Continuity of the Feynman–Kac formula for a generalized parabolic equation

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ABSTRACT

It is well–known that Backward Stochastic Differential Equations provide probabilistic formulae for the solution of (systems of) second order elliptic and parabolic equations, thus providing an extension of the Feynman–Kac formula to semilinear PDEs. This method was applied to the class of PDEs with a nonlinear Neumann boundary condition first by Pardoux and Zhang in 1998. However, the proof of continuity of the extended Feynman–Kac formula with respect to \( x \) (resp. to \((t,x)\)) is not correct in that paper. Here we consider a more general situation, where both the equation and the boundary condition involve the (possibly multivalued) gradient of a convex function. We prove the required continuity. The result for the class of equations studied Pardoux and Zhang in their paper from 1998, as well as those considered by Maticiuc and Răşcanu in their paper from 2010, are Corollaries of our main results.

1. Introduction

There is by now a vast literature on the connection between backward stochastic differential equations (BSDEs) and semilinear parabolic and elliptic systems of second order PDEs, thus generalizing the Feynmann–Kac formula for linear second order PDEs. Pardoux and Zhang \cite{pardoux1999backward} initiated the probabilistic study of semilinear parabolic and elliptic systems of second order partial differential equations with nonlinear Neumann boundary condition. The idea is to prove that an associated Backward Stochastic Differential Equation allows to define a certain function of \((t,x)\) (or in the elliptic case of \( x \) alone), which is continuous, and is a viscosity solution of a certain system of parabolic or elliptic PDEs. Several papers, see \cite{pardoux1999backward,matciuc2010wellposedness,garreau2000existence,garreau2001existence}, have used the above results.

However, the continuity is not really proved in \cite{pardoux1999backward}, nor in \cite{garreau2001existence}. It is claimed that it follows from several estimates given in earlier sections of the paper, but this is not really fair. In \cite{matciuc2009existence} Maticiuc and Rascanu give a proof of the continuity result under some additional assumption. In the very recent paper \cite{garreau2016existence}, the continuity is shown in the case where all coefficients are Lipschitz continuous. The difficulty is that not only the solution of forward SDE depends upon its starting point \( x \) (resp. \((t,x)\)), but also its local time on the boundary, which drives the reflection.
In this paper, we will give a proof of continuity for a class of problems which is more general than the one considered in [11], and deduce the continuity statements from that paper as Corollaries.

Given an open connected bounded set $D$ and continuous mappings $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times k}$, $F : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \to \mathbb{R}^{m \times k}$, $G : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \to \mathbb{R}^m$ and $\kappa : \overline{D} \to \mathbb{R}^m$, we define the collection of second order PDE operators

$$\mathcal{L}_t v(x) = \frac{1}{2} \text{Tr} \left[ g(t,x) g^*(t,x) D^2 v(x) \right] + \langle f(t,x), \nabla v(x) \rangle,$$

and consider the following system of second order semilinear parabolic PDE

$$\begin{cases}
\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_t u(t,x) + F(t,x,u(t,x), (\nabla u(t,x))^* g(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times D, \\
-\frac{\partial u}{\partial n}(t,x) + G(t,x,u(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times Bd(\overline{D}), \\
u(T,x) = \kappa(x), \quad x \in \overline{D}.
\end{cases}$$

In the case of coefficients which do not depend upon time, we consider the following system of second order elliptic PDEs

$$\begin{cases}
\mathcal{L} u(x) + F(x,u(x),(\nabla u_i(x))^* g(x)) = 0, \quad x \in D, \\
\frac{\partial u}{\partial n}(x) = G(x,u(x)), \quad x \in Bd(\overline{D}).
\end{cases}$$

In the parabolic case, we shall in fact consider a more general type of equations. Namely, given two convex functions $\varphi, \psi : \mathbb{R}^m \to (-\infty, +\infty]$, whose subdifferentials $\partial \varphi, \partial \psi$ may be multivalued functions, we consider the system of parabolic PDEs

$$\begin{cases}
\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_t u(t,x) + F(t,x,u(t,x), (\nabla u(t,x))^* g(t,x)) \in \partial \varphi(u(t,x)), \quad (t,x) \in \mathbb{R}_+ \times D, \\
-\frac{\partial u}{\partial n}(t,x) + G(t,x,u(t,x)) \in \partial \psi(u(t,x)), \quad (t,x) \in \mathbb{R}_+ \times Bd(\overline{D}), \\
u(T,x) = \kappa(x), \quad x \in \overline{D},
\end{cases}$$

where

$$(H.1) \begin{cases}
(i) \quad D \text{ is an open connected bounded subset of } \mathbb{R}^d \text{ of the form } \\
\quad \{ x \in \mathbb{R}^d : \phi(x) < 0 \}, \text{ where } \phi \in C_b^2(\mathbb{R}^d), \\
(ii) \quad Bd(\overline{D}) = \{ x \in \mathbb{R}^d : \phi(x) = 0 \} \text{ and } |\nabla \phi(x)| = 1 \forall x \in Bd(\overline{D}).
\end{cases}$$

