An Explanation of the “Pioneer Effect” based on Quasi-Metric Relativity

by

Dag Østvang

Institutt for Fysikk, Norges teknisk-naturvitenskapelige universitet, NTNU
N-7491 Trondheim, Norway

Abstract

According to the so-called “quasi-metric” framework developed elsewhere, the cosmic expansion applies directly to gravitationally bound systems. This prediction has a number of observable consequences, none of which are in conflict with observation. In this paper we compare test particle motion in the nonstatic gravitational field outside a spherically symmetric source (as predicted by a quasi-metric theory of gravity) to test particle motion in the Schwarzchild geometry. It is found that if one incorrectly uses the Schwarzchild geometry (to the relevant accuracy) to represent the nonstatic quasi-metric model, the largest errors result from the mismodelling of null paths. One consequence of this is that using electromagnetic signals to track the motion of a non-relativistic particle results in the illusion that the particle is influenced by an anomalous acceleration of size $cH$ (where $H$ is the Hubble parameter) directed towards the observer. This result naturally explains the apparently anomalous force acting on the Pioneer 10/11, Galileo and Ulysses spacecraft as inferred from radiometric data.

1 Introduction

Some time ago a detailed analysis of the observed versus the calculated orbits of the Pioneer 10, Pioneer 11, Ulysses and Galileo spacecraft was published [1]. The main result was that the observed radiometric data did not agree with calculations based on standard theory; rather the data indicated the existence of an “anomalous”, constant acceleration towards the Sun.

A short summary of the results presented in [1] is as follows: For the Pioneer spacecraft the Doppler frequency shift of the radio carrier wave was recorded and analysed to determine the spacecraft’s orbits. Two independent analyses of the raw data were performed. Both showed an anomalous acceleration towards the Sun, of respectively
(8.09±0.20)×10^{-8} \text{ cm/s}^2\) and \((8.65±0.03)×10^{-8} \text{ cm/s}^2\) for Pioneer 10. For Pioneer 11 only one result is given; an anomalous acceleration of \((8.56±0.15)×10^{-8} \text{ cm/s}^2\) towards the Sun. The acceleration did not vary between 40 – 60 astronomical units, within a sensitivity of \(2×10^{-8} \text{ cm/s}^2\).

For the Galileo and Ulysses spacecraft one also got ranging data in addition to the Doppler data. For Ulysses one had to model the solar radiation pressure in addition to any constant anomalous acceleration. By doing this it was found that Ulysses was influenced by an anomalous acceleration of \((12±3)×10^{-8} \text{ cm/s}^2\) towards the Sun, consistent with both Doppler and ranging data. For Galileo the corresponding result was an anomalous acceleration of \((8±3)×10^{-8} \text{ cm/s}^2\) towards the Sun.

Recently a new comprehensive study of the anomalous acceleration was published [2], including a total error budget for the Pioneer 10 data analysis. The new result reported in [2] is an “experimental” anomalous acceleration of \((7.84±0.01)×10^{-8} \text{ cm/s}^2\) towards the Sun, and including bias and uncertainty terms the final value becomes \((8.74±1.33)×10^{-8} \text{ cm/s}^2\). For Pioneer 11 an experimental value of \((8.55±0.02)×10^{-8} \text{ cm/s}^2\) was given. Also other new results, such as annular and diurnal variations in the anomalous acceleration, were reported in [2].

An interpretation of these results according to the standard general relativistic model indicates the existence of an anomalous, time-independent force acting on the spacecraft. However there are problems with this interpretation since according to the planetary ephemeris there is no indication that such a force acts on the orbits of the planets; the hypothetical force thus cannot be of gravitational origin without violating the weak principle of equivalence. Thus it is speculated that the effect is due to anisotropic radiation of waste heat from the radioactive thermal generators aboard the spacecraft; the design of the spacecraft is such that waste heat may possibly be scattered off the back of the high gain antennae in directions preferentially away from the Sun [3]. Moreover, besides possible anisotropic scattering, an estimate shows that the specific arrangement of waste heat radiators on the surface of the spacecraft may perhaps cause sufficient anisotropy in the radiative cooling to explain the data [4]. However, it seems that these explanations have been effectively refuted [2], [5], [6], [7]. Other possible explanations, such as gas leaks, have been proposed [2], [5], but so far it seems that no satisfactory explanation based on well-known physics exists.

