Lepton and quark mixing patterns with generalized $CP$ transformations

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Abstract

In this work, we have modified a scenario, originally proposed by Grimus and Lavoura, in order to obtain maximal values for atmospheric mixing angle and $CP$ violating Dirac phase of the lepton sector. To achieve this, we have employed $CP$ and some discrete symmetries in a type II seesaw model. In order to make predictions about neutrino mass ordering and the smallness of the reactor angle, we have obtained some conditions on the elements of the neutrino mass matrix of our model. Finally, within the framework of our model, we have studied quark masses and mixing pattern.

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1 Introduction

From the global fits to neutrino oscillation data \cite{1}, it is known that the three mixing angles in lepton sector are close to the tribimaximal (TBM) mixing \cite{2}. In the TBM pattern, the three mixing angles take the following values: \( \sin^2 \theta_{12} = \frac{1}{3} \), \( \sin^2 \theta_{23} = \frac{1}{2} \), \( \sin^2 \theta_{13} = 0 \). On the other hand, the CP violating Dirac phase \( \delta_{CP} \) in lepton sector is yet to be measured precisely. However, from the global fits to neutrino oscillation data \cite{1}, the best fit value for \( \delta_{CP} \) is around \( \pi \left( \frac{3}{2} \pi \right) \) in the case of normal(inverted) ordering of neutrino masses. The TBM value for \( \theta_{23} \) and \( \delta_{CP} = \frac{3}{2} \pi \) are still allowed in the 3\( \sigma \) ranges for these observables in the current neutrino oscillation data \cite{1}. The above mentioned values for \( \theta_{23} \) and \( \delta_{CP} \) are considered to be maximal. To explain the maximal values for \( \theta_{23} \) and \( \delta_{CP} \), Harrison and Scott have proposed \( \mu - \tau \) symmetry in combination with CP symmetry, which together is called \( \mu - \tau \) reflection symmetry \cite{3}. For some works based on \( \mu - \tau \) symmetry and CP symmetry, see Refs. \cite{4, 5}. Ref. \cite{4} is a review article.

In the work by Grimus and Lavoura \cite{6}, it is shown that a mass matrix for light left-handed neutrinos of the following form \cite{7}

\[
M_{\nu} = \begin{pmatrix}
a & r & r^* \\
r & s & b \\
r^* & b & s^*
\end{pmatrix}
\]  
(1)

can yield maximal values for \( \theta_{23} \) and \( \delta_{CP} \). In the above equation, \( a, b \) are real and \( r, s \) are complex. Further, in Ref. \cite{6}, a model is constructed, which is based on \( \mu - \tau \) reflection symmetry and softly broken lepton numbers, in order to obtain a mass matrix of the form of Eq. (1) for light neutrinos. In this model, three Higgs doublets are introduced and light neutrinos acquire masses via type I seesaw mechanism \cite{8}. Lepton number is softly broken by the mass terms for right-handed neutrinos of this model. In this model, in the absence of fine tuning of the parameters, muon and tau leptons can have masses of the same order. To explain the hierarchy in the masses for these leptons, \( K \) symmetry is introduced, under which the muon is massless \cite{6}. Realistic masses for muon and tau leptons are explained in the above mentioned scenario with the soft breaking of the \( K \) symmetry \cite{9}. The work done together in Refs. \cite{6, 9}, which is based on \( \mu - \tau \) reflection symmetry, consistently explains the mixing pattern in lepton sector and also the masses for charged leptons.

Although the work done in Refs. \cite{6, 9} gives a consistent picture about masses and mixing pattern in the lepton sector, there are few limitations for this work which are
explained below. It is argued in Ref. [6] that the mass matrix of Eq. (1), which is obtained from \( \mu - \tau \) reflection symmetry, cannot give predictions about neutrino mass ordering and the mixing angle \( \theta_{12} \). Moreover, in the case that the \( \delta_{CP} \) is maximal, the mass matrix of Eq. (1) cannot predict anything about \( \theta_{13} \). From the current neutrino oscillation data it is known that neutrinos can have either normal or inverted mass ordering, and moreover, we have \( \sin^2 \theta_{12} \sim \frac{1}{3} \) and \( \sin^2 \theta_{13} \sim 10^{-2} \) [1]. Apart from the above mentioned limitations, in Ref. [6], mixing pattern in the quark sector is not addressed. It is known that mixing pattern in the quark sector [10] is quite different from that of lepton sector. One would like to know if the mixing patterns for both quark and lepton sectors can be understood in the same framework.

As stated above, in Ref. [6], a model, which is based on type I seesaw mechanism and \( \mu - \tau \) reflection symmetry, is presented in order to obtain the neutrino mass matrix of the form of Eq. (1). In this work, our aim to see if the matrix of Eq. (1) can be obtained with type II seesaw mechanism [11] in the framework of \( \mu - \tau \) reflection symmetry. In order to achieve this, we construct a model which has three Higgs doublets and one scalar Higgs triplet. In our model, right-handed neutrinos do not exist, and hence, neutrinos acquire masses when the Higgs triplet get vacuum expectation value (VEV). The purpose of Higgs doublets is to give masses to charged leptons via Yukawa couplings. With the \( \mu - \tau \) reflection symmetry in our model, if the VEV of Higgs triplet is real, we show that the light neutrinos will have the mass matrix of the form of Eq. (1). In order to show if the VEV of Higgs triplet can be real, we analyze the scalar potential of our model. We demonstrate that by using an extra discrete symmetry, the VEV of Higgs triplet can be real. While analyzing the scalar potential of our model, we also address the problem of hierarchy in the masses of muon and tau leptons. In the literature, models have been constructed in order to achieve maximal values for \( \theta_{23} \) and \( \delta_{CP} \) using type II seesaw mechanism [12, 13]. However, in these models, multiple Higgs triplets have been introduced in addition to the three Higgs doublets. Hence, the model we have proposed here is economical as compared to the above mentioned models.

As stated previously, the form of mass matrix given in Eq. (1) can predict maximal values for \( \theta_{23} \) and \( \delta_{CP} \). But this matrix cannot give predictions about neutrino mass ordering and the mixing angles \( \theta_{12}, \theta_{13} \). As already pointed before, we have \( \sin^2 \theta_{13} \sim 10^{-2} \) [1], which means \( \theta_{13} \) is a small angle. In this work, we do an analysis, based on some approximation procedure [14], and derive some conditions on the elements of neutrino mass matrix which can predict about neutrino mass ordering and the smallness of \( \theta_{13} \), apart from giving maximal values for \( \theta_{23} \) and \( \delta_{CP} \). In this analysis, we assume \( \sin^2 \theta_{12} \sim \frac{1}{3} \).
To achieve the above mentioned conditions, new mechanisms should be proposed. In this work, we have attempted to give one mechanism in order to achieve one of those conditions. Since in our model three Higgs doublets exist, which give masses to charged leptons, it is worth to know if these scalar doublets can also generate masses and mixing pattern for quarks. Due to CP symmetry in the lepton sector, it is found that these Higgs doublets should transform non-trivially under the CP symmetry. As a result of this, we propose CP transformations for quarks in such a way that the Yukawa couplings for these are invariant under the CP symmetry. It is known that there is a large hierarchy among the masses of quarks. Hence, in order to explain the mixing pattern for quarks, the Yukawa couplings for these should be hierarchically suppressed \[15\]. To explain the realistic mixing pattern for quarks through hierarchically suppressed Yukawa couplings, we have followed the work done in Refs. \[16, 17\]. To know about other works on quark and lepton mixings with generalized CP transformations, see Ref. \[18\]. There are some works which addressed quark and lepton mixings with other symmetries. For example, see Refs. \[19\].

The paper is organized as follows. In the next section, we propose a model for lepton mixing, where maximal values for $\theta_{23}$ and $\delta_{CP}$ can be predicted if the VEV of triplet Higgs is real. In Sec. 3, we analyze the scalar potential of our model and show that the VEV of triplet Higgs can be real if we introduce additional discrete symmetry $Z_3$. In Sec. 4, we obtain some conditions on the elements of the neutrino mass matrix of our model, which can predict about the neutrino mass ordering and the smallness of $\theta_{13}$. In Sec. 5, we study on the quark masses and mixing pattern and demonstrate that they can be explained in the framework of our model. We conclude in the last section. In the Appendix, we attempt to give a mechanism for achieving normal order of neutrino masses.

2 A model for lepton mixing

The model we propose for lepton mixing is similar to that in Ref. \[6\]. We propose scalar Higgs doublets $\phi_i = (\phi_i^+, \phi_i^0)^T$, where $i = 1, 2, 3$, in order to give masses to charged leptons. We denote the lepton doublets and singlets by $D_{\alpha L} = (\nu_{\alpha L}, \alpha_L)^T$ and $\alpha_R$, where $\alpha = e, \mu, \tau$, respectively. The CP transformations on the lepton fields and Higgs doublets...
are defined as \[6\]

\[D_{\alpha L} \rightarrow iS_{\alpha \beta} \gamma^0 C \bar{D}_{\beta L}, \quad \alpha_R \rightarrow iS_{\alpha \beta} \gamma^0 C \bar{\beta}_R, \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\]

\[\phi_{1,2} \rightarrow \phi_{1,2}^*, \quad \phi_3 \rightarrow -\phi_3^*. \quad (2)\]

Here, \(C\) is the charge conjugation matrix. In addition to the invariance under the above mentioned \(CP\) transformations, one needs to impose conservation of \(U(1)_{L_\alpha}\) and \(Z_2\) symmetries. Here, \(U(1)_{L_\alpha}\) is the lepton number symmetry for the individual family of leptons. Under \(Z_2\) symmetry, only the \(e_R\) and \(\phi_1\) change sign. With the above mentioned charge assignments, the invariant Lagrangian for charged lepton Yukawa couplings is given by \[6\]

\[L_Y = -y_e \bar{D}_e L \phi_1 e_R - \sum_{j=2}^3 \sum_{\alpha=\mu, \tau} g_{j\alpha} \bar{D}_{\alpha L} \phi_j R + h.c.. \quad (3)\]

In order for the Lagrangian of Eq. (3) to be invariant under \(CP\) symmetry, we should have \(y_e\) to be real, \(g_{2\mu} = g_{2\tau}^*\) and \(g_{3\mu} = -g_{3\tau}^*\). Since the mass of electron should be real, we take VEV of \(\phi_1\) to be real. On the other hand, the VEVs of \(\phi_{2,3}\) should be complex, which give masses to muon and tau leptons, whose forms are given below \[6\].

\[m_{\mu} = |g_{2\mu} v_2 + g_{3\mu} v_3|, \quad m_\tau = |g_{2\mu} v_2 - g_{3\mu} v_3|. \quad (4)\]

In this work, we are taking \(\langle \phi_0^i \rangle = v_i\) for \(i = 1, 2, 3\). A priori, the VEVs of all Higgs doublets are of the same order. Hence, from the above equations, we can notice that some fine tuning is necessary in order to explain the hierarchy in the muon and tau lepton masses. To reduce this fine tuning, \(K\) symmetry is introduced, under which the non-trivial transformations of the fields are given below \[6\]

\[\mu_R \rightarrow -\mu_R, \quad \phi_2 \leftrightarrow \phi_3. \quad (5)\]

After imposing this \(K\) symmetry in the above described model, one can see that \(g_{2\mu} = -g_{3\mu}\). Using this in Eq. [4], we get

\[\frac{m_{\mu}}{m_\tau} = \frac{|v_2 - v_3|}{|v_2 + v_3|}. \quad (6)\]

Since the scalar potential of this model should also respect the \(K\) symmetry, we should get \(v_2 = v_3\), and hence, \(m_{\mu} = 0\). Now, to explain a non-zero but small \(m_{\mu}\), soft breaking
of $K$ symmetry can be introduced into the scalar potential of this model \cite{9}. Analysis related to this is presented in the next section.

To explain the masses for neutrinos in the above described framework, we introduce the following Higgs triplet into the model.

$$\Delta = \left( \begin{array}{cc} \Delta^+ & \Delta^{++} \\ \overline{\Delta} & -\overline{\Delta}^+ \end{array} \right). \tag{7}$$

$\Delta$ is singlet under $Z_2$, but otherwise transform under $CP$ symmetry as $\Delta \rightarrow \Delta^*$. Now, the Yukawa couplings for neutrinos can be written as

$$\mathcal{L}_Y = \frac{1}{2} \sum_{\alpha,\beta=e,\mu,\tau} Y^\nu_{\alpha\beta} \overline{D}^c_{\alpha L} i \sigma_2 \Delta D_{\beta L} + h.c.. \tag{8}$$

Here, $D^c_{\alpha L}$ is the charge conjugated doublet for $D_{\alpha L}$ and $\sigma_2$ is a Pauli matrix. We can notice that the terms in the above Lagrangian break the lepton number symmetry $U_{L\alpha}$ explicitly. We can consider this technically natural, since in the limit that the symmetry $U_{L\alpha}$ is exact, the neutrino masses become zero in this model. Hence to explain the smallness of neutrino masses we can break $U_{L\alpha}$ symmetry by a small amount. As a result of this, the neutrino Yukawa couplings $Y^\nu_{\alpha\beta}$ can be small in this work. Due to the invariance under $CP$ symmetry, these Yukawa couplings should satisfy

$$SY^\nu S = (Y^\nu)^*. \tag{9}$$

After electroweak symmetry breaking, we can have $\langle \Delta^0 \rangle = v_\Delta$. Now, from Eq. (8), we get the mass matrix for neutrinos, which is given by $M_\nu = Y^\nu v_\Delta$. If $v_\Delta$ is real, using Eq. (9), we get

$$SM_\nu S = M^*_{\nu}. \tag{10}$$

In order to satisfy the above relation, the form for $M_\nu$ should be same as that of Eq. (1). Hence, in the above proposed model, the mixing angle $\theta_{23}$ and the $CP$ violating phase $\delta_{CP}$ are maximal. However, in order to satisfy the relation in Eq. (10), $v_\Delta$ should be real. In the next section, we present an analysis of scalar potential of our model, where we demonstrate that $v_\Delta$ can be real.

