INTEGER DECOMPOSITION PROPERTY OF FREE SUMS OF
CONVEX POLYTOPES

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Abstract. Let \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) be integral convex polytopes of dimension \( d \) and \( e \) which contain the origin of \( \mathbb{R}^d \) and \( \mathbb{R}^e \), respectively. In the present paper, under some assumptions, the necessary and sufficient condition for the free sum of \( P \) and \( Q \) to possess the integer decomposition property will be presented.

Introduction

A convex polytope is called integral if any of its vertices has integer coordinates. Let \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) be convex polytopes and suppose that \( 0_d \in P \) and \( 0_e \in Q \), where \( 0_d \in \mathbb{R}^d \) denotes the origin of \( \mathbb{R}^d \) and \( 0_e \in \mathbb{R}^e \) denotes that of \( \mathbb{R}^e \). We introduce the canonical injections \( \mu : \mathbb{R}^d \to \mathbb{R}^{d+e} \) by setting \( \mu(\alpha) = (\alpha, 0_e) \in \mathbb{R}^{d+e} \) with \( \alpha \in \mathbb{R}^d \) and \( \nu : \mathbb{R}^e \to \mathbb{R}^{d+e} \) by setting \( \nu(\beta) = (0_d, \beta) \in \mathbb{R}^{d+e} \) with \( \beta \in \mathbb{R}^e \). In particular, \( \mu(0_d) = \nu(0_e) = 0_{d+e} \), where \( 0_{d+e} \) denotes the origin of \( \mathbb{R}^{d+e} \). Then \( \mu(P) \) and \( \nu(Q) \) are convex polytopes of \( \mathbb{R}^{d+e} \) with \( \mu(P) \cap \nu(Q) = 0_{d+e} \in \mathbb{R}^{d+e} \). The free sum of \( P \) and \( Q \) is the convex hull of the set \( \mu(P) \cup \nu(Q) \) in \( \mathbb{R}^{d+e} \). It is written as \( P \oplus Q \). One has \( \dim(P \oplus Q) = \dim P + \dim Q \).

For a convex polytope \( P \subset \mathbb{R}^d \) and for each integer \( n \geq 1 \), we write \( nP \) for the convex polytope \( \{n\alpha : \alpha \in P\} \subset \mathbb{R}^d \). We say that an integral convex polytope \( P \subset \mathbb{R}^d \) possesses the integer decomposition property if, for each \( n \geq 1 \) and for each \( \gamma \in nP \cap \mathbb{Z}^d \), there exist \( \gamma^{(1)}, \ldots, \gamma^{(n)} \) belonging to \( P \cap \mathbb{Z}^d \) such that \( \gamma = \gamma^{(1)} + \ldots + \gamma^{(n)} \).

Let \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) be convex polytopes containing the origin (of \( \mathbb{R}^d \) or \( \mathbb{R}^e \)). It is then easy to see that if the free sum of \( P \) and \( Q \) possesses the integer decomposition property, then each of \( P \) and \( Q \) possesses the integer decomposition property. On the other hand, the converse is not true in general. (See Example [0.3].)

The purpose of the present paper is to show the following

Theorem 0.1. Let \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) be integral convex polytopes of dimension \( d \) and \( e \) containing \( 0_d \) and \( 0_e \), respectively. Suppose that \( P \) and \( Q \) satisfy \( \mathbb{Z}(P \cap \mathbb{Z}^d) = \mathbb{Z}^d \), \( \mathbb{Z}(Q \cap \mathbb{Z}^e) = \mathbb{Z}^e \) and

\[
(P \oplus Q) \cap \mathbb{Z}^{d+e} = \mu(P \cap \mathbb{Z}^d) \cup \nu(Q \cap \mathbb{Z}^e),
\]

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Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property if and only if the following two conditions are satisfied:

- each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property;
- either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{f} a_i z_i = b$, where each $a_i$ is an integer, $b \in \{0, 1\}$ and $f \in \{d, e\}$.

An integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ is called a $(0, 1)$-polytope if each vertex of $\mathcal{P}$ belongs to $\{0, 1\}^d$. It then follows that the equality (1) is always satisfied if each of $\mathcal{P}$ and $\mathcal{Q}$ is a $(0, 1)$-polytope. As an immediate corollary of Theorem 0.1, we also obtain the following

**Corollary 0.2.** Let $\mathcal{P} \subset \mathbb{R}^d$ be a $(0, 1)$-polytope of dimension $d$ containing $0_d$ and $\mathcal{Q} \subset \mathbb{R}^e$ an integral convex polytope of dimension $e$ containing $0_e$. Suppose that $\mathcal{P}$ and $\mathcal{Q}$ satisfy $\mathbb{Z}(\mathcal{P} \cap \mathbb{Z}^d) = \mathbb{Z}^d$ and $\mathbb{Z}(\mathcal{Q} \cap \mathbb{Z}^e) = \mathbb{Z}^e$. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ possesses the integer decomposition property if and only if the following two conditions are satisfied:

- each of $\mathcal{P}$ and $\mathcal{Q}$ possesses the integer decomposition property;
- either $\mathcal{P}$ or $\mathcal{Q}$ satisfies that the equation of each facet is of the form $\sum_{i=1}^{f} a_i z_i = b$, where each $a_i$ is an integer, $b \in \{0, 1\}$ and $f \in \{d, e\}$.

**Example 0.3.** Even though $\mathcal{P}$ and $\mathcal{Q}$ possess the integer decomposition property, the free sum $\mathcal{P} \oplus \mathcal{Q}$ may fail to possess the integer decomposition property. For example, let $\mathcal{P} \subset \mathbb{R}^3$ be the $(0, 1)$-polytope with the vertices $(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)$ and $(1, 0, 0)$. Then $\mathcal{P}$ possesses the integer decomposition property, but the free sum $\mathcal{P} \oplus \mathcal{P}$ fails to possess the integer decomposition property. In fact, $z_1 + z_2 + z_3 = 2$ is the equation of a facet of $\mathcal{P}$.

A structure of the present paper is as follows. In Section 1, we will consider the condition for $\mathcal{P}$ and $\mathcal{Q}$ to satisfy the equality (1). In Section 2, a proof of Theorem 0.1 will be given.

### 1. When does the equality (1) hold?

Let $V(\mathcal{P})$ be the set of vertices of $\mathcal{P}$ and let $V(\mathcal{Q})$ be that of $\mathcal{Q}$. First, for $W \subset V(\mathcal{P}) \setminus \{0_d\}$, let

$$\text{int}(W) = (\text{conv}(W \cup \{0_d\}) \setminus \partial \text{conv}(W \cup \{0_d\})) \cap \mathbb{Z}^d.$$  

For $W \subset V(\mathcal{Q}) \setminus \{0_e\}$, $\text{int}(W)$ is also defined in the same way. Next, we define

$$\mathcal{W}(\mathcal{P}) = \{ W \subset V(\mathcal{P}) \setminus \{0_d\} : W \text{ is linearly independent and } \text{int}(W) \neq \emptyset \}.$$  

In the similar way, we also define $\mathcal{W}(\mathcal{Q})$. Last, for any $W = \{w_1, \ldots, w_m\} \in \mathcal{W}(\mathcal{P})$ (similarly, for any $W \in \mathcal{W}(\mathcal{Q})$), let

$$\min(W) = \min \left\{ \sum_{i=1}^{m} r_i : \sum_{i=1}^{m} r_i w_i \in \text{int}(W) \right\}.$$
Then $0 < \min(W) < 1$.

**Proposition 1.1.** Let $\mathcal{P} \subset \mathbb{R}^d$ and $\mathcal{Q} \subset \mathbb{R}^e$ be integral convex polytopes containing $0_d$ and $0_e$, respectively. Then the free sum $\mathcal{P} \oplus \mathcal{Q}$ satisfies the equality (1) if and only if

- $\mathcal{W}(\mathcal{P}) = \emptyset$ or $\mathcal{W}(\mathcal{Q}) = \emptyset$, or
- $\mathcal{W}(\mathcal{P}) \neq \emptyset$, $\mathcal{W}(\mathcal{Q}) \neq \emptyset$ and $\min(F) + \min(G) > 1$ for any $F \in \mathcal{W}(\mathcal{P})$ and $G \in \mathcal{W}(\mathcal{Q})$.