However, it is an intriguing fact that the size of the anomalous acceleration is of the order \(cH\) for all the spacecraft, where \(H\) is the Hubble parameter. Since this seems to be too much of a coincidence one may suspect that the data indicate the existence of new physics rather than a prosaic explanation based on standard theory. This has
been duly noted by others, see [2] and references listed therein for a number of new physics suggestions motivated by the anomaly. But to be acceptable, any non-standard explanation should follow naturally from a general theoretical framework. In this paper we show that such an explanation can be found, thus the data may indeed be taken as evidence for new physics.

2 A short description of the quasi-metric model

In references [8], [9] we defined the so-called “quasi-metric” space-time framework (QMF); this framework is non-metric since it is not based on semi-Riemannian geometry. Briefly the geometrical basis of the QMF consists of a 5-dimensional differentiable manifold with topology $\mathcal{M} \times \mathbb{R}_1$, where $\mathcal{M} = S \times \mathbb{R}_2$ is a Lorentzian space-time manifold, $\mathbb{R}_1$ and $\mathbb{R}_2$ both denote the real line and $S$ is a compact 3-dimensional manifold (without boundaries). Moreover the manifold $\mathcal{M} \times \mathbb{R}_1$ is equipped with a degenerate metric. That is, in addition to the usual time dimension and 3 space dimensions there is an extra degenerate time dimension represented by the global time function $t$. The physical role of the degenerate dimension is to describe global scale changes between gravitational and non-gravitational systems. In particular this yields an alternative description of the expansion of the Universe.

The global time function is unique in the sense that it splits quasi-metric space-time into a unique set of 3-dimensional spatial hypersurfaces called fundamental hypersurfaces (FHSs). Observers always moving orthogonal to the FHSs are called fundamental observers (FOs). The topology of $\mathcal{M}$ indicates that there also exists a unique “preferred” ordinary time coordinate $x^0$. We use this fact to construct the 4-dimensional quasi-metric space-time manifold $\mathcal{N}$ by slicing the submanifold determined by the equation $x^0 = ct$ out of the 5-dimensional differentiable manifold. Thus the 5-dimensional degenerate metric field $\mathbf{g}_t$ may be regarded as a one-parameter family of Lorentzian 4-metrics on $\mathcal{N}$. Note that there exists a set of particular coordinate systems especially well adapted to the geometrical structure of quasi-metric space-time, the global time coordinate systems (GTCSs). A coordinate system is a GTCS iff the time coordinate $x^0$ is related to $t$ via $x^0 = ct$ in $\mathcal{N}$. Besides, for idealized situations it may be possible to find a comoving coordinate system, which by definition is a (non-static) GTCS where the FOs are at rest.

In reference [10] we introduced a model of the gravitational field outside a spherically symmetric, isolated source as predicted by a particular quasi-metric theory of gravity developed in [8], [9] (where detailed descriptions of this theory can be found). According to this theory it was found in [10] that at scales similar to the size of the solar system,
such a gravitational field can be adequately expressed by the one-parameter family \( g_t \) of Lorentzian 4-metrics (expressed in a spherical comoving coordinate system)

\[
ds_t^2 = -B(r)(dx^0)^2 + \left(\frac{t}{t_0}\right)^2\left(A(r)dr^2 + r^2d\Omega^2\right),
\]

where \( r \) is a comoving radial coordinate and \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) is the squared solid angle line element. Furthermore \( t_0 \) represents some arbitrary reference epoch setting the scale of the spatial coordinates. (To accurately model the solar system gravitational field in a cosmological context one must in principle identify the cosmic rest frame with a suitable GTCS and include the gravitational effects of the Galaxy and the cosmological substratum. However, the solar system is so small that we can treat it as an isolated system moving with constant velocity with respect to the cosmic rest frame. We can then transform to a new GTCS where the solar system is at rest.)