Let us get some estimation on the value of $v_\Delta$ in our work. As stated above, the neutrino mass matrix in our work is $M_\nu = Y^\nu v_\Delta$. Since the couplings $Y^\nu$ need to small, because they break the $U_{L\alpha}$ symmetry by a small amount, we take $Y^\nu \sim 10^{-3}$. Now, by fitting $M_\nu$ to neutrino masses, which are obtained from neutrino oscillation data, we
can get some estimation about \( v_\Delta \). Using the neutrino oscillation data, the following mass-square differences have been found \(^1\), where we have given the best fit values.

\[
m_3^2 - m_1^2 = 7.5 \times 10^{-5} \text{ eV}^2, \quad m_a^2 \equiv \begin{cases} m_3^2 - m_1^2 = 2.55 \times 10^{-3} \text{ eV}^2 & \text{(NO)} \\ m_1^2 - m_3^2 = 2.45 \times 10^{-3} \text{ eV}^2 & \text{(IO)} \end{cases} \quad (11)
\]

Here, \( m_{1,2,3} \) are neutrino mass eigenvalues and NO(IO) represents normal(inverted) ordering. Using the above values, we get \( m_\nu \sim 0.0087 \text{ eV} \) and \( m_a \sim 0.05 \text{ eV} \), which correspond to solar and atmospheric neutrino mass scales, respectively. In order to fit these neutrino mass scales in our work, we can take \( v_\Delta \sim 1 - 10 \text{ eV} \).

## 3 Analysis of scalar potential

The scalar fields of the model proposed in the previous section are charged under the symmetry \( CP \times Z_2 \times K \). The invariant scalar potential of this model can be written as

\[
V_{\text{inv}} = V_D + V_T. \quad (12)
\]

Here, \( V_D \) contains potential terms only for the Higgs doublets. \( V_T \) is the scalar potential for the triplet Higgs of our model. The form of \( V_D \) is given by \(^9\)

\[
V_D = -M_1^2 \phi_1^\dagger \phi_1 - M_2^2 (\phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3) + \lambda_1 (\phi_1^\dagger \phi_1)^2 + \lambda_2 ((\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2)
+ \lambda_3 (\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + \lambda_4 (\phi_2^\dagger \phi_2)(\phi_3^\dagger \phi_3) + \lambda_5 ((\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) + (\phi_1^\dagger \phi_3)(\phi_2^\dagger \phi_3))
+ \lambda_6 ((\phi_2^\dagger \phi_3)(\phi_2^\dagger \phi_3) + (\phi_3^\dagger \phi_2)(\phi_3^\dagger \phi_2))
+ \lambda_7 ((\phi_1^\dagger \phi_2)^2 + (\phi_1^\dagger \phi_3)^2) + \lambda_8 ((\phi_2^\dagger \phi_2)^2 + (\phi_3^\dagger \phi_3)^2)
+ i\lambda_9 [(\phi_1^\dagger \phi_2)(\phi_1^\dagger \phi_3) - (\phi_2^\dagger \phi_1)(\phi_3^\dagger \phi_1)] + i\lambda_{10} (\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_2)(\phi_2^\dagger \phi_2 - \phi_3^\dagger \phi_3). \quad (13)
\]

In the above equation, all parameters are real due to either hermiticity or \( CP \) symmetry of the potential. To obtain \( V_T \), we have followed the work of Ref. \(^{20}\). The form of \( V_T \) is given below.

\[
V_T = m_\Delta^2 \text{Tr}(\Delta^\dagger \Delta) + \frac{1}{2} \lambda_\Delta [\text{Tr}(\Delta^\dagger \Delta)]^2 + \lambda_{11} \phi_1^\dagger \phi_1 \text{Tr}(\Delta^\dagger \Delta) + \lambda_{12} (\phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3) \text{Tr}(\Delta^\dagger \Delta)
+ \lambda_{13} \text{Tr}(\Delta^\dagger \Delta) \text{Tr}(\Delta \Delta) + \lambda_{14} \phi_1^\dagger \Delta^\dagger \Delta \phi_1 + \lambda_{15} (\phi_2^\dagger \Delta^\dagger \Delta \phi_2 + \phi_3^\dagger \Delta^\dagger \Delta \phi_3)
+ \kappa_1 (\phi_1^T i\sigma_2 \Delta \phi_1 + h.c.) + \kappa_2 (\phi_2^T i\sigma_2 \Delta \phi_2 + \phi_3^T i\sigma_2 \Delta \phi_3 + h.c.)
+ i\kappa_3 (\phi_2^T i\sigma_2 \Delta \phi_3 - h.c.) \quad (14)
\]
Here, $\tilde{\phi}_k = i\sigma_2 \phi^*_k$, $k = 1, 2, 3$. All the parameters in the above equation are real, due to either hermiticity or $CP$ symmetry of the potential.

As described in the previous section, the VEV of $\phi_1$ is real but the VEVs for $\phi_{2,3}$ should be complex. Although all parameters in Eq. (14) are real, due to complex VEVs of $\phi_{2,3}$, the trilinear terms containing $\kappa_{2,3}$ can contribute complex VEV to $\Delta$. However, it may happen that the phases of the VEVs of $\phi_{2,3}$ can be fine tuned in such a way that the $\kappa_{2,3}$-terms can give a real VEV to $\Delta$. We study these points by minimizing the scalar potential of our model. Before we present that study, we have to estimate the order of magnitudes of the unknown parameters of Eqs. (13) and (14). From the naturalness argument, we take all dimensionless $\lambda$ parameters to be $O(1)$. Since the VEVs of Higgs doublets should be around the electroweak scale of $v_{EW} = 174$ GeV, we take $M_1^2, M_2^2 \sim v_{EW}^2$. Now, we have to determine the order of magnitudes for $m_\Delta^2$ and $\kappa_{1,2,3}$. This is explained below.

After minimizing the potential of Eq. (14) with respect to $\Delta_0$, we naively get

$$v_\Delta \sim \frac{\kappa v_{EW}^2}{m_\Delta^2 + \lambda v_{EW}^2}.$$  (15)

Here, $\kappa \sim \kappa_{1,2,3}$ and $\lambda \sim \lambda_{11,12}$. While obtaining the above equation, we have used $v_\Delta \ll v_{EW}$. In order to get a very small $v_\Delta$, we can consider the following two cases.

$$\text{case I : } m_\Delta \gg v_{EW}, \quad \kappa \sim m_\Delta.$$  
$$\text{case II : } m_\Delta \sim v_{EW}, \quad \kappa \sim v_\Delta.$$  (16)

In case I, the smallness of $v_\Delta$ is explained by taking a large value for $m_\Delta$, which is around $10^{12}$ GeV. In case II, by suppressing the $\kappa$ parameters, one can understand the smallness of $v_\Delta$. In case I, the value of $m_\Delta$ is close to the breaking scale of supersymmetry in supergravity models [21]. Hence, one can motivate case I from supersymmetry. On the other hand, in case II, one has to find a mechanism for the suppression of $\kappa$ parameters.

From the phenomenology point of view, case II can be tested in the LHC experiment, since the masses for the components of scalar triplet Higgs can be around few 100 GeV.

In case I, we can notice that $\langle V_D \rangle \sim \langle V_T \rangle$. Only the terms containing $m_\Delta^2$ and $\kappa$ parameters in $\langle V_T \rangle$ can be of the order of $\langle V_D \rangle$. Other terms in $\langle V_T \rangle$ give negligibly small contribution in comparison to $\langle V_D \rangle$. On the other hand, in case II, $\langle V_T \rangle \ll \langle V_D \rangle$. Because of this difference in the contribution of $V_T$ in both these cases, we minimize the scalar potential of our model separately for these two cases.
3.1 Case I

We parametrize the VEVs of scalar fields as follows.

\[
\langle \phi_1^0 \rangle = v_1, \quad \langle \phi_2^0 \rangle = v_2 = v \cos \sigma e^{i\alpha}, \quad \langle \phi_3^0 \rangle = v_3 = v \sin \sigma e^{i\beta}, \quad \langle \Delta^0 \rangle = v_\Delta = v' e^{i\theta}. \quad (17)
\]

Here, \(v_1, v, v'\) are real. We plug the above parametrizations in the scalar potential of Eq. (12). Since we want \(\langle \Delta^0 \rangle\) to be real, and moreover, \(V_{inv}\) respect \(K\) symmetry, we look for a minimum at

\[
\sigma = \frac{\pi}{4}, \quad \alpha = \beta = \omega, \quad \theta = 0. \quad (18)
\]

Now, we take first derivatives of \(V_{inv}\) with respect to \(\sigma, \alpha, \beta, \theta\) at the values mentioned in Eq. (18). Thereafter, we get the following two conditions.

\[
2\lambda_8 \sin 2\omega + \lambda_9 \cos 2\omega = 0, \quad (19)
\]

\[
2\kappa_2 \sin 2\omega = \kappa_3 \cos 2\omega. \quad (20)
\]

By satisfying the above two conditions, Eq. (18) gives a minimum to our scalar potential. We justify that this a minimum, after computing the second derivatives of the potential.

This analysis is presented shortly later. However, with the minimum of Eq. (18), we get \(v_2 = v_3\). Hence, \(m_\mu = 0\), which follows from Eq. (6). To get non-zero and small \(m_\mu\), one should add \(K\)-violating terms in our model, which break \(K\) symmetry explicitly by a small amount. Here we can see the analogy between \(U_{L_\alpha}\) and \(K\) symmetries of our model. Both of these symmetries are broken explicitly by a small amount in order to generate small masses for neutrinos and muon.

After including the \(K\)-violating terms, the procedure we follow for minimization of the scalar potential is similar to what it is done in Ref. [9]. However, in this work, we write more general form for \(K\)-violating terms as compared to that in Ref. [9]. In Ref. [9], only the soft terms which break the \(K\) symmetry are considered. The general form
for $K$-violating terms in our model, which respect the symmetry $CP \times Z_2$, is given by

$$V_K = i\delta M_2^2(\phi_1^2 \phi_3 - \phi_3^1 \phi_2) + \delta M_2^2 \phi_2^1 \phi_2 + \delta M_3^2 \phi_3^1 \phi_3$$

+ $\delta \kappa_2(\delta_1^1_{\phi_2^2} i\sigma_2 \Delta \phi_2 + h.c.) + \delta \kappa_2'(\delta_3^1_{\phi_2^2} i\sigma_2 \Delta \phi_3 + h.c.)$

+ $\delta \lambda_2(\phi_2^1 \phi_2^2)^2 + \delta \lambda_3(\phi_2^1 \phi_3)(\phi_2^1 \phi_2) + \delta \lambda_4'(\phi_1^1 \phi_1)(\phi_3^1 \phi_3)$

+ $\delta \lambda_5(\phi_1^1 \phi_2)(\phi_2^1 \phi_1) + \delta \lambda_6'(\phi_3^1 \phi_3)(\phi_3^1 \phi_1)$

+ $\delta \lambda_8 \left[(\phi_1^1 \phi_1)^2 + (\phi_1^1 \phi_1)^2\right] + \delta \lambda_8' \left[(\phi_1^1 \phi_3)^2 + (\phi_3^1 \phi_1)^2\right]$

+ $i\delta \lambda_{10}(\phi_2^1 \phi_2)(\phi_2^1 \phi_3 - \phi_3^1 \phi_2) + i\delta \lambda_{10}'(\phi_1^1 \phi_3)(\phi_2^1 \phi_3 - \phi_3^1 \phi_2)$

+ $i\delta \lambda_{12}(\phi_1^1 \phi_1)(\phi_2^1 \phi_3 - \phi_3^1 \phi_2) + i\delta \lambda_{12}'(\phi_1^1 \phi_2)(\phi_1^1 \phi_1) - (\phi_1^1 \phi_1)) (\phi_1^1 \phi_3)$

+ $\delta \lambda_{12}(\phi_1^1 \phi_2) \text{Tr}(\Delta^1 \Delta) + \delta \lambda_{12}'(\phi_3^1 \phi_3 \text{Tr}(\Delta^1 \Delta) + \delta \lambda_{15}(\phi_2^1 \Delta^1 \Delta \phi_2 + \delta \lambda_{15}'(\phi_3^1 \Delta^1 \Delta \phi_3$

+ $i\lambda_{15}(\phi_2^1 \phi_3 - \phi_3^1 \phi_2) \text{Tr}(\Delta^1 \Delta) + i\lambda_{15}'(\phi_2^1 \Delta^1 \Delta \phi_3 - \phi_3^1 \Delta^1 \Delta \phi_2).$  

(21)

All parameters in the above equation are real, due to either hermiticity or $CP$ symmetry of the potential. Terms in the first and second lines are quadratic and trilinear, respectively. Rest of the terms in the above equation are quartic.

In Ref. [9], only the soft terms which are quadratic are given. Moreover, the last two terms in the first line of Eq. (21) are given in Ref. [9], but by taking $\delta M_2^2 = -\delta M_3^2$. We can notice that if $\delta M_2^2 = \delta M_3^2$, sum of the corresponding terms in Eq. (21) is $K$-symmetric. Hence, as long as $\delta M_2^2 \neq \delta M_3^2$, each of these corresponding terms in Eq. (21) is $K$-violating but conserve $CP \times Z_2$. Based on this observation, we have constructed other $K$-violating terms in Eq. (21). Since the terms in Eq. (21) break $K$ symmetry by a small amount, the parameters for these should be small as compared to the corresponding parameters of $V_{inv}$.

