Black Hole Remnant from Gravity’s Rainbow

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In this work, we investigate black hole (BH) physics in the context of gravity rainbow. We investigate this through rainbow functions that have been proposed by Amelino-Camelia, et al. in [1, 2]. This modification will give corrections to both the temperature and the entropy of BH and hence it changes the picture of Hawking radiation greatly when the size of BH approaches the Planck scale. It prevents BH from total evaporation, predicting the existence of BH remnant which may resolve the catastrophic behavior of Hawking radiation as the BH mass approaches zero.

I. INTRODUCTION

One common feature among most of semi-classical approaches to quantum gravity is the departure from the relativistic dispersion relation by redefining the physical momentum or physical energy at the Planck scale, or Lorentz invariance violation. The source of this departure comes from many approaches such as spacetime discreteness [3], spontaneous symmetry breaking of Lorentz invariance in string field theory[4], spacetime foam models [7] or spin-network in Loop quantum gravity (LQG) [5]. Besides, there are other approaches such as non-commutative geometry [6] which predicts a Lorentz invariance violation. All these indications suggest that Lorentz violation may be an essential property in constructing quantum theory of gravity. Theoretically, the departure from Lorentz invariance is expressed in a form of modified dispersion relations (MDR). MDR could be an indication for threshold anomalies that might occur in ultra-high energy cosmic rays and TeV photons [7–9]. Modern observatories recently are gaining the sensitivity needed to measure these effects, and should be improved in coming few years[1]. For a recent detailed review along the mentioned lines can be found in [1].

One of the most interesting class of MDRs, has been suggested in [1, 2] which in the high-energy regime takes the form:

\[ m^2 \simeq E^2 - \tilde{p}^2 + \eta p^2 \left( \frac{E}{E_p} \right)^n \] (1)

where \( E_p \) describes the energy scale at which the dispersion relation is modified and it is taken to be the Planck energy, while \( n \) and \( \eta \) represents a free parameters characterizing the deviation from Lorentz invariance and \( n \) indicates how much strongly the magnitude of the deformation that is suppressed by \( E_p \). This formula is compatible with some of the results obtained in the Loop-Quantum-Gravity approach and reflects some results obtained for theories in \( \kappa \)-Minkowski noncommutative spacetime [1].

For discussion about phenomenological implications of Eq. (1), it is very useful to consult the discussion after Eq. (13) in the detailed review [1].

A theory that predict naturally MDR is known as double special relativity (DSR)[11]. DSR is considered as an extension for special theory of relativity and it extends the invariant quantities to be the Planck energy scale beside the speed of light. The simplest realizations of the idea of DSR are based on a non-linear Lorentz transformation in momentum space, which imply a deformed Lorentz symmetry such that the usual dispersion relations in special relativity may be modified by Planck scale corrections. It should be mentioned that Lorentz invariance violation and Lorentz invariance deformation are in general conceptually different scenarios. Here we are going to adopt Lorentz invariance deformation scenarios by considering DSR and its extension in models of rainbow gravity.

In the framework of DSR the definition of the dual position space suffers a nonlinearity of the Lorentz transformation. To resolve this issue, Magueijo and Smolin [12] proposed a doubly general relativity which assumes that the spacetime background felt by a test particle would depend on its energy. Therefore, there will not be a single metric describing spacetime, but a one parameter family of metrics which depends on the energy (momentum) of these test particles, forming a rainbow of metrics (i.e. rainbow geometry). This approach is known as Gravity Rainbow and can be mathematically constructed as follows; the non-linear of Lorentz transformation leads to the following modified dispersion relation

\[ E^2 f(E/E_p)^2 - p^2 g(E/E_p)^2 = m^2 \] (2)

where \( E_p \) is the Planck energy scale, \( m \) is the mass of the test particle, \( f(E/E_p) \) and \( g(E/E_p) \) are com-
monly known as Rainbow functions and they satisfy
\[ \lim_{E \to 0} f(E/E_p) = 1 \text{ and } \lim_{E \to \infty} g(E/E_p) = 1. \]

A modified equivalence principle was proposed in [12] which requires that one parameter family of energy dependent orthonormal frame fields describe a one parameter family of energy dependent metrics given by
\[ h(E/E_p) = \eta^{ab} e_a(E/E_p) \otimes e_b(E/E_p) \]
where \( e_0(E/E_p) = (1/f(E/E_p)) e_0 \) and \( e_i(E/E_p) = (1/f(E/E_p)) \tilde{e}_i \). But in the limit \( (E/E_p) \to 0 \) general relativity must be recovered. With the definition of one parameter family of energy momentum tensors Einstein’s equations are also modified as
\[ G_{\mu\nu}(E/E_p) = 8\pi G T_{\mu\nu}(E/E_p) \]
Potential investigations on the gravity rainbow can be found in [13].

