ALMOST $G_2$-MANIFOLDS WITH ALMOST TWISTORIAL STRUCTURES

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Abstract. We give the necessary and sufficient conditions for the Penrose-Ward transformation to work on almost $G_2$-manifolds, endowed with natural almost twistorial structures.

Introduction

This paper grew out, in part, of the fact that the method we used to characterise, in [5], the integrability of almost twistorial structures is flawed (see Remark 2.7, below). On the other hand, we continue (see, also, [8]) our study of the differential geometry related to $G_2$, the simply-connected simple complex Lie group of dimension 14 (and to its compact real form).

The main source of Euclidean twistorial structures [3] is provided by the closed orbits, on the Grassmannians of (co)isotropic spaces, of the complexification of a compact Lie group, endowed with a faithful orthogonal representation. For $G_2$, there are two ‘fundamental’ such (generalized) Grassmannians: the hyperquadric $Q$ in the projectivisation of the space $U$ of complex imaginary octonions, and the space $Y$ of anti-self-dual spaces in $U$. See Section 1, from which it, also, follows that $G_2$ has two orbits on each of $\text{Gr}_k^0(U)$, $k = 2, 3$, with the closed orbits given by $Y$ and $Q$, respectively, the latter, thus, leading to the canonical Euclidean twistorial structures on $U$ (cf. [8]). As the dual of this is not maximal (in the sense of [8]) we are led to, also, consider an Euclidean twistorial structure on $\mathbb{C} \times U$. See Section 2, where the integrability of the obtained canonical almost twistorial structures is studied, by using [1].

1. Isotropic spaces closed under the octonionic cross product

We work in the complex analytic category. Let $\mathfrak{g}_2$ be the (complex) simple Lie algebra of dimension 14, and let $G_2$ be the simply-connected simple Lie group whose Lie algebra is $\mathfrak{g}_2$.

Let $U$ be the space of imaginary (complex) octonions. Let $Q \subseteq PU$ be the quadric of isotropic directions, and let $Q'$ be the space of projective planes contained by $Q$. Recall (see [4]) that $Q$ can be embedded into $Q'$ as an orbit of $G_2 (\subset \text{Spin}(7))$.  

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Proposition 1.1. A three-dimensional isotropic subspace \( p \subseteq U \) is closed under the octonionic cross product if and only if \( p \in Q(\subseteq Q') \).

Proof. We claim that the action of \( G_2 \) on \( Q' \) has two orbits, as follows:
(1) \( Q \);
(2) \((\pm i)\)-eigenspaces of the octonionic cross multiplication with a unit vector (this orbit can be identified with \( G_2/\text{SL}(3) \) - the complexified 6-sphere).

Indeed, this follows from the fact that \( Q' \setminus Q \) is the complexified 6-sphere.

Now, let \( p \in Q' \) and denote by \( \alpha \) the linear map from \( \Lambda^2 p \) to \( U \) given by the octonionic cross product. It follows that if \( p \in Q \) then \( \alpha \) has rank 1 and its image is contained in \( p \), whilst, if \( p \in Q' \setminus Q \) then the rank of \( \alpha \) is 3 and if we denote by \( q \) its image then \( p \) and \( q \) are the \((\pm i)\)-eigenspaces of a unit vector orthogonal to \( p + q \).

Remark 1.2. The embedding of \( Q \) into \( Q' \) is determined by the octonionic cross product as follows. Let \( \ell \in Q(\subseteq PU) \), and choose a nondegenerate associative space \( p \) that contains \( \ell \). Then the image of \( \ell \) into \( Q' \) is equal to \( \ell + (\ell \times p^\perp) \), where \( \times \) denotes the octonionic cross product.

We call the points of \( Q \subseteq Q' \) isotropic associative spaces of dimension 3.

Corollary 1.3. (i) Any self-dual space (contained by a nondegenerate coassociative space) is contained by a unique isotropic associative space of dimension 3.
(ii) Any anti-self-dual space is contained by a family of isotropic associative spaces of dimension 3, parametrized by the projective line.
(iii) Any isotropic associative space of dimension 3 contains a family of anti-self-dual spaces, parametrized by the projective line.

