Representation Theory over Tropical Semifield
and
Langlands Correspondence

Anton A. Gerasimov and Dimitri R. Lebedev

Abstract. Recently we propose a class of infinite-dimensional integral representations of classical $\mathfrak{gl}_{t+1}$-Whittaker functions and local Archimedean local $L$-factors using two-dimensional topological field theory framework. The local Archimedean Langlands duality was identified in this setting with the mirror symmetry of the underlying topological field theories. In this note we introduce elementary analogs of the Whittaker functions and the Archimedean $L$-factors given by $U_{t+1}$-equivariant symplectic volumes of appropriate Kähler $U_{t+1}$-manifolds. We demonstrate that the functions thus defined have a dual description as matrix elements of representations of monoids $GL_{t+1}(\mathbb{R})$, $\mathbb{R}$ being the tropical semifield. We also show that the elementary Whittaker functions can be obtained from the non-Archimedean Whittaker functions over $\mathbb{Q}_p$ by taking the formal limit $p \to 1$. Hence the elementary special functions constructed in this way might be considered as functions over the mysterious field $\mathbb{Q}_1$. The existence of two representations for the elementary Whittaker functions, one as an equivariant volume and the other as a matrix element, should be considered as a manifestation of a hypothetical elementary analog of the local Langlands duality for number fields. We would like to note that the elementary local $L$-factors coincide with $L$-factors introduced previously by Kurokawa.

1 Introduction

We introduce a simplified version of the local Archimedean Langlands correspondence as a correspondence between various integral representations of a class of special functions. Under the local Langlands correspondence below we will understood a duality relation between various representations of a class of special functions including local $L$-factors and the Whittaker functions. This new setup arises quite straightforwardly out of an approach to the Archimedean Langlands correspondence on the level of special functions proposed recently [GLO6], [GLO7], [GLO8] (see also [G] for a general overview). It was argued in op. cit. that the Langlands duality between two constructions of the local Archimedean $L$-factors can be considered as an instance of a duality between infinite-dimensional equivariant symplectic geometry and finite-dimensional complex geometry (for generalities on the Langlands correspondence see e.g. [ABV], [B], [L]). Thus the proper setting for Archimedean Langlands duality is the mirror symmetry of two-dimensional topological field theories. In particular two dual constructions of the local Archimedean $L$-factors have a clear interpretation in terms of mirror dual two-dimensional topological field theories.

It is known that one of the consequences of Langlands duality is the existence of two mutually dual constructions of Whittaker functions (see [Sh], [CS] for the non-Archimedean case and [GLO3],
[GLO4], [GLO5] for the Archimedean case). As was demonstrated in [GLO8], the topological field theory approach can be successfully applied to description of the duality pattern. The Langlands dual constructions of the $\mathfrak{g}$-Whittaker functions arise while calculating correlation functions in mirror dual topological field theories of type $A$ and type $B$. In type $B$ topological field theory the correlation function reduces to a finite-dimensional integral which can be identified with an integral form of a matrix element of an infinite-dimensional representation of the Lie algebra $\mathfrak{g}$. On the other hand the integral representation arising in type $A$ topological field theory is given by an infinite-dimensional integral over the space of holomorphic maps of a two-dimensional disk $D$ into a flag space $G^\vee/B^\vee$ where the Lie group $G^\vee$ is dual to the Lie group $G$, $\text{Lie}(G) = \mathfrak{g}$, and $B^\vee \subset G^\vee$ is a Borel subgroup. In the proposed interpretation the infinite-dimensional integral representation of the Whittaker function is identified with the arithmetic side while the finite-dimensional integral representation is identified (via relations between matrix elements of an infinite-dimensional representation of the dual group) with the representation theory side of the Langlands correspondence.

In this note we define a simplified version of the Archimedean Langlands correspondence for a class of special functions obtained by replacing two-dimensional topological field theories by zero-dimensional ones in the constructions of [GLO6], [GLO7], [GLO8]. As a result, this simplified version of the Langlands correspondence is formulated purely in terms of finite-dimensional geometry. We use the adjective “elementary” for constructions arising in this simplified setting. Note that the two-dimensional topological field theories involved have an $S^1$-equivariance parameter $\hbar$ for which the aforementioned reduction to zero dimensions naturally arises in the limit $\hbar \to \infty$. This parameter can also be easily identified in the standard integral representations for the Whittaker functions. In this note we mostly avoid topological field theory considerations by taking the limit $\hbar \to \infty$ in the explicit integral representations for the Whittaker functions [KL], [Giv]. All the standard properties of the Whittaker functions have their elementary counterparts. For instance, the property of $\mathfrak{gl}_{\ell+1}$-Whittaker functions to be a common eigenfunction of $\mathfrak{gl}_{\ell+1}$-Toda chain quantum Hamiltonians is replaced by the property of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function to be an eigenfunction for a quantum billiard associated with $\mathfrak{gl}_{\ell+1}$. We show that the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function is the $U_{\ell+1}$-equivariant symplectic volume of the flag space $GL_{\ell+1}/B$. The dual description is a matrix element of a representation of the monoid $GL_{\ell+1}(\mathbb{R})$ where $\mathbb{R}$ is a tropical semifield. Note that the appearance of the tropical monoid in the elementary setting is directly related with the role played by the monoid of positive elements of $GL_{\ell+1}$ in the Givental type integral representations of the Whittaker functions [GKLO], [GLO1]. Thus the elementary version of mirror symmetry, relating equivariant symplectic volumes of flag spaces $GL_{\ell+1}/B$ and matrix elements of representations of dual Lie monoids over tropical fields, provides an elementary analog of Archimedean Langlands duality. More generally the elementary Langlands correspondence for Whittaker functions relates symplectic finite-dimensional geometry of the flag spaces $G/B$ and representation theory of tropical monoids associated with dual reductive groups $G^\vee$. All the results of [GLO2], [GLO6], [GLO7], [GLO8] have their counterparts in the elementary setting. In particular we provide mutually dual descriptions (i.e. in terms of finite-dimensional symplectic geometry and in terms of tropical geometry) of the eigenfunction property of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions with respect to an elementary version of the Baxter operators [GLO2]. The corresponding eigenvalues are given by elementary analogs of local $L$-factors having both a representation as an equivariant symplectic volume and as an integral over the tropical semifield.

Construction of the elementary Langlands correspondence reveals the fundamental role of geometry over tropical semifields as a dual description of the finite-dimensional symplectic geometry. The notion of a tropical semifield was first proposed by Maslov as a “dequantization” of real numbers in his study of classical asymptotics of quantum amplitudes (see [MS] for detailed explanations).
and has appeared (under various names) in various branches of Mathematics and Physics. Tropical geometry, i.e. geometry over tropical semifields, has, for example, been applied to study mirror symmetry of Calabi-Yau manifolds (the Berkovich geometry approach due to Kontsevich and Soibelman [KS], asymptotic analysis of Fukaya [F]), and to counting complex curves in algebraic manifolds by Viro, Mikhalkin et al (see e.g. [IMS]). See also [GKZ], [Mi], [EKL] for discussions of tropical geometry as a limit of real geometry in the case of toric manifolds.

The meaning of the elementary Whittaker functions can be partially elucidated using the \( q \)-deformed Whittaker functions [GLO3], [GLO4], [GLO5]. These functions provide an interpolation between classical \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions and their non-Archimedean analogs. The elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions can be obtained from the \( q \)-deformed \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions in two different ways. One can first consider a limit corresponding to classical \( \mathfrak{gl}_{\ell+1} \)-Whittaker function [GLO9] and then obtain, by further degeneration, the elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions. Equivalently one can first take a limit of the \( q \)-deformed \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions leading to the non-Archimedean \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions over \( \mathbb{Q}_p \). The elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions are then obtained by taking a formal limit \( p \to 1 \). The last limit has a simple explanation based on the Shintani-Casselman-Shalika formula [Sh], [CS] expressing the non-Archimedean Whittaker functions in terms of characters of finite-dimensional irreducible representations of the dual group. According to the Kirillov philosophy (see e.g. [K]), characters of finite-dimensional irreducible representations of compact Lie groups admit a limit expressed as an integral over corresponding coadjoint orbit. This limiting representation is precisely the expression of the elementary Whittaker functions as equivariant symplectic volume integrals obtained in the limit \( p \to 1 \). Taking the limit \( p \to 1 \) as a way to produce the elementary analogs implies that tropical geometry should be considered as an effective description of geometry over a mysterious field \( \mathbb{Q}_1 \). It is known that amoebas, defined in [GKZ], for variety over non-archimedean fields are identical to tropicalizations of the varieties over complex numbers [Mi], [EKL]. Recently in [CC], among other things, the limit of \( \mathbb{Q}_p \) for \( p \to 1 \) was also discussed and the relation with tropical semifields is stressed. Although there is an obvious similarity with our considerations, we should note that in contrast to [CC], we treat the tropical semifield as a universal target of the valuation map while \( \mathbb{Q}_1 \) being surjectively mapped (as multiplicative monoid) onto tropical semifield \( \mathbb{R} \) can have a larger kernel. We also should note that elementary \( L \)-factors coincide with the \( L \)-factors introduced by Kurokawa [Ku] (see also [Ma]).

Let us note that the relation between quantum billiards and equivariant symplectic volumes is a consequence of the general description of the equivariant cohomology of flag manifolds in terms of representation theory of nil-Hecke algebras [BGG], [KK] (see also [CG], [Gi] for reviews). This points to a hierarchy of generalizations of the results of [GLO6], [GLO7], [GLO8], and of the present article associated with multidimensional analogs of Hecke algebras (for the theory of double Hecke algebras see [Ch]). As a simple straightforward generalization one can consider elementary analogs of spherical functions and their connection with representation theory of graded affine Hecke algebras. Thus elementary analogs of \( GL_{\ell+1} \)-spherical functions have two dual descriptions. On the one hand they can be expressed as \( S^1 \times U_{\ell+1} \)-equivariant symplectic volumes of cotangent bundles \( T^*\mathcal{B} \) where \( S^1 \) acts on the fibres of the projection \( T^*\mathcal{B} \to \mathcal{B} \). On the other hand elementary spherical functions can be realized canonically as spherical functions on the tropical monoid \( G^\#(\mathbb{R}) \). A simple straightforward generalization one can consider elementary analogs of Calogero-Sutherland integrable systems (i.e. an integrable system such that spherical functions are common eigenfunctions of the corresponding quantum Hamiltonians) are systems of quantum particles with delta-function interactions (also known as Yang systems, see e.g. [HO]). The Hecke algebra description, generalizing the discussion in Section 6, is given in terms of the \( G \times \mathbb{C}^* \)-equivariant K-theory \( K_G(\mathcal{B}) \) and the graded affine Hecke algebra \( \mathcal{H}(G) \). Many
other examples of these constructions are known (see e.g. [CG]). We defer detailed discussions of the case of spherical functions and other generalizations for another occasion.

In this note we introduce an elementary analog of a particular manifestation of the local Langlands correspondence expressed as a relation between various representations of special functions e.g. the Whittaker functions. We expect that this can be generalized to an elementary analog of the full-flagged local Langlands correspondence between admissible representations of local reductive groups $G(K)$ and admissible representations of the Weil-Deligne group $W_K$ factored through the homomorphism of $W_K$ into dual reductive group $^L G$. Let us stress that the proposed elementary version of the local Langlands correspondence might be useful for understanding classical Langlands correspondence. In particular one can expect Langlands functoriality to be more accessible in the elementary setting.

The plan of the paper is as follows. In Section 2 we recall relevant facts about classical $\mathfrak{gl}_{\ell+1}$-Whittaker functions, local Archimedean $L$-factors and Baxter integral operators. We also briefly discuss a topological field theory interpretation of a class of Whittaker functions and local Archimedean $L$-factors following [GLO6], [GLO7], [GLO8]. In Section 3 we define elementary analogs of $\mathfrak{gl}_{\ell+1}$-Whittaker functions, local Archimedean $L$-factors and of Baxter integral operators. It is argued that the quantum billiard associated with the root system of $\mathfrak{gl}_{\ell+1}$ plays the role of an elementary analog of the $\mathfrak{gl}_{\ell+1}$-Toda chain. In Section 4 we demonstrate that elementary Whittaker functions can be obtained as a limit of a specialization at $q = 0$ of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions introduced in [GLO3], [GLO4], [GLO5]. In Section 5 elementary analogs of classical functions are identified with equivariant symplectic volumes. Thus the $\mathfrak{gl}_{\ell+1}$-Whittaker functions are identified with $U_{\ell+1}$-equivariant symplectic volumes of the flag spaces $GL_{\ell+1}(\mathbb{C})/B$ and elementary $L$-factors associated with standard representations $\mathbb{C}^{\ell+1}$ of $\mathfrak{gl}_{\ell+1}$ are identified with $U_{\ell+1}$-equivariant symplectic volumes of $\mathbb{C}^{\ell+1}$. We also provide a symplectic geometry interpretation of the eigenfunction property of an elementary Whittaker function with respect to an elementary analog of the Baxter operator. In Section 6 we elucidate some of the previous construction using the Kostant-Kumar description [KK] of equivariant cohomology of flag spaces in terms of representation theory of nil-Hecke algebras. In Section 7 the dual description of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions as matrix elements of infinite-dimensional representations of tropical monoids associated with $GL_{\ell+1}$ is given (Theorem 7.1). Finally, in Section 8 we briefly discuss a relation of our constructions with geometry over the mysterious limiting field $\mathbb{Q}_1$.

**Acknowledgments:** The research was supported by Grant RFBR-09-01-93108-NCNIL-a. The research of AG was also partly supported by Science Foundation Ireland grant. The authors are thankful to Max-Planck-Institut für Mathematik in Bonn for hospitality and excellent working conditions.

## 2 Whittaker functions in classical setting

In [GLO6], [GLO7], [GLO8] we introduced, using a topological field theory framework, infinite-dimensional integral representations of the local Archimedean $L$-factors (given by products of Gamma-functions) and of a class of the Whittaker functions. These integral representations can be reduced to give representations of the special functions as equivariant volumes of the infinite-dimensional spaces of holomorphic maps of a two-dimensional disk into a Kähler manifold such as a flag space $G/P$ or a complex vector space $V = \mathbb{C}^{\ell+1}$. This should be compared with the classical finite-dimensional integral representations of the same special functions e.g. the Euler integral representation of the Gamma-function, the Mellin-Barnes and the Givental integral representations.
of the Whittaker functions [KL], [GKLO]. It was argued in [GLO7], [GLOS] that both the finite dimensional and the infinite-dimensional integral representations arise naturally as correlation functions in two-dimensional topological field theories and are related by a mirror symmetry of the underlying quantum field theories. One can conjecture that analogs of infinite-dimensional integral representations hold for many other special functions such as spherical functions and the Whittaker functions associated with arbitrary pairs \((g, p)\) of Lie algebras \(g\) and their parabolic subalgebras \(p\).

In this Section we review two particular classes of integral representations of the \(gl_{\ell+1}\)-Whittaker functions associated with maximal and minimal parabolic subalgebras. We provide a realizations of the \(gl_{\ell+1}\)-Whittaker functions associated with maximal parabolic subgroups as equivariant volumes of spaces of holomorphic maps [GLOS], recall two integral representations of the local Archimedean \(L\)-factors and its topological field theory interpretation [GLO6], [GLO7].

Let \(E_{ij}, i, j = 1, \ldots, \ell + 1\) be the standard basis of the Lie algebra \(gl_{\ell+1}\). Let \(Z(U(gl_{\ell+1})) \subset U(gl_{\ell+1})\) be the center of the universal enveloping algebra \(U(gl_{\ell+1})\). Let \(B_{\pm} \subset GL_{\ell+1}(\mathbb{C})\) be upper-triangular and lower-triangular Borel subgroups and \(N_{\pm} \subset B_{\pm}\) be upper-triangular and lower-triangular unipotent subgroups. We denote by \(b_{\pm} = \text{Lie}(B_{\pm})\) and \(n_{\pm} = \text{Lie}(N_{\pm})\) their Lie algebras. Let \(h \subset gl_{\ell+1}\) be the diagonal Cartan subalgebra and \(W = \mathcal{S}_{\ell+1}\) be the Weyl group of \(GL_{\ell+1}\). Using the Harish-Chandra isomorphism between \(Z(U(gl_{\ell+1}))\) and the \(\mathcal{S}_{\ell+1}\)-invariant subalgebra of the symmetric algebra \(S^* h\), we identify central characters with homomorphisms \(c : \mathbb{C}[h_1, \ldots, h_{\ell+1}]^{\mathcal{S}_{\ell+1}} \rightarrow \mathbb{C}\). The central characters are in one to one correspondence with \(S_{\ell+1}\)-orbits of elements of the dual space \(h^* \simeq \mathbb{C}^{\ell+1}\). Let \(\pi_{\Delta} : U(gl_{\ell+1}) \rightarrow \text{End}(V_{\Delta})\) be a representation of the universal enveloping algebra \(U(gl_{\ell+1})\) with a central character associated with the \(S_{\ell+1}\)-orbit of \(h^{-1} \Lambda = (h^{-1} \lambda_1, \ldots, h^{-1} \lambda_{\ell+1}) \in i\mathbb{R}^{\ell+1}, h \in \mathbb{R}_+\). We impose an additional condition that the action of the Cartan subalgebra \(h\) can be integrated to an action of the corresponding Cartan subgroup \(H \subset GL_{\ell+1}(\mathbb{C})\) in \(V_{\Delta}\). Let \(V_{\Delta}^{\text{app}}\) be the dual module equipped with the induced action of \(U(gl_{\ell+1}^{\text{app}})\) (universal enveloping algebra of \(gl_{\ell+1}\)) obtained by inverting the signs of the structure constants of \(gl_{\ell+1}\). Denote by \(\langle, \rangle\) the pairing between \(V_{\Delta}^{\text{app}}\) and \(V_{\Delta}\). We suppose that the action of the Cartan subalgebra \(h\) in the representation \(V_{\Delta}\) can be integrated to an action of the corresponding Cartan subgroup \(H \subset GL_{\ell+1}(\mathbb{C})\).

According to Kostant a \(gl_{\ell+1}\)-Whittaker function is defined as a matrix element

\[
\Psi_{\Delta}(z) = e^{\rho(z)} \langle \psi_L | \pi_{\Delta}(e^{\sum_{i=1}^{\ell+1} x_i E_{ii}}) | \psi_R \rangle,
\]

where \(z = (x_1, \ldots, x_{\ell+1})\), \(\rho = (\rho_1, \ldots, \rho_{\ell+1})\) is a vector in \(h^*\) with components \(\rho_k = -1/2(\ell - 2k + 2), k = 1, \ldots, \ell + 1\) (half of the sum of positive roots of \(gl_{\ell+1}\)). The one-dimensional spaces generated by vectors \(|\psi_L\rangle \in V_{\Delta}\) and \(|\psi_R\rangle \in V_{\Delta}\) provide one-dimensional representations of \(U n_{-}\) and \(U n_{+}\) respectively

\[
\langle \psi_L | E_{i+1, i} = -h^{-1} \langle \psi_L |, \quad E_{i, i+1} |\psi_R\rangle = -h^{-1} |\psi_R\rangle, \quad i = 1, \ldots, \ell.
\]

Standard considerations (see e.g. [STS]) show that the matrix element (2.1) is a common eigenfunction of a family of commuting differential operators coming from the action of generators of \(Z(U(gl_{\ell+1}))\) in \(V_{\Delta}\). These differential operators can be identified with quantum Hamiltonians of the \(gl_{\ell+1}\)-Toda chain. The simplest non-trivial quantum Hamiltonian acts on a Whittaker function as the differential operator

\[
\mathcal{H} = -\frac{\hbar^2}{2} \sum_{i=1}^{\ell+1} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{\ell} e^{x_{i+1} - x_i}.
\]

The matrix element representation (2.1) leads to various integral representations of the Whittaker function by using explicit realizations of the universal enveloping algebra representation \(\pi_{\Delta}\) via
difference/differential operators acting on an appropriate space of functions. For example one can consider the principal series representation \( \text{Ind}_{B-}^{GL_{\ell+1}} \chi_{\Delta} \) induced from a one-dimensional representation of \( B_- \) given by the character

\[
\chi_{\Delta}(b) = \prod_{j=1}^{\ell+1} |b_{jj}|^{\ell+1} \delta_{k,j}, \quad b \in B_-.
\]  

(2.3)

Here \( \Delta = (\lambda_1, \ldots, \lambda_{\ell+1}) \) is a vector in \( \mathbb{R}^{\ell+1} \), \( h \) is in \( \mathbb{R}_+ \) and \( \delta_{k,j} = -1/2(\ell - 2k + 2), \quad k = 1, \ldots, \ell + 1 \). The subspace of analytic vectors provides the corresponding representation of \( Ugl_{\ell+1} \). More generally one can consider representations of \( Ugl_{\ell+1} \) realized in the subspace of \( B_- \)-equivariant functions supported on \( B_- \)-stable subvarieties in \( GL_{\ell+1} \). They obviously support an action of \( Ugl_{\ell+1} \) with central character associated to the \( S_{\ell+1} \)-orbit through \( ih^{-1} \Delta \).

In this article we shall be especially interested in the integral representations of \( gl_{\ell+1} \)-Whittaker functions defined in the following two theorems.

**Theorem 2.1** The \( gl_{\ell+1} \)-Whittaker function has the following integral representation:

\[
\Psi_{\Delta}(x, h) = \int_{S} \prod_{n=1}^{\ell+1} \prod_{k=1}^{n} \prod_{m=1}^{n} \frac{\Gamma_1 \left( \frac{\gamma_{n,k+1} - \gamma_{n,m}}{h} \right)}{\Gamma_1 \left( \frac{\gamma_{n,k} - \gamma_{n,m}}{h} \right)} e^{\frac{\ell+1}{\pi} \sum_{n=1}^{\ell} \sum_{j=1}^{n} (\gamma_{n,j} - \gamma_{n-1,j}) x_n} \prod_{n=1}^{\ell} \frac{d\gamma_{n,j}}{2\pi h},
\]

(2.4)

where \( \Gamma_1(z|h) = h^{z} \Gamma(\frac{z}{h}) \), \( \Delta = (\lambda_1, \ldots, \lambda_{\ell+1}) := (\gamma_{\ell+1,1}, \ldots, \gamma_{\ell+1,\ell+1}) \in \mathbb{R}^{\ell+1} \), \( x = (x_1, \ldots, x_{\ell+1}) \) and the domain of integration \( S \) is such that \( \gamma_{k,j} + \epsilon_{k,j} \in \mathbb{R} \) and \( \epsilon_{k,j} \) are fixed real numbers such that the conditions \( \max_j \{\text{Im } \gamma_{k,j}\} < \min_m \{\text{Im } \gamma_{k,m}\} \) for all \( k = 1, \ldots, \ell \) hold. The integral (2.4) converges absolutely. Recall that we assume \( \gamma_{n,j} = 0 \) for \( j > n \).

