Abundant M-fractional optical solitons to the pertubed Gerdjikov–Ivanov equation treating the mathematical nonlinear optics

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Received: 8 October 2021 / Accepted: 9 November 2021 / Published online: 29 November 2021
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Abstract
In this paper, the perturbed Gerdjikov–Ivanov (GI) equation using a truncated M-fractional derivative is studied in mathematical nonlinear optics. We are explored its novel dark and other soliton solutions and compared them with the existing results. To obtain the objective, two particular methods, the modified extended tanh expansion method and the $Exp_a$ function method, are implemented. In this exercise, an arrangement of exact solitons is received as well as verified by utilizing the symbolic soft computations. The dynamical characteristics of the obtained results, along with a fractional parameter, are also discussed via two and three-dimensional graphs. These solutions are suggested that the employed methods are impressive, determined and smooth as compared to many other methods. The work of this paper is of high importance regarding its applications in photonic crystal fibers and mathematical physics.

Keywords M-fractional derivative · Perturbed Gerdjikov–Ivanov Equation · Optical solitons

1 Introduction
Many phenomena are often described in the form of nonlinear Schrödinger (NLS) equations. Then those NLS equations are studied for different purposes with different approaches. The NLS equation is a generic model that governs the wave evolution in

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a broad range of physical systems including water waves, blood flow in blood vessels, nonlinear optics, magnetic films and plasma physics. In recent years, to solve the different NLS equations, different analytical and numerical approaches have been developed (Kumar et al. 2021; Hosseini et al. 2021c; Wazwaz 2009; Arshad et al. 2018; Hosseini et al. 2018b; González-Gaxiola et al. 2017). The NLS equations that are mostly studied are those that have cubic nonlinearity and the perturbed GI equation is one of them (Biswas et al. 2018c; Arshed et al. 2018).

Various approaches have been applied to find different exact solitons of this equation. For instance, the Lie symmetry analysis has produced the conservation laws and the bright soliton solutions for the Gerdjikov–Ivanov equation (Biswas et al. 2017). The extended Kudryashov’s method has been employed to secure dark, bright and singular solitons for the perturbed GI equation with full nonlinearity (Biswas et al. 2018a). Two types of bright wave solutions of perturbed GI equation have been obtained with the help of the semi-inverse variational method (Biswas and Alqahtani 2017). Distinct solitons are investigated by applying the sine-Gordon equation method (Yasar et al. 2018). Biswas et al. have obtained the singular and bright solitons with the use of an extended trial equation approach for the perturbed GI equation (Biswas et al. 2018b). Different solitary wave solutions of the perturbed GI equation have been found by the implementation of the $\exp(\phi(\xi))$-expansion and the Kudryashov techniques (Arshed 2018). Various wave solutions have been obtained by applying the two different methods, $\exp(\phi(\xi))$-expansion method and $(G'/G^2)$-expansion technique (Kaur and Wazwaz 2018). Different optical solitons have been determined by using the $\exp_a$ function method and the modified Kudryashov method (Hosseini et al. 2020). The Kudryashov methods (KMs) have been adopted to investigate a fifth-order nonlinear water wave equation for W-shaped and other solitons (Hosseini et al. 2021a). A $(2 + 1)$-dimensional nonlinear model with the beta time derivative describing the wave propagation in the Heisenberg ferromagnetic spin chain has been explored analytically by the new Kudryashov and exponential methods (Hosseini et al. 2021b). Recently, some novel wave solutions were investigated by using the generalized exponential rational function method (Ghanbari and Baleanu 2020).

Besides these methods, two methods are more reliable, simple and useful named as the modified extended tanh expansion method (ETHEM) and $\exp_a$ function method. The modified extended tanh expansion technique has been used to discuss Biswas and Arshed model with full nonlinearity (Asim Zafar 2020). Optical soliton solutions of the travelling wave nonlinear equations have been determined in Wazwaz (2004), Wazwaz (2008), Fan (2000) and Fan and Hon (2003). Explicit exact solitons of two nonlinear Schrödinger equations have been investigated through two different techniques (Zafar et al. 2020). Similarly, these techniques have been applied to solve the many other NLS equations (Zayed and Al-Nowehy 2017; Zafar 2019; Hosseini et al. 2018a).

