NONLINEAR INTEGRABLE SYSTEMS RELATED TO ARBITRARY SPACE-TIME DEPENDENCE OF THE SPECTRAL TRANSFORM

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Abstract

We propose a general algebraic analytic scheme for the spectral transform of solutions of nonlinear evolution equations. This allows us to give the general integrable evolution corresponding to an arbitrary time and space dependence of the spectral transform (in general nonlinear and with non-analytic dispersion relations). The main theorem is that the compatibility conditions gives always a true nonlinear evolution because it can always be written as an identity between polynomials in the spectral variable $k$. This general result is then used to obtain first a method to generate a new class of solutions to the nonlinear Schrödinger equation, and second to construct the spectral transform theory for solving initial-boundary value problems for resonant wave-coupling processes (like self-induced transparency in two-level media, or stimulated Brillouin scattering of plasma waves or else stimulated Raman scattering in nonlinear optics etc...).
1 Introduction

The spectral transform method to build and solve classes of nonlinear evolution equations (for a field $Q(x,t)$) works as a nonlinear extension of the Fourier transform : it associates to the field $Q(x,t)$ its spectral transform $R(k,x,t)$ which is a distribution in the spectral complex variable $k$ and a function of the real space and time variables $x$ and $t$. The domain of definition for $k$ and the $(x,t)$-dependence of $R$ uniquely determine both the related nonlinear evolution for $Q$ and the class of functions to which it belongs.

For instance when $k$ varies on $\mathbb{C}$ and when $R$ is a $2 \times 2$ off-diagonal matrix depending on $x$ and $t$ through

$$R_x = [R, ik\sigma_3], \quad R_t = [R, ik^2\sigma_3],$$

(1.1)

then $Q$ is a $2 \times 2$ off-diagonal matrix obeying the nonlinear Schrödinger equation

$$i\sigma_3 Q_t = -\frac{1}{2} Q_{xx} + Q^3.$$  

(1.2)

This is the very reason why the spectral transform allows us to solve the nonlinear evolution (1.2) : the corresponding evolution in the spectral space is linear. It is then enough to have a bijection between $Q$ and $R$, which is the set of direct and inverse spectral problems.

The main purpose of this work is to answer the following simple question: can we still build and solve nonlinear evolutions when the $(x,t)$-dependence of the spectral transform is arbitrary. We shall prove in particular that the nonlinear evolutions associated with a local $2 \times 2$ matrix $\bar{\partial}$ problem on the Riemann sphere can always be written in closed form whatever be the space and time dependences of $R$ in a very general class of equations (not necessarily linear).

Here the $\bar{\partial}$ problem will be understood as being the main tool which links $Q(x,t)$ to $R(k,x,t)$ by means of a matrix valued function $\mu(k,x,t)$ solution of

$$\mu(k) = 1 + \frac{1}{2\pi} \int \int \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} \mu(\lambda) R(\lambda)$$

(1.3)

(being parametrically induced by $R(k,x,t)$, the $(x,t)$-dependences can be omitted everywhere in $\mu$ and $R$). If we operate with the $\bar{\partial}$-operator (see next section) on both side of the above equation we obtain the so-called $\bar{\partial}$ problem for $\mu$ :

$$\frac{\partial}{\partial k} \mu(k) = \mu(k) R(k), \quad \mu(k) \to 1 \quad \text{as} \quad k \to \infty$$

(1.4)

It is clear that any dependence of $R(k)$ on external independent (real) parameters $(x$ and $t$) induces a dependence of $\mu(k)$ generally through an overdetermined
system of differential equations: the so-called Lax pair. Then this overdetermination implies some constraints on the coefficients of the Laurent series for $\mu(k)$ (see (2.15) next section). When they form a closed system independent of the variable $k$, these constraints are nothing but the nonlinear evolution equation related to the given space and time dependence of $R(k)$.

The result presented here will undercover some interesting properties of overdetermined systems (for $R(k)$ and for $\mu(k)$) when they are related through a $\partial$ problem like (1.4). First we shall discover that the differential system for $R$ (data of $R_t$ and $R_x$) although very general, has to be written in a special form in order to obtain a non-trivial integrable related evolution. This remark will actually bring into light the necessity of a spectral parameter in the Lax pair. Second, our result will be shown to generate new classes of integrable systems with in particular some very interesting applications to problems of interaction of radiation with matter (for instance stimulated Brillouin and Raman scattering), which are always associated with a mixed initial and boundary value problem.

Although some subcases and applications of the material presented here have been reported elsewhere in [...], the results presented here are mainly new and we shall avoid using directly any previous result. In that purpose, the next section is a brief recall of the basic mathematical tools that we have chosen to present in their light version, reporting to [...] for mathematical details.

The section 3 is the heart of this paper: the main theorem is established and we obtain the general algebraic-analytic scheme of the spectral transform method. Some important consequences of general interest are then discussed.

In section 4 we present an application of the general theory to the nonlinear Schrödinger equation (NLS) and exhibit in particular a new method to generate a more general class of solutions.

The section 5 is devoted to the main interesting application of the general result: the theory is used to build integrable coupled systems for which some arbitrary boundary values are prescribed. As a consequence the time evolution of the spectral transform is in general nonlinear with some very new properties of the solution in physically interesting cases.
2 Notations, definitions and basic tools

The derivative with respect to \( \bar{k} \) (where \( k = k_R + i k_I \)) formally defined as

\[
\frac{\partial}{\partial \bar{k}} = \frac{1}{2} \left( \frac{\partial}{\partial k_R} + i \frac{\partial}{\partial k_I} \right)
\]  

(2.1)

vanishes on the space of analytic functions of \( k \) (it is sometimes called the measure of non-analyticity). It can be conveniently defined by the Cauchy-Green representation of a function in \( \mathbb{C} \):

\[
f(k) = f_0(k) + f_1(k)
\]

\[
f_1(k) = \frac{1}{2\pi i} \iint d\lambda d\bar{\lambda} \frac{\partial f(\lambda)}{\partial \lambda} \delta(\lambda - k)
\]  

(2.2)

where \( f_0(k) \) is the asymptotic behavior (analytic in \( k \)) of \( f(k) \) when \( k \to \infty \).

Here we shall work more particularly in the space of function \( f(k) \) whose behavior \( f_0(k) \) is a polynomial (in \( k \)) which we note \( P_k \), that is

\[
f_0(k) = P_k \{ f \}
\]  

(2.3)

We call \( \mathcal{H}_p \) the space of functions which have the representation (2.2) with the constraint (2.3). A consequence of (2.2) is that we can formally write

\[
\frac{\partial}{\partial \bar{k}} \iint d\lambda d\bar{\lambda} f(\lambda) \delta(\lambda - k) = 2\pi i \delta(\lambda - k)
\]  

(2.4)

where \( \delta \) is the Dirac distribution in \( \mathbb{C} \), namely

\[
\iint d\lambda d\bar{\lambda} f(\lambda) \delta(\lambda - k) = f(k)
\]

\[
= -2i \int_{-\infty}^{\infty} d\lambda R \int_{-\infty}^{+\infty} d\lambda R f(\lambda R + i\lambda I) \delta(\lambda R - k R) \delta(\lambda I - k I).
\]  

(2.5)

We shall also use the distributions \( \delta^\pm \) defined by

\[
\iint d\lambda d\bar{\lambda} f(\lambda) \delta^\pm(\lambda I) = -2i \int_{-\infty}^{+\infty} d\lambda R f(\lambda R \pm i0),
\]  

(2.6)

and make use of the notation

\[
f^\pm(k_R) = f(k_R \pm i0).
\]  

(2.7)

From (2.4) we can deduce

\[
\frac{\partial}{\partial k} \int_{-\infty}^{+\infty} \frac{d\lambda R}{\lambda R - k \pm i0} f(\lambda R) = -\pi f(k) \delta^\pm(k I).
\]  

(2.8)
Let us recall also the Sokhotski-Plemelj formula
\[
\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k ± i0} f(\lambda) = ±i\pi f(k) + P\int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k} f(\lambda),
\] (2.9)
where\( P \) stands for the Cauchy principal value integral. The two above formulae allow us to obtain by direct calculation
\[
\frac{i}{2} \left[ f(k)\delta^+(k_I) - f(k)\delta^-(k_I) \right] = \frac{\partial}{\partial k} f(k),
\] (2.10)
which provides the \( \bar{\partial} \) derivative of a function discontinuous on the real axis but analytic elsewhere.

