Hodge theory for combinatorial geometries

June Huh

CMI, IAS, and Princeton University

with Karim Adiprasito and Eric Katz

SIAM AG15
Three fundamental ideas:

The idea of Bernd Sturmfels that a matroid can be viewed as a piecewise linear object, the tropical linear space.

The idea of Richard Stanley that the Hodge structure on the cohomology of projective toric varieties produces fundamental combinatorial inequalities.

The idea of Peter McMullen that the $g$-conjecture for polytopes can be proved using the ‘flip connectivity’ of simplicial polytopes of given dimension.
A remark concerning applications of algebraic geometry:

I will *not* apply any algebraic geometry.

I will present a purely combinatorial solution to a purely combinatorial problem.
A *graph* is a 1-dimensional space, with vertices and edges.

Graphs are the simplest combinatorial structures.
Hassler Whitney (1932): The *chromatic polynomial* of a graph $G$ is the function

$$\chi_G(q) = \text{(the number of proper colorings of } G \text{ with } q \text{ colors)}.$$ 

**Example**

![Graph](image)

$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q, \quad \chi_G(2) = 2, \ \chi_G(3) = 18, \ \ldots$$

**Read’s conjecture (1968)**

The coefficients of the chromatic polynomial $\chi_G(q)$ form a log-concave sequence for any graph $G$, that is,

$$a_i^2 \geq a_{i-1} a_{i+1} \text{ for all } i.$$
Example

How do we compute the chromatic polynomial? We write

\[
\begin{align*}
\begin{array}{c}
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
| & | & | \\
\bullet & \bullet & \bullet
\end{array}
\end{array}
\end{align*}
\]

\[= \begin{array}{ccc}
\bullet & \bullet & \bullet \\
| & | & \ \ \\
\bullet & \bullet & \bullet
\end{array} - \begin{array}{ccc}
\bullet & \bullet & \bullet \\
| & \ \ & | \\
\bullet & \bullet & \bullet
\end{array}
\]

and use

\[
\chi_{G \setminus e}(q) = q(q - 1)^3
\]

\[
\chi_{G / e}(q) = q(q - 1)(q - 2).
\]

Therefore

\[
\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G / e}(q) = \text{1} q^4 - 4 q^3 + 6 q^2 - 3 q.
\]

This algorithmic description of \( \chi_G(q) \) makes the prediction of the conjecture interesting.
For any finite set of vectors \( A \) in a vector space over a field, define

\[
f_i(A) = \text{(number of independent subsets of } A \text{ with size } i\).
\]

**Example**

If \( A \) is the set of all nonzero vectors in \( \mathbb{F}_2^3 \), then

\[
\begin{align*}
f_0 &= 1, \\
f_1 &= 7, \\
f_2 &= 21, \\
f_3 &= 28.
\end{align*}
\]

How do we compute \( f_i(A) \)? We use

\[
f_i(A) = f_i(A \setminus v) + f_{i-1}(A / v).
\]
Welsh’s conjecture (1969)

The sequence $f_i$ form a log-concave sequence for any finite set of vectors $A$ in any vector space over any field, that is,

$$f_i^2 \geq f_{i-1} f_{i+1} \text{ for all } i.$$
Hassler Whitney (1935).

A matroid $M$ on a finite set $E$ is a collection of subsets of $E$, called independent sets, which satisfy axioms modeled on the relation of linear independence of vectors:

1. Every subset of an independent set is an independent set.

2. If an independent set $A$ has more elements than independent set $B$, then there is an element in $A$ which, when added to $B$, gives a larger independent set.
We write $n + 1$ for the size of $M$, the cardinality of the ground set $E$.

We write $r + 1$ for the rank of $M$, the cardinality of any maximal independent set of $M$.

In all interesting cases, $r < n$. 
1. Let $G$ be a finite graph, and $E$ the set of edges.

Call a subset of $E$ independent if it does not contain a circuit.

This defines a \textit{graphic matroid} $M$.

2. Let $V$ be a vector space over a field $k$, and $A$ a finite set of vectors.

Call a subset of $A$ independent if it is linearly independent.

This defines a matroid $M$ \textit{realizable} over $k$. 
Fano matroid is realizable iff \( \text{char}(k) = 2 \).

Non-Fano matroid is realizable iff \( \text{char}(k) \neq 2 \).

Non-Pappus matroid is not realizable over any field.

How many matroids are realizable over a field?
0% of matroids are realizable.

In other words, almost all matroids are (conjecturally) not realizable over any field.

Testing the realizability of a matroid over a given field is not easy.

