The universality of Hughes-free division rings

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Abstract
Let $E \ast G$ be a crossed product of a division ring $E$ and a locally indicable group $G$. Hughes showed that up to $E \ast G$-isomorphism, there exists at most one Hughes-free division $E \ast G$-ring. However, the existence of a Hughes-free division $E \ast G$-ring $D_{E \ast G}$ for an arbitrary locally indicable group $G$ is still an open question. Nevertheless, $D_{E \ast G}$ exists, for example, if $G$ is amenable or $G$ is bi-orderable. In this paper we study, whether $D_{E \ast G}$ is the universal division ring of fractions in some of these cases. In particular, we show that if $G$ is a residually-(locally indicable and amenable) group, then there exists $D_{E[G]}$ and it is universal. In Appendix we give a description of $D_{E[G]}$ when $G$ is a RFRS group.

Keywords  Locally indicable groups · Universal division ring of fractions · Hughes-free division ring

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1 Introduction
A division $R$-ring $\phi : R \to D$ is called epic if $\phi(R)$ generates $D$ as a division ring. Each division $R$-ring $D$ induces a Sylvester matrix rank function $\text{rk}_D$ on $R$. Given a ring $R$, Cohn introduced the notion of universal division $R$-ring (see, for example, [4, Section 7.2]). In the language of Sylvester rank functions, an epic division $R$-ring $D$ is universal if for every division $R$-ring $\mathcal{E}$, $\text{rk}_D \geq \text{rk}_\mathcal{E}$. By a result of Cohn [3, Theorem 4.4.1], the universal epic division $R$-ring is unique up to $R$-isomorphism. The universal division $R$-ring $D$ is called universal division ring of fractions of $R$ if $D$ is epic and $\text{rk}_D$ is faithful (that is $R$ is embedded in $D$).
If $R$ is a commutative domain, then the field of fractions $\mathcal{Q}(R)$ is the universal division $R$-ring. The situation is much more complicated in the non-commutative setting. For example, Passman [24] gave an example of a Noetherian domain which does not have a universal division ring of fractions. Moreover, we show in Proposition 4.1 that the group algebra $\mathcal{Q}[H]$ does not have a universal division ring of fractions if $H$ is not locally indicable. In this paper we want to study whether a group algebra or, more generally, a crossed product $E \ast G$, where $E$ is a division ring, has a universal division ring of fractions. Thus, from the previous observation it is natural to consider the case of group algebras and crossed products $E \ast G$ where $G$ is locally indicable.

Let $E$ be a division ring and $G$ a locally indicable group. Hughes [11] introduced a condition on an epic division $E \ast G$-rings and showed that up to $E \ast G$-isomorphism, there exists at most one epic division $E \ast G$-ring satisfying this condition. We call this division ring, the Hughes-free division $E \ast G$-ring and denote it by $\mathcal{D}_{E \ast G}$. For simplicity, in this paper the Sylvester matrix rank function $\text{rk}_{\mathcal{D}_{E \ast G}}$ is denoted by $\text{rk}_{E \ast G}$. We say that a locally indicable group $G$ is Hughes-free embeddable if $E \ast G$ has a Hughes-free division ring for every division ring $E$ and every crossed product $E \ast G$.

The existence of a Hughes-free division $E \ast G$-ring is known for several families of locally indicable groups. In the case of amenable locally-indicable groups $G$, $\mathcal{D}_{E \ast G} = \mathcal{Q}(E \ast G)$ is the classical ring of fractions of $E \ast G$, and in the case of bi-orderable groups $G$, $\mathcal{D}_{E \ast G}$ is constructed using the Malcev-Neumann construction [20,23] (see [8]). The existence of $\mathcal{D}_{K[G]}$ is also known for group algebras $K[G]$, where $K$ is of characteristic 0 and $G$ is an arbitrary locally indicable group [15].

In [15, Theorem 8.1] it is shown that if there exists a universal epic division $E \ast G$-ring and a Hughes-free division $E \ast G$-ring, they are isomorphic as $E \ast G$-rings. Following Sánchez (see [25, Definition 6.18]), we say that a locally indicable group $G$ is a Lewin group if it is Hughes-free embeddable and for all possible crossed products $E \ast G$, where $E$ is a division ring, $\mathcal{D}_{E \ast G}$ is universal (in Sect. 3.3 we will see that this definition is equivalent to the Sánchez one). We conjecture that all locally indicable groups are Lewin.

**Conjecture 1** Let $G$ be a locally indicable group, $E$ a division ring and $R = E \ast G$ a crossed product of $E$ and $G$. Then

(A) the Hughes-free division $R$-ring $\mathcal{D}_R$ exists and
(B) it is universal division ring of fractions of $R$.

We want to notice that at this moment it is also an open problem of whether the universal division $E \ast G$-ring of fractions (if exists) should be Hughes-free.

In this paper we study part (B) of the conjecture in some cases where part (A) is known. Using Theorem 3.7 we can show that Conjecture 1 is valid for the following locally indicable groups.

**Theorem 1.1** Locally indicable amenable groups, residually-(torsion-free nilpotent) groups and free-by-cyclic groups are Lewin groups.

In the case of group algebras we can prove a stronger result. The metric space $G_n$ of marked $n$-generated groups consists of pairs $(G; S)$, where $G$ is a group and $S$ is an ordered generating set of $G$ of cardinality $n$. Such pairs are in 1-to-1 correspondence
with epimorphisms $F_n \to G$, where $F_n$ is the free group of rank $n$, and thus the set $G_n$ can be identified with the set of all normal subgroups of $F = F_n$. The distance between two normal subgroups $M_1$ and $M_2$ of $F$ is defined by

$$d(M_1, M_2) = \inf \{ e^{-k} : M_1 \cap B_k(1_F) = M_2 \cap B_k(1_F) \},$$

where $B_k(1_F)$ denotes the closed ball of radius $k$ and center $1_F$.

We say that a sequence of $n$-generated groups $\{G_i\}_{i \in \mathbb{N}}$ converges to an $n$-generated group $G$ if $(G_i; S_i) \to (G; S)$ in $G_n$ for some generating sets $S_i$ of $G_i$ ($i \in \mathbb{N})$ and $S$ of $G$, respectively.

**Theorem 1.2** Let $F$ be a free group freely generated by a finite set $S$ and $M$ and $\{M_i\}_{i \in \mathbb{N}}$ normal subgroups of $F$. We put $G = F/M$ and $G_i = F/M_i$ and assume that $(G_i; S_iM_i/M_i)$ converges to $(G; SM/M)$. Assume that for all $i$, $G_i$ is locally indicable and $D_E[G_i]$ exists. Then $G$ is locally indicable, $D_E[G]$ exists and

$$\text{rk}_E[G] = \lim_{i \to \infty} \text{rk}_E[G_i]$$

as Sylvester matrix rank functions on $E[F]$.

As a corollary we obtain the following consequence.

**Corollary 1.3** Let $G$ be a residually-(locally indicable and amenable) group and let $E$ be a division ring. Then $D_E[G]$ exists and it is the universal division ring of fractions of $E[G]$.

The corollary can be applied to RFRS groups, because they are residually poly-$\mathbb{Z}$.

The notion of RFRS groups arose in a work of Agol [1], in connection with the virtual-fibering of 3-manifolds [2], and it abstracts a critical property of the fundamental groups of special cube complexes. Kielak [18] realizes that the main result of [1] can be stated not only for 3-manifold groups but also for virtually RFRS groups. The proof of Kielak uses a new description of $D_Q[G]$ when $G$ is RFRS. In Sect. 5 we give a description of $D_E[G]$ when $G$ is a RFRS group that generalizes the result of Kielak.

Let us consider now the case of group algebras $K[G]$ where $K$ is a subfield of $\mathbb{C}$ and $G$ is locally indicable. In this case it was shown in [15] that the division closure $D(K[G], U(G))$ of $K[G]$ in the algebra of affiliated operators $U(G)$ is a Hughes-free division $K[G]$-ring. We denote by $\text{rk}_G$ the von Neumann rank function (its definition is recalled in Sect. 2.6), and by $\text{rk}_{[1]}$ the Sylvester matrix rank function on $Q[G]$ induced by the homomorphism $Q[G] \to Q$ that sends all the elements of $G$ to $1$ (in the previous notation $\text{rk}_{[1]}$ is $\text{rk}_Q$). In view of Conjecture 1, it is natural to ask for which groups $G$, $\text{rk}_G \geq \text{rk}_{[1]}$. It follows from [26, Proposition 1.9] that if a group $G$ satisfies the condition $\text{rk}_G \geq \text{rk}_{[1]}$, then $G$ is locally indicable. Thus, we propose also a weak version of Conjecture 1.

**Conjecture 2** Let $G$ be locally indicable group. Then $\text{rk}_G \geq \text{rk}_{[1]}$ as Sylvester matrix rank functions on $Q[G]$.

From the discussion in the paragraph before the conjecture, we conclude that Corollary 1.3 has the following consequence.
Corollary 1.4 Let $G$ be a residually-(locally indicable and amenable) group. Then $\text{rk}_G \geq \text{rk}_{\{1\}}$ as Sylvester matrix rank functions on $\mathbb{Q}[G]$. 

Combining this result with the mentioned above result of Kielak [18], we obtain the following corollary.

Corollary 1.5 Let $G$ be a finitely generated group which is virtually RFRS. Then the following are equivalent.

1. $G$ is virtually fibered, in the sense that it admits a virtual map onto $\mathbb{Z}$ with finitely generated kernel.
2. $G$ admits a virtual map onto $\mathbb{Z}$ whose kernel has finite first Betti number.

Our next result is another consequence of Corollary 1.4 that generalizes a result of Wise [28, Theorem 1.3],

Corollary 1.6 Let $X$ be a compact CW-complex with $\pi_1 X$ non-trivial residually-(locally indicable and amenable) group. Then 

$$b_1^{(2)}(\tilde{X}) \leq b_1(X) - 1 \text{ and } b_p^{(2)}(\tilde{X}) \leq b_p(X) \text{ if } p \geq 2.$$ 

The paper is structured as follows. We introduce the basic notions in Sect. 2. In Sect. 3, we prove Theorem 1.1, Theorem 1.2 and Corollary 1.3. In Sect. 4 we study the consequences of the condition $\text{rk}_G \geq \text{rk}_{\{1\}}$ and, in particular, we prove Corollary 1.5 and Corollary 1.6. In Sect. 5 we give an alternative description of the division ring $D_{E[G]}$ when $G$ is RFRS and $E$ is a division ring.

2 Preliminaries

2.1 Notation and definitions

All rings in this paper are unitary and ring homomorphisms send the identity element to the identity element. By a module we will mean a left module. Let $G$ be a group with trivial element $e$. We say that a ring $R$ is $G$-graded if $R$ is equal to the direct sum $\oplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g$ and $h$ in $G$. If for each $g \in G$, $R_g$ contains an invertible element $u_g$, then we say that $R$ is a crossed product of $R_e$ and $G$ and we will write $R = S \ast G$ if $R_e = S$. In the following if $H$ is a subgroup of $G$, $S \ast H$ will denote the subring of $R$ generated by $S$ and $\{u_h : h \in H\}$.

A ring $R$ may have several different $G$-gradings. It will be always clear from the context what $G$-grading we use. However, under some conditions the grading is unique. Assume that $R \cong E \ast G$, where $E$ is a division ring and $G$ is locally indicable, then by [9], the invertible elements $U(R)$ of $R$ are $\bigcup_{g \in G} R_g \setminus \{0\}$. Hence $R_e$ is the maximal subring in $U(R) \cup \{0\}$ and $G \cong U(R)/(R_e \setminus \{0\})$. Thus, $R$ has a unique grading with $R_e$ is a division ring and $G$ is locally indicable.

An $R$-ring is a pair $(S, \phi)$ where $\phi : R \rightarrow S$ is a homomorphism. We will often omit $\phi$ if it is clear from the context.
2.2 Ordered groups

A total order \( \preceq \) on a group \( G \) is **left-invariant** if for any \( a, b, g \in G \), if \( a \preceq b \) then \( ga \preceq gb \). It is **bi-invariant** if, moreover, we have \( ag \preceq bg \).

Let \( \preceq \) be a left-invariant order on a group \( G \). A subgroup \( H \) is called **convex** if \( H \) contains every element \( g \) lying between any two elements of \( H \) (\( h_1 \preceq g \preceq h_2 \) with \( h_1, h_2 \in H \)). We say that \( \preceq \) is **Conradian** if for all elements \( f, g \geq 1 \), there exists a natural number \( n \) such that \( f g^n > g \). In fact, one may actually take \( n = 2 \) ([6, Proposition 3.2.1]). Recall that a group \( G \) is **locally indicable** if every finitely generated non-trivial subgroup of \( G \) has an infinite cyclic quotient. A useful characterization of locally indicable groups says that they are the groups admitting a Conradian order ([5]). We will need the following important property of a Conradian order.

**Proposition 2.1** [6, Corollary 3.2.28] Let \( (G, \preceq) \) be a group with a Conradian order and let \( N \) be the proper maximal convex subgroup of \( G \). Then there exists an order preserving homomorphism \( \phi : G \to \mathbb{R} \) such that \( N = \ker \phi \).

2.3 Hughes-free division rings

Let \( E \) be a division ring and \( G \) a locally indicable group. Let \( \phi : E \ast G \to D \) be a homomorphism from \( E \ast G \) to a division ring \( D \). We say that a division \( E \ast G \)-ring \((D, \phi)\) is **Hughes-free** if

1. \( D \) is the division closure of \( \phi(E \ast G) \) (\( D \) is epic).
2. For every non-trivial finitely generated subgroup \( H \) of \( G \), a normal subgroup \( N \) of \( H \) with \( H/N \cong \mathbb{Z} \), and \( h_1, \ldots, h_n \in H \) in distinct cosets of \( N \), the sum \( D_N, \phi(u_{h_1}) + \cdots + D_N, \phi(u_{h_n}) \) is direct. (Here \( D_{N,D} = D(\phi(E \ast N), D) \) is the division closure of \( \phi(E \ast N) \) in \( D \).)

