I. INTRODUCTION

The existence of a exotic cosmic fluid with negative pressure, which constitutes about the 70 per cent of the total energy of the universe, has been perhaps the most surprising discovery made in cosmology. This dark energy is supported by the astrophysical data obtained from Wilkinson Microwave Anisotropy Probe (WMAP) (Map) and high redshift surveys of supernovae.

The dark energy is considered a fluid characterized by a negative pressure and usually represented by the equation of state $w = p/\rho$, where $w$ lies very close to $-1$, most probably being below $-1$. Dark energy with $w < -1$, the phantom component of the universe, leads to uncommon cosmological scenarios as it was pointed out in \cite{1}. First of all, there is a violation of the dominant energy condition (DEC), since $\rho + p < 0$. The energy density grows up to infinity in a finite time, which leads to a big rip, characterized by a scale factor blowing up in this finite time. These sudden future singularities are, nevertheless, not necessarily produced by a fluids violating DEC. Barrow \cite{2} has shown, with explicit examples, that exist solutions which develop a big rip singularity at a finite time even if the matter fields satisfy the strong-energy conditions $\rho > 0$ and $\rho + 3p > 0$. A generalization of Barrow’s model has been realized in \cite{3}, giving its Lagrangian description in terms of scalar tensor theory. It was also proved by Chimento et al \cite{4} that exist a duality between phantom and flat Friedmann-Robertson-Walker (FRW) cosmologies with nonexotic fluids. This duality is a form-invariance transformation which can be used for constructing phantom cosmologies from standard scalar field universes. Cosmological solutions for phantom matter which violates the weak energy condition were found in \cite{5}.

The role of the dissipative processes in the evolution of the early universe also has been extensively studied. In the case of isotropic and homogeneous cosmologies, any dissipation process in a FRW cosmology is scalar, and therefore may be modelled as a bulk viscosity within a thermodynamical approach.

A well known result of the FRW cosmological solutions, corresponding to universes filled with perfect fluid and bulk viscous stresses, is the possibility of violating DEC \cite{6}. The bulk viscosity introduces dissipation by only redefining the effective pressure, $P_{eff}$, according to

$$P_{eff} = p + \Pi = p - 3\xi H,$$

where $\Pi$ is the bulk viscous pressure, $\xi$ is the bulk viscosity coefficient and $H$ is the Hubble parameter. Since the equation of energy balance is

$$\dot{\rho} + 3H(\rho + p + \Pi) = 0,$$

the violation of DEC, i.e., $\rho + p + \Pi < 0$ implies an increasing energy density of the fluid that fills the universe, for a positive bulk viscosity coefficient. The condition $\xi > 0$ guarantees a positive entropy production and, in consequence, no violation of the second law of the thermodynamics \cite{7}.

In the present paper we show that the above results are straightforward to obtain from the exact cosmological solutions already found by Barrow in \cite{6}. These solutions were obtained using non causal thermodynamics. Nevertheless, we consider a more physical approach like the full Israel-Stewart-Hiscock causal thermodynamics, showing that it is also possible to obtain big rip type solutions.

The organization of the paper is as follows: in Section II we present the field equations for a flat FRW universe filled with a bulk viscous fluid within the framework of the Eckart theory. We indicate that under a constraint for the parameters of the fluid, one of the Barrow’s solutions presents a future singularity in a finite time. In Section III we obtain big rip solutions using the approach of the full Israel-Stewart-Hiscock causal thermodynamics. In Section IV we discuss our results in relation to the nature of the dark energy of the universe.
II. ECKART THEORY

The FRW metric for an homogeneous and isotropic flat universe is given by

\[ ds^2 = -dt^2 + a(t)^2 \left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right), \]

where \( a(t) \) is the scale factor and \( t \) represents the cosmic time. In the first order thermodynamic theory of Eckart [8], the field equations in the presence of bulk viscous stresses are

\[ \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{\rho}{3}, \]

\[ -\frac{\ddot{a}}{a} = \dot{H} + H^2 = \frac{1}{6} \left(\rho + 3P_{eff}\right), \]

with

\[ P_{eff} = p + \Pi, \]

and

\[ \Pi = -3H\xi. \]

