A high-order discontinuous Galerkin method for the poro-elasto-acoustic problem on polygonal and polyhedral grids

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Abstract

The aim of this work is to introduce and analyze a finite element discontinuous Galerkin method on polygonal meshes for the numerical discretization of acoustic waves propagation through poroelastic materials. Wave propagation is modeled by the acoustics equations in the acoustic domain and the low-frequency Biot’s equations in the poroelastic one. The coupling is introduced by considering (physically consistent) interface conditions, imposed on the interface between the domains, modeling both open and sealed pores. Existence and uniqueness is proven for the strong formulation based on employing the semigroup theory. For the space discretization we introduce and analyze a high-order discontinuous Galerkin method on polygonal and polyhedral meshes, which is then coupled with Newmark-β time integration schemes. A stability analysis both for the continuous problem and the semi-discrete one is presented and error estimates for the energy norm are derived for the semidiscrete problem. A wide set of numerical results obtained on test cases with manufactured solutions are presented in order to validate the error analysis. Examples of physical interest are also presented to test the capability of the proposed methods in practical cases.

1 Introduction

The paper deals with the mathematical model and numerical analysis of the coupled poro-elasto-acoustic differential problem modeling an acoustic/sound wave impacting a poroelastic medium and consequently propagating through it. Coupled poro-elasto-acoustic problems model the combined propagation of pressure and elastic waves through a porous material. Pressure waves propagate through the saturating fluid inside pores, while acoustic ones through the porous skeleton. The theory of propagation of acoustic waves with application to poroelasticity has been developed mainly by Biot [14] in 1956, by introducing general equations and proposing different ways to treat coupling between acoustic and poro-elastic domains. Pioneering advances of Biot’s theory concerned with slow compressional waves, whose study carried on the analysis on fast compressional waves, introduced in 1944 by Frenkel. Coupled poro-elasto-acoustic models find application in many science and engineering fields. For example, in acoustic engineering, for the study of sound propagation through acoustic panels, whose main intent is to intercept and absorb acoustic waves for noise reduction [41]; in civil engineering, for the study of passive control and vibroacoustics, where plastic foams and fibrous or granular materials are mainly used with this intent [29]; in aeronautical engineering, where
air-saturated porous materials are employed [20]; in biomedical engineering, for the study of ultrasound propagation throughout bones to diagnose osteoporosis and study its evolution [26] and to model soft tissues deformation, such as the heart tissue [27], the skin [33] and the aortic tissue [28]. Poro-elasto-acoustic models find a wide strand of literature also in computational geosciences: we refer the reader to [19] for a comprehensive review.

In order to model the poroelastic domain, the concept of pores is necessary. Pores can be seen as “holes” in the (elastic) material where a fluid is able to move. They can be classified into open and sealed (closed) pores: the first ones share a part with the outer surface of the material, while the latter ones are totally locked in, as shown in fig. 1a below. From the modeling viewpoint, the difference between them is the way in which interface conditions between acoustic and poroelastic domains are formulated, as detailed later on. From the numerical viewpoint we mention the Lagrange Multipliers method [36, 1, 24], the finite element method [13, 23] the spectral and pseudo spectral element method [32, 37], the ADER scheme [22, 21] and the finite difference method [30], to cite a few.

The aim of this paper is to propose and analyze a high-order discontinuous Galerkin (dG) method on polygonal and polyhedral grids for the space discretization of a coupled poro-elasto-acoustic problem, by extending the theory carried out in [3], where a coupled system of elasto-acoustic equations is analyzed. We point out that the geometric flexibility due to mild regularity requirements on the underlying computational mesh together with the arbitrary-order accuracy featured by the proposed dG method are crucial within this context as they ensure at the same time a high-level of flexibility in the representation of the geometry and an intrinsic high-level of precision and scalability that are mandatory to correctly represent the solution fields. Moreover, in the proposed semidiscrete dG formulation, the coupling between the acoustic and the poroelastic domains is introduced by considering (physically consistent) interface conditions, naturally incorporated in the scheme based on employing the For early results in the field of dG methods on polygonal and polyhedral grids we refer, for example, to [8, 18, 16, 4] for second-order elliptic problems problems, to [15] for time dependent problems, to [6, 7] for flows in fractured porous media, to [2] for fluid structure interaction problems, to [10] for non-liner sound waves, cf. also [17] for a comprehensive monograph and to [5] for a recent review in the context of geophysical applications.

The remaining part of the paper is structured as follows: in Section 2 we introduce the mathematical model and state an existence and uniqueness for the strong formulation. Moreover, we present the weak formulation of the continuous problem and prove suitable stability estimates. In Section 3 we introduce the the semidiscrete dG approximation on polygonal and polyhedral meshes and prove its stability. Section 4 is devoted to the analysis of the semi-discrete problem and $hp$—version $a$—priori error estimates are shown for the semidiscrete formulation. The time integration schemes based on the Newmark-β method are presented in Section 5. In Section 6 we discuss some two-dimensional numerical experiments to validate the theoretical results and show the performance of the proposed method in some examples of physical interest. Finally, in Section 7 we draw some conclusions. The proofs of the main theoretical results are postponed to Appendix.

2 The physical model and governing equations

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, convex polygonal/polyhedral domain decomposed as the union of two open, convex, polygonal/polyhedral subdomains: $\Omega = \Omega_\ell \cup \Omega_a$, representing the poroelastic and the acoustic domains, respectively, cf. fig. 1b. The two subdomains share part of their boundary, resulting in the interface $\Gamma_I = \partial \Omega_\ell \cap \partial \Omega_a$. The boundary of $\Omega$ is denoted by $\partial \Omega$, and we set $\partial \Omega_\ell = \Gamma_{pD} \cup \Gamma_I$ and $\partial \Omega_a = \Gamma_{aD} \cup \Gamma_I$, with $\Gamma_{pD} \cap \Gamma_I = \emptyset$ and $\Gamma_{aD} \cap \Gamma_I = \emptyset$. Surface measures of $\partial \Omega$, $\partial \Omega_p$, $\partial \Omega_a$ and $\Gamma_I$ are assumed to be strictly positive. The outer unit normal vectors to $\partial \Omega_p$ and $\partial \Omega_a$ are denoted by $\mathbf{n}_p$ and $\mathbf{n}_a$, respectively, so that $\mathbf{n}_p = -\mathbf{n}_a$ on $\Gamma_I$. In the following, for $X \subseteq \Omega$, the notation $L^2(X)$ is adopted in place of $[L^2(X)]^d$, with $d \in \{2, 3\}$. The scalar product in $X$ is denoted by $(\cdot, \cdot)_X$, with associated norm $\| \cdot \|_X$. Similarly, $H^\ell(X)$ is defined as $[H^\ell(X)]^d$, with $\ell \geq 0$, equipped with the norm
(a) Pores classification in a poroelastic domain. (b) $\Omega = \Omega_p \cup \Omega_a$.

Figure 1: (1a) Pores classification in a poroelastic domain: sealed (1) and open (2) pores; (1b) Graphic representation of the domain $\Omega = \Omega_p \cup \Omega_a$ for $d = 2$.

∥·∥_{ℓ,X}. Assuming conventionally that $H^0(X) \equiv L^2(X)$, we set: $∥·∥_X \equiv ∥·∥_{0,X}$. For any integer $k \geq 0$ and a generic Hilbert space $H$, the usual notation $C^k([0,T];H)$ has been adopted for the space of functions $k$-times differentiable in $[0,T]$, belonging to $H$. The notation $x \preceq y$ is introduced in place of $x \leq Cy$, with $C > 0$, independent of the discretization parameters, but possibly dependent on physical coefficients and the final observation time $T$.

2.1 The poro-elasto-acoustic problem

For a final observation time $T > 0$, to model wave propagation in a poro-elastic domain $\Omega_p$ we consider the two-displacement formulation of [31], written in the solid and filtration displacements, denoted by $u$ and $w$, respectively. As in [21], we consider the low-frequency Biot’s equations:

$$\begin{cases}
\rho\ddot{u} + \rho_f\ddot{w} - \nabla \cdot \sigma = f_p, & \text{in } \Omega_p \times (0,T], \\
\rho_f\ddot{u} + \rho_w\ddot{w} + \frac{k}{a}\dot{w} + \nabla p = g_p, & \text{in } \Omega_p \times (0,T].
\end{cases} \tag{1}$$

Here, the average density $\rho$ is given by $\rho = \phi \rho_f + (1 - \phi) \rho_s$, where $\rho_s$ is the solid density, $\rho_f$ is the saturating fluid density, $\rho_w$ is defined as $\rho_w = \frac{\phi}{a} \rho_f$, being $\phi$ the porosity of the material, that is nothing but the percentual of void spaces and satisfy $0 < \phi_0 \leq \phi \leq \phi_1 < 1$ and being $a > 1$ the tortuosity, a measure of the deviation of the pore fluid paths from straight streamlines in porous and saturated media, [38]. In (1) $\eta$ is used to represent the dynamic viscosity of the fluid, a measure of internal resistance and $k$ is the absolute permeability.

Remark 2.1. Notice that, as pointed out in [21], the second equation of system (1) is valid under a constraint on frequencies. In particular, the spectrum of the waves has to lie in the low-frequency range, so that in this analysis will be considered only frequencies lower than $f_c = \eta \phi / (2\pi a k \rho_f)$.

Remark 2.2. We point out that the two-displacement formulation (1) is not the unique possible choice. For example, one could write the equations considering the velocity of the solid skeleton $\dot{u}$ and the filtration velocity $\dot{w}$ as unknowns, cf. [21], or could consider a velocity-pressure $(u - p)$ formulation, as in [1, 12, 34], where $p$ denotes the pores pressure. Here, the two-displacement formulation turns out to be convenient in view of the coupling conditions stated below.

In $\Omega_p \times (0,T]$, we assume the following constitutive laws for the stress tensor $\sigma$ and for the pressure $p$:

$$\sigma(u,p) = C : \varepsilon(u) - \beta p I, \quad p(u,w) = -m(\beta \nabla \cdot u + \nabla \cdot w), \tag{2}$$

where the strain tensor $\varepsilon(\cdot)$ is defined as $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, and $C$ is the fourth-order, symmetric and uniformly elliptic elasticity tensor written as:

$$C = \begin{pmatrix}
\lambda + 2\mu & 0 & \lambda \\
0 & \mu & 0 \\
\lambda & 0 & \lambda + 2\mu
\end{pmatrix}.$$
Here, $\lambda$ and $\mu$ are the Lamé coefficients of the elastic skeleton. In (2) $\beta$ and $m$ are the Biot’s coefficients of the isotropic matrix. It can be shown that the Lamé coefficients of the saturated and dry matrices ($\lambda_f$ and $\lambda$, respectively) are linked, i.e.: $\lambda_f = \lambda + \beta^2 m$. By plugging the constitutive laws (2) into (1), the poro-elastic system in the two-displacement unknowns can be rewritten as:

$$\begin{cases}
\rho \ddot{u} + \rho_f \dot{w} - \nabla \cdot (C : \epsilon(u)) - \beta^2 m \nabla (\nabla \cdot u) - \beta m \nabla (\nabla \cdot w) = f_p, \\
\rho \ddot{u} + \rho_w \dot{w} + \frac{\eta}{k} \dot{w} - \beta m \nabla (\nabla \cdot u) - m \nabla (\nabla \cdot w) = g_p.
\end{cases} \quad (3)$$

In the fluid domain $\Omega_f$, we consider an acoustic wave with constant velocity $c$ and mass density $\rho_a$. For a given source term $f_a$, the acoustic potential $\varphi$ satisfies

$$c^{-2} \ddot{\varphi} - \rho_a^{-1} \nabla \cdot (\rho_a \nabla \varphi) = f_a, \quad \text{in } \Omega_a \times (0,T). \quad (4)$$

