Hardy inequality in variable exponent Lebesgue spaces derived from nonlinear problem

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Abstract

We derive a family of weighted $p(x)$–Hardy inequalities with an additional term of the form
\[
\int_{\Omega} |\xi|^p(x) \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p(x) \mu_2(dx) + \int_{\Omega} |\xi \log \xi|^p(x) \mu_3(dx)
\]
for compactly supported Lipschitz functions. The involved measures depend on a certain solution to the partial differential inequality involving $p(x)$–Laplacian $-\Delta_{p(x)} u \geq \Phi$, where $\Phi$ is a given locally integrable function, and $u$ is defined on an open and not necessarily bounded subset $\Omega \subseteq \mathbb{R}^n$. As a consequence of Caccioppoli–type inequality for the solution $u$ we get Hardy inequality with an additional term in variable exponent Lebesgue spaces.

We present the derivation of the inequality in $\mathbb{R}^n$. We focus on one–dimensional Hardy inequalities implied by the main result. The paper extends the previous results of the second author [41, 42], where classical Hardy and Hardy–Poincaré inequalities are derived with the optimal constants.

∗The author was supported by NCN grant 2011/03/N/ST1/00111.
Keywords: $p(x)$–Laplacian, Hardy inequality, Caccioppoli inequality, variable exponent Lebesgue spaces.

2010 Mathematics Subject Classification: 26D10, 35J60, 35J91.

1 Introduction

In this paper we derive a family of Hardy inequalities with variable exponent of the form

$$
\int_{\Omega} |\xi|^{p(x)}\mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)}\mu_2(dx) + \int_{\Omega} |\log \xi|^{p(x)} \frac{|\nabla p(x)|^{p(x)}}{p(x)^{p(x)}} \mu_2(dx),
$$

where $1 < p^- := \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \text{ess sup}_{x \in \Omega} p(x) < \infty$, $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, and $\Omega$ is an open subset of $\mathbb{R}^n$, not necessarily bounded. The involved measures $\mu_1(dx), \mu_2(dx)$ depend on $p(x)$, a certain parameter $\beta$, a piecewise continuous function $\sigma(x)$, and a nonnegative weak solution $u$ to the PDE

$$
-\Delta_{p(x)} u \geq \Phi \quad \text{in} \quad \Omega,
$$

with a locally integrable function $\Phi$. We admit the functions $\sigma(x)$ and $\Phi$ for which there exists

$$
\sigma_0 := \inf_{x \in \Omega} \left\{ \sigma(x) : \Phi \cdot u + \sigma(x) |\nabla u|^{p(x)} \geq 0 \right\} \in \mathbb{R}.
$$

Hardy-type inequalities are important tools in various fields of analysis. Let us mention such branches as functional analysis, harmonic analysis, probability theory, and PDEs. Nowadays, they are also investigated on their own [28, 29, 30, 34]. They are classically stated in Lebesgue spaces. Recently, many authors take into account their versions generalized in Orlicz setting [6, 7, 8, 24, 25, 43] or for example on Riemannian manifolds [13].

Variable exponent Lebesgue spaces are investigated since 1930s when Orlicz introduced them in [35]. They were discovered later independently by Sharapudinov [40] and Tsenov [44]. Nowadays, they are considered in modelling of some materials with inhomogeneities and nonlinearities, for instance electrorheological fluids [14, 22, 37], which are innovative, ‘smart’ materials changing their viscosity in response to an electric field. They have been used in robotics, space technology, and mechanics in fast acting mechanical device
such as clutches, brakes, shock absorbers and hydraulic valves. Their special properties require more sophisticated framework than classical Lebesgue spaces. For detailed information on the properties of Lebesgue and Sobolev spaces with variable exponent we refer to books \[14, 27\].

Recently, Hardy–type inequalities in variable exponent Lebesgue spaces have become a lively studied topic of analysis \[5, 15, 19, 20, 21, 31, 32, 33, 38, 39\]. The most common idea in these papers is investigating links between validity of Hardy–type inequalities and boundedness of Maximal Operator. One–dimensional case is considered in \[15, 31, 32\], where the exponents are possibly different on the right– and the left–hand side of the inequality. The papers \[38, 39\] are devoted to the inequality with the weights depending on distance from a single point in \(\mathbb{R}^n\), while in \[19\] the weights depend on distance from a boundary in \(\mathbb{R}^n\). Different approach we find in \[5\], where the authors investigate the class of admissible weights for Hardy–type inequality holding for nonincreasing functions.

We point out that in the majority of the above papers the authors deal with the norm version of Hardy inequality. We obtain the modular one. We would like to stress that only in the constant exponent case the both types are equivalent. In the variable exponent case it is not direct to transform one of these types to another. To the authors’ best knowledge the only result of this kind is given by Fan–Zhao \[18, Theorem 1.3\] where it is shown how to derive a norm Hardy inequality from a modular one.

The purpose of this paper is to introduce a new tool for derivation of Hardy inequalities with variable exponent on the basis of nonlinear problems. We find the idea of construction of weighted Hardy inequalities with constant exponent in papers by Barbatis, Filippas, and Tertikas \[1, 2\], where the inequalities are derived on a domain where certain power of distance function is \(p\)–superharmonic. Futhermore, in papers of D’Ambrosio \[10, 11, 12\] the author derives an inequality related to \((1)\) as the consequence of the inequality \(-\Delta_p(u^\alpha) \geq 0\) with a certain constant \(\alpha\).

Our cosiderations are based on the methods intruduced in \[26\] and developed in \[41, 42, 43\]. In \[26\] the authors investigate nonexistence of nontrivial nonnegative weak solutions to \(A\)–harmonic problems starting with derivation of Caccioppoli–type estimate for their weak solutions. As a starting point to derive Hardy inequality we focus on this Caccioppoli–type estimate. We modify the proof of Theorem 4.1 from \[41\], where the investigated PDI reads

\[-\Delta_p u \geq \Phi \quad \text{in} \quad \Omega, \quad (3)\]
with a locally integrable function $\Phi$ being in a certain sense not very negative. As it is shown in [41], the substitution in the derived Caccioppoli–type inequality for solutions implies the family of Hardy inequalities of the form
\[
\int_{\Omega} |\xi|^p \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_{2,\beta}(dx),
\]
where $1 < p < \infty$, $\xi : \Omega \to \mathbb{R}$ is compactly supported Lipschitz function, and $\Omega$ is an open subset of $\mathbb{R}^n$. The involved measures $\mu_{1,\beta}(dx)$, $\mu_{2,\beta}(dx)$ depend on a certain parameter $\beta$ and on $u$ — a nonnegative weak solution to (3). Among other results it implies classical Hardy and Hardy–Poincaré inequalities with optimal constants (see [41, 42], respectively). We retrieve the main result of [41] as a special case here (see Theorem 6.1) and therefore we confirm all the examples from [41, 42].

Our goal now is to extend the techniques from [41] to the more general case when we deal with (2) instead of (3). We derive Hardy inequality in variable exponent Lebesgue spaces on $\mathbb{R}^n$ and then we pay particular attention to one-dimensional case. The paper considering more general inequalities, especially those with radial weights, is in preparation (see [3]). We focus on Hardy inequalities in one dimension because they are easier to interpret and to compare with many existing one-dimensional results (see e.g. [5, 15, 31, 32]). Moreover, higher dimensional problems may be reduced to this case when we assume certain symmetric properties. We hope that our result will be found useful in applied mathematics, especially in investigations on qualitative properties of solutions to nonlinear problems.

The paper is organised as follows. Section 3 is devoted to derivation of Caccioppoli–type inequality for solutions to (2). In Section 4 we derive general $p(x)$–Hardy inequality for compactly supported Lipschitz functions. In Section 5 we concentrate on inequalities in one dimension. In Section 6 we give detailed comparison with the results existing in the literature. We conclude our paper in Section 7 where we pose open problems.

## 2 Preliminaries

By $p(x)$–harmonic problems we understand those which involve $p(x)$–Laplace operator $\Delta_{p(x)} u = \text{div}(|\nabla u|^{p(x)-2} \nabla u)$.

In the sequel we assume that $\Omega \subseteq \mathbb{R}^n$ is an open subset not necessarily bounded. If $f$ is defined on the set $A$ by $f \chi_A$ we understand function $f$
extended by 0 outside $A$. Moreover, every time when we deal with infimum, we set $\inf \emptyset = -\infty$.

