Multidimensional the Ricci-flat spaces  
defined by nonlinear equations  

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1 The Ricci flat 8D metric and KP-equation  
In the papers [1]-[2] was showed that the eight-dimensional metric in local co-
ordinates (x,y,z,t,P,Q,U,V)  

\[ 8ds^2 = 2 \left( -PH_{11} - P\frac{\partial F}{\partial t} - H_{12}Q - 2\Gamma^1_{11}U + H_{22}V \right) dx^2 + \]

\[ + 4(H_{11}Q - H_{12}V) dx dy + 4 \left( -\frac{\partial F}{\partial z} V + \frac{\partial F}{\partial t} U \right) dx dz + 2(H_{11}U - H_{31}V) dy^2 + \]

\[ + 2dxdP + 2dydQ + 2dzdU + 2dtV, \]  
(1)  
is the Ricci-flat R_{ik} = 0 if the conditions on the functions H_{ij} = H_{ij}(y,z,t), \Gamma^3_{11}(y,z,t) and F(y,z,t)  

\[ \frac{\partial H_{12}}{\partial y} - \frac{\partial H_{22}}{\partial t} = 0, \quad \frac{\partial H_{11}}{\partial y} + \frac{\partial H_{21}}{\partial t} = 0, \quad \frac{\partial H_{11}}{\partial z} + \frac{\partial H_{31}}{\partial t} = 0, \]  
(2)  
are hold.  
The metric (1) has the form  

\[ ds^2 = -\Gamma^i_{jk}dx^i dx^j + 2d\xi_k dx^k \]  
(4)  
and it is an example of Riemann extension of affinely-connected the four di-
dimensional space in local coordinates x^i with symmetric connection \Gamma^i_{jk}(x^i) = \Gamma^i_{kj}(x^j). From the system (2) after the substitution [3]  

H_{11} = -\frac{1}{2} u(y,z,t), \quad H_{12} = -\frac{1}{3} v(y,z,t), \quad H_{21} = -\frac{2}{3} v(y,z,t) - \frac{1}{2} \frac{\partial u(y,z,t)}{\partial t} \]

H_{31} = -\frac{3}{4} w(y,z,t) + \frac{3}{8} u(y,z,t)^2 - \frac{\partial v(y,z,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 u(y,z,t)}{\partial t^2}, \]

H_{22} = -\frac{1}{2} w(y,z,t) + \frac{1}{2} u(y,z,t)^2 - \frac{\partial v(y,z,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 u(y,z,t)}{\partial t^2} + \frac{\partial u(y,z,t)}{\partial y}, \]
the famous KP-equation follows
\[
\frac{\partial}{\partial t} \left( \frac{\partial u(y,z,t)}{\partial z} - \frac{3}{2} u(y,z,t) \frac{\partial u(y,z,t)}{\partial t} - \frac{1}{4} \frac{\partial^3 u(y,z,t)}{\partial t^3} \right) = \frac{3}{4} \frac{\partial^2 u(y,z,t)}{\partial y^2}.
\]

(5)

2 4D the Ricci-flat affinely connected subspace and KP-equation

The main result of [2] is the following

Theorem

The four-dimensional affinely connected space with non zero coefficients of connection \( \Gamma^i_{jk} = \Gamma^i_{kj} \) of the form

\[
\Gamma^1_{11} = H_{11} + \frac{\partial F}{\partial t}, \quad \Gamma^2_{11} = H_{12}, \quad \Gamma^3_{11} = \Gamma^3_{11}, \quad \Gamma^2_{12} = H_{11}, \quad \Gamma^3_{13} = -\frac{\partial F}{\partial t}, \quad \Gamma^3_{22} = -H_{11}, \quad \Gamma^4_{11} = -H_{22}, \quad \Gamma^4_{12} = H_{21}, \quad \Gamma^4_{13} = \frac{\partial F}{\partial z}, \quad \Gamma^4_{22} = H_{31}
\]

is a Ricci flat

\[
R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{li} \Gamma^l_{kj} - \Gamma^k_{im} \Gamma^m_{kj} = 0,
\]

if the conditions (2-3) hold.

The problem of metrizability such type of connection is important in theory of 3-dim manifolds now it is open question.

