Global Dynamics for a relativistic charged and colliding plasma in presence of a massive scalar field on the Robertson-Walker spacetime

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Abstract

In this paper we consider the coupled Einstein-Maxwell-Boltzmann system with cosmological constant in presence of a massive scalar field. By combining the energy estimates method with that of characteristics we derive a local (in time) solution of the coupled system. Further under the hypotheses that the data are small in some appropriate norms and that the cosmological constant satisfies $\Lambda > - \frac{4\pi m^2 \Phi_0}{\Phi}$, we derive a unique global solution (Theorem 7.1).

Keywords: Einstein-Maxwell-Boltzmann system; massive scalar field; Sobolev spaces; local existence; global existence, continuity argument.

1 Introduction

The profound knowledge of the universe phenomena is a scientific preoccupation nowadays. A local and a global modelization of the universe are then required. The General Relativity built in 1916 by Albert Einstein is essential to understand, explain and predict the universe phenomena at the macroscopic level. In this paper, we study the global dynamics of a relativistic plasma in the Friedman-Lemaître-Robertson-Walker space time. We will only call it a Robertson -Walker space time. We study the Einstein equations, which are the basic equations of the General Relativity and which describe the gravitational forces coupled to the Maxwell equation, which describe the electromagnetic forces and to the Boltzmann equation, which is one of the basic equation of the kinetic relativistic theory, describing the dynamics of the massive and charged particles, by determining their distribution function $f$, which is a positive scalar function of the position and the momentum of the particles; we suppose that we are in presence of a massive scalar field which is the essential tool to measure the gravitational waves, which can propagate through the space at the speed of the light, even in presence of material bodies, analogously to electromagnetic waves. Recall that the Nobel Prize of physics 1993 was awarded for works on this subject.

Now our motivation for considering the Einstein equations with cosmological constant $\Lambda$, is due to the fact that astrophysical observations based on luminosity via redshift plots of some far away objects such as Supernova-Ia, have made evident the fact that the expansion of the universe is accelerating. A classical
A mathematical tool to model this phenomenon is to include the cosmological constant $\Lambda$ in the Einstein equations. Also recall that the recent Nobel Prize of physics, 2001, was awarded to three Astrophysicists for their research work on this phenomenon of accelerated expansion of the universe: For more details, see [3,9,11,13,15]. In fact, we must point out that, the notion of "dark energy" was introduced in order to provide a physical explanation to universe expansion phenomenon, but the physical structure of this hypothetical form of energy which is unknown in the laboratories remains an open question in modern cosmology; so is the question of "dark matter". Also notice that the scalar fields are considered to be a mechanism producing accelerated models, not only in "inflation", which is a variant of the Big-Bang theory including now a very short period of very high acceleration, but also in the primordial universe.

In the paper, we prove that if $\Lambda > -\beta^2$, where $\beta > 0$ is a constant depending only on the potential of the scalar field, then the coupled Einstein-Maxwell-Boltzmann system with massive scalar field and cosmological constant, has a global in time solution. The papers is organized as follows: In the second Section, we present in details the equations we are interested with; namely, the coupled Einstein-Maxwell-Boltzmann system with massive scalar field and cosmological constant. We specify the assumptions we impose on the shock kernel of the collision operator $Q$. Further the compatibility equations are given and the equation for the potential $\Phi$ of the scalar field with positive mass $m$ is derived. In Section 3, we recall a local existence Theorem for the Boltzmann equation. This gives the opportunity to introduce the functional spaces we are using and to recall some Moser-type substitution inequalities for the collision operator $Q$ which will be use later. In the fourth Section, we first introduce some new coordinates for which the Boltzmann equation has a convenient form and secondly we derive an energy estimate for the solution of an hyperbolic first order PDE: the $H^2$—norm of the unknown is estimated in terms of the $H^2$—norm of the Cauchy data and an integral of the $H^2$—norm of the source term. This will be applied to the Boltzmann equation written in the new coordinates. In Section 5, we introduce a new set of unknown functions for which the Einstein-Maxwell-Boltzmann system with massive scalar field become an equivalent system of first order differential equations; see Equations 5.12-5.17 and then the iterative scheme is set up. In Section 6, by combining the techniques of energy estimates and the method of characteristic, we derive a local existence theorem (Theorem 6.1) to the Einstein-Maxwell-Boltzmann system with massive scalar field as the limit of a suitable sequence in our functional spaces. In the last section (Section 7) global existence of solutions to the Einstein-Maxwell-Boltzmann system with massive scalar field is established under the hypotheses that the norm of the Cauchy data is small enough and the Cosmological constant $\Lambda$ is such that $\Lambda > -\beta^2$.

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2 The Equations

2.1 The detailed equations

We are in the flat Robertson-Walker spacetime $(\mathbb{R}^4,g)$, whose metric $g$ of signature $(-,+,+,+)$ can be written in the canonic coordinates $(x^\alpha)$ of $\mathbb{R}^4$ in which $x^0 = t$ is the time and $(x^i)$, $i = 1,2,3$ the space variables as:

$$g = -dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

where $a > 0$ is an unknown function called the expansion factor. We study an homogeneous case, which means that the unknown functions do not depend on the space variables $(x^i)$, $i = 1,2,3$. We adopt the Einstein summation convention $A_\alpha B^\alpha = \sum_\alpha A_\alpha B^\alpha$, the Greek indices $\alpha$, $\beta$, $\cdots$ vary from 0 to 3, and the
Latin indices $i, j, k, \cdots$ from 1 to 3. The phenomenon is governed by the following system:

\[
\begin{align*}
R_{\alpha\beta} & - \frac{1}{2}R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi (T_{\alpha\beta} + \tau_{\alpha\beta} + K_{\alpha\beta} + H_{\alpha\beta}) \\
\nabla_{\alpha} F^{\alpha \beta} & = J^{\beta} \\
\nabla_{\alpha} F_{\beta \gamma} & + \nabla_{\beta} F_{\gamma \alpha} + \nabla_{\gamma} F_{\alpha \beta} = 0 \\
\mathcal{L}_{\chi} f & = Q(f, f)
\end{align*}
\]

where:

- \( T_{\alpha\beta} \) are the Einstein equations, the basic equations of the General Relativity which describe the gravitational forces with \( \Lambda \) the cosmological constant; \( R_{\alpha\beta} \) is the Ricci tensor and \( R = g^{\alpha\beta} R_{\alpha\beta} \) is the Riemann scalar curvature. \( T_{\alpha\beta}, \tau_{\alpha\beta}, K_{\alpha\beta} \) and \( H_{\alpha\beta} \) are the components of the stress Mass-Energy tensor, source of the gravitational field \( g \), with :

\[
\begin{align*}
T_{\alpha\beta} & = \int_{\mathbb{R}^3} \frac{p_{\alpha} p_{\beta} f(t, \mathbf{p}) a^3}{p^0} dp^1 dp^2 dp^3 \\
\tau_{\alpha\beta} & = -\frac{1}{4} g_{\alpha\beta} F^{\lambda \mu} F_{\lambda \mu} + F_{\alpha \lambda} F^{\lambda \beta} \\
K_{\alpha\beta} & = -\theta_{\alpha\beta} \\
H_{\alpha\beta} & = \nabla_{\alpha} \Phi \nabla_{\beta} \Phi - \frac{1}{2} (\nabla^{\lambda} \Phi \nabla_{\lambda} \Phi + m^2 \Phi^2) g_{\alpha\beta}
\end{align*}
\]

where:

- \( T_{\alpha\beta} \) is generated by the distribution function \( f \) of the charged, massive and colliding particles, which is a positive scalar function of the time \( t = x^0 \) and the momentum \( p = (p^0, \mathbf{p}) = (p^0, p^i) \):

\[
f : T\mathbb{R}^4 \simeq \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^+, (x^0, p^0) \rightarrow f(x^0, p^0) \in \mathbb{R}^+
\]

and \( a^3 = |g|^{\frac{1}{2}} = |\det g|^{\frac{1}{2}} \). We suppose that the massive particles move on the mass hyperboloid \( (P) : g(p, p) = -1 \), and that they are ejective towards the future on the time oriented manifold \( \mathbb{R}^4 \).

From (2.11), we then deduce that :

\[
p^0 = \sqrt{1 + a^2 ((p^1)^2 + (p^2)^2 + (p^3)^2)}.
\]

Equation (2.10) shows that we can always express \( p^0 \) in terms of \( (p^i) \). \( f \) is solution to the Boltzmann equation (2.2) we present later.

- \( \tau_{\alpha\beta} \) is the Maxwell tensor associated to the electromagnetic field \( F = (F_{\alpha\beta}) = (F^{0i}, F_{ij}) \) in which \( F^{0i} \) stands for the electric part and \( F_{ij} \) for the magnetic part. \( F \) is an antisymmetric and closed 2-form, solution to the Maxwell equations (2.3) – (2.4) and describes the electromagnetic forces.

- In Equation (2.3), \( \theta_{\alpha\beta} \) is a symmetric 2-tensor called the pseudo-tensor of pressure. The general form of \( \theta_{\alpha\beta} \) is due to A. Lichnerowicz [0]. The cases of pure matter (\( \theta_{\alpha\beta} = 0 \)) and perfect fluid (\( \theta_{\alpha\beta} = p g_{\alpha\beta} \), where \( p \) is a scalar function representing the pressure) are particular cases. We make on \( \theta_{\alpha\beta} \) the assumptions :

\[
\begin{align*}
\nabla_{\alpha} \theta^{\alpha\beta} & = -\alpha^2 u^\beta \\
g^{ij} \theta_{ij} & = 0
\end{align*}
\]

where in (2.11) \( \alpha > 0 \) is a constant.
$H_{\alpha\beta}$ is the stress-energy tensor defined by the scalar field $\Phi$ with the positive mass $m > 0$ which is an unknown function of $t$.

- In (2.3) which stands for the first group of the Maxwell equations, $J^\beta$ is the Maxwell current generated by the charged particles through the formula:

$$J^\beta = \int_{\mathbb{R}^3} \frac{p^\beta f(t, \mathbf{p}) a^3}{p^0} dp^1 dp^2 dp^3 - e u^\beta$$

where $e \geq 0$ is an unknown function which stands for the elementary electric density. We consider that the particles are comoving, which means $(u^\beta) = (1, 0, 0, 0)$. In the homogeneous case we always have $\nabla_a F^{a0} = 0$, where $\nabla$ is the Levi-Civita connection of $(\mathbb{R}^4, g)$. So, equation (2.3) implies $J^0 = 0$. This determine $e$ to be:

$$e(t) = a^3(t) \int_{\mathbb{R}^3} f(t, \mathbf{p}) d\mathbf{p}.$$  

(2.14)

- Equation (2.4) is the second group of the Maxwell equations and is equivalent to $dF = 0$ since $F$ is a closed 2-form.

- (2.5) is the Boltzmann equation in $f$ we now introduce. In this equation $L_X f$ is the Lie derivative of $f$ with respect to the vector field $X = (\mathcal{P}^\alpha)$ where:

$$\mathcal{P}^\alpha = -\Gamma^\alpha_{\beta\mu} p^\beta p^\mu + e F^\alpha_\lambda p^\lambda$$

with in (2.15), $\Gamma^\alpha_{\beta\mu}$ the Christoffel symbols of the metric $g$. The trajectories of the charged particles are curves in $T\mathbb{R}^4: s \mapsto (x^\alpha(s), p^\alpha(s))$ solutions of the differential system:

$$\begin{cases}
\frac{dx^\alpha}{ds} = p^\alpha \\
\frac{dp^\alpha}{ds} = \mathcal{P}^\alpha
\end{cases}$$

(2.16)

(2.17)

where $\mathcal{P}^\alpha$ is defined by (2.15). (2.16) – (2.17) shows that $X = (p^\alpha, \mathcal{P}^\alpha)$ is tangent to the trajectories of the charged particles. Since $f = f(t, p^\alpha)$, the Boltzmann equation (2.5) writes:

$$p^0 \frac{\partial f}{\partial t} + \mathcal{P}^\alpha \frac{\partial f}{\partial p^\alpha} = Q(f, f).$$

(2.18)

We now introduce the collision operator $Q$ which appears in (2.18). $Q$ is the binary and elastic operator introduced by A. Lichnerowicz and Tchernikov in 1940, and according to which only two particles enter in collision at a point $(t, x^\alpha)$, without destructing each other, and if $(p, q)$ are their momenta before the collision and $(p', q')$ their momenta after the collision, the sums $p + q$ and $p' + q'$ are preserved. Let $f$ and $g$ be two functions in $\mathbb{R}^3$. We have:

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$

(2.19)

with:

$$\begin{cases}
Q^+(f, g) = \int_{\mathbb{R}^3} \frac{a^3(t)}{q^0} dq^0 \int_{S^2} f(q') g(q) B(a, \mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}') d\omega \\
Q^-(f, g) = \int_{\mathbb{R}^3} \frac{a^3(t)}{q^0} dq^0 \int_{S^2} f(q) g(q') B(a, \mathbf{p}, \mathbf{q}, \mathbf{p}', \mathbf{q}') d\omega
\end{cases}$$

(2.20)

(2.21)

We now present the different elements of formulae (2.20) and (2.21) point by point, specifying the assumptions we adopt.
2.2 Compatibility of the equations

a) It is clear that, if \( \mathbf{p} \mapsto f(t, \mathbf{p}) \) is invariant by \( SO_3 \), then the integral in (2.13) is zero if \( \beta = i \). In [9], N. Noutchegueme et al. proved that if \( \mathbf{p} \mapsto f(0, \mathbf{p}) \) is invariant by \( SO_3 \), then \( \mathbf{p} \mapsto f(t, \mathbf{p}) \) is invariant by \( SO_3 \). We adopt the assumption that \( \mathbf{p} \mapsto f(0, \mathbf{p}) \) is invariant by \( SO_3 \), then by (2.3) one has

\[ \nabla_a F^{\alpha i} = 0, \ i = 1, 2, 3. \]
Now $\mathbb{R}^4$ being simply connected, there exists a potential vector $A = (A_\mu)$ such that

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$

then $F_{ij} = \partial_i A_j - \partial_j A_i = 0$ since $A = A(t)$, from where we obtain $F^{ij} = F_{ij} = 0$. This means that the electromagnetic field reduces to its electric part and thus, (2.31) writes:

$$\partial_0 F^{0i} + \Gamma^j_{\beta0} F^{0j} = 0.$$  (2.31)

We deduce from (2.1) that:

$$g^{00} = g_{00} = -1; \ g_{ii} = a^2; \ g^{ii} = a^{-2}; \ g_{0i} = g^{0i} = 0; \ g_{ij} = g^{ij} = 0 \ for \ i \neq j.$$  (2.32)

The usual formula $\Gamma^\lambda_{\alpha\beta} = \frac{1}{2} g^{\lambda\mu} (\partial_\alpha g_{\mu\beta} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta})$ gives, with $\dot{a} = \frac{da}{dt}$,

$$\Gamma^0_{ii} = \dot{a} a; \ \Gamma^i_{0i} = \dot{\Gamma}^i_{0i} = \frac{\dot{a}}{a}; \ \Gamma^0_{00} = 0; \ \Gamma^0_{\alpha\beta} = 0 \ for \ \alpha \neq \beta; \ \Gamma^k_{ij} = 0;$$  (2.33)

and (2.31) is then equivalent to:

$$\partial_0 F^{0i} + 3 \frac{\dot{a}}{a} F^{0i} = 0.$$  (2.34)

The general solution of (2.34) is, with $a_0 = a(0)$:

$$F^{0i}(t) = \left( \frac{a_0}{a} \right)^3 F^{0i}(0).$$  (2.35)

Since $F^{ij} = F_{ij} = 0$, we have directly:

$$F^{\lambda\mu} F_{\lambda\mu} = -2 g_{ij} F^{0i} F^{0j}; \ F_{i\lambda} F^\lambda_j = -2 g_{ik} g_{jl} F^{0k} F^{0l}; \ F_{0\lambda} F^\lambda_0 = 0; \ F_{0\lambda} F^{0\lambda} = g_{ij} F^{0i} F^{0j}$$  (2.36)

and definition (2.7) of $\tau_{\alpha\beta}$ gives:

$$\tau_{00} = \frac{1}{2} g_{ij} F^{0i} F^{0j}; \ \tau_{0j} = 0; \ \tau_{ij} = \left( \frac{1}{2} g_{ij} g_{kl} - g_{ik} g_{jl} \right) F^{0k} F^{0l}.$$  (2.37)

By definition (2.9) of $H_{\alpha\beta}$, we have:

$$H_{00} = \frac{1}{2} (\dot{\Phi}^2 + m^2 \Phi^2); \ H_{0i} = 0; \ H_{ij} = \frac{1}{2} g_{ij} (\ddot{\Phi}^2 - m^2 \Phi^2).$$  (2.38)

b) For the Einstein equations, when $\alpha = 0$ and $\beta = i$, we have: $R_{0i} - \frac{1}{2} R g_{0i} + A g_{0i} = 0$ at the l.h.s. and we must then have at the r.h.s:

$$T_{0i} + \tau_{0i} + K_{0i} + H_{0i} = 0.$$  (2.39)

Since $\overline{\mu} \rightarrow f(t, \overline{\mu})$ is invariant by $SO_3$, we then have for $T_{0\beta}$ given by (2.6), $T_{0i} = 0$. Equations (2.37) and (2.38) give $\tau_{0i} = H_{0i} = 0$ and by definition (2.8) of $K_{\alpha\beta}$ we will have (2.39) if we take $\theta_{0i} = 0$.

c) For $\alpha = i$, $\beta = j$ where $i \neq j$, we have $R_{ij} - \frac{1}{2} R g_{ij} + A g_{ij} = 0$ at the l.h.s. We must then have at the r.h.s:

$$T_{ij} + \tau_{ij} + K_{ij} + H_{ij} = 0, \ i \neq j.$$  (2.40)

Equations (2.38) gives for $i \neq j$: $H_{ij} = 0$; since $\overline{\mu} \rightarrow f(t, \overline{\mu})$ is invariant by $SO_3$, the definition (2.6) of $T_{\alpha\beta}$ gives $T_{ij} = 0$ if $i \neq j$. Expression (2.37) of $\tau_{\alpha\beta}$ gives $\tau_{ij} = 0$. We will then have (2.40) if we take $K_{ij} = -\theta_{ij} = 0, i \neq j$.  

