Stable and chaotic solutions of the complex Ginzburg-Landau equation with periodic boundary conditions

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Abstract

We study, analytically and numerically, the dynamical behavior of the solutions of the complex Ginzburg-Landau equation with diffraction but without diffusion, which governs the spatial evolution of the field in an active nonlinear laser cavity. Accordingly, the solutions are subject to periodic boundary conditions. The analysis reveals regions of stable stationary solutions in the model’s parameter space, and a wide range of oscillatory and chaotic behaviors. Close to the first bifurcation destabilizing the spatially uniform solution, a stationary single-humped solution is found in an asymptotic analytical form, which turns out to be in very good agreement with the numerical results. Simulations reveal a series of stable stationary multi-humped solutions.

1 INTRODUCTION

There is a great current interest in dynamics described by the complex Ginzburg-Landau equation (CGLE) [1], which is a generic nonlinear model with various physical applications, including binary-fluid thermal convection [2], semiconductor lasers [3], nonlinear optical fibers (see Ref. [4] and references therein), etc. In most applications, the boundary conditions (BC) supplementing the CGLE are an inherent type of the underlying physical model. In particular, in the case of the thermal convection and laser cavities, the simplest and most fundamental type of BC are the periodic conditions, as they make it possible to study the intrinsic dynamics of CGLE avoiding obscuring effects of wave reflection from edges.

It should be stressed that the dynamics of the CGLE with periodic BC is principally different from that of an infinite system, as various solutions (even chaotic) which are stable under periodic BC may be easily destabilized in a much longer system [5]. Another important difference from the infinite limit is that the equation subject to periodic BC is equivalent to an infinite chain of evolution equations for amplitudes of the corresponding Fourier expansion. Using this approach, it has been proved that a finite-dimensional attractor exists in the system’s phase space [6] (which is true even in the case of two-dimensional CGLE [7]). Parallel to this, different technical versions of an approximation based on a finite-mode (Galerkin) truncation of the above-mentioned infinite chain of equations were developed, helping to understand the onset of dynamical chaos in the CGLE [8]. Although it was argued that the Galerkin truncation does not always describe adequately the transition to chaos in CGLE [9], it
has been demonstrated that, generally, the finite-mode truncation of the CGLE with periodic BC provides for quite accurate results [10].

For some particular cases, a chain of bifurcations leading to chaos in CGLE under periodic BC was found by means of direct numerical simulations [11]: the chain features transitions limit-cycle $\rightarrow$ two-torus $\rightarrow$ three-torus $\rightarrow$ chaos, $n$-torus being a quasi-periodic dynamical regime with $n$ incommensurate frequencies. Many other nonlinear modes are unstable at onset (i.e., when they appear), but they may be later restabilized through higher-order bifurcations [12].

In this paper, we study analytically and numerically the dynamical behavior and, in particular, stable solutions of the cubic CGLE, subjected to periodic BC, without diffusion, while diffraction (which formally corresponds to an imaginary part of the diffusion coefficient) is present. This situation is typical for laser cavities and other nonlinear optical problems in the spatial domain (see Ref. [13] and [14]), as light beams are subject to transverse diffraction but not diffusion.

A general form of the one-dimensional CGLE with cubic nonlinearity and without diffusion is

$$iu_t + \frac{1}{2}u_{xx} (1 + iR)|u|^2 u = iu.$$  \hspace{1cm} (1)

Here, $u = u(x,t)$ is the complex field which depends on the time $t$ and the spatial coordinate $x$, $R$ is a real positive parameter that represents cubic losses and/or gain saturation. The coefficients of the spatial dispersion, the Kerr nonlinearity and the linear gain (the latter one accounts for the term on the right-hand side of Eq. (1)) are all normalized to 1. In nonlinear optics, where Eq. (1) finds its most important application, it describes the spatial evolution of the electromagnetic field envelope $u(t, x)$ in a planar waveguide or laser cavity with linear gain, cubic loss and Kerr nonlinearity. In this case, the evolutional variable $t$ is, in fact, not time but the propagation distance (which is frequently designated $z$), and the second term in Eq. (1) accounts for the transverse diffraction [13].

Equation (1) also describes the temporal or spatial evolution of the electric field in a (1+1)-dimensional semiconductor laser medium with gain saturation. The second term in Eq. (1) then accounts for the diffraction in space or dispersion in the temporal domain, and the cubic conservative term represents a normalized anti-guiding or linewidth-enhancement factor.