The main task in this research is to search for some new dark and other optical soliton solutions of the perturbed GI equation with truncated M. fractional derivative (SOUZA and OLIVEIRA 2018). The fractional derivative has various definitions that have been published in literature (Khalil et al. 2014), but the most generalized definition is a truncated M-fractional derivative. This definition unifies four existing fractional derivative types, Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Riesz, among others also satisfies the classical properties of integer-order calculus (De Oliveira and Tenreiro Machado 2014). The modified extended tanh expansion method and $\exp_a$ function method are employed to acquire the aforesaid task.
2 The governing model and the mathematical analysis

The perturbed Gerdjikov–Ivanov (GI) equation can be read as (Biswas and Alqahtani 2017; Yasar et al. 2018; Biswas et al. 2018b; Arshed 2018; Kaur and Wazwaz 2018; Hosseini et al. 2020):

\[
ig_t + \tau_1 g_{xx} + \tau_2 |g|^4 g - \iota(\tau_3 g^2 g_x^* + \theta_1 g_x + \theta_2 (|g|^2 g_x + \rho |g|^2 g)) = 0
\]  

(1)

here \( g = g(x, t) \) shows the complex-valued wave function, depends on independent variables \( x \) and \( t \). In Eq. (1), \( \tau_1 \) shows the coefficient of GVD, \( \tau_2 \) represents the coefficient of the quintic non-linearity of the model and \( \tau_3 \) indicates the coefficient of the nonlinear dispersion term. Moreover, the parameters \( \theta_1, \theta_2 \) and \( \rho \) represent the perturbation effects. Finally, the term \( g^* \) shows the complex conjugate of \( g \). Eq. (1) with truncated M-fractional derivative is given as:

\[
iD_M^{\mu, \beta} g + \tau_1 D_M^{2\mu, \beta} g + \tau_2 |g|^4 g = \iota(\tau_3 g^2 D_M^{\mu, \beta} g^* + \theta_1 D_M^{\mu, \beta} g + \theta_2 (|g|^2 g) + \rho g D_M^{\mu, \beta} |g|^2 g), \quad \mu \in (0, 1), \quad \beta > 0,
\]  

(2)

where

\[
D_M^{\mu, \beta} = \lim_{\tau \to 0} \frac{g(x, tE_\beta(\tau t^{1-\mu})) - g(x, t)}{\tau}, \quad \mu \in (0, 1), \quad \beta > 0,
\]  

(3)

that \( E_\beta(.) \) is a truncated Mittag-Leffler function of one parameter (Mainardi and Gorenflo 2000).

Now by using the complex constraints conditions given in the following:

\[
g(x, t) = G(\zeta)e^{i\psi}, \quad \zeta = \frac{1/(\beta+1)}{\mu}(x^\mu - \lambda t^\mu),
\]  

\[
\psi = \frac{\Gamma(\beta+1)}{\mu}(-\sigma_1 x^\mu + \sigma_2 t^\mu),
\]  

(4)

Here \( \lambda \) indicates the phase component, \( \sigma_1 \) shows the frequency of the wave solutions and \( \sigma_2 \) represents the wave number of solitons. By putting the Eq. (4) into the model Eq. (2), we get:

Real part:

\[
\tau_1 G'' - (\sigma_2 + \tau_1 \sigma_1^2 + \theta_1 \sigma_1) G + (\tau_3 - \theta_2) \sigma_1 G^3 + \tau_2 G^5 = 0.
\]  

(5)

Imaginary part:

\[
(\lambda + \theta_1 + 2\tau_1 \sigma_1) + (\tau_3 + 3\theta_2 + 2\rho) G^2 = 0.
\]  

(6)

From Eq. (6), we get:

\[
\lambda = -(\theta_1 + 2\tau_1 \sigma_1); \quad \rho = -\frac{1}{2}(\tau_3 + 3\theta_2)
\]  

(7)

Using the terms \( G'' \) and \( G^5 \) and the homogenous balance approach, we observe \( m = 1/2 \). Therefore, we use the, we use the following transformation to get solution in retrieve form:
\[ Q(\zeta) = G^1(\zeta) \]  

By using the Eq. (8) into Eq. (5), yields

\[ \tau_1 (2QQ' - (Q')^2 - 4(\sigma_2 + \tau_1 \sigma_1^2 + \theta_1 \eta_1)Q^2 + 4(\tau_3 - \tau_2)\eta_1 Q^3 + 4\tau_2 Q^4 = 0 \]  

### 3 Application of the modified EThEM

Here, a quick review of the said method and its implementation both are explained. Let’s assume the below non-linear PDE of the form:

\[ Y(u, u'^2u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \]  

where \( u = u(x, t) \) is a dependent function of independent variables \( x \) and \( t \). Let us assume the below wave transformation:

\[ u(x, t) = U(\zeta), \quad \zeta = x - \nu t \]  

Here \( \nu \) shows the wave velocity. Putting the Eq. (11) into Eq. (10), taking the following nonlinear ODE:

\[ \chi(U, U'^2, U'', \ldots) = 0. \]  

Here primes represents the derivatives w.r.t \( \zeta \).

