Path Integral Solution for an Angle-Dependent Anharmonic Oscillator

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Abstract We have given a straightforward method to solve the problem of noncentral anharmonic oscillator in three dimensions. The relative propagator is presented by means of path integrals in spherical coordinates. By making an adequate change of time we are able to separate the angular motion from the radial one. The relative propagator is then exactly calculated. The energy spectrum and the corresponding wave functions are obtained.

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1 Introduction

As we know the Feynman path integral formalism has played an important role in the comprehension of the quantum phenomena.[1–3] In nonrelativistic quantum mechanics, many problems have been exactly solved starting from their classical origins by the use of path integrals.[4–5] Therefore, it has been proved that this method is powerful in finding exact propagators and wave functions and studying quantum behavior of several systems.[6] At a certain epoch, however, the development of the path integral method was not rapid compared with Schrödinger and Heisenberg approaches. The introduction of the space and time transformation to solve the nonrelativistic Coulomb problem and other basic potentials was the decisive stage for its development.[7–8] Having crossed this crucial stage, the path integral quantization method becomes at the present time more important than the Schrödinger equation. Its rise unveils itself from its applications in several fields of physics such as quantum mechanics,[1–6] statistical physics,[8] quantum field theory,[9–10] condensed matter,[11] cosmology,[12–13] and black hole physics.[14] We can find also a modern introduction to path integrals with significant applications in Refs. [15–16]. This explains the increasing interest in the development of path integration techniques.

However to our knowledge there is no report on the path integral treatment for a class of noncentral potentials which are of interest in understanding the nuclear shell structure within the framework of a modified mean field model.[17] These potentials that describe the rotational-vibrational motion of the nuclear system, have the form

\[ V(r, \theta) = \frac{1}{2} \mu \omega^2 r^2 + q \frac{u(\theta)}{r^2}, \]  

where \( (1/2)\mu \omega^2 r^2 \) is the usual three-dimensional harmonic oscillator that explains the magic numbers for spherical symmetric nucleus, \( u(\theta) \) is a function of \( \theta \) and \( q \) is a deformation parameter. We note that a particular case in two-dimensional space is discussed in Ref. [18].

In the present paper, we use of the Feynman path integral method to solve the problem of a nonrelativistic particle subjected to the following noncentral potential

\[ V(r, \theta) = -V_0 + \frac{1}{2} \mu \omega^2 r^2 + \frac{\alpha \hbar^2}{2\mu r^2} + \frac{\beta \hbar^2 \cos^2 \theta}{2\mu r^2 \sin^2 \theta} \]

\[ + \frac{\gamma \hbar^2}{2\mu r^2 \cos^2 \theta}, \]  

where \( V_0, \alpha, \beta, \) and \( \gamma \) are constants. We note that some particular cases of this potential are studied in Refs. [19–21]. Also some similar potentials have been recently discussed either in nonrelativistic quantum mechanics or in relativistic theory.[22–26]

In the first stage we formulate the problem in spherical coordinates path integrals. Then by the use of a temporal transformation we separate the angular motion and we do integration over \( \theta \) and \( \rho \) to obtain the exact propagator. Finally, we extract the energy spectrum and the wave functions.

2 Path Integral Formulation

In nonrelativistic quantum mechanics, the propagator or the transition amplitude from initial state \( |\vec{r}_a\rangle \) to final state \( |\vec{r}_b\rangle \) for a physical system governed by the Hamiltonian \( \hat{H} \) and the Lagrangian \( L \), is defined, in configuration space, by a matrix element of the evolution operator

\[ K(\vec{r}_b, t_b; \vec{r}_a, t_a) = \langle \vec{r}_b | \exp \left[ -\frac{i}{\hbar} \hat{H} (t_b - t_a) \right] |\vec{r}_a\rangle, \]  

which admits the following functional integral representation

\[ K(\vec{r}_b, t_b; \vec{r}_a, t_a) = \int D\vec{r}(t) \exp \left( \frac{i}{\hbar} S[\vec{r}(t)] \right), \]  

where the action \( S[\vec{r}(t)] \) is a functional of continuous trajectory \( \vec{r}(t) \) connecting space-time point \( (\vec{r}_a, t_a) \) with

