Symmetry relations for multifractal spectra at random critical points

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Abstract. Random critical points are generically characterized by multifractal properties. In the field of Anderson localization, Mirlin et al (2006 Phys. Rev. Lett. 97 046803) have proposed that the singularity spectrum $f(\alpha)$ of eigenfunctions satisfies the exact symmetry $f(2d - \alpha) = f(\alpha) + d - \alpha$. In the present paper, we analyze the physical origin of this symmetry in relation to the Gallavotti–Cohen fluctuation relations of large deviation functions that are well known in the field of non-equilibrium dynamics: the multifractal spectrum of the disordered model corresponds to the large deviation function of the rescaling exponent $\gamma = (\alpha - d)$ along a renormalization trajectory in the effective time $t = \ln L$. We conclude that the symmetry discovered for the specific example of Anderson transitions should actually be satisfied at many other random critical points after an appropriate translation. For many-body random phase transitions, where the critical properties are usually analyzed in terms of the multifractal spectrum $H(a)$ and of the moment exponents $X(N)$ of the two-point correlation function (Ludwig 1990 Nucl. Phys. B 330 639), the symmetry becomes $H(2X(1) - a) = H(a) + a - X(1)$, or equivalently $\Delta(N) = \Delta(1 - N)$ for the anomalous parts $\Delta(N) \equiv X(N) - NX(1)$. We present numerical tests favoring this symmetry for the 2D random $Q$-state Potts model with varying $Q$. 

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1. Introduction

Among the various areas where multifractality appears (see for instance [1]–[7] and references therein), the case of critical points in the presence of frozen disorder is of particular interest. The idea that multifractality occurs at criticality was first established for quantum Anderson localization transitions [8,9] and has been the subject of very detailed studies (see the reviews [10,11]). However multifractality is not limited to such one-particle quantum transitions, and is expected to hold at many random critical points whenever disorder is relevant, in particular in disordered spin models like random ferromagnets [12]–[19], spin glasses and random field spin systems [20]–[22], as well as in disordered polymer models like directed polymers in random media [23] and disordered wetting models [24]. The only exceptions to these multifractal behaviors seem to be the ‘multiscaling’ behaviors [25], which are even stronger than multifractality, that have been
found for some critical correlation functions in disordered quantum spin chains governed by ‘infinite disorder fixed points’ [26]. So the presence of multifractality at criticality seems generic whenever disorder is relevant.

For Anderson localization models, Mirlin et al [27] have proposed that the singularity spectrum \( f(\alpha) \) of critical eigenfunctions satisfies a remarkable exact symmetry (see more details in section 2 below). In this paper, we analyze the physical origin of this symmetry in relation to the well-known symmetry relations of large deviation functions in non-equilibrium dynamical models. We conclude that this symmetry is not specific to Anderson transitions but should be satisfied at many other random critical points, in particular at many-body disordered phase transitions after an appropriate translation that we describe. As an example of an application, we consider the two-dimensional random Potts model with various values of \( Q \) and present numerical tests of the symmetry as regards the multifractal spectrum of the two-point order parameter correlation functions.

The paper is organized as follows. In section 2, we recall the meaning of the symmetry relation for Anderson localization models. In section 3, we analyze the similarity with the well-known fluctuation relations for large deviation functions occurring in the field of non-equilibrium dynamics. This is the central section of the present paper. In section 4, we translate the symmetry relation for \( f(\alpha) \) to one in terms of the multifractal spectrum \( H(\alpha) \) that characterizes the statistics of two-point correlation functions at many-body random critical points. In section 5, we present numerical tests favoring this symmetry for the two-dimensional random \( Q \)-state Potts model with various values of \( Q \). Section 6 contains our conclusions.

2. Symmetry relations at Anderson transitions

2.1. A reminder of the singularity spectrum \( f(\alpha) \) of critical eigenfunctions

For Anderson localization transitions, the multifractal spectrum \( f(\alpha) \) of critical eigenfunctions is defined as follows (for more details see the reviews [10,11]): in a sample of size \( L^d \), the number \( N_L(\alpha) \) of points \( r \) where the weight \( |\psi_L(r)|^2 \) scales as \( L^{-\alpha} \) behaves as

\[
N_L(\alpha) \propto L^{f(\alpha)}. \tag{1}
\]

The inverse participation ratios, which are the most convenient order parameters of the transition, can be then rewritten as an integral over \( \alpha \):

\[
Y_q(L) \equiv \int_{L^d} d^d r |\psi_L(r)|^{2q} \simeq \int d\alpha L^{f(\alpha)} L^{-q\alpha} \simeq L^{-\tau(q)} \tag{2}
\]

where the exponent \( \tau(q) \) can be obtained via a saddle-point calculation in \( \alpha \) to obtain the Legendre transform formula [10,11]

\[
-\tau(q) = \max_\alpha [f(\alpha) - q\alpha]. \tag{3}
\]

The usual normalization condition of eigenfunctions

\[
\int_{L^d} d^d r |\psi_L(r)|^2 = 1 \tag{4}
\]
implies \( Y_{q=1}(L) = 1 \), so \( \tau(q = 1) = 0 \). The weight \(|\psi_L(r)|^2\) has thus the disorder-averaged value

\[
\frac{1}{L^d} \left| \psi_L(r) \right|^2 = 1
\]

(5)

whereas the typical exponent \( \alpha_{\text{typ}} \) governing the typical decay

\[
\ln \left| \psi_L(r) \right|^2 \propto -\alpha_{\text{typ}} \ln L
\]