Proof. (i) If \( p \) is self-dual then \( p \times p \) has dimension 1 and determines (as in Remark 1.2) an isotropic associative space, of dimension 3, containing \( p \).

(ii) Any anti-self-dual space is of the form \( \ell + x \times \ell \) where \( \ell \) is an isotropic direction and \( x \in U \) is nondegenerate and orthogonal onto a nondegenerate associative space containing \( \ell \). Consequently, if \( p \) is anti-self-dual then it is contained by the isotropic associative spaces of dimension 3, determined by the directions contained by \( p \).

(iii) By (i) any isotropic associative space of dimension 3 is of the form \( \ell + p \) where \( \ell \) is an isotropic direction and \( p \) is self-dual such that \( \ell = p \times p \). Then any anti-self-dual space contained by \( \ell + p \) is generated by \( \ell \) and a direction in \( p \).

Theorem 1.4. Let \( p \subseteq U \) be an isotropic subspace of dimension 2. Then the following assertions hold:
(i) \( p \) is self-dual if and only if \( p \times p \neq \{0\} \).
(ii) \( p \) is anti-self-dual if and only if \( p \times p = \{0\} \).

Proof. (i) If \( p \times p \subseteq p \) then we can choose a basis \((a, b)\) of \( p \) such that \( a \times b = a \). Furthermore, any isotropic subspace \( q \), of dimension 3, containing \( p \) is associative (as \( q \cap (q \times q) \neq \{0\} \)). Thus, we can choose a basis \((x, y, z)\) of \( q \) such that \( x \times y = z \),
y \times z = 0 and z \times x = 0 (consequence of Remark 1.2).

Now, let \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\) be the components of \(a\) and \(b\), respectively, with respect to \((x, y, z)\). Then \(a \times b = (\alpha \beta' - \alpha' \beta)z\) which, together with \(a = a \times b\), implies \(\alpha = \beta = 0\), and, hence, \(a \times b = 0\), a contradiction. Therefore \(p + (p \times p)\) is isotropic associative of dimension 3.

(ii) If \(q \subseteq U\) is isotropic, of dimension 3, containing \(p\) then \(q \times q\) has dimension at most 2. Thus, \(q\) is associative, determined by the isotropic direction \(q \times q\). Hence, \(q \times q \subseteq p\) (as, otherwise, we would have \(p \times p \neq \{0\}\)). The proof follows. □

Remark 1.5. 1) Let \(p, q \subseteq U\) be isotropic associative of dimension 3. Then either \(p = q\), or \(p \cap q = \{0\}\), or \(p \cap q\) is an anti-self-dual space.

2) Let \(V \subseteq U\) be a nondegenerate vector subspace of codimension 1, and let \(p \subseteq U\) be the isotropic associative space of dimension 3 determined by the isotropic direction \(\ell \subseteq U\). Then one and only one of the following occurs:

(i) \(p \subseteq V\);
(ii) \(p \cap V\) is anti-self-dual;
(iii) \(p \cap V\) is self-dual.

To understand when each case occurs, endow \(U\) with a conjugation preserving \(V\) and compatible with the octonionic cross product. Then there exists a unique real associative space \(q\) that contains \(\ell\) and the following hold:

(a) Case (i) occurs if and only if \(q \supseteq V^\perp\) and \(\ell \subseteq V\);
(b) Case (ii) occurs if and only if \(q \subseteq V\) (and \(\ell \subseteq V\));
(c) Case (iii) occurs if and only if \(q \not\subseteq V\) and \(\ell \not\subseteq V\).

Consequently, we, also, have the following:

- \(p \subseteq V\) if and only if \(\ell \subseteq V\) and contained by an eigenspace of the orthogonal complex structure on \(V\) (given by the octonionic cross product);
- \(p \cap V\) is anti-self-dual if and only if \(\ell \subseteq V\) but not contained by an eigenspace of the orthogonal complex structure on \(V\);
- \(p \cap V\) is self-dual if and only if \(\ell \not\subseteq V\).

