FREE-FIELD REPRESENTATION OF GROUP ELEMENT FOR SIMPLE QUANTUM GROUPS

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ABSTRACT

A representation of the group element (also known as “universal $T$-matrix”) which satisfies

$$\Delta(g) = g \otimes g,$$

is given in the form

$$g = \left( \prod_{s=1}^{d_B} E_{1/q_i(s)}(\chi^{(s)} T_{-s}(s)) \right) q^2H \left( \prod_{s=1}^{d_B} E_{q_i(s)}(\psi^{(s)} T_{+s}(s)) \right)$$

where $d_B = \frac{1}{2}(d_G - r_G)$, $q_i = q^{||\vec{a}_i||^2/2}$ and $H_i = 2\vec{H}\vec{a}_i/||\vec{a}_i||^2$ and $T_{\pm s}$ are the generators of quantum group associated respectively with Cartan algebra and the simple roots. The “free fields” $\chi, \vec{\phi}, \psi$ form a Heisenberg-like algebra:

$$\psi^{(s)} \psi^{(s')} = q^{-\vec{a}_i(s)\vec{a}_i(s')} \psi^{(s')} \psi^{(s)}, \quad \chi^{(s)} \chi^{(s')} = q^{-\vec{a}_i(s)\vec{a}_i(s')} \chi^{(s')} \chi^{(s)} \quad \text{for } s < s',$$

$$q^{\vec{h} \vec{a}_i} \psi^{(s)} = q^{\vec{h} \vec{a}_i(s)} \psi^{(s)} q^{\vec{h} \vec{a}_i}, \quad q^{\vec{h} \vec{a}_i} \chi^{(s)} = q^{\vec{h} \vec{a}_i(s)} \chi^{(s)} q^{\vec{h} \vec{a}_i},$$

$$\psi^{(s)} \chi^{(s')} = \chi^{(s')} \psi^{(s)} \quad \text{for any } s, s'.$$

We argue that the $d_G$-parametric “manifold” which $g$ spans in the operator-valued universal enveloping algebra, can also be invariant under the group multiplication $g \rightarrow g' \cdot g''$. The universal $\mathcal{R}$-matrix with the property that $\mathcal{R}(g \otimes I)(I \otimes g) = (I \otimes g)(g \otimes I)\mathcal{R}$ is given by the usual formula

$$\mathcal{R} = q^{-\sum_{ij}^{d_G} ||\vec{a}_i||^2 ||\vec{a}_j||^2 (\vec{a} \vec{a})^{-1}} \left( H_i \otimes H_j \prod_{s \geq 0} E_{q_s} \left( -(q_{s_{\vec{a}}} - q_{s_{-\vec{a}}}) T_{\vec{a}} \otimes T_{-\vec{a}} \right) \right).$$
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1 Introduction

The notion of group element is the central one in the theory of Lie algebras and, once available, it should play the same role in the theory \[1, 2, 3\] of quantum groups. In the quantum case parameters, labeling the “group manifold” are no longer commuting \(c\)-numbers. Moreover, a special care is needed to find a parametrization where their commutation relations are simple enough. As usual in modern theoretical physics “simple enough” means the Heisenberg-like (“free-field”) relations

\[[\varphi_a, \varphi_b] = \hbar C_{ab} \quad \text{or} \quad e^{\varphi_a} e^{\varphi_b} = q^{C_{ab}} e^{\varphi_b} e^{\varphi_a}, \quad q = e^{\hbar}\]

with \(\varphi\)-independent matrix \(C_{ab}\) (which can be further diagonalized to bring everything into a form of several independent Heisenberg algebras). Besides its general significance for understanding the algebraic structure, hidden in the notion of quantum group, the Heisenberg-type realization is the simplest one from the technical point of view and usually allows to go much further with any kind of formal manipulations.

For the first time such representation of group element for \(q \neq 1\) was constructed quite recently: by C.Fronsdal and A.Galindo \[5\] in the case of \(G = SL(2)_q\) (see also \[6\]). Their construction for \(G = SL(N)_q\) with \(N > 2\) \[7\], however, is not quite satisfactory, since non-commuting parameters form a sophisticated algebra, which itself still needs to be “bosonized”.

The problem can be also formulated at the level of fundamental representation of \(G = SL(N)_q\). In this case \(g\) reduces to the \(N \times N\) matrix with the standard non-commuting entries, sometime refered to as the elements of “coordinate ring” of \(G\). The question is then about bosonization (free-field representation) of this non-commutative ring. This problem has been recently addressed in \[8\] for the first non-trivial case of \(SL(3)_q\), but the anzats considered there is partly degenerate and not represented in the transparent enough form.

The goal of this paper is to get rid of these drawbacks of the previous approaches and present the simple and transparent expression for the group element of any simple quantum group. This opens the possibility of straightforward analysis of all the properties of the group elements, including the very question of existence of finite-parametric “group manifolds”.

\[1\] For one possible “physical” application see \[4\].
At our present level of understanding it looks like only one of many different parametrizations of group manifold, which are available at the classical level, is adequate for quantization, provided one wants the quantum parameters to form Heisenberg-like algebra. As usual in conformal field/quantum group theory all the adequate representations involve Gauss decomposition

\[ q = glgdgU \]

into a product of lower-triangular, diagonal and upper-triangular parts, so what will be specific is the representation of \( gU \) (and \( gL \)) pieces. In the theory of Wess-Zumino-Novikov-Witten model (see [9]) one would take for the fundamental representation of \( gU \) just

\[
gU = \begin{pmatrix}
1 & \psi_1 & \psi_{12} & .
0 & 1 & \psi_2 & .
0 & 0 & 1 & .
. & . & . & .
\end{pmatrix}
\]

In the group theory the most naive choice (used also in [7]) is

\[
gU = \prod_{\bar{\alpha} > 0}^{d_B} \exp (\psi_{\bar{\alpha}} T_{\bar{\alpha}}) \]

with somehow ordered all the \( d_B = \frac{1}{2}(d_G - r_G) \) positive roots \( \bar{\alpha} \). Both these representations are in fact not good enough for quantization: in both cases the algebra of \( \psi \)-variables is not of the Heisenderg type.

The adequate representation appears to involve only simple roots (which is not a big surprise since the quantum group structure is most naturally introduced in exactly this sector). In order to span the entire Borel subalgebra every simple root should be allowed to appear several times in the product:

\[
gU = \prod_{s=1}^{d_B} \exp \left( \psi^{(s)} T_{\pm i(s)} \right) \]

where \( i = 1, \ldots, r_G \) labels simple roots \( \bar{\alpha}_i \) and the corresponding generators \( T_{\pm i} \equiv T_{\pm \bar{\alpha}_i} \). Index \( s \) labels in some way all the positive roots of \( G \): it can be considered as a double index, \( s = (p,j) \), where \( p \) is the “height” of the root, and for given \( p \) the second index \( j \) is labeling the roots of the “height” \( p \) (there are no more than \( r_G \) of them for any given \( p \)). The map

\[
i(s) = i(p,j) = j.
\]
For example, in the case of $SL(r + 1)$ the sequence $i(s)$ can be

$$r; r - 1, r; r - 2, r - 1, r; \ldots ; 1, 2, \ldots , r - 1, r$$  \hspace{1cm} (1.4)

i.e. $p = r, r - 1, \ldots, 1$ and for given $p$ index $j$ runs from $p$ to $r$ so that for $SL(3)$

$$g_U = e^{\psi(1)}_3 e^{\psi(2)}_2 e^{\psi(3)}_1,$$

and for $SL(4)$

$$g_U = e^{\psi(1)}_3 e^{\psi(2)}_2 e^{\psi(3)}_2 e^{\psi(4)}_1 e^{\psi(5)}_2 e^{\psi(6)}_3,$$

etc., where in the fundamental representation

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \ldots & \ldots & \ldots \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \ldots & \ldots & \ldots \end{pmatrix}, \ldots$$

In other words, for adequate quantization the “canonical basis” in the universal enveloping algebra should be taken in the form

$$\{T_A\} = \left\{ \prod_s T^{n(s)}_{i(s)} \right\}$$  \hspace{1cm} (1.5)

rather than the one associated with (1.1),

$$\{T_A\} = \left\{ \prod_{\vec{\alpha} > 0} T^{n(\vec{\alpha})}_{\vec{\alpha}} \right\}$$  \hspace{1cm} (1.6)

Existence of the “adequate” and “non-adequate” parametrizations is somewhat alarming and should imply that the algebras of $\psi$-variables in other parametrizations, though looking somewhat sophisticated, still can be straightforwardly bosonized. This aspect of the theory, as well as its relation to the theory of quantum double $[2]$, remain to be further clarified.

There is always a self-consistent reduction (restriction) of the free-field representation of the group element:

$$\psi^{(p,j)} = 0 \quad \text{for all } p > 1.$$  \hspace{1cm} (1.7)

This reduction can be of special interest for quantization of KP/Toda integrable hierarchies. It was this reduced (degenerate) representation that was considered in $[8]$.

The rest of this paper is organized as follows:
Section 2 is devoted to notations and basic properties of the $q$-exponents.

The basic formula

\[ \Delta(g) = g \otimes g = (g \otimes I)(I \otimes g) \]

is then proved in Section 3, with some examples in fundamental representation discussed in Section 4.

Section 5 contains explicit evaluation of the group composition rule (“dual comultiplication”) for the case of $SL(2)_q$. Preliminary discussion of $SL(3)_q$ is presented in Appendix B.

Section 6 and Appendix A are devoted to evaluation of

\[ (I \otimes g)(g \otimes I) = \mathcal{R}\Delta(g)\mathcal{R}^{-1} \]

with the universal $\mathcal{R}$-matrix from refs.\cite{2,10}. Full details are presented for all the rank-2 simple Lie groups.
2 Quantum groups and $q$-exponentials

In this section we describe the crucial consistency property between quantum comultiplication and the $q$-exponent. All the notations to be used throughout the paper are also specified here and in eqs.(3.1)-(3.3).

2.1 Comultiplication

Quantization of the universal enveloping algebra, which provides one possible description of the quantum group, is most conveniently described in terms of the Chevalley generators, associated with the simple roots $\vec{\alpha}_i, i = 1, \ldots, r_G$. With every $\vec{\alpha}_i$ there are associated three generators $H_i$, $\hat{T}_\pm^i$ and specific $q$-parameter $q_i \equiv q^{\alpha_{ii}/2}$. Here $\alpha_{ij} \equiv \vec{\alpha}_i \vec{\alpha}_j$ and in the simply-laced case all $\alpha_{ii} = 2$, $q_i = q$. We also use a notation $\vec{h} = \frac{1}{2} \sum_{ij} \vec{\alpha}_i \alpha_{ij}^{-1} \alpha_{jj} h_j$, i.e. $q^{\vec{h} \vec{\alpha}_i} = q_i^{h_i}$ for any vector $\vec{h}$.

Chevalley generators satisfy the following algebra:

$$q_i^{2H_i} \hat{T}_\pm^j = q^{\pm \alpha_{ij}} \hat{T}_\pm^j q_i^{2H_i}, \quad \text{or} \quad q^{2\vec{h} \vec{\xi}} \hat{T}_\pm^j = q_j^{\pm \xi_j} \hat{T}_\pm^j q^{2\vec{h} \vec{\xi}},$$

$$\hat{T}_{+i} \hat{T}_{-j} - \hat{T}_{-j} \hat{T}_{+i} = \frac{q_i^{2H_i} - q_i^{-2H_i}}{q_i - q_i^{-1}} \delta_{ij}; \quad (2.1)$$

$$\Delta(H_i) = I \otimes H_i + H_i \otimes I;$$

$$\Delta(\hat{T}_{\pm i}) = q_i^{H_i} \otimes \hat{T}_{\pm i} + \hat{T}_{\pm i} \otimes q_i^{-H_i}.$$  

They are also subjected to Serre constraints to be explicitly formulated and used in s.3 below.

The adequate Gauss decomposition involves, however, the somewhat different generators:

$$T_{+i} \equiv \hat{T}_{+i} q_i^{-H_i}, \quad (2.2)$$

$$T_{-i} \equiv q_i^{H_i} \hat{T}_{-i}$$

which have asymmetric coproducts:

$$\Delta(T_{+i}) = I \otimes T_{+i} + T_{+i} \otimes q_i^{-2H_i};$$

$$\Delta(T_{-i}) = q_i^{2H_i} \otimes T_{-i} + T_{-i} \otimes I. \quad (2.3)$$

Note that such $h_i$ are not coordinates in an orthonormal basis! For example, $\vec{h} \vec{\xi} = h_\xi = \frac{1}{2} \sum_{ij} \alpha_{ij}^{-1} \alpha_{ii} \alpha_{jj} h_i \xi_j$, in particular for $SL(2)$ $2h \vec{\xi} = h \xi$. 

7
2.2 $q$-exponential

We define $[n] = [n]_q = [n]_{1/q} \equiv \frac{q^n - q^{-n}}{q - q^{-1}}$,

$$E_q(x) \equiv \sum_{n \geq 0} \frac{x^n}{(1-q^{2n})!} = \sum_{n \geq 0} \frac{x^n}{n!} q^{-\frac{1}{2}q^{n(n-1)}}. \quad (2.4)$$

It is a simple combinatorial exercise to show that

$$E_q\left(\frac{x}{q - q^{-1}}\right) = \prod_{n \geq 0} \frac{1}{1 + q^{2n+1}x}, \quad (2.5)$$

thus

$$E_q\left(q^2 \frac{x}{q - q^{-1}}\right) = (1 + qx) E_q\left(\frac{x}{q - q^{-1}}\right) \quad (2.6)$$

The properties of such $q$-exponentials, crucial for our considerations below, are: the antipod relation

$$(E_q(x))^{-1} = E_{1/q}(-x), \quad (2.7)$$

and the Campbell-Hausdorff formula in the particular case of $xy = q^2yx$, when it reduces to especially simple pair of “addition rule”

$$E_q(y)E_q(x) = E_q(x + y), \quad \text{if } xy = q^2yx. \quad (2.8)$$

and Faddeev-Volkov cocycle identity [11]

$$E_q(x)E_q(y) = E_q(x + y + (1 - q^{-2})xy) = E_q(y)E_q\left(1 - q^{-2}\right)E_q(x), \quad \text{if } xy = q^2yx, \quad (2.9)$$

which we shall need in Section 5 in the following form:

$$E_q(y)E_q(x) = E_q(x + y) = E_q(x)E_q\left(\frac{1}{1 + (1 - q^{-2})x} y\right), \quad \text{if } xy = q^2yx \quad (2.10)$$

---

3In general, for any c-number $s$ ($sx = xs$)

$$E_{1/q}(-sx)E_q(x) = \frac{E_q(x)}{E_q(sx)} = \sum_{n \geq 0} \frac{q^{-n(n-1)/2} x^n}{n!} (1-s)(1-q^2s) \cdots (1-q^{2(n-1)}s)$$
2.3 Comultiplication versus $q$-exponential

The basic relation for all our reasoning below is: for any root of the length $||\vec{\alpha}||^2 = 2$

$$\mathcal{E}_q (\psi \Delta(T_+)) = \mathcal{E}_q \left( \psi (I \otimes T_+ + T_+ \otimes q^{-2H}) \right) =$$

$$= \mathcal{E}_q \left( \psi T_+ \otimes q^{-2H} \right) \mathcal{E}_q (\psi I \otimes T_+).$$

This is just an application of (2.8) for $x = I \otimes T_+$ and $y = T_+ \otimes q^{-2H}$. Similarly

$$\mathcal{E}_{1/q} (\chi \Delta(T_-)) = \mathcal{E}_{1/q} \left( \chi (q^{2H} \otimes T_- + T_- \otimes I) \right) =$$

$$= \mathcal{E}_{1/q} (\chi T_- \otimes I) \mathcal{E}_{1/q} (\chi q^{2H} \otimes T_-).$$

For non-simply-laced algebras when roots of different lengths are present, the $q$-parameter of the $q$-exponential should be just changed for $q_{\vec{\alpha}} \equiv q^{||\vec{\alpha}||^2/2}$. In ss. 5,6 below we”ll need some other corollaries of (2.8).
3 Representation of the group element

Let us now take the natural ansatz

\[ g = \left( \prod_s \mathcal{E}_{1/q_i(s)}(\chi^{(s)}T_{-i(s)}) \right) q^{2\vec{\phi}H} \left( \prod_s \mathcal{E}_{q_i(s)}(\psi^{(s)}T_{+i(s)}) \right) \]  \hspace{1cm} (3.1)

with the ordering defined by the map \( i(s) \) (superscripts "<" and ">" stand for the straight and inverse sequences \( i(s) \) respectively), and check the group property

\[ \Delta(g) = g \otimes g = (g \otimes I)(I \otimes g) \]  \hspace{1cm} (3.2)

Note that in the limit \( q \to 1 \) parameters \( \vec{\phi} \) get large: \( \phi \sim 1/\log q \).

In order to simplify the formulas below we shall substitute the index \( i(s) \) by \( s \):

\[ q_s \equiv q_i(s), \quad T_s \equiv T_{i(s)}, \quad H_s \equiv H_{i(s)}, \quad \alpha_{ss'} \equiv \vec{\alpha}_{i(s)}\vec{\alpha}_{i(s')} \]  \hspace{1cm} (3.3)

By definition

\[ \Delta(g) = \prod_s \mathcal{E}_{1/q_i(s)}\left(\chi^{(s)}\Delta(T_{-i})\right) \left( q^{2\vec{\phi}H} \otimes q^{2\vec{\phi}H} \right) \prod_s \mathcal{E}_{q_i(s)}\left(\psi^{(s)}\Delta(T_{+i})\right) \]  \hspace{1cm} (3.4)

Now we use (2.11), (2.12):

\[ \Delta(g) = \prod_s \mathcal{E}_{1/q_s}\left(\chi^{(s)}T_{-s} \otimes I\right) \mathcal{E}_{1/q_s}\left(\chi^{(s)}q_s^{2H_s} \otimes T_{-s}\right) \]

\[ \hspace{1cm} \cdot \left( q^{2\vec{\phi}H} \otimes I \right) \left( I \otimes q^{2\vec{\phi}H} \right) \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}T_{+s} \otimes q_s^{-2H_s}\right) \mathcal{E}_{q_s}\left(\psi^{(s)}I \otimes T_{+s}\right) \]  \hspace{1cm} (3.5)

At the same time

\[ g \otimes g = \left\{ \prod_s \mathcal{E}_{1/q_s}\left(\chi^{(s)}T_{-s} \otimes I\right) \left( q^{2\vec{\phi}H} \otimes I \right) \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}T_{+s} \otimes I\right) \right\} \cdot \left\{ \prod_s \mathcal{E}_{1/q_s}\left(\chi^{(s)}I \otimes T_{-s}\right) \left( I \otimes q^{2\vec{\phi}H} \right) \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}I \otimes T_{+s}\right) \right\} \]  \hspace{1cm} (3.6)

In order for (3.5) and (3.6) to coincide it is enough if:

(a) The \( q \)-exponents in the square brackets in (3.5) can be adequately reordered:

\[ \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}T_{+s} \otimes q_s^{-2H_s}\right) \mathcal{E}_{q_s}\left(\psi^{(s)}I \otimes T_{+s}\right) = \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}T_{+s} \otimes q_s^{-2H_s}\right) \prod_s \mathcal{E}_{q_s}\left(\psi^{(s)}I \otimes T_{+s}\right) \]  \hspace{1cm} (3.7)
and similarly for exponents with $\chi^{(s)}$.

(b) The first factor at the r.h.s. of (3.7) can be pushed to the left through $(I \otimes q^{2\tilde{H}})$ in (3.3), so that

\[
\left( I \otimes q^{2\tilde{H}} \right) \prod_{s}^{<} \mathcal{E}_{qs} \left( \psi^{(s)} T_{+s} \otimes q_{s}^{-2H_{s}} \right) = \prod_{s}^{<} \mathcal{E}_{qs} \left( \psi^{(s)} T_{+s} \otimes I \right) \left( I \otimes q^{2\tilde{H}} \right)
\]

Similarly

\[
\prod_{s}^{>} \mathcal{E}_{1/qs} \left( \chi^{(s)} q_{s}^{2H_{s}} \otimes T_{-s} \right) \left. q^{2\tilde{H}} \otimes I \right) = \left( q^{2\tilde{H}} \otimes I \right) \prod_{s}^{>} \mathcal{E}_{1/qs} \left( \chi^{(s)} I \otimes T_{-s} \right)
\]

(c) The first factor at the r.h.s. of (3.8) commutes with the second factor at the r.h.s. of (3.9).

Conditions (a), (b), (c) are defining the commutation properties of the $\chi$, $\phi$, $\psi$ variables.

Indeed, (a) is fulfilled provided the arguments of the relevant $q$-exponents commute:

\[
\text{for any } s < s' \quad \left[ \psi^{(s')} T_{+s'} \otimes q_{s'}^{-2H_{s'}}, \psi^{(s)} I \otimes T_{+s} \right] = 0
\]

(3.10)

Since

\[
\left( T_{+s'} \otimes q_{s'}^{-2H_{s'}} \right) (I \otimes T_{+s}) = q^{-\alpha_{ss'}} (I \otimes T_{+s}) \left( T_{+s'} \otimes q_{s'}^{-2H_{s'}} \right)
\]

this implies that

\[
\psi^{(s)} \psi^{(s')} = q^{-\alpha_{ss'}} \psi^{(s')} \psi^{(s)}, \quad s < s'.
\]

(3.11)

Similarly

\[
\text{for any } s < s' \quad \left[ \chi^{(s)} T_{-s} \otimes I, \chi^{(s')} q_{s}^{2H_{s'}} \otimes T_{-s'} \right] = 0
\]

(3.12)

and therefore

\[
\chi^{(s)} \chi^{(s')} = q^{-\alpha_{ss'}} \chi^{(s')} \chi^{(s)}, \quad s < s'.
\]

(3.13)

Condition (3.8) is fulfilled if

\[
\left( I \otimes q^{2\tilde{H}} \right) \psi^{(s)} = \psi^{(s)} \left( I \otimes q_{s}^{2H_{s}} \right) \left( I \otimes q^{2\tilde{H}} \right),
\]

(3.14)

i.e. (since $q^{H_{s_{i}}'} = q_{i}^{H_{s_{i}}}$) if for any vector $\vec{h}$

\[
q^{\tilde{h}_{s}} \psi^{(s)} = q^{\tilde{h}_{s}} \psi^{(s)} q^{\tilde{h}_{s}} \quad \text{or} \quad q^{\phi_{s}} \psi^{(s)} = q^{\phi_{s}} \psi^{(s)} q^{\phi_{s}}
\]

(3.15)
Similarly from (3.9)

\[ q \vec{h} \vec{\phi} \chi^{(s)} = q \vec{h} \vec{\phi} \chi^{(s)} q \vec{\phi} \quad \text{or} \quad q^{\phi^i \chi^{(s)}} = q^{\alpha^i \chi^{(s)} q^{\phi^i}} \]  

(3.16)

Finally (c) implies that all the \( \psi \)'s commute with all the \( \chi \)'s.

