Adequate soliton solutions to the time fractional Zakharov-Kuznetsov equation and the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation

M. Nurul Islam, Rehana Parvin, Mst. Rashida Pervin, and M. Ali Akbar

Department of Mathematics, Islamic University, Kushtia, Bangladesh; Department of Quantitative Sciences (Mathematics), International University of Business Agriculture and Technology, Dhaka, Bangladesh; Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh

1. Introduction

Nowadays the fractional nonlinear differential equations (FNLEs) are broadly used as a significant mathematical tool to depict complex physical phenomena. First, in 1819, Lacroix introduced a definition of fractional derivative based on the usual expression for the n-th derivative of the power of a function. After the idea of Lacroix’s definition of fractional derivative, in the history of fractional calculus, within years the fractional calculus and fractional differential equation became a very attractive subject to the mathematicians. Therefore, fractional nonlinear evolution equations (NLEEs) unravel the intricate phenomena. Thus, the study of fractional NLEEs is crucial and much concentration and popularity has gained among the researchers. To recognize further about the fractional NLEEs, the definitions, namely the derivative of modified Riemann-Liouville (Jumarie, 2006), the derivative of conformable (Khalil, Horani, Yousef, & Sababheh, 2014), the derivative Caputo (Caputo & Fabrizio, 2015), etc. are familiar in the modern age. NLEEs come forward in various engineering and scientific areas, vide-lentic quantum mechanics, fluid mechanics, water wave mechanics, chemical kinematics, the modelling of earthquake, biology, optical fibres, electricity, plasma physic, etc. And the exact soliton solutions to nonlinear equations take part in an elementary and crucial role in physical sciences, mathematical physics, engineering, applied sciences.

As a result, the exact solitary wave solutions of fractional NLEEs are very important in order to understand the internal structure of complex physical processes, and many researchers have recently concentrated on finding the closed-form wave solutions of FNLEEs in various fields. Therefore a variety of powerful and modern approaches using computer algebra such as Mathematica or Maple, have recently been developed by several researchers. The available methods are: the auxiliary equation method (Akbar, Ali, & Tanjim, 2019; Akbulut, Kaplan, & Bekir, 2016; Ismail, 2016), the (G'/G)-expansion method (Alam, 2021; Khalil, Horani, Yousef, & Sababheh, 2014; Jumarie, 2006; Caputo & Fabrizio, 2015)
Akbar, & Mohyud-Din, 2014; Bekir & Guner, 2013; Islam & Akbar, 2018a, 2018b, 2018c), the tanh-function method (Ali, 2007), the first integral method (Younis, 2013), the fractional sub-equation method (Alzaidy, 2013), the exp-function method (Zheng, 2013; Yokus, Durur, & Ahmad, 2020; Yokus, Durur, Ahmad, & Yao, 2020), the modified Kudryashov method (Ege & Misirli, 2014), the variational iteration method (He & Latifizadeh, 2020; Neamaty, Agheli, & Darzi, 2015), the homotopy perturbation method (Ain et al., 2020; Anjum & Ain, 2020; Fereidoon, Yaghoobi, & Davoudabadi, 2011; He, 2019a, 2019b; Yu, He, & Garcia, 2019), the modified simple equation method (Islam & Akbar, 2018c), the Jacobian elliptic method (Zheng & Feng, 2014), the differential transformation method (Rabta, Erturk, & Momani, 2010), the modified trial equation method (Bulut, Baskonus, & Pandir, 2013), the finite element method (Deng, 2009), the Adomim decomposition method (Dahmani, Mesmoudi, & Bebbouchi, 2008), the Taylor series method (He et al., 2020), the variational iteration algorithm-I (Ahmad & Khan, 2019; Ahmed, Khan, & Cesarano, 2019) the variational iteration algorithm-II (Ahmad, Seadawy, & Khan, 2020; Ahmad, Seadawy, Khan, & Thounthong, 2020), the Riccati transformation method (Bazighifan, Ahmad, & Yao, 2020), the meshless techniques (Inc et al., 2020), the improve Bernoulli sub-equation function method (Islam & Akbar, 2020), etc.

The wave solutions to fractional NLEEs are noteworthy to analyze real world problems and recently many researchers contemplated to examine the exact wave solutions of fractional NLEEs in different areas. In the literature, the time fractional (2, 2, 2) Zakharov-Kuznetsov(ZK) equation and the space-time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation are investigated through the He’s homotopy perturbation method (Yildirim & Gulkanat, 2010), the modified Kudryashov method (Munro & Parkes, 1999), the first integral method (Hossein, Refahi, & Hadi, 2015), the tanh-coth method (Hossam & Ghany, 2013), the fractional sub-equation method (Ray & Sahoo, 2015), the \( \exp(-\phi(\eta)) \)-expansion method (Ali, Iqbal, & Mohyud-Din, 2016a), the \( \exp(-\phi(\eta)) \)-expansion method (Ali, Iqbal, & Mohyud-Din, 2016b), the \( (G'/G) \)-expansion method (Shakeel & Mohyud-Din, 2015), and the variational iteration method (Mollig, Noorani, Hashim, & Ahmad, 2009; Torvattanabun & Koonprasert, 2017), the \( \exp(-\phi(\eta)) \)-expansion method (Roshid et al., 2014) etc.

To our foremost interpretation the (2, 2, 2) ZK equation and the ZKBBM equation were not examined through the auxiliary equation approach. Therefore, the key objective of this article is to examine the wide-ranging, advanced, compatible and the further general closed form travelling wave solutions to the ZKBBM and the (2, 2, 2) ZK equations by means of the auxiliary equation method and exhibit the physical implication for its definite values of the established solutions. The method is a recently formulated effectual and thriving method to look into advanced and broad-ranging soliton solutions to NLEEs. We also provide graphs of solutions to interpret inner physical mechanisms.

This article is sorted as follows: In section 2, physical meaning of fractional derivative is discussed. In Section 3, we have briefly illustrated the auxiliary equation method. In Section 4, we figured out the solutions. In Section 5, the results are portrayed and discussed. Finally, we provide conclusion in Section 6.

2. Fractional derivative and its explanation

This part covers fractional calculus and fractional derivative (Atangana, 2017; Baleanu, Golmankhaneh, Golmankhaneh, & Nigmatullin, 2010; Brouers, 2014; Brouers & Sotolongo-Costa, 2006; Chen & Liang, 2017; Fan et al., 2015; Golmankhaneh & Baleanu, 2016; Gomez-Aguilar, Razo-Hernandez, & Granados-Lieberman, 2014; He, 2014, 2018; He, Elagan, & Li, 2012; Hu & He, 2016; Liu & He, 2018; Pan, Zheng, Liu, Liu, & Chen, 2018) in order to understand the physical meaning of fractional derivative and demonstrate its geometrical interpretation.