The locally measured Hubble parameter \( H \) is defined as the fractional change of \( t \) as measured by the FOs [10]. That is, from equation (1) we get

\[
H(r, t) = \frac{ct_0}{t} \left(\sqrt{B(r)}\right)^{-1} \frac{d}{dx^0}(\frac{t}{t_0}) = \left(\sqrt{B(r)t}\right)^{-1} = \frac{1}{t} \left(1 + \frac{r_{s0}}{2r} + O(\frac{r_{s0}^2}{r^2})\right),
\]

where \( r_{s0} \) is the Schwarzschild radius at the arbitrary epoch \( t_0 \), and where we have used equation (6) below in the last step. However, for weak gravitational fields the \( r \)-dependent part of \( H \) can be neglected. Hence, since we here consider applications of quasi-metric theory to the solar system only, we will use the approximation \( H = \frac{1}{t} \) for the rest of this paper.

General equations of motion are obtained from the geodesic equation using the non-metric connection [9]. But these equations of motion cannot be obtained from the geodesic equation using any metric connection. Moreover, as shown in [10] special equations of motion for inertial test particles moving in the particular metric family (1) take the form (due to the spherical symmetry we can restrict the motion to the equatorial plane \( \theta = \pi/2 \))

\[
\left(\frac{t}{t_0}\right)^2 A(r) \frac{dr}{cdt}^2 - \frac{1}{B(r)} \frac{J^2}{r^2} = -E,
\]

\[
\frac{t}{t_0} r^2 \frac{d\phi}{cdt} = B(r)J,
\]

\[
d\tau_t^2 = -c^{-2}ds_t^2 = E B^2(r)dt^2,
\]
where $J$ and $E$ are constants of the motion. (Note that the dynamically measured mass of the central object as measured by distant orbiters increases to exactly balance the effect on circle orbit velocities of expanding circle radii, according to equations (3), (4) and (5). For a further explanation of this, see [10].) By setting the scale factor $\frac{t}{t_0} = 1$ in equations (1), (3), (4) and (5) we recover the equations of motion for inertial test particles moving in a spherically symmetric, static gravitational field as obtained from General Relativity (GR). Note that $E = 0$ for photons and $E > 0$ for material particles, which may readily be seen from equation (5).

The functions $A(r)$ and $B(r)$ may be found as series expansions by solving the field equations, this is done approximately in [10]. For our purposes we include terms to post-Newtonian order but not higher. Then we have

$$A(r) = 1 + \frac{r_{s0}}{r} + O\left(\frac{r_{s0}^2}{r^2}\right),$$

$$B(r) = 1 - \frac{r_{s0}}{r} + O\left(\frac{r_{s0}^3}{r^3}\right).$$

(6)

Note that $A(r)$ and $B(r)$ are not inverse functions [10].

3 Comparing the quasi-metric and standard models

Quasi-metric theory predicts that the global cosmic expansion applies directly to gravitationally bound systems [10]. This has a number of observable consequences, some of which we will calculate in the following.

We now explore some of the differences between the non-static system described by equation (1) and the corresponding static system obtained by setting $\frac{t}{t_0} = 1$ in (1) and using GR. To begin with we notice that the shapes of free fall orbits (expressed e.g. as functions of the type $r(\phi)$) are identical for the two cases [10]. Moreover, it can be shown that the time dependence present in equations (3), (4) and (5) does not lead to easily observable perturbations in the paths of non-relativistic particles compared to the static case [10]. However, as we now illustrate, if one considers null paths potential observable consequences appear if one treats $r$ as a static coordinate rather than as a comoving one. To simplify matters we consider purely radial motion, i.e. $J = 0$ (one may easily generalize to $J \neq 0$). Since $E = 0$ for photons we get from equation (3) that radial null curves are described by the equation

$$\frac{dr}{dt} = \pm \frac{ct_0}{t} \sqrt{\frac{B(r)}{A(r)}} = \pm \frac{ct_0}{t} \sqrt{1 - \frac{2r_{s0}}{r} + O\left(\frac{r_{s0}^2}{r^2}\right)} = \pm \frac{ct_0}{t} \left(1 - \frac{r_{s0}}{r} + O\left(\frac{r_{s0}^2}{r^2}\right)\right),$$

