1. Introduction

1.1. Main results. Let $k$ be any field. Consider the hereditary algebra $A = k\Delta$ associated to a finite connected quiver without oriented cycles. A fundamental fact in Representation Theory of Algebras is the distinction of the representation type of $A$: $A$ is representation-finite (that is, there are only finitely many indecomposable $A$-modules, up to isomorphism) exactly when the underlying graph $|\Delta|$ of $\Delta$ is of Dynkin type; $A$ is tame (that is, for each $d \in \mathbb{N}$, the indecomposable $d$-dimensional $A$-modules may be classified in a finite number of one-parameter families of modules) exactly when $|\Delta|$ is of extended Dynkin type; in the remaining cases $A$ is wild (that is, there is an embedding $\text{mod-}k \langle x, y \rangle \rightarrow \text{mod-}A$ which preserves indecomposability and isomorphism classes from the category of finite dimensional modules over the ring in two non-commuting indeterminates into the category of finite dimensional $A$-modules). Consider the Auslander-Reiten translation $\tau_A$ in $\text{mod-}A$ and $P$ an indecomposable projective $A$-module. In case $A$ is tame representation-infinite, the sequence of modules $(\tau_A^{-n}P)_n$ is well defined and the algebra $R(A, P) = \bigoplus_{n=0}^{\infty} \text{Hom}_A(P, \tau_A^{-n}P)$ is an infinite dimensional positively $\mathbb{Z}$-graded surface singularity. The algebra $R(A, P)$ reflects many properties of $A$ and $P$ and has a particularly interesting structure, as shown in [12]:

Assume $A = \mathbb{C}\tilde{\Delta}$ is a tame hereditary algebra where $\tilde{\Delta}$ extends the Dynkin type $|\Delta|$. Let $P$ the indecomposable projective associated to the vertex in $\tilde{\Delta} \setminus \Delta$. Then $R(A, P)$ is isomorphic to the algebra of invariants $\mathbb{C}[x, y]^G$, where $G \subset SL(2, \mathbb{C})$ is a binary polyhedral group of type $|\Delta|$. Accordingly the completion of the graded algebra $R(A, P)$ is isomorphic to the surface singularity of type $|\Delta|$.

In [34], Ringel introduced the canonical algebras $C = C(p, \lambda)$ depending on a weight sequence $p = (p_1, \ldots, p_t)$ of positive integers and a parameter sequence $\lambda =$
\(\lambda_3, \ldots, \lambda_t\) of pairwise distinct non-zero elements from \(k\). In [11] it was shown that \(\text{mod-}C\) is derived equivalent to \(\text{coh} (X)\) the category of coherent sheaves on a weighted projective line \(X = \mathbb{X}(p, \lambda)\).

One of the aims of this work is the introduction of a class of algebras with related interesting properties. Let \(P\) be an indecomposable projective module over a canonical algebra \(C = C(p, \lambda)\), the one-point extension \(A = C[P]\) defined as the matrix algebra

\[
\begin{bmatrix}
k & 0 \\
P & C
\end{bmatrix}
\]

is called an extended canonical algebra. In section 2 we show that for two indecomposable projective \(C\)-modules \(P\) and \(P'\), the algebras \(C[P]\) and \(C[P']\) are derived equivalent. Moreover, if \(C\) is of tame type, the extended canonical algebra is derived equivalent to a wild hereditary or a wild canonical algebra, so essentially \(C[P]\) belongs to a well-studied class of algebras. There are interesting phenomena arising when \(C\) (and hence \(A\)) is of wild type.

Consider the Coxeter transformation of \(A\) as an automorphism \(\varphi_A\) \(K_0(A) \to K_0(A)\) of the Grothendieck group of \(A\), given on the classes of indecomposable projective modules by the formula \(\varphi_A([P(S)]) = -[I(S)]\), where \(P(S)\) (resp. \(I(S)\)) is the projective cover (resp. injective envelope) of a simple module \(S\). The characteristic polynomial \(f_A(T)\) of \(\varphi_A\) is called the Coxeter polynomial of \(A\).

Let \(\chi_X = 2 - \sum_{i=1}^t (1 - 1/p_i)\) be the (orbifold) Euler characteristic of \(X\). As shown in [12], for \(\chi_X > 0\) the classification problem of \(\text{coh} (X)\) is related to the problem of classifying the Cohen-Macaulay modules over a simple surface singularity and is in fact equivalent to the problem of classifying the graded Cohen-Macaulay modules over a corresponding quasi-homogeneous singularity. Assume \(\chi_X < 0\). By [23, 25] we know that for \(A = C[P]\) an extended canonical algebra

\[f_A(T) = P_C(T)f_C(T)\]

where \(P_C(T)\) is the Hilbert-Poincaré series of the positively graded algebra \(R(p, \lambda) = \bigoplus_{n=0}^{\infty} \text{Hom}_C(M, \tau^n_{\mathbb{X}}M)\), where \(M\) is a rank one not preprojective \(C\)-module. Equivalently, \(R(p, \lambda) = \bigoplus_{n=0}^{\infty} \text{Hom}(\mathcal{O}, \tau^n_{\mathbb{X}}\mathcal{O})\), where \(\mathcal{O}\) is the structure sheaf on \(X\). Recall from [21, 23] that in case \(k = \mathbb{C}\), we can interpret \(R(p, \lambda)\) as an algebra of entire automorphic forms associated to the action of a suitable Fuchsian group of the first kind, acting on the upper half plane \(\mathbb{H}_+\).
From [21], we know that the $k$-algebra $R = R(p, \lambda)$ is commutative, graded integral Gorenstein, in particular Cohen-Macaulay, of Krull dimension two. The complexity of the surface singularity $R$ is described by the triangulated category

$$D^Z_{Sg}(R) = \frac{D^b(\text{mod}^Z-R)}{D^b(\text{proj}^Z-R)},$$

where $\text{mod}^Z-R$ (resp. $\text{proj}^Z-R$) denotes the category of finitely generated (resp. finitely generated projective) $\mathbb{Z}$-graded $R$-modules. This category was considered by Buchweitz [5] and Orlov [30], see also Krause’s account [19] for a related, but slightly different approach. For $\chi_X > 0$, where the weight type of $X$ determines a Dynkin quiver $\Delta$, Kajiura, Saito, Takahashi and Ueda [17] have shown that $D^Z_{Sg}(R)$ is equivalent to the derived category of finite dimensional modules over the path algebra $k \Delta$.

For $\chi_X = 0$, the algebra $R$ hence $D^Z_{Sg}(R)$ is not defined, but a close variant $D^{Z(p)}_{Sg}(S)$, as shown by Ueda [35], is equivalent to the derived category $D^b(\text{coh}(X))$ of coherent sheaves on a weighted projective line which is tubular, that is, has weight type $(2,3,6), (2,4,4), (3,3,3)$ or $(2,2,2,2)$.

In section 3 we deal with the case $\chi_X < 0$ and prove that this category, as first observed by Saito and Takahashi (for the field $C$ of complex numbers), is described in the following way.

**Theorem 1.** Let $k$ be an algebraically closed field. Assume $\chi_X < 0$ and let $R$ be the positively $\mathbb{Z}$-graded surface singularity attached to $X$. Then there exists a tilting object $T$ in the triangulated category $D^Z_{Sg}(R)$ whose endomorphism ring is isomorphic to an extended canonical algebra $C[P]$, where $C$ is the canonical algebra associated with $X$.

It follows that the categories $D^Z_{Sg}(R) = T$ and $D^b(\text{mod}(C[P]))$ are equivalent as triangulated categories. In section 3.9 we further introduce the concept of Coxeter-Dynkin algebras and establish their relationship to the Coxeter-Dynkin diagrams from singularity theory.

More precise information on the structure of the ring $R(p, \lambda)$ is obtained by a closer examination of the spectral properties of the Coxeter transformation of the extended canonical algebra $C[P]$.

A sequence of weights $p = (p_1, \ldots, p_t)$ will always satisfy $p_1 \leq p_2 \leq \cdots \leq p_t$. We consider the lexicographical ordering of sequences $(p_1, \ldots, p_t) \leq (q_1, \ldots, q_s)$ if $t = s$
and \( p_i \leq q_i \) for \( 1 \leq i \leq t \). Extend the relation \( p \leq q \) (and say that \( q \) dominates \( p \)) to weight sequences of (possibly) different length by adding 1’s if necessary. The following result is shown in section 4 based on techniques developed in [24] and will be fundamental in the proof of the main results.

**Theorem 2.** Let \( A = C[P] \) be an extended canonical algebra of the wild canonical algebra \( C = C(p, \lambda) \). The following happens:

(a) \( f_A(T) \) has at most 4 roots not in \( S^1 \).
(b) The roots of \( f_A(T) \) lie on the unit circle \( S^1 \) if and only if the weight sequence \( p \) belongs to the 38-member list determined by all \( p < q \) with \( q \) belonging to the following critical list:

\[
\begin{align*}
(t = 3): & \ (2,3,11), (2,4,9), (2,5,8), (2,6,7), \\
& \ (3,3,8), (3,4,7), (3,5,6), \\
& \ (4,4,6), (4,5,5), \\
(t = 4): & \ (2,2,2,7), (2,2,3,6), (2,3,4,4), (3,3,3,4), \\
(t = 5): & \ (2,2,2,2,5), (2,2,2,3,4), (2,2,3,3,3), \\
(t = 6): & \ (2,2,2,2,2,3), \\
(t = 7): & \ (2,2,2,2,2,2).
\end{align*}
\]

We shall say that the algebra \( R(p, \lambda) \) (and also the weight sequence \( p \)) is formally \( n \)-generated if

\[
P_C(T) = \frac{\prod_{i=1}^{n-2} (1 - T^{c_i})}{\prod_{j=1}^{n} (1 - T^{d_j})}
\]

for certain natural numbers \( c_1, \ldots, c_{n-2} \) and \( d_1, \ldots, d_n \), all \( \geq 2 \). The algebra \( R(p, \lambda) \) (and also the weight sequence \( p \)) is formally a complete intersection if \( P_C(T) \) is a rational function \( f_1(T)/f_2(T) \), where each \( f_i(T) \) is a product of cyclotomic polynomials.

**Theorem 3.** Let \( C = C(p, \lambda) \) be a wild canonical algebra with weight sequence \( (p_1, \ldots, p_t) \) and \( A = C[P] \) be an extended canonical algebra. The following are equivalent:

(a) \( R(p, \lambda) \) is formally 3- or 4-generated
(b) \( R(p, \lambda) \) is formally a complete intersection
(c) The roots of $f_A(T)$ lie on $S^1$.

Moreover, for $t = 3$ the algebra $R(p, \lambda)$ is a graded complete intersection of the form $k[X_1, \ldots, X_s]/(\rho_3, \ldots, \rho_s)$ where $s = 3$ or $4$ and $\rho_3, \ldots, \rho_s$ is a homogeneous regular sequence. For $k = \mathbb{C}$ the assertion also holds for $t \geq 4$ for $R(p, \lambda')$ for a suitable choice of parameters. $\lambda' = (\lambda'_3, \ldots, \lambda'_t)$.

We remark that in almost every case $\text{Root} \ f_A(T) \subset S^1$ implies that the Coxeter transformation $\varphi_A$ is periodic. In fact the weight sequences $(3, 3, 3, 3)$ and $(2, 2, 2, 2, 4)$ are the only exceptions (section 4).

**Theorem 4.** Let $C = C(p, \lambda)$ be a wild canonical algebra with weight sequence $p = (p_1, \ldots, p_t)$. Consider $A = C[P]$ an extended canonical algebra. The following are equivalent:

(a) $R(p, \lambda)$ is formally 3-generated

(b) $\varphi_A$ is periodic of period $d$ and there is a primitive $d$-th root of unity which is root of $f_A(T)$.

