Quantum spin models with electrons in Penning traps

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(Dated: June 21, 2008)

Abstract

We propose a scheme to engineer an effective spin Hamiltonian starting from a system of electrons confined in micro-Penning traps. By means of appropriate sequences of electromagnetic pulses, alternated to periods of free evolution, we control the shape and strength of the spin-spin interaction. Moreover, we can modify the effective magnetic field experienced by the particle spin. This procedure enables us to reproduce notable quantum spin systems, such as Ising and XY models. Thanks to its scalability, our scheme can be applied to a fairly large number of trapped particles within the reach of near future technology.

PACS numbers: 03.65.-w, 03.67.Ac, 03.67.Lx, 75.10.Jm
I. INTRODUCTION

Single electrons confined in Penning traps may represent a valid, experimentally viable system for the implementation of a quantum processor [1, 2, 3]. Our proposals have been encouraged by the astonishing results obtained in high precision experiments with a single electron [4, 5, 6, 7, 8] and by the advances in trapping technology, from micro-traps [9] to scalable open planar Penning traps [10, 11]. In this spirit, it has also been put forward how to realize a quantum information channel, based on interacting spin chains, by means of trapped electrons [12, 13].

In this paper we focus on a linear array of electrons, each one confined in a micro-Penning trap. Our aim is to prove that, from the same physical system, we can derive a variety of interacting spin models. In particular, we show how to design and control the relevant terms in the effective spin Hamiltonian. As a result a system of trapped electrons can be exploited to study the dynamics of a wide range of quantum spin models. We recall that these models are very important for the understanding of the rich phenomenology observed in several quantum many-body systems, such as quantum magnets and high temperature superconductors. Moreover quantum spin systems are able to exhibit quantum phase transitions [14]. To this end, it is critical to control and vary system parameters like the applied magnetic field and the spin-spin coupling strength. Our method to shape the effective spin-spin interaction employs sequences of electromagnetic pulses alternated to periods of free evolution. This technique is similar to the refocusing schemes used in nuclear magnetic resonance (NMR) experiments and relies, from the theoretical point of view, on the average Hamiltonian theory [15]. We point out that a system of trapped electrons presents several advantages over NMR implementations. The most important ones are scalability and the possibility to independently adjust the values of relevant quantities, like the spin precession frequencies and the spin-spin coupling. Indeed their values depend on external parameters such as the magnetic field gradient, the inter-particle distance and the voltage applied to the trap electrodes [3, 12].

Also other systems, like linear or planar arrays of trapped ions, enjoy some of these properties and, therefore, it has been proposed to use them as a quantum simulator for interacting spin chains [16, 17, 18]. However working with trapped electrons we naturally have a system of spin one-half particles, without the need for artificially creating an effec-
tive two-level system. Another difference between trapped ions and electrons relies in the typical resonance frequencies. Ions are controlled by means of a sophisticated laser setup, while trapped electrons are manipulated by microwave or radio-frequency fields. In this respect trapped electrons can benefit from the same technology already developed for NMR spectroscopy.

In this paper we consider both the case of electrons with the same spin precession frequency \cite{12} and the case of electrons with different spin precession frequencies \cite{3}. The frequency addressability, which is necessary to manipulate specific particles in the array, is obtained with the insertion of a magnetic field gradient. However this condition is required only to modify the interaction range and topology. In the case of electrons with the same spin precession frequency, we prove that, by flipping the spin state twice, we can effectively reduce or even cancel the spin dynamics due to the external uniform magnetic field. This way the effective spin system is subjected to a weaker magnetic field, whose intensity can go down to zero, without affecting the overall trap stability. The same resonant electromagnetic field, used to flip the spin, is able to produce coherent superpositions of the two spin states by adjusting its phase and duration. These operations are the building blocks of specific pulse sequences that allow to engineer the effective spin Hamiltonian. By iterating such pulse sequences, we obtain various interesting spin Hamiltonians such as the Ising model or the \textit{XY} model. In addition, if we want to customize the interaction range and coordination number, we should apply similar sequences of pulses to selected subsets of spins in the array. The resulting spin system can exhibit a nearest neighbor (NN) as well as a long range interaction. The number of pulses in each sequence is relatively small and, most notably, does not depend on the number of spins in the array, thus making our procedure scalable.

The paper is organized as follows. In Sec. \textbf{II} we briefly present the system of trapped electrons and review the derivation of the effective spin-spin interaction. In Sec. \textbf{III} we describe how to prepare and manipulate, with an additional oscillating magnetic field, the spin state of each electron in the array. In Sec. \textbf{IV} we show how to engineer the spin Hamiltonian by applying appropriate electromagnetic pulse sequences, that allow to control the strength and the range of the interaction. The capability of our technique to reproduce a given Hamiltonian is analyzed in Sec. \textbf{V}. Finally in Sec. \textbf{VI} we summarize our results and discuss future perspectives. The more technical details, concerning the design of the pulse sequences and the estimate of fidelity, are reported, respectively, in Appendices \textbf{A} and \textbf{B}.
II. ARRAY OF TRAPPED ELECTRONS

Let us consider a system of \( N \) electrons confined in an array of micro-Penning traps in the presence of linear magnetic gradients. The Hamiltonian of the system can be written as

\[
H = \sum_{i=1}^{N} H_{i}^{NC} + \sum_{i>j}^{N} H_{i,j}^{C},
\]

where

\[
H_{i}^{NC} = \frac{(p_i - eA_i)^2}{2m_e} + eV_i - \frac{ge\hbar}{4me} \sigma_i \cdot B_i
\]

represents the single electron dynamics inside a trap and

\[
H_{i,j}^{C} = \frac{e^2}{4\pi\varepsilon_0 |r_i - r_j|}
\]

describes the Coulomb interaction between electrons \( i \) and \( j \). In Eqs. (2) and (3) \( m_e, e, g, \) and \( \sigma_i \) are, respectively, the electron mass, charge, gyromagnetic factor, and Pauli spin operators. We assume that the micro-traps are aligned along the \( x \) axis and that \( x_{i,0} \) is the position of the center of the \( i \)-th trap. The electrostatic potential

\[
V_i(x_i, y_i, z_i) \equiv V_0 \frac{z_i^2}{\ell^2} - \frac{[(x_i - x_{i,0})^2 + y_i^2]/2}{\ell^2}
\]

is the usual quadrupole potential of a Penning trap, where \( V_0 \) is the applied potential difference between the trap electrodes and \( \ell \) is a characteristic trap length. The magnetic field

\[
B_i \equiv -\frac{b}{2} [(x_i - x_{i,0})i + y_i j] + (B_{0,i} + b z_i)k
\]

is the sum of the trapping magnetic field \( B_{0,i}k \), providing the radial confinement, with a local linear magnetic gradient \( b \) around the \( i \)-th trap. The associated vector potential

\[
A_i \equiv \frac{1}{2} (B_{0,i} + b z_i) [-y_i i + (x_i - x_{i,0}) j]
\]

preserves the cylindrical symmetry of the unperturbed trapping field.

