On the Reliability Function of Distributed Hypothesis Testing Under Optimal Detection

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Abstract

The distributed hypothesis-testing problem with full side-information is studied. The trade-off (reliability function) between the type 1 and type 2 error exponents under limited rate is studied in the following way. First, the problem of determining the reliability function of distributed hypothesis-testing is reduced to the problem of determining the reliability function of channel-detection codes (in analogy to a similar result which connects the reliability of distributed compression and ordinary channel codes). Second, a single-letter random-coding bound based on an hierarchical ensemble, as well as a single-letter expurgated bound, are derived for the reliability of channel-detection codes. Both bounds are derived for the optimal detection rule. We believe that the resulting bounds are ensemble-tight, and hence optimal within the class of quantization-and-binning schemes.

Index Terms

Binning, channel-detection codes, distributed hypothesis-testing, error exponents, expurgated bounds, hierarchical ensembles, multiterminal data compression, random coding, side information, statistical inference, superposition codes.

I. INTRODUCTION

In the hypothesis-testing (HT) problem, a detector needs to decide between two hypotheses regarding the underlying distribution of some observed data; the hypotheses are commonly known as the null hypothesis and the alternative hypothesis. Two types of error probability are defined - the type 1 error probability of deciding on the alternative hypothesis when the null hypothesis prevails, and type 2 error probability for the opposite event. The celebrated Neyman-Pearson lemma (e.g., [1, Prop. II.D.1]) states that the detector which achieves the optimal trade-off between the two error probabilities takes the form of comparing the likelihood-ratio to a threshold.

In the context of this work, we only consider data which is a sequence $Z_1, \ldots, Z_n$ of independent and identically distributed observations, and therefore the hypotheses correspond to the distribution of a random variable $Z$. In information-theoretic literature (e.g., [2, 3 Ch. 11], [4 Ch. 1], [5 Sec. 2]), large-deviations theory, most notably Sanov’s theorem, is usually applied to this problem. In particular, Stein’s theorem provides the largest exponential
decrease rate of the type 2 error probability when the type 1 error probability is bounded away from one. More generally, the *reliability function*, i.e., the optimal trade-off between the two types of exponents when both are strictly positive, is also known.

As a special case of the HT problem, one may consider a pair of random variables $Z = (X, Y)$, instances of which are fully observed by the detector. If, however, the detector does not directly observe the data sequence, the characterization of the optimal performance is much more challenging. A common model for such a scenario is known as *distributed hypothesis-testing* (DHT), and its reliability function is the subject of this paper.

The DHT model was introduced by Berger [6] as an example for a problem at the intersection of multi-terminal information theory and statistical inference. In this model, one encoder observes $X_1, \ldots, X_n$, while the other observes the corresponding $Y_1, \ldots, Y_n$; they produce codewords to be sent over limited-rate noiseless links to a common detector. The goal is to characterize the optimal detection performance, under a given pair of encoding rates. As a starting point, usually an asymmetric model (sometimes referred to as the *side-information case*) is studied, in which the $Y$-observations are fully available to the decoder, and thus there is only a single rate constraint.

The first major breakthrough on this problem was in a notable paper by Ahlswede and Csiszár [7], who addressed a special scenario termed *testing against independence*. In this case, the null hypothesis states that $(X, Y)$ have a given joint distribution $P_{XY}$, whereas the alternative hypothesis states that they are independent, but with the same marginals as in the null hypothesis. This case is special since the Kullback–Leibler divergence, which is usually associated with Stein’s exponent of HT, can be identified as the mutual information between $X$ and $Y$ under $P_{XY}$, which is naturally related to compression rates. This allowed the authors to use *distributed compression*, specifically quantization-based encoding techniques from [8], [9], to derive Stein’s exponent for the side-information testing-against-independence problem [7, Th. 2]. Quantization-based encoding was also used for an achievable Stein’s exponent for a *general* pair of memoryless hypotheses [7, Th. 5], but without a converse bound.

As summarized in [10, Sec. IV], later progress on this problem for a pair of general hypotheses was non-consecutive, and contributions were made by several groups of researchers. First, in [11], the achievable bound on Stein’s exponent from [7, Th. 5] was improved, and also generalized beyond the side-information case. Then, in [12], achievable bounds on the full trade-off between the two types of exponents were derived. In [13], Stein’s exponent for side-information cases was further significantly improved using binning, as will be described in the sequel.

Interestingly, when either the $X$-marginal or the $Y$-marginal of the hypotheses is different, positive exponents can be obtained even for *zero-rate* encoders [10, Sec. V]. For this case, achievable Stein’s exponents and exponential trade-offs were derived in [10 Th. 5.5] and [11], [12], [15], along with matching converse bounds when some kind of assumptions are imposed.

In the last decade, a renewed interest in the problem arose, aimed both at tackling more elaborated models, as well as at improving the results on the basic model. As for the former, Stein’s exponents under positive rates

\footnote{There is also a variant of *one-bit encoding*, see [10 Sec. V] and [14].}
were explored in successive refinement models \cite{16}, for multiple encoders \cite{17}, for interactive models \cite{18}, \cite{19}, under privacy constraints \cite{20}, combined with lossy compression \cite{21}, over noisy channels \cite{22}, \cite{23}, for multiple decision centers \cite{24}, as well as over multi-hop networks \cite{25}. Exponents for the zero-rate problem were studied under restricted detector structure \cite{14} and for multiple encoders \cite{26}. The finite blocklength and second-order regimes were addressed in \cite{27}.

As for the basic model, which this work also investigates, the encoding approach used in \cite{13} is currently the best known. It is based on quantization and binning, just as used, e.g., for distributed lossy compression (the Wyner-Ziv problem \cite{28, Ch. 11} \cite{29}). First, the encoding rate is reduced by quantization of the source vector to a reproduction vector chosen from a codebook. Second, the rate is further reduced by binning of the reproduction vectors. As the detector is equipped with side information, it can identify the true reproduction vector with high reliability. In the context of DHT, it can then decide on one of the hypotheses using this reproduction and the side information. In \cite{17} it was shown that a quantization-and-binning scheme achieves the optimal Stein’s exponent in a testing against conditional independence problem (as well as in a model inspired by the Gel’fand-Pinsker problem \cite{30}, and also in a Gaussian model). In \cite{31}, the quantization-and-binning scheme was shown to be necessary for the case of DHT with degraded hypotheses. In \cite{32}, \cite{33}, improved exponents were derived (for a doubly-symmetric binary source) by refining the analysis of the effect of binning errors. In addition, a full achievable exponent trade-off was presented, and Körner-Marton coding \cite{34} was used in order to extend the analysis to the symmetric-rate case (for the symmetric source). Finally, \cite{35} suggested an improved detection rule, in which the reproduction vectors in the bin are exhausted one-by-one, and the null hypothesis is declared if a single vector is jointly typical with the side-information vector.

It is evident that the detectors above are all sub-optimal, as they are based upon the two-stage process of reproduction and then testing. However, the decoding of the source vector (or its quantized version) is totally superfluous for the DHT system, as the requirement from the latter is only to distinguish between the hypotheses. In fact, this detection procedure is still sub-optimal even if quantization is used without binning, and only the second stage of the detector is employed. While \cite{35} offered an improved detector, it is still sub-optimal in this work, we investigate the performance of the optimal detector. In fact, the optimal detector directly follows from the standard Neyman-Pearson lemma (see Section \ref{III}). Nonetheless, analyzing its performance is highly non-trivial for DHT.

Specifically, we study the reliability function in the side-information case, and will be guided by the following methodology. Recall that for distributed lossless compression systems (the Slepian-Wolf problem \cite{28, Ch. 10} \cite{37}), the side information helps to fully reproduce the source. The concept of binning the source vectors was originally conceived for this problem, and a common wisdom for this problem states that the source vectors which belong to the same bin should constitute a good channel code for the memoryless channel induced by the conditional distribution

\footnote{Recently in the zero-rate regime, \cite{27} considered the use of an optimal Neyman-Pearson-like detector, rather than the possibly sub-optimal Hoeffding-like detector \cite{36} that was used in \cite{12}.}
of $Y$ given $X$. This intuition was made precise in [38, Th. 1][39], which showed that the reliability function of distributed compression is directly related to the reliability of channel coding. The idea is to use structured binning using a sequence of channel codes which achieves the channel reliability function. At a given blocklength, such a channel code corresponds to one bin of the distributed compression system. All other bins of the system are generated by permuting the source vectors of the first bin. Due to the memoryless nature of the problem, all bins generated this way are essentially as good as the original one, and this allows to directly link the reliability function of distributed compression to that of channel coding. While the reliability of channel codes is itself only known above the critical rate [4, Corollary 10.4], this characterization has two advantages nonetheless. First, analyzing channel codes is simpler than analyzing a distributed compression systems. Second, any bound on the reliability of channel codes immediately translates into a bound on the reliability of distributed compression systems. Specifically, the expurgated bound [4, Problem 10.18] and the sphere-packing bound [4, Th. 10.3] can be immediately used, rather than just a random-coding bound [4, Th. 10.2]. Noting the similarity between the distributed compression problem and the DHT problem, it is natural to ask whether structured binning is useful for the DHT problem, and what are the properties of a “good” bin for the DHT problem?

To address these questions, we introduce the concept of channel-detection (CD) codes. Such codes are not required to carry information, but are rather designed for the task of distinguishing between two possible channel distributions. Namely, a codeword from the code is chosen with a uniform probability over the codewords, and the detector should decide on the prevailing channel, based only on the output vector (and its knowledge of the codebook). It will be evident that this is the same problem encountered by the detector of a DHT system, given the encoded message. It will be shown that optimal DHT systems (in the exponential sense) can be generated by optimal CD codes, just like distributed compression systems are generated from ordinary channel codes. From this observation, the close relation between the reliability of DHT systems and CD codes will be determined. An illustration of the analogy between the relations distributed compression/channel coding and DHT/CD relations appears in Fig. 1 (with all quantities there will be formally defined in the sequel).

This intimate connection allows us to derive bounds on the reliability function of DHT using bounds on the reliability of CD codes. Concretely, we will derive both random-coding bounds and expurgated bounds on the reliability of CD codes under the optimal Neyman-Pearson detector. The analysis goes beyond that of [42] in two senses: first, it is based on a Chernoff distance characterization of the optimal exponents, which leads to simpler single-letter bounds; and second, the analysis is performed for an hierarchical ensemble corresponding

3 The reliability function of distributed lossless compression is the optimal exponential decrease of the error probability as a function of the compression rate.
4 Also mentioned in [17] for the DHT problem, though recognized as inessential.
5 More precisely, this is done type-by-type, i.e., per the subsets of source sequences that share the same empirical distribution.
6 This permutation technique, originally developed in [38], will be useful here too, and will be reviewed in more detail in what follows.
7 In [42], a somewhat different channel-detection setting is considered, where the code is required to simultaneously be a good channel code (in the ordinary sense), as well as a good CD code. In this work, it will only be required that the codewords of the CD code are different from one another.
8 Yielding superposition codes [43], such as the ones used for the broadcast channel, wee, e.g., [28, Ch. 5])
to quantization-and-binning schemes.

The outline of the rest of the paper is as follows. The system model and preliminaries, such as notation conventions and background on ordinary HT, will be given in Section II. The main result of the paper, namely an achievable bound on the reliability function of DHT under optimal detection, will be stated in Section III along with some consequences. For the sake of proving these bounds, the reduction of the DHT reliability problem to the CD reliability problem will be considered in Section IV. While only achievability bounds will ultimately be derived, the reduction to CD codes has both an achievability part as well as a converse part. Derivation of single-letter achievable bounds on the reliability of CD codes will be considered in Section V. From this, the achievability bounds on the DHT reliability will immediately follow. Afterwards, a discussion on computational aspects along with a numerical example are given in Section VI. Several directions for further research are highlighted in Section VII.
II. SYSTEM MODEL

A. Notation Conventions

Throughout the paper, random variables will be denoted by capital letters, specific values they may take will be denoted by the corresponding lower case letters, and their alphabets will be denoted by calligraphic letters. Random vectors and their realizations will be superscripted by their dimension. For example, the random vector $X^n = (X_1, \ldots, X_n)$ (n positive integer), may take a specific vector value $x^n = (x_1, \ldots, x_n) \in \mathcal{X}^n$, the nth order Cartesian power of $\mathcal{X}$, which is the alphabet of each component of this vector. The Cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ (finite alphabets) will be denoted by $\mathcal{X} \times \mathcal{Y}$.

We will follow the standard notation conventions for probability distributions, e.g., $P_X(x)$ will denote the probability of the letter $x \in \mathcal{X}$ under the distribution $P_X$. The arguments will be omitted when we address the entire distribution, e.g., $P_X$. Similarly, generic distributions will be denoted by $Q$, $\overline{Q}$, and in other similar forms, subscripted by the relevant random variables/vectors/conditionings, e.g. $Q_{XY}$, $Q_{X|Y}$. The joint distribution induced by $Q_X$ and $Q_{Y|X}$ will be denoted by $Q_X \times Q_{Y|X}$.

In what follows, we will extensively utilize the method of types [4], [44] and use the following notations. The **type class** of a type $Q_X$ at blocklength $n$, i.e., the set of all $x^n \in \mathcal{X}^n$ with empirical distribution $Q_X$, will be denoted by $\mathcal{T}_n(Q_X)$. The set of all type classes of vectors of length $n$ from $\mathcal{X}^n$ will be denoted by $\mathcal{P}_n(\mathcal{X})$, and the set of all possible types over $\mathcal{X}$ will be denoted by $\mathcal{P}(\mathcal{X}) \overset{\text{def}}{=} \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{X})$. Similar notations will be used for pairs of random variables (and larger collections), e.g., $\mathcal{P}_n(U \times \mathcal{X})$, and $\mathcal{T}_n(Q_{UXY}) \subseteq \mathcal{U}^n \times \mathcal{X}^n \times \mathcal{Y}^n$. The conditional type class of $x^n$ for a conditional type $Q_{Y|X}$, namely, the subset of $\mathcal{T}_n(Q_Y)$ such that the joint type of $(x^n, y^n)$ is $Q_{XY}$, will be denoted by $\mathcal{T}_n(Q_{Y|X}, x^n)$ (sometimes called the $Q$-shell of $x^n$ [4 Definition 2.4]). For a given $Q_X \in \mathcal{P}_n(\mathcal{X})$, the conditional type classes $Q_{Y|X}$ such that $\mathcal{T}_n(Q_{Y|X}, x^n)$ is not empty when $x^n \in \mathcal{T}_n(Q_X)$ will be denoted by $\mathcal{P}_n(\mathcal{Y}, Q_X)$. The probability simplex for an alphabet $\mathcal{X}$ will be denoted by $\mathcal{S}(\mathcal{X})$.

For two distributions $P_X, Q_X$ over the same finite alphabet $\mathcal{X}$, the variational distance ($\mathcal{L}_1$ norm) will be denoted by

$$\|P_X - Q_X\| \overset{\text{def}}{=} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| .$$  \hspace{1cm} (1)

When optimizing a function of a distribution $Q_X$ over a probability simplex $\mathcal{S}(\mathcal{X})$, the explicit display of the constraint will be omitted. For example, for a function $f(Q)$, min$_Q f(Q)$ will be used instead of min$_{Q \in \mathcal{S}(\mathcal{X})} f(Q)$.

The expectation operator with respect to a given distribution, e.g., $Q$, will be denoted by $\mathbb{E}_Q[\cdot]$ where the subscript $Q$ will be omitted if the underlying probability distribution is clear from the context. In general, information-theoretic quantities will be denoted by the standard notation [3], with subscript indicating the distribution of the relevant random variables, e.g. $H_Q(X|Y)$, $I_Q(X;Y)$, $I_Q(X;Y|U)$, under $Q = Q_{UXY}$. As an exception, the entropy of $X$ under $Q$ will be denoted by $H(Q_X)$. The binary entropy function will be denoted by $h_b(q)$ for $0 \leq q \leq 1$. The conditional Kullback–Leibler divergence between conditional two distributions, e.g., $Q_{X|U}$ and $P_{X|U}$, when
averaged with the distribution $Q_U$ will be denoted by $D(Q_X|U||P_X|U|Q_U)$, and in case that $U$ is degenerated, the notation will be simplified to $D(Q_X||P_X)$.

The Hamming distance between two vectors, $x^n \in X^n$ and $\pi^n \in X^n$ will be denoted by $d_H(x^n, \pi^n)$. The complement of a set $A$ will be denoted by $A^c$. For a finite multiset $A$, the number of distinct elements will be denoted by $|A|$. The probability of the event $A$ will be denoted by $P(A)$, and its indicator function will be denoted by $I(A)$.

For two positive sequences, $\{a_n\}$ and $\{b_n\}$ the notation $a_n \asymp b_n$, will mean asymptotic equivalence in the exponential scale, that is, $\lim_{n \to \infty} \frac{1}{n} \log(\frac{a_n}{b_n}) = 0$. Similarly, $a_n \preceq b_n$ will mean $\lim sup_{n \to \infty} \frac{1}{n} \log(\frac{a_n}{b_n}) \leq 0$, and so on. The notation $a_n \asymp b_n$ will mean that $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$. The ceiling function will be denoted by $\lceil \cdot \rceil$. The notation $|t|_+ \defeq \max\{t, 0\}$. Logarithms and exponents will be understood to be taken to the natural base. Throughout, for the sake of brevity, we will ignore integer constraints on large numbers. For example, $\lceil e^{nR} \rceil$ will be written as $e^{nR}$. For $n \in \mathbb{N}$, the set $\{1, \ldots, n\}$ will be denoted by $[n]$.

### B. Ordinary Hypothesis-Testing

Before getting into the distributed scenario, we will shortly review in this section ordinary HT between a pair of hypotheses. Consider a random variable $Z$ over a finite alphabet $Z$, whose distribution according to the hypothesis $H$ (respectively, $\overline{H}$) is given by $P$ (respectively, $\overline{P}$). It is common in the literature to refer to $H$ (respectively, $\overline{H}$) as the null hypothesis (respectively, the alternative hypothesis). However, we will refrain from using such terminology, and the two hypotheses will be considered to have an equal status.

Given $n$ i.i.d. observations $Z^n$, a (possibly randomized) detector

$$\phi : Z^n \to \{H, \overline{H}\},$$

has type 1 and type 2 error probabilities given by

$$p_1(\phi) \defeq P[\phi(Z^n) = \overline{H}],$$

and

$$p_2(\phi) \defeq P[\phi(Z^n) = H].$$

The family of detectors which optimally trades between the two types of error probabilities is given by the Neyman-Pearson lemma [11 Prop. II.D.1], [13 Th. 11.7.1] by

$$\phi^*_{n,T}(z^n) \defeq I \left[ P(z^n) \leq e^{nT} \cdot \overline{P}(z^n) \right],$$

also called the false-alarm probability and misdetection probability in engineering applications.

Since the two hypotheses are assumed to be discrete, randomized tie-breaking should be used if a given constraint on one of the error probabilities should be matched exactly. Nonetheless, this randomization has no effect on the exponential behavior, which is the focus of this paper (in the distributed scenario). Thus, we will not dwell on randomized detectors too much in what follows.
where $T \in \mathbb{R}$ is a threshold parameter. This parameter controls the trade-off between the probability of the two types of error - the larger $T$ is, the type 1 error probability increases and the type 2 error probability decreases, and vice-versa.

To describe bounds on the error probabilities of the optimal detector, let us define the hypothesis-testing reliability function \cite[Section II]{2} as

$$D_2(D_1; P, \overline{P}) \overset{\text{def}}{=} \min_{Q: D(Q\|P) \leq D_1} D(Q\|\overline{P}). \quad (6)$$

For brevity, we shall omit the dependence on $P, \overline{P}$ as they remain fixed and can be understood from context. As is well known \cite[Th. 3]{2}, for a given $D_1 \in (0, D(\overline{P}\|P))$, there exists a proper choice of $T$ such that

$$p_1(\phi_{n,T}^*) \leq \exp(-n \cdot D_1), \quad (7)$$

$$p_2(\phi_{n,T}^*) \leq \exp[-n \cdot D_2(D_1)]. \quad (8)$$

Furthermore, it is also known that this exponential behavior is optimal \cite[Corollary 2]{2}, in the sense that if

$$\liminf_{n \to \infty} \frac{1}{n} \log p_1(\phi_{n,T}^*) \geq D_1 \quad (9)$$

then

$$\limsup_{n \to \infty} \frac{1}{n} \log p_2(\phi_{n,T}^*) \leq D_2(D_1). \quad (10)$$

It should be noted, however, that the detector (5) and the bounds on its error probability (7)-(8) are exactly optimal for any given $n$. In fact, in what follows, we will use these relations for $n = 1$.

The function $D_2(D_1)$ is known to be a convex function of $D_1$, continuous on $(0, \infty)$ and strictly decreasing prior to any interval on which it is constant \cite[Th. 3]{2}. Furthermore, it is known \cite[Th. 7]{2} that it can be represented as

$$D_2(D_1) = \sup_{\tau \geq 0} \{ -\tau \cdot D_1 + (\tau + 1) \cdot d_\tau \}, \quad (11)$$

where

$$d_\tau \overset{\text{def}}{=} -\log \left[ \sum_{z \in \mathbb{Z}} P^{\tau+1}(z) \overline{P}^{\tau+1}(z) \right], \quad (12)$$

is the Chernoff distance between distributions. The representation (11) will be used in the sequel to derive bounds on the reliability of DHT systems.

C. Distributed Hypothesis-Testing

Let $\{(X_i, Y_i)\}_{i=1}^n$ be $n$ independent copies of a pair of random variables $(X, Y) \in (\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite alphabets. Under $H$, the joint distribution of $(X, Y)$ is given by $P_{XY}$, whereas under $\overline{P}$, this distribution is given by $\overline{P}_{XY}$. We assume that both probability measures are absolutely continuous with respect to one another,
Figure 2. A DHT system.

i.e., $P_{XY} \gg P_{XY}$, as well as $P_{XY} \ll P_{XY}$\(^{11}\) and thus it can be assumed without loss of generality (w.l.o.g.) that $\text{supp}(P_X) = \text{supp}(P_X) = \mathcal{X}$ and $\text{supp}(P_Y) = \text{supp}(P_Y) = \mathcal{Y}$. For brevity, we denote the probability of an event $\mathcal{A}$ under $H$ (respectively, $\overline{H}$) by $P(\mathcal{A})$ (respectively, $\overline{P}(\mathcal{A})$).

A DHT system $\mathcal{H}_n \overset{\text{def}}{=} (f_n, \varphi_n)$, as depicted in Fig. 2, is defined by an encoder

$$f_n : \mathcal{X}^n \rightarrow [m_n],$$

which maps a source vector into an index $i = f_n(x^n)$, and a detector (possibly randomized\(^{12}\))

$$\varphi_n : [m_n] \times \mathcal{Y}^n \rightarrow \{H, \overline{H}\}.$$ (14)

The inverse image of $f_n$ for $i \in [m_n]$, i.e.,

$$f_n^{-1}(i) \overset{\text{def}}{=} \{x^n \in \mathcal{X}^n : f_n(x^n) = i\},$$ (15)

is called the bin associated with index $i$. The rate of $\mathcal{H}_n$ is defined as $\frac{1}{n} \log m_n$, the type 1 error probability of $\mathcal{H}_n$ is defined as

$$p_1(\mathcal{H}_n) \overset{\text{def}}{=} P[\varphi_n(f_n(X^n), Y^n) = \overline{H}],$$ (16)

and the type 2 error probability is defined as

$$p_2(\mathcal{H}_n) \overset{\text{def}}{=} \overline{P}[\varphi_n(f_n(X^n), Y^n) = H].$$ (17)

In the sequel, conditional error probabilities given an event $\mathcal{A}$ will be abbreviated as, e.g.,

$$p_1(\mathcal{H}_n | \mathcal{A}) \overset{\text{def}}{=} P[\varphi_n(f_n(X^n), Y^n) = \overline{H} | \mathcal{A}].$$ (18)

\(^{11}\)This implies that both types of Stein’s exponent for this problem are finite.

\(^{12}\)Randomized encoding can also be defined. In this case, the encoder takes the form $f_n : \mathcal{X}^n \rightarrow S([m_n])$, where $f_n(x^n)$ is a probability vector whose $i$th entry is the probability of mapping $x^n$ to the index $i \in [m_n]$. In the sequel, we will also use a rather simple form of randomized encoding, which does not require this general definition. There, the source vector $x^n$ will be used to randomly generate a new source vector $\tilde{X}^n$, and the latter will be encoded by a deterministic encoder (see the proof of the achievability part of Theorem 6 in Appendix B-A).
A sequence of DHT systems will be denoted by \( \mathcal{H} \equiv \{ \mathcal{H}_n \}_{n \geq 1} \). The sequence \( \mathcal{H} \) is associated with two different exponents for each of the two probabilities defined above. The \textit{infimum type 1 exponent} of a sequence of systems \( \mathcal{H} \) is defined by

\[
\liminf_{n \to \infty} -\frac{1}{n} \log p_1(\mathcal{H}_n),
\]

and the \textit{supremum type 1 exponent} is defined by

\[
\limsup_{n \to \infty} -\frac{1}{n} \log p_1(\mathcal{H}_n).
\]

Analogous exponents can be defined for the type 2 error probability.

