Resolution of the piecewise smooth visible-invisible two-fold singularity in \( \mathbb{R}^3 \) using regularization and blowup

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Abstract

Two-fold singularities in a piecewise smooth (PWS) dynamical system in \( \mathbb{R}^3 \) have long been the subject of intensive investigation. The interest stems from the fact that, even when all vector fields are Lipshitz, trajectories which enter the two-fold are associated with forward non-uniqueness. The key questions that arise are: How do we continue orbits forward in time? Are there orbits that are distinguished among all the candidates?

We address these questions by regularizing the PWS dynamical system in the case of a visible-invisible two-fold. We consider, within this framework, perhaps for the first time, a regularization function outside the class of Sotomayor and Teixera. We then undertake a rigorous investigation, using geometric singular perturbation theory and blowup. We show that there is indeed a forward orbit \( U \) that is distinguished amongst all the possible forward orbits leaving the two-fold. For simplicity, we work only with a normal form of the visible-invisible two-fold. We also show that one can obtain attracting limit cycles due to the contraction towards \( U \) upon composition with a global return mechanism and provide some illustrative examples of such return mechanisms for PWS systems.

Our conclusions give rigorous support to the notion that evolution through visible-invisible two-fold into the escaping region for a nonzero length of time can be excluded, in general.

1 Introduction

A piecewise smooth (PWS) dynamical system \([13, 22]\) consists of a finite set of ordinary differential equations

\[
\dot{x} = f_i(x), \quad x \in R_i \subset \mathbb{R}^n
\]
where the smooth vector fields $f_i$, defined on disjoint open regions $R_i$, are smoothly extendable to the closure of $R_i$. The regions $R_i$ are separated by an $(n-1)$-dimensional set $\Sigma$ called the \textit{switching manifold}, which consists of finitely many smooth manifolds intersecting transversely. The union of $\Sigma$ and all $R_i$ covers the whole state space $D \subseteq \mathbb{R}^n$. In this paper, we consider $n = 3$.

Such systems are found in a wealth of applications, including problems in mechanics (impact, friction, backlash, free-play, gears, rocking blocks), electronics (switches and diodes, DC/DC converters, $\Sigma - \Delta$ modulators), control engineering (sliding mode control, digital control, optimal control), oceanography (global circulation models), economics (duopolies) and biology (genetic regulatory networks): see [6, 22] for a full set of references.

Although PWS systems are abundant in applications, they do pose mathematical difficulties as they do not in general define a (classical) dynamical system. In particular, nonuniqueness of solutions cannot always be guaranteed. A prominent example of a PWS system with nonuniqueness is the two-fold in $\mathbb{R}^3$ \cite{5}, which is the subject of the present paper.

Frequently, PWS systems are idealisations of smooth systems with abrupt transitions. It is therefore perhaps natural to view a PWS system as a singular limit of a smooth regularized system. In this paper, we follow this viewpoint and describe the dynamics of a regularization of the PWS visible-invisible two-fold in $\mathbb{R}^3$.

### 1.1 PWS systems in $\mathbb{R}^3$

Let $x = (x, y, z) \in \mathbb{R}^3$ and consider an open set $U$ and a smooth function $f = f(x)$ having 0 as a regular value. Then $\Sigma \subset U$ defined by $\Sigma = f^{-1}(0)$ is a smooth 2D manifold. The manifold $\Sigma$ is our \textit{switching manifold}. It separates the set $\Sigma_+ = \{x \in U | f(x) > 0\}$ from the set $\Sigma_- = \{x \in U | f(x) < 0\}$. We introduce local coordinates so that $f(x) = y$, $\Sigma = \{x \in U | y = 0\}$ and consider a PWS system on $U \subset \mathbb{R}^3$ in the following form

$$ X(x) = \begin{cases} X^-(x) & \text{for } x \in \Sigma_- \\ X^+(x) & \text{for } x \in \Sigma_+ \end{cases}. $$

Here $X^+ = (X_1^+, X_2^+, X_3^+)^T$ and $X^- = (X_1^-, X_2^-, X_3^-)^T$ are smooth vector-fields on $\Sigma_+$ and $\Sigma_-$, respectively. Then $\Sigma$ is divided into two types of region: crossing and sliding:

- $\Sigma_{cr} \subset \Sigma$ is the \textit{crossing region}, where

$$ (X^+ f(x, 0, z))(X^- f(x, 0, z)) = X_2^+(x, 0, z)X_2^-(x, 0, z) > 0. $$

- $\Sigma_{sl} \subset \Sigma$ is the \textit{sliding region}, where

$$ (X^+ f(x, 0, z))(X^- f(x, 0, z)) = X_2^+(x, 0, z)X_2^-(x, 0, z) < 0. $$

Here $X^\pm f = \nabla f \cdot X^\pm$ denotes the Lie-derivative of $f$ along $X^\pm$. Since $f(x) = y$ in our coordinates we have simply that $X^\pm f = X^\pm_2$. On $\Sigma_{sl}$ we will follow the Filippov convention \cite{13} and define the sliding vector-field $X_{sl}(x)$ as the convex combination of $X^+$ and $X^-$

$$ X_{sl}(x) = \sigma X^+(x) + (1 - \sigma)X^-(x), $$

(3)
where $\sigma \in (0, 1)$ is defined so that $X_{sl}(x)$ is tangent to $\Sigma_{sl}$:

$$\sigma = \frac{X^- f(x, 0, z)}{X^- f(x, 0, z) - X^+ f(x, 0, z)}.$$

An orbit of a PWS system can be made up of a concatenation of arcs from $\Sigma$ and $\Sigma_\pm$. The boundaries of $\Sigma_{sl}$ and $\Sigma_{cr}$ where $X^+ f = X^+_2 = 0$ or $X^- f = X^-_2 = 0$ are singularities, sometimes called tangencies. We define two different types of generic tangencies: the fold singularity and the two-fold singularity.

**Definition 1** A point $q \in \Sigma$ is a fold singularity if

$$X^+ f(q) = 0, \quad X^+(X^+ f)(q) \neq 0, \quad \text{and} \quad X^- f(q) \neq 0,$$

or

$$X^- f(q) = 0, \quad X^-(X^- f)(q) \neq 0, \quad \text{and} \quad X^+ f(q) \neq 0.$$

A point $q \in \Sigma$ is a two-fold singularity if both $X^+ f(q) = 0$ and $X^- f(q) = 0$, as well as $X^+(X^+ f)(q) \neq 0$ and $X^-(X^- f)(q) \neq 0$ and if the vectors $X^+(q)$ and $X^-(q)$ are not parallel.

Then we have

**Proposition 1** [17, Proposition 2.2] A two-fold singularity $q$ is the transversal intersection of two lines $l^+$ and $l^-$ of fold singularities satisfying (4) and (5), respectively.

For the fold singularity, we distinguish between the visible and invisible cases.

**Definition 2** [15, Definition 2.1] A fold singularity $q$ with $X^+ f(q) = 0$ or $X^- f(q) = 0$ is visible if

$$X^+(X^+ f)(q) > 0 \quad \text{or} \quad X^-(X^- f)(q) < 0,$$

and invisible if

$$X^+(X^+ f)(q) < 0 \quad \text{or} \quad X^-(X^- f)(q) > 0.$$

Similarly:

**Definition 3** [15, Definition 2.3] The two-fold singularity $q$ is

- visible if the fold lines $l^+$ and $l^-$ are both visible;
- visible-invisible if $l^+$ ($l^-$) is visible and $l^-$ ($l^+$) is invisible;
- invisible if $l^+$ and $l^-$ are both invisible.
1.2 The visible-invisible two-fold

Following [17, Proposition 3.1], the visible-invisible two-fold can be locally described by the following set of normalized equations:

\[
\dot{x} = \begin{cases} 
\beta^{-1}c + O(x + y + z) & \text{for } y > 0 \\
-1 + O(x + y + z) & \text{for } y < 0 
\end{cases}, \\
\dot{y} = \begin{cases} 
bz + O(y + (y + z)(x + y + z)) & \text{for } y > 0 \\
-\beta x + O(y + (x + y)(x + y + z)) & \text{for } y < 0 
\end{cases}, \\
\dot{z} = \begin{cases} 
1 + O(x + y + z) & \text{for } y > 0 \\
b^{-1}\gamma + O(x + y + z) & \text{for } y < 0 
\end{cases},
\]

where \((x, y, z) \in \mathcal{U}\), where \(\mathcal{U}\) is a small neighborhood of the two-fold \(q = (0, 0, 0)\), and 

\[
\beta > 0, \ b > 0.
\] (7)

(In comparison with the more general expressions in [17, Proposition 3.1] we have \(\beta \mapsto -\beta\).) In (6), the fold lines

\[
l^+ : \ y = 0 = z, \\
l^- : \ x = 0 = y,
\] (8)

coincide with the \(x\)- and \(z\)-axis, as shown in Fig. 1. They are visible and invisible, respectively, since from (7)

\[
X^+(X^+f)(x, 0, 0) = b + O(x) > 0, \\
X^-(X^-f)(0, 0, z) = \beta + O(z) > 0,
\]

within \(\mathcal{U}\). In (6) we also have that

\[
\Sigma_{sl} : \ y = 0, xz > 0, \\
\Sigma_{cr} : \ y = 0, xz < 0,
\] (9)

where \(\Sigma_{sl} = \Sigma_{sl}^- \cup \Sigma_{sl}^+\) with

\[
\Sigma_{sl}^- : \ y = 0, x < 0, z < 0, \quad \text{(3rd quadrant of the (x, z)-plane)}
\]

and

\[
\Sigma_{sl}^+ : \ y = 0, x > 0, z > 0, \quad \text{(1st quadrant of the (x, z)-plane)}
\]

are stable and unstable sliding regions, respectively. Similarly, \(\Sigma_{cr} = \Sigma_{cr}^- \cup \Sigma_{cr}^+\) where

\[
\Sigma_{cr}^- : \ y = 0, x > 0, z < 0, \quad \text{(2nd quadrant of the (x, z)-plane)}
\]

and

\[
\Sigma_{cr}^+ : \ y = 0, x < 0, z > 0, \quad \text{(4th quadrant of the (x, z)-plane)}
\]

are regions with crossing downwards and upwards, respectively (see Fig. 1).
We compute the sliding vector-field $X_{sl}$ in (3) within $\Sigma_{sl}$, using (6), to give
\[
\dot{x} = \sigma X_1^+(x,0,z) + (1-\sigma)X_1^-(x,0,z),
\]
\[
\dot{y} = 0,
\]
\[
\dot{z} = \sigma X_3^+(x,0,z) + (1-\sigma)X_3^-(x,0,z),
\]
where
\[
\sigma = \frac{X^-f(x,0,z)}{X^-f(x,0,z) - X^+f(x,0,z)} = \frac{-\beta x + O((x+z))}{-\beta x - bz + O((x+z)^2)}.
\]
The denominator of $\sigma$ vanishes only at the two-fold within $\Sigma_{sl}$. System (10) is therefore singular at $q$. But we can re-parameterize time by multiplying the sliding vector field by $|X^-f(x,0,z) - X^+f(x,0,z)|$ to obtain the following desingularized sliding equations within $\Sigma_{sl}^-$:
\[
\dot{x} = -cx + bz + O((x+z)^2)
\]
\[
\dot{z} = -\beta x - \gamma z + O((x+z)^2).
\]
This transformation slows down time near the two-fold and (12) is now well-defined at $q$, which is, in fact, an equilibrium of (12). Furthermore, the orbits of (12) coincide with the orbits of (10) within $\Sigma_{sl}^+$; one only has to reverse the direction of time for them to agree as trajectories. We may therefore study (12) instead of (10) within $\Sigma_{sl} \cup \{q\}$.

### 1.3 Assumptions

In this paper, we will make the following assumptions.

(A) The following inequalities hold:
\[
c \pm \gamma > \sqrt{(c-\gamma)^2 - 4b\beta}, \quad (c-\gamma)^2 - 4b\beta > 0.
\]

(B) The non-degeneracy condition holds:
\[
\xi = \lambda_+^{-1}\lambda_- \notin \mathbb{N}.
\]
The parameter $\xi$ in (B) is intrinsic to the PWS system under consideration. But its role in understanding the forward nonuniqueness at $q$ will only become clear when we regularize the PWS system in Section 2. By (A), we have $c > 0$ and hence $\dot{x} > 0$ within $\Sigma_+ = \{y > 0\}$ from (6).

### 1.4 Forward nonuniqueness at $q$: the forward orbit $U$ and canard-like orbits

A further consequence of assuming (A), is that solutions of the Filippov system (6) are forward nonunique at the two-fold $q$. Indeed, the two-fold becomes a stable node of (12): the eigenvalues of the linearization of (12) about $(x,z) = 0$ are:
\[
\lambda_{\pm} = -\frac{1}{2}(c+\gamma) \pm \frac{1}{2}\sqrt{(c-\gamma)^2 - 4b\beta}.
\]
Figure 1: Local geometry of the PWS visible-invisible two-fold at \( q = (0, 0, 0) \). The system is forward nonunique at \( q \). We illustrate three possible examples of forward orbits from \( q \) in purple, red and blue. The purple orbit stays in \( \Sigma^{+}_{sl} \) for a finite time, before escaping into \( \Sigma^{+} = \{ y > 0 \} \). The red orbit remains in \( \Sigma^{+}_{sl} \). The blue orbit, denoted by \( U \), does not enter \( \Sigma^{+}_{sl} \), but follows \( X^{+} \) instead. The dotted tangencies are trajectories of \( X^{-} \) with the invisible fold line \( l^{-} \) and the full tangencies are trajectories of \( X^{+} \) with the visible fold line \( l^{+} \). The sections \( \Pi_{in,out} \) are defined in (21) and (22) respectively. \( \Pi_{in} \) is within \( y \geq 0 \) and recall that the point \( u_{out} = U \cap \Pi_{out} \).

with associated eigenvectors spanned by

\[
v_{\pm} = \begin{pmatrix} 1 \\ -\chi_{\pm} \end{pmatrix},
\]

(16)

respectively, where

\[
\chi_{\pm} = -\frac{1}{2b} \left( c - \gamma \pm \sqrt{(c - \gamma)^2 - 4b\beta} \right),
\]

(17)

and

\[
\lambda_{-} < \lambda_{+} < 0, \quad (18)
\]

\[
\chi_{+} < \chi_{-} < 0, \quad (19)
\]

cf. assumption (A). By (19), the span of \( v_{\pm} \) is then contained within \( \Sigma_{sl} \cup \{ q \} \). We therefore have the following proposition (see Fig. 2 and [17, Proposition 4.2]):
Proposition 2 Suppose (A). Then the following holds for the Filippov system (6):

(a) There exists a unique solution of (10): the strong canard $\gamma^s$, that is tangent to $v_-$ at the two fold $q$. It connects $\Sigma^+_sl$ with $\Sigma^-sl$.

(b) There exists a funnel within $\Sigma^-sl$, confined by the strong canard and the invisible fold line $l^-$, consisting of weak canards: orbits of (10) that all pass through the two-fold tangent to $v_+$. The weak canards all connect the stable sliding region with the unstable one.

The forward non-uniqueness of solutions of the Filippov system (6) at the two-fold $q$ is now apparent. We illustrate three possible forward orbits in Fig. 1.

1.5 Resolving the forward nonuniqueness: known results

Previous work has attempted to resolve the forward nonuniqueness associated with the two-fold. In [23], the planar visible two-fold was considered. The ambiguity of forward evolution was removed using separate small perturbations: hysteresis, time-delay and noise. In each case, a probabilistic notion of forward evolution close to the two-fold was developed. In the limit as the perturbation tended to zero, almost all orbits or sample paths followed one of the visible tangencies. Thus the possibility of evolution through the two-fold singularity into the escaping region for a nonzero length of time could be excluded, similar to other results for non-differentiable systems in the zero-noise limit. The author also made the point that the regularization of two-folds may turn out to be futile in the absence of further physical (or biological) information about the problem.

In [4], all three two-folds in $\mathbb{R}^3$ were considered. For the invisible two-fold, the authors asserted that forward evolution through the singularity into the region of unstable sliding is possible. Then after a finite time, evolution away from the unstable sliding region leads to a return mechanism to the stable sliding region and a subsequent forward evolution through the singularity, leading to what they called “nondeterministic chaos”. No such conclusion were made for either the visible or the visible-invisible two-fold, in line with [23].