For $t = 3$ the algebra $R(p, \lambda)$ is always a graded complete intersection of the form $k[X_1, X_2, X_3]/(f)$. Moreover, for $t \geq 4$ and $k = \mathbb{C}$ this also holds for $R(p, \lambda')$ for a suitable choice of parameters $\lambda' = (\lambda'_3, \ldots, \lambda'_t)$.

For the proof of Theorem 3 (resp. Theorem 4) we classify in section 5 all the weight sequences $p$ such that $R(p, \lambda)$ is formally a complete intersection (resp. $R(p, \lambda)$ has 3 homogeneous generators). In the complex case the algebras $R(p, \lambda)$ of Theorem 3 correspond to the Fuchsian singularities which are minimal elliptic [36, Proposition 5.5.1] and the classification is related to Laufer’s [20]. The algebras $R(p, \lambda)$ of Theorem 4 relate to classifications by Dolgachev [6] and Wagreich [37] and include the 14 exceptional unimodal Arnold’s singularities [1]. We refer the reader to the complete account by Ebeling [9].

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1.2. Notation and conventions. Let $Q$ be a finite quiver without oriented cycles. The path algebra $kQ$ has as basis all the oriented paths in $Q$ and product given by juxtaposition of paths. Given an ideal $I$ of $kQ$ which is admissible (that is, $(kQ^+)^m \subset I \subset (kQ^+)^2$ for some $m \geq 2$, where $kQ^+$ is the ideal of $kQ$ generated by the arrows), we consider the finite dimensional $k$-algebra $A = kQ/I$. By ‘module’ we mean a finite dimensional right $A$-module. The category of modules is denoted $\text{mod-}A$. A module is identified with a covariant functor $X: kQ \rightarrow \text{mod-}k$ such that $X(\rho) = 0$ for every $\rho \in I$. Important modules are the simple modules $S$ of $A$.

We denote by $K_0(A)$ the Grothendieck group of $A$. Since $A$ has finite global dimension, the classes $[P_i]$ with $i \in Q_0$ form a basis of $K_0(A)$. Thus the Coxeter transformation $\varphi_A: K_0(A) \rightarrow K_0(A)$, given by $\varphi_A([P_i]) = -[I_i]$ defines an isomorphism. The Grothendieck group $K_0(A)$ is equipped with a bilinear form $\langle -,- \rangle_A: K_0(A) \times K_0(A) \rightarrow \mathbb{Z}$, called the Euler form, defined in the classes of modules $X$ and $Y$ as $\langle [X],[Y] \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_k \text{Ext}_A^i(X,Y)$. In case $A = k\Delta$ is a hereditary algebra, then $\varphi_A([X]) = [\tau_A X]$ for any indecomposable non-projective $A$-module $X$.

For the general situation, we have to look at the derived category $D(A) = \text{D}^b(\text{mod-}A)$ of bounded complexes of $A$-modules.

The derived category $D(A)$ contains a copy $\text{mod-}A[n]$ of $\text{mod-}A$ for each integer $n \in \mathbb{Z}$, with objects written $X[n]$ and satisfying

$$\text{Hom}_{D(A)}(X[n], Y[m]) = \text{Ext}_A^{n-m}(X,Y).$$

We say that an algebra $A$ is derived hereditary (resp. derived canonical) if $D(A)$ is triangle equivalent to $D(H)$ (resp. $D(C)$) for a hereditary algebra $H$ (resp. a canonical algebra $C$).

The category $D(A)$ has Auslander-Reiten triangles which yield a self-equivalence $\tau_{D(A)}$ of $D(A)$, the Auslander-Reiten translation, satisfying $\text{Hom}_{D(A)}(Y, \tau_{D(A)} X[1]) = D \text{Hom}_A(X,Y)$. The natural isomorphism $K_0(A) \rightarrow K_0(D(A))$, $X \mapsto X[0]$ yields $\varphi_A([X]) = [\tau_{D(A)} X]$.

For background material on representations of algebras and derived categories we refer the reader to [14, 34].

For vectors $v, w \in K_0(A)$ we get $\langle v, \varphi_A(w) \rangle_A = -\langle w, v \rangle_A$. 
2. Extended canonical algebras: basic properties

2.1. Let $C = C(p, \lambda)$ be the canonical algebra defined by the weight sequence $p = (p_1, \ldots, p_t)$ with $p_i \geq 2$ and $(\lambda = \lambda_3, \ldots, \lambda_t)$ a sequence of pairwise distinct non-zero elements of $k$, that is, $C$ is defined by the quiver

![Quiver Diagram](image)

satisfying the $t - 2$ equations

$$\alpha_{ip_i} \cdots \alpha_{i2} \alpha_{i1} = \alpha_{2p_2} \cdots \alpha_{22} \alpha_{21} - \lambda_i \alpha_{1p_1} \cdots \alpha_{12} \alpha_{11}, \quad i = 3, \ldots, t.$$

The algebra $C$ is a one-point extension $H[M]$ of the hereditary algebra $H = C/(0)$ by an $H$-module $M$ with dimension vector $[M] = (\dim_k M(i))_i \in K_0(H)$ as follows:

```
1 1 \cdots 1 1
```

In case $t \geq 3$, the module $M$ is an indecomposable $H$-module which is not preprojective or preinjective.

Observe that the underlying graph of $H$ is a star $[p_1, p_2, \ldots, p_t]$ with linear arms having $p_i$ vertices, $i = 1, \ldots, t$. If $H$ is representation-finite, then $[p_1, p_2, \ldots, p_t]$ is a Dynkin diagram (that is, $\sum_{i=1}^{t} \frac{1}{p_i} > t - 2$) and $C$ is tame of domestic type. If $H$ is tame, then $[p_1, p_2, \ldots, p_t]$ is an extended Dynkin diagram (that is, $\sum_{i=1}^{t} \frac{1}{p_i} = t - 2$) and $C$ is tame of tubular type. See [34] for details.
2.2. The representation theory of mod-$C$ for $C = C(p, \lambda)$ a canonical algebra is controlled by the category coh$(\mathbb{X})$ of coherent sheaves on a weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$, since the derived categories $D(C)$ and $D^b(\text{coh}(\mathbb{X}))$ are equivalent as triangulated categories [12]. The complexity of the classification problem for coh$(\mathbb{X})$, and hence for mod-$C$, is essentially determined by the (orbifold) Euler characteristic $\chi_{\mathbb{X}} = 2 - \sum_{i=0}^{t} (1 - 1/p_i)$. Indeed, for $\chi_{\mathbb{X}} > 0$, the algebra $C$ is tame of domestic type and for $\chi_{\mathbb{X}} = 0$, the algebra $C$ is tubular. The wild case $\chi_{\mathbb{X}} < 0$ was carefully studied in [23], a paper which is the basis of the present investigation.

2.3. Let $C = C(p, \lambda)$ be a canonical algebra. Let $P$ be an indecomposable projective or injective $C$-module, then $A = C[P]$ is called an extended canonical algebra. Hence $A$ arises from by adjoining one arrow with a new vertex to an arbitrary vertex of $C$ while keeping the relations for $C$ without introducing any new relations. In particular, the opposite algebra of an extended canonical algebra is again extended canonical.

Lemma. Any extended canonical algebra is wild.

Proof: Recall that for an algebra $B = kQ/I$ where $Q$ has no oriented cycles and $I$ is generated by $\rho_1, \ldots, \rho_s \in \bigcup_{i,j \in Q_0} I(i, j)$, the Tits (quadratic) form $q_B : K_0(B) \to \mathbb{Z}$ is defined by

$$q_B(x) = \sum_{i \in Q_0} x(i)^2 - \sum_{i \rightarrow j} x(i)x(j) + \sum_{i,j \in Q_0} r(i, j)x(i)x(j),$$

where $r(i, j) = \# \{s : \rho_s \in I(i, j)\}$. The Tits form is weakly non-negative (i.e. $q_A(v) \geq 0$ for $v \in \mathbb{N}_0$) if $B$ is a tame algebra [31].

For an extended canonical algebra $A = C[P]$ with extension vertex $\ast$ such that $\text{rad} \, P_\ast = P = P_j$ for some vertex $j$ in $C$, we have the following:

- since $\text{gl dim} \, A = 2$, then $q_A(x) = \langle x, x \rangle_A$;
- the vector $w \in K_0(C) \subset K_0(A)$ with $w(i) = 1$ for every $i$ in $C$, satisfies $q_c(w) = 0$;
- for $e_\ast = [S_\ast]$, we get

$$q_A(2w + e_\ast) = 4q_c(w) - 2w(j) + 1 < 0.$$

Hence $A$ is of wild type. \qed
2.4. The following result is fundamental for introducing the concept of extended canonical algebras.

**Proposition.** Let $X$ and $Y$ be two indecomposable projective or injective $C$-modules over a canonical algebra $C$. Then the extended canonical algebras $C[X]$ and $C[Y]$ are derived equivalent.

In particular, the derived class of an extended canonical algebra is independent of the chosen projective module.

**Proof.** (See [24].) Under the equivalence $D(C) = D^b(\text{coh } (X(p, \lambda)))$ the modules $X$ and $Y$ become line bundles over $X$ (up to translation in $D^b(\text{coh } (X)))$. Since the Picard group of $X$ acts transitively on isomorphism classes of line bundles, there is a self-equivalence of $D(C)$ sending $X$ to $Y$. The assertion follows from [2]. □

2.5. The following remark is useful:

**Proposition.** Let $A$ be an extended canonical algebra of $C = C(p, \lambda)$ with $p = (p_1, \ldots, p_t)$ satisfying $t \geq 3$. Then $A$ is derived equivalent to a one-point extension $H[N]$ of a hereditary algebra $H$ by an indecomposable module $N$.

**Proof.** By (2.3), we may assume that $A$ is the path algebra of the quiver

\[
\begin{array}{ccc}
\alpha_{11} & \rightarrow & \alpha_{12} \\
\alpha_{21} & \rightarrow & \alpha_{22} \\
0 & \rightarrow & \alpha_{t1} \\
\alpha_{11} & \rightarrow & \alpha_{21} \\
& \vdots & \\
& \vdots & \\
& \alpha_{1t} & \rightarrow \\
\end{array}
\]

\[
\begin{array}{ccc}
\cdots & \rightarrow & \cdots \\
\alpha_{1p_1} & \rightarrow & \alpha_{2p_1} \\
\alpha_{2p_2} & \rightarrow & \alpha_{1p_1} \omega \\
\cdots & \rightarrow & \\
\alpha_{tp_t} & \rightarrow & \beta \\
\end{array}
\]

equipped with the canonical relations $\alpha_{ip_i} \cdots \alpha_{i1} = \alpha_{2p_2} \cdots \alpha_{21} - \lambda_i \alpha_{1p_1} \cdots \alpha_{11}$ ($3 \leq i \leq t$). It follows that $A$ is the one-point extension of the path algebra of the star $[2, p_1, p_2, \ldots, p_t]$ by an indecomposable module $N = N(\lambda_3, \ldots, \lambda_t)$ whose restriction to $[p_1, p_2, \ldots, p_t]$ is $M$ as in (2.1) and $N(*) = 0$. □
Remark: An obvious variant of the above statement yields families of one-point extensions of hereditary algebras which are pairwise derived equivalent. For instance, for the canonical type $(2, 3, 7)$, the Proposition yields $10 = (2 - 1) + (3 - 1) + (7 - 1) + 1$ choices of pairs $(H, N)$ of a hereditary algebra $H$ and an indecomposable $H$-module $N$ such that $H[N]$ is derived equivalent to an extended canonical algebra of type $(2, 3, 7)$.

3. The derived category of an extended canonical algebra

3.1. In this section we are going to investigate the nature of the bounded derived category of an extended canonical algebra $A = C[P]$. As it turns out the structure of this triangulated category will sensibly depend on the sign of the (orbifold) Euler characteristic $\chi_X$ of the weighted projective line $X$ associated to $C$.

