\( \mathcal{PT} \) symmetry and renormalisation in quantum field theory

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Abstract.

Quantum systems governed by non-Hermitian Hamiltonians with \( \mathcal{PT} \) symmetry are special in having real energy eigenvalues bounded below and unitary time evolution. We argue that \( \mathcal{PT} \) symmetry may also be important and present at the level of Hermitian quantum field theories because of the process of renormalisation. In some quantum field theories renormalisation leads to \( \mathcal{PT} \)-symmetric effective Lagrangians. We show how \( \mathcal{PT} \) symmetry may allow interpretations that evade ghosts and instabilities present in an interpretation of the theory within a Hermitian framework. From the study of examples \( \mathcal{PT} \)-symmetric interpretation is naturally built into a path integral formulation of quantum field theory; there is no requirement to calculate explicitly the \( \mathcal{PT} \) norm that occurs in Hamiltonian quantum theory. We discuss examples where \( \mathcal{PT} \)-symmetric field theories emerge from Hermitian field theories due to effects of renormalisation. We also consider the effects of renormalisation on field theories that are non-Hermitian but \( \mathcal{PT} \)-symmetric from the start.

1. Introduction

Non-Hermitian Hamiltonians govern systems that in general receive energy from and/or dissipate energy into their environment and so they are typically not in equilibrium. Their energy is not conserved and their energy levels are complex. However, in 1998 [1] non-Hermitian \( \mathcal{PT} \) systems were shown to offer new possibilities for unitary time evolution in quantum mechanics. Since then there has been extensive research activity [2], particularly in material science and optics, which has implemented the ideas of \( \mathcal{PT} \)-symmetric quantum mechanics. \( \mathcal{PT} \)-symmetric quantum mechanics\(^1\) can be considered to be a one-dimensional quantum field theory. To date there have been no clear examples of \( \mathcal{PT} \)-symmetric quantum field theory in

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\(^1\) The term quantum mechanics is used to refer to quantum systems with a finite number of degrees of freedom; the term quantum field theory (QFT) is used to refer to systems with an infinite number of degrees of freedom.
higher dimensions. We examine the possibility that $\mathcal{PT}$ symmetry may emerge when considering the effect of quantum fluctuations in a higher-dimensional field theory. We also examine the converse process where we consider the effect of quantum fluctuations on an initially $\mathcal{PT}$-symmetric quantum field theory.

Using numerical techniques the work of Bender and Boettcher [1] showed that quantum-mechanical Hamiltonians of the form

$$ H = \frac{1}{2} p^2 + x^2 (ix)^\epsilon $$

have a real positive spectrum. Using a correspondence between ordinary differential equations and integrable field theory models, Dorey et al. [3] provided an analytical proof of the result of Bender and Boettcher.

Mostafazadeh [4] considered the framework of pseudo-Hermiticity [5] which contained $\mathcal{PT}$ symmetry as a special case. A Hamiltonian $H$ is pseudo-Hermitian if $H$ is not selfadjoint but

$$ H^\dagger = \eta H \eta^{-1}, $$

where $\eta$ is a positive-definite Hermitian operator. In terms of $\eta$ the Hilbert space of states has a positive-definite inner product given by

$$ \langle \varphi, \eta \chi \rangle $$

where $\langle.,.\rangle$ is a conventional inner product on the Hilbert space of states for the Hermitian part of the Hamiltonian. There is not a universal expression for $\eta$, which is difficult to calculate in general [2]. This would seem to imply an impediment to calculating correlation functions in $\mathcal{PT}$-symmetric field theories. Within the context of quantum mechanics, formulated in terms of functional integrals, Jones and Rivers [6] showed through examples that it was not necessary to calculate $\eta$. They also gave a general field theoretic argument [7] (valid for the case of unbroken $\mathcal{PT}$-symmetry and symmetric Hamiltonians) that the calculation of vacuum expectation values of field operators did not require the explicit calculation of $\eta$ or equivalently the related $C$-operator [2]. This latter argument relied on the relation $A_H = \exp(iHt)A \exp(-iHt)$, where $A_H$, $A$ and $H$ were a general Heisenberg boson field operator, the corresponding Schrödinger operator and a Hamiltonian operator which can be quasi-Hermitian. The versatility of the path-integral calculation of Green’s functions is further confirmed in [8]; there it is shown that in quantum mechanics a given Hamiltonian can have different spectra depending on boundary conditions, which are imposed in the complex plane in Stokes sectors. For a left-right ($\mathcal{PT}$) symmetric choice of two such sectors the spectra can be real. In QFT the boundary conditions are imposed on the path of integration of the functional integral. The Schwinger-Dyson equations for the Greens functions, which are obtained formally from the path integral, do not depend on such details of the path-integration measure. For the $g\phi^4$ scalar field theory there are two solutions one for $g > 0$ and the other for $g < 0$. From the two-point function we can calculate the pole. For $D = 1$ we can also calculate the difference between the energy for the first excited state and the ground state. This difference is found by solving the Schrödinger equation using $\mathcal{PT}$-symmetric Stokes wedges and the Hermitian Stokes wedges. The path-integral computation of the mass and this excitation energy agree. Hence, we shall take the path-integral representation as the definition of a quantum field theory and, except for the Lee model, not consider $\eta$ explicitly. The path integral is a convenient formulation for both perturbative and nonperturbative calculations in our context.

We consider four examples of field theories to illustrate three different aspects of $\mathcal{PT}$ symmetry:

2 We should stress that in other applications of path-integration (such as the computation of S-matrix elements) the $C$-operator cannot be neglected.
(i) the Lee model and the elimination of ghosts [9];
(ii) the role of the top quark and the Higgs instability [10] in QFTs which go beyond the Standard Model (SM) of particle physics [12, 13];
(iii) the new epsilon expansion for $\mathcal{P}\mathcal{T}$-symmetric scalar field theory [14, 15];
(iv) the functional renormalisation group and the preservation of $\mathcal{P}\mathcal{T}$ symmetry in the infrared after renormalisation [16].

2. Role of $\mathcal{P}\mathcal{T}$ symmetry in the Lee model

From the study of dispersion relations in QFT [17, 18] it has been shown that it is possible that a formally Hermitian QFT may contain ghosts. The Lee model [19], which we discuss next, is just such an example where the vertex function and, in particular, coupling-constant renormalisation is related to the emergence of a non-Hermitian Hamiltonian. The role of $\mathcal{P}\mathcal{T}$ symmetry is crucial in banishing the ghosts and making the field theory viable [9].

2.1. Lee model

The Lee model is a field theory which is very different from the Standard Model of particle physics but it has the advantage of being soluble; so it provides a convenient framework for the emergence of non-Hermiticity in the process of renormalisation. Indeed, the model was devised as a field theory whose coupling-constant and wave-function renormalisation could be computed exactly in principle. There are variants of the model. One variant of the model contains fields for infinitely heavy spinless fermions, $V$ and $N$, as well as a neutral scalar field $\theta$. Antifermions are not in the theory and so there is no crossing symmetry. The permitted reaction channel is

$$V \leftrightarrow N + \theta,$$

but the channel

$$N \leftrightarrow V + \theta$$

is not allowed.

