Industrial-Strength Formally Certified SAT Solving

Ashish Darbari\(^1\) and Bernd Fischer\(^1\) and Joao Marques-Silva\(^2\)

\(^{1}\) School of Electronics and Computer Science  
University of Southampton, Southampton, SO17 1BJ, England  
\(^{2}\) School of Computer Science and Informatics  
University College Dublin, Belfield, Dublin 4, Ireland

Abstract. Boolean Satisfiability (SAT) solvers are now routinely used in the verification of large industrial problems. However, their application in safety-critical domains such as the railways, avionics, and automotive industries requires some form of assurance for the results, as the solvers can (and sometimes do) have bugs. Unfortunately, the complexity of modern, highly optimized SAT solvers renders impractical the development of direct formal proofs of their correctness. This paper presents an alternative approach where an untrusted, industrial-strength, SAT solver is plugged into a trusted, formally certified, SAT proof checker to provide industrial-strength certified SAT solving. The key novelties and characteristics of our approach are (i) that the checker is automatically extracted from the formal development, (ii), that the combined system can be used as a standalone executable program independent of any supporting theorem prover, and (iii) that the checker certifies any SAT solver respecting the agreed format for satisfiability and unsatisfiability claims. The core of the system is a certified checker for unsatisfiability claims that is formally designed and verified in Coq. We present its formal design and outline the correctness proofs. The actual standalone checker is automatically extracted from the the Coq development. An evaluation of the certified checker on a representative set of industrial benchmarks from the SAT Race Competition shows that, albeit it is slower than uncertified SAT checkers, it is significantly faster than certified checkers implemented on top of an interactive theorem prover.

1 Introduction

Advances in Boolean satisfiability SAT technology have made it possible for SAT solvers to be routinely used in the verification of large industrial problems, including safety-critical domains that require a high degree of assurance such as the railways, avionics, and automotive industries [9,15]. However, the use of SAT solvers in such domains requires some form of assurance for the results. This assurance can be provided in two different ways.

First, the solver can be proven correct once and for all. However, this approach had limited success. For example, Lescuyer et al. [11] formally designed and verified a SAT solver using the Coq proof-assistant [2], but without any of the techniques and optimizations used in modern solvers. Reasoning about these optimizations makes the
formal correctness proofs exceedingly hard. This was shown by the work of Marić [12], who verified the algorithm used in the ARGO-SAT solver but restricted the verification to the pseudo-code level, and in particular, did not verify the actual solver itself. In addition, the formal verification has to be repeated for every new SAT solver (or even a new version of a solver), or else the user is locked into using the specific verified solver.

Alternatively, a proof checker can be used to validate each individual outcome of the solver independently; this requires the solver to produce a proof trace that is viewed as a certificate justifying the outcome of the solver. This approach was used to design several checkers such as tts, Booleforce, PicoSAT and zChaff [16]. However, these checkers are typically implemented by the developers of the SAT solvers whose output they are meant to check, which can lead to bugs being masked, and none of them was formally designed or verified, which means that they provide only limited assurance.

The problems of both approaches can be circumvented if the checker rather than the solver is proven correct, once and for all. This is substantially simpler than proving the solver correct, because the checker is comparatively small and straightforward. It does not lead to a system lock-in, because the checker can work for all solvers that can produce proof traces (certificates) in the agreed format. This approach was followed by Weber and Amjad [20] in their formal development of a proof checker for zChaff and Minisat proof traces. Their core idea is to replay the derivation encoded in the proof trace inside LCF-style interactive theorem provers such as HOL 4, Isabelle, and HOL Light. Since the design and implementation of these provers is based on a small trusted kernel of inference rules, assurance is very high. However this comes at the cost of usability: their checker can run only inside the supporting prover, and not as a standalone tool. Moreover, performance bottlenecks become prominent when the size of the problems increases.

Here, we follow the same general idea of a formally certified proof checker, but depart considerably from Weber and Amjad in how we design and implement our solution. We describe an approach where one can plug an untrusted, industrial-strength SAT solver into a formally certified SAT proof checker to provide an industrial-strength certified SAT solver. We designed, formalized and verified the SAT proof checker for both satisfiable and unsatisfiable problems. In this paper, we focus on the more interesting aspect of checking unsatisfiable claims; satisfiable certificates are significantly easier to formally certify. Our certified checker SHRUTI is formally designed and verified using
the higher-order logic based proof assistant Coq [2], but we never use Coq as a checker; instead we automatically extract an OCaml program from the formal development that is compiled to a standalone executable that is used independently of Coq. In this regard, our approach prevents the user to be locked-in to a specific proof assistant, something that was not possible with Weber and Amjad’s approach. A high-level architectural view of our approach is shown in Figure 1. Since it combines certification and ease-of-use, it enables the use of certified checkers as regular components in a SAT-based verification workflow.

2 Propositional Satisfiability

2.1 Satisfiability Solving

Given a propositional formula, the goal of satisfiability solving is to determine whether there is an assignment of the Boolean truth values (i.e., True, False) to the variables in the formula such that the formula evaluates to true. If such an assignment exists, the given formula is said to be \textit{satisfiable} or SAT, otherwise the formula is said to be \textit{unsatisfiable} or UNSAT. Many problems of practical interest in system verification involved proving unsatisfiability, one concrete example being bounded model checking [4].