Following the approach described in \[3, 4, 12\] the Hamiltonian, Eq. (2), of a single electron can be written as

\[
H_i^{NC} \simeq -\hbar \omega_{m,i} a_{m,i}^\dagger a_{m,i} + \hbar \omega_{c,i} a_{c,i}^\dagger a_{c,i} + \hbar \omega_{z,i} a_{z,i}^\dagger a_{z,i} + \frac{\hbar}{2} \omega_{s,i} \sigma_i^z + \frac{g}{4} \varepsilon \hbar \omega_z \left( a_{z,i} + a_{z,i}^\dagger \right) \sigma_i^+ - \frac{g}{4} \varepsilon \hbar \omega_z \sqrt{\frac{\omega_z}{\omega_{c,i}}} \left( \sigma_i^+ a_{c,i} + \sigma_i^- a_{c,i}^\dagger \right) \]

(7)
where the annihilation operators $a_{m,i}, a_{c,i}, a_{z,i}$ refer, respectively, to the magnetron, cyclotron and axial oscillators of the $i$-th electron and $\sigma_i^{(\pm)} \equiv (\sigma_i^x \pm i\sigma_i^y)/2$. The frequencies of the different electron motions are $\omega_{m,i} \simeq \omega_z^2/(2\omega_{c,i})$, $\omega_{c,i} \simeq (|e|B_{0,i}/m_e) - \omega_{m,i}$, $\omega_z = \sqrt{2eV_0/(m_e\ell^2)}$ and $\omega_{s,i} \equiv g|e|B_{0,i}/(2m_e)$. The Hamiltonian (7) has been obtained under the assumptions $\omega_{m,i} \ll \omega_z \ll \omega_{c,i}$ and $b|z_i|/B_{0,i} \ll 1$. We also assume that the cyclotron motion is in the ground state and the amplitude of the magnetron motion is sufficiently small (axialization) [19]. The dimensionless parameter

$$\varepsilon \equiv \frac{|e|b}{m_e\omega_z} \sqrt{\frac{\hbar}{2m_e\omega_z}}$$

(8)

represents the coupling, due to the magnetic gradient, between internal and external degrees of freedom of the particle.

Similarly, if the oscillation amplitude of the electrons is much smaller than the inter-trap distance, the part of the Hamiltonian describing the Coulomb interaction can be written as [12]

$$H_{C,i,j} \simeq \frac{\hbar\xi_{i,j}(a_{z,i} + a_{z,j}^\dagger)(a_{z,j} + a_{z,j}^\dagger)}{\omega_{c,i}} - \frac{\hbar\xi_{i,j}}{\omega_{c,i}} \left( a_{c,i}a_{c,j}^\dagger + a_{c,j}^\dagger a_{c,i} \right),$$

(9)

where $\xi_{i,j} \equiv e^2/(8\pi\epsilon_0m_e\omega_zd_{i,j}^3)$ with $d_{i,j}$ being the distance between the $i$-th and $j$-th particle. Now we apply to the system Hamiltonian the unitary transformation [20]

$$S = \sum_{i=1}^N \frac{g}{4} \varepsilon \left[ \sigma_i^x(a_{z,i}^\dagger - a_{z,i}) + \frac{\omega_z}{\omega_{a,i}} \left( \sigma_i^{(-)}a_{c,i}^\dagger - \sigma_i^{(+)}a_{c,i} \right) \right],$$

(10)

with $\omega_{a,i} \equiv \omega_{s,i} - \omega_{c,i}$. This transformation formally removes, to the first order in $\varepsilon$, the interaction between the internal and the external degrees of freedom in Hamiltonian (7) and, at the same time, introduces a coupling between the spin motions of different electrons. Consequently the spin part of the system Hamiltonian can be recast as [12]

$$H_s \simeq \sum_{i=1}^N \hbar \varepsilon \omega_{s,i} \sigma_i^z + \frac{\hbar}{2} \sum_{i>j}^N \left( 2J_{i,j}^x \sigma_i^x \sigma_j^x - J_{i,j}^{xy} \sigma_i^x \sigma_j^y - J_{i,j}^{xy} \sigma_i^y \sigma_j^y \right),$$

(11)

where

$$J_{i,j}^x = \left( \frac{g}{2} \right)^2 \xi_{i,j} \varepsilon^2 = \left( \frac{g}{2} \right)^2 \frac{\hbar e^4}{16\pi\epsilon_0m_e^4} \frac{b^2}{\omega_{a,i}^2d_{i,j}^3},$$

(12)

$$J_{i,j}^{xy} = \left( \frac{g}{2} \right)^2 \xi_{i,j} \varepsilon^2 \frac{\omega_z^4}{4\omega_{a,i}\omega_{c,i}} = \left( \frac{g}{2} \right)^2 \frac{\hbar e^4}{64\pi\epsilon_0m_e^4} \frac{b^2}{\omega_{a,i}^2\omega_{c,i}d_{i,j}^3}.$$ 

(13)
The effective spin Hamiltonian (11) exhibits a long range interaction between all the particles in the chain. The coupling strength decreases with the third power of the distance between particles, i.e. with a dipole-like behavior. Moreover, $J_{i,j}^z$ and $J_{i,j}^{xy}$ depend, respectively, on the axial frequency and the cyclotron and anomaly frequencies. Since the trapping frequencies form a well defined hierarchy, the coupling in the longitudinal and transverse direction can be utterly different. For example, for typical experimental values of the cyclotron and axial frequencies, such as $\omega_c/2\pi \simeq 100$ GHz and $\omega_z/2\pi \simeq 100$ MHz, the ratio $J_{i,j}^{xy}/J_{i,j}^z$ is less than $10^{-6}$. Therefore, for practical purposes $J_{i,j}^{xy}$ is often negligible with respect to $J_{i,j}^z$. In particular, this is true when the difference between the spin frequencies of different particles is much larger than their $xy$ spin-spin coupling strength. In this case the spin Hamiltonian reduces to

$$H_s \simeq \frac{\hbar}{2} \sum_{i=1}^{N} \omega_s, i \sigma_i^z + \hbar \sum_{i>j}^{N} J_{i,j}^z \sigma_i^z \sigma_j^z. \quad (14)$$

In Hamiltonian Eq. (14) we have used the rotating wave approximation (RWA) to neglect the interactions between spins along the $x$ and $y$ directions, since they give rapidly rotating terms. The Hamiltonian (14) is, therefore, similar to the nuclear spin Hamiltonian of the molecules used to perform NMR experiments [15]. However in NMR systems the spin frequency differentiation and the spin-spin couplings are determined by the chemical nature of the molecules, whereas in our system they depend on the value of the applied fields, that are under control of the experimenter.