Finally, we discuss the coupling conditions interface $\Gamma_I$. The poro-elasto-acoustic coupling is prescribed through (physically consistent) interface conditions, cf. [25]. The continuity of stress and velocity fields are imposed in order to ensure the continuity of normal stresses and the conservation of mass at the interface $\Gamma_I$, respectively, while the continuity of the pressure filed is imposed by rewriting the acoustic potential in terms of a pressure. On $\Gamma_I$ we therefore impose

$$\begin{align*}
\sigma n_p &= -\rho_a \dot{\varphi} n_p, \\
(u + \dot{w}) \cdot n_p &= -\nabla \varphi \cdot n_p, \\
\tau[p] &= (\tau - 1) \dot{w} \cdot n_p,
\end{align*} \quad (5-7)$$

where $[\cdot]$ denotes the jump operator the interface $\Gamma_I$, i.e. $[p] = p_c - p_a$, and $\tau = 0,1$ is the hydraulic permeability at the interface and models both open and sealed pores, respectively, cf. fig. 1a, and the stress tensor $\sigma$ and the pressure $p$ obey the constitutive equations (2). If $\tau = 1$ (open pores), equation (7) reduces to the continuity of pressure at the interface, that is: $p = \rho_a \dot{\varphi}$, where $p$ is explicit in terms of $u$ and $w$ through constitutive law (2). If $\tau = 0$ (sealed pores), (7) simplifies to $\dot{w} \cdot n_p = 0$, that implies that (6) imposes a continuity only on the solid velocity, namely: $\dot{u} \cdot n_p = -\nabla \varphi \cdot n_p$, and the filtration velocity is imposed to be null at the interface.

Finally, supplementing the constitutive equations with suitable boundary conditions (here supposed for simplicity to be of homogeneous Dirichlet type), the poro-elasto-acoustic problem reads as: for any time $t \in (0,T]$, find $(u, w, \varphi) : \Omega_p \times \Omega_p \times \Omega_a \rightarrow \mathbb{R}$ such that:

$$\begin{align*}
\rho \ddot{u} + \rho_f \dot{w} - \nabla \cdot (C : \epsilon(u)) - \beta^2 m \nabla (\nabla \cdot u) - \beta m \nabla (\nabla \cdot w) &= f_p, \quad \text{in } \Omega_p, \\
\rho \ddot{u} + \rho_w \dot{w} + \frac{\eta}{k} \dot{w} - \beta m \nabla (\nabla \cdot u) - m \nabla (\nabla \cdot w) &= g_p, \quad \text{in } \Omega_p, \\
\rho_a c^{-2} \ddot{\varphi} - \nabla \cdot (\rho_a \nabla \varphi) &= \rho_a f_a, \quad \text{in } \Omega_a, \\
(C : \epsilon(u) + \beta m (\nabla \cdot u + \nabla \cdot w)) n_p &= -\rho_a \dot{\varphi} n_p, \quad \text{on } \Gamma_I, \\
(u + \dot{w}) \cdot n_p &= -\nabla \varphi \cdot n_p, \quad \text{on } \Gamma_I, \\
-\tau m [\beta \nabla \cdot u + \nabla \cdot w] &= (\tau - 1) \dot{w} \cdot n_p, \quad \text{on } \Gamma_I,
\end{align*} \quad (8)$$

supplemented with initial conditions $u(\cdot, 0) = u_0, w(\cdot, 0) = w_0, \dot{u}(\cdot, 0) = u_1, \dot{w}(\cdot, 0) = w_1$, in $\Omega_p$ and $\varphi(\cdot, 0) = \varphi_0, \dot{\varphi}(\cdot, 0) = \dot{\varphi}_1$ in $\Omega_a$. Notice that the acoustic equation has been multiplied by $\rho_a$.

The existence and uniqueness of a strong solution to (8) follows the lines of [3] and can be inferred in the framework of the Hille-Yosida theory. We define the space $H^2(\Omega_p) = \{v \in L^2(\Omega_p) : \nabla \cdot (C : \epsilon(v)) \in L^2(\Omega_p)\}$, $H^2(\Omega_p) = \{v \in L^2(\Omega_p) : D^2v \in L^2(\Omega_p)\}$, and $H^2(\Omega_a) = \{v \in L^2(\Omega_a) : \Delta v \in L^2(\Omega_a)\}$, where $D^2v$ denotes all the weak second derivatives for the function $v$ and state the following result (its proof is postponed to Appendix).

**Theorem 1** (Existence and uniqueness of (8)). Assume that the initial data have the following regularity: $u_0 \in H^2(\Omega_p) \cap H^1(\Omega_p) \cap H^2_0(\Omega_p), u_1 \in H^1(\Omega_p), w_0 \in H^1(\Omega_p) \cap H^2_0(\Omega_p), w_1 \in H^1(\Omega_p), \varphi_0 \in H^2(\Omega_a) \cap H^1(\Omega_a), \varphi_1 \in H^1(\Omega_a)$, and that the source terms are such that $f_p \in C^1([0,T]; L^2(\Omega_p)), g_p \in C^1([0,T]; L^2(\Omega_p))$ and $f_a \in C^1([0,T]; L^2(\Omega_a))$. Then, problem (8) admits a unique strong solution $(u, w, \varphi)$ s.t.

$$u \in C^2([0,T]; L^2(\Omega_p)) \cap C^1([0,T]; H^1_0(\Omega_p)) \cap C^0([0,T]; H^2(\Omega_p)) \cap C^0([0,T]; H^2(\Omega_a) \cap H^2_0(\Omega_p) \cap H^2_0(\Omega_p)),$$
\( w \in C^2([0,T]; L^2(\Omega_p)) \cap C^1([0,T]; H^1_0(\Omega_p)) \cap C^0([0,T]; H^2(\Omega_p) \cap H^1_0(\Omega_p)), \)
\( \varphi \in C^2([0,T]; L^2(\Omega_a)) \cap C^1([0,T]; H^1_0(\Omega_a)) \cap C^0([0,T]; H^1(\Omega_a) \cap H^1_0(\Omega_a)). \)

### 2.2 Weak formulation and stability estimates

The weak form of (8) reads as: for any \( t \in (0,T] \), find \((u,w,\varphi)\) such that

\[
(\rho \ddot{u}, v_\Omega) + (\rho_f \dot{w}, v_\Omega) + A^p(u,v) + B^p(\beta u + w, \beta v) + C^p(\varphi, v) + (\rho_f \ddot{u}, \xi)_\Omega + (\rho_f \dot{w}, \xi)_\Omega + \eta k^{-1}(\dot{w}, \xi)_\Omega + B^p(\beta u + w, \xi) + \tau C^p(\varphi, \xi) + (\rho a c^{-2} \varphi, \psi)_\Omega \\
+ A^a(\varphi, \psi) + C^a(u, \psi) + \tau C^a(w, \psi) = (f_p, v)_\Omega + (g_p, \xi)_\Omega + (\rho a f_a, \psi)_\Omega, \quad (9)
\]

for all \((v, \xi, \psi)\), where for any \( u, v \in H^1_0(\Omega_p) \) and \( \varphi, \psi \in H^1_0(\Omega_a) \) we have set

\[
A^p(u,v) = (C : \varepsilon(u), \varepsilon(v))_{\Omega_p}, \quad B^p(u,v) = (m \nabla \cdot u, \nabla \cdot v)_{\Omega_p}, \\
A^a(\varphi, \psi) = (\rho_a \nabla \varphi, \nabla \psi)_{\Omega_a}, \quad C^p(\varphi, v) = (\rho_a \varphi n_p, v)_{\Gamma}, \quad C^a(u, \psi) = (\rho_a u \cdot n_a, \psi)_{\Gamma}.
\]

Next, we present a stability estimates for the semi-discrete problem (9). To start with, we define \( \bar{\rho}_s = (1 - \phi)\rho_s, a = 1 + a_0, \) with \( a_0 > 0 \) and \( \bar{\rho}_w = \sqrt{\rho_f a_0} / \phi \) and introduce the following norm:

\[
\| (u, w, \varphi) \|_\mathcal{E}^2 = \| u \|_{E^2,\mathcal{E}}^2 + \| \varphi \|_{E,\mathcal{E},a}^2 + \| m^{1/2} \nabla \cdot (\beta u + w) \|_{\Omega_p}^2 + \| \rho_f^{1/2} (\phi^{1/2} u + \phi^{-1/2} w) \|_{\Omega_p}^2 + \| \bar{\rho}_w w \|_{\Omega_p}^2,
\]

where

\[
\| u \|_{E^2,\mathcal{E}}^2 = \| \bar{\rho}_s \dot{u} \|_{\Omega_p}^2 + \| C^{1/2} \varepsilon(u) \|_{\Omega_p}^2 \quad \forall u \in H^1(\Omega_p), \\
\| \varphi \|_{E,\mathcal{E},a}^2 = \| \varepsilon^{-1} \rho_a^{1/2} \varphi \|_{\Omega_a}^2 + \| \nabla \varphi \|_{\Omega_a}^2 \quad \forall \psi \in H^1(\Omega_a).
\]

The proof of the following result follows taking in (9) as test functions \((\dot{u}, \ddot{u}, \varphi)\), using that \( C_a(u, \varphi) + C_p(\varphi, \dot{u}) = 0 \), integrating in time between 0 and \( t \) and applying the Cauchy-Schwarz inequality together with the Gronwall’s lemma (cf. [35]).

**Theorem 2** (Stability of the continuous formulation). Suppose that the data \( f_p, a_p \) and \( g_p \) are sufficiently regular. For any \( t \in (0,T] \), let \((u, w, \varphi)(t)\) be the solution of (9). Then, the following bound holds

\[
\| (u, w, \varphi)(t) \|_\mathcal{E} \lesssim \| (u, w, \varphi)(0) \|_\mathcal{E} + \int_0^t \left( \| \bar{\rho}_s^{-1/2} f_p \|_{\Omega_p} + \| \bar{\rho}_w^{-1/2} g_p \|_{\Omega_p} + \| \rho a^{1/2} f_a \|_{\Omega_a} \right) dt.
\]

### 3 The semi-discrete formulation and its stability analysis

We introduce a *polytopic* mesh \( \mathcal{T}_h \) made of general polygons (in 2d) or polyhedra (in 3d) and write \( \mathcal{T}_h \) as \( \mathcal{T}_h = \mathcal{T}_h^p \cup \mathcal{T}_h^a \), where \( \mathcal{T}_h^p = \{ \kappa \in \mathcal{T}_h : \kappa \subseteq \Omega_p \} \), with \( i = \{ p, a \} \). Implicit in this decomposition there is the assumption that the meshes \( \mathcal{T}_h^p \) and \( \mathcal{T}_h^a \) are aligned with \( \Omega_p \) and \( \Omega_a \), respectively. Polynomial degrees \( p_{p,\kappa} \geq 1 \) and \( p_{a,\kappa} \geq 1 \) are associated with each element of \( \mathcal{T}_h^p \) and \( \mathcal{T}_h^a \), respectively. The finite-dimensional spaces are introduced as follows: \( V_h^p = [P_{p,\kappa}(\mathcal{T}_h^p)]^d \) and \( V_h^a = [P_{a,\kappa}(\mathcal{T}_h^a)] \) where \( P_{r}(\mathcal{T}_h) \) is the space of square integrable functions \( \psi \) in \( \Omega_i \) that are polynomial of degree less than or equal to \( r \) in any element \( \kappa \) of \( \mathcal{T}_h^i \) with \( i = \{ p, a \} \).

