CR-Calculus and adaptive array theory applied to MIMO random vibration control tests

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Abstract. Performing Multiple-Input Multiple-Output (MIMO) tests to reproduce the vibration environment in a user-defined number of control points of a unit under test is necessary in applications where a realistic environment replication has to be achieved. MIMO tests require vibration control strategies to calculate the required drive signal vector that gives an acceptable replication of the target. This target is a (complex) vector with magnitude and phase information at the control points for MIMO Sine Control tests while in MIMO Random Control tests, in the most general case, the target is a complete spectral density matrix. The idea behind this work is to tailor a MIMO random vibration control approach that can be generalized to other MIMO tests, e.g. MIMO Sine and MIMO Time Waveform Replication. In this work the approach is to use gradient-based procedures over the complex space, applying the so called CR-Calculus and the adaptive array theory. With this approach it is possible to better control the process performances allowing the step-by-step Jacobian Matrix update. The theoretical bases behind the work are followed by an application of the developed method to a two-exciter two-axis system and by performance comparisons with standard methods.

1. Introduction

Multiple-Input Multiple-Output (MIMO) vibration control tests are performed to subject a unit under test to a realistic three-dimensional dynamic environment. Nowadays the common practice, when multiple directions are required for testing the exposure of a structure to a given vibration environment, is to perform sequential single-axis tests. This practice has known drawbacks: on one hand there is certainly a lack of physical meaning in replicating in-service conditions with a single axis test; on the other hand sequential single-axis applications can be time consuming and dangerous, since the test set-up has to be changed multiple times with also the unacceptable risk of damaging the structure. In case of testing heavy slender structures to high overall levels, a single input could also lead to high concentrated loads with the consequence of entering the non-elastic range of the material properties. MIMO tests overcome these limitations and the recent updates to include tailoring guidelines for multi-exciter testing in the United States Military Standards [1] highlight the necessity to improve the knowledge of this innovative practice. Smith and Staffanson in [2] provide an interesting theoretical explanation of how a MIMO application can better reproduce the test article dynamic behaviour: with multiple degrees of freedom it is possible to adequately distribute the input energy in more modes simultaneously and avoid the specimen over or under testing. The more recent work of Daborn et al. [3] presents an experimental case of study showing that, in case of aerodynamic
excitation of a dummy rocket, a single axis standard technique does not reproduce correctly
the relative phase information between the control channels and that the specimen is subject
to far greater loads. Even though the benefits of multi-axial testing are clear and accepted by
the environmental engineering community, their practice still needs to grow. This is basically
due to the control and stability problems that could arise during the multi-axial excitation
and the inherent level of expertise needed to run these tests. In the specific case of MIMO
random vibration control applications, innovative results in the control strategy are mainly due
to the work of Smallwood [4], Underwood [5] and Peeters [6]. In [5] and [7] it has been pointed
out that similarities reside between the different main MIMO vibration control strategies, with
the main differences being the excitations adopted and the target definition. The objective of
these tests is to be able to replicate a certain control response by using as target function a
Power Spectral Density matrix (MIMO random), response spectra (MIMO sine) or time domain
waveforms (MIMO TWR, Time Waveform Replication). In the latter case the time waveforms
can be transformed in the frequency domain by making use of the Fourier Transform. The
control can be performed in all these cases in the frequency domain. The idea behind this
work is to derive a reliable vibration control approach that can, starting from a MIMO Random
application, be further generalized to other MIMO vibration control tests, e.g. MIMO Sine and
MIMO TWR. In this work the possibility of using the Complex-Real Calculus (\(\mathbb{CR}\)-Calculus) to
develop a gradient-based control strategy to update the drives, has been studied for the specific
case of MIMO Random vibration control test. A pioneer in the application of the \(\mathbb{CR}\)-Calculus
to engineering-related topics has been Brandwood [8] by providing a useful application to the
adaptive array theory where nowadays this approach is consistently used. Further insight into
the \(\mathbb{CR}\)-Calculus has been provided by Sorber et al. in [9]. As also shown here, the need for
inverting the Jacobian matrix at each iterations step is common to multi-variable gradient-based
techniques. Broyden [10] explains a method for solving non linear simultaneous equations by
updating the Jacobian matrix instead of recomputing it again during the iterative process.