These commutation relations are nicely consistent with “Hermitean representation”, satisfying \( g^* = g \), where operator \( * \) is defined so, that

\[ T_{\pm \alpha}^* = T_{\mp \alpha}, \]

\[ (q^H)^* = q^H, \]  

(3.17)

and

\[ \forall A, B \quad (AB)^* = B^* A^*, \]  

(3.18)

\[ \forall A \quad (A^*)^* = A. \]

Thus \( \vec{\phi}^* = \vec{\phi}, \chi_i = \psi_i^* \) and explicitly “Hermitean” group element looks like

\[ \prod_{s}^{>} \mathcal{E}_{1/q_s} (\psi_s^* T_s) q^{(\vec{\phi}_s + \vec{\phi}) q^H} \prod_{s}^{<} \mathcal{E}_{q_s} (\psi_s T_s) \]  

(3.19)
4 Examples of fundamental representation

In any fundamental representation of $G_q$

$$\frac{q_i^{2H_i} - q_i^{-2H_i}}{q_i - q_i^{-1}} = 2H_i \quad (4.1)$$

and (2.1), (2.1) (but not (2.1)!) reduce to the ordinary commutation relations of $G$ (with $q = 1$). Thus $T_i$ (or $\hat{T}_i$) can be chosen to be just the ordinary $q$-independent matrices.

4.1 The case of $SL(2)_q (A(1)_q)$

Since $2\vec{\phi}\vec{H} = \phi H = \frac{1}{2}\phi \sigma_3$ and $T_\pm = \sigma_\pm$, eq.(3.1) gives in this case:

$$g_{\text{fund}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = E_{\chi}(\phi_\sigma) q^{\phi_\sigma^3/2} E_{\psi}(\psi_\sigma) =$$

$$= \begin{pmatrix} 1 & 0 \\ \chi & 0 \end{pmatrix} \begin{pmatrix} q^{\phi^2} & 0 \\ 0 & q^{-\phi^2} \end{pmatrix} \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q^{\phi^2} & q^{\phi^2/2} \psi \\ \chi q^{\phi^2} & \chi q^{\phi^2/2} + q^{-\phi^2} \end{pmatrix}, \quad (4.2)$$

and

$$q^{\phi^2/2} \psi = q\psi q^{\phi^2/2},$$

$$q^{\phi^2/2} \chi = q\chi q^{\phi^2/2},$$

$$\psi \chi = \chi \psi \quad (4.3)$$

imply the usual relations:

$$ab = qba, \quad ac = qca,$$

$$bd = qdb, \quad cd = qdc,$$

$$bc = cb, \quad (4.4)$$

$$ad - da = (q - q^{-1})bc,$$

$$ad - qbc = da - q^{-1}bc = 1.$$

Comparison with representation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 + \frac{1}{u_1v_1} \end{pmatrix} \quad (4.5)$$
from \cite{8} with
\[ u_i u_j = q u_j u_i, \quad v_i v_j = q v_j v_i, \quad i < j \] (4.6)
is also straightforward.

4.2 The case of $SL(3)_q (A(2)_q)$

For $N = 3$ we similarly obtain:

\[
g_{\text{fund}} = \begin{pmatrix} a & b & e \\ c & d & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \chi^{(3)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \chi^{(2)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \chi^{(1)} & 1 \end{pmatrix} \begin{pmatrix} q^{\phi(1)} & 0 & 0 \\ 0 & q^{\phi(2)} & 0 \\ 0 & 0 & q^{\phi(3)} \end{pmatrix}. \]

\[ 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \psi^{(1)} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \psi^{(2)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \psi^{(3)} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \]

\[ \begin{pmatrix} \chi^{(2)} & 1 & 0 \\ \chi^{(2)} & \chi^{(3)} & \chi^{(1)} + \chi^{(3)} & 1 \\ 0 & 0 & q^{\phi(3)} & 0 \end{pmatrix} \begin{pmatrix} q^{\phi(1)} & 0 & 0 \\ 0 & q^{\phi(2)} & 0 \\ 0 & 0 & q^{\phi(3)} \end{pmatrix} \begin{pmatrix} 1 & \psi^{(2)} & \psi^{(3)} \\ \psi^{(3)} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} \\ q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} \\ q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} \end{pmatrix} = \begin{pmatrix} \chi^{(2)} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} \\ \chi^{(3)} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} + \chi^{(3)} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} \\ \chi^{(2)} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} + \chi^{(3)} q^{\phi(1)} q^{\phi(2)} q^{\phi(3)} + + (\chi^{(1)} + \chi^{(3)}) q^{\phi(2)} q^{\phi(3)} + (\chi^{(1)} + \chi^{(3)}) q^{\phi(2)} q^{\phi(3)} + + q^{\phi(3)} \end{pmatrix} \]
Now $\vec{\alpha}(1) = \vec{\alpha}(3) = \vec{\alpha}_2$, $\vec{\alpha}(2) = \vec{\alpha}_1$,

$$2\vec{\phi}\vec{H} = \phi(1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \phi(3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =$$

$$= 2\phi(1)H_1 - 2\phi(3)H_2 = 2 \left( \frac{2\phi_1 + \phi_2}{3}H_1 + \frac{\phi_1 + 2\phi_2}{3}H_2 \right),$$

i.e.

$$\phi(1) = \frac{2\phi_1 + \phi_2}{3},$$

$$\phi(2) = -\phi(1) - \phi(3) = -\frac{-\phi_1 + \phi_2}{3},$$

$$\phi(3) = -\frac{-\phi_1 + 2\phi_2}{3}.$$ (4.8)

Again, Heisenberg-like commutation relations

$$\psi^{(1)}\psi^{(2)} = q\psi^{(2)}\psi^{(1)}, \quad \psi^{(2)}\psi^{(3)} = q\psi^{(3)}\psi^{(2)}, \quad \psi^{(1)}\psi^{(3)} = q^{-2}\psi^{(3)}\psi^{(1)},$$

$$q^{\phi(1)}\psi^{(1)} = \psi^{(1)}q^{\phi(1)}, \quad q^{\phi(1)}\psi^{(2)} = q\psi^{(2)}q^{\phi(1)}, \quad q^{\phi(1)}\psi^{(3)} = \psi^{(3)}q^{\phi(1)},$$

$$q^{\phi(2)}\psi^{(1)} = q\psi^{(1)}q^{\phi(2)}, \quad q^{\phi(2)}\psi^{(2)} = q^{-1}\psi^{(2)}q^{\phi(2)}, \quad q^{\phi(2)}\psi^{(3)} = \psi^{(3)}q^{\phi(2)},$$

$$q^{\phi(3)}\psi^{(1)} = q^{-1}\psi^{(1)}q^{\phi(3)}, \quad q^{\phi(3)}\psi^{(2)} = q\psi^{(2)}q^{\phi(3)}, \quad q^{\phi(3)}\psi^{(3)} = q^{-1}\psi^{(3)}q^{\phi(3)},$$ (4.10)

(plus the same with $\psi \to \chi$) reproduce the standard quadratic relations for $a, b, c, \ldots, k$.

In the reduced case of $\chi^{(1)} = \psi^{(1)} = 0$ eq.(4.7) reproduces the result of 8 (with $w_2 = \frac{u_1}{v_1} = \psi^{(2)}$, $w_3 = \frac{u_1}{v_2} = \psi^{(3)}$, $q^{\phi(1)} = u_1v_1$, $q^{\phi(3)} = \frac{u_4u_3}{v_3v_4}$).
5 Group composition rule

The group composition law (the “dual comultiplication”) is defined by the following relation:

\[ g(\Delta^*(\chi), \Delta^*(\phi), \Delta^*(\psi)) = g(\chi \otimes I, \phi \otimes I, \psi \otimes I) g(I \otimes \chi, I \otimes \phi, I \otimes \psi) = \]

\[ = g(\chi', \phi', \psi') g(\chi'', \phi'', \psi'') \]

(5.1)

Given representation (3.1) for \( g(\chi, \phi, \psi) \) it should be just straightforward calculation that leads to explicit formulas for \( \Delta^*(\chi), \Delta^*(\phi), \Delta^*(\psi) \).

5.1 The case of \( SL(2)_q \)

Let us begin, as usual, from the case of a single simple root (i.e. \( SL(2)_q \)). In order to avoid possible confusion we remind that in our notation \( 2\bar{\phi}H = \phi H \). In this case we need to evaluate

\[ g(\chi', \phi', \psi') g(\chi'', \phi'', \psi'') = \]

\[ = \mathcal{E}_{1/q}(\chi T_-) q^{\phi H} \mathcal{E}_q(\psi T_+) \mathcal{E}_{1/q}(\chi'' T_-) q^{\phi'' H} \mathcal{E}_q(\psi'' T_+) \]

(5.2)

1) The first thing to do is permutation of \( \mathcal{E}_q(\psi T_+) \) and \( \mathcal{E}_{1/q}(\chi'' T_-) \).

Sandwiching (2.1) between \( T^a_- \) and \( T^b_- \), we get:

\[ \left[ T^a_+, T^b_+ \right] = \sum_{a+b=n-1} T^a_- q^{2H} q^{-2H} T^b_- = \]

\[ = \left[ n \right] q^{-\left(n-1\right)} \left[ T^a_- q^{2H} q^{-2H} T^b_- \right] \]

(5.3)

Summation over \( n \) with the weight \( \frac{n!}{[n]} q^{n(n-1)/2} \) produces:

\[ \left( T^a_+ + \frac{\chi}{q - q^{-1}} q^{-2H} \right) \mathcal{E}_{1/q}(\chi T_-) = \mathcal{E}_{1/q}(\chi T_-) \left( T^a_+ + \frac{\chi}{q - q^{-1}} q^{2H} \right) \]

This in turn implies that

\[ \mathcal{E}_q \left( \psi' T_+ + \frac{\psi' \chi''}{q - q^{-1}} q^{-2H} \right) \mathcal{E}_{1/q}(\chi'' T_-) = \mathcal{E}_{1/q}(\chi'' T_-) \mathcal{E}_q \left( \psi' T_+ + \frac{\psi' \chi''}{q - q^{-1}} q^{2H} \right) \]

(5.4)

(all the variables with one prime commute with those with two primes).

Now it is time to use (2.8):

\[ \mathcal{E}_q \left( \psi' T_+ + \frac{\psi' \chi''}{q - q^{-1}} q^{2H} \right) = \mathcal{E}_q(\psi' T_+) \mathcal{E}_q \left( \frac{\psi' \chi''}{q - q^{-1}} q^{2H} \right) \]

(5.5)
while
\[ E_q \left( \psi' T_+ + \frac{\psi' \chi''}{q-q^{-1}} q^{-2H} \right) = E_q \left( \frac{\psi' \chi''}{q-q^{-1}} q^{-2H} \right) E_q(\psi' T_+), \] (5.6)
and we obtain:
\[ E_q(\psi' T_+)^2E_{1/q}(\chi'' T_-) = E_{1/q} \left( -\frac{\psi' \chi''}{q-q^{-1}} q^{-2H} \right) E_{1/q}(\chi'' T_-) E_q(\psi' T_+) E_q \left( \frac{\psi' \chi''}{q-q^{-1}} q^{2H} \right) \] (5.7)

2) The next thing to do is to carry \( E_q(\psi' T_+) E_q \left( \frac{\psi' \chi''}{q-q^{-1}} q^{2H} \right) \) to the right through \( q^{\phi'' H} \). This is easy:
\[ E_q(\psi' T_+) E_q \left( \frac{\psi' \chi''}{q-q^{-1}} q^{2H} \right) q^{\phi'' H} = q^{\phi'' H} E_q(\psi' q^{-\phi''} T_+) E_q \left( \frac{\psi' \chi''}{q-q^{-1}} \right) \] (5.8)

Similarly,
\[ q^{\phi'' H} E_{1/q} \left( -\frac{\psi' \chi''}{q-q^{-1}} q^{-2H} \right) E_{1/q}(\chi' T_-) = E_{1/q} \left( -\frac{\psi' \chi''}{q-q^{-1}} \right) E_{1/q}(\chi' q^{-\phi''} T_-) q^{\phi'' H} \] (5.9)

3) Now we need to permute the last two factors at the r.h.s. of (5.8). This is where the Faddeev-Volkov relation (2.10) plays its role:
\[ E_q(\psi' q^{-\phi''} T_+) E_q \left( \frac{\psi' \chi''}{q-q^{-1}} \right) = E_q \left( \frac{1}{1+q^{-1}\psi' \chi''} q^{-\phi''} \psi' T_+ \right) = E_q \left( \frac{1}{1+q^{-1}\psi' \chi''} q^{-\phi''} \psi' T_+ \right) \] (5.10)

Similarly,
\[ E_{1/q} \left( -\frac{\psi' \chi''}{q-q^{-1}} \right) E_{1/q}(\chi' q^{-\phi''} T_-) = E_{1/q} \left( q^{-\phi''/2} \frac{1}{1+q^{-1}\psi' \chi''} q^{-\phi''/2} \chi' T_- \right) E_{1/q} \left( -\frac{\psi' \chi''}{q-q^{-1}} \right) \] (5.11)

4) With the help of (2.8) we now get:
\[ E_q \left( q^{-\phi''/2} \frac{1}{1+q^{-1} \psi' \chi''} q^{-\phi''/2} \psi' T_+ \right) E_q(\psi' T_+) = \]
\[ = E_q \left( \left( \psi'' + q^{-\phi''/2} \frac{1}{1+q^{-1} \psi' \chi''} q^{-\phi''/2} \psi' \right) T_+ \right) = E_q (\Delta^*(\psi) T_+), \] (5.12)
i.e.
\[ \Delta^*_{SL(2)}(\psi) = \psi'' + q^{-\phi''/2} \frac{1}{1+q^{-1} \psi' \chi''} q^{-\phi''/2} \psi' = \]
\[ = I \otimes \psi + \psi \otimes q^{-\phi''/2} \frac{1}{I \otimes I + \psi \otimes \chi} I \otimes q^{-\phi''/2} \] (5.13)
Similarly

\[ \Delta^*_{SL(2)}(\chi) = \chi' + q^{-\phi'/2} \frac{1}{1 + \psi' \chi''} q^{-\phi'/2} \chi'' = \]

\[ = \chi \otimes I + q^{-\phi'/2} \otimes I \frac{1}{I \otimes I + \psi \otimes \chi} q^{-\phi'/2} \otimes \chi \]  

(5.14)

5) What remains to be found is

\[ q^H \Delta^*(\phi) = \mathcal{E}_{1/q} \left( -\frac{\psi' \chi''}{q - q^{-1}} \right) q^{H(\phi' + \phi'')} \mathcal{E}_q \left( \frac{\psi' \chi''}{q - q^{-1}} \right) \]  

(5.15)

In order to evaluate this quantity we can use the trick, usual in the q-bosonization theory \([12]\).

Let us note, that for \( \Phi = \phi' + \phi'' \) and \( z = \psi' \chi'' \) we have \( \Phi z = z(\Phi + 4) \), thus \( \Phi z^n = z^n(\Phi + 4n) \) and for any function \( f(z) \)

\[ \Phi e^{f(z)} = e^{f(z)} \left( \Phi + 4z \frac{df(z)}{dz} \right), \]

or

\[ e^{-f(z)} q^H \Phi e^{f(z)} = q^H(\Phi + 4z df/dz) \]

(5.16)

Thus \( \Delta^*(\phi) = \Phi + 4z \frac{df(z)}{dz} \), where, according to (2.3), \( f(z) = \log \mathcal{E}_q \left( \frac{z}{q - q^{-1}} \right) = -\sum_{k=0}^{\infty} \log(1 + q^{2k+1} z) \), and \( z \frac{df(z)}{dz} = -\sum_{k=0}^{\infty} \frac{q^{2k+1} z}{1 + q^{2k+1} z} = -\sum_{l=1}^{\infty} \frac{(-z)^l}{q^l - q^{-l}} \). Finally,

\[ \Delta^*_{SL(2)}(\phi) = \phi' + \phi'' - 4 \sum_{l} \frac{(-\psi' \chi'')^l}{q^l - q^{-l}} = \]

\[ = \phi \otimes I + I \otimes \phi - \frac{4}{q - q^{-1}} \sum_{l} \frac{(-\psi \otimes \chi)^l}{l} \]  

(5.17)

(It deserves noting that in the classical limit \( q \to 1 \) the field \( \phi \) gets large: \( \phi \sim 1/\log q \).)

6) Alternative description of \( \Delta^*(\phi) \) also deserves mentioning. It is especially useful in considerations of particular representations and of the higher-rank groups.

Let us rewrite (5.13) as

\[ q^H \Delta^*(\phi) = q^H \phi' \mathcal{E}_{1/q} \left( -\frac{\psi' \chi''}{q - q^{-1}} q^{-2H} \right) \mathcal{E}_q \left( \frac{\psi' \chi''}{q - q^{-1}} q^{2H} \right) q^{H\phi''} \]  

(5.18)

and make use of the infinite-product representation (2.3) for the \( q \)-exponentials. This is most straightforward when \( 2H \) has integer eigenvalues (i.e. for the finite-dimensional representations of \( SL(2)_q \)). If \( H = 0 \) eq. (5.18) turns into identity \( 1 = 1 \). Let further \( H = \frac{1}{2} \). Then the product of two exponents at the r.h.s., evaluated with the help of (2.6) is just \( 1 + \psi' \chi' \) and

\[ q^{\Delta^*(\phi)/2} = q^{\phi'/2} (1 + \psi' \chi') q^{\phi''/2} \]
If \( H = -\frac{1}{2} \) we get

\[
q^{-\Delta^*(\phi) / 2} = q^{-\phi'/2} \frac{1}{1 + \psi' \chi''} q^{-\phi''/2} = \left( q^{\Delta^*(\phi) / 2} \right)^{-1}
\]

If, further, \( H = 1 \),

\[
q^{\Delta^*(\phi)} = q^{\phi'} (1 + \frac{1}{q} \psi' \chi'') (1 + q \psi' \chi'') q^{\phi''} = \left( q^{\phi'/2} (1 + \psi' \chi'') q^{\phi''/2} \right)^2
\]

and similarly for any integer \( 2H > 0 \)

\[
q^{\pm H \Delta^*(\phi)} = q^{\pm H \phi'} \left( \prod_{n=1}^{2H} (1 + q^{2n-2H-1} \psi' \chi'') \right) q^{\pm H \phi''} = \left( q^{\phi'/2} (1 + \psi' \chi'') q^{\phi''/2} \right)^{\pm 2H}
\]

(5.19)

For non-integer \( 2H \) this formula can be derived by analytical continuation (though the infinite-product representation (2.5) per se is not of direct use in the general case).

Formulas (5.13), (5.14), (5.17) are not new: they are already found in ref.[5]. They can be easily derived in the fundamental representation for \( T \)-generators, when these are just ordinary \( q \)-independent matrices \( (t_{ab}) \) and \( \Delta^*(t_{ab}) = \sum_c t_{ac} \otimes t_{cb} \). Above derivation demonstrates explicitly that \( \Delta^* \) in coordinates \( \chi, \phi, \psi \) is independent of representation. This actually proves the existence of a closed 3-parametric subgroup in the (operator-valued!) universal enveloping algebra, i.e. justifies - at least for the case of \( SL(2)_q \) - introduction of the notion of “group” for \( q \neq 1 \).

### 5.2 Comments on the general case

For higher-rank groups the first three steps of calculation literally repeat those for \( SL(2)_q \), it is only necessary to make use of the following commutativity properties:

\[
[T_i, T_{-j}] = 0 \quad \text{for } i \neq j, \quad \text{so that} \quad [T_s, T_{-s'}] \neq 0 \quad \text{only for } i(s) = i(s'),
\]

\[
\left[ \psi_s T_{\pm s}, \psi_{s'} q_s^{2 H_{s'}} \right] = 0 \quad \text{for } s < s',
\]

\[
\left[ \chi_s T_{\pm s}, \chi_{s'} q_{s'}^{2 H_{s'}} \right] = 0 \quad \text{for } s < s'
\]

(5.20)

This is enough to get:

\[
g\{\chi_s', \phi_{s'}', \psi'_s\} g\{\chi''_s, \phi''_{s''}, \psi''_s\} = \tilde{g}_L q^{2 H (\phi' + \phi'')} \tilde{g}_U
\]

(5.21)
where \( \tilde{g}_U \) can be represented in one of the four equivalent forms:

\[
\tilde{g}(u) = \left( \prod_{s}^{d_B} \delta_{i(s) i(s')} \left< \mathcal{E}_{q_s} \left( \frac{\psi_s' \chi_{s'}'}{q_s - q_s^{-1}} \right) \right> \right)^{\Delta^* (g_U)} =
\]

\[
= \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} \left( \psi_s' q_s^{-\phi_s'} T_s \right) \right> \prod_{i(s) i(s')}^{d_B} \left< \mathcal{E}_{q_s} \left( \frac{\psi_s' \chi_{s'}'}{q_s - q_s^{-1}} \right) \right> \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} (\psi_s' T_s) \right> =
\]

\[
= \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} \left( \psi_s' q_s^{-\phi_s'} T_s + \frac{1}{q_s - q_s^{-1}} \sum_{i(s') = i(s)} \psi_{s'}' \chi_{s'}' \right) \right> \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} (\psi_s' T_s) \right> =
\]

\[
= \prod_{s}^{d_B} \left< \left( \prod_{i(s') = i(s)}^{d_B} \left< \mathcal{E}_{q_s} \left( \frac{\psi_s' \chi_{s'}'}{q_s - q_s^{-1}} \right) \right> \mathcal{E}_{q_s} \left( q_s^{-\phi_s'/2} \left( 1 + \sum_{i(s') = i(s)}^{d_B} \psi_{s'}' \chi_{s'}' \right)^{-1} q_s^{-\phi_s'/2} \psi_s' T_s \right) \right) \right> \cdot \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} (\psi_s' T_s) \right> 
\]

(5.22)

(Exact ordering in the double products is unessential: what is important is that the item with labels \((s_1, s_1')\) stands before that with \((s_2, s_2')\) whenever both \(s_1 \leq s_2, s_1' \leq s_2'\). In the “mixed” situation the items just commute.) To prove the equivalencies one makes use of relations like

\[
\left[ \psi_s' \chi_{s'}, \ psi_{s'}' q^{-\phi_{s'}} / \right] = 0, \quad \text{for } s < s'
\]

(5.23)
as well as Faddeev-Volkov identity. Similar expressions can be written down for \( \tilde{g}_L \).