Let $T$ be a triangulated $k$-category, see [13, 27, 18] for definition and properties. An object $E$ in $T$ is called exceptional if $\text{End}(E) = k$ and $\text{Hom}(E, E[n]) = 0$ for all integers $n \neq 0$. By $\perp E$ (resp. $E^\perp$) we denote the full triangulated subcategory of $T$ consisting of all objects $X \in T$ (resp. $Y \in T$) satisfying $\text{Hom}(X, E[n]) = 0$ (resp. $\text{Hom}(E[n], Y) = 0$ for each integer $n$. By [3] the inclusion $\perp E \hookrightarrow T$ (resp. $E^\perp \hookrightarrow T$) admits an exact left (resp. right) adjoint $\ell : T \to \perp E$ (resp. $r : T \to E^\perp$).

For the purpose of this paper we call an exceptional object special in $T$ if one of the following two conditions is satisfied:

(i) the left perpendicular category $\perp E$ is equivalent to $D^b(\text{coh}(X))$ for some weighted projective line $X$ and, moreover, the left adjoint $\ell$ maps $E$ to a line bundle in $\text{coh}(X)$.

(ii) the right perpendicular category $E^\perp$ is equivalent to $D^b(\text{coh}(X))$ for some weighted projective line $X$ and, moreover, the right adjoint $r$ maps $E$ to a line bundle in $\text{coh}(X)$.

Again, for the purpose of this paper, an object $T$ of $T$ is called a tilting object in $T$ if (i) $T$ generates $T$ as a triangulated category, (ii) $\text{Hom}(T, T[n]) = 0$ holds for each non-zero integer $n$, and (iii) the endomorphism algebra of $T$ has finite global dimension.

3.2. Our main tool to investigate the shape of $D(A)$ is the following proposition.

Proposition. Let $T$ be a triangulated category having an exceptional object $E$ that is special in $T$. Then there exists a tilting object $\bar{T}$ of $T$ whose endomorphism ring is an extended canonical algebra.
Proof. With the previous notations we assume that $E^\perp = \mathcal{D}b(\text{coh}(\mathcal{X}))$ and $r(E)$ is a line bundle in $\text{coh}(\mathcal{X})$. (The assumption on the left perpendicular $\perp E$ category is treated similarly.) We choose a tilting object $T$ in $\text{coh}(\mathcal{X})$, hence in $\mathcal{D}b(\text{coh}(\mathcal{X}))$ having the line bundle $r(E)$ as a direct summand and such that $\text{End}(T) = C$ is the canonical algebra attached to $\mathcal{X}$ (see [11]). We claim that $\bar{T} = T \oplus E$ is a tilting object in $\mathcal{T}$. Indeed, since $T$ generates $\mathcal{D}b(\text{coh}\mathcal{X})$ and $E$ together with $E^\perp$ generates $\mathcal{T}$, it follows that $\bar{T}$ generates $\mathcal{T}$. Next, we show that $\text{Hom}(\bar{T}, \bar{T}[n]) = 0$ for each nonzero integer $n$. This reduces to show that $\text{Hom}(E[n], T) = 0$ and $\text{Hom}(T, E[n])$ holds for every nonzero $n$. The first assertion holds for each $n$ since $T$ belongs to $E^\perp$. For the second we use that $\text{Hom}(T, E[n]) = \text{Hom}(T[-n], r(E))$ is zero since by construction $r(E)$ is a direct summand of the tilting object $T$. Finally, the endomorphism ring of $\bar{T}$ is given as the matrix ring

$$\begin{pmatrix} C & 0 \\ P & k \end{pmatrix},$$

where $P = \text{Hom}(T, E) = \text{Hom}(T, rE)$ is an indecomposable projective $C$-module, hence $\text{End}(\bar{T}) = C[P]$ is an extended canonical algebra. Moreover, as it is easily seen, $C[P]$ has global dimension two. \qed

Keeping the assumptions on $E$ and $\mathcal{T}$ from the proposition, we obtain.

**Corollary.** If $\mathcal{T}$ is triangle equivalent to a bounded derived category $\mathcal{D}(B)$ for some finite dimensional $k$-algebra $B$ or, more generally, if $\mathcal{T}$ is algebraic in the sense of Keller [18], then $\mathcal{T}$ is triangle equivalent to $\mathcal{D}b(\text{mod-}A)$, for the extended canonical algebra $A = C[P]$.

Proof. The first claim follows from [32], the general version requires [18] or [4]. \qed

3.3. **Positive Euler characteristic: the domestic case.** Consider a canonical algebra $C = C(p_1, \ldots, p_t)$ of domestic type, that is, $\sum_{i=1}^t \frac{1}{p_i} > 1$. Let $\Delta = [p_1, \ldots, p_t]$ be the star corresponding to the weight sequence $p = (p_1, \ldots, p_t)$ and $\tilde{\Delta}$ be the corresponding extended Dynkin diagram. Then $\tilde{\Delta}$ admits a unique positive additive function $\lambda$ assuming value 1, that is, $\lambda: \tilde{\Delta} \to \mathbb{N}$ satisfies the conditions:

(i) $2\lambda(i) = \sum_{j \in i^+} \lambda(j)$, where $i^+$ is the set of neighbors of $i$;

(ii) for some vertex $i \in \tilde{\Delta}_0$, $\lambda(i) = 1$. Such an $i$ is called an extension vertex.
The double extended graph of type $\Delta$, denoted by $\tilde{\Delta}$, is the graph arising from $\tilde{\Delta}$ by adjoining a new edge in an extension vertex. The list of double extended Dynkin graphs is the following:

| weight sequence | double extended Dynkin graph |
|-----------------|-----------------------------|
| $(p, q)$        | ![Graph](image)             |
| $(2, 2, 4)$     | ![Graph](image)             |
| $(2, 3, 3)$     | ![Graph](image)             |
| $(2, 3, 4)$     | ![Graph](image)             |
| $(2, 3, 5)$     | ![Graph](image)             |

**Proposition.** Let $C = C(p, \lambda)$ be a canonical algebra of domestic type $p = (p_1, p_2, p_3)$. Let $\Delta = [p_1, p_2, p_3]$ be the associated Dynkin diagram. For any indecomposable pre-projective $C$-module $N$, the one-point extension $A = C[N]$ is derived equivalent to a hereditary algebra of type $\tilde{\Delta}$. In particular, an extended canonical algebra of weight type $p$ is derived hereditary of type $\tilde{\Delta}$. 
Proof. The algebra $C$ is tilted of a hereditary algebra $H = k\tilde{\Delta}_1$, where $\tilde{\Delta}_1$ is a quiver with underlying graph $\tilde{\Delta}$. There is a derived equivalence $F: D^b(\text{mod-}H) \to D^b(\text{mod-}C)$ sending the indecomposable projective $P_i$ corresponding to an extension vertex $i$ of $\tilde{\Delta}$ into the projective $C$-module $P_\omega$. Observe that $H[P_i] = k\tilde{\Delta}_1$ is a hereditary algebra where $\tilde{\Delta}_1$ is a quiver with underlying graph $\tilde{\Delta}$.

Let $X = X(p, \lambda)$ be a weighted projective line such that $D^b(\text{mod-}C) = D^b(\text{coh}(X))$. As an object in $\text{coh}(X)$ the object $P_\omega$ has rank one (see [11]). Also by [11], there is an equivalence in $D^b(\text{coh}(X))$ sending $P_\omega$ to any indecomposable preprojective $C$-module $N$. By [2], the one-point extension $C[N]$ is derived equivalent to $C[P_\omega]$ which is derived hereditary of type $\tilde{\Delta}$. \hfill \Box

3.4. The converse of Proposition 3.3 also holds.

Proposition. Let $C = C(p, \lambda)$ be a canonical algebra and $P$ be an indecomposable projective $C$-module. The extended canonical algebra $A = C[P]$ is derived hereditary if and only if $C$ is tame domestic.

Proof. If $C$ is tame domestic, then $A$ is derived hereditary by (2.5). For the converse, consider the set of weight sequences $p = (p_1, p_2, \ldots, p_t)$ with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_t$ with the domination order defined in (1.1).

The statement follows by induction on the domination order from the following two facts:

(a) a canonical tubular algebra $C$ is not derived hereditary;
(b) any wild weight sequence dominates a tubular one;
(c) if $M$ is an indecomposable $B$-module such that $B[M]$ is derived hereditary, then $B$ is derived hereditary.

(a): follows from the structure of derived categories of hereditary algebras, see [14].

(b): is clear.

(c): Assume $D^b(\text{mod-}B[M]) = D^b(\text{mod } H)$ for a hereditary algebra $H$. By [12], $D^b(\text{mod-}B)$ is equivalent to the right perpendicular category in $D^b(\text{mod-}H)$ with respect to an exceptional object $E$, that is, $E$ is an indecomposable object satisfying $\text{Ext}^1(E, E) = 0$ and

$$D^b(\text{mod-}B) = E^\perp = \{ X \in D^b(\text{mod-}H) : \text{Hom}(E, X) = 0 = \text{Ext}^1(E, X) \}.$$
Without loss of generality we may assume that $E \in \text{mod-} H$. Then $D^b(\text{mod-} B) \cong D^b(E^\perp)$, where now $E^\perp$ is formed in $\text{mod-} H$. Hence $E^\perp = \text{mod-} H'$ for a hereditary algebra $H'$.

\[ \square \]

\section*{3.5. Euler characteristic zero: the tubular case.}

Consider a canonical algebra $C = C(p, \lambda)$ with weight sequence $p = (p_1, \ldots, p_t)$, we shall assume that $2 \leq p_1 \leq p_2 \leq \cdots \leq p_t$. The module category $\text{mod-} C$ accepts a \textit{separating tubular family} $\mathcal{T} = (T_\lambda)_{\lambda \in \mathbb{P}^1}$, where $T_\lambda$ is a homogeneous tube for all $\lambda$ with the exception of $t$ tubes $T_{\lambda_1}, \ldots, T_{\lambda_t}$ with $T_{\lambda_i}$ of rank $p_i$ ($1 \leq i \leq t$). See [34].

Let $\mathbb{X} = \mathbb{X}(p, \lambda)$ be the weighted projective line such that $\text{mod-} C$ and $\text{coh} (\mathbb{X})$ are derived equivalent. We fix an equivalence $D^b(\text{mod-} C) = D^b(\text{coh} (\mathbb{X}))$. Let $S$ be a simple $C$-module in the mouth of the tube of rank $p_t$ and consider $S$ as an object in $\text{coh} (\mathbb{X})$. The category $S^\perp$ right perpendicular to the object $S$ is the full subcategory of $\text{coh} (\mathbb{X})$ consisting of all $F \in \text{coh} (\mathbb{X})$ satisfying

$$\text{Hom}_{\mathbb{X}}(S, F) = 0 = \text{Ext}^1_{\mathbb{X}}(S, F).$$

By [12], $S^\perp = \text{coh} (\mathbb{X}')$ where $\mathbb{X}' = \mathbb{X}(p', \lambda)$ is a weighted projective line with weight sequence $p' = (p_1, p_2, \ldots, p_{t-1}, p_t - 1)$. Moreover, if $0 \rightarrow \tau S \rightarrow U \rightarrow S \rightarrow 0$ is the almost split sequence in $\text{coh} (\mathbb{X})$, then $U$ is a simple object in $S^\perp$ of $\tau'$-period $p_t - 1$, where $\tau' = \tau_{D^b(\text{coh} (\mathbb{X}'))}$.

\textbf{Proposition.} Let $C = C(p, \lambda)$ be a canonical algebra of tubular type $p = (p_1, \ldots, p_t)$ and $A = C[P]$ be an extended canonical algebra. Then $A$ is derived canonical of type $\bar{p} = (p_1, \ldots, p_{t-1}, p_t + 1)$.

\textit{Proof.} By (2.4), we may choose $P$ to be the simple projective $C$-module. We shall show that $A$ is quasi-tilted of type $\bar{p} = (p_1, \ldots, p_{t-1}, p_t + 1)$, see [15].

