Study of $B_s \to \phi \ell^+ \ell^-$ decays in the PQCD factorization approach with lattice QCD input

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In this paper, we studied systematically the semileptonic decays $B_s \to \phi \ell^+ \ell^-$ with $\ell^- = (e^-, \mu^-, \tau^-)$ by using the perturbative QCD (PQCD) and the “PQCD+Lattice” factorization approach, respectively. We first evaluated all relevant form factors $F_i(q^2)$ in the low $q^2$ region using the PQCD approach, and we also took the available lattice QCD results at the high-$q^2$ region as additional input to improve the extrapolation of $F_i(q^2)$ from the low-$q^2$ region to the endpoint $q_{\text{max}}^2$. We then calculated the branching ratios and many other physical observables: $A_{FB}(\ell)$, $F^\ell_i$, $S_{4,4,7}$, $A_{5,6,8,9}$ and the clean angular observables $P_{1,2,3}$ and $P^\ell_{4,5,6,8}$. From our studies, we find the following points: (a) the PQCD and “PQCD+Lattice” predictions of $B(B_s \to \phi \mu^+ \mu^-)$ are about $7 \times 10^{-7}$, which agree well with the LHCb measured value and the QCD sum rule prediction within one standard deviation; (b) we defined and calculated the ratios of the branching ratios $R^\mu_\phi$ and $R^\tau_\phi$; (c) the PQCD and “PQCD+Lattice” predictions of the longitudinal polarization $F_1$, the CP averaged angular coefficients $S_{4,4,7}$ and the CP asymmetry angular coefficients $A_{5,6,8,9}$, agree with the LHCb measurements in all considered bins within the still large experimental errors; and (d) for those currently still unknown observables $R^\mu_\phi$, $R^\tau_\phi$, $A^\phi_{FB}$, $P_{1,2,3}$ and $P^\ell_{4,5,6,8}$, we suggest LHCb and Belle-II Collaboration to measure them in their experiments.

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I. INTRODUCTION

In the Standard Model (SM) of particle physics, one treats these three generations of the charged leptons $\ell^- = (e^-, \mu^-, \tau^-)$ as exact copies of each other. These charged leptons behave in the same way but differ only in the masses determined by their Yukawa coupling to the Higgs boson. The lepton flavor universality (LFU), i.e. the equality of the coupling to the all electroweak gauge bosons among three families of leptons, has been regarded as an exact symmetry for quite a long time [1]. In recent years, however, some physics observables associated with the flavor-changing neutral current (FCNC) transitions $b \to s \ell \ell$ have exhibited deviations from the SM expectations. These include the LFU-violating (LFUV) ratios $R_K$ and $R_K^\ast$ [2, 3], whose measurements deviates from $\mu - e$ universality [4, 5] by around 2.5$\sigma$. More notably, the measurements of the angular observable $P_4^s$ of $B \to K^\ast \mu^+ \mu^-$ decay in the large recoil region [6–10] as reported by the LHCb [11, 12] and Belle Collaboration [13] point to a deviation of about 3$\sigma$ with respect to the SM prediction [14].

As is well known, the FCNC $b \to s$ transition is forbidden at tree-level, but proceeds by way of loop diagrams with a very low rate. Due to the strong suppression within SM, such kinds of FCNC decays may be sensitive to the possible new physics (NP) effects. Therefore, the semileptonic $b \to s \ell \ell$ decay has received striking attentions by means of measurements of the inclusive $B \to X_s \ell^+ \ell^-$ and/or the exclusive $B \to K^{(*)} \ell^+ \ell^-$ decays and their comparison with the SM predictions. Besides the decay rates, many angular observables of the semileptonic $B \to K^\ast \mu^+ \mu^-$ decays have also been measured previously [11–13]. The precision of the experimental measurements will also be expected to upgrade remarkably in the forthcoming year.

The semileptonic decay $B_s \to \phi \mu^+ \mu^-$, which is closely relevant to the decay $B \to K^\ast \mu^+ \mu^-$, offers an alternative scene to check out the same fundamental quark process, in a different hadronic background. On the theoretical side, various studies on the quark level $b \to s$ transition and the exclusive $B_{(s)} \to V \ell^+ \ell^-$ decays by using rather different

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we present our theoretical predictions of all relevant physical observables. The paper is organized as follows: In Sec. II, we give a short review for the kinematics of the $B_s \to \phi \ell^+ \ell^-$ decays including distribution amplitudes of $B_s$ and $\phi$ mesons. Sec. III is devoted to the theoretical framework including Hamiltonian and transition form factors based on $k_T$ factorization formalism. In Sec. IV, we define explicitly all observables for $B_s \to \phi \ell^+ \ell^-$ decays. In Sec. V we present our theoretical predictions of all relevant physical observables of the considered decay modes, compare these predictions with those currently available experimental measurements and make some phenomenological analysis. A short summary is given in the last section.

II. KINEMATICS AND THE WAVE FUNCTIONS

We treat the $B_s$ meson at rest as a heavy-light system. The kinematics of the semileptonic $B_s \to \phi \ell^+ \ell^-$ decays in the large-recoil (low $q^2$) region will be discussed below, where the PQCD factorization approach is applicable to the considered decays. In the rest frame of $B_s^0$ meson, we define the $B_s^0$ meson momentum $p_1$, the $\phi$ momentum $p_2$ in the light-cone coordinates as Ref. [36]. We also use $x_i$ to denote the momentum fraction of light anti-quark in each meson and set the momentum $p_i$ and $k_i$ (the momenta carried by the spectator quark in $B_s$ and $\phi$ meson) in the following forms:

\[
\begin{align*}
    p_1 &= \frac{m_B}{\sqrt{2}}(1,1,0,0), \\
    p_2 &= \frac{rm_B}{\sqrt{2}}(\eta^+,\eta^-,0,0), \\
    k_1 &= (0, x_1 \frac{m_B}{\sqrt{2}}, k_1), \\
    k_2 &= \frac{m_B}{\sqrt{2}}(x_2 \eta^+, x_2 \eta^-, k_2).
\end{align*}
\]

where the mass ratio $r = m_\phi/m_{B_s}$, and the factor $\eta^\pm$ is defined in the following form:

\[
\eta^\pm = \eta \pm \sqrt{\eta^2 - 1}, \quad \text{with} \quad \eta = \frac{1}{2r} \left[ 1 + r^2 - \frac{q^2}{m_{B_s}^2} \right],
\]

where $q = p_1 - p_2$ is the lepton-pair four-momentum. For the final state $\phi$ meson, its longitudinal and transverse polarization vector $\epsilon_L,T$ can be written as

\[
\epsilon_L = \frac{1}{\sqrt{2}}(\eta^+, -\eta^-, 0), \quad \epsilon_T = (0, 0, 1).
\]
For the $B_s$ meson wave function, we use the same kind of parameterizations as in Refs. [37, 38]

$$\Phi_{B_s} = \frac{i}{\sqrt{2\Lambda_c}}(\bar{b}B_s + m_{B_s})\gamma_5\phi_{B_s}(k_1).$$

(4)

Here only the contribution of the Lorentz structure $\phi_{B_s}(k_1)$ is taken into account, since the contribution of the second Lorentz structure $\bar{\phi}_{B_s}$ is numerically small and has been neglected. We adopted the distribution amplitude of the $B_s$ meson in the similar form as that of $B$-meson in the $SU(3)$ limit being widely used in the PQCD approach

$$\phi_{B_s}(x, b) = N_{B_s} x^2 (1 - x)^2 \exp \left[ -\frac{m_{B_s}^2 x^2}{2x_{B_s}} - \frac{1}{2}(\omega_{B_s} b)^2 \right].$$

(5)

In order to estimate the theoretical uncertainties induced by the variations of $\phi_{B_s}(x, b)$, one usually take $\omega_{B_s} = 0.50 \pm 0.05$ GeV for $B^0_s$ meson. The normalization factor $N_{B_s}$ depends on the values of the shape parameter $\omega_{B_s}$ and the decay constant $f_{B_s}$ and defined through the normalization relation: $\int_0^1 dx \phi_{B_s}(x, 0) = f_{B_s}/(2\sqrt{6})$ [38].

For the vector meson $\phi$, the longitudinal and transverse polarization components can both provide the contribution. Here we adopt the wave functions of the vector $\phi$ as in Ref. [40]:

$$\Phi^{\parallel}_\phi(p, \epsilon_L) = \frac{i}{\sqrt{6}} \left[ \phi_L m_\phi \phi_\phi(x) + \phi_L \phi_\phi^T(x) + m_\phi \phi_\phi^T(x) \right],$$

(6)

$$\Phi^{\perp}_\phi(p, \epsilon_T) = \frac{i}{\sqrt{6}} \left[ \phi_T m_\phi \phi_\phi(x) + \phi_T \phi_\phi^T(x) + m_\phi \epsilon_{\omega\mu\nu\rho} \gamma_5 \gamma_\nu \epsilon_T^\nu \epsilon_T^\nu \phi_\phi(x) \right],$$

(7)

where $p$ and $m_\phi$ are the momentum and the mass of the $\phi$ meson, $\epsilon_L$ and $\epsilon_T$ correspond to the longitudinal and transverse polarization vectors of the vector meson $\phi$, respectively. The twist-2 DAs $\phi_\phi$ and $\phi_\phi^T$ in Eqs. (6, 7) can be reconstructed as a Gegenbauer expansion [40]:

$$\phi_\phi(x) = \frac{3 f_\phi}{\sqrt{6}} x(1 - x) \left[ 1 + \sum_{n=1}^2 a^{\parallel}_{\phi n} C_n^{3/2}(t) \right],$$

$$\phi_\phi^T(x) = \frac{3 f_\phi^T}{\sqrt{6}} x(1 - x) \left[ 1 + \sum_{n=1}^2 a^{\perp}_{\phi n} C_n^{3/2}(t) \right],$$

(8)

where $C_n^{3/2}$ are Gegenbauer polynomials with the coefficients in front called Gegenbauer moments $a^{\parallel}_{\phi n}$ and $a^{\perp}_{\phi n}$ are the longitudinal and transverse components of the decay constants with $f_\phi = 0.231 \pm 0.04$ GeV and $f_\phi^T = 0.20 \pm 0.01$ GeV as given in Ref. [40]. Considering the small uncertainties caused by the scale dependence, the Gegenbauer moments in the DAs are chosen at a starting scale $\mu = 1$ GeV although they can evolve perturbatively to higher scales. In this case, we adopt the Gegenbauer polynomials and moments appeared in above equations are the same ones as those in Refs. [40–43].

$$a^{\parallel}_{1 n} = 0, \ a^{\parallel}_{2 n} = 0.18 \pm 0.08, \ a^{\perp}_{2 n} = 0.14 \pm 0.07, \ C_2^{3/2}(t) = \frac{3}{2} (5t^2 - 1),$$

(9)

The twist-3 DAs $\phi_\phi^T$ and $\phi_\phi^{\parallel}$ in Eqs. (6, 7) are the same ones as those defined in Ref. [40]:

$$\phi_\phi^{\parallel} = \frac{3 f_\phi^T}{2\sqrt{6}} t^2, \ \phi_\phi^T = \frac{3 f_\phi^T}{2\sqrt{6}} (-t), \ \phi_\phi^{\parallel} = \frac{3 f_\phi}{8\sqrt{6}} (1 + t^2), \ \phi_\phi^T = \frac{3 f_\phi}{4\sqrt{6}} (-t),$$

(10)