3 6D the Ricci-flat defined by KP-equation

As particular case we consider the metrics (4) of the form

\[
\begin{align*}
6 \, ds^2 &= 4 \left( \frac{\partial}{\partial x} u(x,y,t) \right) P \, dx \, dt + 2 \, dx \, dP + 4 \left( \frac{\partial}{\partial y} u(x,y,t) \right) P \, dy \, dt + 2 \, dy \, dQ + \\
&+ \left( -2 \, P u(x,y,t) \frac{\partial}{\partial x} u(x,y,t) - 2 \, P \frac{\partial^3}{\partial x^3} u(x,y,t) - 2 \mu \left( \frac{\partial}{\partial y} u(x,y,t) \right) Q + 2 \left( \frac{\partial}{\partial x} u(x,y,t) \right) U \right) dt^2 + \\
&+ 2 \, dt \, dU.
\end{align*}
\]

(6)

The Ricci tensor such type of the metric

\[
R_{ij} =
\]

\[
= 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\frac{\partial}{\partial x} u(x,y,t))^2 + u(x,y,t) \frac{\partial^2}{\partial x^2} u(x,y,t) + \frac{\partial^2}{\partial x \partial y} u(x,y,t) + \frac{\partial^2}{\partial x \partial y} u(x,y,t) + \mu \frac{\partial^2}{\partial y^2} u(x,y,t) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
has one component and it is equal to zero on solutions of the KP-equation
\[ \frac{\partial(u_t + uu_x + ux)}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0. \]

**Remark**

The metric (3) arises from the Riemann extension of the 3D Einstein-Weyl space
\[ 3ds^2 = dy^2 - 4dxdt - 4u(x, y, t)dt^2, \quad \nu = -4uxdt, \]
associated with the DKP-equation
\[ \frac{\partial(u_t - uu_x)}{\partial x} = \frac{\partial^2 u}{\partial y^2}, \]
which is obtained from the Einstein-Weyl conditions on the Ricci tensor
\[ R_{ij} + \frac{1}{2} \nabla_i (\nu_j) + 1/4 \nu_i \nu_j - 1/3 (R + 1/2 \nabla^k \nu_k + 1/4 \nu^k \nu_k) h_{ij} = 0 \]

4 6D the Ricci-flat metrics associated with KdF-equation

3D-Riemann metric of the form
\[ ds^2 = y^2 dx^2 + (l(x, z)) y^2 - 1/2 \ dx \ dz + 2 \ dy \ dz + \]
\[ + \left((l(x, z))^2 y^2 - 2 \left( \frac{\partial}{\partial x} l(x, z) \right) y + l(x, z) \right) dz^2 \]

is a flat \( R_{ijkl} = 0 \) if the function \( l(x, z) \) satisfies the KdF-equation (Dryuma,2006)
\[ \frac{\partial^3 l}{\partial x^3} (x, z) + \frac{\partial}{\partial z} l(x, z) - 3 l(x, z) \frac{\partial}{\partial x} l(x, z) = 0. \]

It has 13 Christoffel symbols. Some of them are in form
\[ \Gamma_{11}^1 = 1/2 \frac{-1 + 2 l(x, z) y^2}{y}, \quad \Gamma_{12}^1 = y^{-1}, \quad \Gamma_{13}^1 = 1/2 \frac{-1 + 2 l(x, z) y^2}{y} \frac{l(x, z)}{y}, \]
\[ \Gamma_{23}^1 = \frac{l(x, z)}{y}, \]
\[ \Gamma_{33}^1 = 1/2 \frac{2 \left( \frac{\partial}{\partial z} l(x, z) \right) y - 4 l(x, z) y \frac{\partial}{\partial z} l(x, z) + 2 \frac{\partial^2}{\partial x^2} l(x, z) - (l(x, z))^2 + 2 \ (l(x, z))^3 y^2}{y}, \]
\[ \Gamma_{11}^2 = -1/4 \frac{4 y^3 \frac{\partial}{\partial z} l(x, z) - 8 l(x, z) y^2 + 1}{y}, \]
\[ \ldots \]
\[ \Gamma_{11}^3 = - y, \quad \Gamma_{13}^3 = - l(x, z) y, \quad \Gamma_{33}^3 = - (l(x, z))^2 y + \frac{\partial}{\partial x} l(x, z). \]

Six-dimensional Riemann extension of the metrics (4) is the space in local coordinates \((x, y, z, \xi_1, \xi_2, \xi_3)\) and it is defined by the expression
\[ ds^2 = -2 \Gamma_{i j k} \xi_i dx^i dx^j + 2 dx^k \xi_k \]
with a given coefficients $\Gamma_{ij}^k$.