**Proof.** "Only if" Assume that there exist $F \in \mathcal{W}(\mathcal{P})$ and $G \in \mathcal{W}(\mathcal{Q})$ such that $\min(F) + \min(G) \leq 1$. Then each of $F$ and $G$ is linearly independent. Let $F = \{v_1, \ldots, v_n\}$ and let $G = \{w_1, \ldots, w_m\}$. Then there are $0 < r_1, \ldots, r_n < 1$, $0 < s_1, \ldots, s_m < 1$ such that $\sum_{i=1}^{n} r_i v_i \in \text{int}(F)$ and $\sum_{i=1}^{m} s_i w_i \in \text{int}(G)$, where $0 < \sum_{i=1}^{n} r_i < 1$ and $0 < \sum_{i=1}^{m} s_i < 1$ with $\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i \leq 1$. Let us consider

$$\alpha = \sum_{i=1}^{n} r_i \mu(v_i) + \sum_{i=1}^{m} s_i \nu(w_i) \in \mathbb{R}^{d+e}.$$ 

Since $\sum_{i=1}^{n} r_i v_i \in \mathbb{Z}^d$, we have $\sum_{i=1}^{n} r_i \mu(v_i) \in \mathbb{Z}^{d+e}$. Similarly, $\sum_{i=1}^{m} s_i \nu(w_i) \in \mathbb{Z}^{d+e}$. Thus, $\alpha \in \mathbb{Z}^{d+e}$. Moreover, since $\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i \leq 1$, we have $\alpha \in \mathcal{P} \oplus \mathcal{Q}$. Hence, $\alpha \in (\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e}$. On the other hand, since $\sum_{i=1}^{n} r_i v_i \neq 0_d$ and $\sum_{i=1}^{m} s_i w_i \neq 0_e$, we see that $\alpha \notin \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$. These mean that the equality (1) is not satisfied.

"If" Assume that (1) is not satisfied. Since the inclusion $(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} \supset \mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e)$ is always satisfied, we may assume that there is $\alpha$ belonging to $(\mathcal{P} \oplus \mathcal{Q}) \cap \mathbb{Z}^{d+e} \setminus (\mu(\mathcal{P} \cap \mathbb{Z}^d) \cup \nu(\mathcal{Q} \cap \mathbb{Z}^e))$. Then $\alpha$ can be written like

$$\alpha = \sum_{i=1}^{n} r_i \mu(v_i) + \sum_{i=1}^{m} s_i \nu(w_i),$$ 

where $v_1, \ldots, v_n \in V(\mathcal{P}) \setminus \{0_d\}$, $w_1, \ldots, w_m \in V(\mathcal{Q}) \setminus \{0_e\}$, $0 \leq r_1, \ldots, r_n \leq 1$, $0 \leq s_1, \ldots, s_m \leq 1$ and $\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i \leq 1$. By Carathéodory’s Theorem (cf. [4, Corollary 7.1]), we can choose $\mu(v_1), \ldots, \mu(v_n), \nu(w_1), \ldots, \nu(w_m)$ as linearly independent vectors of $\mathbb{R}^{d+e}$, that is, $v_1, \ldots, v_n$ are linearly independent in $\mathbb{R}^d$ and so are $w_1, \ldots, w_m$ in $\mathbb{R}^e$. Moreover, if $\sum_{i=1}^{n} r_i = 0$, then $\alpha \in \nu(\mathcal{Q} \cap \mathbb{Z}^e)$, a contradiction. Similarly, if $\sum_{i=1}^{m} s_i = 0$, then $\alpha \in \mu(\mathcal{P} \cap \mathbb{Z}^d)$, a contradiction. Thus, we also assume $\sum_{i=1}^{n} r_i > 0$ and $\sum_{i=1}^{m} s_i > 0$.

