A CLT FOR THE LSS OF LARGE DIMENSIONAL SAMPLE COVARIANCE MATRICES WITH UNBOUNDED DISPERSIONS

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In this paper, we establish the central limit theorem (CLT) for linear spectral statistics (LSS) of large-dimensional sample covariance matrix when the population covariance matrices are not uniformly bounded, which is a non-trivial extension of the Bai-Silverstein theorem (BST) (2004). The latter has strongly stimulated the development of high-dimensional statistics, especially the application of random matrix theory to statistics. However, the assumption of uniform boundedness of the population covariance matrices is found strongly limited to the applications of BST. The aim of this paper is to remove the blockages to the applications of BST. The new CLT, allows the spiked eigenvalues to exist and tend to infinity. It is interesting to note that the roles of either spiked eigenvalues or the bulk eigenvalues or both of the two are dominating in the CLT.

Moreover, the results are checked by simulation studies with various population settings. The CLT for LSS is then applied for testing the hypothesis that a covariance matrix $\Sigma$ is equal to an identity matrix. For this, the asymptotic distributions for the corrected likelihood ratio test (LRT) and Nagao’s trace test (NT) under alternative are derived, and we also propose the asymptotic power of LRT and NT under certain alternatives.

1. Introduction. Consider the general sample covariance matrix $B_n = \frac{1}{n} T_p X_n X_n^* T_p^*$, where $X_n$ is a $p \times n$ matrix with independent and identically distributed (i.i.d.) entries, $T_p$ is a $p \times p$ deterministic matrix, $T_p X_n$ can be considered a random sample from the population with the population covariance matrix $T_p T_p^* = \Sigma$, and $^*$ represents the complex conjugate transpose. In this sequel, we will simply write $B \equiv B_n$, $T \equiv T_p$ and $X \equiv X_n$ when there is no confusion. $\lambda_1, \lambda_2, \ldots, \lambda_p$ denotes the eigenvalues of $B$. For a known kernel function $f$, we call $\sum_{j=1}^{p} f(\lambda_j)$ the linear spectral statistic (LSS) of $B$. As most of the classical test statistics in multivariate statistical analysis are associated with the eigenvalues of sample covariance matrices, LSSs are remarkable tools in many statistical problems (see Anderson (2003); Yao et al. (2015) for details). By extensively studying high-dimensional data, it was found that the performances of the LSSs are significantly different between low dimensions and high-dimensional data. For example, under the low-dimensional setting, Wilks’ theorem (see Wilks (1938)) provides the $\chi^2$ approximation for the likelihood ratio statistics (LRT), which is a kind of LSS. However, when $p$ is large compared with the sample size $n$, the LRTs have Gaussian fluctuations (see Bai et al. (2009); Jiang and Yang (2013)). More generally, Bai and Silverstein (2004) proved the central limit theorem for the LSSs of high-dimensional $B$ under very mild conditions using the random matrix theory (RMT). We refer to this result as the Bai-Silverstein theorem (BST) for brevity. Following the development of Bai and Silverstein (2004), there are many extensions under different settings. Pan and Zhou (2008) generalized the BST by removing the constraint on the fourth moment of the underlying random variables. Zheng (2012) and Yang and Pan (2015) extended the BST to multivariate
Pan (2014) showed the CLT of the LSS for non-centered sample covariance matrices, and Zheng et al. (2015) studied the unbiased sample covariance matrix when the population mean is unknown. Chen and Pan (2015) focused on the ultra-high dimensional case in which the dimension \( p \) is much larger than the sample size \( n \). Gao et al. (2017) and Li et al. (2021) studied the matrices and canonical correlation matrices, respectively. Pan (2014) showed the CLT of \( F \)-eigenvalues may tend to infinity. We use three examples here. Because in many fields, such as economics and wireless communication networks, the leading eigenvalues may tend to infinity. We use three examples here.

- **Panel data model** (Baltagi et al. (2017)): Consider a fixed effect panel data model
  \[ y_{it} = X_{it}' \beta + \mu_i + v_{it}, \quad i = 1, \ldots, n, \quad t = 1, \ldots, T, \]
  where \( i \) is the index of the cross-sectional units, \( t \) is the index of the time series observations, \( \mu_i \) represents the time invariant individual effects, and \( v_{it} \) is the idiosyncratic error term. \( v_t = (v_{1t}, \ldots, v_{nt})' \) are i.i.d. \( N(0, \Sigma) \). Assume that \( v_{it} = \sum_{j=1}^{r} \gamma_{ij} f_{ij} + \epsilon_t \), where \( f_{ij} \) is the factor \( j \) in period \( t \), \( \gamma_{ij} \) is the factor loading of the individual \( i \) for factor \( j \), \( \epsilon_t \) is the error term with i.i.d. \( N(0, \sigma^2) \), and \( r \) is the known number of factors. The sphericity test in the fixed effect panel data model is
  \[
  H_0 : \Sigma = \sigma^2 I_n \quad \text{v.s.} \quad H_1 : \Sigma = \sigma^2 \left( I_n + \sum_{j=1}^{r} \frac{\sigma^2}{\sigma^2} \gamma_j \gamma_j' \right),
  \]
  where \( I_n \) denotes the \( n \)-dimensional identity matrix. \( \gamma_j = (\gamma_{1j}, \ldots, \gamma_{nj})' \) is the vector of factor loading, and \( \sigma^2_j \) is the variance of factor \( f_{ij} \). Many efforts have been made to analyze the asymptotic power of sphericity tests in high-dimensional setups, where the number of cross-sectional units \( n \) in a panel is large, but the number of time series observations \( T \) could also be large. When \( n \) jointly tends to infinity with \( T \), the norm of the perturbation term in the alternative hypothesis is greater than the threshold or even goes to infinity. In this case, the existing methods that assume \( \Sigma \) are bounded are not applicable.

- **Signal detection** (Johnstone and Nadler (2017)): Consider a single signal model
  \[ x = \chi_s h + \nu, \]
  where \( h \) is an unknown \( p \)-dimensional vector, \( u \) is a random variable distributed as \( N(0, 1) \), \( \chi_s \) is the signal strength, \( \sigma \) is the noise level, and \( \nu \) is a random noise vector that is independent of \( u \) and distributed as a multivariate Gaussian \( N_p(0, \Sigma_{\nu}) \). It is easy to check that the covariance matrix of \( x \) is \( \Sigma_x = \sigma^2 \Sigma_{\nu} + \chi_s hh' \). When the noise level is low, while the signal strength is large and sometimes tends to infinity, it is illogical to assume the boundedness of \( \Sigma_x \).

- **m-factor structure** (Li et al. (2020)): Consider the \( m \)-factor model
  \[ X_t = A F_t + E_t \]
  where the factors \( F_t \sim N(0, I_m) \) are independent of the idiosyncratic error terms \( E_t \sim N(0, \sigma^2 I_p) \). The loading matrix \( A_{p \times m} \) is deterministic and of full rank such that \( A^* A \) has eigenvalues \( a_1 > \cdots > a_m > 0 \). The eigenvalues of the population covariance matrix \( \Sigma_p \) of \( X_t \) are
  \[ \text{Spec}(\Sigma_p) = \{ a_1 + \sigma^2, \ldots, a_m + \sigma^2, \sigma^2, \ldots, \sigma^2 \}, \]
which follow the generalized spiked model. Because of the complexity of the real data, when $\sigma$ is small, the signal-to-noise ratio may be large enough to give rise to the large spectral norm of $\Sigma_p/\sigma^2$. Therefore, in this case, we assume the unbounded spectra of the population covariance matrices would be more realistic.

For these reasons, it is of practical value to obtain the asymptotic properties of the LSS when $\Sigma$ is unbounded. Therefore, in this paper, we focus on the CLT for the LSS of a general covariance matrix structure

$$\Sigma = V \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} V^*,$$

where $V$ is a unitary matrix, $D_1$ is a diagonal matrix consisting of the descending unbounded eigenvalues, and $D_2$ is the diagonal matrix of the bounded eigenvalues. The setting (1.1) is attributed to the famous spiked model in which a few large eigenvalues of the population covariance matrix are assumed to be well separated from the rest (Johnstone, 2001). The spiked model has provided the foundations for a rich theory of principal component analysis through the performance of extreme eigenvalues as discussed in Baik and Silverstein (2006); Paul (2007); Bai and Yao (2008); Nadler (2008); Jung and Marron (2009); Bai and Yao (2012); Onatski et al. (2014); Bloemendal et al. (2016); Wang and Yao (2017); Donoho et al. (2018); Perry et al. (2018); Johnstone and Paul (2018); Yang and Johnstone (2018); Yao et al. (2018); Dobriban (2020); Johnstone and Onatski (2020); Cai et al. (2020); Jiang and Bai (2021).

Recently, Li et al. (2020), Yin (2021) and Zhang et al. (2022) investigated the trace of the large sample covariance matrix with the spiked model assumption.

We highlight the main contributions of the present paper. First, we prove a non-trivial extension of the BST to the case in which the population covariance matrices are unbounded in the spectral norm. In particular, we show how the kernel functions and the divergence rate of the population spectral norm affect the new CLT. Second, it is known that Gaussian-like moments, i.e., the first fourth moments, coincide with a standard Gaussian distribution, or the diagonality of the population covariance matrix is necessary for the CLT of the LSSs (e.g., Zheng et al. (2015)). However, we prove that these restrictions can be completely removed by renormalization. More importantly, even if the limit of the LSS variance does not exist, the renormalized CLT still holds. Third, by combining the technical strategy in Bai and Silverstein (2004) and the analysis of the block decomposition of the sample covariance matrix in Jiang and Bai (2021), we prove that the LSSs of the unbounded and bounded parts are asymptotically independent. The proof in this entire paper is built on the decomposition of the LSS, which is divided into two parts, an unbounded part and a bounded part. It is worth noting that the bounded part of the LSS cannot use the result in Bai and Silverstein (2004) directly since off-diagonal sample covariance matrix blocks are not 0, which yields a bias between the bounded part of the LSS and the LSS of the bounded sample covariance matrix blocks. In facing this challenge, we make full use of the RMT and prove for the first time that bias can be measured in probability in the literature.

As an application, the established CLT is employed to study the asymptotic behavior of the LRT and Nagao’s trace (NT) test under the hypothesis

$$H_0 : \Sigma = \mathbf{I}_p \text{ v.s. } H_1 : \Sigma \neq \mathbf{I}_p.$$

It is known that the LRT and NT are typical examples of LSSs with kernel functions $f(x) = x - \log x - 1$ and $x^2 - 2x + 1$. In this paper, we start from a different perspective by studying the asymptotic distribution of the LRT and NT under the alternative that $D_1$ is composed of $M$ spiked eigenvalues tending to infinity with multiplicity, $D_2 = \mathbf{I}_{p-M}$, and then we establish their asymptotic power under the above alternative. Based on previous knowledge (e.g., (Bai et al., 2009)), it seems that when the number of spiked eigenvalues is small, the influence
caused by the spiked part is small, and the distribution of the LSS is mainly decided by the bulk part; however, that is not the case. After simulation, a surprising result is that when the spiked eigenvalues are very large, the LSS will also be affected by the spiked part even though the number of spikes is small.

The remaining sections are organized as follows: Section 2 provides a detailed description of the notation and assumptions. The main results of the CLTs for the LSS of the sample covariance matrix are stated in Section 3. In Section 4, we explore the applications of our main results. We also present the results of our numerical studies in Section 5. Technical proofs of the theorems are presented in Section 6.

2. Notation and assumptions. Throughout the paper, we use bold capital letters and bold lowercase letters to represent matrices and vectors, respectively. Scalars are often in regular letters. \( e \) denotes a standard basis vector whose components are all zero, except the \( i \)-th, which equals 1. We use \( \text{tr}(\mathbf{A}) \), \( \mathbf{A}^t \) and \( \mathbf{A}^* \) to denote the trace, transpose and conjugate transpose of matrix \( \mathbf{A} \), respectively. We also use \( f' \) to denote the derivative of function \( f \), and we use \( \frac{\partial}{\partial z_1} f(z_1, z_2) \) to denote the partial derivative of function \( f \) with respect to \( z_1 \); however, the context is clear enough that there is no risk of ambiguity. Let \( [\mathbf{A}]_{ij} \) denote the \((i, j)\)-th entry of the matrix \( \mathbf{A} \) and \( \oint f(z)dz \) denote the contour integral of \( f(z) \) on the contour \( \mathcal{C} \). Let \( \lambda_i^\mathbf{A} \) denote the \( i \)-th largest eigenvalue of matrix \( \mathbf{A} \). Weak convergence is denoted by \( \overset{d}{\rightarrow} \). Throughout this paper, we use \( o(1) \) (resp. \( o_p(1) \)) to denote a scalar negligible (resp. in probability), and the notation \( \mathcal{C} \) represents some generic constants that may vary from line to line.