For efficiency purposes, SAT solvers represent the propositional formulas in conjunctive normal form (CNF), where the entire formula is a conjunction of \textit{clauses}. Each clause itself denotes a disjunction of \textit{literals}, which are simply (Boolean) variables or negated variables. An efficient CNF representation uses non-zero integers to represent literals. A positive literal is represented by a positive integer, whilst a negated one is denoted by a negative integer. As an example, the (unsatisfiable) formula\footnote{The zeroes are delimiters that separate the clauses from each other.}

\[(a \land b) \lor (\neg a \land b) \lor (a \land \neg b) \lor (\neg a \land \neg b)\]

over two propositional variables \(a\) and \(b\) can thus be represented as

\[
\begin{array}{ccc}
1 & 2 & 0 \\
-1 & 2 & 0 \\
1 & -2 & 0 \\
-1 & -2 & 0
\end{array}
\]

SAT solvers take a Boolean formula, for example represented in the DIMACS notation used here, and produce a SAT/UNSAT claim. A \textit{proof-generating} SAT solver produces additional evidence (or \textit{certificates}) to support its claims. For a SAT claim, the certificate simply consists of an assignment. It is usually trivial to check whether that assignment—and thus the original SAT claim—is correct. One simply substitutes the Boolean values given by the assignment in the formula and then evaluates the overall formula, checking that it indeed is true. For UNSAT claims, the evidence is more complicated, and the solvers need to return a resolution \textit{proof trace} as certificate. Unsurprisingly, checking these UNSAT certificates is more complicated as well.
2.2 Proof Checking

When the solver claims a given problem is UNSAT, we can independently re-play the proofs produced by the solver, to check that the solver’s output is correct. For a given problem, if we can follow the resolution inferences given in the proof trace to derive an empty clause, then we know that the proof trace correctly denotes an UNSAT instance for the problem, and we can conclude that the given problem is indeed UNSAT.

A proof trace consists of the original clauses used during resolution and the intermediate resolvents obtained by resolving the original input clauses. The part of the proof trace that specifies how the input clauses have been resolved in sequence to derive a conflict is organized as chains. These chains are often referred to as the regular input resolution proofs, or the trivial proofs [1,3]. We call the input clauses in a chain its antecedents and its final resolvent simply its resolvent. Designing an efficient checking methodology relies to some extent on the representation of the proof trace produced by the solvers. Representing proof chains as a trivial resolution proof is a key constraint [3]. The trivial resolution proof requires that the clauses generated to form the proof should be aligned in such a manner in the trace so that whenever a pair of clauses is used for resolution, at most one complementary pair of literals is deleted. Another important constraint that is needed for efficiency reasons is that at least one pair of complementary literals gets deleted whenever any two clauses are used in the trace for resolution. This is needed because we want to avoid search during checking, and if the proof trace respects these two criteria we can design a checking algorithm that can validate the proofs in a single pass using an algorithm which is linear in the size of the input clauses.

2.3 PicoSAT Proof Representation

Most proof-generating SAT solvers [3,7,23] preserve these two criterions. We carried out our first set of experiments with PicoSAT [3]. PicoSAT ranked as one of the best solvers in the industrial category of the SAT Competitions 2007 and 2009, and in the SAT Race 2008. Moreover, PicoSAT’s proof representation is in ASCII format. It was reported by Weber and Amjad [22] that some other solvers such as Minisat generate proofs in a more compact, but binary format.

Another advantage of PicoSAT’s proof trace format besides using an ASCII format is its simplicity. It does not record the information about pivot literals as some of the other solvers such as zChaff. It is however straightforward to develop translators for formats used by other SAT solvers [19]. As a proof of concept, we developed a translator from zChaff’s proof format to PicoSAT’s proof format, which can be used for validating unsatisfiability proofs obtained with zChaff.

A PicoSAT proof trace consists of rows representing the input clauses, followed by rows encoding the proof chains. Each row representing a chain consists of an asterisk (*) as place-holder for the chain’s resolvent,3 followed by the identifiers of the clauses involved in the chain. Each chain row thus contains at least two clause identifiers, and

---

3 This is generated by PicoSAT; there is another option of generating proof traces from PicoSAT where instead of the asterisk the actual resolvents are generated delimited by a single zero from the rest of the chain.
denotes an application of one or more of the resolution inference rule, describing a trivial resolution derivation. Each row also starts with a non-zero positive integer denoting the identifier for that row’s (input or resolvent) clause. In an actual trace there are additional zeroes as delimiters at the end of each row, but we remove these before we start proof checking. For the UNSAT formula shown in the previous section, the corresponding proof trace generated from PicoSAT looks as follows:

```
1  1  2
2 -1  2
3  1 -2
4 -1 -2
5  *  3  1
6  *  4  2  5
```

The first four rows denote the input clauses from the original problem (see above) that are used in the resolution, with their identifiers referring to the original clause numbering, whereas rows 5 and 6 represent the proof chains. In row 5, the clauses with identifiers 3 and 1 are resolved using a single resolution rule, whilst in row 6 first the original clauses with identifier 4 and 2 are resolved and then the resulting clause is resolved against the clause denoted by identifier 5 (i.e., the resolvent from the previous chain), in total using two resolution steps.

PicoSAT by default creates a compacted form of proof traces, where the antecedents for the derived clauses are not ordered properly within the chain. This means that there are instances in the chain where we resolve a pair of adjacent clauses and no literal is deleted. This violates one of the constraints we explained above, thus we cannot deduce an existence of an empty clause for this trace unless we re-order the antecedents in the chain.

PicoSAT comes with an uncertified proof checker called Tracecheck that can not only check the outcome of PicoSAT but also corrects the mis-alignment of traces. The outcome of the alignment process is an extended proof trace and these then become the input to the certified checker that we design.

