On the relation between lifting obstructions and ordinary obstructions

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Abstract: We consider partial liftings $k : A \to E$ of maps $f : X \to B$, where $(X, A)$ is a relative CW-complex and $E \to B$ is a fibration. In this situation, we have a primary obstruction to extend the partial lifting to a lifting of $f$ on all of $X$, and there is an obstruction to extend $k$ as an ordinary map into the space $E$. A relation between these two cohomology classes is proved when the fibre of $E \to B$ is an Eilenberg-Mac Lane space $K(\Pi, n)$ and $\pi_i(E) = 0$ for $i \leq q - 1$, where $q \geq n + 2$, that specialises to well-known formulas about secondary obstructions. The result is applied to the Hopf fibration (what includes defect sections in $S^1$-bundles over 4-manifolds) and to the case of a certain $SU(3)$-bundle over $S^4$.

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1 Introduction and summary of results

For a first example, let $M$ be a 4-dimensional closed and oriented manifold and

$$f : M \setminus \Delta \to S^2$$

a continuous map, where $\Delta \subset M$ is a finite set. Following the terminology that is used in the study of defects in ordered media, $f$ will be called a “defect map” and $\Delta$ the set of “point defects” (see [9], [10] and in particular [11] for the physical aspects of the questions treated in this paper). To each point $p \in \Delta$ we can assign its local index $\iota_p(f) \in \pi_3(S^2) = \mathbb{Z}$, given by the restriction of $f$ to the boundary of an embedded ball $D^4 \subset M$ whose center is $p$ and whose intersection with $\Delta$
consists of \( p \) alone (the embedding is supposed to preserve the orientation). Now consider the Hopf fibration \( S^3 \to S^2 \) and a defect lifting
\[
\overline{f}: M\setminus(\Delta \cup \Delta') \to S^3
\]
of \( f \). The defect set \( \Delta' \) can be chosen to be a closed orientable surface in \( M\setminus\Delta \): Let \( \xi := f^*(S^3 \to S^2) \) denote the induced \( S^1 \)-bundle over \( M\setminus\Delta \) and choose a transversal section \( \sigma : M\setminus\Delta \to \xi \times_{S^1} \mathbb{C} \) in the associated complex line bundle that is not zero on a certain neighborhood of \( \Delta \). A defect lifting with defect set \( \Delta' = \sigma^{-1}(0) \) is now given by \( \overline{\sigma} \).

Again, we have local indices that reflect the behavior of \( \overline{f} \) near \( \Delta' \): Let \( \Delta' = \bigcup_i \Delta'_i \) be the decomposition of \( \Delta' \) into its connected components. For each \( i \) choose a point \( \delta_i \in \Delta'_i \) and an embedding \( D^2 \to M\setminus(\Delta \cup \bigcup_{j\neq i} \Delta'_j) \), given by the restriction of a tubular neighborhood of \( \Delta'_i \) to the fibre of the normal bundle of \( \Delta'_i \) over \( \delta_i \). Since the induced bundle \( \xi \) is trivial over \( D^2 \), the lifting \( \overline{f} \) (that is just a section of \( \xi \) on the boundary of \( D^2 \)) is given by a map \( S^1 \to S^1 \), whose homotopy class will be denoted by \( n_i \in \pi_1(S^1) = \mathbb{Z} \) and will be referred to as the \textbf{local index} of \( \overline{f} \) along \( \Delta'_i \). It is easy to see that \( n_i \) does not depend on the tubular neighborhood and the point \( \delta_i \) used to define it (but the sign of \( n_i \) depends on orientations of the normal bundle and \( \Delta'_i \)). If the lifting comes from a transversal section as above, we certainly have \( n_i = \pm 1 \). Finally, we have the pullback \( f^*\eta \) of the canonical bundle over \( S^2 \), its Chern class will be denoted by \( c_1 \in H^2(M\setminus\Delta) = H^2(M) \).