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\[ (\vec{r}_b, t_b) \]

\[
S[\vec{r}(t)] = \int_{t_a}^{t_b} dt L(\vec{r}, \dot{\vec{r}}) ,
\]

with the Lagrangian
\[
L = \frac{\mu}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \hat{V}(x, y, z) ,
\]

and the potential
\[
\hat{V}(x, y, z) = -V_0 + \frac{1}{2} \mu \omega^2 (x^2 + y^2 + z^2) + \frac{(\alpha - \beta) \hbar^2}{2 \mu (x^2 + y^2 + z^2)} + \frac{\beta \hbar^2}{2 \mu (x^2 + y^2)} + \frac{\gamma \hbar^2}{2 \mu z^2} .
\]

In its discrete form \( K(\vec{r}_b, t_b; \vec{r}_a, t_a) \) can be written as
\[
K(\vec{r}_b, t_b; \vec{r}_a, t_a) \equiv K(\vec{r}_b, \vec{r}_a; T) = \int Dx \int Dy \int Dz \exp \left( \frac{1}{\hbar} \sum_n A_n \right) ,
\]

where
\[
A_n = \left( \frac{\mu}{2} (\Delta x_n)^2 + (\Delta y_n)^2 + (\Delta z_n)^2 \right) - \varepsilon \hat{V}(x_n, y_n, z_n) ,
\]

\[
\int Dx \int Dy \int Dz = \lim_{N \to \infty} \prod_{n=1}^{N-1} dx_n \prod_{n=1}^{N-1} dy_n \prod_{n=1}^{N-1} dz_n \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2 \pi \hbar}} \right)^3 .
\]

Here we have used the standard notation
\[
\Delta q_n = q_n - q_{n-1} , \quad q_N = q(t_b) , \quad q_0 = q(t_a) , \quad T = t_b - t_a = N \varepsilon .
\]

Let us now search for a path integral representation in the spherical coordinates
\[
x_n = r_n \sin \theta_n \cos \varphi_n , \quad y_n = r_n \sin \theta_n \sin \varphi_n , \quad z_n = r_n \cos \theta_n .
\]

It is obvious that the measure
\[
\int \prod_{n=1}^{N-1} dx_n \int \prod_{n=1}^{N-1} dy_n \int \prod_{n=1}^{N-1} dz_n \int \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2 \pi \hbar}} \right)^3
\]

will be in spherical coordinates
\[
\int \prod_{n=1}^{N-1} dr_n \int \prod_{n=1}^{N-1} d\theta_n \int \prod_{n=1}^{N-1} d\varphi_n \prod_{n=1}^{N} r_n^2 \sin \theta_n \sqrt{\frac{\mu}{2 \pi \hbar}}^{3N},
\]

and the propagator takes the following form
\[
K(\vec{r}_b, \vec{r}_a; T) = \lim_{N \to \infty} \prod_{n=1}^{N-1} dr_n \int \prod_{n=1}^{N-1} d\theta_n \int \prod_{n=1}^{N-1} d\varphi_n \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2 \pi \hbar}} \right)^3 N
\]

\[
\times \exp \sum_n \left( \frac{i \mu r_n^2 + r_n^2}{2 \hbar} - 2 r_n r_{n-1} \cos \theta_n \cos \theta_{n-1} - \frac{i \mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1}}{\hbar} \cos(\Delta \varphi_n) - \frac{i \hbar}{\varepsilon} \hat{V}(r_n, \theta_n) \right) .
\]

Taking into account that
\[
\prod_{n=1}^{N-1} r_n^2 = \frac{1}{r_b r_a} \prod_{n=1}^{N} (r_n r_{n-1}) ,
\]

and using the following formulae
\[
\exp \left( - i \frac{a}{\varepsilon} \cos(\Delta \varphi_n) \right) = \sum_{m=-\infty}^{+\infty} I_m \left( - i \frac{a}{\varepsilon} \right) e^{i m \Delta \varphi_n} ,
\]

where \( I_m(x) \) is the modified Bessel function, we obtain
\[
K(\vec{r}_b, \vec{r}_a; T) = \frac{1}{r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \prod_{n=1}^{N-1} dr_n \int \prod_{n=1}^{N-1} d\theta_n \int \prod_{n=1}^{N-1} d\varphi_n \prod_{n=1}^{N} (r_n r_{n-1} \sqrt{\sin \theta_n \sin \theta_{n-1}})
\]

\[
\times \sqrt{\frac{\mu}{2 \pi \hbar}}^{3N} \prod_{n=1}^{N} \sum_{m=-\infty}^{+\infty} I_m \left( - i \mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1} / \hbar \right) e^{i m \Delta \varphi_n}
\]

\[
\times \exp \left( \sum_n \left( \frac{i \mu r_n^2 + r_n^2}{2 \hbar} - 2 r_n r_{n-1} \cos \theta_n \cos \theta_{n-1} / \varepsilon - \frac{i \hbar}{\varepsilon} \hat{V}(r_n, \theta_n) \right) \right) .
\]