Fractional calculus (FC) is a useful technique for understanding the evolution of memory systems, which are typically dissipative and complicated (Chen & Liang, 2017; Gomez-Aguilar et al., 2014). This is the key benefit of FC in judgement with the classical integer-order models in which such effects are in fact ignored. In time-domain investigations, the physical meaning of fractional derivatives is useful. This is because fractional differential equations (FDEs) with well-known derivatives behave similarly to ordinary differential equations (ODEs) with a broad scientific understanding. Many physical phenomena can be explained using the fractional-order derivative and therefore it is very important for the FC to provide details (Baleanu et al., 2010; Gomez-Aguilar et al., 2014). FC models physical systems more accurately than traditional systems in a variety of applications. The fractals theory is another big field that necessitates the use of FC (Gomez-Aguilar et al., 2014; He, 2014). The development of the fractals theory has opened more perspective for the theory of fractional derivatives, particularly in describing dynamical processes in self-similar and permeable configurations. Fractional-order derivatives are defined in different ways. The Riemann-Liouville fractional derivative, the Jumarie’s modified Riemann-Liouville fractional derivative (Jumarie, 2006), and the conformal fractional derivative (Khalil et al., 2014),...
the Beta-derivative (Atangan, Baleanu, & Alsaedi, 2016), the Caputo derivative (Caputo & Fabrizio, 2015), etc. are realistic and extensively used. The fractional complex transform was first proposed by Li and He (2010) to transmute a FDE into an ODE. This fractional complex transformation plays a significant role in understanding the physical meaning of the fractional derivative.

Understanding the physical meaning of fractional derivative is important in order to achieve better outcomes in the realistic behalf. The fractional-order model is more functional than the integer-order model, and nonlinear FDE research has piqued the interest of academics in various fields of science, engineering, and mathematical physics.

Fractal geometry, fractal calculus and fractional calculus have been becoming burning topics in both engineering and mathematical physics for non-differential solutions (He, 2018; He et al., 2012; Hu & He, 2016). Fractal theory is the theoretical basis for the fractal mechanics becomes ineffectual to illustrate any phenomena on the porous size scale (He, 2018; Hu & He, 2016; Pan et al., 2018).

Two-scale mathematics is required to disclose the lost information owing to the lower-dimensional method. In general, one scale is identified by usage where standard calculus is used, while a second scale is used to expose lost information where the continuum hypothesis is not allowed and fractal calculus is used (He, 2014). The fractional calculus can simply be transferred into its classical partner via the two-scale transform, making two-scale thermodynamics very promising. The FDE can be translated into traditional differential equations using the two-scale transform, which are easy to solve. The fractal calculus is moderately new; it can efficiently deal with hierarchical forms (Fan et al., 2015) with large success. A fractal space is always not isotropic, i.e. the fractal dimensions in x, y, z-directions are different. By replacing the Equation (2.5) by the following one

\[
\nabla^\alpha x, y, \eta T = \frac{\partial T}{\partial x^\alpha} + \frac{\partial T}{\partial y^\beta} + \frac{\partial T}{\partial z^\eta},
\]

where \( x, \beta, \eta \) are respectively, the fractal dimensions in x, y, z-directions

\[
\frac{\partial T}{\partial x^\alpha} = \Gamma(1 + \alpha) \lim_{x^\alpha - x^\alpha - \zeta_0} \frac{T_B - T_A}{(x_B - x_A)^\alpha},
\]

\[
\frac{\partial T}{\partial y^\beta} = \Gamma(1 + \beta) \lim_{y^\beta - y^\beta - \zeta_0} \frac{T_B - T_A}{(y_B - y_A)^\beta},
\]

\[
\frac{\partial T}{\partial z^\eta} = \Gamma(1 + \eta) \lim_{z^\eta - z^\eta - \zeta_0} \frac{T_B - T_A}{(z_B - z_A)^\eta},
\]

where \( \zeta_0 \), \( \zeta_0 \), \( \zeta_0 \) are the minimal porous sized in x, y, z-directions, respectively.

To establish laws in fractal media, it is essential to set up the idea of fractal velocity, which is presented below (in the Figure 2) (He, 2018):

\[
u = \lim_{x^\alpha - x^\alpha - \zeta_0} \frac{dx^\alpha}{\Gamma(1 + \alpha)\Delta t},
\]

Equation (2.10) can be recognized as an average velocity of a particle moving from A to B in the fractal space (where AB is the discontinuous line in the Figure 1) (He, 2018). The conservation of mass involves

\[
\left( u + \frac{\partial u}{\partial x^\Delta} \Delta^\Delta x \right) \Delta^\Delta y - u \Delta^\Delta x \Delta^\Delta y + \left( v + \frac{\partial v}{\partial y^\Delta} \Delta^\Delta y \right) \Delta^\Delta x - v \Delta^\Delta y \Delta^\Delta x = 0,
\]

The results mass equation in a fractal media in the following:
\[
\frac{\partial u}{\partial x^\alpha} + \frac{\partial v}{\partial y^\beta} = 0, \tag{2.12}
\]

In general case, the mass equation can be shown as follows:
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x^\alpha} + \frac{\partial (\rho v)}{\partial y^\beta} + \frac{\partial (\rho w)}{\partial z^\gamma} = 0, \tag{2.13}
\]
where \(\rho\) is the density of the fluid and \(\alpha, \beta, \gamma\) are the respectively, the fractional dimensions in the in \(x, y, z\)-directions and \(u, v, w\) are the respectively, the fractal velocity in \(x, y, z\)-directions.

The fractal streamlines in a fractal media, it yields:
\[
\frac{dx^\alpha}{\Gamma(1+\alpha) u} = \frac{dy^\beta}{\Gamma(1+\beta) v} = \frac{dz^\gamma}{\Gamma(1+\gamma) w}, \tag{2.14}
\]
where \(u, v, w\) are, the respectively, the fractal velocity in \(x, y, z\)-directions, are defined as,
\[
\frac{dx^\alpha}{\partial t} = \Gamma(1+\alpha) u, \quad \frac{dy^\beta}{\partial t} = \Gamma(1+\beta) v,
\]
\[
\frac{dz^\gamma}{\partial t} = \Gamma(1+\gamma) w.
\]

Introducing the new space \((X, Y, Z)\) and then defined by \(X = \frac{x^\alpha}{\Gamma(1+\alpha)}, Y = \frac{y^\beta}{\Gamma(1+\beta)}\) and \(Z = \frac{z^\gamma}{\Gamma(1+\gamma)}\). i.e. the space \((X, Y, Z)\) can be approximately measured as a smooth one, making the solution much simple and straightforward. We infer that as the order of the fractional derivative increases, the settling time reduces, as does the sensitivity of the settling time to the order of the derivative. As the order of the derivative drops, the rise time increases, becoming more sensitive as the time delay reduces.