### III. THEORETICAL FRAMEWORK

#### A. Effective Hamiltonian for $b \to s\ell^+\ell^-$ decays

The semileptonic decay $B_s \to \phi\ell^+\ell^-$ involves a $b \to s\ell^+\ell^-$ quark-level transition. The process takes place through FCNC transition which is loop suppressed and may be sensitive to the new physics contributions. In the framework of the SM, the corresponding effective Hamiltonian of the $b \to s\ell^+\ell^-$ transitions are expressed as in Refs. [44–48]:

$$\mathcal{H}_{\text{eff}} = \frac{-4G_F}{\sqrt{2}} V_{tb} V_{ts} \left[ C_1(\mu) \mathcal{O}_1(\mu) + C_2(\mu) \mathcal{O}_2(\mu) + \sum_{i=3}^{10} C_i(\mu) \mathcal{O}_i(\mu) \right] + \lambda_i \left[ C_1(\mu) [\mathcal{O}_1(\mu) - \mathcal{O}_1^\mu(\mu)] + C_2(\mu) [\mathcal{O}_2(\mu) - \mathcal{O}_2^\mu(\mu)] \right] + \text{h.c.},$$

(11)
with the Fermi constant $G_F = 1.16638 \times 10^{-5} \, GeV^{-2}$, the CKM ratio $\lambda_u \equiv V_{ub} V_{us}^* / (V_{tb} V_{ts}^*)$, $C_1(\mu)$ and $O_1(\mu)$ are the Wilson coefficients and the 4-fermion operators at the renormalization scale $\mu$. In SM, a suitable basis of the operators $O_1(\mu)$ for $b \to s \ell^+ \ell^-$ transition is given by the current-current operators $O_{1,2}^{q,c}$, the QCD penguin operators $O_{3-6}$, the electromagnetic penguin operator $O_7$ and the chromomagnetic penguin operator $O_8$, as well as the semileptonic operators $O_{9,10}$:

\begin{align}
O_1^f &= (\bar{s}_\alpha \gamma_\mu P_L c_\beta)(\bar{c}_\beta \gamma_\mu P_L b_\alpha), \\
O_1^u &= (\bar{s}_\alpha \gamma_\mu P_L u_\beta)(\bar{u}_\beta \gamma_\mu P_L b_\alpha), \\
O_3 &= (\bar{s}_\alpha \gamma_\mu P_L b_\alpha) \sum_q (\bar{q}_\beta \gamma_\mu P_L q_\beta), \\
O_5 &= (\bar{s}_\alpha \gamma_\mu P_L b_\alpha) \sum_q (\bar{q}_\beta \gamma_\mu P_L q_\beta), \\
O_7 &= \frac{e}{16\pi^2} m_b \bar{s}_\alpha \sigma^{\mu\nu} P_R b_\alpha F_{\mu\nu}, \\
O_9 &= \frac{\alpha_{em}}{4\pi} (\bar{s}_\alpha \gamma_\mu P_L b_\alpha) \sum_q (\bar{\ell}_q \gamma_\mu \ell_q). 
\end{align}

Here $g$ and $\alpha_{em}$ is the strong coupling constant and the fine-structure constant respectively, $P_{L,R} = (1 \mp \gamma_5)/2$ are the chiral projection operators, $T_{a\beta}$ denotes the generators of the $SU(3)_C$ group and $m_b$ is the running $b$ quark mass in the $\overline{MS}$ scheme; $F_{\mu\nu}$ and $G_{\mu\nu}^a$ are the electromagnetic and chromomagnetic tensors, respectively. In Fig. 1, we show the typical Feynman diagrams for the semileptonic decay $B_s \to \phi \ell^+ \ell^-$ in the PQCD approach. The dominant contribution to $b \to s \ell^+ \ell^-$ transitions are given by $O_7$ and $O_{9,10}$, as well as $O_1^{q,c}$. The operator $O_7$ corresponds to the \(\gamma\)-penguin diagram, as shown in Fig. 1(a). The operators $O_{9,10}$ describe the sum of the contributions from the $Z$ penguin in Fig. 2(a) and the $W$ box diagrams in Fig. 2(b). The operators $O_1^{q,c}$ involve a long-distance (LD) contribution, which origins in the real $u\bar{u}, d\bar{s}$ and $c\bar{c}$ intermediate states, namely the $(\rho, \omega, \phi)$ family in Fig 1(c), coupled to the lepton pair via the virtual photon. This contribution is proportional to $C_9$ and can be absorbed into an effective Wilson coefficient $C_9^{eff}$ [49].

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{FeynmanDiagrams.pdf}
\caption{The typical Feynman diagrams for the semileptonic decays $B_s \to \phi(1020)\ell^+\ell^-$ in PQCD approach with the FCNC contributions due to the operators $O_7$ denoted as black squares.}
\end{figure}

Since the contributions from the subleading chromomagnetic penguin, quark-loop and annihilation diagrams are highly suppressed for the considered $b \to s \ell^+ \ell^-$ decays [47], we will neglect them in our calculations. The decay amplitude for $b \to s \ell^+ \ell^-$ loop transition can be decomposed as a product of a short-distance contributions through Wilson coefficients and long-distance contribution which is further expressed in terms of form factors,

\begin{equation}
A(b \to s \ell^+ \ell^-) = \frac{G_F \alpha_{em}}{\sqrt{2} \pi} V_{tb} V_{ts}^* \left\{ C_9^{eff}(q^2) [\bar{s}_\gamma \gamma_\mu P_L b] [\bar{\ell}_\gamma \gamma_\mu \ell] + C_{10} [\bar{s}_\gamma \gamma_\mu P_L b] [\bar{\ell}_\gamma \gamma_\mu \gamma_5 \ell] \\
- 2 m_b C_7^{eff} [\bar{s}_\gamma \sigma_{\mu\nu} \frac{q^\nu}{q^2} P_R b] [\bar{\ell}_\gamma \gamma_\mu \ell] \right\},
\end{equation}

where $C_7^{eff}(\mu)$ and $C_9^{eff}(\mu)$ are the effective Wilson coefficients, defined as in Refs. [38, 50]

\begin{align}
C_7^{eff}(\mu) &= C_7(\mu) + C_{b\to s\gamma}(\mu), \\
C_9^{eff}(\mu, q^2) &= C_9(\mu) + Y_{pert}(s) + Y_{res}(q^2).
\end{align}

The term $C_{b\to s\gamma}$ in above equation is the absorptive part of $b \to s\gamma$ transition and was given in Ref. [50]

\begin{equation}
C_{b\to s\gamma}(\mu) = i\alpha_s \left\{ \frac{2}{9} \mu^{14/3} \left[ \frac{x_t (x_t^2 - 5x_t - 2)}{8 (x_t - 1)^3} + \frac{3x_t^2 \ln x_t}{4(x_t - 1)^4} - 0.1687 \right] - 0.03 C_2(\mu) \right\},
\end{equation}
where \( x_t = m_t^2/m_W^2 \) and \( \eta = \alpha_s(m_W)/\alpha_s(\mu) \). The expressions of all relevant Wilson coefficients \( C_i(\mu) \) in above equations and their running with the renormalization scale \( \mu \) at the NLO level can be found easily for example in Ref. [44].

The term \( Y_{\text{pert}}(s) \) in Eq. (15) describes the short distance contribution from the soft-gluon emission and the one-loop contribution of the four-quark operators \( O_1 - O_6 \). The term \( Y_{\text{res}}(q^2) \) in Eq. (15) includes the contributions of the virtual resonances described by the Breit-Wigner form prescribed in Refs.[48, 51–54].

\[
Y_{\text{pert}}(s) = 0.124 \omega(s) + g(\hat{m}_c, \hat{s})C_0 + \lambda_u \left[ g(\hat{m}_c, \hat{s}) - g(\hat{m}_u, \hat{s}) \right] (3C_1 + C_2) \\
- \frac{1}{2} g(\hat{m}_d, \hat{s}) (C_3 + 3C_4) - \frac{1}{2} g(\hat{m}_b, \hat{s}) (4C_3 + 4C_4 + 3C_5 + C_6) \\
+ \frac{2}{9} (3C_3 + 4C_4 + 3C_5 + C_6),
\]

(17)

\[
Y_{\text{res}}(q^2) = -\frac{3\pi}{\alpha_s^{(m)}} \sum_{V=J/\psi, \psi'} \frac{m_V B(V \rightarrow \ell^+\ell^-) \Gamma_V^{\text{tot}}}{q^2 - m_V^2 + i m_V \Gamma_V^{\text{tot}}} \\
- \lambda_u g(\hat{m}_u, \hat{s}) (3C_1 + C_2) \cdot \sum_{V=\rho, \omega, \phi} \frac{m_V B(V \rightarrow \ell^+\ell^-) \Gamma_V^{\text{tot}}}{q^2 - m_V^2 + i m_V \Gamma_V^{\text{tot}}},
\]

(18)

where where \( C_0 = 3C_1 + C_2 + 3C_3 + C_4 + 3C_5 + C_6 \), \( s = q^2/m_\rho^2 \), \( \hat{m}_q = m_q/m_b \). In Eqs. (17,18), the function \( \omega(s) \) is the soft-gluon correction to the matrix element of operator \( O_0 \), while the functions \( g(\hat{m}_q, \hat{s}) \) describes the one-loop \((q\bar{q})\) contributions to the four-quark operators \( O_1 - O_6 \). One can find the explicit expressions of the function \( \omega(s) \) and \( g(\hat{m}_q, \hat{s}) \) directly for example in Ref. [30] and references therein.

The term \( Y_{\text{res}} \) in Eq. (18) denotes the long-distance resonance contributions from those \( B_s \rightarrow \phi V \rightarrow \phi(V \rightarrow \ell^+\ell^-) \) transitions, where \( V \) stands for the possible intermediate resonance states decaying to lepton pairs. The adopted values of the various resonance state from the Particle Data Group (PDG) [55] are summarized in Table I.

1. The charmless light vector mesons \( V = (\rho, \omega, \phi) \). The kinematic region where the light resonances \( (\rho, \omega, \phi) \) contribute is typically not excluded from the experimental analyses because their effects on branching fractions and other physical observables might be substantial [56].

2. The \( c\bar{c} \) charmonia \( V_{c\bar{c}} = (J/\psi(1S), \psi(2S), \psi(3770), \psi(4040), \psi(4160), \psi(4415)) \). The two lowest charmonium states \( \psi(1S) \) and \( \psi(2S) \) (i.e. \( J/\psi \) and \( \psi' \)), whose masses are below the open charm threshold \((DD)\), have tiny width and can induce large breaking of quark-hadron duality. Hence, the narrow charmonia resonance regions

FIG. 2: Typical Feynman loop diagrams: the \( \gamma \)-penguin (2a) with \( O_7 \), the \( z(\gamma) \)-penguin (2a) and \( W \)-box (2b) with \( O_{9,10} \), and the loops (2c) with \( O_{1,2} \).
are routinely rejected in the theoretical and experimental analysis. For the four higher charmonium resonances, however, they are broad and overlapping throughout the high-$q^2$ region. One usually make the integration over the full high-$q^2$ region.