III. SPIN STATE MANIPULATION

In this section we describe how to prepare and manipulate the spin state with an external oscillatory field. Let us consider a magnetic field $b_p(t)$ oscillating in the $xy$ plane with frequency $\omega$ and phase $\theta$ such that

$$b_p(t) = b_p[i \cos(\omega t + \theta) + j \sin(\omega t + \theta)]. \quad (15)$$

If we add this field to the system, the spin Hamiltonian, Eq. (11), becomes (here and in the rest of the paper we set $\hbar = 1$)

$$H \simeq \frac{1}{2} \sum_{j=1}^{N} \omega_{s,j} \sigma_j^z + \frac{\chi}{2} \sum_{j=1}^{N} [\sigma_j^{(+)} e^{-i(\omega t + \theta)} + \sigma_j^{(-)} e^{i(\omega t + \theta)}], \quad (16)$$
with $\chi \equiv g|e|b_p/(2m_e)$. In deriving the Hamiltonian (16), we assumed that the interaction between the electrons and the oscillating magnetic field is much stronger than the spin-spin coupling. Hence, the terms in Eq. (11) proportional to $J_{ij}^z$ and $J_{ij}^{xy}$ can be neglected. In the case of a system with spin frequency differentiation the field (15), applied for an appropriate time $t$ with frequency $\omega = \omega_{s,j}$, affects only the spin states of the resonant $j$-th electron

$$\ket{\downarrow}_j \rightarrow e^{i(\omega_{s,j}/2)t} \cos \left(\frac{\chi t}{2}\right) \ket{\downarrow}_j - ie^{-i(\omega_{s,j}/2)t} \sin \left(\frac{\chi t}{2}\right) \ket{\uparrow}_j, \quad (17)$$

$$\ket{\uparrow}_j \rightarrow e^{-i(\omega_{s,j}/2)t} \cos \left(\frac{\chi t}{2}\right) \ket{\uparrow}_j - ie^{i(\omega_{s,j}/2)t} \sin \left(\frac{\chi t}{2}\right) \ket{\downarrow}_j. \quad (18)$$

Without spin frequency differentiation, the single qubit addressing with microwave radiation is, of course, no longer possible. Therefore when all the spins have the same precession frequency $\omega_s$, a single resonant pulse suffices to produce the evolution of Eqs. (17) and (18) for each particle in the array. From Eqs. (17) and (18) we see that by changing duration and phase of the applied pulse we can prepare and manipulate at will the spin states of the trapped electrons. In particular, if we apply a pulse for a time $\bar{t} = \pi/\chi$ with $\theta = 0$, we can flip the spin state of each particle

$$\ket{\downarrow}_j \rightarrow -ie^{-i(\omega_{s,j}/2)\bar{t}} \ket{\uparrow}_j, \quad (19)$$

$$\ket{\uparrow}_j \rightarrow -ie^{i(\omega_{s,j}/2)\bar{t}} \ket{\downarrow}_j. \quad (20)$$

We define this transformation as

$$F \equiv \bigotimes_{j=1}^N \{-i[\sigma_j^{(+)} e^{-i(\omega_{s,j}/2)\bar{t}} + \sigma_j^{(-)} e^{i(\omega_{s,j}/2)\bar{t}}]\}. \quad (21)$$

It is not difficult to verify that the inverse transformation $F^{-1}$ is obtained with a pulse of the same duration $\bar{t}$ but with phase $\pi$.

If we move to the interaction picture (IP) with respect to the Hamiltonian $\sum_{i=1}^N (\omega_{s,i}/2)\sigma_i^z$, the system evolution is given by Eqs. (17) and (18) with $\omega_{s,j} = 0$. Consequently, the spin flip operation, Eqs. (19) and (20), turns into

$$\ket{\downarrow}_j \rightarrow -i\ket{\uparrow}_j, \quad (22)$$

$$\ket{\uparrow}_j \rightarrow -i\ket{\downarrow}_j. \quad (23)$$

The above transformations correspond to the application of the operator $-i\sigma_j^x$. In a similar way a pulse applied for the time $\bar{t}$ with $\theta = \pi/2$ produces a transformation corresponding
to the application of \(-i\sigma^y_j\). Furthermore, always working in IP, if the pulse is applied for a time \(\bar{t}/2\), we can obtain the pseudo-Hadamard operations

\[
G_x \equiv \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2}} \left( \mathbb{1} - i\sigma^x_j \right) \quad \text{for } \theta = 0, \tag{24}
\]

\[
G_x^\dagger \equiv \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2}} \left( \mathbb{1} + i\sigma^x_j \right) \quad \text{for } \theta = \pi, \tag{25}
\]

\[
G_y \equiv \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2}} \left( \mathbb{1} - i\sigma^y_j \right) \quad \text{for } \theta = \frac{\pi}{2}, \tag{26}
\]

\[
G_y^\dagger \equiv \bigotimes_{j=1}^{N} \frac{1}{\sqrt{2}} \left( \mathbb{1} + i\sigma^y_j \right) \quad \text{for } \theta = -\frac{\pi}{2}. \tag{27}
\]

The coherent superposition of the spin states \(|\uparrow\rangle, |\downarrow\rangle\) for each particle can be achieved with a single multi-frequency pulse. Hence, an appropriate choice of the frequency, duration and phase of the pulses allows for performing, apart from irrelevant phase factors, single qubit operations on each spin of the array.

**IV. ENGINEERING THE SPIN HAMILTONIAN**

In this section we show that, by using the additional magnetic field (15), we can also adjust and control the form of the effective spin Hamiltonian, starting from the models given by Eqs. (11) and (14). This is achieved by applying to the system specific sequences of pulses alternated to periods of free evolution. Our approach is inspired to the refocusing schemes used in NMR experiments [15]. Similarly to this technique a key point is the choice of the different time scales. Spin operations, operated by means of pulses, should be virtually instantaneous with respect to the free evolution of the system. Therefore, the pulse duration should be much shorter than the free evolution time.

**A. Tuning of the effective magnetic field**

The spin Hamiltonian, Eq. (11), in the case of spins with the same precession frequency can be recast as

\[
H_s \simeq H_0 + H_c, \tag{28}
\]
where

\[ H_0 \equiv \sum_{i=1}^{N} \frac{\omega_s}{2} \sigma_i^z, \quad (29) \]

\[ H_c \equiv \frac{1}{2} \sum_{i>j}^{N} \left( 2J_{ij}^z \sigma_i^z \sigma_j^z - J_{ij}^y \sigma_i^x \sigma_j^x - J_{ij}^y \sigma_i^y \sigma_j^y \right). \quad (30) \]

In the following we shall prove that, by sending resonant pulses of the kind of Eq. (15), it is possible to reduce or even cancel the effects on the spin dynamics of the Hamiltonian term \( H_0 \). This result corresponds to an effective modulation of the external magnetic field, without affecting the trapping stability of the whole set up.