6
d) Consider the case \( \alpha = \beta = i \in \{1, 2, 3\} \): In the Robertson-Walker space time, we have: \( g_{11} = g_{22} = g_{33} = a^2 \), which implies: \( R_{11} = R_{22} = R_{33} \). So the three corresponding Einstein equations have the same l.h.s which is, after calculations:

\[
A = -2\ddot{a} - \dot{a}^2 + \Lambda a^2 .
\] (2.41)

The r.h.s of these equations must be the same. This means that, we must have:

\[
\begin{aligned}
T_{11} + \tau_{11} + H_{11} &= T_{22} + \tau_{22} + K_{22} + H_{22} \\
T_{22} + \tau_{22} + K_{22} + H_{22} &= T_{33} + \tau_{33} + K_{33} + H_{33}
\end{aligned}
\] (2.42)

(2.43)

Since \( p \mapsto f(t, p) \) is invariant by \( SO_3 \) we have: \( T_{11} = T_{22} = T_{33} \). Now if in the solution (2.35) of the equation (2.34) we take:

\[
F^{01}(0) = F^{02}(0) = F^{03}(0)
\] (2.44)

we will then have:

\[
F^{01} = F^{02} = F^{03} .
\] (2.45)

We make the hypothesis (2.12). We then have (2.13) and expression (2.37) of \( \tau_{\alpha\beta} \) implies:

\[
\tau_{11} = \tau_{22} = \tau_{33} .
\] (2.46)

Since \( g_{ii} = a^2 \), expression (2.35) of \( H_{\alpha\beta} \) implies that:

\[
H_{11} = H_{22} = H_{33} .
\] (2.47)

The relations (2.42) - (2.43) then imply, given (2.47) and since \( T_{11} = T_{22} = T_{33} \), and \( K_{\alpha\beta} = -\theta_{\alpha\beta} \):

\[
\theta_{11} = \theta_{22} = \theta_{33} .
\] (2.48)

We use the hypothesis (2.12) which is \( g^{ij} \theta_{ij} = 0 \) to have:

\[
\theta_{11} + \theta_{22} + \theta_{33} = 0
\] (2.49)

and Equations (2.48) and (2.49) imply:

\[
\theta_{11} = \theta_{22} = \theta_{33} = 0 .
\] (2.50)

Hence the Einstein evolution equations reduce to the single equation:

\[
R_{11} - \frac{1}{2}Rg_{11} + \Lambda g_{11} = 8\pi(T_{11} + \tau_{11} + K_{11} + H_{11}) .
\] (2.51)

e) For \( \alpha = \beta = 0 \), we have the Hamiltonian constraint, which writes, after calculations (see for example [11]):

\[
3\left( \frac{\dot{a}}{a} \right)^2 - \Lambda = 8\pi \left[ T_{00} + \tau_{00} + K_{00} + H_{00} \right] .
\] (2.52)

We know that (2.32) will be solved everywhere if and only if it is solved for \( t = 0 \). The only unknown component of \( \theta_{\alpha\beta} \) is \( \theta_{00} = \theta^{00} \) which satisfies, given (2.11) and \( u^0 = 1 \):

\[
\dot{\theta}^{00} + 3\frac{\dot{a}}{a}\theta^{00} = -\alpha^2 ;
\] (2.53)

whose general solution is:

\[
\theta^{00} = \left( \frac{a_0}{a} \right)^3 [\theta^{00}(0) - \alpha^2 \left( \frac{a}{a_0} \right)^3].
\] (2.54)

By (2.53):

\[
(\theta^{00}(0) \leq 0) \Rightarrow (\theta^{00} \leq 0).
\] (2.55)

So we look for a solution \( \theta^{00} \) of (2.43) such that:

\[
\theta^{00} \leq 0 .
\] (2.56)
2.3 Conditions of conservation. Equation for $\Phi$

We always have the identities: $\nabla_\alpha (R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} + \Lambda g^{\alpha\beta}) = 0$. We must then have at the r.h.s of equations (2.3):

$$\nabla_\alpha T^{\alpha\beta} + \nabla_\alpha r^{\alpha\beta} + \nabla_\alpha K^{\alpha\beta} + \nabla_\alpha H^{\alpha\beta} = 0.$$  

(2.57)

But by (1), when $f$ is solution to the Boltzmann equation, we have: $\nabla_\alpha T^{\alpha\beta} = 0$. Then (2.57) reduces to:

$$\nabla_\alpha r^{\alpha\beta} + \nabla_\alpha K^{\alpha\beta} + \nabla_\alpha H^{\alpha\beta} = 0.$$  

(2.58)

From expressions of $r^{\alpha\beta}$ and $H^{\alpha\beta}$ given by (2.7) and (2.10) it is easy to see that

$$\begin{cases}
\nabla_\alpha r^{\alpha\beta} = F^\alpha_\lambda \nabla_\alpha F^{\alpha\lambda} \\
\nabla_\alpha H^{\alpha\beta} = \nabla^\alpha \Phi (\Box g \Phi - m^2 \Phi)
\end{cases}$$  

(2.59)

(2.60)

where $\Box$ is the d’Ambertian operator $\Box g = \nabla_\alpha \nabla^\alpha$ of $g$. But we know that $\nabla_\alpha F^{\alpha\alpha} = 0$ thus, by (2.13) and since $u^i = 0$ we have

$$\nabla_\alpha r^{\alpha\beta} = F^\alpha_\lambda \nabla_\alpha F^{\alpha\lambda} = F^\alpha_\lambda F^\lambda_i = 0.$$  

(2.61)

We now have, given (2.3) and (2.11): $\nabla_\alpha K^{\alpha\beta} = -\nabla_\alpha \theta^{\alpha\beta} = -\alpha^2 u^\beta$. So we deduce from (2.58), using (2.59) and (2.61) that:

$$-\alpha^2 u^\beta + \nabla^\beta \Phi (\Box g \Phi - m^2 \Phi) = 0.$$  

(2.62)

But for $\beta = i$, we have $\nabla^i \Phi = g^{i\lambda} \nabla_\lambda \Phi = g^{i\lambda} \partial_\lambda \Phi = 0$ since $\Phi = \Phi(t)$ and $u^i = 0$. So from (2.61) we have for $\beta = 0$:

$$\nabla^0 \Phi (\Box g \Phi - m^2 \Phi) = \alpha^2.$$  

(2.63)

Further, $\nabla^0 \Phi = g^{0\lambda} \partial_\lambda \Phi = g^{0\lambda} \partial_\lambda \Phi = -\dot{\Phi}$ (since $g^{00} = -1$); and a development of $\Box g \Phi$ gives

$$\Box g \Phi - m^2 \Phi = -\left[\dot{\Phi} + 3\left(\frac{\dot{a}}{a}\right) \Phi + m^2 \Phi\right].$$

From (2.62) we then have the equation for $\Phi$:

$$\dot{\Phi} \left[\dot{\Phi} + 3\left(\frac{\dot{a}}{a}\right) \Phi + m^2 \Phi\right] = \alpha^2.$$  

(2.63)

2.4 The reduced system

Considering (2.2), (2.3), (2.58) and (2.13) divided by $p^0$, we have the following system, using $p_1 = a^2 p^1$:

$$\begin{align*}
3\left(\frac{\dot{a}}{a}\right)^2 - \Lambda &= 8\pi \left[ a^3 \int_{\mathbb{R}^3} p^0 f(p) \hat{m} p + \frac{3}{2} a^2 (F^{01})^2 - \theta_{00} + \frac{a^2}{2} (\Phi^2 + m^2 \Phi^2) \right] \\
-2\bar{a} a - (\dot{a}) a + \Lambda a^2 &= 8\pi \left[ a^7 \int_{\mathbb{R}^3} \left( \frac{1}{p^0} \right)^2 p^0 f(p) \hat{m} p + \frac{a^4}{2} (F^{01})^2 + \frac{a^2}{2} (\Phi^2 + m^2 \Phi^2) \right] \\
F^{01} + 3\frac{\dot{a}}{a} F^{01} &= 0 \\
\theta^{00} + 3\frac{\dot{a}}{a} \theta^{00} &= -\alpha^2 \\
\dot{\Phi} \left[\dot{\Phi} + 3\left(\frac{\dot{a}}{a}\right) \Phi + m^2 \Phi\right] &= \alpha^2 \\
\frac{\partial f}{\partial t} - 2\frac{\dot{a}}{a} p_i \frac{\partial f}{\partial p^i} - \left( a^3 F^{01} \int_{\mathbb{R}^3} f(t, \pi) d\pi \right) \sum_{i=1}^3 \frac{\partial f}{\partial p^i} &= \frac{1}{p^0} \mathcal{Q}(f, f)
\end{align*}$$  

(2.64)  

(2.65)  

(2.66)  

(2.67)  

(2.68)  

(2.69)
3 Local solution for the Boltzmann Equation

In this section we recall a local existence result obtained by NN and KNM in \[9\] for the Boltzmann equation. The functional spaces we will use are the same as those of that reference and they are presented here. We will also recall the most important substitution type inequalities concerning the collision operator \(Q\) the proof of which is given in details in \[7\]. We begin with the functional spaces.

Definition 3.1 Let \(m \in \mathbb{N}, d \in \mathbb{R}^+, T > 0, |\mathcal{P}| = \left(\sum_{i=1}^{3}(p^i)^2\right)^{1/2}\) and define the spaces:

1. \(L_1^d(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}, (1 + |\mathcal{P}|) f \in L^1(\mathbb{R}^3) \}\)
2. \(L_d^2(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}, (1 + |\mathcal{P}|)^d f \in L^2(\mathbb{R}^3) \}\).
3. \(H^m_d(\mathbb{R}^3) = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R}, (1 + |\mathcal{P}|)^{|\beta|} \partial^\beta f \in L^2(\mathbb{R}^3), |\beta| \leq m \}.\) \(H^m_d(\mathbb{R}^3)\) is a separable Hilbert space with the norm :
   \[\|f\|_{H^m_d(\mathbb{R}^3)} = \max_{0 \leq |\beta| \leq 3} \|(1 + |\mathcal{P}|)^{|\beta|} \partial^\beta f\|_{L^2(\mathbb{R}^3)}\].
4. \(H^m_d(0,T,\mathbb{R}^3) = \{ f : [0,T] \times \mathbb{R}^3 \rightarrow \mathbb{R}, f\) continuous, \(f(t,\cdot) \in H^m_d(\mathbb{R}^3), \forall t \in [0,T]\}\). Endowed with the norm :
   \[\|f\|_{H^m_d(0,T,\mathbb{R}^3)} = \sup_{t \in [0,T]} \max_{0 \leq |\beta| \leq 3} \|(1 + |\mathcal{P}|)^{|\beta|} \partial^\beta f(t,\cdot)\|_{L^2(\mathbb{R}^3)}\].

\(H^m_d(0,T,\mathbb{R}^3)\) is a Banach space.

For a fixed \(r > 0\), we set :

\[H^m_{d,r}(0,T,\mathbb{R}^3) = \{ f \in H^m_d(0,T,\mathbb{R}^3), \|f\|_{H^m_d(0,T,\mathbb{R}^3)} \leq r \}.\] (3.1)

Endowed with the norm induced by \(H^m_d(0,T,\mathbb{R}^3)\), \(H^m_{d,r}(0,T,\mathbb{R}^3)\) is a complete metric space.

Remark 3.1 We have the embeddings

\[H^m_d(\mathbb{R}^3) \hookrightarrow L_d^2(\mathbb{R}^3) \hookrightarrow L_1^1(\mathbb{R}^3) \hookrightarrow L_1^1(\mathbb{R}^3); \quad m \in \mathbb{N}, d > 5/2.\] (3.2)

In \[7\] it has been proven that the functions \(\mathcal{P}, \mathcal{Q}, \omega) \rightarrow \partial^\beta b(\mathcal{P}, \mathcal{Q}, \omega), 1 \leq |\beta| \leq 3\), are bounded and therefore important results for the collision operator \(Q\) defined by (2.19), (2.20), and (2.21) are established. We recall these fundamental inequalities.

Proposition 3.1 Let the collision kernel \(B\) satisfies hypotheses \((H_1)\) i.e \((2.2)\), and \(f, g \in H^m_d(0,T,\mathbb{R}^3)\), with \(d > \frac{5}{2}\). Then \(\frac{1}{p^\alpha}Q(f,g)\) is also in \(H^m_d(0,T,\mathbb{R}^3)\) and \((f,g) \mapsto \frac{1}{p^\alpha}Q(f,g)\) is uniformly continuous from \(H^m_d(\mathbb{R}^3) \times H^m_d(\mathbb{R}^3)\) to \(H^m_d(\mathbb{R}^3)\). Namely, the following holds:

\[\left\| \frac{1}{p^\alpha}Q(f,g) \right\|_{H^m_d(\mathbb{R}^3)} \leq C(T)\|f\|_{H^m_d(\mathbb{R}^3)}\|g\|_{H^m_d(\mathbb{R}^3)}, \forall f, g \in H^m_d(\mathbb{R}^3);\] (3.3)

\[\left\| \frac{1}{p^\alpha}Q(f,f) - \frac{1}{p^\alpha}Q(g,g) \right\|_{H^m_d(\mathbb{R}^3)} \leq C(T)(\|f\|_{H^m_d(\mathbb{R}^3)} + \|g\|_{H^m_d(\mathbb{R}^3)})\|f - g\|_{H^m_d(\mathbb{R}^3)}, \forall f, g \in H^m_d(\mathbb{R}^3);\] (3.4)

where \(C(T)\) is a positive constant which only depends on \(T\).

In \[7\], NN and KNM proved, using the Faedo-Galerkin method that the Boltzmann equation has a solution in \(H^3(\mathbb{R}^3)\). This is particularly important since such a solution is of class \(C^1\). We call it a regular solution. We recall here the precise statement of this result.

Proposition 3.2 Let \(f_0 \in H^3_d(\mathbb{R}^3)\). Then there exists \(T > 0\) such that the Boltzmann equation (2.69) has a unique solution \(f\) in \(H^3_d(0,T,\mathbb{R}^3)\) such that \(f(0,\mathcal{P}) = f_0(\mathcal{P})\).

Proof: See \[7\].
4 Energy estimate for the Boltzmann Equation

4.1 Change of coordinates in the Boltzmann equation

We have, ∀k = 1, 2, 3, \( p_k = g_{k,3} \rho \beta = a^2 p^k \). As in [4] let us set:

\[
\begin{align*}
\vec{u} &= (u^1, u^2, u^3) \quad \text{where} \quad u^k = a^2 p^k, \quad u^0 = \sqrt{1 + a^2 |\vec{u}|^2} = p^0 \\
\overrightarrow{v} &= (v^1, v^2, v^3) \quad \text{where} \quad v^k = a^2 q^k, \quad v^0 = \sqrt{1 + a^2 |\overrightarrow{v}|^2} = q^0
\end{align*}
\]  

(4.1)

In the change of variables (2.29), we had the function \( b \) defined by (2.27), with the new scalar product defined by (2.28). With the usual scalar product (\( \cdot \)) it writes:

\[
b(\vec{p}, \vec{q}, \omega) = \frac{2a^2 p^0 q^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p})}{(\hat{e})^2 - a^4 |\omega \cdot (\vec{p} + \vec{q})|^2}.
\]  

(4.2)

The change of variables \( \begin{cases} \vec{p}' = \vec{p} + b(\vec{p}, \vec{q}, \omega) \\
\vec{q}' = \vec{q} - b(\vec{p}, \vec{q}, \omega) \end{cases} \) then writes in terms of \( \vec{u} \), and \( \overrightarrow{v} \):

\[
\begin{align*}
p'^k &= p^k + 2a^2 p^0 q^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) u^k \overline{u} a^0 + 2a^0 v^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) u^k v^0 \overline{v} a^0 = \frac{1}{a^2} (u^k + 2a^2 p^0 q^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) u^k) \\
q'^k &= q^k - 2a^2 p^0 q^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) v^k \overline{u} a^0 - 2a^0 v^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) v^k v^0 \overline{v} a^0 = \frac{1}{a^2} (v^k - 2a^2 p^0 q^0 \hat{e}_\omega \cdot (\vec{q} - \vec{p}) v^k)
\end{align*}
\]

Therefore, if we set:

\[
\begin{align*}
u^k &= a^2 p^k, \quad v^k &= a^2 q^k, \quad u^0 = p^0, \quad v^0 = q^0
\end{align*}
\]  

(4.3)

we obtain:

\[
\begin{align*}
\vec{u}' &= \vec{u} + \hat{b}(\overrightarrow{u}, \overrightarrow{v}, \omega) \\
\overrightarrow{v}' &= \overrightarrow{v} - \hat{b}(\overrightarrow{u}, \overrightarrow{v}, \omega)
\end{align*}
\]  

(4.4) and (4.5)

with:

\[
\hat{b}(\overrightarrow{u}, \overrightarrow{v}, \omega) = \frac{2a^2 u^0 v^0 \hat{e}_\omega \cdot (\overrightarrow{v} - \overrightarrow{u})}{(\hat{e})^2 - |\omega \cdot (\overrightarrow{u} + \overrightarrow{v})|^2}.
\]  

(4.6)

We now write the Boltzmann equation using the variables \( s, u \) and \( v \). Consider the change of variables:

\[
(t, \overrightarrow{u}, \overrightarrow{v}) \longrightarrow (s, \overrightarrow{u}, \overrightarrow{v}) \text{ with } \begin{cases} s = t \\
\overrightarrow{u} = a^2 \overrightarrow{v} \end{cases} \quad \text{with} \quad ds = a^{-6} dt \]

we have:

\[
\begin{align*}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial s} + 2a^0 \frac{\partial f}{\partial u^0} - 2a^2 \frac{a}{a^2} \frac{\partial f}{\partial u^0} + \left( a^3 F_0 \int_{R^3} f(s, \overrightarrow{u}) a^{-6} dv \right) a^2 \sum_{i=1}^3 \frac{\partial f}{\partial u^i} \\
&= \frac{\partial f}{\partial t} - \left( F_0 \int_{R^3} f(t, \overrightarrow{v}) d\overrightarrow{v} \right) \sum_{i=1}^3 \frac{\partial f}{\partial u^i}.
\end{align*}
\]
with, given (2.19), (2.20) and (2.21) \( \tilde{Q} = \tilde{Q}^+ - \tilde{Q}^- \) where:

\[
\begin{align*}
\tilde{Q}^+(f,g)(t,\overline{u}) &= \int_{\mathbb{R}^3} \frac{a^{-3}(t)}{\nu^0} \int_{S^2} f(t,\overline{\nu}) g(t,\overline{\nu}) \tilde{B}(a(t,\overline{\nu},\overline{\nu}',\nu),\omega) d\omega \\
\tilde{Q}^-(f,g)(t,\overline{u}) &= \int_{\mathbb{R}^3} \frac{a^{-3}(t)}{\nu^0} \int_{S^2} f(t,\overline{\nu}) g(t,\overline{\nu}) \tilde{B}(a(t,\overline{\nu},\overline{\nu}',\nu),\omega) d\omega
\end{align*}
\] (4.7)

and where \( \tilde{B} \) is defined as \( B \), in terms of the new variables \( \overline{\nu}, \overline{\nu}', \overline{\nu} \). Thus the Boltzmann equation is equivalent to:

\[
\frac{\partial f}{\partial t} = \left( \frac{F^01}{a} \int_{\mathbb{R}^3} f(t,\overline{\nu}) d\overline{\nu} \right) \sum_{i=1}^{3} \frac{\partial f}{\partial u^i} = \frac{1}{u^0} \tilde{Q}(f,f).
\] (4.8)

**Proposition 4.1** The properties established for \( b \) hold for \( \tilde{b} \) namely, the functions \( (\overline{\nu},\overline{\nu},\nu) \mapsto \tilde{\partial}_{u}^{3} \tilde{b}(\overline{\nu},\overline{\nu},\nu), \ 1 \leq |\beta| \leq 3 \), are bounded.