1.6 Aim of the paper

In this paper, we will carry out a rigorous mathematical analysis of a regularization of the PWS visible-invisible two-fold using geometric singular perturbation theory, with blowup as the essential method. See Section 2 for details on the regularization. In doing so, we will resolve the forward nonuniqueness of the two-fold in the limit as the regularized system approaches the PWS one. For simplicity, we will consider the truncated, piecewise linear versions of (6):

$$
\begin{align*}
\dot{x} &= \begin{cases}
\beta^{-1}c & \text{for } y > 0 \\
-1 & \text{for } y < 0
\end{cases}, \\
\dot{y} &= \begin{cases}
bz & \text{for } y > 0 \\
-\beta x & \text{for } y < 0
\end{cases}, \\
\dot{z} &= \begin{cases}
1 & \text{for } y > 0 \\
b^{-1}\gamma & \text{for } y < 0
\end{cases}.
\end{align*}
$$

(20)
still assuming (7) and (A). The sliding dynamics of (20) are shown in Fig. 2. The funnel mentioned in Proposition 2 is shaded dark grey. (In comparison with Section 1.1, we may now consider \( U = \mathbb{R}^3 \), although we still think of our system as a local one.)

![Figure 2: The sliding flow within \( \Sigma_{\Delta}^+ \) for the truncated, piecewise linear equations (20). The eigenvectors \( v_{\pm} \) are shown in red and orange, respectively. The funnel (dark grey) is confined by the invisible fold line \( l^- \) and the strong canard \( \gamma^s \) within \( \Sigma_{\Delta}^- \). \( I_{\text{in}} \) in purple is a closed interval on the negative \( z \)-axis (with properties described around (23)). The mapping \( \vartheta \) is defined in Lemma 5. The image \( D\vartheta(0)v_+ \) of \( v_+ \) under \( D\vartheta(0) \), shown in blue, is always contained outside the funnel.](image)

### 1.7 Main result

We now describe our main result. For this fix \( \delta > 0 \) and \( \nu > 0 \). We then consider the following sections transverse to the PWS flow, illustrated in Fig. 1:

\[
\Pi_{\text{in}} = \{(x, y, z) \in \mathbb{R}^3 | x = -\delta, (y, z) \in R_{\text{in}}\}, \quad (21)
\]

\[
\Pi_{\text{out}} = \{(x, y, z) \in \mathbb{R}^3 | y = \nu, (x, z) \in R_{\text{out}}\}, \quad (22)
\]

where:

- The set

\[
R_{\text{in}} = [0, \zeta] \times I_{\text{in}}, \quad (23)
\]

is a suitably small rectangle in the \((y, z)\)-plane so that \( \Pi_{\text{in}} \subset \Sigma_+ \cup \Sigma \), where \( \dot{x} > 0 \). In particular, \( I_{\text{in}} \) is a closed interval on the negative \( z \)-axis, see also Fig. 2 (in purple). Furthermore, \( \Pi_{\text{in}} \cap \Sigma \) is contained inside the funnel but does not include the span of
the weak eigendirection $v_+$. $\Pi_{\text{in}}$ is sufficiently thin, that is $\zeta > 0$ in (23) is sufficiently small, so that all points in $\Pi_{\text{in}} \cap \Sigma_+^+$ reach the funnel under the forward flow of $X^+$.

\begin{itemize}
    \item $R_{\text{out}}$ is a suitably small rectangle in the $(x, z)$-plane so that $\Pi_{\text{out}} \subset \{y = \nu\}$ is a small neighborhood of $U \cap \{y = \nu\}$.
\end{itemize}

For our main result in Theorem 1 below, we then show that the local mapping

$$L_\epsilon : \Pi_{\text{in}} \rightarrow \Pi_{\text{out}}, \ (-\delta, y, z) \mapsto (x_+(y, z), \nu, z_+(y, z)), \quad (24)$$

obtained by the forward flow of the regularization, is well-defined and contractive and that the image $L_\epsilon(\Pi_{\text{in}})$ tends to $u_{\text{out}} = U \cap \Pi_{\text{out}}$ as the regularized system approaches the PWS system ($\epsilon \rightarrow 0^+$). Hence it is the forward orbit $U$ that is distinguished amongst all the possible forward orbits leaving $q$.

Since $L_\epsilon$ is contractive, we can also obtain attracting limit cycles by composing the local mapping $L_\epsilon$ with a global return mechanism (see Corollary 1 for details). We provide some simple examples of such return mechanisms for PWS systems in Fig. 5.

### 1.8 Outline of paper

In Section 2, we define the regularization of (20). We also present an initial blowup, see Section 2.1, to describe the regularization and describe different regularization functions. In Section 3 we then present the singular geometry of the regularization using our blowup approach. In Section 4 we formally state our main results. Here we also present some intuition behind our main result, see Section 4.2. The main result Theorem 1 is proven in Section 5 using a separate blowup. We complete our paper in Section 6 with a discussion and conclusion section where, among other things, we will discuss possible extensions of our results.

### 2 Regularization and blowup

In this paper, we consider the following regularizations of the PWS system $X^\pm$:

**Definition 4** A regularization of the PWS system $X^\pm$ is a vector-field

$$X_\epsilon(x) = \frac{1}{2}X^+(x)(1 + \phi(\epsilon^{-1}y)) + \frac{1}{2}X^-(x)(1 - \phi(\epsilon^{-1}y)), \quad (25)$$

for $0 < \epsilon \ll 1$, where the function $\phi : \mathbb{R} \rightarrow [-1, 1]$ is assumed to be sufficiently smooth, monotone: $\phi'(s) > 0$ for $\phi(s) \in (-1, 1)$ and

$$\phi(s) \rightarrow \pm 1 \quad \text{for} \quad s \rightarrow \pm \infty. \quad \square$$

Notice that for $y > 0$, we have by (25) the convergence

$$X_\epsilon(x) \rightarrow \frac{1}{2}X^+(x)(1 + 1) + \frac{1}{2}X^-(x)(1 - 1) = X^+ \quad \text{for} \quad \epsilon \rightarrow 0^+. $$
Similarly, $X_\epsilon(x) \rightarrow X^-$ for $y < 0$.

Using the truncated, piecewise linear system (20), the regularized system becomes:

\[
\begin{align*}
\dot{x} &= \beta^{-1}c(1 + \phi(\epsilon^{-1}y)) - (1 - \phi(\epsilon^{-1}y)), \\
\dot{y} &= b\zeta(1 + \phi(\epsilon^{-1}y)) - \beta x(1 - \phi(\epsilon^{-1}y)), \\
\dot{z} &= (1 + \phi(\epsilon^{-1}y)) + b^{-1}\gamma(1 - \phi(\epsilon^{-1}y)),
\end{align*}
\]

after replacing time $t$ by $2t$. We note that (26) is time-reversible with respect to the symmetry

\[
(t, x, y, z) \mapsto (-t, -x, y, -z).
\]

System (26) is singular at $y = \epsilon = 0$. It will therefore be useful to work with two separate time scales. We will say that $t$ in (26) is the slow time whereas $\tau = t\epsilon^{-1}$ will be referred to as the fast time. In terms of the fast time scale we obtain the following equations

\[
\begin{align*}
\dot{x}' &= \epsilon(\beta^{-1}c(1 + \phi(\epsilon^{-1}y)) - (1 - \phi(\epsilon^{-1}y))), \\
\dot{y}' &= \epsilon(b\zeta(1 + \phi(\epsilon^{-1}y)) - \beta x(1 - \phi(\epsilon^{-1}y))), \\
\dot{z}' &= \epsilon((1 + \phi(\epsilon^{-1}y)) + b^{-1}\gamma(1 - \phi(\epsilon^{-1}y))).
\end{align*}
\]

For $\epsilon = 0$ in (28) we have the trivial dynamics $x' = y' = z' = 0$: all points are equilibrium points.

### 2.1 Initial blowup

To study (28) we proceed as in [17, 18] by considering the following blowup:

\[
y = \pi\hat{y}, \quad \epsilon = \pi\hat{\epsilon}, \quad (\pi, (\hat{y}, \hat{\epsilon})) \in [0, \infty) \times S^1, \quad S^1 = \{(\hat{x}, \hat{y})|\hat{x}^2 + \hat{y}^2 = 1\}.
\]

This transformation $(x, z, \pi, (\hat{y}, \hat{\epsilon})) \mapsto (x, z, y, \epsilon)$ blows up $y = \epsilon = 0$ to a circle $(\hat{y}, \hat{\epsilon}) \in S^1$, as shown in Fig. 3. It gives a vector-field $\overline{X}$ on the blowup space

\[
\overline{P} = \{(x, z, \pi, (\hat{y}, \hat{\epsilon})) \in \mathbb{R}^2 \times [0, \infty) \times S^1\},
\]

by pull-back of the extended vector-field ((28),$\epsilon' = 0$). Here $\overline{X}|_{\pi=0} \neq 0$.

To describe $\overline{P}$ and $\overline{X}$, we could use polar variables:

\[
\hat{y} = \sin \theta, \quad \hat{\epsilon} = \cos \theta.
\]

But in agreement with general theory [19], it is more useful to consider directional charts.

The directional chart obtained by setting $\hat{\epsilon} = 1$ corresponds to $y = \pi\hat{y}, \epsilon = \pi \geq 0$, or simply

\[
\hat{\epsilon} = 1: \quad y = \epsilon\hat{y}, \quad \hat{y} \in \mathbb{R}, \quad \epsilon \geq 0,
\]

after eliminating $\pi$. Therefore by (30) $\hat{y} = \tan \theta, \theta \in (-\pi/2, \pi/2)$. We will use chart (31) to describe a large, but compact, subset with $\hat{\epsilon} > 0$ of the half-circle $S^1 \cap \{\hat{\epsilon} \geq 0\}$. This
chart will bring (28) into a classical slow-fast form (see (41) below). We will then also use the following charts

\[
\bar{y} = 1 : \quad \epsilon = y\dot{\epsilon}, \quad y \geq 0, \quad \dot{\epsilon} \geq 0,
\]

(32)

and

\[
\bar{y} = -1 : \quad \epsilon = y\dot{\epsilon}, \quad y \leq 0, \quad \dot{\epsilon} \leq 0,
\]

(33)

obtained by setting \(\bar{y} = \pm 1\), respectively, in (29), to describe compact subsets with \(\bar{y} \geq 0\) of the half-circle \(S^1 \cap \{\bar{\epsilon} \geq 0\}\). In this way, the charts (31), (32) and (33) cover the half-circle \(S^1 \cap \{\bar{\epsilon} \geq 0\}\) completely. Notice also that

\[
\dot{\epsilon} = \bar{y}^{-1},
\]

(34)

and hence by (30) \(\dot{\epsilon} = \cot \theta, \quad \theta \in (0, \pi/2]\) and \(\theta \in [-\pi/2, 0)\), for \(\bar{y} \leq 0\) in (32) and (33), respectively, with \(\bar{y}\) in (31). Henceforth we refer to the three charts, (31), (32) and (33), as \(\bar{\epsilon} = 1\) and \(\bar{y} = \pm 1\), respectively. For simplicity, we use the same symbols in (32) and (33), but (obviously) the domains are different.

Figure 3: Charts \(\bar{\epsilon} = 1\) (31), \(\bar{y} = 1\) (32) and \(\bar{y} = -1\) (33) in the blowup transformation (29). \(\bar{y} \geq 0\) correspond to the south and the north semicircles, respectively.

We follow the convention that all geometric objects obtained in any of these charts will be given a hat. We will often jump between the charts using (34) but we believe it is clear from the context what variables are used. A geometric object obtained in the charts, say \(\hat{M}\), will be given a bar, \(\overline{M}\), in the blowup variables \((x, z, \pi, (\bar{y}, \bar{\epsilon})) \in \mathbb{R}^2 \times [0, \infty) \times S^1\). Furthermore, an object, say \(\hat{G}\), obtained in \(\bar{\epsilon} = 1\) will often only be partially visible \((\bar{y} \geq 0, \text{ respectively})\) in the charts \(\bar{y} = \pm 1\). For simplicity, we will, however, continue to denote the visible part of \(\hat{G}\) in the charts \(\bar{y} = \pm 1\) by the same symbol.

2.2 Regularization functions

In our previous work [17, 18] we have primarily restricted attention to the following class of regularization functions:
Definition 5 The $C^k$, $k \geq 1$, Sotomayor and Teixeira regularization functions \cite{24}

\[ \phi : \mathbb{R} \to \mathbb{R}, \]

satisfy:

1° Finite deformation:

\[ \phi(\hat{y}) = \begin{cases} 
1 & \text{for } \hat{y} \geq 1, \\
\in (-1, 1) & \text{for } \hat{y} \in (-1, 1), \\
-1 & \text{for } \hat{y} \leq -1,
\end{cases} \quad (35) \]

2° Monotonicity:

\[ \phi'(\hat{y}) > 0 \quad \text{within } \hat{y} \in (-1, 1). \quad (36) \]

The desirable property of this set of regularization functions is the finite deformation property. By 1°

\[ X_\epsilon = X^\pm \quad \text{for } y \gtrless \pm \epsilon. \quad (37) \]

In this paper, we consider the analytic, non-Sotomayor and Teixeira regularization function

\[ \phi(s) = \frac{2}{\pi} \arctan(s), \quad s \in \mathbb{R} \quad (38) \]

It is easier to consider the Sotomayor and Teixeira regularization functions but we aim to push the theory of regularizations of PWS systems beyond these rather artificial functions. Our results generalise to other non-Sotomayor and Teixeira regularization functions but some of the detailed description depends on asymptotic properties of the regularization functions at $\pm \infty$. Therefore we fix $\phi$ as \cite{38} with the following properties

\[ \phi(\hat{y}) = \pm 1 - \frac{2}{\pi} \hat{y}^{-1}(1 + \phi_2(\hat{y}^{-2})) = \pm 1 - \frac{2}{\pi} \hat{\epsilon}(1 + \phi_2(\hat{\epsilon}^2)), \quad (39) \]

using \cite{34}, with $\phi_2 : [0, \infty) \to \mathbb{R}$ smooth satisfying $\phi_2(0) = 0$. For compactness, we define $\phi_+ : [0, \infty) \to \mathbb{R}$ and $\phi_- : (-\infty, 0] \to \mathbb{R}$ as

\[ \phi_\pm(\hat{\epsilon}) = \pm 1 - \frac{2}{\pi} \hat{\epsilon}(1 + \phi_2(\hat{\epsilon}^2)), \quad (40) \]

in charts $\hat{y} = \pm 1$, respectively. In this paper, we will use the charts $\hat{y} = \pm 1$ in conjunction with the standard blowup method \cite{8, 9, 19, 21} to deal with loss of hyperbolicity.

3 Singular geometry of the regularization $X_\epsilon$

Now we present the (singular) geometry of $X_\epsilon$ using the blowup \cite{29} and the charts $\epsilon = 1$ and $\hat{y} = \pm 1$, as $\epsilon \to 0^+$. We consider each of the charts separately.
3.1 The chart $\bar{\epsilon} = 1$: A slow-fast system

Inserting (31) into (28) gives the following set of equations:

\[
\begin{align*}
x' &= \epsilon \left( \beta^{-1} c (1 + \phi(\hat{y})) - (1 - \phi(\hat{y})) \right), \\
\hat{y}' &= bz (1 + \phi(\hat{y})) - \beta x (1 - \phi(\hat{y})), \\
z' &= \epsilon \left( (1 + \phi(\hat{y})) + b^{-1} \gamma (1 - \phi(\hat{y})) \right),
\end{align*}
\]

in terms of the fast time $\tau$. Here $\hat{y} \in \mathbb{R}$, $\epsilon \geq 0$ and $\epsilon' = 0$. This is now a standard slow-fast system. The $\hat{y}$ variable is fast with $O(1)$ velocities in general whereas $x$ and $z$ are slow variables with $O(\epsilon)$ velocities. In slow-fast theory, system (41) is called the **fast system**, whereas

\[
\begin{align*}
x' &= 0, \\
\hat{y}' &= bz (1 + \phi(\hat{y})) - \beta x (1 - \phi(\hat{y})), \\
z' &= 0,
\end{align*}
\]

is called the **slow system**.

**Critical manifold, layer problem and reduced problem**

The limiting system (41)$_{\epsilon=0}$:

\[
\begin{align*}
x' &= 0, \\
\hat{y}' &= bz (1 + \phi(\hat{y})) - \beta x (1 - \phi(\hat{y})), \\
z' &= 0,
\end{align*}
\]

is called the **layer problem**, while (42)$_{\epsilon=0}$:

\[
\begin{align*}
x' &= \beta^{-1} c (1 + \phi(\hat{y})) - (1 - \phi(\hat{y})), \\
0 &= bz (1 + \phi(\hat{y})) - \beta x (1 - \phi(\hat{y})), \\
z' &= (1 + \phi(\hat{y})) + b^{-1} \gamma (1 - \phi(\hat{y})).
\end{align*}
\]

is called the **reduced problem**. Notice that $x$ and $z$ are constant in (43) whereas $\hat{y}$ is slaved in (44).