(7)
the choice of sign depending on whether the motion is outwards or inwards. We may now integrate (7) along a null path (for convenience we choose the positive sign in equation (7)). We get
\[
\int_r^{r+R} \frac{dr'}{1 - \frac{rs_0}{r'}} + O\left(\frac{rs_0}{r^2}\right) = c t_0 \int_t^{t+T} \frac{dt'}{t'} = c t_0 \ln(1 + \frac{T}{t}) = c t_0 \frac{T}{t} \left(1 - \frac{T}{2t} + O\left(\frac{T^2}{t^2}\right)\right),
\]
where \(T\) is the light time along the null path and where \(R\) is the radial coordinate distance between the object and the observer. From equation (8) we find an extra delay, as compared to standard theory, in the time it takes an electromagnetic signal to travel from an object being observed to the observer. To lowest order this extra time delay is \(\frac{T^2}{2t}\), and for weak gravitational fields we may write the extra delay as \(\frac{HR^2}{2c^2}\), where \(H\) is the Hubble parameter as given from equation (2). Besides this extra time delay, the fact that the scale factor in equation (1) increases with time implies that our model predicts an extra redshift, as compared to standard static models, in the Doppler data obtained from any object emitting electromagnetic signals. To lowest order this extra redshift corresponds to a “Hubble” redshift \(\frac{HR}{c}\).

But the velocity at any given time of an observed object cannot model-independently be split up into one “ordinary” piece and one “Hubble” piece. This means that there is no direct way to identify the predicted extra redshift in the Doppler data. Similarly, at any given time there is no direct way to sort out the predicted extra time delay when determining the distance to the object. Rather, to test whether the gravitational field is static or not one should do observations over time and compare the observed motion to a model. In a model one uses a coordinate system and to calculate coordinate accelerations one needs coordinate accelerations. Accordingly we construct the “properly scaled coordinate acceleration” quantity \(a_c\). For photons this is
\[
a_c \equiv t_0 \frac{t}{t_0} \sqrt{A(r)} \frac{d^2r}{dt^2} = \pm \frac{c}{t} + \frac{t_0 \, rs_0 c^2}{t} \frac{r}{r^2} + O\left(\frac{r^2 s_0 c^2}{r^3}\right)
\]
\[
= \pm cH + \frac{t_0 \, rs_0 c^2}{t} \frac{r}{r^2} + O\left(\frac{r^2 s_0 c^2}{r^3}\right).
\]
(9)
The point with this is to show that by treating the comoving coordinate system as a static one and using GR, an “anomalous” term \(\pm cH\) will be missed when modelling coordinate accelerations of photons. We see that the sign of the anomalous term is such that the anomalous acceleration is oriented in the opposite direction to that of the motion of the photons. This means that to sufficient accuracy, treating the comoving coordinate system as a static one is equivalent to introducing a variable “effective” velocity of light \(c_{\text{eff}}\) equal
to

\[ c_{\text{eff}} = c(1 - \int_t^{t+T} \frac{dt'}{t'}) = c(1 - HT + O((HT)^2)). \]  

(10)

The change of \( c_{\text{eff}} \) with \( T \) then yields an anomalous acceleration

\[ a_a = \frac{dc_{\text{eff}}}{dT} = -cH + O(cH^2T), \]  

(11)

along the line of sight of any observed object. That is, if the comoving coordinate system is treated as a static one the coordinate motion of any object will be observed to slow down by an extra amount if the light time, or equivalently, the distance to the observer increases and to speed up by an extra amount if the distance decreases. Hence, judging from its coordinate motion it would seem as if the object were influenced by an anomalous force directed towards the observer.