The conservation equation is

\[ \dot{\rho} + 3H(\rho + p + \Pi) = 0. \]

Assuming that the dark component obey the state equation

\[ p = (\gamma - 1)\rho, \]

where \( 0 \leq \gamma \leq 2 \), we can obtain from equations (4) to (9) a single evolution equation for \( H \):

\[ 2\dot{H} + 3\gamma H^2 = 3\xi H. \]

This equation may be integrated directly as a function of the bulk viscosity. For \( \gamma \neq 0 \) the solution has the form

\[ H(t) = \frac{e^{\frac{2}{3} \int \xi(t) dt}}{C + \frac{2}{3} \gamma \int e^{\frac{2}{3} \int \xi(t) dt} dt}, \]

where \( C \) is an integration constant. From this equation we find the following expression for the scale factor

\[ a(t) = D \left( C + \frac{3}{2} \gamma \int e^{\frac{2}{3} \int \xi(t) dt} dt \right)^{2/(3\gamma)}, \]

where \( D \) is a new integration constant. Thus, for a given \( \xi(t) \) we have the expressions for \( a(t) \), \( \rho(t) \) and \( p(t) \).

For the case \( \gamma = 0 \) we have from Eq. (10)

\[ \xi = \frac{2}{3} \dot{H}, \]

and substituting this expression into Eq. (5) we have

\[ \dot{\rho} = 6\dot{H} \dot{H}, \]

from which we conclude that \( \rho = 3H^2 + const. \) Comparing this expression with (1) we have that the integration constant is zero.

Thus we have that for the state equation \( p = -\rho, \) i.e. for \( \gamma = 0 \), the scale factor is not defined by the field equations. So for a given \( a(t) \) we can write \( H \) and then obtain the expressions for the energy density from Eq. (1) and the bulk viscosity from Eq. (13). Clearly if \( \xi = 0 \) the well known de Sitter scale factor \( a(t) = e^{H_0 t} \) is obtained, where \( p = -\rho \) and both are constants.

Notice that the solution of the field equations may be written through \( \xi(t) \) or \( a(t) \) because there are three independent equations for the four unknown functions \( a(t) \), \( \rho(t) \), \( \xi(t) \) and \( p(t) \).

Now we are interested in the possibility that there are cosmological models with viscous matter which present in its development a big rip singularity.

A. The case for \( \gamma \neq 0 \)

Firstly, let us consider the case \( \gamma \neq 0 \). If the viscous fluid satisfies DEC, then the condition \( 0 \leq \gamma \leq 2 \) must be satisfied. Thus for \( \gamma < 0 \) we have a phantom cosmology. Now from the thermodynamics we know that \( \xi > 0 \), and if \( \gamma < 0 \) the Eq. (12) implies that we can have a big rip singularity at a finite value of cosmic time.

Let us consider some examples to see this more clearly. From Eq. (12) the well known standard case for a perfect fluid, i.e. \( \xi = 0 \), takes the form \( a(t) = D(C + (3/2)\gamma t)^{2/(3\gamma)} \). This scale factor may be rewritten as

\[ a(t) = a_0 \left(1 + \frac{3}{2}H_0\gamma t\right)^{2/(3\gamma)}, \]

and the energy density is given by

\[ \rho = \frac{\rho_0}{\left(1 + \frac{3}{2}H_0\gamma t\right)^2}, \]

where \( \rho_0 = 3H_0^2 \), in order to have \( H(t_o = 0) = H_o > 0 \). If \( \gamma < 0 \) we have a big rip singularity at a finite value of cosmic time \( t_{sr} = -2/(3H_o\gamma) > t_o = 0 \).