In particular, by applying a sequence consisting of a pulse producing the spin flip transformation \( F \), Eq. (21), followed by a period of free evolution \( t \) and by a pulse producing the inverse transformation \( F^{-1} \), we can change the sign of the Hamiltonian term \( H_0 \)

\[ F^{-1} e^{-iHs\ell} F = \exp[-i(-H_0 + H_c)\ell]. \quad (31) \]

To prove Eq. (31) we use the identity

\[ F^{-1} e^{-iHs\ell} F = \exp[-i(F^{-1}H_0F + F^{-1}H_cF)t]. \quad (32) \]

Now we have

\[ F^{-1}H_0F = \sum_{j=1}^{N} [\sigma_j^{(+)} e^{-i(\omega_s/2)\ell} + \sigma_j^{(-)} e^{i(\omega_s/2)\ell}] \left( \frac{\omega_s}{2} \sigma_j^z \right) [\sigma_j^{(+)} e^{-i(\omega_s/2)\ell} + \sigma_j^{(-)} e^{i(\omega_s/2)\ell}] \]

\[ = - \sum_{j=1}^{N} \frac{\omega_s}{2} \sigma_j^z = -H_0. \quad (33) \]

The identity \( F^{-1}H_cF = H_c \) follows from the commutation relation \([H_c,F] = 0\), which can be verified with some algebra. Moreover we observe that \([H_0,H_c] = 0\), because the interaction Hamiltonian preserves the total magnetization \( \sum_{i=1}^{N} \sigma_i^z \). From this last consideration and from Eq. (31) we find

\[ F^{-1} e^{-iHs\ell_1} F e^{-iHs\ell_2} = \exp[-iH_{\text{eff}}(\ell_1 + \ell_2)], \quad (34) \]

with

\[ H_{\text{eff}} \equiv \frac{\ell_1 - \ell_2}{\ell_1 + \ell_2} H_0 + H_c. \quad (35) \]

The left hand side of relation (34) represents a sequence consisting of a period \( \ell_1 \) of free evolution, a pulse producing the transformation \( F \), a period \( \ell_2 \) of free evolution and a pulse
producing $F^{-1}$. From the right hand side of Eq. (34), we see that this sequence is equivalent to the system evolution for the total time $t_1 + t_2$ according to the Hamiltonian $H_{\text{eff}}$. Hence, we can obtain an effective reduction, by a factor $(t_1 - t_2)/(t_1 + t_2)$, of the Hamiltonian term $H_0$. This result can be viewed as a decrease of the magnitude of the uniform magnetic field as far as the electron spin dynamics is concerned. Notice that for $t_1 = t_2$ we can completely suppress the dynamical effects due to the term $H_0$.

\section*{B. Design and control of the spin-spin coupling}

Let us now consider a system with spin frequency differentiation. If we add another field consisting of a superposition of terms resonant with the spin frequencies

$$b_s(t) = \sum_{k=1}^{N} b_s[i\cos(\omega_{s,k}t) + j\sin(\omega_{s,k}t)],$$

the spin Hamiltonian of Eq. (14) becomes in IP with respect to $\sum_{i=1}^{N} (\omega_{s,i}/2)\sigma_i^z$

$$H^{\text{IP}} \simeq H^z + H^{bs},$$

with

$$H^z \equiv \sum_{i>j}^{N} J_{i,j}^z \sigma_i^z \sigma_j^z,$$

$$H^{bs} \equiv \eta \sum_{i=1}^{N} \left( \sigma_i^{(+)} + \sigma_i^{(-)} \right) = \eta \sum_{i=1}^{N} \sigma_i^x,$$

where $\eta \equiv g|e|b_s/(4m_e)$. Hence, the application of the oscillating field (36) gives rise to an effective static transverse magnetic field, whose strength can be controlled and modified, since it depends on the field amplitude $b_s$. This tool may turn out useful in reproducing quantum models like Ising system of spins. In this case the parameter $\eta$ should be comparable to the coupling strength $J_{i,j}^z$ between the spins.

Moreover, we can engineer the spin-spin coupling, that is introduce an effective spin-spin interaction along the $x$ and $y$ axes. This is achieved by means of sequences of pulses, of the kind given in Eqs. (24), (25), (26), and (27), affecting all the spins in the array. Indeed, it can be easily proved that

$$G_x e^{-iH^z t} G_x^\dagger = e^{-iG_x H^z t} G_x^\dagger t = \exp \left( -i \sum_{j>k}^{N} J_{j,k}^z \sigma_j^x \sigma_k^x t \right),$$

$$G_y e^{-iH^z t} G_y^\dagger = e^{-i G_y H^z t} G_y^\dagger = \exp \left( -i \sum_{j>k}^{N} J_{j,k}^z \sigma_j^y \sigma_k^y t \right).$$
Hence a sequence of two specific pulses, alternated to a period of free evolution under the Hamiltonian $H^z$, effectively modifies the direction of the spin-spin coupling. Now, if we combine the three operations (39), (40), and (41) we have that for $t_1, t_2, t_3 \ll \pi/J_{z,i,i+1}, \pi/\eta$

$$e^{-i(H^z + H^{bs})t_1}(G_y e^{-iH^zt_2}G_y^\dagger)(G_x e^{-iH^zt_3}G_x^\dagger) \simeq e^{-iH_{\text{eff}}(t_1 + t_2 + t_3)},$$

with

$$H_{\text{eff}} = \tau_1 \eta \sum_{i=1}^N \sigma_i^x + \tau_1 \sum_{i>j} J_{i,j}^z \sigma_i^z \sigma_j^z + \tau_2 \sum_{i>j} J_{i,j}^z \sigma_i^x \sigma_j^x + \tau_3 \sum_{i>j} J_{i,j}^y \sigma_i^y \sigma_j^y,$$

where $\tau_i = t_i/(t_1 + t_2 + t_3)$. In deriving relation (42) we used the approximate identity [21]

$$e^{-iA_1 t_1} e^{-iA_2 t_2} \ldots e^{-iA_n t_n} \simeq e^{-i(\tau_1 A_1 + \tau_2 A_2 + \ldots + \tau_n A_n)t}$$