3. Methodology

Consider the general fractional NLEE is of the form:
\[
F\left(v, D_\alpha^\alpha v, D_\beta^\beta v, D_\gamma^\gamma v, D_\delta^\delta v, \ldots \right) = 0, \tag{3.1}
\]
where \(v = v(x, y, z, t)\) is wave function, \(F\) is a polynomial in \(v(x, y, z, t)\) and its partial derivatives. In order to determine the solution of Equation (3.1) by means of the auxiliary equation method, we have to present the ensuing steps:

**Step 1:** Let us consider the travelling wave variable \((He et al., 2012; Li & He, 2010)\)
\[
v(x, y, z, t) = V(\xi), \quad \xi = \frac{m x^\alpha}{\Gamma(1+\beta)} + \frac{ny^\beta}{\Gamma(1+\gamma)} + \frac{k t^\gamma}{\Gamma(1+\zeta)}, \tag{3.2}
\]
for real fractional differential equations and \(\alpha, \beta, \gamma, \zeta\) are fractional order derivatives and \(0 < (\alpha, \beta, \gamma, \zeta) \leq 1\) and they may equal or not.

The above wave transformations translate the Equation (3.1) into the following ODE:
\[
H(V, V', V'', \ldots) = 0, \tag{3.3}
\]
where \(H\) is a polynomial in \(V(\xi)\) and its derivatives, wherein \(V'(\xi) = \frac{\partial V}{\partial \xi}\).

**Step 2:** We integrate Equation (3.3) term by term one or more times according to possibility in this step.

**Step 3:** In accordance with auxiliary equation method we reveal the travelling wave solution of Equation (3.3) as:
\[
V(\xi) = \sum_{i=0}^{N} a_i d^{\xi(i)}, \tag{3.4}
\]
where \(a_i\) and \(d\) are constants to be examined, such that \(a_N \neq 0\) and \(f(\xi)\) satisfies the subsequent supportive equation:
\[
f'(\xi) = \frac{1}{ln d} \left[ pd^{-f(i)} + q + rd^{f(i)} \right], \tag{3.5}
\]
wherein the prime stands for derivative with respect to \(\xi\); \(p, q, r\) and \(d\) are parameters.

**Step 4:** The positive integer \(N\) presents in (3.4) can be calculated by balancing the nonlinear and linear terms of the highest order occurring in (3.3).

**Step 5:** Assembling (3.4) together (3.5) into (3.4) and the value of \(N\) obtained in Step 4, we get a polynomial of \(d^{f(i)}\). Collecting all the terms of the similar power \(d^{f(i)}\), where \((i = 1, 2, 3...)\) and setting them to zero yields a system of algebraic equations with the constants \(a_i, p, q, r\) can be established and solving them yields the values of the unknown parameters. As the general solution of (3.5) is known, inserting the values of \(a_i\) \((i = 1, 2, 3...), p, q, r\) and \(d\)
into (3.4), we accomplish broad-spectrum and new exact solitary wave solutions to the Equation (3.3).

**Step 6:** For different values of $p$, $q$ and $r$ and their relationship, (3.5) provides different types of general solutions.

### 4. Extraction of solutions

In this section, we construct compatible, convenient and further general soliton solutions to the fractional (2, 2, 2) ZK equation and the ZKBBM equation by means of the introduced method. Furthermore, we discuss about the graphical representations and physical significance of the attained solutions.

#### 4.1. Analysis of the (2, 2, 2) time-fractional ZK equation

In this sub-section, we determine some new soliton solutions to the (2, 2, 2) time-fractional ZK equation by putting use of the auxiliary equation method. The (2, 2, 2) time-fractional ZK equation (Yıldırım & Gülkanat, 2010) is:

$$D_t^\alpha V + (V^2)_x + \frac{1}{8} (V^2)_{xxx} + \frac{1}{8} (V^2)_{xyy} = 0; \quad 0 < \alpha \leq 1.$$  

(4.1.1)

The fractional wave transformation (3.2) reshapes the Equation (4.1.1) into the following nonlinear equation:

$$k V' + m(V^2)' + \frac{1}{8} m(m^2 + n^2)(V^2)'' = 0.$$  

(4.1.2)

Equation (4.1.2) implies by integrating

$$k V + m V^2 + \frac{1}{8} m(m^2 + n^2)(V^2)'' + L = 0,$$  

(4.1.3)

where $L$ is an integrating constant.

Balancing between the highest order linear and nonlinear terms appearing in Equation (4.1.3), we obtain $N = -2$. Therefore, we employ the transformation $V = w^{-2}$. Then Equation (4.1.3) turns into the following nonlinear ODE:

$$kw^4 + mw^2 + \frac{5}{2} m(m^2 + n^2)(w')^2 - \frac{1}{2} m(m^2 + n^2)ww' + C_2 = 0,$$  

(4.1.4)

where $C_2$ is an integrating constant.

Now, balancing the two highest orders nonlinear terms happening in (4.1.4) yields $N = 1$. Therefore, it is clear that the shape of the solution of Equation (4.1.4) is

$$w = e_0 + e_1 d^{l(\xi)}.$$  

(4.1.5)

Inserting (4.1.5), (3.5) into (4.1.4) and collecting the coefficients the powers of $d^{l(\xi)}$ and setting them zero, we reach a set of algebraic equations (for minimalism which are not gathered here) for $e_0$, $e_1$, $p$, $q$, $r$ and $C_2$. Solving the system of algebraic equations by means of the computer algebra software, such as, Mathematica, offer the solutions as:

$$e_0 = \pm \sqrt{\frac{2C_2}{2C_2 - 5M}}$$  

$$e_1 = \pm \sqrt{\frac{2C_2}{2C_2 - 5M}}$$

$$m = \pm \sqrt{\frac{M}{N}}$$  

$$k = \pm \frac{15M}{64N},$$  

(4.1.6)

where $M = (n^2q^2 - 4n^2pr - 1)$, $N = (4pr - q^2)$, $p$, $q$ and $r$ are free parameters and $C_2$ is an integrating constant.

We establish the subsequent soliton solutions based on the data assembled in (4.1.6) and using the solutions of (3.5):

When $q^2 - 4pr < 0$ and $r \neq 0$, substituting the values of the constants assembled in (4.1.6) into (4.1.5) and simplifying, we attain the following traveling wave solutions

$$w(\xi) = \pm \sqrt{\frac{2C_2}{2C_2 - 5M}} \left[ 1 + \left( -\frac{q}{2r} + \frac{\sqrt{4pr - q^2}}{2r} \tan \left( \frac{\sqrt{4pr - q^2}}{2} \xi \right) \right) \right]$$

(4.1.7)

or

$$w(\xi) = \pm \sqrt{\frac{2C_2}{2C_2 - 5M}} \left[ 1 + \left( -\frac{q}{2r} + \frac{\sqrt{4pr - q^2}}{2r} \cot \left( \frac{\sqrt{4pr - q^2}}{2} \xi \right) \right) \right]$$

(4.1.8)

By means of the transformation $V = w^{-2}$, the solution (4.1.7) and (4.1.8) become

$$V_1(\xi) = \frac{2C_2N - 5M}{2C_2M} \left[ 1 + \left( -\frac{q}{2r} + \frac{\sqrt{4pr - q^2}}{2r} \tan \left( \frac{\sqrt{4pr - q^2}}{2} \xi \right) \right) \right]^{-2},$$