We consider $v = \sum_{i=1}^{n} r_i v_i \in \mathbb{Z}^d$. Since $\sum_{i=1}^{n} r_i > 0$, $\sum_{i=1}^{m} s_i > 0$ and $\sum_{i=1}^{n} r_i + \sum_{i=1}^{m} s_i \leq 1$, we have $0 < \sum_{i=1}^{n} r_i < 1$. Thus, $v \in \mathcal{P} \cap \mathbb{Z}^d$. Let $v_{i_1}, \ldots, v_{i_g}$ be all of $v_i$’s such that $r_i > 0$ and let $S = \{v_{i_1}, \ldots, v_{i_g}\}$. Then $S$ is also linearly independent and $v \in \text{int}(S)$. Hence, $S \in \mathcal{W}(\mathcal{P})$. Similarly, let $w_{j_1}, \ldots, w_{j_h}$ be all of $w_i$’s such
that $s_i > 0$ and let $T = \{w_{j_1}, \ldots, w_{j_h}\}$. Then $T \in \mathcal{W}(Q)$. Now we see
\[
\min(S) + \min(T) \leq \sum_{k=1}^{g} r_{i_k} + \sum_{k=1}^{h} s_{j_k} = \sum_{i=1}^{n} r_{i} + \sum_{i=1}^{m} s_{i} \leq 1,
\]
as required.

\[\square\]

**Example 1.2.** (a) Let $P \subset \mathbb{R}^d$ be a $(0, 1)$-polytope. Then we easily see that $\mathcal{W}(P) = \emptyset$. Thus, if $P$ or $Q$ is a $(0, 1)$-polytope in Proposition 1.1, then the equality (1) always holds.

(b) Let $P = \text{conv}(\{(0, 0), (1, 0), (1, 2)\}) \subset \mathbb{R}^2$ and let $Q = \text{conv}(\{(0, 2)\}) \subset \mathbb{R}^1$. Then $\mathcal{W}(Q) \neq \emptyset$ but $\mathcal{W}(P) = \emptyset$. Thus the equality (1) holds.

(c) Let $P = Q = \text{conv}(\{(0, 0), (2, 1), (1, 2)\}) \subset \mathbb{R}^2$ and consider $W = \{(2, 1), (1, 2)\}$. Then we see that $\mathcal{W}(P) = \{W\}$. On the other hand, we also have $\min(W) = 2/3$. Thus the equality (1) holds.

2. A proof of Theorem 0.1

Let $P \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$. A configuration arising from $P$ is the finite set $A = \{(\alpha, 1) \in \mathbb{Z}^{d+1} : \alpha \in P \cap \mathbb{Z}^d\}$. We say that $A$ is normal if
\[
\mathbb{Z}_{\geq 0} \cdot A = \mathbb{Z} \cdot A \cap \mathbb{Q}_{\geq 0} \cdot A,
\]
where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers and $\mathbb{Q}_{\geq 0}$ is the set of nonnegative rational numbers.

Recall from [3, Chapter IX] what the Ehrhart polynomial of an integral convex polytope is. Let $P \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$ and, for each integer $n \geq 1$, write $i(P, n)$ for the number of integer points belonging to $nP$, i.e., $i(P, n) = \#nP \cap \mathbb{Z}^d$. It is known that $i(P, n)$ is a polynomial in $n$ of degree $d$ with $i(P, 0) = 1$. We call $i(P, n)$ the Ehrhart polynomial of $P$. We then define the integers $\delta_0, \delta_1, \delta_2, \ldots$ by the formula
\[
(1 - \lambda)^{d+1} \left[1 + \sum_{n=1}^{\infty} i(P, n) \lambda^n\right] = \sum_{n=0}^{\infty} \delta_n \lambda^n.
\]
It then follows that $\delta_n = 0$ for $n > d$. The polynomial
\[
\delta(P) = \sum_{n=0}^{d} \delta_n \lambda^n
\]
is called the $\delta$-polynomial of $P$.