**Proof:** The proof is exactly the same as for \( b \) see [7] page 72. \( \square \)

**Remark 4.1** An important consequence of Proposition 4.1 is that if we make for \( \tilde{B} \) the same assumptions as for \( B \), see (2.22) we will have the following results analogous to (3.3) and (3.4): If \( f,g \in H^3_d(0,T;\mathbb{R}^3) \), then \( \frac{1}{u^0} \tilde{Q}(f,g) \in H^3_d(0,T;\mathbb{R}^3) \) and the following hold

\[
\left\| \frac{1}{u^0} \tilde{Q}(f,g) \right\|_{H^3_d(\mathbb{R}^3)} \leq C(T) \| f \|_{H^3_d(\mathbb{R}^3)} \| g \|_{H^3_d(\mathbb{R}^3)} ;
\] (4.9)

\[
\left\| \frac{1}{u^0} \tilde{Q}(f,f) - \frac{1}{u^0} \tilde{Q}(g,g) \right\|_{H^3_d(\mathbb{R}^3)} \leq C(T) \left( \| f \|_{H^3_d(\mathbb{R}^3)} + \| g \|_{H^3_d(\mathbb{R}^3)} \right) \| f - g \|_{H^3_d(\mathbb{R}^3)} .
\] (4.10)

### 4.2 Energy estimate for a first order hyperbolic partial differential equation

Let us consider the first order PDE in \( u = u(t,x) \):

\[
u_t + \sum_{i=1}^{n} a_i(x,t) u_{x_i} + b(x,t) u = f(x,t) \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R} ;
\] (4.11)

with initial data:

\[
u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^n .
\] (4.12)

Here \( b \) and \( f \) are functions defined on \( \mathbb{R}^{n+1} \) and \( a = (a_1, \cdots, a_n) \) a family of functions such that:

\[
\sum_{i=1}^{n} \left( \sup_{(x,t)} |a_i(x,t)| \right) =: |a| \leq \frac{1}{\kappa}.
\] (4.13)

where \( \kappa \) is a positive constant.

**Remark 4.2** We would like the point out the fact that the notations in this section are independent of those of all the other sections of the paper. For examples, the latter a here is used for a collection of real values function and has nothing to do with the expansion factor of the previous section, \( f \) here is the source term of the PDE we are dealing with and must not be confused with the distribution function. We hope that this clash of notations will not confuse the reader.
Proposition 4.2 Let \( a = (a_1, \cdots, a_n) \) be a family of class \( C^1 \) functions with bounded partial derivatives with respect to \( x_i \) defined in \( \mathbb{R}^{n+1} \) and satisfying (4.13). Let \( b \) be a bounded function defined in \( \mathbb{R}^{n+1} \), and \( u \) a solution of the initial value problem (4.11) - (4.12). Then, for every \( T > 0 \), if \( f \in C([0,T];L^2(\mathbb{R}^n)) \) and \( u_0 \in L^2(\mathbb{R}^n) \), we have:

\[
\int_{\mathbb{R}^n} e^{-at} u^2 dx \leq \int_{\mathbb{R}^n} u_0^2 dx + \int_0^t e^{-\alpha s} \|f(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 ds, \quad t \leq T; \tag{4.14}
\]

where \( \alpha \) is a positive constant.

**Proof:** Let \( \overline{t} > 0 \) be given. Define \( D = D_{\kappa, \overline{t}, \overline{t}} = \{ (x, s) \in \mathbb{R}^{n+1}; \kappa|x| < \overline{t} - s, \ 0 < s < t \} \).

Let us denote by \( \Sigma_{\kappa, \overline{t}}, \Sigma_{0, \overline{t}} \) and \( S_{\kappa, \overline{t}} \) respectively, the upper boundary, the lower boundary and the side of \( D \); i.e:

\[
\begin{align*}
\Sigma_{\kappa, \overline{t}} &= \{ (x, t) \in \mathbb{R}^{n+1}; \kappa|x| < \overline{t} - t \} \\
\Sigma_{0, \overline{t}} &= \{ (x, 0) \in \mathbb{R}^{n+1}; \kappa|x| < \overline{t} \} \\
S_{\kappa, \overline{t}} &= \{ (x, s) \in \mathbb{R}^{n+1}; \kappa|x| = \overline{t} - s, \ 0 < s < t \}.
\end{align*}
\]

For \( \alpha > 0 \), we multiply Equation (4.11) by \( 2e^{-at}u \), and obtain an equation which can be written as:

\[
(e^{-at}u^2)_t + \sum_{i=1}^n (e^{-at}a_iu^2)_{x_i} + e^{-at}(\alpha + 2b - \sum_{i=1}^n a_i x_i)u^2 = 2e^{-at}uf. \tag{4.17}
\]

The first two terms of (4.17) can be written as a divergence. If we set \( X = e^{-at}u^2 a\mu \partial_\mu \), then

\[
div X = \sum_{\mu=0}^n (e^{-at}a_\mu u^2)_{x_\mu} = (e^{-at}u^2)_t + \sum_{i=1}^n (e^{-at}a_iu^2)_{x_i}.
\]

with \( a_\mu = (1, a_1, \ldots, a_n) \) and \( x_0 = t \). We integrate (4.17) on \( D \). We have

\[
\int_D \left( (e^{-at}u^2)_t + \sum_{i=1}^n (e^{-at}a_iu^2)_{x_i} + e^{-at}(\alpha + 2b - \sum_{i=1}^n a_i x_i)u^2 \right) dt dx = 2\int_D e^{-at}uf dt dx. \tag{4.18}
\]

By the Stokes theorem, we have:

\[
\int_D \left( (e^{-at}u^2)_t + \sum_{i=1}^n (e^{-at}a_iu^2)_{x_i} \right) dxdt = \int_{\partial D} X \cdot \eta dS = \int_{\Sigma_{\kappa, \overline{t}}} X \cdot \eta dS + \int_{\Sigma_{0, \overline{t}}} X \cdot \eta dS + \int_{S_{\kappa, \overline{t}}} X \cdot \eta dS,
\]

where \( \eta \) is the outward unit normal vector to \( \partial D \) and \( dS \) the surface element on \( \partial D \). But, 

- on \( S_{\kappa, \overline{t}} \), \( \eta = (\eta_1, \eta_2, \cdots, \eta_n) = \frac{1}{\sqrt{1+\kappa^2}} (1, \kappa \overline{t}, \cdots, \kappa \overline{t}) \) from where we have

\[
\int_{S_{\kappa, \overline{t}}} X \cdot \eta dS = \int_{S_{\kappa, \overline{t}}} e^{-at} \left( \sum_{i=1}^n \eta_i a_i + \eta_0 \right) u^2 dS.
\]

- on \( \Sigma_{\kappa, \overline{t}} \) and \( \Sigma_{0, \overline{t}} \) the outward unit normals are respectively \( \eta = (1, \cdots, 0, 0) \) and \( \eta = (-1, \cdots, 0, 0) \), and then

\[
\int_{\Sigma_{\kappa, \overline{t}}} X \cdot \eta dS = \int_{\Sigma_{\kappa, \overline{t}}} e^{-at}u^2 dx \quad \text{and} \quad \int_{\Sigma_{0, \overline{t}}} X \cdot \eta dS = -\int_{\Sigma_{0, \overline{t}}} u_0^2 dx.
\]
Let us observe that, by (4.13) we have \( \sum_{i=1}^{n} \eta_i a_i + \eta_t \geq 0 \) which leads to the following inequality:

\[
\int_D \left( e^{-\alpha t} u^2 \right)_t + \sum_{i=1}^{n} \left( e^{-\alpha t} a_i u^2 \right)_x \right] dxdt \geq \int_{\Sigma_{t,\tau}} e^{-\alpha t} u^2 dx - \int_{\Sigma_{0,\tau}} u_0^2 dx .
\] (4.19)

Now since \( b \) and the partial derivatives of \( a_i \) with respect to \( x_i \) are bounded, we can choose \( \alpha \) such that:

\[
\alpha + 2b - \sum_{i=1}^{n} a_{i,x_i} \geq 1 \text{ in } D
\] (4.20)

From now, we suppose that \( \alpha \) is chosen such that (4.20) holds. We then have

\[
\int_D e^{-\alpha t} (\alpha + 2b - \sum_{i=1}^{n} a_{i,x_i}) u^2 dxdt \geq \int_D e^{-\alpha t} u^2 dt dx .
\] (4.21)

Now, recall the trivial inequality

\[
\int_D 2e^{-\alpha t} u f dxdt \leq \int_D e^{-\alpha t} u^2 dxdt + \int_D e^{-\alpha t} f^2 dx dt .
\] (4.22)

Adding (4.19) and (4.21) and using (4.17), (4.22) give:

\[
\int_{\Sigma_{t,\tau}} e^{-\alpha t} u^2 dx \leq \int_{\Sigma_{0,\tau}} u_0^2 dx + \int_0^t ds \int_{\Sigma_{s,\tau}} e^{-\alpha s} f^2(s, \cdot) dx ;
\] (4.23)

where in 123 \( \Sigma_{s,\tau} = \{(x, s) \in \mathbb{R}^{n+1}; |x| < \tau - s \} \) for fixed \( s \). Now, we let \( \tau \) tend to +\( \infty \), then \( \Sigma_{t,\tau}, \Sigma_{s,\tau} \) et \( \Sigma_{0,\tau} \) tend respectively to \( \{t \} \times \mathbb{R}^n, \{s \} \times \mathbb{R}^n \) and \( \{0 \} \times \mathbb{R}^n \). Therefore (4.23) gives:

\[
\int_{\mathbb{R}^n} e^{-\alpha t} u^2 dx \leq \int_{\mathbb{R}^n} u_0^2 dx + \int_0^t ds \int_{\mathbb{R}^n} e^{-\alpha s} f^2(s, \cdot) dx
\] (4.24)

which is the desired inequality.

\( \square \)

**Remark 4.3** For \( 0 < t \leq T \), inequality (4.24) reads

\[
e^{-\alpha t} \|u(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \leq \|u_0\|_{L^2(\mathbb{R}^n)}^2 + \int_0^t e^{-\alpha s} \|f(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 ds .
\] (4.25)

**Proposition 4.3** Let \( a = (a_1, \ldots, a_n), b \) be a family of smooth functions defined on \( \mathbb{R}^{n+1} \) such that the collection of functions \( a \) satisfies (4.13). Suppose that the partial derivatives up to order \( k \in \mathbb{N}^* \) of \( a \) and \( b \) with respect to the space variables \( x \) are bounded. Assume that \( u \) is a \( C^1 \) solution of the initial value problem (4.11) - (4.12). Then, for every \( T > 0 \), if \( f \in C([0, T); H^k(\mathbb{R}^n)) \) and \( u_0 \in H^k(\mathbb{R}^n) \), then

\[
e^{-\delta_0 t} \|u(t, \cdot)\|_{H^k(\mathbb{R}^n)}^2 \leq \|u_0\|_{H^k(\mathbb{R}^n)}^2 + C_0 \int_0^t \|e^{-\delta_0 s} f(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds ;
\] (4.26)

where \( \delta_0 \) and \( C_0 \) are positive constants.

**Proof:** Note that (4.25) gives (4.26) for \( k = 0 \). Consider the commutator \([L, \partial^\alpha]\) defined by: \([L, \partial^\alpha]u = L\partial^\alpha u - \partial^\alpha Lu\), where \( L \) is the partial differential operator associated to (4.11) (here, \( \partial^\alpha \equiv \partial^\alpha_{x_i} \)). We have, since (4.11):

\[
L\partial^\alpha u = \partial^\alpha f + [L, \partial^\alpha]u .
\] (4.27)
Applying Proposition 4.2 to 4.27 with $u$ replaced by $\partial^\alpha u$ and $f$ replaced by $\partial^\alpha f+[L, \partial^\alpha]u$ shows that there exists $\delta > 0$ such that:

$$e^{-\delta t}\|(\partial^\alpha u)(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \leq \|(\partial^\alpha u)(0, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + \int_0^t e^{-\delta s}\|(\partial^\alpha f+[L, \partial^\alpha]u)(s, \cdot)\|_{L^2(\mathbb{R}^n)}^2 ds$$

$$\leq \|(\partial^\alpha u)(0, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + 2\int_0^t e^{-\delta s}\|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}^2 + 2\int_0^t e^{-\delta s}\|L, \partial^\alpha]u(\cdot, \cdot)\|_{L^2(\mathbb{R}^n)}^2 ds;$$

(4.28)

where $|\alpha| \leq k$. On the other hand,

$$[L, \partial^\alpha]u = L\partial^\alpha u - \partial^\alpha Lu$$

$$= (\partial^\alpha u)_t + \sum_{i=1}^n a_i(\partial^\alpha u)_{x_i} + b\partial^\alpha u - \partial^\alpha (\sum_{i=1}^n a_i u x_i) - \partial^\alpha (bu)$$

$$= \sum_{i=1}^n a_i(\partial^\alpha u)_{x_i} + b\partial^\alpha u - \sum_{i=1}^n \sum_{0 \leq \beta \leq \alpha} C^\beta a_i \partial^{\alpha - \beta} u_{x_i} - \sum_{0 \leq \beta \leq \alpha} C^\beta \partial^\beta b \partial^{\alpha - \beta} u$$

$$= -\sum_{i=1}^n \sum_{0 < \beta \leq \alpha} C^\beta \partial^\beta a_i \partial^{\alpha - \beta} u_{x_i} - \sum_{0 < \beta \leq \alpha} C^\beta \partial^\beta b \partial^{\alpha - \beta} u. \quad (4.29)$$

Since $\partial^\beta a_i$ and $\partial^\beta b$ are bounded, the last equality gives the following estimate

$$\|[L, \partial^\alpha]u\|_{L^2(\mathbb{R}^n)} \leq \|\sum_{i=1}^n \sum_{0 < \beta \leq \alpha} C^\beta \partial^\beta a_i \partial^{\alpha - \beta} u_{x_i}\|_{L^2(\mathbb{R}^n)} + \|\sum_{0 < \beta \leq \alpha} C^\beta \partial^\beta b \partial^{\alpha - \beta} u\|_{L^2(\mathbb{R}^n)}$$

$$\leq C \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)} + C \sum_{|\alpha| \leq k - 1} \|\partial^\alpha u\|_{L^2(\mathbb{R}^n)}$$

$$\leq C\|u\|_{H^k(\mathbb{R}^n)}.$$ 

From (4.28) we deduce that; $\forall \alpha, |\alpha| \leq k$,

$$e^{-\delta t}\|(\partial^\alpha u)(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \leq \|(\partial^\alpha u)(0, \cdot)\|_{L^2(\mathbb{R}^n)}^2 + 2\int_0^t e^{-\delta s}\|(\partial^\alpha f(s, \cdot))\|_{L^2(\mathbb{R}^n)}^2 ds + 2C\int_0^t e^{-\delta s}\|u(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds.$$

Let us take the sum over $|\alpha| \leq k$; we obtain:

$$e^{-\delta t}\|u(t, \cdot)\|_{H^k(\mathbb{R}^n)}^2 \leq \|u(0, \cdot)\|_{H^k(\mathbb{R}^n)}^2 + 2\int_0^t e^{-\delta s}\|f(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds + 2C\int_0^t e^{-\delta s}\|u(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds;$$

and using Gronwall’s Lemma, we have:

$$e^{-\delta t}\|u(t, \cdot)\|_{H^k(\mathbb{R}^n)}^2 \leq \left(\|u(0, \cdot)\|_{H^k(\mathbb{R}^n)}^2 + 2\int_0^t e^{-\delta s}\|f(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds\right)e^{2Ct} ds.$$

Note that $\delta$ and $C$ are two positive constants which depends only on the bounds of the $C^k$-norms of $a$ and $b$. Now setting $\delta_0 = \delta + 2C$ and $C_0 = 2e^{2CT}$ gives

$$e^{-\delta(t)}\|u(t, \cdot)\|_{H^k(\mathbb{R}^n)}^2 \leq \|u_0\|_{H^k(\mathbb{R}^n)}^2 + C_0\int_0^t e^{-\delta s}\|f(s, \cdot)\|_{H^k(\mathbb{R}^n)}^2 ds;$$

which is the desired inequality. □
Corollary 4.1 Let $d$ be a non negative real number and $k$ an integer. Under the hypotheses of Proposition 4.3 assume further that the collections of functions $a = (a_1, \ldots, a_n)$ do not depend on the space variables $(a \equiv a(t))$ and that $b \equiv 0$. If $u$ is a $C^1$ solution of the Cauchy problem (4.11) - (4.12), then for every $T > 0$, $f \in C([0, T); H^k_{\delta}(\mathbb{R}^n))$, $u_0 \in H^k_{\delta}(\mathbb{R}^n)$ and $t \in [0, T]$ we have:

$$e^{-\delta t} \| u(t, \cdot) \|_{L^2(\mathbb{R}^n)}^2 \leq \| u_0 \|_{H^k_{\delta}(\mathbb{R}^n)}^2 + C_1 \int_0^t e^{-\delta s} \| f(s, \cdot) \|_{L^2(\mathbb{R}^n)}^2 \, ds$$

(4.30)

where $\delta_1$ and $C_1$ are positive constants which depend only on $\kappa$.