Moreover, consider the solution of Eq. (12) is of the form:

\[ U(\zeta) = \alpha_0 + \sum_{n=1}^{m} \alpha_n \phi^n(\zeta) + \sum_{n=1}^{m} \beta_n \phi^{-n}(\zeta) \]  

\( \alpha_0, \alpha_n, \beta_n, (n = 1, 2, 3, \ldots, m) \) are undetermined and find to be later. Notice that \( \alpha_n \) and \( \beta_n \) are not both zero at a time. By balancing nonlinear term and the highest derivative in Eq. (12), we get \( m \).

The function \( \phi(\zeta) \) satisfies:

\[ \phi'(\zeta) = \Omega + \phi^2(\zeta) \]  

with \( \Omega \) as a unknown parameter and the Eq. (14) have the below form of solutions (Raslan et al. 2017):

(i) if \( \Omega < 0 \), we get

\[ \phi(\zeta) = -\sqrt{-\Omega} \tanh(\sqrt{-\Omega} \, \zeta), \]  

or

\[ \phi(\zeta) = -\sqrt{-\Omega} \coth(\sqrt{-\Omega} \, \zeta). \]  

(ii) if \( \Omega = 0 \), we get

\[ \phi(\zeta) = \frac{-1}{\zeta} \]
(iii) if \( \Omega > 0 \), we get

\[
\phi(\zeta) = \sqrt{\Omega} \tan(\sqrt{\Omega} \, \zeta). \tag{18}
\]

or

\[
\phi(\zeta) = -\sqrt{\Omega} \cot(\sqrt{\Omega} \, \zeta). \tag{19}
\]

Putting Eq. (13) and its derivatives into Eq. (12) along with Eq. (14), yields polynomials in powers of \( \phi(\zeta) \). Sorting the coefficients of each power of \( \phi(\zeta) \) and by letting each summation equal to 0, we get a set of algebraic equations for \( a_0, \alpha_n, \beta_0, (n = 1, 2, 3, \ldots, m) \), and \( \Omega \). Placing the values of these parameters into Eq. (13) with value of \( m \), yields the solutions to Eq. (10).

Now by using the homogenous balance technique on Eq. (9), we compute \( m = 1 \). Then, Eq. (13) becomes:

\[
Q(\zeta) = \alpha_0 + \alpha_1 \phi(\zeta) + \frac{\beta_1}{\phi(\zeta)} \tag{20}
\]

Here \( \alpha_0, \alpha_1 \) and \( \beta_1 \) are undetermined parameters. By inserting the Eq. (20) and Eq. (14) into Eq. (9), sorting the coefficients of each power of \( \phi(\zeta) \), we obtain the set of equations having \( \alpha_0, \alpha_1, \beta_1 \) and other parameters. By using the soft computation, we gain the following sets:

**Set 1**

\[
\begin{align*}
\alpha_0 &= \frac{3\sigma_1 (\theta_2 - \tau_3)}{8\tau_2}, \\
\alpha_1 &= \mp \frac{i \sqrt{3} \sqrt{\tau_1}}{2 \sqrt{\tau_2}}, \\
\beta_1 &= 0, \\
\sigma_2 &= -\frac{\sigma_1 (-6\theta_2 \sigma_1 \tau_3 + 3\theta_2^2 \sigma_1 + 16\theta_1 \tau_2 + \sigma_1 (3\tau_2^2 + 16\tau_1 \tau_2))}{16\tau_2}, \\
\Omega &= \frac{3\sigma_1^2 (\theta_2 - \tau_3)^2}{16\tau_1 \tau_2}
\end{align*}
\]

(21)

We now using the Eq. (21) and Eqs. (15)–(19) in Eq. (20), and argue the following cases. If \( \Omega < 0 \), then

\[
g_1(x,t) = \left\{ \frac{3\sigma_1 (\theta_2 - \tau_3) \pm 4i \sqrt{3} \sqrt{\tau_1} \sqrt{\tau_2} \sqrt{-\Omega} \tanh\left( \frac{\zeta \sqrt{-\Omega}}{\sqrt{\tau_1}} \right)}{8\tau_2} \right\}^{\frac{1}{2}} \tag{22}
\]

\[
\times \exp \left( \frac{i \Gamma(\beta + 1)}{\mu} (-\sigma_1 x'' + \sigma_2 t'') \right),
\]

or
If $\Omega > 0$, then

$$g_2(x,t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3) \pm 4i\sqrt{3}\sqrt{\tau_1}\sqrt{\tau_2}\sqrt{-\Omega}\coth\left(\frac{\zeta \sqrt{-\Omega}}{\tau_2}\right)}{8\tau_2} \right\}^{\frac{1}{2}} \times \exp\left(\frac{i}{\mu} \left(\frac{\Gamma(\beta + 1)}{\mu}(-\sigma_1 x'' + \sigma_2 t'')\right)\right).$$

