Companion unit lower Hessenberg matrices ∗†

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Abstract

In recent years there has been a growing interest in companion matrices. There is a deep knowledge of sparse companion matrices, in particular it is known that every sparse companion matrix can be transformed into a unit lower Hessenberg matrix of a particularly simple type by any combination of transposition, permutation similarity and diagonal similarity. The latter is not true for the companion matrices that are non-sparse, although it is known that every non-sparse companion matrix is nonderogatory. In this work the non-sparse companion matrices that are unit lower Hessenberg will be described. A natural generalization is also considered.

1 Introduction

For a given field $F$, a matrix $A$ of order $n$ is said to be companion as long as: (i) $A$ has $n^2 - n$ entries that are constants of $F$; (ii) the $n$ remaining entries of $A$ are the variables $x_1, \ldots, x_n$; and (iii) the characteristic polynomial of $A$ is

$$\det(\lambda I_n - A) = \lambda^n - x_1 \lambda^{n-1} - \cdots - x_{n-1} \lambda - x_n. \quad (1)$$

The classical example is the Frobenius companion matrix

$$
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
x_n & x_{n-1} & \cdots & x_2 & x_1
\end{bmatrix}
$$

In 2003 Fiedler [8] introduced new companion matrices, which in turn produced an increased interest in companion matrices. In 2013 Ma and Zhan [10] studied those matrices of order $n$ whose entries are in the field $F(x_1, \ldots, x_n)$ of rational functions in the variables $x_1, \ldots, x_n$ and for which the characteristic polynomial is (1). They showed that such matrices are necessarily irreducible and have at least $2n - 1$ nonzero entries. This permits to define a sparse companion matrix as a companion matrix with $2n - 1$ nonzero entries. In 2016 Garnett et al. [9] characterized the Ma-Zhan matrices.

A square matrix is a unit lower Hessenberg matrix, ULH matrix by short, if all its superdiagonal entries are equal to one and all its entries above the superdiagonal are equal to zero. In 2014

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Eastmann et al. [5, 6] showed that every sparse companion matrix can be transformed into a ULH matrix by any combination of transposition, permutation similarity and diagonal similarity (we will be more precise in Theorems 1 and 2 below). In 2019 Deaett et al. [3] gave one example that shows that this is not necessarily the case for non-sparse companion matrices. On the other hand, they proved that any companion matrix is nonderogatory.

If \( A \) is a companion matrix then \((1, -x_1, \ldots, -x_n)\) are the coordinates of \( \det(\lambda I_n - A) \) with respect to the monomial basis \( \{\lambda^n, \lambda^{n-1}, \ldots, \lambda, 1\} \) of the \((n + 1)\)-dimensional vector space \( \mathbb{F}_n[\lambda] \) of polynomials of degree at most \( n \) on the variable \( \lambda \). The concept of companion matrix can be generalized whenever we consider any polynomial basis of \( \mathbb{F}_n[\lambda] \) other than the monomial basis. For example, consider the Newton basis

\[
\left\{ \prod_{h=1}^{n}(\lambda - \gamma_h), \prod_{h=1}^{n-1}(\lambda - \gamma_h), \ldots, (\lambda - \gamma_1), 1 \right\}
\]

that occurs on interpolation at the points \( \lambda = \gamma_1, \ldots, \lambda = \gamma_n \). Actually, Farahat and Ledermann [7] noted that a Frobenius companion matrix in which we modify the main diagonal,

\[
A = \begin{bmatrix}
\gamma_1 & 1 & 0 & \cdots & 0 \\
0 & \gamma_2 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \gamma_{n-1} & 1 \\
x_n & \cdots & \cdots & x_2 & x_1 + \gamma_n
\end{bmatrix},
\]

has characteristic polynomial

\[
\det(\lambda I_n - A) = \prod_{h=1}^{n}(\lambda - \gamma_h) - x_1 \prod_{h=1}^{n-1}(\lambda - \gamma_h) - \cdots - x_n(\lambda - \gamma_1) - x_n.
\]

Therefore \((1, -x_1, \ldots, -x_n)\) are the coordinates of \( \det(\lambda I_n - A) \) with respect to the Newton basis. The entry \( x_1 + \gamma_n \) causes matrix \( A \) to not meet the condition \((ii)\) of the definition of companion matrix. Nevertheless this can be sidestepped by making \( \gamma_n = 0 \).

By results of [1, 2, 11], to name a few references in this direction, we can go even further and regard

\[
\begin{bmatrix}
d_{11} & 1 & 0 & \cdots & 0 \\
d_{21} & d_{22} & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
d_{n-1,1} & \cdots & d_{n-1,n-2} & d_{n-1,n-1} & 1 \\
x_n & \cdots & \cdots & x_3 & x_2 & x_1
\end{bmatrix},
\]

as a companion matrix with respect to some polynomial basis of \( \mathbb{F}_n[\lambda] \). We will not give the specific polynomial basis for this matrix since in Theorems 13 and 18 we will consider broader families of matrices (that includes (2) as a particular case) for which such a polynomial basis exists. And in the proof of Lemma 12 we will see how the polynomial basis is constructed. These kind of results have motivated us to generalize the concept of companion matrix.\footnote{Garnett et al. in [9] employ the term generalized companion matrix for those matrices of order \( n \) with \( 2n - 1 \) nonzero entries, \( n - 1 \) are ones and \( n \) are elements of the ring \( \mathbb{F}[x_1, \ldots, x_n] \), with characteristic polynomial equal to (1). In the literature there exist other generalizations of companion matrices (see for instance the introduction of the recent work of De Terán and Hernando [3]).}

A matrix \( A \) of order \( n \) is said to be \textit{companion with respect to a polynomial basis, PB-companion} by short, as long as: \((i)\) \( A \) has \( n^2 - n \) entries that are constants of the field \( \mathbb{F} \); \((ii)\) the \( n \) remaining
entries of $A$ are the variables $x_1, \ldots, x_n$; and (iii) the characteristic polynomial of $A$ is
\[
det(\lambda I_n - A) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_{n-1} p_1(\lambda) - x_n p_0(\lambda)
\]
where $\{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\}$ is a polynomial basis of $\mathbb{F}_n[\lambda]$.

A companion matrix is PB-companion with respect to the monomial basis $\{\lambda^n, \ldots, \lambda, 1\}$ of $\mathbb{F}_n[\lambda]$.

The main concern of this work is to describe both, the companion unit lower Hessenberg matrices and the PB-companion unit lower Hessenberg matrices.

## 2 Different sets of ULH matrices to be considered

In the ULH matrices that are candidates to be companion, the positions $(i_1, j_1), \ldots, (i_n, j_n)$ of the variables $x_1, \ldots, x_n$ below the subdiagonal play an essential role. In order to avoid redundancies we define a total order in the set $\mathbb{Z} \times \mathbb{Z}$ by
\[
(i_r, j_r) \prec (i_s, j_s) \text{ if and only if } \begin{cases}
  i_r - j_r < i_s - j_s \\
  \text{or} \\
  i_r - j_r = i_s - j_s \text{ and } i_r < i_s.
\end{cases}
\]

This establishes the order in which $x_1, \ldots, x_n$ should be placed.

For the proper development of this work we will introduce several sets of ULH matrices:

- Let $\mathcal{H}$ be the set of ULH matrices of any order $n$ that have $2n - 1$ nonzero entries: $n - 1$ entries in the superdiagonal equal to one, and $n$ entries below the superdiagonal equal to $x_1, \ldots, x_n$ that are placed according to the order $\prec$. Let us describe the matrices of $\mathcal{H}$ more precisely:

  \[
  H_{(i_1, j_1), \ldots, (i_n, j_n)} \text{ is the ULH matrix of order } n \text{ with } 2n - 1 \text{ nonzero entries} \\
  \text{whose } (i_1, j_1), \ldots, (i_n, j_n) \text{ entries are } x_1, \ldots, x_n \\
  \text{with } (i_1, j_1) \prec \cdots \prec (i_n, j_n) \text{ and } 1 \leq j_k \leq i_k \leq n \text{ for } k = 1, \ldots, n.
  \]

- Let $\mathcal{D}$ be the set composed by those matrices of $\mathcal{H}$ in which the variable $x_1$ is located in the diagonal, $x_2$ in the subdiagonal, $x_3$ in the second subdiagonal, and so forth. Let us describe the matrices of $\mathcal{D}$ more precisely:

  \[
  D_{i_1, i_2, \ldots, i_n} = H_{(i_1, i_1), (i_2, i_2 - 1), \ldots, (i_n, i_n - n + 1)} \text{ whenever } k \leq i_k \text{ for } k = 1, \ldots, n.
  \]