Such metric is a flat on solutions of the KdF-equation $S$.

To obtain an example of non a flat $R_{ijkl} \neq 0$ the six-dimensional metrics associated with the KdF-equation it is necessary to introduce an additional terms into the expression $R$.

As example the change of the component of metric

$$g_{13} := l(x,z)y^2 - 1/2$$

on the following

$$g_{1,3} := l(x,z)y^2 - 1$$

change radically the Ricci-tensor of the space.

From

$$Matrix(3, 3, (1, 1) = 0, (1, 2) = 0, (1, 3) = 0, (2, 1) = 0, (2, 2) = 0, (2, 3) = 0, (3, 1) = 0, (3, 2) = 0, (3, 3) = (-l_{xxx} - l_z + 3ll_x)/y,$$

we obtain

$$Matrix(3, 3, (1, 1) = 0, (1, 2) = 0, (1, 3) = -1/2l_x/y, (2, 1) = 0, (2, 2) = 0, (2, 3) = 0, (3, 1) = -1/2l_x/y, (3, 2) = 0, (3, 3) = (1/2)(-2l_{xxx}y^2 - l_{xx}y^2 + 7ly^2 + ll_x)/y^3).$$

In a six-dimensional case it is possible to change components of metric such a way that metric will be Ricci flat $R_{ik} = 0$ on solutions of the KdF-equation but not a flat $R_{ijkl} \neq 0$.

## 5 6D Heavenly metric and Special Lagrangian equation

We study six-dimensional generalization of the Heavenly metrics

$$ds^2 = dx du + dy dv + dz dw + A(x, y, z, u, v, w)du^2 +
+ 2B(x, y, z, u, v, w)du dv + 2E(x, y, z, u, v, w)du dw + C(x, y, z, u, v, w)dv^2 +
+ 2H(x, y, z, u, v, w)dv dw + F(x, y, z, u, v, w)dw^2. \quad (10)$$

The Ricci tensor of the metric $S$ has a fifteen components.

Nine of them are equal to zero due the conditions

$$\frac{\partial}{\partial u}E(x, y, z, u, v, w) + \frac{\partial}{\partial w}H(x, y, z, u, v, w) + \frac{\partial}{\partial w}F(x, y, z, u, v, w) = 0,$$

$$\frac{\partial}{\partial u}B(x, y, z, u, v, w) + \frac{\partial}{\partial v}C(x, y, z, u, v, w) + \frac{\partial}{\partial w}H(x, y, z, u, v, w) = 0,$$

$$\frac{\partial}{\partial u}A(x, y, z, u, v, w) + \frac{\partial}{\partial v}B(x, y, z, u, v, w) + \frac{\partial}{\partial w}E(x, y, z, u, v, w) = 0. \quad (11)$$

This system of equation has a solutions depending from an arbitrary functions.
In a simplest case we have the solution

\[ A(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial z^2} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial y^2} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, u, v, w) \right)^2, \]

\[ C(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial z^2} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial y^2} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, u, v, w) \right)^2, \]

\[ F(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial x^2} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial z^2} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, u, v, w) \right)^2, \]

\[ E(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial x \partial y} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial z^2} f(x, y, z, u, v, w) \right)^2, \]

\[ B(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial y \partial z} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial x^2} f(x, y, z, u, v, w) \right)^2, \]

\[ H(x, y, z, u, v, w) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, u, v, w) \right) \frac{\partial^2}{\partial x \partial y} f(x, y, z, u, v, w) - \left( \frac{\partial^2}{\partial y^2} f(x, y, z, u, v, w) \right)^2, \]

depending from one arbitrary function.