Let \( \mathbf{X} = (x_1, \ldots, x_n) = (x_{ij}) \), \( 1 \leq i \leq p \), \( 1 \leq j \leq n \), and \( \mathbf{T} \) be a \( p \times p \) deterministic matrix and \( \mathbf{\Sigma} = \mathbf{T} \mathbf{T}^* \). The spectrum of \( \mathbf{\Sigma} \) is formed as \( \rho_1 \geq \cdots \geq \rho_p \). Define the singular value decomposition of \( \mathbf{T} \) as

\[
\mathbf{T} = \mathbf{V} \mathbf{D}^{1/2} \mathbf{U}^* = \mathbf{V} \left( \begin{array}{cc} D_1^{\frac{1}{2}} & 0 \\ 0 & D_2^{\frac{1}{2}} \end{array} \right) \mathbf{U}^* \tag{2.2}
\]

where \( \mathbf{U} \) and \( \mathbf{V} \) are unitary matrices, \( \mathbf{D}_1 = \text{diag}(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_K, \ldots, \alpha_K) \) is a diagonal matrix of the spiked eigenvalues for which the components tend to infinity, and \( \mathbf{D}_2 \) is the diagonal matrix of the eigenvalues with the bounded components. Here, \( \alpha_1 > \cdots > \alpha_K \) denotes the unbounded spiked eigenvalues of \( \mathbf{\Sigma} \) with the multiplicity \( d_k, k = 1, \ldots, K \), and \( d_1 + \cdots + d_K = M \). Moreover, let \( \rho_i = \alpha_k \) if \( i \in J_k \), where \( J_k = \{ j_k + 1, \ldots, j_k + d_k \} \) is the set of ranks of the \( d_k \)-ple eigenvalue \( \alpha_k \). Then, the corresponding sample covariance matrix

\[
\mathbf{B} = \frac{1}{n} \mathbf{T} \mathbf{X} \mathbf{X}^* \mathbf{T}^*
\]

is the so-called generalized spiked sample covariance matrix. \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \) denotes the eigenvalues of \( \mathbf{B} \). Corresponding to the decomposition of \( \mathbf{D} \), we decompose \( \mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2) \), \( \mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2) \), and \( \mathbf{T} = \mathbf{V}_2 \mathbf{D}_2^{1/2} \mathbf{U}_2^* \). \( r_j = \frac{1}{\sqrt{n}} \mathbf{\Gamma} x_j \), \( \mathbf{A}_j = \mathbf{B} - \mathbf{I} - r_j r_j^* \). Let \( \mathbf{E}_j \) be the conditional expectation with respect to the \( \sigma \)-field generated by \( r_1, \ldots, r_j \). For any matrix \( \mathbf{A} \) with real eigenvalues, the empirical spectral distribution of \( \mathbf{A} \) is denoted by

\[
F^{\mathbf{A}}(x) = \frac{1}{p} (\text{number of eigenvalues of } \mathbf{A} \leq x).
\]

For any function of bounded variation \( F \) on the real line, its Stieltjes transform is defined by

\[
m_{F}(z) = \int \frac{1}{\lambda - z} \, dF(\lambda), \quad z \in \mathbb{C}^+ := \{ z \in \mathbb{C} : \Im z > 0 \}.
\]

The assumptions used in the results of this paper are as follows:
ASSUMPTION 1. \( \{x_{ij}, 1 \leq i \leq p, 1 \leq j \leq n\} \) are independent random variables with common moments
\[
E x_{ij} = 0, \quad E |x_{ij}|^2 = 1, \quad \beta_x = E |x_{ij}|^4 - |E x_{ij}^2|^2 - 2, \quad \alpha_x = |E x_{ij}^2|^2,
\]
and satisfy the following Lindeberg-type condition:
\[
\frac{1}{np} \sum_{i=1}^{p} \sum_{j=1}^{n} E \left\{ |x_{ij}|^4 1_{\{|x_{ij}| \geq \eta / \sqrt{n}\}} \right\} \to 0, \quad \text{for any constant } \eta > 0.
\]

ASSUMPTION 2. As \( \min\{p, n\} \to \infty \), the ratio of the dimension-to-sample size (RDS) \( c_n := p/n \to c > 0 \).

REMARK 2.1. Assumptions 1-2 are standard in the RMT. Note that if \( E x_{ij} \neq 0 \), we can use the centralized sample covariance matrices and \( n^{-1} \) instead of \( B_n \) and \( n \), respectively, and the following results also hold. Details can be found in (Zheng et al., 2015). Therefore, in this sequel, we assume \( E x_{ij} = 0 \) without loss of generality.

ASSUMPTION 3. \( T \) is non-random. As \( \min\{p, n\} \to \infty \), \( \alpha_K \to \infty \) and
\[
H_n := F^{\Gamma \Gamma^*} \to H, \quad \text{where } H \text{ is a distribution function on the real line. } M \text{ is fixed.}
\]

REMARK 2.2. It was shown by Silverstein (1995) that under Assumptions 1-3, \( F^B \to F^{c,H} \) almost surely, where \( F^{c,H} \) is a non-random distribution function whose Stieltjes transform \( m := m_{F^{c,H}}(z) \) satisfies equation
\[
m = - \left( z - c \int \frac{t}{1 + t m} dH(t) \right)^{-1},
\]
where \( m := m_{F^{c,H}}(z) \) represents the Stieltjes transform of the LSD of \( B \).

ASSUMPTION 4. Kernel functions \( f_1, \ldots, f_h \) are analytic on an open domain of the complex plan containing the support of \( F^{c,H} \). Moreover, suppose that for any \( l = 1, \ldots, h \),
\[
\lim_{(x_n, y_n) \to \infty} \frac{f_l'(x_n)}{f_l'(y_n)} = 1.
\]

REMARK 2.3. In fact, Assumption 4 is not too restrictive for application, many common functions such as logarithmic and polynomial functions satisfy it. However, it is worth noting that the exponential function may not satisfy this assumption.

ASSUMPTION 5. \( T \) is real or the variables \( x_{ij} \) are complex satisfying \( \alpha_x = 0 \).

ASSUMPTION 6. \( T^* T \) is diagonal or \( \beta_x = 0 \).

REMARK 2.4. Assumption 5 is for the second-order moment condition of \( x_{ij} \), and Assumption 6 is for the fourth-order moment. They were first proposed by Zheng et al. (2015), who proved that the two assumptions are necessary for their results when the Gaussian-like moment conditions in the BST do not hold.
3. Main Results. Now, we are in a position to present our main theorems, and their proofs are provided in Section 6. Note that

\[ \sum_{j=1}^{p} \phi (\lambda_j) = p \int f (x) dF^{B} (x). \]

Thus, for brevity, we define the normalized LSSs as

\[ Y_l = \int f_i (x) dG_n (x) - \sum_{k=1}^{K} d_k f_i (\phi_n (\alpha_k)) - \frac{M}{2 \pi i} \int f_i (z) \frac{m'(z)}{m(z)} dz. \quad l = 1, 2, \ldots, h, \]

where

\[ G_n (x) = p [F^{B} (x) - F^{c_n,H_n} (x)], \quad \phi_n (x) = x \left( 1 + c_n \int \frac{t}{x-t} dH_n (t) \right), \]

and \( F^{c_n,H_n} \) is the LSD \( F^{c,H} \) with \( c, H \) replaced by \( c_n, H_n \). \( F^{c_n,H_n} \) denotes the LSD of matrix \( n^{-1}X^{*}U_2D_2U_2X \), \( U = (U_1, U_2) \), \( U_1 = (u_{ij})_{i=1, \ldots, p, j=1, \ldots, M} \), \( \phi_k = \phi (x) |_{x=\alpha_k} = \alpha_k \left( 1 + c \int \frac{t}{x-t} dH (t) \right), \]

\[ b_n (z) = \frac{1}{1 + n^{-1} \text{tr} \Gamma \Gamma^{*} A_j^{-1} (z)} \], \( U_{t_{i_1}, t_{j_1} t_{i_2} t_{j_2}} = \sum_{t=1}^{p} \pi_{t_{i_1} t_{j_1} t_{i_2} t_{j_2}}, \]

\[ \theta_k = \phi_k^2 m_2 (\phi_k) \], \( \nu_k = \phi_k^2 m_2 (\phi_k) \),

\[ m (\lambda) = \int \frac{1}{\lambda - x} dF^{c,H} (x), \quad m_2 (\lambda) = \int \frac{1}{(\lambda - x)^2} dF^{c,H} (x), \]

\[ c_{nM} = \frac{p - M}{n}, \quad H_{2n} = F^{D_2}, \]

\[ P_n (z) = (1 - c_{nM} \Gamma \Gamma^{*} - z c_{nM} m_{2n0} (z) \Gamma \Gamma^{*} - z I_p)^{-1}. \]

Here, \( m_{2n0} (z) \) is the Stieltjes transform of \( F^{c_{nM},H_{2n}} \) and \( m_{2n0} (z) = -\frac{1-c_{nM}}{z} + c_{nM} m_{2n0} (z) \). For clarification purposes, \( m_{1n0} (z) \) also denotes the Stieltjes transform of \( F^{c_{nM},H_{2n}} \), \( m_n = \frac{1}{p} \text{tr} (B - z I_p)^{-1} \), and \( m_{2n} = \frac{1}{p-M} \text{tr} (S_{2n} - z I_{p-M})^{-1} \), which will be used later in the proof.

We first establish a CLT of the LSS without any restrictions imposed on the Gaussian-like moments or on the structures of the population covariance matrix by renormalizing the LSS.

**Theorem 3.1.** Under Assumptions 1–4, we have

\[ \frac{Y_1 - \mu_1}{\sqrt{\nu_1^2}} \xrightarrow{d} N (0, 1), \]

where the mean function is

\[ \mu_1 = -\frac{\alpha x}{2 \pi i} \cdot \int_{c} \frac{c_{nM} f_1 (z) \int m_{2n0}^3 (z) t^2 (1 + t m_{2n0} (z))^{-3} dH_{2n} (t)}{1 - c_{nM} \int \frac{m_{2n0}^2 (z) t^2}{(1 + t m_{2n0} (z))^{2}} dH_{2n} (t)} dz \]

(3.4)

\[ -\frac{\beta x}{2 \pi i} \cdot \int_{c} \frac{c_{nM} f_1 (z) \int m_{2n0}^3 (z) t^2 (1 + t m_{2n0} (z))^{-3} dH_{2n} (t)}{1 - c_{nM} \int \frac{m_{2n0}^2 (z) t^2}{(1 + t m_{2n0} (z))^{2}} dH_{2n} (t)} dz, \]
and the covariance function is

\[ s^2_k = \sum_{k=1}^{K} \frac{\phi_n^2(\alpha_k)}{n} \left(f_1(\phi_n(\alpha_k)))^2 s_k^2 \right) - \frac{1}{4\pi^2} \int_{C_1} \int_{C_2} f_1(z_1) f_1(z_2) \vartheta_n^2 dz_1 dz_2. \]

Here, \( s^2_k = \sum_{j \in \mathcal{N}} ((\alpha_j + 1) \delta_j + \beta_j U_j, j \nu_k) + \sum_{j \in \mathcal{N}} \beta_j U_j, j \nu_k \), \( \vartheta_n^2 = \Theta_{0,n}(z_1, z_2) + \alpha_x \Theta_{1,n}(z_1, z_2) + \beta_x \Theta_{2,n}(z_1, z_2), \) where

\[ \Theta_{0,n}(z_1, z_2) = \frac{m_{2n0}(z_1) m_{2n0}^*(z_2)}{(m_{2n0}(z_1))^2 - (z_1 - z_2)^2}, \]

\[ \Theta_{1,n}(z_1, z_2) = \frac{\partial}{\partial z_2} \left\{ \frac{\partial A_n(z_1, z_2)}{\partial z_1} \frac{1}{1 - \alpha_x A_n(z_1, z_2)} \right\}, \]

\[ A_n(z_1, z_2) = \frac{z_1 z_2}{n} m_{2n0}^2(z_1) m_{2n0}^*(z_2) \int \Gamma^* \mathbf{P}_n(z_1) \Gamma \mathbf{P}_n(z_2) \Gamma, \]

\[ \Theta_{2,n}(z_1, z_2) = \frac{z_1^2 z_2^2 m_{2n0}^2(z_1) m_{2n0}^*(z_2)}{n} \sum_{i=1}^{p} [\Gamma^* \mathbf{P}_n(z_1) \Gamma]_{ii} [\Gamma^* \mathbf{P}_n(z_2) \Gamma]_{ii}. \]

In the equation above, \( C, C_1 \) and \( C_2 \) are closed contours in the complex plan enclosing the support of the \( F^{c,H} \), and \( C_1 \) and \( C_2 \) are nonoverlapping. It is worth noting that \( \Theta_{1,n}(z_1, z_2) \) and \( \Theta_{2,n}(z_1, z_2) \) may not converge.