We note that the Cauchy-Green integral equation (2.2) can also be used to define the \( \bar{\partial} \) operator in \( \mathcal{H}_p \) as
\[
\bar{\partial}^{-1} \left( \frac{\partial f(k)}{\partial k} \right) = f(k) - \mathcal{P}\{f\},
\] (2.11)
which is nothing but \( f_1(k) \). Hence the k-polynomial part of \( f(k) \) appears as the constant of integration in \( \mathcal{H}_p \) of the \( \bar{\partial} \) operator. Whenever possible we use the short-hand notation
\[
\bar{\partial} f = \frac{\partial}{\partial k} f(k), \quad \bar{\partial}^{-1} f = \frac{1}{2i\pi} \iint d\lambda \wedge d\bar{\lambda} \frac{\partial f(\lambda)}{\lambda - k} f(\lambda)
\] (2.12)
and we have the very important property
\[
\bar{\partial}^{-1} f = O\left( \frac{1}{k} \right) \text{ as } k \to \infty
\] (2.13)

Next we adopt for the Laurent series of \( f(k) \) the following notation
\[
f(k) = f_0(k) + \sum_{n=1}^{\infty} k^{-n} f^{(n)},
\] (2.14)
where from (2.2)
\[
f^{(n)} = -\frac{1}{2i\pi} \iint d\lambda \wedge d\bar{\lambda} \frac{\partial f(\lambda)}{\lambda} \lambda^{n-1}
\] (2.15)

For a complex valued function \( f(k) = a(k) + ib(k) \) of the complex variable \( k = k_R + ik_I \), we note
\[
\bar{f}(k) = a(k_R + ik_I) - ib(k_R + ik_I),
\] (2.16)
\[
f^*(k) = a(k_R - ik_I) - ib(k_R - ik_I) = \bar{f}(\bar{k}).
\] (2.17)
Then, from the definition (2.6), since $\delta^\pm$ are real valued distributions obeying $\delta^+(k_I) = \delta^-(k_I)$, we have
\[
\delta^+(k_I) = (\delta^-(k_I))^*.
\] (2.18)

It is also useful to remember that, when $\lambda$ depends on a real parameter, say $t$, we may write
\[
\frac{\partial}{\partial t} \delta(k - \lambda) = \frac{\delta(k - \lambda)}{k - \lambda} \frac{\partial \lambda}{\partial t},
\] (2.19)
which is meaningful on the space of functions of $k$ that vanish in $k = \lambda$.

Finally, we shall work throughout the paper in the space of complex-valued $2 \times 2$ matrices, say $M$, which elements $m_{ij}$ are differentiable functions of two real variables ($x$ and $t$) and distributions in a complex variable ($k$). It is convenient to split the matrices in their diagonal and anti-diagonal parts, namely
\[
M = M^D + M^A, \quad M^D = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}, \quad M^A = \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}.
\] (2.20)

Using the standard definition of the Pauli matrices
\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (2.21)
we have for instance
\[
[\sigma_3, M] = 2\sigma_3 M^A.
\] (2.22)
3 General theorem

The starting point is the Cauchy-Green integral equation (1.3) or equivalently the \( \partial \bar{\partial} \)-problem (1.4) for the matrix valued complex function \( \mu(k) \):

\[
\frac{\partial}{\partial k} \mu(k) = \mu(k) R(k), \quad \mu(k) = 1 + O\left(\frac{1}{k}\right),
\]

where the given datum \( R(k) \) is a distribution in the set of \( 2 \times 2 \) matrices of vanishing trace:

\[
tr\{R(k)\} = 0.
\]

Theorem 1:

When \( R(k) \) depends on two real parameters \( x \) and \( t \) according to

\[
R_x = [R, \Lambda] + N, \quad R_t = [R, \Omega] + M,
\]

the quantities

\[
U_0 = -\mathcal{P}_k\{\mu\Lambda\mu^{-1}\}, \quad U_1 = \bar{\partial}^{-1}\{\mu(N - \partial\Lambda)\mu^{-1}\},
\]

\[
V_0 = -\mathcal{P}_k\{\mu\Omega\mu^{-1}\}, \quad V_1 = \bar{\partial}^{-1}\{\mu(M - \partial\Omega)\mu^{-1}\}
\]

obey the following equation

\[
U_{0,t} - V_{0,x} + [U_0, V_0] + \mathcal{P}_k\{\mu A \mu^{-1}\} = \mathcal{P}_k\{[V_0, U_1] - [U_0, V_1]\},
\]

where the matrix \( A \) stands for

\[
A = \Lambda_t - \Omega_x + [\Omega, \Lambda],
\]

and where the entries \( \{\Lambda, N, \Omega, M\} \) obey the compatibility constraint

\[
N_t - [N, \Omega] - M_x + [M, \Lambda] = [A, R].
\]

Equation (3.7) is a nonlinear evolution equation because it is an identity between polynomials in \( k \) and then it has to be read as a set of local \( k \)-independent differential equations (each coefficient of a given power of \( k \) vanishes).

Here above the \( 2 \times 2 \) matrices \( \Lambda \) and \( \Omega \) are functions in \( \mathcal{H}_p \) (see sec. 2) and \( M \) and \( N \) are in general distributions. All these 4 given matrices are diferentiable fuctions of the real variables \( x \) and \( t \). Up to now the \( (x, t) \)-dependence of \( R \) is quite arbitrary but we will understand later why this specific structure is important (rather than just letting \( \Lambda = \Omega = 0 \)). Note also that nothing prevents these equations to be nonlinear in \( R \).
The problem is to find the \((x, t)\)-dependence of the solution \(\mu(k)\) of (3.1) when \(R(k)\) is required to solve (3.3) and (3.4). The method proposed in [4] consists in using first the commutativity of the differential operators.

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial k} = 0, \quad \frac{\partial}{\partial x} \frac{\partial}{\partial k} = 0, \tag{3.10}
\]

and second the property that \(\mu(k)\) is a regular matrix. Indeed from (3.2) we easily obtain

\[\frac{\partial}{\partial k} \det \mu(k) = 0 \tag{3.11}\]

and hence, since \(\mu(k) = 1 + 0(1/k)\) for large \(k\), the Liouville theorem implies

\[\det \mu(k) = 1 \tag{3.12}\]

These results allow us to obtain by direct computation the following relations

\[
\bar{\partial} \{(\mu_x - \mu \Lambda) \mu^{-1}\} = \mu(N - \bar{\partial} \Lambda) \mu^{-1} \tag{3.13}
\]

\[
\bar{\partial} \{(\mu_t - \mu \Omega) \mu^{-1}\} = \mu(M - \bar{\partial} \Omega) \mu^{-1} \tag{3.14}
\]

They can be integrated by using (2.11), the result can be written as the following overdetermined system

\[
\mu_x = \mu \Lambda + U \mu, \quad \mu_t = \mu \Omega + V \mu, \tag{3.15}
\]

in which \(U\) and \(V\) are in \(\mathcal{H}_p\), that is

\[
U = U_0 + U_1, \quad V = V_0 + V_1, \tag{3.16}
\]

where \(U_0, U_1, V_0, V_1\) are given in (3.5) and (3.6).