The choice of the Rainbow functions \( f(E/E_p) \) and \( g(E/E_p) \) is important for making predictions. Among different arbitrary choices in [13, 14], many aspects of the theory have been studied with Schwarzshild metric, black holes, FRW universe, and cosmological eras such as inflation and scale invariant fluctuations [15]. Besides, we studied recently the possibility of resolving big bang singularity using gravity rainbow in [16]. In this letter we continue our investigation about the effect of gravity rainbow on BH thermodynamics. We employ the modified dispersion relation of Eq.(1) [1, 2], which fix the rainbow functions \( f(E/E_p) \) and \( g(E/E_p) \) and use these rainbow functions to study the BH thermodynamics and investigate it new properties. We find that end-point of Hawking radiation is not catastrophic anymore in rainbow BH. We found that an existence of BH remnants at which the specific heat vanishes and, therefore, the BH cannot exchange heat with the surrounding space.

An outline of this paper is as follows. In section (II), We review briefly how standard Hawking temperature can be obtained from the Schwarzshild metric and show briefly the catastrophic evaporation of BH. In section (III), we investigate BH thermodynamics in gravity rainbow. We give our conclusions in section (IV).

II. STANDARD HAWKING RADIATION

Let us first review briefly the standard Hawking radiation process. Since Hawking temperature is defined in terms of surface gravity \( \kappa \) [17] as follows:
\[ T_H = \frac{\kappa}{2\pi} \]
where the surface gravity \( \kappa \) is defined as [18]
\[ \kappa = \lim_{r \to R_S} \sqrt{-\frac{1}{4} g^{rr} g^{tt} (g_{tt}, r)^2} \]
where \( R_S = 2GM \) is the Schwarzschild radius. In general relativity the surface gravity is calculated from Eq. (6) for Schwarzschild metric and it is given by
\[ \kappa = \frac{1}{4MG} \]
and hence the Hawking temperature is given by
\[ T_H = \frac{1}{8\pi GM} \]
The BH entropy can be calculated through the first law of BH thermodynamics:
\[ dM = TdS. \]
By integrating Eq. (9) using Eq. (8), one can obtain the the Bekenstein entropy[19] as follows:
\[ S = 4\pi GM^2. \]
The specific heat can be calculated using the thermodynamical relation
\[ C = T \frac{\partial S}{\partial T} = T \frac{\partial S}{\partial M} \frac{\partial M}{\partial T} = \frac{\partial M}{\partial T}, \]
By differentiating Eq. (8) and substituting this into Eq. (11), the specific heat could be given by
\[ C = -8\pi GM^2, \]
The Hawking temperature \( T_H \) can be used in the calculation of the emission rate. The emission rate might be calculated using Stefan-Boltzmann law considering the energy loss was dominated by photons. The emission rate of the BH will be:
\[ \frac{dM}{dt} = -M^3 \frac{\mu'}{t_p} M^{-2}, \]
where \( t_p = G^{1/2} \) is the Planck time in natural units, and the form of \( \mu \) can be found in [20, 21]. The exact calculation should consider the squeezing of the fundamental cell in momentum space, which modify the emission rate equation (13). However, one can neglect this effect in the first order approximation [22]. The decay time of the BH can be obtained by integrating Eq. (13) to give
\[ \tau = \left( \frac{1}{3} \right) \frac{t_p}{M^3} \frac{\mu'}{M} \]
One notice that the calculated Hawking temperature \( T_H \), Bekenstein entropy \( S \), specific heat \( C \), emission rate \( \frac{dM}{dt} \), and decay time \( \tau \) lead to catastrophic evaporation as \( m \to 0 \). This can be explained as following. Since \( C = 0 \) only when \( m = 0 \), the BH will continue to radiate
III. RAINBOW BLACK HOLE

Now we employ the modified dispersion relation which is proposed by the Amelino-Camelia, et al. in Eq. (1), and compare it with Eq. (2). The functions \( f(E/E_p) \) and \( g(E/E_p) \) can be fixed as follows:

\[
f(E/E_p) = 1, \quad g(E/E_p) = \left(1 - \frac{\eta E}{E_p}\right)^n.
\] (15)

