AB}) = 1 - \frac{\det M}{\det M} = 0.
\]

In fact, in this case, the Gaussian channel \( I \otimes \Phi(0, M, \overline{a}) \) maps any Gaussian state \( \rho_{AB} \) to a product state.

Thus, by Theorem 3, we always have \( N^G_F((I \otimes \Phi)\rho_{AB}) = 0 \).

**Remark 3.** If \( M = 0 \), then \( \det K = 1 = \det K^T \), and
\[
N^G_F((I \otimes \Phi)\rho_{AB}) = \sup_{\theta \in [0, \pi/2]} \left\{ 1 - \frac{\det(K(B_0 - C_0^T A_0^{-1} C_0)K^T)}{\det(K(B_0 - C_0^T (I+S\theta)^2/2 A_0^{-1} (I+S\theta)C_0)K^T)} \right\}
\]
\[
= \sup_{\theta \in [0, \pi/2]} \left\{ 1 - \frac{\det(B_0 - C_0^T A_0^{-1} C_0)}{\det(B_0 - C_0^T (I+S\theta)^2/2 A_0^{-1} (I+S\theta)C_0)} \right\}
\]
\[
= \sup_{\theta \in [0, \pi/2]} \left\{ 1 - \frac{\det \begin{pmatrix} A_0 & C_0 \\ C_0^T & B_0 \end{pmatrix}}{\det \begin{pmatrix} A_0 & (I+S\theta)C_0/2 \\ C_0^T (I+S\theta)^2/2 & B_0 \end{pmatrix}} \right\} = N^G_F(\rho_{AB}).
\]

In this case, one can conclude that, after performing the Gaussian operation \( I \otimes \Phi(K, 0, \overline{a}) \), the quantity \( N^G_F \) remains the same for those \((1+1)\)-mode Gaussian states whose CM are of the standard form.

The following result give a kind of local Gaussian operation non-increasing property of \( N^G_F \), which is not possessed by other known similar correlations such as the Gaussian QD (two-mode) [8, 9], Gaussian geometric discord [10], the correlations \( Q, Q_F \) discussed in [13] and the correlation \( \mathcal{N} \) discussed in [16].

**Theorem 7.** Let \( \rho_{AB} \) be a \((1+1)\)-mode Gaussian state. Then, for any Gaussian channel \( \Phi \) performed on the subsystem \( B \), we have
\[
0 \leq N^G_F((I \otimes \Phi)\rho_{AB}) \leq N^G_F(\rho_{AB}).
\]

**Proof.** We first consider the case that the \((1+1)\)-mode Gaussian state \( \rho_{AB} \) has CM \( \Gamma_0 \) of the standard form, that is, \( \Gamma_0 = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & d & 0 & b \end{pmatrix} \). Let \( \Phi = \Phi(K, M, \overline{a}) \) be any Gaussian channel performed on the part \( B \).
with \( K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \) and \( M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \). We have to show that \( N_F^G((I \otimes \Phi)\rho_{AB}) \leq N_F^G(\rho_{AB}) \).

If \( N_F^G(\rho_{AB}) = 0 \), then, by Theorem 3, \( \rho_{AB} \) is a product state. So \( (I \otimes \Phi)\rho_{AB} \) is a product state, and hence \( N_F^G((I \otimes \Phi)\rho_{AB}) = 0 = N_F^G(\rho_{AB}) \).

Assume that \( N_F^G(\rho_{AB}) \neq 0 \). Then \( N_F^G((I \otimes \Phi)\rho_{AB}) \leq N_F^G(\rho_{AB}) \) holds if and only if \( \frac{N_F^G((I \otimes \Phi)\rho_{AB})}{N_F^G(\rho_{AB})} \leq 1 \). Let \( \alpha = (ab - c^2)(ab - d^2), \beta = (ab - c^2/2)(ab - d^2/2), \gamma = a(ab - c^2)n_2 + a(ab - d^2)n_3 + a^2n_4 \) and \( \delta = a(ab - c^2/2)n_2 + a(ab - d^2/2)n_3 + a^2n_4 \) with \( n_2, n_3, n_4 \) as in Theorem 6. Then, according to Theorem 6, we have

\[
\frac{N_F^G((I \otimes \Phi)\rho_{AB})}{N_F^G(\rho_{AB})} \leq 1 \iff 1 - \frac{\alpha n_1 + \gamma}{\beta n_1 + \delta} \leq 1 \iff \frac{\alpha n_1 + \gamma}{\beta n_1 + \delta} \geq \frac{\alpha}{\beta} \iff \gamma \beta - \alpha \delta \geq 0.
\]