Hughes [11] (see also [7]) showed that up to \( E \ast G \)-isomorphism there exists at most one Hughes-free division ring. We denote it by \( D_{E \ast G} \). The uniqueness of Hughes-free division rings implies that for every subgroup \( H \) of \( G \), \( D_H, D_{E \ast G} \) is Hughes-free as a division \( E \ast H \)-ring.

Gräter showed in [8, Corollary 8.3] that \( D_{E \ast G} \) (if it exists) is **strongly Hughes-free**, that it satisfies the following additional condition:

1. For every non-trivial subgroup \( H \) of \( G \), a normal subgroup \( N \) of \( H \) and \( h_1, \ldots, h_n \in H \) in distinct cosets of \( N \), the sum \( D_N, D_{E \ast G} \phi(u_{h_1}) + \cdots + D_N, D_{E \ast G} \phi(u_{h_n}) \) is direct.

In particular, this implies the following result that we will use often without mentioning it explicitly.

**Proposition 2.2** Let \( G \) be a locally indicable group, \( N \) a normal subgroup of \( G \) and \( E \) a division ring. Assume that for a crossed product \( E \ast G \), \( D_{E \ast G} \) exists. Then the ring \( R \) generated by \( D_N, D_{E \ast G} \) and \( G \) has structure of a crossed product \( D_{E \ast N} \ast (G/N) \). In particular,

1. if \( N \) is of finite index in \( G \), then \( D_{E \ast G} = D_{E \ast N} \ast (G/N) \) and
2. if \( G/N \) is abelian, \( D_{E \ast G} \) is isomorphic to the classical Ore ring of fractions of \( D_{E \ast N} \ast (G/N) \).
2.4 Free division $E \ast G$-ring of fractions

Let $G$ be group with a Conradian left-invariant order $\preceq$ (so, $G$ is locally indicable). Let $E$ be a division ring. Let $\varphi : E \ast G \to D$ be a homomorphism from a crossed product $E \ast G$ to a division ring $D$. We say that a division $E \ast G$-ring $(D, \varphi)$ is **free with respect to** $\preceq$ if

1. $D$ is the division closure of $\varphi(E \ast G)$.
2. For every subgroup $H$ of $G$, and the maximal proper convex subgroup $N$ of $H$ (which is normal by Proposition 2.1), and $h_1, \ldots, h_n \in H$ in distinct cosets of $N$, the sum $D_N \cdot D \varphi(u_{h_1}) + \cdots + D_N \cdot D \varphi(u_{h_n})$ is direct.

This notion was introduced by Gräter in [8].

**Remark 2.3** Notice that in part (2) of the definition, we also can assume that $H$ is finitely generated. Indeed, assume (2) holds for finitely generated subgroups, but for some $H$ and $h_1, \ldots, h_n$, there are $d_1, \ldots, d_n \in D_N \cdot D$, not all equal to zero, such that $d_1 \varphi(u_{h_1}) + \cdots + d_n \varphi(u_{h_n}) = 0$. Then we can find a finitely generated subgroup of $N'$ of $N$ such that $d_1, \ldots, d_n \in D_{N'} \cdot D$. Let $H'$ be the subgroup of $G$ generated by $h_1, \ldots, h_n$ and $N'$. Since $n \geq 2$, $N \cap H'$ is the maximal convex subgroup of $H'$. This contradicts our assumption that (2) holds for $H'$.

Gräter proved the following result.

**Proposition 2.4** [8, Corollary 8.3] Let $G$ be a group with a Conradian left-invariant order $\preceq$ and let $E$ be a division ring. A division $E \ast G$-ring is free with respect to $\preceq$ if and only if it is Hughes-free (and so, it is $E \ast G$-isomorphic to $D_E \ast G$).

2.5 Sylvester matrix rank functions

Let $R$ be a ring. A **Sylvester matrix rank function** $\text{rk}$ on $R$ is a function that assigns a non-negative real number to each matrix over $R$ and satisfies the following conditions.

1. $(\text{SMat1})$ $\text{rk}(M) = 0$ if $M$ is any zero matrix and $\text{rk}(1) = 1$;
2. $(\text{SMat2})$ $\text{rk}(M_1M_2) \leq \min\{\text{rk}(M_1), \text{rk}(M_2)\}$ for any matrices $M_1$ and $M_2$ which can be multiplied;
3. $(\text{SMat3})$ $\text{rk} \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices $M_1$ and $M_2$;
4. $(\text{SMat4})$ $\text{rk} \begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \geq \text{rk}(M_1) + \text{rk}(M_2)$ for any matrices $M_1$, $M_2$ and $M_3$ of appropriate sizes.

We denote by $\mathbb{P}(R)$ the set of Sylvester matrix rank functions on $R$, which is a compact convex subset of the space of functions on matrices over $R$. If $\phi : F_1 \to F_2$ is an $R$-homomorphism between two free finitely generated $R$-modules $F_1$ and $F_2$, then $\text{rk}(\phi)$ is $\text{rk}(A)$ where $A$ is the matrix associated with $\phi$ with respect to some $R$-bases of $F_1$ and $F_2$. It is clear that $\text{rk}(\phi)$ does not depend on the choice of the bases.

A useful observation is that a ring homomorphism $\varphi : R \to S$ induces a continuous map $\varphi^*: \mathbb{P}(S) \to \mathbb{P}(R)$, i.e., we can pull back any rank function $\text{rk}$ on $S$ to a rank
function $\varphi^\sharp(\text{rk})$ on $R$ by just defining

$$\varphi^\sharp(\text{rk})(A) = \text{rk}(\varphi(A))$$

for every matrix $A$ over $R$. We will often abuse the notation and write $\text{rk}$ instead of $\varphi^\sharp(\text{rk})$ when it is clear that we speak about the rank function on $R$.

A division ring $D$ has a unique Sylvester matrix rank function which we denote by $\text{rk}_D$. If a Sylvester matrix rank function $\text{rk}$ on $R$ takes only integer values, then by a result of P. Malcolmson [21] there is a division ring $D$ and a homomorphism $\varphi : R \to D$ such that $\text{rk} = \varphi^\sharp(\text{rk}_D)$. Moreover, if $D$ is equal to the division closure of $\varphi(R)$ ($D$ is an epic division $R$-ring), then $\varphi : R \to D$ is unique up to isomorphisms of $R$-rings. We denote the set of integer-valued rank functions on a ring $R$ by $\mathbb{P}_{\text{div}}(R)$.

In the following, if a rank function on $R$ is induced by a homomorphism to $D$ we will also use $\text{rk}_D$ to denote this rank function (in this case the homomorphism will be clear from the context).

Given two Sylvester matrix rank functions on $R, \text{rk}_1$ and $\text{rk}_2$, we will write $\text{rk}_1 \leq \text{rk}_2$ if for any matrix $A$ over $R, \text{rk}_1(A) \leq \text{rk}_2(A)$. In the case where both functions are integer-valued and come from homomorphisms $\varphi_i : R \to D_i$ ($i = 1, 2$) from $R$ to epic division rings $D_1$ and $D_2$, the condition $\text{rk}_{D_1} \leq \text{rk}_{D_2}$ is equivalent to the existence of a specialization from $D_2$ to $D_1$ in the sense of P. M. Cohn ([3, Subsection 4.1]). We say that an epic division $R$-ring $D$ is universal if for every epic division $R$-ring $E$, $\text{rk}_D \geq \text{rk}_E$.

An alternative way to introduce Sylvester rank functions is via Sylvester module rank functions. A Sylvester module rank function $\text{dim}$ on $R$ is a function that assigns a non-negative real number to each finitely presented $R$-module and satisfies the following conditions.

\begin{enumerate}
  \item[(SMod1)] $\text{dim}\{0\} = 0, \text{dim}\ R = 1$;
  \item[(SMod2)] $\text{dim}(M_1 \oplus M_2) = \text{dim}\ M_1 + \text{dim}\ M_2$;
  \item[(SMod3)] if $M_1 \to M_2 \to M_3 \to 0$ is exact then \(\text{dim}\ M_1 + \text{dim}\ M_3 \geq \text{dim}\ M_2 \geq \text{dim}\ M_3\).
\end{enumerate}

There exists a natural bijection between Sylvester matrix and module rank functions over a ring. Given a Sylvester matrix rank function $\text{rk}$ on $R$ and a finitely presented $R$-module $M \cong R^n / R^mA$ ($A$ is a matrix over $R$), we define the corresponding Sylvester module rank function $\text{dim}$ by means of $\text{dim}(M) = n - \text{rk}(A)$. If a Sylvester matrix rank function $\text{rk}_D$ comes from a division $R$-ring $D$, then the corresponding Sylvester module rank function will be denoted by $\text{dim}_D$. Then $D$ is the universal epic division $R$-ring if and only if for every epic division $R$-ring $E$ and every finitely presented $R$-module, $\text{dim}_E(M) \leq \text{dim}_D(M)$.

By a recent result of Li [19], any Sylvester module rank function on $R$ can be extended to a function (satisfying some natural conditions) on arbitrary modules over $R$. In the case of an integer-valued Sylvester module rank function $\text{dim}_D$ and an $R$-module $M$ we simply have $\text{dim}_D(M) = \text{dim}_D(D \otimes_R M)$. 

2.6 Von Neumann rank function

Consider first the case where $G$ is countable. Then $G$ acts by left and right multiplication on the separable Hilbert space $l^2(G)$. A finitely generated Hilbert $G$-module is a closed subspace $V \leq l^2(G)^n$, invariant under the left action of $G$. We denote by $\text{proj}_V : l^2(G)^n \to l^2(G)^n$ the orthogonal projection onto $V$ and we define

$$\dim_G V := \text{Tr}_G(\text{proj}_V) := \sum_{i=1}^{n} \langle (1_i) \text{proj}_V, 1_i \rangle_{l^2(G)^n},$$

where $1_i$ is the element of $l^2(G)^n$ having 1 in the $i$th entry and 0 in the rest of the entries. The number $\dim_G V$ is the von Neumann dimension of $V$.

Let $A \in \text{Mat}_{n \times m}(\mathbb{C}[G])$ be a matrix over $\mathbb{C}[G]$. The action of $A$ by right multiplication on $l^2(G)^n$ induces a bounded linear operator $\phi^A_G : l^2(G)^n \to l^2(G)^m$. We put

$$\text{rk}_G(A) = \dim_G \text{Im} \phi^A_G.$$

If $G$ is not countable then $\text{rk}_G$ can be defined in the following way. Take a matrix $A$ over $\mathbb{C}[G]$. Then the group elements that appear in $A$ are contained in a finitely generated group $H$. We will put $\text{rk}_G(A) = \text{rk}_H(A)$. One easily checks that the value $\text{rk}_H(A)$ does not depend on the subgroup $H$.

Another obvious Sylvester matrix rank function on $G$ arises from the trivial homomorphism $G \to \{1\}$ and it is defined as

$$\text{rk}_{\{1\}}(A) = \text{rk}_\mathbb{C} \overline{A},$$

where $\overline{A}$ is the matrix over $\mathbb{C}$ obtained from $A$ by sending all the elements of $G$ to 1. More generally, if $\overline{G}$ is a quotient of $G$, $\text{rk}_{\overline{G}}(A)$ is denoted to be $\text{rk}_{\overline{G}}(\overline{A})$, where $\overline{A}$ is the matrix over $\mathbb{C}[\overline{G}]$ obtained from $A$ by applying the obvious map $\mathbb{C}[G] \to \mathbb{C}[\overline{G}]$.

2.7 The natural extension

Let $R = E \ast G$ be a crossed product of a division ring $E$ and a group $G$. Let $N$ be a normal subgroup of $G$ such that $G/N$ is amenable. Consider a transversal $X$ of $N$ in $G$. Since $G/N$ is amenable there are finite subsets $X_k$ of $X$ such that $\{N X_k/N\}$ is a Følner sequence in $G/N$ with respect to the right action. Put $X_k = N X_k$.

Let $\text{rk}$ be a Sylvester rank function on $E \ast N$ and assume that $\text{rk}$ is invariant under conjugation by the elements $\{u_g\}_{g \in G}$. Observe that if $\text{rk} = \text{rk}_E$ for some epic division $E \ast N$-ring $E$, then the conjugation of $E \ast N$ by any $u_g(g \in G)$ can be extended to a unique automorphism of $E$. Thus one can consider the crossed product $E \ast G/N$ containing $E \ast G$.

Let $A \in \text{Mat}_{n \times m}(R)$ and let $S$ be the union of supports of the entries of $A$. For any subset $T$ of $G$ we denote $R_T = \bigoplus_{t \in T} R_t$. Let $\phi_k : (R_{X_k})^n \to (R_{X_k S})^m$ be
the homomorphism of finitely generated free $E \ast N$-modules induced by the right multiplication by $A$. Let $\omega$ be a non-principal ultrafilter on $\mathbb{N}$. Then we put

$$\tilde{r}_K(A) = \lim_{\omega} \frac{\text{rk}(\phi_i)}{|X_i|}.$$  \hspace{1cm} (1)

Then $\tilde{r}_K$ is a Sylvester rank function on $R$. The rank function $\tilde{r}_K$ has been already studied previously in different situations (see [14,15,17,27]). In [17] it is shown that $\tilde{r}_K$ does not depend on $\omega$. Therefore in the following we denote $\tilde{r}_K$ by $\tilde{r}$. The Sylvester rank function $\tilde{r}$ is called the natural extension of $\text{rk}$. We describe now the cases that appear in this paper.

**Proposition 2.5** Let $G$ be a group with a normal subgroup $N$ such that $G/N$ is amenable. Let $E$ be a division ring, and assume the previous notation. Then the following holds.