In this paper a first insight of an early stage control strategy developed by making use of the
\(\mathbb{CR}\)-Calculus theoretical principles along with the Broyden’s update is given. The convergence
of the drive correction process is tested on a simple two degrees of freedom model with two
excitation axes and two control channels. Due to the small size of the matrices involved in the
application, the Broyden’s update computational advantages cannot be shown; nevertheless, the
Jacobian matrix update showed the advantage of better control spectral lines that are outliers
in the control process. This property needs to be further proved and leaves room for future
investigations. The article is organized as follows: in section 2 the basic principles of the \(\mathbb{CR}\)-
Calculus and the Jacobian matrix update are explained; in section 3 the application to the MIMO
Random Control tests is presented; in section 4 the control strategy is applied to a simple two
degrees of freedom system, showing the convergence of the process. A comparison with a classical
tested MIMO random control strategy is also presented highlighting the limitation of the early
stage procedure also compared with the commercial MIMO Control software capabilities. The
conclusions and the future directions are finally summarized in section 5.

In this work vectors are denoted by lower case bold letters, e.g. \(\mathbf{a}\), and matrices by upper
case bold letters, e.g. \(\mathbf{A}\). If not specified differently, in section 3 and 4 the arrays are functions of
the angular frequency \(\omega\) [rad/s] whereas time dependency will always be specified. An over-bar
is used to indicate the complex conjugate operation and the apex \(\nexists\) to indicate the complex
conjugate transpose of a matrix, e.g. \(\mathbf{b}\) and \(\mathbf{B}^H\) are the complex conjugate and the complex
conjugate transpose of the vector \(\mathbf{b}\) and the matrix \(\mathbf{B}\), respectively.

2. \(\text{CR-calculus and adaptive array theory applied to vibration testing}\)

The common denominator of all the MIMO vibration control techniques is that the user needs
to excite the structure with multiple individually driven shakers in order to replicate the target’s
specification in a user-defined number of control points within given tolerance boundaries. The whole process needs to be controlled; if the control action is performed in the frequency domain this means to shape the specimen’s inputs (the so called drives) in order to track the real and imaginary part of the response signal’s frequency representation, minimizing a real-valued error functional of a complex vectorial variable \( z \) (representative for the drives). For example, in the MIMO random control specific case, a common choice is to define as error functional the infinity norm or the Frobenius norm of the error matrix between the target Power Spectral Density (PSD) matrix and the control channels PSD matrix, \( E(\omega) = S_{yy}^{\text{ref}}(\omega) - S_{yy}(\omega) \). Thus, the control problem can be seen as an optimization problem over the complex space, i.e. to find the minimum of a function

\[
f(z), \quad \mathbb{C}^m \to \mathbb{R},
\]

where \( m \in \mathbb{N}^+ \) is the number of elements of \( z \). Finding minima of any function as defined in the relation (1) needs a different approach from a classical optimization problem. As a matter of fact, any non-constant real-valued function over the complex space fails the Cauchy-Riemann conditions, meaning that the function (1) is not analytic in \( \mathbb{C}^m \). Thus, a gradient-based technique such as the Newton’s method is not directly applicable. [9] and [11], extending the work of [8], point out, giving some examples, that any non-constant real-valued function (1) can also be represented as

\[
f(\text{Re}(z), \text{Im}(z)), \quad \mathbb{R}^{2m} \to \mathbb{R} \quad (2a)
\]

\[
f'(\mathbf{\bar{z}}) \equiv f(z, \mathbf{\bar{z}}), \quad \mathbb{C}^{2m} \to \mathbb{R}. \quad (2b)
\]

The form in eq. (2b) is referred as representation in Conjugate Coordinates (denoted with \( \mathbb{C} \) and is the basis of the so-called \( \mathbb{C} \)-Calculus [8], [9] and [11]. It is possible to demonstrate that the real-valued functions in Conjugate Coordinates can be holomorphic if considered functions of \( z (\mathbf{\bar{z}}) \) only, keeping \( \mathbf{\bar{z}} (z) \) constant. Complex holomorphic functions are also complex analytic, and by making use of the two operators ([9],[11]) Complex Gradient \( \frac{\partial}{\partial z} \equiv \{ \frac{\partial}{\partial z} \bigg|_{\mathbf{\bar{z}}=\text{const.}}, \frac{\partial}{\partial \mathbf{\bar{z}}} \bigg|_{z=\text{const.}} \}^T \) and a Complex Hessian \( \frac{\partial^2}{\partial z^2} \equiv \left[ \frac{\partial}{\partial z} \left( \frac{\partial}{\partial \mathbf{\bar{z}}} \right) \right] \) is possible to define a convergent power series centered in a point \( \mathbf{\bar{z}}_0 \)

