Autonomous quantized refrigerator: performance beyond the classical bound

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We investigate the principles of autonomous quantized machines that may operate in a dual mode: as “compression” or “absorption” refrigerators. The operation mode is determined by the initial quantum state of the piston. Its coefficient of performance may surpass the classical bound, while adhering to the second law. These general results are illustrated for the minimal quantized model of a refrigerator comprised of a qubit coupled to a harmonic-oscillator piston and two spectrally separated baths.

**Introduction** The importance of studying the refrigeration of extremely small systems, with due account for their quantum properties, is becoming increasingly more apparent for a variety of fundamental and practical reasons: the need for implementing efficient refrigeration of electronic processors miniaturized to the extreme [1]; the strive to operate quantum devices, e.g., sensors, at low temperatures in localized nanosize-regions [2–4]; and the elucidation of the validity of the third law of thermodynamics, i.e., Nernst’s principle of the unattainability of the absolute zero [5–7].

Approaches aimed at examining these questions have been nicknamed quantum refrigerator (QR) models [7–21]. Yet, the major obstacle facing the study of size limits and quantum aspects of refrigeration has been the fact that nearly all of these models are non-autonomous. In these models, the refrigerator power is drawn from an external classical source or a heat bath, whose size and energy consumption may well impede overall miniaturization of the device: Conceptually, since the power source is commonly identified with the piston (work reservoir) that parametrically drives a compression cycle, a classical piston renders the entire refrigerator semiclassical rather than fully quantized. An alternative is an absorption refrigerator [8], where the work reservoir is replaced by a heat or noise reservoir. Endowing such a reservoir with nonclassical (squeezing) properties [22] faces the problem that the bath is not in thermal equilibrium and that the squeezing spectrum is always finite [23], which raises subtle issues concerning the analysis and realization of such models [24].

As part of the upsurge of interest in heat machines for the quantum domain [25], we have recently put forward and analyzed the minimal (simplest) model of a heat machine that can operate as either a quantum heat engine or a QR [26]. It consists of a single two-level system (TLS) with periodically-modulated level distance that is permanently coupled to spectrally separated hot and cold baths. Depending on the modulation rate, it switches from one regime to another. This model is nonautonomous, since the modulation is exerted by a classical field (“piston”). Here we construct a fully quantized analog of the foregoing minimal model of a QR that is as self-contained as possible (autonomous). Our analysis is aimed at addressing several open problems of fundamental and applied interest:

1) Cooling by a time-independent Hamiltonian: What are the proper definitions of cooling in a QR comprised of quantum ingredients, i.e., a system S coupled to a piston P and to hot (H) and cold (C) baths, so that the total Hamiltonian is time independent? The standard formula for the division of energy-exchange between heat Q and work W under classical (parametric) driving of the reduced state of the system, \( \rho_S(t) \), via a cyclic system Hamiltonian \( H_S(t) \) is [27]

\[
Q = \frac{1}{T} \text{tr}\left\{ \hat{\rho}_S \hat{H}_S dt \right\}, \quad W = -\frac{1}{T} \text{tr}\left\{ \hat{\rho}_S \hat{H}_S dt \right\}.
\]

These formulas do not apply here, since \( H_S \) is now time-independent, thus we put forward an alternative analysis. 2) What is the minimal, most compact design of a self-contained (autonomous) QR? We show that it consists of a TLS (S) + harmonic-oscillator (P) + spectrally separated H and C baths. A feasible setup is proposed for its realization. 3) How does the QR performance depend on the initial quantum state of P? We show that, depending on the initial state, the QR may act as either a heat-driven absorption refrigerator or as a work-driven (compression) refrigerator. Remarkably, the initial quantum state may give rise to cooling performance that adheres to the second law yet surpasses the performance allowed by the standard classical bound [19, 28].

**General Principles** We assume that the baths H and C are only coupled to S, whereas S is coupled to P. The total Hamiltonian is then

\[
H_{\text{tot}} = H_S + H_P + H_{SP} + \sum_j \left( H^j_{SB} + H^j_{B}\right), \tag{1}
\]

where \( j = H, C \) is the bath index. This Hamiltonian is time-independent, yet the bath-induced dynamics can drive S to a steady-state and P to a regime of slow change that allows it to cool down the bath C.