This integral representation was first introduced in [KL] and rederived in the framework of representation theory in [GKL]. Another relevant integral representation for the \( gl_{\ell+1} \)-Whittaker function was introduced by Givental [Giv] (see [GKLO] for a representation theoretic derivation).

**Theorem 2.2** The \( gl_{\ell+1} \)-Whittaker function has the following integral representation:

\[
\Psi_{\Delta}(x|h) = \int_{C} \exp \left( \frac{\ell+1}{h} \sum_{k=1}^{\ell} \lambda_k \left( \sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) \right) \times \exp \left\{ -\frac{1}{h} \sum_{k=1}^{\ell} \left( \sum_{i=1}^{k} e^{T_{k+1,i}} - \sum_{i=1}^{k} e^{T_{k+1,i+1}} - T_{k,i} \right) \right\} \prod_{k=1}^{\ell} \prod_{i=1}^{k} dT_{k,i},
\]

(2.5)

where \( \Delta = (\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{R}^{\ell+1} \). Here we set \( x_i = T_{k+1,i}, \ i = 1, \ldots, \ell + 1 \) and the domain of integration \( C \) is a slight deformation of the middle-dimensional real subspace of \( \mathbb{C}^{(\ell+1)/2} \) rendering the integral (2.5) convergent.

In [GLO8] we introduced generalized \( gl_{\ell+1} \)-Whittaker functions associated with parabolic subalgebras \( p \subset gl_{\ell+1} \) (in this sense the standard \( gl_{\ell+1} \)-Whittaker function (2.1) is associated with the Borel subalgebra \( b \subset gl_{\ell+1} \)). For the \( gl_{\ell+1} \)-Whittaker function associated with the maximal parabolic subalgebra \( p_{1,\ell+1} \subset gl_{\ell+1} \) (i.e. such that \( \dim(gl_{\ell+1}/p_{1,\ell+1}) = \ell \)) we coin the term \((1, \ell+1)\)-parabolic Whittaker function. The following analogs of the Mellin-Barnes and the Givental integral representations of a specialization of the \((1, \ell+1)\)-parabolic Whittaker function holds.
Theorem 2.3 The \((1, \ell + 1)\)-parabolic Whittaker function specialized to \(x_1 = x\) and \(x_i = 0, \ i \neq 1\) admits the following integral representations:

\[
\Psi_{\Delta}^{(1, \ell + 1)}(x, 0, \ldots, 0) = \frac{1}{2\pi\hbar} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} \, d\gamma \, e^{ix\gamma} \prod_{j=1}^{\ell+1} \Gamma_1 \left( i(\gamma - \lambda_j) | \hbar \right),
\]

where \(\epsilon > 0\).

(2).

\[
\Psi_{\Delta}^{(1, \ell + 1)}(x, 0, \ldots, 0) = \int_{\mathcal{C}} \prod_{j=1}^{\ell} dt_j \, \exp \left( \frac{t}{\hbar} \left( \sum_{j=1}^{\ell} \lambda_j t_j + \lambda_{\ell+1}(x - \sum_{j=1}^{\ell} t_j) \right) \right) \cdot \exp \left( -\frac{1}{\hbar} \left( \sum_{j=1}^{\ell} e^{-t_j} + e^{\sum_{j=1}^{\ell} t_j - x} \right) \right),
\]

where \(\Delta = (\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{R}^{\ell+1}\) and \(\mathcal{C}\) is a slight deformation of the real subspace of middle dimension in \(\mathbb{C}^\ell\) rendering the integral (2.7) convergent.

Whilst being a common eigenfunction of a family of mutually commuting differential operators, the \(\mathfrak{gl}_{\ell+1}\)-Whittaker function is also a common eigenfunction of a one-parameter family of integral operators [GLO2]. This family of integral operators (called the Baxter operators due to their relation with the Baxter operators in the theory of quantum integrable systems) has a representation theoretic origin. In [GLO2] the Baxter integral operators were defined as generating series for the generators of the Archimedean counterpart of the non-Archimedean spherical Hecke algebra. Explicitly, a one-dimensional family \(Q_{\ell+1}(s), \ s \in \mathbb{C}\) of the Baxter integral operators acting in an appropriate space of functions of \(\ell + 1\) variables has the following integral kernel:

\[
Q_{\ell+1}^s(x, y | s) = \exp \left\{ \frac{t}{\hbar} \sum_{i=1}^{\ell+1} (x_i - y_i) - \frac{1}{\hbar} \sum_{i=1}^{\ell} \left( e^{x_i - y_i} + e^{y_{i+1} - x_i} \right) - \frac{1}{\hbar} e^{x_{\ell+1} - y_{\ell+1}} \right\}.
\]

The Baxter operator \(Q_{\ell+1}^s(s)\) satisfies the following commutativity relations:

\[
Q_{\ell+1}^s(s) \cdot Q_{\ell+1}^{s'}(s') = Q_{\ell+1}^{s'}(s') \cdot Q_{\ell+1}^s(s),
\]

\[
Q_{\ell+1}^s(s) \cdot \mathcal{H}_{r}^{\ell+1} = \mathcal{H}_{r}^{\ell+1} \cdot Q_{\ell+1}^s(s), \quad r = 1, \ldots, \ell + 1,
\]

and the \(\mathfrak{gl}_{\ell+1}\)-Whittaker function (2.4), (2.5) satisfies the following eigenfunction identity:

\[
\int_{\mathbb{R}^{\ell+1}} \prod_{i=1}^{\ell+1} \, dy_i \, Q_{\ell+1}^s(x, y | s) \Psi_{\Delta}^s(y) = L(s, \Delta, h) \Psi_{\Delta}^s(x).
\]

Here \(x = (x_1, \ldots, x_{\ell+1}), \ y = (y_1, \ldots, y_{\ell+1}), \ \Delta = (\lambda_1, \ldots, \lambda_{\ell+1})\) and the eigenvalue is given by the local Archimedean \(L\)-factor attached to the principal series representation of \(GL_{\ell+1}\) associated with the character (2.3) (see e.g. [B], [L])

\[
L(s, \Delta, h) = \prod_{j=1}^{\ell+1} h^{-i\lambda_j} \Gamma \left( \frac{i s - i \lambda_j}{\hbar} \right).
\]
The appearance of the local $L$-factors is not accidental and is related to the fact that the integral operators with kernels (2.8) are realizations of generating functions of elements of the local spherical Archimedean Hecke algebra (see [GLO2] for a detailed discussion). Note that the local Archimedean $L$-factors (2.12) also have integral representations of the Givental type (2.5) given by the Euler integral representations for the Gamma-functions

$$
L(s, \lambda, h) = \int_{\mathbb{R}^{\ell+1}} \prod_{j=1}^{\ell+1} d\tau_j e^{\sum_{j=1}^{\ell+1} \left( \frac{1}{\pi} (s-\lambda_j) \tau_j - \frac{1}{\pi} e^{\tau_j} \right)},
$$

(2.13)

where \(\text{Im}(s) < 0, \ j = 1, \ldots, \ell + 1\).

The expression of the local $L$-factor (2.12) as a product of $\Gamma$-functions is on the other hand an analog of the Mellin-Barnes representation (2.4) of the $\mathfrak{gl}_{\ell+1}$-Whittaker function.

Along with the finite-dimensional integral representations, the Whittaker functions and the local Archimedean $L$-factors have infinite-dimensional integral representations. These integral representations naturally arise from an interpretation of the special functions as equivariant symplectic volumes of infinite-dimensional space of holomorphic maps of a two-dimensional disk into finite-dimensional symplectic spaces [GLO6], [GLO7], [GLO8]. Given a finite-dimensional symplectic manifold $M$ with a symplectic form $\omega$ and Hamiltonian action of a compact Lie group $G$ one defines a $G$-equivariant symplectic volume as the following integral:

$$
Z_M(\lambda) = \int_M e^{\omega_G}.
$$

(2.14)

Here $\omega_G$ is the $G$-equivariant extension of the symplectic form $\omega$ depending on an element $\lambda \in \mathfrak{g}^*$ of the dual space to the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ (see e.g. [Au] for precise definitions and details). In [GLO6], [GLO7], [GLO8] this definition was used for infinite-dimensional spaces of holomorphic maps $\mathcal{M}(D, X)$ of a two-dimensional disk $D = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$ into symplectic $U_{\ell+1}$-spaces $X$ such as $\mathbb{C}^{\ell+1}$, $SL_{\ell+1}/P$, with $P \subset SL_{\ell+1}$ being a parabolic group. There is a natural Hamiltonian action of $S^1 \times U_{\ell+1}$ on $\mathcal{M}(D, X)$, where $S^1$ acts by rotations of $D$ and the action of $U_{\ell+1}$ is induced from the action on $X$. An extension of the definition (2.14) to the infinite-dimensional case requires some care and includes a $\zeta$-function regularization of the infinite-dimensional integrals.

The equivariant symplectic volumes of the spaces $\mathcal{M}(D, X)$ of holomorphic maps can be reformulated as particular correlation functions in two-dimensional equivariant topological sigma models on the disk $D$ with target space $X$. An advantage of this reformulation is that one can invoke mirror symmetry considerations to reformulate the infinite-dimensional integral representations in terms of finite-dimensional ones [GLO6], [GLO7], [GLO8]. For the local Archimedean $L$-factors and the $\mathfrak{gl}_{\ell+1}$-Whittaker functions this leads to the integral representations (2.13), (2.5).

Note that $S^1 \times U_{\ell+1}$-equivariant topological field theory depends on $\lambda \in u_+^{\ast}$ and $h \in \text{Lie}(S^1)$. In the limit $h \to \infty$ one expects that the $S^1 \times U_{\ell+1}$-equivariant symplectic volume of $\mathcal{M}(D, X)$ will be reduced to a $U_{\ell+1}$-equivariant volume integral over $X$. In the following Section we take this limit in the integral representations (2.4), (2.5), (2.13) for the $\mathfrak{gl}_{\ell+1}$-Whittaker functions and local $L$-factors directly and demonstrate that the resulting elementary Whittaker functions and $L$-factors are given by equivariant symplectic volumes of finite-dimensional symplectic spaces. Note that up to now the interpretation of (parabolic) Whittaker functions in terms of equivariant symplectic volumes of infinite-dimensional spaces is not known in full generality (recent progress has however been presented in [O1], [O2]). In cases when the interpretation in terms of topological field theory with a target space $X$ is already established [GLO8] the obtained limiting expression is compatible with the localization to the space of constant maps to $X$ discussed above. Thus the identification
of the $h \to \infty$ limits of the $gl_{\ell+1}$-Whittaker functions with $U_{\ell+1}$-equivariant symplectic volumes of the flag spaces $G/B$ should be considered as additional support for the approach of [GLO6], [GLO7], [GLO8].

3 Elementary analogs of Whittaker functions and local L-factors

In this Section we take the limit $h \to \infty$ of the $gl_{\ell+1}$-Whittaker functions and the local Archimedean $L$-factors and demonstrate that the resulting expression can be interpreted as an equivariant symplectic volume of finite-dimensional Kähler spaces. Let us start by introducing elementary analogs of the $gl_{\ell+1}$-Whittaker functions.

**Definition 3.1** The elementary $gl_{\ell+1}$-Whittaker function is defined as the following absolutely convergent integral:

$$
\Psi_\lambda^{(0)}(x) = \frac{\ell+1}{\pi} \int_{S} \prod_{s=1}^{\ell} \frac{\prod_{n=1}^{i} (\gamma_{ns} - \gamma_{np})}{\prod_{m=1}^{n+1} (\gamma_{nk} - \gamma_{n+1,m})} \prod_{n=1}^{\ell} \frac{d\gamma_{nm}}{2\pi i},
$$

(3.1)

where the domain of integration $S$ is such that $\gamma_{jk} + \epsilon_{jk} \in \mathbb{R}$ and $\epsilon_{jk}$ are fixed real numbers such that the conditions $\max_j \{\text{Im} \gamma_{jk}\} < \min_{k} \{\text{Im} \gamma_{k+1,m}\}$ for all $k = 1, \ldots, \ell$ hold. We set $(\gamma_{\ell+1,1}, \ldots, \gamma_{\ell+1,\ell+1}) := (\lambda_{1}, \ldots, \lambda_{\ell+1}) \in \mathbb{R}^{\ell+1}$ and assume $\gamma_{nj} = 0$ for $j > n$. We also use the notations $x = (x_1, \ldots, x_{\ell+1})$ and $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$.

**Proposition 3.1** The elementary $gl_{\ell+1}$-Whittaker function has the following representation:

$$
\Psi_\lambda^{(0)}(x) = \sum_{s \in \mathcal{S}_{\ell+1}} (-1)^{l(s)} \frac{e^{\sum_{k=1}^{\ell+1} \epsilon_{\lambda(k)} x_k}}{\prod_{i<j} (\lambda_i - \lambda_j)}, \quad x_1 \geq x_2 \geq x_3 \geq \ldots \geq x_{\ell+1},
$$

(3.2)

and

$$
\Psi_\lambda^{(0)}(x) = 0,
$$

(3.3)

otherwise. Here $(s(1), \ldots, s(\ell+1))$ is a permutation of $(1, \ldots, \ell+1)$ corresponding to an element $s \in \mathcal{S}_{\ell+1}$ of the permutation group $\mathcal{S}_{\ell+1}$ and $l(s)$ is the length of the permutation (the number of terms in the minimal decomposition of $s$ into elementary permutations).

**Proof.** 1. In the domain $x_1 \geq x_2 \ldots \geq x_{\ell+1}$ the integration domain $S$ can be deformed so that the integral (3.1) is given by a non-trivial sum of residues. To calculate contributions of the residues let us note that the integrand is symmetric with respect to $\mathcal{S}_{\ell+1} = \mathcal{S}_{2} \times \mathcal{S}_{3} \times \ldots \times \mathcal{S}_{\ell+1}$ acting on $\{\gamma_{ij}\}$ via permutations of the second index. The nontrivial residues corresponding to the poles of the integrand are at the points $\gamma_{k,i} = \gamma_{k+1,j}$, $\gamma_{k,i} \neq \gamma_{k,j}$. Let us first consider the residue contribution at the poles $\gamma_{ij} = \gamma_{kJ}$

$$
\sum_{n=1}^{\ell+1} (\gamma_{nj} - \gamma_{n-1,j}) x_j \rightarrow \sum_{n=1}^{\ell+1} \gamma_{\ell+1,j} x_j.
$$

The contribution of the rational function in the integrand is reduced after cancellations to

$$
\frac{1}{\prod_{s<p} \prod_{i} (\gamma_{\ell+1,s} - \gamma_{\ell+1,p})}.
$$
The integrand is symmetric under action of $\mathcal{G}_{\text{tot}}$ and thus the contributions of other poles $\gamma_{k,i} = \gamma_{k+1,j}$, $\gamma_{k,i} \neq \gamma_{k,j}$ can be obtained by averaging over the permutations. Note that the subgroup $\mathcal{G}_2 \times \mathcal{G}_3 \times \cdots \times \mathcal{G}_\ell \subset \mathcal{G}_{\text{tot}}$ acts trivially on the residue contribution. Thus the averaging over the subgroup cancels the factor $\prod_{i=1}^n (n!)^{-1}$ and the averaging over $\mathcal{G}_{\ell+1}$ gives

$$\sum_{s \in \mathcal{G}_{\ell+1}} e^{\sum_{k=1}^{\ell+1} \lambda_s x_k} = \sum_{s \in \mathcal{G}_{\ell+1}} (-1)^{(s)} e^{\sum_{k=1}^{\ell+1} \lambda_s x_k},$$

2. Outside the dominant domain the integral 3.2 is equal to zero by deformation-of-contour arguments. □

It is easy to see that the representation (3.2) can be written in the following recursive form

$$\Psi^{(0)}_{\lambda}(x_1, \ldots, x_{\ell+1}) = (3.4)$$

$$= \int_{S_{\ell+1}} \prod_{j=1}^{\ell} \frac{d\gamma_{\ell,j}}{2\pi i} \prod_{s \neq p} (\gamma_{\ell,s} - \gamma_{\ell,p}) \Psi^{(0)}_{\lambda_1, \ldots, \lambda_{\ell}}(x_1, \ldots, x_{\ell}),$$

where $S_{\ell+1}$ is defined by the conditions $\max_j \{\text{Im} \, \gamma_{\ell,j}\} < \min_m \{\text{Im} \, \gamma_{\ell+1,m}\}$. We call the functions (3.1) elementary Whittaker functions due the following result.

**Proposition 3.2** The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (3.1) can be represented as the limit

$$\Psi^{(0)}_{\lambda}(x) = \lim_{h \to \infty} (h)^{-\ell(\ell+1)/2} \Psi_{\lambda}(h x, h),$$

where $\Psi_{\lambda}(x, h)$ is defined by (2.4).

*Proof.* Consider the asymptotic behavior of the numerator of (2.4). By definition of the contour $S$ in (2.4), the real parts of arguments of $\Gamma$-functions are positive. Then (3.1) is obtained from (2.4) by applying the following asymptotic form of the $\Gamma$-function:

$$h^{-1} h \Gamma \left( \frac{z}{h} \right) = \int_{-\infty}^{\infty} dx e^{-z x} e^{-h^{-1} e^{-hx}} \to \int_{-\infty}^{\infty} dx e^{-z x} \Theta(x) = \frac{1}{z}, \quad h \to \infty, \quad \text{Re} \, z > 0,$$

where $\Theta(x)$ is the Heaviside function i.e. $\Theta(x) = 1$ for $x \geq 0$ and zero otherwise. □

The integral representation (3.1) is an analog of the Mellin-Barnes integral representation (2.4) and (3.4) is an analog of the fundamental recursive property of the Mellin-Barnes integral representation [KL]. Taking $h \to \infty$ in (2.5) one obtains an elementary analog of the Givental integral representation.

**Proposition 3.3** The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function has the following Givental type integral representation:

$$\Psi^{(0)}_{\lambda}(x) = \int_{\mathbb{R}^{(\ell+1)/2}} \exp \left( \sum_{k=1}^{\ell+1} \lambda_k \left( \sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) \right) \prod_{k=1}^{\ell} \prod_{i=1}^{k} dT_{k,i},$$

where $x_i = T_{\ell+1,i}$, $i = 1, \ldots, \ell+1$ and we assume $T_{k,i} = 0$ for $i > k$. The function $\Theta(x)$ is the Heaviside function i.e. $\Theta(x) = 1$ for $x \geq 0$ and zero otherwise.
Proof. Let us change the variables $T_{k,i} \to \hbar T_{k,i}$, $k = 1, \ldots, \ell + 1$, $i = 1, \ldots, k$ in (2.5). Now we take the limit $\lim_{\hbar \to \infty} \hbar^{-\ell+1/2} \Psi_{\Delta}(hx|\hbar)$ of the Givental integral representation (2.5) using the identity

$$
\lim_{\hbar \to \infty} e^{-\hbar^{-1}e^{-hx}} = \Theta(x).
$$

(3.7)

This gives us (3.6). The second statement is a direct consequence of (3.6). □

Corollary 3.1 The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function is given by

$$
\Psi_\lambda^{(0)}(x) = \int \exp \left( \sum_{k=1}^{\ell+1} \lambda_k \left( \sum_{i=1}^{k} T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i} \right) \right) \prod_{k=1}^{\ell} \prod_{i=1}^{k} dT_{k,i},
$$

(3.8)

and

$$
\Psi_\lambda^{(0)}(x) = 0,
$$

(3.9)

otherwise. Here $\mathcal{D}$ is a convex polytope in $\mathbb{R}^{\ell(\ell+1)/2}$ defined by the inequalities $T_{k,i} \geq T_{k-1,i} \geq T_{k,i+1}$, $k = 1, \ldots, \ell + 1$, $i = 1, \ldots, \ell + 1$. We assume $T_{ki} = 0$ when $i > k$.

Proof. Obviously follows from (3.6). □

Definition 3.2 The elementary $(\ell+1,1)$-Whittaker function specialized at $x = (x,0,\ldots,0)$ is defined as the following integral:

$$
(\ell+1,1)\Psi_\lambda^{(0)}(x,0,\ldots,0) = \int_{\mathbb{R}^{\ell+1}} \frac{d\gamma}{2\pi} e^{ix\gamma} \prod_{j=1}^{\ell+1} \frac{1}{i(\gamma - \lambda_j)},
$$

(3.10)

where $\lambda \in \mathbb{R}^{\ell+1}$ and $\epsilon > 0$.

Proposition 3.4 The following limiting expression holds:

(1).

$$
(\ell+1,1)\Psi_\lambda^{(0)}(x,0,\ldots,0) = \lim_{\hbar \to \infty} (\hbar^{-\ell}) (\ell+1,1)\Psi_{\Delta}(hx,0,\ldots,0|h),
$$

where $(\ell+1,1)\Psi_{\Delta}(x,0,\ldots,0|h)$ is defined by (2.6).

(2). The elementary $(\ell+1,1)$-Whittaker function has the following integral representation:

$$
(\ell+1,1)\Psi_\lambda^{(0)}(x,0,\ldots,0) = \int_{\mathbb{R}^{\ell}} \prod_{j=1}^{\ell} d\tau_j \ e^{\sum_{j=1}^{\ell} \lambda_j \tau_j + i\lambda_{\ell+1}(x-\sum_{j=1}^{\ell} \tau_j)} \Theta(x-\sum_{j=1}^{\ell} \tau_j) \prod_{j=1}^{\ell} \Theta(\tau_j).
$$

Proof. The proof is analogous to the proof of Proposition 3.4. □

Let us note that, in contrast with classical case, the elementary $(\ell+1,1)$-Whittaker function can be obtained as a specialization of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function

$$
\Psi_\lambda^{(0)}(x,0,\ldots,0) = (\ell+1,1)\Psi_\lambda^{(0)}(x,0,\ldots,0).
$$
Classical $\mathfrak{gl}_{\ell+1}$-Whittaker function is a common eigenfunction of a family of mutually commuting differential operators. These differential operators can be identified with quantum Hamiltonians of the $\mathfrak{gl}_{\ell+1}$-Toda chain. Similarly the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions are common eigenfunctions of a family of mutually commuting differential operators defining an elementary analog of the $\mathfrak{gl}_{\ell+1}$-Toda chains. This elementary analog of the $\mathfrak{gl}_{\ell+1}$-Toda chain can be obtained as the $\hbar \to \infty$ limit of the standard $\mathfrak{gl}_{\ell+1}$-Toda chain. We start with the definition of a (well-known) quantum integrable system which plays the role of the elementary $\mathfrak{gl}_{\ell+1}$-Toda chain and then explain how this quantum integrable system arises in the $h \to \infty$ limit from the $\mathfrak{gl}_{\ell+1}$-Toda chain.