In momentum space the Hamiltonian of the Lee model (in a space of finite volume $V$) is

$$H = H_0 + H_{\text{int}},$$

where

$$H_0 = m_V \bar{\psi}_V \psi_V + m_N \bar{\psi}_N \psi_N + \sum_k \omega_k a_k^\dagger a_k$$

and

$$H_{\text{int}} = \delta m_V \bar{\psi}_V \psi_V - g_0 V^{-\frac{1}{2}} \sum_k \frac{u(\omega_k)}{(2\omega_k)^{\frac{1}{2}}} (\bar{\psi}_V \psi_N a_k + \bar{\psi}_N \psi_V a_k^\dagger).$$

$u(\omega_k)$ is a dimensional cut-off function which is chosen to tend to 0 for large $\omega_k$ where $\omega_k = \sqrt{k^2 + \mu^2}$. Standard commutation and anti-commutation rules are assumed:

$$\{\bar{\psi}_V, \psi_V\} = \{\bar{\psi}_N, \psi_N\} = 1$$

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}.$$  

All other commutators and anticommutators vanish. Bare operators and coupling constant appear in $H$. However $m_V, m_N, g_0$ are renormalised parameters and so are determined by experiment; $\delta m_V$ is the mass counterterm for the $V$ particle and is a function of $g_0$. No further mass renormalisation is actually needed. The coupling $g_0$ in principle is determined
from the scattering cross-section of $N$ and $\theta$ although there are interesting complications of non-Hermiticity if the coupling is not small.

However, let us for the moment persist in the view that $g_0$ is real and so $H$ is Hermitian. Let us choose as a basis for the Hilbert space states of the form

$$|\rangle = |n_V, n_N, \{n_k\}\rangle. \quad (11)$$

From $H$ it is clear that there are two conserved quantities $B$ and $Q$:

$$B = n_V + n_N, \quad Q = -\sum_k n_k. \quad (12)$$

where $N_\theta = \sum_k n_k$. Hence the Hilbert space is partitioned into an infinite number of independent (superselection) sectors with fixed $B$ and $Q$. Although this makes the model soluble, the analysis of sectors with large $B$ and $Q$ is complicated. Since our main point is to show how renormalisation can lead to a non-Hermitian Hamiltonian, we simplify our analysis further by considering (i) a nontrivial sector $B = 1$ and $Q = 0$ and (ii) modifying the model so that there is no $k$ dependence. We choose to work with this simplified model since it suffices to illustrate the emergence of non-Hermiticity from renormalisation. (The more complicated field-theoretic model can also be analysed leading to similar results.) The model now is a quantum-mechanical model since the quantum fields in (7) and (8) are replaced by quantum operators. Hence, infinities that arise from summing over an infinite set of modes in quantum field theory are absent; the model remains interesting since some features of coupling-constant renormalisation persist. The resulting Hamiltonian is

$$H = H_0 + H_1, \quad (13)$$

and

$$H_1 = g_0 (V^\dagger Na + a^\dagger N^\dagger V) + \delta m_V V^\dagger V. \quad (14)$$

We will look at the 2-dimensional Hilbert space associated with this sector and consider the eigenstates, which we will denote by

$$|V\rangle = c_{11} |1, 0, 0\rangle + c_{12} |0, 1, 1\rangle, \quad (15)$$

$$|N\theta\rangle = c_{21} |1, 0, 0\rangle + c_{22} |0, 1, 1\rangle, \quad (16)$$

with eigenvalues $m_V$ and $E_{N\theta}$. Let us denote $(m_V + \delta m_V)$ by $m_{V_0}$, the bare mass of $V$. The eigenvalues can be shown to satisfy the equations

$$m_V = \frac{1}{2} (m_N + m_\theta + m_{V_0} - \sqrt{M_0^2 + 4g_0^2}), \quad (17)$$

$$E_{N\theta} = \frac{1}{2} (m_N + m_\theta + m_{V_0} + \sqrt{M_0^2 + 4g_0^2}). \quad (18)$$

where $M_0 = m_N + m_\theta - m_{V_0}$. Following field theory, we define the wave-function renormalisation constant $Z_V$ through the relation

$$1 = \langle 0 | \frac{1}{\sqrt{Z_V}} V | V \rangle, \quad (19)$$

which gives

$$Z_V = \frac{2g_0^2}{\sqrt{M_0^2 + 4g_0^2} (\sqrt{M_0^2 + 4g_0^2} - M_0)}, \quad (20)$$
and the coupling constant renormalisation through

\[ \frac{g^2}{g_0^2} = Z_V. \] (21)

We deduce that

\[ g_0^2 = \frac{g^2}{1 - \frac{g^2}{M^2}}, \] (22)

where \( M \equiv m_N + m_\theta - m_V. \) If \( g \), the experimentally determined value, exceeds \( M \) then \( g_0 \) becomes pure imaginary and the Hamiltonian becomes non-Hermitian. Although the Hamiltonian has become non-Hermitian, it is \( \mathcal{PT} \)-symmetric. Explicitly, the transformations due to \( \mathcal{P} \) are [9]

\[ \mathcal{P} V \mathcal{P} = -V, \quad \mathcal{P} N \mathcal{P} = -N, \quad \mathcal{P} a \mathcal{P} = -a, \] (23)

\[ \mathcal{P} V^\dagger \mathcal{P} = -V^\dagger, \quad \mathcal{P} N^\dagger \mathcal{P} = -N^\dagger, \quad \mathcal{P} a^\dagger \mathcal{P} = -a^\dagger. \] (24)

The transformations due to \( \mathcal{T} \) are

\[ \mathcal{T} V \mathcal{T} = V, \quad \mathcal{T} N \mathcal{T} = N, \quad \mathcal{T} a \mathcal{T} = a \] (25)

\[ \mathcal{T} V^\dagger \mathcal{T} = V^\dagger, \quad \mathcal{T} N^\dagger \mathcal{T} = N^\dagger, \quad \mathcal{T} a^\dagger \mathcal{T} = a^\dagger. \] (26)

It is now straightforward to check that \( i[g_0](V^\dagger N a + a^\dagger N^\dagger V) \) is \( \mathcal{PT} \)-symmetric. This is our first example of a \( \mathcal{PT} \)-symmetric field theory that arises from a Hermitian field theory after renormalisation. On introducing a \( \mathcal{PT} \)-symmetric inner product in the ghost regime of the Lee model, the so-called ghost state (identified within the Hermitian framework) turns out to have a positive norm[9]. The Lee model then can be interpreted as an acceptable quantum field theory.