1. Assume that $N$ and $G/N$ are locally indicable and $\text{rk} = \text{rk}_E$ for some epic division $E \ast N$-ring $E$. Then $\tilde{r}$ coincides with $\text{rk}_{Q(E \ast (G/N))}$, where $Q(E \ast (G/N))$ denotes the classical Ore ring of fractions of $E \ast (G/N)$.

2. Assume $E \ast G = K[G]$, where $K$ is a subfield of $\mathbb{C}$ and $\text{rk} = \text{rk}_N$. Then $\tilde{r}$ is equal to $\text{rk}_G$.

3. Assume $E \ast G = K[G]$, where $K$ is a subfield of $\mathbb{C}$ and $\text{rk} = \text{rk}_{\{1\}}$. Then $\tilde{r}$ is equal to $\text{rk}_{G/N}$.

**Proof** (1) We can extend $\tilde{r}$ to a Sylvester matrix rank function on $E \ast (G/N)$ (which we denote also by $\tilde{r}$) using the formula (1). Since $G/N$ is locally indicable, the ring $E \ast (G/N)$ is a domain. Thus, by the definition of $\tilde{r}$, $\tilde{r}(a) = 1$ for every $0 \neq a \in E \ast (G/N)$. Hence, applying [14, Proposition 5.2], we obtain that $\tilde{r} = \text{rk}_{Q(E \ast (G/N))}$.

The statements (2) and (3) follow from [14, Theorem 12.1]. \hfill \Box

**3 On the universality of $\mathcal{D}_{E \ast G}$**

**3.1 A general criterion of universality**

In this subsection we present a general criterion of universality of a division $R$-ring. The proof of the following lemma is immediate.

**Lemma 3.1** Let $R$ be a ring and $E$ a division $R$-ring. Let $M$ be a finitely generated left $R$-module. Then the following are equivalent.

1. $\dim_{\mathcal{E}}(M) \neq 0$.

2. $E \otimes_{R} M \neq 0$.

3. $\text{Hom}_{R}(M, E) \neq 0$.

The following proposition tells us that in order to check universality of a division $R$-ring $\mathcal{D}$ it is enough to understand the structure of its finitely generated $R$-submodules.
Proposition 3.2  Let $R$ be a ring and $D$ an epic division $R$-ring. Then $\text{rk}_D$ is universal in $\mathcal{D}_{\text{div}}(R)$ if and only if for every finitely generated left $R$-submodule $L$ of $D$ and every division $R$-ring $E$, $\dim_E(L) > 0$.

Proof Assume that $\text{rk}_D$ is universal. Since $\text{Hom}_R(L, D) \neq 0$, by Lemma 3.1, $\dim_D(L) > 0$ and so $\dim_E(L) \geq \dim_D(L) > 0$.

This proves the “only if” part of the proposition.

Now, consider the “if” part. We want to show that for every finitely generated left $R$-module $M$ and every division $R$-ring $E$, $\dim_E(M) \geq \dim_D(M)$. We will do it by induction on $\dim_D(M)$.

Let $\overline{M}$ be the image of the natural $R$-homomorphism $\alpha : M \to D \otimes_R M$ that sends $m \in M$ to $1 \otimes m$. Observe that, since $D \otimes_R M \cong D \otimes_R \overline{M}$, $\dim_D(M) = \dim_D(\overline{M})$. We have also that $\dim_E(\overline{M}) \leq \dim_E(M)$. Thus, without loss of generality, we can assume that $\alpha$ is injective.

Now assume that $\dim_D(M) = 1$. Since $M$ is a submodule of $D$, then $\dim_E(M) > 0$, and so, $\dim_E(M) \geq 1 = \dim_D(M)$. This gives us the base of induction.

Assume that the claim holds if $\dim_D(M) \leq n - 1$. Consider the case $\dim_D(M) = n \geq 2$. Observe that $\dim_E(M) \neq 0$, since $M$ has a nontrivial quotient that lies in $D$. Hence $E \otimes_R M \neq \{0\}$. Let $m \in M$ be such that $1 \otimes m$ is non-trivial in $E \otimes_R M$. Then $\dim_E(M/Rm) = \dim_E(M) - 1$. Since we assume that $\alpha$ is injective, $1 \otimes m$ is non-trivial in $D \otimes_R M$, and so, we also have $\dim_D(M/Rm) = \dim_D(M) - 1$. Applying the inductive assumption we obtain that

$$\dim_D(M) = \dim_D(M/Rm) + 1 \leq \dim_E(M/Rm) + 1 = \dim_E(M).$$

$\Box$

3.2 The universality of $D_{E*G}$ in the amenable case

Let $E$ be a division ring and $G$ a locally indicable group. Proposition 3.2 indicates that in order to prove the universality we have to understand the structure of finitely generated $E*G$-submodules of $D_{E*G}$. If $G$ is amenable, they are isomorphic to finitely generated left ideals of $E * G$. The following result shows that in the latter case the condition of Proposition 3.2 holds.

Proposition 3.3 Let $R = E * G$ be a crossed product of a division ring $E$ and a locally indicable group $G$. Then for every non-trivial finitely generated left ideal $L$ of $R$ and every division $R$-ring $E$, $\dim_E(L) > 0$.

Proof We denote by $R_g$ the $g$th component of $R$ and let $u_g$ be an invertible element of $R_g$. For any element $r = \sum_{g \in G} r_g \in R (r_g \in R_g)$ denote by $\text{supp}(r)$ the elements $g \in G$ for which $r_g \neq 0$ and put $l(r)$ to be equal to the number of non-trivial elements in $\text{supp}(r)$. Thus, $l(r) = 0$ means that $r \in R_e$. For a non-trivial finitely generated left
ideal $L$ of $R$ we put

$$l(L) = \min \{l(r_1) + \cdots + l(r_s) : L = Rr_1 + \cdots + Rr_s\}.$$  

Observe that if a set of generators $\{r_1, \ldots, r_s\}$ of $L$ satisfies the equality $l(L) = l(r_1) + \cdots + l(r_s)$, then for each $i$, $l(r_i) = |\text{supp} (r_i)| - 1$. (If not, we can change $r_i$ by $u_g^{-1}r_i$ with $g \in \text{supp}(r_i)$ and obtain a contradiction.) Moreover, if all $r_i$ are non-trivial and $L \neq R$, then $s \leq l(L)$. Now, we define

$$s(L) = \max \{s : L = Rr_1 + \cdots + Rr_s, l(L) = l(r_1) + \cdots + l(r_s) \text{ and } r_i \text{ are non-trivial}\}.$$  

We will prove the proposition by induction on $l(L)$. If $l(L) = 0$, then $L = R$ and we are done. Now assume that the proposition holds if $l(L) \leq n - 1$, and consider the case $l(L) = n \geq 1$.

We will proceed by inverse induction on $s(L)$. Observe that there is no $L$ such that $s(L) \geq l(L) + 1$, so there is nothing to prove in this case. Assume that we can prove the proposition if $l(L) = n$ and $s(L) \geq k + 1$, and consider the case $l(L) = n$ and $s(L) = k$.

Let $r_1, \ldots, r_k$ be a set of non-zero generators of $L$ such that $n = l(r_1) + \cdots + l(r_k)$. Let $H$ be the group generated by $\bigcup_{i=1}^k \text{supp}(r_i)$. Since $G$ is locally indicable there exists a surjective $\alpha : H \to \mathbb{Z}$. Let $N = \ker \alpha$ and $x \in H$ such that $(x)N = H$. We write

$$r_i = \sum_j u_{ij}^l r_{ij} \text{ with } 0 \neq r_{ij} \in E \ast N.$$  

Let $L'$ be a left ideal of $R$ generated by $\{r_{ij}\}$. Observe that

$$\sum_{i,j} l(r_{ij}) \leq \sum_i l(r_i) \text{ and } |\{r_{ij}\}| > s(L) = k.$$  

Thus, we obtain that either $l(L') < l(L)$ or $l(L') = l(L)$ and $s(L') > s(L)$. Hence we can apply the inductive hypothesis and obtain that $\text{rk}_E(L') > 0$. Thus $\text{Hom}_R(L', E) \neq 0$. Let $0 \neq \phi \in \text{Hom}_R(L', E)$.

Put $S = E \ast H$. Observe that $S \cong E \ast N[x^{\pm 1}; \tau]$, where $\tau$ is conjugation by $u_t$. Let $\tilde{E}$ be the Ore division ring of fractions of $E[x^{\pm}; \tau]$, where $\tau$ is conjugation by $u_t$. Then $\tilde{E}$ has a natural $S$-ring structure. We denote by $\dim_{\tilde{E}}$ the corresponding Sylvester module rank function on $S$. By Proposition 2.5(1), $\text{rk}_{\tilde{E}}$ is equal to the natural extension of the restriction of $\text{rk}_E$ on $E \ast N$.

Let $L_0$ and $L'_0$ be the left ideals of $S$ generated by $\{r_i\}$ and $\{r_{ij}\}$ respectively. We have that $L_0 \leq L'_0$. Every element $m$ of $L'_0$ can be written in a unique way as $m = \sum_j u^l_j m_j$, where $m_j \in E \ast N \cap L'_0$. We define

$$\tilde{\phi}(m) = \sum_j x^j \phi(m_j).$$
This defines a homomorphism of left $S$-modules $\tilde{\phi} : L_0' \to \tilde{\mathcal{E}}$. Since $\phi$ is not trivial, there exists $r_{ij}$ such that $\phi(r_{ij}) \neq 0$. Therefore, $\phi(r_i) \neq 0$. Thus, the restriction of $\tilde{\phi}$ on $L_0$ is not trivial. Hence, by Lemma 3.1, $\dim_{\tilde{\mathcal{E}}}(L_0) > 0$.

Let $\dim'_{\mathcal{E}}$ be the Sylvester module rank function associated to the division $S$-ring $\mathcal{E}$. Since the restrictions of $\text{rk}_{\mathcal{E}}$ and $\text{rk}_{\tilde{\mathcal{E}}}$ on $E \ast N$ coincide, [15, Lemma 8.3] implies that $\text{rk}_{\mathcal{E}} \leq \text{rk}_{\tilde{\mathcal{E}}}$ as Sylvester matrix rank functions on $E \ast H$, and so

$$\dim'_{\mathcal{E}}(L_0) \geq \dim_{\tilde{\mathcal{E}}}(L_0) > 0.$$ 

Now observe that $L \cong R \otimes_S L_0$. Hence

$$\dim_{\mathcal{E}}(L) = \dim'_{\mathcal{E}}(L_0) > 0$$

and we are done. \qed

**Corollary 3.4** Let $G$ be an amenable locally indicable group and let $E$ be a division ring. Then $D_{E \ast G}$ is the universal division ring of fractions of $E \ast G$.

**Proof** Observe that $E \ast G$ satisfies the right Ore condition and so $D_{E \ast G}$ is isomorphic as $E \ast G$-ring to the classical ring of fractions $Q(E \ast G)$. Since any finitely generated left submodule of $Q(E \ast G)$ is isomorphic to a left ideal of $E \ast G$, Proposition 3.2 and Proposition 3.3 imply the desired result. \qed

We remark that Corollary 3.4 can be also proved using arguments similar to the ones used in the proof of [10, Lemma 2.1]. Also it is worth to be mentioned here that, by a result of D. Morris [22], a left orderable amenable group is always locally indicable.

### 3.3 A criterion for a group to be Lewin

In this subsection we will show that in order to prove that a Hughes-free embeddable group $G$ is Lewin, it is enough to consider only group algebras $E[G]$. As before, by $\text{rk}_{E}$ we denote the Sylvester matrix rank function on $E[G]$ induced by the homomorphism $E[G] \to E$ that sends all the group elements from $G$ to 1.

**Proposition 3.5** Let $G$ be a locally indicable group and $E$ a division ring. Assume that for every division ring $E$,

1. $D_{E[G]}$ exists and
2. $\text{rk}_{D_{E[G]}} \geq \text{rk}_{\mathcal{E}}$ as Sylvester matrix rank functions on $E[G]$.

If for a crossed product $E \ast G$, the space $\mathcal{P}_{\text{div}}(E \ast G)$ is not empty, then $E \ast G$ has the Hughes-free division ring $D_{E \ast G}$ and, moreover, $D_{E \ast G}$ is universal.

**Proof** First let us show that $D_{E \ast G}$ exists. Let $\phi : E \ast G \to \mathcal{E}$ be a division $E \ast G$-ring. Write $R = E \ast G = \bigoplus_{g \in G} R_g$. We fix an invertible element $u_g \in R_g$ for each $g \in G$. For every $g_1, g_2 \in G$ we define

$$\alpha(g_1, g_2) = u_{g_1}u_{g_2}u_{g_1 g_2}^{-1} \in E.$$
Observe that $\mathcal{E}$ is a $E \ast G$-bimodule. This allows us to convert the $\mathcal{E}$-space $\tilde{R} = \bigoplus_{g \in G} \mathcal{E} v_g$ into a ring by putting

$$v_g a = (\phi(u_g)a\phi(u_g^{-1}))v_g \quad \text{and} \quad v_g v_h = \phi(\alpha(g, h))v_{gh}, \; g, h \in G, \; a \in \mathcal{E}.$$ 

Clearly the ring $\tilde{R}$ has a structure of a crossed product $\tilde{R} = \mathcal{E} \ast G$. Define the map $\tilde{\phi}: E \ast G \to \tilde{\mathcal{E}} \ast G$ by

$$\tilde{\phi}(\sum_{g \in G} k_g u_g) = \sum_{g \in G} \phi(k_g)v_g, \; k_g \in E.$$ 

Then $\tilde{\phi}$ is a homomorphism.

