Resonance Graphs on Perfect Matchings of Graphs on Surfaces

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Abstract
Let \( G \) be a graph embedded in a surface and let \( \mathcal{F}(G) \) be the set of all faces of \( G \). For a given subset \( \mathcal{F} \) of even faces (faces bounded by an even cycle), the resonance graph of \( G \) with respect to \( \mathcal{F} \), denoted by \( R(G; \mathcal{F}) \), is a graph such that its vertex set is the set of all perfect matchings of \( G \) and two vertices \( M_1 \) and \( M_2 \) are adjacent if and only if the symmetric difference \( M_1 \oplus M_2 \) is a cycle bounding some face in \( \mathcal{F} \). It has been shown that if \( G \) is a plane elementary bipartite graph, the resonance graph of \( G \) with respect to the set of all inner faces is isomorphic to the covering graph of a distributive lattice. However, this result does not hold in general for plane graphs \( G \). The structure properties of resonance graphs in general remain unknown. In this paper, we show that every connected component of the resonance graph of a graph \( G \) on a surface with respect to a given even-face set \( \mathcal{F} \neq \mathcal{F}(G) \) can always be embedded into a hypercube as an induced subgraph. Further, we show that the Clar covering polynomial of \( G \) with respect to \( \mathcal{F} \) is equal to the cube polynomial of the resonance graph \( R(G; \mathcal{F}) \).

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1 Introduction

All graphs considered in this paper are simple and finite, unless stated otherwise. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A perfect matching of $G$ is a set of independent edges of $G$ which covers all vertices of $G$. In other words, the edge induced subgraph of a perfect matching is a spanning 1-regular subgraph, also called 1-factor. Denote the set of all perfect matchings of a graph $G$ by $\mathcal{M}(G)$. The matching polytope, i.e. the convex hull of incidence vectors of matchings of a graph, was characterized by Edmonds [11], which becomes a central part of polyhedral combinatorics and has been used in many combinatorial algorithms (cf. [21]). Lovász [20] characterized the matching lattice defined by the set of all integer linear combinations of incidence vectors of all perfect matchings of a graph, which provides a fundamental view to the structures of matching covered graphs (i.e., every edge belongs to a perfect matching).

A surface in this paper always means a closed surface which is a compact and connected 2-dimensional manifold without boundary. An embedding of a graph $G$ in a surface $\Sigma$ is an injective mapping which maps $G$ into the surface $\Sigma$ such that each vertex of $G$ is mapped to a point, and each edge is mapped to a simple curve connecting two points corresponding to two end-vertices of the edge such that no such curves include points associated with other vertices and two curves never intersect at an interior point of any curve. A face of $G$ embedded in a surface $\Sigma$ is defined as a connected component of $\Sigma \setminus G$. The boundary of a face $f$ is denoted by $\partial f$, the set of edges on the boundary of $f$ is denoted by $E(f)$ and the set of vertices on the boundary of $f$ by $V(f)$. Two distinct faces $f_1$, $f_2$ are disjoint if $\partial f_1 \cap \partial f_2 = \emptyset$. Denote the set of all faces of a graph $G$ on a surface by $\mathcal{F}(G)$. A face $f$ is even if it is bounded by an even cycle and a set of even faces is also called an even-face set. A cycle is a facial cycle if it is the boundary of a face. If there is no confusion, a graph $G$ on a surface always means an embedding of $G$ in the surface, and a face sometime means its boundary cycle.

The structure of perfect matchings of a graph $G$ embedded in a surface can be interpreted in a more intuitive way than Lovász’s matching lattice [20] via so-called resonance graph or Z-transformation graph. Let $G$ be a graph embedded in a surface $\Sigma$ with a perfect matching $M$. A cycle $C$ of $G$ is $M$-alternating if the edges of $C$ appear alternately between $M$ and $E(G) \setminus M$. Moreover, a face $f$ of $G$ is $M$-alternating if $\partial f$ is an $M$-alternating cycle. For a given set of even faces $\mathcal{F} \subseteq \mathcal{F}(G)$, the resonance graph of $G$ with respect to $\mathcal{F}$ (or Z-transformation graph of $G$ [31]), denoted by $R(G; \mathcal{F})$, is a graph with vertex set $\mathcal{M}(G)$ such that two vertices $M_1$ and $M_2$ are adjacent if and only if the symmetric difference $M_1 \oplus M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$ is the boundary of a face $f \in \mathcal{F}$, i.e. $E(f) = M_1 \oplus M_2$. Note that if $\mathcal{M}(G) = \emptyset$, then the resonance graph has no vertices. Therefore, we often assume that $G$ has at least one perfect matching. If a graph has a unique perfect matching (i.e. $|\mathcal{M}(G)| = 1$), then its resonance graph is a single vertex. Another trivial case is $\mathcal{F} = \emptyset$, in which the resonance graph is an empty graph (having no edges).

Resonance graph was introduced by Gründler [15] for hexagonal systems, which are plane 2-connected bipartite graphs with only hexagonal inner faces, and independently by Zhang, Guo and Chen [31] as Z-transformation graph. The properties of resonance
graphs of plane bipartite graphs have been well-studied. Readers may refer to the comprehensive survey on this topic [28].

A graph $G$ is elementary if the edges of $G$ contained in a perfect matching induce a connected subgraph (cf. [21]). An elementary bipartite graph is also matching-covered (or 1-extendable), i.e., every edge is contained in a perfect matching. It is known that a matching-covered graph with at least four vertices is always 2-connected [23]. So is an elementary bipartite graph. It has been shown in [30] that a plane bipartite graph $G$ is elementary if and only if every face boundary of $G$ is an $M$-alternating cycle for some perfect matching $M$ of $G$.

**Theorem 1.1** (Lam and Zhang [19]) Let $G$ be a plane elementary bipartite graph and $\mathcal{F}$ be the set of all inner faces of $G$. Then $R(G; \mathcal{F})$ is the covering graph of a distributive lattice.