Let $(x_1,\ldots,x_{\ell+1})$ be linear coordinates in $\mathbb{R}^{\ell+1}$. The permutation group $\mathcal{S}_{\ell+1}$ acts in $\mathbb{R}^{\ell+1}$ as the group of reflections with respect to the principal diagonals $x_i = x_j$. One defines a quantum billiard associated with the pair $(\mathbb{R}^{\ell+1}, \mathcal{S}_{\ell+1})$ as a free quantum particle moving in the closure of the fundamental domain $\overline{D}_{\ell+1} = \{x = (x_1,\ldots,x_{\ell+1}) \in \mathbb{R}^{\ell+1} | x_i \geq x_{i+1}\}$ of $\mathcal{S}_{\ell+1}$ acting in $\mathbb{R}^{\ell+1}$.

We impose Dirichlet boundary conditions on wave functions at the boundary $\partial \overline{D}_{\ell+1}$. The resulting quantum integrable system is a special case of the well-known series of integrable systems on the fundamental domains of actions of Weyl groups $W$ on Cartan subalgebras $\mathfrak{h}$ (in our case the Lie algebra is $A_\ell$, the Cartan subalgebra is $\mathfrak{h} = \mathbb{R}^{\ell+1}$ and the Weyl group is $W = \mathcal{S}_{\ell+1}$) (see e.g. [I]).

**Proposition 3.5** The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (3.1) is a common eigenfunction of the elementary Toda chain Hamiltonians

$$P_i(\partial_x) \Psi_\lambda(x) = P_i(\lambda) \Psi_\lambda(x), \quad P_i(y) \in \mathbb{C}[y_1,\ldots,y_{\ell+1}]^{\mathcal{S}_{\ell+1}}, \quad x \in \overline{D}_{\ell+1},$$

where $\overline{D}_{\ell+1} = \{x = (x_1,\ldots,x_{\ell+1}) \in \mathbb{R}^{\ell+1} | x_i \geq x_{i+1}\}$ is a compactification of the fundamental domain of the action of $\mathcal{S}_{\ell+1}$ in $\mathbb{R}^{\ell+1}$ and Dirichlet boundary conditions are imposed

$$\Psi_\lambda(x)|_{x_j = x_{j+1}} = 0.$$  

**Proof.** The representation (3.2) of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function as a sum over the Weyl group $\mathcal{S}_{\ell+1}$ implies that the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function is a common eigenfunction of the operators $P_i(\partial_x)$. Taking into account (3.3) we infer that the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function satisfies the boundary condition (3.12) and thus solves the eigenvalue problem of the quantum billiard.

Taking into account (3.2) we obtain an expansion of the quantum billiard eigenfunction

$$\Psi_\lambda(x) = \sum_{s \in \mathcal{S}_{\ell+1}} C(0)(s \cdot \lambda) e^{t(s \cdot \lambda, x)},$$

where

$$C(0)(\lambda) = \prod_{\alpha > 0} \Gamma(0)(\alpha(\lambda, \alpha)) = \prod_{i < j} \Gamma(0)(\alpha_{\lambda_i} - \alpha_{\lambda_j}),$$

the product is over positive roots of $\mathfrak{gl}_{\ell+1}$ and we use the elementary analog $\Gamma(0)(s) = s^{-1}$ of classical Gamma-function. The functions $C(0)(\lambda)$ are elementary analogs of the Harish-Chandra functions for the Whittaker case. Indeed, the expansion (3.13) should be compared with an asymptotic expansion in the region $x_k >> x_{k+1}$, $k = 1,\ldots, \ell$ of classical class one $\mathfrak{gl}_{\ell+1}$-Whittaker function (see e.g. [KL])

$$\Psi_\lambda(x) = \sum_{s \in \mathcal{S}_{\ell+1}} C(s \cdot \lambda) e^{t(s \cdot \lambda, x)} + O \left( \max \left\{ e^{x_{k+1} - x_k} \right\}_{k=1}^{\ell} \right),$$
where

\[
C(\lambda) = h^{-2(\lambda, \rho)/\hbar} \prod_{\alpha > 0} \Gamma \left( \frac{(\lambda, \alpha)}{i \hbar} \right).
\]  

(3.15)

Thus, the \( \mathfrak{gl}_{\ell+1} \)-Whittaker function (2.4) is a common eigenfunction of the \( \mathfrak{gl}_{\ell+1} \)-Toda chain and the elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker function (3.1) is a common eigenfunction of a quantum billiard system associated with \( \mathfrak{gl}_{\ell+1} \). It is natural expect that the quantum billiard can be understood as an \( h \to \infty \) limit of \( \mathfrak{gl}_{\ell+1} \)-Toda chain. Below we demonstrate this relation for the simplest non-trivial case \( \ell = 1 \).

A ring of quantum Hamiltonians of the \( \mathfrak{gl}_2 \)-Toda chain is generated by the two differential operators

\[
\mathcal{H}_1 = -i h \frac{\partial}{\partial x_1} - i h \frac{\partial}{\partial x_2},
\]

\[
\mathcal{H}_2 = -h^2 \frac{\partial^2}{\partial y_1^2} - h^2 \frac{\partial^2}{\partial y_2^2} + e^{x_1 - x_2}.
\]

Let us make the change of variables \( y_i = h^{-1} x_i \) to obtain

\[
\mathcal{H}_2 = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} + e^{y_2 - y_1}.
\]

Elementary analogs of the quantum Hamiltonians obtained by taking the limit \( h \to \infty \) are given by

\[
\mathcal{H}_1^{(0)} = \mathcal{H}_1 = -i \frac{\partial}{\partial y_2} - i \frac{\partial}{\partial y_2},
\]

(3.16)

\[
\mathcal{H}_2^{(0)} = \lim_{h \to \infty} \mathcal{H}_2 = -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} + \Theta_\infty(y_2 - y_1),
\]

(3.17)

where

\[
\Theta_\infty(y) = 0, \quad y > 0, \quad \Theta_\infty(y) = \infty, \quad y < 0.
\]

One can explicitly check that the elementary Whittaker function

\[
\psi_{\lambda_1, \lambda_2}(y_1, y_2) = \frac{e^{i(\lambda_1 y_1 + \lambda_2 y_2)} - e^{i(\lambda_2 y_1 + \lambda_1 y_2)}}{i(\lambda_1 - \lambda_2)} \Theta(y_2 - y_1),
\]

(3.18)

is an eigenfunction of (3.16) and (3.17). Indeed, we have

\[
\left( -\frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial y_2^2} \right) \psi_{\lambda_1, \lambda_2}(y_1, y_2) = (\lambda_1^2 + \lambda_2^2) \psi_{\lambda_1, \lambda_2}(y_1, y_2) + e^{i(\lambda_1 y_1 + \lambda_2 y_2)} \delta(y_1 - y_2).
\]

The function \( \Theta_\infty(y_2 - y_1) \psi_{\lambda_1, \lambda_2}(y_1, y_2) \) is zero for \( y_1 \neq y_2 \) and is infinite at \( y_1 = y_2 \). The precise character of the infinity is fixed by the limiting procedure \( h \to \infty \) and leads to an identification

\[
\Theta_\infty(y_2 - y_1) \frac{e^{i(\lambda_1 y_1 + \lambda_2 y_2)} - e^{i(\lambda_2 y_1 + \lambda_1 y_2)}}{i(\lambda_1 - \lambda_2)} \Theta(y_2 - y_1) = e^{i(\lambda_1 + \lambda_2)(y_1 + y_2)} \delta(y_1 - y_2).
\]

Thus, with this definition of the product \( \Theta_\infty(y_2 - y_1) \psi_{\lambda_1, \lambda_2}(y_1, y_2) \), we obtain that the limit \( h \to \infty \) of \( \mathfrak{gl}_2 \)-Toda chain is indeed given by the quantum \((\mathbb{R}^2, \mathcal{S}_2)\)-billiard. These considerations can
be straightforwardly generalized to the case of an arbitrary rank. We however prefer just to define elementary analogs of $\mathfrak{gl}_{\ell+1}$-Toda chain as quantum $(\mathbb{R}^{\ell+1}, \mathcal{S}_{\ell+1})$-billiard to avoid ill-defined manipulations with $\Theta_\infty(x)$.

Finally we define elementary analogs of the local Archimedean $L$-factors (2.12) and the Baxter operators (2.8).

**Definition 3.3** Let $V = \mathbb{C}^{\ell+1}$ be supplied with the standard action of $U_{\ell+1}$. The elementary local $L$-factor associated with $V$ and with an endomorphism $\Lambda \in \text{End}(V)$ given by a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_{\ell+1})$ is defined as follows:

$$L_V^{(0)}(s, \Lambda) = \prod_{j=1}^{\ell+1} \frac{1}{s - \lambda_j},$$

(3.19)

where $\Lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$, and $s \in \mathbb{C}$.

**Proposition 3.6** The following relation between (3.19) and (2.12) holds:

$$L_V^{(0)}(s, \Lambda) = \lim_{\hbar \to \infty} (\frac{\hbar}{i})^{(\ell+1)} L(s, \Lambda, \hbar).$$

**Proof.** Let $\text{Re} \; z > 0$, then

$$\lim_{\hbar \to \infty} \hbar^{-1} \hbar \star \Gamma \left( \frac{z}{\hbar} \right) = \int_{-\infty}^{\infty} dx \; e^{-x^2} \Theta(x) = \frac{1}{z},$$

and thus

$$\lim_{\hbar \to \infty} (\frac{\hbar}{i})^{(\ell+1)} L(s, \Lambda, \hbar) = i^{\ell+1} \int_{-\infty}^{\infty} \prod_{j=1}^{\ell} dt_j \prod_{j=1}^{\ell} e^{i t_j (s - \lambda_j)} \Theta(t_j) = \prod_{j=1}^{\ell+1} \frac{1}{s - \lambda_j},$$

provided $\text{Im}(s) < 0$.  

The eigenvalues of the Baxter integral operators (2.8) acting on the $\mathfrak{gl}_{\ell+1}$-Whittaker functions (2.4) are given by the local Archimedean $L$-factors. Similar relations hold for their elementary analogs. Consider the one-dimensional family $\mathfrak{gl}_{\ell+1} Q^{(0)}(s)$ of integral operators acting in an appropriate space of functions of $\ell + 1$ variables and having the integral kernel

$$\mathfrak{gl}_{\ell+1} Q^{(0)}(\underline{x}, \underline{y}) s = e^{i s \sum_{i=1}^{\ell+1} (x_i - y_i)} \Theta(y_{\ell+1} - x_{\ell+1}) \prod_{i=1}^{\ell} \Theta(y_i - x_i) \Theta(x_i - y_{i+1}),$$

(3.20)

where we assume $\underline{x} := (x_1, \cdots, x_{\ell+1})$ and $\underline{y} := (y_1, \cdots, y_{\ell+1})$.

**Proposition 3.7** The following identity holds:

$$\mathfrak{gl}_{\ell+1} Q^{(0)}(s) = \lim_{\hbar \to \infty} Q^{\mathfrak{gl}_{\ell+1}}(s).$$

(3.21)

**Proof.** By changing variables $\underline{x} \to h\underline{x}$, $\underline{y} \to h\underline{y}$ the proof reduces to straightforward application of the identity (3.7).  

$\blacksquare$
Corollary 3.2 The elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function has the following eigenfunction property:

$$
q^{(\ell+1)} \int_{\mathbb{R}^{\ell+1}} \prod_{i=1}^{\ell+1} dy_i \, q^{(0)}(x, y; s, \lambda) \, \mathfrak{gl}_{\ell+1} \Psi^{(0)}(y) = L^{(0)}_v(s, \lambda, h) \, q^{(0)}(x, \lambda),
$$

where $x = (x_1, \ldots, x_{\ell+1})$, $y = (y_1, \ldots, y_{\ell+1})$, $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$ and the eigenvalue is equal to the local Archimedean $L$-factor

$$
L^{(0)}(s, \lambda, h) = \prod_{j=1}^{\ell+1} \frac{1}{s - \lambda_j}.
$$

4 A limit of $q$-deformed Whittaker functions

In [GLO3], [GLO4], [GLO5] explicit expressions for $q$-deformation of the $\mathfrak{gl}_{\ell+1}$-Whittaker functions were proposed. The $q$-deformed Whittaker functions are common eigenfunctions of $q$-deformed Toda chains (also known as relativistic Toda chains) [R], [Et] and the standard Whittaker functions arise in the limit $q \to 1$. In the previous Section we defined the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions as a limit of classical $\mathfrak{gl}_{\ell+1}$-Whittaker functions. In this Section we demonstrate that the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function can be obtained directly from the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\ell}(n)$ by specializing at $q = 0$ and taking a limit with respect to spectral variables $z = (z_1, \ldots, z_{\ell+1})$. Similar relations hold between the limits $h \to \infty$ of classical local Archimedean $L$-factors/Baxter operators on the one hand, and the $q = 0$ specialization of $q$-deformed local $L$-factors (introduced in [GLO3], [GLO4], [GLO5])/$q$-deformed Baxter operators on the other hand.

Specialization $q = 0$ of the $q$-deformed class one $\mathfrak{gl}_{\ell+1}$-Whittaker function was already considered in [GLO3]. Under this specialization the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^{\ell}(n)$ is given by a character of a finite-dimensional irreducible representation of $\mathfrak{gl}_{\ell+1}$ corresponding to the partition $n_1 \geq \ldots \geq n_{\ell+1}$

$$
\Psi^{\ell=0}_{z_1, \ldots, z_{\ell+1}}(n_1, \ldots, n_{\ell+1}) = \operatorname{Tr} V_{n_1, \ldots, n_{\ell+1}} z_1^{H_1} \cdots z_{\ell+1}^{H_{\ell+1}},
$$

and is equal to zero for $(n_1, \ldots, n_{\ell+1})$ outside the principal domain $n_1 \geq \ldots \geq n_{\ell+1}$. Using the Weyl character formula (see e.g. [Zh]) we have

$$
\Psi^{\ell=0}_{z_1, \ldots, z_{\ell+1}}(n_1, \ldots, n_{\ell+1}) = \frac{1}{\prod_{1 \leq j < k} (z_k - z_j)} \sum_{s \in \mathcal{S}_{\ell+1}} (-1)^l(s) \prod_{j=1}^{\ell+1} z_j^{n_s(j)+\ell+1-s(j)},
$$

where $\mathcal{S}_{\ell+1}$ is the permutation group identified with the Weyl group of $\mathfrak{gl}_{\ell+1}$ and for an element $s \in \mathcal{S}_{\ell+1}$, $l(s)$ is the length of the minimal product decomposition of $s$. There exists another representation for the characters of irreducible finite-dimensional representations of $\mathfrak{gl}_{\ell+1}$ based on Gelfand-Zetlin bases in finite-dimensional irreducible representations [GZ] (see also [Zh]). Let $\hat{p}^{\ell+1}$ be a set of Gelfand-Zetlin patterns, that is a set of collections $\hat{p} = \{ p_{i,j} \}$, $i = 1, \ldots, \ell + 1$, $j = 1, \ldots, i$ of integers satisfying the conditions $p_{i+1,j} \geq p_{i,j} \geq p_{i+1,j+1}$. An irreducible finite-dimensional representation can be realized in a vector space with the basis $v_{\hat{p}}$ enumerated by the Gelfand-Zetlin patterns $\hat{p}$ with fixed $p_{\ell+1,i}$, $i = 1, \ldots, \ell + 1$. Action of the Cartan generators on $v_{\hat{p}}$ is then given by

$$
z_1^{H_1} \cdots z_{\ell+1}^{H_{\ell+1}} v_{\hat{p}} = \frac{s_1^{s_1} \cdots s_{\ell+1}^{s_{\ell+1}}}{s_{\ell+1}} v_{\hat{p}}, \quad s_k = \sum_{i=1}^{k} p_{ki}.
$$
Let \( x \) be the integer parts of \( x_j / \log t \). In the limit \( t \to 1 \) the function (4.1) reduces to the elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker function (3.1)

\[
\Psi_{\lambda_1, \ldots, \lambda_{\ell+1}}^{(0)}(x_1, \ldots, x_{\ell+1}) = \lim_{t \to 1} \frac{(\log t)^{\ell(\ell+1)/2}}{x_1^{\lambda_1} \cdots x_{\ell+1}^{\lambda_{\ell+1}}} \Psi_{\lambda_1, \ldots, \lambda_{\ell+1}}^{(0)}(n_1(t, x_1), \ldots, n_{\ell+1}(t, x_{\ell+1})).
\]
Proof. Taking the limit of the recursive relations (4.6) we obtain the recursive relations (3.4).

Similar relations hold between $q$-deformed local $L$-factors (introduced in [GLO3], [GLO4], [GLO5]) and elementary local $L$-factors (3.23). Recall that the $q$-deformed local $L$-factor is given by

$$L^q(u|z_1, \ldots, z_{\ell+1}) = \prod_{j=1}^{\ell+1} \prod_{n=0}^{\infty} \frac{1}{1 - u^{-1} z_j q^n}, \quad q < 1.$$ (4.8)

Taking $q = 0$ we have

$$L^{q=0}(u|z_1, \ldots, z_{\ell+1}) = \prod_{j=1}^{\ell+1} \frac{1}{1 - u^{-1} z_j}.$$ (4.9)

Now let $z_i = t^{-\lambda_i}$, $u = t^{-s}$. Taking the limit $t \to 1$ we recover the elementary local $L$-factor

$$L^{(0)}(u|\Delta) = \lim_{t \to 1} (\log t)^{\ell+1} \prod_{j=1}^{\ell+1} \frac{1}{1 - t^{-s-\lambda_j}} = \prod_{j=1}^{\ell+1} \frac{1}{s - \lambda_j}.$$ (4.10)

Local Archimedean $L$-factors are eigenvalues of the Baxter integral operators acting on the Whittaker functions [GLO2]. Similar relations hold for their $q$-deformations. Next we give explicit formulas for $q = 0$ specialization of the $q$-deformed Baxter operators.

**Proposition 4.2** The $q = 0$ specialization (4.1) of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is a common eigenfunction of the one-parameter family of operators

$$Q^{q=0}(u) \cdot f(n_1, \ldots, n_{\ell+1}) = \sum_{m_1, \ldots, m_{\ell+1} \in \mathbb{Z}} Q^{q=0}(u|n_1, \ldots, n_{\ell+1}; m_1, \ldots, m_{\ell+1}) f(m_1, \ldots, m_{\ell+1}),$$ (4.11)

where the kernel is given by

$$Q^{q=0}(u|n_1, \ldots, n_{\ell+1}; m_1, \ldots, m_{\ell+1}) = \sum_{\sum_{i=1}^{\ell+1} (n_i - m_i) \left\{ (m_{\ell+1} - n_{\ell+1}) \prod_{i=1}^{\ell} \Theta(m_i - n_i) \Theta(n_i - m_{i+1}) \right\}.$$ (4.12)

The corresponding eigenvalues are given by local $L$-factors (4.9)

$$Q^{q=0}(u) \cdot \Psi^{q=0}_{z_1, \ldots, z_{\ell+1}}(n_1, \ldots, n_{\ell+1}) = L^{q=0}(u|z_1, \ldots, z_{\ell+1}) \Psi^{q=0}_{z_1, \ldots, z_{\ell+1}}(n_1, \ldots, n_{\ell+1}).$$ (4.13)

Proof. Recall that $\Psi^{q=0}_{z_1, \ldots, z_{\ell+1}}(n_1, \ldots, n_{\ell+1})$ can be interpreted as characters of a finite-dimensional irreducible representation of $\mathfrak{gl}_{\ell+1}$. The local $L$-factor (4.9) can be considered as a character of an infinite-dimensional representation of $\mathfrak{gl}_{\ell+1}$

$$L^{q=0}(u|z_1, \ldots, z_{\ell+1}) = \prod_{j=1}^{\ell+1} \frac{1}{1 - u^{-1} z_j} = \sum_{(k_1, \ldots, k_{\ell+1}) \in \mathbb{Z}_{\ell+1}^+} u^{-(k_1+\ldots+k_{\ell+1})} z_1^{k_1} \cdots z_{\ell+1}^{k_{\ell+1}} = \sum_{(k_1, \ldots, k_{\ell+1}) \in \mathbb{Z}_{\ell+1}^+} u^{-(k_1+\ldots+k_{\ell+1})} z_1^{k_1} \cdots z_{\ell+1}^{k_{\ell+1}}.$$
\[ V_0 = \mathbb{C}[\xi_1, \ldots, \xi_{\ell+1}] \text{ and } H_j = \xi_j \frac{\partial}{\partial \xi_j}. \] Now (4.13) is derived by applying the standard Littlewood-Richardson rules for decomposition of a tensor product of representations (see e.g. [Fu], [Zh])

\[
\chi_{r,0,\ldots,0}(\tilde{z})\chi_{n_1,\ldots,n_{\ell+1}}(\tilde{z}) = \sum_{I_r} \chi_{\mathbb{R} + \nu_{I_r}}(\tilde{z}),
\]

where \( I_r = (i_1 < i_2 < \ldots < i_r) \subseteq \{1, 2, \ldots, \ell + 1\} \) and \( \nu_j = 1 \) or \( \nu_j = 0 \) depending on whether or not \( j \) is in \( I_r \). We also discard all terms on the right hand side for which \( n + \nu_{I_r} \) is not in the principal domain. Now the statement of the Proposition follows from the decomposition

\[
L_{q=0}(t|z_1, \ldots, z_{\ell+1}) = \sum_{k=0}^{\infty} t^{-k} \sum_{k_1 + \ldots + k_{\ell+1} = k} z_1^{k_1} \ldots z_{\ell+1}^{k_{\ell+1}} = \sum_{k=0}^{\infty} t^{-k} \chi_{k,0,\ldots,0}(z_1, \ldots, z_{\ell+1}).
\]

\[ \square \]

Let us consider in detail the eigenfunction equation (4.13) for \( \ell = 0, 1 \). We start with \( \ell = 0 \). Irreducible representations of \( U_1 \) are one-dimensional

\[ e^{i\theta} : V_n \rightarrow e^{in\theta} V_n. \]

