The pre-inflationary and inflationary fast-roll eras and their signatures in the low CMB multipoles

C. Destri (a) and N. G. Sanchez (c) and H. J. de Vega (b,c)

(a) Dipartimento di Fisica G. Occhialini, Università Milano-Bicocca and INFN, sezione di Milano-Bicocca, Piazza della Scienze 3, 20126 Milano, Italia.
(b) LPTHE, Université Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII), Laboratoire Associé au CNRS UMR 7589, Tour 24, 5ème étage, Boîte 126, 4, Place Jussieu, 75252 Paris, Cedex 05, France.
(c) Observatoire de Paris, LERMA. Laboratoire Associé au CNRS UMR 8112, 61, Avenue de l’Observatoire, 75014 Paris, France.

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We study the entire coupled evolution of the inflaton \( \phi(t) \) and the scale factor \( a(t) \) for general initial conditions \( \phi(t_0) \) and \( d\phi(t)/dt \) at a given initial time \( t_0 \). The generic early universe evolution has three stages: decelerated fast-roll followed by inflationary fast-roll and then inflationary slow-roll (an attractor always reached for generic initial conditions). This evolution is valid for all regular inflaton potentials \( \phi(\phi) \). In addition, we find a special (extreme) slow-roll solution starting at \( t = -\infty \) in which the fast-roll stages are absent. At some time \( t = t_\ast \), the evolution backwards in time from \( t_0 \) reaches generically a mathematical singularity where \( a(t) \) vanishes and the Hubble parameter becomes singular. We determine the general behaviour near the singularity. The classical homogeneous inflaton description turns to be valid for \( t - t_\ast > 10t_{Planck} \) well before the beginning of inflation, quantum loop effects are negligible there. The singularity is never reached in the validity region of the classical treatment and therefore it is not a real physical phenomenon here. Fast-roll and slow-roll regimes are analyzed in detail including the equation of state evolution, both analytically and numerically. The characteristic time scale of the fast-roll era turns to be \( t_1 = (1/m) \sqrt{V(0)/3M^4} \sim 10^4 t_{Planck} \) where \( V \) is the double-well inflaton potential, \( m \) is the inflaton mass and \( M \) the energy scale of inflation. The whole evolution of the fluctuations along the decelerated and inflationary fast-roll and slow-roll eras is computed. The Bunch-Davies initial conditions (BDic) are generalized for the present case in which the potential felt by the fluctuations can never be neglected. The fluctuations feel a singular attractive potential near the \( t = t_\ast \) singularity (as in the case of a particle in a central singular potential) with exactly the critical strength \( (-1/4) \) allowing the fall to the centre. Precisely, the fluctuations exhibit logarithmic behaviour describing the fall to \( t = t_\ast \). The power spectrum gets dynamically modified by the effect of the fast-roll eras and the choice of BDic at a finite time through the transfer function \( D(k) \) of initial conditions. The power spectrum vanishes at \( k = 0 \). \( D(k) \) presents a first peak for \( k \sim 2/\eta_0 \) (\( \eta_0 \) being the conformal initial time), then oscillates with decreasing amplitude and vanishes asymptotically for \( k \to \infty \). The transfer function \( D(k) \) affects the low CMB multipoles \( C_\ell \): the change \( \Delta C_\ell/C_\ell \) for \( 1 \leq \ell \leq 5 \) is computed as a function of the starting instant of the fluctuations \( t_0 \). CMB quadrupole observations indicate large suppressions which are well reproduced for the range \( t_0 - t_\ast \gtrsim 0.05/m \approx 10100 t_{Planck} \).

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\*Electronic address: C.Destri@mb.infn.it
\textsuperscript{†}Electronic address: devega@lpthe.jussieu.fr
\textsuperscript{\textcopyright}Electronic address: Norma.Sanchez@obspm.fr
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inflaton and the space-time can be trusted well before the beginning of inflation.

We consider here the double well (broken symmetric) fourth order inflaton potential since it gives the best description
numerical evolution and an analytic approximation, and the whole equation of state evolution in the three regimes.

The scale factor $a$ is never reached in the validity region of the classical treatment and therefore such mathematical singularity
and determine the validity of the classical approximation, namely $H = M$ slow-roll solution starting from
initial conditions. This evolution is valid for all regular inflaton potentials. In addition, we find a particular (extreme)
slow-roll inflationary era and then by a slow-roll inflationary regime which is an attractor always reached for
generic initial conditions, fixed by the values of $\phi(t_0)$ and $d\phi(t_0)/dt$ at a given initial time $t_0$.

We show that the generic early universe evolution has three stages: a decelerated fast-roll stage followed by an
inflationary fast-roll stage and then by a slow-roll inflationary regime which is an attractor always reached for generic
initial conditions. This evolution is valid for all regular inflaton potentials. In addition, we find a particular (extreme)
slow-roll solution starting from $t = -\infty$ in which the fast-roll stages are absent.

The evolution backwards in time from $t_0$ reaches generically a mathematical singularity at some time $t = t_*$ where
the scale factor $a(t)$ vanishes, and the Hubble parameter becomes singular.

We find the general behaviour of the inflaton and the scale factor near the singularity as given by eqs. (2.15)-(2.17)
and determine the validity of the classical approximation, namely $(H/M_{Pl})^2 \ll 1$. It must be stressed that such
mathematical singularity is attained extrapolating the classical treatment where it is no more valid. The singularity
is never reached in the validity region of the classical treatment and therefore such mathematical singularity is not
a real physical phenomenon here.

Quantum loops effects turns to be less than 1% for $t - t_* > 10^{-42}$ sec and therefore the classical treatment of the inflaton and the space-time can be trusted well before the beginning of inflation.

The fast-roll (both decelerated and inflationary) and slow-roll regimes are analyzed in detail, with both the exact
numerical evolution and an analytic approximation, and the whole equation of state evolution in the three regimes.
We consider here the double well (broken symmetric) fourth order inflaton potential since it gives the best description
of the CMB+LSS data within the Ginsburg-Landau effective theory approach we follow.

The characteristic time scale of the fast-roll era turns to be $t_1 = (1/m) \sqrt{V(0)/3 M^4} \sim 10^4 t_{Planck}$ where $V(0)$
is the double well inflaton potential at zero inflaton field, $m$ is the inflaton mass and $M$ the energy scale of inflation.
The time scale of the inflaton in the extreme slow roll solution goes as the inverse of $t_1$, namely $1/[m^2 t_1]$.

We study the whole evolution of the curvature and tensor fluctuations along the three successive regimes: decelerated
fast-roll followed by inflationary fast-roll and then inflationary slow-roll, and compute the power spectrum by the end
of inflation. The fluctuations feel a singular attractive potential near the $t = t_*$ singularity (as in the case of a particle in a central singular potential) with exactly the critical strength $(-1/4)$ for which the fall to the centre
becomes possible. Precisely, the logarithmic behaviour of the fluctuations for \( t \rightarrow t_* \) eq.\([\ref{eq:1.3}]\) describes the fall to \( t = t_* \) for the critical strength of the potential \( W_R \) felt by the fluctuations.

We generalize the Bunch-Davies initial conditions (BDic) to the present case in which the potential felt by the fluctuations can never be neglected.

In general, the mode functions for large \( k \) behave as free modes since the potential \( W_R(\eta) \) vanishes and the fluctuations exhibit a plane wave behaviour for all \( k \) (not necessarily large). This is not the case in strong gravity fields or near curvature singularities as in the present cosmological space-time where \( W_R(\eta) \) can never be neglected at fixed \( k \). However, we can choose Bunch-Davies initial conditions (BDic) at \( \eta = \eta_0 \) (or equivalently, \( t = t_0 \)) by imposing

\[
S_R(k; \eta) = k \rightarrow \infty e^{-i k \eta \sqrt{2/k}} \tag{1.1}
\]

and therefore

\[
\frac{dS_R}{d\eta}(k; \eta_0) = -i k S_R(k; \eta_0).
\]

Here \( \eta \) stands for the conformal time: \( d\eta = dt/a(t) \). Eq.\([\ref{eq:1.1}]\) fulfils the Wronskian normalization (that ensures the canonical commutation relations)

\[
W[S_R, S_R^*] = S_R \frac{dS_R}{d\eta} - \frac{dS_R}{d\eta} S_R^* = i. \tag{1.2}
\]

In asymptotically flat (or conformally flat) regions of the space-time the potential felt by the fluctuations \( W_R(\eta) \) becomes possible. Precisely, the logarithmic behaviour of the fluctuations for \( t \rightarrow \infty \) eq.\([\ref{eq:1.4}]\) imposed there at \( t = t_* \) starts at \( \eta = \eta_0 \) (or equivalently, \( t = t_0 \)) by imposing

\[
\frac{dS_R}{d\eta}(k; \eta_0) = -i k S_R(k; \eta_0) \text{ for all } k. \tag{1.3}
\]

That is, we consider the initial value problem for the mode functions giving the values of \( S_R(k; \eta) \) and \( dS_R/d\eta \) at \( \eta = \eta_0 \). This condition combined with the Wronskian condition eq.\([\ref{eq:1.2}]\) implies that

\[
|S_R(k; \eta_0)| = \frac{1}{\sqrt{2/k}} \quad , \quad \left| \frac{dS_R}{d\eta}(k; \eta_0) \right| = \sqrt{\frac{k}{2}}. \tag{1.4}
\]

which is equivalent to eq.\([\ref{eq:1.1}]\) for large \( k \).

The power spectrum at the end of slow-roll inflation \( P_R(k) \) gets dynamically modified by the effect of the preceding fast-roll eras through the transfer function of initial conditions \( D(k) \):

\[
P_R(k) = P_R^{BD}(k) [1 + D(k)]. \tag{1.5}
\]

\( D(k) \) accounts for the effect of both the initial conditions and the fluctuations evolution during fast-roll (before slow-roll). \( D(k) \) depends on the time \( t_0 \) at which BDic are imposed.

The power spectrum \( P_R^{BD}(k) \) corresponds to start the evolution with pure slow-roll from \( t_0 \rightarrow -\infty \) and with BDic eq.\([\ref{eq:1.3}]\)-eq.\([\ref{eq:1.4}]\) imposed there at \( t_0 \rightarrow -\infty \), that is \( \eta_0 = -\infty \). \( P_R^{BD}(k) \) is given by its customary pure slow-roll expression,

\[
\log P_R^{BD}(k) = \log A_s(k_0) + (n_s - 1) \log \frac{k_{\text{eq}}}{k_0} + \frac{1}{2} n_{\text{run}} \log^2 \frac{k}{k_0} + O \left( \frac{1}{N^2} \right). \tag{1.6}
\]

where \( N \) is the number of inflation efolds since the pivot CMB scale \( k_0 \) exits the horizon. We take here \( N = 60 \).

Actually, BDic can be imposed at \( \eta = \eta_0 = -\infty \) if and only if the inflaton evolution also starts at \( \eta = \eta_0 = -\infty \). This only happens for a particular inflaton solution: the extreme slow-roll solution that we explicitly present and analyze in sec.\([\ref{sec:3.1}]\) In the extreme slow-roll case the fast-roll eras are absent, BDic are imposed at \( t_0 \rightarrow -\infty \) (that is \( \eta_0 = -\infty \)), then \( D(k) = 0 \) and \( P_R(k) = P_R^{BD}(k) \). Only in this case the fluctuation power spectrum at the end of inflation is the usual power spectrum \( P_R^{BD}(k) \) eq.\([\ref{eq:1.24}]\).
When BDic are imposed at finite times $t_0$, the spectrum is not the usual $P_R^{BD}(k)$ but it gets modified by a non-zero transfer function $D(k)$ \textit{eq.(4.24)}. The power spectrum $P_R(k)$ vanishes at $k = 0$ and exhibits oscillations which vanish at large $k$ \textit{[see figs. 6 and 4]}

Generically, the power spectrum vanishes at $k = 0$ and we thus have

\begin{equation}
1 + D(k) \approx 0 \mathcal{O}(k^{n_s+1})
\end{equation}

as shown in sec. \textit{D(k) presents a first peak for $k \sim 2/\eta_0$ and then oscillates asymptotically with decreasing amplitude such that}

\begin{equation}
D(k) \approx 0 \mathcal{O}\left(\frac{1}{k^2}\right)
\end{equation}

We solved numerically the fluctuations equation with the BDic \textit{eq.(4.27)} covering both the fast- and slow-roll regimes, namely for different initial times $t_0$ ranging from the singularity $\tau = \tau_s$ till the transition time $\tau_{trans}$ from fast-roll to slow-roll. That is to say, we solved the fluctuations evolution for BDic imposed at different times in the three eras and we compare the resulting power spectra among them. We computed the corresponding transfer function, $D(k)$ for the BDic imposed at the different eras. We depict $1 + D(k)$ vs. $k$ for the different values of the time $t_0$ where BDic are imposed in figs. \textit{6}

When the BDic are imposed during the fast–roll stage well \textit{before} it ends, $D(k)$ changes much more significantly than along the extreme slow roll solution. This is due to two main effects: the potential felt by the fluctuations is attractive during fast–roll, and $\eta_0$, (far from being almost proportional to $1/a(\eta)$), tends to the constant value $\eta_s$, as $\tau \to \tau_s^+$ and $a(\eta) \to 0$. The numerical transfer functions $1 + D(k)$ obtained from eqs. \textit{4.22} and \textit{4.24} are plotted in figs. \textit{6}

We have also computed $D(k)$ analytically with BDic at finite times $\eta_0$, and a simple form is obtained in the scale invariant case, which is the leading term in the slow-roll expansion:

\begin{equation}
D(k) = \frac{\cos 2x}{x^2} - \frac{\sin 2x}{x^3} + \frac{\sin^2 x}{x^4}, \quad x \equiv k \eta_0.
\end{equation}

Different initial times $t_0$ lead essentially to a rescaling of $k$ in $D(k)$ by a factor $\eta_0$ since the conformal time $\eta$ is almost proportional to $1/a(\eta)$ during slow-roll \textit{[see figs. 6 and below eq.(5.7)]}. By virtue of the dynamical attractor character of slow–roll, the power spectrum when the BDic are imposed at a finite time $t_0$ cannot really distinguish between the extreme slow–roll solution or any other solution which is attracted to slow–roll well before the time $t_0$.