A similar result was obtained independently by Propp [24], see also a result by Felsner [12]. Operations on other combinatorial structures such as, spanning trees, orientations and flows have been discovered to have similar properties [13, 24].

It is natural to ask whether Theorem 1.1 holds for all plane graphs or even more general for graphs embedded in surfaces. However, the answer to this problem is negative. The first author and Žigert Pleteršek found that Theorem 1.1 does not hold for fullerenes in [27], where a fullerene is a plane connected cubic graph in which every face is either a pentagon or a hexagon. They further make the following conjecture, where median graphs form a family of graphs containing distributive lattices as a sub-family (a definition of median graph is given in the next section).

**Conjecture 1.2** (Tratnik and Žigert Pleteršek [27]) Every connected component of the resonance graph of a fullerene is a median graph.

A consequence of Theorem 1.1 shows that the resonance graphs of elementary plane bipartite graphs are median graphs, which holds even for more general plane bipartite graphs [19, 33], extending an older result of Klavžar et al. [18]. Recently, Che [9] gave a different proof of this result on plane bipartite graphs based on a characterization of median graphs (cf [16]).

Note that median graphs are a family of well-studied graphs (see [6, 8]), containing the covering graphs of distributive lattices. Median graphs are a subfamily of cubical graphs [1, 14, 22], which are defined as subgraphs of hypercubes, and have important applications in coding theory, data transmission, and linguistics (cf. [14, 22]).

In this paper, we study the resonance graphs of graphs embedded in a surface in a general manner, and prove the following result.

**Theorem 1.3** Let $G$ be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be an even-face set. Then every connected component of the resonance graph $R(G; \mathcal{F})$ is an induced cubical graph.

A direct corollary of the above result presents a progress toward solving Conjecture 1.2 since a median graph is an induced cubical graph (see Sect. 2 for the detailed definitions).
Corollary 1.4 Every connected component of the resonance graph of a fullerene is an induced cubical graph.

A perfect 2-matching of a graph $G$ embedded in a surface is a spanning subgraph of which every component is either an edge or a cycle. A Clar cover of $G$ is a perfect 2-matching in which every cycle is an even facial cycle. The Clar covering polynomial (or Zhang-Zhang polynomial) of graph $G$ embedded in a surface is a polynomial used to enumerate all Clar covers of $G$. The definition of Clar covering polynomial will be given in the next section. Zhang et al. [35] build the equivalence between the Clar covering polynomial of a hexagonal system and the cube polynomial of its resonance graph, which is further generalized to other plane graphs [4, 27]. In this paper, we show the equivalence between the Clar covering polynomial of a graph $G$ embedded in a surface and the cube polynomial of its resonance graph as follows.

Theorem 1.5 Let $G$ be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be a set of even faces. Then the Clar covering polynomial of $G$ with respect to $\mathcal{F}$ is equal to the cube polynomial of the resonance graph $R(G; \mathcal{F})$.

The paper is organized as follows: some detailed definitions are given in Sect. 2, the proofs of Theorems 1.3 and 1.5 are given in Sects. 3 and 4, respectively. We conclude the paper with some open problems in Sect. 5.

2 Preliminaries

Let $G$ be a graph and let $u, v$ be two vertices of $G$. The distance between $u$ and $v$, denoted by $d_G(u, v)$ (or $d(u, v)$ if there is no confusion) is the length of a shortest path joining $u$ and $v$. A median of a triple of vertices $\{u, v, w\}$ of $G$ is a vertex $x$ that lies on a shortest $(u, v)$-path, on a shortest $(u, w)$-path and on a shortest $(v, w)$-path. Note that $x$ could be one vertex from $\{u, v, w\}$. A graph is a median graph if every triple of vertices has a unique median. Median graphs were first introduced by Avann [2], which arise naturally in the study of ordered sets and distributive lattices. A lattice is a poset such that any two elements have a greatest lower bound and a least upper bound. The covering graph of a lattice $L$ is a graph whose vertex set consists of all elements in $L$ and two vertices $x$ and $y$ are adjacent if and only if either $x$ covers $y$ or $y$ covers $x$. A distributive lattice is a lattice in which the operations of the join and meet distribute over each other. It is known that the covering graph of a distributive lattice is a median graph but not vice versa [10].

The $n$-dimensional hypercube $Q_n$ with $n \geq 1$, is the graph whose vertices are all binary strings of length $n$ and two vertices are adjacent if and only if their strings differ exactly in one position. For convenience, define $Q_0$ to be the one-vertex graph. The cube polynomial of a graph $G$ is defined as follows,

$$Q(G, x) = \sum_{i \geq 0} \alpha_i(G)x^i,$$
where $\alpha_i(G)$ denotes the number of induced subgraphs of $G$ that are isomorphic to the $i$-dimensional hypercube. The cube polynomials of median graphs have been studied by Brešar, Klavžar and Škrekovski [6, 8].