Let us consider the Rainbow Schwarzschild metric for non-rotating and uncharged BH [12]

\[
ds^2 = -\frac{1}{f(E/E_p)^2} \left(1 - \frac{2MG}{r}\right) dt^2 + \frac{1}{g(E/E_p)^2} \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + \frac{r^2}{g(E/E_p)^2} d\Omega^2
\] (16)

By using the rainbow Schwarzschild metric of Eq. (16), the surface gravity of Eq. (6) get the following form:

\[
\kappa' = \frac{g(E/E_p)}{f(E/E_p)^2} \frac{1}{4MG}
\] (17)

If we set \( f(E/E_p) = g(E/E_p) = 1 \), then we will get back the surface gravity for non-rotating and uncharged BH. We find that Eq. (17) introduces the rainbow surface gravity. Then, we can calculate the modified Hawking Temperature in rainbow gravity using the modified surface gravity of Eq. (17) as follows:

\[
T'_H = \frac{\kappa'}{2\pi} = \frac{g(E/E_p)}{f(E/E_p)} \frac{1}{8\pi MG}
\] (18)

Using the identification of Eq. (15), the modified Hawking temperature will be given as follows:

\[
T'_H = \sqrt{1 - \eta \left(\frac{E}{E_p}\right)^n} \frac{1}{8\pi MG}
\] (19)

Let us consider the ordinary uncertainty relation to photons near the event horizon. According to [20, 21, 23, 24] a photon is used to ascertain the position of a quantum particle of energy \( E \) and according to the argument in [25] which states that the uncertainty principle \( \Delta p \geq 1/\Delta x \) can be translated to the lower bound \( E \geq 1/\Delta x \)

\[
E \geq \frac{1}{\Delta x} \approx \frac{1}{2GM}
\] (20)

where the value of \( \Delta x \) has its minimum value taken to be Schwarzschild radius \( R_S \), where this is probably the most sensible choice of length scale in the context of near-horizon geometry [20, 21, 23, 24]. By substituting Eq. (20) into the modified Hawking Temperature in Eq. (19), we get:

\[
T'_H = \frac{1}{4\pi (2GM)^{\frac{n}{2}} \sqrt{(2GM)^n - \eta E_p^n}}
\] (21)

It is clear from Eq. (21) that if we set \( \eta = 0 \) or assume that \( E/E_p \to 0 \) (i.e. \( E_p \to \infty \)), we get back the Hawking temperature of Eq. (8). It is also clear from Eq. (21) that the modified Hawking temperature is physical as far as the BH mass satisfies the following inequality:

\[
M \geq \frac{1}{2} \eta^{1/n} E_p
\] (22)

where we have used the natural units which says \( G = 1/E_p^2 \). This implies that the black hole should have minimum mass \( M_{min} \) given by

\[
M_{min} = \frac{1}{2} \eta^{1/n} E_p
\] (23)

This point to an existence of black hole remnant due to gravity rainbow. We plot in Fig.(1) the relation between the temperature and BH mass for both cases of standard and rainbow BH. It is clear that \( T \) does not blow up as \( M \to 0 \) in rainbow BH in the contrary to the standard BH case. In this figure, we set \( n = 4 \) as an example. However, different values of \( n \) will give similar behavior like in Fig.(1).

The entropy of rainbow BH can be calculated through the first law of BH thermodynamics:

\[
dS = \frac{dM}{T} = \frac{1}{4\pi (2GM)^{\frac{n}{2}} \sqrt{(2GM)^n - \eta E_p^n}}
\] (24)

With putting \( n = 4 \), and taking \( G = 1/E_p^2 \) in natural units, the exact form of the entropy will be

\[
S' = \pi \sqrt{\frac{16M^4 - \eta E_p^4}{E_p^2}}
\] (25)

We show in Fig. (2), that the entropy of rainbow BH reaches a minimal value which represents the information contained in the BH remnant.
Turning to calculate the specific heat of the rainbow BH. It is given by

$$C' = T \frac{\partial S}{\partial T} = \frac{\partial M}{\partial T}$$

$$= 16 \sqrt{2^n \left( \frac{M}{E_p} \right)^n - \eta E_p^{-n} \pi M^2 2^{\frac{n}{2}} \left( \frac{M}{E_p} \right)^{\frac{n}{2}}}$$

$$= -E_p^2 \left( -n \eta E_p^{-n} + 1 + \eta \left( \frac{M}{E_p} \right)^{n - 2} \eta E_p^{-n} \right)$$

(26)

The last expression for the specific heat of Eq. (26) indicates that the specific heat vanishes at $M = M_{\text{min}} = \frac{1}{2} \eta^{1/n} E_p$. Therefore, the BH cannot exchange heat with the surrounding space and hence predicting the existence of black hole remnants.

