Entropy Identity inducing
Non-Equilibrium Thermodynamics of
Relativistic Multi-Component Systems
and their Newtonian Limits*

W. Muschik†
Institut für Theoretische Physik
Technische Universität Berlin
Hardenbergstr. 36
D-10623 BERLIN, Germany

Keywords Special-relativistic multi-component systems · Entropy Identity · Entropy balance of a component of the mixture · Entropy balance of the mixture · Multi-heat relaxation · Equilibrium conditions: 4-temperature’s Killing relation · Newtonian limits of the balances of multi-component systems and their mixture

Abstract Non-equilibrium and equilibrium thermodynamics of an interacting component in a special-relativistic multi-component system is discussed by use of an entropy identity. The special case of the corresponding free component is considered. Equilibrium conditions and especially the multi-component Killing relation of the 4-temperature are discussed. Two axioms characterize the mixture: additivity of the energy momentum tensors and of the 4-entropies of the components generating those of the mixture. The resulting quantities of a component and of the mixture, energy, energy flux, momentum flux, stress tensor, entropy, entropy flux, supply and production and their Newtonian limits in zeroth approximation are derived.

1 Introduction

The treatment of multi-component systems is often restricted to transport phenomena in chemically reacting systems, that means, the mixture consisting of different components is shortly described by 1-component quantities such as temperature, pressure and energy which are not retraced to the corresponding quantities of the several components of the multi-component system. That is the case in non-relativistic physics [1] as well as in

*In memory of Robert Trostel
†Corresponding author: muschik@physik.tu-berlin.de
relativistic physics [2, 3, 4, 5, 6]. In this paper, the single component as an interacting member of the mixture is investigated. Thus, each component of the mixture is equipped with its own temperature, pressure, energy and mass density which all together generate the corresponding quantities of the mixture.

Considering a multi-component system, three items have to be distinguished: one component as a member of the multi-component system which interacts with all the other components of the system, the same component as a free 1-component system separated from the multi-component system and finally the multi-component system itself as a mixture which is composed of its components. Here, all three items are discussed in a special-relativistic framework. For finding out the entropy-flux, -supply, -production and -density, a special tool is used: the entropy identity which constrains the possibility of an arbitrary choice of these quantities [7, 9, 10]. Following J. Meixner and J.U. Keller that entropy in non-equilibrium cannot be defined unequivocally [11, 12, 13], the entropy identity is only an (well set up) ansatz for constructing a non-equilibrium entropy and further corresponding quantities. This fact in mind, a specific entropy and the corresponding Gibbs and Gibbs-Duhem equations are derived. The definition of the energy-momentum tensor and of the rest mass flux density are taken into account as constraints in the entropy identity by introducing fields of Lagrange factors. The physical dimensions of these factors allow to determine their physical meaning. The diffusion flux density is also introduced into the exploitation scheme.

Equilibrium is defined by equilibrium conditions which are divided into basic ones given by vanishing entropy-flux, -supply and -production and into supplementary ones such as vanishing diffusion flux, vanishing heat flux and rest mass production. The Killing relation of the 4-temperature concerning equilibrium is shortly discussed. Constitutive equations are out of scope of this paper.

The paper is organized as follows: After this introduction, the kinematics of a multi-component system is considered in the next two sections for introducing the mass flux and the diffusion flux densities. The energy-momentum tensor is decomposed into its (3+1)-split, and the entanglement of the energy and momentum balances are discussed, followed by non-equilibrium thermodynamics of an interacting component of the mixture and that of the corresponding free component. The equilibrium of both is considered. Thermodynamics of the mixture starts with two axioms: additivity of the energy momentum tensors and of the 4-entropies of the components resulting in those of the mixture. Entropy, entropy flux, -supply and -production are found out. The paper finishes with the Newtonian limit of of the considered relativistic thermodynamics in zeroth approximation, with a summary and an appendix..

2 Kinematics

2.1 The components

We consider a multi-component system consisting of \( Z \) components. The component index \( A \) runs from 1 to \( Z \). Each component has its own rest frame \( B^A \) in which the rest
mass density $\rho^A$ is locally defined. These rest mass densities are relativistic invariants and therefore frame independent\(^1\).

In general, the components have different 4-velocities: $u^A_k$, $A = 1, 2, ..., Z; k = 1, ..., 4$, which all are tensors of first order under Lorentz transformation. We now define the component mass flux density as a 4-tensor of first order and the component mass production term as a scalar

$$N^A_k := \rho^A u^A_k, \quad N^{A, k} = \Gamma^A.$$  \hfill (1)

Here, \(\text{I}_2\) is the mass balance equation of the \(A\)-component. Consequently, we introduce the basic fields of the components

$$\{\rho^A, u^A_k\}, \quad A = 1, 2, ..., Z. \quad \text{(2)}$$

The mass production term has two reasons: an external one by mass supply and one internal one by chemical reactions

$$\Gamma^A = (\text{ex})\Gamma^A + (\text{in})\Gamma^A. \quad \text{(3)}$$

The external mass supply \((\text{ex})\Gamma^A\) depends on the environment of the system, whereas \((\text{in})\Gamma^A\) is determined by chemical reactions depending on the set of frame-independent stoichiometric equations which are discussed in Appendix 10.3.

### 2.2 The mixture

As each component, also the multi-component system has a mass density $\rho$ and a 4-velocity $u_k$ which are determined by the partial quantities of the components. For deriving $\rho$ and $u_k$, we apply the nearly self-evident

- **Mixture Axiom:** The balance equation of a mixture looks like the balance equation of an one-component system.

Especially here, the mixture axiom is postulated for the balance equations of mass, energy-momentum and entropy. According to the mixture axiom, the mass balance of the mixture looks according to \(\text{I}_2\)

$$\sum_A N^A_k = \Gamma, \quad \Gamma = 0, \quad \text{(4)}$$

with vanishing total mass production, if the mass of the mixture is conserved\(^2\).

Now the question arises: which quantities of the components of the mixture are additive? Obviously, neither the mass densities $\rho^A$ nor the 4-velocities $u^A_k$ are additive quantities according to their definitions. Consequently, we demand in accordance with the mixture axiom that the mass flux densities are additive\(^3\)

**Setting I:**

$$\sum_A N^A_k = N_k := \rho u_k = \sum_A \rho^A u^A_k \quad \rightarrow \quad u_k = \sum_A \frac{\rho^A}{\rho} u^A_k. \quad \text{(5)}$$

---

1. more details in Appendix 10.1
2. the mixture as a closed system
3. The sign $\equiv$ stands for a setting and $:= \equiv$ for a definition.
For the present, $\varrho$ and $u_k$ are unknown. Of course, they depend on the basic fields of the components (2). Contraction with $u^k$ and use of (3)$_{2,3}$ results in

$$\varrho = \frac{1}{c^2} \sum_A g^A u^A_k u^k = \frac{1}{c^2} N_k u^k = \frac{1}{c^2} N_k \frac{1}{\varrho} N^k \nrightarrow \varrho = \pm \frac{1}{c} \sqrt{N_k N^k},$$

or in more detail

$$\varrho = \pm \frac{1}{c} \sqrt{\sum_{A,B} g^A g^B u^A_k u^{Bk}}.$$

The mass density $\varrho$ and the 4-velocity $u_k$ of the mixture are expressed by those of the components according to (7) and (5)$_4$. According to (5)$_4$, the 4-velocity of the mixture is a weighted mean value of the 4-velocities of the components. For the mass density, we have according to (6)_1 also a with the Kluitenberg factor $f^A$ weighted mean value of the mass density components [15]

$$f^A := \frac{1}{c^2} u^A_k \nrightarrow \varrho = \sum_A f^A \varrho^A = \sum_A f^A(u^A_k, u^k) \varrho^A,$$

resulting in the entanglement of $\varrho$ and $u_k$ which are not independent of each other

$$\varrho = R(g^A, u^A_k), \quad u_k = U_k(g^A, u^A_k, \varrho).$$

According to [5]$_1$ and [1]$_2$, we obtain the additivity of the mass production terms

$$N^k_{,k} = \sum_A N^{A k}_{,k} = \sum_A \Gamma^A = \Gamma \nrightarrow \sum_A \varrho^A \varrho^A = \sum_A \varrho^A(u^A_k - f^A u_k),$$

Introducing the diffusion flux density using (11)$_2$

$$J^A_k := \varrho^A(u^A_k - f^A u_k) = N^A_k - \varrho^A f^A u_k \nrightarrow \sum_A J^A_k =: J_k = 0,$$

we obtain

$$J^A_k u^k = \varrho^A(u^A_k u^k - f^A c^2) = 0,$$

or

$$J^A_k u^{A k} = c^2 g^A[1 - (f^A)^2] =: c^2 g^A w^A = w^A N^A_k u^{A k},$$

$$1 - w^A \geq 0.$$

---

4Chemical reactions are mass conserving.
By introducing the projectors
\begin{align}
h^{Am}_i := \delta^m_i - \frac{1}{c^2}u^Amu^A_i, \quad h^m_i := \delta^m_i - \frac{1}{c^2}u^mu_i,
\end{align}
we obtain the following properties of the diffusion flux density:
\begin{align}
J^{Am}h^k_m &= J^{Ak} = N^{Am}h^k_m, \\
J^{Am}h^{Ak}_m &= \varrho^A f^A (f^A u^Ak - u^k) \\
J^{Ak} &= J^{Am}h^{Ak}_m + \varrho^A u^A u^Ak = J^{Am}h^k_m + w^A N^{Ak}, \\
J^{Ak}_{,k} &= (J^{Am}h^{Ak}_m)_{,k} + (\varrho^A w^A)_{,k} u^Ak + \varrho^A w^A u^{Ak}_{,k}.
\end{align}

According to (17), the diffusion flux density is that part of the mass flux density which is perpendicular to the 4-velocity of the mixture. The diffusion flux density vanishes in 1-component systems \( (u^A_k \equiv u_k) \) according to \( f^A = f = 1 \) and (12).

\section{The Energy-Momentum Tensor}

\subsection{Free and interacting components}

The energy-momentum tensor \( T^{Akl} \) of the \( A \)-component consists of two parts
\begin{align}
T^{Akl} = \tilde{T}^{Akl} + \sum_B W^{Akl}_B, \quad W^{Bkl}_B = 0.
\end{align}

Here, \( \tilde{T}^{Akl} \) is the energy-momentum tensor of the free \( A \)-component, that is the case, if there are no interactions between the \( A \)-component and the other ones. \( W^{Akl}_B \) describes the interaction between the \( B \)- and the \( A \)-component. The interaction between the external environment and the \( A \)-component is given by the force density \( k^Al \) which appears in the energy-momentum balance equation
\begin{align}
T^{Akl}_{,k} = k^Al = \Omega^Al + \frac{1}{c^2}u^Alu^Amk^Am, \quad \Omega^Alu^A_l = 0,
\end{align}
and in the balance equations of
\begin{align}
\text{energy:} \quad u^A_l T^{Akl}_{,k} &= u^A_l k^Al =: \Omega^A, \\
\text{and momentum:} \quad h^{Am}_l T^{Akl}_{,k} &= h^{Am}_l k^Al =: \Omega^{Am}.
\end{align}

Consequently, the interaction of the \( A \)-component with the other components of the mixture modifies the energy-momentum tensor of the free \( A \)-component. Additionally, its interaction with the environment shows up in the source of the energy-momentum balance. According to its definition, \( T^{Akl} \) is the energy-momentum tensor of the "\( A \)-component in the mixture".
### 3.2 $(3+1)$-split

The $(3+1)$-split of the energy-momentum tensor of the $A$-component is

$$T^{Akl} = \frac{1}{c^2} e^A u^A u^A + u^A p^A + \frac{1}{c^2} q^{Ak} u^A + t^{Akl}. \quad (25)$$

The $(3+1)$-components of the energy-momentum tensor are

$$e^A := \frac{1}{c^2} T^{Ajm} u^A u^m, \quad p^A := \frac{1}{c^2} h^A h^m T^{Ajm} u^A, \quad (26)$$

$$q^{Ak} := h^A h^m T^{Ajm} u^m, \quad t^{Akl} := h^A h^m T^{Ajm} h^m, \quad (27)$$

$$q^{Ak} u^A_k = 0, \quad p^A u^A_i = 0, \quad t^{Akl} u^A_k = 0, \quad t^{Akl} u^A_l = 0. \quad (28)$$

The symmetric part of the energy-momentum tensor $(25)$ is

$$T^{A(kl)} = \frac{1}{c^2} e^A u^A u^A + \frac{1}{2c} u^A \left( c p^A + \frac{1}{c} q^{Ak} \right) u^A + t^{A(kl)}, \quad (29)$$

and its anti-symmetric part is

$$T^{A[kl]} = \frac{1}{2c} u^A \left( c p^A - \frac{1}{c} q^{Ak} \right) - \frac{1}{2c} \left( c p^A - \frac{1}{c} q^{Ak} \right) u^A + t^{A[kl]}. \quad (30)$$

The stress tensor is composed of the pressure $p^A$ and the viscous tensor $\pi^{Akl}$

$$t^{Akl} = -p^A h^{Akl} + \pi^{Akl}, \quad t^{Ak} = -3p^A. \quad (31)$$

We now consider the physical dimensions of the introduced quantities. According to $(16)$ and $(8)$, we have

$$[h^A] = 1, \quad [f^A] = 1. \quad (32)$$

By taking $(31)$, $(32)$ and $(25)$ into account we obtain

$$[t^{Akl}] = [p^A] = [\pi^{Akl}] = [e^A] = [q^{Ak}] \frac{s}{m} = [p^A] \frac{m}{s}, \quad (33)$$

pressure = $[p^A] = \frac{N}{m^2} = \frac{N m}{m^3} = \text{energy density} = \frac{kg}{s^2 m^2} = \frac{kg}{s^2 m^2} = \text{momentum flux density}, \quad (34)$

$$[q^{Ak}] = [e^A] \frac{m}{s} = \frac{N m}{m^3 s} \quad \text{energy flux density}, \quad (35)$$

$$[p^A] = k g \frac{1}{m s} \quad \text{momentum density}. \quad (36)$$

The $(3+1)$-split $(25)$ of the energy-momentum tensor can be written in a more compact form

$$T^{Akl} = \frac{1}{c^2} Q^{Ak} u^A + t^{Akl}, \quad (37)$$

$$Q^{Ak} := e^A u^A + q^{Ak}, \quad t^{Akl} := u^A p^A + t^{Akl}. \quad (38)$$

---

5the $(3+1)$-split is made by taking the physical meaning of $(26)$ and $(27)$ into account, see $(33)$ to $(36)$

6the bracket $[\xi]$ signifies the physical dimension of $\xi$
The energy-momentum tensor (37) is that of the $A$-component in the mixture, that means as discussed in sect. 3.1, the $A$-component is not a free system and the (3+1)-split-components $e^A, q^{Ak}, p^{Al}$ and $t^{Ak}$ include the internal interaction of the $A$-component with all the other ones.

### 3.3 Additivity

We now consider the equivalent-system composed of the $Z$ components: that is the mixture which consists of these $Z$ interacting components. Because this interaction is already taken into account by the (3+1)-split-components, the energy-momentum tensors of the components are additive without additional interaction terms. Consequently, the energy-momentum tensor $T^{kl}$ of the mixture is

Setting II:

$$ T^{kl} := \frac{1}{c^2} Q^k u^l + \tau^{kl} = \sum_A T^{Akl} = \sum_A \left( \frac{1}{c^2} Q^{Ak} u^{Al} + \tau^{Akl} \right). $$ (39)

Multiplication with $u_l$ results by use of \((38)_1\) and \((38)_2\) in

$$ Q^k = \sum_A \left( Q^{Ak} f^A + \tau^{Akl} u_l \right), $$ (40)

and by multiplication with $h^m_l$, \((39)\) results in

$$ \tau^{km} = \sum_A \left( Q^{Ak} g^{Am} + \tau^{Akl} h^m_l \right), $$ (41)

$$ g^{Am} := \frac{1}{c^2} u^{Al} h^m_l = \frac{1}{c^2} (u^{Am} - f^A u^m) = \frac{1}{c^2} g^A J^{Am}. $$ (42)

For an 1-component system ($u^A_k \equiv u_k$), we obtain according to \((42)\) $g^{Am} = g^m = 0$ taking $f^A = f = 0$ into account.