TABLE I: The masses, decay widths and the branching ratios of the considered dilepton decays $V \rightarrow \ell^+\ell^-$ with $\ell = e, \mu$ [55].

| $V$   | $m_V$ (GeV) | $\Gamma^\ell_{tot}$ (MeV) | $BR(V \rightarrow \ell^+\ell^-)$ |
|-------|-------------|----------------------------|----------------------------------|
| $\rho(770)$ | 0.775       | 149.1                      | $4.63 \times 10^{-5}$           |
| $\omega(782)$ | 0.782       | 8.490                      | $7.38 \times 10^{-5}$           |
| $\phi(1020)$ | 1.019       | 4.249                      | $2.92 \times 10^{-4}$           |
| $J/\psi(1S)$ | 3.096       | 0.093                      | $5.96 \times 10^{-2}$           |
| $\psi(2S)$ | 3.686       | 0.294                      | $7.96 \times 10^{-3}$           |
| $\psi(3770)$ | 3.773       | 27.2                       | $9.60 \times 10^{-6}$           |
| $\psi(4040)$ | 4.039       | 80                         | $1.07 \times 10^{-5}$           |
| $\psi(4160)$ | 4.191       | 70                         | $6.90 \times 10^{-6}$           |
| $\psi(4415)$ | 4.421       | 62                         | $9.40 \times 10^{-6}$           |

**B. $B_s \rightarrow \phi$ transition form factors**

For the vector meson $\phi$ with polarization vector $\epsilon^\ast$, as usual, the relevant form factors for $B_s \rightarrow \phi$ transitions are $V(q^2)$ and $A_{0,1,2}(q^2)$ of the vector and axial-vector currents, and $T_{1,2,3}$ of the tensor currents. Between the form factors $A_{0,1,2}(q^2)$ at the point $q^2 = 0$, there is an exact relation $2m_\phi A_0(0) = (m_{B_s} + m_\phi) A_1(0) - (m_{B_s} - m_\phi) A_2(0)$ in order to avoid the kinematical singularity. Between the form factor $T_{1,2}$, there also exist a relation $T_1(0) = T_2(0)$ in an algebraic manner which is implied by the identity $\sigma^{\mu\nu}_2 = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \sigma_{\alpha\beta}$ with the $\epsilon^{0123} = +1$ convention for the the Levi-Civita tensor. Using the well-studied wave functions as given in Sec. II, the PQCD factorization formulas for the relevant form factors of $B_s \rightarrow \phi \ell^+\ell^-$ decays can be calculated and written in the following form:

$$
V(q^2) = 8\pi m_{B_s}^2 C_F (1 + r) \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_s}(x_1) \\
\times \left\{ \left[ -x_2 r \phi_0^\ast(x_2) + \phi_2^T(x_2) + \frac{1 + x_2 r \eta}{\sqrt{\eta^2 - 1}} \phi_0^\ast(x_2) \right] \cdot H_1(t_1) \\
+ \left[ \left( r + \frac{x_1}{2 \sqrt{\eta^2 - 1}} \right) \phi_0^\ast(x_2) - \frac{x_1 - 2 r \eta}{2 \sqrt{\eta^2 - 1}} \phi_0^\ast(x_2) \right] \cdot H_2(t_2) \right\},
$$

(19)

$$
A_0(q^2) = 8\pi m_{B_s}^2 C_F \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_s}(x_1) \times \left\{ \left[ (1 + x_2 r (2 \eta - r)) \phi_0(x_2) \right) \\
+ (1 - 2 x_2 r) \phi_0^\ast(x_2) + \frac{(1 - r \eta) - 2 x_2 r (\eta - r)}{\sqrt{\eta^2 - 1}} \phi_0^\ast(x_2) \right] \cdot H_1(t_1) \\
+ \left[ \left( \frac{x_1}{\sqrt{\eta^2 - 1}} \right) \phi_0(x_2) + \left( \frac{x_1}{2} - x_1 r \eta + r^2 \right) \phi_0^\ast(x_2) \right] \phi_0(x_2) \\
- \frac{x_1 (1 - r \eta) + 2 r (r - \eta)}{\sqrt{\eta^2 - 1}} - x_1 r \right\] \phi_0^\ast(x_2) \cdot H_3(t_2) \right\},
$$

(20)

$$
A_1(q^2) = 16\pi m_{B_s}^2 C_F \frac{r}{1 + r} \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_s}(x_1) \\
\times \left\{ \left[ (1 + x_2 r \eta) \phi_0^\ast(x_2) + (\eta - 2 x_2 r) \phi_2^T(x_2) + x_2 r \sqrt{\eta^2 - 1} \phi_0^\ast(x_2) \right] \cdot H_1(t_1) \\
+ \left[ \left( r \eta - \frac{x_1}{2} \right) \phi_0^\ast(x_2) + \left( r \sqrt{\eta^2 - 1} + \frac{x_1}{2} \right) \phi_0^\ast(x_2) \right] \cdot H_2(t_2) \right\},
$$

(21)
\[ A_2(q^2) = \frac{(1+r)^2(\eta-r)}{2r(\eta^2-1)} A_1(q^2) - 8\pi m_B^2 C_F \frac{1+r}{\eta^2-r} \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_1}(x_1) \]
\[ \times \left\{ \left[ \phi_{B_2}(x_2) + (1+2x_2 r^2 - (1+2x_2) r \eta) \phi_{B_2}(x_2) \right] \cdot H_1(t_1) \right\} \]
\[ + \left[ \phi_{B_2}(x_2) \cdot H_2(t_2) \right], \]
(22)

\[ T_1(q^2) = 8\pi m_B^2 C_F \frac{r}{1-r^2} \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_1}(x_1) \times \left\{ \left[ (1-2x_2) r \phi_{B_2}(x_2) \right] \right\} \]
\[ + \left[ (1+2x_2 r^2 - (1+2x_2) r \eta) \phi_{B_2}(x_2) \cdot H_1(t_1) \right\} \]
\[ + \left[ \phi_{B_2}(x_2) \cdot H_2(t_2) \right], \]
(23)

\[ T_2(q^2) = (1-r)^2(\eta+r) \frac{1-r^2}{\eta^2-1} \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_1}(x_1) \times \left\{ \left[ \phi_{B_2}(x_2) \right] \right\} \]
\[ + \left[ (1-2x_2 r^2 + 2x_2 r - (1+2x_2) r \eta) \phi_{B_2}(x_2) \cdot H_1(t_1) \right\} \]
\[ + \left[ \phi_{B_2}(x_2) \cdot H_2(t_2) \right], \]
(24)

\[ T_3(q^2) = \frac{(1-r)^2(\eta+r)}{2r(\eta^2-1)} T_2(q^2) - 8\pi m_B^2 C_F \frac{1-r^2}{\eta^2-1} \int dx_1 dx_2 \int b_1 b_1 b_2 b_2 \phi_{B_1}(x_1) \times \left\{ \left[ \phi_{B_2}(x_2) \right] \right\} \]
\[ + \left[ (1-2x_2 r^2 + 2x_2 r - (1+2x_2) r \eta) \phi_{B_2}(x_2) \cdot H_1(t_1) \right\} \]
\[ + \left[ \phi_{B_2}(x_2) \cdot H_2(t_2) \right], \]
(25)

where \( r = m_\phi/m_{B_1}, \) the twist-2 DAs (\( \phi_{B_2}^t, \phi_{B_2}^a \)) and the twist-3 DAs (\( \phi_{B_2}^{a,t} \)) have been defined in Eqs. (8,10). The function \( H_i(t_i) \) in above equations are of the following form
\[ H_i(t_i) = h_i(x_1, x_2, b_1, b_2) \cdot \alpha_s(t_i) \cdot S_i(x_2) \exp \left[ -S_{ab}(t_i) \right], \quad \text{for} \quad i = (1, 2). \]
(26)

The hard functions \( h_{1,2}(x_1, x_2, b_1, b_2) \) come form the Fourier transform of virtual quark and gluon propagators and they can be defined by
\[ h_1 = K_0(\beta_1 b_1) \left[ \theta(b_1 - b_2) I_0(\alpha_1 b_2) - \theta(b_1 - b_2) I_0(\alpha_1 b_2) \right], \]
\[ h_2 = K_0(\beta_2 b_1) \left[ \theta(b_1 - b_2) I_0(\alpha_2 b_2) - \theta(b_1 - b_2) I_0(\alpha_2 b_2) \right], \]
(27)

where \( K_0 \) and \( I_0 \) are modified Bessel functions, and
\[ \alpha_1 = m_{B_2} \sqrt{x_2 r \eta^2}, \quad \alpha_2 = m_{B_2} \sqrt{x_2 r \eta^2 - r^2 + r_s^2}, \quad \beta_1 = \beta_2 = m_{B_2} \sqrt{x_1 x_2 r \eta^2}, \]
(28)

where \( r = m_\phi/m_{B_1}, r_s = m_s/m_{B_1}. \)
The hard scales \( t_i \) in Eq. (26) are chosen as the largest scale of the virtuality of the internal particles in the hard \( b \)-quark decay diagram, including \( 1/b(i = 1, 2) \):

\[
t_1 = \max\{\alpha_1, 1/b_1, 1/b_2\}, \quad t_2 = \max\{\alpha_2, 1/b_1, 1/b_2\}. \tag{29}
\]

The threshold resummation factor \( S_t(x) \) in Eq. (26) is adopted from [57],

\[
S_t = \frac{2^{1+2x}\Gamma(3/2 + c)}{\sqrt{\pi t(1 + c)}} [x(1 - x)]^c, \tag{30}
\]

with a fitted parameter \( c(Q^2) = 0.04Q^2 - 0.51Q + 1.87 \) [34] and \( Q^2 = m_H^2 (1 - r^2) \) [58]. The function \( S_t(x) \) is normalized to unity.

The factor \( \exp[-S_{ab}(t)] \) in Eq. (26) contains the Sudakov logarithmic corrections and the renormalization group evolution effects of both the wave functions and the hard scattering amplitude with \( S_{ab}(t) = S_B(t) + S_M(t) \) [57],

\[
S_B(t) = s \left( x_1 \frac{m_B}{\sqrt{2}}, b_1 \right) + \frac{5}{3} \int_{1/b_1}^{t} \frac{d\tilde{\mu}}{\tilde{\mu}} \gamma_q(\alpha_s(\tilde{\mu})),
\]

\[
S_M(t) = s \left( x_2 \frac{m_B}{\sqrt{2}}, r_{\eta^+}, b_2 \right) + s \left( 1 - x_2 \frac{m_B}{\sqrt{2}}, r_{\eta^+}, b_2 \right) + 2 \int_{1/b_2}^{t} \frac{d\tilde{\mu}}{\tilde{\mu}} \gamma_q(\alpha_s(\tilde{\mu})),
\]

where \( \gamma_q = -\alpha_s/\pi \) is the anomalous dimension of the quark. The explicit expressions of the functions \( s(Q, b) \) can be found for example in Appendix A of Ref. [32].

### IV. OBSERVABLES FOR \( B_s \to \phi \ell^+ \ell^- \) DECAYS

In experimental analysis, the \( \bar{B}_s \to \phi \ell^+ \ell^- \) decay is treated as the four body differential decay distribution \( \bar{B}_s \to \phi(\to K^+ K^-) \ell^+ \ell^- \), and has been described in terms of the four kinematic variables [9, 10, 12, 18]: the lepton invariant mass squared \( q^2 \) and the three decay angles \( \Phi = (\cos \theta_K, \cos \theta_\ell, \Phi) \). The angle \( \theta_K \) is the angle between the direction of flight of \( K^+ \) and \( B_s \) meson in the rest frame of \( \phi \), \( \theta_\ell \) is the angle made by \( \ell^- \) with respect to the \( B_s \) meson in the dilepton rest frame and \( \Phi \) is the azimuthal angle between the two planes formed by dilepton and \( K^+ K^- \).