In the special case of \( \xi(t) = \xi_o = const \) we have from Eq. (12) for the scale factor \( a(t) = D(C + \gamma/\xi_o) e^{(3/2)\gamma t^2/(3\gamma)} \). We can rewrite it into the form

\[ a(t) = a_o \left(1 + \frac{H_0}{\xi_o} \gamma \left(e^{3\xi_o t^2/2} - 1\right)\right)^{2/(3\gamma)}, \]

from which we obtain for the energy density

\[ \rho(t) = \rho_o e^{3\xi_o t/\xi_o} \left(1 + \frac{H_0}{\xi_o} \gamma \left(e^{3\xi_o t^2/2} - 1\right)\right)^2, \]

where \( \rho_o = 3H_0^2 \). As before, for \( \gamma < 0 \) we have a big rip singularity at a finite value of cosmic time \( t_{sr} = \frac{2}{3\xi_o} ln(1 - \frac{\xi_o}{H_o\gamma}) > t_o = 0 \).
Note that any additional condition on the system of the field equations will fix the unknown functions. So for instance, for a variable $\xi(t)$ we can take the condition $\xi(t) = \xi(\rho(t))$.

Another example in this line is given by the solution obtained by Barrow [6] for the case $\xi \sim \rho^{1/2}$. Effectively, Barrow [6] assumed that the viscosity has a power-law dependence upon the density $\rho$.

In terms of time $t$, the following expression for the scale factor will fix the unknown functions. So for instance, for a variable $\xi(t)$ we can take the condition $\xi(t) = \xi(\rho(t))$.

From the equation (12) and the parameterized equations (21) and (24) for the scale factor and Hubble parameter, respectively, we obtain the expression for the increasing density of the dark component in terms of scale factor

$$\rho(a) = 3H_0^2 \left(\frac{a}{a_0}\right)^{3(\sqrt{3}\alpha - \gamma)}.$$  (25)

We reproduce completely this solution if we put into the Eq. (12) the bulk viscosity given by

$$\xi(t) = \sqrt{3}\alpha H_0 \left(1 - \frac{t - t_0}{t_{br}}\right)^{-1}.$$  (26)

**B. The case for $\gamma = 0$**

Notice that the structure of equation (10) changes if $\gamma = 0$ and $\xi$ is an arbitrary function of the density, since the quadratic term in $H$ disappears. Nevertheless, if $\xi \sim \rho^{1/2}$, the structure of the equation (10) is the same for any value of $\gamma$ in the range $0 \leq \gamma \leq 2$, except in the case $\sqrt{3}\alpha = \gamma$, where equation (10) becomes $H = 0$. Then, the solution with $\gamma = 0$ can be obtained directly from the general solution given by equation (21). In this case there is a big rip singularity at a finite value of cosmic time $t_{br} = 2/(3\sqrt{3}H_0\alpha) > t_0 = 0$.

## III. ISRAEL-STEWART-HISCOCK THEORY

We now consider the dissipative process in the universe within the framework of the full causal theory of Israel-Stewart-Hiscock. In this case we have the same Friedmann equations but instead of equation (7), we have an equation for the causal evolution of the bulk viscous pressure, which is given by

$$\tau \Pi + \Pi = -3\xi H - \frac{1}{2} \tau \Pi \left(3H + \frac{\dot{\xi}}{\tau} - \frac{\xi}{\tau} - \frac{T}{T}\right).$$  (27)

where $T$ is the temperature and $\tau$ the relaxation time. In order to close the system we have to give the equation specifying $T$

$$T = \beta \rho^\gamma.$$  (28)

The the relaxation time is defined by the expression

$$\tau = \frac{\xi}{\rho} = \alpha \rho^{-1},$$  (29)

where $\beta \geq 0$. This model imposes the constraint

$$r = \frac{\gamma - 1}{\gamma},$$  (30)

in order to have the entropy as an state function. Notice that the above constraint exclude the range $0 < \gamma < 1$, which implies that quintessence fluids are not allowed in this approach. With the above assumptions the field equations and the causal evolution equation for the bulk viscosity lead to the following evolution equation for $H$,
\[\ddot{H} + \frac{3}{2} (1 + (1-r)\gamma) \dot{H} \dot{H} + 3^{1-s} \alpha^{-1} H^{2-2s} \dot{H} - (1+r)H^{-1} \dot{H}^2 + \frac{9}{4} (\gamma - 2)) H^3 + \frac{1}{2} 2^{3-s} \alpha^{-1}\gamma H^{4-2s} = 0. \quad (31)\]

