The paper discusses fundamental problems in mathematical description of social systems based on physical concepts, with so-called statistical social systems being the main subject of consideration. Basic properties of human beings and human societies that distinguish social and natural systems from each other are listed to make it clear that individual mathematical formalism and physical notions should be developed to describe such objects rather than can be directly inherited from classical mechanics and statistical physics. As a particular example systems with motivation are considered. Their characteristic features are analyzed individually and the appropriate mathematical description is proposed. Finally the paper concludes that the basic elements necessary for describing statistical social systems or, more rigorously, systems with motivation are available or partly developed in modern physics and applied mathematics.

Keywords: social systems; statistical systems; motivation; decision-making; prediction; bounded rationality, action points.

1. Introduction

During the last years it has become evident that a novel interdisciplinary branch of science, physics of social systems or sociophysics, is currently under development (for a review of the state of art in this field and its history see, e.g., [1] [2] [3] [4] [5] [6]). Various social processes and phenomena observed in large group of people integrated together by some activity have become the subject matter of this science. Voting behavior in elections, opinion formation, culture and language evolution, cooperative interaction among trade agents, dynamics of traffic and pedestrian flows,
Ihor Lubashevsky and Natalia Plawinska

etc. are typical objects of such investigations.

At the current state of art, the mathematical formalism of sociophysics is based on the notions and methods developed in statistical physics of many-particle systems. One of the fundamental concepts adopted in studying and modeling social systems comprising many individuals (elements) is the principle of local self-averaging. With respect to social systems, it implies that analyzed phenomena are mainly controlled by cooperative interaction of individuals, i.e. based on cumulative contribution of many elements. As a result, the individual properties of similar elements are averaged in some manner during their interaction. The latter feature enables us, first, to introduce the notion of characteristic element (social agent) with properties being the same for all such individuals. Second, in this way the individuality of human beings can be taken into account in terms of random factors characterized by statistical properties being again the same for all the characteristic elements of one type. In addition, the effect of “social influence” [9] stimulates individuals to behave alike. Exactly the principle of local self-averaging is grounds for applying the techniques of statistical physics to describing these social systems which will be referred below as to statistical social systems.

To avoid confusion it should be pointed out that here the principle of local self-averaging is used only to introduce the notion of characteristic element and does not necessary cause the mean field description to hold. For instance, the introduction of aggregate variables in macroeconomics can fail when fluctuations in the behavior of economic agents become crucial (for review of this problem see, e.g. [10] [11] [12]). Nevertheless even in this case the principle of local self-averaging holds if such fluctuations comprise many individuals behaving alike.

In spite of the progress achieved in sociophysics the long-standing question of whether mathematical formalism and physical notions are applicable, at least in principle, to describing social systems is actual up to now. There are widely different opinions on this question, including the well known points of the classics of sociology. Emile Durkheim claimed that from the general point of vies social and natural disciplines are rather similar in methods and approaches [13], whereas Max Weber considered them to be fundamentally distinctive [14]. The present paper makes an attempt to restate the given question and in this way to overcome this contradiction. Assuming the statistical social systems to admit of a description in terms of physical and mathematical notions we pose a question as to what mathematical formalism is appropriate and able to take into account the basic peculiarities of social systems distinguishing them from the natural objects. By way of example so-called systems with motivation will be considered in detail.

It should be noted that the majority of mathematical models proposed for statistical social systems use the notions and concepts inherited directly from physics.

*It should be noted that collective behavior of animal groups, such as schools of fish, flocks of birds or swarms of insects (for a review see, e.g. [7] [8]) is actually a relative problem to the subject under consideration.
(see, e.g., reviews [1, 6]). These models, however, are applicable to social systems as a rough approximation only, at least, on the “microscopic” (most detailed) level because of taking into account just a few of the basic peculiarities of human beings. So, first, let us discuss them.

2. Peculiarities of social systems

Below we will list basic characteristics distinguishing social and physical systems from each other that are essential for the further mathematical constructions.

- **(Individuality and complexity)** Social systems are made up of elements (individuals, agents, decision makers, etc.) with pronounced individuality in behavior and cognition. Besides, from the standpoint of the modeling, human beings are multifactorial objects; the detailed behavior of each of them is a complex outcome of not only social processes but a large number of physiological and psychological ones. So many factors really influencing social phenomena remain uncontrollable and lying beyond the analysis of social systems. By contrast, elements of physical systems are assumed to admit of a complete description at least on the “microscopic” level and elements of one type are identical in properties.