By integrating the anomalous acceleration over the total observation time (i.e. the duration of the experiment) \( T \ll t \) we get an “anomalous” speed

\[ w_a = \int_t^{t+T} a_a dt' = -cHT + O(cH^2T^2), \]  

(12)

towards the observer compared to a model where the coordinates are static rather than comoving. That is, the coordinate motion of any object observed over time indicates an anomalous blueshift compared to a model where the gravitational field is static. *Such an anomalous blueshift may be interpreted as an artefact resulting from a mismodelling of the gravitational field and the mismodelling of null paths in particular.*

Now, since any observer is typically located at the Earth, the quasi-metric model in fact predicts that an anomalous acceleration directed towards the Earth, rather than towards the Sun, should be seen if one incorrectly uses the static GR model to represent the nonstatic gravitational field. But any directional differences will almost average out over time if the observed object moves approximately radially and is located well beyond the Earth’s orbit. However, compared to a static model where the anomalous acceleration is inserted by hand and acts towards the Sun; even if directional differences nearly average out there remains a cumulative difference. We will calculate this below. If, on the other hand, the line of sight to the observed object (e.g. a planet) deviates significantly from the radial direction, observations should not be consistent with an anomalous acceleration directed towards the Sun. Rather the direction of the anomalous acceleration expressed in Sun-centered coordinates would appear to be a complicated function of time.

To estimate the predicted differences between a model where the anomalous acceleration is towards the Sun and the result given by equation (11); for the case where the
observed object moves approximately radially in the ecliptic plane and is located well beyond the Earth’s orbit it is convenient to define the average anomalous acceleration \( \langle a_c \rangle_{\text{in}} \) of a photon away from the Sun during the time of flight \( T \) from the object to the observer. Thus we define

\[
\langle a_c \rangle_{\text{in}} \equiv \frac{1}{T} \int_t^{t+T} a_c dt' = \frac{c}{T} \left[ \int_t^{t+T_1} dt' - \int_{t+T_1}^{t+T} t' \right] = cH \left( 2 \frac{T_1}{T} - 1 + O(HT) \right), \tag{13}
\]

where \( T_1 \) is the moment when the photon crosses a plane through the center of the Sun normal to a line connecting the object and the Sun. (If the Earth is at the same side of this plane as the object, \( T_1 = T \).) Similarly we can define the average \( \langle a_c \rangle_{\text{out}} = -\langle a_c \rangle_{\text{in}} \) of a photon towards the Sun during the time of flight from the observer to the object.

We are now able to estimate the predicted difference \( \delta a \) between a model where the anomalous acceleration is towards the Earth and one where it is towards the Sun. We get

\[
\delta a = -cH - \langle a_c \rangle_{\text{out}} = -2cH \left( 1 - \frac{T_1}{T} + O(HT) \right). \tag{14}
\]

This function has a minimum at solar conjunction and vanishes when the observed object and the Earth are at the same side of the Sun. One may show that \( \delta a \) can be written approximately as a truncated sine function where all positive values are replaced by zero. The period is equal to one year and the amplitude is approximately \( 2cH \frac{R_o}{R_e} \), where \( R_o \) and \( R_e \) are the radial coordinates of the object and of the Earth, respectively.

Anderson et al. [2] (see also [5]) found an annual perturbation on top of the anomalous acceleration \( a_P \) of Pioneer 10. (They claim to see such a perturbation for Pioneer 11 also.) Interestingly, the perturbation consistently showed minima near solar conjunction [2] (i.e. the absolute value of the anomalous acceleration reached maximum near solar conjunction). They fitted an annual sine wave to the velocity residuals coming from the annular perturbation term, using data from Pioneer 10 when the spacecraft was about 60 AU from the Sun. Taking the derivative with respect to time they then found the amplitude of the corresponding acceleration; it was found to be \( a_{a.t.} = (0.215 \pm 0.022) \times 10^{-8} \) cm/s². We may compare this to the corresponding amplitude in \( \delta a \) estimated above. We find an amplitude of about \( 0.29 \times 10^{-8} \) cm/s² (using the value \( 8.74 \times 10^{-8} \) cm/s² for \( cH \)), close enough to be roughly consistent with the data. Besides the annual term a diurnal term was also found in the velocity residuals [2], where the corresponding acceleration amplitude \( a_{d.t.} \) is large compared to \( a_P \). But over one year the contribution from the diurnal term averages out to insignificance (less than \( 0.03 \times 10^{-8} \) cm/s² [2]).

