Fault-tolerant control for delayed interval type-2 fuzzy systems with nonlinear fault input

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Abstract. This paper addresses the stabilization of Takagi-Sugeno (T-S) fuzzy systems with time delay. In particular, the control system with parameter uncertainties are modeled through an interval type-2 (IT2) T-S fuzzy model, in which the uncertainties are handled via lower and upper membership functions. By developing some new techniques, a fault-tolerant controller is designed to ensure that the closed-loop fuzzy time-delay system is asymptotically stable with nonlinear fault input. Finally, a numerical example that demonstrate the effectiveness of the proposed conditions are provided.

1. Introduction

Over the past few decades many Takagi-Sugeno (T-S) fuzzy model [1,2] has became an growing interest in fuzzy control of nonlinear systems. A number of stability analysis and control design for T-S fuzzy model based on the Lyapunov function method has been reported in literature [3]. To mention a few, a stability criteria for fuzzy system with time-varying delays has been derived in [4,5]. In recent years, the stability analysis and control or filtering problems for systems with time delays or missing measurements have become an active research area.

Besides, the type-2 fuzzy set is a extension of type-1 fuzzy set [6]. In [7–9], it is pointed out that the type-2 fuzzy systems have the potential to provide better performance than type-1 fuzzy systems. The authors in [10,11] used IT2 membership functions to capture the nonlinear plants and design state feedback controllers for the IT2 T-S fuzzy systems. It should be pointed out that the controller design results are obtained for continuous-time IT2 T-S fuzzy systems and there lack some modeling, stability analysis, and control design results for discrete-time IT2 T-S fuzzy systems. Furthermore, the stability conditions for IT2 T-S fuzzy control systems are derived by state feedback approach in [12,13].

Nowadays, time delay is an important source of instability and poor performance for a control system, and is frequently encountered in many practical control systems. Less attention has been paid on IT2 T-S fuzzy systems with constant time delays because they can be transformed into fuzzy time delay systems via state augmentation approach in [14,15]. Recently, [16] addressed the stability analysis and controller design for interval type-2 fuzzy systems with time delay.

On the other hand, due to the growing demands of system reliability in practical systems, the study of reliable control which can guarantee the system stability [17,18]. It should be mentioned
that reliable control results under the assumption that the actuator fault is not always a single multiplicative fault, sometimes it couples with a certain nonlinearity, for example, the dead zone or relay is a classic actuator fault \[19, 20\]. Obviously we cannot express these feature as a multiplicative fault. Therefore, for those complicated and practical fault cases, we need to have a further study. Moreover, to the best of our knowledge, the nonlinear fault based problem for IT2 fuzzy system has not been investigated yet in the existing literature which motivates our present study. The main contributions of this paper can be highlighted as follows:

(i) This is the first attempt to investigate the fault-tolerant control design together with nonlinear for IT2 time-delay fuzzy systems.

(ii) By the implementation of Lyapunov stability theory together with fuzzy Lyapunov functional approach, a set of sufficient conditions is proposed in terms of LMIs to assure the asymptotical stability of the IT2 fuzzy time-delay system.

2. Problem Formulation and Preliminaries

A nonlinear plant with time-delay and nonlinear actuator fault subjected to parameter uncertainties is considered and spoken to by the following IT2 fuzzy model with lower and upper boundary membership functions.

**Plant rule** i: IF \(\zeta_1(x(t))\) is \(\tilde{M}_{i1}\), \(\zeta_2(x(t))\) is \(\tilde{M}_{i2}\), \ldots , \(\zeta_p(x(t))\) is \(\tilde{M}_{ip}\), THEN

\[
\dot{x}(t) = (A_i + \Delta A_i(t))x(t) + A_{di}x(t - h(t)) + B_iu^F(t),
\]

\[
x(t) = \phi(t), \quad t \in [-h, 0], \quad i = 1, 2, \cdots, r, \tag{1}
\]

where \(\tilde{M}_{ia}\) is an IT2 fuzzy set of \(i\) th rule corresponding to the function \(\zeta_a(x(t))\), \(a = 1, 2, \cdots, p\), \(i = 1, 2, \cdots, r\), \(p\) is a positive integer, \(x(t) \in \mathbb{R}^n\) is the system state vector, \(u^F(t) \in \mathbb{R}^m\) is the control input vector with fault, \(h(t)\) is the time-varying delay and satisfies \(h(t) \in (0, h]\), \(h(t) \leq \mu\), \(h\) and \(\mu\) are known positive numbers, \(\phi(t)\) is the initial condition. \(A_i \in \mathbb{R}^{n \times n}\), \(A_{di} \in \mathbb{R}^{n \times n}\) and \(B_i \in \mathbb{R}^{n \times m}\) are known input matrices, respectively. Further, \(\Delta A_i(t)\) denotes the parametric uncertainties satisfying \(\Delta A_i(t) = E_i \Delta_i(t) F_i\), where \(E_i\) and \(F_i\) are known constant matrices with appropriate dimensions, and \(\Delta_i(t)\) is an unknown time-varying matrix, which is Lebesque measurable in \(t\) and satisfies \(\Delta_i^T(t) \Delta_i(t) \leq I\). The firing strength of the \(i\) th rule is the following interval sets:

\[
\Theta_i(x(t)) = [\underline{\theta}_i(x(t)), \overline{\theta}_i(x(t))], \quad i = 1, 2, \cdots, r, \tag{2}
\]

where

\[
\underline{\theta}_i(x(t)) = \prod_{\alpha=1}^{p} \mu_{\tilde{M}_{ia}}(\zeta_a(x(t))) \geq 0, \quad \overline{\theta}_i(x(t)) = \prod_{\alpha=1}^{p} \overline{\mu}_{\tilde{M}_{ia}}(\zeta_a(x(t))) \geq 0
\]

in which \(\mu_{\tilde{M}_{ia}}(\zeta_a(x(t)))\) and \(\overline{\mu}_{\tilde{M}_{ia}}(\zeta_a(x(t)))\) denote the lower and upper membership functions respectively satisfying the property \(\overline{\mu}_{\tilde{M}_{ia}}(\zeta_a(x(t))) \geq \mu_{\tilde{M}_{ia}}(\zeta_a(x(t))) \geq 0\), and \(\underline{\theta}_i(x(t))\) and \(\overline{\theta}_i(x(t))\) indicate the lower and upper grades of membership respectively. The inferred IT2 fuzzy model can be defined as follows:

\[
\dot{x}(t) = \sum_{i=1}^{r} \tilde{\theta}_i(x(t)) \bigg( (A_i + \Delta A_i(t))x(t) + A_{di}x(t - h(t)) + B_iu^F(t) \bigg), \tag{3}
\]

where \(\tilde{\theta}_i(x(t))\) denotes the grades of membership of the embedded membership functions and

\[
\tilde{\theta}_i(x(t)) = \alpha_i(x(t)) \underline{\theta}_i(x(t)) + \overline{\mu}_i(x(t)) \overline{\theta}_i(x(t)) \geq 0 \quad \forall \ i,
\]

with \(\sum_{i=1}^{r} \theta_i(x(t)) = 1\) in which \(\alpha_i(x(t)) \in [0, 1]\), \(\overline{\mu}_i(x(t)) \in [0, 1]\) are nonlinear functions with the property that \(\alpha_i(x(t)) + \overline{\mu}_i(x(t)) = 1\).
In practical situations, some actuators of the control system often work in an abnormal status aroused from an integrated factor, such as the linear gain missing, the nonlinearity of dead zone and so on. The fault control input for the actuator deviates the true value in a complicated way including linear and nonlinear part. Here, we consider the nonlinear actuator fault model as

$$u^F(t) = E_1u(t) + g(u(t)),$$

where $0 < E_1 = \text{diag}\{e_1, e_2, \ldots, e_m\} \leq I$ and the vector function $g(u(t)) = [g_1(u(t)), g_2(u(t)), \ldots, g_m(u(t))]^T$ which satisfies $|g_i(u(t))| \leq \sqrt{E_i}|u_i(t)|$, $i \in I = \{i|i = 1, 2, \ldots, m\}$. Further, the inequality (5) can be rewritten as

$$g^T(u(t))g(u(t)) \leq u^T(t)E_2u(t),$$

where $E_2 = \text{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. An IT2 fuzzy controller with $l$ rules of the following format is proposed to stabilize the nonlinear plant spoke to by the IT2 T-S fuzzy model (3).

Control rule $j$: IF $\sigma_1(x(t))$ is $\tilde{N}_{j1}$, $\sigma_2(x(t))$ is $\tilde{N}_{j1}$, $\cdots$, $\sigma_q(x(t))$ is $\tilde{N}_{jq}$, THEN

$$u^F(t) = E_1K_jx(t) + g(u(t)),$$

where $\tilde{N}_{j\beta}$ is an IT2 fuzzy set of $j$th rule corresponding to the function $\sigma_\beta(x(t))$, $\beta = 1, 2, \cdots, q$, $j = 1, 2, \cdots, l$, $q$ is a positive integer, $K_j \in \mathbb{R}^{m \times n}$, $j = 1, 2, \cdots, l$ are the constant feedback gains to be determined. The firing strength of the $j^{th}$ rule is the following interval sets:

$$\eta_j(x(t)) = \left[\eta^L_j(x(t)), \eta^U_j(x(t))\right], \quad j = 1, 2, \cdots, l,$$

where $\eta_j(x(t)) = \prod_{\beta=1}^q \mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0$, $\eta^L_j(x(t)) = \prod_{\beta=1}^q \mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0$ in which $\mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t)))$ and $\mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t)))$ denote the lower and upper membership functions respectively satisfying the property $\mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq \mu_{\tilde{N}_{j\beta}}(\sigma_\beta(x(t))) \geq 0$, and $\eta^L_j(x(t))$ and $\eta^U_j(x(t))$ indicate the lower and upper grades of membership, respectively. The inferred IT2 fuzzy controller is defined as follows:

$$u^F(t) = \sum_{j=1}^l \tilde{\eta}_j(x(t))((E_1K_jx(t) + g(u(t))),$$

where

$$\tilde{\eta}_j(x(t)) = \frac{\beta_j(x(t)) \eta_j(x(t)) + \beta_j(x(t)) \eta_j(x(t))}{\sum_{j=1}^l (\beta_j(x(t)) \eta_j(x(t)) + \beta_j(x(t)) \eta_j(x(t)))} \geq 0$$

for all $j$ with $\sum_{j=1}^l \tilde{\eta}_j(x(t)) = 1$ in which $\beta_j(x(t)) \in [0, 1]$, $\beta_j(x(t)) \in [0, 1]$, $\beta_j(x(t)) + \beta_j(x(t)) = 1$, and $\eta_j(x(t))$ stands for the grades of membership of the embedded membership functions.

From (3)-(9), the plant and controller expression and the property of $\sum_{i=1}^r \tilde{\theta}_i(x(t)) = 1$, $\sum_{j=1}^l \tilde{\eta}_j(x(t)) = 1$, $\sum_{i=1}^r \sum_{j=1}^l \tilde{\theta}_i(x(t)) \tilde{\eta}_j(x(t)) = 1$, we obtain the closed-loop fuzzy system as

$$x(t) = \sum_{i=1}^r \sum_{j=1}^l \tilde{\theta}_{ij}(x(t)) \left( (A_i + \Delta A_i(t) + B_iE_1K_j)x(t) + A_{iq}x(t - h(t)) + B_{iq}g(u(t)) \right),$$

(11)
where $\tilde{v}_{ij}(x(t)) = \tilde{v}_i(x(t))\tilde{v}_j(x(t))$. In addition that $\tilde{v}_{ij}(x(t))$ can be reconstructed as $\gamma_{ij}(x(t))\tilde{v}_{ij}(x(t)) + \gamma_{ij}(x(t))\bar{v}_{ij}(x(t))$, in which $\gamma_{ij}(x(t)) \in [0, 1]$, $\gamma_{ij}(x(t)) \in [0, 1]$ are functions with the property that $\gamma_{ij}(x(t)) + \gamma_{ij}(x(t)) = 1$ and $\bar{v}_{ij}(x(t))$ and $\tilde{v}_{ij}(x(t))$ are the upper and lower bound of $\tilde{v}_{ij}(x(t))$ with definitions below from [7],

$$
\bar{v}_{ij}(x(t)) = \sum_{k=1}^{a} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{i=1}^{n} v_{b_{i_k}}(x_b(t)) \tilde{v}_{ij_{i_1}i_2 \cdots i_n},
$$

$$
\tilde{v}_{ij}(x(t)) = \sum_{k=1}^{a} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{i=1}^{n} v_{b_{i_k}}(x_b(t)) \tilde{v}_{ij_{i_1}i_2 \cdots i_n},
$$

where $0 \leq \tilde{v}_{ij_{1} \cdots i_n} \leq 1$ are constant scalars to be determined, $0 \leq \tilde{v}_{ij}(x(t)) \leq \tilde{v}_{ij}(x(t)) \leq 1$, $v_{b_{i_k}}(x_b(t)) \in [0, 1]$ and $v_{b_{i_k}}(x_b(t)) + v_{b_{i_k}}(x_b(t)) = 1$, otherwise $v_{b_{i_k}}(x_b(t)) = 0$.

As a result, we have

$$
\sum_{k=1}^{a} \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2} \prod_{i=1}^{n} v_{b_{i_k}}(x_b(t)) = 1,
$$

which is used in the stability analysis.

For a simple description, we use the following notation: $\tilde{v}_{ij}(x(t)) = \tilde{v}_{ij}$, where $i = 1, \ldots, r$, $j = 1, \ldots, l$.

We need to visit a fundamental lemma to be used in the following proof.

**Lemma 2.1** [21] For matrix $U = \begin{bmatrix} -R & S \\ * & -R \end{bmatrix} \leq 0$, $h(t) \in (0, h]$, and a vector function $\dot{x} : [-h, 0) \rightarrow \mathbb{R}^r$ such that the integration in the following inequality is well defined, then it holds that

$$
-h \int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq \Omega^T(t)\mathbb{W}\Omega(t),
$$

where $
\Omega^T(t) = [x^T(t) \ x^T(t-h(t)) \ x^T(t-h)], \quad \mathbb{W} = \begin{bmatrix} -R & R+S & -S \\ * & -2R-S-S^T & R+S \\ * & * & -R \end{bmatrix}.$

3. Main Results

In this section, we are in a position to discuss the stability of the IT2 fuzzy system with time-varying delay and nonlinear actuator fault. If we set $\Delta A_i(t) = 0$ in (11), then we obtain the resulting nominal fuzzy system as

$$
\dot{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{l} \tilde{v}_{ij}(x(t)) \left( A_i + B_i\Xi_k K_j x(t) + A_i x(t-h(t)) + B_i g(u(t)) \right).
$$

(12)

Specifically, we need to introduce suitable Lyapunov-Krasovskii functionals to establish new sufficient conditions that can be expressed in terms of LMIs for stability of system (12).