For each $g \in G$ we put $w_g = \phi(u_g^{-1})v_g \in \mathcal{E} \ast G$. Then $w_g$ commutes with the elements from $\mathcal{E}$ and for every $g, h \in G$,

$$w_g w_h = \phi(u_g^{-1})v_g \phi(u_h^{-1})v_h = \phi(u_h^{-1})\phi(u_g^{-1})v_g v_h = \phi(u_g^{-1})\phi(\alpha(g, h))v_{gh} = \phi(u_g^{-1})v_{gh} = w_{gh}.$$ 

Thus, we obtain that $\tilde{R} \cong \mathcal{E}[G]$. In particular $\mathcal{D}_{\mathcal{E} \ast G}$, and so, $\mathcal{D}_{E \ast G}$ exist and $\tilde{\phi}^*(rk_{\mathcal{D}_{E \ast G}})$ is equal to $rk_{\mathcal{D}_{E \ast G}}$.

Now, we want to show that $\mathcal{D}_{E \ast G}$ is universal. In other words we want to show that $rk_{\mathcal{D}_{E \ast G}} \geq \phi^*(rk_{\mathcal{E}})$. Let $\psi: \mathcal{E} \ast G \to \mathcal{E}$ be the map that sends all $w_g$ to 1. Denote by $\mathcal{E}^*$ the Sylvester matrix rank function on $\mathcal{E} \ast G$ induced by $\psi$. By our assumptions, $rk_{\mathcal{E}} \leq rk_{\mathcal{D}_{E \ast G}}$. Now observe that $\phi = \psi \circ \tilde{\phi}$. Hence

$$\phi^*(rk_{\mathcal{E}}) = (\psi \circ \tilde{\phi})^*(rk_{\mathcal{E}}) = \tilde{\phi}^*(\psi^*(rk_{\mathcal{E}})) = \tilde{\phi}^*(rk_{\mathcal{E}})^* \leq \tilde{\phi}^*(rk_{\mathcal{D}_{E \ast G}}) = rk_{\mathcal{D}_{E \ast G}}$$

as Sylvester matrix rank functions on $E \ast G$.

**Corollary 3.6** Any subgroup of a Lewin group is Lewin.

The corollary implies that our definition of Lewin group is equivalent to the one of Sánchez ([25, Definition 6.18]).

### 3.4 Proofs of Theorem 1.2 and Corollary 1.3

Let $F$ be a free group freely generated by a finite set $S$, and let $M$ and $\{M_i\}_{i \in \mathbb{N}}$ be normal subgroups of $F$. We put $G = F/M$ and $G_i = F/M_i$ and assume that $(G_i, SM/M_i)$ converges to $(G, SM/M)$. Assume that for all $i$, $G_i$ is locally indicable and $\mathcal{D}_{E[G_i]}$ exists. Since $G_i$ are quotients of $F$, abusing notation, we will also refer to $rk_{E[G_i]}$ as a Sylvester matrix rank function on $E[F]$.

Let $\omega$ be an arbitrary non-principal ultrafilter on $\mathbb{N}$. We put

$$rk = \lim_{\omega} rk_{\mathcal{D}_{E[G_i]}} \in \mathbb{P}_{div}(E[F]).$$
Observe that for every \( g \in M \), \( \text{rk}(g - 1) = 0 \). Thus, \( \text{rk} \) is also a Sylvester matrix rank function on \( E[G] \). We want to show that \( \text{rk} \) corresponds to the Sylvester matrix rank function of a Hughes-free division \( E \ast G \)-ring. This will prove Theorem 1.2.

For each \( i \) we fix a left-invariant Conradian order \( \preceq_i \) on \( G_i \). Define an order \( \preceq \) on \( G \) by

\[
f M \preceq h M \quad \text{if} \quad \{ i \in \mathbb{N} : f M_i \preceq_i h M_i \} \in \omega.
\]

The definition does not depend on the choice of representatives, because for every \( m \in M \), the set \( \{ i \in \mathbb{N} : m \in M_i \} \) is in \( \omega \). It is also clear that \( \preceq \) is left-invariant and Conradian. In particular, this proves that \( G \) is locally indicable.

Denote by \( \alpha_j \) the canonical homomorphism \( F \to G_j \) and extend it to the homomorphism \( \alpha_j : E[F] \to \mathcal{D}_{E[G_j]} \). The rank function \( \text{rk} \) corresponds to the homomorphism

\[
\alpha = (\alpha_i) : E[F] \to \prod_{\omega} \mathcal{D}_{E[G_i]} : = (\prod_{i \in \mathbb{N}} \mathcal{D}_{E[G_i]})/I_\omega,
\]

with \( I_\omega = \{ (d_i) : \lim_{\omega} \text{rk}_{E[G_j]}(d_i) = 0 \} \). Observe that \( \prod_{\omega} \mathcal{D}_{E[G_i]} \) is a division ring. We denote by \( \mathcal{D} \) the division closure of \( \alpha(E[F]) \) in \( \prod_{\omega} \mathcal{D}_{E[G_i]} \). As we have observed before, for each \( m \in M \), \( \alpha(m - 1) = 0 \). Thus, \( \mathcal{D} \) is a epic division \( E[G] \)-ring. We are going to show that \( \mathcal{D} \) is free with respect to \( \preceq \). For simplicity, in what follows, for each \( j \in \mathbb{N} \), \( \mathcal{D}_{E[G_j]} \) is denoted by \( \mathcal{D}_j \).

Let \( H \) be a finitely generated subgroup of \( G \) and let \( N \) be the maximal convex subgroup of \( H \). Let \( h_1, \ldots, h_n \in H \) be in distinct cosets of \( N \). We want to show that \( \alpha(h_1), \ldots, \alpha(h_n) \) are \( \mathcal{D}_{N, \mathcal{D}_\omega} \)-linearly independent. Without loss of generality we will assume that \( H = G \).

Let \( L_j/M_j \) be the maximal convex subgroup of \( G_j \) with respect to \( \preceq_j \). By Proposition 2.1, since \( \preceq_j \) is Conradian, there exists order-preserving homomorphism \( \phi_j : G_j \to \mathbb{R} \) such that \( \ker \phi_j = L_j/M_j \). Without loss of generality we see \( \phi_j \) as an element of \( H^1(F; \mathbb{R}) \). We can multiply \( \phi_j \) by a scalar in such way that \( \max_{s \in S} |\phi_j(s)| = 1 \). Let \( \phi = \lim_{\omega} \phi_j \in H^1(F; \mathbb{R}) \) and \( L = \ker \phi \). Observe that \( \phi \) is non-trivial, \( M \subseteq \ker \phi \) and \( \phi \) is order-preserving with respect to \( \preceq \) if we consider it as a map \( G \to \mathbb{R} \). In particular, \( N = L/M \).

For each \( i \) choose \( f_i \in F \) such that \( h_i = f_i M \). By way of contradiction, assume that there are \( d_1, \ldots, d_n \in \mathcal{D}_{N, \mathcal{D}} \) such that

\[
d_1 \alpha(f_1) + \cdots + d_n \alpha(f_n) = 0 \quad \text{in} \quad \mathcal{D}\tag{2}
\]

with \( d_i \neq 0 \) for some \( 1 \leq i \leq n \).

Consider the subring \( R \) of \( \mathcal{D} \) generated by \( \mathcal{D}_{[G,G]} \mathcal{D} \) and \( N \). It is a quotient of a crossed product \( \mathcal{D}_{[G,G]} \mathcal{D} \ast (N/[G, G]) \). Since \( N/[G, G] \) is finitely generated abelian, \( \mathcal{D}_{[G,G]} \mathcal{D} \ast (N/[G, G]) \) is left and right Noetherian. Thus, \( R \) is also left and right Noetherian. Since \( R \) is a domain, \( \mathcal{D}_{N, \mathcal{D}} \) is the classical division ring of fractions of \( R \). Hence, without loss of generality we can assume that \( d_i \in R \) in (2). Therefore, there
are \( f_{il} \in L \) and \( d_{il} \in \mathcal{D}_{[G,G], \mathcal{D}} \) such that

\[
d_i = \sum_l d_{il} \cdot \alpha(f_{il}).
\]

Since \( h_1, \ldots, h_n \in H \) belong to distinct cosets of \( N \), all values \( \phi(f_1), \ldots, \phi(f_n) \) are distinct. Let \( \epsilon = \min_{j \neq i} |\phi(f_j) - \phi(f_i)|. \) Since for all \( i, j, \phi(f_{il}) = 0 \), we obtain that

\[
\{ k \in \mathbb{N} : |\phi_k(f_{il})| \leq \frac{\epsilon}{4} \text{ for all } i, l \text{ and } |\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4} \text{ for all } i \neq j \} \in \omega.
\]

Thus, without loss of generality we assume that for every \( k \in \mathbb{N}, |\phi_k(f_{il})| \leq \frac{\epsilon}{4} \) for all \( i, l \) and \( |\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4} \) for all \( i \neq j \).

Since \( d_{il} \in \mathcal{D}_{[G,G], \mathcal{D}}, d_{il} \) are in the division closure of \( \alpha(E[[F, F]]) \). Therefore, we can write

\[
d_{il} = (d_{ilk})_k \quad \text{and} \quad d_i = \left( \sum_l d_{ilk} \alpha_k(f_{il}) \right)_k \in \prod_{ \omega } \mathcal{D}_k, \text{ with } d_{ilk} \in \mathcal{D}_{[G_j, G_j], \mathcal{D}_j}.
\]

Since \( d_1 \alpha(f_1) + \cdots + d_n \alpha(f_n) = 0 \), we obtain that

\[
\{ k \in \mathbb{N} : \sum_{i,l} d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0 \} \in \omega.
\]

Thus, we can assume that \( \sum_{i,l} d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0 \) for all \( k \in \mathbb{N} \). Observe that since

\[
|\phi_k(f_{il})| \leq \frac{\epsilon}{4} \quad \text{and} \quad |\phi_k(f_j) - \phi_k(f_i)| \geq \frac{3\epsilon}{4},
\]

\[
\phi_k(f_{il} \cdot f_i) \neq \phi_k(f_{jl} \cdot f_j) \quad \text{if } i \neq j.
\]

Recall that \( \mathcal{D}_k \) is free with respect to \( \preceq_k \). In particular, this implies that for all \( i, \)

\[
\left( \sum_l d_{ilk} \alpha_k(f_{il}) \right) \alpha_k(f_i) = \sum_l d_{ilk} \alpha_k(f_{il} \cdot f_i) = 0.
\]

Since this holds for all \( k, d_i = 0 \) for all \( i \). This shows that \( \mathcal{D} \) is free with respect to \( \preceq \), and so it is Hughes-free by Proposition 2.4. This finishes the proof of Theorem 1.2.

**Proof of Corollary 1.3** Without loss of generality we may assume that \( G \) is finitely generated. Hence \( G \) is a limit of a collection of locally indicable amenable groups \( \{G_i\} \). Thus, by Theorem 1.2, for every division ring \( \mathcal{E} \), there exists \( \mathcal{D}_{\mathcal{E}[G]} \). Moreover, since by Corollary 3.4, \( \text{rk}_{\mathcal{E}[G_i]} \geq \text{rk}_{\mathcal{E}} \) as Sylvester matrix rank functions on \( \mathcal{E}[G_i] \), Theorem 1.2 also implies that \( \text{rk}_{\mathcal{E}[G]} \geq \text{rk}_{\mathcal{E}} \) as Sylvester matrix rank functions on \( \mathcal{E}[G] \). Now, by Proposition 3.5, we obtain that \( \mathcal{D}_{\mathcal{E}[G]} \) is universal. \( \square \)
3.5 Examples of Lewin groups

The following theorem shows that the groups that appear in Theorem 1.1 are Lewin.

**Theorem 3.7** Let $G$ be a locally indicable group.

1. If all finitely generated subgroups of $G$ are Lewin, then $G$ is also Lewin.
2. Any subgroup of a Lewin group is also Lewin.
3. $G$ is Lewin if $G$ has a normal Lewin subgroup $N$ such that $G/N$ is amenable and locally indicable.
4. Any limit in $G_n$ of Lewin groups which is Hughes-free embeddable is Lewin.
5. A finite direct product of Lewin groups is Lewin.

**Proof** The first statement follows directly from the definition of Lewin groups and the second one from Corollary 3.6. Let us prove now part (3).

First observe that $G$ is Hughes-free embeddable by [12] (see also [25, Theorem 6.10]). Let $E$ be a division ring. Observe that the restriction of $\text{rk}_D E[G]$ on $E[N]$ is equal to $\text{rk}_D E[N]$ and $D E[G] \cong Q(D E[N] * G/N)$ as $E[G]$-rings. Thus, by Proposition 2.5(1),

$$\text{rk}_D E[G] = \text{rk}_Q(E[G/N]).$$

Denote by $\text{rk}_E'$ the Sylvester matrix rank function on $E$ coming from the obvious map $E[N] \to E$. Then, again by Proposition 2.5(1), we obtain that $\text{rk}_D E[G/N] = \text{rk}_E(E[G/N]) = \text{rk}_E'$. Since $N$ is Lewin, $\text{rk}_D E[N] \geq \text{rk}_E'$, and so, $\text{rk}_D E[N] \geq \text{rk}_E$. Thus, $\text{rk}_D E[G] \geq \text{rk}_D E[G/N]$ as Sylvester matrix rank functions on $E[G]$. Since $G/N$ is amenable and locally indicable, Corollary 3.4 implies that $\text{rk}_D E[G/N] \geq \text{rk}_E$. Hence $\text{rk}_D E[G] \geq \text{rk}_E$. Using Proposition 3.5, we obtain (3).

The fourth statement follows from Proposition 3.5 and Theorem 1.2.