The \textit{reliability function} of a DHT system is the optimal trade-off between the two types of exponents achieved by any encoder-detector pair of a given rate \( R \). Specifically, the \textit{infimum DHT reliability function} is defined by

\[
E^{-2}(R, E_1; P_{XY}, \overline{P}_{XY}) \equiv \sup_{\mathcal{H}} \left\{ \liminf_{n \to \infty} -\frac{1}{n} \log p_2(\mathcal{H}_n) : \forall n, m_n \leq e^{nR}, p_1(\mathcal{H}_n) \leq e^{-nE_1} \right\},
\]

and the \textit{supremum DHT reliability function} \( E^{+2}(R, E_1; P_{XY}, \overline{P}_{XY}) \) is analogously defined, albeit with a \( \limsup \). For brevity, the dependence on \( P_{Y|X}, \overline{P}_{Y|X} \) will be omitted henceforth whenever it is understood from context.

### III. Main Result: Bounds on the Reliability Function of DHT

To begin the discussion on the reliability function of DHT systems, we note that for a given encoder \( f_n \), the form of the optimal detector is just a comparison of likelihoods, as in ordinary HT. Indeed, it readily follows from (5) that the optimal detector has the form

\[
\varphi^{*}_{n,T}(i, y^n) \equiv \begin{cases} 
\{ x^n : f_n(x^n) = i \} & \sum_{x^n : f_n(x^n) = i} P_{XY}(x^n, y^n) < e^{nT} \cdot \sum_{x^n : f_n(x^n) = i} \overline{P}_{XY}(x^n, y^n) \\
\emptyset & \text{otherwise}
\end{cases}
\]

for some \( T \in \mathbb{R} \) which sets the trade-off between the two error probabilities.

Hence, the characterization of the DHT reliability function is reduced to finding optimal encoders, to wit, a partition of the alphabet \( \mathcal{X}^n \) into bins, and then, for a given sequence of optimal encoders, finding single-letter expressions for the resulting error exponents, under optimal detection (22). These problems are much more challenging then the characterization of the optimal detector. Nonetheless, just like in distributed compression and channel coding problems mentioned in the introduction, achievability bounds can be derived using random-coding ensembles. Specifically, the main result of this paper, which we next describe, is a random-coding bound and an expurgated bound, obtained under an optimal Neyman-Pearson detector. Before that, we state the trivial converse bound, obtained when \( X^n \) is not compressed, or alternatively, when \( R = \log |\mathcal{X}| \) (immediately deduced from the discussion in Section [I-B]).
Proposition 1. The supremum DHT reliability function is bounded as

\[ E_2^+(R, E_1; P_{XY}, \overline{P}_{XY}) \leq \min_{Q_{XY}: D(Q_{XY}\|P_{XY}) \leq E_1} D(Q_{XY}\|\overline{P}_{XY}). \]  \hspace{1cm} (23)

To state our achievability bound, we will need several additional notations. We denote the Chernoff distance between symbols by

\[ d_\tau(x, \bar{x}) \stackrel{\text{def}}{=} -\log \sum_{y \in \mathcal{Y}} P_{\tau|X}(y|x) \overline{P}_{\tau|X}(y|\bar{x}), \]  \hspace{1cm} (24)

and between vectors by

\[ d_\tau(x^n, \bar{x}^n) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} d_\tau(x_i, \bar{x}_i). \]  \hspace{1cm} (25)

Further, when \((X, \bar{X})\) are distributed according to \(Q_{X\bar{X}}\) we define the average Chernoff distance as

\[ d_\tau(Q_{X\bar{X}}) \stackrel{\text{def}}{=} \mathbb{E}_Q[d_\tau(X, \bar{X})], \]  \hspace{1cm} (26)

and when \(X\) is distributed according to \(Q_X\), we denote, for brevity,

\[ d_\tau(Q_X) \stackrel{\text{def}}{=} \mathbb{E}_Q[d_\tau(X, X)]. \]  \hspace{1cm} (27)

Next, we denote

\[ B'_{rc}(R, R_b, Q_{UX}, \tau) \]

\[ \stackrel{\text{def}}{=} \min_{(Q_{UX}, \overline{Q}_{UX}): Q_{UX} = \overline{Q}_{UX}, Q_V = \overline{Q}_Y} \left\{ \tau \cdot D(Q_{Y|UX}\|P_{Y|X}Q_{UX}) + D(\overline{Q}_{Y|UX}\|\overline{P}_{Y|X}\overline{Q}_{UX}) \right. \]

\[ + \max \left\{ |I_Q(U; Y) - R_b|_+, I_Q(U, X; Y) - H(Q_X) + R \right\} \]

\[ + \tau \cdot \max \left\{ |I_{\overline{Q}}(U; Y) - R_b|_+, I_{\overline{Q}}(U, X; Y) - H(\overline{Q}_X) + R \right\} \}, \]  \hspace{1cm} (28)

and

\[ B''_{rc}(R, R_b, Q_{UX}, \tau) \]

\[ \stackrel{\text{def}}{=} \min_{(Q_{UX}, \overline{Q}_{UX}): Q_{UX} = \overline{Q}_{UX}, Q_V = \overline{Q}_Y, I_Q(U; Y) > R_b} \left\{ \tau \cdot D(Q_{Y|UX}\|P_{Y|X}Q_{UX}) + D(\overline{Q}_{Y|UX}\|\overline{P}_{Y|X}\overline{Q}_{UX}) \right. \]

\[ + |I_Q(X; Y|U) - H(Q_X) + R + R_b|_+ \]

\[ + \tau \cdot |I_{\overline{Q}}(X; Y|U) - H(\overline{Q}_X) + R + R_b|_+ \}, \]  \hspace{1cm} (29)

as well as

\[ B_{rc}(R, R_b, Q_{UX}, \tau) \stackrel{\text{def}}{=} \min \{ B'_{rc}(R, R_b, Q_{UX}, \tau), B''_{rc}(R, R_b, Q_{UX}, \tau) \}, \]  \hspace{1cm} (30)
which are all related to a random-coding based bound on the reliability function. We also denote
\[ B_{\text{ex}}(R, Q_X, \tau) \overset{\text{def}}{=} (\tau + 1) \cdot \min_{Q_{X':Q_X=Q_X', H_Q(X|\tilde{X}) \geq R}} \left\{ d_r(Q_X|\tilde{X}) + R - H_Q(X|\tilde{X}) \right\}, \]
which is related to an expurgated based bound on the reliability function. Finally, we also denote
\[ B(R, Q_X, \tau) \overset{\text{def}}{=} \max \left\{ \sup_{Q_{U|X}} \sup_{R_b: R_b \geq |I_Q(U;X)-R|_+} B_{\text{rc}}(R, R_b, Q_{U|X}, \tau), B_{\text{ex}}(R, Q_X, \tau) \right\}. \]
For brevity, arguments such as \((R, R_b, Q_{U|X}, \tau)\) will sometimes be omitted henceforth.

**Theorem 2.** The infimum DHT reliability function bounded as
\[ E^*_2(R, E_1; P_{XY}, \overline{P}_{XY}) \geq \min_{Q_X} \sup_{\tau \geq 0} \left[ -\tau \cdot E_1 + D(Q_X || \overline{P}_{XY}) + \tau \cdot D(Q_X || P_X) + \min \left\{ (\tau + 1) \cdot d_r(Q_X), B(R, Q_X, \tau) \right\} \right]. \]

As hinted by (32), the best of a random-coding bound and an expurgated bound can be chosen for any given input type \(Q_X\). In the case of a random-coding bound, the achieving scheme is based on quantization and binning. In this respect, for such \(Q_X\) (with \(H(Q_X) > R\)), the conditional type \(Q_{U|X}\) is the test channel for quantizing \(|T_n(Q_X)| = e^{nH(Q_X)}\) source vectors into one of \(e^{nR_q}\) possible reproduction vectors, where the quantization rate \(R_q\) satisfies \(R_q > R\). Then, these reproduction vectors are grouped to bins of size (at most) \(e^{nR_b}\) each, the binning rate \(R_b\) satisfies \(R_b = R_q - R\). Both \(Q_{U|X}\) and \(R_b\) may be optimized, separately for any given \(Q_X\), to obtain the best type 2 exponent. In case the expurgated bound is better than the random-coding bound for \(Q_X\), the scheme which achieves it is based on binning at rate \(R_b = |H(Q_X) - R|_+\), without quantization.

We next discuss several implications of Theorem 2. First, simpler bounds, perhaps at the cost of worse exponents, can be obtained by considering two external choices. To obtain a binning-based scheme, without quantization, we choose \(U\) to be a degenerated random variable (deterministic, i.e., \(|U| = 1\)) and \(R_b = H(Q_X) - R\). We then get that \(B_{\text{rc}}\) dominates the minimization in (30), and
\[ B_{\text{rc}}(R, H(Q_X) - R, Q_{U|X}, \tau) = B_{\text{rc},b}(R, Q_X, \tau) \]
\[ \overset{\text{def}}{=} \min_{(Q_{X,Y}, Q_{Y}): Q_X = Q_X', Q_Y = Q_Y'} \left\{ \tau \cdot D(Q_{Y|X}||P_{Y|X}|Q_X) + D(Q_{Y|X}||\overline{P}_{Y|X}|Q_X) \right. \]
\[ + \left. |R - H_Q(X|Y)|_+ + \tau \cdot |R - H_Q(X|Y)|_+ \right\}. \]
To obtain a quantization-based scheme, without binning, we choose \(R_b = 0\), and limit \(Q_{U|X}\) to satisfy \(R \geq I_Q(U;X)\).

Second, if the rate is large enough then no loss is expected in the reliability function of DHT. We can deduce from Theorem 2 an upper bound on this no-loss rate, as follows.
Corollary 3. Suppose that $R$ is sufficiently large such that
\[ B(R, Q_X, \tau) \geq d_\tau(Q_X) \tag{36} \]
for all $Q_X \in S(X)$ and $\tau \geq 0$. Then,
\[ E_2^-(R, E_1) = E_2^-(\infty, E_1) \tag{37} \]
\[ = D_2(E_1). \tag{38} \]

The proof of this corollary appears in Appendix A.

Third, by setting $E_1 = 0$, Theorem 2 yields an achievable bound on Stein’s exponent, as follows.

Corollary 4. Stein’s exponent is bounded as
\[ E_2^-(R, 0) \]
\[ \geq D(P_X || P_X | X) + \sup_{\tau \geq 0} \min \left\{ (\tau + 1) \cdot d_\tau(P_X), B(R, P_X, \tau) \right\} \tag{39} \]
\[ \geq \min \left\{ D(P_{XY} || P_X \times P_Y | X), D(P_X || P_X | X) + \sup_{Q_{U|X}, R_b : R_b \geq \min(I_{P_X \times Q_{U|X}(U;X) - R})} \lim_{\tau \to \infty} B_{\text{te}}(R, R_b, P_X \times Q_{U|X}, \tau) \right\}. \tag{40} \]

The first term in (40) can be identified as Stein’s exponent when the rate is not constrained at all. The proof of this corollary also appears in Appendix A and it seems that no further significant simplifications are possible. It is worth to note, however, that the resulting bound is quite different from the best known bound by Shimokawa, Han and Amari [10, Th. 4.3], [13] (and its refinement in [32], [33]).

Fourth, it is interesting to examine the case $R = 0$. Using analysis similar to the proof of Corollary 4, it is easy to verify that using a binning-based scheme [i.e., substituting (35) in (33) for $B(R, Q_X, \tau)$] achieves the lower bound
\[ E_2^-(R = 0, E_1) \]
\[ \geq \min_{(Q_{XY}, Q_{XY}) : Q_X = Q_X, Q_Y = Q_Y} \min_{D(Q_{XY} || P_{XY}) \leq E_1} D(Q_{XY} || P_{XY}). \tag{41} \]

As expected, this is the same type 2 error exponent obtained when $y^n$ is also encoded at zero rate, as obtained in [10, Th. 5.4], [11, Th. 6]. For this bound, a matching converse is known [10, Th. 5.5]. When $E_1 = 0$ then $Q_{XY} = P_{XY}$, and then Stein’s exponent is given by
\[ E_2^-(R = 0, E_1 = 0) \]
\[ \geq \min_{Q_{XY} = P_X, Q_Y = P_Y} D(Q_{XY} || P_{XY}). \tag{42} \]

In [15, Th. 2] it was determined that this exponent is optimal (even when $y^n$ is not encoded and given as side-information to the detector).

The rest of the paper is mainly devoted to the proof of Theorem 2 and is based on the following methodology, comprised of two steps. We will introduce CD codes, which, in a nutshell, correspond to a single bin of a DHT.
The first step of the proof is the reduction of the DHT reliability problem to the CD reliability problem which will be considered in Section IV. The second step is derivation of single-letter achievable bounds on the reliability of CD codes, and this will be considered in Section V. The bound of Theorem 2 on the DHT reliability function then follow as easy corollary to these results.

IV. FROM DISTRIBUTED HYPOTHESIS-TESTING TO CHANNEL-DETECTION CODES

In this section, we formulate CD codes, and then use them to characterize the reliability of DHT systems. CD codes were considered in [42] for the problem of joint detection and decoding. For this purpose, the code has to be chosen to allow the receiver to detect the channel conditional probability, as well as for transmitting messages, just like an ordinary channel code. In this paper, each quantization cell of a DHT system will be considered and analyzed as a CD code. Since a DHT system is only required to decide on the hypothesis but not on the actual source vector, the error probability of CD codes (for transmitting messages) is irrelevant in this paper. However, this does not imply that all the codewords of CD code can be chosen to be identical (which is optimal in terms of detection performance), since by definition, the members of a quantization cell are different from one another. Hence, in what follows we will define CD codes of a given cardinality, and enforce their codewords to be different from one another. Here too, for brevity, we will denote the probability of an event \( A \) under \( H \) and \( \overline{H} \) by \( P(A) \) and \( \overline{P}(A) \), respectively. The required definitions for CD codes are quite similar to the ones required for DHT systems, but as some differences do exist, we explicitly outline them in what follows.

A CD code for a type class \( Q_X \in \mathcal{P}_n(X) \) is given by \( C_n \subseteq \mathcal{T}_n(Q_X) \) (where all codewords must be different). An input \( X^n \in C_n \) to the channel is chosen with a uniform distribution over \( C_n \), and sent over \( n \) uses of a DMC which may be either \( P_{Y|X} \) when \( H \) is active or \( \overline{P}_{Y|X} \) when \( \overline{H} \) is. The random channel output is given by \( Y^n \in Y^n \). The detector has to decide based on \( y^n \) whether the DMC conditional probability distribution is \( P_{Y|X} \) or \( \overline{P}_{Y|X} \). As for the DHT problem, we assume that \( P_{Y|X} \ll \overline{P}_{Y|X} \) and \( P_{Y|X} \gg \overline{P}_{Y|X} \). A detector (possibly randomized) for \( C_n \) is given by

\[
\phi_n : Y^n \to \{H, \overline{H}\}. \tag{43}
\]

In accordance, two error probabilities can be defined, to wit, the type 1 error probability

\[
p_1(C_n, \phi_n) \overset{\text{def}}{=} P\left[\phi_n(Y^n) = \overline{H}\right], \tag{44}
\]

and the type 2 error probability

\[
p_2(C_n, \phi_n) \overset{\text{def}}{=} \overline{P}\left[\phi_n(Y^n) = H\right]. \tag{45}
\]

As for the DHT problem, the Neyman-Pearson lemma implies that the optimal detector is given by

\[
\phi^*_{n,T}(y^n) \overset{\text{def}}{=} \mathbb{1}\left\{ \sum_{x^n \in C_n} P_{Y|X}(y^n|x^n) < e^{nT} \cdot \sum_{x^n \in C_n} \overline{P}_{Y|X}(y^n|x^n) \right\}, \tag{46}
\]
Let $Q_X \in \mathcal{P}(X)$ be a given type, and let $\{n_l\}_{l=1}^\infty$ be the subsequence of blocklengths such that $\mathcal{P}_{n}(Q_X)$ is not empty. As for a DHT sequence of systems $\mathcal{H}$, a sequence of CD codes $\mathcal{C} \triangleq \{C_{n_l}\}_{l=1}^\infty$ is associated with two exponents. The \textit{infimum type 1 exponent} of a sequence of codes $\mathcal{C}$ and detector $\{\phi_{n_l}\}_{l=1}^\infty$ is defined as

$$\liminf_{l \to \infty} -\frac{1}{n_l} \log p_1(C_{n_l}, \phi_{n_l})$$

(47)

and the \textit{supremum type 1 exponent} is similarly defined, albeit with a $\limsup$. Analogous exponents are defined for the type 2 error probability. In the sequel, we will construct DHT systems whose bins are good CD codes, for each $Q_X \in \mathcal{P}(X)$. Since to obtain an achievability bound for a DHT system, good performance of CD codes of all types of the source vectors will be simultaneously required, the blocklengths of the components CD codes must match. Thus, the limit inferior definition of exponents must be used, as it assures convergence for all sufficiently large blocklength. For the converse bound, we will use the limit superior definition.

For a given type $Q_X \in \mathcal{P}(X)$, rate $\rho \in [0, H(Q_X))$, and type 1 constraint $F_1 > 0$, we define the \textit{infimum CD reliability function} as

$$F_2^-(\rho, Q_X, F_1; P_{Y|X}, \overline{P}_{Y|X}) \triangleq \sup_{C, \{\phi_{n_l}\}_{l=1}^\infty} \left\{ \liminf_{l \to \infty} -\frac{1}{n_l} \log p_2(C_{n_l}, \phi_{n_l}) : \forall l, C_{n_l} \subseteq \mathcal{T}_{n_l}(Q_X), |C_{n_l}| \geq e^{n_l \rho}, p_1(C_{n_l}, \phi_{n_l}) \leq e^{-n_l \cdot F_1} \right\}$$

(48)

and the \textit{supremum CD reliability function} $F_2^+(\rho, Q_X, F_1; P_{Y|X}, \overline{P}_{Y|X})$ is analogously defined, albeit with a $\limsup$. For brevity, the dependence on $P_{Y|X}, \overline{P}_{Y|X}$ will be omitted whenever it is understood from context. Thus, the only difference in the reliability function of CD codes from ordinary HT, is that in CD codes the distributions are to be optimally designed under the rate constraint $|C_n| \geq e^{n \rho}$. Indeed, for $|C_n| = 1$ symmetry implies that any $x^n \in \mathcal{T}_n(Q_X)$ is an optimal CD code. The detector in this case has no ambiguity regarding the transmitted symbol at any given time point. This, however, does not hold when $|C_n| > 1$, and ambiguity exists at least a single time point. This additional uncertainty complicates the operation of the detector, and reduces the reliability function. Basic properties of $F_2^\pm(\rho, Q_X, F_1)$ are given as follows.

**Proposition 5.** As a function of $F_1$, $F_2^\pm(\rho, Q_X, F_1)$ are non-increasing and have both limit from the right and from the left at every point. They have no discontinuities of the second kind and the set of first kind discontinuities (i.e., jump discontinuity points) is at most countable. Similar properties hold as a function of $\rho \in [0, H(Q_X))$.

\textbf{Proof}: It follows from their definition that $F_2^\pm(\rho, Q_X, F_1)$ are non-increasing in $F_1$. The continuity statements follow from properties of monotonic functions [45] Th. 4.29 and its Corollary, Th. 4.30] (Darboux-Froda’s theorem).

With the above, we can state the main result of this section, which is a characterization of the reliability of DHT systems using the reliability of CD codes.
Theorem 6. The DHT reliability functions $E^\pm_2(R, E_1)$ satisfy:

- **Achievability part:**
  \[
  E^-_2(R, E_1) \geq \lim_{\delta \downarrow 0} \inf_{Q_X \in \mathcal{P}(X)} \left\{ D(Q_X || P_X) + F^-_2(H(Q_X) - R, Q_X, E_1 - D(Q_X || P_X) + \delta) \right\} \quad \text{(49)}
  \]

- **Converse part:**
  \[
  E^+_2(R, E_1) \leq \lim_{\delta \downarrow 0} \inf_{Q_X \in \mathcal{P}(X)} \left\{ D(Q_X || P_X) + F^+_2(H(Q_X) - R + \delta, Q_X, E_1 - D(Q_X || P_X) - \delta) \right\} \quad \text{(50)}
  \]

The achievability and converse part match up to two discrepancies. First, in the achievability (respectively, converse) part the infimum (supremum) reliability function appears. This seems unavoidable, as it is not known if the infimum and supremum reliability functions are equal even for ordinary channel codes [4, Problem 10.7]. Second, the bounds include left and right limits of $E^+_2(R, E_1)$ at rate $R$ and exponent $E_1$. Nonetheless, due to monotonicity, $E^+_2(R, E_1)$ is continuous function of $R$ and $E_1$ for all rates and exponents, perhaps excluding a countable set (Proposition 5). Thus, for any given $(R, E_1)$ there exists an arbitrarily close $(\tilde{R}, \tilde{E}_1)$ such that Theorem 6 holds with $\delta = 0$.

The proof of Theorem 6 appears in Appendix B. The achievability part (Appendix B-A) is proved by first constructing a DHT system for source vectors from a a single type class of the source. The bins are generated by permutations of a “good” CD code. The fact that the two hypotheses are memoryless implies that bins generated this way have approximately the same exponents as the original CD code. Furthermore, since type classes are closed under permutations, they can be covered using permutations of a CD code. A covering lemma by Ahlswede [41, Section 6, Covering Lemma 2] shows that in fact such covering method can be effective, in the sense that the required number of permutations is close to minimal number possible (up to the first order in the exponent). This allows to prove that the encoding rate is as required. Ideas from this spirit were used for the DHT problem in [7], as well as for distributed compression [39], [40], and secure lossy compression [46]. Afterwards, all types of the source are considered simultaneously to generate a DHT system for any possible type of the source vector. This requires proving the uniform convergence of the error probabilities to their asymptotic exponential behavior, uniformly over all possible types.

The proof of the converse part (Appendix B-B) is based upon identifying a sequence of bins whose size and conditional exponents are close to typical values of the entire DHT system. Such a bin corresponds to a CD code, and thus clearly cannot have better exponents than the ones dictated by the reliability function of CD codes. This restriction is then translated back to bound the reliability of DHT systems.

This theorem parallels a similar result of [39], [40] for the reliability functions of distributed compression. By analogy, the reliability function of distributed compression can be characterized by the reliability function of ordinary channel codes (see Fig. 1). The latter can be bounded using well-known classic bounds, such as the random-coding and expurgated achievability bounds, and the sphere-packing, zero-rate, and straight-line converse
bounds. The characterization of Theorem articulates the fact that the reliability of DHT systems is directly connected to the problem of determining the reliability of CD codes. However, even this apparently simpler problem is challenging to solve exactly, and just as in the case of ordinary channel coding, perhaps only lower and upper bounds are at reach. Hence, even though DHT is a source coding problem in nature, for the such reliability functions are known (e.g., Marton’s exponent), Theorem reveals that the reliability of DHT systems depends on the reliability of a channel coding problem, for the such optimal exponents are typically not entirely known, even for the most basic settings. This might manifest the intrinsic difficulty of the DHT problem. Nonetheless, in the next section, we derive concrete bounds on the reliability of CD codes, which, using Theorem lead directly to bounds on the reliability of the DHT problem of Theorem.

V. Bounds on the Reliability of CD Codes

In the previous section, we have shown that the reliability function of DHT can be directly obtained by the reliability of CD codes. In this section, we derive bounds on the latter, using random-coding arguments. A naive approach would be to randomly draw the codewords independently from some distribution. However, as discussed in the introduction, the best known achievable bounds for DHT systems are obtained using quantization-and-binning-based schemes. As CD codes correspond to a single bin of a DHT system, it follows that CD codes can also benefit from adding dependence between the randomly drawn codewords. In more detail, a bin of a DHT system which was constructed by the quantization-and-binning method corresponds to a superposition code. To wit, it will contain a set of reproduction vectors which are sufficiently different from one another, so that the side-information vector enables to decide on the true reproduction vector with high reliability, and each reproduction vector is surrounded by a quantization cell, which correspond to all source vectors which are mapped to it in the quantization process. The quantization cell should be sufficiently “small”, such that as long as the detector correctly decodes the reproduction vector, the detection reliability given the reproduction vector is close to the detection reliability given the true vector. A CD code which corresponds to a binning-based scheme will contain source vectors which correspond to a good channel code for the channels $P_{Y|X}$ and $\overline{P}_{Y|X}$, so that the side-information vector $y^n$ can be used to decode the true vector with high reliability. A CD code which corresponds to a quantization-based scheme will look like a quantization cell of a single reproduction vector. The various types of CD codes are depicted in Fig. 3.

Since a single bin of a quantization-and-binning DHT system corresponds to a superposition code, it should be drawn from an hierarchical ensemble, which we next define:

Definition 7. A fixed-composition hierarchical ensemble for a type $Q_X \in \mathcal{P}_n(X)$ and rate $\rho$ is defined by a conditional type $Q_{U|X} \in \mathcal{P}_n(U,Q_X)$, where $U$ is an auxiliary random variable from a finite alphabet $U$, a cloud-center rate $\rho_c$, and a satellite rate $\rho_s$ such that $\rho = \rho_c + \rho_s$. A codebook $C_n$ from this ensemble is drawn in

13Strictly speaking, this is true only when the type of the source vectors is constant within each bin. Since sending the type class of the source vector requires negligible rate, it can be assumed w.l.o.g. that this is indeed the case, as otherwise, one can further partition the bin into type-classes.
two stages. First, $e^{n \rho c}$ cloud centers $C_{c,n}$ are drawn, independently and uniformly over $T_n(Q_U)$. Second, for each of the cloud centers $u^n \in C_{c,n}$, $e^{n \rho s}$ satellites are drawn independently and uniformly over $T_n(Q_{X|U}, u^n)$. When considered a random entity, the codebook will be denoted by $C_n$.

