A SELF-CONTAINED ACCOUNT OF WHY
THOMPSON’S GROUP $F$ IS OF TYPE $F_\infty$

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Abstract. In 1984 Brown and Geoghegan proved that Thompson’s group $F$ is of type $F_\infty$, making it the first example of an infinite dimensional torsion-free group of type $F_\infty$. Over the decades a different, shorter proof has emerged, which is more streamlined and generalizable to other groups. It is difficult, however, to isolate this proof in the literature just for $F$ itself, with no complicated generalizations considered and no additional properties proved. The goal of this expository note then is to present the “modern” proof that $F$ is of type $F_\infty$, and nothing else.

Introduction and History. A classifying space for a group $G$ is a CW complex $Y$ with $\pi_1(Y) \cong G$ and $\pi_k(Y) = 0$ for all $k \neq 1$. If $G$ admits a classifying space with finite $n$-skeleton, we say $G$ is of type $F_n$. Equivalently, $G$ is of type $F_n$ if it admits a free, cocompact, cellular action on an $(n-1)$-connected CW complex. Being of type $F_1$ is equivalent to being finitely generated, and being of type $F_2$ is equivalent to being finitely presented. We say $G$ is of type $F_\infty$ if it is of type $F_n$ for all $n$.

Thompson’s group $F$ was the first example of a torsion-free group of type $F_\infty$ with no finite dimensional classifying space. The original proof that $F$ is of type $F_\infty$ was given by Brown and Geoghegan in [BG84]. Brown subsequently found a new proof in [Bro87], which generalized more easily to variations of $F$. This proof approach was then simplified and further generalized over the years by Stein [Ste92], Farley [Far03], and others, in a variety of applications to families of “Thompson-like” groups. There are too many examples of this to list here, but lists of such examples can be found in, e.g., [SWZ19, Wit19].

By now a comparatively short, easy proof that $F$ is of type $F_\infty$ exists, thanks to all this work over the years, but isolating it in the literature is difficult. Many (but not all) of the most important steps can be found in [Geo08 Section 9.3] or [Bro92]. Also, one can sort out the full “modern” $F_\infty$ proof for $F$ from the (long) $F_\infty$ proof for the braided Thompson groups in [BFM+16], but this requires quite a bit of effort.

The purpose of this note then is to present the most modern form of the $F_\infty$ proof for Thompson’s group $F$, and only for $F$, with no other groups considered and no other properties proved. The target audience is people interested in understanding the most basic situation, just for $F$, before venturing into more complicated generalizations.

Acknowledgments. Thanks are due to a number of people for encouraging me to write this up, including Brendan Mallery, David Rosenthal, Rachel Skipper, Rob Spahn, and Marco Varisco. This work is supported by grant #635763 from the Simons Foundation.
A: Trees and forests. Throughout this note, a tree will mean a finite rooted binary tree. A forest is a disjoint union of finitely many trees. The roots and leaves of a tree or forest are always ordered. The trivial tree is the tree with 1 leaf (which is also its root). A trivial forest is a forest each of whose trees is trivial. We denote the trivial forest with $n$ roots (and hence $n$ leaves) by $1_n$. If we want to avoid specifying $n$, we will just write $1 = 1_n$.

A caret is a tree with 2 leaves. Given a forest $f$, a simple expansion of $f$ is a forest obtained by adding one new caret to $f$, with the root of the caret identified with a leaf of $f$. If it is the $k$th leaf, this is the $k$th simple expansion of $f$ (see Figure 1). An expansion of $f$ is recursively defined to be $f$ or a simple expansion of an expansion of $f$. Note that if $f$ and $f'$ have the same number of roots then (and only then do) they have a common expansion. For example any two trees have a common expansion.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{tree_expansion.png}
\caption{A tree, and a simple expansion of the tree (namely the 2nd simple expansion).}
\end{figure}

B: The group. A tree pair is a pair $(t_-, t_+)$ where $t_\pm$ are trees with the same number of leaves. A simple expansion of a tree pair is a tree pair $(t'_-, t'_+)$ such that there exists $k$ where $t'_\pm$ is the $k$th simple expansion of $t_\pm$. An expansion of $(t_-, t_+)$ is recursively defined to be $(t_-, t_+)$ or a simple expansion of an expansion of $(t_-, t_+)$. Thompson’s group $F$ is the set of equivalence classes $[t_-, t_+]$ of tree pairs $(t_-, t_+)$, with the equivalence relation generated by $(t_-, t_+) \sim (t'_-, t'_+)$ whenever $(t'_-, t'_+) \text{ is an expansion of } (t_-, t_+)$ (for more details on this and some equivalent definitions of $F$ see, e.g., [CFP96, Bel04]).

