Free products of digroups

Guangliang Zhang, Yuqun Chen and Zerui Zhang

Abstract. We construct the free products of arbitrary digroups, and thus we solve an open problem of Zhuchok.

Mathematics Subject Classification. 16S15, 20M75, 20E06.

Keywords. Disemigroup, Digroup, Free product.

1. Introduction

A digroup is a generalization of a group, and it first implicitly appeared in Loday’s work [6] as a dimonoid with additional conditions. Digroups play an important role in the theory of Leibniz algebras since it is the first approximation to the solution of the so called Coquecigrue problem on finding an appropriate generalization of Lie’s third theorem which associates a (local) Lie group to a Lie algebra. The definition of a digroup was proposed independently by Kinyon [4], Felipe [2], and Liu [5], to provide a partial solution to the Coquecigrue problem for Leibniz algebras. Kinyon made a good progress in terms of digroups to build a partial solution to the above Coquecigrue problem on Leibniz algebras [4, Corollary 5.7].

The study of algebraic properties of digroups has attracted considerable attention. A digroup is a set with two binary associative operations satisfying some additional conditions (Definition 2.2). Kinyon [4] showed that every digroup is a product of a group and a trivial digroup [4]. In [9], Phillips offered an equivalent definition for the variety of digroups. Salazar-Díaz et al. [11] studied a further generalization of the digroup structure and showed an analogous to the first isomorphism theorem. Ongay, Velásquez and Wills-Toro [8] discussed...
the notion of normal subdigroups and quotient digroups, and then established the corresponding analogues of the classical Isomorphism Theorems. In [14], Zhuchok and Zhuchok constructed some interesting digroups. For further investigation on digroups, we refer to Zhuchok [13], Felipe [2] and Liu [5]. One of the open problems raised by Zhuchok in [13] was how to construct the free product of digroups. Another problem on digroups proposed by Zhuchok in [13] was to construct a free digroup. This problem was solved by Zhang and Chen in [12] and for generalized digroups by Rodríguez-Nieto et al. in [10]. And our aim of this article is to construct free products of arbitrary digroups.

2. Free products of arbitrary digroups

We begin with the notion of disemigroups and dimonoids defined in [4]: A disemigroup \((D, \triangleright, \triangleleft)\) is a set \(D\) equipped with two binary operations \(\triangleright\) and \(\triangleleft\) such that the following conditions are satisfied:

\[
\begin{align*}
(D, \triangleright) \text{ and } (D, \triangleleft) \text{ are both semigroups,} \\
a \triangleleft (b \triangleright c) &= a \triangleright (b \triangleleft c), \\
(a \triangleleft b) \triangleright c &= (a \triangleright b) \triangleright c, \\
a \triangleright (b \triangleleft c) &= (a \triangleright b) \triangleleft c,
\end{align*}
\]

(2.1)

for all \(a, b, c \in D\). An element \(e\) in \(D\) is called a bar-unit if \(e \triangleright a = a = e \triangleleft a\) for every \(a \in D\). And a disemigroup \((D, \triangleright, \triangleleft)\) is called a dimonoid if \((D, \triangleright, \triangleleft)\) has a bar-unit. From now on, we shall simply say that \(D\) is a disemigroup if \((D, \triangleright, \triangleleft)\) is.

Now we recall an example of a dimonoid, where each element of the dimonoid is a bar-unit. Moreover, it is in fact a trivial digroup in the sense of [4].

**Example 2.1** [6, Example 1.3.b]. Let \(D = \{e_i \mid i \in I\}\), where \(I\) is an index set. Define \(e_i \triangleright e_j = e_j\) and \(e_j \triangleleft e_i = e_j\) for all \(i, j \in I\). Then \((D, \triangleright, \triangleleft)\) is a disemigroup and every \(e_i\) is a bar-unit of \(D\).

**Definition 2.2** ([4,2,1,5,9,14]). A dimonoid \(D\) with a fixed bar-unit \(e\) is called a digroup with respect to \(e\), denoted by \((D, e)\), if \(D\) satisfies the following condition:

\[(\forall a \in D, \exists a^{-1} \in D) \quad a \triangleright a^{-1} = e = a^{-1} \triangleleft a.\]

In this case, we call \(e\) the fixed bar-unit of the digroup \((D, e)\), and call \(a^{-1}\) the inverse of \(a\) (with respect to \(e\)).

A more precise notation for the inverse might be \(a^{-1}_e\), but since the fixed bar-unit is always clear whenever we write a digroup \((D, e)\), there is no harm to use the notation \(a^{-1}\) instead of \(a^{-1}_e\) for the inverse of \(a\).

In Example 2.1, if we fix an arbitrary element, say \(e_i \in D\), to be the bar-unit, then \((D, e_i)\) becomes a digroup and the inverse of an arbitrary element \(e_j \in D\) (with respect to \(e_i\)) is \(e_i\). It follows that the inverse of an element (with respect to a fixed bar-unit) depends on the bar-unit. We refer
to \([4,1,5,7,11,14]\) for more interesting properties of digroups. For instance, from [1] we easily deduce that the inverse of \(a\) with respect to \(e\) in a digroup \((D,e)\) is unique. We shortly repeat the proof for the convenience of the readers. Suppose that \(a\) and \(b\) are two inverses of \(e\). Then we have
\[
a = e \vdash a = (b \rhd c) \vdash a = b \vdash (c \rhd a) = b \rhd e
\]
Moreover, for an arbitrary digroup \((D,e)\), if \(e\) is the only bar-unit in \(D\), then the two operations \(\vdash\) and \(\rhd\) coincide and thus \((D,\vdash)\) is a group (see [4]). For the convenience of the readers, we briefly offer the main idea of a proof: For every \(a \in D\), since \(a^{-1} \vdash a\) is a bar-unit, we have \(a^{-1} \vdash a = e\) if \(e\) is the unique bar-unit. So we have \(a \vdash e = a \rhd a^{-1} \vdash a = a\) and thus \(a \vdash b = a \rhd b \rhd e = (a \rhd b) \rhd e = a \rhd b\) for all \(a,b \in D\).

Now we recall a useful result which describes the “group part” of a digroup. For the convenience of the readers, we quickly repeat the proof.

**Lemma 2.3** ([4]). Let \((D,e)\) be a digroup, and let \(J = \{a^{-1} \mid a \in D\}\) be the set of all inverses in \(D\) with respect to \(e\). Then we have \(J = \{a \vdash e \mid a \in D\}\). Moreover, \(J\) is a group with the unit \(e\) such that the operations \(\vdash\) and \(\rhd\) coincide in \(J\).

**Proof.** For every \(a \in D\), we have \(a^{-1} \vdash (a \vdash e) = (a^{-1} \vdash a) \vdash e = e \vdash e = e\) and
\[
(a \vdash e) \rhd a^{-1} = (a \vdash (a^{-1} \vdash a)) \rhd a^{-1} = ((a \vdash a^{-1}) \vdash a) \rhd a^{-1}
\]
\[
= (e \vdash a) \rhd a^{-1} = e \rhd (a \vdash a^{-1}) = e \rhd (a \vdash a^{-1}) = e.
\]
So we deduce \(a \vdash e = (a^{-1})^{-1} \in J\) and \(\{a \vdash e \mid a \in D\} \subseteq J\). On the other hand, for every \(a^{-1} \in J\), we have
\[
a^{-1} = e \vdash a^{-1} = (a^{-1} \vdash a) \vdash a^{-1} = a^{-1} \vdash (a \vdash a^{-1}) = a^{-1} \vdash e.
\]
Thus we obtain \(\{a \vdash e \mid a \in D\} = J\).