(6)

is determined by the maximum \( f(\alpha_{\text{typ}}) = d \) of the spectrum \( f(\alpha) \). These scaling behaviors, which concern individual eigenstates \( \psi \), can be translated to ones for the local density of states

\[
\rho_L(E, r) = \sum_n \delta(E - E_n) |\psi_{E_n}(r)|^2
\]

(7)

as follows: for large \( L \), when the \( L^d \) energy levels become dense, the sum of equation (7) scales as

\[
\rho_L(E, r) \propto L^d |\psi_E(r)|^2
\]

(8)

and its moments involve the exponents \( \tau(q) \) introduced in equation (2):

\[
\left[ \rho_L(E, r) \right]^q \propto \frac{1}{L^{\Delta(q)}} \quad \text{with } \Delta(q) = \tau(q) - d(q - 1).
\]

(9)

2.2. The symmetry relation of \( f(\alpha) \)

For any Anderson transition in the so-called ‘conventional symmetry classes’ [11], Mirlin, Fyodorov, Mildenberger and Evers [27] have proposed that the singularity spectrum \( f(\alpha) \) of the critical eigenfunctions satisfies the remarkable exact symmetry

\[
f(2d - \alpha) = f(\alpha) + d - \alpha
\]

(10)

that relates the regions \( \alpha \leq d \) and \( \alpha \geq d \) (see for instance figure 1 of [32] for a pictorial representation of this symmetry). In terms of the exponents \( \tau(q) \) of equation (2), the symmetry of equation (10) with respect to the value \( \alpha_s = d \) becomes a symmetry with respect to the value \( q_s = 1/2 \) [27]:

\[
\tau(q) - \tau(1 - q) = d(2q - 1).
\]

(11)

In terms of the exponents \( \Delta(q) \) of equation (9), the symmetry takes the simpler form

\[
\Delta(q) = \Delta(1 - q).
\]

(12)

As discussed in [27], this type of symmetry was first derived for non-linear sigma models [28], which are \textit{a priori} only valid for weak disorder. Nevertheless, Mirlin, Fyodorov, Mildenberger and Evers have argued in [27] that equation (10) should remain exact even at strong disorder as a consequence of the universality of critical properties. Equation (10) has been checked for the exact expansion in \( d + \epsilon \) up to four-loop order [29], and numerically for various kinds of Anderson transitions, in particular in the power-law random banded matrices [27], in the symplectic 2D Anderson transition [30], in the quantum Hall transition [31], and in the 3D Anderson transition [32] (the only exception

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being, to our knowledge, the spin quantum Hall transition [33] which belongs to the Bogoliubov–de Gennes symmetry class C of the symmetry classification [11]). This symmetry has even been measured in recent experiments [34].

In conclusion, the symmetry of equation (10) seems very well satisfied at most of the Anderson transitions where it has been studied. However its physical origin has remained unclear. In particular, an important issue is whether this symmetry is specific to critical theories of Anderson transitions, or whether it could be satisfied at other random critical points. These questions have been the motivations of the present work.

3. The physical origin of the symmetry relation

The multifractal formalism can be seen as a theory of large deviations in the parameter $\ln L$ (see [35] for a recent review on large deviations in statistical physics). It is thus interesting to discuss in this section the similarity that exists between the Mirlin–Fyodorov–Mildenberger–Evers symmetry of Anderson transitions described in the previous section, and the well-known symmetry relations of large deviation functions that occur in the field of non-equilibrium dynamics.

3.1. Fluctuation relations of large deviation functions in non-equilibrium dynamical models

It is clearly impossible to summarize here all the recent developments as regards the various ‘fluctuation relations’ that have been established in the field of non-equilibrium dynamics recently, and we refer the interested reader to some recent reviews [36]–[42] and references therein. Here we will simply recall some basic definitions that will be useful for our present purposes. The observables $Y_t$ which are expected to become extensive in time in the large time limit usually satisfy some large deviation principle: the probability of having a given time-averaged value $Y_t / t = j$ behaves at large $t$ as

$$\text{Prob} \left( \frac{Y_t}{t} = j \right) \propto t^{-\infty} e^{tG(j)}$$

where $G(j) \leq 0$ is called the large deviation function. The typical value $j_{\text{typ}}$ corresponds to the point where it vanishes, $G(j_{\text{typ}}) = 0$. It has been found that in many cases, the large deviation function $G(j)$ satisfies some symmetry relation of the form

$$G(j) = G(-j) + Kj$$

where the constant $K$ may contain the physical parameters for the problem at hand (like the reservoir fugacities, the temperature, the external applied field, etc). Equivalently, the generating function of the variable $Y_t$ behaves for large time $t$ as

$$\langle e^{\lambda Y_t} \rangle = \int d\lambda e^{t(\lambda j + G(j))} \propto t^{-\infty} e^{t\mu(\lambda)}$$

where $\mu(\lambda)$ is the Legendre transform of $G(j)$:

$$\mu(\lambda) = \max_j [\lambda j + G(j)].$$
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Its series expansion in $\lambda$

$$\mu(\lambda) \equiv \lim_{t \to +\infty} \frac{\ln \langle e^{\lambda Y_t} \rangle}{t} = \lambda j_{\text{typ}} + \frac{\lambda^2}{2} \sigma + o(\lambda^2)$$  \hspace{1cm} (17)

yields the successive cumulants of $Y_t$:

$$j_{\text{typ}} = \lim_{t \to +\infty} \frac{\langle Y_t \rangle}{t}$$

$$\sigma = \lim_{t \to +\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t}.$$  \hspace{1cm} (18)

The symmetry of equation (14) reads for the generating function $\mu(\lambda)$

$$\mu(\lambda) = \mu(-K - \lambda)$$  \hspace{1cm} (19)

