A RESTRICTED SHIFT COMPLETENESS PROBLEM

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Abstract. We solve a problem about the orthogonal complement of the space spanned by restricted shifts of functions in $L^2[0, 1]$ posed by M.Carlsson and C.Sundberg.

Recently, Marcus Carlsson and Carl Sundberg posed the following problem. Let $f \in L^2[0, 1]$. Consider the Fourier transform

$$\hat{f}(\lambda) = \int_0^1 f(x) e^{ix\lambda} dx$$

of $f$ and assume that the zeros of the entire function $\hat{f}$ are simple. Denote the set of these zeros by $\Lambda$. Suppose that $\text{conv}(\text{supp } f) = [0, 1/2]$, and put

$$\mathfrak{A}_f = \text{Clos}_{L^2[0, 1]} \text{Lin}\{\tau_t f : 0 \leq t \leq 1/2\},$$

where $\tau_t f(x) = f(x - t)$. It is clear that $\{e^{i\lambda x}\}_{\lambda \in \Lambda} \perp \mathfrak{A}_f$ in $L^2[0, 1]$. The problem by Carlsson and Sundberg is whether the family

$$\{e^{i\lambda x}\}_{\lambda \in \Lambda} \cup \{\tau_t f\}_{0 \leq t \leq 1/2}$$

is complete in $L^2[0, 1]$. In this article we solve (a slightly more general form of) this problem. Our solution involves two components: a non-harmonic Fourier analysis in the Paley–Wiener space developed recently in [1], and sharp density results of Beurling–Malliavin type from [4, 5].

Theorem 1. Let $0 < a < 1$, $f \in L^2[0, 1]$, and let $\text{conv}(\text{supp } f) = [0, a]$. Denote by $\Lambda = \{(\lambda_k, n_k)\}$ the zero divisor of $\hat{f}$ (i.e., $\hat{f}$ vanishes at $\lambda_k$ with multiplicity $n_k$). Then the family

$$\{x^s e^{i\lambda_k x}\}_{(\lambda_k, n_k) \in \Lambda, 0 \leq s < n_k} \cup \{\tau_t f\}_{0 \leq t \leq 1-a}$$

is complete in $L^2[0, 1]$.

However, in the limit case $a = 1$ it is easy to show that the statement is not true.

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Theorem 2. There exists $f \in L^2[0,1]$ such that $\text{conv}(\text{supp } f) = [0,1]$, $\hat{f}$ has only simple zeros which form a set $\Lambda \subset \mathbb{R}$, and the family
\[ \{e^{i\lambda x}\}_{\lambda \in \Lambda} \cup \{f\} \]
is not complete in $L^2[0,1]$.

Proof of Theorem 1. We apply the Fourier transform and a simple rescaling to reduce our problem to the following one. Let $F$ belong to the Paley–Wiener space $PW_{\pi a}$ (the Fourier image of $L^2[-\pi a, \pi a]$), and let $\Lambda = \{(\lambda_k, n_k)\}$ be the zero divisor of $F$. Then the family
\[ \{F(z)e^{itz}\}_{|t| \leq \pi(1-a)} \cup \{K^s_{\lambda}\}_{(\lambda_k, n_k) \in \Lambda, 0 \leq s < n_k} \]
is complete in $PW_{\pi}$. Here, $K^0_{\lambda}(z) = K_{\lambda}(z) = \frac{\sin[\pi(z-\lambda)]}{\pi(z-\lambda)}$ is the reproducing kernel of the space $PW_{\pi}$, and
\[ K^s_{\lambda} = \left(\frac{d}{d\lambda}\right)^s K_{\lambda} \]
reproduce the $s$-th derivatives:
\[ \langle f, K^s_{\lambda} \rangle_{PW_{\pi}} = f^{(s)}(\lambda), \quad f \in PW_{\pi}, \lambda \in \mathbb{C}, s \geq 0. \]

It is easy to show that for every $\beta \in \mathbb{R}$, the functions
\[ F(z)\frac{\sin[\pi(1-a)(z-\beta)]}{z-\beta-2n(1-a)-1}, \quad n \in \mathbb{Z}, \]
belong to the closed linear span of $\{F(z)e^{itz}\}_{|t| \leq \pi(1-a)}$ in $PW_{\pi}$. We set $G(z) = F(z)\frac{\sin[\pi(1-a)(z-\beta)]}{z-\beta-2n(1-a)-1}$, and fix $\beta$ in such a way that $G$ has only simple zeros. Denote $\Lambda' = \{\beta + \frac{2n}{1-a}\}_{n \in \mathbb{Z}}$. It remains to verify that the family
\[ \left\{\frac{G(z)}{z-\lambda}\right\}_{\lambda \in \Lambda'} \cup \{K^s_{\lambda}\}_{(\lambda_k, n_k) \in \Lambda, 0 \leq s < n_k} \]
is complete in $PW_{\pi}$.