Now we show that the operations \(\vdash\) and \(\rhd\) coincide in \(J\). For all \(a \vdash e, b \rhd e \in J\), we have
\[
(a \vdash e) \rhd (b \rhd e) = (a \vdash b) \rhd e = (a \vdash b) \rhd (b^{-1} \rhd b) = (a \vdash (b \rhd b^{-1})) \rhd b
\]
\[
= (a \vdash e) \rhd (b \rhd e) = (a \vdash e) \rhd (b \rhd e).
\]
Clearly, \(e = e \vdash e \in J\). By the above reasoning, \(J\) is a group with the unit \(e\).

With the notation of Lemma 2.3, the set \(J\) is called the group part of \((D,e)\) and the set
\[
E = \{y \in D \mid y \vdash x = x \vdash y = x\text{ for every }x \in D\}
\]
is called the halo or bar-unit part of the digroup \((D,e)\) (see also [8, Theorem 1]).

Note that the fixed bar-unit \(e\) plays an important role when one describes a digroup. For instance, let \(D\) be the disemigroup in Example 2.1. If we let \(e_i\) be the fixed bar-unit, then \((D,e_i)\) is a digroup and the inverse of \(e_j\) is \(e_i\).
for every $j \in I$. In particular, the group part of $(D, e_i)$ is $\{e_i\}$ and the halo of $(D, e_i)$ is $D$.

Now we show that the product of an element in the group part and an element $b$ not in the group part does not lie in the group part if and only if the product "points to" $b$.

**Lemma 2.4.** Let $(D, e)$ be a digroup with the group part $J$. Then for every $a \in J$ and for every $b \in D \setminus J$, we have
(i) $a \vdash b \in D \setminus J$ and $b \vdash a \in D \setminus J$;
(ii) $a \vdash b \in J$ and $b \vdash a \in J$.

**Proof.** (i) If $a \vdash b \in J$, then we have $a^{-1} \vdash b = (a^{-1} \vdash a) \vdash b = a^{-1} \vdash (a \vdash b) \in J$, which is a contradiction. Thus we deduce $a \vdash b \in D \setminus J$. Similarly, we have $b \vdash a \in D \setminus J$.

(ii) By Lemma 2.3, we know that $\vdash$ and $\dashv$ coincide in $J$ and $e$ is the unit of the group $J$. So we have

$$a \vdash b = a \vdash (b \dashv e) = a \vdash (b \vdash e) = (a \vdash b) \vdash e \in J$$

and

$$b \vdash a = b \vdash (a \vdash e) = (b \vdash a) \vdash e \in J.$$ 

The proof is completed. $\square$

Our aim in this article is to construct the free products of arbitrary digroups. So we first recall some necessary notion and recall the definition of the free product of digroups. A map $\varphi$ from a disemigroup $D_1$ to a disemigroup $D_2$ is called a disemigroup homomorphism if we have $\varphi(a \vdash b) = \varphi(a) \vdash \varphi(b)$ and $\varphi(a \dashv b) = \varphi(a) \dashv \varphi(b)$ for all $a, b \in D_1$. Let $(D_1, e_1)$ and $(D_2, e_2)$ be two digroups. A disemigroup homomorphism $\varphi$ from $D_1$ to $D_2$ is called a digroup homomorphism from $(D_1, e_1)$ to $(D_2, e_2)$ if $\varphi(e_1) = e_2$.

From the definitions above it is plain that digroups form a pointed category: the trivial group, regarded as a digroup, is clearly a zero object (see [11]).

**Definition 2.5.** Let $\{(D_i, e_i) \mid i \in I\}$ be a family of indexed digroups, and let $(D, e)$ be a digroup such that

(i) there exists an injective digroup homomorphism $\varphi_i : (D_i, e_i) \rightarrow (D, e)$ for every $i \in I$;
(ii) if $(D', e')$ is a digroup, and if there exists a digroup homomorphism $\psi_i : (D_i, e_i) \rightarrow (D', e')$ for every $i \in I$, then there exists a unique digroup homomorphism $\eta : (D, e) \rightarrow (D', e')$ such that the diagram

$$\begin{array}{ccc}
(D_i, e_i) & \xrightarrow{\varphi_i} & (D, e) \\
\Psi_i \downarrow & & \downarrow \eta \\
(D', e') & & 
\end{array}$$

commutes for every $i \in I$. 


Then \((D, e)\) is called a free product of \(\{(D_i, e_i) | i \in I\}\).

Clearly, if a free product of digroups \(\{(D_i, e_i) | i \in I\}\) exists, then it is unique up to isomorphism. In general, when one constructs the free products of some objects, say digroups \(\{(D_i, e_i) | i \in I\}\), one can always assume that the digroups are pairwise disjoint.

**Lemma 2.6.** Let \(\{(D_i, e_i) | i \in I\}\) be a family of indexed digroups, and let \(\{(D'_i, e'_i) | i \in I\}\) be another family of indexed digroups such that we have \((D_i, e_i) \cong (D'_i, e'_i)\) for all \(i \in I\). Let \((D, e)\) be the free product of \(\{(D'_i, e'_i) | i \in I\}\). Then \((D, e)\) is the free product of \(\{(D_i, e_i) | i \in I\}\).

**Proof.** Assume that \(\tau_i\) is a digroup isomorphism from \((D_i, e_i)\) to \((D'_i, e'_i)\) for every \(i \in I\). Since \((D, e)\) is the free product of \(\{(D'_i, e'_i) | i \in I\}\), there exists an injective digroup homomorphism \(\varphi'_i : (D'_i, e'_i) \rightarrow (D, e)\) for every \(i \in I\). Define

\[\varphi_i = \varphi'_i \circ \tau_i\]

for every \(i \in I\). It follows immediately that there exists an injective digroup homomorphism \(\varphi_i : (D_i, e_i) \rightarrow (D, e)\) for every \(i \in I\).

\[\begin{align*}
(D_i, e_i) & \xrightarrow{\tau_i} (D'_i, e'_i) & \xrightarrow{\varphi'_i} & (D, e) \\
\psi_i & \downarrow & \downarrow & \downarrow \eta \\
(D', e') & \rightarrow & (D, e) & \rightarrow & (D, e) \\
 & \downarrow & \downarrow \psi_i \circ \tau_i^{-1} & \downarrow \eta & \text{If } (D', e') \text{ is a digroup and if there exists a digroup homomorphism } \psi_i : (D_i, e_i) \rightarrow (D', e') \text{ for every } i \in I, \text{ then } \psi_i \circ \tau_i^{-1} \text{ is a digroup homomorphism from } (D'_i, e'_i) \text{ to } (D', e') \text{ for every } i \in I. \text{ Again, since } (D, e) \text{ is the free product of } \{(D'_i, e'_i) | i \in I\}, \text{ there exists a unique digroup homomorphism } \eta : (D, e) \rightarrow (D', e') \text{ such that we have } \\
\eta \circ \varphi'_i = \psi_i \circ \tau_i^{-1}, \ i \in I.
\end{align*}\]

Therefore, we have

\[\eta \circ \varphi_i = \eta \circ \varphi'_i \circ \tau_i = \psi_i \circ \tau_i^{-1} \circ \tau_i = \psi_i, \ i \in I.\]

It remains to prove the uniqueness of \(\eta\). Suppose that there exists a digroup homomorphism \(\eta'\) from \((D, e)\) to \((D', e')\) such that we have \(\eta' \circ \varphi_i = \psi_i\) for every \(i \in I\). Then we have

\[\eta' \circ \varphi'_i \circ \tau_i = \eta' \circ \varphi_i = \psi_i\]

and thus

\[\eta' \circ \varphi'_i = \psi_i \circ \tau_i^{-1}.\]

Finally, since \((D, e)\) is the free product of \(\{(D'_i, e'_i) | i \in I\}\), we deduce that \(\eta' = \eta\), and the proof is completed. \(\square\)
For every \( i \in I \), we may set \( \overline{D_i} = \{(a, i) | a \in D_i\} \), where the operations on \( (\overline{D_i}, (e_i, i)) \) are introduced naturally by those of \( (D_i, e_i) \). For instance, we have \( (a, i) \triangleright (b, i) = (a \triangleright b, i) \) for all \( a, b \in D_i \). For every \( i \in I \), clearly we have \( (D_i, e_i) \cong (\overline{D_i}, (e_i, i)) \) and \( \overline{D_i} \cap \overline{D_j} = \emptyset \) for all distinct \( i, j \in I \).