2. Obstructions to the integrability of almost twistorial structures

We continue with the same notations as in Section 1. Let \(E\) be the tautological vector bundle over \(Q\) given by its embedding into \(Q'\). Then the tautological line bundle \(L\) over \(Q(\subseteq PU)\) is a subbundle of \(E\) and, on denoting \(U_+ = E/L\), we have an equivariant exact sequence of homogeneous vector bundles

\[
0 \longrightarrow L \longrightarrow E \longrightarrow U_+ \longrightarrow 0.
\]

By using \(\mathbb{H}\), we duce that \(H^j(U_+^\ast) = 0\), for any \(j \in \mathbb{N}\). Together with the Kodaira vanishing theorem and by passing to the exact sequence of cohomology groups of the dual of \((2.1)\), this implies \(H^0(E^\ast) = U_{1,0}\) (equivariantly), and \(H^j(E^\ast) = 0\), for any \(j \in \mathbb{N} \setminus \{0\}\), where \(U_{m,n}\) is the irreducible representation space of \(G_2\) corresponding to
$(m, n) \in \mathbb{N}^2$ (note that, $U_{1,0} = U$ the space of imaginary octonions).

We, also, have the following equivariant exact sequence

\[ (2.2) \quad 0 \rightarrow E_\perp \rightarrow Q \times U_{1,0} \rightarrow E^* \rightarrow 0. \]

Hence, $H^j(E_\perp) = 0$, for any $j \in \mathbb{N}$. Furthermore, by restricting $E_\perp/E$ to any associative conic we obtain that $E_\perp/E$ is a trivial line bundle. Thus, we have an equivariant exact sequence $0 \rightarrow Q \times U_{0,0} \rightarrow (E_\perp)^* \rightarrow E^* \rightarrow 0$, from which we deduce $H^j((E_\perp)^*) = U_{0,0} \oplus U_{1,0}$, and $H^j((E_\perp)^*) = 0$, for any $j \in \mathbb{N} \setminus \{0\}$. Consequently, dualizing (2.2), passing to the cohomology exact sequence, and then dualizing again, we obtain (cf. [4])

\[ (2.3) \quad 0 \rightarrow E_\perp \rightarrow Q \times (U_{0,0} \oplus U_{1,0}) \rightarrow (E_\perp)^* \rightarrow 0, \]

together with the obvious morphism from (2.3) to (2.2). These two equivariant exact sequences give the Euclidean twistorial structures we are interested in.

From Section [1] it follows that $G_2$ has two orbits on $\text{Gr}_{2}^0(U)$: the space $Y$ of anti-self-dual spaces and the space of self-dual spaces. The former is just the twistor space of the Wolf space determined by $G_2$ (the closed adjoint orbit into $P\mathfrak{g}_2$); in particular, $Y$ is compact.

Let $U_-$ be the tautological vector bundle over $Y$. Denote $Z = PU_+ = PU_-$ and let $\pi_\pm$ be the projections from $Z$ onto $Q$ and $Y$, respectively. Let $L_+ = \pi_+^*L$ and, similarly, let $L_-$ be the pull-back by $\pi_-$ of the tautological line bundle over $Y$ (given by the Plücker embedding).

**Proposition 2.1.** (i) $L_+$ is the tautological line bundle over $PU_-$.

(ii) $L_-$ is the tautological line bundle over $P(L_+ \otimes U_+)$.