### 3.4 (3+1)-components of the mixture

Starting with \((38)\), we obtain

$$ Q^{Ak} u^A_k = c^2 e^A, \quad Q^{Ak} h^A_k = q^A, $$ (43)

$$ \tau^{Akm} u^A_k = c^2 p^{Am}, \quad \tau^{Akm} h^A_k = t^{Am}. $$ (44)

According to \((37)\) and \((39)_1\), these relations are analogous for the mixture. Consequently, from \((40)\) follows

$$ Q^k u_k =: c^2 e = \sum_A \left( Q^{Ak} f^A u_k + \tau^{Akl} u_l u_k \right), $$ (45)

resulting with \((43)_1\) in the energy density of the mixture

$$ c^2 e = \sum_A \left( c^2 e^A (f^A)^2 + q^{Ak} f^A u_k + c^2 p^{Al} f^A u_l + t^{Ak} u_l u_k \right). $$ (46)
From (40) follows
\[ Q^k h^m_k =: q^m = \sum_A \left( f^A Q^{Ak} h^m_k + h^m_k \tau^{Akl} u_l \right), \tag{47} \]
resulting with (43), in the energy flux density of the mixture
\[ q^m = \sum_A \left( c^2 e^A f^A g^{Am} + q^{Ak} f^A h^m_k + c^2 p^{Al} g^{Am} u_l + t^{Akl} h^m_k u_l \right). \tag{48} \]
From (41) follows
\[ \tau^{km} u_k =: c^2 p^m = \sum_A \left( Q^{Ak} u_k g^{Am} + \tau^{Akl} h^m_l u_k \right), \tag{49} \]
resulting with (44), in the momentum density of the mixture
\[ c^2 p^m = \sum_A \left( c^2 e^A f^A g^{Am} + q^{Ak} u_k g^{Am} + c^2 p^{Al} f^A h^m_l + t^{Akl} h^m_l u_k \right). \tag{50} \]
And from (44), follows finally
\[ \tau^{km} h^j_k =: t^{jm} = \sum_A \left( Q^{Ak} g^{Am} h^j_k + \tau^{Akl} h^m_l h^j_k \right) \tag{51} \]
which by taking (38) into account results in the stress tensor and the pressure of the mixture
\[ t^{jm} = \sum_A \left( c^2 e^A g^{Aj} g^{Am} + q^{Ak} h^j_k g^{Am} + c^2 p^{Al} g^{Am} + t^{Akl} h^j_k h^m_l \right), \tag{52} \]
\[ p = -\frac{1}{3} t^{jm} h_{jm} = -\frac{1}{3} \sum_A \left( Q^{Ak} g^{Am} h^j_k + \tau^{Akl} h^m_l h^j_k \right) h_{jm} = \]
\[ = -\frac{1}{3} \sum_A \left( \frac{1}{c^2} Q^{Ak} u^Ap_{kp} + \tau^{Akl} h_{kl} \right) = \]
\[ = -\frac{1}{3} \sum_A \left( \frac{1}{c^2} (e^A u^{Ak} + q^{Ak}) u^Ap_{kp} + (u^{Ak} p^{Al} + t^{Akl}) h_{kl} \right). \tag{53} \]
The additivity of the energy-momentum tensors (39) results in (40), (48), (50) and (52), relations which express the (3+1)-components of the energy-momentum tensor of the mixture as those of the components and their velocities
\[ \left\{ e, q^k, p^k, t^{kl} \right\} = F \left( e^A, q^{Ak}, p^{Ak}, t^{Akl}, u^{Ak}, u^A, u_k, q, q^A, u^A_k, u^k, q^A, u^A_k, q \right), \tag{54} \]
\[ T^{kl} = \frac{1}{c^2} e u^k u^l + u^k p^l + \frac{1}{c^2} q^k u^l + t^{kl}, \quad t^{kl} = -ph^{kl} + \pi^{kl}. \tag{55} \]
The 4-velocity \( u^k \) is given by (5). The influence of the additivity of the energy-momentum tensors on the balance equations of energy and momentum is investigated in the next section.
4 Entanglement of Energy and Momentum Balances

If the energy-momentum tensors of the \( A \)-component and of the mixture are \( T_{Akl} \) and \( T_{kl} \), the energy and momentum balances are according to the mixture axiom by use of (23) and (24)

\[
\begin{align*}
\text{energy:} & \quad u_l T_{Akl,k} = \Omega^A & \quad u_l T_{kl,k} = \Omega, \\
\text{momentum:} & \quad h_l^m T_{Akl,k} = \Omega^{Am} & \quad h_l^m T_{kl,k} = \Omega^m.
\end{align*}
\]

The balances (56) and (57) follow from (23) and (24) by the mixture axiom. Here, \( \Omega^A \) and \( \Omega \) are the energy supplies, and \( \Omega^{Am} \) and \( \Omega^m \) the momentum supplies of the \( A \)-component and of the mixture.

The (3+1)-split of the divergence of the energy-momentum tensor of the \( A \)-component results by use of (16) in

\[
\delta^m_l T_{Akl,k} = T_{Akm,k} = h_l^m T_{Akl,k} + \frac{1}{c^2} u^m u_l^A T_{Akl,k}.
\]

If the component index \( A \) is cancelled in (58), we obtain the decomposition of the divergence of the energy-momentum tensor of the mixture. Taking (56) and (57) into account, these divergences can be written as

\[
\begin{align*}
T_{Akm,k} & = \Omega^{Am} + \frac{1}{c^2} u^m u_l^A \Omega^A, & \quad T_{km,k} & = \Omega^m + \frac{1}{c^2} u^m \Omega.
\end{align*}
\]

The additivity of the energy-momentum tensors (39) results in the additivity of the force densities

\[
k^m = \Omega^m + \frac{1}{c^2} u^m \Omega = \sum_A \left( \Omega^{Am} + \frac{1}{c^2} u^m \Omega^A \right) = \sum_A k^{Am}.
\]

Taking (23) and (24) into account, we obtain by multiplication of (60) with \( u_m \), resp. with \( h^p_m \),

\[
\begin{align*}
\Omega & = \sum_A \left( \Omega^{Am} u_m + f^A \Omega^A \right), & \quad \Omega^p & = \sum_A \left( \Omega^{Am} h^p_m + g^A \Omega^A \right).
\end{align*}
\]

Inserting (23) and (24), we obtain in more detail

\[
\begin{align*}
u_l T_{kl,k} & = \sum_A \left\{ \left( h_l^m u_m + f^A u_l^A \right) T_{Akl,k} \right\}, \\
h_l^p T_{kl,k} & = \sum_A \left\{ \left( h_l^m h^p_m + g^A u_l^A \right) T_{Akl,k} \right\}.
\end{align*}
\]

As (61) indicates, the additivity of the energy-momentum tensors causes that the supplies of energy and momentum are entangled, expressed by the inequalities

\[
\begin{align*}
\sum_A f^A \Omega^A & \neq \Omega, & \quad \sum_A \Omega^{Am} h^p_m & \neq \Omega^p.
\end{align*}
\]

\[7\text{this is a strong argument for the validity of Setting II (39).}\]
Also if the total force density and the total momentum supply are zero,

\[ T^{kl}_{,k} = 0 \quad \Rightarrow \quad \Omega_{iso} = 0 \wedge \Omega_{iso}^m = 0, \quad (65) \]

we obtain according to (61)

\[
\sum_A \Omega_{iso}^{Am} u_m = - \sum_A f^A \Omega_{iso}^A \neq 0, \quad (66)
\]

\[
\sum_A \Omega_{iso}^{Am} h^p_m = - \sum_A g^{Ap} \Omega_{iso}^A \neq 0. \quad (67)
\]

As expected, the supplies of energy and momentum remain entangled in a system of vanishing total force and momentum densities. The entanglement vanishes for such isolated systems for which the force and momentum supplies for all \( A \)-components are zero.

5 Thermodynamics of Interacting Components

5.1 The entropy identity

For establishing the entropy balance equation, we use a special procedure starting out with an identity [8, 9], the so-called entropy identity. This tool helps to restrict arbitrariness for defining entropy density, entropy flux density, entropy production and supply. In the sequel, we establish an entropy identity for the \( A \)-component by starting out with the (3+1)-split of the entropy 4-vector

\[ S^{Ak} = s^A u^{Ak} + s^{Ak}. \quad (68) \]

Here \( s^A \) is the entropy density and \( s^{Ak} \) the entropy flux density defined by

\[ s^A := \frac{1}{c^2} S^{Ak} u_k^A, \quad s^{Ak} := S^{Am} h_m^A. \quad (69) \]

From non-relativistic physics, we know the physical dimensions

\[ [s^A] = [e^A] \frac{1}{K} = \frac{Nm}{m^3 K}, \quad [s^{Ak}] = [q^{Ak}] \frac{1}{K} = \frac{Nm m}{m^3 s K}. \quad (70) \]

Before writing down the entropy identity, we have to choose the quantities which are essential for formulating the four entropy quantities mentioned above. The choice is: all quantities appearing in the (3+1)-split of the energy-momentum tensor (25) and in the mass flux density (1) have to be included in the entropy identity which is generated by adding suitable zeros to (68). There is no unequivocal entropy identity [13] and consequently, also no unique entropy density, -flux, -supply and -production:

\[
S^{Ak} \equiv s^A u^{Ak} + s^{Ak} + \kappa^A \left( N^{Ak} - g^A u^{Ak} \right) + \\
+ \Lambda_i^A \left( T^{Akl} - \frac{1}{c^2} e^A u^{Ak} u^{Al} - u^{Ak} p^A l - \frac{1}{c^2} q^{Ak} u^{Al} - t^{Akl} \right). \quad (71)
\]
The Lagrange factors $\kappa^A$ and $\Lambda^A_l$ are field functions whose physical meaning becomes clear in the course of the exploitation of the entropy identity. Here, $\kappa^A$ is a scalar, undefined for the present, and the $(3+1)$-split of the likewise arbitrary vector $\Lambda^A_l$ is

$$
\begin{align*}
\Lambda^A_i &= \lambda^A_i u^A_i + \lambda^A_i, \\
\lambda^A_i u^A_i &= 0, \\
\lambda^A_i h^A_i &= \lambda^A_i, \\
\Lambda^A_i u^A_l &= c^2 \lambda^A_i, \\
\Lambda^A_i h^A_m &= \lambda^A_m.
\end{align*}
\quad (72)
$$

We denote the Lagrange factors $\kappa^A$, $\lambda^A_i$ and $\lambda^A_i$ as *accessory variables* because they help to formulate an entropy identity. An identification of these auxiliary variables is given below after the definitions of entropy flux density, entropy production, density and supply in section 5.3. By use of (72), the entropy identity (71) becomes

$$
S^A_k \equiv u^A_k \left( s^A - \kappa^A e^A - \lambda^A e^A - \lambda^A_i p^A_l \right) + \kappa^A N^A_k + \left( \lambda^A_i u^A_i - \lambda^A_i t^A_k \right) T^A_{kl}.
\quad (74)
$$

This identity transforms in an other one by differentiation and by taking the entropy balance equation

$$
S^A_{k, l} = \sigma^A + \varphi^A
\quad (75)
$$

into account.

$$
S^A_{k, l} \equiv \left[ u^A_k \left( s^A - \kappa^A e^A - \lambda^A e^A - \lambda^A_i p^A_l \right) \right]_l + \left[ s^A_k - \lambda^A k q^A_k - \lambda^A t^A_k \right] T^A_{kl} + \kappa^A \Gamma^A + \kappa^A \Lambda^A_k N^A_k + \left( \lambda^A_i u^A_i \right) T^A_{kl} + \lambda^A_i u^A_i T^A_{kl} + \lambda^A_i T^A_{kl} = \sigma^A + \varphi^A.
\quad (76)
$$

Here, $\sigma^A$ is the entropy production and $\varphi^A$ the entropy supply of the $^A$-component. The identity (76) changes into the entropy production, if $S^A$, $S^A_k$ and $\varphi^A$ are specified below.

Now we look for terms of the third row of (76) which fit into the first row of (76). The shape of these terms is \([u^A_k\text{scalar}]_k\) according to the first term of (76) and $\Psi^A_{k, k}$ (\(\Psi^A_{k, k} u^A_k = 0\)) according to the second term. None of the six terms of the second and third row of (76) have this shape, but inserting the energy-momentum tensor into the last row of (76) may generate such terms. First, the first and the third term of the last row of (76) become

$$
\begin{align*}
\lambda^A_i u^A_k T^A_{kl} &= \left( \lambda^A_i u^A_k + \lambda^A_i u^A_{l, k} \right) \left( \frac{1}{c^2} e^A u^A u^A_l + u^A_k p^A_l + 1 \right) T^A_{kl} = \\
&= \lambda^A_i u^A_k e^A + \lambda^A_i u^A_{l, k} u^A_k p^A_l + \lambda^A_i q^A_k - p^A \lambda^A_i u^A_{k, l} \pi^A_{kl} + \lambda^A_i u^A_{l, k} \pi^A_{kl},
\end{align*}
\quad (77)
$$

$$
\begin{align*}
\lambda^A_i u^A_{l, k} T^A_{kl} &= \lambda^A_i \left( \frac{1}{c^2} e^A u^A u^A_l + u^A_k p^A_l + 1 \right) q^A_k u^A_l + t^A_{kl}.
\end{align*}
\quad (78)
$$

Summing up (77) and (78) results in

$$
\begin{align*}
\left( \lambda^A_i u^A_k \right)_k T^A_{kl} + \lambda^A_i u^A_{l, k} T^A_{kl} &= \lambda^A_i \left( q^A_k + e^A u^A_{l, k} \right) + \lambda^A_i u^A_{l, k} \left( \pi^A_{kl} + u^A_k p^A_l \right) - \\
&\quad - p^A \lambda^A_i u^A_{k, l} + \lambda^A_i \left( \frac{1}{c^2} e^A u^A u^A_l + u^A_k p^A_l + 1 \right) q^A_k u^A_l + t^A_{kl}.
\end{align*}
\quad (79)
$$

\footnote{the signs , , , and mark terms which are related to each other in the sequel}
Evidently, the term \(-p^A\lambda^Au^A_{k,k}\) belongs to the first term of (76). After having performed the derivation and inserted the underlined term of (79), the first term of (76) becomes

\[
u^A_{k,k}\left(s^A - \kappa^A q^A - \lambda^A e^A - \frac{p^A\lambda^A}{1} - \lambda^A p^{Al}\right) + \\
\left(s^A - \kappa^A q^A - \lambda^A e^A - \frac{p^A\lambda^A}{1} - \lambda^A p^{Al}\right) + \left(p^A\lambda^A\right) = \\
\left[u^A\left(s^A - \kappa^A q^A - \lambda^A e^A - \frac{p^A\lambda^A}{1} - \lambda^A p^{Al}\right)\right]_{k} + \left(p^A\lambda^A\right)\cdot \cdot \cdot . \tag{80}
\]

Rearranging the entropy identity (76) results in

\[
S^A_{k,k} \equiv \left[u^A\left(s^A - \kappa^A q^A - \lambda^A e^A - \frac{p^A\lambda^A}{1} - \lambda^A p^{Al}\right)\right]_{k} + \left(p^A\lambda^A\right)\cdot + \\
\left[u^A\left(s^A - \kappa^A q^A - \lambda^A e^A - \frac{p^A\lambda^A}{1} - \lambda^A p^{Al}\right)\right]_{k} + \\
\kappa^A \iota^A + \kappa^A \iota^A \iota^A \iota^A + \\
\lambda^A_{k,k}\left(q^A + e^A u^A\right) + \lambda^A u^A_{t,k}\left(\pi^A_{kl} + u^A_{k} p^{Al}\right) + \\
\lambda^A u^A_{t,k}\left(\frac{1}{c^2} e^A u^A_{k} u^A_{l} + u^A_{k} p^{Al} + \frac{1}{c^2} q^A u^A_{k} u^A_{l} + t^{Akl}\right) + \\
\lambda^A u^A_{t,k} \pi^A_{kl} + \lambda^A u^A_{t,k} \pi^A_{kl} = \sigma^A + \varphi^A. \tag{81}
\]

This entropy identity is incomplete: the multi-heat relaxation is missing which is generated by the different partial temperatures of the components of the mixture. Because of lucidity, the treatment of multi-heat relaxation is postponed and will be considered below in sect 5.4. In the next section, we now specify \(s^A, s^A_{k}, \varphi^A\) and \(\sigma^A\).