**General Lebesgue and Sobolev spaces**

In the sequel we suppose that $p : \Omega \to (1, \infty)$ is a piecewise differentiable function satisfying the following condition

$$1 < p^- := \operatorname{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \operatorname{ess sup}_{x \in \Omega} p(x) < \infty.$$ We denote by $P(\Omega)$ the set of all admissible functions $p$.

We recall some properties of the spaces $L^p(x)(\Omega)$ and $W^{1,p}(x)(\Omega)$, which we call respectively generalized Lebesgue space and generalized Sobolev space (see e.g. [17, 18]). By $E(\Omega)$ we denote the set of all equivalence classes of measurable real functions defined on $\Omega$ being equal almost everywhere. The generalized Lebesgue space is defined as

$$L^p(x)(\Omega) = \{ u \in E(\Omega) : \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \},$$

equipped with the norm

$$\| u \|_{L^p(x)(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.$$ We define the generalized Sobolev space $W^{1,p}(x)(\Omega)$ by

$$W^{1,p}(x)(\Omega) = \{ u \in L^p(x)(\Omega) : \nabla u \in L^p(x)(\Omega; \mathbb{R}^n) \}$$
equipped with the norm $\| u \|_{W^{1,p}(x)(\Omega)} = \| u \|_{L^p(x)(\Omega)} + \| \nabla u \|_{L^p(x)(\Omega)}$.

Then $(L^p(x)(\Omega), \| \cdot \|_{L^p(x)(\Omega)})$ and $(W^{1,p}(x)(\Omega), \| \cdot \|_{W^{1,p}(x)(\Omega)})$ are separable and reflexive Banach spaces (see e.g. [14, 17, 18]).

**Differential inequality**

Our analysis is based on the following differential inequality.

**Definition 2.1.** Let $\Omega$ be any open subset of $\mathbb{R}^n$, $p \in P(\Omega)$, and $\Phi$ be the locally integrable function defined in $\Omega$ such that for every nonnegative compactly supported $w \in W^{1,p}(x)(\Omega)$

$$\int_{\Omega} \Phi w \, dx > -\infty.$$
Let \( u \in W_{\text{loc}}^{1,p(x)}(\Omega) \) and \( u \neq 0 \). We say that
\[-\Delta_{p(x)} u \geq \Phi,\]
if for every nonnegative compactly supported \( w \in W_{\text{loc}}^{1,p(x)}(\Omega) \), we have
\[
\langle -\Delta_{p(x)} u, w \rangle := \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla w \rangle \, dx \geq \int_{\Omega} \Phi w \, dx.
\]

**Remark 2.1.** Note, that \( p(x)-\text{Laplacian} \) is a continuous, bounded and strictly monotone operator defined for every compactly supported function \( w \in W_{\text{loc}}^{1,p(x)}(\Omega) \). In particular, it is well-defined in the distributional sense (see e.g. [16]).

**The crucial condition**

In the sequel we suppose \( u \) and \( \Phi \) are as in Definition \( \ref{def:crucial} \). \( \sigma(x) \) is a bounded and piecewise continuous function such that
\[
\Phi \cdot u + \sigma(x)|\nabla u|^{p(x)} \geq 0 \quad \text{a.e. in } \Omega
\]
and moreover, there exists
\[
\sigma_0 := \inf_{x \in \Omega} \{ \sigma(x) : \sigma(x) \text{ satisfies } (\ref{eq:crucial}) \} \in \mathbb{R}.
\]

**3 Caccioppoli estimate for solution of differential inequality \(-\Delta_{p(x)} u \geq \Phi\)**

Before we formulate the main theorem of this section we state the following useful lemmas.

**Lemma 3.1.** Let \( u \in W_{\text{loc}}^{1,p(x)}(\Omega) \) and \( \phi \) be a nonnegative Lipschitz function with compact support in \( \Omega \) such that the integral \( \int_{\text{supp} \phi} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx \) is finite. We fix \( 0 < \delta < R, \beta > 0 \) and denote
\[
u_{\delta,R}(x) := \min \{ u(x) + \delta, R \}, \quad G(x) := (u_{\delta,R}(x))^{-\beta} \phi(x).
\]
Then \( u_{\delta,R} \in W_{\text{loc}}^{1,p(x)}(\mathbb{R}^n) \) and \( G \in W^{1,p(x)}(\Omega) \).
Remark 3.1. See e.g. [14, Proposition 8.1.9], to obtain $u_{\delta,R} \in W^{1,p(x)}_{loc}(\mathbb{R}^n)$. We note that the truncated function satisfies $\delta \leq u_{\delta,R}(x) \leq R$ and therefore we have $(u_{\delta,R}(x))^\beta \in W^{1,p(x)}_{loc}(\mathbb{R}^n)$. The function $G$ is compactly supported, thus $G \in W^{1,p(x)}(\Omega)$.

Lemma 3.2. Let $p \in P(\Omega)$, $\tau > 0$ and $s_1, s_2 \geq 0$, then for a.e. $x \in \Omega$

$$s_1 s_2^{p(x)-1} \leq \frac{1}{p(x)\tau} \cdot s_1^{p(x)} + \frac{p(x)-1}{p(x)} \cdot \tau \cdot s_2^{p(x)}.$$

Proof. We apply classical Young inequality $ab \leq \frac{a^{p(x)}}{p(x)} + \frac{b^{p(x)-1}}{p(x)}$ with $a = \frac{s_1}{\eta^{p(x)-1}}$, $b = (s_2 \eta)^{p(x)-1}$, where $\eta > 0$, to get

$$s_1 s_2^{p(x)-1} = \left(\frac{s_1}{\eta^{p(x)-1}}\right)^{p(x)} \leq \frac{1}{p(x)} \cdot \left(\frac{s_1}{\eta^{p(x)-1}}\right)^{p(x)} + \frac{p(x)-1}{p(x)} \cdot \frac{p(x)}{p(x)} \cdot \eta^{p(x)} \cdot s_2^{p(x)}.$$

Now it suffices to substitute $\tau = \eta^{p(x)}$. \qed

Lemma 3.3 (e.g. [25], Lemma 3.1). Let $u \in W^{1,1}_{loc}(\Omega)$ be defined everywhere by the formula (see e.g. [3])

$$u(x) := \limsup_{r \to 0} \frac{1}{\int_{B(x,r)} u(y)dy}$$

and let $t \in \mathbb{R}$. Then

$$\{x \in \mathbb{R}^n : u(x) = t\} \subseteq \{x \in \mathbb{R}^n : \nabla u(x) = 0\} \cup N,$$

where $N$ is a set of Lebesgue’s measure zero.

The main goal of this section is the following result.
Theorem 3.1 (Caccioppoli estimate). Assume that \( p \in \mathcal{P}(\Omega) \) and \( u \in W^{1,p(x)}_{\text{loc}}(\Omega) \) is a nonnegative solution to the PDE \( -\Delta_{p(x)} u \geq \Phi \), in the sense of Definition 2.1, with a locally integrable function \( \Phi \) satisfying (5) with a piecewise continuous function \( \sigma(x) \). Assume further that \( \beta > 0 \) is an arbitrary number, such that for all \( x \in \Omega \) we have \( \beta > \sigma(x) \geq \sigma_0 \) with \( \sigma_0 \) as in (6). Then the inequality

\[
\int_{\Omega} \left( \Phi \cdot u + \sigma(x) |\nabla u|^{p(x)} \right) u^{\beta-1} \chi_{\{u > 0\}} \cdot \phi \, dx \leq \int_{\Omega} \left( \frac{p(x) - 1}{p(x) - \sigma(x)} \right)^{p(x)-1} u^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx,
\]

holds for every nonnegative Lipschitz function \( \phi \) with compact support in \( \Omega \) such that the integral \( \int_{\text{supp} \phi} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx \) is finite.

We call (5) Caccioppoli inequality because it involves \( \nabla u \) on the left-hand side and only \( u \) on the right-hand side (see e.g. [9, 23]).

We note that we do not assume that the right-hand side in (8) is finite.

The proof is based on the idea of the proof of Theorem 3.1 from [41] inspired by the proof of Proposition 3.1 from [26].

Proof of Theorem 3.1. The proof follows by three steps.

Step 1. Derivation of a local inequality.

We obtain the following lemma.