**Theorem 3.1** For given positive scalars $\mu$, $h$ and known actuator fault matrix $\Xi_k$, IT2 fuzzy system (12) is asymptotically stable, if there exist symmetric matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$ and $N_{ij} > 0$ of appropriate dimensions such that the following LMIs are satisfied for $i = 1, \ldots, r$, $j = 1, \ldots, l$:

$$
\begin{bmatrix}
-\tilde{R} & \tilde{S} \\
* & -\tilde{R}
\end{bmatrix} < 0,
$$

(13)
Consider the candidate of Lyapunov-Krasovskii functional (LKF) for fuzzy system (12) in the following form

\[
\sum_{i=1}^{r} \sum_{j=1}^{l} \left( \delta_{ij} x_{i-n,k} M_{ij} \right) + \left( \delta_{ij} x_{i-n,k} - \delta_{ij} x_{i-n,k} \right) \left[ \begin{array}{ccc} [\tilde{V}]_{3x3} & 0_{3x3} \\ 0_{3x3} & 0_{3x3} \end{array} \right] < 0
\]

for all \( i,j,k,i_1,i_2,\ldots, i_n \) where \( M_{11} = A_i X + B_i E_1 Y_j + X A_i^T + Y_i E_i^T B_i^T + \tilde{Q} - \tilde{R}/h, \) \( M_{12} = A_i X + (\tilde{R} + \tilde{S})/h, \) \( M_{13} = -\tilde{S}/h, \) \( M_{14} = B_i, \) \( M_{15} = \sqrt{R}(X A_i^T + Y_i E_i^T B_i^T), \) \( M_{16} = Y_i^T, \) \( M_{22} = (\mu - 1)\tilde{Q} - (\tilde{R} + \tilde{S} + \tilde{S}^T)/h, \) \( M_{23} = (\tilde{R} + \tilde{S})/h, \) \( M_{25} = \sqrt{R} A_i^T, \) \( M_{33} = -\tilde{R}/h, \) \( M_{44} = -k^{-1}I, \) \( M_{45} = \sqrt{h} B_i, \) \( M_{55} = \tilde{R} - 2X, \) \( M_{66} = -k E_2^{-1} \) and other parameter positions are zero. Moreover, if the above conditions are satisfied, then the reliable feedback controller gain matrices are given by \( K_j = Y_j X^{-1}. \)

**Proof:** Consider the candidate of Lyapunov-Krasovskii functional (LKF) for fuzzy system (12) in the following form

\[
V(x(t)) = x^T(t) P x(t) + \int_{t-h(t)}^{t} x^T(s) Q x(s) ds + \int_{t-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) d \theta.
\]

Taking the time derivative of \( V(x(t)) \) with respect to \( t \) along the trajectory of fuzzy system (12),

\[
\dot{V}(x(t)) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) + x^T(t) Q x(t) - (1-h(t))x^T(t-h(t)) Q x(t-h(t))
\]

\[
+ h \dot{x}^T(t) R \dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds.
\]

\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{l} \bar{\theta}_i(x(t)) \bar{\eta}_j(x(t)) \left[ \left( A_i + B_i E_1 K_j \right) x(t) + A_{di} x(t-h(t)) + B_i g(u(t)) \right]^T P x(t)
\]

\[
+ x(t)^T P \left( A_i + B_i E_1 K_j \right) x(t) + A_{di} x(t-h(t)) + B_i g(u(t)) \right)^T h R
\]

\[
\times \left( A_i + B_i E_1 K_j \right) x(t) + A_{di} x(t-h(t)) + B_i g(u(t)) \right) + x^T(t) Q x(t)
\]

\[
- (1-\mu) x^T(t-h(t)) Q x(t-h(t)) - \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds.
\]

Applying Lemma 2.1 in (17), the integral part of \( \dot{V}(x(t)) \) can be expressed as

\[
- \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s) ds = \frac{1}{h} \Omega^T(t) \left[ \begin{array}{ccc} -R & R+S & -S \\ * & -2R-S-S^T & R+S \\ * & * & -R \end{array} \right] \Omega(t)
\]

with \( \Omega^T(t) = [x^T(t) \ x^T(t-h(t)) \ x^T(t-h)] \) and subject to \( \left[ \begin{array}{ccc} -R & S \\ * & -R \end{array} \right] \leq 0. \)

