A Hyperbolic System in a One-Dimensional Network

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Abstract. We study a coupled system of Navier-Stokes equation and the equation of conservation of mass in a one-dimensional network. The system models the blood circulation in arterial networks. A special feature of the system is that the equations are coupled through boundary conditions at joints of the network. We prove the existence and uniqueness of the solution to the initial-boundary value problem, discuss the continuity of dependence of the solution and its derivatives on initial, boundary and forcing functions and their derivatives, develop a numerical scheme that generates discretized solutions, and prove the convergence of the scheme.

1 Introduction

In this paper, we study a system of first-order quasilinear hyperbolic partial differential equations defined on one-dimensional networks. By network, we mean a finite collection of smooth curves with finitely many intersections and endpoints. The mathematical system arises from a long time study of fluid dynamical models that simulate blood flow in arterial networks (cf. [5, 8, 10, 11, 12]). Recently, the models have been used in technologies for medical diagnostics ([1, 2, 3, 4]). In particular, a technology called CANVAS, Computer-Assisted Non-invasive Vascular Analysis and Simulation, has been developed to help stroke patients. CANVAS uses data from magnetic resonance imaging to determine volumetric flow within vessels in the patient’s brain [13]. The vessel flows were used to determine the boundary conditions of the model [1]. It is based on a model formulated by Clark and Kufahl [5, 8]. The technology has displayed its capability in helping doctors predict outcomes of major medical procedures. It is the extensive applications of these models that motivate their mathematical study. Of particular importance are whether the mathematical system is well-posed (solution exists, is unique, and is stable), and whether the solutions generated by the computer algorithm really approximate the true solutions.

In this paper, we study a generalization of a model given by [10, 11, 12], prove the existence and uniqueness of the solution, prove the continuous dependence of the solution on the
Figure 1: A schematic diagram of an arterial network

initial, boundary, and forcing functions, and develop a numerical scheme that approximates
the solution.
To explain our system, let us first describe the original model of [10, 11, 12]. Suppose an
arterial network consists of \( n \) vessels. We parameterize each vessel with a spatial variable
\( x \in (0, 1) \). In the vessel, the flow of blood is governed by conservation of mass and Navier-
Stokes momentum:

\[
\frac{\partial Q_i}{\partial x} + \frac{\partial A_i}{\partial t} = 0
\]
\[
\frac{\partial Q_i}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q_i^2}{A_i} \right) = -\frac{A_i}{\rho_i} \frac{\partial P_i}{\partial x} - \frac{8\pi\mu_i Q_i}{\rho_i A_i}, \quad x \in (0, 1), \ t > 0,
\]

(1.1)

where \( Q_i \) is the flow rate, \( P_i \) is the pressure, \( A_i \) is the cross-sectional area of the vessel, and
\( \rho_i, \mu_i \) are positive constants. The initial conditions are given by

\[
P_i(0, x) = P_i^I(x), \quad Q_i(0, x) = Q_i^I(x), \quad i = 1, \ldots, n.
\]

At each end of the vessel, depending on whether it is a source, an internal junction, or a
terminal, a boundary condition is imposed. At a source end, either the pressure

\[
P_i(0, t) = P_i^B(t)
\]

(1.2)
or the flow

\[
Q_i(0, t) = Q_i^B(t)
\]

(1.3)
is specified. Various source ends may have different types of boundary conditions. At an
internal junction, suppose \( j_1, \ldots, j_\nu \) are the incoming vessels and \( j_{\nu+1}, \ldots, j_\mu \) are the outgoing
vessels to the junction. We have mass and pressure continuities at junction given by

$$\sum_{l=1}^{\nu} Q_{jl}(1, t) = \sum_{l' = \nu+1}^{\mu} Q_{j'l'}(0, t),$$

$$P_{jl}(1, t) = P_{j'l'}(0, t), \quad 1 \leq l \leq \nu, \quad \nu + 1 \leq l' \leq \mu.$$  \hfill (1.4)

At a terminal end, we may specify either the pressure,

$$P_i(1, t) = P_i^B(t),$$  \hfill (1.5)

the flow,

$$Q_i(1, t) = Q_i^B(t),$$  \hfill (1.6)

or the impedance. In the last case, the boundary condition takes the form

$$\frac{\partial P_i}{\partial t} - \eta_i \frac{\partial Q_i}{\partial t} + \delta_i P_i - \varepsilon_i Q_i = W_i^B(t), \quad x = 1,$$  \hfill (1.7)

where $\eta_i$, $\delta_i$, and $\varepsilon_i$ are positive constants and $W_i^B$ is a continuous function. This equation arises from the windkessel model of peripheral bed, which simulates the peripheral bed by a circuit that consists of a resistance $R_i^1$ in series with the parallel combination of a resistance $R_i^2$ and a capacitor $C_i$ \cite{8, 10, 12}. (See the diagram below.)

Figure 2: Electric analog of the terminal boundary condition

The resulting equation is

$$C_i \frac{\partial}{\partial t} (P_i - P_i^V) - R_i^1 C_i \frac{\partial Q_i}{\partial t} + \frac{P_i - P_i^V}{R_i^2} - \left(1 + \frac{R_i^1}{R_i^2}\right) Q_i = 0,$$

where $P_i^V$ is the venous pressure. It can be rewritten into (1.7). Again, boundary conditions for different terminals need not be the same.

Finally, the cross-sectional area $A_i$ of the $i$-th vessel is a function of $x$ and $P_i$. A particular example used in \cite{3, 8} is

$$A_i(x, P_i) = A_i^0(x) + \beta \ln \frac{P_i}{P_i^0}$$
where $\beta$ is a positive constant and $A_i^0$ is a positive function which represents the cross-sectional area at certain constant pressure $P_i^0$. This equation is used in [5, 8].

In this paper, we study a more general system which consists of the equations

$$\frac{\partial P_i}{\partial t} + a_i \frac{\partial Q_i}{\partial x} = f_i, \quad x \in (0, 1), \ t > 0 \tag{1.8}$$
$$\frac{\partial Q_i}{\partial t} + b_i \frac{\partial P_i}{\partial x} + 2c_i \frac{\partial Q_i}{\partial x} = g_i,$$

and the initial and boundary conditions described above. For convenience, we also use the vector form

$$(U_i)_t + B_i (U_i)_x = F_i \tag{1.9}$$

where $U_i = (P_i, Q_i)$, $F_i = (f_i, g_i)$ and

$$B_i = \begin{pmatrix} 0 & a_i \\ b_i & 2c_i \end{pmatrix}.$$ 

Eq. (1.1) is a special case of this system where

$$a_i = \frac{1}{(A_i)_P}, \quad b_i = \frac{A_i}{\rho_i} - \frac{Q_i^2 (A_i)_P}{A_i^2}, \quad c_i = \frac{Q_i}{A_i}, \quad f_i = 0, \quad g_i = \frac{Q_i^2 (A_i)_x}{A_i^2} - \frac{8\pi\mu_i Q_i}{OA_i}.$$ 

We do not assume any particular form of these functions though, they are general differentiable functions of $(x, t, P_i, Q_i)$. A basic assumption is $a_i > 0$. Other assumptions will follow.

This problem is interesting not only in fluid mechanics but also in mathematics. Navier-Stokes equations and conservation laws have been studied for over a century. However, rarely have any studies been conducted for systems defined in a network. Unlike the problem of fluid flow in a rigid tube network, the distensibility of vessels greatly increases the complexity of the problem. For example, as is well-known, a first-order quasilinear system of hyperbolic equations on a finite one-dimensional spatial interval needs not have a solution. Even if it has a solution for an interval of time, the solution may not exist for all time. In a network, it is important to know whether the coupling at junctions poses problems to solvability. The effect of the windkessel boundary condition (1.7) on the solvability also needs to be examined.

This paper is divided into two parts. The first part consists of sections 2 and 3. It deals with the problem of solvability using a fixed point approach. Substituting a pair of functions $(p_i, q_i)$ for $(P_i, Q_i)$ in the coefficients $a_i$, $b_i$, $c_i$ and forcing functions $f_i$, $g_i$, the system becomes linear. That is, all the functions $a_i$, etc. are independent of unknowns. If the linear system has a unique solution, then one can establish a mapping from $(p, q)$ to the linear problem solution $(P_i, Q_i)$. If one also shows that this mapping has a unique
fixed point, then the fixed point is necessarily the unique solution of the quasilinear system. Hence, we shall first give a condition for the linear system to have a unique solution, then examine under what conditions the mapping has a unique fixed point. We investigate the first aspect of the problem in Section 2 and the latter in Section 3. We also prove a result on the continuity of dependence of solutions on the initial, boundary and forcing functions for linear and quasilinear systems. Thus, we complete the analysis of the well-posedness of the problem. In the second part, which consists of Section 4 only, we give a numerical scheme that approximates the solution, and prove its convergence. Our scheme is a set of finite-difference equations based on the normal form of the differential equations. Although these approaches are standard in the analysis of quasilinear equations, the network feature of the system and the peculiarities of the boundary conditions make the problem more complicated. In the final section, we give a short discussion.

2 The linear system

In this section, we analyze (1.8) as a linear system with $a_i$, $b_i$, $c_i$, $f_i$ and $g_i$ independent of $P_i$ and $Q_i$. We give conditions for the system to have a unique global solution. The conditions are most naturally given in terms of the eigenvalues of the matrix $B_i$, which have the form

$$
\lambda_i^R = c_i + u_i, \quad \lambda_i^L = c_i - u_i,
$$

where

$$
u_i = \sqrt{c_i^2 + a_i b_i}.
$$

These eigenvalues are real if

$$
c_i^2 + a_i b_i > 0, \quad x \in (0, 1), \quad t > 0, \quad i = 1, \ldots, n. \quad (2.1)
$$

In this case,

$$
\lambda_i^R (x, t) > 0, \quad \lambda_i^L (x, t) < \lambda_i^R (x, t)
$$

and the system is hyperbolic. Under this condition, we show that the linear system has a unique solution if

$$
\lambda_i^L (0, t) < 0, \quad \lambda_i^L (1, t) < 0, \quad i = 1, \ldots, n.
$$

This is clearly equivalent to

$$
a_i b_i > 0, \quad t \geq 0, \quad i = 1, \ldots, n. \quad (2.3)
$$

at $x = 0, 1$ only. It needs not hold for $x \in (0, 1)$. 
Theorem 2.1 Assume that the functions $a_i, b_i, c_i, f_i,$ and $g_i$ are independent of $(P_i, Q_i)$. Suppose these functions and the initial and boundary functions $P_i^L, Q_i^L, P_i^B, Q_i^B$ and $W_i^B$ all have bounded first-order derivatives. Suppose also that $a_i > 0$ and that the conditions (2.1) and (2.3) hold. Then, for any $T > 0$ there is a unique solution in a bounded subset of the space $C([0,1] \times [0,T], \mathbb{R}^{2n})$ to the linear system (1.8) with the initial and boundary conditions given in Section 1.

Proof. We first show that the system has a unique solution for $0 < t < \delta$ for some $\delta > 0$. The proof is based on the method of characteristics and a fixed point principle. For systems defined on only one branch, this is a standard approach. In our case, special care is needed to handle the junction condition (1.4) and the windkessel boundary condition (1.7).

Consider the $i$-th branch. From any point $(\tau, \xi)$ on the left, right, and lower boundary of the rectangle $D =: [0,1] \times [0,T]$, we construct the left-going and right-going characteristic curves $x = x^L_i(t; \xi, \tau)$ and $x = x^R_i(t; \xi, \tau)$ by

$$\frac{dx^L_i}{dt} = \lambda^L_i(x^L_i,t), \quad x^L_i(\tau) = \xi,$$

$$\frac{dx^R_i}{dt} = \lambda^R_i(x^R_i,t), \quad x^R_i(\tau) = \xi,$$

respectively, where $\lambda^L_i$ and $\lambda^R_i$ are the two eigenvalues of the matrix $B_i$. By the uniqueness of solutions of these differential equations, a left-going (resp. right-going) characteristic curve cannot intersect with another left-going (resp. right-going) characteristic curve. Let $X^L_i$ and $X^R_i$ be the right-most left-going and left-most right-going characteristic curves:

$$x = x^L_i(t;1,0) \text{ and } x = x^R_i(t;0,0)$$

starting from the lower boundary of $D$, respectively. It can be shown from (2.2) that the two curves can have at most one intersection. Let $t_i$ be the value of $t$ at the intersection. If the two curves do not intersect in $D$, we simply define $t_i = T$. By condition (2.3), $X^L_i$ cannot reach the right vertical line $x = 1$ at any $t > 0$, and by $\lambda^R_i > 0$, $X^R_i$ cannot reach the vertical line $x = 0$ at any $t > 0$. Thus, the rectangle $D_i =: [0,1] \times [0,t_i]$ can be divided into three parts

$$D_i = D^L_i \cup D^C_i \cup D^R_i,$$

where $D^L_i$ is between the vertical line $x = 0$ and the characteristic curve $X^R_i$, $D^C_i$ is between the two characteristic curves, and $D^R_i$ is between $X^L_i$ and $x = 1$. 

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We show that there is a $\delta_i \leq t_i$ such that the solution $(P_i, Q_i)$ for the $i$-th branch exists in the restriction of $D_i$ to the strip $\{0 \leq t \leq \delta_i\}$.