Lemma 3.4. Assume that \( p \in \mathcal{P}(\Omega) \) and \( u \in W^{1,p(x)}_{\text{loc}}(\Omega) \) is a nonnegative solution to the PDE \( -\Delta_{p(x)} u \geq \Phi \), in the sense of Definition 2.1, with a locally integrable function \( \Phi \). Assume further that \( \beta, \varepsilon > 0 \) are arbitrarily taken numbers. Then, for every \( 0 < \delta < R \), the inequality

\[
\int_{\Omega} \left( \Phi \cdot u + (\beta - \frac{\varepsilon}{p(x)-1}) |\nabla u|^{p(x)} \right) (u + \delta)^{-\beta-1} \chi_{\{u \leq R-\delta\}} \cdot \phi \, dx \leq \int_{\Omega} \left( \frac{1}{p(x) - 1} \right)^{p(x)-1} (u + \delta)^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u \leq R-\delta\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + C(\delta, R),
\]

where

\[
C(\delta, R) = R^{-\beta} \left[ \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \chi_{\{\nabla u \neq 0, u > R-\delta\}} \, dx - \int_{\Omega} \Phi \chi_{\{u > R-\delta\}} \phi \, dx \right],
\]

holds for every nonnegative Lipschitz function \( \phi \) with compact support in \( \Omega \).
**Proof of Lemma 3.4** We take \( w = G \) (see (7)) in the left side of the inequality (11) and note that
\[
L := \int_{\Omega} \Phi \cdot G \, dx = \int_{\Omega} \Phi \cdot (u_{\delta,R})^{-\beta} \phi \, dx = \int_{\Omega \cap \{u \leq R-\delta\}} \Phi \cdot (u + \delta)^{-\beta} \phi \, dx + R^{-\beta} \int_{\Omega \cap \{u > R-\delta\}} \Phi \cdot \phi \, dx.
\]

On the other hand, inequality (11) implies
\[
L := \int_{\Omega} \Phi \cdot G \, dx \leq \langle -\Delta_{p(x)} u, G \rangle = \int_{\Omega \cap \{\nabla u \neq 0\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla G \rangle \, dx = -\beta \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi \, dx + \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} \, dx + R^{-\beta} \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \, dx.
\]

Note that all the above integrals are finite, what follows from Lemma 3.1 (for \( 0 \leq u \leq R - \delta \) we have \( \delta \leq u + \delta \leq R \)). We compute further that
\[
\int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} \, dx \leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)-1} |\nabla \phi| (u + \delta)^{-\beta} \, dx = \int_{\supp \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \left( \frac{|\nabla \phi|}{\phi} (u + \delta) \right) \cdot |\nabla u|^{p(x)-1} (u + \delta)^{-\beta-1} \phi \, dx.
\]

We apply Lemma 3.2 with \( s_1 = \frac{|\nabla \phi|}{\phi} (u + \delta) \), \( s_2 = |\nabla u| \) and arbitrary \( \varepsilon > 0 \), to get
\[
\int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} \, dx \leq \int_{\supp \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \frac{p(x)-1}{p(x)} \varepsilon |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi \, dx + \int_{\supp \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \frac{1}{p(x)\varepsilon^{p(x)-1}} \left( \frac{|\nabla \phi|}{\phi} \right)^{p(x)} (u + \delta)^{p(x)-\beta-1} \phi \, dx.
\]
Combining these estimates we deduce that
\[
L \leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} \left( -\beta + \frac{p(x) - 1}{p(x)} \varepsilon \right) |\nabla u|^{p(x)}(u + \delta)^{-\beta - 1} \phi \, dx + \\
+ \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{1}{p(x) \varepsilon^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + \\
+ R^{-\beta} \int_{\Omega \cap \{\nabla u \neq 0, u > R - \delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \, dx.
\]
This and (11) imply
\[
\int_{\Omega \cap \{u \leq R - \delta\}} \Phi \cdot (u + \delta)^{-\beta} \phi \, dx + \\
+ \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} \left( \beta - \frac{p(x) - 1}{p(x)} \varepsilon \right) |\nabla u|^{p(x)}(u + \delta)^{-\beta - 1} \phi \, dx \leq \\
\leq \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{1}{p(x) \varepsilon^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + C(\delta, R),
\]
where \( C(\delta, R) \) is given by (10). \( \square \)

**Remark 3.1.** Introduction of parameters \( \delta \) and \( R \) was necessary as we needed to move some finite quantities in the estimates to opposite sides of inequalities.

**Step 2. Passing to the limit with \( \delta \searrow 0 \).**

We show that when \( \beta, \varepsilon > 0 \) are arbitrary numbers such that \( \beta - \frac{p(x) - 1}{p(x)} \varepsilon =: \sigma(x) \geq \sigma_0 \), then for any \( R > 0 \)
\[
\int_{\Omega} \left( \Phi \cdot u + \sigma(x)|\nabla u|^{p(x)} \right) u^{-\beta - 1} \chi_{\{0 < u \leq R\}} \cdot \phi \, dx \\
\leq \int_{\Omega} \frac{1}{p(x) \varepsilon^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u \leq R\}} : |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + C(R),
\]
where
\[
C(R) = R^{-\beta} \left[ \int_{\Omega} |\nabla u|^{p(x)-2} |\nabla u| \chi_{\{u \geq \frac{R}{2}\}} : |\nabla \phi| \, dx + \int_{\Omega} \Phi \chi_{\{u \geq \frac{R}{2}\}} \cdot \phi \, dx \right]
\]
holds for every nonnegative Lipschitz function \( \phi \) with compact support in \( \Omega \) such that the integral \( \int_{\text{supp } \phi \cap \{\nabla u \neq 0\}} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx \) is finite. Moreover, all quantities appearing in (12) are finite.
We show first that under our assumptions, when $\delta \searrow 0$, we have
\[
\int_{\Omega \setminus \{u \neq 0\}} \frac{1}{p(x) - p(x) - 1} (u + \delta)^{p(x) - \beta - 1} \chi_{\{u \neq 0, u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx \to (13)
\]
for every nonnegative Lipschitz function $\phi$ with compact support in $\Omega$ such that the integral $\int_{\text{supp} \phi \cap \{u \neq 0\}} \frac{1}{p(x) - p(x) - 1} (u + \delta)^{p(x) - \beta - 1} \chi_{\{u \neq 0, u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx$ is finite.

We note that $(u + \delta)^{p(x) - \beta - 1} \chi_{\{u + \delta \leq R\}} \to 0$ a.e. This follows from Lemma 3.3 (which gives that the set $\{u = 0, |\nabla u| \neq 0\}$ is of measure zero) and the continuity outside zero of the involved functions.

We show (13) independently on separate subsets of domains of integration. Hence, we have
\[
\int_{\Omega \setminus \{u \neq 0\}} \frac{1}{p(x) - p(x) - 1} (u + \delta)^{p(x) - \beta - 1} \chi_{\{u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx =
\]
\[
= \sum_{i=1}^{3} \int_{E_i \setminus \{u \neq 0\}} \frac{1}{p(x) - p(x) - 1} (u + \delta)^{p(x) - \beta - 1} \chi_{\{u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx,
\]
where
\[
E_1 = \{x \in \Omega : p(x) - \beta - 1 = 0\},
E_2 = \{x \in \Omega : p(x) - \beta - 1 < 0\},
E_3 = \{x \in \Omega : p(x) - \beta - 1 \geq 0\}.
\]

Convergence on $E_1$ follows from the Lebesgue’s Monotone Convergence Theorem, as on this set the only expression involving $\delta$ is the characteristic function $\chi_{\{u + \delta \leq R\}}$.

Let us concentrate on the case when $\delta \searrow 0$ on $E_2$. We apply the Lebesgue’s Monotone Convergence Theorem as on this set
\[
(u + \delta)^{p(x) - \beta - 1} \chi_{\{u + \delta \leq R\}} \to u^{p(x) - \beta - 1} \chi_{\{u \leq R\}}.
\]
Indeed, we note first that then for a.e. $x \in \Omega$ such that $u(x) > 0$ we have that $u + \delta \searrow u$. Hence, also $(u + \delta)^{p(x) - \beta - 1} \to u^{p(x) - \beta - 1} \neq 0$. Secondly, we observe that then for a.e. $x \in \Omega$ we have $\chi_{\{0 < u \leq R - \delta\}} \leq \chi_{\{0 < u \leq R\}} \to \chi_{\{0 < u < R\}}$.

In the case of $E_3$, without loss of generality, we assume that $R > 1$. Then we apply the Lebesgue’s Dominated Convergence Theorem as
\[
\int_{E_3 \setminus \{u \neq 0\}} \frac{1}{p(x) - p(x) - 1} (u + \delta)^{p(x) - \beta - 1} \chi_{\{u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx \leq
\]
\[
\leq R^{p(x) - \beta - 1} \frac{\tilde{\epsilon}}{p} \int_{E_3 \setminus \{u \neq 0\}} \chi_{\{u \leq R\}} \cdot |\nabla \phi|^{p(x) - 1} \phi^{1-p(x)} \, dx < \infty,
\]
where $\bar{\varepsilon} = \sup_{x \in \Omega} \varepsilon^{1-p(x)}$. The details are left to the reader.