(4.1.9)

or

$$V_2(\xi) = \frac{2C_2N - 5M}{2C_2M} \left[ 1 + \left( -\frac{q}{2r} + \frac{\sqrt{4pr - q^2}}{2r} \cot \left( \frac{\sqrt{4pr - q^2}}{2} \xi \right) \right) \right]^{-2}.$$

(4.1.10)

For $q^2 - 4pr > 0$ and $r \neq 0$, putting the values arranged in (4.1.6) into (4.1.5) and with the transformation $V = w^{-2}$, we achieve
\[ V_5(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}, \] (4.1.11)

or

\[ V_6(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}. \] (4.1.12)

When \( q^2 + 4p^2 < 0 \), \( r \neq 0 \) and \( r = -p \), then \( M = (n^2q^2 + 4n^2p^2 - 1) \), \( N = (-4p^2 - q^2) \), making use of (4.1.6), (4.1.5) and the transformation \( V = w^{-2} \), the travelling wave solutions turns into

\[ V_5(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}, \] (4.1.13)

or

\[ V_6(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}. \] (4.1.14)

When \( q^2 + 4p^2 > 0 \), \( r \neq 0 \) and \( r = -p \), then \( M = (n^2q^2 + 4n^2p^2 - 1) \), \( N = (-4p^2 - q^2) \). Employing (4.1.6), (4.1.5) and the value \( V = w^{-2} \), we gain

\[ V_5(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}, \] (4.1.15)

or

\[ V_6(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{q}{2} \sqrt{\frac{q^2 - 4pr}{2}} \right)^2 \right\}. \] (4.1.16)

In the case of \( q^2 - 4p^2 < 0 \) and \( r = p \), \( M = (n^2q^2 - 4n^2p^2 - 1) \) and \( N = (4p^2 - q^2) \). Now applying (4.1.6) and (4.1.5) and \( V = w^{-2} \), the solutions provide

\[ V_5(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2p} \sqrt{-\frac{q^2 - 4p^2}{2}} \right)^2 \right\}, \] (4.1.17)

or

\[ V_6(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2p} \sqrt{-\frac{q^2 - 4p^2}{2}} \right)^2 \right\}. \] (4.1.18)

For \( q^2 - 4p^2 > 0 \) and \( r = p \), \( M = (n^2q^2 - 4n^2p^2 - 1) \), \( N = (4p^2 - q^2) \). Putting in use (4.1.6) into (4.1.5) and with the aid of the result \( V = w^{-2} \), we accomplish

\[ V_{11}(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2p} \sqrt{\frac{q^2 - 4p^2}{2}} \right)^2 \right\}, \] (4.1.19)

or

\[ V_{12}(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-q}{2p} \sqrt{\frac{q^2 - 4p^2}{2}} \right)^2 \right\}. \] (4.1.20)

For \( q^2 = 4pr \) then \( M = (4pn^2 - 4n^2p^2 - 1) \), \( N = 0 \) and inserting (4.1.6) into (4.1.5) and using \( V = w^{-2} \), the solution becomes the form

\[ V_{13}(\xi) = \frac{2C_2N - 5M}{2C_2M} \left( 1 - \frac{2q}{2p} \sqrt{\frac{q^2 - 4r^2}{2}} \right)^2. \] (4.1.21)

When \( pr < 0 \), \( q = 0 \) and \( r \neq 0 \), then \( M = -4n^2pr + 1 \), \( N = 4pr \). Now using (4.1.6) and (4.1.5) and the result \( V = w^{-2} \), we derive

\[ V_{14}(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-p}{r} \tan(\sqrt{r} p) \right)^2 \right\}, \] (4.1.22)

or

\[ V_{15}(\xi) = \frac{2C_2N - 5M}{2C_2M} \left\{ 1 + \left( \frac{-p}{r} \coth(\sqrt{r} p) \right)^2 \right\}. \] (4.1.23)

For \( q = 0 \) and \( p = -r \), if found \( M = (4n^2p^2 - 1) \) and \( N = -4p^2 \). By means of (4.1.6), (4.1.5) and the result \( V = w^{-2} \), we examine
Making use of (4.1.6) and (4.1.5) and considering the result $V = w^{-2}$, we determine

$$V_{17}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \cosh(q \xi) + \sinh(q \xi)\right)^{-2}. \tag{4.1.25}$$

When $p = q = h$ and $r = 0$ then $M = n^2 q^2 - 1$, $N = -q^2$. Now applying (4.1.6), (4.1.5) and $V = w^{-2}$, the wave solution becomes

$$V_{18}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + (e^h \xi - 1)\right)^{-2}. \tag{4.1.26}$$

For $q = r = h$ and $p = 0$, then $n^2 h^2 - 1$, $N = -h^2$. By the aid of (4.1.6), (4.1.5) and the result $V = w^{-2}$, we establish

$$V_{19}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \frac{e^h}{1 - e^{-h}}\right)^{-2}. \tag{4.1.27}$$

Applying the condition $q = p + r$, we can get $M = (n^2 q^2 - 4n^2 pq + r^2 - 1)$, $N = (4pq - p^2 - q^2)$. Now setting the constants arranged in (4.1.6) into (4.1.5) and also setting $V = w^{-2}$, we attain

$$V_{20}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 - \frac{1 - pe^{(2p-q)\xi}}{1 - (q-p)e^{(2p-q)\xi}}\right)^{-2}. \tag{4.1.28}$$

For $q = -(p + r)$, we ascertain $M = (n^2 q^2 + 4n^2 p(p + q) - 1)$ and $N = -(4p^2 + 4pq + q^2)$. Applying (4.1.6) and (4.1.5) and $V = w^{-2}$, we achieve

$$V_{21}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \frac{p - e^{(2p+q)\xi}}{-(p + q) - e^{(2p+q)\xi}}\right)^{-2}. \tag{4.1.29}$$

When $p = 0$ then $M = (n^2 q^2 - 1)$, $N = -q^2$. Making use of (4.1.6) and (4.1.5) and the result $V = w^{-2}$, the soliton solution provides

$$V_{22}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \frac{qe^{q\xi}}{1 - re^{r\xi}}\right)^{-2}. \tag{4.1.30}$$

Making use the condition $r = q = p \neq 0$, we get $M = (n^2 q^2 - 4n^2 pr - 1)$, $N = (4pr - q^2)$. Now for $V = w^{-2}$ and putting the values in (4.1.6) into (4.1.5), we ascertain

$$V_{23}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \left\{\frac{1}{2} \left(\sqrt{3} \tan \left(\frac{\sqrt{3}}{2} \, p \, \xi\right) - 1\right)\right\}^{-2}. \tag{4.1.31}$$

For $r = q = 0$, $M = -1$, $N = 0$. Now taking (4.1.6), (4.1.5) and using $V = w^{-2}$, the solution becomes in the form

$$V_{24}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + p \xi\right)^{-2}. \tag{4.1.32}$$

While $p = q = 0$ then $M = -1$, $N = 0$. Now considering (4.1.6) and (4.1.5) and the result $V = w^{-2}$, we obtain

$$V_{25}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 - \frac{1}{r\xi}\right)^{-2}. \tag{4.1.33}$$