Therefore, it suffices to prove that \( \gamma \beta - \alpha \delta \geq 0 \). By some computations, one sees that

\[
\gamma \beta = [a(ab - c^2)n_2 + a(ab - d^2)n_3 + a^2n_4](ab - \frac{c^2}{2})(ab - \frac{d^2}{2}) = a(ab - c^2)(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_2 + a(ab - d^2)(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_3 + a^2(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_4,
\]

and

\[
\alpha \delta = a(ab - c^2)(ab - \frac{c^2}{2})(ab - d^2)n_2 + a(ab - d^2)(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_3 + a^2(ab - c^2)(ab - d^2)n_4.
\]

Note that \( n_1 = k_{11}^2k_{22}^2 + k_{12}^2k_{21}^2 - 2k_{11}k_{12}k_{21}k_{22} = (k_{11}k_{22} - k_{12}k_{21})^2 \geq 0 \) and \( n_4 = m_{11}m_{22} - m_{12}^2 \) det \( M \geq 0 \). Since \( m_{22}k_{21}^2 + m_{11}k_{21}^2 \geq 2\sqrt{m_{22}}\sqrt{m_{11}}k_{21}k_{22} \geq 2m_{12}k_{11}k_{21} \), we have \( n_2 \geq 0 \). One can verify \( n_3 \geq 0 \) by the same way. Also note that \( a, b \geq 1 \) and \( ab \geq c^2(d^2) \) by the constraint condition of the parameters in the definition of the Gaussian state. Now it is clear that

\[
\gamma \beta - \alpha \delta = a(ab - c^2)(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_2 + a(ab - d^2)(ab - \frac{c^2}{2})n_3 + a^2(ab - \frac{c^2}{2})(ab - \frac{d^2}{2})n_4 \geq 0,
\]
as desired. To this end, we come to the conclusion that \( N_F^G((I \otimes \Phi)\rho_{AB}) \leq N_F^G(\rho_{AB}) \), and the equality holds if \( M = 0 \) (See Remark 3 after the proof of Theorem 6).

Now let us consider the general case. Let \( U \otimes V \) be a local Gaussian unitary operation, that is, for some Gaussian unitary operators \( U \) and \( V \) on the subsystems A and B, respectively, so that \( (U \otimes V)(\rho_{AB}) = (U \otimes V)\rho_{AB}(U^\dagger \otimes V^\dagger) \) for each state \( \rho_{AB} \). Then,

\[
(I \otimes \Phi) \circ (U \otimes V) = U \otimes (\Phi \circ V) = (U \otimes I) \circ (I \otimes (\Phi \circ V)).
\]

Note that, \( \Phi \circ V \) is still a Gaussian channel which sends \( \rho_B \) to \( \Phi(V\rho_BV^\dagger) \). Keep this in mind and let \( \rho_{AB} \) be any \((1 + 1)\)-mode Gaussian state. Then there exists a local Gaussian unitary operation \( U \otimes V \) such that
\[ \sigma_{AB} = (U^\dagger \otimes V^\dagger) \rho_{AB} (U \otimes V) \] has CM of the standard form. By what we have proved above and Theorem 2, we see that

\[
N_G^G((I \otimes \Phi) \rho_{AB}) = N_G^G((I \otimes \Phi)((U \otimes V) \sigma_{AB}(U^\dagger \otimes V^\dagger)))
\]

\[
= N_G^G((I \otimes \Phi) \circ (U \otimes V) \sigma_{AB}) = N_G^G((U \otimes I) \circ (I \otimes (\Phi \circ V)) \sigma_{AB})
\]

\[
= N_G^G((I \otimes (\Phi \circ V)) \sigma_{AB}) \leq N_G^G(\sigma_{AB}) = N_G^G(\rho_{AB}),
\]

as desired, which completes the proof.

\[ \square \]

**COMPARISON** \(N_G^G\) **WITH OTHER QUANTIFICATIONS OF THE GAUSSIAN QUANTUM CORRELATIONS**

\(N_G^G\), \(D_G\) and \(Q\) describe the same quantum correlation when they are restricted to Gaussian states because they take value 0 at a Gaussian state \(\rho_{AB}\) if and only if \(\rho_{AB}\) is a product state. In this section, we calculate \(N_G^G(\rho_{AB})\) for all two-mode symmetric squeezed thermal states \(\rho_{AB}\) and compare it with Gaussian geometric discord \(D_G(\rho_{AB})\) and \(Q(\rho_{AB})\) in scale. Our result reveals that \(N_G^G\) is bigger and thus is better to detect the correlation in states. Since the known computation formula of \(D_G(\rho_{AB})\) is only for symmetric squeezed thermal states \(\rho_{AB}\), we compare them on such states.