Without dwelling at this point on the (bath-induced) evolution of S+P, \( \rho_{S+P}(t) \), we consider the reduced state of P, i.e., \( \rho_P = \text{Tr}_SP_{S+P} \), after S has reached steady-state. Although in general the evolution is unknown, we adopt the following parametrization of \( \rho_P(t) \) [29]:
we identify it with a fictitious Gibbs state, \( \rho_P(t) = Z^{-1}e^{-\frac{H_P(t)}{T_P}} \), that has the same entropy, so that we may assign it an effective time-dependent temperature \( T_P(t) \). If \( \rho_P(t) \) is a thermal state, \( T_P(t) \) is its real temperature, otherwise it is merely a parameter that characterizes evolution. What can we learn about refrigeration using this parametrization? We may invoke energy conservation (the first law of thermodynamics)

\[
J_H + J_C - \dot{E}_P = 0,
\]

where \( J_H(C) \) are the heat-flow rates from \( H \) and \( C \) to \( S \), respectively, whereas \( \dot{E}_P \), the piston-energy change-rate, is a combination of heat flow and work-producing power that depends on the piston state.

Spohn’s version of the second law [30] provides the bound for the total entropy-production rate. Considering that \( S \) is at steady-state and the only entropy change is that of the piston, \( S_P \), the Spohn inequality reads

\[
\frac{dS_P}{dt} \geq \frac{J_H}{T_H} + \frac{J_C}{T_C}.
\]

Substituting (2) in (3) yields the cooling threshold of \( C \)

\[
0 \leq J_C \leq \frac{-\dot{E}_P T_C}{T_H - T_C} + \frac{\dot{S}_P T_C T_H}{(T_H - T_C)}.
\]

The coefficient of performance (COP) [25, 28] of the QR obeys, according to (4), the inequality

\[
\text{COP} = \frac{J_C}{-\dot{E}_P} \leq \frac{T_C}{T_H - T_C} - \frac{\dot{S}_P T_C T_H}{E_P(T_H - T_C)}. \tag{5}
\]

The standard bound for the compression QR is recovered for \( \dot{S}_P = 0 \), namely

\[
\text{COP}(\dot{S}_P = 0) \leq \frac{T_C}{T_H - T_C}. \tag{6}
\]

From below, the COP is bounded by the maximum efficiency of an absorption QR completely driven by heat,

\[
\dot{E}_P = T_P \dot{S}_P, \quad \text{COP} \geq \frac{T_C}{T_H - T_C} \left( \frac{T_P - T_H}{T_P} \right). \tag{7}
\]

The compression and absorption QR regimes associated with (5) and (7), respectively, have very different threshold conditions. In what follows we inquire: Are they realizable in the same machine?

**Minimal Model** The general analysis presented above will be illustrated for a quantized analog of the minimal semiclassical model [26]: A harmonic-oscillator \( P \) that is off-resonantly (dispersively) coupled to the \( \sigma_Z \) operator of qubit \( S \):

\[
H_{SP} = g(a + a^+)\sigma_Z, \tag{8}
\]

where \( g \) is the coupling strength and \( a, a^+ \) are respectively the \( P \)-mode annihilation and creation operators (see realization below) [31, 32].

The system-bath (S-B) coupling has the spin-boson form \( H_{SB} = \sigma_X (B_H + B_C) \) where \( B_{H(C)} \) are the multi-mode bath operators. Under the bath-induced dynamics, the reduced density matrix \( \rho_{S+P}(t) \) allows us to compute the heat currents \( J_{C(H)} \) and the effective temperature \( T_P \) that comply with the conditions for QR operation. We subscribe to the description of the bath-induced dynamics by the Lindblad-Gorini-Kossakowski-Sudarshan (LGKS) generator which adheres to the second law [33–35]. In the dressed-state basis that diagonalizes the \( S+P \) Hamiltonian

\[
a \mapsto b = U^\dagger a U, \quad \sigma \mapsto \tilde{\sigma} = U^\dagger \sigma U, \quad U = e^{\frac{i}{2}(a^+ - a)\sigma_Z}, \tag{9}
\]