3 The SHRUTI Certified Proof Checker

Our approach to efficient formally certified SAT solving relies on the use of a certified checker program that has been designed and implemented formally, but can be used in practice, independently of any formal development environment. Rather than verifying a separately developed checker we follow a correct-by-construction approach in which we formally design and mechanically verify a checker using a proof assistant to achieve the highest level of confidence, and then use program extraction to obtain a standalone executable.

Our checker SHRUTI takes an input CNF file which contains the original problem description and a proof trace file and checks whether the two together denote an (i) UNSAT instance, and (ii) that they are “legitimate”. Our focus here is on (i) where we check that each step of the trace is correctly applying the resolution inference rule. However, in order to gain full assurance about the UNSAT claim, we also need to check that all input clauses used in the resolution in the trace are taken from the original
problem, i.e., that the proof trace is legitimate. SHRUTI provides this as an option, but as far as we are aware most checkers do not do this check.

Our formal development follows the LCF style [17], and in particular, only uses definitional extensions, i.e., new theorems can only be derived by applying previously derived inference rules. Axiomatic extensions though possible are prohibited, since one can assume the existence of a theorem without a proof. Thus, we never use it in our own work. We use the the Coq proof assistant [2] as a development tool. Coq is based on the Calculus of Inductive Constructions and encapsulates the concepts of typed higher-order logic. It uses the notion of proofs as types, and allows constructive proofs and use of dependent types. It has been successfully used in the design and implementation of large scale certification of software such as in the CompCert [10] project.

For our development of SHRUTI, we first formalize in Coq the definitions of resolution and its auxiliary functions and then prove inside Coq that these definitions are correct, i.e., satisfy the properties expected of resolution. Once the Coq formalization is complete, OCaml code is extracted from it through the extraction API included in the Coq proof assistant. The extracted OCaml code expects its input in data structures such as tables and lists. These data structures are built by some glue code that also handles file I/O and pre-processes the read proof traces (e.g., removes the zeroes used as separators for the clauses). The glue code wraps around the extracted checker and the result is then compiled to a native machine code executable that can be run independently of the proof-assistant Coq.

3.1 Formalization in Coq

In this section we present the formalization of SHRUTI. Its core logic is formalized as a shallow embedding in Coq. In a shallow embedding we identify the object data types (types used for SHRUTI) with the types of the meta-language, which in our case happens to be the Coq datatypes.

Since most checkers read the clause representation on integers (e.g., using the DIMACS notation) we incorporate integers as first class elements in our formalization, so we do not have to map literals to Booleans. Thus, inside Coq, we denote literals by integers, and clauses by lists of integers. Antecedents (denoting the input clauses) in a proof chain are represented by integers and a proof chain itself by a list of integers. A resolution proof is represented internally in our implementation by a table consisting of (key, binding) pairs. The key for this table is the identifier obtained from the proof chains read from the input proof trace file. The binding of this table is the actual resolvent obtained by resolving the clauses specified (in the proof trace). When the input proof trace is read, the identifier corresponding to the first row of the proof chain becomes the starting point for resolution checking. Once the resolvent is calculated for this the process is repeated for all the remaining rows of the proof chain, until we reach the end of the trace input. If the identifier for the last row of the proof chain denotes an empty resolvent, we conclude that the given problem and its trace represents an UNSAT instance.

We use the usual notation for quantifiers (∀, ∃) and logical connectives (∧, ∨, ¬) but distinguish implication over propositions (⇒) and over types (→) for presentation clarity, though inside Coq they are exactly the same. The notation ⇒ is used during pattern
matching (using \texttt{match} -- \texttt{with}) as in other functional languages. For type annotation
we use \texttt{;}; and for the cons operation on lists we use \texttt{::}. The empty list is denoted by \texttt{nil}.
The set of integers is denoted by \texttt{Z}, the type of polymorphic list by \texttt{list} and the list
of integers by \texttt{list Z}. List containment is represented by \texttt{∈} and its negation by \texttt{∉}. The
function \texttt{abs} computes the absolute value of an integer. We use the keyword \texttt{Definition}
to present our function definitions. It is akin to defining functions in Coq. Main data
structures that we used in the Coq formalization are lists, and finite maps (hashtables
with integer keys and polymorphic bindings).

We define our resolution function \((\triangledown ◁)\) with the help of two auxiliary functions
\texttt{union} and \texttt{auxunion}. Both functions compute the union of two clauses represented
as integer lists, but differ in their behavior when they encounter complementary literals:
whereas \texttt{union removes} both literals and then calls \texttt{auxunion} to process the remainder
of the lists; \texttt{auxunion copies} both the literals into the output and thus produces a tauto-
logical clause. Ideally, if the SAT solver is sound and the proof trace reflects the sound
outcome, then for any pair of clauses that are resolved, there will be only one pair of
complementary literals and we do not need two functions. However in reality, a solver
or its proof trace can have bugs and it can create instances of clauses in the trace with
multiple complementary pair of literals. Hence, we employ the two auxiliary functions
to ensure that the resolution function deals with this in a sound way.

We will later explain in more detail the functionality of the auxiliary functions but
both functions expect the input clauses to respect three well-formedness criteria: there
should be no duplicates in the clauses (\texttt{NoDup}); there should be no complementary
pair of literals within any clause (\texttt{NoCompPair}), and the clauses should be sorted by
absolute value (\texttt{sorted}). The predicate \texttt{Wf} encapsulates these properties.