**Symmetric squeezed thermal states:** Assume that \(\rho_{AB}\) is any two-mode Gaussian state; then its standard CM has the form as in Eq.(3). Recall that the symmetric squeezed thermal states (SSTSs) are Gaussian states whose CMs are parameterized by \(\bar{n}\) and \(\mu\) such that \(a = b = 1 + 2\bar{n}\) and \(c = d = 2\mu \sqrt{\bar{n}(1 + \bar{n})}\), where \(\bar{n}\) is the mean photon number for each part and \(\mu\) is the mixing parameter with \(0 \leq \mu \leq 1\) (ref. [38]). So every SSTS may be denoted by \(\rho_{AB}(\bar{n}, \mu)\).

Thus by Theorem 4, for any SSTS \(\rho_{AB}(\bar{n}, \mu)\), we have

\[
N_G^G(\rho_{AB}(\bar{n}, \mu)) = 1 - \frac{(1 + 2\bar{n})^2 - 4\mu^2 \bar{n}(1 + \bar{n})^2}{((1 + 2\bar{n})^2 - 2\mu^2 \bar{n}(1 + \bar{n}))^2}. \tag{9}
\]

For any two-mode Gaussian state \(\rho_{AB}\), recall that the Gaussian geometric discord of \(\rho_{AB}\) ([10]) was defined as

\[
D_G(\rho_{AB}) = \inf_{\Pi_A} \| \rho_{AB} - \Pi_A(\rho_{AB}) \|_2^2,
\]

where \(\Pi_A = \Pi_A(\alpha)\) runs over all Gaussian positive operator valued measurements of subsystem \(A\), \(\Pi_A(\rho_{AB}) = \int (\Pi_A(\alpha) \otimes I)^{1/2} \rho_{AB}(\Pi_A(\alpha) \otimes I)^{1/2} d^2\alpha\). According to the analytical formula of \(D_G(\rho_{AB})\) provided in [10], for any SSTS \(\rho_{AB}\) with parameters \(\bar{n}\) and \(\mu\), one has

\[
D_G(\rho_{AB}(\bar{n}, \mu)) = \frac{1}{(1 + 2\bar{n})^2 - 4\mu^2 \bar{n}(1 + \bar{n})} - \frac{9}{\sqrt{4(1 + 2\bar{n})^2 - 12\mu^2 \bar{n}(1 + \bar{n}) + (1 + 2\bar{n})^2}}. \tag{10}
\]
FIG. 1: \( z = N^G_F(\rho_{AB}) - D_G(\rho_{AB}) \) with SSTSs, and \( 0 \leq \mu \leq 1, 0 \leq \bar{n} \leq 50 \).

By Eqs. (9)-(10), it is clear that
\[
\lim_{\bar{n} \to \infty} N^G_F(\rho_{AB}(\bar{n}, \mu)) = 1 - \frac{(1 - \mu^2)^2}{(1 - \frac{1}{\tau^2})^2} > 0 \quad \text{for} \quad \mu \in (0, 1),
\]
while
\[
\lim_{\bar{n} \to \infty} D_G(\rho_{AB}(\bar{n}, \mu)) = 0 \quad \text{for} \quad \mu \in (0, 1).
\]

This shows that, for the case \( \mu \neq 0, 1 \), \( N^G_F \) is able to recognize well the quantum correlation in the states with large mean photon number but \( D_G \) is not. It is clear that \( \mu = 0 \) if and only if \( \rho_{AB} \) is a product SSTS, and in this case, \( N^G_F(\rho_{AB}(\bar{n}, 0)) = D_G(\rho_{AB}(\bar{n}, 0)) = 0 \). When \( \mu = 1 \), we have
\[
N^G_F(\rho_{AB}(\bar{n}, 1)) = 1 - \frac{1}{(1 + 2\bar{n} + 2\bar{n}^2)^2}
\]
and
\[
D_G(\rho_{AB}(\bar{n}, 1)) = 1 - \frac{9}{[1 + 2\bar{n} + 2\sqrt{1 + \bar{n} + \bar{n}^2}]^2},
\]
which reveals that we always have
\[
N^G_F(\rho_{AB}(\bar{n}, 1)) > D_G(\rho_{AB}(\bar{n}, 1)).
\]