**Proof:** Let $\beta \in \mathbb{N}^n$ be given such that $|\beta| \leq k$. If we differentiate Equation (4.11) with $\partial^\beta$ (recall $a \equiv a(t)$ and $b \equiv 0$) and then multiply the differentiated equation by the weight $(1 + |x|)^{d + |\beta|}$, then we obtain:

$$(1 + |x|)^{d + |\beta|} \partial^\beta u + \sum_{i=1}^n a_i ((1 + |x|)^{d + |\beta|} \partial^\beta u)_{x_i} = (1 + |x|)^{d + |\beta|} f + \sum_{i=1}^n (d + |\beta|) \frac{x_i a_i}{x} (1 + |x|)^{d + |\beta| - 1} a_i \partial^\beta u.$$  

But the l.h.s is defined by $L$; so we have:

$$L \left[ (1 + |x|)^{d + |\beta|} \partial^\beta u \right] = (1 + |x|)^{d + |\beta|} \partial^\beta f + \sum_{i=1}^n (d + |\beta|) \frac{x_i a_i}{x} (1 + |x|)^{d + |\beta| - 1} a_i \partial^\beta u.$$  

(4.31)

Since the function $a$ is bounded and $\frac{|x|}{|x| + |x|} \leq 1$, we have:

$$\left| \sum_{i=1}^n (d + |\beta|) \frac{x_i a_i}{x} (1 + |x|)^{d + |\beta| - 1} a_i \partial^\beta u \right|_{L^2(\mathbb{R}^n)} \leq C \|(1 + |x|)^{d + |\beta|} u\|_{L^2(\mathbb{R}^n)} \leq C \| u \|_{H^k_{\delta}(\mathbb{R}^n)}.$$  

(4.32)

Now, applying Proposition 4.2 to Equation (4.31) shows that there exists a constant $\delta > 0$ such that:

$$e^{-\delta t} \|(1 + |x|)^{d + |\beta|} \partial^\beta u(t, \cdot)\|_{L^2(\mathbb{R}^n)}^2 \leq$$

$$\|(1 + |x|)^{d + |\beta|} \partial^\beta u_0\|_{L^2(\mathbb{R}^n)}^2 + 2 \int_0^t e^{-\delta s} \| f(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds + 2 C \int_0^t e^{-\delta s} \| u(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds.$$  

(4.33)

where we have used (4.32). Summing over $|\beta| \leq k$, we obtain:

$$e^{-\delta t} \| u(t, \cdot) \|_{L^\infty(\mathbb{R}^n)}^2 \leq \| u_0 \|_{H^k_{\delta}(\mathbb{R}^n)}^2 + 2 \int_0^t e^{-\delta s} \| f(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds + 2 C \int_0^t e^{-\delta s} \| u(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds;$$  

(4.34)

from where using Gronwall’s Lemma, we get:

$$e^{-\delta t} \| u(t, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \leq \left( \| u_0 \|_{H^k_{\delta}(\mathbb{R}^n)}^2 + 2 \int_0^t e^{-\delta s} \| f(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds \right) e^{2 C t}.$$  

Note that $\delta$ and $C$ only depends on the $L^\infty$ norm of $a$ and therefore there exists two positive constants $\delta_1$ and $C_1$ which depends only on $\kappa$ such that:

$$e^{-\delta_1 t} \| u(t, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \leq \| u_0 \|_{H^k_{\delta}(\mathbb{R}^n)}^2 + C_1 \int_0^t e^{-\delta_1 s} \| f(s, \cdot) \|_{H^k_{\delta}(\mathbb{R}^n)}^2 \, ds.$$  

$\square$
5 The Einstein-Maxwell-Boltzmann system with massive scalar field as a first order system. The iterated sequence

5.1 Change of variables in the Einstein-Maxwell-Boltzmann system with massive scalar field

The coupled system writes, using (2.66), (2.67), (2.68), and (4.3); the change of variables \((t, p, q) \mapsto (t, \pi, \psi)\) and \(d\pi = a^{-6}dt\) as:

\[
\begin{aligned}
3 \left( \frac{\dot{a}}{a} \right)^2 - \Lambda &= 8\pi a^{-3} \int_{\mathbb{R}^3} v^0 f(t, \pi)d\pi + 12\pi a^{2}(F^{01})^2 - 8\pi \theta_{00} + 4\pi (\dot{\Phi}^2 + m^2 \Phi^2) \\
-2 \dddot{a} a - \left( \frac{\dot{a}}{a} \right)^2 + \Lambda &= 8\pi a^{-5} \int_{\mathbb{R}^3} \left( \frac{v^1}{v^0} \right)^2 f(t, \pi)d\pi + 4\pi a^2(F^{01})^2 + 4\pi (\dot{\Phi}^2 - m^2 \Phi^2) \\
\dot{F}^{01} + 3\dddot{a} F^{01} &= 0 \\
\dot{\theta}^{00} + 3\dddot{a} \theta^{00} &= -\alpha^2. \\
\dot{\Phi} \left[ \Phi + 3 \left( \frac{\dot{a}}{a} \right) \Phi + m^2 \Phi \right] &= \alpha^2 \\
\frac{\partial f}{\partial t} - \left( \frac{F^{01}}{a} \int_{\mathbb{R}^3} f(t, \pi)d\pi \right) \sum_{i=1}^{3} \frac{\partial f}{\partial u^i} &= \frac{1}{a^6} \dot{Q}(f, f)
\end{aligned}
\] (S)

Note that Equation 5.1 is the Hamiltonian constraint. It is known that it will be satisfied everywhere if it satisfied for \(t = 0\), i.e. if the initial data are subject to the following constraint:

\[
\begin{aligned}
3 \left( \frac{\dot{a}}{a_0} \right)^2 - \Lambda &= 8\pi a_0^{-3} \int_{\mathbb{R}^3} v^0 f_0(\pi)d\pi + 12\pi a_0^{2}(F^{01}(0))^2 - 8\pi \theta_{00}(0) + 4\pi (\dot{\Phi}_0^2 + m^2 \Phi_0^2) \\
with \ a_0 &= a(0); \ \dot{a}_0 = \dot{a}(0); \ f_0(\pi) = f(0, \pi); \ \Phi_0 = \Phi(0); \ \dot{\Phi}_0 = \dot{\Phi}(0); \ F^{01}(0) = F_0^{01}
\end{aligned}
\] (5.7)

So we will suppose that (5.7) holds. Therefore, (5.1) is solved and shall be considered as relation between the unknown functions. In order to have a first order system, we set:

\[
E = \frac{1}{a}, \ U = \frac{\dot{a}}{a}, \ \psi = \frac{1}{2}(\dot{\Phi})^2, \ Z = F^{01}, \ W = \theta^{00}
\] (5.8)

Note \(\alpha \neq 0\) implies using Equation 5.5 that \(\dot{\Phi}\) does not vanish. Since \(\dot{\Phi}\) is continuous, it keeps a constant sign. We choose to look for \(\Phi\) such that:

\[
\dot{\Phi} > 0
\] (5.9)

By (5.9), \(\Phi\) is increasing and we choose:

\[
\Phi(0) := \Phi_0 > 0
\] (5.10)

which implies:

\[
\Phi \geq 0
\] (5.11)

We have:

\[
\begin{aligned}
\psi &= \frac{1}{2}(\dot{\Phi})^2, \ \dot{\Phi} > 0 \implies \dot{\Phi} = \sqrt{2\psi} \\
E &= -\frac{\dot{a}}{a^2} = -\frac{\dot{a}}{a} \times \frac{1}{a} = -UE \\
\dot{U} &= \frac{\ddot{a} - (\frac{\dot{a}}{a})^2}{a^2} = \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 \implies \frac{\ddot{a}}{a} = \dot{U} + U^2
\end{aligned}
\]
Therefore, we deduce from (S) the equivalent system \((S')\) of first order:

\[
\begin{align*}
\dot{E} &= -UE \\
\dot{U} &= \frac{3}{2}U^2 + \frac{\Lambda}{2} - 4\pi E^5 \int_{\mathbb{R}^3} \frac{(u^1)^2}{v^3} f(t, \bar{\nu})d\bar{\nu} - \frac{2\pi}{E^2} Z^2 - 2\pi (2\psi - m^2 \Phi^2) \\
\dot{W} &= -3UW - \alpha^2 \\
\dot{Z} &= -3UZ \\
\dot{\Phi} &= \sqrt{2}\psi \\
\dot{\psi} &= -6U\psi - m^2 \Phi \sqrt{2\psi} + \alpha^2 \\
\frac{\partial f}{\partial t} - (EZ \int_{\mathbb{R}^3} f(t, \bar{\nu})d\bar{\nu}) \sum_{i=1}^{3} \frac{\partial f}{\partial u^i} &= \frac{1}{u^0} \hat{Q}(f,f)
\end{align*}
\]

We will study \((S')\) with the following initial data:

\[
\begin{align*}
E(0) &= E_0 = \frac{1}{\alpha^0}, U(0) = U_0 = \frac{\alpha}{\alpha^0}, W(0) = W_0 < 0, Z(0) = Z_0, \\
\Phi(0) &= \Phi_0 > 0, \psi(0) = \psi_0 \geq 0, f(0,\cdot) = f_0 \in H^3_{d,v}(\mathbb{R}^3) \\
a_0 &= a_0(0), \dot{a}_0 = \dot{a}(0)
\end{align*}
\]

where \(H^3_{d,v}(\mathbb{R}^3)\) is defined by \((5.11)\). We suppose that the initial data are subject to the constraint \((5.7)\). We choose, given \((2.52)\), \(W(0) = \theta^0(0) < 0\) and by \((2.60)\), we have:

\[
W \leq 0.
\]

### 5.2 The iterated sequence

The reader may wonder why one cannot solve directly Equation \((5.18)\) by introducing its equivalent characteristic system. But it is not clear how to derive such system since the equation at hand is an integro-differential equation in which appears the unknown \(f\) and its integral. For this reason, we choose to introduce an iterative scheme in which the characteristic method is used to derive solutions of the linearized equations.

Let \(T > 0\) be given. Define on \([0, T]\), the functions \(E^0, U^0, W^0, Z^0, \Phi^0, \psi^0\) and \(f^0\) by:

\[
E^0(t) = E_0, U^0(t) = U_0, W^0(t) = W_0, Z^0(t) = Z_0, \Phi^0(t) = \Phi_0, \psi^0(t) = \psi_0 \text{ and } f^0(t, \bar{\nu}) = f_0(\bar{\nu}).
\]

Now define \((E^1, U^1, W^1, Z^1, \Phi^1, \psi^1, f^1)\) as solution of the system:

\[
\begin{align*}
\dot{E}^1 &= -U^0 E^0 \\
\dot{U}^1 &= \frac{3}{2}(U^0)^2 + \frac{\Lambda}{2} - 4\pi(E^0)^5 \int_{\mathbb{R}^3} \frac{(u^1)^2}{v^3} f^0(t, \bar{\nu})d\bar{\nu} - \frac{2\pi}{E^0} \frac{(Z^0)^2}{(E^0)^2} - 2\pi (2\psi^0 - m^2 \Phi^0)^2 \\
\dot{W}^1 &= -3U^0 W^0 - \alpha^2 \\
\dot{Z}^1 &= -3U^0 Z^0 \\
\dot{\Phi}^1 &= \sqrt{2}\psi^0 \\
\dot{\psi}^1 &= -6U^0 \psi^0 - m^2 \Phi^0 \sqrt{2\psi^0} + \alpha^2 \\
\frac{\partial f^1}{\partial t} - (E^0 Z^0 \int_{\mathbb{R}^3} f^0(t, \bar{\nu})d\bar{\nu}) \sum_{i=1}^{3} \frac{\partial f^1}{\partial u^i} &= \frac{1}{u^0} \hat{Q}_0
\end{align*}
\]

where \(u^0_0 = \sqrt{1 + (E^0)^2 + |\pi|^2}\) and \(\hat{Q}_0\) stands for the collision operator \(\hat{Q}\) as defined by \(E^0, f^0\) and \(v^0 = \sqrt{1 + (E^0)^2 + |\pi|^2}\), with the initial data:

\[
(E^1, U^1, W^1, Z^1, \Phi^1, \psi^1, f^1)(0) = (E_0, U_0, W_0, Z_0, \Phi_0, \psi_0, f_0).
\]
In fact by direct integrations, equations \([5.21]\) to \([5.26]\) give \((E, U, W, Z, \Phi, \psi)\). Now the partial differential equation \([5.27]\) is equivalent, taking \(t\) as parameter and setting \(h^1(t) = f^1(t, \mathbf{w}(t))\), to its characteristic system:

\[
\begin{aligned}
(S^0) : \quad \frac{du}{dt} + \int_{\mathbb{R}^3} f^0(\mathbf{w}) d\mathbf{w} = -E^0 Z^0 \int_{\mathbb{R}^3} f^0(\mathbf{w}) d\mathbf{w} \\
\frac{du}{dt} = \frac{1}{u_n^0} \tilde{Q}_n(f^0, f^0) \\
i = 1, 2, 3.
\end{aligned}
\]

By simple integration, the characteristic system \((S^0)\) has a unique solution \((\mathbf{w}, h^1)\) which is of class \(C^1\) on \([0, T]\). This gives the unique solution \(f^1\) of \([5.27]\) on \([0, T]\). From there one obtains existence of the solution \((E, U, W, Z, \Phi, \psi, f^1)\) which is in \(C^1([0, T], \mathbb{R})^6 \times C^1([0, T] \times \mathbb{R}^3)\).

**Remark 5.1** Given that the function \((t, \mathbf{w}) \mapsto E^0 Z^0 \int_{\mathbb{R}^3} f^0(t, \mathbf{w}) d\mathbf{w}\) is bounded because

\[
\left|E^0 Z^0 \int_{\mathbb{R}^3} f^0(t, \mathbf{w}) d\mathbf{w}\right| \leq E_0 Z_0 \|f_0\|_{H^2, r}(\mathbb{R}^3) \leq E_0 Z_0 r
\]

and that its derivatives with respect to \(\mathbf{w}\) are zero, by Corollary \([4.7]\), Inequality \([4.30]\) implies that \(f^1\) satisfies:

\[
e^{-\delta_1 t} \|f^1(t, \cdot)\|_{H^2, r}(\mathbb{R}^3) \leq \|f_0\|_{H^2, r}(\mathbb{R}^3) + C_1 \int_0^t e^{-\delta_1 s} \left\|\frac{1}{u_n^0} \tilde{Q}_n(s, \cdot)\right\|^2_{H^2, r}(\mathbb{R}^3) ds
\]

where \(\delta_1\) and \(C_1\) are positive constants which depends \(E_0, |Z_0|\) and \(r\). We deduce from \([5.28]\) given \([4.30]\) that:

\[
e^{-\delta_1 t} \|f^1(t, \cdot)\|_{H^2, r}(\mathbb{R}^3) \leq \|f_0\|_{H^2, r}(\mathbb{R}^3) + C_2 \int_0^t e^{-\delta_1 s} \|f^0(s, \cdot)\|_{H^2, r}(\mathbb{R}^3) ds.
\]

Here \(C_2\) is a positive constant which depends on \(E_0, |Z_0|, r\) and \(T\) and since \(f^0 = f_0 \in H^3, r(\mathbb{R}^3)\) we deduce that the solution \(f^1\) of \([5.27]\) is in \(C([0, T], H^3, r(\mathbb{R}^3))\).

Now we can iterate as follows. Suppose that the set of functions \((E^n, U^n, W^n, Z^n, \Phi^n, \psi^n, f^n)\) are given in \(\left(C^1([0, T], \mathbb{R})\right)^6 \times C^1([0, T], H^3, r(\mathbb{R}^3))\) and define \((E^{n+1}, U^{n+1}, W^{n+1}, Z^{n+1}, \Phi^{n+1}, \psi^{n+1}, f^{n+1})\) as the solution of:

\[
(S^n) : \quad \left\{
\begin{aligned}
\dot{E}^{n+1} &= -E^n U^n \\
\dot{U}^{n+1} &= \frac{3}{2} (U^n)^2 + \frac{\lambda}{2} - 4\pi (E^n)^5 \int_{\mathbb{R}^3} \frac{1}{u_n^0} f^n(t, \mathbf{w}) d\mathbf{w} - 2\pi \left(\frac{Z^n}{E^n}\right)^2 - 2\pi (2\psi^n - m^2(\Phi^n)^2) \\
\dot{W}^{n+1} &= -3U^n W^n - \alpha^2 \\
\dot{Z}^{n+1} &= -3U^n Z^n \\
\dot{\Phi}^{n+1} &= \sqrt{2\psi^n} \\
\dot{\psi}^{n+1} &= -6U^n \psi^n - m^2(\Phi^n)^2 \sqrt{2\psi^n} + \alpha^2 \\
\frac{\partial f^{n+1}}{\partial t} - \left(E^n Z^n \int_{\mathbb{R}^3} f^n(t, \mathbf{w}) d\mathbf{w}\right) \sum_{i=1}^3 \frac{\partial f^{n+1}}{\partial u^i} = \frac{1}{u_n^0} \tilde{Q}_n
\end{aligned}
\right.
\]

where \(u_n^0 = \sqrt{1 + |E^n|^2 |\mathbf{w}|^2}\) and \(\tilde{Q}_n\) stands for the collision operator \(\tilde{Q}\) as defined by \(E^n, f^n\) and \(v_0 = \sqrt{1 + (E^n)^2 |\mathbf{w}|^2}\), with initial data:

\((E^{n+1}, U^{n+1}, W^{n+1}, Z^{n+1}, \Phi^{n+1}, \psi^{n+1}, f^{n+1})(0) = (E_0, U_0, W_0, Z_0, \Phi_0, \psi_0, f_0)\).

We use the method of characteristics as we did before to obtain that system \((S^n)\) has a unique solution \((E^{n+1}, U^{n+1}, W^{n+1}, Z^{n+1}, \Phi^{n+1}, \psi^{n+1}, f^{n+1})\) in \(\left(C^1([0, T], \mathbb{R})\right)^6 \times C^1([0, T]; H^3, r(\mathbb{R}^3))\). We have thus constructed a sequence \((E^n, U^n, W^n, Z^n, \Phi^n, \psi^n, f^n)\) in \(\left(C^1([0, T], \mathbb{R})\right)^6 \times C^1([0, T], H^3, r(\mathbb{R}^3))\) defined on a maximal interval \([0, T_{max}]\), \(T_{max} > 0\) and we want to show that this sequence converges to the solution \((E, U, W, Z, \Phi, \psi, f)\) of the system \((S')\).
6 Local existence of solution to the Einstein-Maxwell-Boltzmann system with massive scalar field

We start this section by stating some boundedness properties of the sequence we have just constructed. We have the following

Proposition 6.1 Let \( f_0 \in L^2(\mathbb{R}^3) \). There exists \( T > 0 \), independent of \( n \), such that, the sequence \((X^n)\) where \( X^n = (E^n, U^n, W^n, Z^n, \Phi^n, \psi^n, f^n)\) is uniformly bounded on \([0, T]\).

Proof: Set

\[
\|X^n(t)\| = |E^n(t)| + |U^n(t)| + |W^n(t)| + |Z^n(t)| + |\Phi^n(t)| + |\psi^n(t)| + \|f^n(t, \cdot)\|_{H^2(\mathbb{R}^3)}
\]

and

\[
C_0 = |E^0| + |U^0| + |W^0| + |Z^0| + |\Phi^0| + |\psi^0| + r.
\]

We will prove by induction that, there exists \( N \) such that, \( \|X^n(t)\| \leq 2C_0, \; \forall n \in \mathbb{N}, \; \forall t \in [0, T] \).

\begin{itemize}
    \item For \( n = 0 \), we have \( \|X^0\| \leq C_0 \leq 2C_0 \).
    \item Let \( n \in \mathbb{N} \) and suppose that \( \forall k \leq n \), we have \( \|X^k(t)\| \leq 2C_0 \). We want to show that \( \|X^{n+1}(t)\| \leq 2C_0 \) for all \( t \in [0, T] \), the choice of \( T \) will be given shortly.
\end{itemize}

- Integrate equation (5.29) on \([0, t]\), we have:

\[
E^{n+1}(t) = E_0 - \int_0^t E^n(s)U^n(s)ds;
\]

then

\[
|E^{n+1}(t)| \leq |E_0| + A_1 t; \tag{a}
\]

where \( A_1 > 0 \) is a constant which only depends on \( C_0 \).

- Integrate equation (5.30) on \([0, T]\), we have

\[
U^{n+1}(t) = U_0 + \int_0^t \left[ -\frac{3}{2} (U^n)^2 + \frac{\Lambda}{2} - 4\pi (E^n)^2 \int_{\mathbb{R}^3} \frac{(\psi^n)^2}{\psi_n^2} f^n(s, \nabla) d\nabla - 2\pi (Z^n)^2 - 2\pi (2\psi^n + m^2(\phi^n)^2) \right] ds \tag{6.1}
\]

We had by (5.29) : \( \dot{E}^n = -U^{n-1}E^{n-1} \) which implies : \( |\dot{E}^n| \leq 4C_0^2 \) i.e. \(-4C_0^2 \leq \dot{E}^n \leq 4C_0^2 \).

