Plateau Instability of Liquid Crystalline Cylinder
in Magnetic Field

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Abstract

The capillary instability of a LC cylinder in magnetic field is considered using an energy approach. The boundary problem is solved in the linear approximation of the anisotropy $\chi_a$ of the magnetic susceptibility $\chi$. The effect of anisotropy, in the region $1 \gg |\chi| > |\chi_a| \gg \chi^2$, can be strong enough to counteract and even reverse the tendency of the field to enhance stabilization by enlarging the cut–off wave number $k_s$ beyond the conventional one set by Rayleigh.

Key words: Plateau Instability, Nematic Liquid Crystal, Magnetic Field, Anisotropy of Susceptibility.

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1 Introduction

Theoretical predictions of the continuum theory of the nematic liquid crystals (LC) were successfully confirmed in many experimental observations [1]. One of the most studied effect is the influence of an external field on the orientational distribution of the LC director field $\mathbf{n}(\mathbf{r})$. The physics involved in a competition between the mechanical and field forces that can be developed in LC media, calls for careful analysis. Well known example is the Freedericksz effect [1], observed when nematic LC cell with initial uniform distribution of $\mathbf{n}(\mathbf{r})$ is subjected to an external magnetic field. In many cases, the stabilizing elastic forces compete with the destabilizing magnetic field giving rise to a critical phenomenon. However, the critical phenomena in LC can be sustained even if both the elastic and magnetic fields are defined as stabilizing. Such critical phenomena exist due to the effect of surface tension which tends to minimize the surface area by distorting the initial shape of the system.

The nematicity of LC’s, being a source for elastic properties, results in enhancement of stability of LC jets [2], as compared to ordinary liquids. A similar stability enhancement appears in ordinary liquid jets with isotropic magnetic permeability (see [3] and [4]) when they are subjected to an external magnetic field. Unlike the elasticity, the external field has a critical value beyond which instability of the jet is completely suppressed for all disturbance wavelengths [4]. Nematic LC’s are usually anisotropic diamagnetics with positive anisotropy $\chi_a$ of the magnetic susceptibility [1]. This poses an additional challenge with respect to the above mentioned phenomena. It is reflected by the extra terms in the LC hydrodynamics due to orientational interaction between the magnetic field $\mathbf{H}$ and LC director $\mathbf{n}$.

The static version of capillary instability in liquid jets is known as the Plateau instability in the liquid cylinders [2], [4]. It dates back to the classical works of J. Plateau [5] who defined the problem of finding a surface of liquid with a minimal area $S$ given its boundary $\partial \Omega$ at fixed volume. The problem relates to the principle of minimum free energy at equilibrium. Further generalization is called for if the excess free energy $W$, of the cylinder, comprises different types of energy that reflect a more complex structure of the liquid (e.g. elasticity [2] etc), as well as its capacity to interact with external fields.

The purpose of this work is to extend the theory of Plateau instability in LC cylinders [2] so as to include the effect of static magnetic fields. Here the motivation is both experimental and theoretical. Experimentally, the question is how to set the initial orientation of director $\mathbf{n}$ collinear with the LC cylinder axis. A weak magnetic field can serve to this end. Theoretically, the framework outlined in [2] can be extended so as to incorporate the influence of external fields on the evolution and stability of the LC cylinder. In particular, the Plateau instability is studied with respect to the effect of the magnetic
anisotropy of the LC cylinder.

2 Free energy of LC cylinder in the presence of magnetic field

Consider an isothermal incompressible LC cylinder in a uniform magnetic field $H_0$ that is applied in free space along the cylinder axis. We assume a rigid boundary condition (BC) where the director is tangentially anchored at the free surface of the LC cylinder. The magnetic susceptibility tensor $\hat{\chi}$ of the LC is assumed anisotropic, symmetric, and independent of the magnetic field. In the reference frame related to the cylinder axis its diagonal terms are $\chi_\parallel, \chi_\perp$ while its off–diagonal term is $\chi_an_zn_r$, where $n_z, n_r$ are the axial and radial components of director $n$, respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{undisturbed_disturbed.png}
\caption{Undisturbed (left) and disturbed (right) homotropic LC cylinder subjected to an axial and uniform external magnetic field.}
\end{figure}

When the LC cylinder (assumed long compared to its diameter) is undisturbed the total free energy $F^0$ of the system is defined by

$$F^0 = \mathcal{E}_s^0 - \chi_\parallel \frac{\mu_0 H_0^2}{2} \cdot \int_{\Omega^0_{cyl}} dv .$$

(1)

where the integral represents the volume $\pi R^2L$ of the undisturbed cylinder which is enclosed by the surface $\partial \Omega^0_{cyl}$. The term $\mathcal{E}_s^0 = \sigma \int_{\partial \Omega^0_{cyl}} ds = 2\pi \sigma RL$ stands for the surface free energy of the undisturbed cylinder, where $R, L$ and $\sigma$ denote its radius, length and surface tension respectively, and $\mu_0$ is permeability of a free space. We specify the commonly used harmonic deformation of the cylinder as $r(z) = R + \zeta_0 \cos kz$, where $k = 2\pi/\Lambda$, $\Lambda$ being the disturbance wavelength. Let the extent of deformation be characterized by a length $\zeta_0$, such that $\zeta_0/R = \epsilon \ll 1$.