After including the $K$-violating terms, the total scalar potential of our model is

$$V_{total} = V_{inv} + V_K.$$  

(22)

Previously, we minimized $V_{inv}$ and argued that the minimum can be at Eq. (18). Now, due to the presence of $V_K$, the above minimum can be shifted by a small amount. Due to this, the minimum for $V_{total}$ in terms of small deviations $\delta_0, \delta_+, \delta_-, \delta_\theta$ can be written as

$$\sigma = \frac{\pi}{4} - \frac{\delta_\theta}{2}, \quad \alpha = \omega + \frac{\delta_-}{2}, \quad \beta = \omega + \frac{\delta_-}{2}, \quad \theta = 0 + \delta_\theta.$$  

(23)

Now, we express $\langle V_{inv} \rangle$ and $\langle V_K \rangle$ as a series summation up to second and first order, respectively, in the above mentioned small deviations. After neglecting the constant terms, we get

$$\langle V_{total} \rangle = \frac{1}{2} \sum_{a,b} F_{ab} \delta_a \delta_b + \sum_a f_a \delta_a.$$  

(24)
Here, $\mathcal{F}_{ab}$ is symmetric in the indices $a, b$ and it corresponds to second derivatives of $V_{\text{inv}}$ calculated at Eq. (18). Non-vanishing elements of $\mathcal{F}_{ab}$ are given below.

$$
\mathcal{F}_{++} = (-8 \lambda_8 \cos 2 \omega + 4 \lambda_9 \sin 2 \omega) v_1^2 v^2 + (8 \kappa_2 \cos 2 \omega + 4 \kappa_3 \sin 2 \omega) v^2 v',
$$

$$
\mathcal{F}_{--} = -2 \lambda_7 v' - 2 \lambda_8 v_1^2 v^2 \cos 2 \omega + 2 \kappa_2 v^2 v' \cos 2 \omega,
$$

$$
\mathcal{F}_{00} = -\frac{1}{2} \lambda v^4 + (\lambda_9 v_1^2 + \kappa_3 v') v^2 \sin 2 \omega,
$$

$$
\mathcal{F}_{\theta \theta} = 2 \kappa_1 v_1^2 v' + 2 \kappa_2 v^2 v' \cos 2 \omega + \kappa_3 v^2 v' \sin 2 \omega,
$$

$$
\mathcal{F}_{0-} = -2 \lambda_8 v_1^2 v^2 \sin 2 \omega + \lambda_{10} v' + 2 \kappa_2 v^2 v' \sin 2 \omega,
$$

$$
\mathcal{F}_{+\theta} = -2 (2 \kappa_2 \cos 2 \omega + \kappa_3 \sin 2 \omega) v^2 v'.
$$

(25)

Here, $\tilde{\lambda} = -2 \lambda_2 + \lambda_4 + \lambda_6 + 2 \lambda_7$. The expressions for $f_a$ are given below.

$$
f_0 = \frac{1}{2} (\delta M_2^2 - \delta M_3^2) v^2 - (\delta \kappa_2 - \delta \kappa_2') v^2 v' \cos 2 \omega + \frac{1}{2} (\delta \lambda_2 - \delta \lambda_2') v^4
$$

$$
+ \frac{1}{2} (\delta \lambda_3 - \delta \lambda_3') + \delta \lambda_5 + \delta \lambda_5' + 2 (\delta \lambda_8 - \delta \lambda_8') \cos 2 \omega] (v_1 v)^2 + \frac{1}{2} (\delta \lambda_{12} - \delta \lambda_{12}') (v v')^2,
$$

$$
f_- = \delta M_2^2 v^2 + (\delta \kappa_2 - \delta \kappa_2') v^2 v' \sin 2 \omega - (\delta \lambda_8 - \delta \lambda_8') \sin 2 \omega (v_1 v)^2 + \frac{1}{2} (\delta \lambda_{10} + \delta \lambda_{10}') v^4
$$

$$
+ (\delta \lambda_8 - \delta \lambda_8') (v_1 v)^2 + \delta \lambda_4 (v v')^2,
$$

$$
f_+ = 2 (\delta \kappa_2 + \delta \kappa_2') v^2 v' \sin 2 \omega - 2 (\delta \lambda_8 + \delta \lambda_8') \sin 2 \omega (v_1 v)^2,
$$

$$
f_\theta = -(\delta \kappa_2 + \delta \kappa_2') v^2 v' \sin 2 \omega.
$$

(26)

Using Eq. (24), the small deviations in the minimum of $V_{\text{total}}$ can be obtained as

$$
\delta = -\mathcal{F}^{-1} f.
$$

(27)

Here, $\delta = (\delta_0, \delta_-, \delta_+, \delta_\theta)^T$, $f = (f_0, f_-, f_+, f_\theta)^T$ and $\mathcal{F}$ is a matrix containing the elements $\mathcal{F}_{ab}$.

Since some elements of $\mathcal{F}_{ab}$ are zero, Eq. (27) can be decomposed into

$$
\begin{pmatrix}
\delta_0 \\
\delta_-
\end{pmatrix} = -\mathcal{F}_1^{-1} \begin{pmatrix}
f_0 \\
f_-
\end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix}
\mathcal{F}_{00} & \mathcal{F}_{0-} \\
\mathcal{F}_{-0} & \mathcal{F}_{--}
\end{pmatrix},
$$

$$
\begin{pmatrix}
\delta_+ \\
\delta_\theta
\end{pmatrix} = -\mathcal{F}_2^{-1} \begin{pmatrix}
f_+ \\
f_\theta
\end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix}
\mathcal{F}_{++} & \mathcal{F}_{+\theta} \\
\mathcal{F}_{+\theta} & \mathcal{F}_{\theta \theta}
\end{pmatrix}.
$$

(28)

We can see that $\mathcal{F}$ is in block diagonal form containing $\mathcal{F}_1$ and $\mathcal{F}_2$. As stated before, the elements of $\mathcal{F}$ correspond to second derivatives of $V_{\text{inv}}$ calculated at Eq. (18). As a result of this, if the eigenvalues of $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive then Eq. (18) give minimum to the scalar potential in the absence of $V_k$. One can see that the unknown $\lambda$ and $\kappa$ parameters...
of $F_{1,2}$ can be chosen in such a way that $F_{1,2}$ yield positive eigenvalues. However, in the presence of $V_K$, the minimum of scalar potential of our model is shifted to Eq. (23). The small deviations of Eq. (23) can be computed from Eq. (28). From Eq. (28), we can see that $\delta_-, \delta_+ \neq 0$. Hence, $v_2 \neq v_3$. Using the expressions for $\delta_-, \delta_+$ in Eq. (6), we can get the required hierarchy between $m_\mu$ and $m_\tau$, provided the parameters of $V_K$ are small.

It can be noticed that the parametrizations we have used in Eq. (23) is similar to that in Ref. [9]. In Ref. [9], it is pointed that $\delta_+ = 0$. In our work, we get $\delta_+ \neq 0$, since $f_+ \neq 0$. This difference is due to the fact that in Ref. [9], $K$-violating quartic terms are not considered.

We have described that with $K$-violating terms of our model, we can explain the required hierarchy between muon and tau lepton masses. However, in doing so, from Eq. (28) we can see that $\delta_+ \neq 0$. This makes $v_\Delta$ complex. One can fine tune the parameters in $F_2, f_+, f_\theta$ in such a way that $\delta_+ = 0$. On the other hand, to get $\delta_+ = 0$, we can take $F_{+\theta} = 0 = f_\theta$. After using Eq. (20), $F_{+\theta} = 0$ implies $\kappa_2 = \kappa_3 = 0$. In order to make $f_\theta = 0$, either we can take $\delta \kappa_2 = -\delta \kappa_3'$ or forbid the trilinear terms of $V_K$. From the above made observations, we can see that, in case I in order to make $v_\Delta$ real without fine tuning the parameters, the trilinear terms of $V_{total}$ containing $\phi_{2,3}$ should be forbidden.

### 3.2 Case II

As explained before, in this case, terms involving triplet Higgs give very small contribution in comparison to that involving only doublet Higgses. As a result of this, minimization of $V_{total}$ in this case proceeds in two steps. In the first step, we minimize $V_{total}$ which contain only doublet Higgses and thereby determine the VEVs of these fields. Later, after using the VEVs of doublet Higgses, we minimize the potential containing the triplet Higgs field.

In the first step of minimization, we can neglect $V_T$ in comparison to $V_D$ and also neglect the terms in $V_K$ which contain triplet Higgs field. In this case also, we parametrize the VEVs for $\phi_{1,2,3}$ and $\Delta$ as given by Eq. (17). Now, after minimizing $V_D$ with respect to $\sigma, \alpha, \beta$, the minimum is given by Eq. (18) with the condition of Eq. (19). Since this minimum gives $m_\mu = 0$, we introduce $V_K$ and parametrize the deviations in $\sigma, \alpha, \beta$ as given by Eq. (23). After doing this, one can notice that the above mentioned deviations can be found from Eq. (27), where, in this case, $F$ and $f$ are $3 \times 3$ and $3 \times 1$ matrices respectively. The components of $F$ and $f$ can be found from Eqs. (25) and (26), where one has to omit the terms containing $v'$. As a result of this, we get $\delta_-, \delta_+ \neq 0$. After using this in Eq. (6), we get small and non-zero $m_\mu$. 

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Since the VEVs of doublet Higgses are determined, we now minimize $V_T$ and try to see if $v_\Delta$ can be real. After using Eq. (17) in $V_T$, we get
\[
\langle V_T \rangle = (m_\Delta^2 + \lambda_{11} v_1^2 + \lambda_{12} v_2^2) v'' + \frac{1}{2} \lambda_\Delta v'' - 2 \kappa_1 v_1^2 v' \cos \theta \\
- 2 \kappa_2 v^2 v'[\cos^2 \sigma \cos(\theta - 2\alpha) + \sin^2 \sigma \cos(\theta - 2\beta)] + \kappa_3 v^2 v' \sin 2\sigma \sin(\theta - \alpha - \beta).
\]
(29)

Since we are looking for a minimum at $\theta = 0$, we do
\[
\frac{\partial \langle V_T \rangle}{\partial \theta} \bigg|_{\theta = 0} = 0 \Rightarrow -2\kappa_2 [\cos^2 \sigma \sin 2\alpha + \sin^2 \sigma \sin 2\beta] + \kappa_3 \sin 2\sigma \cos(\alpha + \beta) = 0.
\]
(30)

As stated before, we have determined $\sigma, \alpha, \beta$ up to first order in $\delta_0, \delta_-, \delta_+$. Plugging the parametrizations for $\sigma, \alpha, \beta$ in the above equation and expanding the terms up to first order in $\delta_0, \delta_-, \delta_+$, we get
\[
2\kappa_2 \sin 2\omega - \kappa_3 \cos 2\omega + 2(2\kappa_2 \cos 2\omega + \kappa_3 \sin 2\omega)\delta_+ = 0.
\]
(31)

Since we have $\delta_+ \neq 0$, after equating the leading and subleading terms of the above equation to zero, we get $\kappa_2 = \kappa_3 = 0$. Hence, in case II, in order to make $v_\Delta$ real, one has to forbid the trilinear terms of $V_T$ which contain $\phi_{2,3}$.

### 3.3 Imposing an extra $Z_3$ symmetry

From the analysis of previous two subsections, we have seen that the trilinear terms in $V_{\text{total}}$ which contain $\phi_{2,3}$ should be forbidden in order to make $v_\Delta$ real. To achieve this, we impose the discrete symmetry $Z_3$ in our model. Under this symmetry, the non-trivial transformations are as follows.

\[
\phi_2 \rightarrow \Omega \phi_2, \quad \phi_3 \rightarrow \Omega \phi_3, \\
\mu_R \rightarrow \Omega^2 \mu_R, \quad \tau_R \rightarrow \Omega^2 \tau_R.
\]
(32)

Here, $\Omega = e^{2\pi i/3}$. Under the above transformations, the Yukawa couplings for leptons are invariant but the following couplings in $V_{\text{total}}$ are forbidden: $\lambda_{8,9}, \kappa_{2,3}, \delta \kappa, \delta \kappa'$. Now, after using the parametrizations of Eq. (17) in $V_{\text{inv}}$, we get
\[
\langle V_{\text{inv}} \rangle = \frac{1}{4}[\tilde{\lambda} - 4\lambda_7 \sin^2 \zeta]v^4 \sin^2 2\sigma + \frac{1}{2} \lambda_{10} v^4 \sin 4\sigma \sin \zeta - 2 \kappa_1 v_1^2 v' \cos \theta.
\]
(33)

Here, $\zeta = \alpha - \beta$. In the above equation, we have neglected constant terms which do not depend on $\sigma, \alpha, \beta, \theta$. We can notice from the above equation that $\theta$ do not mix with $\sigma, \zeta$. 

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Moreover, due to absence of trilinear terms in $V_K$, $\langle V_K \rangle$ do not depend on $\theta$. As a result of this, we can see that $\theta = 0$ is a minimum to $V_{\text{total}}$ if $\kappa_1 v' > 0$. This statement is true for both the cases of I and II. Hence, after imposing the above mentioned $Z_3$ symmetry, $v_\Delta$ can be real in our model.