(23)

If $\Omega < 0$, then

$$g_3(x,t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3) \mp 4i\sqrt{3}\sqrt{\tau_1}\sqrt{\tau_2}\sqrt{\Omega}\tan\left(\frac{\zeta \sqrt{\Omega}}{\tau_2}\right)}{8\tau_2} \right\}^{\frac{1}{2}} \times \exp\left(\frac{i}{\mu} \left(\frac{\Gamma(\beta + 1)}{\mu}(-\sigma_1 x'' + \sigma_2 t'')\right)\right),$$

(24)

or

$$g_4(x,t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3) \mp 4i\sqrt{3}\sqrt{\tau_1}\sqrt{\tau_2}\sqrt{\Omega}\cot\left(\frac{\zeta \sqrt{\Omega}}{\tau_2}\right)}{8\tau_2} \right\}^{\frac{1}{2}} \times \exp\left(\frac{i}{\mu} \left(\frac{\Gamma(\beta + 1)}{\mu}(-\sigma_1 x'' + \sigma_2 t'')\right)\right).$$

(25)

Set 2

$$\left\{ \begin{array}{l}
\alpha_0 = \frac{3\sigma_1(\theta_2 - \tau_3)}{8\tau_2}, \quad \alpha_1 = \mp\frac{i\sqrt{3}}{2\sqrt{\tau_2}}, \quad \beta_1 = \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2}{128\sqrt{\tau_1}\tau_2^{3/2}}, \quad \Omega \\
= -\frac{\sigma_1^2(\theta_2 - \tau_3)^2}{64\tau_1\tau_2} \\
\sigma_2 = -\frac{\sigma_1(-30\theta_2\sigma_1\tau_3 + 15\theta_2^2\sigma_1 + 64\theta_1\tau_2 + \sigma_1(15\tau_3^2 + 64\tau_1\tau_2))}{64\tau_2} \end{array} \right\}. \quad (26)$$

We now using the Eq. (26) and Eqs. (15)–(19) in Eq. (20), and argue the following cases.

If $\Omega < 0$, then
\[ g_1(x, t) = \begin{align*}
\frac{3\sigma_1(\theta_2 - \tau_3)}{8r_2} & \pm \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \coth(\zeta\sqrt{-\Omega})}{128\sqrt{r_2\tau_1\tau_2\sqrt{-\Omega}}} \\
\pm \frac{i\sqrt{3}\sqrt{r_1}\sqrt{-\Omega} \tanh(\zeta\sqrt{-\Omega})}{2\sqrt{r_2}} & \times \exp\left(\frac{i}{\mu}(\Gamma(\beta + 1)(-\sigma_1 x^\mu + \sigma_2 t^\mu)\right) \end{align*} \] (27)

or
\[ g_2(x, t) = \begin{align*}
\frac{3\sigma_1(\theta_2 - \tau_3)}{8r_2} \pm \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \tanh(\zeta\sqrt{-\Omega})}{128\sqrt{r_2\tau_1\tau_2\sqrt{-\Omega}}} \\
\pm \frac{i\sqrt{3}\sqrt{r_1}\sqrt{-\Omega} \coth(\zeta\sqrt{-\Omega})}{2\sqrt{r_2}} & \times \exp\left(\frac{i}{\mu}(\Gamma(\beta + 1)(-\sigma_1 x^\mu + \sigma_2 t^\mu)\right) \end{align*} \] (28)

If \( \Omega > 0 \), then
\[ g_3(x, t) = \begin{align*}
\frac{3\sigma_1(\theta_2 - \tau_3)}{8r_2} & \pm \frac{i\sqrt{3}\sqrt{r_1}\sqrt{\Omega} \tan(\zeta\sqrt{\Omega})}{2\sqrt{r_2}} \\
\pm \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \cot(\zeta\sqrt{\Omega})}{128\sqrt{r_1\tau_1\tau_2\sqrt{\Omega}}} & \times \exp\left(\frac{i}{\mu}(\Gamma(\beta + 1)(-\sigma_1 x^\mu + \sigma_2 t^\mu)\right) \end{align*} \] (29)

or
\[ g_4(x, t) = \begin{align*}
\frac{3\sigma_1(\theta_2 - \tau_3)}{8r_2} & \pm \frac{i\sqrt{3}\sqrt{r_1}\sqrt{\Omega} \cot(\zeta\sqrt{\Omega})}{2\sqrt{r_2}} \\
\pm \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \coth(\zeta\sqrt{-\Omega})}{128\sqrt{r_1\tau_1\tau_2\sqrt{-\Omega}}} & \times \exp\left(\frac{i}{\mu}(\Gamma(\beta + 1)(-\sigma_1 x^\mu + \sigma_2 t^\mu)\right) \end{align*} \] (30)