Note that in order to make sure that the first part of (ii) is true, we should better have that $|\nabla \phi(x)| > 0$ on the boundary of $D$. It is then natural and convenient to assume the second part of (ii), which cannot be true everywhere; in particular, $\nabla \phi(x) = 0$ at any local maximum of $\phi$, which must exist. The outward normal derivative of $u(t,\cdot)$ at the point $x \in Bd(\overline{D})$ is the column vector

$$\frac{\partial u(t,x)}{\partial n} = \left( \frac{\partial u_1(t,x)}{\partial n}, \ldots, \frac{\partial u_m(t,x)}{\partial n} \right)^*,$$

given by

$$\frac{\partial u_i(t,x)}{\partial n} = \sum_{j=1}^d \frac{\partial \phi(x)}{\partial x_j} \frac{\partial u_i(t,x)}{\partial x_j} = \left( \nabla u_i(t,x) \right)^* \nabla \phi(x), \quad i \in \overline{1,m};$$

and consider the following system of second order semilinear parabolic PDE

$$\begin{cases}
\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_t u(t,x) + F(t,x,u(t,x), (\nabla u(t,x))^* g(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times D, \\
-\frac{\partial u}{\partial n}(t,x) + G(t,x,u(t,x)) = 0, \quad (t,x) \in \mathbb{R}_+ \times Bd(\overline{D}), \\
u(T,x) = \kappa(x), \quad x \in \overline{D}.$$
hence
\[ \frac{\partial u(t, x)}{\partial n} = (\nabla u(t, x))^* \nabla \phi(x). \]

In this paper we shall give a probabilistic definition of the quantity \( u(t, x) \) (resp. \( u(x) \)) and prove that it is continuous as a function of \((t, x)\) (resp. as a function of \(x\)).

More precisely, the paper is organized as follows. Section 2 is devoted to a detailed formulation of our assumptions, and the description of the forward–backward SDEs which will allow us to give the probabilistic definition of our (deterministic) function \( u(t, x) \). We shall also collect in the same section a series of estimates from the recent monograph [10] by the authors, which will be useful for the proof of our result. Section 3 is devoted to the proof of the continuity of \( u(t, x) \). Section 4 establishes the same result for the function \( u(x) \). In Section 5, we state precisely in which sense \( u(t, x) \) (resp. \( u(x) \)) solves a system of semilinear parabolic (resp. elliptic) PDEs. Finally Section 1 recalls some well–known facts about convex functions.

### 2. Assumptions and formulation of the problem

#### 2.1. The forward SDE

We fix \( T > 0 \) and consider for each \( 0 \leq t < T \) the stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_s^t)_{s \geq 0})\), where the filtration is generated by a \( k \)-dimensional Brownian motion \((B_r)_{t \geq 0}\) as follows:
\[
\mathcal{F}_s^t = \sigma\{B_r - B_t : t \leq r \leq s\} \vee \mathcal{N}, \quad \text{if } s > t,
\]
where \( \mathcal{N} \) is the family of \( \mathbb{P} \)-negligible subsets of \( \Omega \).

For \( p \geq 0 \), we denote by \( S^p_d[0, T] \) the space of (equivalence classes of) progressively measurable continuous stochastic processes \( X : \Omega \times [0, T] \rightarrow \mathbb{R}^d \) such that additionally in the case \( p > 0 \):
\[
\mathbb{E} \sup_{t \in [0, T]} |X_t|^p < +\infty.
\]
For \( p \geq 0 \), we denote by \( \Lambda^p_d(0, T) \) the space of (equivalence classes of) progressively measurable stochastic processes \( X : \Omega \times ]0, T[ \rightarrow \mathbb{R}^d \) such that:
\[
\mathbb{E} \left[ \left( \int_0^T |X_t|^2 \, dt \right)^{p/2} \right] < +\infty \text{ if } p > 0; \quad \int_0^T |X_t|^2 \, dt < \infty \mathbb{P}-\text{a.s. if } p = 0.
\]

Let \( f(\cdot, \cdot) : \mathbb{R}^+_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) and \( g(\cdot, \cdot) : \mathbb{R}^+_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times k} \) be continuous functions which satisfy: there exist \( \mu_f \in \mathbb{R} \) and \( \ell_g > 0 \) such that for all \( u, v \in \mathbb{R}^d \)
\[(H.2) \left\{ \begin{array}{l}
(\text{i}) \ |u - v, f(t, u) - f(t, v)| \leq \mu_f |u - v|^2, \\
(\text{ii}) \ |g(t, u) - g(t, v)| \leq \ell_g |u - v|.
\end{array} \right.\]
Consider the following reflected (forward) SDE

\[
\begin{align*}
    (j) & \quad X_{t+s}^x \in \overline{D} \text{ and } X_{t+s}^{f,s} = x \text{ for all } s \geq 0, \\
    (jj) & \quad 0 = A_{s}^{t,x} \leq A_{s}^{t,x} \leq A_{s}^{t,x} \text{ for all } 0 \leq u \leq t \leq s \leq v, \\
    (jjj) & \quad X_{t+s}^x + \int_t^s \nabla \phi (X_{r+s}^x) \, dA_{r+s}^x = x + \int_t^s f (r, X_{r+s}^x) \, dr \\
    & \quad + \int_t^s g (r, X_{r+s}^x) \, dB_r, \quad \forall s \geq t, \\
    (jv) & \quad A_{s}^{t,x} = \int_t^s 1_{B_d(D)} (X_{r+s}^x) \, dA_{r+s}^x, \quad \forall s \geq t.
\end{align*}
\]