3. Higgs instability and \( \mathcal{PT} \) symmetry in beyond-the-SM physics

The conventional approach to renormalisation involves the regularisation of loop integrals in Feynman diagrams. This led to scale dependence of the parameters of the theory which can be understood in a different way using the approach [20] of Wilson to renormalisation. Wilson’s approach leads naturally to the concept of effective actions, and the form of the effective actions for some theories in high-energy physics is non-Hermitian and \( \mathcal{PT} \)-symmetric. Low energy and long wavelength approximations of quantum evolution equations obtained from the effective actions can be studied using the methods of \( \mathcal{PT} \)-symmetric quantum mechanics.

The change from the conventional to the Wilson approach can be seen by considering a scalar field \( \Phi \) whose (conventional) action is given by

\[ S_\Lambda [\Phi] = \int d^Dx \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + U_\Lambda (\Phi) \right) \] (27)

where the UV cut-off is \( \Lambda \). If we integrate over Fourier modes with momenta of magnitude \( p \) in the shell \( k \leq p \leq \Lambda \), we arrive at an action which we will denote by \( S_k [\phi] \) where \( \phi \) has only Fourier modes with \( p \leq k \) (a sharp infrared cut-off).The associated partition function is

\[ Z_k [j] \equiv \exp(-W_k [j]) \equiv \int D\phi \exp \left( -S_k [\phi] - \int p_j \phi_{-p} \right) \] (28)

and a Legendre transformation on \( W_k [j] \) leads to the effective action \( \Gamma_k [\phi_c] \)

\[ \Gamma_k [\phi_c] = W_k [j] - \int d^Dx j c \phi_c - \Delta_k [\phi_c]. \] (29)
The renormalisation group flow equation\(^3\) (reviewed in [23]) is

\[
\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left\{ \partial_k R_k \left( \frac{\delta^2 \Gamma_k}{\delta \phi_c(p) \delta \phi_c(q)} + R_k(p) \delta(p+q) \right) \right\}^{-1}.
\]

(30)

There is a systematic (derivative expansion) approximation schemes [21, 24] for solving this integro-differential equation for the effective action.

\[
\Gamma_k[\phi_c] = \int d^D x \left( U_k(\phi_c) + \frac{Z_k(\phi_c)}{2} \partial_\mu \phi_c \partial^\mu \phi_c + Y_k(\phi_c) (\partial_\mu \phi_c \partial^\mu \phi_c)^2 + W_k(\phi_c) (\partial_\mu \partial^\nu \phi_c)^2 + \cdots \right).
\]

(31)

The effective action can be represented (at the lowest level of approximation) by the ansatz

\[
\Gamma_k[\phi_c] = \int d^D x \left( \frac{1}{2} \partial_\mu \phi_c \partial^\mu \phi_c + U_k(\phi_c) \right).
\]

(32)

The counterpart of the classical equation of motion is

\[
\frac{\delta \Gamma_k[\phi_c]}{\delta \phi_c(x)} = 0.
\]

(33)

The effective potential, \(U_k(\phi_c)\), which results from renormalisation, shows the emergence of \(\mathcal{PT}\) symmetry in the next models that we will discuss. For the effective action of (32), in the low energy approximation and small \(k\), equation (33) leads to a Schrödinger equation which we will analyse now. The flow of the effective potential with \(k\) will be analysed in section 5. We consider two theories of current interest using this formalism:

(i) a theory of dynamical breaking of supergravity via a gravitino condensate field [25] \(\varphi_c\) with

\[U(\varphi_c) = -\varphi_c^4 \log(i\varphi_c)\]

for large \(\varphi_c\),

and

(ii) Beyond the Standard Model of particle physics for which \(\varphi_c\) is a pseudoscalar field and

\[U(\varphi_c) = -\varphi_c^4 \log(\varphi_c^2)\]

for large \(\varphi_c\) [10, 11].

We now give brief motivations for the study of the above two models.

- **Motivation for the model underlying (i):** This model was studied in [25] in order to understand the possible role of a supersymmetric (SUSY) phase transition occurring in the early universe in the study of inflation. SUSY is at best a broken symmetry. The breaking of SUSY within the context of a local symmetry, supergravity, can lead to a Starobinsky type model [26] for inflation.

- **Motivation for the model underlying (ii):** Although a scalar Higgs particle has been discovered at the LHC, there is still interest in looking for other Higgs bosons which can exist in models which go beyond the Standard Model (BSM) of particle physics. The interest in BSM comes from trying to understand the origin of Dark Matter and neutrino masses. One such model is the Minimal Supersymmetric Standard Model (MSSM) [13]. The MSSM has two Higgs doublets which leads to five physical Higgs bosons, both \(\text{CP}\) odd and even. Our aim is not phenomenological but to show that the considerations of \(\mathcal{PT}\) symmetry may have a role to play in understanding the scalar sector of extensions of the Standard Model (SM).

\(^3\) Wetterich [21] introduced a smooth infrared cut-off function \(R_k(p^2)\) rather than a sharp cut-off so that

\[S_k[\phi] = \int d^D x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V_k(\phi) \right] + \Delta_k[\phi],\]

where \(\Delta_k[\phi] = \frac{1}{2} \int d^D p \rho_p R_k(p^2) \phi \sim p\). A common choice for \(R_k(p^2)\) is

\[R_k(p^2) = (k^2 - p^2)^\alpha (k^2 - p^2)\] \([22]\).
Both examples, in conventional quantum mechanics, would show unstable behaviour for large $\varphi_c$. Under $\mathcal{P}$ we have $\varphi_c \rightarrow -\varphi_c$ and under $\mathcal{T}$ we have $i \rightarrow -i$. So $U(\varphi_c)$ is $\mathcal{P}\mathcal{T}$ symmetric in both cases.

We study such effective potentials by considering three related quantum-mechanical Hamiltonians $H_i$, $i = 1, 2, 3$:

\begin{align}
H_1 &= p^2 + x^4 \log(ix), \\
H_2 &= p^2 - x^4 \log(ix), \\
H_3 &= p^2 - x^4 \log(x^2),
\end{align}

which are non-Hermitian but $\mathcal{P}\mathcal{T}$-symmetric. We will find that the first two Hamiltonians will show unbroken $\mathcal{P}\mathcal{T}$ symmetry and the Hamiltonian $H_3$ will show broken $\mathcal{P}\mathcal{T}$ symmetry.

3.1. The spectra for $H_1$

We use the WKB method of semiclassical quantum mechanics to determine the energy spectrum of $H_1$:

- Locate turning points.
- Examine the complex classical trajectories on an infinite-sheeted Riemann surface.
- Determine the open or closed nature of the trajectories.

The turning points satisfy the equation

\begin{equation}
E = x^4 \log(ix).
\end{equation}

We take $E = 1.24909$ because this is the numerical value of the ground-state energy obtained separately by solving the Schrödinger equation. (See the table below.)