As a minor price for the applicability enlargement, the new CLT described above only applies to a single LSS. To guarantee that the new CLT applies to multiple normalized LSSs, structural assumptions about the population covariance matrices are needed (Assumptions 5 and 6). The following theorem is a non-trivial extension of the BST:

**Theorem 3.2.** Under Assumptions 1–6, the random vector

\[ \left( \frac{Y_1 - \mu_1}{\sqrt{\sigma_1^2}}, \ldots, \frac{Y_h - \mu_h}{\sqrt{\sigma_h^2}} \right) \overset{d}{\rightarrow} \mathcal{N}_h \left(0, \mathbf{\Psi} \right), \]

with mean function

\[ \mu_l = -\frac{\alpha_x}{2\pi i} \cdot \int_C f_l(z) \frac{c_{nM} \int m_{2n0}^2(z) t^2 (1 + t m_{2n0}(z))^{-3} dH_{2n}(t)} \left( 1 - c_{nM} \int m_{2n0}^2(z) t^2 (1 + t m_{2n0}(z))^{-2} dH_{2n}(t) \right) dz, \]

\[ -\frac{\beta_x}{2\pi i} \cdot \int_C f_l(z) \frac{c_{nM} \int m_{2n0}^2(z) t^2 (1 + t m_{2n0}(z))^{-3} dH_{2n}(t)} \left( 1 - c_{nM} \int m_{2n0}^2(z) t^2 (1 + t m_{2n0}(z))^{-2} dH_{2n}(t) \right) dz, \quad l = 1, \ldots, h, \]

variance function

\[ \sigma_l^2 = \sum_{k=1}^{K} \frac{\phi_n^2(\alpha_k)}{n} \left(f_1(\phi_n(\alpha_k)))^2 s_k^2 \right) + \kappa_{ll}, \quad l = 1, \ldots, h, \]

and covariance matrix \( \mathbf{\Psi} = (\psi_{st})_{h \times h}, \)

\[ \psi_{st} = \frac{\sum_{k=1}^{K} \varpi_{st}^k \varpi_{st}^k s_k^2 + \kappa_{st}}{\sqrt{\sum_{k=1}^{K} (\varpi_{st}^k)^2 s_k^2 + \kappa_{ss}} \sqrt{\sum_{k=1}^{K} (\varpi_{st}^k)^2 s_k^2 + \kappa_{tt}}}, \]
where
\[
\kappa_{st} = - \frac{1}{4\pi^2} \int_{c_1} \int_{c_2} \frac{f_s(z_1) f_t(z_2)}{(m(z_1) - m(z_2))^2} dm(z_1) dm(z_2) - \frac{c_2 \alpha}{4\pi^2} \int_{c_1} \int_{c_2} f_s(z_1) f_t(z_2) \times \left[ \int \frac{t^2}{(m(z_1) t + 1)^2(m(z_2) t + 1)^2} dH(t) \right] dm(z_1) dm(z_2) \\
- \frac{1}{4\pi^2} \int_{c_1} \int_{c_2} f_s(z_1) f_t(z_2) \left[ \frac{\partial^2}{\partial z_1 \partial z_2} \log (1 - a(z_1, z_2)) \right] dz_1 dz_2,
\]
\[
a(z_1, z_2) = \alpha_x \left( 1 + \frac{m(z_1) m(z_2)(z_1 - z_2)}{m(z_2) - m(z_1)} \right).
\]

Note that \(s_n^2\) is defined in Theorem 3.1, and \(\varpi_k^n = \lim_{n \to \infty} \frac{\phi_n(\alpha_k)}{\sqrt{n}} f'_1(\phi_n(\alpha_k))\) is allowed to be infinity.

**Remark 3.1.** If we rewrite the covariance matrix \(\Psi\) in Theorem 3.2 in the form of expressions with \(n\), such as \(\zeta^2\) in Theorem 3.1, then Assumptions 5-6 should be removed analogously. However, in this case, we need to verify the nonsingularity of these covariance matrices, which is difficult unless the kernel functions are linearly related. Thus, we decided not to pursue that direction in this paper.

**Remark 3.2.** It is not difficult to find from the theorem that the asymptotic distributions of the LSSs depend on the divergence rates of \(\phi_n(\alpha_k)\) and \(f'_1(\phi_n(\alpha_k))\). In particular, when \(\frac{\phi_n(\alpha_1)}{\sqrt{n}} f'_1(\phi_n(\alpha_1)) \to 0 (n \to \infty)\), Theorem 3.2 reduces to Theorem 2.1 in Zheng et al. (2015); when \(\frac{\phi_n(\alpha_0)}{\sqrt{n}} f'_1(\phi_n(\alpha_0)) \to \infty (n \to \infty)\), Theorem 3.2 is a non-trivial extension of the CLT derived by Jiang and Bai (2021). Furthermore, Yin (2021) recently obtained a CLT when the kernel functions were polynomial.

**Remark 3.3.** If \(D_2 = I_{p-M}\), then the mean function \(\mu_1\) and \(\kappa_{st}\) in the covariance function of Theorem 3.2 can be simplified from the results in Wang and Yao (2013) and Zheng et al. (2015), i.e.,
\[
\phi_n(x) = x + \frac{x(p - M)}{n(x - 1)}, \quad \mu_1 = \alpha_x I_1(f_1) + \beta_x I_2(f_1),
\]
\[
\kappa_{st} = (\alpha_x + 1) J_1(f_s, f_t) + \beta_x J_2(f_s, f_t),
\]
\[
I_1(f_1) = \lim_{r \downarrow 1} \frac{1}{2\pi i} \oint_{|z|=1} f_1 \left( 1 + \sqrt{cz} \right)^2 \left[ \frac{z}{z^2 - r^2} - \frac{1}{z} \right] dz,
\]
\[
I_2(f_1) = \frac{1}{2\pi i} \oint_{|z|=1} f_1 \left( 1 + \sqrt{cz} \right)^2 \frac{1}{z^3} dz,
\]
\[
J_1(f_s, f_t) = \lim_{r \downarrow 1} \frac{-1}{4\pi^2} \oint_{|z_1|=1} \oint_{|z_2|=1} \frac{f_s \left( 1 + \sqrt{cz_1} \right) f_t \left( 1 + \sqrt{cz_2} \right)}{(z_1 - r z_2)^2} dz_1 dz_2,
\]
\[
J_2(f_s, f_t) = - \frac{1}{4\pi^2} \oint_{|z_1|=1} \oint_{|z_2|=1} \frac{f_s \left( 1 + \sqrt{cz_1} \right)^2}{z_2^2} dz_1 \oint_{|z_2|=1} \frac{f_t \left( 1 + \sqrt{cz_2} \right)^2}{z_2^2} dz_2.
\]
4. Application. In this section, we focus on testing the hypothesis that a high-dimensional covariance matrix $\Sigma$ is equal to an identity matrix, that is,

$$H_0 : \Sigma = I_p \quad \text{vs} \quad H_1 : \Sigma = V \left( \begin{array}{cc} D_1 & 0 \\ 0 & I_{p-M} \end{array} \right) V^*,$$

where $D_1 = \text{diag}(\alpha_1, \ldots, \alpha_1, \alpha_2, \ldots, \alpha_2, \ldots, \alpha_K, \ldots, \alpha_K)$. For this problem, the two most classical test statistics are the likelihood ratio test (LRT) statistic (Wilks, 1938) and Nagao’s trace (NT) test statistic (Nagao, 1973). Specifically, the LR and NT statistics can be formalized as

$$L = \text{tr} \, B - \log |B| - p \quad \text{and} \quad W = \text{tr}(B - I_p)^2,$$

respectively. Under the null hypothesis, we refer to Bai et al. (2009); Jiang and Yang (2013); Ledoit and Wolf (2002); Wang and Yao (2013); Onatski et al. (2013) for the asymptotic properties of the LR and NT statistics for high-dimensional settings. In this section, we mainly focus on the alternative hypothesis $H_1$. However, to provide a better comparison, we also present the asymptotic distributions under the null hypothesis in the following theorems:

**Theorem 4.1 (CLT for LR).** Under Assumptions 1-4 with $c_n = p/n \to c \in (0, 1)$, we have

- (Under $H_0$)

$$\frac{L - p\ell_L - \mu_L}{\sqrt{\varsigma^2_L}} \overset{d}{\to} N(0, 1),$$

where

$$\ell_L = 1 - \frac{c_n - 1}{c_n} \log (1 - c_n), \quad \mu_L = -\frac{1}{2} \log (1 - c_n) \alpha_x + \frac{c_n}{2} \beta_x$$

and

$$\varsigma^2_L = \frac{\alpha_x + 1}{2} (-2 \log (1 - c_n) - 2c_n).$$

- (Under $H_1$)

$$\frac{L - (p - M)\tilde{\ell}_L - \tilde{\mu}_L}{\sqrt{\tilde{\varsigma}^2_L}} \overset{d}{\to} N(0, 1),$$

where

$$\tilde{\ell}_L = 1 - \frac{c_nM - 1}{c_nM} \log (1 - c_nM),$$

$$\tilde{\mu}_L = -\frac{1}{2} \log (1 - c_nM) \alpha_x + \frac{c_nM}{2} \beta_x$$

$$+ \sum_{k=1}^K d_k \left( \phi_n (\alpha_k) - \log \phi_n (\alpha_k) - 1 \right) - M(c_n + \log (1 - c_n))$$

$$\tilde{\varsigma}^2_L = \frac{\alpha_x + 1}{2} (-2 \log (1 - c_nM) - 2c_nM) + \sum_{k=1}^K d_k \frac{2 (\phi_n (\alpha_k) - 1)^4}{n \phi_n^2 (\alpha_k)}$$

$$\phi_n (\alpha_k) = \alpha_k \frac{p - M}{n (\alpha_k - 1)}.$$
Remark 4.1. If \( c > 1 \), then \( \mathbf{B}_n \) is singular for large \( n \), which gives rise to the undefined LR statistic \( L \). Thus, in Theorem 4.1, we add an additional restriction \( c < 1 \).

Theorem 4.2 (CLT for NT). Under Assumptions 1-4, we have

- **(Under \( H_0 \),)**
  \[
  \frac{W - p\ell_W - \mu_W}{\sqrt{\varsigma_W^2}} \xrightarrow{d} N(0, 1),
  \]
  where
  \[
  \ell_W = c_n, \quad \mu_W = \alpha_x c_n + \beta_x c_n \quad \text{and} \quad \varsigma_W^2 = (\alpha_x + 1)(4c_n^3 + 2c_n^2) + 4\beta_x c_n^3.
  \]

- **(Under \( H_1 \),)**
  \[
  \frac{W - (p - M)\tilde{\ell}_W - \tilde{\mu}_W}{\sqrt{\tilde{\varsigma}_W^2}} \xrightarrow{d} N(0, 1),
  \]
  where
  \[
  \tilde{\ell}_W = c_{nM}, \quad \tilde{\mu}_W = \alpha_x c_{nM} + \beta_x c_{nM} + \sum_{k=1}^{K} d_k (\phi_n^2 (\alpha_k) - 2\phi_n (\alpha_k) + 1) - Mc_n^2
  \]
  \[
  \tilde{\varsigma}_W^2 = (\alpha_x + 1)(4c_{nM}^3 + 2c_{nM}^2) + 4\beta_x c_{nM}^3 + \sum_{k=1}^{K} d_k 8(\phi_n (\alpha_k) - 1)^4
  \]
  \[
  \phi_n (\alpha_k) = \alpha_k + \frac{\alpha_k (p - M)}{n(\alpha_k - 1)}.
  \]

Remark 4.2. For illustration, we study the asymptotic properties of the LR and NT statistics under a simplified spiked population model (Johnstone, 2001), where the population covariance matrix is diagonal, with only one spiked eigenvalue, i.e.,

\[
\Sigma = \text{diag}(\alpha_1, 1, \ldots, 1)
\]

Let \( z_a \) be the upper \( a \)% quantile of the standard Gaussian distribution \( \Phi \) and \( \phi_1 = \alpha_1 + \frac{\alpha_1}{\alpha_1 - 1} \).