Consequently the \(\bar{\partial}\) problem (3.1) serves as the link between the two sets of overdetermined differential systems (3.3), (3.4) and (3.15). Computing \(R_{xt}\) in two ways leads to the constraint (3.9) and doing the same for \(\mu_{xt}\) leads to the compatibility condition:

\[U_t - V_x + [U, V] = -\mu A \mu^{-1}. \tag{3.17}\]

We come now to the demonstration of (3.7). This is done first by computing the quantity \(U_t - V_x\) appearing in (3.17) with the definitions (3.11) to (3.16). To arrange conveniently this quantity we have to use the inverse of (3.5) and (3.6) that is

\[
\bar{\partial} U = \mu(N - \bar{\partial} \Lambda) \mu^{-1}, \quad \bar{\partial} V = \mu(M - \bar{\partial} \Omega) \mu^{-1} \tag{3.18}
\]

with this help one can write

\[
U_{1,t} - V_{1,x} = \bar{\partial}^{-1}\left\{\mu(N_t - M_x - [N, \Omega] + [M, \Lambda]) \mu^{-1} - \mu \bar{\partial}(\Lambda_t - \Omega_x + [\Omega, \Lambda]) \mu^{-1} - [U, \bar{\partial} V] - [\bar{\partial} U, V]\right\}. \tag{3.19}
\]
At this stage enters the constraint (3.9) which guarantees the compatibility of the choices for $R_x$ and $R_t$. Then the above equation can be written

$$U_{1,t} - V_{1,x} = -\bar{\partial}^{-1}\{\mu(\bar{\partial}A + [R, A])\mu^{-1} + \bar{\partial}[U, V]\}$$

or else

$$U_{1,t} - V_{1,x} = -\bar{\partial}^{-1}\{\mu A \mu^{-1} + [U, V]\}$$

The inversion formula (2.11) then gives

$$U_{1,t} - V_{1,x} = -\mu A \mu^{-1} - [U, V] + \mathcal{P}_k\{\mu A \mu^{-1} + [U, V]\}$$

which then, inserted in (3.17), easily leads to (3.7) for we have obviously

$$\mathcal{P}_k\{[U, V]\} = [U_0, V_0] + \mathcal{P}_k\{[U_0, V_1] + [U_1, V_0]\}.$$ (3.23)

First we note that this result is far from being trivial. Indeed, looking at the structure of $U$ and $V$ given in (3.5) and (3.6), we realize that the equation (3.17) involves quite complicated functions of $k$ through many $\bar{\partial}^{-1}$ operators (see (2.12)). The relative simplicity of the result (3.7) originates from the fact that all terms in the integral part of (3.17) can actually be written as an exact $\bar{\partial}$-derivative.

Second it is the very reason why the spectral transform works also in the case of singular dispersion relation. In previous works (i.e. in [2] to [6]) the $k$-dependence in (3.17) was seen to cancel out as a result of the particular choice of $\Lambda$ but in no way as a very general property.

Third the result (3.7) justifies the choice of the structure in (3.3), (3.4). Indeed, avoiding any structure in (3.3) and (3.4) by just letting $\Omega = \Lambda = 0$, makes the equation (3.7) trivially solved: we would have from (3.3) and (3.4) $U_0 = 0, V_0 = 0$ and hence (3.17) would not lead to any nonlinear evolution equation.

Note also that another necessary condition that (3.17) be a nonlinear evolution equation is that $U_0$ be nonzero, that is from (3.3) and (3.4) that $P_k\{\Lambda\}$ be non zero. This property can be understood as the necessity of a spectral parameter in the spectral problem (3.15) (here $P_k\{\Lambda\}$ would play the role of the spectral parameter).

We have finally built a set of relations between partial differential equations (obtained by means of the $\bar{\partial}$-problem) which has to be understood as follows: if the set $\{R, \Omega, \Lambda, M, N\}$ obeys the system (3.3), (3.4) and the compatibility constraint (3.9), then the set $\{U_0, U_1, V_0, V_1\}$, given from the solution $\mu$ of the $\bar{\partial}$-problem in (3.3), (3.6), obeys the nonlinear evolution (3.7).

The compatibility constraint (3.9) will be seen to be essential. At this stage it is important to note that the degree of complexity of this constraint is usually...
greater than that of the nonlinear equation itself. Hence the main point to make here is that, in order to build from theorem 1 some practical tools, we have to make a decision, for instance by choosing an elementary solution of the constraint (3.9)
The nonlinear Schrödinger equation

A natural choice for the arbitrary functions \( \Lambda \) and \( \Omega \), for which the constraint (3.9) is independent of the spectral transform \( R \), is

\[
\Lambda_t - \Omega_x + [\Omega, \Lambda] = 0.
\] (4.1)

It is useful to rewrite the theorem in this subcase:

**Theorem 2**

For

\[
R_x = [R, \Lambda] + N, \quad R_t = [R, \Omega] + M,
\] (4.2)

\[
\Lambda_t - \Omega_x + [\Omega, \Lambda] = 0,
\] (4.3)

we have the nonlinear evolution equation

\[
U_{0,t} - V_{0,x} + [U_0, V_0] = \mathcal{P}_k\{[V_0, U_1] - [U_0, V_1]\},
\] (4.4)

with

\[
U_0 = -\mathcal{P}_k\{\mu \Lambda \mu^{-1}\}, \quad U_1 = \bar{\partial}^{-1}\{\mu (N - \bar{\partial} \Lambda) \mu^{-1}\},
\]

\[
V_0 = -\mathcal{P}_k\{\mu \Omega \mu^{-1}\}, \quad V_1 = \bar{\partial}^{-1}\{\mu (M - \bar{\partial} \Omega) \mu^{-1}\},
\]

if \( M \) and \( N \) satisfy the constraint

\[
N_t - [N, \Omega] - M_x + [M, \Lambda] = 0.
\] (4.5)

The simplest non-trivial choice is realized by taking for \( \Lambda \) and \( \Omega \) two diagonal matrices, polynomial in \( k \) with constant coefficients. In that case, the matrix \( A \) given in (3.8) does vanish and a trivial solution to (3.9) is \( M = N = 0 \). Then we recover the usual hierarchies of integrable partial differential equations. To set an example, let us chose

\[
\Lambda = ik \sigma_3, \quad \Omega = ik^2 \sigma_3, \quad M = N = 0.
\] (4.6)

Then from (3.3) and (3.6)

\[
U_0 = 0, \quad V_0 = 0,
\] (4.7)

and from the Laurent series (2.14) for \( \mu(k) \)

\[
U_0 = -ik \sigma_3 + i[\sigma_3, \mu^{(1)}],
\] (4.8)

\[
V_0 = -i \sigma_3 k^2 + ik[\sigma_3, \mu^{(1)}] - i[\sigma_3, \mu^{(1)}] \mu^{(1)} + i[\sigma_3, \mu^{(2)}].
\] (4.9)

The above two polynomials in \( k \) are then inserted in the evolution equation (4.4) and each coefficient of any power of \( k \) is required to vanish. Defining the anti-diagonal matrix \( Q(x, t) \) by

\[
Q = i[\sigma_3, \mu^{(1)}],
\] (4.10)
we easily obtain: at order $k^2$ the equation (4.4) is automatically verified, at
order $k$ this equation gives
\[ i \left[ \sigma_3, Q\mu^{(1)} - i[\sigma_3, \mu^{(2)}] \right] = Q_x. \tag{4.11} \]