Let the smooth function

\[ h : [0, \infty) \to \mathbb{R}, \]

be defined by

\[
h(s) = \phi^{-1} \left( \frac{1 - \beta^{-1} bs}{1 + \beta^{-1} bs} \right).
\]
Proposition 3 [17, Theorem 5.1, Proposition 5.4] The union

\[ \hat{S}_0 = \hat{S}_a \cup \hat{S}_r \cup \hat{q} \]

of the smooth graphs:

\[ \hat{S}_{a,r} : \dot{y} = h(x^{-1}z), \text{ for } (x,0,z) \in \Sigma^\pm, \] (46)

and the line:

\[ \hat{q} = \{(x,\dot{y},z) \in \mathbb{R}^3 | \dot{y} \in \mathbb{R}, x = 0, z = 0\}, \] (47)

is a set of critical points of (43). On \( \hat{S}_{a,r} \) the motion of the slow variables \( x \) and \( z \) is described by (44) which coincides with the sliding equations (10). Also \( \hat{S}_{a,r} \) are both normally hyperbolic, \( \hat{S}_a \) being attracting, \( \hat{S}_r \) being repelling while \( \hat{q} \) is fully nonhyperbolic.

Proof We obtain the set \( \hat{S}_0 \) as the set of equilibria of (43). The hyperbolicity is determined by the linearization of (43). The remainder of the proof then follows from simple calculations. □

Although \( \hat{q} \) and \( \hat{S}_{a,r} \) are only partially visible in the charts \( \tilde{y} = \pm 1 \), we will (as promised) continue for simplicity to denote these objects by the same symbols in these charts.

3.2 The chart \( \tilde{y} = 1 \)

Inserting (32) into (28) we obtain the following equations, using (40):

\[
\begin{align*}
x' &= y(\beta^{-1}c(1 + \phi_+(\dot{\epsilon})) - (1 - \phi_+(\dot{\epsilon}))), \\
y' &= y(bz(1 + \phi_+(\dot{\epsilon})) - \beta x(1 - \phi_+(\dot{\epsilon}))), \\
z' &= y((1 + \phi_+(\dot{\epsilon})) + b^{-1}\gamma(1 - \phi_+(\dot{\epsilon}))), \\
\dot{\epsilon}' &= -\dot{\epsilon}(bz(1 + \phi_+(\dot{\epsilon})) - \beta x(1 - \phi_+(\dot{\epsilon}))),
\end{align*}
\] (48)

where \((y, \dot{\epsilon}) \in [0, \infty)^2\), after division of the right hand side by the common factor \( \dot{\epsilon} \geq 0 \). This division corresponds to a time transformation for \( \dot{\epsilon} > 0 \) and desingularizes the dynamics within the set \( \{\dot{\epsilon} = 0\} \). The flow of the system (48) preserves \( \epsilon = y\dot{\epsilon} \) in (32). Therefore \( \dot{\epsilon} = 0 \) implies either \( y = 0 \) or \( \dot{\epsilon} = 0 \). The corresponding sets \( \{y = 0\} \) and \( \{\dot{\epsilon} = 0\} \) are invariant.

Within \( \{\dot{\epsilon} = 0\} \) we recover the vector-field \( X^+ \):

\[
\begin{align*}
\dot{x} &= \beta^{-1}c, \\
\dot{y} &= bz, \\
\dot{z} &= 1,
\end{align*}
\] (49)

recall (20) \( y > 0 \), after further division of the right hand side by \( 2y \geq 0 \), using \( \phi_+(0) = 1 \). Within \( \{y = 0\} \) we have \( x' = z' = 0 \) and

\[
\dot{\epsilon}' = -\dot{\epsilon}(bz(1 + \phi_+(\dot{\epsilon})) - \beta x(1 - \phi_+(\dot{\epsilon}))),
\]
in agreement with (43) by (34). In particular, the set defined by \( bz(1 + \phi_+(\hat{\epsilon})) - \beta x(1 - \phi_+(\hat{\epsilon})) = 0 \) coincides with \( \hat{S}_0 \cap \{\hat{y} > 0\} \) of Proposition 3 under the coordinate transformation (34), having the same hyperbolicity properties. Here

\[
\hat{q} : x = z = y = 0, \hat{\epsilon} \geq 0,
\]
(extend it to \( \hat{\epsilon} = 0 \) where it remains nonhyperbolic) and

\[
\hat{S}_{a,r} : \hat{\epsilon} = h_+(x^{-1}z), y = 0, \quad x^{-1}z \in (0, b^{-1}\beta),
\]
for \((x, 0, z) \in \Sigma_{ul}^+\), respectively, where \( h_+ : [0, b^{-1}\beta) \to [0, \infty) \) defined by

\[
h_+(s) = \phi_+^{-1} \left( \frac{1 - \beta^{-1}bs}{1 + \beta^{-1}bs} \right).
\]

Notice that

\[
h_+(0) = 0, \quad h_+'(0) = \beta^{-1}\pi b.
\]

But in the chart \( \hat{y} = 1 \) we also obtain the following along the intersection \( \{y = \hat{\epsilon} = 0\} \) of the invariant sets \( \{y = 0\} \) and \( \{\hat{\epsilon} = 0\} \).

**Lemma 1** The set \( \hat{M}^+ = \{y = \hat{\epsilon} = 0\} \) is a set of critical points for (48). It is of saddle-type for \( z \neq 0 \): The linearization about any point in \( \hat{M}^+ \) has only two non-trivial eigenvalues \( \pm 2bz \) with associated eigenvectors

\[
\begin{pmatrix}
\beta^{-1}c \\
bz \\
1 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix},
\]
respectively. The line

\[
\hat{l}^+ = \hat{M}^+ \cap \{z = 0\},
\]

is nonhyperbolic: The linearization about any point in \( \hat{l}^+ \) has only zero eigenvalues. It becomes \( l^+ \) upon returning to the \((x, y, z)\)-variables. \( \square \)

**Proof** Simple calculations.

### 3.3 The chart \( \hat{y} = -1 \)

In this chart we obtain the following equations using (40):

\[
x' = -y(\beta^{-1}c(1 + \phi_-(\hat{\epsilon})) - (1 - \phi_-(\hat{\epsilon}))), \tag{52}
\]

\[
y' = y(bz(1 + \phi_-(\hat{\epsilon})) - \beta x(1 - \phi_-(\hat{\epsilon}))),
\]

\[
z' = -y((1 + \phi_-(\hat{\epsilon})) + b^{-1}\gamma(1 - \phi_-(\hat{\epsilon}))),
\]

\[
\hat{\epsilon}' = \hat{\epsilon}(bz(1 + \phi_-(\hat{\epsilon})) - \beta x(1 - \phi_-(\hat{\epsilon}))).
\]
after division of the right hand side by the common factor $-\hat{\epsilon} \geq 0$. As opposed to the chart $\hat{y} = 1$, here $(y, \hat{\epsilon}) \in (-\infty, 0]^2$. System (52) preserves $\epsilon = y \hat{\epsilon}$, and $\epsilon = 0$ therefore gives either $y = 0$ or $\hat{\epsilon} = 0$. The corresponding sets $\{y = 0\}$ and $\{\hat{\epsilon} = 0\}$ are invariant.

Within $\{\hat{\epsilon} = 0\}$ we recover the vector field $X^-:
\begin{align*}
\dot{x} &= -1, \\
\dot{y} &= -\beta x, \\
\dot{z} &= b^{-1}\gamma,
\end{align*}
(53)
recall (20) $y < 0$, after further division of the right hand side by $-2y \geq 0$, using $\phi^-\left(0\right) = -1$.

Within $\{y = 0\}$ we recover $\hat{S}_0 \cap \{\hat{\epsilon} < 0\}$ from Proposition 3 as a set of critical points, having the same hyperbolicity properties, upon the coordinate change (34). In particular,
\[ \hat{q} : x = z = y = 0, \hat{\epsilon} \leq 0, \]
(54)
( extending it to $\hat{\epsilon} = 0$ where it remains nonhyperbolic) and
\[ \hat{S}_{a,r} : \hat{\epsilon} = h_-(x^{-1}z), y = 0, \quad x^{-1}z \in (0, b^{-1}\beta), \]
for $(x, 0, z) \in \Sigma_{a,l}^\pm$, respectively, with $h_- : [0, b^{-1}\beta) \to (-\infty, 0]$ defined by
\[ h_-(s) = \phi^-\left(1 + \beta^{-1}bs\right). \]
Along $\{y = \hat{\epsilon} = 0\}$ we then obtain the following:

**Lemma 2** The set $\hat{M}^- = \{y = \hat{\epsilon} = 0\}$ is a set of critical points for (52). It is of saddle-type for $x \neq 0$: The linearization about any point in $\hat{M}^-$ has only two non-trivial eigenvalues $\pm 2bx$ with associated eigenvectors
\[
\begin{pmatrix}
1 \\
\beta x \\
-b^{-1}\gamma
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\]
respectively. The line
\[ \hat{l}^- = \hat{M}^- \cap \{x = 0\}, \]
is nonhyperbolic: The linearization about any point in $\hat{l}^-$ has only zero eigenvalues. It becomes $l^-$ upon returning to the $(x, y, z)$-variables for $\epsilon = 0$.

**Proof** Simple calculations.
3.4 Blowup geometry

Hence, in the blowup variables \((x, z, \pi, (\bar{y}, \bar{\epsilon})) \in \mathbb{R}^2 \times [0, \infty) \times S^1\), we obtain the following:

\[
\bar{q} : x = \pi = z = 0, (\bar{y}, \bar{\epsilon}) \in S^1 \cap \{\bar{\epsilon} \geq 0\},
\]
\[
\bar{I}^+ : x \in \mathbb{R}, \pi = z = 0, (\bar{y}, \bar{\epsilon}) = (1, 0) \in S^1,
\]
\[
\bar{I}^- : z \in \mathbb{R}, x = \pi = 0, (\bar{y}, \bar{\epsilon}) = (-1, 0) \in S^1,
\]

as fully nonhyperbolic critical points of \(\bar{X}\). The critical sets

\[
\bar{M}^+ : (x, z) \in \mathbb{R}^2, \pi = 0, (\bar{y}, \bar{\epsilon}) = (1, 0) \in S^1,
\]
\[
\bar{M}^- : (x, z) \in \mathbb{R}^2, \pi = 0, (\bar{y}, \bar{\epsilon}) = (-1, 0) \in S^1,
\]

are, on the other hand, normally hyperbolic for \(x \neq 0\) and \(z \neq 0\), respectively, being each of saddle-type. Recall \(\bar{M}^+ \cap \{x = 0\} = \bar{I}^+, \bar{M}^- \cap \{z = 0\} = \bar{I}^-\). Furthermore, for \((x, z)\) such that \((x, 0, z) \in \Sigma_{sl}^+\) we obtain normally hyperbolic and attracting/repelling critical manifolds \(\bar{S}_{a,r}\) carrying reduced, slow flows on \((x, z)\) which coincide with the Filippov sliding flow on \(\Sigma_{sl}^+\). We illustrate the geometry in Fig. 4. Upon blowing back down and returning to the original \((x, y, z)\)-variables, \(\bar{q}, \bar{I}^\pm, \bar{S}_{a,r}\) all collapse to the two-fold \(q = (0, 0, 0)\), the fold lines \(I^\pm\), and the stable and unstable sliding regions \(\Sigma_{sl}^+\), respectively, for \(\epsilon = 0\).

3.5 Slow manifolds and results from [17]

We now discuss the perturbation of \(\bar{S}_{a,r}\) for \(0 < \epsilon \ll 1\). For this we work in chart \(\bar{\epsilon} = 1, [31]\). Then we have, by Fenichel’s theory [10][11][12], that compact subsets of the normally hyperbolic critical manifolds \(\hat{S}_{a,r}\) perturb to locally invariant, hyperbolic manifolds \(\hat{S}_{a,r,\epsilon}\) for \(0 < \epsilon \leq \epsilon_0\) sufficiently small. In particular, for \((41)\) we have the following:

**Lemma 3** Consider \(\mu > 0\) sufficiently small. Then there exist geometrically unique slow manifolds \(\hat{S}_{a,\epsilon}\) and \(\hat{S}_{r,\epsilon}\) that extend as perturbations of \(\hat{S}_a\) and \(\hat{S}_r\) up to \(x = -\mu^{-1}\sqrt{\epsilon}\) and \(x = \mu^{-1}\sqrt{\epsilon}\), respectively, in the following way: Let \(I_a \subset \mathbb{R}_-, I_r \subset \mathbb{R}_+\) be suitably large intervals and fix \(k > 0\). Then, \(\hat{S}_{a,\epsilon}\) takes the following form

\[
\hat{y} = h(-z_1) + \epsilon_1(bz_1^2 + (\gamma - c)z_1 + \beta)m(z_1, \epsilon_1), \quad (r_1, \epsilon_1, z_1) \in [0, k] \times [0, \mu^2] \times I_a,
\]

with \(m\) smooth, in the projective variables \((r_1, \epsilon_1, z_1) \in [0, \infty)^2 \times (-\infty, 0]\), defined by

\[
x = -r_1, z = r_1z_1, \epsilon = r_1^2\epsilon_1.
\]

Similarly, \(\hat{S}_{r,\epsilon}\) takes the following form

\[
\hat{y} = h(z_3) + \epsilon_3(bz_3^2 - (\gamma - c)z_3 + \beta)m(-z_3, \epsilon_3), \quad (r_3, \epsilon_3, z_3) \in [0, k] \times [0, \mu^2] \times I_r,
\]

in the projective variables \((r_3, \epsilon_3, z_3) \in [0, \infty)^3\), defined by

\[
x = r_3, z = r_3z_3, \epsilon = r_3^2\epsilon_3.
\]
Moreover, both slow manifolds $\hat{S}_{a,r,\epsilon}$ contain the invariant lines

$$\hat{\gamma}^w: \begin{pmatrix} x \\ z \end{pmatrix} = v_+ s, \ s \in \mathbb{R}, \ \hat{y} = h(-\chi_+), \tag{58}$$

$$\hat{\gamma}^s: \begin{pmatrix} x \\ z \end{pmatrix} = v_- s, \ s \in \mathbb{R}, \ \hat{y} = h(-\chi_-).$$

within the subsets defined by $(r_1, \epsilon_1, z_1) \in [0, k] \times [0, \mu^2] \times I_a$ and $(r_3, \epsilon_3, z_3) \in [0, k] \times [0, \mu^2] \times I_r$, respectively.

\begin{proof}

The extension of Fenichel’s slow manifold $\hat{S}_{a,\epsilon}$ follows from [17, Proposition 7.4] using the variables (56). The uniqueness of this manifold in our case follows from the fact that the $r_1$-equation decouples from the $(\hat{y}, z_1, \epsilon_1)$-equations (see (59) below). The result then follows from the uniqueness of $C_{a,1}$ in [17, Proposition 7.4] as an overflowing center manifold. The extended repelling slow manifold, $\hat{S}_{r,\epsilon}$, is obtained by applying the time-reversible symmetry (27).

\end{proof}

Let $\hat{S}^t_{a,\epsilon}$ ($\hat{S}^{-t}_{r,\epsilon}$) denote the forward (backward) flow of $\hat{S}_{a,\epsilon}$ ($\hat{S}_{r,\epsilon}$). In [17], it was shown that $\hat{S}^t_{a,\epsilon}$ and $\hat{S}^{-t}_{r,\epsilon}$ intersect transversally along $\hat{\gamma}^s$. We also show in [17] that the intersection is transverse along $\hat{\gamma}^w$ if and only if the non-degeneracy condition (B) holds. The latter case will be important in the present paper. Hence we now describe this case in more detail. First we set

$$x = r_2 x_2, \ z = r_2 z_2, \ \epsilon = r_2^2. \tag{59}$$

Substituting (59) into (41) gives

$$\dot{x}_2 = \beta^{-1}c(1 + \phi(\hat{y})) - (1 - \phi(\hat{y})), \tag{60}$$

$$\dot{\hat{y}} = b\epsilon_1 x_2 (1 + \phi(\hat{y})) - \beta x_2 (1 - \phi(\hat{y})), \tag{61}$$

$$\dot{z}_2 = 1 + \phi(\hat{y}) + b^{-1}\gamma_x(1 - \phi(\hat{y})),$$

after dividing the right hand sides by $r_2 = \sqrt{\epsilon} > 0$. In these variables, we write $\hat{\gamma}^w$ as $\hat{\gamma}_2^w$ and $\hat{S}_{a,r,\epsilon}$ as $\hat{S}_{a,r,\epsilon,2}$, respectively. Since

$$x_2 = -1/\sqrt{\epsilon_1}, \ z_2 = z_1/\sqrt{\epsilon_1}, \ r_2 = r_1 \sqrt{\epsilon_1},$$

and

$$x_2 = 1/\sqrt{\epsilon_3}, \ z_2 = z_2/\sqrt{\epsilon_3}, \ r_2 = r_3 \sqrt{\epsilon_3},$$

we use Lemma 3 to obtain the following

$$\hat{\gamma}_2^w: \ (x_2, \hat{y}, z_2) = (x_2, h(-\chi_+), -\chi_+ x_2), \ x_2 \in \mathbb{R}, \tag{61}$$

$$\hat{S}_{a,r,\epsilon,2}: \ \hat{y} = h(x_2^{-1} z_2) + O(x^{-2}_2), \tag{62}$$

for

$$(x_2, z_2) \in \{ (x_2, z_2) \in \mathbb{R}^2 | z_2 = -z_1 x_2, \ z_1 \in I_a, \ x_2 \in [-k/\sqrt{\epsilon}, -\mu^{-1}] \},$$
and
\[
\hat{S}_{r,\epsilon,2} : \quad \dot{y} = h(x_2^{-1}z_2) + O(x_2^{-2}),
\]
for
\[
(x_2, z_2) \in \{(x_2, z_2) \in \mathbb{R}^2 | z_2 = z_3x_2, z_3 \in I_r, x_2 \in [\mu^{-1}, k/\sqrt{\epsilon}] \}.
\]
Notice that the expressions for \(\hat{S}_{a,\epsilon,2}\) and \(\hat{S}_{r,\epsilon,2}\) are independent of \(r_2 = \sqrt{\epsilon}\).