It is straightforward from Theorem 4.1 that the asymptotic power of the LR statistic under the alternative hypothesis (4.6) is

\[
\Phi\left(-c + (\phi_1 - \log \phi_1 - 1) - z_a \sqrt{\frac{(\alpha_1 + 1)(-2 \log(1 - c) - 2c)}{2 \left(\frac{\phi_1 - 1}{\phi_1}\right)}}\right).
\]

Analogously, the asymptotic power of the NT statistic under the alternative hypothesis (4.6) can be formulated as

\[
\Phi\left(-2c + (\phi (\alpha_1) - 1)^2 - c^2 - z_a \sqrt{(\alpha_x + 1)(4c^3 + 2c^2) + 4\beta_x c^3}\right).
\]

It is easy to find that if \( \alpha_1 \to \infty \), the asymptotic power of the LR and NT statistics tend to 1 of order \( \min\{\alpha_1, \sqrt{n}\} \) and \( \min\{\alpha_1^2, \sqrt{n}\} \), respectively.
5. Simulation. In this section, we conducted a number of simulation studies to examine the asymptotic distributions of statistics $L$ and $W$ under $H_0$ and $H_1$ in Section 4. For brevity, under $H_1$, we focus on a simplified spiked population with only one spiked eigenvalue $\alpha_1$. In our experiments, we define $Z = TX$, where $X = (x_{ij})_{p \times n}$, and $x_{ij}$ are i.i.d. standard normality distributions. All the simulations are based on $c = \frac{1}{3}$ and 1,000 repetitions.

The theories in Section 4 reveal that for the LR statistic $L = \text{tr}B - \log |B| - p$, when the spiked eigenvalue is weak, there is only a constant shift in the mean term, but the covariance term remains unchanged. When the spiked eigenvalue divergences of order $n^{1/2}$, the mean term has a divergence shift, and the covariance term has a constant shift. When $\alpha_1/n^{1/2}$ tends to infinity, the asymptotic distribution is dominated by the spiked eigenvalue. For these properties, we set three cases in the simulation, which are:

Case 1: $\alpha_1 = 3$,  Case 2: $\alpha_1 = n^{1/2}$,  Case 3: $\alpha_1 = n^{2/3}$.

Similar to the discussion of the LR statistic, we know that the distinguishing divergence rate of $\alpha_1$ in the NT statistic $W = \text{tr}(B - I_p)^2$ is $n^{1/4}$. Thus, to examine the properties of $W$, we set three cases:

Case 4: $\alpha_1 = 3$,  Case 5: $\alpha_1 = n^{1/4}$,  Case 6: $\alpha_1 = n^{1/3}$.

All the results are presented in Figures 1-6. Note that the kernel density and normal density under each setting are compared using the same color.

We highlight two observations from Figures 1-6. First, all the curves fit the limiting distribution well when the dimension is sufficiently large, which is consistent with Theorem 3.1 and Theorem 3.2. Second, by comparing Figures 1-3 with Figures 4-6, we find that the NT statistic is more powerful than the LR statistic for the spiked alternative hypothesis, which indicates that different kernel function choices are also important for the testing problem.

6. Technical proofs. In this section, we present the proofs of Theorems 3.1, 3.2, 4.1 and 4.2. First, we truncate and renormalize the random variables to ensure the existence of their higher order moments.
6.1. Truncation and renormalization. Let \( \hat{x}_{ij} = x_{ij} \mathbb{1}_{\{x_{ij} < \eta_n \sqrt{n}\}} \) and \( \check{x}_{ij} = \frac{x_{ij} - \mathbb{E}[\hat{x}_{ij}]}{\hat{\sigma}_n} \), where \( \hat{\sigma}_n^2 = \mathbb{E}[\hat{x}_{ij} - \mathbb{E}[\hat{x}_{ij}]]^2 \) and \( \eta_n \to 0 \) with a slow rate. Correspondingly, define \( \check{B} = \frac{1}{n} T \check{X} \check{X}^* T^* \) and \( \check{\hat{B}} = \frac{1}{n} T \check{X} \check{X}^* T^* \), where \( \check{X} = (\check{x}_{ij}) \) and \( \check{\hat{X}} = (\check{x}_{ij}) \). \( \hat{G}_n \) and \( \check{G}_n \) denote the analogues of \( G_n \) with the matrix \( B \) replaced by \( \check{B} \) and \( \check{\hat{B}} \), respectively. Next, we demonstrate that the entries of \( X \) in the LSSs are equivalent to and can be replaced by the truncated and renormalized entries.

According to the Lindeberg-type condition in Assumption 1, we obtain that as \( \min\{n, p\} \to \infty \),
\[
P(B \neq \check{B}) \leq \sum_{i,j} P\left( |x_{ij}| \geq \eta_n \sqrt{n} \right) \to 0.
\]
It follows from the definition of LSSs that for any \( l = 1, \ldots, h \),
\[
\left| \int f_l(x) d\hat{G}_n - \int f_l(x) d\check{G}_n(x) \right| = \sum_{i=1}^p \left| f_l(\lambda_{i0}) - f_l(\lambda_{i0}) \right|
\]
(6.7)
\[
\leq \sum_{i=1}^M \left| f_l(\lambda_{i0}) - f_l(\lambda_{i0}) \right| + \sum_{i=M+1}^p \left| f_l(\lambda_{i0}) - f_l(\lambda_{i0}) \right|.
\]
Using the same discussion in Bai and Silverstein (2004), we can easily obtain that the second term of (6.7) tends to 0 in probability. For the first term of (6.7), from the arguments in Supplement B of Jiang and Bai (2021), we know that
\[
\left| \lambda_{i0} - \lambda_{i0} \right| = o_p(n^{-\frac{1}{2}} \rho_i).
\]
Then, for brevity, we denote \( \beta_i = (\lambda_{i0} - \lambda_{i0})/\rho_i \) and obtain that
\[
f_l(\lambda_{i0}) - f_l(\lambda_{i0}) = \int_0^{\beta_i \rho_i} f'_l(t + \lambda_{i0}) dt
\]
\[
= \int_0^1 \beta_i \rho_i f'_l(\beta_i \rho_i + \lambda_{i0}) ds = \beta_i \rho_i f'_l(\rho_i) \int_0^1 \frac{f'_l(\beta_i \rho_i + \lambda_{i0})}{f'_l(\rho_i)} ds,
\]
which, together with Assumption 4 and (6.8), implies
\[
\sqrt{n} \sum_{i=1}^M \left| f_l(\lambda_{i0}) - f_l(\lambda_{i0}) \right| = o_p(1).
\]
Therefore, in the following proofs, we can safely assume that \( |x_{ij}| < \eta_n \sqrt{n} \).

6.2. Some primary definitions and lemmas. In this section, we provide some useful results that will be used later in the proofs of Theorems 3.1 and 3.2. For the population covariance matrix \( \Sigma = T T^* \), consider the corresponding sample covariance matrix \( B = T S_x T^* \), where \( S_x = \frac{1}{n} X X^* \). By singular value decomposition of \( T \) (see (2.2)),
\[
B = V \begin{pmatrix}
D_{11}^{\frac{1}{2}} U^*_1 S^*_x U_1 D_{11}^{\frac{1}{2}} & D_{12}^{\frac{1}{2}} U^*_1 S^*_x U_2 D_{22}^{\frac{1}{2}} \\
D_{21}^{\frac{1}{2}} U^*_2 S^*_x U_1 D_{12}^{\frac{1}{2}} & D_{22}^{\frac{1}{2}} U^*_2 S^*_x U_2 D_{22}^{\frac{1}{2}}
\end{pmatrix} V^*.
\]
Note that
\[
S = \begin{pmatrix}
D_{11}^{\frac{1}{2}} U^*_1 S^*_x U_1 D_{11}^{\frac{1}{2}} & D_{12}^{\frac{1}{2}} U^*_1 S^*_x U_2 D_{22}^{\frac{1}{2}} \\
D_{21}^{\frac{1}{2}} U^*_2 S^*_x U_1 D_{12}^{\frac{1}{2}} & D_{22}^{\frac{1}{2}} U^*_2 S^*_x U_2 D_{22}^{\frac{1}{2}}
\end{pmatrix} = \begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}.
\]
Note that $\mathbf{B}$ and $\mathbf{S}$ have the same eigenvalues.

Recall that $\mathbf{B} = \frac{1}{n} \mathbf{X}' \mathbf{T}' \mathbf{T} \mathbf{X}$ (the spectral of which differs from that of $\mathbf{B}$ by $|n - p|$ zeros). Its limiting spectral distribution is $F^c, H$, $L_c, H \equiv (1 - c) \mathbf{1}_{(0, \infty)} + c F^c, H$, and its Stieltjes transform is $m(z)$. Let $\lambda_j$ be the eigenvalues of $\mathbf{S}_{22}$ so that the LSS of $\mathbf{S}_{22}$ is $\sum_{j=1}^{p-M} f(\lambda_j)$. Correspondingly, recall that $c_n M := \frac{p - M}{n}$, $H_{2n} :=FD_2$ and $m_{2n0} := m_{2n0}(z)$ are the Stieltjes transforms of $F_{c_n M, H_{2n}}$. Then, we have the following preliminary results:

**Lemma 6.1.** Under Assumptions 1-4,

$$(p - M) \int f(x) dF_{c_n M, H_{2n}} = p \int f(x) dF_{c_n, H_n}(x).$$

**Proof.** By the Cauchy integral formula,

$$p \int f(x) dF_{c_n, H_n} = -\frac{p}{2\pi i} \oint_C f(z) m_{1n0} dz = -\frac{n}{2\pi i} \oint_C f(z) m_{1n0} dz,$$

$$(p - M) \int f(x) dF_{c_n M, H_{2n}} = -\frac{p - M}{2\pi i} \oint_C f(z) m_{2n0} dz = -\frac{n}{2\pi i} \oint_C f(z) m_{2n0} dz,$$

where $m_{1n0}$ and $m_{2n0}$ are the Stieltjes transforms of $F_{c_n, H_n}$ and $F_{c_n M, H_{2n}}$, respectively. Then,

$$(p - M) \int f(x) dF_{c_n M, H_{2n}} - p \int f(x) dF_{c_n, H_n} = \frac{n}{2\pi i} \oint_C f(z) (m_{1n0} - m_{2n0}) dz.$$

Next, we prove that $m_{1n0} = m_{2n0}$.

Note that $m_{1n0}$ and $m_{2n0}$ are the unique solutions to

$$(6.9) \quad z = -\frac{1}{m_{1n0}} + c_n \int \frac{tdH_n(t)}{1 + tm_{1n0}}$$

$$(6.10) \quad z = -\frac{1}{m_{2n0}} + c_n M \int \frac{tdH_{2n}(t)}{1 + tm_{2n0}},$$

respectively, where $m_{1n0} = -\frac{1 - c_n}{z} + c_n m_{1n0}$ and $m_{2n0} = -\frac{1 - c_n M}{z} + c_n M m_{2n0}$. Since

$$H_n(t) = \frac{1}{p} \sum_{i=1}^{p} \mathbf{1}_{\{0 \leq t\}} + \sum_{i=M+1}^{p} \mathbf{1}_{\{t \leq \lambda_i\}} = \frac{M}{p} + \frac{1}{p} \sum_{i=M+1}^{p} \mathbf{1}_{\{t \leq \lambda_i\}}$$

and $H_{2n}(t) = \frac{1}{p-M} \sum_{i=M+1}^{p} \mathbf{1}_{\{t \leq \lambda_i\}}$, (6.9) can be written as

$$z = -\frac{1}{m_{1n0}} + \frac{p}{n} \int \frac{td\left(\frac{M}{p} + \frac{1}{p} \sum_{i=M+1}^{p} \mathbf{1}_{\{t \leq \lambda_i\}}(t)\right)}{1 + tm_{1n0}}$$

$$= -\frac{1}{m_{1n0}} + \frac{1}{n} \sum_{i=M+1}^{p} \frac{\alpha_i}{1 + \alpha_i m_{1n0}}.$$

Similarly, equation (6.10) can be written as

$$z = -\frac{1}{m_{2n0}} + \frac{1}{n} \sum_{i=M+1}^{p} \frac{\alpha_i}{1 + \alpha_i m_{2n0}}.$$
Thus, according to the fact that \( m_{1n0} \) and \( m_{2n0} \) are the unique solutions of (6.11) and (6.12), respectively, we have \( m_{1n0} = m_{2n0} \), which completes the proof of this lemma.