Let \( H \) be a generator of \( \text{Lie}(U_1) \). The \( q = 0 \) specialization of the \( q \)-deformed \( \mathfrak{gl}_1 \)-Whittaker function is given by the character

\[ \Psi_t^{q=0}(n) = \text{Tr}_{V_n} t^H = t^n, \quad t \in \mathbb{C}^*, \quad n \in \mathbb{Z}. \]

The local Archimedean \( L \)-factor has the following trace representation:

\[ L_{q=0}(u|t) = \frac{1}{1 - u^{-1}t} = \sum_{n=0}^{\infty} (t/u)^n = \text{Tr}_{\mathbb{C}[z]} (t/u)^H = \sum_{n=0}^{\infty} u^{-n} \Psi_t^{q=0}(n), \]

where \( H \) acts in \( \mathbb{C}[z] \) as \( H = z\partial_z \). We have the following decomposition of the product of characters of \( V_n \) and \( H \):

\[ \text{Tr}_{V_n} t^H \times \text{Tr}_{\mathbb{C}[z]} (u^{-1}t)^H = \sum_{m=0}^{\infty} \text{Tr}_{V_{m+n}} t^H u^{-m} t^H = \sum_{m \in \mathbb{Z}} \Theta(m-n) u^{-m} \text{Tr}_{V_m} t^H. \]

This can be rewritten as the action of an integral operator on the Whittaker function \( \Psi_t^{q=0}(n) = t^n \)

\[ Q(u) \cdot \Psi_t^{q=0}(n) = L(u|t) \Psi_t^{q=0}(n), \]

where \( Q(u) \) acts on functions on \( \mathbb{Z} \) as follows:

\[ Q(u) \cdot f(n) = \sum_{m \in \mathbb{Z}} \Theta(m-n) u^{-m} f(m). \]

Now consider the case \( \ell = 1 \). Let \( V_{n_1,n_2} \) be the finite-dimensional irreducible representation of \( \mathfrak{gl}_2 \) corresponding to a partition \((n_1,n_2), n_1 \geq n_2\). Let \( H_i, i = 1, 2 \) be generators of the diagonal
Cartan subalgebra. The $q = 0$ specialization of the $q$-deformed $\mathfrak{gl}_2$-Whittaker function is expressed via characters of irreducible finite-dimensional representations as

$$\Psi_{t_1, t_2}^{q=0}(n_1, n_2) = \text{Tr}_{V_{n_1, n_2}} t_1^{H_1} t_2^{H_2} = (t_1 t_2)^{n_2} \left( \sum_{m_1 + m_2 = n_1 - n_2} t_1^{m_1} t_2^{m_2} \right) = \frac{t_1^{n_1 + 1} t_2^{n_2} - t_1^{n_2} t_2^{n_1 + 1}}{t_1 - t_2}. $$

The local $L$-factor has the following representation:

$$L^{q=0}(u|t_1, t_2) = \frac{1}{(1 - u^{-1} t_1)(1 - u^{-1} t_2)} = \text{Tr}_C[z_1, z_2](u^{-1} t_1) H_1 (u^{-1} t_2) H_2 = \sum_{r=0}^{\infty} u^{-r} \text{Tr}_{V_r} t_1^{H_1} t_2^{H_2},$$

where $H_j$, $j = 1, 2$ act in $C[z_1, z_2]$ as $H_j = z_j \partial_{z_j}$. The following identity (a particular instance of the Richardson-Littlewood rule) holds:

$$\chi_{(r_1, r_2)} \cdot \chi_{(k, 0)} = \sum_{I, k \in \mathbb{Z}^2} \chi_{p_1, p_2},$$

where $\chi_{n_1, n_2}(t_1, t_2) := \text{Tr}_{V_{n_1, n_2}} t_1^{H_1} t_2^{H_2}$ and the sum goes over the subset $I, k \in \mathbb{Z}^2$ for which

$$p_2 + p_1 = r_1 + r_2 + k, \quad p_1 \geq r_1 \quad p_2 \geq r_2, \quad r_1 \geq p_2.$$ 

This can be rewritten as follows:

$$\chi_{n_1, n_2}(t_1, t_2) \chi_{S^r \mathbb{C}^2}(u^{-1} t_1, u^{-1} t_2) = \sum_{m_1, m_2 \in \mathbb{Z}^2} \Theta(m_1 - n_1) \Theta(n_1 - m_2) \Theta(m_2 - n_2) u^{n_1 + n_2 - m_1 - m_2} \chi_{m_1, m_2}(z_1, z_2).$$

Thus we recover a special case of (4.13) with the kernel of the Baxter operator given by

$$Q(u|n_1, n_2; m_1, m_2) = \Theta(m_1 - n_1) \Theta(n_1 - m_2) \Theta(m_2 - n_2) u^{n_1 + n_2 - m_1 - m_2}.$$ 

Let us finally note that the elementary Baxter operator (3.6) can be obtained as a limit of its $q = 0$ counterpart.

**Proposition 4.3** Let $u = t^\lambda$ and let $n_i(t, x_i)$, $m_j(t, y_j)$ be the corresponding integer parts of $x_j/\log t$, $y_j/\log t$ respectively. Then the following identity holds:

$$Q^{(0)}(x_1, \ldots x_{\ell+1}; y_1, \ldots y_{\ell+1}|\lambda) = \lim_{t \to 1} Q^{q=0}(u|n_1(t, x_1), \ldots n_{\ell+1}(t, x_{\ell+1}); m_1(t, y_1), \ldots, m_{\ell+1}(t, y_{\ell+1})).$$

**Proof.** The proof is straightforward. □

### 5 Elementary special functions as symplectic volumes

In this Section we represent the elementary special functions defined in the previous Section as equivariant volumes of finite-dimensional symplectic spaces. As was already noticed at the end of Section 2 this representation is not surprising, given the existence of a similar representation of classical Whittaker functions and local $L$-factors as equivariant volumes of infinite-dimensional
symplectic spaces of holomorphic maps of a two-dimensional disk into finite-dimensional symplectic spaces [GLO6], [GLO7], [GLO8]. In the limit when the $S^1$-equivariance parameter $h$ corresponding to disk rotations goes to infinity, the corresponding equivariant volume of the space of holomorphic maps of the disk into $X$ tends to the equivariant volume of $X$ given by an elementary analog of the corresponding special function. In this Section we demonstrate directly that $U_{\ell+1}$-equivariant volumes of flag spaces $B_{\ell+1} = GL_{\ell+1}(\mathbb{C})/B$ are equal to elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions 
(2.4). The same relation also holds for $(\ell + 1, 1)$-Whittaker functions and local $L$-factors. Thus, for elementary analogs, we prove the relation between symplectic volumes and special functions for a more general case than was done in [GLO8]. In this Section we also provide a reformulation of the eigenfunction property (2.11) of the elementary Whittaker function in terms of symplectic geometry.

Let $B_{\ell+1} = GL_{\ell+1}(\mathbb{C})/B$ be a flag space of $GL_{\ell+1}$. It can be identified with the factor $U_{\ell+1}/H$ of the unitary group $U_{\ell+1}$ over the Cartan torus $U_{1}^{\ell+1}$ of $U_{\ell+1}$. The space $U_{\ell+1}/H$ on the other hand is obviously a coadjoint orbit $\mathcal{O}_{u_{0}}$ of a regular element $u_{0} \in u_{\ell+1}^{*}$. This interpretation provides a family of Kirillov-Kostant symplectic structures on the $B_{\ell+1}$ parameterized by positive cone in the dual to the Cartan subalgebra $\mathfrak{h} \in u_{\ell+1}$ (see e.g. [K]). In the following we identify both $u_{\ell+1}^{*}$ with $u_{\ell+1}$ as well as $\mathfrak{h}^{*}$ with $\mathfrak{h}$ via the Killing quadratic form on $u_{\ell+1}$.

Let $u_{0} = \iota \text{diag}(x_{1}, \ldots, x_{\ell+1}) \in u_{\ell+1}^{*}$ with $x_{1} > x_{2} > \ldots > x_{\ell+1}$. The Kirillov-Kostant symplectic form $\omega$ on an open part of the coadjoint orbit $\mathcal{O}_{u_{0}}$ can be written explicitly as follows. Consider the closed two-form on $U_{\ell+1}$

$$\omega^{(0)}_{u_{0}} = \delta \text{Tr} u_{0} \delta g g^{-1}, \quad g \in U_{\ell+1},$$

where trace is taken in the standard representation $u_{\ell+1} \rightarrow \text{End}(\mathbb{C}^{\ell+1})$. This two-form is a lift of the Kirillov-Kostant closed non-degenerate two-form $\omega_{u_{0}}$ on $\mathcal{O}_{u_{0}}$ along a projection $U_{\ell+1} \rightarrow \mathcal{O}_{u_{0}}$, such that $g \rightarrow u = g^{-1}u_{0}g$. The action of the group $U_{\ell+1}$ on $(\mathcal{O}_{u_{0}}, \omega_{u_{0}})$ is Hamiltonian and the corresponding momentum map is given by

$$H(g, u_{0}) := u(g, u_{0}) = g^{-1}u_{0}g. \quad (5.1)$$

**Proposition 5.1** The following representation for the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (3.1) holds:

$$\psi^{(0)}_{\lambda_{1}, \ldots, \lambda_{\ell+1}}(x_{1}, \ldots, x_{\ell+1}) = \int_{B_{\ell+1}} e^{\omega_{u_{0}} + \sum_{j=1}^{\ell+1} \lambda_{j} H_{jj}} e^{\sum_{j=1}^{\ell+1} \lambda_{j} x_{j}}, \quad u_{0} = \iota \text{diag}(x_{1}, \ldots, x_{\ell+1}), \quad (5.2)$$

where $H_{jj}$ are diagonal components of $H(u_{0}, g)$.

**Proof.** The integral (5.2) can be calculated using the Harish-Chandra formula for orbit integrals

$$I(u_{0}, \Lambda) = \int_{g \in U_{\ell+1}/H} e^{\text{Tr}(g^{-1}u_{0}g\Lambda)} \text{vol}_{U_{\ell+1}/H}(g) = \Delta^{-1}(x) \Delta^{-1}(t\lambda) \det||e^{x_{j}\lambda_{j}}|| = \quad (5.3)$$

$$= \Delta^{-1}(x) \Delta^{-1}(t\lambda) \sum_{s \in \mathcal{E}_{\ell+1}} (-1)^{l(s)} e^{\sum_{j} x_{j}(s)\lambda_{j}},$$

where $\Lambda = \text{diag}(\lambda_{1}, \ldots, \lambda_{\ell+1})$ with $\lambda_{1} > \cdots > \lambda_{\ell+1}$ and the measure $\text{vol}_{U_{\ell+1}/H}$ is the canonical volume form on the factor $U_{\ell+1}/H$ induced by the Killing form on $u_{\ell+1}$. Here we also denote $\Delta(z) = \prod_{1 \leq i < j \leq \ell+1} (z_{i} - z_{j})$. Thus, taking into account (5.1), to derive (5.2) from (5.3) one should
compare $\mathrm{vol}_{U_{\ell+1}/H}$ with the Liouville measure $\omega_{u_0}^d/d!$ where $d = \frac{1}{2} \dim(U_{\ell+1}/H)$. Both $\mathrm{vol}_{U_{\ell+1}/H}$ and $\omega_{u_0}^{\dim(U_{\ell+1}/H)/2}$ are $U_{\ell+1}$-invariant top-dimensional forms on $B$ and thus it is enough to compare these measures near the projection of the unit element $e \in U_{\ell+1}$. Simple calculation gives

$$\omega_{u_0}^d = d! \Delta(x) \mathrm{vol}_{U_{\ell+1}/H},$$

Thus we obtain

$$\int_B e^{\omega_{u_0} + \sum_{j=1}^{\ell+1} \lambda_j H_{jj}(u_0)} = \Delta^{-1}(t\lambda) \sum_{s \in \Sigma_{\ell+1}} (-1)^{l(s)} e^{\sum_{j} t x_{s(j)} \lambda_j}. $$

Taking into account (3.2) we obtain (5.2). \(\square\)

The identity (5.2) can also be proved explicitly using a Gelfand-Zetlin type parametrization on an open part of the regular coadjoint orbit $O_{u_0}$ of $U_{\ell+1}$ [AFS]. Recall that an open part of the regular coadjoint orbit $O_{u_0}$ allows a parametrization in Darboux coordinates $\{T_{ij}, \theta_{ij}\}, 1 \leq j \leq i < \ell + 1$ so that the symplectic form $\omega_{u_0}$ is given by

$$\omega_{u_0} = \sum_{i \geq j} \delta T_{ij} \wedge \delta \theta_{ij}. $$

Here $\theta_{ij}$ are periodic coordinates $\theta_{ij} \sim \theta_{ij} + 1$ and $T_{ij} \in \mathbb{R}, 1 \leq j \leq i < \ell + 1$ satisfy the Gelfand-Zetlin conditions

$$T_{i+1,j} \geq T_{i,j} \geq T_{i+1,j+1}, \quad 1 \leq j \leq i \leq \ell + 1, $$

and we identify $T_{\ell+1,j} := x_j, j = 1, \ldots, \ell + 1$. Thus the image of an open part of $O_{u_0}$ under the projection along an $\ell(\ell + 1)/2$-dimensional torus parameterized by $\theta_{ij} \in \mathbb{R}, 1 \leq j \leq i < \ell + 1$ is a convex Gelfand-Zetlin polytope $P_{\ell+1}$ in $\mathbb{R}^{\ell(\ell+1)/2}$ defined by the conditions (5.4). In coordinates $(T, \theta)$ the components of the momentum map of the action of the diagonal Lie subalgebra $u_1^{\ell+1} \subset u_{\ell+1}$ are given by

$$H_{jj} = \sum_{i=1}^{j} i T_{ji} - \sum_{i=1}^{j-1} i T_{j-1,i}. $$

Thus the integral, after integration over $\theta_{ij}$, can be written in the following form:

$$\int_{P_{\ell+1}} e^{\omega_{u_0} + \sum_{j=1}^{\ell+1} \lambda_j H_{jj}} = \int_{P_{\ell+1}} e^{\sum_{ij} \lambda_j (T_{ij} - T_{i-1,j})} \prod_{1 \leq j \leq \ell} dT_{ij}, $$

where $T_{ij} := 0$ for $i < j$. This integral representation coincides with the Givental type integral representation (3.8) and thus we again recover the identity (5.2).

Remark 5.1 The appearance of the Gelfand-Zetlin polytope $P_{\ell+1}$ in the Givental type integral representation (3.8) is related with a deep duality relation between the Gelfand-Zetlin and the Givental realizations of representations of $U \mathfrak{gl}_{\ell+1}$ discussed in [GLO2].

The relation (5.2) can also be understood as a limit of the identification of the $q = 0$ $\mathfrak{gl}_{\ell+1}$-Whittaker functions with the characters of irreducible representations of $\mathfrak{gl}_{\ell+1}$. Indeed, according to the Kirillov philosophy, unitary irreducible representations of a Lie group $G$ can be obtained by quantization of coadjoint orbits of $G$ [K]. In the other direction, a classical limit of the character of
an irreducible representation is given by a $G$-equivariant symplectic volume of the corresponding coadjoint orbit. The precise relation in our case is as follows. The orbit integral (5.3) is equal to a limit of the character (4.4) of an irreducible finite-dimensional representation of $\mathfrak{gl}_{\ell+1}$

$$
\int_{S_{\ell+1}} e^{\omega(u_0)} \sum_{j=1}^{\ell+1} \chi_{e^{i\lambda_j}} = \Delta(x) \lim_{\ell \to 0} \frac{\chi_{e^{i\lambda_1}, \ldots, e^{i\lambda_{\ell+1}}}}{\chi_{e^{i\lambda_1}, \ldots, e^{i\lambda_{\ell+1}}}}.
$$

where

$$
\chi_{e^{i\lambda_1}, \ldots, e^{i\lambda_{\ell+1}}} = \prod_{i<j} (-1)^{l(i)} \prod_{j=1}^{\ell+1} e^{i(x_{s(j)} + (\ell+1-s)(j)) \lambda_j}.
$$

A similar interpretation holds for the elementary $(\ell+1,1)$-Whittaker functions (3.10). Note that these functions can be obtained either by specialization of the previous formulas or as equivariant symplectic volumes of the partial flag space $GL_{\ell+1}/P_{\ell+1} = \mathbb{P}^\ell$.

**Proposition 5.2** The elementary $(\ell+1,1)$-Whittaker function associated with the partial flag space $\mathbb{P}^\ell$ has the following integral representations:

$$
W_{\lambda_1, \ldots, \lambda_{\ell+1}}(x) = \int_{R_{\ell+1}} \prod_{j=1}^{\ell+1} dt_j e^{it_j \lambda_j} \delta(\sum_{j=1}^{\ell+1} t_j - x) = \int_{\Delta_\ell(x)} \prod_{j=1}^{\ell+1} dt_j e^{it_j \lambda_j},
$$

where $\Delta_\ell(x)$ is a simplex defined by the equation $\sum_{j=1}^{\ell+1} t_j = x$ in $R_{\ell+1}$.

**Proof.** This representation can be derived straightforwardly from (3.10) taking into account $\int_0^\infty dt e^{-at} = a^{-1}$, $a > 0$. □

The simplex $\Delta_\ell$ in (5.6) can be understood as an image of the projective space $\mathbb{P}^\ell$ under the $U_1$-momentum map. In this respect it is an analog of the Gelfand-Zetlin polytope $\mathcal{P}_{\ell+1}$ in (5.5). Indeed the Darboux coordinates $(T, \theta)$ on an open part $O_{u_0}$ of the coadjoint orbit $O_{u_0}$ define the Hamiltonian action of the torus $T^{\ell+1}/2$ acting by rotation on $\theta_{ij}$. The momentum map for this action maps $O_{u_0}$ onto the Gelfand-Zetlin polytope $\mathcal{P}_{\ell+1}$.

Now we provide a similar interpretation of the elementary local $L$-factors (3.19) as equivariant symplectic volumes of non-compact symplectic spaces. Let us equip the vector space $V = \mathbb{C}^{\ell+1}$ with the standard symplectic structure

$$
\omega = \frac{i}{2} \sum_{j=1}^{\ell+1} dz^j \wedge d\bar{z}^j,
$$

where $(z^1, \ldots, z^{\ell+1})$ are complex linear coordinates on $V$. The standard action of $U_{\ell+1}$ on $V$ is Hamiltonian. Explicitly the action of the diagonal subgroup $U_1^{\ell+1} \subset U_{\ell+1}$ on $V$ is generated by vector fields

$$
v_j = \left( \bar{z}_j \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial \bar{z}_j} \right).
$$

The corresponding momenta $H_j$ (i.e. solutions of equations $\iota_{v_j} \omega = dH_j$) are given by

$$
H_j = \frac{i}{2} |\bar{z}^j|^2, \quad j = 1, \ldots, (\ell+1).
$$
The $U_1^{\ell+1}$-equivariant volume of $V$ is defined as the integral

$$Z(\lambda_1, \ldots, \lambda_{\ell+1}) = \frac{1}{(2\pi)^{\ell+1}} \int_{C^{\ell+1}} e^{\omega+\sum_{j=1}^{\ell+1} H_j \lambda_j} = \frac{1}{(2\pi)^{\ell+1}} \int_{C^{\ell+1}} e^{\sum_{j=1}^{\ell+1} H_j \lambda_j}. \quad (5.7)$$

**Lemma 5.1** The elementary local $L$-factor associated with the vector space $V = \mathbb{C}^{\ell+1}$ is expressed through the equivariant volume (5.7) as follows:

$$L_V^{(0)}(s) = Z(\lambda_1 - s, \ldots, \lambda_{\ell+1} - s).$$

**Proof.** Direct calculation of the Gaussian integral (5.2). $\square$

Finally let us give an interpretation of the eigenfunction property (3.22) of the elementary Whittaker function with respect to the action of the elementary Baxter operator. We consider the simplest cases $\ell = 0, 1$ leaving the general case for another occasion. Let $(M, \omega, G, \mu)$ be a quantizable $G$-symplectic space $M$ with a symplectic form $\omega$ and a fixed momentum map $\mu : M \to \mathfrak{g}^*$ where $\mathfrak{g}^*$ is dual to $\mathfrak{g} = \text{Lie}(G)$. Here quantizable means that one can naturally associate with $(M, \omega, G, \mu)$ a unitary $G$-module $V_M$. In general the representation $V_M$ associated with the quantizable symplectic manifold $(M, \omega, G, \mu)$ has a non-trivial decomposition on irreducible $G$-representation

$$V_M = \bigoplus_{\alpha \in \text{Rep}(G)} V_\alpha \otimes W_\alpha, \quad (5.8)$$

where $W_\alpha = \text{Hom}_G(V_\alpha, V_M) = V_\alpha^* \otimes_G V_M$ are multiplicity spaces. Let us recall a realization of this decomposition (5.8) via symplectic geometry of the underlying symplectic space $M$ (see e.g. [GLS]). Given quantizable symplectic spaces $(M_1, G, \omega_1, \mu_1)$, $(M_2, G, \omega_2, \mu_2)$ with corresponding unitary $G$-modules $V_{M_1}$ and $V_{M_2}$, the space of linear $G$-maps $\text{Hom}_G(V_{M_1}, V_{M_2})$ can be obtained by quantization of a symplectic space associated with $(M_1, G, \omega_1, \mu_1)$, $(M_2, G, \omega_2, \mu_2)$ as follows. Consider the space $(M_1 \times M_2, G, \omega_1 - \omega_2, \mu_1 - \mu_2)$ where $G$ acts diagonally on $M_1 \times M_2$. The vector space of $G$-maps $\text{Hom}_G(V_2, V_1)$ can be obtained by quantization of the Hamiltonian reduction of the space $(M_1 \times M_2, G, \omega_1 - \omega_2, \mu_1 - \mu_2)$ over zero values of the $G$-momentum map $\mu^{\text{tot}} = \mu_1 - \mu_2$. Let us denote the result of this reduction by $N(M_1, M_2)$.