- **(Uncertainty)** The individuality and complexity of human beings endow social systems with original variability and partial uncertainty. As a result both the regular and random factors are present in their dynamics on any level of detail. The classical dynamics of natural objects is deterministic on the “microscopic” level and the probabilistic formalism used in the statistical physics is caused by their reduced description.

- **(Memory and time constraints)** The laws of social systems can change as the human society evolves. Thereby, first, a specific implementation of these laws should have its onset and a finite lifetime. Second, in studying the regularities of social systems reproducing the initial conditions could be hampered or even impossible. Third, a priory, it is not clear how long the memory of social objects is. In other words, how long time span should separate events in the past from the present instant in order to ignore their effects within the most detailed description. Natural systems, by contrast, are characterized by the reproducibility; under the same initial and external conditions either their dynamics or probabilistic characteristics are identical on all the trails. In this meaning, the history of natural systems does not matter.

- **(Motivation and value factors)** The human behavior is governed by many motives for achieving individual goals as well as obeys the social and cultural norms. There is, typically, a set of possible strategies of behavior.

\[\text{b}^{\text{b}}\text{The quantum uncertainty reflecting the wave-particle dualism is unrelated to the subject under discussion.}\]
among which a decision maker chooses the appropriate one. In doing so he applies to various value factors that reflect his individual preferences and the social and cultural meaning as well. These notions are just inapplicable to natural systems.

- **(Information deficiency and breakdown of the explicit means-end relationships)** The decision-making environment involves many factors, external and internal ones, that, on one hand, are hidden, i.e. are not recognizable and controllable in principle for decision makers. On the other hand, these factors affect substantially the dynamics of social systems. Therefore, first, decision makers seldom have perfect information about the choice alternatives and their consequences. Second, if a cause and its effect are separated by a significant time interval it could be difficult to recognize and establish their relationship even within a very thorough analysis. For the disciplines studying natural objects the existence of the direct means-end relationships is one of their cornerstones.

- **(Learning, prediction, and social norms)** To choose an appropriate behavior under the information deficiency of social system states human beings draw on either their own experience or the experience of the society. The former one is gained during the learning process based on predicting the results of their actions. The latter one is aggregated in various social norms of human behavior. In particular, due to effects of the human prediction the dynamics of a social system is affected substantially not only by its history and the current state, but also by its possible future development existing in the human mind. These notions are also inapplicable to natural systems.

The aforementioned features enable us to claim that the description of statistical social systems requires individual physical notions and mathematical formalism to be developed. In what follows we will discuss these points in detail with respect to a certain specific class of such objects, namely, statistical ensembles of elements whose dynamics is governed by motives for their actions, the evaluation of possible behavior strategies and making the appropriate choice with its further correction. We will call such ensembles systems with motivation. Traffic and pedestrian flows, interacting market agents, as well as in some sense bird flocks and fish schools can be regarded as characteristic examples of the systems under consideration.

However before passing directly to the main subject of the paper let us consider from the general point of view the decision-making process. It plays the essential role in a large variety of social systems and its properties should be taken into account in the mathematical description of any statistical social system.

### 3. Decision-making process

The classical theory of making decisions is based on the notion of the preference relation and the utility function quantifying this relation (see, e.g., [15]). The concept of the perfect rationality assumes the human choice or decision to be determined
by the most preferable result meeting the maximum of the utility function. The related theory of making decisions under uncertainty also deals with some utility function aggregating in itself the realization of various environmental conditions in a probabilistic way. However, such an approach encounters obstacles caused by the fundamental properties of human beings described above. For example, the possibility of introducing the preference relation with respect to the final goals seems to be doubtful whereas local aims that are similar in value can be indistinguishable for human beings in making decisions.

In order to overcome these obstacles the concepts of bounded rationality [16, 17] and limited cognition [18] have been developed (see also [19, 20, 21]). In particular, it has been proposed [21] that the decision-making process (at least in statistical social systems) should be mainly based on selection of possible behavior strategies rather than final goals. Such a strategy is a certain sequence of local actions, i.e. a collection of steps of achieving subsequent intermediate aims. These strategies are formed in the trial-and-error process and evolve during the adaptation of individuals to the decision-making environment under uncertainty of the information about the social system states. Following [21] we will call these strategies heuristics.