The symmetry of equation (14) or equation (19) is usually called a Gallavotti–Cohen fluctuation relation. The full domain of validity of this type of symmetry in the field of non-equilibrium dynamics is not easy to state, since it has been derived in various contexts with different assumptions on the dynamics, which can be either stochastic or deterministic but sufficiently chaotic (see the various presentations in the reviews [36]–[42]). Let us recall here the most important ideas for the simplest case: for a stochastic dynamics defined by some Markov chain with transition probabilities $k(C \to C')$ between configurations

$$P_{t+1}(C) = \sum_{C'} P_t(C') k(C' \to C)$$  \hspace{1cm} (20)

the symmetry relation of large deviation has usually for its origin some ‘generalized detailed balance relation’ of the form

$$k_y(C \to C') = k_{-y}(C' \to C) e^{Ky}$$  \hspace{1cm} (21)

where $y$ denotes the increase of the dynamical quantity $Y_t$ for the jump $C \to C'$, and $(-y)$ denotes the increase of the dynamical quantity $Y_t$ for the jump $C' \to C$. Equation (21) implies that a given dynamical trajectory $\text{Traj} = \{C_0, C_1, \ldots, C_t\}$ characterized by the value $Y_t(\text{Traj}) = y_1 + y_2 + \cdots + y_t$, and the reverse trajectory $(-\text{Traj}) = \{C_t, C_{t-1}, \cdots, C_0\}$ characterized by the opposite value $Y_t(-\text{Traj}) = -Y_t(\text{Traj})$ have probabilities related by the simple relation

$$\frac{P(\text{Traj})}{P(-\text{Traj})} = \frac{k_{y_1}(C_0 \to C_1) k_{y_2}(C_1 \to C_2) \cdots k_{y_t}(C_{t-1} \to C_t)}{k_{-y_1}(C_t \to C_{t-1}) \cdots k_{-y_2}(C_2 \to C_1) k_{-y_t}(C_1 \to C_0)} = e^{K Y_t(\text{Traj})}.$$  \hspace{1cm} (22)

The generating function of equation (15) reads

$$\langle e^{\lambda Y_t} \rangle \equiv \sum_{\text{Traj}} P(\text{Traj}) e^{\lambda Y_t(\text{Traj})} = \sum_{\text{Traj}} P(-\text{Traj}) e^{(K+\lambda)Y_t(\text{Traj})}.$$  \hspace{1cm} (23)

Using the one-to-one change of variable $\text{Traj}' = -\text{Traj}$ and the antisymmetry relation $Y_t(-\text{Traj}) = -Y_t(\text{Traj})$, one obtains

$$\langle e^{\lambda Y_t} \rangle = \sum_{\text{Traj}'} P(\text{Traj}') e^{-(K+\lambda)Y_t(\text{Traj}')} = \langle e^{-(K+\lambda)Y_t} \rangle$$  \hspace{1cm} (24)
which corresponds to the symmetry relation of equation (19) for the generating function introduced in equation (15).

To derive a symmetry relation for a large deviation function of some dynamical observable \( Y_t \), it is sufficient to justify some ‘generalized detailed balance relation’ of the form of equation (21) for the elementary transition probabilities. We refer the reader to the reviews [36]–[42] for the description of various physical situations where this can be done (a system in contact with two reservoirs at different temperatures, a system in an external applied field, etc). However, as explained in [43], for any stochastic process there is always a quantity \( B_t \) for which it works by construction: it is the quantity \( B_t \) whose increment \( b(C, C') \) is defined by equation (21) with \( K = 1 \), i.e. it corresponds to the logarithm of the ratio of the two transition rates [43]:

\[
b(C, C') = \ln \left( \frac{k(C \rightarrow C')}{k(C' \rightarrow C)} \right) .
\]

The functional \( B_t \) of the trajectory \( \text{Traj} = \{C_0, C_1, \ldots, C_t\} \) for which the fluctuation relation holds by construction then reads [43]

\[
B_t(\text{Traj} = \{C_0, C_1, \ldots, C_t\}) = \sum_{m=1}^{t} \ln \left( \frac{k(C_{m-1} \rightarrow C_m)}{k(C_m \rightarrow C_{m-1})} \right) .
\]

We stop this brief reminder on large deviation functions occurring in non-equilibrium dynamics here, and refer the interested reader to [43] and to the reviews [36]–[42] for the physical interpretation in terms of irreversibility and entropy production. We now analyze the analogy with multifractal spectra at random critical points.

### 3.2. The dictionary of the analogy

To make the formal analogy between equation (14) and equation (10) complete, it is convenient to replace the singularity exponent \( \alpha \) by its distance to \( d \) which represents the ‘homogeneous’ value under rescaling:

\[
\gamma \equiv \alpha - d.
\]

Then the negative function

\[
g(\gamma) \equiv f(\alpha = d + \gamma) - d
\]

satisfies (equation (10))

\[
g(\gamma) = g(-\gamma) + \gamma.
\]

This equation is now exactly similar to equation (14) with \( K = 1 \). The relation between the generating function \( \mu(\lambda) \) of equation (16) and the exponents \( \Delta(q) \) of equation (9) reads

\[
\mu(\lambda) = \max_{\gamma} [\lambda \gamma + g(\gamma)]
\]

\[
= \max_{\alpha} [\lambda(\alpha - d) + f(\alpha) - d] = -\tau(-\lambda) - d\lambda - d
\]

\[
= -\Delta(-\lambda)
\]

so the symmetry relation of equation (19) with \( K = 1 \) is equivalent to equation (12).

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Since the physical situations of a non-equilibrium dynamical system and of a static disordered system at criticality are a priori completely different, this analogy may seem completely formal at first sight. It is however very suggestive: the large time $t$ of the dynamical system corresponds to the logarithm of the large linear scale $L$ for the random critical system

$$t = \ln L.$$  

(31)

The ‘effective dynamics’ for the random system thus corresponds to the renormalization process on the scale $\ln L$.

