HARMONIC APPROXIMATION OF DIFFERENCE OPERATORS

MARKUS KLEIN AND ELKE ROSENBERGER

Abstract. For a general class of difference operators \( H_\varepsilon = T_\varepsilon + V_\varepsilon \) on \( \ell^2((\varepsilon\mathbb{Z})^d) \), where \( V_\varepsilon \) is a multi-well potential and \( \varepsilon \) is a small parameter, we analyze the asymptotic behavior as \( \varepsilon \to 0 \) of the (low-lying) eigenvalues and eigenfunctions. We show that the first \( n \) eigenvalues of \( H_\varepsilon \) converge to the first \( n \) eigenvalues of the direct sum of harmonic oscillators on \( \mathbb{R}^d \) located at the several wells. Our proof is microlocal.

1. Introduction

The central topic of this paper is the investigation of a rather general class of families of difference operators \( H_\varepsilon \) on the Hilbert space \( \ell^2((\varepsilon\mathbb{Z})^d) \), as the small parameter \( \varepsilon > 0 \) tends to zero. The operator \( H_\varepsilon \) is given by

\[
H_\varepsilon = (T_\varepsilon + V_\varepsilon), \quad \text{where} \quad T_\varepsilon = \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} a_\gamma \tau_\gamma,
\]

\[
(\tau_\gamma u)(x) = u(x + \gamma) \quad \text{and} \quad (a_\gamma u)(x) := a_\gamma(x, \varepsilon)u(x) \quad \text{for} \quad x, \gamma \in (\varepsilon\mathbb{Z})^d
\]

where \( V_\varepsilon \) is a multiplication operator, which in leading order is given by \( V_0 \in \mathcal{C}^\infty(\mathbb{R}^d) \).

We will show that the low lying spectrum of \( H_\varepsilon \) on \( \ell^2((\varepsilon\mathbb{Z})^d) \) is in the limit \( \varepsilon \to 0 \) asymptotically given by the spectrum of an adapted harmonic oscillator on \( \mathcal{L}^2(\mathbb{R}^d) \). We remark that the limit \( \varepsilon \to 0 \) is analogous to the semiclassical limit \( \hbar \to 0 \) for the Schrödinger operator \( -\hbar^2\Delta + V \). The central result of this paper (the validity of the harmonic approximation) is the first basic step in any WKB-theory for the Schrödinger operator (see e.g. Simon [20], Helffer-Sjöstrand [12]). In our case, this basic step is considerably more difficult. The discrete kinetic operator \( T_\varepsilon \) is not a local operator (in particular, not a differential operator). Furthermore, \( H_\varepsilon \) and the approximating harmonic oscillator act on different spaces. We remark that this is in fact crucial: Letting \( H_\varepsilon \) act on \( L^2(\mathbb{R}^d) \) would lead to infinite multiplicity of the point spectrum. In addition, the proofs for the Schrödinger operator in Simon [20], Helffer-Sjöstrand [12] use special identities for harmonic operators of second order. To overcome these difficulties, we use a microlocal approach.

The basic theorems necessary for our analysis are proven in Appendix A.

This paper is based on the thesis Rosenberger [19]. It is the second in a series of papers (see Klein-Rosenberger [15]): the aim is to develop an analytic approach to the semiclassical eigenvalue problem and tunneling for \( H_\varepsilon \) which is comparable in detail and precision to the well known analysis for the Schrödinger operator (see Simon [20], [6] and Helffer-Sjöstrand [12]). Our motivation comes from stochastic problems (see Klein-Rosenberger [15], Bovier-Eckhoff-Gayraud-Klein [3], [4], Baake-Bovier-Klein [2]). A large class of discrete Markov chains analyzed in [4] with probabilistic techniques falls into the framework of difference operators treated in this article.

We assume

Hypothesis 1.1 (a) The coefficients \( a_\gamma(x, \varepsilon) \) in (1.3) are functions
\[
a : (\varepsilon\mathbb{Z})^d \times \mathbb{R}^d \times (0, 1] \to \mathbb{R}, \quad (\gamma, x, \varepsilon) \mapsto a_\gamma(x, \varepsilon),
\]
satisfying the following conditions:

(i) They have an expansion
\[
a_\gamma(x, \varepsilon) = a_\gamma^{(0)}(x) + \varepsilon a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x, \varepsilon),
\]

where \( a_\gamma^{(i)} \in \mathcal{C}^\infty(\mathbb{R}^d) \) and \( |a_\gamma^{(j)}(x) - a_\gamma^{(j)}(x + h)| = O(|h|) \) for \( j = 0, 1 \) uniformly with respect to \( \gamma \in (\varepsilon\mathbb{Z})^d \) and \( x \in \mathbb{R}^d \). Furthermore \( R_\gamma^{(2)} \in \mathcal{C}^\infty(\mathbb{R}^d \times (0, 1]) \) for all \( \gamma \in (\varepsilon\mathbb{Z})^d \).

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Hypothesis 1.1

Remark 1.2

A combination of the expansion \( (d) \) (b) (i) (c) (1.4) \( \varepsilon \) \( \varepsilon \) \( B \) where

\[
\sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a^{(0)}_\gamma(x) \leq 0 \quad \text{for} \quad \gamma \neq 0
\]

\[
a_\gamma(x, \varepsilon) = a_{-\gamma}(x + \gamma, \varepsilon) \quad \text{for} \quad x \in \mathbb{R}^d, \; \gamma \in (\varepsilon \mathbb{Z})^d
\]

(iv) For any \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^d \) there exists a \( C > 0 \) such that for \( j = 0, 1 \) uniformly with respect to \( x \in (\varepsilon \mathbb{Z})^d \) and \( \varepsilon \)

\[
\| (\varepsilon \delta_x a^{(j)}_\gamma(x)) \|_{L^2((\varepsilon \mathbb{Z})^d)} \leq C \quad \text{and} \quad \| (\varepsilon \delta_x R^{(2)}(x, \varepsilon)) \|_{L^2((\varepsilon \mathbb{Z})^d)} \leq C \varepsilon^2.
\]

(v) \( \text{span} \{ \gamma \in (\varepsilon \mathbb{Z})^d | a^{(0)}_\gamma(x) < 0 \} = \mathbb{R}^d \) for all \( x \in \mathbb{R}^d \).

(b) (i) The potential energy \( V_\varepsilon \) is the restriction to \((\varepsilon \mathbb{Z})^d\) of a function \( \tilde{V}_\varepsilon \in C^\infty(\mathbb{R}^d) \), which has an expansion

\[
\tilde{V}_\varepsilon(x) = V_0(x) + \varepsilon V_1(x) + R_2(x; \varepsilon),
\]

where \( V_0, V_1 \in C^\infty(\mathbb{R}^d), \; R_2 \in C^\infty(\mathbb{R}^d \times (0, \varepsilon_0]) \) for some \( \varepsilon_0 > 0 \) and for any compact set \( K \subset \mathbb{R}^d \) there exists a constant \( C_K \) such that \( \sup_{x \in K} |R_2(x; \varepsilon)| \leq C_K \varepsilon^2 \).

(ii) \( V_\varepsilon \) is polynomially bounded and there exist constants \( R, C > 0 \) such that \( V_\varepsilon(x) < C \) for all \( |x| \geq R \) and \( \varepsilon \in (0, \varepsilon_0] \).

(iii) \( V_0 \geq 0 \) and it takes the value 0 only at a finite number of points \( \{ x_j \}_{j=1}^m \), where its Hessian

\[
\left( \partial_{x_\mu}^2 V_0 \right) := \frac{1}{2} \left( \frac{\partial^2 V_0}{\partial x_\mu \partial x_\nu}(x_j) \right)
\]

is positive definite (i.e. the absolute minima are non-degenerate). We call the minima \( \{ x_j \}_{j=1}^m \) of \( V_0 \) potential wells.

We set for \( \varepsilon \in (0, 1] \)

\[
t(x, \xi, \varepsilon) := \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a_\gamma(x, \varepsilon) \exp \left( \frac{-i}{\varepsilon} \gamma \cdot \xi \right), \quad x \in \mathbb{R}^d, \xi \in T^d := \mathbb{R}^d / (2\pi \mathbb{Z})^d,
\]

and denote the function on \( \mathbb{R}^{2d} \times [0, 1) \), which is \( 2\pi \)-periodic with respect to \( \xi \) by \( t \) as well. The expansion (1.4) of \( a_\gamma(x, \varepsilon) \) leads to the definition

\[
t(x, \xi, \varepsilon) = t_0(x, \xi) + \varepsilon t_1(x, \xi) + t_2(x, \xi; \varepsilon), \quad \text{with} \quad t_j(x, \xi) := \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a_\gamma(x) e^{-\frac{j}{\varepsilon} \gamma \cdot \xi}, \quad j = 0, 1
\]

\[
t_2(x, \xi, \varepsilon) := \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} R_\gamma(x, \varepsilon) e^{-\frac{2}{\varepsilon} \gamma \cdot \xi}.
\]

Hypothesis 1.1

(c) At the minima \( x_j \) of \( V_0 \), we assume that \( t_0 \) defined in (1.9) fulfills

\[
t_0(x_j, \xi, \varepsilon) > 0, \quad \text{if} \quad |\xi| > 0.
\]

Remark 1.2

It follows from (the proof of) Klein-Rosenberger [15], Lemma 1.2, that under the assumptions given in Hypothesis (1.4)

(a) \( t \in C^\infty(\mathbb{R}^d \times T^d \times [0, 1)) \) and \( \sup_{x, \xi} |\partial_\xi^\alpha t(x, \xi, \varepsilon)| \leq C_{\alpha, \beta} \) for all \( \alpha, \beta \in \mathbb{N}^d \) uniformly with respect to \( \varepsilon \). Moreover \( t_0 \) and \( t_1 \) are bounded and \( \sup_{x, \xi} |t_2(x, \xi, \varepsilon)| = O(\varepsilon^2) \).

(b) At \( \xi = 0 \), for fixed \( x \in \mathbb{R}^d \) the function \( t_0 \) defined in (1.9) has an expansion

\[
t_0(x, \xi) = \langle \xi, B(x) \xi \rangle + O (|\xi|^4) \quad \text{as} \quad |\xi| \to 0,
\]

where \( B : \mathbb{R}^d \to \mathcal{M}(d \times d, \mathbb{R}) \) is positive definite and symmetric. By straightforward calculations one gets

\[
B_{\mu \nu}(x) = -\frac{1}{2\varepsilon^2} \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a^{(0)}_\gamma(x) \gamma_\mu \gamma_\nu.
\]

(c) By Hypothesis (1.4) (a)(iii) and since the \( a_\gamma \) are real, the operator \( T_\varepsilon \) defined in (1.11) is symmetric. In the probabilistic context, which is our main motivation, the latter is a standard reversibility condition while the former is automatic for a Markov chain. Moreover, \( T_\varepsilon \) is bounded (uniformly in \( \varepsilon \)) by condition (a)(iv) and bounded from below by \( -C \varepsilon \) for some \( C > 0 \) by condition (a)(iv),(iii) and (ii).

(d) A combination of the expansion (1.4) and the reversibility condition (a)(iii) leads to the fact that the \( 2\pi \)-periodic function \( \mathbb{R}^d \ni \xi \mapsto t_0(x, \xi) \) is even.
(e) Since $T_\varepsilon$ is bounded, $H_\varepsilon = T_\varepsilon + V_\varepsilon$ defined in (1.11) possesses a self adjoint realization on the maximal domain of $V_\varepsilon$. Abusing notation, we shall denote this realization also by $H_\varepsilon$ and its domain by $\mathcal{D}(H_\varepsilon) \subset L^2((\varepsilon\mathbb{Z})^d)$. The associated symbol is denoted by $h(x;\xi;\varepsilon)$. Clearly, $H_\varepsilon$ commutes with complex conjugation.

We will use the notation
\[ \tilde{a} : \mathbb{Z}^d \times \mathbb{R}^d \ni (\eta,x) \mapsto \tilde{a}_\eta(x) := a^{(0)}_{\eta}(x) \in \mathbb{R} \] (1.12)
and set
\[ \tilde{h}_0(x,\xi) := -h_0(x,i\xi) = \tilde{t}_0(x,\xi) - V_0(x) : \mathbb{R}^{2d} \rightarrow \mathbb{R} , \] (1.13)
where by Remark 1.2 (d)
\[ \tilde{t}_0(x,\xi) := -t_0(x,i\xi) = - \sum_{\eta \in \mathbb{Z}^d} \tilde{a}_\eta(x) \cosh(\eta \cdot \xi) . \] (1.14)

The main result of this paper is the following theorem:

**Theorem 1.3** Let $H_\varepsilon$ be an operator satisfying Hypothesis 1.1 and let $A^j := B_j^\frac{1}{2} \hat{A}^j B_j^\frac{1}{2}$, where $\hat{A}^j$ is given in (1.7) and $B_j = B(x_j)$ is defined in (1.6). We denote by
\[ K_j := -\Delta + \langle x,A^j x \rangle + V_1(x_j) + t_1(x_j,0) , \quad j = 1, \ldots, m \] (1.15)
the self adjoint operators on $L^2(\mathbb{R}^d)$ defined by Friedrich extension and set $K := \bigoplus_{j=1}^m K_j$ (which is self adjoint on $\bigoplus_{j=1}^m L^2(\mathbb{R}^d)$).

Then for any fixed $n \in \mathbb{N}^*$ and $\varepsilon$ sufficiently small, $H_\varepsilon$ has at least $n$ eigenvalues. Counting multiplicity, we denote for $k \in \mathbb{N}^*$ the $k$-th eigenvalue of $K$ by $e_k$ and the $k$-th eigenvalue of $H_\varepsilon$ by $E_k(\varepsilon)$ (ordered by magnitude). Then, as $\varepsilon \to 0$,
\[ E_k(\varepsilon) = \varepsilon e_k + O(\varepsilon^{\frac{2}{5}}) . \] (1.16)

We remark that (under additional assumptions) Theorem 1.3 considerably sharpens Theorem 1 in Baake-Baake-Bovier-Klein [2].

The strategy of the proof of Theorem 1.3 is to restrict the Hamilton operator $H_\varepsilon$ to small $\varepsilon^{\frac{2}{5}}$-scaled neighborhoods of its critical points in $x$ and $\xi$, i.e. to neighborhoods of $\{(x_j,0)\}_{j=1}^m$ in phase space. Then restricted to these regions, the difference operator can be compared with a corresponding differential operator acting on $L^2(\mathbb{R}^d)$.

We follow in part the ideas of the proof of Theorem 11.1 in Cycon-Froese-Kirsch-Simon [6] on the quasi-classical eigenvalue limit of a Schrödinger operator. But in contrast to this proof, our difference operator $T_\varepsilon$ depends on both position and momentum and acts on a different space than the harmonic oscillator. The first step of the proof consists in localizing the operator simultaneously with respect to $x$ and $\xi$, which is done by use of a version of microlocal calculus adapted to the discrete setting as introduced in Definition 2.1. These localized operators still act on $\ell^2((\varepsilon\mathbb{Z})^d)$. The second step consists in comparing the localized operators on $\ell^2((\varepsilon\mathbb{Z})^d)$ with the associated localized operators on $L^2(\mathbb{R}^d)$, which are standard pseudo-differential operators. With these preparations, the remaining part of the proof follows closely the arguments in Simon [20].

The plan of the paper is as follows. We introduce in Section 2 some notations, define symbol-spaces on $\mathbb{R}^d \times \mathbb{T}^d$ and associated operators and state some essential results concerning these symbols and operators. In Section 3 we state and prove lemmata, which are essential ingredients for the proof of Theorem 1.3 Proposition 3.1 Lemma 3.3 and Lemma 3.6 contain the main estimates on the error introduced by localizing the relevant operators on $\ell^2((\varepsilon\mathbb{Z})^d)$ and $L^2(\mathbb{R}^d)$. Lemma 3.7 estimates the difference between these operators. The proof of Theorem 1.3 is finally given in Section 4. Appendix A is concerned with pseudo-differential operators in the discrete setting. In particular, we collect some properties of symbols and prove the $\ell^2$-continuity of pseudodifferential operators with symbols in $S^0(1)(\mathbb{R}^d \times \mathbb{T}^d)$ (see Definition 2.1). In Section 4 we show an analog of the Theorem of Persson for some class of difference operators.
2. Notations and Preliminaries

For $\varepsilon > 0$, we consider $\ell^2 \left( (\varepsilon \mathbb{Z})^d \right)$, the space of square summable functions on the $\varepsilon$-scaled lattice, with scalar product

$$
(u, v)_{\ell^2} := \sum_{x \in (\varepsilon \mathbb{Z})^d} \hat{u}(x)v(x), \quad u, v \in \ell^2 \left( (\varepsilon \mathbb{Z})^d \right). \quad (2.1)
$$

Denoting the $d$-dimensional torus by $\mathbb{T}^d := \mathbb{R}^d/(2\pi \mathbb{Z})^d$, we identify functions in $L^2(\mathbb{T}^d)$ with periodic functions in $L^2_{\text{loc}}(\mathbb{R}^d)$. Then

$$
(f, g)_{\mathbb{T}} := \int_{[-\pi, \pi]^d} \tilde{f}(\xi)g(\xi)\,d\xi,
$$

(2.2)
denotes the scalar product in $L^2(\mathbb{T}^d)$. We denote the associated norms by $\| \cdot \|_{\ell^2}$ and $\| \cdot \|_{\mathbb{T}}$.

The discrete Fourier transform $\mathcal{F}_\varepsilon : L^2(\mathbb{T}^d) \rightarrow \ell^2 \left( (\varepsilon \mathbb{Z})^d \right)$ is defined by

$$
(f)_{\mathbb{T}} := \frac{1}{\sqrt{2\pi}} \int_{[-\pi, \pi]^d} e^{-ix\cdot\xi}f(\xi)\,d\xi, \quad f \in L^2(\mathbb{T}^d),
$$

(2.3)
with inverse $\mathcal{F}_\varepsilon^{-1} : \ell^2 \left( (\varepsilon \mathbb{Z})^d \right) \rightarrow L^2(\mathbb{T}^d)$,

$$
(\mathcal{F}_\varepsilon^{-1}v)(\xi) := \frac{1}{\sqrt{2\pi}} \sum_{x \in (\varepsilon \mathbb{Z})^d} e^{ix\cdot\xi}v(x), \quad v \in \ell^2 \left( (\varepsilon \mathbb{Z})^d \right),
$$

(2.4)
where $x \cdot y := (x, y) := \sum_{j=1}^d x_jy_j$ for $x, y \in \mathbb{R}^d$ and points in $\mathbb{T}^d$ are identified with points in $[-\pi, \pi]^d$. Then $\mathcal{F}_\varepsilon$ is an isometry, i.e.,

$$
\langle v, u \rangle_{\ell^2} = \langle \mathcal{F}_\varepsilon^{-1}v, \mathcal{F}_\varepsilon^{-1}u \rangle_{\mathbb{T}}, \quad u, v \in \ell^2 \left( (\varepsilon \mathbb{Z})^d \right).
$$

(2.5)

On $L^2(\mathbb{R}^d)$ we denote by $\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^d} f(\xi)g(\xi)\,d\xi$ the standard scalar product and we introduce the $\varepsilon$-scaled Fourier transform

$$
(F_\varepsilon^{-1}f)(\xi) := (\varepsilon \sqrt{2\pi})^{-d} \int_{\mathbb{R}^d} e^{i\xi\cdot x}f(x)\,dx \quad (2.6)
$$

and

$$
(F_\varepsilon u)(x) := (\sqrt{2\pi})^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x}u(\xi)\,d\xi,
$$

where compared to the usual Fourier transform the roles of $x$ and $\xi$ are interchanged. We notice that for any $f, g \in L^2(\mathbb{R}^d)$

$$
\langle F_\varepsilon^{-1}f \mid F_\varepsilon^{-1}g \rangle_{L^2(\mathbb{R}^d)} = \varepsilon^{-d} \langle f \mid g \rangle_{L^2(\mathbb{R}^d)}.
$$

(2.7)

We write $\langle x \rangle := \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^d$.

We introduce the symbol-spaces $S(m)(\mathbb{R}^d \times \mathbb{T}^d)$ and $S^k(m)(\mathbb{R}^d \times \mathbb{T}^d)$ depending on the small parameter $\varepsilon \in (0, 1]$ following Dimassi-Sj" ostrand [7]. A corresponding symbolic calculus is introduced in Appendix A.

**Definition 2.1**
(a) A function $m : \mathbb{R}^d \times \mathbb{T}^d \rightarrow [0, \infty)$ is called an order function, if there exist constants $C_0, N_1 > 0$ such that

$$
m(x, \xi) \leq C_0 \langle x - y \rangle^{-N_1}m(y, \eta), \quad x, y \in \mathbb{R}^d, \xi, \eta \in \mathbb{T}^d.
$$

(b) For $\delta \in [0, 1]$, the space $S^\delta(m)(\mathbb{R}^d \times \mathbb{T}^d)$ consists of functions $a(x, \xi; \varepsilon)$ on $\mathbb{R}^d \times \mathbb{T}^d \times (0, 1]$, such that there exist constants $C_{\alpha, \beta} > 0$ such that for all $\varepsilon \in (0, 1], (x, \xi) \in \mathbb{R}^d \times \mathbb{T}^d$

$$
|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; \varepsilon)| \leq C_{\alpha, \beta}m(x, \xi)\varepsilon^{-\delta((|\alpha|+|\beta|)}.
$$

(2.8)

The best constants $C_{\alpha, \beta}$ in (2.8) are denoted by $||a||_{\alpha, \beta}$ and endow $S^\delta(m)(\mathbb{R}^d \times \mathbb{T}^d)$ with a Fréchet-topology.

(c) Let $a_j \in S^\delta_k(m), k_j \nearrow \infty$, then we write $a := \sum_{j=0}^{\infty} a_j$ if $a - \sum_{j=0}^{N} a_j \in S^{\delta N+1}(m)$ for every $N \in \mathbb{N}$. 