In the following, we assume that \( \mathcal{C}, \rho_a \) and \( m \) are element-wise constant and we define \( \overline{\mathcal{C}}_\kappa = (|C^{1/2}||^2)_{\kappa}, \overline{m}_\kappa = (m)_{\kappa} \) for all \( \kappa \in \mathcal{T}_h^p \) and \( \overline{\rho}_{a,\kappa} = \rho_a|\kappa| \) for all \( \kappa \in \mathcal{T}_h^a \). The symbol \( | \cdot |_2 \) stands for the norm induced by the \( L^2 \)-norm on \( \mathbb{R}^n \), where \( n \) is the dimension of the space of second-order symmetric tensors, so that \( n = 3 \) if \( d = 2 \) and \( n = 6 \) if \( d = 3 \). In order to deal with polygonal and polyhedral elements, we define an *interface* as the intersection of the \( (d-1) \)-dimensional faces of any two neighboring elements of \( \mathcal{T}_h \). If \( d = 2 \), an interface/facet is a line segment and the set of all
interfaces/faces is denoted by $F_h$. When $d = 3$, an interface can be a general polygon that we assume could be further decomposed into a set of planar triangles collected in the set $F_h$. We decompose $F_h$ as $F_h = F_{h,t} \cup F_{h}^0 \cup F_{h}^a$, where $F_{h,t} = \{ F \in F_h : F \subset \partial \kappa^p \cap \partial \kappa^a, \kappa^p \in T_h^p, \kappa^a \in T_h^a \}$, and $F_h^0$ and $F_h^a$ denote all the faces of $T_h^0$ and $T_h^a$, respectively, not lying on $\Gamma_I$. Finally, the faces of $T_h^p$ and $T_h^a$ can be further written as the union of internal (i) and boundary (b) faces, respectively, i.e.: $F_h^p = F_h^{p,i} \cup F_h^{p,b}$ and $F_h^a = F_h^{a,i} \cup F_h^{a,b}$.

Following [17], we next introduce the main assumption on $T_h$.

**Definition 3.1.** A mesh $T_h$ is said to be polytopic-regular if for any $\kappa \in T_h$, there exists a set of non-overlapping $d$-dimensional simplices, $d = 2, 3$, contained in $\kappa \{ S^F_\kappa \}_{F \subset \partial \kappa}$ such that for any face $F \subset \partial \kappa$, the following condition holds:

$$
h_\kappa \lesssim \frac{d|S^F_\kappa|}{|F|}.
$$

(10)

This definition allows thus to introduce the following assumption.

**Assumption 3.1.** The sequence of meshes $\{T_h\}_h$ is assumed to be uniformly polytopic regular in the sense of definition 3.1.

Note that, as pointed out in [17], the above assumption does not impose any restriction on either the number of faces per element nor on their measure relative to the diameter of the element they belong to; we refer to [17] for more details. Under this assumption, the following trace-inverse inequality holds:

$$
||v||_{L^2(\partial\kappa)} \lesssim p h^{-1/2}_\kappa ||v||_{L^2(\kappa)} \quad \forall \kappa \in T_h \forall v \in P_p(\kappa).
$$

(11)

In the following, to avoid technicalities we also make the following assumption.

**Assumption 3.2.** For any pair of neighboring elements $\kappa^\pm \in T_h$. The following $hp$-local bounded variation property holds: $h_{\kappa^+} \lesssim h_{\kappa^-} \lesssim h_{\kappa^+}, \quad p_{\kappa^+} \lesssim p_{\kappa^-} \lesssim p_{\kappa^+}$.

Finally, following [11], for sufficiently piecewise smooth scalar-, vector- and tensor-valued fields $\psi$, $v$ and $\tau$, respectively, we define the averages and jumps on each interior face $F \in F_h^{i,i} \cup F_h^{i,j} \cup F_h^{j,i}$ shared by the elements $\kappa^\pm \in T_h^p$ as follows:

$$
\{\psi\} = \psi^\pm \mathbf{n}^+ + \psi^- \mathbf{n}^-, \quad \{v\} = v^\pm \otimes \mathbf{n}^+ + v^- \otimes \mathbf{n}^-, \quad \{\tau\} = \tau^\pm \mathbf{n}^+ + \tau^- \mathbf{n}^-,
$$

where $\mathbf{a} \otimes \mathbf{b}$ is the tensor product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, $\pm$ is the trace on $F$ taken within the interior of $\kappa^\pm$, respectively, and $\mathbf{n}^\pm$ is the outer unit normal vector to $\partial \kappa^\pm$, respectively. Accordingly, on boundary faces $F \in F_h^{i,b} \cup F_h^{a,b}$, we set $\{\psi\} = \psi \mathbf{n}$, $\{\psi\} = \psi$, $\{v\} = v \otimes \mathbf{n}$, $\{v\} = v$, $\{\tau\} = \tau \mathbf{n}$, $\{\tau\} = \tau$.

### 3.1 Semi-discrete dG formulation

We are now ready to introduce the semi-discrete dG formulation: for any time $t \in [0, T]$, find $\mathbf{E}_h(t) = (u_h, w_h, \varphi_h) \in V_h^p \times V_h^a \times V_h^a$, s.t.

$$
(\rho \ddot{u}_h, \dot{v}_h)_{\Omega_p} + (\rho f \dot{w}_h, \dot{v}_h)_{\Omega_p} + A_p^0(u_h, v_h) + B_p^0(\beta u_h + w_h, \dot{v}_h)
$$

$$
+ C_h^p(\varphi_h, \dot{v}_h) + (\rho \dot{w}_h, \dot{\xi}_h)_{\Omega_p} + (\rho \omega \dot{w}_h, \dot{\xi}_h)_{\Omega_p} + \eta \kappa^{-1} \dot{w}_h, \dot{\xi}_h)_{\Omega_p} + B_h^p(\beta u_h + w_h, \xi_h)
$$

$$
+ \gamma \kappa^{-1} \dot{w}_h, \dot{\xi}_h + (\rho c \kappa^{-1} \varphi_h, \dot{v}_h)_{\Omega_a} + A_a^0(\varphi_h, \dot{v}_h) + C_a^0(u_h, \dot{v}_h) + \tau C_h^a(\dot{w}_h, \dot{\xi}_h)
$$

$$
= (f_p, v_h)_{\Omega_p} + (g_p, \xi_h)_{\Omega_p} + (\rho \alpha a, \psi_h)_{\Omega_a}
$$

(12)

for all $(v_h, \xi_h, \psi_h) \in V_h^p \times V_h^a \times V_h^a$. Defining $\nabla_h$ and $\nabla_h^-$ to be the broken and divergence operators, respectively, setting $\epsilon_h(v) = \nabla_h v + \nabla_h v^\tau$, $\sigma_h(v) = C : \epsilon_h(v)$, and using the short-hand notation
\( \langle \cdot, \cdot \rangle_\Omega = \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \cdot \) and \( \langle \cdot, \cdot \rangle_{\mathcal{F}_h} = \sum_{F \in \mathcal{F}_h} \int_F \cdot \), the bilinear forms appearing in the above formulation are given by

\[
A^p_h(u, v) = (\sigma_h(u), e_h(v))_{\Omega_p} - \langle \{\sigma_h(u)\}, [v] \rangle_{\mathcal{F}^p_h} - \langle [u], \{\sigma_h(v)\} \rangle_{\mathcal{F}^p_h} + \langle \eta[u], [v] \rangle_{\mathcal{F}^p_h},
\]

\[
B^p_h(u, v) = (m \nabla_h \cdot u, \nabla_h \cdot v)_{\Omega_p} - \langle \{m(\nabla_h \cdot u)I\}, [v] \rangle_{\mathcal{F}^p_h} - \langle [u], m(\nabla_h \cdot v)I \rangle_{\mathcal{F}^p_h} + \langle \gamma[u], [v] \rangle_{\mathcal{F}^p_h},
\]

\[
A^a_h(\varphi, \psi) = (\rho_a \nabla_h \varphi, \nabla_h \psi)_{\Omega_a} - \langle \{\rho_a \nabla_h \varphi\}, [\psi] \rangle_{\mathcal{F}^a_h} - \langle [\varphi], \{\rho_a \nabla_h \psi\} \rangle_{\mathcal{F}^a_h} + \langle \chi[\varphi], [\psi] \rangle_{\mathcal{F}^a_h},
\]

and

\[
C^p_h(\varphi, v) = \langle \rho_a \varphi n_p, v \rangle_{\mathcal{F}^p_h}, \quad C^a_h(u, \psi) = \langle -\rho_a u \cdot n_p, \psi \rangle_{\mathcal{F}^a_h},
\]

for all \( u, v \in V^p_h \) and \( \varphi, \psi \in V^a_h \). Moreover, the stabilization functions \( \eta \in L^\infty(\mathcal{F}^p_h) \), \( \gamma \in L^\infty(\mathcal{F}^p_h) \) and \( \chi \in L^\infty(\mathcal{F}^a_h) \), are defined as follows:

\[
\eta|_F = \begin{cases} 
    c_1 \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left( \frac{p_{\kappa, p, p, \kappa}}{h_\kappa} \right) & \forall F \in \mathcal{F}^{p,i}_h, \quad F \subseteq \partial \kappa^+ \cap \partial \kappa^-,
    \\
    \frac{\bar{p}_{\kappa, p, p, \kappa}}{h_\kappa} & \forall F \in \mathcal{F}^{p,b}_h, \quad F \subseteq \partial \kappa,
\end{cases}
\]

(14)

\[
\gamma|_F = \begin{cases} 
    c_2 \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left( \frac{m_{\kappa, p, p, \kappa}}{h_\kappa} \right) & \forall F \in \mathcal{F}^{p,i}_h, \quad F \subseteq \partial \kappa^+ \cap \partial \kappa^-,
    \\
    \frac{\bar{m}_{\kappa, p, p, \kappa}}{h_\kappa} & \forall F \in \mathcal{F}^{p,b}_h, \quad F \subseteq \partial \kappa,
\end{cases}
\]

(15)

\[
\chi|_F = \begin{cases} 
    c_3 \max_{\kappa \in \{\kappa^+, \kappa^-\}} \left( \frac{\bar{p}_{\kappa, p, \kappa}}{h_\kappa} \right) & \forall F \in \mathcal{F}^{a,i}_h, \quad F \subseteq \partial \kappa^+ \cap \partial \kappa^-,
    \\
    \frac{\bar{p}_{\kappa, p, \kappa}}{h_\kappa} & \forall F \in \mathcal{F}^{a,b}_h, \quad F \subseteq \partial \kappa,
\end{cases}
\]

(16)

with \( c_1, c_2, c_3 > 0 \) positive constants, to be properly chosen.

By fixing a basis for the discrete spaces \( V^p_h \) and \( V^a_h \) and denoting by \( U, W, \Phi \) the vector of the expansion coefficients in the chosen basis of the unknowns \( u_h, w_h \) and \( \varphi_h \), respectively, , the semi-discrete formulation (12) can be written equivalently as:

\[
\begin{bmatrix}
M^p \quad M^p_{zf} \quad 0 \\
M^p_{zf} \quad M^p_{zw} \quad 0 \\
0 \quad 0 \quad M^a
\end{bmatrix}
\begin{bmatrix}
\dot{U} \\
\dot{W} \\
\dot{\Phi}
\end{bmatrix}
+ \begin{bmatrix}
0 \quad 0 \quad C^p \\
0 \quad \eta^{-1} M^p \quad \tau C^p \\
0 \quad \tau C^a
\end{bmatrix}
\begin{bmatrix}
U \\
W \\
\Phi
\end{bmatrix}
= \begin{bmatrix}
A^p + \beta^2 B^p \quad \beta B^p \quad 0 \\
\beta B^p \quad B^p \quad 0 \\
0 \quad 0 \quad A^a
\end{bmatrix}
\begin{bmatrix}
F^p \\
W \\
G^a
\end{bmatrix}
\]

(17)

with initial conditions \( U(0) = U_0, \quad W(0) = W_0, \quad \Phi(0) = \Phi_0, \quad \dot{U}(0) = U_1, \quad \dot{W}(0) = W_1, \quad \dot{\Phi}(0) = \Phi_1 \).