3.3. The stability of the multifractal spectrum upon renormalization

In critical phenomena, it is well known that critical properties are stable under coarse-graining. This explains their universal character (independence with respect to microscopic details) and why renormalization is an appropriate framework. Similarly for random critical points, the multifractal spectrum is expected to be stable under coarse-graining, i.e. the formulations that we have written above in terms of the microscopic observables (like the eigenfunction weight $\psi^2(r)$ in Anderson localization models) can be reformulated in terms of coarse-grained observables as follows [10]: if the system of volume $L^d$ is decomposed into boxes of volume $L_b^d$ one introduces the weight $w_L^{L_b}(r)$ of the box of volume $L_b^d$ around the point $r$:

$$w_L^{L_b}(r) \equiv \int_{L_b^d} d^d r' \psi_L^2(r + r')$$  

(32)

that generalizes the microscopic weight $\psi_L^2(r)$ corresponding to $L_b = 1$. Then all previous multifractal notions apply to the box weights if one replaces $L$ by the ratio $L/L_b$ of the two length scales. Equation (1) becomes

$$\text{Prob} \left[ w_L^{L_b}(r) \sim \left( \frac{L_b}{L} \right)^{-\alpha} \right] \sim \left( \frac{L}{L_b} \right)^{f(\alpha)-d}$$  

(33)

and equivalently, their moments behave as

$$\left[ w_L^{L_b}(r) \right]^q \sim \int d\alpha \left( \frac{L}{L_b} \right)^{f(\alpha)-q\alpha} \sim \left( \frac{L}{L_b} \right)^{-d-\tau(q)}.$$  

(34)

This formulation with coarse-grained boxes is also sometimes used numerically, in particular to measure correctly the scaling behaviors of negative moments $q < 0$ [27,32].

3.4. Analysis of the renormalization process

To go from the microscopic scale $l = 1$ to the macroscopic scale $l = L$ of the whole system, we may introduce intermediate scales $l_m$ regularly placed on a logarithmic scale. For definiteness, let us consider a discrete system with $L = 2^M$ and introduce the intermediate scales

$$l_m = 2^m \quad \text{with} \quad m = 0, 1, \ldots, M.$$  

(35)
We are interested in the flow of microscopic weight \( \psi^2(r) \) upon renormalization. Up to now, we had always assumed the usual normalization of equation (4) for the full sample of volume \( L^d \), but here to analyze the coarse-graining process using samples of various sizes, it is more convenient to work with fields \( \psi^2(r) \) free of any global normalization constraint.

To characterize the microscopic weight \( \psi^2(r) \) within the box of volume \( l_m^d \) surrounding the point \( r \), we introduce the variables
\[
W_m \equiv \frac{l_m^d \psi^2(r)}{\int_{l_m^d} d^d r' \psi^2(r + r')}.
\]
(36)

If \( \psi^2(r + r') \) were constant within the box of volume \( l_m^d \), one would have \( W_m = 1 \). The lowest scale \( l_0 = 1 \) is characterized by \( W_0 = \psi^2(r)/\psi^2(r) = 1 \). We are interested in the renormalization process
\[
W_0 = 1 \rightarrow W_1 \rightarrow W_2 \rightarrow \cdots \rightarrow W_M \equiv \frac{L^d \psi^2(r)}{\int_{L^d} d^d r' \psi^2(r + r')}.
\]
(37)

Since the box of size \( l_m^d \) contains the box of size \( l_{m-1}^d \), it is convenient to introduce the positive exponent \( \hat{\alpha}_m \geq 0 \):
\[
2^{\hat{\alpha}_m} \equiv \frac{\int_{l_m^d} d^d r' \psi^2(r + r')}{\int_{l_{m-1}^d} d^d r' \psi^2(r + r')}
\]
(38)

to characterize the coarse-graining step from \( l_{m-1}^d \) to \( l_m^d \). The ratio of two successive rescaled variables \( W_m \) of equation (36) then reads
\[
\frac{W_m}{W_{m-1}} = 2^d \frac{\int_{l_{m-1}^d} d^d r' \psi^2(r + r')}{\int_{l_m^d} d^d r' \psi^2(r + r')} = 2^{-\hat{\gamma}_m}
\]
(39)
in terms of the exponent
\[
\hat{\gamma}_m \equiv \hat{\alpha}_m - d.
\]
(40)

The variable \( W_m \) can be then written as
\[
W_m = 2^{-\Gamma_m}
\]
(41)
in terms of the accumulated value
\[
\Gamma_m \equiv \sum_{n=1}^{m} \hat{\gamma}_n
\]
(42)
along the RG trajectory. The multifractal definition for the weight
\[
\text{Prob} \left( \frac{\psi^2(r)}{\int_{L^d} d^d r' \psi^2(r + r')} \sim \frac{1}{L^{d+\gamma}} \right) \xrightarrow{L \to +\infty} L^{g(\gamma)}
\]
(43)
in terms of the function \( g(\gamma) \) introduced in equation (28) corresponds to the following large deviation behavior for the variable \( \Gamma_M \) as \( M = \ln L/\ln 2 \) becomes large:
\[
\text{Prob} \left( \frac{\Gamma_M}{M} = \gamma \right) \xrightarrow{M \to +\infty} 2^{Mg(\gamma)}.
\]
(44)

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Equivalently, the definition of the exponents $\Delta(q)$ from the weight moments at the Anderson transition

$$\left( \frac{L^d \psi^2(r)}{\int L^d \, d^d \psi^2(r + r')} \right) \bigg|_{L \rightarrow +\infty} \propto L^{-\Delta(q)}$$