The point of expansions is that one can multiply equivalence classes $[t_-, t_+]$ and $[u_-, u_+]$ by expanding until without loss of generality $t_+ = u_-$, and then $[t_-, t_+][u_-, u_+] := [t_-, u_+]$. In this way, $F$ is a group. The identity is $[1_1, 1_1]$ and the inverse of an element $[t_-, t_+]$ is $[t_+, t_-]$.

C: The groupoid. A groupoid is a set with all the axioms of a group except the product $gh$ need not necessarily be defined for every pair of elements $g, h$. A standard example is the set of all square matrices, where two elements can be multiplied if and only if they have the same dimension. Thompson’s group $F$ naturally lives in the groupoid where we generalize trees to forests, which we describe now.

A forest pair is a pair $(f_-, f_+)$ where $f_\pm$ are forests with the same number of leaves. An expansion of a forest pair is defined analogously to an expansion of a tree pair, and we define equivalence of forest pairs similarly to equivalence of tree pairs. Let $F$ be the set
of all equivalence classes \([f_-, f_+]\) of forest pairs \((f_-, f_+)\). Since any two forests with the same number of roots have a common expansion, we can multiply two elements \([f_-, f_+]\) and \([e_-, e_+]\) of \(\mathcal{F}\) provided the number of roots of \(f_+\) and \(e_-\) are the same. In this case we expand until \(f_+ = e_-\) and then \([f_-, f_+][e_-, e_+] := [f_-, e_+]\). In this way, \(\mathcal{F}\) is a groupoid. Note that the group \(F\) is a subgroupoid of \(\mathcal{F}\).

**D: The poset.** Define a *split* to be an element of \(\mathcal{F}\) of the form \([f, 1]\). For \([f_-, f_+] \in \mathcal{F}\), declare that \([f_-, f_+] \leq [f_-, f_+][f, 1]\) for any split \([f, 1]\) such that this product is defined.

**Lemma 1.** The relation \(\leq\) is a partial order.

*Proof.* Clearly \(\leq\) is reflexive, since any \([1_n, 1_n]\) is a split. A product of splits is itself a split, because any forest with \(n\) roots is an expansion of \(1_n\), so \(\leq\) is transitive. Finally, a product of non-trivial splits is non-trivial since any expansion of a non-trivial forest is non-trivial, so \(\leq\) is antisymmetric. \(\Box\)

Let \(\mathcal{F}_1\) be the subset of \(\mathcal{F}\) consisting of all \([t, f]\) for \(t\) a tree (and \(f\) a forest with the same number of leaves as \(t\)). The groupoid product on \(\mathcal{F}\) restricts to a left action of \(F\) on \(\mathcal{F}_1\). It is clear that \(\leq\) restricts to \(\mathcal{F}_1\), and that this partial order on \(\mathcal{F}_1\) is \(F\)-invariant, since left multiplication by an element of \(F\) commutes with right multiplication by a split. In this way, \(\mathcal{F}_1\) is an \(F\)-poset.

The *geometric realization* \(|\mathcal{P}|\) of a poset \(\mathcal{P}\) is the simplicial complex with a simplex for every chain \(x_0 < \cdots < x_k\) of elements \(x_i \in \mathcal{P}\), with face relation given by taking subchains. A poset is *directed* if any two elements have a common upper bound. It is a standard fact that the geometric realization of a directed poset is contractible.

**Lemma 2.** The poset \(\mathcal{F}_1\) is directed, and so the geometric realization \(|\mathcal{F}_1|\) is contractible.

*Proof.* Note that \([t, f][f, 1] = [t, 1]\), so any element of \(\mathcal{F}_1\) has an upper bound of the form \([t, 1]\) for \(t\) a tree. Given two such elements \([t, 1]\) and \([u, 1]\), let \(v\) be a common expansion of \(t\) and \(u\), and now \([v, 1]\) is a common upper bound of \([t, 1]\) and \([u, 1]\). \(\Box\)

Since the action of \(F\) on \(\mathcal{F}_1\) is order preserving, it induces a simplicial action of \(F\) on the contractible complex \(|\mathcal{F}_1|\).