We first observe that the initial conditions alone determine the solution completely in the central region $D^C_i$. This follows from the theory of first-order linear hyperbolic systems and the fact that from any point $(x, t) \in D^C_i$, the two characteristic curves, followed backwards, must land on the horizontal line $t = 0$. (The latter is a consequence of (2.2).) To extend the solution to other parts of $D_i$, we make a change of unknowns and derive a set of integral equations. Note that $l^R_i := (\lambda^L_i a_i, a_i)$ and $l^L_i := (\lambda^R_i a_i, a_i)$ are the left eigenvectors of $B_i$ corresponding to $\lambda^R_i$ and $\lambda^L_i$, respectively. Introduce new unknowns

$$r_i = l^R_i U_i \equiv -\lambda_i^L P_i + a_i Q_i, \quad s_i = l^L_i U_i \equiv -\lambda_i^R P_i + a_i Q_i. \quad (2.4)$$

The system (1.8) can be written in terms of $r_i$ and $s_i$ by multiplying the left eigenvectors to (1.9) and substituting in

$$P_i = \frac{1}{2u_i} (r_i - s_i), \quad Q_i = \frac{1}{2u_i a_i} (\lambda_i^R r_i - \lambda_i^L s_i). \quad (2.5)$$

This results in the equations

$$\partial_t^R r_i = F_i^R (x, t, r_i, s_i), \quad \partial_t^L s_i = F_i^L (x, t, r_i, s_i), \quad (2.6)$$

where

$$\partial_t^R = \frac{\partial}{\partial t} + \lambda_i^R \frac{\partial}{\partial x}, \quad \partial_t^L = \frac{\partial}{\partial t} + \lambda_i^L \frac{\partial}{\partial x}, \quad (2.7)$$

and

$$F_i^R (x, t, r_i, s_i) = l_i^R F_i + (\partial_t^R l_i^R) U_i, \quad F_i^L (x, t, r_i, s_i) = l_i^L F_i + (\partial_t^L l_i^L) U_i. \quad (2.8)$$

(A differential operator acting on a vector means that it acts on each component of the vector.) Let $(x, t) \in D_i$. We integrate the first equation of (2.6) along the right-going characteristic curve $x^R (t; \xi, \tau)$ which passes through $(x, t)$ and reaches the left or lower
boundary of \( D_i \) at \((\xi, \tau)\). It can be shown that for \((x, t) \in D_i^C \cup D_i^R \), \(\tau = 0\), and for \((x, t) \in D_i^L \), \(\xi = 0\). In the former case, we obtain
\[
  r_i (x, t) = r_i^L (\xi) + \int_0^t F_i^R (x_i^R (t'; \xi, 0), t', r_i, s_i) \, dt'
\]
(2.9)

In the latter case, we have
\[
  r_i (x, t) = r_i (0, \tau) + \int_\tau^t F_i^R (x_i^R (t'; 0, \tau), t', r_i, s_i) \, dt'.
\]
(2.10)

Similarly, by integrating the second equation of (2.6) along the left-going characteristic curve
\( x_i^L (t; \xi, \tau) \) that passes through both \((x, t)\) and \((\xi, \tau)\) (which is on either the right or lower boundary of \( D_i \)), the equations are
\[
  s_i (x, t) = s_i^L (\xi) + \int_0^t F_i^L (x_i^L (t'; \xi, 0), t', r_i, s_i) \, dt'
\]
(2.11)

if \((x, t) \in D_i^L \cup D_i^C \) and
\[
  s_i (x, t) = s_i (1, \tau) + \int_\tau^t F_i^L (x_i^L (t'; 1, \tau), t', r_i, s_i) \, dt'
\]
(2.12)

if \((x, t) \in D_i^R \). These are the integral equations we need.

For any \(\delta_i \leq t_i\) we use \( D_i^{L,\delta_i} \), \( D_i^{C,\delta_i} \) and \( D_i^{R,\delta_i} \) to denote the restrictions of \( D_i^L \), \( D_i^C \) and \( D_i^R \) to the strip \( \{0 \leq t \leq \delta_i\} \), respectively. We first extend the solution to a left region \( D_i^{L,\delta_i} \), where \(\delta_i\) is to be determined. For this, we need the boundary condition on the left end of the branch. The left end is either a source or a junction. For a source with the boundary condition (1.2), we define \( \hat{s}_i = s_i/\varepsilon \) where \(\varepsilon < 1\) is any constant. Using the first equation of (2.3) in the integral equations (2.10) and (2.11),
\[
  \left( \begin{array}{c}
  r_i (x, t) \\
  \hat{s}_i (x, t)
  \end{array} \right) = \left( \begin{array}{c}
  2u_i (0, \tau) \, P_i^B (\tau) + \varepsilon \hat{s}_i (0, \tau) + \int_\tau^t F_i^R (x_i^R (t'; 0, \tau), t', r_i, \varepsilon \hat{s}_i) \, dt' \\
  \frac{1}{\varepsilon} s_i^L (\xi) + \frac{1}{\varepsilon} \int_0^t F_i^L (x_i^L (t'; \xi, 0), t', r_i, \varepsilon \hat{s}_i) \, dt'
  \end{array} \right)
\]
(2.13)

This is a fixed point equation for \((r_i, \hat{s}_i)\) if we define the right hand side as a mapping of an operator \( K \) on \((r_i, \hat{s}_i)\) in a bounded subset of \( C(D_i^{L,\delta_i} \cup D_i^{C,\delta_i}, \mathbb{R}^2) \). In a standard approach, it can be shown that \( K \) is a contraction mapping if \(\delta_i\) is sufficiently small. Hence, the fixed point exists and is unique. Therefore, the solution \((r_i, s_i)\) can be uniquely extended to \( D_i^{L,\delta_i} \cup D_i^{C,\delta_i} \).
For a source with the boundary condition \((1.3)\), we define \(\hat{s}_i = s_i/\varepsilon\), where \(\varepsilon > 0\) and is so small such that

\[
\varepsilon \left| \frac{\lambda^L_i (0, \tau)}{\lambda^R_i (0, \tau)} \right| < 1, \quad \tau \in (0, t_i).
\]

The fixed point equation is then

\[
\begin{pmatrix}
  r_i (x, t) \\
  \hat{s}_i (x, t)
\end{pmatrix} = \begin{pmatrix}
  2a_i u_i (0, \tau) \lambda^L_i (0, \tau) Q_i^B (\tau) + \frac{\lambda^L_i (0, \tau)}{\lambda^R_i (0, \tau)} \varepsilon \hat{s}_i (0, \tau) + \int_{\tau}^{t} F_i^R \left( x_i^R (t', 0, \tau), t', r_i, \varepsilon \hat{s}_i \right) dt' \\
  \frac{1}{\varepsilon} s_i^l (\xi) + \frac{1}{\varepsilon} \int_{0}^{t} F_i^L \left( x_i^L (t', \xi, 0), t', r_i, \varepsilon \hat{s}_i \right) dt'
\end{pmatrix}.
\]

By a similar argument, the solution can again be uniquely extended.

If the left end of the branch is a junction, we shall extend the solution on all the branches that are connected to the same junction simultaneously. Thus, also extend the solution to \(D^R_{i, \delta_i}\), on the branches incoming to the junction. Let \(j_1, j_2, \ldots, j_\nu\) be the incoming and \(j_{\nu+1}, \ldots, j_\mu\) the outgoing branches to the junction. Equations \((1.4)\) and \((2.5)\) give rise to a \(2 \mu \times \mu\) homogenous system of linear equations for \(r_i (1, \tau), s_i (1, \tau), i = j_1, \ldots, j_\nu\) and \(r_i (0, \tau), s_i (0, \tau), i = j_{\nu+1}, \ldots, j_\mu\):

\[
\begin{align*}
\frac{1}{u_i(1, \tau)} (r_1 (1, \tau) - s_1 (1, \tau)) - \frac{1}{u_i(1, \tau)} (r_i (1, \tau) - s_i (1, \tau)) &= 0, \quad i = j_2, \ldots, j_\nu, \\
\frac{1}{u_i(1, \tau)} (r_1 (1, \tau) - s_1 (1, \tau)) - \frac{1}{u_i(0, \tau)} (r_i (0, \tau) - s_i (0, \tau)) &= 0, \quad i = j_{\nu+1}, \ldots, j_\mu, \\
\sum_{l=1}^\nu \frac{1}{u_{jl}^*} (\lambda^R_{jl} r_{jl} - \lambda^L_{jl} s_{jl}) (1, \tau) - \sum_{l'=\nu+1}^\mu \frac{1}{u_{jl}^*} \lambda^R_{jl'} s_{jl'} (0, \tau) &= 0.
\end{align*}
\]

This system can be solved for \(s_{j_1} (1, \tau), \ldots, s_{j_\nu} (1, \tau), r_{j_{\nu+1}} (0, \tau), \ldots, r_{j_\mu} (0, \tau)\) because the coefficient matrix

\[
\begin{pmatrix}
  -\frac{1}{u_{j_1}(1, \tau)} & \frac{1}{u_{j_2}(1, \tau)} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  -\frac{1}{u_{j_1}(1, \tau)} & 0 & \cdots & \frac{1}{u_{j_\mu}(0, \tau)} \\
  -\frac{1}{u_{j_1} a_{j_1}(1, \tau)} & -\frac{\lambda^L_{j_1}}{u_{j_2} a_{j_2}(1, \tau)} & \cdots & -\frac{\lambda^L_{j_\mu}}{u_{j_\mu} a_{j_\mu}(0, \tau)}
\end{pmatrix}
\]

has the determinant

\[
(-1)^{\nu+1} \prod_{l=1}^\nu a_{jl} (1, \tau) \prod_{l'=\nu+1}^\mu u_{jl'} (0, \tau) \left( -\sum_{l=1}^\nu \frac{\lambda^L_{jl}}{a_{jl} (1, \tau)} + \sum_{l'=\nu+1}^\mu \frac{\lambda^R_{jl'}}{a_{jl'} (0, \tau)} \right).
\]

Since \(\lambda^L_j < 0 < \lambda^R_j\) at the junction, the determinant is not zero. Hence, we can express \(s_{j_1} (1, \tau), \ldots, s_{j_\nu} (1, \tau), r_{j_{\nu+1}} (0, \tau), \ldots, r_{j_\mu} (0, \tau)\) in terms of other unknowns as

\[
s_i (1, \tau) = \sum_{l=1}^\nu m^L_{jl} (\tau) r_{jl} (1, \tau) + \sum_{l'=\nu+1}^\mu m^L_{jl'} (\tau) s_{jl'} (0, \tau), \quad i = j_1, \ldots, j_\nu,
\]
for some functions \( m_j^i \), \( n_j^i \). Choose an \( \varepsilon > 0 \) such that

\[
\varepsilon \max \left\{ \sum_{l=1}^{\mu} |m_j^i(\tau)|, \sum_{l=1}^{\mu} |n_j^i(\tau)| \right\} < 1, \quad i = j_1, \ldots, j_\mu, \quad \tau \in [0, t_i]
\]

and introduce

\[
\dot{r}_{ji} = \frac{r_{ji}}{\varepsilon}, \quad \dot{s}_{ji} = \frac{s_{ji}}{\varepsilon}, \quad l = 1, \ldots, \nu, \quad l' = \nu + 1, \ldots, \mu.
\]

Then, from (2.9)–(2.12), the integral equations for the 2\( \mu \) unknowns \( \dot{r}_{ji}, s_{ji}, r_{ji'}, s_{ji'} \), \( l = 1, \ldots, \nu, \quad l' = \nu + 1, \ldots, \mu \) constitute a fixed point equation,

\[
w = \left( \dot{r}_{j_1}, \ldots, \dot{r}_{j_\nu}, s_{j_1}, \ldots, s_{j_\nu}, r_{j_{\nu+1}}, \ldots, r_{j_\mu}, \dot{s}_{j_{\nu+1}}, \ldots, \dot{s}_{j_\mu} \right)
\]

and

\[
Kw = \left( \frac{1}{\varepsilon} \int_0^t F_j^R \left( x_{j_1}, t', \varepsilon \dot{r}_{j_1}, s_{j_1} \right) dt', \ldots, \\
\varepsilon \left( \sum_{k=1}^{\mu} m_{j_k}^1 \dot{r}_{j_k} (1, \tau) + \sum_{k'=\nu+1}^{\mu} m_{j_k'}^1 \dot{s}_{j_k'} (0, \tau) \right) + \int_0^t F_j^L \left( x_{j_1}, t', \varepsilon \dot{r}_{j_1}, s_{j_1} \right) dt', \ldots, \\
\varepsilon \left( \sum_{k=1}^{\mu} n_{j_k}^1 \dot{r}_{j_k} (1, \tau) + \sum_{k'=\nu+1}^{\mu} n_{j_k'}^1 \dot{s}_{j_k'} (1, \tau) \right) + \int_0^t F_{j_{\nu+1}}^R \left( x_{j_{\nu+1}}, t', \varepsilon \dot{s}_{j_{\nu+1}}, s_{j_{\nu+1}} \right) dt', \ldots, \\
\frac{1}{\varepsilon} s_{j_{\nu+1}}^1 \left( \xi_{j_{\nu+1}} \right) + \frac{1}{\varepsilon} \int_0^t F_{j_{\nu+1}}^L \left( x_{j_{\nu+1}}, t', r_{j_{\nu+1}}, \varepsilon \dot{s}_{j_{\nu+1}} \right) dt', \ldots \right).
\]

It can be shown by a standard argument that \( K \) is a contraction mapping in the space

\[
X_j =: \prod_{l=1}^{\nu} C \left( D^{C, \delta_j}_{j_l, \delta_j} \cup D^{R, \delta_j}_{j_l, \delta_j}, \mathbb{R}^2 \right) \times \prod_{l=\nu+1}^{\mu} C \left( D^{L, \delta_j}_{j_l, \delta_j} \cup D^{R, \delta_j}_{j_l, \delta_j}, \mathbb{R}^2 \right)
\]

if \( \delta_j \) is sufficiently small. Hence, it has a unique fixed point in \( X_j \). This extends the solution \((r_i, s_i)\) for the neighboring branches of the junction.