At this conditions the six-dimensional metric looks as

\[ 6 ds^2 = \left( \frac{\partial^2}{\partial w^2} K(\vec{x}) \frac{\partial^2}{\partial v^2} K(\vec{x}) - \left( \frac{\partial^2}{\partial v \partial w} K(\vec{x}) \right)^2 \right) dx^2 + \]

\[ + 2 \left( \frac{\partial^2}{\partial u \partial w} K(\vec{x}) \frac{\partial^2}{\partial v \partial w} K(\vec{x}) - \frac{\partial^2}{\partial u^2} K(\vec{x}) \frac{\partial^2}{\partial v \partial w} K(\vec{x}) \right) dxdy + \]

\[ + \left( \frac{\partial^2}{\partial u^2} K(\vec{x}) \frac{\partial^2}{\partial v^2} K(\vec{x}) - \left( \frac{\partial^2}{\partial u \partial v} K(\vec{x}) \right)^2 \right) dy^2 + \]

\[ + 2 \left( \frac{\partial^2}{\partial v \partial w} K(\vec{x}) \frac{\partial^2}{\partial u \partial w} K(\vec{x}) - \frac{\partial^2}{\partial u^2} K(\vec{x}) \frac{\partial^2}{\partial v \partial w} K(\vec{x}) \right) dxz + \]

\[ + \left( \frac{\partial^2}{\partial u^2} K(\vec{x}) \frac{\partial^2}{\partial v^2} K(\vec{x}) - \left( \frac{\partial^2}{\partial u \partial v} K(\vec{x}) \right)^2 \right) dz^2 + \]

\[ + 2 \left( \frac{\partial^2}{\partial u \partial w} K(\vec{x}) \frac{\partial^2}{\partial u \partial v} K(\vec{x}) - \frac{\partial^2}{\partial u^2} K(\vec{x}) \frac{\partial^2}{\partial v \partial w} K(\vec{x}) \right) dzy + \]

\[ + dxdx + dydy + dzdz \]  

(12)

where \( K(\vec{x}) = K(x, y, z, u, v, w) \) is arbitrary function.

The Ricci tensor \( R_{ij} \) of the metric has a six components.
All equations \( R_{ij} = 0 \)
after the substitution
\[
K(x, y, z, u, v, w) = \phi(y + v + x, z + w + x)
\]
are reduced to the one equation
\[
-\left( \frac{\partial^4}{\partial \xi^4} \right) \phi(\xi, \rho) + 2 \left( \frac{\partial^3}{\partial \xi^3 \partial \rho} \right) \phi(\xi, \rho)^2 - 2 \left( \frac{\partial^3}{\partial \rho^3} \right) \phi(\xi, \rho) -
\]
\[
- \left( \frac{\partial^2}{\partial \rho^2} \phi(\xi, \rho) \right) \frac{\partial^4}{\partial \xi^4} \phi(\xi, \rho) + 2 \left( \frac{\partial^2}{\partial \rho \partial \xi} \phi(\xi, \rho) \right) \frac{\partial^4}{\partial \xi^4} \phi(\xi, \rho) +
\]
\[
+ 2 \left( \frac{\partial^3}{\partial \rho^3} \phi(\xi, \rho) \right)^2 - \left( \frac{\partial^2}{\partial \rho^2} \phi(\xi, \rho) \right) \frac{\partial^4}{\partial \xi^4} \phi(\xi, \rho) - 2 \left( \frac{\partial^3}{\partial \rho^3} \phi(\xi, \rho) \right) \frac{\partial^3}{\partial \rho \partial \xi} \phi(\xi, \rho) -
\]
\[
- \left( \frac{\partial^2}{\partial \rho^2} \phi(\xi, \rho) \right) \frac{\partial^4}{\partial \xi^4} \phi(\xi, \rho) + 2 \left( \frac{\partial^2}{\partial \rho \partial \xi} \phi(\xi, \rho) \right) \frac{\partial^4}{\partial \xi^4} \phi(\xi, \rho) = 0, \quad (13)
\]
where
\[
\xi = x + y + v, \quad \rho = z + x + w.
\]
In compact form this equation can be rewritten as
\[
\Delta \psi(\xi, \rho) = 0
\]
where
\[
\psi(\xi, \rho) = \left( \frac{\partial^2}{\partial \xi^2} \phi(\xi, \rho) \right) \frac{\partial^2}{\partial \rho^2} \phi(\xi, \rho) - \left( \frac{\partial^2}{\partial \xi \partial \rho} \phi(\xi, \rho) \right)^2
\]
and
\[
\Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \rho^2}
\]
is the Laplace operator.

Its solutions give the Ricci-flat examples of the metric (5).

### 5.1 The Beltrami parameters

To the investigation of the properties of the metrics (5) can be considered two invariant equations defined by the first
\[
\Delta \psi = g^{ij} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j}
\]
and the second
\[
\Box \psi = g^{ij} \left( \frac{\partial^2}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial}{\partial x^k} \right) \psi
\]

Beltrami parameters.