Let $\mathbb{X}$ be a weighted projective line with $D^b(\text{coh} (\mathbb{X})) = D^b(\text{mod-} C)$ and let $\bar{\mathbb{X}}$ denote a weighted projective line of type $\bar{p}$ such that $\text{coh} (\bar{\mathbb{X}})$ is the perpendicular category $S^\perp$ formed in $\text{coh} (\bar{\mathbb{X}})$ for a simple $S$ from the tube of rank $p_t + 1$. Let $U$ be the middle term of the almost split sequence $0 \rightarrow \tau S \rightarrow U \rightarrow S \rightarrow 0$ in $\text{coh} (\bar{\mathbb{X}})$. Then $U$ is a simple in $S^\perp = \text{coh} (\mathbb{X})$ belonging to the largest tube in $\text{coh} (\mathbb{X})$.

By hypothesis, $\mathbb{X}$ has tubular type. By [22], there is a tilting object $\mathcal{T}$ in $\text{coh}_0(\mathbb{X}) \vee \text{coh}_+(\mathbb{X})[1]$ such that $\text{End} (\mathcal{T}) = C$, where $\text{coh}_0(\mathbb{X})$ (resp. $\text{coh}_+(\mathbb{X})$) denotes the full
subcategory of coh \((\mathcal{X})\) formed by the sheaves of rank 0 (resp. positive rank). Therefore, \(\mathcal{T} \cup \{S\} \subset \text{coh}_0(\mathcal{X}) \cup \text{coh}_+ (\mathcal{X})[1]\) is a tilting complex for \(\mathcal{X}\) whose endomorphism ring is isomorphic to \(C[P] = A\).

\(\square\)

3.6. **Negative Euler characteristic: the wild case.** For negative Euler characteristic the derived category of modules over an extended canonical algebra \(C[P]\) relates to the study of the \(\mathbb{Z}\)-graded surface singularity \(R\) associated with \(C\) and the weighted projective line \(\mathcal{X}\) associated to \(C\). We refer to [11, 21, 12] for further details.

The weighted projective line \(\mathcal{X} = \mathcal{X}(p, \lambda)\) for a weight sequence \(p = (p_1, \ldots, p_t)\) and a parameter sequence \(\lambda = (\lambda_3, \ldots, \lambda_t)\) was introduced in [11] by means of the algebra

\[
S = S(p, \lambda) := k[x_1, \ldots, k_t]/(x_i^p = x_2^{p_2} - \lambda_i x_1^{p_1}), \quad i = 3, \ldots, t.
\]

The algebra \(S\) is naturally graded over the abelian group \(\mathbb{L}(p)\) with generators \(\bar{x}_1, \ldots, \bar{x}_t\), and relations \(p_i \bar{x}_1 = \cdots = p_t \bar{x}_t =: \bar{c}\) by giving each \(x_i\) the degree \(\bar{x}_i\). Here, \(\bar{c}\) is called the canonical element of \(\mathbb{L}(p)\). The group \(\mathbb{L}(p)\) is isomorphic to the direct sum of the group \(\mathbb{Z}\) of integers and some finite group. A quick way to arrive at the category coh \((\mathcal{X})\) of coherent sheaves on \(\mathcal{X}\) is by putting

\[
\text{coh} (\mathcal{X}) = \frac{\text{mod}_{\mathbb{L}(p)} S}{\text{mod}_{\mathbb{L}(p)}^0 S},
\]

where the quotient category on the right is formed in the sense of [10] and the categories \(\text{mod}_{\mathbb{L}(p)} S\) (resp. \(\text{mod}_{\mathbb{L}(p)}^0 S\)) are the categories of finitely generated \(\mathbb{L}(p)\)-graded \(S\)-modules (resp. those of finite length).

The element \(\bar{\omega} = (t - 2)\bar{c} - \sum_{i=0}^t \bar{x}_i\) from \(\mathbb{L}(p)\) is called the **dualizing element**. Its importance comes from the fact that Serre duality for coh \((\mathcal{X})\) holds in the form

\[
\text{DExt}^1(X, Y) = \text{Hom}(Y, X(\bar{\omega})), \quad X \mapsto X(\bar{\omega})
\]

is the self-equivalence of coh \((\mathcal{X})\) induced by grading shift \(M \mapsto M(\bar{\omega})\), given by \(M(\bar{\omega})_{\bar{x}} = M_{\bar{x} + \bar{\omega}}\).

Assume that \(\chi_{\mathcal{X}} < 0\). Then the **graded surface singularity** \(R = R(p, \lambda)\) attached to the weighted projective line \(\mathcal{X}\) (or the canonical algebra \(C\)) with data \((p, \lambda)\) is defined as

\[
R = \bigoplus_{n=0}^{\infty} R_n, \quad \text{where } R_n = S_{n\bar{c}}.
\]

It follows immediately that \(R\) is a finitely generated, i.e. affine, \(k\)-algebra where each \(R_n\) is finite dimensional over \(k\) and, moreover, \(R_0 = k\) and \(R_1 = 0\). The next theorem illustrates the role of \(R\), and shows in particular that the algebra \(R\) keeps all information on \(\mathcal{X}\). For the proofs we refer to [21, 12].
Theorem. Assume $\chi_X < 0$. Then the following holds:

(a) The algebra $R = R(p, \lambda)$ is a positively $\mathbb{Z}$-graded isolated surface singularity which is graded Gorenstein of Gorenstein index $-1$.

(b) There is a natural equivalence $\text{coh}(X) \to \text{mod}^{\mathbb{Z}}-R/\text{mod}_{\mathbb{L}}^{\mathbb{Z}}-R$, induced by restricting the grading from $\mathbb{L}(p)$ to $\mathbb{Z} = \mathbb{Z}\omega$.

(c) For $k = \mathbb{C}$, the algebra $R(p, \lambda)$ is the positively $\mathbb{Z}$-graded algebra of automorphic forms on the (upper) complex half-plane $\mathbb{H}_+$ with respect to the action of a Fuchsian group $G$ of the first kind of signature $(0; p_1, \ldots, p_t)$.

Concerning (a) we note that — restricting to the case of Krull dimension two — the Gorenstein index $d$ can be defined through the minimal graded injective resolution

$$0 \to R \to E^0 \to E^1 \to E^2 \to 0$$

of the $R$-module $R$, where the term $E^2$ is the graded injective hull of $k(d)$ and generally the grading shift $(n)$ is defined by $M(n)_m = M_{n+m}$. In this situation, Serre duality holds for $\text{coh}(X)$ in the form $D\text{Ext}^1(X, Y) = \text{Hom}(Y, X(-d))$, such that the Auslander-Reiten translation comes from the grading shift $X \mapsto X(-d)$.

Concerning (c) we remark that $G$ is the orbifold fundamental group of $X$, having a presentation $\langle \sigma_1, \ldots, \sigma_t \mid \sigma_1^{p_1} = \ldots = \sigma_t^{p_t} = \sigma_1 \cdots \sigma_t \rangle$ acting by covering transformations on the (branched) universal cover $\mathbb{H}_+$ of $X$. We refer to [26, 28] for the associated rings of automorphic forms.

3.7. For a variety $X$ Orlov investigated in [29] the triangulated category $D_{Sg}(X)$ of the singularities of $X$ defined as the quotient of the bounded derived category $D^b(\text{coh}(X))$ of coherent sheaves modulo the full subcategory of perfect complexes. If $X$ is affine with coordinate algebra $R$ this category $D_{Sg}(R)$ is just the quotient $D^b(\text{mod}-R)/D^b(\text{proj}-R)$, where $\text{proj}-R$ is the category of finitely generated projective $R$-modules. In [30] Orlov further introduced a graded variant

$$D^Z_{Sg}(R) = D^b(\text{mod}^{\mathbb{Z}}-R)/D^b(\text{proj}^{\mathbb{Z}}-R)$$

called the \textit{triangulated category of the graded singularity} $R$ which will play a central role in this section.

Under the name \textit{stabilized derived category of} $R$ the categories $D_{Sg}(R)$ were introduced by Buchweitz in [5]. His results easily extend to the graded case and yields for an $R$ that is graded Gorenstein an alternative description of $D^Z_{Sg}(R)$ as the \textit{stable}
category of graded maximal Cohen-Macaulay modules $\text{MCM}^{Z}-R$. More precisely, he showed that the category $\text{MCM}^{Z}-R$ of maximal graded Cohen-Macaulay $R$-modules is a Frobenius-category, hence inducing — in Keller’s terminology [18] — on the attached stable category $\text{MCM}^{Z}-R$ of graded maximal Cohen-Macaulay modules modulo projectives, the structure of an algebraic triangulated category. For a related approach measuring the complexity of a singularity by a triangulated category we refer to Krause’s account [19].

Let $R = \bigoplus_{n \geq 0} R_n$, $R_n = S_{n\omega}$, be the positively $Z$-graded Gorenstein singularity attached to the weighted projective line $\mathbb{X}$. It follows from [21, 5.6] that $R$ has Krull dimension two and Gorenstein index $-1$. We fix some notation: Let $\mathcal{M} = D^b(\text{mod}^{Z}-R)$ and $\mathcal{M}_+ = D^b(\text{mod}^{Z+}-R)$. Let $\mathcal{P}_+$ be the triangulated subcategory of $\mathcal{M}_+$ generated by all $R(-n)$, $n \geq 0$ and $\mathcal{T}$ its left perpendicular category $\perp \mathcal{P}_+$ formed in $\mathcal{M}_+$. Denote further by $\mathcal{S}_+$ the triangulated subcategory of $\mathcal{M}_+$ generated by all $k(-n)$, $n \geq 0$ and $\mathcal{D}$ its right perpendicular category $\mathcal{S}_+^\perp$ formed in $\mathcal{M}_+$. Finally let $\mathcal{D}(-1) = \mathcal{S}_+(-1)^\perp$. Then [30, 2.5] implies the following proposition.

**Proposition.** Assume that $\chi_{\mathbb{X}} < 0$ and let $R$ be the positively $Z$-graded singularity attached to $\mathbb{X}$. Then the following holds:

(a) The natural functor $\mathcal{T} \hookrightarrow \mathcal{M} \xrightarrow{\mathcal{q}} D_{\text{SG}}(R)$, where $\mathcal{q}$ is the quotient functor, is an equivalence of triangulated categories.

(b) The $R$-module $k$ is an exceptional object in $\mathcal{T}$ with $\perp k = \mathcal{D}(-1)$. Moreover, the category $D^b(\text{coh}(\mathbb{X}))$ is naturally equivalent to $\mathcal{D}(-1)$ under the functor $Y \mapsto (R\Gamma_+(Y))(-1)$.

**Proof.** For the convenience of the reader we sketch the argument. Using that $R$ is Gorenstein of Gorenstein index -1, and invoking Gorenstein duality $R\text{Hom}_R^*(\cdot, R)$ of $\mathcal{M}$ one sees that $T^\perp \subset \mathcal{D}(-1)^\perp$ and hence $\mathcal{D}(-1)$ is a full subcategory of $\mathcal{T}$. Further we see that $\perp k = \mathcal{D}(-1)$. It is well-known that

$$\Gamma_+: \text{coh}(\mathbb{X}) \to \text{mod}^{Z+}-R, \quad Y \mapsto \bigoplus_{n=0}^{\infty} \text{Hom}(O, Y(n))$$

is a full embedding having sheafification, that is, the quotient functor $q_+: \text{mod}^{Z+}-R \to \text{coh}(\mathbb{X})$ as an exact left adjoint and such that composition $q \Gamma_+$ is the identity functor on $\text{coh}(\mathbb{X})$, compare [11, 1.8], [21, 5.7]. It follows that $R\Gamma_+: D^b(\text{coh}(\mathbb{X})) \to \mathcal{M}_+$ is a full embedding having $q_+: \mathcal{M}_+ \to \mathcal{M}_+/\mathcal{S}_+$ as a left adjoint, and $q_+ R\Gamma_+ = 1$. 
Since $R$ is positively graded with $R_0 = k$, it follows that $k$ is exceptional in $\mathcal{M}$ and hence in $\mathcal{M}_+$. Invoking the minimal graded injective resolution $0 \to R \to E^0 \to E^1 \to E^2 \to 0$, where $E^0$ and $E^1$ are socle-free and $E^2$ is the graded injective hull of $k(-1)$, it follows that $k$ belongs to $\mathcal{T}$ and then also to $\mathcal{D}(-1)$. It is straightforward to check that $\mathcal{D}(-1)^\perp$ equals the triangulated subcategory $\langle k \rangle$ generated by $k$, and hence $\perp k = \mathcal{D}(-1)$ in $\mathcal{T}$.