We now extend the solution \((r_i, s_i)\) to a right region \( D^{R, \delta_j}_{j_i, \delta_j} \). This has been done if the right end is a junction. Thus, only terminal ends need to be discussed. For the boundary condition of either (1.3) or (1.6) type, the argument is similar to the above discussion about source ends. We only sketch the steps in these two cases. The boundary condition of (1.7) type, however, requires more effort.

If condition (1.3) is assumed, then, by (2.3),

\[
s_i (1, t) = r_i (1, t) - 2u_i P_i^B (t).
\]

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Let \( \hat{r}_i = r_i/\varepsilon \) with \( 0 < \varepsilon < 1 \). Then, the fixed point equation for \((\hat{r}_i, s_i)\) has the form

\[
\begin{pmatrix}
\hat{r}_i (x, t) \\
s_i (x, t)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\varepsilon} r_i^L(\xi) + \frac{1}{\varepsilon} \int_0^t F_i^R (t', x_i^R (t'; \xi, 0), \varepsilon \hat{r}_i, s_i) \, dt' \\
\varepsilon \hat{r}_i (1, \tau) - 2u_i (1, \tau) P_i^R (\tau) + \int_\tau^t F_i^L (t', x_i^L (t'; 1, \tau), \varepsilon \hat{r}_i, s_i) \, dt'
\end{pmatrix}.
\]

As before, the mapping defined by the right hand side is contractive if \( \delta_i \) is small enough. Hence, the solution is uniquely extended into \( D_{i, \delta_i}^R \). If condition (1.6) is assumed, we find again from (2.5) that

\[
\lambda_i^L s_i (1, t) = \lambda_i^R r_i (1, t) - 2u_i (1, t) a_i (1, t) Q_i^B (t).
\]

Since \( \lambda_i^L (1, t) < 0 \), the equation can be uniquely solved for \( s_i \). Choose \( \varepsilon > 0 \) sufficiently small such that

\[
\varepsilon \frac{\lambda_i^R (1, \tau)}{\lambda_i^L (1, \tau)} < 1 \quad \text{for} \quad \tau \in [0, t_i]
\]

and let \( \hat{r}_i = r_i/\varepsilon \). The fixed point equation for \((\hat{r}_i, s_i)\) has the form

\[
\begin{pmatrix}
\hat{r}_i (x, t) \\
s_i (x, t)
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\varepsilon} r_i^L(\xi) + \frac{1}{\varepsilon} \int_0^t F_i^R (x_i^R (t'; \xi, 0), t', \varepsilon \hat{r}_i, s_i) \, dt' \\
\lambda_i^L (1, \tau) \varepsilon \hat{r}_i (1, \tau) - 2a_i u_i (1, \tau) + 2a_i u_i (1, \tau) - \int_\tau^t F_i^L (x_i^L (t'; 1, \tau), t', \varepsilon \hat{r}_i, s_i) \, dt'
\end{pmatrix}.
\]

Again, the mapping is contractive in a bounded subset of \( C \left( D_{i, \delta_i}^C \cup D_{i, \delta_i}^R, \mathbb{R}^2 \right) \) if \( \delta_i \) is sufficiently small. The solution is thus, uniquely extended to \( D_{i, \delta_i}^R \).

In the case where the boundary condition (1.7) is assumed, we integrate it with respect to \( t \) to obtain

\[
(P_i - \eta_i Q_i) (1, t) = (P_i^I - \eta_i Q_i^I) (1) + \int_0^t (W_i^R (t') - \delta_i P_i (1, t') + \varepsilon_i Q_i (1, t')) \, dt'.
\]

Substituting (2.5) into this equation, we can write

\[
m_i (t) r_i (1, t) - n_i (t) s_i (1, t) = m_i (0) r_i^I (1) - n_i (0) s_i^I (1) + \int_0^t H_i (t', r_i (1, t'), s_i (1, t')) \, dt'
\]

where

\[
m_i (t) = \frac{a_i (1, t) - \eta_i \lambda_i^R (1, t)}{2a_i u_i (1, t)}, \quad n_i (t) = \frac{-a_i (1, t) + \eta_i \lambda_i^L (1, t)}{2a_i u_i (1, t)}
\]

and

\[
H_i (t, r, s) = W_i^R (t) + \frac{\varepsilon_i \lambda_i^R (1, t) - \delta_i a_i (1, t)}{2a_i u_i (1, t)} r - \frac{\varepsilon_i \lambda_i^L (1, t) - \delta_i a_i (1, t)}{2a_i u_i (1, t)} s.
\]
Since \( a_i > 0, u_i > 0, \eta_i > 0 \) and \( \lambda_i^L(1, t) < 0 \), it follows that \( n_i(t) < 0 \). Hence, there exists \( \varepsilon > 0 \) such that
\[
\varepsilon \left( \frac{m_i(\tau)}{n_i(\tau)} \right) < 1 \quad \text{for} \quad \tau \in [0, t_i].
\]

Let \( \hat{r}_i = r_i/\varepsilon \). The integral equations for \( \hat{r}_i \) and \( s_i \) then have the form
\[
\dot{r}_i(x,t) = \frac{1}{\varepsilon} r_i^L(\xi) + \frac{1}{\varepsilon} \int_0^t F_i^R(x_i^R(t', \xi), t', \varepsilon\hat{r}_i, s_i) \, dt',
\]
\[
s_i(x,t) = \varepsilon \frac{m_i(\tau)}{n_i(\tau)} \hat{r}_i(1, \tau) - \frac{1}{n_i(\tau)} \left( M_i + \int_0^t H_i(t', \varepsilon\hat{r}_i(1, t'), s_i(1, t')) \, dt' \right)
+ \int_0^t F_i^L(x_i^L(t', 1, \tau), t', \varepsilon\hat{r}_i, s_i) \, dt',
\]

where \( M_i = m_i(0) r_i^L(1) - n_i(0) s_i^L(1) \) is a constant. The extension of the solution to \( D_{i,\delta} \) is thus, guaranteed.

Finally, if we let \( \delta \) be the minimum of all \( \delta_i \) occurring above, we see that \( \delta > 0 \) and the solution exists and is unique in \( (x,t) \in D_\delta =: [0,1] \times [0,\delta] \). Observe that \( \delta \) depends only on the bounds of the system functions \( a_i, \) etc., the initial and boundary functions \( P_i^L, \) etc., and their first-order derivatives in \( D = [0,1] \times [0,T] \). Hence, it is independent of \( t \), and we can extend the solution successively in the time intervals \( [0,\delta], [\delta,2\delta], \) etc. In this way, the solution is obtained in \( D \) in finitely many steps. ■

It can be seen from the above proof that the linear system needs not have a solution if condition (2.3) fails at any end point of a branch. In the quasilinear case, since \( a_i \) and \( b_i \) depend on the unknowns \( P_i \) and \( Q_i \), this condition may fail at a future moment. Therefore the solution does not generally exist for all time.

We next derive an estimate of the deviation of solution in terms of the deviations of the initial, boundary and forcing functions. This estimate is needed in the next section. For any vector function \( v = (v_1, \ldots, v_k) \) defined in \( C(X; \mathbb{R}^k) \), we use \( |v|_X \) to denote the norm \( \max_i \{ |v_i|_{C(X)} \} \), where \( X \) represents a closed subset of either \( \mathbb{R} \) or \( \mathbb{R}^2 \).

**Lemma 2.1** Let \( U = (P,Q) \) and \( \tilde{U} = (\tilde{P}, \tilde{Q}) \) be two solutions of the linear problem (1.9) with different initial, boundary, and forcing functions. Suppose the conditions of Theorem 2.1 hold for both solutions. Then, there exists a constant \( M > 0 \), independent of initial, boundary and forcing functions, such that
\[
|U - \tilde{U}|_{C(D_\delta)} \leq M \left( \left| P^I - \tilde{P}^I \right|_{C[0,1]} + \left| Q^I - \tilde{Q}^I \right|_{C[0,1]} + \left| P^B - \tilde{P}^B \right|_{C[0,\delta]} + \left| Q^B - \tilde{Q}^B \right|_{C[0,\delta]} + \delta \left| f - \tilde{f} \right|_{C(D_\delta)} + \delta \left| g - \tilde{g} \right|_{C(D_\delta)} + \delta \left| W - \tilde{W} \right|_{C[0,\delta]} \right). \tag{2.20}
\]
Proof. We need only prove \((2.20)\) for a \(\delta \leq \min_i \{\delta_i\}\), where \(\delta_i\) represents the constants occurring in the proof of Theorem \(2.1\). This is because for larger \(\delta\), we can divide the interval \([0, \delta]\) into subintervals, each has a length less than \(\min_i \{\delta_i\}\), and apply \((2.20)\) in each subinterval. We can then take the maximum on each side of the inequalities to derive the inequality of in \([0, \delta]\). In the sequel, \(D^C_\delta, D^L_\delta\) and \(D^R_\delta\) are the restrictions of \(D^C_i, D^L_i\) and \(D^R_i\) to the strip \(\{0 \leq t \leq \delta\}\), respectively.

By linearity, \(U - \hat{U}\) is the solution of the system with the initial, boundary and forcing functions \(P^I_i - \hat{P}^I_i, Q^I_i - \hat{Q}^I_i, P^B_i - \hat{P}^B_i, Q^B_i - \hat{Q}^B_i, W^B_i - \hat{W}^B_i, f_i - \hat{f}_i\) and \(g_i - \hat{g}_i\). Let \(r_i, \hat{r}_i, s_i, \hat{s}_i\) be defined as in the proof of Theorem \(2.1\), corresponding to \(U - \hat{U}\). We show that these quantities have upper bounds in the form of the right hand side of \((2.20)\) in \(D^C_\delta, D^L_\delta\) and \(D^R_\delta\).

In \(D^C_\delta\), \((2.9)\) and \((2.11)\) hold. Notice that the functions \(F^R_i\) and \(F^L_i\) are linear in \(r_i\), and \(s_i\). Hence, there exists a constant \(M\) (we will use \(M\) generically for any constant bounds that are independent of solutions) such that

\[
R^C_i (t) + S^C_i (t) \leq |r^I_i|_{C[0,1]} + |s^I_i|_{C[0,1]} + M \int_0^t \left( R^C_i (t') + S^C_i (t') + T^C_i (t') \right) dt',
\]

where

\[
R^C_i (t) = \sup_{x: (x,t) \in D^C_\delta} |r_i (x,t)|, \quad S^C_i (t) = \sup_{x: (x,t) \in D^C_\delta} |s_i (x,t)|,
\]

and

\[
T^C_i (t) = \sup_{x: (x,t) \in D^C_\delta} \left( |f_i (x,t) - \hat{f}^i (x,t)| + |g_i (x,t) - \hat{g}^i (x,t)| \right).
\]

Hence, by Gronwall’s inequality (see, e.g. [1] p.327),

\[
R^C_i (t) + S^C_i (t) \leq M \left( |r^I_i|_{C[0,1]} + |s^I_i|_{C[0,1]} + \delta \sup_{t \in (0, \delta)} T^C_i (t) \right)
\]

for \(t \in [0, \delta]\). This proves that \(R^C_i\) and \(S^C_i\) have upper bounds in the form of the right side of \((2.20)\). In \(D^L_\delta\), if the left end is a source, we use either \((2.13)\) or \((2.14)\) according to the type of the boundary condition. The resulting inequality has the form

\[
R^L_i (t) + \hat{S}^L_i (t) \leq \sigma \hat{S}^L_i (t) + M \left( |s^I_i|_{C[0,1]} + |s^B_i|_{C[0,\delta]} + \int_0^t \left( R^L_i (\tau) + \hat{S}^L_i (\tau) + T^L_i (\tau) \right) d\tau \right)
\]

where \(\xi^B_i\) is either \(P^B_i\) or \(Q^B_i\) depending on the boundary condition, and \(R^L_i, \hat{S}^L_i\) and \(T^L_i\) are defined in the same way as in \((2.21)\), \((2.22)\), with \(D^C_\delta\) substituted by \(D^L_\delta\) and \(D^R_\delta\), and \(\sigma > 0\) is a positive constant such that \(\sigma = \varepsilon\) if the boundary condition is \((1.2)\) and

\[
\sigma = \varepsilon \sup_{t \in (0, \delta)} \left| \frac{\lambda^L_i (0, t)}{\lambda^R_i (0, t)} \right| < 1.
\]
if the boundary condition is (1.3). Replacing $M$ by $(1 - \sigma) M$, we can write

$$R_i^L (t) + \hat{S}_i^L (t) \leq M \left( |s_i^I|_{C[0,1]} + |\xi_i^B|_{C[0,\delta]} + \int_0^t \left( R_i^L (\tau) + \hat{S}_i^L (\tau) + T_i^L (\tau) \right) d\tau \right).$$

Hence, by Gronwall’s inequality

$$R_i^L (t) + \hat{S}_i^L (t) \leq M \left( |s_i^I|_{C[0,1]} + |\xi_i^B|_{C[0,\delta]} + \delta \max_{t \in (0,\delta)} T_i^L (t) \right).$$

This proves that both $R_i^L (t)$ and $S_i^L (t)$ have upper bounds in the form of the right hand side of (2.21).