Let $H$ and $G$ be two graphs. A function $\ell : V(H) \rightarrow V(G)$ is called an embedding of $H$ into $G$ if $\ell$ is injective and, for any two vertices $x, y \in V(H)$, $\ell(x) \ell(y) \in E(G)$ if $xy \in E(H)$. If such a function $\ell$ exists, we say that $H$ can be embedded in $G$. In other words, $H$ is isomorphic to a subgraph of $G$. Moreover, if $\ell$ is an embedding such that for any two vertices $x, y \in V(H)$, $\ell(x) \ell(y) \in E(G)$ and only if $xy \in E(H)$, then $H$ can be embedded in $G$ as an induced subgraph. An embedding $\ell$ of the graph $H$ into $G$ is called an isometric embedding if for any two vertices $x, y \in V(H)$, $d_H(x, y) = d_G(\ell(x), \ell(y))$. A graph $H$ is (induced) cubical if $H$ can be embedded into $Q_n$ for some integer $n \geq 1$ (as an induced subgraph), and $H$ is called a partial cube if $H$ can be isometrically embedded into $Q_n$ for some integer $n \geq 1$. For more information and properties on cubical graphs, readers may refer to [1, 5, 7, 14]. It holds that a median graph is a partial cube (in fact, even stronger result is true, i.e. median graphs are retracts of hypercubes, see [3]). Therefore, we have the nested relations for these interesting families of graphs:

$$\{\text{covering graphs of distributive lattices}\} \subset \{\text{median graphs}\} \subset \{\text{partial cubes}\} \subset \{\text{induced cubical graphs}\} \subset \{\text{cubical graphs}\}.$$  

In the following, let $G$ be a graph embedded in a surface $\Sigma$ and let $f$ be a face bounded by a cycle of $G$. If $G$ has two perfect matchings $M_1$ and $M_2$ such that the symmetric difference $M_1 \oplus M_2$ is a cycle which bounds face $f$, then we say that $M_1$ can be obtained from $M_2$ by rotating the edges of $f$. Therefore, two perfect matchings $M_1$ and $M_2$ of $G$ are adjacent in the resonance graph $R(G; \mathcal{F})$ if and only if $M_1$ can be obtained from $M_2$ by rotating the edges of some face $f \in \mathcal{F}$. We sometimes also say that edge $M_1M_2$ corresponds to face $f$ or face $f$ corresponds to edge $M_1M_2$.

A Clar cover of $G$ is a spanning subgraph $S$ of $G$ such that every component of $S$ is either the boundary of an even face or an edge. Let $\mathcal{F} \subseteq \mathcal{F}(G)$ be an even-face set. The Clar covering polynomial of $G$ with respect to $\mathcal{F}$ (also called the Zhang-Zhang polynomial) was defined by Zhang and Zhang [29] as follows,

$$C_\mathcal{F}(G, x) = \sum_{k \geq 0} z_k(G, \mathcal{F}) x^k,$$

where $z_k(G, \mathcal{F})$ is the number of Clar covers of $G$ with exactly $k$ faces and all the $k$ faces belong to $\mathcal{F}$. Note that $z_0(G, \mathcal{F})$ equals the number of perfect matchings of $G$, i.e., the number of vertices of the resonance graph $R(G; \mathcal{F})$.

### 3 Resonance Graphs and Cubical Graphs

Let $G$ be a graph embedded in a surface and let $\mathcal{F}$ be a set of even faces of $G$ such that $\mathcal{F} \neq \mathcal{F}(G)$. In this section, we investigate the resonance graph $R(G; \mathcal{F})$ and show that
every connected component of \( R(G; \mathcal{F}) \) is an induced cubical graph. The following lemma is a generalization of similar results from [30, 31].

**Lemma 3.1** Let \( G \) be a graph embedded in a surface and let \( \mathcal{F} \neq \mathcal{F}(G) \) be an even-face set. Assume that \( C = M_0M_1 \ldots M_{t-1}M_0 \) is a cycle of the resonance graph \( R(G; \mathcal{F}) \). Let \( f_i \) be the face of \( G \) corresponding to the edge \( M_iM_{i+1} \) for \( i \in \{0, 1, \ldots, t - 1\} \) where subscripts take modulo \( t \). Then every face of \( G \) appears an even number of times in the face sequence \( (f_0, f_1, \ldots, f_{t-1}) \).

**Proof** Let \( f \) be a face of \( G \), and let \( \delta(f) \) be the number of times \( f \) appears in the face sequence \( (f_0, f_1, \ldots, f_{t-1}) \). It suffices to show that \( \delta(f) \equiv 0 \pmod{2} \). Since \( C = M_0M_1 \ldots M_{t-1}M_0 \) is a cycle of \( R(G; \mathcal{F}) \) and \( f_i \) is the corresponding face of the edge \( M_iM_{i+1} \), it follows that \( M_i \oplus M_{i+1} = E(f_i) \) for \( i \in \{0, 1, \ldots, t - 1\} \). So

\[
E(f_0) \oplus E(f_1) \oplus \cdots \oplus E(f_{t-1}) = \bigoplus_{i=0}^{t-1}(M_i \oplus M_{i+1}) = \emptyset
\]  

where all subscripts take modulo \( t \).

Let \( f \) and \( g \) be two faces of \( G \) such that \( E(f) \cap E(g) \neq \emptyset \), and let \( e \in E(f) \cap E(g) \). Since \( e \) is contained by only \( f \) and \( g \), and the total number of faces in the sequence \( (f_0, f_1, \ldots, f_{t-1}) \) containing \( e \) is even by (1), it follows that \( \delta(f) + \delta(g) \equiv 0 \pmod{2} \). So \( \delta(f) \equiv \delta(g) \pmod{2} \). Since the dual graph is connected, it follows that all faces \( f \) of \( G \) have the same parity for \( \delta(f) \).

Note that \( \mathcal{F} \neq \mathcal{F}(G) \). So \( G \) has a face \( g \notin \mathcal{F} \). Hence \( g \) does not appear in the face sequence. It follows that \( \delta(g) = 0 \). Hence \( \delta(f) \equiv \delta(g) \equiv 0 \pmod{2} \) for any face \( f \) of \( G \). This completes the proof. \( \square \)

The following proposition follows immediately from Lemma 3.1.

**Proposition 3.2** Let \( G \) be a graph embedded in a surface, and let \( \mathcal{F} \neq \mathcal{F}(G) \) be a set of even faces. Then the resonance graph \( R(G; \mathcal{F}) \) is bipartite.