5.2 Exploitation of the entropy identity

5.2.1 Entropy density, Gibbs and Gibbs-Duhem equations

We now define the entropy rest density \(s^A\) according to the first round bracket in (81)

\[
\text{Setting III:} \quad s^A \bullet \kappa^A q^A + \lambda^A e^A + p^A \lambda^A + \lambda^A p^{Al}, \tag{82}
\]

resulting in the specific rest entropy

\[
\frac{s^A}{q} = \kappa^A \frac{q^A}{q} + \lambda^A \frac{e^A}{q} + p^A \lambda^A \frac{1}{q} + \lambda^A p^{Al} \frac{1}{q}. \tag{83}
\]

A non-equilibrium state space –which is spanned by the independent variables– contains the equilibrium variables \(q^A,\) \(\varphi^A\) and \(e^A\) and beyond them the non-equilibrium variables \(p^{Al}\)

---

\(\bullet\) is the "component time derivative" \(\bullet \bullet \bullet \bullet  \bullet  \bullet  := \frac{\partial}{\partial t} u^A_{k,k}\)
extending the equilibrium sub-space in the sense of Extended Thermodynamics\textsuperscript{10}\textsuperscript{16,17}. Consequently, we choose the state space

\[ z^A = \left( \frac{c^A}{\varrho}, \frac{\varrho^A}{\varrho}, p^A \right), \quad c^A := \frac{\varrho^A}{\varrho} \]  

(84)

The corresponding Gibbs equation according to (83) and (84) is

\[ \left( \frac{\dot{S}^A}{\varrho} \right) = \kappa^A \dot{c}^A + \lambda^A \left( \frac{\dot{\varrho}^A}{\varrho} \right) + p^A \lambda^A \left( \frac{1}{\varrho} \right) + \lambda^A \left( \frac{\dot{p}^A}{\varrho} \right). \]  

(85)

Differentiation of (83) results in the Gibbs-Duhem equation by taking (85) into account

\[ 0 = \kappa^A \dot{c}^A + \lambda^A \left( \frac{\dot{\varrho}^A}{\varrho} \right) + (p^A \lambda^A) \frac{1}{\varrho} + \lambda^A \frac{\dot{p}^A}{\varrho}, \]  

(86)

resulting in

\[ (p^A \lambda^A)^* = - \kappa^A \dot{\varrho}^A - \lambda^A \dot{c}^A - \lambda^A \dot{p}^A. \]  

(87)

Taking (87) and (82) into account, the entropy identity (81) becomes

\[ S_{Ak,k} \equiv - \kappa^A \dot{\varrho}^A - \lambda^A \dot{c}^A - \lambda^A \dot{p}^A + \left[ s_{Ak} - \lambda^A \dot{q}^A - \lambda^A \dot{t}^{Ak} \right] + \]  

\[ + \kappa^A (\text{ex}) \Gamma^A + \kappa^A \dot{N}^{Ak} + \]  

\[ + \lambda^A (\dot{q}^A + \dot{c}^A \dot{u}^{Ak}) + \lambda^A u_{\ell,k} \left( \pi^{Ak} + u^{Ak} \dot{p}^A \right) + \]  

\[ + \lambda^A u_{\ell,k} \left( \frac{1}{c^2} c^A u^{Ak} u^{Al} + \dot{u}^{Ak} \dot{p}^A + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Akt} \right) + \]  

\[ + \lambda^A u_{A1} u^{Ak} + \lambda^A \dot{N}^{Ak} = \sigma^A + \varphi^A. \]  

(88)

The marked terms cancel each other. Taking (12) and (19) into account, we consider

\[ 0 = - \kappa^A \lambda^A - \kappa^A \lambda^A N^{Ak} = - \kappa^A \lambda^A \left( J^{Ak} - \dot{J}^{Ak} \right) = \]  

\[ = - \kappa^A \dot{J}^{Ak} + \kappa^A \lambda^A J^{Am} h^{Ak} + \sigma^A w^{A} u^{Ak} = \]  

\[ = - \kappa^A \lambda^A \left( J^{Ak} - \dot{w}^A N^{Ak} \right) + \left( \kappa^A J^{Am} h^{Ak} \right)_{,k} - \kappa^A \left( J^{Am} h^{Ak} \right)_{,k}. \]  

(89)

This zero contains the diffusion flux which does not appear up to here in the entropy identity (71). That means, the diffusion is missing in (71), and we will not ignore the

\textsuperscript{10}\textsuperscript{If the energy-momentum tensor is presupposed to be symmetric – consequently $p^A = (1/c^2)q^A$ is valid according to (39) – the momentum density is replaced by energy flux density which in non-relativistic Extended Thermodynamics is set as a non-equilibrium variable, even if the stress tensor is non-symmetric.}
Underbraced terms in (89), but we insert (89) into (88). Consequently, the entropy identity results in
\[ S_{Ak,k} \equiv \left[ s^{Ak} - \lambda^A q^{Ak} - \lambda^A t^{Akl} + \kappa^A J^{Am} t^{Am}_{m,k} \right] + \]
\[ + \kappa^A (ex) A - (in) A - (J^{Am} h^A_m) \right) - \kappa^A J^{Am} h^A_m + \]
\[ + \lambda^A, k q^{Ak} + \lambda^A u^{A,l,k} \left( \pi^{Akl} + u^{A} p^{Al} \right) + \]
\[ + \lambda^A, t,k \left( \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Akl} \right) + \]
\[ + \lambda^A u^A k^{Akl} + \lambda^A k^{Akl} = \sigma^A + \phi^A. \] (90)

The underlined terms belong to the interaction of the system with its environment. We now specify the entropy flux density \( s^{Ak} \) and the entropy supply \( \phi^A \) in the next section.

5.2.2 Entropy flux, –supply and –production

According to the first row of (90), we define the entropy flux density

\[ S^{Ak} \equiv \lambda^A q^{Ak} + \lambda^A t^{Akl} - \kappa^A J^{Am} h^A_m. \] (91)

We now split the entropy identity (88) into the entropy production and the entropy supply. For this end, we need a criterion to distinguish between entropy production and supply. Such a criterion is clear for discrete systems: a local isolation suppresses the entropy supply but not the entropy production. Isolation means: the three underlined terms in (90) vanish, if the \( A \)-component is isolated from the exterior of the mixture. Consequently, we define the entropy supply as follows

\[ \phi^A \equiv k^A (ex) A - (in) A - (J^{Am} h^A_m). \] (92)

with the result that the entropy identity (90) transfers into the entropy production density by taking (91) and (92) into account

\[ \sigma^A = \kappa^A (ex) A - (in) A - (J^{Am} h^A_m) \right) - \kappa^A J^{Am} h^A_m + \]
\[ + \lambda^A, k q^{Ak} + \lambda^A u^{A,l,k} \left( \pi^{Akl} + u^{A} p^{Al} \right) + \]
\[ + \lambda^A, t,k \left( \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Akl} \right). \] (93)

As expected, the entropy production is composed of terms which are a product of ”forces” and ”fluxes” as in the non-relativistic case\[11\]. The expressions \( s^A \), \( s^{Ak} \), \( \phi^A \) and \( \sigma^A \) contain accessory variables which are introduced for formulating the entropy identity (71) playing up to here the role of place-holders. Their physical meaning is discussed in the next section.

---

11 The mass production \( (in) A \) due to chemical reactions can be expressed by the time rate of the reaction velocity, see (337) in sect. 10.3.
5.3 Accessory variables

Starting out with (52), we have the following equation of physical dimensions

\[ [s^A] = [\lambda^A][e^A]. \]  (94)

Taking (70) and (33) into account, we obtain

\[ \frac{N}{m^2 K} \cdot \frac{1}{K} = [\lambda^A] \frac{N}{m^2} \rightarrow [\lambda^A] = \frac{1}{K}. \]  (95)

that means, \( \lambda^A \) is a reciprocal temperature belonging to the \( A \)-component. Therefore, we accept the following

**Setting VI:**

\[ \lambda^A \cdot \frac{\nu^A}{\Theta^A}, \]  (96)

with the partial temperature \( \Theta^A \) of the \( A \)-component{\(^{12}\)} and a scalar \( \nu^A \) which is suitably chosen below.

Starting out with (91), we have the following equation of physical dimensions

\[ [s^{Ak}] = [\kappa^A][J^{Am}][h_{m}^{Ak}]. \]  (97)

Taking (70)\(_2\), (12)\(_2\) and (32)\(_1\) into account, we obtain

\[ \frac{N}{ms K} \cdot \frac{1}{K} = [\kappa^A] \frac{k g m^3}{m^3 s} \rightarrow [\kappa^A] = \frac{m^2}{s^2 K}. \]  (98)

We know from the non-relativistic Gibbs equation that the chemical potentials \( \mu^A \) have the physical dimension of the specific energy \( e^A/\varrho^A \)

\[ [\mu^A] = \frac{[e^A]}{[\varrho^A]} = \frac{N m^3}{m^2 k g} = K[\kappa^A]. \]  (99)

Consequently, we make the following choice by taking (99) into consideration

**Setting VII:**

\[ \kappa^A \cdot \frac{\mu^A}{\Theta^A}. \]  (100)

Starting out with (91) and (33)\(_1\), we have the following equation of physical dimensions

\[ [s^{Ak}] = [\lambda^{Ak}[l^{Aki}] = [\lambda^{Ak}][p^A]. \]  (101)

Taking (70)\(_2\) and (34) into account, we obtain

\[ \frac{N}{ms K} \cdot \frac{1}{K} = [\lambda^{Ak}] \frac{N}{m^2} \rightarrow [\lambda^{Ak}] = \frac{m}{s K}. \]  (102)

\(^{12}\)This temperature is a non-equilibrium one, the contact temperature [18] [19] which should not be confused with the thermostatic equilibrium temperature \( \Theta_{eq} = T, \wedge A \).
that means, $\lambda^{Ak}$ is proportional to a velocity and at the same time perpendicular to $u^{Ak}$ according to (72). Consequently, the velocity $u^m$ of the mixture remains for defining $\lambda^{Ak}$

Setting VIII:

$$\lambda^{Ak} = \frac{1}{\Theta^A} u^m h^{Ak}_m. \tag{103}$$

Inserting the accessory variables into the expression of entropy density (82), of entropy flux density (91) and of entropy supply (92), we obtain

$$s^A = \frac{1}{\Theta^A} \left( \mu^A q^A + \nu^A (e^A + p^A) + u_m p^Am \right), \tag{104}$$

$$s^{Ak} = \frac{1}{\Theta^A} \left( \nu^A q^{Ak} - \mu^A J^{Am} h^{Ak}_m + u_m t^{Ak m} \right), \tag{105}$$

$$\varphi^A = \frac{1}{\Theta^A} \left( \mu^{A(ex)} \Gamma^A + \nu^A u^A_{ml} k^{Al} + u_p h^{Ap}_{m k} \right). \tag{106}$$

The entropy production density (93) results by use of (100), (103) and (19)

$$\sigma^A = \frac{\mu^A}{\Theta^A} \left( \Gamma^{A(m)} - (J^{Am} h^{Ak}_m)_{,k} \right) - \left( \frac{\mu^A}{\Theta^A} \right)_{,k} J^{Am} h^{Ak}_m + \left( \frac{\nu^A}{\Theta^A} \right)_{,k} q^{Ak} + \left( \frac{\nu^A}{\Theta^A} \right)_{,k} u^A_{ml} \left( \pi^{A_{k l}} + u^{Ak p} A^l \right) + \left( \frac{1}{\Theta^A} u^m h^{Am}_{ml} \right)_{,k} \left( \frac{1}{c^2} e^A u^{Ak u^A} + \frac{1}{c^2} q^{Ak u^A} + t^{A_{k l}} \right). \tag{107}$$

The first four terms of the entropy production describe the four classical reasons of irreversibility: chemical reactions, diffusion, heat conduction and internal friction with a modified non-equilibrium viscous tensor. The last term of (107)\(^{13}\) is typical for an interacting $^A$-component as a part of the mixture according to its LHS factor.

Up to now, a further phenomenon of irreversibility was not taken into consideration: the multi-heat relaxation which is discussed in the next section.

### 5.4 Multi-heat relaxation and the partial temperatures

Because the different components of the mixture have different partial (reciprocal) temperatures $\lambda^A$, $A = 1, 2, ..., Z$, a multi-heat relaxation\(^{14}\) takes place which is an irreversible phenomenon. Consequently, multi-heat relaxation has to be taken into account in the entropy identity by adding a suitable zero as done in (71).

A heat transfer $H^{ABk}$ between two components of the mixture $-A$ and $B$- takes place by multi-heat relaxation, if the corresponding temperatures of the components are different from each other. Consequently, the entropy exchange between these two components is

Setting IX:

$$G^{ABk} := H^{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right), \quad H^{BBk} \equiv 0. \tag{108}$$

\(^{13}\) which vanishes in equilibrium and for free 1-component systems, as we will see below

\(^{14}\) do not take multi-heat relaxation for the heat conduction which is caused by temperature gradients
As the energy flux density \( \mathbf{27} \) and \( \mathbf{35} \), the multi-heat transfer satisfies
\[
[\mathbf{109}] \quad H_{ABk} u^A_k = 0, \quad H_{ABk} u^B_k = 0, \quad H_{ABk} = -H_{BAk},
\]
\[
[\mathbf{110}] \quad [\mathbf{q}^{ABk}] = \left[ \mathbf{e}^A \right] M \frac{m}{s}, \quad [\mathbf{G}^{ABk}] = \left[ \mathbf{e}^A \right] M \frac{1}{s K} = [\mathbf{s}^A].
\]

For the \( A \)-component, this results according to \( \mathbf{109} \) in
\[
[\mathbf{111}] \quad H^{Ak} := \sum_B H_{ABk}, \quad \sum_{AB} H_{ABk} = 0,
\]
\[
[\mathbf{112}] \quad G^{Ak} := \sum_B H_{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) = H^{Ak} \frac{1}{\Theta^A} - \sum_B H_{ABk} \frac{1}{\Theta^B}, \quad G^{Ak} u^A_k = 0,
\]
\[
[\mathbf{113}] \quad \sum_A G^{Ak} = \sum_{AB} H_{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) \neq 0, \text{ if } \Theta^A \neq \Theta^B.
\]

The entropy exchange of the \( A \)-component according to multi-heat exchange \( \mathbf{112} \) has now to be introduced into the entropy identity \( \mathbf{71} \). Because \( G^{Ak} \) has the same physical dimension as \( s^A_k \), the zero
\[
[\mathbf{114}] \quad 0 = \left( G^{Ak} - \sum_B H^{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) \right)_{,k} =
\]
\[
= G^{Ak}_{,k} - \sum_B H^{ABk}_{,k} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) - \sum_B H^{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right),
\]

\[
\text{can be directly introduced into the entropy identity without defining an additional accessory variable. According to \( \mathbf{71} \), the three terms of \( \mathbf{114} \) are attached as follows}
\]
\[
[\mathbf{115}] \quad + \sum_B H^{ABk}_{,k} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) \quad \rightarrow \quad \text{entropy production density},
\]
\[
[\mathbf{116}] \quad + \sum_B H^{ABk}_{,k} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right) \quad \rightarrow \quad \text{entropy supply},
\]
\[
[\mathbf{117}] \quad -G^{Ak}_{,k} \quad \rightarrow \quad \text{entropy exchange density}.
\]

Introducing these terms as drawn in sect.\( \mathbf{5.1} \) \( \mathbf{104} \) to \( \mathbf{107} \) yield
\[
[\mathbf{118}] \quad s^A = \frac{1}{\Theta^A} \left( \mu^A \phi^A + \nu^A (\epsilon^A + p^A) + u_m \rho^A m \right),
\]
\[
[\mathbf{119}] \quad s^{Ak} = \frac{1}{\Theta^A} \left( \nu^A q^{Ak} - \mu^A J^A m h^A k + u_m \epsilon^{Ak} \right) + G^{Ak},
\]
\[
[\mathbf{120}] \quad \varphi^A = \frac{1}{\Theta^A} \left( \mu^A (\epsilon^A) \Gamma^A + \nu^A \epsilon^{Ak} \Gamma^A l^A k + u_m \epsilon^{Ak} m \right) + \sum_B H^{ABk}_{,k} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right),
\]
\[
[\mathbf{121}] \quad \sigma^A = \frac{1}{\Theta^A} \left( \mu^A (\epsilon^A) \Gamma^A - (J^A m h^A k)_{,k} \right) + \left( \frac{\mu^A}{\Theta^A} \right)_{,k} J^A m h^A k +
\]

17
\[ + \left( \frac{\nu^A}{\Theta^A} \right)_k q^{Ak} + \frac{\nu^A}{\Theta^A} u^A_{t,k} \left( \pi^{Akl} + u^{Ak} p^{Al} \right) + \sum_B H^{ABk} \left( \frac{1}{\Theta^A} - \frac{1}{\Theta^B} \right)_k + \]
\[ + \left( \frac{1}{\Theta^A} u^m h^{Al}_m \right)_k \left( \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Akl} \right). \]  

The six terms of the entropy production density (121) have the following meaning:

- diffusion modified chemical reaction: \((\mu^A/\Theta^A) \left( \Gamma^A + (J^{Am} h^{Ak}_m) \right)_k\)
- diffusion: \((\mu^A/\Theta^A) \left( J^{Am} h^{Ak}_m \right)\)
- heat conduction: \((\nu^A/\Theta^A) \left( q^{Ak} \right)\)
- multi-component modified internal friction: \((\nu^A/\Theta^A) u^{Al,k} \left( \pi^{Akl} + u^{Ak} p^{Al} \right)\)
- multi-heat relaxation: \(\sum_B H^{ABk} \left( \left(1/\Theta^A \right) - \left(1/\Theta^B \right) \right)_k\)
- multi-component interaction\footnote{This term vanishes in equilibrium and for 1-component systems: see sect.6} \((u^m h^{Al}_m/\Theta^A)_k \left( \frac{1}{c^2} e^A u^{Ak} u^{Al} + \frac{1}{c^2} q^{Ak} u^{Al} + t^{Akl} \right)\).