Evidently, codewords which pertain to the same cloud are dependent, whereas codewords from different clouds are independent. Further, the ordinary ensemble, in which each codeword is drawn uniformly over $T_n(Q_X)$, independently of all other codewords, can be obtained as a special case by choosing $U = X$ and $\rho_c = \rho$. On the other hand, when $\rho_s = \rho$, then a single cloud center is drawn, and all codewords are satellites of this center. This corresponds to the bin of a DHT system which is based only on quantization, without binning. More generally, for source vectors of type $Q_X$, a quantization-and-binning scheme of rate $R$, binning rate $R_b$, and quantization rate $R_q = R + R_b$, leads to CD codes of rate $\rho = H(Q_X) - R$, cloud-center rate $\rho_c = R_b$ and satellite rate $\rho_s = H(Q_X) - R_q$. When used for a DHT system, the cloud centers $C_{c,n}$ are the reproduction vectors. The joint distribution of any source vector and its reproduction vector is exactly $Q_{UX}$, and the choice of the test channel $Q_{U|X}$ is used to control the distortion of the quantization.

In this section, it will be more convenient to use the parameter $\lambda \overset{\text{def}}{=} \frac{1}{\tau+1}$ (so that $\tau = \frac{1-\lambda}{\lambda}$). Using this convention, we will use, e.g., $d_{\lambda}$ instead of $d_{\tau}$ for the Chernoff distance. Note also that $\lambda \in [0, 1].$

To state a random-coding bound on the reliability of CD codes, we denote

$$A'_{rc}(\rho, \rho_c, Q_{UX}, \lambda) \overset{\text{def}}{=} \min_{(Q_{UXY}, Q_{UXY}) \colon Q_{U|X} = \overline{Q_{U|X}}, Q_{Y} = \overline{Q_{Y}}} \left\{ (1 - \lambda) \cdot D(Q_{Y|UX}||P_{Y|X}Q_{UX}) + \lambda \cdot D(\overline{Q_{Y|UX}}||\overline{P_{Y|X}}\overline{Q_{UX}}), \right\}$$

Or a sub-exponential number of cloud centers.

Usually in rate-distortion theory, the test channel is used to control the average distortion $E_Q[d(X,U)]$ for some additive distortion measure $d : X \times U \rightarrow \mathbb{R}_+$. Here, $x^n$ is always chosen from $T_n(Q_{X|U}, u^n)$ and therefore the distortion between $u^n$ and $x^n$ is constant depending on $Q_{UX}$.
+ \lambda \cdot \max \left\{ |I_Q(U; Y) - \rho_c|_+, \ I_Q(U, X; Y) - \rho \right\} + (1 - \lambda) \cdot \max \left\{ |I_{\overline{Q}}(U; Y) - \rho_c|_+, \ I_{\overline{Q}}(U, X; Y) - \rho \right\},
\end{align}

and

\begin{align}
A_{rc}''(\rho, \rho_c, Q_{UX}, \lambda) \equiv & \min_{(Q_{UX}, \overline{Q}_{UXY}) : Q_{U|X} = \overline{Q}_{U|X}, \ Q_{UY} = \overline{Q}_{UY}, \ I_Q(U; Y) > \rho_c} \left\{ (1 - \lambda) \cdot D(Q_{Y|UX} \| P_{Y|X}Q_{UX}) + \lambda \cdot D(\overline{Q}_{Y|UX} \| \overline{P}_{Y|X}\overline{Q}_{UX}),
\right.
\left.
+ \lambda \cdot |I_Q(X; Y|U) - \rho + \rho_c|_+ + (1 - \lambda) \cdot |I_{\overline{Q}}(X; Y|U) - \rho + \rho_c|_+ \right\},
\end{align}

as well as

\begin{align}
A_{rc}(\rho, \rho_c, Q_{UX}, \lambda) \equiv \min \left\{ A_{rc}'(\rho, \rho_c, Q_{UX}, \lambda), \ A_{rc}''(\rho, \rho_c, Q_{UX}, \lambda) \right\}.
\end{align}

For brevity, when can be understood from context, the dependency on \((\rho, \rho_c, Q_{UX}, \lambda)\) will be omitted. Our random-coding bound is as follows.

**Theorem 8.** The infimum CD reliability function is bounded as

\begin{align}
F_2^{-}(\rho, Q_X, F_1) \geq \max_{0 \leq \lambda \leq 1} \left\{ \frac{1 - \lambda}{\lambda} \cdot F_1 + \frac{1}{\lambda} \cdot \min \left[ d_{\lambda}(Q_X), \sup_{Q_{UX}} \sup_{\rho_c \geq |\rho - H_Q(X|U)|_+} \ A_{rc}(\rho, \rho_c, Q_{UX}, \lambda) \right] \right\}.
\end{align}

The proof of Theorem 8 appears entirely in Appendix C and here we provide a brief outline. The main idea of the proof is that when a CD code is used, the output distribution under \(H\) is given by the induced distribution

\begin{align}
P_{Y_n|X}^{(C_n)}(y^n) \overset{\text{def}}{=} \frac{1}{|C_n|} \sum_{x^n \in C_n} P_{Y_n}(y^n|x^n).
\end{align}

Based on the output vector \(y^n\), the detector should distinguish between two distributions \(P_{Y_n}^{(C_n)}\) and \(\overline{P}_{Y_n}^{(C_n)}\) [the latter defined similarly to \(\overline{Q}_{UX}\)]. This corresponds to an ordinary HT problem between the \(P_{Y_n}^{(C_n)}\) and \(\overline{P}_{Y_n}^{(C_n)}\), and therefore the exponents of this HT problem are simply given by (7) and (8) (when letting \(Z = Y_n\)). In turn, using the characterization (11), the reliability function can be expressed using the Chernoff distance between \(P_{Y_n}^{(C_n)}\) and \(\overline{P}_{Y_n}^{(C_n)}\). As the CD code which maximizes this Chernoff distance is difficult to find, we turn to a random-coding approach, and evaluate its ensemble average, namely

\begin{align}
\mathbb{E} \left\{ \sum_{y^n \in Y^n} \left[ P_{Y_n}^{(C_n)}(y^n) \right]^{1-\lambda} \cdot \left[ \overline{P}_{Y_n}^{(C_n)}(y^n) \right]^{\lambda} \right\}.
\end{align}

In the same spirit, we will evaluate the number of unique codewords of a randomly chosen CD code. In fact, we will show that under proper conditions, it is asymptotically equal to the number of drawn codewords, i.e., \(e^{n\rho}\). Then, a sequence of codebooks, such that each codebook has effectively \(e^{n\rho}\) unique codewords, and the Chernoff distance is typical to the ensemble average will be considered. For this sequence, it will then be shown that removing
duplicates, i.e., keeping only a single instance of codewords which were drawn more than once (thus making it a valid CD code), may only cause a negligible loss in the achieved exponent. This will prove a single-letter random-coding bound on the reliability function.

We make the following comments:

1) Loosely speaking, in (53), the exponent $A'_{rc}$ corresponds to the contribution to the error probability from codewords which belong to different cloud centers, whereas the exponent $A''_{rc}$ corresponds to the contribution to the error probability from codewords which belong to the same cloud center as the transmitted codeword. Thus, for a given rate $\rho$, $A'_{rc}$ is monotonically non-increasing with $\rho_c$, while $A''_{rc}$ is monotonically non-decreasing with $\rho_c$ (or, monotonically non-increasing with $\rho_s$). The cloud-center rate $\rho_c$ and the test channel $Q_{U|X}$ therefore should be chosen to optimally balance between these two contributions to the error probability.

2) For a given $Q_{U|X}$, the random-coding analysis of the ensemble average of the Chernoff distance is based on the type-enumeration method [48, Sec. 6.3], which leads to ensemble-tight bounds in many problems. While we cannot rigorously claim ensemble-tightness here, our analysis still avoids any use of bounds such as Jensen’s inequality. Thus, we conjecture that our bounds are ensemble-tight, and cannot be improved.

3) In comparison to [42], we have generalized the random-coding analysis of the detection error exponents to hierarchical ensembles, and also obtained simpler expressions using the ensemble average of the exponent of the Chernoff distance.

4) In fact, a stronger claim than the one appears in Theorem 8 can be made. It can be shown that there exists a single sequence of CD codes \( \{C^*_{nl}\}_{l=1}^{\infty} \) such that

\[
\liminf_{l \to \infty} -\frac{1}{n_l} \log \min_{T: p_1(C^*_n, \phi^*_n, T) \leq e^{-nF_1}} p_2(C^*_n, \phi^*_n, T) \geq \max_{0 \leq \lambda \leq 1} \left\{ -\frac{1 - \lambda}{\lambda} \cdot F_1 + \frac{1}{\lambda} \cdot \min \left\{ d_\lambda(Q_{X}), \sup_{Q_{U|X} \rho_c: \rho_c \geq \sup_{\rho - H_Q(X|U)} A_{rc} \right\} \right\},
\]

simultaneously for all $F_1$. Thus, when using such a CD code, the operating point along the trade-off curve between the two exponents can be determined solely by the detector, and can be arbitrarily changed from block to block. For details regarding the proof of this claim, see Remark 18.

Next, we state our expurgated exponent, and to this end we denote

\[
A_{ex}(\rho, Q_{X}, \lambda) \overset{\text{def}}{=} \min_{Q_{X}: Q_{X} = Q_{X}, I_Q(X|\tilde{X}) \leq \rho} \left\{ d_\lambda(Q_{X}, \tilde{X}) + I_Q(X; \tilde{X}) - \rho \right\}.
\]

**Theorem 9.** The infimum CD reliability function is bounded as

\[
F_2^{-}(\rho, Q_{X}, F_1) \leq \max_{0 \leq \lambda \leq 1} \left\{ -\frac{1 - \lambda}{\lambda} \cdot F_1 + \frac{1}{\lambda} \cdot \min \{ d_\lambda(Q_{X}), A_{ex}(\rho, Q_{X}, \lambda) \} \right\}.
\]

**Proof of Theorem 9.** As was shown in [42, Appendix E], for any $\delta > 0$ and all $n$ sufficiently large, there
exists a CD code $C_n^*$ (of rate $\rho$) such that
\[
\sum_{y^n \in Y_n} \left[ P_{Y_n}^{(C_n^*)} (y^n) \right]^{1-\lambda} \left[ \mathcal{T}^{(C_n^*)}_{Y_n} (y^n) \right]^{\lambda} \leq \exp \left[ -n \cdot \min \{ d_{\lambda}(Q_X), A_{\text{ex}}(\rho, Q_X, \lambda) \} \right].
\] (60)

Substituting this bound to the characterization (11), taking $n \to \infty$ and $\delta \downarrow 0$ completes the proof of the theorem.  

We conclude with two comments:

1) A similar expurgated bound can be derived for hierarchical ensembles. However, when optimizing the rates $(\rho_s, \rho_c)$ for this expurgated bound, it turns out that choosing $\rho_s = 0$ is optimal. Thus, the resulting bound exactly equals the bound of Theorem 9, which corresponds to an ordinary ensemble.

2) Since the expurgated exponent only improves the random-coding exponent of the ordinary ensemble (which is inferior in performance to the hierarchical ensemble), it is anticipated that expurgation does not play a significant role in this problem, compared to the channel coding problem. This might be due to the fact that the aim of expurgation is to remove codewords which have “close” neighbors, while this is not actually required in the DHT problem. This can also be attributed to the bounding technique of the expurgated bound, which is based on pairwise Chernoff distances.

After deriving the bounds on the reliability of CD codes, we return to the DHT problem, and conclude the section with a short proof of Theorem 2.

Proof: Up to the arbitrariness of $\delta > 0$, in Theorem 6 we have obtained
\[
E_2^-(R, E_1) = \inf_{Q_X \in \mathcal{P}(X)} \left\{ D(Q_X||P_X) + F_2^- (H(Q_X) - R, Q_X, E_1 - D(Q_X||P_X)) \right\}.
\] (61)

Further, the random-coding bound of Theorem 8 and the expurgated bound of Theorem 9 both imply that
\[
F_2^- (H(Q_X) - R, Q_X, E_1 - D(Q_X||P_X))
\geq \max \left\{ \max_{0 \leq \lambda \leq 1} \left[ -\frac{1-\lambda}{\lambda} \cdot E_1 + \frac{1-\lambda}{\lambda} \cdot D(Q_X||P_X) \right] + \frac{1}{\lambda} \cdot \min \left\{ d_{\lambda}(Q_X), \sup_{Q_{U\mid X}} R_b : R_b \geq |I_{Q}(U;X)-R|_+ \sup A_{\text{rc}}(H(Q_X) - R, R_b, Q_{UX}, \lambda) \right\} \right\},
\]
\[
\max_{0 \leq \lambda \leq 1} \left[ -\frac{1-\lambda}{\lambda} \cdot E_1 + \frac{1-\lambda}{\lambda} \cdot D(Q_X||P_X) + \frac{1}{\lambda} \cdot \min \left\{ d_{\lambda}(Q_X), A_{\text{ex}}(H(Q_X) - R, Q_X, \lambda) \right\} \right]
\] (62)
\[
= \sup_{Q_{U\mid X}} R_b : R_b \geq |I_{Q}(U;X)-R|_+ \max_{0 \leq \lambda \leq 1} \left\{ -\frac{1-\lambda}{\lambda} \cdot E_1 + \frac{1-\lambda}{\lambda} \cdot D(Q_X||P_X) + \frac{1}{\lambda} \cdot \min \left\{ d_{\lambda}(Q_X), \max A_{\text{rc}}(H(Q_X) - R, R_b, Q_{UX}, \lambda), A_{\text{ex}}(H(Q_X) - R, Q_X, \lambda) \right\} \right\}.
\] (63)
The bound of Theorem 2 is obtained by substituting in (63) in (61), while changing variables from $\lambda$ to $\tau \overset{\text{def}}{=} \frac{1-\lambda}{\lambda}$ and from $\rho_c$ to $R_b$, as well as using the definitions of $B_{rc}$ (30) and $B_{ex}$ (31) and $B$ (32).

VI. COMPUTATIONAL ASPECTS AND A NUMERICAL EXAMPLE

The bound of Theorem 2 is rather involved, and therefore it is of interest to discuss how to compute it efficiently. Evidently, the main computational task is the computation of $B'_{rc}$ and $B''_{rc}$ for a given $(R, R_b, Q_{UX}, \tau)$. To this end, it can be seen that the objective functions of both $B'_{rc}$ and $B''_{rc}$ are strictly convex functions of $(Q_{Y|UX}, Q_{Y|UX}^*)$ (as $P_{Y|X} \ll P_{Y|X}$ was assumed).

Furthermore, the feasible set of $B'_{rc}$ is a convex set (only has linear constraints) and thus the computation of $B'_{rc}$ is a convex optimization problem, which can be solved efficiently [49]. However, the feasible set of $B''_{rc}$ is not convex, due to the additional constraint $I_Q(U; Y) > R_b$ beyond the linear constraints. Nevertheless, the value of $B_{rc}$ can be computed efficiently, by solving only convex optimization problems, in the following algorithm:

1) Solve the optimization problem (28) defining $B'_{rc}$, and let the optimal value be $v'$.
2) Solve the optimization problem (29) defining $B''_{rc}$, but without the constraint $I_Q(U; Y) > R_b$ (this is a convex optimization problem). Let the solution be $(Q_{U|XY}, Q_{U|XY}^*)$ and the optimal value be $v''$. If $I_Q^*(U; Y) > R_b$
   then set $B''_{rc} = v''$, and otherwise, set $B''_{rc} = \infty$.
3) The result is $B_{rc} = \min\{v', v''\}$.

This algorithm is correct due to the following argument. It is easily verified that if $I_Q^*(U; Y) > R_b$ then the constraint $I_Q(U; Y) > R_b$ is inactive, and therefore can be omitted. Thus, in this case $B''_{rc} = v''$. However, if this is not the case, then the solution must be on the boundary, i.e., must satisfy $I_Q(U; Y) = R_b$. This is because the objective in $B''_{rc}$ is a strictly convex function. In the latter case, it can be easily seen that $B'_{rc} \leq B''_{rc}$, and as $B_{rc}$ is the minimum between the two, $B''_{rc} = \infty$ can be set. Computing $B_{ex}(R, Q_X, \tau)$ of (31) is a convex optimization problem, over $Q_{X|X}$.

Given the value of $B_{rc}$, the next step is to optimize over $Q_{U|X}$ and $\rho_c$. While this should be done exhaustively, any specific choice of $Q_{U|X}$ and $\rho_c$ (or a restricted optimization set for them) leads to an achievable bound on $E_2^-(R, E_1)$. It should be mentioned, however, that it is not clear to us how to apply standard cardinality-bounding techniques (based on the support lemma [4], p. 310) to bound $|U|$ in this problem. Thus, in principle, the cardinality of $U$ is unrestricted, improved bounds are possible.

Finally, both $\tau$ and $Q_X$ should be optimized, which is feasible when $X$ is not very large and exhausting the simplex $S(\mathcal{X})$ in search of the minimizer $Q_X$ is possible. Furthermore, as we have seen in Corollary 4, when only Stein’s exponent is of interest, i.e., $E_1 = 0$, the minimal value in (33) must be attained for $Q_X = P_X$. Thus, there is no need to minimize over $Q_X \in S(\mathcal{X})$, but rather only on $Q_X = P_X$. We can also set $\tau \rightarrow \infty$ if the weak version

16Indeed, the divergence terms and $I(U; X; Y)$ are convex in $Q_{Y|UX}$. The term $I(U; Y)$ is also convex in $Q_{Y|UX}$, as a composition of a linear function which maps $Q_{Y|UX}$ to $Q_{Y|U}$ and the mutual information $I(U; Y)$. The pointwise maximum of two convex functions is also a convex function (note that $|f(t)|_+ = \max\{0, f(t)\}$).

17Or by any other general-purpose global optimization algorithm.
Figure 4. A binary example.

Theorem 4 of Corollary 4 is used as a bound. More generally, the minimizer of $Q_X$ must satisfy $D(Q_X||P_X) \leq E_1$, and this can decrease the size of the feasible set of $Q_X$ whenever the required $E_1$ is not very large.

A simple example for using the above methods to compute bounds on the DHT reliability function is given as follows.

**Example 10.** Consider the case $\mathcal{X} = \mathcal{Y} = \{0, 1\}$, and $P_X = P_X = (1/2, 1/2)$, where $P_{Y|X}$ and $P_{Y|X}$ are binary symmetric channels with crossover probabilities $10^{-1}$ and $10^{-2}$, respectively. We have used an auxiliary alphabet of size $|\mathcal{U}| = |\mathcal{X}| + 1 = 3$ since this cardinality suffices for the best known bounds [13], and due to the symmetry in the problem, we have only optimized over symmetric $Q_{U|X}$. Achievable bounds on the reliability of the DHT are shown in Fig. 4 for two different rates.

**VII. Conclusion and Further Research**

In this paper, we have considered the trade-off between the two types of error exponents of a DHT system, with full side-information. We have shown that its reliability is intimately related to the reliability of CD codes, and thus the latter simpler problem should be considered. Achievable bounds on the reliability of CD codes were derived, under the optimal Neyman-Pearson detector.

There are multiple directions in which our understanding of the problem can be broadened:

1) **Variable-rate coding:** As was noted in the past, the reliability of compression systems can be increased when *variable-rate* is allowed, either with an average rate constraint [39], or under excess-rate exponent constraint [40]. It is straightforward to use the techniques developed there to the DHT problem studied here (see also the informal discussion in [46, Appendix]).
2) **Computation of the bounds:** As discussed in the paper, the main challenge in computing the random-coding bound computation is the optimization over the test channel $Q_{U|X}$. First, deriving cardinality bounds on the auxiliary random variable alphabet $U$ is of interest. Second, finding an efficient algorithm to optimize the test channel, perhaps an alternating-maximization algorithm in the spirit of the Csiszár-Tusnády [50] and the Blahut-Arimoto algorithms [51], [52], [53], could be developed for this problem. As was noted in [16], [21], the Stein exponent of the DHT problem of testing against independence is identical to the information bottleneck problem [54], for which such alternating-maximization algorithm was developed.

3) **Converse bounds:** While we have shown that to obtain converse bounds on the reliability of DHT systems, it suffices to obtain converse bounds on the reliability of CD codes, no concrete bounds were derived. To obtain converse bounds which explicitly depend on the rate (in contrast to Proposition 1), two challenges are visible. First, it is tempting to conjecture that the Chernoff characterization (11) characterizes the reliability of CD codes, in the sense that

$$F_2^+(\rho, Q_X, F_1; P_{Y|X}, \overline{P}_{Y|X}) = \limsup_{l \to \infty} \sup_{T_n(\rho)} \max_{C_n} \sup_{\tau \geq 0} \frac{-\tau \cdot F_1 - (\tau + 1) \cdot \frac{1}{n_l} \log \left\{ \sum_{y^{n_l} \in Y^{n_l}} \left[ P(Y^{n_l} | C_n) \right]^{1/1+\tau} \cdot \left[ \overline{P}(C_n) Y^{n_l} \right]^{1/1+\tau} \right\}}{n_l},$$

just as a similar quantity was used to derive the random-coding and expurgated bounds. Second, even if this conjecture holds, the value of

$$\left\{ \sum_{y^{n_l} \in Y^{n_l}} \left[ P(Y^{n_l} | C_n) \right]^{1/1+\tau} \cdot \left[ \overline{P}(C_n) Y^{n_l} \right]^{1/1+\tau} \right\}$$

should be lower bounded for all CD codes whose size is larger than $e^{n \rho}$. As this term can be identified as a Rényi divergence [55], [56], the problem of bounding its value is a Rényi divergence characterization. This problem seems formidable, as the methods developed in [8] for the entropy characterization problem rely heavily on the chain rule of mutual information; a property which is not naturally satisfied by Rényi entropies and divergences. Hence, the problem of obtaining a non-trivial converse bound for the reliability of DHT systems with general hypotheses and a positive encoding rate remains an elusive open problem.

4) **Rate constraint on the side-information:** The reliability of a DHT systems in which the side-information vector $y^n$ is also encoded at a limited rate should be studied under optimal detection. Such systems will naturally lead to multiple-access CD codes, as studied for ordinary channel coding (see [57], [58] and references therein). However, for such a scenario, it was shown in [33] that the use of linear codes (a-la Körner-Marton coding) dramatically improves performance. Thus, it is of interest to analyze DHT systems with both linear codes and optimal detection. However, it is not yet known how to apply the type-enumeration method, used here for analysis of optimal detection, to linear codes. Hence, either the type-enumeration method should be refined,
or an alternative approach should be sought after.

5) **Generalized hypotheses:** Hypotheses regarding the distributions of continuous random variables, or regarding the distributions of sources with memory can be considered. Furthermore, the case of composite hypotheses, in which the distribution under each hypotheses is not exactly known (e.g., belongs to a subset of a given parametric family), and finding universal detectors which operate as well as the for simple hypotheses can also be considered. For preliminary results along this line see [15], [42].

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**Appendix A**

**Proofs of Corollaries to Theorem 2**

**Proof of Corollary 3.** Suppose that in the bound of Theorem 2 the inner minimization is dominated by $(\tau + 1) \cdot d_\tau(Q_X)$ for all $Q_X$. Then, the bound (33) reads

$$
E_2^-(R, E_1; P_{XY}, \overline{P}_{XY}) \geq \min_{Q_X} \sup_{\tau \geq 0} \left\{ -\tau \cdot E_1 + D(Q_X||\overline{P}_X) + \tau \cdot D(Q_X||P_X) + (\tau + 1) \cdot d_\tau(Q_X) \right\} 
$$

(A.1)

$$
\equiv \min_{Q_X} \sup_{\tau \geq 0} \min_{Q_{Y|X}} \left\{ -\tau \cdot E_1 + D(Q_X||\overline{P}_X) + \tau \cdot D(Q_X||P_X) + \tau \cdot D(Q_{Y|X||P_X}) \right\} + D(Q_{Y|X||P_X}) 
$$

(A.2)

$$
= \min_{Q_X} \sup_{\tau \geq 0} \min_{Q_{Y|X}} \left\{ -\tau \cdot E_1 + D(Q_X||\overline{P}_X) + \tau \cdot D(Q_{Y|X||P_X}) \right\} 
$$

(A.3)

$$
= \min_{Q_{XY}: D(Q_{XY}||P_{XY}) \leq E_1} D(Q_{XY}||\overline{P}_{XY}), 
$$

(A.5)

where (a) follows since as can be easily shown that (see (C.31) in the proof of Lemma 16 and the discussion surrounding it)

$$
(\tau + 1) \cdot d_\tau(Q_X) = \min_{Q_{Y|X}} \left[ \tau \cdot D(Q_{Y|U_X}||P_{Y|X}Q_{UX}) + D(Q_{Y|U_X}||\overline{P}_Y|X) \right] 
$$

(A.6)

$$
= \min_{Q_{Y|X}} \left[ \tau \cdot D(Q_{Y|X}||P_{Y|X}Q_X) + D(Q_{Y|X}||\overline{P}_Y|X) \right], 
$$

(A.7)

and (b) follows since the objective function is linear in $\tau$ (and hence concave) and convex in $Q_{Y|X}$, and therefore the minimization and maximization order can be interchanged [59]. Thus, the achievability bound of Theorem 2 coincides with the converse bound of Proposition 1, the latter obtained when the rate of the DHT system is not constrained at all and given by the reliability function of ordinary HT problem between $P_{XY}$ and $\overline{P}_{XY}$.