Set 3
\[
\begin{aligned}
\alpha_0 &= \frac{3\sigma_1 (\theta_2 - \tau_3)}{8\tau_2}, \quad \alpha_1 = \mp \frac{i\sqrt{3}\sqrt{\tau_1}}{2\sqrt{\tau_2}}, \\
\beta_1 &= \pm \frac{3i\sqrt{3}\sigma_1^2 (\theta_2 - \tau_3)^2}{128\sqrt{\tau_1 \tau_2}^{3/2}}, \\
\Omega &= \frac{3\sigma_1^2 (\theta_2 - \tau_3)^2}{64\tau_1 \tau_2} \\
\sigma_2 &= -\frac{\sigma_1 (-6\theta_2 \sigma_1 \tau_3 + 3\theta_2^2 \sigma_1 + 16\theta_1 \tau_2 + \sigma_1 (3\tau_3^2 + 16\tau_1 \tau_2))}{16\tau_2}.
\end{aligned}
\]

(31)

We now using the Eq. (31) and Eqs. (15)–(19) in Eq. (20), and argue the following cases.

If \(\Omega < 0\), then

\[
\begin{aligned}
g_1(x, t) &= \left\{ \frac{3\sigma_1 (\theta_2 - \tau_3)}{8\tau_2} \mp \frac{3i\sqrt{3}\sigma_1^2 (\theta_2 - \tau_3)^2 \coth \left( \zeta \sqrt{-\Omega} \right)}{128\sqrt{\tau_1 \tau_2} \sqrt{-\Omega}} \right. \\
&\quad \pm \left( \frac{i\sqrt{3}\sqrt{\tau_1} \sqrt{-\Omega}}{2\sqrt{\tau_2}} \tanh \left( \zeta \sqrt{-\Omega} \right) \right)^{1/2} \\
&\times \exp \left( \frac{i}{\mu} \left( -\sigma_1 \chi^\mu + \sigma_2 \tau^\mu \right) \right) \right\}
\end{aligned}
\]

(32)

or

\[
\begin{aligned}
g_2(x, t) &= \left\{ \frac{3\sigma_1 (\theta_2 - \tau_3)}{8\tau_2} \mp \frac{3i\sqrt{3}\sigma_1^2 (\theta_2 - \tau_3)^2 \tanh \left( \zeta \sqrt{-\Omega} \right)}{128\sqrt{\tau_1 \tau_2} \sqrt{-\Omega}} \right. \\
&\quad \pm \left( \frac{i\sqrt{3}\sqrt{\tau_1} \sqrt{-\Omega}}{2\sqrt{\tau_2}} \coth \left( \zeta \sqrt{-\Omega} \right) \right)^{1/2} \\
&\times \exp \left( \frac{i}{\mu} \left( -\sigma_1 \chi^\mu + \sigma_2 \tau^\mu \right) \right) \right\}
\end{aligned}
\]

(33)

If \(\Omega > 0\), then
Abundant M-fractional optical solitons to the pertubed Gerdjikov–…

We now using the Eq. (36) and Eqs. (15)–(19) in Eq. (20), and argue the following cases.

If \( \Omega < 0 \), then

\[
g_3(x, t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3)}{8\tau_2} \pm \frac{(i\sqrt{3}\sqrt{\tau_1})\sqrt{\Omega}}{2\sqrt{\tau_2}} \cot \left( \sqrt[3]{\Omega} \right) \left( 3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \right) \right\}^{1/2}
\]

\[
\times \exp \left( i\frac{\Gamma(\beta + 1)}{\mu} (\sigma_1 x^\mu + \sigma_2 t^\mu) \right)
\]

or

\[
g_4(x, t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3)}{8\tau_2} \pm \frac{(i\sqrt{3}\sqrt{\tau_1})\sqrt{\Omega}}{2\sqrt{\tau_2}} \tan \left( \sqrt[3]{\Omega} \right) \left( 3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2 \right) \right\}^{1/2}
\]

\[
\times \exp \left( i\frac{\Gamma(\beta + 1)}{\mu} (\sigma_1 x^\mu + \sigma_2 t^\mu) \right)
\]

Set 4

\[
\begin{align*}
\alpha_0 &= \frac{3\sigma_1(\theta_2 - \tau_3)}{8\tau_2}, \quad \alpha_1 = 0, \quad \beta_1 = \mp \frac{3i\sqrt{3}\sigma_1^2(\theta_2 - \tau_3)^2}{32\sqrt{\tau_1}\tau_2^{3/2}}, \\
\sigma_2 &= -\frac{\sigma_1(-6\theta_2\sigma_1\tau_3 + 3\theta_2^2\sigma_1 + 16\theta_1\tau_2 + \sigma_1(3\tau_3^2 + 16\tau_1\tau_2))}{16\tau_2}, \quad \Omega = \frac{3\sigma_1^2(\theta_2 - \tau_3)^2}{16\tau_1\tau_2}
\end{align*}
\]

We now using the Eq. (36) and Eqs. (15)–(19) in Eq. (20), and argue the following cases.