One turning point lies on the negative imaginary-$x$ axis. To find this point we set $x = -ir$ ($r > 0$) and obtain the algebraic equation $E = r^4 \log r$. Solving this equation by using Newton’s method, we find that the turning point lies at $x = -1.39316i$. To find the other turning points we seek solutions to (37) in polar form $x = re^{i\theta}$ ($r > 0$, $\theta$ real). Substituting for $x$ in (37) and taking the imaginary part, we obtain

\begin{equation}
\log r = -(2k\pi + \theta + \pi/2) \cos(4\theta)/\sin(4\theta),
\end{equation}

where $k$ is the sheet number in the Riemann surface of the logarithm. (We choose the branch cut to lie on the positive-imaginary axis.) Using (38), we simplify the real part of (37) to

\begin{equation}
E = -r^4(2k\pi + \theta + \pi/2)/\sin(4\theta).
\end{equation}

We then use (38) to eliminate $r$ from (39) and use Newton’s method to determine $\theta$. For $k = 0$ and $E = 1.24909$, two $\mathcal{P}\mathcal{T}$-symmetric (left-right symmetric) pairs of turning points lie at $\pm 0.93803 - 0.38530i$ and at $\pm 0.32807 + 0.75353i$. For $k = 1$ and $E = 1.24909$ there is a turning point at $-0.53838 + 0.23100i$; the $\mathcal{P}\mathcal{T}$-symmetric image of this turning point lies on sheet $k = -1$ at $0.53838 + 0.23100i$.

The turning points determine the shape of the classical trajectories. Two topologically different kinds of classical paths are shown in Figs. 1 and 2. All classical trajectories are closed and left-right symmetric, and this implies that the quantum energies are all real [27].

The WKB quantization condition is a complex path integral on the principal sheet of the logarithm ($k = 0$). On this sheet a branch cut runs from the origin to $+i\infty$ on the imaginary
axis; this choice of branch cut respects the $\mathcal{PT}$ symmetry of the configuration. The integration path goes from the left turning point $x_L$ to the right turning point $x_R$ [28]:

$$\left( n + \frac{1}{2} \right) \pi \sim \int_{x_L}^{x_R} dx \sqrt{E - V(x)} \quad (n \gg 1). \quad (40)$$

If the energy is large ($E_n \gg 1$), then from (38) with $k = 0$ we find that the turning points lie slightly below the real axis at $x_R = re^{i\theta}$ and at $x_L = re^{-\pi - i\theta}$ with

$$\theta \sim -\pi/(8 \log r) \quad \text{and} \quad r^4 \log r \sim E. \quad (41)$$

We choose the path of integration in (40) to have a constant imaginary part so that the path is a horizontal line from $x_L$ to $x_R$. Since $E$ is large, $r$ is large and thus $\theta$ is small. We obtain the simplified approximate quantization condition

$$\left( n + \frac{1}{2} \right) \pi \sim r^3 \log r \int_{-1}^{1} dt \sqrt{1 - t^4}, \quad (42)$$

which leads to the WKB approximation for $n \gg 1$:

$$\frac{E_n}{[\log(E_n)]^{1/3}} \sim \left[ \frac{\Gamma(7/4)(n + 1/2)\sqrt{\pi}}{\Gamma(5/4)\sqrt{2}} \right]^{4/3}. \quad (43)$$

### Calculation of values of energy

| Energy level $n$ | $E_n$    | $\frac{E_n}{[\log(E_n)]^{1/3}}$ | WKB       | % error |
|------------------|----------|---------------------------------|-----------|---------|
| 0                | 1.24909  | 2.06161                         | 0.54627   | 73.5028 |
| 3                | 13.7383  | 9.96525                         | 7.31480   | 26.5969 |
| 6                | 31.6658  | 20.9458                         | 16.6979   | 20.2804 |
| 9                | 52.9939  | 33.4674                         | 27.6956   | 17.2463 |
| 12               | 76.9748  | 47.1776                         | 39.9324   | 15.3573 |
Analysis of the supergravity model Hamiltonian $H_2$: The classical trajectories for the Hamiltonian $H_2$ are plotted in Figs. 3 and 4. Like the classical trajectories for the Hamiltonian $H_1$, these trajectories are closed, which implies that all the eigenvalues for $H_2$ are real.

Analysis of the Higgs model Hamiltonian $H_3$: To make sense of $H_3$ we again introduce a parameter $\epsilon$ and we define $H_3$ as the limit of $H = p^2 + x^2(ix)^\epsilon \log(x^2)$ as $\epsilon : 0 \to 2$. This case is distinctly different from that for $H_2$. Figure 5 shows that the $\mathcal{PT}$ symmetry is broken for all $\epsilon \neq 0$. When $\epsilon = 2$, there are only four real eigenvalues: $E_0 = 1.1054311$, $E_1 = 4.577736$, $E_2 = 10.318036$, and $E_3 = 16.06707$. To confirm this result we plot a classical trajectory for $\epsilon = 2$ in Fig. 6. In contrast with Fig. 4, the trajectory is open and not left-right symmetric.

Figure 3. Three nested classical trajectories for $H_2$ with $E = 2.07734$.

Figure 4. Complex classical trajectory for $H_2$ with $E = 2.07734$.

Figure 5. Energies of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon \log(x)$ plotted versus $\epsilon$. 

Figure 6. Classically invariant classical trajectory for $H_2$ with $E = 2.07734$. 

Analysis of the Higgs model Hamiltonian $H_3$: To make sense of $H_3$ we again introduce a parameter $\epsilon$ and we define $H_3$ as the limit of $H = p^2 + x^2(ix)^\epsilon \log(x^2)$ as $\epsilon : 0 \to 2$. This case is distinctly different from that for $H_2$. Figure 5 shows that the $\mathcal{PT}$ symmetry is broken for all $\epsilon \neq 0$. When $\epsilon = 2$, there are only four real eigenvalues: $E_0 = 1.1054311$, $E_1 = 4.577736$, $E_2 = 10.318036$, and $E_3 = 16.06707$. To confirm this result we plot a classical trajectory for $\epsilon = 2$ in Fig. 6. In contrast with Fig. 4, the trajectory is open and not left-right symmetric.
4. Renormalisation and canonical scalar field theory

We have shown examples of $\mathcal{PT}$-symmetric quantum field theories arising from the process of renormalisation in field theories. The properties of $\mathcal{PT}$-symmetric field theories have yet to be established. In quantum mechanics much progress in understanding $\mathcal{PT}$ symmetry has been made through the study of the Hamiltonian in (1). A natural generalisation to an Euclidean field theory Hamiltonian in $D$-dimensions is to consider the Lagrangian

\begin{equation}
L = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{1}{2} g \mu_0^2 \phi^2 (i \mu_0^2 \phi)^\epsilon \tag{44}
\end{equation}

where $\phi$ is a dimensional pseudoscalar field, $\delta = 2 - D$ and $\epsilon \geq 0$. It is natural to study this class of field theories since we can then compare our findings in certain limits with those found in quantum mechanical systems. For noninteger $\epsilon$ the interaction is nonpolynomial and the methods that are normally used for calculating Greens functions in quantum field theory do not apply. The method we will use to set up a quantum field theory involves

(i) rewriting of the interaction in terms of a formal series of polynomial interactions;

(ii) creating modified Feynman rules in a nonperturbative expansion;

(iii) consideration of infinite contributions and renormalisation.