**Proof of Corollary 4.** It can be seen that the outermost minimum in (33) must be attained for $Q_X = P_X$. Intuitively, since we are only interested in negligible type 1 exponent, any event with $Q_X \neq P_X$ has exponentially
decaying probability \( \exp[-nD(Q_X||P_X)] \), and does not affect the exponent. More rigorously, if \( Q_X \neq P_X \) then by taking \( \tau \to \infty \) the objective function becomes unbounded. Hence (39) immediately follows. Further simplifications are possible if the bound is weakened by ignoring the expurgated term, i.e., setting \( B_{\alpha}(R, Q_X, \tau) = 0 \) in (32). In this case, since

\[
(\tau + 1) \cdot d_\tau(P_X) = \min_{Q_{Y|X}} \left[ \tau \cdot D(Q_{Y|X}||P_{Y|X}|P_X) + D(Q_{Y|X}||P_{Y|X}|P_X) \right],
\]

(A.8)

[see (A.7)], and since \( \tau \) only multiplies positive terms in the objective functions of \( B_{\alpha}' \), \( B_{\alpha}'' \) [see (28) and (29)], it is evident that the supremum in (39) is obtained as \( \tau \to \infty \). Hence, the supremum and minimum in (39) can be interchanged to yield the bound

\[
E_2^-(R, 0) = D(P_X||\overline{P}_X) + \min \left\{ \lim_{\tau \to \infty} (\tau + 1) \cdot d_\tau(P_X), \sup_{Q_{U|X}} \sup_{R_b: R_b \geq |R \times Q_{U|X}|(U;X)-R} \lim_{\tau \to \infty} B_{\alpha}(R, R_b, P_X \times Q_{U|X}, \tau) \right\}
\]

(A.9)

\[
= \min \left\{ D(P_{XY}||P_X \times \overline{P}_{Y|X}), D(P_X||\overline{P}_X) + \sup_{Q_{U|X}} \sup_{R_b: R_b \geq |R \times Q_{U|X}|(U;X)-R} \lim_{\tau \to \infty} B_{\alpha}(R, R_b, P_X \times Q_{U|X}, \tau) \right\}
\]

(A.10)

where (a) follows since

\[
\sup_{\tau \geq 0} (\tau + 1) \cdot d_\tau(P_X) = \sup_{\tau \geq 0} \min_{Q_{Y|X}} \left[ \tau \cdot D(Q_{Y|X}||P_{Y|X}|P_X) + D(Q_{Y|X}||\overline{P}_{Y|X}|P_X) \right]
\]

(A.11)

\[
= \sup_{\tau \geq 0} \min_{Q_{Y|X}} \sup_{\tau \geq 0} \left[ \tau \cdot D(Q_{Y|X}||P_{Y|X}|P_X) + D(Q_{Y|X}||\overline{P}_{Y|X}|P_X) \right]
\]

(A.12)

\[
= D(P_{Y|X}||\overline{P}_{Y|X}|P_X).
\]

(A.13)

---

**APPENDIX B**

**Proof of Theorem 6**

**A. Proof of the Achievability Part**

We begin by showing that an asymptotically good CD code must have a large subcode, such that the error probabilities of any of the subcodes of the latter are not much worse than the corresponding probabilities of the entire code.

**Lemma 11.** Let \( C_n \) be a CD code, and \( \phi_n \) be a detector. Then, there exists a CD code \( \tilde{C}_n \) with \( |\tilde{C}_n| \geq |C_n|/3 \) which
satisfies the following: For any subcode \( \mathcal{C}_n \subseteq \mathcal{C}_n \), there exists a of detector \( \bar{\phi}_n \) such that

\[
p_i(\mathcal{C}_n, \bar{\phi}_n) \leq 3 \cdot p_i(\mathcal{C}_n, \phi_n)
\]  

(B.1)

holds for both \( i = 1, 2 \).

\textbf{Proof:} Note that the error probabilities in (44) and (45) are averaged over the transmitted codeword \( X^n \in \mathcal{C}_n \). We first prove that by expurgating enough codewords from a codebook with good average error probabilities, a codebook with maximal (over the codewords) error probabilities can be obtained (for both types of error). The proof follows the standard expurgation argument from average error probability to maximal error probability (which in turn follows from Markov’s inequality). Denoting the conditional type 1 error probability by\(^{19}\)

\[
p_1(\mathcal{C}_n, \phi_n | X^n = x^n) \overset{\text{def}}{=} P \left[ \phi_n(Y^n) = \overline{H} | X^n = x^n \right],
\]

(B.2)

we may write

\[
p_1(\mathcal{C}_n, \phi_n) = \sum_{x^n \in \mathcal{C}_n} \mathbb{P}(X^n = x^n) \cdot p_1(\mathcal{C}_n, \phi_n | X^n = x^n)
\]

(B.3)

\[
= \frac{1}{|\mathcal{C}_n|} \sum_{x^n \in \mathcal{C}_n} p_1(\mathcal{C}_n, \phi_n | X^n = x^n).
\]

(B.4)

Thus, at least \( 2/3 \) of the codewords in \( x^n \in \mathcal{C}_n \) satisfy

\[
p_1(\mathcal{C}_n, \phi_n | X^n = x^n) \leq 3 \cdot p_1(\mathcal{C}_n, \phi_n).
\]

(B.5)

Using a similar notation for the conditional type 2 error probability, and repeating the same argument, we deduce that there exists \( \mathcal{C}_n \subseteq \mathcal{C}_n \) such that \( |\mathcal{C}_n| \geq |\mathcal{C}_n|/3 \) and both (B.5) as well as

\[
p_2(\mathcal{C}_n, \phi_n | X^n = x^n) \leq 3 \cdot p_2(\mathcal{C}_n, \phi_n),
\]

(B.6)

hold for any \( x^n \in \mathcal{C}_n \). Let us now consider any \( \mathcal{C}_n \subseteq \mathcal{C}_n \). For the code \( \mathcal{C}_n \), the detector \( \phi_n \) is possibly sub-optimal, and thus might be improved. Using the standard Neyman-Pearson lemma [1, Prop. II.D.1], one can find a detector \( \hat{\phi}_n \) (perhaps randomized) to match any prescribed type 1 error probability value, which is optimal in the sense that if any other detector \( \hat{\phi}_n \) satisfies \( p_1(\mathcal{C}_n, \hat{\phi}_n) \leq p_1(\mathcal{C}_n, \phi_n) \) then \( p_2(\mathcal{C}_n, \hat{\phi}_n) \geq p_2(\mathcal{C}_n, \phi_n) \). Specifically, let us require that

\[
p_1(\mathcal{C}_n, \bar{\phi}_n) = 3 \cdot p_1(\mathcal{C}_n, \phi_n),
\]

(B.7)

and choose \( \hat{\phi}_n = \phi_n \). Then, as (B.5) holds for any \( x^n \in \mathcal{C}_n \),

\[
p_1(\mathcal{C}_n, \phi_n) = \sum_{x^n \in \mathcal{C}_n} \mathbb{P}(X^n = x^n) \cdot p_1(\mathcal{C}_n, \phi_n | X^n = x^n)
\]

(B.8)

\(^{19}\)Note that conditioned on \( X^n = x^n \), \( p_1(\mathcal{C}_n, \phi_n | X^n = x^n) \) depends on the code \( \mathcal{C}_n \) only if the detector \( \phi_n \) depends on the code. In this lemma, the detector \( \phi_n \) is arbitrary, and the use of this notation is therefore just for the sake of consistency.
and as $\bar{\phi}_n$ is optimal in the Neyman-Pearson sense, (B.6) implies that

$$p_2(\bar{C}_n, \bar{\phi}_n) \leq p_2(\bar{C}_n, \phi_n)$$

(B.11)

$$= \sum_{x^n \in \bar{C}_n} \mathbb{P}(X^n = x^n) \cdot p_2(C_n, \phi_n|X^n = x^n)$$

(B.12)

$$\leq 3 \cdot p_2(C_n, \phi_n).$$

(B.13)

The result follows from (B.7) and (B.13). Note that $\bar{C}_n = \tilde{C}_n$ is a valid choice.

Next, we construct optimal DHT systems for a single type class of the source.

**Lemma 12.** Let $\delta > 0$ and $Q_X \in \mathcal{P}(\mathcal{X})$ be given, and let $\{n_i\}$ the subsequence of blocklengths such that $\mathcal{T}_n(Q_X)$ is not empty. Further, let $\mathcal{C}$ be a sequence of CD codes of type $Q_X$ and rate $\rho$, and $\{\phi_{n_i}\}_{i=1}^\infty$ be a sequence of detectors. Then, there exist a sequence of DHT systems $\mathcal{H}$ of rate $H(Q_X) - \rho$ such that

$$\liminf_{l \to \infty} -\frac{1}{n_l} \log p_i(\mathcal{H}_{n_l}|X^{n_l} \in \mathcal{T}_{n_l}(Q_X)) \geq \liminf_{l \to \infty} -\frac{1}{n_l} \log p_i(C_{n_l}, \phi_{n_l}) - \delta,$$

(B.14)

holds for both $i = 1, 2$.

**Proof:** We only need to focus on of $x^n \in \mathcal{T}_n(Q_X)$. For notational simplicity, let us assume that $n$ is always such that $\mathcal{T}_n(Q_X)$ is not empty. Let us first extract from $\mathcal{C}$ the sequence of CD codes $\tilde{\mathcal{C}}$ whose existence is assured by Lemma 11. The rate of $\tilde{\mathcal{C}}$ is chosen to be larger than $\rho - \delta$ (for all sufficiently large $n$), and for any given codeword, the error probability of each type is assured to be up to a factor of 3 of its average error probability.

Now, for a given permutation $\pi$ of $[n]$, we define the permutation of $x^n = (x_1, x_2, \ldots, x_n)$ as

$$\pi(x^n) \overset{\text{def}}{=} (x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}),$$

(B.15)

and for a set $\mathcal{D}_n \overset{\text{def}}{=} \{x^n(0), \ldots, x^n(|\mathcal{D}_n| - 1)\}$, we define

$$\pi(\mathcal{D}_n) \overset{\text{def}}{=} \{\pi(x^n(0)), \ldots, \pi(x^n(|\mathcal{D}_n| - 1))\}.$$

(B.16)

As $\tilde{\mathcal{C}}_n \in \mathcal{T}_n(Q_X)$, clearly so is $\pi(\tilde{\mathcal{C}}_n) \in \mathcal{T}_n(Q_X)$. Thus, there exists a set of permutations $\{\pi_{n,i}\}_{i=1}^{\kappa_n}$ such that

$$\bigcup_{i=1}^{\kappa_n} \pi_{n,i}(\tilde{\mathcal{C}}_n) = \mathcal{T}_n(Q_X).$$

(B.17)

By a simple counting argument, the minimal number of permutations required $\kappa_n$ is at least $|\mathcal{T}_n(Q_X)|/|\tilde{\mathcal{C}}_n|$. This is achieved when the permuted sets are pairwise disjoint, i.e., $\pi_{n,i}(\tilde{\mathcal{C}}_n) \cap \pi_{n,i'}(\tilde{\mathcal{C}}_n) = \phi$, for all $i \neq i'$. While this property is difficult to assure, Ahlswede’s covering lemma [41, Section 6, Covering Lemma 2] implies that up to the first order in the exponent, this minimal number can be achieved. To wit, there exists a set of permutations
\{\pi_{n,i}^*\}_{i=1}^{\kappa_n^*} \text{ such that }
\kappa_n^* \leq \frac{|\mathcal{T}_n(Q_X)|}{|\mathcal{C}_n|} \cdot e^{n}\delta \leq \frac{e^{n[H(Q_X)+\delta]}}{e^{n(\rho-\delta)}} \cdot e^{n}\delta = e^{n[H(Q_X)-\rho+3\delta]},
(B.18)

for all \( n \) sufficiently large. W.l.o.g., we assume that \( \pi_{n,1}^* \) is the identity permutation, and thus \( \mathcal{C}_{n,1}^* = \tilde{\mathcal{C}}_n \). Further, for \( 2 \leq i \leq \kappa_n^* \), let
\[ \mathcal{C}_{n,i}^* = \pi_{n,i}^*(\tilde{\mathcal{C}}_n) \setminus \left\{ \bigcup_{j=1}^{i-1} \pi_{n,j}^*(\tilde{\mathcal{C}}_n) \right\}. \]
(B.19)

In words, the code \( \mathcal{C}_{n,i}^* \) is the permutation \( \pi_{n,i}^* \) of the code \( \tilde{\mathcal{C}}_n \), excluding codewords which belong to a permutation of \( \tilde{\mathcal{C}}_n \) with smaller index. Thus, \( \{\mathcal{C}_{n,i}^*\}_{i=1}^{\kappa_n^*} \) forms a disjoint partition of \( \mathcal{T}_n(Q_X) \). Moreover, recall that from Lemma [II] for any given
\[ \overline{\mathcal{C}}_{n,i} = [\pi_{n,i}^*]^{-1}(\mathcal{C}_{n,i}^*) \subseteq \tilde{\mathcal{C}}_n \]
(B.20)

(where \( \pi^{-1} \) is the inverse permutation of \( \pi \)), one can find a detector \( \overline{\phi}_{n,i} \) such that
\[ p_1(\overline{\mathcal{C}}_{n,i}, \overline{\phi}_{n,i}) = 3 \cdot p_1(\mathcal{C}_n, \phi_n), \]
(B.21)

and
\[ p_2(\overline{\mathcal{C}}_{n,i}, \overline{\phi}_{n,i}) \leq 3 \cdot p_2(\mathcal{C}_n, \phi_n). \]
(B.22)

Now, let
\[ \phi_{n,i}^*(y^n) \overset{\text{def}}{=} \overline{\phi}_{n,i} \left[ [\pi_{n,i}^*]^{-1}(y^n) \right]. \]
(B.23)

Due to the memoryless nature of the hypotheses, the permutation does not change the probability distributions. Indeed, for an arbitrary CD code \( \mathcal{C}'_n \), a detector \( \phi'_n \) and a permutation \( \pi \),
\[ p_1(\mathcal{C}'_n, \phi'_n) = \sum_{x^n \in \mathcal{C}'_n} \mathbb{P}(X^n = x^n) \cdot p_1(x^n, \phi'_n) \]
(B.24)
\[ = \sum_{x^n \in \mathcal{C}'_n} \mathbb{P}(X^n = x^n) \cdot \sum_{y^n: \phi'_n(y^n) = \pi} P(Y^n = y^n | X^n = x^n) \]
(B.25)
\[ = \frac{1}{|\mathcal{C}'_n|} \sum_{x^n \in \mathcal{C}'_n} \sum_{y^n: \phi'_n(y^n) = \pi} P(Y^n = y^n | X^n = x^n) \]
(B.26)
\[ = \frac{1}{|\mathcal{C}'_n|} \sum_{x^n \in \mathcal{C}'_n} \sum_{y^n: \phi'_n(y^n) = \pi} P \left[ Y^n = \pi(y^n) | X^n = \pi(x^n) \right] \]
(B.27)
\[ = \frac{1}{|\mathcal{C}'_n|} \sum_{x^n \in \pi^{-1}(\mathcal{C}'_n)} \sum_{y^n: \phi'_n[y^n] = \pi^{-1}(y^n)} P(Y^n = y^n | X^n = x^n) \]
(B.28)
\[ = p_1(\pi^{-1}(\mathcal{C}'_n), \phi'_{n,\pi}). \]
(B.29)
where \( \phi'_{n,\pi}(y^n) \overset{\text{def}}{=} \phi'_n[\pi^{-1}(y^n)] \). Hence,

\[
p_1(\mathcal{C}_{n,i}, \overline{\phi}_{n,i}) = p_1(C^*_{n,i}, \phi^*_{n,i}),
\]

and

\[
p_2(\mathcal{C}_{n,i}, \overline{\phi}_{n,i}) = p_2(C^*_{n,i}, \phi^*_{n,i}).
\]

We thus construct \( \mathcal{H}_n = (f_n, \varphi_n) \) as follows. The codes \( \{C^*_{n,i}\}_{i=1}^{\kappa^*_n} \) will serve as the bins of \( f_n \), and detectors \( \{\phi^*_{n,i}\}_{i=1}^{\kappa^*_n} \) as the decision function, given that the bin index is \( i \). As said above, only \( x^n \in \mathcal{T}_n(Q_X) \) will be encoded. More rigorously, the encoding of \( x^n \in \mathcal{T}_n(Q_X) \) is given by \( f_n(x^n) = i \) whenever \( x^n \in C^*_{n,i} \), and by \( f_n(x^n) = 0 \) whenever \( x^n \not\in \mathcal{T}_n(Q_X) \). Clearly, the rate of the code is less than

\[
\frac{1}{n} \log \kappa^*_n \leq H(Q_X) - \rho + 3\delta,
\]  

for all \( n \) sufficiently large. The detector \( \varphi_n \) is given by \( \varphi_n(i, y^n) = \phi^*_{n,i}(y^n) \). The conditional type 1 error probability of this DHT system is given by

\[
\begin{align*}
P[\varphi_n(f_n(X^n), Y^n) = \overline{\mathcal{P}} | X^n \in \mathcal{T}_n(Q_X)] &= \sum_{i=1}^{\kappa^*_n} \mathbb{P}[f_n(X^n) = i] \cdot P[\varphi_n(f_n(X^n), Y^n) = \overline{\mathcal{P}} | f_n(X^n) = i] \\
&= \sum_{i=1}^{\kappa^*_n} \mathbb{P}[f_n(X^n) = i] \cdot \mathcal{P}[f_n(X^n) = i] \\
&= \sum_{i=1}^{\kappa^*_n} \mathbb{P}[f_n(X^n) = i] \cdot p_1(C^*_{n,i}, \phi^*_{n,i}) \\
& \overset{(a)}{=} \sum_{i=1}^{\kappa^*_n} \mathbb{P}[f_n(X^n) = i] \cdot p_1(C^*_{n,i}, \phi^*_{n,i}) \\
& \overset{(b)}{=} \sum_{i=1}^{\kappa^*_n} \mathbb{P}[f_n(X^n) = i] \cdot p_1(\mathcal{C}_{n,i}, \overline{\phi}_{n,i}) \\
& \overset{(c)}{=} 3 \cdot p_1(C_n, \phi_n),
\end{align*}
\]

where \((a)\) follows because given \( f_n(X^n) = i \) the source vector \( X^n \) is distributed uniformly over \( C^*_{n,i} \), \((b)\) follows from \((B.30)\), and \((c)\) follows from \((B.21)\). Similarly, the conditional type 2 error probability is upper bounded as

\[
\overline{\mathcal{P}}[\varphi_n(f_n(X^n), Y^n) = H | X^n \in \mathcal{T}_n(Q_X)] \leq 3 \cdot p_2(C_n, \phi_n).
\]  

The factor 3 in the error probabilities is negligible asymptotically.

The prove the achievability part of Theorem 6 the DHT system of Lemma 12 can be used to encode a single type class. It still remains, however, to assure convergence of the error probabilities to their exponential bounds, uniformly over all types. In other words, the DHT system of Lemma 12 is designed for a given \( Q_X \), and as such, the minimal blocklength for which the error probabilities are close to their exponential bounds may depend on \( Q_X \). However, as the blocklength \( n \) gets larger, source vectors types which did not exist at smaller blocklengths are also emitted by the source. For such type class, the blocklength required to achieve good error exponents may be much
larger. It may be that these types dominate the type 1 error probability, and thus prevent convergence. The method to solve this problem (see also [40], [46]) is to define a finite grid of types. CD codes are constructed only for types in the grid, and as the grid is finite, uniform convergence is assured (by taking the maximum required blocklength over all types within the grid). Now, if the type of \( x^n \) belongs to the grid, it can be encoded using the CD which pertains to its type. Otherwise, \( x^n \) is slightly modified to a different vector \( \tilde{X}^n \), where the type of the latter does belong to the grid. Then, \( \tilde{X}^n \) is encoded using the CD code which pertain to its type. Since the CD codes are designed for the pair \((X^n, Y^n)\), rather than for \((\tilde{X}^n, Y^n)\), the side-information vector \( Y^n \) is also modified to a vector \( \tilde{Y}^n \), using additional information sent from the encoder. To analyze the effect of this modification on the error probabilities, we will need the following partial mismatch lemma.

**Lemma 13.** Let \( C_n \) be a CD code, and \( \phi_n \) a detector. For \( d \overset{\text{def}}{=} \delta n \in [n] \), assume that \( \tilde{Y}^n = (\tilde{Y}_1^d, \tilde{Y}_{d+1}^n) \) is drawn as follows: Given \( x^n \in C_n \), under the hypothesis \( H \), \( \tilde{Y}^n = Y^n \sim P_{Y|X}(\cdot|x^n) \), and under the hypothesis \( \overline{H} \), \( \tilde{Y}_1^d \sim P_{Y|X}(\cdot|x_1^d) \) and \( \tilde{Y}_{d+1}^n \sim P_{Y|X}(\cdot|x_{d+1}^n) \). Then,

\[
\mathbb{P}\left[ \phi_n(\tilde{Y}^n) = H \right] = p_1(C_n, \phi_n), \tag{B.39}
\]

and

\[
\mathbb{P}\left[ \phi_n(\tilde{Y}^n) = \overline{H} \right] \leq e^{n\Omega} \cdot p_2(C_n, \phi_n) \tag{B.40}
\]

where

\[
\Omega \overset{\text{def}}{=} \max_{x \in X, y \in Y} \left| \log \frac{P_{Y|X}(y|x)}{P_{Y|X}(y|x)} \right| < \infty. \tag{B.41}
\]

**Proof:** Fix some blocklength \( n \). The statement regarding the type 1 error probability is trivial. As for the type 2 error probability, we will show that the “wrong” distribution of \( Y^n \) in the first \( d \) coordinates does not change likelihoods and probabilities significantly. Indeed, conditioning on \( Y_1^d = y_1^d \)

\[
\mathbb{P}\left[ \phi_n(\tilde{Y}^n) = H | Y_1^d = y_1^d \right] = \mathbb{P}\left[ \phi_n(Y^n) = H | Y_1^d = y_1^d \right]. \tag{B.42}
\]

Then, since,

\[
\mathbb{P}\left( Y_1^d = y_1^d \right) = \frac{1}{|C_n|} \sum_{x^n \in C_n} \mathbb{P}_{Y|X}(y_1^d|x_1^d) \leq e^{d\Omega} \cdot \frac{1}{|C_n|} \sum_{x^n \in C_n} P_{Y|X}(y_1^d|x_1^d) = e^{d\Omega} \cdot \mathbb{P}\left( Y_1^d = y_1^d \right), \tag{B.43}
\]

we obtain

\[
\mathbb{P}\left[ \phi_n(Y^n) = H \right] = \sum_{y_1^d \in Y_1^d} \mathbb{P}\left( Y_1^d = y_1^d \right) \cdot \mathbb{P}\left[ \phi_n(\tilde{Y}^n) = \overline{H} | Y_1^d = y_1^d \right]. \tag{B.46}
\]
\[
\leq \sum_{y_i^d \in Y^d} e^{d \Omega} \cdot P \left( Y_1^d = y_1^d \right) \cdot P \left[ \phi_n(Y^n) = \overline{H} Y_1^d = y_1^d \right] \\
= e^{d \Omega} \cdot p_2(C_n, \phi_n).
\]

We will also use the following simple lemma:

**Lemma 14.** Let \( Q_X, \tilde{Q}_X \in \mathcal{P}_n(\mathcal{X}) \) and assume that \( \|Q_X - \tilde{Q}_X\| = \frac{2d}{n} \) where \( d > 0 \). If \( x^n \in T_n(Q_X) \) then

\[
\min_{\tilde{x}^n \in T_n(Q_X)} d_H(\tilde{x}^n, x^n) \leq d.
\]

We are now ready to prove the achievability part. Since a similar method was applied in [40, Appendix A, proof of Th. 5][46, Sec. VI.C], we will shorten the details of some of the arguments.

**Proof of the achievability part of the Theorem**: The encoded message begins with a description of the type \( Q_X \) of \( X^n \). Then, to encode a given \( X^n \in T_n(Q_X) \), a DHT system which is based on good CD codes, as in Lemma [12] will be used.

Given \( \epsilon > 0 \) (to be specified later), we choose the grid as \( \mathcal{P}_{n_0}(\mathcal{X}) \), and denote

\[
\Phi_\epsilon(Q_X) \overset{\text{def}}{=} \arg\min_{\tilde{Q}_X \in \mathcal{P}_{n_0}(\mathcal{X})} \|Q_X - \tilde{Q}_X\|.
\]

The blocklength \( n_0 \) is chosen to be sufficiently large, such that \( \Phi_\epsilon(Q_X) \leq \frac{\epsilon}{2} \) for any \( Q_X \in \mathcal{P}(\mathcal{X}) \). Note that \( \Phi_\epsilon(Q_X) = Q_X \) whenever \( Q_X \in \mathcal{P}_{n_0}(\mathcal{X}) \). We shall only consider blocklength of the form \( n = kn_0 \) for some \( k \in \mathbb{N}_+ \). For \( n = kn_0 + k_0 \) with \( 1 \leq k_0 < n_0 \), the last \( k_0 \) symbols of \( (X^n, Y^n) \) can be simply ignored by both the encoder and detector. The loss in exponents is a factor \( \frac{n}{k_0n_0} \), which is negligible for sufficiently large \( k_0 \).