We now describe how the tangent space \(T\hat{S}_{a,\epsilon,2}\) of \(\hat{S}_{a,\epsilon,2}\) twists upon forward flow application of the variation of (60) along \(\gamma_2^w\). For this we first replace time by \(x_2\) by dividing the equations for \(\hat{y}\) and \(z_2\) by \(\dot{x}_2\) and then linearize about \(\gamma_2^w\). This gives the following set of variational equations:

\[
\frac{du}{dx_2} = -\lambda_+^{-1}\beta (\psi ux_2 + bv) \tag{63}
\]

\[
\frac{dv}{dx_2} = -b^{-1}\lambda_+^{-1}\lambda_- \psi u.
\]

Here

\[
\psi \equiv \frac{1}{2}\beta(1 - b\chi_+)^2\phi\left(\frac{1 + b\chi_+}{1 - b\chi_+}\right).
\]

Notice that \(\psi > 0\). We then have the following:

**Lemma 4** Suppose (A) and (B). Let \(n = \lfloor \xi \rfloor \in \mathbb{N}\) so that \(n < \xi < n + 1\). Then \(\hat{S}_{a,\epsilon,2}\) and \(\hat{S}_{r,\epsilon,2}\) also intersect transversally along \(\hat{\gamma}_2^w\). In particular, the tangent space of \(\hat{S}_{a,\epsilon,2}\) twists along \(\hat{\gamma}_2^w\) in the following way: Consider \(\mu\) sufficiently small and the sections \(\Gamma_{\pm}^2 = \{(x, \hat{y}, z) \in \mathbb{R}^3 | x = \pm\mu^{-1}\sqrt{\epsilon}, z \in \sqrt{\epsilon}I_2\}\), \(I_2 \subset \mathbb{R}_-\) interval containing \(-\chi_+,\) transverse to \(\hat{\gamma}_2^w\). Then the tangent vector of \(\hat{S}_{a,\epsilon} \cap \Gamma_{-}^2\) at \(\hat{\gamma}_2^w \cap \Gamma_{-}^2\):

\[
\varpi_{\text{in}} = \begin{pmatrix}
0 \\
\frac{2\beta^{-1}b(1 + b\chi_+) \phi'}{(1 - b\chi_+) \phi(1 + b\chi_+)} \\
1
\end{pmatrix} \mu + O(\mu^2) \tag{64}
\]

is under the forward flow of the variational equations (63), transformed to a tangent vector \(\varpi_{\text{out}}\) of \(\hat{S}_{a,\epsilon} \cap \Gamma_{+}^2\) at \(\hat{\gamma}_2^w \cap \Gamma_{+}^2\), which is transverse to the tangent space of \(\hat{S}_{r,\epsilon} \cap \Gamma_{+}^2\) at \(\hat{\gamma}_2^w \cap \Gamma_{+}^2\), satisfying

\[
\varpi_{\text{out}} = \frac{\varpi_{\text{out}}(\mu)}{\varpi_{\text{out}}(\mu)} = (0, \zeta + o(1), o(1)) \tag{65}
\]

as \(\mu \to 0\), where

\[
\zeta = \begin{cases}
-1 & \text{if } n \text{ is odd} \\
1 & \text{if } n \text{ is even}
\end{cases} \tag{66}
\]
The result follows from [17, Lemma 7.8] and the fact that (63) can be written as the Weber equation

\[
\frac{d^2 v}{d\bar{x}_2^2} - \bar{x}_2 \frac{dv}{d\bar{x}_2} + \xi v = 0,
\]

by replacing \( x_2 \) by

\[
\bar{x}_2 = (-\psi/\beta \lambda^1_+)^{1/2} x_2,
\]

and eliminating \( u \). The form of \( \varpi_{in} \) is obtained by transforming (62) into the scaled variables in (59), valid for \( x_2 \leq -\mu^{-1} \). Differentiating the resulting expression with respect to \( z_2 \) gives (64).

3.6 The section \( \Pi_{in} \)

The section \( \Pi_{in} \), as defined for the PWS system in (21), intersects \( \Sigma \) in \( \Pi_{in} \cap \Sigma \), which is therefore blown up by (29), as shown in Fig. 4. We restrict this blowup of \( \Pi_{in} \cap \Sigma \) to extend from \( (\bar{y}, \epsilon) = (1, 0) \in S^1 \) down to include a small neighborhood of the critical manifold only, so that the regularized vector-field is transverse \( (\dot{x} > 0) \) to this section. (We could avoid this re-definition of \( \Pi_{in} \), if \( R_{in} \) were a neighborhood of \( y = 0 \): \( R_{in} = [-\zeta, \zeta] \times I_{in} \). But then \( \dot{x} < 0 \) for \( y < 0 \) while \( \dot{x} > 0 \) for \( y > 0 \) cf. (20), which presents us with further difficulties.) Hence we redefine \( \Pi_{in} \) for the regularization as

\[
\Pi_{in} = \{(x, y, z) \in U | x = -\delta, y \in [\epsilon(h(-\delta^{-1}z) - \varsigma), \zeta], z \in I_{in}\},
\]

for \( \varsigma > 0 \) sufficiently small (recall (23), (31) and (46)). We keep using the same symbol \( \Pi_{in} \) to refer to this new section in the \((x, y, z)\)-variables.

![Figure 4: Illustration of blowup geometry: the nonhyperbolic critical points \( \bar{q} \) and \( \bar{l}^{\pm} \) of \( \bar{X} \), the normally hyperbolic critical manifolds \( \bar{S}_{a,r} \) and the redefined section \( \Pi_{in} \).](image-url)
4 Main results

4.1 Local dynamics: passage through the two-fold

Our main technical result is the following (recall from Fig. 1 that $u_{out} = U \cap \Pi_{out}$).

**Theorem 1** Suppose (A) and (B). Consider (26) with the regularization function (38) and the local mapping $L_\epsilon: \Pi_{in} \to \Pi_{out}$, $(y, z) \mapsto (x(y, z) + \nu, z(y, z))$, recall (21), (22) and (24), obtained by the forward flow. Then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, $L_\epsilon$ is well-defined. Moreover,

$$L_\epsilon(y, z) = u_{out} + o(1), \quad DL_\epsilon(y, z) = o(1),$$

as $\epsilon \to 0^+$. $\blacksquare$

4.2 Discussion

One can interpret the theorem in the following way: the forward orbit $U$ is distinguished amongst all the possible forward orbits leaving $q$. Intuitively, this makes sense because $U$ is the only stable exit from the two-fold of the PWS system. The other forward orbits in Fig. 1 are very fragile due to the unstable sliding region. Furthermore, if we consider an initial condition of $X^+$ just above $\Sigma^+_sl$ close to $q$, then the forward flow follows close to $U$.

But if we start below $\Sigma^-_{sl}$ and follow $X^-$, then we return to $\Sigma$, since $l^-$ is invisible, and potentially even $\Sigma^-_{sl}$. One could then imagine that this rotation could continue indefinitely so that the forward orbit of the regularized system would never leave a small vicinity of the two-fold. A related phenomenon occurs in PWS systems with intersecting discontinuity sets [14]. We shall see that the following lemma excludes this behaviour:

**Lemma 5** Let

$$\vartheta(x, z) = (x_+(x, z), z_+(x, z)), \quad (x, z) \in (0, \infty) \times \mathbb{R},$$

be so that $(x_+(x, z), 0, z_+(x, z))$ is the first return of $(x, 0, z) \in \Sigma \cap \{x > 0\}$ to $\Sigma$ under the forward flow of $X^-$. Then

$$\vartheta(x, z) = (-x, z + \frac{2\gamma}{b} x), \quad (x, 0, z) \in \{x > 0, y = 0\}. \quad (69)$$

In particular, $\vartheta$ has a smooth extension to $(x, z) \in [0, \infty) \times \mathbb{R}$, so that

$$(x, 0, z) \mapsto (x_+(x, z), 0, z_+(x, z)),$$

leaves the invisible fold line $l^-$ fixed: $(0, 0, z) \mapsto (0, 0, z)$, or simply $\vartheta(0, z) = (0, z)$, for all $z \in \mathbb{R}$. Furthermore,

$$D\vartheta(0)v_+ = \begin{pmatrix} -1 \\ z_1^* \end{pmatrix},$$

with $v_+ (16)$, the weak direction of the node of (12), and

$$z_1^* \equiv -\chi + \frac{2\gamma}{b} > \chi^-.$$  

$\blacksquare$
Proof See Appendix A.

(A similar local result holds for $X^-$ in (6)$_{y<0}$.) The consequence of (71) is that the half-line
\[
\{(x, y, z) \in \Sigma \mid (x, z) = sD\vartheta(0)v_+ = s(-1, z_1^*), s \in (0, \infty)\},
\]
obtained from (70), is contained outside the funnel of the PWS Filippov system (see Fig. 2). Therefore we have the following: Initial conditions within $\Sigma_{st}$ with $(x, z) = sv_+, s > 0$ small, will upon following $X^-$, return to $\Sigma$ with $(x, z) = s(-1, z_1^*)$ cf. (69) and then from there either:

- For $z_1^* > 0$: Follow $X^+$ through crossing.
- For $z_1^* \leq 0$: Follow $X_{st}$ up to the visible fold line $l^+$ and then from there subsequently follow $X^+$.

For $s$ small, the resulting forward orbit within $\Sigma^+$ will then in both cases remain close to $U$. This is the key intuition behind our main theoretical result.

4.3 Global dynamics

Suppose the following:

(C) There exists a smooth (global) mapping $G_\epsilon : \Pi_{out} \to \Pi_{in}$ of the PWS system.

The following result is then by the Contraction Mapping Theorem a simple corollary of Theorem 1.

Corollary 1 Suppose (A), (B), (C) and consider the Poincaré return mapping
\[
P_\epsilon = L_\epsilon \circ G_\epsilon : \Pi_{out} \to \Pi_{out}, (x, z) \mapsto (x_+(x, z), \nu, z_+(x, z)).
\]
Then there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon \leq \epsilon_0$, the mapping $P_\epsilon$ has a unique and attracting fixed point, which is $o(1)$-close to $u_{out}$.

A similar result is known for the passage through folded nodes in slow-fast systems with two slow variables and one fast, see [2, Theorem 4.1]. We will discuss this connection further in our final conclusion Section 6. In Fig. 5 we illustrate some PWS examples with global returns to the two fold. Corollary 1 shows that the regularization of these systems can produce attracting limit cycles that follow the singular PWS cycle (illustrated in Fig. 5 using thick lines).

5 Proof of Theorem 1

To prove Theorem 1 we first work with (28) and the blowup (29) in the phase space
\[
\mathcal{P} = \{(x, z, \pi, (\bar{y}, \bar{\epsilon})) \in \mathbb{R}^2 \times [0, \infty) \times S^1\}.
\]
Here
\[
\bar{q} : x = \pi = z = 0, (\bar{y}, \bar{\epsilon}) \in S^1,
\]
is a circle of nonhyperbolic critical points. In this section, we blowup this circle by applying the blowup method [7, 8, 9], in the formulation of Krupa and Szmolyan [19, 20].
Figure 5: Different (global) PWS systems with a visible-invisible two-fold, satisfying (A) and (B), where the segment $U$ returns to $q$ under the forward flow of $X^\pm$ and $X_{sl}$: (a) the segment $U$ intersects the funnel directly; (b) the segment $U$ intersects $\Sigma_{cr}$ and then follows $X^{-}$ up to the funnel; (c) as in (b), but the orbit segment of $X^{-}$ returns outside the funnel, then follows $X_{sl}$ up until the visible fold line $l^+$, and then the orbit of $X^+$ returns the funnel to complete the closed cycle. Corollary \[\Box\] shows that the regularization of these systems can produce attracting limit cycles.

5.1 Blowup of $\bar{q}$

We apply the following blowup transformation $(x, \pi, z, (\bar{y}, \bar{\epsilon})) \mapsto (r, (\bar{x}, \bar{\pi}, \bar{z}), (\bar{y}, \bar{\epsilon}))$ defined by

$$x = r\bar{x}, \quad \pi = r^2\bar{\pi}, \quad z = r\bar{z}, \quad r \geq 0, (\bar{x}, \bar{\pi}, \bar{z}) \in S^2 = \{\bar{x}^2 + \bar{\pi}^2 + \bar{z}^2 = 1\}. \quad (73)$$
This transformation blows up (72) to a circle of spheres $S^1 \times S^2$:

\[
\bar{q} : r = 0, (\bar{x}, \bar{\pi}, \bar{z}) \in S^2, (\bar{y}, \bar{\epsilon}) \in S^1,
\]

as illustrated in Fig. 6. We use double-bar to indicate that the two-fold $q$ has now been blown up twice. Henceforth we therefore consider the following phase space

\[
\overline{P} = \{(r, (\bar{x}, \bar{\pi}, \bar{z}), (\bar{y}, \bar{\epsilon})) \in [0, \infty) \times S^2 \times S^1\}.
\]

The transformation (73) pulls back $X$ on $P$ to a vector-field $\overline{X}$ on $\overline{P}$. Here $\overline{X}_{r=0} = 0$. But by following [19], the exponents of $r$ in the blowup: 1, 2 and 1 have been chosen so that $\overline{X} = r^{-1}X$ is well-defined and nontrivial. It is $\overline{X}$ that we shall study in the following. For $\overline{X}$ we will have gained partial hyperbolicity on the blowup of $\bar{q}$. This will allow us to apply standard perturbation theory arguments to study the perturbation of the nonhyperbolic circle $\bar{q} = 0$ and eventually to prove Theorem 1.

Figure 6: Illustration of the blowup of $\bar{q}$, a circle of nonhyperbolic critical points defined in (72), to $\bar{q}$, a circle of spheres $S^1 \times S^2$ defined in (74).