With the hadronic and leptonic amplitudes defined in Eq. (13), we write down the four fold differential distribution of four-body \( B_s \to \phi(\to K^+ K^-) \ell^+ \ell^- \) decay [12, 47, 59, 60],

\[
\frac{d^4 \Gamma}{dq^2 d\vec{\Omega}} = \frac{9}{32\pi} I(q^2, \vec{\Omega}), \quad d\vec{\Omega} = d\cos \theta_K d\cos \theta_\ell d\Phi,
\]

where the functions \( I(q^2, \vec{\Omega}) \) can be written in terms of a set of angular coefficients and trigonometric functions [59]:

\[
I(q^2, \vec{\Omega}) = \sum_i I_i(q^2) f_i(\vec{\Omega}) = I_1 \sin^2 \theta_K + I_2 \cos^2 \theta_K + (I_3^2 \sin^2 \theta_K + I_4^2 \cos^2 \theta_K) \cos 2\theta_\ell + I_5 \sin \theta_K \cos \theta_\ell \cos \Phi + I_6 \cos \theta_K \sin \theta_\ell \cos \Phi + I_7 \sin \theta_\ell \sin \Phi + I_8 \cos \theta_\ell \sin \Phi + I_9 \sin \theta_K \sin \theta_\ell \sin \Phi. \tag{34}
\]

For the CP-conjugated mode \( B_s \to \phi(\to K^- K^+) \ell^+ \ell^- \), the corresponding expression of the angular decay distribution is

\[
\frac{d^4 \tilde{\Gamma}}{dq^2 d\vec{\Omega}} = \frac{9}{32\pi} I(q^2, \vec{\Omega}), \tag{35}
\]

where the function \( I(q^2, \vec{\Omega}) \) is obtained from \( I(q^2, \vec{\Omega}) \) in Eq. (34) by making the complex conjugation for all weak phases in \( I_i \) [59], and numerically by the following substitution:

\[
I_{1,2,3,4,7} \to \tilde{I}_{1,2,3,4,7}, \quad I_{5,6,8,9} \to -\tilde{I}_{5,6,8,9}. \tag{36}
\]

The minus sign in Eq. (36) is a result of the convention that, under the previous definitions of three angles \( (\theta_K, \theta_\ell, \Phi) \), a CP transformation interchanges the lepton and anti-lepton, i.e., leading to the transformation \( \theta_\ell \to \theta_\ell - \pi \) and \( \Phi \to -\Phi \).
The angular coefficients $I_i$, which are functions of $q^2$, only, are usually expressed in terms of the transverse amplitudes \cite{7, 12}. In the limit of massless leptons, there are six such complex amplitudes: $A_{0}^{L,R}$, $A_{\perp}^{L,R}$ and $A_{\perp}^{L,L}$, where $L$ and $R$ refer to the chirality of the leptonic current. For the massive case, an additional complex amplitude $A_\ell$ is required, where the timelike component of the virtual gauge boson (which can later decay into dilepton) couple to an axial-vector current.

In Table II, we summarize the treatment of the angular distribution by decomposition of the angular coefficients $I_i(q^2)$ into seven transverse amplitude $A_{\perp}^{L,R}$ and $A_\ell$ as well as the corresponding trigonometric factor $f_\ell(\bar{\Omega})$. Here we will not consider scalar contribution to facilitate the comparison with Ref. \cite{30}. Notice that the distribution including lepton masses (but neglecting scalar $I_5 = 0$) contains eleven $I_i$, where only 10 of them are independent \cite{7, 61}. In the limit of massless leptons, it is easy to obtain the relations $I_1^\perp = 3I_2^\perp$ and $I_5^\perp = -I_7^\perp$ \cite{59}.

| $I_i(q^2)$ | $f_\ell(\bar{\Omega})$ |
|-----------|-----------------|
| $I_1(q^2)$ | $\sin^2 \theta_K$ |
| $I_2(q^2)$ | $\cos^2 \theta_K$ |
| $I_3(q^2)$ | $\sin^2 \theta_K \cos 2\theta_t$ |
| $I_4(q^2)$ | $\sin \theta_K \sin \theta_t \cos 2\Phi$ |
| $I_5(q^2)$ | $\sin 2\theta_K \sin \theta_t \cos \Phi$ |
| $I_6(q^2)$ | $\sin^2 \theta_K \theta_t \cos 2\Phi$ |
| $I_7(q^2)$ | $\sin 2\theta_K \sin \theta_t \sin \Phi$ |
| $I_8(q^2)$ | $\sin \theta_K \sin^2 \theta_t \cos 2\Phi$ |

The seven transverse amplitudes $A_{0}^{L,R}$, $A_{\perp}^{L,R}$, $A_\ell$ and $A_{\perp}^{L,L}$ of $B_{s} \to \phi \ell^+ \ell^-$ decay, in turn can be parameterized by means of the relevant form factors \cite{59, 62}:

$$A_{\perp}^{L,R} = -N_\ell \sqrt{2N_\phi} \sqrt{\lambda} \left[ (C_9^{\text{eff}} + C_{10}) \frac{V(q^2)}{m_{B_s} + m_{\phi}} + 2\hat{m}_b C_7^{\text{eff}} T_1(q^2) \right],$$

$$A_{\perp}^{L,L} = N_\ell \sqrt{2N_\phi} \left[ (C_9^{\text{eff}} + C_{10}) (m_{B_s} + m_{\phi}) A_1(q^2) + 2\hat{m}_b C_7^{\text{eff}} (m_{B_s}^2 - m_{\phi}^2) T_2(q^2) \right],$$

$$A_{0}^{L,R} = \frac{N_\ell \sqrt{N_\phi}}{2m_{B_s} \sqrt{q^2}} \left\{ (C_9^{\text{eff}} + C_{10}) \left[ m_{B_s}^2 - m_{\phi}^2 - q^2 (m_{B_s} + m_{\phi}) A_1(q^2) - \frac{\lambda}{m_{B_s} + m_{\phi}} A_2(q^2) \right] 
+ 2\hat{m}_b C_7^{\text{eff}} \left[ m_{B_s}^2 + 3m_{\phi}^2 - q^2 T_2(q^2) - \frac{\lambda}{m_{B_s}^2 - m_{\phi}^2} T_3(q^2) \right] \right\},$$

$$A_{\ell} = 2N_\ell \sqrt{N_\phi} \sqrt{\lambda} C_{10} A_0(q^2),$$

where $\lambda = (m_{B_s}^2 - m_{\phi}^2 - q^2)^2 - 4m_{\phi}^2 q^2$, $\hat{m}_b = m_b / q^2$ and the normalization constants are given as:

$$N_\ell = \frac{i\alpha_{em} G_F}{4\sqrt{2} \pi} V_{tb} V_{ts}^*, \quad N_\phi = \frac{8\sqrt{\lambda} q^2}{3 \times 256 \pi^3 m_{B_s} \sqrt{1 - 4m_{\phi}^2 / q^2}} B(\phi \to K^+ K^-).$$

In numerical calculations, we take $B(\phi \to K^+ K^-) = 0.492$ from PDG 2018 \cite{55}. It is easy to see that the narrow width approximation works well in the case of $\phi$ meson since $\Gamma_\phi / m_\phi = 4.17 \times 10^{-3} \approx 0$.

Analogous to Ref. \cite{59}, to separate CP-conserving and CP-violating effects, one can define the CP averaged angular coefficients $S_i$ and CP asymmetry angular coefficients $A_i$ normalized by the differential (CP-averaged) decay rate to
reduce the theoretical uncertainties,

\[ S_i = \frac{I_i + \bar{I}_i}{d(\Gamma + \Gamma)/dq^2}, \quad A_i = \frac{I_i - \bar{I}_i}{d(\Gamma + \Gamma)/dq^2}, \quad (42) \]

where \( I_i \) and \( \bar{I}_i \) have been defined in Eqs. (34,35,36) and Table II, and the differential decay rate reads (analogously for \( \bar{\Gamma} \)),

\[ \frac{d\Gamma}{dq^2} = \frac{1}{4} \left( 3I_1 + 6I_1^* - I_2^* - 2I_2^* \right). \quad (43) \]

Based on the definition of \( S_i \), one can find the relation \( 3S_1^* + 6S_2^* - I_2^* - 2S_2^* = 4 \).

Consequently, all established observables can be expressed in terms of \( S_i \) and \( A_i \):

1. The CP asymmetry

\[ A_{CP}(q^2) = \frac{d\Gamma/dq^2 - d\bar{\Gamma}/dq^2}{d\Gamma/dq^2 + d\bar{\Gamma}/dq^2} = \frac{1}{4} (3A_1^* + 6A_1^* - A_2^* - 2A_2^*). \quad (44) \]

2. The lepton forward-backward (CP) asymmetry:

\[ A_{FB}(q^2) = \frac{\int_0^1 - \int_{-1}^0 d\cos\theta_i \frac{d^2(\Gamma - \Gamma)}{dq^2 d\cos\theta_i}}{d(\Gamma + \bar{\Gamma})/dq^2} = \frac{3}{4} S_6^* , \quad (45) \]

\[ A_{FB}^{CP}(q^2) = \frac{\int_0^1 - \int_{-1}^0 d\cos\theta_i \frac{d^2(\Gamma + \bar{\Gamma})}{dq^2 d\cos\theta_i}}{d(\Gamma + \bar{\Gamma})/dq^2} = \frac{3}{4} A_6^*. \quad (46) \]

3. The \( \phi \) polarization fractions:

\[ F_L(q^2) = \frac{1}{4} (3S_1^* - S_2^*), \quad F_T(q^2) = \frac{1}{2} (3S_1^* - S_2^*). \quad (47) \]

In the massless limit, since the CP-averaged observable \( S_{1,2}^* \) obey the relations \( S_1^* = 3S_2^* \) and \( S_1^* = -S_2^* \), the definitions of the polarization fractions can be simplified directly as:

\[ F_L(q^2) = S_1^* = -S_2^*, \quad F_T(q^2) = \frac{4}{3} S_1^* = 4S_2^*. \quad (48) \]

4. The clean (no S-wave pollution) observables \( P_{1,2,3} \) and \( P_{4,5,6}^* \) in the natural basis can be defined in terms of the coefficients \( S_i \) through the following relations [60, 63]:

\[ P_1 = \frac{S_3}{2S_2^*}, \quad P_2 = \beta \frac{S_6}{8S_2^*}, \quad P_3 = -\frac{S_9}{4S_2^*}, \quad (49) \]

\[ P_4' = \frac{S_4}{\sqrt{S_1^*S_2^*}}, \quad P_5' = \frac{\beta S_5}{2\sqrt{S_1^*S_2^*}}, \quad P_6' = -\frac{\beta S_7}{2\sqrt{S_1^*S_2^*}}, \quad P_8' = -\frac{S_8}{\sqrt{S_1^*S_2^*}}, \quad (50) \]

where \( \beta = \sqrt{1 - 4m_t^2/q^2} \).

5. In the massless limit of leptons, the optimised observables \( P_i^{(ij)} \) [7] can be transformed as the following form:

\[ P_1 = \frac{2S_3}{F_T} = A_3^{(2)}, \quad P_2 = \frac{S_6}{2F_T} = \frac{1}{2} \text{Re}[A_T], \quad P_3 = -\frac{S_9}{F_T} = -\frac{1}{2} \text{Im}[A_T], \quad (51) \]

\[ P_4' = \frac{2S_4}{\sqrt{F_L(1 - F_L)}}, \quad P_5' = \frac{S_5}{\sqrt{F_L(1 - F_L)}}, \quad P_6' = -\frac{S_7}{\sqrt{F_L(1 - F_L)}}, \quad P_8' = -\frac{2S_8}{\sqrt{F_L(1 - F_L)}}, \quad (52) \]
One should know that our definitions of the CP averaged angular coefficients $S_i$ , the CP asymmetry angular coefficients $A_i$ and the clean observable $P_{1,2,3}$ and $P'_{4,5,6}$ differ from those adopted by the LHCb collaboration. To be specific, the reasons are the following:

(1) Our conventions for the angles to define the $B_s \to \phi \ell^+ \ell^-$ kinematics are identical to the Ref. [59] but different from the LHCb choices [12, 29]. The corresponding relations are the following:

$$\theta_K^{\text{LHCb}} = \theta_K, \quad \theta_{\ell}^{\text{LHCb}} = \pi - \theta_{\ell}, \quad \Phi^{\text{LHCb}} = -\Phi.$$  

(53)

Some angular coefficients $I_i$ , $S_i$ and $A_i$, consequently, will have different signs:

$$I_{4,6s,7,9}^{\text{LHCb}} = -I_{4,6s,7,9}, \quad S_{4,6s,7,9}^{\text{LHCb}} = -S_{4,6s,7,9}, \quad A_{4,6s,7,9}^{\text{LHCb}} = -A_{4,6s,7,9}.$$  

(54)

Other remaining coefficients $I_i$ ($S_i$ and $A_i$), however, have the same sign in both conventions.