**Lemma 3.** The action of \(F\) on \(|\mathcal{F}_1|\) is free.

*Proof.* The action of \(F\) on \(|\mathcal{F}_1|^{(0)} = \mathcal{F}_1\) is free, since it is an action of a subgroup of a groupoid on the groupoid by left translation. Since the action of \(F\) on \(\mathcal{F}_1\) is order preserving, the stabilizer of the simplex \(x_0 < \cdots < x_k\) lies in the stabilizer of \(x_0\), hence is trivial. \(\Box\)
E: The Stein complex. In [Bro87] Brown used the action of $F$ on $|F_1|$ to give a new proof that $F$ is of type $F_\infty$, which generalized to many additional groups. The topological analysis in [Bro87] was still quite complicated though. The complex $|F_1|$ deformation retracts to a smaller, more manageable subcomplex $X$ now called the Stein complex. This complex was first constructed by Stein in [Ste92] (also see [Bro92]), and simplified the $F_\infty$ proof for $F$ in [Bro87] quite a bit.

To define $X$ we need the notion of “elementary” forests, splits, and simplices. First, call a forest $f$ elementary if every tree in $f$ is either trivial or a single caret (see Figure 2). Call a split $[f, 1]$ elementary if $f$ is an elementary forest. If $x \in F_1$ and $s$ is a split, so $x \leq xs$, then write $x \preceq xs$ if $s$ is an elementary split. (Note that $\preceq$ is reflexive and antisymmetric, but not transitive.) Call a simplex $x_0 < \cdots < x_k$ in $|F_1|$ elementary if $x_i \preceq x_j$ for all $i < j$.

The elementary simplices form a subcomplex $X$, called the Stein complex. Note that $X$ is invariant under the action of $F$.

Figure 2. An example of an elementary forest and a non-elementary forest.

**Proposition 4.** The Stein complex $X$ is homotopy equivalent to $|F_1|$, hence is contractible.

**Proof.** Given a forest $f$, there is a unique maximal elementary forest with $f$ as an expansion, namely the elementary forest whose $k$th tree is non-trivial (hence a caret) if and only if the $k$th tree of $f$ is non-trivial, for each $k$. Call this the elementary core of $f$, denoted core($f$). Note that if $f$ is non-trivial then so is core($f$). If $\epsilon = \text{core}(f)$, call $[\epsilon, 1]$ the elementary core of $[f, 1]$, and write core($[f, 1]$) := $[\epsilon, 1]$. Now let $x \leq z$ with $x \not\leq z$, and consider $(x, z) := \{y \mid x < y < z\}$. Since any $y \in (x, z)$ is of the form $xs$ for $s$ a non-trivial split, we can define a map $\phi: (x, z) \to (x, z)$ via $\phi(xs) := x \text{core}(s)$. This is clearly a poset map that restricts to the identity on its image, and satisfies $\phi(y) \leq y$ for all $y$. Finally, note that $\phi(y) \leq \phi(z) \in (x, z)$ for all $y$. Standard poset theory (see, e.g., [Qui78, Section 1.5]) now tells us that $(x, z)$ is contractible (intuitively, $\phi$ “retracts” it to a cone on the point $\phi(z)$).

Now our goal is to build up from $X$ to $|F_1|$ by gluing in the missing simplices, in such a way that whenever we add a new simplex it is along a contractible relative link, which will imply that $X \simeq |F_1|$. The missing simplices are precisely the non-elementary ones. Let us actually glue in all the non-elementary simplices in chunks, by gluing in (contractible) subcomplexes of the form $\{|y \mid x \leq y \leq z\}$ for $x < z$ non-elementary. We glue these in, in order of increasing $f(z) - f(x)$ value, where $f: F_1 \to \mathbb{N}$ sends $[t, f]$ to the number of roots of $f$. (Think of $f(z) - f(x)$ as the number of carets in the split taking $x$ to $z$.) When we glue in $\{|y \mid x \leq y \leq z\}$, the relative link is $\{|y \mid x \leq y < z\} \cup \{|y \mid x < y \leq z\}$. This
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is the suspension of $|\{y \mid x < y < z\}|$, which is contractible by the first paragraph of the proof.

\[\square\]

Note that the action of $F$ on $|\mathcal{F}_1|$ restricts to an action of $F$ on $X$.