\texttt{Definition \ Wf \ c \ = \ NoCompPair \ c \ ∧ \ NoDup \ c \ ∧ \ sorted \ c}

The assumptions that there are no duplicates and no complementary pair of literal
within a clause are essentially the constraints imposed on input clauses when the reso-
lution function is applied in practice. Sorting is enforced by us to keep the complexity
of our algorithm linear.

The union function takes a pair of sorted (by absolute value) lists of integers, and an
accumulator list, and computes the resolvent by doing a pointwise comparison on input
literals.

\texttt{Definition \ union \ (c_1 \ c_2 : list \ Z)(acc : list \ Z) =}
\texttt{match \ c_1, \ c_2 \ with}
\texttt{\ | \ nil, \ c_2 \ ⇒ \ app (rev \ acc) \ c_2}
\texttt{\ | \ c_1, \ nil \ ⇒ \ app (rev \ acc) \ c_1}
\texttt{\ | \ x :: xs, \ y :: ys \ ⇒ \ if \ (x + y = 0) \ then \ auxunion \ xs \ ys \ acc}
\texttt{\ | \ else \ if \ (abs \ x < abs \ y) \ then \ union \ xs \ (y :: ys)(x :: acc)}
\texttt{\ | \ else \ if \ (abs \ y < abs \ x) \ then \ union \ (x :: xs) \ ys \ (y :: acc)}
\texttt{\ | \ else \ union \ xs \ ys \ (x :: acc)\)}

We already pointed out above that this function and the auxiliary union function --
\texttt{auxunion} (shown below) that it employs are different in behaviour if the literals being
compared are complementary. However, when the literals are non-complementary, if
they are equal, only one copy is put in the resolvent whilst when they are unequal both
are kept in the resolvent. Once a single run of any of the clauses is finished, the other clause is merged with the accumulator. Actual sorting in our case is done by simply reversing the accumulator (since all elements are in descending order).

The resolvent is created by matching the two clauses with the accumulator. The accumulator is reversed when the elements are in descending order.

\[
\text{Definition } \text{auxunion}(c_1, c_2 : \text{list } Z)(\text{acc} : \text{list } Z) = \\
\text{match } c_1, c_2 \text{ with} \\
| \text{nil, } c_2 \Rightarrow \text{app (rev acc) } c_2 \\
| c_1, \text{nil} \Rightarrow \text{app (rev acc) } c_1 \\
| x :: xs, y :: ys \Rightarrow \text{if } (\text{abs } x < \text{abs } y) \text{ then auxunion } xs \ (y :: ys) \ (x :: acc) \\
\text{else if } (\text{abs } y < \text{abs } x) \text{ then auxunion } (x :: xs) \ ys \ (y :: acc) \\
\text{else if } x = y \text{ then auxunion } xs \ ys \ (x :: acc) \\
\text{else auxunion } xs \ ys \ (x :: y :: acc)
\]

We can now show the actual resolution function denoted by \(\bowtie\) below. It makes use of the \text{union} function.

\[
\text{Definition } c_1 \bowtie c_2 = (\text{union } c_1 \ c_2 \ \text{nil})
\]

Given a problem representation in CNF form and a proof trace that respects the well-formedness criterion and is a trivial resolution proof for the given problem, SHRUTI will deduce the empty clause and thus validate the solver’s UNSAT claim. Conversely, whenever SHRUTI validates a claim, the problem is indeed UNSAT—SHRUTI will never deduce an empty clause for a SAT instance and will thus never give a false positive.

If SHRUTI cannot deduce the empty clause, it invalidates the claim. This situation can be caused by three different reasons. First, counter to the claim the problem is SAT. Second, the problem may be UNSAT but the resolution proof may not represent this because it may have bugs. Third, the traces either do not respect the well-formedness criteria, or do not represent a trivial resolution proof. In this case both the problem and the proof may represent an UNSAT instance but our checker cannot validate it as such. In this respect our checker is incomplete.

### 3.2 Soundness of the resolution function

Here we formalize the soundness criteria for our checker and present the soundness theorem stating that the definition of our resolution function is sound. We need to prove that the resolvent of a given pair of clauses is logically entailed by the two clauses. Thus at a high-level we need to prove that:

\[
\forall c_1, c_2, c_3 \cdot (c_3 = c_1 \bowtie c_2) \supset \{ c_1, c_2 \} \models c_3
\]

where \(\models\) denotes the logical entailment.

We can use the deduction theorem

\[
\forall a, b, c \cdot \{ a, b \} \models c \iff (a \land b \supset c)
\]

to re-state what we intuitively would like to prove:

\[
\forall c_1, c_2 \cdot (c_1 \land c_2) \supset (c_1 \bowtie c_2)
\]
However instead of proving the above theorem directly we prove its contraposition:

\[ \forall c_1, c_2 \cdot \neg(c_1 \bowtie c_2) \supset \neg(c_1 \land c_2) \]

In our formalization a clause is denoted by a list of non-zero integers, and a conjunction of clauses is denoted by a list of clauses. We now present the definition of the logical disjunction and conjunction functions that operate on the integer list representation of clauses. We do this with the help of an interpretation function that maps an integer to a Boolean value.

The function Eval shown below maps a list of integers to a Boolean value by using the logical disjunction ∨ and an interpretation I of the type \( Z \rightarrow \text{Bool} \).