**Proof** Let \( C = M_0M_1 \ldots M_{t-1}M_0 \) be a cycle of \( R(G; \mathcal{F}) \) and let \( f_i \) be the face corresponding to the edge \( M_iM_{i+1} \) for \( i \in \{0, 1, \ldots, t - 1\} \) (subscripts take modulo \( t \)). By Lemma 3.1, every face \( f \) of \( G \) appears an even number of times in the sequence \( (f_0, f_1, \ldots, f_{t-1}) \). So \( C \) is a cycle of even length. Therefore, \( R(G; \mathcal{F}) \) is a bipartite graph. \( \square \)

The above proposition shows that the resonance graph \( R(G; \mathcal{F}) \) is bipartite if \( \mathcal{F} \neq \mathcal{F}(G) \). However, if \( \mathcal{F} = \mathcal{F}(G) \), then \( R(G; \mathcal{F}) \) may not be bipartite. For example, the graph \( G \) on the left in Fig. 1 is a plane graph with three faces \( f_1, f_2 \) and \( f_3 \). If \( \mathcal{F} = \{f_1, f_2, f_3\} \), then its resonance graph \( R(G; \mathcal{F}) \) is a triangle as shown on the right in Fig. 1. Note that the resonance graphs of plane bipartite graphs with \( \mathcal{F} = \mathcal{F}(G) \) have already been studied in [32].

It is known that a resonance graph \( R(G; \mathcal{F}) \) may not be connected [27]. In the following, we focus on a connected component \( H \) of \( R(G; \mathcal{F}) \), and always assume \( \{f_1, \ldots, f_k\} \) to be the set of all the faces that correspond to the edges of \( H \), which is a subset of \( \mathcal{F} \). Denote the set of all the edges of \( H \) that correspond to the face \( f_i \) by \( E_i \) for \( i \in \{1, \ldots, k\} \). In the rest of this section, \( H \setminus E_i \) denotes the graph obtained from \( H \) by deleting all the edges from \( E_i \).
Lemma 3.3 Let $R(G; \mathcal{F})$ be the resonance graph of a graph $G$ on a surface with respect to a set of even faces $\mathcal{F} \neq \mathcal{F}(G)$, and let $H$ be a connected component of $R(G; \mathcal{F})$. Assume that $M_1M_2 \in E_i$ where $E_i$ is the set of all the edges of $H$ corresponding to some face $f_i \in \mathcal{F}$. Then $M_1$ and $M_2$ belong to different components of $H \setminus E_i$.

Proof Suppose to the contrary that $M_1$ and $M_2$ belong to the same component of $H \setminus E_i$. Then $H \setminus E_i$ has a path $P$ joining $M_1$ and $M_2$. In other words, $H$ has a path $P$ joining $M_1$ and $M_2$ such that $E(P) \cap E_i = \emptyset$. Let $C = P \cup \{M_1M_2\}$ be a cycle of $H$. Then $f_i$ appears exactly once in the face sequence corresponding to the edges in the cycle $C$, which contradicts Lemma 3.1. The contradiction implies that $M_1$ and $M_2$ belong to different components of $H \setminus E_i$. □

By Lemma 3.3, the graph $H \setminus E_i$ is disconnected for any face $f_i$ of $G$ which corresponds to the edges in $E_i$. Define the quotient graph $\mathcal{H}_i$ of $H$ with respect to $f_i$ to be a graph obtained from $H$ by contracting all edges in $E(H) \setminus E_i$ and replacing any set of parallel edges by a single edge. So a vertex of $\mathcal{H}_i$ corresponds to a connected component of $H \setminus E_i$.

Lemma 3.4 Let $R(G; \mathcal{F})$ be the resonance graph of a graph $G$ on a surface with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$. Moreover, let $f_i$ be a face of $G$ that corresponds to some edge of a connected component $H$ of $R(G; \mathcal{F})$. Then the quotient graph $\mathcal{H}_i$ with respect to $f_i$ is bipartite.

Proof Suppose to the contrary that $\mathcal{H}_i$ has an odd cycle. Then $H$ has a cycle $C$ which contains an odd number of edges corresponding to the face $f_i$. In other words, the face $f_i$ appears an odd number of times in the face sequence corresponding to edges of $C$, which contradicts Lemma 3.1. Therefore, $\mathcal{H}_i$ is bipartite. □

Recall that $\{f_1, \ldots, f_k\}$ is the set of all the faces that correspond to the edges of a connected component $H$ of $R(G; \mathcal{F})$, and $\mathcal{H}_i$ is the quotient graph of $H$ with respect to the face $f_i$ for $i \in \{1, \ldots, k\}$. By Lemma 3.4, let $(A_i, B_i)$ be the bipartition of $\mathcal{H}_i$, and let $\mathcal{M}_{A_i}$ and $\mathcal{M}_{B_i}$ be the sets of perfect matchings of $G$ which are vertices of connected components of $H \setminus E_i$ corresponding to vertices of $\mathcal{H}_i$ in $A_i$ and $B_i$, respectively. Define a function $\ell_i : V(H) \rightarrow \{0, 1\}$ as follows, for any $M \in V(H)$,

$$\ell_i(M) = \begin{cases} 0; & M \in \mathcal{M}_{A_i} \\ 1; & M \in \mathcal{M}_{B_i}. \end{cases}$$
Further, define a function $\ell : V(H) \to \{0, 1\}^k$ such that, for any $M \in V(H)$,

$$\ell(M) = (\ell_1(M), \ldots, \ell_k(M)).$$

(2)

Our main result, Theorem 1.3, follows directly from the following result — Theorem 3.5.

**Theorem 3.5** Let $G$ be a graph embedded in a surface, and let $H$ be a connected component of the resonance graph $R(G; \mathcal{F})$ of $G$ with respect to an even-face set $\mathcal{F} \neq \mathcal{F}(G)$. Then the function $\ell : V(H) \to \{0, 1\}^k$ defines an embedding of $H$ into a $k$-dimensional hypercube as an induced subgraph.