As in the non causal case we will choose \( s = 1/2 \) and the above equation becomes
\[\ddot{H} + bH \dot{H} - \left(2 - \frac{1}{\gamma}\right) H^{-1} \dot{H}^2 + aH^3 = 0, \quad (32)\]
where \( a \) is defined by
\[a = \frac{9}{4} \left(1 + \frac{2}{\sqrt{3\alpha}}\right) (\gamma - 2), \quad (33)\]
and \( b \) by
\[b = 3 \left(1 + \frac{1}{\sqrt{3\alpha}}\right). \quad (34)\]

Solutions of equation \((32)\), were obtained in [10]. In this work only was considered \( \gamma \) in the range \( 1 \leq \gamma \leq 2 \). Some of these solutions presents an increasing energy density and accelerated expansion.

Inspired in the solution for the Hubble parameter given by equation \((24)\) in the non causal scheme, we use the following Ansatz, where for simplicity we take \( t_0 = 0 \)
\[H(t) = A (\tau_{br} - t)^{-1}, \quad (35)\]
where \( A \equiv H_0 \tau_{br} \). With this Ansatz the scale factor \( a(t) \) evolves as
\[a(t) \sim (\tau_{br} - t)^{-A}, \quad (36)\]
and the energy density, \( \rho \), of the dark component as a function of the scale factor becomes
\[\rho(a) \sim a^{2/A}. \quad (37)\]

Using the Ansatz \((35)\) in equation \((32)\) we obtain a second grade equation for \( A \)
\[aA^2 + bA + \frac{1}{\gamma} = 0. \quad (38)\]
The solutions for \( A \) are given by
\[2A_{\pm} = -\frac{b}{a} \pm \sqrt{\Delta}, \quad (39)\]
where the discriminant \( \Delta \) has the expression:
\[\Delta = \left(\frac{b}{a}\right)^2 - 4 \left(\frac{1}{a\gamma}\right). \quad (40)\]

Since we are interested only in positive solutions for \( A \), the coefficient \( a \) must be negative. We have two cases of interest.

**Case** \( a < 0; \gamma > 0 \). In this case only \( A_+ \) correspond to a solution with big rip. The parameters \( \alpha \) and \( \gamma \) satisfy the following constraint
\[\sqrt{3\alpha} > \gamma \left(1 - \frac{\gamma}{2}\right). \quad (41)\]

Notice that there is no big rip solution if the cosmic fluid representing the dark component is stiff matter (\( \gamma = 2 \)). The factor \( \left(1 - \frac{\gamma}{2}\right) \) is the correction introduced by the causal thermodynamics to the constraint given by equation \((22)\). The solution for \( A_+ \) is given by
\[A_+ = \frac{1}{3} \left(1 + \frac{1}{\sqrt{3\alpha}}\right) + \left(\frac{1}{3\alpha^2} + \frac{2}{\gamma}\right)^{1/2} \left(1 - \left(1 + \frac{2}{\sqrt{3\alpha}}\right)^{2}\right)^{1/2}, \quad (42)\]
which implies that a big rip will occurs at a time
\[\tau_{br} = A_+ H_0^{-1}. \quad (43)\]
The expressions for \( a = a(t) \) and \( \rho = \rho(a) \) can be easily evaluate from equations \((39)\) and \((37)\), respectively.