- Let $\mathcal{C}$ be the set composed by those matrices of $\mathcal{D}$ in which the variables $x_1, \ldots, x_n$ lay in the rectangular submatrix with top-right vertex the $(i_1, i_1)$ entry and with bottom-left vertex the $(n, 1)$ entry. Let us describe the matrices of $\mathcal{C}$ more precisely:

  \[
  C_{i_1, \ldots, i_n} = D_{i_1, \ldots, i_n} \text{ whenever } i_k - k + 1 \leq i_1 \leq i_k \text{ for } k = 1, \ldots, n. \tag{3}
  \]

- Let $\mathcal{G}$ be the set composed by those matrices of $\mathcal{C}$ which have at least one variable on each one of the rows $i_1, \ldots, n$ and on each one of the columns $1, \ldots, i_1$. Let us describe the matrices of $\mathcal{G}$ more precisely:

  \[
  G_{i_1, \ldots, i_n} = C_{i_1, \ldots, i_n} \text{ whenever } \{i_1, \ldots, i_n\} = \{i_1, \ldots, n\} \text{ and} \\
  \{i_1, i_2 - 1, \ldots, i_n - n + 1\} = \{1, \ldots, i_1\}.
  \]
• And finally, let $\mathcal{F}$ be the set composed by those matrices of $\mathcal{G}$ such that for $k = 2, \ldots, n$ the variable $x_k$ is in the same row or in the same column than the variable $x_{k-1}$. So in a matrix of $\mathcal{F}$ the variables $x_1, \ldots, x_n$ form a lattice path starting on the diagonal with $x_1$ and ending on the bottom-left corner with $x_n$. Let us describe the matrices of $\mathcal{F}$ more precisely:

$$F_{i_1, \ldots, i_n} = G_{i_1, \ldots, i_n}$$

whenever either $i_k = i_{k-1}$ or $i_k = i_{k-1} + 1$ for $k = 2, \ldots, n$.

As the following examples shows, we have $\mathcal{H} \supseteq \mathcal{D} \supseteq \mathcal{C} \supseteq \mathcal{G} \supseteq \mathcal{F}$.

| $H_{(1,1),(3,3),(2,1),(5,4),(5,2)} \in \mathcal{H} \setminus \mathcal{D}$ | $D_{4,5,3,5,5} \in \mathcal{D} \setminus \mathcal{C}$ | $C_{3,4,3,4,5} \in \mathcal{C} \setminus \mathcal{G}$ |
|---|---|---|
| $\begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \\ 0 & x_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & x_5 & 0 & x_4 \\ \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_3 & 0 & x_1 & 1 \\ 0 & 0 & 0 & 1 \\ x_5 & x_4 & 0 & x_2 \\ \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x_3 & 0 & x_1 & 1 \\ x_4 & 0 & x_2 & 0 \\ x_5 & 0 & 0 & 0 \\ \end{bmatrix}$ |

**Observations:**
1. The choice of letters for the sets of matrices was made according to the following mnemonics rule: $\mathcal{H}$ for Hessenberg (this is the largest set to consider), $\mathcal{D}$ for diagonal (each variable is in a different diagonal), $\mathcal{C}$ for companion (these are the sparse companion ULH matrices), $\mathcal{G}$ for gather (rows $i_1$ to $n$ and columns $1$ to $i_1$ gather all variables with each row and each column having at least one), and $\mathcal{F}$ for Fiedler (these are the Fiedler companion matrices).

2. Each matrix $A$ of order $n$ has associated a digraph of $n$ vertices in which the vertex $i$ is joined with the vertex $j$ by an oriented edge if and only if the $(i, j)$ entry of $A$ is nonzero. If $A_1 \in \mathcal{H} \setminus \mathcal{D}$, $A_2 \in \mathcal{D} \setminus \mathcal{C}$, $A_3 \in \mathcal{C} \setminus \mathcal{G}$, $A_4 \in \mathcal{G} \setminus \mathcal{F}$, and $A_5 \in \mathcal{F}$, then it is not difficult to see that the digraph associated to $A_i$ is not isomorphic to the digraph associated to $A_j$ for $1 \leq i < j \leq 5$. In [3] the authors studied the digraphs associated to matrices of $\mathcal{C}$.

## 3 Companion ULH matrices

From now on we will frequently use the following definitions and notations:

a) Given a matrix $A$, if nothing is mentioned, it will be assumed that it is square of order $n$.

b) Instead of $\det(\lambda I_n - A)$ we will use the shorter notation $P_A(\lambda)$ to refer to the characteristic polynomial of $A$.

c) The matrix $A$ of order 0 is the empty matrix, with $\det(A) = 1$ and $P_A(\lambda) = 1$.

d) The constant part of a matrix $A$ whose entries are elements of the ring $\mathbb{F}[x_1, \ldots, x_n]$ is the constant matrix obtained from $A$ by doing $x_1 = \cdots = x_n = 0$.

e) For $k = 0, 1, \ldots, n - 1$ the $k$-subdiagonal of $A$ is $(a_{k+1,1}, a_{k+2,2}, \ldots, a_{n,n-k})$.

f) $s_k(A)$ is the sum of the entries on the $k$-subdiagonal of $A$. 

4
g) \( A[i_1, \ldots, i_r; j_1, \ldots, j_s] \) is the submatrix of all entries in rows \( i_1, \ldots, i_r \) and columns \( j_1, \ldots, j_s \).

h) The \( k \)-block of \( A \) is the submatrix \( A[k_1, \ldots, n; 1, \ldots, k] \) of \( A \) with top-right corner at position \( (k, k) \) and bottom-left corner at position \( (n, 1) \).

i) \( A[k_1, \ldots, k_r] \) is \( A[k_1, \ldots, k_r; k_1, \ldots, k_r] \).

j) The \( k \)-leading principal submatrix of \( A \) is \( A[1, \ldots, k] \).

k) The \( k \)-trailing principal submatrix of \( A \) is \( A[n - k + 1, \ldots, n] \).

l) \( A[i, \ldots, j] \) is the empty matrix if \( i, \ldots, j \) is not a valid range for \( A \).

m) A companion ULH matrix is a companion matrix which is also ULH.

n) If starting with a matrix \( A \) we obtain \( B \) by changing zero entries of \( A \) by nonzero constants, we will say that \( B \) is a superpattern of \( A \). This terminology comes from [5].

o) If starting with a ULH matrix \( A \) we obtain \( B \) by changing zero entries of \( A \) below the superdiagonal by nonzero constants, we will say that \( B \) is a ULH superpattern of \( A \).

p) The set of all ULH superpatterns of a ULH matrix \( A \) is denoted by \( \tilde{A} \).

q) The set of all ULH superpatterns of all matrices of, respectively, \( \mathcal{H}, \mathcal{D}, \mathcal{C}, \mathcal{G}, \) and \( \mathcal{F} \) is denoted by, respectively, \( \tilde{\mathcal{H}}, \tilde{\mathcal{D}}, \tilde{\mathcal{C}}, \tilde{\mathcal{G}}, \) and \( \tilde{\mathcal{F}} \).

3.1 From sparse to non-sparse

Eastman et al. characterized the sparse companion matrices.

**Theorem 1.** [6] A is a sparse companion matrix if and only if \( A \) can be obtained from some matrix of \( \mathcal{C} \) by any combination of transposition, permutation similarity and diagonal similarity.

And they also determined which are the sparse companion matrices within \( \mathcal{H} \). Note that the set of sparse companion matrices of \( \mathcal{H} \) is the same as the set of sparse companion ULH matrices. So we adapt their result to the notation we have introduced.

**Theorem 2.** [5, Theorem 4.1] The sparse companion ULH matrices are the matrices of \( \mathcal{C} \).

When we broaden our gaze to non-sparse companion matrices we will no longer obtain a result like Theorem [1] This is so because Deaett et al. [3] showed one non-sparse companion matrix that can not be transformed into a ULH matrix by any combination of transposition, permutation similarity and diagonal similarity. So we will focus our efforts to obtain a result like Theorem [2] also valid for non-sparse companion ULH matrices. The following theorem tells us where we should look for the companion ULH matrices and how to characterize them.