By the above reasoning and by Lemma 2.6, we can assume that the digroups under consideration are pairwise disjoint, but then the construction of their free product will become lengthy. The situation for constructing the free products of arbitrary digroups generalizes those for groups, and so we generalize the method used for the construction of the free products of arbitrary groups, where a common unit is assumed. It is natural to assume that the digroups under consideration have a common bar-unit: Suppose that \( (D, e) \) have \( (D_i, e_i) \) for every \( i \in I \). Then there exists an injective digroup homomorphism \( \varphi_i : (D_i, e_i) \rightarrow (D, e) \) for every \( i \in I \). In particular, we have \( (D_i, e_i) \cong \varphi_i((D_i, e_i)) \) and \( \varphi_i(e_i) = e \). Therefore, we may assume \( (D'_i, e) = \varphi_i((D_i, e_i)) \). Then we obtain that

\[
(D'_i, e) \cong (D_i, e_i).
\]

Finally, by Lemma 2.6, we know that the free product of \( \{(D'_i, e) | i \in I\} \) is the free product of \( \{(D_i, e_i) | i \in I\} \). Therefore, to simplify the formulas in the article, we assume that they have a common fixed bar-unit. To fix the notations, we state the following remark.

**Remark 2.7.** From now on, we assume that \( \{(D_i, e_i) | i \in I\} \) is a family of disjoint digroups and we shall identify \( (D_i, e_i) \) with \( (((D_i \setminus \{e_i\}) \cup \{e\}, e) \) for every \( i \in I \) with the obvious multiplications, where \( e \) is a symbol that is not in \( \bigcup_{i \in I} D_i \). We denote by \( (D'_i, e) \) the digroup \( (((D_i \setminus \{e_i\}) \cup \{e\}, e) \) and denote by \( J_i \) the group part of \( (D'_i, e) \). Then \( (D_i, e_i) \) and \( (D'_i, e) \) are isomorphic as digroups. We shall construct the free product of \( \{(D'_i, e) | i \in I\} \) in Theorem 2.16. Now we fix the notations \( D \) and \( J \) defined by

\[
D = \bigcup_{i \in I} D'_i \text{ and } J = \bigcup_{i \in I} J_i.
\]

Let \( J^+ \) (resp. \( J^* \)) be the free semigroup (resp. monoid) generated by \( J \). Note that \( J_i \) is a group for every \( i \in I \), and for all distinct \( i, t \in I \), we have \( J_i \cap J_t = \{e\} \). Denote by \( \Pi^*_{i \in I} J_i \) the free product of the groups \( \{J_i | i \in I\} \). Moreover, by the construction of the free products of groups [3, Section 2.9], we can consider \( J_i \) as a subgroup of \( \Pi^*_{i \in I} J_i \) for every \( i \in I \).

Now we introduce two special kinds of words over \( D \), which resemble the elements in the free product of semigroups or groups.

**Definition 2.8.** For every \( a \in D \setminus \{e\} \), there exists a unique index \( i \in I \) such that \( a \in D_i \), we refer to \( i \) as the index of \( a \), and denote it by \( \sigma(a) \). A word \( u \) is called a good word over \( D \), if \( u = e \), or if \( u = c_1c_2 \ldots c_n, n \geq 1 \), each \( c_i \) lies in \( D \setminus \{e\} \) and \( \sigma(c_p) \neq \sigma(c_{p+1}) \) for every integer \( p \leq n - 1 \). Denote by \( W(D) \) the set of all good words over \( D \). A word \( u \) (over \( D \)) is called a reduced word in \( J^+ \) if \( u \) lies in \( W(D) \cap J^+ \).
We also note that, the set of all reduced words in $J^+$ forms a set of normal forms of elements in $\Pi_{i \in I} J_i$. And this is why we use the name “reduced words”. More precisely, let $\theta$ be a semigroup homomorphism from $J$ to $\Pi_{i \in I} J_i$ such that $\theta(a) = a$ for every $a \in J$. Then clearly $\theta$ is a surjective homomorphism. So there exists a congruence $\rho$ on $J^+$ such that we have $J^+/\rho \cong \Pi_{i \in I} J_i$. By the construction of the free products of arbitrary groups, for every word $u \in J^+$, there exists a unique reduced word $\bar{u}$ in $J^+$ such that we have $u \rho = \bar{u} \rho$ in $J^+/\rho$. For instance, let $a \in J_1$, $b \in J_2 \setminus \{e\}$ such that $aa = c \neq e$ in $J_1$. Then we have $aab = cb$. In particular, if $u$ is a reduced word, then we have $u = \bar{u}$ in $J^+$. Moreover, it follows that $\bar{u}$ is a map from $J^+$ to $J^+$ such that for all $u, v \in J^+$, we have

$$ u \bar{v} = \bar{uv} = \bar{u} \bar{v} \text{ and } \bar{e}v = \bar{ue} = u. $$

Moreover, for all $a, b \in D_i$, we have $(a \vdash b) \vdash e = (a \vdash b) \vdash e = (a \vdash e) \vdash (b \vdash e)$ in $J_i$. So it follows that $(a \vdash b) \vdash e = (a \vdash e)(b \vdash e)$ in $J^+$. 

Now we introduce a map $\sharp$ from $D^*$ to $D^*$, where $D^*$ is the free monoid generated by the set $D$. For every $a \in D$, we define

$$ a^\sharp = a \vdash e. $$

Note that, if $a \in D_i$, then $a \vdash e$ means the product in $D'_i$ and we have $a \vdash e \in J_i$. Since the sets $D'_i \setminus \{e\}$ and $D'_j \setminus \{e\}$ are disjoint for all $i \neq j \in I$, the notation $a \vdash e$ makes sense and $a \vdash e$ means an element in $D'_i$. By Lemma 2.3, we have $a^\sharp = a \vdash e \in J \subseteq J^+$. Moreover, for every word $u = a_1a_2 \ldots a_m \in D^+$, where $D^+$ is the free semigroup generated by $D$ and every $a_t$ lies in $D$, we define

$$ u^\sharp = a_1^\sharp a_2^\sharp \ldots a_m^\sharp $$

to be an element in $J^+ \subseteq D^*$. Clearly, $J^+$ is a subsemigroup of $D^*$. Then we obtain that $u^\sharp = a_1^\sharp a_2^\sharp \ldots a_m^\sharp$ is a reduced word in $J^+$. Because $e_i^\prime \vdash e = e$ for all bar-unit $e_i^\prime \in D'_i$ and $i \in I$, it is natural to extend $\sharp$ by defining

$$ e^\sharp = e, $$

where $e$ is the empty word. Then $\sharp$ becomes a map from $D^*$ to $D^*$ such that the image of $D^*$ is contained in $J^+$.

The following result is an obvious corollary of Lemma 2.3, and thus the proof is omitted.

**Lemma 2.9.** The map $\sharp$ is a semigroup homomorphism from $D^*$ to $D^*$ such that for every $a \in J$, we have $a^\sharp = a$. Moreover, for every $u \in D^*$, we have $(u^\sharp)^\sharp = u^\sharp \in J^+$.

