Reductions related with Hopf maps

Vahagn Yeghikyan

Yerevan State University, 1 Alex Manoogian St., Yerevan, 0025, Armenia

Abstract

We consider the reductions of $2p$-dimensional particle system ($p = 2, 4, 8$), associated with the Hopf map. For the third Hopf map we explicitly construct the functions associated to the symmetry related to the rotations in the fiber.

1 Introduction

It is known that the systems describing motion of particle in the field of Dirac an Yang monopoles can be constructed using the reduction procedure associated with the first and second Hopf maps [1]. The Hopf maps (fibrations) are fibrations of spheres over spheres with the fiber-sphere [2]. There are four Hopf maps: $S^{2n-1}/S^{n-1} = S^n$, $(n = 1, 2, 4, 8)$. The Dirac and Yang monopoles are related with the first and the second ones respectively. Zero Hopf map is related with anyons (or magnetic vortices) [3], while for the last Hopf map the mentioned procedure does not exist. Moreover, it is even unclear which sort of monopole should arise after the reduction. The problem comes from the fact that the algebra of the octonions is not associative (equivalently the fiber of the last fibration is not a group manifold). Therefore, the transformation, which leave invariant the coordinates of base of the third Hopf fibration are not isometries of bundle space.

The goal of current paper is to investigate the problems arising while trying to construct the reduction procedure related to the third Hopf map. For this purpose we formulate the reduction procedures associated with the Hopf maps ([1, 4]) in terms of real coordinates and Clifford algebras.

Together with geometric methods (for review, see e.g. [5]) the algebraic understanding of the nature of the Hopf maps leaves to no surprise that important differences are encountered between the Hopf maps.

The paper is arranged as follows.

In the Second Section we present an explicit description of the Hopf maps in terms needed for our purposes.

In the Third section we employ the Hopf maps to reduce the bosonic free-particle systems to lower dimensional systems with magnetic $SU(2)$ monopoles.

2 Hopf maps

The Hopf maps (or Hopf fibrations) are the fibrations of the sphere over a sphere, $S^{2p-1}/S^{p-1} = S^p$, $p = 1, 2, 4, 8$. These fibrations reflect the existence of normed division algebras: real ($\mathbb{R}$, $p = 1$), complex ($\mathbb{C}$, $p = 2$), quaternionic ($\mathbb{H}$, $p = 4$) and octonionic ($\mathbb{O}$, $p = 8$) numbers.
2.1 Normed Division algebras

Any element of normed division algebras can be expressed via the generating elements of the algebra $e_{\mu}$.

\[ x = x^n + x^\mu e_\mu, \quad \mu = 1, \ldots, n-1, \quad n = 1, 2, 4, 8, \quad (2.1) \]

where the generating elements satisfy the following multiplicative rule:

\[ e_{\mu} e_{\nu} = -\delta_{\mu\nu} + C_{\mu\nu\lambda} e_{\lambda}, \quad (2.2) \]

where $C_{\mu\nu\lambda}$ are constants antisymmetric under any permutations of indices. Here and in the further we will use bold style to denote the elements of algebra and normal style for real elements.

The conjugation and norm are defined by analogy with complex numbers ($n = 2$):

\[ \bar{x} = x^n - x^\mu e_\mu, \quad |x| \equiv \sqrt{x \bar{x}} = \sqrt{x_a x_a}. \]

The greek symbols $\mu, \nu, \lambda$ run $1, \ldots, n-1$, while the latin symbols $a, b, c = 1, \ldots, n$.

It was proven \([6]\), that one can construct the constants $C_{\mu\nu\lambda}$ so that the algebras have division operation only for the dimensions $n = 1, 2, 4, 8$. It is clear that for real and complex numbers we have $C_{\mu\nu\lambda} = 0$. For quaternionic numbers we define $C_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda}$, where $\varepsilon_{\mu\nu\lambda}$ are the elements of totally antisymmetric tensor and for the octonions we have

\[ C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1, \quad (2.3) \]

while all other non-vanishing components are determined by the total antisymmetry.