For small \( \varepsilon \), the modified Bessel function behaves as follows
\[
I_m \left( - i \frac{a}{\varepsilon} \right) \approx \left( \frac{i \varepsilon}{2 \pi a} \right)^{1/2} \times \left[ - i \frac{a}{\varepsilon} - i \varepsilon \left( m^2 - \frac{1}{4} \right) \right] ,
\]
which permits us to write
\[
K(\vec{r}_b, \vec{r}_a; T) = \frac{1}{2\pi r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \int_{N=1}^{\infty} \prod_{n=1}^{N-1} dr_n \int_{N=1}^{\infty} \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} (r_n r_{n-1})^{1/2} \sqrt{\frac{\mu}{2\pi i \hbar \varepsilon}} \int_{N=1}^{\infty} \prod_{n=1}^{N} \frac{d\varphi_n}{2\pi} \prod_{n=1}^{N} \int_{m=-\infty}^{+\infty} \exp \left[ im\Delta \varphi_n + \frac{i}{2} \frac{r_n^2 + r_{n-1}^2 - 2r_n r_{n-1} \cos(\Delta \theta_n)}{\varepsilon} \right] - \frac{i}{\hbar} V(r_n, \theta_n) \varepsilon \right] .
\] (18)

Now by using the Taylor development up to order 4
\[
\cos \Delta \theta_n = 1 - \frac{1}{2}(\Delta \theta_n)^2 + \frac{1}{24}(\Delta \theta_n)^4 + \cdots ,
\] (19)
we have
\[
K(\vec{r}_b, \vec{r}_a; T) = \frac{1}{2\pi r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \int_{N=1}^{\infty} \prod_{n=1}^{N-1} dr_n \int_{N=1}^{\infty} \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} \frac{(\mu r_n r_{n-1})^{1/2}}{2\pi i \hbar \varepsilon} \sqrt{\frac{\mu}{2\pi i \hbar \varepsilon}} \int_{N=1}^{\infty} \prod_{n=1}^{N} \frac{d\varphi_n}{2\pi} \prod_{n=1}^{N} \int_{m=-\infty}^{+\infty} \exp \left[ \frac{i}{\hbar} \left( m\hbar \Delta \varphi_n + \frac{\mu (\Delta \varphi_n)^2}{2\varepsilon} + \frac{\mu}{2} \frac{(\Delta \theta_n)^2}{\varepsilon} \right) \right]
\]
\[
- \frac{\mu r_n r_{n-1}}{24 \varepsilon} (\Delta \theta_n)^4 - \frac{(m^2 - 1/4)\hbar^2 \varepsilon}{2\mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1}} - V(r_n, \theta_n) \varepsilon \right] .
\] (20)

According to McLaughlin-Shulman procedure,[27] the term with \((\Delta \theta_n)^4\) leads to a quantum correction that can be calculated with the help of the property
\[
\int_{N=1}^{N} d\theta_n = \int_{N=2}^{N} d(\Delta \theta_n),
\] (21)
and the use of the integral
\[
\int du u^4 \exp[iau^2] = \left( -\frac{3}{4a^2} \right) \int du \exp[iau^2] .
\] (22)
This correction is
\[
\langle (\Delta \theta_n)^4 \rangle \approx 3 \left( \frac{i\hbar \varepsilon}{\mu r_n r_{n-1}} \right)^2 .
\] (23)