**Proof** If $G$ has no perfect matching or a unique perfect matching, the result holds trivially. So, in the following, assume that $G$ has at least two perfect matchings.

First, we show that the function $\ell : V(H) \to \{0, 1\}^k$ defined in (2) is injective, i.e., for any $M_1, M_2 \in V(H)$, it holds that $\ell(M_1) \neq \ell(M_2)$ if $M_1 \neq M_2$. Let $P = M_1X_1 \ldots X_{t-1}M_2$ be a shortest path of $H$ between $M_1$ and $M_2$. Moreover, let $g_1, \ldots, g_t$ be the faces corresponding to the edges of $P$ such that $g_j$ corresponds to $X_{j-1}X_j$ for $j \in \{1, \ldots, t\}$ (where $X_0 = M_1$ and $X_t = M_2$). Note that, some faces $g_i$ and $g_j$ may be the same face for different $i, j \in \{1, \ldots, t\}$. If every face of $G$ appears an even number of times in the sequence $(g_1, \ldots, g_t)$, then $M_1 = M_1 \oplus E(g_1) \oplus E(g_2) \oplus \cdots \oplus E(g_t) = M_1$, contradicting that $M_1 \neq M_2$. Therefore, there exists a face appearing an odd number of times in the face sequence $(g_1, \ldots, g_t)$. Without loss of generality, assume the face is $f_i$. By Lemma 3.3, two end-vertices of an edge in $E_i$ belong to different connected components of $H \setminus E_i$. Since the face $f_i$ appears an odd number of times in the face sequence $(g_1, \ldots, g_t)$, it follows that if we contract all edges of $P$ not in $E_i$, the resulting walk $P'$ of $H_i$ joining the two vertices corresponding to the two components containing $M_1$ and $M_2$ has an odd number of edges. Note that $H_i$ is bipartite by Lemma 3.4. So one of $M_1$ and $M_2$ belongs to $M_{A_i}$, and the other belongs to $M_{B_i}$. So $\ell_i(M_1) \neq \ell_i(M_2)$. Therefore, $\ell(M_1) \neq \ell(M_2)$.

Next, we show that $\ell$ defines an embedding of $H$ into a $k$-dimensional hypercube. It suffices to show that for any edge $M_1M_2 \in E(H)$, it holds that $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position. Assume that $M_1M_2$ corresponds to a face $f_i \in \mathcal{F}$. In other words, the symmetric difference of two perfect matchings $M_1$ and $M_2$ is the boundary of the face $f_i$. For any $j \in \{1, \ldots, k\}$ and $j \neq i$, the edge $M_1M_2 \in E(H \setminus E_j)$ because $M_1M_2 \in E_i$ and $E_i \cap E_j = \emptyset$. Therefore $M_1$ and $M_2$ belong to the same connected component of $H \setminus E_j$. Hence $\ell_j(M_1) = \ell_j(M_2)$ for any $j \in \{1, \ldots, k\}, j \neq i$. Since $\ell_i(M_1) \neq \ell_i(M_2)$, it follows that $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position, the $i$-th position. Hence, $\ell$ defines an embedding of $H$ into a $k$-dimensional hypercube.

Finally, we are going to show that $\ell$ embeds $H$ in a $k$-dimensional hypercube as an induced subgraph. It suffices to show that, for any $M_1, M_2 \in V(H)$, $M_1M_2 \in E(H)$ if $\ell(M_1)$ and $\ell(M_2)$ differ in exactly one position. Without loss of generality, assume that $\ell_i(M_1) = 0$ and $\ell_i(M_2) = 1$ but $\ell_j(M_1) = \ell_j(M_2)$ for any $j \in \{1, \ldots, k\} \setminus \{i\}$. By the definition of the function $\ell$, we have $M_1 \in M_{A_i}$, and $M_2 \in M_{B_i}$. Let $P$ be a path of $H$ joining $M_1$ and $M_2$. Then contract all edges of $P$ not in $E_i$ and the resulting walk $P'$ of $H_i$ joins two vertices from different partitions of $H_i$. So $P'$ has an odd number of
edges. In other words, \(|P \cap E_j|\) is odd. But, for any \(j \in \{1, \ldots, k\}\) and \(j \neq i\), contract all edges of \(P\) not in \(E_j\) and the resulting walk \(P''\) joins two vertices from the same partition of \(\delta(p)\). So \(|P \cap E_j| \equiv 0 \pmod{2}\). Therefore, for any \(e \in E(G)\), the edge \(e\) appears an odd number of times along path \(P\) if \(e \in E(f_i)\), but an even number of times if \(e \notin E(f_i)\). It follows that \(M_1 \oplus M_2 = E(f_i)\), which implies \(M_1M_2 \in E(H)\). This completes the proof.

It is worth mentioning that the proofs of previous results, that the resonance graphs are median for plane bipartite graphs [19, 33] and carbon nanotubes [26], are based on distributive structure, but the proofs of the result for catacondensed even ring systems (which include hexagonal systems) [17, 18] and plane bipartite graphs given in [9] are based on structural characterization of median graphs and special structural properties of plane bipartite graphs. However, in general, these methods may not be applicable to graphs on surfaces.

### 4 Clar Covers and the Cube Polynomial

In this section, we show that the Clar covering polynomial of a graph \(G\) on a surface with respect to an even-face set \(\mathcal{F} \neq \mathcal{F}(G)\) is equal to the cube polynomial of the resonance graph \(R(G; \mathcal{F})\), which generalizes the main results on some plane bipartite graphs [4, 35] and fullerenes [27]. However, this general setting of our result requires some new ideas and additional insights into the role and structure of the resonance graph. The following essential lemma generalizes results of [34, 35] originally proved for plane bipartite graphs.