According to Kirillov (see e.g. [K]) an irreducible representation $V_\lambda$ of a Lie group $G$ characterized by a weight $\lambda$ shall be associated with a coadjoint symplectic orbit $O_\lambda$ of an element $\lambda \in \mathfrak{g}^*$ supplied with the Kirillov-Kostant symplectic structure $\omega_\lambda$. Thus for the special case $M_2 = O_\lambda$ the reduced space $N(M, O_\lambda)$ is given by a factor of the zero momentum subset

$$\mu_M - \mu_O = 0, \quad (5.9)$$

over the diagonal action of $G$. The locus (5.9) can be identified with $M_0 = \mu_M^{-1}(O_\lambda)$ and thus the symplectic counterpart of the multiplicity space $\text{Hom}_G(V_\lambda, V_M)$ is obtained by Hamiltonian reduction of $M$ over the coadjoint orbit $O_\lambda$

$$N(M, O_\lambda) = M/\mu_M=\lambda G := \mu_M^{-1}(O_\lambda)/G. \quad (5.10)$$

The symplectic counterpart of the decomposition (5.8) is given by a realization of $M$ as a stratified symplectic bundle (see e.g. [GLS]) with base $\mathfrak{h}^*$, $\mathfrak{h} = \text{Lie}(H)$, and with fibres over $\lambda \in \mathfrak{h}^*$ being $H_\lambda \times O_\lambda \times (M/\mu=\lambda G)$. Here $H_\lambda \subset H$ is a subgroup of the Cartan subgroup $H \subset G$ depending on $\lambda$. Note that one can have $H_\lambda = \emptyset$. Let us now apply these considerations to the construction of symplectic counterparts to the Baxter operator for low ranks $\ell = 0, 1$. 

23
For $\ell = 0$ we consider two $U_1$-modules $V_n$ and $V_C = \mathbb{C}[z]$ such that the generator $H$ of Lie($U_1$) acts as

$$H|_{V_n} = n, \quad H|_{V_C} = z\partial z.$$  

The modules $V_n$ and $V_C$ can be obtained by quantization of the symplectic $U_1$-spaces ($pt, U_1, \omega = 0, \mu_n = n$) and $(\mathbb{C}, U_1, \omega_C, \mu = \frac{1}{2}|z|^2)$ where $\omega_C = \frac{i}{2}dz \wedge d\bar{z}$. The Hamiltonian action of $U_1$ is given by $e^{\theta z} : z \mapsto e^{\theta z}$.

Recall that the $q=0$ specialization (4.11) of the $q$-deformed Baxter operator is associated with the decomposition of the product $V_n \otimes H$ into irreducible representations

$$V_n \otimes V_C = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_{n+m}, \quad V_C = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m. \quad (5.11)$$

We would like to find a symplectic geometry counterpart of this decomposition. The multiplicity spaces are given by

$$Hom_{U_1}(V_m, V_n \otimes V_C) = \mathbb{C}, \quad m \geq n,$$

$$Hom_{U_1}(V_m, V_n \otimes V_C) = 0, \quad m < n.$$  

Using (5.10) we obtain for symplectic analogs $N_\lambda = \mathbb{C}/\mu=\lambda U(1)$ of the multiplicity spaces $Hom_{U_1}(V_m, V_n \otimes V_C)$,

$$N_{\lambda \geq 0} = pt, \quad N_{\lambda < 0} = \emptyset.$$  

The symplectic geometry analog of the decomposition (5.11) is a bundle over the stratified space $\mathbb{R} = \mathbb{R} < \lambda \cup \{\lambda\} \cup \mathbb{R} > \lambda$ such that the fibre over $\mathbb{R} < \lambda$ is the empty set, the fibre over $\{\lambda\}$ is a point and the fibre over $\mathbb{R} > \lambda$ is $S^1$. This indeed provides a model for $\mathbb{C}$ via momentum map projection $\mu_{U(1)} \mathbb{C} \to \mathbb{R}$. Classical counterpart of the decomposition (5.11) for the equivariant symplectic volume integral can thus be written in the following form:

$$\int_{\mathbb{C}} \tau^{\omega_C + \tau \mu_{U_1}} \times e^{-i\lambda \tau} = \int_{\mathbb{C}} \tau^{\omega_C + \tau \mu_{U_1} - i\lambda} = \int_{T^*S^1} d\theta dp \Theta(\rho - \lambda)e^{-ir\rho}.$$

Let us now consider the case of $\ell = 1$. We have two $U_2$-modules $V_{n_1, n_2}$ and $V_{C^2} = \mathbb{C}[z_1, z_2]$ such that diagonal generators $H_j$, $j = 1, 2$ of Lie($U_2$) act as $H_j|_{V_{n_1, n_2}} = n_j$, $H_j|_{V_{C^2}} = z_j\partial z_j$. The Hamiltonian action of $U_2$ on $\mathbb{C}^2$ is given by $g : z_i \mapsto \sum_j^2 g_{ij} z_j$ and the corresponding momenta are

$$(\mu_{C^2})_{ij} = \frac{i}{2}z_i \bar{z}_j,$$

with respect to the symplectic structure

$$\omega_{C^2} = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

The modules $V_{n_1, n_2}$ and $V_{C^2}$ can be obtained by quantization of the symplectic $U_2$-spaces ($\mathbb{P}^1, U_2, \omega_{s_1, s_2} = (s_1 - s_2)\omega_{FS}, \mu_{FS}$) and $(\mathbb{C}^2, U_2, \omega_{C^2}, \mu_{C^2})$, where $\omega_{FS}$ is the Fubini-Studi symplectic form on $\mathbb{P}^1$, and the momentum map in stereographic coordinates is

$$\mu_{FS} =� \frac{s_1 + s_2|z|^2}{(1 + |z|^2)}.$$  

The $q=0$ Baxter $Q$-operator is associated with the decomposition of the product $V_{n_1, n_2} \otimes V_{C^2}$ into irreducible $U_2$-representations. The Baxter eigenfunction equation (4.15) can be represented in the following form

$$V_{n_1, n_2} \otimes V_{C^2} = \bigoplus_{m_1 + m_2 = n_1 + n_2 + \tau} \Theta(m_1 - n_1)\Theta(n_1 - m_2)\Theta(m_2 - n_2) \ V_{m_1, m_2}. \quad (5.12)$$
The symplectic geometry counterpart of this decomposition is a representation of the product of a $U_2$ coadjoint orbit $O_{s_1,s_2}$, $s_1 > s_2$ and $\mathbb{C}^2$ as a bundle over a stratified space $u_1 \oplus u_1 = \mathbb{R}^2$ with generic non-empty fibre $S^1 \times S^1 \times O_{\lambda_1,\lambda_2} \times N(s_1,s_2|\lambda_1,\lambda_2)$, where $\lambda_1 > \lambda_2$. The symplectic space $N(s_1,s_2|\lambda_1,\lambda_2)$ is a symplectic counterpart of the multiplicity space $\text{Hom}_{U_2}(V_{m_1,m_2},V_{n_1,n_2} \otimes V_{\mathbb{C}^2})$.

We construct this representation by rewriting the $U_2$-equivariant symplectic volume integral

$$I_{O_{s_1,s_2} \times \mathbb{C}^2}(\xi;\tau_1,\tau_2) = \int_{O_{s_1,s_2} \times \mathbb{C}^2} \omega_{s_1,s_2} d^2 z_1 d^2 z_2 e^{\tau_1 \mu_{11} + \tau_2 \mu_{22} + \xi \mu}, \quad (5.13)$$

where $\mu = \mu_{\mathbb{C}^2} + \mu_{U_1}$ is a momentum map $O_{s_1,s_2} \times \mathbb{C}^2 \to u_1^*$, $\mu_* = (\mu_{\mathbb{C}^2})_{11} + (\mu_{U_1})_{22}$ and $\omega_{s_1,s_2}$ is the Kirillov symplectic form on $O_{s_1,s_2} = \mathbb{P}^1$. The integral (5.13) can be rewritten as follows:

$$I_{O_{s_1,s_2} \times \mathbb{C}^2}(\xi;\tau_1,\tau_2) = \int_{O_{s_1,s_2} \times \mathbb{C}^2 \times u_1^*} \omega_{s_1,s_2} d^2 z_1 d^2 z_2 d^4 u \prod_{i,j=1}^2 \delta(\mu_{ij} - u_{ij}) e^{\tau_1 \mu_{11} + \tau_2 \mu_{22} + \xi \mu}. $$

A generic element $u \in \mathfrak{g}^*$ can be represented as $u = g^{-1}u_0 g$, $g \in U_2/U_1^2$ and $u_0 = \delta \text{diag}(\lambda_1,\lambda_2)$ such that $\lambda_1 > \lambda_2$ via a projection $\mathfrak{g}^* \to \mathfrak{t}^*/W$. We have the following relation between the integration measures (c.f. the Weyl integration formula):

$$d^4 u = \frac{1}{2} \Delta^2(\lambda) \text{vol}_{U_2/U_1^2}(g) \wedge d\lambda_1 \wedge d\lambda_2, \quad \Delta(\lambda) = \prod_{1 \leq i < j \leq 2} (\lambda_i - \lambda_j),$$

where $\text{vol}_{U_2/U_1^2}$ is the induced volume form on the coadjoint orbit $U_2/U_1^2$. Thus we obtain

$$I_{O_{s_1,s_2} \times \mathbb{C}^2}(\xi;\tau_1,\tau_2) = \frac{1}{2} \int_{\mathbb{P}^1 \times \mathbb{R}^2} \text{vol}_{U_2/U_1^2}(g) \ w \lambda_1 \ w \lambda_2 \Delta(\lambda) J_{O_{s_1,s_2} \times \mathbb{C}^2}(g,\lambda_1,\lambda_2;\tau_1,\tau_2),$$

where

$$J_{O_{s_1,s_2} \times \mathbb{C}^2}(g,\lambda_1,\lambda_2;\xi;\tau_1,\tau_2) = \Delta(\lambda) \int_{O_{s_1,s_2} \times \mathbb{C}^2} \omega_{s_1,s_2} d^2 z_1 d^2 z_2 \prod_{i,j=1}^2 \delta(\mu_{ij} - u_{ij}) e^{\tau_1 \mu_{11} + \tau_2 \mu_{22} + \xi \mu} =$$

$$= e^{\tau_1 u_{11}(\mu_{0,0}) + \tau_2 u_{22}(\mu_{0,0})} \Delta(\lambda) \int_{O_{s_1,s_2} \times \mathbb{C}^2} \omega_{s_1,s_2} d^2 z_1 d^2 z_2 \prod_{i,j=1}^2 \delta(\mu_{ij} - \lambda_i \delta_{ij}) e^{\frac{\mu}{2}(|z_1|^2 + |z_2|^2)}. $$

In the last formula we use $SU(2)$-invariance of the integral. Thus we obtain

$$I_{O_{s_1,s_2} \times \mathbb{C}^2}(\xi;\tau_1,\tau_2) = \int_{\mathbb{R}^2} d\lambda_1 \ w \lambda_2 \ J(s_1,s_2;\lambda_1,\lambda_2) F(\lambda_1,\lambda_2;\tau_1,\tau_2), \quad (5.14)$$

where

$$J(s_1,s_2;\lambda_1,\lambda_2) = \frac{1}{2} \Delta(\lambda) \int_{O_{s_1,s_2} \times \mathbb{C}^2} \omega_{s_1,s_2} d^2 z_1 d^2 z_2 \prod_{i,j=1}^2 \delta(\mu_{ij} - \lambda_i \delta_{ij}) e^{\frac{\mu}{2}(|z_1|^2 + |z_2|^2)},$$

and

$$F(\lambda_1,\lambda_2;\tau_1,\tau_2) = \int_{O_{\lambda_1,\lambda_2}} \omega_{\lambda_1,\lambda_2} e^{\tau_1 u_{11} - \tau_2 u_{22}}, \quad (5.15)$$

where we use $\omega_{\lambda_1,\lambda_2} = \Delta(\lambda) \text{vol}_{U_2/U_1^2}$. The integral (5.15) is the $U_2$-equivariant symplectic volume of the coadjoint orbit $O_{\lambda_1,\lambda_2}$ and is classical counterpart of the character of the irreducible representation associated with the coadjoint orbit. On the other hand the function $J(s_1,s_2;\xi;\lambda_1,\lambda_2)$ shall be considered to be classical analog of the multiplicity function $\dim \text{Hom}_{U_2}(V_{m_1,m_2},V_{n_1,n_2} \otimes V_{\mathbb{C}^2})$.  

25
Proposition 5.3 One has the following expression for the integral:

\[
J(s_1, s_2; \xi; \lambda_1, \lambda_2) = \frac{1}{2} \Delta(\lambda) \int_{\mathcal{O}_{s_1, s_2} \times \mathbb{C}^2} \omega_{s_1, s_2} d^2 z_1 d^2 z_2 \prod_{i,j=1}^{2} \delta(\mu_{ij} - \lambda_i \delta_{ij}) e^{\frac{\lambda_i}{2} (|z_1|^2 + |z_2|^2)} = (5.16)
\]

\[
= e^{i \xi (s_1 + s_2 - \lambda_1 - \lambda_2)} \Theta(\lambda_1 - s_1) \Theta(s_1 - \lambda_2) \Theta(\lambda_2 - s_2),
\]

where \( \lambda_1 > \lambda_2 \) and \( s_1 > s_2 \).

\[\text{Proof.}\] The calculation for general \( s_1, s_2 \) can easily be reduced to the case \( s_2 = 0 \) and \( s_1 = s \). In this case the integral can be calculated using the representation of the integral over the orbit \( \mathcal{O} = \mathbb{P}^1 \) via the integral over \( \mathbb{C}^2 \)

\[
J(s_1, s_2; \xi; \lambda_1, \lambda_2) = \Delta(\lambda) \int_{\mathbb{C}^2 \times \mathbb{C}^2} d^2 w_1 d^2 w_2 d^2 z_1 d^2 z_2 \times \\
\times \prod_{i,j=1}^{2} \delta(z_i \bar{z}_j + w_i \bar{w}_j - (u_0)_{i,j}) \delta(|w_1|^2 + |w_2|^2 - (s_1 - s_2)) e^{\frac{\lambda_i}{2} (|z_1|^2 + |z_2|^2)}
\]

\[
= \Delta(\lambda) \int_{\mathbb{C}^2 \times \mathbb{C}^2} d^2 w_1 d^2 w_2 d^2 z_1 d^2 z_2 \times \delta(|z_1|^2 + |w_1|^2 - \lambda_1) \delta(|z_2|^2 + |w_2|^2 - \lambda_2) \times \\
\times \delta(z_1 \bar{z}_2 + w_1 \bar{w}_2) \delta(|z_1 z_2 + \bar{w}_1 \bar{w}_2|) \delta(|w_1|^2 + |w_2|^2 - s) e^{\frac{\lambda_i}{2} (|z_1|^2 + |z_2|^2)}.
\]

Let us introduce the variable \( \xi = z_1 \bar{z}_2 + w_1 \bar{w}_2 \) and integrate over \( \xi \) to obtain

\[
\Delta(\lambda) \int d^2 w_1 d^2 w_2 \frac{d^2 z_2}{|z_2|^2} \delta \left( \frac{|w_1|^2 |w_2|^2}{|z_2|^2} + |w_1|^2 - \lambda_1 \right) \delta(|z_2|^2 + |w_2|^2 - \lambda_2) \delta(|w_1|^2 + |w_2|^2 - s) e^{i \xi (\lambda_1 + \lambda_2 - s)}.
\]

Integration over \( w_1 \) gives

\[
e^{i \xi (\lambda_1 + \lambda_2 - s)} \frac{\Delta(\lambda)}{\lambda_2} \int d^2 w_2 d^2 z_2 \delta(|z_2|^2 + |w_2|^2 - \lambda_2) \delta \left( \frac{\lambda_1}{\lambda_2} |z_2|^2 + |w_2|^2 - s \right),
\]

and further integration over \( z_2 \) provides

\[
e^{i \xi (\lambda_1 + \lambda_2 - s)} \frac{\Delta(\lambda)}{\lambda_2} \int d^2 w_2 \Theta(\lambda_2 - |w_2|^2) \delta \left( \frac{\lambda_1}{\lambda_2} (\lambda_2 - |w_2|^2) + |w_2|^2 - s \right) = \\
= e^{i \xi (\lambda_1 + \lambda_2 - s)} \frac{\Delta(\lambda)}{\lambda_2} \int_0^\infty dt \Theta(\lambda_2 - t) \delta \left( \lambda_1 + \frac{\lambda_2 - \lambda_1}{\lambda_2} \lambda_2 - s \right).
\]

Finally after integration over \( t \) and taking into account \( \lambda_1 > \lambda_2 \) we arrive at

\[
J(0, s; \lambda_1, \lambda_2) = e^{\frac{i \xi}{2} (\lambda_1 + \lambda_2 - s)} \Theta(\lambda_2) \Theta(\lambda_1 - s) \Theta(s - \lambda_2).
\]

This finishes the proof of the Proposition. \( \square \)

The integral representation (5.16) is the symplectic integral representation of the Baxter operator kernel (4.12) for \( \ell = 1 \).
6 Equivariant symplectic volumes and nil-Hecke algebras

In Section 3 we constructed an elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function and demonstrated that it provides a solution of the eigenfunction problem for the quantum billiard associated with the Lie algebra $\mathfrak{gl}_{\ell+1}$. Further, in Section 5 the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions were identified with $U_{\ell+1}$-equivariant symplectic volumes of flag spaces $B_{\ell+1} = GL_{\ell+1}/B$. Thus we have managed to express eigenfunctions of the quantum billiard via equivariant symplectic volumes of flag spaces. This connection between quantum billiard eigenfunctions and equivariant symplectic volumes can be considered as a manifestation of a general relation between quantum many-body integrable systems, representation theory of the Hecke algebras and (generalized) equivariant cohomology of $G$-spaces (see e.g. [CG], [Ch] for detailed discussions). Thus a description of $G$-equivariant cohomology $H_G(B, \mathbb{C})$ of the flag space $B = G/B$ in terms of the nil-Hecke algebra $H^{nil}(G^\vee)$ associated with the dual Lie group $G^\vee$ was proposed in [KK] (see [BGG] for non-equivariant case). Below we consider the special case $G = G^\vee = GL_{\ell+1}$ and demonstrate that the results of [KK] are compatible with the considerations of our previous Sections.

Let us first recall some standard facts on the nil-Hecke algebra associated with the Lie algebra $\mathfrak{gl}_{\ell+1}$ (see e.g. [CG]). Let $\Phi$ be the root system of $\mathfrak{gl}_{\ell+1}$. We identify the diagonal Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{\ell+1}$ with $\mathbb{R}^{\ell+1}$ and fix a basis $\{e_i\}, i = 1, \ldots, \ell+1$ in $\mathfrak{h}$, orthonormal with respect to the bilinear form $(\ , \ )$ induced by the Killing form on $\mathfrak{gl}_{\ell+1}$. We define coroots as $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$. Using the Killing form we identify $\mathfrak{h}$ and its dual $\mathfrak{h}^\ast$. The positive simple roots of $\Phi$ are given then by $\alpha_i = e_i+1 - e_i, i = 1, \ldots, \ell$. The Weyl group acts on $\mathfrak{h}$ by reflections

$$s_\alpha : y \mapsto y - \langle \alpha, y \rangle \alpha^\vee, \quad \alpha \in \Phi,$$

and is isomorphic to the permutation group $S_{\ell+1}$ generated by elementary permutations $s_i = \sigma_i\ell+1$ acting on the basis $\{e_j\}, j = 1, \ldots, (\ell+1)$ in $\mathbb{R}^{\ell+1}$ via permutations of the indexes. Let $s_{\text{max}} \in S_{\ell+1}$ be the element with reduced decomposition of maximal length. It can be written for example as follows

$$s_{\text{max}} = s_1s_2s_1s_3s_2s_1 \cdots s_\ell s_{\ell-1} \cdots s_1.$$

**Definition 6.1** The nil-Hecke algebra $H^{nil}_{\ell+1}$ associated with the root system $\Phi$ of $\mathfrak{gl}_{\ell+1}$ is an associative algebra generated by $R_s$, $s \in S_{\ell+1}$, $D_i, i = 1, \ldots, \ell+1$ and a central element $c$ with the following relations:

$$R_s \cdot R_{s'} = \begin{cases} R_{ss'} & \text{if } l(s) + l(s') = l(ss') \\ 0 & \text{if } l(s) + l(s') \neq l(ss'), \end{cases}$$

$$D_i D_j - D_j D_i = 0,$$

$$R_i \cdot D_j - D_{s_i(j)} : R_i = c \delta_{ij}. $$

The generators $R_i := R_{s_i}$ corresponding to elementary permutations $s_i$ satisfy the following relations:

$$R_i R_j = R_j R_i, \quad |i - j| \geq 2,$$

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1},$$
\( R_i^2 = 0, \)

and generate the nil-Coxeter subalgebra \( W^{nil} \subset \mathcal{H}^{nil}_{\ell+1}. \) In particular operators

\[
R_s = R_{s_{i_1}} R_{s_{i_2}} \cdots R_{s_{i_t}}, \quad s = s_{i_1} s_{i_2} \cdots s_{i_t},
\]

(6.7)

are zero unless the product representation of \( s \) in terms of elementary reflections has minimal length. The corresponding element (6.7) is independent of the choice of the product decomposition of \( s. \)

The nil-Hecke algebra \( \mathcal{H}^{nil}_{\ell+1} \) is a semidirect product of \( W^{nil} \) and the algebra \( \mathbb{C}[h] \) of polynomial functions on \( h. \) The center \( Z_{\ell+1} \) of \( \mathcal{H}^{nil}_{\ell+1} \) is isomorphic to the algebra of \( \mathcal{S}_{\ell+1} \) symmetric polynomials of the generators \( D_i, \ i = 1, \ldots, (\ell + 1). \) Irreducible representations of \( \mathcal{H}^{nil}_{\ell+1} \) can be characterized by their central characters i.e. homomorphisms \( Z_{\ell+1} \to \mathbb{C} \) and a homomorphism \( Z_{\ell+1} \to \mathbb{C} \) is uniquely defined by \( \mathcal{S}_{\ell+1} \)-orbits of an element \( \lambda \in \mathfrak{h}. \)

Let \( \mathcal{W}_\lambda \) be the linear space generated by the exponential functions \( \Psi_{\lambda,s}(x) = \exp(i(s \cdot \lambda, x)) \) on \( \mathfrak{h}^* \) and let \( \{s \cdot \lambda | s \in \mathfrak{S}_{\ell+1}\} \) be the orbit through a fixed element \( \lambda \in \mathfrak{h}. \) \( \mathcal{W}_\lambda \) may be identified with the space of common eigenfunctions of the ring \( \text{Diff}_{\mathfrak{c}}^{\mathfrak{S}_{\ell+1}} \) of \( \mathfrak{S}_{\ell+1} \)-invariant differential operators on \( \mathfrak{h}^* \) with constant coefficients

\[
\mathcal{D}_\chi \cdot \Psi_{\lambda,s}(x) = \chi(i\lambda) \Psi_{\lambda,s}(x),
\]

(6.8)

where \( \mathcal{D}_\chi \in \text{Diff}_{\mathfrak{c}}^{\mathfrak{S}_{\ell+1}} \) is the differential operator corresponding to \( \chi \in \mathbb{C}[h]^{\mathfrak{S}_{\ell+1}} \) via the isomorphism \( \text{Diff}_{\mathfrak{c}}^{\mathfrak{S}_{\ell+1}} = \mathbb{C}[h]^{\mathfrak{S}_{\ell+1}}. \) The space of solutions has cardinality \( |\mathfrak{S}_{\ell+1}| \) and is given by linear combinations of the exponents. The following integral/differential operators provide a realization of the irreducible representation \( \pi : \mathcal{H}^{nil}_{\ell+1} \to \text{End}(\mathcal{W}_\lambda) : \)

\[
\pi(R_j) \cdot g(x) = \int_0^{(x,\alpha_j)} g(x - t\alpha_j) dt,
\]

(6.9)

\[
\pi(D_j) \cdot g(x) = c \frac{\partial g(x)}{\partial x_j},
\]

(6.10)

where \( g \in \mathcal{W}_\lambda. \) Equivalently these operators can be written as follows:

\[
\pi(R_j) e^{i(\lambda,x)} = \frac{e^{i(\lambda,x)} - e^{i(s_j \cdot \lambda,x)}}{i(\lambda,\alpha_j)},
\]

(6.11)

\[
\pi(D_j) e^{i(\lambda,x)} = ic(\lambda,\alpha_j) e^{i(\lambda,x)}.
\]

(6.12)

From now on we shall abuse notations and use the symbols \( R_j \) and \( D_j \) for the images \( \pi(R_j) \) and \( \pi(D_j) \) of the generators in the representation \( \mathcal{W}_\lambda. \)

We call a vector \( \Psi_{\lambda}^{(0)} \in \mathcal{W}_\lambda \) to be of class one if it satisfies the conditions

\[
R_i \Psi_{\lambda}^{(0)}(x) = 0, \quad i = 1, \ldots, (\ell + 1).
\]

(6.13)

This condition is a nil-Hecke analog of the sphericity condition for affine Hecke algebras [CG] and uniquely defines a class one vector in \( \mathcal{W}_\lambda. \) Taking into account that \( l(s_i s_{\text{max}}) < l(s_i) + l(s_{\text{max}}) \) and the relations (6.3) for \( R_i \cdot R_j, \) we infer that the class one vector can be written as follows

\[
\Psi_{\lambda}^{(0)}(x) = R_{s_{\text{max}}} e^{i(\lambda,x)}.
\]

(6.14)
Proposition 6.1 The class one vector in the representation $W_\lambda$ of $H_{\ell+1}^{\text{nil}}$ solves the eigenfunction problem of the quantum billiard (3.11), (3.12) associated with the root system of $\mathfrak{gl}_{\ell+1}$.