These heuristics aggregate and accumulate the information about the previous actions, successful and failed ones. That is why the history of a social system impacts on its dynamics. There are at least two distinct ways of the heuristics formation. The first one is the individual learning, i.e. the process of gaining the knowledge about the successful rules of behavior via the personal experience or the experience of individuals directly related to a given one. In particular, the idea that the individual learning plays the leading role in the heuristics formation has been developed in [22, 23] (see also references therein). The second way deals with the cooperative interaction of many individuals forming large units of human society. It is implemented via the formation of the social norms and cultural values aggregating all the fragments of information about the human society for a rather long time interval. The human societies possess own mechanisms governing the social norms and keeping up the social order (see, e.g., [21] and references therein). There are at least two types of models for mechanisms via which the social norms and cultural values arise and evolve. One of them is based on emulating the behavior of the most successful persons, i.e. the social interdependence via significant others [24]. The other type models, interdependence via reference groups [25], go beyond the individualistic level of social interdependence. They relate the social and cultural proclivities of human behavior to some large groups or their typical representatives that have high social rank.

4. Systems with motivation

In what follows we will confine our consideration to systems with motivation. The main purpose of the remaining part is to develop for such systems the main physical notions and concepts of mathematical formalism that allow for the basic features of
human behavior discussed above to be implemented in the appropriate models. In this way we demonstrate, in particular, the feasibility of overcoming the discrepancy between the disciplines studying natural and social systems.

It should be underlined beforehand, that we do not intend to construct a self-consistent and complete theory of systems with motivation; it goes far beyond the scope the present analysis. Our goal is to consider their main features individually and to formulate the conceptual basis for constructing mathematical models for specific phenomena.

4.1. Extended phase space

As a system with motivation is concerned, by the term “phase space” we mean a collection of variables \(\{w\}\) that completely characterize its state at the current moment of time \(t\). In contrast to physical systems, these variables taken at the current moment of time are not necessary to determine the system dynamics, i.e. the rate of their time variations. However, known the values of these variables at all the previous moments of time, the system dynamics has to be determined completely, may be in some probabilistic way. In other words, it is assumed that the rates \(\{\delta_tw\}\) of time variations in the phase variables are certain functionals on themselves rather then some functions,

\[
\{w[t']\}_{t'<t} \longrightarrow \{\delta_tw\}.
\]

(1)

Here the square brackets at the symbol \(w\) stand for the function \(w\) of the argument \(t'\) rather than its value taken at \(t'\) and the symbol \(\delta_tw\) denotes the time derivative \(\dot{w}\) of the corresponding variable if it is continuous one or, otherwise, specifies step-like jumps between its possible values.

Due to the active cognitive behavior of the elements the phase space \(\{w\}\) of a system with motivation comprises variables of two types, objective and subjective ones, \(\{w\} = \{q, h\}\). Let us discuss these types of phase variables individually.

**Objective phase space:** There is assumed to be a collection of variables \(\{q\}_\alpha\), discrete or continuous ones, that completely characterize the possible states of a given element \(\alpha\) from the standpoint of the other elements. The information about the state of the element \(\alpha\) is necessary for them to make the appropriate decisions in governing their own states. We have used the term “objective” in order to underline that the characteristics \(\{q\}_\alpha\) of the element \(\alpha\) are detectable for the other elements. They are not related to the intentions, plans, wishes of the element \(\alpha\) which are hidden for external observers. It should be noted that the variables \(\{q\}_\alpha\) are accessible for external observers only in principle. As noted above in a social system getting information about the states of its elements can be hampered. The quantities \(\{q\}_\alpha\) will be referred to as the objective phase variables ascribed to the element \(\alpha\) and their combination for all the elements, \(\{q\}\), makes up the objective phase space of the given system. For example, the spatial coordinates of pedestrians, the direction of motion, and may be their velocities form the objective phase space of the pedestrian ensemble, the coordinates and the velocities of vehicles on a highway make
up the objective phase space of traffic flow. The set of personal opinions makes the objective phase space of voting process, cultural features with preferences ascribed to every individual can be regarded as the objective phase space of the cultural dynamics. The production and comprehensive matrices characterizing the frequency of using and associating words to the corresponding objects by individuals can be considered in the same way in describing the evolution of languages (see [1] and references therein).

Among the objective phase variables a special group of controllable variables should be singled out. Elements of social systems try to control their own states, which influences the system dynamics. This self-control is determined, on one hand, by individual motives, desires, wishes, goals, plans etc. and, on the other hand, by the social order and the rules of behavior. It is implemented via maintaining or changing the objective phase variables or a certain group of them enabling this action directly. The subscript \( c \) will be added to the corresponding quantities \( \{q_c\} \) to underline the given feature. Time variations of the remaining quantities are determined by these controllable variables and, may be, some natural regularities. For example, in traffic flow a driver can change directly only the velocity of his car. Therefore the velocities of cars are the controllable variables, whereas the coordinates of their position on highways are not so.

Keeping in mind traffic flow and a set of voters as examples of systems with motivation, it is naturally to assume that if the objective phase space contains together with a variable \( w \) also its time derivative \( \dot{w} \), then the latter is a controllable phase variable whereas the variable \( w \) itself is not it. Otherwise, when only the variable \( w \) enters the phase space it should be a controllable variable.