For \( n = kn_0, \mathcal{P}_{n_0}(\mathcal{X}) \subset \mathcal{P}_n(\mathcal{X}) \), and thus if \( Q_X \) is the type of the source vector \( x^n \), then either \( Q_X \in \mathcal{P}_{n_0}(\mathcal{X}) \) or \( Q_X \in \mathcal{P}_n(\mathcal{X}) \setminus \mathcal{P}_{n_0}(\mathcal{X}) \). In the former case, \( (X^n, Y^n) \) remains unchanged, and in the latter case, it is modified as follows. First, the encoder will randomly generate \( \tilde{X}^n \in T_n(\Phi_\epsilon(Q_X)) \) with a uniform distribution over the set

\[
\left\{ \tilde{x}^n \in T_n(\Phi_\epsilon(Q_X)) : d_H(x^n, \tilde{x}^n) = \frac{n \epsilon}{4} \right\}.
\]

Lemma [14] assures that this set is not empty. Further, due to the permutation symmetry of type classes, if \( X^n \) is distributed uniformly over \( T_n(Q_X) \) then \( \tilde{X}^n \) is distributed uniformly over \( T_n(\Phi_\epsilon(Q_X)) \). Now, as \( x^n \) is altered to \( \tilde{x}^n = \tilde{x}^n \), a similar change has to be made to the side-information vector. Indeed, given \( x^n \), the actual distribution of \( Y^n \) is either \( P_{Y|X}(\cdot|x^n) \) or \( \overline{P}_{Y|X}(\cdot|x^n) \) rather than either \( P_{Y|X}(\cdot|\tilde{x}^n) \) or \( \overline{P}_{Y|X}(\cdot|\tilde{x}^n) \). To account for this discrepancy, the encoder will send the location of the \( \frac{2n \epsilon}{d} \) coordinates in which \( x^n \) and \( \tilde{x}^n \) differ, and the values of \( \tilde{x}^n \) at those coordinates. Given this knowledge, the detector draws a new side-information vector \( \tilde{Y}^n \) in the following way. If \( x_i = \tilde{x}_i \) then \( \tilde{Y}_i = Y_i \), and if \( x_i \neq \tilde{x}_i \) then \( \tilde{Y}_i \sim P_{Y|X}(\cdot|\tilde{x}_i) \). Note that if the hypothesis \( P_{Y|X} \) is active then

20If the minimizer is not unique, one of the minimizers can be arbitrarily and consistently chosen.

21As \( P_{XY} \bowtie P_{XY} \) the exponents achieved by the DHT system are always finite.
\( \bar{Y}^n \sim P_{Y|X}(\cdot|\bar{x}^n) \) as required. If, however, the hypothesis \( \bar{P}_{Y|X} \) is active, then \( n - \lceil \frac{n}{n} \rceil \) coordinates of \( \bar{Y}^n \) are drawn “correctly” according to \( Y_i \sim \bar{P}_{Y|X}(\cdot|x_i) \), whereas \( \frac{n}{n} \) coordinates are drawn according to the “wrong” distribution \( \bar{Y}_i \sim P_{Y|X}(\cdot|x_i) \). As we will see, Lemma 13 assures that this mismatch has small effect on the type 1 error probability. At this point, the pair \((\bar{X}^n, \bar{Y}^n)\) can be encoded and detected using a DHT for the type \( \Phi_t(Q_X) \). For the sake of the brevity of next derivations, let us assume that \((\bar{X}^n, \bar{Y}^n)\) are fictitiously generated even when \( Q_X \in \mathcal{P}_{n_0}(\mathcal{X}) \). Thus, the encoded vector is always \( \bar{x}^n \), and the detector always uses the side-information vector \( \bar{y}^n \).

We next turn to discuss the encoding of the modified source vector. We construct DHT subsystems \( \mathcal{H}_{n,\hat{Q}_X} = (f_{n,\hat{Q}_X}, \varphi_{n,\hat{Q}_X}) \) for all \( \hat{Q}_X \in \mathcal{P}_{n_0}(\mathcal{X}) \), according to Lemma 12. Each DHT subsystem is constructed from a constituent CD code whose type 1 error exponent is tuned to meet a constraint of \( E_1 - D(Q_X||P_X) \). Now, as \( |\mathcal{P}_{n_0}(\mathcal{X})| \) is finite, uniform convergence over \( \mathcal{P}_{n_0}(\mathcal{X}) \) is assured, to wit, there exists subsystems \( \mathcal{H}_{n,\hat{Q}_X} \) and \( k \) sufficiently large such that

\[
\begin{align*}
p_1(f_{n,\hat{Q}_X}, \varphi_{n,\hat{Q}_X}) &\leq \exp \left[ -n \cdot \left( E_1 - D(Q_X||P_X) - \delta \right) \right], \\
p_2(f_{n,\hat{Q}_X}, \varphi_{n,\hat{Q}_X}) &\leq \exp \left\{ -n \cdot \left[ E_2^{-1}(R, \hat{Q}_X, E_1 - D(Q_X||P_X)) - \delta \right] \right\},
\end{align*}
\]

for all \( \hat{Q}_X \in \mathcal{P}_{n_0}(\mathcal{X}) \). A vector \( \hat{x}^n \) of type \( \hat{Q}_X \in \mathcal{P}_{n_0}(\mathcal{X}) \) is encoded using a DHT subsystem \( \mathcal{H}_{n,\hat{Q}_X} \). Let us denote the constructed system by \( \mathcal{H}_n = (f_n, \varphi_n) \) which includes both the randomly generated \((\hat{X}^n, \hat{Y}^n)\) and the subsystems \( \mathcal{H}_{n,\hat{Q}_X} \). It is required to analyze its asymptotic rate and both types of error exponent.

To analyze the rate, let us now summarize the different parts of the encoded message:

- An index of the type class within \( |\mathcal{P}_n(\mathcal{X})| \). As \( |\mathcal{P}_n(\mathcal{X})| \leq (n + 1)^{|\mathcal{X}|} \), it can be encoded in no more than \( \lceil |\mathcal{X}| \cdot \log(n + 1) \rceil \) nats.
- The location of the \( \frac{n}{n} \) coordinates in which \( \hat{x}^n \) (might) differ from \( x^n \). The number of sets of possible \( \frac{n}{n} \) coordinates satisfies

\[
\binom{n}{\frac{n}{n}} \leq e^{nh_b(\frac{x}{n})},
\]

and thus can be encoded in no more than \( nh_b(\frac{x}{n}) \) nats.
- The value of \( x^n \) in the possibly changed \( d \) coordinates. Each letter can be encoded using \( \lceil \log|\mathcal{X}| \rceil \) nats.
- The \( nR \) encoded nats of the DHT system for \( \hat{x}^n \).

Hence, the required rate is upper bounded by

\[
\frac{1}{n} \lceil |\mathcal{X}| \cdot \log(n + 1) \rceil + h_b \left( \frac{\epsilon}{4} \right) + \frac{\epsilon}{4} |\log|\mathcal{X}| | + R.
\]

By choosing \( \epsilon > 0 \) sufficiently small, the required rate can be made less than \( R + \delta \) for all \( n \) sufficiently large.
For the type 1 error exponent, note that for all $n = kn_0$ sufficiently large

$$p_1(H_n) = \sum_{Q_X \in \mathcal{P}_n(X)} \mathbb{P}[X^n \in \mathcal{T}_n(Q_X)] \cdot P[\varphi_n(Y^n) = \mathcal{H}|X^n \in \mathcal{T}_n(Q_X)]$$

(B.56)

$$(a) \leq e^{n\delta} \cdot \max_{Q_X \in \mathcal{P}_n(X)} e^{-nD(Q||P_X)} \cdot P[\varphi_n(Y^n) = \mathcal{H}|X^n \in \mathcal{T}_n(Q_X)]$$

(B.57)

$$(b) = e^{n\delta} \cdot \max_{Q_X \in \mathcal{P}_n(X)} e^{-nD(Q||P_X)} \cdot P\left[\varphi_n,\Phi_\epsilon(Q_X)(\hat{Y}^n) = \mathcal{H}\left|X^n \in \mathcal{T}_n(\Phi_\epsilon(Q_X))\right.\right]$$

(B.58)

$$(c) \leq e^{n\delta} \cdot \max_{Q_X \in \mathcal{P}_n(X)} e^{-nD(Q||P_X)} \cdot P[\varphi_n,\Phi_\epsilon(Q_X)(Y^n) = \mathcal{H}|X^n \in \mathcal{T}_n(\Phi_\epsilon(Q_X))]$$

(B.59)

$$(d) \leq \max_{Q_X \in \mathcal{P}_n(X)} \exp\left\{-n \cdot \min_{Q_X \in \mathcal{P}_n(X)} [D(Q||P_X) + E_1 - D(\Phi_\epsilon(Q_X)||P_X) - \delta]\right\}$$

(B.60)

$$(e) \leq \exp\left[-n \cdot (E_1 - 2\delta)\right],$$

(B.61)

where $(a)$ follows since $|\mathcal{P}_n(X)| \leq (n + 1)^{|X|} \leq e^{n\delta}$ and $\mathbb{P}[X^n \in \mathcal{T}_n(Q_X)] \leq \exp[-n \cdot D(Q||P_X)]$, $(b)$ follows from the definition of the system $\mathcal{H}_n$, $(c)$ follows from Lemma 13 (the trivial part regarding the type 1 error exponent), $(d)$ follows from (B.52), and $(e)$ follows from the fact that $D(Q||P_X)$ is a continuous function of $Q_X$ in $\mathcal{S}(X)$, and thus uniformly continuous.

To analyze the type 2 error exponent, first note that, as for the type 1 error probability,

$$p_2(H_n)$$

(B.62)

$$= \sum_{Q_X \in \mathcal{P}_n(X)} \mathbb{P}[X^n \in \mathcal{T}_n(Q_X)] \cdot \mathcal{P}[\varphi_n(Y^n) = H|X^n \in \mathcal{T}_n(Q_X)]$$

$$\leq \exp\left(-n \cdot \min_{Q_X \in \mathcal{P}_n(X)} \left\{D(Q||P_X) - \frac{1}{n} \log \mathcal{P}[\varphi_n,\Phi_\epsilon(Q_X)(\hat{Y}^n) = H|X^n \in \mathcal{T}_n(\Phi_\epsilon(Q_X))] - \delta\right\}\right).$$

(B.63)

Now, recalling that $n = kn_0$

$$\lim_{k \to \infty} -\frac{1}{kn_0} \log p_2(H_n)$$

$$= \lim_{k \to \infty} \min_{Q_X \in \mathcal{P}_n(X)} \left\{D(Q||P_X) - \frac{1}{n} \log \mathcal{P}[\varphi_n,\Phi_\epsilon(Q_X)(\hat{Y}^n) = H|X^n \in \mathcal{T}_n(\Phi_\epsilon(Q_X))] - \delta\right\}$$

(B.64)

$$(a) \geq \lim_{k \to \infty} \min_{Q_X \in \mathcal{P}_n(X)} \left\{D(Q||P_X) - \frac{1}{n} \log p_2[H_n,\Phi_\epsilon(Q_X),X^n \in \mathcal{T}_n(\Phi_\epsilon(Q_X))] - \delta - \frac{\epsilon\Omega}{4}\right\}$$

(B.65)

$$(b) \geq \lim_{k \to \infty} \min_{Q_X \in \mathcal{P}_n(X)} \left\{D(Q||P_X) + F_2^- [H(\Phi_\epsilon(Q_X)) - R,\Phi_\epsilon(Q_X),E_1 - D(Q||P_X)]\right\}$$

$$- \frac{\epsilon\Omega}{4} - 2\delta$$

(B.66)

$$(c) \geq \lim_{k \to \infty} \min_{Q_X \in \mathcal{P}_n(X)} \left\{D(\Phi_\epsilon(Q_X)||P_X) + F_2^- [H(\Phi_\epsilon(Q_X)) - R,\Phi_\epsilon(Q_X),E_1 - D(\Phi_\epsilon(Q_X)||P_X)] + \delta_1\right\}$$

$$- 2\delta - \frac{\epsilon\Omega}{4} - \delta_1$$

(B.67)

$$= \min_{Q_X \in \mathcal{P}_n_0} \left\{D(Q||P_X) + F_2^- [H(Q_X) - R,Q_X,E_1 - D(Q||P_X) + \delta_1] - 2\delta - \frac{\epsilon\Omega}{4} - \delta_1\right\}$$

(B.68)
Further, conditioned on the type class $Q$, vectors from the true type class, it can be assumed w.l.o.g. that each bin contains only sequences from a unique inequality implies the type of $x$ by $n$ as for all $n$ sufficiently small, the loss in exponents can be made arbitrarily small.

\[ \inf_{Q_X \in \mathcal{P}_n(X)} \left\{ D(Q_X \| P_X) + F_2^{-1} [H(Q_X) - R, Q_X, E_1 - D(Q_X \| P_X) + \delta_1] - 2\delta - \frac{\epsilon \Omega}{4} - \delta_1 \right\}, \tag{B.69} \]

where $(a)$ follows from Lemma 13 (and the way $\tilde{Y}^n$ was defined), $(b)$ follows from the uniform convergence of the type 1 error exponents of the DHT systems for $\Phi(Q_X) \in \mathcal{P}_m(X)$ [see (B.53)]. Passage $(c)$ holds for some $\delta_1 > 0$ that satisfies $\delta_1 \downarrow 0$ as $\epsilon \downarrow 0$, and follows from the fact that $D(Q_X \| P_X)$ is a continuous function of $Q_X$ in $\mathcal{S}(\mathcal{X})$, and thus uniformly continuous. Finally, by choosing $\epsilon > 0$ sufficiently small, and then $\delta > 0$ sufficiently small, the loss in exponents can be made arbitrarily small.

\[ \lim_{n \to \infty} \frac{\delta_n}{n} \leq \frac{\epsilon}{2} \mu_{Q_X} \frac{1}{\epsilon}, \]

where $\mu_{Q_X}$ is the average of $\frac{1}{\epsilon} \mu_{Q_X}$. As $n \to \infty$, $\frac{\delta_n}{n}$ converges to $\frac{\epsilon}{2} \mu_{Q_X}$, and thus

\[ \lim_{n \to \infty} \frac{\delta_n}{n} = \frac{\epsilon}{2} \mu_{Q_X} \frac{1}{\epsilon} = \frac{1}{\mu_{Q_X}}. \]

\[ \frac{1}{\mu_{Q_X}} \leq \mu_{Q_X} \frac{1}{\epsilon}, \]

as for all $n$ sufficiently large, $m_n \leq n^{n(R+\delta)}$, and thus clearly $m_{n,Q_X} \leq e^{n(R+\delta)}$. Hence, for any $\gamma > 1$, Markov’s inequality implies

\[ \mathbb{P} \left[ I_n : |f_n^{-1}(I_n)| \geq \frac{1}{\gamma \cdot \mu_{Q_X}} \left| X^n \in \mathcal{T}_n(Q_X) \right| \right] = \mathbb{P} \left[ I_n : |f_n^{-1}(I_n)|^{-1} \leq \gamma \cdot \mu_{Q_X} \left| X^n \in \mathcal{T}_n(Q_X) \right| \right] \geq 1 - \frac{1}{\gamma}, \tag{B.77} \]
Thus, using (B.75), conditioned on $X^n \in \mathcal{T}_n(Q_X)$

$$|f_n^{-1}(I_n)| \geq \frac{1}{\gamma \cdot \mu_{Q_X}} \geq \frac{1}{\gamma} \cdot e^{n[H(Q_X) - R - 2\delta]},$$  \hspace{1cm} (B.78)

with probability larger than $1 - \frac{1}{\gamma} > 0$. Now, assume by contradiction that the statement of the theorem does not hold. This implies that there exists an increasing subsequence of blocklengths $\{n_k\}_{k=1}^\infty$ and $\delta > 0$ such that

$$p_1(\mathcal{H}_{n_k}) \leq \exp \{-n_k \cdot [E_1 - \delta]\},$$  \hspace{1cm} (B.79)

and

$$p_2(\mathcal{H}_k) \leq \exp \{-n_k \cdot [E_2^+ (R - 3\delta, E_1 - 3\delta) + 5\delta]\},$$  \hspace{1cm} (B.80)

for all $k$ sufficiently large. For brevity of notation, we assume w.l.o.g. that these bounds hold for all $n$ sufficiently large, and thus omit the subscript $k$. Let $\delta > 0$ be given. Then, for all $n$ sufficiently large [which only depends on $(\delta, |X|)$],

$$p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] \leq \exp \{-n \cdot [E_1 - D(Q_X||P_X) - 2\delta]\},$$  \hspace{1cm} (B.81)

for all $Q_X$ such that $\mathcal{T}_n(Q_X)$ is not empty. Indeed, for all $n$ sufficiently large, it holds that

$$\exp [-n \cdot (E_1 - \delta)] \geq p_1(\mathcal{H}_n) \geq \sum_{Q_X \in \mathcal{P}_n(X)} \mathbb{P}[X^n \in \mathcal{T}_n(Q_X)] \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] \geq \sum_{Q_X \in \mathcal{P}_n(X)} e^{-n[D(Q_X||P_X)+\delta]} \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] \geq \max_{Q_X \in \mathcal{P}_n(X)} e^{-n[D(Q_X||P_X)+\delta]} \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)],$$  \hspace{1cm} (B.82) (B.83) (B.84) (B.85)

and (B.81) is obtained by rearranging. Writing

$$p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] = \sum_{i \in \mathcal{M}_{n,q_X}} \mathbb{P}[I_n = i|X^n \in \mathcal{T}_n(Q_X)] \cdot p_1(\mathcal{H}_n|I_n = i),$$  \hspace{1cm} (B.86)

Markov’s inequality implies

$$\mathbb{P}\left\{I_n : p_1(\mathcal{H}_n|I_n) \geq e^{n\delta} \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)]\right\} \leq \frac{\mathbb{E}[p_1(\mathcal{H}_n|I_n)]}{e^{n\delta} \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)]} = \sum_{i \in \mathcal{M}_{n,q_X}} \mathbb{P}[I_n = i|X^n \in \mathcal{T}_n(Q_X)] \cdot p_1(\mathcal{H}_n|I_n = i) \cdot e^{n\delta} \cdot p_1[\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] = e^{-n\delta},$$  \hspace{1cm} (B.87) (B.88) (B.89)

The same arguments can be applied for the type 2 exponent. Thus, from the above and (B.78), with probability
larger than \( 1 - \gamma^{-1} - 2 \cdot e^{-n\delta} \), which is strictly positive for all sufficiently large \( n \), the bin index satisfies (B.78),

\[
p_1 (\mathcal{H}_n|I_n) \leq e^{n\delta} \cdot p_1 [\mathcal{H}_n|X^n \in \mathcal{T}_n(Q_X)] \leq \exp \{ -n \cdot [E_1 - D(Q_X||P_X) - 3\delta] \},
\]

(B.90)

as well as

\[
p_2 (\mathcal{H}_n|I_n) \leq \exp \{ -n \cdot [E_2^+(R + 3\delta, E_1 - 3\delta) - D(Q_X||\mathcal{P}_X) + 3\delta] \}.
\]

(B.92)

Now, let \( Q^*_X \in \mathcal{P}(\mathcal{X}) \) be chosen to achieve \( E_2^+(R + 3\delta, E_1 - 3\delta) \) up to \( \delta \), i.e., to be chosen such that

\[
E_2^+(R + 3\delta, E_1 - 3\delta) - D(Q^*_X||\mathcal{P}_X) \geq F_2^+(H(Q^*_X) - R - 3\delta, Q^*_X, E_1 - D(Q^*_X||P_X) - 3\delta) - \delta,
\]

(B.93)

and let \( \{n_l\}_{l=1}^{\infty} \) be the subsequence of blocklengths such that \( \mathcal{T}_n(Q^*_X) \) is not empty. From the above discussion, there a sequence of bin indices \( \{i^*_n\}_{n=1}^{\infty} \) such that (B.78), (B.91) and (B.92) hold for \( Q^*_X \). Consider the sequence of bins \( C^*_n = f_{n_l}^{-1}(i^*_n) \) to be a sequence of CD codes, whose rate is larger than \( H(Q_X) - R - 3\delta \), its detectors are induced by the DHT system detector as \( \phi^*_n(y^{n_l}) = \varphi_{n_l}(i^*_n, y^{n_l}) \), and such that

\[
p_1 (C^*_n, \phi^*_n) \leq \exp \{ -n_l \cdot [E_1 - D(Q^*_X||P_X) - 3\delta] \},
\]

(B.94)

and

\[
p_2 (C^*_n, \phi^*_n) \leq \exp \{ -n_l \cdot [E_2^+(R + 3\delta, E_1 - 3\delta) - D(Q^*_X||\mathcal{P}_X) + 3\delta] \}
\leq \exp \{ -n_l \cdot [F_2^+(H(Q^*_X) - R - 3\delta, Q^*_X, E_1 - D(Q^*_X||P_X) - 3\delta) + 2\delta] \},
\]

(B.96)

where the last inequality follows from (B.93). However, this is a contradiction, since whenever (B.94) holds the definition of CD reliability function implies that

\[
p_2 (C^*_n, \phi^*_n) \geq \exp \{ -n_l \cdot [F_2^+(H(Q^*_X) - R - 3\delta, Q^*_X, E_1 - D(Q^*_X||P_X) - 3\delta) + \delta] \}
\]

(B.97)

for all \( l \) sufficiently large.

\[\text{\textbf{APPENDIX C}}\]

\[\text{\textbf{PROOF OF THEOREM}}\]

We will follow the outline which appears right after the statement of the theorem. We will consider CD codes drawn from the fixed-composition hierarchical ensemble defined in Definition with conditional distribution \( Q_{U|X} \), cloud-center rate \( \rho_c \), and satellite rate \( \rho_s \) (which satisfy \( \rho = \rho_c + \rho_s \)). In the course of the proof, we shall consider various types of the form \( Q_{UXY} \) and \( Q_{U,XY} \). All of them are assume to have \( (U, X) \) marginal \( Q_{UX} = Q_{UX} \) even if it is not explicitly stated. Furthermore, we shall assume that the blocklength \( n \) is such that \( \mathcal{T}_n(Q_{UX}) \) is not empty.
In this case, the notation for exponential equality (or inequality) needs to be clarified as follows. We will say that
\[ a_n \overset{\text{def}}{=} b_n \text{ if } \lim_{n \to \infty} \frac{1}{n_l} \log \frac{a_{n_l}}{b_{n_l}} = 1, \]  
(C.1)
where \( \{n_l\}_{l=1}^\infty \) is the subsequence of blocklengths such that \( T_n(Q_{UX}) \) is not empty.

The proof of Theorem 8 relies on the following result, which is stated and proved by means of the type-
enumeration method (see [43, Sec. 6.3]). Specifically, for a given \( y^n \), we define type-class enumerators for a random CD code \( \mathcal{C}_n \) by
\[
M_{y^n}(Q_{UXY}) \overset{\text{def}}{=} \left| \{ x^n \in \mathcal{C}_n : \exists u^n \text{ such that } x^n \in \mathcal{C}_{n,s}(u^n), (u^n, x^n, y^n) \in T_n(Q_{UXY}) \} \right|. \tag{C.2}
\]
To wit, \( M_{y^n}(Q_{UXY}) \) counts the random number of codewords which have joint type \( Q_{UXY} \in P_n(U \times X \times Y) \) with their cloud center \( u^n \) and \( y^n \). To derive a random-coding bound on the achievable CD exponents, we will need to evaluate the exponential order of \( \mathbb{E}[M_{y^n}^{1-\lambda}(Q_{UXY})M_{y^n}^{\lambda}(\overline{Q}_{UXY})] \) for an arbitrary sequence of \( \{y^n\} \) taken from \( T_n(Q_Y) = T_n(\overline{Q}_Y) \).

The result is summarized in the following proposition, which is interesting on its own right. To this end, it will be convenient to denote
\[
\Delta_\lambda(Q_{UXY}; \overline{Q}_{UXY}) \overset{\text{def}}{=} (1 - \lambda) \cdot [\rho - I_Q(U, X; Y)] - \lambda \cdot \max \left\{ |I_Q(U; Y) - \rho_c|_+, I_Q(U, X; Y) - \rho \right\} 
+ \lambda \left\{ \rho - I_{\overline{Q}}(U, X; Y) \right\} - (1 - \lambda) \cdot \max \left\{ |I_{\overline{Q}}(U; Y) - \rho_c|_+, I_{\overline{Q}}(U, X; Y) - \rho \right\}. \tag{C.3}
\]

**Proposition 15.** Let \( Q_{UXY}, \overline{Q}_{UXY} \in P_{n_0}(U \times X \times Y) \) be given for some \( n_0 \), with \( Q_{UX} = \overline{Q}_{UX} \) and \( Q_Y = \overline{Q}_Y \). Also let \( \{n_l\} \) be the subsequence of blocklengths such that \( T_n(Q_{UXY}) \) and \( T_n(\overline{Q}_{UXY}) \) are both non-empty, and let \( \{y^{n_l}\}_{l=1}^\infty \) satisfy \( y^{n_l} \in T_n(Q_Y) \) for all \( l \). Then, for any \( \lambda \in (0, 1) \)
\[
\lim_{n_l \to \infty} \frac{1}{n_l} \log \mathbb{E} \left[ M_{y^{n_l}}^{1-\lambda}(Q_{UXY})M_{y^{n_l}}^{\lambda}(\overline{Q}_{UXY}) \right] = \begin{cases} 
\rho - I_Q(U, X; Y), & Q_{UXY} = \overline{Q}_{UXY} \\
\Delta_\lambda(Q_{UXY}; \overline{Q}_{UXY}), & Q_{UY} \neq \overline{Q}_{UY} \text{ and } Q_{UXY} \neq \overline{Q}_{UXY} \\
\Delta_\lambda(Q_{UXY}; \overline{Q}_{UXY}) - |I_Q(U; Y) - \rho_c|_+, & Q_{UY} = \overline{Q}_{UY} \text{ (and } Q_{UXY} \neq \overline{Q}_{UXY})
\end{cases}. \tag{C.4}
\]

It is interesting to note that the expression (C.4) is not continuous, as, say, \( \overline{Q}_{UXY} \to Q_{UXY} \). The proof of Proposition 15 is of technical nature, and thus relegated to Appendix D. For the rest of the proof, no knowledge of the type enumeration method is required.