5.5 The 4-entropy

We need the 4-entropy of the \(A\)-component for describing thermodynamics of a mixture. Starting out with (68), (104), (117) and (105), we obtain

\[ S^{Ak} = \frac{1}{\Theta^A} u^{Ak} \left\{ \mu^A q^A + \nu^A \left( e^A + p^A \right) + u^m p^{Am} + \frac{1}{\Theta^A} \left\{ \nu^A q^{Ak} - \mu^A J^{Am} h^{Ak}_m + u^m t^{Ak}_m \right\} \right\} + G^{Ak} = \]
\[ = \frac{1}{\Theta^A} \left\{ \nu^A \left( e^A u^{Ak} + q^{Ak} \right) + u^m \left( u^{Ak} p^{Am} + t^{Ak}_m \right) \right\} + \frac{\mu^A}{\Theta^A} N^{Ak} + \frac{\nu^A}{\Theta^A} p^A u^{Ak} - \frac{\mu^A}{\Theta^A} J^{Am} h^{Ak}_m + G^{Ak}. \]  

(122)

Inserting (38) and (19)\footnote{2}, (122) results in

\[ S^{Ak} = \frac{1}{\Theta^A} \left\{ \nu^A Q^{Ak} + u^m r^{Ak}_m \right\} + \frac{\mu^A}{\Theta^A} \left( (1 + w^A) N^{Ak} - J^{Ak} \right) + \frac{\nu^A}{\Theta^A} p^A u^{Ak} + G^{Ak}. \]  

(123)

The transition from the interacting \(A\)-component to the free 1-component system is considered in sect.6 and that to the mixture in sect.7. All quantities introduced up to here are non-equilibrium ones, because we did not consider equilibrium conditions up to now. This will be done in the next section.

5.6 Equilibrium

5.6.1 Equilibrium conditions

Equilibrium is defined by equilibrium conditions which are divided into basic and supplementary ones\footnote{9, 10}. The basic equilibrium conditions are given by vanishing entropy
production, vanishing entropy flux density and vanishing entropy supply.

\[ \sigma_{eq}^A = 0 \quad \land \quad \dot{s}_{eq}^A = 0 \quad \land \quad \varphi_{eq}^A = 0. \]  

(124)

A first supplementary equilibrium condition is the vanishing of all diffusion flux densities. According to [12], we obtain

\[ J_k^{Aeq} = 0 \quad \rightarrow \quad u_k^{Aeq} = f_{eq}^A u_k^{eq} \quad \rightarrow \quad c^2 = f_{eq}^A u_k^{eq} u_k^{Aeq}. \]  

(125)

Taking (8)\textsubscript{1} into account, (125)\textsubscript{3} results in

\[ (f_{eq}^A)^2 = 1 \quad \rightarrow \quad f_{eq}^A = \pm 1. \]  

(126)

Consequently, we have to demand beyond (125)\textsubscript{1} the supplementary equilibrium condition that the mass densities are additive in equilibrium. We obtain according to (8)\textsubscript{2} and (14)\textsubscript{2}

\[ \varrho_{eq} = \sum_A \varrho_{eq}^A \rightarrow f_{eq}^A = 1 \quad \rightarrow \quad w_{eq}^A = 0. \]  

(127)

Taking (125)\textsubscript{2} and (103) into account, (127)\textsubscript{2} yields

\[ u_k^{Aeq} = u_k^{eq} \rightarrow \lambda_{eq}^A = 0, \quad g_{eq}^{Am} = 0. \]  

(128)

Further supplementary equilibrium conditions are given by vanishing covariant time derivatives, except that of the four-velocity:

\[ \mathbf{\Theta}_{eq}^A = 0, \quad \mathbf{\Theta} \neq u^l, \]  

(129)

that means \( u_{eq}^l \) is in general not zero in equilibrium. Consequently, the time derivatives of all expressions which contain the 4-velocity must be calculated separately, as we will see below.

According to (129)\textsubscript{1}, we obtain

\[ \dot{q}_{eq}^A = 0, \quad \left( \frac{\nu^A}{\Theta^A} \right)_{eq} = 0, \]  

(130)

and the (3+1)-components of the energy-momentum tensor, (26) and (27), satisfy

\[ \dot{e}_{eq}^A = 0, \quad \dot{p}_{eq}^{Al} = 0, \quad \dot{q}_{eq}^{Ak} = 0, \quad \dot{p}_{eq}^A = 0, \quad \dot{\pi}_{eq}^{Akl} = 0. \]  

(131)

Starting out with (8)\textsubscript{1}, we have

\[ \dot{f}_{eq}^A = \frac{1}{c^2} \left( \dot{u}_{eq}^A u_{eq}^m u_{eq}^m + u_{eq}^A \dot{u}_{eq}^m u_{eq}^m \right). \]  

(132)

Taking (128)\textsubscript{1} into account, this results in

\[ \dot{f}_{eq}^A = 0 \quad \rightarrow \quad \dot{w}_{eq}^A = 0. \]  

(133)

\textsuperscript{16}The sign \( \doteq \) stands for a setting which implements an equilibrium condition.
In equilibrium, we have according to (128) and (16)

\[ h_{eq}^{Am} = h_{eq}^{m}, \]  

(134)

according to (103) resulting in

\[ \lambda^{Aeq}_{l,k} = \left( \frac{1}{\Theta^{A}_{eq}} u_m h_{eq}^{Am} \right)_{,k} = 0. \]  

(135)

Consequently, the time derivatives of the accessory variables vanish in equilibrium, and according to (85) and (86), Gibbs and Gibbs–Duhem equations are identically satisfied in equilibrium.

Another supplementary equilibrium condition is the vanishing of the mass production terms in (103)

\[ (ex) \Gamma^{A}_{eq} = 0 \land (in) \Gamma^{A}_{eq} = 0 \rightarrow \Gamma^{A}_{eq} = 0 \]  

(136)

Thus, we obtain from (130) and (136)

\[ \varrho^{A}_{,k} u^{A}_{k} + \varrho^{A}_{,} u^{A}_{k} = \Gamma^{A} \rightarrow u^{A}_{eq,k} = 0. \]  

(137)

Taking (127), (128) and (26) into account, the entropy density (118) becomes in equilibrium

\[ s^{A}_{eq} = \frac{1}{\Theta^{A}_{eq}} \left( \mu^{A}_{eq} + \nu^{A}_{eq} (e^{A}_{eq} + p^{A}_{eq}) \right). \]  

(138)

Except of \( \nu^{A}_{eq} \), this is the usual expression for the entropy density in thermostatics\(^{17}\).

The energy density and the pressure are here defined by the (3+1)-decomposition (25) of the energy-momentum tensor. The chemical potential is as well as the temperature introduced as an accessory variable.

The equilibrium temperature in (138) is characterized by vanishing multi-heat relaxation

\[ \Theta^{A}_{eq} = \Theta^{B}_{eq} = \Theta^{C}_{eq} = ... =: \Theta_{eq}. \]  

(139)

Often one can find in literature\(^{20}\) the case of equilibrium of multi-heat relaxation: although out of equilibrium, only one temperature is considered in multi-component systems. This case is realistic, if the relaxation of multi-heat relaxation to equilibrium is remarkably faster than that of the other non-equilibrium variables\(^{21}\). As demonstrated, the non-equilibrium entropy density (118) does not depend on multi-heat relaxation, a reason why often equilibrium of multi-heat relaxation is presupposed without any remark.

Taking (124), (125) and (128) into account, the entropy flux density (119) vanishes in equilibrium

\[ 0 = q^{A}_{eq}, \]  

(140)

and finally taking (136), (124), (127), (125) and (128) into account, the entropy supply (120) results in

\[ 0 = u^{A}_{eq}_{l,k}. \]  

(141)

\(^{17}\)Below, we will see that \( \nu^{A}_{eq} = 1 \)
that means, the power exchange vanishes in equilibrium.

The entropy production (121) has to vanish in equilibrium according to the basic equilibrium condition (124). Taking (136), (125), (140), (131), (128) and (139) into account and using (129), (121) results in

$$0 = u^{Aeq}_{l,k} n^{Akl}_{eq}.$$  \hfill (142)

As demonstrated, equilibrium of an \( A \)-component in the mixture is described by three basic equilibrium conditions (124) and six supplementary ones: (125), (127), (129), (136), (131), (128) and (139). Often, we can find in literature [22, 23] equilibrium conditions which are different from those postulated here. The reason for that is, that entropy production and supply and the entropy flux as starting-points for the basic equilibrium conditions differ from the expressions (118) to (121). Such different equilibrium conditions and their derivations are considered in the next two sections.

### 5.6.2 Killing relation of the 4-temperature

Starting out with (77), we now consider the following relations

\[
\begin{align*}
(\lambda^A_{,iu}^A + \lambda^A_{,iu}^A) \frac{1}{c^2} e^A u^A u^A & = \dot{\lambda}^A e^A, \\
(\lambda^A_{,iu}^A + \lambda^A_{,iu}^A) \frac{1}{c^2} e^A u^A u^A & = -\lambda^A_{,iu}^A p^A, \\
-\lambda^A_{,iu}^A + \lambda^A_{,iu}^A & = \lambda^A_{,iu}^A p^A, \\
-\lambda^A_{,iu}^A + \lambda^A_{,iu}^A & = \lambda^A_{,iu}^A n^{Akl}, \\
-\lambda^A_{,iu}^A + \lambda^A_{,iu}^A & = \lambda^A_{,iu}^A n^{Akl}.
\end{align*}
\]

Taking (144), (145) and (147) into account, we obtain from (77)

\[
(\lambda^A_{,iu}^A)_{,k} \left( T^{Akl} \frac{1}{c^2} e^A u^A u^A + p^A n^{Akl} \right) = \lambda^A_{,iu}^A p^A + \lambda^A_{,iu}^A n^{Akl} + \lambda^A_{,iu}^A n^{Akl} + \lambda^A_{,iu}^A n^{Akl}.
\]  \hfill (148)

Replacing the second row of (121) by (148) yields for the entropy production of vanishing multi-heat relaxation

\[
\sigma^A = \lambda^A \mu^A \left( (in) \Gamma^A - (J^{Am} h^{A}_{m}, k) - (\lambda^A_{,iu}^A)_{,k} J^{Am} h^{A}_{m} + \right. \\
\left. + \left( \lambda^A_{,iu}^A \right)_{,k} \left( T^{Akl} \frac{1}{c^2} e^A u^A u^A + p^A n^{Akl} \right) + \right) \\
\left. + \left( \frac{1}{\Theta^A} u^A h^{A}_{m}, k \right) \left( \frac{1}{c^2} e^A u^A u^A + \frac{1}{c^2} q^A u^A + t^A \right). \right)
\]  \hfill (149)

Evident is that

\[
(\lambda^A_{,iu}^A)_{,k} \left( T^{Akl} \frac{1}{c^2} e^A u^A u^A + p^A n^{Akl} \right) = 0.
\]  \hfill (150)
is not a sufficient condition for equilibrium because the equilibrium conditions (136), (125), and (128) are not necessarily satisfied and the entropy production (149) does not vanish.

If the energy-momentum tensor is symmetric, (150) results in

\[ T^{Akl} = T^{Alk} : \left( \lambda^A u^A_l \right)_k + (\lambda^A u^A_k)_l \left( T^{Akl} - \frac{1}{c^2} e^A u^A k u^A l + p^A h^{Akl} \right) = 0, \]  \hspace{1cm} (151)

an expression which as well as (150) is not sufficient for equilibrium. Consequently, the Killing relation of the 4-temperature \( \lambda^A u^A_l \)

\[ \left( \lambda^A u^A_l \right)_k + (\lambda^A u^A_k)_l = 0 \]  \hspace{1cm} (152)

is also not sufficient for equilibrium. If equilibrium is presupposed, the equilibrium conditions (136), (125), and (128) are satisfied, the entropy production vanishes and

\[ (\lambda^A u^A_l)^{eq} \left( T^{Akl} - \frac{1}{c^2} e^A u^A k u^A l + p^A h^{Akl} \right)^{eq} = 0, \]  \hspace{1cm} (153)

\[ T^{Akl}^{eq} = T^{Alk}^{eq} : \left( (\lambda^A u^A_l)^{eq} \right)_k + (\lambda^A u^A_k)^{eq}_l \left( T^{Akl}^{eq} - \frac{1}{c^2} e^A u^A k u^A l + p^A h^{Akl} \right)^{eq} = 0 \]  \hspace{1cm} (154)

are necessary conditions for equilibrium according to (149). There are different possibilities to satisfy (153) and (154) which are discussed in the next section.

5.6.3 The gradient of the 4-temperature

The necessary condition for equilibrium (153) can be differently satisfied generating different types of equilibria. There are three possibilities:

If equilibrium exists, one of the following three conditions is valid:

\[ (\lambda^A u^A_l)^{eq} \left( \frac{1}{c^2} e^A u^A k u^A l - p^A h^{Akl} \right)^{eq} = 0, \]  \hspace{1cm} (155)

\[ T^{Akl}^{eq} = \frac{1}{c^2} e^A u^A k u^A l - p^A h^{Akl} \]  \hspace{1cm} (156)

\[ (\lambda^A u^A_l)^{eq} \neq 0 \land \left( T^{Akl}^{eq} \neq \frac{1}{c^2} e^A u^A k u^A l - p^A h^{Akl} \right), \]  \hspace{1cm} (157)

Multiplier of (155) with \( u^A l \) results in

\[ \lambda^A_{eq} = 0 \land u^A_{l eq} = 0, \]  \hspace{1cm} (158)

that means, (155) represents an intensified equilibrium because additionally to the usual equilibrium conditions mentioned in sect 5.6 (158) is valid.

If (156) is valid, the equilibrium exists in a perfect material whose entropy production is zero. If the considered material is not perfect and if the equilibrium is not intensified,
is valid, and the question arises, whether \((153)\) can be valid under these constraints. To answer this question, we consider \((143)\) to \((147)\) in equilibrium. According to the equilibrium conditions, we obtain

\[
\begin{align*}
(\lambda^A_{,k} u^A_{l} + \lambda^A u^A_{l,k}) \frac{1}{c^2} A^A_{eq} u^A_{eq} &= 0, \\
(\lambda^A_{,k} u^A_{l} + \lambda^A u^A_{l,k}) \frac{1}{c^2} A^A_{eq} p^A_{eq} &= 0, \\
(\lambda^A_{,k} u^A_{l} + \lambda^A u^A_{l,k}) \frac{1}{c^2} A^A_{eq} u^A_{eq} &= 0, \\
-(\lambda^A_{,k} u^A_{l} + \lambda^A u^A_{l,k}) \frac{1}{c^2} A^A_{eq} u^A_{eq} &= 0, \\
(\lambda^A_{,k} u^A_{l} + \lambda^A u^A_{l,k}) \frac{1}{c^2} A^A_{eq} u^A_{eq} &= 0.
\end{align*}
\]

Summing up \((159)\) to \((163)\) yields

\[
(\lambda^A u^A_{l})_{,k} T^A_{eq} = 0.
\]

Consequently, \((153)\) is satisfied because each of the three terms vanishes for its own, thus being compatible with \((157)\). If an \(A\)-component of a mixture is in equilibrium, two types of equilibria can occur: one in an arbitrary material showing the usual equilibrium conditions and another one which has beyond the usual equilibrium conditions vanishing temperature gradient and vanishing 4-velocity gradient according to \((158)\).

Evident is that an 1-component system which does not interact with other components is as a special case included in the theory of an \(A\)-component in the mixture. This case is discussed in the next section.

### 6 Special Case: Thermodynamics of 1-Components

#### 6.1 Entropy flux, -supply and -density

An 1-component system\(^{20}\) can be described by setting equal all component indices of a multi-component system

\[
A, B, C, ..., Z \rightarrow 0,
\]

and for shortness, we omit this common index 0. Then the basic fields of an 1-component system are according to \((2)\)

rest mass density and 4-velocity: \(\{\varrho, u_k\}\).

The equations \((3)\) of Setting I change into identities. According to \((5)_1\), \((12)_1\), \((14)_2\) and \((22)\), we have

\[
f = 1, \quad J_k = 0, \quad w = 0, \quad T^{kl}_{,k} = k^l.
\]

The accessory variables become according to \((96)\), \((100)\) and \((103)\)

\[
\lambda = \frac{\nu}{\Theta}, \quad \kappa = \frac{\mu}{\Theta}, \quad \lambda^k = 0.
\]

\(^{20}\)that is not a mixture which is a multi-component system by definition
The entropy density (118) and the state space (84) are as in equilibrium of the $A$-component (138)
\[ s = \frac{1}{\Theta} \left( \mu \rho + \nu (e + p) \right), \quad z = (\rho, e). \] (169)
The entropy flux (119), the entropy supply (120) and the entropy production (149) are
\[ s^k = \frac{\nu}{\Theta} q^k, \quad \varphi = \frac{1}{\Theta} \left( \mu (ex) \Gamma + \nu u_k k^l \right), \quad \sigma = \left( \frac{\nu}{\Theta} u_l \right)_k \left( T^{kl} - \frac{1}{c^2} e u^k u^l + ph^{kl} \right). \] (170)
According to sect.3.4, the (3+1)-components of the mixture change into the corresponding quantities of the 1-component system. The necessary equilibrium conditions of a 1-component system are equal to those of an $A$-component in the mixture.