We have

**Proposition 2.1:** For any \((t, x) \in [0, T] \times \overline{D}\), the reflected SDE (3) has a unique continuous progressively measurable solution \((X^{t,x}, A^{t,x}) : \Omega \times \mathcal{T} \mapsto \mathbb{R}^d \times \mathbb{R}\). Moreover for all \(p \geq 1\), there exists \(C_p\) such that for all \((t, x),\)

\[\mathbb{E} \sup_{r \in [0, T]} |X_{r+t}^x|^p \leq C_p (1 + |x|^p),\]

and we have the identity

\[A_{s}^{t,x} = \int_t^s \mathcal{L}_r (X_{r+s}^x) \, dr + \int_t^s \left[ \nabla \phi (X_{r+s}^x), g (r, X_{r+s}^x) \, dB_r \right] - \left[ \phi (X_{r+s}^x) - \phi (x) \right],\]

with \(\mathcal{L}_r\) defined by (1).

**Proof:** Existence and uniqueness follow from Theorem 4.54, Proposition 4.55 and Corollary 4.56 in Pardoux and Răşcanu [10], while (4) follows from (4.112) in [10].

We now collect some additional properties of the pair \((X^{t,x}, A^{t,x})\) for further reference. For every \(p \geq 1\), by Proposition 4.55 and Corollary 4.56 from [10],

\[\begin{align*}
    (j) & \quad (t, x) \mapsto (X_{t+s}^x, A_{t+s}^x) : [0, T] \times D \rightarrow S^p_d[0, T] \times S^p_t[0, T] \text{ is } \mathbb{P}\text{-a.s. a continuous mapping and } \\
    (jj) & \quad \sup_{(t, x) \in [0, T] \times D} \left( \sup_{s \in [0, T]} \mathbb{E} e^{\lambda A_{s}^{t,x}} \right) \leq \exp \left[ C (1 + \lambda^2) \right],
\end{align*}\]

for some \(C > 0\) independent of \((t, x)\) and every \(\lambda > 0\). Moreover, exploiting the fact that the process \(X^{t,x}\) takes values in the bounded set \(\overline{D}\), we have that for every pair of continuous functions \(h_1, h_2 : [0, T] \times \overline{D} \rightarrow \mathbb{R}\) the mapping

\[\begin{align*}
    (t, x) \mapsto \mathbb{E} \int_t^T h_1 (s, X_{s+t}^x) \, ds + \mathbb{E} \int_t^T h_2 (s, X_{s+t}^x) \, dA_{s+t}^x : [0, T] \times \overline{D} \rightarrow \mathbb{R}
\end{align*}\]

is continuous.

By the Kolmogorov criterion (choosing a proper version)

\[\begin{align*}
    (t, x, s) \mapsto (X_{s+t}^x, A_{s+t}^x) : [0, T] \times \overline{D} \times [0, T] \rightarrow \mathbb{R}^d \times \mathbb{R}\]

is continuous, \(\mathbb{P} - a.s.\).
and consequently if \((t_n, x_n) \to (t, x)\), then (based also on (3-j)), the boundedness of \(\hat{D}\) and (5-ji)) we infer that for all \(q > 0\), as \(n \to \infty\),
\[
\left| X_{t_n}^{t_n, x_n} - X_t^{t, x} \right| + \left| A_{t_n}^{t_n, x_n} - A_t^{t, x} \right| \to 0, \quad \mathbb{P} - \text{a.s. and in } L^q \left( \Omega, \mathcal{F}, \mathbb{P} \right).
\]

**Remark 2.2:** Following Section 3.7.2 from Pardoux and Răşcanu [10] we have the following results in the case \(D = \mathbb{R}^d\) (no boundary conditions). If the assumptions \((H.2)\) are satisfied and for every \(T > 0\) there exist \(M, m > 0\) such that
\[
|f(s, u)| + |g(s, u)| \leq M \left(1 + |u|^m\right), \quad \forall (s, u) \in [0, T] \times \mathbb{R}^d,
\]
then the SDE (3), with \(A^{t, x} = 0\), has a unique solution \(X_t^{t, x} \in S_d^0[0, T], (t, x, s) \mapsto X_s^{t, x} : [0, T] \times \mathbb{R}^d \times [0, T] \to \mathbb{R}^d\) is continuous, \(\mathbb{P} - \text{a.s.},\) and there exists \(C\) such that for all \((t, x) \in [0, T] \times \mathbb{R}^d,\)
\[
\mathbb{E} \sup_{r \in [0, T]} |X_r^{t, x}|^p \leq C \left(1 + |x|^{mp}\right).
\]

### 2.2. The backward SDE

Let \(T > 0\) and \((t, x) \in [0, T] \times \hat{D}\) be fixed throughout this section. We want to consider the following backward stochastic variational inequality (BSVI), which is a generalized version of a BSDE:
\[
dY_t + F(s, X^t_s, Y^t_s, Z^t_s, Z^t_s) \, ds + G(s, X^t_s, Y^t_s, Z^t_s) \, dA^t_s - Z^t_s \, dB_s \in \partial \varphi (Y^t_s) \, ds + \partial \psi (Y^t_s) \, dA^t_s,
\]
where \(F, G\) and \(\kappa\) are continuous functions, and \(\varphi, \psi\) convex functions satisfying some assumptions which we will formulate below.