We remark that \( F^p, G^p \) and \( F^a \) are the vector representations of the linear functionals \( (f_p, v_h)_{\Omega_p}, (g_p, \xi_h)_{\Omega_p} \) and \( (\rho_a f_a, \psi_h)_{\Omega_a} \), respectively.
3.2 Stability analysis

To carry out the stability analysis, we introduce the following energy norm

$$
\|(v, z, \psi)\|_E^2 = \|v\|_{E,v}^2 + \|\beta v + z\|_{E,p}^2 + \|\psi\|_{E,a}^2,
$$

(18)

for all \((v, z, \psi) \in V_h^p \times V_h^p \times V_h^a\), where

$$
\|v\|_{E,v}^2 = \left(\bar{\rho}_s^{1/2} \dot{v}\right)_{\Omega_p}^2 + \|v\|_{dG,e}^2,
$$

(19)

$$
\|\beta v + z\|_{E,p}^2 = \|\beta v + z\|_{dG,p}^2 + \left(\phi^{1/2} \dot{v} + \phi^{-1/2} \dot{z}\right)_{\Omega_p}^2 + \|\bar{\rho}_w \dot{z}\|_{\Omega_p}^2,
$$

(20)

$$
\|\psi\|_{E,a}^2 = \left(\rho_a^{-1/2} \psi\right)_{\Omega_a}^2 + \|\psi\|_{dG,a}^2,
$$

(21)

and where

$$
\|v\|_{dG,e}^2 = \left(\mathcal{C}^{1/2} e_h(v)\right)_{\Omega_p}^2 + \left(\eta^{1/2} \|v\|_{F_h^p} \right)^2, \quad \forall v \in V_h^p \\
\|z\|_{dG,p}^2 = \left(m^{1/2} \nabla_h \cdot z\right)_{\Omega_p}^2 + \left(\gamma^{1/2} \|z\|\right)_{\Omega_p}^2, \quad \forall z \in V_h^p, \\
\|\psi\|_{dG,a}^2 = \left(\rho_a^{1/2} \nabla_h \psi\right)_{\Omega_a}^2 + \left(\chi^{1/2} \|\psi\|\right)_{\Omega_a}^2, \quad \forall \psi \in V_h^a.
$$

The main stability result is stated in the following theorem.

**Theorem 3** (Stability of the semi-discrete formulation). Let assumption 3.1 and assumption 3.2 be satisfied. For sufficiently large penalty parameter \(c_1, c_2\) and \(c_3\) in (14), (15) and (16), respectively, let \((u_h, w_h, \varphi_h)(t)\) be the solution of (12) for any \(t \in [0, T]\). Then,

$$
\|(u_h, w_h, \varphi_h)(t)\|_E \lesssim \|(u_h, w_h, \varphi_h)(0)\|_E + \int_0^t \left(\left(\|\bar{\rho}_s^{-1/2} f_p\|_{\Omega_p} + \|\bar{\rho}_w^{-1/2} g_p\|_{\Omega_p} + \|\rho_a^{-1/2} f_a\|_{\Omega_a}\right)\right) d\tau.
$$

(22)

**Proof.** By taking \((v_h, \xi_h, \psi_h) = (u_h, \dot{w}_h, \dot{\varphi}_h)\) in (12), using the definition of the bilinear forms \(A_h^p, A_h^a\) and \(B_h^p\) and the skew-symmetry of the coupling bilinear forms (13), we obtain:

$$
\frac{1}{2} \frac{d}{dt} \left[\|\rho^{1/2} \dot{u}_h\|_{\Omega_p}^2 + \|u_h\|_{dG,e}^2 - 2\langle \sigma_h(u_h), u_h \rangle_{F_h^p} + \left(\rho_a^{-1/2} \varphi_h\right)_{\Omega_a}^2\right] \\
+ \|\dot{\varphi}_h\|_{dG,a}^2 - 2\langle \nabla_h \varphi_h, [\varphi_h]\rangle_{F_h^p} + \left|\rho_w^{1/2} \dot{w}_h\right|_{\Omega_a}^2 + 2\left(\rho_f \dot{w}_h, \dot{u}_h\right)_{\Omega_p} + 2\|\beta u_h + w_h\|_{dG,p}^2 \\
- 2\langle \nabla_h (\beta u_h + w_h), f \rangle_{F_h^p} + \|\beta u_h + w_h\|_{F_h^p}^2 + \eta^{-1}\|\dot{w}_h\|_{\Omega_p}^2

= (f_p, \dot{u}_h)_{\Omega_p} + (g_p, \dot{w}_h)_{\Omega_p} + (\rho_a f_a, \dot{\varphi}_h)_{\Omega_a}.
$$

(23)

We next observe that \(\|\rho^{1/2} \dot{u}_h\|_{\Omega_p}^2 = \|\bar{\rho}_s^{1/2} \dot{u}_h\|_{\Omega_p}^2 + \|\phi^{1/2} \rho_f^{1/2} \dot{u}_h\|_{\Omega_p}^2\), and, writing the tortuosity \(a > 1\) as \(a = 1 + a_0\), with \(a_0 > 0\), and setting \(\bar{\rho}_s = (1 - \phi) \rho_s\) and \(\bar{\rho}_w = a_0 \rho_f / \phi\), we obtain

$$
\|\rho_w^{1/2} \dot{w}_h\|_{\Omega_p}^2 = \|(\rho_f / \phi)^{1/2} \dot{w}_h\|_{\Omega_p}^2 + \|(a_0 \rho_f / \phi)^{1/2} \dot{w}_h\|_{\Omega_p}^2.
$$

According to the above notation, it follows that

$$
\|\rho^{1/2} \dot{u}_h\|_{\Omega_p}^2 + \|\rho_w^{1/2} \dot{w}_h\|_{\Omega_p}^2 + 2\left(\rho_f \dot{w}_h, \dot{u}_h\right)_{\Omega_p} = \\
\|\bar{\rho}_s^{1/2} \dot{u}_h\|_{\Omega_p}^2 + \|\phi^{1/2} \rho_f^{1/2} \dot{u}_h + \phi^{-1/2} \rho_f^{1/2} \dot{w}_h\|_{\Omega_p}^2 + \|\bar{\rho}_w^{1/2} \dot{w}_h\|_{\Omega_p}^2.
$$
From definitions (18), (19), (20), it follows that the left hand side of (23) can be written as: \( \frac{1}{2} \frac{d}{dt} \| I \| + \eta k^{-1} \| \dot{\mathbf{w}}_h \|_{\Omega_p}^2 \) where
\[
I = \| u_h \|_{E_h}^2 + \| \varphi_h \|_{E_a}^2 + \| \beta u_h + w_h \|_{F_h}^2 - 2 \{ \{ \sigma_h(u_h) \} \} \| u_h \|_{F_h}^p \\
- 2 \{ \{ \nabla_h \varphi_h \} \| \varphi_h \|_{F_h} - 2 (m \{ \nabla_h \cdot (\beta u_h + w_h)I \}, \| \beta u_h + w_h \|_{F_h}^p).
\]
By integrating between 0 and \( t \) we therefore obtain
\[
\left[ I \right]_0^t + 2 \eta k^{-1} \int_0^t \| \dot{\mathbf{w}}_h \|_{\Omega_p}^2 (\tau) d\tau = 2 \int_0^t (f_p, \dot{u}_h)_{\Omega_p} + (g_p, \dot{w}_h)_{\Omega_p} + (\rho_a f_a, \dot{\varphi}_h)_{\Omega_a} d\tau.
\]
By using lemma A.3 in the appendix, the Cauchy-Schwarz inequality and since
\[
\| (u_h, w_h, \varphi_h)(t) \|_{E_h}^2 \lesssim \| (u_h, w_h, \varphi_h)(0) \|_{E_h}^2 + 2 \eta k^{-1} \int_0^t \| \dot{\mathbf{w}}_h \|_{\Omega_p}^2 (\tau) d\tau,
\]
it follows
\[
\| (u_h, w_h, \varphi_h)(t) \|_{E_h}^2 \lesssim \| (u_h, w_h, \varphi_h)(0) \|_{E_h}^2 + \int_0^t \left( \| \dot{\mathbf{w}}_h \|_{\Omega_p}^2 \right) \| (u_h, w_h, \varphi_h)(t) \|_{E_h} d\tau,
\]
for \( t \in (0, T] \). The thesis follows by applying Gronwall’s Lemma (see [35]). \( \square \)

## 4 Error analysis for the semi-discrete formulation

In this section we prove an a-priori error estimate for the semi-discrete problem (12). We first observe that by setting, for any time \( t \in [0, T] \), \( e^u(t) = (u-u_h)(t) \), \( e^w(t) = (w-w_h)(t) \), and \( e^\varphi(t) = (\varphi-\varphi_h)(t) \) and by using the strong consistency of the semi-discrete formulation (12), the error equation reads as follows
\[
(p \dot{e}^u, v)_{\Omega_p} + (p \dot{e}^w, v)_{\Omega_p} + A_h^p(e^u, v) + B_h^p(\beta e^u + e^w, v) + C_h^p(e^\varphi, v) \\
+ (p \dot{e}^\varphi, \xi)_{\Omega_p} + (p \dot{e}^\varphi, \xi)_{\Omega_p} + \eta k^{-1} (\dot{e}^u, \xi)_{\Omega_p} + B_h^p(\beta e^u + e^w, \xi) \\
+ \tau C_h^p(e^\varphi, \xi) + (p \dot{\varphi}, \dot{\varphi})_{\Omega_a} + A_h^a(e^\varphi, \psi) + C_h^a(e^u, \psi) + \tau C_h^a(e^w, \psi) = 0
\]
for any \( (v, \xi, \psi) \in V_h^p \times V_h^p \times V_h^a \).

Next, we introduce the following definition and a further mesh assumption; cf [18, 17].

**Definition 4.1.** A covering \( \mathcal{T}_\# \) of the polytopic mesh \( \mathcal{T}_h \) is a set of regular shaped \( d \)-dimensional simplices \( \mathcal{K}, d = 2, 3, \) s.t. \( \forall \mathcal{K} \in \mathcal{T}_h, \mathcal{K} \in \mathcal{T}_\# \) s.t. \( \mathcal{K} \subseteq \mathcal{K} \).

**Assumption 4.1.** Any mesh \( \mathcal{T}_h \) admits a covering \( \mathcal{T}_\# \) in the sense of definition 4.1 of such that i) \( \max_{\mathcal{K} \in \mathcal{T}_h} \text{card}\{ \mathcal{K}' \in \mathcal{T}_h : \mathcal{K}' \cap \mathcal{K} \neq \emptyset, \mathcal{K} \cap \mathcal{T}_h \} \lesssim 1 \); ii) \( h_\mathcal{K} \lesssim h_\kappa \) for each pair \( \mathcal{K} \in \mathcal{T}_h, \mathcal{K} \in \mathcal{T}_\# \) with \( \kappa \subseteq \mathcal{K} \).

