Weakly convex biharmonic hypersurfaces in Euclidean spaces are minimal

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Abstract

A submanifold $M^m$ of a Euclidean space $\mathbb{R}^{m+p}$ is said to have harmonic mean curvature vector field if $\Delta \vec{H} = 0$, where $\vec{H}$ is the mean curvature vector field of $M \hookrightarrow \mathbb{R}^{m+p}$ and $\Delta$ is the rough Laplacian on $M$. There is a famous conjecture named after Bangyen Chen which states that submanifolds of Euclidean spaces with harmonic mean curvature vector fields are minimal. In this paper we prove that weakly convex hypersurfaces (i.e. hypersurfaces whose principle curvatures are nonnegative) with harmonic mean curvature vector fields in Euclidean spaces are minimal.

1 Introduction

Let $\vec{x} : M^m \rightarrow \mathbb{R}^{m+p}$ be an immersion from a Riemmanian manifold $M$ of dimension $m$ to a Euclidean space of dimension $m + p$, $p \geq 1$. Denote by $\vec{x}, \vec{H}, \Delta$ respectively the position vector of $M$, the mean curvature vector of $M$ and the Laplacian operator with respect to the induced metric $g$ on $M$. Then it is well known that (see for example [3])

$$\Delta \vec{x} = -n \vec{H}.$$  

This shows that $M$ is a minimal submanifold if and only if its coordinates functions are harmonic functions. According to this equation, a submanifold in a Euclidean space with harmonic mean curvature vector field, i.e.

$$\Delta \vec{H} = 0,$$  \hspace{1cm} (1.1)

if and only if

$$\Delta^2 \vec{x} = 0.$$  \hspace{1cm} (1.2)

Therefore a submanifold with harmonic mean curvature vector field is called a biharmonic submanifold.

There is also a variational description of biharmonic submanifolds as follows. Assume that $\vec{x} : M \rightarrow \mathbb{R}^{m+p}$ is as before, then the biharmonic energy of $\vec{x}$ is defined by

$$E_2(\vec{x}) = \int_M |\tau(\vec{x})|^2 dM,$$  \hspace{1cm} (1.3)

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where $\tau(\vec{x})$ is the tension field of $\vec{x}$. The critical points of the functional $E_2$ satisfy the following E-L equation (see [8])

$$\Delta \vec{H} = 0.$$  \hfill (1.4)

As is easy to see, minimal submanifolds of Euclidean spaces are biharmonic submanifolds. It is nature to arise the question whether the space of biharmonic submanifolds is strictly larger than the space of minimal submanifolds. Concerning this problem, Chen conjectured that the answer is no.

**Chen’s Conjecture.** Suppose that $\vec{x} : M^m \to R^{m+p}, p \geq 1,$ satisfies

$$\Delta \vec{H} = 0.$$  \hfill (1.5)

Then $\vec{H} = 0$.

Since Chen’s conjecture arose, it remains to be open with very little progress even for hypersurfaces with dimensions grater that 4. But in recent years it attracts many attentions and there are some partial answers to this conjecture. Now we give an overview of them, as to the author’s knowledge. Chen’s conjecture is proved:

- For Surfaces in $R^3$ in [4] and [8] and in an unpublished work by Chen himself, as reported in [2].
- For hypersurfaces in $R^4$ in [7] and a different proof in [5].
- For proper hypersurfaces which admit at most 2 distinct principle curvatures;
  - For curves;
  - For submanifolds $M^m$ which are pseudo-umbilic and $m \neq 4$;
  - For submanifolds of finite type.
- For submanifolds which are complete and proper, i.e. any preimage of compact subsets are compact in [1].
- For submanifolds whose square integral of the mean curvature vector field satisfies certain decay condition at infinity (see [11]).
- For submanifolds whose $L^p, p \geq 2$, integral of the mean curvature vector field satisfies certain decay condition at infinity (see [9]).

In this paper we prove Chen’s conjecture for weakly convex biharmonic hypersurfaces.

**Theorem 1.1.** Assume that $\vec{x} : M^m \to R^{m+1}$ is a weakly convex biharmonic submanifold in $R^{m+1}$, i.e. $\Delta \vec{H} = 0$. Then $\vec{H} = 0$.