We now look at the map \( \overline{f} \) as an ordinary defect map from \( M \to S^3 \) with regular defect set \( \Delta \cup \Delta' \). It is possible to replace the surface \( \Delta' \) by point defects by just altering the map \( \overline{f} \) on certain neighborhoods of the \( \Delta'_i \): For each \( i \), choose a tubular neighborhood \( N_i \subset M\setminus\Delta \) for \( \Delta'_i \) with \( N_i \cap N_j = \emptyset \) for \( i \neq j \), and let \( DN_i \) denote the disk bundle, \( SN_i \) its boundary. Because we have \( \pi_1(S^3) = \pi_2(S^2) = 0 \), we can find a finite set \( \Delta''_i \) for each \( i \), lying in the interior \( DN_i \) of \( DN_i \), and a defect map \( \overline{\eta} : M\setminus(\Delta \cup \bigcup_j \Delta''_j) \to S^3 \) that coincides with \( \overline{f} \) outside \( DN = \bigcup_i DN_i \). For each defect point \( q \in \Delta'' := \bigcup_j \Delta''_j \) we now have a local index \( \iota_q(\overline{\eta}) \in \pi_3(S^3) = \mathbb{Z} \).

In this example, the main question that will be examined in this paper appears as the following problem: Are there any relations between the points in \( \Delta \) with their indices, the points in \( \Delta'' \) and some geometric data of the surfaces \( \Delta'_i \)? In this simple case, there is in fact a concrete formula, that will be derived from a more general result presented later on:

**Proposition 1** For each \( i \), let \( \chi_i \) be the self-intersection number of \( \Delta'_i \). Then
\[
\sum_{q \in \Delta''} \iota_q(\overline{\eta}) = \pm n_i^2 \chi_i,
\]
where the sign is the same for all \( i \), and \( \sum_i n_i^2 \chi_i = c_1^2 \).

The sign in this formula does not depend on \( M, \Delta' \) or \( f \), it only depends on the sign in a certain equation in the group \( H^1(\mathbb{Z}, 2; \mathbb{Z}) \) (see the proof for details).
Note that this result also applies to defect sections in $S^1$-bundles over $M$, because up to point defects, every complex line bundle over $M$ can be obtained as a pullback of the canonical bundle over $S^2$, associated to the Hopf fibration.

As already mentioned above, Proposition 1 will be obtained as a special case of the main result of this paper that applies to the following situation: Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fibration whose fibre $F$ is of the type $(\Pi, n)$ for some $n \geq 1$, i.e. $\pi_i(F) = 0$ for $i \neq n$ and $\pi_n(F) = \Pi$, where $\Pi$ is an abelian group, and whose base space $B$ is a CW-complex. Let $q$ be the smallest integer with $\pi_q(E) \neq 0$, and suppose $q \geq n + 2$. Using the homotopy sequence of the fibration, we obtain $\pi_{n+1}(B) = \Pi$, the isomorphism given by the boundary map, $\pi_i(B) = 0$ for $1 \leq i \leq n$ and $n + 2 \leq i \leq q - 1$ and $\pi_q(B) = \pi_q(E)$. Let $G$ denote the groups $\pi_q(E)$ and $\pi_q(B)$ that will be identified in the sequel.

Now let $X$ be a connected CW-complex with a non-empty subcomplex $A$, considered as a relative CW-complex $(X, A)$ (using the terminology of Whiteheads book [4]), so $X_k$ will denote the union of all $k$-cells not lying in $A := X_{-1}$ with $A$). Suppose there are given two continuous maps $f, g : X_q \to B$ that coincide on $A$ and a partial lifting $k : A \to E$. Now we have a well defined primary obstruction $\overline{\gamma}^{n+1}(k; f) \in H^{n+1}(X, A; \Pi)$ to extend $k$ to a lifting of $f$ (originally, the obstruction is lying in the group $H^{n+1}(X_q, A; \Pi)$, but will be considered as an element of the isomorphic group $H^{n+1}(X, A; \Pi)$) and the corresponding obstruction $\overline{\gamma}^{n+1}(k; g)$ to extend $k$ as a lifting of $g$ (the coefficient group really is $\Pi = \pi_n(F)$ because the base space $B$ is simply connected). On the other hand, we have ordinary obstructions $\overline{\gamma}^{q+1}(f), \overline{\gamma}^{q+1}(g) \in H^{q+1}(X, A; G)$ to extend $f$ and $g$ to the $(q + 1)$-skeleton. Finally, there is a unique cohomology operation $\Theta : H^{n+1}(\cdot; \Pi) \to H^{q+1}(\cdot; G)$ with $\Theta(i^{n+1}) = k^{q+1}(B)$, where $i^{n+1} \in H^{n+1}(\Pi, n+1; \Pi)$ denotes the characteristic class and $k^{q+1}(B) \in H^{q+1}(\Pi, n + 1; G)$ the first non-trivial Postnikov invariant of $B$. We now have the following