**Theorem 3.** Any companion ULH matrix belongs to \( \tilde{\mathcal{C}} \). Moreover, if \( A \in \tilde{\mathcal{C}}_{i_1, \ldots, i_n} \) then \( A \) is a companion matrix if and only if the following conditions are met:

(i) The constant part of \( A \) is a nilpotent matrix;

(ii) \( A[1, \ldots, i_k - k] \) is a constant nilpotent matrix or the empty matrix for \( k = 1, \ldots, n \);

(iii) \( A[i_k + 1, \ldots, n] \) is a constant nilpotent matrix or the empty matrix for \( k = 1, \ldots, n \).

If the only nonzero entries of the \( i_1 \)-block of \( A \) are \( x_1, \ldots, x_n \) then condition (i) can be omitted.
Proof. The set of companion ULH matrices is the same as the set of companion matrices of \( \tilde{H} \), so in order to prove that any companion ULH matrix belongs to \( \tilde{C} \) it is enough to demonstrate that any matrix of \( \tilde{H} \setminus \tilde{C} \) is not companion.

If \( B \in \tilde{H} \setminus \tilde{D} \) then there are two variables \( x_r \) and \( x_s \) on the same \( k \)-subdiagonal of \( B \). In the characteristic polynomial of \( B \) appear the terms \( x_r \lambda^{n-k-1} \) and \( x_s \lambda^{n-k-1} \). So \( B \) is not companion.

If \( B \in \tilde{D} \setminus \tilde{C} \) then \( x_1 \) is the \((i_1, i_1)\) entry of \( B \) and there exists some \( x_s \) out of the \( i_1 \)-block of \( B \). As \( x_s \) is on the \((s-1)\)-subdiagonal of \( B \) then in the characteristic polynomial of \( B \) appears the term \( x_1 x_s \lambda^{n-1-s} \). So \( B \) is not companion.

Now we go to the second part of the theorem. Recall the notation for the characteristic polynomial of a matrix that we introduced in \( \text{[3]} \) at the beginning of Section \( \text{[3]} \). If \( A \) is a matrix of \( \tilde{C}_{i_1, \ldots, i_n} \) then all the terms of \( P_A(\lambda) \) in which variables \( x_1, \ldots, x_n \) do not appear conform \( P_{A_0}(\lambda) \) where \( A_0 \) is the constant part of \( A \). On the other hand, as \( A \) is a ULH matrix and \( x_k \) is the \((i_k, i_k - k + 1)\) entry of \( A \) then \( x_k \) appears in \( P_A(\lambda) \) only in the term

\[-x_k P_{[1, \ldots, i_k-k]}(\lambda) P_{[i_k+1, \ldots, n]}(\lambda).\]

Note that \( A[1, \ldots, i_k-k] \) is the empty matrix if \( i_k = k \) and \( A[i_k+1, \ldots, n] \) is the empty matrix if \( i_k = n \). If we add the fact that the variables \( x_1, \ldots, x_n \) are in the \( i_1 \)-block of \( A \) then \( A[1, \ldots, i_k-k] \) is a submatrix of the constant matrix \( A[1, \ldots, i_1-1] \) and \( A[i_k+1, \ldots, n] \) is a submatrix of the constant matrix \( A[i_1+1, \ldots, n] \).

Therefore

\[ P_A(\lambda) = P_{A_0}(\lambda) - \sum_{k=1}^{n} x_k P_{[1, \ldots, i_k-k]}(\lambda) P_{[i_k+1, \ldots, n]}(\lambda). \]

(4)

Note that if some \( P_{[1, \ldots, i_k-k]}(\lambda) P_{[i_k+1, \ldots, n]}(\lambda) \) contained a variable, then \( \text{[4]} \) would not be true.

So \( A \) is a companion matrix if and only if equations \( \text{[3]} \) and \( \text{[4]} \) match, that is, if and only if

\[ \lambda^n = P_{A_0}(\lambda); \]

\[ \lambda^{n-k} = P_{[1, \ldots, i_k-k]}(\lambda) P_{[i_k+1, \ldots, n]}(\lambda) \quad \text{for} \quad k = 1, \ldots, n; \]

if and only if

\[ \lambda^n = P_{A_0}(\lambda); \]

\[ \lambda^{k-k} = P_{[1, \ldots, i_k-k]}(\lambda) \quad \text{for} \quad k = 1, \ldots, n; \]

\[ \lambda^{n-i_k} = P_{[i_k+1, \ldots, n]}(\lambda) \quad \text{for} \quad k = 1, \ldots, n; \]

if and only if

\begin{itemize}
  
  \item \( A_0 \) is nilpotent;
  \item \( A[1, \ldots, i_k-k] \) is a constant nilpotent matrix or the empty matrix for \( k = 1, \ldots, n; \)
  \item \( A[i_k+1, \ldots, n] \) is a constant nilpotent matrix or the empty matrix for \( k = 1, \ldots, n. \)
\end{itemize}

Let us prove the second part. If the only nonzero entries of the \( i_1 \)-block of \( A \) are \( x_1, \ldots, x_n \) then

\[ A_0 = \begin{bmatrix}
  A[1, \ldots, i_1-1] & e_{i_1-1} & 0 \\
  0 & 0 & e_1^T \\
  0 & 0 & A[i_1+1, \ldots, n]
\end{bmatrix}. \]

So, if \( A \) meets conditions \( (ii) \) and \( (iii) \) then \( A_0 \) is nilpotent in all three possible cases:

\begin{itemize}
  \item \( (a) \) If \( 1 < i_1 < n \) then \( A[1, \ldots, i_1-1] \) and \( A[i_1+1, \ldots, n] \) are nilpotent. So it is \( A_0 \).
  \item \( (b) \) If \( i_1 = 1 \) then \( A_0 = \begin{bmatrix} 0 & e_1^T \\ 0 & A[2, \ldots, n] \end{bmatrix} \) and \( A[2, \ldots, n] \) is nilpotent. So it is \( A_0 \).
  \item \( (c) \) If \( i_1 = n \) then \( A_0 = \begin{bmatrix} A[1, \ldots, n-1] & e_{n-1} \\ 0 & 0 \end{bmatrix} \) and \( A[1, \ldots, n-1] \) is nilpotent. So it is \( A_0 \).
\end{itemize}

\( \square \)
3.2 Nested nilpotent ULH matrices

Let \( n_1, \ldots, n_s \) be integers with \( 0 \leq n_1, \ldots, n_s \leq n \). We will denote by \( \mathcal{N}_L(n_1, \ldots, n_s) \) the set of constant ULH matrices of order \( n \) such that for each \( h = 1, \ldots, s \) with \( n_h \neq 0 \) the \( n_h \)-leading principal submatrix is nilpotent. And we will denote by \( \mathcal{N}_T(n_1, \ldots, n_s) \) the set of ULH matrices of order \( n \) such that for each \( h = 1, \ldots, s \) with \( n_h \neq 0 \) the \( n_h \)-trailing principal submatrix is nilpotent. By convention we have included the possibility that the sequence \( n_1, \ldots, n_s \) contains repeated elements or elements equal to zero.

If \( J_n = \begin{bmatrix} 0 & \cdots & 1 \\ 1 & \ddots & 0 \\ \vdots & \ddots & \ddots \end{bmatrix} \) is the backward identity matrix then

\[ \mathcal{N}_L(n_1, \ldots, n_s) = (J_n \mathcal{N}_L(n_1, \ldots, n_s) J_n)^T. \]

The transposition is necessary to transform ULH matrices into ULH matrices. An example:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
3 & -5 & 1 & 0 \\
15 & -28 & 5 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix} \in \mathcal{N}_L(1, 3, 4) \text{ and } (J_n A J_n)^T = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 5 & 1 & 0 \\
0 & -28 & -5 & 1 \\
0 & 15 & 3 & 0
\end{bmatrix} \in \mathcal{N}_L(1, 3, 4).
\]

Now we will state Theorem 3 in terms of nested nilpotent matrices.

**Theorem 4.** Any companion ULH matrix belongs to \( \tilde{C} \). Moreover, if \( A \in \tilde{C}_{i_1, \ldots, i_n} \) then \( A \) is a companion matrix if and only if the following conditions are met:

(i) The constant part of \( A \) is a nilpotent matrix;

(ii) \( A[1, \ldots, i_1-1] \) belongs to \( \mathcal{N}_L^{i_1-1}(i_1-1, i_2-2, \ldots, i_n-n) \);

(iii) \( A[i_1+1, \ldots, n] \) belongs to \( \mathcal{N}_T^{n-i_1}(n-i_1, n-i_2, \ldots, n-i_n) \).

If the only nonzero entries of the \( i_1 \)-block of \( A \) are \( x_1, \ldots, x_n \) then condition (i) can be omitted.