**Definition**

\[
\text{Eval} \, \text{nil} \, I = \text{False} \\
\text{Eval} \, (x :: xs) \, I = \text{Eval} \, x \, I \lor (\text{Eval} \, xs \, I)
\]

We now define what it means to perform a conjunction over a list of clauses. The function And shown below takes a list of clauses and an interpretation I (with type \( Z \rightarrow \text{Bool} \)) and returns a Boolean which denotes the conjunction of all the clauses in the list.

**Definition**

\[
\text{And} \, \text{nil} \, I = \text{True} \\
\text{And} \, (x :: xs) \, I = (\text{Eval} \, x \, I) \land (\text{And} \, xs \, I)
\]

The interpretations that we are interested in are the logical interpretations which means that if we apply an interpretation I on a negative integer the value returned is the logical negation of the value returned when the same interpretation is applied on the positive integer.

**Definition**

\[
\text{Logical} \, I = \forall (x : Z) \cdot I(-x) = \neg(I \, x)
\]

Thus we can now state the precise statement of the soundness theorem that we proved for our checker as:

**Theorem 1. Soundness theorem**

\[ \forall c_1, c_2 \cdot \forall I \cdot \text{Logical} \, I \supset \neg(\text{Eval} \, (c_1 \bowtie c_2) \, I) \supset \neg(\text{And} \, [c_1, c_2] \, I) \]

**Proof.** The proof begins by structural induction on \( c_1 \) and \( c_2 \). The first three sub-goals are easily proven by term rewriting and simplification by unfolding the definitions of \( \bowtie \), Eval and And. The last sub-goal is proven by doing a case split on if-then-else and then using a combination of induction hypothesis and generating conflict among some of the assumptions. A detailed transcription of the Coq proof is available from http://sites.google.com/site/certifiedsat/.

### 3.3 Correctness of Implementation

Further to ensure that the formalization of our checker is correct we check that the union function is implemented correctly. We check that it preserves the following properties of the trivial resolution function. These properties are:

1. A pair of complementary literals is deleted in the resolvent obtained from resolving a given pair of clauses (Theorem 2).
2. All non-complementary pair of literals that are unequal are retained in the resolvent (Theorem 3).

3. For a given pair of clauses, if there are no duplicate literals within each clause, then for a literal that exists in both the clauses of the pair, only one copy of the literal is retained in the resolvent (Theorem 4).

We have proven these properties in Coq. The actual proof consists of proving several small and big lemmas - in total about 4000 lines of proof script in Coq (see the proofs online).

The general strategy is to use structural induction on clauses \( c_1 \) and \( c_2 \). For each theorem, this results in four main goals, three of which are proven by contradiction since for all elements \( \ell, \ell \notin \text{nil} \). For the remaining goal a case-split is done on if-then-else, thereby producing sub-goals, some of whom are proven from induction hypotheses, and some from conflicting assumptions arising from the case-split.

**Theorem 2.** A pair of complementary literals is deleted.
\[
\forall c_1 c_2 \cdot \text{Wf } c_1 \supset \text{Wf } c_2 \supset \text{UniqueCompPair } c_1 c_2 \supset \\
\forall \ell_1 \ell_2 \cdot (\ell_1 \in c_1) \supset (\ell_2 \in c_2) \supset (\ell_1 + \ell_2 = 0) \supset \\
(\ell_1 \notin (c_1 \bowtie c_2)) \land (\ell_2 \notin (c_1 \bowtie c_2))
\]

Note that to ensure that only a single pair of complementary literals is deleted we need to assume that there is a unique complementary pair (UniqueCompPair). The above theorem will not hold for the case with multiple complementary pairs.

For the following theorem we need to assert in the assumption that for any literal in one clause there exists no literal in the other clause such that the sum of two literals is 0. This is defined by the predicate NoCompLit.

**Theorem 3.** All non-complementary, unequal literals are retained.
\[
\forall c_1 c_2 \cdot \text{Wf } c_1 \supset \text{Wf } c_2 \supset \\
\forall \ell_1 \ell_2 \cdot (\ell_1 \in c_1) \supset (\ell_2 \in c_2) \supset \\
(\text{NoCompLit } \ell_1 c_2) \supset (\text{NoCompLit } \ell_2 c_1) \supset \\
(\ell_1 \neq \ell_2) \supset (\ell_1 \in (c_1 \bowtie c_2)) \land (\ell_2 \in (c_1 \bowtie c_2))
\]

**Theorem 4.** Only one copy of equal literals is retained (factoring).
\[
\forall c_1 c_2 \cdot \text{Wf } c_1 \supset \text{Wf } c_2 \supset \\
\forall \ell_1 \ell_2 \cdot (\ell_1 \in c_1) \supset (\ell_2 \in c_2) \supset (\ell_1 = \ell_2) \supset \\
((\ell_1 \in (c_1 \bowtie c_2)) \land (\text{count } \ell_1 (c_1 \bowtie c_2) = 1))
\]

In order to check the resolution steps for each row, one has to collect the actual clauses corresponding to their identifiers and this is done by the findClause function. The function findClause takes a list of clause identifiers (dlst), an accumulator (acc) to collect the list of clauses, and requires as input a table that has the information about all the input clauses (ctbl). If a clause id is processed, then its resolvent is fetched from the resolvent table (rtbl), else obtained from ctbl. If there is no entry for a given id in the resolvent table and in the clause table, an error is signalled. This error denotes the fact that there was an input/output problem with the proof trace file due to which some input clauses in the proof trace could not be accessed properly. This could have happened either because the proof trace was ill-formed accidentally or wilfully tampered with.
The function that uses the ⊲ function recursively on a list of input clause chain is called \textit{chainResolution} and it simply folds the ⊲ function from left to right for every row in the proof part of the proof trace file.