Equipped with this result we proceed to analyze typical properties of randomly generated codes. As the CD problem corresponds to an HT problem between \( P_{Y^n}^{(C_n)} \) and \( P_Y^{(C_n)} \), and the performance of the latter is determined by the Chernoff distance [see (1)], we next analyze the average exponent of the Chernoff distance over a random choice of CD codes.
Lemma 16. Let $\mathcal{C}_n$ be drawn randomly from the hierarchical ensemble of Definition 7 with conditional distribution $Q_{U|X}$, cloud-center rate $\rho_c$, and satellite rate $\rho_s$ (which satisfy $\rho = \rho_c + \rho_s$). Then,

$$
\mathbb{E} \left\{ \sum_{y^n \in \mathcal{Y}^n} \left[ P_{Y^n}(\mathcal{C}_n)(y^n) \right]^{-1-\lambda} \cdot \left[ \overline{P}_{Y^n}(\mathcal{C}_n)(y^n) \right]^\lambda \right\} \leq \exp \left[ -n \cdot \min \{ d_\lambda(Q_X), A_{rc} \} \right].
$$

(C.5)

where $d_\lambda(Q_X)$ is defined in (7) when setting $\tau = \frac{1-\lambda}{\lambda}$, and $A_{rc}$ is defined in (33).

Proof: For $(x^n, y^n) \in \mathcal{T}_n(Q_{XY})$, let us denote the log-likelihood of $P_{Y|X}$ by

$$
L(Q_{XY}) \overset{\text{def}}{=} -\frac{1}{n} \log P_{Y|X}(y^n|x^n)
$$

(C.6)

and the log-likelihood of $\overline{P}_{Y|X}$ by $\overline{L}(Q_{XY})$ (with $\overline{P}_{Y|X}$ replacing $P_{Y|X}$). For any given $n$,

$$
\mathbb{E} \left\{ \sum_{y^n \in \mathcal{Y}^n} \left[ \sum_{x^n \in \mathcal{X}_n} P_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^{-1-\lambda} \cdot \left[ \sum_{x^n \in \mathcal{X}_n} \overline{P}_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^\lambda \right\}
$$

(C.7)

$$
= \frac{1}{e^{\lambda \rho}} \sum_{y^n \in \mathcal{Y}^n} \mathbb{E} \left\{ \left[ \sum_{x^n \in \mathcal{X}_n} P_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^{-1-\lambda} \cdot \left[ \sum_{x^n \in \mathcal{X}_n} \overline{P}_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^\lambda \right\}
$$

(C.8)

$$
= \frac{1}{e^{\lambda \rho}} \sum_{Q_Y \in \mathcal{P}_n(\mathcal{Y})} \sum_{y^n \in \mathcal{T}_n(Q_Y)} \mathbb{E} \left\{ \left[ \sum_{x^n \in \mathcal{X}_n} P_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^{-1-\lambda} \cdot \left[ \sum_{x^n \in \mathcal{X}_n} \overline{P}_{Y^n}(\mathcal{C}_n)(y^n|x^n) \right]^\lambda \right\}
$$

(C.9)

$$
\leq |\mathcal{P}_n(\mathcal{Y})| \cdot \max_{Q_{UXY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} e^{n[H(Q_{XY}) - \rho]} \cdot \mathbb{E} \left\{ \left[ \sum_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n\overline{L}(Q_{XY})} \right]^{-1-\lambda} \cdot \left[ \sum_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n\overline{L}(Q_{XY})} \right]^\lambda \right\}
$$

(C.10)

$$
\leq |\mathcal{P}_n(\mathcal{Y})| \cdot |\mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})|^2
$$

$$
\times \max_{Q_{UXY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} e^{n[H(Q_{XY}) - \rho]} \cdot \mathbb{E} \left\{ \max_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n(1-\lambda)\overline{L}(Q_{XY})} \cdot \max_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n\lambda\overline{L}(Q_{XY})} \right\}
$$

(C.12)

$$
\leq |\mathcal{P}_n(\mathcal{Y})| \cdot |\mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})|^2
$$

$$
\times \max_{Q_{UXY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} e^{n[H(Q_{XY}) - \rho]} \cdot \mathbb{E} \left\{ \sum_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n(1-\lambda)\overline{L}(Q_{XY})} \cdot \sum_{Q_{UXY}} M_{y^n}(Q_{UXY}) e^{-n\lambda\overline{L}(Q_{XY})} \right\}
$$

(C.13)

$$
\leq |\mathcal{P}_n(\mathcal{Y})| \cdot |\mathcal{P}_n(\mathcal{U} \times \mathcal{X} \times \mathcal{Y})|^2
$$

$$
\times \max_{Q_{UXY} \in \mathcal{P}_n(\mathcal{X} \times \mathcal{Y})} e^{n[H(Q_{XY}) - \rho]} \sum_{Q_{UXY}} \sum_{Q_{UXY}} \mathbb{E} \left[ M_{y^n}(Q_{UXY}) \cdot M_{y^n}(Q_{UXY}) \cdot e^{-n(1-\lambda)\overline{L}(Q_{XY}) + n\lambda\overline{L}(Q_{XY})} \right]
$$

(C.14)
where (a) follows since by symmetry, the expectation only depends on the type of \( y^n \), and by using the definitions of the enumerators in (C.2), and the log-likelihood in (C.6). After passage (a) and onward, \( y^n \) is an arbitrary member of \( \mathcal{T}_n(Q_Y) \), and the sums and maximization operators are over \((Q_{UXY}, \overline{Q}_{UXY})\) restricted to the set

\[
Q_n \overset{\text{def}}{=} \{(Q_{UXY}, \overline{Q}_{UXY}) \in \mathcal{P}_2(U \times X \times Y) : Q_Y = \overline{Q}_Y, Q_{XU} = \overline{Q}_{XU}\}.
\]

(C.17)

In the last equality we have implicitly defined \( c_n \) and \( \zeta_n(Q_{UXY}, \overline{Q}_{UXY}) \). By defining

\[
\overline{Q} \overset{\text{def}}{=} \{(Q_{UXY}, \overline{Q}_{UXY}) \in S^2(U \times X \times Y) : Q_Y = \overline{Q}_Y, Q_{XU} = \overline{Q}_{XU}\},
\]

(C.18)

and using standard arguments (e.g., as in the proof of Sanov’s theorem [3, Theorem 11.4.1]) we get

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left\{ \sum_{y^n \in \mathcal{Y}^n} \left[ \mathcal{P}_n(y^n) \right]^{1-\lambda} \cdot \left[ \overline{\mathcal{P}}_n(y^n) \right]^\lambda \right\} \geq \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q}} \liminf_{n \to \infty} -\frac{1}{n} \log \zeta_n(Q_{UXY}, \overline{Q}_{UXY}).
\]

(C.19)

Hence, to obtain the exponential upper bound required (C.5), we may minimize over \((Q_{UXY}, \overline{Q}_{UXY})\) the exponential order of \( \zeta_n(\cdot) \), denoted as

\[
\Lambda_\lambda(Q_{UXY}, \overline{Q}_{UXY}) \overset{\text{def}}{=} \liminf_{n \to \infty} -\frac{1}{n} \log \zeta_n(Q_{UXY}, \overline{Q}_{UXY})
\]

(C.20)

\[
= -H(Q_Y) + \rho - \liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) M_{y^n}^\lambda(\overline{Q}_{UXY}) \right]
\]

\[
+ \left[ (1 - \lambda) \cdot L(Q_{XY}) + \lambda \cdot \overline{L}(Q_{XY}) \right].
\]

(C.21)

We now evaluate this expression in three cases, which correspond to the three cases of Proposition 15. In each one, we substitute for \( \liminf_{n \to \infty} -\frac{1}{n} \log \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) M_{y^n}^\lambda(\overline{Q}_{UXY}) \right] \) the appropriate result as follows:

**Case 1.** For \( Q_{UXY} = \overline{Q}_{UXY} \),

\[
\Lambda_\lambda(Q_{UXY}, \overline{Q}_{UXY})
\]

\[
= -H(Q_Y) + \rho - [\rho - I_Q(U, X; Y)] + (1 - \lambda) \cdot L(Q_{XY}) + \lambda \cdot \overline{L}(Q_{XY})
\]

\[
= (1 - \lambda) \cdot [Q_Y(X, U) + L(Q_{XY})] + \lambda \cdot [-H_Q(Y|X, U) + \overline{L}(Q_{XY})]
\]

\[
\overset{(a)}{=} (1 - \lambda) \cdot D(Q_{Y|UX}||P_{Y|X}Q_{UX}) + \lambda \cdot D(Q_{Y|UX}||\overline{P}_{Y|X}\overline{Q}_{UX}),
\]

(C.22)

(C.23)

(C.24)

\[22\] In fact, Proposition 15 implies that the limit inferior of this sequence is a proper limit.
2. For Case it is evident that follows from (C.25) again and rearranging.

By defining the distribution

\[
P_Y^{(\lambda,x)}(y) \overset{\text{def}}{=} \frac{P_{Y|X}^{1-\lambda}(y|x)P_Y^{\lambda}(y|x)}{\sum_{y' \in \mathcal{Y}} P_{Y|X}^{1-\lambda}(y'|x)P_Y^{\lambda}(y'|x)}
\]

and observing that

\[
\begin{align*}
&\min_{Q_{Y|U:X}} [(1 - \lambda) \cdot D(Q_{Y|U:X}||P_Y|X|Q_{U:X}) + \lambda \cdot D(Q_{Y|U:X}||P_Y|X|Q_{U:X})] \\
&\quad = \min_{Q_{Y|X}} \left\{ D(Q_Y|X||P_{Y|X}^{(\lambda,x)}|Q_X) - \sum_{x \in \mathcal{X}} Q_X(x) \log \left[ \sum_{y' \in \mathcal{Y}} P_{Y|X}^{1-\lambda}(y'|x)P_Y^{\lambda}(y'|x) \right] \right\} \\
&\quad = -\sum_{x \in \mathcal{X}} Q_X(x) \log \left[ \sum_{y \in \mathcal{Y}} P_{Y|X}^{1-\lambda}(y|x)P_Y^{\lambda}(y|x) \right],
\end{align*}
\]

it is evident that

\[
\Lambda_{\lambda,1}(Q_{U:XY}, \overline{Q}_{U:XY}) = \min_{Q_{Y|X}} [(1 - \lambda) \cdot D(Q_{Y|U:X}||P_Y|X|Q_{U:X}) + \lambda \cdot D(Q_{Y|U:X}||P_Y|X|Q_{U:X})] \\
\quad = -\sum_{x \in \mathcal{X}} Q_X(x) \log \left[ \sum_{y \in \mathcal{Y}} P_{Y|X}^{1-\lambda}(y|x)P_Y^{\lambda}(y|x) \right] \\
\quad = d_\lambda(Q_X).
\]

Case 2. For \(Q_{U:XY} \neq \overline{Q}_{U:XY}\) and \(Q_{U:Y} \neq \overline{Q}_{U:Y}\)

\[
\Lambda_{\lambda}(Q_{U:XY}, \overline{Q}_{U:XY}) \\
\quad = -H(Q_Y) + \rho + (1 - \lambda) \cdot L(Q_{XY}) + \lambda \cdot L(\overline{Q}_{XY}) + (1 - \lambda) \cdot [\rho - I_Q(U;X;Y)] + \lambda \cdot \max \left\{ \left| I_Q(U;Y) - \rho_c \right|_+, (1 - \lambda) \cdot [\rho - I_{\overline{Q}}(U;X;Y)] \right\} \\
\quad - \lambda \left[ \rho - I_{\overline{Q}}(U;X;Y) \right] + \rho + (1 - \lambda) \cdot \max \left\{ \left| I_{\overline{Q}}(U;Y) - \rho_c \right|_+, (1 - \lambda) \cdot [I_{\overline{Q}}(U;X;Y) - \rho] \right\}
\]

\[
\overset{(a)}{=} (1 - \lambda) \cdot D(Q_{Y|U:X}||P_Y|X|Q_{U:X}) + \lambda \cdot D(\overline{Q}_{Y|U:X}||P_Y|X|\overline{Q}_{U:X}) + \rho + (1 - \lambda) \cdot \max \left\{ \left| I_Q(U;Y) - \rho_c \right|_+, (1 - \lambda) \cdot [I_Q(U;X;Y) - \rho] \right\}
\]

\[
\overset{\text{def}}{=} \Lambda_{\lambda,2}(Q_{U:XY}, \overline{Q}_{U:XY}),
\]

where \((a)\) follows from (C.25) again and rearranging.
Case 3. For \( Q_{UXY} \neq \overline{Q}_{UXY} \) and \( Q_{UY} = \overline{Q}_{UY} \)

\[
\Lambda_\lambda(Q_{UXY}, \overline{Q}_{UXY}) \\
= \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}) - |I_Q(U;Y) - \rho_c| + \\
\overset{\text{def}}{=} \Lambda_{\lambda,3}(Q_{UXY}, \overline{Q}_{UXY}).
\] (C.36)

Hence, the required bound on the Chernoff distance is given by

\[
\min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q}} \Lambda_\lambda(Q_{UXY}, \overline{Q}_{UXY}) \\
= \min \left\{ \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q}} \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}), \right. \\
\left. \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UY}=\overline{Q}_{UY}} \Lambda_{\lambda,3}(Q_{UXY}, \overline{Q}_{UXY}) \right\}. 
\] (C.38)

Observing (C.37), we note that the third term in (C.38) satisfies

\[
\min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UXY}=\overline{Q}_{UXY}} \Lambda_{\lambda,3}(Q_{UXY}, \overline{Q}_{UXY}) \\
= \min \left\{ \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UXY}=\overline{Q}_{UXY}, I_Q(U;Y) \leq \rho_c} \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}), \right. \\
\left. \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UXY}=\overline{Q}_{UXY}, I_Q(U;Y) > \rho_c} \{ \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}) - I_Q(U;Y) + \rho_c \} \right\} 
\] (C.39)

\[
\overset{(a)}{=} \min \left\{ \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UXY}=\overline{Q}_{UXY}, I_Q(U;Y) > \rho_c} \{ \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}) - I_Q(U;Y) + \rho_c \} \right\},
\] (C.40)

where in (a) we have removed the constraint \( I_Q(U;Y) \leq \rho_c \) in the first term of the outer minimization, since for \( Q_{UY} \) with \( I_Q(U;Y) > \rho_c \) the second term will dominate the minimization. Thus, the first term in (C.40) may be unified with the second term of (C.38). Doing so, the constraint \( Q_{UXY} \neq \overline{Q}_{UXY} \) may be removed in the second term of (C.38). Consequently, the required bound on the Chernoff distance is given by

\[
\min \left\{ d_\lambda(Q_X), \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q}} \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}), \right. \\
\left. \min_{(Q_{UXY}, \overline{Q}_{UXY}) \in \overline{Q} Q_{UXY}=\overline{Q}_{UXY}, I_Q(U;Y) > \rho_c} \{ \Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}) - I_Q(U;Y) + \rho_c \} \right\}. 
\] (C.41)

The second term in (C.41) corresponds to \( A_{rc}' \) defined in (51). The third term corresponds to \( A_{rc}'' \) defined in (52), when using \( \rho_b = \rho - \rho_c \), and noting that when \( I_Q(U;Y) > \rho_c \), it is equal to

\[
\Lambda_{\lambda,2}(Q_{UXY}, \overline{Q}_{UXY}) - I_Q(U;Y) + \rho_c
\]
\[= (1 - \lambda) \cdot D(Q_Y|UX) \parallel P_Y|X|QUX) + \lambda \cdot D(Q^\lambda_Y|UX) \parallel Q_Y^\lambda|UX)\]
\[+ \lambda \cdot |I_Q(X; Y|U) - \rho_s|_+ + (1 - \lambda) \cdot |I_Q^\lambda(X; Y|U) - \rho_s|_+. \quad \text{(C.42)}\]

Using the definition of \(A_{rc}\) as the minimum of the last two cases, (C.5) is obtained.

Next, recall that by its definition, a valid CD code of rate \(\rho\) is comprised of at least \(e^{n\rho}\) distinct codewords. However, when the codewords are independently drawn, some of them might be identical. Nonetheless, the next lemma shows that the average number of distinct codewords is asymptotically close to \(e^{n\rho}\).

**Lemma 17.** Let \(C_n\) be drawn randomly from the hierarchical ensemble of Definition 7 with conditional distribution \(Q_{UX}\), cloud-center rate \(\rho_c\), and satellite rate \(\rho_s\) (which satisfy \(\rho = \rho_c + \rho_s\)). If \(\rho_c + H_Q(X|U) \geq \rho\) then

\[E[|C_n|] \geq e^{n\rho}. \quad \text{(C.43)}\]

**Proof:** Let us enumerate the random cloud centers as \(\{U^n(i)\}_{i=1}^{e^{n\rho_c}}\) and the random satellite codebooks as \(\{C_{n,s}(U^n(i))\}_{i=1}^{e^{n\rho_c}}\). When a random satellite codebook \(C_{n,s}(u^n)\) is drawn, then for any given \((u^n, x^n) \in \mathcal{T}_n(Q_{UX})\)

\[P[x^n \in C_{n,s}(u^n)] = 1 - \left(1 - \frac{1}{|\mathcal{T}_n(Q_{UX}, u^n)|}\right)e^{n\rho_s}
\geq \frac{1}{2} \cdot \min\left\{1, \frac{e^{n\rho_s}}{|\mathcal{T}_n(Q_{UX}, u^n)|}\right\}
\geq \exp\left(-n \cdot \{|H_Q(X|U) - \rho_s|_+ + \delta\}\right), \quad \text{(C.46)}\]

using \(1 - (1 - t)^K \geq \frac{1}{2} \cdot \min\{1, tK\}\) [Lemma 1]. Further, for a random cloud center \(U^n\) and \(x^n \in \mathcal{T}_n(Q_X)\)

\[P[x^n \in C_{n,s}(U^n)] = P[(x^n, U^n) \in \mathcal{T}_n(Q_{UX})] \cdot P[x^n \in C_{n,s}(U^n) | (x^n, U^n) \in \mathcal{T}_n(Q_{UX})] \]
\[\overset{(a)}{=} P[(x^n, U^n) \in \mathcal{T}_n(Q_{UX})] \cdot P[x^n \in C_{n,s}(U^n)] \]
\[\geq \frac{|\mathcal{T}_n(Q_{UX}, x^n)|}{|\mathcal{T}_n(Q_X)|} \cdot \exp\left(-n \cdot \{|H_Q(X|U) - \rho_s|_+ + \delta\}\right) \]
\[\geq \exp\left(-n \cdot \{I_Q(U; X) + |H_Q(X|U) - \rho_s|_+ + 2\delta\}\right) \]
\[\overset{\text{def}}{=} e^{-n(\xi + 2\delta)}, \quad \text{(C.51)}\]

where (a) is due to symmetry, and the definition

\[\xi \overset{\text{def}}{=} I_Q(U; X) + |H_Q(U|X) - \rho_s|_+ \]
\[\overset{(C.53)}{=} \max \{I_Q(U; X), H(Q_X) - \rho_s\}. \quad \text{(C.52)}\]

Therefore, the average number of distinct codewords in the random CD code \(C_n\) is lower bounded as

\[E[|C_n|] = E\left[\sum_{x^n \in \mathcal{T}_n(Q_X)} I(x^n \in C_n)\right] \quad \text{(C.54)}\]
where \((a)\) holds since for a given set of \(K\) pairwise independent events \(\{A_k\}_{k=1}^{K}\) \[61\] Lemma A.2

\[
\Pr \left[ \bigcup_{k=1}^{K} A_k \right] \geq \frac{1}{2} \min \left\{ 1, \sum_{k=1}^{K} \Pr(A_k) \right\}. \tag{C.61}
\]

The passage \((b)\) follows from the assumptions \(\rho < H(Q_X)\) and \(\rho_c + H_Q(X|U) \geq \rho\). Thus, on the average, a randomly chosen \(C_n\) has more than \(e^{n(\rho - 3\delta)}\) distinct codewords.

We are now ready to prove Theorem 8.

Proof of Theorem 8: As noted in Section IV, given a CD code \(C_n\), the detector faces an ordinary HT problem between the distributions \(P^{(C_n)}_{Y^n}\) and \(\overline{P}^{(C_n)}_{Y^n}\), and thus the bounds of Section III-B can be used. Specifically, using (11) (with \(\tau = \frac{1-\lambda}{\lambda}\)), the type 2 error exponent under a type 1 error exponent constraint of \(F_1\), can be lower bounded for any given \(n\) by

\[
\max_{C_n \subseteq T_n(Q_X); |C_n| \geq e^{n\rho}} \max_{0 \leq \lambda \leq 1} \left\{ -\frac{1}{\lambda} \cdot F_1 - \frac{1}{\lambda} \cdot \frac{1}{n} \log \left( \sum_{y^n \in Y^n} \left[ P^{(C_n)}_{Y^n}(y^n) \right]^{1-\lambda} \cdot \left[ \overline{P}^{(C_n)}_{Y^n}(y^n) \right]^{\lambda} \right) \right\}. \tag{C.62}
\]

Instead of maximizing over \(C_n\), we use ensemble averages. To this end, note that up to this point, we have obtained (Lemma 16)

\[
\mathbb{E} \left\{ \sum_{y^n \in Y^n} \left[ P^{(C_n)}_{Y^n}(y^n) \right]^{1-\lambda} \cdot \left[ \overline{P}^{(C_n)}_{Y^n}(y^n) \right]^{\lambda} \right\} = \exp \left[ -n \cdot \min \{ d_{\lambda}(Q_X), A_{rc} \} \right], \tag{C.63}
\]

and showed that the average number of distinct codewords in a randomly chosen codebook is \(|C_n| \geq e^{n\rho}\) (Lemma 17). It remains to prove the existence of a CD code, whose codewords are all distinct, and its Chernoff distance exponent is close to the ensemble average. To this end, let \(\delta > 0\) be given, and consider the events

\[
A_1 \overset{\text{def}}{=} \left\{ |C_n| \geq \frac{1}{2} \mathbb{E}[|C_n|] \right\}, \tag{C.64}
\]

and

\[
A_2(\lambda) \overset{\text{def}}{=} \left\{ \sum_{y^n \in Y^n} \left[ P^{(C_n)}_{Y^n}(y^n) \right]^{1-\lambda} \cdot \left[ \overline{P}^{(C_n)}_{Y^n}(y^n) \right]^{\lambda} \leq e^{4n\delta} \cdot \exp \left[ -n \cdot \min \left\{ d_{\lambda}(Q_X), \tilde{A}_{rc} \right\} \right] \right\}. \tag{C.65}
\]
Note that since $\mathbb{P}(\{|C_n| \leq e^{n\delta} \cdot \mathbb{E}[|C_n|]\}) = \mathbb{P}(\{|C_n| \leq e^{n\rho}\}) = 1$, for all $n$ sufficiently large, the reverse Markov inequality\textsuperscript{23} Section 9.3, p. 159] implies that

$$\mathbb{P}[A_1] \geq \frac{1 - 1/2}{e^{n\delta} - 1/2} \geq e^{-2n\delta}.$$  \hfill (C.66)

Further, for any given $\lambda \in [0, 1]$, Markov's inequality implies that for all $n$ sufficiently large

$$\mathbb{P}[A_2(\lambda)] \geq 1 - e^{-n^3\delta}. \hfill (C.67)$$

Then, we note that

$$\mathbb{P}[A_1 \cap A_2(\lambda)] \geq 1 - \mathbb{P}[A_1^c] - \mathbb{P}[A_2^c(\lambda)] \geq 1 - e^{-3n\delta} - \left(1 - e^{-2n\delta}\right)$$

$$\geq e^{-2n\delta} - e^{-3n\delta} \hfill (C.69)$$

$$> 0, \hfill (C.70)$$

and thus deduce that there exists a CD code $C_n^*(\lambda)$ such that $|C_n^*(\lambda)| \geq \frac{1}{4} e^{n(\rho - \delta)} \geq e^{n(\rho - 2\delta)}$ and $A_2(\lambda)$ holds for all $n$ sufficiently large. Let the CD code obtained after keeping only the unique codewords of $C_n^*(\lambda)$ be denoted as $C_n^{**}(\lambda)$. It remains to show that the exponent of the Chernoff distance of $C_n^{**}(\lambda)$ is asymptotically equal to that of $C_n^*(\lambda)$. Indeed,

$$\sum_{y^n \in Y^n} \left( P_{Y^n}^{(C_n^*(\lambda))}(y^n) \right)^{1-\lambda} \cdot \left[ \overline{P}_{Y^n}^{(C_n^*(\lambda))}(y^n) \right]^\lambda$$

$$= \sum_{y^n \in Y^n} \left[ \sum_{x^n \in C_n^*(\lambda)} \frac{1}{e^{n\rho} P_{Y^n}^{(C_n^*(\lambda))}(x^n)} \right]^{1-\lambda} \cdot \left[ \sum_{\bar{x}^n \in C_n^*(\lambda)} \frac{1}{e^{n\rho} \overline{P}_{Y^n}^{(C_n^*(\lambda))}(\bar{x}^n)} \right]^\lambda$$

$$\geq (a) \sum_{y^n \in Y^n} \left[ \sum_{x^n \in C_n^{**}(\lambda)} \frac{1}{e^{n\rho} P_{Y^n}^{(C_n^{**}(\lambda))}(x^n)} \right]^{1-\lambda} \cdot \left[ \sum_{\bar{x}^n \in C_n^{**}(\lambda)} \frac{1}{e^{n\rho} \overline{P}_{Y^n}^{(C_n^{**}(\lambda))}(\bar{x}^n)} \right]^\lambda$$

$$\geq e^{-2n\delta} \cdot \sum_{y^n \in Y^n} \left[ \sum_{x^n \in C_n^{**}(\lambda)} \frac{1}{|C_n^{**}(\lambda)|} P_{Y^n}^{(C_n^{**}(\lambda))}(x^n) \right]^{1-\lambda} \cdot \left[ \sum_{\bar{x}^n \in C_n^{**}(\lambda)} \frac{1}{|C_n^{**}(\lambda)|} \overline{P}_{Y^n}^{(C_n^{**}(\lambda))}(\bar{x}^n) \right]^\lambda, \hfill (C.74)$$

where $(a)$ follows since $C_n^{**}(\lambda) \subseteq C_n^*(\lambda)$, and $(b)$ follows since $|C_n^{**}(\lambda)| \geq e^{n(\rho - 2\delta)}$, and therefore

$$\sum_{y^n \in Y^n} \left( P_{Y^n}^{(C_n^{**}(\lambda))}(y^n) \right)^{1-\lambda} \cdot \left[ \overline{P}_{Y^n}^{(C_n^{**}(\lambda))}(y^n) \right]^\lambda \leq e^{6n\delta} \cdot \exp \left[-n \cdot \min \left\{d_\lambda(Q_X), \overline{A}_{cc}(\rho, \rho_c, Q_U, X, \lambda)\right\}\right]. \hfill (C.75)$$

As all the above holds for any $\lambda \in [0, 1]$, \textsuperscript{62} implies

$$\max_{0 \leq \lambda \leq 1} \max_{C_n \subseteq \mathcal{C}(Q_X): |C_n| \geq e^{n\rho}} \left\{ -\frac{1 - \lambda}{\lambda} \cdot F_1 - \frac{1}{\lambda} - \frac{1}{n} \log \left( \sum_{y^n \in Y^n} \left[ P_{Y^n}^{(C_n)}(y^n) \right]^{1-\lambda} \cdot \left[ \overline{P}_{Y^n}^{(C_n)}(y^n) \right]^\lambda \right) \right\}$$

\textsuperscript{23}The reverse Markov inequality states that if $\mathbb{P}(0 \leq X \leq \alpha \mathbb{E}[X]) = 1$ for some $\alpha > 1$. Then, for any $\beta < 1, \mathbb{P}(X > \beta \mathbb{E}[X]) \geq \frac{1 - \beta}{\alpha - \beta}$.
can be established for any blocklength such that 

\[ T_n \] blocklength exists a CD code such that the event and this bound can be used in lieu of (C.67) in the proof. This will prove the simultaneous achievability with this, we have proved the existence of a CD code such that sufficiently large \( n \) in \( 0 \) for all enumerators for the cloud centers by 17).

\[ \delta \] and \( \lambda \) and allowing to maximize \( A_{rc}(\rho, \rho_c, Q_{UX}, \lambda) \) over \( Q_{UX} \) and \( \rho_c \) (with a constraint set dictated by Lemma 17).