With \( a \in S^k_f(m) \), \( \mathbb{R}^d \times \mathbb{T}^d \) we associate a pseudo-differential operator \( \text{Op}_e^T(a) \) formally given by
\[
\text{Op}_e^T(a) \psi(x) := (2\pi)^{-d} \sum_{y \in (\mathbb{Z})^d} e^{i(\pi y \cdot \xi)} a(x, \xi, y) \psi(y) \ dx.
\]
This is denoted by \( \text{Op}_e^T(a) \).

We show in Appendix A that \( \text{Op}_e^T(a) \) is a continuous function on 
\[
s((\mathbb{Z})^d) \setminus \{0\} := \left\{ u : (\mathbb{Z})^d \to C \mid \| u \|_\alpha := \sup_{x \in (\mathbb{Z})^d} |x^\alpha u(x)| < \infty, \alpha \in \mathbb{N}^d \right\}.
\]
equipped with the Fréchet-topology associated to the family of seminorms \( \| \cdot \|_\alpha \). By standard arguments, \( s((\mathbb{Z})^d) \) is dense in \( \mathcal{L}^2((\mathbb{Z})^d) \).

Moreover, we define the \#-product
\[\# : \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d) \times \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d) \ni (a, b) \mapsto a \# b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d)\]
by
\[a \# b = a \circ D_x a \cdots D_x a (x, \xi, \eta).
\]
Corollary A.3 tells us that this product has a bilinear continuous extension to symbol spaces:
\[\#: \mathcal{S}^{r_1}(m_1) \times \mathcal{S}^{r_2}(m_2) \to \mathcal{S}^{r_1+r_2}(m_1 m_2) \ni (a, b) \mapsto \text{Op}_e^T(a) \circ \text{Op}_e^T(b).
\]

Let \( t \) denote the function defined in (1.8), then \( t \in S^0_0(1) \). A straightforward calculation gives \( \text{Op}_e^T(e^{-\frac{x^2}{2}}) = \tau \), and thus
\[\text{Op}_e^T(t) = T_e.
\]

REMARK 2.2 Any function \( f \in \mathcal{C}_0^\infty(\mathbb{R}^d) \), which is supported in \((-\pi, \pi)^d\), admits a unique \( \mathcal{C}_0^\infty \) periodic continuation to \( \mathbb{R}^d \). Thus any such \( f \) can be considered as a function on the torus \( \mathbb{T}^d \). We shall denote this function on \( \mathbb{T}^d \) by \( \hat{f} \).

Let \( k \in \mathcal{C}_0^\infty(\mathbb{R}^d) \) be a cut-off function on \( \mathbb{R}^d \) such that \( |k(\xi)| = 1 \) for \( |\xi| \leq 2 \) and \( \text{supp } k \subset (-\pi, \pi)^d \). Then the truncated quadratic approximation of \( t \) by \( \tilde{t}_{\pi, q}(x, \xi) := \langle (\xi, B(x) \xi + \pi t_1(x, 0) \rangle k(\xi), \quad \xi \in \mathbb{R}^d, x \in \mathbb{R}^d,
\]
defines a function \( \tilde{t}_{\pi, q} \in S^0_0(1)(\mathbb{R}^d \times \mathbb{T}^d) \) with the notation of Remark 2.2. The associated bounded operator on the lattice (see (2.9)) is denoted by \( \text{Op}_e^T(\tilde{t}_{\pi, q}) =: T_{\pi, q} \).

Moreover, we define for a potential well \( x_j \) of \( V_0 \) in the sense of Hypothesis 1.1
\[\tilde{t}_{\pi, q, j}(x, \xi) := \tilde{t}_{\pi, q}(x_j, \xi) \quad \text{and} \quad T_{\pi, q, j} := \text{Op}_e^T(\tilde{t}_{\pi, q, j}). \]

To compare \( H_{\xi} \) with an harmonic oscillator on \( L^2(\mathbb{R}^d) \), we associate to \( t \) (considered as an element of \( S^0_0(1)(\mathbb{R}^d) \)) the translation operator on \( L^2(\mathbb{R}^d) \)
\[\tilde{T}_e := \text{Op}_e(t) = \sum_{\gamma \in (\mathbb{Z})^d} a_\gamma(x, \xi) \tau_\gamma, \quad x \in \mathbb{R}^d,
\]
A one-dimensional Hermite polynomial

\[ \hat{H}_\varepsilon u(x) := \sum_{\gamma \in (\varepsilon \mathbb{Z})^d} a_\gamma(x, \varepsilon)(u(x + \gamma)) + \hat{V}_\varepsilon(x)u(x), \quad u \in \mathcal{D}(\hat{H}_\varepsilon). \]  

(2.14)

Setting \( t_q(x, \xi) := (\xi, B(x)\xi) + \varepsilon t_1(x, 0) \) on \( \mathbb{R}^d \times \mathbb{R}^d \), we have

\[ \hat{T}_q := \text{Op}_\varepsilon(t_q) = -\varepsilon^2 \sum_{\nu, \mu=1}^d B_{\nu\mu}(x)\partial_\nu\partial_\mu + \varepsilon t_1(x, 0). \]  

(2.15)

For \( x_j \in \mathbb{R}^d \) as above we set

\[ t_{q,j}(\xi) := t_q(x_j, \xi), \quad (\xi \in \mathbb{R}^d) \quad \text{and} \quad \text{Op}_\varepsilon(t_{q,j}) =: \hat{T}_{q,j}. \]  

(2.16)

Remark 2.3 We denote by \( \mathcal{G}_{\varepsilon_0} = (\varepsilon \mathbb{Z})^d + x_0 \) the \( \varepsilon \)-scaled lattice, shifted to the point \( x_0 \in \mathbb{R}^d \). Then \( x + \gamma \in \mathcal{G}_{\varepsilon_0} \) for any \( x \in \mathcal{G}_{\varepsilon_0}, x_0 \in \mathbb{R}^d \) and \( \gamma \in (\varepsilon \mathbb{Z})^d \). If \( 1_{\mathcal{G}_{\varepsilon_0}} \) is defined as the restriction map to the lattice \( \mathcal{G}_{\varepsilon_0} \), it follows that \( \tau_\gamma \) commutes with \( 1_{\mathcal{G}_{\varepsilon_0}} \). Then \( H_\varepsilon = \hat{H}_\varepsilon 1_{\mathcal{G}_{\varepsilon_0}} \) and \( H_{\varepsilon, x_0} := \hat{H}_\varepsilon 1_{\mathcal{G}_{\varepsilon_0}} \) defines a natural realization of \( H_\varepsilon \) on \( \ell^2(\mathcal{G}_{\varepsilon_0}) \).

By Hypothesis [11] at a potential well \( x_j \), for \( |x - x_j| \to 0 \), the potential energy \( \hat{V}_\varepsilon \) has the expansion

\[ \hat{V}_\varepsilon(x) = V_\varepsilon^j(x) + \varepsilon O(|x - x_j|) + O(|x - x_j|^2) + R_\varepsilon(x, \varepsilon) \]

where

\[ V_\varepsilon^j(x) := V_\varepsilon^j(x) + \varepsilon V_1(x_j), \quad \text{and} \quad V_\varepsilon^j(x) := \langle x - x_j, \hat{A}(x - x_j) \rangle. \]  

(2.17)

Remark 2.4 The operators \( K_j \) defined in [11] are harmonic oscillators with the additional additive constant \( V_1(x_j) + t_1(x_j, 0) \). Denoting by \( (\omega_\nu^2)^2 \) for \( \omega_\nu^2 > 0 \) the eigenvalues of the matrix \( A^j \), the eigenvalues of the operator \( K_j \) are given by

\[ \sigma(K_j) = \left\{ e_{\alpha,j} = \sum_{\nu=1}^d (\omega_\nu^2(2\alpha_\nu + 1)) + V_1(x_j) + t_1(x_j, 0) \mid \alpha \in \mathbb{N}^d \right\}. \]  

(2.18)

The spectrum \( \sigma(K) \) of \( K \) is the union \( \sigma(K) = \bigcup_{j=1}^m \sigma(K_j) \) of the spectra \( \sigma(K_j) \) for all \( j \).

The normalized eigenfunctions of the operators \( K_j \) associated to an eigenvalue \( e_{\alpha,j} \) are given by

\[ g_{\alpha,K_j}(x) = h_\alpha(x)e^{-\varphi_\nu(x)}, \quad \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d, \]  

(2.19)

where

\[ h_\alpha(x) = h_{\alpha_1}(\langle x, y_1^\nu \rangle) \cdot h_{\alpha_2}(\langle x, y_2^\nu \rangle) \cdots h_{\alpha_d}(\langle x, y_d^\nu \rangle) \]  

(2.20)

(\( y_\nu^\nu \in \mathbb{R}^d, (\nu = 1, \ldots, d) \) denotes an orthonormal basis in \( \mathbb{R}^d \) of eigenvectors of \( A^j \)), and each \( h_{\alpha_\nu} \) is a one-dimensional Hermite polynomial

\[ h_k(t) = \frac{(-1)^k}{\sqrt{2^k k!}} e^{t^2} \left( \frac{d}{dt} \right)^k e^{-t^2} \]  

(2.21)

with \( k = \alpha_\nu \). We assume \( h_\alpha \) to be normalized in the sense that \( \| g_{\alpha,K_j} \|_{L^2} = 1 \). The phase function in [2.19] is given by

\[ \varphi_\nu(x) := \frac{1}{2} \sum_{\nu=1}^d \omega_\nu^2(x, y_\nu^\nu)^2. \]  

(2.22)

3. Localization estimates

The starting point of the proof lies in choosing a partition of unity. This permits us to treat separately the neighborhoods of the minima and the region outside of these neighborhoods.

By standard arguments, there exists \( \chi \in \mathcal{C}_0^\infty (\mathbb{R}^d) \) such that

(a) \( 0 \leq \chi \leq 1 \),
(b) \( \chi(x) = 1 \) if \( |x| \leq 1 \) and \( \chi(x) = 0 \) if \( |x| \geq 2 \),
(c) \( \sqrt{1 - \chi^2} \in \mathcal{C}_0^\infty (\mathbb{R}^d) \).
We define for \( s > 0 \) functions which localize in \( \varepsilon^s \)-scaled neighborhoods of the minima \( x_j, \ 1 \leq j \leq m \), by
\[
\chi_{j, \varepsilon, s}(x) := \chi \left( \varepsilon^{-s}(x - x_j) \right), \quad x \in \mathbb{R}^d.
\]
(3.1)
For \( \varepsilon \) sufficiently small, \( \text{supp} \chi_{j, \varepsilon, s} \cap \text{supp} \chi_{k, \varepsilon, s} = \emptyset \) for \( k \neq j \) and thus by (c)
\[
\chi_{0, \varepsilon, s} := \left( 1 - \sum_{j=1}^{m} \chi_{j, \varepsilon, s}^2 \right) \in \mathcal{C}_0^\infty \left( \mathbb{R}^d \right) \quad \text{and} \quad \sum_{j=0}^{m} \chi_{j, \varepsilon, s}^2 = 1.
\]
Furthermore we set for \( j = 0, 1, \ldots, m \)
\[
\varepsilon \chi_{j, \varepsilon} \cdot \quad \text{(3.2)}
\]
Using this partition of unity, we obtain modulo \( O \left( \varepsilon^2 \right) \) for \( 1 \leq j \leq m \) with the notation \( V_j(x) := V_j(x_j) \left( \text{using (1.10) and (2.14)} \right)
\[
\left\| \chi_{j, \varepsilon} \left( \tilde{V}_\varepsilon - V_j \right) \chi_{j, \varepsilon} \right\|_\infty = \left\| \chi_{j, \varepsilon} \left( \left( V_0 - V_j \right) + \varepsilon \left( V_1 - V_j \right) \right) \chi_{j, \varepsilon} \right\|_\infty \leq \sup_{x \in \text{supp} \left( \chi_{j, \varepsilon} \right)} \left| \left( V_0 - V_j \right)(x) + \varepsilon \left( V_1 - V_j \right)(x) \right| = O \left( \varepsilon^\frac{2}{3} \right),
\]
(3.3)
where the last estimate follows from \( \left( V_0(x) - V_j(x) \right) = O \left( |x - x_j|^3 \right) \) and \( \left( V_1 - V_j \right)(x) = O \left( |x - x_j| \right) \) as \( x \to 0 \) and from \( |x - x_j| = O \left( \varepsilon^\frac{2}{3} \right) \) for \( x \in \text{supp} \left( \chi_{j, \varepsilon} \right) \). We shall now simultaneously localize \( T_\varepsilon \) around \( \xi = 0 \) and \( x = x_j \), which gives the main contribution to the low-lying spectrum. To this end we define a partition of unity by
\[
\phi_{0, \varepsilon, s}(\xi) := \chi(\varepsilon^{-s}\xi), \quad \xi \in \mathbb{R}^d
\]
(3.4)
and \( \phi_{1, \varepsilon, s} := \sqrt{1 - \phi_{0, \varepsilon, s}^2} \). To \( \phi_{0, \varepsilon, s} \in \mathcal{C}_0^\infty \left( \mathbb{R}^d \right) \) we associate \( \hat{\phi}_{0, \varepsilon, s} \in \mathcal{C}_0^\infty \left( T^d \right) \) (see Remark 2.2).
Then \( \hat{\phi}_{1, \varepsilon, s}(\xi) := \sqrt{1 - \phi_{0, \varepsilon, s}^2} \in \mathcal{C}_0^\infty \left( T^d \right) \) satisfies \( \hat{\phi}_{0, \varepsilon, s} + \hat{\phi}_{1, \varepsilon, s} = 1 \). The functions \( \hat{\phi}_{k, \varepsilon, s} \) can be considered as elements of \( S_{\delta}^0 \left( \mathbb{R}^d \times T^d \right) \) with associated operator \( \text{Op}_\varepsilon^\top(\hat{\phi}_{k, \varepsilon, s}) \). As above we set
\[
\hat{\phi}_{k, \varepsilon} := \hat{\phi}_{k, \varepsilon} \cdot \frac{1}{\varepsilon^\frac{2}{3}} \quad \text{and} \quad \hat{\phi}_{k, \varepsilon} := \hat{\phi}_{k, \varepsilon} \cdot \frac{1}{\varepsilon^\frac{2}{3}}, \quad k = 0, 1.
\]
(3.5)
PROPOSITION 3.1 Let \( T_\varepsilon \) be a translation operator on the lattice \( (\varepsilon \mathbb{Z})^d \) as described in Hypothesis [17] with the symbol \( t \) and let \( T_{\varepsilon, q, j} \) denote the quadratic approximation of \( T_\varepsilon \), associated to the symbol \( t_{\varepsilon, q, j} \) defined in \( [2, 12] \). Let \( \chi_{j, \varepsilon}, 1 \leq j \leq m \), and \( \hat{\phi}_{0, \varepsilon} \) be the cut-off-functions defined in \( (3.2) \) and \( (3.3) \) respectively. Then
\[
\| P \| = O \left( \varepsilon^\frac{2}{3} \right), \quad \text{where} \quad P := \chi_{j, \varepsilon} \text{Op}_\varepsilon^\top(\hat{\phi}_{0, \varepsilon})(T_\varepsilon - T_{\varepsilon, q, j}) \text{Op}_\varepsilon^\top(\hat{\phi}_{0, \varepsilon})\chi_{j, \varepsilon}.
\]
(3.6)
Proof. By Proposition A.6 we only need to show that \( p \in S_{\delta}^0 \left( 1 \right) \) for some \( 0 \leq \delta \leq \frac{1}{2} \), where \( P = \text{Op}_\varepsilon^\top(p) \). First we remark that for two symbols \( a, b \in S_{\delta}^0 \left( m \right) \), \( \delta < \frac{1}{2} \), \( b \) has compact support, and a function \( \psi \in \mathcal{C}_0^\infty \left( \mathbb{R}^d \times T^d \right) \) with \( \psi|_{\text{supp} b} = 1 \), we have by Corollary A.5
\[
a \# b(x, \varepsilon) = a \psi \# b(x, \xi, \varepsilon) + O(\varepsilon^\infty).
\]
Now choose cut-off-functions \( \hat{\phi}_{0, \varepsilon}(\xi) \) and \( \hat{\chi}_{j, \varepsilon} \) constructed as above from \( \hat{\chi} \in \mathcal{C}_0^\infty \left( \mathbb{R}^n \right) \) with \( \hat{\chi} = 1 \) for \( |x| \leq 2 \) and \( \hat{\chi} = 0 \) for \( |x| \geq 3 \). By Lemma A.12 and (3.7) it suffices to show that \( \hat{p} \in S_{\delta}^0 \left( 1 \right) \), where
\[
\hat{p}(x, \xi, \varepsilon) := \left( \chi_{j, \varepsilon} \# \hat{\phi}_{0, \varepsilon} \# (t - t_{\varepsilon, q, j}) \hat{\phi}_{0, \varepsilon} \hat{\chi}_{j, \varepsilon} \# \hat{\phi}_{0, \varepsilon} \# \chi_{j, \varepsilon} \right)(x, \xi, \varepsilon).
\]
We first determine the symbol class of \( (t - t_{\varepsilon, q, j}) \hat{\phi}_{0, \varepsilon} \hat{\chi}_{j, \varepsilon} \). Let \( \alpha, \beta \in \mathbb{N}^d \), then for \( \alpha_1 + \alpha_2 = \alpha \) and \( \beta_1 + \beta_2 = \beta \)
\[
\left| \partial_{x}^\alpha \partial_{\xi}^\beta (t - t_{\varepsilon, q, j}) \hat{\phi}_{0, \varepsilon} \hat{\chi}_{j, \varepsilon} \right| \\
= \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} \left| \partial_{x}^{\alpha_1} \partial_{\xi}^{\beta_1} (t - t_{\varepsilon, q, j}) \hat{\phi}_{0, \varepsilon} \hat{\chi}_{j, \varepsilon} \right| \left( \partial_{\xi}^{\beta_2} \hat{\phi}_{0, \varepsilon} \right) \left( \partial_{x}^{\alpha_2} \hat{\chi}_{j, \varepsilon} \right).
\]
(3.8)
Writing \( t - t_{\pi,q,j} = t - t_{\pi,q} + t_{\pi,q} - t_{\pi,q,j} \), we use that by the definition of \( t_1 \), Hypothesis 1.1 (a), (i) and Remark 1.2 (a) for each \( x \in \mathbb{R}^d \)

\[
(t - t_{\pi,q})(x,\xi,\varepsilon) = O(|\xi|^4) + \varepsilon O(|\xi|) + O(\varepsilon^2) \quad \text{and} \quad (3.9)
\]

\[
(t_{\pi,q} - t_{\pi,q,j})(x,\xi,\varepsilon) = (\xi, (B(x) - B(x_j))\xi) + \varepsilon(t_1(x,0) - t_1(x_j,0)) = O(|\xi|^2)O(|x|) + \varepsilon O(|x|).
\]

The scaling in the definition of the cut-off-functions yields \(|x - x_j| = \varepsilon \hat{\varphi}(\hat{x}) = |\xi|\), therefore by (3.9)

\[
\sup_{|\xi| \in \text{supp}(\hat{\varphi}_{0,\varepsilon})} \sup_{|x| \in \text{supp}(\hat{\chi}_{j,\varepsilon})} \left| \partial_\alpha^\beta \xi (t - t_{\pi,q,j})(x,\xi,\varepsilon) \right| \leq C\varepsilon_\alpha^\beta |\xi|^{-|\alpha| - |\beta|}. \quad (3.10)
\]

By construction \( \hat{\varphi}_{0,\varepsilon}, \hat{\chi}_{j,\varepsilon} \in S_\xi^0(1) \), thus inserting (3.10) in (3.8) shows

\[
\left| \partial_\alpha^\beta \xi (t - t_{\pi,q,j})\hat{\varphi}_{0,\varepsilon} \hat{\chi}_{j,\varepsilon}(x,\xi,\varepsilon) \right| \leq C\varepsilon_\alpha^\beta |\xi|^{-|\alpha| - |\beta|}
\]

and therefore \((t - t_{\pi,q,j})\hat{\varphi}_{0,\varepsilon} \hat{\chi}_{j,\varepsilon} \in S_\xi^0(1)\). The cut-off-functions \( \chi_{j,\varepsilon} \) and \( \hat{\varphi}_{0,\varepsilon} \) are both elements of \( S_\xi^0(1) \), thus by Corollary 1.3 we get \( p \in S_\xi^0(1)(\mathbb{R}^d \times T^d) \). The estimate of the norm of the associated operator in \( L^2((\varepsilon\mathbb{Z})^d) \) follows by use of Proposition 1.6

\[
\Box
\]

**Remark 3.2** Using the symbolic calculus introduced in Dimassi-Sjöstrand [7], in particular Proposition 7.7, Theorem 7.9 and Theorem 7.11, it is possible to show by similar considerations as in the lattice case that for \( \tilde{T}, \tilde{T}_q \) defined in (2.13) and (2.16) respectively, with the cut-off functions \( \chi_{j,\varepsilon}, \hat{\varphi}_{k,\varepsilon} \) defined in (3.12) and (3.13), one has the norm estimate

\[
\| \chi_{j,\varepsilon}(x) \hat{\varphi}_{0,\varepsilon}(\varepsilon D)(T_\varepsilon - T_{\varepsilon q})(\varepsilon D)\chi_{j,\varepsilon}(x) \|_{\infty} = O(\varepsilon^{\frac{5}{4}}) \quad (3.11)
\]

(3.11) suggests to define (see (2.17))

\[
\hat{H}_j := \tilde{T}_{q,j} + V_0^j + \varepsilon V_1^j(x_j) = \hat{T}_{q,j} + V_0^j
\]

(3.12) as an approximating operator of \( \hat{H}_j \) and \( H_j \) respectively on \( L^2(\mathbb{R}^d) \). By means of the unitary transformation \( Uf(x) := \sqrt{|\det B_j^{-1/2}|} f(B_j^{-1/2}x) \), the operator \( \hat{H}_j \) is unitarily equivalent to

\[
H_j := -\varepsilon^2 \Delta + \langle (x - x_j), A^j(x - x_j) \rangle + \varepsilon (V_1^j(x_j) + t_1(x_j,0)) = U^{-1}\hat{H}_j U, \quad (3.13)
\]

where \( A^j, B_j \) are defined as in Theorem 1.3. Furthermore, by scaling, \( H_j \) is unitarily equivalent to \( \varepsilon K_j \). Thus the spectrum of \( H_j \) and \( \hat{H}_j \) is given by \( \varepsilon \sigma(K_j) \). The eigenfunctions of \( H_j \) and \( \hat{H}_j \) are

\[
g_{a,j}(x) = \varepsilon^{-\frac{d}{2}} h_a \left( \frac{x - x_j}{\varepsilon} \right) e^{-\varepsilon^2 \left( \frac{x - x_j}{\sqrt{\varepsilon}} \right)} \quad \text{and} \quad g_{a,j} := U g_{a,j} \text{ respectively}.
\]

(3.14)

We will show now that modulo terms of order \( \varepsilon^{\frac{5}{4}} \) one can decompose \( H_j \) with respect to the partition of unity introduced above into a sum of Dirichlet operators. This is a generalization of the IMS-localization formula for Schrödinger operators described for example in Cycon-Froese-Kirsch-Simon [6].