Using the transfer function $D(k)$ we obtained, we computed the change on the CMB multipoles $\Delta C_\ell/C_\ell$ for $\ell = 1, 2$ and 3 as functions of the starting instant of the fluctuations $t_0$. We plot $\Delta C_\ell/C_\ell$ for $1 \leq \ell \leq 5$ vs. $t_0 - t_s$ in fig. \textit{6}. We see that $\Delta C_\ell/C_\ell$ is positive for small $t_0 - t_s$ and decreases with $t_0$ becoming then negative. The CMB quadrupole observations indicate a large suppression thus indicating that $t_0 - t_s \gtrsim 0.05/m \simeq 101000 t_{Planck}$.

The fact that choosing BDic leads to a primordial power and its respective CMB multipoles which correctly reproduce the observed spectrum justifies the use of BDic.

Besides finding a CMB quadrupole suppression in agreement with observations \textit{2, 3}, we provide here predictions for the dipole and $\ell \leq 5$-multipole suppressions. Forthcoming CMB observations can provide better data to confront our CMB multipole suppression predictions. It will be extremely interesting to measure the primordial dipole and compare with our predicted value.

**II. THE PRE-INFLATIONARY AND INFLATIONARY FAST-ROLL ERAS**

The current WMAP data are validating the single field slow-roll scenario \textit{1}. Single field slow-roll models provide an appealing, simple and fairly generic description of inflation. This inflationary scenario can be implemented using a scalar field, the \textit{inflaton} with a Lagrangian density (see for example ref. \textit{2})

\begin{equation}
\mathcal{L} = a^3(t) \left[ \frac{\varphi^2}{2} - \frac{(\nabla \varphi)^2}{2 a^2(t)} - V(\varphi) \right],
\end{equation}

\textit{figs. 6.
where $V(\varphi)$ is the inflaton potential. Since the universe expands exponentially fast during inflation, gradient terms are exponentially suppressed and can be neglected. At the same time, the exponential stretching of spatial lengths classically the physics and permits a classical treatment. One can therefore consider an homogeneous and classical inflaton field $\varphi(t)$ which obeys the evolution equation

$$\ddot{\varphi} + 3 H(t) \dot{\varphi} + V'(\varphi) = 0$$  

(2.2)

in the isotropic and homogeneous FRW metric which is sourced by the inflaton

$$ds^2 = dt^2 - a^2(t) \, d\vec{x}^2$$  

(2.3)

$H(t) \equiv \dot{a}(t)/a(t)$ stands for the Hubble parameter. The energy density and the pressure for a spatially homogeneous inflaton are given by

$$\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi) \quad , \quad p = \frac{\dot{\varphi}^2}{2} - V(\varphi) .$$  

(2.4)

Therefore, the scale factor $a(t)$ obeys the Friedmann equation,

$$H^2(t) = \frac{1}{3M^2_{Pl}} \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right] .$$  

(2.5)

In order to have a finite number of inflation e-folds, the inflaton potential $V(\varphi)$ must vanish at its absolute minimum

$$V'(\varphi_{\text{min}}) = V(\varphi_{\text{min}}) = 0$$  

(2.6)

Otherwise, inflation continues forever.

We formulate inflation as an effective field theory within the Ginsburg-Landau spirit[2,10,17]. The theory of the second order phase transitions, the Ginsburg-Landau theory of superconductivity, the current-current Fermi theory of weak interactions, the sigma model of pions, nucleons (as skyrmions) and photons are all successful effective field theories. Our work shows how powerful is the effective theory of inflation to predict observable quantities that can be or will be soon contrasted with experiments.

The effective theory of inflation should be the low energy limit of a microscopic fundamental theory not yet precisely known. The energy scale of inflation $M$ should be at the Grand Unified Theory (GUT) energy scale in order to reproduce the amplitude of the CMB anisotropies[2]. Therefore, the microscopic theory of inflation is expected to be a GUT in a cosmological space-time. Such a theory of inflation would contain many fields of various spins. However, in order to have a homogeneous and isotropic universe the expectation value of the energy-momentum tensor of the fields must be homogeneous and isotropic. The inflaton field in the effective theory may be a coarse-grained average of fundamental scalar fields, or a composite (bound state) of fundamental fields of higher spin, just as in superconductivity. The inflaton does not need to be a fundamental field, for example it may emerge as a condensate of fermion-antifermion pairs $<\bar{\Psi}\Psi>$ in a GUT in the cosmological background. In order to describe the cosmological evolution is enough to consider the effective dynamics of such condensates. The relation between the effective field theory of inflation and the microscopic fundamental GUT is akin to the relation between the effective Ginzburg-Landau theory of superconductivity and the microscopic BCS theory, or like the relation of the $O(4)$ sigma model, an effective low energy theory of pions, photons and chiral condensates with quantum chromodynamics (QCD) [16].

Vector fields have been considered to describe inflation in ref. [14]. The results for the inflaton should not be very different from the effective inflaton description since the energy-momentum tensor of the vector field is to be taken homogeneous and isotropic. Namely, we are always in the presence of a scalar condensate.

Since the mass of the inflaton is given by $M^2/M_{Pl} \sim 10^{13}$GeV[2], massless fields alone cannot describe inflation which leads to the observed amplitude of the CMB anisotropies.

The classical inflaton potential $V(\varphi)$ gets modified by quantum loop corrections. We computed relevant quantum loop corrections to inflationary dynamics in ref. [2,15]. A thorough study of the effect of quantum fluctuations reveals that these loop corrections are suppressed by powers of $(H/M_{Pl})^2 \sim 10^{-9}$ where $H$ is the Hubble parameter during inflation [2,15]. Therefore, quantum loop corrections are very small, a conclusion that validates the reliability of the classical approximation and of the effective field theory approach to inflationary dynamics. In particular, the (small) one-loop corrections to the potential in an inflationary universe are very different from the Coleman-Weinberg form [2,15].

We choose the inflaton field initially homogeneous which ensures it is always homogeneous. The fluctuations around are small and give small corrections to the homogeneity of the Universe. The rapid expansion of the Universe, in the inflationary regimes, takes care of the classical fluctuations, quickly flattening an eventually non-homogeneous condensate.
A. The complete inflaton evolution through the different eras

It is convenient to use the dimensionless variables to analyze the inflaton evolution equations eqs. (2.2)-(2.5), [2]:

\[ \tau = m t, \quad h \equiv \frac{H}{m}, \quad \phi = \frac{\varphi}{M_{Pl}}. \] (2.7)

The inflaton potential has then the universal form

\[ V(\varphi) = M^4 v \left( \frac{\varphi}{M_{Pl}} \right), \] (2.8)

where \( M \) is the energy scale of inflation and \( v(\phi) \) is a dimensionless function. Without loss of generality we can set \( v'(0) = 0 \) [2]. Moreover, provided \( V''(0) \neq 0 \) we can set without loss of generality \( |v''(0)| = 1/2 \). Namely, we have for small fields,

\[ v(\phi) \varphi \rightarrow 0 = v(0) \pm \frac{1}{2} \phi^2 + O(\phi^3) \] (2.9)

where the minus sign in the quadratic term corresponds to new inflation and the plus sign to chaotic inflation.

In these dimensionless variables, the energy density and the pressure for a spatially homogeneous inflaton are given from eq.(2.4) by

\[ \rho M^4 = \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + v(\phi), \quad p M^4 = \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 - v(\phi), \] (2.10)

and the coupled inflaton evolution equation (2.2) and the Friedmann equation (2.5) take the form [2],

\[ \frac{d^2\phi}{d\tau^2} + 3 h \frac{d\phi}{d\tau} + v'(\phi) = 0, \quad h^2(\tau) = \frac{1}{3} \left[ \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + v(\phi) \right]. \] (2.11)

These coupled nonlinear differential equations completely define the time evolution of the inflaton field and the scale factor once the initial conditions are given at the initial time \( \tau_0 \). Namely, the initial conditions are fixed by giving two real numbers, the values of \( \phi(\tau_0) \) and \( d\phi(\tau_0)/d\tau \).

It follows from eqs. (2.11) that

\[ \frac{d^2a}{d\tau^2} = \frac{1}{3} \left[ v(\phi) - \left( \frac{d\phi}{d\tau} \right)^2 \right] = -\frac{1}{2} \left( p + \frac{1}{3} \rho \right). \] (2.12)

When \( d^2a/d\tau^2 > 0 \) the expansion of the universe accelerates and it is then called inflationary.

The derivative of the Hubble parameter is always negative:

\[ \frac{dh}{d\tau} = -\frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2. \] (2.13)

Therefore \( h(\tau) \) decreases monotonically with increasing \( \tau \). Conversely, if we evolve the solution backwards in time from \( \tau_0 \), \( h(\tau) \) will generically increase without bounds. Namely, at some time \( \tau = \tau_*, \) \( h(\tau) \) can exhibit a singularity where simultaneously \( a(\tau_*) \) vanishes.

In fact, the equations (2.11) admit the singular solution for \( \tau \rightarrow \tau_* \),

\[ \phi(\tau) \rightarrow \frac{-b}{3} \log \frac{\tau - \tau_*}{b} \rightarrow -\infty, \quad h(\tau) \equiv \frac{d}{d\tau} \log a(\tau) \rightarrow +\infty, \quad \frac{1}{3 (\tau - \tau_*)} \rightarrow +\infty, \] (2.14)

where \( b \) is an integration constant. The energy density \( \epsilon(\tau) \) and equation of state take the limiting form,

\[ \rho(\tau) \rightarrow \frac{1}{3 (\tau - \tau_*)^2} \rightarrow +\infty, \quad \frac{p(\tau)}{\rho(\tau)} \rightarrow 1. \] (2.15)
Namely, the limiting equation of state is \( p \tau \rightarrow \tau^* + \rho \).

We have in this regime

\[
a(\tau)^{\tau \rightarrow \tau^*} C (\tau - \tau_*)^{\frac{4}{9}} \rightarrow 0 ,
\]

where \( C \) is some constant. That is, the geometry becomes singular for \( \tau \rightarrow \tau_* \). The behaviour near \( \tau_* \) is non-inflationary, namely decelerated, since

\[
d^2 a \over d\tau^2 \tau \rightarrow \tau^* = -2 \left( \tau - \tau_* \right)^{-\frac{2}{9}} \rightarrow -\infty .
\]

(2.17)

For \( \tau \rightarrow \tau_* \), near the singularity, the potential \( v(\phi) \) becomes negligible in eqs. (2.11). Therefore, eqs. (2.14)-(2.17) are valid for all regular potentials \( v(\phi) \).

The evolution starts thus by this decelerated fast-roll regime followed by an inflationary fast-roll regime and then by a slow-roll inflationary regime. See [2]. Recall that the slow-roll regime is an attractor [4], and therefore the inflaton always reaches a slow-roll inflationary regime for generic initial conditions. We display in fig. 1 the inflaton flow in phase space, namely \( d\phi/d\tau \) vs. \( \phi \) for different initial conditions.

The number of efolds of slow-roll inflation \( N_{sr} \) is determined by the time when the inflaton trajectory reaches the red quasi-horizontal line of slow-roll regime [see fig. 1]. We see that \( d\phi/d\tau \) decreases steeply with \( \phi \). This implies that \( N_{sr} \) is mainly determined by the initial value of \( \phi \) with a mild (logarithmic) dependence on the initial value of \( d\phi/d\tau \).

The inflaton flow described by eq. (2.14) results

\[
\dot{\phi}(\tau)^{\tau \rightarrow \tau^*} = \sqrt{2 \over 3} e^{-\sqrt{2 \over 3} \phi(\tau)}
\]

(2.18)

which well reproduce the almost vertical blue and green lines in fig. 1.

The inflationary regimes are characterized by the slow-roll parameters \( \epsilon_v \) and \( \eta_v \) [2]

\[
\epsilon_v = {1 \over 2 \hbar^2} \left( {d\phi \over d\tau} \right)^2 , \quad \eta_v = {v''(\phi) \over v(\phi)} .
\]

(2.19)

The slow-roll behaviour is defined by the condition \( \epsilon_v < 1/N \). Typically, \( \epsilon_v \approx 1/N \) during slow-roll. More generally accelerated expansion (inflation) happens for \( \epsilon_v < 1 \) while we have decelerated expansion for \( \epsilon_v > 1 \) as follows from eqs. (2.10)-(2.12) and (2.19).

The parameter \( \eta_v \) is also of the order \( 1/N \) during slow-roll and it is generically of order \( 1/N \) during fast-roll except when the potential \( v(\phi) \) vanishes.

Eq. (2.13) implies a monotonic decreasing of the expansion rate of the universe. There are four stages in the universe evolution described by eqs. (2.11):

- The non-inflationary fast-roll stage starting at the singularity \( \tau = \tau_* \) and ending when \( d^2 a/d\tau^2 \) becomes positive [see eq. (2.12)].
- The inflationary fast-roll stage starts when \( d^2 a/d\tau^2 \) becomes positive and ends at \( \tau = \tau_{trans} \) when \( \epsilon_v \) becomes smaller than \( 1/N \) [see eq. (2.19)].
- The inflationary slow-roll stage follows, and it continues as long as \( \epsilon_v < 1/N \) and \( d^2 a/d\tau^2 > 0 \). It ends when \( d^2 a/d\tau^2 \) becomes negative at \( \tau = \tau_{end} \).
- A matter-dominated stage follows the inflationary era.

The four stages described above correspond to the evolution for generic initial conditions or, equivalently, starting from the singular behaviour eqs. (2.14). In addition, there exists a special (extreme) slow-roll solution starting at \( \tau = -\infty \) where the fast-roll stages are absent. We derive this extreme slow-roll solution in sec. III A.

As shown in refs. [2, 7] the double well (broken symmetric) fourth order potential

\[
V(\varphi) = {1 \over 4} \lambda \left( \varphi^2 - {m^2 \over \lambda} \right)^2 = -{1 \over 2} m^2 \varphi^2 + {1 \over 4} \lambda \varphi^4 + {m^4 \over 4 \lambda},
\]

(2.20)
FIG. 1: The complete inflaton flow in phase space. \( \frac{d\phi}{d\tau} \) vs. \( \phi \) for different initial conditions. We see that the inflaton always reaches a slow-roll regime for generic initial conditions represented by a red quasi-horizontal line. Hence, the slow-roll line is an attractor. Ultimately the inflaton reaches asymptotically the absolute minima \( \frac{d\phi}{d\tau} = 0, \phi = \phi_{\text{min}} = \sqrt{8N/y} = 19.52 \ldots \). The number of e-folds of slow-roll inflation \( N_{sr} \) increases for decreasing initial \( \phi > 0 \) when \( \frac{d\phi}{d\tau} > 0 \) initially. The \( \phi > 0, \frac{d\phi}{d\tau} > 0 \) trajectories corresponding to \( N_{sr} > 63 \) are colored in green.