To the metric (5) the equation \( \Box \phi = 0 \) looks as \( K = \phi(\vec{x}) \)
\[
\frac{\partial^2}{\partial u \partial x} \phi(\vec{x}) + \frac{\partial^2}{\partial v \partial z} \phi(\vec{x}) + \frac{\partial^2}{\partial v \partial y} \phi(\vec{x}) - \left( \frac{\partial^2}{\partial y^2} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial z^2} \rho(\vec{x}) \right) +
\]
\[
\frac{\partial^2}{\partial u \partial x} \phi(\vec{x}) + \frac{\partial^2}{\partial v \partial z} \phi(\vec{x}) + \frac{\partial^2}{\partial v \partial y} \phi(\vec{x}) - \left( \frac{\partial^2}{\partial y^2} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial z^2} \rho(\vec{x}) \right) +
\]
The equation (5.1) is reduced to the form

\[ +2 \left( \frac{\partial^2}{\partial x \partial z} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial x \partial z} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) - 2 \left( \frac{\partial^2}{\partial x \partial z} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial y^2} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) + \]

\[ +2 \left( \frac{\partial^2}{\partial y \partial z} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial y \partial z} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) - \left( \frac{\partial^2}{\partial z^2} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial y^2} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) - \]

\[ - \left( \frac{\partial^2}{\partial z^2} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial y^2} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) - 2 \left( \frac{\partial^2}{\partial x \partial y} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial x \partial z} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) + \]

\[ + \left( \frac{\partial^2}{\partial x^2} \phi(\vec{x}) \right) \left( \frac{\partial^2}{\partial y \partial z} f(\vec{x}) \right)^2 \left( \frac{\partial^2}{\partial \eta^2} f(\vec{x}) \right)^2 + \]

\[ + \frac{\partial^2}{\partial x^2} \phi(\vec{x}) \left( \frac{\partial^2}{\partial y \partial z} f(\vec{x}) \right)^2 = 0. \quad (14) \]

In particular case the equation (5.1) after the substitution

\[ \phi(\vec{x}) = f(\vec{x}) \]

takes the form

\[ \frac{\partial^2}{\partial \eta \partial x} f(\vec{x}) + \frac{\partial^2}{\partial \eta \partial z} f(\vec{x}) + \frac{\partial^2}{\partial y \partial \eta} f(\vec{x}) - 3 \left( \frac{\partial^2}{\partial x^2} f(\vec{x}) \right) \left( \frac{\partial^2}{\partial \eta^2} f(\vec{x}) \right) \frac{\partial^2}{\partial z^2} f(\vec{x}) + \]

\[ +3 \left( \frac{\partial^2}{\partial x \partial z} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial \eta^2} f(\vec{x}) - 6 \left( \frac{\partial^2}{\partial x \partial \eta} f(\vec{x}) \right) \left( \frac{\partial^2}{\partial y \partial \eta} f(\vec{x}) \right) \frac{\partial^2}{\partial \eta^2} f(\vec{x}) + \]

\[ +3 \left( \frac{\partial^2}{\partial x \partial y} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial \eta^2} f(\vec{x}) + 3 \left( \frac{\partial^2}{\partial y \partial \eta} f(\vec{x}) \right)^2 \frac{\partial^2}{\partial \eta^2} f(\vec{x}) = 0. \quad (15) \]

After the change of variables

\[ f(\vec{x}) = f(x, y, z, u, v, w) = h(x + u, v + y, w + z) = h(\eta, \xi, \rho) \]

the equation (5.1) is reduced to the form

\[ \frac{\partial^2}{\partial \eta^2} h(\eta, \xi, \rho) - \frac{\partial^2}{\partial \rho^2} h(\eta, \xi, \rho) + \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) + 3 \left( \frac{\partial^2}{\partial \eta \partial \rho} h(\eta, \xi, \rho) \right)^2 \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) - \]

\[ -6 \left( \frac{\partial^2}{\partial \eta \partial \rho} h(\eta, \xi, \rho) \right) \left( \frac{\partial^2}{\partial \rho \partial \xi} h(\eta, \xi, \rho) \right) \frac{\partial^2}{\partial \eta^2} h(\eta, \xi, \rho) + 3 \left( \frac{\partial^2}{\partial \eta \partial \rho} h(\eta, \xi, \rho) \right)^2 \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) = \]