**Lemma 3.3** Let \( H_\varepsilon = T_\varepsilon + V_\varepsilon \) satisfy Hypothesis 1.1 and denote by \( V_\varepsilon^j \) the quadratic approximation of \( V_\varepsilon \) defined in (2.17).

Let \( \chi_{j,\varepsilon}, 0 \leq j \leq m \) and \( \tilde{\varphi}_{k,\varepsilon}, k = 0, 1 \) be given by (3.12) and (3.13) respectively. Then the following estimates hold in operator norm.

(a)

\[
H_\varepsilon = \sum_{j=0}^{m} \chi_{j,\varepsilon} H_\varepsilon \chi_{j,\varepsilon} + O \left( \varepsilon^{\frac{5}{4}} \right).
\]

(b)

\[
T_\varepsilon + V_\varepsilon^j = \text{Op}_{\varepsilon}^\uparrow (\tilde{\varphi}_{0,\varepsilon})(T_\varepsilon + V_\varepsilon^j) \text{Op}_{\varepsilon}^\uparrow (\tilde{\varphi}_{0,\varepsilon}) + \text{Op}_{\varepsilon}^\uparrow (\tilde{\varphi}_{1,\varepsilon})(T_\varepsilon + V_\varepsilon^j) \text{Op}_{\varepsilon}^\uparrow (\tilde{\varphi}_{1,\varepsilon}) + O \left( \varepsilon^{\frac{5}{4}} \right).
\]
Proof. (a): $H_\varepsilon$ can be written as
\[
H_\varepsilon = \frac{1}{2} \sum_{j=0}^{m} \chi_{j,\varepsilon}^2 H_\varepsilon + \frac{1}{2} H_\varepsilon \sum_{j=0}^{m} \chi_{j,\varepsilon} = \sum_{j=0}^{m} \chi_{j,\varepsilon} H_\varepsilon \chi_{j,\varepsilon} + \frac{1}{2} \sum_{j=0}^{m} [\chi_{j,\varepsilon}, [\chi_{j,\varepsilon}, H_\varepsilon]] ,
\]
therefore we have to estimate the double commutators on the right hand side of (3.15). Since $t \in S^0_5(1)$ and $\chi_{j,\varepsilon} \in S^0_5(1)$, $j = 0, \ldots, m$, it follows at once from Lemma A.8 that $[\chi_{j,\varepsilon}, [\chi_{j,\varepsilon}, t]]_{\#} \in S^0_5(1)$, which leads to (a) by Proposition A.6.

(b): The arguments are quite similar to (a), but we need to consider the expansions for the symbolic double commutator, since the quadratic potential $V_2^\varepsilon$ is not bounded, but $V_2^\varepsilon \in S^0_5(|x|^2)$. Thus the general result on the symbol class of the double commutator given in Lemma A.8 does not allow to use Proposition A.6 directly. By Lemma A.8, the double commutator in the symbolic calculus with $\alpha, \alpha_1, \alpha_2 \in \mathbb{N}^d$ for $k = 0, 1$ can be written as
\[
[\hat{\phi}_{k,\varepsilon}(\xi), [\hat{\phi}_{k,\varepsilon}(\xi), (t + V_2^\varepsilon)(x, \xi)]_{\#}]_{\#} = \sum (i\varepsilon^{\alpha}) [\partial_x^\alpha (t + V_2^\varepsilon)](x, \xi) \sum_{\alpha_1 + \alpha_2 = \alpha} \left( \partial_x^\alpha \hat{\phi}_{k,\varepsilon} \right) \left( \partial_x^\alpha \hat{\phi}_{k,\varepsilon} \right) (\xi) + R_3 .
\]

Now we use that $t \in S^0_5(1)$ and $\hat{\phi}_{k,\varepsilon} \in S^0_5(1)$ and furthermore that the second derivative of the quadratic term $V_2^\varepsilon$ is constant. Thus all the summands are bounded, of order $\varepsilon^{2-\frac{d}{5}}$ and the $\varepsilon$-order in lowered by $\hat{\phi}_\varepsilon$ with each differentiation, i.e., they are elements of $S^0_\varepsilon(1)$. By Lemma A.8 the remainder $R_3$ depends linearly on a finite number of derivatives $\partial_x^\beta (h + V_2^\varepsilon)$ with $|\beta| \geq 3$ (which is bounded) and $\left( \partial_x^\beta \hat{\phi}_{k,\varepsilon} \right) \left( \partial_x^\beta \hat{\phi}_{k,\varepsilon} \right)$ with $|\beta_1| + |\beta_2| \geq 3$. Thus it is an element of $S^0_\varepsilon(1)$. We therefore get $[\hat{\phi}_{k,\varepsilon}(\xi), [\hat{\phi}_{k,\varepsilon}(\xi), (t + V_2^\varepsilon)(x, \xi)]_{\#}]_{\#} \in S^0_\varepsilon(1)$, yielding by Proposition A.6 the stated norm estimate for the associated operator.

We shall now restrict the eigenfunctions $g_{n\varepsilon}$ of $\tilde{H}$ introduced in (3.14) to the lattice $(\varepsilon \mathbb{Z})^d$. We denote these restrictions by $\hat{g}_{n\varepsilon}$ and we shall use them as approximate eigenfunctions for $H_\varepsilon$.

**Lemma 3.4** Let $f, g$ denote eigenfunctions of $\tilde{H}$ as defined in (3.14) and $f^\varepsilon, g^\varepsilon$ their restriction to $(\varepsilon \mathbb{Z})^d$. Then
\[
\langle f^\varepsilon, g^\varepsilon \rangle_{L^2} = \varepsilon^{-d} \left( \langle f, g \rangle_{L^2} + O(\sqrt{\varepsilon}) \right) .
\]

**Proof.** We use $\varepsilon^d = \int_{[x, x + \varepsilon]^d} dx$ to write
\[
\langle f^\varepsilon, g^\varepsilon \rangle_{L^2} = I_1 + I_2 + I_3 ,
\]
where
\[
I_1 = \varepsilon^{-d} \sum_{x \in (\varepsilon \mathbb{Z})^d} \int_{[x, x + \varepsilon]^d} (f(x) - f(y)) g(x) dy
\]
\[
I_2 = \varepsilon^{-d} \sum_{x \in (\varepsilon \mathbb{Z})^d} \int_{[x, x + \varepsilon]^d} f(y) (g(x) - g(y)) dy
\]
and
\[
I_3 = \varepsilon^{-d} \int_{\mathbb{R}^d} f(y)g(y) dy = \varepsilon^{-d} \langle f, g \rangle_{L^2} .
\]

It thus remains to show that $I_1$ and $I_2$ are of order $\varepsilon^{-d+\frac{d}{5}}$. By the scaling of $f$ and since $f = O(\varepsilon^{-\frac{d}{5}})$
\[
\sup_{x \in (\varepsilon \mathbb{Z})^d} \sup_{y \in [x, x + \varepsilon]^d} |f(x) - f(y)| \leq \varepsilon \sup_{z \in \mathbb{R}^d} |\nabla f(z)| \leq C \varepsilon^{-\frac{d}{5}} e^{\frac{d}{5}} .
\]

Thus, setting $g(x) = \varepsilon^{-\frac{d}{5}} \hat{g}(\frac{x - x}{\varepsilon})$ for some $j \in \{1, \ldots, m\}$, we have by (3.18) and (3.19)
\[
|I_1| \leq C \varepsilon^{-\frac{d}{5}} \sum_{y \in \varepsilon \mathbb{Z}^d} \hat{g} \left( y - \frac{x}{\varepsilon} \right) = O \left( \varepsilon^{\frac{d}{5}} \right) ,
\]
\[
|I_2| \leq C \varepsilon^{-\frac{d}{5}} \sum_{y \in \varepsilon \mathbb{Z}^d} \hat{g} \left( y - \frac{x}{\varepsilon} \right) = O \left( \varepsilon^{\frac{d}{5}} \right) .
\]
where in the last step we used that by the definition of the Riemann Integral
\[
\lim_{\epsilon \to 0} \epsilon^\frac{d}{2} \sum_{y \in (\sqrt{\epsilon} \Z)^d} \left| \tilde{g}(y - \frac{x_i}{\epsilon}) \right| = \int_{\R^d} |\tilde{f}(u)| \, du,
\] (3.21)
which is a constant independent of \(\epsilon\). The estimates for \(I_2\) are analogous. \(\square\)

The functions \(g_{\alpha_j}\) defined in (3.14) are localized near the well \(x_j\) for \(j = 1, \ldots, m\) and decrease exponentially fast. We need the following localization estimates.

**Lemma 3.5** For \(s < \frac{1}{2}\) let \(\chi_{j, \epsilon}, \chi_{j, \epsilon, s}, 1 \leq j \leq m\) and \(\tilde{\phi}_{0, \epsilon, s}\), denote the cut-off functions defined in (3.22), (3.24) and below (3.4) respectively. Let \(g_{\alpha_j}(\epsilon)\) denote the eigenfunctions of the harmonic oscillator defined in (3.14) (or their restriction to the lattice). Then for \(\epsilon \to 0\):

(a) There exists a constant \(C > 0\) such that
\[
\left| \left\langle g_{\alpha_j}, (1 - \chi_{j, \epsilon, s}^2) g_{\alpha_j} \right\rangle \right|_{L^2} = O \left( e^{-C\epsilon^{2s-1}} \right).
\]
(b) For all \(N \in \N\)
\[
\left| \left\langle \mathcal{F}_{\epsilon}^{-1} \left( \chi_{j, \epsilon} g_{\alpha_j}^\epsilon \right), \phi_{1, \epsilon, s}^2 \mathcal{F}_{\epsilon}^{-1} \left( \chi_{j, \epsilon} g_{\alpha_j}^\epsilon \right) \right\rangle \right|_T = O \left( \epsilon^N \right).
\]

**Proof.** (a): Estimating \(\left\langle g_{\alpha_j}, (1 - \chi_{j, \epsilon, s}^2) g_{\alpha_j} \right\rangle \) by 1 on its support gives
\[
\left| \left\langle g_{\alpha_j}, (1 - \chi_{j, \epsilon, s}^2) g_{\alpha_j} \right\rangle \right|_{L^2} \leq \int_{|x - x_j| \geq \epsilon^s} |g_{\alpha_j}(x)|^2 \, dx.
\]
Using \(g_{\alpha_j} = O(\epsilon^{-\frac{d}{2}})\) and the exponential decay of \(g_{\alpha_j}\), the right hand side can be estimated from above by
\[
\epsilon^{-\frac{d}{2}} \int_{|x - x_j| \geq \epsilon^s} p(|u|) e^{-C\epsilon^{2s-2}x_j^2} \, dx = O \left( e^{-C\epsilon^{2s-1}} \right)
\] (3.22)
for some \(c, C > 0\) and some polynomial \(p\), proving (a).

(b): To prove this statement, we sum by parts. Setting \(v := \mathcal{F}_{\epsilon}^{-1} \left( \chi_{j, \epsilon} g_{\alpha_j}^\epsilon \right)\) and replacing the function \(\tilde{\phi}_{1, \epsilon, s}\) on its support by 1, we get
\[
\left| \left\langle v, \phi_{1, \epsilon, s}^2 v \right\rangle_T \right| \leq \int_{[\epsilon^s, \epsilon^d] \times [\epsilon^s, \epsilon^d]} |v(\xi)|^2 \, d\xi.
\] (3.23)
We now estimate \(|v(\xi)|^2\). By the definition (2.4) of the inverse Fourier transform,
\[
v(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{y \in (\epsilon \Z)^d} e^{\frac{x}{\epsilon^{2s}} y} \chi_{j, \epsilon}(y) g_{\alpha_j}^\epsilon(y).
\] (3.24)
To analyze \(v\) and \(\tilde{v}\), we use summation by parts and the discrete Laplace operator \(\Delta_{\epsilon}\)
\[
(\Delta_{\epsilon} f)(x) = \left( \sum_{\nu = 1}^d (\tau_{\epsilon x_{\nu}} + \tau_{-\epsilon x_{\nu}}) - 2d \right) f(x).
\] (3.25)
The operator \(\Delta_{\epsilon}\) is symmetric in \(\ell^2((\epsilon \Z)^d)\), i.e.,
\[
\left\langle f, \Delta_{\epsilon} h \right\rangle_{\ell^2} = \left\langle \Delta_{\epsilon} f, h \right\rangle_{\ell^2}, \quad f, h \in \ell^2((\epsilon \Z)^d).
\] (3.26)
By (3.2a) we have
\[
e^{\pm x_{\frac{d}{2}} \cdot \xi} = - \left( 2d - 2 \sum_{\nu = 1}^d \cos(\xi_{\nu}) \right)^{-1} \Delta_{\epsilon} e^{\pm x_{\frac{d}{2}} \cdot \xi}.
\] (3.27)
Combining (3.24), (3.27) and (3.26) for any \(N \in \N\) leads to
\[
\sqrt{2\pi} \cdot v(\xi) = - \left( 2d - 2 \sum_{\nu = 1}^d \cos(\xi_{\nu}) \right)^{-N} \sum_{x \in (\epsilon \Z)^d} \delta_{\epsilon}^N \chi_{j, \epsilon} g_{\alpha_j}^\epsilon)(x) e^{\pm x_{\frac{d}{2}} \cdot \xi}.
\] (3.28)
We shall estimate the first factor on the right hand side of (3.28) for \( \xi \in \mathcal{M}_\varepsilon := \{ \xi \in [-\pi, \pi]^d \mid |\xi| \geq \varepsilon^n \} \). From the inequality \( \pi^2 (1 - \cos \xi) \geq \xi^2 \) for \( |\xi| \leq \pi \) it follows that

\[
\frac{1}{\sum_{\nu=1}^{d} (2 - 2 \cos (\xi_{\nu}))} \leq \frac{\pi^2}{2 \sum_{\nu=1}^{d} \xi_{\nu}^2} = \frac{\pi^2}{2 |\xi|^2}, \quad \xi \in \mathcal{M}_\varepsilon
\]

and therefore

\[
\left( \sum_{\nu=1}^{d} (2 - 2 \cos (\xi_{\nu})) \right)^{-N} \leq \left( \frac{\pi^2}{2 e^{2\pi}} \right)^N = O \left( \varepsilon^{-2Nn} \right), \quad \xi \in \mathcal{M}_\varepsilon. \tag{3.29}
\]

To find an estimate for the remaining series on the right hand side of (3.28), we use the differentiability of the functions \( \chi_{j,\varepsilon}g_{\alpha j} \). We set \( u := \varepsilon^4 \chi_{j,\varepsilon}g_{\alpha j} \), then by the chain rule and the scaling of \( g_{\alpha j} \) and \( \chi_{j,\varepsilon} \)

\[
\partial_{\varepsilon}^\alpha u(x) = O \left( \varepsilon^{-1} \right), \quad x \in \mathbb{R}^d.
\]

Thus Taylor expansion gives

\[
\Delta_{\varepsilon} u(x) = \sum_{\nu=1}^{d} (u(x + \varepsilon e_{\nu}) - u(x)) + (u(x - \varepsilon e_{\nu}) - u(x)) \tag{3.30}
\]

\[
= \varepsilon^2 \sum_{\nu=1}^{d} \int_0^1 (\partial_{\varepsilon}^\alpha u(x + t\varepsilon e_{\nu}) + \partial_{\nu}^n u(x - t\varepsilon e_{\nu})) \, dt
\]

\[
= O(\varepsilon).
\]

Iterating (3.30) gives

\[
\Delta_{\varepsilon}^N u(x) = O \left( \varepsilon^{-N} \right). \tag{3.31}
\]

Inserting (3.31) and (3.29) into (3.28) gives

\[
|u(\xi)|^2 = O \left( \varepsilon^{-\frac{2}{N(1-2n)}} \right), \quad \xi \in \mathcal{M}_\varepsilon. \tag{3.32}
\]

Inserting (3.32) into (3.23) shows (b). \( \square \)

In the following lemma we use the above results to analyze the difference of matrix elements for \( H_\varepsilon \), \( V^j_\varepsilon \) and \( T_\varepsilon \) and their localized approximations in the case \( s = \frac{1}{2} \).

**Lemma 3.6** Let \( H_\varepsilon \) and \( T_\varepsilon \) be given as in Hypothesis \( \text{H 1} \). \( V^j_\varepsilon \) be given in (2.17) and \( \tilde{T}_{q,i} \) in (2.16). Let \( \tilde{\phi}_{0,\varepsilon} \), \( \phi_{0,\varepsilon} \) and \( \chi_{j,\varepsilon}, \) \( 1 \leq j \leq m \), denote the cut-off functions defined in (3.15) and (3.22) respectively. Let \( g_{\alpha j}^{(\varepsilon)} \) denote the eigenfunctions of \( \tilde{H}^j \) defined in (3.14) (or their restriction to the lattice). Then for \( \varepsilon \to 0 \):

(a)

\[
\left| \left\langle g_{\alpha j}^{(\varepsilon)} , H_\varepsilon g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} - \left\langle \chi_{j,\varepsilon}g_{\alpha j}^{(\varepsilon)} , H_\varepsilon \chi_{j,\varepsilon}g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} \right| = O \left( \varepsilon^{\frac{5}{4}} \right). \tag{3.33}
\]

(b) There exists a constant \( c > 0 \) such that

\[
\left| \left\langle g_{\alpha j} , V^j_\varepsilon g_{\beta j} \right\rangle_{L^2} - \left\langle \chi_{j,\varepsilon}g_{\alpha j} , V^j_\varepsilon \chi_{j,\varepsilon}g_{\beta j} \right\rangle_{L^2} \right| = O \left( \varepsilon^{-c - \frac{1}{2}} \right). \tag{3.34}
\]

(c)

\[
\left| \left\langle \chi_{j,\varepsilon}g_{\alpha j}^{(\varepsilon)} , T_\varepsilon \chi_{j,\varepsilon}g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} - \left\langle \text{Op}_{\varepsilon}^{\tilde{T}}(\tilde{\phi}_{0,\varepsilon})\chi_{j,\varepsilon}g_{\alpha j}^{(\varepsilon)} , T_\varepsilon \text{Op}_{\varepsilon}^{\tilde{T}}(\tilde{\phi}_{0,\varepsilon})\chi_{j,\varepsilon}g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} \right| = O \left( \varepsilon^{\frac{5}{4}} \right). \tag{3.35}
\]

(d)

\[
\left| \left\langle g_{\alpha j} , \tilde{T}_{q,j} g_{\beta j} \right\rangle_{L^2} - \left\langle \text{Op}_{\varepsilon}(\phi_{0,\varepsilon})\chi_{j,\varepsilon}g_{\alpha j} , \tilde{T}_{q,j} \text{Op}_{\varepsilon}(\phi_{0,\varepsilon})\chi_{j,\varepsilon}g_{\beta j} \right\rangle_{L^2} \right| = O \left( \varepsilon^{\frac{5}{4}} \right). \tag{3.36}
\]

**Proof.** (a):

By Lemma 3.3

\[
\left| \left\langle g_{\alpha j}^{(\varepsilon)} , H_\varepsilon g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} - \left\langle \chi_{j,\varepsilon}g_{\alpha j}^{(\varepsilon)} , H_\varepsilon \chi_{j,\varepsilon}g_{\beta j}^{(\varepsilon)} \right\rangle_{L^2} \right| = \sum_{k \neq j} \left| \chi_{k,\varepsilon}(x)g_{\alpha j}^{(\varepsilon)} , (T_\varepsilon + V_\varepsilon)\chi_{k,\varepsilon}(x)g_{\beta j}^{(\varepsilon)} \right|_{L^2} + O \left( \varepsilon^{\frac{5}{4}} \right). \tag{3.35}
\]
We consider the kinetic and potential term separately, starting with the potential term $V_\varepsilon$. By estimating $(1 - \chi^2_{j,\varepsilon})$ on its support by 1 and using $g_{\beta l}(t) = O(\varepsilon^{-\frac{4}{5}})$, we get for some $C > 0$
\[
\left| \left< g^\varepsilon_{\alpha j}, (1 - \chi^2_{j,\varepsilon}) V_\varepsilon g^\varepsilon_{\beta l} \right> \right| \leq C \varepsilon^{-\frac{4}{5}} \sum_{\varepsilon(x) \neq \varepsilon(x) \neq \varepsilon} \left| V_\varepsilon(x) g^\varepsilon_{\alpha j}(x) \right| .
\]
$V_\varepsilon$ is by Hypothesis [14] polynomially bounded, thus the right hand side is bounded from above by

\[
C \varepsilon^{-\frac{4}{5}} \sum_{|x-x_j| \leq \varepsilon} |p(|x-x_j|)|e^{-c|\frac{x-x_j}{\varepsilon}|^2}
\]
for some $c, C > 0$ and some polynomial $p$. This yields for some $c > 0$
\[
\left| \left< g^\varepsilon_{\alpha j}, (1 - \chi^2_{j,\varepsilon}) V_\varepsilon g^\varepsilon_{\beta l} \right> \right| = O \left( e^{-c\varepsilon^{-\frac{1}{5}}} \right) .
\] (3.36)

The boundedness of $T_\varepsilon$ together with Lemma [3.35] yields

\[
\sum_{k \neq j} \left< \chi_{k,\varepsilon}(x) g^\varepsilon_{\alpha j}, T_\varepsilon \chi_{k,\varepsilon}(x) g^\varepsilon_{\beta j} \right> \leq C \sum_{k \neq j} \left< \chi_{k,\varepsilon} g^\varepsilon_{\alpha j} \right> = O \left( e^{-c\varepsilon^{-\frac{1}{5}}} \right) .
\] (3.37)
for some $c > 0$. Inserting (3.36) and (3.37) in (3.35) shows the stated estimate.