Proof. Let us note that it is possible to write $s_{\text{max}}$ as a minimal length product of the elementary reflections starting with any $s_i$, and thus obtain

$$\Psi_\lambda^{(0)}(x) = R_{s_i} R_{s_*} e^{i\langle \lambda, x \rangle}. \quad (6.15)$$

Now using the realization (6.9) for $R_i := R_{s_i}$ we infer that

$$\Psi_\lambda^{(0)}(x)|_{x_j = x_j + 1} = 0, \quad j = 1, \ldots, \ell. \quad (6.16)$$

Thus taking into account (6.8) we prove the statement of the Proposition. $\square$

In previous Sections we show that the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi_\lambda^{(0)}(x)$ (3.2), (3.3) is a quantum billiard eigenfunction. Now we explain how the relation (5.2) between elementary $\mathfrak{gl}_{\ell+1}$-Whittaker functions and $U_{\ell+1}$-equivariant symplectic volumes of flag spaces $\mathcal{B}_{\ell+1}$ arises via a relation between nil-Hecke algebras and equivariant cohomology of flag manifolds.

The class one vector (6.14) solving the set of equations (6.8) can be expressed through $U_{\ell+1}$-equivariant symplectic volume of the flag space $\mathcal{B}_{\ell+1}$. This follows from a general connection between equivariant cohomology of flag spaces and nil-Hecke algebras which we review below following [KK] (see also [CG]). Let us start with the non-equivariant case. Recall that the flag space $\mathcal{B}_{\ell+1}$ has a Schubert cell decomposition, with cells $O_s$ enumerated by elements $s \in \mathfrak{S}_{\ell+1}$ of the corresponding Weyl group. The homology classes $[O_s]$ provide a basis in the homology groups $H_\ast(\mathcal{B}_{\ell+1}, \mathbb{Q})$ and we denote by $\sigma_s \in H_\ast(\mathcal{B}_{\ell+1})$ the dual cohomology classes. Thus for example $\sigma_{\text{id}} = 1$ is dual to the fundamental class of $\mathcal{B}_{\ell+1}$ and $\sigma_{s_{\text{max}}}$ is dual to the unique zero homology class. The following orthogonality relation follows from the standard intersection product relations

$$(\sigma_s, \sigma_{s'}) = \delta_{s, s_{\text{max}}}, \quad (\omega) := \int_{\mathcal{B}_{\ell+1}} \omega.$$

There exists another description of the cohomology groups of flag manifolds due to Borel

$$H_\ast(\mathcal{B}_{\ell+1}, \mathbb{C}) = \mathbb{C}[y_1, \ldots, y_{\ell+1}] / J, \quad (6.17)$$

where the ideal $J$ is generated by $f \in \mathbb{C}[y_1, \ldots, y_{\ell+1}]^{\mathfrak{S}_{\ell+1}}$, $f(0) = 0$. Here the element $f \in \mathfrak{h}_{\mathbb{Z}}^*$ in the right hand side of (6.17) corresponds to the image of the first Chern class $c_1(L_\varphi)$ of the line bundle $L_\varphi$ associated with the weight $\varphi$. Let us denote by $\bar{f}$ the element of $H_\ast(\mathcal{B}_{\ell+1}, \mathbb{C})$ corresponding to $f$ under the isomorphism (6.17).

The pairing of an element $\bar{f} \in H_\ast(\mathcal{B}_{\ell+1}, \mathbb{C})$ with the fundamental class of $\mathcal{B}_{\ell+1}$ can be expressed in the integral form

$$(\bar{f}) = \frac{1}{(2\pi i)^{\ell+1}} \int_{C_0} dy_1 \cdots dy_{\ell+1} \frac{f(y)}{\prod_{j=1}^{\ell+1} e_j(y)}, \quad \bar{f} \in H_\ast(\mathcal{B}_{\ell+1}), \quad (6.18)$$

where $e_j(y)$ are elementary symmetric functions

$$\sum_{j=0}^{\ell+1} (-1)^j z^{\ell+1-j} e_j(y) = \prod_{j=1}^{\ell+1} (z - y_j), \quad (6.19)$$

29
and the integration domain $C_0$ encloses the poles of the denominator.

The representatives of the dual Schubert classes $\sigma_w \in H^*(B_{\ell+1})$ in the polynomial ring $\mathbb{Q}[y_1, \ldots, y_{\ell+1}]$ are called Schubert polynomials. For example one can take
\[
\sigma_{s_{\max}}(y) = \frac{1}{|G_{\ell+1}|} \prod_{1 \leq i < j \leq \ell+1} (y_i - y_j).
\] (6.20)

The cohomology $H^*(B_{\ell+1})$ of the flag space admits the structure of a module over nil-Hecke algebra $\mathcal{H}_{\ell+1}^{nil}$. Precisely, the following operators
\[
\tilde{R}_i \cdot P(y) = \frac{P(y) - P(s_i \cdot y)}{i\langle \alpha_i, y \rangle},
\] (6.21)
\[
\tilde{D}_i \cdot P(y) = ic \langle \alpha_i, y \rangle P(y),
\] (6.22)
define an action of the nil-Hecke algebra $\mathcal{H}_{\ell+1}^{nil}$ on the space of polynomials $P \in \mathbb{C}[y_1, \ldots, y_{\ell+1}]$. This action commutes with the multiplication on $G_{\ell+1}$-invariant polynomials and thus descends to the Borel realization (6.17) of $H^*(B_{\ell+1})$ to provide an action of $\mathcal{H}_{\ell+1}^{nil}$ on $H^*(B_{\ell+1})$. The action of the nil-Hecke algebra provides expressions for generic Schubert polynomials, thus
\[
\sigma_s(y) = \tilde{R}_{s_{\max}} \sigma_{s_{\max}}(y).
\] (6.23)

The pairing of the product of an element $\tilde{f} \in H^*(B_{\ell+1})$ and a Schubert class $\sigma_s$ with the fundamental class $[B_{\ell+1}]$ is given by
\[
(f, \sigma_s) = \tilde{R}_{s_{\max}} \cdot \tilde{f}|_{y=0}.
\] (6.24)

Let $\omega_i = \tilde{y}_i$ be generators of $H^2(B_{\ell+1})$. Then, using the general formula (6.24) in the special case $f(x) = \exp(i \sum_{j=1}^{\ell+1} y_j x_j)$, we obtain the following representation for the symplectic volume of $B_{\ell+1}$:
\[
Z_{B_{\ell+1}}(x) := \int_{B_{\ell+1}} e^{\sum_{j=1}^{\ell+1} \omega_j x_j} = \tilde{R}_{s_{\max}} \cdot e^{\sum_{j=1}^{\ell+1} y_j x_j}|_{y=0}.
\] (6.25)

**Example 6.1** For $\ell = 1$ the element of the Weyl group with maximal length is $s_{\max} = s_1$, interchanging $y_1$ and $y_2$. Thus we have
\[
Z_{B_2}(x) = \int_{B_2} e^{i \omega_1 x_1 + i \omega_2 x_2} = \tilde{R}_{s_1} \cdot e^{i y_1 x_1 + i y_2 x_2}|_{y=0} = \frac{e^{i y_1 x_1 + i y_2 x_2} - e^{i y_2 x_1 + i y_1 x_2}}{i(y_1 - y_2)}|_{y=0} = (x_1 - x_2).
\]

Now let us consider an equivariant analog of the relation between the symplectic volume of $B_{\ell+1}$ and representation theory of $\mathcal{H}_{\ell+1}^{nil}$. Let $T_{\ell+1} \subset U_{\ell+1}$ be the diagonal Cartan torus. The $T_{\ell+1}$-equivariant cohomology of the flag space $B_{\ell+1}$ has the following description generalizing (6.17):
\[
H_{T_{\ell+1}}^*(B_{\ell+1}, \mathbb{C}) = \mathbb{C}[y_1, \ldots, y_{\ell+1}, \lambda_1, \ldots, \lambda_{\ell+1}]/J,
\] (6.26)
where the ideal $J$ is generated by functions $f(y) - f(\lambda)$ where $f(y_1, \ldots, y_{\ell+1})$ is an arbitrary $G_{\ell+1}$-symmetric polynomial function. Linear functions $\langle \omega, y \rangle = \sum_{j=1}^{\ell+1} \omega_j y_j$, correspond to $T_{\ell+1}$-equivariant extensions of the first Chern classes $c_1(L_\omega)$ of the line bundles $L_\omega$ associated with the
integral weights $\omega$ of the Lie algebra $\mathfrak{gl}_{\ell+1}$. The $U_{\ell+1}$-equivariant cohomology $H^*_U(B_{\ell+1})$ can be similarly realized as follows:

$$H^*_U(B_{\ell+1}, \mathbb{C}) = \mathbb{C}[y_1, \ldots, y_{\ell+1}] \otimes \mathbb{C}[\lambda_1, \ldots, \lambda_{\ell+1}] / J,$$

where the ideal $J$ is the same as above. In the equivariant case the pairing with the fundamental cycle $[B_{\ell+1}]$ is given by

$$(\bar{f})_\lambda = \frac{1}{(2\pi i)^{\ell+1}} \oint_{\mathcal{C}} dy_1 \cdots dy_{\ell+1} \frac{f(y, \lambda)}{\prod_{j=1}^{\ell+1} (e_j(y) - e_j(\lambda))}, \quad \bar{f} \in H^*_U(B_{\ell+1}), \quad (6.27)$$

where $e_j(z)$ are elementary symmetric polynomials (6.19) and the integration domain $\mathcal{C}$ encloses all singularities of the denominator. The integral reduces to a sum over the residues and is given by

$$(\bar{f})_\lambda = \frac{1}{\Delta(\lambda)} \sum_{s \in \mathfrak{S}_{\ell+1}} (-1)^{f(s)} f(s \cdot y)|_{y_i=\lambda_i}, \quad \Delta(\lambda) := \prod_{1 \leq i, j \leq \ell+1} (\lambda_i - \lambda_j).$$

Define a pairing between the space $\mathcal{F}(\mathfrak{h}^*)$ of analytic functions on $\mathfrak{h}^*$ and polynomial functions $\mathbb{C}[\mathfrak{h}]$ as follows:

$$(f, g) = [f(\partial_y) \cdot g(y)]_0, \quad f \in \mathbb{C}[\mathfrak{h}], \quad g \in \mathcal{F}(\mathfrak{h}^*), \quad (6.28)$$

where $[f]_0$ is the constant term of the series expansion of $f$ in a neighborhood of $y = 0$. The space $\mathcal{W}_\lambda$ defined by (6.8) is dual to the cohomology groups given by (6.26) with respect to the pairing (6.28) i.e.

$$\mathcal{W}_\lambda = \{ g \in \mathbb{C}((\mathfrak{h}^*)) | (f(\partial_y) - f(\lambda)) \cdot g(y) = 0, \text{ for any } f \in J \},$$

where $J \subset \mathbb{C}[\mathfrak{h}]$ is the subspace of $\mathfrak{S}_{\ell+1}$-invariant polynomials. Note that operators $\widetilde{R}_i$ and $\widetilde{D}_j$ given by (6.21), (6.22) are conjugate to the operators $R_i$ and $D_j$ given by (6.9), (6.10) with respect to the pairing (6.28). Let us define the basis $\{D_s\}, s \in \mathfrak{S}_{\ell+1}$ in $\mathcal{W}_\lambda$ to be dual to the basis of the Schubert polynomials $\{\sigma_s\}$ in $H^*(B_{\ell+1})$.

**Proposition 6.2** One has

$$(\bar{f})_\lambda = \widetilde{R}_{s_{\text{max}}} \cdot f(y)|_{y_i=\lambda_i}, \quad (6.29)$$

**Proof.** First we have

$$(\bar{f} \sigma_{s_{\text{max}}})_\lambda = \frac{1}{|\mathfrak{S}_{\ell+1}|} \frac{1}{(2\pi i)^{\ell+1}} \oint_{\mathcal{C}} dy_1 \cdots dy_{\ell+1} \frac{f(y) \prod_{i<j} (y_i - y_j)}{\prod_{j=1}^{\ell+1} (e_j(y) - e_j(\lambda))} = \frac{1}{|\mathfrak{S}_{\ell+1}|} \sum_{s \in \mathfrak{S}_{\ell+1}} f(y)|_{y_{s(i)}=\lambda_i},$$

where in the last step we replace the integral by the sum over residues. Now we have

$$(f)_\lambda = ((f, D_{id})) = ((f, R_{s_{\text{max}}} D_{s_{\text{max}}})) = ((\widetilde{R}_{s_{\text{max}}} f, D_{s_{\text{max}}})),$$

where we use the dual version of (6.23) for $s = \text{id}$. Using the duality between $\sigma_{s_{\text{max}}}$ and $D_{s_{\text{max}}}$ we obtain

$$(f)_\lambda = ((\widetilde{R}_{s_{\text{max}}} f) \sigma_{s_{\text{max}}})_\lambda = \widetilde{R}_{s_{\text{max}}} f(y)|_{y_i=\lambda_i}.$$

Here at the last step we take into account that $\widetilde{R}_{s_{\text{max}}} f(y)$ is a symmetric function. □
Proposition 6.3 Suppose that \( \omega_i(\lambda) \) are the equivariant cohomology classes represented by the generator \( y_i \) in the Borel realization (6.26). Then we have for equivariant symplectic volume

\[
Z(x, \lambda) := \int_{B_{\ell+1}} e^{\sum_{i=1}^{\ell+1} \omega_j(\lambda) x_j} = (e^{\sum_{j=1}^{\ell+1} y_j x_j})_\lambda = \tilde{R}_{s_{max}} \cdot e^{\sum_{j=1}^{\ell+1} y_j x_j} \big|_{y_i = \lambda_i} = R_{s_{max}} \cdot e^{\sum_{j=1}^{\ell+1} \lambda_j x_j}.
\] (6.30)

Proof. The result follows from (6.29) applied to the special case \( f(y) = \exp(t \sum_{j=1}^{\ell+1} y_j x_j) \). □

Example 6.2 For \( \ell = 1 \) we have \( s_{max} = s_1 \) interchanging \( y_1 \) and \( y_2 \). Thus for the equivariant symplectic volume of \( B_2 = \mathbb{P}^1 \) we obtain

\[
Z_{B_2}(x, \lambda) = \int_{\mathbb{P}^1} e^{i(\omega_1 + \lambda_1) x_1 + i(\omega_2 + \lambda_2) x_2} = R_{s_1} \cdot e^{iy_1 x_1 + iy_2 x_2} \big|_{y = \lambda} = R_{s_1} \cdot e^{i \lambda_1 x_1 + iy_2 x_2 - iy_2 x_1 + i \lambda_2 x_2 - e^{i \lambda_2 x_1 + i \lambda_1 x_2} \frac{e^{i \lambda_1 x_1 + iy_2 x_2} - e^{i \lambda_2 x_1 + iy_1 x_1}}{t(y_1 - y_2)} \big|_{y = \lambda} = R_{s_1} \cdot e^{i \lambda_1 x_1 + iy_2 x_2 - iy_2 x_1 + i \lambda_2 x_2 - e^{i \lambda_2 x_1 + i \lambda_1 x_2} \frac{e^{i \lambda_1 x_1 + iy_2 x_2} - e^{i \lambda_2 x_1 + iy_1 x_1}}{t(\lambda_1 - \lambda_2)}}.
\] (6.31)

Comparing (6.30) and (6.14) we obtain the equality of the class one vector \( \Psi^{(0)}(x) \in W_\lambda \) and the equivariant symplectic volume \( Z_{B_{\ell+1}} \)

\[
\Psi^{(0)}_\lambda(x) = Z_{B_{\ell+1}}(x, \lambda).
\] (6.32)

Finally note that the expression (6.14) for the elementary \( gl_{\ell+1} \)-Whittaker function \( \Psi^{(0)}(x) \) considered as an eigenfunction of the quantum billiard provides an integral representation for \( \Psi^{(0)}_\lambda(x) \). For example for \( \ell = 1 \) the representation (6.14) with \( R_{s_1} \) realized by the integral operator (6.9) gives the following:

\[
\Psi^{(0)}_{\lambda_1, \lambda_2}(x_1, x_2) = R_{s_1} \cdot e^{i(\lambda_1 x_1 + \lambda_2 x_2)} = \int_0^{x_2 - x_1} e^{i((x_1 + t) \lambda_1 + (x_2 - t) \lambda_2)} dt = \int_{x_1}^{x_2} e^{i(t \lambda_1 + (x_2 + x_1 - t) \lambda_2)} dt.
\] (6.33)

This integral representation shall be considered as the special case \( g = gl_1 \) of an elementary version of the integral representation of a \( g \)-Whittaker function introduced in [GLO1] for an arbitrary semisimple Lie algebra. Note that the integral representation introduced in [GLO1] shares with the Givental integral representation the property of positivity. In the \( \ell = 1 \) case the integral representation (6.33) accidentally coincides with the Givental representation (3.8).

7 Elementary Whittaker function as a matrix element

In the previous Sections we prove that \( U_{\ell+1} \)-equivariant symplectic volume of flag space \( B_{\ell+1} = GL_{\ell+1}/B \) is expressed through the elementary \( gl_{\ell+1} \)-Whittaker function. This identification may be considered as a limit of the identification of classical \( gl_{\ell+1} \)-Whittaker functions with \( S^1 \times U_{\ell+1} \)-equivariant volumes of the spaces of holomorphic maps of a two-dimensional disk \( D \) into \( B_{\ell+1} \).
[GLO8]. Recall that the $S^1 \times U_{\ell+1}$-equivariant volume of the space of holomorphic maps $D \to B_{\ell+1}$ can be succinctly described as a correlation function for the equivariant type A topological sigma model [GLO8], thus providing an infinite-dimensional integral representation of the classical Whittaker function. It was argued in [GLO8] that the mirror dual description in terms of the type B equivariant topological Landau-Ginzburg sigma model leads to a finite-dimensional integral representation of classical $\mathfrak{gl}_{\ell+1}$-Whittaker function. This finite-dimensional integral representation has an interpretation as a matrix element of an infinite-dimensional representation of $\mathcal{U}\mathfrak{gl}_{\ell+1}$ written in an integral form. This pair of infinite-dimensional and finite-dimensional integral representations of classical Whittaker functions was advocated in [GLO8] as a manifestation of the local Archimedean Langlands correspondence.

In this Section we provide an elementary analog of the mirror symmetry by constructing a representation of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (3.1) as a matrix element of a monoid $GL_{\ell+1}(R)$ where $R$ is the tropical semifield (for the definition of a tropical semifield see e.g. [MS], [IMS]). The correspondence between realizations of the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function as a $U_{\ell+1}$-equivariant volume of $B_{\ell+1}$ and as a matrix element of a representation of the monoid $GL_{\ell+1}(R)$ can be considered as an elementary analog of the local Archimedean Langlands correspondence.