**Lemma 6.2.** Under Assumptions 1-4,

\[
\sum_{j=M+1}^{p} f(\lambda_j) - \sum_{j=1}^{p-M} f(\tilde{\lambda}_j) - \frac{M}{2\pi i} \oint_C f(z) \frac{m'(z)}{m(z)} \, dz = o_p(1).
\]

**Proof.** Note that

\[ L_1 := \sum_{j=M+1}^{p} f(\lambda_j). \]

By the Cauchy integral formula, we have

\[ L_1 = -\frac{p}{2\pi i} \oint_C f(z) m_n(z) \, dz, \]

where \( m_n = \frac{1}{p} \text{tr} (S - zI_p)^{-1} = \frac{1}{p} \text{tr} (B - zI_p)^{-1} \). Analogously, we have

\[ L_2 := \sum_{j=1}^{p-M} f(\tilde{\lambda}_j) = -\frac{p-M}{2\pi i} \oint_C f(z) m_{2n}(z) \, dz, \]

where \( m_{2n} = \frac{1}{p-M} \text{tr} (S_{22} - zI_{p-M})^{-1} \). By applying the block matrix inversion formula to \( m_n \), we can obtain

\[
(\text{6.13}) \quad L_1 - L_2 = -\frac{1}{2\pi i} \oint_C f(z) (T_1 - T_2) \, dz, \]

where

\[
T_1 = \text{tr} \left( S_{11} - zI_M - S_{12} (S_{22} - zI_{p-M})^{-1} S_{21} \right)^{-1}, \]

\[
T_2 = -\text{tr} \left[ (S_{11} - zI_M - S_{12} (S_{22} - zI_{p-M})^{-1} S_{21})^{-1} S_{12} (S_{22} - zI_{p-M})^{-2} S_{21} \right].
\]

Note that for any matrix \( Z \),

\[ Z (Z^* Z - \lambda I)^{-1} Z^* = I + \lambda (ZZ^* - \lambda I)^{-1}, \]

which, together with the notation \( Y_n := \frac{1}{n} \text{D}_i^\frac{1}{2} U_i^* X \left( \frac{1}{n} X^* U_2 D_2 U_2^* X - z I_n \right)^{-1} X^* U_1 D_1^\frac{1}{2} \), implies that

\[
T_1 = -z^{-1} \text{tr} (I_M + Y_n)^{-1},
\]

\[
T_2 = z^{-1} \text{tr} \left[ (I_M + Y_n)^{-1} S_{12} (S_{22} - zI_{p-M})^{-2} S_{21} \right].
\]

\( m_{2n} = m_{2n}(z) \) denotes the Stieltjes transform of \( F_{\frac{1}{n} X^* U_2 D_2 U_2^* X} \). Thus, we have that \( m_{2n}(z) - m(z) = o_p(1) \) for any \( z \in C \). From Theorem 3.1 of (Jiang and Bai, 2021), we know that

\[
(\text{6.14}) \quad \frac{1}{n} U_1^* X \left( \frac{1}{n} X^* U_2 D_2 U_2^* X - z I_n \right)^{-1} X^* U_1 = m_{2n}(z) I_M + o_p(n^{-\frac{1}{2}}).
\]
Thus, under Assumption 3, we obtain that
\begin{equation}
D_1^{1/2}(I_M + \U_n)^{-1}D_1^{1/2} = \frac{1}{m(z)}I_M + o_p(1),
\end{equation}
which yields
\begin{equation}
T_1 = o_p(1).
\end{equation}

It follows that
\begin{align}
D_1^{-1/2}S_{12}(S_{22} - zI_{p-M})^{-2}S_{21}D_1^{-1/2} \\
= \frac{1}{n} \text{tr} \left[ (S_{22} - zI_{p-M})^{-2}S_{22} \right] I_M + O_p(n^{-\frac{1}{2}}) \\
= \frac{1}{n} \text{tr} (S_{22} - zI_{p-M})^{-1}I_M + \frac{z}{n} \text{tr} (S_{22} - zI_{p-M})^{-2}I_M + O_p(n^{-\frac{1}{2}}) \\
= cm(z)I_M + cm'(z)I_M + o_p(1)
\end{align}
\begin{equation}
= m(z)I_M + zm'(z)I_M + o_p(1),
\end{equation}
where the last equality is derived from \( m = -\frac{1-\epsilon}{z} + cm(z) \). Therefore, according to (6.15) and (6.17), we obtain
\begin{equation}
T_2 = M \frac{m(z) + zm'(z)}{zm(z)} + o_p(1),
\end{equation}
which, together with (6.13) and (6.16), implies that
\begin{equation}
L_1 - L_2 = \frac{M}{2\pi i} \oint_C f(z) \frac{m(z) + zm'(z)}{zm(z)} dz + o_p(1) = \frac{M}{2\pi i} \oint_C f(z) \frac{m'(z)}{m(z)} dz + o_p(1).
\end{equation}
Therefore, the proof of this lemma is complete. 

Define random vector \( \gamma_k = (\gamma_{kj})' = \left( \sqrt{n} \frac{\lambda_j - \phi_k(\alpha_k)}{\phi_k(\alpha_k)}, j \in J_k \right)' \), where \( J_k \) is the indicator set of a packet of \( d_k \) consecutive sample eigenvalues. Then, we present the following lemma, which is borrowed from Jiang and Bai (2021) and characterizes the limiting distribution of the spiked eigenvalues of the sample covariance matrix.

**Lemma 6.3.** (Jiang and Bai (2021)) Under Assumptions 1-4, random vector \( \gamma_k \) converges weakly to the joint distribution of \( d_k \) eigenvalues of a Gaussian random matrix
\begin{equation}
-\frac{1}{\theta_k} [\Omega_{\phi_k}]_{kk},
\end{equation}
where
\begin{equation}
\theta_k = \phi_k^2 m_2(\phi_k), \quad m_2(\lambda) = \int \frac{1}{(\lambda - x)^2} dF^{c,H}(x)
\end{equation}
with \( F^{c,H} \) being the LSD of matrix \( n^{-1}X^*U_2D_2U_2^*X \), \( \phi_k = \alpha_k \left( 1 + c \int \frac{1}{\alpha_k - x} dH(t) \right) \). \( [\Omega_{\phi_k}]_{kk} \) is the \( k \)th diagonal block of matrix \( \Omega_{\phi_k} \). The variances and covariances of the elements \( \omega_{ij} \) of \( \Omega_{\phi_k} \) are:
\begin{equation}
\text{Cov}(\omega_{i_1,j_1}, \omega_{i_2,j_2}) = \left\{ \begin{array}{ll}
(\alpha_x + 1)\theta_k + \beta_x U_{ii}U_{jj} \nu_k, & i_1 = j_1 = i_2 = j_2 = i \\
\theta_k + \beta_x U_{ij}U_{jj} \nu_k, & i_1 = i_2 = i \neq j_1 = j_2 = j \\
\beta_x U_{ii}U_{jj} \nu_k, & \text{other cases}
\end{array} \right.
\end{equation}
where \( \beta_x U_{ij}U_{jj} = \sum_{l=1}^p \tilde{u}_{il}u_{lj}u_{il}u_{lj} \beta_x, \quad u_i = (u_{i1}, \ldots, u_{ip})' \) are the \( i \)th column of the matrix \( U_1 \), \( \nu_k = \phi_k^2 m_2(\phi_k) \).
Recall that $\lambda_j$ is the eigenvalue of $\mathbf{B}$, and $\tilde{\lambda}_j$ is the eigenvalue of $\mathbf{S}_{22}$. The following lemma shows the independence between $\sum_{j=1}^M f(\lambda_j)$ and $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$.

**Lemma 6.4.** Under Assumptions 1-4, $\sum_{j=1}^M f(\lambda_j)$ and $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$ are asymptotically independent.

**Proof.** It is sufficient to prove that for a given $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$, the asymptotic limiting distribution of $\sum_{j=1}^M f(\lambda_j)$ does not depend on the random part of $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$, that is, it only depends on its limit.

First, we consider $f(x) = x$. From the proof of Theorem 3.1 in Jiang and Bai (2021), we have that

$$0 = |[\Omega_M(\phi_k)]_{kk} + \lim_{\gamma_k \rightarrow 0} \{ \phi_k^2 m_2(\phi_k) \} I_d|,$$

where $\Omega_M(\phi_k)$ is

$$\frac{\phi_k}{\sqrt{n}} \left[ \text{tr} \left( \left( \phi_k \mathbf{I} - \frac{1}{n} \mathbf{X}^* \mathbf{U}_2 \mathbf{D}_2 \mathbf{U}_2^* \mathbf{X} \right)^{-1} \right) \mathbf{I} - \mathbf{U}_1^* \mathbf{X} \left( \phi_k \mathbf{I} - \frac{1}{n} \mathbf{X}^* \mathbf{U}_2 \mathbf{D}_2 \mathbf{U}_2^* \mathbf{X} \right)^{-1} \mathbf{X}^* \mathbf{U}_1 \right],$$

and $m_2(\phi_k)$ is the limit of $\text{tr} \left( \phi_k \mathbf{I} - \frac{1}{n} \mathbf{X}^* \mathbf{U}_2 \mathbf{D}_2 \mathbf{U}_2^* \mathbf{X} \right)^{-2}$. Then, we know that $\gamma_k$ has the same asymptotic distribution with eigenvalues of $-\frac{[\Omega_M(\phi_k)]_{kk}}{m_2(\phi_k)}$ in order. Given $\mathbf{U}_2^* \mathbf{X}$, from Jiang and Bai (2021), we could suppose that $\mathbf{U}_2^* \mathbf{X}$ and $\mathbf{U}_1^* \mathbf{X}$ are independent. Then, the limiting distribution of $\gamma_k$ only depends on the limit of $\text{tr} \left( \phi_k \mathbf{I} - \frac{1}{n} \mathbf{X}^* \mathbf{U}_2 \mathbf{D}_2 \mathbf{U}_2^* \mathbf{X} \right)^{-1}$, that is, $m_2(\phi_k)$, and has nothing to do with the random part. Therefore, the independence between $\sum_{j=1}^M f(\lambda_j)$ and $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$ is obtained when $f(x) = x$.

When $f(x) \neq x$, by using the Newton-Leibniz formula, we have

$$\sum_{j=1}^M f(\lambda_j) - \sum_{k=1}^K d_k f(\phi_n(\alpha_k)) = \sum_{k=1}^K \sum_{j \in J_k} (f(\lambda_j) - f(\phi_n(\alpha_k)))$$

$$= \sum_{k=1}^K \sum_{j \in J_k} \int_0^1 \frac{\phi_n(\alpha_k)}{\sqrt{n}} \gamma_k f'(t + \phi_n(\alpha_k)) dt$$

$$= \sum_{k=1}^K \sum_{j \in J_k} \int_0^1 \frac{\phi_n(\alpha_k)}{\sqrt{n}} \gamma_k f'(\phi_n(\alpha_k)) \left( 1 + \frac{\gamma_k}{\sqrt{n}} \right) f'(\phi_n(\alpha_k)) ds$$

$$\to \sum_{k=1}^K \sum_{j \in J_k} \int_0^1 \gamma_k s \varpi^k ds = \sum_{k=1}^K \sum_{j \in J_k} \varpi^k \gamma_k = 0,$$

where (6.18) is true due to Assumption 4, and $\varpi^k = \lim_{\sqrt{n}} \phi_n(\alpha_k) f'(\phi_n(\alpha_k))$. Thus, we turn it into a function of $\gamma_k$. Since we have proven above that given $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$, the limiting distribution of $\gamma_k$ is only concerned with the limit of $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$, as is $\sum_{k=1}^K \sum_{j \in J_k} \varpi^k \gamma_k$, accordingly, we can conclude that $\sum_{j=1}^M f(\lambda_j)$ and $\sum_{j=1}^{p-M} f(\tilde{\lambda}_j)$ are asymptotically independent. The proof is complete. \(\square\)
The following lemma derives the asymptotic distribution of the LSS generated from sub-matrix $S_{22}$.

**Lemma 6.5.** Define $Q_1 = \sum_{j=1}^{p-M} f_1(\tilde{\lambda}_j) - (p - M) \int f_1(x) dF^{c_n,M,H_{2n}}$; then, under Assumptions 1-4, we have

$$\kappa^{-1}_1 (Q_1 - \mu_1) \overset{d}{\to} N(0,1)$$

with mean function

$$\mu_1 = -\frac{\alpha}{2\pi i} \cdot \oint f_1(z) \frac{c_{nM} \int m_2(z) t^2 (1 + tm_{2n}(z))^{-3} dH_{2n}(t) \left(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t)\right)}{(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t))} \, dz$$

$$-\frac{\beta}{2\pi i} \cdot \oint f_1(z) \frac{c_{nM} \int m_2(z) \int m_2(z) t^2 (1 + tm_{2n}(z))^{-3} dH_{2n}(t) \left(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t)\right)}{(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t))} \, dz,$$

and the covariance function is

$$\kappa^2 = -\frac{1}{4\pi^2} \oint f_1(z_1) f_1(z_2) \, \vartheta^2_n \, dz_1 \, dz_2,$$

where $\vartheta^2_n = \Theta_{0,n}(z_1, z_2) + \alpha_x \Theta_{1,n}(z_1, z_2) + \beta_x \Theta_{2,n}(z_1, z_2)$,

$$\Theta_{0,n}(z_1, z_2) = \frac{m_{2n}(z_1)m_{2n}(z_2)}{(m_{2n}(z_1) - m_{2n}(z_2))^2 - (z_1 - z_2)^2},$$

$$\Theta_{1,n}(z_1, z_2) = \frac{\partial}{\partial z_2} \left\{ \frac{\partial A_n(z_1, z_2)}{\partial z_1} \frac{1}{1 - \alpha_x A_n(z_1, z_2)} \right\},$$

$$A_n(z_1, z_2) = \frac{z_1z_2}{n} m_{2n}(z_1)m_{2n}(z_2) \text{tr} \Gamma^* P_n(z_1) \Gamma^* P_n(z_2) \Gamma,$$

$$\Theta_{2,n}(z_1, z_2) = \frac{z_1^2z_2^2}{n} m_{2n}(z_1)m_{2n}(z_2) \sum_{i=1}^{p} \left[ \Gamma^* P_n^2(z_1) \Gamma \right]_{ii} \left[ \Gamma^* P_n^2(z_2) \Gamma \right]_{ii},$$

and the definitions of $P_n$, $\Gamma$, and $m_{2n}$ are defined in Section 3.