Assume the converse. Then there exists $h \in PW_{\pi} \setminus \{0\}$ such that
\[ \left(\frac{G(z)}{z-\lambda}, h\right) = 0, \quad \lambda \in \Lambda', \]
\[ (h, K^s_{\lambda}) = 0, \quad (\lambda_k, n_k) \in \Lambda, 0 \leq s < n_k. \]

For $0 \leq \gamma < 1$, we expand $h$ with respect to the orthogonal basis $K_{n+\gamma}$:
\[ h = \sum_{n \in \mathbb{Z}} \tilde{a}_{n,\gamma} K_{n+\gamma}, \quad \{a_{n,\gamma}\} \in \ell^2. \]
Then (2)-(3) can be rewritten as
\[
\sum_{n \in \mathbb{Z}} \frac{a_{n,\gamma}G(n + \gamma)}{n + \gamma - \lambda} = 0, \quad \lambda \in \Lambda', \\
\sum_{n \in \mathbb{Z}} \frac{\bar{a}_{n,\gamma}(-1)^n}{(n + \gamma - \lambda_k)^s} = 0, \quad (\lambda_k, n_k) \in \Lambda, \quad 0 < s \leq n_k.
\]
Changing \(\gamma\) if necessary we can assume that \(a_{n,\gamma} \neq 0, \ G(n + \gamma) \neq 0, \ n \in \mathbb{Z}\). Therefore there exist entire functions \(S_{\gamma}\) and \(T_{\gamma}\) such that
\[
\sum_{n \in \mathbb{Z}} a_{n,\gamma}G(n + \gamma) = \frac{T_{\gamma}(z) \sin[\pi(1 - a)(z - \beta)]}{\sin[\pi(z - \gamma)]},
\]
\[
\sum_{n \in \mathbb{Z}} \bar{a}_{n,\gamma}(-1)^n = \frac{S_{\gamma}(z)F(z)}{\sin[\pi(z - \gamma)]} = \frac{h(z)}{\sin[\pi(z - \gamma)]}.
\]
Since \(h = FS_{\gamma}\) does not depend on \(\gamma\), we write in what follows \(S = S_{\gamma}\).

Put \(V_{\gamma} = ST_{\gamma}\). Comparing the residues in equations (4)-(5) at the points \(n + \gamma, \ n \in \mathbb{Z}\), we conclude that
\[
V_{\gamma}(n + \gamma) = (-1)^n|a_{n,\gamma}|^2, \quad n \in \mathbb{Z}.
\]
By construction, \(V_{\gamma}\) is of at most exponential type \(\pi\). Therefore, we have the representation
\[
V_{\gamma}(z) = Q_{\gamma}(z) + \sin[\pi(z - \gamma)]R_{\gamma}(z),
\]
where
\[
Q_{\gamma}(z) = \sin \pi(z - \gamma) \sum_{n \in \mathbb{Z}} \frac{|a_{n,\gamma}|^2}{z - n - \gamma},
\]
and \(R_{\gamma}\) is a function of zero exponential type. Thus, the conjugate indicator diagram of \(V_{\gamma}\) is \([-\pi, \pi]\), and hence, the conjugate indicator diagram of \(T_{\gamma}\) and \(S\) are \([-\pi a, \pi a]\) and \([-\pi(1 - a), \pi(1 - a)]\) correspondingly. Therefore, each of the functions \(V_{\gamma}^*/V_{\gamma}, \ T_{\gamma}^*/T_{\gamma}, \) and \(S^*/S\) is a ratio of two Blaschke products. Here we use the notation \(H^*(z) = H(\bar{z})\).

It follows from (5) that
\[
\frac{S(z)F(z)}{\sin[\pi(z - \gamma)]} \cdot \frac{S^*(z)}{S(z)} = \sum_{n \in \mathbb{Z}} \bar{a}_{n,\gamma}(-1)^n\frac{S^*(n + \gamma)}{n + \gamma - z} + H(z)
\]
for some entire function \(H\). Since \(FS^* \in \mathcal{P}W_{\pi}\), we conclude that \(H\) is of zero exponential type and tends to 0 along the imaginary axis. Thus, \(H = 0\).