(b): This is analogue to the proof of Lemma [3.35] since $V_\varepsilon^j$ just changes the polynomial term in (3.22).

(c): By Lemma [3.3]
\[
\left< \chi_{j,\varepsilon} g^\varepsilon_{\alpha j}, T_\varepsilon \chi_{j,\varepsilon} g^\varepsilon_{\beta j} \right> = \left< \text{Op}_T^\varepsilon (\tilde{\phi}_0, \varepsilon) \chi_{j,\varepsilon} g^\varepsilon_{\beta j} \right> = \left< \text{Op}_T^\varepsilon (\tilde{\phi}_0, \varepsilon) \chi_{j,\varepsilon} g^\varepsilon_{\beta j} \right> = O \left( \varepsilon^\frac{5}{2} \right) \] (3.38)

Since by (A.12) $T_\varepsilon \text{Op}_T^\varepsilon (\tilde{\phi}_0, \varepsilon) = \text{Op}_T^\varepsilon (t\tilde{\phi}_0, \varepsilon)$, we have by the isometry of $F_\varepsilon$
\[
\left< \text{Op}_T^\varepsilon (\tilde{\phi}_0, \varepsilon) \chi_{j,\varepsilon} g^\varepsilon_{\beta j} \right> \leq C \| \tilde{\phi}_0 F_\varepsilon^{-1}(X_{j,\varepsilon} g^\varepsilon_{\beta j}) \|_T = O (\varepsilon^\infty) ,
\] (3.39)

where the second estimate follows from the boundedness of $t$ and the last from Lemma [3.35] (b).

(d): We set $P := \text{Op}_\varepsilon (\phi_{0,\varepsilon}) \tilde{T}_{q,j}$, with symbol $p(\xi) = \phi_{0,\varepsilon}^2(\xi) q_{j}(\xi)$. Then, using (2.7),
\[
\left< g_{\alpha j}, \tilde{T}_{q,j} g_{\beta l} \right> - \left< g_{\alpha j}, P g_{\beta l} \right> = \varepsilon^d \left< F_\varepsilon^{-1} g_{\alpha j}, \phi_{0,\varepsilon}^2 t_{j,q} F_\varepsilon^{-1} g_{\beta l} \right> \leq \int_{|k| \leq 2^8} \left| F_\varepsilon^{-1} (\xi) q_{j}(\xi) \right| d\xi \leq C \varepsilon^{-\frac{5}{2}} ,
\] (3.40)

where we used that
\[
\left| \left< F_\varepsilon^{-1} g_{\alpha j} \right> (\xi) \right| \leq C \varepsilon^{-N} \left| q(\xi) e^{-c|\frac{\xi}{\varepsilon}|^2} \right|
\]
for some $N \in \mathbb{N}, C, c > 0$ and some polynomial $q(\xi)$. Next observe that by (3.15)
\[
P - \sum_{j=0}^m \chi_{j,\varepsilon} P \chi_{j,\varepsilon} = \frac{1}{2} [\chi_{j,\varepsilon}, [\chi_{j,\varepsilon}, P]] = O \left( \varepsilon^\frac{5}{2} \right) ,
\] (3.41)

since $p \in S^\frac{4}{5}_2(1)$ and $\chi_{j,\varepsilon} \in S^0_{\frac{4}{5}}(1)$, using PDO-calculus on $\mathbb{R}^d$, in particular the Theorem of Calderon and Vaillancourt (see [17]). Furthermore
\[
\sum_{k \neq j} \left| \left< g_{\alpha j}, \chi_{k,\varepsilon} P \chi_{k,\varepsilon} g_{\beta l} \right> \right| \leq \sum_{k \neq j} \left< \chi_{k,\varepsilon} g_{\alpha j} \right> \left< P \chi_{k,\varepsilon} g_{\beta l} \right> = O \left( e^{-c\varepsilon^{-\frac{1}{5}}} \right) .
\] (3.42)
by Lemma 3.5 (a). Combining
$$\tilde{T}_{q,j} - \chi_{j,\varepsilon} P \chi_{j,\varepsilon} = (\tilde{T}_{q,j} - P) + \left( P - \sum_{k=0}^{m} \chi_{k,\varepsilon} P \chi_{k,\varepsilon} \right) + \sum_{k\neq j} \chi_{k,\varepsilon} P \chi_{k,\varepsilon}$$
with (3.40), (3.41) and (3.42) proves (d).

Since Theorem 1.3 compares the eigenvalues of a self adjoint unbounded operator on $l^2((\varepsilon \mathbb{Z})^d)$ with the eigenvalues of the harmonic oscillator, which is an unbounded self adjoint operator on $L^2(\mathbb{R}^d)$, we have to compare some matrix elements with respect to the scalar product $\langle \cdot, \cdot \rangle_{\varepsilon^2}$ with those with respect to $\langle \cdot, \cdot \rangle_{l^2}$. How this can be done is shown in the next lemma, giving an estimate for the difference of these terms.

**Lemma 3.7** Let $T_{\varepsilon, q,j}$ and $\tilde{T}_{q,j}$ be defined in (2.12) and (2.16) respectively and let $V^j_{\varepsilon}$ be given by (2.17). Let $f, g \in L^2(\mathbb{R}^d)$ denote normalized eigenfunctions of $\tilde{H}^j$ given in (3.12) (of the form (3.14)) and $f^\varepsilon, g^\varepsilon \in l^2((\varepsilon \mathbb{Z})^d)$ their restrictions to the lattice. Let $\chi_{j,\varepsilon}, 1 \leq j \leq m, \tilde{\phi}_{0,\varepsilon}$ and $\phi_{0,\varepsilon}$ be the cut-off functions defined in (3.32) and (3.33). Then for $\varepsilon$ sufficiently small

(a) for any $\alpha < \frac{1}{2}$
$$\left\langle \chi_{j,\varepsilon} f^\varepsilon, \mathcal{O}_\varepsilon^j(\tilde{\phi}_{0,\varepsilon}) T_{\varepsilon, q,j} \mathcal{O}_\varepsilon^j(\tilde{\phi}_{0,\varepsilon}) \chi_{j,\varepsilon} g^\varepsilon \right\rangle_{l^2} = \varepsilon^{-d} \left( \left\langle \chi_{j,\varepsilon} f, \mathcal{O}_\varepsilon(\phi_{0,\varepsilon}) \tilde{T}_{q,j} \mathcal{O}_\varepsilon(\phi_{0,\varepsilon}) \chi_{j,\varepsilon} g \right\rangle_{L^2} + O(\varepsilon^{1+\alpha}) \right),$$

(b) 
$$\left\langle f^\varepsilon, \chi_{j,\varepsilon} V^j_{\varepsilon} \chi_{j,\varepsilon} g^\varepsilon \right\rangle_{l^2} = \varepsilon^{-d} \left( \left\langle f, \chi_{j,\varepsilon} V^j_{\varepsilon} \chi_{j,\varepsilon} g \right\rangle_{L^2} + O(\varepsilon^{\frac{3}{2}}) \right).$$

**Remark 3.8** The estimate in (b) is a rough Corollary of Lemma 3.7.

**Proof.** (a):

Let $\hat{t}_{\pi, q,j}$ and $t_{q,j}$ be defined in (2.12) and (2.16) respectively. Then we observe that $\tilde{\phi}_{0,\varepsilon}^j \hat{t}_{\pi, q,j}$ on $T^d$ can be identified with the function $G := \tilde{\phi}_{0,\varepsilon}^j t_{q,j}$ on $\mathbb{R}^d$, since supp $G \in (-\pi, \pi)^d$ for $\varepsilon$ sufficiently small. Setting
$$u_1 := \mathcal{F}^{-1}(\chi_{j,\varepsilon} f^\varepsilon), \quad u_2 := \mathcal{F}^{-1}(\chi_{j,\varepsilon} f),$$
$$v_1 := \mathcal{F}^{-1}(\chi_{j,\varepsilon} g^\varepsilon), \quad v_2 := \mathcal{F}^{-1}(\chi_{j,\varepsilon} g),$$
we obtain by use of (3.41) that the left hand side of (a) is given by
$$\langle u_1, G v_1 \rangle_{L^2} = I_1 + I_2 + I_3,$$
where
$$I_1 = \langle u_1 - u_2, G v_1 \rangle_{L^2}, \quad I_2 = \langle u_2, G(v_1 - v_2) \rangle_{L^2}$$
and
$$I_3 = \langle u_2, G v_2 \rangle_{L^2} = \varepsilon^{-d} \langle \chi_{j,\varepsilon} f, \mathcal{O}_\varepsilon(\phi_{0,\varepsilon}) \tilde{T}_{q,j} \mathcal{O}_\varepsilon(\phi_{0,\varepsilon}) \chi_{j,\varepsilon} g \rangle_{L^2},$$
with the last equality follows from the “Parseval” relation (2.7) for the $\varepsilon$-Fourier transform $F_\varepsilon$ defined in (2.6). We claim that for any $\alpha < \frac{1}{2}$ and for $j = 1, 2$
$$|I_j| = O(\varepsilon^{d+1+\alpha}),$$
which together with (3.40) proves (a).

Using Cauchy-Schwarz in (3.45), we obtain by Lemma 3.5 (b), and the boundedness of $G$ for any $s < \frac{1}{2}$
$$|I_1| \leq \|u_1 - u_2\|_{L^2} \|G v_1\|_{L^2} = \|h_1 + h_2\|_{L^2(\varepsilon^2 \mathbb{R}^d)} \|G v_1\|_{L^2(\varepsilon^2 \mathbb{R}^d)} + O(\varepsilon^{\infty}),$$
where, setting $Q_\varepsilon := [x, x + \varepsilon]^d$,
$$h_1(\xi) = (\varepsilon \sqrt{2\pi})^{-d} \sum_{x \in (\varepsilon \mathbb{Z})^d} \int_{Q_\varepsilon} \left( e^{\hat{\xi} \cdot x \varepsilon} - e^{\hat{\xi} \cdot y \varepsilon} \right) \chi_{j,\varepsilon} f(x) dy$$
$$h_2(\xi) = (\varepsilon \sqrt{2\pi})^{-d} \sum_{x \in (\varepsilon \mathbb{Z})^d} \int_{Q_\varepsilon} e^{\hat{\xi} \cdot y \varepsilon} h(y) dy,$$
with
\[ h(y) = (\chi_{j, \eps} f)(y) - (\chi_{j, \eps} f)(x), \quad y \in Q_x. \]
Thus we have \( h_2(\xi) = F_{\eps}^{-1} h(\xi) \), giving by (3.51)
\[ \|h_2\|_{L^2} = \|F_{\eps}^{-1} h\|_{L^2} = \frac{\varepsilon^{-\frac{d}{4}}}{2} \|h\|_{L^2}. \]
Using Lemma (a), we obtain for any \( s < \frac{1}{2} \) and \( \varepsilon \) sufficiently small
\[ \|h\|^2_{L^2} = \int_{|x-x_j| < \varepsilon^s} |h(y)|^2 \, dx + O(\varepsilon^\infty). \]  
In the domain of integration we have \( \chi_{j, \varepsilon} = 1 \) for \( s \in (\frac{2}{3}, \frac{1}{2}) \) and \( \varepsilon \) sufficiently small. This gives, applying the chain rule to the scaled function \( f \),
\[ |h(y)| \leq \varepsilon \sup_{z \in \mathbb{R}^d} |\nabla f(z)| \leq C \varepsilon^{-\frac{d}{4}} \varepsilon^{\frac{1}{2}}. \]
Thus
\[ \left( \int_{|x-x_j| < \varepsilon^s} |h(y)|^2 \, dy \right)^{\frac{1}{4}} \leq C \varepsilon^{-d(\frac{1}{4} - s) + \frac{1}{4}} \]  
Combining (3.51), (3.52) and (3.51) gives, taking \( \frac{1}{2} - s \) small,
\[ \|h_2\|_{L^2} \leq C \varepsilon^{-\frac{d}{4}} \varepsilon^{d(\frac{1}{4} - s) + \frac{1}{4}} = O\left( \varepsilon^{-\frac{d}{4}} \varepsilon^\alpha \right) \quad \text{for any} \quad \alpha < \frac{1}{2}. \]
To estimate \( h_1(\xi) \), observe that for \( y \in Q_x \)
\[ \left| \varepsilon^{\frac{d}{4}} x - \varepsilon^{\frac{d}{4}} y \right| \leq \sup_{y \in Q_x} \left| \frac{1}{2} (x - y) \cdot \xi \right| \leq C |\xi| \]
uniformly in \( x \) and \( \xi \). Inserting this into (3.49) and setting
\[ f(x) = \varepsilon^{-\frac{d}{4}} \tilde{f} \left( \frac{x - x_j}{\sqrt{\varepsilon}} \right) \]
gives by (3.21)
\[ |h_1(\xi)| \leq C \sum_{x \in (\varepsilon^2)^d} |\xi| |f(x)| \leq C |\xi| \varepsilon^{-\frac{d}{4}} \sum_{y \in (\sqrt{\varepsilon})^d} \left| \tilde{f} \left( y - \frac{x_j}{\sqrt{\varepsilon}} \right) \right| \leq C |\xi| \varepsilon^{-\frac{d}{4}}. \]
From (3.57), we obtain
\[ \|h_1\|^2_{L^2(\{\xi \leq \varepsilon^s\})} \leq C \varepsilon^{-\frac{d}{4}} \int_{|\xi| \leq \varepsilon^s} |\xi|^2 \, d\xi \leq C \varepsilon^{-d} \varepsilon^{d(\frac{1}{4} - s) + 2\alpha}. \]  
Thus, taking \( \frac{1}{2} - s \) small, we get for any \( \alpha < \frac{1}{2} \)
\[ \|h_1\|_{L^2(\{\xi \leq \varepsilon^s\})} \leq C \alpha \varepsilon^{-\frac{d}{4} + \alpha}. \]
Furthermore, since \( \sup_{|\xi| < \varepsilon^s} |\xi_j(\xi)| \leq C \varepsilon^{2\alpha} \), we get using (3.43) and (2.17)
\[ \|G_{\psi_j}\|_{L^2(\{\xi \leq \varepsilon^s\})} \leq C \varepsilon^{2\alpha} \|\chi_{j, \varepsilon} g^\varepsilon\|_{L^2} = O\left( \varepsilon^{-\frac{d}{4} + 2\alpha} \right). \]
Combining (3.58), (3.58), (3.58) and (3.41) proves (3.44) for \( I_1 \). The estimate for \( I_2 \) is similar.
(b):
Using the identity \( \varepsilon^d = \int_{Q_x} \, dy \) and setting \( W := \chi_{j, \varepsilon} V_{\varepsilon}^j \chi_{j, \varepsilon} \), the left hand side of (b) can analog to (3.44) be written as
\[ \langle f^\varepsilon, W g^\varepsilon \rangle_{L^2} = I_1 + I_2 + I_3, \]
where
\[ I_1 = \frac{1}{\varepsilon^d} \sum_{x \in (\varepsilon^2)^d} \int_{Q_x} (f(x) - f(y)) W g(x) \, dy \]
\[ I_2 = \frac{1}{\varepsilon^d} \sum_{x \in (\varepsilon^2)^d} \int_{Q_x} f(y) (W g(x) - W g(y)) \, dy \]
and
\[ I_3 = \varepsilon^{-d} \int_{\mathbb{R}^d} f(y) W g(y) \, dy = \varepsilon^{-d} \langle f, \chi_{j, \varepsilon} V_{\varepsilon}^j \chi_{j, \varepsilon} g \rangle_{L^2}. \]
Then, similar to the proof of (a), it remains to estimate $I_1$ and $I_2$. We claim that
\[
|I_j| = O \left( \varepsilon^{\frac{5}{2} - d} \right), \quad j = 1, 2.
\] (3.63)

By the scaling of $\chi_{j,\varepsilon}$
\[
\sup_{x \in \mathbb{R}^d} |W(x)| \leq \sup_{|x| \leq \varepsilon^\frac{1}{2}} |V_j^j(x)| = O \left( \varepsilon^\frac{1}{2} \right),
\] (3.64)
since $V_j^j$ is quadratic in $x$. Combining (3.20) and (3.64) shows (3.63) for $j = 1$. The proof for $I_2$ is similar. \hfill \Box

We still need one more estimate for the proof of Theorem 3.3. It concerns replacing the $x$-dependent quadratic approximation $\hat{T}_q$ of the kinetic energy by the operator $\hat{T}_{q,j}$ fixed at the well $x_j$.

**Lemma 3.9** Let $\hat{T}_q$ and $\hat{T}_{q,j}$ be given by (2.15) and (2.16) respectively for $1 \leq j \leq m$. Let $\chi_{j,\varepsilon}$ be the cut-off function defined in (3.2) and $f, g$ denote normalized eigenfunctions of $\hat{H}^j$ given in (3.12), then
\[
\left| \langle f, \chi_{j,\varepsilon} \hat{T}_q \chi_{j,\varepsilon} g \rangle_{L^2} - \langle f, \chi_{j,\varepsilon} \hat{T}_{q,j} \chi_{j,\varepsilon} g \rangle_{L^2} \right| = O \left( \varepsilon^\frac{1}{2} \right).
\]

**Proof.** By the definition of the operators $\hat{T}_q$ and $\hat{T}_{q,j}$
\[
\left| \langle f, \chi_{j,\varepsilon} (\hat{T}_q - \hat{T}_{q,j}) \chi_{j,\varepsilon} g \rangle_{L^2} \right| = \left| \langle f, \chi_{j,\varepsilon} \left[ \varepsilon^2 \sum_{\nu = 1}^{n} (B_{\nu\mu}(x) - B_{\nu\mu}(x_j)) \partial_{\nu} \partial_{\mu} + \varepsilon (t_1(x, 0) - t_1(x_j, 0)) \right] \chi_{j,\varepsilon} g \rangle_{L^2} \right|.
\]

As $g$ is scaled by $\varepsilon^{-\frac{1}{2}}$,
\[
\left\| \varepsilon^2 \partial_{\nu} \partial_{\mu} \chi_{j,\varepsilon} g \right\|_{L^2} = O(\varepsilon).
\] (3.65)

Since $|x - x_j| \leq 2 \varepsilon^\frac{1}{2}$ in the support of $\chi_{j,\varepsilon}$, we have by Hypothesis 1.1(a),(i) that $B_{\nu\mu}(x) - B_{\nu\mu}(x_j) = O \left( \varepsilon^\frac{1}{2} \right)$ and $t_1(x, 0) - t_1(x_j, 0) = O(\varepsilon^\frac{1}{2})$. Together with (3.65), this estimate proves the lemma by use of the Schwarz inequality. \hfill \Box

4. **Proof of Theorem 1.3**

Following Simon [20], we prove equality in (1.16) by proving an upper and a lower estimate. For the sake of the reader, we give a complete proof.