**Proof.** From Zheng et al. (2015), we have that under Assumptions 1-4, the random variable $(\kappa^0_n)^{-1} (Q_1 - \mu_1) \overset{d}{\to} N(0,1)$, with mean function

$$\mu_1 = -\frac{\alpha}{2\pi i} \cdot \oint f_1(z) c_{nM} \int m_2(z) t^2 (1 + tm_{2n}(z))^{-3} dH_{2n}(t) \left(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t)\right) \, dz$$

$$-\frac{\beta}{2\pi i} \cdot \oint f_1(z) c_{nM} \int m_2(z) \int m_2(z) t^2 (1 + tm_{2n}(z))^{-3} dH_{2n}(t) \left(1 - c_{nM} \int \frac{m_2^2(z)t^2}{(1+tm_{2n}(z))^2} dH_{2n}(t)\right) \, dz,$$

and the covariance function is

$$(\kappa^0_n)^2 = -\frac{1}{4\pi^2} \oint f_1(z_1) f_1(z_2) (\vartheta^0_n)^2 \, dz_1 \, dz_2,$$

where

$$(\vartheta^0_n)^2 = \frac{n^2}{b_n(z_1)b_n(z_2)} \sum_{j=1}^{n} \text{tr} E_j \Gamma^{*} A_j^{-1}(z_1) E_j \left( \Gamma^{*} A_j^{-1}(z_2) \right)$$
where \( b_n(z) = \frac{1}{1 + n^{-1} \Theta^* \Lambda^{-1}(z)} \). The notation \( A_j, e_i \) is defined in Section 2. Moreover Najim and Yao (2016) provided an estimation \( \bar{\vartheta}_n^2 \) for \( (\vartheta_n^0)^2 \) and proved that \( (\vartheta_n^0)^2 \) is close to \( \bar{\vartheta}_n^2 \) in the Lévy–Prohorov distance, where \( \bar{\vartheta}_n^2 = \Theta_{0,n}(z_1, z_2) + \alpha_x \Theta_{1,n}(z_1, z_2) + \beta_x \Theta_{2,n}(z_1, z_2) \),

\[
\Theta_{0,n}(z_1, z_2) = \frac{m_{2n0}(z_1) m_{2n0}'(z_2)}{(m_{2n0}(z_1) - m_{2n0}(z_2))^2} - \frac{1}{(z_1 - z_2)^2},
\]

\[
\Theta_{1,n}(z_1, z_2) = \frac{\partial}{\partial z_2} \left\{ \frac{\partial A_n(z_1, z_2)}{\partial z_1} \right\},
\]

\[
A_n(z_1, z_2) = \frac{z_1 z_2}{n} \left\{ m_{2n0}'(z_1) m_{2n0}'(z_2) \right\} \sum_{i=1}^p \left[ \Gamma^* P_n(z_1) \Gamma P_n(z_2) \right]_{ii},
\]

\[
\Theta_{2,n}(z_1, z_2) = \frac{z_1 z_2}{n} \left\{ m_{2n0}'(z_1) m_{2n0}'(z_2) \right\} \sum_{i=1}^p \left[ \Gamma^* P_n^2(z_1) \right]_{ii} \left[ \Gamma^* P_n^2(z_2) \right]_{ii},
\]

The definitions of \( P_n, \Gamma, \) and \( m_{2n0} \) are defined in Section 3. Notably, if \( \Gamma \) is not real, the convergence of \( \Theta_{1,n}(z_1, z_2) \) is not granted. However, if \( \Gamma \) and entries \( x_{ij} \) are real, that is, \( \alpha_x = 1 \), then it can be easily proven that \( \Theta_{0,n}(z_1, z_2) = \Theta_{1,n}(z_1, z_2) \). Similarly, the convergence of \( \Theta_{2,n}(z_1, z_2) \) depends on the assumption that \( \Gamma^* \Gamma \) is diagonal; thus, under Assumptions 1-4, \( \Theta_{1,n}(z_1, z_2) \) and \( \Theta_{2,n}(z_1, z_2) \) may have no limits.

Thus, the covariance term \( (\kappa_0^2)^2 \) is estimable, and the estimation is \( \kappa_1^2 \), with

\[
\kappa_1^2 = -\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f_1(z_1) f_1(z_2) \vartheta_n^2 dz_1 dz_2.
\]

Therefore, the proof is finished. \( \square \)

6.3. **Proof of Theorem 3.1.** The proof of Theorem 3.1 builds on the decomposition analysis of the LSSs and is divided into part (I) \( \sum_{j=1}^M f(\lambda_j) \) and part (II) \( \sum_{j=M+1}^p f(\lambda_j) \). Enlightened by the BST in Bai and Silverstein (2004), we have

\[
\sum_{j=1}^p f(\lambda_j) - p \int f(x) dF_{\xi_0,n}H_n
\]

\[
= \sum_{j=1}^M f(\lambda_j) + \sum_{j=M+1}^p f(\lambda_j) - p \int f(x) dF_{\xi_0,n}H_n
\]

\[
= \sum_{j=1}^M f(\lambda_j) + \sum_{j=1}^{p-M} f(\bar{\lambda}_j) - (p - M) \int f(x) dF_{\xi_0,m,n,M}H_{2n} + \sum_{j=M+1}^p f(\lambda_j) - \sum_{j=1}^{p-M} f(\bar{\lambda}_j)
\]

\[
+ (p - M) \int f(x) dF_{\xi_0,m,n,M}H_{2n} - p \int f(x) dF_{\xi_0,n}H_n.
\]
Since Lemma 6.1 has shown the difference between \( p - M \) \( \int f(x) dF_{\text{cn},H_2} \) and \( p \int f(x) dF_{\text{cn},H_2} \) is 0. Moreover, in Lemma 6.2 we have proved

\[
\sum_{j=M+1}^{p} f(\lambda_j) - \sum_{j=1}^{p-M} f(\lambda_j) = \frac{M}{2\pi i} \int_{C} f(z) \frac{m'(z)}{m(z)} dz + o_p(1).
\]

It follows that

\[
\sum_{j=1}^{p} f(\lambda_j) - p \int f(x) dF_{\text{cn},H_2} = \sum_{j=1}^{M} f(\lambda_j) + \sum_{j=1}^{p-M} f(\lambda_j) - (p - M) \int f(x) dF_{\text{cn},H_2} + \frac{M}{2\pi i} \int_{C} f(z) \frac{m'(z)}{m(z)} dz + o_p(1),
\]

which yields

(6.19)

\[
\sum_{j=1}^{p} f(\lambda_j) - p \int f(x) dF_{\text{cn},H_2} = \sum_{j=1}^{M} f(\lambda_j) - \sum_{k=1}^{K} d_k f(\phi_n(\alpha_k)) - \frac{M}{2\pi i} \int_{C} f(z) \frac{m'(z)}{m(z)} dz
\]

(6.20)

\[
= \sum_{j=1}^{M} f(\lambda_j) - \sum_{k=1}^{K} d_k f(\phi_n(\alpha_k)) + \sum_{j=1}^{p-M} f(\lambda_j) - (p - M) \int f(x) dF_{\text{cn},H_2} + o_p(1).
\]

The analysis below is executed by dividing (6.20) into two parts: (I) \( \sum_{j=1}^{M} f(\lambda_j) - \sum_{k=1}^{K} d_k f(\phi_n(\alpha_k)) \) and (II) \( \sum_{j=1}^{p-M} f(\lambda_j) - (p - M) \int f(x) dF_{\text{cn},H_2} \), where we ignore the impact of \( o_p(1) \) on the asymptotic distribution. Since we have derived the asymptotic distribution of part (II) in Lemma 6.5, we only need to consider the asymptotic distribution of part (I) \( \sum_{j=1}^{M} f(\lambda_j) - \sum_{k=1}^{K} d_k f(\phi_n(\alpha_k)) \). From the proof of Lemma 6.4, \( \sum_{j=1}^{M} f(\lambda_j) - \sum_{k=1}^{K} d_k f(\phi_n(\alpha_k)) \) has the same limiting distribution as \( \sum_{k=1}^{K} \frac{\phi_n(\alpha_k)}{\sqrt{n}} f'(\phi_n(\alpha_k)) \sum_{j \in J_k} \gamma_{kj} \).

From Lemma 6.3, we have \( (\gamma_{kj}, j \in J_k)' \frac{d}{\Upsilon_k} - \frac{1}{\Upsilon_k} [\Omega_{\phi_k}]_{kk} \), so \( \sum_{j \in J_k} \gamma_{kj} \frac{d}{\Upsilon_k} - \frac{1}{\Upsilon_k} \text{tr} [\Omega_{\phi_k}]_{kk} \). Recall that \( \omega_{ij} \) is the element of \( \Omega_{\phi_k} \), and \( \text{tr} [\Omega_{\phi_k}]_{kk} \) is the summation of the diagonal element, that is, \( \sum_{j \in J_k} \omega_{jj} \). Because the diagonal elements are i.i.d., \( \mathbb{E} \left( \sum_{j \in J_k} \omega_{jj} \right) = 0 \),

\[
\text{Var} \left( \sum_{j \in J_k} \omega_{jj} \right) = \sum_{j \in J_k} \text{Var} (\omega_{jj}) + \sum_{j_1 \neq j_2} \text{cov} (\omega_{jj}, \omega_{j_1j_2}) = \sum_{j \in J_k} ((\alpha_x + 1) \theta_k + \beta_2 \mathcal{U}_{j,j,j,j} \nu_k) + \sum_{j_1 \neq j_2} \beta_2 \mathcal{U}_{j_1,j_1,j_2,j_2} \nu_k.
\]

Therefore, from Lemma 6.3, we have that the asymptotic distribution of \( \sum_{j \in J_k} \gamma_{kj} \) is a Gaussian distribution with

\[
\mathbb{E} \left( \sum_{j \in J_k} \gamma_{kj} \right) = 0,
\]

\[
s_k^2 \triangleq \text{Var} \left( \sum_{j \in J_k} \gamma_{kj} \right) = \sum_{j \in J_k} ((\alpha_x + 1) \theta_k + \beta_2 \mathcal{U}_{j,j,j,j} \nu_k) + \sum_{j_1 \neq j_2} \beta_2 \mathcal{U}_{j_1,j_1,j_2,j_2} \nu_k \frac{\phi_n(\alpha_k)}{\sqrt{n}} f'(\phi_n(\alpha_k)) \sum_{j \in J_k} \gamma_{kj}
\]

and then, we directly derive that the mean function of \( \sum_{k=1}^{K} \frac{\phi_n(\alpha_k)}{\sqrt{n}} f'(\phi_n(\alpha_k)) \sum_{j \in J_k} \gamma_{kj} \) is 0 and that its covariance function is

\[
\text{Var} (Y_{f_1}) = \sum_{k=1}^{K} \frac{\phi_n^2(\alpha_k)}{n} (f'_1(\phi_n(\alpha_k)))^2 s_k^2.
\]
Finally, we focus on the asymptotic distribution of equation (6.20). Because of Lemma 6.4, the two LSSs are asymptotically independent; thus, the random variable

\[ \zeta^{-1}_1 (Y_1 - \mathbb{E}Y_1) \overset{d}{\rightarrow} N(0,1) \]

with mean function \( \mathbb{E}Y_1 = \mu_1 \) being

\[ -\frac{\alpha_1}{2\pi i} \cdot \int_{\mathcal{C}} f_1(z) \frac{c_{nM} \int m^3_{2n0}(z) t^2 (1 + t m_{2n0}(z))^{-3} dH_{2n}(t)}{(1 - c_{nM} \int m^2_{2n0}(z) t^2 dH_{2n}(t))} \] \[ - \frac{\beta_1}{2\pi i} \cdot \int_{\mathcal{C}} f_1(z) \frac{c_{nM} \int m^3_{2n0}(z) t^2 (1 + t m_{2n0}(z))^{-3} dH_{2n}(t)}{1 - c_{nM} \int m^2_{2n0}(z) t^2 (1 + t m_{2n0}(z))^{-2} dH_{2n}(t)} \] \[ \sum_{k=1}^{K} \phi_n^2(\alpha_k) \left( f_1'(\phi_n(\alpha_k)) \right)^2 \sum_{j \in J_k} \gamma_{kj}, \ldots, \sum_{k=1}^{K} \phi_n^2(\alpha_k) \left( f_1'(\phi_n(\alpha_k)) \right)^2 \sum_{j \in J_k} \gamma_{kj} \right) \]

and

\[ \left( \sum_{j=1}^{p-M} f_1(\tilde{\lambda}_j) - (p-M) \int f_1(x) dF_{c_{nM},H_{2n}}(x) \right) \]