This model of new inflation yields as most probable values: \( n_s \simeq 0.964, r \simeq 0.051 \) [2, 7]. This value for \( r \) is within reach of forthcoming CMB observations. For \( y > 0.431946 \ldots \) and in particular for the best fit value \( y \simeq 1.26 \), the inflaton field exits the horizon in the negative concavity region \( V''(\phi) < 0 \) intrinsic to new inflation [2]. We find for the best fit [2, 7],

\[
M = 0.543 \times 10^{16} \text{ GeV for the scale of inflation and } \quad m = 1.21 \times 10^{13} \text{ GeV for the inflaton mass.} \quad (2.22)
\]

provides a very good fit for the CMB+LSS data, while at the same time being particularly simple, natural and stable in the Ginsburg-Landau sense. This is a new inflation model with the inflaton rolling from the vicinity of the local maxima of \( V(\varphi) \) at \( \varphi = 0 \) towards the absolute minimum \( \varphi = m/\sqrt{\lambda} \).

The inflaton mass \( m \) and coupling \( \lambda \) are naturally expressed in terms of the two relevant energy scales in this problem: the energy scale of inflation \( M \) and the Planck mass \( M_{Pl} = 2.43534 \times 10^{18} \text{ GeV} \),

\[
m = \frac{M^2}{M_{Pl}}, \quad \lambda = \frac{y}{8N \left( \frac{M}{M_{Pl}} \right)^4}. \quad (2.21)
\]

Here \( N \sim 60 \) is the number of e-folds since the cosmologically relevant modes exit the horizon till the end of inflation and \( y \sim 1 \) is the quartic coupling.

The MCMC analysis of the CMB+LSS data combined with the theoretical input above yields the value \( y \simeq 1.26 \) for the coupling [2, 7]. \( y \) turns to be order one consistent with the Ginsburg-Landau formulation of the theory of inflation [2].
We consider from now on the quartic broken symmetric potential eq. (2.20) which becomes using eq. (2.8):

\[ v(\phi) = \frac{g}{4} \left( \phi^2 - \frac{1}{g} \right)^2 = -\frac{1}{2} \phi^2 + \frac{g}{4} \phi^4 + \frac{1}{4g} \quad \text{where} \quad g = \frac{y}{8N}. \]  

(2.23)

We have two arbitrary real coefficients characterizing the initial conditions. We can choose them as \( b \) and \( \tau^* \) [see eq. (2.14)]. A total number of slow-roll inflation efolds \( N_{sr} \approx 63 \) permits to explain the CMB quadrupole suppression [2, 5, 6]. Such requirement fixes the value of \( b \) for a given coupling \( y \).

We integrated numerically eqs. (2.11) with eq. (2.14) as initial conditions. We find that \( b = 4.745272 \ldots \) yields 63 efolds of inflation during the slow-roll era for \( y = 1.26 \), the best fit to the CMB and LSS data. We find that \( b \) is a monotonically increasing function of the coupling \( y \) for fixed number of slow-roll efolds. At fixed coupling, \( b \) increases with the number of slow-roll efolds.

We display in fig. 2 \( b \) as a function of \( y \) and the number of slow–roll inflation efolds \( N_{sr} \).

![Graphs](image)

FIG. 2: Left panel: the coefficient \( b \) characterizing the initial conditions vs. the quartic coupling \( y \) for \( N_{sr} = 63 \) efolds of slow-roll inflation. Right panel: \( b \) vs. \( N_{sr} \) for \( y = 1.26 \). The preferred values \( y = 1.26 \) and \( N_{sr} \) are highlighted in both panels.

For this value of \( y \) and 63 efolds of inflation during the slow-roll, fast-roll ends by \( \tau = \tau_{trans} = 0.2487963 \ldots \). In figures 3 we depict \( \log a(\tau) \), \( \log h(\tau) \), \( \phi(\tau) \), \( \log |\dot{\phi}(\tau)| \), \( \log |N \epsilon_v(\tau)| \) and \( p(\tau)/\rho(\tau) \) vs. \( \tau \) till a short time after the end of inflation. We define the time \( \tau_{end} \) when inflation ends by the condition \( \ddot{a}(\tau_{end}) = 0 \) which gives \( (\tau_{end} - \tau^*) = 18.2547816 \ldots \).

Furthermore, we study in this paper the curvature and tensor fluctuations during the whole inflaton evolution in its three successive regimes: non-inflationary fast-roll, inflationary fast-roll and inflationary slow-roll.

The equation for the scalar curvature fluctuations take in conformal time \( \eta \) and dimensionless variables the form

\[ \left[ \frac{d^2}{d\eta^2} + k^2 - W_R(\eta) \right] S_R(k; \eta) = 0. \]  

(2.24)

where \( d\eta = d\tau/a(\tau) \),

\[ W_R(\eta) \equiv \frac{1}{z} \frac{d^2 z}{d\eta^2} \quad \text{and} \quad z(\eta) \equiv \frac{\alpha(\eta)}{h(\eta)} \frac{d\phi}{d\tau}. \]  

(2.25)
FIG. 3: Time evolution during the three eras: non-inflatonary fast-roll, inflationary fast-roll and slow-roll and beyond the end of inflation (MD era). log $a(\tau)$, log $h(\tau)$, $\phi(\tau)$, $\log(\phi(\tau))$, $\log[N\epsilon_v(\tau)]$ and $p(\tau)/\rho(\tau)$ vs. $\tau$. $a(\tau)$ grows monotonically reaching 63 efolds by the end of inflation. $h(\tau)$ diverges for $\tau \to \tau_s = -0.8499574\ldots$ according to eq. (2.14) and decreases fast during fast-roll ($\tau \leq \tau_{trans} = 0.2487963\ldots$). Then, $h(\tau)$ decreases slowly during slow-roll as discussed in sec. IIIB. We depict $h(\tau)$ for short times $(0 < \tau - \tau_s < 0.3)$ in fig. 4. $\phi(\tau)$ diverges for $\tau \to \tau_s$ according to eq. (2.14) and decreases fast during fast-roll becoming very small during slow-roll. After the fast-roll stage where the inflaton field grows according to eq. (2.14), $\phi(\tau)$ slowly rolls toward its absolute minimum at $\phi_{end} = \sqrt{8N/y} = 19.52\ldots$ according to eq. (2.19). $\epsilon_v(\tau)$ decreases during fast-roll becoming of the order 1/$N$. We define the end of fast-roll (and beginning of slow-roll) by the condition $N\epsilon_v(\tau) \equiv 1$ which gives $\tau_{trans} - \tau_s = 0.2487963\ldots$. The equation of state $p(\tau)/\rho(\tau)$ fastly decreases during fast-roll from the value $p/\rho = +1$ for $\tau \to \tau_s$ [see eq. (2.15)] passing through $p/\rho = -1/3$ at the beginning of fast-roll inflation [see eq. (2.12)], $\tau = \tau_s = \tau_s = 0.0573$, and reaching $p/\rho = -1$ by the beginning of slow-roll. $p/\rho$ vanishes again near the end of slow-roll inflation by $\tau_{end} = \tau_s + 18.698\ldots$.

In cosmic time $\tau$, eq. (2.24) takes the form

$$\left[\frac{d^2}{d\tau^2} + h(\tau) \frac{d}{d\tau} + \frac{k^2}{a^2(\tau)} - V_R(\tau)\right]S_R(k; \tau) = 0. \quad (2.26)$$

where

$$V_R(\tau) \equiv \frac{W_R(\tau)}{a^2(\tau)} = h^2(\tau) \left[2 - 7\epsilon_v + 2\epsilon_v^2 - \sqrt{8\epsilon_v} \frac{v'(\phi)}{h^2(\tau)} - \eta_v (3 - \epsilon_v)\right] = h^2(\tau) \left[2 - 7\epsilon_v + 2\epsilon_v^2 - 2 \frac{d\phi}{d\tau} \left(\frac{v'(\phi)}{h(\tau)}\right) - \eta_v (3 - \epsilon_v)\right], \quad (2.27)$$

and $\epsilon_v$ and $\eta_v$ are given by eq. (2.19).

We display $V_R(\tau)$ vs. $\tau$ in fig. 4 for the best fit value of the coupling $y = 1.26$ and 63 efolds of slow-roll inflation.
The equation for the tensor fluctuations take in conformal time $\eta$ and dimensionless variables the form

$$S''_T(k; \eta) + \left[k^2 - \frac{a''(\eta)}{a(\eta)}\right] S_T(k; \eta) = 0 .$$

(2.28)

![Graphs and equations](image)

**FIG. 4**: The potential $V_R(\tau)$ felt by the fluctuations. Upper plot: $V_R(\tau)$ vs. $(\tau - \tau_*)$ in the stage where $V_R(\tau)$ is repulsive ($V_R(\tau) > 0$) which happens for $(\tau - \tau_*) > 0.114$. Notice that $V_R(\tau)$ slowly decreases during the slow-roll stage as $V_R(\tau) \simeq 2 h^2(\tau) + 1 + O(1/N)$ according to eq.(2.27) and fig. 3. Lower plots: Comparison of the exact (numerical) evolution and the analytic approximations eq.(2.43) during fast-roll and slow-roll. Left lower plot: $(\tau - \tau_*)^2 V_R(\tau)$ vs. $\tau - \tau_*$ in the stage where $V_R(\tau)$ is attractive ($V_R(\tau) < 0$) from the exact (numerical) calculation and from the analytic approximation eq.(2.43). This happens for $0 \leq (\tau - \tau_*) < 0.114$. Notice that $\lim_{\tau \to \tau_*} (\tau - \tau_*)^2 V_R(\tau) = -1/9$ according to eq.(4.1). Lower right plot: $V_R(\tau)$ vs. $\tau - \tau_*$ when $V_R(\tau) > 0$ from the exact (numerical) calculation and from the analytic approximation eq.(2.43).

**B. Inflaton and scale factor behaviour near the initial mathematical singularity**

In order to find the behaviour of $\phi(\tau)$ and $a(\tau)$ near the initial singularity we write

$$\phi(\tau) = \sqrt{\frac{2}{3}} \log \frac{\tau - \tau_*}{b} + \phi_1(\tau) \quad , \quad h(\tau) = \frac{1}{3 (\tau - \tau_*)} + h_1(\tau) .$$

(2.29)
Inserting now eqs. (2.29) into eqs. (2.14) yields for \( \phi_1(\tau) \) and \( h_1(\tau) \) the non-autonomous differential equations

\[
\frac{\dot{\phi}_1}{6} + \left( \frac{1}{\tau - \tau_s} + 3 \right) \dot{h}_1 + \sqrt{6} \frac{\dot{h}_1}{\tau - \tau_s} h_1 - \phi_1 - \sqrt{\frac{2}{3}} \log \frac{\tau - \tau_s}{b} + g \left( \frac{2}{3} \log \frac{\tau - \tau_s}{b} + \phi_1 \right)^3 = 0
\]

\[
h_1^2 + \frac{2}{3 (\tau - \tau_s)} h_1 - \frac{\dot{\phi}_1}{6} + \frac{1}{6} \left( \frac{2}{3} \log \frac{\tau - \tau_s}{b} + \phi_1 \right)^2 - \frac{g}{12} \left( \frac{2}{3} \log \frac{\tau - \tau_s}{b} + \phi_1 \right)^4 - \frac{1}{12 g} = 0,
\]

where \( \dot{\phi} \) stands for \( d\phi/d\tau \).

The asymptotic solution of eqs. (2.30) for \( \tau \to \tau_s \) turns to have the dominant form

\[
\phi_1(\tau) \xrightarrow{\tau \to \tau_s} (\tau - \tau_s)^2 P^\phi_4 \left( \log \frac{\tau - \tau_s}{b} \right), \quad h_1(\tau) \xrightarrow{\tau \to \tau_s} (\tau - \tau_s)^2 P^h_4 \left( \log \frac{\tau - \tau_s}{b} \right)
\]

where \( P^\phi_4(z) \) and \( P^h_4(z) \) are fourth degree polynomials in their arguments. The polynomials turn to be of fourth degree because the inflaton potential is of fourth degree. Their explicit expressions follow after calculation

\[
\phi_1(\tau) \xrightarrow{\tau \to \tau_s} - \frac{(\tau - \tau_s)^2}{\sqrt{6}} \left[ \frac{\theta}{18} \left( \frac{4}{b} \log \frac{\tau - \tau_s}{b} + \frac{2}{3} \log^3 \frac{\tau - \tau_s}{b} - \frac{11}{3} \log^2 \frac{\tau - \tau_s}{b} + \frac{49}{9} \log \frac{\tau - \tau_s}{b} - \frac{439}{54} \right) - \frac{1}{6} \left( \log^2 \frac{\tau - \tau_s}{b} + \frac{1}{3} \log \frac{\tau - \tau_s}{b} - \frac{7}{8} \right) + \frac{1}{8 g} \right],
\]
As a consequence, the scale factor near the singularity takes the form
\[ a(\tau) \overset{\tau \rightarrow \tau_*}{\sim} C (\tau - \tau_*)^{\frac{1}{3}} \left[ 1 + (\tau - \tau_*)^2 P_4^a \left( \log \frac{\tau - \tau_*}{b} \right) \right]. \] (2.33)

where the coefficients of the fourth order polynomial \( P_4^a \) can be obtained from eqs. (2.14) and (2.32).

C. Quantum loop effects and the validity of the classical inflaton picture

When \( \tau \to \tau_* \), quantum loop corrections are expected to become very large spoiling the classical description. More precisely, quantum loop corrections are of the order \( (H/M_{Pl})^2 \). From eqs. (2.7) and (2.14) the quantum loop corrections are of the order
\[ \left( \frac{H}{M_{Pl}} \right)^2 (\tau - \tau_*) \leq \frac{1}{9} \left( \frac{\tau_{Planck}}{\tau - \tau_*} \right)^2 \]

where we used \( m = 1.21 \times 10^{13} \) GeV \( [2] \).