**Proof.** Assertion (i) is obvious. To prove (ii), note that, $\pi_+^*E/\pi_-^*(U_-)$ is the dual of the tautological line bundle over $P(U_+^*)$. As $U_+^* = (\Lambda^2U_+^*) \otimes U_+ = L_+^* \otimes U_+$ and $\pi_+^*E/\pi_-^*(U_-) = L_+^* \otimes L_-$, the proof follows quickly. $\square$

**Corollary 2.2.** For any $m, n \in \mathbb{N}$, we have $U_{m,n} = H^0((L_+^*)^m \otimes (L_-^*)^n)$.

**Proof.** By using a description of $\mathfrak{g}_2$ from [7] (see [5]) we deduce that $Z$ is the generalized complete flag manifold of $G_2$. The proof follows from the Borel-Weil theorem. $\square$

**Proposition 2.3.** (i) $H^0(\Lambda^2E^*) = U_{1,0} \oplus U_{0,1}$.

(ii) $H^0(\Lambda^2(E_\perp)^*) = U_{1,0} \oplus U_{0,1}$.

**Proof.** Obviously, $\pi_+^*(U_-)$ is a subbundle of $\pi_+^*E$. Also, the projection from $\Lambda^2U_{1,0}$ onto $U_{0,1}$ decomposes as the composition of an equivariant linear map from the former to $H^0(\Lambda^2E^*)$ followed by the (equivariant) linear map between the spaces of sections of $\Lambda^2E^*$ and $\Lambda^2U_{1,0}$. Furthermore, the kernel of the linear map between the spaces of sections of $\Lambda^2E^*$ and $\Lambda^2U_{1,0}$ is formed of the sections of $\Lambda^2U_{1,0}^* = L^*$, and the proof of (i) quickly follows.
Assertion (ii) follows from (i) and the fact that $E^*$ is the kernel of the vector bundles morphisms from $\Lambda^2(E^\perp)^*$ onto $\Lambda^2E^*$.

\[ \square \]

Remark 2.4. We can improve Proposition 2.3 as follows. Firstly, as rank $E = 3$, we have $\Lambda^2E^* = (\Lambda^3E^*) \otimes E = (L_+^*)^2 \otimes E$. Together with (2.1), this gives

\[ 0 \longrightarrow L_+^* \longrightarrow (L_+^*)^2 \otimes E \longrightarrow (L_+^*)^2 \otimes U_+ \longrightarrow 0, \]

whose cohomology exact sequence gives (i) of Proposition 2.3, by using [1]; moreover, we have the following exact sequence

\[ (2.4) \quad 0 \longrightarrow E^* \longrightarrow \Lambda^2(E^\perp)^* \longrightarrow (L_+^*)^2 \otimes E \longrightarrow 0 \]

we, similarly, obtain (ii) of Proposition 2.3 and $H^j(\Lambda^2(E^\perp)^*) = 0$, for any $j \geq 1$.

Consequently, from

\[ (2.5) \quad H^0(E^* \otimes \Lambda^2(E^\perp)^*) = U_{1,0} \oplus kU_{0,1} \oplus 3U_{2,0} \oplus U_{1,1}, \]

and

\[ H^j((E^*)^* \otimes \Lambda^2(E^\perp)^*) = 3U_{1,0} \oplus (k + 1)U_{0,1} \oplus 3U_{2,0} \oplus U_{1,1}, \]

where $k \in \{2, 3\}$.

\[ \text{Proof.} \quad \text{We have } \otimes^2E^* = (\otimes^2E^*) \oplus (\Lambda^2E^*), \text{ where } \otimes \text{ denotes the symmetric product.} \]

From (2.1), we obtain that $L_+ \otimes E$ is the kernel of $\otimes^2E \to \otimes^2U_+$, and, consequently, we have the following exact sequence

\[ (2.6) \quad 0 \longrightarrow \otimes^2U_+^* \longrightarrow \otimes^2E^* \longrightarrow L_+^* \otimes E^* \longrightarrow 0. \]

By using [1], we obtain that $H^1(\otimes^2U_+^*) = U_{0,1}$ and $H^j(\otimes^2U_+^*) = 0$, for any $j \neq 1$. Similarly, from $0 \longrightarrow L_+^* \otimes U_+^* \longrightarrow L_+^* \otimes E^* \longrightarrow (L_+^*)^2 \otimes E \longrightarrow 0$ (consequence of (2.1)), we deduce that $H^0(L_+^* \otimes E^*) = U_{0,1} \oplus U_{2,0}$ and $H^j(L_+^* \otimes E^*) = 0$, for any $j \geq 1$.

