Superposed solutions for a generalized shallow water wave equation

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Abstract

In the context of the emergence of new classes of superposed solutions for a variety of nonlinear equations we present some additional ones for a generalized shallow water wave equation.

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Physical systems described by nonlinear evolution equations are commonplace in nature. To understand the properties of such systems and to derive exact solutions for their guiding equations extensive research has been carried out (see, for discussions, [1, 2, 3]) and in the process several important techniques developed such as the inverse scattering method [4], Lax pair formulation [5], Bäcklund transformations [6] and the bilinear approach of Hirota and Satsuma [7, 8] to name a few.

Normally, nonlinear equations do not yield solutions that would superpose making tractability of the systems they represent rather difficult. However, sometime ago, a useful procedure was laid out [9] for finding linear superposition of solutions for a class of such equations using cyclic identities of Jacobi elliptic functions. Subsequently, this means of generating exact solutions has triggered off much theoretical activity [10, 11]. Very recently, Khare and Saxena, in a series of works [12, 13], have discussed a large body of nonlinear equations that admit not only exact periodic solutions given in terms of Jacobi elliptic functions $cn(x, m)$ and $dn(x, m)$, where $m$ is the modulus of the elliptic function, but showed that the combinations $dn(x, m) \pm cn(x, m)$ also emerge as viable solutions.

However, an intriguing aspect pointed out by them is that for certain nonlinear equations when their solutions involve the type $dn^2(x, m)$ there also exist additional superposed solutions like $dn^2(x, m) \pm \sqrt{m}dn(x, m)cn(x, m)$ even though $dn(x, m)cn(x, m)$ is not a specific solution.

In this note we expand the latter list of superposed solutions by demonstrating that such a feature (that cannot be technically called a superposition) also holds for the solutions of a fourth-order generalized shallow water wave (GSWW) equation not with regard to the $dn^2$ but for the elliptic integral function of second kind $E(\phi, m)$. Indeed we demonstrate that while the GSWW equation admits $E(\phi, m)$ as a solution, the combinations

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E(φ, m) ± \sqrt{m}\text{sn}(x, m) may also be its solutions. Note that \text{sn}(x, m) is not a solution of GSWW equation.

The GSWW equation reads \cite{14}

\[ u_{xxxt} + \alpha u_x u_t + \beta u_t u_{xx} - u_{xx} - u_{xt} = 0, \]

where \( \alpha, \beta \in \mathbb{R} - \{0\} \). Equation (1) was derived from a classical study of water waves under the Boussinesq approximation. Investigation of Painlevé tests reveals that its complete integrability holds if and only if \( \alpha = \beta \) or \( \alpha = 2\beta \). Exact solutions of (1) have been obtained \cite{14, 15, 16} including the ones \cite{15} in terms of quasi-periodic elliptic integral function and doubly-periodic Jacobian elliptic function.

The reduction of (1) to the following form is straightforward by carrying out a two-step integration: We seek travelling wave solution of equation (1) in the form

\[ v = \frac{du}{d\xi} : \frac{d^2v}{d\xi^2} + av^2 + bv + c = 0, \quad \xi = \gamma(x - Vt + \delta), \]

where \( a = \frac{\alpha + \beta}{\gamma} \), \( b = \frac{1 - \frac{V}{\gamma}}{\gamma} \) and \( c \) is an integrating constant.

Equation (2) admits of an exact solution

\[ v = \frac{du}{d\xi} = A\text{dn}^2(\gamma(x - Vt + \delta), m), \]

provided

\[ A = \frac{6}{a}, \quad V = \frac{1}{1 - 4\gamma^2(2 - m)}. \]

Then from (3) an exact solution for (1) is given by

\[ u(\gamma(x - Vt + \delta), m) = AE(\phi, m), \]

where \( \sin \phi = \text{sn}(\gamma(x - Vt + \delta), m) \) and \( E(\phi, m) \) stands for the integral \cite{17}

\[ E(\phi, m) = \int_0^\phi \sqrt{1 - m \sin^2 w} \, dw. \]

For the limit case \( m \to 1^- \), the solution (5) becomes

\[ u = A\tanh[\gamma(x - \frac{1}{1 - 4\gamma^2}t + \delta)] \]

while for the limit \( m \to 0^+ \), (5) goes over to

\[ u = A\gamma(x - \frac{1}{1 - 8\gamma^2}t + \delta). \]

Interestingly it turns out that there also exists for (2) a superposed solution of the type

\[ v = \frac{du}{d\xi} = \frac{A}{2} [\text{dn}^2(\gamma(x - Vt + \delta), m) \pm \sqrt{m} \, \text{cn}(\gamma(x - Vt + \delta), m) \text{dn}(\gamma(x - Vt + \delta), m)] \]

subject to \( V = \frac{1}{1 - \gamma^2(5 - m)} \). Of course, the latter is different from its value in (11) except for \( m = 1 \).

The solutions (3) and (9) mimick the situation encountered in \cite{12, 13} where \( \text{dn}^2 \pm \text{cn} \text{dn} \) superposed solutions were obtained for various nonlinear systems like the KdV, coupled NLS-MKdV and coupled NLS-KdV models.
From (9) a further integration gives way to the following superposed solution of the GSWW equation

\[ u = \frac{A}{2} [E(\phi, m) \pm \sqrt{m} \text{sn}(\gamma(x - \frac{1}{1 - \gamma^2(5 - m)} t + \delta), m)], \quad (10) \]

For the limit case \( m \to 1^- \), solution (10) reduces to the tanh-form in (7). On the other hand, for the limit \( m \to 0^+ \), solution (10) becomes

\[ u = \frac{A\gamma}{2} [x - \frac{1}{1 - 5\gamma^2} t + \delta]. \quad (11) \]

From (5) and (10) we therefore conclude that while the GSWW enjoys \( E(\phi, m) \) as a solution, it also admits \( E(\phi, m) \pm \sqrt{m} \text{sn}(\gamma(x - \frac{1}{1 - \gamma^2(5 - m)} t + \delta), m) \) as new solutions although \( \text{sn}(\gamma(x - \frac{1}{1 - \gamma^2(5 - m)} t + \delta), m) \) in itself is not a solution.

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