Dimension of spacetime from the viewpoint of different fields

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Abstract

In this paper a new geometric perspective on gravity is considered, proposing a different geometric feature of gravitational effects on Minkowski spacetime which can be seen as classification of the spacetime into some equivalence classes. By introducing some geometric objects based on noncommutative geometry, one can switch between field-picture and geometry-picture representing gravity and conclude dimensionally dependence of gravitational field equation and consequently emerging different features of gravity in spacetimes with different dimensions. Furthermore in the case of interaction of gravity with an external field, one can deduce that from the viewpoint of different fields, spacetime can be seen as an object with different dimensions. The proposal makes the possibility of resolving blackhole singularity and also observing some similarities with Kaluza-Klein-like theories where a reduction from higher to lower dimensional gravity is associated with emerging some different fields.

Keywords: Quantum Gravity, Non Commutative Geometry, Bargmann-Wigner Formalism

1 Introduction

As we know, Einstein’s general relativity provides a description of gravity as a geometric property of spacetime where effects of gravity in a flat spacetime can be replaced by considering a free theory on a curved geometry. In

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this work a different viewpoint has been considered. We preserve Minkowski
spacetime structure, aiming to show a completely different geometric version
of the gravitational effects appearing in Minkowski spacetime structure, which
will not be interpreted as curvature on geometry. As required mathematical
framework, we choose Connes’ noncommutative geometric construction
[1, 2] which seems may provide an appropriate framework to achieve quantum
geometry. In section 2, by means of some elementary concepts of the noncom-
mutable geometry (NCG), we introduce some mathematical constructions
to present quantum spacetime structure, including a new geometric concept
-quantum points (q-points)- which can be interpreted as superposition of ge-
ometric points (g-points). Afterward in sections 3 and 4, we apply our new
ground concepts through Bargmann-Wigner formalism where an appropriate
tree toward gravity can be achieved. In our formalism, effects of pure
gravity can be seen as classification of 4-dimensional Minkowski spacetime into
$(4 - 1)$-dimensional foliations. This idea can be extended to a $D$-dimensional
Minkowski spacetime providing a framework to indicate spacetime dimension-
dependence of gravitational field spin and therefore emerging different features
of gravity in spacetimes with different dimensions. Furthermore as explained
in section 5, in the case where gravity coupled to an external field, the equiva-
ence classes change such that from the viewpoint of different external fields,
spacetime seems to have different dimensions.
In last section, dimensionally dependence of gravitational field spin explains
preventing the formation of singularity inside blackholes. Also observing some
similarities with Kaluza-Klein theory, can suggest interpreting the spin-1 field
emerged after compactification in this theory, as a different aspect of gravita-
tional field.

2 Quantization of metric

In Connes’ formulation of NCG, all information on the geometry of a (non)
commutative geometry (NCG), we introduce some mathematical constructions
of quantum spacetime structure, including a new geometric concept
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2 Quantization of metric

In Connes’ formulation of NCG, all information on the geometry of a (non)
commutative space are encoded in spectral triple $(A, H, D)$, where $A$ is a
C*-algebra represented on Hilbert space $H$ by linear operators, and $D$ is an
unbounded self-adjoint operator called Dirac operator which is not in $A$ but
acts on $H$, satisfying precise conditions [1, 2].

Given a spectral triple $(A, H, D)$, one can find distance between any two states
on the algebra $A$ by Connes’ distance function defined by:

$$d(\phi, \psi) = \sup\{|\phi(a) - \psi(a)|; a \in A, \|[D, a]\| \leq 1\},$$

(A state on the algebra $A$ is a linear functional $\phi : A \rightarrow \mathbb{C}$ which is positive, i.e.
$\forall a \in A, \phi(a^*a) \geq 0$ and of norm one, i.e. $||\phi|| = 1$.}