Recall that for every disemigroup $M$, for all $a_1, \ldots, a_t \in M$, every parenthesizing of

$$ a_1 \vdash a_2 \ldots \vdash a_m \vdash \ldots \vdash a_t $$

gives the same element in $M$ [6]. In light of this, we define a formal expression $u \bar{a} \bar{v}$ as follows: for all reduced words $u, v \in J^+$ and for every $a \in D$,
say $u = a_1 \ldots a_m$ and $v = b_1 \ldots b_n$, where all $a_1, \ldots, a_m, b_1, \ldots, b_n$ lie in $J$, $m \geq 0$ and $n \geq 0$, we define

$$u \dot{a} v = a_1 \ldots a_m \dot{a} b_1 \ldots b_n,$$

where if $m$ (resp. $n$) is 0, that is, when $u$ (resp. $v$) is the empty word, it means that $u$ (resp. $v$) does not appear. In particular, if $u$ and $v$ are both the empty word, then by $u \dot{a} v$ we mean $\dot{a}$. Therefore, for all words $u, v \in D^*$ and for every $a \in D$, the formula $u^* \dot{a} v^*$ makes sense.

For the moment, the formula $a_1 \ldots a_m \dot{a} b_1 \ldots b_n$ is just a formal expression that does not lie in any digroup. But we shall construct a set consisting of some of such elements and define certain products on the elements of the set to make it a digroup such that we have

$$a_1 \ldots a_m \dot{a} b_1 \ldots b_n = a_1 \vdash \ldots \vdash a_m \vdash a \vdash b_1 \vdash \ldots \vdash b_n.$$

Suppose that we already have the free product of $\{(D'_i, e) \mid i \in I\}$. Then for all $a_i \in D'_i$ and $a_j \in D'_j$, in the free product, we have

$$a_i \vdash a_j = (a_i \vdash e) \vdash a_j = (a_i \vdash e) \vdash a_j = a_i^* \vdash a_j$$

and

$$a_i \vdash a_j = a_i \vdash (a_j \vdash e) = a_i \vdash (a_j \vdash e) = a_i \vdash a_j^*.$$

Therefore, for an arbitrary element $a_1 \vdash \ldots \vdash a_m \vdash a \vdash b_1 \vdash \ldots \vdash b_n$ in the free product of $\{(D'_i, e) \mid i \in I\}$, we may always assume $a_p \in J$ and $b_t \in J$ for all $p \leq m$ and $t \leq n$. This motivates us to introduce the notion of good center-words as follows. Recall that $W(D)$ is the set of all good words over $D$ (Definition 2.8). We call $u \dot{a} v$ a good center-word over $D$ if we have

(i) $u, v \in J^*$, $a \in D$, $uav \in W(D)$;
(ii) if $a \in J$, then $u$ is the empty word.

Denote by $CW(D)$ the set of all good center-words over $D$, that is,

$$CW(D) = \{u \dot{a} v \mid u, v \in J^*, a \in D, uav \in W(D); u = \varepsilon \text{ if } a \in J\}. \quad (2.2)$$

By the definition of $W(D)$, it follows that if $u \dot{a} v$ lies in $CW(D)$ and $a = e$, then we have $u = v = \varepsilon$ and thus $u \dot{a} v = \dot{a}$. Now we introduce an extended reduction rule on certain “center-words” as follows. For all reduced words $u = a_1 \ldots a_m, v = b_1 \ldots b_n \in J^+$, where all $a_1, \ldots, a_m, b_1, \ldots, b_n$ lie in $J$, and for every $a \in D \setminus J$, we define
\[
[u \circ v] = \begin{cases} 
  u \circ v, & \text{if } a_m, a \text{ do not lie in the same set } D'_i \text{ for any } i \in I, \text{ and } a, b_1 \text{ do not lie in the same set } D'_j \text{ for any } j \in I; \\
  a_1 \ldots a_{m-1} c v, & \text{if } a_m, a \in D'_i \text{ for some index } i \in I, \text{ and } a_m \vdash a = c \in D'_i, \text{ } b_1 \notin D'_i; \\
  u \circ c b_2 \ldots b_n, & \text{if } a, b_1 \in D'_i \text{ for some index } i \in I, \text{ and } a \vdash b_1 = c \in D'_i, \text{ } a_m \notin D'_i; \\
  a_1 \ldots a_{m-1} c b_2 \ldots b_n, & \text{if } a_m, a, b_1 \in D'_i \text{ for some index } i \in I, \\
  \text{and } (a_m \vdash a) \vdash b_1 = c \in D'_i. 
\end{cases}
\]

By Lemma 2.4, it follows that \([u \circ v]\) is a good center-word in \(CW(D)\). Finally, for every reduced word \(u = a_1 \ldots a_m \in J^+, \text{ where } a_1, \ldots, a_m \text{ lie in } J\), we define

\[ [u] = \hat{a}_1 \ldots a_m. \]

In particular, for every word \(u \in D^*\), the formula \([u^\sharp]\) makes sense.

The next lemma shows that the extended reduction rule is compatible with the reduction rule in \(\Pi_i^*\) in the following sense.

**Lemma 2.10.** For all reduced words \(u = a_1 \ldots a_m \) and \(v = b_1 \ldots b_n \in J^+\), where all \(a_i, b_j \) lie in \(J\), \(m \geq 0\) and \(n \geq 0\), for every \(a \in D \setminus J\), if \([u \circ v] = c_1 \ldots c_p \circ c_d_1 \ldots d_q\), then we have \(a_1^\sharp \ldots a_m^\sharp a_1^\sharp \ldots a_m^\sharp \circ c_1 \ldots c_p^\sharp c_d_1^\sharp \ldots d_q^\sharp\).

**Proof.** For every index \(i \in I\), we have the following three observations:

(i) If \(a_m \vdash a = c \text{ in } D'_i\), then since by Lemma 2.3 the operations \(\vdash \) and \(\dvdash\) coincide in \(J_i\), we have

\[ c^\sharp = (a_m \vdash a) \vdash e = (a_m \vdash e) \vdash (a \vdash e) = a_m^\sharp a_1^\sharp. \]

(ii) If \(a \dvdash b_1 = c \text{ in } D'_i\), then we have

\[ c^\sharp = (a \dvdash b_1) \vdash e = (a \vdash e) \dvdash (b_1 \vdash e) = a^\sharp b_1^\sharp. \]

(iii) If \((a_m \vdash a) \dvdash b_1 = c \text{ in } D'_i\), then we have

\[ c^\sharp = ((a_m \vdash a) \dvdash b_1) \vdash e = (a_m \vdash e) \dvdash (a \vdash e) \dvdash (b_1 \vdash e) = a_m^\sharp a_1^\sharp b_1^\sharp. \]

The proof of the lemma follows immediately by the definition of \([u \circ v]\). \(\square\)

With the above notations, we can define new operations \(\vdash\) and \(\dvdash\) on the set of all good center-words over \(D\) in Definition 2.11. Since the operations extend those of \(D'_i\) for every \(i \in I\), we use the same symbol \(\vdash\) and \(\dvdash\) but not \(\vdash'\), \(\dvdash'\). We shall prove that the following definition turns out to be a way of constructing the free products of arbitrary digroups in Theorem 2.16.

**Definition 2.11.** With the notion of Remark 2.7, let \(CW(D)\) be the set of all good center-words over \(D\) defined in (2.2). We define two operations \(\vdash\) and \(\dvdash\) on \(CW(D)\) as follows. For all \(u_1 \hat{a}_1 v_1, u_2 \hat{a}_2 v_2 \in CW(D)\), we define

\[ u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2 = \begin{cases} 
  [u_1^\sharp a_1^\sharp v_1^\sharp u_2^\sharp \hat{a}_2^\sharp v_2^\sharp], & \text{if } a_2 \in D \setminus J; \\
  [u_1^\sharp a_1^\sharp v_1^\sharp u_2^\sharp a_2^\sharp v_2^\sharp], & \text{if } a_2 \in J, 
\end{cases} \]
and
\[ u_1 \hat{a}_1 v_1 + u_2 \hat{a}_2 v_2 = \begin{cases} \{ u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2^2 \}, & \text{if } a_1 \in D \setminus J; \\ \{ u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2^2 \}, & \text{if } a_1 \in J. \end{cases} \]

Before showing that \((CW(D), \vdash, \dashv)\) is a digroup, we first establish several useful properties of \(\vdash\) and \(\dashv\). Now we show that, when the product “points to” an element, we may assume that the other element is of certain form.