\begin{verbatim}
Definition chainResolution lst =
    match (lst : list (list Z)) with
    | nil ⇒ nil
    | (x :: xs) ⇒ List.fold_left (⊳) xs x
\end{verbatim}

The function \textit{findAndResolve} is our last function defined in Coq world for Unsat checking and provides a wrapper on other functions. Once the input clause file and proof trace files are opened and read into different tables, \textit{findAndResolve} starts the checking process by first obtaining the clause ids from the proof part of the proof trace file, and then invoking \textit{findClause} to collect all the clauses for each row in the proof part of the proof trace file. Once all the clauses are obtained the function \textit{chainResolution} is called and applied on the list of clauses row by row. For each row the resolvent is stored in a separate table. The checker then simply checks if the last row has an empty clause, and if there is one, it agrees with the sat solver and says yes, the problem is UNSAT, else no.

If the proof trace part of the trace file contains nothing (ill-formed) then there would be no entry for an identifier in the trace table (ttbl), and this is signalled by an error state consisting of a list with a single zero. Since zeros are otherwise prohibited to be a legal part of the CNF problem description, we use them to signal error states. Similarly, if the clause and trace table both are empty, then a list with two zeros is output as an error.

\subsection{3.4 Program Extraction}

We extract the OCaml code by using the built-in extraction API in Coq. At the time of extraction we mapped several Coq datatypes and data structures to equivalent OCaml ones. For optimization we made the following replacements:

1. Coq Booleans by OCaml Booleans.
2. Coq integers (Z) by OCaml int.
3. Coq lists by OCaml lists.
4. Coq finite map by OCaml’s finite map.
5. The combination of \textit{app} and \textit{rev} on lists in the function \textit{union}, and \textit{auxunion} was replaced by the tail-recursive List.rev_append in OCaml.

Replacing Coq Zs with OCaml integers gave a performance boost by a factor of 7-10. Making minor adjustments by replacing the Coq finite maps by OCaml ones and using tail recursive functions gave a further 20% improvement. An important consequence of our extraction is that only some datatypes, and data structures get mapped to OCaml’s; the key logical functionality is unmodified. The decisions for making changes in data types and data structures are a standard procedure in any extraction process using Coq [2].
Table 1: Comparison of our results with HOL 4 and Tracecheck. Number of Resolutions (inferences) shown for HOL 4 is the number that HOL 4 calculated from the proof trace obtained from running ZVerify - the uncertified checker (for zChaff) that Amjad used to obtain the proof trace. SHRUTI Resolution count is obtained from the proof trace generated by the uncertified checker Tracecheck. In terms of inferences/second, we are 1.5 to 32 times faster than Amjad’s HOL 4 checker, whilst a factor 2.5 slower than Tracecheck. All times shown are total times for all the three checkers. The symbol z? denotes that zChaff timed out after an hour.

| No. | Benchmark                  | HOL 4 Resolutions | HOL 4 Time | inf/sec | SHRUTI Resolutions | SHRUTI Time | inf/s | Tracecheck Resolutions | Tracecheck Time | inf/s |
|-----|---------------------------|-------------------|------------|---------|-------------------|------------|-------|------------------------|----------------|-------|
| 1   | een-tip-uns-numsv-t5.B   | 89136             | 4.61       | 19335   | 122816            | 0.86       | 142809| 341155                 |                 |       |
| 2   | seen-pico-prop01-75       | 205807            | 5.70       | 36106   | 246430            | 1.67       | 147562| 4813395                |                 |       |
| 3   | seen-pico-prop05-50       | 180493            | 58.41      | 30901   | 2804173           | 20.76      | 135075| 8.11 345767            |                 |       |
| 4   | hoons-vbmc-lucky7         | 3460518           | 24.76      | 58013   | 4359478           | 35.18      | 123919| 12.95 336639           |                 |       |
| 5   | ibm-2002-26r-k45         | 1448              | 24.76      | 58      | 1105              | 11.05      | 540447| 2.59 27625             |                 |       |
| 6   | ibm-2004-26-k25          | 1020              | 11.78      | 86      | 1132              | 8.6        | 27625 | 0.04 281300            |                 |       |
| 7   | ibm-2004-26-k25          | 1589429           | 24.17      | 65760   | 1093933           | 7.42       | 327982| 3.02 334236           |                 |       |
| 8   | ibm-2002-11-k60          | z?                | z?         | z?      | 1398258           | 133.05     | 235912| 59.27 235912           |                 |       |
| 9   | ibm-2004-26-k25          | 82890             | 2.12       | 39099   | 245222           | 1.59       | 490444| 0.50 490444           |                 |       |
| 10  | ibm-2004-26-k25          | 700084            | 26.79      | 26132   | 265931           | 1.81       | 492464| 0.54 492464           |                 |       |
| 11  | ibm-2004-26-k25          | 36682             | 11.23      | 3266    | 395897           | 2.60       | 482801| 0.82 482801           |                 |       |
| 12  | ibm-2004-26-k25          | z?                | z?         | z?      | 458042           | 3.06       | 381701| 1.21 381701           |                 |       |
| 13  | ibm-2004-26-k25          | 325509            | 8.82       | 36905   | 788790           | 5.40       | 398378| 1.98 398378           |                 |       |
| 14  | manol-pipe-g6bi          | 198446            | 3.15       | 62998   | 863749           | 6.29       | 345499| 2.50 345499           |                 |       |
| 15  | manol-pipe-c9midw_s      | 104401            | 5.07       | 20591   | 1058871          | 7.89       | 356522| 2.97 356522           |                 |       |
| 16  | manol-pipe-c10midw_s     | 806583            | 13.76      | 58617   | 4666001          | 38.03      | 300257| 5.54 300257           |                 |       |
| 17  | manol-pipe-c10midw_s     | 824716            | 14.31      | 57632   | 4901713          | 42.31      | 272317| 18.27 272317           |                 |       |
| 18  | manol-pipe-g10bidw       | 735665            | 23.21      | 33416   | 6009262          | 50.82      | 289035| 21.08 289035           |                 |       |
| 19  | manol-pipe-g10bidw       | 2719959           | 52.90      | 51416   | 7827637          | 64.69      | 291532| 26.85 291532           |                 |       |
| 20  | manol-pipe-g7idw         | 956072            | 35.17      | 27184   | 7665865          | 68.14      | 249377| 30.74 249377           |                 |       |
| 21  | manol-pipe-g7idw         | 4107275           | 125.82     | 32644   | 14776611        | 134.92     | 216888| 68.13 216888           |                 |       |