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where $\phi$ and $\psi$ are two states on the algebra. In the case where $A$ is commutative, states can be interpreted as points. For instance consider a space made of $n$ points $p_1, ..., p_n$. For this space, the algebra is the direct sum $A = \oplus_n \mathbb{C}$ and any element $a \in A$ is identified with $n$ complex numbers $(a(p_1)...a(p_n))$, with $a(p_i)$ the corresponding value of $a$ at the point $p_i$, similar to the algebra of functions on a manifold $M$. In matrix representation, we may represent elements of $A$ as diagonal matrices,

$$A \ni a \rightarrow \pi(a) = \begin{pmatrix} a(p_1) \\ \vdots \\ a(p_n) \end{pmatrix},$$

where by $\pi(a)$, we mean matrix representation of $a$ and $a(p_i)$ is the eigenvalue corresponding to eigenstate $|p_i\rangle$ of $a$, i.e. $a|p_i\rangle = a(p_i)|p_i\rangle$. So one can interpret $|p_i\rangle$ as point $p_i$ which we name them geometric points (g-points) and now distance between any two states on the algebra can be replaced by distance between the corresponding two g-points which is given by:

$$d(p_i,p_j) = \sup\{|a(p_i) - a(p_j)|; a \in A, \| [D, a] \| \leq 1 \}.$$  \hspace{1cm} (3)

Here, we consider space of eigenstates of Dirac operator $\{ |\psi_i\rangle \}$, instead of eigenstates of the algebra $\{ |p_i\rangle \}$, where $|\psi_i\rangle = \sum_k c_{ik} |p_k\rangle; c_{ik} \in \mathbb{C}$, and interpret these states as quantum points (q-points) which obviously are superposition of the g-points$^2$. With this definition we no longer have specific distance between any two q-points which obviously can be interpreted as quantization of the distance.

Before applying our defined geometric concept to describe quantum geometry of spacetime, let have a brief review of Bargmann-Wigner formalism which seems can suggest a possible route for investigating gravitational effects.

### 3 Bargmann-Wigner formalism

In particle physics Bargmann-Wigner (B.W) formalism are used to derive equation of motion for a spin $s$ particle. According to B.W formalism, a field of rest mass $m$ and spin $s \geq 1/2$ in Minkowski spacetime is represented by a totally symmetric $2s$-rank multispinor $\Psi_{\alpha\beta...\tau}(x)$, satisfying $2s$ Dirac-type equations in all indices $^4$. As an example for $s = 2$, the equations reduce to

$^2$In this article by g-point we mean candidate for eigenstate of the algebra and q-point as candidate for eigenstate of Dirac operator.
the following set of 4 equations:

\[(i\gamma^\mu\partial_\mu - m)_{\alpha\alpha}\Psi_{\alpha\beta\gamma\delta}(x) = 0,\]
\[(i\gamma^\mu\partial_\mu - m)_{\beta\beta}\Psi_{\alpha\beta\gamma\delta}(x) = 0,\]
\[(i\gamma^\mu\partial_\mu - m)_{\gamma\gamma}\Psi_{\alpha\beta\gamma\delta}(x) = 0,\]
\[(i\gamma^\mu\partial_\mu - m)_{\delta\delta}\Psi_{\alpha\beta\gamma\delta}(x) = 0.\] (4)

We can go to the rest frame and find four independent positive energy plane-wave solutions. For example in presentation of the \(\gamma\)-matrices \(\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), and \(\gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}\), the solutions are \(U_i(k)e^{i\mathbf{K}\cdot\mathbf{x}-iE_kt}\), where \(k = (E_k, \mathbf{K})\) is energy-momentum four-vector, and

\[
U_1 = \begin{pmatrix} A \\ B \\ 0 \\ 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} C \\ D \\ 0 \\ 0 \end{pmatrix}, \quad U_3 = \begin{pmatrix} B \\ C \\ 0 \\ 0 \end{pmatrix}, \quad U_4 = \begin{pmatrix} D \\ E \\ 0 \\ 0 \end{pmatrix},
\]

where \(A, B, C, D\) and \(E\) are determined with completely symmetry requirement in all indices of the spinor \(\Psi_{\alpha\beta\gamma\delta}(x)\), which enforces the indices take only two values 1 and 2.