If the left end is a junction, the solutions on the branches $j_1, \ldots, j_\mu$ connecting to the junction constitute a fixed point of the operator $K$, which is defined in (2.16). Let

$$W (t) = \sum_{l=1}^\nu \left( \hat{R}_{jl}^L (t) + S_{jl}^L (t) \right) + \sum_{l'=\nu+1}^\mu \left( R_{jl'}^L (t) + \hat{S}_{jl'}^L (t) \right),$$

where $\hat{R}_{jl}^L$ and $S_{jl}^L$ are defined as in (2.21) with $D_{\delta}^C$ substituted by $D_{\delta}^C \cup D_{\delta}^R$. Then, from $w = Kw$, we can deduce

$$W (t) \leq \sigma \left( \sum_{l=1}^\nu \hat{R}_{jl}^L (t) + \sum_{l'=\nu+1}^\mu \hat{S}_{jl'}^L (t) \right)$$

$$+ M \left( \sum_{l=1}^\nu |r_{jl}^I|_{C[0,1]} + \sum_{l'=\nu}^\mu |s_{jl'}^I|_{C[0,1]} + \int_0^t (W (\tau) + T (\tau)) d\tau \right),$$

where

$$T (\tau) = \sum_{l=1}^\nu T_{jl}^R (\tau) + \sum_{l'=\nu+1}^\mu T_{jl'}^L (\tau)$$

and $T_{jl}^R (t)$ is defined as in (2.22) with $D_{\delta}^C$ substituted by $D_{\delta}^C \cup D_{\delta}^R$. Replacing $M$ by $(1 - \sigma) M$, we obtain

$$W (t) \leq M \left( \sum_{l=1}^\nu |r_{jl}^I|_{C[0,1]} + \sum_{l'=\nu}^\mu |s_{jl'}^I|_{C[0,1]} + \int_0^t (W (\tau) + T (\tau)) d\tau \right).$$

Hence, by Gronwall’s inequality,

$$W (t) \leq M \left( \sum_{l=1}^\nu |r_{jl}^I|_{C[0,1]} + \sum_{l'=\nu}^\mu |s_{jl'}^I|_{C[0,1]} + \delta \max_{t \in (0,\delta)} T (t) \right).$$
This leads to an upper bound in the form of the right hand side of (2.20) for \( R_i^R(t), S_i^R(t), \)
i = \( j_1, \ldots, j_\nu \), and \( \tilde{R}_i^R(t), \tilde{S}_i^R(t), i = j_{\nu+1}, \ldots, j_{\mu} \).

The only remaining case is when the right end of the branch is a terminal. The fixed point equation to be used is either (2.17), (2.18) or (2.19) depending on the type of the boundary condition. In the former two cases, the treatment is similar to that for sources. Hence, we only consider the third case. From (2.19), we obtain

\[
\hat{R}_i^R(t) + \hat{S}_i^R(t) \leq \sigma \hat{R}_i^R(t) + M \left( |r_i^I|_{C[0,1]} + \int_0^t \left( \hat{R}_i^R(t') + \hat{S}_i^R(t') + |W_i^B(t')| + T_i^R(t') \right) dt' \right)
\]

where

\[
\sigma = \varepsilon \max_{t \in [0,\delta]} \left| \frac{m_i(t)}{n_i(t)} \right| < 1.
\]

Hence, by Gronwall’s inequality,

\[
\hat{R}_i^R(t) + \hat{S}_i^R(t) \leq M \left( |r_i^I|_{C[0,1]} + \delta \max_{t \in (0,\delta)} T_i^R(t) + \delta \max_{t \in (0,\delta)} |W_i^B(t)| \right),
\]

which gives the desired upper bounds of \( R_i^R \) and \( S_i^R \).

We have thus obtained an upper bound in the form of the right hand side of (2.20) for the quantities \( |r_i - \tilde{r}_i|_{C(D_\delta)} \) and \( |s_i - s_i|_{C(D_\delta)} \). The conclusion of the lemma follows now from (2.5).

3 The quasilinear system

In this section, we study the quasilinear system where the coefficients \( a_i, b_i, c_i, f_i \) and \( g_i \) depend on both \( (x, t) \) and \( (P_i, Q_i) \). Under certain conditions, we show that the system has a unique local solution. We then present a theorem on the continuity of dependence of the solution on initial, boundary and forcing function.

The basic idea in the proof of the existence of solution is to construct an iterative sequence. Substituting any vector function \((p_i, q_i)\) for \((P_i, Q_i)\) in \( a_i \), etc., the system becomes linear. Thus, we can use Theorem 2.1 to get a solution \((P_i, Q_i)\). This defines a mapping \( S \) from \( u = (p_i, q_i) \) to \( U = (P_i, Q_i) \), and the solution for the quasilinear system is a fixed point of \( S \). If there is a subset of a Banach space that is invariant under \( S \), then, we can construct a sequence

\[
u_{k+1} = Su_k, \quad k = 0, 1, \ldots.
\]

In the case where the limit exists and is unique, it gives rise to fixed point of \( S \). This is our approach in this section.

In this approach, conditions (2.1) and (2.3) are repeatedly used. One might want to impose them for all the values of the variables. This would give the existence and uniqueness
for the global solution, as in the case of the linear system. However, such a requirement is so restrictive that even the original system (1.1) cannot meet it. Therefore, we will impose them only for \( t = 0 \), and obtain the local solution for the quasilinear system.

**Theorem 3.1** Assume that the initial and boundary functions \( P_i^I, Q_i^I, P_i^B, Q_i^B, W_i^B \) and the system functions \( a_i, b_i, c_i, f_i, g_i \) all have continuous first-order derivatives with respect to each variable. Suppose that \( a_i > 0 \) for all the values of its arguments, and that conditions (2.1)–(2.3) hold at \( t = 0 \). Suppose also that the initial functions \( P_i^I, Q_i^I \) satisfy any relevant boundary conditions at \( t = 0 \). Then, for some \( \delta > 0 \), there is a unique solution for \( 0 \leq t < \delta \) to the quasilinear system (1.8) with the initial and boundary conditions described in Section 7.

**Proof.** We first consider the simpler case where \( U^I = (P^I, Q^I) = 0 \). Let \( v = \{v_i\} \), \( v_i = (p_i, q_i) \) be a family of vector functions (not necessarily constitutes a solution) that satisfy the initial and boundary conditions. Substitute \( v \) for \( U \) in the functions \( a_i, b_i, c_i, f_i \) and \( g_i \). Then, the system becomes linear and we can invoke Theorem 2.1 to obtain a solution \( U \) to the linear system. This defines a mapping \( S : v \mapsto U \). A solution of the quasilinear system is then a fixed point of \( S \). We will choose a subset \( X_{\delta,M_0} \) of a Banach space such that

1. \( SX_{\delta,M_0} \subset X_{\delta,M_0} \), and
2. \( S \) is contracting in \( X_{\delta,M_0} \). For any scalar or vector function \( f \in C^k(D_\delta) \), let \( |f|_{k,\delta} \) denote the maximum norm of all the \( k \)-th order derivatives of \( f \) in \( D_\delta \). (If \( f \) is a vector function, \( |f|_{k,\delta} = \max \left\{ |f_i|_{k,\delta} \right\} \).) Let \( C_B(D_\delta, \mathbb{R}^{2n}) \) denote the subset of the vector-valued functions in \( C(D_\delta, \mathbb{R}^{2n}) \) that satisfy the initial and boundary conditions. We seek \( X_{\delta,M_0} \) in the form

\[
X_{\delta,M_0} = \left\{ v \in C_B(D_\delta, \mathbb{R}^{2n}) : |v|_{0,\delta} \leq M_0, |v|_{1,\delta} \leq M_1 \right\}
\]  

(3.1)

where \( M_0 \) is an arbitrary positive constant and \( M_1 \) is a constant to be determined. Note that by the vanishing initial condition, for any \( M_1, |U|_{1,\delta} \leq M_1 \) implies \( |U|_{0,\delta} \leq M_1 \). Hence, for any \( M_0 \), we can ensure \( |U|_{0,\delta} \leq M_0 \) by reducing \( \delta \). It remains, therefore, only to show that for \( M_1 \) sufficiently large and \( \delta \) sufficiently small, \( |v|_{1,\delta} \leq M_1 \) implies \( |Sv|_{1,\delta} \leq M_1 \). Throughout this proof, we use \( M \) to represent any positive constant that may depend on \( M_1 \) but is otherwise independent of \( v \) and \( \delta \), and use \( \bar{M} \) for any constant that is independent of \( M_1, v \) and \( \delta \). The values of \( M \) or \( \bar{M} \) in different occurrences need not be equal.

Let \( U = Sv \) and let \( r_i \) and \( s_i \) be defined by (2.4). On each branch, we show that

\[
\max \left\{ |(r_i)_x|, |(s_i)_x| \right\} \leq M_1
\]

(3.2)

and

\[
\max \left\{ |(r_i)_t|, |(s_i)_t| \right\} \leq M_1
\]

(3.3)
Hence, we obtain from Lemma 2.1 with \(\tilde{\eta}\) conditions gives

\[
\max_i \left\{ \left| P_i^B \right|_{C[0,\delta]}, \left| Q_i^B \right|_{C[0,\delta]} \right\} \leq M\delta.
\]

Hence, we obtain from Lemma 2.1 with \(\tilde{U} = 0\) that

\[
|U|_{0,\delta} \leq M\delta. \tag{3.4}
\]

From (2.6) and (2.8), there are constants \(\bar{M}\) and \(M\) such that

\[
|\partial^R_i r_i| \leq |l_i^R F_i| + |\partial^R_i l_i^R | |U_i| \leq \bar{M} + M\delta,
\]

\[
|\partial^L_i s_i| \leq |l_i^L F_i| + |\partial^L_i l_i^L | |U_i| \leq \bar{M} + M\delta
\]

for each \(i = 1, \ldots, n\). Hence, (3.3) follows from (3.2), (3.5) and the definition of \(\partial^L_i\) and \(\partial^R_i\) in (2.7). We also note that (2.3) and (3.3) imply

\[
|\partial^R_i U_i|_{0,\delta} \leq \bar{M} + M\delta, \quad |\partial^L_i U_i|_{0,\delta} \leq \bar{M} + M\delta
\]

for all \(i\). This will be used later.

We first consider the middle region \(D^C_\delta\), where the solution \((r_i, s_i)\) satisfies the integral equations (2.9) and (2.11) with \(r_i^L = s_i^L = 0\). Differentiating the equations with respect to \(x\), we have

\[
(r_i)_x = (l_i^R)_x U_i (x, t) + \int_0^t \left[ ((l_i^R F_i)_x + (\partial_i^R l_i^R) (U_i)_x - (l_i^R)_x (\partial_i^R U_i)) \right] (x_i^R)_x dt,
\]

\[
(s_i)_x = (l_i^L)_x U_i (x, t) + \int_0^t \left[ (l_i^L F_i)_x + (\partial_i^L l_i^L) (U_i)_x - (l_i^L)_x (\partial_i^L U_i)) \right] (x_i^L)_x dt.
\]

Here, we used an identity from \([3, p.469]\):

\[
\frac{d}{d\xi} \int_a^b f (x (t), t) Dg (x (t), t) dt = f (x (b), b) g_x (x (b), b) - f (x (a), a) g_x (x (a), a)
\]

\[
+ \int_a^b [f_x (x (t), t) Dg (x (t), t) - Df (x (t), t) g_x (x (t), t)] dt
\]

where \(x (t)\) is a function such that \(x (b) = \xi\) and \(D = \frac{\partial}{\partial t} + x' (t) \frac{\partial}{\partial x}\). Let

\[
R^C_i (t) = \sup_{x : (x, t) \in D^C_\delta} \{(r_i)_x (x, t)\}, \quad S^C_i (t) = \sup_{x : (x, t) \in D^C_\delta} \{(s_i)_x (x, t)\}. \tag{3.9}
\]
From (3.4), (3.6) and (3.7), we derive

\[ R^C_i(t) + S^C_i(t) \leq M\delta + M \int_0^t (1 + R^C_i(t') + S^C_i(t')) \, dt' \]

for \( t \in [0, \delta] \). Hence, Gronwall’s inequality gives

\[ |(r_i)_x| \leq M\delta e^{M\delta}, \quad |(s_i)_x| \leq M\delta e^{M\delta} \]

in \( D^C_\delta \). This proves (3.2) in \( D^C_\delta \) if \( M_1 \) is sufficiently large and \( \delta \) is sufficiently small.