If \( \Omega < 0 \), then

\[
g_1(x, t) = \left\{ \frac{3\sigma_1(\theta_2 - \tau_3)}{8\tau_2} \pm \frac{i\sqrt{3}\sigma_1(\theta_2 - \tau_3)^2}{4\sqrt{\tau_1}\sqrt{-\Omega}} \coth \left( \sqrt[3]{\Omega} \right) \right\}^{1/2}
\]

\[
\times \exp \left( i\frac{\Gamma(\beta + 1)}{\mu} (\sigma_1 x^\mu + \sigma_2 t^\mu) \right)
\]

or
If $\Omega > 0$, then

$$g_2(x, t) = \left(3\sigma_1(\theta_2 - \tau_3)\right) \left(1 \pm \frac{i\sqrt{3}\sigma_1(\theta_2 - \tau_3) \tanh(\zeta \sqrt{-\Omega})}{4\sqrt{\tau_2 \tau_1 \sqrt{-\Omega}}} \right) \frac{1}{2} \times \exp \left(\frac{\Gamma(\beta + 1)}{\mu} (-\sigma_1 x^\mu + \sigma_2 t^\mu) \right).$$

$$g_3(x, t) = \left(3\sigma_1(\theta_2 - \tau_3)\right) \left(1 \mp \frac{i\sqrt{3}\sigma_1(\theta_2 - \tau_3) \cos(\zeta \sqrt{-\Omega})}{4\sqrt{\tau_2 \tau_1 \sqrt{-\Omega}}} \right) \frac{1}{2} \times \exp \left(\frac{\Gamma(\beta + 1)}{\mu} (-\sigma_1 x^\mu + \sigma_2 t^\mu) \right),$$

or

$$g_4(x, t) = \left(3\sigma_1(\theta_2 - \tau_3)\right) \left(1 \pm \frac{i\sqrt{3}\sigma_1(\theta_2 - \tau_3) \tan(\zeta \sqrt{-\Omega})}{4\sqrt{\tau_2 \tau_1 \sqrt{-\Omega}}} \right) \frac{1}{2} \times \exp \left(\frac{\Gamma(\beta + 1)}{\mu} (-\sigma_1 x^\mu + \sigma_2 t^\mu) \right).$$

### 3.1 Application of $Exp_\alpha$ function approach

Here, we recall the main points of the aforesaid approach and then its demonstration has been exercised for required solutions. Let’s assume we have a NPDE in the following form:

$$G(u, u^2, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0$$

The above PDE given in the Eq. (41) may be obtained in the below form of NODE:

$$\Lambda(U, U^', U'', \ldots) = 0,$$

by implementing the below wave transformations:

$$u(x, t) = U(\tau), \ \tau = bx + rt,$$

Let us suppose a solution of Eq. (42) is of the below form Zayed and Al-Nowehy (2017); Zafar (2019):
\[
U(\tau) = \frac{\alpha_0 + \alpha_1 d^\xi + \cdots + \alpha_m d^{m\xi}}{\beta_0 + \beta_1 d^\xi + \cdots + \beta_m d^{m\xi}}, \quad d \neq 0, 1, \quad (44)
\]

Here \(\alpha_i\) and \(\beta_i\) (\(0 \leq i \leq m\)) are unknown constants and to be find later. Positive integer \(m\) is obtained by using the homogenous balance technique and balancing the highest derivative and nonlinear term in the Eq. (42). Putting Eq. (44) into non-linear Eq. (42), yields

\[
\varphi (d^\xi) = \ell_0 + \ell_1 d^\xi + \cdots + \ell_i d^{i\xi} = 0.
\]

Putting \(\ell_i (0 \leq i \leq t)\) in Eq. (45) equal to zero, a set of algebraic equations is gained as follows.

\[
\ell_i = 0, \quad \text{where} \quad i = 0, \ldots, t. \quad (46)
\]

By the obtained sets, we get the nontrivial solutions of the NPDE (41).

