SOME IDENTITIES ON THE \(q\)-BERNSTEIN POLYNOMIALS, \(q\)-STIRLING NUMBERS AND \(q\)-BERNOULLI NUMBERS

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Abstract In this paper, we consider the \(q\)-Bernstein polynomials on \(\mathbb{Z}_p\) and investigate some interesting properties of \(q\)-Bernstein polynomials related to \(q\)-Stirling numbers and Carlitz’s type \(q\)-Bernoulli numbers.

1. Introduction

Let \(C[0,1]\) denote the set of continuous functions on \([0,1]\). Then Berstein operator for \(f \in C[0,1]\) is defined as

\[ B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \]

for \(k, n \in \mathbb{Z}_+\), where \(B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}\) is called the Bernstein polynomial of degree \(n\) (see [1, 9, 10]).

In [9], Phillips introduced the \(q\)-extension of Bernstein polynomials and Kim-Jang-Yi proposed the modified \(q\)-Bernstein polynomials of degree \(n\), which are different \(q\)-Bernstein polynomials of Phillips (see [1]).

Let \(q\) be regarded as either a complex number \(q \in \mathbb{C}\) or a \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\), then we always assume that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we normally assume that \(|1-q|_p < p^{-\frac{1}{p-1}}\), which yields the relation \(q^n = \exp(x \log q)\) for \(|x|_p \leq 1\) (see [1-11]).

Here, the symbol \(\cdot|_p\) stands for the \(p\)-adic absolute on \(\mathbb{C}_p\) with \(|1|_p = 1/p\).

Let \(p\) be a fixed prime number. Throughout this paper, the symbol \(\mathbb{Z}_p\), \(\mathbb{Q}_p\), \(\mathbb{C}\) and \(\mathbb{C}_p\) denote the ring of \(p\)-adic integers, the field of \(p\)-adic rational numbers, the complex number field, and the completion of algebraic closure of \(\mathbb{Q}_p\), respectively. Let \(N\) be the set of natural numbers and \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\} \).

2000 Mathematics Subject Classification : 11B68, 11B73, 41A30, 05A30, 65D15.

Key words and phrases : Bernstein polynomial, Bernstein operator, Bernoulli numbers and polynomials, Stirling number.
Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [4, 5, 6]).}$$

Carlitz’s $q$-Bernoulli number $\beta_{k,q}$ can be defined respectively by $\beta_{0,q} = 1$ and by the rule that $q(q\beta + 1)^k - \beta_{k,q}$ is equal to 1 if $k = 1$ and to 0 if $k > 1$ with the usual convention of replacing $\beta^i$ by $\beta_{i,q}$ (see [5]). As was shown in [5], Carlitz’s $q$-Bernoulli numbers can be represented by $p$-adic $q$-integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} [x]_q^n q^x = \beta_{n,q}, \quad n \in \mathbb{Z}_+. \quad (1)$$

The $k$-th order factorial of the $q$-number $[x]_q$, which is defined by

$$[x]_{k,q} = [x]_q[x-1]_q \cdots [x-k+1]_q = \frac{(1-q^2)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k},$$

is called the $q$-factorial of $x$ of order $k$ (see [3]). Thus, we note that $\left( \begin{array}{c} x \\ k \end{array} \right)_q = \frac{[x]_{k,q}}{[k]_q!}$.

In this paper, we consider $q$-Bernoulli polynomials on $\mathbb{Z}_p$ and we investigate some interesting properties of $q$-Bernstein polynomials related $q$-Stirling numbers and Carlitz’s $q$-Bernoulli numbers.

2. $q$-Bernstein polynomials related to $q$-Stirling numbers and $q$-Bernoulli numbers

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{m-1}}$. For $f \in UD(\mathbb{Z}_p)$, we consider $q$-Bernstein type operator on $\mathbb{Z}_p$ as follows:

$$\mathbb{B}_{n,q}(f)(x) = \sum_{k=0}^{n} f\left( \frac{k}{n} \right) \left( \begin{array}{c} n \\ k \end{array} \right)_q [x]_{k,q}[1-x]_{n-k,q} \quad (2)$$

for $k, n \in \mathbb{Z}_+$, where $B_{k,n}(x,q) = \left( \begin{array}{c} n \\ k \end{array} \right)_q^k [x]_{q}^k [1-x]_{q}^{n-k}$ is called $q$-Bernstein type polynomials of degree $n$ (see [1]).