Each time we pass from a lower dimensional algebra to the next one, we lose some symmetry. Hence, first we lose the fact that every element is its own conjugate, then we lose commutativity, then we lose associativity \([7]\). However, the last-octonionic algebra has a weaker property called alternativity which implies any subalgebra consisting of two elements is associative (for associativity we should have three). For more information about modern status of the theory of normed division algebras see an excellent review \([7]\).

One can consider the elements (2.1) as columns with $n$ real elements. Using (2.2) one can write down the multiplicative rule for this columns:

\[ (xy)_a = x_c \gamma_{ab}^c y_b, \quad (\gamma^c)_{ab} = -\delta_{an} \delta_{\mu b} + \delta_{a\mu} \delta_{bn} + \delta_{n\mu} \delta_{ab} - C_{\mu \nu}^{\rho} \equiv (\gamma^c)_{ab}, \quad (2.4) \]

where we have chosen $C_{abc} = 0$ if at least one index is equal to $n$. Since we deal with Euclidean space, there is no difference between upper and lower indices. Here and further we will denote the columns by the normal letters without indices.

One can see, that from requirement $|xy| = |x||y|$ ($\forall x, y \in \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) and the definition (2.4) it follows, that

\[ \{\gamma^\mu, \gamma^\nu\} = -\delta^{\mu\nu}, \quad \gamma^n = 1_{nn}, \quad (\gamma^\mu)^T = -\gamma^\mu, \quad (2.5) \]

where $\{., .\}$ denotes anticommutator and $T$-transpose of matrices (See e.g. \([8]\)). This is all we need to know about the normed division algebras. Now, let us pass to the description of Hopf maps.

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**Remark:**

The Greek letters $\mu, \nu, \lambda$ run $1, \ldots, n-1$, while the Latin letters $a, b, c$ run $1, \ldots, n$. This notation is consistent with the convention used in the literature on division algebras. The choice of these indices ensures that the antisymmetry properties of the $C_{\mu\nu\lambda}$ constants are maintained, which is crucial for the construction of the algebras. The use of bold and normal styles for elements of the algebra and real elements, respectively, is also a standard practice in the field, helping to distinguish between the two types of elements. The equality $|xy| = |x||y|$ is a consequence of the normed division algebra structure, and the expression for $(\gamma^c)_{ab}$ is derived from the multiplication rule (2.2) using the properties of the $C_{\mu\nu\lambda}$ constants. The anticommutator and transpose properties in (2.5) are essential for understanding the algebraic structure of these algebras.
2.2 Hopf maps (Normed division algebras)

Let us describe the Hopf maps using normed division algebras. For this purpose, we consider the functions

\[
(x(u_\alpha, \overline{u}_\alpha), x_{p+1}(u_\alpha, \overline{u}_\alpha))
\]

where \( u_1, u_2 \) are complex numbers for the \( n = 2 \) case (first Hopf map), quaternionic numbers for the \( n = 4 \) case (second Hopf map) and octonionic numbers for the \( n = 8 \) case (third Hopf map) (see, e.g. [5]). One can consider them as coordinates of the \( 2n \)-dimensional space \( \mathbb{R}^{2n} \) for \( n = 2 \) (complex numbers), \( n = 4 \) (quaternionic numbers), \( n = 8 \) (octonionic numbers). In all cases \( x_{n+1} \) is a real number while \( x \) is, respectively, a complex number (\( n = 2 \)), a quaternionic one (\( n = 4 \)), and octonionic one (\( n = 8 \)).

\[
x \equiv x^n + e_\mu x^\mu.
\]

(2.7)

One could immediately check that the following equation holds:

\[
r^2 \equiv \overline{x}x + (x^{n+1})^2 = (\overline{u}_1 u_1 + \overline{u}_2 u_2)^2 \equiv R^4.
\]

(2.8)

Thus, defining the \((2n-1)\)-dimensional sphere in \( \mathbb{R}^{2n} \) of radius \( R \), \( \overline{u}_\alpha u_\alpha = R^2 \), we will get the \( p \)-dimensional sphere in \( \mathbb{R}^{n+1} \) with radius \( r = R^2 \).