Consider now the fifth claim. First let us prove that the direct product $G = G_1 \times G_2$ of two Lewin groups $G_1$ and $G_2$ is again Lewin. By [12], $G$ is Hughes-free embeddable. Let $E$ be a division ring. Consider the natural homomorphisms

$$\phi_1 : E[G] \to E[G_1], \quad \phi_2 : E[G_1] \to E \quad \text{and} \quad \phi_3 = \phi_2 \circ \phi_1 : E[G] \to E.$$ 

Since $G_2$ is Lewin,

$$\text{rk}_D E[G_1][G_2] \geq \text{rk}_D E[G_1] \quad \text{in} \quad P(D E[G_1][G_2]).$$

Therefore, since $D E[G] = D D E[G_1][G_2]$,

$$\text{rk}_D E[G] \geq \phi_1^\#(\text{rk}_D E[G_1]) \quad \text{in} \quad P(E[G]).$$

Since $G_1$ is Lewin,

$$\text{rk}_D E[G_1] \geq \phi_2^\#(\text{rk}_E) \quad \text{in} \quad P(E[G_1]).$$
Hence, we conclude that
\[ \text{rk}_{D[G]} \geq \phi_1^\#(\text{rk}_{D[E[G]]}) \geq \phi_1^\#(\phi_2^\#(\text{rk}_E)) = \phi_3^\#(\text{rk}_E) \quad \text{in } \mathbb{P}(E[G]). \]

Since $E$ is arbitrary, applying Proposition 3.5, we obtain that $G$ is Lewin. The case of two groups implies that (5) holds for an arbitrary finite product of Lewin groups. \qed

4 Universality of $\text{rk}_G$

As we have already mentioned in Introduction, when $G$ is locally indicable $\text{rk}_G = \text{rk}_{D[C[G]]}$. In this section we compare $\text{rk}_G$ with other natural Sylvester matrix rank functions on $C[G]$.

4.1 The condition $\text{rk}_G \geq \text{rk}_{[1]}$

In this subsection we will see several consequences of the condition $\text{rk}_G \geq \text{rk}_{[1]}$. Recall that $\text{rk}_{[1]}$ is an alternative expression for $\text{rk}_C$ that has appeared in the previous sections. We start with the following useful proposition.

Proposition 4.1 Let $H$ be a finitely generated group and assume that $H$ is not indicable. Then $\text{rk}_{[1]}$ is maximal in $\mathbb{P}(Q[H])$. In particular, any group $G$ for which $Q[G]$ has a universal division ring of fractions, is locally indicable.

Proof Suppose that $H$ has the following presentation.

\[ H = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots \rangle. \]

Reordering the relations $\{r_i\}$ of $H$, without loss of generality we can assume that the abelianization of the group

\[ \tilde{H} = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots, r_d \rangle \]

is already finite.

Let $F$ be a free group generated by $x_1, \ldots, x_d$. For each $1 \leq i \leq d$, we write $r_i - 1 = \sum_{j=1}^{d} a_{ij} (x_j - 1)$, where $a_{ij} \in \mathbb{Z}[F]$. Let

\[ A = (a_{ij}) \in \text{Mat}_d(\mathbb{Z}[F]) \quad \text{and} \quad B = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_d - 1 \end{pmatrix} \in \text{Mat}_{d \times 1}(\mathbb{Z}[F]). \]

Denote by $\tilde{A}$ and $\tilde{B}$ the matrices over $\mathbb{Z}[H]$ obtained from $A$ and $B$, respectively, by applying the obvious homomorphism $\mathbb{Z}[F] \to \mathbb{Z}[H]$. Since $\tilde{H}$ has finite abelianization, we obtain that

\[ \text{rk}_{[1]}(A) = d - \dim_{\mathbb{Q}} H_1(\tilde{H}; \mathbb{Q}) = d. \]
Let \( rk \in \mathbb{P}(\mathbb{Q}[H]) \) satisfy \( rk \geq rk_{1} \). In particular,
\[
rk(\overline{A}) \geq rk_{1}(\overline{A}) = rk_{1}(A) = d.
\]
Since \( AB = \begin{pmatrix} r_{1} - 1 \\ \vdots \\ r_{d} - 1 \end{pmatrix} \), we obtain that \( A\overline{B} = 0 \). Thus, by [13, Proposition 5.1(3)], \( rk(\overline{B}) = 0 \). Therefore, \( rk(a) = 0 \) for every \( a \in I \), where \( I \) is the augmentation ideal of \( \mathbb{Q}[H] \). Since \( \mathbb{Q}[H]/I \) is a division ring and so it has only one Sylvester matrix rank function, \( rk = rk_{1} \). This shows the first part of the proposition.

Assume now that \( \mathbb{Q}[G] \) has a universal division ring of fractions \( D \). Let \( H \) be a finitely generated subgroup of \( G \). If \( H \) is not indicable, then, as we have just proved, the restriction of \( rk_{D} \) on \( \mathbb{Q}[H] \) is equal to \( rk_{1} \). Since \( rk_{D} \) is faithful, \( H = \{1\} \). \( \square \)

In the next proposition we will show that the condition \( rk_{G} \geq rk_{1} \) implies that \( rk_{G} \geq rk_{\overline{G}} \) for any amenable quotient \( \overline{G} \) of \( G \).

**Proposition 4.2** Let \( G \) be a group and \( N \) a normal subgroup with \( G/N \) amenable. Let \( K \) be a subfield of \( \mathbb{C} \). Assume that \( rk_{N} \geq rk_{1} \) in \( \mathbb{P}(K[N]) \). Then \( rk_{G} \geq rk_{G/N} \) as Sylvester matrix rank functions on \( K[G] \).

**Proof** By Proposition 2.5, \( rk_{G} \) is the natural extension of \( rk_{N} \) and \( rk_{G/N} \) is the natural extension of \( rk_{1} \). Since \( rk_{N} \geq rk_{1} \) in \( \mathbb{P}(K[N]) \), we obtain that \( rk_{G} \geq rk_{G/N} \) in \( \mathbb{P}(K[G]) \). \( \square \)

**Corollary 4.3** Let \( G \) be a group and \( N \) a normal subgroup with \( G/N \) residually amenable. Let \( K \) be a subfield of \( \mathbb{C} \). If \( rk_{G} \geq rk_{1} \) in \( \mathbb{P}(K[G]) \), then \( rk_{G} \geq rk_{G/N} \) holds as well.

**Proof** Without loss of generality we may assume that \( G \) is finitely generated. Then there exists a chain \( G = N_{0} > N_{1} > N_{2} > \cdots \) of normal subgroups of \( G \) such that \( G/N_{k} \) is amenable and \( \cap N_{k} = N \). By [13, Theorem 1.3],
\[
rk_{G/N} = \lim_{k \to \infty} rk_{G/N_{k}} \text{ in } \mathbb{P}(K[G]).
\]
By Proposition 4.2, \( rk_{G} \geq rk_{G/N_{k}} \) in \( \mathbb{P}(K[G]) \) for every \( k \). Hence \( rk_{G} \geq rk_{G/N} \) holds as well. \( \square \)

We conjecture that the corollary holds without the condition that \( G/N \) is residually amenable.

**Conjecture 3** Let \( G \) be a group and let \( K \) be a subfield of \( \mathbb{C} \). Assume that \( rk_{G} \geq rk_{1} \) in \( \mathbb{P}(K[G]) \). Then \( rk_{G} \geq rk_{\overline{G}} \) in \( \mathbb{P}(K[G]) \) for any quotient \( \overline{G} \) of \( G \).

**4.2 Proof of Corollary 1.5**

It is clear that part (1) of of Corollary 1.5 implies part (2). Kielak proved in [18] that in order to show (1), it is enough to prove that the first \( L^2 \)-Betti number of \( G \) is zero.
Using Theorem 1.1, we will show that the condition (2) of Corollary 1.5 implies that the first $L^2$-Betti number of $G$ is zero.

First, let us recall the definition of RFRS groups. A group $G$ is called **residually finite rationally solvable** or **RFRS** if there exists a chain $G = H_0 > H_1 > \cdots$ of finite index normal subgroups of $G$ with trivial intersection such that $H_{i+1}$ contains a normal subgroup $K_{i+1}$ of $H_i$ satisfying that $H_i/K_{i+1}$ is torsion-free abelian. The following proposition implies that RFRS groups are residually poly-$\mathbb{Z}$.

**Proposition 4.4** Let $G$ be a finitely generated group, and let 

$$G = H_0 > H_1 > H_2 > \cdots > H_n > \cdots$$

be a chain of finite index normal subgroups of $G$ with $\bigcap_{n=0}^{\infty} H_n = 1$. Suppose that for every $n \geq 0$ there exists a subgroup $K_{n+1} \triangleleft H_n$ such that $K_{n+1} \leq H_{n+1}$ and $H_n/K_{n+1}$ is poly-$\mathbb{Z}$. Then $G$ is residually poly-$\mathbb{Z}$.

**Proof** A pro-$p$ version of this result is proved in [16, Proposition 5.1]. The same proof works in our case. We include it for the convenience of the reader.

For $n \geq 1$ let

$$\tilde{K}_n = \bigcap_{g \in G/H_{n-1}} gK_ng^{-1} \triangleleft G$$

be the normal core of $K_n$ in $G$. Since the direct product of poly-$\mathbb{Z}$-groups is poly-$\mathbb{Z}$ and a subgroup of a poly-$\mathbb{Z}$ group is poly-$\mathbb{Z}$, the group $H_{n-1}/\tilde{K}_n$ is poly-$\mathbb{Z}$ as well.

For every $n \geq 1$ set

$$L_n = \bigcap_{i \leq n} \tilde{K}_i \triangleleft G$$

and note that since $\bigcap_{n=0}^{\infty} H_n = 1$, this is a chain of subgroups that satisfies

$$\bigcap_{n=1}^{\infty} L_n \subseteq \bigcap_{n=1}^{\infty} \tilde{K}_n \subseteq \bigcap_{n=1}^{\infty} K_n \subseteq \bigcap_{n=1}^{\infty} H_{n-1} = 1.$$

We shall argue, by induction on $n$, that $G/L_n$ is poly-$\mathbb{Z}$. For $n = 1$ we have

$$G/L_1 = G/\tilde{K}_1 = H_0/\tilde{K}_1$$

is poly-$\mathbb{Z}$.

Once $n \geq 2$ we have $L_n = L_{n-1} \cap \tilde{K}_n$, and by induction $G/L_{n-1}$ is poly-$\mathbb{Z}$. Thus, since an extension of two poly-$\mathbb{Z}$ groups is poly-$\mathbb{Z}$, it suffices to show that $L_{n-1}/L_n$ is poly-$\mathbb{Z}$. Indeed, since $K_{n-1} \leq H_{n-1}$, we have that

$$L_{n-1}/L_n = L_{n-1}/L_{n-1} \cap \tilde{K}_n \cong L_{n-1} \tilde{K}_n/\tilde{K}_n \leq H_{n-1}/\tilde{K}_n$$

is poly-$\mathbb{Z}$. Therefore, we conclude by recalling that a subgroup of a poly-$\mathbb{Z}$ group is poly-$\mathbb{Z}$. $\square$
Now let us prove that the condition (2) of Corollary 1.5 implies that the first $L_2$-Betti number of $G$ is zero. Let $H$ be a subgroup of finite index such that there exists a normal subgroup $N$ of $H$ with $H/N \cong \mathbb{Z}$ and $H_1(N; \mathbb{Q})$ is finite-dimensional.

Assume that $H$ has the following presentation.

$$H = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots \rangle.$$ 

Observe that $H_1(N; \mathbb{Q}) \cong H_1(H; \mathbb{Q}[H/N]).$

Let $F$ be a free group generated by $x_1, \ldots, x_d$ and consider $\mathbb{Q}[H/N]$ as an $F$-module. Then $H_1(F; \mathbb{Q}[H/N]) \cong \mathbb{Q}[H/N]^{d-1}$ as a $\mathbb{Q}[H/N]$-module. Since $\mathbb{Q}[H/N]$ is a PID, we can reorganize the relations $\{r_i\}$ and without loss of generality we can assume that $H_1(\tilde{H}; \mathbb{Q}[\tilde{H}/\tilde{N}])$ is finite-dimensional, where

$\tilde{H} = \langle x_1, \ldots, x_d \mid r_1, r_2, \ldots, r_{d-1} \rangle.$

$\phi : \tilde{H} \to H$ is the canonical surjection and $\tilde{N} = \phi^{-1}(N).$

For each $1 \leq i \leq d - 1$, we write $r_i - 1 = \sum_{j=1}^{d} a_{ij}(x_j - 1)$, where $a_{ij} \in \mathbb{Z}[F].$

Let

$$A = (a_{ij}) \in \text{Mat}_{(d-1) \times d}(\mathbb{Z}[F])$$

and

$$B = \begin{pmatrix} x_1 - 1 \\ \vdots \\ x_d - 1 \end{pmatrix} \in \text{Mat}_{d \times 1}(\mathbb{Z}[F]).$$

Denote by $\overline{A}$ and $\overline{B}$ the matrices over $\mathbb{Z}[H]$ obtained from $A$ and $B$, respectively, by applying the obvious homomorphism $\mathbb{Z}[F] \to \mathbb{Z}[H].$ Since $H_1(\tilde{H}; \mathbb{Q}[\tilde{H}/\tilde{N}])$ is finite-dimensional, we obtain that

$$\text{rk}_{H/N}(\overline{A}) = \text{rk}_{H/N}(A) = \text{rk}_{\tilde{H}/\tilde{N}}(A) = d - 1.$$

By Proposition 4.4, $H$ is residually poly-$\mathbb{Z}$. By Corollary 4.3, $\text{rk}_H \geq \text{rk}_{[1]}$ in $\mathbb{P}(\mathbb{Q}[H])$. Thus, by Corollary 4.3, $\text{rk}_H(\overline{A}) \geq \text{rk}_{H/N}(A) = d - 1.$ Hence, since $H$ is infinite, the sequence

$$l^2(H)^{d-1} \xrightarrow{\phi_H^\overline{A}} l^2(H)^d \xrightarrow{\phi_H^\overline{B}} l^2(H) \to 0$$

is weakly exact. Therefore, the first $L^2$-Betti number of $H$ vanishes, and so the first $L^2$-Betti number of $G$ vanishes as well.