**Proof.** It is enough to prove that conditions (ii) and (iii) in Theorems 3 and 4 are equal. Taking into account (3) we conclude that \( 0 \leq i_k - k \leq i_1 - 1 \) for each \( k = 1, \ldots, n \) and so conditions (ii) in both theorems are equal. Again, taking into account (3) we conclude that \( i_1 \leq i_k \leq n \) for each \( k = 1, \ldots, n \) and so conditions (iii) in both theorems are equal. \( \square \)

In Section 3.4 we will show how to parametrize the sets \( \mathcal{N}_L(n_1, \ldots, n_s) \) and \( \mathcal{N}_T(n_1, \ldots, n_s) \). Previously we analyze a specially simple case.

**Lemma 5.** \( \mathcal{N}_L(1, \ldots, n) = \mathcal{N}_T(1, \ldots, n) = \{ U_n \} \), where \( U_n \) is the upper shift matrix with ones on the superdiagonal and zeroes elsewhere.

**Proof.** We will prove by induction that \( \mathcal{N}_L(1, \ldots, n) = \{ U_n \} \). For \( n = 1 \), \( \mathcal{N}_L(1) = \{ [0] \} = \{ U_1 \} \). Assume that it is true for \( k \), that is, \( \mathcal{N}_L(1, \ldots, k) = \{ U_k \} \). Let \( A \) be a matrix of \( \mathcal{N}_L^{k+1}(1, \ldots, k+1) \). As \( A[1, \ldots, k] \in \mathcal{N}_L(k, 1, \ldots, k) \) then \( A[1, \ldots, k] = U_k \). As \( A \) is a ULH matrix then

\[
A = \begin{bmatrix}
& & & U_k & \\
& & a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} \\
& \vdots & & \ddots & \ddots & \ddots \\
& & & 0 & \ddots & \ddots \\
& & & & 0 & \ddots \\
& & & & & 1
\end{bmatrix}.
\]

As \( A \) is nilpotent then \( a_{k+1,1} = \cdots = a_{k+1,k+1} = 0 \). So \( \mathcal{N}_L^{k+1}(1, \ldots, k+1) = \{ U_{k+1} \} \).

On the other hand,

\[
\mathcal{N}_T(1, \ldots, n) = (J_n \mathcal{N}_L(n_1, \ldots, n_r) J_n)^T = ((J_n U_n J_n)^T = \{ U_n \}.
\]

\( \square \)
3.3 Description of the companions of $\tilde{G}$

Even though Theorems 3 and 4 characterize the companion matrices of $\tilde{C}$, we can give a nice and much easier to check description of the companion matrices of the subset $\tilde{G}$ of $\tilde{C}$.

First we define the sets $\hat{C}$, $\hat{G}$, and $\hat{F}$ composed by those matrices of, respectively, $\tilde{C}$, $\tilde{G}$, and $\tilde{F}$ such that all its nonzero entries are either in the superdiagonal (and are equal to 1) or in the $i_1$-block determined by the position of the variable $x_1$. Note that $\hat{C} \supseteq \hat{G} \supseteq \hat{F}$.

This structure makes that if $A$ is a matrix of $\hat{C}$ then

$$P_A(\lambda) = \lambda^n - s_0(A)\lambda^{n-1} - \cdots - s_{n-2}(A)\lambda - s_{n-1}(A)$$

where $s_k(A)$ is the sum of the entries on the $k$-subdiagonal of $A$. Matching (8) with (1) we obtain the following characterization of the companion matrices of $\hat{C}$.

**Lemma 6.** $A \in \hat{C}$ is companion if and only if $s_{k-1}(A) = x_k$ for $k = 1, \ldots, n$.

Let us see how to use this result to characterize the companion matrices of $\tilde{G}$.

**Theorem 7.** $A \in \tilde{G}$ is companion if and only if $A \in \hat{G}$ and $s_{k-1}(A) = x_k$ for $k = 1, \ldots, n$.

**Proof.** The sufficiency follows from Lemma 6.

The necessity. By Lemma 6, it is enough to prove that $A \in \hat{G}$. Note that $A \in \tilde{G}_{i_1, \ldots, i_n}$ for some $i_1, \ldots, i_n$. Then there exists at least one variable on each of the rows $i_1, i_1 + 1, \ldots, n$ and one variable on each of the columns $1, \ldots, i_1$. From Theorem 4 and Lemma 5 it follows that

$$A[i_1, \ldots, i_1 - 1] \in \mathcal{N}_T^{i_1 - 1}(1, \ldots, i_1 - 1) = \{U_{i_1-1}\};$$

$$A[i_1 + 1, \ldots, n] \in \mathcal{N}_T^{n-i_1}(1, \ldots, n - i_1) = \{U_{n-i_1}\}.$$

So $A$ meets the conditions to be a matrix of $\tilde{G}$.

The interest of Theorem 7 is that we are able to parametrize all the ULH superpatterns of a considerable number of matrices of $\hat{C}$. Now we give an example of such a parametrization.

**Example 8.** Consider the matrix $G_{4,4,6,5,5,7,7} \in \mathcal{G}$. According to Theorem 7 the parameterization of the set of all the companion matrices within $\tilde{G}_{4,4,6,5,5,7,7}$ is

$$
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
b & a & x_2 & x_1 & 1 & 0 & 0 \\
x_5 & x_4 & 1 - a & 0 & 0 & 1 & 0 \\
0 & d & c & x_3 & 0 & 0 & 1 \\
x_7 & x_6 & 1 - d & 1 - b - c & 0 & 0 & 0
\end{bmatrix}
$$

A consequence of Theorem 7 is related with an open question stated by Deaett et al. [3]: “We wonder if, in producing a companion matrix by changing some zero entries of a Fiedler companion matrix $F_{i_1, \ldots, i_n}$ by nonzero constants, the extra nonzero entries are always restricted to the submatrix corresponding to the $i_1$-block”. They partially confirmed that supposition.

**Theorem 9.** [3, Theorem 5.4] Let $A$ be a matrix obtained from the Fiedler companion matrix $F_{i_1, \ldots, i_n}$ by changing zero entries that are not in the $i_1$-block. Then $A$ is not companion.

We contribute solving the case in which the change of zero entries in the Fiedler companion matrix is below the superdiagonal. Indeed, this is particular case of Theorem 7. “$A \in \tilde{F}$ is companion if and only if $A \in \hat{F}$ and $s_{k-1}(A) = x_k$ for $k = 1, \ldots, n$”. We write this in the same language as Theorem 9.
Theorem 10. Let $A$ be a matrix obtained from the Fiedler companion matrix $F_{i_1,...,i_n}$ by changing zero entries that are below the superdiagonal. Then $A$ is companion if and only if $A$ is obtained by only changing zero entries that are in the $i_1$-block and $s_{k-1}(A) = x_k$ for $k = 1, \ldots, n$.

It remains unknown if exists a companion matrix which is obtained from some Fiedler companion matrix $F_{i_1,...,i_n}$ by changing at least one zero entry of the $i_1$-block and at least one zero entry above the superdiagonal. We have computational evidence that this is not the case for $n \leq 7$.

3.4 Parameterization of the companions of $\widetilde{C} \setminus \widetilde{G}$

When we have a matrix $C_{i_1,...,i_n} \in C \setminus G$ the parametrization of the set of all the companion matrices within $\widetilde{C}_{i_1,...,i_n}$ is not so easy as with matrices of $G$. We will show it with an specific example. Let

$$C_{7,7,9,7,8,9,10,8,9,10} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_4 & 0 & 0 & x_1 & 1 & 0 & 0 \\
x_8 & 0 & 0 & x_5 & 0 & 0 & 0 & 0 & 1 & 0 \\
x_9 & 0 & 0 & x_6 & 0 & 0 & x_3 & 0 & 0 & 1 \\
x_{10} & 0 & 0 & x_7 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
U_6 & e_6 & 0 \\
X & e_1 & e_3^T \\
U_3
\end{pmatrix}.$$ 

The set $\widetilde{C}_{7,7,9,7,8,9,10,8,9,10}$ of its ULH superpatterns is composed by the matrices

$$C = \begin{pmatrix}
A & e_6 & 0 \\
Y & e_1 & e_3^T \\
B
\end{pmatrix}$$

where $A$ is a ULH superpattern of $U_6$, $B$ is a ULH superpattern of $U_3$, and $Y$ is a superpattern of $X$. By Theorem 4, $C$ is companion if and only if the following conditions are met:

(i) The constant part of $C$ is a nilpotent matrix;

(ii) $A$ belongs to $\mathcal{M}_L^6(i_1 - 1, i_2 - 2, \ldots, i_{10} - 10) = \mathcal{M}_L^6(6, 5, 6, 3, 3, 3, 3, 0, 0, 0) = \mathcal{M}_L^6(3, 5, 6)$;

(iii) $B \in \mathcal{M}_T^3(10 - i_1, 10 - i_2, \ldots, 10 - i_{10}) = \mathcal{M}_T^3(3, 3, 1, 3, 2, 1, 0, 2, 1, 0) = \mathcal{M}_T^3(1, 2, 3)$.