(2) Our definitions of the clean observables $P_{1,2,3}$ and $P'_{4,5,6,8}$ in Eq. (50) in terms of $S_i$ may be different from those defined and used by the LHCb Collaboration for example in Ref. [12]. The resultant differences of the sign and normalization are of the following:

$$P_1^{\text{LHCb}} = P_1, \quad P_{2,3}^{\text{LHCb}} = -P_{2,3}, \quad P_{4,8}^{\text{LHCb}} = -\frac{1}{2}P_{4,8}', \quad P_{5,6}^{\text{LHCb}} = P_{5,6}'.$$  

(55)

### V. NUMERICAL RESULTS AND DISCUSSIONS

In the numerical calculations we use the following input parameters (here masses and decay constants are in units of GeV) [55]:

$$A_{MS}^{f=4} = 0.250, \quad \tau_B^0 = 1.509 ps, \quad m_b = 4.8, \quad m_W = 80.38, \quad m_\phi = 1.019$$

$$m_{B_s} = 5.367, \quad m_c = 0.00511, \quad m_\mu = 0.105, \quad m_\tau = 1.777,$$

$$f_{B_s} = 0.23, \quad f_\phi = 0.231 \pm 0.004, \quad f_\phi^T = 0.200 \pm 0.01,$$

$$a_{2\phi}^{||} = 0.18 \pm 0.08, \quad a_{2\phi}^\perp = 0.14 \pm 0.07.$$  

(56)

For the CKM matrix elements and angles, we adopt the independently measured parametrization with the updated parameters as [55]:

$$V_{tb} = 1.019 \pm 0.025, \quad V_{us} = 0.2243 \pm 0.0005,$$

$$V_{ts} = |V_{ts}| e^{-i\beta_s}, \quad |V_{ts}| = (39.4 \pm 2.3) \times 10^{-3}, \quad 2\beta_s = 0.021 \pm 0.031,$$

$$V_{ub} = |V_{ub}| e^{-i\gamma}, \quad |V_{ub}| = (3.94 \pm 0.36) \times 10^{-3}, \quad \gamma = (73.5^{+9.2}_{-8.1})^\circ.$$  

(57)

### A. The form factors and their extrapolations

For the considered semileptonic decays, the differential decay rates and other physical observables strongly rely on the value and the shape of the relevant form factors $V(q^2), A_{0,1,2}(q^2)$ and $T_{1,2,3}(q^2)$ for $B_s \to \phi \ell^+ \ell^-$ decays. These form factors have been calculated in rather different theories or models [22, 31–33, 40]. Since the PQCD predictions for the considered form factors are valid only at the large hadronic recoil (low-$q^2$) region, we usually calculate explicitly the values of the relevant form factors at the low-$q^2$ region, say $0 \leq q^2 \leq m_\tau^2$, and then make an extrapolation for all relevant form factors from the low-$q^2$ towards the high-$q^2$ region by using the pole model parametrization [64, 65] or other different methods.

In Refs. [66–68], we developed a new method: the so-called “PQCD+Lattice” approach. Here we still use the PQCD approach to evaluate the form factors at the low $q^2$ region, but take those currently available lattice QCD results for the relevant form factors at the high-$q^2$ region as the lattice QCD input to improve the extrapolation of the form factors up to $q^2_{\text{max}}$. In Refs. [67, 68], we used the Bourrely-Caprini-Lellouch (BCL) parametrization method [69, 70] instead of the traditional pole model parametrization motivated by the analytical properties and asymptotic behaviors of the heavy-to-light form factors.

In Table III, for the $B_s \to \phi$ transition form factors ($V(q^2), A_{0,1}(q^2), T_2(q^2)$), we quote directly the values of the lattice QCD results at three reference points of the high $q^2$ region, say $q^2 = 12, 16$ GeV$^2$ and $q^2_{\text{max}} = (m_{B_s} - m_\phi)^2 \approx$...
18.9 GeV$^2$, as listed in Table XXXI of Ref. [71]. In Ref. [71], the authors defined the helicity form factors $A_{12}(q^2)$ and $T_{23}(q^2)$ from the ordinary form factors $A_{1,2}(q^2)$ and $T_{2,3}(q^2)$:

\[ A_{12}(q^2) = \frac{(m_{B_\pm} + m_\phi)^2(m_{B_\pm}^2 - m_\phi^2 - q^2)A_1(q^2) - \lambda A_2(q^2)}{16m_{B_\pm}m_\phi^2(m_{B_\pm} + m_\phi)}, \]

\[ T_{23}(q^2) = \frac{m_{B_\pm} + m_\phi}{8m_{B_\pm}m_\phi^2} \left[ (m_{B_\pm}^2 + 3m_\phi^2 - q^2)T_2(q^2) - \frac{\lambda T_3(q^2)}{m_{B_\pm}^2 - m_\phi^2} \right], \quad (58) \]

where the kinematic variable $\lambda = (t_+ - q^2)(t_- - q^2)$ with $t_\pm = (m_{B_\pm} \pm m_\phi)^2$. From above two equations and the numerical values of $(A_1(q^2), T_2(q^2), A_{12}(q^2), T_{23}(q^2))$ as given in Table XXXI of Ref. [71], we can find the corresponding lattice QCD results of $A_2(q^2)$ and $T_3(q^2)$ at the two points $q^2 = (12, 16)$ GeV$^2$ by direct numerical calculations. Since when $q^2 \to q_{\text{max}}^2$, the parameter $\lambda(q_{\text{max}}^2)$ in Eq. (58) is also approaching zero simultaneously. One therefore can not determine $A_2(q_{\text{max}}^2)$ and $T_3(q_{\text{max}}^2)$ reliably from the values of $(A_{12}(q_{\text{max}}^2)$ and $T_{23}(q_{\text{max}}^2))$ as given in Ref. [71]. For the sake of the reader, we list in Table III the lattice QCD results of all relevant form factors at three reference points of high $q^2$ region.

### Table III: The values for the lattice QCD results of relevant $B_\pm \to \phi$ transition form factors at three reference points of $q^2$: $q^2 = 12, 16$ GeV$^2$ and $q_{\text{max}}^2 = (m_{B_\pm} - m_\phi)^2 \approx 18.9$ GeV$^2$ [71].

| $q^2$ | $V(q^2)$ | $A_0(q^2)$ | $A_1(q^2)$ | $A_2(q^2)$ | $T_1(q^2)$ | $T_2(q^2)$ | $T_3(q^2)$ |
|-------|----------|------------|------------|------------|------------|------------|------------|
| 12    | 0.77(6)  | 0.90(6)    | 0.44(3)    | 0.48(4)    | 0.69(4)    | 0.45(3)    | 0.46(4)    |
| 16    | 1.19(7)  | 1.32(7)    | 0.52(3)    | 0.54(4)    | 0.99(5)    | 0.53(3)    | 0.70(5)    |
| 18.9  | 1.74(10) | 1.85(10)   | 0.62(3)    |             | 1.36(8)    | 0.62(3)    |             |

In this work, we will use both the PQCD factorization approach and the “PQCD+Lattice” approach to evaluate all relevant form factors over the whole range of $q^2$.

1. In the PQCD approach, we use the definitions and formulae as given in Eqs.(19-25) to calculate the values of all relevant form factors $V(q^2)$, $A_{0,1,2}(q^2)$ and $T_{1,2,3}(q^2)$ in the low $q^2$ region: $0 \leq q^2 \leq m_\phi^2$. We then make the extrapolation for these form factors to the large $q^2$ region up to $q_{\text{max}}^2$ by using the selected parametrization method.

2. In the “PQCD+Lattice” approach, we take the lattice QCD results for the form factors at some large $q^2$ points as input and then make a combined fit to the PQCD and the lattice QCD results at the low and high $q^2$ region.

3. For both approaches, we always use the model-independent $z$-series parametrization, which is based on a rapidly converging series in the parameter $z$, as in Refs. [22, 47] to make the extrapolation. The entire cut $q^2$-plane will be mapped onto the unit disc $|z(q^2)| \leq 1$ under the conformal transformation as [72]

\[ z(q^2) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_- - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_- - t_0}} \]

where $t_\pm = (m_{B_\pm} \pm m_\phi)^2$ and $0 \leq t_0 < t_-$ is a auxiliary parameter which can be optimised to reduce the maximum value of $|z(q^2)|$ in the physical range of the form factors and will be taken in the same way as in Ref. [73]: $t_0 = t_-(1 - \sqrt{1 - t_- / t_+})$. The form factors are finally parameterized in the BCL version of the $z$-series expansion [69]

\[ F_{B_\pm \to \phi}^i(q^2) = \frac{F_{B_\pm \to \phi}^i(0)}{1 - q^2/m_{i,\text{pole}}^2} \left\{ 1 + \sum_{k=1}^{N} b_k^i [z(q^2, t_0)^k - z(0, t_0)^k] \right\}. \quad (60) \]

The input values of the various $s\bar{b}$-resonance masses with different quantum numbers $J^P$ from PDG-2018 [55] are summarized in Table IV. With the optimised value for $t_0$, the form factors can be well described by Eq.(60) which is truncated at $N = 1$ for the purpose of fitting the coefficients $b_k^i$ practically. The further discussions on the systematic uncertainties due to the dependence of truncation schemes and on the implementation of the strong unitary constraints can be seen in Refs. [73, 74].
FIG. 3: Theoretical predictions of the relevant form factors for $B_s \to \phi$ transition in the PQCD (red curves) and “PQCD+Lattice” (blue curves) approach with an extrapolation to the entire kinematical region by applying the $z$-series parameterizations. The red (blue) shaded band represents the theory uncertainties. The lattice data points with error in high-$q^2$ region can be found in Table III.

(4) In Fig. 3, we show the theoretical predictions of the form factors $V(q^2), A_{0,1,2}(q^2)$ and $T_{1,2,3}(q^2)$ for $B_s \to \phi$ transition using the PQCD (red curves) and “PQCD+Lattice” (blue curves) approach with an extrapolation to the entire kinematical region by applying the $z$-series parameterizations. The red and blue shaded band represent the theory uncertainties in two approaches. The black error bars in low-$q^2$ and the high-$q^2$ region are the PQCD results and the lattice QCD inputs, respectively.

In Table V, as a comparison, we show the central values of all relevant form factors in this work and other theoretical predictions as given in Refs. [19, 20, 38, 40, 75–81] at the scale $q^2 = 0$. There exist always some differences even
among the authors using the same approach. Taking the calculations based on the LCSR method as an example, the authors of Ref. [22] introduced the hadronic input parameters, Ball and Zwicky considered the one-loop radiative corrections [19], Yilmaz included the radiative and higher twist corrections and SU(3) breaking effects [76]. In Fig. 3, we show the PQCD and "PQCD+Lattice" predictions for the form factors $V(q^2)$, $A_{0,1,2}(q^2)$ and $T_{1,2,3}(q^2)$ in the whole range of $0 \leq q^2 \leq g_{max}$, where the theory uncertainties are obtained by adding all the separate uncertainties in quadrature. Although there are some real differences between the theoretical predictions obtained by employing different approaches or models, they are generally consistent with each other within the still large theoretical uncertainties.