**F: The Stein–Farley cube complex.** The Stein complex $X$ is easier to use than $|\mathcal{F}_1|$, but there is one further simplification that makes it still easier, namely, the simplices of $X$ can be glommed together into cubes, making $X$ a cube complex. This was observed by Stein in [Ste92] and further developed in [Far03], where $X$ was shown to even be a CAT(0) cube complex.

Given $x \preceq z$, say $z = xs$ for $s = [e, 1]$ an elementary split, the set $\{y \mid x \leq y \leq z\}$ is a boolean lattice. This is because the forests $e'$ with $x[e', 1] \leq x[e, 1]$ are all obtained by assigning a 0 or a 1 to each caret in $e$ and including said caret in $e'$ if and only if it was assigned a 1. The geometric realizations of these boolean lattices, which are metric cubes, cover $X$, and any non-empty intersection of such cubes is itself such a cube, so in this way $X$ has the structure of a cube complex. When we view $X$ as a cube complex instead of a simplicial complex, we will call it the Stein–Farley complex.

The action of $F$ on $X$ takes cubes to cubes, so $F$ acts cellularly on the Stein–Farley complex.

**G: Sublevel complexes.** At this point we have a free cellular action of $F$ on the contractible cube complex $X$. If the action were cocompact, then we would be done proving $F$ is of type $F_\infty$. In fact the action is not cocompact, but $X$ does admit a natural filtration into cocompact subcomplexes that are increasingly highly connected, as we now explain.

Let $f : \mathcal{F}_1 \to \mathbb{N}$ be the function from the proof of Proposition 4, so $f([t, f])$ equals the number of roots of $f$. Note that $f$ is $F$-invariant. For each $m \in \mathbb{N}$ let $X^{f \leq m}$ be the full subcomplex of $X$ spanned by vertices $x \in X^{(0)} = \mathcal{F}_1$ with $f(x) \leq m$. The $X^{f \leq m}$ are called sublevel complexes. Note that the $X^{f \leq m}$ are nested, and their union is all of $X$, so they form a filtration of $X$. Each $X^{f \leq m}$ is $F$-invariant.

**Lemma 5.** Each $X^{f \leq m}$ is cocompact under the action of $F$.

**Proof.** Since there are only finitely many elementary forests with a given number of roots or a given number of leaves, $X$ is locally finite. Hence it suffices to show $X^{f \leq m}$ has finitely many $F$-orbits of vertices, and for this we claim that if $x, x' \in X^{(0)}$ with $f(x) = f(x')$ then $F.x = F.x'$. Indeed, $f(x) = f(x')$ ensures that $x'.x^{-1}$ is an allowable product in $\mathcal{F}$, and clearly $x'.x^{-1} \in F$ with $(x'.x^{-1})x = x'$.

To summarize, for each $m \in \mathbb{N}$, $F$ acts freely, cocompactly, and cellularly on $X^{f \leq m}$. To show that $F$ is of type $F_\infty$, i.e., of type $F_n$ for all $n$, it just remains to show that for each $n$ there exists $m$ such that $X^{f \leq m}$ is $(n - 1)$-connected.

Let $\nu(m) := \left\lfloor \frac{m - 2}{3} \right\rfloor$.

**Proposition 6.** The complex $X^{f \leq m}$ is $(\nu(m) + 1) - 1)$-connected.
We will prove Proposition 6 shortly. First let us see why we will be done after this.

**Theorem 7.** \( F \) is of type \( F_{\infty} \).

**Proof.** For each \( m \in \mathbb{N} \), \( F \) acts freely, cocompactly, and cellularly on the \((\nu(m + 1) - 1)\)-connected complex \( X f \leq m \). Hence \( F \) is of type \( F_{\nu(m+1)} \) for all \( m \). Since \( \nu(m + 1) \) goes to \( \infty \) as \( m \) goes to \( \infty \), \( F \) is of type \( F_{\infty} \). \( \square \)

**H: Descending links.** To prove Proposition 6 we will use Bestvina–Brady Morse theory (see [BB97]). This is admittedly a slight violation of our claim that this note is “self-contained”, but the machinery is very standard by now. Given an affine cell complex \( Y \), e.g., a simplicial or cube complex, a map \( h: Y \to \mathbb{R} \) is a *Morse function* if \( h \) is affine on cells, non-constant on edges, and discrete on vertices. Given a Morse function \( h: Y \to \mathbb{R} \) and a cell \( C \) in \( Y \), \( h \) achieves its maximum value on \( C \) at a unique vertex, called the *top* of \( C \). The *descending link* \( \text{lk}^D y \) of a vertex \( y \in Y^{(0)} \) is the link of \( y \) in all the cells with \( y \) as their top. The point of Morse theory is that a sufficient understanding of descending links can translate into knowledge about sublevel complexes (see [BB97, Corollary 2.6]).