For Eq. (33), the minimum in terms of $\sigma, \zeta$ can be at

$$\sigma = \frac{\pi}{4}, \quad \zeta = 0.$$  \hspace{1cm} (34)

Since $\zeta = 0$ corresponds to $m_\mu = 0$, we introduce $K$-violating terms into the model. As a result of this, the above mentioned minimum can be shifted by small deviations $\delta_0, \delta_\zeta$ as

$$\sigma = \frac{\pi}{4} - \frac{\delta_0}{2}, \quad \zeta = 0 + \delta_\zeta.$$  \hspace{1cm} (35)

Now, after imposing $Z_3$ symmetry in Eq. (21) and after following the procedure for minimizing $V_{\text{total}}$, which is described in Sec. 3.1, we get

$$\begin{pmatrix} \delta_0 \\ \delta_\zeta \end{pmatrix} = \mathcal{F}^{-1} \begin{pmatrix} f_0 \\ f_\zeta \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} -\frac{1}{2} \lambda & \lambda_{10} \\ \lambda_{10} & -2\lambda_7 \end{pmatrix} v^4,$$

$$f_0 = \frac{1}{2} (\delta M_2^2 - \delta M_3^2) v^2 + \frac{1}{2} (\delta \lambda_2 - \delta \lambda'_2) v^4 + \frac{1}{2} (\delta \lambda_3 - \delta \lambda'_3 + \delta \lambda_5 - \delta \lambda'_5) (v_1 v)^2$$

$$+ \frac{1}{2} (\delta \lambda_{12} - \delta \lambda'_{12}) (v v')^2$$

$$f_\zeta = \delta M_2^2 v^2 + \frac{1}{2} (\delta \lambda_{10} + \delta \lambda'_{10}) v^4 + (\delta \lambda_4 - \delta \lambda'_4) (v_1 v)^2 + \delta_t (v v')^2$$ \hspace{1cm} (36)

We can see that $\delta_0, \delta_\zeta \neq 0$. After using these in the parametrizations for $v_{2,3}$, from Eq. (6), we get

$$\frac{m_\mu}{m_\tau} = \frac{1}{2} |\delta_0 + i\delta_\zeta|. \hspace{1cm} (37)$$

Using the above equation, the required hierarchy between muon and tau leptons can be explained if we take $\delta_0, \delta_\zeta \sim 0.1$.

In Sec. 2 we have described our model for lepton sector by introducing additional fields and symmetries. In the current section, we have introduced one more symmetry, $Z_3$, in order to make the triplet Higgs VEV to be real. In Tab. 1 we summarize the additional fields and symmetries, which are needed for our model, in the lepton sector.

### 4 Neutrino mass ordering and the smallness of $\theta_{13}$

After showing that the triplet Higgs can acquire real VEV, the neutrino mass matrix of the model proposed in Sec. 2 satisfy Eq. (10). As a result of this, after diagonalizing
additional field | role  
--- | ---  
$\phi_1$ | to generate the mass of electron  
$\phi_2, \phi_3$ | to generate masses for $\mu$ and $\tau$  
$\Delta$ | to generate masses for neutrinos  

| additional symmetry | role  
--- | ---  
$CP$ symmetry | to get $\mu - \tau$ form for neutrino mass matrix  
$Z_2$ | forbids unwanted Yukawa couplings among charged leptons  
$U(1)_{L\alpha}$ | to get diagonal masses for charged leptons  
$K$ symmetry | to reduce the fine-tuning in muon and tau masses  
$Z_3$ | to make the VEV of $\Delta$ to be real  

Table 1: Additional fields and symmetries, which are introduced in the lepton sector of our model. The roles of these fields and symmetries are also described here.

$M_\nu, \theta_{23}$ and $\delta_{CP}$ would be maximal [6]. However, the form of $M_\nu$ doesn’t give predictions about $\theta_{12}, \theta_{13}$ and also about neutrino mass ordering. In this section, we do an analysis and give a procedure which can give predictions about neutrino mass ordering and the smallness of $\theta_{13}$ in our model.

In the model proposed in Sec. 2, the charged lepton masses are in diagonal form. Hence, the unitary matrix which diagonalizes $M_\nu$ can be written as

$$U = \tilde{U} U_{PMNS}, \quad \tilde{U} = \text{diag}(1,1,-1),$$

$$U_{PMNS} = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13}
\end{pmatrix}. \quad (38)$$

Here, $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. $U_{PMNS}$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix which is parameterized in terms of the three lepton mixing angles and the $CP$ violating Dirac phase, according to the convention of PDG [10]. Diagonal elements of $\tilde{U}$ can be absorbed into the charged lepton fields. Now, the relation for diagonalizing $M_\nu$ can be expressed as

$$M_\nu = U^* \text{diag}(m_1, m_2, m_3) U^\dagger. \quad (39)$$

While solving the above equation, we can use an approximation procedure [14] which is related to neutrino masses and the mixing angle $\theta_{13}$. This procedure is explained below.
In the expression for $U_{PMNS}$ one can have Majorana phases. These phases cannot be
determined from neutrino oscillation data. But they can affect the life-time of neutrinoless
double beta decay, since neutrinos in our model are Majorana particles. However, so far
no concrete evidence is there for this decay \cite{10}, and as a result, the Majorana phases can
be anywhere between 0 and $2\pi$. Hence, in our analysis, for the sake of simplicity, we have
chosen these phases to be zero. On the other hand, by taking some specific values for
Majorana phases in the below described procedure, one can study the conditions which
can give rise for neutrino Yukawa couplings of our model. However, we reserve this study
for future.

In $U_{PMNS}$, we put $\theta_{23} = \frac{\pi}{4}$ and $\delta_{CP} = \frac{3\pi}{2}$. From the neutrino oscillation data, we
have $s_{12}^2 \sim \frac{1}{3}$ and $s_{13}^2 \sim 2 \cdot 10^{-2}$ \cite{11}. Here we can notice that $s_{13}^2$ is negligibly small in
comparison to unity, and hence $s_{13} \sim 0.15$ can be treated as small variable. On the other
hand, $s_{12}^2$ and $s_{23}^2$ are of order one. Since $s_{13}$ is the only small variable in $U_{PMNS}$, we
expand $U_{PMNS}$ up to first order in $s_{13}$. Expression for this is given below.

$$U_{PMNS} = U_0 + \delta U,$$

$$U_0 = \begin{pmatrix}
    c_{12} & s_{12} & 0 \\
    -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
    \frac{s_{12}}{\sqrt{2}} & -\frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}, \quad \delta U = \begin{pmatrix}
    0 & 0 & 1 \\
    \frac{c_{12}}{\sqrt{2}} & \frac{s_{12}}{\sqrt{2}} & 0 \\
    \frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & 0
\end{pmatrix} i s_{13}. \quad (40)$$

From the neutrino oscillation data, two mass-squared differences for neutrinos are found,
which are given in Eq. (11). From this equation we can notice that $m_{23}^2 \sim s_{13}^2$, which is
negligibly small in comparison to unity. This indicates, an approximation with respect
to neutrino masses can also be applied while solving the Eq. (39). In order to fit the
mass-square differences of Eq. (11), we can take the neutrino masses as follows.

$$\text{NO : } m_1 \lesssim m_s, \quad m_2 = \sqrt{m_s^2 + m_1^2}, \quad m_3 = \sqrt{m_a^2 + m_1^2}.$$  

$$\text{IO : } m_3 \lesssim m_s, \quad m_1 = \sqrt{m_a^2 + m_3^2}, \quad m_2 = \sqrt{m_s^2 + m_3^2}. \quad (41)$$

Now we can notice that $\frac{m_1}{m_a} \lesssim s_{13}$ in the case of NO. Whereas, $\frac{m_3}{m_a} \sim 1$ in the case of IO.
Similar conclusions can be made about $\frac{m_2}{m_a}$ and $\frac{m_s}{m_a}$.

Using the approximation scheme described in the previous paragraph, we can expand
$\frac{1}{m_a} M_\nu$ in powers of $s_{13}, \frac{m_1}{m_a}$. After neglecting the second and higher order corrections in
$s_{13}, \frac{m_s}{m_a}$, for both the cases of NO and IO, the elements of $\frac{1}{m_a} M_\nu$ are given below.

\[
\frac{1}{m_a} M_\nu = \frac{1}{2m_a} \begin{pmatrix}
x & z & z^* \\
z & w & y \\
z^* & y & w^*
\end{pmatrix},
\]

NO : $x = 2c_{12}^2 m_1 + 2s_{12}^2 m_2, \quad z = \sqrt{2}c_{12}s_{12}(m_2 - m_1) - i\sqrt{2}m_3 s_{13},$

$w = m_3 + c_{12}^2 m_2 + s_{12}^2 m_1, \quad y = -m_3 + c_{12}^2 m_2 + s_{12}^2 m_1.$

IO : $x = 2m_1, \quad z = -\sqrt{2}i s_{13} m_1, \quad w = m_1 + m_3, \quad y = m_1 - m_3.$ \quad (42)

Using the above relations, in order for the matrix $M_\nu$ to predict about neutrino mass ordering and smallness of $\theta_{13}$, the Yukawa couplings in Eq. (8) should satisfy the following conditions.

- To predict NO and smallness of $\theta_{13}$:
  
  (i) $Y_{\nu ee}, Y_{\nu e\mu}$ should be suppressed by about 0.1 as compared to that of $Y_{\nu \mu\mu}, Y_{\nu \mu\tau}$.

  (ii) $Y_{\nu \mu\mu}$ should be real.

- To predict IO and smallness of $\theta_{13}$:

  (i) $Y_{\nu e\mu}$ should be purely imaginary and its magnitude is suppressed by about 0.1 as compared to other elements of $Y^\nu$.

  (ii) $Y_{\nu \mu\mu}$ should be real.

It is to remind here that $Y^\nu$ is a symmetric matrix and satisfies Eq. (9). Hence, not all elements of $Y^\nu$ are independent. As a result of this, while describing the above conditions, we have considered $Y^\nu_{ee}, Y^\nu_{e\mu}, Y^\nu_{\mu\mu}, Y^\nu_{\mu\tau}$ as independent elements of $Y^\nu$. Another point to mention here is that the above mentioned conditions are true after neglecting second and higher order corrections in $\frac{1}{m_a} M_\nu$.

The condition (ii) described for the cases of NO and IO is trivially satisfied if one uses relations of Eq. (42). The non-trivial condition to check is the condition (i) in the cases of both NO and IO. The suppression factor mentioned in this condition is arising due to $s_{13} \sim \frac{m_s}{m_a} \sim 0.15$. We have checked this suppression factor for the case of NO by computing the following ratios: $\frac{|Y_{\nu ee}|}{|Y_{\nu \mu\mu}|}, \frac{|Y_{\nu e\mu}|}{|Y_{\nu \mu\mu}|}, \frac{|Y_{\nu \mu\mu}|}{|Y_{\nu \mu\tau}|}$. While for the case of IO, the following ratios are computed in order to check condition (i): $\frac{|Y_{\nu e\mu}|}{|Y_{\nu ee}|}, \frac{|Y_{\nu e\mu}|}{|Y_{\nu \mu\mu}|}, \frac{|Y_{\nu \mu\mu}|}{|Y_{\nu \mu\tau}|}$. One can notice that the neutrino Yukawa couplings are proportional to the elements of $M_\nu$, which are given in Eq. (42). The neutrino masses in Eq. (42) are computed by using Eq. (41).
and also by varying $m_s^2, m_a^2$ over their allowed $3\sigma$ ranges. The mass of lightest neutrino is varied from 0 to $m_s$ in both the cases of NO and IO. While computing the above mentioned ratios, we have also varied $s^2_{12}$ and $s^2_{13}$ over their allowed $3\sigma$ ranges. We have tabulated the allowed $3\sigma$ ranges for the above mentioned variables in Tab. 2.

| parameters     | allowed range                  |
|----------------|--------------------------------|
| $m_s^2$        | $(6.94-8.14) \times 10^{-5}$ eV$^2$ |
| $m_a^2$ (NO)   | $(2.47-2.63) \times 10^{-3}$ eV$^2$ |
| $m_a^2$ (IO)   | $(2.37-2.53) \times 10^{-3}$ eV$^2$ |
| $s^2_{12}$     | $0.271-0.369$                  |
| $s^2_{13}$ (NO)| $0.0200-0.02405$              |
| $s^2_{13}$ (IO)| $0.02018-0.02424$             |

Table 2: Allowed $3\sigma$ ranges of the neutrino oscillation observables [1], which are used in our analysis.

While doing the above described analysis, we have also checked if the sum of the three neutrinos is less than 0.12 eV, which is a constraint obtained from the cosmological observations [22]. As already described in the previous paragraph, the suppression in the ratios of various Yukawa couplings should be around $s_{13} \sim \frac{m_s}{m_a} \sim 0.15$. However, in the analysis we have found that some of these ratios can become as large as 0.5, and thus, invalidate the approximation procedure that we are using here. Hence, in the analysis we have restricted all these ratios to be less than or of the order of 0.2. Selected plots from this analysis are presented in Fig. 1. From this figure we can see that the mass of lightest neutrino, $m_{\text{lightest}}$, is constrained due to above mentioned restriction on the ratios of Yukawa couplings. We can notice that $m_{\text{lightest}}$ has narrow allowed region in the case of NO as compared to that of IO. Apart from the $m_{\text{lightest}}$ in Fig. 1, $s^2_{13}$ is also constrained to be in the range of 0.02 to 0.023, in the case of NO. But otherwise, this variable is not constrained in the case of IO. As for $s^2_{12}$, we have found that it can take the full $3\sigma$ range in both the cases of NO and IO of Fig. 1. Although we have presented selected plots in Fig. 1, we have got similar kind of plots for other ratios of Yukawa couplings, which are mentioned in the previous paragraph. These plots justify the approximation procedure that we are using here and also verify the condition (i) for the cases of NO and IO, which are mentioned below Eq. (42).