Because of equation (6.20), the random vector \( (Y_1, \ldots, Y_h) \) shares the same asymptotic distribution with the summation of two random vectors

Both random vectors are asymptotically independent; thus, the random vector

\[(Y_1 - \mathbb{E}Y_1, \ldots, Y_h - \mathbb{E}Y_h) ^{\dagger} \overset{d}{\rightarrow} N_h(0, \Omega), \]

where \( \phi_n^2(\alpha_k) = \frac{1}{n} f_1'(\phi_n(\alpha_k)) \to \varkappa_k^2 \) as \( n \to \infty \). Moreover, the asymptotic distribution of the second random vector is derived in Zheng et al. (2015). Because of Lemma 6.4, two random vectors are asymptotically independent; thus, the random vector

\[(Y_1 - \mathbb{E}Y_1, \ldots, Y_h - \mathbb{E}Y_h) ^{\dagger} \overset{d}{\rightarrow} N_h(0, \Omega), \]

where \( \phi_n^2(\alpha_k) = \frac{1}{n} f_1'(\phi_n(\alpha_k)) \to \varkappa_k^2 \) as \( n \to \infty \). Moreover, the asymptotic distribution of the second random vector is derived in Zheng et al. (2015).
with mean function $EY_t$ is the same as $\mu_t$, and the covariance matrix is $\Omega$ with its entries

$$\omega_{st} = \sum_{k=1}^{K} \omega^k_s \omega^k_t s_k^2 + \kappa_{st},$$

where

$$\kappa_{st} = - \frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} \frac{s(z_1)f_t(z_2)}{(m(z_1) - m(z_2))^2} dm(z_1)dm(z_2) - \frac{c \beta_x}{4\pi^2} \oint_{C_1} \oint_{C_2} s(z_1)f_t(z_2)$$

$$\left[ \int \frac{t}{(m(z_1)t + 1)^2} \times \frac{t}{(m(z_2)t + 1)^2} dH(t) \right] dm(z_1)dm(z_2)$$

$$\int \frac{t}{(m(z_1)t + 1)^2} \times \frac{t}{(m(z_2)t + 1)^2} dH(t) \right] dm(z_1)dm(z_2)$$

$$-\frac{1}{4\pi^2} \oint_{C_1} \oint_{C_2} f_s(z_1)f_t(z_2) \left[ \frac{\partial^2}{\partial z_1 \partial z_2} \log (1 - a(z_1, z_2)) \right] dz_1dz_2,$$

$$a(z_1, z_2) = \alpha_x \left( 1 + \frac{m(z_1)m(z_2)(z_1 - z_2)}{m(z_2) - m(z_1)} \right).$$

Then, we obtain the random vector

$$\left( \frac{Y_1 - EY_1}{\sqrt{\sigma^2_1}}, \ldots, \frac{Y_h - EY_h}{\sqrt{\sigma^2_h}} \right)$$

which has a mean function that is the same as that in Theorem 3.1, and variance function

$$\sigma^2_l = \sum_{k=1}^{K} \text{C}(\alpha_k)^2 \left( f'_l(\phi_n(\alpha_k)) \right)^2 s_k^2 + \kappa_{ll}, \quad l = 1, \ldots, h,$$

and the covariance matrix $\Psi = (\psi_{st})_{h \times h}$ is the correlation coefficient matrix of random vector $(Y_1, \ldots, Y_h)'$ with its entries

$$\psi_{st} = \frac{\sum_{k=1}^{K} \omega^k_s \omega^k_t s_k^2 + \kappa_{st}}{\sqrt{\sum_{k=1}^{K} (\omega^k_s)^2 s_k^2 + \kappa_{ss}}}.$$

Note that renormalization is necessary to guarantee that elements in the correlation coefficient matrix $\Psi$ are limited. Therefore, the proof is finished.

6.5. **Proof of Theorem 4.1.** The result under $H_0$ is a direct result of applying Theorem 4.1 in Zheng et al. (2015) using the substitution principle. Therefore, we omit the proof here. Next, we focus on the result under $H_1$.

Recall that

$$G_n(x) = p \left[ F_B(x) - F_{c_n,H_n}(x) \right],$$

$$Y = \int f_L(x) dG_n(x) - \sum_{k=1}^{K} d_k f_L(\phi_n(\alpha_k)) - \frac{M}{2\pi i} \oint_{C} f_L(z) \frac{m'(z)}{m(z)} dz,$$

when $f_L(x) = x - \log x - 1$. After some calculations, we obtain

$$\int f_L(x) dG_n(x) = \text{tr} B - \log |B| - p - p \int f_L(x) dF_{c_n,H_n}(x) = L - p \int f_L(x) dF_{c_n,H_n}(x),$$
where (6.21) is obtained from Lemma 6.1 and Bai et al. (2009). For consistency, we present the proof of (6.22) in Appendix A. According to Theorem 3.1, when (6.22) is obtained from Lemma 6.1 and Bai et al. (2009). For consistency, we present the proof of (6.22) in Appendix A. According to Theorem 3.1, when $f_L(x) = x - \log x - 1$, we have

\[
\frac{L - p \int f_L(x) dF_{c_n,H_n}(x) - \mu_L}{\sqrt{\varsigma_L^2}} \overset{d}{\rightarrow} N(0,1),
\]

where the mean function is $\mu_L = -\frac{\log(1-c_{nM})}{2} \alpha_x + \frac{c_{nM}}{2} \beta_x + \sum_{k=1}^{K} d_k (\phi_n(\alpha_k) - \log \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$, and the covariance function $\varsigma_L^2$ is $\frac{\alpha_x}{2} (\phi_n(\alpha_k) - (1-c_n)) - 2 \log(1-c_{nM}) + \phi_n(\alpha_k) - 1) - M(c_n + \log(1-c_n))$. Now, we consider the power of the hypothesis. Let $a$ be the size of the hypothesis in Section 4; $z_a$ is the upper $a\%$ quantile of the standard Gaussian distribution $\Phi$. Since

\[
P_{H_1}(|L| > w) = P_{H_0}\left(\frac{|L| - L_0}{\varsigma_L} > \frac{w - L_0}{\varsigma_L}\right),
\]

here, $w$ is a threshold of the critical region $L_0 = p \int f_L(x) dF_{c_n,H_n} + \mu_L$. Then, we have $w = \varsigma_L z_a + L_0$. For brevity, we use the notation $L_1 = (p - M) \int f_L(x) dF_{c_{nM},H_{2n}} + \mu_L$. Therefore, the power of the hypothesis is

\[
P_{H_1}(|L| > w) = P_{H_0}\left(\frac{|L| - L_1}{\varsigma_L} > \frac{w - L_1}{\varsigma_L}\right) = \Phi\left(\frac{L_1 - L_0 - \varsigma_L z_a}{\varsigma_L}\right) = \Phi\left(\frac{L_1 - L_0}{\varsigma_L} - z_a \frac{\varsigma_L}{\varsigma_L}\right).
\]

When $M = 1$,

\[
L_1 - L_0 = (p - 1)(1 - \frac{c_{n1} - 1}{c_{n1}} \log(1-c_{n1})) - p(1 - \frac{c_n - 1}{c_n} \log(1-c_n)) + (\phi_n(\alpha_1) - \log \phi_n(\alpha_1)) - (c_n + \log(1-c_n)) - \frac{\log(1-c_{n1})}{2} \alpha_x + \frac{\log(1-c_n)}{2} \alpha_x + \frac{c_{n1}}{2} \beta_x - \frac{c_n}{2} \beta_x,
\]

after some elementary calculations, $\phi(\alpha_1) = \alpha_1 + \frac{\alpha_1}{\alpha_1 - 1}$, we obtain as $n \rightarrow \infty$,

\[
L_1 - L_0 \rightarrow -c + (\phi_1 - \log \phi_1 - 1),
\]

\[
\varsigma_L \rightarrow \sqrt{\frac{\alpha_x + 1}{2} (-2 \log(1-c) - 2c)},
\]

\[
\varsigma_L \rightarrow \sqrt{\frac{\alpha_x + 1}{2} (-2 \log(1-c) - 2c) + \frac{2}{n} \left(\frac{\phi_1 - 1}{2}\right)^4}.
\]
Therefore, the asymptotic power of LRT is
\[
\Phi\left(-c + \left(\phi_1 - \log \phi_1 - 1\right) - za \sqrt{\frac{a+1}{2}(\log(1 - c) - 2c)}\right).
\]

Thus, the proof of Theorem 4.1 is finished.

6.6. Proof of Theorem 4.2. First, we focus on the results under \( H_0 \). From Remark 3.3, we have
\[
I_1(f_W) = c, \tag{6.23}
\]
\[
I_2(f_W) = c, \tag{6.24}
\]
\[
J_1(f_W, f_W) = 4c^3 + 2c^2, \tag{6.25}
\]
\[
J_2(f_W, f_W) = 4c^3, \tag{6.26}
\]
which then yields
\[
\mu_W = \alpha_x I_1(f_W) + \beta_x I_2(f_W) = \alpha_x c + \beta_x c,
\]
\[
\varsigma_W^2 = (\alpha_x + 1)J_1(f_W, f_W) + \beta_x J_2(f_W, f_W) = (\alpha_x + 1)(4c^3 + 2c^2) + 4\beta_x c^3.
\]
The results are still valid if \( c \) is replaced by \( c_n \). Moreover, the center term
\[
\int f_W(x) dF_{c_n,H_n} = c_n, \tag{6.27}
\]
is a direct result of Lemma 2.2 in Wang and Yao (2013). The proofs of (6.23), (6.24), (6.25) and (6.26) are presented in Appendix A. Therefore, from Zheng et al. (2015) or Wang and Yao (2013), we have
\[
\frac{W - p \int f_W(x) dF_{c_n,H_n} - \mu_W}{\sqrt{\varsigma_W^2}} \xrightarrow{d} N(0,1).
\]

Then, we focus on the results under \( H_1 \). Note that
\[
Y = \int f_W(x) dG_n(x) - \sum_{k=1}^{K} d_k f_W(\phi_n(\alpha_k)) - \frac{M}{2\pi i} \oint_C f_W(z) \frac{m'(z)}{m(z)} dz.
\]

After some calculations, we obtain
\[
\int f_W(x) dG_n(x) = \text{tr}(B - I_p)^2 - p \int f_W(x) dF_{c_n,H_n} = W - p \int f_W(x) dF_{c_n,H_n}, \tag{6.28}
\]
\[
p \int f_W(x) dF_{c_n,H_n} = (p - M) \int f_W(x) dF_{c_n,M,H_{2n}} = (p - M)c_nM
\]
\[
\sum_{k=1}^{K} d_k f_W(\phi_n(\alpha_k)) = \sum_{k=1}^{K} d_k (\phi_n^2(\alpha_k) - 2\phi_n(\alpha_k) + 1),
\]
\[
\frac{M}{2\pi i} \oint_C f_W(z) \frac{m'(z)}{m(z)} dz = -Mc_n^2. \tag{6.29}
\]
For consistency, we present the proof of (6.29) in Appendix A. Therefore, from Theorem 3.1, we have

$$
\frac{W - (p - M)\tilde{t}_W - \tilde{\mu}_W}{\sqrt{\xi^2_W}} \xrightarrow{d} N(0, 1),
$$

where

$$
\tilde{t}_W = c_nM, \quad \tilde{\mu}_W = \alpha_x c_nM + \beta_x c_nM + \sum_{k=1}^{K} d_k \left( \phi_n^2(\alpha_k) - 2\phi_n(\alpha_k) + 1 \right) - Mc_n^2,
$$

$$
\xi^2_W = (\alpha_x + 1)(4c_n^2 + 2c_n^2) + 4\beta_x c_n^2 + \sum_{k=1}^{K} \frac{8(\phi_n(\alpha_k) - 1)^4}{n}.
$$

Moreover, the power analysis for NT is similar to that for the LR; thus, we omit the detailed proof here. Therefore, the proof of Theorem 4.2 is complete.