**4.1. Estimate from above.**

\[
\frac{E_n(\varepsilon)}{\varepsilon} \leq e_n + O \left( \varepsilon^\frac{1}{2} \right) \quad \text{as} \quad \varepsilon \to 0.
\] (4.1)

At first we use the points (a) and (c) of Lemma 3.6 leading to the estimate
\[
\left\langle g_{\alpha_j}^e, H_{\varepsilon} g_{\beta_l}^e \right\rangle_{L^2} = \left\langle g_{\alpha_j}^e, \chi_{j,\varepsilon} H_{\varepsilon} \chi_{j,\varepsilon} g_{\beta_l}^e \right\rangle_{L^2} + O \left( \varepsilon^\frac{1}{2} \right) \quad \text{(4.2)}
\]

By Proposition 3.1 and by (3.9) (quadratic approximation of $T_\varepsilon$ localized at $\xi = 0, x = x_j$ and of $V_\varepsilon$ localized at $x = x_j$) we have
\[
\left\langle g_{\alpha_j}^e, H_{\varepsilon} g_{\beta_l}^e \right\rangle_{L^2} = \left\langle g_{\alpha_j}^e, \chi_{j,\varepsilon} \left( O_{\varepsilon}^\varepsilon (\phi_{\alpha_j}) T_\varepsilon \chi_{j,\varepsilon} g_{\beta_l}^e \right) \right\rangle_{L^2} + O \left( \varepsilon^\frac{1}{2} \right)
\] (4.3)

where for the second step, the transition from (functions and scalar product in) $\ell^2 (\varepsilon \mathbb{Z})^d$ to $L^2 (\mathbb{R}^d)$, we used Lemma 3.7(a) and (b). Point (b) and (d) of Lemma 3.6 and (3.12) yield
\[
\varepsilon^d \left( \text{rhs of (1.3)} \right) = \left\langle g_{\alpha_j}, \hat{H}^j g_{\beta_l} \right\rangle_{L^2} + O \left( \varepsilon^\frac{1}{2} \right) = \left\langle g_{\alpha_j}^e, \hat{H}^j g_{\beta_l}^e \right\rangle_{L^2} + O \left( \varepsilon^\frac{1}{2} \right),
\] (4.4)
where the second equality follows from the fact that $H^j$ and $\tilde{H}^j$ are unitarily equivalent (see [33]). Since by definition $H^j g_{\alpha,j}^\varepsilon = \varepsilon e_{n(\alpha,j)} g_{\alpha,j}^\varepsilon$, the estimates (4.5) and (4.6) can be combined to give

$$\left(g_{\alpha,j}^\varepsilon, H^j g_{\beta}^\varepsilon\right)_{L^2} = \varepsilon^{-d} \left(\varepsilon e_{n(\alpha,j)} g_{n(\alpha,j), n(\beta,l)} + O\left(\varepsilon^{\frac{d}{2}}\right)\right),$$

where $n(\alpha,j)$ denotes the number of the eigenvalue corresponding to the pair $(\alpha,j)$. We shall show that (4.1) leads to (4.1) by use of the Min-Max-principle. Choose $\zeta_1, \ldots, \zeta_{n-1}$ in the domain of $H^\varepsilon$ and define

$$Q(\zeta_1, \ldots, \zeta_{n-1}) := \inf \left\{ \langle \psi, H^\varepsilon \psi \rangle_{L^2} \mid \psi \in \mathcal{D}(H^\varepsilon), \|\psi\| = 1, \psi \in [\zeta_1, \ldots, \zeta_{n-1}] \right\}$$

and

$$\mu_n(\varepsilon) := \sup_{\zeta_1, \ldots, \zeta_{n-1}} Q(\zeta_1, \ldots, \zeta_{n-1}).$$

For $\lambda > 0$ we can choose $\zeta_1, \ldots, \zeta_{n-1}$, such that

$$\mu_n(\varepsilon) \leq Q(\zeta_1, \ldots, \zeta_{n-1}) + \lambda.$$  

It follows from Lemma [3.4] that for $\varepsilon > 0$ sufficiently small the functions $g_{n(\alpha,j)}^\varepsilon := \varepsilon \frac{d}{2} g_{\alpha,j}^\varepsilon$ satisfy

$$\langle g_n^\varepsilon, g_m^\varepsilon \rangle_{L^2} = \delta_{n,m} + O(\varepsilon),$$

in particular they are linearly independent and $\mathcal{M}_n := \text{span}\{g_m^\varepsilon \mid m \leq n\}$ has dimension $n$. Then $\mathcal{N} := \mathcal{M}_n \cap [\zeta_1, \ldots, \zeta_{n-1}]$ is at least one dimensional. Thus there exists a function $\psi \in \mathcal{N}$ with $\|\psi\|_{L^2} = 1$ and it follows from (4.6), (4.7) and (4.8) that

$$Q(\zeta_1, \ldots, \zeta_{n-1}) \leq \langle \psi, H^\varepsilon \psi \rangle_{L^2} \leq \varepsilon e_n + O\left(\varepsilon^{\frac{d}{2}}\right)$$

Since $\lambda$ is arbitrary, we have by (4.8) and (4.10)

$$\mu_n(\varepsilon) \leq \varepsilon e_n + O\left(\varepsilon^{\frac{d}{2}}\right).$$

By Theorem [3.1] Hypothesis [1.1] ensures that $\inf \sigma_{ess}(H^\varepsilon) \geq c > 0$ uniformly in $\varepsilon$ for $\varepsilon$ sufficiently small. Since $\mu_n(\varepsilon)$ is by (4.11) of order $\varepsilon$, for $\varepsilon$ small enough it follows from the Min-Max-principle that $\mu_n(\varepsilon)$ belongs to the discrete spectrum and coincides with $E_n(\varepsilon)$.

### 4.2. Estimate from below.

$$\frac{E_n(\varepsilon)}{\varepsilon} \geq e_n + O\left(\varepsilon^{\frac{d}{2}}\right) \quad \text{as} \quad \varepsilon \to 0$$

For $n > 1$, let $l \leq n-1$ be such that $e_n = e_{n-1} = \ldots = e_{l+1} > e_l$ and set $e \in (e_l, e_n)$, for $n = 1$ choose $e < e_1$ (in particular $e \not\in \sigma(\bigoplus_j K_j)$). Then we claim that there exists a constant $C > 0$ such that

$$\langle \psi, H^\varepsilon \psi \rangle_{L^2} \geq \varepsilon e(\psi, \psi)_{L^2} + \langle \psi, \psi \rangle_{L^2} - C\varepsilon^\frac{d}{2} \|\psi\|^2_{L^2}, \quad \psi \in \mathcal{D}(H^\varepsilon),$$

for some symmetric operator $R_l$ with rank $R_l \leq l$. This implies (4.12). To see this implication, let $\psi \in E_n := \text{span}\{h_k \in L^2((\varepsilon^2)^{\delta}) \mid h_k \text{ is the } k \text{th eigenfunction of } H^\varepsilon, \|h_k\|_{L^2} = 1, 1 \leq k \leq n\}$. From the Min-Max-formula it follows that

$$\langle \psi, H^\varepsilon \psi \rangle_{L^2} \geq \langle \psi, H^\varepsilon \psi \rangle_{L^2}.$$  

On the other hand there exists a $\psi \in E_n \cap \ker R_l$, since $\dim \ker (R_l|E_n) \geq 1$. For this $\psi$ the inequality (4.13) yields

$$\langle \psi, H^\varepsilon \psi \rangle_{L^2} \geq \varepsilon e + O\left(\varepsilon^{\frac{d}{2}}\right),$$

which together with (4.11) gives (4.12). It therefore suffices to show (4.13).

By Lemma [3.3] $H^\varepsilon$ splits as

$$H^\varepsilon = \sum_{j=1}^{m} \chi_{j,\varepsilon} H^\varepsilon \chi_{j,\varepsilon} + \chi_{0,\varepsilon} H^\varepsilon \chi_{0,\varepsilon} + O\left(\varepsilon^{\frac{d}{2}}\right),$$

where the estimate on the error term in the following estimates is understood with respect to operator norm. $\chi_{0,\varepsilon}$ is supported in the region outside of the wells, thus $|x - x_j| > \varepsilon^{\frac{d}{2}}$ for $1 \leq j \leq m$ and $x \in \text{supp} \chi_{0,\varepsilon}$. Since the kinetic term is positive modulo terms of order $\varepsilon$ and the potential is of second order in $x$ or of order $\varepsilon$, we have for $\varepsilon$ sufficiently small, $e < e_n$ and some constant $C > 0$

$$\chi_{0,\varepsilon} H^\varepsilon \chi_{0,\varepsilon} \geq \chi_{0,\varepsilon} V \chi_{0,\varepsilon} \geq (-C\varepsilon + \tilde{C}\varepsilon^{\frac{d}{2}}) \chi_{0,\varepsilon}^2 \geq \varepsilon e \chi_{0,\varepsilon}^{2}.$$  

(4.17)
In the neighborhoods of the wells, (3.3) allows to approximate the potential by the quadratic term, therefore (3.3) and (4.17) give
\[
H_\varepsilon \geq \sum_{j=1}^{m} \chi_{j,\varepsilon}(T_\varepsilon + V_\varepsilon^j)\chi_{j,\varepsilon} + \varepsilon \ v \chi_{0,\varepsilon}^2 + O\left(\varepsilon^\frac{3}{2}\right).
\] (4.18)

In the first summand we introduce the partition of unity \(\hat{\phi}_{k,\varepsilon}, k = 1, 0\) in momentum space, defined in (3.5), and get by Lemma 3.3
\[
\sum_{j=1}^{m} \chi_{j,\varepsilon}(T_\varepsilon + V_\varepsilon^j)\chi_{j,\varepsilon} = \sum_{j=1}^{m} \chi_{j,\varepsilon}(x) \text{Op}_\varepsilon^\tau(\hat{\phi}_{0,\varepsilon})(T_\varepsilon + V_\varepsilon^j) \text{Op}_\varepsilon^\tau(\hat{\phi}_{0,\varepsilon}) \chi_{j,\varepsilon}(x) +
\]
\[
+ \sum_{j=1}^{m} \chi_{j,\varepsilon}(x) \text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon})(T_\varepsilon + V_\varepsilon^j) \text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon}) \chi_{j,\varepsilon}(x) + O\left(\varepsilon^\frac{3}{2}\right). \tag{4.19}
\]

By Proposition 3.1 modulo terms of order \(O\left(\varepsilon^\frac{3}{2}\right)\), it is possible to replace \(T_\varepsilon\) by \(T_{\varepsilon,q,j}\) near \(\xi = 0\) and \(x = x_j, j = 1, \ldots, m\). The function \(\hat{\phi}_{1,\varepsilon}\) is supported in the exterior region with \(|\xi| > \varepsilon^{\frac{3}{2}}\), thus we have by arguments similar to those leading to (4.17)
\[
\text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon})(T_\varepsilon + V_\varepsilon^j) \text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon}) \geq \varepsilon \ v \ \text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon})^2. \tag{4.20}
\]

Substituting (4.20) in (4.19), replacing \(T_\varepsilon\) by \(T_{\varepsilon,q,j}\) in the first summand of (4.19) and substituting the resulting equation in (4.18) yields
\[
H_\varepsilon \geq M + \varepsilon \sum_{j=1}^{m} \chi_{j,\varepsilon}(x) \left(\text{Op}_\varepsilon^\tau(\hat{\phi}_{1,\varepsilon})\right)^2 \chi_{j,\varepsilon}(x) + \varepsilon \ v \chi_{0,\varepsilon}^2 + O\left(\varepsilon^\frac{3}{2}\right), \quad \text{where} \tag{4.21}
\]
\[
M := \sum_{j=1}^{m} \chi_{j,\varepsilon}(x) \text{Op}_\varepsilon^\tau(\hat{\phi}_{0,\varepsilon})(T_{\varepsilon,q,j} + V_\varepsilon^j) \text{Op}_\varepsilon^\tau(\hat{\phi}_{0,\varepsilon}) \chi_{j,\varepsilon}(x)
\]

By the isometry of the Fourier transform
\[
\langle \psi, M\psi \rangle_{L^2} = \sum_{j=1}^{m} \left\langle \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi), (t_{\varepsilon,q,j} + \mathcal{F}_\varepsilon^{-1}V_\varepsilon^j\mathcal{F}_\varepsilon)\phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi) \right\rangle_{L^2}, \tag{4.22}
\]
\[
= \sum_{j=1}^{m} \left\langle \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi), (F_\varepsilon^{-1}H^j F_\varepsilon)\phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi) \right\rangle_{L^2}.
\]

In the last step we used that for \(\varepsilon\) sufficiently small we can replace the scalar product in \(L^2(\mathbb{R}^d)\) by the scalar product in \(L^2(\mathbb{T}^d)\), if we simultaneously replace \(\hat{\phi}_{0,\varepsilon}\) by \(\phi_{0,\varepsilon}\) and \(t_{\varepsilon,q,j}\) by \(t_{\varepsilon,q,j}\). This follows from the fact that the range of the integral is both cases restricted to the support of \(\phi_{0,\varepsilon}\). Moreover changing variables allows to replace \(H^j = \hat{T}_{\varepsilon,q,j} + V_\varepsilon^j\) by \(H^j\) (see (3.13)) and \(\mathcal{F}_\varepsilon^{-1}V_\varepsilon^j\mathcal{F}_\varepsilon = F_\varepsilon^{-1}V_\varepsilon^j F_\varepsilon\) and \(F_\varepsilon(\xi, \xi)F_\varepsilon^{-1} = -\varepsilon^2 \Delta\).

We introduce the spectral decomposition of \(F_\varepsilon^{-1}H^j F_\varepsilon\). Denote by \(e_{k,j}\) the kth eigenvalue of \(H^j\) and by \(l_j\) the number of eigenvalues of \(H^j\) not exceeding \(\varepsilon\). Thus \(e_l \leq e < e_l\) for all \(j\) and \(\sum_{j=1}^{m} l_j = l\). By replacing all eigenvalues \(e_{k,j} > e\) of \(H^j\) by \(e\) we get
\[
(F_\varepsilon^{-1}H^j F_\varepsilon) = \varepsilon \sum_{k} \sum_{k \leq l_j} e_{k,j} \Pi_{k}^{l} \geq \varepsilon \sum_{k \leq l_j} e_{k,j} \Pi_{k}^{l} + \varepsilon (1 - \sum_{k \leq l_j} \Pi_{k}^{l}), \tag{4.23}
\]

where \(\Pi_{k}^{l}\) denotes the projection on the eigenspace of \(e_{k,j}\). Inserting (4.23) into the right hand side of (4.22) and replacing \(\phi_{0,\varepsilon}\) by \(\hat{\phi}_{0,\varepsilon}\) and \(\langle \cdot, \cdot, \cdot \rangle_{L^2}\) by \(\langle \cdot, \cdot, \cdot \rangle_{\mathbb{T}}\) yields
\[
\langle \psi, M\psi \rangle_{\mathbb{T}} \geq \sum_{j=1}^{m} \left\{ \langle \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi), \varepsilon \sum_{k \leq l_j} (e_{k,j} - e) \Pi_{k}^{l} \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi) \rangle_{\mathbb{T}}
\]
\[
+ \varepsilon \varepsilon \langle \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi), \phi_{0,\varepsilon}\mathcal{F}_\varepsilon^{-1}(\chi_{j,\varepsilon}\psi) \rangle_{\mathbb{T}} \right\}. \tag{4.24}
\]
Thus by (4.24) together with (4.21) there exists a constant $C > 0$ such that

\[
\langle \psi, H_\varepsilon \psi \rangle_{L^2} \geq \varepsilon e \sum_{j=1}^{m} \left( \tilde{\phi}_0, \mathcal{F}_\varepsilon^{-1}(\chi_j, \varepsilon) \psi \right) + \left( \mathcal{F}_\varepsilon^{-1} \psi, R_{l} \mathcal{F}_\varepsilon^{-1} \psi \right)_{L^2} + \varepsilon e \sum_{j=1}^{m} \left( \tilde{\phi}_0, \mathcal{F}_\varepsilon^{-1}(\chi_j, \varepsilon) \psi \right)_{L^2} + \left( \mathcal{F}_\varepsilon^{-1} \psi, R_{l} \mathcal{F}_\varepsilon^{-1} \psi \right)_{L^2} - C \varepsilon^\frac{2}{3} \| \psi \|^2_{L^2},
\]

where

\[
R_{l} := \sum_{j=1}^{m} (\mathcal{F}_\varepsilon^{-1} \chi_j, \varepsilon) \tilde{\phi}_0, A_j \tilde{\phi}_0, (\mathcal{F}_\varepsilon^{-1} \chi_j, \varepsilon) \mathcal{F}_\varepsilon^{-1}, \quad A_j := \sum_{k \leq l} (\varepsilon(e_{k,j}-e)\Pi_k^l).
\]

Since $\operatorname{rank}(A + B) \leq \operatorname{rank}A + \operatorname{rank}B$ and rank $\Pi_k^l = 1$, the operator $A_j$ has rank at most $l_j$. Conjugation does not increase the rank and moreover $\sum_{j=1}^{m} l_j = l$, thus we get rank $R_l \leq l$. Introducing $\tilde{\phi}_0^2 + \tilde{\phi}_1^2 = 1$ in the fourth summand, we combine this term with the first and third summand and rewrite the rhs of (4.25) as

\[
\varepsilon e \sum_{j=0}^{m} \left( \tilde{\phi}_0, \mathcal{F}_\varepsilon^{-1}(\chi_j, \varepsilon) \psi \right) + \left( \mathcal{F}_\varepsilon^{-1} \psi, R_{l} \mathcal{F}_\varepsilon^{-1} \psi \right)_{L^2} + \varepsilon e \sum_{j=0}^{m} \left( \tilde{\phi}_0, \mathcal{F}_\varepsilon^{-1}(\chi_j, \varepsilon) \psi \right)_{L^2} + \left( \mathcal{F}_\varepsilon^{-1} \psi, R_{l} \mathcal{F}_\varepsilon^{-1} \psi \right)_{L^2} - C \varepsilon^\frac{2}{3} \| \psi \|^2_{L^2}.
\]

Again the first and third summand can be combined so that the cut-off functions in both spaces add up to 1. We thus get by (4.25) and (4.27), for some $C > 0$,

\[
\langle \psi, H_\varepsilon \psi \rangle_{L^2} \geq \varepsilon e \langle \psi, \psi \rangle_{L^2} + \langle \psi, B_l \psi \rangle_{L^2} - C \varepsilon^\frac{2}{3} \| \psi \|^2_{L^2},
\]

where $B_l := \mathcal{F}_\varepsilon R_{l} \mathcal{F}_\varepsilon^{-1}$ is again an operator of rank at most $l$. Thus (4.13) holds. Combined with (4.11), this completes the proof of Theorem 1.3

**Appendix A. Pseudo-differential operators in the discrete setting**

In the following, some properties of the symbols given in Definition 2.1 and of the associated operators are collected. For the sake of the reader, we recall the definitions of the $h$-scaled symbol classes $S^k_h(m)(\mathbb{R}^d)$ and of the associated pseudo-differential operators (see Dimassi-Sjöstrand [7] and Robert [13]).

**Definition A.1**

(a) A function $m : \mathbb{R}^d \to [0, \infty)$ is called an order function, if there exist constants $C_0 > 0$ and $N_0 > 0$ such that

\[
m(x) \leq C_0 (x - y)^{-N_0} m(y), \quad x, y \in \mathbb{R}^d.
\]

(b) For $\delta \in [0, 1]$, the space $S^k_h(m)(\mathbb{R}^d)$ consists of functions $a(x; \varepsilon)$ on $\mathbb{R}^d \times (0, 1]$, such that there exists constants $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d, \varepsilon \in (0, 1]$

\[
|\partial^\alpha_x a(x; \varepsilon)| \leq C_\alpha m(x)e^{k-\delta|\alpha|}.
\]

(c) Let $a_j \in S^k_h(m)(\mathbb{R}^d)$, $k_j \to \infty$, then $a \sim \sum_{j=0}^{\infty} a_j$ means that $a - \sum_{j=0}^{N} a_j \in S^{k_{N+1}}(m)(\mathbb{R}^d)$ for every $N \in \mathbb{N}$.

(d) A pseudo-differential operator $\operatorname{Op}_\varepsilon : \mathcal{C}_0^\infty(\mathbb{R}^d) \to \mathcal{C}_0^\infty(\mathbb{R}^d)$ associated to a symbol $a \in S^k_h(m)(\mathbb{R}^d)$ is defined by

\[
\operatorname{Op}_\varepsilon(a) u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x, \xi; \varepsilon) u(y) \, dy d\xi, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^d).
\]

We start this section showing that by use of the identification of functions on the torus $\mathbb{T}^d$ with $2\pi$-periodic functions on $\mathbb{R}^d$, the discrete operator $\operatorname{Op}_\varepsilon^T(a)$ associated to the symbol $a \in S^k_h(m)(\mathbb{R}^d \times \mathbb{T}^d)$ can be understood as a special case of the operator $\operatorname{Op}_\varepsilon(b)$ associated to the symbol $b \in S^k_h(m)(\mathbb{R}^d)$. 

First we notice (see for example Hörmander [13]) that for any $2\pi$-periodic function $g \in C^\infty (\mathbb{R}^d)$ the Fourier transform $F \cdot g$ defined in (2.4) satisfies

$$F \cdot g = \left( \frac{\varepsilon}{\sqrt{2\pi}} \right)^d \sum_{z \in (\mathbb{Z})^d} \delta_z c_z, \quad \text{where } c_z := \int_{|\pi|} e^{-\frac{i}{\varepsilon} \pi \mu} g(\mu) \, d\mu. \tag{A.2}$$

Thus for $a \in S_1^0 (\mathbb{R}^d)$ with $a(x, \xi + 2\pi \eta) = a(x, \xi)$ for any $\eta \in \mathbb{Z}^d$, $x, \xi \in \mathbb{R}^d$ and $u \in \mathcal{F}(\mathbb{R}^d)$ by (A.1)

$$\text{Op}_\varepsilon (a) u(x) = \frac{1}{(2\pi)^d} \left( F^{\varepsilon} (x, \xi \rightarrow x) a \right) (x) = \frac{1}{(2\pi)^d} \sum_{y \in \mathbb{Z}} \int_{|\pi|} e^{i\eta \cdot (x-y)} a(x, \xi \varepsilon) u(y) \, d\xi, \tag{A.3}$$

where, as in Remark 2.3, $\mathcal{G}_\varepsilon = (\varepsilon \mathbb{Z})^d + x$. If we denote by $r : \mathcal{F}(\mathbb{R}^d) \to \mathcal{C}((\varepsilon \mathbb{Z})^d)$ the restriction to the lattice $(\varepsilon \mathbb{Z})^d$, (A.3) implies

$$r \circ \text{Op}_\varepsilon (a) u = \text{Op}_\varepsilon (a) ru, \quad u \in \mathcal{F}(\mathbb{R}^d). \tag{A.4}$$

**Lemma A.2** Let $a \in S_0^0 (\mathbb{R}^d \times T^d)$, then, for fixed $\varepsilon > 0$, $\text{Op}_\varepsilon (a)$ defined in (2.3) is continuous : $s((\varepsilon \mathbb{Z})^d) \rightarrow s((\varepsilon \mathbb{Z})^d)$, where the space $s((\varepsilon \mathbb{Z})^d)$ with its natural Fréchet topology is defined in (2.10).