(45)
corresponds, in the large deviation theory in the variable $\Gamma_M$, to the following statement for its generating function:

$$\langle 2^{-\gamma M} \rangle \propto \begin{cases} 2^{-M \Delta(q)} & M \rightarrow +\infty \end{cases}$$

(46)

The symmetry property of equation (29) for $g(\gamma)$ means that the probability of obtaining a value $\Gamma_M = M\gamma$ and the probability of obtaining the opposite value $\Gamma_M = -M\gamma$ are related via

$$\frac{\text{Prob}(\Gamma_M = \gamma M)}{\text{Prob}(\Gamma_M = -\gamma M)} \propto \frac{2^M [g(\gamma) - g(-\gamma)]}{2^M} = 2^M = 2^\Gamma_M.$$  

(47)

We now come to our central assumption. It seems physically natural to consider that the renormalization process of equation (37) represents some Markov chain, i.e. one expects that the probability of seeing $W_m$ at scale $l_m$ will depend on $W_n$ with $n < m - 1$ that describe the finer structure at smaller scales inside the box of volume $l_{m-1}^d$. The analogy with the analysis of non-equilibrium stochastic dynamics recalled in section 3.1 then suggests that the physical origin of equation (47) is that the increment $\gamma$ of the functional $\Gamma_M$ considered exactly measures the ratio of two opposite transition probabilities as in equation (25), i.e. with our present notation,

$$2^\gamma = \frac{k_\gamma(W \rightarrow W')}{{k_{-\gamma}}(W' \rightarrow W)}.$$  

(48)

And indeed, using equation (39), if one writes an elementary transition probability as

$$k_\gamma(W \rightarrow W') = \delta(W' - 2^{-\gamma}W)$$  

(49)
one obtains, using the properties of the $\delta$-function, that equation (48) is satisfied:

$$k_\gamma(W \rightarrow W') = 2^\gamma \delta(W - 2^\gamma W') = 2^{\gamma} k_{-\gamma}(W' \rightarrow W).$$  

(50)

This interpretation suggests that the symmetry relation of equation (47) has an origin which is completely independent of the physical problem under consideration, so it should be valid not only for Anderson transitions, but also for other random critical points. We should stress that for each random critical point, the large deviation function $g(\gamma)$ is of course non-trivial and depends on the critical model, but the symmetry of equation (29) seems generic and model independent.

The physical meaning of equation (48) is that the variable $\gamma$ characterizes the irreversibility of the RG flow. In a pure homogeneous system, the stability of the fixed point corresponds to $\gamma = 0$, so the forward RG and the backward RG are equivalent. In the presence of quenched disorder however, the exponent $\gamma$ becomes distributed, and its typical value $\gamma_{\text{typ}}$ corresponding to $g(\gamma_{\text{typ}}) = 0$ is strictly positive:

$$\gamma_{\text{typ}} > 0.$$  

(51)

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This introduces some asymmetry between the forward RG flow and the backward RG flow. The strong multifractality limit where $\gamma_{\text{typ}}$ is far from zero corresponds to the far-from-equilibrium limit for dynamical systems, whereas the weak multifractality limit where $\gamma_{\text{typ}}$ is near zero corresponds to the close-to-equilibrium limit for dynamical systems. Of course for a given random critical point, the disorder strength which has already been tuned to reach criticality cannot be tuned again to reach the regime of weak multifractality, in contrast with the case for dynamical systems where one may always imagine applying a weak external field to remain close to equilibrium. However, for a given type of random transition, there usually exists a parameter that allows one to interpolate between weak multifractality and strong multifractality. For the usual Anderson transitions for instance, it is the space dimension $d$ that leads to weak multifractality for $d = 2 + \epsilon$ and to strong multifractality for large $d$ [11]. Similarly for the two-dimensional random Potts models that will be considered in section 5, it is the number $Q$ of possible states of an individual spin that allows one to interpolate weak multifractality for $Q = 2 + \epsilon$ and strong multifractality for large $Q$.

4. The symmetry relation for many-body random critical points

In the previous section, we have interpreted the symmetry of the multifractal spectrum discovered for the specific example of Anderson transitions as a Gallavotti–Cohen symmetry relation for a renormalization procedure. This suggests that this symmetry should be satisfied at other random critical points after an appropriate translation. In the remainder of this paper, we thus consider the case of many-body random critical points, like disordered spin models, where the multifractal properties are usually defined in terms of the statistics of the two-point correlation.

4.1. The multifractal spectrum $H(\alpha)$ of the two-point correlation function

Following the notation of Ludwig [12], let us call $\phi(r)$ the local order parameter, and $X(N)$ the scaling dimensions of its disorder-averaged moments $\phi^N(r)$. Then the two-point correlation function

$$C(r, r + R) \equiv \langle \phi(r)\phi(r + R) \rangle$$

will present the following scaling behaviors for its disorder-averaged moments:

$$C^N(r, r + R) = \int dC C^N P_R(C) \propto \frac{1}{R^{2X(N)}}.$$  \hspace{1cm} (53)

Equivalently, the probability distribution $P_R(C)$ is described by the multifractal spectrum $H(\alpha) \geq 0$ such that

$$dCP_R\left( C \sim \frac{1}{R^{2\alpha}} \right) \propto R^{-2H(\alpha)} \, da.$$  \hspace{1cm} (54)

The saddle-point calculation of the integral in equation (53) yields that $X(N)$ is the Legendre transform of $H(\alpha)$:

$$-2X(N) = \max_\alpha [-2Na - 2H(\alpha)].$$  \hspace{1cm} (55)