To describe $\overline{P}$ and $\overline{X}$ we use charts (31), (32) and (33) along with the following directional charts

\[
\kappa_1 : x = -r_1, \pi = r_1^2 \bar{\pi}_1, z = r_1 \bar{z}_1,
\]

\[
\kappa_2 : x = r_2 \bar{x}_2, \pi = r_2^2, z = r_2 \bar{z}_2,
\]

\[
\kappa_3 : x = r_3, \pi = r_3^2 \bar{\pi}_3, z = r_3 \bar{z}_3.
\]

for $r_i \geq 0$, obtained by setting $\bar{x} = -1$, $\bar{\pi} = 1$, and $\bar{x} = 1$, respectively, in (73), as suggested by [19]. Within chart (31) where $y = \epsilon \hat{y}$, $\epsilon \geq 0$, $\hat{y} \in \mathbb{R}$ this becomes

\[
(\bar{\epsilon} = 1, \kappa_1) : x = -r_1, \epsilon = r_1^2 \epsilon_1, z = r_1 \epsilon_1, \quad r_1 \geq 0, \epsilon_1 \geq 0,
\]

\[
(\bar{\epsilon} = 1, \kappa_2) : x = r_2 \bar{x}_2, \epsilon = r_2^2, z = r_2 \bar{z}_2, \quad r_2 \geq 0,
\]

\[
(\bar{\epsilon} = 1, \kappa_3) : x = r_3, \epsilon = r_3^2 \epsilon_3, z = r_3 \epsilon_3, \quad r_3 \geq 0, \epsilon_3 \geq 0.
\]
We refer to these chart as \((\check{\epsilon} = 1, \kappa_i), i = 1, 2, 3\), namely the \textit{entry chart}, the \textit{scaling chart} (since \(r_2 = \sqrt{\check{\epsilon}}\)), and the \textit{exit chart}, respectively. The coordinate changes between the \((\check{\epsilon} = 1, \kappa_i)\)-charts are defined by

\[
\begin{align*}
(\check{\epsilon} = 1, \kappa_1) &\rightarrow (\check{\epsilon} = 1, \kappa_2) : \quad x_2 = -1/\sqrt{\epsilon_1}, r_2 = r_1 \sqrt{\epsilon_1}, z_2 = z_1/\sqrt{\epsilon_1}, \\
(\check{\epsilon} = 1, \kappa_2) &\rightarrow (\check{\epsilon} = 1, \kappa_3) : \quad r_3 = r_2 x_2, z_3 = x_2^{-1} z_2, \epsilon_3 = x_2^{-2},
\end{align*}
\]

(78, 79)

defined for \(\epsilon_1 > 0\) and \(x_2 > 0\), respectively, and their respective inverses. Similarly in charts \((\hat{\epsilon} = 1, \kappa_i), i = 1, 2, 3\), respectively\(^1\). Now, the coordinate changes between the \((\hat{\epsilon} = 1, \kappa_i)\)-charts are defined by

\[
\begin{align*}
(\hat{\epsilon} = 1, \kappa_1) &\rightarrow (\hat{\epsilon} = 1, \kappa_2) : \quad x_2 = -1/\sqrt{\pm y_1}, r_2 = r_1 \sqrt{\pm y_1}, z_2 = z_1/\sqrt{\pm y_1}, \\
(\hat{\epsilon} = 1, \kappa_2) &\rightarrow (\hat{\epsilon} = 1, \kappa_3) : \quad r_3 = r_2 x_2, z_3 = x_2^{-1} z_2, y_3 = \pm x_2^{-2},
\end{align*}
\]

(80, 81, 82, 83, 84)

defined for \(y_1 \geq 0\) and \(x_2 > 0\), respectively, and their respective inverses. Notice that the coordinate change from \((\check{\epsilon} = 1, \kappa_i)\) to \((\hat{\epsilon} = 1, \kappa_i), i = 1, 3\), is obtained from

\[
y_1 = \hat{y} \epsilon_1, \quad y_3 = \hat{y} \epsilon_3, \quad \hat{\epsilon} = \hat{y}^{-1}.
\]

(85)

Fixing appropriate compact subsets in each chart, these charts then cover the relevant part of the cylinder \(\overline{q} \cap \{\overline{\pi} \geq 0\} \cap \{\epsilon \geq 0\}\) completely.

We will follow the (standard) convention that a geometric object obtained in \((\check{\epsilon} = 1, \kappa_i)\) or \((\hat{\epsilon} = 1, \kappa_i)\) will be given a hat and a subscript \(i\). Such an object, say \(\widehat{G}_i\), will be denoted by \(\overline{G}\) in the blowup variables \((73)\) of either of the charts \(\check{\epsilon} = 1, \hat{\epsilon} = \pm 1\). In the full blowup space \(\overline{P}\), this object will be denoted by \(\overline{G}\). As above, objects, say \(\widehat{G}_i\), obtained in \((\check{\epsilon} = 1, \kappa_i)\) will frequently only be partially visible \((\hat{\epsilon} \geq 0, \text{respectively})\) in the charts \((\hat{\epsilon} = 1, \kappa_i)\). For simplicity, we will, however, (again) continue to denote the visible part of \(\widehat{G}_i\) in the charts \((\hat{\epsilon} = 1, \kappa_i)\) by the same symbol.

### 5.2 Outline of the proof and figures of singular orbits

The proof of the theorem is naturally divided into two separate cases. These are defined as follows:

**Definition 6** Let \(n = \lfloor \xi \rfloor\) be the greatest integer less than \(\xi > 1\), the ratio of the eigenvalues, see \((14)\). Then we define the following two cases:

- **Case (a):**
- \( \Pi_{\text{in}} \cap \Sigma \) is between \( \hat{\gamma}^w \) and \( \hat{\gamma}^s \) and \( n \) is even,  

or  

- \( \Pi_{\text{in}} \cap \Sigma \) is between \( l^- \) and \( \hat{\gamma}^w \) and \( n \) is odd.  

\[ \text{Case (b):} \]

- \( \Pi_{\text{in}} \cap \Sigma \) is between \( \hat{\gamma}^w \) and \( \hat{\gamma}^s \) and \( n \) is odd,  

or  

- \( \Pi_{\text{in}} \cap \Sigma \) is between \( l^- \) and \( \hat{\gamma}^w \) and \( n \) is even.

In case (a), the forward flow of \( \Pi_{\text{in}} \) will rotate around the weak canard \( \gamma^w \) near \( q \) and subsequently be repelled from the slow manifold \( S_{r,\epsilon} \) and move upwards (increasing values of \( y \)). In case (b), on the other hand, the forward flow of \( \Pi_{\text{in}} \) will be repelled downwards (decreasing values of \( y \)) by \( S_{r,\epsilon} \).

We prove the theorem by first describing the initial passage of \( q \) in Section 5.3. During this part, there will be a contraction towards the weak canard (that ultimately produce the contractive properties of the mapping \( L_\epsilon \) in Theorem 1). Together with the rotation of the tangent spaces, described by Lemma 4, the contraction towards the weak canard will also allow us to separate the cases (a) and (b) and carry the forward flow of \( \Pi_{\text{in}} \) towards \( \hat{\gamma}^w \) within \( q \). From there we then follow the orbits by initially working in the charts \((\hat{y} = \pm 1, \kappa_3)\) and \((\hat{y} = \pm 1, \kappa_i)\). Here our approach is to successively identify hyperbolic segments of \( \bar{X} \) on \( \bar{y} \) through our blowup approach. We will denote these segments in the blowup space \( \bar{\mathcal{P}} \) by \( \bar{Q}_j,a \) and \( \bar{Q}_j,b \), for \( j = 1, 2, \ldots \), in case (a) and case (b), respectively. We then complete the proof by perturbing away from these segments at \( \epsilon = 0 \) using standard local hyperbolic methods of dynamical systems theory in the appropriate charts \((\epsilon, \kappa_i)\) and \((\hat{y} = \pm 1, \kappa_i)\). The result is then that the forward flow of \( \Pi_{\text{in}} \) under \( \bar{X} \) converges to the union of \( \bar{\gamma}^w \) within \( q \): \( r = 0 \) and the segments

- In case (a): \( \bar{Q}_1,a, \bar{Q}_2,a \) and \( \bar{U} \);
- In case (b): This case further depends upon the sign of \( z_1^* \):
  - For \( z_1^* \leq 0 \): \( \bar{Q}_1,b, \bar{Q}_2,b, \bar{Q}_3,b, \bar{Q}_4,b, \bar{Q}_5,b \) and \( \bar{U} \);
  - For \( z_1^* > 0 \): \( \bar{Q}_1,b, \bar{Q}_2,b, \bar{Q}_3,b, \bar{Q}_4,b, \bar{Q}_5,b \) and \( \bar{U} \);

as \( \epsilon \to 0 \). We illustrate these singular segments in Fig. 7 and Fig. 8, focussing in case (b) on the more complicated situation where \( z_1^* < 0 \). Note that the orbits are defined in the 4 dimensional space \( \bar{\mathcal{P}} \). We therefore use an artistic projection in the \( \epsilon = 1 \) chart of the blowup (29). Notice, in particular, that we have folded the half-circle of (hemi-)spheres \( \bar{q} \) out to the double-infinite strip \( \hat{y} \in [-\infty, \infty] \) of discs! There are three important discs: \( \hat{y} = h(-\chi_+) \), which contains \( \bar{\gamma}^w \), and is illustrated in Fig. 7 and Fig. 8 using orange (boundaries); \( \hat{y} = +\infty \), illustrated using cyan (boundaries); \( \hat{y} = -\infty \), illustrated using purple (boundaries).
Figure 7: Illustration of the singular orbit for case (a) using a projection in the $\bar{\epsilon} = 1$ chart of the blowup (29). In this chart the half-circle of (hemi-)spheres $\tilde{q}$ becomes a double-infinite strip $\hat{y} \in [-\infty, \infty]$ of discs.

5.3 Initial passage through $q$

In this section we describe a mapping from $\Pi_{in}$ to $\{\hat{y} = \pm \delta^{-1}\}$ with $\delta > 0$ sufficiently small. We describe the image of this mapping using the coordinates in chart $(\bar{\epsilon} = 1, \kappa_3)$ (77). More specifically, in the chart $(\bar{\epsilon} = 1, \kappa_3)$ we define the case-dependent section $\hat{\Lambda}_{out,3}$ as follows:

- case (a): $\hat{\Lambda}_{out,3}: \hat{y} = \delta^{-1}, (r_3, z_3, \epsilon_3) \in \hat{R}_{3, out} \cap \{r_3 \geq 0, \epsilon_3 \geq 0\}$,
- case (b): $\hat{\Lambda}_{out,3}: \hat{y} = -\delta^{-1}, (r_3, z_3, \epsilon_3) \in \hat{R}_{3, out} \cap \{r_3 \geq 0, \epsilon_3 \geq 0\}$,

with $\hat{R}_{3, out}$ a small neighborhood of $(0, -\chi_+, 0)$. Then we have the following.
Figure 8: Illustration of the singular orbit for case (b) with $z_1^* < 0$. The segments $Q^{2,b} \subset \{ \hat{y} = -\infty \}$ and $Q^{3,b}$ re-inject the orbit into the region of stable sliding in this case.

**Proposition 4** The mapping

\[ \hat{L}_{3,\epsilon} : \Pi_{\text{in}} \to \hat{\Lambda}_{\text{out},3}, (y, z) \mapsto (r_3, \hat{y}, z_3, \epsilon_3) = (r_3(y, z), \pm \delta^{-1}, z_3(y, z), \epsilon_3(y, z)) \]  

where $r_3^2\epsilon_3 = \epsilon$, obtained by the forward flow, is well-defined. In fact, $\hat{L}_{3,\epsilon}$ is contractive for $\epsilon > 0$ sufficiently small: That is, fix any $\nu \in (0, 1)$, $K > 0$ sufficiently large and consider the set

\[ \hat{D}_3 = \left\{ (r_3, \hat{y}, z_3, \epsilon_3) \in \hat{\Lambda}_{\text{out},3} | r_3^2\epsilon_3 = \epsilon, K^{-1} \ln^{-1} \epsilon^{-1} \leq z_3 + \chi_+ + \epsilon_3 \leq K\ln^{-1} \epsilon^{-1} \right\}, \]

contained within $\hat{\Lambda}_{\text{out},3}$. Then $\hat{L}_\epsilon(\Pi_{\text{in}}) \subset \hat{D}_3$. Furthermore, the Jacobian

\[ D \left( \begin{pmatrix} \hat{z}_3+ \\ \epsilon_3+ \end{pmatrix} \right) (y, z), \]
has one eigenvalue of $\mathcal{O}(e^{-c/\epsilon})$ and another one of $\mathcal{O}(e^{(\xi-1)^{1/2}})$ with $\xi$ defined in (B). □

Proof From $\Pi_{in}$ (68) all orbits initially contract towards $S_{a,\epsilon}$. We will therefore first work in the entry chart ($\bar{\epsilon} = 1, \kappa_1$) and the scaling chart ($\bar{\epsilon} = 1, \kappa_2$). In both of these charts, $S_{a,\epsilon}$ [55] will be visible. We denote this unique slow manifold by $S_{a,\epsilon,1}$ and $S_{a,\epsilon,2}$ in the charts ($\bar{\epsilon} = 1, \kappa_1$) and ($\bar{\epsilon} = 1, \kappa_2$), respectively. We then have the following.

**Lemma 6** Consider the section

$$\hat{\Gamma}_{in,2} : x_2 = -\mu^{-1}, (\hat{y}, z_2) \in \hat{R}_{in,2},$$

within ($\bar{\epsilon} = 1, \kappa_2$), for a suitably large rectangle $\hat{R}_{in,2}$ in the $(\hat{y}, z_2)$-plane, and the associated mapping

$$\hat{L}_{2,\epsilon} : \Pi_{in} \to \hat{\Gamma}_{in,2}, (y, z) \mapsto (x_2, \hat{y}, z_2) = (-\mu^{-1}, \hat{y}_+(y, z), z_2+(y, z)),$$

(87)

with $r_2 = \sqrt{\epsilon}$, obtained by the forward flow. Then $\hat{L}_{2,\epsilon}$ is contractive for $\epsilon > 0$ sufficiently small: That is, fix any $v \in (0, 1)$. Then the image $\hat{L}_{2,\epsilon}(\Pi_{in})$ is a $\mathcal{O}(e^{-c/\epsilon})$-thin strip around $S_{a,\epsilon,2} \cap \hat{\Gamma}_{in,2}$ with $c > 0$. The width of the strip is $\mathcal{O}(e^{(\xi-1)^{1/2}})$. In particular, let $A_{\hat{\gamma}_2}$ be an annulus in $\hat{\Gamma}_{in,2}$ with inner radius $K^{-1}e^{(\xi-1)/(2v)}$ and outer radius $Ke^{(\xi-1)^{1/2}}$, with $K$ sufficiently large, centered around $\hat{\gamma}_2 \cap \hat{\Gamma}_{in,2}$. Moreover, consider $N_{S_{a,\epsilon,2}}$ as the $c_1^{-1}e^{-c_1/\epsilon}$-neighborhood of $S_{a,\epsilon,2} \cap \hat{\Gamma}_{in,2}$ within $\hat{\Gamma}_{in,2}$:

$$N_{S_{a,\epsilon,2}} = \{(x_2, \hat{y}, z_2) \in \hat{\Gamma}_{in,2} | \text{dist}((x_2, \hat{y}, z_2), S_{a,\epsilon,2} \cap \hat{\Gamma}_{in,2}) \leq c_1^{-1}e^{-c_1/\epsilon}\},$$

with $c_1 > 0$ sufficiently small. Then for $\epsilon > 0$ sufficiently small,

$$\hat{L}_{2,\epsilon}(\Pi_{in}) \subset A_{\hat{\gamma}_2} \cap N_{S_{a,\epsilon,2}}.$$

(88)

Furthermore, the Jacobian

$$D \begin{pmatrix} \hat{y}_+ \\ \hat{z}_2+ \end{pmatrix} (y, z),$$

has one eigenvalue of $\mathcal{O}(e^{-c/\epsilon})$, $c > 0$, and another one of $\mathcal{O}(e^{(\xi-1)^{1/2}})$. □

Proof In the chart ($\bar{\epsilon} = 1, \kappa_1$) we obtain the following equations

$$\dot{r}_1 = -r_1 \epsilon_1 G(\hat{y}),$$

$$\dot{\hat{y}} = b_{21}(1 + \phi(\hat{y})) + \beta(1 - \phi(\hat{y})),$$

$$\dot{z}_1 = \epsilon_1 (H(\hat{y}) + z_1 G(\hat{y})), $$

$$\dot{\epsilon}_1 = 2\epsilon_1^2 G(\hat{y}),$$

by inserting (75) into (41) and dividing the right hand side by the common factor $r_1$. Here we have introduced the functions

$$G(\hat{y}) = \beta^{-1}c(1 + \phi(\hat{y})) - (1 - \phi(\hat{y})), \quad H(\hat{y}) = (1 + \phi(\hat{y})) + b^{-1}\gamma(1 - \phi(\hat{y})).$$

(90)
The line
\[
\hat{L}_{a,1} : r_1 = \epsilon_1 = 0, \quad \hat{y} = h(-z_1), \quad z_1 \in \mathbb{R}_-, \tag{91}
\]
is a normally hyperbolic set of critical points of \( \hat{\Pi}_{a,1} \). Let \( \mu, k \) and \( I_a \) be as in Lemma 3. Then by the partial hyperbolicity of \( \hat{L}_{a,1} \), we obtain a unique center manifold \( \hat{\Pi}_{a,1} \), we obtain a unique center manifold \( \hat{S}_{a,\epsilon,1} \) in \( \hat{\Pi}_{a,1} \). See \( \hat{S}_{a,\epsilon,1} \) we obtain the reduced problem
\[
\begin{align*}
\hat{z}_1 &= (c - bz_1)^{-1} \left( bz_1^2 + (c - \gamma)z_1 + \beta \right) (1 + \epsilon_1 n_1(z_1, \epsilon_1)), \\
\hat{\epsilon}_1 &= 2\epsilon_1,
\end{align*}
\tag{92}
\]
with \( n_1 \) smooth, after division of the right hand side by \( (\beta^{-1}c(1 + \phi(\hat{y}))) - (1 - \phi(\hat{y})) \) \( \epsilon_1 \). This quantity is positive inside the funnel and therefore corresponds to a nonlinear transformation of time. Since the \( r_1 \)-equation decouples, \( \epsilon_1 \) can be determined by the conservation of \( \epsilon_1 \): \( r_1^2 = \epsilon_1^{-1}\epsilon_1 \). The point \( z_1 = \chi_+ < 0, \epsilon_1 = 0 \) is a saddle for \( (92) \), the linearization having the eigenvalues
\[
1 - \xi, 2.
\]
Notice \( 1 - \xi < 0 \) by (A). The result then follows by (a) simple estimation through Gronwall’s inequality, (b) the exponential contraction towards \( \hat{S}_{\epsilon,1} \), (c) applying the coordinate transformation \( \hat{\Pi}_{a,1} \).