The conditions mentioned below Eq. (42), for both the cases of NO and IO, cannot be
achieved just with the $CP$ symmetry. Additional mechanism should be proposed in order to satisfy these conditions. In an attempt towards this, we have given one mechanism to achieve condition (i) for the case of NO, where the necessary suppression in the Yukawa couplings is explained through non-renormalizable terms within the framework of $CP$ symmetry. This mechanism is presented in the Appendix. In this work, we do not have a mechanism to achieve condition (i) for the case of IO and also to achieve condition (ii) for both NO and IO. We work on these problems in future. As already described, with generalized $CP$ transformations and $\mu - \tau$ symmetry, $\theta_{23}$ and $\delta_{CP}$ will be maximal. However, in future neutrino oscillation experiments, $\theta_{23}$ and $\delta_{CP}$ may be found to be away from their maximal values. In such a case, one needs to device a mechanism for the breaking of $\mu - \tau$ reflection symmetry in order to explain the non-maximal values for the above mentioned observables. These topics are outside the scope of this paper.

5 Quark mixing

In the model we proposed for lepton mixing, three Higgs doublets exist. Since these Higgs doublets can also give masses to quarks, it is interesting to see if quark mixing can also be explained with $CP$ and other symmetries of our model. As already described in Sec. 1, since there is a hierarchy among quark masses, the mixing pattern for quarks can be explained if their Yukawa couplings are hierarchically suppressed. Babu and Nandi have
proposed one model \cite{16} for explaining quark mixing through hierarchically suppressed Yukawa couplings. Later, this model has been modified in Ref. \cite{17}, where the suppression in Yukawa couplings is explained with a singlet scalar field. We follow the work of Refs. \cite{16,17} in order to explain quark mixing in our framework.

5.1 Model for quark masses and mixing

We denote the three families of quark doublets, up- and down-type singlets as $Q_{jL}$, $u_{jR}$ and $d_{jR}$, respectively. We propose a scalar field $X$ which is singlet under standard model gauge group. We assume all the quark doublets to be singlets under the symmetry $K \times Z_2 \times Z_3$. On the other hand, all the right-handed quark fields are singlets under $K \times Z_3$ but they are odd under $Z_3$ symmetry. $X$ field is singlet under $K \times Z_3$ but is odd under $Z_2$. Both the quark and $X$ fields transform under $CP$ symmetry as

$$Q_{jL} \to i\gamma^0 C\bar{Q}^T_{jL}, \quad u_{jR} \to i\gamma^0 C\bar{u}^T_{jR}, \quad d_{jR} \to i\gamma^0 C\bar{d}^T_{jR}, \quad X \to X^*.$$  \hspace{1cm} (43)

Now, with the above mentioned transformations and fields, we consider the following effective Lagrangian for quark masses.

$$\mathcal{L}_Y = h^u_{33} \bar{Q}_{3L} \phi_1 u_{3R} + \left(\frac{X}{M}\right)^2 [h^d_{33} \bar{Q}_{3L} \phi_1 d_{3R} + h^u_{22} \bar{Q}_{2L} \phi_1 u_{2R} + h^u_{23} \bar{Q}_{2L} \phi_1 u_{3R} + h^u_{32} \bar{Q}_{3L} \phi_1 d_{2R} + h^u_{13} \bar{Q}_{1L} \phi_1 u_{1R} + h^u_{31} \bar{Q}_{3L} \phi_1 u_{1R}] + \left(\frac{X}{M}\right)^4 [h^d_{22} \bar{Q}_{2L} \phi_1 d_{2R} + h^d_{23} \bar{Q}_{2L} \phi_1 d_{3R} + h^d_{32} \bar{Q}_{3L} \phi_1 d_{2R} + h^u_{21} \bar{Q}_{2L} \phi_1 u_{1R} + h^u_{12} \bar{Q}_{1L} \phi_1 u_{2R} + h^u_{13} \bar{Q}_{1L} \phi_1 u_{3R} + h^u_{31} \bar{Q}_{3L} \phi_1 u_{1R}] + \left(\frac{X}{M}\right)^6 [h^d_{11} \bar{Q}_{1L} \phi_1 u_{1R} + h^d_{21} \bar{Q}_{2L} \phi_1 u_{1R} + h^d_{12} \bar{Q}_{1L} \phi_1 d_{1R} + h^d_{13} \bar{Q}_{1L} \phi_1 d_{3R} + h^d_{31} \bar{Q}_{3L} \phi_1 u_{1R}] + \left(\frac{X}{M}\right)^8 [h^d_{21} \bar{Q}_{2L} \phi_1 u_{1R}] + \left(\frac{X}{M}\right)^{10} [h^d_{31} \bar{Q}_{3L} \phi_1 u_{1R}] + h.c. \hspace{1cm} (44)$$

The above Lagrangian is valid below a mass scale of $M$. The non-renormalizable terms of this Lagrangian can be motivated from the UV completion of this model, which is presented in the next subsection. According to this UV completion, we propose a flavor symmetry $U(1)_F$ and heavy vector-like quark (VLQ) fields above the scale $M$. After integrating the heavy fields of our model, below the scale $M$, the non-renormalizable terms of Eq. (44) can appear. Here we can see that $M$ represents the mass scale of heavy VLQs. Since new particles can be probed at LHC experiment if their masses are around 1 TeV, so we take $M \sim 1$ TeV.

Due to $CP$ symmetry, the Yukawa couplings $h^u_{jk}$ should be real in Eq. (44). After $X$ acquires VEV, for $\langle X \rangle \leq M$, we can see that $\frac{X}{M}$ gives suppression to effective quark
Yukawa couplings. Since the Yukawa couplings of Eq. \([44]\) are real, we assume \((X)\) is complex and this can be the source for \(CP\) violation in the quark sector. In the Lagrangian of Eq. \([44]\), only the doublet \(\phi_1\) generates Yukawa couplings for quark fields. The other doublets \(\phi_{2,3}\) do not generate these Yukawa couplings due to the presence of \(CP \times K\) symmetry.

After electroweak symmetry breaking, using Eq. \([44]\), the matrices for up- and down-type quarks can be written, respectively, as

\[
M_u = \begin{pmatrix}
h^u_{11} \epsilon^6 & h^u_{12} \epsilon^4 & h^u_{13} \epsilon^4 \\
h^u_{21} \epsilon^4 & h^u_{22} \epsilon^2 & h^u_{23} \\
h^u_{31} \epsilon^4 & h^u_{32} \epsilon^2 & h^u_{33} \\
\end{pmatrix} v_1, \quad M_d = \begin{pmatrix}
h^d_{11} \epsilon^6 & h^d_{12} \epsilon^6 & h^d_{13} \epsilon^6 \\
h^d_{21} \epsilon^{10} & h^d_{22} \epsilon^4 & h^d_{23} \epsilon^4 \\
h^d_{31} \epsilon^{10} & h^d_{32} \epsilon^4 & h^d_{33} \epsilon^2 \\
\end{pmatrix} v_1. \tag{45}
\]

Here, \(\epsilon = \frac{(X)}{M}\). The form of \(M_{u,d}\) is similar to the corresponding matrices of Refs. \([16,17]\). However, the only difference is that the elements 21 and 31 of \(M_d\) are generated at higher order as compared to that of Refs. \([16,17]\). It is argued in Ref. \([17]\) that the above mentioned elements do not affect quark masses and mixing if they are generated at higher order. Hence, after diagonalizing the above matrices, the masses and mixing angles for quarks, up to leading order in \(|\epsilon|\), are given by

\[
(m_t, m_c, m_u) \approx (|h^u_{33}|, |h^u_{22}| |\epsilon|^2, |h^u_{11} - h^u_{12} h^u_{21}/h^u_{22}| |\epsilon|^6) v_1,
\]

\[
(m_b, m_s, m_d) \approx (|h^d_{33}| |\epsilon|^2, |h^d_{22}| |\epsilon|^4, |h^d_{11}| |\epsilon|^6) v_1,
\]

\[
|V_{us}| \approx \frac{h^d_{12}}{h^d_{22}} - \frac{h^d_{12}}{h^d_{22}} |\epsilon|^2,
\]

\[
|V_{cb}| \approx \frac{h^d_{13}}{h^d_{33}} - \frac{h^d_{13}}{h^d_{33}} |\epsilon|^2,
\]

\[
|V_{ub}| \approx \frac{h^d_{13}}{h^d_{33}} - \frac{h^d_{12} h^d_{23}}{h^d_{22} h^d_{33}} - \frac{h^u_{13}}{h^u_{33}} |\epsilon|^4,
\]

\[
\text{arg}(V_{ub}) \approx 4\text{arg}(\epsilon). \tag{46}
\]

Due to three Higgs doublets in our model, we have \(|v_1|^2 + |v_2|^2 + |v_3|^2 \approx (174 \text{ GeV})^2\). To satisfy this, we take \(v_1, v \sim 174/\sqrt{2} \text{ GeV}\). With this value for \(v_1\), we have fitted the expressions of Eq. \([46]\) to the following best fit values \([10]\).

\[
(m_t, m_c, m_u) = (172.76, 1.27, 2.16 \times 10^{-3}) \text{ GeV},
\]

\[
(m_b, m_s, m_d) = (4.18 \times 10^3, 93, 4.67) \text{ MeV},
\]

\[
(|V_{us}|, |V_{cb}|, |V_{ub}|) = (0.2245, 0.041, 0.00382),
\]

\[
\text{arg}(V_{ub}) = -1.196 \tag{47}
\]
After doing the above mentioned fitting, below we have given a sample set of numerical values with $|\epsilon| = 1/5.5$.

$$
(h_{33}^u, h_{22}^u, h_{11}^u - h_{12}^u h_{21}^u/h_{22}^u) \approx (1.4, 0.31, 0.49),
$$

$$
(h_{33}^d, h_{22}^d, h_{11}^d) \approx (1.03, 0.69, 1.05),
$$

$$
(h_{12}^d, h_{12}^u, h_{23}^u, h_{23}^d, h_{13}^u, h_{13}^d) \approx (1.49, -1.45, 0.69, -0.8, 1.12, 1.0),
$$

$$
\arg(\epsilon) \approx -0.3
$$

From the numerical values given above, we can see that the magnitudes of all Yukawa couplings are less than about 1.5. We have tried the numerical values with $|\epsilon| = 1/6$. However, in this case some of the Yukawa couplings can become larger than 2.0. Hence, with $|\epsilon| = 1/5.5$ and $O(1)$ Yukawa couplings, we can explain the quark masses and mixing pattern in our model. Since we expect new physics to appear around 1 TeV, we can take the cut-off scale of Eq.(44) to be $M \sim 1$ TeV. Now, for $|\epsilon| = 1/5.5$, we get $|\langle X \rangle| \sim 181$ GeV.

### 5.2 UV completion

Here we present the UV completion for our model in order to explain the origin of non-renormalizable terms of Eq. (44). To achieve this UV completion, we follow the works of Refs. [17, 23]. The idea of this UV completion is to explain non-renormalizable terms following from a theory which is renormalizable at a high scale. Hence, we assume our model is renormalizable at and above the scale $M$ and propose a flavor symmetry $U(1)_F$ which is exact above $M$. To generate non-renormalizable terms below $M$, we propose additional fields like flavons and VLQs, which transform under $U(1)_F$. The standard model quarks are charged under the $U(1)_F$ symmetry. But the Higgs doublets and $X$ field are singlets under $U(1)_F$. The $U(1)_F$ is spontaneously broken when the flavons acquire VEVs around $M$, which is also the mass scale of VLQs. Here, we can see that our model should respect the symmetry $CP \times K \times Z_2 \times Z_3 \times U(1)_F$ above the scale $M$. However, below $M$, after integrating the heavy VLQs and flavon fields, our model should generate non-renormalizable terms of Eq.(44), which respect the symmetry $CP \times K \times Z_2 \times Z_3$.

Under the $U(1)_F$, we denote the charges for $Q_{jL}$, $u_{jR}$ and $d_{jR}$ as $q_{jj}$, $u_{jj}$ and $d_{jj}$, respectively. We propose only two flavon fields, $F_1$ and $F_2$, whose charges under $U(1)_F$ are $f_1$ and $f_2$, respectively. Flavons are charged under $CP$ symmetry, but otherwise are singlets under $K \times Z_2 \times Z_3$. Under the $CP$ symmetry, flavons transform like the $X$ field of our model. Now, to generate non-renormalizable terms for up-type quarks of Eq. (44),
we introduce VLQs $K_{jL}$ and $K_{jR}$, which are color triplets and their hypercharges are same as that of right-handed singlet up-quarks. Analogous to $K_{jL}$ and $K_{jR}$, we introduce $G_{jL}$ and $G_{jR}$, which generate non-renormalizable terms for down-type quarks. The above VLQs are singlets under $SU(2)$ symmetry of standard model and $K \times Z_3$. These fields are charged under $Z_2$ symmetry. Under the $CP$ symmetry, they transform like the quark fields.

After describing the field content and their charge assignments in the UV completion of our model, below we explain the generation of non-renormalizable terms of Eq. (44). The $h_{33}^u$ term of Eq. (44) is renormalizable, which can be generated in our model by taking $q_{3f} = u_{3f}$. To generate $h_{32}^u$ term of Eq. (44), we consider the below invariant terms in the UV completion of our model.

\[ L_{u_{32}} = \bar{Q}_3 \tilde{\phi}_1 K_{1R} + F_1^* \tilde{K}_{1R} K_{1L} + X \tilde{K}_{1L} K_{2R} + F_2 \tilde{K}_{2R} K_{2L} + X \tilde{K}_{2L} u_{2R} + h.c. \]  

(49)

Since the terms in the above equation are invariant under $CP$ symmetry, the dimensionless Yukawa couplings should be real. These Yukawa couplings are $O(1)$, which we have not written explicitly here. The $U(1)_F$ charges for $K_{jL}, K_{jR}$ can be fixed in terms of corresponding charges of quarks and flavons in such a way that the above equation is invariant under $U(1)_F$. Similarly, the $Z_2$ charges for these VLQs can be assigned so that the above equation is invariant under $Z_2$. The $U(1)_F \times Z_2$ charges for VLQs of $K$-type are given in Eq. (57). Now, when the flavons acquire VEVs, the VLQs in Eq. (49) acquire masses of the order of $M$. After integrating these heavy VLQs, terms in Eq. (49) generate the $h_{32}^u$ term of Eq. (44).