The first part of the method, used in (i), was introduced decades ago and has been recently developed in (ii) and (iii) within the context of $\mathcal{PT}$-symmetric field theory. The procedure of (i) allows us to use Wick’s theorem and the linked-cluster theorem in conjunction with (ii).

In a conventional field theory described by a Lagrangian $\mathcal{L}$ within polynomial interactions in a field $\varphi$, we calculate connected Green’s functions using a partition functional $Z(J)$ defined through a path integral as

\begin{equation}
Z(J) = \int \mathcal{D}\varphi \exp\left(- \int d^Dx (\mathcal{L} + J\varphi)\right), \tag{45}
\end{equation}

where $J$ is a source. It is well known that the normalised partition functional $\frac{Z(J)}{Z(0)}$ satisfies

\begin{equation}
\ln \frac{Z(J)}{Z(0)} = \exp\left(\sum \text{ Connected source to source diagrams} \right). \tag{46}
\end{equation}
This result simplifies our calculations in (ii) and (iii). First, we note that formally, on writing $\psi = \mu_0^2 \varphi$ (and suppressing the argument of $\psi$)

$$\psi^2(i\psi)^r = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \sum_{r=0}^{n} \left( \frac{n!}{r!} \right) \psi^2 \left( \frac{1}{2} \ln \psi^2 \right)^{n-r} = \sum_{n=1}^{\infty} \epsilon^n L_n + \psi^2$$

(47)

we can rewrite the partition function in (45) as

$$Z[J] = \int D\varphi \exp \left( - \int d^Dx \left( \sum_{n=0}^{\infty} \epsilon^n L_n + J\varphi \right) \right).$$

(51)

We are only interested in the connected Green’s functions $G_c(x_1, x_2, \cdots, x_n)$. Expanding the integrand in the path integral in powers of $\epsilon$ we obtain products of $L_n$ which consist of integrals of powers of $\varphi$ and the operations denoted by $I_w$ and $F^0_{N,s}$. On passing the operators $I_w$ and $F^0_{N,s}$ from inside the path integral to outside the path integral, we are left with a path integral which can be evaluated using Wick’s theorem. This is a well-defined procedure (explored in two recent papers [14, 15]) which has the advantage that the path integral is performed along the real $\varphi$-axis. Thus, we need not be concerned with the hopelessly complicated (infinite-dimensional) integration paths that terminate in complex Stokes sectors.

4.1. Results to $O(\epsilon)$

In general $Z = \exp(-E_0 V)$ where $E_0$ is the ground-state energy density and $V$ is the spacetime volume. Using the above method to $O(\epsilon)$ we find for general $D$ that

$$\Delta E = \frac{1}{4} \epsilon (4\pi)^{-D/2} \Gamma(1 - \frac{1}{2} D) \left\{ \ln[2(4\pi)^{-D/2} \Gamma(1 - \frac{1}{2} D)] + \psi(3/2) \right\}.$$  

(52)

As a check, for $D = 0$ and first order in $\epsilon$, we have

$$\Delta E = \frac{\epsilon}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi \exp \left( -\frac{1}{2} \phi^2 \right) \phi^2 \ln(i\phi) = \frac{\epsilon}{4} \left[ \psi(3/2) + \ln(2) \right].$$

(53)
which agrees with (52). For $D = 1$, $\Delta E$ is the expectation of the interaction Hamiltonian to $O(\epsilon)$ in the unperturbed (Gaussian) ground state

$$\Delta E = \frac{\epsilon}{2} \int_{-\infty}^{\infty} dx \exp(-x^2) x^2 \ln(i x) / \int_{-\infty}^{\infty} dx \exp(-x^2) = \frac{\epsilon}{8} \psi(3/2). \quad (54)$$

The calculated higher order connected Green’s function to $O(\epsilon)$ is

$$G_c(y_1, \cdots, y_n) = -\frac{\epsilon}{2} (-i)^n \Gamma \left( \frac{n}{2} - 1 \right) \left[ \frac{1}{2} (4\pi)^{-D/2} \Gamma (1 - \frac{1}{2} D) \right]^{1-n/2} \int d^n x \prod_{k=1}^{n} \triangle_1 (y_k - x), \quad (55)$$

where the free propagator $\triangle_\lambda(x)$ is associated with $L_0 = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \lambda^2 \phi^2$ and obeys the equation

$$(-\nabla^2 + \lambda^2) \triangle_\lambda (x) = \delta^{(D)} (x). \quad (56)$$

The solution of (56) is

$$\triangle_\lambda (x) = \lambda^{D/2-1} |x|^{1-D/2} (2\pi)^{-D/2} K_{1-D/2} (\lambda |x|) \quad (57)$$

and

$$\triangle_\lambda (0) = \lambda^{D-2} (4\pi)^{-D/2} \Gamma (1 - \frac{D}{2}) \sim \frac{1}{2\pi \delta} \quad (\delta \to 0). \quad (58)$$

From (55) it is clear that as $\delta \to 0+$, the connected Green’s functions $G_c(y_1, \cdots, y_n) \to 0$ for $n \geq 3$. This indicates that at least to $O(\epsilon)$ the theory is noninteracting at $D = 2$. When we consider $O(\epsilon^2)$ contributions, we will reexamine this issue.

Turning to $n = 1$ and $n = 2$ we have

$$G_1 = -i\epsilon \sqrt{\frac{1}{2} \pi (4\pi)^{-D/2} \Gamma (1 - \frac{D}{2})} \quad (59)$$

and the two-point function $\tilde{G}_2 (p)$ in momentum space is

$$\tilde{G}_2 (p) = 1 / [p^2 + 1 + \epsilon K + O(\epsilon^2)] \quad (60)$$

where $K = \frac{3}{2} - \frac{1}{2} \gamma + \frac{1}{2} \ln [\frac{1}{2} (4\pi)^{-D/2} \Gamma (1 - \frac{D}{2})]$. Thus the renormalised mass to $O(\epsilon)$ is

$$M_R^2 = 1 + K \epsilon + O(\epsilon^2). \quad (61)$$

So near $D = 2$

$$G_1 \sim -i\epsilon \frac{1}{2\sqrt{\delta}} \quad (62)$$

and

$$M_R^2 \sim -\epsilon \ln \delta + A, \quad (63)$$

where $A = 1 + \epsilon [\frac{3}{2} - \frac{1}{2} - \frac{1}{2} \ln (4\pi)]$. Because of the singularities in $G_1$ and $M_R^2$ as $\delta \to 0$ some renormalisation is needed to remove these infinities. The question is whether perturbative renormalisation can be performed in the context of the novel $\epsilon$-expansion method. We can remove the divergence in $G_1$ by introducing in the Lagrangian a linear counter term $i v \phi$ where $v$ has dimension (mass)$^{1+D/2}$ and $v = v_1 \epsilon + v_2 \epsilon^2 + v_3 \epsilon^3 + \cdots$. Since $v$ is real such a term
is compatible with $\mathcal{PT}$-symmetry. Adding also a mass counter term $\mu$ we can consider the Lagrangian density

$$L = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{1}{2} g \mu_0^2 \phi^2 (i \mu_0^{1-D/2}) \phi^*,$$