Remark 18. As mentioned after Theorem 8 the random-coding bound can be achieved by a single sequence of CD codes, simultaneously for all type 1 error exponent constraint \( F_1 \). This can be proved by showing that there exists a CD code such that the event \( A_2(\lambda) \) defined in (C.67) holds for all \( \lambda \in [0, 1] \). To show the latter, we uniformly quantize the interval \([0, 1]\) to \( \{\lambda_i\}_{i=0}^K \) with \( \lambda_i = \frac{i}{K} \) and a fixed \( K \). Then, using the union bound, for all \( n \) sufficiently large

\[
\mathbb{P} \left[ \bigcap_{i=0}^K A_2(\lambda_i) \right] \geq 1 - \sum_{i=0}^K \mathbb{P} [A_2(\lambda_i)] \\
\geq 1 - (K + 1) \cdot e^{-3n\delta} \\
\geq 1 - e^{-2n\delta},
\]

and this bound can be used in lieu of (C.67) in the proof. This will prove the simultaneous achievability of \( A_2(\lambda_i) \) for all \( 0 \leq i \leq K \). Then, utilizing the continuity of \( A_{rc}(\rho, \rho_c, Q_{UX}, \lambda) \), by taking \( K \) to increase sub-exponentially in \( n \) the same result can be established to the entire \([0, 1]\) interval.

APPENDIX D

THE TYPE-ENUMERATION METHOD AND THE PROOF OF PROP\( \text{O\textsc{osition 15}} \)

We begin with a short review of the type-enumeration method [48 Sec. 6.3]. To begin, let us define type-class enumerators for the cloud centers by

\[
N_{y^n}(Q_{UY}) \overset{\text{def}}{=} \left| \{u^n \in \mathcal{C}_{c,n} : (u^n, y^n) \in \mathcal{T}_n(Q_{UY})\} \right|.
\]
To wit, \( N_y^n(Q_{UY}) \) counts the random number of cloud centers which have joint type \( Q_{UY} \in \mathcal{P}_n(U \times Y) \) with \( y^n \). While \( M_y^n(Q_{UXY}) \) defined in (C.2) is an enumerator of an hierarchical ensemble, \( N_y^n(Q_{UY}) \) is an enumerator of an ordinary ensemble, and thus simpler to analyze. Furthermore, the analysis of \( M_y^n(Q_{UXY}) \) depends on the properties of \( N_y^n(Q_{UY}) \), and thus we begin by analyzing the latter.

As the cloud centers in the ensemble are drawn independently, \( N_y^n(Q_{UY}) \) is a binomial random variable. It pertains to \( e^{n\rho_c} \) trials and probability of success of the exponential order of \( \exp[-n \cdot I_Q(U; Y)] \), and consequently, \( \mathbb{E}[N_y^n(Q_{UY})] = \exp\{n \cdot [\rho_c - I_Q(U; Y)]\} \). In the sequel, we will need refined properties of the enumerator, and specifically, its large-deviations behavior and the moments \( \mathbb{E}[N_y^n(Q_{UY})] \). To this end, we note that as \( N_y^n(Q_{UY}) \) is just an enumerator for a code drawn from an ordinary ensemble, the analysis of [48, Sec. 6.3] [63, Appendix A.2] holds. As was shown there, when \( I_Q(U; Y) \leq \rho_c \), \( \mathbb{E}[N_y^n(Q_{UY})] \) increases exponentially with \( n \) as \( \exp\{n \cdot [\rho_c - I_Q(U; Y)]\} \), and \( N_y^n(Q_{UY}) \) concentrates double-exponentially rapidly around this average. Specifically, letting

\[
\mathcal{B}_n(Q_{UY}, \delta) \overset{\text{def}}{=} \left\{ e^{-n\delta} \cdot \mathbb{E}[N_y^n(Q_{UY})] \leq N_y^n(Q_{UY}) \leq e^{n\delta} \cdot \mathbb{E}[N_y^n(Q_{UY})] \right\}, \tag{D.2}
\]

then for any \( \delta > 0 \) sufficiently small

\[
\mathbb{P}[\mathcal{B}_n^c(Q_{UY}, \delta)] \leq \exp\left[-e^{n\delta}\right]. \tag{D.3}
\]

When \( I_Q(U; Y) > \rho_c \) holds, \( \mathbb{E}[N_y^n(Q_{UY})] \) decreases exponentially with \( n \) as \( \exp\{-n \cdot [I_Q(U; Y) - \rho_c]\} \), and \( N_y^n(Q_{UY}) = 0 \) almost surely. Furthermore, the probability that even a single codeword has joint type \( Q_{UY} \) with \( y^n \) is exponentially small, and the probability that there is an exponential number of such codewords is double-exponentially small. Specifically, for all sufficiently large \( n \)

\[
\mathbb{P}\{N_y^n(Q_{UY}) \geq 1\} \leq \exp\{-n \cdot [I_Q(U; Y) - \rho_c]\}, \tag{D.4}
\]

and for any given \( \delta > 0 \)

\[
\mathbb{P}\left\{ N_y^n(Q_{UY}) \geq e^{2n\delta} \right\} \leq \exp\left[-e^{n\delta}\right]. \tag{D.5}
\]

Using the properties above, it can be easily deduced that

\[
\mathbb{E}\left[N_y^n(Q_{UY}) \right] = \begin{cases} 
\exp\{n\lambda \cdot [\rho_c - I_Q(U; Y)]\}, & I_Q(U; Y) \leq \rho_c, \\
\exp\{-n \cdot [I_Q(U; Y) - \rho_c]\}, & I_Q(U; Y) > \rho_c
\end{cases}, \tag{D.6}
\]

or, in an equivalent and more compact form,

\[
\mathbb{E}\left[N_y^n(Q_{UY}) \right] = \exp\left(n \cdot [\lambda \cdot \rho_c - I_Q(U; Y)] - (1 - \lambda) \cdot [I_Q(U; Y) - \rho_c]\right). \tag{D.7}
\]

We can now turn to analyze the behavior of the more complicated enumerator \( M_y^n(Q_{UXY}) \). To this end, note that conditioned on the event \( N_y^n(Q_{UY}) = e^{nu} \), \( M_y^n(Q_{UXY}) \) is a binomial random variable pertaining to \( \exp[n(\nu + \rho_s)] \).
and the conditional large-deviations behavior of $M_{y^n}(Q_{UXY})$ is identical to the large-deviations behavior of $N_{y^n}(Q_{UY})$, with $\nu + \rho_s$ and $I_Q(X;Y|U)$ replacing $\rho_c$ and $I_Q(U;Y)$, respectively. The next lemma provides an asymptotic expression for the unconditional moments of $M_{y^n}(Q_{UXY})$. It can be easily seen that (D.7) is obtained as a special case, when setting $U = X$ and $\rho = \rho_c$.

**Lemma 19.** For $\lambda > 0$

\[
\mathbb{E} \left[ M_{y^n}^{\lambda}(Q_{UXY}) \right] \doteq \exp \left( n \cdot \left[ \lambda [\rho - I_Q(U,X;Y)] - (1 - \lambda) \cdot \max \{ |I_Q(U;Y) - \rho_c|, I_Q(U,X;Y) - \rho \} \right] \right).
\]  

(D.9)

**Proof:** Let

\[
\delta \in \left( 0, \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{E}[N_{y^n}(Q_{UY})] \right),
\]

(D.10)

define the events

\[
A_n^0(Q_{UY}) \doteq \{ N_{y^n}(Q_{UY}) = 0 \},
\]  

(D.11)

\[
A_n^{-1}(Q_{UY}, \delta) \doteq \{ 1 \leq N_{y^n}(Q_{UY}) \leq e^{2n\delta} \},
\]

(D.12)

\[
A_n^{\geq 1}(Q_{UY}, \delta) \doteq \{ N_{y^n}(Q_{UY}) \geq e^{2n\delta} \},
\]

(D.13)

and recall the definition of the event $B_n(Q_{UY}, \delta)$ in (D.2). We will consider four cases depending on the relations between the rates and mutual information values. For the sake of brevity, only the first case will be analyzed with a strictly positive $\delta > 0$ and then the limit $\delta \downarrow 0$ will be taken. In all other three cases, we shall derive the expressions for the moments assuming $\delta = 0$, with the understanding that upper and lower bounds can be derived in a similar manner to the first case. For notational convenience, when the expressions are derived assuming $\delta = 0$ we will omit $\delta$ from the notation of the events defined above [e.g., $A_n^{-1}(Q_{UY})$]. We will use the moments (D.6), the strong concentration relation (D.3) and the large-deviations bound (D.4), for both $N_{y^n}(Q_{UY})$ and $M_{y^n}(Q_{UXY})$ [when the latter is conditioned on the value of $N_{y^n}(Q_{UY})$].

**Case 1.** If $I_Q(U;Y) > \rho_c$ and $I_Q(X;Y|U) > \rho_s$, then for all $\delta > 0$ sufficiently small, $I_Q(U;Y) > \rho_c + \delta$ and $I_Q(X;Y|U) > \rho_s + \delta$ and thus,

\[
\mathbb{E} \left[ M_{y^n}^{\lambda}(Q_{UXY}) \right] \leq \mathbb{P} \left[ A_n^0(Q_{UY}) \right] \cdot 0 + \mathbb{P} \left[ A_n^{-1}(Q_{UY}, \delta) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(Q_{UXY}) | A_n^{-1}(Q_{UY}, \delta) \right] + \mathbb{P} \left[ A_n^{\geq 1}(Q_{UY}, \delta) \right] \cdot e^{n\lambda \rho}
\]

(D.14)
\[ \leq 0 + \exp \{ -n \cdot [I_Q(U; Y) - \rho_c - \delta] \} \cdot \exp \{ -n \cdot [I_Q(X; Y|U) - \rho_s - 2\delta] \} + \exp \left[ -e^{n\delta} \right] \cdot e^{n\lambda \rho} \]  
(D.15)

\[ \equiv \exp \{ n \cdot [\rho - I_Q(U, X; Y) + 3\delta] \}. \]  
(D.16)

Similarly, it can be shown that

\[ \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] \geq \exp \{ n \cdot [\rho - I_Q(U, X; Y) - 3\delta] \}. \]  
(D.17)

As \( \delta \geq 0 \) is arbitrary, we obtain

\[ \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] \equiv \exp \{ n \cdot [\rho - I_Q(U, X; Y)] \}. \]  
(D.18)

**Case 2.** If \( I_Q(U; Y) > \rho_c \) and \( I_Q(X; Y|U) < \rho_s \) then

\[ \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] \leq \mathbb{P} \left[ A_{n}^{\geq 0}(Q_{UY}) \right] \cdot 0 + \mathbb{P} \left[ A_{n}^{= 1}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] \cdot A_{n}^{= 1}(Q_{UY}) \]

\[ + \mathbb{P} \left[ A_{n}^{\geq 1}(Q_{UY}) \right] \cdot e^{n\lambda \rho} \]  
(D.19)

\[ \equiv 0 + \exp \{ -n \cdot [I_Q(U; Y) - \rho_c] \} \cdot \exp \{ n \cdot \lambda \cdot [\rho_s - I_Q(X; Y|U)] \} + 0 \]  
(D.20)

\[ = \exp \{ n \cdot [\rho_c - I_Q(U; Y) + \lambda \cdot (\rho_s - I_Q(X; Y|U))] \} \]  
(D.21)

\[ = \exp \{ n \cdot [\rho_c - I_Q(U; Y) + \rho_s - I_Q(X; Y|U) - (1 - \lambda) \cdot (\rho_s - I_Q(X; Y|U))] \} \]  
(D.22)

\[ = \exp \{ n \cdot [\rho - I_Q(U, X; Y) - (1 - \lambda) (\rho_s - I_Q(X; Y|U))] \}. \]  
(D.23)

**Case 3.** If \( I_Q(U; Y) < \rho_c \) and \( I_Q(X; Y|U) > \rho_c - I_Q(U; Y) + \rho_s \) then

\[ \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] = \mathbb{P} \left[ \mathcal{B}_n(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) | \mathcal{B}_n(Q_{UY}) \right] \]

\[ + \mathbb{P} \left[ \mathcal{B}_n^{c}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) | \mathcal{B}_n^{c}(Q_{UY}) \right] \]  
(D.24)

\[ \equiv 1 \cdot \exp \{ n \cdot [\rho_c - I_Q(U; Y) + \rho_s - I_Q(X; Y|U)] \} \} + 0 \]  
(D.25)

\[ = \exp \{ n \cdot [\rho - I_Q(U, X; Y)] \}. \]  
(D.26)

**Case 4.** If \( I_Q(U; Y) < \rho_c \) and \( I_Q(X; Y|U) < \rho_c - I_Q(U; Y) + \rho_s \) then

\[ \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) \right] = \mathbb{P} \left[ \mathcal{B}_n(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) | \mathcal{B}_n(Q_{UY}) \right] \]

\[ + \mathbb{P} \left[ \mathcal{B}_n^{c}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_c^{\lambda}(Q_{U|XY}) | \mathcal{B}_n^{c}(Q_{UY}) \right] \]  
(D.27)

\[ \equiv 1 \cdot \exp \{ n \cdot \lambda \cdot [\rho_c - I_Q(U; Y) + \rho_s - I_Q(X; Y|U)] \} + 0 \]  
(D.28)
\[= \exp \{ n \cdot \lambda \cdot [\rho - I_Q(U, X; Y)] \} \]
\[= \exp \{ n \cdot [\rho - I_Q(U, X; Y) - (1 - \lambda) [\rho - I_Q(U, X; Y)] \} . \]

Noting that \( I_Q(X; Y|U) > \rho_c - I_Q(U; Y) + \rho_s \) is equivalent to \( I_Q(U, X; Y) > \rho \), it is easy to verify that
\[
\mathbb{E} \left[ M_{y^n}^\lambda (Q_{UXY}) \right] \doteq \exp \left( n \cdot \{ \rho - I_Q(U, X; Y) - (1 - \lambda) \, |\rho_s - I_Q(X; Y|U) + |\rho_c - I_Q(U; Y)|_+ \} \right) \] (D.31)

matches all four cases. The final expression is obtained from the identities
\[
\rho - I_Q(U, X; Y) - (1 - \lambda) \cdot |\rho_s - I_Q(X; Y|U) + |\rho_c - I_Q(U; Y)|_+ |_+ \\
\overset{(a)}{=} \rho - I_Q(U, X; Y) - (1 - \lambda) \, |\rho - I_Q(U, X; Y) + I_Q(U; Y) - \rho_c|_+ |_+ , \\
\overset{(b)}{=} \lambda [\rho - I_Q(U, X; Y)] - (1 - \lambda) \, |I_Q(U; Y) - \rho_c|_+ - (1 - \lambda) \, |I_Q(U, X; Y) - \rho - I_Q(U; Y) - \rho_c|_+ |_+ \\
\overset{(c)}{=} \lambda [\rho - I_Q(U, X; Y)] - (1 - \lambda) \cdot \max \{ |I_Q(U; Y) - \rho_c|_+, I_Q(U, X; Y) - \rho \} , 
\] (D.32) (D.33) (D.34)

where (a) follows from the identity \(|t|_+ = t + |t|_+\) with \( t = \rho_c - I_Q(U; Y) \), (b) follows from the same identity with \( t = \rho - I_Q(U, X; Y) + I_Q(U; Y) - \rho_c |_+ \), and (c) follows from \( t + |s - t|_+ = \max \{ t, s \} \) with \( t = |I_Q(U; Y) - \rho_c|_+ \) and \( s = I_Q(U, X; Y) - \rho \).

To continue, we will need to separate the analysis according to whether \( Q_{UY} \neq Q_{UY} \) (Lemma \text{[22]} or \( Q_{UY} = Q_{UY} \) (Lemma \text{[25]}). In the former case \( M_{y^n}(Q_{UXY}) \) and \( M_{y^n}(\overline{Q}_{UXY}) \) count codewords which pertain to different cloud centers, and as we shall next see, \( M_{y^n-\lambda}(Q_{UXY}) \) and \( M_{y^n}(\overline{Q}_{UXY}) \) are asymptotically uncorrelated. In the later case, these enumerators may count codewords which pertain to the same cloud center, and correlation between \( M_{y^n-\lambda}(Q_{UXY}) \) and \( M_{y^n}(\overline{Q}_{UXY}) \) is possible. We will need two auxiliary lemmas.

**Lemma 20.** Let \( f(t) \) be a monotonically non-decreasing function, let \( B \sim \text{Binomial}(N, p) \) and \( \tilde{B} \sim \text{Binomial}(\tilde{N}, p) \) with \( \tilde{N} > N \). Then,
\[
\mathbb{E} [f(B)] \leq \mathbb{E} [f(\tilde{B})] . \] (D.35)

**Proof:** Let \( L \overset{\text{def}}{=} N - \tilde{N} \) and \( A \sim \text{Binomial}(L, p) \), independent of \( B \). As the sum of independent binomial random variables with the same success probability is also binomially distributed, we have \( B + A \) is equal in distribution to \( \tilde{B} \sim \text{Binomial}(\tilde{N}, p) \). Thus,
\[
\mathbb{E} [f(\tilde{B})] = \mathbb{E} [f(B + A)] \\
\overset{}{\geq} \mathbb{E} [f(B)] , \] (D.36) (D.37)

where the inequality holds pointwise for any given \( A = a \), and thus also under expectation.

As is well known from the method of types, \( \mathbb{P}[ (U^n, y^n) \in T_n(Q_{UY}) ] \doteq \exp[-nI_Q(U; Y)] \). The next lemma shows that this probability can be upper bounded asymptotically even when \( I_Q(U; Y) = 0 \), and the large-deviations behavior does not hold.
Lemma 21. Let $Q_{UY} \in \mathcal{P}(U \times Y)$ be given such that $\text{supp}(Q_U) \geq 2$ and $\text{supp}(Q_Y) \geq 2$. Also let $y^n \in \mathcal{T}_n(Q_Y)$ and assume that $U^n$ is distributed uniformly over $\mathcal{T}_n(Q_U)$. Then, for any given $\epsilon > 0$ there exists $n_0(Q_U, Q_Y)$ such that for all $n \geq n_0(Q_U, Q_Y)$

$$P[(U^n, y^n) \in \mathcal{T}_n(Q_{UY})] \leq \epsilon. \quad (D.38)$$

Proof: Using Robbins’ sharpening of Stirling’s formula (e.g., [4, Problem 2.2]), it can be shown that the size of a type class satisfies

$$|\mathcal{T}_n(Q_X)| \approx \exp \left\{ n \cdot \left[ H(Q_X) - \left( \frac{\text{supp}(Q_X) - 1}{2} \right) \log \frac{n}{n} \right] \right\} n^{-\frac{1}{2}[\text{supp}(Q_X)-1]} \quad (D.39)$$

Hence,

$$P[(U^n, y^n) \in \mathcal{T}_n(Q_{UY})] \overset{(a)}{=} P[(U^n, Y^n) \in \mathcal{T}_n(Q_{UY})] \quad (D.40)$$

$$= \frac{\mathcal{T}_n(Q_{UY})}{|\mathcal{T}_n(Q_{UY})| |\mathcal{T}_n(Q_{Y})|} \quad (D.41)$$

$$\approx e^{-n \cdot I_Q(U; Y)} \cdot n^{-\frac{1}{2}[\text{supp}(Q_{UY})-\text{supp}(Q_U)-\text{supp}(Q_Y)+1]}, \quad (D.42)$$

where $(a)$ holds by symmetry [assuming that $Y^n$ is drawn uniformly over $\mathcal{T}_n(Q_Y)$].

Now, by the Kullback-Csiszár-Kemperman-Pinsker inequality [3, Lemma 11.6.1][4, Exercise 3.18]

$$I_Q(U; Y) = D(Q_{UY} || Q_U \times Q_Y) \quad (D.43)$$

$$\geq \frac{1}{2 \log 2} \| Q_{UY} - Q_U \times Q_Y \|^2, \quad (D.44)$$

and thus for any given $\delta > 0$

$$\left\{ \tilde{Q}_{UY} : \tilde{Q}_U = Q_U, \tilde{Q}_Y = Q_Y, I_{\tilde{Q}}(U; Y) \leq \delta \right\}$$

$$\subseteq \left\{ \tilde{Q}_{UY} : \tilde{Q}_U = Q_U, \tilde{Q}_Y = Q_Y, \| \tilde{Q}_{UY} - Q_U \times Q_Y \| \leq \eta \right\} \quad (D.45)$$

$$\overset{\text{def}}{=} \mathcal{J}(\eta, Q_U, Q_Y), \quad (D.46)$$

where $\eta \overset{\text{def}}{=} \sqrt{2\delta \log 2}$. Choose $\eta(Q_U, Q_Y)$ such that $\text{supp}(\tilde{Q}_{UY}) = \text{supp}(Q_U \times Q_Y) = \text{supp}(Q_U) \cdot \text{supp}(Q_Y)$ for all $\tilde{Q}_{UY} \in \mathcal{J}(\delta_0, Q_U, Q_Y)$. We consider two cases:

Case 1. If $Q_{UY} \in \mathcal{J}(\delta_0, Q_U, Q_Y)$ then it is elementary to verify that in this event, as $\text{supp}(Q_U) \geq 2$ and $\text{supp}(Q_Y) \geq 2$ was assumed,

$$\text{supp}(Q_{UY}) - \text{supp}(Q_U) - \text{supp}(Q_Y)$$

$$= \text{supp}(Q_U) \cdot \text{supp}(Q_Y) - \text{supp}(Q_U) - \text{supp}(Q_Y) \quad (D.47)$$

$$\geq 0, \quad (D.48)$$
and thus (D.42) implies that
\[
P[(U^n, y^n) \in \mathcal{T}_n(Q_{UY})] \leq \frac{1}{\sqrt{n}}. \tag{D.49}
\]

**Case 2.** If \( Q_{UY} \notin \mathcal{J}(\delta, Q_U, Q_Y) \) then \( I_Q(U; Y) \geq \delta = \frac{1}{2 \log 2} \eta^2 \) and
\[
P[(U^n, y^n) \in \mathcal{T}_n(Q_{UY})] \leq e^{-n \frac{1}{2 \log 2} \eta^2}. \tag{D.50}
\]

This completes the proof.