3.8. **Proof of Theorem 1** We are now in a position to clarify the structure of the category $\mathcal{D}_{\mathcal{Sg}}^Z(R)$. The result was first observed by K. Saito and A. Takahashi (personal communication); it is not yet published, and uses the technique of matrix factorizations as in [17].

By Proposition 3.2 it suffices to show that the left adjoint $\ell : \mathcal{T} \to \perp k$ to the inclusion $j : \perp k \hookrightarrow \mathcal{T}$ maps $k$ to a line bundle in $\mathcal{D}(-1) = \mathcal{D}^b(\text{coh}(X))$ up to translation in $\mathcal{D}(-1)$. We put $A = (R \Gamma_+ (\mathcal{O}(\vec{\omega})))(-1)$ and construct a morphism $\gamma : k \to A[1]$ such that $\text{Hom}(\gamma, Y) : \text{Hom}(A[1], Y) \to \text{Hom}(k, Y)$ is an isomorphism for each $Y \in \perp k$ such that $\ell(k) = A[1]$.

The claim is proved in two steps. Put $R_+ = \bigoplus_{n \geq 1} R_n$, then the exact sequence $0 \to R_+ \to R \to k$ yields an exact triangle $R \to k \to R_+[1]$ in $\mathcal{M}$, where $\text{Hom}(\alpha, Y)$ is an isomorphism for each $Y \in \perp k$. Note for this that $R$ belongs to $\perp \mathcal{D}(-1)$.

For the next step it is useful to identify the derived category $\mathcal{M}_+$ with the full subcategory of $\mathcal{D}^b(\text{Mod}_{\mathcal{Z}^+}^Z - R)$ consisting of all complexes with cohomology in $\text{mod}_{\mathcal{Z}^+}^Z - R$. Here, $\text{Mod}_{\mathcal{Z}^+}^Z - R$ denotes the category of all graded $R$-modules. Let $0 \to R(-1) \to E^0 \to E^1 \to E^2 \to 0$ be the minimal graded injective resolution of $R(1)$ such that $E^2$ equals the graded injective envelope of $k$. (This uses that $R$ has Gorenstein index $-1$.) Sheafification yields the minimal injective resolution $0 \to \mathcal{O}(\vec{\omega}) \to \tilde{E}^0 \to \tilde{E}^1 \to 0$ of $\mathcal{O}(\vec{\omega})$. Accordingly $R \Gamma_+ (\mathcal{O}(\vec{\omega}))$ is given by the complex

$$A : \cdots \to 0 \to E^0_+ (-1) \to E^2_+ (-1) \to 0 \cdots,$$

whose cohomology is concentrated in degrees zero and one and given by

$$\text{H}^0(A) = R_+, \quad \text{H}^1(A) = k(-1).$$

It follows the existence of an exact triangle

$$k(-1)[-2] \to R_+ \xrightarrow{\beta} A \to k(-1)[-1],$$
in \( \mathcal{M}_+ \) where, by construction, \( A \) belongs to \( \mathcal{D}(-1) \). For \( Y \) from \( \mathcal{D}(-1) \) we have, in particular, that \( Y \) belongs to \( k(-1) \) implying that \( \text{Hom}(\beta, Y) \) is an isomorphism. To summarize: The morphism \( \gamma = [k \xrightarrow{\alpha} R_+[1] \xrightarrow{\beta[1]} A[1]] \) yields isomorphisms \( \text{Hom}(\gamma, Y) \) for each \( Y \in \mathcal{D}(-1) \). Hence \( \ell(k)[-1] = A = R\Gamma_+(\mathcal{O}(\vec{\omega}))(1) \) is a line bundle, as claimed. \( \square \)

3.9. The Coxeter-Dynkin algebras of a singularity. In the theory of singularities the attached Coxeter-Dynkin diagrams, see for instance [8, 7], play an important role, in particular, since they establish a link to Lie theory.

**Definition.** Let \( k \) be an algebraically closed field, and \( R = R(p, \lambda) \) be the \( \mathbb{Z} \)-graded singularity attached to the weighted projective line \( X(p, \lambda) \).

(a) By the Coxeter-Dynkin algebra of hereditary type we mean the path algebra \( D[p] \) of the hereditary star \( [p_1, \ldots, p_t] \) having a unique sink.

(b) By the Coxeter-Dynkin algebra of canonical type we mean the algebra \( D(p, \lambda) \) given in terms of the quiver

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_t \\
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_t \\
\end{array}
\]

with the two relations \( \sum_{i=2}^{t} \alpha_i \beta_i = 0 \) and \( \alpha_1 \beta_1 = \sum_{i=3}^{t} \lambda_i \alpha_i \beta_i \).

(c) By the Coxeter-Dynkin algebra of extended-canonical type we mean the one-point extension \( \hat{D}(p, \lambda) \) of the Coxeter-Dynkin algebra \( D(p, \lambda) \) of canonical type above, introducing a new arrow at the sink vertex and keeping the relations.

The link to singularity theory is given by the following well known result, compare [7, 8].

**Theorem.** For \( k = \mathbb{C} \) the Coxeter-Dynkin diagram of the singularity \( R(p, \lambda) \) is the underlying digraph of the Coxeter-Dynkin algebra

(a) of hereditary type, if \( \chi_X > 0 \), and then \([p_1, \ldots, p_t]\) is Dynkin.
(b) of canonical type, if \( \chi_X = 0 \), and then \( p \) is tubular.
(c) of extended canonical type, if \( \chi_X < 0 \).

Recall that the digraph of a finite dimensional algebra \( A \) has the underlying graph of the quiver of \( A \) as solid edges and the minimal number of relations between vertices \( i \) and \( j \) as dotted edges.

**Proposition.** Assume the number of weights \( > 1 \) is at least two. Then the following holds:

(a) The Coxeter-Dynkin algebra \( D(p, \lambda) \) of canonical type is derived equivalent to the canonical algebra \( C(p, \lambda) \).
(b) The Coxeter-Dynkin algebra \( \hat{D}(p, \lambda) \) of extended canonical type is derived equivalent to the extended canonical algebra \( A = C[P] \), where \( C = C(p, \lambda) \).

**Proof.** Let \( X \) be the weighted projective line with \( t \) weighted points \( x_1, \ldots, x_t \) of weight \( p_1, \ldots, p_t \), respectively. For each \( i = 1, \ldots, t \) denote by \( U_i \) the unique indecomposable sheaf of length \( p_i - 1 \) concentrated in \( x_i \) such there exists an epimorphism \( \mathcal{O} \to U_i \) in \( \text{coh}(X) \), where \( \mathcal{O} \) is the structure sheaf on \( X \). Moreover, let

\[
S_i = U_i^{(1)} \subset U_i^{(2)} \subset \cdots \subset U_i^{(p_i-1)} = U_i
\]

be the complete system of subobjects of \( U_i \). It is straightforward to verify that the object

\[
T = \mathcal{O}(\bar{c}) \oplus \bigoplus_{i=1}^t \left( U_i^{(1)} \oplus U_i^{(2)} \oplus \cdots \oplus U_i^{(p_i-1)} \right) \oplus \mathcal{O}(-\bar{\omega})[1]
\]

is a tilting object in \( \text{D}^b(\text{coh}(X)) \). Moreover, it is not difficult to see that the endomorphism ring of \( T \) is isomorphic to the Coxeter-Dynkin algebra \( D(p, \lambda) \).

This proves (a). Assertion (b) follows from (a) noting that \( \mathcal{O}(-\bar{\omega}) \) is a line bundle by applying [2] or arguing as in Proposition 3.2. \( \square \)

3.10. **The triangulated category of singularities for nonnegative Euler characteristic.** In this paper we mainly concentrate on the case \( \chi_X < 0 \). To complete the picture we review the situation for \( \chi_X \geq 0 \).

Assume that \( k = \mathbb{C} \). For \( \chi_X > 0 \) the Dynkin diagram given by the weight type and the Coxeter-Dynkin diagram of the singularity \( R = R(p, \lambda) \) agree. Moreover, it is shown in [17] that the category \( \text{D}^b_{\text{sg}}(R) \) is equivalent to the bounded derived category \( \text{D}^b(k\Delta) \) of the path algebra of a quiver of the same Dynkin type.
Next we deal with the case $\chi_X = 0$. First, there is no $\mathbb{Z}$-graded Gorenstein algebra $R$ such that $C = \text{mod}^\mathbb{Z}R/\text{mod}^\mathbb{Z}_0R$ is equivalent to $\text{coh}(X)$. Assume, indeed, that such an algebra $R$ would exist. Let $d$ denote its Gorenstein index. Since the Auslander-Reiten translation $\tau(X) = X(−d)$ has finite order 2, 3, 4 or 6 for any weighted projective line $X$ of tubular type, it follows that $d = 0$, hence $\tau$ is the identity, contradiction.

Hence for $\chi_X = 0$ it is more natural to investigate the $L(p)$-graded singularity $S = S(p, \lambda)$ and its triangulated category of singularities $D_{\text{Sg}}^{L(p)}(S)$ which in the tubular case is equivalent to $D^b(\text{coh}(X))$ due to recent work of Ueda [35].

4. The Coxeter polynomial of an extended canonical algebra

4.1. Let $A$ be a finite dimensional $k$-algebra of finite global dimension. The Coxeter transformation $\varphi_A: K_0(A) \to K_0(A)$ is the automorphism induced by the Auslander-Reiten translation $\tau_{D^b(\text{mod}A)}: D^b(\text{mod}A) \to D^b(\text{mod}A)$. We shall consider $\varphi_A$ as a $n \times n$ integral matrix where $n$ is the rank of $K_0(A)$. From the Introduction, we recall that $f_A(T) = \det(T\text{id} - \varphi_A)$ is called the Coxeter polynomial of $A$.

For a one-point extension $A = B[P]$ of an algebra $B$ by an indecomposable projective a simple calculation shows (see [24, 34]):

$$f_A(T) = (1 + T)f_B(T) - T f_C(T).$$

Of particular interest is the hereditary case where any algebra can be constructed by repeated one-point extensions using indecomposable projective modules. In fact, for a star $H$ of type $[p_1, \ldots, p_t]$ the above formula yields:

$$f_{[p_1, \ldots, p_t]}(T) := f_H(T) = \left( T + 1 - T \sum_{i=1}^t \frac{v_{p_{i-1}}}{v_{p_i}} \right) \prod_{i=1}^t v_{p_i},$$

where we set $v_n(T) = T^n - T^{n-1} = \sum_{i=0}^{n-1} T^i$.