The expressions (2.6) can be easily inverted by the use of equality (2.8)

\[
u_\alpha = g r_\alpha,
\]

(2.9)

where

\[
r_1 = \sqrt{\frac{r + x^{n+1}}{2}}, \quad r_2 \equiv r_+ = \frac{x}{\sqrt{2(r + x^{n+1})}}, \quad gg = 1.
\]

It follows from the last equation in (2.9) that \( g \) parameterizes the \((n - 1)\)-dimensional sphere of unit radius.

Using above equations, it is easy to describe the first three Hopf maps. Indeed, for \( n = 1, 2, 4 \) the functions \( x, x_{n+1} \) remain invariant under the transformations

\[
u_\alpha \to G u_\alpha, \quad \text{where} \quad GG = 1
\]

(2.10)

Therefore, \( G \) parameterizes the spheres \( S^{n-1} \) of unit radius. Taking into account the isomorphism between these spheres and the groups, for \( n = 1, 2, 4; S^0 = \mathbb{Z}_2, S^1 = U(1), S^3 = SU(2) \), we get that (2.6) is invariant under \( G \)-group transformations for \( n = 1, 2, 4 \) (where \( G = \mathbb{Z}_2 \) for \( n = 1 \), \( G = U(1) \) for \( n = 2 \), and \( G = SU(2) \) for \( n = 4 \)).

For the octonionic case \( n = 8 \) situation is more complicated. Because of losing associativity the standard transformation \( u_\alpha \) that leaves invariant coordinates \( x, x_9 \) will not be just (2.10). Instead, we should write
Its modification can be easily obtained using (2.9):

\[ u_\alpha \mapsto (Gg)(\bar{g}u_\alpha) = \frac{(Gu_1)(\bar{u}_1u_\alpha)}{\bar{u}_1u_1}. \]  

(2.11)

Also, for the \( n = 8 \) case the bundle \( S^7 \) is not isomorphic with any group. So, one can expect further troubles in the extensions of the constructions, related with the lower Hopf maps to the third one.

### 2.3 Hopf maps (Spinor representation)

One can consider a \( 2n \)-dimensional column \( U \) consisting of the real coordinates \( u_{\alpha,a} \), \( \alpha = 1, 2, a = 1, \ldots, n \):

\[ U = (u_{1,1}, \ldots, u_{1,n}, u_{2,1}, \ldots, u_{2,n}). \]

Using this denotation one can rewrite (2.6) in the following form:

\[ x^A = U\Gamma^A U, \quad A = 1, \ldots, n+1 \]  

(2.12)

where

\[ \Gamma^\mu = \begin{pmatrix} 0 & \lambda^\mu \\ -\lambda^\mu & 0 \end{pmatrix}, \quad \Gamma^n = \begin{pmatrix} 0 & 1_{n \times n} \\ 1_{n \times n} & 0 \end{pmatrix}, \quad \Gamma^{n+1} = \begin{pmatrix} -1_{n \times n} & 0 \\ 0 & 1_{n \times n} \end{pmatrix}, \]

(2.13)

where

\[ (\lambda^\mu)_{ab} = -\delta_{an}\delta_b^\mu + \delta_a^\mu \delta_{bn} + C_{\mu ab} \]  

(2.14)

(compare with (2.4)). This matrices satisfy the relations (2.5). One can check, that the matrices \( \Gamma^A \) are the Euclidean gamma-matrices satisfying the anticommutational relations:

\[ \{ \Gamma^A, \Gamma^B \} = \delta^{AB}1_{2n \times 2n}. \]  

(2.15)

In [9] all the three Hopf maps were explicitly constructed using spinor representations of \( SO(1, n+1) \). For the complex and quaternionic \( (n = 2, 4) \) case it was shown the direct connection between this description and the one using normed division algebras.