\[ -3 \left( \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) \right) \left( \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) \right) + 3 \left( \frac{\partial^2}{\partial \rho^2} h(\eta, \xi, \rho) \right)^2 \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) = 0. \quad (16) \]

or

\[ \Delta h(\eta, \xi, \rho) - 3 \det \begin{bmatrix} \frac{\partial^2}{\partial \eta^2} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \rho^2} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \xi^2} h(\eta, \xi, \rho) \\ \frac{\partial^2}{\partial \eta \partial \rho} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \rho \partial \xi} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \eta \partial \rho} h(\eta, \xi, \rho) \\ \frac{\partial^2}{\partial \eta \partial \xi} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \rho \partial \xi} h(\eta, \xi, \rho) & \frac{\partial^2}{\partial \rho^2} h(\eta, \xi, \rho) \end{bmatrix} = 0, \]
where

\[ \Delta = \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \rho^2} \]

is a three-dimensional Laplace operator.

The equation (5.1) is a famous Harvey-Lawson "Special Lagrangian" equation having an important applications in theory of Calabi-Yau manifolds and mirror symmetry.

### 5.2 The simplest solutions

To obtain particular solutions of the partial nonlinear differential equation

\[ F(x, y, z, f_x, f_y, f_z, f_{xx}, f_{xy}, f_{yy}, f_{xxy}, f_{yyyy}, ...) = 0 \] (17)

can be applied a following approach.

We use the following parametric presentation of the functions and variables

\[ f(x, y, z) \rightarrow u(x, t, z), \quad y \rightarrow v(x, t, z), \quad f_x \rightarrow u_x - \frac{u_t}{v_t} v_z, \]

\[ f_z \rightarrow u_z - \frac{u_t}{v_t} v_z, \quad f_y \rightarrow \frac{u_t}{v_t}, \quad f_{yy} \rightarrow \frac{(u_t)^2}{v_t}, \quad f_{xy} \rightarrow \frac{(u_x - \frac{u_t}{v_t} v_x)}{v_t}, \ldots \] (18)

where variable \( t \) is considered as parameter.

Remark that conditions of the type

\[ f_{xy} = f_{yx}, \quad f_{xz} = f_{zx}, \ldots \]

hold at the such type of presentation.

In result instead of equation (17) one get the relation between the new variables \( u(x, t, z) \) and \( v(x, t, z) \) and their partial derivatives

\[ \Psi(u, v, u_x, u_z, v_x, v_z, v_t, ...) = 0. \] (19)

This relation coincides with initial p.d.e at the condition \( v(x, t, z) = t \) and takes more general form after presentation of the functions \( u, v \) in form \( u(x, t, z) = F(\omega, \omega_t, ... \) and \( v(x, t, z, s) = \Phi(\omega, \omega_t, ... \) with some function \( \omega(x, t, z) \).

In result of change variables and function with accordance of (5.2) the equation (5.1)

\[ \frac{\partial^2}{\partial x^2} h(x, y, z) + \frac{\partial^2}{\partial z^2} h(x, y, z) + \frac{\partial^2}{\partial y^2} h(x, y, z) + 3 \left( \frac{\partial^2}{\partial x \partial z} h(x, y, z) \right)^2 \frac{\partial^2}{\partial y^2} h(x, y, z) - 6 \left( \frac{\partial^2}{\partial x \partial z} h(x, y, z) \right) \left( \frac{\partial^2}{\partial y \partial z} h(x, y, z) \right) + \frac{\partial^2}{\partial x^2} h(x, y, z) + 3 \left( \frac{\partial^2}{\partial x \partial y} h(x, y, z) \right)^2 \frac{\partial^2}{\partial z^2} h(x, y, z) - 3 \left( \frac{\partial^2}{\partial y^2} h(x, y, z) \right) \left( \frac{\partial^2}{\partial z^2} h(x, y, z) \right) + \frac{\partial^2}{\partial y \partial z} h(x, y, z) + 3 \left( \frac{\partial^2}{\partial y \partial z} h(x, y, z) \right)^2 \frac{\partial^2}{\partial x^2} h(x, y, z) = 0 \]

is transformed into the relation (19).