We next consider the left triangular region \( D^L_\delta \) in the case where the branch is connected to a source. Let \( \hat{s}_i = s_i/\varepsilon \) for any \( \varepsilon > 0 \). Then, the pair \((r_i, \hat{s}_i)\) satisfies the fixed point equations of either (2.13) or (2.14), depending on the type of the boundary condition. Differentiating the equations with respect to \( x \) and using a slightly modified version of (3.8), we have

\[ (r_i)_x = (\zeta_i - l^R_i F_i - (\partial^R_i l^R_i) U_i - (l^R_i)_x U_i) (0, \tau) \tau_x + (l^R_i)_x U_i (x, t) \]
\[ + \int_\tau^t \left[ (l^R_i F_i)_x + \partial^R_i (l^R_i) U_i \right] (x^R_i)_x \, dt, \quad (3.10) \]
\[ (\hat{s}_i)_x = \frac{1}{\varepsilon} (l^L_i)_x U_i (t, x) + \frac{1}{\varepsilon} \int_0^t \left[ (l^L_i F_i)_x + \partial^L_i (l^L_i) (U_i)_x - (l^L_i)_x (\partial^L_i U_i) \right] (x^L_i)_x \, dt, \]

where

\[ \zeta_i = 2 \left( u_i P_i^B \right)_t + \varepsilon (\hat{s}_i)_t \]

if the boundary condition is given by (1.2), and

\[ \zeta_i = 2 \left( a_i u_i \lambda^R_i \right)_t + \varepsilon (\hat{s}_i)_t + \varepsilon \left( \frac{\lambda^L_i}{\lambda^R_i} \right) (\hat{s}_i)_t \]

if the boundary condition is given by (1.3). (Modification of (3.8) is caused by the lower limit of the integral in the first equation of (3.10) which also depends on \( x \).) This equation is valid for any \( \varepsilon \). So, we may choose \( \varepsilon \) so small such that

\[ \sigma =: \varepsilon |\lambda^L_i \tau_x(0, t)| \max \left\{ 1, \left| \frac{\lambda^L_i (0, t)}{\lambda^R_i (0, t)} \right| \right\} < 1, \quad t \in [0, \delta]. \]

To proceed further, we need an estimate of \( |\tau_x(0, t)| \). Observe that \( \tau (x) \) satisfies the equation

\[ x^R_i (\tau; x, t) = 0 \]

where \( x^R_i (\tau; x, t) \) is the solution of the initial value problem

\[ \frac{dx^R_i}{ds} = \lambda^R_i (x^R_i, s), \quad x^R_i (t; x, t) = x. \]
By differentiation,
\[ \lambda_i^R (0, \tau (x)) \tau_x + \frac{\partial x_i^R}{\partial x} \bigg|_{(\tau(x),x,t)} = 0. \tag{3.11} \]

Let \( w_i = \partial x_i^R / \partial x \). Then, \( w_i \) is the solution of the linear equation
\[ \frac{dw_i}{ds} = \left( \lambda_i^R \right)_x (x_i^R (s; x, t), s) w_i, \quad w_i (t) = 1. \]

Solving the equation,
\[ w_i (s) = \exp \left( \int_t^s \left( \lambda_i^R \right)_x (x_i^R (s'; x, t), s') \, ds' \right). \]

Returning to (3.11), we find
\[ \tau_x = \frac{-1}{\lambda_i^R (0, \tau (x))} \exp \left( \int_t^{\tau(x)} \left( \lambda_i^R \right)_x (x_i^R (s'; x, t), s') \, ds' \right). \]

Observe that \( 0 < \tau (x) < t \leq \delta \) and the integrand is bounded. Hence,
\[ |\tau_x| \leq \tilde{M} e^{M\delta}. \tag{3.12} \]

This is the estimate we need. By this estimate, for any \( M_1 \), we can choose \( \delta \) small enough such that the constants \( \sigma \) and \( \varepsilon \) are independent of \( M_1 \). Let \( R_i^L (t) \) and \( \hat{S}_i^L (t) \) be defined as in (3.9) except that \( s_i \) is substituted by \( \hat{s}_i \) and \( D^C_\delta \) is substituted by \( D^L_\delta \cup D^C_\delta \). We derive from (3.10) and the identity
\[ (\hat{s}_i)_t = \partial_i^L \hat{s}_i - \lambda_i^L (\hat{s}_i)_x \]
that
\[ R_i^L (t) + \hat{S}_i^L (t) \leq \sigma \hat{S}_i^L (t) + \tilde{M} + M\delta + M \int_0^t \left( 1 + R_i^L (t') + \hat{S}_i^L (t') \right) dt'. \]

Replacing \( M \) and \( \tilde{M} \) by \( M(1 - \sigma) \) and \( \tilde{M}(1 - \sigma) \), respectively, and applying Gronwall’s inequality, we obtain
\[ R_i^L (t) + \hat{S}_i^L (t) \leq \left( \tilde{M} + M\delta \right) e^{M\delta}. \]

Since \( |s_i| \leq |\hat{s}_i| \), it follows that
\[ \max \{|(r_i)_x|, |(s_i)_x|\} \leq \left( \tilde{M} + M\delta \right) e^{M\delta} \]
in \( D^L_\delta \cup D^C_\delta \). This proves (3.2) in \( D^L_\delta \cup D^C_\delta \) if \( M_1 \) is large and \( \delta \) is small.

We next consider the case where the left end of the branch is a junction. As before, we shall consider the branches that are connected to the same junction simultaneously. This
also includes the right triangular regions \( D_3^R \) for the branches that are connected to the junction from left. We consider the fixed point equation \( w = Kw \) where \( w \) and \( Kw \) are defined in (2.13) and (2.14), respectively. Differentiating the equations, we obtain (3.10) in \( D_1^C \cup D_2^C \) for \( i = j_{\nu + 1}, \ldots, j_\mu \) and

\[
\begin{align*}
(\hat{r}_i)_x &= \frac{1}{\varepsilon} (l_i^R) \ x U_i(t, x) + \frac{1}{\varepsilon} \int_0^t \left[ (l_i^R F_i)_x + (\partial_t l_i^R) (U_i)_x - (l_i^R)_x \ (\partial_t^R U_i) \right] (x_i^R)_x \ dt, \\
(s_i)_x &= (\theta_i - l_i^L F_i - (\partial_t l_i^L) U_i - (l_i^L)_x U_i) (1, \tau)_x + (l_i^L)_x U_i(x, t) \\
&\quad + \int_0^t \left[ (l_i^L F_i)_x + (\partial_t l_i^L) (U_i)_x - (l_i^L)_x (\partial_t l_i^L U_i) \right] (x_i^L)_x \ dt,
\end{align*}
\tag{3.13}
\]

in \( D_1^C \cup D_2^R \) for \( i = j_1, \ldots, j_\nu \), where

\[
\begin{align*}
\zeta_i &= \varepsilon \sum_{k_1 = 1}^\nu \ (n^i_{j_1})_t \ (\hat{r}_i)_x (1, \tau) + n^i_{j_1} (\hat{r}_i)_t (1, \tau) + \varepsilon \sum_{l' = \nu + 1}^\mu \ (n^i_{j_\nu'})_t \ (\hat{r}_i)_x (0, \tau) + n^i_{j_\nu'} (\hat{r}_i)_t (0, \tau), \\
\theta_i &= \varepsilon \sum_{k_1 = 1}^\nu \ (m^i_{j_1})_t \ (\hat{r}_i)_x (1, \tau) + m^i_{j_1} (\hat{r}_i)_t (1, \tau) + \varepsilon \sum_{l' = \nu + 1}^\mu \ (m^i_{j_\nu'})_t \ (\hat{r}_i)_x (0, \tau) + m^i_{j_\nu'} (\hat{r}_i)_t (0, \tau),
\end{align*}
\]

and \( m^i_{j_1}, n^i_{j_1} \) are defined in the proof of Theorem 2.1. Note that the estimate (3.12) holds for \( \tau_x \) in both (3.11) and (3.13), although in the latter case, \( \tau \) is the \( t \)-coordinate of the intersection of the left-going characteristic curve \( x^L_t \) with the vertical line \( x = 1 \). The derivation is identical. Hence, there is a constant \( \varepsilon, \) independent of \( M_1 \), such that

\[
\varepsilon |\tau_x| \left( \sum_{k_1 = 1}^\nu \ |m^i_{j_k}(t)| + \sum_{k_1 = \nu + 1}^\mu \ |m^i_{j_k'}(t)| \right) < 1, \\
\varepsilon |\tau_x| \left( \sum_{k_1 = 1}^\nu \ |n^i_{j_k}(t)| + \sum_{k_1 = \nu + 1}^\mu \ |n^i_{j_k'}(t)| \right) < 1
\]

in \([0, \delta]\). Let \( \sigma \) be the maximum of the quantities on the left hand side of the above inequalities. Define \( \hat{R}^R_{j_1}, S^R_{j_1}, \hat{R}^L_{j_1} \) and \( \hat{S}^L_{j_1} \) as in (3.9) with obvious modifications. We see that the function

\[
W(t) = \sum_{l_1 = 1}^\nu \ (\hat{R}^R_{j_1}(t) + S^R_{j_1}(t)) + \sum_{l' = \nu + 1}^\mu \ (\hat{R}^L_{j_\nu'}(t) + \hat{S}^L_{j_\nu'}(t))
\]

satisfies the inequality

\[
(1 - \sigma) W(t) \leq \sum_{l_1 = 1}^\nu \ (1 - \sigma) \ (\hat{R}^R_{j_1}(t) + S^R_{j_1}(t)) + \sum_{l' = \nu + 1}^\mu \ (\hat{R}^L_{j_\nu'}(t) + (1 - \sigma) \hat{S}^L_{j_\nu'}(t)) \\
\leq \bar{M} + M \delta + M \int_0^t (1 + W(t')) \ dt'.
\]
Hence, by rescaling and using Gronwall’s inequality, we achieve

\[ W(t) \leq \left( \tilde{M} + M\delta \right) e^{M\delta}. \]

This proves that

\[ \max \{ |(r_i)_x|, |(s_i)_x| \} \leq \tilde{M} + M\delta \]

in \( D_\delta^R \) for \( i = j_1, \ldots, j_\nu \) and in \( D_\delta^L \) for \( i = j_{\nu+1}, \ldots, j_\mu \) if \( M_1 \) is sufficiently large and \( \delta \) is sufficiently small. We have thus proved (3.2) in this case.

It remains to treat the branches that are connected to terminals. If the terminal boundary condition is either (1.5) or (1.6), the argument is parallel to the one given above for sources. Hence, we only consider the case where the boundary condition is (1.7). The fixed point equation in this case is (2.19). Differentiating (2.19) with respect to \( x \) gives (3.13) with \( \zeta_i = \varepsilon \left( \frac{m_i}{n_i} \right)_t \tau_x r_i (1, \tau) + \varepsilon \left( \frac{m_i}{n_i} \right)_t \tau_x r_i (1, \tau) - \left( \frac{1}{n_i} \right)_t \int_0^t H_i (t', r_i (1, t'), s_i (1, t')) dt'. \)

Let \( \delta \) be sufficiently small such that \( |\tau_x| \) is bounded by a constant independent of \( M_1 \). Choose \( \varepsilon > 0 \) such that

\[ \sigma =: \varepsilon \left| \lambda_i (1, t) \right| \left| \frac{m_i}{n_i} \tau_x (1, t) \right| < 1 \]

for \( t \in [0, \delta] \). Note that \( \left( \frac{m_i}{n_i} \right)_t \) and \( \left( \frac{1}{n_i} \right)_t \) are bounded (by a constant depending on \( M_1 \)). Hence,

\[ \hat{R}_i^R (t) + S_i^R (t) \leq \sigma \hat{R}_i^R (t) + \tilde{M} + M\delta + M \int_0^t \left( 1 + \hat{R}_i^R (t') \right) dt'. \]

This leads to

\[ \hat{R}_i^R (t) + S_i^R (t) \leq \left( \tilde{M} + M\delta \right) e^{M\delta} \]

in \( D_\delta^R \) upon rescaling of constants. Hence, (3.2) holds in \( D_\delta^R \).

This completes the proof of (3.2) in all cases. By choosing appropriate values of \( M_1 \) and \( \delta \), we thus obtain a set \( X_{\delta,M_0} \) in the form of (3.1) which is invariant under the mapping \( S \).

We now show that \( S \) is a contraction in \( X_{\delta,M_0} \). Let \( U = Sv, \tilde{U} = S\tilde{v} \) for some \( v, \tilde{v} \in X_\delta \), and let \( W = U - \tilde{U} \). \( W \) satisfies the vanishing initial and external boundary conditions and its differential equations takes the form of (1.8) with the coefficients

\[ a_i = a_i (x, t, v), \ b_i = b_i (x, t, v), \ c_i = c_i (x, t, v) \]

and the forcing functions \( f_i \) and \( g_i \) replaced by

\[ \hat{f}_i =: f_i (x, t, v) - f_i (x, t, \tilde{v}) + (a_i (x, t, v) - a_i (x, t, \tilde{v})) \frac{\partial \tilde{Q}_i}{\partial x} \]

(3.14)
and
\[ \hat{g}_i =: g_i(x, t, \bar{v}) - g_i(x, t, v) + (b_i(x, t, v) - b_i(x, t, \bar{v})) \frac{\partial \tilde{P}_i}{\partial x} \]
\[ + 2(c_i(x, t, v) - c_i(x, t, \bar{v})) \frac{\partial \tilde{Q}_i}{\partial x}, \]
respectively. By the Lipschitz property and the boundedness \( |\hat{U}|_{1, \delta} \leq M_1 \), there is a constant \( M \) such that
\[ |\hat{f}|_{0, \delta} \leq M |v - \bar{v}|_{0, \delta}, \quad |\hat{g}|_{0, \delta} \leq M |v - \bar{v}|_{0, \delta}. \]
Hence, by Theorem 2.1,
\[ |Sv - S\bar{v}|_{0, \delta} \leq M \delta |v - \bar{v}|_{0, \delta}. \]
Therefore, \( S \) is contracting in \( X_{\delta, M_0} \) if \( \delta \) is sufficiently small.

The rest is standard (cf. e.g., [6]). Starting with a \( v_0 \in X_{\delta, M_0} \), we generate an iterative sequence \( v_{k+1} = Sv_k \). Clearly, each \( v_k \) lies in \( X_{\delta, M_0} \) and the sequence converges uniformly. The limit then satisfies the integral equations in the proof of Theorem 2.1, and hence, is differentiable. Therefore, it is the solution of the quasilinear differential equations. This proves the existence and uniqueness of the solution when \( U^I = 0 \).

