SUPERGRAVITY DRESSING OF $\beta$-FUNCTIONS IN $N=1$ AND $N=2$ SUPERSYMMETRIC MODELS

M. T. Grisaru

Physics Department, Brandeis University, Waltham, MA 02254, USA

and

D. Zanon

Dipartimento di Fisica dell’Università di Milano and INFN, Sezione di Milano,
Via Celoria 16, I-20133 Milano, Italy

ABSTRACT

We discuss the dressing of one-loop $\sigma$-model $\beta$-functions by induced supergravity, for both $N = 1$ and $N = 2$ supersymmetric theories. We obtain exact results by a superconformal gauge argument, and verify them in the semi-classical limit by explicit perturbative calculations in the light-cone gauge. We find that for $N = 2$ theories there is no dressing of the one-loop $\beta$-functions.
Recently, a number of papers have appeared dealing with the gravitational dressing of one-loop $\beta$-functions for two-dimensional systems away from their conformal point \[1, 2, 3, 4\]. Although using different approaches, they conclude that the effect of induced gravity \[5\] is to rescale the one-loop $\beta$-functions by an overall factor

$$\beta_G = \frac{\kappa + 2}{\kappa + 1} \beta$$

where $\kappa$ is the central charge of the gravitational $SL(2R)$ current algebra. An alternative way of writing this result is

$$\beta_G = -\frac{2}{\alpha_+ Q} \beta$$

where

$$Q = \sqrt{\frac{25 - c}{3}}$$

$$\alpha_+ = \frac{1}{2} \left[ -Q + \sqrt{Q^2 - 8} \right]$$

This result appears to be universal, holding for conformal field theories perturbed by some marginal operator, or for two-dimensional $\sigma$-models away from their fixed point. Beyond one loop the situation is less clear, although indications exist that the dressing may not be universal \[3\].

The one-loop result has been obtained by presenting induced gravity in conformal gauge, i.e. in its Liouville incarnation \[1, 3\], in light-cone gauge using the nonlocal form of the induced action \[4\], and by studying the problem in $2 + \epsilon$ dimensions \[4\]. The first method attributes the effect to the difference between the scale defined by the fiducial metric, and that defined by the physical metric determined by the Liouville mode. The second method relies on the induced gravity Ward identities \[7\] and the corresponding dressing of correlation functions. The third method is similar to the first, insofar as it relies on the presence of the (induced) renormalized cosmological constant.

In this work we extend the above results to the cases of (1, 1) and (2, 2) supergravity. We present a general argument, similar in spirit to that of the above references, relying on the general KPZ \[8\] and DDK \[9, 10, 11\] results, and we supplement it with perturbative verifications.

The general argument in (super)conformal gauge, for ordinary, $N = 0$, gravity, and for $N = 1$ or $N = 2$ supergravity is based on the following idea: the dressed $\beta$-functions are defined by the response of systems to changes in the physical scale.
which in the presence of the Liouville field gets modified with respect to the standard renormalization scale. In two-dimensions, a sensible definition of the physical scale is provided by the cosmological constant, the only dimensionful object in the theory. In light-cone gauge the cosmological term is just a $c$-number, so that the usual scaling is physical and modifications of the matter $\beta$-functions arise through new divergent contributions due to the gravitational couplings. In conformal gauge instead, the one-loop matter divergence does not receive gravitational corrections. However, in this case, for $N = 0$ and $N = 1$ theories the cosmological constant is renormalized by quantum corrections:

$$\Lambda_0 \int d^2 z d^2 N \theta \ e^\phi \rightarrow \Lambda_R \int d^2 z d^2 N \theta : e^{\alpha_+ \phi} :$$

where

$$\Lambda_0 = \mu^{2s} \Lambda_R Z$$

In the above relations $\mu$ is the renormalization mass, $Z = (\mu^2 a^2)^{\alpha^2 \over 2}$, and $\alpha_+$ is the positive root of

$$-\frac{1}{2} \alpha (\alpha + Q) = s$$

i.e.