Deformation of the cylinder shape changes the magnetic field $H(r)$ over all space $\mathbb{R}^3$, while the director field $n(r)$ is changed only within the internal domain $\Omega_{cyl}$. Following Plateau, we assume conservation of the cylinder volume

$$\int_{\Omega^0_{cyl}} dv = \int_{\Omega_{cyl}} dv .$$

(2)
The total free energy $F$ of the disturbed cylinder takes the following form,

$$ F = \mathcal{E}_s + \mathcal{E}_n + \mathcal{E}_H^\in + \mathcal{E}_H^\ex. $$

(3)

The first term in (3), which stands for the interfacial energy of the disturbed cylinder, is classically known due to Plateau [5]

$$ \mathcal{E}_s = \sigma_0 \int_{\partial \Omega_{cyl}} ds = 2\pi \sigma RL + \sigma \frac{\pi c^2_0}{2R} (k^2 R^2 - 1). $$

(4)

The second term in (3) is due to the elastic deformation of the director field $n(r)$, and in the single elastic approximation is given by

$$ \mathcal{E}_n = \frac{K}{2} \int_{\Omega_{cyl}} (\text{div}^2 n + \text{rot}^2 n) \, dv, $$

where $K$ is the elastic modulus. The last two terms in (3) correspond to the effect of the magnetic fields in the internal $\Omega_{cyl}$ and external $\mathbb{R}^3 \setminus \Omega_{cyl}$ domains

$$ \mathcal{E}_H^\in = \frac{\mu_0}{2} \int_{\Omega_{cyl}} H_0^2 \, dv - \frac{\mu_0}{2} \int_{\Omega_{cyl}} \mu_{jk} \, \text{in} H_j \, \text{in} H_k \, dv, \quad \mathcal{E}_H^\ex = \frac{\mu_0}{2} \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} H_0^2 \, dv - \frac{\mu_0}{2} \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \text{ex} H^2(r) \, dv, $$

(6)

where $\text{in} H(r)$ and $\text{ex} H(r)$ are the internal and external magnetic fields, respectively. If the deviations of the director $n = n^0 + n^1$ from its initial orientation $n^0$ along the $z$ direction are small, then

$$ n^0_r = 0, \quad n^0_z = 1, \quad 1 \gg n^1_r, \quad |n^1_z| \sim (n^1_z)^2. $$

(7)

The magnetic energy density in the second term of $\mathcal{E}_H^\in$ (scaled by $\mu_0/2$) reads

$$ \mu_{jk} \, \text{in} H_j \, \text{in} H_k = (1 + \chi_\parallel) \, (\text{in} H_z)^2 + (1 + \chi_\perp) \, (\text{in} H_r)^2 + 2\chi_a n^1_r n^1_z \, \text{in} H_r \, \text{in} H_z, $$

(8)

where $\mu_{jk}$ is the LC relative permeability tensor: $\mu_{zz} = 1 + \chi_\parallel$, $\mu_{rr} = 1 + \chi_\perp$, $\mu_{zr} = \chi_a n^1_r n^1_z$. The excess free energy $W$ of the system is defined as,

$$ W = F - F^0. $$

(9)

From the mathematical standpoint, the variational problem for minimization of $W$, supplemented with constraint (2) for all smooth surfaces $\partial \Omega_{cyl}$, is known as the isoperimetric problem. The cylinder instability can be studied assuming small perturbation in its shape. In this case the Plateau problem becomes solvable in closed form. The fields $\text{in} H(r)$ and $\text{ex} H(r)$, which must satisfy Maxwell equations, can be presented as small perturbations of $H_0$,

$$ \text{in} H(r) = H_0 + \text{in} H^1(r) = (H_0 + \text{in} H^1_z, \text{in} H^1_r), \quad \text{ex} H(r) = H_0 + \text{ex} H^1(r) = (H_0 + \text{ex} H^1_z, \text{ex} H^1_r), $$

(10)
where according to the assumption $\epsilon \ll 1$ the following approximations apply

$$\{ {\text{in}} H^1_r, {\text{ex}} H^1_r, {\text{in}} H^1_z, {\text{ex}} H^1_z \} = \{ {\text{in}} h^1_r, {\text{ex}} h^1_r, {\text{in}} h^1_z, {\text{ex}} h^1_z \} \times \epsilon H_0 .$$

The dimensionless fields $\text{in,ex} h^1_{r,z}(r, z)$ are dependent on the coordinates as indicated By virtue of translational invariance of the problem

$${\text{in,ex}} h^1_{r,z}(r, z + \Lambda) = {\text{in,ex}} h^1_{r,z}(r, z)$$

we set $L = \Lambda$ and evaluate the free energy per unit wave length. Substituting (1), (4)–(8) and (10) into (9) we obtain in the $\epsilon^2$–approximation

$$\frac{1}{L} W = \frac{\epsilon^2 \pi}{2} \sigma R (k^2 R^2 - 1) + \frac{K}{2L} \int_{\Omega_{cyl}} \left( \text{div}^2 \mathbf{n} + \text{rot}^2 \mathbf{n} \right) d\mathbf{v} - \frac{\mu_0}{2L} U ,$$

where the magnetic part was calculated in Appendix A

$$U = \int_{\Omega_{cyl}} \left\{ (1 + \chi_\parallel) \left( {\text{in}} H^1_z \right)^2 + (1 + \chi_\perp) \left( {\text{in}} H^1_r \right)^2 \right\} d\mathbf{v} + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left\{ \left( {\text{ex}} H^1_z \right)^2 + \left( {\text{ex}} H^1_r \right)^2 \right\} d\mathbf{v} + 2H_0 \left( \chi_a \int_{\Omega_{cyl}} n_r n_z {\text{in}} H^1_z d\mathbf{v} + (1 + \chi_\parallel) \int_{\Omega_{cyl}} {\text{in}} H^1_z d\mathbf{v} + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} {\text{ex}} H^1_z d\mathbf{v} \right) .$$