6.2 Equilibrium and reversibility

Considering an 1-component system, (150) results in
\[ (\lambda u_l)_k \left( T^{kl} - \frac{1}{c^2} e u^k u^l + ph^{kl} \right) = 0. \] (171)
In contrast to (150), (171) is sufficient and necessary for vanishing entropy production in 1-component systems according to (149) and (170). But concerning equilibrium, (171) is only necessary but not sufficient for it, because further equilibrium conditions according to (170) may not be satisfied. Vanishing entropy production out of equilibrium belongs to reversible processes and vice versa. Consequently, reversible processes in 1-component systems satisfy
\[ \sigma^{rev} = (\lambda u_l)^{rev}_k \left( T^{kl} - \frac{1}{c^2} e u^k u^l + ph^{kl} \right)^{rev} = 0 \] (173)
which has the same structure as (153) in case of equilibrium. Thus, all results of sect.5.6.3 change into those of an 1-component system, if the component index $A$ is omitted, $eq$ is replaced by $rev$, equilibrium is not presupposed and the generated expressions belong to reversible processes and vice versa. Consequently, the derivative of the 4-temperature and the Killing relation of the 4-temperature
\[ (\lambda u_l)^{rev}_k = 0, \quad or \quad T^{kl} = T^{lk} : \left( (\lambda u_l)_k + (\lambda u_k)_l \right)^{rev} = 0 \] (174)
are rather conditions for reversible processes in 1-component systems because the entropy production is enforced to be zero without existing equilibrium. Summarized: (171) is necessary for equilibrium and additionally sufficient and necessary for reversible processes in 1-component systems. Independently of the 4-temperature, we obtain the well-known fact [24] that all processes of perfect materials are reversible in 1-component systems according to (170) and
\[ T^{kl}_{per} := \frac{1}{c^2} e u^k u^l - ph^{kl} \to \sigma_{per} = 0. \] (175)

---

21 There are no chemical reactions in 1-component systems.
22 In equilibrium: "all cats are grey"
7 Thermodynamics of a Mixture

According to the mixture axiom in sect. 2.2, the balance equations of a mixture look like those of a 1-component system. But a mixture as a whole behaves differently from the interacting $^A$-component in the mixture and also differently from an 1-component system which both were discussed in sect. 5 and sect. 6. Because the interaction between the components is still existing in the mixture, the diffusion fluxes and also the multi-heat relaxation do not vanish as in 1-component systems. Because component indices $^A$ do not appear in the description of mixtures, they are summed up in contrast to 1-component systems for which they vanish. The Settings I and II enforce the mixture axiom resulting in

\begin{align}
\text{mass balance:} & \quad N_{^A}^{Ak} \rightarrow N_{^A}^{k,k} = \Gamma, \\
\text{energy balance:} & \quad u_{^A}^l T_{^A}^{Ak},k \rightarrow u_{^A}^l T_{^A}^{kl},k = \Omega, \\
\text{momentum balance:} & \quad h_{^A}^m T_{^A}^{Ak},k \rightarrow h_{^A}^m T_{^A}^{kl},k = \Omega_{^A}^m, \tag{176}
\end{align}

According to (62), (63) and (22), we obtain

\begin{align}
 u_{^A}^l k^l = \sum_{^A} \left\{ h_{^A}^m u_{^A}^l + f_{^A}^l u_{^A}^l \right\} k^l, & \quad h_{^A}^p k^l = \sum_{^A} \left\{ h_{^A}^m h_{^A}^p + g_{^A}^p u_{^A}^l \right\} k^l. \tag{179}
\end{align}

According to \([62], [63]\) and \([22]\), we obtain

\begin{align}
 S_{^A}^{Ak} = \sum_{^A} \left\{ \left( h_{^A}^m u_{^A}^l + f_{^A}^l u_{^A}^l \right) k^l \right\}, & \quad S_{^A}^{Ak} = \sum_{^A} \left\{ \left( h_{^A}^m h_{^A}^p + g_{^A}^p u_{^A}^l \right) k^l \right\}. \tag{180}
\end{align}

Consequently, the mixture axiom –represented by (176) to (178)– does not demand in contrast to the non-relativistic case the additivity of the balances of the $^A$-components.

Setting III to Setting VIII are related to the $^A$-component itself, whereas Setting I and II describe additivity properties in the set of the $^A$-components. Both kinds of setting do not contain the entropy of the mixture. Obviously, we need an additional setting concerning the entropy of the mixture which will be formulated in the next section.

7.1 Additivity of 4-entropies

7.1.1 Entropy density and -flux

Like the additivity of the mass flux densities \([51]\) and the energy-momentum tensors \([39]\) of the $^A$-components, we demand that also of the 4-entropies are additive

\begin{align}
\text{Setting X:} & \quad S^k = \sum_{^A} S^{Ak}. \tag{180}
\end{align}

Consequently, we obtain from (123)

\begin{align}
 S^k = \sum_{^A} \left\{ \frac{1}{\Theta_{^A}} \left( \nu_{^A}^l Q_{^A}^{Ak} + u_{^A} m T_{^A}^{Ak,m} \right) + \frac{\mu_{^A}}{\Theta_{^A}} \left( 1 + w_{^A} \right) N_{^A}^{Ak} - J_{^A}^{Ak} \right\} + \frac{\nu_{^A}}{\Theta_{^A}} P_{^A} u_{^A}^{Ak} + G_{^A}^{Ak}. \tag{181}
\end{align}
According to (69), we obtain the entropy density and the entropy flux density of the mixture by use of (13), (8) and (17)

\[ S^k u_k = c^2 s = \sum_A \left\{ \frac{1}{\Theta A} \left( \nu^A Q^A k + u_p A k \right) u_k + \right. \]
\[ \left. + \frac{\mu^A}{\Theta A} \left( 1 + w^A \right) f^A f^A + \frac{\nu^A}{\Theta A} p^A c^2 f^A + G^A k u_k \right\}, \quad (182) \]

\[ S^k h_m^m = s^m = \sum_A \left\{ \frac{1}{\Theta A} \left( \nu^A Q^A k + u_p A k \right) h_m^m + \right. \]
\[ \left. + \frac{\mu^A}{\Theta A} w^A J^m + \frac{\nu^A}{\Theta A} p^A c^2 g^A m + G^A k h_m^m \right\}. \quad (183) \]

Taking (40) into consideration, we introduce by comparing with (182) and (183) the Setting XI:

\[ \nu^A \cdot = f^A. \quad (184) \]

With this setting, the expressions of the entropy density and the entropy flux of the mixture correspond to those which are generated by the additivity of the energy-momentum tensors: (45) to (52).

Finally, we obtain the entropy and entropy flux density of the mixture

\[ s = \sum_A \left\{ \frac{1}{c^2} \frac{1}{\Theta A} \left( f^A Q^A k + u_p A k \right) u_k + \right. \]
\[ \left. + \frac{\mu^A}{\Theta A} \left( 1 + w^A \right) f^A f^A + \frac{f^A}{\Theta A} p^A f^A + \frac{1}{c^2} G^A k u_k \right\}, \quad (185) \]

\[ s^m = \sum_A \left\{ \frac{1}{\Theta A} \left( f^A Q^A k + u_p A k \right) h_m^m + \right. \]
\[ \left. + \frac{\mu^A}{\Theta A} w^A J^m + \frac{f^A}{\Theta A} p^A c^2 g^A m + G^A k h_m^m \right\}. \quad (186) \]

These expressions of the entropy and entropy flux densities of the mixture are direct results of Setting X (180). They will be discussed in sect 7.2

7.1.2 Entropy production density and -supply

From (180) and (75) follows the entropy balance equations of the mixture

\[ S^k, k = \sum_A S^A k, k = \sum_A \left( \sigma^A + \varphi^A \right) = \varphi + \varphi, \quad (187) \]

satisfying the mixture axiom. Accepting the additivity of the entropy supplies of the \( A \)-component,\(^\text{23}\)

Setting XII:

\[ \varphi \cdot = \sum_A \varphi^A. \quad (188) \]

\(^\text{23}\)Supplies are caused by external influences, productions by internal ones. That the reason why they do not mix up.
we obtain the additivity of the entropy productions of the $A$-components
\[ \sigma = \sum_A \sigma^A. \] (189)

We obtain the entropy supply of the mixture from (120), (188) and (184)
\[ \varphi = \sum_A \left\{ \frac{1}{\Theta_A} \left( \mu^{A(ex)} A - f^A u^k A^k + u^p h^p_m A^m \right) + \sum_B H^{ABk} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right) \right\}. \] (190)

The entropy production of the mixture follows from (121), (189) and (184)
\[ \sigma = \sum_A \left\{ \frac{\mu^A}{\Theta_A} \left( (in) A - \left( J^{Am} h^A_m \right)_k \right) - \frac{\mu^A}{\Theta_A} J^{Am} h^A_m + \left( \frac{f^A}{\Theta_A} \right)_k q^{Ak} + \frac{f^A}{\Theta_A} u^A_{1,k} \left( \pi^{Akl} + u^{Ak} p^A \right) + \sum_B H^{ABk} \left( \frac{1}{\Theta_A} - \frac{1}{\Theta_B} \right)_k + \right\} \] (191)

7.2 Entropy, entropy flux densities and partial temperatures

We now consider the two expressions
\[ \frac{1}{c^2 u_k} \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) \] and \[ h^m_k \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) \] (192)

which appear in the entropy density (182) and in the entropy flux density (183). A comparison with (45) and (47)
\[ \frac{1}{c^2 u_k} \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) = e, \] (193)
\[ h^m_k \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) = q^m \] (194)

results in the fact, that the partial temperatures $\Theta^A$ in (192) prevent from introducing the mixture quantities energy density (193) and the energy flux density (194) into the entropy density (185) and the entropy flux density (186) of the mixture.

Beyond that, the temperature of the mixture $\Theta$ cannot be defined properly: even the mean value theorem does not help
\[ \frac{1}{c^2 u_k} \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) \equiv \frac{1}{c^2 \Theta} u_k \sum_A \left( Q^{Ak} f^A + u p^A \right) = \frac{1}{\Theta} e, \] (195)
\[ h^m_k \sum_A \frac{1}{\Theta_A} \left( Q^{Ak} f^A + u p^A \right) \equiv \frac{1}{\Theta(m)} h^m_k \sum_A \left( Q^{Ak} f^A + u p^A \right) = \frac{1}{\Theta(m)} q^m. \] (196)
In the first case (195), it is not for sure that the presupposition
\[ u_k\left(Q^{Ak}f^A + u_p\tau^{Ak}\right) \text{ definite for all } A \] (197)
is satisfied which is sufficient for the validity of the mean value theorem. The second case (196) results in mean values which may depend on the tensor component index, quantities which are useless for defining a common temperature.

There exists another argumentation that a temperature of a mixture cannot be defined properly: the partial temperatures are internal contact variables [25] measured by thermometers which are selective for the temperature \(\Theta^A\) of the corresponding \(A\)-component. Evident is, that the temperature of the mixture is a certain mean value of the partial temperatures of the components of the mixture [26, 27, 28]

\[ \Theta = \sum_B \alpha_B \Theta^B, \quad \alpha_B \geq 0, \quad \sum_B \alpha^B = 1. \] (198)

But this mean value depends on the selectivity of the used thermometer, and consequently the measured temperature is not unequivocal. Different definitions of the mixture temperature can be found in literature [29]. But a unique mixture temperature –independent of thermometer selectivities or arbitrary definitions– is given in the case of multi-heat relaxation equilibrium (139). This case is silently presupposed, if only one temperature is used in multi-component non-equilibrium systems. Only this sure case is considered in the sequel.

### 7.3 Multi-heat relaxation equilibrium

#### 7.3.1 Entropy and entropy flux densities

Presupposing multi-heat relaxation equilibrium (139),

\[ G^A_{eq} = 0 \] (199)

follows according to (108), and the expressions (192) transfer according to (193) and (194) to

\[ \frac{1}{c^2} u_k \sum_A \frac{1}{\Theta^A} \left(Q^{Ak}f^A + u_p\tau^{Ak}\right) = \frac{1}{c\Theta} \Theta^e, \] (200)

\[ h_k^m \sum_A \frac{1}{\Theta^A} \left(Q^{Ak}f^A + u_p\tau^{Ak}\right) = \frac{1}{c\Theta} \Theta^q_m. \] (201)

Entropy density (185) and entropy flux density (186) become

\[ s = \frac{1}{\Theta} \Theta^e + \frac{1}{\Theta} \sum_A \left\{ \mu^A g^A \left(1 + w^A\right) + p^A f^A \right\} f^A, \] (202)

\[ s^m = \frac{1}{\Theta} \Theta^q_m + \frac{1}{\Theta} \sum_A \left\{ \mu^A w^A f^Am + f^A p^A c^2 g^Am \right\}. \] (203)
The first term in the sum of (202) can be exploited by use of the mean value theorem because in contrast to (197), the presupposition for its validity is here satisfied according to (8)

\[ \sum_A \mu^A f^A = \frac{\partial}{\partial \bar{\varrho}} \sum_A f^A \varrho^A = \varrho \mu, \quad (204) \]

Consequently, the chemical potential of the mixture is

\[ \varrho \mu := \sum_A \mu^A \varrho^A / \bar{\varrho}, \quad (205) \]

and the entropy density of the mixture yields

\[ s = \frac{1}{\vartheta} \varepsilon + \frac{1}{\vartheta} \varrho \mu + \frac{1}{\vartheta} \sum_A \left\{ \mu^A \varrho^A w^A f^A + p^A (f^A)^2 \right\}, \quad (206) \]

Introducing an effective "pressure"

\[ p^{\text{eff}} := \sum_A \left\{ \mu^A \varrho^A w^A f^A + p^A (f^A)^2 \right\}, \quad (207) \]

which is different from \( p \) in (53). The entropy density of the mixture in multi-heat relaxation equilibrium has the usual shape (169) as for an 1-component system

\[ s = \frac{1}{\vartheta} \left( \varrho \mu + e + p^{\text{eff}} \right). \quad (208) \]

We now consider the equilibrium values of \( p \) and \( p^{\text{eff}} \). Starting with (53) and taking (134) and (127) into account, we obtain

\[ p^{\text{eq}} = -\frac{1}{3} \sum_A t^{Akl} h^{A}_{kl} = \sum_A p^A = p^{\text{eff}} \quad (209) \]

that means, in equilibrium appears the pressure of the mixture in its entropy density, whereas in non-equilibrium this pressure is replaced by an effective quantity which has the physical dimension of a pressure, but which is different from it.

### 7.3.2 Entropy production, density and supply

In multi-heat relaxation equilibrium, entropy supply (190) and entropy production density (191) of the mixture become

\[ \varphi = \frac{1}{\vartheta} \sum_A \mu^A \Gamma^A + f^A u^A k^A + u^A p_{\lambda m k}^A + \left( \mu^A \varrho^A w^A f^A + p^A (f^A)^2 \right), \quad (210) \]

\[ \bar{\varphi} = \sum_A \left\{ \frac{\mu^A}{\vartheta} \left( \Gamma^A - \left( J^A m^A_{\lambda m} \right)_{k} \right) - \left( \frac{\mu^A}{\vartheta} \right)_{k} J^A m^A_{\lambda m} + \left( \frac{f^A}{\vartheta} \right)_{k} q^A_{\lambda k} + \frac{f^A}{\vartheta} u^A l_{\lambda k} \left( u^A f^A_{\lambda k} + u^A p^A_{\lambda k} \right) + \left( \frac{1}{c^2} u^A l_{\lambda m} \right)_{k} \left( \frac{1}{c^2} u^A q^A_{\lambda m} + \frac{1}{c^2} q^A_{\lambda m} f^A_{\lambda m} + \pi^{Akl} \right) \right\}. \quad (211) \]

\[ ^{24} \varphi \text{ does not depend on } \varrho^A \text{ and } \mu^A, \text{ whereas } p^{\text{eff}} \text{ is independent of } e^A, q^A_{\lambda k}, p^A_{\lambda k}. \]

\[ ^{25} \nu = f = 1 \]
The meaning of each individual term of (211) was already discussed with regard to the \( A \)-component according to (121).

Entropy density (208), entropy flux density (203), entropy supply (210), entropy production density (211) and chemical potential (205) of the mixture are represented by sums of quantities of the \( A \)-components. As expected, the \((3+1)\)-components of the energy-momentum tensors cannot represent the mentioned thermodynamical quantities because diffusion fluxes, chemical potentials and temperature are not included in the energy-momentum tensor. From them, only the energy density \( e \) and the energy flux density \( q^m \) of the mixture appear in entropy and entropy flux densities. All other thermodynamical quantities, such as the chemical potential \( \mu \) (205) and the effective pressure \( \rho^{\text{eff}} \) (207) are composed of \( A \)-component quantities which do not belong to the energy-momentum tensor. A temperature \( \Theta \) of the mixture exists in multi-heat relaxation equilibrium by definition, but it is not composed by the partial temperatures \( \Theta^A \) of the \( A \)-components but rather by the component sensitivity of a thermometer.