$$\alpha_+ = \frac{1}{2} \left[ -Q + \sqrt{Q^2 - 8s} \right]$$

For the $N = 0, 1$ theories one has

$$\begin{align*}
N &= 0 : \quad s = 1 \quad Q = \sqrt{\frac{25 - c}{3}} \\
N &= 1 : \quad s = \frac{1}{2} \quad Q = \sqrt{\frac{9 - c}{2}}
\end{align*}$$

From Eq. (6) we obtain

$$\frac{\partial \ln \Lambda_R}{\partial \ln \mu} = -(2s + \alpha_+^2) = \alpha_+ Q$$

Therefore

$$\beta_G = \frac{\partial \ln \mu}{\partial \ln (\Lambda_R)^{1/2\pi}} \beta = -\frac{2s}{\alpha_+ Q} \beta$$

Using the above expressions, one finds for the ordinary gravity case, $N = 0$, the result in Eq. (1). For $N = 1$, using also the expression for the level $\kappa$ of the light-cone supergravity Kač-Moody algebra

$$N = 1 : \quad \kappa + \frac{3}{2} = \frac{1}{8} \left[ c - 5 - \sqrt{(1 - c)(9 - c)} \right]$$
one finds
\[ \beta_G = \frac{\kappa + 3\beta}{\kappa + 1} \] (13)

For the \( N = 2 \) theories the cosmological term, written as a chiral integral, is not renormalized, and therefore the physical scale coincides with the renormalization scale. Hence for \( N = 2 \) there is no supergravity dressing of \( \beta \)-functions.

These results can be verified perturbatively. We calculate in the semiclassical limit \( c \to -\infty \) when the predicted dressing becomes

\[ \begin{align*}
N &= 0 : \quad \beta_G \to (1 + \frac{6}{c})\beta \\
N &= 1 \quad \beta_G \to (1 + \frac{2}{c})\beta \\
N &= 2 \quad \beta_G \to \beta
\end{align*} \] (14)

We concentrate on the gravitational dressing of one-loop \( \beta \)-functions for \( \sigma \)-models and to begin with we consider the case of bosonic theories. We work in light-cone gauge where the only nonvanishing component of the gravitational field is \( h_{-\bar{z}} \equiv h \) and the induced action is

\[ S_{\text{ind}} = -\frac{c}{24\pi} \int d^2x \left( \frac{\partial^2 h_{-\bar{z}}}{\partial \bar{z}^2} \right) \frac{1}{1 - \partial^{-1}_{-\bar{z}} \partial^{-1}_{\bar{z}}} \partial^z \partial^{-1}h_{-\bar{z}} \]

\[ = -\frac{c}{24\pi} \int d^2x \left[ h_{-\bar{z}} \partial^z h - h \left( \partial^z h \right)^2 - \left( h \partial^z h \right) \partial^z \left( h \partial^z h \right) + \cdots \right] \] (15)

We are using the conventions of ref. \[12\] with space-time light-cone coordinates denoted by \( x^\xi \) and \( x^\bar{z} \).

We consider a bosonic \( \sigma \)-model parametrized by scalar fields \( \phi^i(x) \) and a target manifold metric \( g_{ij}(\phi) \), and described by the action (in the presence of light-cone induced gravity)

\[ S[\phi] = -\frac{1}{2} \int d^2x \ g_{ij}(\phi) \left( \partial^z \partial^\xi - h_{-\bar{z}} \partial^z \phi^i \right) \partial^z \phi^j \] (16)