Let us first discuss some special features of the integral representations (2.5) of the $\mathfrak{gl}_{\ell+1}$-Whittaker function. The integral form (2.5) of the Whittaker function arises from the general expression (2.1) using a particular representation of $\mathcal{U}\mathfrak{gl}_{\ell+1}$. Let $\chi_\Delta$ be a character of $B_-$ given by (2.3). Then the representation is realized in a subspace $\mathcal{V}_\Delta \subset \text{Ind}_{B_-}^{GL_{\ell+1}} \chi_\Delta$ of $B_-$-equivariant functions supported at $N_+^\times \times B_- \subset GL_{\ell+1}$ where $N_+^\times$ is the subset of positive elements of the maximal unipotent subgroup $N_+ \subset GL_{\ell+1}$. Recall that positive elements of $GL_{\ell+1}(R)$ are the elements realized in the standard matrix representation by positive matrices i.e. matrices with all minors positive (see e.g. [Lu2]). Similarly, positive elements of $N_+$ are the elements realized in the standard matrix representation by matrices with all non-identically zero minors positive. The following parametrization of positive subset $N_+^\times$ was introduced in [GKLO] using the results of [Lu2] (see also [BFZ]). Let $\epsilon_{i,j}$ be the elementary $(\ell + 1) \times (\ell + 1)$-matrix with unit in the $(i,j)$-place and all other entries zero. Consider the set consisting of the diagonal matrices

$$U_k = \sum_{i=1}^{k} e^{-x_{k,i}} \epsilon_{i,i} + \sum_{i=k+1}^{\ell+1} \epsilon_{i,i},$$

and of their upper-triangular deformations

$$\tilde{U}_k = \sum_{i=1}^{k} e^{-x_{k,i}} \epsilon_{i,i} + \sum_{i=k+1}^{\ell+1} \epsilon_{i,i} + \sum_{i=1}^{k-1} e^{-x_{k-1,i}} \epsilon_{i,i+1}. \quad (7.1)$$

The factorized parametrization of $N_+^\times$ follows from the fact that the image of any generic unipotent element $v \in N_+^\times$ in the tautological representation $\pi_{\ell+1} : \mathfrak{gl}_{\ell+1} \to \text{End}(\mathbb{C}_{\ell+1})$ can be represented in the form

$$\pi_{\ell+1}(v) = \tilde{U}_2 U_2^{-1} \tilde{U}_3 U_3^{-1} \cdots \tilde{U}_\ell U_\ell^{-1} \tilde{U}_{\ell+1}, \quad (7.2)$$

where we assume that $x_{\ell+1,i} = 0, \; i = 1, \ldots, \ell + 1$. Using this parametrization the following realization of $\mathcal{U}\mathfrak{gl}_{\ell+1}$ by differential operators was constructed in [GKLO].
Proposition 7.1 The following differential operators define a realization of representation \( \pi_\lambda \) of \( \mathfrak{gl}_{\ell+1} \) in \( \mathcal{V}_\mu \) in the space of functions on \( N^+_+ \):

\[
E_{i,i} = \mu_i - \sum_{k=1}^{i-1} \frac{\partial}{\partial x_{\ell+1+k-i,k}} + \sum_{k=i}^{\ell} \frac{\partial}{\partial x_{k,i}},
\]

\[
E_{i,i+1} = -\sum_{n=1}^{i} e^{x_{\ell-i+n-n}_{-x_{\ell-i-1+n-n}}} \sum_{k=1}^{n} \left\{ \frac{\partial}{\partial x_{\ell-i+k,k}} - \frac{\partial}{\partial x_{\ell-i+k-1,k}} \right\},
\]

\[
E_{i+1,i} = \sum_{n=1}^{\ell-i} e^{x_{i+1-i-x_{k+i-1,k}}} \left[ \mu_i - \mu_{i+1} + \sum_{k=1}^{n} \left\{ \frac{\partial}{\partial x_{i+k-1,i}} - \frac{\partial}{\partial x_{i+k-1,i+1}} \right\} \right],
\]

where \( \mu_k = \hbar^{-1} \lambda_k - \rho_k \). Here \( E_{i,j} = \pi_\lambda(e_{i,j}) \), \( i, j = 1, \ldots, \ell + 1 \) are the images of the standard generators \( e_{ij} \) of \( \mathfrak{gl}_{\ell+1} \) satisfying the relations

\[
[e_{ij}, e_{km}] = \delta_{jk} e_{im} - \delta_{mi} e_{kj}.
\]

The matrix element (2.1) written explicitly using this realization of the representation \( \mathcal{V}_\mu \) is given by the Givental integral formula (2.5). Note that the matrix element (2.1) is defined for representations such that the action of the Cartan subalgebra can be integrated to an action of the corresponding group. The integration of the action of the whole Lie algebra \( \mathfrak{gl}_{\ell+1} \) is not necessary and actually is not possible for the representation leading to (2.5). This is obvious taking into account that the integration in (2.5) is over a subset \( N^+_+ \subset N_+ \) of positive elements of \( N_+ \). Thus the function (2.5) does not naturally extend to a function on the group for which

\[
\Psi_\lambda(n-\eta n) = \psi^+(n_+) \psi^-(n_-) \Psi_\lambda(g), \quad g \in \text{GL}_{\ell+1}(\mathbb{R}), \quad n_\pm \in N_+, \tag{7.4}
\]

where the characters of \( N_\pm \) are the Lie group version of the characters in (2.2):

\[
\psi^\pm(n_\pm) = \exp \left( -\frac{1}{\hbar} \sum_{j=1}^{\ell+1} \frac{1}{(1+\frac{j}{1+1})} \right), \quad n_\pm \in N_+. \tag{7.5}
\]

The proper analog of (7.4) for (2.5) is defined as follows. In [Lu2] Lusztig constructed a monoid \( G^\circ \) of positive elements for an arbitrary reductive Lie group \( G \) (recall that a monoid has a multiplication operation and a unity but an inverse element is not defined). Consider the monoid of positive elements \( \text{GL}_{\ell+1}^+ (\mathbb{R}) \subset \text{GL}_{\ell+1}(\mathbb{R}) \). Elements of \( \text{GL}_{\ell+1}^+ (\mathbb{R}) \) can be represented via the Gauss decomposition as \( g = f h n \) where \( n \) is an upper-triangular positive matrix, \( f \) is a lower-triangular positive matrix and \( h \) is a diagonal matrix with positive entries [Lu2]. The monoid \( \text{GL}_{\ell+1}^+ (\mathbb{R}) \) naturally acts on the positive subset \( N^+_+ \) of the flag space \( B_{\ell+1} \). Indeed, the monoid property implies that the product of two positive elements of \( \text{GL}_{\ell+1}(\mathbb{R}) \) is again a positive element. The Gauss decomposition of the product of \( g \in \text{GL}_{\ell+1}^+ (\mathbb{R}) \) and \( n \in N^+_+ \) has the following form:

\[
n \cdot g = \tilde{f} \cdot \tilde{h} \cdot \tilde{n}, \tag{7.6}
\]

where \( \tilde{n} \in N^+_+, \tilde{h} \in H^+ \) and \( \tilde{f} \in N^+_+ \). We define an action \( \text{GL}_{\ell+1}^+ (\mathbb{R}) \) on the positive subset \( N^+_+ \) of the flag space \( B_{\ell+1} \) by taking \( \tilde{n} \) as the result of the action of \( g \) on \( n \in N^+_+ \).

Let us consider a monoid \( GL_{\ell+1}^+ \) over a ring \( \mathbb{R}[\delta_1, \ldots, \delta_N] \) of nilpotent elements \( \delta_1, \ldots, \delta_N \) such that non-diagonal elements of the lower and upper triangular parts in the Gauss decomposition
\( g = f h n \) are in nilpotent subalgebras of \( R_{nil}^{\geq} = \mathbb{R}_{\geq}[3_1, \ldots, 3_N] \) and the diagonal elements of the Cartan part are invertible elements of \( R_{nil}^{\geq} \). Let \( nil GL_{\ell+1}^{\geq} \) be a submonoid of \( GL_{\ell+1}^{\geq}(R_{nil}) \) such that all strictly lower-triangular elements \( f_{i,j}, i > j \) of \( f \) are nilpotent i.e. \( f_{ij}^M = 0 \) for some large \( M \in \mathbb{Z}_+ \).

**Proposition 7.2** The \( gl_{\ell+1} \)-Whittaker function (2.5) can be lifted to a function on the monoid \( nil GL_{\ell+1}^{\geq} \) such that the following functional equation holds:

\[
\Psi_\lambda(fgn) = \psi^+(n)\psi^-(f)\Psi_\lambda(g), \quad g \in nil GL_{\ell+1}^{\geq}, \quad n \in nil N_{\ell+1}^{\geq}, \quad f \in nil N_{\ell+1}^{\geq},
\]

where the functions \( \psi^\pm \) are given by (7.5).

**Proof.** The equivariance property (7.7) with respect to \( nil N_{\ell+1}^{\geq} \) follows from properties of the right Whittaker vector. Equivariance with respect to \( nil N_{\ell+1}^{\geq} \) may be reduced to equivariance with respect to the Lie algebra and thus follows from properties of the left Whittaker vector. \( \square \)

Now we are ready to provide a matrix element interpretation of the elementary \( gl_{\ell+1} \)-Whittaker functions (3.6). Let us start by recalling the definition of a tropical semifield (see e.g. [MS], [IMS]).

**Definition 7.1** A tropical semifield \( \mathcal{R} \) is a set isomorphic to \( \mathbb{R} \) with the following operations

\[
\alpha \times \beta = \alpha + \beta, \quad \alpha \hat{+} \beta = \min(\alpha, \beta).
\]

We also introduce the notation \( \alpha / \beta := \alpha \hat{+}(-\beta) \). The tropical semifield \( \mathcal{R} \) can be understood as a degeneration of the standard semifield structure on the positive subset \( \mathbb{R}_+ \subset \mathbb{R} \) of real numbers. Indeed, consider the semifield \( \mathbb{R}_+^{(h)} \) with the following operations

\[
a \times_h b = a \times b, \quad a +_h b = (a^h + b^h)^{1/h}.
\]

The semifield \( \mathbb{R}_+^{(h)} \) is isomorphic to \( \mathbb{R}_+^{(1)} = \mathbb{R}_+ \) via the map \( a \rightarrow a^h \). Let \( a = e^{-\alpha}, b = e^{-\beta}, \alpha, \beta \in \mathbb{R} \). Then in the limit \( h \rightarrow +\infty \) the operations (7.9) are transformed into the tropical operations (7.8):

\[
\alpha \times \beta := - \lim_{h \rightarrow +\infty} \log \left( e^{-\alpha} \times_h e^{-\beta} \right) = \alpha + \beta,
\]

\[
\alpha \hat{+} \beta = - \lim_{h \rightarrow +\infty} \log \left( e^{-\alpha} +_h e^{-\beta} \right) = \min(\alpha, \beta),
\]

and the semifield \( \mathbb{R}_+^{(h)} \) turns into the tropical semifield \( \mathcal{R} \). The set of matrices \( \text{Mat}_{\ell+1}(\mathcal{R}) \) has a monoid structure arising in the limit \( h \rightarrow +\infty \) from the monoid structure on the subset of positive elements \( GL_{\ell+1}(\mathbb{R}_+^{(h)}) \). Note that invertibility automatically holds in the tropical case and thus in the following we can identify the matrix monoid \( \text{Mat}_{\ell+1}(\mathcal{R}) \) with \( GL_{\ell+1}(\mathcal{R}) \).

Consider a principal series representation \( \pi_\Lambda^{(0)} \) of \( GL_{\ell+1}(\mathcal{R}) \) in \( \mathcal{V}_\Lambda = \text{Ind}_{B_-(\mathcal{R})}^{GL_{\ell+1}(\mathcal{R})} \chi_\Lambda^{(0)} \) of the monoid \( GL_{\ell+1}(\mathcal{R}) \) induced from a character \( \chi_\Lambda^{(0)} \) of the submonoid \( B_-(\mathcal{R}) \) of upper triangular matrices

\[
\chi_\Lambda^{(0)}(b) = \prod_{j=1}^{\ell+1} \chi_{\lambda_j}^{(0)}(b_{jj}), \quad b \in B_-(\mathcal{R}).
\]

where \( \Lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) \) and \( \chi_\lambda^{(0)}(\tau) = \exp(-i\tau \lambda), \tau \in \mathbb{R}; \lambda \in \mathbb{C} \) is a multiplicative character of \( \mathcal{R} \).

\[
\chi_\lambda^{(0)}(\tau_1 \hat{+} \tau_2) = \chi_\lambda^{(0)}(\tau_1)\chi_\lambda^{(0)}(\tau_2).
\]
Example 7.1 The following formulas

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \cdot f(\tau) = f(\tau + \alpha),
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \cdot f(\tau) = f(\tau \cdot \tau_2 / \tau_1) e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)},
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \cdot f(\tau) = e^{-i(\lambda_1 - \lambda_2)(\min(0,\beta + \tau))} f(-\min(-\tau, \beta)),
\]

(7.10)

define an action of the tropical monoid \( GL_2(\mathbb{R}) \) on the space

\[ V_\Delta = \text{Ind}_{B_-(\mathbb{R})}^{GL_2(\mathbb{R})} \chi_\Delta = \{ f(g) \in \text{Fun}(GL_2(\mathbb{R})) | f(b_-g) = \chi^{(0)}_\Delta(b_-)f(g), \ b_- \in B_-(\mathbb{R}) \}, \]

where

\[
\chi^{(0)}_\Delta \begin{pmatrix} \tau_1 & 0 \\ \beta & \tau_2 \end{pmatrix} = e^{-i(\tau_1 \lambda_1 + \lambda_2 \tau_2)},
\]

and \( \alpha, \beta, \tau_1, \tau_2 \in \mathbb{R} \).

The explicit formulas in the example above can be obtained as a limit of the analogous formulas for the principal series representation \( \pi_\Delta \) of \( GL_2^>(\mathbb{R}) \) acting on \( V_\Delta = \text{Ind}_{B_-(\mathbb{R})}^{GL_2(\mathbb{R})} \chi^{(0)}_\Delta \), where

\[
\chi^{(0)}_\Delta(b) = |b_{11}|^{\lambda_1/\hbar} |b_{22}|^{\lambda_2/\hbar}.
\]

Thus we have

\[
\pi^{(0)}_\Delta \begin{pmatrix} 1 & e^{-h\alpha} \\ 0 & 1 \end{pmatrix} \cdot f(e^{-h\tau}) = f(e^{-h(\tau + \alpha)}),
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} e^{-h\tau_1} & 0 \\ 0 & e^{-h\tau_2} \end{pmatrix} \cdot f(e^{-h\tau}) = f(e^{-h(\tau_2 - \tau_1 + \tau)} e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)}),
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} 1 & 0 \\ e^{-h\beta} & 1 \end{pmatrix} \cdot f(e^{-h\tau}) = (1 + e^{-h(\beta + \tau)} \frac{\lambda_1 - \lambda_2}{1 + e^{-h(\tau + \beta)}}) f\left( \frac{e^{-h\tau}}{1 + e^{-h(\tau + \beta)}} \right).
\]

Taking the limit \( h \to +\infty \) we obtain

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \cdot f(\tau) = f(\tau + \alpha),
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \cdot f(\tau) = f(\tau_2 - \tau_1 + \tau) e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)},
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \cdot f(\tau) = e^{-i(\lambda_1 - \lambda_2)(\min(0,\beta + \tau))} f(-\min(-\tau, \beta)),
\]

where in the last case we use the identity

\[
-\min(-\alpha, \beta) = \alpha - \min(0, \beta + \alpha).
\]

Thus we recover the formulas (7.10) in the limit \( h \to +\infty \).
Now we construct elementary analogs of the Whittaker vectors (2.2) entering the construction of the matrix element (2.1). Precisely, we would like to construct vectors $\psi_+^0(x)$ and $\psi_-^0(x)$ in $\mathcal{V}_\Delta = \text{Ind}_{H_+^\infty(R)}^{GL_{\ell+1}(R)} \chi_\Delta$ satisfying the functional relations

$$\psi_+^0(x \times n_+) = \psi_0^+(n_+) \psi_+^0(x), \quad \psi_-^0(x \times n_-) = \psi_0^-(n_-) \psi_-^0(x), \quad n_+ \in N_+(R), \quad (7.11)$$

where

$$\psi_0^+(n_+) = \prod_{j=1+\frac{\ell}{2}(1+1)}^\ell \psi_0(n_{j,j\pm1}). \quad (7.12)$$

Here $\psi_0(x)$ is an additive character of $R$

$$\psi_0(x) = \Theta(x), \quad \psi_0(x_1 \times x_2) = \psi_0(x_1) \psi_0(x_2), \quad x_i \in R. \quad (7.13)$$

**Proposition 7.3** The following expressions for Whittaker vectors hold:

$$\psi_+^0(\tau) = \prod_{i=1}^\ell \prod_{k=1}^{\ell+1-i} \Theta(\tau_{k+i-1,i} - \tau_{k+i,i+1}), \quad (7.14)$$

$$\psi_-^0(\tau) = e^\Delta \sum_{k=1}^k \sum_{i=1}^i (\lambda_k - \lambda_{k+1})(\tau_{k,i} - \tau_{k+1,i}) \prod_{i=1}^\ell \prod_{k=1}^{\ell+1-i} \Theta(\tau_{k+i,k} - \tau_{k+i-1,k}). \quad (7.15)$$

**Proof.** Recall that all the constructions of [GKLO] are formulated in terms of positive elements of $GL_{\ell+1}(R)$ and thus can be reformulated in terms of $GL_{\ell+1}^+(R)$. This allows the definition of the $h \to +\infty$ limit (see e.g. the derivation of (7.10) above) and so the obtaining of explicit expressions for Whittaker vectors as the $h \to \infty$ limit of the Whittaker vectors constructed previously (see eqs. (3.12), (3.14), (3.15) and (3.16) in [GKLO])

$$\psi_+(T) = \exp \left\{ - \frac{1}{h} \sum_{i=1}^\ell \sum_{n=1}^{\ell+1-i} e^{T_{n+i,i+1} - T_{n+i-1,i}} \right\}, \quad (7.16)$$

$$\psi_-(T) = \exp \left\{ \frac{\lambda}{h} \sum_{k=1}^k \sum_{i=1}^i (\lambda_k - \lambda_{k+1})T_{k,i} \right\} \exp \left\{ - \frac{1}{h} \sum_{i=1}^\ell \sum_{k=1}^{\ell+1-i} e^{T_{k+i-1,k} - T_{k+i,k}} \right\}. \quad (7.17)$$

Using the variables $\tau_{i,j}(\tau) = h^{-1} \tau_{i,j}$ and taking the limit $h \to \infty$, we obtain

$$\psi_+^0(\tau) = \lim_{h \to +\infty} \psi_+(T(\tau)) = \prod_{i=1}^\ell \prod_{k=1}^{\ell+1-i} \Theta(\tau_{k+i-1,i} - \tau_{k+i,i+1}), \quad (7.18)$$

and

$$\psi_-^0(\tau) = \lim_{h \to +\infty} \psi_-(T(\tau)) = e^\Delta \sum_{k=1}^k \sum_{i=1}^i (\lambda_k - \lambda_{k+1})(\tau_{k,i} - \tau_{k+1,i}) \prod_{i=1}^\ell \prod_{k=1}^{\ell+1-i} \Theta(\tau_{k+i,k} - \tau_{k+i-1,k}). \quad (7.19)$$

Here we use the following relation

$$\lim_{h \to +\infty} e^{-\frac{\Delta}{h} T_{k+1,k}} = \Theta(\tau).$$
Example 7.2 For \( \ell = 1 \) the Whittaker vectors satisfying the equations

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \cdot \psi^{(0)}_+ (\tau) = \Theta(\alpha) \psi^{(0)}_+ (\tau),
\]

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \cdot \psi^{(0)}_- (\tau) = \Theta(\beta) \psi^{(0)}_- (\tau),
\]

are given by

\[
\psi^{(0)}_+ (\tau) = \Theta(\tau), \quad \psi^{(0)}_- (\tau) = e^{-\iota \tau (\lambda_1 - \lambda_2)} \Theta(-\tau).
\]

Indeed we have

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \cdot \psi^{(0)}_+ (\tau) = \psi^{(0)}_+ (\tau + \alpha) = \Theta(\alpha) \psi^{(0)}_+ (\tau), \quad \tau + \alpha = \text{min}(\tau, \alpha).
\]

For the Whittaker vector \( \psi^{(0)}_- \) we have

\[
\pi^{(0)}_\Delta \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \psi^{(0)}_- (\tau) = e^{-\iota \tau (\lambda_1 - \lambda_2)(\text{min}(0, \beta + \tau))} e^{-\iota (\tau - (\text{min}(0, \beta + \tau))(\lambda_1 - \lambda_2)) \Theta(\text{min}(\tau, \beta))} = \Theta(\beta) \psi^{(0)}_- (\tau).
\]

Let us define an analog of the monoid \( \text{nil} GL_{\ell+1}(\mathbb{R}) \) used in the formulation of Proposition 7.2. We define elements of a submonoid \( \text{nil} GL_{\ell+1}(\mathcal{R}) \) of \( GL_{\ell+1}(\mathcal{R}) \) via the Gauss decomposition \( g = \text{fhn} \). Precisely we impose the additional condition that \( f \in N_-(\mathcal{R}_+) \) where \( \mathcal{R}_+ \) is modeled on \( \mathbb{R}_+ \) with operations \( \times, \dot{+} \).

Theorem 7.1 (i) From (3.6), the elementary \( gl_{\ell+1} \)-Whittaker function

\[
\Psi^{(0)}_\Delta (x) = \int_{\mathbb{R}^{(\ell+1)/2}} \exp \left( i \sum_{k=1}^{\ell+1} \lambda_k \left( \sum_{i=1}^k \tau_{k,i} - \sum_{i=1}^{k-1} \tau_{k-1,i} \right) \right) \prod_{i=1}^\ell \prod_{k=1}^i \Theta(\tau_{i,k} - \tau_{i+1,k+1}) \prod_{k=i}^\ell \Theta(\tau_{i+1,k} - \tau_{i,k}) \prod_{k=1}^\ell \prod_{i=1}^k d\tau_{k,i},
\]

with \( x_i = \tau_{\ell+1,i}, \ i = 1, \ldots, \ell \) and \( \tau_{k,i} = 0 \) for \( i > k \), allows the following matrix element representation:

\[
\Psi^{(0)}_\Delta (x) = \langle \psi_L, \pi^{(0)}_\Delta (g(x)) \psi_R \rangle, \quad g(x) = \text{diag}(x_1, \ldots, x_{\ell+1}),
\]

where \( \psi_L = \psi^{(0)}_+ \) and \( \psi_R = \psi^{(0)}_- \) are defined by (7.11) and the pairing is given by

\[
\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{R}^{(\ell+1)/2}} d\mu^\times (\tau) \overline{\psi_1(\tau)} \psi_2(\tau).
\]

Here

\[
d\mu^\times (\tau) = \prod_{k=1}^\ell \prod_{i=1}^k d\tau_{k,i},
\]

is a product of multiplicative measures \( d\mu^\times (\tau) = d\tau \) on \( \mathcal{R} \).