**Subjective phase space:** By definition, the subjective variables \( \{h\} \) describe time variations in the controllable phase variables, namely, \( \{h := \delta q_c\} \). The collection of quantities \( \{h\}_\alpha \) ascribed to a given element \( \alpha \) will be referred to as the subjective (hidden) phase variables of the element \( \alpha \) and the combination of all these quantities will be called the subjective phase space of the given system.

The quantities \( \{h\}_\alpha \) characterize active behavior of the element \( \alpha \) in managing its state and are related to its motives, wishes, goals, etc. in making decisions. So they are accessible only for the element \( \alpha \) and hidden for the others. For every element \( \alpha \) its subjective phase variables \( \{h\}_\alpha \) are valuable in their own right. This is due to the fact that internal processes accompanying the decision-making themselves take effort in order, for example, to get a decision of changing the current state of the element \( \alpha \). In addition, time variations in the quantities \( \{q_c\} \) can affect this element in some physical way. Therefore, in making decision the preferences are determined directly by the objective and subjective variables simultaneously. That is why the phase space of systems with motivation is made up of both the types of the phase variables, \( \{w\} = \{q, h\} \). For example, the accelerations of cars moving on a highway have to be treated as the subjective phase variables of traffic flow and the functions quantifying the quality of individual car motion should contain the car acceleration in the list of their arguments [26, 27, 28]. Figure 1 illustrates the
structure of the extended phase space.

extended phase space \( \{x, v, a\} \)

![Diagram](image_url)

Fig. 1. Illustration of the structure of the extended phase space for systems with motivation.

4.2. Decision-making and heuristics choice

The decision making process governs time dependence of the controllable objective variables, which in turn via physical regularities determines the system dynamics as a whole. Symbolically we write this in terms of time increment in the phase variables

\[
\text{decision-making} \quad \{w\} = \{q, h\} \\
\{h = \delta_t q_c\} \oplus \text{natural regularities} \quad (2)
\]

As noted in Sec. 3 the decision-making is reduced to the choice of local heuristics because of the bounded capacity of human cognition and the variety of factors uncontrollable and hidden for elements of a social system. These heuristics, i.e., the local strategies of the element behavior are sequences of actions focused on achieving local aims. Since in statistical social systems the explicit means-end relationships can be broken the specific actions of elements are evaluated by local motives rather
than intentions of getting final goals. The latter goals can only single out some rather general class of the element actions. Moreover the final goals are typically stated in a rather general form without particular details.

In order to describe the heuristics choice we introduce an imaginary phase space \( \{ \omega \}_\alpha \) in addition to the real one \( \{ w \} \) which is ascribed individually to each element \( \alpha \). So, in fact, we have introduced the set of spaces existing in the human mind. The imaginary phase spaces enable us to describe a hypothetical dynamics of the system in the near feature that can be expected by its elements based on the available information. Every imaginary phase space

\[
\{ \omega \}_\alpha = \{ \theta, \eta \}_\alpha
\]  

comprises the objective variables \( \{ \theta \}_\alpha \) of all the elements and the subjective variables \( \{ \eta \}_\alpha \) of the given element \( \alpha \). These quantities specify the hypothetical states of the elements in the “mind” of the element \( \alpha \). Therefore the symbol \( \theta_{\alpha' : \alpha} \) is written in bold to underline its dependence on two indices, meaning the phase variables of the element \( \alpha' \) in the “mind” of the element \( \alpha \). Collection (3) does not contain the subjective phase variables of the other elements because they are hidden for the given element \( \alpha \).

In these terms a possible strategy of behavior of the element \( \alpha \) is represented as a certain time dependence \( \{ \eta[t''] \}_{t'' > t} \) of its subjective phase variables in the near future. The hypothetical time dependence \( \{ \theta[t'''] \}_{t''' > t} \) of its objective variables is determined by the given strategy of behavior. The hypothetical time dependence \( \{ \theta[t'''] \}_{t''' > t} \) of the objective variables ascribed to another element \( \alpha' \neq \alpha \) is constructed in the “mind” of the element \( \alpha \) based on the available information. We also will use the notation \( \{ \omega[t'''] \}_{t'''} \) to denote this strategy as the hypothetical motion of the system in the space \( \{ \omega \}_\alpha \). The symbol \( \{ \omega[t'''] \} \) without the element index stands for the heuristics as whole.