To complete the proof of Step 2 we note that (13) says that, when $\delta \searrow 0$, the first integral on the right–hand side of (9) is convergent to the first integral of the right–hand side of (12). To deal with the second expression note that for $\delta \leq R^2$, we have

$$|C(\delta, R)| \leq \left| R^{-\beta} \int_{\Omega} |\nabla u|^{p(x)-2}\langle \nabla u, \nabla \phi \rangle \chi_{\{u > R-\delta\}} \, dx \right| + \left| R^{-\beta} \int_{\Omega} \Phi \chi_{\{u > R-\delta\}} \cdot \phi \, dx \right| \leq C(R).$$

It suffices now to pass to the limit with $\delta \searrow 0$ on the left–hand side of (9). We do it due to the Lebesgue’s Monotone Convergence Theorem as the expression in brackets is nonnegative. Indeed, the condition (5) implies

$$\Phi \cdot u + \sigma(x)|\nabla u|^{p(x)} \geq 0 \quad \text{a.e. on } \Omega \cap \{u > 0\} \quad \text{where } \sigma(x) \geq \sigma_0.$$

**Step 3. We let $R \to \infty$ and finish the proof.**

Without loss of generality we can assume that the integral in the right–hand side of (8) is finite, as otherwise the inequality follows trivially. Note that since $|\nabla u|^{p(x)-2}\langle \nabla u, \nabla \phi \rangle$ and $\Phi \phi$ are integrable we have $\lim_{R \to \infty} C(R) = 0$. Therefore, (8) follows from (12) by the Lebesgue’s Monotone Convergence Theorem (note that $\varepsilon = \frac{p(x)(\beta-\sigma(x))}{p(x)-1}$).

4 General $p(x)$–Hardy inequality

In the proof of $p(x)$–Hardy inequality we need the following lemma.

**Lemma 4.1.** Let $p \in \mathcal{P}(\Omega)$ and $s_1, s_2 \geq 0$, then the following inequality holds for a.e. $x \in \Omega$

$$(s_1 + s_2)^{p(x)} \lesssim 2^{(p(x)-1)\chi_{\{s_1 \neq 0\}}} \left( s_1^{p(x)} + s_2^{p(x)} \right). \quad (14)$$

**Remark 4.1.** Note that in this lemma the role of $s_1$ is not the same as $s_2$. If $s_1 = 0$, then (13) becomes $s_2^{p(x)} = s_2^{p(x)}$. This is necessary to retrieve Theorem 4.1 from [41] (concerning constant exponent case) with the best constant via our investigations (see Theorem 6.1 here).

Now we state our main result.
Theorem 4.1 \((p(x)-\text{Hardy inequality})\). Assume that \(p \in \mathcal{P}(\Omega)\) and \(u \in W_{\text{loc}}^{1,p(x)}(\Omega)\) is a nonnegative solution to the \(PDI - \Delta_{p(x)} u \geq \Phi\), in the sense of Definition 2.1, with a locally integrable function \(\Phi\) satisfying \([5]\) with a piecewise continuous function \(\sigma(x)\). Assume further that \(\beta > 0\) is an arbitrary number such that \(\beta > \sigma(x) \geq \sigma_0\) with \(\sigma_0\) as in \([6]\). Then, for every Lipschitz function \(\xi\) with compact support in \(\Omega\), we have

\[
\int_{\Omega} |\xi|^{p(x)} \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_2(dx) + \int_{\Omega} |\xi| \log \xi |^{p(x)} \cdot \frac{|\nabla p(x)|^{p(x)} p(x)_{p(x)} \mu_2(dx)},
\]

where

\[
\mu_1(dx) = (\Phi \cdot u + \sigma(x)|\nabla u|^{p(x)}) \cdot u^{-\beta-1} \chi_{\{u > 0\}} \ dx,
\]

\[
\mu_2(dx) = \left(\frac{p(x) - 1}{\beta - \sigma(x)} \right)^{p(x)-1} 2^{p(x)-1} \chi_{\{|\nabla u| \neq 0\}} u^{p(x)-\beta-1} \chi_{\{|\nabla u| \neq 0\}} \ dx.
\]

**Proof.** We are going to apply Theorem 3.1 and, after substituting a certain \(\sigma\) piecewise continuous function \(\xi\) \((\beta - \sigma(x))\), we apply Lemma 4.1 to \((18)\) (with \(\xi\), \(\sigma\) piecewise continuous function \(\xi\)) to get

\[
\int_{\Omega} |\xi|^{p(x)} \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_2(dx) + \int_{\Omega} |\xi| \log \xi |^{p(x)} \cdot \frac{|\nabla p(x)|^{p(x)} p(x)_{p(x)} \mu_2(dx)},
\]

\[
\begin{align*}
\mu_1(dx) & = (\Phi \cdot u + \sigma(x)|\nabla u|^{p(x)}) \cdot u^{-\beta-1} \chi_{\{u > 0\}} \ dx, \\
\mu_2(dx) & = \left(\frac{p(x) - 1}{\beta - \sigma(x)} \right)^{p(x)-1} 2^{p(x)-1} \chi_{\{|\nabla u| \neq 0\}} u^{p(x)-\beta-1} \chi_{\{|\nabla u| \neq 0\}} \ dx.
\end{align*}
\]

We take \(\xi(x) = (\phi(x))^{\frac{1}{p(x)}}\). Then whenever \(\phi > 0\), we have

\[
|\nabla \xi| = \frac{1}{p(x)} \phi^{\frac{1}{p(x)}} - 1 \nabla \phi - \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x).
\]

Equivalently, we have

\[
\phi^{\frac{1}{p(x)}} - 1 \nabla \phi = p(x) \nabla \xi + \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x).
\]

\[
(18)
\]

We observe that

\[
\left\{ \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} |\nabla p(x)| \neq 0 \right\} \subseteq \{ |\nabla p(x)| \neq 0 \} =: P.
\]

We apply Lemma 1.1 to \((18)\) (with \(s_1 = \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} |\nabla p(x)|\)) and \(s_2 = p(x) |\nabla \xi|\) to get

\[
\left| \phi^{\frac{1}{p(x)}} - 1 \nabla \phi \right|^{p(x)} = \left| p(x) \nabla \xi + \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x) \right|^{p(x)} \leq \left| 2 p(x)^{1} \chi_{P} |p(x)|^{p(x)} + 2 p(x)^{1} \chi_{P} \left| \phi^{\frac{1}{p(x)}} \nabla p(x) \right|^{p(x)} \right|.
\]

\[
13
\]
We substitute $\xi^{p(x)} = \phi$ on the right-hand side of (19) to obtain

$$|\nabla \phi|^{p(x)} \phi^{1-p(x)} = \left| \phi \frac{1}{p(x)-1} \nabla \phi \right|^{p(x)} \leq$$

$$\leq 2^{(p(x)-1)\chi_p} |p(x)\nabla \xi|^{p(x)} + 2^{(p(x)-1)\chi_p} \left| \frac{\log(\xi)}{p(x)}^{\xi} \nabla p(x) \right|^{p(x)} =$$

$$= 2^{(p(x)-1)\chi_p} |p(x)\nabla \xi|^{p(x)} + 2^{(p(x)-1)\chi_p} |\xi \log \xi \nabla p(x)|^{p(x)}. \quad (20)$$

We recall that $\mu_1$ is given in (16) and let us denote $\mu$ as follows

$$\mu(dx) = \frac{(p(x) - 1)^{p(x)-1}}{p(x)^{p(x)}(\beta - \sigma(x))^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{|\nabla u| \neq 0\}} \, dx.$$

Applying (20), we get

$$\int_{\Omega} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \mu(dx) = \int_{\Omega} \left| \phi \frac{1}{p(x)-1} \nabla \phi \right|^{p(x)} \mu(dx) \leq$$

$$\leq \int_{\Omega} 2^{(p(x)-1)\chi_p} \left(|p(x)\nabla \xi|^{p(x)} + |\xi \log \xi \nabla p(x)|^{p(x)}\right) \mu(dx) =$$

$$= \int_{\Omega} |\nabla \xi|^{p(x)} \mu_2(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \frac{\nabla p(x)|^{p(x)}{p(x)^{p(x)}}} \mu_2(dx),$$

where $\mu_2(dx)$ is given by (17).