### 4.3 Proof of Corollary 1.6

Consider the cellular chain complex of $\tilde{X}$

$$\mathcal{C}(\tilde{X}) : \cdots \to \mathbb{Z}[\mathcal{C}_{p+1}(\tilde{X})] \xrightarrow{\partial_{p+1}} \mathbb{Z}[\mathcal{C}_p(\tilde{X})] \xrightarrow{\partial_p} \mathbb{Z}[\mathcal{C}_{p-1}(\tilde{X})] \to \cdots \to \mathbb{Z} \to 0.$$
Since \( G \) acts freely on \( \tilde{X} \) and \( X = \tilde{X}/G \) is of finite type, we obtain that \( \mathbb{Z}[C_p(\tilde{X})] \cong \mathbb{Z}[G]^{n_p} \) is a free \( \mathbb{Z}[G] \)-module of finite rank and the connected morphisms \( \partial_p \) are represented by multiplication by matrices \( A_p \) over \( \mathbb{Z}[G] \). Hence we obtain the following equivalent representation of \( C(\tilde{X}) \):

\[
C(\tilde{X}) : \mathbb{Z}[G]^{n_p+1} \times A_{p+1} \mathbb{Z}[G]^{n_p} \mathbb{Z}[G]^{n_p-1} \cdots \mathbb{Z} \to 0.
\]

Therefore, if \( p \geq 1 \) the \( p \)th Betti number of \( X \) and the \( p \)th \( L^2 \)-Betti number of \( \tilde{X} \) can be expressed in the following way.

\[
b_p(X) = n_p - (\text{rk}_{[1]}(A_p) + \text{rk}_{[1]}(A_{p+1})) \text{ and } b_2^{(2)}(\tilde{X}) = n_p - (\text{rk}_G(A_p) + \text{rk}_G(A_{p+1})).
\]

Thus, Corollary 1.4 implies that \( b_2^{(2)}(\tilde{X}) \leq b_p(X) \) if \( p \geq 2 \). If \( p = 1 \), then \( \text{rk}_G(A_1) = 0 \) and \( \text{rk}_{[1]}(A_1) = 0 \). Therefore \( b_1^{(2)}(\tilde{X}) \leq b_1(X) - 1 \).

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5 Appendix: The universal division ring of fractions of group rings of division rings and RFRS groups

In this section \( G \) is assumed to be a finitely generated RFRS group and \( E \) is a division ring. By Proposition 4.4, \( G \) is residually poly-\( \mathbb{Z} \). Therefore, Corollary 1.3 implies that \( D_{E[G]} \) exists and it is universal. In this section we will give an alternative description of \( D_{E[G]} \) (see Theorem 5.10). Our proof follows essentially the argument of Kielak [18], where this description is done when \( E = \mathbb{Q} \).

5.1 Characters

A character of \( G \) is a homomorphism from \( G \) to the additive group of real numbers \( \mathbb{R} \). The set of characters \( \text{Hom}(G, \mathbb{R}) \) is denoted also by \( H^1(G; \mathbb{R}) \). A character \( \phi \) is called irrational if \( \ker \phi / [G, G] \) is a torsion group.

If \( H \) is a subgroup of finite index of \( G \) then the restriction map embeds \( H^1(G; \mathbb{R}) \) into \( H^1(H; \mathbb{R}) \). In what follows, we will often consider \( H^1(G; \mathbb{R}) \) as a subset of \( H^1(H; \mathbb{R}) \).
If $H$ is a normal subgroup of $G$ then $G$ acts on $H^1(H; \mathbb{R})$: for $\phi \in H^1(H; \mathbb{R})$ and $g \in G$, we denote by $\phi^g$ the character that sends $h \in H$ to $\phi(ghg^{-1})$.

Let $G = H_0 > H_1 > H_2 > \cdots$ be a chain of subgroups of $G$ of finite index and $n \geq 0$. For any $U \subset H^1(H_n; \mathbb{R})$ we denote

$$U_n = U^o \text{ and } U_{k-1} = (\overline{U_k})^o \cap H^1(H_{k-1}; \mathbb{R}) \text{ when } 1 \leq k \leq n.$$ 

We say that $U$ is $(G, \{H_i\}_{i \geq 0})$-rich if $U_0$ contains all the irrational characters of $G$. When $G$ and $\{H_i\}_{i \geq 0}$ are clear from the context, we will simply say that $U$ is rich.

**Lemma 5.1** Let $G = H_0 > H_1 > H_2 > \cdots$ be a chain of subgroups of $G$ of finite index.

1. If $U$ is rich in $H^1(H_n; \mathbb{R})$ and $g \in G$, then $U^g$ is also rich.
2. The intersection of two rich subsets of $H^1(H_n; \mathbb{R})$ is again rich.

**Proof** Claim (1) is clear. Let us show the second claim.

First observe that if $U$ and $V$ are two open subsets of $\mathbb{R}^k$, then

$$\overline{U \cap V} = \overline{U} \cap \overline{V}.$$  

Indeed, let $x \in \overline{U} \cap \overline{V}$ and let $O(x)$ be a neighborhood of $x$ such that

$$O(x) \subseteq \overline{U} \cap \overline{V}.$$ 

Consider $y \in O(x)$, and let $O(y)$ be an arbitrary neighborhood of $y$ such that

$$O(y) \subseteq \overline{U} \cap \overline{V}.$$ 

In particular, there exists $z \in U \cap O(y)$. Recall that $U$ is open. Consider an arbitrary neighborhood $O(z)$ of $z$ such that $O(z) \subseteq U \cap \overline{V}$. Since $V$ is open, $O(z) \cap U \cap V$ is not empty. Hence $z \in U \cap \overline{V}$, and so, $y \in U \cap \overline{V}$ as well. Thus, $O(x) \subseteq \overline{U} \cap \overline{V}$ and $x \in (\overline{U \cap V})^o$.

Now let $U$ and $V$ be two rich subset of $H^1(H_n; \mathbb{R})$ and let $W = U \cap V$. We put

$$U_n = U^o \text{ and } U_{k-1} = (\overline{U_k})^o \cap H^1(H_{k-1}; \mathbb{R}) \text{ when } 1 \leq k \leq n,$$

and similarly we define $V_k$ and $W_k$.

Then we have that $W_n = U_n \cap V_n$. Now, assume that we have proved that $W_k = U_k \cap V_k$ for some $k \leq n$. Then we obtain that

$$W_{k-1} = (\overline{W_k})^o \cap H^1(H_{k-1}; \mathbb{R}) = (\overline{U_k} \cap \overline{V_k})^o \cap H^1(H_{k-1}; \mathbb{R}) = (\overline{U_{k-1} \cap V_{k-1}})^o.$$ 

In particular, $W_0$ contains all the irrational characters of $G$, and so, $W$ is rich.

We will need the following criterion of richness.

\[\square\]
Lemma 5.2 Let \( G = H_0 > H_1 > H_2 > \cdots \) be a chain of subgroups of \( G \) of finite index. Take non-negative integers \( n \geq k \geq 0 \). Let \( U \) be an open subset of \( H^1(H_k; \mathbb{R}) \) and let \( V \) be an open subset of \( H^1(H_n; \mathbb{R}) \). Assume that \( U \) is rich and all the irrational characters of \( U \) belong to \( V \). Then \( V \) is also rich.

Proof We put \( V_n = V^o \) and \( V_{i-1} = (V_i)^o \cap H^1(H_{i-1}; \mathbb{R}) \) when \( 1 \leq i \leq n \). Then by the inverse induction on \( i \), we obtain that all the irrational characters of \( U \) belong also to \( V_i \) for \( n \leq i \leq k \). Hence \( \overline{U} \subseteq \overline{V_k} \). This clearly implies that \( V \) is rich. \( \square \)

5.2 Novikov rings

Let \( S \ast G \) be a crossed product and let \( \phi \in H^1(G; \mathbb{R}) \). Denote by \( \| | \|_\phi \) a norm on \( S \ast G \) defined by

\[
\| \sum_i s_i \bar{g}_i \|_\phi = \max \{ 2^{-\phi(g_i)} : s_i \neq 0 \}.
\]

Our convention is that \( 0 \|_\phi = 0 \). Let \( \hat{S} \ast \hat{G} \) be the completion of \( \hat{S} \ast \hat{G} \) with respect to the metric induced by the norm \( \| | \|_\phi \). The ring \( \hat{S} \ast \hat{G} \) is called the Novikov ring of \( S \ast G \) with respect to \( \phi \).

Let \( N = \ker \phi \). Then \( \phi \) is also a character of \( G/N \) and \( \hat{S} \ast \hat{G} \) is canonically isomorphic to \( (S \ast N) \ast G/N^\phi \). We will not distinguish between these two rings.

Any element of \( \hat{S} \ast \hat{G} \) can be represented in the following form \( \sum_{i=1}^\infty a_i g_i \), where \( a_i \in S \ast N \) and \( \{ \phi(g_i) \}_{i \in \mathbb{N}} \) is an increasing sequence tending to the infinity.

We would like to construct an environment, where we can calculate the intersection \( D_E[G] \cap \hat{E}[G]^\phi \). In order to do this, consider the following commutative diagram of injective homomorphisms of rings.

\[
\begin{array}{ccc}
E[G] & \hookrightarrow & D_E[G] \\
\downarrow & & \downarrow \alpha_{G,\phi} \\
\hat{E}[G]^\phi & \hookrightarrow & \hat{D}_{E[N]} \ast G/N^\phi,
\end{array}
\]

where the maps are defined as follows.

Notice that \( \hat{D}_{E[N]} \ast G/N^\phi \) is a division ring and \( D_E[G] \) is the classical Ore ring of fractions of \( D_{E[N]} \ast G/N \). Therefore, the map \( \alpha_{G,\phi} \) is the unique extension of the embedding \( D_{E[N]} \ast G/N \hookrightarrow \hat{D}_{E[N]} \ast G/N^\phi \).

Since Hughes-free division ring is unique, for every subgroup \( H \) of \( G \), the division ring \( D_{E[H]} \) can be identified with the division closure of \( E[H] \) in \( D_E[G] \). Thus, the ring \( \hat{D}_{E[N \cap H]} \ast (H/(N \cap H))^\phi \) can be identified with the closure of \( D_{E[N \cap H]} \ast (H/N \cap H) \cong D_{E[N \cap H]} \ast (HN/N) \subset D_{E[N]} \ast G/N \).
in \( \mathcal{D}_{E[N]} \ast G/N^n \) with respect to the topology induced by \( \| \|_\phi \). Using this identifications, we obtain that \( \alpha_{H, \phi} \) is the restriction of \( \alpha_{G, \phi} \). Therefore, in the following we will simply write \( \alpha_\phi \) instead of \( \alpha_{G, \phi} \).

The map \( \beta_{G, \phi} \) can be defined as the the continuous (with respect to \( \| \|_\phi \)) extension of the map \( E[G] = E[N] * G/N \cong \mathcal{D}_{E[N]} * G/N \).

Let \( H \) be a normal subgroup of \( G \) of finite index. Then the restriction of \( \phi \) on \( H \) is a character of \( H \) and \( E[H]^{\phi} \) can be identified with the closure of \( E[H] \) in \( E[G]^{\phi} \) with respect to the topology induced by \( \| \|_\phi \). It follows from the definitions that \( \beta_{H, \phi} \) is the restriction of \( \beta_{G, \phi} \) on \( E[H]^{\phi} \). Thus, in the following we will simply write \( \beta_\phi \) instead of \( \beta_{G, \phi} \).

For any subset \( S \) of \( H^1(G; \mathbb{R}) \) we put

\[
\mathcal{D}_{E[G],S} = \{ x \in \mathcal{D}_{E[G]} : \alpha_\phi(x) \in \text{Im} \beta_\phi \text{ for every } \phi \in S \}. \tag{5}
\]

If \( \phi \in H^1(G; \mathbb{R}) \), we will simply write \( \mathcal{D}_{E[G],\phi} \) instead of \( \mathcal{D}_{E[G],\{\phi\}} \). Therefore, by our definition,

\[
\mathcal{D}_{E[G],S} = \bigcap_{\phi \in S} \mathcal{D}_{E[G],\phi}.
\]

**Proposition 5.3** Let \( H \) be a normal subgroup of \( G \) of finite index and let \( S \) be a subset of \( H^1(G; \mathbb{R}) \). Then \( \mathcal{D}_{E[H],S} \) is \( G \)-invariant and \( \mathcal{D}_{E[G],S} \) is equal to the ring generated by \( \mathcal{D}_{E[H],S} \) and \( G \). In particular \( \mathcal{D}_{E[G],S} \) is a crossed product \( \mathcal{D}_{E[H],S} * G/H \).

**Proof** It is clear that \( \mathcal{D}_{E[H],S} \) and \( G \) are contained in \( \mathcal{D}_{E[G],S} \).

Now let \( x \in \mathcal{D}_{E[G],S} \). Let \( Q \) be a transversal of \( H \) in \( G \). Since \( \mathcal{D}_{E[G]} = \mathcal{D}_{E[H]} * G/H \), we can write

\[
x = \sum_{q \in Q} x_q q
\]

with \( x_q \in \mathcal{D}_{E[H]} \). We want to show that

\[
x_q \in \mathcal{D}_{E[H],S} \text{ for all } q \in Q. \tag{6}
\]

This will prove the proposition. Observe that this claim does not depend on the choice of \( Q \), because \( H \subset \mathcal{D}_{E[H],S} \).

In order to prove (6), it is enough to show that for every \( \phi \in S \), \( x_q \in \mathcal{D}_{E[H],\phi} \). Put \( N = \ker \phi \) and \( T = HN \). Let \( Q_1 \) be a transversal of \( H \) in \( T \) and \( Q_2 \) a transversal of \( T \) in \( G \). We assume that \( Q = Q_1 Q_2 \). Thus, we obtain that

\[
x = \sum_{q_2 \in Q_2} y_{q_2} q_2 \text{ where } y_{q_2} = \sum_{q_1 \in Q_1} x_{q_1 q_2} q_1.
\]
Since $\ker \phi \leq T$ and $T$ has finite index in $G$,

$$E[G]^{\phi} = \bigoplus_{q_2 \in Q} E[T]^{\phi} q_2.$$  

Thus, for all $q_2 \in Q_2$, $y_{q_2} \in D_{E[T], \phi}$.