with $t = \sum_{i=1}^n t_i$ and $\tau_i = t_i/t$, which is valid, to first order in $t$, for $t_i$ much shorter than the typical time scale of the dynamics due to the Hamiltonian $A_i$. However, more elaborate sequences of pulses (see Appendix [13]) give approximations to higher orders in $t$ [22]. A recursive application of the sequence (42) determines an effective evolution under the Hamiltonian $H_{\text{eff}}$. We point out that, in this Hamiltonian, Eq. (43), we can independently control and change the values of the parameters $\tau_i$’s and $\eta$, since they depend, respectively, on the free evolution times $t_i$’s and on the pulse amplitude $b_s$. Consequently we obtain an Hamiltonian $H_{\text{eff}}$ with a variable relative strength of the spin-spin coupling in the $x$, $y$, and $z$ directions and a tunable transverse magnetic field. Notice that we can also set $\tau_i = 0$ for any desired $i$ or $\eta = 0$. This is achieved by simply choosing $t_i = 0$ or switching off the external field $b_s(t)$. In this way various interesting quantum spin models can be derived from Hamiltonian (42). For example for $\tau_2 = \tau_3 = 0$ we obtain the Ising model, whereas for $\tau_1 = 0$ ($\tau_2 = 0$) we obtain the XY model in its usual (rotated) basis.

C. How to modify the interaction range and topology

The spin Hamiltonian (38) can be written as

$$\sum_{i>j}^N J_{i,j}^z \sigma_i^z \sigma_j^z \equiv H_1^z + H_2^z + \ldots,$$

where

$$H_n^z = \sum_{i=1}^{N-n} J_{i,i+n}^z \sigma_i^z \sigma_{i+n}^z$$

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represents the coupling between the \( n \)-th nearest neighbor spins. In our system the interaction between spins has a dipole-like nature, i.e. it decreases with the third power of the inter-particle distance. Consequently only the first few terms \( H_n \) at the right hand side of Eq. (45) play a significant role. In the following we are going to outline a procedure to independently control and modify, in a relatively simple way, the strength and the sign of the relevant \( H_n \) terms. In other words we can design the interaction topology by enhancing or suppressing the coupling between the \( n \)-th nearest neighbors. This is achieved by iteratively applying to the system appropriate sequences consisting of pulses alternated to periods of free evolution. In our scheme each pulse affects simultaneously a specific subset of spins in the array. Notice that if a particular pulse sequence \( S \) modifies the spin Hamiltonian \( H^z \), we can extend the same kind of coupling to the other directions by simply performing, according to Eqs. (40) and (41), the sequences \( G_y S G_y^\dagger \) and \( G_x S G_x^\dagger \). Therefore, we restrict ourselves to the transformations affecting the spin-spin coupling along the \( z \) direction. Furthermore, we are going to prove that the number of pulses in each sequence does not depend on the number of spins in the array, thus making our technique scalable with the system size.

As an example, we describe how to suppress the second nearest neighbor interaction by making use of three different transformations, defined as \( \sigma^x_o \), \( \sigma^x_{c_1} \) and \( \sigma^x_{c_2} \). Each of them can be performed by a single multi-frequency pulse. The transformation \( \sigma^x_o \) consists in the simultaneous application of \( \sigma^x \) to all the spins in the odd sites of the array. The transformations \( \sigma^x_{c_1} \) and \( \sigma^x_{c_2} \) flip, instead, alternated couples of neighboring spins. In particular, \( \sigma^x_{c_1} \) affects the spin couples \( \{1, 2\}, \{5, 6\}, \ldots \), whereas \( \sigma^x_{c_2} \) affects the couples \( \{2, 3\}, \{6, 7\}, \ldots \).

We prove in Appendix A that the sequence

\[
\sigma^x_{c_2} e^{-iH^z t} \frac{1}{2} \sigma^x_o e^{-iH^z t} \frac{1}{2} \sigma^x_{c_1} e^{-iH^z t}
\]

(47)

corresponds to the system evolution for a time \( t \) under the effective Hamiltonian \( H_{\text{eff}} \approx H^z_1 + H^z_3 \), where the coupling of each spin with its second nearest neighbors has been removed. Indeed, since the term \( H^z_3 \) is small, the above sequence well approximates an effective NN Hamiltonian. The transformations \( \sigma^x_o \), \( \sigma^x_{c_1} \) and \( \sigma^x_{c_2} \) are the building blocks to construct other sequences which realize different kinds of spin Hamiltonians. For example, as described in Appendix A we can easily invert the sign of \( H^z \), thus switching from an anti-ferromagnetic to a ferromagnetic interaction, or make the coupling strength between first and second nearest neighbors equal. This last case corresponds to an effective change in the array topology,
FIG. 1: Schematic drawing of a linear chain of \( N \) spins (upper part). When the first and second nearest neighbor coupling strengths are made equal, the linear chain becomes equivalent to a planar array (lower part).

since the number of nearest neighbors passes from two (linear chain) to four (see Fig. 1).

In order to affect the coupling between neighboring spins of order higher than three, we should apply simultaneously the transformation \( \sigma^z \) to selected subsets of three or more spins along the chain. For example, as we show in Appendix A with a seven pulse sequence we can suppress the interaction between a spin and all its neighbors from the second up to the sixth nearest neighbors. In such a way we improve our approximation of a NN interacting spin chain.

V. FIDELITY

In this section we discuss the performances of the scheme, based on the detailed analysis reported in Appendix B. We emphasize that our treatment focuses on the limitations due to the mapping of the system of trapped electrons into the desired target system of interacting spins. Hence, most experimental imperfections are not considered here.

As described in the previous section, to derive the effective spin Hamiltonian we make use of the approximate identity \( \equiv \) or of more sophisticated approximations \( \equiv \). As a consequence we introduce an error \( \equiv \)

\[
\mathcal{E} \equiv \| U - U' \| \equiv \max_{|\psi\rangle : ||\psi|| = 1} (|U - U'| |\psi\rangle),
\]

which measures the distance between the desired evolution \( U \) and the approximated evolution \( U' \). For instance, in our case the target unitary operator \( U = \exp(-iH_{\text{eff}} t) \), with the effective spin Hamiltonian \( H_{\text{eff}} \) of the kind of Eq. \( \equiv \), is approximated to the fourth order in \( t \) by the sequence \( S \), Eq. \( \equiv \). By using some algebra (see Appendix B), we can
bound the error associated to the application of a single sequence $S$ with

$$E_S = (J^z t)^5 f(N),$$

(49)

where $J^z \equiv J^z_{i,i+1}$ is the NN coupling strength and $f(N)$ is, in good approximation, an increasing linear function of the number $N$ of electrons in the array. The exact form of $f(N)$ depends on the specific target spin Hamiltonian. From this result, we see that the error is small whenever the time evolution is much shorter than the flipping time, i.e. $t \ll \pi/J^z$. We prove in Appendix [12] that, if we iterate $m$ times the sequence $S$ the total error is $E \leq mE_S$. Therefore, if we want to simulate the system evolution for a given time $T = mt$, to keep the accuracy high we should apply the same sequence $S$ $m$ times

$$E \leq \frac{(J^zT)^5 f(N)}{m^4}.\quad (50)$$

For a given simulation time $T$ and coupling strength $J^z$, the error $E$ decreases with the number of iterations $m$ and, therefore, with the total number of pulses.