7.4 Equilibrium

Evident is that the equilibrium conditions of a mixture follow from those of the \( A \)-components which we considered in sect.5.6. Consequently, to demand additional equilibrium conditions for mixtures is not necessary. Starting with (199), we obtain with (124), (128)\(_1\), (127)\(_2\) and (138)\(_{26}\)

\[
\sum_{\mu} s_{eq}^m = 0, \quad s_{eq} = \sum_A s^A_{eq} = \sum_A \lambda^A_{eq} \left( \mu^A_{eq} \rho^A_{eq} + e^A_{eq} + p^A_{eq} \right) \tag{212}
\]

According to (188), (189) and (124)\(_{1,3}\), we obtain

\[
^o \varphi_{eq} = 0, \quad ^o \sigma_{eq} = 0. \tag{213}
\]

From (48) follows with (128)\(_1\) and (212)\(_1\)

\[
q^m_{eq} = 0. \tag{214}
\]

From (46), (50) and (52) follows with (128)\(_1\) and (131)

\[
e_{eq} = \sum_A e^A_{eq}, \quad p^A_{eq} = \sum_A \rho^A_{eq}, \quad t^{jm}_{eq} = \sum_A t^{AJm}_{eq}. \tag{215}
\]

According to (10)\(_3\), (3) and (136), we obtain

\[
^{(ex)} \Gamma_{eq} = 0, \quad ^{(in)} \Gamma_{eq} = 0, \quad \Gamma_{eq} = 0. \tag{216}
\]

From (4) follows in equilibrium analogously to (137)

\[
u^k_{eq,k} = 0. \tag{217}
\]

The vanishing entropy supply of the mixture (213)\(_1\) is according to (210) satisfied by the equilibrium conditions (136)\(_1\), (141) and (131). The vanishing entropy production of the mixture (213)\(_2\) is according to (211) satisfied by the equilibrium conditions (136)\(_2\), (125)\(_1\), (140), (142), (129) and (134).

\[\nu_{eq} ^A = f^A_{eq} = 1\]
8 Verification: The non-relativistic case

The non-relativistic case\(^{27}\) is defined by the formal setting\(^{28}\)

\[
\frac{v}{c} \approx 0, \quad \text{or more formally: } c \to \infty
\]

which generates the non-relativistic Newtonian shape of the thermodynamical expressions of sect. 7.3.

The approximation\(^{218}\) represents the zeroth step of developing the relativistic square-root

\[
\sqrt{1 - \left(\frac{v}{c}\right)^2} \approx 1. \quad (219)
\]

The energy-momentum tensor\(^{25}\) and the 4-entropy\(^{68}\) are chosen in such a way that their \((3+1)\)-components are all of the same order of approximation, namely of zeroth order. Consequently, a “book keeping”\(^{31,32,33}\) fixing different conditions of approximation orders for these \((3+1)\)-components is not required in the case of zeroth approximation generating the Newtonian limit of continuum thermodynamics.

8.1 Velocity and mass balance

The norm of the 4-velocity \((k=1,...,4; \alpha=1,...,3)\)

\[
u^k u_k = u^\alpha u_\alpha + u^4 u_4 = u_m g^{mk} u_k = u_\alpha g^{\alpha \beta} u_\beta + u_4 g^{4 \beta} u_\beta + u_4 g^{44} u_4. \quad (220)
\]

can be decomposed into its space-like, time-like and mixed parts \((\alpha = 1, 2, 3)\). We now introduce two frames having a special Galilean metric\(^{29}\)

\[
g^{\alpha \beta} \equiv \mp \delta^{\alpha \beta}, \quad g^{44} \equiv \pm 1, \quad g^{4 \alpha} \equiv g^{\alpha 4} \equiv 0, \quad \to \quad \text{signature}(g^{mk}) = \mp 2. \quad (221)
\]

The corresponding parts of the 4-velocity are chosen as\(^{30}\)

\[
u^\alpha \equiv \frac{\nu^\alpha}{\sqrt{1 - (v/c)^2}} , \quad \nu_\alpha \equiv \mp \frac{\nu_\alpha}{\sqrt{1 - (v/c)^2}} \quad (222)
\]

\[
u^4 \equiv \frac{c}{\sqrt{1 - (v/c)^2}} , \quad \nu_4 \equiv \pm \frac{c}{\sqrt{1 - (v/c)^2}} \quad (223)
\]

Consequently,\(^{220}\) results in

\[
u^k u_k = u^\alpha u_\alpha + u^4 u_4 \equiv \mp \frac{\nu \cdot \nu}{1 - v^2/c^2} \pm \frac{c^2}{1 - v^2/c^2} = \pm c^2. \quad \sqrt{224}
\]

\(^{27}\)Do not take this case for the post-Newtonian limit. Here we are not interested in relativistic corrections of the non-relativistic equations as done in\(^{7,30,31}\). Here we need the non-relativistic equations for a verification of the relativistic ones.

\(^{28}\)\(\approx\) characterizes the approximation step which results in the non-relativistic Newtonian limit.

\(^{29}\)\(\equiv\) marks the introduction of Galilean co-ordinates (which may have different signature.

\(^{30}\)a component index \(A\) is suppressed
By taking
\[ \frac{\partial}{\partial t}(1 - (v/c)^2)^{-1/2} = -\frac{1}{2}(1 - (v/c)^2)^{-3/2} \frac{\partial t}{c} \approx 0 \] (225)
to account, the relativistic mass balance changes into the usual non-relativistic one
\[ N^{k,k} = (\rho u^k)_{,k} = (\rho u^\alpha)_{,\alpha} + (\rho u^4)_{,4} = \nabla \cdot \left( \frac{\rho v}{\sqrt{1 - (v/c)^2}} \right) + \frac{1}{c} \frac{\partial}{\partial t} \left( \frac{\rho c}{\sqrt{1 - (v/c)^2}} \right) \approx \nabla \cdot (\rho v) + \partial_t (\rho) = \Gamma \approx \Gamma. \] (226)

### 8.2 The momentum balance

Because the energy-momentum tensors of the mixture (39), of one component in the mixture (37) and of a 1-component system have the same shape, we can start with (37) and (38) without using the component index \( A \)

\[ T^{kl,k} = \frac{1}{c^2} \left( q^k + eu^k \right) u^l_{,k} + \frac{1}{c^2} \left( q^k + eu^k \right) u^l_{,k} + \left( t^{kl} + u^k u^j \right)_{,k}, \] (227)

resulting in

\[ h^m_{,i} T^{kl,k} = \frac{1}{c^2} \left( q^k + eu^k \right) u^m_{,k} + \left( t^{kl} + u^k u^j \right)_{,k} \left( \delta^m_l + \frac{1}{c^2} u^m u_l \right). \] (228)

Inserting
\[ u^l_{,k} h^m_{,i} = u^m_{,k}, \] (229)

the RHS of the momentum balance equation (227) becomes

\[ h^m_{,i} T^{kl,k} = \frac{1}{c^2} \left( q^k + eu^k \right) u^m_{,k} + \left( t^{kl} + u^k u^j \right)_{,k} \left( \delta^m_l + \frac{1}{c^2} u^m u_l \right). \] (230)

In Galilean co-ordinates, we have for the first term of (230)

\[ \frac{1}{c^2} \left( q^k + eu^k \right) u^m_{,k} \equiv \frac{1}{c^2} \left( q^\alpha + e \frac{u^\alpha}{\sqrt{1 - (v/c)^2}} \right) u^m_{,\alpha} + \frac{1}{c^2} \left( q^4 + e \frac{c}{\sqrt{1 - (v/c)^2}} \right) u^m_{,4}. \] (231)

According to (225), we have with (222)_2 and (223)_2

\[ q^k u_k = 0 \equiv \mp q \cdot \frac{v}{\sqrt{1 - (v/c)^2}} \pm q^4 \frac{c}{\sqrt{1 - (v/c)^2}} \rightarrow q^4 = q \cdot \frac{v}{c} \approx 0. \] (232)

According to (225), the following non-relativistic limits are valid

\[ u^\beta_{,\alpha} \approx \nabla v, \quad u^4_{,\alpha} \approx \nabla c = 0, \quad u^\alpha_{,4} \approx \frac{1}{c} \partial_\ell v = 0, \quad u^4_{,4} \approx \frac{1}{c} \partial_\ell c = 0. \] (233)
Consequently, we obtain for the first term of (230) according to (231) to (233)

\[
\begin{align*}
\frac{1}{c^2} (q^\alpha + eu^\alpha) u^\alpha & \approx 0, \\
\frac{1}{c^2} (q^4 + eu^4) u^4 & \approx 0, \\
\rightarrow \frac{1}{c^2} (q^k + eu^k) u^m, & \approx 0.
\end{align*}
\]

(234)

(235)

(236)

The second term of (230) is in Galilean co-ordinates

\[
\left( t^m + u^k p^m \right)_{,k} \equiv \left( t^m + \frac{v^\alpha}{\sqrt{1 - (v/c)^2}} p^m \right)_{,\alpha} + \left( t^4m + \frac{c}{\sqrt{1 - (v/c)^2}} p^m \right)_{,4}.
\]

(237)

Starting with (28)_3 and (28)_4, we obtain by taking (222)_2 and (223)_2 into account

\[
\begin{align*}
\sum_{k} u^k t^k & = 0 \equiv \pm \frac{v_a}{\sqrt{1 - (v/c)^2}} t^a \pm \frac{c}{\sqrt{1 - (v/c)^2}} t^4 \rightarrow t^a = \frac{v_a}{c} t^a \approx 0, \\
\sum_{k} t^k u^k & = 0 \equiv \pm \frac{v_a}{\sqrt{1 - (v/c)^2}} t^a \pm \frac{c}{\sqrt{1 - (v/c)^2}} t^4 \rightarrow t^4 = \frac{v_4}{c} t^4 \approx 0.
\end{align*}
\]

(238)

(239)

Especially, setting \( l = \beta, 4 \) and \( k = \alpha \) in (238) and (239), we obtain

\[
t^4 = \frac{v}{c} t \cdot \frac{v}{c} \approx 0.
\]

(240)

According to (28)_2, we have

\[
p^k u_k = 0 \equiv \pm p \cdot \frac{v}{\sqrt{1 - (v/c)^2}} \pm p^4 \frac{c}{\sqrt{1 - (v/c)^2}} \rightarrow p^4 = \frac{p \cdot v}{c} \approx 0.
\]

(241)

Taking (28)_3 to (241) into account, the non-relativistic limit of the second term of (230) becomes according to (237)

\[
\begin{align*}
\left( t^{\alpha \beta} + u^\alpha p^\beta \right)_{,\alpha} \approx \nabla \cdot (t + vp), \\
\left( t^{\alpha 4} + u^\alpha p^4 \right)_{,4} \approx 0, \\
\rightarrow \left( t^{km} + u^k p^m \right)_{,k} \approx \partial_t p + \nabla \cdot (t + vp).
\end{align*}
\]

(242)

(243)

(244)

The third term of (230) is by taking (243) into account

\[
\pm \frac{1}{c^2} \left( t^{kl} + u^k p^l \right)_{,k} u^m m = \pm \frac{1}{c^2} \left[ \left( t^{\alpha \beta} + u^\alpha p^\beta \right)_{,\alpha} + \left( t^{44} + u^4 p^4 \right)_{,4} \right] u_\beta u^m.
\]

(245)

Because of

\[
\frac{1}{c^2} u_\beta u^m \approx 0,
\]

(246)
the non-relativistic limit of the third term of (230) vanishes, and we obtain from (236), (242) and (243)
\begin{align}
\hat{h}_i^\alpha T_{kl} \approx \partial_\alpha p + \nabla \cdot (vp + t), \\
\hat{h}_4^4 T_{kl} \approx 0.
\end{align}

We now consider the RHS of the momentum balance equation (24) by taking (246) into account
\begin{align}
\hat{h}_m^m T_{kl} = \hat{h}_m^m k^l = 1/c^2 u^m u^l k^l, \\
\hat{h}_i^\alpha k^l = k^\alpha \mp 1/c^2 u^\alpha u t^l \approx k^\alpha, \\
\hat{h}_4^4 k^l = k^4 \mp 1/c^2 u^4 \left( u^\alpha k^\alpha + u^4 k^4 \right) \approx k^4 - k^4 = 0,
\end{align}
corresponding to (248). Consequently, the non-relativistic momentum balance equation
\begin{equation}
\partial_\alpha p + \nabla \cdot (vp + t) = k
\end{equation}
is decoupled from the energy balance equation and – as expected – is independently valid of the signature of the Galilean co-ordinates. The RHS of (252) \( k \) is the external force density. The corresponding 4-component \( k^\alpha \) is not restricted by the momentum balance equation according to (251).

If the energy momentum-tensor is symmetric, we have according to (30)
\begin{equation}
T_{kl} = T_{lk} \rightarrow p^m = 1/c^2 q^m \approx 0.
\end{equation}
Inserting (253) into (244), the momentum balance equation (252) results in the equilibrium condition
\begin{equation}
T^{kl} = T^{lk} \rightarrow \nabla \cdot t = k, \quad t = t^\top.
\end{equation}

### 8.3 The energy balance

Starting with (227), we obtain the energy balance equation by taking (28)_{2,4} into account
\begin{equation}
u_i T_{kl} \approx \pm \left( q^k + eu^k \right)_{,k} + \left( t^{kl} + u^k p^l \right)_{,k} = u_k k^l.
\end{equation}
Decomposing the first term of (255), we obtain by use of (222)_{1, 193} and (211)_3
\begin{align}
\pm (q^k + eu^k)_{,k} &= \pm (q^4 + eu^4)_{,4} \pm (q^\alpha + eu^\alpha)_{,\alpha} \\
&= \pm \frac{1}{c} \partial_\alpha \left( q \cdot \frac{v}{c} + e \frac{c}{\sqrt{1 - (v/c)^2}} \right) \pm \nabla \cdot \left( q + e \frac{v}{\sqrt{1 - (v/c)^2}} \right) \approx \\
&\approx \pm \partial_\alpha e + \nabla \cdot \left( q + ev \right).
\end{align}

34
According to (243), the second term of (255) results in by taking (242) and (243) into account
\[
\left( t^{kl} + u^k p^l \right)_{,k} u_l = \left[ \left( t^{\alpha\beta} + u^\alpha p^\beta \right)_{,\alpha} + \left( t^{4\beta} + u^4 p^\beta \right)_{,\alpha} \right] u_\beta + \\
+ \left[ \left( t^{\alpha 4} + u^\alpha p^4 \right)_{,\alpha} + \left( t^{44} + u^4 p^4 \right)_{,\alpha} \right] u_4 \approx \\
\approx \mp \left[ \nabla \cdot \left( t + v p \right) + \partial_t p \right] \cdot v \pm \nabla \cdot \left( t \cdot v + v(p \cdot v) \right) + \partial_t (p \cdot v),
\]
(257)

Using the momentum balance equation (252), the second term of (255) results in
\[
\left( t^{kl} + u^k p^l \right)_{,k} u_l \approx \mp \mathbf{k} \cdot \mathbf{v} \pm \left[ \nabla \cdot \left( t \cdot \mathbf{v} \right) + \nabla \cdot \left( v p \cdot \mathbf{v} \right) + \partial_t \left( p \cdot \mathbf{v} \right) \right] \approx - \left( t^{kl} + u^k p^l \right) u_{l,k}.
\]
(258)

The RHS of the energy balance equation (255) is
\[
u_{l,k} = u_{\alpha k}^\alpha + u_4^k 4^4 \approx \mp \mathbf{v} \cdot \mathbf{k} \pm r, \quad r := c k^4.
\]
(259)

Consequently, the energy balance equation (255) becomes with (256), (258) and (259)
\[
\partial_t e + \nabla \cdot \left( q + e \mathbf{v} \right) + \nabla \cdot \left( t \cdot \mathbf{v} \right) + \partial_t \left( p \cdot \mathbf{v} \right) + \nabla \cdot \left( v p \cdot \mathbf{v} \right) = r.
\]
(260)

Taking into account
\[
\nabla \cdot \left( t \cdot \mathbf{v} \right) = \nabla \cdot \mathbf{t} + \left( \nabla \cdot t \right) \cdot \mathbf{v},
\]
(261)
\[
\partial_t \left( p \cdot \mathbf{v} \right) = \partial_t p \cdot \mathbf{v} + \partial_t \mathbf{v} \cdot p,
\]
(262)
\[
\nabla \cdot \left( v p \cdot \mathbf{v} \right) = \left( \nabla \cdot v p \right) \cdot \mathbf{v} + \nabla \cdot \mathbf{v} : p v,
\]
(263)

and taking in mind that \( e \) is independent of \( u^k \) according to its definition (25), and consequently also independent of \( \mathbf{v} \), the energy balance equation (260) results in the local rest frame
\[
v \mp 0 : \quad \partial_t e \mp \nabla \cdot \left( q + e \mathbf{v} \right) + \nabla \cdot \mathbf{t} + \partial_t \mathbf{v} \cdot \mathbf{p} = r.
\]
(264)

Consequently, \( e \) is the internal energy, a fact which can be read off also from the Gibbs equation (85). Beyond that, (264) is in accordance with the balance of the internal energy which is not based on relativistic presuppositions [31].