**Theorem** $\overline{\gamma}^{q+1}(f) - \overline{\gamma}^{q+1}(g) = \Theta(-\overline{\gamma}^{n+1}(k; f)) - \Theta(-\overline{\gamma}^{n+1}(k; g))$.

There is a certain relationship between this equation and known results about secondary obstructions. Consider for example the case $A = \ast$ and a map $f : X_{i+1} \to S^i$ defined on the $(i + 1)$-skeleton of $X$, $i \geq 3$. By using an embedding of $S^i$ into $K(\mathbb{Z}, i)$ (that may be constructed by attaching cells) and restricting the path-space fibration over $K(\mathbb{Z}, i)$ to $S^i$, we obtain a fibration $E \to S^i$ as described above. If we choose the lifting $k$ to be the constant map and use the simple fact that $k^{q+2}(S^i)$ is the additive operation $Sq^2 \circ \rho_*$, where $\rho_*$ denotes reduction mod 2 (see the proof of Proposition 2), we obtain

$$\overline{\gamma}^{q+2}(f) - \overline{\gamma}^{q+2}(g) = Sq^2(\overline{\gamma}^i(f) - \overline{\gamma}^i(g) \mod 2).$$

Using the definitions, it is easy to verify that the difference $\overline{\gamma}^i(f) - \overline{\gamma}^i(g)$ is nothing else but the primary difference $\overline{\gamma}^i(f, g) \in H^i(X; \mathbb{Z})$ (here again we make use of
the boundary map in the homotopy sequence to identify the coefficient groups),
and we obtain the result

\[ \overline{c}_i + 2(f) - \overline{c}_i + 2(g) = Sq^2(d(i)(f, g) \mod 2), \]

that has been proved by Steenrod in his paper [7]. So well known facts about
the secondary obstruction can be obtained as a special case of the result above.
It should be possible to prove the Theorem by using known generalisations of
Steenrods result and the above relationship between the primary difference and
the lifting obstructions, but our proof avoids the difficulties that arise when the
operation \( \Theta \) fails to be additive, because we do not make use of the primary
difference between the maps \( f \) and \( g \) into the base \( B \).

Finally, a second example is considered, where the fibre is not of Eilenberg-
Mac Lane type, but the preceding results can be applied with the help of an
appropriate decomposition of the fibration: Let \( E \to S^4 \) denote the unique
\( SU(3) \)-principal bundle whose second Chern class \( c_2(P) \) is the orientation. This
bundle can be obtained by gluing together the trivial bundles over the northern
and southern hemisphere, using a representative of \( 1 \in \pi_3(SU(3)) = \mathbb{Z} \)
as gluing map. It is well-known that \( \pi_1(SU(3)) = \pi_2(SU(3)) = 0 \), \( \pi_3(SU(3)) = \mathbb{Z} \) and \( \pi_4(SU(3)) = 0 \) (see [4]). From this, it is easy to deduce
\( \pi_i(E) = 0 \) for \( i \leq 4 \) and \( \pi_5(E) = H_5(E) = \mathbb{Z} \), using the exact homotopy sequence
of \( E \to S^4 \), the Wang sequence and the Hurewicz Theorem.