The above equation can then be shown to be equivalent to the diagonal part of
the $k$-independent term in (4.4), and it actually defines the quantity $i[\sigma_3, \mu^{(2)}]$ from $Q$. Provided this information, the anti-diagonal part of (4.4) finally gives,
again by using (4.11), the nonlinear Schrödinger equation
\[ i\sigma_3 Q_t + \frac{1}{2} Q_{xx} - Q^3 = 0. \tag{4.12} \]

At this point there is the important remark that $\Lambda$ and $\Omega$ can be modified in
such a way as to keep the evolution equation unchanged (that is to keep $U_0$ and $V_0$ unchanged) but to change the $(x, t)$-dependence of $R(k, x, t)$. This will then
generate, through the solution of (1.4) and of (4.10), a larger class of solutions
of the NLS equation. This can be done by choosing
\[ \mathcal{P}_k\{\Lambda\} = ik\sigma_3, \quad \mathcal{P}_k\{\Omega\} = ik^2\sigma_3, \tag{4.13} \]
\[ N = \bar{\partial}\Lambda, \quad M = \bar{\partial}\Omega, \tag{4.14} \]
for which $U_0$ is still given by (1.8) and $V_0$ by (1.9), and for which $U_1 = V_1 = 0$. Note that the constraint (4.5) is automatically verified from (4.3) and (4.14).

While $R(k, x, t)$ now obeys a much more general $(x, t)$-dependence, the related
nonlinear evolution is still the above NLS equation (4.12), and hence we have
indeed here a method to generate a larger class of solutions to the NLS
equation. The method to perform this can be sketched as follows:
Step 1: get some explicit functions $\Lambda$ and $\Omega$ by means of
\[ \begin{bmatrix} \Lambda_t - \Omega_x + [\Omega, \Lambda] = 0 \\
\mathcal{P}_k\{\Lambda\} = ik\sigma_3, \quad \mathcal{P}_k\{\Omega\} = ik^2\sigma_3 \end{bmatrix} \longrightarrow \{\Lambda(k, x, t), \Omega(k, x, t)\}. \tag{4.15} \]
Step 2: solve the $(x, t)$-dependences for $R(k, x, t)$
\[ \begin{bmatrix} \Lambda_t - \Omega_x + [\Omega, \Lambda] = 0 \\
\mathcal{P}_k\{\Lambda\} = ik\sigma_3, \quad \mathcal{P}_k\{\Omega\} = ik^2\sigma_3 \end{bmatrix} \longrightarrow \{\Lambda(k, x, t), \Omega(k, x, t)\}. \tag{4.16} \]
Step 3: solve the linear Cauchy integral equation for $\mu(k, x, t)$
\[
\left[ \mu(k) = 1 + \frac{1}{2\pi i} \oint \frac{d\lambda}{\lambda-k} \mu(\lambda) R(\lambda) \right] \longrightarrow \{\mu(k, x, t)\}. \tag{4.17} \]
Step 4: compute the solution $Q(x, t)$ of the NLS equation by means of $\mu(k, x, t)$
\[ Q(x, t) = i[\sigma_3, \mu^{(1)}(x, t)]. \tag{4.18} \]
Steps 3 and 4 hereabove are the usual problems to be solved when a solution to an integrable nonlinear evolution is seeked and step 2 is a linear differential equation which is solved through standard methods. The novelty here lies in the step 1 which solution actually reduces to solving the nonlinear integrable hierarchy generated through the potentials $\Lambda$ and $\Omega$ by the compatibility condition (4.3). The problem is that of the stringent constraints

$$
\Lambda = ik\sigma_3 + O\left(\frac{1}{k}\right), \quad \Omega = ik^2\sigma_3 + O\left(\frac{1}{k}\right).
$$

The technical complexity of the proposed method is such that we have not been able to find an explicit example. Note in particular that, when $\Lambda$ is not simply $ik\sigma_3$, the scattering problem (the x-part of the Lax pair (3.15)) has to be rebuilt completely. And this is a necessary step in order to find the constants of integration for the two differential equations in (4.16).

In this context, the last point to check is the unicity of the solution of the Cauchy problem for the NLS eq. when $Q(x,0)$ is in the Schwartz space. This unicity will hold if, when analyzing $Q(x,0)$ through the Zakharov-Shabat spectral problem, in other words with

$$
\Lambda = ik\sigma_3,
$$

we would obtain the usual time evolution of the spectral transform, in other words

$$
\Omega = ik^2\sigma_3.
$$

This result is obtained here by first remarking that the compatibility equation (4.3) in which $\Lambda = ik\sigma_3$ gives by induction (successive powers of $1/k$)

$$
\Omega^A = 0.
$$

Second, we use the property given in the appendix that the spectral transform $R$ is an anti-diagonal matrix. Then the time evolution in (4.2) can be splitted in its diagonal and anti-diagonal parts to give

$$
\bar{\partial}\Omega^D = [\Omega^A, R], \quad R_t = [R, \Omega^D] + \bar{\partial}\Omega^A.
$$

The first of the above equations can be explicitly integrated thanks to (4.13) and (4.22) and we have

$$
\Omega^D = ik^2\sigma_3
$$

which achieves the proof. Indeed, (4.23) now reads

$$
R_t = [R, ik^2\sigma_3].
$$
5 Integrable asymptotic boundary value problems

5.1 Statement of the problem

The general theorem 1 is shown here to serve as the basic tool for demonstrating the following important result.

Theorem 3:

The system of coupled equations for the three fields $q(x, t)$, $a_1(k, x, t)$ and $a_2(k, x, t)$:

$$
q_t = \int_{-\infty}^{+\infty} g(k) a_1 \bar{a_2},
$$

$$
a_{1,x} = qa_2,
$$

$$
a_{2,x} - 2ika_2 = \sigma \bar{qa_1},
$$

(5.1)

(with $\sigma = \pm$, $x, k \in \mathbb{R}$, $t > 0$ and $g = g(k)$ an arbitrary function in $L^2$ which may also depend on $t$), is integrable for the arbitrary asymptotic boundary values $a_j(k, \pm\infty, t)$.

More precisely we will give the evolution of the spectral transform in the following four cases:

$$
a_1 \rightarrow I_1(k, t), \quad a_2 \rightarrow I_2(k, t)e^{2ikx},
$$

(5.2)

$$
a_1 \rightarrow J_1(k, t), \quad a_2 \rightarrow J_2(k, t)e^{2ikx},
$$

(5.3)

$$
a_1 \rightarrow K_1(k, t), \quad a_2 \rightarrow K_2(k, t)e^{2ikx},
$$

(5.4)

$$
a_1 \rightarrow L_1(k, t), \quad a_2 \rightarrow L_2(k, t)e^{2ikx}.
$$

(5.5)

Here above the asymptotic boundary values $(I_1, I_2, J_1, ...)$ are arbitrary functions of the real variable $k$ in $L^2$ with an arbitrary dependence on the real parameter $t$. Of course, due to the linearity of the equation for $a_j$ in (5.1), any of the above behaviors can actually be obtained from a particular one. However we will see that it is quite useful to have the four evolutions of the spectral transform separately as the link between these cases can well be very complicated. Note also that to each of the above 4 cases there are two choices $\sigma = \pm$ for which the corresponding evolutions of the spectral transform will be seen to be quite different.
The above system will be understood as the equation for the two envelopes $a_1$ and $a_2$ of the two components of a high frequency field, interacting resonantly with a low frequency field of envelope $q$. The parameter $k$ actually measures a frequency mismatch and $g(k)$ measures the relative intensity of the coupling of the waves for each value of $k$. Hence $g(k)$ has to be positive, symmetric around the proper resonance $k = 0$ (even function of $k$) and of finite support. Consequently the system (5.1) is in three variables and is constituted of a differential eq. in the variable $x$ running on the infinite line, a differential eq. in the variable $t$ running on an arbitrary compact support and an integral eq. in the variable $k$ running on a given compact support (the support of the measure $g(k)$). Given $g(k)$, $q(x,0)$ and the boundary values for the $a_j$’s, the system is closed and the problem of solving it is well posed.