Then, it follows from inequality (6) that

\[
\varphi^T(t) \left[ k^{-1} u^T(t) E_2 u(t) - k^{-1} g^T(u(t)) g(u(t)) \right] \varphi^T(t) \geq 0.
\]
where \( \varphi^T(t) = [x^T(t) \ x^T(t-h(t)) \ x^T(t-h) \ g^T(u(t)) \ u^T(t)] \) and \( k \) is any positive scalar. Applying Schur complement to (17) and combine with inequality (17), we can obtain the inequality as follows:

\[
\dot{V}(x(t)) \leq \sum_{i=1}^{r} \sum_{j=1}^{l} \tilde{\theta}_{ij}(x(t)) \varphi^T(t)[U_{ij}]_{5 \times 5} \varphi(t),
\]

where \( U_{1,1} = PA_i + PB_i E_1 K_j + A_i^T P + K_j^T E_1^T B_i^T P + Q - R/h + h(A_i + B_i E_1 K_j)^T R(A_i + B_i E_1 K_j)^T, \) \( U_{1,2} = PA_{ii} + (R + S)/h + h(A_i + B_i E_1 K_j)^T R A_{ii}, \) \( U_{1,3} = -S/h, \) \( U_{1,4} = PB_i + h(A_i + B_i E_1 K_j)^T R B_i, \) \( U_{1,5} = K_j^T, \) \( U_{2,2} = (\mu - 1)Q - (R + S + S^T)/h + h A_{ii}^T R A_{ii}, \) \( U_{2,3} = (R + S)/h, \) \( U_{2,4} = h A_{ii}^T R B_i, \) \( U_{2,5} = -k \mathbb{E}^{-1} \).

By using Remark 3 in [12], the above inequality can be obtain that

\[
\dot{V}(x(t)) \leq \sum_{i=1}^{r} \sum_{j=1}^{l} \tilde{\varphi}_{ij}(t) U_{ij} \varphi(t)
\]

with \( \tilde{N}_{ij} \geq 0. \) The stability condition for the closed-loop system (12) can be written as

\[
\sum_{i=1}^{r} \sum_{j=1}^{l} \varphi^T(t)[\tilde{\varphi}_{ij} U_{ij} + (\tilde{\varphi}_{ij} - \tilde{\varphi}_{ij}) N_{ij}] \varphi(t) < 0. \tag{20}
\]

Define \( X = P^{-1}, \) \( \tilde{Q} = Q X, \) \( \tilde{R} = R X, \) \( \tilde{S} = S X, \) \( -X R^{-1} X \leq R - 2P, \) \( Y_j = K_j X \) and then pre- and post-multiplying the above matrix \( U_{ij} \) by \( \text{diag}\{X, X, X, I, I\} \) and \( \tilde{N}_{ij} = \tilde{X} \tilde{N}_{ij} \tilde{X} , \tilde{X} = \text{diag}\{X, X, X\} \), the resulting inequality (20) is equivalent to (17). Thus, the closed-loop fuzzy system (12) is asymptotically stable, which completes the proof. \( \square \)

Now, we further extend the result in Theorem 3.1 to discuss reliable control design for uncertain closed-loop fuzzy system (11).

**Theorem 3.2** For given positive scalars \( \mu, h \) and known actuator fault matrix \( E_1 \), the uncertain IT2 fuzzy system (11) is asymptotically stable, if there exist symmetric matrices \( P > 0, Q > 0, R > 0, S > 0 \) and \( \tilde{N}_{ij} > 0 \) of appropriate dimensions and positive scalars \( \epsilon_i \) such that the following LMI\( s \) are satisfied for \( i = 1, \ldots, r, j = 1, \ldots, l \) together with (13) holds:

\[
\begin{bmatrix}
[\mathcal{M}_{ij}]_{6 \times 6} & \epsilon_i E_i & \mathcal{F}_i \\
* & -\epsilon_i I & 0 \\
* & * & -\epsilon_i I
end{bmatrix}
\begin{bmatrix}
\tilde{N}_{ij} & 0_{3 \times 5} \\
0_{5 \times 3} & 0_{5 \times 5}
end{bmatrix}
< 0, \quad \forall \ i, j
\tag{21}
\]

\[
\sum_{i=1}^{r} \sum_{j=1}^{l} \tilde{\varphi}_{ij}(x(t)) \begin{bmatrix}
[\mathcal{M}_{ij}]_{6 \times 6} & \epsilon_i E_i & \mathcal{F}_i \\
* & -\epsilon_i I & 0 \\
* & * & -\epsilon_i I
end{bmatrix}
+ \begin{bmatrix}
\tilde{\varphi}_{ij}(x(t)) - \tilde{\varphi}_{ij}(x(t)) \end{bmatrix}
\begin{bmatrix}
\tilde{N}_{ij} & 0_{3 \times 5} \\
0_{5 \times 3} & 0_{5 \times 5}
end{bmatrix}
< 0
\tag{22}
\]
for all $i, j, k, i_1, i_2, \ldots, i_n$, where $\mathcal{E}_i = [E_i^T 0 \ldots 0]^T$, $\mathcal{F}_i = [F_i^T X 0 \ldots 0 \sqrt{h} F_i X 0]^T$ and the other parameter positions are defined as in Theorem 3.1. Moreover, the desired robust reliable feedback controller gain matrices are given by $K_j = Y_j X^{-1}$.