**Lemma 4.1** Let \(G\) be a graph embedded in a surface. If the resonance graph \(R(G; \mathcal{F})\) of \(G\) with respect to an even-face set \(\mathcal{F} \neq \mathcal{F}(G)\) contains a 4-cycle \(M_0M_1M_2M_3M_0\), then \(M_0 \oplus M_1 = M_2 \oplus M_3\) and \(M_0 \oplus M_3 = M_1 \oplus M_2\). Further, the two faces bounded by \(M_0 \oplus M_1\) and \(M_0 \oplus M_3\) are disjoint.

**Proof** Since \(M_0M_1M_2M_3M_0\) is a 4-cycle in the resonance graph \(R(G; \mathcal{F})\), let \(f_i\) be the face of \(G\) such that \(E(f_i) = M_i \oplus M_{i+1}\) where \(i \in \{0, 1, 2, 3\}\) and subscripts take modulo 4. Since \(M_i \neq M_{i+2}\), it follows that \(f_i \neq f_{i+1}\), where \(i \in \{0, 1, 2, 3\}\) and subscripts take modulo 4. So \(f_i \neq f_{i+1}\) and \(f_i \neq f_{i-1}\). By Lemma 3.1 and \(\mathcal{F} \neq \mathcal{F}(G)\), any face appears an even number of times in the face sequence \((f_0, f_1, f_2, f_3)\) and therefore, \(f_i = f_{i+2}\) for \(i \in \{0, 1, 2, 3\}\). So it follows that \(f_0 = f_2\) and \(f_1 = f_3\). In other words, \(M_0 \oplus M_1 = M_2 \oplus M_3\) and \(M_0 \oplus M_3 = M_1 \oplus M_2\).

To finish the proof, we need to show that the faces \(f_0\) and \(f_1\) are disjoint. Suppose to the contrary that \(\partial f_0 \cap \partial f_1 \neq \emptyset\). Note that \(f_0 \neq f_1\). So every component of \(\partial f_0 \cap \partial f_1\) is a path on at least two vertices. Let \(v\) be an end vertex of some component of \(\partial f_0 \cap \partial f_1\). Therefore, \(v\) is incident with three edges \(e_1, e_2\) and \(e_3\) such that \(e_1, e_2 \in E(f_0)\) but \(e_1, e_3 \in E(f_1)\). Since both \(f_0\) and \(f_1\) are \(M_1\)-alternating, it follows that \(e_1 \in M_1\). Note that \(M_0 = M_1 \oplus E(f_0)\). So \(e_1 \notin M_0\). Since \(f_1 = f_3\), both \(f_0\) and \(f_3\) are \(M_0\)-alternating. Hence \(e_1 \in M_0\), contradicting \(e_1 \notin M_0\). This completes the proof. \(\square\)
Remark If $\mathcal{F} = \mathcal{F}(G)$, then Lemma 4.1 does not hold because Lemma 3.1 fails. For example, the resonance graph $R(G; \mathcal{F}(G))$ of the plane graph in Fig. 2 (left) has a 4-cycle $M_1 M_2 M_3 M_4 M_1$ which does not satisfy the property of Lemma 4.1, where $\mathcal{F}(G) = \{f_1, \ldots, f_4\}$.

Now, we are going to prove Theorem 1.5.

Proof of Theorem 1.5 Let $G$ be a graph on a surface and let $R(G; \mathcal{F})$ be the resonance graph of $G$ with respect to $\mathcal{F}$. If $G$ has no perfect matching, then the result holds trivially. So, in the following, we always assume that $G$ has a perfect matching.

For an nonnegative integer $k$, let $Z_k(G, \mathcal{F})$ be the set of all Clar covers of $G$ with exactly $k$ faces such that all these faces are included in $\mathcal{F}$, and let $Q_k(R(G; \mathcal{F}))$ be the set of all induced subgraphs of $R(G; \mathcal{F})$ that are isomorphic to the $k$-dimensional hypercube. For a Clar cover $S \in Z_k(G, \mathcal{F})$, let $M_1, M_2, \ldots, M_t$ be all the perfect matchings of $G$ such that all faces in $S$ are $M_i$-alternating and all isolated edges of $S$ belong to $M_i$ for all $i \in \{1, \ldots, t\}$. Define a mapping

$$m_k : Z_k(G, \mathcal{F}) \longrightarrow Q_k(R(G; \mathcal{F}))$$

such that $m_k(S)$ is the subgraph of $R(G; \mathcal{F})$ induced by the vertex set $\{M_1, M_2, \ldots, M_t\}$. Since the subgraph induced by $\{M_1, \ldots, M_t\}$ is unique, the mapping $m_k$ is well-defined, which follows from the following claim.

Claim 1. For each Clar cover $S \in Z_k(G, \mathcal{F})$, the image $m_k(S) \in Q_k(R(G; \mathcal{F}))$.

Proof of Claim 1 It is sufficient to show that $m_k(S)$ is isomorphic to the $k$-dimensional hypercube $Q_k$. Let $f_1, f_2, \ldots, f_k$ be the faces in the Clar cover $S$. Then $\{f_1, \ldots, f_k\} \subseteq \mathcal{F}$. So each $f_i$ with $i \in \{1, \ldots, k\}$ is even and hence has two perfect matchings labelled by “0” and “1” respectively. For any vertex $M$ of $m_k(S)$, let $b(M) = (b_1, b_2, \ldots, b_k)$, where $b_i = \alpha$ if $M \cap E(f_i)$ is the perfect matching of $\partial f_i$ with label $\alpha \in \{0, 1\}$ for each $i \in \{1, 2, \ldots, k\}$. It is obvious that $b : V(m_k(S)) \rightarrow V(Q_k)$ is a bijection. For $M' \in V(m_k(S))$, let $b(M') = (b'_1, b'_2, \ldots, b'_k)$. If $M$ and $M'$ are adjacent in $m_k(S)$ then $M \oplus M' = E(f_i)$ for some $i \in \{1, 2, \ldots, k\}$. Therefore, $b_j = b'_j$ for each $j \neq i$ and $b_i \neq b'_i$, which implies $(b_1, b_2, \ldots, b_k)$ and $(b'_1, b'_2, \ldots, b'_k)$ are adjacent in $Q_k$. Conversely, if $(b_1, b_2, \ldots, b_k)$ and $(b'_1, b'_2, \ldots, b'_k)$ are adjacent in $Q_k$, it follows that
$M$ and $M'$ are adjacent in $m_k(S)$. Hence $b$ is an isomorphism between $m_k(S)$ and $Q_k$. This completes the proof of Claim 1.