Now let \( M \) be a manifold of dimension 6 and \( f : M \to S^4 \) a continuous map.
Suppose there is a lifting \( \overline{f} : M \setminus \Delta \to E \) outside a closed surface \( \Delta \). Again, we
have a local index for each connected component \( \Delta_i \), but since we do not longer
assume \( M \) and \( \Delta \) to be orientable, the local index is well-defined only up to sign.
Because of that, we only consider its reduction mod 2, that will be denoted by \( n_i \).
Because the total space \( E \) is 4-connected, we again can replace the defect set
\( \Delta \) by point defects if we just look at \( \overline{f} \) as an ordinary map into the total space \( E \).
Similar to the above, we can find a map \( \overline{g} : M \setminus \Delta' \to E \), where \( \Delta' \) is a finite union
of finite sets \( \Delta'_i \), each lying in the interior of a tubular neighborhood \( DN_i \subset M \)
of \( \Delta_i \), and \( \overline{g} \) is supposed to coincide with \( \overline{f} \) outside \( DN = \bigcup_i DN_i \). Finally we
have local indices \( \iota_q(\overline{g}) \) for each defect point \( q \in \Delta' \), and because \( \pi_5(E) = \mathbb{Z} \) and
\( M \) is not assumed to be oriented, this index is well-defined in the group \( \mathbb{Z}_2 \).

**Proposition 2** For each \( i \), let \( w_2(N_i) \in H^2(\Delta_i, \mathbb{Z}_2) = \mathbb{Z}_2 \) denote the second
Stiefel-Whitney-class of the normal bundle of \( \Delta_i \). Then, for all \( i \)

\[ \sum_{q \in \Delta'_i} \iota_q(\overline{g}) = n_i \cdot w_2(N_i) \in \mathbb{Z}_2. \]
2 Proof of the Theorem

During this proof, we again make use of the notations introduced on page 3. Consider a Postnikov decomposition \( \{B^k, f_k\} \) of \( B \), i.e., the space \( B^k \) is obtained by attaching cells of dimension \( \geq k + 2 \) at \( B \), \( \pi_i(B^k) = 0 \) for \( i > k \) and there is a diagram

\[
\ldots \to B^{k+1} \xrightarrow{f_{k+1}} B^k \xrightarrow{f_k} B^{k-1} \to \ldots
\]

where the inclusions \( \iota_k \) induce isomorphisms \( \pi_i(B^k) = \pi_i(B) \) for \( i \leq k \). We can assume that the mappings \( f_k \) are fibrations, whose fibres \( F_k \) are of the type \( (\pi_k(B), k) \) and the inclusions \( \pi_k(F_k) \to \pi_k(B^k) \) are isomorphisms (see the discussion in [9] p. 431). Furthermore it is well known that \( k^{q+1}(B) \) is just the primary obstruction to find a section in \( B^q \to B^{q-1} = K(\Pi, n+1) \).

Now consider the path space fibration \( P = \{\alpha : [0, 1] \to B^{q-1} | \alpha(0) = f(*) = g(*)\} \to B^{q-1} \), where \( * \in A \) is a distinguished base point and the projection is given by sending a path \( \alpha \) to \( \alpha(1) \) (see [3] p. 31). It is clear that \( P \) is contractible. Using standard arguments (CW-approximation), we can assume \( E \) to be a CW-complex and apply obstruction theory to prove the existence of a lifting of \( \iota_{q-1} \circ p \), so we have the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\eta} & P \\
| \quad p & \quad \downarrow p' \quad |
\end{array}
\]

\[
\begin{array}{ccc}
B & \xrightarrow{\iota_q} & B^q \\
| \quad \iota_{q-1} & \quad \downarrow \iota_{q-1} \quad |
\end{array}
\]

The homotopy sequence of the fibration \( P \to B^{q-1} \) shows that the fibre \( F' \) of \( p' \) is of the type \( (\Pi, n) \), where the isomorphism \( \Pi \cong \pi_n(F') \) is induced by the restriction of \( \eta \) to the fibre. An easy computation using the definition of the boundary map in the homotopy sequence proves that the primary obstruction to a section in \( P \to B^{q-1} \) is just \(-\iota^{n+1} \in H^{n+1}(B^{q-1}, \Pi)\). Since \( P \) is contractible, there is a homotopy

\[
\hat{h} : A \times [0, 1] \to P
\]