By introducing more VLQs of $K$-type, the process described in the previous paragraph can be applied in order to generate other non-renormalizable terms of Eq. (44). Below we have given the invariant Lagrangians of the form $L_{ij}^u$, which generate the $h_{ij}^u$ term of Eq. (44), after integrating the heavy VLQs and flavons. The $U(1)_F \times Z_2$ charges for the VLQs in these Lagrangians can be seen in Eq. (57).

\[ L_{31}^u = \bar{Q}_{3L} \tilde{\phi}_1 K_{1R} + F_1^* \tilde{K}_{1R} K_{1L} + X \tilde{K}_{1L} K_{2R} + F_2 \tilde{K}_{2R} K_{2L} + X \tilde{K}_{2L} u_{1R} + h.c. \]  

(50)

\[ L_{23}^u = \bar{Q}_{2L} \tilde{\phi}_1 K_{5R} + M \tilde{K}_{5R} K_{5L} + X \tilde{K}_{5L} K_{6R} + F_1 \tilde{K}_{6R} K_{6L} + X \tilde{K}_{6L} u_{3R} + h.c. \]  

(51)

\[ L_{22}^u = \bar{Q}_{2L} \tilde{\phi}_1 K_{5R} + M \tilde{K}_{5R} K_{5L} + X \tilde{K}_{5L} K_{7R} + F_2 \tilde{K}_{7R} K_{7L} + X \tilde{K}_{7L} u_{2R} + h.c. \]  

(52)
\[ L_{21}^a = \bar{Q}_{2L} \tilde{\phi}_1 K_{5R} + M \bar{K}_{5R} K_{5L} + X \bar{K}_{5L} K_{7R} + F_2 \bar{K}_{7R} K_{7L} + X \bar{K}_{7L} K_{8R} + M \bar{K}_{8R} K_{8L} + X \bar{K}_{8L} K_{9R} + F_2 \bar{K}_{9R} K_{9L} + X \bar{K}_{9L} u_{1R} + h.c. \]  
\[ (53) \]

\[ L_{11}^a = \bar{Q}_{1L} \tilde{\phi}_1 K_{10R} + F_2 \bar{K}_{10R} K_{10L} + X \bar{K}_{10L} K_{11R} + F_2 \bar{K}_{11R} K_{11L} + X \bar{K}_{11L} K_{12R} + F_2 \bar{K}_{12R} K_{12L} + X \bar{K}_{12L} K_{13R} + F_2 \bar{K}_{13R} K_{13L} + X \bar{K}_{13L} u_{2R} + h.c.. \]  
\[ (54) \]

\[ L_{12}^a = \bar{Q}_{1L} \tilde{\phi}_1 K_{10R} + F_2 \bar{K}_{10R} K_{10L} + X \bar{K}_{10L} K_{11R} + F_2 \bar{K}_{11R} K_{11L} + X \bar{K}_{11L} K_{12R} + F_2 \bar{K}_{12R} K_{12L} + X \bar{K}_{12L} K_{16R} + F_1 \bar{K}_{16R} K_{16L} + X \bar{K}_{16L} u_{3R} + h.c.. \]  
\[ (55) \]

\[ L_{13}^a = \bar{Q}_{1L} \tilde{\phi}_1 K_{10R} + F_2 \bar{K}_{10R} K_{10L} + X \bar{K}_{10L} K_{11R} + F_2 \bar{K}_{11R} K_{11L} + X \bar{K}_{11L} K_{12R} + F_2 \bar{K}_{12R} K_{12L} + X \bar{K}_{12L} K_{16R} + F_1 \bar{K}_{16R} K_{16L} + X \bar{K}_{16L} u_{3R} + h.c.. \]  
\[ (56) \]

\[ U(1)_F : K_{1R} \rightarrow q_3 f, \quad K_{1L}, K_{2R} \rightarrow q_3 f + f_1, \quad K_{2L}, K_{3R} \rightarrow q_3 f + f_1 - f_2, \quad K_{3L}, K_{4L}, K_{4R} \rightarrow q_3 f + f_1 - 2f_2, \quad K_{5R}, K_{5L}, K_{7R} \rightarrow q_2 f, \quad K_{7L}, K_{8L}, K_{9R} \rightarrow q_2 f - f_2, \quad K_{6L} \rightarrow q_2 f - f_1, \quad K_{9L} \rightarrow q_2 f - 2f_2, \quad K_{10R} \rightarrow q_1 f, \quad K_{10L}, K_{11R} \rightarrow q_1 f - f_2, \quad K_{11L}, K_{12R} \rightarrow q_1 f - 2f_2, \quad K_{12L}, K_{13R}, K_{16R} \rightarrow q_1 f - 3f_2, \quad K_{13L}, K_{14R} \rightarrow q_1 f - 4f_2, \quad K_{14L}, K_{15R}, K_{15L} \rightarrow q_1 f - 5f_2, \quad K_{16L}, u_{3R} \rightarrow q_1 f - 3f_2 - f_1. \]  
\[ (57) \]

Since the Lagrangians of Eqs. (49) - (56) are invariant under \( U(1)_F \), we get relations among the \( U(1)_F \) charges of quarks and flavons. These relations can be consistently solved. Taking \( q_3 f, f_1 \) and \( f_2 \) as independent variables, the above mentioned relations can be expressed as

\[ q_{2f} = q_3 f + f_1, \quad q_{1f} = q_3 f + f_1 + 3f_2, \quad u_{3f} = q_3 f, \quad u_{2f} = q_3 f + f_1 - f_2, \quad u_{1f} = q_3 f + f_1 - 2f_2. \]  
\[ (58) \]

The procedure described above has been applied in order to generate non-renormalizable terms for down-type quarks of Eq. (44). In this case, we introduce VLQs \( G_{1R}, G_{1L} \), where
\(i = 1, \ldots, 30\). Below we have given invariant Lagrangians in the form of \(\mathcal{L}^{d}_{ij}\), which generate the \(h^{d}_{ij}\) term of Eq. (44), after integrating the heavy VLQs and flavons. Since these Lagrangians are invariant under the \(U(1)_{F} \times Z_{2}\), the charges of VLQs under this symmetry have been fixed in terms of corresponding charges of quarks and flavons. These charges are given in Eq. (68).

\[
\mathcal{L}^{d}_{33} = \bar{Q}_{3L}\phi_{1}G_{1R} + F_{1}\bar{G}_{1R}G_{1L} + X\bar{G}_{1L}G_{2R} + F_{1}\bar{G}_{2R}G_{2L} + X\bar{G}_{2L}d_{3R} + h.c. \tag{59}
\]

\[
\mathcal{L}^{d}_{32} = \bar{Q}_{3L}\phi_{1}G_{1R} + F_{1}\bar{G}_{1R}G_{1L} + X\bar{G}_{1L}G_{2R} + F_{1}\bar{G}_{2R}G_{2L} + X\bar{G}_{2L}G_{3R} + MG_{3R}G_{3L} + XG_{3L}G_{4R} + F_{2}\bar{G}_{4R}G_{4L} + XG_{4L}d_{2R} + h.c. \tag{60}
\]

\[
\mathcal{L}^{d}_{23} = \bar{Q}_{2L}\phi_{1}G_{5R} + F_{1}\bar{G}_{5R}G_{5L} + X\bar{G}_{5L}G_{6R} + F_{1}\bar{G}_{6R}G_{6L} + X\bar{G}_{6L}G_{7R} + F_{1}\bar{G}_{7R}G_{7L} + X\bar{G}_{7L}G_{8R} + MG_{8R}G_{8L} + \bar{G}_{8L}d_{3R} + h.c. \tag{61}
\]

\[
\mathcal{L}^{d}_{22} = \bar{Q}_{2L}\phi_{1}G_{5R} + F_{1}\bar{G}_{5R}G_{5L} + X\bar{G}_{5L}G_{6R} + F_{1}\bar{G}_{6R}G_{6L} + X\bar{G}_{6L}G_{7R} + F_{1}\bar{G}_{7R}G_{7L} + X\bar{G}_{7L}G_{9R} + F_{2}\bar{G}_{9R}G_{9L} + X\bar{G}_{9L}d_{2R} + h.c. \tag{62}
\]

\[
\mathcal{L}^{d}_{13} = \bar{Q}_{1L}\phi_{1}G_{10R} + F_{1}\bar{G}_{10R}G_{10L} + X\bar{G}_{10L}G_{11R} + F_{1}\bar{G}_{11R}G_{11L} + X\bar{G}_{11L}G_{12R} + F_{1}\bar{G}_{12R}G_{12L} + X\bar{G}_{12L}G_{13R} + F_{2}\bar{G}_{13R}G_{13L} + X\bar{G}_{13L}G_{14R} + F_{2}\bar{G}_{14R}G_{14L} + X\bar{G}_{14L}G_{15R} + F_{2}\bar{G}_{15R}G_{15L} + X\bar{G}_{15L}d_{3R} + h.c. \tag{63}
\]

\[
\mathcal{L}^{d}_{12} = \bar{Q}_{1L}\phi_{1}G_{10R} + F_{1}\bar{G}_{10R}G_{10L} + X\bar{G}_{10L}G_{11R} + F_{1}\bar{G}_{11R}G_{11L} + X\bar{G}_{11L}G_{12R} + F_{1}\bar{G}_{12R}G_{12L} + X\bar{G}_{12L}G_{13R} + F_{2}\bar{G}_{13R}G_{13L} + X\bar{G}_{13L}G_{14R} + F_{2}\bar{G}_{14R}G_{14L} + X\bar{G}_{14L}G_{16R} + MG_{16R}G_{16L} + X\bar{G}_{16L}d_{2R} + h.c. \tag{64}
\]

\[
\mathcal{L}^{d}_{11} = \bar{Q}_{1L}\phi_{1}G_{10R} + F_{1}\bar{G}_{10R}G_{10L} + X\bar{G}_{10L}G_{11R} + F_{1}\bar{G}_{11R}G_{11L} + X\bar{G}_{11L}G_{12R} + F_{1}\bar{G}_{12R}G_{12L} + X\bar{G}_{12L}G_{13R} + F_{1}\bar{G}_{13R}G_{13L} + X\bar{G}_{13L}G_{14R} + F_{1}\bar{G}_{14R}G_{14L} + X\bar{G}_{14L}G_{19R} + F_{2}\bar{G}_{19R}G_{19L} + X\bar{G}_{19L}d_{1R} + h.c. \tag{65}
\]

\[
\mathcal{L}^{d}_{21} = \bar{Q}_{2L}\phi_{1}M_{5R} + F_{1}\bar{G}_{5R}G_{5L} + X\bar{G}_{5L}G_{6R} + F_{1}\bar{G}_{6R}G_{6L} + X\bar{G}_{6L}G_{7R} + F_{1}\bar{G}_{7R}G_{7L} + X\bar{G}_{7L}G_{9R} + F_{2}\bar{G}_{9R}G_{9L} + X\bar{G}_{9L}G_{20R} + F_{2}\bar{G}_{20R}G_{20L} + X\bar{G}_{20L}G_{21R} + F_{2}\bar{G}_{21R}G_{21L} + X\bar{G}_{21L}G_{22R} + F_{2}\bar{G}_{22R}G_{22L} + X\bar{G}_{22L}G_{23R} + F_{1}\bar{G}_{23R}G_{23L} + X\bar{G}_{23L}G_{24R} + F_{1}\bar{G}_{24R}G_{24L} + X\bar{G}_{24L}G_{25R} + MG_{25R}G_{25L} + X\bar{G}_{25L}d_{1R} + h.c. \tag{66}
\]
\[ \mathcal{L}_{31}^d = \mathcal{L}_{31} \]

\[ + F_1 \mathcal{G}_{2R} G_{26L} + X \mathcal{G}_{26L} G_{27R} + F_1 \mathcal{G}_{27R} G_{27L} + X \mathcal{G}_{27L} G_{28R} + F_2 \mathcal{G}_{28R} G_{28L} + X \mathcal{G}_{28L} G_{29R} + F_2 \mathcal{G}_{29R} G_{29L} + X \mathcal{G}_{29L} G_{30R} + F_2 \mathcal{G}_{30R} G_{30L} + X \mathcal{G}_{30L} G_{31R} + F_2 \mathcal{G}_{31R} G_{31L} + X \mathcal{G}_{31L} d_{1R} + h.c.. \]  

(67)