The proof of the integrability of (5.1) has been first given in [5] and follows a large series of papers devoted to the self-induced transparency (SIT) equations of McCall and Hahn [3] which were given a Lax pair and soliton solutions by Lamb [9]. The sharp line case was then shown to be completely integrable in [10] where the N-soliton solution is displayed. Then the inhomogeneously broadened case was also shown to be completely integrable in [11] where in particular the evolution of the continuous part of the spectrum has been obtained, which has allowed the authors to demonstrate mathematically that this simple model does contain the property of transparency (the radiation was shown to vanish exponentially as the input laser pulse propagates in the medium). Later, more general asymptotic boundary conditions (a subcase of (5.2) for $|I_1|^2 + |I_2|^2 = 1$) have been studied in [12] and shown to lead to interesting physical consequences. Simultaneously the scientific community became aware of the importance of this system as a fundamental model for resonant wave coupling processes in the slowly varying amplitude approximation (SVEA), role which is played by the nonlinear Schrödinger equation in the scalar case (no wave coupling). Indeed, the system (5.1) appears as a universal limit in many different physical systems, not only for the Maxwell-Bloch system in two-level one-dimensional media (the starting equations for SIT). It appears for instance in nonlinear optics to describe stimulated Raman scattering [13], in diatomic chains of coupled harmonic oscillators [14], in plasma physics for stimulated Brillouin scattering [15]... The point is that to each individual physical situation, there correspond different asymptotic boundary values [6]. Consequently, the integrability of (5.1) has not a unique common implication on the behaviour of the solution and, as we shall see, the properties of the solution can well be drastically different from a problem to another.

The integrability of the system (5.1) is proved in three steps. First we obtain a general integrable system of a structure similar to (5.1) for a particular choice of the entries $(N, \Lambda, M, \Omega)$ in theorem 1 (sec. 3). Second we consider some convenient reductions of the obtained evolution equation. Third the values of the arbitrary functions occurring in the chosen set $(N, \Lambda, M, \Omega)$ are determined.
in terms of the asymptotic boundary values. The proof will actually only be sketched for the details can be found in [6].

5.2 General integrable system

Let us choose

\( N = 0, \quad \Lambda = i k \sigma_3, \)  

\( M = \begin{pmatrix} 0 & m^-(k, t) \\ m^+(k, t) & 0 \end{pmatrix} \exp[2 i k \sigma_3 x], \)

(5.6)

\( \bar{\partial} \Omega = p(k, t) \sigma_3, \quad \mathcal{P}_k \{ \Omega \} = 0. \)

(5.7)

For that simple choice the compatibility constraint (3.9) holds (note that \( A = 0 \)) and the potentials \( U \) and \( V \) can be computed out of (3.5) (3.6) and read

\( U_0 = -i k \sigma_3 + i [\sigma_3, \mu^{(1)}], \quad U_1 = 0, \)  

(5.8)

\( V_0 = 0, \quad V_1 = \bar{\partial}^{-1} \{ \mu (M - \bar{\partial} \Omega) \mu^{-1} \}. \)

(5.9)

The 2 \( \times \) 2 matrix \( \mu(k, x, t) \) obeys the differential equation (3.15) namely here:

\( \mu_x = -i k [\sigma_3, \mu] + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \mu, \)

(5.10)

where we have defined

\( \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} = i [\sigma_3, \mu^{(1)}]. \)

Finally the evolution equation (3.7) of theorem 1 can be written

\( q_t = -\frac{1}{\pi} \int d k \wedge d \bar{k} (2 \mu_{11} \mu_{12} p + \mu_{11}^2 m^- e^{-2 ikx} - \mu_{12}^2 m^+ e^{2 ikx}), \)  

\( r_t = -\frac{1}{\pi} \int d k \wedge d \bar{k} (2 \mu_{21} \mu_{22} p + \mu_{21}^2 m^- e^{-2 ikx} - \mu_{22}^2 m^+ e^{2 ikx}), \)

(5.11)

(5.12)

(5.13)

where of course the \( \mu_{ij} \)'s are the matrix elements of \( \mu(k, x, t) \).

The equations (5.11) and (5.13) constitute a system of 6 coupled fields to which we have to associate the initial data \( (q(x, 0), \quad r(x, 0)) \) and some boundary values for the \( \mu_{ij} \)'s (for instance at \( x = \infty \)) for all \( t \).
5.3 Constraints

We shall work out three stages of simplifications of the system (5.11) (5.13). The first one is a result of the constraint on the space of functions to which belongs the initial datum (we call it a structural constraint). The second one is a constraint on the nature of the coupling term in the evolution equation to be solved (we call it a natural constraint). The last one is a reduction of the 6-field system to a 3-field one by assuming a simple relation between \( q \) and \( r \) (we call it a reduction constraint).

\[ R(k, x, t) = \frac{i}{2} \begin{pmatrix} 0 & -\alpha^-(k, t)\delta^-(k_I) \\ \alpha^+(k, t)\delta^+(k_I) & 0 \end{pmatrix} e^{2ik\sigma_3 x} + 2\pi \sum_{n=1}^{N^\pm} \left( C^+_n(t)\delta(k - k^+_n) - C^-_n(t)\delta(k - k^-_n) \right) e^{2ik\sigma_3 x} \]  

(5.14)

when we chose two basic solutions \( \mu^\pm(k, x, t) \) of (5.11) according to the following behaviors for \( k \) real

\[ \begin{pmatrix} \beta^+ & -e^{-2ikx}\alpha^-/\beta^- \\ 0 & 1/\beta^+ \end{pmatrix} \xrightarrow{x \to -\infty} \begin{pmatrix} \mu^+ & 0 \\ e^{2ikx}\alpha^+ & 1 \end{pmatrix} \xrightarrow{x \to +\infty} \begin{pmatrix} 1 & e^{-2ikx}\alpha^- \\ 0 & 1 \end{pmatrix}. \]  

(5.15)

\[ \begin{pmatrix} 1/\beta^- & 0 \\ -e^{2ikx}\alpha^+ / \beta^+ & \beta^- \end{pmatrix} \xrightarrow{x \to -\infty} \begin{pmatrix} \mu^- & 1 \\ e^{2ikx}\alpha^- & 0 \end{pmatrix} \xrightarrow{x \to +\infty} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

(5.16)

It should be clear that the behaviors at \( x = -\infty \) are actually a consequence of the choice of the behaviors at \( x = +\infty \). Here above the new coefficients \( \beta^\pm \) are explicitly defined from the datum of \( R \) in the appendix, we simply need here to remark that, from the property (5.12), we have the so called unitarity relation

\[ \alpha^+\alpha^- + \beta^+\beta^- = 1. \]  

(5.17)

The main constraint resulting from the above structure of the distribution \( R \), imposed on \( M \) to ensure self consistency of the equation (3.4), read

\[ m^\pm(k, t) = m^\pm_0(k, t)\delta^\pm(k_I) + \sum m^\pm_n(t)\delta(k - k^\pm_n). \]  

(5.18)

No such constraint applies on \( \Omega \) because it is a function multiplying \( R \).
ii) **Natural constraints**