To find the trajectories of non-relativistic particles we may set \( E \equiv 1 - \frac{w^2}{c^2} \), where \( \frac{w^2}{c^2} \)
is small. Then equation (3) yields

\[
\frac{dr}{dt} = \pm \frac{ct_0}{t} \sqrt{\frac{B(r)}{A(r)} \left(1 + \left(\frac{w^2}{c^2} - 1\right)B(r)\right)} = \pm \frac{ct_0}{t} \sqrt{\frac{r s_0^2}{r} + \frac{w^2}{c^2} + O\left(\frac{r^2}{r^2}\right)},
\]  

and the properly scaled coordinate acceleration for non-relativistic particles is

\[
a_c = \mp \frac{c}{t} \sqrt{\frac{r s_0^2}{r} + \frac{w^2}{c^2} + O\left(\frac{r^2}{r^2}\right)} - \frac{t_0}{t} \frac{r s_0^2 c^2}{2r^2} + O\left(\frac{r^2}{r^2}\right).
\]

We see that for non-relativistic particles the effect on coordinate accelerations of treating the comoving coordinates as static ones and using GR, is a factor \(\sqrt{\frac{r s_0}{r} + \frac{w^2}{c^2}}\) smaller than the corresponding effect for photons. This means that the trajectories of non-relativistic particles do not depend crucially on the fact that the gravitational field is non-static.

On the other hand the paths of photons depend more significantly on whether the gravitational field is static or not and this yields the illusion of an anomalous acceleration. That is, if one receives electromagnetic signals from some freely falling object located e.g. in the outer parts of the solar system, the coordinate acceleration of the object as inferred from the signals should not agree with the “real” coordinate acceleration of the object if one treats the comoving coordinates as static ones. Rather, from equation (11) we see that it would seem as if the object were influenced by an attractive anomalous acceleration of size \(cH\). The relevance of this is apparent when modeling the orbits of spacecraft and comparing to data obtained from radio signals received from the spacecraft; in particular this applies to the analyses performed in [1], [2] and [5]. An extra bonus for the model considered in this paper is that it predicts small deviations during the year if the data are compared to a model where the anomalous acceleration is directed towards the Sun rather than towards the Earth. And as we have seen, this prediction seems to be consistent with the data.

4 Cosmic expansion and the PPN-formalism

Orbit analysis of objects moving in the solar system must be based on some assumptions of the nature of space-time postulated to hold there. The standard framework used for this purpose is the parameterized post-Newtonian (PPN) formalism applicable for most metric theories of gravity. But the standard PPN-framework does not contain any terms representing expanding space via a global scale factor as shown in equation (1) since it is inherently assumed that the solar system is decoupled from the cosmic expansion. One may try to overcome this by inventing some other sense of “expanding space” where the
scale factor varies in space rather than in time. But such a model must necessarily be
different from our quasi-metric model, and we show below that it cannot work. Thus,
to illustrate the inadequacy of the PPN-framework to model expanding space we now
consider a specific model where suitable terms are added by hand in the metric. We may
then compare to the change in light time obtained from our quasi-metric model.