**Lemma 2.12.** With the notation of Definition 2.11, for all \(u_1 \hat{a}_1 v_1, u_2 \hat{a}_2 v_2 \in CW(D)\), we have \(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2 = [u_1^2 \hat{a}_1 v_1^2] \vdash u_2 \hat{a}_2 v_2\) and \(u_1 \hat{a}_1 v_1 \dashv u_2 \hat{a}_2 v_2 = u_1 \hat{a}_1 v_1 + [u_2^2 a_2^2 v_2^2]\).

**Proof.** We just prove \(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2 = [u_1^2 \hat{a}_1 v_1^2] \vdash u_2 \hat{a}_2 v_2\) because the other one can be proved similarly. If \(a_2 \in J\), then we obtain
\[ u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2 = [u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2] = [u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2^2] = [u_1^2 \hat{a}_1 v_1^2] \vdash u_2 \hat{a}_2 v_2. \]

If \(a_2 \notin J\), then we have
\[ u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2 = [u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2] = [u_1^2 \hat{a}_1 v_1^2 u_2^2 a_2^2 v_2^2] = [u_1^2 \hat{a}_1 v_1^2] \vdash u_2 \hat{a}_2 v_2. \]

The proof is completed. \(\square\)

The next two lemmas show that the extended reduction rule is compatible with the operations \(\vdash\) and \(\dashv\).

**Lemma 2.13.** For all reduced words \(u, v \in J^+\), and for every \(a \in D \setminus J\), we have \([uv\hat{a}] = [u] \vdash [v\hat{a}]\) and \([\hat{a}uv] = [\hat{a}u] \dashv [v]\). Moreover, for every reduced word \(w \in J^+\), if \(aw\) is a good word or if \(w = e\), then we have
\[ [uv\hat{a}w] = [u] \vdash [v\hat{a}w]; \]
and if \(wa\) is a good word or if \(w = e\), then we have
\[ [w\hat{a}uv] = [w\hat{a}u] \vdash [v]. \]

**Proof.** We first show \([uv\hat{a}] = [u] \vdash [v\hat{a}]\). Assume that \(u = a_1 \ldots a_m, v = b_1 \ldots b_n\), where all \(a_p, b_q\) lie in \(J\) and \(m, n\) are positive integers. There are several cases to consider:

Case 1: If \(a, b_n \notin D'_j\) for any \(j \in I\), then by Definition 2.11, we have \([u] \vdash [v\hat{a}] = \hat{a}_1 \ldots a_m \vdash v\hat{a} = [uv\hat{a}]\).

Case 2: If \(a, b_n \in D'_j\) for some \(j \in I\) such that \(b_n \vdash a = b\) in \(D'_j\), and if \(uv = ub_1 \ldots b_{n-1} b_n\) as words in the free semigroup \(J^+\), then by Definition 2.11, we have
\[ [u] \vdash [v\hat{a}] = \hat{a}_1 \ldots a_m \vdash b_1 \ldots b_{n-1} \hat{b} = [ub_1 \ldots b_{n-1} b_n] = [uv\hat{a}]. \]

Case 3: If \(a, b_n \in D'_j\) for some \(j \in I\) such that \(b_n \vdash a = b\) in \(D'_j\), and if \(uv \neq ub_1 \ldots b_{n-1} b_n\) as words in the free semigroup \(J^+\), then we deduce
\[ a_1 \ldots a_m b_1 \ldots b_{n-1} = a_1 \ldots a_{m-n} a_{m-n+1} \]
as words in the free semigroup $J^+$, and we have $a_{m-n+1}, b_n, a \in D'_i$. Moreover, for every $0 \leq k \leq n-2$, we have $b_{k+1} = a_{m-k}^{-1} \in J_{i_k}$ for some $i_k \in I$.

If $b_n = a_{m-n+1}^{-1}$, then we have $uv = a_1 \ldots a_{m-n}$ and

$$a_{m-n+1} \vdash b = a_{m-n+1} \vdash (b_n \vdash a) = (a_{m-n+1} \vdash b_n) \vdash a = e \vdash a = a.$$

So we deduce

$$[u] \vdash [v\hat{a}] = \hat{a}_1 \ldots a_m \vdash b_1 \ldots b_{n-1} \hat{b} = [a_1 \ldots a_m b_1 \ldots b_{n-1} \hat{b}] = [a_1 \ldots a_{m-n} a_{m-n+1} \hat{b}] = [a_1 \ldots a_{m-n} \hat{a} \hat{c}] = [uv\hat{a}].$$

If $b_n \neq a_{m-n+1}^{-1}$, then we have $a_{m-n+1} \vdash b_n = c \neq e$. Assume $c \vdash a = d$. Then we have $uv = a_1 \ldots a_{m-n} c$ in the free semigroup $J^+$ and

$$a_{m-n+1} \vdash b = a_{m-n+1} \vdash (b_n \vdash a) = (a_{m-n+1} \vdash b_n) \vdash c \vdash a = d.$$

So we deduce

$$[u] \vdash [v\hat{a}] = \hat{a}_1 \ldots a_m \vdash b_1 \ldots b_{n-1} \hat{b} = [a_1 \ldots a_m b_1 \ldots b_{n-1} \hat{b}] = [a_1 \ldots a_{m-n} \hat{a} \hat{c}] = [uv\hat{a}].$$

Similarly, we can show $[\hat{a}uv] = [\hat{a}u] \vdash [v] = [u] \vdash [\hat{a}v]$.

As for what remained, if $w = e$, then we have $[uaw] = [\hat{a}u]$ and $[uw\hat{a}] = [\hat{a}u]$. So the claim follows by the above proof; if $w \neq e$, then by Lemma 2.4, what remained can be proved similarly by noting that $w$ is never involved in the above reasoning on the extended reduction rule.

**Lemma 2.14.** For all reduced words $u, v \in J^+$, and for every $a \in D \setminus J$, we have $[uav] = [u\hat{a}] \vdash [v] = [u] \vdash [\hat{a}v]$.

**Proof.** By the definition of $[uav]$ and by Lemma 2.4, we clearly have $[uav] = [u\hat{a}] \vdash [v]$. Similarly, by the fact that $(b \vdash c) \vdash d = b \vdash (c \vdash d)$ for all $b, c, d$ in a disemigroup, we easily deduce $[uav] = [u] \vdash [\hat{a}v]$. 

Now we show that $CW(D)$ is a disemigroup with respect to the operations defined in Definition 2.11.

**Lemma 2.15.** For all $u_i \hat{a}_i v_i \in CW(D)$, $1 \leq i \leq 3$, we have

(i) $(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2) \vdash u_3 \hat{a}_3 v_3 = (u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2) \vdash u_3 \hat{a}_3 v_3$;

(ii) $u_1 \hat{a}_1 v_1 \vdash (u_2 \hat{a}_2 v_2 \vdash u_3 \hat{a}_3 v_3) = u_1 \hat{a}_1 v_1 \vdash (u_2 \hat{a}_2 v_2 \vdash u_3 \hat{a}_3 v_3)$;

(iii) $(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2) \vdash u_3 \hat{a}_3 v_3 = u_1 \hat{a}_1 v_1 \vdash (u_2 \hat{a}_2 v_2 \vdash u_3 \hat{a}_3 v_3)$;

(iv) $(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2) \vdash u_3 \hat{a}_3 v_3 = u_1 \hat{a}_1 v_1 \vdash (u_2 \hat{a}_2 v_2 \vdash u_3 \hat{a}_3 v_3)$;

(v) $(u_1 \hat{a}_1 v_1 \vdash u_2 \hat{a}_2 v_2) \vdash u_3 \hat{a}_3 v_3 = u_1 \hat{a}_1 v_1 \vdash (u_2 \hat{a}_2 v_2 \vdash u_3 \hat{a}_3 v_3)$.