This relation after the substitution

\[ u(x, t, z) = t \frac{\partial}{\partial t} \omega(x, t, z) - \omega(x, t, z), \quad v(x, t, z) = \frac{\partial}{\partial t} \omega(x, t, z). \] (20)
takes the form of p.d.e.

\[-3 \left( \frac{\partial^2}{\partial x^2} \omega(x, t, z) \right) \frac{\partial^2}{\partial z^2} \omega(x, t, z) + 3 \left( \frac{\partial^2}{\partial x \partial z} \omega(x, t, z) \right)^2 - \left( \frac{\partial^2}{\partial t^2} \omega(x, t, z) \right) \frac{\partial^2}{\partial x^2} \omega(x, t, z) +
\]

\[+ 1 + \left( \frac{\partial^2}{\partial t \partial z} \omega(x, t, z) \right)^2 + \left( \frac{\partial^2}{\partial t \partial x} \omega(x, t, z) \right)^2 - \left( \frac{\partial^2}{\partial t \partial z} \omega(x, t, z) \right) \frac{\partial^2}{\partial z^2} \omega(x, t, z) = 0.\]

The equation (21) consists from the 2D M-A equations with respect the variables \((x, t), (x, z)\) and \((t, z)\).

Let us consider some examples of its solutions.

1. To the equation

\[\mu \Delta (f) + \text{Hess}(f) = 0\]  (22)

The substitution into (22)

\[\omega(x, t, z) = A(x^2 + z^2, t)\]

lead to the equation on the function \(A(\eta, t), (\eta = x^2 + z^2)\)

\[-4 \mu \left( \frac{\partial^2}{\partial t \partial \eta} A(\eta, t) \right)^2 \eta - 8 \left( \frac{\partial^2}{\partial \eta^2} A(\eta, t) \right) \frac{\partial}{\partial \eta} A(\eta, t) - 4 \left( \frac{\partial}{\partial \eta} A(\eta, t) \right)^2 +
\]

\[+ 4 \mu \left( \frac{\partial^2}{\partial \eta^2} A(\eta, t) \right) \frac{\partial}{\partial \eta} A(\eta, t) - \mu + 4 \mu \left( \frac{\partial^2}{\partial \eta^2} A(\eta, t) \right) \frac{\partial^2}{\partial \eta^2} A(\eta, t) = 0.\]

Its particular solution is

\[A(\eta, t) = B(t) + \eta e^{kt},\]

\[B(t) = \frac{e^{kt}}{\mu k^2} + 1/4 \frac{e^{-kt}}{k^2} - C1 t + C2.\]

Now elimination of the parameter \(t\) from the system

\[y - \frac{\partial}{\partial t} \omega(x, t, z) = 0, \quad f(x, y, z) - \frac{\partial}{\partial t} \omega(x, t, z) + \omega(x, t, z) = 0\]

lead to the solution of the equation (22)

\[f(x, y, z) = -\left( 1 - \ln \left( \frac{y \mu + \sqrt{T}}{1 + \mu x^2 + \mu z^2} \right) y \sqrt{T} + \ln \left( \frac{2 y^2 \mu - \ln \left( \frac{y \mu + \sqrt{T}}{1 + \mu x^2 + \mu z^2} \right) y^2 \mu}{y \mu + \sqrt{T}} \right)^{-1} +
\]

\[+ \left( y^2 \mu + \mu x^2 \right) + \left( \ln \left( \frac{2 y \mu + \sqrt{T} + \mu z^2 + y \sqrt{T}}{y \mu + \sqrt{T}} \right) \right)^{-1},\]

where

\[\mu \left( y^2 + 1 + \mu x^2 + \mu z^2 \right) = T\]

2. The substitution

\[\omega(x, t, z) = A(x + t, z)\]

into (22) lead to the equation on the function \(A(\text{eta}, z) (\eta = x + z)\)

\[- \left( \frac{\partial^2}{\partial \eta^2} A(\eta, z) \right) \frac{\partial^2}{\partial z^2} A(\eta, z) - \mu \left( \frac{\partial^2}{\partial \eta \partial z} A(\eta, z) \right)^2 + \mu \left( \frac{\partial^2}{\partial \eta^2} A(\eta, z) \right) \frac{\partial^2}{\partial z^2} A(\eta, z) -
\]
\[-\mu + \left( \frac{\partial^2}{\partial \eta \partial z} A(\eta, z) \right)^2 \]