### TABLE V

The theoretical predictions for the central values of the form factors of the $B_s \rightarrow \phi$ transitions at $q^2 = 0$ obtained by using rather different theories or models.

|                | $V(0)$ | $A_0(0)$ | $A_1(0)$ | $A_2(0)$ | $T_{1,2}(0)$ | $T_3(0)$ |
|----------------|--------|----------|----------|----------|--------------|----------|
| This work      | 0.31   | 0.26     | 0.25     | 0.24     | 0.26         | 0.20     |
| PQCD[40]       | 0.25   | 0.30     | 0.19     | –        | –            | –        |
| LCSR[22]       | 0.387  | 0.389    | 0.296    | –        | 0.309        | –        |
| LCSR[19]       | 0.434  | 0.474    | 0.311    | 0.234    | 0.349        | 0.175    |
| LCSR[76]       | 0.433  | 0.382    | 0.296    | 0.255    | 0.348        | 0.254    |
| QCDSR[77]      | 0.45   | 0.30     | 0.32     | 0.30     | 0.33         | 0.22     |
| RDA[78]        | 0.44   | 0.42     | 0.34     | 0.31     | 0.38         | 0.26     |
| RQM[79]        | 0.406  | 0.322    | 0.320    | 0.318    | 0.275        | 0.133    |
| SCET[80]       | 0.329  | 0.279    | 0.232    | 0.210    | 0.276        | 0.170    |
| HQEFT[20]      | 0.339  | 0.269    | 0.271    | 0.212    | 0.299        | 0.191    |
| SQEH[81]       | 0.259  | 0.311    | 0.194    | –        | –            | –        |
| CQM[16]        | 0.31   | 0.28     | 0.27     | 0.27     | 0.27         | 0.18     |
TABLE VI: Theoretical predictions for the total branching fractions $B(B_s \to \phi \ell^+ \ell^-)$ ( in units of $10^{-7}$ ) in the PQCD (the first row) and “PQCD+Lattice” (the second row) approaches. As a comparison, we also list the LHCb measured value for muon channel [29] and the QCDSR predictions for all three channels [77].

| BFs $B(B_s \to \phi e^+ e^-)$ | PQCD / “PQCD+Lattice” | QCDSR [77] | LHCb [29] |
|-----------------------------|-------------------------|------------|-----------|
| $B(B_s \to \phi e^+ e^-)$   | 8.65^{+0.05}_{-0.71} (FFs) $\pm$ 0.15($\mu$) $\pm$ 0.42(V_{tb}) $\pm$ 0.65(V_{tb}) | 7.12 $\pm$ 1.40 | 7.97^{+0.81}_{-0.80} |
| $B(B_s \to \phi \mu^+ \mu^-)$ | 8.44^{+0.12}_{-0.15} (FFs) $\pm$ 0.14($\mu$) $\pm$ 0.41(V_{tb}) $\pm$ 0.63(V_{tb}) | 7.06 $\pm$ 1.59 | 7.97^{+0.81}_{-0.80} |
| $B(B_s \to \phi \tau^+ \tau^-)$ | 7.16^{+0.27}_{-0.25} (FFs) $\pm$ 0.12($\mu$) $\pm$ 0.38(V_{tb}) $\pm$ 0.53(V_{tb}) | 6.95^{+0.47}_{-0.15} (FFs) $\pm$ 0.11($\mu$) $\pm$ 0.33(V_{tb}) $\pm$ 0.52(V_{tb}) | 0.35 $\pm$ 0.17 |

B. Observables for $B_s \to \phi \ell^+ \ell^-$

We now proceed to explore the phenomenological aspects of the cascade decays $B_s \to \phi (\to K^- K^+) \ell^+ \ell^-$, which allows us to define and compute a number of physical observables and compare them with those measured by experiments. We first compare our results for the branching ratios and angular observables with the experimental data reported by the LHCb Collaboration [29]. As studied systematically in last section, the physical observables accessible in the semileptonic decays $B_s \to \phi \ell^+ \ell^-$ [29] are the CP averaged differential branching fraction $dB/dq^2$, the CP-averaged $\phi$ meson longitudinal polarization fraction $F_1$, the forward-backward asymmetry $A_{FB}$, the angular coefficients $S_i$ and $A_i$, and the optimized observables $P_l$ and $P_l'$ [23]. The CP asymmetry angular coefficients $A_{5,6,8,9}$ in the SM are induced by the weak phase from the CKM matrix. For the $b \to s$ transition, the CP asymmetries proportional to $\text{Im}(V_{ub}V_{us}^*/V_{ub}V_{us})$, which is of order $10^{-2}$ [18] as measured by the LHCb Collaboration (see Table 3 in Ref. [29]), but the statistical uncertainties are still large. For these reasons, we will focus on the CP averaged quantities when taking the binned observables into consideration.

We begin with the branching ratios of the decays $B_s \to \phi \ell^+ \ell^-$. From the differential decay rates as defined in Eq. (43), it is straightforward to make the integration over the range of $4m_{\ell}^2 \leq q^2 \leq (m_{B_s} - m_{\phi})^2$. In order to be consistent with the choices made by LHCb Collaboration in their data analysis, we here also cut off the regions of dilepton-mass squared around the charmonium resonances $J/\psi(1S)$ and $\psi(2S)$: i.e., $8.0 < q^2 < 11.0 \text{GeV}^2$ and $12.5 < q^2 < 15.0 \text{GeV}^2$ for $\ell = (e, \mu, \tau)$ cases. We display the PQCD and “PQCD+Lattice” predictions for the differential branching ratios $dB/dq^2$ in Fig. 4 for the cases of $l = (\mu, \tau)$, including currently available LHCb results in six bins of $q^2$ [29] indicated by the crosses for $B_s \to \phi \mu^+ \mu^-$. From Fig. 4 one can see that both PQCD and “PQCD+Lattice” predictions for the differential branching ratios do agree well with the LHCb results within the still large errors. Since the theoretical prediction for the differential branching ratio of the electron mode is almost identical with the one of the muon, we do not draw the figure of $dB(B_s \to \phi e^+ e^-)/dq^2$ in Fig. 4.

In Table VI we present the theoretical predictions of the total branching fractions for $B_s \to \phi \ell^+ \ell^-$ with $\ell = (e, \mu, \tau)$ obtained by the integration over the six $q^2$ bins using the PQCD (the first row) and “PQCD+Lattice” approach (the second row), respectively. The major theoretical errors from different sources, such as the form factors (FFs), the scale $\mu$, the CKM matrix element $V_{tb}$ and $V_{ts}$, are also listed. As in Ref. [29], a correction factor $f_{\text{Veto}} = 1.52$ is applied to account for the contribution in the veto $q^2$ bins for $\ell = (e, \mu)$ cases. As a comparison, we also show the LHCb result $B(B_s \to \phi \mu^+ \mu^-) = (7.97^{+0.81}_{-0.80}) \times 10^{-7}$ and the QCDSR predictions $B(B_s \to \phi \ell^+ \ell^-)$ for all three decay modes [77]. For $B_s \to \phi \mu^+ \mu^-$ decay, for instance, the theoretical predictions and the LHCb measurement [29] (in unit of $10^{-7}$) are the following:

$$B(B_s \to \phi \mu^+ \mu^-) = \begin{cases} 
7.16^{+3.47}_{-2.36}, & \text{in PQCD}, \\
6.95^{+1.60}_{-1.31}, & \text{in PQCD + Lattice}, \\
7.06^{+1.59}_{-1.59}, & \text{in QCDSR[77]}, \\
7.97^{+0.81}_{-0.80}, & \text{LHCb[29]}. 
\end{cases}$$

From the numerical results in above equation and Table VI, one can see that

1. The PQCD and “PQCD+Lattice” predictions for the branching ratio $B(B_s \to \phi \ell^+ \ell^-)$ with $\ell = (e, \mu, \tau)$ do agree well with each other within the errors, while the “PQCD+Lattice” predictions of $B(B_s \to \phi \ell^+ \ell^-)$ have smaller errors than those of the PQCD predictions.

2. Both PQCD and “PQCD+Lattice” predictions of $B(B_s \to \phi \mu^+ \mu^-)$ do agree well with currently available LHCb measured value [29] within errors. For the electron and tau mode, however, we have to wait for the future experimental measurements.
(3) For all three decay modes, our theoretical predictions of the branching ratios do agree well with the theoretical predictions obtained from the QCD sum rule [77].

Since the large theoretical uncertainties of the branching ratios could be largely canceled in the ratio of the branching ratios of \( B_s \rightarrow \phi \ell^+ \ell^- \) decays, one can define and check the physical observables \( R^\mu \) and \( R^\tau \) [5]. In the region \( q^2 < 4m^2_\mu \), where only the \( e^+e^- \) modes are allowed, there is a large enhancement due to the \( 1/q^2 \) scaling of the photon penguin contribution [82]. In order to remove the phase space effects in the ratio \( R^\mu \) and keep consistent with other analysis [5], we here also use the lower cut of \( 4m^2_\mu \) for both the electron and muon modes in the definition of the ratio \( R^\mu_\phi \) as in Ref. [5]:

\[
R^\mu_\phi = \frac{\int_{q^2_{min}}^{q^2_{max}}dq^2 \frac{d\mathcal{B}(B_s \rightarrow \phi \mu^+ \mu^-)}{dq^2}}{\int_{q^2_{min}}^{q^2_{max}}dq^2 \frac{d\mathcal{B}(B_s \rightarrow \phi e^+ e^-)}{dq^2}} = \begin{cases} 
0.992 \pm 0.002, & \text{in PQCD,} \\
0.991 \pm 0.002, & \text{in PQCD + Lattice,}
\end{cases}
\]

For the case of the ratio \( R^\tau_\phi \) we have

\[
R^\tau_\phi = \frac{\int_{q^2_{min}}^{q^2_{max}}dq^2 \frac{d\mathcal{B}(B_s \rightarrow \phi \mu^+ \mu^-)}{dq^2}}{\int_{q^2_{min}}^{q^2_{max}}dq^2 \frac{d\mathcal{B}(B_s \rightarrow \phi e^+ e^-)}{dq^2}} = \begin{cases} 
0.100 \pm 0.004, & \text{in PQCD,} \\
0.086 \pm 0.009, & \text{in PQCD + Lattice,}
\end{cases}
\]

where the total error is the combination of the individual errors in quadrature. We suggest the LHCb and Belle-II to measure these two ratios.

For \( B_s \rightarrow \phi \mu^+ \mu^- \) decay, we show the PQCD and “PQCD+Lattice” predictions for the \( q^2 \)-dependence of the longitudinal polarization \( F_4(q^2) \), the CP averaged angular coefficients \( S_{3,4,7}(q^2) \) and the CP asymmetry angular coefficients \( A_{5,6,8,9}(q^2) \) in Fig. 5. As a comparison, the currently available LHCb measurements for these observables of \( B_s \rightarrow \phi \mu^+ \mu^- \) decay in the six \( q^2 \) bins [29] are also shown by those crosses explicitly. One can see from the Fig. 5 that:

1. For the longitudinal polarization \( F_4(q^2) \), although both PQCD and “PQCD+Lattice” predictions all agree well with the LHCb measurements in the six bins, our theoretical predictions in the region of the fourth and fifth bin are little larger than the measured ones.

2. For the CP averaged angular coefficients \( S_{3,4,7}(q^2) \), the PQCD and “PQCD+Lattice” predictions agree very well with each other, and are consistent with the LHCb results within the still large experimental errors. For the last two high \( q^2 \) bins, the LHCb results of \( S_3 \) (\( S_7 \)) is a little larger (smaller) than our theoretical predictions.

3. For the CP asymmetry angular coefficients \( A_{5,6,8,9}(q^2) \), the PQCD and “PQCD+Lattice” predictions are very small: in the range of \( \sim 10^{-4} \) to \( 10^{-2} \). For the LHCb measurements in the six bins, they are clearly consistent with our theoretical predictions due to still large experimental errors!

In Fig. 6, we show the PQCD and “PQCD+Lattice” predictions for the \( q^2 \)-dependence of the forward-backward asymmetry \( A_{FB}(q^2) \), the optimized observables \( P_{1,2,3}(q^2) \) and \( P_{4,5,6,8}(q^2) \) for \( B_s \rightarrow \phi \mu^+ \mu^- \) decay. Unfortunately, there exist no any experimental measurements for these observables. We have to wait for future LHCb and Belle-II measurements. Analogous to Fig. 4, the vertical grey blocks in both Fig. 5 and 6 also denote the two experimental veto regions of \( q^2 \): \( 8.0 < q^2 < 11.0 \text{ GeV}^2 \) and \( 12.5 < q^2 < 15.0 \text{ GeV}^2 \).