We also introduce the following norms
\[
\| v \|_{2_{dG,e}}^R = \| v \|_{dG,e}^2 + \| \eta^{-1/2} \{ \{ C : e_h(v) \} \} \|_{F_h}^2, \quad \forall v \in H^2(\mathcal{T}_h^p),
\]
\[
\| \psi \|_{2_{dG,a}}^R = \| \psi \|_{dG,a}^2 + \| \chi^{-1/2} \{ \rho_a \nabla_h \psi \} \|_{F_h}^2, \quad \forall \psi \in H^2(\mathcal{T}_h^a),
\]
\[
\| z \|_{2_{dG,p}}^R = \| z \|_{dG,p}^2 + \| \gamma^{-1/2} \{ (\nabla_h \cdot z) I \} \|_{F_h}^2, \quad \forall z \in H^2(\mathcal{T}_h^p).
\]
For an open bounded polytopic domain $\Sigma \subset \mathbb{R}^d$ and a generic polytopic mesh $T_h$ over $\Sigma$ satisfying assumption 4.1, as in [18], we can introduce the Stein extension operator $\hat{\mathbf{E}} : H^m(\kappa) \to H^m(\mathbb{R}^d)$ [39], for any $\kappa \in T_h$ and $m \in \mathbb{N}_0$, such that $\hat{\mathbf{E}} v|_\kappa = v$ and $\|\hat{\mathbf{E}} v\|_{m,\mathbb{R}^d} \lesssim \|v\|_{m,\kappa}$. The corresponding vector-valued version mapping $H^m(\kappa)$ onto $H^m(\mathbb{R}^d)$ acts component-wise and is denoted in the same way. Reasoning as in [3], we can obtain the following interpolation bounds.

**Lemma 1.** For any $(v, z, \psi) \in H^m(T_h^p) \times H^\ell(T_h^p) \times H^n(T_h^p)$, with $m, \ell, n \geq 2$, there exists $(v_I, z_I, \psi_I) \in V^p_h \times V^p_h \times V^a_h$ such that

$$
\|v - v_I\|^2 \lesssim \sum_{\kappa \in T_h^p} \frac{h^{2(s \kappa - 1)}_{\kappa}}{p^{s \kappa - 3}_{\kappa}} \|\hat{\mathbf{E}} v\|^2_{m,\kappa},
$$

$$
\|z - z_I\|^2 \lesssim \sum_{\kappa \in T_h^p} \frac{h^{2(r \kappa - 1)}_{\kappa}}{p^{2r \kappa - 3}_{\kappa}} \|\hat{\mathbf{E}} z\|^2_{\ell,\kappa},
$$

$$
\|\psi - \psi_I\|^2 \lesssim \sum_{\kappa \in T_h^p} \frac{h^{2(q \kappa - 1)}_{\kappa}}{p^{q \kappa - 3}_{\kappa}} \|\hat{\mathbf{E}} \psi\|^2_{n,\kappa},
$$

where $s_\kappa = \min(m, p_{\kappa \kappa} + 1)$, $r_\kappa = \min(\ell, p_{\kappa \kappa} + 1)$ and $q_\kappa = \min(n, p_{\kappa \kappa} + 1)$. Moreover, if $(u, w, \varphi) \in C^1([0, T] ; H^m(T_h^p)) \times C^1([0, T] ; H^\ell(T_h^p)) \times C^1([0, T] ; H^n(T_h^p))$, with $m, \ell, n \geq 2$, there exists $(u_I, w_I, \varphi_I) \in V^p_h \times V^p_h \times V^a_h$ s.t.:

$$
\|(u - u_I, w - w_I, \varphi - \varphi_I)\|_{2} \lesssim \sum_{\kappa \in T_h^p} \frac{h^{2(s \kappa - 1)}_{\kappa}}{p^{s \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} u\|^2_{m,\kappa} + \|\hat{\mathbf{E}} u\|^2_{m,\kappa} \right)
$$

$$
+ \sum_{\kappa \in T_h^p} \frac{h^{2(r \kappa - 1)}_{\kappa}}{p^{2r \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} + \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} \right)
$$

$$
+ \sum_{\kappa \in T_h^p} \frac{h^{2(q \kappa - 1)}_{\kappa}}{p^{q \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} + \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} \right).
$$

We are now ready to state the main result of this section.

**Theorem 4** (A-priori error estimates). Let assumption 3.1, assumption 3.2 and assumption 4.1 hold. Assume that the exact solution of problem (8) is such that $u \in C^1([0, T] ; H^2(\Omega_p) \cap H^m(T_h^p))$, $w \in C^2([0, T] ; H^2(\Omega_p) \cap H^\ell(T_h^p))$ and $\varphi \in C^2([0, T] ; H^2(\Omega_p) \cap H^n(T_h^p))$, with $m, n, \ell \geq 2$ and let $(u_h, w_h, \varphi_h) \in C^2([0, T] ; V^p_h) \times C^2([0, T] ; V^p_h) \times C^2([0, T] ; V^a_h)$ be the solution of the semi-discrete problem (12), with sufficiently large penalty parameters $c_1$, $c_2$ and $c_3$. Then, for any $t \in [0, T]$

$$
\|(e^u, e^w, e^\varphi)(t)\|_{2} \lesssim \sum_{\kappa \in T_h^p} \frac{h^{s \kappa - 1}_{\kappa}}{p^{s \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} u\|^2_{m,\kappa} + \|\hat{\mathbf{E}} u\|^2_{m,\kappa} + \int_0^t \left( \|\hat{\mathbf{E}} u\|^2_{m,\kappa} + \|\hat{\mathbf{E}} u\|^2_{m,\kappa} \right) \right)
$$

$$
+ \sum_{\kappa \in T_h^p} \frac{h^{r \kappa - 1}_{\kappa}}{p^{r \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} + \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} + \int_0^t \left( \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} + \|\hat{\mathbf{E}} w\|^2_{\ell,\kappa} \right) \right)
$$

$$
+ \sum_{\kappa \in T_h^p} \frac{h^{q \kappa - 1}_{\kappa}}{p^{q \kappa - 3}_{\kappa}} \left( \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} + \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} + \int_0^t \left( \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} + \|\hat{\mathbf{E}} \varphi\|^2_{n,\kappa} \right) \right).
$$

**Proof.** For any time $t \in [0, T]$, let $(u_I, w_I, \varphi_I)(t) \in V^p_h \times V^p_h \times V^a_h$ be the interpolants defined in lemma 1. We split the error $E(t) = (e^u, e^w, e^\varphi)(t)$ as $E(t) = E_I(t) - E_h(t)$, where

$$
E_I(t) = (e^u_I, e^w_I, e^\varphi_I)(t) = (u - u_I, w - w_I, \varphi - \varphi_I)(t),
$$

$$
E_h(t) = (e^u_h, e^w_h, e^\varphi_h)(t) = (u - u_h, w - w_h, \varphi - \varphi_h)(t).
$$
\( E_h(t) = (e^u_h, e^w_h, c^v_h)(t) = (u_h - u_I, w_h - w_I, \varphi_h - \varphi_I)(t). \)

From the triangle inequality we have \( \| E(t) \|_E^2 \leq \| E_h(t) \|_E^2 + \| E_I(t) \|_E^2 \), and lemma 1 can be used to bound the term \( \| E_I(t) \|_E \). As for the term \( \| E_h \|_E \), by taking \( (v, \xi, \psi) = (e^u_h, \dot{e}^u_h, e^v_h) \) as test functions in (24), taking into account that \( E = E_I - E_h \), neglecting the coupling terms thanks to skew-symmetry, collecting a first time derivative and defining the recall of the energy norm, identity (24) can be rewritten as

\[
\frac{1}{2} \frac{d}{dt} \mathcal{N}(t) + \eta k^{-1} \| \dot{e}^w_h \|_{\Omega_p}^2 = 1
\]

(25)

where

\[
(1) = (\rho \dot{e}^u_h, \dot{e}^w_h)_{\Omega_p} + (\rho f \dot{e}^u_h, \dot{e}^v_h)_{\Omega_p} + A^p_h(e^u_h, \dot{e}^u_h) + B^p_h(\beta e^u_h + e^w_h, \dot{e}^u_h) + C^p_h(e^u_h, \dot{e}^w_h)_{\Omega_p}
\]


(26)

and

\[
\mathcal{N}(t) - \mathcal{N}(0) + 2 \int_0^t \eta k^{-1} \| \dot{e}^w_h \|_{\Omega_p}^2 \, dt \lesssim \int_0^t (1) \, dt,
\]

(27)

where in the second inequality we have used lemma A.3 to bound from below the left hand side and that \( E_h(0) \) since \( e^u_h(0) = e^v_h(0) = e^w(0) = 0 \). Next, we bound each of the terms appearing in \( (1) \), cf. (26). By employing the Cauchy-Schwarz inequality, it follows

\[
| (\tilde{\rho}_s \dot{e}^u_h, \dot{e}^w_h)_{\Omega_p} | \lesssim \| \dot{e}^u_h \|_{\Omega_p} \| \dot{e}^w_h \|_{\Omega_p}, \quad | (\rho_c \dot{e}^u_h, \dot{e}^w_h)_{\Omega_p} | \lesssim \| \dot{e}^u_h \|_{\Omega_p} \| \dot{e}^w_h \|_{\Omega_p},
\]

and

\[
(\rho f \dot{e}^u_h, \dot{e}^v_h)_{\Omega_p} + (\rho f \dot{e}^u_h, \dot{e}^v_h)_{\Omega_p} + (\rho f \dot{e}^u_h, \dot{e}^v_h)_{\Omega_p} + (\rho f \phi^{-1} \dot{e}^w_h, \dot{e}^w_h)_{\Omega_p} + (\tilde{\rho}_u \dot{e}^u_h, \dot{e}^w_h)_{\Omega_p}
\]

\[
= \left( \phi^{1/2} \rho_f^{1/2} \dot{e}^u_h + \rho_f^{1/2} \phi^{-1/2} \dot{e}^w_h, \phi^{1/2} \rho_f^{1/2} \dot{e}^u_h + \rho_f^{1/2} \phi^{-1/2} \dot{e}^w_h \right)_{\Omega_p} + (\tilde{\rho}_u \dot{e}^u_h, \dot{e}^w_h)_{\Omega_p}
\]

\[
\lesssim \left\| \phi^{1/2} \rho_f^{1/2} \dot{e}^u_h + \rho_f^{1/2} \phi^{-1/2} \dot{e}^w_h \right\|_{\Omega_p} \left\| \phi^{1/2} \rho_f^{1/2} \dot{e}^u_h + \rho_f^{1/2} \phi^{-1/2} \dot{e}^w_h \right\|_{\Omega_p}
\]

\[
+ \left\| \tilde{\rho}_u \dot{e}^u_h \right\|_{\Omega_p} \left\| \tilde{\rho}_u \dot{e}^w_h \right\|_{\Omega_p}.
\]

For the \( A^p_h \) and \( A^0_h \)- terms in (26), by employing the continuity estimates in lemma A.2 and observing that \( \| \cdot \|_{dG, \ast} \leq \| \cdot \|_{\Omega_p} \), it follows

\[
A^p_h(e^u_h, \dot{e}^u_h) + A^0_h(e^v_h, \dot{e}^v_h) = \frac{d}{dt} \left( A^p_h(e^u_h, \dot{e}^u_h) + A^0_h(e^v_h, \dot{e}^v_h) \right) - A^p_h(e^u_h, \dot{e}^u_h) + A^0_h(e^v_h, \dot{e}^v_h)
\]

\[
\lesssim \frac{d}{dt} \left( \| e^u_h \|_{dG, e} \| e^u_h \|_{\Omega_p} + \| e^v_h \|_{dG, e} \| e^v_h \|_{\Omega_p} \right) + \left( \| e^u_h \|_{dG, e} \| e^u_h \|_{\Omega_p} + \| e^v_h \|_{dG, e} \| e^v_h \|_{\Omega_p} \right).
\]