the LGKS generator involves the bath response at the Hamiltonian eigenvalues[36]: the qubit (S) resonant frequency, \( \omega_0 \), and \( \nu = \omega_0 \pm \nu \) where \( \nu \) is the piston (P) frequency. Namely,

\[
\frac{d\rho_{S+P}(t)}{dt} = \sum_{q=0,\pm 1} (\mathcal{L}^H_q + \mathcal{L}^C_q)\rho_{S+P}(t), \tag{10}
\]

where \( q = 0, \pm 1 \) labels \( \omega_0 + q\nu \) of \( S + P \), and the generators associated with harmonic \( q\nu \) in the two baths are \( \mathcal{L}_j^q \) (\( j = H, C \)). The explicit form of these generators (see SI) implies that the diagonal and off-diagonal matrix elements evolve independently.

**Steady-state** The dynamics has two different time scales for \( \frac{q}{\nu} \ll 1 \): the fast change of the TLS states and the slow change of the piston state. Subsequent to the fast TLS equilibration, the slow evolution of the diagonal elements of \( P \) obeys, upon denoting its Fock-state population by \( \langle \rho_{nn} \rangle_p \equiv \rho_n \), the rate equation (see SI)
\[ \dot{\rho}_n = r_+ (n+1) \rho_{n+1} + r_- n \rho_{n-1} - (r_+ n + r_- (n+1)) \rho_n, \]

where \( r_+ \) and \( r_- \), the death and birth rates of \( \rho_n \), respectively, have the following form to second order in \( g/\nu \)

\[ r_\pm = \left( \frac{g}{\nu} \right)^2 \left( G(\nu_\pm) \dot{\rho}^{sp} + G(-\nu_\pm) \dot{\rho}^{bg} \right). \]

Here \( G(\omega) = G_C(\omega) + G_H(\omega) \), \( G_C(H) \) being the C(H)-bath spectral response; \( \dot{\rho}^{sp(bg)} = \frac{G(\mp \omega_H)}{G(-\omega_H) + G(\omega_H)} \) are the TLS stationary level populations.

**Cooling conditions** The heat current flowing from C to S is explicitly given by

\[ J_C = \sum_{q=0\pm 1} \text{Tr}\left( \frac{1}{2} \omega_0 \sigma_Z + \nu b^\dagger b \right) \mathcal{L}_q \rho_{S+P}. \]

We assume spectral separation of H and C, as in Fig. 1a. Then (11)-(13) yield

\[ J_C = \left( \frac{g}{\nu} \right)^2 \frac{G_C(\nu_-) G_H(\omega_0)}{N} \left( e^{-\frac{\nu_-}{\gamma_0}} \langle n_P \rangle - e^{-\frac{\nu_+}{\gamma_0}} \langle n_P + 1 \rangle \right). \]

Here \( N \) is a positive normalizing factor and \( \langle n_P \rangle \) is the expected value of the P population. The QR condition \( (J_C > 0) \) then amounts to \( 0 < \frac{\nu_-}{\gamma_0} < \frac{T_H}{T_C} \) and

\[ (e^{-\frac{\nu_-}{\gamma_0}} - 1)^{-1} \equiv \langle n_P \rangle_{\text{min}} < \langle n_P \rangle. \]

Hence, \( \langle n_P \rangle_{\text{min}} \) limits QR action to \( \langle n_P \rangle \) above a lower threshold. The more energetic is the initial state of the piston, the more cooling it delivers, via energy loss of P, which requires a positive drift in (11) \( \gamma \equiv r_+ - r_- > 0 \).

Let us assume that condition (15) holds and analyze the resulting cooling for different initial states of P (Fig. 2):

i) **Initial pure state**: The work stored by a pure state is equal to its energy [29]. P is then the work source of the QR which operates as a compression refrigerator. The refrigerator stops cooling when Eq. (15) no longer holds, having provided \( \nu \langle n_P(0) \rangle \) of work. At the same time, P, initially at \( T_P = 0 \), absorbs heat from the refrigerator, and ends up in a thermal state with the critical temperature

\[ (T_P)_{\text{crit}} = \frac{\nu T_H T_C}{-\nu_T H + \omega_0 T_C}. \]

This \( (T_P)_{\text{crit}} \) can be controlled by changing the P frequency: \( (T_P)_{\text{crit}} \rightarrow \infty \) if the upper bound of (15) holds.