At the \( \mu = -1/3 \) we get the M-A equation

\[-\left( \frac{\partial^2}{\partial \eta^2} A(\eta, z) \right) \frac{\partial^2}{\partial z^2} A(\eta, z) + \left( \frac{\partial^2}{\partial \eta \partial z} A(\eta, z) \right)^2 + 1/4 = 0 \quad (23)\]

which after the \((u,v)\)-transformation is reduced to the Laplace equation

\[4 \frac{\partial^2}{\partial \eta^2} \theta(\eta, \xi) + \frac{\partial^2}{\partial \xi^2} \theta(\eta, \xi) = 0.\]

Its general solution

\[\theta(\eta, \xi) = M(\eta + 2I\xi) + N(\eta - 2I\xi)\]

allow us to construct particular solutions of the equation \((23)\) and corresponding “Special lagrangian equation”.

### 6 Heavenly metrics and Yang-Mills equation

We consider following generalization of Heavenly metrics

\[ds^2 = dx \, du + dy \, dv + dz \, dw + A(x, y, z, p, q) \, du^2 + 2B(x, y, z, p, q) \, du \, dv + \]

\[+ 2E(x, y, z, p, q) \, du \, dw + C(x, y, z, p, q) \, dv^2 + 2H(x, y, z, p, q) \, dv \, dw + \]

\[+ F(x, y, z, p, q) \, dw^2 + dp \, dq, \quad (24)\]

having a following components

\[A(x, y, z, p, q) = \left( \frac{\partial^2}{\partial z^2} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial y^2} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \right)^2,\]

\[C(x, y, z, p, q) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial y^2} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \right)^2,\]

\[F(x, y, z, p, q) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right)^2,\]

\[E(x, y, z, p, q) = \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right)^2,\]

\[B(x, y, z, p, q) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \right)^2,\]

\[H(x, y, z, p, q) = \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) - \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right)^2.\]

At these conditions from 21 components of the Ricci-tensor of the metric (21) only six components

\[R_{uu} \neq 0, \quad R_{uv} \neq 0, \quad R_{uw} \neq 0, \quad R_{uv} \neq 0, \quad R_{vw} \neq 0, \quad R_{ww} \neq 0.\]
are different from zero.

The equation

\[ g^{ij} \left( \frac{\partial^2 \Psi}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \Psi}{\partial x^k} \right) = 0, \]

defined by the Laplace-Beltrami operator of the metric (6) depends on the function \( f(x, y, z, p, q) \) and after the substitution \( f = \Psi \) takes the form

\[ \frac{\partial^2}{\partial p \partial q} f(x, y, z, p, q) + \\
+ 3 \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right)^2 \frac{\partial^2}{\partial z^2} f(x, y, z, p, q) - \\
- 6 \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right) \left( \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) + \\
+ 3 \left( \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial x^2} f(x, y, z, p, q) + \\
- 3 \left( \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \right) \frac{\partial^2}{\partial y^2} f(x, y, z, p, q) = 0 \]

or

\[ \frac{\partial^2}{\partial p \partial q} f(x, y, z, p, q) - \\
- 3 \left[ \begin{array}{ccc}
\frac{\partial^2}{\partial x^2} f(x, y, z, p, q) & \frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) & \frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) \\
\frac{\partial^2}{\partial x \partial y} f(x, y, z, p, q) & \frac{\partial^2}{\partial y^2} f(x, y, z, p, q) & \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) \\
\frac{\partial^2}{\partial x \partial z} f(x, y, z, p, q) & \frac{\partial^2}{\partial y \partial z} f(x, y, z, p, q) & \frac{\partial^2}{\partial z^2} f(x, y, z, p, q)
\end{array} \right] = 0. \tag{25} \]

The equation (25) is of Monge-Ampere type and after the substitution

\[ f(x, y, z, p, q) = U(p, q) E(x, y, z) \]

takes the form

\[ \frac{\partial^2}{\partial p \partial q} U(p, q) - \frac{3}{(U(p, q))^3} \text{Hess}(E(x, y, z)) = 0. \]

So it has particular solutions defined by the equations

\[ 3 \text{Hess}(E) - \mu E = 0 \]

and

\[ \frac{\partial^2 U}{\partial p \partial q} - \mu U^3 = 0. \]

Both of equations have an important applications.

The second equation is reduction of the \( SU(2) \) Yang-Mills equation and the first is a Monge-Ampere type of equation having applications in theory of Calabi-Yau manifolds.
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