For the canonical algebra $C = C(p, \lambda)$ of type $p = (p_1, \ldots, p_t)$, we get

$$f_{(p_1, \ldots, p_t)}(T) := f_C(T) = (T - 1)^2 \prod_{i=1}^t v_{p_1}(T).$$

In particular, all the eigenvalues of $\varphi_C$ lie on the unit circle $\mathbb{S}^1$. For these calculations we refer the reader to [23].
Lemma [24]. An extended canonical algebra $A = C[P]$ where $C = C(p, \lambda)$ with $p = (p_1, \ldots, p_t)$ has Coxeter polynomial:

$$\hat{f}_{(p_1,\ldots,p_t)}(T) := f_A(T) = (T + 1)(T - 1)^2 \prod_{i=1}^{t} v_{p_i}(T) - T f_{[p_1,\ldots,p_t]}(T) \quad \square$$

4.2. For later use we recall some facts on cyclotomic polynomials.

The $n$-cyclotomic polynomial $\phi_n(T)$ is inductively defined by the formula

$$T^n - 1 = \prod_{d|n} \phi_d(T).$$

Recall that the Möbius function is defined as follows:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a square} \\ (-1)^r & \text{if } n = p_1\ldots p_r \text{ is a factorization into distinct primes.} \end{cases}$$

A more explicit expression for the cyclotomic polynomials is given by:

**Lemma.** For each $n \geq 2$, we have

$$\phi_n(T) = \prod_{1 \leq d < n \atop d|n} v_{n/d}(T)^{\mu(d)}$$

4.3. Following [24] we say that a polynomial $p(T) \in \mathbb{Z}[T]$ is represented by $q(T) \in \mathbb{Z}[T]$ if

$$p(T^2) = q^*(T) := T^{\deg q} q(T + T^{-1}).$$

The interest in the representability of polynomials is due to the relation between the set of roots of $p(T)$ and $q(T)$ whenever $p(T^2) = q^*(T)$. Indeed, in that case $\text{Root } p(T) \subset S^1$ (resp. $S^1 \setminus \{1\}$) if and only if $\text{Root } q(T) \subset [-2, 2]$ (resp. $(-2, 2)$).

In [24] is shown that the Coxeter polynomial $\hat{f}_{(p_1,\ldots,p_t)}(T)$ of an extended canonical algebra of type $(p_1, \ldots, p_t)$ is represented by

$$q_{(p_1,\ldots,p_t)}(T) = T(T^2 - 1) \prod_{i=1}^{t} v_{p_i}(T) - \chi_{[p_1,\ldots,p_t]}(T),$$

where $\chi_{[p_1,\ldots,p_t]}(T)$ is the characteristic polynomial of the adjacency matrix of the star graph of type $[p_1, \ldots, p_t]$.

Using the above expressions the following was recently shown by the authors:

**Theorem** [24]. Let $A = C[P]$ be an extended canonical algebra of type $(p_1, \ldots, p_t)$ and $A'$ be an extended canonical algebra of type $(p_1, \ldots, p_t, p_t + 1)$ and $A'$ be an extended canonical algebra of type $(p_1, \ldots, p_t)$. Then the following holds:
(a) \( \varphi_A \) accepts at most 4 eigenvalues outside \( S^1 \)
(b) If \( \text{Root } f_A \subset S^1 \), then also \( \text{Root } f_{A'} \subset S^1 \)

Sketch of proof: The polynomials \( q_{(p_1, \ldots, p_t)}(T) \) satisfy a Chebyshev recursion formula as follows:
\[ q_{(p_1, \ldots, p_t+1)}(T) = T q_{(p_1, \ldots, p_t)}(T) - q_{(p_1, \ldots, p_t-1)}(T). \]

A version of Sturm’s Theorem applies to assure that for any real interval \( [\alpha, \beta] \), if \( q_{(p_1, \ldots, p_t+1)}(T) \) has roots \( \lambda_1 \leq \cdots \leq \lambda_s \) in \( [\alpha, \beta] \), then \( q_{(p_1, \ldots, p_t)}(T) \) has roots \( \lambda'_1 \leq \cdots \leq \lambda'_{s-1} \) in \( [\alpha, \beta] \) satisfying
\[ \lambda_1 \leq \lambda'_1 \leq \lambda_2 \leq \lambda'_2 \leq \cdots \leq \lambda_{s-1} \leq \lambda'_{s-1} \leq \lambda_s. \]

(a) and (b) follow easily from these facts. \( \square \)

4.4. According to (4.3), to prove Theorem 2 we need to calculate the minimal weight sequences \( p \) such that \( \hat{f}_p(T) \) is not contained in \( S^1 \). This is done by systematically computing the roots of Coxeter polynomials of extended canonical algebras.

Recall that the spectral radius of the Coxeter transformation \( \varphi_A \) is by definition \( \rho_A(\varphi_A) = \max \{ |\lambda| : \lambda \in \text{Root } f_A(T) \} \). In the following list \( A \) is an extended canonical algebra of weight type \( p \). The invariant called Dynkin index is explained in section 5.
| Weight $\rho(\varphi_A)$ | Dynkin index |
|--------------------------|--------------|
| $t=3$ | | |
| $(2, 3, 11)$ | $1 + T - T^3 - T^4 + T^6 + T^7 + T^9 + T^{10} - T^{12} - T^{13} + T^{15} + T^{16}$ | $1.1064$ |
| $(2, 4, 9)$ | $\phi_2 \phi_5(T^{10} - T^9 + T^5 + T + 1)$ | $1.1329$ |
| $(2, 5, 8)$ | $1 + T + T^4 + T^5 + T^6 + 2T^8 + T^9 + T^{10} + T^{11} + T^{14} + T^{15}$ | $1.1574$ |
| $(2, 6, 7)$ | $1 + T + T^4 + 2T^5 + 2T^6 + T^7 + T^8 + 2T^9 + 2T^{10} + T^{11} + T^{14} + T^{15}$ | $1.1669$ |
| $(3, 3, 8)$ | $1 + T + T^2 + T^5 + 2T^6 + 3T^7 + 2T^8 + T^9 + T^{12} + T^{13} + T^{14}$ | $1.1498$ |
| $(3, 4, 7)$ | $1 + T + T^2 + T^3 + T^4 + 2T^5 + 3T^6 + 3T^7 + 3T^8 + 2T^9 + T^{10} + T^{11} + T^{12} + T^{13} + T^{14}$ | $1.1847$ |
| $(3, 5, 6)$ | $\phi_3(T^{12} + T^9 + T^7 + T^8 + T^5 + T^4 + T^3 + T + 1)$ | $1.1966$ |
| $(4, 4, 6)$ | $\phi_2 \phi_4(T^{10} - T^9 + T^8 + T^6 + T^4 + T^2 - T + 1)$ | $1.2175$ |
| $(4, 5, 5)$ | $\phi_5(T^{10} + T^7 + T^6 + T^5 + T^4 + T^3 + 1)$ | $1.2277$ |
| $t=4$ | | |
| $(2, 2, 2, 7)$ | $\phi_2^3(T^{10} + T^9 + T^5 + T^4 + 1)$ | $1.1670$ |
| $(2, 2, 3, 6)$ | $\phi_2^2 \phi_3(T^8 - T^7 + T^6 + T^4 + T^2 - T + 1)$ | $1.2196$ |
| $(2, 3, 3, 4)$ | $\phi_2 \phi_4(T^8 + T^5 + T^3 + T^2 + 1)$ | $1.2874$ |
| $(3, 3, 3, 4)$ | $\phi_3^2(T^8 + T^6 + 2T^5 + 2T^3 + T^2 + 1)$ | $1.3307$ |
| $t=5$ | | |
| $(2, 2, 2, 5)$ | $\phi_2^3(T^8 + T^6 + T^3 + 2T^4 + T^3 + T^2 + 1)$ | $1.2874$ |
| $(2, 2, 3, 4)$ | $\phi_4^3(T^8 + 2T^6 + T^5 + 3T^4 + 2T^2 + 1)$ | $1.3351$ |
| $(2, 2, 3, 3, 3)$ | $\phi_2 \phi_3^2(T^8 + T^4 + 2T^3 + T^2 + 1)$ | $1.3765$ |
| $t=6$ | | |
| $(2, 2, 2, 2, 3)$ | $\phi_2^4(T^9 + 2T^4 + 3T^3 + 2T^2 + 1)$ | $1.3305$ |
| $t=7$ | | |
| $(2, 2, 2, 2, 2, 2)$ | $\phi_2^5(T^4 - T^3 + 3T^2 - T + 1)$ | $1.5392$ |

**Table 1.** Critical weight sequences.

4.5. To complete the proof of Theorem [2] we shall show that any weight sequence $\rho' < \rho$ with $\rho$ in Table 1 has all its roots on $S^1$. This is computed in the following Table 2.

As above $A = C[P]$ is an extended canonical algebra of weight type $\rho = (p_1, \ldots, p_t)$. In Table 2, the marks $\bullet$ and $\circ$ refer to the case $k = \mathbb{C}$: those weight sequences marked
by $\bullet$ or $\square$ correspond to algebras $R(p, \lambda)$ associated to hypersurface singularities, in those cases $R(p, \lambda)$ is formally 3-generated. The marks $\bullet$ correspond to Arnold’s 14 exceptional unimodal singularities in the theory of singularities of differentiable maps [1]. Among those weight sequences $p = (p_1, p_2, p_3)$ (that is $t = 3$), Arnold’s singularities are exactly those rings of automorphic forms having three generators [37].
| Weight sequence | Factorization of $f_A(T)$ | Poincaré series | Period of $\varphi_A$ |
|-----------------|--------------------------|----------------|--------------------|
| (2, 3, 7)       | $\phi_{42}$              | (6, 14, 21) (42) | 42                 |
| (2, 3, 8)       | $\phi_2 \cdot \phi_{10} \cdot \phi_{30}$ | (6, 8, 15) (30) | 30                 |
| (2, 3, 9)       | $\phi_3 \cdot \phi_{12} \cdot \phi_{24}$ | (6, 8, 9) (24) | 24                 |
| (2, 3, 10)      | $\phi_2 \cdot \phi_{16} \cdot \phi_{18}$ | (6, 8, 9, 10) (16, 18) | 72 |
| (2, 4, 5)       | $\phi_2 \cdot \phi_6 \cdot \phi_{30}$ | (4, 10, 15) (30) | 30                 |
| (2, 4, 6)       | $\phi_2^2 \cdot \phi_{22}$ | (4, 6, 11) (22) | 22                 |
| (2, 4, 7)       | $\phi_2 \cdot \phi_9 \cdot \phi_{18}$ | (4, 6, 7) (18) | 18                 |
| (2, 4, 8)       | $\phi_2^2 \cdot \phi_4 \cdot \phi_{12} \cdot \phi_{14}$ | (4, 6, 7, 8) (12, 14) | 84 |
| (2, 5, 5)       | $\phi_5 \cdot \phi_{20}$ | (4, 5, 10) (20) | 20                 |
| (2, 5, 6)       | $\phi_2 \cdot \phi_8 \cdot \phi_{16}$ | (4, 5, 6) (16) | 16                 |
| (2, 5, 7)       | $\phi_{11} \cdot \phi_{12}$ | (4, 5, 6, 7) (11, 12) | 132 |
| (2, 6, 6)       | $\phi_2^2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_{10} \cdot \phi_{12}$ | (4, 5, 6, 6) (10, 12) | 60 |
| (3, 3, 4)       | $\phi_4 \cdot \phi_{24}$ | (3, 8, 12) (24) | 24                 |
| (3, 3, 5)       | $\phi_2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_{18}$ | (3, 5, 9) (18) | 18                 |
| (3, 3, 6)       | $\phi_2^2 \cdot \phi_{15}$ | (3, 5, 6) (15) | 15                 |
| (3, 3, 7)       | $\phi_2 \cdot \phi_3 \cdot \phi_4 \cdot \phi_{10} \cdot \phi_{12}$ | (3, 5, 6, 7) (10, 12) | 60 |
| (3, 4, 4)       | $\phi_2 \cdot \phi_4 \cdot \phi_{16}$ | (3, 4, 8) (16) | 16                 |
| (3, 4, 5)       | $\phi_{13}$ | (3, 4, 5) (13) | 13                 |
| (3, 4, 6)       | $\phi_2 \cdot \phi_3 \cdot \phi_9 \cdot \phi_{10}$ | (3, 4, 5, 6) (9, 10) | 90 |
| (3, 5, 5)       | $\phi_2 \cdot \phi_5 \cdot \phi_8 \cdot \phi_{10}$ | (3, 4, 5, 5) (8, 10) | 40 |
| (4, 4, 4)       | $\phi_2^2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_{12}$ | (3, 4, 4) (12) | 12                 |
| (4, 4, 5)       | $\phi_2 \cdot \phi_4 \cdot \phi_8 \cdot \phi_{9}$ | (3, 4, 5, 5) (8, 9) | 72 |
| (2, 2, 2, 3)    | $\phi_2^2 \cdot \phi_{18}$ | (2, 6, 9) (18) | 18                 |
| (2, 2, 2, 4)    | $\phi_2^2 \cdot \phi_{14}$ | (2, 4, 7) (14) | 14                 |
| (2, 2, 2, 5)    | $\phi_2^2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_{12}$ | (2, 4, 5) (12) | 12                 |
| (2, 2, 2, 6)    | $\phi_2^2 \cdot \phi_8 \cdot \phi_{10}$ | (2, 4, 5, 6) (8, 10) | 40 |
| (2, 2, 3, 3)    | $\phi_2 \cdot \phi_3 \cdot \phi_4 \cdot \phi_{12}$ | (2, 3, 6) (12) | 12                 |
| (2, 2, 3, 4)    | $\phi_2^2 \cdot \phi_5 \cdot \phi_{10}$ | (2, 3, 4) (10) | 10                 |
| (2, 2, 3, 5)    | $\phi_2 \cdot \phi_7 \cdot \phi_8$ | (2, 3, 4, 5) (7, 8) | 56 |
| (2, 2, 4, 4)    | $\phi_2^2 \cdot \phi_4 \cdot \phi_6 \cdot \phi_{8}$ | (2, 3, 4, 4) (6, 8) | 24 |
| (2, 3, 3, 3)    | $\phi_2^2 \cdot \phi_0$ | (2, 3, 3) (9) | 9                  |
| (2, 3, 3, 4)    | $\phi_2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_7$ | (2, 3, 3, 4) (6, 7) | 42 |
| (3, 3, 3, 3)    | $\phi_2 \cdot \phi_3 \cdot \phi_6 \cdot \phi_7$ | (2, 3, 3, 3) (6, 6) | $\infty$ |
| (2, 2, 2, 2, 2) | $\phi_2^2 \cdot \phi_{10}$ | (2, 2, 5) (10) | 10                 |
| (2, 2, 2, 2, 3) | $\phi_2^2 \cdot \phi_4 \cdot \phi_8$ | (2, 2, 3) (8) | 8                  |
| (2, 2, 2, 2, 4) | $\phi_2^2 \cdot \phi_3 \cdot \phi_5 \cdot \phi_6$ | (2, 2, 3, 4) (6, 6) | $\infty$ |
| (2, 2, 2, 3, 3) | $\phi_2^2 \cdot \phi_3 \cdot \phi_5 \cdot \phi_6$ | (2, 2, 3, 3) (5, 6) | 30                 |
| (2, 2, 2, 2, 2) | $\phi_2^2 \cdot \phi_4 \cdot \phi_6$ | (2, 2, 2, 3) (4, 6) | 12                 |