Integrating this last inequality gives \( E_0 - 4C_0^2 t \leq E^n(t) \). But \( E_0 = \frac{1}{a_0} > 0 \). So if we take \( t \) such that \( 0 < 4C_0^2 t < \frac{E_0}{2} \), then \( E_0 - 4C_0^2 t \geq \frac{E_0}{2} \); then \( \frac{E_0}{2} \leq E^n(t) \) i.e. \( \frac{1}{E^n(t)} \leq \frac{2}{E_0} \). This proves that one can find two constants \( t_1 > 0 \) and \( A_2 > 0 \) such that, identity (6.1) gives the following:

\[
|U^{n+1}(t)| \leq |U_0| + A_2 t, \forall 0 \leq t \leq t_1. \tag{b}
\]

- Integrate equation (5.31) on \([0, t]\), we have: \( W^{n+1}(t) = W_0 - \int_0^t (3U^n(s)W^n(s) + \alpha^2)ds \). Then there exists a constant \( A_3 > 0 \) such that:

\[
|W^{n+1}(t)| \leq |W_0| + A_3 t. \tag{c}
\]

- Integrate equation (5.32) on \([0, t]\), we have: \( Z^{n+1}(t) = Z_0 - 3 \int_0^t U^n(s)Z^n(s)ds \). Then there exists a constant \( A_4 > 0 \) such that:

\[
|Z^{n+1}(t)| \leq |Z_0| + A_4 t. \tag{d}
\]
- Integrate equation \((5.33)\) on \([0, t]\), we have: \(\Phi^{n+1}(t) = \Phi_0 + \int_0^t 2\psi^n(s)ds\). This shows that there exists a constant \(A_5 > 0\) such that:

\[|\Phi^{n+1}(t)| \leq |\Phi_0| + A_5 t .\]  

\((e)\)

- Integrate equation \((5.34)\) on \([0, t]\), we have:

\[\psi^{n+1}(t) = \psi_0 - \int_0^t (6U^n(s)\psi^n(s) + m^2\Phi^n(s)\sqrt{2\psi^n(s) - \alpha^2})ds .\]

This shows that there exists a constant \(A_6 > 0\) such that:

\[|\psi^{n+1}(t)| \leq |\psi_0| + A_6 t .\]  

\((f)\)

- Now we want to use Corollary \((4.1)\) to obtain a bound for \(\|f^{n+1}(t, \cdot)\|_{H^2_0(\mathbb{R}^3)}^2\). Observe that Equation \((6.36)\) is of the form \((4.1)\) with \(a = a(t) = E^nZ^n \int_{\mathbb{R}^3} f^n(t, \vec{\nu})d\vec{\nu}, \ b = 0\) and since by induction hypothesis

\[|E^nZ^n \int_{\mathbb{R}^3} f^n(t, \vec{\nu})d\vec{\nu}| \leq 4C_0^2 \|f^n(t, \cdot)\|_{L^1(\mathbb{R}^3)} \leq C(C_0) \|f^n(t, \cdot)\|_{H^2_0(\mathbb{R}^3)} \leq C(C_0) ;\]

one can use inequality \((4.30)\) of Corollary \((4.1)\) (with \(\frac{\gamma}{4} = C(C_0), \ k = n = 3\)) to obtain

\[\|f^{n+1}(t, \cdot)\|_{H^2_0(\mathbb{R}^3)}^2 \leq \left( \|f_0\|_{H^2_0(\mathbb{R}^3)}^2 + C(C_0) \int_0^t \|f^n(s, \cdot)\|_{H^2_0(\mathbb{R}^3)}^2 ds \right) e^{C(C_0)t} .\]

This proves that:

\[\|f^{n+1}(t, \cdot)\|_{H^2_0(\mathbb{R}^3)}^2 \leq e^{C(C_0)t} \left( \|f_0\|_{H^2_0(\mathbb{R}^3)}^2 + C(C_0)t \right) .\]

We take \(t_2 > 0\) such that \(e^{C(C_0)t} \leq 1, \forall t \in [0, t_2]\). Then, there exists a constant \(A_7 > 0\), such that for \(t \in [0, t_2]\):

\[\|f^{n+1}(t)\|_{H^2_0(\mathbb{R}^3)} \leq \|f_0\|_{H^2_0(\mathbb{R}^3)} + A_7 \sqrt{t} .\]  

\((g)\)

Now we add inequalities \((a), (b), (c), (d), (e), (f), (g)\) to obtain, for \(t \leq \min(t_1, t_2)\):

\[\|X^{n+1}(t, \cdot)\| \leq C_0 + \left( \sum_{i=1}^{7} A_i \right) (t + \sqrt{t}) .\]

Now choose \(t_3 > 0\) such that for \(\forall 0 \leq t \leq t_3\) we have \(\left( \sum_{i=1}^{7} A_i \right) (t + \sqrt{t}) \leq C_0\). Finally, by setting \(T = \min(t_1, t_2, t_3)\) we obtain that

\[\|X^{n+1}(t, \cdot)\| \leq 2C_0 \quad \text{for all} \quad 0 \leq t \leq T .\]

\(\Box\)

We have the following

**Lemma 6.1** Let \(n\) be a non negative integer, then we have the following inequalities

\[\left| E^n \right|^5 \int_{\mathbb{R}^3} \frac{|\nu|^2}{\nu_n} f^n(\vec{\nu})d\vec{\nu} - \left| E^{n-1} \right|^5 \int_{\mathbb{R}^3} \frac{|\nu|^2}{\nu_{n-1}} f^{n-1}(\vec{\nu})d\vec{\nu} \leq C(C_0) \left( \left| E^n - E^{n-1} \right| + \left| f^n - f^{n-1} \right|_{H^2_0(\mathbb{R}^3)} \right) .\]

\((6.2)\)

and

\[\left\| \frac{1}{\nu_n} \tilde{Q}_n - \frac{1}{\nu_{n-1}} \tilde{Q}_{n-1} \right\|_{H^2_0(\mathbb{R}^3)} \leq C(C_0) \left( \left| E^n - E^{n-1} \right| + \left| f^n - f^{n-1} \right|_{H^2_0(\mathbb{R}^3)} \right) .\]

\((6.3)\)
Proof: First, we prove \[6.2\]. We have:

\[ \left| E^0 f_n - E^{n-1} f_n \right| \int_{R^3} \frac{|v|^2}{v_n^0} f - \left| E^{n-1} f_n \right| \int_{R^3} \frac{|v|^2}{v_n^0} f^{n-1} \\]

\[ = \left| (E^0 - E^{n-1}) \right| \int_{R^3} \frac{|v|^2}{v_n^0} f + \left| E^{n-1} \right| \int_{R^3} \frac{|v|^2}{v_n^0} f^{n-1} \\]

\[ \leq C(C_0) \left| E^0 - E^{n-1} \right| + \left| E^{n-1} \right| \int_{R^3} \frac{1}{v_n^0} \left| f - \left| f^{n-1} \right| \right| d\mathbf{v} \]

\[ \leq C(C_0) \left( \left| E^0 - E^{n-1} \right| + \left| f - f^{n-1} \right| \right) . \]

Secondly we prove \[6.3\]. From now, we choose to write \( \tilde{Q}(E^0, f_n, f_n) \) for \( \tilde{Q}_n \) to take advantage of the presence of \( E^0 \) in \( \tilde{Q}_n \). We have

\[ \frac{1}{u_n^0} \tilde{Q}_n - \frac{1}{u_{n-1}^0} \tilde{Q}_{n-1} = \frac{1}{u_n^0} \left( \tilde{Q}(E^0, f_n, f_n) - \tilde{Q}(E^{n-1}, f_n, f_n) \right) + \left( \frac{1}{u_n^0} - \frac{1}{u_{n-1}^0} \right) \tilde{Q}(E^{n-1}, f_n, f_n) \]

\[ + \frac{1}{u_{n-1}^0} \left( \tilde{Q}(E^{n-1}, f_n, f_n) - \tilde{Q}(E^{n-1}, f^{n-1}, f^{n-1}) \right) . \quad (6.4) \]

As far as the first term of \[6.4\] is concerned, we have:

\[ \frac{1}{u_n^0} \left( \tilde{Q}(E^0, f_n, f_n) - \tilde{Q}(E^{n-1}, f_n, f_n) \right) = \frac{1}{u_n^0} \left[ \tilde{Q}^+(E^0, f_n, f_n) - \tilde{Q}^+(E^{n-1}, f_n, f_n) \right] \]

\[ - \frac{1}{u_n^0} \left[ \tilde{Q}^-(E^0, f_n, f_n) - \tilde{Q}^-(E^{n-1}, f_n, f_n) \right] =: (I) + (II) . \]

Let us estimate the first term. We have

\[ (I) = \frac{1}{u_n^0} \left[ \int_{R^3} \frac{(E^0)^3}{v_n^0} d\mathbf{v} \int_{S^2} f^n(t, \mathbf{v}) d\mathbf{w} - \int_{R^3} \frac{(E^{n-1})^3}{v_{n-1}^0} d\mathbf{v} \int_{S^2} f^n(t, \mathbf{v}) d\mathbf{w} \right] \]

\[ = \frac{1}{u_n^0} \left[ \int_{R^3} \frac{(E^0)^3}{v_n^0} d\mathbf{v} \int_{S^2} f^n(t, \mathbf{v}) d\mathbf{w} - \int_{R^3} \frac{(E^{n-1})^3}{v_{n-1}^0} d\mathbf{v} \int_{S^2} f^n(t, \mathbf{v}) d\mathbf{w} \right] \]

\[ = (I_1) + (I_2) . \]

Note that

\[ \frac{(E^0)^3}{v_n^0} - \frac{(E^{n-1})^3}{v_{n-1}^0} = \frac{(E^0)^3 (v_n^0 - v_{n-1}^0)}{v_n^0 v_{n-1}^0} + v_n^0 ( (E^0)^3 - (E^{n-1})^3 ) \]

\[ = \frac{(E^0)^3 (E^{n-1})^3}{v_n^0 v_{n-1}^0} + \frac{(E^0)^3}{v_n^0} \sqrt{1 + (E^{n-1})^2 \| \mathbf{v} \|^2} - \frac{1}{(E^0)^2 \| \mathbf{v} \|^2} \]

\[ = \frac{E^n - E^{n-1}}{v_n^0} \left[ (E^0)^2 + E^n E^{n-1} + (E^{n-1})^2 + \frac{(E^0)^3 (E^n + E^{n-1}) \| \mathbf{v} \|^2}{v_n^0 (v_n^0 + v_{n-1}^0)} \right] ; \]

\[ := \xi_n \]

\[ I_2 = \left( E^n - E^{n-1} \right) \frac{1}{u_n^0} \int_{R^3} \frac{E^0 d\mathbf{v}}{v_n^0} \int_{S^2} f^n(t, \mathbf{v}) d\mathbf{w} . \quad (6.5) \]
Before continuing, we point out the following Lemma

**Lemma 6.2** If \( f \) and \( g \) are functions such that the partial derivatives of \( f \) up to order \( k \) are uniformly bounded then 
\[
\|fg\|_{H^2(\mathbb{R}^3)} \leq C\|g\|_{H^3(\mathbb{R}^3)}
\]
where the positive constant \( C \) only depends on the bounds of \( f \) and its derivatives.

This Lemma shows that in order to control \( H^2 - \text{norm} \) of \((I_2)\), we then need to show that the function \( \tilde{u} \mapsto \frac{u_n^0}{u_n^0} \) and its derivatives up to order two are uniformly bounded. Note that for all \( n \in \mathbb{N}, \ 0 < \frac{E_n}{2} \leq E^\infty \leq 2C_0 \) thus we have
\[
\frac{u_n^0}{u_n^0} = \frac{\sqrt{1 + (E_n^{-1})^2|u|^2}}{\sqrt{1 + (E^n)^2|u|^2}} \leq C(C_0, E_0);
\]
and
\[
\left| \partial_i \left( \frac{u_n^0}{u_n^0} \right) \right| = \left| \frac{(E_n^{-1})^2u_i^0}{u_n^0u_n^0} - \frac{(E^n)^2u_i^0u_n^0}{(u_n^0)^3} \right| \leq C(C_0, E_0);
\]
and
\[
\left| \partial_{ij} \left( \frac{u_n^0}{u_n^0} \right) \right| = \left| \frac{(E_n^{-1})^2\delta_{ij}^0}{u_n^0u_n^0} - \frac{2(E_n^{-1})^2u_i^0u_j^0}{(u_n^0)^3u_n^0} - \frac{(E^n)^2u_i^0u_j^0}{(u_n^0)^3} - \frac{\delta_{ij}^0(E^n)^2u_n^0}{(u_n^0)^3} + \frac{3(E^n)^4u_i^0u_j^0u_n^0}{(u_n^0)^5} \right| \leq C(C_0, E_0) .
\]

Lemma [6.2] implies that
\[
\|(I_2)\|_{H^2(\mathbb{R}^3)} \leq C(C_0)|E^n - E^{n-1}| \int_{\mathbb{R}^3} \int_{S^2} \left| f_n(t, \mathbf{u})f_n(t, \mathbf{v})\tilde{B}(E^{n-1}, \mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}')d\omega \right|_{H^2(\mathbb{R}^3)} .
\]
Since \( \xi^\infty \) is bounded and does not depend on \( \tilde{u} \), the \( H^2(\mathbb{R}^3) \)-norm at the r.h.s. of the previous inequality will give an estimate similar to (6.9). More precisely, we have
\[
\|(I_2)\|_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| |f_n|^2_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| .
\]
In order to obtain an estimate for the \( H^2(\mathbb{R}^3) \)-norm of the term \((I_1)\), we proceed exactly as in the proof of Proposition 3.6 of [7], page 88. The term \( B \) in that reference is replaced by the difference \( (\tilde{B}(E^{n-1}, \mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}') - \tilde{B}(E^n, \mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}')) \) and we use instead the property that \( B \) and its derivatives are Lipschitz continuous. This leads to and estimate of the form
\[
\|(I_1)\|_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| |f_n|^2_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| ;
\]
and then
\[
\left\| \frac{1}{u_n^0} \left[ \hat{Q}^+(E^n, f^n, f^n) - \hat{Q}^+(E^{n-1}, f^n, f^n) \right] \right\|_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| .
\]
Similarly we have:
\[
\|(III)\|_{H^2(\mathbb{R}^3)} = \left\| \frac{1}{u_n^0} \left[ \hat{Q}^-(E^n, f^n, f^n) - \hat{Q}^-(E^{n-1}, f^n, f^n) \right] \right\|_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| ;
\]
and we deduce that:
\[
\left\| \frac{1}{u_n^0} \left[ \hat{Q}(E^n, f^n, f^n) - \hat{Q}(E^{n-1}, f^n, f^n) \right] \right\|_{H^2(\mathbb{R}^3)} \leq C|E^n - E^{n-1}| .
\]
As far as the second term of (6.4) is concerned, we have:

\[
\left\| \left( \frac{1}{u_n^0} - \frac{1}{u_{n-1}^0} \right) \tilde{Q}(E^{n-1}, f^n, f^n) \right\|_{H^2_2(\mathbb{R}^3)} = \left\| \frac{u^0_{n-1} - u^0_n}{u^0_n} \tilde{Q}(E^{n-1}, f^n, f^n) \right\|_{H^2_2(\mathbb{R}^3)} = (E^{n-1})^2 - (E^n)^2 \left\| \frac{|\tilde{u}|^2}{u^0_{n-1} + u^0_n} \tilde{Q}(E^{n-1}, f^n, f^n) \right\|_{H^2_2(\mathbb{R}^3)} \leq C \left| E^{n-1} - E^n \right| \left\| \frac{1}{u^0_{n-1}} \tilde{Q}(E^{n-1}, f^n, f^n) \right\|_{H^2_2(\mathbb{R}^3)} \leq C \left| E^{n-1} - E^n \right| f^n \right\|_{H^2_2(\mathbb{R}^3)}^2 \leq C \left| E^{n-1} - E^n \right| f^n \right\|_{H^2_2(\mathbb{R}^3)} \right\|_{H^2_2(\mathbb{R}^3)} \leq C \int_0^T \left( |E^n(s) - E^{n-1}(s)|^2 + |U^n(s) - U^{n-1}(s)|^2 \right) ds . \tag{6.12}
\]

Finally, adding (6.9) - (6.11) gives (6.3) and the proof is complete. \qed

Proposition 6.2 The hypotheses are those of the previous Proposition. Set \( Y^n = (E^n, U^n, W^n, Z^n, \Phi^n, \psi^n) \) then the sequences \((X^n), (Y^n)\) and \((\partial^n f^n)\) are Cauchy sequences respectively in the Banach spaces \((C^0([0, T]; \mathbb{R}))^6 \times C^0([0, T]; H^2_2(\mathbb{R}^3)), (C^1([0, T]; \mathbb{R}))^6 \) and \( C^0([0, T]; H^2_4(\mathbb{R}^3)) \) possibly for smaller \( T \).

Proof: In what follows, the constant \( C \) only depends on \( C_0 \) and \( T \) and may be different from line to line.

1. We first prove that \((X^n)\) is a Cauchy sequence in \((C^0([0, T]; \mathbb{R}))^6 \times C^0([0, T]; H^2_2(\mathbb{R}^3))\).

