THE KODAIRA DIMENSION OF LEFSCHETZ FIBRATIONS

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Abstract. In this note, we verify that the complex Kodaira dimension $\kappa^h$ equals the symplectic Kodaira dimension $\kappa^s$ for smooth 4–manifolds with complex and symplectic structures. We also calculate the Kodaira dimension for many Lefschetz fibrations.

Key words. Kodaira Dimension, symplectic topology, Lefschetz fibration, Lefschetz pencil.

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1. Introduction. Let $M$ be a smooth 4-manifold. The manifold $M$ can be endowed with different structures: complex, symplectic, Lefschetz fibration, etc. If given a complex structure, the (complex) Kodaira dimension $\kappa^h(M, J)$ is defined within the framework of birational classification of complex manifolds. This notion has proven to be very useful, at least in complex dimension 2, in which we have a detailed birational classification. Similarly, given a symplectic structure, there is the concept of a (symplectic) Kodaira dimension $\kappa^s(M, \omega)$ (see [28], [17] and [21]). One would like to achieve a similar classification as in the complex case for symplectic manifolds. Such a classification is very clear when the symplectic Kodaira dimension is $-\infty$. Moreover, in symplectic Kodaira dimension 0, T.-J. Li ([21] and [22]) gives a classification of symplectic 4 manifolds up to their rational homology groups (see also [31] for a similar result). In Section 5 (Prop. 5.5), we will give some evidence to support his conjecture, that the classification in [22] is complete.

The main goal of this paper is to show the relationship between $\kappa^h(M, J)$ and $\kappa^s(M, \omega)$ on manifolds admitting complex and symplectic structures and the role they play in the classification of Lefschetz fibrations (and pencils).

In Section 2, we state the definitions of the Kodaira dimension in the complex and symplectic cases including some useful background information. We then define the Kodaira dimension $\kappa^l(g, h, n)$ for Lefschetz fibrations with base genus $h \geq 1$, fiber genus $g$ and $n$ singular points (Def 2.7).

In Section 3, we show that the symplectic Kodaira dimension $\kappa^s(M, \omega)$ as defined in [21] is the same as the complex Kodaira dimension for all smooth 4-manifolds admitting complex and symplectic structures. That is, we have the following theorem:

**Theorem 1.1.** Let $M$ be a smooth 4-manifold which admits a symplectic structure $\omega$ as well as a complex structure $J$. Then $\kappa^s(M, \omega) = \kappa^h(M, J)$.

The symplectic structure $\omega$ is not necessarily compatible with or even tamed by the complex structure $J$. In the Kähler case, this result is known to T.-J. Li ([21]). This allows us to give the Kodaira dimension of a complex or symplectic 4-manifold without ambiguity, we denote it simply as $\kappa(M)$.

In the following, we address the equivalence of $\kappa(M)$ and the Lefschetz Kodaira dimension $\kappa^l(g, h, n)$. This is broken into three cases: Excluding the exceptional case

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