(64)

where $\phi$ is dimensionful. Using this Lagrangian we obtain

$$G_1 = - \frac{i e g}{m^2} \mu_0^{D/2-1} \sqrt{\frac{1}{2} \pi m^{D-2} \Delta (0)} + \frac{i e v_1}{\mu_0^2 m^2},$$

(65)

and for the renormalised mass

$$M_R^2 = (m \mu_0^2) + \frac{1}{2} \epsilon g \mu_0^2 \{3 - \gamma + \ln[\frac{1}{2} m^{D-2} \Delta (0)]\}.$$  

(66)

In both expressions we have introduced the dimensionless quantity $m^2 = g + \mu^2/\mu_0^2$ and as $\delta \to 0$

$$G_1 \sim \frac{i e}{g \mu_0^2 + \mu^2} (v_1 - \frac{g \mu_0^2}{2 \sqrt{\delta}}),$$

(67)

and

$$M_R^2 \sim \mu^2 - \frac{1}{2} \epsilon g \mu_0^2 \ln \delta + A,$$

(68)

where $A = g \mu_0^2 \{1 + \epsilon[\frac{3}{2} - \frac{7}{2} - \frac{1}{2} \ln(4\pi)]\}$ is a finite quantity. By setting $v_1 = \frac{g \mu_0^2}{2 \sqrt{\delta}}$ we have a finite $G_1 = 0$. $M_R$ is logarithmically divergent in $\delta$. We absorb this divergence into $\mu$ by setting

$$\mu^2 = B + \frac{\epsilon}{2} g \mu_0^2 \ln \delta,$$

(69)

so that $M_R^2 = A + B$ and $B$ is a finite quantity determined in principle from experiment.

4.2. Calculations to second order in $\epsilon$

In second order the calculations become much more involved. The $O(\epsilon^2)$ contribution to the connected part of $G_1$, which we denote by $G_{1,2}$, can be shown to be

$$G_{1,2} = - \frac{1}{2} i g m^{-2} \sqrt{\frac{1}{2} \pi \Delta (0)} \left\{ [\ln[2 \mu_0^{2-D} \Delta (0)] + \psi(2)] + \frac{1}{8} i g^2 m^{-4} \sqrt{\frac{1}{2} \pi \Delta (0)} \left\{ (\ln[2 \mu_0^{2-D} \Delta (0)] + \psi(3/2)) (6 - D) + 2D - 4 + 4 \Delta (0) \mu_0^2 m^2 \int d^D x (1 + \frac{\Delta (x)}{\Delta (0)}) \ln[1 + \frac{\Delta (x)}{\Delta (0)}] \right\} \right\},$$

(70)

where $\Delta (0)$ denotes for brevity $\Delta_{\mu mun}(0)$. As $\delta \to 0$,

$$G_{1,2} \sim - \frac{i}{4} g m^{-2} \delta^{-1/2} [\psi(2) - \ln(\pi \delta)] + \frac{i}{4} g^2 m^{-4} \delta^{-1/2} [1 + \psi(3/2) - \ln \pi - \ln \delta] + O(\delta).$$

(71)

So the algebraic divergence $\delta^{-1/2}$ persists to $O(\epsilon^2)$. The divergence can be removed through $v_2$. Similarly the $\ln \delta$ divergence persists for $G_2$ at second order. Interestingly, the higher-order Green’s functions continue to vanish for $D = 2$. Avoidance of a noninteracting theory remains an open question in this approach [15].
5. Renormalisation group flows of $PT$-symmetric theories

So far we have considered whether a Hermitian theory can lead to a non-Hermitian $PT$-symmetric theory due to the effect of renormalisation. We ask the opposite question in this section: Can a $PT$-symmetric field theory retain its $PT$ symmetry as the Lagrangian flows due to renormalisation?

We review some preliminary work on this question. In its full generality this is an intractable problem. We turn to the framework of the functional renormalisation group [20], which combines the functional formulation of quantum field theory with the Wilsonian renormalisation group. As mentioned in Sec. 3, it is possible to make some progress in solving the functional equations using the simpler approach developed by Wetterich [21] and Morris [24] which, in the local potential approximation (32), leads to a nonlinear partial differential equation (PDE) rather than a functional equation. This simplification enables substantial progress in understanding the effective potential in arbitrary spacetime dimensions. In $D$ dimensions this PDE is

$$\partial_k U_k(\phi_c) = \frac{1}{\pi^D} \frac{k^{D+1}}{k^2 + U''_k(\phi_c)}, \quad (72)$$

where $\pi_D = \frac{D(2\pi)^D}{S_{D-1}}$ and $S_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ is the surface area of a unit $D$-dimensional sphere. We may assume that the equations for $U_k(\phi_c)$ involve dimensionless quantities. If this were not so, we could achieve dimensionlessness by introducing a mass scale $M$. For example, for $D = 1$ the dimensionless variables, denoted by a tilde, are $\tilde{\phi} = M^{1/2}\phi$, $\tilde{g} = M^{-3}g$, $\tilde{k} = M^{-1}k$, and $\tilde{\mu} = M^{-1}\mu$. The Wetterich equation (72) can be thought of as being in terms of such dimensionless variables.

To avoid difficulties in numerical analysis associated with boundary conditions, in the past the PDE (72) has been analyzed by approximating $U_k(\phi_c)$ as a finite series in powers of the field $\phi_c$. This ansatz leads to a sequence of coupled nonlinear ordinary differential equations. The consistency of such a procedure has not been established.

For $D = 1$ (the quantum-mechanical case), (72) becomes

$$\partial_k U_k(\phi_c) = \frac{k^2}{32\pi^2 k^2 + U''_k(\phi_c)}. \quad (73)$$

Even in this one-dimensional setting, no exact solution to this nonlinear PDE is known and only numerical solutions have been discussed.

We depart from the above treatment by performing an asymptotic analysis for large values of the cut-off $k$ [16]. The novelty of the approach used here is that it avoids the appearance of coupled nonlinear ordinary differential equations. To leading order the results of this analysis are qualitatively different depending on whether the space-time dimension $D$ is greater or less than 2. Of course, we are interested in the $k \to 0$ limit which we will, as we shall see, approach through the method of Padé approximants.