**Remark 2.3:** There are two aspects which distinguish the above BSVI from standard BSDEs. First the appearance of the integrator \(dA^t_s\) (which is singular with respect to Lebesgue’s measure \(ds\)). Second, and more importantly, the appearance of the subdifferentials \(\partial \varphi\) and \(\partial \psi\), which are possibly multivalued functions. This is the reason for the sign \(\in\) replacing the standard \(=\).

The following are some references to BSVI’s: [4–10].

We now give a rigorous formulation of our BSVI. We are looking for a quadruplet \((Y_r^{t, x}, Z_r^{t, x}, U_r^{t, x}, V_r^{t, x})_{r \in [t, T]}\), which is an \(\mathbb{R}^m \times \mathbb{R}^{m \times k} \times \mathbb{R}^m \times \mathbb{R}^m\)-valued stochastic process such that for all \(s \in [t, T], \mathbb{P}\)-a.s.,
\[
\left\{ Y_s^{t, x} + \int_s^T (U_r^{t, x} \, dr + V_r^{t, x} \, dA_r^{t, x}) = \kappa(X_T^{t, x}) + \int_s^T F(r, X_r^{t, x}, Y_r^{t, x}, Z_r^{t, x}) \, dr, \right.
\]
\[
\left. \quad + \int_s^T G(r, X_r^{t, x}, Y_r^{t, x}) \, dA_r^{t, x} - \int_s^T Z_r^{t, x} \, dB_r, \right. \]
\[
\text{and for all } u, v \in [t, T], u \leq v, \text{ for all continuous stochastic process } S, \mathbb{P}\text{-a.s.}:
\]
\[
\left. \int_u^v (U_r^{t, x}, S_r - Y_r^{t, x}) \, dr + \int_u^v \varphi(Y_r^{t, x}) \, dr \leq \int_u^v \varphi(S_r) \, dr, \right.
\]
\[
\left. \int_u^v (V_r^{t, x}, S_r - Y_r^{t, x}) \, dA_r^{t, x} + \int_u^v \psi(Y_r^{t, x}) \, dA_r^{t, x} \leq \int_u^v \psi(S_r) \, dA_r^{t, x}. \right. \]
We assume that $F : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \times \mathbb{R}^{m \times k} \to \mathbb{R}^m$, $G : \mathbb{R}_+ \times \overline{D} \times \mathbb{R}^m \to \mathbb{R}^m$ and $\kappa : \overline{D} \to \mathbb{R}^m$ are continuous, and that there exist $b_F, b_G, \ell_F > 0$ and $\mu_F, \mu_G \in \mathbb{R}$ such that for all $t \in [0, T], x \in \overline{D}, y, \tilde{y} \in \mathbb{R}^m, z, \tilde{z} \in \mathbb{R}^{m \times k},$

$$
\begin{align*}
&(i) \ |y - \tilde{y}, F(t, x, y, z) - F(t, x, \tilde{y}, z)| \leq \mu_F |y - \tilde{y}|^2, \\
&(ii) \ F(t, x, y, z) - F(t, x, \tilde{y}, z) \leq \ell_F |z - \tilde{z}|, \\
&(iii) \ F(t, x, y, 0) \leq b_F (1 + |y|), \\
&(iv) \ |y - \tilde{y}, G(t, x, y) - G(t, x, \tilde{y})| \leq \mu_G |y - \tilde{y}|^2, \\
&(v) \ G(t, x, y) \leq b_G (1 + |y|).
\end{align*}
$$

(H.3)

We also assume that

$$
\begin{align*}
&(i) \ \varphi, \psi : \mathbb{R}^m \to (-\infty, +\infty] \ are \ proper \ convex \ l.s.c. \ functions \\
&(ii) \ \exists u_0 \in \text{int } (\text{Dom } (\varphi) \cap \text{int } (\text{Dom } (\psi))) \ such \ that \\
&\quad \varphi (y) \geq \varphi (u_0) \ and \ \psi (y) \geq \psi (u_0), \ \forall \ y \in \mathbb{R}.
\end{align*}
$$

(H.4)

where $\text{Dom } (\varphi) = \{ y \in \mathbb{R}^m : \varphi (y) < \infty \}$ and similarly for $\text{Dom } (\psi)$. We also introduce some compatibility conditions:

there exists $M > 0$ and $c > 0$ such that for all $\varepsilon > 0$, $t \in [0, T], x \in \overline{D}, y \in \mathbb{R}^m, z \in \mathbb{R}^{m \times k},$

$$
\begin{align*}
&(a) \ \sup_{x \in \overline{D}} |\varphi (\kappa (x))| + \sup_{x \in \overline{D}} |\psi (\kappa (x))| = M < \infty \\
&(b) \ \langle \nabla \varphi_\varepsilon (y), \nabla \psi_\varepsilon (y) \rangle \geq 0, \\
&(c) \ \langle \nabla \varphi_\varepsilon (y), G(t, x, y) \rangle \leq c |\nabla \psi_\varepsilon (y)| \left[ 1 + |G(t, x, y)| \right], \\
&(d) \ \langle \nabla \psi_\varepsilon (y), G(t, x, y, z) \rangle \leq c |\nabla \varphi_\varepsilon (y)| \left[ 1 + |F(t, x, y, z)| \right], \\
&(e) \ -\langle \nabla \varphi_\varepsilon (y), F(t, x, u_0) \rangle \leq c \langle \nabla \psi_\varepsilon (y), G(t, x, u_0) \rangle \left[ 1 + |G(t, x, u_0)| \right], \\
&(f) \ -\langle \nabla \psi_\varepsilon (y), F(t, x, u_0, 0) \rangle \leq c \langle \nabla \varphi_\varepsilon (y), F(t, x, u_0, 0) \rangle \left[ 1 + |F(t, x, u_0)| \right]
\end{align*}
$$