Let $(Eh)(x) = h(x+1)$ be the shift operator. Consider the $q$-difference operator as follows:

$$\Delta_q^n = \prod_{i=1}^{n} (E - q^{i-1}I), \quad (3)$$

where $(Ih)(x) = h(x)$. From (3), we note that

$$f(x) = \sum_{n \geq 0} \left( \begin{array}{c} x \\ n \end{array} \right)_q \Delta_q^n f(0), \quad \text{(see [3]),} \quad (4)$$

where

$$\Delta_q^n f(0) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right)_q (-1)^k q(\frac{k}{n}) f(n-k). \quad (5)$$
The $q$-Stirling number of the first kind is defined by
\[\prod_{k=1}^{n} (1 + [k]_q z) = \sum_{k=0}^{n} S_1(n, k : q) z^k,\] (6)
and the $q$-Stirling number of the second kind is also defined by
\[\prod_{k=1}^{n} \frac{1}{1 + [k]_q z} = \sum_{k=0}^{n} S_2(n, k : q) z^k,\] (see [3]). (7)

By (3), (4), (5), (6) and (7), we see that
\[S_2(n, k : q) z^k = q^{-\binom{k}{2}} [k]_q! \Delta_k [0]_q^n,\] (see [1]). (8)

For $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$, we have that for $k, n \in \mathbb{Z}_+$,
\[F^{(k)}_q(t, x) = \frac{t^k e^{[1-x]_q t} [x]_q^k}{k!} \sum_{n=0}^{\infty} \binom{n+k}{n} [1-x]_q^n [n+k]_q \frac{t^{n+k}}{(n+k)!}\]
\[= \sum_{n=k}^{\infty} \binom{n}{k} [x]_q^n [1-x]_q^{n-k} \frac{t^n}{n!}\]
\[= \sum_{n=0}^{\infty} B_{k,n}(x, q) \frac{t^n}{n!},\] (see [1]).

Thus, we note that \(\frac{t^k e^{[1-x]_q t} [x]_q^k}{k!}\) is the generating function of $q$-Berstein polynomials (see [1]). It is easy to show that
\[[1-x]_q^{n-k} = \sum_{m=0}^{\infty} \sum_{l=0}^{n-k} \binom{l+m-1}{m} \binom{n-k}{l} (-1)^{l+m} q^l [x]_q^{l+m} (q-1)^m.\] (9)

By (1), (2) and (9), we obtain the following theorem.

**Theorem 1.** For $k, n \in \mathbb{Z}_+$ with $n \geq k$, we have
\[\int_{\mathbb{Z}_p} B_{k,n}(x, q) d\mu_q(x)\]
\[= \sum_{m=0}^{\infty} \sum_{l=0}^{n-k} \binom{l+m-1}{m} \binom{n-k}{l} (-1)^{l+m} q^l (q-1)^m \beta_{l+m+k,q},\]
where $\beta_{n,q}$ are the $n$-th Carlitz $q$-Bernoulli numbers.

In [3], it is known that
\[[x]_q^n = \sum_{k=0}^{n} q^{-\binom{k}{2}} \binom{x}{k} [k]_q! S_2(n, k : q),\] (10)
and
\[\sum_{k=i-1}^{n} \binom{k}{i} B_{k,n}(x, q) \frac{[x]_q + (1-x]_q^{n-i}}{([x]_q + (1-x]_q)^{n-i}} = [x]_q^i,\] for $i \in \mathbb{N}$, (see [1]). (11)
By (2) and (11), we see that for $i \in \mathbb{N}$,
\[
[x]_q^i = \sum_{m=0}^{\infty} \sum_{k=i-1}^{n} \sum_{l=0}^{m+n-k} \sum_{p=0}^{\infty} \binom{k}{n} \binom{n}{p} \binom{l+p-1}{p} \times \left( \frac{m + n - k}{l} \right) \left( \frac{n - i + m - 1}{m} \right) (-1)^{l+p+m} q^l (q-1)^p [x]_q^{n-i-m+k+p+l}.
\] (12)

By (12), we obtain the following theorem.