For an initial coherent state, \( \hat{\rho}_n = \frac{1}{\sqrt{N}} \hat{n}_P^{(0)} \). Namely, its COP is determined by \( r_- < \nu_- \), the diffusion rate, \( \gamma \), the drift rate in (11), and the initial energy \( \langle n_P(0) \rangle \). Strikingly, the COP for an initial Fock state with \( n_P(0) \) may well surpass that of a coherent state with the same \( \langle n_P(0) \rangle \) or its classical counterpart (Eq. (6), Fig. 2a). The reason is that a Fock state is much more unstable and prone to thermalization, which allows it to both deliver work and absorb heat more efficiently than a coherent state.

ii) **Initial thermal state**: Since the work content of a thermal state is zero, using (3) for QR action implies that \( T_P > T_H \): P should then be hotter than H. In the course of its evolution, P will remain in a thermal state, but \( T_P \) will decrease until it attains \( (T_P)_{\text{crit}} \) and stops cooling. The corresponding maximal COP conforms to that of an absorption refrigerator (7), which is always lower than the compression-QR COP.

**Realization** The S-P coupling in Eq. (8) is realizable,
e.g., in the well-investigated setup of a superconducting flux qubit that is off-resonantly coupled to P, a high-Q cavity mode (Fig.1-c) [31, 32]. The coupling of the qubit to spectrally separated baths (Fig. 1a) is achievable in such a setup by spectrally filtered, local heat-pump and heat-dump. The spectral filtering procedure is analyzed in [26]. In a superconducting coplanar microwave cavity, it is possible to have $\gamma t_{coh} \gg 1$, where $t_{coh} \geq (\frac{2}{\nu P}) \sim 1/\nu Q$ Msec is the P-mode coherence time [3, 31]. This allows P to evolve from its initial state to the final state characterized by (15) and (16) at a rate $\gamma$, before it leaks out of the cavity.

**Conclusions** Our analysis has produced several striking conclusions concerning fully quantized refrigerators, both generally and in an experimentally feasible setup: a) While the semiclassical limit of piston dynamics coincides with that of an externally (classically) driven machine, once the piston is also quantized, the QR action is dramatically modified: Remarkably, it proves advantageous for the quantum piston to change its temperature and entropy during the process, so that its energy exchange with the system is a combination of heat and work. The instability (rapid entropy increase) of an initial Fock state compared to an initial coherent state, make Fock states temporarily superior in terms of cooling. Our most striking conclusion is that the COP for an initial Fock state exceeds the classical or coherent-state COP, but only at short times (Fig. 2a) b) The quantized piston may act as a “quantum battery” that, once charged, does not require any external energy source to drive the QR. This could be relevant in situations where external power is scarce or where miniaturization of the power source is an advantage. c) This autonomous QR may act in dual mode: The work stored in an initial pure state of the piston may be used to run the machine as a compression QR (driven by work) as long as the piston is below the critical temperature (16). Alternatively, an initial thermal state of the piston may drive the machine as an absorption QR provided the piston temperature is above (16).

This research reveals genuine quantum aspects of refrigeration, beyond the scope of studies which have thus far been restricted to QR driven by an external classical field or heat bath, whose energy or entropy changes in the course of cooling are imperceptible. On the applied side, this study is expected to lay the ground for the design of machines powered by work and heat stored in a quantum device. On the fundamental side, it provides deeper understanding of the rapport between thermodynamics and quantum mechanics, as it is concerned with the simplest, fully quantized model of a refrigerator. Effects of breaking the detailed balance in coherent multilevel [37, 38] or entangled multipartite systems [39], as well as in engineered non-Markovian environments [21, 40–42], may elucidate additional aspects of this fundamental rapport. The third law [5–7] should also be revisited in the framework of this model, and so should quantum fluctuation effects [43].