**Proof:** Replacing $A_i$ with $A_i + E_i \Delta_i(t) F_i$ in LMIs (14) and (15), then by following the similar steps of Theorem 3.1, we can easily get

$$\mathcal{M}_{ij} + \mathcal{E}_i \Delta_i(t) \mathcal{F}_i + \mathcal{F}_i^T \Delta_i(t)^T \mathcal{E}_i^T < 0.$$  

(23)

By Lemma 2.6 [20], a sufficient condition guaranteeing (23) for system (11) is that there exists a positive number $\epsilon > 0$ such that

$$\mathcal{M}_{ij} + \epsilon \mathcal{E}_i \mathcal{E}_i^T + \epsilon^{-1} \mathcal{F}_i^T \mathcal{F}_i < 0.$$  

(24)

By the virtue of Schur complement Lemma in [20], it can be concluded that (24) is equivalent to LMIs (21) and (22) which completes the proof of this theorem. □

Based on the method proposed in [1] with $\mathcal{E}_i = I$ (fault is absence) and $g(u(t)) = 0$ (nonlinear existence), the following corollaries can be easily obtained.

**Corollary 3.3** For given positive scalars $\mu$ and $h$, IT2 fuzzy system (12) is asymptotically stable, if there exist symmetric matrices $P > 0, Q > 0, R > 0, S > 0$ and $N_{ij} > 0$ of appropriate dimensions such that the following LMIs are satisfied for $i = 1, \ldots, r, j = 1, \ldots, l$ together with (23) holds:

$$\begin{bmatrix} \mathcal{M}_{ij} \\ \delta_{ij12\ldots ink} \end{bmatrix} 6 \times 6 \begin{bmatrix} [\hat{N}_{ij}] 3 \times 3 & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} < 0, \ \forall \ i, j$$  

(25)

$$\sum_{i=1}^{r} \sum_{j=1}^{l} \begin{bmatrix} \delta_{ij12\ldots ink} \end{bmatrix} 6 \times 6 \begin{bmatrix} \mathcal{M}_{ij} \\ \delta_{ij12\ldots ink} \end{bmatrix} 6 \times 6 + \left( \delta_{ij12\ldots ink} - \delta_{ij12\ldots ink} \right) \begin{bmatrix} [\hat{N}_{ij}] 3 \times 3 & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} < 0$$  

(26)

for all $i, j, k, i_1, i_2, \ldots, i_n$ and the parameters are defined as in Theorem 3.1. Moreover, the feedback controller gain matrices are given by $K_j = Y_j X^{-1}$.

**Proof:** Substituting the actuator fault matrix $\mathcal{E}_i = I$ and nonlinear $g(u(t)) = 0$ in (12), then by following the similar steps of Theorem 3.1, we omit it here. □

**Corollary 3.4** For given positive scalars $\mu$ and $h$, the uncertain IT2 fuzzy system (11) is asymptotically stable, if there exist symmetric matrices $P > 0, Q > 0, R > 0, S > 0$ and $\hat{N}_{ij} > 0$ of appropriate dimensions and positive scalars $\epsilon_i$ such that the following LMIs are satisfied for $i = 1, \ldots, r, j = 1, \ldots, l$ together with (13) holds:

$$\begin{bmatrix} [\mathcal{M}_{ij}] 6 \times 6 \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} \epsilon_i \mathcal{E}_i & \mathcal{F}_i \\ -\epsilon_i I & 0 \\ 0 & -\epsilon_i I \end{bmatrix} + \begin{bmatrix} [\hat{N}_{ij}] 3 \times 3 & 0_{3 \times 5} \\ 0_{3 \times 5} & 0_{5 \times 5} \end{bmatrix} < 0, \ \forall \ i, j$$  

(27)

$$\sum_{i=1}^{r} \sum_{j=1}^{l} \left( \begin{bmatrix} \mathcal{M}_{ij} 6 \times 6 \\ * \\ * \\ * \end{bmatrix} \begin{bmatrix} \epsilon_i \mathcal{E}_i & \mathcal{F}_i \\ -\epsilon_i I & 0 \\ 0 & -\epsilon_i I \end{bmatrix} + \left( \delta_{ij12\ldots ink} - \delta_{ij12\ldots ink} \right) \begin{bmatrix} [\hat{N}_{ij}] 3 \times 3 & 0_{3 \times 5} \\ 0_{3 \times 5} & 0_{5 \times 5} \end{bmatrix} \right) < 0$$  

(28)

for all $i, j, k, i_1, i_2, \ldots, i_n$, where $\mathcal{E}_i = [E_i^T 0 \ldots 0]^T$, $\mathcal{F}_i = [F_i^T X 0 \ldots 0 \sqrt{h} F_i^T X 0]^T$ and the other parameter positions are defined as in Theorem 3.1. Moreover, the desired robust feedback controller gain matrices are given by $K_j = Y_j X^{-1}$.