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The minimum $a_{\text{typ}}$ of $H(a)$ where $H(a_{\text{typ}}) = 0$ governs the typical correlation
\[
\ln C(r, r + R) = -2a_{\text{typ}} \ln R
\] (56)
whereas the averaged correlation is governed by $X(1)$
\[
\frac{C(r, r + R)}{R^{2X(1)}} \xrightarrow{R \to +\infty} 1
\] (57)

4.2. Translation of the symmetry of $f(\alpha)$ into a symmetry relation for $H(a)$

To translate the symmetry relation that holds for the singularity spectrum $f(\alpha)$ in Anderson localization models to the many-body transitions, we should first rephrase the statements concerning two-point functions within an infinite system into statements concerning one-point functions within a finite size system. In a system of size $L^d$, the local order parameter $\phi_L(r)$ is characterized by the scaling dimensions $X(N)$ of equation (53) that govern the moments
\[
\phi_L^N(r) \xrightarrow{L \to +\infty} \frac{1}{L^{X(N)}}
\] (58)
or by the multifractal spectrum $H(a)$ of equation (54):
\[
d\phi P_R \left( \phi \sim \frac{1}{L^a} \right) \xrightarrow{L \to +\infty} L^{-H(a)} da.
\] (59)
Now to make the link with the Anderson localization framework or more generally with the usual multifractal formalism, which is defined in terms of a normalized probability measure [1], it is convenient to construct a probability measure from the non-normalized observables that one is interested in [24, 44, 45]. We thus introduce the normalized local order parameter
\[
w_L(r) \equiv \frac{\phi_L(r)}{\int_{L^d} d^d r' \phi_L(r')}
\] (60)
where the denominator scaling is governed by the exponent $X(1)$ of equation (58) as a consequence of the equivalence between spatial averaging and disorder averaging at the scaling level:
\[
\int_{L^d} d^d r' \phi_L(r) \sim L^{d-X(1)}.
\] (61)
The $f(\alpha)$ analogous to the Anderson localization definition of equation (1) is that the probability of having $w_L(r) \sim 1/L^a$ behave as
\[
dw P_R \left( w_L(r) \sim \frac{1}{L^a} \right) \sim L^{f(\alpha)-d} \, da.
\] (62)
If $w_L(r) \sim 1/L^a$, the one-point function decays as $\phi_L(r) \sim 1/L^a$ with (see equations (60) and (61))
\[
a = \alpha - d + X(1)
\] (63)

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and thus the relation between the multifractal spectra \( f(\alpha) \) and \( H(a) \) reads (equations (62) and (59))

\[
f(\alpha) - d = -H(a = \alpha - d + X(1)).
\]

(64)

Equivalently, the relation between their respective Legendre transforms \( \tau(q) \) and \( X(N) \) reads

\[
\tau(N) = X(N) - NX(1) + d(N - 1),
\]

(65)
i.e. the exponents \( \Delta(N) \) introduced in equation (9)

\[
\Delta(N) = \tau(N) - d(N - 1) = X(N) - NX(1)
\]

(66)
directly measure the non-linearity of \( X(N) \).

The symmetry relation of equation (10) translates for \( H(a) \) into

\[
H(2X(1) - a) = H(a) + a - X(1),
\]

(67)
i.e. a symmetry with respect to \( a_s = X_1 \). In terms of the Legendre transform \( X(N) \), the symmetry becomes

\[
X(N) - X(1 - N) = (2N - 1)X(1)
\]

(68)
which is analogous to equation (11). Finally, in terms of \( \Delta(N) \) of equation (66), the symmetry is given by equation (12) again:

\[
\Delta(N) = \Delta(1 - N).
\]

(69)

5. Tests of the symmetry for the 2D random Potts model

The random many-body transition for which multifractal properties have been the most studied is the disordered two-dimensional ferromagnetic Potts model \([12]–[19]\): it is a lattice spin model defined by the Hamiltonian

\[
-\beta H = \sum_{(i,j)} J_{ij} \delta_{\sigma_i,\sigma_j}
\]

(70)
where the spin variables can take \( Q \) different values \( \sigma_i \in \{0, \ldots, Q - 1\} \). The sum extends over nearest neighbors on the lattice. The exchange couplings \( J_{ij} \) are quenched variables. In the following, we have considered two different kinds of disorder realizations: (i) self-dual random bond realizations, i.e. two different nonzero ‘ferromagnetic’ values of the nearest neighbor interaction distributed with the same probability \( \frac{1}{2} \), and (ii) dilute bonds, i.e. nonzero values distributed over a fraction of the links only. The two kinds of disorder realizations belong to the same universality class. The phase transition is of second order for any value of \( Q \) in the disordered case (whereas in the pure case, the transition is of second order for \( Q \leq 4 \) and of first order for \( Q > 4 \)).

The moments of the spin–spin correlation function are expected to decay algebraically:

\[
\langle \delta_{\sigma_i,\sigma_j} \rangle^N \sim |\vec{r}_i - \vec{r}_j|^{-2X(N)}
\]

(71)
where we keep the prescription that \( \cdots \) denotes the average over disorder while \( \langle \cdots \rangle \) is the thermal average. The exponents \( X(N) \) are the exponents that have been introduced in the previous section in equation (53) in the general case.

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5.1. Numerical tests of the symmetry of the multifractal spectrum for $2 < Q \leq 8$

We now present numerical data obtained using the connectivity transfer matrix introduced by Blöte and Nightingale [46]. In this formulation, even non-integer values of $Q$ can be considered because the number $Q$ of states enters only as a parameter. Furthermore large strips can be considered because the dimension of the transfer matrix is smaller in the connectivity space than in the spin space where it grows as $Q^L \times Q^L$ for strip width $L$. Finally the spin–spin correlation functions can be expressed in terms of connectivity and thus be computed very efficiently. In the following, the data have been averaged over 80 000 disorder realizations. Further details of the numerical procedure can be found in [16].