### 3 Boundary problem and its solution

The magnetostatics of the disturbed LC cylinder is governed by Maxwell equations for the internal $\text{in} H(r)$ and external $\text{ex} H(r)$ magnetic fields and the Euler–Lagrange equation apply for the director field $\mathbf{n}(r)$,

$$\text{rot} \, \text{in} H = \text{rot} \, \text{ex} H = 0 , \quad \text{div} \, \text{in} B = \text{div} \, \text{ex} B = 0 , \quad \left\{ \frac{\partial}{\partial r} \frac{\partial}{\partial (\partial_r n_r)} + \frac{\partial}{\partial z} \frac{\partial}{\partial (\partial_z n_r)} - \frac{\partial}{\partial n_r} \right\} (\mathcal{E}_n + \mathcal{E}_H^{\text{in}}) = 0 , \quad \partial_z = \partial_x .$$

where $\text{in} B$ and $\text{ex} B$ denote internal and external magnetic inductions, respectively

$$\text{in} B_j = \mu_0 \mu_{jk} \text{in} H_k , \quad \mu_{jk} = (1 + \chi_\perp) \delta_{jk} + \chi_a n_j n_k , \quad \text{ex} B_j = \mu_0 \text{ex} H_j , \quad \chi_a = \chi_\parallel - \chi_\perp .$$

Equations (15), (16) must be supplemented with boundary conditions (BC) at the interface $r = R$,

$$\langle \text{in} H, \mathbf{t} \rangle = \langle \text{ex} H, \mathbf{t} \rangle , \quad \langle \text{in} B, \mathbf{e} \rangle = \langle \text{ex} B, \mathbf{e} \rangle , \quad \langle \mathbf{e}, \mathbf{n} \rangle = 0 ,$$

where $\mathbf{t}$ and $\mathbf{e}$ stand for tangential and normal unit vectors to the surface, respectively. Since the surface deformation is small, linearization can be applied,

$$t_r = -e_z = \partial \zeta / \partial z , \quad t_z = e_r = \sqrt{1 - (\partial \zeta / \partial z)^2} \approx 1 .$$
A standard way to solve the problem is to introduce a director potential \( \Theta(r) \) and two magnetic potentials \( \Phi_{\text{in}}(r) \) and \( \Phi_{\text{ex}}(r) \) as follows

\[
n_r = \frac{\partial \Theta}{\partial r}, \quad \text{in} \, H^1(r) = -\nabla \Phi_{\text{in}}, \quad \text{ex} \, H^1(r) = -\nabla \Phi_{\text{ex}}, \quad |\nabla \Phi_{\text{in}}|, |\nabla \Phi_{\text{ex}}| \ll H_0.
\]  

(20)

\( \Phi_{\text{in}}(r) \) and \( \Phi_{\text{ex}}(r) \) satisfy the first two equations in (15). The last two equations in (15) yield,

\[
(1 + \chi_\parallel) \frac{\partial^2 \Phi_{\text{in}}}{\partial z^2} + (1 + \chi_\perp) \Delta_2 \Phi_{\text{in}} = \chi_a H_0 \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Theta}{\partial r} \right), \quad \frac{\partial^2 \Phi_{\text{ex}}}{\partial z^2} + \Delta_2 \Phi_{\text{ex}} = 0,
\]

where \( \Delta_2 = \partial^2/\partial r^2 + 1/r \partial/\partial r \) is the two–dimensional Laplacian. The variational equation (16) gives,

\[
K \left( \Delta_2 - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial \Theta}{\partial r} - 2 \mu_0 \chi_a H_0 \frac{\partial \Phi_{\text{in}}}{\partial r} = 0.
\]

(22)

Making use of the commutation rules

\[
\left( \Delta_2 - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial \Theta}{\partial r} = \frac{\partial}{\partial r} \left( \Delta_2 + \frac{\partial^2}{\partial z^2} \right) \Theta,
\]

we finally arrive at

\[
\left( \Delta_2 + \frac{\partial^2}{\partial z^2} \right) \Phi_{\text{ex}} = 0, \quad \left( \Delta_2 + \alpha^2 \frac{\partial^2}{\partial z^2} \right) \Phi_{\text{in}} = (\alpha^2 - 1) H_0 \Delta_2 \Theta,
\]

(23)

\[
\left( \Delta_2 + \frac{\partial^2}{\partial z^2} \right) \Theta = 2 \frac{\mu_0 \chi_a H_0}{K} \Phi_{\text{in}}, \quad 0 \leq \alpha^2 - 1 = \frac{\chi_a}{1 + \chi_\perp} \approx \chi_a \ll 1.
\]

BC (18) can be reformulated as follows

\[
\frac{\partial \Phi_{\text{ex}}}{\partial z} = \frac{\partial \Phi_{\text{in}}}{\partial z}, \quad \frac{\partial \Phi_{\text{ex}}}{\partial r} - (1 + \chi_\perp) \frac{\partial \Phi_{\text{in}}}{\partial r} = \chi_\perp H_0 \frac{\partial \zeta}{\partial z}, \quad \frac{\partial \Theta}{\partial r} = \frac{\partial \zeta}{\partial z}.
\]

(24)