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**SUPPLEMENTARY INFORMATION**

A. Dressing transformation

The model Hamiltonian (cf. (8))

\[ H_S = \frac{1}{2}\omega_0 \sigma_Z, \quad H_{SB} = \sigma_X (B_H + B_C), \]
\[ H_{SP} = g a \sigma_Z (a^+ + a). \quad (S1) \]

The analysis of the model is simplified by using a new set of canonical operators \( b, b^+ \) and Pauli matrices \( \tilde{\sigma}_k \) obtained from \( a, a^+, \sigma_k \) by a unitary dressing transformation that diagonalizes the S-P Hamiltonian

\[ a \rightarrow b = U a U^\dagger, \quad \sigma_k \rightarrow \tilde{\sigma}_k = U \sigma_k U^\dagger, \quad U = e^{2i(a^+ - a)\sigma_Z}. \quad (S2) \]

Under this transformation \( \sigma_Z \rightarrow \tilde{\sigma}_Z \), therefore the relevant Hamiltonian can be written as

\[ H_S + H_{SP} + H_P = \frac{1}{2}\omega_0 \sigma_Z + \nu b^+ b - \frac{g^2}{2}\frac{1}{\nu} \quad (S3) \]

and the dressing transformation can be expressed in terms of new variables

\[ \tilde{U} = e^{\frac{2i}{\nu}(b^+ - b)\sigma_Z}. \quad (S4) \]

The Pauli matrix \( \sigma_X \), which appears in the \( H_{SB} \) interaction Hamiltonian (S1) is given in terms of the new dynamical variables as

\[ \sigma_X = \tilde{\sigma}_+ e^{\frac{2i}{\nu}(b^+ - b)} + \tilde{\sigma}_- e^{-\frac{2i}{\nu}(b^+ - b)} \quad (S5) \]

From now on we use this dressed system representation, so that \( S \) is the TLS described by \( \tilde{\sigma}_k \) and \( P \) the one described by the operators \( b \) and \( b^+ \).

The Heisenberg-picture Fourier decomposition of \( \sigma_X \) within the lowest order approximation with respect to the small parameter \( g/\nu \), can be obtained from

\[ \sigma_+(t) = e^{iH_S t} \sigma_+ e^{-iH_S t} = e^{i\omega_0 t} \tilde{\sigma}_+ e^{\frac{2i}{\nu}(b^+ e^{-i\omega_0 t} - b e^{i\omega_0 t})} \sim \]
\[ \tilde{\sigma}_+ e^{i\omega_0 t} + \frac{g}{\nu} (S^+_1 e^{i(\omega_0 + \nu)t} - S^-_1 e^{i(\omega_0 - \nu)t}), \quad (S6) \]

where

\[ S^+_1 = \tilde{\sigma}_+ b^+_1, S^-_1 = \tilde{\sigma}_+ b. \quad (S7) \]

The approximation made in (S6) is valid for

\[ \frac{g}{\nu} \sqrt{\langle b^+ b \rangle} << 1, \quad (S8) \]

i.e., for mean piston excitation that is not too large.

B. The rate equation for the piston

The slow-change equations can be written in the following equivalent operator form

\[ \dot{\rho}_P = \frac{r^+_e}{2} ([b, \rho b^+_1] + [b \rho, b^+_1]) + \frac{r^-_e}{2} ([b^+_1, \rho b] + [b^+_1, \rho b]). \quad (S9) \]

The LGKS generators derived from Eq. (S9) are:

\[ L^e_0 \rho_{SP} + P = \frac{1}{2} \left\{ G_j(\omega_0) \left[ (\sigma_- - \rho_{SP} + \sigma_+) + (\sigma_+ - \rho_{SP} + \sigma_-) + G_j(-\omega_0) \left[ (\sigma_- + \rho_{SP} + \sigma_+) + (\sigma_+ + \rho_{SP} + \sigma_-) \right] \right] \right\}, \quad (S10a) \]
\[ L^g_0 \rho_{SP} + P = \frac{g^2}{\nu} \left\{ G_j(\omega_0 + q\nu) \left[ (S_q \rho_{SP} + S^+_q + S^-_q) + (S^+_q \rho_{SP} + S^+_q) \right] \right\} + G_j(-\omega_0 - q\nu) \left[ (S^+_q \rho_{SP} + S_q + [S^+_q, \rho_{SP} + S_q]) \right], \quad q = \pm 1, \quad (S10b) \]