Therefore the cohomology exact sequence of (2.6) gives

\[ 0 \longrightarrow H^0(\otimes^2E^*) \longrightarrow U_{0,1} \oplus U_{2,0} \longrightarrow U_{0,1} \longrightarrow H^1(\otimes^2E^*) \longrightarrow 0, \]

and $H^j(\otimes^2E^*) = 0$, for any $j \geq 2$. Thus, either $H^1(\otimes^2E^*) = 0$ and $H^0(\otimes^2E^*) = U_{2,0}$, or $H^1(\otimes^2E^*) = U_{0,1}$ and $H^0(\otimes^2E^*) = U_{0,1} \oplus U_{2,0}$.

Together with Remark 2.4, this gives $H^j(\otimes^2E^*) = 0$, for any $j \geq 2$, and only one of the following

1. $H^1(\otimes^2E^*) = 0$ and $H^0(\otimes^2E^*) = U_{1,0} \oplus U_{0,1} \oplus U_{2,0}$,
2. $H^1(\otimes^2E^*) = U_{0,1}$ and $H^0(\otimes^2E^*) = U_{1,0} \oplus 2U_{0,1} \oplus U_{2,0}$.

Further, by tensorising the dual of (2.1) with $(L_+^*)^2 \otimes U_+$ we obtain

\[ (2.7) \quad 0 \longrightarrow (L_+^*)^2 \otimes U_+ \otimes U_+^* \longrightarrow (L_+^*)^2 \otimes U_+ \otimes E^* \longrightarrow (L_+^*)^3 \otimes U_+ \longrightarrow 0. \]

As $(L_+^*)^2 \otimes U_+ \otimes U_+^* = L_+^* \otimes U_+ \otimes (\otimes^2U_+^*) + (L_+^*)^2$, by using [1], we deduce $H^0((L_+^*)^2 \otimes U_+ \otimes U_+^*) = U_{2,0}$, $H^1((L_+^*)^2 \otimes U_+ \otimes U_+^*) = U_{0,0}$, and $H^j((L_+^*)^2 \otimes U_+ \otimes U_+^*) = 0$, for any $j \geq 2$. 
Similarly, $H^0((L^*_+)^3 \otimes U_+) = U_{1,1}$ and $H^j((L^*_+)^3 \otimes U_+) = 0$, for any $j \geq 1$.

Therefore from (2.7) we obtain

$$0 \to U_{2,0} \to H^0((L^*_+)^2 \otimes U_+ \otimes E^*) \to U_{1,1} \to U_{0,0} \to H^1((L^*_+)^2 \otimes U_+ \otimes E^*) \to 0,$$

and $H^j((L^*_+)^2 \otimes U_+ \otimes E^*) = 0$, for any $j \geq 2$; consequently,

$$H^0((L^*_+)^2 \otimes U_+ \otimes E^*) = U_{0,0},$$
\hspace{1cm}
$$H^j((L^*_+)^2 \otimes U_+ \otimes E^*) = U_{2,0} \oplus U_{1,1}.
$$

(2.8)

Also, by using (2.1) and [1], we obtain

$$H^0(L^*_+ \otimes E^*) = U_{0,1} \oplus U_{2,0},$$
\hspace{1cm}
$$H^j(L^*_+ \otimes E^*) = 0,$$ for any $j \geq 1.