Then the propagator \(K(\vec{r}_b, \vec{r}_a; T)\) takes the following path integral representation
\[
K(\vec{r}_b, \vec{r}_a; T) = \frac{1}{2\pi r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \int_{N=1}^{\infty} \prod_{n=1}^{N-1} dr_n \int_{N=1}^{\infty} \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} \left( \frac{\mu r_n r_{n-1}}{2\pi i \hbar \varepsilon} \right)^{1/2} \int_{N=1}^{\infty} \prod_{n=1}^{N} \frac{d\varphi_n}{2\pi} \prod_{n=1}^{N} \int_{m=-\infty}^{+\infty} \exp \left[ \frac{i}{\hbar} \left( m\hbar \Delta \varphi_n + \frac{\mu (\Delta \varphi_n)^2}{2\varepsilon} + \frac{\mu}{2} \frac{(\Delta \theta_n)^2}{\varepsilon} \right) \right]
\]
\[
- \frac{\hbar^2}{8\mu r_n r_{n-1}} \varepsilon - \frac{(m^2 - 1/4)\hbar^2 \varepsilon}{2\mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1}} - V(r_n, \theta_n) \varepsilon \right] .
\] (24)

At this level, we see that only the integration over \(\varphi\) is straightforward. One cannot do integration over \(\theta\) and \(r\) because of the position dependent kinetic term \((\mu/2)r_n r_{n-1}(\Delta \theta_n)^2\) and the term \(\hbar^2(2\mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1})^{-1}\). Let us in that case do integration over \(\varphi\). We obtain
\[
K(\vec{r}_b, \vec{r}_a; T) = \sum_{m=-\infty}^{+\infty} \frac{e^{im(\varphi_b - \varphi_a)}}{2\pi} K_m(r_b, \theta_b, r_a, \theta_a; T) ,
\] (25)
where the novel propagator \(K_m(r_b, \theta_b; r_a, \theta_a; T)\) is given by
\[
K_m(r_b, \theta_b; r_a, \theta_a; T) = \frac{1}{r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \int_{N=1}^{\infty} \prod_{n=1}^{N-1} dr_n \int_{N=1}^{\infty} \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} \left( \frac{\mu r_n r_{n-1}}{2\pi i \hbar \varepsilon} \right)^{1/2} \int_{N=1}^{\infty} \prod_{n=1}^{N} \frac{d\varphi_n}{2\pi} \prod_{n=1}^{N} \int_{m=-\infty}^{+\infty} \exp \left[ \frac{i}{\hbar} \left( \frac{(\Delta \varphi_n)^2}{\varepsilon} + \frac{(\Delta \theta_n)^2}{\varepsilon} \right) \right]
\]
\[
+ \frac{\hbar^2}{8\mu r_n r_{n-1}} \varepsilon - \frac{(m^2 - 1/4)\hbar^2 \varepsilon}{2\mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1}} - V(r_n, \theta_n) \varepsilon \right] .
\] (26)

Here we note that the terms \(\hbar^2/(8\mu r_n r_{n-1})\) and \(\hbar^2/(2\mu r_n r_{n-1} \sin \theta_n \sin \theta_{n-1})\) represent an effective potential that describes the quantum corrections resulting from the passage to spherical coordinates. In the next step we separate the angle dependence from the radial one by using a simple time transformation.
3 Separation of Variables

Having formulated the problem of angle-dependant anharmonic oscillator in the framework of path integrals in spherical coordinates, let us proceed to find exact solutions by doing separation of variables. First we define the fixed energy amplitude $\hat{K}_m(r_b, \theta_b; r_a, \theta_a; E)$ to be the Fourier transform of $K_m(r_b, \theta_b; r_a, \theta_a; T)$

$$K_m(r_b, \theta_b; r_a, \theta_a; T) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-i(E/\hbar)T} \hat{K}_m(r_b, \theta_b; r_a, \theta_a; E), \quad (27)$$

with

$$\hat{K}_m(r_b, \theta_b; r_a, \theta_a; E) = \frac{1}{r_b r_a \sqrt{\sin \theta_b \sin \theta_a}} \int_{-\infty}^{+\infty} dT e^{i(E/\hbar)T} \int \prod_{n=1}^{N-1} dr_n \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} \left( \frac{\mu \sqrt{r_n r_{n-1}}}{2\pi \hbar} \right) \exp \left[ i \sum_n \left( \frac{\Delta r_n}{\epsilon} + \frac{\mu}{2} \frac{(\Delta \theta_n)^2}{\epsilon} + \frac{\hbar^2}{8 \mu r_n r_{n-1}} \right) \right]. \quad (28)$$