To complete the proof of Proposition \( \hat{\Pi}_{a,1} \) we subsequently describe the mapping
\[
\hat{\mathcal{L}}_{2,\epsilon}(\Pi_{in}) \subset \hat{\Gamma}_{in,2} \rightarrow \hat{\Gamma}_{out,2}, \tag{93}
\]
from \( \hat{\mathcal{L}}_{2,\epsilon}(\Pi_{in}) \) to
\[
\hat{\Gamma}_{out,2} : x_2 = \mu^{-1}, (\hat{y}, z_2) \in \hat{R}_{in,2},
\]in the chart \( (\epsilon = 1, \kappa_2) \). By the previous lemma, we can do this by considering the variational equations \( (63) \) about \( \gamma_2^w \). Indeed, the image \( \hat{\mathcal{L}}_{2,\epsilon}(\Pi_{in}) \) is close to the tangent space of \( \hat{S}_{a,\epsilon,2} \cap \hat{\Gamma}_{in,2} \) for \( 0 < \epsilon \ll 1 \). By \( (92) \), the case when \( \Pi_{in} \cap \Sigma \) is between \( \hat{\gamma}_w \) and \( \hat{\gamma}_s \), then corresponds to variations in the positive direction of \( \omega_{in} \), recall \( (64) \). By Lemma 3, in particular \( (66) \), the forward flow of \( \hat{\mathcal{L}}_{2,\epsilon}(\Pi_{in}) \) therefore intersects \( \hat{\Gamma}_{out,2} \) below (above) the unique slow manifold \( S_{r,\epsilon,2} \), see \( (57) \) for \( x_2 = 1/\sqrt{\epsilon_3} \geq \mu^{-1} \), when \( n \) is even (odd), respectively. On the other hand, the case when \( \Pi_{in} \cap \Sigma \) is between \( \lim^{-} \) and \( \lim^{+} \), corresponds to variations in the negative direction of \( \omega_{in} \) and the forward flow of \( \hat{\mathcal{L}}_{2,\epsilon}(\Pi_{in}) \) therefore intersects \( \hat{\Gamma}_{out,2} \) above (below) the unique slow manifold \( S_{r,\epsilon,2} \) when \( n \) is even (odd), respectively. Under the \( O(1) \)-time application of the forward flow in chart \( \kappa_2 \), the image of \( (93) \) remains cf. \( (88) \) sufficiently bounded away from the weak canard and \( S_{r,\epsilon,2} \) at \( \hat{\Gamma}_{out,2} \).
In the chart \((\epsilon = 1, \kappa_3)\), we obtain the following equations:

\[
\begin{align*}
\dot{r}_3 &= r_3 \epsilon_3 G(\hat{y}), \\
\dot{\hat{y}} &= b z_1 (1 + \phi(\hat{y})) - \beta (1 - \phi(\hat{y})), \\
\dot{z}_3 &= \epsilon_3 (H(\hat{y}) - z_3 G(\hat{y})), \\
\dot{\epsilon}_3 &= -2 \epsilon_3^2 G(\hat{y}),
\end{align*}
\]  
(94)


recall (90). The line

\[
\hat{L}_{r,3} : r_3 = \epsilon_3 = 0, \quad \hat{y} = h(z_3), \quad z_3 \in (0, \infty),
\]  
(95)

is partially hyperbolic and, as in chart \(\kappa_1\), this produces a unique center manifold:

\[
W^c(\hat{L}_{r,3}|_{z_3 \in I_r}) : \hat{y} = h(z_3) + \epsilon_3 (bz_3^2 - (\gamma - c)z_3 + \beta)m(-z_3, \epsilon_3),
\]

\[
(r_3, \epsilon_3, z_3) \in [0, k] \times [0, \mu^2] \times I_r,
\]

with \(\mu, k\) and \(I_r\) as in Lemma 3, which upon restriction to the invariant set \(r_3^2 \epsilon_3 = \epsilon\) gives \(S_{r,\epsilon,3}\) (57). Here \(\hat{\gamma}_3^w\) intersects \(\hat{L}_{r,3}\) in

\[
\hat{\gamma}_3^w : r_3 = 0, \epsilon_3 = 0, \quad z_3 = -\chi_+, \quad \hat{y} = h(-\chi_+).
\]  
(96)

for \(r_3 = 0\). The unstable manifold of (96) for (94) is the union of the two sets

\[
\hat{Q}_3^{1,a} : r_3 = \epsilon_3 = 0, \quad z_3 = -\chi_+, \quad \hat{y} \geq h(-\chi_+),
\]  
(97)

\[
\hat{Q}_3^{1,b} : r_3 = \epsilon_3 = 0, \quad z_3 = -\chi_+, \quad \hat{y} \geq h(-\chi_+).
\]  
(98)

Using the initial conditions at \(\hat{\Gamma}_{\text{out},3} \subset \{\epsilon_3 = \mu^2\}\) it is then relatively easy to finish the proof of Proposition 4 e.g. by using Fenichel’s normal form, and follow the forward flow up/down to the section \(\Lambda_{\text{out},3} \subset \{\hat{y} = \pm \delta^{-1}\}\) in cases (a) and (b), respectively. The result shows that the forward orbits in \((\epsilon = 1, \kappa_3)\) follow the union of \(\hat{\gamma}_3^w\) within \(\{r_3 = 0\}\) and: \(\hat{Q}_3^{1,a}\) in case (a) or \(\hat{Q}_3^{1,b}\) in case (b) within \(\{r_3 = \epsilon_3 = 0\}\) as \(\epsilon \to 0\).

\[\blacksquare\]

### 5.4 Case (a)

We now focus attention on case (a) of Definition 6 starting at \(\hat{L}_{3,\epsilon}(\Pi_{in}) \subset \{\hat{y} = \delta^{-1}\}\). For this we note that \(\hat{y}\) is increasing on \(\hat{Q}_3^{1,a}\setminus\{\hat{y} = h(-\chi_+)\}\). Therefore to follow \(\hat{Q}_3^{1,a}\) forward we move to chart \((\hat{y} = 1, \kappa_3)\) using the coordinate change (85).

**Chart \((\hat{y} = 1, \kappa_3)\)**

Here we obtain the following equations from (48):

\[
\begin{align*}
\dot{r}_3 &= r_3 y_3 J(\hat{\epsilon}), \\
\dot{y}_3 &= y_3 (K_3(z_3, \hat{\epsilon}) - 2 y_3 J(\hat{\epsilon})), \\
\dot{z}_3 &= y_3 (L(\hat{\epsilon}) - z_3 J(\hat{\epsilon})), \\
\dot{\hat{\epsilon}} &= - \hat{\epsilon} K_3(z_3, \hat{\epsilon}),
\end{align*}
\]  
(99)
where
\[ J(\hat{\epsilon}) = \frac{1}{2} \beta^{-1} c (1 + \phi_+(\hat{\epsilon})) - \frac{1}{2} (1 - \phi_+(\hat{\epsilon})), \]
\[ K_3(z_3, \hat{\epsilon}) = \frac{1}{2} b z_3 (1 + \phi_+(\hat{\epsilon})) - \frac{1}{2} \beta (1 - \phi_+(\hat{\epsilon})), \]
\[ L(\hat{\epsilon}) = \frac{1}{2} (1 + \phi_+(\hat{\epsilon})) + \frac{1}{2} b^{-1} \gamma (1 - \phi_+(\hat{\epsilon})), \]
so that
\[ J(0) = \beta^{-1} c, \quad K_3(z_3, 0) = b z_3, \quad L(0) = 1, \] (100)
using \( \phi_+(0) = 1 \). In the \((\bar{y} = 1, \kappa_3)\)-chart, \( \hat{Q}_{3}^{1,a} \) from (97) becomes
\[ \hat{Q}_{3}^{1,a} : r_3 = y_3 = 0, z_3 = -\chi_+, \text{ and } \hat{\epsilon} \in [0, h(-\chi_+)^{-1}] \text{ for } h(-\chi_+) > 0, \]
or \( \hat{\epsilon} \in [0, \infty) \) for \( h(-\chi_+) \leq 0, \)
(remaining it to \( \hat{\epsilon} = 0 \)). Then we have

**Lemma 7** The set
\[ \hat{M}_3^+ : r_3 \geq 0, y_3 = 0, z_3 \in (0, \infty), \hat{\epsilon} = 0, \]
is a set of critical points of (99) of saddle-type. The linearization about any point \((r_3, y_3, z_3, \hat{\epsilon}) \in \hat{M}_3^+ \) has only two non-zero eigenvalues
\[ \pm b z_3, \]
with corresponding eigenvectors:
\[
\begin{pmatrix}
\beta^{-1} c r_3 \\
 b z_3 \\
1 - \beta^{-1} c z_3 \\
0
\end{pmatrix}

\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix},
\]
respectively. Let
\[ \hat{q}_{3}^{1,a} = (0, 0, -\chi_+, 0) \in \hat{M}_3^+ \cap \{r_3 = 0\}. \]

Then \( \hat{Q}_{3}^{1,a} \) is the stable manifold of \( \hat{q}_{3}^{1,a} \). On the other hand, the unstable manifold of \( \hat{q}_{3}^{1,a} \) is
\[ \hat{Q}_{3}^{2,a} = \left\{ (r_3, y_3, z_3, \hat{\epsilon}) | r_3 = \hat{\epsilon} = 0, z_3 = c^{-1} \beta \left( -1 + e^{-\beta^{-1} c s} \left( 1 + \beta^{-1} c \chi_+ \right) \right), \right. \]
\[ y_3 = \frac{1}{2} c^{-2} b \beta^2 \left( 1 + 2 \beta^{-1} c \chi_+ \right) e^{2 \beta^{-1} c s} + 1 - 2 \left( 1 + \beta^{-1} c \chi_+ \right) e^{-\beta^{-1} c s}, \]
for \( s \in [0, \infty) \), (101)
tangent to \((0, -b\chi_+, 1 + \beta^{-1}c\chi_+, 0)^T\) at \(\hat{q}_3^{1,a}\). 

\(\hat{Q}_3^{2,a}\) is contained in the stable manifold of the hyperbolic equilibrium

\[
\hat{u}_3 = (0, \frac{1}{2}c^{-2}b\beta^2, c^{-1}\beta, 0).
\]  

(102)

\(\hat{u}_3\) has a 1D unstable manifold:

\[
\hat{U}_3 = \{(r_3, y_3, z_3, \hat{\epsilon})| \hat{\epsilon} = 0, r_3 \geq 0, y_3 = \frac{1}{2}c^{-2}b\beta^2, z_3 = c^{-1}\beta\}.
\]  

(103)

Proof Straightforward calculations. In particular, we obtain \(\hat{Q}_3^{2,a}\) by setting \(r_3 = \hat{\epsilon} = 0\) in (99):

\[
\begin{align*}
\dot{y}_3 &= K_3(z_3) - 2y_3J(0) = bz_3 - 2y_3\beta^{-1}c, \\
\dot{z}_3 &= L(0) - z_3J(0) = 1 - z_3\beta^{-1}c,
\end{align*}
\]  

(104)

using (100), after division of the right hand side by \(y_3 \geq 0\). Solving this linear system with the initial conditions

\[
y_3(0) = 0, z_3(0) = -\chi_+,
\]

gives the parametrization in (101) by the time \(s\) in (104). Letting \(s \to \infty\) gives (102). ■

By blowing back down to chart \(\hat{y} = 1\) and the variables \((x, y, z, \hat{\epsilon})\), we realize (see Appendix B) that \(\hat{U}_3\) becomes \(\hat{U}\), as desired. Therefore, using e.g. Fenichel’s normal form along the normally hyperbolic set \(\hat{M}_3^+\) and the fact that \(\epsilon = r_3^2y_3\hat{\epsilon}\) in \((\hat{y} = 1, \kappa_3)\) along with more standard methods near \(\hat{u}_3\), it is then possible to guide the image \(\hat{L}_{3,\text{out}}\) along \(\hat{Q}_3^{1,a}, \hat{Q}_3^{2,a}\) and finally \(\hat{U}_3\) for \(0 < \epsilon \ll 1\) and then upon blowing back down, complete the proof of Theorem 1 in case (a). We omit the simple, but lengthy details.

5.5 Case (b)

We now turn our attention to case (b) of Definition 6 starting at \(\hat{L}_{3,\epsilon}(\Pi_{in}) \subset \{\hat{y} = -\delta^{-1}\}\). For this we note that \(\hat{y}\) decreases along \(\hat{Q}_3^{1,b}\backslash\{\hat{y} = h(-\chi_+)\}\), see (98) and (99). Therefore to follow \(\hat{Q}_3^{1,b}\) forward we move to chart \((\hat{y} = -1, \kappa_3)\) using the coordinate change in (85).

Chart \((\hat{y} = -1, \kappa_3)\)

Here we obtain the following equations from (52):

\[
\begin{align*}
\dot{r}_3 &= -r_3y_3\Psi(\hat{\epsilon}), \\
\dot{y}_3 &= -y_3(\Delta_3(z_3, \hat{\epsilon}) - 2y_3\Psi(\hat{\epsilon})), \\
\dot{z}_3 &= -y_3(\Omega(\hat{\epsilon}) - z_3\Psi(\hat{\epsilon})), \\
\dot{\hat{\epsilon}} &= \hat{\epsilon}\Delta_3(z_3, \hat{\epsilon}),
\end{align*}
\]  

(105)
where
\[
\Psi(\hat{\epsilon}) = \frac{1}{2} \beta^{-1} c(1 + \phi(-\hat{\epsilon})) - \frac{1}{2} (1 - \phi(-\hat{\epsilon})),
\]
\[
\Delta_3(z_3, \hat{\epsilon}) = \frac{1}{2} b z_3 (1 + \phi(-\hat{\epsilon})) - \frac{1}{2} \beta (1 - \phi(-\hat{\epsilon})),
\]
\[
\Omega(x, z, \hat{\epsilon}) = \frac{1}{2} (1 + \phi(-\hat{\epsilon})) + \frac{1}{2} b^{-1} \gamma (1 - \phi(-\hat{\epsilon})),
\]

so that
\[
\Psi(0) = -1, \quad \Delta(z_3, 0) = -\beta, \quad \Omega(0) = b^{-1} \gamma,
\]
using that \(\phi_-(0) = -1\). In the \((\bar{y} = -1, \kappa_3)\)-chart, \(\hat{Q}^{1,b}_3\) from \((\bar{\epsilon} = 1, \kappa_3)\), see (98), becomes
\[
\hat{Q}^{1,b}_3 : r_3 = y_3 = 0, \quad z_3 = -\chi_+, \quad \text{and } \hat{\epsilon} \in [h(-\chi_+)^{-1}, 0] \text{ for } h(-\chi_+) < 0,
\]
or \(\hat{\epsilon} \in (-\infty, 0] \text{ for } h(-\chi_+) \geq 0,
\]
(Extending it to \(\hat{\epsilon} = 0\)). Notice that \(\hat{Q}^{1,b}_3\) from \((\bar{\epsilon} = 1, \kappa_3)\) is only partially visible in the \((\bar{y} = -1, \kappa_3)\)-chart for \(h(-\chi_+) \geq 0\). But as promised, we will continue to use the same symbol for this object in the new chart.