We perform a conventional \( \beta \)-function calculation by expanding the action in normal coordinates

\[ S = S[\phi] - \frac{1}{2} \int d^2x g_{ij} \left( \partial^z \xi^i - h_{-\bar{z}} \partial^z \xi^i - 2h \partial^z \phi^i \right) \partial^z \xi^j \]

\[ - \frac{1}{2} \int d^2x R_{ik\ell j} \left( \partial^z \phi^i - h_{-\bar{z}} \partial^z \phi^i \right) \partial^z \phi^j \xi^k \xi^\ell \]

\[ - \frac{1}{3} \int d^2x R_{ik\ell j} \left( \partial^z \phi^i \partial^z \xi^j + \partial^z \phi^j \partial^z \xi^i \right) \xi^k \xi^\ell + \cdots \] (17)
We note a subtlety: in a conventional quantum-background splitting of the covariantized lagrangian
\[ g_{ij} \nabla_a \phi^i \nabla_a \phi^j \rightarrow g_{ij} [\nabla_a \phi^i \nabla_a \phi^j + 2 \nabla_a \phi^i \nabla_a \xi^j + \nabla_a \xi^i \nabla_a \xi^j + \cdots] \]
one would be tempted to drop the middle term because it is linear in the quantum field \( \xi \). However, the covariant derivative contains the quantum field \( h \); hence the middle term is quadratic in quantum fields and must be kept.

As it is standard in \( \sigma \)-model quantum calculations we refer the \( \xi^i \) fields to tangent space frames, i.e. \( \xi^a = e^a_i(\phi) \xi^i \). The momentum space propagators are (in our light-cone conventions \( i\partial \rightarrow \frac{1}{2} q \))
\[
<\xi^a \xi^b> = 4\delta^{ab} \frac{1}{q^\pm}
\]
\[
<h h> = -\frac{4\pi}{c} \frac{q^\pm}{q^3}
\]
(18)

The one-loop \( \beta \)-function in the absence of gravity is obtained from the divergence of the tadpole diagram in Fig.1a, generated by Wick-contracting the two quantum fields in the second line of Eq. (17):
\[
\Gamma_0 = -\frac{1}{2} \int d^2 x R_{ij} \partial_\xi \phi^i \partial_\xi \phi^j \int d^2 q \frac{4}{(2\pi)^2} \frac{4}{q^\pm}
\]
(19)

The tadpole integral, after separating out an IR divergence, leads to the usual \( 1/\epsilon \) UV divergence.

![Fig. 1. Loop diagrams for the calculation of beta-functions in the N=0 and N=1 theories.](image)

We obtain gravitational corrections to \( \mathcal{O}(1/c) \) by evaluating the diagrams in Fig.1b,c,d. However, it is trivial to verify (and follows from Lorentz invariance) that the contribution from the diagram in Fig.1c vanishes. The contribution from Fig.1b
is given by

$$\Gamma^{(b)}_1 = -\frac{1}{2} \int d^2 x R_{ij} \partial_i \phi^j \partial_j \phi^i$$

\[
\times \int \frac{d^2 q d^2 p}{(2\pi)^4} \left( -\frac{48\pi}{c} \right) \frac{4}{q_- (q-p)^-} p_- (q-p)^+ (q-p)^- 
\]

(20)

We perform the $p$ integral using the methods and the table of integrals of ref. [12], in particular (A.5)

\[
\int d^2 p \frac{(q-p)^\pm_p}{(q-p)^\pm_p} = \frac{\pi q_-}{2q_+} 
\]

(21)

and we find that $\Gamma^{(b)}_1 = -(6/c) \Gamma_0$.

From Fig. 1d we obtain the contribution

$$\Gamma^{(d)}_1 = \frac{1}{3} \int d^2 x R_{ij} \partial_i \phi^j \partial_j \phi^i$$

\[
\times \int \frac{d^2 q d^2 p}{(2\pi)^4} \left( -\frac{48\pi}{c} \right) \frac{4}{q_- (p-q)^-} p_- \left[ \frac{(k-p)^\pm}{p^\pm} - \frac{1}{2} (k-p)^- \right] 
\]

(22)

which leads to $\Gamma^{(d)}_1 = (12/c) \Gamma_0$.