**U(1)\(_F\)** :  
\[ G_{1R} \rightarrow q_{3f}, \ G_{1L}, G_{2R} \rightarrow q_{3f} - f_1, \ G_{2L}, G_{3R}, G_{3L}, G_{4R}, G_{25R} \rightarrow q_{3f} - 2f_1, \]
\[ G_{4L} \rightarrow q_{3f} - 2f_1 + f_2, \ G_{5R} \rightarrow q_{2f}, \ G_{5L}, G_{6R} \rightarrow q_{2f} - f_1, \]
\[ G_{6L}, G_{7R} \rightarrow q_{2f} - 2f_1, \ G_{7L}, G_{8R}, G_{8L}, G_{9R} \rightarrow q_{2f} - 3f_1 \]
\[ G_{9L}, G_{2O} \rightarrow q_{2f} - 3f_1 + f_2, \ G_{10R} \rightarrow q_{1f}, \ G_{10L}, G_{11R} \rightarrow q_{1f} - f_1, \]
\[ G_{11L}, G_{12R} \rightarrow q_{1f} - 2f_1, \ G_{12L}, G_{13R}, G_{17R} \rightarrow q_{1f} - 3f_1, \]
\[ G_{13L}, G_{14R} \rightarrow q_{1f} - 3f_1 - f_3, \ G_{14L}, G_{15R}, G_{16L}, G_{16R} \rightarrow q_{1f} - 3f_1 - 2f_2 \]
\[ G_{15L} \rightarrow q_{1f} - 3f_1 - 3f_3, \ G_{17L}, G_{18R} \rightarrow q_{1f} - 4f_1, \ G_{18L}, G_{19R} \rightarrow q_{1f} - 5f_1, \]
\[ G_{19L} \rightarrow q_{1f} - 5f_1 - f_2, \ G_{20L}, G_{21R} \rightarrow q_{2f} - 3f_1 + 2f_2, \]
\[ G_{21L}, G_{22R} \rightarrow q_{2f} - 3f_1 + 3f_2, \ G_{22L}, G_{23R} \rightarrow q_{2f} - 3f_1 + 4f_2, \]
\[ G_{23L}, G_{24R} \rightarrow q_{2f} - 4f_1 + 4f_2, \ G_{24L}, G_{25L}, G_{25R} \rightarrow q_{2f} - 5f_1 + 4f_2, \]
\[ G_{26L}, G_{27R} \rightarrow q_{3f} - 3f_1, \ G_{27L}, G_{28R} \rightarrow q_{3f} - 4f_1, \ G_{28L}, G_{29R} \rightarrow q_{3f} - 4f_1 + f_2, \]
\[ G_{29L}, G_{30R} \rightarrow q_{3f} - 4f_1 + 2f_2, \ G_{30L}, G_{31R} \rightarrow q_{3f} - 4f_1 + 3f_2, \]
\[ G_{31L} \rightarrow q_{3f} - 4f_1 + 4f_2. \]

\[ \mathcal{Z}_2 \] :  
\[ G_{1L}, G_{1R}, G_{3L}, G_{3R}, G_{5L}, G_{5R}, G_{7L}, G_{7R}, G_{10L}, G_{10R}, G_{12L}, G_{12R}, G_{14L}, G_{14R}, G_{18L}, G_{18R}, G_{20L}, G_{20R}, G_{22L}, G_{22R}, G_{24L}, G_{24R}, G_{26L}, G_{26R}, G_{28L}, G_{28R}, G_{30L}, G_{30R} \rightarrow \text{odd.} \]
\[ G_{2L}, G_{2R}, G_{4L}, G_{4R}, G_{6L}, G_{6R}, G_{8L}, G_{8R}, G_{9L}, G_{9R}, G_{11L}, G_{11R}, G_{13L}, G_{13R}, G_{15L}, G_{15R}, G_{16L}, G_{16R}, G_{17L}, G_{17R}, G_{19L}, G_{19R}, G_{21L}, G_{21R}, G_{23L}, G_{23R}, G_{25L}, G_{25R}, G_{27L}, G_{27R}, G_{29L}, G_{29R}, G_{31L}, G_{31R} \rightarrow \text{even.} \]

(68)

Since the Lagrangians of Eqs. (59) - (67) are invariant under \( U(1)_F \), nine relations exist among the \( U(1)_F \) charges of quarks and flavons. These relations can be solved consistently along with Eq. (58). After doing this, the \( U(1)_F \) charges of singlet down-type quarks can be expressed as

\[ d_{3f} = q_{3f} - 2f_1, \quad d_{2f} = q_{3f} - 2f_1 + f_2, \quad d_{1f} = q_{3f} - 4f_1 + 4f_2. \]

(69)

In this section, we have described our model for quark sector and also the UV completion to this model. As part of this whole construction, we have introduced extra fields
and symmetries into our model. We have summarized these fields and symmetries in Tab.

| additional field | role |
|------------------|------|
| $X$              | to generate hierarchy in quark masses and also $CP$ violation in quark sector |
| $F_1, F_2$       | to generate masses for VLOs in the UV completion of our model |
| $K_{iL}, K_{iR}(i = 1, \cdots, 16)$ | to generate effective Yukawa couplings for up-type quarks from UV completion our model |
| $G_{iL}, G_{iR}(i = 1, \cdots, 31)$ | to generate effective Yukawa couplings for down-type quarks from UV completion our model |

Table 3: Additional fields and symmetry, along with their roles, in the quark sector of our model.

6 Full scalar potential

In Sec. 3, we have given the analysis of scalar potential for the model described in Sec. 2. However, the model in Sec. 2 addresses problems related to masses of leptons. Later, within the framework of the model of Sec. 2, we have addressed hierarchy in the masses of quark fields in Sec. 5. While addressing the hierarchy in quark masses, we have introduced additional singlet scalar fields: $X, F_1, F_2$. These additional scalar fields can give extra terms with the doublet and triplet Higgses in the scalar potential. These extra terms may change the results derived in Sec. 3. For this purpose, in this section, we give the full scalar potential of our model. After minimizing the full scalar potential, we demonstrate that the above mentioned singlet scalar fields do not change the main conclusions of the analysis of Sec. 3. It is to remind that the following are the main conclusions of Sec. 3: (i) triplet Higgs acquire real VEV, (ii) VEVs of doublet Higgses $\Phi_{2,3}$ explain the hierarchy between $m_\mu$ and $m_\tau$.

The full scalar potential of our model is

$$V_{\text{full}} = V_{\text{inv}} + V_{X,F_1,F_2} + V_{K} + V'_{K}$$  \hspace{1cm} (70)
Here, $V_{X,F_1,F_2}$ is the invariant scalar potential of our model, arising due to the singlet fields $X, F_1, F_2$. $V'_K$ contain potential terms due to $X, F_1, F_2$, which violate $K$-symmetry explicitly. It is to remind here that minimization of $V_{inv} + V'_K$ has been discussed in Sec. 3. First we find a minimum after minimizing $V_{inv} + V_{X,F_1,F_2}$. Later we study the shift in this minimum due to the presence of $K$-violating terms. In this regard, the minimization of $V_{inv}$, after applying the $Z_3$ symmetry, has been studied in Sec. 3.3. Now, let us see if this minimization can be affected due to $V_{X,F_1,F_2}$. The form for this potential is given below.

$$V_{X,F_1,F_2} = -m_X^2(X^*X) - m_{F_1}^2(F_1^*F_1) - m_{F_2}^2(F_2^*F_2) + \lambda_X(X^*X)^2 + \lambda_{F_1}(F_1^*F_1)^2 + \lambda_{F_2}(F_2^*F_2)^2 + 3A(X^2 + X^{*2}) + B(X^4 + X^{*4}) + \lambda_X'(X^3X^* + X^{*3}X)
+ \lambda_{F_1,F_2}(F_1^*F_1)(F_2^*F_2) + \lambda_{F_1,X}(F_1^*F_1)(X^*X) + \lambda_{F_2,X}'(F_2^*F_2)(X^2 + X^{*2})
+ \lambda_{F_2,X}(F_2^*F_2)(X^*X) + \lambda_{F_1,X}'(F_2^*F_2)(X^2 + X^{*2}) + \lambda_{\phi_1,X}(\phi_1^*\phi_1)(X^*X)
+ \lambda_{\phi_2,X}(\phi_2^*\phi_2 + \phi_3^*\phi_3)(X^2 + X^{*2}) + \lambda_{\Delta X} Tr(\Delta^1\Delta)(X^*X)
+ \lambda_{\Delta X} Tr(\Delta^1\Delta)(X^2 + X^{*2}) + \lambda_{F_1,\phi_1}(F_1^*F_1)(\phi_1^*\phi_1) + \lambda_{F_2,\phi_1}(F_2^*F_2)(\phi_1^*\phi_1)
+ \lambda_{F_1,\phi_2}(F_1^*F_1)(\phi_2^*\phi_2 + \phi_3^*\phi_3) + \lambda_{F_2,\phi_2}(F_2^*F_2)(\phi_2^*\phi_2 + \phi_3^*\phi_3)
+ \lambda_{F_1,\Delta} Tr(\Delta^1\Delta)(F_1^*F_1) + \lambda_{F_2,\Delta} Tr(\Delta^1\Delta)(F_2^*F_2).$$

(71)

In the above equation, all the parameters are real due to hermiticity and $CP$ symmetry.

In Eq. (71), $F_1$ and $F_2$ appear in the form of $F_1^*F_1$ and $F_2^*F_2$, respectively. As a result of this, we can take the VEVs of $F_1$ and $F_2$ to be real. Hence, we can parameterize the VEVs for $X, F_1, F_2$ as

$$\langle X \rangle = v_X e^{i\theta_X}, \quad \langle F_1 \rangle = v_{f_1}, \quad \langle F_2 \rangle = v_{f_2}. \quad (72)$$

Here, $\theta_X$ is the phase in the VEV of $X$. After using the above VEVs and also Eq. (17) in Eq. (71), we get

$$\langle V_{X,F_1,F_2} \rangle \equiv 2v_X^2 [A + \lambda_X' v_X^2 + \lambda_{\phi_1,X} v_1^2 + \lambda_{\phi_2,X} v_2^2 + \lambda_{\Delta X} v^2 + \lambda_{F_1,X} v_{f_1}^2 + \lambda_{F_2,X} v_{f_2}^2] \cos 2\theta_X
+ 2B v_X^4 \cos 4\theta_X. \quad (73)$$

In the above equation, we have not written constant terms which do not contain phases of the VEVs of the fields. From the above equation we can see that $\theta_X$ do not mix with the phases in the VEVs of $\phi_{2,3}$ and $\Delta$. Hence, the minimization of $\langle V_{inv} \rangle$, which is presented in Sec. 3.3, is not affected due to $\langle V_{X,F_1,F_2} \rangle$. As a result of this, $\Delta$ can acquire real
VEV, and moreover, Eq. (34) is still valid. Now, from the minimization of \( V_{X,F_1,F_2} \) with respect to \( \theta_X \), we get

\[
\cos 2\theta_X = -\frac{1}{4Bv_X^2} [A + \lambda'_X v_X^2 + \lambda'_{\phi_1} v_1^2 + \lambda'_{\phi_2} v_2^2 + \lambda'_{\Delta X} v'^2 + \lambda'_{F_1 X} v'^2 + \lambda'_{F_2 X} v'^2]
\]

(74)

The above relation corresponds to the minimum for \( \theta_X \), provided the below condition is satisfied.

\[
16B^2v_X^4 \geq [A + \lambda'_X v_X^2 + \lambda'_{\phi_1} v_1^2 + \lambda'_{\phi_2} v_2^2 + \lambda'_{\Delta X} v'^2 + \lambda'_{F_1 X} v'^2 + \lambda'_{F_2 X} v'^2]_2.
\]

(75)

Here we have shown that \( X \) can acquire complex VEV. This is necessary to achieve, in order to generate the \( CP \) violation in quark sector, which is discussed in Sec. 5.

After minimizing \( \langle V_{inv} \rangle + \langle V_{X,F_1,F_2} \rangle \), we have shown that the minimum can be given by Eqs. (34) and (74). Now, this minimum can be shifted by small amount due to \( K \)-violating terms. Terms in \( V_{K'} \) are presented in Sec. 3. Below we give the form for \( V_{K'} \).

\[
V_{K'} = \delta\lambda_{\phi_2}(\phi_2^\dagger\phi_2)(X^*X) + \delta\lambda_{\phi_3}(\phi_3^\dagger\phi_3)(X^*X) + \delta\lambda_{\phi X}(\phi_2^\dagger\phi_2)(X^2 + X'^2) + \delta\lambda'_{\phi_2}(\phi_2^\dagger\phi_2)(X^2 + X'^2) + \delta\lambda'_{\phi_3}(\phi_3^\dagger\phi_3)(X^2 + X'^2) + \delta\lambda'_{\phi X}(\phi_2^\dagger\phi_2)(X^2 + X'^2) + \delta\lambda'_{\phi_2}(\phi_2^\dagger\phi_2)(X^2 + X'^2) + \delta\lambda'_{\phi_3}(\phi_3^\dagger\phi_3)(X^2 + X'^2) + \delta\lambda'_{\phi X}(\phi_2^\dagger\phi_2)(X^2 + X'^2) + \delta\lambda_{\phi_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_1}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{F_2}}(\phi_3^\dagger\phi_3)(F_1^2 + F_2^2) + \delta\lambda_{\phi'_{X}}(\phi_2^\dagger\phi_2)(F_1^2 + F_2^2).\]

(76)

All the parameters in the above equation are real due to either \( CP \) symmetry or hermiticity of the potential. These parameters should be small as compared to the parameters in \( V_{inv} + V_{X,F_1,F_2} \), since the above potential violates \( K \) symmetry by a small amount. We can see that the VEVs of \( X, F_1, F_2 \) in \( V_{K'} \) can give additional contribution to \( f_0 \) and \( f_\zeta \) of Eq. (36). Since the parameters of \( V_{K'} \) are small, we can notice that the contribution due to \( X, F_1, F_2 \) can be of the same order of the terms which are already obtained for \( f_0 \) and \( f_\zeta \) in Eq. (36). As a result of this, the hierarchy in \( m_\mu \) and \( m_\tau \) can be explained in our framework.