Since we are interested in evolutions (5.1) where the integral on the r.h.s runs on the real axis only, it is readily seen on (5.13) that it is necessary to set first

\[ m_{\pm}^0(t) = 0. \]  
(5.19)

Then it will be necessary that \( p = \bar{\partial} \omega \) be proportional to the distributions \( \delta^\pm(k_I) \), hence

\[ p(k,t) = p_0^+(k_R,t)\delta^+(k_I) + p_0^-(k_R,t)\delta^-(k_I). \]  
(5.20)

Inserting now the set of constraints (5.18)(5.19)(5.20) into the evolution (5.13), we obtain the following system

\[ q_t = \frac{2i}{\pi} \int 2p_0^+ \mu_{11}^+ \mu_{12}^+ + 2p_0^- \mu_{11}^- \mu_{12}^- + m_0^- (\mu_{11}^-)^2 e^{-2ikx} - m_0^+ (\mu_{12}^+)^2 e^{2ikx}, \]

\[ r_t = \frac{2i}{\pi} \int 2p_0^+ \mu_{21}^+ \mu_{22}^+ + 2p_0^- \mu_{21}^- \mu_{22}^- + m_0^- (\mu_{21}^-)^2 e^{-2ikx} - m_0^+ (\mu_{22}^+)^2 e^{2ikx}, \]

where of course the matrix \( \mu = \{ \mu_{ij} \} \) solves (5.11), and where the notation \( \mu_{ij}^\pm \) is defined in (2.7). It turns out (see appendix) that indeed the functions \( \mu_{ij}^\pm \) do obey the behaviors (5.15) (5.16).

### iii) Reduction constraints

It is convenient for physical applications to look for reduction of the above 6-field system to a simpler one (actually a 3-field system). This is done by assuming the following reduction

\[ r(x,t) = \sigma \bar{q}(x,t), \quad \sigma = \pm \]  
(5.22)

for which from (5.11)

\[ \bar{\mu}_{11}^\pm = \mu_{22}^\pm, \quad \bar{\mu}_{12}^\pm = \sigma \mu_{21}^\pm, \]  
(5.23)

and consequently using (5.14) in (3.1)

\[ \alpha^+ = \sigma \bar{\alpha}^-, \quad \beta^+ = \bar{\beta}^-, \quad k_+^+ = k_-^- \quad C_+^n = \sigma C_0^- \]  
(5.24)

The evolutions (3.4) together with (5.21) then imply the following constraints

\[ m_0^+ (k,t) = \sigma \bar{m}_0^- (\bar{k},t), \quad p_0^+ (k,t) = -\bar{p}_0^- (\bar{k},t). \]  
(5.25)

It is convenient to define lighter notations through

\[ m_0 = m_0^+, \quad p_0 = p_0^+, \quad \alpha = \alpha^+, \quad \beta = \beta^+, \quad C_n = C_n^+, \quad k_n = k_n^+. \]  
(5.26)
Finally, considering all the simplifications resulting from the above three stages of constraints, the system (5.21)(5.11) reads
\[
q_t = 2i \pi \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k + i0} p_0^\dagger (\lambda, t) - 2i \pi \int_{-\infty}^{+\infty} \frac{d\lambda}{\lambda - k - i0} p_0 (\lambda, t),
\]
(5.27)
\[\mu_{11, x} = \sigma \mu_{21}, \quad \mu_{21, x} - 2i k \mu_{21} = \sigma \bar{\mu}_{11}, \quad \mu_{12}^\pm = \sigma \bar{\mu}_{21}^\mp.
\]

Since the relevant datum for the evolution of \(q(t, t)\) is the function \(p_0(k, t)\), we need to compute \(\omega(k, t)\) out of (5.20)
\[\omega(k, t) = -i(p_0 + \bar{p}_0) - \frac{1}{\pi} P \int \frac{d\lambda}{\lambda - k} (p_0 - \bar{p}_0).
\]
(5.29)

It results that \(\omega\) is purely imaginary which makes the constraints (5.24) compatible with the evolution (3.4).

5.4 Evolution of the spectral transform

To achieve the proof of integrability of the system (5.1) with arbitrary boundary values for the fields \((a_1, a_2)\), it is necessary to find the transformation from (5.1) to (5.27) by expanding the vector \((a_1, a_2)\) on a suitable basis constructed from the vectors \((\mu_{11}^\pm, \mu_{21}^\pm)\). To do that it is enough to compare their asymptotic boundary values (they both solve the same first order differential linear system), and hence, from (5.13) (5.16) we can deduce for each of the cases (5.4) (5.5)
\[
\begin{align*}
(a_1) & = I_1 \mu_1^+ + I_2 \mu_2^+ e^{2ikx}, \\
(a_2) & = K_1 \beta^+ \mu_1^+ + K_2 \beta^- \mu_2^- e^{2ikx}, \\
(a_1) & = J_1 \mu_1^+ + J_2 \mu_2^+ e^{2ikx}, \\
(a_2) & = L_1 \beta^+ \mu_1^+ + L_2 \beta^- \mu_2^- e^{2ikx}.
\end{align*}
\]
(5.30) (5.31) (5.32) (5.33)

Next it is necessary to compute the product \(a_1 \bar{a}_2\) appearing in the equation (5.1) and to express it in terms of the only four products appearing in (5.27).
namely $\mu^+_{11}\mu^+_{12}, \mu^-_{11}\mu^-_{12}, (\mu^-_{11})^2 e^{-2ikx}$ and $(\mu^+_{12})^2 e^{2ikx}$. This is possible by repeatedly using the Riemann-Hilbert problem for the function $\mu$, as shown in the appendix. Actually we discover there that the real part of $p_0$ can be set to zero. Once this is done, it is the sufficient to compare the equations (5.1) and (5.27) and to equate the different coefficients of the above mentionned terms. The result reads:

i) In the case (5.2) with (5.30):
\[
p_0 = -\frac{i\pi}{8} g(\sigma |I_1|^2 + |I_2|^2),
\]
\[
m_0 = \frac{i\pi}{2} g[\sigma I_1 I_2 - \frac{1}{2} \alpha (\sigma |I_1|^2 + |I_2|^2)].
\] (5.34)

ii) In the case (5.3) with (5.31):
\[
p_0 = -i\frac{\pi}{8} g[\sigma |J_1|^2 (1 + \sigma |\alpha|^2) + |J_2|^2 |\beta|^2 + J_1 \bar{J}_2 \alpha \bar{\beta} + \bar{J}_1 J_2 \alpha \beta],
\]
\[
m_0 = \frac{i\pi}{2} g[\sigma \bar{J}_1 J_2 \beta (1 - \frac{1}{2} \sigma |\alpha|^2) - \frac{1}{2} \bar{\beta}(\alpha)^2 J_1 J_2 + \frac{1}{2} \alpha |\beta|^2 (\sigma |J_1|^2 - |J_2|^2)].
\] (5.35)

iii) In the case (5.4) with (5.32):
\[
p_0 = -i\frac{\pi}{8} g[(\sigma |K_1|^2 + |K_2|^2) \frac{1 + \sigma |\alpha|^2}{1 - \sigma |\alpha|^2} + K_1 \bar{K}_2 \frac{\alpha}{\beta^2} + \bar{K}_1 K_2 \frac{\bar{\alpha}}{\bar{\beta}^2}],
\]
\[
m_0 = \frac{i\pi}{2} g[\sigma \bar{K}_1 K_2 \beta (\bar{\beta})^{-1} + \frac{1}{2} \alpha (\sigma |K_1|^2 + |K_2|^2)].
\] (5.36)

iv) In the case (5.5) with (5.33):
\[
p_0 = -i\frac{\pi}{8} g[(\sigma |L_1|^2 (1 - \frac{1 + \sigma |\alpha|^2}{1 - \sigma |\alpha|^2}) + |L_2|^2 + L_1 \bar{L}_2 \frac{\alpha}{\beta} + \bar{L}_1 L_2 \frac{\bar{\alpha}}{\bar{\beta}}],
\]
\[
m_0 = \frac{i\pi}{2} g[\sigma \bar{L}_1 L_2 (1 - \frac{1 - \frac{1}{2} \sigma |\alpha|^2}{\frac{1}{2} \sigma |\alpha|^2}) \bar{L}_1 L_2 + \frac{1}{2} \alpha (\sigma |L_1|^2 - |L_2|^2)].
\] (5.37)

With these values in hand, the evolution of the spectral transform defined in (5.14) is given by
\[
\alpha_t = 2\omega \alpha - 2im_0,
\]
\[
k_{n,t} = 0,
\]
\[
C_{n,t} = 2\omega(k_n)C_n,
\] (5.38)
where $\omega(k, t)$ is given from $p_0(k, t)$ in (5.29).