A weak decoupling of the equations (23) makes it possible to solve the boundary problem in closed form. Assuming

\[
\{ \Phi_{\text{in}}(r, z), \Phi_{\text{ex}}(r, z), \Theta(r, z) \} = \{ \phi_{\text{in}}(r), \phi_{\text{ex}}(r), \theta(r) \} \times \sin k z
\]

(25)

we find

\[
(\Delta_2 - k^2) \phi_{\text{ex}} = 0, \quad (\Delta_2 - \alpha^2 k^2) \phi_{\text{in}} = \chi_a H_0 \Delta_2 \theta, \quad (\Delta_2 - k^2) \theta = 2 \frac{\mu_0 \chi_a H_0}{K} \phi_{\text{in}},
\]

(26)

The following BC exist at \( r = R \)

\[
\phi_{\text{ex}} = \phi_{\text{in}}, \quad (1 + \chi_\perp) \frac{\partial \phi_{\text{in}}}{\partial r} - \frac{\partial \phi_{\text{ex}}}{\partial r} = \chi_\perp H_0 \kappa_0, \quad \frac{\partial \theta}{\partial r} = -k \zeta_0.
\]

(27)

The two last equations in (26) can be represented through the determinant equation

\[
[\Delta_2^2 - D_1 \Delta_2 + D_0] \begin{pmatrix} \phi_{\text{in}} \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{where}
\]

\[
D_1 = (2 + \chi_a) k^2 + \chi_a g^2, \quad D_0 = (1 + \chi_a) k^4, \quad g^2 = 2 \frac{\mu_0 \chi_a H_0^2}{K}.
\]

(28, 29)
Factorization of the differential operator in (28) gives
\[
\Delta^2_2 - D_1 \Delta_2 + D_0 = (\Delta_2 - l_1^2) (\Delta_2 - l_2^2), \quad \text{where} \quad l_{1,2}^2 = \frac{1}{2} \left( D_1 \pm \sqrt{D} \right), \quad (30)
\]
\[
\mathcal{D} = D_1^2 - 4D_0 = \chi_a \left[ \chi_a (k^4 + g^4) + 2(2 + \chi_a)k^2g^2 \right]. \quad (31)
\]
The fundamental solutions of (28) which are finite at \( r = 0 \) are the following
\[
\phi_m(r) = c_1 I_0(l_1r) + c_2 I_0(l_2r), \quad \theta(r) = b_1 I_0(l_1r) + b_2 I_0(l_2r), \quad \frac{\chi_a H_0}{1 + \chi} \delta = \frac{l_2^2 - \alpha^2 k^2}{l_j^2} \delta, \quad j = 1, 2, \quad (32)
\]
where \( I_m(x) \) is a modified Bessel function of the 1st kind and order \( m \), and \( c_j, b_j \) are indeterminates.
The first equation in (26) produces a solution in the exterior domain which is finite at \( r = \infty \)
\[
\phi_{ex}(r) = c_3 K_0(kr), \quad (33)
\]
where \( K_m(x) \) is a modified Bessel function of the 2nd kind and order \( m \). All indeterminates \( c_1, c_2, c_3 \) can be found from BC (27) by substitution therein the expressions (32), (33). Further simplification comes after substitution of \( c_3 = c_1 I_0(l_1R)/K_0(kR) + c_2 I_0(l_2R)/K_0(kR) \) and using \( 1 + \chi \approx 1 \)
\[
c_1 \left[ l_1 I_1(l_1R) + \frac{k K_1(kR)}{K_0(kR)} I_0(l_1R) \right] + c_2 \left[ l_2 I_1(l_2R) + \frac{k K_1(kR)}{K_0(kR)} I_0(l_2R) \right] = \chi \frac{\chi_0 H_0}{\chi} \delta, \quad (34)
\]
where the identities \( I'_0(x) = I_1(x) \) and \( K'_0(x) = -K_1(x) \) for the derivatives were used. Straightforward calculations give
\[
c_j = k \chi_0 H_0 \frac{\Gamma_j}{\Gamma_0}, \quad b_j = k \chi_0 \frac{\Gamma_j}{\chi_0 \Gamma_0} \frac{l_j^2 - \alpha^2 k^2}{l_j^2}, \quad j = 1, 2, \quad (35)
\]
where \( \Gamma_j, j = 0, 1, 2 \) are determinants of \( (2 \times 2) \) matrices
\[
\begin{align*}
\Gamma_0 &= \alpha \left( l_1^2 - l_2^2 \right) I_1(l_1R)I_1(l_2R) + \frac{k K_1(kR)}{K_0(kR)} \left[ (\alpha l_1 - l_2) I_0(l_1R)I_1(l_2R) - (\alpha l_2 - l_1) I_0(l_2R)I_1(l_1R) \right] \\
\Gamma_1 &= \left( \chi \left( l_2^2 - \chi \alpha^2 k^2 \right) \right) \frac{l_2}{l_2} + \chi \frac{K_1(kR)}{K_0(kR)} I_0(l_2R), \\
\Gamma_2 &= \left( \chi \left( l_1^2 - \chi \alpha^2 k^2 \right) \right) \frac{l_1}{l_1} + \chi \frac{K_1(kR)}{K_0(kR)} I_0(l_1R) \quad (36)
\end{align*}
\]