(2.9)

By tensorising (2.1) with $(L^*_+)^2 \otimes E^*$ we obtain the exact sequence

$$0 \to L^*_+ \otimes E^* \to (L^*_+)^2 \otimes E \otimes E^* \to (L^*_+)^2 \otimes U_+ \otimes E^* \to 0,$$

which, together with (2.8) and (2.9), gives the following two exact sequences

$$0 \to U_{0,1} \oplus U_{2,0} \to H^0((L^*_+)^2 \otimes E \otimes E^*) \to U_{2,0} \oplus U_{1,1} \to 0,$$
\hspace{1cm}
$$0 \to H^1((L^*_+)^2 \otimes E \otimes E^*) \to U_{0,0} \to 0.
$$

(2.10)

We have, thus, proved the following

$$H^0((L^*_+)^2 \otimes E \otimes E^*) = U_{0,1} \oplus 2U_{2,0} \oplus U_{1,1},$$
\hspace{1cm}
$$H^1((L^*_+)^2 \otimes E \otimes E^*) = U_{0,0},$$
\hspace{1cm}
$$H^j((L^*_+)^2 \otimes E \otimes E^*) = 0,$$ for any $j \geq 2.

(2.11)

Now, by tensorising (2.4) with $E^*$ we obtain

$$0 \to \otimes^2 E^* \to E^* \otimes \Lambda^2(E^\perp)^* \to (L^*_+)^2 \otimes E \otimes E^* \to 0.$$

(2.12)

From the cohomology exact sequence of (2.11), together with (2.10) and the two possibilities (1) and (2), above, it follows the first relation of (2.5).

To prove the second relation of (2.5), we use the exact sequence $0 \to Z \times U_{0,0} \to (E^\perp)^* \to E^* \to 0$, which implies

$$0 \to \Lambda^2(E^\perp)^* \to (E^\perp)^* \otimes \Lambda^2(E^\perp)^* \to E^* \otimes \Lambda^2(E^\perp)^* \to 0.
$$

Finally, the cohomology exact sequence of (2.12), together with Remark (2.4), and the first relation of (2.5), quickly completes the proof.

Let $M$ be a manifold, with dim $M = 7, 8$, endowed with an almost $G_2$-structure, through the representations $U_{1,0}$ and $U_{0,0} \oplus U_{1,0}$, respectively. Any compatible connection $\nabla$ on $M$ induces an almost twistorial structure on $M$, by using the Euclidean twistorial structure given by (2.2) and (2.3), respectively, which we call the canonical almost twistorial structure of $(M, \nabla)$.
Corollary 2.6. Let \( M \) be a manifold, with \( \dim M = 7, 8 \), endowed with an almost \( G_2 \)-structure, through the representations \( U_{1,0} \) and \( U_{0,0} \oplus U_{1,0} \), respectively. Let \( \nabla \) be a compatible connection on \( M \), and denote by \( R \) and \( T \) its curvature form and torsion tensor field, respectively.

Then the following assertions are equivalent:

(i) The canonical almost twistorial structure of \( (M, \nabla) \) is integrable and the Penrose-Ward transformation can be applied to it (locally).

(ii) \( R = 0 \) and \( T \in kU_{0,0} \oplus kU_{1,0} \), at each point, where \( k = 1, 2 \) if \( \dim M = 7, 8 \), respectively.

Proof. We prove the \( \dim M = 7 \) case, the proof of the other one is similar.

By using [2, 2.7.3] we obtain

\[
(\Lambda^2 U_{1,0}) \otimes U_{1,0} = U_{0,0} \oplus 2U_{1,0} \oplus U_{0,1} \oplus 2U_{2,0} \oplus U_{1,1}.
\]

Assertion (i) is equivalent to the fact that, at each point, the sections given by restricting \( R \) and \( T \) to \( \Lambda^2 E^\perp \) and \( \Lambda^2 (E^\perp) \otimes E \), respectively, are zero (here \( R \) is seen as a bundle valued 2-form, and \( T \) as the tensor field of degree \( (0,3) \) given by the torsion of \( \nabla \) and the underlying Riemannian metric on \( M \)).

The proof follows from Proposition 2.3, Theorem 2.5, and (2.13). □

Remark 2.7. Corollary 2.6 corrects statements of [5, §2], and the results therein can be straightforwardly formulated in the current setting.

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