We set \(\bar{b}_{n,\gamma} = \bar{a}_{n,\gamma}S^*(n+\gamma)/S(n+\gamma)\), and obtain
\[
\sum_{n \in \mathbb{Z}} \bar{b}_{n,\gamma}(-1)^n\frac{S^*(z)F(z)}{n + \gamma - z} = \frac{\sin \pi(z - \gamma)}{z - n - \gamma}.
\]
Analogously, using the fact that the function $z \mapsto T_\gamma(z) \sin[p(1-a)(z-\beta)]$ belongs to $PW_\pi$ and the fact that $ST_\gamma$ is real on $\mathbb{Z} + \gamma$, we deduce from (4) that
\[
\sum_{n \in \mathbb{Z}} b_{n,\gamma} G(n + \gamma) \frac{n + \gamma - z}{n + \gamma - z} = \frac{T_*^\gamma(z) \sin[p(1-a)(z-\beta)]}{\sin[p(z-\gamma)]}.
\]
Thus, the function
\[
g = \sum_{n \in \mathbb{Z}} \bar{b}_n K_{n+\gamma}
\]
is orthogonal to the system (1), whence the elements $h + g$, $ih - ig$ are also orthogonal to (1), and correspond to the pairs $(S + S^*, T_\gamma + T_\gamma^*)$, $(iS - iS^*, -iT_\gamma + iT_\gamma^*)$. Therefore, from now on we assume that $S$, $T_\gamma$, and hence, $V_\gamma$ are real on the real line.

Now it follows from (6) that the function $V_\gamma$ has at least one zero in every interval $(n + \gamma, n + 1 + \gamma)$, $n \in \mathbb{Z}$. By (7), the zeros of $V_\gamma$ coincide with the zeros of the function
\[
R_\gamma(\lambda) + \sum_{n \in \mathbb{Z}} |a_{n,\gamma}|^2 \lambda - n - \gamma.
\]
Next we fix $\gamma \in [0, 1)$ and a sufficiently small $\delta > 0$ for which there exist two subsets $\Sigma, \Sigma_1$ of the zero set $\mathcal{Z}(S)$ of the function $S$ with the following properties:
- $\Sigma$ has exactly one point in those intervals where $\mathcal{Z}(S) \cap [n + \gamma, n + 1 + \gamma) \neq \emptyset$, and
  \[\text{dist}(x, \mathbb{Z} + \gamma) > \frac{\delta}{1 + x^2}, \quad x \in \Sigma;\]
- $\Sigma_1$ has positive upper density, and $\text{dist}(x, \mathbb{Z} + \gamma) > \delta$, $x \in \Sigma_1$.

From now on, we use the notations $R = R_\gamma$, $a_n = a_{n,\gamma}$, $V = V_\gamma$, $T = T_\gamma$. We need to consider three cases. If $R$ is a nonzero polynomial, then the zeros of the function (8) approach $\mathbb{Z} + \gamma$ and we obtain a contradiction to the existence of $\Sigma_1$. If $R = 0$, then [1, Proposition 3.1] implies that the density of $\Sigma_1$ is zero. Finally, if $R$ is not a polynomial, we can divide it by $(z-z_1)(z-z_2)$, where $z_1$ and $z_2$ are two arbitrary zeros of $R$, $z_1, z_2 \notin \Sigma$, to get a function $R_1$ of zero exponential type which is bounded on $\Sigma$.

Next, we obtain some information on $\Sigma$. For a discrete set $X = \{x_n\} \subset \mathbb{R}$ we consider its counting function $n_X(t) = \text{card}\{n : x_n \in [0, t)\}$, $t \geq 0$, and $n_X(t) = -\text{card}\{n : x_n \in (-t, 0)\}$, $t < 0$. If $f$ is an entire function and $X$ is the set of its real zeros (counted according to multiplicities), then there exists a branch of the argument of $f$ on the real axis, which is of the form $\arg f(t) = \pi n_X(t) + \psi(t)$, where $\psi$ is a smooth function. Such choice of the argument is unique up to an additive constant and in what follows we always assume that the argument is chosen to be of this form.
We use the (easy to show) fact that for every function \( f \in \mathcal{PW}_\pi \) with the conjugate indicator diagram \([-\pi, \pi]\) and all zeros in \( \mathbb{C}_+ \), one has

\[
\arg f = \pi x + \tilde{u} + c,
\]

where \( u \in L^1((1 + x^2)^{-1}dx) \), \( c \in \mathbb{R} \). Here \( \tilde{u} \) denotes the conjugate function (the Hilbert transform) of \( u \),

\[
\tilde{u}(x) = \frac{1}{\pi} v.p. \int_{\mathbb{R}} \left( \frac{1}{x - t} + \frac{t}{t^2 + 1} \right) u(t) dt.
\]