### 3.1 \( \chi_a \)-expansion of the solutions

The complexity of expressions (32) in conjunction with (35) and (36) makes further evaluation of the problem excessively difficult. Therefore, we develop in this Section another approach for solution of the amplitude equations (26) endowed with BC (27). Bearing in mind that for most nematic LCs the
anisotropy is small $|\chi_a| < |\chi_\perp|, |\chi_\parallel|$ we seek the linear in $\chi_a$ representation of the functions $\phi_n(r)$ and $\theta(r)$

$$\phi_n(r) = \bar{\phi}_n(r) + \chi_a \phi_m(r), \quad \theta(r) = \bar{\theta}(r) + \chi_a \bar{\theta}(r).$$  \hspace{1cm} (37)

The isotropic parts $\bar{\phi}_n(r), \bar{\theta}(r)$ together with the external potential $\phi_{ex}$, satisfy the following equations

$$(\Delta_2 - k^2) \phi_{ex} = (\Delta_2 - k^2) \bar{\phi}_n = (\Delta_2 - k^2) \bar{\theta} = 0,$$  \hspace{1cm} (38)

supplemented with the BC at $r = R$

$\phi_{ex} = \bar{\phi}_n, \quad (1 + \chi_\perp) \frac{\partial \bar{\phi}_n}{\partial r} - \frac{\partial \phi_{ex}}{\partial r} = \chi_\perp H_0 K_0, \quad \frac{\partial \bar{\theta}}{\partial r} = -k \zeta_0.$  \hspace{1cm} (39)

The solutions $\bar{\phi}_n(r), \bar{\theta}(r), \phi_{ex}(r)$ of (38) were found in [2] and [4]

$$\bar{\phi}_n(r) = A_1 I_0(kr), \quad \bar{\theta}(r) = A_2 I_0(kr), \quad \phi_{ex}(r) = A_3 K_0(kr),$$  \hspace{1cm} (40)

where the coefficients $C_i$ are given by

$$A_1 = \zeta_0 \chi_\perp H_0 k R \frac{K_0(k R)}{T(k R, \chi_\perp)}, \quad A_2 = -\frac{\zeta_0}{I_1(k R)}, \quad A_3 = \zeta_0 \chi_\perp H_0 k R \frac{I_0(k R)}{T(k R, \chi_\perp)},$$  \hspace{1cm} (41)

and $T(x, a) = 1 + ax I_1(x) K_0(x)$. Henceforth, $T(k R, \chi_\perp) \approx 1$ in accordance with $\chi_\perp \ll 1$. The amplitude equations for the remaining functions $\bar{\phi}_n(r)$ and $\bar{\theta}(r)$ can be found by inserting (37) into (26) and making use of (38)

$$(\Delta_2 - k^2) \bar{\phi}_n = A_\phi I_0(kr), \quad A_\phi = (A_1 + A_2 H_0) k^2 = -\frac{\zeta_0 k^2 H_0}{I_1(k R)};$$  \hspace{1cm} (42)

$$(\Delta_2 - k^2) \bar{\theta} = A_\theta I_0(kr), \quad A_\theta = 2 A_1 \frac{\mu H_0}{K} = 2 \chi_\perp \zeta_0 k R K_0(k R) \frac{\mu H_0^2}{K}.$$

The BC for $\bar{\phi}_n(r)$ and $\bar{\theta}(r)$ is,

$$\bar{\phi}_n(R) = \bar{\theta}(R) = 0.$$  \hspace{1cm} (43)

After simple calculations (see Appendix B) we obtain

$$\bar{\phi}_n(kr) = \frac{A_\phi}{k^2} G(kr) I_0(kr), \quad \bar{\theta}(kr) = \frac{A_\theta}{k^2} G(kr) I_0(kr),$$  \hspace{1cm} (44)

where

$$G(kr) = \frac{k^2}{4} (r^2 - R^2) + \frac{1}{2} \int_R^{kr} \frac{I^2(y)}{I^2_0(y)} y dy, \quad \frac{G_1'(kr)}{k} = \frac{kr}{2} \left(1 - \frac{I^2_1(kr)}{I^2_0(kr)} \right).$$  \hspace{1cm} (45)

1The characteristic magnitudes of the magnetic susceptibility $\chi$ and its anisotropy $\chi_a$ for the classical nematic LC's 4-metorybenziliden-4-butilalin (MBBA) and para-azoxyanisole (PAA) can be found in [6] : $\chi_\perp \approx \chi_\parallel \approx 10^{-5}$ and $\chi_a \approx 10^{-6}$. 

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Recalling the definition (20) of potentials $\Phi_n(r)$, $\Phi_{ex}(r)$ and $\Theta(r)$ we get

\[ n^1_r(r, z) = k A_2 I_1(kr) \left\{ 1 + \chi_a \frac{A_{\theta}}{A_2 k^2} \left[ G(kr) + \frac{G'(kr)}{k} \frac{I_0(kr)}{I_1(kr)} \right] \right\} \sin k z , \quad (46) \]

\[ -i m H^1_r(r, z) = k A_1 I_1(kr) \left\{ 1 + \chi_a \frac{A_{\phi}}{A_1 k^2} \left[ G(kr) + \frac{G'(kr)}{k} \frac{I_0(kr)}{I_1(kr)} \right] \right\} \sin k z , \quad (47) \]

\[ -i m H^1_z(r, z) = k A_1 I_0(kr) \left\{ 1 + \chi_a \frac{A_{\phi}}{A_1 k^2} G(kr) \right\} \cos k z , \quad (48) \]

\[ e^{\pm} n^1_m(r, z) = k A_3 K_1(kr) \sin k z , \quad e^{\pm} H^1_m(r, z) = -k A_3 K_0(kr) \cos k z . \quad (49) \]

where

\[ \chi_a \frac{A_{\theta}}{A_2 k^2} \simeq -2 \chi_a \chi_{\perp} \frac{\mu_0 H_0^2 R^2 I_1(kR) K_0(kR)}{K} , \quad \chi_a \frac{A_{\phi}}{A_1 k^2} \simeq -\frac{\chi_a}{\chi_{\perp}} \frac{1}{kRI_1(kR)K_0(kR)} . \quad (50) \]

Before going to further calculation of the excess free energy $W$ by (13), let us estimate the main contributions of the anisotropy $\chi_a$ to the distribution of the fields $n(r, z)$ and $i m H^1(r, z)$ in accordance with (49) and (50).