**Case** \( a < 0; \gamma < 0 \). Since we need \( \Delta \geq 0 \) in order to have real solutions, the parameters \( \alpha \) and \( \gamma \) must satisfy the following constraint
\[\sqrt{3\alpha} \leq \frac{\sqrt{2}}{2}. \quad (44)\]

If \( \Delta = 0 \), i.e., \( \sqrt{3\alpha} = \sqrt{\frac{1}{2}} \), the solution for \( A \), which we shall call \( A_0 \), has the following expression
\[A_0 = \frac{2}{3} \left(1 + \frac{1}{\sqrt{3\alpha}}\right) \frac{1}{\sqrt{\left|\gamma\right|}}. \quad (45)\]

If \( \Delta > 0 \), the solutions for \( A \) can be written as
\[A_\pm = A_0 \pm \frac{1}{2} \sqrt{\Delta}. \quad (46)\]

### IV. DISCUSSION

Within the framework of the non causal thermodynamics we have showed that the power law solution, found by Barrow in [11] for dissipative universes with \( \xi = \alpha \rho^{1/2} \), yields cosmologies which present big rip singularity when the constraint given in equation \((22)\) holds. If we consider that the dark component is quintessence, i.e., \( 0 \leq \gamma \leq 2/3 \), with a sufficiently large bulk viscosity will make this quintessence behaves like a phantom energy. In the range \( 2/3 > \gamma \geq 2 \) it is possible, at least
from the mathematical point of view, to obtain solutions with big rip even with a matter fluid. It is not clear for us how can be interpreted a radiation fluid, for example, with a large bulk viscosity leading to high negative pressures and increasing densities.

At the boundary between the quintessence sector and the phantom sector, i.e. $\gamma = 0$ or $p = -\rho$, also there exist cosmologies with a big rip singularity.

Using a more accurate approach like the full causal theory of Israel-Stewart-Hiscock, we have also found cosmological solutions with big rip.

If $1 \leq \gamma \leq 2$ the parameters $\alpha$ and $\gamma$ satisfy the constraint given in equation (41). Due to the constraint stiff matter is not allowed. As we mentioned above this correspond to matter fluids that can lead to a phantom behavior. Quintessence region are not allowed. If $\gamma < 0$ the cosmological solutions can be computed directly from equations (45) and (46). The main conclusion, in the context of the full causal thermodynamics, is that in order to obtain physically reasonable big rip solutions, the dark component must be phantom energy.

Note added: While this manuscript was being written we noticed about the work of Brevik and Gorbunova [11]. The authors also consider the possibility of big rip in viscous fluids with $p = w\rho$, by a different formalism. They consider the case where the bulk viscosity is proportional to the scalar expansion. This is equivalent to the Barrow’s choice $\xi(t) \propto \sqrt{\rho}$, and then they also reobtained the Barrow’s solution, for $\gamma \neq 0$, considered here. In the framework of the standard Eckart theory citeEckart, the authors show that fluids which lie in the quintessence region ($w > -1$) can reduce its thermodynamical pressure and cross the barrier $w = -1$, and behave like a phantom fluid ($w < -1$) with the inclusion of a sufficiently large bulk viscosity. The case for $\gamma = 0$ was not considered by these authors.

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[1] R. R. Cadwell, M. Kamionkowski and N. N. Weinberg, Phys. Rev. Lett. 91, 071301 (2003).
[2] J. D Barrow, Class. Quantum Grav. 21, L79 (2004); J. D Barrow, Class. Quantum Grav. 21, 5619 (2004).
[3] S. Nojiri and S. Odintsov, Phys. Rev. 70, 103522 (2004).
[4] L. Chimento and R. Lazkoz, Phys. Rev. Lett. 91, 211301 (2003).
[5] M. P. Dabrowski, T. Stachowiak and M. Szydlowski, hep-th/0307128
[6] J. D Barrow, Phys. Lett. B 180, 335-339 (1987); J. D Barrow, Nucl. Phys. B 310, 743 (1988).
[7] W. Zimdahl and D. Pavón, Phys. Rev. 61, 108301 (2000).
[8] C. Eckart, Phys. Rev. 58, 919 (1940).
[9] R. Maartens, Class. Quantum Grav. 12, 1455 (1995).
[10] M.K. Mak and T. Harko, Gen. Rel. Grav. 30, 1171 (1998).
[11] I. Brevik and O. Gorbunova, gr-qc/0504001