Altogether we obtain therefore

$$\beta_0 + \beta_1 = \left( 1 + \frac{6}{c} \right) \beta_0$$

(23)

which agrees in the large $c$ limit with the value obtained from the exact analysis.

We consider now the dressing of $\beta$-functions in $N = 1$ supersymmetric $\sigma$-models coupled to induced supergravity. In light-cone gauge $N = 1$ supergravity is described by the superfield $H^\pm$ [13]. The other geometrical quantities are given by

$$\nabla_+ = D_+$$

$$\nabla_- = D_- + iH^\pm \partial_\mp - \frac{1}{2} (D_+ H^\pm) D_+ + i(\partial_\mp H^\pm) M$$

$$R = iD_+ \partial_\mp H^\pm$$

$$E = 1$$

(24)

($M$ is a Lorentz generator) with $\nabla_\pm = -i(\nabla_\pm)^2$ and $\nabla_\mp = -i(\nabla_\mp)^2$. The induced action has the form

$$S_{ind} = -\frac{c}{4\pi} \int d^2 x d^2 \theta R \frac{\nabla_+ \nabla_-}{\Box - \nabla_\alpha R \nabla_\alpha} R$$

1Note some misprints: in the right hand side of (A.3) the exponent should be $n-1$ and the right hand side of (A.7) should contain a factor $1/p_-$; the right hand side of (A.9) should be multiplied by $(-1)^n p_-^{n-1}$. 

5
\[ S = -\frac{1}{2} \int d^2 x d^2 \theta g_{ij} \nabla_+ \Phi^i \nabla_- \Phi^j \]
\[ \rightarrow -\frac{1}{2} \int d^2 x d^2 \theta g_{ij} \left( \nabla_+ \xi^i \nabla_- \xi^j + \nabla_+ \Phi^i \nabla_- \xi^j - \nabla_- \Phi^i \nabla_+ \xi^j \right) \]
\[ -\frac{1}{2} \int d^2 x d^2 \theta R_{ik\ell j} \nabla_+ \Phi^i \nabla_- \Phi^j \xi^k \xi^\ell \]
\[ -\frac{1}{3} \int d^2 x d^2 \theta R_{ik\ell j} \left( \nabla_+ \Phi^i \nabla_- \xi^j - \nabla_- \Phi^i \nabla_+ \xi^j \right) \xi^k \xi^\ell + \cdots \]  

We emphasize again that the terms linear in \( \xi \) do lead to quadratic quantum field contributions and must be kept.

The propagators are
\[ < \xi^a \xi^b > = 4 \delta^{ab} \frac{D_+ D_-}{q_{*)/=} \]
\[ < H^*_+ H^*_+ > = \frac{64 \pi}{c} \frac{D_+ D_-}{p^3_+} \]  

and the relevant diagrams are again the ones in Fig. 1.

The standard one-loop \( \beta \)-function is obtained from the tadpole diagram in Fig.1a which, after \( D \)-algebra, gives
\[ \Gamma_0 = -\frac{1}{2} \int d^2 x d^2 \theta R_{ij} \nabla_+ \Phi^i \nabla_- \Phi^j \int \frac{d^2 q}{(2\pi)^2} \frac{4}{q_{*)/=} q_{*}^2 \]  