The characteristic time is here the Planck time
\[ \tau_{Planck} = m t_{Planck} = \frac{m}{M_{Pl}} = 2.703 \times 10^{-43} \text{ sec} \times m = 4.97 \times 10^{-6}. \]

Namely, the quantum loop corrections are less than 1% for times
\[ (\tau - \tau_*) > \frac{10}{3} \tau_{Planck} = 1.66 \times 10^{-5}. \] (2.34)

Therefore, for times \( (\tau - \tau_*) > 10^{-5} \) the classical treatment of the inflaton and the space-time presented in sec. [11] and [16] can be trusted and we see that the classical description has a wide domain of validity.

The use of a classical and homogeneous inflaton field is justified in the out of equilibrium field theory context as the quantum formation of a condensate during inflation. This condensate turns to obey the classical evolution equations of an homogeneous inflaton [11].

We see from eq. (2.14) that the inflaton field becomes negative for \( \tau \to \tau_* \). But since a condensate field should be always positive, the classical and homogeneous inflaton picture requires
\[ \tau - \tau_* > b \]

For the best fit coupling \( y = 1.26 \) and 63 e-folds of inflation we have \( b = 4.745272 \ldots \) \( 10^{-5} = 9.55 \tau_{Planck} \) which is consistent with eq. (2.34). By comparing this value of \( b \) with eq. (2.34) we see that the quantum loop corrections are negligible in the stage where the condensate is already formed.

We can obtain a lower bound on \( b \) since \( b \) increases with the number of inflation e-folds \( N_{ef} \) at fixed inflaton potential and since \( N_{ef} \) cannot be smaller than the lower bound provided by flatness and entropy [2].

Although all inflationary solutions obtained evolving backwards in time from the slow-roll stage do reach a zero of the scale factor, such mathematical singularity is attained extrapolating the classical treatment where it is no more valid. In fact, one never reaches the singularity in the validity region of the classical treatment. In summary, the classical singularity at \( \tau = \tau_* \) is not a real physical phenomenon here.

The classical description with the homogeneous inflaton is very good for \( \tau - \tau_* > 10 \tau_{Planck} \) well before the beginning of inflation.
D. The fast-roll regime: analytic approach

As we see from fig. 3 the inflaton field $\phi(\tau)$ is much smaller than $d\phi/d\tau$ during fast-roll. We can therefore approximate the coupled inflaton evolution equation and Friedmann equation eqs. (2.11) as

$$
\frac{d^2\phi}{d\tau^2} + 3 \frac{h}{d\tau}\frac{d\phi}{d\tau} = 0,
$$

$$
h^2(\tau) = \frac{1}{3} \left[ \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{4g} \right].
$$

(2.35)

Or, in a compact form,

$$
\frac{d^2\phi}{d\tau^2} + \sqrt{\frac{3}{2}} \frac{d\phi}{d\tau} \sqrt{\left( \frac{d\phi}{d\tau} \right)^2 + \frac{1}{2g}} = 0,
$$

(2.36)

which has the exact solution

$$
\frac{d\phi}{d\tau} = \sqrt{\frac{2}{3}} \frac{1}{\tau_1 \sinh \left( \frac{\tau - \tau_\ast}{\tau_1} \right)} , \quad \phi(\tau) = \sqrt{\frac{2}{3}} \log \left[ \frac{2\tau_1}{b} \tanh \left( \frac{\tau - \tau_\ast}{2\tau_1} \right) \right] ,
$$

(2.37)

where $\tau_1$ turns out to be the characteristic time scale

$$
\tau_1 = 2 \sqrt{\frac{g}{3}} = \sqrt{\frac{y}{6N}}.
$$

(2.38)

We find for the best fit to CMB and LSS data, $y = 1.26$ and $N = 60$,

$$
\tau_1 = 0.0592 = 11910 \tau_{\text{Planck}},
$$

(2.39)

well after the Planck scale $\tau_{\text{Planck}} = 4.97 \times 10^{-6}$.

The integration constant in eq. (2.37) matches with the small $\tau - \tau_\ast$ behaviour eq. (2.14). The Hubble parameter and the scale factor are here

$$
h(\tau) = \frac{1}{3 \tau_1} \coth u , \quad a(\tau) = C \left| \tau_1 \sinh u \right|^{1/2} , \quad u = \frac{\tau - \tau_\ast}{\tau_1},
$$

(2.40)

where the integration constant was chosen to fulfill eq. (2.16). The scale factor eq. (2.40) interpolates between the non-inflationary power law behaviour eq. (2.16) for $\tau - \tau_\ast \to 0$ and the eternal inflationary de Sitter behaviour for $\tau - \tau_\ast \gg \tau_1$. Since we have set $v(\phi)$ equal to constant, slow-roll De Sitter inflation never stops in this approximation. Namely, neither matter-dominated nor radiation-dominated eras are reached in this approximation.

We can eliminate the variable $u$ between $\phi$ and $d\phi/d\tau$ in eq. (2.37) with the result

$$
\frac{d\phi}{d\tau} = \sqrt{\frac{2}{3}} \left[ e^{-\sqrt{\frac{2}{3}} \phi(\tau)} - \frac{b}{4 \tau_1^2} e^{\sqrt{\frac{2}{3}} \phi(\tau)} \right].
$$

(2.41)

This equation generalizes eq. (2.18) which corresponds to the first term here and describes the behaviour for $\tau - \tau_\ast$. Notice that

$$
-\infty < \phi(\tau) < \sqrt{\frac{2}{3}} \log \left[ \frac{2\tau_1}{b} \right], \quad 0 < \frac{d\phi}{d\tau} < +\infty
$$

and that $b/[2\tau_1] = 4.0105 \times 10^{-3}$.

The evolution described by eqs. (2.37)-(2.40) starts from the mathematical singularity at $\tau = \tau_\ast$ with monotonically decreasing $d\phi/d\tau$ and $h(\tau)$ and a monotonically increasing $\phi(\tau)$ from its initial value $\phi(\tau_\ast) = -\infty$.

Slow-roll is reached asymptotically for large $\tau$ since $d\phi/d\tau$ vanishes for $\tau - \tau_\ast \to \infty$. 

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[Note: The page ends here, but the rest of the content is not transcribed.]

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We find for the parameter $\epsilon_v$ [eq. (2.19)] and for the equation of state,

$$
\epsilon_v(\tau) = \frac{3}{1 + \sinh^2 u}, \quad \frac{p(\tau)}{\rho(\tau)} = \frac{2}{\cosh^2 u} - 1 .
$$

(2.42)

We see that $\epsilon_v(\tau)$ monotonically decreases with $\tau$ and vanishes for $\tau - \tau_s \to \infty$. The equation of state $p/\rho$ smoothly interpolates between $+1$ at $\tau = \tau_s$ (extreme non-inflationary fast-roll) and $-1$ (slow-roll inflation) for $\tau - \tau_s \to \infty$, passing by $p/\rho = -1/3$ (the beginning of fast-roll inflation) at $\tau - \tau_s = 0.0573$.

The potential $V_R(\tau)$ eq.(2.24) felt by the fluctuations takes here the form

$$
V_R(\tau) = \frac{1}{6} g \left[ 1 - \frac{1}{2} \frac{\cosh^2 u}{\sinh^2 u} - \frac{9}{\cosh^2 u} \right] , \quad u = \frac{\tau - \tau_s}{\tau_1} .
$$

(2.43)

The limiting values of $h(\tau)$, $\phi(\tau)$ and $V_R(\tau)$ for $\tau \to \infty$ give a reasonable approximation to the numerical results. We have

$$
h(\infty) = \frac{1}{3} \frac{\tau_1}{\tau}, \quad \phi(\infty) = \sqrt{\frac{2}{3}} \log \left[ \frac{2 \tau_1}{b} \right], \quad \frac{d\phi}{d\tau}(\infty) = 0 , \quad V_R(\infty) = \frac{4 N}{3 y} .
$$

(2.44)

The characteristic time scale $\tau_1$ is generically a small number since according to eq.(2.38) $\tau_1 \sim 1/\sqrt{N}$. The value of $\tau_1$ for the best fit value for $y$ is given in eq.(2.39).

The end of fast-roll $\tau_{trans}$ can be estimated in this approximation by using eq.(2.42) for $\epsilon_v(\tau)$ setting $\epsilon_v(\tau_{trans}) = 1/N$. This gives,

$$
\epsilon_v(\tau) \approx 12 e^{-2\tau_{trans}/\tau_1} = \frac{1}{N} , \quad \tau_{trans} \approx \frac{1}{2} \tau_1 \ln(12 N) = 0.195 .
$$

This approximated value for $\tau_{trans}$ should be compared with the exact numerical result $\tau_{trans} = 0.2487963 \ldots$ $h(\tau_{trans})$ and $V_R(\tau_{trans})$ differ in less than 1% from their values at $\tau = \infty$ given by eq.(2.44).

In figs. 3 we plot $\ln a(\tau)$, $\ln h(\tau)$, $\ln |\phi(\tau)|$, $\epsilon_v(\tau)$ and $p(\tau)/\rho(\tau)$ computed numerically and computed using the analytic expressions eqs.(2.37)-(2.42). We compare in figs. 4 the exact potential $V_R(\tau)$ with the analytic approximation eq.(2.43).

We see that the simple analytic formulas eqs.(2.37)-(2.43) provide a very good approximation during the fast-roll regime $\tau \leq \tau_{trans} = 0.2487963 \ldots$ In particular, eq.(2.37) provides an excellent approximation to $\phi(\tau)$ as shown in fig. 6. In particular, the analytic formulas eqs.(2.37)-(2.43) become exact near the singularity at $\tau = \tau_s$.

E. The fast-roll regime: numerical solution

To construct a singular solution we can integrate eqs. (2.11) backwards in time starting from initial conditions of strong non-inflationary fast–roll type, namely

$$
K = \frac{\dot{\phi}^2}{2v(\phi)} \gg 1 ,
$$

producing a given total number $N_{e\tau}$ of slow–roll inflationary efolds. For instance, we start from some $\phi$ and $\dot{\phi}$ such that $K = 10^4$. The time extent backwards from this moment has to be limited so that, integrating back and forth, the required relative accuracy of $10^{-12}$ is preserved. We furthermore impose that $N_{e\tau} = 63$.

We adopt the convention that conformal time $\eta$ vanishes from below when inflation ends and that $a(\tau = 0) = 1$ when there are still $N = 60$ efolds till the end of inflation. This choice of the scale factor normalization seems the most natural. Then, $\eta$ has a finite non-zero limit $\eta_s$ as $\tau$ approaches the time $\tau_s$ of the singularity, since $a(\tau) \simeq C (\tau - \tau_s)^{1/3}$ as $\tau \to \tau_s$ according to eq. (2.19). That is,

$$
\eta = \int_{\tau_{end}}^{\tau} \frac{d\tau'}{a(\tau')} = \eta_s + \int_{\tau_s}^{\tau} \frac{d\tau'}{a(\tau')} .
$$

The numerics of a fast–roll solution of this type are in Table I where a relative accuracy of $10^{-12}$ is preserved.
Using the asymptotic behaviour eq. \((2.14)\) as \(\tau \to \tau^*_+\) we obtain from Table I:

\[
\tau_\ast = -0.8499574\ldots, \quad b = 4.745272\ldots10^{-5} \quad \text{and} \quad \eta_\ast = -15.605614\ldots.
\]

Slow-roll begins at \(\tau_{\text{trans}} = \tau_\ast + 0.2487963\ldots = -0.6011611\ldots\).

The initial value of the ratio

\[
\frac{d\varphi}{dt} = m \frac{\dot{\phi}}{\phi}
\]

has the dimension of mass. The natural mass scale in the problem is here the energy scale of inflation \(M\). Therefore, assuming this ratio of the order \(M\) yields

\[
\frac{\dot{\phi}}{\phi} < M \sim 10^3.
\]

Hence, it is natural to start the fast-roll evolution with \(\dot{\phi}/\phi < 10^3\).

| \(K = 5.3458 \ldots 10^3\) | \(K = 10^4\) | inflation start: \(\ddot{a} = 0\) | fast-roll \(\to\) slow-roll | \(a = 1\) | inflation end: \(\ddot{a} = 0^+\) |
|---|---|---|---|---|---|
| \(\tau\) | -0.8499493 | -0.8494993 | -0.7746949 | -0.6011611 | 0 | 17.408422 \ldots |
| \(\phi\) | -1.4410123 | 2.6909604 | 5.3942489 | 6.4783577 | 6.7484076 | 18.558653 \ldots |
| \(\dot{\phi}\) | 10.039135 \ldots | 13.6505241 | 8.8601670 | 0.9182661 | 0.3974015 | 0.9415055 \ldots |
| \(\log a\) | -7.9325621 \ldots | -5.599353 | -3.9142151 | -2.9909999 | 0 | 60 |
| \(h\) | 40.984.4689 | 55.7.30817 | 6.2650841 | 5.0295509 | 4.9653990 | 0.6657449 \ldots |
| \(\eta\) | -15.6050091 \ldots | -15.376218 \ldots | -15.354999 \ldots | -4.0169827 \ldots | -0.2090609 \ldots | 0 |

**TABLE I**: Fast-roll solution with \(N_{sr} = 63\) efolds of slow-roll inflation. Recall that \(\tau = 4.97 \times 10^{-6} (t/t_{\text{Planck}})\).

### III. THE SLOW-ROLL INFLATIONARY ERA

#### A. The extreme slow-roll solution

There always exist a special solution of eqs.\((2.11)\) that starts at \(\tau = -\infty\) with vanishing inflaton, vanishing scale factor but nonzero Hubble parameter. More precisely, eqs.\((2.11)\) can be approximated for small \(\phi\) and \(\dot{\phi}\) as

\[
\frac{d^2 \phi}{d\tau^2} + 3 \frac{d\phi}{d\tau} - \phi = 0, \quad h^2(\tau) = \frac{1}{3} v(0),
\]

where we used eqs.\((2.9)\) and \((2.11)\).