First, as follows from Figure 2, $G(kr)$ and $G(kr) + \frac{G'(kr)}{k} \frac{I_0(kr)}{I_1(kr)}$ which are both continuous monotone growing functions, are bounded as follows

\[-0.21 < G(kr) < 0 , \quad \text{and} \quad 0.75 < G(kr) + \frac{G'(kr)}{k} \frac{I_0(kr)}{I_1(kr)} < 0.9 . \quad (51)\]

In order to simplify further calculation we consider, henceforth, these functions as constant $N_1$ and $N_2$, respectively

\[ G(kr) = N_1 , \quad G(kr) + \frac{G'(kr)}{k} \frac{I_0(kr)}{I_1(kr)} = N_2 , \quad (52) \]

The next simplification comes for $n^1_i(r, z)$. Indeed, bearing in mind (50) we conclude that for the influence of $\chi_a$ on $n^1_i(r, z)$ to be significant a huge magnetic field $H_0$ is required

\[ H_0 > H_\bullet = \frac{1}{\sqrt{|\chi_a \chi_{\perp}|}} \times \frac{1}{R} \sqrt{\frac{K}{\mu_0}} , \quad (53) \]
The magnitude of this field can be as high as \(10^8 \text{A/m}\) for classical LC's with radius \(R \approx 10 \mu\text{m}\). In fields which are significantly lower than \(H_\bullet\), the behaviour of the director \(\mathbf{n}(\mathbf{r})\) is dictated primarily by competition between bulk elasticity and surface tension of the LC's, and governed by dimensionless parameter \(\varkappa = K/\sigma R\) [2]. Recasting (46), (47) and (48) gives

\[
 n_1^r(r, z) = kA_2 I_1(kr) \sin kz, \quad \epsilon^x H_1^r(r, z) = kA_3 K_1(kr) \sin kz, \quad \epsilon^x H_1^z(r, z) = -kA_3 K_0(kr) \cos kz,
\]

\[
 -i^n H_1^r(r, z) = kA_1 I_1(kr) \left(1 + \chi a \frac{A_\delta}{A_1 k^2} N_2\right) \sin kz, \quad -i^n H_1^z(r, z) = kA_1 I_0(kr) \left(1 + \chi a \frac{A_\delta}{A_1 k^2} N_1\right) \cos kz.
\]

Now we are in position to calculate the magnetic part of the excess free energy \(W\) according to (14), in the limit \(\chi_a < \chi_\perp \ll 1\) (see Appendix A).

\[
 -U = \pi L (\zeta_0 H_0)^2 \chi_\perp k R \frac{I_0(kR)}{I_1(kR)} \left(\chi_\perp k R I_1(kR) K_0(kR) + 2N_1 \frac{\chi_a}{\chi_\perp}\right). \tag{54}
\]

The elastic part of \(W\) was found in [2]

\[
 \mathcal{E}_n = \pi L k^2 \zeta_0^2. \tag{55}
\]

Inserting (54) and (55) into (13) we get

\[
 W = \frac{\pi L \sigma R}{2} \left(\frac{\zeta_0}{R}\right)^2 f(kR, \chi, H_0), \tag{56}
\]

where

\[
 f(kR, \chi, H_0) = (kR)^2(1 + 2\varkappa) - 1 + \frac{\mu_0 RH_0^2 k R I_0(kR)}{\sigma I_1(kR)} \left(\chi_\perp k R I_1(kR) K_0(kR) + 2N_1 \chi_a\right). \tag{57}
\]

All the terms in (57), except the last one, describe the stabilization of a LC cylinder due to the existence of isotropic susceptibility irrespective of its sign [4] and due to the elasticity of the LC phase [2]. The influence of the last term in (57) which accounts for the anisotropy of \(\chi\) can be significant and even dominating. The latter occurs if

\[
 |\chi_a| \gg \chi_\perp^2, \tag{58}
\]

as indeed is the case in classical LC materials (MBBA, PAA). Here the physical situation changes completely. The cylinder is destabilized with the corresponding cut-off \(k_s R\)

\[
 k_s R \simeq 1 + 2\chi_a |N_1| \frac{\mu_0 RH^2}{\sigma}. \tag{59}
\]

The interesting property of (59) is the fact that the cut-off \(k_s R\) extends beyond the range \((0 \leq k_s R \leq 1)\) of the classical Rayleigh instability. This kind of extension cannot be obtained as a field or elastic effects in the absence of anisotropy \(\chi_a\) of the magnetic susceptibility.
4 Conclusion

- The capillary instability of a LC cylinder in magnetic field is considered using an energy approach. The excess free energy, which includes terms due to surface, LC’s elasticity, and magnetic field is used to find extremum conditions associated with instability. The boundary problem is solved and then expanded in terms of the anisotropy $\chi_a$ of the magnetic susceptibility.