**Proof.** We will deduce the continuity of $\text{Op}_\varepsilon^\tau (a)$ on $s((\varepsilon \mathbb{Z})^d)$ from the continuity of $\text{Op}_\varepsilon$ on $\mathcal{F}(\mathbb{R}^d)$, which is proven e.g. in Grigs-Sjöstrand [7]. To this end, we consider a cut-off function $\zeta \in C^\infty_0 (\mathbb{R}^d)$ such that $\zeta(0) = 1$ and $\zeta(x) = 0$ for $|x| \geq 1$. We set $\zeta_{\varepsilon, z} := \zeta \left( \frac{1}{\varepsilon} (x-z) \right)$ and define

$$j : s((\varepsilon \mathbb{Z})^d) \rightarrow \mathcal{F}((\varepsilon \mathbb{Z})^d), \quad u \mapsto j u := \sum_{z \in (\varepsilon \mathbb{Z})^d} u(z) \zeta_{\varepsilon, z}.$$

Then $r \circ j = 1_s$ and $\text{Op}_\varepsilon^\tau (a) = r \circ \text{Op}_\varepsilon (a) \circ j$ by (A.4). It remains to show that $r$ and $j$ are continuous, which is straightforward with $\|rf\|_\alpha \leq \|f\|_{\alpha, 0}$ and $\|jv\|_{\alpha, \beta} \leq C_{\beta} \varepsilon^{-|\beta|} \|v\|_\alpha$ for some $C_{\beta} > 0$.

We define for $u \in \mathcal{F}(\mathbb{R}^{2d})$

$$e^{i\varepsilon D_x D_\xi} u(x, \xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{-\frac{i}{\varepsilon} \pi \eta} u(x - z, \xi - \eta) \, dz \, d\eta. \tag{A.5}$$

The following lemma is an adapted and more detailed version of Dimassi-Sjöstrand [7], Proposition 7.6.

**Lemma A.3** Let $0 \leq \delta \leq \frac{1}{2}$ and $m$ be an order function. Then $e^{i\varepsilon D_x D_\xi} : \mathcal{F}(\mathbb{R}^d \times T^d) \to \mathcal{F}(\mathbb{R}^d \times T^d)$ is continuous : $S_0^0 (m) \rightarrow S_0^0 (m)$. If $\delta < \frac{1}{2}$, then

$$e^{i\varepsilon D_x D_\xi} b(x, \xi) \sim \sum_{j = 0}^\infty \frac{1}{j!} \left( i \varepsilon D_x D_\xi \right)^j b \tag{A.6}$$

in $S_0^0 (m) \times T^d$. If we write $e^{i\varepsilon D_x D_\xi} b = \sum_{j = 0}^{N-1} \frac{(i \varepsilon D_x D_\xi)^j b}{j!} + R_N(b)$, the remainder $R_N(b)$ is an element of the symbol class $S_0^{N+1} (m)$ and the Fréchet-semimodules of $R_N$ depend (linearly) only on finitely many $\|b\|_{\alpha, \beta}$ with $|\alpha|, |\beta| \geq N$:

$$\|R_N(b)\|_{\alpha, \beta} \leq \sum_{N \leq |\alpha'|, |\beta'| \leq M} C_{\alpha, \beta, \alpha', \beta'} \|b\|_{\alpha', \beta'}, \tag{A.7}$$

for some $M \in N$ and $C_{\alpha, \beta, \alpha', \beta'} > 0$ independent of $\varepsilon \in (0, 1)$.

**Proof.** Since $S_0^0 (m) \times T^d$ injects continuously into $S_0^0 (m) \times \mathbb{R}^{2d}$, by [7], Prop.7.6, $e^{i\varepsilon D_x D_\xi}$ maps $S_0^0 (m) \times T^d$ continuously into $S_0^0 (m) \times T^d$. Thus, to prove continuity, it remains to show that $e^{i\varepsilon D_x D_\xi} a$ is periodic with respect to $\xi$ for $a \in S_0^0 (m) \times T^d$. Since $e^{i\varepsilon D_x D_\xi} : \mathcal{F}(\mathbb{R}^{2d}) \to \mathcal{F}(\mathbb{R}^{2d})$ is defined by $e^{i\varepsilon D_x D_\xi} \phi := u \left( e^{i\varepsilon D_x D_\xi} \phi \right)$, it suffices to prove on $\mathcal{F}(\mathbb{R}^{2d})$

$$e^{i\varepsilon D_x D_\xi} \tau_\gamma = \tau_\gamma e^{i\varepsilon D_x D_\xi}, \quad \gamma \in (2\pi \mathbb{Z})^d, \tag{A.8}$$
where \( \tau_\gamma \phi(x, \xi) := \phi(x, \xi + \gamma) \). But, since by (A.5) \( e^{i \varepsilon D_x D_\xi} \) is a convolution operator, it commutes with all translations, which shows (A.3).

Thus it remains to show that \( R_N(b) \) is in \( S^{r+N(1-2\delta)}(m)(\mathbb{R}^{2d}) \) - for then it is in \( S^{r+2N(1-2\delta)}(m)(\mathbb{R}^d \times \mathbb{T}^d) \) - and depends only on Fréchet-seminorms \( \|b\|_{\alpha, \beta} \) with \( |\alpha|, |\beta| \geq N \). We sketch the proof of these statements, since the standard proofs of (A.6) for \( b \in S^r(m)(\mathbb{R}^{2d}) \) or some similar classes (see e.g. Dimassi-Sjöstrand [7], Grigis-Sjöstrand [9], Martinez [16]) do not directly lead to these more refined remainder estimates.

First one proves the statement for \( b \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \). Then the integral in (A.5) converges absolutely and

\[
R_N(b) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \left( e^{-\frac{x^2}{\varepsilon^2}} - \sum_{j=0}^{N-1} \frac{1}{j!} \left( \frac{x^2}{\varepsilon^2} \right)^j \right) b(x-z, \xi-\eta) \, dz \, d\eta.
\]  

(A.9)

Thus it is formally obvious that \( \|R_N\|_{\alpha, \beta} \) depends only on Fréchet-seminorms \( \|b\|_{\alpha', \beta'} \) for \( |\alpha'|, |\beta'| \geq N \). One needs integration by parts and standard arguments to show that \( R_N \in S^{r+2N(1-2\delta)}(m)(\mathbb{R}^d \times \mathbb{T}^d) \) and that (A.7) holds with constants \( C_{\alpha, \beta, \alpha', \beta'} \) independent of \( b \).

Now let \( b \in S^r(m)(\mathbb{R}^{2d}) \) and choose a cut-off function \( h \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \) with \( h = 1 \) on the ball with radius 1 and supp \( \tilde{h} \) contained in the ball with radius 2. Set \( h_R(x) := h(\frac{x}{R}) \) and \( b_R := h_R \cdot b \) for \( R > 1 \). One readily verifies that the family \( b_R \) is uniformly bounded in \( S^r(m)(\mathbb{R}^{2d}) \). By standard arguments it follows that \( b_R \) converges to \( b \) in the topology of \( S^r(\hat{m})(\mathbb{R}^{2d}) \) for \( \hat{m}(x, \xi) = \langle x \rangle^m \langle \xi \rangle^m \) (see e.g. Grigis-Sjöstrand [9]). Furthermore, \( R_N(b_R) \) is uniformly bounded in \( S^{r+N(1-2\delta)}(m)(\mathbb{R}^{2d}) \) - using the dominated convergence theorem after integration by parts and the fact that \( R_N(D_x^a D_\xi^b) \) is \( D_x^a D_\xi^b R_N(b) \) for all symbols \( b \) - and converges pointwise to some symbol \( r_N \in S^{r+N(1-2\delta)}(m)(\mathbb{R}^{2d}) \). Again, \( R_N(b_R) \) converges to \( r_N \) in the topology of \( S^{r+N(1-2\delta)}(\hat{m})(\mathbb{R}^{2d}) \). Using the continuity of \( e^{i \varepsilon D_x D_\xi} : S^r(\hat{m})(\mathbb{R}^{2d}) \to S^r(\hat{m})(\mathbb{R}^{2d}) \), it follows that

\[
e^{i \varepsilon D_x D_\xi} b = \sum_{j=0}^{N-1} \frac{(i \varepsilon D_x \cdot D_\xi)^j}{j!} b + r_N.
\]

Thus \( r_N = R_N(b) \in S^{r+N(1-2\delta)}(m)(\mathbb{R}^{2d}) \), which completes the proof of Lemma (A.3).

Remark A.4 The rougher standard argument in e.g. Grigis-Sjöstrand [9] and Martinez [16] splits \( b = \tilde{b} + (1 - h) \cdot b \) with \( h \in \mathcal{C}_0^\infty(\mathbb{R}^{2d}) \). By stationary phase, \( e^{i \varepsilon D_x D_\xi} (1-h) \cdot b \in S^{-\infty}(m)(\mathbb{R}^{2d}) \), but its Fréchet-seminorms depend on all Fréchet-seminorms of \( b \). Of course, the relevant terms in the estimate for \( e^{i \varepsilon D_x D_\xi} (1-h) \cdot b \) are precisely cancelled by corresponding terms in the estimate for \( e^{i \varepsilon D_x D_\xi} \tilde{b} \). This cancellation, however, is not evident from the estimates stated [9] and [16].

The following corollary concerns the composition of symbols.

Corollary A.5 The map

\[
\mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d) \times \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d) \ni (a, b) \mapsto a \# b \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{T}^d)
\]

with

\[
(a \# b)(x, \xi; \varepsilon) := \left( e^{i \varepsilon D_x D_\xi} a(x, \xi; \varepsilon) b(y, y; \varepsilon) \right) \big|_{y=x, \eta=\varepsilon}
\]

(A.10)

has a bilinear continuous extension :

\[
S^{r_1}(m_1)(\mathbb{R}^d \times \mathbb{T}^d) \times S^{r_2}(m_2)(\mathbb{R}^d \times \mathbb{T}^d) \to S^{r_1+r_2}(m_1 m_2)(\mathbb{R}^d \times \mathbb{T}^d)
\]

for all \( \delta_k \in [0, \frac{1}{2}] \), \( k = 1, 2 \) and all order functions \( m_1, m_2 \), where \( \delta := \max(\delta_1, \delta_2) \). If in addition \( \delta_1 + \delta_2 < 1 \),

\[
(a \# b)(x, \xi; \varepsilon) \sim \sum_{\alpha \in \mathbb{N}^d} \frac{(i \varepsilon)^{|\alpha|}}{|\alpha|!} \left( \partial_\alpha^2 a(x, \xi; \varepsilon) \right) \left( \partial_\alpha^2 b(x, \xi; \varepsilon) \right)
\]

(A.11)

(with respect to \( k_N = r_1 + r_2 + N(1 - \delta_1 - \delta_2) \)). Writing

\[
a \# b(x, \xi; \varepsilon) = \sum_{|\alpha| = 0}^{N-1} \frac{(i \varepsilon)^{|\alpha|}}{|\alpha|!} \left( \partial_\alpha^2 a(x, \xi; \varepsilon) \right) \left( \partial_\alpha^2 b(x, \xi; \varepsilon) \right) + R_N(a, b, \varepsilon),
\]
the remainder \( R_N \) is an element of the symbol class \( \mathcal{S}_N^{r_1+r_2+N(1-\delta_1-\delta_2)}(m_1m_2) \) and it depends linearly on a finite number of derivatives of the single symbols \( a \) and \( b \). Furthermore it depends only on derivatives of \( a \) and \( b \) with respect to \( \xi \) and \( x \) respectively which are at least of order \( N \).

Proof. By the Leibnitz rule, the map

\[
S_{\delta_1}(m_1) \times S_{\delta_2}(m_2) \ni (a, b) \mapsto a \cdot b \in S_{\delta_1+r_2}(m_1m_2)
\]

is continuous, since each Fréchet-norm of the product depends only on a finite number of Fréchet-norms of \( a \) and \( b \). The same is true for the restriction map. The main part follows from Lemma A.3 by doubling the dimension of the space.

It is shown in [9] that the \#-product of symbols reflects the composition of the associated operators. In particular for \( a \in S_{\delta_1}(m_1) (\mathbb{R}^d \times \mathbb{T}^d) \), \( b \in S_{\delta_2}(m_2) (\mathbb{R}^d \times \mathbb{T}^d) \) with \( 0 \leq \delta_k \leq \frac{1}{2} \),

\[
\left( \text{Op}_\varepsilon^T(a) \right) \circ \left( \text{Op}_\varepsilon^T(b) \right) = \text{Op}_\varepsilon^T(a \# b) .
\]

The following proposition is an adapted version of the Calderon-Vaillancourt-Theorem (see Calderon-Vaillancourt [5]). The proof is inspired by the proof of the Calderon-Vaillancourt-Theorem given by Hwang [14].

**Proposition A.6** Let \( a \in S_{\delta}(1)(\mathbb{R}^d \times \mathbb{T}^d) \) with \( 0 \leq \delta \leq \frac{1}{2} \). Then there exists a constant \( M > 0 \) such that, for the associated operator \( \text{Op}_\varepsilon^T(a) \) given by (2.3), the estimate

\[
\left\| \text{Op}_\varepsilon^T(a)u \right\|_{\mathcal{L}(\varepsilon(T\mathbb{Z})^d)} \leq M \varepsilon^r \left\| u \right\|_{\mathcal{L}(\varepsilon(T\mathbb{Z})^d)}
\]

holds for any \( u \in s((\varepsilon\mathbb{Z})^d) \) and \( \varepsilon > 0 \). \( \text{Op}_\varepsilon^T(a) \) can therefore be extended to a continuous operator:

\[
\ell^2((\varepsilon\mathbb{Z})^d) \to \ell^2((\varepsilon\mathbb{Z})^d) \text{ with } \| \text{Op}_\varepsilon^T(a) \| \leq M \varepsilon^r .
\]

Moreover \( M \) can be chosen depending only on a finite number of Fréchet-seminorms of the symbol \( a \).

**Remark A.7** There is a dual approach to the operators \( \text{Op}_\varepsilon^T(a) \), starting from pseudo-differential calculus on the torus \( \mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d \) (see e.g. Gérard-Nier [8]). We denote by \( j : \bigcup_{k \in \mathbb{Z}} H^k(\mathbb{T}^d) \to \mathcal{S}'(\mathbb{R}^d) \) the injection defined by periodic continuation, where \( H^k(\mathbb{T}^d) \) is the Sobolev-space of order \( k \) on the torus. Then we define the \( \varepsilon \)-quantization of a periodic symbol \( a \), i.e. \( a(k+\mu,\eta) = a(k,\eta) \) for all \( \mu \in 2\pi\mathbb{Z}^d \), in some Hörmander class \( S(m_1) \) by

\[
\text{Op}_{\varepsilon,T}(a) = j^{-1} \circ \text{Op}_{\varepsilon,T}(a) \circ j ,
\]

where \( \text{Op}_{\varepsilon,T}(b) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d) \) is induced from

\[
\text{Op}_{\varepsilon,T}(b)(u)(x) := (\varepsilon^2 2\pi)^{-d} \int_{\mathbb{R}^d} e^{\frac{i}{\varepsilon^2 2\pi}(k-k')\eta} b(tk + (1-t)k', \eta) u(k') \, dk' \, d\eta , \quad u \in \mathcal{S}(\mathbb{R}^d)
\]

(cf. Robert [13] and Dimassi-Sjöstrand [7]). \( \text{Op}_{\varepsilon,T}(a) \) is well defined, since by the periodicity of \( a \), the operator \( \text{Op}_{\varepsilon,T}(a) \) commutes with all translations \( \tau_\gamma \), \( \gamma \in \mathbb{T}^d \). Essentially, this is the approach in Gérard-Nier [8]. One now observes that (A.2) may be rewritten as

\[
F_{\varepsilon} \circ j = \varepsilon^{d-r^*} \circ F : \mathcal{S}(\mathbb{R}^d) \to s((\varepsilon\mathbb{Z})^d) , \quad \text{for } r^* (u) = \sum_{\gamma \in (\varepsilon\mathbb{Z})^d} u(\gamma) \delta_\gamma , \quad u \in s^*((\varepsilon\mathbb{Z})^d)
\]

where \( r^* \) is the adjoint of the restriction map \( r : \mathcal{S}(\mathbb{R}^d) \to s((\varepsilon\mathbb{Z})^d) \). Furthermore, a straightforward calculation gives

\[
\text{Op}_{\varepsilon}(b) \circ F_{\varepsilon} = F_{\varepsilon} \circ \text{Op}_{\varepsilon,0}(b) \quad \text{for } b(x) := b(x, \xi) .
\]

Thus, for \( b \in S_{\delta}(m_1)(\mathbb{R}^d \times \mathbb{T}^d) \), the symbol \( \hat{b} \) is periodic in the sense mentioned before (A.14).

Moreover, taking adjoints in (A.4) gives on \( s^*((\varepsilon\mathbb{Z})^d) \)

\[
r^* \circ \text{Op}_{\varepsilon}^T(a) = \text{Op}_{\varepsilon}(a) \circ r^*
\]

for all \( a \in S_{\delta}(m_1)(\mathbb{R}^d \times \mathbb{T}^d) \), since \( (\text{Op}_{\varepsilon}^T(a))^* = \text{Op}_{\varepsilon}^T(a^\#) \) for \( a^\#(x, \xi) = e^{i\xi D_x D_\xi} \hat{a}(x, \xi) \). By Lemma A.3 \( a^\# \in S_{\delta}(m_1)(\mathbb{R}^d \times \mathbb{T}^d) \) for \( a \) in this class, if \( \delta \leq \frac{1}{2} \). Combining (A.14), (A.13), (A.16) and (A.17) gives for \( a \in S_{\delta}(m_1)(\mathbb{R}^d \times \mathbb{T}^d) \)

\[
\text{Op}_{\varepsilon}^T(a) \circ F = F_{\varepsilon,0} \circ \text{Op}_{\varepsilon}^T(a) ,
\]
since \( r^* \) is injective. Since \( \mathcal{F}_r \) is unitary, \( L^2((\varepsilon \mathbb{Z})^d)-\)boundedness of \( \text{Op}_r^T(a) \) is equivalent to \( L^2(\mathbb{T}^d)-\)boundedness of \( \text{Op}_r^T(\hat{a}) \).

Under the additional assumption that
\[
|\partial_{\alpha}^r a(k, \eta)| \leq C_\alpha \langle \eta \rangle^{-|\alpha|},
\]
\( L^2(\mathbb{T}^d) \)-boundedness of \( \text{Op}_r^T(a) \) follows from the standard Calderon-Vaillancourt-Theorem for \( \text{Op}_r(a) \) in \( \mathbb{R}^d \) and integration by parts (see Gérard-Nier \[8\] for a simple proof in the case \( \varepsilon = 1, t = 1; \) the proof works for any \( t \in [0, 1] \)).

Proof. Since \( \varepsilon^{-r} a \in S^0_0(1) \), we can restrict the proof to the case \( r = 0 \). It suffices to show that for all \( u, v \) with compact support the estimate
\[
\left| \left\langle u, \text{Op}_r^T(a)v \right\rangle_{L^2} \right| \leq M \|u\|_{L^2} \|v\|_{L^2}
\]
holds, where \( M \) depends only on a finite number of Fréchet-seminorms \( \|\partial_{\alpha}^r \epsilon^\beta a\|_\infty \). We assume that \( \text{supp} a \) is compact. The general case then follows by standard techniques (approximating \( a \) by a compactly supported sequence \( a_n \) with \( \text{Op}_r^T(a_n) \to \text{Op}_r^T(a) \) strongly). We have
\[
\left\langle u, \text{Op}_r^T(a)v \right\rangle_{L^2} = (2\pi)^{-\frac{3d}{4}} \sum_{x \in (\varepsilon \mathbb{Z})^d} \int d\eta e^{-\frac{i}{2} \eta x} \left( d\mathcal{F}_r^{-1} u \right) (\eta) \sum_{y \in (\varepsilon \mathbb{Z})^d} \int d\xi e^{-\frac{i}{2} (x-y) \xi} a(x, \xi; \varepsilon) v(y).
\]
By the assumption on \( a \), the iterated integrals (and sums) in \( (A.20) \) can be understood as integrals on the product space (thus Fubini’s Theorem holds). Let \( \zeta \in C_0^\infty(\mathbb{R}) \) be a cut-off function with \( \zeta = 1 \) at zero, then we split the right hand side of \( (A.20) \) in two summands by introducing \( \zeta \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right) \) and \( (1 - \zeta) \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right) \). It then suffices to show that the part multiplied with \( (1 - \zeta) \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right) \), which we denote by \( I_1 \), is an element of \( S^\infty(1) \) and the part multiplied with \( \zeta \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right) \), denoted by \( I_2 \), is bounded by a constant independent of \( \varepsilon \).