We would like to point out that for closed (compact without boundary) bi-harmonic submanifolds Chen’s conjecture is easily proved to be true by an argument due to Jiang (see [8]) or by directly using an integration by parts argument.

Chen’s conjecture belongs to the research field of classification of biharmonic submanifolds in Riemannian or pseudo-Riemannian manifolds. Nowadays it is an active research field and for readers who have interest with it we refer to a survey paper [10] by Montaldo and Oniciuc and references therein.

**Organization.** In section 2 we give a brief sketch of submanifolds geometry. Theorem 1.1 is proved in section 3.
2 Preliminaries

Assume that \( \vec{x} : M^m \rightarrow N^{m+p} \) is an immersion to a Riemannian manifold \( N \) with Riemannian metric \( \langle \cdot, \cdot \rangle \), which is a bilinear form on \( TN \otimes TN \), the tensor of the tangent vector space of \( N \). Then \( M \) inherits a Riemannian metric from \( N \) by \( g_{ij} := \langle \partial_i \vec{x}, \partial_j \vec{x} \rangle \) and a volume form by \( \sqrt{\det g_{ij}} dx \). The second fundamental form of \( M \hookrightarrow N \), \( h : T M \otimes T M \rightarrow N M \), is defined by

\[
h(X, Y) := D_X Y - \nabla_X Y,
\]

for any \( X, Y \in TM \), where \( D \) is the covariant derivative with respect to the Levi-Civita connection on \( N \), \( \nabla \) is the Levi-Civita connection on \( M \) with respect to the induced metric and \( NM \) is the normal bundle of \( M \). For any normal vector field \( \eta \) the Weingarten map \( A_\eta : TM \rightarrow TM \) is defined by

\[
D_X \eta = -A_\eta X + \nabla_X^\perp \eta,
\]

where \( \nabla^\perp \) is the normal connection and as is well known that \( h \) and \( A \) are related by

\[
\langle h(X, Y), \eta \rangle = \langle A_\eta X, Y \rangle.
\]

For any \( p \in M \), let \( \{e_1, e_2, ..., e_m, e_{m+1}, ..., e_{m+p}\} \) be a local orthonormal basis of \( N \) such that \( \{e_1, ..., e_m\} \) is an orthonormal basis of \( T_p M \). Then \( h \) is decomposed at \( p \) as

\[
h(X, Y) = \sum_{\alpha = m+1}^{m+p} h_\alpha(X, Y)e_\alpha.
\]

The mean curvature vector field is defined as

\[
\vec{H} := \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i) = \sum_{\alpha = m+1}^{m+p} H_\alpha e_\alpha,
\]

where

\[
H_\alpha := \frac{1}{m} \sum_{i=1}^{m} h_\alpha(e_i, e_i).
\]

3 Proof of theorem 1.1

It is well known that for a submanifold \( M^m \) in a Euclidean space to be biharmonic, i.e. \( \Delta \vec{H} = 0 \) if and only if (2)

\[
\Delta^\perp \vec{H} - \sum_{i=1}^{m} h(A_{\vec{H}} e_i, e_i) = 0,
\]

\[
m \nabla |\vec{H}|^2 + 4 \sum_{i=1}^{m} A_{\nabla^\perp}^\perp \vec{H} e_i = 0,
\]

where \( \Delta^\perp \) is the (nonpositive) Laplace operator associated with the normal connection \( \nabla^\perp \).
Assume that $H = H^{}\nu$, where $\nu$ is the unit normal vector field on $M$. Note that by the assumption that $M$ is weakly convex, we have $H \geq 0$. Define

$$B := \{ p \in M : H(p) > 0 \} \quad (3.3)$$

We will prove that $B$ is an empty set by a contradiction argument, and so $M$ is minimal and we are done.

If $B$ is not empty, we see that $B$ is an open subset of $M$. We assume that $B_1$ is a nonempty connect component of $B$. We will prove that $H \equiv 0$ in $B_1$, thus a contradiction.

We prove it in two steps.