In either case, we supplement Eq. (1) with periodic BC, , where $L$ is the system’s length (in the transverse direction, in the case of the planar waveguide). In its literal form, Eq. (1) with periodic BC applies to a planar waveguide closed into a cylindrical surface. Such a configuration can be easily implemented in Vertical Cavity Surface-Emitting Lasers (VCSEL) [3]. However, the most widespread application of Eq. (1) subject to periodic BC is the description of a ring cavity laser, in which some ingredients of the system (e.g., the saturable amplifier and nonlinear waveguide) may be separated. In this context, Eq. (1) has the meaning of an average master equation that govern the evolution of the electromagnetic field in the cavity. Thus, the model is controlled by two dimensionless parameters, $R$ and $L$. Despite a number of results obtained in
the above-mentioned works for CGLE of a more general type with periodic
BC, the case of zero diffusion is fundamentally important for applications to
nonlinear optics and must be considered separately.

Equation (1) has an obvious continuous-wave (CW), i.e., stationary spatially
uniform solution,
\[ u_0 = R^{-1/2} \exp(iR^{-1}t). \] (2)

A straightforward stability analysis, taking into account the periodic BC,
shows that the CW solution (2) is unstable to small perturbations if the length
L exceeds a critical value:
\[ L_{cr} = \pi \sqrt{R}. \] (3)

A primary objective of this work is to study the model’s dynamics in the
case \( L > L_{cr} \). To this end, we solved Eq. (1) numerically, using the uniform
solution (2) perturbed by a small-amplitude harmonic with the fundamental
wave-number \( (2\pi/L) \) as initial conditions. We have found that the long-time
evolution of the solutions exhibits a wide variety of behaviors (stable, oscillating,
chaotic), depending on values of the parameters \( R \) and \( L \).

The rest of the paper is organized as follows. In section 2, we display a phase
diagram, which summarizes the dynamical behavior of the system in terms of the
\((R, L)\) parametric plane. In this section, we also find, in an asymptotic analytical
form, the first spatially nonuniform stable stationary solution which sets in after
the stability loss of (2), and compare it to numerical results. Although the
procedure leading to the analytical solution is quite simple, the solution itself
appears to be a new one. In section 3, we study higher-order stable stationary
solutions which have a multi-humped structure. In section 4, we study various
oscillatory dynamical regimes and section 5 concludes the paper.

All these dynamical regimes admit straightforward interpretation in terms
of field patterns in the laser cavity which is modeled by Eq. (1). In fact, as it was
already mentioned above, we study in detail two vast classes of regular stable
solutions: stationary and oscillatory (we also find chaotic dynamical regimes,
but, from the viewpoint of applications, they are interesting mainly in the sense
of choosing parameter values in order to avoid chaos). The physical relevance of
both types of regular dynamical regimes is quite obvious. In particular, we will
find different parametric regions that feature stable stationary states with one
or several humps (local maxima). This finding has straightforward applications
in terms of the laser cavity, making it possible, for instance, to simultaneously
generate several output light beams. Stable oscillatory regimes are relevant
too, as they can help the laser to generate a temporally modulated (pulsed)
beam with various types of the modulation, depending on the quasi-harmonic
or strongly anharmonic character of the oscillations (we find stable solutions of
both types).
2 A DIAGRAM OF DYNAMICAL REGIMES AND BASIC STATIONARY STATES

Figure 1 displays a diagram of different dynamical regimes in the parametric plane \((L, R)\) generated by numerical simulations of Eq. (1). The figure shows several regions of regular (stationary or oscillatory) solutions, which are separated by chaotic layers. Regular solutions existing in different regions exhibit different spatial patterns. Each region is divided into sub-regions of stationary and oscillating solutions.

In region I, which is \(R < R_c\), the uniform solution (2) is stable. Direct simulations demonstrate that, indeed, there is no other attractor in this region. The first stable stationary non-uniform solution found beyond the boundary (3) is shown in Fig. 2. Close to the boundary, this solution can be approximated analytically as a combination of a uniform field and the first two spatial harmonics:

\[
 u(x, t) = R^{-1/2} \exp \left(i R^{-1} t - i \omega_2 t\right) \cdot \left[1 + a_0 + a_1 \cos(kx) + a_2 \cos(2kx)\right] \quad (4)
\]

where \(k = 2\pi/L\), \(a_0\), \(a_1\) and \(a_2\) are, respectively, small amplitudes of the fundamental, zeroth, and second harmonics, \(\omega_2\) is a frequency shift, and it is expected that \(a_0\), \(a_1\) and \(\omega_2\) are \(\sim a_1^2\).