We wish to parameterize the companions matrices of type (3). We divide it in three cases:

1. $A = U_6$ and $B = U_3$. So we consider the companion matrices of type

$$C_1 = \begin{pmatrix}
U_6 & e_6 & 0 \\
Y & e_1 & e_3^T \\
U_3
\end{pmatrix}.$$ 

2. $A$ is a superpattern of $X$. The solution is given in Lemma 3, $C_1$ is a companion matrix if and only if $s_{k-1}(C_1) = x_k$ for each $k = 1, \ldots, n$. So

$$Y = \begin{pmatrix}
h & f & d & x_4 & a & x_2 & x_1 \\
x_8 & i & g & x_5 & b & -a & 0 \\
x_9 & j & -h - i & x_6 & e & c & x_3 \\
x_{10} & 0 & -j & x_7 & -f - g & -d - e & -b - c
\end{pmatrix}$$

where $a, b, \ldots, j \in \mathbb{F}$.
Recall that a matrix

\[ C_2 = \begin{bmatrix} A & e_6 & 0 \\
                      & e^T & 0 \\
                      & X & B \end{bmatrix} \]

where \( A \) is a constant ULH matrix of order 6, and \( B \) is a constant ULH matrix of order 3. By Theorem 4, \( C_2 \) is a companion matrix if and only if \( A \in \mathcal{N}_6^4(3,5,6) \) and \( B \in \mathcal{N}_3^2(1,2,3) \).

By Lemma 4, \( \mathcal{N}_5^2(1,2,3) = \{ U_3 \} \). So \( B = U_3 \).

On the other hand, as any matrix of \( \mathcal{N}_6^4(3,5,6) \) is a ULH matrix then

\[ A = \begin{bmatrix}
    a_{11} & 1 & 0 & 0 & 0 & 0 \\
    a_{21} & a_{22} & 1 & 0 & 0 & 0 \\
    a_{31} & a_{32} & a_{33} & 1 & 0 & 0 \\
    a_{41} & a_{42} & a_{43} & a_{44} & 1 & 0 \\
    a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & 1 \\
    a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\
\end{bmatrix} \]

where \( A[1,2,3], A[1,2,3,4,5] \) and \( A[1,2,3,4,5,6] \) are nilpotent. The ULH submatrices \( A[1] \) and \( A[1,2] \) have no restriction. So, assume that we have assigned arbitrary constant values to \( a_{11}, a_{21}, a_{22} \). As \( A[1,2,3] \) is a nilpotent ULH matrix, then \( a_{31}, a_{32}, a_{33} \) should take values that make the characteristic polynomial of \( A[1,2,3] \) to be \( \lambda^3 \). Corollary that we will state later, implies that these values are unique and can be obtained in a practical way since \( (1, a_{33}, a_{32}, a_{31}) \) are the coordinates of \( \lambda^3 \) with respect to certain basis of \( \mathbb{F}_3[\lambda] \). Let us go to \( A[1,2,3,4] \), for which we have no further restrictions. So, assume that we have assigned arbitrary constant values to \( a_{41}, a_{42}, a_{43}, a_{44} \). As \( A[1,2,3,4,5] \) is a nilpotent ULH matrix, then we repeat the arguments above in order to obtain the adequate values for \( a_{51}, a_{52}, a_{53}, a_{54}, a_{55} \). And, finally, \( A[1,2,3,4,5,6] \) is also a nilpotent ULH matrix which we get by doing \( a_{61} = \cdots = a_{66} = 0 \) since \( A[1,2,3,4,5] \) is nilpotent and has characteristic polynomial \( \lambda^5 \). Moreover, by Corollary this solution is unique. So, doing the corresponding calculations, we obtain

\[ A = \begin{bmatrix}
    a & 1 & 0 & 0 & 0 & 0 \\
    c & b & 1 & 0 & 0 & 0 \\
    -a^3 - 2ac - bc & -a^2 + ab + b^2 - c & -a \cdot b & 1 & 0 & 0 \\
    g & f & d & e & 1 & 0 \\
    a_{51} & a_{52} & ae + be + de - f & -d^2 - e & -d & 1 \\
    0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

with \( a, b, \ldots, g \in \mathbb{F} \)

where \( a_{51} = a^3e + 2ace - ag + bce - cf - df \) and \( a_{52} = a^2e + abe + b^2e - bf + ce - df - g \).

(3) \( Y \neq X \) and \( (A, B) \neq (U_6, U_3) \). It is possible to give a partial parametrization for this case, although we do not have the complete parametrization. In Example 4.5 of [3] the authors gave, for \( C_{3,4,3,4,7,6,7} \), a partial parametrization for the analogous case.

## 4 The PB-companion ULH matrices

Recall that a matrix \( A \) of order \( n \) is said to be \textit{PB-companion} or \textit{companion with respect to a polynomial basis}, as long as: (i) \( A \) has \( n^2 - n \) entries that are constants of a field \( \mathbb{F} \); (ii) the \( n \) remaining entries of \( A \) are the variables \( x_1, \ldots, x_n \); and (iii) the characteristic polynomial of \( A \) is

\[ \det(\lambda I_n - A) = p_n(\lambda) - x_1p_{n-1}(\lambda) - \cdots - x_{n-1}p_1(\lambda) - x_np_0(\lambda) \]
where \( \{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\} \) is a polynomial basis of \( \mathbb{F}_n[\lambda] \).

The issue we are concerned about in this section is the characterization of the PB-companion ULH matrices or, what is the same, the PB-companion matrices within the set \( \tilde{\mathcal{H}} \). We will see that the set of the PB-companion matrices of \( \tilde{\mathcal{H}} \) is greater than \( \tilde{\mathcal{C}} \) and strictly smaller that \( \mathcal{H} \), and that the intermediate set \( \mathcal{D} \) is not useful since as \( \mathcal{D} \setminus \mathcal{C} \) does not contain any PB-companion matrix. So we need a new set between \( \mathcal{H} \) and \( \tilde{\mathcal{C}} \) in which we will locate the PB-companion ULH matrices.

Let \( \mathcal{B} \) be the set composed by those matrices of \( \mathcal{H} \) in which the variables \( x_1, \ldots, x_n \) are on the \( t \)-block for a certain \( t \in \{1, \ldots, n\} \) (there may be more than one such \( t \)). Note that \( \mathcal{B} \cap \mathcal{D} = \mathcal{C} \). Let us describe the matrices of \( \mathcal{B} \) more precisely:

\[
B_{(i_1,j_1),\ldots,(i_n,j_n)} = H_{(i_1,j_1),\ldots,(i_n,j_n)} \quad \text{whenever} \quad 1 \leq j_1, \ldots, j_n \leq i_1, \ldots, i_n \leq n.
\]

The variables \( x_1, \ldots, x_n \) are on the \( t \)-block of \( B_{(i_1,j_1),\ldots,(i_n,j_n)} \) if and only if

\[
\max\{j_1, \ldots, j_n\} \leq t \leq \min\{i_1, \ldots, i_n\}.
\]

As the following examples shows, we have \( \mathcal{H} \supseteq \mathcal{B} \supseteq \mathcal{C} \).

| \( \mathcal{H}_{(1,1),(3,3),(2,1),(5,4),(5,2)} \in \mathcal{H} \setminus \mathcal{B} \) | \( \mathcal{B}_{(3,1),(4,2),(4,1),(5,2),(5,1)} \in \mathcal{B} \setminus \mathcal{C} \) |
|---|---|
| \[
\begin{bmatrix}
x_1 & 0 & 0 & 0 & 0 \\
x_3 & 0 & 1 & 0 & 0 \\
0 & 0 & x_2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & x_5 & 0 & x_4 & 0 \\
\end{bmatrix}
\] | \[
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
x_1 & 0 & 0 & 1 & 0 \\
x_3 & x_2 & 0 & 0 & 1 \\
x_5 & x_4 & 0 & 0 & 0 \\
\end{bmatrix}
\] |

Let us see that \( \tilde{\mathcal{B}} \) is the right place where we should look for PB-companion ULH matrices.