In particular, $CW(D)$ is a disemigroup.

**Proof.** We first prove Point (i). For every $\delta \in \{\vdash, \vdash\}$, assume

$$(u_1 \hat{a}_1 v_1)\delta(u_2 \hat{a}_2 v_2) = u\hat{av}.$$ 

Then by Lemma 2.10 and by Definition 2.11, we have

$$u^\sharp a^\sharp v^\sharp = u_1^\sharp a_1^\sharp v_1^\sharp u_2^\sharp a_2^\sharp v_2^\sharp.$$
and thus
\[ [u^xa^v^y] = [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y]. \]

So by Lemma 2.12, we deduce
\[
(u_1^a1^v_1)\delta(u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3 = u\dot{a}v \vdash u_3^a\dot{a}_3^v_3
\]
\[
= [u^xa^v^y] \vdash u_3^a\dot{a}_3^v_3 = [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y] \vdash u_3^a\dot{a}_3^v_3.
\]

In particular, we have
\[
(u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3 = (u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3.
\]

Point (iii) can be proved similarly.

Now we prove Point (iii). If \( a_3 \notin J \), then by the above reasoning and by Lemmas 2.12 and 2.13, we have
\[
u_1^a1^v_1 \vdash (u_2^a2^v_2 \vdash u_3^a3^v_3) = u_1^a1^v_1 \vdash [u_2^a2^v_2 u_3^a3^v_3]
\]
\[
= [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y] \vdash u_3^a\dot{a}_3^v_3
\]
\[
= [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y] \vdash u_3^a\dot{a}_3^v_3 \quad \text{(by Lemma 2.13)}
\]
\[
= (u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3.
\]

If \( a_3 \in J \), then by Lemma 2.12 and by the above reasoning, we have
\[
u_1^a1^v_1 \vdash (u_2^a2^v_2 \vdash u_3^a3^v_3) = u_1^a1^v_1 \vdash [u_2^a2^v_2 u_3^a3^v_3]
\]
\[
= [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y u_3^a3^v_3] \vdash u_3^a\dot{a}_3^v_3
\]
\[
= (u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3.
\]

Point (iv) can be proved similarly.

Finally we prove Point (v). If \( a_2 \in J \), then clearly we have
\[
(u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3 = [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y u_3^a3^v_3] \vdash (u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3).
\]

If \( a_2 \notin J \), then by Lemma 2.14, we have
\[
u_1^a1^v_1 \vdash u_2^a2^v_2 = [u_1^xa_1^yv_1^y u_2^xa_2^yv_2^y] = [u_1^xa_1^yv_1^y u_2^a2^v_2] \vdash [v_2^y]
\]
and
\[
u_2^a2^v_2 \vdash u_3^a3^v_3 = [u_2^a2^v_2 u_3^a3^v_3] = [u_2^a2^v_2 u_3^a3^v_3] \vdash [v_3^y] \vdash [\dot{a}_2^a2^v_2 u_3^a3^v_3].
\]

By Points (iii) and (iv), and by Lemma 2.14 again, we deduce
\[
(u_1^a1^v_1 \vdash u_2^a2^v_2) \vdash u_3^a\dot{a}_3^v_3 = ([u_1^xa_1^yv_1^y u_2^a2^v_2] \vdash [v_2^y]) \vdash u_3^a\dot{a}_3^v_3
\]
\[
= [u_1^xa_1^yv_1^y u_2^a2^v_2] \vdash [v_2^y] = [u_1^xa_1^yv_1^y u_2^a2^v_2] \vdash [v_2^y u_3^a3^v_3]
\]
\[
= [u_2^a2^v_2 u_3^a3^v_3] = (u_1^a1^v_1 \vdash [v_2^y]) \vdash [\dot{a}_2^a2^v_2 u_3^a3^v_3]
\]
\[
= u_1^a1^v_1 \vdash [v_2^y] \vdash [\dot{a}_2^a2^v_2 u_3^a3^v_3] = u_1^a1^v_1 \vdash (u_2^a2^v_2 \vdash u_3^a\dot{a}_3^v_3).
The proof is completed. □

Now we are ready to prove the main result of our article, in which we construct the free products of arbitrary digroups.

**Theorem 2.16.** With Definition 2.11, \((CW(D), \dot{\epsilon})\) is the free product of the family of indexed digroups \(\{(D'_i, e) \mid i \in I\}\). Moreover, the group part of \((CW(D), \dot{\epsilon})\) is \(\{w \mid w\) is a reduced word in \(J^+\}\), which is isomorphic to \(\Pi_{i \in I} J_i\), and the halo of \((CW(D), \dot{\epsilon})\) is \(\{u \dot{a} v e \in CW(D) \mid u^\#a^\#v^\# = e\}\).

**Proof.** We first show that \((CW(D), \dot{\epsilon})\) is a digroup. By Lemma 2.15, we know that \(CW(D)\) becomes a disemigroup. We proceed to show that \((CW(D), \dot{\epsilon})\) is a digroup. Clearly \(\dot{\epsilon}\) is a bar-unit in the disemigroup \(CW(D)\). Moreover, for every element \(u \dot{a} v e \in CW(D)\), since \(\Pi_{i \in I} J_i\) is a group, by the construction of the free products of groups, there exists a reduced word \(w \in J^+\) such that

\[
u^\#a^\#v^\#w = wu^\#a^\#v^\# = e.
\]

Thus we obtain

\[
\begin{align*}
u \dot{a} v &\vdash [w] = [u^\#a^\#v^\#w^\#] = [w^\#a^\#v^\#w] = [e] = \dot{\epsilon}, \\
[w] &\vdash u \dot{a} v = [w^\#u^\#a^\#v^\#] = [wu^\#a^\#v^\#] = [e] = \dot{\epsilon}.
\end{align*}
\]

Therefore, \((CW(D), \dot{\epsilon})\) is a digroup.

Now we show that \((CW(D), \dot{\epsilon})\) is the free product of \(\{(D'_i, e) \mid i \in I\}\). For every \(i \in I\), assume that \(\varphi_i : (D'_i, e) \rightarrow (CW(D), \dot{\epsilon})\) is the injective di-group homomorphism which is defined by \(\varphi_i(x) = [x] = \dot{x}\) for every \(x \in D'_i\). Let \((D', e')\) be an arbitrary digroup such that there exists a digroup homomorphism \(\psi_i : (D'_i, e) \rightarrow (D', e')\) for every \(i \in I\). Recall that for every element \(d \in D \setminus \{e\}\), the notation \(\sigma(d)\) means the index \(i \in I\) such that \(d\) lies in \(D'_i\). Then we define a map \(\eta\) from \((CW(D), \dot{\epsilon})\) to \((D', e')\) by the rule \(\eta(\dot{\epsilon}) = e'\) and

\[
\eta(a_1 \ldots a_m \dot{a} b_1 \ldots b_n) = \psi_{\sigma(a_1)}(a_1) \vdash \ldots \vdash \psi_{\sigma(a_m)}(a_m) \vdash \psi_{\sigma(a)}(a) \vdash \psi_{\sigma(b_1)}(b_1) \\
\vdash \ldots \vdash \psi_{\sigma(b_n)}(b_n),
\]

where \(\epsilon \neq a_1 \ldots a_m \dot{a} b_1 \ldots b_n \in CW(D), m \geq 0\) and \(n \geq 0\). If \(m = 0\) (resp. \(n = 0\)), then \(\psi_{\sigma(a_1)}(a_1) \vdash \ldots \vdash \psi_{\sigma(a_m)}(a_m) \vdash \) (resp. \(\psi_{\sigma(b_1)}(b_1) \vdash \ldots \vdash \psi_{\sigma(b_n)}(b_n)\)) does not appear. In particular, we have \(\eta(\dot{a}) = \psi_{\sigma(a)}(a)\) for every \(a \in D \setminus \{e\}\).