The term \( B^p_h \) can be bounded in the same way, i.e.,

\[
B^p_h(\beta e^u_h + e^v_h, \dot{e}^u_h + e^v_h) \lesssim \frac{d}{dt} \left( \| \beta e^u_h + e^v_h \|_{dG, \ast} \| \beta e^u_h + e^v_h \|_{dG, \ast} \right)
\]
A direct application of the Young’s inequality gives
\[ p \leq \phi \left( \frac{\|e_h \|_{\Omega_p}^2}{h^2} \right) \]
Collecting all the previous bounds, we get
\[ \phi \left( \int \frac{\|e_h \|_{\Omega_p}^2}{h^2} \right) \]
for $\tau = \{0, 1\}$. Now, recalling the definitions of the coupling bilinear forms, using the Cauchy-Schwarz inequality, the trace-inverse inequality (11), the definition of the energy norm we get
\[ C_h(p, e, \dot{e}) = C_h(p, e, \dot{e}^w) + C_h(p, e, \dot{e}^w) + C_h(p, e, \dot{e}^y) = C_h(p, e, \dot{e}^y) + C_h(p, e, \dot{e}^y), \]
where in the last bound we have also used assumption 3.2. Then, we have
\[ \int_0^t C_h(p, e, \dot{e}) \, dt \lesssim \sum_{p \in T_h} \|\rho_e \dot{e}_f \|_F \|\dot{e}_h \|_F \lesssim \sum_{p \in T_h} \|\dot{e}_f \|_{\partial_\Omega_e} \|\dot{e}_h \|_{\partial_\Omega_e} \]
In the same way, we can conclude that
\[ \int_0^t C_h(p, e, \dot{e}) \, dt \lesssim \sum_{p \in T_h} \|\rho_e \dot{e}_f \|_F \|\dot{e}_h \|_F \]
Collecting all the previous bounds, we get
\[ \|E_h\|_{\mathcal{E}}^2 \lesssim \|\dot{e}_f \|_{\mathcal{E}, e} + \|\dot{e}_h \|_{\mathcal{E}, h} + \|\dot{e}_i \|_{\mathcal{E}, i} \]
A direct application of the Young’s inequality gives
\[ \|E_h\|_{\mathcal{E}}^2 \lesssim C^2(e, e, e) + \int_0^t D(e, e, e) \|E_h\|_{\mathcal{E}} \, dt, \]
(28)
The term $C$ can be bounded based on employing the estimates in lemma 1. Analogously, to estimate term $D$, we make use of lemma 1 and the following ones for $I_h^0(\dot{e}_T^n)$ and $I_h^0(\dot{e}_T^n + \tau \dot{e}_T^n)$

$$I_h^0(\dot{e}_T^n) \lesssim \sum_{k \in T_h} \frac{h_k^{n-1}}{m-3/2} \| \dot{\mathcal{E}} \|_{n,K}$$

$$I_h^0(\dot{e}_T^n + \tau \dot{e}_T^n) \lesssim \sum_{k \in T_h} \frac{h_k^{n-1}}{m-3/2} \| \mathcal{E} \|_{m,K} + \sum_{k \in T_h} \frac{h_k^{n-1}}{p_{p,k}} \| \mathcal{E} \|_{\ell,K}.$$  

Finally, by employing Gronwall’s lemma we get

$$\| E_k \|_E \lesssim C(e_T^n, e_T^n, e_T^n) + \int_0^t D(e_T^n, e_T^n, e_T^n) dt,$$

and the thesis follows.

5 Time discretization

In order to discretize in time equation (17), we employ the Newmark-$\beta$ method, as described in the following. We discretize the time interval $[0,T]$ by introducing a timestep $\Delta t > 0$, such that $\forall k \in \mathbb{N}$, $t_{k+1} - t_k = \Delta t$ and define $X^k$ as $X^k = X(t_k)$, with $X = [U, W, \Phi]^T$. We rewrite equation (17) in compact form as $A \ddot{X} + B \dot{X} + C X = F$ and get

$$\ddot{X} = A^{-1}(F - B \dot{X} - C X) = A^{-1} F - A^{-1} B \dot{X} - A^{-1} C X = \mathcal{L}(t, X, \dot{X}),$$

The Newmark scheme is defined by introducing a Taylor expansion for displacement and velocity, respectively:

$$\begin{cases}
X^{k+1} = X^k + \Delta t Z^k + \Delta t^2 (\beta N L^{k+1} + (\frac{1}{2} - \beta N) L^k), \\
Z^{k+1} = Z^k + \Delta t (\gamma N L^{k+1} + (1 - \gamma N) L^k),
\end{cases} \tag{29}$$

where $Z^k = \dot{X}(t_k)$, $L^k = L(t_k, X^k, Z^k)$ and the Newmark parameters $\beta N$ and $\gamma N$ satisfy the following constraints $0 \leq \gamma N \leq 1, 0 \leq 2\beta N \leq 1$. The typical choices of parameters are $\gamma N = 1/2$ and $\beta N = 1/4$, for which the scheme is unconditionally stable and second order accurate. Finally, by plugging the definition of $\mathcal{L}$ into (29), for $k \geq 0$, the time integration reduces to:

$$\begin{bmatrix}
A + \Delta t^2 \beta N C \\
\gamma N \Delta t C
\end{bmatrix}
\begin{bmatrix}
X^{k+1} \\
Z^{k+1}
\end{bmatrix} =
\begin{bmatrix}
A + \Delta t^2 \beta N C \\
\gamma N \Delta t C
\end{bmatrix}
\begin{bmatrix}
X^k \\
Z^k
\end{bmatrix} +
\begin{bmatrix}
\Delta t^2 \beta N F^{k+1} \\
\gamma N \Delta t F^k
\end{bmatrix},$$

where $\tilde{\beta}_N = (\frac{1}{2} - \beta N)$ and $\tilde{\gamma}_N = (1 - \gamma N)$.

6 Numerical results

Numerical implementation has been carried out through the software MATLAB. Meshes have been generated through the polymeshier software, cf.[40].
The model problem is solved in $\Omega = (-1,1) \times (0,1)$, on a sequence of polygonal meshes as the one shown in fig. 2, and with physical parameters shown in table 1. For the first test case, we choose as exact solution

$$u(x,y;t) = \left( x^2 \cos\left(\frac{\pi x}{2}\right) \sin(\pi x) \right) \cos(\sqrt{2} \pi t), \quad w(x,y;t) = -u(x,y;t),$$

in order to have a null pressure in the whole poroelastic domain. Since the solution together with its first $x$, $y$- and $t$-derivatives are identically zero at the interface $\Gamma = 0 \times (0,1)$, interface coupling conditions are consequently null. This suggests to test both the open pores ($\tau = 0$) and the sealed pores ($\tau = 1$) cases with the same manufactured solution. A sequence of uniformly refined polygonal meshes have been considered, with uniform polynomial degree $p_{\kappa} = p_{a,\kappa} = p = 1,2,3$. The final time $T$ has been set equal to 0.25, considering a timestep of $\Delta t = 10^{-4}$ for the Newmark-$\beta$ scheme, $\gamma_{N} = 1/2$ and $\beta_{N} = 1/4$. The penalty parameters $c_{1}, c_{2}$ and $c_{3}$ appearing in the definition (14)–(16) have been chosen equal to 10. In fig. 3 we report the computed errors as a function of the inverse of the mesh-size (log-log scale), for the case $p = 3$. As predicted by theorem 4 the errors decays proportionally to $h^3$. Moreover, we have also computed the $L^2$-errors on the pressure field $p$. These results are reported fig. 4 and show a convergence rate proportional to $h^3$, as expected. We point out
the that discrete pressure has been computed through equation (2). Finally, we compute the $L^2$ norm

$$
\|p - p_h\|_{\Omega}
$$

of the error fixing a computational mesh of $N = 400$ polygons and varying the polynomial degree $p = 1, 2, \ldots, 5$. The computed errors are reported in fig. 5 (semi-log scale), and an exponential decay of the error is clearly attained.

Test case 2. Oblique interface

The second test cases consider a domain $\Omega = (0, 400) \times (0, 400)$ m$^2$, with a straight interface with slope 60$^\circ$, cf. fig. 6a. Physical and dimensional parameters have been chosen as in [21] and listed in table 2. Boundary and initial conditions have been set equal to zero both for the poroelastic and the acoustic domain. Forcing terms are null in $\Omega_p$, while in $\Omega_a$ a forcing term is imposed until $t = 0.05$ s, by considering the following load: $f_a = r(x, y)h(t)$, where

$$
h(t) = \begin{cases}
\sum_{k=1}^{4} \alpha_k \sin(\gamma_k \omega_0 t), & \text{if } 0 < t < \frac{1}{f_0} \\
0, & \text{otherwise,}
\end{cases}
$$

(30)

with coefficients defined as: $\alpha_1 = 1$, $\alpha_2 = -21/32$, $\alpha_3 = 63/768$, $\alpha_4 = -1/512$, $\gamma_k = 2^{k-1}$, $\omega_0 = 2\pi f_0$ Hz, $f_0 = 20$ Hz. The function $r(x, y)$ is defined as $r(x, y) = 1$, if $(x, y) \in \bigcup_{i=1}^{4} B(x_i, r)$, while

![Figure 4: Test case 1. Computed error, at the final time $T$, $\|p - p_h\|_{\Omega}$ as a function of $h$ ($p = 3$).](image)

![Figure 5: Test case 2. Physical parameters.](image)
Sealed pores ($\tau = 0$)

Open pores ($\tau = 1$)

Figure 5: Test case 1. Computed errors, at the final time $T$, as a function of the polynomial degree $p$ on a computational mesh of $N = 400$ polygons.

$r(x, y) = 0$, otherwise, where $B(x_i, r)$ is the circle centered in $x_i$ and with radius $r$. Here, we set $x_1 = (250, 100)$ m, $x_2 = (250, 150)$ m, $x_3 = (250, 200)$ m, $x_4 = (250, 250)$ m and $r = 10$ m. Notice that, the support of the function $r(x, y)$ has been reported in fig. 6a, superimposed with a sample of one of the computational meshes employed. Simulations have been carried out by considering a polygonal mesh consisting in $N = 6586$ polygons, subdivided into $N_a = 3564$ and $N_p = 3022$ polygons for the acoustic and poroelastic domain, respectively, a Newmark scheme with time step $\Delta t = 10^{-3}$ s and $\gamma_N = 1/2$ and $\beta_N = 1/4$. In a time interval $[0, 0.15]$ s and a polynomial degree $p_{a, \kappa} = p_{a, \kappa} = p = 4$.

In fig. 7 we show the absolute value of the computed acoustic potential $\varphi_h$ in the acoustic domain and of the computed displacement wave field $u_h$ in the poroelastic one. As one can see part of the acoustic wave propagates through the poroelastic domain where the main wavefront is aligned with the oblique interface. Moreover, in fig. 8 we report the computed pressure $p_h$ along the horizontal line $y = 200$ m. It is possible to notice that the pressure wave correctly propagates from the acoustic domain to the poroelastic one: the continuity at the interface boundary can be appreciated. Remark that $p_h = \rho_a \dot{\varphi}$ in the acoustic domain while $p_h = -m(\beta \nabla \cdot u + \nabla \cdot w)$ in the poroelastic one.