It follows from (4)–(5) that \( FV \in \mathcal{PW}_{\pi a + \pi} \). Now let us replace all zeros \( \lambda \) of the functions \( h, F, S, T, \) and \( V \) in \( \mathbb{C}_- \) by \( \bar{\lambda} \). Since the Paley–Wiener space is closed under division by Blaschke products, we still have for the new functions \( h, F, S, T, \) and \( V \) (which we denote by the same letters) that \( h \in \mathcal{PW}_\pi \) and \( FV \in \mathcal{PW}_{\pi a + \pi} \). Recall that the function \( V \) has at least one zero in each of the intervals \((n + \gamma, n + 1 + \gamma)\), \( n \in \mathbb{Z} \). Let us consider its representation \( V = V_0 H \), where the zeros of \( V_0 \) are simple, interlacing with \( \mathbb{Z} + \gamma \) and \( V_0|_\Sigma = 0 \). It is clear that \( \arg V_0 = \pi x + O(1) \). Since, by (9),

\[
\arg(FV) = \pi ax + \pi x + \tilde{u} + c,
\]

we conclude that

\[
\arg(FH) = \pi ax + \tilde{u} + O(1).
\]

Consider the equality \( h = FHS/H \) and note that

\[
\arg \left( \frac{S}{H} \right) = \pi n_\Sigma - \alpha,
\]

where \( \alpha \) is some nondecreasing function on \( \mathbb{R} \). This follows from the fact that \( S/H \) vanishes only on a subset of the real axis which contains \( \Sigma \) and \( \frac{S}{H} \) is a Blaschke product. Applying the representation (9) to \( h \), we conclude that

\[
\pi n_\Sigma(x) = \pi (1 - a)x + \tilde{u} + v + \alpha,
\]

where \( u \in L^1((1 + x^2)^{-1}dx) \), \( v \in L^\infty(\mathbb{R}) \), and \( \alpha \) is nondecreasing.

Summing up, we have an entire function \( R_1 \) of zero exponential type which is not a polynomial, and which is bounded on a set \( \Sigma \subset \mathbb{R} \) satisfying (10).

To deduce a contradiction from this, we use some information on the classical Polya problem and on the second Beurling–Malliavin theorem. We say that a sequence \( X = \{x_n\} \subset \mathbb{R} \) is a *Polya sequence* if any entire function of zero exponential type which is bounded on \( X \) is a constant. We say that a disjoint sequence of intervals \( \{I_n\} \) on the real
line is a long sequence of intervals if

\[ \sum_n \frac{|I_n|^2}{1 + \text{dist}^2(0, I_n)} = +\infty. \]

A complete solution of the Polya problem was obtained by Mishko Mitkovski and Alexei Poltoratski \[5\]. In particular, a separated sequence \(X \subset \mathbb{R}\) is not a Polya sequence if and only if there exists a long sequence of intervals \(\{I_n\}\) such that

\[ \frac{\text{card}(X \cap I_n)}{|I_n|} \to 0. \]

Applying this result to our \(R\) and \(\Sigma\) (formally speaking, \(\Sigma\) is not a separated sequence but by construction it is a union of two separated sequences which are interlacing), we find a long system of intervals \(\{I_n\}\) such that

\[ \frac{\text{card}(\Sigma \cap I_n)}{|I_n|} \to 0. \]

Given \(I = [a, b]\), denote \(I^- = [a, (2a + b)/3]\), \(I^+ = [(a + 2b)/3, b]\),

\[ \Delta_I^* = \inf_{I^+}[\pi(1-a)x - \pi n_\Sigma(x) + v] - \sup_{I^-}[\pi(1-a)x - \pi n_\Sigma(x) + v]. \]

Now, for a long system of intervals \(\{I_n\}\) and for some \(c > 0\) we have

\[ \Delta_{I_n}^* \geq c|I_n|. \]

Next we use a version of the second Beurling–Malliavin theorem given by Nikolai Makarov and Alexei Poltoratskii in \[4, Theorem 5.9\]. This theorem (or rather its proof) gives that if the function \(\pi(1-a)x - \pi n_\Sigma(x) + v\) may be represented as \(-\alpha - \tilde{u}\) for \(\alpha\) and \(u\) as above, then there is no such long family of intervals. This contradiction completes the proof. \(\square\)

**Proof of Theorem 2.** Let \(\Lambda_1 = \{\pi 2^n\}_{n \geq 1}\), \(\Lambda = 2\pi \mathbb{Z} \setminus \Lambda_1\). Consider the entire function \(F\) of zero order with zero set \(\Lambda_1\), and denote \(G(z) = e^{iz/2} \cdot \frac{\sin(z/2)}{f(z)}\). Then \(G = \hat{f}\) for some \(f \in L^2[0, 1]\), and \(L^2[0, 1] \ominus \{e^{i\lambda x}\}_{\lambda \in \Lambda}\) is of infinite dimension. \(\square\)

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1 The “only if” part of this statement is implicitly contained in the results of Louis de Branges in the 1960-s: \[2, Theorem XI, 3, Theorems 66, 67\]; see also \[5, Remark, page 1068\].
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