Since the homogenous balance technique on Eq. (9) implies \(m = 1\), therefore Eq. (44) reduces to:

\[
Q(\zeta) = \frac{\alpha_0 + \alpha_1 d^\xi}{\beta_0 + \beta_1 d^\xi}, \quad (47)
\]

where \(\alpha_0, \alpha_1, \beta_0\) and \(\beta_1\) are unknowns. By putting Eq. (47) in the Eq. (5) and collecting the coefficients of each power of \(d^\xi\), yields the set of equations having \(\alpha_0, \alpha_1, \beta_1\) and \(\beta_1\) and other parameters. By using the soft computations, we gain the following sets:

**Set 1**

\[
\begin{cases}
\alpha_0 = \alpha_0, \alpha_1 = 0, \beta_0 = -\frac{2ia_0 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)} , \beta_1 = \beta_1, \sigma_1 = \frac{2i \sqrt{\tau_1} \sqrt{\tau_2} \log(d)}{\sqrt{3} (\theta_2 - \tau_3)}, \\
\sigma_2 = \frac{\sqrt{\tau_1} \log(d)}{12 (\theta_2 - \tau_3)^2} (3\theta_2^2 \sqrt{\tau_1} \log(d) + \theta_2 \left(-6 \sqrt{\tau_1} \tau_3 \log(d) - 8i \sqrt{3} \theta_1 \sqrt{\tau_2}\right) + 16 \tau_2 \tau_1^{3/2} \log(d) + 3 \sqrt{\tau_1} \tau_3^2 \log(d) + 8i \sqrt{3} \theta_1 \sqrt{\tau_2} \tau_3) \}.
\end{cases}
\]

\[
g(x, t) = \left\{ \frac{\alpha_0}{\beta_1 d^\xi - \frac{2ia_0 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)}} \right\}^{1/3} \times \exp \left( i \frac{\Gamma(\beta + 1)}{\mu} \left(-\sigma_1 x'' + \sigma_2 t''\right) \right). \quad (49)
\]

**Set 2**
\[
\begin{align*}
\begin{cases}
\alpha_0 = \alpha_0, \quad \alpha_1 = 0, \quad \beta_0 = \frac{2i\alpha_0 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)}, \quad \beta_1 = \beta_1, \quad \sigma_1 = -\frac{2i \sqrt{\tau_1} \sqrt{\tau_2} \log(d)}{\sqrt{3}(\theta_2 - \tau_3)},
\end{cases}
\end{align*}
\]
\[\sigma_2 = \frac{\sqrt{\tau_1} \log(d)}{12(\theta_2 - \tau_3)^2}(3\theta_2^2 \sqrt{\tau_1} \log(d) + \theta_2 \left(-6 \sqrt{\tau_1} \tau_3 \log(d) + 8i \sqrt{3 \theta_1} \sqrt{\tau_2}\right) + 16 \tau_2 \tau_1^{3/2} \log(d) + 3 \sqrt{\tau_1} \tau_3^2 \log(d) - 8i \sqrt{3 \theta_1} \sqrt{\tau_2} \tau_3)\right).}
\]
\[g(x, t) = \left\{ \frac{\alpha_0}{\beta_1 d^\epsilon + \frac{2i\alpha_0 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)}} \right\}^{1 \over 2} \times \exp\left( \frac{\Gamma(\beta + 1)}{\mu} \left(-\sigma_1 x^\mu + \sigma_2 t^\mu\right) \right). \tag{51}
\]

\[\begin{align*}
\begin{cases}
\alpha_0 = 0, \quad \alpha_1 = \alpha_1, \quad \beta_0 = \beta_0, \quad \beta_1 = -\frac{2i\alpha_1 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)}, \quad \sigma_1 = \frac{2i \sqrt{\tau_1} \sqrt{\tau_2} \log(d)}{\sqrt{3}(\theta_2 - \tau_3)},
\end{cases}
\end{align*}
\]
\[\sigma_2 = \frac{\sqrt{\tau_1} \log(d)}{12(\theta_2 - \tau_3)^2}(3\theta_2^2 \sqrt{\tau_1} \log(d) + \theta_2 \left(-6 \sqrt{\tau_1} \tau_3 \log(d) - 8i \sqrt{3 \theta_1} \sqrt{\tau_2}\right) + 16 \tau_2 \tau_1^{3/2} \log(d) + 3 \sqrt{\tau_1} \tau_3^2 \log(d) + 8i \sqrt{3 \theta_1} \sqrt{\tau_2} \tau_3)\right).}
\]
\[g(x, t) = \left\{ \frac{\alpha_1 d^\epsilon}{\beta_0 - \frac{2i\alpha_1 \sqrt{\tau_2} d^\epsilon}{\sqrt{3} \sqrt{\tau_1} \log(d)}} \right\}^{1 \over 2} \times \exp\left( \frac{\Gamma(\beta + 1)}{\mu} \left(-\sigma_1 x^\mu + \sigma_2 t^\mu\right) \right). \tag{52}
\]