It is convenient also, for future use, to obtain the evolutions of the coefficient $\beta$ and of the quantity $|\alpha|^2$. From the definition of $\beta^\pm$ given in the appendix, it is easy to derive

$$\beta_t = -\beta \frac{1}{2i\pi} \int \frac{d\lambda}{\lambda - k - i0} \frac{(\sigma|\alpha|^2)_t}{1 - \sigma|\alpha|^2}$$

(5.39)

and from the above evolution (5.38)

$$((|\alpha|^2)_t = 2i(\bar{\alpha}_0\alpha - m_0\bar{\alpha}).$$

(5.40)

Remember finally that, within the reduction (5.22), the unitarity relation (5.17) becomes

$$\sigma|\alpha|^2 + |\beta|^2 = 1.$$  

(5.41)

5.5 Comments

We have shown here above that the time evolution (5.4) of the spectral transform can be computed explicitly in terms of the asymptotic boundary values (5.2)–(5.5) in the two cases $\sigma = \pm$, when $q(x, t)$ obeys the coupled system (5.1). As can be seen in the preceding subsection, these evolutions can be quite complicated and possibly not explicitly solvable. In such a case the property of integrability of the system (5.3) is weakened by the constraints of the boundary values. Still the spectral transform remains a quite useful tool as it maps the problem of the solution of (5.1) (3 variables with one on the infinite line) into the simpler problem of solution of (5.38) (2 variables both on compact support).

We have displayed in [6] a few applications of the present results for physical problems of resonant wave interaction. The important result to emphasize here is the strong dependence of the nature of the solution on the chosen boundary values. For instance, in the case of self-induced transparency, we obtain that $|\alpha|^2 \to 0$ as $t \to \infty$ and this is precisely the transparency property. For stimulated Brillouin scattering in plasmas we get $|\alpha|^2 \to 1$ as $t \to \infty$, and this is precisely the property of total reflexivity. We also give an example of special boundary values in stimulated Raman scattering for which $|\alpha|^2 \to \infty$ as $t \to t_s$ (finite), in that case the general solution blows up in finite time. There are other physical situations (under study now [16]) for which the time evolutions (5.38) (5.39) and (5.40) are not explicitly solvable. In that case, although the system possesses a Lax pair and a spectral transform, its integrability is partially destroyed by the constraints due to the boundary values. For instance, such a situation would be encountered here if considering the evolutions (5.38) (5.39) and (5.40) in the case (5.32) with $J_1 J_2 \neq 0$ and $J_i \text{ external}$ (not functionals of $\alpha, \beta...$).
As a last comment it is instructive to consider the case of self-induced transparency which corresponds to the choice

\[ \sigma = -, \quad K_1 = 1, \quad K_2 = 0 \]  

in (5.36). Hence we get

\[ p_0 = i \pi g \frac{1 - |\alpha|^2}{1 + |\alpha|^2}, \quad m_0 = -i \frac{\pi}{4} g \alpha, \]  

and the evolution (5.38) reads

\[ \alpha_t = -\pi g \alpha - \frac{1}{2} \alpha P \int d\lambda g \frac{1 - |\alpha|^2}{\lambda - k + i0}, \]  

We have also from (5.40)

\[ (|\alpha|^2)_t = -\pi g |\alpha|^2, \]  

which proves the transparency: \(|\alpha|^2\) vanishes exponentially as \(t \to \infty\). The interesting (curious) fact is that, originally in [11], the time evolution of the spectral transform has been written for the quantity

\[ \rho = \sigma \bar{\alpha} \beta(\bar{\beta})^{-1}, \]  

which is, in the language of the scattering theory, the reflection coefficient to the left. It turns out that, by using successively (5.44) (5.45) and (5.38), the evolution of \(\rho(k, t)\) can be written with help of (2.9):

\[ \rho_t = \rho i \int \frac{d\lambda}{\lambda - k + i0} g(\lambda). \]  

The point is that this evolution is linear while the equivalent one for \(\alpha\) is not. This property lead the people thinking that the situation for coupled systems was much the same as that of scalar systems, namely that the difference concerns only the dispersion relation which from polynomial becomes a singular function of \(k\). This has produced a lot of works [9] [10] [11] [12] [17] [18], including those of the author [2] [3] [4], where the linearity of the evolution of the spectral transform has been taken \textit{a priori}, which implicitly imposes constraints on the boundary values. We see here that the linearity of the evolution can occur in particular situations, but is never a general property. The first instance of a system of coupled waves corresponding to a nonlinear evolution of the spectral transform has been given by KAUP in [19] in the case when the spectral problem is the Schrödinger scattering problem. This problem has then been further studied in [20] and an extension to arbitrary boundary values is also feasible [21].
6  Appendix: spectral and $\bar{\partial}$ problems

6.1 Spectral problem and basic solutions

We briefly recall here the basic results on the Zakharov-Shabat spectral problem which we write for the $2 \times 2$ matrix $\mu(k, x, t)$

$$\mu_x + ik[\sigma_3, \mu] = Q\mu, \quad Q = \begin{pmatrix} 0 & q(x, t) \\ r(x, t) & 0 \end{pmatrix}.$$  \hfill (A.1)

Its two fundamental solutions, say $\mu^\pm$, are determined by (a $t$-dependence is understood everywhere)

$$\begin{pmatrix} \mu_{11}^+(k, x) \\ \mu_{21}^+(k, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} - \int_x^\infty d\xi q(\xi)\mu_{21}^+(k, \xi) \\ \int_x^\infty d\xi r(\xi)\mu_{11}^+(k, \xi)e^{2ik(x-\xi)} \end{pmatrix}$$  \hfill (A.2)

$$\begin{pmatrix} \mu_{12}^+(k, x) \\ \mu_{22}^+(k, x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \int_x^\infty d\xi q(\xi)\mu_{22}^+(k, \xi)e^{-2ik(x-\xi)} \\ \int_x^\infty d\xi r(\xi)\mu_{12}^+(k, \xi) \end{pmatrix}$$  \hfill (A.3)

$$\begin{pmatrix} \mu_{11}^-(k, x) \\ \mu_{21}^-(k, x) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} \int_x^\infty d\xi q(\xi)\mu_{21}^-(k, \xi)e^{2ik(x-\xi)} \\ \int_x^\infty d\xi r(\xi)\mu_{11}^-(k, \xi) \end{pmatrix}$$  \hfill (A.4)

$$\begin{pmatrix} \mu_{12}^-(k, x) \\ \mu_{22}^-(k, x) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \int_x^\infty d\xi q(\xi)\mu_{22}^-(k, \xi)e^{-2ik(x-\xi)} \\ -\int_x^\infty d\xi r(\xi)\mu_{12}^-(k, \xi) \end{pmatrix}$$  \hfill (A.5)