(H.5)

where $\nabla \varphi_\varepsilon (y), \nabla \psi_\varepsilon (y)$ are the unique solutions $u$ and $v$, respectively, of the equations

$$
\partial \varphi (y - \varepsilon u) \ni u \ and \ \partial \psi (y - \varepsilon v) \ni v.
$$

(the Moreau–Yosida approximations: see Section 1 below).

**Remark 2.4:** The compatibility assumptions are satisfied e.g. in the two following cases.

(a) $\varphi = \psi$.

(b) $m = 1$ and $\varphi, \psi : \mathbb{R} \to (-\infty, +\infty]$ are the following convex indicator functions

$$
\varphi (y) = \begin{cases} 
0, & \text{if } y \in [a, \infty), \\
+\infty, & \text{if } y \notin [a, \infty),
\end{cases}
\quad \text{and} \quad
\psi (y) = \begin{cases} 
0, & \text{if } y \in (-\infty, b], \\
+\infty, & \text{if } y \notin (-\infty, b],
\end{cases}
$$

where $-\infty \leq a < b \leq +\infty$, then

$$
\nabla \varphi_\varepsilon (y) = -\frac{(a - y)^+}{\varepsilon} \quad \text{and} \quad \nabla \psi_\varepsilon (y) = \frac{(y - b)^+}{\varepsilon}.
$$
In this case the compatibility assumptions (H.4) are satisfied in particular if there exists \( u_0 \in (a, b) \) such that for all \( (t, x) \in [0, T] \times \overline{D} \) and for all \( z \in \mathbb{R}^{1 \times k} \):

\[
G(t, x, y) \geq 0, \text{ for all } y < a, \\
F(t, x, y, z) \leq 0, \text{ for all } y > b, \\
G(t, x, u_0) \leq 0 \text{ and } F(t, x, u_0, 0) \geq 0,
\]

The following follows readily from Theorem 5.69 in [10].

**Proposition 2.5:** Under the assumptions (H.3), (H.4) and (H.5), the BSVI (8) has a unique progressively measurable solution \((Y^{t,x}, Z^{t,x}, U^{t,x}, V^{t,x})\), with \(Y^{t,x}\) having continuous trajectories, such that for all \( \lambda \geq 0 \), \((t, x) \in [0, T] \times \overline{D}\),

\[
\mathbb{E} \sup_{r \in [t, T]} e^{2\lambda A^r_t} |Y_r^{t,x}|^2 + \mathbb{E} \left( \int_t^T e^{2\lambda A^r_s} |Z_r^{t,x}|^2 \, dr \right) < \infty.
\]

We extend the stochastic processes from (8) on \( \mathbb{R}^m \)-valued continuous stochastic process \( S \), \( \mathbb{P} \)-a.s.,

\[
Y_s^{t,x} + \int_s^T U_s^{t,x} \, dr + \int_s^T V_s^{t,x} \, dr = Y_t^{t,x} \quad \forall s \in [0, T],
\]

\[
U_t^{t,x} = \partial \varphi (Y_r^{t,x}) \text{ and } V_t^{t,x} = \partial \psi (Y_r^{t,x}) \quad \text{a.e. on } [0, T].
\]

Now we can write (8) as follows: for all \( s \in [t, T] \), \( \mathbb{P} \)-a.s.,

\[
\begin{cases}
Y_s^{t,x} + \int_s^T (U_s^{t,x} \, dr + V_s^{t,x} \, dA_s^{t,x}) = \kappa (X_T^{r,t}) + \int_s^T 1_{[t, T]}(r) F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \, dr \\
+ \int_s^T 1_{[t, T]}(r) G(r, X_r^{t,x}, Y_r^{t,x}) \, dA_s^{t,x} - \int_s^T Z_s^{t,x} \, dB_r, \\
\text{and for all } 0 \leq u \leq v \leq T,
\end{cases}
\]

\[
\text{for any } \mathbb{R}^m \text{-valued continuous stochastic process } S, \mathbb{P} \text{-a.s.,}
\]

\[
\begin{cases}
\int_u^v U_r^{t,x} (S_r - Y_r^{t,x}) \, dr + \int_u^v \varphi (Y_r^{t,x}) \, dr \leq \int_u^v \varphi (S_r) \, dr, \\
\int_u^v V_r^{t,x} (S_r - Y_r^{t,x}) \, dA_r^{t,x} + \int_u^v \psi (Y_r^{t,x}) \, dA_r^{t,x} \leq \int_u^v \psi (S_r) \, dA_r^{t,x}.
\end{cases}
\]