For the conditions $r = p$ and $q = 0$, then $M = (n^2 q^2 - 4n^2 p^2 - 1)$, $N = (4p^2 - q^2)$. Making use of (4.1.6) and (4.1.5) and the result $V = w^{-2}$, the traveling wave solution is given below

$$V(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \tan(p\xi)\right)^{-2}. \tag{4.1.34}$$

When $r = 0$, then $M = (n^2 q^2 - 1)$, $N = -q^2$. Substituting the values scheduled in (4.1.6) into (4.1.5) and applying the result $V = w^{-2}$, we gain

$$V_{27}(\xi) = \frac{2C_2 N - 5M}{2C_2 M} \left(1 + \left(e^{q\xi} - \frac{q}{p}\right)^{-2}. \tag{4.1.35}$$

Here, for all above solutions, $\xi = \frac{K}{\Gamma(1+\alpha)} + \frac{m\nu}{\Gamma(1+\alpha)} + \frac{n^2}{\Gamma(1+\alpha)}k = \pm \frac{15m^2}{\Gamma(1+2\alpha)} + m = \pm \frac{M}{N} n, p, q, r$ and $C_2$ are arbitrary constant.

The erstwhile accomplished solutions are compatible, reliable and capable to interpret the demandable model in the ion-acoustic waves in plasma, the sound waves, the electromagnetic field, the optical fibres, the material science, the signal processing wave, the mathematical finances, the traffic flow, etc.

### 4.2. Solutions to the space-time fractional ZKBBM equation

Let us consider the nonlinear ZKBBM equation given below (Hosseini et al., 2015):

$$D_{t}^{\alpha} u + D_{x}^{\alpha} u - 2auD_{t}^{\alpha} u - bD_{x}^{\alpha} (D_{x}^{\alpha} u) = 0; \quad 0 < \alpha \leq 1, \tag{4.2.1}$$

where $a$ and $b$ are physical parameters.

Making use of the travelling wave variable (3.2), Equation (4.2.1) converts to the ODE and integrating once, yields

$$(k + m)u - amu^2 - bkm^2 u'' + C_1 = 0, \tag{4.2.2}$$

where $C_1$ stands for an integral constant.

We balance the highest order derivative $u''$ for linear term and the highest order nonlinear term $u^2$ taking place in (4.2.2), gives $N = 2$. Therefore, the solution outline of Equation (4.2.2) is:
\[ u = a_0 + a_1 a^{d(t)} + a_2 a^{d^2(t)}. \]  

(4.2.3)  

Since \( a_0, a_1, a_2 \) are unknown, they must be determined.

Inserting the solution (4.2.3) together with (3.5) into (4.2.2) and setting the coefficients of \( d^i(t) \) of its different power to zero, a set of simultaneous algebraic equations are attained (for simplicity these algebraic equations are not present here) for \( a_0, a_1, a_2, p, q, r \) and \( C_1 \). Solving the equations by the aid of symbolic computation software Maple, we extract the solution of unknowns as

\[
C_1 = \frac{(b^2 k^2 m^4 (q^4 + 16r^2 p^2 - 8rq p) - k^2 - 2km - m^2)}{4am},
\]

(4.2.4)

\[
a_0 = \frac{(S - bkm T)}{2am}, \quad a_1 = -\frac{6rmbk}{a}, \quad a_2 = -\frac{6mbkr^2}{a},
\]

where \( T = q^2 + 8rp, \ S = k + m. \)

We evaluate the following soliton solutions using the values gathered in (4.2.4) and the solutions of (3.5).

When \( q^2 - 4pr < 0 \) but \( r \neq 0 \), inserting the values assembled in (4.2.4) into (4.2.3) and simplifying, we attain the soliton solutions

\[
u_1(\xi) = \frac{(S - bkm T)}{2am} - \frac{6rmbk}{a} \left\{ -q + \frac{\sqrt{4pr - q^2}}{2r} \tan \left( \frac{\sqrt{4pr - q^2}}{2r} \xi \right) \right\}
- \frac{6mbkr^2}{a} \left\{ -q - \frac{\sqrt{4pr - q^2}}{2r} \tan \left( \frac{\sqrt{4pr - q^2}}{2r} \xi \right) \right\}^2,
\]

(4.2.5)

or

\[
u_2(\xi) = \frac{(S - bkm T)}{2am} - \frac{6rmbk}{a} \left\{ -q + \frac{\sqrt{4pr - q^2}}{2r} \cot \left( \frac{\sqrt{4pr - q^2}}{2r} \xi \right) \right\}
- \frac{6mbkr^2}{a} \left\{ -q + \frac{\sqrt{4pr - q^2}}{2r} \cot \left( \frac{\sqrt{4pr - q^2}}{2r} \xi \right) \right\}^2.
\]

(4.2.6)

When \( q^2 - 4pr > 0 \) and \( r \neq 0 \), using (4.2.3) and (4.2.4), the solutions turns into

\[
u_3(\xi) = \frac{(S - bkm T)}{2am} - \frac{6rmbk}{a} \left\{ \frac{q - \sqrt{q^2 - 4pr}}{2r} \tanh \left( \frac{\sqrt{q^2 - 4pr}}{2r} \xi \right) \right\}
- \frac{6mbkr^2}{a} \left\{ \frac{q - \sqrt{q^2 - 4pr}}{2r} \tanh \left( \frac{\sqrt{q^2 - 4pr}}{2r} \xi \right) \right\}^2,
\]

(4.2.7)

or

\[
u_4(\xi) = \frac{(S - bkm T)}{2am} - \frac{6rmbk}{a} \left\{ \frac{q + \sqrt{q^2 - 4pr}}{2r} \coth \left( \frac{\sqrt{q^2 - 4pr}}{2r} \xi \right) \right\}
- \frac{6mbkr^2}{a} \left\{ \frac{q + \sqrt{q^2 - 4pr}}{2r} \coth \left( \frac{\sqrt{q^2 - 4pr}}{2r} \xi \right) \right\}^2.
\]

(4.2.8)

If \( q^2 + 4p^2 < 0, r \neq 0 \) and \( r = -p \), then \( T = q^2 - 8p^2 \). Making use of (4.2.4) and (4.2.3), we secure

\[
u_5(\xi) = \frac{(S - bkm T)}{2am} + \frac{6pqmbk}{a} \left\{ \frac{q + \sqrt{-(q^2 + 4p^2)}}{2p} \tan \left( \frac{\sqrt{-(q^2 + 4p^2)}}{2p} \xi \right) \right\}
- \frac{6mbkp^2}{a} \left\{ \frac{q + \sqrt{-(q^2 + 4p^2)}}{2p} \tan \left( \frac{\sqrt{-(q^2 + 4p^2)}}{2p} \xi \right) \right\}^2,
\]

(4.2.9)

or

\[
u_6(\xi) = \frac{(S - bkm T)}{2am} + \frac{6pqmbk}{a} \left\{ \frac{q - \sqrt{-(q^2 + 4p^2)}}{2p} \cot \left( \frac{\sqrt{-(q^2 + 4p^2)}}{2p} \xi \right) \right\}
- \frac{6mbkp^2}{a} \left\{ \frac{q - \sqrt{-(q^2 + 4p^2)}}{2p} \cot \left( \frac{\sqrt{-(q^2 + 4p^2)}}{2p} \xi \right) \right\}^2.
\]