**Proof of Proposition 6.** We can extend \( f: X^{(0)} \to \mathbb{N} \) to a map \( f: X \to \mathbb{R} \) by extending affinely to each cube, and this is a Morse function. Since \( X \) is contractible, it now suffices by [BB97, Corollary 2.6] to prove that for every \( x \in X^{(0)} \) with \( f(x) > m \), the descending link \( \text{lk}^D x \) is \((\nu(m + 1) - 1)\)-connected. The descending link of \( x \) is the simplicial complex with a \( k \)-simplex for each \( x' = x[1, \epsilon] \), where \( \epsilon \) is an elementary forest with \( k + 1 \) caret and \( f(x) \) leaves, with face relation given by removing caret. (For this, it is important that we are using the cubical structure on \( X \), not the simplicial structure.) If \( f(x) = n \) then this is isomorphic to the matching complex on the graph \( L_n \). Here \( L_n \) is the graph with vertex set \{1, \ldots, n\} and an edge \{i, i + 1\} for each \( 1 \leq i \leq n - 1 \), and the *matching complex* \( \mathcal{M}(\Gamma) \) of a graph \( \Gamma \) is the simplicial complex with a simplex for each non-empty finite collection of pairwise disjoint edges of \( \Gamma \) with face relation given by inclusion (see Figure 3).

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**Figure 3.** The correspondence between \( \text{lk}^D x \) with \( f(x) = 5 \) (the top picture, with the forests \( \epsilon \) representing the simplices) and \( \mathcal{M}(L_5) \) (the bottom picture).
Since $n > m$, it now suffices to show that $\mathcal{M}(L_n)$ is $(\nu(n) - 1)$-connected. This is well known (see, e.g., [Koz08, Proposition 11.16]), but it is easy to prove so we present a proof here. We will induct on $n$ to prove that this holds, and that moreover $\mathcal{M}(L_n)$ is contractible whenever $n \equiv 2 \mod 3$, and that the inclusion $\mathcal{M}(L_{n-1}) \to \mathcal{M}(L_n)$ is a homotopy equivalence whenever $n \equiv 1 \mod 3$. As a base case we can check “by hand” that $\mathcal{M}(L_n)$ is non-empty (i.e., $(-1)$-connected) for $n \geq 2$, $\mathcal{M}(L_2)$ is contractible, and $\mathcal{M}(L_3) \to \mathcal{M}(L_4)$ is a homotopy equivalence. Now assume $n \geq 5$. Clearly $\mathcal{M}(L_n)$ is isomorphic to $\mathcal{M}(L_{n-1})$ union the star of $\{n-1, n\}$, and the intersection of $\mathcal{M}(L_{n-1})$ with this star is $\mathcal{M}(L_{n-2})$. Hence $\mathcal{M}(L_n)$ is homotopy equivalent to the mapping cone of the inclusion $\mathcal{M}(L_{n-2}) \to \mathcal{M}(L_{n-1})$. If $n \equiv 0, 1 \mod 3$ then $\nu(n-1) = \nu(n)$, so $\mathcal{M}(L_{n-1})$ is $(\nu(n)-1)$-connected, and moreover $\mathcal{M}(L_{n-2})$ is $(\nu(n)-2)$-connected, so $\mathcal{M}(L_n)$ is $(\nu(n)-1)$-connected (this follows for example from Van Kampen, Mayer–Vietoris, and Hurewicz). If $n \equiv 2 \mod 3$ then the inclusion $\mathcal{M}(L_{n-2}) \to \mathcal{M}(L_{n-1})$ is a homotopy equivalence, so $\mathcal{M}(L_n)$ is contractible. Lastly, if $n \equiv 1 \mod 3$ then $\mathcal{M}(L_{n-2})$ is contractible, so the inclusion $\mathcal{M}(L_{n-1}) \to \mathcal{M}(L_n)$ is a homotopy equivalence. □

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