In Table VII, we list the theoretical predictions for the values of the observables \( F_4^\phi, A_{FB}, S_{3,4,7}, A_{5,6,8,9}, P_{1,2,3} \) and \( P_{4,5,6,8} \), obtained after the integrations over the whole kinematic region of \( q^2 \) for the semileptonic decays \( B_s \rightarrow \phi \ell^+ \ell^- \) with \( \ell = (e, \mu, \tau) \) in the PQCD (the first row) and “PQCD+Lattice” (the second row) approaches, respectively. Of course, the regions corresponding to resonance \( J/\psi(1S) \) and \( \psi(2S) \), say \( 8.0 < q^2 < 11.0 \text{ GeV}^2 \) and \( 12.5 < q^2 < 15.0 \text{ GeV}^2 \) numerically, are also cut off here. The total errors are the combinations of the individual errors from the form factors, the renormalization scales and the relevant CKM matrix elements. The above theoretical predictions should be tested in the near future LHCb and Belle-II experiments. For the considered \( B_s \) meson decays, one should consider the effects from the \( B_s \rightarrow B_s \) mixing. The theoretical framework for examining the time-dependent decays with the inclusion of such mixing effects can be found in Ref. [83]. The authors of Ref. [83] proved that the mixing effects on the values of decay rates and CP averaged observables are generally within a few percent and could be neglected.

C. The \( q^2 \)-binned observables

For \( B_s \rightarrow \phi \mu^+ \mu^- \) decay mode, the LHCb Collaboration has reported their experimental measurements for many physical observables in several \( q^2 \) bins [29]. In order to compare our theoretical predictions with the LHCb results
and $29$.

We calculate and list the theoretical predictions of these observables bin by bin for the cases of $A_3, A_5, A_6, A_8$ and $P_3$ and $P_6$ at present, they are relatively large in size and may be measured in the near future LHCb and Belle-II experiments, so we calculate and list the theoretical predictions of these observables bin by bin for the cases of $\ell = (\mu, \tau)$ in Table IX.

The definitions of the $q^2$-binned observables are the following:

$$B(q^2_1, q^2_2) = \frac{\int_{q^2_1}^{q^2_2} dq^2 d\mathcal{B}(B_s \to \phi \ell^+ \ell^-)}{q^2_2 - q^2_1}, \quad F_L(q^2_1, q^2_2) = \frac{\int_{q^2_1}^{q^2_2} dq^2 [3(I^c_1 + \bar{I}^c_1) - (I^c_2 + \bar{I}^c_2)]}{4 \int_{q^2_1}^{q^2_2} dq^2 [d(\Gamma + \bar{\Gamma})/dq^2]},$$

(64)
and in Table VIII and IX, we find the following points about the relevant physical observables of the considered $B_s \to \phi \mu^+ \mu^-$ decays in bins:

(1) For $B_s \to \phi \mu^+ \mu^-$ decay, besides the good consistency between the theory and the LHCb data for the integrated total branching ratio $\mathcal{B}(B_s \to \phi \mu^+ \mu^-)$ as listed in Eq. (61), the PQCD and “PQCD+Lattice” predictions for $\mathcal{B}(B_s \to \phi \mu^+ \mu^-)$ in most bins do agree well with the measured ones within $2\sigma$ errors. For the first low-$q^2$ bin...
0.1 < q^2 < 2 (GeV^2), however, the central value of the LHCb result 1.11 ± 0.16 is larger than the theoretical ones by roughly a factor of three. The LHCb results of B(B_s → φμ^+μ^-) in different bins of q^2 as listed in the third column of Table VIII are obtained from the results in Table I of Ref. [29] by multiplying the LHCb measured values of differential decay rate dB(B_s → φμ^+μ^-)/dq^2 with the width of the corresponding bin (q^2_2 − q^2_1). The theoretical errors of our theoretical predictions of the branching ratios in bins are still relatively large, while the differences between the PQCD and “PQCD+Lattice” predictions for B(B_s → φμ^+μ^-) with ℓ = (μ, τ) are small.

(2) In the first low-q^2 bin 0.1 < q^2 < 2 (GeV^2), the central value of LHCb result F_L^φ(ℓ = μ) = 0.20^{+0.08}_{-0.07} ± 0.02 is smaller than the theoretical one 0.46 ± 0.01 in the PQCD approach or 0.49 ± 0.01 in the “PQCD+Lattice”

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### Table VII: Theoretical predictions for the observables F_L^φ, A_{FB}, S_{3,4,7}, A_{5,6,8,9}, P_{1,2,3} and P_{5,6,8}^d integrated over the whole kinematic region for B_s → φτ^+τ^- decays in the PQCD (the first row) and “PQCD+Lattice” (the second row) approaches, respectively.

| Obs. | ℓ = e | ℓ = μ | ℓ = τ |
|------|-------|-------|-------|
| F_L^φ | 0.395^{+0.005}_{-0.004} | 0.46^{+0.006}_{-0.007} | 0.398^{+0.002}_{-0.003} |
| S_{3} | 0.462^{+0.012}_{-0.011} | 0.548^{+0.001}_{-0.002} | 0.444^{+0.001}_{-0.002} |
| P_{1} | 0.114^{+0.006}_{-0.005} | 0.137^{+0.005}_{-0.004} | 0.082^{+0.004}_{-0.003} |
| P_{2} | 0.907^{+0.008}_{-0.007} | 0.117^{+0.007}_{-0.006} | 0.076^{+0.006}_{-0.007} |
| P_{3} | 0.208^{+0.004}_{-0.003} | 0.255^{+0.004}_{-0.004} | 0.102^{+0.004}_{-0.003} |
| P_{4} | 0.200^{+0.012}_{-0.012} | 0.246^{+0.011}_{-0.009} | 0.099^{+0.009}_{-0.009} |
| P_{5} | 0.346^{+0.004}_{-0.003} | 0.366^{+0.012}_{-0.012} | 0.023^{+0.004}_{-0.003} |
| P_{6} | 0.383^{+0.023}_{-0.021} | 0.411^{+0.036}_{-0.036} | 0.031^{+0.001}_{-0.001} |
| 10^5S_{7} | 0.347^{+0.005}_{-0.005} | 0.402^{+0.012}_{-0.012} | 0.040^{+0.004}_{-0.004} |
| 10^5A_{8} | 0.367^{+0.021}_{-0.021} | 0.429^{+0.038}_{-0.036} | 0.036^{+0.001}_{-0.001} |
| 10^5A_{9} | 0.273^{+0.003}_{-0.001} | 0.324^{+0.002}_{-0.012} | 0.071^{+0.003}_{-0.001} |
| 10^5A_{10} | 0.249^{+0.012}_{-0.012} | 0.396^{+0.030}_{-0.020} | 0.060^{+0.001}_{-0.001} |
| 10^5A_{11} | 0.327^{+0.022}_{-0.022} | 0.862^{+0.026}_{-0.027} | 0.012^{+0.001}_{-0.001} |
| 10^5A_{12} | 0.270^{+0.090}_{-0.090} | 0.967^{+0.077}_{-0.071} | 0.015^{+0.000}_{-0.000} |
| 10^5A_{13} | 0.133^{+0.004}_{-0.004} | 0.171^{+0.003}_{-0.003} | 0.013^{+0.002}_{-0.002} |
| 10^5A_{14} | 0.136^{+0.003}_{-0.003} | 0.175^{+0.010}_{-0.010} | 0.015^{+0.002}_{-0.002} |

### Table VIII: Theoretical predictions for the q^2-binned observables B(B_s → φμ^+μ^-) (in unit of 10^-1) and F_L^φ with ℓ = (μ, τ) in the PQCD (the first low) and “PQCD+Lattice” (the second row) approach, respectively. For a comparison, we also list the experiment measurements for the q^2-binned results from the LHCb Collaboration [29].

| q^2 bin (GeV^2) | B(ℓ = μ) LHCb | B(ℓ = τ) LHCb |
|----------------|----------------|----------------|
| [0.1, 2.0]    | 0.40^{+0.16}_{-0.11} | 1.11 ± 0.16    |
| [2.0, 5.0]    | 0.52^{+0.22}_{-0.15} | 0.77 ± 0.14    |
| [5.0, 8.0]    | 1.00^{+0.45}_{-0.31} | 0.96 ± 0.15    |
| [10.0, 12.5]  | 0.77^{+0.13}_{-0.11} | 0.71 ± 0.12    |
| [15.0, 17.0]  | 1.12^{+0.61}_{-0.41} | 0.90 ± 0.13    |
| [17.0, 19.0]  | 0.78^{+0.40}_{-0.34} | 0.75 ± 0.13    |
| [19.0, 21.0]  | 0.60^{+0.05}_{-0.04} | 0.55 ± 0.12    |
| [21.0, 23.0]  | 0.92^{+0.29}_{-0.27} | 1.29 ± 0.19    |
| [23.0, 25.0]  | 1.09^{+0.34}_{-0.31} | 1.33 ± 0.19    |
| [25.0, 27.0]  | 1.98^{+1.02}_{-0.66} | 1.62 ± 0.20    |
| [27.0, 29.0]  | 1.59^{+0.16}_{-0.15} | 0.65^{+0.06}_{-0.05} |

| q^2 bin (GeV^2) | F_L^φ(ℓ = μ) LHCb | F_L^φ(ℓ = τ) LHCb |
|----------------|-----------------|-----------------|
| [0.1, 2.0]    | 0.46^{+0.007}_{-0.008} | 0.20 ± 0.09    |
| [2.0, 5.0]    | 0.49^{+0.006}_{-0.007} | 0.68 ± 0.15    |
| [5.0, 8.0]    | 0.58^{+0.007}_{-0.010} | 0.54 ± 0.10    |
| [10.0, 12.5]  | 0.68^{+0.008}_{-0.008} | 0.29 ± 0.11    |
| [15.0, 17.0]  | 0.36^{+0.001}_{-0.014} | 0.23 ± 0.09    |
| [17.0, 19.0]  | 0.41^{+0.004}_{-0.004} | 0.40 ± 0.14    |
| [19.0, 21.0]  | 0.28^{+0.002}_{-0.002} | 0.36^{+0.001}_{-0.001} |
| [21.0, 23.0]  | 0.71^{+0.007}_{-0.008} | 0.63 ± 0.09    |
| [23.0, 25.0]  | 0.77^{+0.008}_{-0.008} | 0.29 ± 0.07    |
| [25.0, 27.0]  | 0.76^{+0.008}_{-0.008} | 0.29 ± 0.07    |
| [27.0, 29.0]  | 0.39^{+0.003}_{-0.003} | 0.44 ± 0.01    |
errors in the ratios. The theoretical predictions for the binned observables $S_{3,4,7}$ of the decays $B_s \rightarrow \phi \mu^+ \mu^-$ in the PQCD (the first row) and “PQCD+Lattice” (the second row) approaches. For a comparison, we also list the LHCb measured values [29].