If we define

\[
K_s^{t,x} = \int_0^s (U_r^{t,x} \, dr + V_r^{t,x} \, dA_r^{t,x}), \quad s \in [0, T],
\]

then as measures on \([0, T]\) we have

\[
dK_r^{t,x} = U_r^{t,x} \, dr + V_r^{t,x} \, dA_r^{t,x} + \partial \varphi (Y_r^{t,x}) \, dr + \partial \psi (Y_r^{t,x}) \, dA_r^{t,x}
\]

and from the monotonicity of the subdifferential operators we have for all \((t, x), (\tau, y) \in [0, T] \times \overline{D}\),

\[
(Y_r^{t,x} - Y_r^{\tau,y}, dK_r^{t,x} - dK_r^{\tau,y}) \geq 0, \text{ as measure on } [0, T].
\]

We now list a series of estimates which will be useful below. Note that the various constants \( C_p, C_1, \ldots, C_4 \) below will not depend upon the value of \((t, x) \in [0, T] \times \overline{D}\), a fact which will be critical for us. We first deduce from [6], or Proposition 5.46 in [10] that for
every \( p \geq 2 \) there exists a positive constant \( C_p \) depending only upon \( p \) such that for all \( t \in [0, T], x \in \overline{D}, s \in [t, T] \) and \( \lambda \geq \max \{ (\mu_F + \ell_F^2), \mu_G \} \)

\[
\mathbb{E} \left[ \sup_{r \in [0,T]} e^{p\lambda(r+A_t^x)} \left| Y_r^t - u_0 \right|^p \right] + \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| Z_r^t \right|^2 \, dr \right)^{p/2} \right] \\
+ \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| \varphi \left( Y_r^t \right) - \varphi \left( u_0 \right) \right| \, dr \right)^{p/2} \right] \\
+ \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| \psi \left( Y_r^t \right) - \psi \left( u_0 \right) \right| \, dA_r^x \right)^{p/2} \right] \\
\leq C_p \mathbb{E} \left[ e^{\lambda(T+A_T^x)} \left| \kappa \left( X_T^x \right) - u_0 \right|^p + \left( \int_0^T e^{\lambda(r+A_t^x)} \left| \psi \left( Y_r^t \right) - \psi \left( u_0 \right) \right| \, dA_r^x \right)^p + \left( \int_0^T e^{\lambda(r+A_t^x)} \left| G \left( r, X_r^x, u_0 \right) \right| \, dr \right) \right].
\] (12)

Since \([0, T] \times \overline{D}\) is bounded, \(X_t^x \in \overline{D}\) for all \( r \in [0, T]\) and the functions \( \kappa, F \) and \( G \) are continuous, there exists a constant \( C_1 \) such that for all \( r \in [0, T]\)

\[
|\kappa \left( X_T^x \right)| + |F \left( r, X_r^x, u_0, 0 \right)| + |G \left( r, X_r^x, u_0 \right)| \leq C_1, \quad \mathbb{P} - \text{a.s.} \] (13)

Combining (12) with the estimate (5-jj), we deduce that for every \( \lambda \geq (\mu_F + \ell_F^2) \vee \mu_G \) and \( p > 0 \) there exists a constant \( C_2 \) such that

\[
\mathbb{E} \left( \sup_{r \in [0,T]} e^{p\lambda(r+A_t^x)} \left| Y_r^t \right|^p \right) + \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| Z_r^t \right|^2 \, dr \right)^{p/2} \right] \\
+ \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| \varphi \left( Y_r^t \right) \right| \, dr \right)^{p/2} \right] \\
+ \mathbb{E} \left[ \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| \psi \left( Y_r^t \right) \right| \, dA_r^x \right)^{p/2} \right] \\
\leq C_2
\] (14)

We deduce from (5.164) in Theorem 5.66 in [10] that there exists a constant \( C_3 \) such that

\[
\mathbb{E} \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| U_r^t \right|^2 \, dr \right) + \mathbb{E} \left( \int_0^T e^{2\lambda(r+A_t^x)} \left| V_r^t \right|^2 \, dA_r^x \right) \leq C_3
\] (15)

Since \( |G(t, x, y)| \leq b_G (1 + |y|) \) and \( |F(t, x, y, z)| \leq \ell_F |z| + b_F (1 + |y|) \), we deduce from (12) that for every \( p > 0 \) and \( \lambda > 0 \) large enough, there exists a positive constant \( C_4 \) such
that for all \((t, x), (\tau, y) \in [0, T] \times D\),
\[
\mathbb{E} \left[ \left( \int_0^T e^{2\lambda (r + A^{t,x}_r)} \left| F \left( r, X_{t}^{t,x}, Y_{r}, Z_{t}^{t,y} \right) \right|^2 \, dr \right)^p \right] + \mathbb{E} \left[ \left( \int_0^T e^{2\lambda (r + A^{t,x}_r)} \left| G \left( r, X_{t}^{t,x}, Y_{r} \right) \right|^2 \, dA^{t,x}_r \right)^p \right] \leq C_4
\]
(16)

It is clear that the inequalities (14)–(16) are satisfied for all \(\lambda \geq 0\).