In Fig. 1, we compare \( N^G_F(\rho_{AB}) \) with \( D_G(\rho_{AB}) \) for SSTSs \( \rho_{AB} \) by considering \( N^G_F(\rho_{AB}) - D_G(\rho_{AB}) \) for \( \bar{n} \leq 50 \). Fig. 1 shows that \( N^G_F(\rho_{AB}) - D_G(\rho_{AB}) \geq 0 \) and
\[
N^G_F(\rho_{AB}) > D_G(\rho_{AB})
\]
for SSTSs \( \rho_{AB} \) with bigger \( \bar{n} \) and bigger \( \mu < 1 \) near 1. For example, considering the state \( \rho_{AB} \) with \( \bar{n} = 49 \) and \( \mu = 0.9 \), we have \( D_G(\rho_{AB}) \approx 0.000356 \), which is very close to 0 and difficult to judge whether or not \( \rho_{AB} \) contains the correlation. However, \( N^G_F(\rho_{AB}) \approx 0.897995 > 0 \), which guarantees that \( \rho_{AB} \) does
FIG. 2: $z=\mathcal{N}_F^G(\rho_{AB}) - D_G(\rho_{AB})$ with SSTs, and $0 \leq \mu \leq 1$, $100000 \leq \bar{n} \leq 100500$.

contain the quantum correlation. For large mean photon number, for example, $\bar{n} = 10000$, taking $\mu = 0.9$, we have $\mathcal{N}_F^G(\rho_{AB}) \approx 0.89803 \gg 0$, but $D_G(\rho_{AB}) \approx 8.72518 \times 10^{-11} \approx 0$. Fig.2 gives a picture of $\mathcal{N}_F^G(\rho_{AB}) - D_G(\rho_{AB})$ for $\bar{n} \in (100000, 100500)$ and $\mu \in (0, 1)$, which also shows that $\mathcal{N}_F^G(\rho_{AB}) \geq D_G(\rho_{AB})$. Moreover, we randomly chose 100000 pairs of $(\bar{n}, \mu)$ with $\bar{n} \in (0, 10000000000000)$ and $\mu \in (0, 1)$, and the numerical results shows that $\mathcal{N}_F^G(\rho_{AB}(\bar{n}, \mu)) > D_G(\rho_{AB}(\bar{n}, \mu))$. All of these suggest that

$$\mathcal{N}_F^G(\rho_{AB}(\bar{n}, \mu)) > D_G(\rho_{AB}(\bar{n}, \mu))$$

for all SSTs $\rho_{AB}(\bar{n}, \mu)$ with $\mu \neq 0$. Hence $\mathcal{N}_F^G$ is better than $D_G$ in detecting the quantum correlation contained in any SST.

$Q$ is a quantum correlation for $(m + n)$-mode continuous-variable systems defined in terms of average distance between the reduced states under the local Gaussian positive operator valued measurements [13]:

$$Q(\rho_{AB}) := \sup_{\Pi^A} \int p(\alpha) \| \rho_B - \rho_B^{(\alpha)} \|^2_2 d^{2m} \alpha,$$

where $\Pi^A = \Pi^A(\alpha)$ runs over all Gaussian positive operator valued measurements of subsystem $A$, $\Pi^A = \{\Pi^A(\alpha)\}$ on the subsystem $H_A$, $\rho_B = \text{Tr}_A(\rho_{AB})$, $p(\alpha) = \text{Tr}[(\Pi^A(\alpha) \otimes I_B)\rho_{AB}]$ and $\rho_B^{(\alpha)} = \frac{1}{p(\alpha)} \text{Tr}_A[(\Pi^A(\alpha) \otimes I_B)^{\frac{1}{2}} \rho_{AB}(\Pi^A(\alpha) \otimes I_B)^{\frac{1}{2}}]$.