In order to be able to separate angular motion from the radial one we use the method of Ref. [28]. To begin we change the evolution time from $t$ to $s$, with

$$ds = \frac{dt}{r^2}. \quad (29)$$

This transformation is equivalent to

$$\varepsilon = \sigma_n r_n r_{n-1}, \quad (30)$$

with a time interval

$$S = \int_{0}^{T} \frac{dt}{r^2}. \quad (31)$$

To incorporate this changes in the path integral representation of $\hat{K}_m(r_b, \theta_b; r_a, \theta_a; E)$, we start form the identity

$$\int dS \delta(S - \int_{0}^{T} \frac{dt}{r^2}) = 1, \quad (32)$$

and by the use of

$$\delta(y - f(x)) = \frac{1}{f'(x)} \delta(f^{-1}(y) - x), \quad (33)$$

we get

$$\int dS \delta(T - \int_{0}^{S} \frac{dt}{r^2} ds) = 1. \quad (34)$$

By inserting the later identity in Eq. (28) and being aware of $r(T) = r_b$, we can write $\hat{K}_m(r_b, \theta_b; r_a, \theta_a; E)$ as an integral of two independent kernels

$$\hat{K}_m(r_b, \theta_b; r_a, \theta_a; E) = \int_{0}^{+\infty} dS P_E(r_b, r_a; S) Q_m(\theta_b, \theta_a; S), \quad (35)$$

where

$$P_E(r_b, r_a; S) = \frac{r_b}{r_a} \int \prod_{n=1}^{N-1} dr_n \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \sigma_n r_n r_{n-1}}} \right) \exp \left\{ i \sum_n \left[ \frac{\mu}{2} \frac{(\Delta r_n)^2}{\sigma_n r_n r_{n-1}} \right] \right\}, \quad (36)$$

$$Q_m(\theta_b, \theta_a; S) = \frac{1}{\sqrt{\sin \theta_b \sin \theta_a}} \int \prod_{n=1}^{N-1} d\theta_n \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \sigma_n}} \right) \exp \left\{ i \sum_n \left[ \frac{\mu}{2} \frac{(\Delta \theta_n)^2}{\sigma_n} - \frac{\hbar^2}{2\mu} \left( \frac{\lambda^2 - 1/4}{\sin^2 \theta_n} + \frac{k^2 - 1/4}{\cos^2 \theta_n} \right) \sigma_n \right] \right\}, \quad (37)$$

with

$$\lambda = \sqrt{3 + m^2}, \quad (38)$$

$$k = \sqrt{\gamma + \frac{1}{4}}. \quad (39)$$

In the continuous limit the angular propagator $Q_m(\theta_b, \theta_a; S)$ has the form

$$Q_m(\theta_b, \theta_a; S) = \frac{1}{\sqrt{\sin \theta_b \sin \theta_a}} \int d\theta \exp \left\{ i \frac{1}{\hbar} \int_{0}^{S} \left( \frac{\mu}{2} \dot{\theta}^2 - \frac{\hbar^2}{2\mu} \left( \frac{\lambda^2 - 1/4}{\sin^2 \theta} + \frac{k^2 - 1/4}{\cos^2 \theta} \right) \right) \right\} ds, \quad (40)$$

which has some resemblance to the Pöschl-Teller problem. $Q_m(\theta_b, \theta_a; S)$ is then integrable and the result is$^{29–31}$

$$Q_m(\theta_b, \theta_a; S) = \sum_{n=0}^{\infty} e^{-i/\hbar} \epsilon(n_b) \Theta_{n_b, m}^{*}(\theta_a) \Theta_{n_b, m}(\theta_b), \quad (41)$$

where $\epsilon(n_\theta)$ is given by

$$\epsilon(n_\theta) = \frac{\hbar^2}{2\mu}(2n_\theta + k + \lambda + 1)^2, \quad (42)$$
and angular wave functions are
\[ \Theta_{n_\mu,m}(\theta) = N_{n_\mu,m}(\sin \theta)^\lambda (\cos \theta)^{k+1/2} P^{(\lambda,k)}_{n_\mu}(\cos 2\theta) . \]
with the normalization constant
\[ N_{n_\mu,m} = \sqrt{2(2n_\theta + k + \lambda + 1) \frac{n_\theta \Gamma(n_\theta + k + \lambda + 1)}{\Gamma(n_\theta + k + 1) \Gamma(n_\theta + \lambda + 1)}} . \]