To proceed with the analytical consideration of the solution (4), we eliminate \(a_0\), \(a_1\) and \(\omega_2\), substituting Eq. (4) into Eq. (1) and equating terms at order \(a_1^2\). We notice that the ansatz (4) allows one to assume that \(a_0\) is purely real.

Obviously, \(a_0\) is real too, while \(a_1^2\) and \(a_2^2\) may be complex:

\[
 a_1 \equiv a_{11} + ia_{12}, \quad a_2 \equiv a_{21} + ia_{22}. \quad (5)
\]

Substituting (5) into (4) and solving for both real and imaginary parts, we obtain the following expressions at order \(a_1^2\):

\[
 \omega_2 = \left(1 + \frac{1}{R^2}\right) a_{11} a_{12}, \quad a_0 = \frac{3}{4} a_{11}^2 - \frac{1}{4} a_{12}^2 - \frac{1}{2 R} a_{11} a_{12},
\]

\[
 a_{21} = \frac{1}{4} a_{11}^2 + \frac{1}{12} a_{12}^2 - \frac{R}{6} a_{11} a_{12}, \quad a_{22} = \frac{R}{4} a_{11}^2 + \frac{R}{12} a_{12}^2 - \left(\frac{R^2}{24} - \frac{1}{8}\right) a_{11} a_{12}. \quad (6)
\]

Next, we equate coefficients in front of the fundamental harmonic (obtained after the substitution of (4) into (1)) to find the amplitude \(a_1\) at order \(a_1^2\). It is convenient to introduce a small positive parameter \(\delta\) (overcriticality) which measures proximity of the system to the bifurcation point (3),

\[
 \delta \equiv 4R^{-1} - k^2 \quad (7)
\]

(recall \(k = 2\pi/L\)). The amplitude \(a_1\) can then be found as a function of \(\delta\),

\[
 a_{12} = Ra_{11}, \quad a_{11}^2 = \frac{3R}{R^4 - 9R^2 + 30\delta}. \quad (8)
\]

Figures 3 and 4 present comparison between the analytical solution (4) - (8) and a numerical one for different values of \(R\) and \(L\). Up to relatively large values
of the overcriticality ($\delta = 0.30$ in Fig. 3), fairly good agreement between the numerical and analytical solutions is evident. A very good agreement was also found between the analytically calculated frequency shift and the shift that was extracted from numerical simulations. With further increase of $\delta$, a deviation between the analytical and the numerical solutions becomes visible (at $\delta \approx 0.40$ in Fig. 4).

The parameter range, in which this stationary solution is stable, is a part of region II in Fig. 1. Although the region is narrow, it has an internal structure, being divided into three subregions, hosting the above stationary solution and oscillatory ones. A detailed consideration of the subregions is given in section 4. Above the region II (at larger $L$ and $R < 2.6$), the simulations reveal solutions which are chaotic in time.

3 HIGHER-ORDER STABLE STATIONARY SOLUTIONS

Another region partly filled with stable stationary solutions is found as $L$ is increased, namely, region III in Fig. 1. In contrary to the solutions considered above, the profile of the solution in the region III, a typical example of which is depicted in Fig. 5, has two maxima (with different values) and two minima, being similar to solutions discussed, in the framework of a similar model, in Ref. [15]. In this region, the amplitude of the second harmonic is not small in comparison with that of the fundamental harmonic, giving rise to the double-humped profile of the solution. A minimum value of the nonlinear-loss coefficient for which region III exists is $R = 1.06$, the corresponding value of the system’s length being $L = 5.9$. Regions II and III merge at $R > 2.6$ and form a single region (see Fig. 1). The dotted line inside the united II - III region in Fig. 1 indicates a border between the above-mentioned single-humped and double-humped stationary solutions. This border is a direct continuation of the upper border of region II at lower values of $R$, strongly suggesting that the transition between the single- and double-humped patterns is caused by destabilization of the single-humped solution. In fact, region III, as well as II, is also divided into several subregions exhibiting stationary and oscillating regular solutions.