For any SST $\rho_{AB}$ with parameters $\bar{n}$ and $\mu$, by [13],

$$Q(\rho_{AB}(\bar{n}, \mu)) = \frac{1}{1 + 2\bar{n}(1 - \mu^2)} - \frac{1}{1 + 2\bar{n}}. \quad (11)$$

Obviously,

$$\lim_{\bar{n} \to \infty} Q(\rho_{AB}(\bar{n}, \mu)) = 0, \quad \text{for } \mu \in (0, 1),$$
FIG. 3: $z = N^G_F(\rho_{AB}) - Q(\rho_{AB})$ with SSTSs, and $0 \leq \mu \leq 1$, $0 \leq \bar{n} \leq 50$.

FIG. 4: $z = N^G_F(\rho_{AB}) - D_G(\rho_{AB})$ with SSTSs, and $0 \leq \mu \leq 1$, $100000 \leq \bar{n} \leq 100500$.

which reveals that $Q$ is not valid for those SSTSs with $\mu \in (0, 1)$ and large mean photon number. For the case $\mu = 1$, we have

$$Q(\rho_{AB}(\bar{n}, 1)) = 1 - \frac{1}{1 + 2\bar{n}} < N^G_F(\rho_{AB}(\bar{n}, 1))$$

for any $\bar{n}$.

The difference of $N^G_F(\rho_{AB})$ and $Q(\rho_{AB})$ for SSTSs is showed in Fig.3 for $\bar{n} \leq 50$. It reveals that $N^G_F(\rho_{AB}(\bar{n}, \mu)) > Q(\rho_{AB}(\bar{n}, \mu))$ if $\mu > 0$ and $N^G_F(\rho_{AB}) \gg Q(\rho_{AB})$ for those SSTSs $\rho_{AB}$ with large mean photon number $\bar{n}$ and larger mixing parameter $\mu < 1$. Consider the states $\rho_{AB}$ with respectively $(\bar{n}, \mu) = (49, 0.9)$ and $(\bar{n}, \mu) = (10000, 0.9)$, the same examples as above. We have respectively $Q(\rho_{AB}) \approx 0.040867 < N^G_F(\rho_{AB}) \approx 0.897955$ and $Q(\rho_{AB}) \approx 0.000021 \ll N^G_F(\rho_{AB}) \approx 0.89803$, which means that applying $N^G_F$ is much more better than $Q$ to guarantees that $\rho_{AB}$ contains the quantum correlation. Fig.4 demonstrates that $N^G_F(\rho_{AB}) - Q(\rho_{AB}) > 0$ also holds for these $\bar{n} \in (100000, 100500)$ and $\mu \in (0, 1)$. Furthermore, for random pairs $(\bar{n}, \mu)$ with $\bar{n} \in (0, 10^{13})$ and $\mu \in (0, 1)$, $10^5$ numerical results illustrate that $N^G_F(\rho_{AB}(\bar{n}, \mu)) > Q(\rho_{AB}(\bar{n}, \mu))$. So, we have