- The excess magnetic free energy, which was founded to be a function of the isotropic susceptibility ($\chi$) squared proved to have an anisotropic part linear in $\chi_a$. This indicates that the effect of anisotropy can turn dominant provided that $1 \gg |\chi_\perp| > |\chi_a| \gg \chi_\perp^2$. This means that the effect of anisotropy can be strong enough to counteract and even reverse the tendency of the field to enhance stabilization by extending the cut–off $k_s$ wave number beyond the conventional range set by Rayleigh.

- As the existence of magnetic anistropy is not limited to complex fluids such as LC’s, the result of this work can be considered of a more general nature.

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A Contribution of the magnetic field to free energy

Evaluate the contribution $\mu_0 U/2$ of the magnetic field inside $\Omega_{cyl}$ and outside $\mathbb{R}^3 \setminus \Omega_{cyl}$ of the disturbed liquid cylinder to the excess free energy $W$

$$ U = \chi \int_{\Omega_{cyl}} \left( \mathbf{in} \mathbf{H}_z \right)^2 dv + \int_{\Omega_{cyl}} \left[ \left( \mathbf{in} \mathbf{H}_z \right)^2 + (1 + \chi_\perp) \left( \mathbf{in} \mathbf{H}_r \right)^2 \right] dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left[ \left( \mathbf{ex} \mathbf{H}_z \right)^2 + \left( \mathbf{ex} \mathbf{H}_r \right)^2 \right] dv - \int_{\mathbb{R}^3} \mathbf{H}^2 dv + \int_{\Omega_{cyl}} n_r n_z \mathbf{in} \mathbf{H}_r \mathbf{in} \mathbf{H}_z dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left[ \left( \mathbf{ex} \mathbf{H}_z \right)^2 + \left( \mathbf{ex} \mathbf{H}_r \right)^2 \right] dv - \int_{\mathbb{R}^3} \mathbf{H}^2 dv \tag{A1} $$

$$ = \chi \int_{\Omega_{cyl}} \left( H_0 + \mathbf{in} \mathbf{H}_z \right)^2 dv + \int_{\Omega_{cyl}} \left( H_0 + \mathbf{in} \mathbf{H}_z \right) \mathbf{in} \mathbf{H}_r dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left( \mathbf{ex} \mathbf{H}_z \right)^2 dv - \int_{\mathbb{R}^3} \mathbf{H}^2 dv $$

$$ = \int_{\Omega_{cyl}} \chi \int_{\Omega_{cyl}} \left( \mathbf{in} \mathbf{H}_r \right)^2 dv + \int_{\Omega_{cyl}} \left( (1 + \chi_\parallel) \left( \mathbf{in} \mathbf{H}_r \right)^2 \right) dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left( \mathbf{ex} \mathbf{H}_r \right)^2 dv $$

$$ = \int_{\Omega_{cyl}} \chi \int_{\Omega_{cyl}} \left( \mathbf{in} \mathbf{H}_r \right)^2 dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left( \mathbf{ex} \mathbf{H}_r \right)^2 dv.$$ 

For the aims, discussed in section 3.1, we also give the linear in $\chi_a$ representations

$$ U = U_0 + \chi_a U_1, \quad \mathbf{in} \mathbf{H}_r = \mathbf{in} \mathbf{H}_r + \mathbf{in} \mathbf{H}_r, \quad \mathbf{in} \mathbf{H}_z = \mathbf{in} \mathbf{H}_z + \mathbf{in} \mathbf{H}_z, \quad \tag{A2} $$

where

$$ U_0 = (1 + \chi_\perp) \int_{\Omega_{cyl}} \left[ \left( \mathbf{in} \mathbf{H}_r \right)^2 + \left( \mathbf{in} \mathbf{H}_r \right)^2 \right] dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \left[ \left( \mathbf{ex} \mathbf{H}_r \right)^2 + \left( \mathbf{ex} \mathbf{H}_r \right)^2 \right] dv$$

$$ = 2H_0 \left( (1 + \chi_\perp) \int_{\Omega_{cyl}} \mathbf{in} \mathbf{H}_r dv + \int_{\mathbb{R}^3 \setminus \Omega_{cyl}} \mathbf{ex} \mathbf{H}_r dv \right),$$

$$ U_1 = \frac{2(1 + \chi_\perp)}{\chi_a} \int_{\Omega_{cyl}} \left[ \left( \mathbf{in} \mathbf{H}_z \right)^2 + \left( \mathbf{in} \mathbf{H}_r \right)^2 \right] dv + \int_{\Omega_{cyl}} \left[ \left( \mathbf{in} \mathbf{H}_z \right)^2 + 2n_r n_z H_0 \mathbf{in} \mathbf{H}_r + 2H_0 \mathbf{in} \mathbf{H}_r \right] dv \tag{A4} $$

In the last formulas we introduced the following notations

$$ \mathbf{in} \mathbf{H}_r = -kA_1 I_1(kr) \sin kz, \quad \mathbf{in} \mathbf{H}_z = -\chi_a \frac{A_\phi}{k} N_2 I_1(kr) \sin kz, \quad \tag{A5} $$

$$ \mathbf{in} \mathbf{H}_r = -kA_1 I_0(kr) \cos kz, \quad \mathbf{in} \mathbf{H}_z = -\chi_a \frac{A_\phi}{k} N_1 I_0(kr) \cos kz. $$
In fact, $U_0$ was calculated in [4]

$$U_0 = -\pi L \left( \chi \zeta_0 k R H_0 \right)^2 I_0(kR) K_0(kR) .$$

(A6)