\[\begin{align*}
\begin{cases}
\alpha_0 = 0, \quad \alpha_1 = \alpha_1, \quad \beta_0 = \beta_0, \quad \beta_1 = \frac{2i\alpha_1 \sqrt{\tau_2}}{\sqrt{3} \sqrt{\tau_1} \log(d)}, \quad \sigma_1 = -\frac{2i \sqrt{\tau_1} \sqrt{\tau_2} \log(d)}{\sqrt{3}(\theta_2 - \tau_3)},
\end{cases}
\end{align*}
\]
\[\sigma_2 = \frac{\sqrt{\tau_1} \log(d)}{12(\theta_2 - \tau_3)^2}(3\theta_2^2 \sqrt{\tau_1} \log(d) + \theta_2 \left(-6 \sqrt{\tau_1} \tau_3 \log(d) + 8i \sqrt{3 \theta_1} \sqrt{\tau_2}\right) + 16 \tau_2 \tau_1^{3/2} \log(d) + 3 \sqrt{\tau_1} \tau_3^2 \log(d) + 8i \sqrt{3 \theta_1} \sqrt{\tau_2} \tau_3)\right).}
\]
\[g(x, t) = \left\{ \frac{\alpha_1 d^\epsilon}{\beta_0 + \frac{2i\alpha_1 \sqrt{\tau_2} d^\epsilon}{\sqrt{3} \sqrt{\tau_1} \log(d)}} \right\}^{1 \over 2} \times \exp\left( \frac{\Gamma(\beta + 1)}{\mu} \left(-\sigma_1 x^\mu + \sigma_2 t^\mu\right) \right). \tag{53}
\]
4 Results and discussion with graphical representation

In this portion, we commenced notable graphs in 2-Dimension and 3-Dimension to explain the results given above. All of our results are consistent with the results found in Ghanbari and Baleanu (2020); Hosseini et al. (2020). The graph of (22) using the modified extended tanh expansion approach at \( \mu = 1, \lambda = 0.5, \beta = 1, \sigma_1 = 0.1, \theta_1 = -0.1, \theta_2 = 2, \tau_1 = -0.1, \tau_2 = 1, \tau_3 = 0.1 \) is introduced in Fig. 1. We shown the graph of (22) using the modified extended tanh expansion approach at \( \mu = 0.5, \lambda = 0.1, \theta_1 = -0.1, \theta_2 = 2, \tau_1 = -0.1, \tau_2 = 1, \tau_3 = 0.1 \) with different values of \( \mu \) and \( \beta \) in Fig. 2. In Fig. 3 we show the graph of (37) using the modified extended tanh expansion approach at \( \lambda = 0.5, \sigma_1 = 0.1, \theta_1 = -0.1, \theta_2 = 2, \tau_1 = -0.1, \tau_2 = 1, \tau_3 = 0.1 \) with different values of \( \mu \) and \( \beta \) in Fig. 2. In Fig. 3 we show the graph of (37) using the modified extended tanh expansion approach at \( \lambda = 0.5, \sigma_1 = 0.1, \theta_1 = -0.1, \theta_2 = 2, \tau_1 = -0.1, \tau_2 = 1, \tau_3 = 0.1 \) with different values of \( \mu \) and \( \beta \) in Fig. 2. In Fig. 3 we show the graph of (37) using the modified extended tanh expansion approach at \( \lambda = 0.5, \sigma_1 = 0.1, \theta_1 = -0.1, \theta_2 = 2, \tau_1 = -0.1, \tau_2 = 1, \tau_3 = 0.1 \) with different values of \( \mu \) and \( \beta \) in Fig. 2.
The graph of (55) using the $Exp_a$ function approach at $\lambda = 0.5, \beta = 1, \alpha_0 = 0.01, \theta_1 = 0.1, \theta_2 = 0.3, \tau_3 = 0.1, \beta_1 = 2, d = 3$ is introduced in Fig. 7. Finally, we introduced the graph of (55) using the $Exp_a$ function approach at $\lambda = 0.5, \alpha_0 = 0.01, \theta_1 = 0.1, \theta_2 = 0.3, \tau_1 = 0.01, \tau_2 = 0.1, \tau_3 = 0.1, \beta_1 = 2, d = 3$ with different values of $\mu$ and $\beta$ in Fig. 8.
5 Conclusion

In this paper, the perturbed Gerdjikov–Ivanov (GI) equation with truncated $M$-fractional derivative has been worked out and found its dark and other optical soliton solutions. For
this, a fractional wave transformation was used for reducing the perturbed GI equation with truncated $M$-fractional derivative into a nonlinear ODE. Then, by applying modified extended tanh expansion and the $Exp_a$ function methods, the novel soliton solutions are obtained. The achieved results are verified by soft computations as well as explained with the help of 2-dimensional and 3-dimensional graphs.

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