Above region III, there is a well-defined stripe in which the behavior of the system is chaotic. For higher values of $L$, still another range of stable stationary solutions, region IV in Fig. 1, is revealed. Stationary solutions in this region also have two lobes, like in region III, but its second-harmonic component is larger, giving rise to a profile which is similar to the solution discussed in Ref. [16] (a cnoidal-wave-type, see Fig. 6). It should be noted, however, that region IV is rather narrow.

The next stability range for stationary solutions is found directly above region IV, at $R > 1.72$ and $L > 11.8$ (region V in Fig. 1). In this region, stable solutions have four maxima (two pairs of local maxima with of different heights), see Fig. 7. It seems plausible that this solution was obtained via a
doubling bifurcation of the double-humped solution found above in region II, especially since the minimal system length for this region \((L = 11.8)\) is exactly twice the minimal length for region III \((L = 5.9)\).

This pattern of changing the structure of stationary solutions develops at larger values of \(L\), with stability regions featuring stationary solutions that appear to be produced by tripling and quadrupling the basic stationary solution found in region III (see Fig. 5). As \(L\) is increased, reaching region VI in Fig. 1, a new stable stationary solution of Eq. (1) appears, characterized by local maxima with three different amplitudes, see Fig. 8. The minimum value of \(R\) for which this solution appears is \(R = 1.55\), at \(L = 18\). This solution evolves gradually from the four-lobed solution found in region V and described above, see Fig. 7. As \(L\) is increased, the two small maxima move outwards; simultaneously, two additional humps evolve between the small maxima and the central high maxima (see Fig. 9). Eventually, these two humps evolve into two additional maxima.

4 OSCILLATORY SOLUTIONS

As it was mentioned before, each region containing stable regular solutions is actually divided into several subregions that exhibit a wide variety of dynamical behaviors. As an example, we present here a detailed description of the intrinsic structure of region II.

Figure 10 illustrates the intrinsic structure of region II for \(0 < R < 2.2\). Although this region is relatively narrow, it is divided into three subregions. In the first subregion (IIa), the established solution is always the stationary one that was described in section 2. For larger values of \(L\) (subregion IIb), the solution does not remain stationary; instead, it becomes an oscillatory function of \(t\) (see Fig. 11). Further increase of \(L\) results in a change of the character of the oscillations in the subregion IIc: while the oscillations in the subregion IIb were quasi-harmonic, in the subregion IIc they acquire a more complex anharmonic structure. Clear characterization of the oscillations of different types is provided by examining the corresponding time dependence of the average field power, which is defined in a straightforward way,

\[
P(t) = L^{-1} \int_0^L |u(x, t)|^2 dx,
\]

and also by the temporal Fourier transform (power spectrum) of the time-dependent power (9). In particular, in Fig. 12 it is easy to see distinction between the aforementioned quasi-harmonic and strongly anharmonic oscillations, in terms of the dependences \(P(t)\).

Figure 13 shows the power spectrum of \(P\) obtained at \(R = 0.7\), while varying \(L\) between \(L = 2.9\) and beyond the upper border of region II \((L = 3.15)\). In this figure, the dc (zero-harmonic) component of the Fourier transform was removed.
in order to stress the dynamical behavior. As $L$ crosses the upper border of region IIa, two symmetric harmonics with opposite frequencies appear, indicating that the solution is oscillating quasi-harmonically. As $L$ is increased further, the frequency of the oscillations drops to a minimum value, after which several higher-order harmonics appear, and the dominant frequency increases with $L$. At the point with the lowest value of the oscillation frequency, a transition between two different dynamical regimes, viz., quasi-harmonic oscillations whose frequency decreases with $L$ and complex multi-frequency oscillations whose dominant frequency increases with $L$, takes place.

Further increase of $L$ eventually results in a transition to dynamical chaos. A similar behavior is observed when decreasing $R$ at constant $L$.

It should be emphasized that all the other stability regions marked in Fig. 1 are also divided into subregions which exhibit various oscillatory dynamical regimes, see examples shown in Fig. 14 for region V, and in Fig. 15 for region VI. However, a detailed analysis of these subregions is beyond the scope of the present paper.