The elements are assumed to evaluate and choose the desired strategies of behavior \( \{ \omega_{\text{op}[t'']} \} \) in some optimal way, which determines the system dynamics. It should be noted that in this choice every element \( \alpha \) evaluates possible strategies of its own behavior \( \{ \omega_{\text{op}[t'']} \}_\alpha \) only, the behavior of the other elements is regarded by it as given beforehand or predictable with some probability. These features of the heuristics choice enable us to represent symbolic expression (1) as

\[
\{ \omega[t'], \omega[t'''] \}_{t''' < t} \xrightarrow{\text{individual choice of system elements}} \{ \omega_{\text{op}[t''']} \}_{t'''' > t} \xrightarrow{\delta \{ w \}} \{ \delta[t''''] \}_{t'''' > t}
\]  

It should be pointed out that in choosing the heuristics the elements can predict the system dynamics extrapolating the time variations of the phase variables in some simple way, for example, fixing them or supposing the linear time dependence to hold in the near future. Figure illustrates this.
Concluding the present subsection we state that the laws governing systems with motivation should be based on some variational principles dealing with trajectories \( \{\eta'[t'']\}^{t''>t} \) in the imaginary subjective spaces of the corresponding elements. As a result the governing equations have to belong to a certain class of temporally boundary value problems because these trajectories join the current state of the system with “desirable” ones (cf. [27]). Some implementation of these variational principles gives us the time variations \( \{\delta_t w\} \) of the phase variables at the current time \( t \). In this sense the “imaginary future” of the system affects its dynamics at present.

4.3. The approximation of perfect rationality

The implementation of the variational principles mentioned in the previous subsection requires some measure for quantifying the heuristics with respect to their value at least approximately. This measure can be constructed in a certain limit case called the perfect rationality. It comes into being when, first, analyzed situations are repeated many times, with the environment conditions being the same.
Thereby the time restrictions affecting the decision-making process are removed and the complete information about the system becomes accessible. Second, the elements are able to correct their states continuously.

Under such conditions the individual choice of the optimal heuristics $\{\eta_{op}[t'']\}_{t'' > t}^t$ by a given element $\alpha$ is reduced to finding the maximum of a certain preference functional

$$ U_{\alpha} := U_{\alpha}\{\{\varpi[t'']\}_{t'' > t}\} $$

with respect to its own strategy of behavior $\{\eta[t'']\}_{t'' > t}^t$. In the limit of perfect rationality functional (5a) depends only on the trial trajectory of the system motion in the imaginary space $\{\varpi[t'']\}_{t'' > t}^t$. Besides, all the objective phase spaces in the individual imaginary spaces of the elements are identical. The following expression

$$ U_{\alpha} := \int_0^\infty dt'' e^{-(t''-t)/T_u} u_{\alpha}(\varpi_{\alpha}[t'']) $$

is an example of functional (5a), where all the moments of time contribute independently of each other with the weight $\exp\{-(t''-t)/T\}$ decreasing exponentially as the time interval $t''-t$ increases, the scale $T$ specifies the temporal horizon of predicting the system dynamics, and the function $u_{\alpha}(\varpi_{\alpha}[t''])$ measures the contribution of individual time moments.

Functional (5) quantifies the preferences of the element $\alpha$ in the choice of its own heuristics, provided the behavior of the other elements is known. In other words, within the frameworks of the perfect rationality the optimal strategy of behavior $\{\eta_{op}[t'']\}_{t'' > t}^t$ is specified by the expression

$$ \{\eta_{op}[t'']\}_{t'' > t}^t \iff \max_{\{\eta[t'']\}_{t'' > t}^t} U_{\alpha}. $$

By way of example, we note that in the limit of perfect rationality schema (4) for traffic flow gives rise to Newtonian type models [27]. It is the case where the concept of social forces [6] holds. If the driver behavior is not perfect then the description of traffic dynamics goes beyond the notions of Newtonian mechanics [28].

In the given case the found governing equation (6b) is of the second order with respect to the objective phase variables, whereas the initial conditions specifying the initial system state can contain only the objective variables and, thereby, does not determine uniquely the system dynamics. By this rather simple example we have demonstrated the fact that some kind terminal conditions should be imposed on the system to determine its dynamics.
dynamics of the analyzed systems belong to a certain class of temporal boundary value problems. Another fact demonstrated by this example is the influence of the prediction horizon \( T \) on the system dynamics. If the parameter \( T \) is rather small then the governing equation (6b) is practically of the first order with respect to the objective variables \( \{q\} \) and in this limit the system dynamics can be considered to be some initial value problem. In the opposite limit, i.e. when the parameter \( T \) is large the effect of the terminal conditions is crucial.