It is obvious, that for \( n = 2, 4 \) we have reducible representation of Clifford algebra (we have \( n = 2, 4 \) generating elements and 4 and 8 dimensional representation respectively).

\( n = 8 \) is the only case for which the constructed matrices form an irreducible representation of Clifford algebra. E.g. nine matrices \( \Gamma^A \) have dimension 16 = 2\(^9/2\). It is clear, that in this representation all the matrices \( \Gamma^A \) are symmetric. These is, in fact, such representation, where the matrix of charge conjugation is identity matrix:

\[ (C^T)^{-1}\Gamma^A C = (\Gamma^A)^T = \Gamma^A, \quad C = 1_{16 \times 16} \]

In [9] it was shown, that the infinitesimal transformation (2.11) can be presented in the following form:

\[ \delta U = -\frac{1}{6} \omega_{AB}(U^T \Gamma^{ABCD} U)\Gamma^{CD} U \]

(2.16)
3 Hopf maps and reductions

Let us apply the obtained formulae for the Hopf fibrations to reduce the $2n$-dimensional free particle system to a lower dimensional one. For this reason we consider Lagrangian in terms of the coordinates of fiber and base of Hopf fibrations:

$$\mathcal{L}_{2n} = \frac{g(r_a)}{2} (\dot{r}_a \dot{r}_a + 2 \text{Re} \left( (\dot{r}_a \dot{g})(\dot{g} r_a) \right) + r \dot{g} \dot{g}) = \frac{g}{2} (\dot{r}_a \dot{r}_a + 2 r v_a A_{ab} \dot{v}_b + r \dot{v}_a \dot{v}_a), \quad a, b, c, d = 1, \ldots, 8$$

(3.17)

where

$$A_{ab} = \frac{x_c (\Sigma_{cd})_{ab} \dot{x}_d}{2r (r + x_9)}, \quad \Sigma^{\mu\nu} = \frac{[\lambda^\mu, \lambda^\nu]}{2}, \quad \Sigma^{\mu\nu} = -\Sigma^{\nu\mu} = \lambda^\nu, \quad \mu, \nu = 1, \ldots, 7$$

(3.18)

is precisely the potential of $U(1)$ Dirac, $SU(2)$ Yang and $S0(8)$ monopoles [10] with $\lambda_i$ be the two, four or eight-dimensional gamma-matrices (for $n = 2, 4, 8$ respectively). The functions $v_i$ are the Euclidean coordinates of the fiber of Hopf fibrations: $S^1$ for the first, $S^3$ for the second and $S^7$ for the third Hopf map. $g = v_8 + e_a v_a$, $v_a v_a = 1$ and express via projective coordinates of $S^{2n-1}$ as follows:

$$v_n = 1 - \frac{y^2}{1 + y^2}, \quad v_\mu = \frac{2y_\mu}{1 + y^2}, \quad y^2 = \sum_{\mu=1}^{n-1} y_\mu^2$$

(3.19)

Taking into account the last expressions, we can represent the Lagrangian in the following form:

$$\mathcal{L} = \frac{g}{2} \left( \dot{r}_a \dot{r}_a + 2 r D_\mu \dot{y}_\mu + 2 r \frac{\dot{y}_\mu \dot{y}_\mu}{(1 + y^2)^2} \right),$$

where

$$D_\mu = \frac{1}{(1 + y^2)^2} \left( A_{\mu\nu} (1 - y^2) + y_{\nu} A_{\mu\nu} + 2 y_{\nu} A_{\nu\mu} y_\mu \right), \quad n = 2, 4, 8.$$ 

(3.20)

Let us replace our Lagrangian by the variationally equivalent one, performing Legendre transformation of the "isospin" varyables $y_\mu$.