5 CONCLUSIONS

We have studied the dynamical properties of the solutions of the complex cubic Ginzburg-Landau equation without diffusion, which is subject to periodic boundary conditions. This equation is a fundamental model of nonlinear-optical cavities characterized by saturable gain. Systematic investigation of its dynamics is necessary, as it is essentially different from both a more general Ginzburg-Landau equation with its entire coefficients complex (i.e., including also diffusion), and from its infinite-length counterpart.

We found that the model exhibits a complex structure, including regions of stable stationary, oscillatory, and chaotic solutions in its parameter plane. For a case of strong nonlinear loss, the system features a multi-layered structure in the parameter plane, with a different dynamical behavior in each layer. A boundary between adjacent layers implies the existence of a bifurcation destabilizing the solution in one of layers. This could be easily seen by examining the boundary between regions II and III in Fig. 1, i.e., the upper boundary of region II at $R > 2.6$: it is a direct continuation of the upper boundary of region II which, at $R < 2.6$, separates it from a chaotic zone. It should be stressed that we did not observe hysteresis (overlapping) between stable solutions of different types.

We have found an asymptotic analytical single-humped solution for the case when the system is close to the first bifurcation destabilizing the spatially uniform CW solution. Even when the system is not really close to the bifurcation point, the analytical solution is found to be in good agreement with numerical results. Numerically, we have also found a series of stable stationary solutions of a multi-humped type.

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Figure Captions

Fig. 1. The dynamical phase diagram of the system, showing regions of regular and chaotic behavior in the parameter plane \((R, L)\), see detailed description of the regions in the text.

Fig. 2. A stable stationary nonuniform solution found for \(R = 1.0\) and \(L = 3.4\) (inside region II of Fig. 1), which has a single maximum and a single
minimum. In this figure and in Figs. 3 - 9 below, shown is $|u(x)|$ in the interval $0 < x < L$.

Fig. 3. Comparison between the analytical (circles) and the numerical (solid) stationary solutions for $R = 0.7$ and $L = 2.7$. At these values of $R$ and $L$, Eq. (7) gives the value of the overcriticality parameter $\delta = 0.30$, and Eq. (3) yields the critical value beyond which the nontrivial stationary solution may exist, $L_{cr} = 2.628$.

Fig. 4. Comparison between the analytical (dashed) and numerical (solid) stationary solutions for $R = 1.4$ and $L = 4.01$. At this value of $R$, one has $L_{cr} = 3.717$ and $\delta = 0.40$.

Fig. 5. A typical profile of the stable stationary solution found in region III, for $R = 2.0$ and $L = 6.0$. The solution exhibits two maxima and two minima.

Fig. 6. The profile of the stable stationary solution found in region IV for $R = 3.2$ and $L = 11.8$, featuring a cnoidal-wave-like solution with two maxima.

Fig. 7. The profile of the stable stationary solution found in region V for $R = 2.5$ and $L = 12.0$, which exhibits a double-period solution having local maxima and minima with different amplitudes.

Fig. 8. The profile of the stable stationary solution obtained in region VI for $R = 1.8$ and $L = 18.8$, with three maxima having different amplitudes.

Fig. 9. The evolution of the stable stationary solution observed when crossing from region V into VI, for $R = 2.0$ and $L = 13.2$ (solid), 17.0 (dotted), 19.0 (dashed), 21.0 (dashed-dotted).

Fig. 10. The inner structure of region II which reveals three subregions that feature, respectively, stable stationary patterns and quasi-harmonic or strongly anharmonic oscillatory solutions, as it is described in the text in detail.

Fig. 11. The oscillatory solution as a function of $t$, found in region IIb (for $R = 0.3$ and $L = 1.85$).

Fig. 12. Comparison between the time dependence of the average field power $P(t)$ in regions IIb (for $R = 0.7$ and $L = 2.97$, solid line) and IIc (for $R = 0.7$ and $L = 3.02$, dashed line).

Fig. 13. The evolution of the power spectrum of $P(t)$ inside the interval $2.9 < L < 3.15$ at fixed $R = 0.7$, indicating at the existence of two different types of oscillatory regimes.

Fig. 14. Examples of the temporal evolution of the field corresponding to different types of oscillatory dynamical regimes in region V: (a) $R = 4.0$, $L = 17.5$; (b) $R = 3.0$, $L = 17.4$.

Fig. 15. Field evolution in time in region VI, for $R = 1.7$ and $L = 16.1$. 
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