Summing up, by Theorem 3.1 we obtain

$$\int_{\Omega} \xi^{p(x)} \mu_1(dx) = \int_{\Omega} \phi \mu_1(dx) \leq \int_{\Omega} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \mu(dx) \leq$$

$$\leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_2(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \frac{\nabla p(x)|^{p(x)}{p(x)^{p(x)}}} \mu_2(dx),$$

which completes the proof. \qed

## 5 One–dimensional case

We start this section with the reduced version of Theorem 4.1, when $\Omega = I$ is an open interval in $\mathbb{R}$ (not necessarily finite). Then we give a few original examples indicating that our conditions on admissible functions $p(x)$ are not very restrictive.

**Theorem 5.1** (One–dimensional inequality). *Let $p \in \mathcal{P}(I)$, $I \subseteq \mathbb{R}$, and $u \in W^{-1,p(x)}(I) \cap W^{2,1}_{\text{loc}}(I)$ be a nonnegative function. Moreover, suppose that there*
exists a piecewise continuous function $\sigma(x)$ for which the following condition is satisfied

$$g(x) := \sigma(x)(u')^2 - p'(x)uu' \log |u'| \cdot \text{sgn}(x) - (p(x) - 1)uu'' \geq 0$$

(21)

for a.e. $x \in I$. Assume further that $\beta > 0$ is an arbitrary number such that for every $x$, we have $\beta > \sigma(x) \geq \sigma_0 := \inf_{x \in I \cap \{u(x) > 0\}} \{\sigma(x) : g(x) \geq 0\} > -\infty$.

Then, for every Lipschitz function $\xi$ with compact support in $I$, we have

$$\int_I |\xi|^{p(x)} \mu_1(dx) \leq \int_I |\xi'|^{p(x)} \mu_2(dx) + \int_I |\xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_2(dx),$$

(22)

where

$$\mu_1(dx) = \frac{|u'|^{p(x)-2}}{u^{\beta+1}}g(x)\chi_{\{u>0\}}dx,$$

$$\mu_2(dx) = \left(\frac{p(x)-1}{\beta - \sigma(x)}\right)^{p(x)-1}2^{p(x)-1}\chi_{\{p' \neq 0\}}u^{p(x)-\beta-1}\chi_{\{|u'| \neq 0\}}dx.$$ 

Proof. It suffices to apply Theorem 4.1 with $u = u(x)$, $x \in I$. Suppose $\tilde{I}$ is the set where $u''$ is well defined, then

$$\Delta_{p(x)}u = (|u'|^{p(x)-2}u')' = (|u'|^{p(x)-2}u')' + |u'|^{p(x)-2}u'' \quad \text{on} \quad \tilde{I}$$

and thus

$$-\Delta_{p(x)}u = -|u'|^{p(x)-2}|p'(x) \cdot \text{sgn}(x) \cdot u' \log |u'| + (p(x) - 2)u'' + uu'' \quad \text{on} \quad \tilde{I}.$$ 

We set

$$\Phi = \begin{cases} -\Delta_{p(x)}u & \text{if} \quad u \in \tilde{I}, \\ 0 & \text{if} \quad u \in I \setminus \tilde{I}, \end{cases}$$

which satisfies all the restrictions of Theorem 4.1.

Direct computations gives inequality (22). \qed

We present several particular results in one dimension. We start with the inequality with power–type weights.
**Corollary 5.1.** Suppose $\beta > 0$ and $\alpha \in \mathbb{R}$ are arbitrary numbers and $I \subset \mathbb{R}^+_+$ is an open subset. Let $p \in \mathcal{P}(I)$ and $\sigma(x)$ be a piecewise continuous function bounded by $\beta$ such that for a.e. $x \in I$ we have

$$\mathcal{g}(x) := \sigma(x)\alpha^2 - p'(x)x\alpha \log|\alpha x^{\alpha^{-1}}| + (p(x) - 1)\alpha(1 - \alpha) \geq 0.$$  

(23)

Then, for every Lipschitz function $\xi$ with compact support in $I$, we have

$$\int_I |\xi|^{p(x)} \mu_1(dx) \leq \int_I |\xi'|^{p(x)} \mu_2(dx) + \int_I |\xi \log |\xi||^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_2(dx),$$  

(24) with

$$\mu_1(dx) = |\alpha|^{p(x)-2}x^{\alpha(p(x)-\beta-1)-p(x)} \cdot \mathcal{g}(x) \, dx,$$

$$\mu_2(dx) = x^{\alpha(p(x)-\beta-1)} \left(2 \cdot \frac{p(x)-1}{\beta - \sigma(x)}\right)^{p(x)-1} \, dx.$$  

Proof. We apply Theorem 5.1 with the function $u = x^\alpha$. We note that $u' = \alpha x^{\alpha-1}$ and $u'' = \alpha(\alpha - 1)x^{\alpha-2}$ and thus according to (21) we have

$$g(x) = x^{2\alpha-2} \left[\sigma(x)\alpha^2 - p'(x)x\alpha \log|\alpha x^{\alpha^{-1}}| + (p(x) - 1)\alpha(1 - \alpha)\right],$$

which is nonnegative due to (23). Direct computations gives inequality (24) with the desired measures. \hfill \Box

Applying directly Theorem 4.1 instead of Theorem 5.1 we may consider $u = c|x|$ with $c \geq 1$ on $\mathbb{R}$. In this case $u' = c sgn(x)$, $u'' \equiv 0$ a.e. Hence, we obtain the following result.

**Corollary 5.2.** Suppose $c \geq 1$ and $\beta > 0$ are arbitrary numbers and $I \subset \mathbb{R}$ is an open subset. Let $p \in \mathcal{P}(I)$ and $\sigma(x)$ be a piecewise continuous function bounded by $\beta$ and

$$\sigma(x) \geq |x|p'(x) \log c \text{ on } I.$$

Then for every Lipschitz function $\xi$ with compact support in $I$, we have

$$\int_I |\xi|^{p(x)} \mu_1(dx) \leq \int_I |\xi'|^{p(x)} \mu_2(dx) + \int_I |\xi \log |\xi||^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_2(dx),$$

where

$$\mu_1(dx) = c^{p(x)}|x|^{-\beta-1} \left[\sigma(x) - |x|p'(x) \log c\right] \, dx,$$

$$\mu_2(dx) = c^{p(x)}|x|^{p(x)-\beta-1} \left[2 \cdot \frac{p(x)-1}{\beta - \sigma(x)}\right]^{p(x)-1} \, dx.$$
Remark 5.1. We give examples of triplets of $p(x)$, $\sigma(x)$, and the interval $I$ admissible in Corollary 5.2.

- For arbitrary numbers $d > 1$, $d_1, d_2, M > 0$ such that $d_1 < d_2$ and $c > 1$, we may take
  
  $$p(x) = \frac{d_1}{\log c} \log |x| + d$$

  and any piecewise continuous function $\sigma(x)$ such that $-d_1 \leq \sigma(x) \leq d_2$ on the set $(-M, M)$ (e.g. $\sigma(x) \equiv d_2$).

- For arbitrary numbers $1 < p^- \leq p^+ < \infty$, $d > 1$, and $c \geq 1$, we may take
  
  $$p(x) = \frac{p^+ x + d}{x + d} + p^- - 1$$

  and any function $\sigma(x)$ such that $\sigma(x) \geq \frac{d(p^+ - 1) |x| \log c}{(x + d)^2} \log c$ (e.g. $\sigma(x) \equiv d(p^+ - 1) \log c$), which is piecewise continuous and bounded on $\mathbb{R}$.

As an another example we give the following inequality.

Corollary 5.3. Suppose $a, \beta > 0$ are arbitrary numbers and $I \subseteq \mathbb{R}_+$ is an open subset. Let $p \in \mathcal{P}(I)$ and $\sigma(x)$ be a piecewise continuous function bounded by $\beta$ such that

$$g(x) := \sigma(x) + p'(x) x \log \frac{a}{x^2} - 2p(x) + 2 \geq 0.$$  

Then for every Lipschitz function $\xi$ with compact support in $I$, we have

$$\int_I |\xi|^{p(x)} \mu_1(dx) \leq \int_I |\xi'|^{p(x)} \mu_2(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{p'(x)|p(x)}{p(x)^{p(x)}} \mu_2(dx), \quad (25)$$

where

$$\mu_1(dx) = \left( \frac{a}{x} \right)^{p(x) - \beta - 1} x^{-p(x)} \cdot g(x) \, dx,$$

$$\mu_2(dx) = \left( \frac{a}{x} \right)^{p(x) - \beta - 1} \left( 2 \cdot \frac{p(x) - 1}{\beta - \sigma(x)} \right)^{p(x) - 1} \, dx.$$

Proof. We apply Theorem 5.1 with the function $u = \frac{a}{x}$. Direct computations gives inequality (25) with the desired measures. \qed
We obtain also an inequality with exponential–type weights.