$$N^G_F(\rho_{AB}) > Q(\rho_{AB})$$

for all SSTSs with $\mu \neq 0$ and $N^G_F$ is better than $Q$. 
CONCLUSION

In this paper, based on fidelity $F(\rho, \sigma) = \frac{(tr\rho\sigma)^2}{tr\rho tr\sigma}$, we propose a new kind of quantum correlation $N^G_F$ by local Guassian unitary operations for any states in $(n + m)$-mode continuous-variable systems. Though, when restricted to the Gaussian states, $N^G_F$ describes the same nonclassical correlation as all known correlations such as Gaussian QD, Gaussian geometric discord $D_G$ and the nonlocality $Q$, it is comparatively much easier to be computed and estimated. Furthermore, $N^G_F$ has several nice properties that other known quantifications of such correlation do not possess: $N^G_F$ is a quantum correlation without ancilla problem; $N^G_F((I \otimes \Phi)\rho_{AB}) \leq N^G_F(\rho_{AB})$ holds for any $(1 + 1)$-mode Gaussian state $\rho_{AB}$ and any Gaussian channel $\Phi$, that is, undergoing a local Gaussian channel performed on the unmeasured part, the quantity $N^G_F$ will not increase. We guess that this nice property is still valid for $(n + m)$-mode systems. We give a computation formula of $N^G_F$ for any $(1 + 1)$-mode Gaussian states and an upper bound for any $(n + m)$-mode Gaussian states, which are simple and easily calculated. Furthermore, by comparing $N^G_F(\rho_{AB})$ with $D_G(\rho_{AB})$ and $Q(\rho_{AB})$ for two-mode symmetric squeezed thermal states, we find that $N^G_F$ is greater than $D_G$ and $Q$, and so is better in detecting quantum correlation in Gaussian states.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] H. Ollivier, W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
[3] B. Dakić, V. Vedral, Č. Brukner, Phys. Rev. Lett. 105, 190502 (2010).
[4] S. Luo, S. Fu, Phys. Rev. A 82, 034302 (2010).
[5] A. Miranowicz, P. Horodecki, R. W. Chhajlany, et.al., Phys. Rev. A 86, 042123 (2012).
[6] S. Luo, S. Fu, Phys. Rev. Lett. 106, 120401 (2011).
[7] S. Luo, Phys. Rev. A 77, 022301 (2008).
[8] P. Giorda, M. G. A. Paris, Phys. Rev. Lett. 105, 020503 (2010).
[9] G. Adesso, A. Datta, Phys. Rev. Lett. 105, 030501 (2010).
[10] G. Adesso, D. Girolami, Int. J. Quantum Inf. 09, 1773-1786 (2011).
[11] L. Mišta, Jr., R. Tatham, D. Girolami, N. Korolkova, G. Adesso, Phys. Rev. A 83, 042325 (2011).
[12] R.-F. Ma, J.-C. Hou, X.-F. Qi, Int. J. Theor. Phys. 56, 1132-1140 (2017).
[13] R.-F. Ma, J.-C. Hou, X.-F. Qi, Y.-Y. Wang, Quantum Inf. Process. 17:98 (2018).
[14] A. Farace, A. De. Pasquale1, L. Rigovacca, V. Giovannetti, New J. Phys. 16, 073010 (2014).
[15] L. Rigovacca, A. Farace, A. D. Pasquale, V. Giovannetti, Phys. Rev. A 92, 042331 (2015).
[16] Y.-Y. Wang, J.-C. Hou, X.-F. Qi, Entropy 20, 266 (2018).
[17] L. Fu, Europhys. Lett. 75, 1 (2006).
[18] A. Datta, S. Gharibian, Phys. Rev. A 79, 042325 (2009).
[19] S. Gharibian, Phys. Rev. A 86, 042106 (2012).
[20] Y. Huang, New J. Phys. 16, 033027 (2014).
[21] S. L. Braunstein, P. van Loock, Rev. Mod. Phys. 77, 513 (2005).
[22] X.-B. Wang, T. Hiroshima, A. Tomin, M. Hayashi, Phys. Rep. 448, 1-111 (2007).
[23] C. Weedbrook, S. Pirandola, et al., Rev. Mod. Phys. 84, 621 (2012).
[24] J. Anders, arXiv:quant-ph/0610263 (2006).
[25] H. Lutkenpohl, Handbook of Matrices. John Wiley and son’s Ltd, Chichester. (1996).
[26] R. A. Horn, C. R. Johnson, Matrix Analysis, Cambridge university Press, Cambridge, United Kingdom (2012).
[27] R. Jozsa, Journal of Modern Optics, 41, 2315 (1994).
[28] N. Gisin, S. Massar, Phys. Rev. Lett. 79, 2153-2156 (1997).
[29] G.-F. Zhang, Phys. Rev. A 75, 034304 (2007).
[30] Yu. I. Bogdanov, G. Brida, M. Genovese, S. P. Kulik, E. Moreva, A. p shurupov, Phys. Rev. Lett. 105, 010404 (2010).
[31] T. Gorin, T. Prosen, H. Seligman, M. Znidaric, Phys. Rep. 435, 33-156 (2006).
[32] S.-J. Gu, Int. J. Mod. Phys. B 24, 4371 (2010).
[33] A. Gilchrist, N. K. Langford, M. A. Nielsen, Phys. Rev. A 71, 062310 (2005).
[34] X. Wang, C.-S. Yu, X.-X. Yi, Phys. Lett. A 373, 58-60 (2008).
[35] R. Muthuganesan, R. Sankaranarayanan, Phys. Lett. A 381, 3028-3032 (2017).
[36] P. Marian, T. A. Marian, Phys. Rev. A 86, 022340 (2012).
[37] G. J. Milburn, J. Phys. A 17, 737-745 (1984).
[38] W. P. Bowen, R. Schnabel, P. K. Lam, T. C. Ralph, Phys. Rev. A 69, 012304 (2004).