\(^{31}\partial_t \mathbf{v} \cdot \mathbf{p} \) corresponds to \( \epsilon : \mathbf{T} \) in [31]; both terms are generated by the non-symmetry of the energy-momentum tensor.
8.4 The entropy balance

8.4.1 The $A$-component of the mixture

Starting with (68) and (75), we obtain

$$S_{Ak}^A = (s^A u^A),_A + (s^A u^A),_4 + s^A_{A},_A + s^A_{A4} = \sigma^A + \varphi^A. \quad (265)$$

Analogously to (232)

$$s^A u^A_k = 0 \rightarrow s^A \approx 0 \quad (266)$$

is valid. In Galilean coordinates follows for the non-relativistic limit the usual balance equation of the entropy

$$S_{Ak}^A \approx \nabla \cdot (v^A s^A + s^A) + \partial_t s^A = \sigma^A + \varphi^A. \quad (267)$$

Especially, we consider the case of multi-heat relaxation equilibrium and signature $(-2)$ in the sequel. From (118) and (1) follows

$$\nu^A = f^A = \frac{1}{c^2}(u^A u^A + u^A u^4) \approx 1. \quad (268)$$

The non-relativistic limits of entropy density $s^A$, entropy flux density $s^A$, entropy production $\sigma^A$ and entropy supply $\varphi^A$ follow for a component in the mixture from (118) to (121) by (108). Taking (241) into account, the entropy density of the $A$-component (118) results in

$$s^A = \frac{1}{\Theta} \left( \mu^A g^A + f^A(e^A + p^A) + u_A p^{A\alpha} + u_4 p^{A4} \right) \approx \frac{1}{\Theta} \left( \mu^A g^A + e^A + p^A - v \cdot p^A + p^A \cdot v^A \right) =$$

$$= \frac{1}{\Theta} \left( \mu^A g^A + e^A + p^A + p^A \cdot (v^A - v) \right). \quad (269)$$

The diffusion flux (18) becomes by taking (268) into account

$$J^{A}_{Am} h^A_m \approx g^A(v^A - v) = J^A, \quad J^{A}_{Am} h^A_{4m} \approx 0, \quad (270)$$

and with (240), we obtain

$$u_m t^{A3m} = u_{3A} t^{A3\alpha} + u_{4A} t^{A4} \approx -t \cdot v + t \cdot v = 0, \quad (271)$$

$$u_m t^{A4m} = u_{4A} t^{A4\alpha} + u_{4A} t^{A44} \approx 0. \quad (272)$$

Inserting (270) to (272) into (119), the non-relativistic limit of the entropy flux density becomes taking (232) into account

$$s^A = \frac{1}{\Theta} \left( q^A - \mu^A J^A \right). \quad (273)$$
According to (259), the energy supply of the $A$-component is
\[ r^A = u^A_k k^A = u^A_k A + u^A_4 k^A_4 \approx -v^A \cdot k^A + c k^A_4 \] \tag{274}

Taking (268) and (274) into account, two terms of the entropy supply (120) become
\[ \nu^A u^A_k k^A + u_p h^A_{m, k} \approx \nu^A (u^A_k + u^A_4) k^A_m = \nu^A (u^A_k + u^A_4) k^A_m = -v^A \cdot k^A + c k^A_4 \] \tag{275}

Taking (275) into account, the non-relativistic limit of the entropy supply (120) is in multi-heat relaxation equilibrium
\[ \varphi^A \approx \frac{1}{\Theta} \left( \mu^A (v^A + r^A + (v^A - v) \cdot k^A) \right). \] \tag{276}

We now will find out the different non-relativistic terms of the entropy production (121). We start with the first term taking (18) into account
\[ -(J^{Am} h^A_{mk})_k = - \left( \frac{1}{\Theta} \frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{1}{\Theta} \right) \approx -\nabla \cdot \left( \frac{\partial}{\partial x} \right) \approx -\nabla \cdot J. \] \tag{277}

The second term of (121) yields by taking (270) into account
\[ -\left( \frac{\mu^A}{\Theta^A} \right)_k J^{Am} h^A_{mk} = - \left( \frac{\mu^A}{\Theta^A} \right) J^{Am} h^A_{mk} - \left( \frac{\mu^A}{\Theta^A} \right)_k J^{Am} h^A_{mk} \approx -J^A \cdot \nabla \left( \frac{\mu^A}{\Theta^A} \right). \] \tag{278}

Taking (232) into account, the third term of (121) results in
\[ \left( \frac{1}{\Theta^A} \right)_k q^A_k \approx \nabla \left( \frac{1}{\Theta} \right) \cdot q^A. \] \tag{279}

A term which is similar to the fourth term of (121) is calculated in (258). Consequently, we obtain
\[ \frac{\nu^A}{\Theta^A} u^A_{l, k} \left( \pi^{Akl} + u^A_k p^A_l \right) \approx -\frac{1}{\Theta} \left( \nabla v^A : \pi^{A^T} + (d_t v^A) \cdot p^A \right). \] \tag{280}

Taking
\[ \left( \frac{1}{\Theta^A} u^m h^A_{ml} \right)_k = \left( \frac{1}{\Theta^A} (u^m - f^A u^A_m) \right)_k \] \tag{281}

into account, the non-relativistic limit of the last term of (121) is
\[ \left[ \frac{1}{\Theta^A} (u^m - f^A u^A_m) \right]_k \left( \frac{1}{c^2} e^A u^A_k u^A_l + \frac{1}{c^2} q^A_k u^A_l + t^A_{kl} \right) \approx -\nabla \left[ \frac{1}{\Theta} (v - v^A) \right] : t^{AT}. \] \tag{282}

37
Inserting (277) to (280) and (282) into (121) results in the non-relativistic limit of the entropy production

\[ \sigma^A \approx \frac{\mu^A}{\Theta} \left( (\Gamma^A - \nabla \cdot J^A) - J^A \cdot \nabla \left( \frac{\mu^A}{\Theta} \right) + \nabla \left( \frac{1}{\Theta} \right) \cdot q^A - \frac{1}{\Theta} \left( \nabla v^A : \pi^{A \top} + (dt^A) \cdot p^A \right) + \nabla \left[ \frac{1}{\Theta} (v^A - v) \right] : t^{A \top} \right). \]  

(283)

and rearranging (283), we obtain

\[ \sigma^A \approx \frac{\mu^A}{\Theta} (\Gamma^A - \nabla \cdot \left( \frac{\mu^A}{\Theta} J^A \right) + \nabla \left( \frac{1}{\Theta} \right) \cdot q^A - \frac{1}{\Theta} \left( \nabla v^A : \pi^{A \top} + p^A v^A + \partial_t v^A \cdot p^A \right) + \nabla \left[ \frac{1}{\Theta} (v^A - v) \right] : t^{A \top} \right). \]  

(284)

The first three terms of the RHS of (284) belong to the usual effects: chemical reactions, diffusion and heat conduction. The fourth term belonging to viscosity is coupled to the momentum balance by a modified viscosity tensor and an additional expression which vanishes with the symmetry of the energy-momentum tensor. The last term of (284) is typical for a component of a multi-component system in non-equilibrium.

### 8.4.2 The mixture

Relativistic thermodynamics of a mixture is characterized by two items: the additivity of the energy-momentum tensors and of the 4-entropies of the components of the mixture resulting in the corresponding quantities of the mixture itself, (39) and (180). These two settings are the starting-points for determining the (3+1)-components of the energy-momentum tensor of the mixture in sect 3.4 and of the entropy related quantities in sect 7.3. Here, we are looking for their corresponding non-relativistic limits.

First of all, we need the non-relativistic limits of \( h_{k\alpha}^m \), (16), and \( g^{Am} \), (42):

\[ h_{\alpha \beta}^3 \equiv \delta_{\alpha \beta} + \frac{1}{c^2} \frac{v_{\alpha} v_{\beta}}{\sqrt{1 - (v/c)^2}} \approx \delta_{\alpha \beta}, \]  

(285)

\[ h_{\alpha \beta}^4 \equiv 0 - \frac{1}{c^2} \frac{v_{\alpha} v_{\beta}}{\sqrt{1 - (v/c)^2}} \approx 0, \]  

(286)

\[ h_{\alpha \beta}^4 \equiv 0 + \frac{1}{c^2} \frac{v_{\alpha} v_{\beta}}{\sqrt{1 - (v/c)^2}} \approx 0, \]  

(287)

\[ h_{\alpha \beta}^4 \equiv 1 - \frac{1}{c^2} \frac{v_{\alpha} v_{\beta}}{\sqrt{1 - (v/c)^2}} \approx 0, \]  

(288)

\[ g^{A \alpha} = \frac{1}{c^2} \left( u^{A \beta} h_{\beta \alpha}^3 + u^{A 4} h_{\beta 4}^3 \right) \approx 0, \]  

(289)

\[ g^{A 4} = \frac{1}{c^2} \left( u^{A \beta} h_{\beta 4}^3 + u^{A 4} h_{4 4}^3 \right) \approx 0. \]  

(290)

Consequently, the (3+1)-components of the mixture’s energy-momentum tensor become as follows: The third term of (46) is by taking (241) and (265) into account

\[ f^A \left( p^{A \alpha} u_{\alpha} + p^{A 4} u_{4} \right) \equiv f^A \left( - \frac{p^A \cdot v}{\sqrt{1 - (v/c)^2}} + \frac{p^A \cdot v^A}{c} \frac{v}{\sqrt{1 - (v/c)^2}} \right) \approx \]  

38
\[ \approx \mathbf{p}^A \cdot (\mathbf{v}^A - \mathbf{v}). \]  

Consequently, the internal energy of the mixture \[40\] results in

\[ e \approx \sum_A \left( e^A + \mathbf{p}^A \cdot (\mathbf{v}^A - \mathbf{v}) \right). \]  

The spatial component of the energy flux density \[48\] is

\[ q^\alpha = \sum_A \left( c^2 e^A f^A g^{A\alpha} + q^{Ak} f^A h_k^\alpha + c^2 p^A g^{A\alpha} u_l + t^{Akil} h_k^\alpha u_l \right) = \sum_A \left( c^2 g^{A\alpha}(e^A f^A + p^A u_l) + h_k^\alpha (q^{Ak} f^A + t^{Akil} u_l) \right). \]  

Taking \[42\] and \[81\] into account, special terms of \[293\] are:

\[ c^2 g^{A\alpha} \approx \frac{v^{A\alpha}}{\sqrt{1 - (v^A/c)^2}} - \frac{f^A}{\sqrt{1 - (v/c)^2}} \approx v^A - v, \]  

\[ c^2 g^{A4} \approx \frac{c}{\sqrt{1 - (v^A/c)^2}} - \frac{f^A}{\sqrt{1 - (v/c)^2}} \approx 0, \]  

and by taking \[291\] into account

\[ c^2 g^{A\alpha}(e^A f^A + p^A u_l) \approx (v^A - \mathbf{v})(e^A + \mathbf{p}^A \cdot (\mathbf{v}^A - \mathbf{v})). \]  

By taking \[285\], \[286\] and \[238\] to \[240\] into account, parts of the second term of \[293\] result in

\[ h_k^\alpha t^{Akil} u_l = h_k^\alpha t^{A\beta\gamma} u_\gamma + h_k^\alpha t^{A4\gamma} u_\gamma + h_k^\alpha t^{4\beta\alpha} u_\gamma + h_k^\alpha t^{44\gamma} u_\gamma \approx -t^A \cdot \mathbf{v} + t^A \cdot \mathbf{v} - 0 + 0 = 0. \]  

Taking \[290\] to \[298\] into account, the non-relativistic limit of the spatial part of the energy flux density is

\[ q^\alpha \approx \sum_A \left[ q^A + \left( e^A + \mathbf{p}^A \cdot (\mathbf{v}^A - \mathbf{v}) \right) (\mathbf{v}^A - \mathbf{v}) \right]. \]  

The temporal component of the energy flux density \[48\] is according to \[293\]

\[ q^4 = \sum_A \left( c^2 g^{A4}(e^A f^A + p^A u_l) + h_k^4 (q^{Ak} f^A + t^{Akil} u_l) \right). \]  

According to \[294\] to \[298\], we obtain

\[ c^2 g^{A4}(e^A f^A + p^A u_l) \approx 0, \quad h_k^4 (q^{Ak} f^A + t^{Akil} u_l) \approx 0 \rightarrow q^4 \approx 0, \]  

\[ 39 \]
The spatial component of the momentum flux density \((50)\) is

\[
p^\alpha = \sum_A \left( e^A f^A g^{A\alpha} + \frac{1}{c^2} q^{Ak} u_k g^{A\alpha} + p^{Al} f^A h_i^{\alpha} + \frac{1}{c^2} t^{Ak\beta} h_i^{\beta} u_k \right) = \sum_A \left( g^{A\alpha} (e^A f^A + \frac{1}{c^2} q^{Ak} u_k) + h_i^{\alpha} (p^{Al} f^A + \frac{1}{c^2} t^{Ak\beta} u_k) \right). \tag{302}
\]

According to \((289)\), we obtain

\[
g^{A\alpha} (e^A f^A + \frac{1}{c^2} q^{Ak} u_k) \approx 0. \tag{303}
\]

Taking \((285), (286)\) and \((298)\) into account, the second term of \((302)\) becomes

\[
h_i^{\alpha} p^{Al} f^A = h_i^{\alpha\beta} p^{A\beta} f^A + h_i^{\alpha} p^{A4} f^A \approx p^4, \tag{304}
\]

\[
h_i^{\alpha} \frac{1}{c^2} t^{Ak\beta} u_k \approx 0, \tag{305}
\]

resulting in

\[
p^\alpha \approx \sum_A p^A. \tag{306}
\]

The temporal component of the momentum flux density \((50)\) is according to \((302)\)

\[
p^4 = \sum_A \left( g^{A4} (e^A f^A + \frac{1}{c^2} q^{Ak} u_k) + h_i^{4\alpha} p^{Al} f^A + \frac{1}{c^2} t^{Ak\beta} u_k \right) \approx 0, \tag{307}
\]

according to \((286), (287)\) and \((290)\).

The spatial components of the energy momentum tensor \((52)\) of the mixture are

\[
t^{\alpha\beta} = \sum_A \left( c^2 g^{A\alpha} (e^A g^{A\beta} + p^{Al} h_i^{\beta}) + h_i^{\alpha\beta} (q^{Ak} g^{A\beta} + t^{Ak\beta} h_i^{\beta}) \right). \tag{308}
\]

The first term is by taking \((294), (290), (285)\) and \((286)\) into account

\[
c^2 g^{A\alpha} (e^A g^{A\beta} + p^{Al} h_i^{\beta}) \approx (v^A - v) p^A. \tag{309}
\]

Taking \((285)\) and \((286)\) into account, the second term of \((308)\) becomes

\[
h_k^{\alpha} (q^{Ak} g^{A\beta} + t^{Ak\beta} h_i^{\beta}) = h_k^{\alpha\beta} q^{Ak} g^{A\beta} + h_k^{\alpha} q^{A4} g^{A\beta} + h_k^{\alpha} t^{Ak\beta} h_i^{\beta} + h_k^{\alpha} t^{A4\beta} h_i^{\beta} + h_k^{\alpha} t^{A44} h_i^{\beta} \approx t^A. \tag{310}
\]

Consequently, the non-relativistic limit of the stress tensor \((308)\) of the mixture is

\[
t^{\alpha\beta} \approx \sum_A \left( t^A + (v^A - v) p^A \right). \tag{311}
\]
The three other components of the energy momentum tensor of the mixture are as follows:

\[ t^\alpha_4 = \sum_A \left( c^2 g^{A\alpha} (e^A g^A_4 + p^A h^A_4) + h^\alpha_k (q^{Ak} g^A_4 + t^{AKl} h^A_4) \right), \quad (312) \]

\[ t^{A\beta} = \sum_A \left( c^2 g^{A\beta} (e^A g^A_\beta + p^A h^A_\beta) + h^\beta_k (q^{Ak} g^A_\beta + t^{AKl} h^A_\beta) \right), \quad (313) \]

\[ t^{44} = \sum_A \left( c^2 g^A_4 (e^A g^A_4 + p^A h^A_4) + h^4_k (q^{Ak} g^A_4 + t^{AKl} h^A_4) \right). \quad (314) \]

Taking (295), (287) and (288) into account, (312) to (314) result in

\[ t^\alpha_4 \approx \sum_A \left( h^\alpha_k (q^{Ak} g^A_4 + t^{AKl} h^A_4) \right), \quad (315) \]
\[ t^{A\beta} \approx \sum_A \left( h^\beta_k (q^{Ak} g^A_\beta + t^{AKl} h^A_\beta) \right), \quad (316) \]
\[ t^{44} \approx \sum_A \left( h^4_k (q^{Ak} g^A_4 + t^{AKl} h^A_4) \right). \quad (317) \]

Taking (295) and (287) into account, we obtain

- \[ t^\alpha_4 \approx 0, \quad t^{A\beta} \approx 0, \quad t^{44} \approx 0. \quad (318) \]

The non-relativistic limit of the pressure of the mixture (53) is

- \[ p \approx -\frac{1}{3} \sum_A t^A_k = \sum_A p^A. \quad (319) \]

9 Summary

A multi-component system is formed by its components which are characterized by own individual quantities, such as velocity, density, chemical potential, stress tensor, temperature, heat flux and entropy flux densities, entropy production and supply and further items. All these individual quantities determine those of the multi-component system which is described as a mixture. Individual temperatures of the components result in multi-heat relaxation towards the corresponding equilibrium generating a common temperature of all components and the mixture. A temperature of the mixture in multi-heat relaxation non-equilibrium depends on the used thermometer and cannot be defined unequivocally.