(4.2.10)

When \( q^2 + 4p^2 > 0, r \neq 0 \) and \( r = -p \), then \( T = q^2 - 8p^2 \). Now, embedding the values accumulated in (4.2.4) into (4.2.3), we ascertain
\[ u_7(\xi) = \frac{(S - bm^2 T)}{2am} + \frac{6pqmbk}{a} \left\{ \frac{q}{2} + \frac{\sqrt{(q^2 + 4 p^2)}}{2} \tanh \left( \frac{\sqrt{(q^2 + 4 p^2)}}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{q}{2} + \frac{\sqrt{(q^2 + 4 p^2)}}{2} \tanh \left( \frac{\sqrt{(q^2 + 4 p^2)}}{2} \xi \right) \right\}^2. \]

or
\[ u_8(\xi) = \frac{(S - bkm^2 T)}{2am} - \frac{6pqmbk}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{-(q^2 - 4 p^2)}}{2} \tan \left( \frac{-q^2 - 4 p^2}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{-(q^2 - 4 p^2)}}{2} \tan \left( \frac{-(q^2 - 4 p^2)}{2} \xi \right) \right\}^2. \]

While \( q^2 - 4 p^2 < 0 \) and \( r = p \), then \( T = q^2 + 8p^2 \). Thus, from (4.2.4) and (4.2.3), we accomplish
\[ u_9(\xi) = \frac{(S - bkm^2 T)}{2am} - \frac{6pqmbk}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{-(q^2 - 4 p^2)}}{2} \cot \left( \frac{-q^2 - 4 p^2}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{-q}{2} - \frac{\sqrt{-(q^2 - 4 p^2)}}{2} \cot \left( \frac{-(q^2 - 4 p^2)}{2} \xi \right) \right\}^2. \]

For the relation \( q^2 - 4 p^2 > 0 \) and \( r = p \), it is found \( T = q^2 + 8p^2 \). Inserting (4.2.4) into (4.2.3), we attain the soliton solutions as
\[ u_{10}(\xi) = \frac{(S - bkm^2 T)}{2am} - \frac{6pqmbk}{a} \left\{ \frac{-q}{2} - \frac{\sqrt{(q^2 - 4 p^2)}}{2} \cot \left( \frac{-q^2 - 4 p^2}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{(q^2 - 4 p^2)}}{2} \cot \left( \frac{-(q^2 - 4 p^2)}{2} \xi \right) \right\}^2. \]

or
\[ u_{11}(\xi) = \frac{(S - bkm^2 T)}{2am} - \frac{6pqmbk}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{(q^2 - 4 p^2)}}{2} \tanh \left( \frac{q^2 - 4 p^2}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{-q}{2} - \frac{\sqrt{(q^2 - 4 p^2)}}{2} \tanh \left( \frac{-(q^2 - 4 p^2)}{2} \xi \right) \right\}^2. \]

or
\[ u_{12}(\xi) = \frac{(S - bkm^2 T)}{2am} - \frac{6pqmbk}{a} \left\{ \frac{-q}{2} - \frac{\sqrt{(q^2 - 4 p^2)}}{2} \cot \left( \frac{q^2 - 4 p^2}{2} \xi \right) \right\} \]
\[ - \frac{6mbkp^2}{a} \left\{ \frac{-q}{2} + \frac{\sqrt{(q^2 - 4 p^2)}}{2} \cot \left( \frac{-(q^2 - 4 p^2)}{2} \xi \right) \right\}^2. \]

While \( q^2 = 4 rp \) then \( T = 12rp \) and applying (4.2.3) and (4.2.4), we find out
\[ u_{13}(\xi) = \frac{(S - bkm^2 T)}{2am} + \frac{6pqmbk}{a} \left( \frac{2 + q^2}{2r^2} \right) - \frac{6mbkp r}{a} \left( \frac{2 + q^2}{2r^2} \right)^2. \]

When \( pr < 0, q = 0 \) and \( r \neq 0 \), then \( T = 8pr \). Setting the result scheduled in (4.2.4) into (4.2.3), the soliton turns out to be
\[ u_{14}(\xi) = \frac{(S - bkm^2 T)}{2am} + \frac{6r q k p r}{a} \sqrt{\frac{p}{r}} \tanh \left( \sqrt{-r \frac{p}{r}} \xi \right) \]
\[ + \frac{6 m b k p r}{a} \tanh^2 \left( \sqrt{-r \frac{p}{r}} \xi \right), \]

or
\[ u_{15}(\xi) = \frac{(S - bkm^2 T)}{2am} + \frac{6r q k p r}{a} \sqrt{\frac{p}{r}} \coth \left( \sqrt{-r p} \xi \right) \]
\[ + \frac{6 m b k p r}{a} \coth^2 \left( \sqrt{-r p} \xi \right). \]
For \( r = h \) and \( p = 0 \), then \( T = h^2 + 8ph \). Now using (4.2.4) and (4.2.3), we achieve
\[
\begin{align*}
\frac{u_{17}}{a} &= \frac{(S - bkm^2 T)}{2am} - \frac{6mbk p^2}{a} \left( e^{\frac{q}{a}} - 1 + e^{2q/a} \right)^2. \\
\end{align*}
\]
(4.2.21)

By applying the condition \( q = r + p \), we attain \( T = q^2 + 8p(q - p) \) and with the aid of the values scheduled in (4.2.4) into (4.2.3), we carry out
\[
\begin{align*}
\frac{u_{18}}{a} &= \frac{(S - bkm^2 T)}{2am} + \frac{6mbk (q - p)}{a} \left( 1 - (q - p) e^{(2q - q)/a} \right) - \frac{6mbk (q - p)^2}{a} \left( \frac{1 - (q - p) e^{(2q - q)/a}}{1 - (q - p) e^{(2q - q)/a}} \right)^2. \\
\end{align*}
\]
(4.2.22)

For \( q = -(p + r) \), it becomes \( T = q^2 - 8p(p + q) \). Now inserting the values managed in (4.2.4) into (4.2.3), we attain
\[
\begin{align*}
\frac{u_{19}}{a} &= \frac{(S - bkm^2 T)}{2am} + \frac{6mbk (p + q)}{a} \left( 1 - \frac{p - e^{(2p + q)/a}}{(p - q) - e^{2(p + q)/a}} \right) - \frac{6mbk (p + q)^2}{a} \left( \frac{1 - \frac{p - e^{(2p + q)/a}}{(p - q) - e^{2(p + q)/a}}}{1 - \frac{p - e^{(2p + q)/a}}{(p - q) - e^{2(p + q)/a}}} \right)^2. \\
\end{align*}
\]
(4.2.23)

If \( p = 0 \) then \( T = q^2 \). Putting the values into (4.2.3) from (4.2.4), the soliton solution becomes
\[
\begin{align*}
\frac{u_{20}}{a} &= \frac{(S - bkm^2 T)}{2am} - \frac{6mbkq}{a} \left( q e^{a/2} - 1 - re^{a/2} \right) - \frac{6mbk}{a} \left( q e^{a/2} - 1 - re^{a/2} \right)^2. \\
\end{align*}
\]
(4.2.24)

The soliton solution turns into the following form, for \( r = q = \neq 0 \) and by means of (4.2.4) into (4.2.3)
the literature, as well as several primal solutions are originated that were not revealed in the previous study. The upstretched solutions can be used to investigate signal processing in optical fibres, gravitational waves, water wave mechanics, turbulent motion, and fluid driving flow, etc.