\[
f(\mathbf{\bar{z}}_0 + \mathbf{c}^z, \mathbf{\bar{z}}_0) \approx f(\mathbf{\bar{z}}_0) + \mathbf{c}^z \cdot \mathbf{\Delta} z_T \cdot \frac{\partial f(\mathbf{\bar{z}})}{\partial \mathbf{z}} \bigg|_{\mathbf{\bar{z}}_0} + \frac{1}{2} \mathbf{c}^z T \cdot \frac{\partial^2 f(\mathbf{\bar{z}})}{\partial \mathbf{z}^2} \bigg|_{\mathbf{\bar{z}}_0} \mathbf{\Delta} z \quad (3)
\]

where the approximation is needed since the \( \mathbf{c}^T \mathbf{\Delta} z \mathbf{\Delta} z^T \) terms have been neglected. The expression (3) can be used to find the minimum of the function (2b) deriving with respect to \( \mathbf{\Delta} z \) and imposing the stationary point condition. This operation leads to the following expression to be satisfied

\[
\frac{\partial f(\mathbf{\bar{z}})}{\partial \mathbf{z}} \bigg|_{\mathbf{\bar{z}}_0} + \frac{\partial^2 f(\mathbf{\bar{z}})}{\partial \mathbf{z}^2} \bigg|_{\mathbf{\bar{z}}_0} \mathbf{c}^z \mathbf{\Delta} z = 0 \quad (4)
\]

Eq. (4) is very general and can be used to define a drive update for all the MIMO Vibration Control problems by adequately choosing the target function and the drive vector. It is non-trivial to demonstrate that eq. (4) leads to two equally valid (conjugate) expressions that closely
resembles the purely complex Taylor power series centred in \( z_0 \) and \( \bar{z}_0 \), respectively. Without losing generality the expression of \( z_0 \) will be considered

\[
g(z_0 + \Delta z) = g(z_0) + \frac{\partial g(z)}{\partial z} \bigg|_{z_0} \Delta z
\]

(5)

where (i) the function \( g(z) \) depends on the MIMO application and comes from the first term in the left hand side of eq. (4), (ii) \( \frac{\partial g(z)}{\partial z} \bigg|_{z_0} \) is the Jacobian matrix \( J \) of the transformation \( g(z) \) evaluated in \( z_0 \), where the component-wise derivatives are taken by keeping \( z \) constant. At this point it is possible to obtain the Newton’s method by writing eq. (5) as an iteration

\[
z_{i+1} = z_i + J_i^{-1}[g(z_{i+1}) - g(z_i)]
\]

(6)

Starting from an initial estimation \( z_0 \), the eq. (6) can be used to step forward and find the minimum of the function (1). To notice from eq. (6) that the Jacobian matrix needs to be inverted at each iteration step. This is a time consuming operation that can directly influence the process convergence. In order to avoid this inversion, according to [10] a rank one update of the Jacobian matrix computed at previous iteration steps is a possible solution, thus performing the inversion just once or once any few iterations. This operation can be carried on by making use of the update proposed in [10]

\[
J_i^{-1} = J_{i-1}^{-1} - \frac{(z_i - z_{i-1}) - J_{i-1}^{-1}[g(z_i) - g(z_{i-1})]}{(z_i - z_{i-1})^T J_{i-1}^{-1}[g(z_i) - g(z_{i-1})]} (z_i - z_{i-1})^T J_{i-1}^{-1}.
\]

(7)

The basic principles of the Complex-Real Calculus will be applied in order to try to develop a convergent control method for MIMO random vibration control application. The possibility of applying the formula (7) to estimate the Jacobian matrix to be used in the next iteration step in order to enhance the process behaviour will also be considered.