### 7 Phenomenology of our model

In Secs. 3 and 6 we have analyzed the minimum of the scalar potential of our model. One needs to study if this minimum corresponds to global or local minimum. Following
the studies made in Refs. [24], we expect some additional conditions to be imposed on
the parameters of our model in order for the minimum of the potential in this work to be
global. We work on the vacuum stability of our scalar potential in future.

The scalar fields, which are proposed in our model, are: three Higgs doublets, one
Higgs triplet and three singlet scalar fields. We can choose the $U(1)_F$ symmetry of our
model be gauged. As a result of this, after electroweak symmetry breaking, the following
fields remain in the theory: one doubly charged scalar, three singly charged scalars, seven
neutral scalars and five pseudo scalars. In the case I, which is described in Sec. 3, the scalar
fields belonging to the triplet Higgs can have masses around $10^{12}$ GeV. But otherwise, we
can choose the parameters in the scalar potential of our model in such a way that all the
scalar fields can have masses less than or about 1 TeV. In the case where the masses for the
scalar fields are less than 1 TeV, one can study the collider phenomenology. For this study,
one needs to know the interaction of the scalar fields with the standard model particles.
We can see that the scalar components of Higgs doublets and Higgs triplet have gauge
interactions. For the field $\phi_1$, it has Yukawa interactions with quark fields. Hence, the
scalars belonging to doublet and triplet Higgses can be produced at the LHC experiment
either via gauge or strong interactions. After production, subsequently they will decay
into standard model fields. In the case of singlet scalars $X, F_1, F_2$, the flavons have Yukawa
interactions with VLQs. Moreover, these flavons interact with doublet and triplet Higgses
in scalar potential. As for the $X$ field, it has Yukawa interactions containing a VLQ and
a right-handed quark field. Moreover, $X$ has interactions with Higgs fields in the scalar
potential. Since VLQs are color triplets, they can be produced at the LHC experiment via
strong interactions. From the decay of these VLQs, one can produce the above mentioned
singlet scalars in the LHC experiment. Studying the collider phenomenology of this model
is the beyond the scope of this work.

In the lepton sector of our model, the Yukawa couplings for charged leptons are di-
agonal. Hence, these Yukawa interactions are flavor conserving. On the other hand, the
Yukawa couplings for neutrinos are flavor violating. As a result of this, the singly and
doubly charged triplet Higgs fields can drive flavor violating decays of the form $\ell \rightarrow 3\ell'$
and $\ell \rightarrow \ell'\gamma$. However, it is stated in Sec. 2 that the Yukawa couplings for neutrinos are
suppressed by about $10^{-3}$. Hence, the branching ratios for the above mentioned decays
are suppressed even if the components of triplet Higgs can have masses around few hun-
dred GeV. As a result of this, constraints due to non-observation of charged lepton flavor
violating decays [10] are satisfied in our model.

The phenomenology of our model in quark sector is similar to that discussed in Ref.
In this regard, the $X$ field can cause flavor changing neutral currents at tree level in our model. As a result of this, there can be mass splitting in the $K^0 - \bar{K}^0$ and $D^0 - \bar{D}^0$ due to the mediation of $X$. We have estimated the above mentioned mass splittings, using the procedures described in Refs. [16, 17]. For this purpose, we define $\beta = \frac{v_1}{M}$. In our calculations we have taken $\epsilon \sim \beta = \frac{1}{5}$. We have chosen $h_{21}^d \sim 1$ and $h_{21}^u = -0.7$. Using the above set of parameters, we have found $\Delta m_K \approx 10^{-16}$ GeV and $\Delta m_D \approx 10^{-15}$ GeV. These numerical values are smaller than the corresponding current experimental values, which are as follows: $\Delta m_K = 3.5 \times 10^{-15}$ GeV and $\Delta m_D = 2.35 \times 10^{-14}$ GeV [10]. Hence, our model satisfies the constraints due to the mass splitting in $K^0 - \bar{K}^0$ and $D^0 - \bar{D}^0$.

By choosing $U(1)_F$ be gauged, the gauge boson corresponding to this symmetry, $Z'$, can be massive. The mixing between $Z - Z'$ is constrained to be very small. In this regard, phenomenology due to $Z'$ can be studied in our model. To know some phenomenology on $Z'$, see Refs. [25, 26].

8 Conclusions

In this work, we have proposed a model which explains the maximal values for $\theta_{23}$ and $\delta_{CP}$ in the lepton sector. To achieve this purpose we have introduced three Higgs doublets and one Higgs triplet. This model is based on $\mu - \tau$ reflection symmetry and type II seesaw mechanism. In this model, to explain the above observables, the VEV of triplet Higgs should be real. Moreover, due to $\mu - \tau$ reflection symmetry, the masses for muon and tau can be of the same order. After introducing the $K$ symmetry and explicit violation of it by a small amount, we have studied the minimization of scalar potential of our model. Thereafter, we have shown that the VEV of triplet Higgs can be real, apart from explaining the hierarchy in the muon and tau masses. In addition to predicting the above observables, the mass matrix for neutrinos in our model can also predict about neutrino mass ordering and smallness of $\theta_{13}$, if the elements of this matrix satisfy certain conditions. These conditions are given in Sec. 4. To explain these conditions, one has to propose a new mechanism in addition to $CP$ symmetry. Although we do not have a mechanism to explain all the conditions given in Sec. 4, we have attempted to give one mechanism to explain condition (i) for the case of NO. This mechanism is presented in the Appendix.

Since in our model three Higgs doublets exist, we have studied the Yukawa couplings between quarks and these doublets by proposing $CP$ transformations for quark fields. After employing a certain texture for these Yukawa couplings, we have consistently ex-
explained the quark masses and mixing pattern. To employ this texture in the quark sector of our model, we have introduced additional fields like VLQs and singlet scalars. One of these singlet scalars should acquire complex VEV in order to generate the $CP$ violating phase in quark sector. Finally, we have analyzed the scalar potential containing the singlet scalars and the above mentioned Higgs fields. After this analysis, we have demonstrated that masses and mixing pattern in lepton and quark sectors can be consistently explained.

Appendix: A model for achieving condition (i) in the case of NO

In Sec. 4, we have described some conditions on the elements of neutrino mass matrix, which can predict the case of NO or IO and also about the smallness of $\theta_{13}$. These conditions are purely phenomenological and cannot be achieved with just the $CP$ symmetry. Additional mechanism should be proposed in order to satisfy these conditions. For the case of NO, condition (i) can be achieved if we propose an extra $U(1)_S$ symmetry and the singlet scalar fields $S_1, S_2$. Under the $U(1)_S$ symmetry, we consider the following charge assignments, where $l$ is some non-zero rational number:

$D_e L \rightarrow l, S_1 \rightarrow -2l, S_2 \rightarrow -l.$

Under $U(1)_S$, $e_R$ should transform like $D_{eL}$ and rest of the fields in our model are singlets. $S_1, S_2$ transform under the $CP$ symmetry as $S_1, S_2 \rightarrow S_1^*, S_2^*$. With the above charge assignments, the Yukawa terms for $D_{eL}$ in Eq. (8) are forbidden. Now, these terms can be effectively generated by the following invariant terms.

$$Y_e \frac{S_1}{M} D_{eL} i \sigma_2 \Delta D_{eL} + Y_\mu \frac{S_2}{M} D_{eL} i \sigma_2 \Delta D_{\mu L} + Y_\tau \frac{S_2}{M} \bar{D}_{eL} i \sigma_2 \Delta D_{\tau L} + h.c..$$  \hspace{1cm} (77)

Here, $M$ is a mass scale which is analogous to that in the quark sector Lagrangian of Eq. (44). The above non-renormalizable terms can be generated by studying the UV completion for these terms, where one can propose heavy vector-like leptons whose masses are around $M$. The process of this UV completion is analogous to what we describe in Sec. 5.2. In order for Eq. (77) to be invariant under $CP$ symmetry, $Y_e$ should be real and $Y_\mu = Y_\tau^*$. After $U(1)_S$ symmetry is spontaneously broken, terms in Eq. (77) effectively generate the Yukawa couplings $Y_{ee}, Y_{e\mu}, Y_{e\tau}$. Moreover, by taking $\langle S_1, S_2 \rangle \sim 0.1$, condition (i) for the case of NO is satisfied. Since $Y_{ee}$ is real and $Y_{e\mu} = (Y_{e\tau})^*$, $\langle S_1 \rangle$ and $\langle S_2 \rangle$ should be real. We justify this statement by studying the scalar potential for these fields.

The scalar potential, which is invariant under $CP \times Z_2 \times Z_3 \times K \times U(1)_F \times U(1)_S$
and containing $S_{1,2}$ can be written as

$$V_{S_{1,2}} = -m_{S_{1}}^2(S_{1}^*S_{1}) - m_{S_{2}}^2(S_{2}^*S_{2}) + \lambda_{S_{1}}(S_{1}^*S_{1})^2 + \lambda_{S_{2}}(S_{2}^*S_{2})^2 + \lambda_{S_{1}S_{2}}(S_{1}^*S_{1})(S_{2}^*S_{2})$$
$$+ \lambda_{\phi_{S_{1}}}((\phi_{S_{1}}^*\phi_{S_{1}}))(S_{1}^*S_{1}) + \lambda_{\phi_{S_{2}}}((\phi_{S_{2}}^*\phi_{S_{2}} + \phi_{S_{3}}^*\phi_{S_{3}}))(S_{1}^*S_{1}) + \lambda_{\phi_{S_{2}}}((\phi_{S_{1}}^*\phi_{S_{1}}))(S_{2}^*S_{2})$$
$$+ \lambda_{\phi_{S_{2}}}((\phi_{S_{2}}^*\phi_{S_{2}} + \phi_{S_{3}}^*\phi_{S_{3}}))(S_{2}^*S_{2}) + \lambda_{\Delta_{S_{1}}}Tr(\Delta_{S_{1}}^\dagger\Delta_{S_{1}})(S_{1}^*S_{1}) + \lambda_{\Delta_{S_{2}}}Tr(\Delta_{S_{2}}^\dagger\Delta_{S_{2}})(S_{2}^*S_{2})$$
$$+ \lambda_{S_{1}X}(S_{1}^*S_{1})(X^\dagger X) + \lambda_{S_{2}X}(S_{2}^*S_{2})(X^\dagger X) + \lambda_{S_{1}X}((S_{1}^*S_{1})(X^2 + X^\dagger)^2$$
$$+ \lambda_{S_{2}X}((S_{2}^*S_{2})(X^2 + X^\dagger)^2) + \lambda_{F_{1}S_{1}}((F_{1}^\dagger F_{1}))(S_{1}^*S_{1}) + \lambda_{F_{1}S_{2}}((F_{1}^\dagger F_{1}))(S_{2}^*S_{2})$$
$$+ \lambda_{F_{2}S_{1}}((F_{2}^\dagger F_{2}))(S_{1}^*S_{1}) + + \lambda_{F_{2}S_{2}}((F_{2}^\dagger F_{2}))(S_{2}^*S_{2}) + a(S_{1}^*S_{2} + S_{2}^*S_{1}).$$

(78)

Now, the $K$-violating terms containing $S_{1,2}$ can be written as

$$V''_{K} = \delta\lambda_{\phi_{S_{1}}}((\phi_{S_{1}}^*\phi_{S_{1}}))(S_{1}^*S_{1}) + \delta\lambda_{\phi_{S_{1}}}((\phi_{S_{1}}^*\phi_{S_{1}}))(S_{1}^*S_{1}) + \delta\lambda_{\phi_{S_{2}}}((\phi_{S_{2}}^*\phi_{S_{2}}))(S_{2}^*S_{2})$$
$$+ \lambda_{\phi_{S_{2}}}((\phi_{S_{2}}^*\phi_{S_{2}}))(S_{2}^*S_{2}) + i\delta\lambda_{25}(\phi_{S_{3}}^*\phi_{S_{3}} - \phi_{S_{3}}^*\phi_{S_{3}})(S_{3}^*S_{3})$$
$$+ i\delta\lambda_{26}(\phi_{S_{3}}^*\phi_{S_{3}} - \phi_{S_{3}}^*\phi_{S_{3}})(S_{3}^*S_{3}).$$

(79)

In the above two potentials, all the parameters are real due to hermiticity or $CP$ symmetry. Analogous to what we describe in Sec. 6, we can see that the potential terms in $V_{S_{1,2}}$ and $V''_{K}$ do not alter the main conclusions of Sec. 3.3. This means, even with the fields $S_{1,2}$, $\Delta$ acquires real VEV, the VEVs of $\phi_{2,3}$ explain the hierarchy in $m_{\mu}$ and $m_{\tau}$.

Here we demonstrate that $S_{1,2}$ can acquire real VEVs. In this regard, we can see that the last term of $V_{S_{1,2}}$ can only contain the phases in the VEVs of $S_{1,2}$. Hence, after parameterizing $\langle S_{1} \rangle = v_s e^{\theta_{s_1}}$ and $\langle S_{1} \rangle = v_s e^{\theta_{s_2}}$, we get

$$\langle V_{S_{1,2}} \rangle \equiv +2av_{s_1}v_{s_2}^2 \cos(2\theta_{s_2} - \theta_{s_1})$$

(80)

The above term has a minimum at $2\theta_{s_2} - \theta_{s_1} = 0$ when $av_{s_1} < 0$. To satisfy this minimum we can choose $\theta_{s_1} = \theta_{s_2} = 0$. Now, the minimum at $\theta_{s_1} = \theta_{s_2} = 0$ cannot be shifted by the terms of $V''_{K}$, since $S_{1,2}$ appear in the form of $S_{1}^*S_{1}$ and $S_{2}^*S_{2}$ in $V''_{K}$. Hence, there exist a parameter region where the VEVs of $S_{1}$ and $S_{2}$ are real in this model.

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