The first column vector $\mu_{11}^+$ of the matrix $\mu^+$ is meromorphic in $\text{Im}(k) > 0$ where it has a finite number $N^+$ of poles $k_n^+$ (assumed to be simple). The second vector $\mu_{21}^+$ is holomorphic in $\text{Im}(k) > 0$. The vector $\mu_{12}^+$ is holomorphic in $\text{Im}(k) < 0$ while the second one $\mu_{22}^+$ is meromorphic in $\text{Im}(k) < 0$ where it has a finite number $N^-$ of pole $k_n^-$ (simple). One can check directly on the above integral equations that we have the relations

$$\text{Res}_{k=k_n^+} \mu_{11}^+(k) = iC_n^+\mu_{12}^+(k_n^+),$$  \hfill (A.6)

$$\text{Res}_{k=k_n^-} \mu_{21}^-(k) = -iC_n^-\mu_{22}^-(k_n^-),$$  \hfill (A.7)

which define the normalization coefficients $C_n^\pm$.

The function $\mu(k)$ defined as $\mu^+$ in the upper half plane and $\mu^-$ in the lower is then discontinuous on the real $k$-axis. Its discontinuity can be expressed simply in terms of $\mu$ itself as

$$\mu_1^+ - \mu_1^- = e^{2ikx_0} \alpha^+ \mu_2^+, \quad (A.8)$$

$$\mu_2^+ - \mu_2^- = -e^{-2ikx_0} \alpha^- \mu_1^-,$$  \hfill (A.9)

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which define the reflection coefficients \( \alpha^{\pm}(k) \):

\[
\alpha^{+}(k) = \int_{-\infty}^{+\infty} d\xi \, r(\xi) \, \mu_{11}^{+}(k, \xi) e^{-2i k \xi}, \\
\alpha^{-}(k) = \int_{-\infty}^{+\infty} d\xi \, q(\xi) \, \mu_{22}^{-}(k, \xi) e^{2i k \xi}.
\]  

(A.10)

(A.11)

We can also define the transmission coefficients \( \beta^{\pm} \):

\[
\beta^{+}(k) = 1 - \int_{-\infty}^{+\infty} d\xi \, q(\xi) \, \mu_{21}^{+}(k, \xi), \\
\beta^{-}(k) = 1 - \int_{-\infty}^{+\infty} d\xi \, r(\xi) \, \mu_{12}^{-}(k, \xi).
\]  

(A.12)

(A.13)

The integral equations (A.2) to (A.5) give the following behaviors at large \( x \):

\[
\begin{pmatrix}
\beta^{+} & -e^{-2i k x} \alpha^{-} / \beta^{-} \\
0 & 1 / \beta^{+}
\end{pmatrix}
\xlongequal{\rightarrow}^{-\infty}_{-\infty} \mu^{+} \xlongrightarrow{\rightarrow}_{+\infty}^{+\infty}
\begin{pmatrix}
1 & 0 \\
e^{2i k x} \alpha^{+} & 1
\end{pmatrix}
\]  

(A.14)

\[
\begin{pmatrix}
1 / \beta^{-} & 0 \\
e^{-2i k x} \alpha^{+} / \beta^{+} & -\beta^{-}
\end{pmatrix}
\xlongequal{\rightarrow}^{-\infty}_{-\infty} \mu^{-} \xlongrightarrow{\rightarrow}_{+\infty}^{+\infty}
\begin{pmatrix}
1 & 0 \\
e^{-2i k x} \alpha^{-} & 1
\end{pmatrix}
\]  

(A.15)

where we have used that from (A.1), \( \det(\mu) \) is \( x \)-independent. Consequently we have also the unitarity relation

\[
\alpha^{+} \alpha^{-} + \beta^{+} \beta^{-} = 1.
\]  

(A.16)

To obtain the asymptotic behaviors \( -e^{-2i k x} \alpha^{-} / \beta^{-} \) of \( \mu_{12}^{+} \) and \( e^{-2i k x} \alpha^{+} / \beta^{+} \) of \( \mu_{12}^{-} \) as \( x \) goes to \(-\infty\), one must use the relations (A.8) and (A.9) repeatedly in (A.2) and (A.4) respectively.

Solving the direct scattering problem consists in solving for given \( Q(x) \) the integral equations (A.2) to (A.5) and then calculating the spectral data \( S \):

\[
S = \{ \alpha^{\pm}(k), k \in \mathbb{R}; k_{n}^{\pm}, C_{n}^{\pm}, n = 1, \ldots, N^{\pm} \}.
\]  

(A.17)

6.2 \( \bar{\partial} \)-problem and spectral transform

Since the work [22], we know that the solution of the inverse problem, i.e. the reconstruction of \( Q \) from \( S \), is given by a Cauchy-Green integral equation which solves a \( \bar{\partial} \)-problem for the function \( \mu(k) \) previously defined. Actually the \( \bar{\partial} \)-problem is simply the formula which summarizes the analytical properties of \( \mu(k) \), it reads

\[
\frac{\partial}{\partial k} \mu(k) = \mu(k) R(k)
\]  

(A.18)
where the distribution $R(k)$ is the spectral transform and is given from the spectral data $S$ by

$$R(k) = \frac{i}{2} \begin{pmatrix} 0 & -\alpha^-(k)\delta^-(k) \\ \alpha^+(k)\delta^+(k) & 0 \end{pmatrix} e^{2ik\sigma_3x} +$$

$$+ 2\pi \sum_{n=1}^{N^\pm} \begin{pmatrix} 0 & C_n^\delta(k - k_n^-) \\ C_n^\delta(k - k_n^+) & 0 \end{pmatrix} e^{2ik\sigma_3x}. \quad (A.19)$$

This formula can be demonstrated simply by noting that

$$k \in \mathbb{R} : \bar{\partial}_\mu(k) = \frac{i}{2} \mu(k)[\delta^+(k) - \delta^-(k)]$$

$$\text{Im} \ k > 0 : \bar{\partial}_\mu(k) = \left( -2i\pi \sum_{k_n^+} \delta(k - k_n^+) \text{Res} \mu_1^+(k) , 0 \right) \quad (A.20)$$

$$\text{Im} \ k < 0 : \bar{\partial}_\mu(k) = \left( 0 , 2i\pi \sum_{k_n^-} \delta(k - k_n^-) \text{Res} \mu_2^-(k) \right).$$

The above $\bar{\partial}$-problem is completed by the asymptotic behavior of $\mu$ at large $k$, which is obtained from $\text{(A.2)}$ to $\text{(A.3)}$ by integration by parts and reads

$$k \to \infty \Rightarrow \mu(k) = 1 + \mathcal{O}\left(\frac{1}{k}\right). \quad (A.21)$$

Finally the solution of the equation $\text{(A.18)}$ obeying the above behavior is obtained by solving the following Cauchy-Green integral equation:

$$\mu(k) = 1 + \frac{1}{2\pi i} \int \frac{d\lambda \wedge d\bar{\lambda}}{\lambda - k} \mu(\lambda)R(\lambda). \quad (A.22)$$

By comparison of the different powers of $1/k$ in the equation $\text{(A.1)}$, we readily obtain

$$Q = i[\sigma_3, \mu^{(1)}], \quad (A.23)$$

where we have defined $\mu^{(1)}$ through

$$\mu(k) = 1 + \sum_{1}^{\infty} k^{-n}\mu^{(n)}. \quad (A.24)$$

Therefore the inverse problem is solved by the integral equation $\text{(A.22)}$. 

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