**APPENDIX A: SOME DERIVATIONS AND CALCULATIONS**

This section contains proof of formulas stated in the proof of Theorems 4.1 and 4.2, and we begin by deriving formula (6.22). First, we consider

$$
\int_c f_L(z) \frac{m'(z)}{m(z)}dz.
$$

$$\begin{align*}
\int_c f_L(z) \frac{m'(z)}{m(z)}dz &= \int_c f_L(z) d\log m(z) = -\int_c f_L(z) \log m(z) dz \\
&= \int_{a(c)}^{b(c)} f_L(z) \left[ \log m(x + i\varepsilon) - \log m(x - i\varepsilon) \right] dx
\end{align*}
$$

(1.30) $$= 2i \int_{a(c)}^{b(c)} f_L(z) \Im \log m(x + i\varepsilon) dx$$

Here, $$a(c) = (1 - \sqrt{c})^2$$ and $$b(c) = (1 + \sqrt{c})^2$$. Since

$$m(z) = -\frac{1 - c}{z} + cm(z),$$

under $$H_1$$, we have

$$m(z) = \frac{(z + 1 - c) + \sqrt{(z - 1 - c)^2 - 4c}}{2z}.$$

As $$z \to x \in [a(c), b(c)]$$, we obtain

$$m(x) = -\frac{(x + 1 - c) + \sqrt{4c - (x - 1 - c)^2}i}{2x}.$$

Therefore,

$$\int_{a(c)}^{b(c)} f_L(x) \Im \log m(x + i\varepsilon) dx$$

$$= \int_{a(c)}^{b(c)} f_L(x) \tan^{-1} \left( \frac{\sqrt{4c - (x - 1 - c)^2}}{-x + 1 - c} \right) dx$$

$$= \left[ \tan^{-1} \left( \frac{\sqrt{4c - (x - 1 - c)^2}}{-x + 1 - c} \right) f_L(x) \right]_{a(c)}^{b(c)} - \int_{a(c)}^{b(c)} f_L(x) d\tan^{-1} \left( \frac{\sqrt{4c - (x - 1 - c)^2}}{-x + 1 - c} \right).$$
It is easy to verify that the first term is 0, and we now focus on the second term,

$$\int_{a(c)}^{b(c)} f_L(x) d\tan^{-1} \left( \frac{\sqrt{4c - (x - 1 - c)^2}}{-(x + 1 - c)} \right)$$

(1.31) $$= \int_{a(c)}^{b(c)} \frac{(x - \log x - 1)}{1 + \frac{4c - (x - 1 - c)^2}{2(x + 1 - c)^2}} \cdot \frac{\sqrt{4c - (x - 1 - c)^2} + \frac{(x - 1 - c)(x + 1 - c)}{\sqrt{4c - (x - 1 - c)^2}}}{(x + 1 - c)^2} \, dx.$$

By substituting $$x = 1 + c - 2\sqrt{c}\cos(\theta)$$, we obtain

$$\int_{0}^{2\pi} (1 + c - 2\sqrt{c}\cos(\theta) - \log (1 + c - 2\sqrt{c}\cos(\theta)) - 1) \frac{c - \sqrt{c}\cos(\theta)}{1 + c - 2\sqrt{c}\cos(\theta)} \, d\theta$$

(1.32) $$= \frac{1}{2} \int_{0}^{2\pi} \left[ 1 - \frac{\log (1 + c - 2\sqrt{c}\cos(\theta)) + 1}{\sqrt{c}\cos(\theta)} \right] (c - \sqrt{c}\cos(\theta)) \, d\theta$$

It is easy to obtain that the first term of (1.32) is $$\pi c$$; then, we consider the second term. By substituting $$\cos \theta = \frac{z + z^{-1}}{2}$$, we turn it into a contour integral on $$|z| = 1$$

$$\frac{1}{2} \int_{0}^{2\pi} \frac{\log (1 + c - 2\sqrt{c}\cos(\theta))}{1 + c - 2\sqrt{c}\cos(\theta)} \, d\theta$$

$$= \frac{1}{4} \int_{|z|=1} \log |1 - \sqrt{c}z|^2 \cdot \frac{c - \sqrt{c} \frac{z + z^{-1}}{2}}{1 + c - 2\sqrt{c} \cdot \frac{z + z^{-1}}{2}} \, dz$$

$$= \frac{1}{4} \int_{|z|=1} \log |1 - \sqrt{c}z|^2 \cdot \frac{2cz - \sqrt{c}(z^2 + 1)}{(z - \sqrt{c})(z + \sqrt{c})} \, dz$$

When $$c < 1$$, 0 and $$\sqrt{c}$$ are poles, by using the residue theorem, the integral is $$-\pi \log(1 - c)$$. The same argument also holds for the third term, and the integral is 0 after some calculation.

Therefore,

$$\frac{M}{2\pi i} \oint_{C} f_L(z) \frac{m'(z)}{m(z)} \, dz = -M(c + \log(1 - c)),$$

and the result is still valid if $$c$$ is replaces $$c_n$$; therefore, formula (6.22) holds.

Now, we prove (6.29). Since $$z = -\frac{1}{m} + \frac{c}{1 + m}$$, we have, for $$c > 1$$,

$$\oint_{C} f_W(z) \frac{m'(z)}{m(z)} \, dz = \oint_{C_1} f_W(z) \frac{m'(z)}{m(z)} \, dz + \oint_{C_2} f_W(z) \frac{m'(z)}{m(z)} \, dz,$$

where $$C_1$$ is a contour that includes the interval $$((1 - \sqrt{c})^2, (1 + \sqrt{c})^2)$$, and $$C_2$$ is a contour that includes the origin. Using $$C_m$$ to denote the contour of $$m$$, we obtain

$$\oint_{C_1} f_W(z) \frac{m'(z)}{m(z)} \, dz = \oint_{C_m} (-\frac{1}{m} + \frac{c}{1 + m} - 1) \frac{m'(z)}{m(z)} \, dz \, dm$$

$$= \oint_{C_m} \left( -\frac{1 + m}{m} + \frac{c}{1 + m} \right) \frac{1}{m} \, dm = \oint_{C_m} \left( \frac{1}{m^3} + \frac{c^2}{(1 + m)^2m} - \frac{2c}{m^2} \right) \, dm$$
Since the $z$ contour cannot enclose the origin, neither can the resulting $m$ contour. Thus, the only pole is $-1$, the residue is $-c^2$ by residue theorem, and we obtain the integral as $-2\pi ic^2$.

Then, we focus on the second integral $\oint_{c_0} f_W (z) \frac{m'(z)}{m(z)} dz$. When $z = 0$, we obtain $m = \frac{1}{c-1}$; since $c > 1$, $\frac{1}{c-1} > 0$. Both $m = 0$ and $m = -1$ are not in the contour. Thus, the integrand $\left(\frac{1+cz}{m^2} + \frac{c^2}{(1+m)^2 m - 2 \frac{m}{m^2}}\right)$ is analytic in the contour. The integral is $0$. Therefore, when $c > 1$, $M = \frac{\sqrt{c - 1}}{2\pi \sqrt{c}} \oint_{c_0} f_W (z) \frac{m'(z)}{m(z)} dz = -M \sqrt{c}$. When $c < 1$, the contour integral $\oint_{c_0} f_W (z) \frac{m'(z)}{m(z)} dz$ reduces to $\oint_{c_1} f_W (z) \frac{m'(z)}{m(z)} dz$, and the result is also the same as above. When $c = 1$, the result is still true by continuity in $c$. The results above are still valid if $c$ replaces $c_n$. Therefore, the proof of (6.29) is complete.

We now detail the calculations of (6.23), (6.24), (6.25), and (6.26). They are all based on the formula provided in Remark 3.3 and repeated use of residue theorem.

**Proof of (6.23):**

$$I_1(f_W) = \lim_{r \to 1} \frac{1}{2\pi i} \oint_{|z|=1} (|1 + \sqrt{c}z| - 1)^2 \left[ \frac{z}{z^2 - r^2} - \frac{1}{z} \right] dz,$$

$$= \lim_{r \to 1} \frac{1}{2\pi i} \oint_{|z|=1} \frac{(\sqrt{c} + cz + \sqrt{c}z^2)^2}{z^2} \left[ \frac{z}{z^2 - r^2} - \frac{1}{z} \right] dz,$$

$$= \lim_{r \to 1} \frac{1}{2\pi i} \oint_{|z|=1} \frac{(\sqrt{c} + cz + \sqrt{c}z^2)^2}{z} \left[ \frac{1}{(z + \frac{1}{2})(z - \frac{1}{2})} \right] dz.$$

In the first integral, the poles are $0$, $-\frac{1}{2}$, and $\frac{1}{2}$. The residues are $-r^2c$, $\frac{1}{2}(\sqrt{c} - \frac{c}{r} + \frac{\sqrt{c}}{r})^2$, and $\frac{1}{2}(\sqrt{c} + \frac{c}{r} + \frac{\sqrt{c}}{r})^2$.

In the second integral, the pole is 0, and the residue is $c^2 + 2c$. Then, by using residue theorem, the first part of $I_1(f_W)$ is $3c + c^2$, and the second part is $c^2 + 2c$; thus, $I_1(f_W) = c$.

**Proof of (6.24):**

$$I_2(f_W) = \frac{1}{2\pi i} \oint_{|z|=1} (|1 + \sqrt{c}z| - 1)^2 \frac{1}{z^3} dz,$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \frac{(\sqrt{c} + cz + \sqrt{c}z^2)^2}{z^5} dz,$$

The pole is 0, and the residue is $c$; then, by using residue theorem, we obtain $I_2(f_W) = c$.

**Proof of (6.25):**

$$J_1(f_W, f_W) = \lim_{r \to 1} \frac{-1}{4\pi i} \oint_{|z|=1} \oint_{|z|=1} \left[ (1 + \sqrt{c}z_1) - 1 \right]^2 \left[ (1 + \sqrt{c}z_2) - 1 \right]^2 \frac{dz_1 dz_2}{(z_1 - rz_2)^2}.$$

$$= \lim_{r \to 1} \frac{-1}{4\pi i} \oint_{|z|=1} \oint_{|z|=1} \left[ (\sqrt{c} + cz_1 + \sqrt{c}z_1^2)^2 \right] \frac{dz_1 dz_2}{z_1^2 (z_1 - rz_2)^2}.$$

First, we focus on $\oint_{|z_1|=1} \frac{(\sqrt{c} + cz_1 + \sqrt{c}z_1^2)^2}{z_1^2 (z_1 - rz_2)^2} dz_1$. Since $|z_1| = |rz_2| = |r| > 1$, the pole is only 0. By using the residue theorem, the integral is $2\pi i \left( 2\frac{2c^2}{r^2} (\sqrt{c} + cz_2 + \sqrt{c}z_2^2)^2 + 2c (\sqrt{c} + cz_2 + \sqrt{c}z_2^2) \right)$. Therefore,

$$J_1(f_W, f_W) = \lim_{r \to 1} \frac{-1}{4\pi i} \oint_{|z_1|=1} 2\pi i \frac{2c^2}{r^2} \left( \sqrt{c} + cz_2 + \sqrt{c}z_2^2 \right)^2 dz_2 +$$
\[
\lim_{r \downarrow 1} -\frac{1}{4\pi^2} \oint_{|z_1|=1} \frac{2c (\sqrt{c} + cz_2 + \sqrt{c}z_2^2)^2}{z_2^5} dz_2.
\]

For the first integral above, the pole is 0, and the residue is \(2c^3\). Then, by using residue theorem, the integral is \(2c^3\); thus, the first part of \(J_1(f_W, f_W)\) is \(4c^3\). For the second integral, the pole is also 0, and the residue is \(c\). Similarly, the second integral is \(2c^2\). Therefore, \(J_1(f_W, f_W) = 4c^3 + 2c^2\).

**Proof of (6.26):**

\[
J_2(f_W, f_W) = -\frac{1}{4\pi^2} \oint_{|z_1|=1} \left(\frac{(1 + \sqrt{c}z_1^2)^2 - 1)2^2}{z_1^2} dz_1 \oint_{|z_2|=1} \left(\frac{(1 + \sqrt{c}z_2^2)^2 - 1)^2}{z_2^2} dz_2.
\]

First, we calculate the first integral \(\oint_{|z_1|=1} \left(\frac{(1 + \sqrt{c}z_1^2)^2 - 1)2^2}{z_1^2} dz_1\). Since

\[
\oint_{|z_1|=1} \left(\frac{(1 + \sqrt{c}z_1^2)^2 - 1)2^2}{z_1^2} dz_1 = \oint_{|z_1|=1} \frac{(\sqrt{c} + cz_1 + \sqrt{c}z_1^2)^2}{z_1^4} dz_1,
\]

the pole is 0, and the residue is \(2c^3\). Then, by using the residue theorem, the integral is \(4\pi ic^3\).

The same calculations also hold for the second integral and \(\oint_{|z_2|=1} \left(\frac{(1 + \sqrt{c}z_2^2)^2 - 1)^2}{z_2^2} dz_2 = 4\pi ic^3\). Therefore, \(J_2(f_W, f_W) = 4c^3\). The proof is finished.

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