3. MIMO random control strategy

The block scheme of a general MIMO random vibration control test is illustrated in figure 1. The structure under test is excited driving \( m \) electrodynamic or hydraulic shakers and the system’s response is recorded in \( \ell \geq m \) control points. In the hypothesis of the structure under test behaving linearly and being time invariant, the system is represented by the Frequency Response Function (FRF) matrix \( H \in C^{\ell \times m} \)

\[
Y = HU
\]

(8)

where \( Y \in C^{\ell \times 1} \) and \( U \in C^{m \times 1} \) are the spectra of the control points output \( y(t) = \{y_1(t), \ldots, y_\ell(t)\}^T \) and the input drives \( u(t) = \{u_1(t), \ldots, u_m(t)\}^T \), respectively. In case of rectangular systems, i.e. \( \ell \geq m \), the impedance matrix \( Z \) is generally obtained via a Moore-Penrose pseudo-inverse, \( Z = H^T \in C^{m \times \ell} \). In all the vibration control tests, a System Identification pre-test phase is needed to estimate the system transfer function; this is usually performed by running a low-level random test and using the so called \( H_t \) estimator

\[
\hat{H} = \hat{S}_{yu} \hat{S}_{uu}^{-1}
\]

(9)

where \( \hat{S}_{yu} \in C^{\ell \times m} \) and \( \hat{S}_{uu}^{-1} \in C^{m \times m} \) are spectral density matrices estimated via the Welch’s averaged periodogram.

The Objective of MIMO Random Control vibration tests is to replicate a full Power Spectral Density (PSD) matrix \( S_{yy}^{ref} \), with the off-diagonal terms representing the Cross Spectral Densities (CSDs) between the control channels. It is possible to obtain the input PSD matrix
that in the ideal case will directly solve the MIMO random control problem,

$$S_{yy} = HS_{uu}H^H = \hat{H}\hat{Z}S_{yy}^\text{ref}(\hat{H}\hat{Z})^H \approx S_{yy}^\text{ref}$$ (11)

According to Smallwood in [12], it is always possible to obtain a specified full spectral matrix by coupling sets of independent white noise sources with unity amplitude spectrum $W$, through a lower triangular matrix. A spectral matrix with a physical meaning is hermitian and positive semi-definite: these properties allow to decompose the input PSD matrix in the product of two triangular hermitian matrices via the Cholesky Decomposition [13]

$$S_{uu} = DD^H$$ (12)

The matrix $D$ is called Cholesky Factor of $S_{uu}$ and is the coupling matrix used to generate the drives via the Smallwood’s method. From eq. (12) is possible to see that if a sufficient number of averages is considered, the inputs time histories with the specified PSD matrix $S_{uu}$ can be obtained by transforming the frequency domain quantity $U = DW$ to the time domain via an inverse Fourier transform [6], [12]

$$S_{uu} = DD^H = D\hat{d}D^H = DWW^HD^H$$ (13)

This procedure generates the so called first drives. Eq. (11) clearly shows that in the ideal squared case no control action is needed since the product $H\hat{Z}$ does not approximate but is strictly equal to the identity matrix. Unfortunately this is never the case because noise on the
input and the output is always present and non linearities could be excited during the test. Thus an error correction is needed. The error matrix at the iteration \(i\) is
\[
E_i = S_{yy}^{ref} - S_{yy}^i
\]
whose infinity norm needs to be minimized. The approach followed in this work is to apply a correction on the \(D\) matrix in the frequency domain. Particularly, a vectorial form \(d\) of the matrix \(D\) will be used as independent variable in the Newton’s update (6). The error defined in equation (14) can be seen as an error on the input PSD matrix
\[
E_{uu_i} = ZS_{yy}^{ref}Z^H - ZS_{yy}^iZ^H.
\]
In the hypothesis of reaching the target at the next iteration step, a vectorized form \(e_{uu}(d)_i\) of the error (15) is the function error to be used to step forward in the next iteration via the Newton’s formula (6) and reduce the infinity norm of the matrix (14). The update is thus given by
\[
d_{i+1} = d_i + \alpha J_i^{-1}[e_{uu_i}(d)]
\]
where the gain \(0 < \alpha \leq 1\) has been introduced to improve the control stability. The matrix \(J_i\) appearing in the eq. (16) is the Jacobian matrix evaluated at the iteration \(i\) by keeping \(\bar{d}\) constant, as explained in section 2. This matrix is evaluated at each iteration and then inverted to generate the needed drive correction. The possibility of using the formula (7) has been also investigated in order to try to use the information at the current and previous iteration step to update \(J^{-1}\) for the next iteration
\[
J_{i+1}^{-1} = J_i^{-1} + \frac{(d_i - d_{i-1}) - J_i^{-1}e_{uu_i}(d_i - d_{i-1})^T J_i^{-1} e_{uu_i}(d_i - d_{i-1})}{(d_i - d_{i-1})^T J_i^{-1} e_{uu_i}(d_i - d_{i-1})}
\]
The new vector \(d_{i+1}\) is then used in a matrix form \(D_{i+1}\) to generate the next drive via the method described in [6] and [12].