Eqs.\((3.1)\) admit the asymptotic solution for \(\tau \to -\infty\)

\[
\phi(\tau) \xrightarrow{\tau \to -\infty} C_0 e^\alpha \tau \to 0, \quad h(\tau) \xrightarrow{\tau \to -\infty} \sqrt{\frac{v(0)}{3}}, \quad a(\tau) \xrightarrow{\tau \to -\infty} e^{\sqrt{\frac{2h}{3h^2}}} \tau \to 0,
\]

where \(C_0\) is an integration constant, \(v(0) = 2N/y\) for the double-well potential eq.\((2.23)\) and

\[
\alpha = \frac{1}{2} \left[ \sqrt{3v(0)} + 4 - \sqrt{3v(0)} \right] > 0.
\]

Notice that \(\alpha\) can be expressed in terms of the fast-roll characteristic time-scale \(\tau_1\) [eq.\((2.38)\)],

\[
\alpha = \frac{1}{2} \tau_1 \left[ \sqrt{1 + 4 \tau_1^2} - 1 \right] \simeq \tau_1
\]
It must be noticed that the characteristic time scale of the inflaton evolution in the extreme slow-roll solution for early times [see eq. (3.2)]

\[ \frac{1}{\alpha} \simeq \frac{1}{\tau_1} \gg 1, \]

turns to be the inverse of the characteristic time scale \( \tau_1 \) of the fast-roll solution and to be very large.

On the contrary, the characteristic time scale of the scale factor evolution in the same regime is very short

\[ \sqrt{3} v(0) = 3 \tau_1 \ll 1. \]

The fast-roll stages both non-inflationary and inflationary are absent in this solution. The extreme slow-roll solution only possesses the slow-roll inflationary stage followed by the matter dominated era.

| inflation start | a = 1 | inflation end: \( \ddot{a} = 0 \) |
|------------------|-------|----------------------------------|
| \( \tau \)      | -344.9514017 \ldots | 0 | 17.40482446 \ldots |
| \( \phi \)      | 10^{-8} | 6.7484118 \ldots | 18.5586530 \ldots |
| \( \dot{\phi} \) | \( 10^{-8} = 5.8937108453 \ldots \times 10^{-10} \) | 0.3973384 \ldots | 0.94150557 \ldots |
| log \( a \)     | -1938.4867948 \ldots | 0 | 60 |
| \( h \)         | \( \sqrt{2 N/(3 y)} = 5.6361006 \ldots \) | 4.9653973 \ldots | 0.6657449 \ldots |
| \( \eta \)      | \( -\infty \) (f.a.p.p) | -0.0200610 \ldots | 0 |

TABLE II: Relevant quantities of the extreme slow-roll inflaton solution for the coupling \( y = 1.2592226 \ldots \). We adopt the convention that \( a(\tau = 0) = 1 \) when there are still \( N = 60 \) e-folds till the end of inflation. Recall that \( \tau = 4.97 \times 10^{-6} \) (\( t/t_{\text{Planck}} \)).

For the value of the coupling \( y = 1.2592226 \ldots \), we get for the extreme slow-roll solution

\[ \alpha = 0.058937108 \ldots \quad \phi_{\text{end}} = 18.5586530 \ldots \quad \dot{\phi}_{\text{end}} = 0.9415055 \ldots \] (3.3)

In table II we display the values of the relevant magnitudes for this extreme slow-roll solution.

Except for the extreme slow-roll solution, all solutions are of fast-roll type and come from singular values of \( \phi \) and \( h \) according to eq. (2.14) as \( \tau \to \tau_1^+ \) for some finite \( \tau_1 \) characteristic of each particular solution. The slow-roll stage (which starts when \( \epsilon_v = 1/N \) from above, and ends when again \( \epsilon_v = 1/N \) from below) of all distinct solutions turns to be almost identical to that of the extreme slow-roll case as one could expect for an attractor.

### B. The inflaton during slow-roll inflation: analytical solution

In the slow-roll regime higher time derivatives can be neglected in the evolution eqs. (2.11) with the result

\[ 3 h(\tau) \dot{\phi} + v'(\phi) = 0 \quad , \quad h^2(\tau) = \frac{v(\phi)}{3}. \] (3.4)

These first order equations can be solved in closed form as

\[ N[\phi] = -\int_{\phi}^{\phi_{\text{end}}} v(\phi') \frac{d\phi'}{dv} d\phi'. \] (3.5)

where \( N[\phi] \) is the number of e-folds since the field \( \phi \) exits the horizon till the end of inflation (where it takes the value \( \phi_{\text{end}} \)).

Eq. (3.5) indicates that \( N[\phi] \) scales as \( \phi^2 \) and hence the field \( \phi \) is of the order \( \sqrt{N} \sim \sqrt{60} \). Therefore, we proposed as universal form for the inflaton potential \( v(\phi) = N M^4 w(\chi) \),

\[ v(\phi) = N M^4 w(\chi), \] (3.6)
where \( \chi \) is the dimensionless, slowly varying field

\[
\chi = \frac{\varphi}{\sqrt{N} M_{Pl}} = \frac{\varphi}{\sqrt{N}}. \tag{3.7}
\]

The equations of motion (3.11) in the field \( \chi \) become

\[
\mathcal{H}^2(\hat{\tau}) = \frac{1}{3} \left[ \frac{1}{2N} \left( \frac{d\chi}{d\hat{\tau}} \right)^2 + w(\chi) \right] \quad \text{with} \quad \mathcal{H} = \frac{h}{\sqrt{N}},
\]

\[
\frac{1}{N} \frac{d^2\chi}{d\hat{\tau}^2} + 3 \mathcal{H} \frac{d\chi}{d\hat{\tau}} + w'(\chi) = 0 \quad . \tag{3.8}
\]

and \( \hat{\tau} \) stands for the rescaled dimensionless time

\[
\hat{\tau} \equiv \frac{\tau}{\sqrt{N}} = \frac{m t}{\sqrt{N}} .
\]

To leading order in the slow-roll approximation (neglecting \( 1/N \) corrections), eqs. (3.8) are solvable in terms of quadratures

\[
\hat{\tau} - \hat{\tau}_{trans} = - \int_{\chi(\hat{\tau}_{trans})}^{\chi} d\chi' \frac{\sqrt{3 w'(\chi')}}{w''(\chi')} , \tag{3.9}
\]

where \( \hat{\tau}_{trans} \) stands for the beginning of slow-roll inflation and we used that

\[
\mathcal{H}(\hat{\tau}) = \sqrt{\frac{w'(\chi)}{3}} + \mathcal{O}\left(\frac{1}{N}\right) , \tag{3.10}
\]

For the broken symmetric potential eq. (2.20), from eqs. (2.10), (3.9) and (3.10), we find

\[
\chi(\hat{\tau}) = \chi(\hat{\tau}_{trans}) e^{\sqrt{\frac{N}{3}} (\hat{\tau} - \hat{\tau}_{trans})} + \mathcal{O}\left(\frac{1}{N}\right) = \sqrt{\frac{N}{3}} e^{-\sqrt{\frac{N}{3}} (\hat{\tau}_{end} - \hat{\tau})} + \mathcal{O}\left(\frac{1}{N}\right) , \tag{3.11}
\]

\[
\mathcal{H}(\hat{\tau}) = \sqrt{\frac{2}{3y}} \left[ 1 - e^{-\sqrt{\frac{N}{3}} (\hat{\tau}_{end} - \hat{\tau})} \right] + \mathcal{O}\left(\frac{1}{N}\right) ,
\]

\[
\frac{p}{\rho}(\hat{\tau}) = -1 + \frac{y}{6N} \sin^2 \left[ \frac{1}{\sqrt{6}} (\hat{\tau}_{end} - \hat{\tau}) \right] + \mathcal{O}\left(\frac{1}{N^2}\right) , \tag{3.12}
\]

for \( \hat{\tau}_{trans} \leq \hat{\tau} \leq \hat{\tau}_{end} = \sqrt{\frac{3}{2y}} \ln \left[ \frac{8}{\chi^2(\hat{\tau}_{trans})} \right] + \mathcal{O}\left(\frac{1}{N}\right) . \tag{3.13}
\]

Inflation ends when the equation of state becomes \( \rho/p = -1/3 \) [see eq. (2.12)]. According to eq. (3.12), this happens when \( \hat{\tau}_{end} - \hat{\tau} \sim \mathcal{O}\left(1/\sqrt{N}\right) \). Therefore, expressions eqs. (3.11), (3.12) are valid as long as

\[
\hat{\tau}_{trans} \leq \hat{\tau} \leq \hat{\tau}_{end} = \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \quad \text{where} \quad \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) > 0 .
\]

That is, eqs. (3.11) hold while the inflaton is not very near the minimum of the potential \( \chi_{end} = \sqrt{8/y} \).

By integrating the Hubble parameter \( \mathcal{H}(\hat{\tau}) \) we obtain for the scale factor \( a(\hat{\tau}) \)

\[
\log \frac{a(\hat{\tau})}{a(\hat{\tau}_{trans})} = \sqrt{\frac{2}{3y}} N (\hat{\tau} - \hat{\tau}_{trans}) - \frac{N}{8} \chi^2(\hat{\tau}_{trans}) \left[ e^{\sqrt{\frac{N}{3}} (\hat{\tau} - \hat{\tau}_{trans})} - 1 \right] = \tag{3.14}
\]

\[
= \sqrt{\frac{2}{3y}} N m (t - t_{trans}) - \frac{1}{8} \left[ \frac{\varphi(t_{trans})}{M_{Pl}} \right]^2 \left[ e^{\sqrt{\frac{N}{3}} m (t - t_{trans})} - 1 \right] ,
\]

where we used eqs. (2.7) and (3.11). It must be noticed that \( a(\hat{\tau}) \) is not exactly a de Sitter scale factor, even in the large \( N \) limit at fixed \( \hat{\tau} \).
At the end of inflation the number of efolds is \( \ln a \approx 64 \), the inflaton is near its minimum
\[
\chi = \sqrt{\frac{8}{y}} \approx 2.52,
\]
\( \dot{\chi} \) starts to oscillate around zero and \( \mathcal{H}(\tilde{\tau}) \) begins a rapid decrease (see figs. 3). At this time the inflaton field is no longer slowly coasting in the \( w''(\chi) < 0 \) region but rapidly approaching its equilibrium minimum. When inflation ends, the inflaton is at its minimum value up to corrections of order \( 1/\sqrt{N} \). Therefore, we see from the Friedmann eq.(3.8) and eqs.(3.11) that
\[
\frac{1}{N} \left( \frac{d\chi}{d\tilde{\tau}} \right)^2 (\tilde{\tau}_{\text{end}}) = \mathcal{O} \left( \frac{1}{N} \right) , \quad w(\chi(\tilde{\tau}_{\text{end}})) = \mathcal{O} \left( \frac{1}{N} \right) \quad \text{and therefore,} \quad \mathcal{H}(\tilde{\tau}_{\text{end}}) = \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) , \quad (3.15)
\]
while \( \mathcal{H}(\tilde{\tau}_{\text{trans}}) = \mathcal{O}(1) \). Namely, the Hubble parameter decreases by a factor of the order \( \sqrt{N} \sim 8 \) during slow-roll inflation. We see in fig. 3 that the exact \( \mathcal{H}(\tilde{\tau}) \) decreases by a factor six during slow-roll inflation, confirming the slow-roll analytic estimate.

We can compute the total number of inflation efolds \( N_{\text{tot}} \) to leading order in slow-roll inserting the analytic formula for \( \tilde{\tau}_{\text{end}} \) eq.(3.13) in eq.(3.14) with the result,
\[
N_{\text{tot}} = \frac{N}{y} \left\{ \ln \left[ \frac{8}{\chi^2(\tilde{\tau}_{\text{trans}})} \right] - 1 + \frac{1}{8} \ y \chi^2(\tilde{\tau}_{\text{trans}}) \right\} + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) . \quad (3.16)
\]
We have verified the slow-roll analytical results eqs.(3.11)-(3.16) comparing them with the numerical solution of eqs.(2.11). Both results are concordant up to the error estimation in each case: \( \mathcal{O}(1/N) \) or \( \mathcal{O}(1/\sqrt{N}) \).

The field \( \phi \) as a function of the dimensionless time \( \tau \) eq.(3.11) takes the form
\[
\phi(\tau) = \phi(\tau_{\text{trans}}) \ e^{\sqrt{\frac{y}{6N}} (\tau-\tau_{\text{trans}})}
\]
and then
\[
\dot{\phi}(\tau) = \sqrt{\frac{y}{6N}} \phi(\tau) .
\]
For \( y \approx 1.26 \) and \( N = 60 \) we get \( \sqrt{y/[6N]} = 0.0577 \) in agreement with the slope of the red quasi-horizontal slow-roll line in the phase space flow fig. 1.

IV. COMPLETE FLUCTUATIONS EVOLUTION AND FAST-ROLL EFFECTS ON THE POWER SPECTRUM.

A. Scalar and tensor fluctuations near the initial singularity.

In order to study the curvature and tensor fluctuations in this regime, it is important to evaluate the parameter \( \epsilon_v \) and the potential felt by the fluctuations \( V_R \). Inserting eqs. (2.29) and (2.31) into eqs. (2.19) and (2.27) yields near the initial singularity
\[
V_R(\tau) \ \tau^{-2\tau_*} - \frac{1}{9} (\tau - \tau_*)^2 P_4^V \left( \log \frac{\tau - \tau_*}{b} \right) , \quad \epsilon_v \ \tau^{-2\tau_*} 3 \left[ 1 + (\tau - \tau_*)^2 P_4^v \left( \log \frac{\tau - \tau_*}{b} \right) \right] \quad (4.1)
\]
\[
W_R(\eta) \ \eta^{-\eta_*} - \frac{1}{4} \eta^2 \left[ 1 + \eta^3 P_4^W (\log \eta) \right] . \quad (4.2)
\]
where
\[
\eta \ \tau^{-\tau_*} 3 \left( \tau - \tau_* \right)^{\frac{3}{2}}
\]
is the conformal time for \( \tau \to \tau_* \) and \( P_4^V(x), P_4^v(x) \) and \( P_4^W(x) \) are polynomials of degree four in \( x \).
We see that the fluctuations feel a singular attractive potential near the \( \eta = 0 \) singularity. Actually, the behaviour of \( W_R(\eta) \) for \( \eta \to 0 \) is exactly the critical strength \((-1/4)\) for which the fall to the centre becomes possible in a central and attractive singular potential [3].