5. Graphical representation and discussion

5.1. Graphical representation of the solutions

We will demonstrate the profile of the graphs and describe the physical features of the resulting solutions to the space-time fractional ZKBBM equations and the (2, 2, 2) time fractional ZK equation in this paragraph.

In addition, for more details, we have illustrated a variety of figures (Figures 3–10) of the solution (4.1.25) in the interval $-12 \leq x, t \leq 12$ of the ZK equation and for the solution (4.2.6) in the interval $0 \leq x, t \leq 10$ of the ZKBBM equation for the values of all involved parameters are fixed but the fractal...
parameter $\alpha$ varies. Consequently, the effect of the fractional order is shown in the following figures. For different fractional orders, the profiles of the solution (4.1.25) of the ZK equation are:

As can be seen from the portrayals above, the fractional order $\alpha$ has a substantial impact on the profile of solitary waves. When $\alpha = 0.99$, the shape shown in Figure 11 of the solution (4.1.25) of the ZK equation is a multiple singular kink soliton. And the profiles are marked in Figures 12–16 for the values $\alpha = 0.80, 0.70, 0.60, 0.50, 0.35$, respectively, and it is seen that the profiles change with the change of the fractional order $\alpha$. 
The wave profile presented for the solution (4.2.6) of the ZKBBM equation, the influence of fractional order is shown in the underneath:

We observe that the shape is a singular periodic soliton when \( a = 0.99 \). The profiles are identified in Figures 18–22 for the values \( a = 0.80, 0.70, 0.60, 0.50, 0.45 \), respectively. Therefore, we can conclude that the fractal order \( P \) has a significant impact on wave profiles.

5.2. Physical significance of the solutions

In this earlier section, we portrayed some 3D representations using symbolic computing software Mathematica to figure out the studied solutions for both equations. For minimalism, we have depicted some graphs from the obtained adequate solutions, videlicet the solutions (4.1.9), (4.1.17), (4.1.23), (4.1.31) and (4.1.34) of the ZK equation. The remaining obtained solutions of this equation are sketched in the same way which are similar of the above illustrated figures. Therefore, these graphs are not shown here for the sake of simplicity.

Also for simplicity, we have sketched a few graphs of the ZKBBM equation, namely for the solutions (4.2.6), (4.2.11), (4.2.12), (4.2.14) and (4.2.24). We also make out that the other figures of the remaining solutions which are alike to above outlined figures. Therefore, for simplicity these figures are not been displayed here.

We also observe that, the sketched graphs are different nature of well-known shapes of wave solutions inasmuch as, kink shape wave solution, singular kink shape solutions, spike shape wave solutions and singular periodic solutions, etc.

6. Conclusion

In this article, we have established wide-ranging, useful and further developed closed-form soliton solutions to the space-time fractional ZKBBM equation and the time fractional (2, 2, 2) ZK equation. The closed-form wave solutions are established in terms of trigonometric, exponential, hyperbolic, and rational functions, as well as their integration with a number of free parameters and the fractal dimension \( a \). Theoretically, the fractional dimension \( a \) has a role in the nature of waves. To demonstrate this, we have sketched the solution (4.1.25) and (4.2.6) for different values of \( a \), keeping all the other parameters constant. The figures confirm that as the value of \( a \) changes, so does the nature of the wave, establishing the effect of \( a \). The attained solutions might play vital role in ion acoustic waves, electro-hydro-dynamical waves in the local electric field, shallow water waves, turbulent motion, driving flow of fluid, heat transfer, etc. The results obtained demonstrate that the auxiliary equation method is further developed, effective algorithm, powerful and can be employed to unravel further fractional nonlinear equations in physical science and engineering.

Acknowledgement

The authors are grateful to the anonymous referees for their insightful remarks and ideas on how to enhance the article.
Disclosure statement
No potential conflict of interest was reported by the authors.