Therefore only four independent \(\Psi\)s are responsible to represent a spin 2 field. On the other hand, it can be shown that with a change of variable in the 4-rank symmetric spinor, one can derive Einstein’s equation for spin 2 field, describing equation of motion for a spin 2 particle in Minkowski spacetime which has a proper geometrical description in the context of General Relativity [5]. So in the field-perspective, the four independent solutions of \(B.W\) equations contain gravitational information. Furthermore, according to our geometry-perspective, the Dirac eigenstates are geometric objects (q-points) as representative of spacetime structure. Therefore the gravitational field and spacetime are, in fact, indistinguishable. So considering the four independent q-points containing gravitational information brings up a question that how gravity affects Minkowski spacetime structure. In the next section we propose a different geometrical interpretation of gravity through B.W formalism rather than GR.

\[3^\text{As explained in chapter 2 of reference [6].}\]
4 Gravity from the viewpoint of B.W equations and spacetime dimension-dependence of gravitational field spin

Now let consider B.W formalism in the framework of NCG. In B.W equations the Dirac operator is $i\gamma^\mu \partial_\mu$ and the four independent $\Psi$s, containing gravitational information, play the role of q-points. So it will be interesting to study how gravity acts on geometry of spacetime in the viewpoint of q-points. Therefore we consider the plane wave solutions $\Psi(x) = U(p)e^{ip.x}$ as the q-points where $p$ is an arbitrary four vector which satisfy $p^2 = m^2$ and $U(p)$ is spinorial part of $\Psi(x)$. Here all the q-points are degenerate with the same eigenvalue $m$. This correspondence comes from $e^{ip.x}$ part in the plane wave solution. For a specific four vector $p$, infinite number of position four-vectors $4$ can result a fixed value for $p.x$ and so all of these g-points with the same $p.x$ value, correspond to the same q-point $\Psi(x)$. Actually this observation reflects the idea which a q-point is superposition of some g-points as we expected previously. On the other hand all of the g-points like $x$ which satisfy the equation $p.x = b$, with fixed values of $p$ and $b$, are located in a 3D subspace of our 4D spacetime. This observation may force us to define an equivalence relation between g-points of spacetime as $x \sim x'$ if $p.x = p.x' = b$, for specific $p$ and $b$. By fixing $p$, spacetime is foliated by some 3-dimensional equivalence classes corresponding to different values of $b$ such that from the viewpoint of $\Psi$, the g-points of each equivalence class are indistinguishable. So according to B.W formalism, gravity without external sources is a phenomenon which partitions 4D space to disjointed $(4-1)$D subspaces$^5$ and consequently from the viewpoint of the gravitational field $\Psi$, space can effectively be seen one-dimensional, i.e., the resulting quotient space is one-dimensional. On the other hand from introductory geometry we know that any (flat) $n$-dimensional subspace embedded in a (flat) $D$-dimensional space can be specified by $n + 1$ (geometrically) independent points which according to B.W formalism, can be the 4 independent q-points (4 independent spinorial parts) for each 3D foliation. If we postulate that the result can be extended to spacetime with arbitrary dimension, namely the gravitational field $\Psi$ always treats the spacetime effectively as a one-dimensional space, then in a $D$-dimensional spacetime only $D$ (geometrically) independent points (q-points) are required to specify $D-1$ dimensional equivalence classes representative of gravitational effects. In other words in a $D$-dimensional space, $D$ independent q-points are solu-

$^4$Each position four-vector plays the role of a spacetime g-point.

$^5$In the case of interaction with an external field, we expect to have different equivalence classes.
tions to $D$ Dirac-type equations of B.W formalism. By considering the fact that the number of B.W equations are $2s$, a direct consequence is that for a $D$-dimensional space, the spin of gravitational field is $D/2$.