Table 2. Weights $p$ with $\rho(\varphi_A) = 1$. 
4.6. **Proof of Theorem 2.** (a) follows from (4.4) and (4.5) using Theorem (4.3). Part (b) is shown in [24], see (4.3). □

5. **The Poincaré series of an extended canonical algebra**

5.1. Let $C = C(p, \lambda)$ be a wild canonical algebra with weight sequence $p = (p_1, \ldots, p_t)$. Let $P$ be an indecomposable projective $C$-module. We define the Poincaré series $\hat{P}_C = \hat{P}(p_1, \ldots, p_t) \in \mathbb{Z}[[T]]$ by

$$\hat{P}_C(T) = \sum_{n=0}^{\infty} \langle [P], \varphi^n_C[P] \rangle c T^n.$$ 

Recall that $\varphi^n_C[P] = [\tau^n_{\text{D}b(\text{mod-}C)}P]$ in $K_0(\text{D}b(\text{mod-}C)) = K_0(C)$. Moreover, observe that $T + \hat{P}_C(T)$ is the Hilbert-Poincaré series $P_C(T)$, as defined in the Introduction, for the graded algebra

$$R(p, \lambda) = \bigoplus_{n=0}^{\infty} \text{Hom}(L, \tau^n_X L)$$

where $X = \mathbb{X}(p, \lambda)$ is a weighted projective line, $\tau_X$ is the Auslander-Reiten translation in $\text{coh} \ (X)$ and $L$ is any rank one bundle, see [23] and [25]. In particular, $\hat{P}_C(T)$ does not depend on the choice of $P$.

**Proposition.** With the notation above, let $A = C[P]$ be an extended canonical algebra. Then

(a) [25] Cor 3.6: $f_A(T) = P_C(T)f_C(T)$

(b) [23] Th. 8.6: $P_C(T) = 1 + T - \frac{T}{f(p_1; \ldots, p_t)(T)}$

(c) [23] Prop. 4.3: $P_C(T) = T + \frac{1}{1-T} + (t-2) \frac{T}{(1-T)^2} - \sum_{i=1}^{t} \frac{T}{(1-T)(1-T^{p_i})}$. □

5.2. We recall from [23] the following concepts.

**Definition** [23]. Assume $p = (p_1, \ldots, p_t)$ is a weight sequence of wild type. The *Dynkin label* of $p$ is the Dynkin diagram of one of the extended Dynkin graphs $[2, 2, 2, 2]$, $[3, 3, 3]$ [2, 4, 4] or [2, 3, 6] specified as follows:

(a) if $t \geq 4$, then the label is of type $[2, 2, 2, 2]$

(b) if $t = 3$, then the label is of type $[a, b, c]$ if $[a, b, c] \leq [p_1, p_2, p_3]$ and $a + b + c$ is minimal.

We say that $p$ has *Dynkin index* 2, 3, 4 or 6 if its Dynkin label is $[2, 2, 2, 2]$, $[3, 3, 3]$, $[2, 4, 4]$ or $[2, 3, 6]$ respectively.
Consider the graded algebra $R = R(p, \lambda)$. In [23] the support monoid $M(p)$ was introduced as the set of those $n \in \mathbb{N}$ with $R_n \neq 0$. Clearly, $M(p)$ is an additive semigroup in $\mathbb{N}$ generating $\mathbb{Z}$ as a group.

**Proposition** [23]. The support monoid $M(p)$ is finitely generated with at most 6 generators. The smallest element in $M(p)$ is the Dynkin index of $p$. \qed

5.3. In a preliminary version of [23], the authors displayed the list of all possible support monoids $M(p)$. This list of 22 semigroups is essential for the proofs of Theorems 3 and 4 and we reproduce it below. For a weight sequence $p$ the largest integer $n$ such that $n$ does not belong to $M(p)$ is called the Frobenius number of $M(p)$ and it is denoted by $\alpha(p)$. Of particular interest is the fact that $p \leq q$ implies $M(p) \subset M(q)$. 
5.4. For a given weight sequence \( p = (p_1, \ldots, p_t) \), the Coxeter polynomials \( f_p(T) \) and \( \hat{f}_p(T) \) are readily computed by (3.2). In case Root \( \hat{f}_p(T) \subset S^1 \), then using (3.2), the Poincaré series \( P_T(T) \) can be written as a rational function

\[
P_T(T) = \frac{m}{\prod_{j=1}^m (1-T^{d_j}) (1-T)^r}
\]

### Table 3. Semigroups of \((\mathbb{N}, +)\) with the form \( M(p) \).

| Weight type \( p \) | Frobenius number \( \alpha(p) \) | generators of \( \mathbb{M}(p) \) |
|---------------------|-----------------------------|-----------------------------|
| \((2, 3, 7)\)       | 43                          | \{6, 14, 21\}               |
| \((2, 3, 8)\)       | 25                          | \{6, 8, 15\}               |
| \((2, 3, 9)\)       | 19                          | \{6, 8, 9\}               |
| \((2, 3, 10)\)      | 13                          | \{6, 8, 9, 10\}           |
| \((2, 3, 11)\)      | 13                          | \{6, 8, 9, 10, 11\}       |
| \((2, 3, 12)\)      | 13                          | \{6, 8, 9, 10, 11\}       |
| \((2, 3, 13)\)      | 7                           | \{6, 8, 9, 10, 11, 13\}   |
| \((2, 3, \infty)\)  | 7                           | \{6, 8, 9, 10, 11, 13\}   |
| \((2, 4, 5)\)       | 21                          | \{4, 10, 15\}              |
| \((2, 4, 6)\)       | 13                          | \{4, 6, 11\}               |
| \((2, 4, 7)\)       | 9                           | \{4, 6, 7\}                |
| \((2, 4, 8)\)       | 9                           | \{4, 6, 7\}                |
| \((2, 4, 9)\)       | 5                           | \{4, 6, 7, 9\}             |
| \((2, 4, \infty)\)  | 5                           | \{4, 6, 7, 9\}             |
| \((2, 5, 5)\)       | 11                          | \{4, 5\}                   |
| \((2, 5, 6)\)       | 7                           | \{4, 5, 6, 7\}             |
| \((2, 5, 7)\)       | 3                           | \{4, 5, 6, 7\}             |
| \((2, 5, \infty)\)  | 3                           | \{4, 5, 6, 7\}             |
| \((2, 6, 6)\)       | 7                           | \{4, 5, 6\}                |
| \((2, 6, 7)\)       | 3                           | \{4, 5, 6, 7\}             |
| \((2, 6, \infty)\)  | 3                           | \{4, 5, 6, 7\}             |
| \((2, \infty, \infty)\) | 3                     | \{4, 5, 6, 7\}         |
| \((3, 3, 4)\)       | 13                          | \{3, 8\}                   |
| \((3, 3, 5)\)       | 7                           | \{3, 5\}                   |
| \((3, 3, 6)\)       | 7                           | \{3, 5\}                   |
| \((3, 3, 7)\)       | 4                           | \{3, 5, 7\}                |
| \((3, 3, \infty)\)  | 3                           | \{3, 5, 7\}                |
| \((3, 3, 4)\)       | 5                           | \{3, 4\}                   |
| \((3, 3, 5)\)       | 2                           | \{3, 4, 5\}                |
| \((3, 3, 6)\)       | 5                           | \{3, 4\}                   |
| \((3, 3, 7)\)       | 2                           | \{3, 4, 5\}                |
| \((\infty, \infty, \infty)\) | 2          | \{3, 4\}                   |
| \((2, 2, 2, 3)\)    | 7                           | \{2, 9\}                   |
| \((2, 2, 2, 4)\)    | 5                           | \{2, 7\}                   |
| \((2, 2, 2, 5)\)    | 3                           | \{2, 5\}                   |
| \((2, 2, 2, \infty)\) | 3                     | \{2, 5\}                   |
| \((2, 2, 3, 3)\)    | 1                           | \{2, 3\}                   |
| \((2, 2, 3, \infty)\) | 1                      | \{2, 3\}                   |
| \((\infty, \infty, \infty, \infty)\) | 1              | \{2, 3\}                   |
| \((2, 2, 2, 2, 2)\) | 3                           | \{2, 3\}                   |
| \((2, 2, 2, 2, 3)\) | 1                           | \{2, 3\}                   |
| \((2, 2, 2, 2, \infty)\) | 1                   | \{2, 3\}                   |
| \((\infty, \infty, \ldots, \infty)\) | 1         | \{2, 3\}                   |
for sequences \((d_1, \ldots, d_n)\) and \((c_1, \ldots, c_m)\) of natural numbers \(\geq 2\) and some \(r \in \mathbb{Z}\). In case \(p\) is of wild type, then by (4.1)

\[
\hat{f}_{(p_1, \ldots, p_t)}(1) = -f_{[p_1, \ldots, p_t]}(1) = \left( t - 2 \right) - \sum_{i=1}^{t} \frac{1}{p_i} \prod_{i=1}^{t} p_i > 0
\]

and by (5.1), \(P_C(T)\) has a pole of order 2 at \(T = 1\), that is \(m - n + r = -2\). Moreover, developing the series \(P_C(T)\) we readily see that \(r \neq 0\) implies that the semigroups \(M(p)\) is \(N\), but [23, Th. 10.4] claims that \(1 + \frac{1}{t - 2} \leq \alpha(p)\), that is, \(\alpha(p) > 1\) and therefore \(r = 0\). We state these considerations in the following.