Let $K[t_1^{-1}, \ldots, t_d^{-1}, s]$ denote the Laurent polynomial ring in $d+1$ variables over a field $K$. If $\alpha = (\alpha_1, \ldots, \alpha_d) \in P \cap \mathbb{Z}^d$, then we write $u_{\alpha}$ for the Laurent monomial $t_1^{\alpha_1} \cdots t_d^{\alpha_d} \in K[t_1^{-1}, \ldots, t_d^{-1}]$. The toric ring of $A$ is the subring $K[A]$ of $K[t_1^{-1}, \ldots, t_d^{-1}, s]$ which is generated by those Laurent monomials $u_{\alpha}s$ with $\alpha \in P \cap \mathbb{Z}^d$. Let $K[\{x_{\alpha}\}_{\alpha \in P \cap \mathbb{Z}^d}]$ be the polynomial ring in $|P \cap \mathbb{Z}^d|$ variables
over $K$ with each $\deg x_\alpha = 1$. We then define the surjective ring homomorphism $\pi: K[[x_\alpha, \alpha \in P \cap Z^d]] \to K[A]$ by setting $\pi(x_\alpha) = u_\alpha s$ for each $\alpha \in P \cap Z^d$.

Finally, the Hilbert function of the toric ring $K[A]$ of the configuration $A$ arising from an integral convex polytope $P \subset \mathbb{R}^d$ of dimension $d$ is introduced. We write $(K[A])_n$ for the subspace of $K[A]$ spanned by those Laurent monomials of the form

$$(u_\alpha(1)s)(u_\alpha(2)s)\cdots(u_\alpha(n)s)$$

with each $\alpha^{(i)) \in P \cap Z^d$. In particular $(K[A])_0 = K$ and $(K[A])_1 = \sum_{\alpha \in P \cap Z^d} K u_\alpha s$.

The Hilbert function of $K[A]$ is the numerical function

$$H(K[A], n) = \dim_K(K[A])_n, \quad n = 0, 1, 2, \ldots.$$ 

Thus in particular $H(K[A], 0) = 1$ and $H(K[A], 1) = |P \cap Z^d|$. We then define the integers $h_0, h_1, h_2, \ldots$ by the formula

$$(1 - \lambda)^{d+1} \left[ \sum_{n=0}^{\infty} H(K[A], n)\lambda^n \right] = \sum_{n=0}^{\infty} h_n\lambda^n.$$ 

A basic fact [1, Theorem 11.1] of Hilbert functions guarantees that $h_n = 0$ for $n \gg 0$. We say that the polynomial

$$h(K[A]) = \sum_{n=0}^{\infty} h_n\lambda^n$$

is the $h$-polynomial of $K[A]$.

**Lemma 2.1.** Let $P \subset \mathbb{R}^d$ be an integral convex polytope of dimension $d$ and $A \subset \mathbb{Z}^{d+1}$ the configuration arising from $P$. Suppose that $P$ satisfies $\mathbb{Z}(P \cap \mathbb{Z}^d) = \mathbb{Z}^d$. Then the following conditions are equivalent:

(i) $P$ possesses the integer decomposition property;

(ii) $A$ is normal;

(iii) $\delta(P) = h(K[A]).$

**Proof.** It follows that $P$ possesses the integer decomposition property if and only if, for $\alpha \in nP \cap \mathbb{Z}^d$, one has $(\alpha, n) \in \mathbb{Z}_{\geq 0}A$. Since $\mathbb{Z}(P \cap \mathbb{Z}^d) = \mathbb{Z}^d$, i.e., $\mathbb{Z}A = \mathbb{Z}^{d+1}$, it follows that $A$ is normal if and only if $\mathbb{Z}_{\geq 0}A = \mathbb{Z}^{d+1} \cap \mathbb{Q}_{\geq 0}A$. Moreover, for $\alpha \in \mathbb{Q}^d$, one has $\alpha \in nP$ if and only if $(\alpha, n) \in \mathbb{Q}_{\geq 0}A$. Hence (i) $\iff$ (ii) follows.