If \( U^I \neq 0 \), we regard \( U^I \) as a vector function of \( x \) and \( t \) and introduce \( \tilde{U} = U - U^I \). It follows that \( \tilde{U} \) is a solution of the quasilinear equations (1.8) with the forcing functions \( \tilde{f}_i \) and \( \tilde{g}_i \) given by
\[ \tilde{f}_i = f_i - (Q^I_i)_x a_i, \quad \tilde{g}_i = g_i - (P^I_i)_x b_i - (Q^I_i)_x 2c_i \]
and the boundary functions are given by
\[ \tilde{P}^B_i = P^B_i - P^I_i, \quad \tilde{Q}^B_i = Q^B_i - Q^I_i, \quad \tilde{W}^B_i = W^B_i - \delta_i P^I_i + \varepsilon_i Q^I_i. \]
Since \( \tilde{U} \) has the vanishing initial values, it can be uniquely solved for an interval of \( t \in [0, \delta] \). This gives rise to a solution \( U \). 

**Remark:** Examples can be constructed to show that if the condition (2.3) fails at \( t = 0 \), then, the local solution need not exist or may be not unique. In particular, if (2.3) fails at a source end, then, the system is under-determined, and if it fails at a terminal end, the system is over-determined. See Section 5 for further discussion.

We give next a result for the continuity of dependence of the solution and its derivatives on the initial, boundary and forcing functions and their derivatives. This follows from an argument similar to the proofs of Lemma 2.1 and Theorem 3.1.

**Corollary 3.1** Let \( U = (P, Q) \) and \( \tilde{U} = \left( \tilde{P}, \tilde{Q} \right) \) be two solutions of the quasilinear problem of Theorem 3.1. Suppose the conditions of that theorem hold for the initial and boundary
functions of both solutions. Then, there exists a constant $M > 0$, independent of initial, boundary and forcing functions, such that

$$
|U - \tilde{U}|_{k,\delta} \leq M \left( |P^I - \tilde{P}^I|_{C^k[0,1]} + |Q^I - \tilde{Q}^I|_{C^k[0,1]} + |P^B - \tilde{P}^B|_{C^k[0,\delta]} + |Q^B - \tilde{Q}^B|_{C^k[0,\delta]} + \delta |f - \tilde{f}|_{C^k(\overline{D})} + \delta |g - \tilde{g}|_{C^k(\overline{D})} + \delta |W - \tilde{W}|_{C^k[0,\delta]} \right),
$$

for $k = 0, 1$.

**Proof.** For $k = 0$, the result follows from substituting one of the solutions into the coefficients, modifying the forcing functions by (3.14)–(3.15), and using Lemma 2.1. For $k = 1$, we differentiate the equations and apply the lemma to the resulting equations for the derivatives of the solution. The process is standard and is omitted.

## 4 A finite-difference scheme

In this section, we present a finite-difference scheme that computes discretized solutions, and prove the convergence of the scheme.

The scheme is based on the equations in (2.6). Substituting (2.4) and (2.8) into (2.6), we obtain the normal form of the equations

$$
-\lambda^L_i P_{i,t} + a_i Q_{i,t} + \lambda^R_i (-\lambda^L_i P_{i,x} + a_i Q_{i,x}) = d^R_i,
$$

$$
-\lambda^R_i P_{i,t} + a_i Q_{i,t} + \lambda^L_i (-\lambda^R_i P_{i,x} + a_i Q_{i,x}) = d^L_i,
$$

where

$$
d^R_i (x, t, P_i, Q_i) = -\lambda^L_i f_i + a_i g_i, \quad d^L_i (x, t, P_i, Q_i) = -\lambda^R_i f_i + a_i g_i.
$$

Let $h$ and $k$ be the spatial and temporal step sizes, respectively. Hence, $hN = 1$ for some integer $N$. We impose the finite-difference equations as

$$
\frac{1}{k} \left[ -\lambda^L_{i,n} \left( p_{i,n+1}^m - p_{i,n}^m \right) + a_i^m \left( q_{i,n+1}^m - q_{i,n}^m \right) \right]
+ \frac{\lambda^L_{i,n}}{h} \left[ -\lambda^L_{i,n} \left( p_{i,n}^m - p_{i,n-1}^m \right) + a_i^m \left( q_{i,n}^m - q_{i,n-1}^m \right) \right] = d_{i,n}^R,
$$

for $n = 1, \ldots, N$ and

$$
\frac{1}{k} \left[ -\lambda^R_{i,n} \left( p_{i,n+1}^m - p_{i,n}^m \right) + a_i^m \left( q_{i,n+1}^m - q_{i,n}^m \right) \right]
+ \frac{\lambda^R_{i,n}}{h} \left[ -\lambda^R_{i,n} \left( p_{i,n}^m - p_{i,n-1}^m \right) + a_i^m \left( q_{i,n}^m - q_{i,n-1}^m \right) \right] = d_{i,n}^L,
$$

for $n = 1, \ldots, N$. 

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for \( n = 0, \ldots, N - 1 \), where \( a_{i,n}^m \), etc. are the values of the respective functions \( a_i \), etc. at the point \((nh, mk, p_{i,n}^m, q_{i,n}^m)\). (In this section, \( n \) is always the running index for the spatial variable, not the number of branches.) The initial condition is simply

\[
p_{i,n}^0 = P_i^T(nh), \quad q_{i,n}^0 = Q_i^T(nh).
\]

If for a fixed \( m \) the quantities \( p_{i,n}^m \) and \( q_{i,n}^m \) are constructed for \( n = 0, \ldots, N \), then, equations (4.1) and (4.2) determine \( p_{i,n}^{m+1} \) and \( q_{i,n}^{m+1} \) for \( n = 1, \ldots, N - 1 \). The quantities for \( n = 0 \) and \( N \) are determined by boundary conditions. At a source end, if the boundary condition is given by (1.2), we impose

\[
p_{i,0}^{m+1} = P_i^B((m+1)k) \quad (4.4)
\]

and solve \( q_{i,0}^{m+1} \) from (4.2) with \( n = 0 \). If the boundary condition is (1.3), we impose

\[
q_{i,0}^{m+1} = Q_i^B((m+1)k) \quad (4.5)
\]

and solve \( p_{i,0}^{m+1} \) from (4.2). At a junction with \( j_1, \ldots, j_\nu \) incoming branches and \( j_{\nu+1}, \ldots, j_\mu \) outgoing branches, we prescribe

\[
p_{j_{l',N}}^{m+1} = p_{j_{l',0}}^{m+1} =: p_{j_{l',N}}^{m+1} \quad (4.6)
\]

for \( l = 1, \ldots, \nu \), \( l' = \nu + 1, \ldots, \mu \), and

\[
\sum_{l=1}^{\nu} q_{j_l,N}^{m+1} = \sum_{l'=\nu+1}^{\mu} q_{j_{l'},0}^{m+1}. \quad (4.7)
\]

These equations are solved jointly with equation (4.1) at \( n = N \) for \( i = j_1, \ldots, j_\nu \) and with equation (4.2) at \( n = 0 \) for \( i = j_{\nu+1}, \ldots, j_\mu \). The reason that the quantities \( p_{j_{l',N}}^{m+1}, q_{j_{l',0}}^{m+1} \) can be uniquely solved is that the coefficient matrix

\[
\begin{pmatrix}
0 & R_1 & R_2 \\
-S_1 & \frac{1}{k} A_1 & 0 \\
-S_2 & 0 & \frac{1}{k} A_2
\end{pmatrix}
\]

with

\[
R_1 = (1, \ldots, 1), \quad R_2 = (-1, \ldots, -1),
\]

\[
S_1 = \left(\lambda_{j_{l',N}}^{L,m}, \ldots, \lambda_{j_{l',N}}^{L,m}\right)^T, \quad S_2 = \left(\lambda_{j_{l',0}}^{R,m}, \ldots, \lambda_{j_{l',0}}^{R,m}\right)^T,
\]

\[
A_1 = \text{diag}(a_{j_{l',N}}^m, \ldots, a_{j_{l',N}}^m), \quad A_2 = \text{diag}(a_{j_{l',0}}^m, \ldots, a_{j_{l',0}}^m)
\]

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has the determinant
\[
\frac{1}{k^\mu} \left( -\sum_{l=1}^{\nu} \frac{\lambda_{L,m,l}^{i,N}}{a_{L,m,l}^{i,N}} + \sum_{\nu=1}^{\mu} \frac{\lambda_{R,m,\nu}^{i,l}}{a_{R,m,\nu}^{i,l}} \right) \prod_{l=1}^{\nu} a_{L,m,l}^{i,N} \prod_{\nu=1}^{\mu} a_{R,m,\nu}^{i,l} > 0.
\]
(We used the fact \( \lambda^L_i < 0 \), \( \lambda^R_i > 0 \) and \( a_i > 0 \) here.) At a terminal end with the boundary condition (1.5) resp. (1.6), we impose
\[
p_{i,N}^{m+1} = P_B^i ((m+1)k) \quad \text{resp.} \quad q_{i,N}^{m+1} = Q_B^i ((m+1)k)
\]
and solve the other quantity from (4.1) with \( n = N \). If the boundary condition is (1.7), we impose
\[
p_{i,N}^{m+1} - p_{i,N}^m - \frac{\eta_i}{k} (q_{i,N}^{m+1} - q_{i,N}^m) + \frac{\delta_i}{k} (p_{i,N}^{m+1} + p_{i,N}^m) - \frac{\varepsilon_i}{2} (q_{i,N}^{m+1} + q_{i,N}^m) = W_B^i \left( \left( m + \frac{1}{2} \right) k \right).
\]
Together with (4.1) for \( n = N \), the values of \( p_{i,N}^{m+1} \) and \( q_{i,N}^{m+1} \) are uniquely determined. This is because the coefficient matrix has the determinant
\[
\det \left( -\frac{\lambda_{L,m}^{i,N}}{1} + \frac{a_{R,m}^{i,N}}{1} \begin{bmatrix} \frac{\eta_i}{k} & \frac{\delta_i}{k} & \frac{\varepsilon_i}{2} & 0 \\ -\frac{\eta_i}{k} & \frac{\delta_i}{k} & -\frac{\varepsilon_i}{2} & 0 \end{bmatrix} \right) < 0.
\]
(One might suspect that the simpler condition
\[
\frac{1}{k} (p_{i,N}^{m+1} - p_{i,N}^m) - \frac{\eta_i}{k} (q_{i,N}^{m+1} - q_{i,N}^m) + \delta_i p_{i,N}^m - \varepsilon_i q_{i,N}^m = W_B^i (mk).
\]
would also suffice. It indeed can determine unique values of \( p_{i,N}^{m+1} \) and \( q_{i,N}^{m+1} \). However, we are unable to prove the convergence of the scheme with this condition. This will be clear from the proof of the next theorem.)

It is clear that for any step-sizes \( h \) and \( k \), this scheme generates a discretized solution as long as \( \lambda^L_i \) remains negative at \( x = 0 \) and \( x = 1 \). We show that if the ratio \( k/h \) is fixed and sufficiently small, then, in a time interval the solutions for the finite-difference equations converge to the solution to the original system of differential equations (1.8) as \( h \to 0 \).

**Theorem 4.1** Suppose that the conditions of Theorem 3.1 holds and that
\[
a_i (x,t,p,q) > 0, \quad \lambda_i^L (x,t,p,q) < 0
\]
for all \( (x,t) \in [0,1] \times [0,\delta] \) and \((p,q) \in \mathbb{R}^2\), where \( \delta > 0 \) appears in Theorem 3.1. Suppose also that the initial and boundary functions \( P_i^L, Q_i^L, P_i^B, Q_i^B \) and \( W_i^B \) have continuous second derivatives. Let \( \sigma > 0 \) be a positive constant such that
\[
\sigma \max \left\{ |\lambda_i^L|_{0,\delta}, |\lambda_i^R|_{0,\delta} \right\} < 1,
\]
(4.11)
and let the ratio \( k/h = \sigma \) be fixed. Then, there is a constant \( \delta_0 > 0 \) such that, as \( h \to 0 \), the solutions of the finite-difference scheme described above converges to the solution of the differential equation (1.8) in the strip \( 0 \leq t \leq \delta_0 \).

**Remark:** The condition of \( a_i > 0, \lambda_i^L < 0 \) for all \((p, q)\) is stronger than needed. One may only require that the inequalities hold in a certain range of \((p, q)\) containing the solution \((P_i, Q_i)\) in its interior. The theorem is stated as above to simplify the argument.