Lemma. Let \(p = (p_1, \ldots, p_t)\) be a weight sequence of wild type and \(C = C(p, \lambda)\) be a canonical algebra. Then \(\text{Root } \hat{f}_{(p_1, \ldots, p_t)}(T) \subset S^1\) if and only if \(R(p, \lambda)\) is formally \(n\)-generated, that is

\[
P_C(T) = \frac{\prod_{i=1}^{n-2} (1 - T^{c_i})}{\prod_{j=1}^{n} (1 - T^{d_j})}
\]

for numbers \(c_1, \ldots, c_{n-2}\) and \(d_1, \ldots, d_n\), all \(\geq 2\), satisfying

\[
1 + \sum_{j=1}^{n} d_j = \sum_{i=1}^{n-2} c_i.
\]

Proof. If \(\text{Root } \hat{f}_{(p_1, \ldots, p_t)}(T) \subset S^1\), we showed that \(P_C(T)\) has the desired form. Moreover,

\[
\sum_{j=1}^{n} d_j + \deg \hat{f}_{(p_1, \ldots, p_t)}(T) = \sum_{i=1}^{n-2} c_i + \deg f_C(T).
\]

The converse follows from \(\hat{f}_{(p_1, \ldots, p_t)}(T) = P_C(T)f_C(T)\). \(\square\)

In Table 2, for a weight sequence \(p = (p_1, \ldots, p_t)\) of wild type and \(C = C(p, \lambda)\) the corresponding canonical algebra we have calculated (under the column ‘Poincaré series’) the sequences \((d_1, \ldots, d_n)\) and \((c_1, \ldots, c_{n-2})\) corresponding to \(P_C(T) = \frac{f_{(p_1, \ldots, p_t)}(T)}{f_{(p_1, \ldots, p_t)}(T)}\).

5.5. Proof of Theorem 3. Let \(C = C(p, \lambda)\) be a canonical algebra of wild type and \(A = C[P]\) be an extended canonical algebra. Implications (a) ⇒ (b) ⇒ (c) are clear.

(c) ⇒ (a): Assume \(\text{Root } f_A(T) \subset S^1\). By (5.4), \(P_C(T)\) is \(n\)-generated. Hence \(M(p)\) is at most \(n\)-generated. By Table 3, if \(n \geq 5\) then \(p = (2, 3, p_3)\) with \(p_3 \geq 11\). But a calculation shows (see Table 2) that \(\text{Root } \hat{f}_{(2,3,11)}(T)\) is not contained in \(S^1\). Then
Theorem 2 implies that Root $\hat{f}_{(2,3,p_3)}(T)$ is not contained in $\mathbb{S}^1$ for $p_3 \geq 11$, which shows that $n \leq 4$.

For $k = \mathbb{C}$, the list entries in Table 2 correspond to Fuchsian singularities which are minimal elliptic as classified in [36]. These rings are graded complete intersection domains.

5.6. As a consequence of Theorem 3 and the classification given in Table 2 we get the following.

**Corollary.** Let $C = C(p, \lambda)$ be a canonical algebra of wild type. Let $A = C[P]$ be an extended canonical algebra. The following are equivalent:

(a) $\varphi_A$ is periodic.

(b) $\text{Spec} \varphi_A \subset \mathbb{S}^1$ and $p$ is not $(3,3,3,3)$ or $(2,2,2,2,4)$.

(c) $f_A(T) = \prod_{i=1}^m \phi_{s_i}(T)^{e_i}$ for $1 \leq s_1 < s_2 < \cdots < s_m$ and $e_i \geq 1$ ($1 \leq i \leq m$), with $e_m = 1$.

6. Graded integral domains with 3 homogeneous generators

6.1. The following simple remark is well-known.

**Lemma.** Let $R$ be a graded complete intersection integral $k$-algebra of Krull dimension 2. Then $R$ is generated by 3 homogeneous elements if and only if $R = k[x_1, x_2, x_3]/(f)$ with $\deg x_i = d_i$ ($1 \leq i \leq 3$) and $f$ a homogeneous prime polynomial. In this case the Poincaré-Hilbert series of $R$ has the form

$$\sum_{n=0}^{\infty} (\dim_k R_n)T^n = \frac{1-T^c}{(1-T^{d_1})(1-T^{d_2})(1-T^{d_3})}$$

for some natural numbers $c, d_1, d_2, d_3$ satisfying $1 + d_1 + d_2 + d_3 = c$.

**Proof.** Assume we have a graded surjection $g: k[x_1, x_2, x_3] \to R$ such that $y_i = g(x_i)$ is homogeneous of degree $d_i$, $1 \leq i \leq 3$. Since $R$ is graded integral, then $I = \ker g$ is a prime ideal. Since $R$ has Krull-dimension two, the ideal $I$ has height one, hence it is principal. Let $I = (f)$ and $\deg (f) = c$. Then the Poincaré series has the desired form and $1 + d_1 + d_2 + d_3 = c$ by (4.4).
6.2. Proof of Theorem \([4]\). Let \(C = C(p, \lambda)\) be a wild canonical algebra and \(A = C[P]\) a corresponding extended canonical algebra.

(a) \(\Rightarrow\) (b): Assume \(R(p, \lambda)\) is formally 3-generated. By \([5, 4]\), Root \(f_A \subset S^1\). The result follows from the list given in Table 2.

(b) \(\Rightarrow\) (a): follows as above from Table 2 and \([5, 4]\).

Assume \(R = R(p, \lambda)\) is formally 3-generated with

\[
\sum_{n=0}^{\infty} (\dim_k R_n)T^n = \frac{1-T^{d_1}}{(1-T^{d_2})(1-T^{d_3})}
\]

with \((d_1, d_2, d_3), (c)\) according to Table 2. We shall consider two distinguished situations:

if \(t = 3\), then \(R\) is a quasi-homogeneous complete intersection of the form \(k[x_1, x_2, x_3]/(f)\) with \(\deg x_i = d_i\) and \(f\) a homogeneous relation as displayed in \([6, 3]\).

The case \(t \geq 4\) and \(k = C\) is considered in \([6, 4]\).

\[ \square \]

6.3. Theorem \([21]\). Let \(C = C(p, \lambda)\) be a canonical algebra of wild weight type \(p = (p_1, p_2, p_3)\) such that the graded algebra \(R = R(p, \lambda)\) is formally 3-generated. Then \(R\) has the form

\[
R = k[x, y, z] = k[X, Y, Z]/(F)
\]

where the relation \(F\), the degree triple \(\deg (x, y, z)\) and \(\deg (F)\) are displayed in Table 4:

| \(p\) | \(\deg (x, y, z)\) | relation \(F\) | \(\deg (F)\) | Name |
|---|---|---|---|---|
| Index = 6 | | | | |
| (2, 3, 7) | (6, 14, 21) | \(Z^2 + Y^3 + X^7\) | 42 | \(E_{12}\) |
| (2, 3, 8) | (6, 8, 15) | \(Z^2 + X^5 + XY^3\) | 30 | \(Z_{11}\) |
| (2, 3, 9) | (6, 8, 9) | \(Z^2 + XZ^2 + X^4\) | 36 | \(Q_{10}\) |
| 4 | | | | |
| (2, 4, 5) | (4, 10, 15) | \(Z^2 + Y^3 + X^2Y\) | 30 | \(E_{13}\) |
| (2, 4, 6) | (4, 6, 11) | \(Z^2 + X^4Y + ZY^3\) | 22 | \(Z_{12}\) |
| (2, 4, 7) | (4, 6, 7) | \(Y^3 + X^3Y + XZ^2\) | 18 | \(Q_{11}\) |
| (2, 5, 5) | (4, 5, 10) | \(Z^2 + Y^2Z + X^5\) | 20 | \(W_{12}\) |
| (2, 5, 6) | (4, 5, 6) | \(XZ^2 + Y^2Z + X^4\) | 16 | \(S_{11}\) |
| 3 | | | | |
| (3, 3, 4) | (3, 8, 12) | \(Z^2 + Y^3 + X^2Z\) | 24 | \(E_{14}\) |
| (3, 3, 5) | (3, 5, 9) | \(Z^2 + XY^3 + X^3Z\) | 18 | \(Z_{13}\) |
| (3, 3, 6) | (3, 5, 6) | \(Y^3 + X^3Z + XZ^2\) | 15 | \(Q_{12}\) |
| (3, 4, 4) | (3, 4, 8) | \(Z^2 - Y^2Z + X^4Y\) | 16 | \(W_{13}\) |
| (3, 4, 5) | (3, 4, 5) | \(X^3Y + XZ^2 + Y^2Z\) | 13 | \(S_{12}\) |
| (4, 4, 4) | (3, 4, 4) | \(X^4 - YZ^2 + Y^2Z\) | 12 | \(U_{12}\) |
As observed in [21], these 14 equations are equivalent to Arnold’s exceptional unimodal singularities. The equations are slightly different to those of the singularity theory classification, but equivalent for \( k = \mathbb{C} \).

6.4. In view of the identification of \( R(p, \lambda) \) with a ring of automorphic forms [6,5] in case \( k = \mathbb{C} \), we get:

**Theorem** [9]. Let \( C = C(p, \lambda) \) be a canonical algebra of wild type \( p = (p_1, \ldots, p_t) \) with \( t \geq 4 \) over the complex numbers \( \mathbb{C} \) and \( R = R(p, \lambda) \) be the associated graded algebra. Then the following are equivalent:

(a) \( R(p, \lambda) \) is formally 3-generated;

(b) \( 9 \leq \sum_{i=1}^{t} p_i \leq 11 \)

(c) there is a parameter sequence \( \lambda' = (\lambda'_3, \ldots, \lambda'_t) \) such that the algebra \( R(p, \lambda') \) is of the form

\[
\mathbb{k}[x, y, z] = \mathbb{k}[X, Y, Z]/(F)
\]

where the relation \( F \), the degree sequence \( \deg (x, y, z) \) and \( \deg (F) \) are displayed below:

| \( t = 4 \) | \( p \) | \( \deg (x, y, z) \) | relation \( F \) | \( \deg (F) \) | Name |
|---|---|---|---|---|---|
| \( (2, 2, 2, 3) \) | \( (2, 6, 9) \) | \( Z^2 + Y^3 + X^9 \) | 18 | \( J_{3,0} \) |
| \( (2, 2, 2, 4) \) | \( (2, 4, 7) \) | \( Z^2 + XY^3 + X^7 \) | 14 | \( Z_{1,0} \) |
| \( (2, 2, 2, 5) \) | \( (2, 4, 5) \) | \( Y^3 + XZ^2 + X^6 \) | 12 | \( Q_{2,0} \) |
| \( (2, 2, 3, 3) \) | \( (2, 3, 6) \) | \( Z^2 + Y^4 + X^6 \) | 12 | \( W_{1,0} \) |
| \( (2, 2, 3, 4) \) | \( (2, 3, 4) \) | \( YZ^2 + XZ^2 + X^5 \) | 10 | \( S_{1,0} \) |
| \( (2, 3, 3, 3) \) | \( (2, 3, 3) \) | \( Z^3 + Y^3 + X^3Y \) | 9 | \( U_{1,0} \) |

| \( t = 5 \) | \( p \) | \( \deg (x, y, z) \) | relation \( F \) | \( \deg (F) \) |
|---|---|---|---|---|
| \( (2, 2, 2, 2, 2) \) | \( (2, 2, 5) \) | \( Z^2 + Y^5 + X^6 \) | 10 | \( NA_{1,0,0} \) |
| \( (2, 2, 2, 2, 3) \) | \( (2, 2, 3) \) | \( YZ^2 + Y^4 + X^4 \) | 8 | \( VNA_{1,0,0} \) |

Table 5.

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