Thus the problem is essentially reduced to the study of Borel elements and bringing them to our “standard form”:

\[
\Delta^* (g_L) = \prod_{s}^{d_B} \left< \mathcal{E}_{1/q_s} (\Delta^* (\chi_s) T_{-s}) \right>,
\]

\[
\Delta^* (g_U) = \prod_{s}^{d_B} \left< \mathcal{E}_{q_s} (\Delta^* (\psi_s) T_s) \right>
\]

(5.24)
Also in the Cartan sector one has:

\[
q^{2\Delta^\vee}(\tilde{\omega})\tilde{H} = \left( \prod_{i(s)=i(s')}^{d_B} \mathcal{E}_{1/q_s} \left( \frac{-\psi_s^I \chi_{s'}^I}{q_s - q_s^{-1}} \right) \right) q^{2(\tilde{\omega} + \tilde{\omega}')} \tilde{H} \left( \prod_{i(s)=i(s')}^{d_B} \mathcal{E}_{q_s} \left( \frac{\psi_s^I \chi_{s'}^I}{q_s - q_s^{-1}} \right) \right) = q^{2\tilde{\omega}'} \tilde{H} \left( \prod_{i(s)=i(s')}^{d_B} \mathcal{E}_{1/q_s} \left( \frac{-\psi_s^I \chi_{s'}^I}{q_s - q_s^{-1}} q^{-2H_s} \right) \right) \left( \prod_{i(s)=i(s')}^{d_B} \mathcal{E}_{q_s} \left( \frac{\psi_s^I \chi_{s'}^I}{q_s - q_s^{-1}} q^{2H_s} \right) \right) q^{2\tilde{\omega}'} \tilde{H}
\]

Explicit resolution of these equations makes use of the Serre identities and related generalizations of the \(q\)-exponential identities like (2.9). Detailed discussion remains beyond the scope of the present paper. As already mentioned, it is of crucial importance for the proof of existence of the notion of the \(d_G\)-dimensional “group manifold” for \(q \neq 1\) and will be addressed elsewhere. For illustrative purposes we consider the example of \(SL(3)_q\) in Appendix B at the end of this paper.
6 Universal R-matrix

Since we now possess an explicit representation of the group element, all other relations from the theory of quantum groups can be straightforwardly derived. The (universal) R-matrix should not be an exclusion. Non-trivial R-matrix appears because of the non-commutativity of the χ, φ, ψ-parameters, implying that

\[(I \otimes g)(g \otimes I) \neq (g \otimes I)(I \otimes g) = g \otimes g = \Delta(g).\]

Instead

\[\Delta(g) = R^{-1}(I \otimes g)(g \otimes I)R.\]  \hspace{1cm} (6.1)

Thus in order to determine R one should evaluate \((I \otimes g)(g \otimes I)\).

6.1 From group to algebra

Given (3.1), this is a straightforward exercise:

\[
(I \otimes g)(g \otimes I) = \\
= \left( \prod_s^> \mathcal{E}_{1/q_s} \left( \chi^{(s)} T_{-s} \otimes q^{2\vec{\hat{H}}_s} \right) \prod_s^< \mathcal{E}_{q_s} \left( \psi^{(s)} I \otimes T_{+s} \right) \right) \cdot \\
\cdot \left( \prod_s^> \mathcal{E}_{1/q_s} \left( \chi^{(s)} T_{-s} \otimes I \right) \left( q^{2\vec{\hat{H}}_s} \otimes I \right) \prod_s^< \mathcal{E}_{q_s} \left( \psi^{(s)} T_{+s} \otimes I \right) \right)
\]  \hspace{1cm} (6.2)

Of the 6 different factors on the r.h.s. the 4-th and the 3-rd can be freely permuted. Further,

\[
(I \otimes q^{2\vec{\hat{H}}}) \mathcal{E}_{1/q_s} \left( \chi^{(s)} T_{-s} \otimes I \right) = \\
= \mathcal{E}_{1/q_s} \left( \chi^{(s)} T_{-s} \otimes q^{2H_s} \right) \left( I \otimes q^{2\vec{\hat{H}}_s} \right)
\]  \hspace{1cm} (6.3)

and

\[
\mathcal{E}_{q_s} \left( \psi^{(s)} I \otimes T_{+s} \right) \left( q^{2\vec{\hat{H}}_s} \otimes I \right) = \\
= \left( q^{2\vec{\hat{H}}_s} \otimes I \right) \mathcal{E}_{q_s} \left( \psi^{(s)} q_s^{2H_s} \otimes T_{+s} \right)
\]  \hspace{1cm} (6.4)

Thus

\[
(I \otimes g)(g \otimes I) = \\
= \prod_s^> \mathcal{E}_{1/q_s} \left( \chi^{(s)} I \otimes T_{-s} \right) \prod_s^> \mathcal{E}_{1/q_s} \left( \chi^{(s)} T_{-s} \otimes q^{2H_s} \right) \cdot \\
\cdot \left( q^{2\vec{\hat{H}}_s} \otimes q^{2\vec{\hat{H}}_s} \right) \prod_s^< \mathcal{E}_{q_s} \left( \psi^{(s)} q_s^{2H_s} \otimes T_{+s} \right) \prod_s^< \mathcal{E}_{q_s} \left( \psi^{(s)} T_{+s} \otimes I \right)
\]  \hspace{1cm} (6.5)
The two next steps are already familiar from the Section 3 above: the two first and the two last products can be reordered, e.g.

\[
\prod_s \mathcal{E}_{q_s} \left( \psi^{(s)} q_s^{2H_s} \otimes T_{+s} \right) \prod_s \mathcal{E}_{q_s} \left( \psi^{(s)} T_{+s} \otimes I \right) =
\]

\[
= \prod_s \left\{ \mathcal{E}_{q_s} \left( \psi^{(s)} T_{+s} \otimes I \right) \mathcal{E}_{q_s} \left( \psi^{(s)} q_s^{2H_s} \otimes T_{+s} \right) \right\}
\]

(6.6)

(this is because \( [\psi^{(s)} T_{+s} \otimes I, \psi^{(s)} q_s^{2H_s'} \otimes T_{+s'}] = 0 \) for \( s < s' \)), and further (2.8) can be used to rewrite every pair product in the square brackets in (6.6) as a single \( q \)-exponent:

\[
\mathcal{E}_{q_s} \left( \psi^{(s)} T_{+s} \otimes I \right) \mathcal{E}_{q_s} \left( \psi^{(s)} q_s^{2H_s} \otimes T_{+s} \right) =
\]

\[
= \mathcal{E}_{q_s} \left( \psi^{(s)} \left( T_{+s} \otimes I + q_s^{2H_s} \otimes T_{+s} \right) \right)
\]

(6.7)

Finally we get:

\[
\Delta(g) =
\]

\[
= \prod_s \mathcal{E}_{1/q_s} \left( \chi^{(s)} \Delta(T_{-s}) \right) \left( q^{2\tilde{\delta}H} \otimes q^{2\tilde{\delta}H} \right) \prod_s \mathcal{E}_{q_s} \left( \psi^{(s)} \Delta(T_{+s}) \right) =
\]

\[
= \prod_s \mathcal{E}_{1/q_s} \left( \chi^{(s)} \left( q_s^{2H_s} \otimes T_{-s} + T_{-s} \otimes I \right) \right) \cdot
\]

\[
\cdot \left( q^{2\tilde{\delta}H} \otimes q^{2\tilde{\delta}H} \right) \prod_s \mathcal{E}_{q_s} \left( \psi^{(s)} \left( I \otimes T_{+s} + T_{+s} \otimes q_s^{2H_s} \right) \right) =
\]

\[
= \mathcal{R}^{-1} (I \otimes g) (g \otimes I) \mathcal{R} =
\]

\[
= \mathcal{R}^{-1} \prod_s \mathcal{E}_{1/q_s} \left( \chi^{(s)} \left( I \otimes T_{-s} + T_{-s} \otimes q_s^{2H_s} \right) \right) \cdot
\]

\[
\cdot \left( q^{2\tilde{\delta}H} \otimes q^{2\tilde{\delta}H} \right) \prod_s \mathcal{E}_{q_s} \left( \psi^{(s)} \left( q_s^{-2H_s} \otimes T_{+s} + T_{+s} \otimes I \right) \right) \mathcal{R}.
\]

(6.8)

It is clear now that the task of \( \mathcal{R} \)-matrix is to perform the transformation:

\[
\Delta(T_{+i}) = I \otimes T_{+i} + T_{+i} \otimes q_i^{-2H_i} = \mathcal{R}^{-1} \left( q_i^{-2H_i} \otimes T_{+i} + T_{+i} \otimes I \right) \mathcal{R},
\]

\[
\Delta(T_{-i}) = q_i^{2H_i} \otimes T_{-i} + T_{-i} \otimes I = \mathcal{R}^{-1} \left( I \otimes T_{-i} + T_{-i} \otimes q_i^{2H_i} \right) \mathcal{R}
\]

(6.9)

for all the simple roots \( i = 1, \ldots, r_G \), provided

\[
\mathcal{R} \left( q^{2\tilde{\delta}H} \otimes q^{2\tilde{\delta}H} \right) = (q^{2\tilde{\delta}H} \otimes q^{2\tilde{\delta}H}) \mathcal{R}.
\]

(6.10)

Thus we see that the universal \( \mathcal{R} \)-matrix for the group elements, defined in (5.1), is just the same as the ordinary one, introduced by (6.9), (6.10) for the generators \( T \) of the algebra (see also [6, 7, 7] for a more abstract reasoning). For the sake of completeness in the remainder of this section we’ll briefly describe the solution [2, 11] to the relations (6.3), (6.10).
6.2 The case of $SL(2)_q$

Condition (6.11) is satisfied, if $\mathcal{R}$ contains the generators $T_\alpha$ only in the combinations $T_{+\alpha} \otimes T_{-\alpha}$, what will be always the case below. In accordance with (6.9) $\mathcal{R}$ is naturally decomposed, $\mathcal{R} = \hat{\mathcal{Q}} \hat{\mathcal{R}}$, so that the role of $\hat{\mathcal{Q}}$ is to move the factors $q_i^{\pm 2H}$ to the “right place”:

$$
\hat{\mathcal{Q}}^{-1} \left( q_i^{-2H} \otimes T_{+i} + T_{+i} \otimes q_i^{-2H} \right) \hat{\mathcal{Q}} = \left( I \otimes T_{+i} + T_{+i} \otimes q_i^{-2H} \right),
$$

$$
\hat{\mathcal{Q}}^{-1} \left( I \otimes T_{-i} + T_{-i} \otimes q_i^{2H} \right) \hat{\mathcal{Q}} = \left( q_i^{2H} \otimes T_{-i} + T_{-i} \otimes I \right),
$$

while the task of $\hat{\mathcal{R}}$ is to reverse the sign of $2H_i$:

$$
\hat{\mathcal{R}}^{-1} \left( I \otimes T_{+i} + T_{+i} \otimes q_i^{+2H} \right) \hat{\mathcal{R}} = \left( I \otimes T_{+i} + T_{+i} \otimes q_i^{-2H} \right),
$$

$$
\hat{\mathcal{R}}^{-1} \left( q_i^{-2H} \otimes T_{-i} + T_{-i} \otimes I \right) \hat{\mathcal{R}} = \left( q_i^{2H} \otimes T_{-i} + T_{-i} \otimes I \right).
$$

Solution to (6.11) is obviously given by

$$
\hat{\mathcal{Q}} = q^{- \sum_i \alpha_i \alpha_j (\alpha^{-1})_{ij} H_i \otimes H_j}, \quad \alpha_{ij} = \bar{\alpha}_i \bar{\alpha}_j
$$

Construction of $\hat{\mathcal{R}}$ is somewhat more complicated. The basic relation is the following. Let us fix $i$ and take $T_\pm = T_{\pm i}$, $q = q_i$. Then

$$
[(I \otimes T_+)(T_+ \otimes T_-)^n] = T_+^n \otimes \left( \frac{q^{2H} - q^{-2H}}{q - q^{-1}} T_+^{n-1} + T_- \frac{q^{2H} - q^{-2H}}{q - q^{-1}} T_-^{n-2} + \ldots \right) = \frac{[n] q^{n-1}}{q - q^{-1}} \left( (T_+ \otimes q^{2H})(T_+ \otimes T_-)^{n-1} - (T_+ \otimes T_-)^{n-1}(T_+ \otimes q^{2H}) \right)
$$

Summation over $n$ with the weight $\frac{(-q^{-q-1})^n}{[n]!} q^{-\frac{q}{2} n(n-1)}$ gives now:

$$
(I \otimes T_+ + T_+ \otimes q^{2H}) \mathcal{E}_q \left( -(q - q^{-1}) T_+ \otimes T_- \right) = \mathcal{E}_q \left( -(q - q^{-1}) T_+ \otimes T_- \right) (I \otimes T_+ + T_+ \otimes q^{-2H})
$$

This looks just as (6.12) for given $i$ and

$$
\hat{\mathcal{R}}_i = \mathcal{E}_{q_i} \left( -(q_i - q_i^{-1}) T_{+i} \otimes T_{-i} \right)
$$

Relation (6.13) is also satisfied.

This provides the complete answer for universal $\mathcal{R}$-matrix in the case of $SL(2)_q$ (when there is only one value the index $i$ can take):

$$
\mathcal{R}^{SL(2)} = \hat{\mathcal{Q}} \hat{\mathcal{R}} = q^{-2H \otimes H} \mathcal{E}_q \left( -(q - q^{-1}) T_+ \otimes T_- \right).
$$
For groups of higher rank there is additional complication because $\hat{R}_j$ acts non-trivially on some other $T_i$'s with $i \neq j$. Moreover this action produces the generators $T_{\bar{\alpha}}$, associated with all the non-simple roots $\bar{\alpha}$, which can be further eliminated by conjugation with the corresponding $\hat{R}_{\bar{\alpha}}$ matrices, so that finally

$$\mathcal{R} = \hat{Q} \prod_{\bar{\alpha} > 0} \hat{R}_{\bar{\alpha}} \quad (6.19)$$

Note, that the product here - in variance with (3.1) - is over all the positive roots $\bar{\alpha}$, and, as an obvious generalization of (6.17),

$$\hat{R}_{\bar{\alpha}} = \mathcal{E}_{q_{\bar{\alpha}}} \left( -(q_{\bar{\alpha}} - q_{\bar{\alpha}}^{-1}) T_{\bar{\alpha}} \otimes T_{-\bar{\alpha}} \right), \quad (6.20)$$

where $q_{\bar{\alpha}} = q^{||\bar{\alpha}||^2/2}$ and generators $T_{\bar{\alpha}}$ for non-simple $\bar{\alpha}$ still need to be defined.

To make la raison d’etre of this general formula more clear let us consider the simplest example of the rank-2 groups. This example is enough to clarify all the subtle points which could cause confusion in explicit check of (6.19) for any simple group $G_q$.

### 6.3 Serre relations and specification of generators for $q \neq 1$

Before we can proceed to actual calculations we need the explicit form of Serre identities:

- If $\alpha_{ij} = 0$, $T_i T_j = T_j T_i$;
- If $\alpha_{ij} = -\frac{1}{2} \alpha_{ii}$, $T_i^2 T_j - (q_i + q_i^{-1}) T_i T_j T_i + T_j T_i^2 = 0$;
- If $\alpha_{ij} = -\alpha_{ii}$, $T_i^3 T_j - (q_i^2 + 1 + q_i^{-2})(T_i^2 T_j T_i - T_i T_j T_i^2) - T_i T_j^3 = 0$;
- If $\alpha_{ij} = -\frac{3}{2} \alpha_{ii}$, $T_i^4 T_j - \left( \begin{array}{c} 1 \\ 4 \end{array} \right)_i (T_i^3 T_j T_i + T_i T_j T_i^3) + \left( \begin{array}{c} 2 \\ 4 \end{array} \right)_i T_i^2 T_j T_i^2 = 0$,

with $\left( \begin{array}{c} 1 \\ 4 \end{array} \right)_i = [4]_i = q_i^3 + q_i + q_i^{-1} + q_i^{-3}$ and $\left( \begin{array}{c} 2 \\ 4 \end{array} \right)_i = \frac{[4]_i!}{[2]_i! [2]_i!} = (q_i^2 + 1 + q_i^{-2})(q_i^2 + q_i^{-2})$. The same is true for negative generators:

- If $\alpha_{ij} = 0$, $T_{-i} T_{-j} = T_{-j} T_{-i}$;
- If $\alpha_{ij} = -1$, $T_{-i}^2 T_{-j} - (q_i + q_i^{-1}) T_{-i} T_{-j} T_{-i} + T_{-j} T_{-i}^2 = 0$;

etc
Whenever \( \alpha_{ij} \neq 0 \) for \( i \neq j \) the commutator of the two generators \( T_i \) and \( T_j \) in the classical \((q = 1)\) case is equal to \( T_{ij} = T_{\bar{a}_i + \bar{a}_j} \). Serre identities imply that the multiple commutator, which would produce \( T_{(1-a_{ij})\bar{a}_i + \bar{a}_j} \) with \( a_{ij} \equiv \frac{2\alpha_{ij}}{\alpha_{ii}} \) (\( -a_{ij} = 0, 1, 2, 3 \) for simple groups), vanishes. When \( q \neq 1 \) the definition of \( T_{\bar{a}} \) for non-simple roots \( \alpha \) involves \( q \)-commutators and needs some care. Let us make some notational agreements. Namely, assume that the simple roots of the bigger length are ascribed smaller numbers \( i, \alpha_{ii} \geq \alpha_{jj} \) for \( i < j \). This guarantees that for \( i < j \), \( \alpha_{ij} \) is either vanishing or \( \alpha_{ij} = \frac{-1}{2} \alpha_{ii} \), while the ratio \( 2\alpha_{ij}/\alpha_{jj} \) can still be equal to 0, \(-1\), \(-2\), \(-3\). Everywhere below \([n]_{\bar{a}} = [n]_{q_{\alpha}}, [n]_{j} = [n]_{q_{j}}\).