To analyze \( I_1 \), we use the operators
\[
L_1 := \frac{1 - \varepsilon \Delta_\xi}{\sqrt{2} \varepsilon (x - y)} \quad \text{and} \quad L_2 := \frac{\varepsilon \Delta^{\varepsilon}}{2d - 2 \sum_{\nu} \cos \left( \frac{1}{\sqrt{2} \varepsilon} (\xi + \eta_\nu) \right)},
\]
where \( -\Delta^{\varepsilon}_{\xi} := 2d - \sum_{\nu} \left( \tau_{\sqrt{2} \varepsilon \xi} + \tau_{\sqrt{2} \varepsilon \eta_\nu} \right) \) is a scaled version of the discrete Laplacian \( \Delta_\xi \) defined in \( (3.25) \). Then \( L_1 \) and \( L_2 \) leave \( e^{-\frac{1}{2} \varepsilon ((x-y)\xi + x \eta)} \) invariant, and we have by the symmetry of \( \Delta^{\varepsilon}_{\xi} \) and \( 1 - \varepsilon \Delta_\xi \) (using Fubini)
\[
I_1 = (2\pi)^{-\frac{3d}{4}} \sum_{y, x \in (\varepsilon \mathbb{Z})^d} \int \int d\eta d\xi \left( L_2 L_1 e^{-\frac{1}{2} ((x-y)\xi + x \eta)} \right)
\times \left( d\mathcal{F}_r^{-1} u \right) (\eta) \left( 1 - \zeta \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right) \right) a(x, \xi; \varepsilon) v(y)
\begin{align*}
&= (2\pi)^{-\frac{3d}{4}} \sum_{y, x \in (\varepsilon \mathbb{Z})^d} \int \int d\eta d\xi e^{-\frac{1}{2} ((x-y)\xi + x \eta)} \left( d\mathcal{F}_r^{-1} u \right) (\eta) \frac{v(y)}{\sqrt{2} \varepsilon (x - y)}
\times \sum_{|\alpha| \leq 2l} K_{k, \alpha}(\xi, \eta; \varepsilon) P_{\alpha}(\sqrt{2} \varepsilon D_\xi) \left( -\Delta^{\varepsilon}_{\xi} \right)^k a(x, \xi; \varepsilon),
\end{align*}
\]
where
\[
K_{k, \alpha}(\xi, \eta; \varepsilon) = Q_{\alpha}(\sqrt{2} \varepsilon D_\xi) \frac{1 - \zeta \left( \frac{1}{\sqrt{2} \varepsilon} |\xi + \eta| \right)}{2d - 2 \sum_{\nu} \cos \left( \frac{1}{\sqrt{2} \varepsilon} (\xi + \eta_\nu) \right)}.
\]
and $Q_\alpha$ and $P_\alpha$ denote polynomials with $\deg Q_\alpha + \deg P_\alpha = 2l$. With the notation
\begin{equation}
G_{k,\alpha}(x, \xi; \varepsilon) := \mathcal{F}_\varepsilon \left[ \mathcal{F}_\varepsilon^{-1} \tilde{u}(\cdot) (\cdot) K_{k,\alpha}(\xi, \cdot; \varepsilon) \right](x)
\end{equation}
we have
\begin{equation}
I_1 = (2\pi)^{-\frac{d}{2}} \sum_{x \in (\mathbb{Z})^d} \int_{[-\pi, \pi]^d} d\xi \, e^{-\frac{\varepsilon}{2} x \cdot \xi} \sum_{|\alpha| \leq 2l} G_{k,\alpha}(x, \xi) P_\alpha(\sqrt{\varepsilon} D_\xi) \left(-\Delta_x \varepsilon^2\right)^k a(x, \xi; \varepsilon).
\end{equation}
Thus by the Schwarz-inequality
\begin{equation}
|I_1| \leq \sum_{|\alpha| \leq 2l} \sup_{x, \xi} \left| P_\alpha(\sqrt{\varepsilon} D_\xi) \left(-\Delta_x \varepsilon^2\right)^k a(x, \xi; \varepsilon) \right| \|F_i\|_{l^2_\varepsilon x T^d}^2 \|G_{k,\alpha}\|_{l^2_\varepsilon x T^d}^2.
\end{equation}
By the isometry of $\mathcal{F}_\varepsilon$
\begin{equation}
\|F_i\|_{l^2_\varepsilon x T^d}^2 = \|\mathcal{F}_\varepsilon F_i\|^2_{l^2_\varepsilon x T^d}
\end{equation}
we have
\begin{equation}
\|G_{k,\alpha}\|^2_{l^2_\varepsilon x T^d} = \|\mathcal{F}_\varepsilon^{-1} G_{k,\alpha}\|^2_{l^2_\varepsilon x T^d}
\end{equation}
with $t = x - y$, where the last estimate follows from (3.21) for $l$ big enough. For $G_{k,\alpha}$, we have by the isometry of $\mathcal{F}_\varepsilon^{-1}$
\begin{equation}
\|G_{k,\alpha}\|^2_{l^2_\varepsilon x T^d} = \|\mathcal{F}_\varepsilon^{-1} G_{k,\alpha}\|^2_{l^2_\varepsilon x T^d}
\end{equation}
Using $\sqrt{\varepsilon} D_\xi f(x/\sqrt{\varepsilon}) = O(1)$ for any smooth function with bounded derivative and $\pi^2 (1 - \cos(\sqrt{\varepsilon} \tau)) \geq \frac{\pi^2}{\varepsilon}$ for $|\frac{\tau}{\sqrt{\varepsilon}}| \leq \pi$, we have for $\tau = \xi + \eta$
\begin{equation}
|K_{k,\alpha}(\xi, \eta; \varepsilon)|^2 \leq \tilde{C}_{k,\alpha} \left| \frac{\tau}{\sqrt{\varepsilon}} \right|^{-4k}.
\end{equation}
Since for $k$ large enough
\begin{equation}
\int_{\supp(1-\zeta)(\frac{\tau}{\sqrt{\varepsilon}})} \left| \frac{\tau}{\sqrt{\varepsilon}} \right|^{-4k} d\tau \leq C_k \varepsilon^{\frac{d}{2}},
\end{equation}
we get by inserting (A.28) into (A.27)
\begin{equation}
\|G_{k,\alpha}\|^2_{l^2_\varepsilon x T^d} \leq C_k \varepsilon^{\frac{d}{2}} \|u\|^2_{l^2_\varepsilon((\mathbb{Z})^d)}.
\end{equation}
To analyze $P_\alpha(\sqrt{\varepsilon} D_\xi) \left(-\Delta_x \varepsilon^2\right)^k a(x, \xi; \varepsilon)$, we use that by Taylor expansion
\begin{equation}
\Delta_x \varepsilon^2 a(x, \xi; \varepsilon) = -\sum_{\nu=1}^d \varepsilon \partial_{x_{2\nu}}^2 a(x, \xi; \varepsilon) + \frac{\varepsilon^2}{3!} \int_0^1 \partial_{x_{2\nu}}^2 a(x + t\sqrt{\varepsilon} \nu; \varepsilon) dt.
\end{equation}
By iteration, we have for $a \in S_0^0(1)$
\begin{equation}
\sup_{x, \xi} \left| P_\alpha(\sqrt{\varepsilon} D_\xi) \left(-\Delta_x \varepsilon^2\right)^k a(x, \xi; \varepsilon) \right| \leq \tilde{M}_{k,\alpha} \varepsilon^{2k(\frac{d}{2} - \delta)},
\end{equation}
where $\tilde{M}_{k,\alpha}$ depends only on Fréchet-seminorms of $a$ up to order $3k + |\alpha|$. Inserting (A.26), (A.29) and (A.30) in (A.25) yields for any $k \in \mathbb{N}$
\begin{equation}
|I_1| \leq M \varepsilon^{2k(\frac{d}{2} - \delta)} \|u\|_{l^2_\varepsilon((\mathbb{Z})^d)} \|v\|_{l^2_\varepsilon((\mathbb{Z})^d)}.
\end{equation}
Thus $I_1 = O(\varepsilon^\infty)$. 
To get an estimate for the modulus of $I_2$, which denotes the integral over the support of $\zeta$, we use $L_1$ given in \ref{A.21} to get by integration by parts and similar arguments
\[
I_2 = (2\pi)^{-\frac{d}{2}} \sum_{x,y \in \mathbb{Z}^d} \frac{v(y)}{\frac{1}{2\pi}(x-y)} \int_{[-\pi,\pi]^d} d\eta d\xi e^{-\frac{i}{\eta}(x-y)(\xi+\eta)} \chi \times (\mathcal{F}_\varepsilon^{-1}\hat{u})(\eta) (1 - \varepsilon \Delta_\xi)^j \zeta \left( \frac{1}{\sqrt{\varepsilon}}|\xi + \cdot| \right) a(x,\xi;\varepsilon).
\]

Setting, for $P_\alpha$ and $Q_\alpha$ as above,
\[
G_\alpha(x,\xi;\varepsilon) := \mathcal{F}_\varepsilon \left( (\mathcal{F}_\varepsilon^{-1}\hat{u})(\cdot)Q_\alpha(\sqrt{\varepsilon}D_\xi)\zeta \left( \frac{1}{\sqrt{\varepsilon}}|\xi + \cdot| \right) \right)(x)
\]
we have, with $F_l$ as in \ref{A.23},
\[
I_2 = (2\pi)^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} \int_{[-\pi,\pi]^d} d\xi e^{-\frac{i}{\eta}\xi} F_l(x,\xi;\varepsilon) \sum_{|\alpha| \leq 2l} G_\alpha(x,\xi;\varepsilon) P_\alpha(\sqrt{\varepsilon}D_\xi)a(x,\xi;\varepsilon).
\]
By the isometry of $\mathcal{F}_\varepsilon$ and the arguments leading to \ref{A.29}, we have
\[
\|G_\alpha\|_{L^2}^2 \leq \|\mathcal{F}_\varepsilon^{-1}G_\alpha\|^2_{L^2} \leq \left( \int_{[-\pi,\pi]^d} d\eta \left( |(\mathcal{F}_\varepsilon^{-1}\hat{u})(\eta)|^2 \int_{[-\pi,\pi]^d} d\xi \left| Q_\alpha(\sqrt{\varepsilon}D_\xi)\zeta \left( \frac{1}{\sqrt{\varepsilon}}|\xi + \eta| \right) \right|^2 \right) \right)^{1/2}
\leq C_l \|u\|^2_{L^2(\mathbb{Z}^d)} \int_{\text{supp} \zeta(\cdot)} d\xi \leq \varepsilon^{\frac{d}{2}} C_l \|u\|^2_{L^2(\mathbb{Z}^d)},
\]
where the estimate in the last line follows from the scaling of $\zeta$. Analog to \ref{A.25} we get by \ref{A.33}, \ref{A.26}, and \ref{A.30} for $k = 0$
\[
|I_2| \leq M \|u\|_{L^2(\mathbb{Z}^d)} \|v\|_{L^2(\mathbb{Z}^d)}
\]
and therefore we finally get \ref{A.19}. \hfill \Box

For $a \in S^\infty_{\delta_1}(m_{\alpha})(\mathbb{R}^d \times \mathbb{T}^d)$ and $b \in S^\infty_{\delta_2}(m_{\alpha})(\mathbb{R}^d \times \mathbb{T}^d)$ let $[a,b] = a\#b - b\#a$ denote the commutator in symbolic calculus. Then by \ref{A.12}
\[
\text{Op}_{\varepsilon}^\tau([a,b]_{\#}) = \left[ \text{Op}_{\varepsilon}^\tau(a), \text{Op}_{\varepsilon}^\tau(b) \right].
\]
The following lemma, which gives the resulting symbol class of double commutators, is an application of Corollary \ref{A.5.} and \ref{A.12}.

**Lemma A.8** Let $h(x,\xi) \in S^\infty_{\delta_1}(m_{\alpha})(\mathbb{R}^d \times \mathbb{T}^d)$, $\delta_2 < \frac{1}{2}$ and let $\chi, \phi \in S^\infty_{\delta_2}(m_{\alpha})(\mathbb{R}^d \times \mathbb{T}^d)$, $\delta_1 < \frac{1}{2},$ where $\chi$ does not depend on $\xi$ and $\phi$ does not depend on $x$. Then for $\alpha, \alpha_1, \alpha_2 \in \mathbb{N}^d$ with $\alpha_1 + \alpha_2 = 2\delta$ and $|\alpha_k| \geq 1, k = 1, 2,$ for $\delta := \max \{\delta_1, \delta_2\}$ and for any $N \in \mathbb{N}, N \geq 3$: 

(a) $[\chi(h), [\chi, h]\#]_{\#} = \sum_{2 \leq |\alpha| < N} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} (\partial_x^{\alpha_1} h)(x,\xi) \sum_{\alpha_1, \alpha_2} (\partial_x^{\alpha_2} \chi)(x) \left( \partial_x^{\alpha_2} h \right)(x) + R_N.$

(b) $[\phi(h), [\phi, h]\#]_{\#} = \sum_{2 \leq |\alpha| < N} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} (\partial_x^{\alpha} h)(x,\xi) \sum_{\alpha_1, \alpha_2} \chi \left( \partial_x^{\alpha_1} \phi \right)(x) \left( \partial_x^{\alpha_2} \phi \right)(x) + \tilde{R}_N,$

where $R_N$ and $\tilde{R}_N$ are elements of the symbol class $S^N_{\delta_1}(1 - \delta_1 - \delta_2)(m_{\alpha}^2 m_{\alpha})$ and depend linearly on a finite number of Fréchet-seminorms of the single symbols. Furthermore they depend only on the derivatives of $h$, which are at least of order $N$ and of the product of derivatives of $\chi$ and $\phi$ respectively, which are of order $N_1$ and $N_2$, such that $N_1 + N_2 \geq N$.

**Proof.** (a): The double commutator is given by
\[
[\chi(h), [\chi, h]\#]_{\#} = \chi\#\chi\#h(x,\xi) + h\#\chi\#h(x,\xi) - 2\chi\#h\#\chi(x,\chi).
\]
By Lemma \(^{A.3}\) these terms are for \(\alpha \in \mathbb{N}^d\) given by
\[
\chi^\# \chi^\# h(x, \xi) = \chi \cdot \chi \cdot h(x, \xi)
\]
\[
h^\# \chi^\# h(x, \xi) = \sum_{|\alpha| \leq N-1} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} (\partial^2_{\xi} h) \left( \partial^2_{\xi} \chi \right) (x, \xi) + R_N(x, \xi; \varepsilon)
\]
\[
\chi h^\# \chi h^\# (x, \chi) = \sum_{|\alpha| \leq N-1} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} \chi \left( \partial^2_{\xi} h \right) \left( \partial^2_{\xi} \chi \right) (x, \xi) + \tilde{R}_N(x, \xi; \varepsilon).
\]
where \(R_N, \tilde{R}_N \in S_0^{N(1-\delta_1-\delta_2)}(m_1^2m_2)\). The terms with \(|\alpha| = 0\) and \(|\alpha| = 1\) cancel in \(^{A.3}\).
Furthermore all terms with \(2 \chi \partial^2_{\xi} \chi_j\) cancel. Thus the Leibnitz formula gives the expansion
\[
[\chi(x), [\chi(x), h(x, \xi)]_\#]_\# = \sum_{2 \leq |\alpha| \leq N-1} (i\varepsilon)^{|\alpha|} \left( \partial^2_{\xi} h \right) \sum_{|\alpha_1|, |\alpha_2| \in \mathbb{N}^d} \frac{1}{|\alpha_1|!|\alpha_2|!} \left( \partial^2_{\xi} \chi \right)(x, \xi) + R_N(x, \xi; \varepsilon)
\]
where the second sum runs over \(\alpha_1 + \alpha_2 = \alpha\) with \(|\alpha_k| \geq 1, k = 1, 2\) and \(R_N \in S_0^{N(1-\delta_1-\delta_2)}(m_1^2m_2)\). The statement on the symbol class follows at once from this expansion, since each summand is at least of order \(\varepsilon^{2(1-\delta_1-\delta_2)}\) and by use of the Leibnitz rule.

(b):
As above the double commutator consists of the terms
\[
[\phi(\xi), [\phi(\xi), h(x, \xi)]_\#]_\# = \phi^\# \phi^\# h(x, \xi) + h^\# \phi^\# \phi(x, \xi) - 2\phi^\# h^\# \phi(x, \chi)
\]
and the summands have for \(\alpha \in \mathbb{N}^d\) the expansions
\[
\phi^\# \phi^\# h(x, \xi) = \sum_{|\alpha| \leq N-1} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} \left( \partial^2_{\xi} h \right) \left( \partial^2_{\xi} \phi^2 \right) (x, \xi) + R_N(x, \xi; \varepsilon)
\]
\[
h^\# \chi^\# \chi(x, \xi) = h \cdot \phi \cdot \phi(x, \xi)
\]
\[
\chi h^\# \chi h^\# (x, \chi) = \sum_{|\alpha| \leq N-1} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} \phi \left( \partial^2_{\xi} h \right) \left( \partial^2_{\xi} \phi \right) (x, \xi) + \tilde{R}_N(x, \xi; \varepsilon),
\]
where \(R_N, \tilde{R}_N \in S_0^{N(1-\delta_1-\delta_2)}(m_1^2m_2)\). Therefore, as discussed in (a), \(^{A.36}\) gives with
\[
[\phi(\xi), [\phi(\xi), h(x, \xi)]_\#]_\# \sim \sum_{2 \leq |\alpha| \leq N-1} \frac{(i\varepsilon)^{|\alpha|}}{|\alpha|!} \left( \partial^2_{\xi} h \right) \sum_{\alpha_1, \alpha_2 \in \mathbb{N}^d, \alpha_1+\alpha_2=\alpha} \left( \partial^2_{\xi} \phi \right) \left( \partial^2_{\xi} \phi \right)(x, \xi) + R_N(x, \xi; \varepsilon)
\]
with \(\alpha_1 + \alpha_2 = \alpha\), \(|\alpha_k| \geq 1, k = 1, 2\) and \(R_N \in S_0^{N(1-\delta_1-\delta_2)}(m_1^2m_2)\). The statement on the symbol class follows from this expansion as discussed in (a).

The additional properties of \(R_N\) and \(\tilde{R}_N\) respectively follow immediately from the properties of remainder in Corollary \(^{A.3}\).}

**APPENDIX B. PERSSON’S THEOREM IN THE DISCRETE SETTING**

In this section we will prove a theorem on the inﬁmum of the essential spectrum of \(H_\varepsilon\) acting in \(l^2(\mathbb{Z}^d)\), which is similar to Persson’s Theorem for Schrödinger operators. The proof follows the proof of Persson’s Theorem in the Schrödinger setting given in Helffer \(^{11}\) and Agmon \(^{1}\) respectively.

**THEOREM B.1** Let \(H_\varepsilon = T_\varepsilon + V_\varepsilon\) satisfy Hypothesis \(^{L.1}\), denote by \(\sigma_{ess}(H_\varepsilon)\) the essential spectrum of \(H_\varepsilon\) and deﬁne
\[
\Sigma(H_\varepsilon) := \sup_{K \subset (\varepsilon \mathbb{Z})^d} \inf_{\phi \in c_0 \left( (\varepsilon \mathbb{Z})^d \setminus K \right)} \frac{(H_\varepsilon \phi, \phi)^2}{\|\phi\|_{l^2}^2},
\]
where \(c_0(D)\) denote the space of real-valued functions on \((\varepsilon \mathbb{Z})^d\) with compact, i.e., ﬁnite, support in \((\varepsilon \mathbb{Z})^d \setminus D\). Then
\[
\inf \sigma_{ess}(H_\varepsilon) = \Sigma(H_\varepsilon).
\]

The proof of Theorem \(^{B.1}\) is divided in two Lemmata and the main part.
Lemma B.2 For $x \in (\mathbb{Z})^d$ and $R > 0$ let $B_x(R) := \{ y \in (\mathbb{Z})^d \mid |x - y| < R \}$ denote the ball around $x$ with radius $R$ and

$$\Lambda_R(x, H_z) := \inf \left\{ \frac{\langle H_z \phi, \phi \rangle_{L^2}}{\| \phi \|_{L^2}} ; \phi \in c_0 (B_x(R)) \right\}. \quad (B.2)$$

Then for all $\delta > 0$ there exists a radius $R_\delta > 0$ such that for all $R > R_\delta$ and $\phi \in c_0 ((\mathbb{Z})^d)$

$$\langle H_z \phi, \phi \rangle_{L^2} \geq \sum_{x \in (\mathbb{Z})^d} (\Lambda_R(x, H_z) - \delta) |\phi(x)|^2. \quad (B.5)$$

Proof of Lemma B.2. Let $\rho \in C_0^\infty (\mathbb{R}^d)$ be real valued with $\rho(x) = 0$ for $|x| \geq \frac{1}{2}$ and $\int_{\mathbb{R}^d} |\rho(x)|^2 \, dx = 1$ and define

$$\rho_{y,R} := \rho \left( \frac{y - x}{R} \right).$$

Then $\rho_y, \phi \in c_0 (B_y (\frac{R}{2}))$ and therefore by the definition of $\Lambda_R$

$$\langle H_z \rho_y, \phi, \phi \rangle_{L^2} \geq \sum_{x \in (\mathbb{Z})^d} \Lambda_R(x, H_z) |\rho_y, \phi \rangle_{L^2}^2. \quad (B.6)$$

Since $B_y (\frac{R}{2}) \subset B_x(R)$ for $|x - y| < \frac{R}{2}$ and thus $\Lambda_y (x) \geq \Lambda_R(x)$, we get the estimate

$$\langle H_z \rho_y, \phi, \phi \rangle_{L^2} \geq \sum_{x \in (\mathbb{Z})^d} \Lambda_R(x, H_z)(\rho_y, \phi \rangle_{L^2}^2 \quad (B.3)$$

To analyze the scalar product we use that $T_\varepsilon$ is self adjoint and $\phi, \rho$ are real valued, yielding

$$\langle T_\varepsilon \rho_y, \phi, \phi \rangle_{L^2} = \frac{1}{2} (\langle T_\varepsilon \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} + \langle T_\varepsilon \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} + \langle T_\varepsilon \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2})$$

$$= \langle T_\varepsilon \phi, \rho_y \rangle_{L^2} + \frac{1}{2} \langle [T_\varepsilon, \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} \rangle_{L^2}.$$ \quad (B.4)

Since $[T_\varepsilon, \rho_y, \phi]$ it follows that

$$\langle T_\varepsilon \rho_y, \phi \rangle_{L^2} = \langle T_\varepsilon \phi, \rho_y \rangle_{L^2} + \frac{1}{2} \langle [T_\varepsilon, \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} + \langle \rho_y, \phi \rangle_{L^2} \rangle_{L^2} \quad (B.5)$$

To analyze the double commutator, we use the symbolic calculus introduced in Section A. By Lemma A.3 the symbol associated to the operator $[\rho_y, \phi, [T_\varepsilon, \rho_y, \phi]]$ is given by

$$\rho_{y,R}(x), [t(x, \xi), \rho_y, \phi(x)]_{\#} \#$$

$$= \sum_{\alpha, \beta \leq |\alpha| + |\beta|} \frac{(ix)^{|\alpha|}}{\alpha!} \frac{(i\xi)^{|\beta|}}{\beta!} \sum_{|\alpha_1| + |\alpha_2| = |\alpha|} \left( \partial_{x_1}^{\alpha_1} \rho_y (x) \right) \left( \partial_{x_2}^{\alpha_2} \rho_y (x) \right) + R_N(t, \rho_y, \phi), \quad (B.6)$$

where $R_N$ depends on a finite number of derivatives of $\rho_y, \phi$, which are at least of order $N$. By the scaling of $\rho_y, \phi$, it follows that $|\nabla_x \rho_y, \phi(x)| \leq \frac{C}{R^2}$ for $C$ suitable. Since all terms in the finite sum in $[\rho_y, \phi, [T_\varepsilon, \rho_y, \phi]]$ depend on a product of two (at least first order) derivatives of $\rho_y, \phi$, any Fréchet semi-norm of the symbol of the double commutator is of order $\frac{1}{R^2}$, By Proposition A.6 the same statement follows for the operator-norm of the associated operator, thus there is a constant $C > 0$ such that

$$\| [\rho_y, \phi, [T_\varepsilon, \rho_y, \phi]] \|_\infty \leq \frac{C}{R^2} \quad (B.7)$$

By the Cauchy-Schwarz inequality, we get by inserting (B.3) and (B.6) in (B.4)

$$\langle H_z \phi, \rho_y^2 \phi \rangle_{L^2} \geq \sum_{x \in (\mathbb{Z})^d} \Lambda_R(x, H_z) |\rho_y, \phi \rangle_{L^2}^2 - \frac{C}{R^2} \sum_{x \in B_y(R)} |\phi(x)|^2. \quad (B.8)$$

We remark that by setting $z = \frac{y - x}{R}$

$$\int_{\mathbb{R}^d} \rho_y^2 (y) \, dy = R^d \int_{\mathbb{R}^d} \rho_z^2 (z) \, dz = R^d \quad (B.9)$$
Proof of Lemma B.3.
We split the proof in two parts showing the two fundamental inequalities.