ORCID
M. Ali Akbar https://orcid.org/0000-0001-5688-6259

References
Ahmad, H., & Khan, T. A. (2019). Variational iteration algorithm-I with an auxiliary parameter for wave-like vibration equations. Journal of Low Frequency Noise, Vibration and Active Control, 38(3–4), 1113–1132. doi:10.1177/1461348418823126
Ahmed, H., Khan, T. A., & Cesarano, C. (2019). Numerical solutions of coupled Burgers’ equations. Axioms, 8(4), 119.
Ahmad, H., Seadawy, A. R., & Khan, T. A. (2020). Study on numerical solution of dispersive wave water phenomena by using a reliable modification of variational iteration algorithm. Mathematics and Computers in Simulation, 177, 13–23. doi:10.1016/j.matcom.2020.04.005
Ahmad, H., Seadawy, A. R., Khan, T. A., & Thounthong, P. (2020). Analytic approximate solutions for some nonlinear parabolic dynamical wave equations. Journal of Taibah University for Science, 14(1), 346–358. doi:10.1080/16583655.2020.1741943
Ain, Q. T., He, J. H., Anjum, N., & Ali, M. (2020). The fractional complex transform: A novel approach to the time-fractional Schrodinger equation. Fractals, 28(07), 2050141. doi:10.1142/S0218348X20501418
Akbar, M. A., Ali, N. H. M., & Tanjim, T. (2019). Outset of solutions of coupled Burgers and the coupled Burgers equation. Journal of Physics Communications, 3(9), 095013. doi:10.1088/2399-6528/ab3615
Akbulut, A., Kaplan, M., & Bekir, A. (2016). Auxiliary equation method for fractional differential equations with modified Riemann-Liouville derivative. International Journal of Nonlinear Science and Numerical Simulation, 17(4), 13–20.
Alam, M. N., Akbar, M. A., & Mohyud-Din, S. T. (2014). A novel \((G'/G)\)-expansion method and its application to the Boussinesq equations. Chinese Physics B, 23(2), 020203. doi:10.1088/1674-1056/23/2/020203
Ali, A., Iqbal, A., & Mohyud-Din, S. T. (2016a). Traveling wave solutions of generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony and simplified modified form of Camassa-Holm equation method by using exp\((-\phi(\eta))\)-expansion method. Egyptian Journal of Basic and Applied Sciences, 3(2), 134–140. doi:10.1016/j.ejbas.2016.01.001
Ali, A., Iqbal, M. A., & Mohyud-Din, S. T. (2016b). Solitary wave solutions Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation. Journal of the Egyptian Mathematical Society, 24(1), 44–48. doi:10.1016/j.joems.2014.10.008
Ali, A. H. A. (2007). The modified extended tanh-function method for solving coupled mKdV and coupled Hirota-Satsuma coupled KdV equations. Physics Letters A, 363(5–6), 420–425. doi:10.1016/j.physleta.2006.11.076
Alzaidy, J. F. (2013). The fractional sub-equation method and exact analytical solutions for some nonlinear fractional PDEs. British Journal of Mathematics & Computer Science, 3(2), 153–163. doi:10.9734/BJMCS/2013/2908
Anjum, N., & Ain, Q.T. (2020). Application of He’s fractional derivative and fractional complex transform for time fractional Camassa-Holm equation. Thermal Science, 24(5 Part A), 3023–3030. doi:10.2298/TSC190930450A
Atangana, A., Baleanu, D., & Alsaeedi, A. (2016). Analysis of time-fractional Hunter-Saxton equation: A model of neumatic liquid crystal. Open Physics, 14(1), 145–149. doi:10.1515/phys-2016-0010
Atangana, A. (2017). Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system. Chaos, Solitons & Fractals, 102, 396–406. doi:10.1016/j.chaos.2017.04.027
Baleanu, D., Golmankhaneh, A. K., Golmankhaneh, A. K., & Nigmatullin, R. R. (2010). Newtonian law with memory. Nonlinear Dynamics, 60(1–2), 81–86. doi:10.1007/s10755-009-9581-1
Bazighifan, O., Ahmad, H., & Yao, S.-W. (2020). New oscillation criteria for advanced differential equations of fourth order. Mathematics, 8(5), 728. doi:10.3390/math8050728
Bekir, A., & Guner, O. (2013). Exact solutions of nonlinear fractional differential equation by \((G'/G)\)-expansion method. Chinese Physics B, 22(11), 110202–110206. doi:10.1088/1674-1056/22/11/110202
Brouers, F. (2014). The fractal (BSF) kinetics equation and its approximations. Journal of Modern Physics, 05(16), 1594–1601. doi:10.4236/jmp.2014.516160
Brouers, F., & Al-Musawi, T. J. (2018). Brouers-Sotolongo fractal kinetics versus fractional derivative kinetics: A new strategy to analyze the pollutants sorption kinetics in porous materials. Journal of Hazardous Materials, 350, 162–168. doi:10.1016/j.jhazmat.2018.02.015
Brouers, F., & Sotolongo-Costa, O. (2006). Generalized fractal kinetics in complex systems (application to biophysics and biotechnology). Physica A: Statistical Mechanics and Its Applications, 368(1), 165–175. doi:10.1016/j.physa.2005.12.062
Bulut, H., Baskonus, H. M., & Pandir, Y. (2013). The modified trial equation method for fractional wave equation and time fractional generalized Burgers equation. Abstract and Applied Analysis, 2013, 1–8. doi:10.1155/2013/636802
Caputo, M., & Fabrizio, M. A. (2015). A new definition of fractional derivatives without singular kernel. Progress in Fractional Differentiation and Applications, 1, 73–85.
Chen, W., & Liang, Y. (2017). New methodologies in fractional and fractal derivatives modeling. Chaos, Solitons & Fractals, 102, 72–77. doi:10.1016/j.chaos.2017.03.066
Dahmani, Z., Mesmoudi, M. M., & Bebbouchi, R. (2008). The foam-drainage equation with time and space fractional derivative solved by the ADM method. Journal of Qualitative Theory of Differential Equations, 30, 1–10.
Deng, W. (2009). Finite element method for the space and time fractional Fokker-Planck equation. SIAM Journal on Numerical Analysis, 47(1), 204–226. doi:10.1137/08071430E
Ege, S. M., & Misirli, E. (2014). Solutions of space-time fractional foam drainage equation and the fractional Klein-Gordon equation by use of modified Kudryashov method. International Journal of Research in Advent Technology, 2(3), 384–388.
Fan, J., Wang, L.-L., Liu, F.-J., Liu, Z., Liu, Y., & Zhang, S. (2015). Model of moisture diffusion in fractal media. Thermal Science, 19(4), 1161–1166. doi:10.2298/TSC1504161F
Fereidoon, A., Yaghoobi, H., & Davoudabadi, M. (2011). Application of the homotopy perturbation method for solving the foam drainage equation. International Journal of Differential Equations, 2011, 1–13. doi:10.1155/2011/864023
Islam, M. N., & Akbar, M. A. (2020). Stable wave solutions to the Landau-Ginzburg-Higgs equation and the modified equal width wave equation using the IBSF method. *Arab Journal of Basic and Applied Sciences*, 27(1), 270–278. doi:10.1080/25765299.2020.1791466

Islam, M. N., & Akbar, M. A. (2018a). Exact traveling wave solutions for the (2+1)-dimensional ZK-BBM equation by exp(−φ(η))-expansion method. *Global Journal of Pure and Applied Sciences, Part A, Basic and Applied Sciences*, 14(2), 1–5.

Islam, M. N., & Akbar, M. A. (2018b). Closed form solutions to the coupled space-time fractional evolution equations in mathematical physics through analytical method. *Journal of Mechanics of Continua and Mathematical Sciences*, 13(2), 1–23. doi:10.26782/jmcs.2018.06.00001

Islam, M. N., & Akbar, M. A. (2018c). Closed form exact solutions to the higher dimensional fractional Schrodinger equation via the modified simple equation method. *Journal of Applied Mathematics and Physics*, 06(01), 90–102. doi:10.4236/jamp.2018.61009
Yokus, A., Durur, H., & Ahmad, H. (2020). Hyperbolic type solutions for the couple Boiti-Leon-Pempinelli system. *Facta Universitatis, Series: Mathematics and Informatics*, 35(2), 523–531. doi:10.22190/FUMI2002523Y

Yokus, A., Durur, H., Ahmad, H., & Yao, S.-W. (2020). Construction of different types analytic solutions for the Zhiber-Shabat equation. *Mathematics*, 8(6), 908. doi:10.3390/math8060908

Younis, M. (2013). The first integral method for time-space fractional differential equations. *Journal of Advanced Physics*, 2(3), 220–223. doi:10.1166/jap.2013.1074

Yıldırım, A., & Gulkanat, Y. (2010). Analytical approach to fractional Zakharov-Kuznetsov equations by He’s homotopy perturbation method. *Communication and Theatrical Physics*, 53(6), 1005–1010.

Yu, D.N., He, J.H., & Garcia, A.G. (2019). Homotopy perturbation method with an auxiliary parameter for nonlinear oscillators. *Journal of Low Frequency Noise, Vibration and Active Control*, 38(3–4), 1540–1554. doi:10.1177/1461348418811028

Zheng, B. (2013). Exp-function method for solving fractional partial differential equations. *TheScientificWorldJournal*, 2013, 465723. doi:10.1155/2013/465723

Zheng, B & Feng, Q. (2014). The Jacobi elliptic equation method for solving fractional partial differential equations. *Abstract and Applied Analysis*, 2014, 1–9. doi:10.1155/2014/249071