The system consists in an infinitely long strip of finite width $L$ with periodic boundary conditions in the transverse direction, i.e. it is an asymptotically one-dimensional one. Therefore the correlations decay exponentially rather than algebraically at the critical point and, as a consequence of conformal invariance, the average correlation functions and its moments involve the critical exponents $X(N)$ in a very simple manner:

$$C^N(u, u + R) \equiv \int dC C^N P_R(C) \sim e^{-(2\pi/L)X(N)R}. \quad (72)$$

In the strip geometry, the system thus exhibits an even closer relation to the dynamical model of section 3.1, with the role of time $t$ now played by the coordinate $u$ along the strip. In figure 1, we have plotted $\Delta(N)$ obtained from the exponents $X(N)$ using the definition of equation (66). The original data are plotted as filled black symbols while the exponents $\Delta(N)$ after the transformation $1 - N \rightarrow N$ are plotted as open red symbols. Within numerical accuracy, the symbols fall on top of each other as expected from equation (69).

The spectral function $H(a)$ follows from equation (72) for the spin–spin correlation function on the strip:

$$H(a) = X(N) - aN. \quad (73)$$

We present in figure 2 the multifractal spectrum $H(a)$ for the self-dual random bond Potts model. This spectrum has been obtained using equation (73) and the exponents $X(N)$ computed by interpolation of the moments of the spin–spin correlation function with equation (72). On top of the multifractal spectrum $H(a)$, full symbols in the figure, we have plotted using open symbols the image of $H(a)$ under the transformations $H(a) + a - X(1) \rightarrow H(a)$ and $2X(1) - a \rightarrow a$. If the symmetry holds for the random bond Potts model, the two curves should fall on top of each other according to equation (67). We have emphasized with black and red colors in the figure the region where the two curves superimpose. The numerical data are in very good agreement with equation (67) for $Q$ from 2.25 to 3.

We now present another test of the symmetry using an alternative determination of the multifractal spectrum $H(a)$. The definition (72), written in terms of the variable $Y = -\ln C$, reads as a Laplace transform:

$$\int dY P_R(Y)e^{-NY} \sim e^{-(2\pi/L)X(N)R}. \quad (74)$$
The inverse Laplace transform evaluated close to the saddle-point approximation leads to (see [17])

\[ P_R(a) \sim \left( \frac{2\pi R}{L} \right)^{1/2} e^{-(2\pi R/L)H(a)} \]  

(75)

with \( a = -\ln C/(2\pi R/L) \). The spectral function is thus directly given in terms of the probability distribution \( P_R(-\ln C/(2\pi R/L)) \). Using again the connectivity transfer matrix, we have computed the probability distribution of the spin–spin correlation function for the diluted Potts model [17]. The multifractal spectrum was then obtained by interpolation of the numerical data with equation (75). In figure 3, the multifractal spectrum is plotted in black for three different values of the number of states \( Q \). In red, we have plotted the data after the transformations \( H(a) + a - X(1) \rightarrow H(a) \) and \( 2X(1) - a \rightarrow a \). Again, the data fall nicely on top of each other for small and intermediate values of \( a \). For large values of \( a \), the two curves significantly deviate from each other, but...
Figure 2. Numerical test of the spectrum symmetry as calculated from the set of exponents $X(N)$ obtained from the decay of the moments $\langle \sigma_i \sigma_{i+u} \rangle^N$ for the self-dual random bond Potts model for $Q \leq 3$. The filled circles correspond to $H(a)$ versus $a$, while the open squares correspond to $H(a) + a - X(1)$ versus $2X(1) - a$. The parts of the curves which superimpose (black and red) are emphasized.

at the same time, the data dispersion which measures the uncertainty also increases. According to (73), this region, where the multifractal spectrum increases with $a$, is associated with negative moments $N$ of the spin–spin correlation function. The latter are dominated by rare events corresponding to weak spin–spin correlation and thus strongly diluted realizations of disorder. Numerically, one can hope to sample correctly events whose probability is of order $1/N_{\text{Dis}}$, where $N_{\text{Dis}}$ is the number of disorder realizations considered. Very rare events are unlikely to be generated. We may thus interpret the deviation observed in figure 3 as due to an underestimate of the tail of the probability distribution $P(C) \lesssim 10^{-5}$.

5.2. A reminder of existing perturbative expansions in $(Q - 2)$

The Harris criterion indicates that disorder is relevant at the transition for $Q > 2$ but only marginal for $Q = 2$: the point $Q = 2$ can thus be used as a starting point for a
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Figure 3. Numerical test of the spectrum symmetry as calculated from the probability distribution of the correlation function of the dilute bond Potts model for \( Q \geq 3 \). The full lines (black) correspond to \( H(a) \) versus \( a \), while the filled circles (red) correspond to \( H(a) + a - X(1) \) versus \( 2X(1) - a \).

The exponents \( X(N) \) governing the correlation functions have been computed at first order in \( y_H \) by Ludwig [12]:

\[
X(N) = NX(1) - \frac{N(N-1)}{16} y_H + O(y_H^2). \tag{77}
\]

The corresponding Gaussian multifractal spectrum \( H(\alpha) \) can be found from Ludwig [12], but the symmetry is much more easily tested using the form of equation (69). According to the definition of equation (66), the expansion of equation (77) leads to

\[
\Delta(N) = X(N) - NX(1) = -\frac{N(N-1)}{16} y_H + O(y_H^2) \tag{78}
\]

which is manifestly invariant under the transformation \( N \rightarrow 1 - N \).