   - We integrate Equation (5.29) and obtain:

\[
|E^{n+1}(t) - E^n(t)|^2 \leq C \int_0^t \left( |E^n(s) - E^{n-1}(s)|^2 + |U^n(s) - U^{n-1}(s)|^2 \right) ds . \tag{6.12}
\]

   - From Equation (5.30) we have:

\[
U^{n+1}(t) - U^n(t) = - \int_0^t \left[ \frac{3}{2} \left( (U^n(s))^2 - (U^{n-1}(s))^2 \right) + 2 \pi \left( (Z^n)^2(s) - (E^n)^2(s) \right) - (Z^{n-1})^2(s) \right] ds .
\]

which implies (since 6.2) that

\[
|U^{n+1}(t) - U^n(t)| \leq C \int_0^t \left[ |E^n(s) - E^{n-1}(s)| + |U^n(s) - U^{n-1}(s)| + |Z^n(s) - Z^{n-1}(s)| + |\Phi^n(s) - \Phi^{n-1}(s)| + |\psi^n(s) - \psi^{n-1}(s)| + \| f^n(s, \cdot) - f^{n-1}(s, \cdot) \|_{H^2_2(\mathbb{R}^3)} \right] ds .
\]

Thus

\[
|U^{n+1}(t) - U^n(t)|^2 \leq C \int_0^t \left[ |E^n(s) - E^{n-1}(s)|^2 + |U^n(s) - U^{n-1}(s)|^2 + |Z^n(s) - Z^{n-1}(s)|^2 + |\Phi^n(s) - \Phi^{n-1}(s)|^2 + |\psi^n(s) - \psi^{n-1}(s)|^2 + \| f^n(s, \cdot) - f^{n-1}(s, \cdot) \|_{H^2_2(\mathbb{R}^3)}^2 \right] ds . \tag{6.13}
\]
Integrating Equation (5.31) gives:

$$\frac{1}{2} \int_0^t (|U^n(s) - U^{n-1}(s)|^2 + |W^n(s) - W^{n-1}(s)|^2) \, ds \geq C \int_0^t (|U^n(s) - U^{n-1}(s)|^2 + |W^n(s) - W^{n-1}(s)|^2) \, ds .$$  \hfill (6.14)

Similarly, Equation (5.32) gives:

$$\frac{1}{2} \int_0^t (|Z^n(s) - Z^{n-1}(s)|^2) \, ds \leq C \int_0^t (|U^n(s) - U^{n-1}(s)|^2 + |Z^n(s) - Z^{n-1}(s)|^2) \, ds .$$  \hfill (6.15)

Now, Equation (5.33) gives:

$$\frac{1}{2} \int_0^t \left( \frac{1}{2} \psi^n(s) - \frac{1}{2} \psi^{n-1}(s) \right) \, ds \geq C \int_0^t \left( \frac{1}{2} \psi^n(s) - \frac{1}{2} \psi^{n-1}(s) \right) \, ds .$$

In order to get rid of the denominator of the right hand side, we use Equation (5.34) which we recall is $\psi^{n+1} = -6U^n \psi^n - m^2 \Phi^n \sqrt{2 \psi^n} - \alpha^2$. Since the sequence $(X^n)$ is uniformly bounded, on $[0, T]$, we have $| -6U^n \psi^n - m^2 \Phi^n \sqrt{2 \psi^n} - \alpha^2 | \leq C_1$ and thus,

$$\frac{d\psi^{n+1}}{dt} \geq -C_1 .$$

The last inequality implies that $\psi^{n+1}(t) \geq \psi_0 - C_1 t$. Since $\psi_0 > 0$, we take $T$ sufficiently small so that $0 \leq C_1 t \leq \frac{\psi_0}{2}$, $0 \leq t \leq T$ and obtain that $\psi^{n+1}(t) \geq \frac{\psi_0}{2}$ and $\frac{1}{\sqrt{2\psi^{n+1}(t)}} \leq \frac{1}{\sqrt{\psi_0}}$. From there, we deduce that there exists a constant $C > 0$ such that:

$$\frac{1}{2} \int_0^t \left( \frac{1}{2} \psi^n(s) - \frac{1}{2} \psi^{n-1}(s) \right) \, ds \leq C(C_0) \int_0^t |\psi^n(s) - \psi^{n-1}(s)|^2 \, ds .$$  \hfill (6.16)

From Equation (5.34) we have:

$$\psi^{n+1}(t) - \psi^n(t) = \int_0^t \left( \frac{1}{2} \psi^n(s) - \frac{1}{2} \psi^{n-1}(s) \right) \, ds .$$

Since $\frac{1}{\sqrt{2\psi^{n+1}(t)}} \leq \frac{1}{\sqrt{\psi_0}}$, we deduce that there exists a constant $C > 0$ such that:

$$|\psi^{n+1}(t) - \psi^n(t)|^2 \leq C \int_0^t \left( |U^n(s) - U^{n-1}(s)|^2 + |\Phi^n(s) - \Phi^{n-1}(s)|^2 + |\psi^n(s) - \psi^{n-1}(s)|^2 \right) \, ds .$$  \hfill (6.17)

Finally, from Equation (5.35) we get:

$$\frac{\partial (f^{n+1} - f^n)}{\partial t} + \left( E^n Z^n \int_{\mathbb{R}^3} f^n(t, \tau) \, d\tau \right) \sum_{i=1}^3 \frac{\partial (f^{n+1} - f^n)}{\partial u^i}$$

$$= \left( E^n Z^n \int_{\mathbb{R}^3} f^n(t, \tau) \, d\tau - E^{n-1} Z^{n-1} \int_{\mathbb{R}^3} f^{n-1}(t, \tau) \, d\tau \right) \sum_{i=1}^3 \frac{\partial f^n}{\partial u^i} + \frac{1}{u^0_n} \hat{Q}_n - \frac{1}{u^0_{n-1}} \hat{Q}_{n-1} .$$

But since the sequence $(X^n)$ is bounded, we have:

$$\left| E^n(t) Z^n(t) \int_{\mathbb{R}^3} f^n(t, \tau) \, d\tau - E^{n-1}(t) Z^{n-1}(t) \int_{\mathbb{R}^3} f^{n-1}(t, \tau) \, d\tau \right|$$

$$= \left| (E^n(t) - E^{n-1}(t)) Z^n(t) \int_{\mathbb{R}^3} f^n(t, \tau) \, d\tau + (Z^n(t) - Z^{n-1}(t)) E^{n-1} \int_{\mathbb{R}^3} f^n(t, \tau) \, d\tau + E^{n-1}(t) Z^{n-1}(t) \int_{\mathbb{R}^3} (f^n - f^{n-1})(t, \tau) \, d\tau \right|$$

$$\leq C \left( |E^n(t) - E^{n-1}(t)| + |Z^n(t) - Z^{n-1}(t)| + ||f^n(t, \tau) - f^{n-1}(t, \tau)||_{L^2(\mathbb{R}^3)} \right) .$$
Since \( f^n \in H^2_d(\mathbb{R}^3) \) the second term in the r.h.s. of Equation (6.18) is an element of \( H^2_d(\mathbb{R}^3) \) and then,

\[
\begin{align*}
\left\| \left( E^n(t)Z^n(t) \right) \int_{\mathbb{R}^3} f^n(t, \varphi) d\sigma - E^{-1}(t)Z^{-1}(t) \right\| \left( \int_{\mathbb{R}^3} f^n(t, \varphi) d\sigma \right) \sum_{i=1}^{3} \partial f^n(t, \cdot) \right\|_{H^2_d(\mathbb{R}^3)} \\
\leq C \left( \left| E^n(t) - E^{-1}(t) \right| + \left| Z^n(t) - Z^{-1}(t) \right| + \left\| f^n(t, \cdot) - f^{-1}(t, \cdot) \right\|_{H^2_d(\mathbb{R}^3)} \right). \quad (6.18)
\end{align*}
\]

Further (see inequality (6.13) of Lemma 6.1),

\[
\left\| \frac{1}{u_{n}} \hat{Q}_{n} - \frac{1}{u_{n-1}} \hat{Q}_{n-1} \right\|_{H^2_d(\mathbb{R}^3)} \leq C \left( \left| E^n - E^{-1} \right| + \left\| f^n - f^{-1} \right\|_{H^2_d(\mathbb{R}^3)} \right).
\]

Equation (6.18) has a form to which Corollary 4.1 applies. Thus, for \( k = 2 \) and \( u = f^{n+1} - f^n \) in Inequality (4.30) and using the last two estimates, we have:

\[
\| f^{n+1}(s, \cdot) - f^n(s, \cdot) \|^2_{H^2_d(\mathbb{R}^3)} \leq C \int_0^t \left( \left| E^n(s) - E^{-1}(s) \right|^2 + \left| Z^n(s) - Z^{-1}(s) \right|^2 + \left\| f^n(s, \cdot) - f^{-1}(s, \cdot) \right\|^2_{H^2_d(\mathbb{R}^3)} \right) ds. \quad (6.19)
\]

Consider the space

\[
\Sigma := \left( \mathcal{C}([0, T], \mathbb{R}) \right)^6 \times \mathcal{C}([0, T], H^2_d(\mathbb{R}^3)).
\]

Endowed with the norm

\[
\| X \| := \sum_{i=1}^{6} \sup_{0 \leq t \leq T} \| X_i(t) \| + \sup_{0 \leq t \leq T} \| X_7(t) \|_{H^2_d(\mathbb{R}^3)}
\]

where \( X = (X_i)_{1 \leq i \leq 7} \in \Sigma \), \( \Sigma \) is a Banach space. We want to show that there exists a constant \( 0 < \alpha < 1 \) which depends only upon \( C_0 \) and \( T \) such that \( \| X^{n+1} - X^n \| \leq \alpha \| X^n - X^{n-1} \| \) if \( T \) is small enough. Summing up inequalities (6.12 6.17) and (6.19) gives:

\[
\begin{align*}
&\left| E^{n+1}(t) - E^n(t) \right|^2 + \left| U^{n+1}(t) - U^n(t) \right|^2 + \left| W^{n+1}(t) - W^n(t) \right|^2 + \left| Z^{n+1}(t) - Z^n(t) \right|^2 \\
&\quad + \left| \Phi^{n+1}(t) - \Phi^n(t) \right|^2 + \left| \psi^{n+1}(t) - \psi^n(t) \right|^2 + \left\| f^{n+1}(t, \cdot) - f^n(t, \cdot) \right\|^2_{H^2_d(\mathbb{R}^3)} \\
&\leq C \int_0^t \left( \left| E^n(s) - E^{-1}(s) \right|^2 + \left| U^n(s) - U^{-1}(s) \right|^2 + \left| W^n(s) - W^{-1}(s) \right|^2 \\
&\quad + \left| Z^n(s) - Z^{-1}(s) \right|^2 + \left| \Phi^n(s) - \Phi^{-1}(s) \right|^2 + \left| \psi^n(s) - \psi^{-1}(s) \right|^2 \\
&\quad + \left\| f^n(s, \cdot) - f^{-1}(s, \cdot) \right\|^2_{H^2_d(\mathbb{R}^3)} \right) ds. \quad (6.20)
\end{align*}
\]

This last inequality implies that

\[
\| X^{n+1} - X^n \| \leq \sqrt{C(C_0)T} \| X^n - X^{n-1} \|. \quad (6.21)
\]

Now, we choose \( T \) small enough such that \( C(C_0)T \leq 1 \), we obtain from (6.21) that the sequence \( (X^n) \) is a Cauchy sequence in the Banach space \( \Sigma \).

2. Next, we show that \( \left( \frac{dY^n}{dt} \right) \) is a Cauchy sequence in \( \left( C^0([0, T], \mathbb{R}) \right)^6 \).

Since the sequence \( (X^n) \) is bounded, from equations (6.20 6.21) we deduce that there exists a constant \( C > 0 \) which only depends upon \( C_0 \) and \( T \) such that:

\[
\left\| \frac{dY^{n+1}}{dt} - \frac{dY^n}{dt} \right\| \leq C \left\| Y^n - Y^{n-1} \right\|. \quad (6.22)
\]

25
Note that \( ||Y^n - Y^{n-1}|| \leq ||X^n - X^{n-1}|| \), thus inequality (6.21) shows that:

\[
\left| \frac{dY^{n+1}}{dt} - \frac{dY^n}{dt} \right| \leq (C_2 \sqrt{C(C_0)T})^{n-1} ||X^1 - X^0||. \tag{6.23}
\]

As we did before, the constant \( C_2 \sqrt{C(C_0)T} \) can be assumed to be less than one even if it means to shrink \( T \) again. This shows that \( \left( \frac{dY^n}{dt} \right) \) is a Cauchy sequence and thus, \( (Y^n) \) is a Cauchy sequence in \( (C^1([0,T], \mathbb{R})) \).

3. Let us now prove that \( \left( \frac{d\tilde{f}^n}{dt} \right) \) is a Cauchy sequence in \( C^0([0,T]; H^1_d(\mathbb{R}^3)) \). From Equation (6.18) we have:

\[
\left\| \frac{\partial (f^{n+1} - f^n)}{\partial t} \right\|_{H^1_d(\mathbb{R}^3)} \leq C \left\{ \| f^{n+1} - f^n \|_{H^2_d(\mathbb{R}^3)} + \| f^n - f^{n-1} \|_{H^2_d(\mathbb{R}^3)} + |E^n - E^{n-1}| + |Z^n - Z^{n-1}| \right\}.
\]

Thus,

\[
\left\| \frac{\partial (f^{n+1} - f^n)}{\partial t} \right\|_{C^0([0,T]; H^1_d(\mathbb{R}^3))} \leq C \left\{ \| X^{n+1} - X^n \| + \| X^n - X^{n-1} \| \right\} \tag{6.24}
\]

\[
\leq C \alpha^n \| X^1 - X^0 \|. \tag{6.25}
\]

which shows \( \left( \frac{d\tilde{f}^n}{dt} \right) \) is a Cauchy sequence in \( C^0([0,T]; H^1_d(\mathbb{R}^3)) \) and ends the proof.

\[ \square \]

**Remark 6.1** Note that \( (f^n) \) is a Cauchy sequence in \( C^0([0,T]; H^2_d(\mathbb{R}^3)) \cap C^1([0,T]; H^1_d(\mathbb{R}^3)) \).

**Remark 6.2** We notice for later use that the sequence \( \left( \frac{d\tilde{f}^n}{dt} \right) \) is uniformly bounded in \( H^2_d(\mathbb{R}^3) \). In fact we have (recall that \( (X^n) \) is uniformly bounded in \( (C^1([0,T]) \times C^0([0,T]; H^2_d(\mathbb{R}^3))) \):

\[
\left\| \frac{\partial f^n}{\partial t}(t, \cdot) \right\|_{H^2_d(\mathbb{R}^3)} = \left\| (E^{n-1} Z^{n-1} \int_{\mathbb{R}^3} f^{n-1}(t, \varpi) d\varpi, \sum_{i=1}^{3} \frac{\partial f^n}{\partial x^i} + \frac{1}{u_{n-1}^3} \tilde{Q}_{n-1} \right\|_{H^2_d(\mathbb{R}^3)}
\]

\[
\leq \left( (E^{n-1} Z^{n-1} \| f^{n-1} \|_{H^2_d(\mathbb{R}^3)}) \| f^n \|_{H^2_d(\mathbb{R}^3)} + \| f^{n-1} \|^2_{H^2_d(\mathbb{R}^3)} \right)
\]

\[
\leq C(C_0). \]

We are now ready to prove existence and uniqueness of the solution of the Einstein-Maxwell-Boltzmann-massive scalar field system with data described in [5,19]. Before stating the main theorem of this paper let us recall some classical facts about Sobolev spaces \( H^{(s)} \) with real \( s \) (see [12]).

**Definition 6.1** Let \( n \) be a positive integer, \( s \) a real number. Denote by \( S^l(\mathbb{R}^n) \) the set of all temperate distributions on \( \mathbb{R}^n \). We say that \( u \in H^{(s)}(\mathbb{R}^n) \) if its Fourier transform \( \hat{u} \) is a measurable function such that \( (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \) is square integrable. If \( u \in H^{(s)}(\mathbb{R}^n) \) we define the norm

\[
\| u \|_{(s)} := \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.
\]

**Remark 6.3** \( H^{(s)}(\mathbb{R}^n) \) is a Hilbert space and we have \( H^{(s)}(\mathbb{R}^n) \hookrightarrow H^{(r)}(\mathbb{R}^n) \) for \( s \leq r \). Further, \( H^{(0)}(\mathbb{R}^n) = L^2(\mathbb{R}^n) \) and \( H^{(s)}(\mathbb{R}^n) \equiv H^s(\mathbb{R}^n) \) when \( s \) is an integer.
We recall now an important inequality of functional analysis called interpolation inequality.

**Lemma 6.3** Let $s_1 < s_2 < s_3$ be real numbers and assume that $u \in H^{(s_3)}(\mathbb{R}^n)$. Then we have the following inequality

$$
\|u\|_{(s_2)} \leq \|u\|^{s_2-s_1}_{(s_1)} \times \|u\|^{s_3-s_2}_{(s_3)}. 
$$

(6.26)

**Theorem 6.1** Let $r$ and $d$ be two positive real numbers such that $d > 5/2$. The system of partial differential equations \[5.12\] - \[5.18\] with Cauchy data $(E_0, U_0, W_0, Z_0, \Phi_0, \psi_0, f_0)$ satisfying the Hamiltonian constraint \[7.7\] and which are such that

$$
E_0 > 0, \quad W_0 < 0, \quad \Phi_0 > 0, \quad \psi_0 > 0, \quad f_0 > 0 \quad \text{and} \quad f_0 \in H^3_d,\mathbb{R}^3
$$

has a unique local (in time) solution $(E, U, W, Z, \Phi, \psi, f)$ defined on a set $[0, T]$ (for a positive constant $T$ which only depends on $d$ and the size of the data) such that

$$(E, U, W, Z, \Phi, \psi, f) \in (C^1([0, T]))^6 \times C^1([0, T] \times \mathbb{R}^3).$$

Furthermore,

$$f \in C^0([0, T]; H^3_d,\mathbb{R}^3).$$

(6.28)

Consequently, the coupled system Einstein-Maxwell-Boltzmann-massive scalar field has unique local solution.

**Remark 6.4** Property\[6.28\] will be the key property when deriving global solutions for small data since we will use the continuity argument.

**Proof:** 1)- Existence: As the first step towards existence of solution we prove that the sequence $(X^n)$ converges in the space $(C^1([0, T]))^6 \times C^1([0, T] \times \mathbb{R}^3)$. From Proposition\[6.2\] we know that the sequence $(Y^n)_n$ is a Cauchy sequence in the Banach space $(C^1([0, T]; \mathbb{R}))^6$ thus there exists a set of functions $Y = (E, U, W, Z, \Phi, \psi)$ such that $(Y^n)$ converges towards $Y$ in $(C^1([0, T]; \mathbb{R}))^6$. Secondly, Proposition\[6.2\] also tells us that $(f^n)_n$ is a Cauchy sequence in the Banach space $C^0([0, T]; H^3_d,\mathbb{R}^3))$. It follows that there exists a function $f$ such that $(f^n)_n$ converges to $f$ in the space $C^0([0, T]; H^2_d,\mathbb{R}^3))$. But the space $C^0([0, T]; H^2_d,\mathbb{R}^3))$ embeds continuously in $C^0([0, T]; H^2,\mathbb{R}^3))$, therefore $(f^n)$ is a Cauchy sequence in $C^0([0, T]; H^2,\mathbb{R}^3))$. Now from interpolation inequality \[6.20\] for any real number $2 < s < 3$ we have:

$$
\|f^n(t, \cdot) - f^p(t, \cdot)\|_{(s)} \leq \|f^n(t, \cdot) - f^p(t, \cdot)\|_{H^3,\mathbb{R}^3} \times \|f^n(t, \cdot) - f^p(t, \cdot)\|_{H^2,\mathbb{R}^3}.
$$

(6.29)

Since $(f^n(t, \cdot))_n$ is a uniformly bounded in $H^3_d,\mathbb{R}^3$ and then in $H^3,\mathbb{R}^3$, inequality \[6.20\] shows the $(f^n)_n$ is a Cauchy sequence in $C^0([0, T]; H^2,\mathbb{R}^3))$ for any $2 < s < 3$.