with \( \hat{h}_0 = \eta \circ k \) and \( \hat{h}_1 = * \). Let \( h := p' \circ \hat{h} \), so \( h_0 = \iota_{q-1} \circ f|A = \iota_{q-1} \circ g|A \) and \( h_1 = * \). Finally, let \( \overline{h} : A \times [0, 1] \to B^q \) be a lifting of \( h \) with \( \overline{h}_0 = \iota_q \circ f|A \). Since \( \overline{h}_1 \) and \( * \) are two liftings of the trivial map \( * \), there is a well-defined primary difference

\[
d := d'(\overline{h}_1, *) \in H^q(A; G) \quad (1)
\]
between these two maps as liftings of \( \ast : A \to B^{q-1} \).

Since \( B^{q-1} \) is an Eilenberg-Mac Lane space, we can extend \( \iota_{q-1} \circ f \) over all of \( X \), and using the homotopy extension property of \( (X, A) \), we can find a homotopy \( H : X \times [0, 1] \to B^{q-1} \) with \( H_0 \mid X_q = \iota_{q-1} \circ f \) that coincides with \( h \) on \( A \times [0, 1] \). Let \( f' := H_1 : (X, A) \to (B^{q-1}, \ast) \). Now we have a well-defined primary difference between the two liftings \( \overline{h}_1 \) and \( \ast \) of \( f' \mid A = \ast \). Note that this difference is exactly the class \( d \in H^q(A; G) \) defined in equation (11), since it only depends on the restriction of \( f' \) to \( A \) which is the constant map \( \ast \).

There are primary obstructions \( \overline{\gamma}^{q+1}(\overline{h}_1; f') \) (resp. \( \overline{\gamma}^{q+1}(\overline{h}_0, H_0) \)) to extend the partial lifting of \( f' \) (\( H_0 \)), given by \( \overline{h}_1 (\overline{h}_0) \). From the coboundary formula (see [3], 36.7) we have

\[
\overline{\gamma}^{q+1}(\overline{h}_1; f') - \overline{\gamma}^{q+1}(\ast; f') = \delta d,
\]

where \( \delta : H^q(A; G) \to H^{q+1}(X, A; G) \) denotes the boundary. Using this, the homotopy invariance and the naturality properties of the obstructions and we can conclude

\[
\overline{\gamma}^{q+1}(\overline{h}_0; H_0) = \overline{\gamma}^{q+1}(\overline{h}_1; f') = \overline{\gamma}^{q+1}(\ast; f') + \delta \overline{d}(\overline{h}_1, \ast)
\]

\[
= \overline{\gamma}^{q+1}(\ast; f') + \delta d = f'^* \overline{\gamma}^{q+1}(\ast; id) + \delta d.
\]

But as mentioned above, the primary obstruction \( \overline{\gamma}^{q+1}(\ast; id) \) is nothing else but the Postnikov invariant, and it is well known (see [3] p. 450) that the class \( \overline{\gamma}^{q+1}(\overline{h}_0; H_0) \) coincides with the obstruction \( \overline{c}^{q+1}(f) \) (remember that \( H_0 \) is an extension of \( \iota_{q-1} \circ f \)). So we obtain the relation

\[
\overline{c}^{q+1}(f) = f'^* k^{q+1}(B) + \delta d.
\]

On the other hand, we can consider the fibration \( P \to B^{q-1} \) with primary lifting obstruction \( -\nu^{n+1} \in H^{n+1}(X, A; \Pi) \), and we have

\[
f'^* \nu^{n+1} = -f'^* \overline{\gamma}^{n+1}(\ast; id) = -\overline{\gamma}^{n+1}(\ast; H_1) = -\overline{\gamma}^{n+1}(\eta \circ k; H_0),
\]

but the latter class is (remember that we identified the homotopy of the fibre of \( P \to B^{q-1} \) with that of \( F \), using the map \( \eta \)) exactly \( -\overline{\gamma}^{n+1}(k; f) \), so we finally have

\[
f'^* \nu^{n+1} = -\overline{\gamma}^{n+1}(k; f).
\]