4. Two-exciter two-axis MIMO random test simulation

In order to test the control capabilities, an application on a simple system has been studied. The system is the two degrees of freedom (dof) lumped mass illustrated in figure 2, with all the parameters reported in table 1. The two degrees of freedom are coupled via a diagonal spring and a damper. The drives are applied as two forces that excite the mass in the \(x\) and \(y\) directions; the resulting displacements in these directions are used as control channels. To introduce some non ideal situation (i) an offset in the whole frequency range in the drive generation has been introduced and (ii) a noise free FRF has been used as output of the System Identification phase, while the real system has been simulated by adding 10% of noise on the FRFs. The FRF matrix between the controls and the inputs is the two by two matrix shown in figure 4. The target PSDs, i.e. the two diagonal terms of \(S_{yy}^{ref}\) (namely \(S_{xx}\) and \(S_{yy}\)), have the frequency profile reported in figure 3. The CSD terms \(S_{xy}\) and \(S_{yx} = \bar{S}_{xy}\), are obtained setting zero phase \(\phi_{xy}\) and a coherence \(\gamma_{xy}^2 = 0.5\) between the control channels and using the coherence definition
\[
\gamma_{xy}^2 = \frac{|S_{xy}|^2}{S_{xx}S_{yy}}
\]
\[
S_{xy} = |S_{xy}|exp(j\phi_{xy}) = \sqrt{\gamma_{xy}^2 S_{xx}S_{yy}}exp(j\phi_{xy})
\]
where \(j\) is the imaginary unit. As also pointed out in [6], the target spectrum physical realizability is guaranteed, for a two outputs case of study, by setting \(\gamma_{xy}^2 \leq 1\). MIMO random test specifications require the response spectra to be bounded in defined abort limits. Usually
Table 1. Two degrees of freedom system’s parameters.

| Stiffness | Damping | Resonance frequency |
|-----------|---------|---------------------|
| $k_x$     | 100     | 0.1                 | $x$ mode 1.66 |
| $k_y$     | 200     | 0.1                 | $y$ mode 2.31 |
| $k_d$     | 10      | 0.05                |               |
| Mass (m)  | 1 kg    |                     |               |

Figure 2. Lumped mass system used to run the simulations.

Figure 3. Reference PSDs (blue) with ±3 dB alarm limits (orange).

the abort threshold is set to ±6 dB from the target while a lower limit (±3 dB) generates an alarm during the test.

The developed procedure will be tested via two simulations. The first simulation is run correcting the drives using the Newton’s formula (16), as explained in section 3. In the second simulation the Jacobian matrix update using the relation (17) formula is also taken into account. The control gain $\alpha$ is set to the value 0.3 in both simulations. To evaluate the process’s performances, the infinity norm of the error defined in eq. (14) will be plotted frequency line per frequency line, normalized to the target infinity norm. The mean error over the frequency lines at each iteration will be used to estimate the error convergence and the output spectrum, the coherence and the phase between the output channels will be shown as simulated test results. The same case study will be run using an implementation of the well tested correction procedure discussed in [4] and [6] and a comparison between the approaches will be performed.

4.1. Simulation 1: drives correction recomputing the Jacobian matrix

The convergence of the algorithm has been tested during this simulation. Figure 5 shows the PSD $S_{xx}$ obtained exciting the structure with the first drives; the output profile is above the target in all the frequency range due to the target offset introduced in the simulation phase that needs to be corrected by the control algorithm.