Similarly, since the sequence $\left(\frac{\partial f^n}{\partial t}\right)$ is uniformly bounded (see Remark\[6.2\]) and is a Cauchy sequence in $C^0([0, T]; H^1_d,\mathbb{R}^3))$, the interpolation inequality shows that it is a Cauchy sequence in $C^0([0, T]; H^{(s)},\mathbb{R}^3))$ for any $1 < s < 2$. We then obtain that

$$(f^n) \text{ is a Cauchy sequence in } C^0([0, T]; H^{(s+1)},\mathbb{R}^3) \cap C^1([0, T]; H^{(s)},\mathbb{R}^3)); \quad 1 < s < 2.$$

(6.30)

Now, from Sobolev embedding inequality we know that

$$
C^0([0, T]; H^{(s+1)},\mathbb{R}^3) \cap C^1([0, T]; H^{(s)},\mathbb{R}^3)) \hookrightarrow C^1_b([0, T] \times \mathbb{R}^3) \quad \text{for any } \quad s > \frac{3}{2}.
$$

(6.31)

Therefore choosing a particular $s$ in \[6.30\] such that $\frac{3}{2} < s < 2$ shows that $(f^n)$ is a Cauchy in $C^1_b([0, T] \times \mathbb{R}^3)$ and thus converges towards a function $\tilde{f}$ in $C^1_b([0, T] \times \mathbb{R}^3)$ and the embedding $C^0([0, T]; H^3_d,\mathbb{R}^3)) \hookrightarrow C^0([0, T] \times \mathbb{R}^3)$ shows that $f = \tilde{f}$. This shows that the collection of functions $X = (E, U, W, Z, \Phi, \psi, f)$ is the limit of the sequence $(X^n)$ in the space $(C^1_b([0, T]))^6 \times C^1([0, T] \times \mathbb{R}^3)$.  

27
As the second step towards existence, we now prove that \( X \) is indeed a solution of \( 5.12 - 5.18 \). Since \((Y^n)\) converges towards \( Y = (E, U, W, Z, \Phi, \psi) \) in \((C^1([0, T]; \mathbb{R}))^n\) taking the limit pointwise in Equations \( 5.29, 5.31 - 5.34 \) shows that \( E, U, W, Z, \Phi \) and \( \psi \) satisfy \( 5.12 - 5.17 \). It remains to show that integrals
\[
\int_{\mathbb{R}^3} f^n(t, \tau) d\tau \quad \text{and} \quad \int_{\mathbb{R}^3} |\tau|^2 \, f^n(t, \tau) d\tau
\]
converge respectively to
\[
\int_{\mathbb{R}^3} f(t, \tau) d\tau \quad \text{and} \quad \int_{\mathbb{R}^3} |\tau|^2 \, f(t, \tau) d\tau, \quad \forall t \in [0, T]
\]
as \( n \) goes to infinity. We notice that these last two integrals are convergent since \( \forall t \in [0, T], \ f(t, \cdot) \in H^2_0(\mathbb{R}^3) \). We have
\[
\int_{\mathbb{R}^3} f^n(t, \tau) d\tau - \int_{\mathbb{R}^3} f(t, \tau) d\tau \leq \int_{\mathbb{R}^3} |f^n(t, \tau) - f(t, \tau)| d\tau \leq \|f^n(t, \cdot) - f(t, \cdot)\|_{H^2_0(\mathbb{R}^3)}.
\]
This shows that \( \int_{\mathbb{R}^3} f^n(t, \tau) d\tau \to \int_{\mathbb{R}^3} f(t, \tau) d\tau, \quad \forall t \in [0, T] \). Similarly, from \( 6.2 \) we have
\[
\int_{\mathbb{R}^3} |\tau|^2 \, f^n(t, \tau) d\tau - \int_{\mathbb{R}^3} |\tau|^2 \, f(t, \tau) d\tau \leq C (\|E^n - E\| + \|f^n(t, \cdot) - f(t, \cdot)\|_{H^2_0(\mathbb{R}^3)}) \to 0.
\]
Finally let us prove that \( \frac{1}{u_n} \tilde{Q}(f^n, f^n) \to \frac{1}{u} \tilde{Q}(f, f) \). As we did in the proof of \( 6.3 \) we have
\[
\left\| \frac{1}{u_n} \tilde{Q}(f^n, f^n) - \frac{1}{u} \tilde{Q}(f, f) \right\|_{H^2_0(\mathbb{R}^3)} \leq C(T) \left( \|f^n\|_{H^2_0(\mathbb{R}^3)} + \|f\|_{H^2_0(\mathbb{R}^3)} \right) (\|E^n - E\| + \|f^n - f\|_{H^2_0(\mathbb{R}^3)}) \leq C(T) \left( C_0 + \|f\|_{H^2_0(\mathbb{R}^3)} \right) (\|E^n - E\| + \|f^n - f\|_{H^2_0(\mathbb{R}^3)}).
\]
Thus, \( \frac{1}{u_n} \tilde{Q}_n \) converges towards \( \frac{1}{u} \tilde{Q}(f, f) \) in the space \( H^2_0(\mathbb{R}^3) \) and since \( H^2_0(\mathbb{R}^3) \hookrightarrow C^0_b(\mathbb{R}^3) \) it follows that \( \frac{1}{u_n} \tilde{Q}_n \) converges towards \( \frac{1}{u} \tilde{Q}(f, f) \) in \( C^0_b(\mathbb{R}^3) \).

We have thus proved that the limit can also be taken pointwise in the remaining Equations \( 5.30, 5.34 \) to obtain that \( X = (E, U, W, Z, \Phi, \psi, f) \) also satisfies \( 5.13 \) and \( 5.18 \). Therefore \( X \) is a local solution of \( 5.12 - 5.18 \).

2) Uniqueness: Suppose that there exists two sets of functions \( X_i = (E_i, U_i, W_i, Z_i, \Phi_i, \psi_i, f_i), \ i = 1, 2 \) which solve the system \( 5.12 - 5.18 \) with the same Cauchy data. We proceed exactly as we did in the proof of Proposition \( 6.2 \) and obtain an estimate of the form \( 6.23 \) where \( X^{n+1} - X^n \) and \( X^n - X^{n-1} \) replaced by \( X_2 - X_1 \). Applying Gronwall’s inequality to the obtained estimate proves that \( X_1 = X_2 \).

3) Now, we prove that \( \forall t \in [0, T], \ f(t, \cdot) \in H^2_0(\mathbb{R}^3) \). Recall, the sequence \( (f^n(t, \cdot)) \) is uniformly bounded in the Hilbert space \( H^3_0(\mathbb{R}^3) \). But it is well known that any bounded sequence in a Hilbert space has a weakly convergent subsequence (see for example [3], Theorem 5.4.2 page 151). Therefore there exists a subsequence \( (f^n(t, \cdot))_n \) and a function \( g(t, \cdot) \in H^3_0(\mathbb{R}^3) \) such that \( (f^n(t, \cdot)) \) converges to \( g(t, \cdot) \) in \( H^3_0(\mathbb{R}^3) \) endowed with its weak topology which continuously embeds in \( H^2_0(\mathbb{R}^3) \) endowed with its weak topology. Moreover, recall again, \( (f^n(t, \cdot))_n \) converges to \( f(t, \cdot) \) in the space \( H^2_0(\mathbb{R}^3) \) thus this convergence also holds in \( H^2_0(\mathbb{R}^3) \) endowed with its weak topology. Since the weak topology is Hausdorff, we thus conclude that \( f(t, \cdot) = g(t, \cdot) \) and then \( f(t, \cdot) \in H^2_0(\mathbb{R}^3) \). Note that
\[
\|f(t, \cdot)\|_{H^2_0(\mathbb{R}^3)} \leq \liminf \|f^n(t, \cdot)\|_{H^2_0(\mathbb{R}^3)} \leq C, \quad \text{uniformly in } t.
\]

4) Let us show that \( 6.25 \) holds. We proceed as in [12]. The following Lemma will be needed, its proof is given at the end of the paper.

**Lemma 6.4** The space \( C_\infty^\infty(\mathbb{R}^3) \) of Compactly supported smooth functions defined on \( \mathbb{R}^3 \) is dense in the space \( H^2_0(\mathbb{R}^3) \).
Weak continuity: First, let us prove that the solution is weakly continuous. Let $F$ be in the dual of $H_0^3(\mathbb{R}^3)$. Then, due to the Riesz representation theorem there exists $\varphi_F \in H_0^3(\mathbb{R}^3)$ such that for all $h \in H_0^3(\mathbb{R}^3)$,

$$F(h) = \langle h, \varphi \rangle_{H_0^3(\mathbb{R}^3)} = \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha h(\vec{v}) \cdot (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha \varphi_F(\vec{v}) d\vec{v}.$$  

We then have,

$$F(f^n(t, \cdot)) - F(f(t, \cdot)) = \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha f^n(1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha \varphi_F d\vec{v} - \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha f(1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha \varphi_F d\vec{v}.$$  

Consequently, if $(\varphi_j)$ is a sequence of compactly supported smooth functions converging to $\varphi_F$ in $H_0^3(\mathbb{R}^3)$, we obtain:

$$F(f^n(t, \cdot)) - F(f(t, \cdot)) = \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha (f^n - f)(t, \vec{v}) \cdot (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha (\varphi_F - \varphi_j)(\vec{v}) d\vec{v}$$

$$+ \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha (f^n - f)(t, \vec{v}) \cdot (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha \varphi_j(\vec{v}) d\vec{v}.$$  

We then deduce since, the sequence $(f^n)$ is bounded in $H_0^3(\mathbb{R}^3)$ that:

$$|F(f^n(t, \cdot)) - F(f(t, \cdot))| \leq C\|\varphi_F - \varphi_j\|_{H_0^3(\mathbb{R}^3)} + \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha (f^n - f)(t, \vec{v}) \cdot (1 + |\vec{v}|)^{d+|\alpha|} \partial^\alpha \varphi_j(\vec{v}) d\vec{v}.$$  

Letting $j$ be large enough that the first term on the right-hand side is less than or equal to $\frac{s}{2}$, and then choosing $n$ large enough, depending on $j$, so that the second term is less than $\frac{s}{2}$, we conclude that the right-hand side is less than $\varepsilon$. We conclude that $F(f^n(t, \cdot))$ converges uniformly to $F(f(t, \cdot))$ which proves that the solution $f$ is weakly continuous.

**Strong continuity:** Let $t_0 \in [0, T]$. We want to prove that $f : [0, T] \rightarrow H_0^3(\mathbb{R}^3)$ is continuous at $t_0$ i.e.

$$\lim_{t \rightarrow t_0} \|f(t, \cdot) - f(t_0, \cdot)\|_{H_0^3(\mathbb{R}^3)} = 0.$$  

Using the inner product $\langle \cdot, \cdot \rangle$ on $H_0^3(\mathbb{R}^3)$, we can write for $t \in [0, T]$:

$$\langle f(t, \cdot) - f(t_0, \cdot), f(t, \cdot) - f(t_0, \cdot) \rangle = \langle f(t, \cdot), f(t, \cdot) \rangle - 2\langle f(t, \cdot), f(t_0, \cdot) \rangle + \langle f(t_0, \cdot), f(t_0, \cdot) \rangle.$$  

(6.33)

Note that the last term on the right-hand side is $\|f(t_0, \cdot)\|^2_{H_0^3(\mathbb{R}^3)}$. Due to the weak continuity of $f$, the limit as $t$ goes to $t_0$ of second term on the right-hand side is $-2\|f(t_0, \cdot)\|^2_{H_0^3(\mathbb{R}^3)}$.

For the first term on the right-hand side, we suppose that $t > t_0$ and use the fact that there exists $\delta > 0$ such that (see estimate (4.30))

$$e^{-\delta(t-t_0)}\|f(t, \cdot)\|^2_{H_0^3(\mathbb{R}^3)} \leq \|f(t_0, \cdot)\|^2_{H_0^3(\mathbb{R}^3)} + C \int_{t_0}^t e^{-\delta s} \|f(s, \cdot)\|^2_{H_0^3(\mathbb{R}^3)} ds$$  

and the fact that $f(t, \cdot)$ is uniformly bounded (see (6.32)), to have $\lim_{t \rightarrow t_0} \langle f(t, \cdot), f(t, \cdot) \rangle \leq \|f(t_0, \cdot)\|^2_{H_0^3(\mathbb{R}^3)}$.

Combine these observations with (6.33), to have $\lim_{t \rightarrow t_0} \langle f(t, \cdot) - f(t_0, \cdot), f(t, \cdot) - f(t_0, \cdot) \rangle \leq 0$, and conclude, since $\langle f(t, \cdot) - f(t_0, \cdot), f(t, \cdot) - f(t_0, \cdot) \rangle \geq 0$, $\forall t \in [0, T]$ that

$$\lim_{t \rightarrow t_0} \langle f(t, \cdot) - f(t_0, \cdot), f(t, \cdot) - f(t_0, \cdot) \rangle = 0.$$  

29
i.e. \( f: [0, T] \rightarrow H^3_b(\mathbb{R}^3) \) is right continuous on \([0, T]\). By time reversal one obtains left continuity and thus continuity of \( f \).

5)- Finally, since the Einstein-Maxwell-Boltzmann-massive scalar field system is equivalent to the system of first order partial differential equations \([5.12]-[5.18]\), we have therefore proved that the Einstein-Maxwell-Boltzmann-massive scalar field equations have a unique local (in time) solution.

\[ \square \]

7 Global existence of solutions to the Einstein-Maxwell-Boltzmann system with massive scalar field

7.1 The method

In this section, we prove under further assumptions on the data that the local solution obtained in Section 6 is in fact a global solution. We will use the well known continuation criterium (see for example [14, Proposition 1.5 p. 365]) which says that, the breakdown of a classical solution \( u \) of a system of hyperbolic partial differential equations must involve a blow-up of either \( \sup_x |u(t, x)| \) or \( \sup_x |\nabla_x u(t, x)| \). In other words, if the \( C^1 \)-norm of a solution \( u(t, \cdot) \) on an interval \([0, T)\) is uniformly bounded then this solution can be extended beyond \( T \). Let us sketch out the method we adopt. Denote \([0, T_*]\), \( T_* > 0 \), the maximal existence time interval of the solution of system \([5.12]-[5.18]\), with initial data defined by \([5.19]\), subject to the Hamiltonian constraint \([5.7]\) and satisfying \([6.27]\). Assume by contradiction that \( T_* < \infty \) (otherwise \( T_* = +\infty \) and there is nothing to do). Then we will prove using a continuity type argument that the solution \((E, U, W, Z, \Phi, \psi, f)\) is uniformly bounded on \([0, T_*]\) by a constant depending only on the initial data, \( T_* \), \( m \), \( r \) and \( \Lambda \). It will then follow by the continuation criterium that this solution can be extended to a larger time interval \([0, T']\) thus contradicting the maximality of \( T_* \). This will imply that \( T_* = +\infty \) and the solution is global. Before doing this, let us give some useful estimates on the obtained local solution.

7.2 A priori estimates and global solution

**Lemma 7.1** In addition to hypotheses of Theorem [6.1] assume \( U_0 > 0 \) and that the cosmological constant satisfies \( \Lambda > -4\pi m^2 \Phi_0 \). Then the solution \((E, U, W, Z, \Phi, \psi)\) defined on \([0, T_*]\) of system \([6.12]-[6.17]\) with Cauchy data given in \([6.27]\) satisfies the set of inequalities

\[
\begin{cases}
\left( \frac{4}{3} + \frac{2r}{\pi m^2} \Phi^2(0) \right)^{1/2} \leq U(t) \leq U_0; \quad 0 \leq E(t) \leq E_0; \quad 0 \leq \frac{1}{2} + \frac{1}{2}e^{U_0 T} \\
|Z(t)| \leq |Z_0|; \quad |\Phi| \leq \left( \frac{30U_0^6 - \Lambda}{8\pi} \right)^{1/2}; \quad 0 \leq \psi \leq \frac{30U_0^6 - \Lambda}{8\pi}; \quad |W| \leq |W_0| + \frac{\alpha^2}{3U_0} e^{3U_0 T}
\end{cases}
\]

**Proof:** With the change of functions \([5.13]\), the Hamiltonian constraint \([5.1]\) writes :

\[3U^2 - \Lambda = 8\pi E^3 \int_{\mathbb{R}^3} v^0 f(t, \tau) d\tau + 12\pi \frac{Z^2}{E^2} - 8\pi W + 4\pi (2\psi + m^2 \Phi^2). \tag{7.2}\]

Taking \((-2) \times [5.13] + [7.2]\) gives :

\[\dot{U} = -4\pi \left[ E^5 \int_{\mathbb{R}^3} \frac{(v^1)^2}{v^0} f(t, \tau) d\tau + E^3 \int_{\mathbb{R}^3} v^0 f(t, \tau) d\tau \right] - 8\pi \frac{E^2}{Z^2} + 4\pi W - 8\pi \psi. \tag{7.3}\]

But since by \([5.20]\) we have \( W \leq 0 \), \([7.3]\) implies that \( \dot{U} \leq 0 \). So \( U \) is decreasing. The Hamiltonian constraint \([7.2]\) implies, since \( \Phi^2 \) is increasing :

\[U^2 \geq \frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2_0. \tag{7.4}\]

30
But by hypothesis, $\Lambda \geq -4\pi m^2 \Phi_0^2$ thus, (7.3) is equivalent to:
\[
(U - \sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)})(U - \sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)}) \geq 0 ;
\]
which implies:
\[
U \leq -\sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)} \quad \text{or} \quad U \geq \sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)} .
\]
Now, from our hypotheses, $U_0 > 0$ and since $U$ is a continuous function we will only have:
\[
U \geq \sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)} , \quad (7.5)
\]
and then ($U$ is decreasing):
\[
\sqrt{\frac{\Lambda}{3} + \frac{4\pi}{3} m^2 \Phi^2(0)} \leq U \leq U_0 . \quad (7.6)
\]
By (7.6), $U$ is bounded. Now we have by (5.12) : $\dot{E} = -UE$ since $E = \frac{1}{a}$ $\geq 0$ and $U > 0$, this implies $\dot{E} < 0$
so $E$ is decreasing and:
\[
0 \leq E \leq E_0 = \frac{1}{a_0} . \quad (7.7)
\]
Now by (5.15) we have $\dot{Z} = -3UZ$; since $U \geq 0$ and bounded, we deduce that :
\[
|Z(t)| \leq |Z_0| . \quad (7.8)
\]
Moreover, Equation (5.12) gives after integration on $[0, t], t > 0$:
\[
0 \leq \frac{1}{E(t)} \leq \frac{1}{E_0} e^{U_0 t} \leq \frac{1}{E_0} e^{U_0 T} .
\]
Further, the Hamiltonian constrain (7.2) implies
\[
8\pi \psi \leq 3U^2 - \Lambda \quad \text{and} \quad 4\pi m^2 \Phi^2 \leq 3U^2 - \Lambda .
\]
Since $0 \leq U(t) \leq U_0$ and $\Phi, \psi > 0$, we have
\[
0 < \psi \leq \frac{3U_0^2 - \Lambda}{8\pi} \quad \text{and} \quad 0 < \Phi \leq \sqrt{\frac{3\Psi_0^2 - \Lambda}{4\pi m^2}} .
\]
Finally, integrating on $[0, t]$ Equation (5.14) gives:
\[
W(t) = e^{-\int_0^t 3U(s) ds} \left[ W(0) - \alpha^2 \int_0^t e^{\int_0^\tau 3U(\tau) d\tau} d\tau \right] .
\]
Once more we use $0 \leq U \leq U_0$ and get:
\[
|W(t)| \leq |W(0)| + \alpha^2 \int_0^t e^{3U_0 s} ds \leq |W_0| + \alpha^2 \frac{e^{U_0 t}}{3U_0} ;
\]
and the proof is complete. $\square$

We are now ready to state and prove existence of a unique global solution of the Einstein-Maxwell-Boltzmann equations with massive scalar field and cosmological constant.