Before showing that \(\eta\) is a digroup homomorphism, we first note that the map \(\eta\) is compatible with the extended reduction rule in the following sense: For all \(a_1, \ldots, a_m, b_1, \ldots, b_n\) in \(J\) and for every \(a \in D\), assume that \(a_t \delta a_{t+1} = c\) in \(D'_i\) for some \(i \in I\), where \(\delta \in \{-, \}\). Then clearly we have

\[
\begin{align*}
\eta(\dot{a}_1) \vdash \ldots \vdash \eta(\dot{a}_m) &\vdash \eta(\dot{a}) \dashv \eta(\dot{b}_1) \dashv \ldots \dashv \eta(\dot{b}_n) \\
&= \eta(\dot{a}_1) \dashv \ldots \dashv \eta(\dot{a}_{t-1}) \vdash (\psi_i(a_t) \delta \psi_i(a_{t+1})) \dashv \ldots \dashv \eta(\dot{a}) \dashv \ldots \dashv \eta(\dot{b}_n) \\
&= \eta(\dot{a}_1) \dashv \ldots \dashv \eta(\dot{a}_{t-1}) \dashv (\psi_i(a_t \delta a_{t+1})) \dashv \ldots \dashv \eta(\dot{a}) \dashv \eta(\dot{b}_1) \dashv \ldots \dashv \eta(\dot{b}_n) \\
&= \eta(\dot{a}_1) \dashv \ldots \dashv \eta(\dot{a}_{t-1}) \dashv \eta(\dot{c}) \dashv \ldots \dashv \eta(\dot{a}_m) \vdash \eta(\dot{a}) \dashv \eta(\dot{b}_1) \dashv \ldots \dashv \eta(\dot{b}_n).
\end{align*}
\]
Similarly, if $a_m$ and $a$ lie in the same digroup, or if $b_1, a$ or $b_t, b_{t+1}$ lie in the same digroup, then a similar result holds. Therefore, if $a_1 \ldots a_m$ and $b_1 \ldots b_n$ are words in $J^+$ and if $a \in D \setminus J$, then no matter whether $a_1 \ldots a_m \hat{a}_b b_1 \ldots b_n$ is a good center-word or not, we have

$$
\eta([a_1 \ldots a_m \hat{a}_b b_1 \ldots b_n]) = \eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{b}_1) \vdash \ldots \vdash \eta(\hat{b}_n).
$$

In particular, we deduce

$$
\eta([a_1 \ldots a_m]) = \eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \tag{2.3}
$$

for all $a_1, \ldots, a_m \in J$.

We also note that, for every $c_1 \in D'_1$, we have

$$
\eta([c_1]) = \psi_i(c_1) = \psi_i(c_1 \mapsto e) = \psi_i(c_1) \mapsto e' = \eta(\hat{c}_1) \mapsto e'.
$$

By Lemma 2.3, it follows that $\psi_i(c_1 \mapsto e)$ lies in the group part of $D'$, and in particular, for every $a_p \in J$, the element $\eta(\hat{a}_p)$ lies in the group part of $D'$. So by (2.3) and by Lemma 2.3, we obtain

$$
\eta([a_1 \ldots a_m]) = \eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m)
$$

for every $t$ such that $1 \leq t \leq m - 1$.

Now we are ready to show that $\eta$ is a digroup homomorphism. For all $u_1 \hat{c}_1 v_1$ and $u_2 \hat{c}_2 v_2$ in $CW(D)$, where $u_1 = a_1 \ldots a_m$, $v_1 = b_1 \ldots b_n$, $u_2 = d_1 \ldots d_p$ and $v_2 = f_1 \ldots f_q$, if $c_2 \in J$, then we have $p = 0$ and

$$
\eta(u_1 \hat{c}_1 v_1 \vdash u_2 \hat{c}_2 v_2) = \eta([u_1^x c_1^x v_1^x u_2^x c_2^x v_2^x])
$$

$$
= \eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{c}_1) \vdash \ldots \vdash \eta(\hat{c}_2) \vdash \ldots \vdash \eta(\hat{f}_q)
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{c}_1) \vdash \ldots \vdash \eta(\hat{c}_2) \vdash \ldots \vdash \eta(\hat{f}_q))
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash (\eta(\hat{c}_1) \mapsto e') \vdash \ldots \vdash \eta(\hat{f}_q))
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{c}_1) \mapsto \ldots \vdash \eta(\hat{f}_q)) \vdash \eta(u_2 \hat{c}_2 v_2)
$$

$$
= \eta(u_1 \hat{c}_1 v_1) \vdash \eta(u_2 \hat{c}_2 v_2);
$$

On the other hand, if $c_2 \in D \setminus J$, then we have

$$
\eta(u_1 \hat{c}_1 v_1 \vdash u_2 \hat{c}_2 v_2) = \eta([u_1^x c_1^x v_1^x u_2^x c_2^x v_2^x])
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{c}_1) \vdash \ldots \vdash \eta(\hat{c}_2) \vdash \ldots \vdash \eta(\hat{f}_q))
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash (\eta(\hat{c}_1) \mapsto e') \vdash \ldots \vdash \eta(\hat{f}_q))
$$

$$
= (\eta(\hat{a}_1) \vdash \ldots \vdash \eta(\hat{a}_m) \vdash \eta(\hat{c}_1) \vdash \ldots \vdash \eta(\hat{f}_q)) \vdash \eta(u_2 \hat{c}_2 v_2)
$$

$$
= \eta(u_1 \hat{c}_1 v_1) \vdash \eta(u_2 \hat{c}_2 v_2).
$$

Similarly, we have $\eta(u_1 \hat{c}_1 v_1 \vdash u_2 \hat{c}_2 v_2) = \eta(u_1 \hat{c}_1 v_1) \vdash \eta(u_2 \hat{c}_2 v_2)$. It follows that $\eta$ is a digroup homomorphism from $(CW(D), \hat{e})$ to $(D', e')$.

Since the set $\{\hat{a} \mid a \in D\}$ generates $(CW(D), \hat{e})$, it is clear that $\eta$ is the unique digroup homomorphism from $(CW(D), \hat{e})$ to $(D', e')$ such that $\eta \varphi_i = \psi_i$ for every $i \in I$. Therefore, $(CW(D), \hat{e})$ is the free product of the family of digroups $\{(D'_i, e) \mid i \in I\}$. 
Assume that $uav$ lies in the group part of $(CW(D), \hat{e})$. Then by Lemma 2.3, we have

$$uav = uav \vdash \hat{e} = [u^2a^2v^2e^2] = [u^2a^2v^2],$$

which holds if and only if $u$ is the empty word and $av$ lies in $CW(D)$ satisfying $av \in J^+$, namely, $uav = [av]$ and $av$ is a reduced word in $J^+$. Now the isomorphism is clear by the constructions of $\Pi_{i \in I} J_i$ and $(CW(D), \hat{e})$.

Assume that $uav$ lies in the halo of $(CW(D), \hat{e})$. Then we have

$$uav \vdash \hat{e} = [u^2a^2v^2e^2] = [u^2a^2v^2] = \hat{e}.$$

It follows immediately that we have $\underline{u^2a^2v^2} = e$. On the other hand, by the construction of $(CW(D), \hat{e})$, it is clear that every $uav \in CW(D)$ satisfying $\underline{u^2a^2v^2} = e$ lies in the halo of $(CW(D), \hat{e})$. \hfill $\square$

Theorem 2.16 shows that the free product of digroups extends the notion of the free product of groups in the sense that, if every digroup $(D'_i, e)$ is a group, $i \in I$, then $(CW(D), \hat{e})$ is the free product of the groups $\{D'_i \mid i \in I\}$.