**Proof.** By Theorem 3.1, the system of differential equations has a solution \((P_i, Q_i)\) in \( D_\delta \) for some \( \delta > 0 \). Since the initial and boundary functions have continuous second derivatives, it can be shown using standard arguments that the solution \((P_i, Q_i)\) has continuous second order derivatives in \( D_\delta \). (Reduce \( \delta \) if necessary.) By Taylor’s theorem and \( k = \sigma h \), we can write

\[
\frac{1}{k} \left[ -\tilde{\lambda}_{i,n}^L (P_{m+1}^{i,n} - P_{i,n}^m) + \tilde{a}_{i,n}^m (Q_{i,n}^{m+1} - Q_{i,n}^m) \right] + \frac{\lambda_{i,n}^R}{h} \left[ -\tilde{\lambda}_{i,n}^R (P_{i,n}^m - P_{i,n-1}^m) + \tilde{a}_{i,n}^m (Q_{i,n}^m - Q_{i,n-1}^m) \right] = \tilde{d}_{i,n}^{R,m} + O(h) \tag{4.12}
\]

for \( n = 1, \ldots, N \), and

\[
\frac{1}{k} \left[ -\tilde{\lambda}_{i,n}^R (P_{m+1}^{i,n} - P_{i,n}^m) + \tilde{a}_{i,n}^m (Q_{i,n}^{m+1} - Q_{i,n}^m) \right] + \frac{\lambda_{i,n}^L}{h} \left[ -\tilde{\lambda}_{i,n}^L (P_{i,n}^m - P_{i,n-1}^m) + \tilde{a}_{i,n}^m (Q_{i,n}^m - Q_{i,n-1}^m) \right] = \tilde{d}_{i,n}^{L,m} + O(h) \tag{4.13}
\]

for \( n = 0, \ldots, N - 1 \), where \( P_{i,n}^m \) and \( Q_{i,n}^m \) are the values of the corresponding functions at the point \((nh, mk)\), and \( \tilde{\lambda}_{i,n}^L \) etc. represent the values of the corresponding functions at the point \((nh, mk, P_{i,n}, Q_{i,n})\). Let

\[ u_{i,n}^m = P_{i,n}^m - P_{i,n}, \quad v_{i,n}^m = Q_{i,n}^m - Q_{i,n}. \]

Our task is to show

\[ u_{i,n}^m \to 0, \quad v_{i,n}^m \to 0 \]

as \( h \to 0 \) and \( k = \sigma h \). We prove it by showing that there are positive constants \( \delta_0, h_0 \) and \( M \), independent of \( m \), such that

\[
|u_{i,n}^m| \leq M h, \quad |v_{i,n}^m| \leq M h, \tag{4.14}
\]

if \( h \leq h_0, \quad k = \sigma h \) and \( 0 \leq mk \leq \delta_0 \).
We first derive some recursive relations. Subtract (4.1) and (4.2) from (4.12) and (4.13), respectively, and use the Lipschitz property and the boundedness of the derivatives of $P_l$ and $Q_i$, we obtain

$$\frac{1}{k} \left[ -\lambda_{i,n}^{L_m} (u_{i,n}^{m+1} - u_{i,n}^m) + a_{i,n}^m (v_{i,n}^{m+1} - v_{i,n}^m) \right] + \frac{\lambda_{i,n}^{R_m}}{h} \left[ -\lambda_{i,n}^{L_m} (u_{i,n}^{m} - u_{i,n-1}^m) + a_{i,n}^m (v_{i,n}^{m} - v_{i,n-1}^m) \right] = O \left(h \right) + O \left(u_{i,n}^m \right) + O \left(v_{i,n}^m \right) \tag{4.15}$$

and, similarly,

$$\frac{1}{k} \left[ -\lambda_{i,n}^{R_m} (u_{i,n}^{m+1} - u_{i,n}^m) + a_{i,n}^m (v_{i,n}^{m+1} - v_{i,n}^m) \right] + \frac{\lambda_{i,n}^{L_m}}{h} \left[ -\lambda_{i,n}^{R_m} (u_{i,n}^{m} - u_{i,n-1}^m) + a_{i,n}^m (v_{i,n}^{m} - v_{i,n-1}^m) \right] = O \left(h \right) + O \left(u_{i,n}^m \right) + O \left(v_{i,n}^m \right) \tag{4.16}$$

Introduce

$$r_{i,n}^m = -\lambda_{i,n}^{L_m-1} u_{i,n}^m + a_{i,n}^{m-1} v_{i,n}^m, \quad s_{i,n}^m = -\lambda_{i,n}^{R_m-1} u_{i,n}^m + a_{i,n}^{m-1} v_{i,n}^m. \tag{4.17}$$

One can show that (4.14) is equivalent to

$$|r_{i,n}^m| \leq Mh, \quad |s_{i,n}^m| \leq Mh. \tag{4.17}$$

(Throughout the proof of this theorem, we use $M$ to denote any positive constant that is independent of $m$.) Using the identity

$$-\lambda_{i,n}^{L_m} u_{i,l}^m + a_{i,n}^{m} v_{i,l}^m = r_{i,l}^m + \left( \lambda_{i,l}^{L_m-1} - \lambda_{i,n}^{L_m} \right) u_{i,l}^m + \left(a_{i,l}^{m-1} - a_{i,n}^{m} \right) v_{i,l}^m,$$

together with

$$\lambda_{i,l}^{L_m-1} - \lambda_{i,n}^{L_m} = O \left(k \right) + O \left(p_{i,l}^{m-1} - p_{i,n}^m \right) + O \left(q_{i,l}^{m-1} - q_{i,n}^m \right),$$

$$a_{i,l}^{m-1} - a_{i,n}^{m} = O \left(k \right) + O \left(p_{i,l}^{m-1} - p_{i,n}^m \right) + O \left(q_{i,l}^{m-1} - q_{i,n}^m \right),$$

and

$$p_{i,l}^{m-1} - p_{i,n}^m = -u_{i,l}^{m-1} + u_{i,n}^m + \left(p_{i,l}^{m-1} - p_{i,n}^m \right),$$

$$q_{i,l}^{m-1} - q_{i,n}^m = -v_{i,l}^{m-1} + v_{i,n}^m + \left(q_{i,l}^{m-1} - q_{i,n}^m \right),$$

for $l = n - 1, n, n + 1$, we can write

$$-\lambda_{i,n}^{L_m} u_{i,l}^m + a_{i,n}^{m} v_{i,l}^m = r_{i,l}^m + u_{i,l}^m O \left(k \right) + v_{i,l}^m O \left(k \right) + u_{i,l}^m O \left(u_{i,l}^{m-1} , u_{i,n}^m , v_{i,l}^{m-1} , v_{i,n}^m \right) + v_{i,l}^m O \left(v_{i,l}^{m-1} , u_{i,n}^m , v_{i,l}^{m-1} , v_{i,n}^m \right),$$

$$-\lambda_{i,n}^{R_m} u_{i,l}^m + a_{i,n}^{m} v_{i,l}^m = s_{i,l}^m + u_{i,l}^m O \left(k \right) + v_{i,l}^m O \left(k \right) + u_{i,l}^m O \left(u_{i,l}^{m-1} , u_{i,n}^m , v_{i,l}^{m-1} , v_{i,n}^m \right) + v_{i,l}^m O \left(v_{i,l}^{m-1} , u_{i,n}^m , v_{i,l}^{m-1} , v_{i,n}^m \right).$$
where
\[ O\left(u_{i,l}^{m-1}, u_{i,n}^{m}, v_{i,l}^{m-1}, v_{i,n}^{m}\right) = O\left(u_{i,l}^{m-1}\right) + O\left(u_{i,n}^{m}\right) + O\left(v_{i,l}^{m-1}\right) + O\left(v_{i,n}^{m}\right). \]

Substituting these relations into (4.15) and (4.16), we obtain
\[ r_{i,n}^{m+1} = r_{i,n}^{m} - \sigma \lambda_{i,n}^{R,m} \left(r_{i,n}^{m} - r_{i,n}^{m-1}\right) + O_{i,n,n-1}^{m}, \quad n = 1, \ldots, N, \]
\[ s_{i,n}^{m+1} = s_{i,n}^{m} - \sigma \lambda_{i,n}^{L,m} \left(s_{i,n}^{m} - s_{i,n}^{m-1}\right) + O_{i,n,n+1}^{m}, \quad n = 0, \ldots, N - 1, \]
for \( m \geq 1 \), where
\[ O_{i,n,n-1}^{m} = O\left(h^2\right) + h \left(O\left(u_{i,n}^{m}\right) + O\left(v_{i,n}^{m}\right)\right) + u_{i,n-1}^{m}O\left(h\right) + v_{i,n-1}^{m}O\left(h\right) \]
\[ + u_{i,n-1}^{m}O\left(u_{i,n}^{m-1}, u_{i,n}, v_{i,n}^{m-1}, v_{i,n}\right) + v_{i,n-1}^{m}O\left(u_{i,n}^{m-1}, u_{i,n}, v_{i,n}^{m-1}, v_{i,n}\right) \]
and \( O_{i,n,n+1}^{m} \) is defined similarly with \( n-1 \) substituted by \( n+1 \). These are the recursive relations we need.

We now prove (4.17). Assume \( \delta_0 < \sigma/2 \). Then, \( mk \leq \delta_0 \) implies \( m < N - m \). The proof will be divided into three cases: (1) \( m \leq n \leq N - m \), (2) \( 0 \leq n < m \) and (3) \( N - m < n \leq N \). It may be helpful to compare the argument below with the proof of Theorem 2.1, in which the region \( D_i \) is divided into \( D_i^C \), \( D_i^L \) and \( D_i^R \).

**Case 1:** \( m \leq n \leq N - m \). Let
\[ e_m = \max_{m \leq n \leq N - m} \left\{ \left|r_{i,n}^m\right|, \left|s_{i,n}^m\right| \right\}. \]
In view of (4.11), the coefficients of \( r_{i,n}^m, r_{i,n-1}^m, s_{i,n}^m \) and \( s_{i,n+1}^m \) in (4.18) are all nonnegative. Hence, from (4.18),
\[ e_{m+1} \leq e_m + C \left(h^2 + he_m + e_m e_{m-1} + e_m^2\right), \quad m \geq 1 \]
where \( C > 0 \) is a constant. By initial condition (4.3),
\[ u_{i,n}^0 = v_{i,n}^0 = 0. \]
Thus, \( e_0 = 0 \). Also, by (4.18) with \( m = 0 \),
\[ r_{i,n}^1 = O\left(h^2\right) \quad \text{for } n = 1, \ldots, N, \]
\[ s_{i,n}^1 = O\left(h^2\right) \quad \text{for } n = 0, \ldots, N - 1. \]
(4.20)
This implies \( e_1 = O\left(h^2\right) \). Consider the linear difference equation with initial condition
\[ E_{m+1} = (1 + 3Ch) E_m + Ch^2, \quad m \geq 1, \quad E_1 = C_0 h^2. \]
where \( C_0 \) is so large that \( e_1 \leq C_0 h^2 \). It has the solution
\[
E_{m+1} = C_0 h^2 (1 + 3Ch)^m + \frac{h}{3} ((1 + 3Ch)^m - 1)
\]
\[
\leq h \left( C_0 h e^{3Chm} + \frac{1}{3} e^{3Chm} - 1 \right).
\]

Let \( \delta_0 \) be so small that \( e^{3C\delta_0/\sigma} < 4 \). Then, there is an \( h_0 > 0 \) such that \( E_m \leq h \) for all \( h \leq h_0 \) and \( mk \leq \delta_0 \). This implies that
\[
E_{m+1} \geq E_m + C \left( h^2 + hE_m + E_mE_{m-1} + E_m^2 \right), \quad E_1 \geq e_1.
\]

Hence,
\[
e_m \leq E_m \leq h,
\]
which leads to (4.17) with \( M = 1 \) in Case 1.

**Case 2:** \( 0 \leq n < m \). The proof in this case depends on the type of the boundary condition at the left end of the branch. Suppose the end is a source with the boundary condition (4.4). Let
\[
e_m = \max_{0 \leq n \leq N_m} \left\{ |r_{i,n}^m|, |s_{i,n}^m| \right\}.
\]
(As was the case in the proof of Theorem 2.1, it is more convenient to include the central trapezoidal part \( m \leq n \leq N - m \).) Hence, from (4.18)
\[
|r_{i,n}^{m+1}| \leq |e_m| + C \left( h^2 + hE_m + e_m e_{m-1} + e_m^2 \right) \quad \text{for } n = 1, \ldots, N - m,
\]
\[
|s_{i,n}^{m+1}| \leq |e_m| + C \left( h^2 + hE_m + e_m e_{m-1} + e_m^2 \right) \quad \text{for } n = 0, \ldots, N - m.
\]

Since by (4.4), \( u_{i,0}^m = 0 \), it follows that \( r_{i,0}^m = s_{i,0}^m \) for all \( m \). Therefore, \( e_m \) satisfies the same difference inequality (4.19). We also have \( e_1 = O(h^2) \) by (4.20). Thus, the above analysis gives \( e_m \leq h \).