We define
\[
u(t, x) = Y_{t}^{t,x}, \quad (t, x) \in [0, T] \times D,
\]
which is a deterministic quantity since \(Y_{t}^{t,x}\) is \(F_t\)-measurable. In the next section we shall prove that \((t, x) \mapsto \nu(t, x) : [0, T] \times D \to \mathbb{R}^m\) is a continuous function.

We remark that from the Markov property, we have
\[
u(s, X_{t}^{t,x}) = Y_{s}^{t,x}.
\]

Remark 2.6: We note that in the particular case where \(\varphi = \psi \equiv 0\), we are in the situation which was studied in [11].

Remark 2.7: By Theorem 5.52 from Pardoux and Răşcanu [10], we have in case \(D = \mathbb{R}^d\) \((A^{t,x}_r \equiv 0\), there is no \(G\) nor \(\psi = 0\) and the assumptions (H.3-(iii), (iv)) and (H.5-(a) are replaced by: there exist \(M, m > 0\) such that for all \((s, u) \in [0, T] \times \mathbb{R}^d\)
\[
|F(s, u, y, 0)| + |G(s, u, y)| + |\kappa(u)| + |\varphi(\kappa(u))| + |\psi(\kappa(u))| \leq M \left( 1 + |u|^m + |y| \right),
\]
that the BSVI (8) has a unique solution \((Y^{t,x}, Z^{t,x}, U^{t,x}, V^{t,x})\) and the boundedness properties (14)–(16) hold uniformly with respect to \((t, x)\) in any compact subset of \([0, T] \times \mathbb{R}^d\), the constants \(C_2, C_3\) and \(C_4\) depending possibly upon the compact set.

2.3. A backward stochastic inequality

We state a result which is a minor variant of Proposition 6.80 (Annex C) in Pardoux and Răşcanu [10].

Lemma 2.8: Let \((Y, Z) \in S_{m}^0 \times \Lambda_{m \times k}^0\) satisfy
\[
Y_t = Y_T + \int_t^T d\mathcal{K}_r - \int_t^T Z_r dB_r, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.,
\]
where \(\mathcal{K} \in S_{m}^0\) and \(\mathcal{K}_r(\omega) \in BV \left([0, T]; \mathbb{R}^m\right), \mathbb{P} - a.s.\)

Assume that we are given

- \(L\) a non-decreasing stochastic process, \(L_0 = 0\),
- \(R\) a stochastic process, \(R_0 = 0\) and \(R(\omega) \in BV \left([0, T]; \mathbb{R}^m\right), \mathbb{P} - a.s.\),
- \(V\) a continuous stochastic process, \(V_0 = 0\), \(V(\omega) \in BV \left([0, T]; \mathbb{R}^m\right), \mathbb{P} - a.s., such that
\[
\mathbb{E} \left( \int_0^T e^{2V_r} \, dR_r \right)^- < \infty
\]
If

(i) \( \langle Y_r, dK_r \rangle \leq \frac{a}{2} |Z_r|^2 \, dr + \left( |Y_r|^2 |dV_r + |Y_r|dL_r + dR_r \right) \) as measures on \([0, T]\),

(ii) \( \mathbb{E} \sup_{t \in [0, T]} |r^{-2} |Y_r|^2 < \infty \),

with \( a < 1 \), then there exists a positive constant \( C_a \), depending only \( a \), such that

\[
\mathbb{E} \left( \sup_{r \in [0, T]} \left| e^{V_r} Y_r \right|^2 \right) + \mathbb{E} \left( \int_0^T e^{2V_r} |Z_r|^2 \, dr \right) \leq C_a \mathbb{E} \left[ |e^{V_T} Y_T|^2 + \left( \int_0^T e^{V_r} dL_r \right)^2 + \int_0^T e^{2V_r} dR_r \right].
\]

The proof of Lemma 2.8 is almost identical to the proof of Proposition 6.80 [10], with a single small change: in the definition of the localizing stopping time, we delete the term containing \( R \), and therefore we do not need to restrict us to the case where \( R \) is non-decreasing.

3. Continuity of \( u(t, x) \)

We present here the main result of this paper. The proof will rely upon several Lemmas which will be proved later in this section.

**Theorem 3.1:** Under the assumptions \((H.1), \ldots, (H.5)\), the mapping \((t, x) \mapsto u(t, x) = Y_t^{t, x} : [0, T] \times \bar{D} \to \mathbb{R}^m\) is continuous.

**Proof:** Let \((t_n, x_n)_{n \geq 1}, (t, x) \in [0, T] \times \bar{D}\) be such that \((t_n, x_n) \to (t, x)\), as \( n \to \infty \).