After 10 iterations in the correction process, figure 6 shows that the output PSD matrix matches the reference profile and is well below the abort limits in all the frequency range. Also the phase and the coherence between the control channels (figure 7) match the target, with just few spectral lines where the difference is not negligible. In figure 8 the mean value over the frequencies of the error infinity norm is plotted for 50 iterations; although the convergence is not strictly decreasing, the figure shows that within 10 iterations the mean error decreases and start to oscillate around a stable value. These oscillations indicate the algorithm’s attempts to control the small random errors that occur in the whole frequency range at each iteration. However, looking at the infinity norm error frequency line per frequency line (figure 10) during
4.2. Simulation 2: drives correction updating the Jacobian matrix

In this simulation the Jacobian matrix update described in the eq. (17) has been used to adapt the Jacobian for the next iteration. Since the number of inputs is limited in the discussed application and the size of the Jacobian matrix strictly depends on the number of exciters, computational efficiency studies have not been performed. Nevertheless, interesting results followed by adapting the Jacobian matrix at each iteration step and will be presented in this subsection. In order to be able to compare the results of the two simulations, the same random sequence has been used in the two different runs. It is interesting to notice the difference between figure 10 and 11 where no significant outliers are present in the infinity norm error frequency line per frequency line during the control procedure. At all the iterations, the output PSD does not exceed the abort limits. Figure 12 shows this PSD matrix at the iteration 7; particularly, the auto-power spectral density of the x channel (top left corner) in the whole band has no
Figure 6. Simulation 1, output PSD matrix (red) after 10 iterations. Reference profile (blue) and alarm (orange) and abort (magenta) limits. Phase angles in [deg].

Figure 7. Simulation 1, coherence and phase between the output channels after 10 iterations.

Figure 8. Simulation 1, trend (dash) of the frequency error mean (solid).

Figure 9. Simulation 1, x channel output PSD after 7 iterations (red). Reference profile (blue) and alarm (orange) and abort (magenta) limits. Phase angles in [deg].
Figure 10. Simulation 1, infinity norm of the relative error for all the frequency lines.

Figure 11. Simulation 2, infinity norm of the relative error for all the frequency lines.

Figure 12. Simulation 2, iteration 7: output PSD matrix (red). Reference profile (blue) and alarm (orange) and abort (magenta) limits. Phase angles in [deg].

Figure 14 shows a comparison between simulation 1 and 2 in terms of mean value of the error over the frequency lines. As expected, the curve that represents simulation 2 is lower especially at the fourth iteration and at the iterations from 6 to 10 (to notice from figure 10 that at these iteration some frequency lines exhibit a large error). The results of this simulation have shown that the Jacobian matrix update can cover an important role in controlling the spectral lines in the neighbourhood of the system’s resonances that appear to be outliers in the gradient-based approach followed in this work. This capability needs to be further proved and leaves room for additional investigations.
4.3. Results from existing methods
In this case, a current software implementation of the method proposed by Smallwood in [4] has been tried on the two degrees of freedom system of figure 2. An insight of this method is outside the aim of the work but detailed discussions can be found in [4] and the more recent work of Peeters [6]. In figure 15 is reported the $x$ channel power spectral density at the seventh iteration. It is possible to see that the spectrum is far from the abort limits and within the alarm boundaries in all the frequency range. The Smallwood method’s application is able to control accurately the system’s response and uncontrolled peaks do not appear during the process. An indication of this trend is shown in figure 16 where no outliers are visible in the infinity norm of the error frequency line per frequency line.

5. Conclusions
In this work an early stage implementation of a MIMO random vibration control strategy based on the Complex-Real Calculus and the theories widely used in adaptive arrays has been presented. The theoretical bases that reside behind the strategy can further bring an extension of the algorithm also to other MIMO vibration tests, e.g. MIMO sine or MIMO Time Waveform Replication. Simulated results using a simple two degrees of freedom system have been studied to evaluate the control convergence trend and to investigate the method’s capabilities. The simple application showed that the method is able to generate drives that replicate the target PSD matrix at the control locations and to reduce the error during the process. Nevertheless, challenges have been encountered in controlling the responses in the neighbourhood of the system resonances. The possibility of adapting the Jacobian matrix inverse during the process has also been taken into account and tested in the same application. The results have shown that the Jacobian matrix update can cover an important role in controlling the spectral lines in the neighbourhood of the system’s resonances. Even though the global convergence of the process is
Figure 15. Smallwood method’s application, iteration 7: output spectrum (red), reference spectrum (blue) and alarm (orange) and abort (magenta) limits. Phase angles in [deg].

Figure 16. Smallwood method’s application, infinity norm of the relative error for all the frequency lines.

just slightly influenced by the aforementioned limitation, the application of a standard technique on the same case of study and the more general results of commercial software’s applications highlight the necessity and the possibility of tailoring improvements in the control strategy that can overcome this limitation.

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