In appendix [13] we provide the explicit expression of $f(N)$ for the $XY$ and NN Ising models. Consequently we are able to estimate the upper bound of $E$ in both cases. In our analysis, we also take into account the error introduced by the derivation of the effective spin-spin coupling [12]. In particular, the error $E_c$ due to the canonical transformation (10) satisfies the relation

$$E_c \leq N \left( \bar{k} + \frac{1}{2} \right) \varepsilon^2,$$

(51)

where $\bar{k}$ is the mean axial oscillator excitation number. We consider an array of 50 electrons with inter-particle distance of $100 \mu m$, $\omega_s/2\pi = 100$ GHz, $\omega_z/2\pi = 160$ MHz, and a magnetic gradient $b \simeq 200$ T/m. With these parameters we obtain, according to Eq. (12), a NN coupling constant $J^z = 10$ Hz. We also assume that the spin frequencies of neighboring electrons differ of about 2 MHz, each pulse has a duration of the order of $\mu s$ and the axial motion is cooled to the ground state. In order to simulate the $XY$ model (NN Ising model) for a time $T = 1$ s with fidelity of 99%, we need to iterate the specific sequence $S$ about 100 (50) times. In particular the simulation of the $XY$ model requires about 3000 pulses, whereas the Ising model with NN coupling requires about 2000 pulses. Notice that the Ising model with dipole-like coupling requires no pulse sequence, since it is obtained, directly, by applying the field (36). Therefore, in this case we only take into account, as a source of error, the thermal excitation of the axial oscillator which, according to Eq. (51), is of the order of $10^{-3}$. 

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VI. CONCLUSIONS

In this paper we have proposed a scalable technique for easily controlling and adjusting the effective Hamiltonian of a system of interacting spins. The underlying physical system consists of an array of trapped electrons in micro-Penning traps. The electron spin is prepared and manipulated with an external resonant magnetic field. These spin operations, applied to all the particles or subsets of them, are alternated to periods of free evolution in a fashion similar to NMR refocusing schemes. To selectively address the electrons in the array, it is necessary to introduce a detuning between the characteristic spin frequencies by means of a magnetic gradient. In particular, we have shown that, in the case of a system without spin frequency differentiation, a two pulse sequence permits to reduce or even cancel the effect on the spin dynamics of the uniform magnetic field, without affecting the overall trap stability. This is potentially useful for the observation of quantum phase transitions \cite{14}, where it is important to modulate the ratio between the external magnetic field and the spin-spin coupling strength. In the case of a system with spin frequency differentiation, we have proved that with a repeated application of appropriate pulse sequences we can modify and control the interaction terms in the effective spin Hamiltonian. As a result a wide range of spin Hamiltonians can be obtained, such as the Ising model and the XY models. Moreover, specific pulse sequences allow to control the sign and strength of the coupling between the $k$-th nearest neighbors for any significant value of $k$ (first, second, \ldots , nearest neighbors). As an example, we provide a prescription to obtain an Hamiltonian with substantially only NN coupling starting from a dipole-like interaction. In our scheme the number of pulses in each sequence is relatively small and does not depend on the number of spins in the array. We derive an analytical formula to estimate the fidelity of our method for simulating the effective spin Hamiltonian, as a function of the coupling strength, the simulation time and the number of particles. Our estimates show that it is feasible to simulate the Ising, with NN coupling, and the $XY$ model with fidelity of 99\% for a system of 50 electrons with a coupling strength $J_z = 10$ Hz. Of course, the evaluation of the performances of a real experiment would require a closer analysis of all the possible sources of errors and decoherence. This is, however, beyond the scope of the present work.
In this appendix, we are going to prove that spin flip operations, applied to subsets of particles in the array, result in an effective sign change in the interaction between neighbors of arbitrary order. The starting point is represented by the relation

$$\sigma_i^x \sigma_i^z \sigma_i^x = -\sigma_i^z,$$  \hspace{1cm} (A1)

which reverts the sign of the $i$-th spin operator. We define the following operators

$$\sigma_o^x \equiv \bigotimes_{i=1}^{N/2} \sigma_{2i-1}^x,$$ \hspace{1cm} (A2)

$$\sigma_{c_k}^x \equiv \bigotimes_{i \in c_k} \sigma_i^x \sigma_{i+1}^x \quad \text{with} \quad c_k = \{k, k+4, k+8, \ldots\} \quad \text{for} \quad k = 1, 2,$$ \hspace{1cm} (A3)

$$\sigma_{T_k}^x \equiv \bigotimes_{i \in T_k} \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \quad \text{with} \quad T_k = \{k, k+6, k+12, \ldots\} \quad \text{for} \quad k = 1, 2, 3,$$ \hspace{1cm} (A4)

$$\sigma_{Q_k}^x \equiv \bigotimes_{i \in Q_k} \sigma_i^x \sigma_{i+1}^x \sigma_{i+2}^x \sigma_{i+3}^x \quad \text{with} \quad Q_k = \{k, k+8, k+16, \ldots\} \quad \text{for} \quad k = 1, 2, 3, 4,$$ \hspace{1cm} (A5)

that affect simultaneously different subsets of spins. Moreover, we observe that

$$(\sigma_o^x)^2 = (\sigma_{c_k}^x)^2 = (\sigma_{T_k}^x)^2 = (\sigma_{Q_k}^x)^2 = 1$$ \hspace{1cm} (A6)

for any possible value of $k$, so that we can use the identity

$$Ae^B A = \exp (ABA),$$ \hspace{1cm} (A7)

which holds true for any pair of operators $A$ and $B$, whenever $A^2 = 1$.

From the relation (A11) and the definition of $\sigma_o^x$ it follows that

$$\sigma_o^x \sigma_i^z \sigma_o^x = (-1)^{i+j} \sigma_i^z \sigma_o^x,$$ \hspace{1cm} (A8)

which amounts to a sign change in the interaction between spins with different parity. Consequently, given the Hamiltonian $H^z$ of Eq. (38), the transformation $\sigma_o^x H^z \sigma_o^x$ inverts the coupling between neighbors of odd orders

$$\sigma_o^x e^{-iH^z t} \sigma_o^x = e^{-i\sigma_o^x H^z \sigma_o^x t} = \exp \left[-i \sum_{k=1}^{N} (-1)^k H^z_k t \right].$$ \hspace{1cm} (A9)

This property allows us to make equal in strength the coupling between first and second nearest neighbors

$$\sigma_o^x e^{-iH^z t} \sigma_o^x e^{-iH^z t} \simeq \exp \left[ -i J' \sum_{j=1}^{N-2} \sigma_j^z (\sigma_{j+1}^z + \sigma_{j+2}^z) t \right],$$ \hspace{1cm} (A10)
with \( J' \equiv (2/9)J_{i,i+1}^{z} \).