We plot in Fig. (3) the specific heat versus the BH mass (putting $n = 4$ as an example) and found that the specific heat of rainbow BH diverges at a point at which the BH temperature reaches its maximum value and then it decreases to zero when the mass of the BH reaches its minimal value (i.e., remnant). At this point, specific heat vanishes and hence the BH do not exchange heat with the surrounding space leaving a remnant.

Now, we use the rainbow Hawking temperature to calculate the emission rate of the rainbow BH using Stefan-Boltzmann law considering the energy loss was dominated by photons. This is found to be

$$\frac{dM}{dt} \text{ (rainbow)} = \frac{dM}{dt} \text{ (standard)} \left( 1 - \eta \frac{E_p^n}{(2M)^n} \right)^2,$$

(27)

This means that the emission rate of the rainbow BH vanishes when the BH reaches its minimal value $M = M_{\text{min}} = \frac{1}{2} \eta^{1/n} E_p$. We show in Fig. (4) (putting $n = 4$) how the picture of the emission rate of the BH got changed in gravity rainbow. In the standard picture, the emission rate goes to infinity as the mass of the BH tends to zero. In rainbow gravity, the emission rate of rainbow BH do not diverge at all, and it just go to zero when the rainbow BH reaches its minimum value which can be called as a BH remnant. Since remnant of black holes need not possess the horizon, we think that our result may ameliorate information loss problem[27].

It is worth comparing our results with previous analysis of black hole evaporation with MDR and GUP that has been investigated in [26]. It has been found in [26] that the modified temperature of evaporating black hole due to MDR diverges when the mass of the black hole decreases to a Planck-scale value instead of having divergence only when the mass tends to zero in the standard case. It is clear this picture is different from the picture that we obtained in our work in Eq. (21) in which the temperature do not suffer any divergence as the mass of the black hole approaches Planck scale. On the contrary, the Hawking temperature for rainbow BH in Eq. (21) increases as the mass of the black hole decreases until it reaches maximum value, and then decreasing to be zero when the black hole reaches a remnant case at which the black hole cannot exchange heat with the surrounding space.
a catastrophic because rainbow functions that are proposed by Amelino-Camelia, et al. in [1, 2] imply the existence of BH remnants at which the specific heat vanishes and, therefore, the BH cannot exchange heat with the surrounding space. The gravity rainbow prevents BHs from evaporating completely, just like the standard uncertainty principle prevents the hydrogen atom from collapsing[20, 21]. Our result agree with results obtained in the framework of generalized uncertainty principle and black hole physics which also predicted the existence of black hole remnants[20, 21]. Since remnant of black holes does not possess the horizon, we think that our result may ameliorate information loss problem[27]. In the future, it would be appropriate to generalize the calculations in extra dimensions to investigate the possibilities to see the remnants of rainbow black holes at LHC.