**Theorem 11.** Any PB-companion ULH matrix belongs to \( \tilde{\mathcal{B}} \).

**Proof.** The problem of determining the PB-companion ULH matrices is the same that the problem of determining the PB-companion matrices within \( \mathcal{H} \).

Let \( A \in \tilde{\mathcal{H}}_{(i_1,j_1),\ldots,(i_n,j_n)} \) be a matrix of \( \tilde{\mathcal{H}} \setminus \tilde{\mathcal{B}} \). As \( A \) is not in \( \tilde{\mathcal{B}} \) then

\[
\max\{j_1, \ldots, j_n\} = j_r > i_s = \min\{i_1, \ldots, i_n\}.
\]

for some \( r, s \in \{1, \ldots, n\} \). Therefore \( A[j_s, \ldots, i_s] \) contains \( x_s \), \( A[j_r, \ldots, i_r] \) contains \( x_r \), and both matrices are disjoint. So in \( P_A(\lambda) \) the term \( x_r x_s \lambda^{n-(i_s-j_s+1)-(i_r-j_r+1)} \) appears, and this term can not cancel out. We conclude that \( A \) is not a companion matrix.

In the proof of the previous result we have seen that in the characteristic polynomial of a matrix of \( \mathcal{H} \setminus \tilde{\mathcal{B}} \) inevitably the product of two variables \( x_r x_s \) appear. For matrices of \( \tilde{\mathcal{B}} \) no product of two variables appear, as we will show in the next Lemma. Moreover, the polynomials \( p_{n-k}(\lambda) \) that accompany each variable \( x_k \) are crucial for knowing when a matrix of \( \tilde{\mathcal{B}} \) is PB-companion. We will show some useful properties of those polynomials.

**Lemma 12.** Let \( A \) be a matrix of \( \tilde{\mathcal{B}}_{(i_1,j_1),\ldots,(i_n,j_n)} \). Then the following is true:

\( i \) The characteristic polynomial of \( A \) can be written as

\[
P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda)
\]

where \( p_n(\lambda), p_{n-1}(\lambda), \ldots, p_0(\lambda) \) are monic polynomials of \( \mathbb{F}_n[\lambda] \).

\( ii \) \( \deg(p_n(\lambda)) = n \) and \( \deg(p_{n-k}(\lambda)) = n - i_k + j_k - 1 \) for \( k = 1, \ldots, n. \)
(iii) \( n = \deg(p_n(\lambda)) \geq \deg(p_{n-1}(\lambda)) \geq \ldots \geq \deg(p_0(\lambda)) \).

(iv) If \( A \) is PB-companion then for \( k = 0, 1, \ldots, n \) the degree of \( p_{n-k}(\lambda) \) is at least \( n - k \).

(v) If \( A \) is PB-companion then for \( k = 1, \ldots, n \) the variable \( x_k \) is above the \( k \)-subdiagonal.

Proof. (i) To show that the characteristic polynomial of \( A \) can be written as in (7) we use the same arguments that in the proof of Theorem 3. This leads us to conclude that

\[
P_A(\lambda) = P_{A_0}(\lambda) - \sum_{k=1}^{n} x_k P_{A[1, \ldots, j_k - 1]}(\lambda) P_{A[\lambda t + 1, \ldots, n]}(\lambda),
\]

where \( A_0 \) is the constant part of \( A \). Since \( x_1, \ldots, x_n \) are in some \( t \)-block of \( A \) then for \( k = 1, \ldots, n \) the matrix \( A[1, \ldots, j_k - 1] \) is empty or a submatrix of the constant matrix \( A[1, \ldots, t - 1] \), and the matrix \( A[i_k + 1, \ldots, n] \) is empty or a submatrix of the constant matrix \( A[t + 1, \ldots, n] \). Since the variables \( x_1, \ldots, x_n \) do not appear in either \( P_{A[1, \ldots, j_k - 1]}(\lambda) \) or \( P_{A[i_k + 1, \ldots, n]}(\lambda) \), then we can match (8) and (7) to conclude that \( p_n(\lambda) = P_{A_0}(\lambda) \) and \( p_{n-k}(\lambda) = P_{A[1, \ldots, j_k - 1]}(\lambda) P_{A[i_k + 1, \ldots, n]}(\lambda) \) for \( k = 1, \ldots, n \) are monic polynomials of \( \mathbb{F}_n[\lambda] \).

(ii) \( \deg(p_n(\lambda)) = \deg(P_{A_0}(\lambda)) = n \) and for \( k = 1, \ldots, n \)

\[
\deg(p_{n-k}(\lambda)) = \deg(P_{A[1, \ldots, j_k - 1]}(\lambda)) \deg(P_{A[i_k + 1, \ldots, n]}(\lambda)) = j_k - 1 + n - i_k.
\]

(iii) It follows from (i) by taking into account that the variables \( x_1, \ldots, x_n \) are placed in \( A \) according to the order \( \prec \), which implies that \( i_1 - j_1 \leq i_2 - j_2 \leq \cdots \leq i_n - j_n \).

(iv) If \( A \) is PB-companion then \( \{p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)\} \) is a basis of \( \mathbb{F}_n[\lambda] \). Therefore the inequality \( \deg(p_{n-k}(\lambda)) \geq n - k \) for each \( k = 1, \ldots, n \) follows from (iii).

(v) If \( x_k \) is on the \( h \)-subdiagonal then, by (iii) and (iv), \( \deg(p_{n-k}(\lambda)) = n - h - 1 \geq n - k \). So \( h < k \). \( \square \)

Lemma 12 helps us to find a first great set of PB-companion ULH matrices.

**Theorem 13.** Each matrix of \( \widetilde{C} \) is PB-companion.

**Proof.** If \( A \in \widetilde{C}(i_1, j_1) \ldots (i_n, j_n) \) then \( j_k = i_k - k + 1 \) for \( k = 1, \ldots, n \). From Lemma 12 (ii) it follows that \( P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_n p_0(\lambda) \) with \( \deg(p_n(\lambda)) = n \) and

\[
\deg(p_{n-k}(\lambda)) = n - i_k + j_k - 1 = n - i_k + i_k - k + 1 - 1 = n - k \quad \text{for} \quad k = 1, \ldots, n.
\]

So \( \{p_n(\lambda), \ldots, p_1(\lambda), p_0(\lambda)\} \) is a basis of \( \mathbb{F}_n[\lambda] \) and \( A \) is PB-companion. \( \square \)

### 4.1 A criterion for PB-companion matrices

Lemma 12 permits us to introduce a constant matrix associated to each matrix of \( \widetilde{B} \).

**Definition 14.** Let \( A \) be a matrix of \( \widetilde{B} \) whose characteristic polynomial is

\[
P_A(\lambda) = p_n(\lambda) - x_1 p_{n-1}(\lambda) - \cdots - x_{n-1} p_1(\lambda) - x_n p_0(\lambda).
\]

We will denote by \( \mathfrak{M}_A \) the constant matrix \([p_{ij}]_{i,j=0}^n\) of order \( n + 1 \) such that the entries on its rows are taken from \( p_i(\lambda) = p_{00} + p_{10} \lambda + \cdots + p_{0n} \lambda^n \) for \( i = 0, 1, \ldots, n \).
The main objective of Section 4 is the characterization of the PB-companion ULH matrices. Relying on Theorem 11, we will achieve this objective in the next result with the characterization of the PB-companion matrices of \( \widetilde{B} \).

**Theorem 15.** Any PB-companion ULH matrix belongs to \( \widetilde{B} \). Moreover, if \( A \in \widetilde{B} \) then \( A \) is PB-companion if and only if \( \mathcal{M}_A \) is nonsingular.