With a three-pulse sequence

\[
\sigma_{c_{2}}^{x}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{o}^{x}}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{c_{1}}^{x}} = \exp \left[ -i \sum_{k=1}^{N/2} (-1)^{k}H_{2k}^{z} t \right],
\]

(A11)

we remove the coupling between odd order neighbors and alternatively change the sign of the coupling in \( H^{z} \) between even order neighbors. To prove Eq. (A11) we use the identity \( \sigma_{o}^{x} = \sigma_{c_{2}}^{x}\sigma_{c_{1}}^{x} \) and the commutation relation \( [\sigma_{c_{2}}^{x}H_{c_{1}}^{z}\sigma_{c_{1}}^{x},\sigma_{c_{1}}^{x}H_{c_{1}}^{z}\sigma_{c_{1}}^{x}] = 0 \) in order to obtain

\[
\sigma_{c_{2}}^{x}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{o}^{x}}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{c_{1}}^{x}} = \sigma_{c_{2}}^{x}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{c_{2}}^{x}\sigma_{c_{1}}^{x}}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{c_{1}}^{x}} = e^{-i(\sigma_{c_{2}}^{x}H_{c_{1}}^{z}\sigma_{c_{2}}^{x}+\sigma_{c_{1}}^{x}H_{c_{1}}^{z}\sigma_{c_{1}}^{x})/2}.
\]

(A12)

The transformation \( \sigma_{c_{k}}^{x}H_{c_{k}}^{z}\sigma_{c_{k}}^{x} \) selectively changes the sign in \( H^{z} \) to the operators \( \sigma_{j}^{x} \) and \( \sigma_{j+1}^{x} \) with \( j \in c_{k} \), according to Eq. (A3). Consequently we have

\[
\sigma_{c_{1}}^{x}H_{c_{1}}^{z}\sigma_{c_{1}}^{x} = \sum_{i=1}^{N-1} (-1)^{i+1}J_{i,i+1}^{z}\sigma_{i}^{x}\sigma_{i+1}^{x} - \sum_{i=1}^{N-2} J_{i,i+2}^{z}\sigma_{i+2}^{x}\sigma_{i}^{x} + \sum_{i=1}^{N-3} (-1)^{i}J_{i,i+3}^{z}\sigma_{i+3}^{x}\sigma_{i}^{x} + \ldots \quad \text{(A13)}
\]

\[
\sigma_{c_{2}}^{x}H_{c_{2}}^{z}\sigma_{c_{2}}^{x} = \sum_{i=1}^{N-1} (-1)^{i}J_{i,i+1}^{z}\sigma_{i}^{x}\sigma_{i+1}^{x} - \sum_{i=1}^{N-2} J_{i,i+2}^{z}\sigma_{i+2}^{x}\sigma_{i}^{x} + \sum_{i=1}^{N-3} (-1)^{i+1}J_{i,i+3}^{z}\sigma_{i+3}^{x}\sigma_{i}^{x} + \ldots \quad \text{(A14)}
\]

Hence, to demonstrate Eq. (A11), we sum Eq. (A13) and Eq. (A14) obtaining

\[
\sigma_{c_{2}}^{x}H_{c_{2}}^{z}\sigma_{c_{2}}^{x} + \sigma_{c_{1}}^{x}H_{c_{1}}^{z}\sigma_{c_{1}}^{x} = -2H_{2}^{z} + 2H_{4}^{z} + \ldots = 2 \sum_{k=1}^{N/2} (-1)^{k}H_{2k}^{z}
\]

(A15)

Notice that in the sum the coupling between nearest neighbors of odd orders cancels out.

By combining the sequences (A9) and (A11), we can invert the sign of the coupling up to third nearest neighbors

\[
(\sigma_{c_{2}}^{x}e^{-iH_{z}^{x}\sigma_{o}^{x}}e^{-iH_{z}^{x}\sigma_{c_{1}}^{x}})(\sigma_{o}^{x}e^{-iH_{z}^{x}\sigma_{o}^{x}}) \simeq \exp \left[ -i(-H_{1}^{z} - H_{2}^{z} - H_{3}^{z}) t \right],
\]

(A16)

thus turning a ferromagnetic interaction into an anti-ferromagnetic one and vice versa. Another consequence of Eq. (A11) is

\[
\sigma_{c_{2}}^{x}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{o}^{x}}e^{-iH_{z}^{x}\frac{1}{2}\sigma_{c_{1}}^{x}} e^{-iH_{z}^{x}t} \simeq \exp \left[ -i(H_{1}^{z} + H_{3}^{z}) t \right],
\]

(A17)

where the coupling between second nearest neighbors has been removed.

The approach described so far can be extended in order to cancel coupling terms of higher order. For example, to remove both the second and third nearest neighbor couplings in \( H^{z} \) we make use of the transformations defined in Eq. (A4). They simultaneously affect sets of
three nearest neighbors in alternate succession. With arguments similar to those used for verifying Eq. (A11), we can demonstrate the following identity

\[
(\sigma_{T_3}^x e^{-iH_z 3/4} \sigma_{T_2}^x)(\sigma_{T_2}^x e^{-iH_z 3/4} \sigma_{T_1}^x)(\sigma_{T_1}^x e^{-iH_z 3/4} \sigma_{T_3}^x) = e^{-i(H_1^z - H_2^z - 3H_3^z - H_4^z + ...)/3}.
\] (A18)

By combining Eq. (A11) and Eq. (A18) we prove that the sequence

\[
(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x) e^{-iH_z t}
\] (A19)

corresponds to the evolution for a time \((4/3)t\) under the Hamiltonian \((H_1^z + H_2^z + ...\)), where the coupling between second and third nearest neighbors has been removed. The implementation of this sequence requires six pulses, since each couple of consecutive transformations in Eq. (A19) is equivalent to a single transformation affecting simultaneously a specific subset of spins in the array.