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[1] G. Amelino-Camelia, Living Rev. Rel. 16, 5 (2013) [arXiv:0806.0339 [gr-qc]].
[2] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos and D. V. Nanopoulos, Int. J. Mod. Phys. A 12, 607 (1997) [hep-th/9605211].
[3] G. ’t Hooft, Class. Quant. Grav. 13, 1023 (1996) [gr-qc/9601014].
[4] V. A. Kostelecký and S. Samuel, Phys. Rev. D 39, 683 (1989).
[5] R. Gambini and J. Pullin, Phys. Rev. D 59, 124021 (1999) [gr-qc/9809038].
[6] S. M. Carroll, J. A. Harvey, V. A. Kostelecký, C. D. Lane and T. Okamoto, Phys. Rev. Lett. 87, 141601 (2001) [hep-th/0105082].
[7] G. Amelino-Camelia, J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos and S. Sarkar, Nature 393, 763 (1998) [astro-ph/9712103].
[8] G. Amelino-Camelia, J. Lukierski and A. Nowicki, Phys. Atom. Nucl. 61, 1811 (1998) [Yad. Fiz. 61, 1925 (1998)] [hep-th/9706031].
[9] G. Amelino-Camelia, J. Lukierski and A. Nowicki, Int. J. Mod. Phys. A 14, 4575 (1999) [gr-qc/9903066].
[10] G. Amelino-Camelia, New J. Phys. 6, 188 (2004) [gr-qc/0212002].
[11] G. Amelino-Camelia, Int. J. Mod. Phys. D 11, 35 (2002) [gr-qc/0012051]; J. Magueijo and L. Smolin, Phys. Rev. D 67, 044017 (2003) [gr-qc/0207085].
[12] J. Magueijo and L. Smolin, Class. Quant. Grav. 21, 1725 (2004) [gr-qc/0305055].
[13] P. Galan and G. A. Mena Marugan, Phys. Rev. D 70, 124003 (2004) [gr-qc/0411089]; J. Hackett, Class. Quant. Grav. 23, 3833 (2006) [gr-qc/0509103]; F. Girelli, S. Liberati and L. Sindoni, Phys. Rev. D 75, 064015 (2007) [gr-qc/0611024]; C. -Z. Liu and J. -Y. Zhu, Gen. Rel. Grav. 40, 1899 (2008) [gr-qc/0703055 [GR-QC]]; H. Li, Y. Ling and X. Han, Class. Quant. Grav. 26, 065004 (2009) [arXiv:0809.4819 [gr-qc]]; R. Garattini and G. Mandanici, Phys. Rev. D 85, 023507 (2012) [arXiv:1109.6563 [gr-qc]]; R. Garattini and F. S. N. Lobo, Phys. Rev. D 85, 024043 (2012); R. Garattini and G. Mandanici, Phys. Rev. D 83, 084021 (2011); J. -J. Peng and S. -Q. Wu, Gen. Rel. Grav. 40, 2619 (2008) [arXiv:0709.0167 [hep-th]].
[14] Y. Ling, JCAP 07080, 017 (2007) [gr-qc/0609129]; Y. Ling and Q. Wu, Phys. Lett. B 687, 103 (2010) [arXiv:0811.2615 [gr-qc]].
[15] J. D. Barrow and J. Magueijo, arXiv:1310.2072 [astro-ph.CO]; G. Amelino-Camelia, M. Arzano, G. Guiboutsi and J. Magueijo, Phys. Rev. D 88, 041303 (2013) [arXiv:1307.0745 [gr-qc]].
[16] A. Awad, A. F. Ali and B. Majumder, JCAP 1310, 052 (2013) [arXiv:1308.4343 [gr-qc]].
[17] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)]; S. W. Hawking, Phys. Rev. D 13, 191 (1976); J. D. Bekenstein, Lett. Nuovo Cim. 4, 737 (1972).
[18] Oyvind Gron, Einstein’s general theory of relativity: with modern applications in cosmology. Springer, 2007; R.M.Wald, General relativity (University of Chicago Press; Chicago, 1984).
[19] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[20] R. J. Adler, P. Chen, D. I. Santiago, Gen. Rel. Grav. **33**, 2101-2108 (2001). [gr-qc/0106080].
[21] M. Cavaglia, S. Das, R. Maartens, Class. Quant. Grav. **20**, L205-L212 (2003). [hep-ph/0305223]; M. Cavaglia, S. Das, Class. Quant. Grav. **21**, 4511-4522 (2004). [hep-th/0404050].
[22] J. C. Niemeyer, Phys. Rev. D **65**, 083505 (2002) [astro-ph/0111479]; A. Kempf, J.Phys. A **30** (1997) 2093 [arXiv:hep-th/9604045].
[23] A. J. M. Medved, E. C. Vagenas, Phys. Rev. D **70**, 124021 (2004). [hep-th/0411022].
[24] B. Majumder, Phys. Lett. B **703**, 402 (2011). [arXiv:1106.0715 [gr-qc]].
[25] G. Amelino-Camelia, M. Arzano and A. Procaccini, Phys. Rev. D **70**, 107501 (2004) [gr-qc/0405084]; E.M. Lifshitz, L.P. Pitaevskii and V.B. Berestetskii, Landau-Lifshitz Course of Theoretical Physics, Volume 4: Quantum Electrodynamics, (Reed Educational and Professional Publishing, 1982).
[26] G. Amelino-Camelia, M. Arzano, Y. Ling and G. Mandanici, Class. Quant. Grav. **23**, 2585 (2006) [gr-qc/0506110].
[27] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, Phys. Rev. Lett. **108**, 031101 (2012) [arXiv:1110.5249 [gr-qc]].