We find from eqs. (2.26) and (4.1) for the fluctuations near the singularity

\[
S_R(k; \eta) \xrightarrow{\eta, \eta_0 \to 0} \sqrt{\frac{\eta}{\eta_0}} \left[ A_R(k) + B_R(k) \log \frac{\eta}{\eta_0} \right], \tag{4.3}
\]

where \( \eta_0 \) is the time when the initial conditions will be imposed, \( A_R \) and \( B_R \) are complex constants constrained by the Wronskian condition (that ensures the canonical commutation relations) [2]

\[
W[S_R, S^*_R] = S_R \frac{dS^*_R}{d\eta} - \frac{dS_R}{d\eta} S^*_R = i. \tag{4.4}
\]

Namely,

\[
2 \text{ Im}[A_R B^*_R] = \eta_0. \tag{4.5}
\]

Precisely, the logarithmic behaviour for \( \eta \to 0 \) of the wave function eq.(4.3) describes the fall to \( \tau - \tau^* = 0 \) for the critical strength of the potential \( W_R(\eta) \). For larger attractive strengths the wave function eq.(4.3) shows up an oscillatory behaviour [3]. Notice, however the physical nature of the process: here we have a time evolution near a classical singularity at a given time while in the potential case one has particles falling (or emerging) from a point in space where the potential is singular.

In general, the mode functions for large \( k \) must behave as free modes (plane waves) since the potential \( W_R(\eta) \) in eq.(2.24) becomes negligible in this limit except at the singularity \( \tau = \tau^* \). One can then impose Bunch-Davies conditions for large \( k \) which corresponds to assume an initial quantum vacuum Fock state, empty of curvature excitations [2]

\[
S_R(k; \tau) \xrightarrow{k \to \infty} e^{-i k \eta} \sqrt{\frac{2}{k}}, \tag{4.6}
\]

and therefore

\[
\frac{dS_R}{d\eta}(k; \eta_0) \xrightarrow{k \to \infty} -i k S_R(k; \eta_0). \tag{4.7}
\]

Eq.(4.6) fulfils the Wronskian normalization eq.(4.4).

In asymptotically flat (or conformally flat) regions of the space-time the potential felt by the fluctuations vanish and the fluctuations exhibit a plane wave behaviour for all \( k \) (not necessarily large). This is not the case near strong gravity fields or curvature singularities as in the present cosmological space-time where \( W_R(\eta) \) can never be neglected at fixed \( k \). However, we can choose Bunch-Davies initial conditions (BDic) at \( \eta = \eta_0 \) by imposing

\[
\frac{dS_R}{d\eta}(k; \eta_0) = -i k S_R(k; \eta_0) \quad \text{for all } k. \tag{4.7}
\]

That is, we consider the initial value problem for the mode functions giving the values of \( S_R(k; \eta) \) and \( dS_R/d\eta \) at \( \eta = \eta_0 \).

Notice that eq.(4.7) combined with the Wronskian condition eq.(4.4) implies that

\[
|S_R(k; \eta_0)| = \frac{1}{\sqrt{2/k}}, \quad \left| \frac{dS_R}{d\eta}(k; \eta_0) \right| = \sqrt{\frac{k}{2}}.
\]

which is equivalent to eq.(4.6) for large \( k \).

Since the mode functions \( S_R(k; \eta) \) are defined up to an arbitrary constant phase we can write eq.(4.3) valid near the metric singularity as

\[
S_R(k; \eta) \xrightarrow{\eta, \eta_0 \to 0} \sqrt{\frac{\eta}{2k \eta_0}} \left[ 1 - \left( \frac{1}{2} + i k \eta_0 \right) \log \frac{\eta}{\eta_0} \right]. \tag{4.8}
\]
In eq. (4.3) this corresponds to the coefficients,

\[ A_R = \frac{1}{\sqrt{2}} k, \quad B_R = \frac{1}{\sqrt{2}} k \left( \frac{1}{2} + i \ k \eta_0 \right) \]

We have in cosmic time,

\[ S_R(k; \tau) \xrightarrow{\tau \to \tau_*} \frac{1}{\sqrt{2}} k \left( \frac{\tau - \tau_*}{\tau_0} \right)^{\frac{1}{2}} \left[ -1 + \left( \frac{1}{3} + i \ k \tau_0^3 \right) \log \frac{\tau - \tau_*}{\tau_0} \right], \quad (4.9) \]

where \( \tau_0 - \tau_* = (2 \eta_0/3)^{2} \) for \( \tau \to \tau_* \).

Namely, imposing the BD initial condition (BDic) eq. (4.7) at small \( \eta_0 \) where the small \( \eta \) behavior eq. (4.3) applies, yields specific values for the coefficients of the linearly independent solutions \( \sqrt{\eta} \) and \( \sqrt{\eta} \log \eta \) that we can read from eqs. (4.8)-(4.9).

For general \( \tau_0 \) (i.e., \( \tau_0 \) not near \( \tau_* \)), the mode functions for \( \tau \to \tau_* \) take the form

\[ S_R(k; \tau) \xrightarrow{\tau \to \tau_*} \frac{1}{\sqrt{2}} k \left( \frac{\tau - \tau_*}{\tau_0} \right)^{\frac{1}{2}} \left[ X(k; \tau_0) - \left( \frac{Y(k; \tau_0)}{X(k; \tau_0)} \right) \left( \frac{i \ k \tau_0^3}{X(k; \tau_0)} \log \frac{\tau - \tau_*}{\tau_0} \right) \right], \quad (4.10) \]

where we imposed eq. (4.5) and we have from eq. (4.9),

\[ X(k; \tau_0) = 1 \quad \text{and} \quad Y(k; \tau_0) = \frac{1}{3}. \]

Notice that \( X(k; \tau_0) > 0 \) for \( \tau_0 \to \tau_* \) as we see from eq. (4.9). Our numerical calculations show that \( X(k; \tau_0) > 0 \) for all \( \tau_0 \) and \( k \).

**B. The primordial power spectrum, Scalar curvature fluctuations and the CMB+LSS data.**

The power spectrum of curvature perturbations \( \mathcal{R} \) is given by the expectation value \( \langle R^2 \rangle \) in the state with general initial conditions,\(^2\)

\[ \langle R^2(\vec{x}, \eta) \rangle = \left( \frac{m}{M_{PL}} \right)^2 \int_0^\infty \frac{|S_R(k; \eta)|^2}{z^2(\eta)} \frac{k^2 \, dk}{2 \pi^2}. \quad (4.11) \]

where \( z(\eta) \) is given by eq. (2.25). Notice in eq. (4.11) the factor \( (m/M_{PL})^2 \) in the physical power spectrum expressed in terms of the dimensionless quantities used here.

The power spectrum at time \( \eta \) is customary defined as the power per unit logarithmic interval in \( k \)

\[ \langle R^2(\vec{x}, \eta) \rangle = \int_0^\infty \frac{dk}{k} P_R(k, \eta). \]

Therefore, the scalar power for general initial conditions is given by the fluctuations behavior by the end of inflation,\(^2\)

\[ P_R(k) = \left( \frac{m}{M_{PL}} \right)^2 \frac{k^3}{2 \pi^2} \lim_{\eta \to 0^-} \left| \frac{S_R(k; \eta)}{z(\eta)} \right|^2. \quad (4.12) \]
The mode functions $S_R(k; \eta)$ obey the fluctuations equation (2.24) where the potential $W_R(\eta)$ [eq. (2.27)] during slow-roll and to leading order in $1/N$ takes the simple form [2],

$$W_R(\eta) = \frac{2}{\eta^2} \left[ 1 + \frac{3}{2} (3 \epsilon_v - \eta_v) \right] = \frac{\nu_R^2 - \frac{1}{2}}{\eta^2} , \quad \nu_R = \frac{3}{2} + 3 \epsilon_v - \eta_v + O \left( \frac{1}{N^2} \right) .$$

(4.13)

In the slow-roll regime we can consider $\epsilon_v$ and $\eta_v$ [see eq. (2.19)] constants in time in eq. (4.13). During slow-roll, the general solution of eq. (2.24) is then given by

$$S_R(k; \eta) = A(k) g_{\nu_R}(k; \eta) + B_R(k) g^*_{\nu_R}(k; \eta) ,$$

(4.14)

with

$$g_{\nu}(k; \eta) = \frac{1}{2} \nu^{\nu+\frac{1}{2}} \sqrt{-\pi \eta} H^{(1)}_{\nu}(\eta) ,$$

(4.15)

$A(k), B_R(k)$ are constants determined by the initial conditions and $H^{(1)}_{\nu}(z)$ is a Hankel function.

The Wronskian of the solutions $S_R, S_R^*$ is given by eq. (4.11) and

$$W[g_{\nu}, g^*_\nu] = i$$

This generically determines that

$$|A(k)|^2 - |B_R(k)|^2 = 1 .$$

(4.16)

For wavevectors deep inside the Hubble radius $|k \eta| \gg 1$ the mode functions $g_{\nu}(k; \eta)$ have the asymptotic behavior

$$g_{\nu}(k; \eta) \overset{\eta \rightarrow -\infty}{=} \frac{1}{\sqrt{2k}} e^{-i k \eta} , \quad g^*_\nu(k; \eta) \overset{\eta \rightarrow -\infty}{=} \frac{1}{\sqrt{2k}} e^{i k \eta} ,$$

(4.17)

while for $\eta \rightarrow 0^-$, they behave as:

$$g_{\nu}(k; \eta) \overset{\eta \rightarrow 0^-}{=} \frac{\Gamma(\nu)}{\sqrt{2 \pi k}} \left( \frac{2}{i k \eta} \right)^{-\frac{\nu}{2}} .$$

(4.18)

In particular, in the scale invariant case $\nu = \frac{3}{2}$ which is the leading order in the slow-roll expansion, the mode functions eqs. (4.15) simplify to

$$g_{\frac{3}{2}}(k; \eta) = \frac{e^{-i k \eta}}{\sqrt{2k}} \left[ 1 - \frac{i}{k \eta} \right] .$$

(4.19)

As we see from eq. (2.27), $z(\eta)$ obeys eq. (2.21) for $k = 0$ and therefore $z(\eta)$ in the slow-roll regime behaves as

$$z(\eta) = \frac{z_0}{(-k_0 \eta)^{\nu_\eta - \frac{3}{2}}} ,$$

(4.20)

where $z_0$ is the value of $z(\eta)$ when the pivot scale $k_0$ exits the horizon, that is at $\eta = -1/k_0$. Combining this result with the small $\eta$ limit eq. (4.18) we find from eqs. (4.12) and (4.20),

$$P_R(k) = P^BD_R(k) [1 + D(k)] ,$$

(4.21)

where we introduced the transfer function for the initial conditions of curvature perturbations:

$$D(k) = 2 |B_R(k)|^2 - 2 \text{Re} \left[ A_R(k) B^*_R(k) i^{2 \nu_\eta - 3} \right] .$$

(4.22)

$D(k)$ is obtained imposing BDic at $\tau = \tau_0$ according to eq. (4.7).

Notice as shown in sec. V A that the transfer function $D(k)$ enjoys the properties

$$1 + D(k) \overset{k \rightarrow 0}{=} O(k^{\nu_\eta + 1}) , \quad D(k) \overset{k \rightarrow \infty}{=} O \left( \frac{1}{k^2} \right) .$$

(4.23)
$D(k)$ accounts for the effect in the power spectrum both of the initial conditions and of the fluctuations evolution during fast-roll (before slow-roll). $D(k)$ depends on the time $\tau_0$ at which BDic are imposed.

If one chooses the extreme slow-roll solution presented in sec. II A and imposes BDic at $\tau_0 = -\infty$ (that is, $\eta_0 = -\infty$) then $D(k) = 0$ and the fluctuation power spectrum at the end of inflation is the usual power spectrum $P_R(k) = P^{BD}_R(k)$.

$P^{BD}_R(k)$ is given by its customary slow-roll expression,

$$
\log P^{BD}_R(k) = \log A_s(k_0) + (n_s - 1) \log \frac{k}{k_0} + \frac{1}{2} n_{run} \log^2 \frac{k}{k_0} + \mathcal{O}\left(\frac{1}{N_s^3}\right).
$$

(4.24)

We solved numerically the fluctuations equation (2.26) in cosmic time with the BDic eq.(4.7) covering both the fast-roll and slow-roll regimes. We started at initial times $\tau_0$ ranging from the vicinity of $\tau = \tau_*$ till the transition time $\tau_{trans} = 0.2487963 \ldots$ from fast-roll to slow-roll. We computed the transfer function $D(k)$ from the mode functions behaviour deep during slow-roll inflation from eqs.(4.12) and (4.21) \cite{2}. In figs. 6 we depict $1 + D(k)$ vs. $k$ for twelve values of the time $\tau_0$ where BDic are imposed.

![Plot of $1 + D(k)$ vs. $k$](image)

FIG. 6: Numerical transfer function $1 + D(k)$. Lower left panel: Numerical transfer function $1 + D(k)$ for BDic at $\tau = \tau_0 = \tau_* + \Delta \tau$ for different $\Delta \tau$ values as given in the picture. We see here that the peak of $1 + D(k)$ grows and moves for larger $k$ as $\tau_0$ increases. Here $N_{sr} = 63$. Lower right panel: The transfer function $1 + D(k)$ when the BDic eq.(4.7) are imposed during slow–roll at finite times $\tau_0$ and $N_{sr}$ efolds of slow–roll have still to occur. Upper panels: Numerical transfer function $1 + D(k)$ for BDic at $\tau = \tau_0 = \tau_* + \Delta \tau$, for different values of $\Delta \tau$ as given in the picture. We get stronger oscillations in $1 + D(k)$ for decreasing $\tau_0$ in the range $\Delta \tau < 0.04$. Here $N_{sr} = 63$.

Notice that when BDic are imposed at finite times $\tau_0$, the spectrum is not the usual $P^{BD}_R(k)$ but it gets modified by a non-zero transfer function $D(k)$ eq.(4.21). The power spectrum $P_R(k)$ vanishes at $k = 0$ and exhibits oscillations which vanish at large $k$ [see figs. 6 and 7].
FIG. 7: Power spectrum with BDic eq.(4.17) imposed during slow–roll when $N_{sr}$ efolds of slow–roll inflation have still to occur. We see here the decrease of the power spectrum $P_R$ as $k^{n_s-1}$ multiplied by the oscillations of $1 + D(k)$. See eqs.(4.21) and (4.24) and figs. 6. The non-oscillatory black curve corresponds to the usual power with BDic at $\eta_0 = -\infty$ eq.(4.24) decreasing as $k^{n_s-1}$. The later are imposed the BDic, the smaller is the number of slow-roll efolds $N_{sr}$ and the whole $k$-spectrum shifts to larger $k$.