**Theorem 2.** For $k, n \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we have
\[
\beta_{i,q} = \sum_{m=0}^{\infty} \sum_{k=i-1}^{n} \sum_{l=0}^{m+n-k} \sum_{p=0}^{\infty} \binom{k}{n} \binom{n}{p} \binom{l+p-1}{p} \times \left( \frac{m + n - k}{l} \right) \left( \frac{n - i + m - 1}{m} \right) (-1)^{l+p+m} q^l (q-1)^p [x]_q^{n-i-m+k+p+l}.
\]

From (10) and (11), we note that
\[
\sum_{n=k}^{\infty} \binom{k}{n} B_{k,n}(x, q) (x)_q^n = \sum_{k=0}^{\infty} q^{\binom{k}{2}} \binom{n}{k} [k]_q! S_2(i, k : q).
\] (13)

In (3), it is known that
\[
\int_{\mathbb{Z}_p} \binom{x}{n} q^x \, d\mu_q(x) = \frac{(-1)^n \binom{n}{n+1} q^{(n+1)-\binom{n+1}{2}}}{n+1}.
\] (14)

By (13), (14) and Theorem 2, we have
\[
\beta_{n,q} = q \sum_{k=0}^{m} \frac{[k]_q!}{[k+1]_q} (-1)^k S_2(k, n-k : q).
\]

For $S_2(n, k : q)$, we see that
\[
S_2(n, k : q) = \frac{1}{(1-q)^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+n}{k-j} \binom{j+n}{j} q^j S_2(j, k-j : q),
\] (see [3]),

and
\[
\binom{n}{k}_q = \sum_{j=0}^{n} \binom{n}{j}_q (q-1)^{-j} S_2(k, j-k : q).
\]

By simple calculation, we show that
\[
q^{nx} = \sum_{k=0}^{n} (q-1)^k \binom{n}{k}_q [x]_{k,q}
\] (16)

and
\[
\int_{\mathbb{Z}_p} q^{nx} q^x \, d\mu_q(x) = \sum_{m=0}^{\infty} \sum_{k=m}^{n} (q-1)^k \binom{n}{k}_q S_1(k, m : q) [x]_q^m.
\]

and
\[
\int_{\mathbb{Z}_p} q^{nx} d\mu_q(x) = \sum_{m=0}^{\infty} \sum_{k=m}^{n} (q-1)^m \beta_{m,q}.
\] (17)
By (16) and (17), we see that
\[
\binom{n}{m}_q = \sum_{k=m}^{n} (q - 1)^{-m+k} \binom{n}{k}_q S_1(k, m : q).
\]
Therefore, we obtain the following theorem.

**Theorem 3.** For \(k, n \in \mathbb{Z}_+\), we have
\[
B_{k,n}(x, q) = \sum_{m=k}^{n} (q - 1)^{-k+m} \binom{n}{m}_q S_1(m, k : q)[x]^k_q[1 - x]^{n-k}_q.
\]

From the definition of the \(q\)-Stirling numbers of the first kind, we derive
\[
\binom{n}{m}_q \left[x\right]_q! = [x]_{n,q} \binom{n}{m}_q = \sum_{k=0}^{n} S_1(n, k : q)[x]^k_q.
\]  
(18)

By (13) and (18), we obtain the following theorem.

**Theorem 4.** For \(k, n \in \mathbb{Z}_+\) and \(i \in \mathbb{N}\), we have
\[
\sum_{k=0}^{n} \binom{k}{i} \frac{B_{k,n}(x, q)}{([1 - x]_q + [x]_q)^{n-1}} = \sum_{k=0}^{i} \sum_{l=0}^{k} S_1(n, l : q)S_2(i, k : q)[x]^l_q.
\]

By Theorem 2 and Theorem 4, we obtain the following result.

**Corollary 5.** For \(i \in \mathbb{N}\), we have
\[
\beta_{i,q} = \sum_{k=0}^{i} \left(\sum_{l=0}^{k} S_1(n, l : q)S_2(i, k : q)\beta_{i,q}\right),
\]
where \(\beta_{i,q}\) are the \(i\)-th Carlitz \(q\)-Bernoulli numbers.