After some work we find

$$\mathcal{L}_{\text{int}} = p_\mu \dot{y}_\mu + \frac{g}{2} \dot{r}_a \dot{r}_a - (1 + y^2) \frac{(p_\mu - 2 gr D_\mu)^2}{8rg}$$

(3.21)

The generator of transformations (2.11) is defined, in these terms, as follows

$$I^\mu = \frac{1}{2} (1 + y^2) S_\nu^\mu D_\nu, \quad \text{where} \quad S_\nu^\mu = \frac{1}{1 + y^2} \left( 2 y_\mu y_\nu + (1 - y^2) \delta_\nu^\mu + 2 y_\lambda C_{\nu\lambda}^\mu \right),$$

(3.22)

with $C_{\mu\nu\lambda}$ be structure constants of complex, quaternionic and octonionic algebra.
Remark. It is obvious, that for the case of associative algebras (complex and quaternionic) the transformation \( (2.11) \) form the symmetry of the initial Lagrangian \( (3.17) \). However, for the case of octonions the lack of associativity leads to the fact, that this transformations do not preserve the Lagrangian and, therefore, the quantities \( I_\mu \) are not the integrals of motion of the system. We will discuss this below.

Taking into account the equalities

\[
SS^T = 1_{n-1}, \quad \frac{I_\mu I_\nu}{2gr} = (1+y^2)^2 \frac{p^2}{16rg}, \quad \dot{r}_A \dot{r}_A - r D_\mu D_\nu (1+y^2)^2 = \frac{\dot{x}_A \dot{x}_A}{4r} \tag{3.23}
\]

we can represent the Lagrangian in very transparent form

\[
\mathcal{L}_{int} = g \frac{\dot{x}_A \dot{x}_A}{8r} + p_\mu \dot{y}_\mu + rg J_{ab} A_{ab} - \frac{1}{4} \frac{I_\mu I_\nu}{2gr}, \tag{3.24}
\]

where

\[
J_{\mu\nu} = y_\mu p_\nu - y_\nu p_\mu, \quad J_{\mu n} = -J_{n\mu} = \frac{1-y^2}{2} p_\mu + (y_\nu p_\nu) y_\mu, \quad n = 2, 4, 8. \tag{3.25}
\]

are the generators of \( SO(n) \) rotations.

### 3.1 \( n = 2 \) complex case

In this case we have \( a, b = 1, 2 \) and, therefore, one element \( J_{ab} : J_{12} = -J_{21} = p \). It is easy to check, that this element is a constant of motion of the system and therefore we can fix its value to be equal to a constant \( s \). The term with \( \dot{y} \) disappears because it becomes full time derivative and finally we find the reduced Lagrangian:

\[
\mathcal{L}_3 = g \frac{\dot{x}_A \dot{x}_A}{8r} + rs A_D - \frac{s^2}{2gr}, \quad A = 1, 2, 3, \tag{3.26}
\]

where \( A_D \) is the vector-potential of Dirac monopole.

### 3.2 \( n = 4 \) quaternionic case

We have already mentioned, that for \( n = 4 \) the representation of Clifford algebra is not minimal and, therefore not all the components of \( A_{ab} \) are independent. Using the properties of \( \varepsilon_{\mu\nu\lambda} \) one can find the following connection between this elements:

\[
\varepsilon_{\lambda\mu\nu} A_{\mu\nu} = 2 A_\lambda \tag{3.27}
\]

And, therefore, we find

\[
J_{ab} A_{ab} = P_\mu \tilde{A}_\mu \tag{3.28}
\]

where

\[
\tilde{A}_\lambda = \frac{1}{2} \varepsilon_{\lambda\mu\nu} A_{\mu\nu}, \quad P_\lambda = J_{n\lambda} - \frac{1}{2} \varepsilon_{\lambda\mu\nu} J_{\mu\nu}. \tag{3.29}
\]
Let us mention that the following identity obeys:

\[ I_\mu = -J_{n\lambda} - \frac{1}{2} \varepsilon_{\lambda \mu \nu} J_{\mu \nu} \]  

(3.30)

Using this denotations, one can rewrite the Lagrangian (3.24) as follows:

\[ L_8 = g \frac{\dot{x}_A \dot{x}_A}{8r} + p_\mu \dot{y}_\mu - 4rgP_\mu \dot{A}_\mu - \frac{1}{4} I_\mu I_\mu \]  