During slow-roll different initial times $\tau_0$ lead essentially to a rescaling of $k$ in $D(k)$ by a factor $\eta_0$ since the conformal time $\eta$ is almost proportional to $1/a(\eta)$ during slow-roll [see figs. 7[6] and below eq.(5.7)]. By virtue of the dynamical attractor character of slow–roll, the power spectrum when the BDic are imposed at a finite time $\tau_0$ cannot really distinguish between the extreme slow–roll solution (for which slow–roll starts from the very beginning $\eta_0 = -\infty$) or any other solution which is attracted to slow–roll well before the time $\tau_0$.

C. Accurate numerical computation of the power spectrum and the transfer function $D(k)$ of initial conditions.

In order to accurately calculate $n_s$ we proceed as follow. We match the solution $S_R(k; \eta)$ with the slow–roll solution $g_{\nu R}(k; \eta)$ eq.(4.15) at the time $\tau_0$ when $N_{sr}$ efolds of slow–roll have still to occur. $\eta$ and $\nu_R$ are computed at this time $\tau_0$. In practice, this corresponds to setting $A_R(k) = 1$, $B_R(k) = 0$ (and therefore $D_R(k) = 0$) in the Bogoliubov transformation eq.(4.14).

Then, we integrate numerically the fluctuations equations eq.(2.26). By construction, this produces the standard spectra $P_R^{BD}(k)$ eq.(4.22) that quickly stabilize as $N_{sr}$ is increased a few efolds above $N = 60$.

It is convenient to introduce the quantity

$$L_s \equiv \log \left( \frac{M_{PL}}{m} \right)^2 A_s(k_0 = m),$$

with $k_0 = m$ when $a(\eta) = 1$, that is $N = 60$ efolds before inflation ends. In table IV we provide $L_s$ for several values of $N_{sr}$. 

\[4.25\]
To transform this $k_0$ in a wavenumber today we need:

- the total redshift from 60 efolds before inflation ends till today [since we choose $a(\tau = 0) = 1$ when there are still $N = 60$ efolds till the end of inflation].
- the value of $m$ as determined by the observed value of the amplitude $A_s(k_0)$.

Let $k_{0\,\text{CMC}}$ be the value of the pivot scale of CosmoMC [that is 50 (Gpc)$^{-1}$ today] 60 efolds before the end of inflation. Then, we have from eqs. (4.24) and (4.25),

$$
\log A_s(k_0 = m) = L_s + 2 \log \frac{m}{M_{PL}} = L_s^{\text{CMC}} + \left( n_s^{\text{CMC}} - 1 \right) \log \frac{m}{k_0^{\text{CMC}}} + \frac{1}{2} n_{\text{run}} \left[ \log \frac{m}{k_0^{\text{CMC}}} \right]^2 + \mathcal{O} \left( \frac{1}{N^2} \right),
$$

where $L_s^{\text{CMC}} \equiv \log A_s^{\text{CMC}}(k_0^{\text{CMC}})$ and $n_s^{\text{CMC}}$ are best fit values in a given CosmoMC run. Since the running index $n_{\text{run}}$ is $O(1/N^2)$, we get for $m$,

$$
\left( \frac{m}{M_{PL}} \right)^2 = \left( \frac{m}{k_0^{\text{CMC}}} \right)^{n_s^{\text{CMC}} - 1} \exp \left( L_s^{\text{CMC}} - L_s \right) \left[ 1 + \mathcal{O} \left( \frac{1}{N^2} \right) \right].
$$

The wavevectors at $a = 1$ (60 efolds before inflation ends) and today are related by

$$
k^{a=1} = \frac{e^{60}}{a_r} k^{\text{today}},
$$

where $a_r$ is the scale factor by the end of inflation

$$
a_r = 2.5 \times 10^{-29} \sqrt{\frac{10^{-4} M_{PL}}{H_{60}}},
$$

and $H_{60}$ is the Hubble parameter 60 efolds before inflation ends. We thus have for the pivot wavenumber at $a = 1$

$$
k_0^{\text{CMC}} \simeq 1.46 \ldots \sqrt{\frac{H_{60}}{10^{-4} M_{PL}}} \times 10^{15} \text{ GeV}
$$

and

$$
\left( \frac{m}{M_{PL}} \right)^{2 - (n_s^{\text{CMC}} - 1)/2} = \left( \frac{16.67 \ldots}{\sqrt{h_{60}}} \right)^{n_s^{\text{CMC}} - 1} \exp \left( L_s^{\text{CMC}} - L_s \right), \quad \text{where} \quad h_{60} = \frac{H_{60}}{m}.
$$

Notice the small $1/N$ correction $(n_s^{\text{CMC}} - 1)/2$ in the exponent of $m/M_{PL}$. Eq. (4.26) yields for the best fit CosmoMC run $L_s^{\text{CMC}} = -19.9808 \ldots$ and $n_s = 0.9635 \ldots$ [2]:

$$
m \simeq 4.8114 \ldots 10^{-6} M_{PL} = 1.171 \ldots 10^{13} \text{ GeV}
$$

The exact values given above in Table IV

$$
A_s = \left( \frac{m}{M_{PL}} \right)^2 \exp(L_s), \quad n_s \quad \text{and} \quad n_{\text{run}}
$$

| $N_{sr}$ | $L_s$ | $n_s$ | $n_{\text{run}}$ |
|---------|-------|-------|-----------------|
| 61      | 4.6585381... | 0.9637013... | -0.0000701... |
| 63      | 4.6583004... | 0.9641135... | -0.0001639... |
| 65      | 4.6584371... | 0.9642483... | -0.0002165... |
| 67      | 4.6584463... | 0.9642444... | -0.0002165... |
| 69      | 4.6584469... | 0.9642448... | -0.0002167... |
are obtained taking into account the fast-roll and slow–roll stages in the numerical calculation. We can compare them to their slow–roll (leading $1/N$) analytic counterparts for the double-well quadratic plus quartic potential. $^2$

$$A_s = \frac{N^2}{12\pi^2} \left( \frac{m}{M_{PL}} \right)^2 \frac{(1-z)^4}{y^2 z}, \quad n_s = 1 - \frac{y}{N} \frac{3z + 1}{(1-z)^2}, \quad n_{\text{run}} = \frac{y^2 z}{N^2 (1-z)^4} (24z^2 - 35z + 3)$$

where $N = 60$, $z = 0.117446$ and $y = z - 1 - \log z = 1.2592226\ldots$, that is

$$A_s = \frac{N^2}{12\pi^2} \left( \frac{m}{M_{PL}} \right)^2 \exp(4.59536898\ldots), \quad n_s = 0.9635620\ldots, \quad n_{\text{run}} = -0.0000664\ldots$$

The figure in the exponent is to be compared with the $L_s$ values in Table IV. The agreement with Table IV is quite good, especially for $n_s$.

![Image of plots showing difference between transfer functions](image-url)

**FIG. 8:** Upper Left panel: Difference between the (approximate) transfer function $\tilde{D}(k \eta_0)$ eqs.(5.7)-(5.9) for $\nu_R = 1.5182189\ldots$ and the numerical (exact at least to a $10^{-7}$ relative error) transfer function $D(k)$, when $N_{sr} = 63$. Upper Right panel: Difference between the (approximate) transfer function $\tilde{D}(k \eta_0)$ eqs.(5.7)-(5.9) for $\nu_R = 3/2$ (the scale-invariant value) and the numerical (exact) transfer function. We see that the difference in the right panel eqs.(5.9) is $< 0.014$ while in the left panel the difference of the analytic formula eq.(5.7) is much smaller, $< 0.0005$. Lower Left panel: difference between the exact (numerical) $D(k)$ computed for the fast-roll inflaton solution of table II and for the extreme slow–roll inflaton solution of table I when BDic are imposed 63 e-folds before the end of inflation. Lower Right panel: difference between the exact (numerical) fast–roll $D(k)$ and the approximate $\tilde{D}(k \eta_0)$ calculated with $\nu_R = 3/2$ and $\eta_0 = -4.0169827\ldots$. We see that the differences are small in both cases.
We now find the exact (numerical) transfer function \( D(k) \) for the initial conditions, by simply taking in eq.\([4.21]\) the ratio of the two power spectra: \( P_X(k) \) with BDic at time \( \tau_0 \) and \( P_R^{BD}(k) \). In the case of BDic at finite times the result is given in fig 6. At the largest value \( k/m = 100 \) of the wavenumber interval considered, we have

\[
1 + D(100 \, m) = 0.9996994 \ldots, \quad 1.0000061 \ldots, \quad 1.0000001 \ldots
\]

This provides a good check of the accuracy of the calculation.

In figs. 8 we compare the numerically computed \( D(k) \) against \( \tilde{D}(k \, \eta_0) \) analytically computed for BDic imposed at time \( \eta_0 \) during slow–roll in eq.\([5.7]\), sec. \([5.8]\). The comparison is performed for BDic imposed when \( N_{sr} = 63 \) on the extreme slow roll solution, which corresponds to \( \eta_0 = -4.0202308 \ldots \). We consider two values of \( \nu_R : \nu_R = 2 - n_s/2 = 1.5182189 \ldots, \quad n_s = 0.9635620 \ldots \) corresponding to slow–roll at leading \( 1/N \) order, and the exactly scale–invariant case \( \nu_R = 3/2 \). Notice that in the latter case \( \tilde{D}(k \, \eta_0) \) has the explicit simple analytic form eq.\([6.9]\).

The maximum of the numerical transfer function \( 1 + D(k) \) is located at \( k/m = 0.68755 \ldots \) and has the value 1.13218 \ldots. The maximum of \( 1 + \tilde{D}(k \, \eta_0) \), when \( \nu_R = 3/2 \) is in \( k/m = 0.68755 \ldots \) and has the value 1.13009 \ldots. Recall that these values of \( k/m \) have the scale fixed by the choice \( a = 1 \) when \( N = 60 \) efolds lack before inflation ends.

Let us now consider the fluctuations on the fast–roll solution of Table II. Since \( \eta \) has a finite lower limit, the choice \( A_R(k) = 1, \quad B_R(k) = 0 \) has little meaning and BDic can be imposed only at a finite time \( \tau_0 \) later than the singularity time \( \tau_s \). If \( \tau_0 \) is exactly the transition time \( \tau_{\text{trans}} \) when \( \epsilon_0 = 1/N, \) fast–roll and slow–roll begins, (to proceed for \( N_{sr} = 63 \) efolds), then \( D(k) \) does not differ too much from that computed with the extreme slow roll solution. This comparison is performed in the lower left panel of fig. 8. In the right panel \( D(k) \) is compared to the \( \tilde{D}(k \, \eta_0) \) for \( \nu_R = 3/2 \) and \( \eta_0 = -4.0169827 \ldots \), which is the value of the conformal time at the onset of slow–roll (see Table II).

When the BDic are imposed during the fast–roll stage well before it ends, \( D(k) \) changes much more significantly than along the extreme slow roll solution. This is due to two main effects: the potential felt by the fluctuations is attractive during fast–roll and \( \eta_0 \), far from being almost proportional to \( 1/a(\eta) \), tend to the constant value \( \eta_s \) as \( \tau \to \tau_s^+ \) and \( a(\eta) \to 0 \). The numerical transfer functions \( 1 + D(k) \) obtained from eqs.\([4.12]\) and \( [4.21] \) are plotted in figs. 6.

The fact that choosing BDic leads to a primordial power and its respective CMB multipoles which correctly reproduce the observed spectrum justifies the use of BDic for the scalar curvature fluctuations.

### D. The effect of the fast-roll stage on the low multipoles of the CMB

In the region of the Sachs-Wolfe plateau for \( l \lesssim 30 \), the matter-radiation transfer function can be set equal to unity and the CMB multipole coefficients \( C_l \)'s are given by

\[
C_l = \frac{4\pi}{9} \int_0^\infty \frac{dk}{k} P_X(k) \left\{ j_l(k(\eta_0 - \eta_{LSS})) \right\}^2 ,
\]

where \( P_X \) is the power spectrum of the corresponding perturbation, \( X = R \) for curvature perturbations and \( X = T \) for tensor perturbations, \( j_l(x) \) are spherical Bessel functions \([8]\) and \( \eta_0 - \eta_{LSS} \) is the comoving distance between today and the last scattering surface (LSS) given by

\[
\eta_0 - \eta_{LSS} = \frac{1}{H_0} \int_{z_{LSS}}^1 \frac{da}{\sqrt{\Omega_r + \Omega_M + \Omega_\Lambda}} ,
\]

where \( \Omega_r, \Omega_M \) and \( \Omega_\Lambda \) stand for the fraction of radiation, matter and cosmological constant in today’s Universe. We find using \( z_{LSS} = 1100 \),

\[
\eta_0 - \eta_{LSS} = \frac{3.296}{H_0}.
\]

Notice that \( k/H_0 \sim d_H/\lambda_{\text{phys}}(t_0) \) is the ratio between today’s Hubble radius and the physical wavelength. The power spectrum for curvature (\( R \)) perturbations \( P_R(k) \) is given by eqs.\([4.21]\)–\([4.24]\).
FIG. 9: The change $\Delta C_{\ell}/C_{\ell}$ on the CMB multipoles for $\ell = 1, \ldots, 5$. Upper plot: $\Delta C_{\ell}/C_{\ell}$ vs. $\tau_0 - \tau_*$ for $0 < \tau_0 - \tau_* < 0.2487963\ldots$. Lower plot: $\Delta C_{\ell}/C_{\ell}$ vs. $\tau_0 - \tau_*$ for $0.0193 < \tau_0 - \tau_* < 0.2487963\ldots$. $\tau_0$ is the time when the BDic eq. (4.7) are imposed to the fluctuations. We choose $\tau_0$ inside the fast-roll stage. $\Delta C_{\ell}/C_{\ell}$ is positive for small $\tau_0 - \tau_*$ and decreases with $\tau_0$ becoming then negative. The CMB quadrupole observations indicate a large suppression thus indicating that $\tau_0 - \tau_* \gtrsim 0.05 \simeq 10^{100} \tau_{Planck}$. Our predictions here for the quadrupole and octupole suppressions are to be confronted with forthcoming CMB observations. It will be extremely interesting to measure the primordial dipole and compare with our predicted value.