Let us further define:

for \( i < j \), \( \alpha_{ij} = -\alpha_{ii}/2 \):

\[
T_{ij} \equiv T_i T_j - q_i T_j T_i, \tag{6.23}
\]

\[
T_{ijj} \equiv \frac{1}{[2]_j} \left( T_{ij} T_j - \frac{q_i}{q_j^2} T_j T_{ijj} \right), \tag{6.24}
\]

\[
T_{ijjj} \equiv \frac{1}{[3]_j} \left( T_{ijj} T_j - \frac{q_i}{q_j^3} T_j T_{ijjj} \right), \tag{6.25}
\]

\[
T_{ijjjj} \equiv \frac{1}{[4]_j} \left( T_{ijjj} T_j - \frac{q_i}{q_j^4} T_j T_{ijjjj} \right) \tag{6.26}
\]

and also

\[
T_{iij} \equiv T_i T_{ij} - q_i^{-1} T_{ij} T_i, \tag{6.27}
\]

\[
T_{iijj} \equiv \frac{1}{[3]_j} \left( T_{ij} T_{ijj} - \frac{q_i}{q_j^3} T_{ijj} T_{ij} \right) \tag{6.28}
\]

Similarly for negative generators

for \( i < j \), \( \alpha_{ij} = -\alpha_{ii}/2 \)

\[
T_{-ji} \equiv T_{-j} T_{-i} - \frac{1}{q_i} T_{-i} T_{-j}, \tag{6.29}
\]

\[
T_{-jjj} \equiv \frac{1}{[2]_j} \left( T_{-j} T_{-ji} - \frac{q_j}{q_i} T_{-ji} T_{-j} \right), \tag{6.30}
\]

\[
T_{-jjjj} \equiv \frac{1}{[3]_j} \left( T_{-j} T_{-jjj} - \frac{q_j^2}{q_i} T_{-jjj} T_{-j} \right) \tag{6.31}
\]
\[ T_{-jjji} \equiv \frac{1}{[4]_j} \left( T_{-j} T_{-jjji} - \frac{q_i^6}{q_i} T_{-jjji} T_{-j} \right) \] (6.32)

and

\[ T_{-jji} \equiv T_{-ji} T_{-i} - q_i T_{-i} T_{-ji}, \]

\[ T_{-jjjii} \equiv \frac{1}{[3]_j} \left( T_{-jjj} T_{-i} - \frac{q_i^3}{q_i} T_{-ji} T_{-jjj} \right) \] (6.33)

Then Serre identities can be represented in the following form:

for \( i < j \), \( \alpha_{ij} = -\alpha_{ii}/2 \):

always \( T_{iij} = 0 \) : \hspace{1cm} (6.34)

if \( \alpha_{ij} = -\frac{1}{2} \alpha_{jj} \), thus \( q_i = q_j \), then \( T_{ijj} = 0 \), \hspace{1cm} (6.35)

if \( \alpha_{ij} = -\alpha_{jj} \), thus \( q_i = q_j^2 \), then \( T_{ijjj} = 0 \), \hspace{1cm} (6.35)

if \( \alpha_{ij} = -\frac{3}{2} \alpha_{jj} \), thus \( q_i = q_j^3 \), then \( T_{ijjjj} = 0 \)

and similarly for negative generators. As direct corollary of Serre identities we also have:

if \( \alpha_{ij} = -\frac{1}{2} \alpha_{jj} \) or \( \alpha_{ij} = -\alpha_{jj} \), \hspace{1cm} (6.36)

\[ T_{iijj} = T_{-jjjii} = 0 \]

This set of relations, together with the basic definitions (2.1), (2.1) imply natural definitions of all the generators \( T_{\vec{\alpha}} \) and their commutations properties. It is our next task to list the relevant relations.4

### 6.4 Commutation relations between \( T_{\vec{\alpha}} \) for rank-2 groups

To begin with, one more remark about notations. According to our general agreement in the non-simply laced case the root “1” is the bigger one. In most formulas \( q_1 \) and \( q_2 \) are preserved as different variables, so that the formulas can be used in more general framework (beyond consideration of the simple rank-2 algebras). However, some formulas, involving the

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4 It can deserve noting that above definitions of the generators \( T_{\vec{\alpha}} \) for \( q \neq 1 \) is not the only possible one, consistent with representation of the Serre identities in the simple form (5.35). This choice is, however, motivated by the desire to obtain the ingredients of the universal \( R \) matrix in the simple form (1.20).
generators $T_{1222}$ and $T_{11222}$, are written in the form where $q_1 = q_2^3$ is already substituted. Otherwise the formulas would look too ugly. This restriction is of now harm, at least for considerations of simple groups, since $G_2$ is the only example when $q_i = q_j^3$ (in simply-laced case always $q_i = q_j$ and for $B,C$-series sometime also $q_i = q_j^2$). In formulas where substitution $q_1 = q_2^3$ has been made we write “≈” instead of “=”.

\[ T_{1222} - T_{1222}T_1 - T_{12} \equiv 0, \quad (6.37) \]
\[ T_{12} - \frac{1}{q_1} T_{12}T_1 \equiv 0, \quad (6.38) \]
\[ T_{1222} - \frac{1}{q_1} T_{12}T_1 = \frac{1}{q_1[2]_2} \left( 1 - \frac{q_1^2}{q_2^2} \right) T_{12}^2, \quad (6.39) \]
\[ T_{1222} - q_1 T_{1222}T_1 + \frac{q_2^2}{q_1} T_{11222} = \]
\[ = \frac{1}{[3]_2} \left( 1 - \frac{q_1^2}{q_2^2} \right) \left( \frac{q_2^2}{q_1} T_{12}T_{1222} + T_{1222}T_1 \right), \quad (6.40) \]
\[ T_{11222} - \frac{1}{q_1} T_{11222}T_1 = \frac{1}{q_1[2][3]_2} \left( 1 - \frac{q_1^2}{q_2^2} \right) \left( 1 - \frac{q_1^2}{q_2^2} \right) T_{12}^3, \quad (6.41) \]
\[ T_{12} - \frac{q_1}{q_2} T_{12}T_1 - [2]_2 T_{1222} \equiv 0, \quad (6.42) \]
\[ T_{1222} - \frac{q_1}{q_2} T_{12}T_1 - [3]_2 T_{1222} \equiv 0, \quad (6.43) \]
\[ T_{1222} - \frac{q_1}{q_2^6} T_{12}T_1 = 0, \quad (6.44) \]
\[ T_{11222} - T_2 T_{11222} \approx - \left( 1 - \frac{1}{q_2^2} \right) T_{1222}^2, \quad (6.45) \]
\[ T_{12} - \frac{q_1}{q_2} T_{12}T_1 - [3]_2 T_{11222} \equiv 0, \quad (6.46) \]

5 Examples of what happens when $q_1$ is independent of $q_2$ are given by the following relations:

\[ T_{11222} - \frac{q_1^2}{q_2^2} T_{11222} = T_{12}T_{1222} - \frac{q_1^2}{q_2^2} T_{1222}T_{12}, \]
\[ \left( \frac{q_1^2}{q_2^2} [3]_2 - q_2[2]_2 \right) T_{1222}T_{12} = q_1 \left( [3]_2 - \frac{q_1^2}{q_2^2} q_2[2]_2 \right) T_{1222}T_{12} \]

etc.
\[ T_{12}T_{1122} - T_{1222}T_{12} \approx - \left( 1 - \frac{1}{q_2^2} \right) T_{122}^2, \quad (6.47) \]
\[ T_{12}T_{1122} - \frac{1}{q_2^2} T_{1122}T_{12} \approx 0; \quad (6.48) \]
\[ T_{122}T_{1122} - \frac{1}{q_2^3} T_{1122}T_{122} \approx 0, \quad (6.49) \]
\[ T_{122}T_{1122} - q_2^3 T_{1122}T_{122} \approx 0; \quad (6.50) \]
\[ T_{1222}T_{1122} - q_2^3 T_{11222}T_{1222} \approx \frac{(q_2 - q_2^{-1})^2}{3_2} T_{1122}^3 \quad (6.51) \]

This list involves all the relations between positive-root generators of $G(2)_q$, when $q_1 = q_2^2$. In order to truncate the case of $B(2)_q \cong C(2)_q$ it is enough to omit relations with the generators $T_{1222}$ and $T_{11222}$ and put $q_1 = q_2^2$. Further truncation to $A(2)_q = SL(3)_q$ case implies that relations with $T_{122}$ are also omitted and $q_1 = q_2$.

The list is not the complete set of commutation relations for rank-2 algebras. Commutation relations among negative generators are easily obtained from those for positive ones: by “transponing” $T_{i\ldots j}T_{k\ldots l} \rightarrow T_{l\ldots k}T_{j\ldots i}$ and changing $q \rightarrow q^{-1}$. Commutation rules with Cartan generators $q^{\pm H_2}$ are trivially deduced from (2.1): for $\alpha = k_1 \alpha_1 + k_2 \alpha_2$
\[ q^{\pm 2H_1} T_\alpha = q^{\pm (k_1 \alpha_1 + k_2 \alpha_2)} T_\alpha T_\alpha q^{\pm 2H_1} = q_1^{\pm (k_1 - k_2)} T_\alpha q_1^{\pm 2H_1}, \]
\[ q^{\pm 2H_2} T_\alpha = q^{\pm (k_1 \alpha_1 + k_2 \alpha_2)} T_\alpha T_\alpha q^{\pm 2H_2} = q_2^{\pm k_1} q_2^{\pm 2k_2} T_\alpha q_2^{\pm 2H_2} \quad (6.52) \]

Relation between negative and positive generators are simple corollaries of (2.1). Of these we’ll need only the following:
\[ [T_2, T_{-21}] = \frac{q_1 - q_1^{-1}}{q_2 - q_2^{-1}} T_{-1} q_2^{2H_2}, \quad (6.53) \]
\[ [T_2, T_{-22}] = \frac{q_1 / q_2 - q_2 / q_1}{q_2 - q_2^{-1}} T_{-21} q_2^{2H_2}, \quad (6.54) \]
\[ [T_2, T_{-222}] = \frac{q_1 / q_2 - q_2 / q_1^2}{q_2 - q_2^{-1}} T_{-221} q_2^{2H_2}, \quad (6.55) \]
\[ [T_2, T_{-2221}] = \frac{q_1^2}{q_2^2 [2]} \frac{(1 - q_2^2 / q_1^2) (1 - q_1^2 / q_2^2)}{1 - 1 / q_2^2} T_{-21} q_2^{2H_2} \approx \quad (6.56) \]
\[ \approx q_2 (q_2 - q_2^{-1}) T_{-21} q_2^{2H_2} \]
Also:

\[
[T_1, T_{-21}] = q_1^{-2H_1} T_{-2}, \quad (6.57)
\]

\[
[T_1, T_{-221}] = \frac{q_1}{[2]^2} \left( 1 - \frac{q_2}{q_1} \right) q_1^{-2H_1} T_{-2}^2, \quad (6.58)
\]

\[
[T_1, T_{-2221}] = \frac{q_1^2}{[2][3]^2} \left( 1 - \frac{q_2}{q_1} \right) \left( 1 - \frac{q_4}{q_1} \right) q_1^{-2H_1} T_{-3}^2, \quad (6.59)
\]

\[
[T_1, T_{-22211}] = \frac{1}{[3]^2} \left( q_1 \left( 1 - \frac{q_2}{q_1} - \frac{q_4^2}{q_1^2} \right) T_{-22} T_{-221} + \left( 1 + q_2 - \frac{q_4^2}{q_1^2} \right) T_{-221} T_{-2} \right); \quad (6.60)
\]

\[
[T_{12}, T_{-21}] = \frac{1}{q_2 - q_2} \left( q_1^{-2H_1} q_2^{2H_2} - q_1^{-2H_1} q_2^{-2H_2} \right) - \frac{1}{q_1 - q_1} \left( q_1^{-2H_1} q_2^{2H_2} - q_1^{-2H_1} q_2^{-2H_2} \right) \quad (6.61)
\]

(for \( SL(3)_q \), when \( q_1 = q_2 = q \), the r.h.s. turns just into \( \frac{1}{q - q^{-1}} (q^{2H_1} q^{2H_2} - q^{-2H_1} q^{-2H_2}) \)).

After all these specifications we are finally in a position to check (6.5).

### 6.5 The case of the rank-2 groups

We are going to prove now that

\[
\hat{R} = \hat{R}_1 \hat{R}_{ns} \hat{R}_2 \quad (6.62)
\]

satisfies eqs. (6.12), (6.13) for \( i = 1, 2 \), provided

\[
\hat{R}_{ns} = \hat{R}_{12} \quad \text{for } A(2)_q,
\]

\[
\hat{R}_{ns} = \hat{R}_{12} \hat{R}_{122} \quad \text{for } B(2)_q \cong C(2)_q,
\]

\[
\hat{R}_{ns} = \hat{R}_{12} \hat{R}_{1222} \hat{R}_{122} \hat{R}_{1222} \quad \text{for } G(2)_q
\]

Here \( 1 \ldots 12 \ldots 2 \) denote the (non-simple) roots \( \vec{\alpha}_1 + \ldots + \vec{\alpha}_1 + \vec{\alpha}_2 + \ldots + \vec{\alpha}_2 \) and \( T_{\vec{\alpha}}, T_{-\vec{\alpha}} \) in (6.20) are defined in (6.23)-(6.33).

\[\text{Note that the coefficients at the r.h.s. of (6.60) are exactly the same as in (6.40), if the latter one is rewrinen as:}\]

\[
T_1 T_{122} - q_1 T_{122} T_1 = \frac{q_1}{q_2^{[1]}[2]} \left( q_1 \left( 1 - \frac{q_2}{q_1} - \frac{q_4^2}{q_1^2} \right) T_{122} T_1 + \left( 1 + q_2 - \frac{q_4^2}{q_1^2} \right) T_{122} T_{12} \right)
\]

(In variance with (6.40) the r.h.s. here is not vanishing in the limit \( q \to 1 \).)
Since we already know the property (6.16) of \( \hat{R} \), it follows that (6.12) and (6.13) are valid with \( \hat{R} \) in (6.62) provided

\[
\hat{R}_{ns} \hat{R}_2 \left( I \otimes T_1 + T_1 \otimes q_1^{-2H_1} \right) \hat{R}_2^{-1} \hat{R}_{ns}^{-1} = \left( I \otimes T_1 + T_1 \otimes q_1^{-2H_1} \right), \quad (6.64)
\]

\[
\hat{R}_{ns} \hat{R}_2 \left( q_1^{2H_1} \otimes T_{-1} + T_{-1} \otimes I \right) \hat{R}_2^{-1} \hat{R}_{ns}^{-1} = \left( q_1^{2H_1} \otimes T_{-1} + T_{-1} \otimes I \right); \quad (6.65)
\]

\[
\hat{R}_{ns}^{-1} \hat{R}_1^{-1} \left( I \otimes T_2 + T_2 \otimes q_2^{2H_2} \right) \hat{R}_1 \hat{R}_{ns} = \left( I \otimes T_2 + T_2 \otimes q_2^{2H_2} \right), \quad (6.66)
\]

\[
\hat{R}_{ns}^{-1} \hat{R}_1^{-1} \left( q_2^{-2H_2} \otimes T_{-2} + T_{-2} \otimes I \right) \hat{R}_1 \hat{R}_{ns} = \left( q_2^{-2H_2} \otimes T_{-2} + T_{-2} \otimes I \right) \quad (6.67)
\]

We present calculations only for eq. (6.66), the three other cases can be analyzed similarly with the same result.

### 6.5.1 Conjugation by \( \hat{R}_\alpha \)

We shall consider separately conjugation of \( I \otimes T_2 \) and \( T_2 \otimes q_2^{2H_2} \) by each of the relevant operators \( \hat{R} \)-matrices \( \hat{R}_\alpha = \mathcal{E}_{q_\alpha} \left( -(q_\alpha - q_\overline{\alpha})V_\alpha \right), V_\alpha \equiv T_\alpha \otimes T_{-\alpha} \). Every such conjugation is evaluated in four steps:

1) Let

\[
U^R_\alpha = [I \otimes T_2, V_\alpha] = (I \otimes T_2)(T_\alpha \otimes T_{-\alpha}) - (T_\alpha \otimes T_{-\alpha})(I \otimes T_2),
\]

\[
U^L_\alpha = -[T_2 \otimes q_2^{2H_2}, V_\alpha] = (T_\alpha \otimes T_{-\alpha})(T_2 \otimes q_2^{2H_2}) - (T_2 \otimes q_2^{2H_2})(T_\alpha \otimes T_{-\alpha}) \quad (6.68)
\]

Here commutators are the ordinary ones (not \( q \)-commutators).

2) With our choice of generators \( T_\alpha \) we always have (see Appendix A):

\[
U^R_\alpha V_\alpha = \sigma^R_\alpha \left( V_\alpha U^R_\alpha + W^R_\alpha \right),
\]

\[
W^R_\alpha V_\alpha = \rho^R_\alpha V_\alpha W^R_\alpha, \quad (6.69)
\]

with

\[
\sigma^R_\alpha = \frac{1}{q_\alpha^2}, \quad \text{and} \quad \rho^R_\alpha = (\sigma^R_\alpha)^2 = \frac{1}{q_\alpha^4} \quad (6.70)
\]

while

\[
U^L_\alpha V_\alpha = \sigma^L_\alpha \left( V_\alpha U^L_\alpha - W^L_\alpha \right),
\]

\[
W^L_\alpha V_\alpha = \rho^L_\alpha V_\alpha W^L_\alpha, \quad (6.71)
\]
with

\[ \sigma^L_{\tilde{\alpha}} = q^2_{\tilde{\alpha}}, \quad \text{and} \quad \rho^L_{\tilde{\alpha}} = (\sigma^L_{\tilde{\alpha}})^2 = q^4_{\tilde{\alpha}} \]  
(6.72)

3) It now follows that

\[ V^a_{\tilde{\alpha}} U^R_{\tilde{\alpha}} = q^{2a}_{\tilde{\alpha}} U^R_{\tilde{\alpha}} V^a_{\tilde{\alpha}} - q^{3a-3}[a]_{\tilde{\alpha}} W^R_{\tilde{\alpha}} V^{a-1}_{\tilde{\alpha}} \]  
(6.73)

and

\[ U^L_{\tilde{\alpha}} V^b_{\tilde{\alpha}} = q^{2b}_{\tilde{\alpha}} V^b_{\tilde{\alpha}} U^L_{\tilde{\alpha}} - q^{3b-1}[b]_{\tilde{\alpha}} V^{b-1}_{\tilde{\alpha}} W^L_{\tilde{\alpha}} \]  
(6.74)

so that finally:

\[ [I \otimes T_2, V^a_{\tilde{\alpha}}] = \sum_{a+b=n-1} V^a_{\tilde{\alpha}} U^R_{\tilde{\alpha}} V^b_{\tilde{\alpha}} = q^{n-1}_{\tilde{\alpha}} [n]_{\tilde{\alpha}} U^R_{\tilde{\alpha}} V^{n-1}_{\tilde{\alpha}} - \frac{q^{n-1}_{\tilde{\alpha}}}{[2]_{\tilde{\alpha}}} [n]_{\tilde{\alpha}} q^{n-2}_{\tilde{\alpha}} [n-1]_{\tilde{\alpha}} W^R_{\tilde{\alpha}} V^{n-2}_{\tilde{\alpha}} \]  
(6.75)

while

\[ [T_2 \otimes q_2^2 H^2, V^a_{\tilde{\alpha}}] = - \sum_{a+b=n-1} V^a_{\tilde{\alpha}} U^L_{\tilde{\alpha}} V^b_{\tilde{\alpha}} = -q^{n-1}_{\tilde{\alpha}} [n]_{\tilde{\alpha}} V^{n-1}_{\tilde{\alpha}} U^L_{\tilde{\alpha}} + \frac{q^{n-1}_{\tilde{\alpha}}}{[2]_{\tilde{\alpha}}} q^{n-2}_{\tilde{\alpha}} [n]_{\tilde{\alpha}} q^{n-2}_{\tilde{\alpha}} [n-1]_{\tilde{\alpha}} V^{n-2}_{\tilde{\alpha}} W^L_{\tilde{\alpha}} \]  
(6.76)

4) Finally summation over \( n \) with the weight \( \frac{(-q_\alpha^{-1} - q_\alpha^{-2})^n}{[n]_{\tilde{\alpha}}} q_{\tilde{\alpha}}^{-(n-1)/2} \) gives:

\[ \hat{\mathcal{R}}^{-1}_{\tilde{\alpha}} \left( I \otimes T_2 + (q_\alpha - q_\alpha^{-1}) U^R_{\tilde{\alpha}} + \frac{1}{q_\alpha [2]_{\tilde{\alpha}}} (q_\alpha - q_\alpha^{-1})^2 W^R_{\tilde{\alpha}} \right) \hat{\mathcal{R}}_{\tilde{\alpha}} = I \otimes T_2 \]  
(6.77)

and

\[ \hat{\mathcal{R}}^{-1}_{\tilde{\alpha}} \left( T_2 \otimes q_2^2 H^2 \right) \hat{\mathcal{R}}_{\tilde{\alpha}} = T_2 \otimes q_2^2 H^2 + (q_\alpha - q_\alpha^{-1}) U^L_{\tilde{\alpha}} + \frac{q^{n-1}_{\tilde{\alpha}}}{[2]_{\tilde{\alpha}}} (q_\alpha - q_\alpha^{-1})^2 W^L_{\tilde{\alpha}} \]  
(6.78)

### 6.5.2 The rank-2 groups

We now list operators \( U \) and \( W \) for all the positive roots of \( A(2)_q, B(2)_q \cong C(2)_q, G(2)_q \). They can be found - and properties \( (6.69), (6.71) \) checked - with the help of the commutation relations from subsection \( 6.4 \) (see Appendix A for more details).