For \( P_E(r_b, r_a; S) \) we can not do integration directly because the kinetic term \((\mu/2 r_a r_n - 1)((\Delta r_n)^2/\sigma_n)\) contains an inconvenient variable dependent mass. It is the necessary to restore the original time \( t \). To this aim let us proceed as follows. First we incorporate the result of \( Q_m(\theta_b, \theta_a; S) \) in Eq. (35) to get
\[ \tilde{K}_m(r_b, \theta_b; r_a, \theta_a; E) = \sum_{n_\mu=0}^\infty \Theta^*_{n_\mu,m}(\theta_a) \Theta_{n_\mu,m}(\theta_b) \tilde{K}_{n_\mu,m}(r_b, r_a; E) , \]
where \( \tilde{K}_{n_\mu,m}(r_b, r_a; E) \) is given by
\[ \tilde{K}_{n_\mu,m}(r_b, r_a; E) = \int_0^{+\infty} dS e^{- i/\hbar (n_\mu S) \int_{r_a}^{r_b} P_E(r_b, \theta_a; S) , \]
and has the following path integral representation
\[ \tilde{K}_{n_\mu,m}(r_b, r_a; E) = \frac{r_b}{r_a} \int_0^{+\infty} dS \int_0^{+\infty} dT \int_0^{+\infty} \prod_{n=1}^{N-1} \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \epsilon}} \right) \left( 1 + \frac{i}{\hbar} \sum_{n} \left( \frac{\mu}{2} \frac{(\Delta r_{2n})^2}{\epsilon} \right) \right) \}
\]
Then we make the change \( r_n r_{n-1} \sigma_n \rightarrow \epsilon \) to obtain for \( \tilde{K}_{n_\mu,m}(r_b, r_a; E) \) the following form
\[ \tilde{K}_{n_\mu,m}(r_b, r_a; E) = \frac{1}{r_b r_a} \int_0^{+\infty} dT \int_0^{+\infty} \prod_{n=1}^{N-1} \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \epsilon}} \right) \left( 1 + \frac{i}{\hbar} \sum_{n} \left( \frac{\mu}{2} \frac{(\Delta r_{2n})^2}{\epsilon} \right) \right) \]
For \( K_{n_\mu,m}(r_b, r_a; T) \) we have
\[ K_{n_\mu,m}(r_b, r_a; T) = \frac{1}{r_b r_a} \int_0^{+\infty} dT \int_0^{+\infty} \prod_{n=1}^{N-1} \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \epsilon}} \right) \left( 1 + \frac{i}{\hbar} \sum_{n} \left( \frac{\mu}{2} \frac{(\Delta r_{2n})^2}{\epsilon} \right) \right) \]
In the next section we show how to find solutions by integrating \( K_{n_\mu,m}(r_b, r_a; T) \).

4 Integration of the Radial Propagator

Having shown how to do separation of variables and integration over angular ones let us do integration over radial variable to find the final solution to our problem. Starting from the \( K_{n_\mu,m}(r_b, r_a; T) \) that can be written in the form
\[ K_{n_\mu}(r_b, r_a; T) = \frac{1}{r_b r_a} \int_0^{+\infty} dT \int_0^{+\infty} \prod_{n=1}^{N-1} \prod_{n=1}^{N} \left( \sqrt{\frac{\mu}{2\pi \hbar \epsilon}} \right) \left( 1 + \frac{i}{\hbar} \sum_{n} \left( \frac{\mu}{2} \frac{(\Delta r_{2n})^2}{\epsilon} \right) \right) \]
with
\[ \tilde{\ell} = (\sqrt{k + \lambda + 2n_\theta + 1})^2 + (\alpha - \beta) - 1 . \]
This path integral is the ordinary path integral of the radial oscillator, the solution is known. We can find in Ref. [6] that \( K_{n_\mu}(r_b, r_a; T) \) takes the form
\[ K_{n_\mu}(r_b, r_a; T) = \frac{1}{r_b r_a} e^{i V_\mu T} \frac{\mu \omega}{\hbar \sin(\omega T)} I_{\tilde{\ell}+1/2} \left( \frac{\mu \omega r_b r_a}{\hbar \sin(\omega T)} \right) \exp \left( i \frac{\mu \omega}{2 \hbar} (r_b^2 + r_a^2) \cot(\omega T) \right) . \]
Now to obtain the radial wave functions and the energetic levels we use the Hille–Hardy formula\[32\]
\[ \frac{s}{1-s^2} \exp \left[ - \frac{1}{2} (X+Y)^2 \frac{s^2}{1-s^2} \right] \frac{\Gamma_{\tilde{\ell}+1/2} (2\sqrt{XY})}{(1-s^2)} \sum_{n=0}^{\infty} \frac{s^{2n+\tilde{\ell}+3/2} \hbar^{2n+1} e^{(1/2)(X+Y)}}{\Gamma(n+a+1)} \]
and we take \( s = e^{-i\omega T}; X = (\mu\omega/\hbar) r^2_b; Y = (\mu\omega/\hbar) r^2_a \), to get the spectral decomposition of \( K_{n,a,m}(r_b, r_a; T) \)