**Lemma 8** The set
\[
\hat{M}^-_3 : r_3 \geq 0, \quad y_3 = 0, \quad z_3 \in \mathbb{R}, \quad \hat{\epsilon} = 0,
\]
is a set of critical points of (105) of saddle-type. The linearization about any point \((r_3, y_3, z_3, \hat{\epsilon}) \in \hat{M}^-_3\) has only two non-zero eigenvalues
\[
\pm \beta,
\]
with corresponding eigenvectors:
\[
\begin{pmatrix}
-r_3 \\
-\beta \\
z_3 + b^{-1} \gamma \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]
respectively. Let
\[
\hat{q}^{1,b}_3 = (0, 0, -\chi_+, 0) \in \hat{M}^-_3 \cap \{r_3 = 0\}. \tag{107}
\]
Then \(\hat{Q}^{1,b}_3\) is the stable manifold of \(\hat{q}^{1,b}_3\). On the other hand, the unstable manifold of \(\hat{q}^{1,b}_3\) is
\[
\hat{Q}^{2,b}_3 = \left\{(r_3, y_3, z_3, \hat{\epsilon}) | r_3 = \hat{\epsilon} = 0, \quad z_3 = -b^{-1} \left(\gamma - (\gamma - b \chi_+) \sqrt{-2 \beta^{-1} y_3 + 1}\right), \quad y_3 \leq 0 \right\}. \tag{108}
\]

tangent to \((0, -\beta, -\chi_+ + \gamma, 0)^T\) at \(\hat{q}^{1,b}_3\).
Proof. Straightforward calculations. In particular, we obtain $\hat{Q}^{2,b}_3$ by setting $r_3 = \hat{\epsilon} = 0$ in (99):

$$
\begin{align*}
\dot{y}_3 &= \Delta_3(z_3, 0) - 2y_3\Psi(0) = -\beta + 2y_3, \\
\dot{z}_3 &= \Omega(0) - z_3\Psi(0) = b^{-1}\gamma + z_3,
\end{align*}
$$

(109)

using (106), after division of the right hand side by $-y_3 \geq 0$. Solving this linear system with the initial conditions

$$
y_3(0) = 0, \quad z_3(0) = -\chi_+,
$$

gives the parametrization in (108) upon elimination of time.

$\hat{y}_3$ decreases unboundedly along $\hat{Q}^{2,b}_3$ in (108). Working in chart $(\bar{y} = -1, \kappa_2)$, we can follow $\hat{Q}^{2,b}$ into chart $(\bar{y} = -1, \kappa_1)$.

Chart $(\bar{y} = -1, \kappa_1)$

In this chart, $\hat{Q}^{2,b}$ becomes

$$
\hat{Q}^{2,b}_1 = \left\{ (r_1, y_1, z_1, \hat{\epsilon}) | r_1 = \hat{\epsilon} = 0, \quad z_1 = -b^{-1}\left(\gamma - (\gamma - b\chi_+)\sqrt{2\beta^{-1}y_1 + 1}\right), \quad y_1 \leq 0 \right\},
$$

(110)

(Extending it to $y_1 = 0$). We then have

Lemma 9. In chart $(\bar{y} = -1, \kappa_1)$, the set

$$
\hat{M}^-_1 : \quad r_1 \geq 0, \quad y_1 = 0, \quad z_1 \in \mathbb{R}, \quad \hat{\epsilon} = 0,
$$

is a set of critical points of (105) of saddle-type. The linearization about any point $(r_1, y_1, z_1, \hat{\epsilon}) \in \hat{M}^-_1$ has only two non-zero eigenvalues

$$
\pm \beta,
$$

with corresponding eigenvectors:

$$
\begin{pmatrix}
\beta r_1 \\
-\beta \\
z_1 - b^{-1}\gamma \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix},
$$

respectively. Let

$$
\hat{q}^{2,b}_1 = (0, 0, z^*_1, 0) \in \hat{M}^-_1 \cap \{r_1 = 0\}.
$$

(111)

Then $\hat{Q}^{2,b}_1$ (110) is the stable manifold of $\hat{q}^{2,b}_1$, tangent to $(0, -\beta, z^*_1 - b^{-1}\gamma, 0)^T$ at $\hat{q}^{2,b}_1$. On the other hand, the unstable manifold of $\hat{q}^{2,b}_1$ is

$$
\hat{Q}^{3,b}_1 = \left\{ (r_1, y_1, z_1, \hat{\epsilon}) | r_1 = y_1 = 0, \quad z_1 = z^*_1, \quad \hat{\epsilon} \in [-\zeta, 0] \right\},
$$

(112)

$\zeta > 0$ sufficiently small, tangent to $(0, 0, 0, 1)^T$ at $\hat{q}^{2,b}_1$.

$\square$
Proof In \((\bar{y} = -1, \kappa_1)\), we have from (52) that

\[
\begin{align*}
\dot{r}_1 &= r_1 y_1 \Psi(\hat{\epsilon}), \\
\dot{y}_1 &= -y_1 (\Delta_1(z_1, \hat{\epsilon}) + 2y_1 \Psi(\hat{\epsilon})), \\
\dot{z}_1 &= -y_1 (\Omega(\hat{\epsilon}) + z_1 \Psi(\hat{\epsilon})), \\
\dot{\hat{\epsilon}} &= \hat{\epsilon} \Delta_1(z_1, \hat{\epsilon}),
\end{align*}
\]

where

\[
\Delta_1(z_1, \hat{\epsilon}) = \frac{1}{2} bz_1 (1 + \phi_-(\hat{\epsilon})) + \frac{1}{2} \beta (1 - \phi_-(\hat{\epsilon}))
\]

so that

\[
\Delta_1(z_1, 0) = \beta,
\]

using that \(\phi_-(0) = -1\). We then obtain the result through simple calculations. □

Remark 1 Notice that \(\hat{Q}^{2,b}\) is a heteroclinic connection on the sphere \(\bar{q} \cap \{(\bar{y}, \bar{\epsilon}) = (-1, 0)\}\), connecting \(\hat{q}_3^{1,b}\) (107) in chart \((\bar{y} = -1, \kappa_3)\) with \(\hat{q}_1^{2,b}\) (111) in \((\bar{y} = -1, \kappa_1)\). This connection is in agreement with Lemma 5 and (70). Indeed, the linear mapping \(D\theta(0)\) in Lemma 5 maps \(v_+\) to \((-1, \chi_+)^T\). In the projective variables, \(z_3 = x^{-1}z, x > 0\) and \(z_1 = -x^{-1}z, x < 0\), this assignment becomes \(z_3 = -\chi_+ \mapsto z_1 = z_1^*,\) or simply \(\hat{q}_3^{1,b} \mapsto \hat{q}_1^{2,b}\) for \(r = 0\). □

The variable \(\hat{\epsilon} \leq 0\) decreases along \(\hat{Q}_1^{2,b}\). The extension of this manifold by the forward flow now depends, in agreement with the discussion proceeding Lemma 5, on the sign of \(z_1^*\). We postpone the details of the separate cases \(z_1^* < 0, z_1^* = 0\) and \(z_1^* > 0\) to Appendix C, Appendix D and Appendix E, respectively. The case \(z_1^* > 0\) is similar to case (a). \(z_1^* \leq 0\) is more involved due to the nonhyperbolicity of

\[
\hat{q}_1^{4,b} : r_1 = y_1 = z_1 = \hat{\epsilon} = 0,
\]

in chart \((\bar{y} = 1, \kappa_1)\). This point corresponds, using the coordinates in the chart \((\bar{y} = 1, \kappa_1)\), to the intersection of the nonhyperbolic line (of visible folds) \(\bar{l}^+\), see (51) and recall Lemma 1, with the \(\bar{x} < 0\) subset of the blown up two-fold \(\bar{q}\). We will therefore have to blowup this point in Appendix C to obtain a complete, hyperbolic, singular picture.

6 Discussion and conclusion

In this paper, we viewed the PWS visible-invisible two-fold in the normal form (20) satisfying (A) as a singular limit system of a regularized system, see (26). Restricting attention to the regularization function \(\phi(s) = \frac{2}{\pi} \arctan(s)\), and assuming a non-degeneracy condition (B), our main result, Theorem 1, then states that there is a distinguished forward trajectory \(U\) among all the candidates leaving the two-fold as the regularized system limits to the PWS one.
The non-degeneracy condition (B) is independent of the regularization function. The parameter \( \xi \) determines, together with the position of \( R_\text{in} \) in relation to the span of \( v_+ \), whether the forward flow of the regularization follows \( X_+ \) directly beyond \( q \) or whether a twist occurs where the forward orbit first follows \( X^- \) before returning to \( \Sigma^+ \) and \( X^+ \) (see cases (a) and (b) in Definition 6). The case \( \xi \in \mathbb{N} \), which are excluded by (B), are at the boundaries of these two separate cases. Here the forward flow may follow the unstable sliding region for an extended period of time.

Our approach to the problem is new, as we combine two separate blowups. The first blowup (29) resolves the singularity at \( y = 0, \epsilon = 0 \). In the blowup space we then obtain the two-fold as a circle of nonhyperbolic critical points. The second blowup (73) is a blowup in the sense of Dumortier, Roussarie, Krupa and Szmolyan [8, 9, 19] to study nonhyperbolic critical points. We blow up the circle of nonhyperbolic critical points to a circle of spheres and, by selecting appropriate weights associated with the blowup, use desingularization to gain hyperbolicity. The use of consecutive blowups can be used to study other singular perturbation phenomenon in different regularizations of piecewise smooth systems, see e.g. [16].

Our result can be extended in a number of ways. For example, the result holds true for other regularization functions, including the Sotomayor and Teixeira regularization functions, see Definition 5. This involves only minor modifications. In fact, for the Sotomayor and Teixeira regularization functions the scaling chart (31) associated with the blowup (29) is enough to prove the theorem. For \( \phi(s) = \tanh(s) \) one would have to use the method in [16] to generalise Appendix C and the blowup of \( \hat{\gamma}_{4b} \) to the exponential decay of tanh. The standard blowup approach can only generalise (118) to regularization functions with algebraic decay at \( s \to \infty \).

It is also possible to relax the assumption (A) by considering \( c - \gamma \leq \sqrt{(c - \gamma)^2 - 4b\beta} \) and still obtain the main result. In this case \( \chi^- \geq 0 \) in (17) and hence \( z_1^* > 0 \) always. In this case there is no strong canard \( \hat{\gamma}^s \) for the PWS system and any orbit of \( X_{\delta} \) is tangent to \( v_+ \) at \( q \). In contrast, for the more complicated case defined by (A), both \( z_1^* > 0 \) and \( z_1^* \leq 0 \) are possible.

Furthermore, we can also replace the piecewise linear system (20) with the full nonlinear PWS system (6). In [17], we showed (see Theorem 7.1) that if the non-degeneracy condition (B) holds then the lines \( \hat{\gamma}_{4b} \) in Lemma 3 perturb into a weak canard \( \hat{\gamma}_w(\epsilon) \) and a strong canard \( \hat{\gamma}_s(\epsilon) \), respectively, for \( \epsilon > 0 \) sufficiently small. These orbits are transverse intersections of extended versions of the (now non-unique) Fenichel slow manifolds \( \hat{S}_{a,\epsilon} \) and \( \hat{S}_{r,\epsilon} \), similar to (55) and (57). Their projections onto the \((x, z)\)-plane have tangents at \((x, z) = 0\) that are \( O(\sqrt{\epsilon}) \)-close to the eigenvectors \( v_{\pm} \).

For the regularization of (6), the strong canard tends to the unique solution of the sliding equations that are tangent to the strong eigenvector \( v_- \) at the two-fold, as \( \epsilon \to 0 \); compare Proposition 2(a). But the limit of the weak canard \( \hat{\gamma}_w(\epsilon) \) is more complicated. There is a whole funnel of singular weak canard candidates, recall Proposition 2(b). Hence a priori, for general initial conditions within the funnel, it is impossible to determine on what side of the canard the initial conditions are. This is important for the generalisation of our results to the regularization of (6). To handle this, we propose to add a condition of the form...
(D) There exists a $K > 0$ sufficiently large so that

$$\text{dist}(\Pi_{\infty}, \hat{\gamma}(\epsilon)) \geq K^{-1} > 0,$$

for all $0 < \epsilon \leq \epsilon_0$ sufficiently small.

Interestingly, condition (D) will most likely depend upon the choice of regularization function. We do not need condition (D) when we use the truncation (20) because there the weak canard $\hat{\gamma}$ is explicitly known and independent of $\epsilon$. Similar issues arise with weak canards of folded nodes in standard slow-fast systems in $\mathbb{R}^3$, see [2, 25]. The authors of [2] also (implicitly) assume a condition like (D) in their Theorem 4.1.

Our conclusions give rigorous support to the notion that evolution through visible-invisible two-fold into the escaping region for a nonzero length of time can be excluded, in general.

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A Proof of Lemma 5

ϑ is well-defined since $l^- = \{(x, y, z) \in \mathbb{R}^3 | x = y = 0\}$ is an invisible fold line. See Fig. 1. The first part of the result therefore follows from simple calculations. (70) is obtained by differentiating (69). The inequality for $z^*_1$ in (71) is obtained from (17), the positivity of $b$ and using (A). Indeed

$$b(z^*_1 - \chi-) = 2\gamma + c - \gamma = c + \gamma > \sqrt{(c - \gamma)^2 - 4b\beta} > 0$$

using (13) in the last two inequalities.

B A Lemma

Lemma 10 The forward orbit $U$ of $X^+$, becomes

$$\hat{U} = \{(x, y, z, \hat{\epsilon}) = (r_3, \frac{1}{2}r_3^2c^{-2}b\beta^2, r_3c^{-1}\beta, 0) \mid r_3 \in [0, \infty)\} \quad (114)$$

or equivalently

$$\hat{U} = \{(x, y, z, \hat{\epsilon}) = (r_2\beta^{-1}c\sqrt{2b-1}, r_2^2, r_2\sqrt{2b-1}, 0) \mid r_2 \in [0, \infty)\} \quad (115)$$

in the chart $\bar{y} = 1$.

Proof Solving (49) with $(x(0), y(0), z(0)) = q = (0, 0, 0)$ gives

$$x(t) = \beta^{-1}ct, \\
y(t) = \frac{b}{2}t^2, \\
z(t) = t,$n

for $t \geq 0$. Eliminating time by $x(t) = r_3$ we obtain (114). Similarly, setting $y(t) = r_2^2$ gives (115).

C Case (b) with $z^*_1 < 0$

Consider first chart ($\bar{\epsilon} = 1, \kappa_1$) and the equations (89). Then in the case under consideration, we have $\bar{y}' = 0$ for $\bar{y} = h(-z^*_1) \in \mathbb{R}$ within $r_1 = \epsilon_1 = 0, z_1 = z^*_1$. Therefore $\hat{Q}^{3,b}_1 : r_1 = \epsilon_1 = 0, z_1 = z^*_1, \bar{y} \in (-\infty, h(-z^*_1)]$ (extending it to $\bar{y} = h(-z^*_1)$, in the chart $(\bar{\epsilon} = 1, \kappa_1)$, is a hyperbolic fiber of

$$\hat{q}^{3,b}_1 : r_1 = \epsilon_1 = 0, z_1 = z^*_1, \bar{y} = h(-z^*_1), \quad (116)$$

belonging to the normally hyperbolic and attracting line $\hat{L}_{a,1}$ (91). On $\hat{L}_{a,1}$, we obtain a slow flow by (92)$_{\epsilon_1=0}$. Now, $z_1 = \chi_-$ is an unstable node for (92)$_{\epsilon_1=0}$. But then since $z^*_1 > \chi_-$,
recall (71), we have $z_1' > 0$ at (116). We therefore obtain the following subsequent singular orbit segment

$$Q_{1}^{4,b} : r_1 = \epsilon_1 = 0, \hat{y} = h(-z_1), z_1 \in [z_1^*, 0).$$

The variable $\hat{y}$ increases unboundedly on $Q_{1}^{4,b}$ since $z_1$ increases by the slow flow. We therefore move to the chart $(\bar{\epsilon} = 1, \kappa_1)$.

**Chart** $(\bar{\epsilon} = 1, \kappa_1)$

In this chart, the dynamics is described by (132). Within $r_1 = y_1 = 0$ we then re-discover the normally hyperbolic and attracting line of critical points

$$\hat{L}_{a,1} : r_1 = y_1 = 0, \dot{\epsilon} = h_+(-z_1), z_1 \in (-b^{-1}\beta, 0),$$

containing $Q_{1}^{4,b}$. The slow flow on $\hat{L}_{a,1}$, described by (92)$_{\epsilon_1=0}$, reaches the boundary point at $z_1 = 0$:

$$q_{1}^{4,b} : r_1 = y_1 = z_1 = \epsilon = 0,$$

in finite time. The point $q_{1}^{4,b}$ is due to the nonhyperbolicity of $\hat{t}^+$ (51) also nonhyperbolic. To describe the dynamics near $q_{1}^{4,b}$, we apply the following blowup transformation $(r_1, y_1, z_1, \dot{\epsilon}) \mapsto (r_1, \rho, (\tilde{y}_1, \tilde{z}_1, \tilde{\epsilon}))$ defined by:

$$y_1 = \rho^2 \tilde{y}_1, z_1 = \rho \tilde{z}_1, \dot{\epsilon} = \rho \tilde{\epsilon}, \rho \geq 0, (\tilde{y}_1, \tilde{z}_1, \tilde{\epsilon}) \in S^2,$$

and apply desingularization through the division of the right hand side by $\rho$. Notice that the $r_1$-equation decouples in our simplified setting and that the blowup does not involve $r_1$. We describe the blowup using the directional charts

$$(\bar{y} = 1, \kappa_{11}) : y_1 = \rho_1^2 y_{11}, z_1 = \rho_1 z_{11}, \dot{\epsilon} = \rho_1,$$

$$(\bar{y} = 1, \kappa_{12}) : y_1 = \rho_2^2, z_1 = \rho_2 z_{12}, \dot{\epsilon} = \rho_2 \tilde{\epsilon}_2,$$

obtained by setting $\tilde{\epsilon} = 1$ and $\tilde{y}_1 = 1$, respectively. We have the following coordinate change:

$$y_{11} = \tilde{\epsilon}_2^{-2}, z_{11} = z_{12} \tilde{\epsilon}_2^{-1}, \rho_2 = \rho_1 \sqrt{y_{11}},$$

for $\tilde{\epsilon}_2 > 0$. We consider each of the charts in the following.