Inserting eq. (4.21) into eq. (4.31) yields the $C_{\ell}$ as the sum of two terms

$$C_{\ell} = C_{\ell}^{BD} + \Delta C_{\ell}, \quad \frac{\Delta C_{\ell}}{C_{\ell}} = \frac{\int_{0}^{\infty} D(k x) f_{l}(x) \, dx}{\int_{0}^{\infty} f_{l}(x) \, dx}, \quad x = k(\eta_0 - \eta_{LSS}) = k/\kappa,$$

where from eq. (4.33), $\kappa \equiv H_0/3.296\ldots$,

$$f_{l}(x) = x^{n_{l} - 2} [j_{l}(x)]^2.$$

and $j_{l}(x)$ stand for the spherical Bessel functions.

The $C_{\ell}^{BD}$s correspond to the standard BD power spectrum $P_{R}^{BD}(k)$ eq. (4.24) and the $\Delta C_{\ell}$ exhibit the effect of the transfer function $D(k)$ on the $C_{\ell}$.

Using the transfer function $D(k)$ obtained above eq. (4.22), we computed the change on the CMB multipoles $\Delta C_{\ell}/C_{\ell}$ for $\ell = 1, \ldots, 5$ as functions of the starting instant of the fluctuations $\tau_0$. We plot $\Delta C_{\ell}/C_{\ell}$ for $1 \leq \ell \leq 5$ vs. $\tau_0 - \tau_*$ in fig. 9. We see that $\Delta C_{\ell}/C_{\ell}$ is positive for small $\tau_0 - \tau_*$ and decreases with $\tau_0$ becoming then negative. The CMB quadrupole observations indicate a large suppression thus indicating that $\tau_0 - \tau_* \gtrsim 0.05 \simeq 10^{100} \tau_{Planck}$.

Being $D(k) < 0$ for low $k$ as depicted in figs. 6 the primordial power at large scales is then suppressed and the low $C_{\ell}$ decrease as seen from eq. (4.34).
\[ \Delta C_{\ell}/C_{\ell} \] mainly originates from the peak of \( D(k) \) displayed in figs.\( \square \) whose position moves to smaller \( k \) for decreasing \( \tau_0 \). Therefore, the primordial power suppression is less important for decreasing \( \tau_0 \) and the CMB multipole suppression \( \Delta C_{\ell}/C_{\ell} \) less important as depicted in figs.\( \square \).

For small \( \tau_0 - \tau_s \lesssim 0.05 \) the peak of \( D(k) \) grows significantly and \( \Delta C_{\ell}/C_{\ell} \) become positive, namely the low CMB multipoles are enhanced.

It should be recalled that the observation of a low CMB quadrupole sparked many different proposals to explain that suppression [18].

Besides finding a CMB quadrupole suppression in agreement with observations [2]-[6], we provide here predictions for the dipole and octupole suppressions. Forthcoming CMB observations can provide better data to confront our quadrupole and octupole suppression predictions. It will be extremely interesting to measure the primordial dipole and compare with our predicted value.

V. ANALYTIC FORMULAS FOR THE TRANSFER FUNCTION \( D(k) \).

It is very important to dispose of analytic formulas for the transfer function \( D(k) \) in order to better understand the physical origin of its oscillations and properties as well as in the perspective of the MCMC data analysis.

However, the mode equations (2.24) are not solvable in closed form for \( k \neq 0 \), not even for the approximated inflation solution eq. (2.37) which leads to the potential \( V_R(\tau) \) eq. (2.43).

The function \( D(k) \) must obey the general properties eq. (4.23).

A. The primordial power spectrum vanishes for \( k \to 0 \) and becomes the BD power spectrum for \( k \to \infty \).

The fluctuations equation (2.24) can be solved explicitly for \( k = 0 \)

\[
 s(\eta) = c_1 z(\eta) + c_2 \int_{\eta_0}^{\eta} \frac{d\eta'}{z^2(\eta')} ,
\]

(5.1)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

The BDic eq. (4.7) introduce for \( k \to 0 \) a \( 1/\sqrt{2k} \) singularity in the mode functions. Thus, the mode functions must have the behaviour

\[
 S_R(k; \eta) \xrightarrow{k \to 0} \frac{s(\eta)}{\sqrt{2k}} [1 + \mathcal{O}(k)]
\]

(5.2)

where \( s(\eta) \) is given by eq. (5.1).

Inserting eq. (5.2) into the BDic eq. (4.7) yields for \( k \to 0 \),

\[
 s(\eta_0) = 1 , \quad \frac{ds(\eta_0)}{d\eta} = 0 ,
\]

which determines the coefficients \( c_1 \) and \( c_2 \) in eq. (5.1). We finally obtain

\[
 s(\eta) = \frac{z(\eta)}{z(\eta_0)} - \frac{1}{z(\eta_0)} \int_{\eta_0}^{\eta} \frac{d\eta'}{z^2(\eta')}
\]

(5.3)

and using eq. (4.20) valid for \( \eta \to 0^- \) when slow–roll applies

\[
 \lim_{\eta \to 0^-} \frac{s(\eta)}{z(\eta)} = \frac{1}{z(\eta_0)} .
\]

(5.4)

The primordial power spectrum for \( k \to 0 \) follows by inserting eq. (5.2) and eq. (5.4) into the general expression eq. (4.12),

\[
 P_R(k) \xrightarrow{k \to 0} \left( \frac{m}{M_{PL}} \right)^2 \frac{k^3}{2 \pi^2} \lim_{\eta \to 0^-} \frac{|S_R(k; \eta)|^2}{z(\eta)} \xrightarrow{k \to 0} \left( \frac{m}{M_{PL}} \right)^2 \left( \frac{k}{2 \pi z(\eta_0)} \right)^2
\]
We thus find in general that the power spectrum vanishes as $k^2$ for $k \to 0$ and therefore

$$1 + D(k) \xrightarrow{k \to 0} O(k^{\nu_s+1})$$

as stated in eq. (4.23). This property is generally true except for the extreme slow-roll inflaton solution (sec. IIIA) with BDic imposed at $\eta_0 = -\infty$ in which case $D(k)$ vanishes identically for all $k$.

For growing $k$ the modes exit the horizon later on, during the slow–roll regime where eq. (4.14) applies. For large $k$ the mode functions $S_{\mathcal{R}}$ as well as $g_{\nu_{\mathcal{R}}}$ behave as plane waves [eqs. (4.6) and (4.18)] and therefore

$$A_{\mathcal{R}}(k) = 1 \quad , \quad B_{\mathcal{R}}(k) = 0 \quad .$$

Hence $D(k) \xrightarrow{k \to \infty} 0$.

### B. The transfer function $D(k)$ when BDic are imposed during slow–roll.

When the BDic eq. (4.7) are imposed during slow–roll at a finite time $\eta_0$ we can use eq. (4.14) for the mode functions at $\eta = \eta_0$ and we obtain,

$$e^{-i k \eta_0} \sqrt{2} = A_{\mathcal{R}}(k) \ g_{\nu_{\mathcal{R}}}(k; \eta_0) + B_{\mathcal{R}}(k) \ g'_{\nu_{\mathcal{R}}}(k; \eta_0)$$

$$-i k \ e^{-i k \eta_0} \sqrt{2} = A_{\mathcal{R}}(k) \ g'_{\nu_{\mathcal{R}}}(k; \eta_0) + B_{\mathcal{R}}(k) \ g_{\nu_{\mathcal{R}}}(k; \eta_0)$$

which determines

$$A_{\mathcal{R}}(k) = \frac{e^{-i k \eta_0}}{i \sqrt{2}} \left[ g^{*}_{\nu_{\mathcal{R}}}(k; \eta_0) + i k \ g_{\nu_{\mathcal{R}}}(k; \eta_0) \right] , \quad B_{\mathcal{R}}(k) = \frac{-i k \eta_0}{i \sqrt{2}} \left[ g'_{\nu_{\mathcal{R}}}(k; \eta_0) + i k \ g_{\nu_{\mathcal{R}}}(k; \eta_0) \right].$$

These coefficients satisfy eq. (4.16) and

$$|A_{\mathcal{R}}(k)|^2 + |B_{\mathcal{R}}(k)|^2 = \frac{1}{k} \left[ \left| g'_{\nu_{\mathcal{R}}}(k; \eta_0) \right|^2 + k^2 \left| g_{\nu_{\mathcal{R}}}(k; \eta_0) \right|^2 \right]$$

Notice that the function $g_{\nu}(k; \eta)$ eq. (4.15) and the $k$ factors in eq. (5.6) combine to produce functions $A_{\mathcal{R}}(k) \equiv \hat{A}_{\mathcal{R}}(k; \eta_0)$ and $B_{\mathcal{R}}(k) \equiv \hat{B}_{\mathcal{R}}(k; \eta_0)$ that only depend on the product $k \eta_0$.

We find from eqs. (4.22) and (5.6) the corresponding transfer function which is a function of $k \eta_0$ too,

$$1 + \hat{D}(k \eta_0) = \frac{1}{k} \left\{ \left| g_{\nu_{\mathcal{R}}}(k; \eta_0) \right|^2 + k^2 \left| g_{\nu_{\mathcal{R}}}(k; \eta_0) \right|^2 - \text{Re} \left[ i^{3-2 \nu_{\mathcal{R}}} \left( g'_{\nu_{\mathcal{R}}}(k; \eta_0) + k^2 g^2_{\nu_{\mathcal{R}}}(k; \eta_0) \right) \right] \right\}$$

The functional dependence on $k \eta_0$ confirms the assertion in sec. IVB that different initial times $\tau_0$ lead to a rescaling in $k$.

In the $k \eta_0 \to \infty$ limit two types of vanishing terms show up in $\hat{D}(k \eta_0)$: (a) terms that strongly oscillate as $e^{\pm 2 i k \eta_0}$ as they tend to zero and (b) non-oscillatory decreasing terms. Under integrals on $k$, the terms of type (a) yield convergent expressions. We derive the non-oscillatory decreasing terms (b) by inserting the asymptotic behaviour of the Hankel functions eq. (4.15) [8] in eq. (5.7) with the result

$$\hat{D}(k \eta_0) \xrightarrow{k \to \infty} \frac{(\nu_{\mathcal{R}} - 1)^2}{8 (k \eta_0)^3} + \text{terms oscillating as } e^{\pm 2 i k \eta_0}.$$ 

However, this approximation will not be valid for large enough $k$ since the modes at small enough wavelength will exit the horizon after the end of slow–roll where eq. (5.3) does not apply anymore. We recall that the occupation number $|B_{\mathcal{R}}(k)|^2$ (and therefore $D(k)$) must decrease faster than $1/k^4$ for $k \to \infty$ in order to ensure finite UV values for the expectation value of the energy-momentum fluctuations [2, 13].

The case $\nu_{\mathcal{R}} = 3/2$ is a good approximation which simplifies the expressions above. We obtain in this scale invariant case:

$$A_{\mathcal{R}}(k) = 1 + \frac{i}{k \eta_0} - \frac{1}{2 k^2 \eta_0^2} \quad , \quad B_{\mathcal{R}}(k) = -\frac{e^{-2 i k \eta_0}}{2 k^2 \eta_0^2}.$$
The transfer function is in this case,
\[ \tilde{D}(x) = \frac{\cos 2x}{x^2} - \frac{\sin 2x}{x^3} + \frac{\sin^2 x}{x^4} , \quad \nu_R = 3/2 , \quad x \equiv k \eta_0 . \] (5.9)

Eq.(5.8) for \( \nu = 3/2 \) coincides with eq.(5.9) in the \( x \to \infty \) limit, as it must be.

Notice that the simple formula eq.(5.9) obeys the general properties eq.(4.23). In particular,
\[ \tilde{D}(x) \xrightarrow{x \to 0} -1 + \frac{4}{9} x^2 + O(x^4) . \]

VI. FIXING THE TOTAL NUMBER OF INFLATION E-FOLDS AND THE BOUND FROM ENTROPY

It is very useful to plot the comoving scales of the cosmological fluctuation wavenumbers and the comoving Hubble radius together [see fig. 10]. One sees in this way how and when the cosmological fluctuations cross out and in the Hubble radius. The comoving Hubble radius is defined by \( R_H \equiv 1/[a(\tau) \ H(\tau)] \). We display in Table VI the dependence of \( R_H \) on the scale factor \( a \) for all the relevant eras of the universe.

| Expansion stage       | Dependence of \( R_H \) on \( a \) |
|-----------------------|--------------------------------------|
| Extreme Fast-roll     | \( a^2 \)                            |
| Fast-roll             | \( a^2/\sqrt{a^6} + \text{constant} \) |
| Slow–Roll inflation   | \( 1/a \)                            |
| Radiation Dominated   | \( a \)                              |
| Matter Dominated      | \( \sqrt{a} \)                        |

TABLE V: Dependence of the comoving Hubble radius \( R_H = 1/[a \ H] \) on the scale factor \( a \) for the relevant eras of the universe.

The observed CMB quadrupole suppression can be easily explained if it exited the horizon by the end of fast-roll [5, 6]. In that case, the modes which are horizon size today had wavenumbers \( k_Q \simeq 11.5 \ m \) at horizon exit [6]. Combining this value of \( k_Q \) with the redshift since the pivot wavenumber exited the horizon, eqs. (4.28), (4.29) and (4.30), determines the total redshift since the beginning of inflation to be
\[ z_{tot} = 0.9 \ 10^{56} \simeq e^{129} . \]

Combining this value with the value of \( 1 + z_r \simeq 4 \ 10^{28} \simeq e^{66} \) by the end of inflation eq.(4.29) yields a total number of \( N_{tot} = 63 \) inflation efolds. This value is very close to the minimal number of inflation efolds required to explain the entropy of the present universe due to photons and neutrinos [2]:
\[ N_{tot} \geq 62.4 . \]

Namely, this is the minimum number of inflation efolds compatible with the present entropy of the universe.

In summary, assuming that the CMB quadrupole is suppressed because it exited the horizon by the end of fast-roll inflation fixes the total number of inflation efolds which turns to be
\[ N_{tot} \simeq 63 . \]

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FIG. 10: The logarithm of the comoving scales and the logarithm of the comoving Hubble radius $R_H = 1/|a H|$ vs. $\log a$. 

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