Now we offer three easy examples of the free product of a family of digroups. By Remark 2.7, we may always assume that the considered digroups are disjoint.

**Example 2.17.** Let $(D_1, e_1)$ and $(D_2, e_2)$ be two disjoint digroups such that the group part of $(D_2, e_2)$ contains exactly one element. Then with the conventions and notations of Remark 2.7, we have $J_2 = \{e\}$, $J = J_1$ and $D = D'_1 \cup D'_2$. Moreover, the set of all good center-words over $D$ is

$$CW(D) = \{uav \mid u, v \in (J_1 \cup \{e\}) \setminus \{e\}, a \in D, \text{ if } a \in D'_1 \text{ then } u = v = e\}.$$

With Definition 2.11, $(CW(D), \hat{e})$ is the free product of $(D'_1, e)$ and $(D'_2, e)$, which is isomorphic to the free product of $(D_1, e_1)$ and $(D_2, e_2)$.

**Example 2.18.** Let $\{(D_i, e_i) \mid i \in I\}$ be a family of disjoint digroups such that the group part of $(D_i, e_i)$ contains exactly one element for every $i \in I$. Then with the conventions and notations of Remark 2.7, we have $J = J_i = \{e\}$ for every $i \in I$ and $D = \bigcup_{i \in I} D'_i$. Moreover, the set of all good center-words over $D$ is

$$CW(D) = \{a \mid a \in D\}.$$

With Definition 2.11, $(CW(D), \hat{e})$ is the free product of $\{(D'_i, e) \mid i \in I\}$, which is isomorphic to the free product of $\{(D_i, e_i) \mid i \in I\}$.

In the above examples, a trivial digroup (namely, every element in the digroup is a bar-unit) is involved. So we conclude the article with another example in which no trivial digroup is involved in general.

**Example 2.19.** Let $\{G_i \mid i \in I\}$ be a family of disjoint groups such that $1_i$ is the unit of $G_i$, and let $S_i$ be a set on which $G_i$ acts on the left for every $i \in I$. Suppose that for every $i \in I$, there exists a distinguished fixed point $s_i \in S_i$, that is, $gs_i = s_i$ for all $g \in G_i$ (and we distinguish $s_i$ in case there are more than one fixed elements). For all $(s, g), (s', g') \in S_i \times G_i$, define

$$(s, g) \cdot (s', g') = (gs', gg'); \quad (s, g) \cdot (s', g') = (s, gg').$$
Then with respect to the above operations, \((S_i \times G_i, (s_i, 1_i))\) is exactly the digroup constructed in [4, Example 4.2] such that the halo of \((S_i \times G_i, (s_i, 1_i))\) is \(S_i \times \{1_i\}\). (In [4, Example 4.2], it is assumed that \(G_i\) acts transitively on \(S_i \setminus \{s_i\}\), but if we have no specific assumptions on certain property of the digroup \((S_i \times G_i, (s_i, 1_i))\), then this condition is not necessary, see also [7, Proposition 3] and [8, Theorem 2].) Denote \(S_i \times G_i\) by \(D_i\) and denote \((s_i, 1_i)\) by \(e_i\) for every \(i \in I\). Then with the notations of Remark 2.7, and by Lemma 2.3, we easily obtain

\[
D = \bigcup_{i \in I} D'_i \quad \text{and} \quad J = \bigcup_{i \in I} J_i = \bigcup_{i \in I} \{(s_i) \times (G_i \setminus \{1_i\})\} \cup \{e\}.
\]

Moreover, the set of all good center-words over \(D\) is

\[
CW(D) = \{\hat{e}\} \cup \Omega,
\]

where

\[
\Omega = \{(s_{i_1}, b_{i_1}) \ldots (s_{i_n}, b_{i_n}) \bar{d}(s_{i_{n+1}}, b_{i_{n+1}}) \ldots (s_{i_{n+m}}, b_{i_{n+m}}) \mid n, m \geq 0, \\
i_p \neq i_{p+1} \text{ for } p \leq n - 1 \text{ or } n + 1 \leq p \leq n + m - 1, \\
d = (s, a) \in (S_j \times G_j) \setminus \{(s_j, 1_j)\}, \quad i_n \neq j, \ j \neq i_{n+1}, \\
j, i_1, \ldots, i_{n+m} \in I, \ s_{i_q} \text{ is the distinguished fixed element in } S_{i_q}, \\
b_{i_q} \in G_{i_q} \setminus \{1_{i_q}\} \text{ for } 0 \leq q \leq n + m, \text{ and if } s = s_j \text{ then } n = 0\}.
\]

With Definition 2.11, \((CW(D), \hat{e})\) is the free product of \(\{(D'_i, e) \mid i \in I\}\), which is isomorphic to the free product of \(\{(S_i \times G_i, (s_i, 1_i)) \mid i \in I\}\).

**Acknowledgements**

We thank very much the anonymous referee for his/her extremely careful reading of the original version of this article. The referee’s insightful comments and suggestions greatly improve the exposition of the article.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**

[1] Felipe, R.: Generalized Loday algebras and digroups. Comunicaciones del CIMAT. No. I-04-01/21-01-2004 (2004)

[2] Felipe, R.: Digroups and their linear representations. East West J. Math. 8, 27–48 (2006)

[3] Jacobson, N.: Basic algebra II. Dover Publications Inc, New York (2009)

[4] Kinyon, M.K.: Leibniz algebras, Lie racks, and digroups. J. Lie Theory 17, 99–114 (2004)

[5] Liu, K.: A class of group-like objects. arXiv:math.RA/0311396v1
[6] Loday, J.L.: Dialgebras, dialgebras and related operads. Lecture Notes in Mathematics, vol. 1763, pp. 7–66. Springer, Berlin (2001)

[7] Ongay, F.: On the notion of digroup. CIMAT I-10-04 (MB) (2010)

[8] Ongay, F., Velásquez, R., Wills-Toro, L.A.: Normal subdigroups and the isomorphism theorems for digroups. Algebra Discrete Math. 22, 262–283 (2016)

[9] Phillips, J.D.: A short basis for the variety of digroups. Semigroup Forum 70, 466–470 (2005)

[10] Rodriguez-Nieto, J.G., Salazar-Díaz, O.P., Velásquez, R.: Augmented free and tensor generalized digroups. Open Math. 17, 71–88 (2019)

[11] Salazar-Díaz, O.P., Velásquez, R., Wills-Toro, L.A.: Generalized digroups. Commun. Algebra 44, 2760–2785 (2016)

[12] Zhang, G., Chen, Y.: A construction of the free digroup. Semigroup Forum 102, 553–567 (2021)

[13] Zhuchok, A.V.: Structure of relatively free dimonoids. Commun. Algebra 45, 1639–1656 (2017)

[14] Zhuchok, A.V., Zhuchok, Y.V.: On two classes of digroups. São Paulo J. Math. Sci. 11, 240–252 (2017)

Guangliang Zhang
School of Mathematics and Systems Science
Guangdong Polytechnic Normal University
Guangzhou 510631
People’s Republic of China
e-mail: zgl541@163.com

Yuqun Chen and Zerui Zhang
School of Mathematical Sciences
South China Normal University
Guangzhou 510631
People’s Republic of China
e-mail: yqchen@scnu.edu.cn

and

International Center for Mathematics
SUSTech
Shenzhen
People’s Republic of China
e-mail: zeruizhang@scnu.edu.cn

Received: 30 January 2021.
Accepted: 18 October 2021.