In general, one has $i(P, n) \geq H(K[A], n)$ for $n \in \mathbb{Z}_{\geq 0}$. Furthermore, it follows that $i(P, n) = H(K[A], n)$ for all $n \in \mathbb{Z}_{\geq 0}$ if and only if $P$ possesses the integer decomposition property. Hence (i) $\iff$ (iii) follows. \qed

**Lemma 2.2.** Let $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ be integral convex polytopes of dimension $d$ and $e$ which contain the origin of $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. Let $A \subset \mathbb{Z}^{d+1}$ and $B \subset \mathbb{Z}^{e+1}$ be the configurations arising from $P$ and $Q$, respectively. Let $A \oplus B \subset \mathbb{Z}^{d+e+1}$ denote the configuration arising from the free sum $P \oplus Q \subset \mathbb{R}^{d+e}$. Suppose that

$$(P \oplus Q) \cap \mathbb{Z}^{d+e} = \mu(P \cap \mathbb{Z}^d) \cup \nu(Q \cap \mathbb{Z}^e).$$
Then
\[ h(K[A \oplus B]) = h(K[A])h(K[B]). \]

Furthermore, if \( P \oplus Q \) possesses the integer decomposition property, then
\[ \delta(P \oplus Q) = \delta(P)\delta(Q). \]

**Proof.** Let \( K[A] \subset K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s] \) and \( K[B] \subset K[t'_1, t'_1^{-1}, \ldots, t'_e, t'_e^{-1}, s'] \). Then \( K[A \oplus B] = (K[A] \otimes K[B])/(s - s') \). Hence \( h(K[A \oplus B]) = h(K[A] \otimes K[B]) = h(K[A])h(K[B]) \), as desired.

If, furthermore, \( P \oplus Q \) possesses the integer decomposition property, then each of \( P \) and \( Q \) possesses the integer decomposition property. Lemma 2.1 then says that \( \delta(P \oplus Q) = h(K[A \oplus B]) \), \( \delta(P) = h(K[A]) \) and \( \delta(Q) = h(K[B]) \). Hence \( \delta(P \oplus Q) = \delta(P)\delta(Q) \), as required. \( \square \)

We also recall the following theorem.

**Theorem 2.3** ([2, Theorem 1.4]). Let \( P \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^e \) be integral convex polytopes containing the origin (of \( \mathbb{R}^d \) or \( \mathbb{R}^e \)). Then the equality \( \delta(P \oplus Q) = \delta(P)\delta(Q) \) holds if and only if either \( P \) or \( Q \) satisfies that the equation of each facet is of the form \( \sum_{i=1}^f a_iz_i = b \), where each \( a_i \) is an integer, \( b \in \{0, 1\} \) and \( f \in \{d, e\} \).

We are now in the position to give a proof of Theorem 0.1.

**Proof of Theorem 0.1.** Assume that each of \( P \) and \( Q \) possesses the integer decomposition property and either \( P \) or \( Q \) satisfies the condition on its facets described in Theorem 0.1. It then follows from Theorem 2.3 that the condition on the facets is equivalent to satisfying that
\[ \delta(P \oplus Q) = \delta(P)\delta(Q). \]

Moreover, since each of \( P \) and \( Q \) possesses the integer decomposition property, we have the equalities \( \delta(P) = h(K[A]) \) and \( \delta(Q) = h(K[B]) \) by Lemma 2.1. In particular, one has
\[ \delta(P)\delta(Q) = h(K[A])h(K[B]). \]

Furthermore, since the equality (1) is satisfied, it follows from Lemma 2.2 that
\[ h(K[A \oplus B]) = h(K[A])h(K[B]), \]
where \( A \oplus B \subset \mathbb{Z}^{d+e+1} \) denotes the configuration arising from \( P \oplus Q \subset \mathbb{R}^{d+e} \). Hence, by (2), (3) and (4), we obtain
\[ \delta(P \oplus Q) = h(K[A \oplus B]). \]

Therefore, from Lemma 2.1 we conclude that \( P \oplus Q \) possesses the integer decomposition property.
On the other hand, suppose that $P \oplus Q$ possesses the integer decomposition property. Then it is easy to see that each of $P$ and $Q$ possesses the integer decomposition property. Moreover, since $P \oplus Q \subseteq \mathbb{R}^{d+e}$ satisfies (1), the equality $\delta(P \oplus Q) = \delta(P)\delta(Q)$ holds by Lemma 2.2. Therefore, by Theorem 2.3, either $P$ or $Q$ satisfies the condition on its facets described in Theorem 0.1, as required. □

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