**Proof:** Substituting the actuator fault matrix $\mathcal{E}_i = I$ and nonlinear $g(u(t)) = 0$ in (11), then by following the similar steps of Theorem 3.2, we omit it here. □
4. Numerical Example

In this section, we give a numerical results with simulation for the closed-loop systems for both, the Type-1 and the interval Type-2 fuzzy time delay systems with nonlinear actuator fault.

Consider a three-rule IT2 fuzzy model with time-varying delay and nonlinear actuator fault as follows

\[ \dot{x}(t) = \sum_{i=1}^{3} \sum_{j=1}^{2} \tilde{\varphi}_{ij}(x(t)) \left( (A_i + \Delta A_i(t) + B_i E_1 K_j) x(t) + A_{d_i} x(t - h(t)) + B_i g(u(t)) \right), \quad (29) \]

where

\[
A_1 = \begin{bmatrix} 2.78 & -5.63 \\ 0.01 & 0.33 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & -3.22 \\ 0.35 & 0.12 \end{bmatrix}, \quad A_3 = \begin{bmatrix} a & -6.63 \\ 0.45 & 0.15 \end{bmatrix},
\]

\[
A_{d_2} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{d_3} = \begin{bmatrix} 0.4 & 0 \\ 0 & -0.6 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -b + 6 \\ 0 \end{bmatrix},
\]

where \(22 \leq a \leq 30\) and \(20 \leq b \leq 25\). In addition, we choose \(E_1 = E_2 = E_3 = [1 \ 0]^T\), \(F_1 = F_2 = F_3 = [0 \ -0.1]\), \(\Delta_1(t) = \Delta_2(t) = \Delta_3(t) = 0.1 \cos(t)\) are uncertainties and time delay \(h(t) = 0.1(1 + 9 \sin^2 t)\). Moreover, the nonlinear assume that \(g(u(t)) = E_2 u(t) \sin(u(t))\).

On stability of interval Type-2 fuzzy systems compare with Type-1 fuzzy systems. There are two simulation tasks performed for fuzzy system (29).

**Task 1: Type-1 fuzzy system**

The IT2 fuzzy system can reduce to T1 T-S fuzzy system by setting \(\tilde{\varphi}_{ij}(x(t)) = \theta_i(x(t))\) in system (29), here \(\theta_i(x(t))\) is T1 membership function. The type-1 T-S fuzzy model is

\[ \dot{x}(t) = \sum_{i=1}^{3} \theta_i(x(t)) \left( (A_i + \Delta A_i(t)) B_i E_1 K_j x(t) + A_{d_i} x(t - h(t)) + B_i g(u(t)) \right), \quad (30) \]

where \(\theta_1(x(t)) = 0, \theta_2(x(t)) = 0.5, \theta_3(x(t)) = 0.5\) and other parameter values are defined as in system (29). Figs. 2(a) and 2(b) are indicates the solid line for state and control responses of the T1-T-S fuzzy systems and stable. Choosing \(h = 0.041\) and applying Theorem 3.2 the reliable fuzzy controller gain matrices can be found as follows: \(K_1 = [-7.9680 \ 6.9638], K_2 = [-14.4867 \ 7.5217]\) and \(K_3 = [9.9511 \ -5.9216]\).

**Task 2: Interval Type-2 fuzzy system**

The lower and upper membership functions in (2) of the IT2 fuzzy system (plant) are defined in TABLE 1. By setting the constants \(\alpha_1(x(t)) = 0.4, \alpha_2(x(t)) = 0.5, \alpha_3(x(t)) = 0.6, \bar{\alpha}_1(x(t)) = 0.6, \bar{\alpha}_2(x(t)) = 0.5\) and \(\bar{\alpha}_3(x(t)) = 0.4\), we can obtain the membership functions of the IT2 fuzzy system (3) according to the representation in (29), which are shown in Fig 1(a). Next, according to the description in (8), the lower and upper membership function in (8) of the IT2 fuzzy controller are defined in TABLE 2. From (10), by choosing the constants \(\beta_j(x(t)) = 0.5\) and \(\bar{\beta}_j(x(t)) = 0.5\) \((j = 1, 2)\), we can acquire the membership functions of the IT2 fuzzy state feedback controller, which are appeared in Fig 1(b).