| $q^2$ bin (GeV$^2$) | Theor. | LHCb | Theor. | LHCb | Theor. (10$^{-3}$) | LHCb |
|---------------------|--------|--------|--------|--------|-------------------|--------|
| [0.1, 2.0]         | 0.003 $^{+0.000}_{-0.000}$ | $-0.05 \pm 0.13$ | $-0.057_{-0.001}^{+0.001}$ | $-0.27 \pm 0.23$ | 1.536 $^{+0.001}_{-0.002}$ | $-0.04 \pm 0.12$ |
| [2.0, 5.0]         | 0.003 $^{+0.000}_{-0.000}$ | $-0.06 \pm 0.21$ | 0.194 $^{+0.004}_{-0.004}$ | 0.47 $^{+0.37}_{-0.37}$ | 0.979 $^{+0.003}_{-0.003}$ | 0.03 $\pm 0.21$ |
| [5.0, 8.0]         | $-0.025_{-0.003}^{+0.002}$ | $-0.10 \pm 0.25$ | 0.272 $^{+0.003}_{-0.002}$ | 0.10 $^{+0.17}_{-0.17}$ | 0.377 $^{+0.003}_{-0.003}$ | $-0.04 \pm 0.18$ |
| [11.0, 12.5]       | $-0.125_{-0.003}^{+0.004}$ | $-0.19 \pm 0.21$ | 0.296 $^{+0.001}_{-0.001}$ | 0.47 $^{+0.25}_{-0.25}$ | 0.157 $^{+0.002}_{-0.001}$ | 0.00 $\pm 0.16$ |
| [15.0, 17.0]       | $-0.116_{-0.001}^{+0.002}$ | 0.303 $^{+0.001}_{-0.001}$ | 0.03 $^{+0.15}_{-0.15}$ | 0.054 $^{+0.001}_{-0.001}$ | 0.12 $^{+0.15}_{-0.15}$ |
| [15.0, 19.0]       | $-0.283_{-0.002}^{+0.001}$ | $-0.07 \pm 0.25$ | 0.325 $^{+0.001}_{-0.001}$ | 0.39 $^{+0.30}_{-0.30}$ | 0.020 $^{+0.001}_{-0.001}$ | $-0.20 \pm 0.26$ |
| [1.0, 6.0]         | $-0.024_{-0.002}^{+0.002}$ | $-0.02 \pm 0.13$ | 0.184 $^{+0.003}_{-0.002}$ | 0.19 $^{+0.14}_{-0.14}$ | 0.971 $^{+0.001}_{-0.001}$ | 0.03 $\pm 0.14$ |
| [1.0, 15.0, 19.0]  | $-0.254_{-0.002}^{+0.001}$ | $-0.09 \pm 0.12$ | 0.318 $^{+0.001}_{-0.001}$ | 0.14 $^{+0.11}_{-0.11}$ | 0.046 $^{+0.002}_{-0.001}$ | $-0.13 \pm 0.11$ |
| [1.0, 15.0, 19.0]  | $-0.239_{-0.001}^{+0.001}$ | 0.325 $^{+0.001}_{-0.001}$ | 0.058 $^{+0.001}_{-0.001}$ |

Table X: Theoretical predictions for the $q^2$-binned observables $A_{FB}^{\mu}$, $P_{1,2}$ of the decays $B_s \rightarrow \phi \ell^+ \ell^-$ with $\ell = (\mu, \tau)$ in the PQCD (the first row) and “PQCD+Lattice” (the second row) approaches.

| $q^2$ bin (GeV$^2$) | $A_{FB}^{\mu}$ | $A_{FB}^{\tau}$ | $P_{1}(\ell = \mu)$ | $P_{1}(\ell = \tau)$ | $P_{2}(\ell = \mu)$ | $P_{2}(\ell = \tau)$ |
|---------------------|---------------|----------------|-------------------|-------------------|-------------------|-------------------|
| [0.1, 2.0]         | 0.123 $^{+0.003}_{-0.003}$ | $-0.006_{-0.001}^{+0.001}$ | 0.016 $^{+0.001}_{-0.001}$ | $-0.006_{-0.001}^{+0.001}$ | 0.204 $^{+0.001}_{-0.001}$ |
| [2.0, 5.0]         | $-0.074_{-0.004}^{+0.004}$ | $-0.209_{-0.007}^{+0.007}$ | $-0.209_{-0.003}^{+0.003}$ | $-0.209_{-0.003}^{+0.003}$ |
| [5.0, 8.0]         | $-0.058_{-0.002}^{+0.001}$ | $-0.239_{-0.006}^{+0.005}$ | $-0.239_{-0.003}^{+0.003}$ | $-0.239_{-0.003}^{+0.003}$ |
| [11.0, 12.5]       | $-0.361_{-0.001}^{+0.001}$ | $-0.441_{-0.010}^{+0.010}$ | $-0.441_{-0.003}^{+0.003}$ | $-0.441_{-0.003}^{+0.003}$ |
| [15.0, 17.0]       | $-0.308_{-0.001}^{+0.001}$ | $-0.192_{-0.001}^{+0.001}$ | $-0.696_{-0.009}^{+0.009}$ | $-0.707_{-0.005}^{+0.005}$ | $-0.326_{-0.003}^{+0.003}$ | $-0.328_{-0.003}^{+0.003}$ |
| [17.0, 19.0]       | $-0.212_{-0.002}^{+0.003}$ | $-0.154_{-0.002}^{+0.001}$ | $-0.868_{-0.005}^{+0.005}$ | $-0.875_{-0.003}^{+0.003}$ | $-0.216_{-0.001}^{+0.001}$ | $-0.224_{-0.001}^{+0.001}$ |

approach. For other remaining bins, both the PQCD and “PQCD+Lattice” predictions of $F_L^\phi$ for muon mode do agree very well with currently available LHCb measured values [29] within 2σ errors. It is worth of remaining that our theoretical predictions of $F_L^\phi$ have a little error of $\sim 2\%$ due to the strong cancellation of the theoretical errors in the ratios. The theoretical predictions for $\mathcal{B}(B_s \rightarrow \phi \ell^+ \ell^-)$ and $F_L^\phi(\ell = \tau)$ in different bins of $q^2$ as listed in Table VIII will be tested by future experimental measurements.

(3) For the observables $S_{3,4,7}$, as listed in Table IX, the PQCD and “PQCD+Lattice” predictions for their values in all bins are in the range of $10^{-3} - 10^{-1}$, and show a good agreement with the LHCb measured values [29]. The theoretical errors of the PQCD and “PQCD+Lattice” predictions are also very small, $\sim 2\%$ in magnitude, because of their nature as the ratios. In all bins, the LHCb measured values of $S_{3,4,7}$ are still consistent with zero due to their still large errors, which is a clear feature as can be seen easily from the numerical values in Table IX and the crosses in Fig. 5.

In Table X and XI, we show the PQCD and “PQCD+Lattice” predictions for the physical observables $A_{FB}^{\mu,\tau}$ and $P_{4,5}(\ell = e, \mu, \tau)$ in six bins. Although there exist no experimental measurements for these physical observables
TABLE XI: Theoretical predictions for the $q^2$-binned optimized observables $P'_1$ and $P'_5$ of the decays $B_s \to \phi \ell^+\ell^-$ with $\ell = (e, \mu, \tau)$ in the PQCD (the first row) and “PQCD+Lattice” (the second row) approach.

| $q^2$ (GeV$^2$) | $P'_1(\ell = e)$ | $P'_1(\ell = \mu)$ | $P'_1(\ell = \tau)$ | $P'_2(\ell = e)$ | $P'_2(\ell = \mu)$ | $P'_2(\ell = \tau)$ |
|----------------|------------------|------------------|------------------|------------------|------------------|------------------|
| [0.1, 2.0]     | $-0.342^{+0.005}_{-0.005}$ | $-0.308^{+0.005}_{-0.005}$ | $-0.308^{+0.005}_{-0.005}$ | $0.445^{+0.001}_{-0.002}$ | $0.418^{+0.001}_{-0.002}$ | $-0.435^{+0.001}_{-0.002}$ |
| [2.0, 5.0]     | $0.931^{+0.007}_{-0.007}$ | $0.932^{+0.007}_{-0.007}$ | $-0.684^{+0.004}_{-0.004}$ | $-0.685^{+0.004}_{-0.004}$ | $-0.839^{+0.005}_{-0.006}$ | $-0.839^{+0.005}_{-0.006}$ |
| [5.0, 8.0]     | $1.118^{+0.005}_{-0.005}$ | $1.118^{+0.005}_{-0.005}$ | $-0.814^{+0.003}_{-0.003}$ | $-0.813^{+0.003}_{-0.003}$ | $-0.719^{+0.005}_{-0.007}$ | $-0.719^{+0.005}_{-0.007}$ |
| [15.0, 17.0]   | $1.302^{+0.004}_{-0.004}$ | $1.302^{+0.004}_{-0.004}$ | $1.306^{+0.004}_{-0.004}$ | $-0.505^{+0.005}_{-0.007}$ | $-0.505^{+0.005}_{-0.007}$ | $-0.52^{+0.006}_{-0.008}$ |
| [17.0, 19.0]   | $1.314^{+0.003}_{-0.003}$ | $1.314^{+0.003}_{-0.003}$ | $1.317^{+0.003}_{-0.003}$ | $-0.481^{+0.005}_{-0.006}$ | $-0.481^{+0.005}_{-0.006}$ | $-0.495^{+0.006}_{-0.006}$ |

at present, we do believe that these predictions could be tested in the near future LHCb and Belle-II experiments.

VI. SUMMARY

In this paper, we made a systematic study of the semileptonic decays $B_s \to \phi \ell^+\ell^-$ with $\ell = (e, \mu, \tau)$ using the PQCD and the “PQCD+Lattice” factorization approach respectively. We first evaluated all relevant form factors in the low $q^2$ region using the PQCD approach, and also took currently available lattice QCD results at the high-$q^2$ points $q^2 = (12, 16, 18.9)$ GeV$^2$ as additional input to improve the extrapolation of the form factors from the low to the high-$q^2$ region. We calculated the branching ratios $B(B_s \to \phi \ell^+\ell^-)$, the CP averaged $\phi$ longitudinal polarization fraction $F_L(q^2)$, the forward-backward asymmetry $A_{FB}(q^2)$, the CP averaged angular coefficients $S_{1,4,7}(q^2)$, the CP asymmetry angular coefficients $A_{5,6,8,9}(q^2)$, the optimized observables $P_{1,2,3}(q^2)$ and $P'_{1,5,6,8}(q^2)$. For $B_s \to \phi \mu^+\mu^-$ decay mode, we calculated the binned values of the branching ratio $B(B_s \to \phi \mu^+\mu^-)$, the observables $F'_L$ and $S_{3,4,7}$ in the same bins as defined by LHCb Collaboration [29] in order to compare our theoretical predictions with those currently available LHCb measurements bin by bin directly.

Based on the analytical evaluations, the numerical results and the phenomenological analysis, we found the following main points:

(1) For $B_s \to \phi \mu^+\mu^-$ decay, both PQCD and “PQCD+Lattice” predictions of $B(B_s \to \phi \mu^+\mu^-)$ are about $7 \times 10^{-7}$, which agree well with the LHCb measured value $(7.97^{+1.20}_{-0.81}) \times 10^{-7}$ and the QCDSR prediction $(7.06 \pm 1.59) \times 10^{-7}$ within one standard deviation. For the electron and tau mode, our theoretical predictions for their decay rates are also well consistent with the corresponding QCDSR predictions and to be tested by future experimental measurements.

(2) For the ratios of the branching ratios $R^\phi$ and $R'^\phi$, the PQCD and “PQCD+Lattice” predictions agree with each other and with small theoretical errors because of the strong cancellation of the theoretical errors in such ratios. We suggest the LHCb and Belle-II collaboration to measure these ratios.

(3) For the longitudinal polarization $F_L$, both PQCD and "PQCD+Lattice" predictions agree with the LHCb measurements in the considered bins within the errors. For the CP averaged angular coefficients $S_{3,4,7}$, the PQCD and "PQCD+Lattice" predictions in all bins are small in magnitude, in the range of $10^{-3} - 10^{-1}$, and agree well with the LHCb results within the still large experimental errors. For the CP asymmetry angular coefficients $A_{5,6,8,9}$, the PQCD and "PQCD+Lattice" predictions are very small, in the range of $10^{-4} - 10^{-2}$, and clearly consistent with the LHCb measurements in the six bins.

(4) For the physical observables $A'_{FB}, P_{1,2,3}$ and $P'_{5,6,8}$, the experimental measurements are still absent now, we think that the PQCD and "PQCD+Lattice" predictions for these physical observables will be tested in the near future LHCb and Belle-II experiments.
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