(ii) The function (7.20) can be naturally lifted to a function on the monoid \( \text{nil} GL_{\ell+1}(\mathcal{R}) \) satisfying the functional relations

\[
\Psi^{(0)}_\Delta (fgn) = \psi_0^+ (n) \psi_0^- (f) \Psi^{(0)}_\Delta (g), \quad g \in \text{nil} GL_{\ell+1}(\mathcal{R}), \ n \in N_+(\mathcal{R}), \ f \in N_-(\mathcal{R}_+)
\]

where \( \psi_0^\pm \) are given by (7.12).
Proof. The first part of the Theorem is a direct consequence of the explicit formulas (7.14), (7.15) for Whittaker vectors. The second part of the Theorem follows from Proposition 7.2 by taking the limit \( h \to \infty. \) □

**Example 7.3** Let us directly check the second part of Theorem 7.1 for the simplest nontrivial case of \( \ell = 1. \) The only property that is not obvious is the relation

\[
\langle \psi_L, \pi^{(0)}_{<, \lambda} \left( \begin{array}{cc} 0 & 0 \\ \beta & 0 \end{array} \right) \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) \psi_R \rangle = \Theta(\beta, \pi) \langle \psi_L, \pi \left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) \psi_R \rangle, \quad \beta > 0.
\]

Using the integral representation for the pairing and the explicit form of the Whittaker vectors we obtain

\[
\int d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) \pi^{(0)}_{=, \lambda} \left( \begin{array}{cc} 0 & 0 \\ \beta & 0 \end{array} \right) \Theta(\tau + \tau_2 - \tau_1)e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)}.
\]

Taking into account the explicit form of the action

\[
\pi^{(0)}_{=, \lambda} \left( \begin{array}{cc} 0 & 0 \\ \beta & 0 \end{array} \right): f(\tau) = e^{-i(\lambda_1 - \lambda_2)(\min(0, \beta + \tau))} f(-\min(-\tau, \beta)),
\]

we obtain the following integral:

\[
I = \int_{\mathbb{R}} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) e^{-i(\lambda_1 - \lambda_2)(\min(0, \beta + \tau + \tau_2 - \tau_1))} \Theta(-\min(-\tau + \tau_2 + \tau_1, \beta))e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)}.
\]

Let us represent this integral as the sum

\[
I = I_1 + I_2,
\]

with

\[
I_1 = \int_{\tau < \tau_1 - \tau_2 - \beta} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) e^{-i(\lambda_1 - \lambda_2)(\min(0, \beta + \tau + \tau_2 - \tau_1))} \Theta(-\min(-\tau + \tau_2 + \tau_1, \beta))e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)},
\]

\[
I_2 = \int_{\tau > \tau_1 - \tau_2 - \beta} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) e^{-i(\lambda_1 - \lambda_2)(\min(0, \beta + \tau + \tau_2 - \tau_1))} \Theta(-\min(-\tau + \tau_2 + \tau_1, \beta))e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)}.
\]

Taking into account the condition \( \beta > 0 \) for the first integral \( I_1 \) we have

\[
I_1 = \int_{\tau < \tau_1 - \tau_2 - \beta} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) e^{-i(\lambda_1 - \lambda_2)(\beta + \tau + \tau_2 - \tau_1)} \Theta(-\beta)e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)} = 0.
\]

The second integral \( I_2 \) is given by

\[
I_2 = \int_{\tau > \tau_1 - \tau_2 - \beta} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) \Theta(\tau + \tau_2 - \tau_1)e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)} =
\]

\[
= \int_{\mathbb{R}} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) \Theta(\tau + \tau_2 - \tau_1)e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)} \Theta(\tau + \tau_1 + \tau_2 + \beta).
\]

Note that if \( \tau - \tau_1 + \tau_2 > 0 \) then obviously \( \tau - \tau_1 + \tau_2 + \beta > 0 \) when \( \beta > 0. \) Thus we obtain

\[
I_2 = \int_{\mathbb{R}} d\tau e^{i(\lambda_1 - \lambda_2)\tau} \Theta(-\tau) \Theta(\tau + \tau_2 - \tau_1)e^{-i(\lambda_1 \tau_1 + \lambda_2 \tau_2)}, \quad \beta > 0.
\]

To recapitulate: the elementary \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions shall be considered as the \( \mathfrak{gl}_{\ell+1} \)-Whittaker functions over the tropical semifield \( \mathcal{R}. \)
8 From $\mathbb{Q}_p$ to $\mathbb{Q}_1$

In Section 4 the elementary analog of the Whittaker functions (3.1) defined in Section 3 were obtained as a limit of the $q$-deformed Whittaker functions specialized at $q = 0$. In [GLO3], [GLO4], [GLO5] it was demonstrated that the $q$-deformed Whittaker functions interpolate the Whittaker functions over the Archimedean field $\mathbb{R}$ and the Whittaker functions over the non-Archimedean fields $\mathbb{Q}_p$, $p$ is a prime number. The representation (4.1) can be understood as an expression of the non-Archimedean Whittaker functions in terms of characters of irreducible finite-dimensional representations of the dual reductive Lie algebra and thus is an instance of the Shintani-Casselman-Shalika formula [Sh], [CS]. Note that the Shintani-Casselman-Shalika formula provides an explicit realization of the local non-Archimedean Langlands duality in terms of the Whittaker functions.

The elementary Whittaker functions (3.1) are obtained by further degeneration of the $q = 0$ Whittaker functions. Equivalently the elementary Whittaker functions can be obtained as the limit $p \to 1$ of the non-Archimedean Whittaker functions. Taking into account Theorem 7.1, the elementary $\mathfrak{gl}_{\ell+1}$-Whittaker function (3.1) should be formally considered as the $\mathfrak{gl}_{\ell+1}$-Whittaker function over a mysterious field which we may call $\mathbb{Q}_1$. A similar relation holds between elementary $L$-factors (3.23) and non-Archimedean local $L$-factors. Thus elementary $L$-factors shall be considered as local $L$-factors corresponding to $\mathbb{Q}_1$. Below we will argue that these local $L$-factors have a natural interpretation over the tropical semifield $\mathcal{R}$ considered as a domain of the valuation map for $\mathbb{Q}_1$. In this Section we briefly discuss this interpretation for the simplest $L$-functions, leaving detailed considerations for another occasion.

To explain the appearance of the tropical semifield $\mathcal{R}$ in the $p \to 1$ limit of constructions over $\mathbb{Q}_p$, we first recall the notion of a valuation on a non-Archimedean field. A valuation on a local non-Archimedean field $\mathcal{K}$ is a map $\nu : \mathcal{K} \to \mathbb{R}$ such that

$$\nu(x) = 0 \iff x = 0, \quad (8.1)$$
$$\nu(xy) = \nu(x) + \nu(y), \quad (8.2)$$
$$\nu(x+y) \geq \min(\nu(x),\nu(y)). \quad (8.3)$$

These conditions can be succinctly summarized as follows. A non-Archimedean valuation is a partial morphism $\nu : \mathcal{K} \to \mathbb{R}$ of $\mathcal{K}$ considered as a semifield (i.e taking into account only addition, multiplication and division operations) to the tropical semifield $\mathcal{R}$. The term partial here means that the morphism property does not necessarily hold for sums of elements of equal norms $\nu$. In the case of the p-adic field $\mathbb{Q}_p$ a non-Archimedean valuation can be defined as follows:

$$\nu_p(p^na) = n, \quad (p,a) = 1.$$ 

Thus the image of the partial morphism $\nu_p : \mathbb{Q}_p \to \mathcal{R}$ is a discrete subsemifield $(\mathbb{Z}, \times, +) \in \mathcal{R}$. Note that the valuation $\nu_p$ has a large kernel $\mathbb{Z}_p^*$ consisting of invertible p-adic integers. Now we can partially clarify what meaning one can assign to a limit of the field $\mathbb{Q}_p$ when $p \to 1$. It is clear that the field $\mathbb{Q}_1$, considered as a semifield, should allow a partial epimorphism $\nu_1$ onto $\mathcal{R}$. What the kernel of $\nu_1$ is not quite clear. Fortunately many constructions over $\mathbb{Q}_p$ can be reformulated directly in terms of the semifield $(\mathbb{Z}, \times, +)$ considered as an image domain of the valuation map $\nu_p$. Hence these constructions over $\mathbb{Q}_p$ allow a limit $p \to 1$ formulated in terms of the image of $\nu_1$ identified with the tropical semifield $\mathcal{R}$. Below we illustrate this phenomena for the simple case of the local $L$-factors.
Let us define a monoid which is a non-Archimedean analog of the monoid $GL_{\ell+1}(\mathbb{R})$ introduced in [Lu2] and discussed in the previous Section. Define on the set $\mathcal{R}_p = p\mathbb{Z}$ a monoid structure with multiplication and addition

$$p^{n_1} \cdot p^{n_2} = p^{n_1+n_2}, \quad p^{n_1} + p^{n_2} = p^{\min(n_1,n_2)}.$$

(8.4)

We consider the subset $GL_{\ell+1}(\mathbb{R}_p) \subset GL_{\ell+1}(\mathbb{Q}_p)$ consisting of matrices with all entries of the form $p^n$, $n \in \mathbb{Z}$. The monoid structure on $GL_{\ell+1}(\mathbb{R}_p)$ is defined using (8.4). In the limit $p \rightarrow 1$ the monoid $GL_{\ell+1}(\mathbb{R}_p)$ reduces to the tropical monoid $GL_{\ell+1}(\mathbb{R})$ considered in the previous Section.

Local non-Archimedean $L$-factors admit integral representations which can be rewritten as sums over a set of points of the varieties defined over $\mathbb{R}_p$. Below we illustrate how the elementary local $L$-factors arise as $L$-functions over $\mathbb{Q}_1$. Consider an additive character $\psi_p(x)$ of $\mathbb{Q}_p$

$$\psi_p(x + y) = \psi_p(x)\psi_p(y),$$

given by a step-function (characteristic function of the subset $\mathbb{Z}_p \subset \mathbb{Q}_p$)

$$\psi_p(x) = \Theta(v_p(x)).$$

(8.5)

Fix a multiplicative character $\chi_s(x)$ of $\mathbb{Q}_p$

$$\chi_s(x \cdot y) = \chi_s(x)\chi_s(y), \quad \chi_s(x) = p^{-sv(x)}.$$  

(8.6)

The local non-Archimedean $L$-factor

$$L_p(s) = \frac{1}{1 - p^{-s}}.$$  

has the following standard integral representation (see e.g. [W], [MaP]):

$$L_p(s) = \int_{\mathbb{Q}_p^\times} d\mu^\times(x) \psi_p(x)\chi_s(x) = \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1 - p^{-s}}.$$  

(8.7)

where $d\mu^\times$ is the multiplicative measure on $\mathbb{Q}_p^\times$ for which $\text{vol}(\mathbb{Z}_p^\times) = 1$. Indeed, integrating over the fibres $\mathbb{Z}_p^*$ of the projection $\mathbb{Q}_p^* \rightarrow p\mathbb{Z}$, one obtains a sum over $\mathbb{Z}$. Taking into account the restriction imposed by $\psi_p(x)$, the sum reduces to a sum over $\mathbb{Z}_{\geq 0}$ and reproduces the right hand side of (8.7).

It is easy to check that the $p \rightarrow 1$ limit of the integral representation (8.7) recovers the integral representation of the elementary local $L$-factors (see (4.10) for $t = p^{-1}$, $\lambda_j = 0$ and $\ell = 0$). It is instructive to consider the limit of the integral representation (8.7). The limits of the additive and multiplicative characters (8.5), (8.6) are multiplicative and additive characters of $\mathbb{R}$

$$\psi_0(\tau) = \Theta(\tau), \quad \chi_s^{(0)}(\tau) = e^{-st},$$

where $\Theta(\tau)$ is the Heaviside function

$$\Theta(\tau) = 1, \quad \tau \geq 0, \quad \Theta(\tau) = 0, \quad \tau < 0.$$  

Thus for the integral representation of the elementary $L$-factor we obtain

$$L_\infty(s) = \int_{\mathbb{R}} d\tau \; \psi_0(\tau)\chi_s^{(0)}(\tau) = \frac{1}{s}.$$  

(8.8)
This is precisely the elementary local Archimedean $L$-factor (3.19) for $\ell = 0$ and $\lambda = 0$. To recapitulate, let us stress that we start with the integral representation (left hand side of (8.7)) formulated in terms of $\mathbb{Q}_p$ and rewrite it in terms of the image of $\mathbb{Q}_p$ under the valuation map $\nu_p$ (the sum representation in the right hand side of (8.7)). Then we take the limit $p \to 1$ and obtain the integral representation (8.8) for elementary $L$-factor. This integral can be also understood as an integral over tropical semifield $\mathcal{R}$ of the product of additive and multiplicative characters of $\mathcal{R}$. Thus the integral (8.8) over tropical semifield can be formally considered as an analog for $\mathbb{Q}_1$ of the intermediate step in (8.7).

References

[ABV] J. Adams, D. Barbash, and D.A. Vogan Jr., The Langlands Classification and Irreducible Characters of Real Reductive Groups, Progr. Math., 104, Birkhäuser, 1992.

[AFS] A. Alekseev, L. Faddeev, S. Shatashvili, Quantization of symplectic orbits of compact Lie groups by means of the functional integral, J. Geom. Phys. 5, (1988), 391–406.

[Au] M. Audin, Torus actions on symplectic manifolds, Progress in Math, Birkhäuser, 2004.

[BFZ] A. Berenstein, S. Fomin, A. Zelevinsky, Parametrization of canonical bases and totally positive matrices, Advances in Math. 122 (1996), 49–149.

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Schubert cells, and the cohomology of the spaces $G/P$, Russian Math. Surveys 28 (1973), no. 3, 1–26

[B] D. Bump, Automorphic Forms and Representations, Cambridge Univ. Press, Cambridge, 1998.

[CS] W. Casselman, J. Shalika, The unramified principal series of $p$-adic groups II. The Whittaker function. Comp. Math. 41 (1980) 207–231.

[Ch] I. Cherednick, Double affine Hecke algebras, London Math. Soc., Lecture Note Ser., 319.

[CG] N. Chriss, V. Ginzburg, Representation theory and complex geometry, Birkhäuser, 1997.

[CC] A. Connes, K. Consani, Characteristic one, entropy and the absolute point, [arXiv:0911.3537].

[EKL] M. Einsiedler, M. Kapranov, D. Lind, Non-archimedean amoebas and tropical varieties, [arXiv:math.AG/0408311].

[Et] P.I. Etingof, Whittaker functions on quantum groups and $q$-deformed Toda operators, Differential topology, infinite-dimensional Lie algebras, and applications, 9-25, Amer. Math. Soc. Transl. Ser. 2, 194, Amer. Math. Soc., Providence, RI, 1999, [arXiv:math.QA/9901053].

[F] K. Fukaya, Multivalued Morse theory, asymptotic analysis and mirror symmetry, in Graphs and Patterns in Mathematics and Theoretical Physics, ed: M. Lyubich, L. Takhtajan, Proceedings of Symposia in Pure Mathematics, 73, 2005.

[Fu] W. Fulton, Young Tableaux: With Applications to Representation Theory and Geometry, Cambridge University Press, 1997.

42
[GZ] I.M. Gelfand, M.L. Tsetlin, *Finite-dimensional representations of the group of unimodular matrices*, Dokl.Akad. Nauk SSSR 71 (1950), 825–828.

[GKZ] I. Gelfand, M, Kapranov, A. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, 1994.

[GKL] A. Gerasimov, S. Kharchev, D. Lebedev, *Representation Theory and Quantum Inverse Scattering Method: the open Toda chain and the hyperbolic Sutherland model*, Int. Math. Res. Notices, 17 (2004) 823–854, [arXiv:math/0204206].

[GKLO] A. Gerasimov, S. Kharchev, D. Lebedev, S. Oblezin, *On a Gauss-Givental representation of quantum Toda chain wave function*, Int. Math. Res. Notices, (2006), ID:96489, [arXiv:math.RT/0505310].

[GLO1] A. Gerasimov, D. Lebedev, S. Oblezin, *New integral representations of Whittaker functions for classical groups*, Uspechi Math. Nauk, 67 number 1(403), (2012), 1–94, [arXiv:math.RT/0705.2886].

[GLO2] A. Gerasimov, D. Lebedev, S. Oblezin, *Baxter operator and archimedean Hecke algebras*, Commun. Math. Phys. 284(3), (2008), 867–896; [arXiv:0706.3476].

[GLO3] A. Gerasimov, D. Lebedev, S. Oblezin, *On q-deformed $gl_{l+1}$-Whittaker functions I*, Commun. Math. Phys. 294 (2010), 97–119, [arXiv:0803.0145].

[GLO4] A. Gerasimov, D. Lebedev, S. Oblezin, *On q-deformed $gl_{l+1}$-Whittaker functions II*, Commun. Math. Phys. 294 (2010), 121–143, [arXiv:0803.0970].

[GLO5] A. Gerasimov, D. Lebedev, S. Oblezin, *On q-deformed $gl_{l+1}$-Whittaker functions III*, Lett. Math. Phys. DOI 10.1007/s11005-011-0468-y, [arXiv:0805.3754].

[GLO6] A. Gerasimov, D. Lebedev, S. Oblezin, *Archimedean L-factors and Topological Field Theories I*, Communications in Number Theory and Physics, v 5, (2011) no 1, 57–100, [arXiv:0906.1065].

[GLO7] A. Gerasimov, D. Lebedev, S. Oblezin, *Archimedean L-factors and Topological Field Theories II*, Communications in Number Theory and Physics, v 5, (2011) no 1, 101–133, [arXiv:0909.2016].

[GLO8] A. Gerasimov, D. Lebedev, S. Oblezin, *Parabolic Whittaker Functions and Topological Field Theories I*, Communications in Number Theory and Physics, v 5, (2011) no 1, 135–201, [arXiv:1002.2622].

[GLO9] A. Gerasimov, D. Lebedev, S. Oblezin, *On a classical limit of q-deformed Whittaker functions*, Lett. Math. Phys. DOI 10.1007/s11005-012-0545-x, [arXiv:1101.4567].

[G] A. Gerasimov, *A Quantum Field Theory Model of Archimedean Geometry*, talk at Rencontres Itzykson 2010: New trends in quantum integrability, 21–23 June, 2010, IPhT Saclay, France (see link to slides on the webpage of the conference).

[GL] A. Gerasimov, D. Lebedev, *On topological field theory representation of higher analogs of classical special functions*, JHEP v.2011(2011) no. 9, p.76, DOI:10.1007/JHEP(2011), 076, [arXiv:1011.0403].
[Gi] V. Ginzburg, Geometric methods in the representation theory of Hecke algebras and quantum groups, [arXiv:math.AG/9802004].

[Giv] A. Givental, Stationary Phase Integrals, Quantum Toda Lattices, Flag Manifolds and the Mirror Conjecture. Topics in Singularity Theory, Amer. Math. Soc. Transl. Ser., 2 180, AMS, Providence, Rhode Island, 1997, 103–115 [arXiv:alg-geom/9612001].

[GLS] V. Guillemin, E. Lerman, S. Sternberg, Symplectic Fibrations and Multiplicity Diagrams, Cambridge University Press, 1996.

[HO] G.J. Heckman, E.M. Opdam, Yang’s system of particles and Hecke algebras, Annals of Math. 145 (1997), 139.

[I] C. Itzykson, Simple Integrable Systems and Lie Algebras, International Journal of Modern Physics A, 1, 01, 1986, pp. 65–115.

[IMS] I. Itenberg, G. Mikhalkin, E. Shustin, Tropical algebraic geometry, Birkhäuser, 2009.

[KL] S. Kharchev, D. Lebedev, Eigenfunctions of GL(N, R) Toda chain: The Mellin-Barnes representation, JETP Lett. 71, (2000), 235–238, [arXiv:hep-th/0004065].

[K] A.A. Kirillov, Lectures on the Orbit Method, Graduate Studies in Mathematics, 64, 2004.

[KS] M. Kontsevich and Y Soibelman, Integral affine structures In: The Unity of Mathematics in honor of the 90th birthday of I.M. Gelfand, Progress in Mathematics 244, Birkhäuser (2005), 321–386.

[KK] B. Kostant, S. Kumar, T-equivariant K-theory of generalized flag varieties, J. Differential Geom., 32, 2 (1990), 549–603.

[Ku] N. Kurokawa, Zeta functions over $F_1$, Proc. Japan Acad., 81, Ser. A (2005),180–184.

[L] An introduction to the Langlands program, Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001. Edited by J. Bernstein and S. Gelbart. Birkhäuser Boston, Inc., Boston, MA, 2003.

[Lu1] G. Lusztig, Introduction to Quantum Groups, Progress in Mathematics, 110, Birkhäuser Boston, Inc., Boston, MA, 1993.

[Lu2] G. Lusztig Total positivity in reductive groups, in Lie Theory and Geometry: In Honor of B. Kostant, Progr. Math. 123, Birkhäuser, 1994, 531–568.

[Ma] Yu.I. Manin, Lectures on zeta functions and motives (according to Deninger and Kurokawa), In: Columbia University Number Theory Seminar, Asterisque, 228 (1995), 121–164.

[MaP] Y. Manin, A. Panchishkin, Introduction to modern number theory, 2ed., Springer, 2005.

[MS] Idempotent analysis, V.P. Maslov (ed.) S.N. Samborskii (ed.) , Amer. Math. Soc. (1992).

[Mi] G. Mikhalkin, Amoebas of algebraic varieties and tropical geometry, [arXiv:math.AG/0403015].

[O1] S. Oblezin, On parabolic Whittaker functions, [arXiv:math.AG/1011.4250].

[O2] S. Oblezin, On parabolic Whittaker functions II, [arXiv:math.AG/1107.2998].
[R]  S.N.M. Ruijsenaars, *Relativistic Toda system*, Comm. Math. Phys. 133 (1990), 217–247.

[Sh]  T. Shintani, *On an explicit formula for class 1 Whittaker functions on GL_n over p-adic fields*. Proc. Japan Acad. 52 (1976), 180–182.

[Skl]  E. Sklyanin, *The quantum Toda chain*, in Lecture Notes in Phys., 226, Springer, New York, 1985, pp. 196–233.

[STS]  M. Semenov-Tian-Shansky, *Quantization of open Toda lattice*, in Encyclopedia of Math. Sciences, 16, Springer Verlag, 1994, 226–259.

[W]  A. Weil, *Basic of Number theory*, Springer, 1967.

[Zhi]  D.P. Zhelobenko *Compact semisimple Lie groups and their representations*, Amer. Maths. Soc., Translations of Mathematical monographs, 40.

---

A.G. Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia; School of Mathematics, Trinity College Dublin, Dublin 2, Ireland; Hamilton Mathematics Institute, Trinity College Dublin, Dublin 2, Ireland; E-mail address: anton@maths.tcd.ie

D.L. Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia; E-mail address: lebedev.dm@gmail.com