**Corollary 5.4.** Let $\beta > 0$ be arbitrary number, $I \subseteq \mathbb{R}$ is an open subset and $p \in \mathcal{P}(I)$. Suppose there exists a piecewise continuous function $\sigma(x)$ bounded by $\beta$ such that

$$
\bar{g}(x) := \sigma(x) - p'(x)x \cdot \text{sgn}(x) + p(x) - 1 \quad \text{on} \quad I.
$$

Then for every Lipschitz function $\xi$ with compact support in $I$, we have

$$
\int_I |\xi|^{p(x)} \mu_1(dx) \leq \int_I |\xi'|^{p(x)} \mu_2(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_2(dx),
$$

where

$$
\mu_1(dx) = \bar{g}(x)e^{x(p(x) - \beta - 1)} \, dx,
$$

$$
\mu_2(dx) = \left(2 \cdot \frac{p(x) - 1}{\beta - \sigma(x)}\right)^{p(x)-1} e^{x(p(x) - \beta - 1)} \, dx.
$$

**Proof.** We apply Theorem 5.1 with the function $u = e^x$. We note that according to (21) we have

$$
\bar{g}(x) = e^{2x} (\sigma(x) - p'(x)x \cdot \text{sgn}(x) - p(x) + 1),
$$

which is nonnegative due to (26). Direct computations gives inequality (27) with the desired measures $\mu_1$ and $\mu_2$.

**Remark 5.2.** We give examples of triplets of $p(x), \sigma(x)$ and the interval $I$ admissible in Corollary 5.4.

- For arbitrary $d > 0$, we may take $p(x) = 1 + \frac{d}{|x|+1}$ and any function $\sigma(x)$, which is nonnegative, piecewise continuous, and bounded on any interval $I \subseteq \mathbb{R}$.

- We may take $p(x) = e^x$, any piecewise continuous function $\sigma(x)$ such that

  $$
  \sigma(x) \geq (x \text{sgn}(x) + 1)e^x - 1
  $$

  which is bounded on a finite interval $I \subseteq \mathbb{R}_+$.

- We may take $p(x) = 2 - e^{-x^2}$ and $\sigma(x) \geq e^{-x^2}(2x^2 \text{sgn}(x) - 1) + 1$ (e.g. $\sigma(x) \equiv 2e^{-3/2} + 1$) on $I = (0, \infty)$. 

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6 Links with the existing results

In this section we present several applications of Theorem 4.1. We start with re–obtaining the main result of Skrzypczak [41], which deals with constant function \( p(x) \) and implies classical Hardy inequality with optimal constant (see [41], Theorem 5.1). Then we concentrate on the comparison with the results of Harjulehto–Hästö–Koskenoja [19] and Mashiyev–Çekić–Mamedov–Ogras [31]. We mention also the related papers by Diening–Samko [15], Rafeiro–Samko [36] and by Harman [20].

Our next paper focused on \( n \)-dimensional inequalities, in particular with radial weights, is in preparation (see [3]).

Results of Skrzypczak [41, 42]

When we consider \( 1 < p(x) \equiv p < \infty \) in Theorem 4.1, we retrieve the main result of [41], implying the classical Hardy inequality with the optimal constant (see [41] for the details and the numerous other examples). Moreover, the following theorem leads to Hardy–Poincaré inequalities with the weights of a type \( \left( 1 + |x|^\frac{p}{p-1} \right)^\alpha \), where the constants are proven to be optimal for sufficiently big parameter \( \alpha > 0 \) (see [42] for the details).

Corollary 6.1 ([41, Theorem 4.1]). Assume that \( 1 < p < \infty \) and \( u \in W^{1,p}_\text{loc}(\Omega) \) is a nonnegative solution to the PDI \( -\Delta_p u \geq \Phi \), in the sense of Definition 2.1, where \( \Phi \) is locally integrable and satisfies the condition (6) with \( \sigma_0 \in \mathbb{R} \). Assume further that \( \beta \) and \( \sigma \) are arbitrary numbers such that \( \beta > 0 \) and \( \beta > \sigma \geq \sigma_0 \). Then, for every Lipschitz function \( \xi \) with compact support in \( \Omega \), we have

\[
\int_\Omega |\xi|^p \mu_1(dx) \leq \int_\Omega |\nabla \xi|^p \mu_2(dx),
\]

where

\[
\mu_1(dx) = \left( \frac{\beta - \sigma}{p - 1} \right)^{p-1} (\Phi \cdot u + \sigma |\nabla u|^p) \cdot u^{-\beta-1} \chi_{\{u>0\}} \, dx,
\]

\[
\mu_2(dx) = u^{p-\beta-1} \chi_{\{|\nabla u|\neq 0\}} \, dx.
\]

Remark 6.1. See Remark 4.1 to realise how Lemma 4.1 is useful in the proof of Theorem 4.1. It enables us to retrieve Corollary 6.1 with exactly the same constant and thus it is necessary to obtain classical Hardy inequality with the optimal constant.
Results of Harjulehto–Hästö–Koskenoja [19]

Paper [19] concerns the $n$–dimensional norm version of Hardy inequality, but also the one–dimensional case is specially emphasized therein. Let us mention the following result.

**Theorem 6.1** ([19, Theorem 5.2]). Let $I = [0, M) \, \text{for} \, M < \infty$, the variable exponent $p : I \to [1, \infty)$ be bounded, $p(0) > 1$ and

$$\limsup_{x \to 0^+} (p(x) - p(0)) \log \frac{1}{x} < \infty. \quad (28)$$

Moreover, suppose $\text{ess sup}_{x \in (0,x_0)} p(x) = p(0)$ for some $x_0 \in (0, 1)$. If $a \in [0, 1 - \frac{1}{p(0)}]$, then Hardy inequality

$$\|\xi(x)x^{a-1}\|_{L^p(x)(I)} \leq C\|\xi'(x)x^a\|_{L_p(I)} \quad (29)$$

holds for every $\xi \in W^{1,p(x)}(I)$ with $\xi(0) = 0$.

We have the following related result.

**Corollary 6.2.** Suppose $a, \beta > 0$ are arbitrary numbers and $I \subseteq \mathbb{R}_+$ is an open subset. Let $p \in \mathcal{P}(I)$ be continuous and assume that there exists a continuous function $A(x)$ such that

$$a\beta + (1 - a)xp'(x) \log x + (a - 3)(p(x) - 1) \geq A(x) \geq 0. \quad (30)$$

Then, for every Lipschitz function $\xi$ with compact support in $I$, we have

$$\int_I |x^{a-1}\xi|^{p(x)}\mu_1(dx) \leq \int_I |x^{a}\xi'|^{p(x)}\mu_2(dx) + \int_I \left(x^a|\xi|\log \frac{|\xi'(x)|}{p(x)}\right)^{p(x)}\mu_2(dx), \quad (31)$$

where

$$\mu_1(dx) = x^{-a(\beta+1)}A(x) \, dx,$$

$$\mu_2(dx) = x^{-a(\beta+1)} \, dx.$$

**Proof.** We apply $u = \frac{1}{a}x^a$ in Theorem 4.1 and we obtain inequality (13) with measures $\tilde{\mu}_1, \tilde{\mu}_2$. We simplify the right–hand side measure $\tilde{\mu}_2$ by taking $\sigma(x) = \beta - \frac{2}{a}(p(x) - 1)$ which is bounded.
We ensure the condition (5) by (30). Indeed, we estimate the expression in the left–hand side measure \( \tilde{\mu}_1 \) from below as follows

\[
a \beta - 2(p(x) - 1) + (1 - a) [x \log xp'(x) - (p(x) - 1)] = \]
\[
= a \beta + (1 - a)x \log xp'(x) + (a - 3)(p(x) - 1) \geq A(x) \geq 0.
\]

We reach the goal by dividing both sides by \( a \beta \).

**Remark 6.2** (Comparison of Theorem 6.1 and Corollary 6.2). Inequalities (29) and our (31) are similar, however there are some differences. Inequality (29) is a norm version, while (31) is a modular one. Moreover, our inequality, unlike (29), involves the additional term as well as the weights \( \mu_1 \) and \( \mu_2 \) of power type with strictly negative exponents.