\[
K_{n,a,m}(r_b, r_a; T) = 2 \left( \frac{\mu\omega}{\hbar} \right)^{3/2} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + \ell + 3/2)} e^{-i[\omega(2n+\ell+3/2)-V_0]T} \times \exp \left( -\frac{\mu\omega}{2\hbar} \frac{r^2_a}{a^2} \right) \frac{L_n^{\ell+1/2} \left( \frac{\mu\omega}{\hbar} r^2_a \right)}{L_n^{\ell+1/2} \left( \frac{\mu\omega}{\hbar} r^2_a \right)},
\]

or in closed form

\[
K_{n,a,m}(r_b, r_a; T) = \sum_{n=0}^{\infty} e^{-i(h/E_n)E_n,a,m} R_{n,a,m}^* R_{n,a,m} (r_a) R_{n,a,m} (r_b),
\]

where the energy spectrum is given by

\[
E_{n,a,m} = \left( 2n + \ell + \frac{3}{2} \right) \hbar \omega - V_0,
\]

and the radial wave function reads

\[
R_{n,a,m}(r) = \sqrt{2 \left( \frac{\mu\omega}{\hbar} \right)^{3/2} \frac{n!}{\Gamma(n + \ell + 3/2)}} \exp \left( -\frac{\mu\omega}{2\hbar} \frac{r^2_a}{a^2} \right) \left( \sqrt{\frac{\mu\omega}{\hbar}} \right)^{\ell} L_n^{\ell+1/2} \left( \frac{\mu\omega}{\hbar} r^2_a \right).
\]

Finally the relative propagator can be expressed in the following spectral decomposition

\[
K(\vec{r}_b, \vec{r}_a; T) = \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{-i/hE_{n,a,m}T} \psi^*_{n,a,m}(\vec{r}_a) \psi_{n,a,m}(\vec{r}_b),
\]

where

\[
\psi_{n,m}(\vec{r}) = N_{n,a,m} \exp \left( -\frac{\mu\omega}{2\hbar} \frac{r^2}{a^2} \right) \left( \sqrt{\frac{\mu\omega}{\hbar}} \right)^{\ell} L_n^{\ell+1/2} \left( \frac{\mu\omega}{\hbar} r^2 \right) \times (\sin \theta)^{\lambda} (\cos \theta)^{k+1/2} P_{\lambda}^{(k)}(\cos 2\theta) e^{im\varphi},
\]

with the normalization constant

\[
N_{n,a,m} = \sqrt{\frac{2}{\pi} \left( \frac{\mu\omega}{\hbar} \right)^{3/2} (2n_0 + k + \lambda + 1)} \frac{n!n_0 \Gamma(n_0 + k + \lambda + 1)}{\Gamma(n_0 + k + 1) \Gamma(n_0 + \lambda + 1) \Gamma(n + \ell + 3/2)}.
\]

Let us remark that when we consider the particular cases studied in Refs. [19–21], we see that our results coincide with these results.

5 Conclusion

In this paper we have given a straightforward method to solve the problem of noncentral anharmonic oscillator in three dimensions. In the first stage we have expressed the relative propagator by means of path integrals in spherical coordinates. Then by making an adequate change of time we are able to separate the angular motion from the radial one. The angular part path integration is reduced to the well-known Pöschl-Teller problem and the resulting radial path integral is written in the form of three dimensional isotropic harmonic oscillator. Then we have exactly calculated the relative propagator and we have extracted the bound states energies and the corresponding wave functions.

We remark also that the presented method is useful for all potentials of the form

\[
V(r, \theta) = v(r) + \frac{u(\theta)}{r^2},
\]

and there is no need to use other systems of coordinates.

Through the formulation given above and the obtained energies and wave functions we conclude that the path integral formulation is a powerful method to study quantum dynamics of particles in nonrelativistic theory.

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