**Proof.** The first sentence is Theorem 11. Now, let \( A \) be a matrix of \( \widetilde{B} \). Then \( A \) is PB-companion if and only if \( \{p_0(\lambda), p_1(\lambda), \ldots, p_n(\lambda)\} \) is a basis of \( \mathbb{F}_n[\lambda] \) if and only if \( \mathcal{M}_A \) is nonsingular. \( \square \)

**Example 16.** We wish to determine if it is PB-companion the matrix

\[
A = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & -2 & 1 & 0 & 0 & 0 \\
4 & 0 & 2 & -1 & 1 & 0 & 0 \\
x_5 & 1 & -3 & x_2 & x_1 & 1 & 0 \\
x_7 & x_6 & 2 & 5 & x_3 & 3 & 1 \\
1 & 0 & 4 & x_4 & 1 & 4 & 2
\end{bmatrix} \in \widetilde{B}_{(5,5),(5,4),(6,5),(7,4),(5,1),(6,2),(6,1)}.
\]

The characteristic polynomial of \( A \) is

\[
P_A(\lambda) = p_7(\lambda) - x_1 p_6(\lambda) - \cdots - x_6 p_1(\lambda) - x_7 p_0(\lambda)
\]

where

\[
x_7 \rightarrow p_0(\lambda) = P_{A[7]}(\lambda) = -2 + \lambda; \\
x_6 \rightarrow p_1(\lambda) = P_{A[1]}(\lambda) P_{A[7]}(\lambda) = 4 - 4\lambda + \lambda^2; \\
x_5 \rightarrow p_2(\lambda) = P_{A[6,7]}(\lambda) = 2 - 5\lambda + \lambda^2; \\
x_4 \rightarrow p_3(\lambda) = P_{A[1,2,3]}(\lambda) = 17 - 7\lambda - 3\lambda^2 + \lambda^3; \\
x_3 \rightarrow p_4(\lambda) = P_{A[1,2,3,4]}(\lambda) P_{A[7]}(\lambda) = -2 - 39\lambda + 44\lambda^2 - 8\lambda^3 - 4\lambda^4 + \lambda^5; \\
x_2 \rightarrow p_5(\lambda) = P_{A[1,2,3,4]}(\lambda) P_{A[6,7]}(\lambda) = 34 - 99\lambda + 46\lambda^2 + 10\lambda^3 - 8\lambda^4 + \lambda^5; \\
x_1 \rightarrow p_6(\lambda) = P_{A[1,2,3,4]}(\lambda) P_{A[6,7]}(\lambda) = 2 + 35\lambda - 123\lambda^2 + 76\lambda^3 - 7\lambda^5 + \lambda^6; \\
1 \rightarrow p_7(\lambda) = P_{A_0}(\lambda) = 208 - 317\lambda + 168\lambda^2 - 129\lambda^3 + 73\lambda^4 - 7\lambda^5 + \lambda^7.
\]

The matrix \( \mathcal{M}_A = [p_{ij}]_{i,j=0}^7 \) is

| \( \lambda^0 \) | \( \lambda^1 \) | \( \lambda^2 \) | \( \lambda^3 \) | \( \lambda^4 \) | \( \lambda^5 \) | \( \lambda^6 \) | \( \lambda^7 \) |
|---|---|---|---|---|---|---|---|
| \( p_0(\lambda) \) | -2 | 1 | 0 | 0 | 0 | 0 | 0 |
| \( p_1(\lambda) \) | 4 | -4 | 1 | 0 | 0 | 0 | 0 |
| \( p_2(\lambda) \) | 2 | -5 | 1 | 0 | 0 | 0 | 0 |
| \( p_3(\lambda) \) | 17 | -7 | -3 | 1 | 0 | 0 | 0 |
| \( p_4(\lambda) \) | -2 | -39 | 44 | -8 | -4 | 1 | 0 |
| \( p_5(\lambda) \) | 34 | -99 | 46 | 10 | -8 | 1 | 0 |
| \( p_6(\lambda) \) | 2 | 35 | -123 | 76 | 0 | -7 | 1 |
| \( p_7(\lambda) \) | 208 | -317 | 168 | -129 | 73 | 0 | -7 | 1 |

By Theorem 15 \( A \) is PB-companion since \( \mathcal{M}_A[0,1,2], \mathcal{M}_A[3], \mathcal{M}_A[4,5], \mathcal{M}_A[6], \) and \( \mathcal{M}_A[7] \) (the grey blocks in the diagonal) are nonsingular.

In general, to determine if a given matrix \( A \) of \( \widetilde{B} \) is PB-companion needs some cumbersome calculations to obtain the polynomials that go into the characteristic polynomial of \( A \). Nevertheless, in some cases the calculations are going to simplify so much that we will conclude easily if \( A \) is or it is not PB-companion. This is because we do not need to known all the entries on the matrix \( \mathcal{M}_A \) to conclude that it is nonsingular. We only need to known the submatrices that were marked in gray in the previous example.
Theorem 17. Let $A$ be a matrix of $\tilde{B}$ with $P_A(\lambda) = p_n(\lambda) - x_1p_{n-1}(\lambda) - \cdots - x_np_0(\lambda)$ and let the concatenation associated to $A$ be

$$(0, 1, \ldots, n) = (b_1, \ldots, e_1) \cdots (b_s, \ldots, e_s),$$

where $e_1, \ldots, e_s$ are the $t \in \{0, 1, \ldots, n\}$ such that the degree of $p_t(\lambda)$ is $t$.

Then $A$ is PB-companion if and only if $M_A[b_h, \ldots, e_h]$ is nonsingular for $h = 1, \ldots, s$.

Proof. From Lemma 12 (iii) and (iv), if $A$ is PB-companion then $M_A$ is a lower triangular block matrix with $M_A[b_1, \ldots, e_1], \ldots, M_A[b_s, \ldots, e_s]$ as the blocks on its diagonal. From Theorem 13 $A$ is PB-companion if and only if $M_A$ is nonsingular if and only if $M_A[b_h, \ldots, e_h]$ is nonsingular for each $h = 1, \ldots, s$. □

4.2 Concatenations with components of length at most two

If $A \in \tilde{B}$ then $(0, 1, \ldots, n) = (0) \cdots (n)$ is the concatenation associated to $A$ if and only if $\deg(p_t(\lambda)) = t$ for $t = 0, 1, \ldots, n$ if and only if $A$ belongs to $\tilde{C}$. So, the following is a restatement of Theorem 13 in terms of the length of the components of the concatenation: Each matrix of $\tilde{B}$ whose associated concatenation has only components of length one is PB-companion.

In the next result we characterize the PB-companion matrices of $\tilde{B}$ whose associated concatenation has all its components of length at most two.

Theorem 18. Let $A$ be a matrix of $\tilde{B}$ with

$P_A(\lambda) = p_n(\lambda) - x_1p_{n-1}(\lambda) - \cdots - x_np_0(\lambda).$

We define the concatenation associated to $A$ as

$$(0, 1, \ldots, n) = (b_1, \ldots, e_1) \cdots (b_s, \ldots, e_s)$$

where $e_1, \ldots, e_s$ are the $t \in \{0, 1, \ldots, n\}$ such that the degree of $p_t(\lambda)$ is $t$.

Then $A$ is a PB-companion matrix such that the length of the components of the concatenation associated to $A$ is either one or two if and only if on each $k$-subdiagonal of $A$

I. the only variable is $x_{k+1}$; or

II. the only two variables are $x_{k+1}$ and $x_{k+2}$, and

$$\sum_{r = i - k}^{i} a_{rr} \neq \sum_{r = i' - k}^{i'} a_{rr} \text{ if } x_{k+1} \text{ is the } (i, i - k) \text{ entry}$$

of $A$ and $x_{k+2}$ is the $(i', i' - k)$ entry of $A$; or

III. no variable appears.

Proof. ⇒ Assume that $A$ is a PB-companion matrix of $\tilde{B}$ such that the length of the components of the concatenation associated to $A$ is either one or two.

We will consider all the possibilities for each $k$-subdiagonal of $A$:

1) On the $k$-subdiagonal of $A$ there are no variables. This is case III.

2) On the $k$-subdiagonal the variable with smallest index is $x_h$ with $h \leq k$. Then $x_h$ is on or below the $h$-subdiagonal. A contradiction with Lemma 12 (iv).

3) On the $k$-subdiagonal the variable with smallest index is $x_h$ with $h \geq k+2$. Then neither $x_{k+1}$ is on the $k$-subdiagonal nor $x_{k+2}$ is on the $(k+1)$-subdiagonal. Equivalently, $\deg(p_{n-k-2}(\lambda)) \neq n-k-2$ and $\deg(p_{n-k-1}(\lambda)) \neq n-k-1$. So the concatenation has a component of length at least three. A contradiction with the hypothesis.
4) On the \( k \)-subdiagonal the variable with smallest index is \( x_{k+1} \). We consider three subcases:

i) The only variable on the \( k \)-subdiagonal is \( x_{k+1} \). This is case (I).

ii) The only variables on the \( k \)-subdiagonal are \( x_{k+1} \) and \( x_{k+2} \). Three new possibilities appear:
   a) The variable \( x_{k+3} \) is on the \((k + 1)\)-subdiagonal. The same situation as in (II).
   b) The variable \( x_{k+3} \) is on the \((k + h)\)-subdiagonal with \( h \geq 3 \). The same situation as in (II).
   c) The variable \( x_{k+3} \) is on the \((k + 2)\)-subdiagonal. So we have that

   \[
   \deg(p_{n-k-3}(\lambda)) = n - k - 3, \quad \deg(p_{n-k-2}(\lambda)) \neq n - k - 2, \quad \text{and} \quad \deg(p_{n-k-1}(\lambda)) = n - k - 1.
   \]

   Therefore \( \sim (n - k - 2, n - k - 1) \sim \) is a component of the concatenation of length two.