Expressions (6) specifies the optimal heuristics \( \{\eta_{\text{op}}(t''|t')\}_{t''>t} \) as certain trajectories. Therefore, if the elements choose these optimal strategies of behavior at the current moment of time and follow them, then further correction of the system motion will be not necessary. The latter is the essence of the Nash equilibrium.

As a particular case let us adopt an additional assumption that the analyzed system admitting the limit case of perfect rationality possesses also a special point \( Q \) (or a set of points with equal values of the controllable variables, \( \{q_c = \text{const}\} \)) in the objective phase space. This point united with the origin \( \{\eta = 0\} \) of the subjective phase space matches the steady-state dynamics of the given system under stationary external conditions in the limit of perfect rationality. It means that if the system is initially located at the given point, it will not leave this point further. For example, traffic flow where all the cars move with the same speed and at some optimal headway distance matches this situation. If the system during its motion governed by expression (6a) tends to the point \( Q \) or the corresponding set of points, it will be referred to as an attractor of rational dynamics.

4.4. Bounded rationality and action points

As discussed previously, the time constraints together with the bounded capacity of human cognition endow the choice of heuristics and, thereby, the system dynamics with random properties. If two strategies of behavior are rather close to each other in value then it can be tough to order them by preference and to choose one in a rational way. We apply the notion of perception threshold \( \Theta \) to tackle this problem. The perception threshold as well as the preference functional depends generally on the type of elements, which here is not labeled directly to simplify the notations.

Let us make use of the preference functional (5a). Two strategies of behavior \( \{\eta_1(t'')\}_{t''>t}^\alpha \) and \( \{\eta_2(t'')\}_{t''>t}^\alpha \) are considered to be equivalent, with the other environment conditions being the same, if the corresponding magnitudes of the preference functional meet the inequality \( |U_{1,\alpha} - U_{2,\alpha}| \lesssim \Theta \). It is the point where the form of the preference functional (5a) becomes determined. The matter is that any increasing function applied to the preference functional gives rise to a new preference functional describing the same set of the optimal heuristics. Introducing the perception threshold we actually fix its form.

When a currently chosen strategy of behavior \( \{\eta(t'')\}_{t''>t}^\alpha \) is close to the optimal one in the given sense, the element \( \alpha \) has no motives to change it. Roughly speaking, if it is not clear what to do, to change nothing is quite adequate. If the difference
in the magnitudes of the preference functional (5a) for the two strategies becomes remarkable in comparison with the perception threshold, the element α recognizes the necessity of correcting its current state. Exactly this choice of new more proper heuristics is the point where the random factors enter the system directly. Indeed, since all the strategies of behavior that are close to the optimal heuristics in terms of the perception threshold are regarded as equivalent then the choice of some of them is a random event. The time moment when this choice arises is also a random quantity.

It should be pointed out that the perception threshold Θ characterizes probabilistic properties of the element behavior rather than the step-like dynamics. Namely, let us consider two heuristics, the strategy of behavior \{η_{c}[t'\prime\prime]\}_α^{t'\prime\prime} > tα that is followed by the element α at the current moment of time t and the optimal one \{η_{op}[t'\prime\prime]\}_α^{t'\prime\prime} > tα which is actually hidden for it. When the difference in the corresponding magnitudes of the preference functional (5a) becomes equal to the threshold, |U_{c,α} - U_{op,α}| = Θ, the events of correcting the current state by the element α just arise most often rather than exhibits a stepwise behavior. In the cases |U_{c,α} - U_{op,α}| \ll Θ the element cannot recognize the fact of the system deviating from the optimal dynamics. States matching the opposite inequality |U_{c,α} - U_{op,α}| \gg Θ cannot be reached because the element would respond earlier. In particular, according to empirical data for traffic flow such events of correcting the car motion are distributed rather widely near the corresponding threshold [29].

**Action points:** Let us introduce the notion of action points in order to describe the dynamics of systems with motivation. An action point is an event of changing the current strategy of behavior by some element in correcting its state. Every action point is associated with this strategy and the time moment of changing it. When the dynamics of a given system is optimal within the human perception characterized by the threshold Θ its elements do not correct their heuristics. At these moments of time the system motion is not controlled by the elements and proceeds according to natural regularities affecting the system and the strategy chosen previously. When the system motion deviates from the optimal one substantially the elements recognize this fact and correct their individual strategies of behavior. In doing so an element selects some new strategy of behavior in a neighborhood of the optimal heuristics whose thickness quantified by the preference functional (5a) is less than or of the order of the perception threshold. Then the given element follows the selected heuristics until it recognize the necessity of correcting its state again. We note that the notion of action points for the car-following process was introduced for the first time in [30] to denote the moments of time when drivers correct the motion of their vehicles by pressing the gas or braking pedals.