Calculate the integrals in (A4) taking in mind $\chi \ll 1$

$$\frac{1}{\chi a} \int_{\Omega_{cyl}} \int_{H_z} \int_{H_z} dv = A_1 A_2 N_1 \int_{\Omega_{cyl}} I_2^2(kr) \cos^2 k z dv = \frac{\pi L}{2} A_1 A_2 N_1 R^2 \left[ I_0^2(kR) - I_1^2(kR) \right]$$

$$= -\frac{\pi L}{2} \chi (\zeta_0 H_0)^2 \left[ I_0^2(kR) - I_1^2(kR) \right] K_0(kR) I_1(kR) ,$$

(A7)

$$\frac{1}{\chi a} \int_{\Omega_{cyl}} \int_{H_z^2} \int_{H_z^2} dv = A_1 A_2 N_2 \int_{\Omega_{cyl}} I_1^2(kr) \sin^2 k z dv = \frac{\pi L}{2} A_1 A_2 N_2 R^2 \left[ I_0^2(kR) - I_0(kR) I_2(kR) \right]$$

$$= -\frac{\pi L}{2} \chi (\zeta_0 H_0)^2 \left[ I_0^2(kR) - I_0(kR) I_2(kR) \right] K_0(kR) I_1(kR) ,$$

(A8)

$$\frac{H_0}{\chi a} \int_{\Omega_{cyl}} \int_{H_z} \int_{H_z} dv = A_1 N_1 H_0 \int_{\Omega_{cyl}} I_0(kr) \cos k z dv = \pi L \zeta_0 R A_1 N_1 H_0 I_0(kR) =$$

$$= -\pi L N_1 (\zeta_0 H_0)^2 k R I_0(kR) I_1(kR) ,$$

(A9)

$$\int_{\Omega_{cyl}} \left( \int_{H_z^2} \right) dv = k^2 A_1^2 \int_{\Omega_{cyl}} I_0^2(kr) \cos^2 k z dv = \frac{\pi L}{2} A_1^2(kR)^2 \left[ I_0^2(kR) - I_1^2(kR) \right]$$

$$= \frac{\pi L}{2} \chi (\zeta_0 H_0)^2 \left[ I_0^2(kR) - I_1^2(kR) \right] K_0^2(kR) ,$$

(A10)

$$H_0 \int_{\Omega_{cyl}} n_r n_z \int_{H_z^2} dv = -k^2 A_1 A_2 H_0 \int_{\Omega_{cyl}} I_1^2(kr) \sin^2 k z dv$$

$$= -\frac{\pi L}{2} A_1 A_2 (kR)^2 H_0 \left[ I_1^2(kR) - I_0(kR) I_2(kR) \right]$$

$$= -\frac{\pi L}{2} \chi (\zeta_0 H_0)^2 \left[ I_1^2(kR) - I_0(kR) I_2(kR) \right] K_0(kR) I_1(kR) ,$$

(A11)

$$H_0 \int_{\Omega_{cyl}} \int_{H_z^2} dv = -k A_1 H_0 \int_{\Omega_{cyl}} I_0(kr) \cos k z dv = \pi L A_1 H_0 \zeta_0 k R I_0(kR)$$

$$= \pi L \chi (\zeta_0 H_0)^2 (kR)^3 I_0(kR) K_0(kR) .$$

(A12)

It is quite surprising that among all above integrals there is only one (A9) which dominates over the others in the region $\chi \ll 1$. Thus, in this limit we finally have

$$-U = \pi L (\zeta_0 H_0)^2 \chi k R \frac{I_0(kR)}{I_1(kR)} \left( \chi k R I_1(kR) K_0(kR) + 2 N_1 \frac{\chi a}{\chi} \right) .$$

(A13)

The last expression shows that the anisotropic part of $U$ can prevail over the isotropic one provided that

$$|N_1 \chi a| > \chi^2 .$$

(A14)
B On the solution of Equation (43).

The non–homogeneous equation

\[
\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - 1 \right) \Psi = CI_0(x) , \quad \Psi(x) = CG(x)I_0(x) , \quad \Psi(x_0) = 0 ,
\]  

(B1)

leads to the non–homogeneous equation for the amplitude function \( G(x) \)

\[
\frac{d^2G}{dx^2} + \left( 2 \frac{I_1(x)}{I_0(x)} + \frac{1}{x} \right) \frac{dG}{dx} = 1 , \quad G(x_0) = 0 ,
\]  

(B2)

or, after substitution \( \frac{dG}{dx} = S(x) \), essentially simplifies the problem

\[
\frac{dS}{dx} + \left( 2 \frac{I_1(x)}{I_0(x)} + \frac{1}{x} \right) S = 1 .
\]  

(B3)

Its solution reads

\[
S(x) = \exp \left( - \int \left( 2 \frac{I_1(x)}{I_0(x)} + \frac{1}{x} \right) \, dx \right) \int \exp \left( \int \left( 2 \frac{I_1(t)}{I_0(t)} + \frac{1}{t} \right) \, dt \right) \, dy = \frac{1}{xI_0^2(x)} \int yI_0^2(y) \, dy = \frac{x}{2} \left( 1 - \frac{I_1^2(x)}{I_0^2(x)} \right) ,
\]

and finally

\[
G(x) = \frac{1}{4} \left( x^2 - x_0^2 \right) - \frac{1}{2} \int_{x_0}^{x} \frac{I_1^2(y)}{I_0^2(y)} \, dy .
\]  

(B4)