\( \tilde{\alpha} = \tilde{\alpha}_1, \quad q_{\tilde{\alpha}} = q_1 \):

\[ U^R_1 = 0, \quad W^R_1 = 0, \]  
(6.79)

\[ U^L_1 = T_{12} \otimes T_{-1} q_2^2 H^2, \quad W^L_1 = 0 \]
\[ \vec{\alpha} = \vec{\alpha}_1 + \vec{\alpha}_2, \quad q_{12} = q_2: \]

\[ U_{12}^R = \frac{q_1 - q_1^{-1}}{q_2 - q_2^{-1}} T_{12} \otimes T_{-21} q_2^{2H_2}, \quad W_{12}^R = 0, \]
\[ U_{12}^L = [2]_2 T_{122} \otimes T_{-21} q_2^{2H_2}, \quad W_{12}^L = [2][3]_2 T_{1122} \otimes T_{21} q_2^{2H_2} \]

(6.80)

\[ \vec{\alpha} = 2\vec{\alpha}_1 + 3\vec{\alpha}_2, \quad q_{1122} = \frac{q_2^4}{q_1^2} \approx q_1: \]

\[ U_{1122}^R \approx (q_2^2 - 1) T_{1122} \otimes T_{-21} q_2^{2H_2}, \quad W_{1122}^R = 0, \]
\[ U_{1122}^L = -(1 - q_2^{-2}) T_{122} \otimes T_{-2211} q_2^{2H_2}, \quad W_{1122}^L = 0 \]

(6.81)

In this case we shall need one more statement:
\[ \left[ T_{12} \otimes T_{-21} q_2^{2H_2}, V_{1122} \right] = 0, \]
thus
\[ \hat{R}_{1122}^{-1} \left( T_{12} \otimes T_{-21} q_2^{2H_2} \right) \hat{R}_{1122} = \left( T_{12} \otimes T_{-21} q_2^{2H_2} \right) \]

(6.82)

\[ \vec{\alpha} = \vec{\alpha}_1 + 2\vec{\alpha}_2, \quad q_{122} = \frac{q_2^2}{q_1^2}, \]

\[ U_{122}^R = \frac{q_1/q_2 - q_2^2/q_1}{q_2 - q_2^{-1}} T_{122} \otimes T_{-21} q_2^{2H_2}, \]
\[ W_{122}^R = -\frac{q_1/q_2 - q_2^2/q_1}{q_2 - q_2^{-1}} [3]_2 T_{122} \otimes T_{-2211} q_2^{2H_2} \approx [2][3]_2 T_{122} \otimes T_{-2211} q_2^{2H_2}, \]
\[ U_{122}^L = [3]_2 T_{122} \otimes T_{-221} q_2^{2H_2}, \]
\[ W_{122}^L = 0 \]

(6.84)

\[ \vec{\alpha} = \vec{\alpha}_1 + 3\vec{\alpha}_2, \quad q_{1222} = \frac{q_2^3}{q_1^3} \approx q_1: \]

\[ U_{1222}^R = \frac{q_1/q_2 - q_2^3/q_1}{q_2 - q_2^{-1}} T_{1222} \otimes T_{-221} q_2^{2H_2} \approx T_{1222} \otimes T_{-221} q_2^{2H_2}, \]
\[ W_{1222}^R = U_{1222}^L = W_{1222}^L = 0 \]

(6.85)
Applying now (6.77) and (6.78) we get the following chain of transformations:

\[ \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \{ I \otimes T_2 + T_2 \otimes q^{2H_2} \} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} = \]  
\[ = \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \{ I \otimes T_2 + T_2 \otimes q^{2H_2} + (q_1 - q_1^{-1})T_{12} \otimes T_{-1}q_2^{2H_2} \} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} = \]  
\[ = \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \{ I \otimes T_2 + T_2 \otimes q^{2H_2} + (q_{12} - q_{12}^{-1}) \} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} = \]  
\[ = \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \{ I \otimes T_2 + T_2 \otimes q^{2H_2} + (q_{12} - q_{12}^{-1}) \} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} = \]  
\[ = \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \hat{R}_{122}^{-1} \{ I \otimes T_2 + T_2 \otimes q^{2H_2} + (q_{12} - q_{12}^{-1}) \} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} \hat{R}_{122} = \]  

This chain is written down for the most complicated \(G(2)_q\) case, for \(A(2)_q\) and \(B(2)_q \cong C(2)_q\) cases some steps are just omitted (by putting irrelevant \(T_\alpha = 0\) and \(\hat{R}_\alpha = I \otimes I\)). In these formulas expressions (6.79)-(6.85) are used with the “≈” equalities (valid only if \(q_1 = q_2^3\)) - but only in the terms which are irrelevant for the \(A, B, C\) groups.
This completes the proof of (6.66) for all the rank-2 simple groups. The other three
relations (6.64), (6.65), (6.67) can be verified just in the same way. Generalization to higher-
rank groups is also straightforward. It does not involve anything new, except for the choice
of ordering of positive roots $\vec{\alpha}$ in (3.19), which is always obvious (see [10] for the case of
$A$-series).

\[ T_{i...j}T_{\pm k...l} \rightarrow T_{\mp l...k}T_{-j...i} \text{ and } q \rightarrow q^{-1}. \]

The proof of (6.64), (6.65) is somewhat more tedious: one should
deal not only with double, but also with 3- and 4-fold $q$-commutators at intermediate stages of calculation.

\[ \text{In order to obtain (6.67) it is enough to switch between the positive and negative roots, i.e. change} \]

\[ T_{i...j}T_{\pm k...l} \rightarrow T_{\mp l...k}T_{-j...i} \text{ and } q \rightarrow q^{-1}. \]

\[ \text{The proof of (6.64), (6.65) is somewhat more tedious: one should} \]

\[ \text{deal not only with double, but also with 3- and 4-fold } q\text{-commutators at intermediate stages of calculation.} \]
7 Conclusion

We described a simple explicit formula for the group element of any simple quantum group. Non-commuting “coordinates” on the “group manifold” satisfy a Heisenberg-type algebra and only the Chevalley generators, associated with simple roots are involved. There is no reference to particular representation of Chevalley generators themselves, which can be substituted not only as matrices but, for example, in the form of difference operators etc. The most obscure feature of the formalism is the need to choose a map $s$ from all the positive roots to simple ones, which is ambiguously defined, thus giving rise to somewhat different representations, when $r_G > 1$. A closely related problem is to find out the adequate group composition rule for $r_G > 1$ and explicit formulas for $\Delta^*(\chi, \phi, \psi)$. Probably some representations of group elements involving non-simple roots can be constructed as well.

We described also an explicit check of the $\mathcal{RTT}$ relation, involving explicit $q$-exponential representation of the universal $\mathcal{R}$-matrix as a product of “elementary” $\mathcal{R}_{\vec{\alpha}}$-matrices, associated with all the positive roots $\vec{\alpha}$. A detailed presentation was given of the rank-2 case.

All this provides certain evidence that a well-defined notion of $d_G$-parametric “group manifold” can emerge for $q \neq 1$, just as it exists in the classical case of $q = 1$. See Appendix B below for more comments on this issue.

The two straightforward directions to further develop this technique is to work out explicit formulas for quantum affine (Kac-Moody) algebras and to describe the realization of quantum groups with the trivial $\mathcal{R}$ matrix, $\mathcal{R} = q^{\sum_{\alpha=1}^{d_G} T_{\alpha} \otimes T_{\alpha}}$, but non-trivial $q$-associator.

Another task is to find a Lagrangian/functional integral description, i.e. an adequate version of the geometrical quantization and deformation of the WZNW model. Given the simple algebra of the $\chi$, $\phi$, $\psi$-variables (fields) this should not be a very complicated problem. Once solved, it can open the way to complete bosonization of the quantum group, including also some natural choice of Heisenberg-like representation of the generators $T_{\vec{\alpha}}$.

---

8 E.g. for $SL(2)_q$ one can substitute in (3.1)

\[
T_+ = \frac{q}{t} \frac{1 - M_{t^{-2}}}{q - q^{-1}}, \quad q^{\pm 2H} = q^{\pm 2\lambda} M_{t^{-2}}, \quad T_- = t \frac{q^{2\lambda} - q^{-2\lambda} M_{t^{-2}}}{q - q^{-1}};
\]

see [3], [14] and especially [15] for general description of such representations.
8 Acknowledgements

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9 APPENDIX A. R-matrix for rank-2 algebras

This appendix contains some details of the derivation of formulas (6.79)-(6.85) and (6.86)-(6.95).

9.1 $I \otimes T_2$ versus $\hat{R}_1$

Since $[T_2, T_{-1}] = 0$, see (2.1),

$$U^R_1 = [I \otimes T_2, T_1 \otimes T_{-1}] = 0,$$

then also $W^R_1 = 0$, and the steps 2)-4) from subsection 5.5.1 are trivial:

$$\hat{R}_1^{-1} (I \otimes T_2) \hat{R}_1 = I \otimes T_2$$

9.2 $T_2 \otimes q_2^{2H_2}$ versus $\hat{R}_1$

1) This time

1a) : $q_2^{2H_2} T_{-1} = q_1 T_{-1} q_2^{2H_2}$

(since $q^{-\alpha_{21}} = q^{\alpha_{11}/2} = q_1$) and, according to (6.37),

1b) : $T_2 T_1 = \frac{1}{q_1} (T_1 T_2 - T_{12})$ (9.4)

When combined these two relations imply:

$$U^L_1 = - \left[ (T_2 \otimes q_2^{2H_2}), (T_1 \otimes T_{-1}) \right] = T_{12} \otimes T_{-1} q_2^{2H_2}$$

2) Since:

2a) = 1a) : $(T_{-1} q_2^{2H_2}) T_{-1} = q_1 T_{-1} (T_{-1} q_2^{2H_2})$

and, according to (6.38),

2b) : $T_{12} T_1 = q T_1 T_{12}$

It is important that the coefficients at the r.h.s. in (9.3) and (9.4) cancel each other. It is this cancelation that (given the universal form of the $R$-matrix) selects the relevant definition of $T_{12}$, against the alternative $\hat{T}_{12} = T_1 T_{-2} - \frac{1}{q} T_2 T_1$. (Things work similarly for all other $T_{\vec{\alpha}}$ with non-simple roots $\vec{\alpha}$.) If coefficients do not cancel, $U^L$ in (9.5) would be some $q$-commutator instead of an ordinary one. This will produce extra $n$-dependence in (6.76) and destroy the reasoning, which leads to relations (6.77), (6.78).
we have:
\[(T_{12} \otimes T_{-1}q_2^{2H_2})(T_1 \otimes T_{-1}) = \sigma_1^2(T_1 \otimes T_{-1})(T_{12} \otimes T_{-1}q_2^{2H_2})\] (9.8)
i.e. \(W_1^L = 0\) and indeed \(\sigma_1^L = q_1^2\), as stated in (6.72).

Steps 3) and 4) deserve no comments: we just apply (6.78) to obtain:
\[
\hat{R}_{12}^{-1} (T_2 \otimes q_2^{2H_2}) \hat{R}_{12} = (T_2 \otimes T_{-21}q_2^{2H_2} + (q_1 - q_1^{-1})T_{12} \otimes T_{-1}q_2^{2H_2}) \] (9.9)
Together with (9.2) this formula describes the transition from (6.86) to (6.87). The result coincides also with (6.88) because \(q_1 = q_2\) and \((q_1 - q_1^{-1})q_2^{-1} = q_1 - q_1^{-1}\).

### 9.3 \(I \otimes T_2\) versus \(\hat{R}_{12}\)

1) From (6.53) we get:
\[
U_{12}^R = [I \otimes T_2, T_{12} \otimes T_{-21}] = T_{12} \otimes [T_2, T_{-21}] = \frac{q_1 - q_1^{-1}}{q_2 - q_2^{-1}}T_{12} \otimes T_{-1}q_2^{2H_2} 
\] (9.10)

2) Since
\[
2a) : \quad q_2^{2H_2}T_{-21} = \frac{q_1}{q_2}T_{-21}q_2^{2H_2} \] (9.11)
(because \(q^{-\alpha_{22} - \alpha_{21}} = q^{-\alpha_{22} + \alpha_{11}/2} = q_1/q_2^2\)) and (according to the Serre identity \(T_{-ji} = 0\))
\[
2b) : \quad T_{-1}T_{-21} = q_1^{-1}T_{-21}T_{-1} \] (9.12)
we have:
\[
(T_{12} \otimes T_{-1}q_2^{2H_2})(T_{12} \otimes T_{-21}) = \frac{1}{q_2^2}(T_{12} \otimes T_{-21})(T_{12} \otimes T_{-1}q_2^{2H_2}) \] (9.13)
Thus \(W_{12}^R = 0\) and indeed \(\sigma_{12}^R = q_2^{-2} = q_{12}^{-2}\) - as stated in (6.7)(i).

It follows from (6.77) now, that
\[
\hat{R}_{12}^{-1} [I \otimes T_2 + (q_1 - q_1^{-1})T_{12} \otimes T_{-1}q_2^{2H_2}] \hat{R}_{12} = (I \otimes T_2) \] (9.14)

### 9.4 \(T_2 \otimes q_2^{2H_2}\) versus \(\hat{R}_{12}\). The end of the proof for \(SL(3)_q\)

1) Now
\[
1a) = \text{eq.(9.11)} : \quad q_2^{2H_2}T_{-21} = \frac{q_1}{q_2}T_{-21}q_2^{2H_2} \] (9.15)
and, according to (6.42),

\[ 1b) : \quad T_2 T_{12} = \frac{q_2^2}{q_1} (T_{12} T_2 - [2]_2 T_{122}) \quad (9.16) \]

Thus

\[ U^L_{12} = - \left[ T_2 \otimes q_2^{2H_2}, T_{12} \otimes T_{-21} \right] = [2]_2 T_{122} \otimes T_{-21} q_2^{2H_2} \quad (9.17) \]

Since for \( SL(3)_q = A(2)_q \) the Serre identity implies that \( T_{122} = 0 \), this is the end of our calculations in this case: steps 2)-4) are now trivial,

for \( SL(3)_q \):

\[ \hat{R}^{-1}_{12}(T_2 \otimes q_2^{2H_2}) \hat{R}_{12} = T_2 \otimes q_2^{2H_2} \quad (9.18) \]

which together with (9.14) gives the desired (6.66), i.e. the proof of (6.66) for \( A(2)_q \) is already completed. We, however, proceed further with the non-simply-laced case, when \( T_{122} \neq 0 \).

2) From (9.11) it follows that

\[ 2a) : \quad (T_{-21} q_2^{2H_2}) T_{-21} = \frac{q_1^3}{q_2} T_{-21} (T_{-21} q_2^{2H_2}) \quad (9.19) \]

and, due to (6.46)

\[ 2b) : \quad T_{122} T_{12} = \frac{q_1^4}{q_2} (T_{12} T_{122} - [3]_2 T_{11222}) \quad (9.20) \]

we have:

\[ [2]_2 (T_{122} \otimes T_{-21} q_2^{2H_2})(T_{12} \otimes T_{-21}) = q_2^2 [2]_2 \left( (T_{12} \otimes T_{-21})(T_{122} \otimes T_{-21} q_2^{2H_2}) - [3]_2 T_{11222} \otimes T_{21}^2 q_2^{2H_2} \right), \]

i.e. \( \sigma^L_{12} \) is indeed \( q_{122}^2 = q_2^2 \), and

\[ W^L_{12} = [2]_2 [3]_2 T_{11222} \otimes T_{21}^2 q_2^{2H_2} \quad (9.22) \]

This quantity is non-vanishing for \( G(2)_q \) only. This allows to use \( \approx \) formulas (i.e. valid only for \( q_1 = q_2^3 \)) in the next lines. Because of the same (9.11)

\[ 2c) : \quad (T_{-21}^2 q_2^{2H_2}) T_{-21} = \frac{q_1^3}{q_2^3} T_{-21} (T_{-21}^2 q_2^{2H_2}) \approx q_2 T_{-21} (T_{-21}^2 q_2^{2H_2}) \quad (9.23) \]

and of (6.48)

\[ 2d) : \quad T_{11222} T_{12} \approx q_2^3 T_{12} T_{11222} \quad (9.24) \]
we have:

\[
\left( T_{11222} \otimes T_{21}^2 q_2^{2H_2} \right) \left( T_{12} \otimes T_{-21} \right) \approx q_2^4 \left( T_{12} \otimes T_{-21} \right) \left( T_{11222} \otimes T_{21}^2 q_2^{2H_2} \right)
\]  
\[\text{Eq.(9.25)}\]

and indeed \( \rho_{12}^L = q_2^4 = q_{12}^4 \) as stated in (6.72).

Eq.(6.78) now says that

\[
\hat{R}_{12}^{-1} \left( T_2 \otimes q_2^{2H_2} \right) \hat{R}_{12} = T_2 \otimes q_2^{2H_2} + [2]_2(q_2 - q_2^{-1})T_{12} \otimes T_{-21} q_2^{2H_2} + [3]_2 q_2(q_2 - q_2^{-1})^2 T_{11222} \otimes T_{-21}^2 q_2^{2H_2}
\]  
\[\text{Eq.(9.26)}\]

Together with (9.14) this describes the transition from (6.87) to (6.89). The next step - to (6.90) - is to note that the last term at the r.h.s. of (9.26) is present only in \( G(2)_q \) case, when \( q_1 = q_2^3 \) and thus one can substitute \( q_{11222} - q_{11222}^{-1} = q_1 - q_1^{-1} \approx [3]_2(q_2 - q_2^{-1}) \).

As mentioned above calculation for \( A(2)_q \) is already finished. The next conjugation - by \( \hat{R}_{11222} \) - is present only in the \( G(2)_q \) case. It will eliminate the last term at the r.h.s. of (9.26) and gives rise instead to another one - also present only in the case of \( G(2)_q \). The second term in (9.26) is however, present in the \( B(2)_q \cong C(2)_q \) case as well - thus it is important that it is left intact by the \( \hat{R}_{11222} \) conjugation.

**9.5 \( T_{122} \otimes T_{21} q_2^{2H_2} \) versus \( \hat{R}_{11222} \)**

According to (6.48),

\[
T_{11222} T_{12} \approx q_2^3 T_{12} T_{11222},
\]

thus

\[
T_{-21} T_{-22211} \approx q_2^{-3} T_{-22211} T_{-21}
\]

Together with (6.50),

\[
T_{12} T_{11222} \approx q_2^3 T_{11222} T_{122},
\]

and orthogonality property \( \alpha_{11222} \alpha_{2} = 2 \alpha_{12} + 3 \alpha_{22} = 3 \alpha_{22} - \alpha_{11} \approx 0 \) which guarantees that

\[
q_2^{2H_2} T_{-22211} = T_{-22211} q_2^{2H_2} \]  
\[\text{(9.27)}\]

this implies that

\[
\left[ T_{12} \otimes T_{21} q_2^{2H_2}, \ T_{11222} \otimes T_{-22211} \right] = 0 \]  
\[\text{(9.28)}\]

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### 9.6 $I \otimes T_2$ versus $\hat{\mathcal{R}}_{11222}$

1) From (6.56)

$$U_{12}^R = [I \otimes T_2, T_{11222} \otimes T_{-22211}] = T_{11222} \otimes [T_2, T_{-22211}] \approx (q_2^2 - 1)T_{11222} \otimes T_{-21}^2 q_2^{2H_2}$$

(9.29)

2) From (6.48)

$$T_{11222}T_{12} \approx q_2^3 T_{12} T_{11222}$$

Thus

$$T_{-21}T_{-22211} \approx q_2^{-3} T_{-22211} T_{21}$$

and, combined with eq.(9.27) this means that

$$(T_{11222} \otimes T_{-21}^2 q_2^{2H_2})(T_{11222} \otimes T_{-22211}) = q_2^{-6}(T_{11222} \otimes T_{-22211})(T_{11222} \otimes T_{-21}^2 q_2^{2H_2})$$

(9.30)

We get $W_{11222}^R = 0$ and $\sigma_{11222}^R = q_2^{-6} = q_{11222}^2$, in accordance with (6.70).

Finally

$$\hat{\mathcal{R}}_{11222}^{-1}(I \otimes T_2 + (q_{11222} - q_{11222}^{-1})(q_2 - q_2^{-1})q_2 T_{11222} \otimes T_{-21}^2 q_2^{2H_2}) \hat{\mathcal{R}}_{11222} =$$

$$= I \otimes T_2$$

(9.31)

### 9.7 $T_2 \otimes q_2^{2H_2}$ versus $\hat{\mathcal{R}}_{11222}$

1) $U_{11222}^L \neq 0$ only in the $G(2)_q$ case so again equations with “$\approx$” can be used. Since $q^{-3\alpha_{22} - 2\alpha_{12}} = \frac{q_2}{q_2^6} \approx 1$,

1a) $q_2^{2H_2} T_{-22211} = \frac{q_2}{q_2^6} T_{-22211} q_2^{2H_2}$,

(9.32)

and due to (5.45)

1b) $T_2 T_{11222} \approx T_{11222} T_2 + (1 - \frac{1}{q_2})T_{122}^2$ (9.33)

we have

$$U_{11222}^L = - [T_2 \otimes q_2^{2H_2}, T_{11222} \otimes T_{-22211}] \approx$$

$$\approx -(1 - \frac{1}{q_2}) T_{122}^2 \otimes T_{-22211} q_2^{2H_2}$$

(9.34)
2) Again (9.32) implies

\[ 2a) : \quad T_{-2211} (q_2^2 T_{-22211}) = \frac{q_1^2}{q_2^3} T_{-22211} (T_{-22211} q_2^2), \]  

and from (5.50) follows:

\[ 2b) : \quad T_2^2 T_{1122} \approx q_2^4 T_1122 T_2^2 = q_11222 T_{11222} T_{122}^2, \]  

so that

\[ (T_2^2 \otimes T_{-22211} q_2^2) (T_{11222} \otimes T_{-22211}) = \]

\[ = q_11222 (T_{11222} \otimes T_{-22211}) (T_2^2 \otimes T_{-22211} q_2^2), \]  

i.e. indeed \( \sigma_{1122}^L = q_11222 \) and \( W_{1122} = 0 \).

Thus we get

\[ \tilde{R}_{1122}^{-1} \left( T_2 \otimes q_2^2 \right) \tilde{R}_{1122} = T_2 \otimes q_2^2 - (q_{11222} - q_{11222}^{-1}) \frac{q_2 - q_2^{-1}}{q_2} T_2^2 \otimes T_{-22211} q_2^2. \]  

Together with (9.28) and (9.31) this describes transition from (6.90) to (6.91). In order to perform the next transformation to (6.92) we need to make two substitutions.

First, \([2]_2 (q_2 - q_2^{-1}) = q_2^2 - q_2^{-2}\) is actually equal to \( (q_2^4 - q_2^2 - q_1^4) (q_1/q_2 - q_2/q_1) \) for both \( q_1 = q_2^2 \) (the case of \( B(2) \)) and \( q_1 = q_3^2 \) (the case of \( G(2) \)).

Second, in the case of \( G(2) \), when \( q_{11222} \), \( q_2 \) we can also change \( (q_{11222} - q_{11222}^{-1}) \frac{q_2 - q_2^{-1}}{q_2} \approx \frac{1}{q_2} [3]_2 (q_2 - q_2^{-1})^2 \) for \( [2]_2 [3]_2 \) \( (q_{11222} - q_{11222}^{-1})^2 \).

9.8 \( I \otimes T_2 \) versus \( \tilde{R}_{122} \)

1) Because of (5.54)

\[ U_{122}^R = \left[ I \otimes T_2, T_{122} \otimes T_{-2211} \right] = T_{122} \otimes [T_2, T_{-2211}] = \]

\[ = \frac{q_1/q_2 - q_2/q_1}{q_2 - q_2} T_{122} \otimes T_{-221} q_2^2, \]  

2) First,

\[ 2a) : \quad T_2^2 H T_{-221} = \frac{q_1}{q_2} T_{-221} T_{-221} \]  

Second, since \( T_{12} T_{122} = \frac{q_1}{q_2} T_{12} T_{12} + T_{11222} \) (see (6.40)),

\[ 2b) : \quad T_{-21} T_{-221} = \frac{q_1}{q_2} (T_{-221} - T_{-22111}) \]  

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and therefore

\[(T_{122} \otimes T_{-21}q_2^{2H_2})(T_{122} \otimes T_{-221}) = \]
\[= q_{122}^{-2} \left( (T_{122} \otimes T_{-221})(T_{122} \otimes T_{-21}q_2^{2H_2}) - [3]_2 T_{122}^2 \otimes T_{-22211}q_2^{2H_2} \right) \]

so that \(\sigma_{122}^R = q_{122}^{-2}\) and

\[W_{122}^R = [3]_2 \frac{q_1/q_2 - q_2/q_1}{q_2 - q_1} T_{122}^2 \otimes T_{-22211}q_2^{2H_2} \]

is as usual, non-vanishing only in the case of \(G(2)_q\).