As \( x_{k+1} \) is on the \( k \)-subdiagonal then \( x_{k+1} \) is the \((i, i - k)\) entry of \( A \) for some \( i \) and

\[
p_{n-k-1}(\lambda) = P_{A[1,...,i-k-1]}(\lambda) P_{A[i+1,...,n]}(\lambda)
= (\lambda^{i-k-1} - \sum_{r=1}^{i-k-1} a_{rr} \lambda^{i-k-2} + \cdots) (\lambda^{n-i} - \sum_{r=i+1}^{n} a_{rr} \lambda^{n-i-1} + \cdots)
= \lambda^{n-k-1} - \sum_{r \in \{1,...,n\} \setminus \{i-k,...,i\}} a_{rr} \lambda^{n-k-2} + \cdots.
\]

Analogously, as \( x_{k+2} \) is on the \( k \)-subdiagonal then \( x_{k+2} \) is the \((i', i' - k)\) entry of \( A \) for some \( i' \) and

\[
p_{n-k-2}(\lambda) = \lambda^{n-k-1} - \sum_{r \in \{1,...,n\} \setminus \{i'-k,...,i'\}} a_{rr} \lambda^{n-k-2} + \cdots.
\]

Therefore

\[
\mathcal{M}_A[n-k-2, n-k-1] = \begin{bmatrix} p_{n-k-2,n-k-2} & p_{n-k-2,n-k-1} \\ p_{n-k-1,n-k-2} & p_{n-k-1,n-k-1} \end{bmatrix} = \begin{bmatrix} -\sum_{r \in \{1,...,n\} \setminus \{i-k,...,i\}} a_{rr} & 1 \\ -\sum_{r' \in \{1,...,n\} \setminus \{i'-k,...,i'\}} a_{rr'} & 1 \end{bmatrix}.
\]

By Theorem (II) \( \mathcal{M}_A[n-k-2, n-k-1] \) is nonsingular. And this is so if and only if

\[
\sum_{r=i-k}^{i} a_{rr} \neq \sum_{r=i'-k}^{i'} a_{rr}. \quad \text{This is case (II)}
\]

iii) On the \( k \)-subdiagonal there are, at least, three variables \( x_{k+1}, x_{k+2} \) and \( x_{k+3} \). In this case neither \( x_{k+2} \) is on the \((k + 1)\)-subdiagonal of \( A \) nor \( x_{k+3} \) is on the \((k + 2)\)-subdiagonal of \( A \). Equivalently, \( \deg(p_{n-k-3}(\lambda)) \neq n - k - 3 \) and \( \deg(p_{n-k-2}(\lambda)) \neq n - k - 2 \). So the concatenation has a component of length at least three. A contradiction with the hypothesis.

\[ \iff \text{Assume that each } k \text{-subdiagonal verifies (I) or (II) or (III). It is useful to note that each variable } x_h \text{ is either on the } (h - 1) \text{ or on the } (h - 2) \text{-subdiagonal of } A. \] (10)

Let us prove that the \( k \)-subdiagonal verifies (I) if and only if the \((k + 1)\)-subdiagonal verifies (III).

For the sufficiency assume that the \( k \)-subdiagonal verifies (I) that is, that on the \( k \)-subdiagonal the only variables are \( x_{k+1} \) and \( x_{k+2} \). Then \( x_{k+2} \) is not on the \((k + 1)\)-subdiagonal which, by discarding non-viable options, implies that the \((k + 1)\)-subdiagonal verifies (III). For the necessity, assume that the \((k + 1)\)-subdiagonal verifies (III) that is, that the \((k + 1)\)-subdiagonal has no variables. Then \( x_{k+2} \) is on the \( k \)-subdiagonal because of (II), what makes the only viable option that \( x_{k+1} \) is on the \( k \)-subdiagonal and that the \( k \)-subdiagonal verifies (I).

By what we have just shown, if we traverse the subdiagonals in order we will sometimes find consecutive subdiagonals that verify (II) and (III) and the rest of subdiagonals must verify (I). The translation into components of the concatenation (II) is that two consecutive subdiagonals that verify (II) and (III) correspond to a component of length two, and a subdiagonal that verifies (I) corresponds to a component of length one. Let us see why this is so:
Example 19. The matrix of $B_{(5,5),(5,4),(6,5),(5,2),(6,3),(6,1)}$ given by

$$A = \begin{bmatrix} a_{11} & 1 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 1 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 1 & 0 \\ a_{51} & x_4 & a_{53} & x_2 & x_1 & 1 \\ x_6 & a_{62} & x_5 & a_{63} & x_3 & a_{66} \end{bmatrix}.$$ 

is, by Theorem 18, PB-companion if and only if $a_{44} \neq a_{66}$ and $a_{22} \neq a_{66}$.

Let us see it in detail. If the characteristic polynomial of $A$ is

$$P_A(\lambda) = p_6(\lambda) - x_1p_5(\lambda) - x_2p_4(\lambda) - x_3p_3(\lambda) - x_4p_2(\lambda) - x_5p_1(\lambda) - x_6p_0(\lambda)$$

then

$$\begin{align*}
x_6 & \to p_0(\lambda) = P_A[0](\lambda) = 1, \\
x_5 & \to p_1(\lambda) = P_A[1,2](\lambda) = \lambda^2 + (-a_{11} - a_{22})\lambda + \cdots; \\
x_4 & \to p_2(\lambda) = P_A[1](\lambda) P_A[6](\lambda) = \lambda^2 + (-a_{11} - a_{66})\lambda + \cdots; \\
x_3 & \to p_3(\lambda) = P_A[1,2,3,4](\lambda) = \lambda^4 + (-a_{11} - a_{22} - a_{33} - a_{44})\lambda^3 + \cdots; \\
x_2 & \to p_4(\lambda) = P_A[1,2,3](\lambda) P_A[6](\lambda) = \lambda^4 + (-a_{11} - a_{22} - a_{33} - a_{66})\lambda^3 + \cdots; \\
x_1 & \to p_5(\lambda) = P_A[1,2,3,4](\lambda) P_A[6](\lambda) = \lambda^5 + \cdots; \\
1 & \to p_6(\lambda) = P_A[0](\lambda) = \lambda^6 + \cdots; \end{align*}$$
And therefore the matrix $\mathcal{M}_A$ of coefficients is

|     | $1$ | $\lambda$ | $\lambda^2$ | $\lambda^3$ | $\lambda^4$ | $\lambda^5$ | $\lambda^6$ |
|-----|-----|-----------|-------------|-------------|-------------|-------------|-------------|
| $p_0(\lambda)$ | 1   | 0         | 0           | 0           | 0           | 0           | 0           |
| $p_1(\lambda)$ | *   | $-a_{11} - a_{22}$ | 1           | 0           | 0           | 0           | 0           |
| $p_2(\lambda)$ | *   | $-a_{11} - a_{66}$ | 1           | 0           | 0           | 0           | 0           |
| $p_3(\lambda)$ | *   | *         | $-a_{11} - a_{22} - a_{33} - a_{44}$ | 1 | 0 | 0 | 0 |
| $p_4(\lambda)$ | *   | *         | *           | $-a_{11} - a_{22} - a_{33} - a_{66}$ | 1 | 0 | 0 | 0 |
| $p_5(\lambda)$ | *   | *         | *           | *           | *           | 1           | 0           |
| $p_6(\lambda)$ | *   | *         | *           | *           | *           | 1           |             |

Note that the concatenation associated to $A$ is

$$(0, 1, 2, 3, 4, 5, 6) = (0) \mathbin{|} (1, 2) \mathbin{|} (3, 4) \mathbin{|} (5) \mathbin{|} (6),$$

where $0, 2, 4, 5, 6$ are all $t \in \{0, 1, 2, 3, 4, 5, 6\}$ such that $\deg(p_t(\lambda)) = t$. So $A$ is PB-companion if and only if $\det(\mathcal{M}_A[1, 2]) = -a_{22} + a_{66} \neq 0$ and $\det(\mathcal{M}_A[3, 4]) = -a_{44} + a_{66} \neq 0$.

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