Our restriction on the exponent \( p(x) \) is of different type than in Theorem 6.1. We do not expect that \( p(x) \) satisfies the log–Hölder condition (see (28)). We require piecewise differentiable function \( p(x) \), which may even not be continuous. Furthermore, we formulate inequality (31) for every Lipschitz and compactly supported function \( \xi \) in \( I \), while (29) is stated for \( \xi \in W^{1,p(x)}(I) \) with \( \xi(0) = 0 \). Moreover, we allow infinite interval \( I \) and a bit different range of parameter \( a \).

**Remark 6.3.** The following functions are admissible for \( p(x) \) in Corollary 6.2 and do not satisfy the log–Hölder condition at the origin (required in Theorem 6.1).

- If \( k > 0 \) and \( d_1 > d_2 > 0 \), we take \( p(x) = \frac{k + d_1 x}{k + d_2 x} \) (\( I \) may be infinite).
- If \( \gamma \geq 1 \), we take \( p(x) = 2 - \frac{1}{x+\gamma} \) (\( I \) may be infinite).
- If \( I = (0, M) \) with \( M < \infty \) and \( k > 0 \), we take \( p(x) = k + \log(e + x) \).

**Remark 6.4.** The following functions \( p(x) \) are admissible both in Corollary 6.2 and in Theorem 6.1. Let us consider the inequality on the interval \( I = (0, M) \) with \( M < \infty \).

- If \( \gamma > 1 \), we take \( p(x) = x + \gamma \).
- If \( \gamma \geq 0 \), we take \( p(x) = 1 + \frac{1}{x+\gamma} \).
- If \( \gamma \geq 0 \), we take \( p(x) = \gamma + e^x \).

In each example of the above ones, in \( \mu_1 \) we may choose \( A(x) \) separated from zero (i.e. there exists a constant \( A_0 \) such that \( A(x) \geq A_0 > 0 \) for every \( x \)).
Results of Mashiyev-Çekiç-Mamedov-Ogras [31]

In [31] the authors prove the following extension of Hardy inequality from [19] by Harjulehto–Hästö–Koskenoja described above.

**Theorem 6.2 ([31, Theorem 3]).** Suppose $p(x), q(x)$ and $\alpha(x)$ are log–Hölder continuous at the origin and at the infinity, i.e. there exist constants $C_i$, $i = 1, 2$, such that the following conditions hold

$$
|p(x) - p(0)| \log \frac{1}{x} \leq C_1, \quad \text{where } x \in (0, 1/2] \tag{32}
$$

and

$$
|p(x) - \lim_{|x| \to \infty} p(x)| \log(e + x) \leq C_2, \quad \text{where } x \in (0, \infty), \tag{33}
$$

with $1 < p^- \leq p(x) \leq q(x) \leq q^+ < \infty$ and $-\infty < \alpha^- \leq \alpha(x) < \infty$ for $x \in (0, \infty)$. Then there exists a constant $C > 0$ such that for every function $\xi$, absolutely continuous on $[0, \infty)$, with $\xi(0) = 0$ we have

$$
\|\xi(x)x^{\alpha(x)-\frac{1}{p(x)}}\|_{L^q(I_0, \infty)} \leq C\|\xi'(x)x^{\alpha(x)}\|_{L^p(I_0, \infty)}. \tag{34}
$$

In [19] the authors prove (34) with constant $\alpha$, $q(x) = p(x) > 1$, on a finite interval $I$, and without the assumption $p(0) \leq p(x)$ for small $x$’s.

We have the following related result.

**Remark 6.5.** When in Corollary [6.2] we assume additionally that $A(x)$ is separated from zero ($A(x) \geq A_0 > 0$ for every $x$), we take $I = \mathbb{R}_+$, we put $\alpha(x) = a \left(1 - \frac{\beta+1}{p(x)}\right)$, and we rearrange power–type terms, we obtain

$$
A_0 \left(\int_0^\infty |x^{\alpha(x)-1}\xi|^{p(x)} \, dx \right) \leq \int_0^\infty |x^{\alpha(x)}\xi'|^{p(x)} \, dx + \left(x^{\alpha(x)}|\xi\log|\xi|\xi'|^{p(x)} \right)\, dx
$$

for every Lipschitz function $\xi$ with compact support in $\mathbb{R}_+$.

The comparison of Theorem 6.2 with our above inequality is similar as in the case of theorem by Harjulehto–Hästö–Koskenoja [19] (see Remark 6.2).

**Summary of the above comparison**

We would like to emphasize that we supply a new tool, different from the existing ones, especially from [19] and [32]. Our inequality is a modular one and involves the additional term. We do not assume log–Hölder continuity of $p(x)$. In particular, it can even be not continuous, if only it is piecewise differentiable and bounded. Moreover, we allow either finite or infinite interval $I$. 

22
Results of Diening–Samko \[15\], Rafeiro–Samko \[36\] and Harman \[20\]

In \[15\] the derived Hardy inequality is connected with Hardy operator. For \(p \in \mathcal{P}(0, \infty)\) satisfying conditions analogous to (32) and (33) (related to log–Hölder continuity), the authors prove the following inequality

\[
\left\| x^{\alpha(x)+\mu(x)-1} \int_0^x \frac{\xi(y)}{y^{\alpha(y)}} dy \right\|_{L^p(x)(0,\infty)} \leq C \left\| \xi \right\|_{L^p(x)(0,\infty)},
\]

where \(x \in (0, \infty)\), an exponent \(q \in \mathcal{P}(0, \infty)\) is any function such that \(\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0)\) with \(\mu(0) \in (0, \frac{1}{p(0)})\), \(\frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty)\) with \(\mu(\infty) \in (0, \frac{1}{p(\infty)})\) and \(\alpha(0) < \frac{1}{p(0)}, \alpha(\infty) < \frac{1}{p(\infty)}\). Exponents \(q(x), \mu(x)\) and \(\alpha(x)\) are also supposed to satisfy local log–Hölder condition in zero and infinity.

In \[36\] by Rafeiro–Samko the derived Hardy inequality is connected with the Riesz potential. It is stated on a bounded domain \(\Omega \subset \mathbb{R}^n\), which complement \(\mathbb{R}^n \setminus \Omega\) has the cone property. Similar Hardy inequality is considered in i.e. \[33\].

The inequality corresponding to results of \[36\], but involving Hardy operator \(Hv(x) = \int_0^x v(t) dt\), is proved in \[20\]. The authors derive the following inequality which holds for every nonnegative and locally integrable function \(\xi\)

\[
\left\| Hv(x)^{\alpha(x)-1} \right\|_{L^p(x)(0,l)} \leq \left\| x^{\alpha(x)} \right\|_{L^p(x)(0,l)},
\]

where \(l > 0\), functions \(\alpha, p : (0, l) \to \mathbb{R}\) are measurable and such that \(-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < \infty\) and \(-\infty < p^- \leq p(x) \leq p^+ < \infty\). Moreover, the author indicate the necessary condition for validity of Hardy inequality \(35\) (see \[20\] Theorem 3, 4)).

7 Open questions

We find it interesting to investigate the following ideas.

Erasing the additional term

Our first question is, when the additional term can be erased? It may be obtained by Sobolev–type embedding theorem stating

\[
\int_{\Omega} |\xi \log \xi|^{p(x)} \cdot \left| \nabla p(x) \right|^{p(x)} \mu_2(dx) \leq c_1 \int_{\Omega} \left| \nabla \xi \right|^{p(x)} \mu_2(dx),
\]

(36)
where $c_1 > 0$ and $\mu_2(dx)$ is given by (17).

Inequality of the form (36) could be applied to (15) in order to estimate the left–hand side by one term and to obtain inequality of the following form

$$\int_\Omega |\xi|^{p(x)} \mu_1(dx) \leq c_2 \int_\Omega |\nabla \xi|^{p(x)} \mu_2(dx),$$

(37)

where $c_2 > 0$ and $\mu_1(dx), \mu_2(dx)$ are given by (16), (17), respectively. Eventually, it may be satisfying to prove (37) with a worse measure instead of $\mu_2(dx)$.

Such a result would substantially improve our result.

**Improving the right–hand exponent**

We find it deserving attention to improve an exponent on the right–hand side of (15). When is it possible to prove an inequality

$$\int_\Omega |\xi|^{q(x)} \mu_1(dx) \leq \int_\Omega |\nabla \xi|^{p(x)} \mu_2(dx) + \int_\Omega |\xi \log \xi|^{p(x)} \mu_3(dx),$$

with $q(x) > p(x)$?