When $k = \mathbb{Q}$, this is equivalent to Hilbert’s tenth problem over $\mathbb{Q}$ (Sturmfels):

“Is there an algorithm to decide whether a given polynomial equation
with $\mathbb{Q}$ coefficients has a solution over $\mathbb{Q}$?”
One can define the characteristic polynomial of a matroid by the recursion

\[ \chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M/e}(q). \]

**Rota’s conjecture (1970)**

The coefficients of the characteristic polynomial \( \chi_M(q) \) form a log-concave sequence for any matroid \( M \), that is,

\[ \mu_i^2 \geq \mu_{i-1} \mu_{i+1} \quad \text{for all } i. \]

This implies the conjecture on \( G \) and the conjecture on \( A \) (Brylawski).
How to show that a given sequence is log-concave?
Any nonconstant homogeneous polynomial $h \in \mathbb{C}[z_0, \ldots, z_r]$ defines a sequence of 'Milnor numbers' $\mu^0(h), \ldots, \mu^r(h)$ with the following properties:

1. $\mu^i(h)$ is the number of $i$-dimensional cells in a CW-model of the complement
   $$D(h) := \{ x \in \mathbb{P}^r \mid h(x) \neq 0 \}.$$
2. $\mu^i(h)$ form a log-concave sequence, and
3. if $h$ is product of linear forms, then the attaching maps are homologically trivial:
   $$\mu^i(h) = b_i(D(h)).$$

When $h$ defines a hyperplane arrangement $\mathcal{A}$, this gives
   $$\mu^i(h) = \mu_i(\mathcal{A}) := (\text{the } i\text{-th coefficient of the characteristic polynomial of } \mathcal{A}),$$
justifying the log-concavity for matroids realizable over a field of characteristic zero.
Matroids on $[n] = \{0, 1, \ldots, n\}$ are related to the geometry of the $n$-dimensional permutohedron, the convex hull of an orbit of the symmetric group $S_{n+1}$.

The above is, in fact, the picture of flags in the Boolean lattice of $[n]$. 
The rays of the dual fan $\Delta_{A_n}$ correspond to nonempty proper subsets of $[n]$.

More generally, $k$-dimensional cones of $\Delta_{A_n}$ correspond to flags of nonempty proper subsets of $[n]$:

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$ 

The “extra symmetry” of the permutohedron maps a flag

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k.$$ 

to the flag of complements

$$[n] \setminus S_1 \supseteq [n] \setminus S_2 \supseteq \cdots \supseteq [n] \setminus S_k.$$
A matroid $M$ of rank $r + 1$ on $[n]$ can be viewed as an $r$-dimensional subfan

$$\Delta_M \subseteq \Delta_{A_n}$$

which consists of cones corresponding to flags of flats of $M$:

$$F_1 \subset F_2 \subset \cdots \subset F_r.$$

The fan $\Delta_M$ is the Bergman fan of $M$, or the tropical linear space associated to $M$. 
In a recent joint work with Karim Adiprasito and Eric Katz, we obtained inequalities that imply Rota’s log-concavity conjecture in its full generality.

What we show is that the tropical variety $\Delta_M$ has a “cohomology ring” which has the structure of the cohomology ring of a smooth projective variety.
Let $X$ be a smooth projective variety of dimension $r$, $k \leq r/2$, and let $C^k(X)$ be the image of cycle class map in $H^{2k}(X, \mathbb{Q}_l)$.

Grothendieck's standard conjectures say that we should have the following:

1. **Hard Lefschetz:** Any hyperplane class $\ell$ defines an isomorphism 
   \[ C^k(X) \longrightarrow C^{r-k}(X), \quad h \longmapsto \ell^{r-2k} \cdot h. \]

2. **Hodge-Riemann:** Any hyperplane class $\ell$ defines a definite form of sign $(-1)^k$ 
   \[ PC^k(X) \times PC^k(X) \longrightarrow C^r(X) \cong \mathbb{Q}, \quad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2, \]
   where $PC^k(X) \subseteq C^k(X)$ is the kernel of the multiplication by $\ell^{r-2k+1}$. 

June Huh
A motivating observation is that the toric variety of $\Delta_M$ is, in the realizable case, 'Chow equivalent' to a smooth projective variety (Feichtner-Yuzvinsky):

There is a map from a smooth projective variety

$$V \rightarrow X_{\Delta_M}$$

which induces an isomorphism between Chow cohomology rings

$$A^*(X_{\Delta_M}) \rightarrow A^*(V).$$

It is tempting to think this as a 'Chow homotopy'.

(When the base field is $\mathbb{C}$, it is important not to think this as the usual homotopy.)
In fact, the converse also holds.

**Theorem**

The toric variety $X_{\Delta M}$ is Chow equivalent to a smooth projective variety over $k$ if and only if $M$ is realizable over the field $k$.

We show that, even in the non-realizable case, $A^*(M) := A^*(X_{\Delta M})$ has the structure of the cohomology ring of a smooth projective variety (of dimension $r$).
Our argument is a good advertisement for tropical geometry to pure combinatorialists:

For any two matroids on \([n]\) with the same rank, there is a diagram

\[
\begin{align*}
\Delta_M &\xrightarrow{\text{"flip"}} \Delta_1 \xrightarrow{\text{"flip"}} \Delta_2 \xrightarrow{\text{"flip"}} \cdots \xrightarrow{\text{"flip"}} \Delta_{M'},
\end{align*}
\]

and each flip preserves the validity of the ‘Kähler package’ in their cohomology rings.