Further, from (9.40)

2c) : \(q_2^{2H_2}T_{-221} = \frac{q_1}{q_2} T_{-221}q_2^{2H_2} \approx 1 T_{-221}q_2^{2H_2} \) (9.44)

and from (6.50)

\[T_{-22211} \approx 1 q_2 T_{-221} T_{-22211} \]

so that

\[(T_{122}^2 \otimes T_{-22211}q_2^{2H_2})(T_{122} \otimes T_{-221}) = \]
\[= \frac{1}{q_2^2}(T_{122} \otimes T_{-221})(T_{122}^2 \otimes T_{-22211}q_2^{2H_2}), \]

i.e. \(\rho_{122}^R \approx q_{122}^{-4}\), in accordance with (5.70). Note that \(q_{122} = q_2\) only for \(G(2)_q\), for \(B(2)_q \approx C(2)_q\) instead \(q_{122} = q_1 = q_2^2\), but for this group \(W_{122}^R = 0\).

Thus

\[\hat{R}_{122}^{-1} \left( I \otimes T_2 + (q_{122} - q_{122}^{-1}) \frac{q_1/q_2 - q_2/q_1}{q_2 - q_1} T_{122} \otimes T_{-21}q_2^{2H_2} - \right. \]
\[- \left. \frac{(q_{122} - q_{122}^{-1})^2}{q_{122}[2]_2} [2]_2 [3]_2 T_{122}^2 \otimes T_{-22211}q_2^{2H_2} \right) \hat{R}_{122} = I \otimes T_2 \]

(9.47)

### 9.9 \(T_2 \otimes q_2^{2H_2}\) versus \(\hat{R}_{122}\). The end of the proof for \(B(2)_q = C(2)_q\)

1) Using (9.40),

1a) : \(q_2^{2H_2}T_{-221} = \frac{q_1}{q_2} T_{-221}q_2^{2H_2} \) (9.48)

and (6.43),

1b) : \(T_2 T_{122} = \frac{q_4^3}{q_1} (T_{122} T_2 - [3]_2 T_{1222}) \),

(9.49)

we get:

\[U_{122}^L = -\left[ T_2 \otimes q_2^{2H_2}, T_{122} \otimes T_{-221} \right] = [3]_2 T_{1222} \otimes T_{-221}q_2^{2H_2} \]

(9.50)
2) Since $U^L_{122}$ is non-vanishing only for $G(2)_q$, we can make use of (6.49) to obtain:

\[(T_{1222} \otimes T_{-221}q_2^{2H_2})(T_{122} \otimes T_{-221}) \approx \]

\[\approx q_2^2(T_{122} \otimes T_{-221})(T_{1222} \otimes T_{-221}q_2^{2H_2})\]

with $\sigma^L_{122} = q_{122}^2 \approx q_2^2$ and $W^L_{122} = 0$. Thus

\[\hat{\mathcal{R}}_{122}^{-1}(T_2 \otimes q_2^{2H_2}) \hat{\mathcal{R}}_{122} = \]

\[= T_2 \otimes q_2^{2H_2} + [3]_2(q_{122} - q_{122}^{-1})T_{1222} \otimes T_{-221}q_2^{2H_2}\]

and, when added to (9.47), this describes the transition from (6.92) to (6.93). The next step to (6.94) is just the substitution $[3]_2(q_2 - q_2^{-1}) \approx q_{1122} - q_{1122}^{-1}$.

At this stage the proof of (6.66) is finished for $B(2)_q \cong C(2)_q$, since the second term in (9.52) - to be eliminated by the last conjugation by $\hat{\mathcal{R}}_{1222}$ - is absent for this group.

9.10 $I \otimes T_2$ versus $\hat{\mathcal{R}}_{1222}$

1) From (6.55)

\[U^R_{1222} = [I \otimes T_2, T_{1222} \otimes T_{-2221}] = T_{1222} \otimes [T_2, T_{-2221}] = \]

\[= \frac{q_1/q_2 - q_2^2/q_1}{q_2 - q_2^{-1}} T_{1222} \otimes T_{-221}q_2^{2H_2} \approx T_{1222} \otimes T_{-221}q_2^{2H_2}\]

(9.53)

2) Next,

2a): \[q_2^{2H_2}T_{-2221} = \frac{q_1}{q_2^2}T_{-2221}q_2^{2H_2} = \frac{1}{q_{1222}}T_{-2221}q_2^{2H_2}\]

(9.54)

and eq.(6.49), $T_{1222}T_{122} = q_2^3T_{122}T_{1222}$, implies

2b): \[T_{-221}T_{-2221} = \frac{1}{q_2^2}T_{-2221}T_{-2221} \approx \frac{1}{q_{1222}}T_{-2221}T_{-2221}\]

(9.55)

Thus

\[(T_{1222} \otimes T_{-221}q_2^{2H_2})(T_{122} \otimes T_{-221}) \approx \]

\[\approx \frac{1}{q_{1222}}(T_{1222} \otimes T_{-2221})(T_{1222} \otimes T_{-221}q_2^{2H_2})\]

(9.56)

i.e. $W^R_{1222} = 0$ and indeed $\sigma^R_{1222} = q_{1222}^{-2}$, so that (6.77) is applicable:

\[\hat{\mathcal{R}}_{1222}^{-1}(I \otimes T_2 + (q_{1222} - q_{1222}^{-1})T_{1222} \otimes T_{-221}q_2^{2H_2}) \hat{\mathcal{R}}_{1222} = I \otimes T_2\]

(9.57)
9.11 \( T_2 \otimes q_2^{2H_2} \) versus \( \hat{R}_{1222} \)

1) In this case

\[1a) \quad q_2^{2H_2} T_{-2221} = \frac{q_1}{q_2} T_{-2221} q_2^{2H_2} \quad (9.58)\]

and (6.44),

\[1b) \quad T_2 T_{1222} = \frac{q_2^6}{q_1} T_{1222} T_2, \quad (9.59)\]

so that

\[U_{1222}^L = - \left[ T_2 \otimes q_2^{2H_2}, T_{1222} \otimes T_{-2221} \right] = 0 \quad (9.60)\]

Thus it follows that

\[\hat{R}_{1222}^{-1} \left( T_2 \otimes q_2^{2H_2} \right) \hat{R}_{1222} = T_2 \otimes q_2^{2H_2}, \quad (9.61)\]

and together with (9.57) this describes the last transition from (6.94) to (6.95).

This ends the proof of (6.66) for all the three simple rank-2 quantum groups.
10 APPENDIX B. Group multiplication in the case of $SL(3)_q$

In section 5 of the main text we did not actually prove that the group multiplication closes after restriction to a $d_G$-parametric “manifold” in the (operator-valued) universal enveloping algebra. We are not aware of any simple proof in the general situation, but by tedious calculation one can usually perform an explicit check. In order to demonstrate the very idea of the calculation, we consider briefly the example of $SL(3)_q$. Even in this case we are not describing exhaustive proof, but rather some pieces of it, which, however, highlight the main steps of complete proof. More detailed presentation is hardly necessary, because some more straightforward kind of arguments should be found to prove the statement, which can be also considered as an adequate formulation of the Campbell-Hausdorff formula for quantum groups.

Below in this Appendix $G = SL(3)_q$.

Our logic in this Appendix is as follows. Assuming that the group property is true, one can get the expression for $\Delta^*(\chi,\phi,\psi)$ from computations in the fundamental representation, just manipulating $3 \times 3$ matrices (section 10.1). If these formulas are substituted into the l.h.s. of (5.22) and (5.25) we obtain the relations (Campbell-Hausdorff formula for $SL(3)_q$) which should be proved to hold just in the same form in any representation (i.e. for any $H_i, T_i$ with the right commutation properties and satisfying Serre identities). These relations are formulated and proved for Cartan part (5.25) in s.10.2 and just formulated (not proved) for Borel part (5.22) in s.10.3. Instead of proving relation for Borel part we consider a strongly simplified particular case (when many of $\chi, \psi$ variables are taken to be zero) in ss. 10.4-10.6. This allows to make the role and interplay of $q$-exponential properties and Serre identities transparent. Complete proof can be obtained in the same way but it will be unacceptably lengthy for such a conceptually simple statement (as existence of the group). Finally in s.10.7 we comment briefly on the Campbell-Hausdorff formula for $q$-exponentials (still to be discovered), of which (2.8) and (2.9) should be particular simple examples, and which, with the $T$-generators as arguments should provide the Campbell-Hausdorff relations for quantum groups.
10.1 The answer from fundamental representation

The final formulas for $\Delta^*(\chi, \phi, \psi)$ are most easily deduced from explicit multiplication of two matrices \([4, 7]\) in the fundamental representation:

$$
\begin{pmatrix}
q^{\Delta^*(\phi_{(1)})} & q^{\Delta^*(\phi_{(1)})}\Delta^*(\psi_{(2)}) \\
\Delta^*(\chi_{(2)})q^{\Delta^*(\phi_{(1)})} & \Delta^*(\chi_{(2)})q^{\Delta^*(\phi_{(1)})}\Delta^*(\psi_{(2)}) + q^{\Delta^*(\phi_{(2)})} \\
\Delta^*(\chi_{(3)})\Delta^*(\chi_{(2)})q^{\Delta^*(\phi_{(1)})} & \Delta^*(\chi_{(3)})\Delta^*(\chi_{(2)})q^{\Delta^*(\phi_{(1)})}\Delta^*(\psi_{(2)}) + \\
& +(\Delta^*(\chi_{(1)}) + \Delta^*(\chi_{(3)}))q^{\Delta^*(\phi_{(2)})}
\end{pmatrix}
$$

$$= 
\begin{pmatrix}
q^{\phi_{(1)}} & q^{\phi_{(1)}}\psi_{(2)} & q^{\phi_{(1)}}\psi_{(2)}\psi_{(3)} \\
x_{(2)}q^{\phi_{(1)}} & x_{(2)}q^{\phi_{(1)}}\psi_{(2)} + q^{\phi_{(2)}} & x_{(2)}q^{\phi_{(1)}}\psi_{(2)}\psi_{(3)} + q^{\phi_{(2)}}(\psi_{(1)} + \psi_{(3)}) \\
x_{(3)}x_{(2)}q^{\phi_{(1)}} & x_{(3)}x_{(2)}q^{\phi_{(1)}}\psi_{(2)} + (x_{(1)} + x_{(3)})q^{\phi_{(2)}} & x_{(3)}x_{(2)}q^{\phi_{(1)}}\psi_{(2)}\psi_{(3)} + (x_{(1)} + x_{(3)})q^{\phi_{(2)}}(\psi_{(1)} + \psi_{(3)}) + q^{\phi_{(3)}}
\end{pmatrix}.
$$

$$
\begin{pmatrix}
q^{\phi'_{(1)}} & q^{\phi'_{(1)}}\psi'_{(2)} & q^{\phi'_{(1)}}\psi'_{(2)}\psi'_{(3)} \\
x_{(2)}q^{\phi'_{(1)}} & x_{(2)}q^{\phi'_{(1)}}\psi'_{(2)} + q^{\phi'_{(2)}} & x_{(2)}q^{\phi'_{(1)}}\psi'_{(2)}\psi'_{(3)} + q^{\phi'_{(2)}}(\psi'_{(1)} + \psi'_{(3)}) \\
x_{(3)}x_{(2)}q^{\phi'_{(1)}} & x_{(3)}x_{(2)}q^{\phi'_{(1)}}\psi'_{(2)} + (x'_{(1)} + x'_{(3)})q^{\phi'_{(2)}} & x_{(3)}x_{(2)}q^{\phi'_{(1)}}\psi'_{(2)}\psi'_{(3)} + (x'_{(1)} + x'_{(3)})q^{\phi'_{(2)}}(\psi'_{(1)} + \psi'_{(3)}) + q^{\phi'_{(3)}}
\end{pmatrix}
$$
\[ q^{\Delta^*(\phi(1))} = q^{\phi(1)}(1 + \psi'_{(2)} \chi''_{(2)} + \psi'_{(2)} \psi'_{(3)} \chi''_{(3)} \chi''_{(2)}) q^{\phi(1)} = \]
\[ = q^{\phi(1)}(1 + z_{(22)} + q z_{(33)} z_{(22)}) q^{\phi(1)}, \]
\[ \Delta^*(\psi_{(2)}) = \psi''_{(2)} + q^{-\Delta^*(\phi(1))} \left( q^{\phi(1)} \psi'_{(2)} \{1 + \psi'_{(2)} \chi''_{(2)}(1 + \chi''_{(3)})\} q^{\phi(2)}\right) = \]
\[ = \psi''_{(2)} + q^{-\phi(1)} \frac{1}{1 + z_{(22)} + q z_{(33)} z_{(22)}}(1 + z_{(31)} z_{(33)}) q^{\phi''(2)}, \]
\[ \Delta^*(\psi_{(3)}) = \psi''_{(3)} + \]
\[ + \left( q^{\Delta^*(\phi(1))} \Delta^*(\psi_{(2)}) \right)^{-1} q^{\phi(1)} \left\{ \psi'_{(2)} (1 + \psi'_{(2)} \chi''_{(2)}(1 + \chi''_{(3)})) q^{\phi(2)} \psi''_{(1)} + \psi'_{(2)} \psi'_{(3)} \right\} = \]
\[ = \psi''_{(3)} + \left\{ (1 + z_{(22)} + q z_{(33)} z_{(22)}) q^{-\phi(1)} \psi''_{(2)} + \psi'_{(2)} (1 + z_{(31)} + z_{(33)}) q^{\phi''(2)} \right\}^{-1}. \]
\[ \cdot \left\{ \psi'_{(2)} (1 + z_{(31)} + z_{(33)}) q^{\phi''(2)} \psi''_{(1)} + \psi'_{(2)} \psi'_{(3)} \right\}, \]
\[ \ldots \]

Here \( z_{(s_1 s_2)} \equiv \psi'_{s_1} \chi''_{s_2}. \)

This calculation, though straightforward, is in fact not very interesting: by itself it is not enough to guarantee that \( \Delta^* \) indeed exists, i.e. is independent on particular representation. Still, if the group property is valid, \( \Delta^* \) can be evaluated in any representation, and it is under this assumption that \( (10.39) \) make sense. Our main goal is to check the assumption. For this we come back to representation-independent discussion of section 5.

### 10.2 Cartan part

We begin with the relatively simple transformation of the formula (5.25). The task is to generalize the reasoning of p.6) of section 5.1 to the case of \( SL(3)_q \). In this case each double product in (5.25) consists of 5 factors:

\[ q^{2 \Delta^*(\phi)} \hat{H} = P^{-1} q^{2 \phi} \hat{H} q^{2 \phi''} \hat{H} P, \]  

(10.2)

where

\[ P = P_{(1)} P_{(13)} P_{(21)} P_{(31)} P_{(33)} = E_q \left( \hat{z}_{(11)} + \hat{z}_{(13)} \right) E_q \left( \hat{z}_{(22)} \right) E_q \left( \hat{z}_{(31)} + \hat{z}_{(33)} \right), \]

and condensed notation are introduced:

\[ P_{(s_1 s_2)} \equiv E_q \left( \hat{z}_{(s_1 s_2)} \right), \quad \hat{z}_{(s_1 s_2)} \equiv \frac{z_{(s_1 s_2)}}{q - q^{-1}} = \frac{\psi'_{(s_1)} \chi''_{(s_2)}}{q - q^{-1}} \]

\[ \ldots \]

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and $\psi^{(s)}, \chi^{(s)}, s = 1, 2, 3$ are the same as in the section 4.2. (Note that $\mathcal{P}_{(13)}, \mathcal{P}_{(31)}, \mathcal{P}_{(22)}$ mutually commute.)

In this subsection it will be convenient to expand $\vec{H}$ in the basis of fundamental weights $\vec{\lambda}_i (i = 1 \ldots r_G), \vec{\lambda}_i \vec{a}_j = \delta_{ij}$: $\vec{H} = \frac{1}{2} \sum_{i=1}^{r_G} n_i \vec{\lambda}_i$. In finite-dimensional representations all $n_i$ are integers. As in the case of $SL(2)_q$ we perform all the calculations in this subsection for integer $n_i$’s and, then, if necessary, the result can be analytically continued to all non-integer values.

Decomposition in fundamental weights is convenient because it diagonalises the commutation relations:

$$q^2 \vec{\varphi} \vec{\lambda}_i z_{s_1s_2} = q^{\delta_{i(1)s_1}} z_{s_1s_2} q^2 \vec{\varphi} \vec{\lambda}_i, \quad q^2 \vec{\varphi} \vec{\lambda}_j z_{s_1s_2} = q^{\delta_{i(2)s_2}} z_{s_1s_2} q^2 \vec{\varphi} \vec{\lambda}_j$$

Let us remind that in our notation for $SL(3)_q$ (see section 4.2) $i(1) = i(3) = 2$, $i(2) = 1$, and $\phi_1 = \vec{\varphi} \vec{\lambda}_1, \phi_2 = -\vec{\varphi} \vec{\lambda}_2$ (in general case $\vec{\varphi} \vec{\lambda}_j = \sum_{k=1}^{r_G} \alpha^{-1}_{jk} \frac{q^k}{2} \phi_k$).

In p.6) of section 5.1 we essentially showed that

$$\mathcal{E}_{1/q}(-z_{kk})q^{2(\vec{\varphi}' + \vec{\varphi}'')} \vec{H} \mathcal{E}_{q}(z_{kk}) = q^{(\vec{\varphi}' + \vec{\varphi}'') \sum_{j \neq k} n_j \vec{\lambda}_j} \left(q^{2 \vec{\varphi} \vec{\lambda}_k} (1 + z_{kk}) q^{2 \vec{\varphi} \vec{\lambda}_k} \right)^{n_k} \quad (10.3)$$

Keeping all this in mind we can split evaluation of the r.h.s. of (10.2) into several steps. First,

$$\left( \mathcal{E}_{q}(\hat{z}_{(11)} + \hat{z}_{(13)}) \right)^{-1} q^{2(\vec{\varphi}' + \vec{\varphi}'')} \vec{H} \mathcal{E}_{q}(\hat{z}_{(11)} + \hat{z}_{(13)}) =$$

$$= q^{n_1(\vec{\varphi}' + \vec{\varphi}'') \vec{\lambda}_1} \left(q^{2 \vec{\varphi} \vec{\lambda}_2} \{1 + z_{(11)} + z_{(13)}\} q^{2 \vec{\varphi} \vec{\lambda}_2} \right)^{n_2} \quad (10.4)$$

Let us now add conjugation by $\mathcal{E}_{q}(\hat{z}_{(22)})$ (we remind that $n_1, n_2$ are assumed to be integers):

$$\left( \mathcal{E}_{q}(\hat{z}_{(22)}) \right)^{-1} q^{n_1(\vec{\varphi}' + \vec{\varphi}'') \vec{\lambda}_1} \left\{q^{2 \vec{\varphi} \vec{\lambda}_2} (1 + z_{(11)} + z_{(13)}) q^{2 \vec{\varphi} \vec{\lambda}_2} \right\}^{n_2} \mathcal{E}_{q}(\hat{z}_{(22)}) =$$

$$= \left\{ \mathcal{E}_{q}(\hat{z}_{(22)}) \right\}^{-1} q^{n_1(\vec{\varphi}' + \vec{\varphi}'') \vec{\lambda}_1} \mathcal{E}_{q}(\hat{z}_{(22)}) \} \cdot \left(q^{2 \vec{\varphi} \vec{\lambda}_2} (\mathcal{E}_{q}(\hat{z}_{(22)}) \right)^{-1} (1 + z_{(11)} + z_{(13)}) \mathcal{E}_{q}(\hat{z}_{(22)}) q^{2 \vec{\varphi} \vec{\lambda}_2} \right\}^{n_2} =$$

$$= \left\{ q^{2 \vec{\varphi} \vec{\lambda}_1} (1 + z_{(22)} \right\}^{n_1} \left\{ q^{2 \vec{\varphi} \vec{\lambda}_2} (1 + (1 + q z_{(22)}) z_{(11)} + z_{(13)}) q^{2 \vec{\varphi} \vec{\lambda}_2} \right\}^{n_2} \quad (10.5)$$

At the last stage the following transformation was made in the second bracket:

$$\left( \mathcal{E}_{q}(\hat{z}_{(22)}) \right)^{-1} (1 + z_{(11)} + z_{(13)}) \mathcal{E}_{q}(\hat{z}_{(22)}) =$$

$$= \left( \mathcal{E}_{q}(\hat{z}_{(22)}) \right)^{-1} \left( \mathcal{E}_{q}(\hat{z}_{(22)})(1 + z_{(13)}) + \mathcal{E}_{q}(q^2 \hat{z}_{(22)}) z_{(11)} \right) \quad (10.6)$$

and application of (2.6) gives $1 + (1 + q z_{(22)}) z_{(11)} + z_{(13)}$. 