Test case 3: Sinusoidal interface

Finally, with the same data of test case 2, we consider a square domain $\Omega = [-1500, 1500]^2$ m$^2$ and a sinusoidal interface $\Gamma$ defined through the relation $\Gamma(x) = 40 \sin \left( \frac{\pi}{100} x \right)$, cf. fig. 6b. The number of polygons composing the mesh is $N = 5441$, subdivided into $N_a = 2713$ and $N_p = 2728$ polygons for the acoustic and poro-elastic subdomains, respectively. Moreover, as shown in fig. 6b, we have set the initial conditions on the acoustic domain, by defining $h(t)$ as before and $r(x, y) = 1$, if $(x, y) \in B(x_i, r)$, and equal to 0, otherwise, with $x_1 = (0, 150)$ m and $r = 50$ m. In fig. 9 we show the propagation of the discrete pressure at the time instants $t = 0.2, 0.4, 0.5$ s and $t = 0.6$ s. Observe how the sinusoidal interface contributes to the diffraction of the acoustic wave in the poroelastic domain. In particular, we can observe the main wave front traveling towards the rigid walls of the domain followed by waves having smaller amplitude originated by the sinusoidal shape of the contact boundary.
Figure 6: Test cases 2 and 3. Computational domains and computational grids. The support of $r(x, y)$ is also superimposed in cyan over the mesh.

Figure 7: Test case 2. Oblique interface. Computed potential $\varphi_h$ in the acoustic domain and the computed displacement wave field $u_h$ at three time instants ($t = 0.04, 0.08, 0.12$ s), with $\Delta t = 10^{-3}$ s.

Figure 8: Test case 2. Oblique interface. Computed pressure wave $p_h$ along the line $y = 200$ m at the time instants $t = 0.04$ s (left), $t = 0.08$ s (center) and $t = 0.12$ s (right).
Figure 9: Test case 4. computed pressure $p_h$ at the time instants $t = 0.2$ s (top-left), $t = 0.4$ s (top-right), $t = 0.5$ s (bottom-left) and $t = 0.6$ s (bottom-right) with $\Delta t = 10^{-3}$ s.
7 Conclusions

In this work we have presented and analyzed a dG approximation to the coupled poro-elasto-acoustic problem on polygonal and polyhedral grids. Existence and uniqueness of the (strong) solution to the continuous problem has been proven based on employing the semigroup theory. We have proved a stability result for both the continuous and the semi-discrete formulations together with an a priori $hp$-version error estimates for the semidiscrete solution in a suitable energy norm. Finally, a wide set of two-dimensional numerical simulations have been carried out.

A Appendix

The proof of theorem 1 as well as some technical results used for the stability and error analysis are presented below.

Proof. (Theorem 1) The proof follows the lines of [3, Theorem 3.1]. Let $v = \dot{u}$, $z = \dot{w}$, $\lambda = \dot{\varphi}$ and let $\mathcal{U} = (u, v, w, z, \varphi, \lambda)$. We introduce the following Hilbert space $\mathbb{H} = H^1_0(\Omega_p) \times L^2(\Omega_p) \times H^1_0(\Omega_p) \times L^2(\Omega_p) \times H^1_0(\Omega_a) \times L^2(\Omega_a)$, equipped with the scalar product

\[
\langle (\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} \rangle = (\tilde{\rho} u_1, u_2)_{\Omega_p} + (\tilde{\rho} w z_1, z_2)_{\Omega_p} + \left( \rho_f [\phi^{1/2} v_1 + \phi^{-1/2} z_1], \phi^{1/2} v_2 + \phi^{-1/2} z_2 \right)_{\Omega_p} + (\mathcal{C} : \mathbf{e}(u_1), \mathbf{e}(u_2))_{\Omega_p} + (m \nabla \cdot (\beta u_1 + w_1), \nabla \cdot (\beta u_2 + w_2))_{\Omega_p} + (\rho \alpha c^{-2} \lambda_1, \lambda_2)_{\Omega_a} + (\rho \varphi_1, \nabla \varphi_2)_{\Omega_a},
\]

where $\tilde{\rho} = (1 - \phi) \rho_f$, $\tilde{\rho} = a_0 \rho_f / \phi$ and $\alpha = 1 + a_0$. We define the operator

\[
A : \mathcal{D}(A) \subset \mathbb{H} \to \mathbb{H}
\]

\[
A \mathcal{U} = \begin{pmatrix}
-\frac{1}{\rho_T} \left( \rho w \nabla \cdot \sigma + \rho_f (\frac{\eta}{k} z + \nabla p) \right) \\
-\frac{1}{\rho_T} (\rho_f \nabla \cdot \sigma + \rho (\frac{\eta}{k} z + \nabla p)) \\
-\lambda \\
-c^2 \Delta \varphi
\end{pmatrix},
\]

with $\rho_T = \rho w - \rho_f^2 > 0$, and where

\[
\mathcal{D}(A) = \{ \mathcal{U} \in \mathbb{H} : u \in H^1_0(\Omega_p) \cap H^2(\Omega_p), v \in H^1_0(\Omega_p), w \in H^2(\Omega_p), \\
z \in H^1_0(\Omega_p), \varphi \in H^2(\Omega_a), \lambda \in H^1_0(\Omega_a); (\sigma + \rho \lambda I) \cdot n_p = 0, \text{ on } \Gamma_I, \\
p - \rho \lambda n_p = 0, \text{ on } \Gamma_1, (\nabla \varphi + v + z) \cdot n_a = 0, \text{ on } \Gamma_1 \}.
\]

With the above notation, problem (8) can be reformulated as follows: given $F \in C^1([0, T] ; \mathbb{H})$ defined as $F(t) = (0, (\rho w f_p - \rho_f g_p) / \rho_T, 0, (\rho g_p - \rho_f f_p) / \rho_T, 0, c^2 f_a)$ and $\mathcal{U}_0 \in \mathcal{D}(A)$, find $\mathcal{U} \in C^1([0, T] ; \mathbb{H}) \cap C^0([0, T] ; \mathcal{D}(A))$ such that

\[
\begin{aligned}
\frac{d \mathcal{U}}{dt} + A \mathcal{U}(t) &= F(t), \quad t \in (0, T], \\
\mathcal{U}(0) &= \mathcal{U}_0.
\end{aligned}
\]

Owing to the Hille-Yosida theorem, the above problem is well-posed provided $A$ is maximal monotone, i.e. $(A \mathcal{U}, \mathcal{U})_{\mathbb{H}} \geq 0 \forall \mathcal{U} \in \mathcal{D}(A)$ and $I + A$ is surjective from $\mathcal{D}(A)$ to $\mathbb{H}$. The first condition follows by definition of the scalar product in $\mathbb{H}$, the definition of the domain $\mathcal{D}(A)$ and integrating by parts, i.e.

\[
(A \mathcal{U}, \mathcal{U})_{\mathbb{H}} = - \left( \rho w \nabla \cdot \sigma + \rho_f \frac{\eta}{k} z + \rho \frac{\eta}{k} \nabla p, v \right)_{\Omega_p} - (\mathcal{C} : \mathbf{e}(v), \mathbf{e}(u))_{\Omega_p} \]

\[
+ \left( \rho_f \frac{\eta}{k} \nabla \cdot \sigma + \rho \frac{\eta}{k} z + \rho \frac{\eta}{k} \nabla p, v \right)_{\Omega_p} - (\rho \alpha \Delta \varphi, \lambda)_{\Omega_a}.
\]
(31a), (31c) and (31e). Then \( U \in \mathcal{D} \).

Milgram Lemma. Indeed, \( A \) and \( D \) can be proved based on employing assumption 3.1 and the trace inverse inequality (11).

\[
\phi := v_0 \quad \text{and} \quad \lambda := v_1, \quad \text{respectively, so that} \quad (31a)-(31f) \quad \text{can be rewritten as:}
\]

\[
\begin{align*}
\rho u + \rho_f w - \nabla \cdot \sigma &= \rho (\mathcal{F}_1 + \mathcal{F}_2) + \rho_f (\mathcal{F}_3 + \mathcal{F}_4) = G_1, \\
\rho_f u + \rho_w w + \frac{\eta}{k} w + \nabla p &= \rho_f (\mathcal{F}_1 + \mathcal{F}_2) + \rho_w (\mathcal{F}_3 + \mathcal{F}_4) + \frac{\eta}{k} \mathcal{F}_3 = G_2, \\
\rho_a c^{-2} \varphi - \rho_a \Delta \varphi &= \rho_a c^{-2} (\mathcal{F}_5 + \mathcal{F}_6) = G_3.
\end{align*}
\]

Since \( n_p = -n_a \) on \( \Gamma_I \), using (31a), (31e) and (31e) and the transmission conditions on \( \Gamma_I \) embedded in the definition of \( \mathcal{D}(A) \), the variational formulation of the above problem reads: find \( (u, w, \varphi) \) \( \in \mathbf{H}^1_0(\Omega_p) \times \mathbf{H}^1_0(\Omega_p) \times H^1_0(\Omega_a) \) s.t.

\[
\mathcal{A}((u, w, \varphi), (v, z, \lambda)) = \mathcal{L}(v, z, \lambda),
\]

for any \( (v, z, \lambda) \in \mathbf{H}^1_0(\Omega_p) \times \mathbf{H}^1_0(\Omega_p) \times H^1_0(\Omega_a) \), where

\[
\mathcal{A}((u, w, \varphi), (v, z, \lambda)) = (\rho u, v)_{\Omega_p} + (\rho_f w, v)_{\Omega_p} + (\mathcal{C} e(u), e(v))_{\Omega_a} + (m \nabla \cdot (\beta u + w), \nabla \cdot (\beta v + z))_{\Omega_p} + (\eta k^{-1} w, z)_{\Omega_p}
\]

\[
+ (\rho_a c^{-2} \varphi, \lambda)_{\Omega_a} + (\rho_a \nabla \varphi, \nabla \lambda)_{\Omega_a} + (\rho_f u, z)_{\Omega_p} + (\rho_w w, z)_{\Omega_p} + (\rho_a \varphi n_p, v + z)_{\Gamma_I} - (\rho_a (u + w) \cdot n_p, \lambda)_{\Gamma_I},
\]

and \( \mathcal{L}(v, z, \lambda) = (G_1, v)_{\Omega_p} + (G_2, z)_{\Omega_p} + (G_3, \lambda)_{\Omega_a} \). This problem is well posed thanks to the Lax-Milgram Lemma. Indeed, \( \mathcal{A} \) is coercive since the interface contribution vanish when \( v = u, z = w \) and \( \lambda = \varphi \). In addition, thanks to (31b), \( u \in \mathbf{H}^2(\Omega_p) \cap H^1_0(\Omega_p), w \in \mathbf{H}^2(\Omega_p) \cap H^1_0(\Omega_p) \) and \( \varphi \in H^2(\Omega_a) \cap H^1_0(\Omega_a) \). Moreover, this gives \( (v, z, \lambda) \in \mathbf{H}^1_0(\Omega_p) \times \mathbf{H}^1_0(\Omega_p) \times H^1_0(\Omega_a) \), thanks to (31a), (31e) and (31e). Then \( \mathcal{U} \in \mathcal{D}(A) \) and the proof is complete.

Finally, we conclude the Appendix with some technical results needed in the analysis. The first lemma can be proved based on employing assumption 3.1 and the trace inverse inequality (11).

**Lemma A.1.** The following bounds hold:

\[
\left\| \eta^{-1/2} \{ \sigma_h(v) \} \right\|_{\mathcal{F}^1_h} \leq \frac{1}{\sqrt{c_1}} \left\| \mathbf{C}^{1/2} \epsilon_h(v) \right\|_{\Omega_p} \quad \forall v \in \mathbf{V}^p_h,
\]

(33)
where $c_1$, $c_2$ and $c_3$ are the stability parameters appearing in (14), (15) and (16), respectively.

The next result follows based on employing lemma A.1.