Together with the above relation, we obtain

\[
\Theta(-\overline{\gamma}^{n+1}(k; f)) = \Theta(f'^* \nu^{n+1}) = f'^* k^{q+1}(B) = \overline{c}^{q+1}(f) - \delta d.
\]

Replacing \( f \) by \( g \) in the above construction yields the formula

\[
\Theta(-\overline{\gamma}^{q+1}(k; g)) = \overline{c}^{q+1}(g) - \delta d
\]

(note that we can use the same homotopies \( \overline{h} \) and \( \hat{h} \) and so we have the same class \( d \), since \( f \) and \( g \) coincide on \( A \)). Subtraction of these equations give the desired result. \( \square \)
3 Proof of Proposition 1

Now we specialise to the case where $E \to B$ is the Hopf fibration, $q = 3$, $n = 1$ and the pair $(X, A)$ is $(DN_i, SN_i)$, with the notations already used in the introduction ($i$ will be fixed in the sequel). The map $f$ is the restriction of the point-defect map $f : M \setminus \Delta \to S^2$. Since $DN_i$ does not contain defect points, we have $\overline{\tau}^{i+1}(f) = 0$. Let $g := \pi \circ \overline{\gamma}$, where $\pi$ denotes the projection $S^3 \to S^2$. It is easy to see (using the identification $\pi_3(S^3) = \pi_3(S^2) = \mathbb{Z}$ induced by $\pi$ and a triangulation of $DN_i$ whose 3-skeleton does not intersect $\Delta_i''$), that the obstruction $\overline{\tau}^{i+1}(g) \in H^4(DN_i, SN_i) = \mathbb{Z}$ is just the sum $\sum_{q \in \Delta_i''} \iota_q(\overline{\gamma})$. Now choose orientations of the normal bundle and $\Delta_i'$ with the property that under the Thom isomorphism, the orientation of the surface $\Delta_i'$ corresponds to the orientation of $DN_i$, induced by that of $M$ (since the formula contains the square $n_i^2$, we can suppose that these orientations were used to define the index $n_i$). Then the class $\overline{\tau}^{i+1}(k; f)$ is a certain multiple of the Thom class $\tau$, in fact it is $n_i \tau$, and certainly $\overline{\tau}^{i+1}(k; g) = 0$. Finally, the operation $\Theta$ associated to the Postnikov invariant $k^{i+1}(S^2)$ is just the cup square, up to sign. Since the square $\tau_2$ is $\chi_i[DN]$, where $[DN]$ denotes the orientation, we obtain the first part of Proposition 1. The second part is just the fact that certainly $\sum_{p \in \Delta} \iota_p(f) + \sum_{q \in \Delta_i} \iota_q(\overline{\gamma}) = 0$, and the first sum is just $\pm c_1^2$, where the sign is the same as before (this follows if we apply the Theorem with $(X, A) = (M, \ast)$ and $g = \ast$).

4 Proof of Proposition 2

First, we have to compute the Postnikov invariant $k^6(S^4) \in H^6(\mathbb{Z}, 4; \mathbb{Z}_2)$. It is a result of Eilenberg and Mac Lane (\cite{Eilenberg-MacLane} p. 122) that this group is $\mathbb{Z}_2$. The non-zero element is given by the operation $Sq^2 \circ \rho_*$, where $\rho$ denotes the reduction mod 2: Because the operation is stable, $Sq^2 \circ \rho_* = 0$ in dimension 4 would imply $Sq^2 \circ \rho_* = 0 : H^2(X; \mathbb{Z}) \to H^4(X; \mathbb{Z})$, what is easily seen not to be true, just consider the example $X = \mathbb{C}P^n$. A similar argument, using a suspension formula for the Postnikov invariants (\cite{Eilenberg-MacLane} p. 439), tells us that the operation $k^6(S^4)$ is not zero (this would give $\rho_*k^4(S^2) = 0$, but $k^4(S^2)$ is just the cup square). Hence $k^6(S^4) = Sq^2 \circ \rho$ (in fact we have proved $k^{i+2}(S^i) = Sq^2 \circ \rho_*$ for all $i \geq 4$).