In [3], the \(q\)-Bernoulli polynomials of order \(k\) (\(\in \mathbb{Z}_+\)) are defined by
\[
\beta_{n,q}^{(k)}(x) = \frac{1}{(1 - q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q \int_{x_p}^{x_q} \cdots \int_{x_p}^{x_q} q^{\sum_{i=1}^{k}(k-l+i)x_l}dx_d(x_1)\cdots dx_d(x_k).
\]  
(19)

From (19), we note that
\[
\beta_{n,q}^{(k)} = \frac{1}{(1 - q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q \frac{(i+k)\cdots(i+1)}{[i+k]_q\cdots[i+1]_q}x^i_q, \quad \text{see [3]}.
\]

The inverse \(q\)-Bernoulli polynomial of order \(k\) are also defined by
\[
\beta_{n,q}^{(-k)}(x) = \frac{1}{(1 - q)^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i}_q x^i q \int_{x_p}^{x_q} \cdots \int_{x_p}^{x_q} q^{\sum_{i=1}^{k}(k-l+i)x_l}dx_d(x_1)\cdots dx_d(x_k).
\]  
(20)

(see [3]). In the special case \(x = 0\), \(\beta_{n,q}^{(k)} = \beta_{n,q}^{(k)}(0)\) are called the \(n\)-th \(q\)-Bernoulli numbers of order \(k\) and \(\beta_{n,q}^{(-k)} = \beta_{n,q}^{(-k)}(0)\) are called the \(n\)-th inverse \(q\)-Bernoulli numbers of order \(k\).
By (20), we see that
\[
\beta_{k,q}^{(-n)} = \frac{1}{(1-q)^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} [j+n]_q \cdots [j+1]_q \frac{(j)^n}{(j+n)\cdots(j+1)}
\]
\[
= \frac{1}{(1-q)^k} \sum_{j=0}^{k} (-1)^j \frac{(k+n)^n}{(k+n)} \frac{(j)^n}{q^n \cdot \cdots \cdot (j+n)^n} [j+n]_q \frac{(j)^n}{(j+n)\cdots(j+1)}
\]
\[
= \frac{1}{(1-q)^k} \sum_{j=0}^{k} (-1)^j \frac{(k+n)^n}{(k+n)} \frac{(j)^n}{q^n \cdot \cdots \cdot (j+n)^n} [j+n]_q \frac{(j)^n}{(j+n)\cdots(j+1)}
\]
\[
= \frac{[n]_q!}{(k+n)^n} \sum_{j=0}^{k} (-1)^j \left( \frac{k+n}{k+n-j} \frac{(j+n)^n}{(j+n)} \right) \frac{[j+n]_q!}{q^n \cdot \cdots \cdot (j+n)^n} \{n\}.
\] (21)

From (15) and (21), we note that
\[
S_2(n, k : q) = \left( \frac{k+n}{n} \right) \frac{[n]_q!}{n!} \beta_{k,q}^{(-n)}.
\] (22)

By (13) and (22), we obtain the following theorem.

**Theorem 6.** For \( k, n \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \), we have
\[
\sum_{k=0}^{n} \binom{k}{i} \frac{B_{k,n}(x,q)}{([1-x]_q + [x]_q)^n} = \sum_{k=0}^{n} \binom{k}{i} \frac{[k]_q!}{i!} \frac{1}{x^k} \beta_{k,q}^{(-i)}.
\]

It is not difficult to show that
\[
q \binom{x}{n}_q = \frac{1}{[n]_q!} \prod_{k=0}^{n-1} ([x]_q - [k]_q)
\] (23)
\[
= \frac{1}{[n]_q!} \sum_{k=0}^{n} (-1)^k [x]_q^{n-k} S_1(n-1, k : q).
\] (24)

By Theorem 4 and (23), we obtain the following result.

**Corollary 7.** For \( k, n \in \mathbb{Z}_+ \) and \( i \in \mathbb{N} \), we have
\[
\sum_{k=0}^{n} \binom{k}{i} \frac{B_{k,n}(x,q)}{([1-x]_q + [x]_q)^n} = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j [x]_q^{n-j} S_1(k-1, j : q) \frac{[i]_q!}{i!} \beta_{k,q}^{(-i)}.
\]

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