Starting out with the rest mass densities of the components of the multi-component system, the mass flux densities of the components are defined by introducing their different 4-velocities. The mixture of the components is characterized by several settings. The first one is the additivity of the component’s mass flux densities to the mass flux density of the mixture. In combination with the mixture axiom, this setting allows to define mass
density and 4-velocity of the mixture and the diffusion fluxes of the components. The non-symmetric energy-momentum tensor of one component interacting with the mixture is introduced, and its (3+1)-split together with the component’s mass and diffusion flux densities are generating the entropy identity \([9]\). The exploitation of the entropy identity requires additional settings: the entropy density, flux and supply. These settings result in physical interpretations of entropy density, flux and supply. The entropy production follows from the entropy identity which restricts possible arbitrariness of defining.

The use of the entropy identity introduces so-called accessory variables which are Lagrange parameters concerning the constraints taken into consideration. These are temperature, chemical potential and an additional non-equilibrium variable which characterizes the considered component to be a part of the mixture. Beside the classical irreversible processes –diffusion, chemical reactions, heat conduction and friction– an additional irreversible process –multi-heat relaxation– appears due to the embedding of the considered component into the mixture. Different from the classical case, the mass production term, the heat flux density and the viscous tensor are modified by so-called effective quantities.

Equilibrium is defined by equilibrium conditions which are divided into necessary and supplementary ones \([9,10,12]\). The necessary equilibrium conditions are given by vanishing entropy production, vanishing entropy flux density and vanishing entropy supply. Supplementary equilibrium conditions are: vanishing diffusion flux densities, vanishing component time derivatives\(^{32}\) and vanishing of the mass production terms. Presupposing these equilibrium conditions, we obtain: all components have the same 4-velocity, all heat flux densities are zero, the power as well as the divergence of the 4-velocity of each component vanish, and the viscous tensor is perpendicular to the velocity gradient.

The corresponding free component is defined by undistinguishable component indices\(^{33}\). This 1-component system represents the easiest classical case serving as a test, if the interacting component in the mixture is correctly described. The vanishing of the entropy production in equilibrium is shortly investigated: the so-called Killing relation of the vector of 4-temperature is neither a necessary nor a sufficient condition for equilibrium. Also the statement that materials are perfect in equilibrium cannot be confirmed.

**Acknowledgement** Discussions with H.-H. v. Borzeszkowski are gratefully acknowledged.

**10 Appendices**

**10.1 Rest mass densities**

Consider two frames, \(B^A\) and \(B^B\). \(B^A\) is the rest frame of the \(A\)-component and \(B^B\) that of the \(B\)-component. The corresponding rest mass densities are

rest densities: \(\varrho^A\) in \(B^A\), \(\varrho^B\) in \(B^B\). \hspace{1cm} (320)

\(^{32}\)except that of the 4-velocity
\(^{33}\)that is not the mixture which is a multi-component system
By definition, the rest mass densities do not depend on the frame, that means, the rest mass densities are relativistic invariants which should not be confused with the measured densities in non-resting frames
\[
B^B: \quad \varrho^A_B = \frac{\varrho^A}{1 - v^2_{AB}/c^2}, \quad B^A: \quad \varrho^B_A = \frac{\varrho^B}{1 - v^2_{BA}/c^2}, \quad v_{AB} = -v_{BA}. \tag{321}
\]
Here \(\varrho^A_B\) is the density of the \(A\)-component in the rest frame of the \(B\)-component, and \(v_{AB}\) is the translational 3-velocity of \(B^A\) in the frame \(B^B\). These densities are out of scope in this paper.

We now consider (8) in the rest frame \(B_0\) of the mixture which is defined by \(u^k_0 = (0, 0, 0, c)\). Consequently, we obtain
\[
f^A_0 = \frac{1}{c^2} u^A_{40} c = \frac{1}{c} \sqrt{1 - v^2_{AO}/c^2}. \tag{322}
\]
Inserting (322) into (8) results in the mass density of the mixture in its rest frame
\[
B_0: \quad \varrho_0 = \sum_A f^A_0 \varrho^A_0 = \sum_A \frac{1}{c} \sqrt{1 - v^2_{AO}/c^2} \frac{\varrho^A}{1 - v^2_{AO}/c^2} = \sum_A \frac{\varrho^A}{(1 - v^2_{AO}/c^2)^{3/2}}. \tag{323}
\]
The same result is obtained, if (5) is written down for the rest system of the mixture.

10.2 Example: Uniform component velocities

If there exists a common rest frame \(B^0\) for all \(A\)-components
\[
u^A_k \doteq u^0_k, \quad \land A. \tag{324}
\]
According to (5), we obtain
\[
\varrho u_k = v^0_k \sum_A \varrho^A \longrightarrow \varrho c^2 = u^k_0 \sum_A \varrho^A \land \varrho u_k u^{0k} = c^2 \sum_A \varrho^A, \tag{325}
\]
and with (8) follows
\[
\varrho c^2 = c^2 f^0 \sum_A \varrho^A \land \varrho c^2 f^0 = c^2 \sum_A \varrho^A \longrightarrow (f^0)^2 = 1, \tag{326}
\]
resulting in
\[
f^0 = \pm 1. \tag{327}
\]
We obtain from (8)
\[
\varrho = f^0 \sum_a \varrho^A = \pm \sum_a \varrho^A \longrightarrow f^0 = +1, \tag{328}
\]
and taking (325) into account
\[
u_k = u^0_k. \tag{329}
\]
As expected, the 4-velocity of the mixture is identical with the uniform component velocities.
10.3 Stoichiometric equations

The system of the relativistic stoichiometric equations runs as follows

\[ \sum_A \nu^A_\alpha M^A_0 = 0, \]  
\[ \text{(330)} \]

component index: \( A = 1, 2, ..., Z \), reaction index: \( \alpha = 1, 2, ..., \Omega \).

The stoichiometric coefficients \( \nu^A_\alpha \) are scalars, and the partial rest mole mass \( M^A_0 \) is defined using the scalar mole number \( n^A \) and the mole concentration \( \zeta^A \) of the \( A \)-component

\[ M^A_0 := \frac{m^A_0}{n^A} = \frac{V_0}{n^A} g^A = \frac{g^A}{\zeta^A}, \quad \zeta^A := \frac{n^A}{V_0} \]  
\[ \text{(331)} \]

according to (320) and (??). The stoichiometric coefficients \( \nu^A_\alpha \) are determined by the partial rest mole masses \( M^A_0, A = 1, 2, ..., Z \), before and after the \( \alpha \)th reaction.

The time derivative of the mole number is determined by the reaction velocities \( \xi_\alpha \)

\[ \dot{n}^A = \sum_\alpha \nu^A_\alpha \xi_\alpha. \]  
\[ \text{(332)} \]

Multiplication with \( M^A_0 \) results by use of (330) in

\[ M^A_0 \dot{n}^A = \sum_\alpha \nu^A_\alpha M^A_0 \dot{\xi}_\alpha \rightarrow \sum_A M^A_0 \dot{n}^A = 0. \]  
\[ \text{(333)} \]

Starting out with the physical dimensions

\[ [\nu^A_\alpha] = \text{mol}, \quad [n] = \text{mol}, \quad [M^A_0] = \frac{kg}{mol}, \quad [\zeta^A] = \frac{mol}{m^3}, \quad [\xi_\alpha] = \frac{1}{s}, \]  
\[ \text{(334)} \]

that of the mass production term in the first row of (107) is evidently

\[ [(\text{in}) \Gamma^A] = \frac{kg}{m^3 s}. \]  
\[ \text{(335)} \]

A comparison with

\[ [M^A_0 \dot{n}^A] = \frac{kg}{s} \]  
\[ \text{(336)} \]

shows that \( (\text{in}) \Gamma^A \) is the density which belongs to the mass production (333). Because according to (??), all rest mass densities are referred to the relativistic invariant \( V_0 \), we obtain from (336) and (335) with (333)

\[ (\text{in}) \Gamma^A = \frac{1}{V_0} \sum_\alpha \nu^A_\alpha M^A_0 \dot{\xi}_\alpha \rightarrow \sum_A (\text{in}) \Gamma^A = 0. \]  
\[ \text{(337)} \]
10.4 Remark: Constitutive equations and the 2nd Law

Up to here, a special material was not taken into account: all considered relations are valid independently of the material which is described by constitutive equations supplementing the balance equations. Especially, the entropy productions \( \text{(121)} \) of the \( A \)-component and \( \text{(211)} \) of the mixture are not specified for particular materials. There are different possibilities for introducing constitutive equations \(^{34}\). Because constitutive equations are not in the center of our considerations, we restrict ourselves on the easiest ansatz which only serves for elucidation.

The entropy production of the \( A \)-component is a sum of two-piece products whose factors are so-called fluxes and forces. According to (107), the fluxes are

\[
\mathcal{Y}^A = \left\{ (\text{in}) \Gamma^A - \left( J^{Am} h^m_{Ak} \right)_{,k}, \left( J^{Am} h^m_{Ak} \right)_{,k}, q^{Ak}, \pi^{Akl} + u^{Ak} p^A, \right. \\
\left. H^{ABk}, \frac{1}{c^2} c^A u^{Ak} u^A + \frac{1}{c^2} q^{Ak} u^A + t^{Akl} \right\},
\]

and the corresponding forces are

\[
\mathcal{X}^A = \left\{ \mu^A \Theta^A, \left( \mu^A \right)_{,k}, \left( f^A \right)_{,k}, \left( f^A \right)_{,k}, \mu^A \Theta^A u^A_{,k}, \left( \frac{1}{c^A} - \frac{1}{c^A} \right)_{,k}, \left( \frac{1}{c^A} u^m h^m_{Ak} \right)_{,k} \right\}.
\]

The entropy production density \( \text{(107)} \) writes symbolically

\[
\sigma^A = \mathcal{Y}^A \cdot \mathcal{X}^A.
\]

The material is described by the dependence of the fluxes on the forces, by the constitutive equations

\[
\mathcal{Y}^A = F^A(\mathcal{X}^A),
\]
resulting in

\[
\sigma^A = F^A(\mathcal{X}^A) \cdot \mathcal{X}^A \quad \longrightarrow \quad \sigma^A = \sum_A F^A(\mathcal{X}^A) \cdot \mathcal{X}^A \geq 0.
\]

The inequality is caused by the Second Law which states that the entropy production of the mixture is not negative for all events after having inserted the constitutive equations into the general expression \( \text{(211)} \). The entropy production of an \( A \)-component \( \text{(121)} \) is not necessarily positive semi-definite. There are different methods for exploiting the dissipation inequality \( \text{(342)} \) which are beyond this paper.

References

[1] S.R. de Groot, P. Mazur: Non-Equilibrium Thermodynamics, North-Holland, Amsterdam 1962, Chap. II and III

\(^{34}\)as an ansatz, or better by construction procedures \[35, 36\]
[2] G.A. Kluitenberg, S.R. de Groot, P. Mazur: Relativistic thermodynamics of irreversible processes I. Heat conduction, diffusion, viscous flow and chemical reactions; Formal part. Physica 19 (1953) 689-794

[3] G.A. Kluitenberg, S. R. de Groot and P. Mazur: Relativistic thermodynamics of irreversible processes II. Heat conduction and diffusion; Physical part. Physica 19 (1953) 1079-1094

[4] W. Israel: Nonstationary irreversible thermodynamics: A causal relativistic theory. Annals of Physics 100 (1976) 310-331

[5] P. Havas, R.J. Swenson: Relativistic thermodynamics of fluids. I. Annals of Physics 118 (1979) 259-306

[6] G. Neugebauer: Relativistische Thermodynamik, Akademie-Verlag, Berlin 1980

[7] H.-H. v. Borzeszkowski, T. Chrobok: On special- and general-relativistic thermodynamics. Atti dell’Accademia Peloritana dei Pericolanti, Classe I di Science Fis.Mat. e Nat., Vol. LXXXI-LXXXII, A01001 (2003-04)

[8] Muschik, W.: In: Relativittstage Berlin, Jena, 2022 June 1990, TU Berlin, Verhandl. DPG (VI) 25, p. 314 (1990)

[9] W. Muschik, H.-H. v. Borzeszkowski: Entropy identity and equilibrium conditions in relativistic thermodynamics. Gen Relativ Gravit 41 (2009) 1285-1304

[10] W. Muschik, H.-H. v. Borzeszkowski: Entropy production and equilibrium conditions of general-covariant spin systems. Entropy 17 (2015) 8325-8340

[11] J. Meixner: The entropy concept in non-equilibrium thermodynamics. J. Phys. Soc. Japan 26 (1969) Suppl., Proc. Int. Conf. Stat. Mech. 1968

[12] J. Meixner: Entropie im Nichtgleichgewicht. Rheologica Acta 7 (1968) 8-13

[13] J.U. Keller: Über den 2. Hauptsatz der Thermodynamik irreversibler Prozesse. Acta Phys. Austriaca 35 (1972) 321-330

[14] W. Kern: Zur Vieldeutigkeit der Gleichgewichtsentropie in kontinuierlichen Medien. Dissertation, Aachen (Germany) 1972

[15] G.A. Kluitenberg, S.R. de Groot, P. Mazur: Relativistic thermodynamics of irreversible processes I-III. Physica 19 (1953) 689-794, 1079-1094; 20 (1954) 199-209

[16] D. Jou, J. Casas-Vázquez, G. Lebon: Extended Irreversible Thermodynamics. Springer Berlin 1993

[17] I. Müller, T. Ruggeri: Extended Thermodynamics. Springer New York 1993

[18] W. Muschik: Empirical foundation and axiomatic treatment of non-equilibrium temperature. Arch.Rat.Mech.Anal. 66 (1977) 379-401
[19] W. Muschik, G. Brunk: A concept of non-equilibrium temperature. Int. J. Engng. Sci. 15 (1977) 377-389

[20] G.D.C. Kuiken: Thermodynamics of Irreversible Processes. Wiley, Chichester 1994, Sect. 4.3

[21] W. Muschik: Internal Variables in Non-Equilibrium Thermodynamics. J. Non-Equilib. Thermodyn. 15 (1990) 127-137

[22] I. Müller: Thermodynamics. Pitman, Boston 1985, Sect. 11.3.2

[23] B.C. Eu: Kinetic Theory and Irreversible Thermodynamics. Wiley, New York 1992, Sect. 7.5

[24] H. Stephani: General Relativity. Cambridge University Press, London 1982

[25] W. Muschik: Contact temperature and internal variables: A glance back, 20 years later. J. Non-Equilib. Thermodyn. 39 (2014) 113-121

[26] I. Müller: A thermodynamic theory of mixtures of fluids. Arch. Rational Mech. Anal. 28 (1968) 1-39

[27] N.T. Dunwoody, I. Müller: A thermodynamic theory of two chemically reacting ideal gases with different temperatures. Arch. Rational Mech. Anal. 29 (1968) 344-369

[28] R.M. Bowen, D.J. Garcia: On the thermodynamics of mixtures with several temperatures. Int. J. Engng. Sci. 8 (1970) 63-83

[29] J. Casas-Vázquez, D. Jou: Temperature in non-equilibrium states: A review of open problems and current proposals. Rep. Prog. Phys. 66 (2003) 1937-2023

[30] S. Chandrasekhar: The post-Newtonian equations of hydrodynamics in general relativity. Astrophysical Journal 142 (1965) 1488-1512

[31] G.O. Schellstede: Die Kompatibilität der Newtonschen und der post-Newtonsschen Näherungen der relativistischen Kontinuumstheorie und der klassischen Kontinuumsthermodynamik. Diplomarbeit, Fachbereich Physik, Freie Universität Berlin (Germany)

[32] C. Will: Theory and Experiment in Gravitational Physics. Cambridge University Press 1993

[33] G. O. Schellstede, H.-H. von Borzeszkowski, T. Chrobok, W. Muschik: The relation between relativistic and non-relativistic continuum thermodynamics. Gen Relativ Gravit 46 (2014) 1640 26ps

[34] W. Muschik: A sketch of continuum thermodynamics. J. Non-Newtonian Fluid Mech. 96 (2001) 255-290; Sect. 2.2.6

[35] I.S. Liu: Method of Lagrange multipliers for exploitation of the entropy principle. Arch. Rat. Mech. Anal. 46 (1972) 131-148

47
[36] W. Muschik, H. Ehrentraut: An amendment to the secon law. J. Non-Equilib. Thermodyn. 21 (1996) 175-192

[37] V. Triani, C. Papenfuss, V. Cimmelli, W. Muschik: Exploitation of th second law: Coleman-Noll and Liu procedure in comparison. J. Non-Equilib. Thermodyn. 33 (2008) 47-60

[38] W. Muschik, V. Triani, C. Papenfuss: Exploitation of the dissipation inequality, if some Balances are missing. J. Mech. Mat. Structures 3 (2008) 1125-1133