**Chart** $(\bar{y} = 1, \kappa_{11})$

In this chart, we obtain the following equations:

$$\dot{r}_1 = -r_1 \rho_1 y_{11} J(\rho_1),$$

$$\dot{y}_{11} = 2y_{11}(K_{11}(z_{11}, \rho_1) + \rho_1 y_{11} J(\rho_1)),$$

$$\dot{z}_{11} = y_{11}(K_{11}(z_{11}, \rho_1) + \rho_1 y_{11} J(\rho_1)) + z_{11} K_{11}(z_{11}, \rho_1),$$

$$\dot{\rho}_1 = -\rho_1 K_{11}(z_{11}, \rho_1),$$

where

$$K_{11}(z_{11}, \rho_1) = b z_{11}(1 - \pi^{-1} \rho_1 (1 + \phi_2(\rho_1))) + \pi^{-1} \beta (1 + \phi_2(\rho_1)),$$

cf. (40).
Lemma 11 The unique slow manifold $S_{a,\epsilon,1}$ (see (55) for the expression of $S_{a,\epsilon,1}$ in chart $(\bar{\epsilon} = 1, \kappa_1)$) can be extended into $\kappa_{11}$ as a hyperbolic and attracting invariant manifold:

$$z_{11} = -\beta(b\pi)^{-1}(1 + l_1(\rho_1))) + y_{11}(\pi\beta^{-1} + y_{11}l_2(y_{11}, \rho_1)), \quad r_1 \in [0, \delta], \rho_1 \in [0, \mu], y_{11} \in [0, \xi],$$

(122)

for $\mu > 0$ and $\xi > 0$ sufficiently small. The intersection of (122) with the invariant sub-space $\{r_1 = \rho_1 = 0\}$:

$$z_{11} = -\beta(b\pi)^{-1} + y_{11}(\pi\beta^{-1} + y_{11}l_2(y_{11}, 0)), \quad r_1 = \rho_1 = 0, y_{11} \in [0, \xi],$$

(123)

is unique. $y_{11}$ increases on (122).

Proof The point

$$q_{11}^{4b} = \{(r_1, y_{11}, z_{11}, \rho_1)|\rho_1 = 0, y_{11} = 0, z_{11} = -\beta(b\pi)^{-1}\},$$

is partially hyperbolic, the linearization having eigenvalues $-\beta/\pi, 0, 0, 0$ and associated eigenvectors:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi^{-1}\beta \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By simple calculations we obtain (122). We recover $S_{a,\epsilon,1}$ at $\rho_1 = \mu$ and can therefore select the non-unique (122) so that it coincides with this unique manifold there. Finally, the manifold (123) is overflowing since $y_{11}' > 0$ and therefore it is unique.

Chart $(\bar{y} = 1, \kappa_{12})$

Using (121), (123) becomes

$$z_{12} = \hat{\epsilon}_2 \left(-\beta(b\pi)^{-1} + \hat{\epsilon}_2^2(\pi\beta^{-1} + \hat{\epsilon}_2^2l_2(\hat{\epsilon}_2^2, 0))\right), \quad r_1 = \rho_2 = 0, \hat{\epsilon}_2 \geq 1/\sqrt{\xi},$$

(124)

Within the invariant $\{r_1 = 0\}$-subspace, we also obtain the following equations

$$\dot{\rho}_2 = \frac{1}{2}\rho_2(K_{12}(z_{12}, \rho_2, \hat{\epsilon}_2) + 2\rho_2J(\rho_2\hat{\epsilon}_2)), \quad (125)$$

$$\dot{z}_{12} = L(\rho_2\hat{\epsilon}_2) - \frac{1}{2}z_{12}K_{12}(z_{12}, \rho_2, \hat{\epsilon}_2),$$

$$\hat{\epsilon}_2 = -\hat{\epsilon}_2 \left(\frac{3}{2}K_{12}(z_{12}, \rho_2, \hat{\epsilon}_2) + \rho_2J(\rho_2\hat{\epsilon}_2)\right),$$

where

$$K_{12}(z_{12}, \rho_2, \hat{\epsilon}_2) = bz_{12}(1 - \pi^{-1}\rho_2\hat{\epsilon}(1 + \phi_2(\rho_2\hat{\epsilon}))) + \pi^{-1}\beta\hat{\epsilon}_2(1 + \phi_2(\rho_2\hat{\epsilon}_2)).$$
Lemma 12 Consider the invariant subspace \( \{ r_1 = 0 \} \). Then the point
\[
\hat{q}_{12}^{5,b} : (r_1, \rho_2, z_{12}, \hat{\epsilon}_2) = (0, 0, \sqrt{2b^{-1}}, 0),
\] (126)
is a hyperbolic equilibrium of (125), with eigenvalues \(-3\sqrt{b}/2, -\sqrt{b} \) and \( \sqrt{b}/2 \). The set
\[
W^s(\hat{q}_{12}^{5,b}) = \{ r_1 = 0, \rho_2 = 0, (z_{12}, \hat{\epsilon}_2) \neq (-\sqrt{2b^{-1}}, 0), \hat{\epsilon}_2 \geq 0 \},
\] is the associated global 2D-stable manifold while
\[
W^u(\hat{q}_{12}^{5,b}) = \{ r_1 = 0, z_{12} = \sqrt{2b^{-1}}, \hat{\epsilon}_2 = 0, \rho_2 \geq 0 \},
\] is the associated global 1D-stable manifold.

Proof The first part of the lemma follows easily. The fact that (126) attracts all points in \( W^s \) follow from a simple phase plane analysis of the \( \{ r_1 = \rho_2 = 0 \} \)-subsystem:
\[
\dot{z}_{12} = 1 - \frac{1}{2} z_{12}(bz_{12} + \pi^{-1}\beta \hat{\epsilon}_2),
\]
\[
\dot{\hat{\epsilon}}_2 = -\hat{\epsilon}_2(bz_{12} + \pi^{-1}\beta \hat{\epsilon}_2). \tag{127}
\]
The remainder of the proof then follows from straightforward calculations.

On the unstable set \( W^u(\hat{q}_{12}^{5,b}) \), \( \rho_2 \) (and hence \( y_1 \) by (120)) increases. We can therefore return to the chart \((\bar{y} = 1, \kappa_1)\). Here \( W^u(\hat{q}_{12}^{5,b}) \) becomes
\[
\hat{Q}_{1}^{5,b} = \{ (r_1, y_1, z_1, \hat{\epsilon}) | r_1 = \hat{\epsilon} = 0, y_1 = \rho_2^2, z_1 = \rho_2 \sqrt{2b^{-1}}, \rho_2 \geq 0 \}. \tag{128}
\]
Since \( y_1 \) increases along \( \hat{Q}_{1}^{5,b} \) we then move to chart \((\bar{y} = 1, \kappa_2)\).

Chart \((\bar{y} = 1, \kappa_2)\)
In this chart we obtain the following equations
\[
\dot{x}_2 = J(\hat{\epsilon}) - \frac{1}{2} x_2 K_2(x_2, z_2, \hat{\epsilon}),
\]
\[
\dot{r}_2 = \frac{1}{2} r_2 K_2(x_2, z_2, \hat{\epsilon}),
\]
\[
\dot{z}_2 = L(\hat{\epsilon}) - \frac{1}{2} z_2 K_2(x_2, z_2, \hat{\epsilon}),
\]
\[
\dot{\hat{\epsilon}} = -\hat{\epsilon} K_2(x_2, z_2, \hat{\epsilon}), \tag{130}
\]
where
\[
K_2(x_2, z_2, \hat{\epsilon}) = \frac{1}{2} bz_2(1 + \phi_+(\hat{\epsilon})) - \frac{1}{2} \beta x_2(1 - \phi_+(\hat{\epsilon})).
\]
Notice that \( K_2(x_2, z_2, 0) = bz_2 \) for all \( z_2 \in \mathbb{R} \). We then have that
Lemma 13 In chart \((\bar{y} = 1, \kappa_2)\),
\[
\hat{Q}_{2}^{5, b} = \{ (x_2, r_2, z_2, \hat{\epsilon}) | r_2 = \hat{\epsilon} = 0, x_2 \in (-\infty, \beta^{-1}c\sqrt{2b^{-1}}], z_2 = \sqrt{2b^{-1}} \}, \tag{131}
\]
is contained within the stable manifold of the hyperbolic equilibrium
\[
\hat{u}_2 = \left( \beta^{-1}c\sqrt{2b^{-1}}, 0, \sqrt{2b^{-1}}, 0 \right).
\]

The 1D unstable manifold of \(\hat{u}_2\) is
\[
\hat{U}_2 = \{ (x_2, r_2, z_2, \hat{\epsilon}) | x_2 = \beta^{-1}c\sqrt{2b^{-1}}, z_2 = \sqrt{2b^{-1}}, r_2 \geq 0, \hat{\epsilon} = 0 \}.
\]

**Proof** Transforming \((128)\) into chart \((\bar{y} = 1, \kappa_2)\) gives
\[
\{ (x_2, r_2, z_2, \hat{\epsilon}) | r_2 = \hat{\epsilon} = 0, x_2 = -1/\rho_2, \, , z_2 = \sqrt{2b^{-1}}, \rho_2 > 0 \}.
\]
The set \(r_2 = \hat{\epsilon} = 0, z_2 = \sqrt{2b^{-1}}\) is invariant and on this line we obtain
\[
\dot{x}_2 = \beta^{-1}c - \frac{1}{2} x_2 \beta^{-1}c \sqrt{2b^{-1}} = \sqrt{\frac{\beta}{2}} \left( \beta^{-1}c \sqrt{2b^{-1}} - x_2 \right).
\]

Here \(x_2 = \beta^{-1}c\sqrt{2b^{-1}}\) is a stable node. The remainder of the proof then follows from straightforward calculations.

By blowing back down to chart \(\bar{y} = 1\) and the variables \((x, y, z, \hat{\epsilon})\), we realize (see Appendix B) that \(\hat{U}_2\) becomes \(\hat{U}\) in \((115)\), as desired. (Clearly, \(\hat{u}_2\) and \(\hat{U}_2\) coincide with \(\hat{u}_3\) and \(\hat{U}_3\) in \((102)\) and \((103)\), respectively, upon the coordinate change \((84)\).) Therefore, using standard hyperbolic methods, it is then possible to complete the proof of Theorem 1 in this case \((b)\) with \(z_1^* < 0\) and guide the image \(\hat{L}_{3, \mathrm{out}}(\Pi_{\mathrm{in}})\) along \(\hat{Q}_{1}^{4, b}, \hat{Q}_{1}^{2, b}, \hat{Q}_{3}^{1, b}, \hat{Q}_{4}^{1, b}, \hat{Q}_{5}^{1, b}\) and finally \(\hat{U}\), by working in the appropriate charts of the blowup \((73)\). We omit the simple, but lengthy details.

**D Case (b) with \(z_1^* = 0\)**

In the case under consideration, \(\hat{Q}_{1}^{3, b}\) in the chart \((\bar{y} = 1, \kappa_1)\) is asymptotic to the nonhyperbolic point
\[
\hat{q}_{1}^{4, b} : r_1 = y_1 = z_1 = \hat{\epsilon} = 0,
\]
see also \((117)\). Following the analysis for the case \(z_1^* < 0\), it is then, by working with the blowup \((118)\) and the charts \((119)\) and \((120)\), respectively, possible to connect \(\hat{Q}_{1}^{3, b}\), to \(\hat{Q}_{2}^{5, b}\), see \((131)\), in chart \((\bar{y} = 1, \kappa_2)\). The proof of Theorem 1 can then be completed in this case too. In comparison with \(z_1^* < 0\), the singular orbit for \(z_1^* = 0\) is obtained by simply removing the slow segment \(\hat{Q}_{1}^{4, b}\) from the case \(z_1^* < 0\).
E Case (b) with \( z_1^* > 0 \)

In the case under consideration, we have \( \dot{y}' > 0 \) in chart \((\bar{c} = 1, \kappa_1), \) see (89), (112) therefore becomes \( \hat{Q}_1^{3,b} : r_1 = \epsilon_1 = 0, \ z_1 = z_1^*, \ \dot{y} \in (-\infty, \infty). \) Hence we move to chart \((\bar{y} = 1, \kappa_1). \)

**Chart \((\bar{y} = 1, \kappa_1). \)**

In this chart we obtain the following equations

\[
\begin{align*}
\dot{r}_1 &= -r_1 y_1 J(\bar{\epsilon}), \\
\dot{y}_1 &= y_1 (K_1(z_1, \bar{\epsilon}) + 2y_1 J(\bar{\epsilon})), \\
\dot{z}_1 &= y_1 (L(\bar{\epsilon}) + z_1 J(\bar{\epsilon})), \\
\dot{\bar{\epsilon}} &= -\bar{\epsilon} K_1(z_1, \bar{\epsilon}),
\end{align*}
\]

where

\[
K_1(z_1, \bar{\epsilon}) = \frac{1}{2} b z_1 (1 + \phi(\bar{\epsilon})) + \frac{1}{2} \beta (1 - \phi(\bar{\epsilon})).
\]

In particular,

\[
K_1(z_1, 0) = b z_1.
\]

using that \( \phi(0) = 1. \) Similar to case (a), in particular Lemma 7, we obtain the following:

**Lemma 14** The set

\[
\hat{M}_1^+ : r_1 \geq 0, \ y_1 = 0, \ z_1 \in \mathbb{R}_+, \ \bar{\epsilon} = 0,
\]

is a set of critical points of \((99)\) of saddle-type: The linearization about any point \((r_1, y_1, z_1, \bar{\epsilon}) \in \hat{M}_1^+\) has only two non-zero eigenvalues

\[
\pm b z_1,
\]

with corresponding eigenvectors:

\[
\begin{pmatrix}
-\beta^{-1} c r_1 \\
 b z_1 \\
1 + \beta^{-1} c z_1 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix},
\]

respectively. Let

\[
\hat{q}_1^{3,b} = (0, 0, z_1^*, 0) \in \hat{M}_1^+ \cap \{r_1 = 0\}.
\]

Then \( \hat{Q}_1^{3,b} \) is the stable manifold of \( \hat{q}_1^{3,b}. \) On the other hand, the unstable manifold of \( \hat{q}_1^{3,b} \) is

\[
\hat{Q}_1^{4,b} = \left\{ (r_1, y_1, z_1, \bar{\epsilon}) | r_1 = \bar{\epsilon} = 0, \ z_1 = c^{-1} \beta \left(-1 + e^{\beta^{-1}c s} (1 + \beta^{-1} c z_1^*)\right) \right\},
\]

\[
y_1 = \frac{1}{2} c^{-2} b \beta^2 \left(1 + 2 \beta^{-1} c z_1^*\right) e^{2\beta^{-1}c s} + 1 - 2 \left(1 + \beta^{-1} c z_1^*\right) e^{\beta^{-1}c s},
\]

for \( s \in \mathbb{R}_+ \).

\[
\hat{q}_1^{3,b} \text{ tangent to } (0, b z_1^*, 1 + \beta^{-1} c z_1^*, 0)^T \text{ at } \hat{q}_1^{3,b}.
\]
Working in chart ($\bar{y} = 1, \kappa_2$) we can carry $\bar{Q}^{1,b}$ into the chart ($\bar{y} = 1, \kappa_3$). Here it is contained within the stable manifold of the hyperbolic equilibrium $\hat{u}_3$ (102). From here we obtain $\hat{U}_3$ as the $1D$ unstable manifold.

Now using standard hyperbolic methods, it is then possible to complete the proof of Theorem 1 in this case (b) with $z_1^* > 0$ and guide the image $\hat{\mathcal{L}}_{3,\text{out}}$ along $\bar{Q}^{1,b}, \bar{Q}^{2,b}, \bar{Q}^{3,b}, \bar{Q}^{4,b}$ and finally $\bar{U}$, by working in the appropriate charts. We omit the simple, but lengthy details.