**Lemma A.2.** Let assumption 3.1 and assumption 3.2 be satisfied. Then,

$$
\begin{align*}
A_h^p(u, v) &\lesssim \|u\|_{dG,e}^2 \|v\|_{dG,e} \quad A_h^p(u, u) \gtrsim \|u\|_{dG,e}^2 & \forall u, v \in V^p_h, \\
B_h^p(u, v) &\lesssim \|u\|_{dG,p} \|v\|_{dG,p} \quad B_h^p(u, u) \gtrsim \|u\|_{dG,p}^2 & \forall u, v \in V^p_h, \\
A_h^p(\varphi, \psi) &\lesssim \|\varphi\|_{dG,a} \|\psi\|_{dG,a} \quad A_h^p(\varphi, \varphi) \gtrsim \|\varphi\|_{dG,a}^2 & \forall \varphi, \psi \in V^a_h, \\
A_h^p(\varphi, \psi) &\lesssim \|\varphi\|_{dG,a} \|\psi\|_{dG,a} \quad B_h^p(w, z) \lesssim \|w\|_{dG,p} \|z\|_{dG,p} & \forall w \in H^2(T^p_h), \forall z \in V^p_h.
\end{align*}
$$

The coercivity bounds hold provided that the stability parameters $c_1$, $c_2$ and $c_3$ appearing in (14),(15) and (16), respectively, are chosen sufficiently large.

**Lemma A.3.** It holds:

$$
\begin{align*}
\|u_h\|_{L^2(E,e)}^2 &\lesssim \|u_h\|_{L^2(E,e)}^2 - 2 \{\sigma_h(u_h)\} \cdot [u_h]_{F^p_h} \lesssim \|u_h\|_{L^2(E,e)}^2 & \forall u_h \in C^1([0, T]; V^p_h), \\
\|\varphi_h\|_{L^2(E,a)}^2 &\lesssim \|\varphi_h\|_{L^2(E,a)}^2 - 2 \{\nabla_h \varphi_h\} \cdot [\varphi_h]_{F^p_h} \lesssim \|\varphi_h\|_{L^2(E,a)}^2 & \forall \varphi_h \in C^1([0, T]; V^p_h), \\
\|z_h\|_{0, dG,p}^2 &\lesssim \|z_h\|_{0, dG,p}^2 - 2m \{\langle \nabla_h \cdot z_h I \rangle\} \cdot [z_h]_{F^p_h} \lesssim \|z_h\|_{0, dG,p}^2 & \forall \varphi_h \in C^1([0, T]; V^a_h).
\end{align*}
$$

The lower bounds hold provided that the penalization constants $c_1$, $c_2$ and $c_3$ appearing in equations (14), (15) and (16) are chosen sufficiently large.

**Proof.** For the proof of the first two bounds, cf [9] and [3, Lemma A.2]. Concerning the last bound, it follows with the same arguments.

**References**

[1] I. Ambartsumyan, E. Khattatov, I. Yotov, and P. Zunino, *A Lagrange multiplier method for a Stokes–Biot fluid–poroelastic structure interaction model*, Numerische Mathematik, 140 (2018), pp. 513–553.

[2] P. Antonietti, M. Verani, C. Vergara, and S. Zonca, *Numerical solution of fluid-structure interaction problems by means of a high order Discontinuous Galerkin method on polygonal grids*, Finite Elem. Anal. Des., 159 (2019), pp. 1–14.

[3] P. F. Antonietti, F. Bonaldi, and I. Mazzieri, *A high-order discontinuous Galerkin approach to the elasto-acoustic problem*, Comput. Methods Appl. Mech. Engrg., 358 (2020), pp. 112634, 29.

[4] P. F. Antonietti, A. Cangiani, J. Collis, Z. Dong, E. H. Georgoulis, S. Giani, and P. Houston, *Review of discontinuous Galerkin finite element methods for partial differential equations on complicated domains*, in Building bridges: connections and challenges in modern approaches to numerical partial differential equations, vol. 114 of Lect. Notes Comput. Sci. Eng., Springer, [Cham], 2016, pp. 279–308.
[5] P. F. Antonietti, C. Facciolà, P. Houston, I. Mazzieri, G. Pennesi, and M. Verani, High-order discontinuous Galerkin methods on polyhedral grids for geophysical applications: seismic wave propagation and fractured reservoir simulations, in D. Di Pietro, L. Formaggia, and R. Masson (eds.), Polyhedral Methods in Geosciences, SEMA-SIMAI Springer series, to appear, 2020.

[6] P. F. Antonietti, C. Facciolà, A. Russo, and M. Verani, Discontinuous Galerkin Approximation of Flows in Fractured Porous Media on Polytopic Grids, SIAM J. Sci. Comput., 41 (2019), pp. A109–A138.

[7] P. F. Antonietti, C. Facciolà, and M. Verani, Unified analysis of discontinuous Galerkin approximations of flows in fractured porous media on polygonal and polyhedral grids, Mathematics in Engineering, 2 (2020), pp. 340–385.

[8] P. F. Antonietti, S. Giani, and P. Houston, hp-version composite Discontinuous Galerkin methods for elliptic problems on complicated domains, SIAM J. Sci. Comput., 35 (2013), pp. A1417–A1439.

[9] P. F. Antonietti and I. Mazzieri, High-order discontinuous Galerkin methods for the elastodynamics equation on polygonal and polyhedral meshes, Comput. Methods Appl. Mech. Engrg., 342 (2018), pp. 414–437.

[10] P. F. Antonietti, I. Mazzieri, M. Muhr, V. Nikolić, and B. Wohlmuth, A high-order discontinuous Galerkin method for nonlinear sound waves, J. Comput phys, 415 (2020), p. 109484.

[11] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM Journal on Numerical Analysis, 39 (2001/02), pp. 1749–1779.

[12] R. L. Berge, I. Berre, E. Keilegavlen, J. M. Nordbotten, and B. Wohlmuth, Finite volume discretization for poroelastic media with fractures modeled by contact mechanics, International Journal for Numerical Methods in Engineering, 121 (2020), pp. 644–663.

[13] A. Bermúdez, R. Rodríguez, and D. Santamarina, Finite element approximation of a displacement formulation for time-domain elastoacoustic vibrations, Journal of Computational and Applied Mathematics, 152 (2003), pp. 17 – 34.

[14] M. A. Biot, General theory of three-dimensional consolidation, Journal of applied physics, 12 (1941), pp. 155–164.

[15] A. Cangiani, Z. Dong, and E. H. Georgoulis, hp-version space-time discontinuous Galerkin methods for parabolic problems on prismatic meshes, SIAM J. Sci. Comput., 39 (2017), pp. A1251–A1279.

[16] A. Cangiani, Z. Dong, E. H. Georgoulis, and P. Houston, hp-version discontinuous Galerkin methods for advection-diffusion-reaction problems on polytopic meshes, ESAIM Math. Model. Numer. Anal., 50 (2016), pp. 699–725.

[17] A. Cangiani, Z. Dong, E. H. Georgoulis, and P. Houston, hp-version discontinuous Galerkin methods on polytopic meshes, SpringerBriefs in Mathematics, Springer International Publishing, 2017.

[18] A. Cangiani, E. H. Georgoulis, and P. Houston, hp-version discontinuous Galerkin methods on polygonal and polyhedral meshes, Mathematical Models and Methods in Applied Sciences, 24 (2014), pp. 2009–2041.

[19] J. Carcione, Wave Fields in Real Media, vol. 38, Elsevier Science, 2014.
[20] B. Castagne, A. Aknine, M. Melon, and C. Depollier, *Ultrasonic characterization of the anisotropic behavior of air-saturated porous materials*, Ultrasonics, 36 (1998), pp. 323–341.

[21] G. Chiavassa and B. Lombard, *Time domain numerical modeling of wave propagation in 2d acoustic / porous media*, J Comput. Phys., 230 (2011), pp. 5288–5309.

[22] J. de la Puente, M. Dumbser, M. Käser, and H. Igel, *Discontinuous Galerkin methods for wave propagation in poroelastic media*, Geophysics, 73 (2008), pp. T77–T97.

[23] B. Flemisch, M. Kaltenbacher, S. Tribenbacher, and B. Wohlmuth, *The equivalence of standard and mixed finite element methods in applications to elasto-acoustic interaction*, SIAM J. Sci. Comput., 32 (2010), p. 1980–2006.

[24] B. Flemisch, M. Kaltenbacher, and B. Wohlmuth, *Elasto-acoustic and acoustic-acoustic coupling on non-matching grids*, Internat. J. Numer. Methods Engrg., 67 (2006), pp. 1791–1810.

[25] B. Gurevich and M. Schoenberg, *Interface conditions for Biot’s equations of poroelasticity*, J. Acoust. Soc. Am., 105 (1999), pp. 2585–2589.

[26] T. Haire and C. Langton, *Biot theory: a review of its application to ultrasound propagation through cancellous bone*, Bone, 24 (1999), pp. 291 – 295.

[27] J. M. Huyghe, D. H. van Campen, T. Arts, and R. M. Heethaar, *A two-phase finite element model of the diastolic left ventricle*, Journal of biomechanics, 24 (1991), pp. 527–538.

[28] G. Jayaraman, *Water transport in the arterial wall—a theoretical study*, Journal of biomechanics, 16 (1983), pp. 833–840.

[29] B. Krishnan, D. M., S. Raja, and K. Venkataramana, *Structural and Vibroacoustic Analysis of Aircraft Fuselage Section with Passive Noise Reducing Materials: A Material Performance Study*, 03 2015.

[30] B. Lombard and J. Piraux, *Numerical treatment of two-dimensional interfaces for acoustic and elastic waves*, Journal of Computational Physics, 195 (2004), pp. 90 – 116.

[31] P. J. Matuszyk and L. F. Demkowicz, *Solution of coupled poroelastic/acoustic/elastic wave propagation problems using automatic hp-adaptivity*, Comput. Methods Appl. Mech. Engrg., 281 (2014), pp. 54–80.

[32] C. Morency and J. Tromp, *Spectral-element simulations of wave propagation in porous media*, Geophysical Journal International, 175 (2008), pp. 301–345.

[33] C. Oomens, D. Van Campen, and H. Grootenboer, *A mixture approach to the mechanics of skin*, Journal of biomechanics, 20 (1987), pp. 877–885.

[34] P. J. Phillips and M. F. Wheeler, *A coupling of mixed and discontinuous Galerkin finite-element methods for poroelasticity*, Computational Geosciences, 12 (2008), pp. 417–435.

[35] A. Quarteroni, *Numerical models for differential problems*, vol. 8, Springer-Verlag Mailand, 2014.

[36] R. T. Rockafellar, *Lagrange multipliers and optimality*, SIAM review, 35 (1993), pp. 183–238.

[37] R. Sidler, J. M. Carcione, and K. Holliger, *Simulation of surface waves in porous media*, Geophysical Journal International, 183 (2010), pp. 820–832.

[38] M. Souzanchi, L. Cardoso, and S. Cowin, *Tortuosity and the averaging of microvelocity fields in poroelasticity*, Journal of applied mechanics, 80 (2013).

[39] E. M. Stein, *Singular integrals and differentiability properties of functions*, vol. 2, Princeton university press, 1970.
[40] C. Talischi, G. H. Paulino, A. Pereira, and I. F. Menezes, *Polymesher: a general-purpose mesh generator for polygonal elements written in Matlab*, Structural and Multidisciplinary Optimization, 45 (2012), pp. 309–328.

[41] S. Triebenbacher, M. Kaltenbacher, B. Wohlmuth, and B. Flemisch, *Applications of the mortar finite element method in vibroacoustics and flow induced noise computations*, Acta Acustica united with Acustica, 96 (2010), pp. 536–553(18).