The dressing is provided by the diagrams in Fig.1b,c,d but again Fig.1c gives no contribution. For Fig.1b the relevant vertices are \(-\frac{1}{2} R_{ik\ell j} \nabla_+ \Phi^i \nabla_- \Phi^j \xi^k \xi^\ell \) and \( \frac{1}{2} i H^*_+ D_+ \xi^i \partial^*_+ \xi^i \). This diagram, because of three distinct Wick contractions gives three contributions which, after \( D \)-algebra, lead to the following result:
\[ \Gamma_1^{(b)} = \frac{1}{2} \int d^2 x d^2 \theta R_{ij} \nabla_+ \Phi^i \nabla_- \Phi^j \int \frac{d^2 q d^2 p}{(2\pi)^4} \frac{4 q_{*}}{q_{*}^2} \]
\[ \times \frac{32 \pi}{c} \left[ \frac{(q - p)_+}{q_{*} p^2_+ (q - p)_=} + \frac{q_{*}}{q_{*}^2 (q - p)_=} + 2 \frac{(q - p)_+}{q_{*} p^2_+ (q - p)_=} \right] \]
\[ = \frac{1}{2 c} \int d^2 x d^2 \theta R_{ij} \nabla_+ \Phi^i \nabla_- \Phi^j \int \frac{d^2 q}{(2\pi)^2} \frac{8}{q_{*)/=} \]
\[ = -\frac{2}{c} \Gamma_0 \]  

\[ = \frac{ic}{16\pi} \int d^2 x d^2 \theta H^*_+ \frac{\partial^2}{\partial^2 z} D_+ D_- H^*_+ + \cdots \]  

The \( \sigma \)-model action and normal coordinate expansion take a form similar to that of the bosonic model:
For the diagram in Fig. 1d the relevant vertices are \( \frac{1}{3} R_{ik\ell j} \nabla_i \Phi^j D_+ \xi^i \xi^k \xi^\ell \), 
\( -\frac{i}{2} \nabla_+ \Phi^i \partial_+ \Phi^j \xi^i H_\Phi \), and 
\( -\frac{i}{2} D_+ \xi^i \partial_+ \Phi^j H_\Phi \). The diagram, after doing the \( D \)-algebra and keeping only divergent contributions gives

\[
\frac{1}{4} \int d^2 x d^2 \theta R_{ij} \nabla_i \Phi^j \nabla_i \Phi^j \int d^2 q d^2 p \left( \frac{-128\pi}{c} \right) \frac{1}{q=p^2(p-q)}
\]

Therefore

\[
\beta_0 + \beta_1 = \left( 1 + \frac{2}{c} \right) \beta_0
\]

which agrees with the exact result in the \( c \to -\infty \) limit.

Finally, we consider the \( N = 2 \) case. The \( N = 2 \) \( \sigma \)-model is described as usual by a Kähler potential and, including the coupling to supergravity, the action takes the form

\[
S = \int d^2 x d^4 \theta E^{-1} K(e^{iH\cdot \partial}, e^{-iH\cdot \bar{\partial}})
\]

The induced supergravity action is

\[
S_{ind} = \frac{c}{2\pi} \int d^2 x d^4 \theta R \frac{1}{\Box + \ldots \bar{R}}
\]

where the \( \cdots \) indicate curvature dependent terms. At the linearized level we have the explicit expressions \[14\] \[34\]

\[
E^{-1} = 1 - [\bar{D}_+, D_+] H_\pm - [\bar{D}_-, D_-] H_\pm
\]

\[
R = 4 \bar{D}_+ \bar{D}_- [\bar{\sigma} + D_+ D_+ H_\pm + D_- \bar{D}^- H_\pm]
\]

\[
\bar{R} = 4 D_+ D_- [\sigma - \bar{D}_+ \bar{D}_+ H_\pm - \bar{D}_- \bar{D}^- H_\pm]
\]

where the vector superfield \( H \) and the chiral compensator \( \sigma \) are the supergravity prepotentials. The linearized gauge transformations are

\[
\delta H_\pm = D_+ \bar{L}_+ - \bar{D}_+ L_+
\]

\[
\delta H_\pm = D_- \bar{L}_- - \bar{D}_- L_-
\]

\[
\delta \sigma = \bar{D}^2 (D_+ L_+ - D_- L_-)
\]

\[
\delta \bar{\sigma} = D^2 (\bar{D}_+ \bar{L}_- - \bar{D}_- \bar{L}_+)
\]