31
The hypotheses are those of Lemma 7.1. If the real number \( r \) (the norm of \( f_0 \) in the space \( H^3_\delta(\mathbb{R}^3) \)) is small enough then the local solution obtained in Theorem 6.1 is global. Consequently, the Einstein-Maxwell-Boltzmann equations with massive scalar field and cosmological constant have a unique global solution for small data.

**Proof:** As we mentioned earlier it will suffice to prove that the \( C^1 \)-norm of the set of functions \( X = (E, U, W, Z, \Phi, \psi, f) \) is uniformly bounded on \([0, T_*] \). Since from Lemma 7.1, the function \( Y = (E, U, W, Z, \Phi, \psi) \) is bounded it will suffice to prove that if \( r \) is sufficiently small then there exists a constant \( M > 0 \) such that

\[
\sup_{t \in [0, T_\ast)} \|e^{-\delta t / 2} f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq M.
\]

We use a continuity argument. By hypothesis \( f_0 \in H^3_{d, r}(\mathbb{R}^3) \) thus \( \|f_0\|_{H^2_\delta(\mathbb{R}^3)} \leq r \) and by continuity

\[
e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq 4 r \tag{7.9}
\]
on a sub-interval of \([0, T_\ast)\). Denote by \([0, T_0] \) the largest time interval on which \((7.9)\) still holds and let us show that if \( r \) is sufficiently small then on the interval \([0, T_0] \) inequality \((7.9)\) implies the same inequality with the constant 4 replaced by 2. It will then follow by continuity that there exists a real \( \epsilon > 0 \) such that \((7.9)\) still holds on \([0, T_0 + \epsilon] \) which contradicts the maximality of \( T_0 \) and then \( T_0 = T_* \).

Recall \( f \) is a \( C^1 \)-solution of the hyperbolic Equation (5.18) which satisfies hypotheses of Corollary (4.1) with \( n = k = 3 \). Therefore inequality \((4.30)\) reads instead:

\[
e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq \|f_0\|_{H^2_\delta(\mathbb{R}^3)}^2 + C_1 \int_0^t e^{-\delta s} \left\| \frac{1}{u_0} Q(f, f) \right\|^2_{H^2_\delta(\mathbb{R}^3)} ds.
\]

This last inequality implies, using \((4.30)\) and \((7.9)\) that

\[
e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq \|f_0\|_{H^2_\delta(\mathbb{R}^3)}^2 + 16C_1 r^2 \int_0^t e^{\delta s} \cdot e^{-\delta t / 2} \|f(s, \cdot)\|_{H^2_\delta(\mathbb{R}^3)}^2 ds.
\]

Using again Gronwall’s inequality, one has:

\[
e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq \|f_0\|_{H^2_\delta(\mathbb{R}^3)}^2 e^{16C_1 r^2 \cdot \frac{\epsilon t \cdot \eta \cdot 70 - 1}{\eta}} \leq r^2 e^{16C_1 r^2 \cdot \frac{\epsilon t \cdot \eta \cdot 70 - 1}{\eta}}.
\]

Note that \( \lim_{r \to 0^+} e^{16C_1 r^2 \cdot \frac{\epsilon t \cdot \eta \cdot 70 - 1}{\eta}} = 1 \) thus there exists a small \( r_0 > 0 \) such that

\[
0 < r \leq r_0 \quad \Rightarrow \quad e^{16C_1 r^2 \cdot \frac{\epsilon t \cdot \eta \cdot 70 - 1}{\eta}} \leq 4 \quad \text{i.e.} \quad e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq 2 r.
\]

Now fix a real number \( r \) such that \( 0 < r \leq r_0 \) and pick \( f_0 \in H^3_{d, r}(\mathbb{R}^3) \) then \( \sup_{t \in [0, T_0]} e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq 2 r \).

By continuity, there exists a real \( \epsilon > 0 \) such that \( \sup_{t \in [0, T_0 + \epsilon]} e^{-\delta t / 2} \|f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq 4 r \) which contradicts the maximality of \( T_0 \) and then \( T_0 = T_* \). We have thus proved that

\[
\sup_{t \in [0, T_\ast]} \|e^{-\delta t / 2} f(t, \cdot)\|_{H^2_\delta(\mathbb{R}^3)} \leq 4 r. \tag{7.10}
\]

This proves that the \( C^1 \)-norm of \( f \) and then of \( X = (E, U, W, Z, \Phi, \psi, f) \) is uniformly bounded on \([0, T_*] \) and from the continuation criterium can be extended as a \( C^1 \)-solution beyond \( T_* \) which in turn contradicts the maximality of \( T_* \) and thus \( T_* = +\infty \) i.e. the solution is global. □
A Proof of Lemma 6.4

Lemma A.1 The space $C_c^\infty(\mathbb{R}^3)$ of Compactly supported smooth functions defined on $\mathbb{R}^3$ is dense in the space $H^k_0(\mathbb{R}^3)$.

Proof: The proof follows closely that of the density of $C_c^\infty(\mathbb{R}^3)$ in the usual Sobolev space $H^k(\mathbb{R}^3)$ and will be done in two steps.

First step:
Set $\mathcal{T} = \{ f \in H^3_0(\mathbb{R}^3); \text{supp}(f) \text{ compact} \}$ and let us show that $\mathcal{T}$ is dense in $H^3_0(\mathbb{R}^3)$. Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that $\varphi(x) = 1$ for $|x| \leq 1$; $0 \leq \varphi \leq 1$ and $\text{supp}(\varphi) \subset B_{\mathbb{R}^3}(0,2)$. For every integer $j \geq 1$ set $\varphi_j(x) = \varphi\left(\frac{x}{j}\right)$ then,

$$\varphi_j \in C_c^\infty(\mathbb{R}^3), \ 0 \leq \varphi_j \leq 1, \ \text{supp} \varphi_j \subset B(0,2j).$$

Moreover, $\forall \alpha \in \mathbb{N}^3$, $D^\alpha \varphi_j$ is uniformly bounded; more precisely:

$$\forall j \in \mathbb{N}^*, \ |D^\alpha \varphi_j| \leq \frac{C_\alpha}{j^{|\alpha|}} \leq C_\alpha \text{ where } C_\alpha = \sup_{x \in \mathbb{R}^3} |D^\alpha \varphi(x)|.$$

Since $\varphi_j(x) = 1$ for $|x| \leq j$, the sequence $(\varphi_j)_j$ converges pointwise to 1 as $j$ goes to $\infty$. Now let $f \in H^3_0(\mathbb{R}^3)$ and set $f_j = \varphi_j f$. Since $\varphi_j \in C_c^\infty(\mathbb{R}^3)$, $f_j \in \mathcal{T}$. Let us prove that $(f_j)$ converge towards $f$ in $H^3_0(\mathbb{R}^3)$.

We have

$$\|f - f_j\|_{H^3_0(\mathbb{R}^3)} = \|f - \varphi_j f\|_{H^3_0(\mathbb{R}^3)} = \left( \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} (1 + |x|)^{2d + 2|\alpha|} |D^\alpha (f(1 - \varphi_j))|^2 \, dx \right)^{\frac{1}{2}}. \quad (A.1)$$

From Leibniz's formula and the convexity of the function $t \mapsto t^2$,

$$|D^\alpha(f(1 - \varphi_j))| \leq \sum_{|\beta| \leq |\alpha|} C_\beta^\alpha |D^\alpha - D^\beta f| |D^\beta(1 - \varphi_j)| \leq \left( \sum_{|\beta| \leq |\alpha|} (C_\beta^\alpha)^2 \right)^{\frac{1}{2}} \left( \sum_{|\beta| \leq |\alpha|} |D^\alpha - D^\beta f|^2 |D^\beta(1 - \varphi_j)|^2 \right)^{\frac{1}{2}};$$

thus,

$$(1 + |x|)^{2d + 2|\alpha|} |D^\alpha(f(1 - \varphi_j))|^2 \leq C_\alpha \sum_{|\beta| \leq |\alpha|} (1 + |x|)^{2d + 2|\alpha|} |D^\alpha - D^\beta f|^2 |D^\beta(1 - \varphi_j)|^2. \quad (A.2)$$

Note that

$$\sum_{|\beta| \leq |\alpha|} (1 + |x|)^{2d + 2|\alpha|} |D^\alpha - D^\beta f|^2 |D^\beta(1 - \varphi_j)|^2 = (1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2 |1 - \varphi_j|^2$$

$$+ \sum_{1 \leq |\beta| \leq |\alpha|} (1 + |x|)^{2d + 2|\alpha|} |D^\alpha - D^\beta f|^2 |D^\beta(1 - \varphi_j)|^2.$$

i) $\varphi_j \to 1$ and $|1 - \varphi_j| \leq 1 + |\varphi_j| \leq 2$, thus,

$$\left\{ \begin{array}{l}
(1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2 |1 - \varphi_j|^2 \to 0, \text{ as } j \to +\infty \text{ pointwise} \\
(1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2 |1 - \varphi_j|^2 \leq 4(1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2
\end{array} \right.;$$

but $f \in H^3_0(\mathbb{R}^3)$, thus $(1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2$ is integrable and we can thus apply the Lebesgue dominated convergence theorem to obtain

$$\int_{\mathbb{R}^3} (1 + |x|)^{2d + 2|\alpha|} |D^\alpha f|^2 |1 - \varphi_j|^2 dx \to 0 \text{ as } j \to +\infty.$$
ii) Recall that $|D^{\beta} \varphi_j| \leq \frac{C_\varphi}{j^\beta}$, $\forall |\beta| \geq 1$ and $supp \varphi_j \subset B(0, 2j)$, thus,

$$(1 + |x|^{2+2|\alpha|}) |D^{\alpha-\beta} f|^2 |D^{\beta} \varphi_j|^2 \leq C_\varphi^2 \frac{(1 + 2j)^{2|\beta|}}{j^{2|\beta|}} \leq C_\varphi (1 + j^{2|\beta|}) \leq 2C_\varphi$$

and then,

$$\begin{align*}
(1 + |x|)^{2d+2|\alpha|} |D^{\alpha-\beta} f|^2 |D^{\beta} \varphi_j|^2 & \leq (1 + |x|)^{2d+2|\alpha|} |D^{\alpha-\beta} f|^2 \frac{C_\varphi}{j^{\beta}} \rightarrow 0, \text{ as } j \rightarrow +\infty \\
(1 + |x|)^{2d+2|\alpha|} |D^{\alpha-\beta} f|^2 |D^{\beta} \varphi_j|^2 & \leq 2C_\varphi (1 + |x|)^{2d+2|\alpha-\beta|} |D^{\alpha-\beta} f|^2.
\end{align*}$$

Again, $f \in H^3_2(\mathbb{R}^3)$, and then $(1 + |x|)^{2d+2|\alpha-\beta|} |D^{\alpha-\beta} f|^2$ is integrable. We use once more the dominated convergence theorem and obtain that $\forall \beta$ such that $1 \leq |\beta| \leq |\alpha|$,

$$\int_{\mathbb{R}^3} (1 + |x|)^{2d+2|\alpha|} |D^{\alpha-\beta} f|^2 |D^{\beta} \varphi_j|^2 dx \rightarrow 0, \text{ as } j \rightarrow +\infty.$$

and thus

$$\sum_{1 \leq |\beta| \leq |\alpha|} \int_{\mathbb{R}^3} (1 + |x|)^{2d+2|\alpha|} |D^{\alpha-\beta} f|^2 |D^{\beta} \varphi_j|^2 dx \rightarrow 0, \text{ as } j \rightarrow +\infty.$$

Finally we obtain from \((A.2)\) that

$$\forall \alpha, |\alpha| \leq 3, \int_{\mathbb{R}^3} (1 + |x|)^{2d+2|\alpha|} |D^{\alpha} (f(1 - \varphi_j))|^2 dx \rightarrow 0 \text{ as } j \rightarrow +\infty$$

and conclude from \((A.1)\), that $f_j \rightarrow f$ in the topology of $H^3_2(\mathbb{R}^3)$. Thus, $\mathcal{T}$ is dense in $H^3_2(\mathbb{R}^3)$.

### Second step:

We prove that $C^\infty_c(\mathbb{R}^3)$ is dense in $\mathcal{T}$ endowed with the topology of $H^3_2(\mathbb{R}^3)$. Let $\theta$ be a standard mollifier, which means that $\theta$ is a positive $C^\infty$ function in $\mathbb{R}^3$ supported in the unit ball and such that $\int_{\mathbb{R}^3} \theta(x) dx = 1$.

For every $j \in \mathbb{N}^*$, we define $\theta_j$ by

$$\theta_j(x) = j^3 \theta(jx).$$

Then, the sequence $(\theta_j)_{j \in \mathbb{N}^*}$ has the following properties

$$\theta_j \geq 0; \quad \theta_j \in C^\infty_c(\mathbb{R}^3); \quad \int_{\mathbb{R}^3} \theta_j(x) dx = 1; \quad supp \theta_j \subset B(0, \frac{1}{j}), \quad j \in \mathbb{N}^*.$$

Now let $f \in \mathcal{T}$, and let $R$ be a positive real number such that $supp(f) \subset B(0, R)$. Consider the following convolution product $f_j = f \ast \theta_j$, then $supp(f_j)$ is compact since $f$ and $\theta_j$ are compactly supported, more precisely, $\theta_j \in C^\infty_c(\mathbb{R}^3)$. We shall prove that $(f_j)$ converges towards $f$ in $H^3_2(\mathbb{R}^3)$. For this purpose, it will be sufficient to prove that

$$\forall \alpha, |\alpha| \leq 3, (1 + |x|)^{d+|\alpha|} D^\alpha f_j \rightarrow (1 + |x|)^{d+|\alpha|} D^\alpha f \text{ in } L^2(\mathbb{R}^3).$$

Let $\alpha, |\alpha| \leq 3$, we have: $D^\alpha f_j = D^\alpha f \ast \theta_j$ and $D^\alpha f - D^\alpha f_j = D^\alpha f - D^\alpha f \ast \theta_j$. Now

$$D^\alpha f \ast \theta_j(x) = \int_{\mathbb{R}^3} D^\alpha f(y) \theta_j(x - y) dy$$

and

$$D^\alpha f(x) = D^\alpha f(x) \int_{\mathbb{R}^3} \theta_j(x - y) dy = \int_{\mathbb{R}^3} D^\alpha f(x) \theta_j(x - y) dy.$$
thus
\[
(1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f_j(x)|^2 \leq (1 + |x|)^{2\alpha+2|\alpha|} \left( \int_{\mathbb{R}^3} |D^{\alpha} f(x) - D^{\alpha} f(y)| \theta_j(x - y) dy \right)^2 .
\]  
(A.3)

Applying Hölder’s inequality gives the following estimates
\[
(1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f_j(x)|^2 \leq (1 + |x|)^{2\alpha+2|\alpha|} \int_{\mathbb{R}^3} |D^{\alpha} f(x) - D^{\alpha} f(y)|^2 \theta_j(x - y) dy.
\]
which after integration on $\mathbb{R}^3$, gives:
\[
\int_{\mathbb{R}^3} (1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f_j(x)|^2 dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f(y)|^2 \theta_j(x - y) dx dy .
\]  
(A.4)

Consider on $\mathbb{R}^{2n}$ the change of variables $u = x - y$, $v = y$, we have:
\[
\int_{\mathbb{R}^3} (1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f_j(x)|^2 dx \leq \int_{|u| \leq \frac{1}{2}} \int_{|v| < R+1} (1 + |u+v|)^{2\alpha+2|\alpha|} |D^{\alpha} f(u+v) - D^{\alpha} f(v)|^2 \theta_j(u) du dv ,
\]
which can be written as
\[
\int_{\mathbb{R}^3} (1 + |x|)^{2\alpha+2|\alpha|} |D^{\alpha} f(x) - D^{\alpha} f_j(x)|^2 dx \leq \int_{|u| \leq \frac{1}{2}} \theta_j(u) du \int_{|v| < R+1} (1 + |u+v|)^{2\alpha+2|\alpha|} |D^{\alpha} f(u+v) - D^{\alpha} f(v)|^2 dv .
\]  
(A.5)

then, for $|u| \leq \frac{1}{2} < 1$, we have:
\[
\int_{\mathbb{R}^3} (1 + |u+v|)^{2\alpha+2|\alpha|} |D^{\alpha} f(u+v) - D^{\alpha} f(v)|^2 dv \leq C_{\alpha} \int_{\mathbb{R}^3} |D^{\alpha} f(u+v) - D^{\alpha} f(v)|^2 dv
\]  
(A.6)

where $C_{\alpha} = (3 + R(1+|x|))^{2\alpha+2|\alpha|}$. Recall that $f \in T \subset H^3_3(\mathbb{R}^3) \subset H^3(\mathbb{R}^3)$ thus $D^\alpha f \in L^2(\mathbb{R}^3)$ and by the continuity of the $L^2$-norm, one has:
\[
\int_{\mathbb{R}^3} |D^\alpha f(y+z) - D^\alpha f(y)|^2 dy \rightarrow 0 \text{ as } z \rightarrow 0
\]
and it follows from (A.6) that:
\[
\int_{\mathbb{R}^3} (1 + |y+z|)^{2\alpha+2|\alpha|} |D^\alpha f(y+z) - D^\alpha f(y)|^2 dy \rightarrow 0 \text{ as } z \rightarrow 0 .
\]

Now let $\varepsilon > 0$, there exists $\exists \delta > 0$ such that
\[
\forall z \in \mathbb{R}^3, |z| \leq \delta \implies \int_{\mathbb{R}^3} (1 + |y+z|)^{2\alpha+2|\alpha|} |D^\alpha f(y+z) - D^\alpha f(y)|^2 dy < \varepsilon .
\]
Since $\lim_{j \to \infty} \frac{1}{j} = 0$, there exists $j_0 > 0$ such that $\forall j \in \mathbb{N}^*, j > j_0 \implies \frac{1}{j} < \delta$ thus, $j > j_0$ implies $|z| < \frac{1}{j} < \delta$ and then
\[
\int_{\mathbb{R}^3} (1 + |y+z|)^{2\alpha+2|\alpha|} |D^\alpha f(y+z) - D^\alpha f(y)|^2 dy < \varepsilon ;
\]
from where we obtain
\[
j > j_0 \implies \int_{\mathbb{R}^3} (1 + |x|)^{2\alpha+2|\alpha|} |D^\alpha f(x) - D^\alpha f_j(x)|^2 dx \leq \varepsilon \int_{|z| \leq \frac{1}{j}} \theta_j(z) dz < \varepsilon ;
\]
thus,
\[
\forall \alpha, |\alpha| \leq 3, (1 + |x|)^{d+|\alpha|} D^\alpha f_j \rightarrow (1 + |x|)^{d+|\alpha|} D^\alpha f \text{ in } L^2(\mathbb{R}^3) ;
\]
and consequently, $f_j \rightarrow f$ in $H^3_d(\mathbb{R}^3)$ which proves that $C^\infty_c(\mathbb{R}^3)$ is dense in $T$ endowed with the topology of $H^3_d(\mathbb{R}^3)$.  
\[
\square
\]

35
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