Suppose the boundary condition is given by (4.5), then, \( u_{i,0}^m = 0 \) and
\[
r_{i,0}^m = \frac{\lambda_{L,m-1}^{i,0}}{\lambda_{R,m-1}^{i,0}} s_{i,0}^m
\]
for all \( m \geq 1 \). Let \( \hat{r}_{i,n}^m = r_{i,n}^m / M \) where \( M \) is sufficiently large such that
\[
M > \max_m \left\{ \left| \frac{\lambda_{L,m}^{i,0}}{\lambda_{R,m}^{i,0}} \right| \right\}.
\]
Then, (4.18) still holds with \( r \) substituted by \( \hat{r} \). Let

\[
e_m = \max_{0 \leq n \leq N-m} \left\{ \left| \hat{r}^m_{i,n} \right|, \left| s^m_{i,n} \right| \right\}.
\]

We again have (4.21) and

\[
|\hat{r}^m_{i,0}| \leq |s^m_{i,0}| \leq e_m + C (h^2 + h e_m + e_m e_{m-1} + e_m^2).
\]

Hence, \( e_m \) satisfies (4.19) again. Therefore,

\[
|r^m_{i,n}| \leq Mh, \quad |s^m_{i,n}| \leq h.
\]

Suppose the left end is a junction. We shall treat all the branches connected to the same junction simultaneously. Let \( j_1, \ldots, j_\nu \) be the incoming branches and \( j_{\nu+1}, \ldots, j_\mu \) the outgoing branches. It is easy to see that the boundary conditions (4.6)–(4.7) are satisfied if \( p \) and \( q \) are substituted by \( u \) and \( v \), respectively. Using the identities

\[
u^m_{i,n} = \frac{r^m_{i,n} - s^m_{i,n}}{\lambda^m_{i,n}}, \quad \lambda^m_{i,n} = \lambda^{R,m}_{i,n} - \lambda^{L,m}_{i,n} > 0,
\]

the equations for \( r \) and \( s \) have the form

\[
\frac{1}{\lambda^m_{j_1,N}} (r^m_{j_1,N} - s^m_{j_1,N}) - \frac{1}{\lambda^m_{i,N}} (r^m_{i,N} - s^m_{i,N}) = 0, \quad i = j_2, \ldots, j_\nu,
\]

\[
\frac{1}{\lambda^m_{j_1,N}} (r^m_{j_1,N} - s^m_{j_1,N}) - \frac{1}{\lambda^m_{i,0}} (r^m_{i,0} - s^m_{i,0}) = 0, \quad i = j_{\nu+1}, \ldots, j_\mu,
\]

\[
\sum_{l=1}^\nu \frac{1}{a^m_{j_l,N}\lambda^m_{j_l,N}} \left( \lambda^{R,m}_{j_l,N} r^m_{j_l,N} - \lambda^{L,m}_{j_l,N} s^m_{j_l,N} \right) - \sum_{l=1}^\mu \frac{1}{a^m_{j_l,N}\lambda^m_{j_l,N}} \left( \lambda^{R,m}_{j_l,N} r^m_{j_l,0} - \lambda^{L,m}_{j_l,N} s^m_{j_l,0} \right) = 0.
\]

The system can be solved for \( s^m_{j_1,N}, \ldots, s^m_{j_\nu,N}, r^m_{j_{\nu+1},0}, \ldots, r^m_{j_\mu,0} \) because the coefficient matrix

\[
\begin{pmatrix}
\frac{1}{\lambda^m_{j_1,N}} & \frac{1}{\lambda^m_{j_2,N}} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
-\frac{1}{\lambda^m_{j_1,N}} & 0 & \cdots & -\frac{1}{\lambda^m_{j_\mu,0}} \\
-\frac{\lambda^m_{1,N}}{\lambda^m_{j_1,N}a^m_{j_1,N}} & \frac{\lambda^m_{L,m}}{\lambda^m_{j_1,N}a^m_{j_1,N}} & \cdots & -\frac{\lambda^m_{L,m}}{\lambda^m_{j_\mu,0}a^m_{j_\mu,0}}
\end{pmatrix}
\]

has the determinant

\[
\prod_{l=1}^\nu (-1)^{\nu+1} \frac{(-1)^{\nu+1}}{\prod_{l=1}^\nu \lambda^m_{j_l,N} \prod_{l'=\nu+1}^\mu \lambda^m_{j_{l'},0}} \left( -\sum_{l=1}^\nu \lambda^{L,m}_{j_l,N} a^m_{j_l,N} + \sum_{l'=\nu+1}^\mu \lambda^{R,m}_{j_{l'},0} a^m_{j_{l'},0} \right) \neq 0.
\]
(We used here $\lambda_{i,n}^m > 0$, $a_{i,n}^m > 0$, $\lambda_{i,n}^{R_m} > 0$ and $\lambda_{i,n}^{L_m} < 0$.) Let the solution be written as

$$s_{i,N}^{m+1} = \sum_{l=1}^{\nu} m_{ji}^l s_{j_l,N}^{m+1} + \sum_{l'=\nu+1}^{\mu} m_{j_l^r}^i s_{j_l^r,0}^{m+1}, \quad i = j_1, \ldots, j_{\nu},$$

$$r_{i,0}^{m+1} = \sum_{l=1}^{\nu} n_{ji}^l r_{j_l,0}^{m+1} + \sum_{l'=\nu+1}^{\mu} n_{j_l^r}^i s_{j_l^r,0}^{m+1}, \quad i = j_{\nu+1}, \ldots, j_{\mu}.$$  \hfill (4.23)

Choose a constant $M$ such that

$$M > \max_{i=j_1 \ldots j_{\nu}} \left\{ \sum_{l=1}^{\mu} |m_{ji}^l|, \sum_{l=1}^{\mu} |n_{ji}^l| \right\}$$

and introduce

$$\hat{s}_{ji,n}^{m} = s_{ji,n}^{m}/M, \quad \hat{r}_{ji,r,n}^{m} = r_{ji,r,n}^{m}/M$$

for $l = 1, \ldots, \nu$, $l' = \nu+1, \ldots, \mu$. Equations in (4.18) still hold if $\hat{s}_{ji,n}^{m}$ and $\hat{r}_{ji,r,n}^{m}$ are substituted for $s_{ji,n}^{m}$ and $r_{ji,r,n}^{m}$, respectively. Let $e_m$ denote the maximum of the quantities

$$\max_{1 \leq l \leq \nu} \left\{ |r_{ji,n}^{m+1}|, |s_{ji,n}^{m}| \right\}, \quad \max_{0 \leq n \leq N-m} \left\{ |r_{ji,n}^{m+1}|, |s_{ji,n}^{m}| \right\}.$$  

(Notice again the inclusion of the middle part $m \leq n \leq N - m$.) Since the coefficients of $r$ and $s$ are all positive, it is easy to see that

$$|r_{ji,n}^{m+1}| \leq e_m + C (h^2 + h e_m + e_m e_{m-1} + e_m^2)$$

for $l = 1, \ldots, \nu$, $n = m, \ldots, N$ and

$$|s_{ji,n}^{m+1}| \leq e_m + C (h^2 + h e_m + e_m e_{m-1} + e_m^2)$$

for $l = \nu+1, \ldots, \mu$, $n = 0, \ldots, m$. Similar inequalities can be derived for $|\hat{s}_{ji,n}^{m+1}|$, $l = 1, \ldots, \nu$, $n = m, \ldots, N - 1$ and for $|\hat{r}_{ji,n}^{m+1}|$, $l' = \nu+1, \ldots, \mu$, $n = 1, \ldots, m$. Furthermore, by (4.23)

$$|\hat{s}_{ji,N}^{m+1}| = \frac{1}{M} \left| \sum_{l=1}^{\nu} m_{ji}^l r_{j_l,N}^{m+1} + \sum_{l'=\nu+1}^{\mu} m_{j_l^r}^i s_{j_l^r,0}^{m+1} \right| \leq \max_{1 \leq l \leq \nu} \left\{ |r_{ji,l,N}^{m+1}|, |s_{ji,l,0}^{m+1}| \right\},$$

$$|\hat{r}_{ji,N}^{m+1}| = \frac{1}{M} \left| \sum_{l=1}^{\nu} n_{ji}^l r_{j_l,N}^{m+1} + \sum_{l'=\nu+1}^{\mu} n_{j_l^r}^i s_{j_l^r,0}^{m+1} \right| \leq \max_{1 \leq l \leq \nu} \left\{ |r_{ji,l,N}^{m+1}|, |s_{ji,l,0}^{m+1}| \right\}.$$  

Therefore, we achieve again the difference inequality (4.19) for $e_m$. Hence, $e_m \leq h$, and consequently,

$$|r_{i,n}^{m}| \leq h, \quad |s_{i,n}^{m}| \leq M h.$$  

This not only proves (4.17) for Case 2, but also for the part of Case 3 where the right endpoint is a junction.
Case 3: $N - m \leq n \leq N$. It only remains to discuss the case where the right end is a terminal. If the boundary condition is given by (4.8), the results follow from similar arguments in Case 2, when the source end boundary condition is either (1.4) or (1.5). Thus, we shall only discuss the case when the boundary condition is given by (4.9), which corresponds to the windkessel-type boundary condition (1.7) for the differential equations.

From (1.7), we derive

$$
\frac{1}{k} (P_{i,N}^{m+1} - P_{i,N}^m) - \eta_i \frac{1}{k} (Q_{i,N}^{m+1} - Q_{i,N}^m) + \frac{\delta_i}{2} (P_{i,N}^{m+1} + P_{i,N}^m)
$$

$$
- \frac{\varepsilon_i}{2} (Q_{i,N}^{m+1} + Q_{i,N}^m) = W_i \left( \left( m + \frac{1}{2} \right) k \right) + O \left( k^2 \right).
$$

Subtracting (4.9) from above yields

$$
\frac{1}{k} (u_{i,N}^{m+1} - u_{i,N}^m) - \eta_i \frac{1}{k} (v_{i,N}^{m+1} - v_{i,N}^m) + \frac{\delta_i}{2} (u_{i,N}^{m+1} + u_{i,N}^m) - \frac{\varepsilon_i}{2} (v_{i,N}^{m+1} + v_{i,N}^m) = O \left( k^2 \right).
$$

Let

$$
f^m = \left( 1 + \frac{\delta_i k}{2} \right) u_{i,N}^m - \left( \eta_i + \frac{\varepsilon_i k}{2} \right) v_{i,N}^m, \quad m = 0, 1, \ldots.
$$

The equation for $f^m$ has the form

$$
f^{m+1} = f^m + k (\varepsilon_i v_{i,N}^m - \delta_i u_{i,N}^m) + O \left( k^3 \right).
$$

Since $f^0 = 0$, the difference equation has the solution

$$
f^{m+1} = k \sum_{j=0}^{m} (\varepsilon_i v_{i,N}^j - \delta_i u_{i,N}^j) + O \left( k^2 \right).
$$

From (4.22), we obtain

$$
s_{i,N}^{m+1} = \frac{M_i^m}{N_i^m} p_{i,N}^{m+1} - \frac{k}{N_i^m} \sum_{j=0}^{m} (\varepsilon_i v_{i,N}^j - \delta_i u_{i,N}^j) + O \left( k^2 \right). \quad (4.24)
$$

where

$$
M_i^m = \frac{1}{\lambda_{i,n}^m} \left( 1 + \frac{\delta_i k}{2} - \left( \eta_i + \frac{\varepsilon_i k}{2} \right) \frac{\lambda_{i,n}^{R,m}}{a_{i,n}^m} \right),
$$

$$
N_i^m = \frac{1}{\lambda_{i,n}^m} \left( 1 + \frac{\delta_i k}{2} - \left( \eta_i + \frac{\varepsilon_i k}{2} \right) \frac{\lambda_{i,n}^{L,m}}{a_{i,n}^m} \right).
$$
(Notice that $N_j^m > 0$, hence (1.24) is valid.) Let $\tilde{s}_{i,n}^m = s_{i,n}^m/M$ where $M$ is a constant to be determined later. Also let

$$e_m = \max_{0 \leq j \leq m} \left\{ |r_{i,n}^j| : |\tilde{s}_{i,n}^j| \right\}.$$ 

Unlike previous cases where $e_m$ depends on the $m$-th level quantities, here it is more convenient to let $e_m$ be the maximum of all the lower level quantities. Then, by (4.18) modified with $\tilde{s}$ substituted for $s$,

$$|r_{i,n}^{m+1}| \leq e_m + C \left( h^2 + he_m + e_m e_{m-1} + e_m^2 \right)$$

for $n = m, \ldots, N$ and

$$|\tilde{s}_{i,n}^{m+1}| \leq e_m + C \left( h^2 + he_m + e_m e_{m-1} + e_m^2 \right)$$

for $n = m, \ldots, N - 1$, where $C$ is a positive constant. Also, by (1.24) and the relation $mk \leq \delta_0$,

$$|\tilde{s}_{i,N}^{m+1}| \leq \frac{1}{M} \left| \frac{M_i^m}{N_i^m} \right| |r_{i,N}^{m+1}| + \delta_0 C'e_m + O(h^2)$$

where $C' > 0$ is constant. Hence, from (1.23) we see that if $M$ is sufficiently large and $\delta_0$ is sufficiently small, we can ensure

$$|\tilde{s}_{i,N}^{m+1}| \leq e_m + C \left( h^2 + he_m + e_m e_{m-1} + e_m^2 \right).$$

(This is where the boundary condition (4.10) fails. Instead of $O(h^2)$, it can only provide $O(h)$, which is inconsistent with (4.19).) Thus, $e_m$ satisfies the relation (4.19), which leads to $e_m \leq h$. We have thus shown that

$$|r_{i,n}^m| \leq h, \quad |\tilde{s}_{i,n}^m| \leq Mh.$$ 

This completes the proof of Case 3, and also the entire theorem. ■

5 Discussion

We have given a rather thorough treatment to the initial-boundary value problem of the first-order quasilinear system (1.8) with various source and terminal boundary conditions. From our results, it can be seen that the junction condition (1.4), which stems from the conservation of mass and Navier-Stokes momentum, is consistent with the differential equations. Also, the windkessel-type terminal boundary condition does not cause problems to the solvability. However, due to the nature of the first-order hyperbolic equations, the existence of global solution generally is not guaranteed. This problem may disappear if more
accurate models are used. For example, in (1.8) and its special case (1.1), only the effect of viscosity on the wall of the vessels is taken into consideration. If we include viscosity more comprehensively, a term of $\mu \nabla^2 Q_i$ appears in the right side of the second equations of (1.8) and (1.1). The system then becomes parabolic, instead of hyperbolic. It is well-known that parabolic systems have better regularity properties than hyperbolic ones. Therefore, it may be possible to prove the existence of global solutions. We are currently investigating this issue.

We have developed a numerical scheme for the computation of solutions and proved its convergence. Although our scheme uses a nonstaggered method similar to the one developed by Raines, et al [11, 12], they are substantially different. (By nonstaggered, we mean the values of $P_i$ and $Q_i$ are approximated at the same mesh points, unlike the staggered method developed in [5, 8].) This is because ours is based on the normal form of the equations and takes into account of the characteristic directions. This may explain why our scheme converges even if the network has loops while the other can break down (cf. [8]).

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