Letting $z = k^{2+D}$ and $U_k(\phi_c) = U(z, \phi)$, we can rewrite (72) as

$$U_z(z, \phi) = \frac{1}{(2 + D)\pi D} \frac{1}{z^{2+2\pi} + U_{\phi\phi}(z, \phi)}, \quad (74)$$

where the subscripts on $U$ indicate partial derivatives. We now assume that for large $z$ we can neglect the $U_{\phi\phi}$ term in the denominator. (The consistency of this assumption is easy to verify when $D < 2$.) Then, for large $z$ we have to leading order in our approximation scheme

$$U_z(z, \phi) \sim \frac{1}{(2 + D)\pi D} z^{-2} \quad (z \gg 1). \quad (75)$$
On incorporating a correction $\epsilon$ to this leading behavior

$$U(z, \phi) = \frac{1}{D\pi D} z^{\frac{D}{2}} + \epsilon(z, \phi),$$

(76)

we get to order $O(\epsilon^2)$

$$\epsilon_z(z, \phi) = -\frac{1}{(D+2)\pi D} z^{-\frac{D}{2} + \frac{2}{D}} \epsilon_{\phi\phi}(z, \phi).$$

(77)

On making the further change of variable

$$t = \frac{D+2}{D} \frac{D-2}{2} z^{\frac{D-2}{D}},$$

(77) becomes

$$\epsilon_t(t, \phi) = \frac{1}{(D+2)\pi D} \epsilon_{\phi\phi}(t, \phi).$$

(78)

The variable $t$ is positive for $D < 2$ and negative for $D > 2$ and is not defined at $D = 2$. Thus, (78) is a conventional diffusion equation for $D < 2$ but is a backward diffusion equation for $D > 2$. The backward diffusion equation is an inverse problem that is ill-posed. The problems associated with this ill-posedness may be connected with difficulties in solving (72) numerically when $D = 4$.

In the preliminary study these issues were not addressed further. The case $D = 1$ was considered and some simple $\mathcal{PT}$-symmetric theories were studied [16]. From (72), on defining $\hat{U}_k \equiv U_k - \frac{k^2}{D\pi D}$, we can deduce that

$$\partial_k \hat{U}_k = -\frac{k^{D-4} \hat{U}''_k}{\pi D \left( k^2 + \hat{U}''_k \right)}.$$ 

(79)

From (79) it is consistent to assume that $\hat{U} \to V(\phi)$ and $\hat{U}''_k \to 0$ as $k \to \infty$. (For simplicity of notation we have dropped the suffix $k$ in $\hat{U}_k$.) Let us write the correction as $V(\phi) + \frac{1}{k} U_1(\phi)$. On substituting in (79), we obtain

$$U_1(\phi) = \frac{1}{\pi} V''(\phi).$$

(80)

We can proceed in this way and write the next correction as $\hat{U}(\phi) = V(\phi) + \frac{1}{k^2} V''(\phi) + \frac{1}{k^4} U_2(\phi)$. This leads to $U_2(\phi) = \frac{1}{2\pi^2} V^{(4)}(\phi)$ where $V^{(4)}(\phi) \equiv \frac{d^4}{d\phi^4} V(\phi)$. Repeating this procedure, we get

$$U_3(\phi) = \frac{1}{3} \left( \frac{1}{2\pi^2} V^{(6)}(\phi) - V^{(2)}(\phi)^2 \right).$$

This procedure can be formalized: On writing $\delta = \pi/k$ (not to be confused with $\delta$ in the last section) and $x = \pi \phi$, (79) becomes

$$\frac{\partial}{\partial \delta} \hat{U} = \frac{\partial^2}{\partial x^2} \hat{U} \frac{1}{1 + \delta^2 \frac{\partial^2}{\partial x^2} U}$$

(81)

and $\hat{U} = \sum_{n=0}^{\infty} \delta^n U_n(\phi)$.

When $\delta$ is small, the scale $k$ is large and we are probing the microscopic potential. As $\delta \to \infty$ we probe the infrared limit of the effective potential. However, the analysis outlined above was based on a perturbation theory in $\delta$. There is a parallel to the theory of phase transitions, where a physical entity is expanded in inverse powers of the temperature $T$, and then an extrapolation procedure is used to find the critical behavior at lower $T$. This often-used extrapolation technique is based on Padé approximation.

So far, our treatment has been for a general potential; we will now specialize $V(x)$ to two cases $V(x) = gx^3$ and $V(x) = gx^4$, which are $\mathcal{PT}$-symmetric for $g = i$ and $g$ real. We also include a mass term. Our purpose is to consider the series $\sum_{n=0}^{N} \delta^n U_n(x)$ generated from the series relevant to these two cases and to consider the $\delta \to 0$ limit.
Figure 7. Effective potential flow for massive $i\phi^3$ with the mass parameter $\mu = 1$. Shown is a plot of the real and imaginary parts of the $P_5^5$ approximant plotted as functions of real $\phi$. Observe that there are no poles. The real part of the potential is right-side-up, so there is no instability. Furthermore, apart from small fluctuations near the origin, the imaginary part of the potential behaves like $i\phi^3$ for large $|\phi|$.  

5.1. Cubic potentials

We consider both massless and massive cubic potentials. In the former case $U_0(\phi) = g\phi^3$ and in the latter case $U_0(\phi) = \mu^2\phi^2 + g\phi^3$. For the massless case the solution to (81) is

$$
\begin{align*}
U_1(\phi) &= 6g\phi, \\
U_2(\phi) &= 0, \\
U_3(\phi) &= -12g^2\phi^2, \\
U_4(\phi) &= -6g^3, \\
U_5(\phi) &= \frac{216}{5}g^3\phi^3, \\
U_6(\phi) &= \frac{436}{9}g^3\phi^5, \\
U_7(\phi) &= \frac{-1296}{25}g^4\phi^4, \\
U_8(\phi) &= \frac{-34868}{25}g^4\phi^2, \\
U_9(\phi) &= \frac{7716}{9}g^5\phi^5 - \frac{8946}{315}g^4.
\end{align*}
$$

These expressions, which are valid for pure imaginary $g$ as well as for real $g$, are cumbersome but manageable. We stress that there has been no truncation of the function space on which $\hat{U}(\phi)$ has support. This contrasts with the usual approach which requires a truncation at the
onset of the calculation of the renormalisation group flow. This absence of truncation continues to be a feature for the massive case.

