A note on discrete light cone quantization

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Abstract

In this brief note we would like to discuss, in a simple model system, the conditions under which the discrete light cone quantization framework should be trusted as an approximation scheme, with regard, in particular, to the size and mass of the system. Specifically, we are going to discuss “quark-antiquark” bound states in 1+1 dim., for which a natural size is provided by analogy with a “two points and a spring” system, and show that the condition for obtaining a reliable estimate is the same as the one derived in a recent paper for black holes in matrix theory.

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The light cone frame \[ \text{[1]} \] is a choice of coordinates, which for some problems can be extremely convenient, consisting in keeping unmodified all but one spatial dimensions (called transverse) and replacing the other space coordinate (longitudinal) and the time coordinate with

\[
X^\pm = \frac{x^1 \pm x^0}{\sqrt{2}} \tag{1}
\]

The \( X^+ \) is to be thought of as a new “time” coordinate (meaning that we codify a Cauchy problem by assigning initial data on a surface \( X^+ = \text{constant} \)); the hamiltonian which describes time evolution is required to have the form

\[
H = \frac{p_-^2}{2p_-} + \frac{M^2}{2p_-} \tag{2}
\]

thus giving a natural identification of the light cone description of our (relativistic) system with another, which has non relativistic energy and a term of potential energy which turns out to be Galilean invariant and proportional to \( 1/p_- \). We want, however, to stress that light cone quantization allows to bypass the complexity and the subtleties of a quantum field theory by relating it to a system described by a non relativistic type Schrödinger equation, which makes it (in the situations in which it can be handled) very powerful. A variant of the method is the so-called discrete light cone quantization, which consists in a compactification in the lightlike (!) direction \( X^- \) over a circle of length \( 2\pi R \), thus yielding a discrete spectrum for \( p_- \):

\[
p_- = \frac{N}{R} \tag{3}
\]

Let us now turn to describe the ’t Hooft model for 1+1 dimensional “mesons” \[ \text{[2]} \]. The theory contains “quarks” and non abelian “gluons”; the gauge group is \( U(n) \) with \( n \) very large (allowing perturbative treatment in \( \frac{1}{n} \)). One chooses a description in light cone coordinates (since the system is 1+1 dim. the transverse coordinates are just absent), and, furthermore, the light cone gauge condition

\[
A_- = 0 \tag{4}
\]

decouples the ghosts and linearizes the gauge field Lagrangian. There is a subtlety concerning this gauge condition in the compactified theory. It is well known that the compactified gauge theories have gauge invariant global Wilson loop degrees of freedom which in this case have the form

\[
W = Tr P e^{i \int A_- dx^-} \tag{5}
\]

Obviously if \( W \) is not equal to 1 there is an obstruction to setting \( A_- \) to zero. However it is possible to include \( W \) in the hamiltonian with a large coefficient in such a way as to
energetically force $W$ to 1. This eliminates the Wilson loop from the dynamics and allows the choice of gauge (4).

The Lagrangian is
\begin{equation}
\mathcal{L} = -\frac{1}{2} \text{Tr}(\partial_- A_+)^2 - \bar{q}^a(\gamma_\mu \partial^\mu + m(a) + g\gamma_- A_+)q^a
\end{equation}
(following the notation of [2] it is convenient in the following to define $g_0^2 \equiv g^2 n$) and we remind that, being in 1+1 dim., the coupling constant $g$ has the dimension of a mass.

Since the $\gamma_+$, $\gamma_-$ matrices are nilpotent, the propagator of the quark is given by the expression $\frac{-i k_+}{m^2(a) + 2k_+ k_- - i\epsilon}$; the gluon propagator is $\frac{1}{k_-^2}$ and the quark-quark-gluon vertex is $-2g$. (There are no $\gamma$ matrices left). Considering the limit $n \to \infty$, $g^2 n$ fixed, the only contributions come from planar diagrams (we neglect the possible subtleties which may arise in presence of asymptotic states and lines which close at infinity, since we will be studying bound states) which have the shape of ladders with, possibly, self energy insertions for the “quark” propagator. It is possible to compute the dressed propagator for the “quark” (see [2]):

\begin{equation}
\frac{-i}{k_+ + \frac{m^2}{2k_-} - \frac{g_0^2}{2\pi} + \frac{g_0^2}{2\pi\lambda}(\text{sgn } k_-) - i\epsilon - \epsilon'}{2k_-}
\end{equation}

where $\lambda$ is an infrared cutoff: $\lambda < |k_-| < \infty$. (We won’t care about the limit $\lambda \to 0$ since the final results don’t depend on $\lambda$; let us notice, however, that the pole of the propagator when $\lambda \to 0$ is moved towards very big $k_+$. This corresponds to existence of a “confining” potential).