For Theorem 3.2 uncertain case with nonlinear actuator fault, if we set \(h = 0.0327, k = 0.2, E_1 = 0.7 I, E_2 = 10 \) and \(\mu = 0.1\), then by Theorem 3.2 the state feedback controller gains can be obtained as \(K_1 = [-0.9199 \ 1.4740]\) and \(K_2 = [-2.7097 \ 0.6634]\). Under the initial condition \(x(0) \in [-10 \ 10]^T\), Fig 3(a) and 3(b) shows the state and control response of the closed-loop system.
system (29) which is asymptotically stable. Fig. 3(a) depicts the state responses of the open-loop system in (29) is not stable. In Fig. 3(b), the broken line gives the upper bound value of $h$ corresponding to different $\mu$. Moreover, TABLE III present the upper bound value of $h$ for different values of $\mu$.

For Theorem 3.1 nominal case with nonlinear actuator fault, it is assumed that $h = 0.0300$ and the remaining parameters are are selected as same in Fig 2. Then, the LMIs (13)-(15) in Theorem 3.1 can be solved recursively by using MATLAB and a set of feasible solutions is obtained. From the obtained solutions, the state feedback control gain matrices are calculated as $K_1 = \begin{bmatrix} -0.7422 & 1.0956 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -2.4048 & 0.5776 \end{bmatrix}$. Fig. 4, the broken line gives the state and control response of the closed-loop system (29) except for $E_i \Delta_i(t) F_i$ ($i = 1, 2, 3$).

For Corollary 3.3 nominal case, if we set $h = 0.4211$ and the remaining parameters are selected as same in Fig 2, then the state feedback controller gains can be obtained as $K_1 = \begin{bmatrix} -0.8508 & 0.9184 \end{bmatrix}$ and $K_2 = \begin{bmatrix} -0.7416 & 0.3126 \end{bmatrix}$. In Fig. 4, simulation results for closed-loop system (29) with $\Delta A_i(t) = 0$ and absence of nonlinear fault is indicates solid line.

| Table 1. | Lower membership functions | Upper membership functions |
|-----------------|----------------------------|-----------------------------|
| $\theta_1(x_1) = 1 - 1/(1 + e^{-(x_1+4+0.25)})$ | $\theta_1(x_1) = 1 - 1/(1 + e^{-(x_1+4-0.25)})$ |
| $\theta_2(x_1) = 1 - \theta_1(x_1) - \theta_3(x_1)$ | $\theta_2(x_1) = 1 - \theta_1(x_1) - \theta_3(x_1)$ |
| $\theta_3(x_1) = 1/(1 + e^{-(x_1-4-0.25)})$ | $\theta_3(x_1) = 1/(1 + e^{-(x_1-4+0.25)})$ |

| Table 2. | Lower membership functions | Upper membership functions |
|-----------------|----------------------------|-----------------------------|
| $\eta_1(x_1) = 1 - \frac{1}{e^{-(x_1+0.15)}}$ | $\eta_1(x_1) = 1 - \frac{1}{e^{-(x_1-0.15)}}$ |
| $\eta_2(x_1) = 1 - \eta_1(x_1)$ | $\eta_2(x_1) = 1 - \eta_1(x_1)$ |

| Table 3. | Calculated $h$ for various values of $\mu$ (Theorem 3.2) |
|-----------------|-----------------------------|
| $\mu$ | 0.02 | 0.04 | 0.06 | 0.08 | 0.10 |
| $h$ | 0.0295 | 0.0307 | 0.0316 | 0.0321 | 0.0327 |

**Remark 4.1** In this example we show that our control design provides a special case for Theorem 3.1 and Corollary 3.3 control design is shown in Fig. 4(b). Comparison results for Theorems and Corollaries are given below:

| Actuator fault | Nonlinear | Uncertain |
|----------------|-----------|-----------|
| Theorem 3.2    | ✓         | ✓         | ✓         |
| Theorem 3.3    | ✓         | ✓         | ✓         |
| Corollary 3.3  | ×         | ×         | ×         |
| Corollary 3.4  | ×         | ×         | ✓         |
Figure 1. Simulation results for the membership functions of the plant and controller

Figure 2. Simulation results for closed-loop fuzzy system

Figure 3. Simulation results for fuzzy system
5. Conclusion
This paper focuses on the challenges of theoretical underpinning including notation, learning type-2 sets and systems. The problem of state-feedback controller design for a class of IT2 fuzzy time delay system with nonlinear actuator fault has been investigated. By using the Lyapunov-Krasovskii functional approach and LMIs techniques, a sufficient condition is derived such that the closed loop IT2 T-S fuzzy systems asymptotically stable under the nonlinear actuator fault. From the given example, one can see that the IT2 T-S fuzzy system with nonlinear actuator failure can still maintain a certain control performance by using the designed fault-tolerant controller. Another objective of this paper is to derive a novel uncertainty measure for IT2 membership functions with clearer presentation. In the future, the practical application and standard measures of type-2 fuzzy sets will be an in-depth study in order to solve more and more practical problems.

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