Lemma B.3
Let $\Lambda_R(x, H_\varepsilon)\rho_{y,R}^2(x)\phi^2(x) - \frac{C}{R^2} \sum_{x \in (\varepsilon \mathbb{Z})^d} 1_{|x-y|<R} |\phi(x)|^2$ dy
by (B.12).

Thus integration of the left hand side of (B.7) with respect to $y$ yields by (B.8)
$$\int_{\mathbb{R}^d} \langle H_\varepsilon \phi, \rho_{y,R}^2 \phi \rangle_{L^2} dy = \langle H_\varepsilon \phi, \int_{\mathbb{R}^d} \rho_{y,R}^2 dy \phi \rangle_{L^2} = R^d \langle H_\varepsilon \phi, \phi \rangle_{L^2}. \tag{B.10}$$

If we integrate the right hand side of (B.7) with respect to $y$ and use (B.9), we get
$$\int_{\mathbb{R}^d} \left( \sum_{x \in (\varepsilon \mathbb{Z})^d} \Lambda_R(x, H_\varepsilon)\rho_{y,R}^2(x)\phi^2(x) - \frac{C}{R^2} \sum_{x \in (\varepsilon \mathbb{Z})^d} 1_{|x-y|<R} |\phi(x)|^2 \right) dy$$
$$= R^d \left( \sum_{x \in (\varepsilon \mathbb{Z})^d} \Lambda_R(x, H_\varepsilon)\phi^2(x) - \frac{C'}{R^2} \sum_{x \in (\varepsilon \mathbb{Z})^d} |\phi(x)|^2 \right). \tag{B.11}$$

The Integration of both sides of (B.7) with respect to $y$ and division by $R^d$ gives by (B.10) and (B.11)
$$\langle H_\varepsilon \phi, \phi \rangle_{L^2} \geq \sum_{x \in (\varepsilon \mathbb{Z})^d} \left( \Lambda_R(x, H_\varepsilon) - \frac{C'}{R^2} \right) |\phi(x)|^2. \tag{B.12}$$

By choosing for $\delta > 0$ the radius $R_\delta = \sqrt{\frac{C}{\delta}}$, the statement of Lemma B.2 follows for all $R > R_\delta$ by (B.12).

The family $\Lambda_R(x, H_\varepsilon)$ describes the lowest eigenvalue of the Dirichlet problem with respect to the ball $B_x(R)$. The next lemma relates this family with $\Sigma(H_\varepsilon)$.

Lemma B.3 Let $\Lambda_R(x, H_\varepsilon)$ and $\Sigma(H_\varepsilon)$ defined in (B.2) and (B.4), respectively, then
$$\Sigma(H_\varepsilon) = \lim_{R \to +\infty} \liminf_{|x| \to \infty} \Lambda_R(x, H_\varepsilon). \tag{B.13}$$

Proof of Lemma B.3 We split the proof in two parts showing the two fundamental inequalities.

Step 1: Estimate from above
$$\Sigma(H_\varepsilon) \leq \lim_{R \to +\infty} \liminf_{|x| \to \infty} \Lambda_R(x, H_\varepsilon) \tag{B.14}$$

Let $K \subset (\varepsilon \mathbb{Z})^d$ compact and $R > 0$ fixed. Then $B_x(R) \subset (\varepsilon \mathbb{Z})^d \setminus K$ for $|x|$ large enough and thus
$$\inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{L^2}}{\|\phi\|_{L^2}^2}; \phi \in c_0((\varepsilon \mathbb{Z})^d \setminus K) \right\} \leq \inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{L^2}}{\|\phi\|_{L^2}^2}; \phi \in c_0(B_x(R)) \right\} (= \Lambda_R(x, H_\varepsilon)).$$

This inequality is satisfied for all $|x|$ large enough and the left hand side is independent of $x$, thus
$$\inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{L^2}}{\|\phi\|_{L^2}^2}; \phi \in c_0((\varepsilon \mathbb{Z})^d \setminus K) \right\} \leq \liminf_{|x| \to \infty} \Lambda_R(x, H_\varepsilon).$$

The left hand side of this inequality is independent of $R$ and the right hand side understood as a function in $R$ is monotonically decreasing and bounded from below, thus the limit $R \to \infty$ is well defined and
$$\inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{L^2}}{\|\phi\|_{L^2}^2}; \phi \in c_0((\varepsilon \mathbb{Z})^d \setminus K) \right\} \leq \lim_{R \to +\infty} \liminf_{|x| \to \infty} \Lambda_R(x, H_\varepsilon).$$

Now the right hand side is independent of the choice of $K$, thus we can take the supremum over all compact sets $K \subset (\varepsilon \mathbb{Z})^d$ and by the definition of $\Sigma(H_\varepsilon)$, this shows (B.14).

Step 2: Estimate from below
$$\Sigma(H_\varepsilon) \geq \lim_{R \to +\infty} \liminf_{|x| \to \infty} \Lambda_R(y, H_\varepsilon). \tag{B.15}$$

By the definition of $\liminf$, it follows that for all $\delta > 0$ and all $R > 0$ there exists an $R_0$ such that for all $|x| > R_0$
$$\Lambda_R(x, H_\varepsilon) \geq \liminf_{|x| \to \infty} \Lambda_R(x, H_\varepsilon) - \delta.$$
It follows immediately that for all \( \phi \in c_0 \left((\varepsilon \mathbb{Z})^d \setminus \overline{B_0(R_0)}\right) \)
\[
\sum_{x \in (\varepsilon \mathbb{Z})^d} \Lambda_R(x, H_x) |\phi(x)|^2 \geq \left(\liminf_{|x| \to \infty} \Lambda_R(x, H_x) - \delta\right) \|\phi\|_{\ell^2}^2 .
\] (B.16)

By Lemma B.2 we know that for all \( \delta > 0 \) and \( \phi \in c_0 \left((\varepsilon \mathbb{Z})^d\right) \) there exists \( R_\delta \) such that for all \( R > R_\delta \)
\[
\langle H_\varepsilon \phi, \phi \rangle_{\ell^2} \geq \sum_{x \in (\varepsilon \mathbb{Z})^d} (\Lambda_R(x, H_x) - \delta) |\phi(x)|^2 .
\] (B.17)

Inserting (B.17) in (B.16) it follows that for all \( \delta > 0 \) there exists \( R_\delta \) such that for all \( R > R_\delta \) there exists \( R_0 \) such that for all \( \phi \in c_0 \left((\varepsilon \mathbb{Z})^d \setminus \overline{B_0(R_0)}\right) \)
\[
\frac{\langle H_\varepsilon \phi, \phi \rangle_{\ell^2}}{\|\phi\|_{\ell^2}^2} \geq \liminf_{|x| \to \infty} \Lambda_R(x, H_x) - 2\delta .
\] (B.18)

By the definition of \( \Sigma(H_\varepsilon) \) it follows directly that
\[
\Sigma(H_\varepsilon) \geq \inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{\ell^2}}{\|\phi\|_{\ell^2}^2} \mid \phi \in c_0 \left((\varepsilon \mathbb{Z})^d \setminus \overline{B_0(R_0)}\right) \right\} .
\] (B.19)

The equation (B.17) holds for all \( \phi \in c_0 \left((\varepsilon \mathbb{Z})^d \setminus \overline{B_0(R_0)}\right) \), thus we can take on the left hand side the infimum over all these functions, which together with (B.19) yields
\[
\Sigma(H_\varepsilon) \geq \liminf_{|x| \to \infty} \Lambda_R(x, H_x) - 2\delta .
\] (B.20)

The left hand side is independent of \( R \) and since the relation holds for all \( R > R_\delta \), it is possible to take the limit \( R \to \infty \), which yields for all \( \delta > 0 \)
\[
\Sigma(H_\varepsilon) \geq \lim_{R \to +\infty} \liminf_{|x| \to \infty} \Lambda_R(x, H_x) - 2\delta .
\]

Thus in the limit \( \delta \) the estimate (B.19) follows. \( \square \)

Proof of Theorem B.7. We discuss the cases \( \Sigma(H_\varepsilon) = \infty \) and \( \Sigma(H_\varepsilon) < \infty \) separately.

Case 1: \( \Sigma(H_\varepsilon) < \infty \):
As in the preceding proof, we conclude the equality by showing that both inequalities hold.

Step 1: Estimate from below
\[
\inf \sigma_{ess}(H_\varepsilon) \geq \Sigma(H_\varepsilon)
\] (B.21)

As a function of \( R \), the term \( \lim_{|x| \to \infty} \Lambda_R(x, H_x) \) is monotonically decreasing, thus it follows by Lemma B.3 that for fixed \( R > 0 \)
\[
\Sigma(H_\varepsilon) \leq \liminf_{|x| \to \infty} \Lambda_R(x, H_x)
\]
and thus for all \( \delta > 0 \) there exists \( a_\delta \) such that for all \( x \in (\varepsilon \mathbb{Z})^d \) with \( |x| > a_\delta \)
\[
\Sigma(H_\varepsilon) - \frac{\delta}{2} \leq \Lambda_R(x, H_x) .
\] (B.22)

On the other hand denoting by \( \sigma(H_\varepsilon) \) the spectrum of \( H_\varepsilon \), it is clear by the definition of \( \Lambda_R(x, H_x) \) and the Min-Max-principle that
\[
\Lambda_R(x, H_x) \geq \inf \sigma(H_\varepsilon) .
\] (B.23)

Since \( H_\varepsilon \) is bounded from below, it follows by (B.22) and (B.23) that there exists a constant \( C > 0 \) such that for all \( x \in (\varepsilon \mathbb{Z})^d \)
\[
\Lambda_R(x, H_x) \geq \Sigma(H_\varepsilon) - C .
\] (B.24)
We choose a function $W \in c_0 ((\varepsilon \mathbb{Z})^d)$ such that $W(x) \geq C$ for $|x| < a_\delta$ and $W(x) \geq 0$ everywhere. Then for $H_\varepsilon + W$ it follows by Lemma 3.22 and 3.24 that for $\phi \in c_0 ((\varepsilon \mathbb{Z})^d)$

$$\langle (H_\varepsilon + W)\phi, \phi \rangle_{L^2} \geq \sum_{x \in (\varepsilon \mathbb{Z})^d} (W(x) - \Lambda_R(x, H_\varepsilon) - \frac{\delta}{2})|\phi(x)|^2$$

$$\geq \sum_{|x| \leq a_\delta} (\Sigma(H_\varepsilon) - \frac{\delta}{2})|\phi(x)|^2 + \sum_{|x| > a_\delta} (W(x) + \Sigma(H_\varepsilon) - \delta)|\phi(x)|^2$$

$$\geq (\Sigma(H_\varepsilon) - \delta) \sum_{x \in (\varepsilon \mathbb{Z})^d} |\phi(x)|^2 .$$

Thus

$$\inf \sigma_{ess}(H_\varepsilon + W) \geq \inf \sigma(H_\varepsilon + W) \geq \Sigma(H_\varepsilon) - \delta ,$$

where the first estimate follows directly by the definition of the spectra. The perturbation $W$ is compactly supported, thus each $u \in \ell^2((\varepsilon \mathbb{Z})^d)$ is mapped by $W$ to a lattice function with compact support, i.e. which is non-zero only at finitely many lattice points. Thus $W$ is a finite rank operator and in particular compact. Using Weyl’s theorem (see e.g. Reed-Simon [17]), it follows that

$$\sigma_{ess}(H_\varepsilon + W) = \sigma_{ess}(H_\varepsilon)$$

and since (B.25) holds for all $\delta > 0$ the estimate (B.21) is shown.

**Step 2:** Estimate from above

$$\inf \sigma_{ess}(H_\varepsilon) \leq \Sigma(H_\varepsilon)$$

Fix $\mu < \inf \sigma_{ess}(H_\varepsilon)$ and denote by $\Pi_\mu := \Pi_{(-\infty, \mu]}$ the spectral projection to the eigenspace of energies smaller or equal to $\mu$. Since $\mu$ lies below the essential spectrum and $H_\varepsilon$ is semi-bounded from below, it follows that $\Pi_\mu$ has finite rank. Thus there exists an orthonormal system of eigenfunctions $\psi_1, \ldots, \psi_n \in \ell^2((\varepsilon \mathbb{Z})^d)$ such that

$$\Pi_\mu = \sum_{j=1}^n \langle \cdot, \psi_j \rangle_{L^2} \psi_j$$

and for all $\delta > 0$ there exists an $R_\delta$ such that

$$\sum_{|x| > R_\delta} |\psi_j(x)|^2 \leq \delta .$$

Therefore (by the Cauchy-Schwarz inequality) for all $\phi \in c_0 ((\varepsilon \mathbb{Z})^d \setminus B_0(R_\delta))$

$$||\Pi_\mu \phi(x)||_{L^2}^2 = \sum_{j=1}^n |\langle \phi, \psi_j \rangle_{L^2}|^2 \leq ||\phi||_{L^2}^2 \sum_{j=1}^n \sum_{|x| > R_\delta} |\psi_j(x)|^2 \leq \delta ||\phi||_{L^2}^2 .$$

By the definition of $\Pi_\mu$ and since there exists a constant $C > 0$ such that $H_\varepsilon \geq -C$, we have

$$\langle H_\varepsilon \phi, \phi \rangle_{L^2} \geq \mu \langle (1 - \Pi_\mu) \phi, (1 - \Pi_\mu) \phi \rangle_{L^2} - C \langle \Pi_\mu \phi, \Pi_\mu \phi \rangle_{L^2} .$$

Therefore

$$\Sigma(H_\varepsilon) \geq \inf \left\{ \frac{\langle H_\varepsilon \phi, \phi \rangle_{L^2}}{||\phi||_{L^2}^2} \mid \phi \in c_0 ((\varepsilon \mathbb{Z})^d \setminus B_0(R_\delta)) \right\}$$

$$\geq \inf \left\{ \frac{\mu ||(1 - \Pi_\mu) \phi||^2}{||\phi||_{L^2}^2} - C ||\Pi_\mu \phi||^2 \mid \phi \in c_0 ((\varepsilon \mathbb{Z})^d \setminus B_0(R_\delta)) \right\}$$

$$= \inf \left\{ \mu - (C + \mu) \frac{||\Pi_\mu \phi||^2}{||\phi||_{L^2}^2} \mid \phi \in c_0 ((\varepsilon \mathbb{Z})^d \setminus B_0(R_\delta)) \right\}$$

and by (B.27)

$$\Sigma(H_\varepsilon) \geq \mu - (C + \mu)\delta .$$

The left hand side is independent of $\delta$, thus for $\delta \rightarrow 0$ we get

$$\Sigma(H_\varepsilon) \geq \mu$$

for any $\mu < \inf \sigma_{ess}(H_\varepsilon)$ and thus in the limit $\mu \rightarrow \inf \sigma_{ess}(H_\varepsilon)$ the estimate (B.26) follows and thus Theorem 3.1 is proven.
Case 2: $\Sigma(H_x) = \infty$.

By Lemma B.3 it follows at once that $\lim_{|x| \to \infty} \Lambda_R(x, H_x) = \infty$, because $\Lambda_R(x, H_x)$ is monotonically decreasing with respect to $R$. Thus for all $M > 0$ there exists an $a_M$ such that for all $x \in (\varepsilon \mathbb{Z})^d$ with $|x| > a_M$ the estimate $\Lambda_R(x, H_x) \geq M$ holds. On the other hand by (B.23) and the semi-boundedness of $H_x$ it follows that there exists a constant $C > 0$ such that

$$\Lambda_R(x, H_x) \geq -C, \quad \text{for all } x \in (\varepsilon \mathbb{Z})^d.$$  

We can choose a function $W \in c_0 ((\varepsilon \mathbb{Z})^d)$ such that $W(x) \geq C + M$ for $|x| < a_M$ and $W(x) \geq 0$ everywhere. Then

$$\langle (H_x + W)\phi, \phi \rangle_{L^2} \geq \langle (W + \Lambda_R(\cdot, H_x) - \frac{\delta}{2})\phi, \phi \rangle_{L^2} \geq \left( M - \frac{\delta}{2} \right) \|\phi\|_{L^2}^2$$

and thus for all $M > 0$ there exists a function $W \in c_0 ((\varepsilon \mathbb{Z})^d)$ such that

$$\sigma_{ess}(H_x + W) \geq \sigma(H_x + W) \geq M.$$  

As in the case $\Sigma(H_x) < \infty$ we have $\sigma(H_x + W) = \sigma(H_x)$ and therefore $\sigma_{ess}(H_x) \geq M$ for all $M > 0$ and thus $\sigma_{ess}(H_x) = \infty$. \hfill $\square$

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References

[1] S. Agmon: Lectures on Exponential Decay of Solutions of Second-order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrödinger Operators, Mathematical Notes 29, Princeton University Press, 1982

[2] E. Baake, M. Baake, A. Bovier, M. Klein: An asymptotic maximum principle for essentially linear evolution models, J. Math. Biol. 50 no.1, p. 83-114, 2005

[3] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein: Metastability in stochastic dynamics of disordered mean-field models, Probab. Theory Relat. Fields 119, p. 99-161, 2001

[4] A. Bovier, M. Eckhoff, V. Gayrard, M. Klein: Metastability and low lying spectra in reversible Markov chains, Comm. Math. Phys. 228, p. 219-255, (2002)

[5] A. P. Calderon, R. Vaillancourt: On the Boundedness of Pseudo-Differential Operators, J.Math.Soc. Japan 23,2, p. 374-378, 1971

[6] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon: Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Springer, 1987

[7] M. Dimassi, J. Sjöstrand: Spectral Asymptotics in the Semi-Classical Limit, London Mathematical Society Lecture Note Series 268, Cambridge University Press, 1999

[8] C. Gérard, F. Nier: Scattering theory for the perturbations of periodic Schrödinger operators J. Math. Kyoto Univ. 38 (1998), no. 4, 595-634.

[9] A. Grigis, J. Sjöstrand: Microlocal Analysis for Differential Operators, London Mathematical Society, Lecture Note Series 196, Cambridge University Press, 1994

[10] B. Helffer: Semi-Classical Analysis for the Schrödinger Operator and Applications, LNM 1336, Springer, 1988

[11] B. Helffer: Spectral Theory and application, Cours de DEA 1999-2000

[12] B. Helffer, J. Sjöstrand: Multiple wells in the semi-classical limit I, Comm. in P.D.E. 9 (1984), p. 337-408

[13] L. Hörmander: The Analysis of Linear Partial Differential Operators I, Springer-Verlag Berlin, 1983

[14] I. L. Hwang: The $L^2$-Boundedness of Pseudodifferential Operators, Trans.Amer.Math.Soc. 302, p. 55-76, 1987

[15] M. Klein, E. Rosenberger: Agnon-Type Estimates for a class of Difference Operators, to appear in Ann Inst. H. Poincare

[16] A. Martinez: An Introduction to Semiclassical and Microlocal Analysis, Springer-Verlag, 2002

[17] M. Reed, B. Simon: Methods of Modern Mathematical Physics 4, Academic Press, 1979

[18] D. Robert: Autour de l’Approximation Semi-Classique, Progr. in Math.68. Birkhäuser, 1987

[19] E. Rosenberger: Asymptotic Spectral Analysis and Tunneling for a class of Difference Operators, Thesis, [http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-7393](http://nbn-resolving.de/urn:nbn:de:kobv:517-opus-7393)

[20] B. Simon: Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions, Ann. Inst. H. Poincare Phys. Theor. 38, p. 295 - 308, 1983

MARKUS KLEIN, UNIVERSITÄT POTSDAM, INSTITUT FÜR MATHEMATIK, AM NEUEN PALAIS 10, 14469 POTSDAM

E-mail address: mklein@math.uni-potsdam.de

ELKE ROSENBERGER, UNIVERSITÄT POTSDAM, INSTITUT FÜR MATHEMATIK, AM NEUEN PALAIS 10, 14469 POTSDAM

E-mail address: eroesen@rz.uni-potsdam.de