We now decompose our fibration using the following Lemma (that can be seen as the first stage of the so called “Moore-Postnikov decomposition” , see \cite{Moore}, Chapter 8 for details):

**Lemma 1:** Let $F \leftrightarrow E \xrightarrow{p} B$ be a fibration with $(n-1)$-connected $E$, $n \geq 2$, simply connected base space $B$ and simple and connected fibre $F$. The total space $E$ is supposed to be of the homotopy type of a CW-complex. Let $q$ be an integer, $1 \leq q < n$. 


Then there are fibrations $F' \hookrightarrow S \xrightarrow{\pi} B$ and $F'' \hookrightarrow \tilde{E} \xrightarrow{q'} S$, together with maps $q : E \to S$ and $\lambda : E \to \tilde{E}$ such that the diagram

![Diagram](image)

commutes and the following holds:

1. $\lambda$ is a homotopy equivalence,
2. $q_* : \pi_i(F) \to \pi_i(F')$ is an isomorphism for $i \leq q - 1$ and $\pi_i(F') = 0$ for $i \geq q$,
3. $S$ is $q$-connected, $\pi_* : \pi_i(S) \to \pi_i(B)$ is an isomorphism for $i \geq q + 1$, and
4. $\pi_i(F'') = 0$ for $i \leq q - 1$, and $\pi_i(F'') \cong \pi_i(F)$ for $i \geq q$.

**Proof:** By attaching cells of dimension $\geq q + 2$, we can construct a cellular extension $B^*$ of $B$ with $\pi_i(B^*) = 0$ for $i \geq q + 1$. Let $P' \to B^*$ be the path space fibration and $S$ be the total space of its restriction to $B$. It is easy to see that there is a lifting $q : E \to S$ of $E \to B$, using the assumptions made on the homotopy groups of $E$, and the existence of the claimed homotopy equivalence $\lambda$ and the fibration $\tilde{E} \to S$ is based on a standard tool in homotopy theory, see [9] p. 42. It is now a straightforward computation to verify the above properties, using the homotopy sequences for the involved fibrations. \[\square\]

Now choose a decomposition as in the Lemma for the $SU(3)$-bundle $E \to S^4$ with $q = 4$. Then $\pi_5(S) = \pi_5(S^4) = \mathbb{Z}_2$, the fibre $F'$ is of the type $(\mathbb{Z}, 3)$, and with the identifications $\pi_5(E) = \mathbb{Z}$, $\pi_5(S) = \mathbb{Z}_2$, the map $q_* : \pi_5(E) \to \pi_5(S)$ is easily seen to be an epimorphism, so it is just the canonical map $\mathbb{Z} \to \mathbb{Z}_2$. We can suppose that $\Delta$ is connected and $M = DN$. Now let $\overline{\gamma}^4(q \circ \overline{f}; f) \in H^4(DN, SN; \mathbb{Z})$ be the primary obstruction to extend the lifting $q \circ \overline{f}$ of $f$ to $DN$. It is easy to see that $\overline{\gamma}^4(q \circ \overline{f}; f)$ mod $2 = n\tau$, where $\tau$ denotes the $\mathbb{Z}_2$ Thom class of the normal bundle $N$. Let $\overline{e}^6(q \circ \overline{f}) \in H^6(DN, SN; \mathbb{Z}_2)$ denote the primary obstruction to extend $q \circ \overline{f}$ as an ordinary map into $S$. Applying the above calculation of $k^6(S^4)$ and the Theorem, similar to the proof of Proposition 1, we obtain $\overline{e}^6(q \circ \overline{f}) = Sq^2(n\tau) = nSq^2(\tau)$. But clearly $\overline{e}^6(q \circ \overline{f}) \in H^6(DN, SN; \mathbb{Z}_2) = \mathbb{Z}_2$ is just the sum $\sum_{q \in \Delta'} t_q(\overline{f})$, and because $Sq^2(\tau)$ is identified with the second
Stiefel-Whitney class $w_2(N)$ under the Thom isomorphism (see [3], Chapter 17, 9.1), we obtain the desired result $\sum_{q \in \Delta'} \iota_q(\overline{\gamma}) = nw_2(N)$. □

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