**Qualitative properites of solutions to nonlinear problems**

$p(x)$–Hardy inequalities of the type (15) may be applied in investigations on certain problems stated in variable exponent Lebesgue spaces such as models of electrorheological fluids (see references in Introduction) or other problems involving $p(x)$–Laplacian e.g. of the type (2). Among other qualitative properties of solutions to nonlinear problems, nonexistence to (2) may be proven due to the method introduced in [26].

**Obtaining a new type of inequalities via our approach**

It would be interesting to extend our approach to link certain nonlinear eigenvalue problem, possibly involving more general operator than $p(x)$–Laplacian, with the results of Diening–Samko [15], Rafeiro–Samko [36] and Harman [20].

**Acknowledgments**

The authors would like to thank Agnieszka Kalamajska and Tomasz Adamowicz for discussions.
References

[1] G. Barbatis, S. Filippas, A. Tertikas, *A unified approach to improved $L^p$ Hardy inequalities with best constants*, Trans. Amer. Math. Soc. 356 (2004), no. 6, 2169–2196.

[2] G. Barbatis, S. Filippas, A. Tertikas, *Series expansion for $L^p$ Hardy inequalities*, Indiana Univ. Math. J. 52 (2003), no. 1, 171–190.

[3] S. Barnaś, I. Skrzypczak, *On some Hardy inequality in variable exponent Lebesgue spaces and its applications*, in preparation.

[4] B. Bojarski, P. Hajłasz, *Pointwise inequalities for Sobolev functions and some applications*, Studia Math. 106 (1993), 77–92.

[5] S. Boza, J. Soria, *Weighted Hardy modular inequalities in variable $L^p$ spaces for decreasing functions*, J. Math. Anal. Appl. 348 (2008) 383–388.

[6] S. M. Buckley, R. Hurri–Syrjänen, *Iterated log–scale Orlicz–Hardy inequalities*, Ann. Acad. Sci. Fenn. Math. 38, No. 2 (2013), 757–770.

[7] S. M. Buckley, P. Koskela, *Orlicz–Hardy inequalities*, Ill. J. Math. 48, No. 3 (2004), 787–802.

[8] P. L. Butzer, F. Fehér, *Generalized Hardy and Hardy–Littlewood inequalities in rearrangement–invariant spaces*, Comment. Math. Prace Mat. Tomus Specialis in Honorum Ladislai Orlicz 1 (1978), 41–64.

[9] R. Caccioppoli, *Limitazioni integrali per le soluzioni di un’equazione lineare ellittica a derivate parziali*, Giorn. Mat. Battaglini (4) 4 (80) (1951), 186–212.

[10] L. D’Ambrosio, *Hardy Type Inequalities Related to Degenerate Elliptic Differential Operators*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. ser. 5, IV (2005), 451–486.

[11] L. D’Ambrosio, *Some Hardy inequalities on the Heisenberg group*, (Russian) Differ. Uravn. 40 (2004), No. 4, 509–521, 575; translation in Differ. Equ. 40 (2004), No. 4, 552–564.
[12] L. D’Ambrosio, *Hardy inequalities related to Grushin type operators*, Proc. Amer. Math. Soc. 132 (2004), No. 3, 725–734.

[13] L. D’Ambrosio, S. Dipierro, *Hardy inequalities on Riemannian manifolds and applications*, Ann. I.H. Poincare–AN (2013), http://dx.doi.org/10.1016/j.anihpc.2013.04.004.

[14] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer–Verlag, Heidelberg, 2011.

[15] L. Diening, S. Samko, *Hardy inequality in variable exponent Lebesgue spaces*, Fract. Calc. Appl. Anal., 10 (1) (2007), 1–17.

[16] X. Fan, Q. Zhang, *Existence of solutions for p(x)–Laplacian Dirichlet problem*, Nonlinear Anal. 52 (2003), 1843–1852.

[17] X. Fan, D. Zhao, *On the generalized Orlicz–Sobolev space W^{k,p(x)}(Ω)*, J. Gansu Educ. College 12 (1) (1998), 1–6.

[18] X. Fan, D. Zhao, *On the spaces L^{p(x)}(Ω) and W^{m,p(x)}(Ω)*, J. Math. Anal. Appl. 263 (2001), 424-446.

[19] P. Harjulehto, P. Hästö and M. Koskenoja, *Hardy’s inequality in a variable exponent Sobolev spaces*, Georgian Math. J. 12 (2005), no. 3, 431–442.

[20] A. Harman, *On Necessary Condition for the Variable Exponent Hardy Inequality*, J. Func. Sp. Appl. 2012 (2012), http://dx.doi.org/10.1155/2012/385925

[21] A. Harman, F. I. Mamedov, *On Boundedness of Weighted Hardy Operator in L^{p(·)} and Regularity Condition*, J. Ineq. Appl. 2010 (2010), http://dx.doi.org/10.1155/2010/837951

[22] T. C. Halsey, *Electrorheological fluids*, Science 258 (1992), 761–766.

[23] T. Iwaniec, C. Sbordone, *Caccioppoli estimates and very weak solutions of elliptic equations*. Renato Caccioppoli and modern analysis. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 14 (2003), No. 3 (2004), 189–205.

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[24] A. Kalamajska, K. Pietruska–Paluba, *On a variant of the Hardy inequality between weighted Orlicz spaces*, Studia Math. 193 (2009), 1–28.

[25] A. Kalamajska, K. Pietruska–Paluba, *New Orlicz variants of Hardy type inequalities with power, power–logarithmic, and power–exponential weights*, Cent. Eur. J. Math. 10, No. 6 (2012), 2033–2050.

[26] A. Kalamajska, K. Pietruska–Paluba, I. Skrzypczak, *Nonexistence results for differential inequalities involving A–Laplacian*, Adv. Diff. Eqs. (Vol. 17) no 3–4 (2012), 307–336.

[27] O. Kovácik and J. Rákosník, *On spaces L^{p(x)} and W^{1,p(x)},* Czechoslovak Math. J. 41(116) (1991), 592–618.

[28] A. Kufner, L. Maliganda, L. E. Persson, *The Hardy inequality. About its history and some related results*, Vydavatelský Servis, Plzen, 2007.

[29] A. Kufner, B. Opic, *Hardy–type inequalities*, Longman Scientific and Technical, Harlow, 1990.

[30] A. Kufner, L. E. Persson, *Weighted inequalities of Hardy type*, World Sci., River Edge, NJ, 2003.

[31] R. Mashiyev, B. Çekiç, S. Ogras, *On Hardy’s inequality in L^{p(x)}(0, ∞)* JIPAM. J. Inequal. Pure Appl. Math 7 (3), 1–5.

[32] R. Mashiyev, B. Çekiç, F. I. Mamedov, S. Ogras, *Hardy’s inequality in power–type weighted L^{p(x)}(0, ∞),* J. Math. Anal. Appl., 334 (1) (2007), 289–298.

[33] F. I. Mamedov, A. Harman, *On a weighted inequality of Hardy type in spaces L^{p(c)},* J. Math. Anal. Appl., 353 (2) (2009), 521–530.

[34] E. Mitidieri, *A simple approach to Hardy inequalities*, Mat. Zametki 67 (2000) 563–572.

[35] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. 3 (1931), 200–211.

[36] H. Rafeiro, S. Samko, *Hardy type inequality in variable Lebesgue spaces*, Annales Academ. Scien. Fen. Mathematica 34 (2009), 279–289.
[37] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer–Verlag, Berlin, 2000.

[38] S. Samko, *Hardy inequality in the generalized Lebesgue spaces*, Fract. Calc. Appl. Anal. 6(4) (2003), 355–362.

[39] S. Samko, *Hardy–Littlewood–Stein–Weiss inequality in the Lebesgue spaces with variable exponent*, Fract. Calc. Appl. Anal. 6 (4) (2003), 421–440.

[40] I. I. Sharapudinov, *On the topology of the space $L^{p(t)}([0; 1])$*, Matem. Zametki 26 (1978), no. 4, 613–632.

[41] I. Skrzypczak, *Hardy–type inequalities derived from p–harmonic problems*, Nonlinear Analysis TMA Vol. 93, (2013), 30–50.

[42] I. Skrzypczak, *Hardy–Poincaré type inequalities derived from p–harmonic problems*, Banach Center Publ. 101 Calculus of variations and PDEs (2014), 223–236.

[43] I. Skrzypczak, *Hardy inequalities resulted from nonlinear problems dealing with A–Laplacian*, NoDEA Nonlinear Differential Equations Appl. (2014), DOI 10.1007/s00030-014-0269-y.

[44] I. V. Tsenov, *Generalization of the problem of best approximation of a function in the space $L^p$*, Uch. Zap. Dagestan Gos. Univ. 7 (1961), 25–37.