where \( S_q \) and \( \tilde{\sigma}_\pm \) were defined in Eqs. (S6) and (S7). The transition operators \( S_{\pm 1} \) describe the relaxation of \( S \) accompanied by the respective excitation or deexcitation of \( P \), while \( S^+_1 \) describe their time-reversed counterparts. The effects of the baths in Eqs. (S10) are here described by the Fourier transforms of the autocorrelation functions

\[ G_j(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} \langle B_j(t) B_j(0) \rangle dt = e^{i\omega T_j} G_j(-\omega). \quad (S11) \]

For \( \rho_P \) labeled by Fock states \((n,m)\) and the S levels \((e,g)\)

\[ \dot{\rho}^e_{nm} = -\rho^e_{nm} \times \]
\[ \left[ G(\omega_0) + \frac{g^2}{2\nu^2} \left( G(\omega_0 - \nu)(2 + n + m) + G(\omega_0 + \nu)(m + n) \right) \right] \]
\[ + G(-\omega_0) \rho^e_{nm} + \frac{g^2}{\nu} [G(-\omega_0 - \nu) \sqrt{n} \sqrt{m} \rho^e_{n-1,m-1} + G(-\omega_0 + \nu) \sqrt{n + 1} \sqrt{m + 1} \rho^e_{n+1,m+1}], \]
\[ \dot{\rho}^g_{nm} = -G(\omega_0) \rho^g_{nm} + \frac{g^2}{\nu} \left( G(\omega_0 - \nu) \sqrt{n} \sqrt{m} \rho^g_{n-1,m-1} + G(\omega_0 + \nu) \sqrt{n + 1} \sqrt{m + 1} \rho^g_{n+1,m+1} \right). \quad (S12) \]

This process is governed by:

\[ \dot{\rho}^e_{nm} = -G(\omega_0) \rho^e_{nm} + G(-\omega_0) \rho^g_{nm}, \]
\[ \dot{\rho}^g_{nm} = -G(-\omega_0) \rho^e_{nm} + G(\omega_0) \rho^g_{nm}, \]

where \( G(\omega) = \sum_{j \in H,C} G_j(\omega) \). The steady-state TLS populations are

\[ \rho^e_{nm} = \sum_n \rho^e_{nm} = \frac{G(-\omega_0)}{G(-\omega_0) + G(\omega_0)} \]
\[ \rho^g_{nm} = \sum_n \rho^g_{nm} = \frac{G(\omega_0)}{G(-\omega_0) + G(\omega_0)}. \quad (S13) \]

The combined S+P steady-state populations are
\[
\rho_{nm}^{ee} = \rho_{nm} \frac{G(-\omega_0)}{G(-\omega_0) + G(-\omega_0)}, \\
\rho_{nm}^{gg} = \rho_{nm} \frac{G(\omega_0)}{G(-\omega_0) + G(-\omega_0)},
\]

where \( \rho_{nm} \) are the P matrix elements.

C. Effective temperature and entropy-production rate by the piston

(i) For an initial coherent state:

\[
T_P = \frac{1}{\nu} \log \left( \frac{1 + \frac{r_+}{\gamma} (1 - e^{-\gamma t})}{\frac{r_-}{\gamma} (1 - e^{-\gamma t})} \right),
\]

\[
\dot{S}_P = \frac{r_+ e^{-\gamma t}}{T_P}.
\]  

(ii) For an initial thermal state:

\[
T_P = \frac{1}{\nu} \log \left( \frac{1 + \frac{r_+}{\gamma} (1 - e^{-\gamma t})}{\frac{r_-}{\gamma} (1 - e^{-\gamma t})} \right),
\]

\[
\dot{S}_P = \frac{(r_- - \gamma \langle n(0) \rangle_P) e^{-\gamma t}}{T_P}.
\]