Without loss of generality we can also assume that $Q_1 \subset N$. Thus $Q_1$ is also a transversal of $N \cap H$ in $N$.

For each $r \in \phi(T) = \phi(H)$, choose, $h_r \in H$ such that $\phi(h_r) = r$. Then there are $r_1 > r_2 > r_3 > \cdots$ such that we can write

$$\alpha_\phi(x_q) = \sum_{i=1}^{\infty} h_{r_i} a_{i,q} \quad \text{with} \quad a_{i,q} \in D_{E[N \cap H]}.$$  

For each $q_2 \in Q_2$, we obtain that

$$\alpha_\phi(y_{q_2}) = \sum_{i=1}^{\infty} h_{r_i} \left( \sum_{q_1 \in Q_1} a_{i,q_1,q_2} q_1 \right).$$  

Since $\alpha_\phi(y_{q_2}) \in \text{Im} \beta_\phi$, we obtain that for each $i \geq 1$,

$$\sum_{q \in Q} a_{i,q_1,q_2} q_1 \in E[N].$$  

Therefore, for each $i \geq 1$ and $q \in Q$, $a_{i,q} \in E[N \cap H]$. This implies, that $\alpha_\phi(x_q) \in \text{Im} \beta_\phi$, and so, $x_q \in D_{E[H], \phi}$ for every $q$.

Let $H$ be a normal subgroup of finite index of $G$ and let $S$ be a subset of $H^1(H; \mathbb{R})$. Then we put

$$D_{E[G],S} = \sum_{g \in G} D_{E[H],S} g.$$  

In view of Proposition 5.3, this definition is coherent with the previous definition of $D_{E[G],S}$ in (5).

Observe that if $S$ is $G$-invariant, then $g^{-1} D_{E[H],S} g \subseteq D_{E[H],S}$ for all $g$, and so, $D_{E[G],S}$ is equal to the subring of $D_{E[G]}$ generated by $G$ and $D_{E[H],S}$. In this case $D_{E[G],S}$ has a structure of a crossed product $D_{E[H],S} \ast G / H$. For arbitrary $S$, $D_{E[G],S}$ is not always a subring of $D_{E[G]}$.

Let $\phi \in H^1(H; \mathbb{R})$. We denote by $\phi^G$ the $G$-orbit in $H^1(H; \mathbb{R})$. Then $D_{E[G],\phi}$ is a right $D_{E[G],\phi^G}$-module. Let $N = \ker \phi$. As in (4) we have

$$E[H] \hookrightarrow D_{E[H]} \downarrow^{\phi},$$

$$E[H]^{\phi} \hookrightarrow \beta_\phi D_{E[N]} * H / N^{\phi}, \quad (7)$$
which induces another commutative diagram

\[
\begin{array}{ccc}
E[G] & \leftrightarrow & \mathcal{D}_{E[G]} \\
\downarrow & & \downarrow \alpha_\phi \\
\overline{E[H]} \otimes_{E[H]} E[G] & \leftrightarrow & \mathcal{D}_{E[N]} \ast H/N \otimes \mathcal{D}_{E[H],\phi^G} \mathcal{D}_{E[G],\phi^G}
\end{array}
\]  

where \( \alpha_\phi \) and \( \tilde{\beta}_\phi \) are homomorphisms of right \( E[G] \)-modules defined in the following way. Fix a right transversal \( Q \) of \( H \) in \( G \). Then \( \tilde{\beta}_\phi \) is defined on a basic tensor by

\[
\tilde{\beta}_\phi(b \otimes q) = \beta_\phi(b) \otimes q.
\]

In order to define \( \alpha_\phi \), we write an element \( a \in \mathcal{D}_{E[G]} \) as \( a = \sum_{q \in Q} a_q q \), with \( a_q \in \mathcal{D}_{E[H]} \), and define

\[
\tilde{\alpha}_\phi(a) = \sum_{q \in Q} \alpha_\phi(a_q) \otimes q.
\]

Observe that with this new notation we also have

\[
\mathcal{D}_{E[G],\phi} = \{ x \in \mathcal{D}_{E[G]} : \tilde{\alpha}_\phi(x) \in \text{Im} \tilde{\beta}_\phi \}.  
\]  

5.3 Continuity of \( \| \cdot \|_\phi \)

Let \( \phi \in H^1(G; \mathbb{R}) \) and \( x \in \mathcal{D}_{E[G]} \). Then we put

\[
\| x \|_\phi = \| \alpha_\phi(x) \|_\phi.
\]

**Proposition 5.4** Let \( x \in \mathcal{D}_{E[G]} \). Then the map \( H^1(G; \mathbb{R}) \to \mathbb{R} \) defined by

\[
\phi \mapsto \| x \|_\phi
\]

is continuous.

**Proof** Let \( G/K \) be the maximal torsion-free abelian quotient of \( G \). Let \( R \) be a subring of \( \mathcal{D}_{E[G]} \) generated by \( \mathcal{D}_{E[K]} \) and \( G \). Then the ring \( \mathcal{D}_{E[G]} \) is isomorphic to the classical Ore ring of fractions of \( R \). Thus, there are \( y \in R \) and \( 0 \neq z \in R \) such that \( x = yz^{-1} \). Since \( \| x \|_\phi = \| y \|_\phi \| z^{-1} \|_\phi \), it is enough to prove the proposition in the case \( x \in R \). Thus, let us assume that \( x \in R \).

Let \( A \) be a transversal of \( K \) in \( G \). Then we can write \( x = \sum_{a \in A_0} x_a a \), where \( A_0 \) is a finite subset of \( A \), and, for each \( a \in A_0, x_a \in \mathcal{D}_{E[K]} \). Observe that

\[
\| x \|_\phi = \max\{ \| a \|_\phi : a \in A_0 \} = \max\{ 2^{-\phi(a)} : a \in A_0 \}.
\]

This clearly implies that \( \| x \|_\phi \) is a continuous function in \( \phi \). \( \Box \)
5.4 Invertibility over Novikov rings

Let $H$ be a normal subgroup of $G$ of finite index and $\phi \in H^1(H; \mathbb{R})$. In this subsection we want to give a sufficient condition for $x \in \mathcal{D}_{E[G],\phi}$ to have its inverse in $\mathcal{D}_{E[G],\phi}$.

Let $G_0$ be a subgroup of $G$ containing $H$ and let $Q$ be a transversal of $H$ in $G_0$. Observe that

$$\phi^{G_0} = \{ \phi^g : g \in G_0 \} = \{ \phi^g : g \in Q \} = \phi^Q.$$

We can decompose any $x \in \mathcal{D}_{E[G_0]}$ as $x = \sum_{q \in Q} x_q q$ with $x_q \in \mathcal{D}_{E[H]}$. The $(Q, \phi)$-norm of $x$ is defined by

$$\|x\|_{\phi, Q} = \max\{ \|x_q\|_{\psi} \|q\|_{\phi}^{1/|Q|} : \psi \in \phi^Q, q \in Q \}.$$

By the definition, $\|\|_{\phi, Q}$ has the following properties.

**Lemma 5.5** Let $z_1, z_2 \in \mathcal{D}_{E[H]}$ and $q \in Q$. Then

1. $\|z_1 z_2 q\|_{\phi, Q} \leq \|z_1\|_{\phi, Q} \|z_2\|_{\phi, Q}$.
2. $\|z_1 q\|_{\phi, Q} = \|z_1\|_{\phi, Q} \|q\|_{\phi, Q}$.

Observe that if $\phi \in H^1(G_0; \mathbb{R}) \subseteq H^1(H; \mathbb{R})$ is a restriction of some character of $G_0$, then $\|x\|_{\phi, Q} = \|x\|_{\phi}$, and so, in this case $\|\|_{\phi, Q}$ is multiplicative. However, if $\phi$ is an arbitrary character of $H^1(H; \mathbb{R})$, then $\|\|_{\phi, Q}$ is not multiplicative in general. This motivates the notion of the defect of $\|\|_{\phi, Q}$.

$$\text{def}_Q(\phi) = \max \left\{ \frac{\|q_1 q_2\|_{\phi, Q}}{\|q_1\|_{\phi, Q} \|q_2\|_{\phi, Q}} : q_1, q_2 \in Q \right\}.$$

Observe that if $q_1 \in H$, then by Lemma 5.5, $\|q_1 q_2\|_{\phi, Q} = \|q_1\|_{\phi, Q} \|q_2\|_{\phi, Q}$. Thus, $\text{def}_Q(\phi)$ is always at least 1. We have the following consequence of Proposition 5.4.

**Corollary 5.6** Let $H$ be a normal subgroup of finite index of $G$, $H \leq G_0 \leq G$ and $Q$ a transversal of $H$ in $G_0$. Let $x \in \mathcal{D}_{E[G_0]}$. Then the following functions on $H^1(H; \mathbb{R})$,

$$\phi \mapsto \|x\|_{\phi, Q} \quad \text{and} \quad \phi \mapsto \text{def}_Q(\phi),$$

are continuous.

We will use the following properties of $\|\|_{\phi, Q}$.

**Proposition 5.7** Let $H$ be a normal subgroup of finite index of $G$, $H \leq G_0 \leq G$ and $Q$ a transversal of $H$ in $G_0$. Let $\phi \in H^1(H; \mathbb{R})$. Then for every $w, z \in \mathcal{D}_{E[G_0]}$,

$$\|z+w\|_{\phi, Q} \leq \max\{ \|z\|_{\phi, Q}, \|w\|_{\phi, Q} \} \quad \text{and} \quad \|z \cdot w\|_{\phi, Q} \leq \|z\|_{\phi, Q} \cdot \|w\|_{\phi, Q} \cdot \text{def}_Q(\phi).$$
Proof If \( g \in G_0 \), let \( \bar{g} \in Q \) be such that \( Hg = H\bar{g} \). We write \( z = \sum_{q \in Q} z_q q \) and \( w = \sum_{q \in Q} w_q q \), with \( z_q, w_q \in D_{E[H]} \). Then

\[
z + w = \sum_{q \in Q} (z_q + w_q)q \quad \text{and} \quad z \cdot w = \sum_{q \in Q} \left( \sum_{q = q_1q_2} z_{q_1}(w_{q_2})^{q_1^{-1}} q_1 q_2 \right).
\]

Let \( \psi \in \phi^Q \). Since \( \|z_q + w_q\|_\psi \leq \max\{\|z_q\|_\psi, \|w_q\|_\psi\} \), we obtain that \( \|z + w\|_{\phi, Q} \leq \max\{\|z\|_{\phi, Q}, \|w\|_{\phi, Q}\} \).

Observe that

\[
\|z_q (w_{q_2})^{q_1^{-1}} q_1 q_2\|_{\phi, Q} \leq \|z_{q_1}\|_{\phi, Q} \|w_{q_2}\|_{\phi, Q} \|q_1 q_2\|_{\phi, Q} \text{def } Q(\phi)
\]

Lemma 5.5 \( \leq \|z_{q_1}\|_{\phi, Q} \|w_{q_2}\|_{\phi, Q} \|q_1 q_2\|_{\phi, Q} \text{def } Q(\phi) \).

Therefore \( \|z \cdot w\|_{\phi, Q} \leq \|z\|_{\phi, Q} \cdot \|w\|_{\phi, Q} \cdot \text{def } Q(\phi) \). \( \square \)

Corollary 5.8 Let \( H \) be a normal subgroup of finite index of \( G \), \( H \leq G_0 \leq G \) and \( Q \) a transversal of \( H \) in \( G_0 \). Let \( \phi \in H^1(H; \mathbb{R}) \) and let \( w, y \in D_{E[G_0], \phi^Q} \). Assume that \( w \) is invertible in \( D_{E[G_0], \phi^Q} \) and \( \|y\|_{\phi, Q} \cdot \|w^{-1}\|_{\phi, Q} < \text{def } Q(\phi)^{-2} \).

Then \( w + y \neq 0 \) and \( (w + y)^{-1} \in D_{E[G_0], \phi} \).

Proof By Proposition 5.7,

\[
(w + y)w^{-1} = 1 - z \quad \text{with} \quad \|z\|_{\phi, Q} < \text{def } Q(\phi)^{-1}.
\]

In particular \( w + y \neq 0 \).

Let us put \( \epsilon = \|z\|_{\phi, Q} \text{def } Q(\phi) \). Then \( \epsilon < 1 \) and, by Proposition 5.7,

\[
\|z^n\|_{\phi, Q} \leq \frac{\epsilon^n}{\text{def } Q(\phi)}.
\]

Thus, if we write

\[
z^n = \sum_{q \in Q} z_{q,n} q, \quad \text{with} \quad z_{q,n} \in D_{E[H], \phi^Q},
\]

then we obtain that for every \( \psi \in \phi^Q \),

\[
\|z_{q,n}\|_\psi \leq \frac{\|z^n\|_{\phi, Q}}{\|q^{|Q|}\|_{\phi}^{1/|Q|}} \frac{\epsilon^n}{\text{def } Q(\phi)\|q^{|Q|}\|_{\phi}^{1/|Q|}}.
\] (10)
Consider
\[ v = \sum_{q \in Q} \left( \sum_{n=0}^{\infty} z_{q,n} \right) \otimes q, \]
and observe that, by (10), \( v \in \text{Im} \tilde{\beta}_\psi \). On the one hand we have that
\[ v(1-z) = \left( \sum_{q \in Q} \left( \lim_{k \to \infty} \sum_{n=0}^{k} z_{q,n} \right) \otimes q \right) (1-z) \]
\[ = \left( \lim_{k \to \infty} \tilde{\beta}_\psi \left( \sum_{n=0}^{k} z^n \right) \right) (1-z) = \lim_{k \to \infty} \tilde{\beta}_\psi (1-z^{k+1}) = 1 \otimes 1. \]

On the other hand,
\[ \tilde{\alpha}_\psi ((1-z)^{-1})(1-z) = \tilde{\alpha}_\psi (1) = 1 \otimes 1. \]
Thus, \( \tilde{\alpha}_\psi ((1-z)^{-1}) = v \). By (9), we conclude that \((1-z)^{-1} \in \mathcal{D}_{E[G],\phi}\), and so, \((w+y)^{-1} \in \mathcal{D}_{E[G],\phi}\).