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5.2.2. The second-order contribution. The perturbative expansion of Ludwig has been extended to the second order in a replica symmetric scenario by Lewis [13]:

\[
X(N) = NX(1) - \frac{N(N-1)}{16} \left[ y_H + \left( \frac{11}{12} - 4 \ln 2 + \frac{N - 2}{24} \left( 33 - \frac{29\pi}{\sqrt{3}} \right) \right) y_H^2 \right] + \mathcal{O}(y_H^3).
\]  

(79)

The presence of the \((N-2)\) term explicitly breaks the symmetry \(N \rightarrow 1 - N\). As a comparison, the equivalent weak multifractality perturbative expansion for Anderson localization can be found in [29] up to order \(\epsilon^4\) in \(\epsilon = d - 2\) and satisfies the multifractal symmetry.

If the result of equation (79) is really true, this means that the multifractal symmetry is not realized in the random Potts model. However, we believe that one should be cautious as regards the result of equation (79), in spite of previous numerical confirmations. It is a perturbative replica result that has been derived within the so-called replica symmetric scenario. Indeed in some disordered systems, the replica symmetry is known to be spontaneously broken, so one should in principle study the various fixed points corresponding to various replica symmetry properties and then decide which one corresponds to the physical fixed point. For the random Potts model, two different kinds of fixed points have been compared in [47]: the replica symmetric fixed point, used to derive equation (79) and the broken replica symmetry fixed point based on the Parisi block diagonal matrix ansatz. We refer the reader to [48] for a detailed discussion of the possible physical meanings of replica symmetry breaking at the critical point of random ferromagnets. According to [47], the differences in critical exponents between these two fixed points appear only at second order \(y_H^2\), so the first-order contribution of equation (77), although derived within the replica symmetric fixed point, seems to be of larger validity in the space of possible replica fixed points, whereas the second-order contribution of equation (79) seems specific to the replica symmetric fixed point. Since it has not been possible up to now to determine on theoretical grounds which fixed point describes the scaling limit of the disordered lattice models of equation (70), very precise numerical computations [16,49] have been necessary to discriminate between the two fixed points studied in [47]: these numerical studies [16,49] have concluded (i) that the replica symmetric broken fixed point based on the Parisi block diagonal matrix ansatz is excluded, and (ii) that the replica symmetric fixed point is thus the physical one. However we believe that (i) does not directly imply (ii), because only two particular fixed points have been studied. Since the Parisi block diagonal matrix ansatz comes from spin glasses in the mean-field limit, whose physics is completely different from that of two-dimensional random ferromagnets, one may argue that another type of replica symmetry breaking could be realized in critical random ferromagnets. For instance, it has been proposed in [50] that in some disordered systems, it was appropriate to consider some vector symmetry breaking scheme instead of the usual Parisi matrix sector breaking ansatz. Very recently, yet another pattern of replica symmetry breaking has been identified to account for a pre-freezing phenomenon of multifractal exponents in a disordered system [51]. As a consequence, the proper identification of the true physical fixed point among the possible fixed points in replica space remains the important issue for deciding whether equation (79) is really true and sufficient to rule out the multifractal symmetry.
5.3. Discussion

In summary, we cannot give a definite conclusion on the existence of multifractal symmetry in the random Potts model, but we would like to stress the following two points:

(1) From a purely numerical point of view, the data presented in the figures above satisfy the symmetry within the error bars for all values $2 < Q \leq 8$ that have been studied. This statement is valid for the self-dual disordered model and in the dilute case. Moreover, the differences between the numerical data and the second-order perturbative expansion of equation (79) plotted as dashed lines in figure 1, are significantly larger than the deviation from a symmetric $\Delta(N)$.

(2) From a theoretical point of view, it would be interesting to study other fixed points that break the replica symmetry using various schemes, to see whether one can identify a type of scheme that would preserve the multifractal symmetry.

6. Conclusions and perspectives

In this paper, we have analyzed the physical origin of the symmetry relation $f(2d - \alpha) = f(\alpha) + d - \alpha$ proposed by Mirlin et al [27] for the singularity spectrum $f(\alpha)$ of critical eigenfunctions at Anderson transitions. We have explained the analogy with the Gallavotti–Cohen symmetry of large deviation functions that are well known in the field of non-equilibrium dynamics: the multifractal spectrum of the disordered model corresponds to the large deviation function of the rescaling exponent $\gamma = (\alpha - d)$ for a renormalization procedure considered as a Markov chain in the ‘effective time’ $t = \ln L$.

We have concluded that the symmetry discovered in the specific example of Anderson transitions should actually be satisfied at many other random critical points after an appropriate translation. For many-body random phase transitions, where the critical properties are usually analyzed in terms of the multifractal spectrum $H(a)$ and of the moments exponents $X(N)$ of two-point correlation function, we have obtained that the symmetry becomes $H(2X_1 - a) = H(a) + a - X(1)$, or equivalently $\Delta(N) = \Delta(1 - N)$ for the anomalous parts $\Delta(N) \equiv X(N) - NX(1)$. We have presented detailed numerical tests favoring these relations, or at least compatible with them, for the two-dimensional random Potts model with various values of $Q > 2$. Although currently available numerical results for the scaling dimensions favor the RG results deduced from the replica symmetric scenario (which does not satisfy the multifractal symmetry) and rule out the Parisi matrix sector breaking ansatz, we cannot exclude the possibility that other fixed points would equally be compatible with the numerical results, and also with an exact multifractal symmetry. It would be interesting in the future to test this multifractal symmetry at other random critical points. At a more general level, we hope that the analogy with the field of non-equilibrium dynamics will give new insights into the properties of renormalization flows in disordered systems.

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