It is worth to point out that with a seven-pulse sequence we can approximate the NN model in a very accurate way, i.e. we can suppress the interaction between a spin and all its neighbors from the second up to the sixth nearest neighbors. This is achieved by using the four transformations defined in Eq. (A5), that simultaneously affect alternated sets of four nearest neighbors. It can be proved that

\[
(\sigma_{Q_4}^x e^{-iH_z 3/8} \sigma_{Q_1}^x)(\sigma_{Q_3}^x e^{-iH_z 3/8} \sigma_{Q_3}^x)(\sigma_{Q_2}^x e^{-iH_z 3/8} \sigma_{Q_2}^x)(\sigma_{Q_1}^x e^{-iH_z 3/8} \sigma_{Q_1}^x) e^{-i(H_1^z - H_3^z - 2H_4^z + H_6^z + ...)/t}.
\] (A20)

From Eq. (A11) and Eq. (A20) we have that the sequence

\[
(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x e^{-iH_z 3/4} \sigma_{c_1}^x)(\sigma_{c_2}^x e^{-iH_z 3/4} \sigma_{c_1}^x e^{-iH_z 3/4} \sigma_{c_1}^x) e^{-iH_z t}
\] (A21)

corresponds to the evolution for a time \(2t\) under the Hamiltonian \((H_1^z + H_2^z + ...\)), where the coupling between nearest neighbors from the second up to the sixth order has been removed. The implementation of the above sequence requires just seven pulses, since each couple of consecutive transformations in Eq. (A21) is equivalent to a single transformation affecting simultaneously a specific subset of spins in the array.
APPENDIX B

The sequence of unitary operators to the left hand side of relation (44) approximates the evolution under the target Hamiltonian

$$H = \sum_{i=1}^{n} \tau_i A_i$$  \hspace{1cm} (B1)

to first order in $t$. However, more elaborate combinations of unitary operators provide better approximations. For example the sequence

$$S \equiv \bar{S}_1 S_1 \bar{S}_2 S_1 \bar{S}_1 S_1 S_1 S_1 S_1 \bar{S}_1 S_1 \bar{S}_1 S_1$$  \hspace{1cm} (B2)

with

$$S_k \equiv e^{-i \frac{k}{12} t_1 A_1} e^{-i \frac{k}{12} t_2 A_2} \cdots e^{-i \frac{k}{12} t_n A_n}$$  \hspace{1cm} (B3)

and

$$\bar{S}_k \equiv e^{-i \frac{k}{12} t_n A_n} \cdots e^{-i \frac{k}{12} t_2 A_2} e^{-i \frac{k}{12} t_1 A_1},$$  \hspace{1cm} (B4)

approximates the unitary operator $e^{-i H t}$, with $t \equiv \sum_{i=1}^{n} t_i$ and $\tau_i \equiv t_i / t$, to the fourth order in $t$.

The error introduced by approximating the unitary operator $U$ with the unitary operator $U'$ can be measured by the quantity

$$E \equiv \| U - U' \| \equiv \max_{\| \psi \| = 1} | \langle \psi | (U - U') | \psi \rangle |.$$  \hspace{1cm} (B5)

Now if we want to approximate the evolution under the Hamiltonian Eq. (B1) for a time $T = mt$ we can apply $m$ times the sequence (B2). From the generalization of the inequality

$$\| AB - CD \| \leq \| A - C \| + \| B - D \|,$$  \hspace{1cm} (B6)

verified for any unitary operator $A$, $B$, $C$, $D$, we have that the error $E$ of our approximation satisfies the inequality $E \leq \sum_{i=1}^{m} E_i$, where $E_i$ is the error introduced by the $i$-th application of the sequence (B2). More specifically we evaluate the error

$$E_i \equiv \| e^{-i (\sum_{i=1}^{n} \tau_i A_i) t} - S \|,$$  \hspace{1cm} (B7)

when the operators $A_i$'s are typically of the kind $\sum_{i \neq j} J_{i,j}^{k} \sigma_i^k \sigma_j^k$, with $k = x, y, z$. We explicitly expand each operator to the right hand side and find that their difference is proportional to
$t^5$, because the sequence $S$ approximates the desired unitary evolution $\exp(-i \sum_{i=1}^{n} \tau_i A_i t)$ to the fourth order in $t$. Moreover, we make use of the inequality
\[
\|\alpha A + \beta B\| \leq |\alpha| \|A\| + |\beta| \|B\|, \quad (B8)
\]
which holds true for any pair of operators $A$ and $B$ and complex numbers $\alpha$, $\beta$. Finally, we observe that $\|C\| = 1$ if $C$ is any product of Pauli operators $\sigma_i^k$. This approach lead us to the following estimate for the error, defined in Eq. (B7),
\[
E_i \leq (J^z t)^5 f(N), \quad (B9)
\]
where $J^z \equiv J^z_{i,i+1}$ is the nearest neighbor coupling strength and $f(N)$ is, in good approximation, an increasing linear function of the number $N$ of electrons in the array, depending on the specific form of the spin Hamiltonian.

When we apply $m$ times the sequence $S$, from the previous discussion it follows that $E \leq m (J^z t)^5 f(N)$. Now if we indicate with $T = mt$ the total simulation time, we obtain
\[
E \leq \frac{(J^z T)^5 f(N)}{m^4} \quad (B10)
\]
or, equivalently,
\[
m \leq \sqrt[4]{\frac{(J^z T)^5 f(N)}{E}}. \quad (B11)
\]
Equation (B11) gives an upper bound to the number of iterations required to mimic the desired evolution with an error $E$. For example, to approximate the XY model in the rotated basis, we choose
\[
A_1 = \frac{\eta}{2} \sum_{i=1}^{N} \sigma_i^x + \sum_{i>j}^{N} J^z_{i,j} \sigma_i^z \sigma_j^z, \quad (B12)
\]
\[
A_2 = \frac{\eta}{2} \sum_{i=1}^{N} \sigma_i^x - \sum_{i>j}^{N} J^z_{i,j} \sigma_i^y \sigma_j^y. \quad (B13)
\]
In this case, the outlined approach gives for $N > 5$, $f(N) \simeq 0.25N - 0.85$. The simulation of the NN Ising model is achieved with
\[
A_1 = \frac{\eta}{2} \sum_{i=1}^{N} \sigma_i^x + \sum_{i>j}^{N} J^z_{i,j} \sigma_i^z \sigma_j^z, \quad (B14)
\]
\[
A_2 = \frac{\eta}{2} \sum_{i=1}^{N} \sigma_i^x - \sum_{i=1}^{N-2} J^z_{i,i+2} \sigma_i^z \sigma_{i+2}^z, \quad (B15)
\]
and, for $N > 5$, $f(N) \simeq 0.015N - 0.035$.  

ACKNOWLEDGMENTS

This research was supported by the European Commission through the Specific Targeted Research Project \textit{QUELE}, the Integrated Project FET/QIPC \textit{SCALA}, and the Research Training Network \textit{CONQUEST}.

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