In these terms the dynamics of a system with motivation can be represented as a sequence of action points, i.e., jumps between various strategies of the element behavior. The particular strategies of behavior joined by these jump-like transitions and their time moments are random quantities. These random transitions are the
cause of the stochasticity in the dynamics of systems with motivation. Between the action points the system dynamics is not controlled by its elements at all and is regular or affected by random factors of natural origin. Symbolically this feature is represented by the following diagram generalizing the previous one:

\[
\begin{align*}
\{u[t'], \varpi[t'']\}^{t'' > t}_{t < t} & \xrightarrow{\text{individual choice}} \{\varpi_{\text{op}}[t'']\}^{t'' > t} \\
\{\delta_t w\} & \xleftarrow{\text{action points}} \{\delta_t h\}
\end{align*}
\]

The concept of action points is illustrated in Fig. 3.

\[\text{Fig. 3. Illustration of the system dynamics governed by elements with bounded rationality.}\]

**Dynamical traps:** If such a system possesses an attractor of rational dynamics it can exhibit a new type of cooperative phenomena. Indeed, by definition, the point \(Q\) in the objective phase space matches the steady-state dynamics of a system with perfect rationality. Then the elements with bounded rationality will regard the system motion in the vicinity of this attractor also optimal. To discuss the given feature in more detail let us introduce the notion of dynamical traps.

Using the preference functional \((5a)\) we construct a certain neighborhood of the set \(Q \otimes \{h = 0\}\) in the space of heuristics \(\{\varpi[t'']\}^{t'' > t}\) of thickness \(\Theta\) and then project it onto the objective phase space \(\{w\}\). In this way we obtain a certain neigh-
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neighborhood $\mathcal{D}_Q$ of the set $\Omega$ called the dynamical trap region. When the system with bounded rationality enters this region its elements consider the system dynamics optimal and the correction of their state unnecessary. Since the system dynamics in the region $\mathcal{D}_Q$ is really close to the optimal one, a time span between two action points could rather prolonged in comparison with that of the system dynamics far from $\mathcal{D}_Q$. In other words such fragments of system motion inside the dynamical trap region can be regarded as long-lived states [32, 31]. Their origin is due to the stagnation of the element active behavior for a relatively long time. Therefore dynamical traps are able to induce nonequilibrium phase transitions of a new type that should be widely met in social systems [32, 31, 33] rather than in natural ones. We note that the dynamical traps for Hamiltonian systems was introduced in [34, 35] (see also a review [36]) and for systems with nonlinear oscillations it was done in [37]. A simple example of nonequilibrium phase transitions induced by dynamical traps is illustrated in Fig. 4.

4.5. Memory effects: individual learning and formation of social and cultural norms

Because of the bounded capacity of human cognition, gaining the knowledge about the proper strategies of behavior is crucial. As noted in Sec. 8 there are two channels
of accumulating and aggregating such information. One is the individual learning of the elements based on own experience or local interaction with the other elements. In some sense it is a typical mechanism of cooperative phenomena widely met in natural systems and caused by local or quasi-local interaction of their particles. It seems to be possible to describe the individual learning process using the introduced phase space and perception thresholds. No additional variables are necessary to do this. Indeed, let us ascribe an individual perception threshold $\Theta_\alpha$ to every element $\alpha$. Then the individual learning is represented as the evolution of the perception thresholds $\{\Theta_\alpha\}$ caused by some interaction of the elements. Symbolically it takes the form

$$\{w[t]\}_t \Rightarrow \{\Theta_\alpha(t)\}. \quad (8)$$

In these terms the individual learning is reduced to the time decrease of the perception thresholds $\{\Theta_\alpha\}$ due to the accumulation and aggregation of information about the system properties.

The second channel is related to a unique collective interaction of all the elements in a social system in addition to their individual interrelations of various types. It arises via the formation of social and cultural norms of behavior. These norms affect directly the heuristics and their preference, and involve all the members of a social system or their large groups independently of their relationships and distance in space and time. The social and cultural norms aggregate the information about the properties and features of a social system during a long time interval and make up the basis for finding general rules of successful strategies of behavior. So in order to describe the effect of the social and cultural norms on the system dynamics some additional variables, the space of cultural features $\{\chi\}$, should be introduced. We presume that the cultural features cannot be ascribed to individual persons in any way, they have their own carriers, e.g., books, newspapers, magazines, movies, broadcasts, and other types of mass media. Symbolically the formation of social and cultural norms can be written as

$$\{w[t]\}_v \Rightarrow \{\chi(t)\}, \quad (9)$$

where, as it is rather natural to assume, the father a given event in the past, the weaker its influence on the present. In order to include the effect of these norms on the social system dynamics we generalize diagram 17 as follows
which is the final diagram presenting the essence of the mathematical components and notions describing the systems with motivation.