50
It remains to perform the last conjugation by \( P_{(31)} P_{(33)} = \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) \):

\[
(\mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}))^{-1} (\ldots)^{n_1} (\ldots)^{n_2} \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) = \\
= \left\{ \left( \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) \right)^{-1} (\ldots) \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) \right\}^{n_1} \cdot \\
\cdot \left\{ \left( \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) \right)^{-1} (\ldots) \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) \right\}^{n_2} 
\]

\((10.7)\)

The entry of the first bracket actualy is:

\[
q^{\partial^2 \bar{\lambda}_1} \left( \mathcal{E}_q(\hat{\zeta}_{(33)}) \right)^{-1} \left( \mathcal{E}_q(\hat{\zeta}_{(31)}) \right)^{-1} (1 + z_{(22)}) \mathcal{E}_q(\hat{\zeta}_{(31)}) \mathcal{E}_q(\hat{\zeta}_{(33)}) q^{\partial^2 \bar{\lambda}_1} = \\
= q^{\partial^2 \bar{\lambda}_1} \left\{ 1 + (1 + qz_{(33)}) z_{(22)} \right\} q^{\partial^2 \bar{\lambda}_1}. 
\]

\((10.8)\)

Evaluation of that of the second bracket is a little more tedious:

\[
(\mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}))^{-1} q^{\partial^2 \bar{\lambda}_2} (\ldots) q^{\partial^2 \bar{\lambda}_2} \mathcal{E}_q(\hat{\zeta}_{(31)} + \hat{\zeta}_{(33)}) = \\
= q^{\partial^2 \bar{\lambda}_2} \left( \mathcal{E}_q(q^{-1} \hat{\zeta}_{(33)}) \right)^{-1} \left( \mathcal{E}_q(q^{-1} \hat{\zeta}_{(31)}) \right)^{-1}. \\
\cdot \left\{ 1 + (1 + qz_{(22)}) z_{(11)} + z_{(13)} \right\} \mathcal{E}_q(q \hat{\zeta}_{(31)}) \mathcal{E}_q(q \hat{\zeta}_{(33)}) q^{\partial^2 \bar{\lambda}_2} = \\
= q^{\partial^2 \bar{\lambda}_2} \left( \mathcal{E}_q(q^{-1} \hat{\zeta}_{(33)}) \right)^{-1} \left\{ (1 + z_{(31)})(1 + z_{(13)}) + (1 + qz_{(22)}) z_{(11)} \right\} \mathcal{E}_q(q \hat{\zeta}_{(33)}) q^{\partial^2 \bar{\lambda}_2} = \\
= q^{\partial^2 \bar{\lambda}_2} \left\{ 1 + z_{(31)} + z_{(13)} + z_{(33)} + qz_{(22)} z_{(11)} + (z_{(11)} + z_{(13)} z_{(31)} \frac{1}{1 + z_{(33)}} \right\} q^{\partial^2 \bar{\lambda}_2} 
\]

\((10.9)\)

Combination of (10.7), (10.8) and (10.9) finally gives:

\[
q^{2 \Delta^\ast (\partial) \bar{H}} = q^{\Delta^\ast (\partial \bar{\lambda}_1) n_1} q^{\Delta^\ast (\partial \bar{\lambda}_2) n_2} = \left\{ q^{\partial^2 \bar{\lambda}_1} \left( 1 + z_{(22)} + qz_{(33)} z_{(22)} \right) q^{\partial^2 \bar{\lambda}_1} \right\}^{n_1} \cdot \\
\cdot \left\{ q^{\partial^2 \bar{\lambda}_2} \left( 1 + z_{(31)} + z_{(13)} + z_{(33)} + qz_{(22)} z_{(11)} + (z_{(11)} + z_{(13)} z_{(31)} \frac{1}{1 + z_{(33)}} \right) q^{\partial^2 \bar{\lambda}_2} \right\}^{n_2} 
\]

\((10.10)\)

in accordance with (10.39). The two expressions in brackets at the r.h.s. are of course mutually commuting.

We now proceed to more sophisticated analysis of the Borel part.
10.3 Elimination of $\mathcal{P}$-factors

The last of the four representations (10.22) in the case of $SL(3)_q$ looks like:

$$
\Delta^*(g_U) = \mathcal{P}_{(33)}^{-1}\mathcal{P}_{(31)}^{-1}\mathcal{P}_{(22)}^{-1}\mathcal{E}_q\left(\frac{q^{-\phi''/2}}{1+\psi'(1)\chi''(1)+\psi'(1)\chi''(3)}q^{-\phi''/2}\psi'(1)T_2\right)\mathcal{P}_{(22)},
$$

$$
\cdot\mathcal{E}_q\left(\frac{q^{-\phi''/2}}{1+\psi'(2)\chi''(2)}q^{-\phi''/2}\psi'(2)T_1\right)\mathcal{P}_{(31)}\mathcal{P}_{(33)},
$$

$$
\cdot\mathcal{E}_q\left(\frac{q^{-\phi''/2}}{1+\psi'(3)\chi''(1)+\psi'(3)\chi''(3)}q^{-\phi''/2}\psi'(3)T_2\right)\mathcal{E}_q(\psi'(1)T_2)\mathcal{E}_q(\psi'(2)T_1)\mathcal{E}_q(\psi'(3)T_2)
$$

Our first goal now is to get rid of the factors $\mathcal{P}_{(22)}, \mathcal{P}_{(31)}$ and $\mathcal{P}_{(33)}$, which are still present at the r.h.s. of (10.11):

$$
\Delta^*(g_U) = \mathcal{E}_q\left(T_2\Delta^*(\psi(1))\right)\mathcal{E}_q\left(T_1\Delta^*(\psi(2))\right)\mathcal{E}_q\left(T_2\Delta^*(\psi(3))\right) = \mathcal{E}_q(\xi(1)T_2)\mathcal{E}_q(\xi(2)T_1)\mathcal{E}_q(\xi(3)T_2)\mathcal{E}_q(\psi'(1)T_2)\mathcal{E}_q(\psi'(2)T_1)\mathcal{E}_q(\psi'(3)T_2)
$$

where

$$
\xi(1) = \mathcal{P}_{(33)}^{-1}\mathcal{P}_{(31)}^{-1}\mathcal{P}_{(22)}^{-1}\left(\frac{q^{-\phi''/2}}{1+z(11)+z(13)}q^{-\phi''/2}\psi'(1)\right)\mathcal{P}_{(22)}\mathcal{P}_{(31)}\mathcal{P}_{(33)},
$$

$$
\xi(2) = \mathcal{P}_{(33)}^{-1}\mathcal{P}_{(31)}^{-1}\left(\frac{q^{-\phi''/2}}{1+z(22)}q^{-\phi''/2}\psi'(2)\right)\mathcal{P}_{(31)}\mathcal{P}_{(33)},
$$

$$
\xi(3) = q^{-\phi''/2}\frac{1}{1+z(31)+z(33)}q^{-\phi''/2}\psi'(3) = q^{-\phi''/2}\frac{1}{1+\psi'(3)\chi''(1)+\psi'(3)\chi''(3)}q^{-\phi''/2}\psi'(3)
$$

(10.13)

It is easy to evaluate $\xi(2)$ explicitly: since

$$
(\psi'(2)q^{-\phi''})z(33) = q^2z(33)(\psi'(2)q^{-\phi''}), \quad (\psi'(2)q^{-\phi''})z(31) = q^2z(31)(\psi'(2)q^{-\phi''})
$$

and

$$
z(22)z(33) = q^2z(33)z(22), \quad z(22)z(31) = z(31)z(22)
$$

we get:

$$
\left(\mathcal{E}_q(\hat{z}(33))\right)^{-1}\left(\mathcal{E}_q(\hat{z}(31))\right)^{-1}\frac{1}{1+q^{-1}z(22)}\mathcal{E}_q(q^2\hat{z}(31))\mathcal{E}_q(q^2\hat{z}(33)) = \frac{1}{(1+q^{-1}z(22))\mathcal{E}_q(\hat{z}(33))}\left(\mathcal{E}_q(\hat{z}(31))\right)^{-1}\mathcal{E}_q(q^2\hat{z}(31))\mathcal{E}_q(q^2\hat{z}(33)) = \mathcal{E}_q(\hat{z}(33)) + \frac{1}{q\mathcal{E}_q(q^2\hat{z}(33))z(22)}\left(\mathcal{E}_q(\hat{z}(31))\right)^{-1}\mathcal{E}_q(q^2\hat{z}(31))\mathcal{E}_q(q^2\hat{z}(33))
$$

(10.14)
It now remains to apply (2.6) and obtain (taking into account that \(z_{(31)}z_{(33)} = q^{-2}z_{(33)}z_{(31)}\) and \(\phi_1 = \phi_{(1)} - \phi_{(2)}\) :

\[
\xi_{(2)} = \frac{1}{1 + q^{-1}(1 + qz_{(31)})z_{(22)}} \bigg( \mathcal{E}_q(q\hat{z}_{(33)}) \bigg)^{-1} \bigg( 1 + qz_{(31)} \bigg) \mathcal{E}_q(q^2\hat{z}_{(33)}) q^{-\phi'_{(2)}} \psi'_{(2)} =
\]
\[
= \frac{1}{1 + q^{-1}(1 + qz_{(31)})z_{(22)}} (1 + qz_{(31)} + qz_{(33)}) q^{-\phi''_{(2)}} \psi'_{(2)} =
\]
\[
= q^{-\phi''_{(2)}} \frac{1}{1 + \psi'_{(2)} \chi''_{(2)} + \psi'_{(2)} \chi''_{(3)} \chi''_{(2)}} (1 + \psi''_{(3)} \chi''_{(1)} + \psi''_{(3)} \chi''_{(3)}) q^{\phi''_{(2)}} \psi'_{(2)}
\]

Similarly,

\[
\xi_{(1)} = \bigg( \mathcal{E}_q(q\hat{z}_{(33)}) \bigg)^{-1} \bigg( \mathcal{E}_q(q\hat{z}_{(31)}) \bigg)^{-1} \bigg( \mathcal{E}_q(q\hat{z}_{(22)}) \bigg)^{-1}. \frac{1}{1 + q^{-1}z_{(11)} + q^{-1}z_{(13)}} \mathcal{E}_q(q^2\hat{z}_{(22)}) \mathcal{E}_q(q^{-2}\hat{z}_{(31)}) \mathcal{E}_q(q^{-4}\hat{z}_{(33)})
\]

The same trick can be now used, with putting inverse exponentials into denominator, which is then transformed to:

\[
\left\{ 1 + q^{-1}z_{(11)} + q^{-1}z_{(13)} \right\} \mathcal{E}_q(q\hat{z}_{(22)}) \mathcal{E}_q(q\hat{z}_{(31)}) \mathcal{E}_q(q\hat{z}_{(33)}) =
\]
\[
\mathcal{E}_q(q\hat{z}_{(22)}) \mathcal{E}_q(q^{-2}\hat{z}_{(31)}) \mathcal{E}_q(q^{-4}\hat{z}_{(33)}) \cdot \left\{ (1 + q^{-1}z_{(11)})(1 + q^{-3}z_{(33)})(1 + q^{-1}z_{(13)} + q^{-1}z_{(33)}) +
\right.
\]
\[
+ q^{-1}(1 + qz_{(22)} + q^{-2}z_{(33)}z_{(22)})z_{(11)} \right\}
\]

After that in the numerator we get:

\[
\bigg( \mathcal{E}_q(q^{-4}\hat{z}_{(33)}) \bigg)^{-1} \bigg( \mathcal{E}_q(q^{-2}\hat{z}_{(31)}) \bigg)^{-1} \bigg( \mathcal{E}_q(q\hat{z}_{(22)}) \bigg)^{-1} \mathcal{E}_q(q^2\hat{z}_{(22)}) \mathcal{E}_q(q^{-4}\hat{z}_{(31)}) \mathcal{E}_q(q^{-4}\hat{z}_{(33)}) =
\]
\[
\bigg( \mathcal{E}_q(q^{-4}\hat{z}_{(33)}) \bigg)^{-1} \frac{1}{1 + q^{-3}z_{(31)}} (1 + qz_{(22)}) \mathcal{E}_q(q^{-4}\hat{z}_{(33)}) =
\]
\[
= \frac{1}{1 + q^{-3}z_{(31)} + q^{-5}z_{(33)}} \frac{1}{1 + q^{-5}z_{(33)}} (1 + qz_{(22)} + q^{-2}z_{(33)}z_{(22)})
\]

and finally

\[
\xi_{(1)} = \left\{ (1 + q^{-1}z_{(31)})(1 + q^{-3}z_{(33)})(1 + q^{-1}z_{(13)} + q^{-1}z_{(33)}) +
\right.
\]
\[
+ q^{-1}(1 + qz_{(22)} + q^{-2}z_{(33)}z_{(22)})z_{(11)} \right\} \cdot \frac{1}{1 + q^{-3}z_{(31)} + q^{-5}z_{(33)}} \frac{1}{1 + q^{-5}z_{(33)}} (1 + qz_{(22)} + q^{-2}z_{(33)}z_{(22)})
\]

As an intermediate check let us note that in the fundamental representation (10.12) turns
into

\[
\Delta^*(g_U) = \begin{pmatrix}
1 & \Delta^*(\psi_2) & \Delta^*(\psi_2)\Delta^*(\psi_3) \\
0 & 1 & \Delta^*(\psi_1) + \Delta^*(\psi_3) \\
0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & \xi_2 & \xi_2\xi_3 \\
0 & 1 & \xi_1 + \xi_3 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & \psi''_2 & \psi''_2\psi''_3 + \xi_2(\psi''_1 + \psi''_3) + \xi_2\xi_3 \\
0 & 1 & \psi''_1 + \psi''_3 + \xi_1 + \xi_3 \\
0 & 0 & 1
\end{pmatrix}
\]

and we have:

\[
\Delta^*(\psi_2) = \psi''_2 + \xi_2,
\]

\[
\Delta^*(\psi_3) = \psi''_3 + \frac{1}{\psi''_2 + \xi_2}(\psi''_1 + \xi_3),
\]

\[
\Delta^*(\psi_1) = \psi''_1 + \xi_1 + \xi_3 - \frac{1}{\psi''_2 + \xi_2}(\psi''_1 + \xi_3).
\]

Substituting our expression for \(\xi\)'s one can recognize (10.39).

Our next goal is to demonstrate that (10.12), with (10.20) substituted into the r.h.s., holds without reference to any particular representation. The only properties of the generators \(T_{1,2}\) which can be used are their commutation relations and Serre identities.

### 10.4 Toy-problem

We shall actually consider now a much simpler problem, which will help to emphasize the role of the Serre identities and their consistency property with the q-exponentials. Namely, let us put all \(\chi = 0\) and also \(\psi'_1 = \psi'_3 = 0\). (this is of course consistent with the commutation relations between \(\chi, \phi, \psi\) variables). In this case all \(z_{(s_1 s_2)} = 0\), thus \(\xi_2 = q^{-\phi''_1}\psi''_2, \xi_1 = \xi_3 = 0\)
\( \xi_3 = 0. \) Then \((10.12)\) - with \((10.20)\) substituted into it - becomes:

\[
\Delta^*(g) = E_q \left( T_2 \Delta^*(\psi(1)) \right) E_q \left( T_1 \Delta^*(\psi(2)) \right) E_q \left( T_2 \Delta^*(\psi(3)) \right) = \\
E_q \left( \frac{1}{\psi''(2)} + q^{-\phi''(2)} \psi''(1) T_2 \right) E_q \left( \left( \psi''(2) + q^{-\phi''(2)} \psi''(1) T_1 \right) \right).
\]

(10.21)

and we need to prove that this relation is indeed true.

In order to analyze this simplified problem introduce first a more reasonable notation: let

\[
u \equiv \xi(2) = \psi(2), \quad v \equiv \psi''(2),
\]

\[
x \equiv \psi''(3), \quad y \equiv \psi''(1)
\]

Then the identity to be proved is

\[
E_q \left( \frac{1}{u + v} vyT_2 \right) E_q \left( (u + v)T_1 \right) E_q \left( \left( x + \frac{1}{u + v} uy \right) T_2 \right) = \\
= E_q(uT_1)E_q(yT_2)E_q(vT_1)E_q(xT_2)
\]

(10.22)

provided

\[
uv = \frac{1}{q^2} vu, \quad xy = q^2 yx,
\]

\[
ux = qxu, \quad uy = qyu,
\]

\[
vx = q xv, \quad vy = \frac{1}{q} yv
\]

(10.23)

Addition formula \((2.8)\) can be immediately applied to the last two \(q\)-exponentials at the l.h.s. of \((10.22)\), and \(x\)-dependence is completely eliminated:

\[
E_q \left( \frac{1}{u + v} vyT_2 \right) E_q(uT_1)E_q(vT_1)E_q \left( \frac{1}{u + v} uyT_2 \right) = \\
= E_q(uT_1)E_q(yT_2)E_q(vT_1)
\]

(10.24)

10.5 On the proof of the simplified identity. Algebraic level

Let us begin from expanding \((10.24)\) in a seria in powers of \(v\). The first non-trivial term is:

\[
E_q(uT_1) \left[ vT_1, E_q(yT_2) \right] + \left[ \frac{1}{u} v y T_2, E_q(uT_1) \right] E_q(yT_2) = 0
\]

(10.25)
In order to handle this relation we need the simplest version of the Campbell-Hausdorff formula for $q$-exponentials \[16\],

$$\forall A, B, \quad \mathcal{E}_{1/q}(-B) A \mathcal{E}_q(B) = (\mathcal{E}_q(B))^{-1} A \mathcal{E}_q(B) = $$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \ldots \left[ [A, B], B \right]_q \ldots, B \right]_{q^{n-1}} = $$

$$= A + [A, B] + \frac{1}{2} \left[ [A, B], B \right]_q + \frac{1}{3!} \left[ \left[ [A, B], B \right]_q, B \right]_{q^2} + \ldots,$$

where

$$[A, B]_{q^k} \equiv \frac{1}{q^k} AB - q^k BA$$

(10.27)

(so that $[A, B] \equiv AB - BA = [A, B]_1$ and $[A, B]_q = -[B, A]_{1/q}$). Similarly,

$$\mathcal{E}_q(B) A \mathcal{E}_{1/q}(-B) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ B, \ldots \left[ B [B, A] \right]_q \ldots, B \right]_{q^{n-1}} = $$

$$= A + [B, A] + \frac{1}{2} \left[ B, [B, A] \right]_q + \frac{1}{3!} \left[ B [B, A] \right]_{q^2} + \ldots,$$

Let us now apply these results to (10.25):

$$[vT_1, \mathcal{E}_q(yT_2)] \mathcal{E}_{1/q}(-yT_2) = $$

$$= - \left( [yT_2, vT_1] + \frac{1}{2} [yT_2, [yT_2, vT_1]]_q + \ldots \right)$$

(10.29)

The first commutator at the r.h.s. is

$$[yT_2, vT_1] = yvT_2T_1 - vyT_1T_2 = -vyT_{12},$$

where $T_{12} = T_1T_2 - qT_2T_1$ is our familiar generator (5.23), associated with the non-simple positive root of the $SL(3)_q$. The next commutator is

$$\frac{1}{2} [yT_2, [yT_2, vT_1]]_q = -\frac{1}{2} [yT_2, vyT_{12}]_q = -qvy^2T_{122}$$

(10.30)

But Serre identity (5.35) for $SL(3)_q$ states that $T_{122} = \frac{1}{12} \left( T_{122}T_2 - \frac{1}{q} T_2T_{12} \right)$ is vanishing! Because of the apparent recurrent structure of (10.26) this implies, that instead of the infinite sum we have just\[10\]

$$[vT_1, \mathcal{E}_q(yT_2)] = vyT_{12} \mathcal{E}_q(yT_2)$$

(10.31)

\[10\] Of course, in the non-simply-laced case there will be more terms at the r.h.s.: with $T_{122}$ and $T_{1222}$.
Similarly, because of the second Serre identity (6.34) for \( SL(3)_q \), 
\[ T_{112} = T_1 T_{12} - \frac{1}{q} T_{12} T_1 = 0, \]
and these two equations together imply that (10.25) is indeed true.

One can of course consider higher terms of the expansion, as well as expansions in powers of \( u \) and \( y \). This is, however, not the best way to proceed.

10.6 On the proof of the simplified identity. Group level

In the case of \( q \neq 1 \) the way to reduce the proof of the commutant-type (involving permutation of exponentials, i.e. “group elements”) relation like (10.24) to that of the commutator-type like (10.26) or (10.28) (involving permutation of algebra elements or that of an algebra element with the group element) is provided by differentiation. For \( q \neq 1 \) one needs to use shift operators \( M^\pm_k f(z) = f(q^\pm k z) \) instead of derivatives.

10.6.1 Difference equations

Coming back to our identity (10.24) let us denote its l.h.s. and r.h.s. through \( F(u,v,y) \) and \( G(u,v,y) \) respectively. Then, due to (2.6),
\[
M^+_u G(u,v,y) = G(q^2 u,v,y) = \left(1 + (q^2 - 1) u T_1 \right) G(u,v,y),
\]
\[
M^+_v G(u,v,y) = G(u,q^2 v,y) = G(u,v,y) \left(1 + (q^2 - 1) v T_1 \right),
\]
\[
M^+_y G(u,v,y) = G(u,v,q^2 y) = E_q(uT_1)E_q(yT_2) \left(1 + (q^2 - 1) y T_1 \right) E_q(yT_1) = G(u,v,y) \left(1 + (q^2 - 1) (y T_2 - vy T_{12}) \right)
\]
In the last case one has to evaluate \( E_1/q(-vT_1) y T_2 E_q(vT_1) = y T_2 - vy T_{12} \) with the help of (10.26) and the Serre identity, just as we did in the previous section 10.5.