The iterative solution to (81) for the massive case is similar to that for the massless case, but the expressions for $U_n(\phi)$ have many more terms. For example, the coefficient of $\delta^9$ is

$$U_9(\phi) = \frac{8}{315}(34020 g^5 \phi^5 + 56700 g^4 \mu^2 \phi^4 - 11187 g^4 + 37800 g^3 \mu^4 \phi^3$$

$$+ 12600 g^2 \mu^6 \phi^2 + 2100 g \mu^8 \phi + 140 \mu^{10}),$$

which in the massless limit $\mu \to 0$ reduces to the two-term expression in (82). We refrain from listing the coefficients explicitly and instead proceed directly to the large-$\delta$ behavior of the diagonal Padé approximants. We denote the diagonal Padé approximants in the massless case by $P^0_N(\delta)$ and in the massive case by $P^M_N(\delta)$. Let us examine some low-order Padé approximants in the limit $\delta \to \infty$. For example,

$$\lim_{\delta \to \infty} P^2_2(\delta) = \frac{18 g^3 \phi^5 + 3 g^2 (10 \mu^2 \phi^4 - 9) + 14 g \mu^4 \phi^3 + 2 \mu^6 \phi^2}{2 (3 g \phi + \mu^2)^2}.$$

[As a check, when $\mu = 0$ this expression agrees with the corresponding expression $g^3 \frac{3}{2 \phi^2}$ for $P^0_2(\delta)$]. For $N = 3$ we obtain

$$\lim_{\delta \to \infty} P^3_3(\phi) = \frac{g_3 (736 g^5 \phi^5 + 1705)}{25 - 800 g \phi^5}$$

and

$$P^3_3(\phi) = -u_3/d_3,$$

where

$$u_3 = 1600632 g^8 \phi^8 + 729 g^7 \phi^7 (5888 \mu^2 \phi^4 + 5115) + 243 g^6 \mu^2 \phi^2 (23008 \mu^2 \phi^4 + 15345)$$

$$+ 1296 g^5 \mu^4 \phi (3376 \mu^2 \phi^4 + 945) + 15120 g^4 \mu^6 (14 \mu^2 \phi^4 + 9) + 658944 g^3 \mu^{10} \phi^3$$

$$+ 121824 g^2 \mu^{12} \phi^2 + 12288 g \mu^{14} \phi + 512 \mu^{16},$$

$$d_3 = 225 g^2 [7776 g^5 \phi^5 + 81 g^4 (160 \mu^2 \phi^4 - 3) + 8640 g^3 \mu^4 \phi^3 + 2880 g^2 \mu^6 \phi^2 + 480 g \mu^8 \phi + 32 \mu^{10}].$$

[As a check, we have $\lim_{\mu \to 0} P^3_3(\phi) = P^0_3(\phi).$]

Note that for the massless case $P^0_2(\delta)$ has a double pole for both real and imaginary $g$. In fact, the same is true for $P^0_n$ for even $n$. This pole is an artifact of the Padé approximation and is not present when $n$ is odd, so we will only consider the behavior of these approximants for odd $n$. In general, the odd-$n$ diagonal Padé approximants have no singularities at all on the real-$\phi$ axis when $g$ is imaginary but singularities occur for the case of real $g$. These findings indicate that the $\mathcal{PT}$-symmetric effective potential is well behaved in the infrared limit. From the expressions for the diagonal Padé approximants we see that for large $|\phi|$, the leading behavior of the imaginary part of the effective potential is exactly $i \phi^3$. Consequently the $\mathcal{PT}$ nature of the interaction is preserved under renormalisation.

A similar investigation of quartic potentials also shows that $\mathcal{PT}$-symmetry is maintained under renormalisation [16].

6. Conclusions

The study of $\mathcal{PT}$-symmetric quantum field theory is still in its infancy. We have shown that there is a link between renormalisation of some Hermitian field theories and $\mathcal{PT}$-symmetric field theories. This provides a motivation for the study of $\mathcal{PT}$-symmetric field theories. Often, the usual tools of quantum field theory cannot be applied directly. We have reviewed some interesting and promising approaches to studying these new field theories. There are many opportunities for future work in these approaches. The models we chose were based on simplicity but, the SM Lagrangian has many types of fields and interactions which can lead to the emergence of other $\mathcal{PT}$-symmetric QFTs.
Acknowledgments
CMB thanks the Alexander von Humboldt and Simons Foundations for financial support. SS and CMB thank the UK Engineering and Physical Sciences Research Council for financial support.

References
[1] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[2] Bender C M 2019 PT Symmetry in Quantum and Classical Physics (Singapore: World Scientific)
[3] Dorey P E, Dunning C and Tateo R 2001 J. Phys. A 34 5679
[4] Mostafazadeh A 2002 J. Math. Phys. 43, 205 ; 2003 J. Phys. A 36 7081
[5] Scholtz F G, Greyer H B and Hahne F J W 1992 Ann. Phys. 213 74
[6] Jones H F and Rivers R J 2007 Phys. Rev. D 75 025023
[7] Jones H F and Rivers R J 2009 Phys. Lett. A 373 3304
[8] Bender C M and Klevansky S P 2009 Phys. Rev. Lett. 103 3304
[9] Bender C M, Brandt S F, Chen J-H and Wang Q 2005 Phys. Rev. D 71 025014
[10] Sher M 2010 Phys. Rept. 179 031601
[11] Coleman S and Weinberg E J 1973 Phys. Rev. D 8 1888
[12] Bender C M, Hook D , Mavromatos N E and Sarkar S 2016 J. Phys. A 49 45LT01
[13] Haber H E and Kane G L 1985 Phys. Rept. 117 75
[14] Bender C M, Hassanpour N, Klevansky S P and Sarkar S 2018 Phys. Rev. D 98 125003
[15] Felski A, Bender C M, Klevansky S P and Sarkar S 2021 Towards perturbative renormalization of φ^2(iφ)^n quantum field theory Preprint (Preprint hep-th/2103.07577)
[16] Bender C M and Sarkar S 2018 J. Phys. A 51 225202
[17] Leutmann H, Symanzik K and Zimmermann W 1955 Nuovo Cimento 1 205
[18] Barton G 1963 Introduction to Advanced Field Theory (New York: Wiley & Sons)
[19] Lee T D 1954 Phys. Rev. 95 1329
[20] Gies H 2012 Lect. Notes Phys. 852 287
[21] Wetterich C 2001 Int. J. Mod. Phys. A 16 1951
[22] Litim D 2001 Phys. Rev. D 64 105007
[23] Wipf A 2013 Statistical Approach to Quantum Field Theory (Lect. Notes Phys. 864) (Berlin : Springer)
[24] Morris T R 1994 Phys. Lett. B 329 241
[25] Alexandre J , Houston N and Mavromatos N E 2013 Phys. Rev. D 88 125017; 2015 Int. J. Mod. Phys. D 24 1541004
[26] Starobinsky A A 1990, Phys. Lett. B 91, 99
[27] Bender C M, Holm D D and Hook D W 2007 J. Phys. A 40 F81; Brody D C and Hook D W 2008 J. Phys. A 41 352003; Fring A and Bagchi B 2011 J. Phys. A 44 325201
[28] Bender C M and Orszag S A 1978 Advanced Mathematical Methods for Scientists and Engineers (New York: McGraw Hill)