The intermediate objects are tropical varieties with good cohomology rings, but not in general associated to a matroid (unlike in McMullen’s case of polytopes).
The cohomology ring $A^*(M)$ can be described explicitly by generators and relations, which can be taken as a definition.

**Definition**

The cohomology ring of $M$ is the quotient of the polynomial ring

$$A^*(M) := \mathbb{Z}[x_F]/(I_1 + I_2),$$

where the variables are indexed by nonempty proper flats of $M$, and

$$I_1 := \text{ideal} \left( \sum_{i_1 \in F} x_{F_{i_1}} - \sum_{i_2 \in F} x_{F_{i_2}} \mid i_1 \text{ and } i_2 \text{ are distinct elements of } [n] \right),$$

$$I_2 := \text{ideal} \left( x_{F_1} x_{F_2} \mid F_1 \text{ and } F_2 \text{ are incomparable flats of } M \right).$$
Theorem

The Chow ring $A^*(M)$ is a Poincaré duality algebra of dimension $r$:

(1) **Degree map:** There is an isomorphism

$$\deg_M : A^r(M) \longrightarrow \mathbb{Z}, \quad \prod_{i=1}^{r} x_{F_i} \mapsto 1,$$

for any complete flag of nonempty proper flats $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r$ of $M$.

(2) **Poincaré duality:** For any nonnegative integer $k \leq r$, the multiplication defines the perfect pairing

$$A^k(M) \times A^{r-k}(M) \longrightarrow A^r(M) \simeq \mathbb{Z},$$

Note that the underlying simplicial complex of $\Delta_M$, the *order complex* of $M$, is not Gorenstein in general.
Let $\mathcal{K}_n$ be the convex cone of linear forms with real coefficients

$$\sum_S c_S x_S$$

that satisfy, for any two incomparable nonempty proper subsets $S_1, S_2$ of $[n]$,

$$c_{S_1} + c_{S_2} > c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \quad (c_\emptyset = c_n = 0).$$

**Definition**

The *ample cone* of $M$, denoted $\mathcal{K}_M$, is defined to be the image

$$\mathcal{K}_n \rightarrow \mathcal{K}_M \subseteq A^1(M)_{\mathbb{R}},$$

where all the non-flats of $M$ are mapped to zero.
Main Theorem

Let $\ell$ be an element of $\mathcal{K}_M$ and let $k$ be a nonnegative integer $\leq r/2$.

(1) **Hard Lefschetz:** The multiplication by $\ell$ defines an isomorphism

$$A^k(M)_{\mathbb{R}} \longrightarrow A^{r-k}(M)_{\mathbb{R}}, \quad h \longmapsto \ell^{r-2k} \cdot h.$$ 

(2) **Hodge-Riemann:** The multiplication by $\ell$ defines a definite form of sign $(-1)^k$

$$PA^k(M)_{\mathbb{R}} \times PA^k(M)_{\mathbb{R}} \longrightarrow A^r(M)_{\mathbb{R}} \cong \mathbb{R}, \quad (h_1, h_2) \longmapsto \ell^{r-2k} \cdot h_1 \cdot h_2,$$

where $PA^k(M)_{\mathbb{R}} \subseteq A^k(M)_{\mathbb{R}}$ is the kernel of the multiplication by $\ell^{r-2k+1}$. 
Why does this imply the log-concavity conjecture?

Let $i$ be an element of $[n]$, and consider the linear forms

\[ \alpha(i) := \sum_{i \in S} x_S, \]

\[ \beta(i) := \sum_{i \notin S} x_S. \]

Note that these linear forms are ‘almost’ ample:

\[ c_{S_1} + c_{S_2} \geq c_{S_1 \cap S_2} + c_{S_1 \cup S_2} \quad (c_{\emptyset} = c_{[n]} = 0). \]

Their images in the cohomology ring $A^*(M)$ does not depend on $i$; they will be denoted by $\alpha$ and $\beta$ respectively.
Proposition

Under the isomorphism \( \text{deg} : A^r(M) \to \mathbb{Z} \), we have

\[
\alpha^r \beta^k \longmapsto (k\text{-th coefficient of the reduced characteristic polynomial of } M).
\]

While neither \( \alpha \) nor \( \beta \) are in the ample cone \( \mathcal{K}_M \), we may take the limit

\[
\ell_1 \to \alpha, \quad \ell_2 \to \beta, \quad \ell_1, \ell_2 \in \mathcal{K}_M.
\]

This may be one reason why direct combinatorial reasoning for log-concavity
was not easy.