Let us note, that also
\[
M^+_y G(u,v,y) = G(u,v,q^2 y) = \left\{1 + (q^2 - 1) (y T_2 + uy T_{12}) \right\} G(u,v,y)
\]
where \( \tilde{T}_{12} \equiv T_1 T_{2} - \frac{1}{q} T_{12} T_1 \) is the alternative choice of the generator \( T_{\vec{a}_{12}} \). If expressed in

\[ 11 \] In section 6 we dealt with \( T_{12} \) only, since it is \( T_{12} \) and not \( \tilde{T}_{12} \) that appears in the simple universal formula (6.20) for the \( R \)-matrix. This does not prevent \( \tilde{T}_{12} \) from appearing in other contexts, as we shall see below in this section.
terms of $\bar{T}_{12}$ the same Serre identities for $SL(3)_q$ state that

$$\bar{T}_{12}T_2 = qT_2\bar{T}_{12}, \quad T_1\bar{T}_{12} = q\bar{T}_{12}T_1$$  \hspace{1cm} (10.34)

10.6.2 The action of $M^2_y$ on $F(u, v, y)$

In order to prove (10.24) one should now demonstrate that its l.h.s., i.e. $F(u, v, y)$ satisfies the same eqs. (10.33). Let us begin from the last equation in (10.33).

$$M^2_y F(u, v, y) = F(u, v, q^2 y) =$$

$$= \mathcal{E}_q \left( \frac{1}{u+v} vyT_2 \right) \left\{ 1 + (q^2 - 1) \frac{1}{u+v} vyT_2 \right\} \mathcal{E}_q ((u + v)T_1) \cdot$$

$$\cdot \left\{ 1 + (q^2 - 1) \frac{1}{u+v} uyT_2 \right\} \mathcal{E}_q \left( \frac{1}{u+v} uyT_2 \right)$$

((2.6) was used at this stage). Since $(vy)(u+v) = q(u+v)(vy)$ is a multiplicative commutation relation one can effectively use (10.26) and Serre identity to show that

$$\left\{ 1 + (q^2 - 1) \frac{1}{u+v} vyT_2 \right\} \mathcal{E}_q ((u + v)T_1) =$$

$$= \mathcal{E}_q ((u + v)T_1) \left\{ 1 + (q^2 - 1) \left( \frac{1}{u+v} vyT_2 - vyT_{12} \right) \right\}$$

and it remains to prove that

$$\left( 1 + (q^2 - 1) \left( \frac{1}{u+v} vyT_2 - vyT_{12} \right) \right) \left( 1 + (q^2 - 1) \frac{1}{u+v} uyT_2 \right) =$$

$$= \mathcal{E}_q \left( \frac{1}{u+v} uyT_2 \right) \left( 1 + (q^2 - 1) (yT_2 - vyT_{12}) \right) \left( \mathcal{E}_q \left( \frac{1}{u+v} uyT_2 \right) \right)^{-1}$$  \hspace{1cm} (10.37)

This time one should apply (10.28) at the r.h.s. and show:

$$(q^2 - 1) \left\{ \frac{1}{u+v} vy - \frac{1}{u+v} uy T^2_2 - vy \frac{1}{u+v} uy T_{12}T_2 \right\} =$$

$$= \left[ \frac{1}{u+v} uy, y \right] T^2_2 - \left[ \frac{1}{u+v} uyT_2, vyT_{12} \right] + \ldots$$  \hspace{1cm} (10.38)
where ". . ." stands for double and higher commutators. Since $u y$ commutes with $v y$, we have 

$$ \left( \frac{1}{u + v} u y \right) (v y) = q(v y) \left( \frac{1}{u + v} u y \right), $$

and the second commutator at the

$$ q^{\Delta^* (\phi(1))} = q^{\phi(1)} (1 + \psi'_{(2)} \chi''_{(2)} + \psi'_{(3)} \chi''_{(3)} \chi''_{(2)}) q^{\phi(1)} = $$

$$ q^{\phi(1)} (1 + z_{(22)} + q z_{(33)} z_{(22)}) q^{\phi(1)}, $$

$$ \Delta^* (\psi_{(2)}) = \psi''_{(2)} + q^{-\Delta^* (\phi(1))} \left( \psi'_{(2)} \left( 1 + \psi'_{(3)} (\chi''_{(1)} + \chi''_{(3)}) \right) \right) q^{\phi(2)} = $$

$$ \psi''_{(2)} + q^{-\phi(1)} \frac{1}{1 + z_{(22)} + q z_{(33)} z_{(22)}} (1 + z_{(31)} z_{(33)}) q^{\phi(2)}, $$

$$ \Delta^* (\psi_{(3)}) = \psi''_{(3)} + $$

$$ + q^{-\Delta^* (\phi(1))} \left( \psi''_{(2)} \psi''_{(3)} \left( 1 + \psi'_{(3)} (\chi''_{(1)} + \chi''_{(3)}) \right) \right) q^{\phi(2)} \psi'_{(1)} + \psi'_{(2)} \psi'_{(3)} q^{\phi(3)} \right)$$

$$ = \psi''_{(3)} + \left\{ (1 + z_{(22)} + q z_{(33)} z_{(22)}) q^{\phi(1)} \psi''_{(3)} + \psi'_{(2)} (1 + z_{(31)} + z_{(33)}) q^{\phi(3)} \right\}^{-1} \cdot \left\{ \psi''_{(2)} (1 + z_{(31)} + z_{(33)}) q^{\phi(2)} \psi''_{(3)} + \psi'_{(2)} \psi'_{(3)} q^{\phi(3)} \right\}, $$

$$ \ldots $$

Here $z_{(s_1 s_2)} \equiv \psi_{s_1} \chi''_{s_2}$.

r.h.s. of (10.38) is equal to $v y \frac{1}{u + v} u y (q T_1 T_{12} - T_{12} T_2)$. It remains to use Serre identity, $T_2 T_{12} = q T_{12} T_2$, in order to see that this coincides with the second term at the l.h.s. of (10.38). The first terms at both sides also coincide, as a result of identity

$$ \left[ \frac{1}{u + v} u y, y \right] = (q^2 - 1) \frac{1}{u + v} v y \frac{1}{u + v} u y, $$

(10.40)

which is most easily derived just by expansion in powers of $v$:

$$ \sum_{n \geq 0} \left\{ (1 - \frac{1}{u}) y - y (-\frac{1}{u}) y \right\} = -(q^2 - 1) \sum_{k, l \geq 0} (-\frac{1}{u})^{k+1} y (-\frac{1}{u})^{l} $$

Since $y (\frac{1}{u}) = q^2 (\frac{1}{u}) y$ the both sides here are indeed equal:

$$ \sum_{n \geq 0} (1 - q^{2n}) (-\frac{1}{u}) y = -(q^2 - 1) \sum_{n \geq 1} (-\frac{1}{u})^{n} y \sum_{l=0}^{n-1} q^{2l} $$

In the same manner one can easily show that double commutators and thus all the other terms, substituted by ". . ." in (10.38) are vanishing and this completes the check of one of the last equation (10.33) for $F(u, v, y)$. 

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In order to check the other two equations one should first bring $F(u, v, y)$ to the form which is adequate for application of shift operators. Let us note that Faddeev-Volkov identity (2.9)\footnote{Of course (10.41) can itself be proved by exactly the same method of finite-difference equations. Namely, the commutant $K(x, y) = \mathcal{E}_1(y)\mathcal{E}_q(y)\mathcal{E}_q(y)$ satisfies:}

$$\mathcal{E}_q(x + y + (1 - q^{-2})xy) = \mathcal{E}_q(x)\mathcal{E}_q(y), \quad \text{if} \quad xy = q^2yx$$  \hspace{1cm} (10.41)

with the help of (2.8) can be transformed into

$$(\mathcal{E}_q(x))^{-1} \mathcal{E}_q\left\{(1 + (1 - q^{-2})x\} \mathcal{E}_q(x) = \mathcal{E}_q(y)$$

or

$$\mathcal{E}_q(y)\mathcal{E}_q\left(x\{1 + (1 - q^{-2})y\} \right) (\mathcal{E}_q(y))^{-1} = \mathcal{E}_q(x)$$

which after the change of variables

$$\{1 + (1 - q^{-2})x\} y \to y, \quad x \to \frac{x}{1 - q^{-2}} = \frac{qx}{q - q^{-1}}$$

or

$$x\{1 + (1 - q^{-2})y\} = x\{1 + q^{-2}(q^2 - 1)y\} \to y, \quad y \to \frac{x^{-1}}{q^2 - 1} = \frac{q^{-1}x^{-1}}{q - q^{-1}}$$

In this calculation (10.26) and (10.28) are used and “…” stand for the terms which contain $[x, y]_q = 0$ (this is why condition $xy = q^2yx$ implies tremendous simplifications). Also $1 + (q^2 - 1)(x + [x, y] + [x[xy]] = \{1 + (q^2 - 1)x\} [1 + (q^2 - 1)x]$, and we get:

$$M_x^{q^2}K(x, y) = K(x, y)\{1 + (q^2 - 1)[x, y]\}$$

Similarly

$$M_y^{q^2}K(x, y) = \{1 + (q^2 - 1)[x, y]\}K(x, y)$$

Thus for $xy = q^2yx$

$$K(x, y) = \mathcal{E}_q ([x, y]) = \mathcal{E}_q ((1 - q^{-2})xy)$$

Application of addition rule (2.8) now gives the Faddeev-Volkov identity in the form (10.41).
respectively implies the two formulas (of which only one was so far used in this paper):

\[
\mathcal{E}_q \left( \frac{1}{1+x} \right) = \mathcal{E}_{1/q} \left( -\frac{qx}{q-q^{-1}} \right) \mathcal{E}_q(y) \mathcal{E}_q \left( \frac{qx}{q-q^{-1}} \right), \\
\mathcal{E}_q \left( \frac{1}{1+x^{-1}} \right) = \mathcal{E}_q \left( \frac{1}{1+q^{-2}x^{-1}} \right) = \mathcal{E}_q \left( \frac{-1x^{-1}}{q-q^{-1}} \right) \mathcal{E}_q(y) \mathcal{E}_{1/q} \left( -\frac{q^{-1}x^{-1}}{q-q^{-1}} \right) 
\]

(10.42)

Since \((v^{-1}u)y = q^2y(v^{-1}u)\) and \(u^{-1}v = (v^{-1}u)^{-1}\) we can use these two identities to get:

\[
\mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) = \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) = \mathcal{E}_q \left( \frac{q^{-1}u^{-1}v}{q-q^{-1}} \right) \mathcal{E}_q(yT_2) \mathcal{E}_{1/q} \left( -\frac{q^{-1}u^{-1}v}{q-q^{-1}} \right) 
\]

(10.43)

so that

\[
F(v, u, y) = \text{l.h.s. of (10.24)} = \\
= \mathcal{E}_{1/q} \left( -\frac{q^{-1}u}{q-q^{-1}} \right) \mathcal{E}_q(yT_2) \mathcal{E}_q(yT_2) \mathcal{E}_{1/q} \left( -\frac{q^{-1}u^{-1}v}{q-q^{-1}} \right) \mathcal{E}_q(yT_2) \mathcal{E}_{1/q} \left( -\frac{q^{-1}u^{-1}v}{q-q^{-1}} \right) 
\]

(10.44)

10.6.4 The action of \(M_u^2\) on \(F(u, v, y)\)

Now, according to (2.6), the shift operators \(M_{u,v}\) act on (10.44) by multiplying or dividing the relevant items by linear functions of the arguments, and it is easy to show that, for example,

\[
M_u^2 \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) = \mathcal{E}_q \left( \frac{1}{q^2u + vyT_2} \right) = \\
= \left( 1 + (q^2 - 1) \frac{1}{(u + v)^2} \right) \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) 
\]

(10.45)

and

\[
M_u^2 \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) = \mathcal{E}_q \left( \frac{1}{q^2u + vyT_2} \right) = \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) = \frac{1}{1 + (q^2 - 1) \frac{1}{(u + v)^2} \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right)} 
\]

or

\[
\mathcal{E}_q \left( \frac{1}{q^2u + vyT_2} \right) \left( 1 + (q^2 - 1) \frac{1}{(u + v)^2} \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) \right) \mathcal{E}_q \left( \frac{1}{u + vyT_2} \right) 
\]

(10.46)

Since

\[
M_u^2 F(u, v, y) = F(q^2u, v, y) = \\
= \mathcal{E}_q \left( \frac{1}{q^2u + vyT_2} \right) \mathcal{E}_q(q^2uT_1) \mathcal{E}_q(vT_1) \mathcal{E}_q \left( \frac{1}{q^2u + vyT_2} \right) 
\]

(10.47)
we substitute (10.45) and push the new factor to the left through $T_1$-containing exponentials (with the help of (10.26) and Serre identities):

\[
E_q(q^2 u T_1) E_q(v T_1) \left\{ 1 + (q^2 - 1)v \frac{1}{(u + v)^2} u y T_2 \right\} = \\
\left\{ 1 + (q^2 - 1) \left( u T_1 + v \frac{1}{(u + v)^2} u y T_2 + v \frac{1}{u + v} u y \bar{T}_{12} \right) \right\} E_q(u T_1) E_q(v T_1)
\]

(10.48)

Note that $\bar{T}_{12} \equiv T_1 T_1 - \frac{1}{q} T_2 T_1$ appears in this calculation instead of $T_{12} \equiv T_1 T_2 - q T_2 T_1$, and the appropriate Serre identities are (10.34). In order to obtain the desired relation $M^2_u F(u, v, y) = (1 + (q^2 - 1)u T_1) F(u, v, y)$ one still needs to show that

\[
(1 + (q^2 - 1)u T_1) E_q \left( \frac{1}{u + v} v y T_2 \right) = \\
E_q \left( \frac{1}{q^2 u + v} v y T_2 \right) \left\{ 1 + (q^2 - 1) \left( u T_1 + v \frac{1}{(u + v)^2} u y T_2 + v \frac{1}{u + v} u y \bar{T}_{12} \right) \right\}
\]

(10.49)

This is an easy corollary of (10.46), in fact (10.49) is a sum of two identities: (10.46) itself and

\[
u T_1 E_q \left( \frac{1}{u + v} v y T_2 \right) = E_q \left( \frac{1}{q^2 u + v} v y T_2 \right) \left( u T_1 + v \frac{1}{u + v} u y \bar{T}_{12} \right)
\]

(10.50)

This last identity can be proved by expanding $q$-exponential in a power series and using the fact that due to (10.34) $T_1 T_2^n = \frac{1}{q^n} T^n_2 T_1 + [n] T_2^{n-1} \bar{T}_{12}$.

This completes the proof of one more equation for $F(u, v, y)$. The last one - for the action of $M^2_v$ - can be checked just the same way.
10.7 On the Campbell-Hausdorff formula

Campbell-Hausdorff formula for the $q$-exponential functions states that $\forall A, B$

$$
\mathcal{E}_q(A)\mathcal{E}_q(B) = \mathcal{E}_q\left(A + B + \frac{1}{[2]!}[A, B]_{1/q} + \frac{1}{[3]!/[2]}\left\{[A, [A, B]_{1/q}]_q + [[A, B]_{1/q}, B]_q\right\} + \\
+ \frac{1}{[4]!/2}\left\{- [A, [A, [A, B]_{1/q}]_q]_q + [A, [A, [A, B]_{1/q}]_q]_q - \\
- [[A, [A, B]_{1/q}]_q, B]_q + [[A, B]_{1/q}, B]_q + \\
+ [[A, [A, B]_{1/q}]_q, B]_q^3 + [A, [[A, B]_{1/q}, B]_q]_{1/q}^3 - \\
- [[A, [A, B]_{1/q}, B]_q]_{1/q}^2 - [A, [[A, B]_{1/q}, B]_q]_{1/q} + \\
+ [[A, [A, B]_{1/q}, B]_q]_{1/q} + \ldots\right\}
$$

(10.51)

This is just a pure combinatorial relation, no assumption is made about the properties of “$A$” and “$B$”. □

\[13\] Identities (2.8) - for $[A, B]_{1/q} = qAB - \frac{1}{q}BA = 0$ - and (2.9) - for $[A, B]_q = \frac{1}{q}AB - qBA = 0$ - would be immediate corollaries of (10.51), if all the higher commutators in the argument of exponential at the r.h.s. were proportional to $[A, [A, B]_{1/q}]_q = [A, [A, B]_{1/q, B}]_q$ or $[[A, B]_{1/q}, B]_q = [[A, B]_{1/q}, B]_{1/q}$. However, the story is far less simple, as is clear from occurrence of the item $[[A, [A, B]_{1/q}, B]_q]_q$ in (10.51), which also vanishes in both cases, but does not contain any vanishing double commutator. Representations can be easily found which contain $q^{\pm 1}$-commutators only, but instead the coefficients in front of them are no longer $\pm 1$. In particular, the $A^2B^2$-contribution at the r.h.s. of (10.51) can be alternatively presented as \[17\]

$$
\frac{1}{[4!/2]}\left\{\frac{1}{[4!/2]^2}\left\{[[A, [A, B]_{1/q}]_q, B]_{1/q} + [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q} - [[A, [A, B]_{1/q}]_q, B]_{1/q} - [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q} + \\
+ \frac{3}{4/2}\left\{[[A, [A, B]_{1/q}]_q, B]_{1/q} + [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q} - [[A, [A, B]_{1/q}]_q, B]_{1/q} - [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q} + \\
+ \frac{q^2 - 1 + q^{-2}}{2}\left\{[[A, [A, B]_{1/q}]_q, B]_{1/q} + [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q} + \\
+ \frac{1}{2}\left\{[[A, [A, B]_{1/q}]_q, B]_{1/q} + [A, [[A, B]_{1/q}, B]_{1/q}]_{1/q}\right\}\right\}
$$

Because of occurrence of terms like $[[[, ]_1/q]_q]_{1/q}$ and $[[[, ]_1/q]_q]_{q^{\pm 1}}$, this does not make (2.8) or (2.9) more transparent. Moreover, now these identities result from cancelation between different terms. Complicated coefficients (nothing to say about their dependence on the non-$q$-numbers “2”, “4” and “$q^2 - 1 + q^{-2}$”) is another manifestation that such representations should in fact be irrelevant.
In the case of \( q = 1 \) the full expression at the r.h.s. (see, for example, \([15]\)) can be written down in a readable form with the help of operation \( \text{ad} \): \( \forall A, B \quad \text{ad}_A B \equiv [A, B] \). To make notations simpler let us also assume that \((\text{ad}_A)I \equiv A\). Then

\[
e^A e^B = \exp \left( \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} \left( \sum_{(k_l, l_i = 1, \ldots, m)} \frac{(\text{ad}_A)^{k_1}(\text{ad}_B)^{l_1} \cdots (\text{ad}_A)^{k_m}(\text{ad}_B)^{l_m} I}{k_1!l_1! \cdots k_m!l_m! \sum (k_l + l_i)} \right) \right) = \exp \left( \int_0^1 ds \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n} \left( e^{s\text{ad}_A} e^{s\text{ad}_B} - 1 \right)^{n-1} \left( A + e^{s\text{ad}_B} B \right) \right) = \exp \left( \int_0^1 ds \frac{\log (e^{s\text{ad}_A} e^{s\text{ad}_B})}{e^{s\text{ad}_A} e^{s\text{ad}_B} - 1} e^{s\text{ad}_A} (A + B) \right) = \exp \left( \int_0^1 ds \frac{\log (e^{s\text{ad}_A} e^{s\text{ad}_B})}{e^{s\text{ad}_A} e^{s\text{ad}_B} - 1} e^{s\text{ad}_A} (A + B) \right)
\]

\((10.52)\)

The crucial feature of \((10.52)\) is that exponent at the r.h.s. contains only commutators.

In application to Lie algebras, when \( A \) and \( B \) are linear functions of generators of the algebra this implies that exponents map the algebra into the group: group multiplication is again an exponent of a linear combination of generators. If one starts not from the whole set of \( d_G \) generators, but only from those associated with simple roots, then only finite number \((2(d_G - r_G))\) of new linear independent combinations (\(T_a\)'s) will be produced - and this is guaranteed by Serre identities. Thus, if Lie algebra is taken to be the original object, it is the Campbell-Hausdorff formula (supplemented by Serre identities) that ensures the existence of \( d_G\)-dimensional group manifold - a small subset of the universal enveloping algebra, which is invariant under multiplication.\(^{13}\)

One more identity can be useful for understanding of the structure of \((10.51)\):

\[
[[A, [A, B]]_q, B]_{1/q^2} = [A, [[A, B]_q, B]_{1/q}]_{1/q}
\]

It is easy to describe the appearance of the “highest commutators” in \((10.53)\). Let the argument of exponent at the r.h.s. be \( A + B + \frac{1}{[2]} (qAB - q^{-1} BA) + \sum_{k \geq 2} \gamma_k (A^k B + AB^k) + \ldots \), where \( \ldots \) denote all the other more complicated terms. Then coefficients \( \gamma_k \) are expressed through Bernulli numbers: \( \gamma(z) = 1 + \frac{1}{1!} z + \sum_{k \geq 2} \frac{\gamma_k z^k}{k} \).

\[z^k = \frac{\zeta_k(z)}{\zeta_k(1)} = \frac{1 + \frac{1}{2} z + \frac{1}{[2]} z^2 - \frac{1}{[3]} z^3 + \frac{1}{[4]} z^4 - \frac{1}{[5]} z^5 + \ldots}{1 - \frac{1}{[2]} z + \frac{1}{[3]} z^2 - \frac{1}{[4]} z^3 + \frac{1}{[5]} z^4 + \ldots} z^k + \ldots \]

Note, that while for \( q = 1 \) all the \( \gamma_{2k+1} = 0 \) for \( k \geq 1 \), this is no longer true for \( q \neq 1 \). Degrees of the polynomials in \( q, q^{-1} \) which appear in the numerators are easily controlled by consideration of asymptotics \( q^{\pm 1} \to 0 \), which is also useful for examination of other properties of \((10.51)\), which drastically simplifies in these limits.

\(^{14}\) When \( q \neq 1 \), there is no such distinguished \( \text{ad} \): of equal importance are \( q^n\)-commutators with various \( n \)'s (also they appear differently in different applications: say, in \((10.51)\) and in expression for commutants).

If extra gradation with the help of a new variable \( t \) is introduced, of real importance is the operation defined by \( t^{n+1} [A, B]_{q^n} = (A \hat{M}_t^n) (Bt^n) - (Bt^n) (A \hat{M}_t^n) t \).

\(^{15}\) In order to avoid possible confusion it deserves mentioning that in the fundamental representation of
In order to prove the existence of $d_G$-parametric “group” (with operator-valued “coordinates”) when $q \neq 1$ one needs a “quantum” Campbell-Hausdorff formula. The new thing is that while for $q = 1$ it was just enough to obtain commutators and nothing else at the r.h.s. of (10.51), for $q \neq 1$ it is important that the right $q$-commutators appear at the right places: they should be adjusted in order to match with the Serre identities. Moreover, such matching occurs only after all the $q$-factors coming from permutations of $\chi, \phi, \psi$ variables are taken into account. This Appendix, refering to $SL(3)_q$ as the simplest example, presents some fragments of the future construction, and can probably convince the reader that the entire construction can also be built and the notion of “quantum group” can indeed make sense.

\textit{GL}(N)$ the statement can look trivial, just because the set of $N^2$ generators form a full basis in the linear space of $N \times N$ matrices, and it does not seem surprising that a product of exponents of generators is again such exponent with some sophisticated mapping of parameters. However, this argument can not explain, why the same feature will be preserved in any other representation, when the entire universal envelopping algebra is no longer \textit{linearly} generated by original generators of the Lie algebra. Moreover, at the level of fundamental representation exponential functions do not look distinguished: the reasoning would work for \textit{any} other function. In fact it is the Campbell-Hausdorff formula that provides real explanations.
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