**5.5 A description of \( \mathcal{D}_{E[G]} \).**

For any \( x \in \mathcal{D}_{E[G]} \) and any normal subgroup \( H \) of finite index in \( G \) we put
\[ U_H(x) = \{ \phi \in H^1(H; \mathbb{R}) : x \in \mathcal{D}_{E[G],\phi} \}. \]
Informally, \( U_H(x) \) consists of the set of characters of \( H \) such that \( x \) can be represented as a matrix over \( E[H]^\phi \).

**Lemma 5.9** Let \( H_2 \leq H_1 \) be two normal subgroups of \( G \) of finite index. Let \( A \) be a transversal of \( H_1 \) in \( G \). Consider \( x \in \mathcal{D}_{E[G]} \) and write \( x = \sum_{a \in A} x_a a \) with \( x_a \in \mathcal{D}_{E[H_1]} \).

Then
\[ U_{H_2}(x) = \bigcap_{a \in A} U_{H_2}(x_a). \]

**Proof** Let \( \phi \in H^1(H_2; \mathbb{R}) \). By the definition,
\[ \mathcal{D}_{E[G],\phi} = \sum_{g \in G} \mathcal{D}_{E[H_2],\phi g} \] and \( \mathcal{D}_{E[H_1],\phi} = \sum_{g \in H_1} \mathcal{D}_{E[H_2],\phi g} \).
Therefore, \( \mathcal{D}_{E[G],\phi} = \sum_{a \in A} \mathcal{D}_{E[H_1],\phi a} \). Thus, \( x \in \mathcal{D}_{E[G],\phi} \) if and only if \( x_a \in \mathcal{D}_{E[H_1],\phi} \) for all \( a \in A \). Hence, \( U_{H_2}(x) = \bigcap_{a \in A} U_{H_2}(x_a) \). \( \square \)
Since $G$ is RFRS, there exists a chain $G = H_0 > H_1 > \cdots$ of finite index normal subgroups of $G$ with trivial intersection such that $H_{i+1}$ contains a normal subgroup $K_i$ of $H_i$ satisfying $H_i/K_i$ is torsion free abelian. The chain $\{H_i\}$ satisfying this property is called witnessing. We fix a witnessing chain $\{H_i\}$ in $G$. Let $K_{E[G]}$ denotes the set of all $x \in D_{E[G]}$ such that for some $k \geq 0$, $U_{H_n}(x)$ is $(G, \{H_i\})$-rich for every $n \geq k$.

In this section we prove the following theorem. This is the main result of Appendix.

**Theorem 5.10** We have that $K_{E[G]} = D_{E[G]}$.

First let us see that $K_{E[G]}$ is a subring of $D_{E[G]}$. Indeed, if $a, b \in K_{E[G]}$, using Lemma 5.1, we obtain that there exists $k \geq 0$ such that for every $n \geq k$ there is a $G$-invariant rich subset $U_n$ of $H_1(H_n; R)$ with $a, b \in D_{E[G], U_n}$. Since $D_{E[G], U_n}$ is a subring of $D_{E[G]}$, $a + b, ab \in D_{E[G]}$. Hence $K_{E[G]}$ a subring of $D_{E[G]}$.

Thus, in order to show that $K_{E[G]} = D_{E[G]}$, we have to prove that for any $0 \neq x \in K_{E[G]}$, $x^{-1} \in K_{E[G]}$. First let us consider the case where $x \in E[G]$.

**Proposition 5.11** Let $0 \neq x \in E[G]$. Then $x$ is invertible in $K_{E[G]}$.

**Proof** Write $x = \sum_{g \in G} \alpha_g g$ and denote by supp $x = \{g \in G : \alpha_g \neq 0\}$. We will show that $x^{-1} \in K_{E[G]}$ by induction on $|\text{supp} \, x|$. The base of induction is clear. Let us assume that $|\text{supp} \, x| > 1$. There exists $k \geq 0$ such that

$$||(g H_k : g \in \text{supp} \, x)|| = 1 \quad \text{and} \quad ||(g H_{k+1} : g \in \text{supp} \, x)|| \geq 2.$$ 

Let $A$ be a transversal of $H_{k+1}$ in $H_k$. Hence, there exists $g \in G$ such that we can write

$$x = \sum_{a \in A} x_a g, \quad \text{with} \quad x_a \in E[H_{k+1}].$$

Since $g, g^{-1} \in K_{E[G]}$, without loss of generality we may assume that $g = 1$. In particular, $x \in E[H_k]$.

For each $i \geq k$ we fix a transversal $Q_i$ of $H_i$ in $H_k$. For any $a \in A$, we put

$$V_{i, a} = \{\phi \in H^1(H_i; \mathbb{R}) : ||x - x_a a\|_{\phi, Q_i} \cdot ||(x_a a)^{-1}\|_{\phi, Q_i} < \text{def}_{Q_i}(\phi)^{-2}\}.$$ 

Let $V_i = \bigcup_{a \in A} V_{i, a}$.

**Claim 5.12** For each $i \geq k$, the set $V_i$ is rich in $H^1(H_i; \mathbb{R})$.

**Proof** First observe that Corollary 5.6 implies that $V_{i, a}$, and so, $V_i$ are open in $H^1(H_i; \mathbb{R})$. Let $\phi$ be an irrational character of $H^1(H_k; \mathbb{R})$. Since $\{H_i\}$ is a witnessing chain and $\phi$ is irrational, $\ker \phi \leq H_{k+1}$. Therefore, there exists $a \in A$ such that

$$||x - x_a a\|_{\phi, Q_i} = ||x - x_a a\|_{\phi} < ||(x_a a)^{-1}\|_{\phi} = \frac{1}{||x_a a\|_{\phi}} = \frac{1}{||x_a a\|_{\phi}}.$$  

Since \( \text{def} \, Q_i(\phi) = 1 \), we obtain that \( \phi \in V_{i,a} \) for all \( i \geq k \), and so \( V_i \) contains all irrational characters of \( H_k \). Now the claim follows from Lemma 5.10. \( \square \)

By the inductive assumption, \( x_a a \) is invertible in \( K_{E[G]} \). Thus, there exists \( n \geq k \) such that for every \( i \geq n \) and \( a \in A \), \( U_{H_i}((x_a a)^{-1}) \) is rich in \( H^1(H_i, \mathbb{R}) \). We put

\[
W_i = \bigcap_{q \in Q_i} \left( V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1}) \right)^q.
\]

By Lemma 5.1, \( W_i \) is rich. Let \( \phi \in W_i \). Observe that \( W_i \) is \( H_k \)-invariant. Hence \( \phi^{Q_i} \subseteq V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1}) \). There exists \( a \in A \) such that \( \phi \in V_{i,a} \). Observe that \( x - x_a a, x_a a, (x_a a)^{-1} \in D_{E[H_k], \phi} \). By Corollary 5.8, \( x^{-1} \in D_{E[H_k], \phi} \subseteq D_{E[G], \phi} \).

Thus, \( W_i \subseteq U_{H_i}(x^{-1}) \) and we are done. \( \square \)

Now, we consider the general case.

**Proof of Theorem 5.10** We will show that \( x^{-1} \in K_{E[G]} \) for every \( 0 \neq x \in K_{E[G]} \) by induction on the level \( l(x) \) of \( x \), that is defined as follows.

\[
l(x) = \min\{n - k : x \in D_{E[H_k]} \text{ and } U_{H_i}(x) \text{ is rich for every } i \geq n\}.
\]

Consider first the case \( l(x) \leq 0 \). Then \( x \in D_{E[H_k]} \) and \( U_{H_i}(x) \) is rich for every \( i \geq k \). Let \( H_k/K \) be the maximal torsion-free abelian quotient of \( H_k \). Let \( R \) be the subring of \( D_{E[H_k]} \) generated by \( D_{E[K]} \) and \( H_k \). Since \( D_{E[H_k]} \) is the classical ring of quotients of \( R \), we can write \( x = yz^{-1} \) with non-zero \( y, z \in R \). Let \( A \) be a transversal of \( K \) in \( H_k \). Then there are finite subsets \( A_0 \) and \( B_0 \) of \( A \) such that

\[
y = \sum_{a \in A_0} y_a a, \quad z = \sum_{a \in B_0} z_a a \quad \text{with non-zero } y_a, z_a \in D_{E[K]}.
\]

Let \( \phi \) be an irrational character of \( H_k \). Observe that \( \phi \) takes different values on the elements of \( A_0 \) and on the elements of \( B_0 \). Therefore, there are unique \( a_\phi \in A_0 \) and \( b_\phi \in B_0 \) such that

\[
\phi(a_\phi) = \min\{\phi(a) : a \in A_0\} \quad \text{and} \quad \phi(b_\phi) = \min\{\phi(b) : b \in B_0\}.
\]

**Claim 5.13** Let \( \phi \) be an irrational character of \( H_k \) and \( w = (y_\phi a_\phi)(z_\phi b_\phi)^{-1} \). Then \( \|x\|_\phi = \|w\|_\phi > \|x - w\|_\phi \). Moreover, if \( x \in D_{E[H_k], \phi} \), then \( w \in E[H_k] \).

**Proof** The claim follows directly from the definitions. \( \square \)

Let

\[
T = \{w_{a,b} = (y_a a)(z_b b)^{-1} : a \in A_0, b \in B_0\} \cap E[H_k].
\]

Since \( T^{-1} \subseteq K_{E[G]} \) (Proposition 5.11), there exists \( n \) such that \( U_{H_i}(w^{-1}) \) is rich for every \( w \in T \) and \( i \geq n \).
For each $i \geq n$ let $Q_i$ be a transversal of $H_i$ in $H_k$. For each $w \in T$ and $i \geq n$ we put

$$V_{i,w} = \{ \phi \in H^1(H_i; \mathbb{R}) : \| x - w \|_{\phi,Q} \cdot \| w^{-1} \|_{\phi,Q} < \text{def}_{Q_i}(\phi)^{-2} \}$$

and $V_i = \bigcup_{w \in T} V_{i,w}$. Observe that $V_i$ are open and if $\phi \in H^1(H_k; \mathbb{R})$, $\text{def}_{Q_i}(\phi) = 1$. Thus, by Claim 5.13, for all $i \geq n$, $V_i$ contains all the irrational characters of $(U_{H_k}(x))^{o}$. Since $(U_{H_k}(x))^{o}$ is rich, Lemma 5.2 implies that $V_i$ is rich for $i \geq n$.

For each $i \geq n$ we define

$$W_i = \bigcap_{q \in Q_i} \left( V_i \cap U_{H_i}(x) \cap \bigcap_{w \in T} U_{H_i}(w^{-1}) \right)^q.$$

By Lemma 5.1, $W_i$ is rich. Let $\phi \in W_i$. Observe that $W_i$ is $H_k$-invariant. Hence $\phi^{Q_i} \subseteq V_i \cap \bigcap_{w \in T} U_{H_i}(w^{-1})$. There exists $w \in T$ such that $\phi \in V_{i,w}$. Observe that $x - w, w, (w)^{-1} \in D_{E[H_k],\phi,Q_i}$. By Corollary 5.8, $x^{-1} \in D_{E[H_k],\phi \subseteq D_{E[G],\phi}}$. Thus, $W_i \subseteq U_{H_i}(x^{-1})$. Thus, $x^{-1} \in K_{E[G]}$.

Now, we assume that $l(x) > 0$ and that the non-zero elements of $K_{E[G]}$ of level less than of $l(x)$ are invertible in $K_{E[G]}$. There are $n$ and $k$ such that $l(x) = n - k$, $x \in D_{E[H_k]}$ and $U_{H_i}(x)$ is rich for every $i \geq n$.

Let $A$ be a transversal of $H_{k+1}$ in $H_k$. Hence, we can write

$$x = \sum_{a \in A} x_a a, \quad \text{with} \quad x_a \in D_{E[H_{k+1}]}.$$

By Lemma 5.9, for every $a \in A, x_a \in K_{E[G]}$ and $l(x_a) < l(x)$.

For each $i \geq k$ we fix a transversal $Q_i$ of $H_i$ in $H_k$. For any $a \in A$ we put

$$V_{i,a} = \{ \phi \in H^1(H_i; \mathbb{R}) : \| x - x_a a \|_{\phi,Q_i} \cdot \| (x_a a)^{-1} \|_{\phi,Q_i} < \text{def}_{Q_i}(\phi)^{-2} \}.$$

Let $V_i = \bigcup_{a \in A} V_{i,a}$. Arguing as in the proof of Claim 5.12, we obtain that all $V_i$ are rich. By the inductive assumption, $x_a a$ is invertible in $K_{E[G]}$. Thus, there exists $n \geq k$ such that for every $i \geq n$ and $a \in A$, $U_{H_i}((x_a a)^{-1})$ is rich in $H^1(H_i, \mathbb{R})$. We put

$$W_i = \bigcap_{q \in Q_i} \left( V_i \cap U_{H_i}(x) \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1}) \right)^q.$$

By Lemma 5.1, $W_i$ is rich. Let $\phi \in W_i$. Observe that $W_i$ is $H_k$-invariant. Hence $\phi^{Q_i} \subseteq V_i \cap \bigcap_{a \in A} U_{H_i}((x_a a)^{-1})$. There exists $a \in A$ such that $\phi \in V_{i,a}$. Observe that $x - x_a a, x_a a, (x_a a)^{-1} \in D_{E[H_k],\phi,Q_i}$. By Corollary 5.8, $x^{-1} \in D_{E[H_k],\phi \subseteq D_{E[G],\phi}}$. Thus, $W_i \subseteq U_{H_i}(x^{-1})$ and we are done.
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