(3.31)

The quantities \( P_\mu \) together with \( I_\mu \) form \( so(4) = so(3) \times so(3) \) algebra of symmetries of \( S^3 \). E.g. they obey the following commutation relations:

\[ \{ P_\mu, I_\nu \} = 0, \quad \{ P_\mu, P_\nu \} = \varepsilon_{\mu \nu \lambda} P_\lambda, \quad \{ I_\mu, I_\nu \} = \varepsilon_{\mu \nu \lambda} I_\lambda, \quad I_\mu I_\mu = P_\mu P_\mu \]  

(3.32)

Now, we are ready to fix the values of the integrals of motion and hence, to perform the reduction. Without loss of generality we can fix

\[ I_1 = I_2 = 0, \quad I_3 = s \]  

(3.33)

Because of the relations (3.32) we can denote:

\[ P_+ = P_2 + iP_1 = -is \frac{\bar{z}}{1 + z\bar{z}} \equiv -ish_-, \quad P_- = \bar{P}_+ = is \frac{z}{1 + z\bar{z}} \equiv ish_+ \]  

(3.34)

\[ P_3 = -s \frac{1 - z\bar{z}}{1 + z\bar{z}} \equiv -sh_3, \quad \{ z, \bar{z} \} = (1 + z\bar{z})^2 \]

and the Lagrangian (3.31) will take the following form:

\[ \mathcal{L}_{\text{red}} = \frac{\tilde{g} \dot{x}_A \dot{x}_A}{2} - is \frac{\bar{z} \dot{z} - z \dot{\bar{z}} - sh_\mu(z, \bar{z}) A_\mu}{1 + z\bar{z}} - \frac{s^2}{2r^2 \tilde{g}}, \quad \tilde{g} \equiv \frac{g}{2r}, \quad \mu = 1, \ldots, 5, \]  

(3.35)

where the quantities \( h_\pm, h_3 \) are defined by (3.34).

The second term in the above reduced Hamiltonian is the one-form defining the symplectic (and Kähler) structure on \( S^2 \), while \( h_\mu \) given in (3.31) are the Killing potentials defining the isometries of the Kähler structure. We have in this way obtained the Lagrangian describing the motion of a five-dimensional isospin particle in the field of an \( SU(2) \) Yang monopole. The metric of the configuration space is defined by the expressions \( \tilde{g}_{\mu \nu} = \tilde{g} \delta_{\mu \nu} \). For a detailed description of the dynamics of the isospin particle we refer to [11].

### 3.3 \( n = 8 \) octonionic case

It was already mentioned that because of lack of associativity of the octonionic algebra the transformations (2.11) and therefore the functions \( I_\mu \) do not form isometries of the Lagrangian (3.17).
It is seen from (3.23), that \( I_a I_a = J_{ij} J_{ij} \) defines constant of motion of the system, in complete analogy with the lower Hopf map. Reducing the system by this constant of motion, we shall get the system with \( 30(= 2 \cdot 9 + 12) \)-dimensional phase space, which describes the interaction of the 8-dimensional isospin particle with \( S0(8) \) monopole field. The dimensionality of the internal phase space of the particle is equal to 12.

Now, we should proceed the last step: we need to modify the Lagrangian by adding the specific term (vanishing for the lower Hopf maps), in such a way, that not only \( I^2 \), but each \( I_a \) will be the constant of motion of our system.

4 Conclusion

We have presented the reduction procedure associated with the first and second Hopf map in the Lagrangian approach. For the last- the third Hopf fibration we have presented the explicit formulae of the Lagrangian in coordinates of base and fiber. Since we deal with irreducible representation of \( SO(8) \) algebra it is impossible to construct the motion integrals corresponding to respective ones for the first and second Hopf maps. The only way to avoid this problem seems to be modifying the initial Lagrangian or considering non- Lie algebras of motion integrals.

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