In order to show $C(G, x) = Q(R(G; F), x)$, it suffices to show that mapping $m_k$ is bijective for any $k$. Note that in the case of $k = 0$, a Clar cover $S$ is a perfect matching of $G$ and hence $m_k(S)$ is a vertex of $R(G; F)$. So the mapping $m_k$ is obviously bijective for $k = 0$. In the following, assume that $k$ is a positive integer.

First, we show that $m_k$ is injective. Let $S$ and $S'$ be two different Clar covers from $Z_k(G; F)$. If $S \cap F = S' \cap F$, then the isolated edges of $S$ and $S'$ are different. So a perfect matching of $S$ is different from a perfect matching of $S'$. Therefore, the vertex sets of $m_k(S)$ and $m_k(S')$ are disjoint and hence, $m_k(S)$ and $m_k(S')$ are disjoint induced subgraphs of $R(G; F)$. So $m_k(S) \neq m_k(S')$. Now suppose that $S \cap F \neq S' \cap F$. Note that $|S \cap F| = |S' \cap F| = k$. So $S \cap F$ has a face $f \notin S' \cap F$. Note that the faces adjacent to $f$ do not all belong to $S'$ since the faces in $S'$ are independent. Hence the face $f$ contains at least one edge $e$ which does not belong to $S'$. From the definition of the function $m_k$, the edge $e$ does not belong to those perfect matchings of $G$ that correspond to the vertices of $m_k(S')$. For any perfect matching $M$ corresponding to a vertex of $m_k(S)$, the face $f$ is $M$-alternating. Hence either $M$ or $M' = M \oplus E(f)$ contains $e$. Without loss of generality, assume that $e \in M'$. So $M'$ is not a vertex of $M_k(S')$. Since both $M$ and $M'$ are perfect matchings of $S$, both $M$ and $M'$ are vertices of $m_k(S)$. So $m_k(S) \neq m_k(S')$. This shows that $m_k$ is injective.

In the following, we are going to show that $m_k$ is surjective. Let $Q \in Q_k(R(G; F))$. Then every vertex $u$ of $Q$ can be represented by a binary string $(u_1, u_2, \ldots, u_k)$ such that two vertices of $Q$ are adjacent in $Q$ if and only if their binary strings differ in precisely one position. Label the vertices of $Q$ by $M^0 = (0, 0, 0, \ldots, 0)$, $M^1 = (1, 0, 0, \ldots, 0)$, $M^2 = (0, 1, 0, \ldots, 0)$, ..., $M^k = (0, 0, 0, \ldots, 1)$. So $M^0 M^1$ is an edge of $R(G; F)$ for every $i \in \{1, \ldots, k\}$. By definition of $R(G; F)$, the symmetric difference of perfect matchings $M^0$ and $M^1$ is the boundary of an even face in $F$, denoted by $f_i$. Then we have a set of faces $\{f_1, \ldots, f_k\} \subseteq F$. Note that $f_i \neq f_j$ for $i, j \in \{1, \ldots, k\}$ and $i \neq j$ since $M^i \neq M^j$. Hence, all faces in $\{f_1, \ldots, f_k\}$ are distinct. In order to show that $m_k$ is surjective, it is sufficient to show that $G$ has a Clar cover $S$ such that $S \cap F = \{f_1, \ldots, f_k\}$.

**Claim 2. All faces in $\{f_1, \ldots, f_k\}$ are pairwise disjoint.**

**Proof of Claim 2.** Let $f_i, f_j \in \{f_1, \ldots, f_k\}$ with $i \neq j$ and let $W$ be a vertex of $Q$ having exactly two 1’s which are in the $i$-th and $j$-th position. Then $M^0 M^1 W M^j M^0$ is a 4-cycle such that $E(f_i) = M^0 \oplus M^j$ and $E(f_j) = M^0 \oplus M^j$. Then by Lemma 4.1, it follows that $f_i$ and $f_j$ are disjoint. This completes the proof of Claim 2.

By Claim 2, we only need to show that $G - \bigcup_{i=1}^k V(f_i)$ has a perfect matching. Consider the perfect matching $M^0$ corresponding to the vertex of $Q$ labelled by the string with $k$ zeros. Recall that $E(f_i) = M^0 \oplus M^j$ and hence every $f_i$ is $M^0$-alternating for any $i \in \{1, 2, \ldots, k\}$. Therefore, $M := M^0 \setminus (\bigcup_{i=1}^k E(f_i))$ is a perfect matching of $G - \bigcup_{i=1}^k V(f_i)$. So $S = M \cup \{f_1, \ldots, f_k\}$ is a Clar cover of $G$ such that $m_k(S) = Q$. This completes the proof of that $m_k(G)$ is surjective.

From the above, $m_k$ is a bijection between the set of all Clar covers of $G$ with $k$ facial cycles and the set of all induced subgraphs isomorphic to the $k$-dimensional
hypercube for any integer \( k \). Therefore, we have \( C_F(G, x) = Q(R(G; \mathcal{F}), x) \) and this completes the proof of Theorem 1.5.