One may try a post-Newtonian metric of the type
\[
ds^2 = - \left(1 - \frac{r_s}{r} + \frac{2H_0 r}{c} + O\left(\frac{r_s^3}{r^3}\right)\right)(dx^0)^2 + \left(1 + \frac{r_s}{r} - \frac{2H_0 r}{c} + O\left(\frac{r_s^2}{r^2}\right)\right)dr^2 + r^2d\Omega^2,
\]
where \(H_0\) is a constant, to describe expanding space within the PPN-framework. (To show
that this metric yields a spatially variable scale factor, transform to isotropic coordinates.)
It may be readily shown that the metric (17) yields a constant anomalous acceleration
\(cH_0\) towards the origin. But the problem with all metrics of this type is that they represent a
“real” anomalous acceleration of gravitational origin, and this is observationally excluded
from observations of planetary orbits [2].

Anyway we may calculate the change in light time \(\Delta T\) due to the terms containing
\(H_0\) in equation (17) by integrating a radial null path from \(r_o\) to \(r_e\) (let \(r_o > r_e\), say). This
yields
\[
\Delta T = -c^{-2}H_0(r_o^2 - r_e^2) + \cdots \approx -H_0T^2,
\]
where the light time \(T\) is equal to \(c^{-1}(r_o - r_e)\) to first order and where the last approxima-
tion is accurate only if \(r_o\) is large. Note that \(\Delta T\) is negative; this is quite counterintuitive
for a model representing expanding space.

Anderson et al. [2] have considered a phenomenological model representing “expand-
ing space” by adding a quadratic in time term to the light time in order to determine
the coefficient of the quadratic by comparing to data. To sufficient accuracy this model
may be represented by the transformation
\[
T \rightarrow (1 + a_{\text{quad}}t)T \equiv T + \Delta T,
\]
where \(a_{\text{quad}}\) is a “time acceleration” term. This model fits both Doppler and range very
well [2].

If we compare equations (18) and (19) we see that the value \(a_{\text{quad}} = -H_0\frac{T}{T}\) corresponds
to the change in light time calculated from the metric (17). This is far too small (of order
\(10^{-30}\) s\(^{-1}\) for a light time of a few hours) to be found directly from the tracking data.
Thus, the fact that \(a_{\text{quad}}\) was estimated to be zero based on the tracking data alone [2]
in no way favours a constant acceleration model over a time acceleration model. They
are in fact equivalent as far as the data are concerned.
To compare differences in $a_{quad}$ we may use equation (8) to find the extra delay
\[ \frac{H}{2c^2}(r_o - r_e)^2 = \frac{HT^2}{2} \]
in the light time compared to the static case. This corresponds to a value $a_{quad} = \frac{HT^2}{2}$ for the time acceleration term. But a determination of $a_{quad}$ directly from the tracking data still reflects the model-dependency explicitly present in the orbit determination process. This means that a determination of $a_{quad}$ directly from the tracking data should in principle be consistent with the model (17) and not with our quasi-metric model. However, the quantity $a_{quad}$ is so small that it is not feasible to check this. But the fact that a phenomenological model of the type (19) works so well should be taken to mean that the explanation of the anomalous acceleration given in this paper is sufficient.

5 Conclusion

We conclude that a natural explanation of the data is that the gravitational field of the solar system is not static with respect to the cosmic expansion. This also explains why any orbit analysis program based on the PPN-formalism is insufficient for the task and how the largest errors arise from the mismodelling of null paths. (In fact, using the PPN-formalism is equivalent to introducing a variable “effective” velocity of light as shown in equation (10).) But these explanations, while not involving any ad hoc assumptions, are based on the premise that space-time is quasi-metric. That is, rather than being described by one single Lorentzian metric, the gravitational field of the solar system should be modeled (to a first approximation) by the metric family shown in equation (1). From a theoretical point of view this premise is radical; thus it is essential that the subject is further investigated to make certain that more mundane explanations may be eliminated. However, so far no such explanations based on well-known physics have been found. But the facts are that the model presented in this paper follows from first principles and fits the data very well. Moreover there exists independent observational evidence in favour of the prediction that the Earth-Moon system is not static with respect to the cosmic expansion, and quasi-metric gravity predicts these observations from first principles as well [10]. Thus the fact is that several observations in the solar system seem to involve the Hubble parameter. This should not be dismissed as a coincidence, and indicates that explanations based on quasi-metric relativity should be taken seriously.

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