It is worth pointing out that eq. (7) has exactly the right form in order to identify our system with an auxiliary non relativistic one. If we consider the non relativistic amplitude for a particle which is in $x$ for $\tau = 0$ to be in $x'$ at $\tau = t$

\begin{equation}
\theta(t)\langle x|e^{i\mathcal{H}(p)t}|x'\rangle = F(x, x'; t)
\end{equation}

where $\mathcal{H}(p)$ does not depend on the time component of $p_\mu$, and Fourier-transform the matrix elements, we get the well-known Green function (if we like, the non relativistic propagator

\begin{equation}
\frac{1}{\omega - \mathcal{H}(p) + i\epsilon}
\end{equation}

If we keep in mind the interpretation of $k_+$ as the generator of the $x^+$ (“time”) translations, it is clear that the two first terms in the denominators of (4) and (8) should be identified ($\omega$ arises from the $e^{i\omega t}$ in the Fourier transform); as for $\mathcal{H}(p)$, in our context it should be a function of the $k_-$ (“space translation” generators) only. The imaginary part of the pole is simply renamed, $\epsilon' \equiv \frac{1}{2k_-}$; we don’t expect any subtleties in the limit $k_- \to 0,
because the DLCQ keeps us away from the danger and, besides, the final results obtained by means of an i. r. cutoff do not contain it, so we trust our computations all the same. Notice, incidentally, that within the non relativistic interpretation the sum of ladder diagrams becomes a computation in ordinary time independent perturbation theory for the Schrödinger equation eigenvalues/eigenfunctions in quantum mechanics.

Reference [2] computes the approximate eigenfunctions and eigenvalues for the two-particle bound state “quark-antiquark” in the regime of small quark masses:

\[ \varphi^K(x) \simeq \sin(K \pi x) \]  
\[ M_{B.S.}^2 = \pi g_0^2 K \]

where \( K \in \mathbb{N} \), \( M_{B.S.} \) mass of the \( q\bar{q} \) bound state.

Does the criterion of reference [3] for a good DLCQ approximation, namely that the extension of the system fits inside the compactification region, apply to this system?

The linear extension of the bound state is

\[ R_{B.S.} \sim \frac{M_{B.S.}}{g_0^2} \]  
\[ \text{(12)} \]

To see it, consider a quark pair separated by a distance \( L \). Gauss’ law tells us that there is an electric flux between them and that its energy density is \( g^2 n = g_0^2 \). Thus the potential energy is \( g_0^2 L \) (a wierd “linear spring” in place of the usual quadratic one). In the rest frame of the bound state, where the total energy is \( M_{B.S.} \), the energy is both kinetic and potential. The maximum size of the system is reached (classically) when all the energy is potential, that is when \( L g_0^2 = M_{B.S.} \).

The condition \( R_{B.S.} \lesssim R_{COMP} \) gives immediately, by multiplying by \( p_- = M \) and using \( R p_- = N \),

\[ N \gtrsim M_{B.S.} R_{B.S.} = \frac{M_{B.S.}^2}{g_0^2} \sim K \]  
\[ \text{(13)} \]

which is exactly the form of the condition in [3].

It is clear at once that this is the correct criterion: if we want to identify correctly which is the eigenfunction in eq. (10) (that is, if we want to resolve the quantum number of the system), we need to make sure that we are sampling at least as many Fourier components as \( K \), or the reconstruction will simply miss the oscillations of the wave function and fail altogether.

We have shown in this article that in a very controllable context, the criterion for the validity of the approximation introduced in ref. [3] is satisfied, namely that a good approximation to the properties of a system in DLCQ only requires \( N \) to be of order \( MR \). We have chosen to examine the ’t Hooft model because of its simplicity but it should not be difficult to see that the criterion applies to a wider set of examples. For example replacing the fermionic quarks by bosons should not make significant differences for states of high excitation.
Acknowledgments

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References

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