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5 Concluding Remarks

Let \( G \) be a graph embedded in a surface \( \Sigma \) and let \( \mathcal{F} \) be an even-face set. Assume that \( H \) is a connected component of \( R(G; \mathcal{F}) \). Let \( G_H \) be the subgraph of \( G \) induced by all the faces corresponding to edges of \( H \).

**Proposition 5.1** Let \( G \) be a graph embedded in a surface \( \Sigma \) and let \( \mathcal{F} \) be an even-face set. If a perfect matching \( M \) is a vertex of a connected component \( H \) of \( R(G; \mathcal{F}) \) with at least one edge, then \( M \cap E(G_H) \) is a perfect matching of \( G_H \).

**Proof** Let \( M \) be a perfect matching of \( G \) corresponding to a vertex of \( H \) and let \( v \) be an arbitrary vertex of \( G_H \). Then \( G_H \) has a face \( f \in \mathcal{F} \) containing \( v \), which corresponds to an edge of \( H \). Then \( f \) is \( M' \)-alternating for some perfect matching \( M' \) which is a vertex of \( H \). Obviously, \( v \) is incident with exactly one edge of \( M' \). Since \( H \) is connected, there is a path of \( H \) joining \( M \) and one of \( M' \) and \( M' \oplus E(f) \). Therefore, \( H \) has a path \( P \) joining one of them and containing an edge corresponding to \( f \) which joins \( M' \) and \( M' \oplus f \). Assume that the faces corresponding to the edges of \( P \) are \( f_1 = f, \ldots, f_k \). Then \( M = M' \oplus E(f_1) \oplus \cdots \oplus E(f_k) \), and the symmetric difference does not change the number of edges incident with \( v \). It follows that \( v \) is incident with exactly one edge of \( M \) and that this edge belongs to at least one of the faces \( f_1, \ldots, f_k \). Note that \( E(f_i) \subseteq E(G_H) \) for all \( i \in \{1, \ldots, k\} \). So the edge of \( M \) incident with \( v \) is an edge of \( G_H \). Therefore, \( M \cap E(G_H) \) is a perfect matching of \( G_H \). This completes the proof.

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\Box
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For a connected component \( H \) of \( R(G; \mathcal{F}) \), if the union of all faces (including boundaries) corresponding to edges of \( H \) is homeomorphic to a closed disc, then \( G_H \) with the embedding inherited from the embedding from \( G \) in \( \Sigma \) is a plane elementary bipartite graph. By Theorem 1.1, we have the following proposition.

**Proposition 5.2** Let \( G \) be a graph on a surface \( \Sigma \) and \( \mathcal{F} \) be an even-face set. Assume that \( H \) is a connected component of \( R(G; \mathcal{F}) \). If the union of all faces corresponding to edges of \( H \) is homeomorphic to a closed disc, then \( H \) is the covering graph of a distributive lattice.

It has been evident in [27] that, if the subgraph induced by faces in \( \mathcal{F} \) is non-bipartite, a connected component of \( R(G; \mathcal{F}) \) may not be the covering graph of a distributive lattice. But the condition in Proposition 5.2 is not a necessary condition. It has been shown in [26] that a connected component of the resonance graph of a plane bipartite graph could be the covering graph of a distributive lattice if \( \mathcal{F} \) contains all faces but two. It is natural to ask what is the necessary and sufficient condition for \( \mathcal{F} \) so that every connected component of \( R(G; \mathcal{F}) \) is the covering graph of a distributive lattice.

But so far, in all examples we have, a connected component of \( R(G; \mathcal{F}) \) is always a median graph. Therefore, we risk the following conjecture.
Conjecture 5.3. Let $G$ be a graph embedded in a surface and let $\mathcal{F} \neq \mathcal{F}(G)$ be a set of even faces. Then every connected component of the resonance graph $R(G; \mathcal{F})$ is a median graph.

Conjecture 5.3 holds for catacondensed even ring systems [18], plane elementary bipartite graphs [9, 19, 33], and carbon nanotubes [26]. It would be interesting to show that every connected component of $R(G; \mathcal{F})$ can be embedded into a hypercube as an isometric subgraph, which would be a result weaker than Conjecture 5.3.

In order to prove the above conjecture or improve Theorem 1.3, some new ideas different from what we have in the proof of Theorem 3.5 may be needed.

Note that, the embedding function $\ell$ given in Equation (2) and used in the proof of Theorem 3.5 is not always an isometric embedding. In the following, we use the resonance graph of the coronene to demonstrate this. Note that, the resonance graph of the coronene is a very commonly used example and this graph can be found in [25, 33]. For convenience, we also include it here. Let $\mathcal{F}$ be the set of all inner faces of the coronene, i.e., $\mathcal{F} = \{h_1, \ldots, h_7\}$. Obviously, by Theorem 1.1 the resonance graph $R(G; \mathcal{F})$ is the covering graph of a distributive lattice.

Let $E_i$ be the set of edges of $R(G; \mathcal{F})$ corresponding to the face $h_i$ for $i \in \{1, \ldots, 7\}$. Note that the subgraph $R(G; \mathcal{F}) \setminus E_i$ for $i \in \{1, \ldots, 6\}$ has exactly two connected components and the vertices $u$ and $v$ of $R(G; \mathcal{F})$ belong to different connected components. Therefore, the binary strings $\ell(u)$ and $\ell(v)$ differ in the first six positions. But the subgraph $R(G; \mathcal{F}) \setminus E_7$ has three connected components and vertices $u$ and $v$ belong to two components that are not connected by any edge in the quotient graph. Therefore, the binary strings $\ell(u)$ and $\ell(v)$ have the same number in the last position. So $\ell(u)$ and $\ell(v)$ differ exactly in six positions. However, the distance between $u$ and $v$ in $R(G; \mathcal{F})$ is eight. So the embedding $\ell$ is not isometric. Therefore, the proof of Theorem 3.5 may not be adapted to show that a connected component of $R(G; \mathcal{F})$ is a partial cube, nor a median graph.

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