4 Experimental Results

We evaluated our certified checker SHRUTI on a set of benchmarks from the SAT Races of 2006 and 2008 and the SAT Competition of 2007. We present our results on a sample of the SAT Race Benchmarks in Table 1. The results for SHRUTI shown in the table are for validating proof traces obtained from the PicoSAT solver. Our experiments were carried out on a server running Red Hat on a dual-core 3 GHz, Intel Xeon CPU with 28GB memory.

The HOL 4 and Isabelle based checkers [22] were evaluated on the SAT Race Benchmarks shown in the table [21]. Isabelle reported segmentation faults on most
of the problems, whilst HOL 4’s results are summarized along with our’s in Table 1. HOL 4 was run on an AMD dual-core 3.2 GHz processor running Ubuntu with 4GB of memory. We also compare our timings with that obtained from the uncertified checker Tracecheck. Since the size of the proof traces obtained from zChaff is substantially different than the size of the traces obtained from Tracecheck on most problems, we decided to compare the speed of our checker with HOL 4 and Tracecheck in terms of resolutions (inferences) solved per second. We observe that in terms of inferences/sec we are 1.5 to 32 times faster than HOL 4 and 2.5 times slower than Tracecheck. Times shown for all the three checkers in the table are the total times including time spent on actual resolution checking, file I/O and garbage collection. Amjad reported that the version of the checker he has used on these benchmarks is much faster than the one published in [22].

As a proof of concept we also validated the proof traces from zChaff by translating them to PicoSAT’s trace format. The performance of SHRUTI in terms of in/sec on the translated proof traces (from zChaff to PicoSAT) was similar to the performance of SHRUTI when it checked PicoSAT’s traces obtained directly from the PicoSAT solver – something that is to be expected.

4.1 Discussion

The Coq formalization consisted of 8 main function definitions amounting to nearly 160 lines of code, and 4 main theorems shown in the paper and 4 more that are about maps (not shown here due to space). Overall the proof in Coq was nearly 4000 lines consisting of the proofs of several big and small lemmas that were essential to prove the 4 main theorems. The extracted OCaml code was approximately 2446 lines, and the OCaml glue code was 324 lines.

We found that there is no implementation of the array data type which meant that we had to use the type of list. Since lists are defined inductively, it is easier to do reasoning with them, although implementing very fast and efficient functions on these is impossible. In a recent development related to Coq, there has been an emergence of a tool called Ynot [14] that can deal with arrays, pointers and file related I/O in a Hoare Type Theory. Future work in certification using Coq should definitely investigate the relevance and use of this.

We noticed that the OCaml compiler’s native code compilation does produce efficient binaries but the default settings for automatic garbage collection were not useful. We observed that if we do not tune the runtime environment settings of OCaml by setting the values of OCAMLRUNPARAM, as soon as the input proof traces had more than a million inferences, garbage collection would kick in so severely that it will end up consuming (and thereby delaying the overall computation) as much as 60% of the total time. By setting the initial size of major heap to a large value such as 2 GB and making the garbage collection less eager, we noticed that the computation times of our checker got reduced by up to a factor of 7 on proof traces with over 1 million inferences.

5 Related Work

Recent work on checking the result of SAT solvers can be traced to the work of Zhang & Malik [23] and Goldberg & Novikov [8], with additional insights provided in recent
work [1,19]. Besides Weber and Amjad, others who have advocated the use of a checker include Bulwahn et al [5] who experimented with the idea of doing reflective theorem proving in Isabelle and suggested that it can be used for designing a SAT checker.

In a recent paper [12], Marić presented a formalization of SAT solving algorithms in Isabelle that are used in modern day SAT solvers. An important difference is that whereas we have formalized a SAT checker and \textit{extracted} an executable code from the formalization itself, Marić formalizes a SAT solver (at the abstract level of state machines) and then implements the verified algorithm in the SAT solver \textit{off-line}.

An alternative line of work involves the formal development of SAT solvers. Examples include the work of Smith & Westfold [18] and the work of Lescuyer and Conchon [11]. Lescuyer and Conchon have formalized a simplified SAT solver in Coq and extracted an executable. However, the performance results have not been reported on any industrial benchmarks. This is because they have not formalized several of the key techniques used in modern SAT solvers. The work of of Smith & Westfold involves the formal synthesis of a SAT solver from a high level description. Albeit ambitious, the preliminary version of the SAT solver does not include the most effective techniques used in modern SAT solvers.