At this stage a natural question is that: “Is there any evidence for this amazing conclusion?” Actually we live in a 4D-world, but we think that in this 4D-space, some regions might be found with effectively less than 4 dimensions. As an example, at the center of a blackhole we may find such places. On the other hand in some theoretical models which possesses dimension reduction mechanism, such as Kaluza-Klein and ADS/CFT theories, we can find the emergence of different fields with different spins after dimensional reduction. Of course since these theories are considered in different frameworks, the emerged fields usually interpreted as completely new fields. In the last section we will consider some of these titles.

5 The role of Dirac operator in the classification of Minkowski spacetime

In our formalism, gravitational effects can be seen through equivalence classes emerged in Minkowski spacetime. However in general relativity, gravitational effects appear through curved geometry and energy-momentum tensor acts as the source of spacetime curvature. Therefore a question may arise: “what is reasonable candidate for the source of the gravitational effects which can make some changes in structure of the equivalence classes?” As mentioned before through B.W equations, pure gravity forces to classify Minkowski spacetime into some equivalence classes. Therefore it is expected that Dirac operator play important role in this classification. For considering the source of gravity in our formalism, we can define generalized Dirac operator with the notion of considering bundle description over Minkowski spacetime and consequently see appearing different equivalence classes as different gravitational effects. Also we can bring the source of gravity through Dirac eigenstates without touching the dirac operator, as is mentioned in reference [5]. The comparison of our formalism with the Einstein’s framework can be seen in the following table.
| Einstein’s formalism | Our formalism |
|----------------------|---------------|
| Einstein’s equation  | B.W equations |
| Source of gravity    | Connection on principle bundle over Minkowski spacetime |
| (energy-momentum tensor) | The equivalence classes over Minkowski spacetime |
| Curved space         |               |

As a special case, not necessarily a general case, a gauge field can be considered as a source of gravity.

To generalize the Dirac operator, as simplest case we can consider Dirac equation in an external Yang-Mills gauge field $A^a_\mu(x)$. So we have the extended Dirac operator $i\gamma^\mu(\partial_\mu + igA^a_\mu(x)T_a)$ in the equation:

$$\{i\gamma^\mu(\partial_\mu + igA^a_\mu(x)T_a) - m\} \Psi(x) = 0;$$

where $g$ is coupling constant and $T_a$s are generators of $SU(N)$ group. According to reference [7] we have:

$$\Psi(x) = \frac{i\gamma^\mu(\partial_\mu + igA^a_\mu(x)T_a) + m}{2m} \Phi(x),$$

where $\Phi(x)$ has the following form:

$$\Phi(x) \equiv \psi_{\sigma\alpha}(x, p) = e^{ip.x} \exp\{\frac{g^2}{2N} \int_0^\phi d\phi'(A^a_\mu(\phi')A^a_\mu(\phi')) \}$$

$$\times \cos(\theta) \{(1 - igT_\mu \frac{\tan\theta}{(p.k)} \int_0^\phi d\phi'(A^{a\mu}(p')\mu)) + \frac{g(\gamma^\mu k_\mu)(\gamma^\nu A^a_\nu)}{2p.k} \}$$

$$\times \frac{\tan\theta}{\theta} T_a + \frac{g}{(p.k)} \frac{1}{2N} \int_0^\phi d\phi'(A^{a\mu}(p')\mu)(-i\frac{\tan\theta}{\theta}) + \frac{g}{(p.k)} \frac{\theta - \tan\theta}{\theta^3}$$

$$\times T_b \int_0^\phi d\phi'(A^{b\mu}(p')\mu))\} u_\sigma(p) \cdot v_\alpha; \quad (5)$$

where the 4-vectors $k^\mu$ and $p^\mu$ are such that $k^\mu k_\mu = 0$, $p^\mu p_\mu = m^2$, $u_\sigma(p)$ and $v_\alpha$ are some spinors. $\theta = \sqrt{\frac{g^2}{2N} \int_0^\phi d\phi'(A^a_\mu(\phi')p') \int_0^\phi d\phi''(A^a_\mu(\phi'')p'')^{1/2},}$

$\phi \equiv k.x$ and there is a gauge field condition $A^a_\mu(x) = A^a_\mu(k.x)$. 