As an illustrative simple example let us consider the implementation of the memory effects in describing the dynamics of a car following a lead car moving, e.g., at a fixed speed $V$ in the frameworks of the social force model. The social force model [6] relates the acceleration $a$ of the following car to its current velocity $v$ and the distance between the two cars (headway distance) $h$,

$$a = \mathcal{F}(h, v).$$

Hear $\mathcal{F}(h, v)$ is a certain function which takes into account two stimuli in car driving, the necessity of maintaining safe headway and zero value relative velocity. It should be underlined that model (11) deals with the acceleration $a$ and velocity $v$ of the following car as well as the headway distance $h$ taken at the current moment of time $t$. However, because of the bounded capacity of human cognition drivers cannot recognize the necessity to correct the car motion immediately. The information about the state of motion should be aggregated and accumulated during some time for a driver to make the proper conclusion about correcting its motion. Keeping in mind expressions similar to (8) and (9) let us represent the relationship between the acceleration $a$ and the social force $\mathcal{F}(h, v)$ in the following functional form

$$a(t) = \int_{-\infty}^{t} dt' K(t - t') \mathcal{F}(h[t'], v[t']).$$

with a kernel $K(t - t')$ decreasing as the analyzed point in the past, $t'$, goes away from the current moment of time $t$. The general speculations prompt us that the effect of the past should be scale free, causing this kernel $K(t - t')$ to be a power function of the form $K(t - t') \propto (t - t')^{-\gamma}$ with $0 < \gamma < 1$. The latter inequality has to hold because, otherwise, functional (12a) would be reduced to a local relationship without memory effects. This claim is partly justified by the observed memory effects in the scale-free foraging by primates [38, 39, 40] or insects [41, 42, 43], as well as scale-free pattern of human memory retrieval [44]. In this case equation (12a) reads

$$a(t) = \frac{1}{\Gamma(1 - \gamma)} \int_{-\infty}^{t} \frac{dt'}{\tau^{1-\gamma}(t - t')^{\gamma}} \mathcal{F}(h[t'], v[t']).$$

Hear $\Gamma(\ldots)$ is the Gamma function and $\tau$ is some time scale characterizing delay in the driver behavior. The right-hand side of (12b) is the Riemann-Liouville fractional integral of order $1 - \gamma$ (see, e.g., [45]). It can be inverted using the formalism of fractional calculus, yielding

$$\frac{d^{1-\gamma}a}{dt^{1-\gamma}} = \mathcal{F}(h[t], v[t]),$$

(12c)
where the left-hand side is the Riemann-Liouville fractional derivative of order \(1 - \gamma\) defined by the expression

\[
\frac{d^{1-\gamma}}{dt^{1-\gamma}} := \frac{1}{\Gamma(\gamma)} \frac{d}{dt} \int_{-\infty}^{t} \frac{a(t')}{\tau^{\gamma}(t-t')^{1-\gamma}} dt'.
\]

Comparing formulae (11) and (12c) we can claim that the fractional calculus is likely to be an appropriate formalism for describing memory effects in systems with motivation.

5. Conclusion

The paper has considered systems with motivation as a typical example of statistical social systems. First, the elements of such a system are characterized by motivated behavior and the decision-making process governs the system dynamics. Second, all the elements can be divided into large groups by their properties and similarity. Therefore the local self-averaging holds in this system, enabling us to introduce the notion of the characteristic elements. Their regular properties describe the common features of the elements, whereas the random ones take into account the individuality of the elements as well as unpredictable factors of their behavior.

There have been two purposes of the present paper. The first has been to propose an answer to one of the fundamental questions in this field, namely, whether mathematical formalism and physical notions are applicable to describing and modeling various phenomena in systems of social nature, at least, the so-called statistical social systems. To demonstrate the given question to be not trivial up to know we have listed the basic properties of social systems distinguishing them from physical ones or, more generally, natural systems. The key point is to restate this question as to what mathematical formalism and physical notions should be developed that allow for these properties to be implemented in the appropriate mathematical models.

The second purpose has been to development, at least, partly such notions and mathematical constructions for systems with motivation. Not intending to create a self-consistent and complete theory of systems with motivation we have analyzed their characteristic features caused by the basic properties of human beings and human societies and demonstrated the feasibility of this idea for them individually. As the final conclusion we state that practically all the basic elements necessary for describing statistical social systems or, more rigorously, systems with motivation are available or partly developed in modern physics and applied mathematics.

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