There has been a recent surge in the area of certifying SMT solvers. M. Moskal recently provided an efficient certification technique for SMT solvers [13] using term-rewriting systems. The soundness of the proof checker is guaranteed through a formalization using inference rules provided in a term-rewriting formalism. L. de Moura and N. Bjørner [6] presented the proof and model generating features of the state-of-the-art SMT solver Z3.

6 Conclusion

In this paper we presented a methodology for performing efficient yet formally certified SAT solving. The key feature of our approach is that we did a one-off formal design and reasoning of the checker using Coq proof-assistant and extracted an OCaml program which was used as a standalone executable to check the outcome of industrial-strength SAT solvers such as PicoSAT and zChaff. Our certified checker can be plugged in with any proof generating SAT solver with previously agreed certificates for satisfiable and unsatisfiable problems. On one hand our checker provides much higher assurance as compared to uncertified checkers such as Tracecheck and on the other it enhances usability and performance when compared to the certified checkers implemented in HOL 4 and Isabelle. In this regard our approach provides an arguably optimal middle ground between the two extremes. We are investigating on optimizing the performance aspects of our checker even further so that the slight difference in overall performance between uncertified checkers and us can be further minimized.

Acknowledgements. We thank H. Herbelin, Y. Bertot, P. Letouzey, and many more people on the Coq mailing list who helped us with Coq questions. We also thank T. Weber and H. Amjad for answering our questions on their work and also carrying out industrial benchmark evaluation on their checker. A. P. Landells helped out with server issues. This work was partially funded by EPSRC Grant EP/E012973/1, and EU Grants ICT/217069 and IST/033709.
References

1. P. Beame, H. A. Kautz, and A. Sabharwal. Towards understanding and harnessing the potential of clause learning. *J. Artif. Intell. Res. (JAIR)*, 22:319–351, 2004.
2. Y. Bertot and P. Castéran. Interactive theorem proving and program development. Coq’Art: The calculus of inductive constructions, 2004.
3. A. Biere. PicoSAT essentials. *Journal on Satisfiability, Boolean Modeling and Computation*, 4:75–97, 2008.
4. A. Biere, A. Cimatti, E. Clarke, O. Strichman, and Y. Zhu. *Advances in Computers*, chapter Bounded Model Checking. Academic Press, 2003.
5. L. Bulwahn, A. Krauss, F. Haftmann, L. Erkök, and J. Matthews. Imperative functional programming with Isabelle/HOL. In *Theorem Proving in Higher Order Logics*, pages 134–149, 2008.
6. L. M. de Moura and N. Bjørner. Proofs and refutations, and Z3. In *Proceedings of the LPAR 2008 Workshops*, 2008.
7. N. Een and N. Sorensson. An extensible sat-solver. In E. Giunchiglia and A. Tacchella, editors, *SAT*, volume 2919 of *Lecture Notes in Computer Science*, pages 502–518. Springer, 2003.
8. E. I. Goldberg and Y. Novikov. Verification of proofs of unsatisfiability for CNF formulas. In *Design, Automation and Test in Europe Conference*, pages 10886–10891, March 2003.
9. J. Hammarberg and S. Nadjm-Tehrani. Formal verification of fault tolerance in safety-critical reconfigurable modules. *International Journal on Software Tools for Technology Transfer (STTT)*, 7(3):268–279, 2005.
10. X. Leroy and S. Blazy. Formal Verification of a C-like Memory Model and its uses for Verifying Program Transformations. *Journal of Automated Reasoning*, Jan 2008.
11. S. Lescuyer and S. Conchon. A reflexive formalization of a SAT solver in Coq. In *Emerging Trends of TPHOLs*, 2008.
12. F. Marić. Formalization and implementation of modern SAT solvers. *Journal of Automated Reasoning*, 43(1):81–119, June 2009.
13. M. Moskal. Rocket-fast proof checking for SMT solvers. In *Tools and Algorithms for the Construction and Analysis of Systems*, pages 486–500, 2008.
14. A. Nanevski, G. Morrisett, A. Shinnar, P. Govereau, and L. Birkedal. Ynot: dependent types for imperative programs. In *International Conference on Functional Programming*, pages 229–240, 2008.
15. M. Penicka. Formal approach to railway applications. In *Formal Methods and Hybrid Real-Time Systems*, pages 504–520, 2007.
16. SAT Competition. [http://www.satcompetition.org/](http://www.satcompetition.org/).
17. D. S. Scott. A type-theoretical alternative to ISWIM, CUCH, OWHY. *Theor. Comput. Sci.*, 121(1-2):411–440, 1993.
18. D. R. Smith and S. J. Westfold. Synthesis of propositional satisfiability solvers. Technical report, Kestrel Institute, April 2008.
19. A. Van Gelder. Verifying propositional unsatisfiability: Pitfalls to avoid. In *Theory and Applications of Satisfiability Testing*, pages 328–333, May 2007.
20. T. Weber. Efficiently checking propositional resolution proofs in Isabelle/HOL. *6th International Workshop on the Implementation of Logics*, 2009.
21. T. Weber and H. Amjad. Private communication.
22. T. Weber and H. Amjad. Efficiently checking propositional refutations in HOL theorem provers. *Journal of Applied Logic*, 7(1):26–40, 2009.
23. L. Zhang and S. Malik. Validating SAT solvers using an independent resolution-based checker: Practical implementations and other applications. In *Design, Automation and Test in Europe Conference*, pages 10880–10885, March 2003.