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As a special case, not necessarily a general case, a gauge field can be considered as a source of gravity.
By considering the solution, here the equivalence classes should be identified with two conditions \( p.x = \text{constant} \) and \( k.x = \text{constant} \). For specific values of \( p \) and \( k \), the equivalence classes are intersection of two 3D subspaces in 4D Minkowski spacetime which consequently are 2-dimensional. Namely from the viewpoint of the gauge field, spacetime is a 2-dimensional quotient space. Therefore it seems that from the viewpoint of different fields, spacetime can be seen as an object with different dimensions, as from the viewpoint of matter field, spacetime is 4-dimensional\(^6\). The result has some similarities with the one obtained in reference [8], where in the context of loop quantum gravity, dimensionally transition of spacetime from lower to higher dimension, is associated with adding some new fields to pure gravity, interpreted as matter fields.

6 Some physical consequences

- **The Kaluza-Klein theory**

In Kaluza-Klein theory [9, 10], which is a classical theory, general relativity is extended to a five dimensional spacetime. By compactification of one of the spatial dimensions, an ordinary theory of gravity in four dimensions together with a Maxwellian theory of electromagnetism can be achieved. But appearing the both different types of forces with the same order of magnitude can not be justified. On the other hand, the present approach predicts that with a change in spacetime dimension, gravity will represent itself in a different feature. Comparing the result with emerging spin-1 field by compactification in Kaluza-Klein theory, one might find a similarity between the present quantum approach and the Kaluza-Klein approach which both predict emerging different fields depending on spacetime dimension. Therefore the appeared electromagnetic-type force might be interpreted as a different aspect of gravity not a different type of force.

- **Blackhole singularity**

Curvature singularity is predicted by Einstein’s general relativity for gravitation. Such a prediction indicates that the geometrical description offered by general relativity fails to give a consistent picture at spacetime singularities. The spacetime singularities must be resolved with a theory of gravity in which blackhole singularity and Bigbang singularity

\(^6\)One might predict that in 5D or higher dimensional spacetime, other kinds of matter fields should be searched, which from the viewpoint of those fields, spacetime can be seen as the higher dimensional object.
are controlled by quantum effects [6, 11, 12, 13, 14].

Now consider a static blackhole. The Schwarzschild metric, 
\[ ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\Omega^2, \]
describes the spacetime curvature around the blackhole with a curvature singularity located at \( r = 0 \). As \( r \to 0 \) one obtains the metric independent of the angular coordinate \( \Omega \).

So in this limit we effectively have two dimensional spacetime and according to what the present approach proposed, the spin of gravitational field should change to 1 and so gravitational force law changes and one can see a different aspect of gravity as a repulsive force in the same order of magnitude of attractive force arising from the spin 2 field, so forming a singularity is prevented in the proposed model.\(^7\)

7 Conclusion

We introduce a new geometrical interpretation of spacetime structure with defining geometric concept of q-points in the framework of NCG. Applying this geometric construction through B.W equations, we propose that gravitational effects on Minkowski spacetime can be seen as a completely different geometric version which classifies the spacetime into some equivalence classes. This classification may change in the case of introducing interaction of gravity with an external field, such that from the viewpoint of different external fields, spacetime can be seen dimensionally different. Furthermore the model proposes that spin of the field introducing gravity depends on spacetime dimension. Indeed in dimensionally different spacetimes, gravitational force law is different. Comparing the result with some theories where a change of spacetime dimension assumed, might exhibit some precious physical features. For instance by observing some similarities with Kaluza-Klein theory, the appeared spin 1 field in this theory can be interpreted as a different aspect of gravity. Also the result can explain the absence of singularity in blackholes.

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\(^7\) A similar procedure is shown in reference [15], where a Yang-Mills gauge theory coupled to Einstein gravity, can make a repulsive force to balance gravitational attractive force preventing spacetime singularity.
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