We are now ready to state and prove the asymptotic uncorrelation lemma for \( Q_{UY} \neq \overline{Q}_{UY} \).

**Lemma 22.** Let \((Q_{UXY}, \overline{Q}_{UXY})\) be given such that \( Q_{UY} \neq \overline{Q}_{UY} \). Then,
\[
\mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY})M_{yn}^{\lambda}(\overline{Q}_{UXY}) \right] \doteq \mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E}\left[ M_{yn}^{\lambda}(\overline{Q}_{UXY}) \right]. \tag{D.51}
\]

**Proof:** We begin by lower bounding the correlation. To this end, let us decompose
\[
M_{yn}(Q_{UXY}) = M_{yn}(Q_{UXY}, 1) + M_{yn}(Q_{UXY}, 2), \tag{D.52}
\]

where \( M_{yn}(Q_{UXY}, 1) \) is the enumerator of to the subcode of \( \mathcal{C}_n \) of codewords which pertain to half of the cloud centers (say, for cloud centers with odd indices) and \( M_{yn}(Q_{UXY}, 2) \) is the enumerator pertaining to the rest of the codewords. Note that \( M_{yn}(Q_{UXY}, 1) \) and \( M_{yn}(\overline{Q}_{UXY}, 2) \) are independent for any given \((Q_{UXY}, \overline{Q}_{UXY})\). Hence,
\[
\mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY})M_{yn}^{\lambda}(\overline{Q}_{UXY}) \right]
\]
\[
= \mathbb{E}\left\{ [M_{yn}(Q_{UXY}, 1) + M_{yn}(Q_{UXY}, 2)]^{1-\lambda} \cdot [M_{yn}(\overline{Q}_{UXY}, 1) + M_{yn}(\overline{Q}_{UXY}, 2)]^{\lambda} \right\} \tag{D.53}
\]
\[
\geq \mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY}, 1) \cdot M_{yn}^{\lambda}(\overline{Q}_{UXY}, 2) \right] \tag{D.54}
\]
\[
= \mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY}, 1) \right] \cdot \mathbb{E}\left[ M_{yn}^{\lambda}(\overline{Q}_{UXY}, 2) \right] \tag{D.55}
\]
\[
= \mathbb{E}\left[ M_{yn}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E}\left[ M_{yn}^{\lambda}(\overline{Q}_{UXY}) \right], \tag{D.56}
\]

where \((a)\) follows since enumerators are positive, and \((b)\) follows since \( M_{yn}(Q_{UXY}, 1) \) and \( M_{yn}(\overline{Q}_{UXY}, 2) \) are independent for any given \((Q_{UXY}, \overline{Q}_{UXY})\). \((c)\) holds true since \( M_{yn}(Q_{UXY}, i) \ (i = 1, 2) \) pertain to codebooks of cloud-center size \( \frac{1}{2} e^{n\rho_c} \doteq e^{n\rho_c} \) and satellite rate \( \rho_s \), and thus clearly \( \mathbb{E}[M_{yn}^{\lambda}(Q_{UXY}, i)] = \mathbb{E}[M_{yn}^{\lambda}(Q_{UXY})] \).

To derive an upper bound on the correlation, we first note two properties. First, as the codewords enumerated by \( M_{yn}(\overline{Q}_{UXY}) \) necessarily correspond to different cloud centers from the codewords enumerated by \( M_{yn}(Q_{UXY}) \), the following Markov relation holds:
\[
M_{yn}(Q_{UXY}) - N_{yn}(Q_{U}) - N_{yn}(\overline{Q}_{U}) - M_{yn}(\overline{Q}_{UXY}). \tag{D.57}
\]

Second, conditioned on \( N_{yn}(Q_{U}) \), \( N_{yn}(\overline{Q}_{U}) \) is a binomial random variable pertaining to \( e^{n\rho_c} - N_{yn}(Q_{U}) \leq e^{n\rho_c} \).
trials with a probability of success given by
\[
\mathbb{P} \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \bigg| (U^n, y^n) \notin \mathcal{T}_n(\overline{Q}_{UY}) \right].
\] (D.58)

However, this probability is not significantly different than the unconditional probability. More rigorously, for any \( \epsilon \in (0, 1) \), and all \( n > n_0(Q_U, Q_Y) \) sufficiently large
\[
\mathbb{P} \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \right] \leq \mathbb{P} \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \bigg| (U^n, y^n) \notin \mathcal{T}_n(\overline{Q}_{UY}) \right] \leq \frac{1}{1 - \epsilon} \cdot \mathbb{P} \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \right].
\] (D.59)

To see this, note that since \( Q_{UY} \neq \overline{Q}_{UY} \) we must have \( \text{supp}(Q_Y) \geq 2 \), and thus Lemma 21 implies that for any \( \epsilon \in (0, 1) \), and all \( n \) sufficiently large
\[
\mathbb{P} \left[ (U^n, x^n, y^n) \notin \mathcal{T}_n(\overline{Q}_{UXY}) \right] \geq \mathbb{P} \left[ (U^n, x^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right].
\] (D.63)

Equipped with the Markov relation (D.57) and the bound on the conditional probability (D.59), we can derive an upper bound on the correlation that asymptotically matches the lower bound (D.56), and thus prove the lemma. Indeed,
\[
\mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \cdot M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \right]
\]
\[
\overset{(a)}{=} \mathbb{E} \left\{ \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \cdot M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(Q_{UY}), N_{y^n}(\overline{Q}_{UY}) \right] \right\}
\]
\[
\overset{(b)}{=} \mathbb{E} \left\{ \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(\overline{Q}_{UY}) \right] \right\}
\]
\[
\overset{(c)}{=} \mathbb{E} \left\{ \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(\overline{Q}_{UY}) \right] \bigg| N_{y^n}(Q_{UY}) \right\}
\]
\[
= \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(\overline{Q}_{UY}) \right] \left| N_{y^n}(Q_{UY}) \right]
\]
\[
\overset{(d)}{=} \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(\overline{Q}_{UY}), N_{y^n}(Q_{UY}) = 0 \right]
\]
\[
= \mathbb{E} \left[ M^{1-\lambda}_{y^n}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M^{\lambda}_{y^n}(\overline{Q}_{UXY}) \bigg| N_{y^n}(\overline{Q}_{UY}), N_{y^n}(Q_{UY}) = 0 \right],
\] (D.69)
where \( (a) \) and \( (c) \) follows from the law of total expectation, and \( (b) \) follows from the Markov relation \((D.57)\). To see \((d)\) note that \( E[M_{y^n}(\overline{Q}_{UXY}) | N_{y^n}(\overline{Q}_{UY}) = \overline{s}] \) is a non-decreasing function of \( \overline{s} \) [see \((D.8)\)]. In addition, conditioned on \( N_{y^n}(Q_{UY}) = s, N_{y^n}(\overline{Q}_{UY}) \) is a binomial random variable pertaining to less \( e^{n\rho_c} - N_{y^n}(Q_{UY}) \leq e^{n\rho_c} \) trials. Thus, \( (d) \) follows from Lemma \[20\]. We now note that

\[
E \left( E \left[ M_{y^n}(\overline{Q}_{UXY}) \left| N_{y^n}(\overline{Q}_{UY}), N_{y^n}(Q_{UY}) = 0 \right. \right] \right) \quad (D.70)
\]

and

\[
E \left( E \left[ M_{y^n}(\overline{Q}_{UXY}) \left| N_{y^n}(\overline{Q}_{UY}) \right. \right] \right) = E \left[ M_{y^n}(\overline{Q}_{UXY}) \right] \quad (D.71)
\]

are both moments of binomial random variables with the same number of trials, but the former has a success probability

\[
P \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \right] \doteq e^{-nH(U;Y)},
\]

and the latter has a success probability

\[
P \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \right] \ni (U^n, y^n) \not\in \mathcal{T}_n(\overline{Q}_{UY}).
\]

However, \((D.59)\) shows that the latter success probability has the same exponential order. In turn, the proof of Lemma \[19\] shows that the exponential order of this expectation only depends on the exponential order of the success probability. Consequently,

\[
E \left( E \left[ M_{y^n}(\overline{Q}_{UXY}) \left| N_{y^n}(\overline{Q}_{UY}), N_{y^n}(Q_{UY}) = 0 \right. \right] \right) = E \left( E \left[ M_{y^n}(\overline{Q}_{UXY}) \left| N_{y^n}(\overline{Q}_{UY}) \right. \right] \right) \quad (D.74)
\]

\[
= E \left[ M_{y^n}(\overline{Q}_{UXY}) \right]. \quad (D.75)
\]

Using this in \((D.69)\) completes the proof.

Next, we move to the case where the cloud centers may be the same, i.e., \( Q_{UY} = \overline{Q}_{UY} \). In this case, correlation between \( M_{y^n}^{1-\lambda}(Q_{UXY}) \) and \( M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \) is possible even asymptotically. Apparently, this is due to the fact that \( N_{y^n}(Q_{UY}) = 0 \) with high probability whenever \( I_Q(U;Y) > \rho_c \), and thus, in this case, \( M_{y^n}(Q_{UXY}) = M_{y^n}(\overline{Q}_{UXY}) = 0 \) with high probability. However, as we will next show, \( M_{y^n}^{1-\lambda}(Q_{UXY}) \) and \( M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \) are asymptotically uncorrelated when conditioned on \( N_{y^n}(Q_{UY}) \).

To show this, we first need a result analogous to Lemma \[21\]. To this end, we first need to exclude possible \( Q_{UXY} \) from the discussion. Let us say that \( (Q_{UX}, Q_{UY}) \) is a joint-distribution-dictator (JDD) pair if it determines \( Q_{UXY} \) unambiguously. For example, when \( |U| = |X'| = |Y'| = 2 \), \( Q_{X'|U} \) corresponds to a \( Z \)-channel and \( Q_{Y'|U} \) corresponds to an \( S \)-channel. \( Q_{UX}, Q_{UY} \) is a JDD pair. Clearly, in this case no \( Q_{UXY} \neq \overline{Q}_{UXY} \) exists with the same \( (U, X) \)

\(^{24}\)That is \( Q_{Y'|U}(0|1) = Q_{X'|U}(1|0) = 0 \) and all other transition probabilities are non-zero.
and \((U, Y)\) marginals, and thus such \(Q_{UY}\) are of no interest to the current discussion.

By carefully observing the Z-channel/S-channel example above, it is easy to verify if for all \(u \in \mathcal{U}\) either \(\text{supp}(Q_X|U=u) < 2\) or \(\text{supp}(Q_Y|U=u) < 2\) then \((Q_X, Q_{UY})\) is a JDD pair. Therefore, if \((Q_X, Q_{UY})\) is not a JDD pair then there must exist \(u^* \in \mathcal{U}\) such that both \(\text{supp}(Q_X|U=u^*) \geq 2\) and \(\text{supp}(Q_Y|U=u^*) \geq 2\). This property will be used in the proof of the following lemma.

**Lemma 23.** Let \(Q_{UXY} \in \mathcal{P}(U \times X \times Y)\) be given such that \(\text{supp}(Q_X) \geq 2\), \(\text{supp}(Q_Y) \geq 2\), and \((Q_X, Q_{UY})\) is not a JDD pair. Also, let \((u^n, y^n) \in \mathcal{T}_n(Q_{UXY})\) and assume that \(X^n\) is distributed uniformly over \(\mathcal{T}(Q_X, u^n)\).

Then, for any given \(\epsilon > 0\) there exists \(n_0(Q_{UX}, Q_{UY})\) such that for all \(n \geq n_0(Q_{UX}, Q_{UY})\)

\[
\Pr[(u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY})] \leq \epsilon. \tag{D.76}
\]

**Proof:** Had \(X^n\) been distributed uniformly over \(\mathcal{T}_n(Q_X)\), the claim would follow directly from Lemma 21, where \(y^n\) and \(U^n\) there are replaced by \((u^n, y^n)\) and \(X^n\), respectively. However, since \(X^n\) is distributed uniformly over \(\mathcal{T}_n(Q_X, u^n)\) the proof is not immediate. Nonetheless, it follows the same lines, and thus we will only highlight the required modifications.

Just as in \((D.39)\), the size of a conditional type class can be shown to satisfy

\[
|\mathcal{T}_n(Q_X|U, u^n)| \approx \prod_{u \in \text{supp}(Q_U)} \exp \left\{ nQ_U(u) \cdot H_Q(X|U = u) \right\} \cdot [nQ_U(u)]^{-\frac{1}{2}[\text{supp}(Q_X|U=u) - 1]} \tag{D.77}
\]

and thus

\[
\Pr[(u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY})] = \frac{|\mathcal{T}_n(Q_X|U, u^n)|}{|\mathcal{T}_n(Q_X|U)|} \tag{D.78}
\]

\[
\overset{(a)}{=} \prod_{(u,y) \in \text{supp}(Q_{UY})} \exp \left\{ nQ_{UY}(u, y) \cdot H_Q(X|U = u, Y = Y) \right\} \cdot [nQ_{UY}(u, y)]^{-\frac{1}{2}[\text{supp}(Q_X|U=u, Y=y) - 1]} \cdot \prod_{u \in \text{supp}(Q_U)} \exp \left\{ nQ_U(u) \cdot H_Q(X|U = u) \right\} \cdot [nQ_U(u)]^{-\frac{1}{2}[\text{supp}(Q_X|U=u) - 1]} \tag{D.79}
\]

\[
= e^{-nI_Q(X;Y|U)} \cdot n^{\frac{1}{2} \sum_{u \in \text{supp}(Q_U)} [\text{supp}(Q_X|U=u) - 1]} \cdot \frac{\prod_{(u,y) \in \text{supp}(Q_{UY})} Q_{UY}(u, y)}{\prod_{u \in \text{supp}(Q_U)} Q_U(u)^{-\frac{1}{2}[\text{supp}(Q_X|U=u) - 1]}} \cdot c(Q_{UY}), \tag{D.80}
\]

where

\[
c(Q_{UY}) \overset{\text{def}}{=} \frac{\prod_{(u,y) \in \text{supp}(Q_{UY})} Q_{UY}(u, y)}{\prod_{u \in \text{supp}(Q_U)} Q_U(u)^{-\frac{1}{2}[\text{supp}(Q_X|U=u) - 1]}} \tag{D.81}
\]

Now, suppose that \(I_Q(X;Y|U) = 0\). Then,

\[
\frac{1}{2} \sum_{u \in \text{supp}(Q_U)} [\text{supp}(Q_X|U=u) - 1] - \frac{1}{2} \sum_{(u,y) \in \text{supp}(Q_{UY})} [\text{supp}(Q_X|U=u, Y=y) - 1] \overset{(a)}{=} \frac{1}{2} \sum_{u \in \text{supp}(Q_U)} [\text{supp}(Q_X|U=u) - 1] - \frac{1}{2} \sum_{(u,y) \in \text{supp}(Q_{UY})} [\text{supp}(Q_X|U=u) - 1] \tag{D.82}
\]
\[
\begin{align*}
\frac{1}{2} \sum_{u \in \text{supp}(Q_U)} \left\{ \left[ \text{supp}(Q_X|U=u) - 1 \right] - \sum_{y \in \text{supp}(Q_Y|U=u)} \left[ \text{supp}(Q_X|U=u) - 1 \right] \right\} \\
= \frac{1}{2} \sum_{u \in \text{supp}(Q_U)} \left[ 1 - \text{supp}(Q_X|U=u) \right] \left[ \text{supp}(Q_X|U=u) - 1 \right]
\end{align*}
\]

\[
(b) \frac{1}{2} \sum_{u \in \text{supp}(Q_U)} \left[ 1 - \text{supp}(Q_Y|U=u) \right] \left[ \text{supp}(Q_X|U=u) - 1 \right]
\]

\[
\leq - \frac{1}{2},
\]

where (a) follows since \(Q_X|U=u = Q_X|U=u, Y=y\) for all \(u \in \text{supp}(Q_U)\), and (b) follows since \((Q_{UX}, Q_{UY})\) is not a JDD pair, and thus there must exist \(u^* \in \text{supp}(Q_U)\) such that both \(\text{supp}(Q_X|U=u^*) \geq 2\) and \(\text{supp}(Q_X|U=u^*) \geq 2\) (as noted before the statement of the lemma). Thus, when \(I_Q(X;Y|U) = 0\) we get

\[
P \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right] \leq c(Q_{UY}) \frac{1}{\sqrt{n}}.
\]

The proof then may continue as the proof of Lemma 21. One can find \(\eta > 0\) sufficiently small such that any \(\tilde{Q}_{UXY} \in \left\{ \tilde{Q}_{UXY} : \|\tilde{Q}_{UXY} - Q_U \times Q_X|U \times Q_Y|U\| \leq \eta \right\}\) has the same support as \(Q_U \times Q_X|U \times Q_Y|U\) (at least conditioned on \(u^*\)). Further, one can find \(\delta(\eta, Q_{UX}, Q_{UY}) > 0\) such that

\[
\left\{ \tilde{Q}_{UXY} : I_Q(X;Y|U) \leq \delta \right\} \subseteq \left\{ \tilde{Q}_{UXY} : \|\tilde{Q}_{UXY} - Q_U \times Q_X|U \times Q_Y|U\| \leq \eta \right\},
\]

and the two cases considered in Lemma 21 can be considered here as well. In the first case, \(I_Q(X;Y|U)\) may vanish, but \(\text{supp}(Q_{UXY}) = \text{supp}(Q_U \times Q_X|U \times Q_Y|U)\) and thus holds. In the second case \(I_Q(X;Y|U) \geq \delta\), and thus for any given \(\epsilon > 0\) there exists \(n_0(Q_{UX}, Q_{UY})\) such that

\[
P \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right] \leq e^{-nI_Q(X;Y|U) \cdot n|U||X|} \cdot c(Q_{UY}) \leq \epsilon
\]

for all \(n \geq n_0(Q_{UX}, Q_{UY})\).

Lemma 24. Let \((Q_{UXY}, \overline{Q}_{UXY})\) be given such that \(Q_{UY} = \overline{Q}_{UY}\). Then,

\[
\mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY})M_{y^n}(\overline{Q}_{UXY}) \right] = \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}(\overline{Q}_{UXY}) \right].
\]

Proof: The proof follows the same lines of the proof of Lemma 22 and so we only provide a brief outline. For a lower bound on the conditional correlation, one can decompose

\[
M_{y^n}(Q_{UXY}) = M_{y^n}(Q_{UXY}, 1) + M_{y^n}(Q_{UXY}, 2)
\]

where \(M_{y^n}(Q_{UXY}, 1)\) [respectively, \(M_{y^n}(Q_{UXY}, 2)\)] corresponds to codewords of odd (even) satellite indices (say).

For a asymptotically matching upper bound on the correlation, we note that similarly to \(D.59\), when \(X^n\) is drawn uniformly over \(\mathcal{T}_n(Q_X|U, u^n)\),

\[
P \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right] (u^n, X^n, y^n) \notin \mathcal{T}_n(\overline{Q}_{UXY})
\]
is close to the unconditional probability, in the sense that for any given $\epsilon > 0$,

$$
\mathbb{P} \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right] \leq \mathbb{P} \left[ (U^n, y^n) \in \mathcal{T}_n(Q_{UY}) \right] \mathbb{P} \left[ (U^n, y^n) \notin \mathcal{T}_n(Q_{UXY}) \right] \leq \frac{1}{1 - \epsilon} \cdot \mathbb{P} \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right]. \tag{D.92}
$$

To prove this, a derivation similar to (D.62) can be used, while noting that $(Q_{UX}, Q_{UY})$ is not a JDD pair, and so according to Lemma 23

$$
\mathbb{P} \left[ (u^n, X^n, y^n) \in \mathcal{T}_n(Q_{UXY}) \right] \leq \epsilon \tag{D.93}
$$

for all $n$ sufficiently large. Equipped with these results, we get

$$
\mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right] \\
\overset{(a)}{=} \mathbb{E} \left\{ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}), M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \bigg| N_{y^n}(Q_{UY}) \right\} \tag{D.94}
$$

$$
\overset{(b)}{=} \mathbb{E} \left\{ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}), M_{y^n}^{\lambda}(\overline{Q}_{UXY}) = 0 \right] \bigg| N_{y^n}(Q_{UY}) \right\} \tag{D.95}
$$

$$
\overset{(c)}{=} \mathbb{E} \left\{ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \bigg| N_{y^n}(Q_{UY}) \right\} \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \bigg| M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right], \tag{D.96}
$$

where $(a)$ follows from the law of total expectation. For $(b)$ note that conditioned on both $N_{y^n}(Q_{UY}), M_{y^n}(\overline{Q}_{UXY})$, $M_{y^n}(Q_{UXY})$ is a binomial random variable pertaining to $N_{y^n}(Q_{UY}) e^{n c} - M_{y^n}(\overline{Q}_{UXY}) \leq N_{y^n}(Q_{UY}) e^{n c}$ trials. Thus, $\mathbb{E}[M_{y^n}^{1-\lambda}(Q_{UXY}) | N_{y^n}(Q_{UY}), M_{y^n}^{\lambda}(\overline{Q}_{UXY}) = \overline{\pi}]$ is a non-increasing function of $\pi$ [see (D.8)], and $(b)$ follows from Lemma 20. For $(c)$, we note that from (D.92), the conditioning on $M_{y^n}(\overline{Q}_{UXY}) = 0$ does not change the exponential order of the success probability of $M_{y^n}(\overline{Q}_{UXY})$. As evident from (D.8), this conditioning can be removed without changing the exponential order of the expression.

Proceeding with the case of $Q_{UY} = \overline{Q}_{UY}$, we next evaluate the expectation over $N_{y^n}(Q_{UY})$. We show that the asymptotic uncorrelation result of Lemma 22 holds, albeit with a correction term required when $I_Q(U; Y) > \rho_c$.

**Lemma 25.** Let $\delta > 0$, and $(Q_{UXY}, \overline{Q}_{UXY})$ be given such that $Q_{UY} = \overline{Q}_{UY}$. Then,

$$
\mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] = \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \cdot e^{n I_Q(U; Y) - \rho_c a}. \tag{D.97}
$$

**Proof:** We consider two cases separately. First suppose that $I_Q(U; Y) \leq \rho_c$. In this case, $N_{y^n}(Q_{UY})$ concentrates double-exponentially fast around its expected value, where the latter equals $\exp[n(\rho_c - I_Q(U; Y))]$ up to the first order in the exponent. Thus, the conditional expectation and the unconditional expectation are equal up to the first order in the exponent. More rigorously, let $\delta > 0$ be given and recall the definition of the event $B_n(Q_{UY}, \delta)$ in (D.2). Then,

$$
\mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] = \mathbb{P} \left[ N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right] \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \bigg| N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right].
$$
Thus, 

\[
\mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] = \mathbb{P} \left[ N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right] \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[= \mathbb{P} \left[ N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right] \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[(a) \leq e^{-n\delta} \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[(b) \leq e^{-n\delta} \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[\leq e^{2n\delta} \cdot \mathbb{P} \left[ N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right] \]

\[\times \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \geq \mathbb{E} \left[ N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[\leq e^{4n\delta} \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] ,
\]

where \((a)\) follows from the fact \(\mathbb{P}[B_n(Q_{UY}, \delta)]\) decays double-exponentially [see (D.3)], and \((b)\) follows from (D.3). Similarly

\[
\mathbb{E} \left[ M_{y^n}^{\lambda}(Q_{UXY}) \right] \geq e^{-n\delta} \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(Q_{UXY}) \right] \]

\[
\mathbb{E} \left[ M_{y^n}(Q_{UXY}) \right] = \mathbb{E} \left[ N_{y^n}(Q_{UY}) \right] = \mathbb{E} \left[ N_{y^n}(Q_{UY}) \right] \]

Thus,

\[(c) \leq e^{2n\delta} \cdot \mathbb{P} \left[ N_{y^n}(Q_{UY}) \in B_n(Q_{UY}, \delta) \right] \]

\[\times \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \geq \mathbb{E} \left[ N_{y^n}(Q_{UY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

\[(d) \leq e^{4n\delta} \cdot \mathbb{E} \left[ M_{y^n}^{1-\lambda}(Q_{UXY}) \right] \cdot \mathbb{E} \left[ M_{y^n}^{\lambda}(\overline{Q}_{UXY}) \right] \]

where \((a)\) and \((d)\) follow from (D.3), \((b)\) follows from Lemma 24 (c) from (D.8), and \((c)\) from (D.100) and (D.101).

We next address the case \(I_Q(U; Y) > \rho_c\). In this case, \(N_{y^n}(Q_{UY}) = 0\) with high probability, \(1 \leq N_{y^n}(Q_{UY}) \leq e^{2n\delta}\) with probability \(\exp\{ -n[I_Q(U; Y) - \rho_c] \}\), and \(N_{y^n}(Q_{UY}) \geq e^{2n\delta}\) with probability double-exponentially small [see (D.4) and (D.5)]. For brevity, we will use the definitions of \(\mathcal{A}_n^0(Q_{UY}), \mathcal{A}_n^1(Q_{UY})\) and \(\mathcal{A}_n^2(Q_{UY}, \delta)\) in
and the result follows from Lemma 19. Similarly, the third case follows from Lemmas 25 and 19. Specifically, the result is just as in the second case, except for the correction term |I_{Q(Y)} - \rho_{c}| to the exponent. Standard manipulations lead to the expression shown in the third case.
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