We go to a partial light-cone gauge by gauging away \( H_\pm \) and the compensator \( \sigma, \bar{\sigma} \). In this gauge the quadratic part of the induced action takes the form

\[
S^{(2)}_{ind} = \frac{2c}{\pi} \int d^2 x d^4 \theta H_\pm \frac{\partial_\pm}{\partial_\pm} \bar{D}^2 D^2 H_\pm
\]

\[2\]The general solution of the constraints of \( N = 2 \) supergravity is given in Ref. \[14\].
It still has a residual gauge invariance but, for the present purpose, instead of using it to gauge away a part of $H_\pm$, we prefer to maintain explicit $N=2$ supersymmetry and do ”covariant” gauge fixing. This leads to propagating ghosts as well; however, since they do not couple to the fields $\Phi$, they play no role in our calculation. After gauge-fixing we obtain

$$S^{(2)}_{\text{ind}} = \frac{c}{\pi} \int d^2x d^4\theta \ H_\pm \bar{\partial}_\Phi^2 H_-$$  \hspace{1cm} (37)$$

We recall [16] that in the absence of supergravity the computation of $\beta$-functions for $N=2 \sigma$-models involves a straightforward background-field expansion of the Kähler potential $K(\Phi, \bar{\Phi}) \rightarrow K(\Phi + \xi, \bar{\Phi} + \bar{\xi})$ leading to a quadratic quantum-field term $K_{\Phi \Phi} \xi \bar{\xi} = \xi \bar{\xi} + (K_{\Phi \Phi} - 1)\xi \bar{\xi}$ with the usual chiral propagator

$$\langle \xi \bar{\xi} \rangle = -\frac{4D^2 \bar{D}^2}{q \bar{q}}$$  \hspace{1cm} (38)$$

As discussed in ref. [16], in order to compute divergences all the $D$’s and $\bar{D}$’s have to stay inside the loops, and the summation over the $K_{\Phi \Phi} - 1$ vertices leads to an effective propagator which contains one factor of the inverse Kähler metric $K^{-1}_{\Phi \Phi}$.

At one-loop the $\beta$-function is obtained from the UV divergent contribution to the Kähler potential

$$\Gamma_0 = \int d^2x d^4\theta \sum_{n=1}^{\infty} \frac{(-1)^n (K_{\Phi \Phi} - 1)^n}{n} \int \frac{d^2q}{(2\pi)^2} \frac{4}{q \bar{q}}$$

$$= -\int d^2x d^4\theta \ln(K_{\Phi \Phi}) \int \frac{d^2q}{(2\pi)^2} \frac{4}{q \bar{q}}$$  \hspace{1cm} (39)$$

Fig. 2. Two-loop diagram for the N=2 theory.

We now consider the gravitational couplings and discuss the two-loop situation with one supergravity-field exchange. The relevant interaction, to lowest order in $H_\pm$ is [14]

$$L_{\text{int}} = 2H_\pm D_+ \xi \bar{D}_+ \bar{\xi} K_{\Phi \Phi}$$  \hspace{1cm} (40)$$

The diagram of interest is given in Fig. 2 where we have explicitly indicated the dependence on the background fields at the vertices and in the matter effective propagators. (The diagram similar to that in Fig. 1d gives no divergent contributions.)
Since, as emphasized above, one obtains a divergence only when spinor and space-time derivatives stay in the loops, irrespective of the details of the loop integrations the dependence on the $\Phi$ fields cancels completely and no correction to the Kähler potential is produced by the supergravity coupling. Thus in accordance with the general argument presented earlier, there is no correction to the one-loop $\beta$-function in the $N = 2$ $\sigma$-model.

Acknowledgments This research is partially supported by the National Science Foundation under grant PHY-92-22318. D. Zanon thanks NSF and MURST, and M. T. Grisaru thanks INFN for support. D. Zanon thanks the Physics Department of Harvard University and M.T. Grisaru the Physics Department of Università di Milano for hospitality during the period when some of this work was done.

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