**Step 1.** $H$ is a constant in $B_1$.

Let $p \in B_1$ be a point. Around $p$ we choose a local orthonormal frame $\{ e_k, k = 1, \ldots, m \}$ such that $h(e_i, e_k)$ is a diagonal matrix at $p$, where $\nu$ is the unit normal vector field of $M$.

For any $1 \leq k \leq m$ we have at $p$

$$\langle \sum_{i=1}^{m} A^{\perp e_i} e_i, e_k \rangle = \sum_{i=1}^{m} \langle h(e_i, e_k), \nabla_{e_i}^{} H e_k \rangle = \langle h(e_k, e_k), \nabla_{e_k}^{} H \rangle.$$ 

By equation (3.2) we have

$$0 = m \nabla_{e_k}^{} |H|^2 + 4 \sum_{i=1}^{m} A^{\perp e_i} e_i, e_k \rangle = m \nabla_{e_k}^{} |H|^2 + 4 \langle h(e_k, e_k), \nabla_{e_k}^{} H \rangle = 2mH \nabla_{e_k}^{} H + 4 \lambda_k \langle \nu, \nabla_{e_k}^{} H \rangle = (2mH + 4\lambda_k) \nabla_{e_k}^{} H,$$

where $\lambda_k := h(e_k, e_k)$ is the $k$th principle curvature of $M$ at $p$, which is nonnegative by the assumption that $M$ is weakly convex.

By $2mH + 4\lambda_k > 0$ at $p$, one gets

$$\nabla_{e_k}^{} H = 0 \quad \text{at} \quad p, \quad (3.4)$$

for any $k = 1, \ldots, m$, which implies that

$$\nabla H = 0 \quad \text{at} \quad p. \quad (3.5)$$

Because $p$ is an arbitrary point in $B_1$, we see that

$$\nabla H = 0 \quad \text{in} \quad B_1. \quad (3.6)$$

Therefore we get that $H$ is a constant in $B_1$.

**Step 2.** $H$ is zero in $B_1$.

Let $p \in B_1$, by step 1, we see that

$$\Delta |H|^2(p) = 0. \quad (3.7)$$
On the other hand, by equation (3.1), we have
\[ \Delta|\vec{H}|^2 = 2|\nabla^\perp \vec{H}|^2 + 2\langle \vec{H}, \Delta \vec{H} \rangle \geq 2 \sum_{i=1}^{m} \langle h(\vec{H}e_i, e_i), \vec{H} \rangle \]
\[ = 2 \sum_{i=1}^{m} \langle A\vec{H}e_i, A\vec{H}e_i \rangle \]
\[ = 2 \sum_{i=1}^{m} H^2 \langle A\nu e_i, A\nu e_i \rangle \geq 2n \nu H^4. \] (3.8)

From (3.7)-(3.8), we get \( H(p) = 0 \). Because \( p \) is an arbitrary point in \( B_1 \), we see that \( H \equiv 0 \) in \( B_1 \).

**An alternative proof of step 2.** By step 1 we have at \( p \in B_1 \), \( \Delta|\vec{H}|^2 = 0 \).

On the other hand at \( p \)
\[ \Delta|\vec{H}|^2 = 2|\nabla \vec{H}|^2 + 2\langle \vec{H}, \Delta \vec{H} \rangle \]
\[ = 2|\nabla \vec{H}|^2. \]

Therefore \( \nabla \vec{H}(p) = 0 \). Now we choose an orthogonal basis \( \{e_i, i = 1, ..., m\} \) of \( T_pM \). Computing directly one gets at \( p \)
\[ 0 = \langle \nabla_{e_i} \vec{H}, e_j \rangle = H \langle \nabla_{e_i} \nu, e_j \rangle = -H \langle \nu, \nabla_{e_i} e_j \rangle = H \langle h(e_i, e_j), \nu \rangle, \] (3.9)

for any \( 1 \leq i, j \leq m \). Taking trace over this equality we get
\[ H^2(p) = 0. \]

Therefore \( H(p) = 0 \). For \( p \) is an arbitrary point in \( B_1 \) we have \( H \equiv 0 \) in \( B_1 \).

This completes the proof of theorem 1.1. \( \square \)

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