Denote \( \Theta^n_s = \Theta^{t_n, x_n}_s \) and \( \Theta^0_s = \Theta^{t, x}_s \) for \( \Theta = X, Y, Z, U, V, K \). We have

\[ Y^n_s - Y_s = \kappa \left( X^n_T \right) - \kappa \left( X_T \right) + \int_s^T dK^n_r - \int_s^T \left( Z^n_r - Z_r \right) dB_r \]

where

\[ dK^n_r = d\left( K_r - K^n_r \right) + \left[ 1_{t_n, T^n} \left( r \right) F \left( r, X^n_r, Y^n_r, Z^n_r \right) - 1_{[t, T]} \left( r \right) F \left( r, X_r, Y_r, Z_r \right) \right] dr \]

\[ + \left[ 1_{t_n, T^n} \left( r \right) G \left( r, X^n_r, Y^n_r \right) dA^n_r - 1_{[t, T]} \left( r \right) G \left( r, X_r, Y_r \right) dA_r \right]. \]

with \( dK^n_r = U^n_r \, dr + V^n_r dA^n_r \in \partial \varphi \left( Y^n_r \right) \, dr + \partial \psi \left( Y^n_r \right) dA^n_r \) and \( dK_r = U_r \, dr + V_r dA_r \in \partial \varphi \left( Y_r \right) \, dr + \partial \psi \left( Y_r \right) dA_r \). Remark that by (11) it holds

\[ \langle Y^n_r - Y_r, dK_r - dK^n_r \rangle \leq 0, \text{ as a signed measure on } [0, T]. \]

It is easy to verify that:

\[ \langle Y^n_r - Y_r, 1_{t_n, T^n} \left( r \right) F \left( r, X^n_r, Y^n_r, Z^n_r \right) - 1_{[t, T]} \left( r \right) F \left( r, X_r, Y_r, Z_r \right) \rangle \, dr \leq \langle Y^n_r - Y_r, 1_{t_n, T^n} \left( r \right) \left[ F \left( r, X^n_r, Y^n_r, Z^n_r \right) - F \left( r, X^n_r, Y^n_r, Z_r \right) \right] \rangle \, dr \]
\begin{align*}
&+ \{Y^n_r - Y_r, 1_{\mathfrak{fr}, T \not= \emptyset} (r) \left[ F \left( r, X^n_r, Y^n_r, Z_r \right) - F \left( r, X^n_r, Y_r, Z_r \right) \right] \} \, dr \\
&+ \{Y^n_r - Y_r, 1_{\mathfrak{fc}, T \not= \emptyset} (r) \left( F \left( r, X^n_r, Y_r, Z_r \right) - 1_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right) \} \, dr \\
&\leq \ell_F \left| Y^n_r - Y_r \right| \left| Z^n_r - Z_r \right| \, dr + \mu_F \left| Y^n_r - Y_r \right|^2 \, dr \\
&+ \{Y^n_r - Y_r \left| 1_{\mathfrak{fc}, T \not= \emptyset} (r) \left( F \left( r, X^n_r, Y_r, Z_r \right) - 1_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right) \right| \} \, dr \\
&\leq (\mu_F + \ell_F^2) \left| Y^n_r - Y_r \right|^2 \, dr + \frac{1}{4} \left| Z^n_r - Z_r \right|^2 \, dr \\
&+ \{Y^n_r - Y_r \left| 1_{\mathfrak{fc}, T \not= \emptyset} (r) \left( F \left( r, X^n_r, Y_r, Z_r \right) - 1_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right) \right| \} \, dr \\
\end{align*}

and

\begin{align*}
\left\langle Y^n_r - Y_r, 1_{\mathfrak{fc}, T \not= \emptyset} (r) \ G \left( r, X^n_r, Y^n_r \right) \ dA^n_r - 1_{[t, T]} (r) \ G \left( r, X_r, Y_r \right) \ dA_r \right\rangle \\
\leq \left\langle Y^n_r - Y_r, 1_{\mathfrak{fc}, T \not= \emptyset} (r) \left[ G \left( r, X^n_r, Y^n_r \right) - G \left( r, X^n_r, Y_r \right) \right] \ dA^n_r \right\rangle \\
+ \left\langle Y^n_r - Y_r, 1_{\mathfrak{fc}, T \not= \emptyset} (r) \ G \left( r, X^n_r, Y_r \right) - 1_{[t, T]} (r) G \left( r, X_r, Y_r \right) \ dA^n_r \right\rangle \\
+ \left\langle Y^n_r - Y_r, 1_{[t, T]} (r) G \left( r, X_r, Y_r \right) \left( dA^n_r - dA_r \right) \right\rangle \\
\leq \mu_G \left| Y^n_r - Y_r \right|^2 \, dA^n_r \\
+ \left\langle Y^n_r - Y_r \left| 1_{\mathfrak{fc}, T \not= \emptyset} (r) \ G \left( r, X^n_r, Y_r \right) - 1_{[t, T]} (r) G \left( r, X_r, Y_r \right) \ dA^n_r \right| \right\rangle \\
+ \left\langle Y^n_r - Y_r, 1_{[t, T]} (r) G \left( r, X_r, Y_r \right) \left( dA^n_r - dA_r \right) \right\rangle
\end{align*}

Hence for \( \lambda \geq (\mu_F + \ell_F^2) \vee \mu_G \)

\begin{align*}
\left\langle Y^n_r - Y_r, dK^n_r \right\rangle \leq \frac{1}{4} \left| Z^n_r - Z_r \right|^2 \, dr + \left| Y^n_r - Y_r \right|^2 \lambda \left( dr + dA^n_r \right) \\
+ \left| Y^n_r - Y_r \right| \left| dL_r^{(n)} + dR_r^{(n)} \right|
\end{align*}

with

\begin{align*}
dL_r^{(n)} &= \left\langle 1_{\mathfrak{fc}, T \not= \emptyset} (r) F \left( r, X^n_r, Y_r, Z_r \right) - 1_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right| \, dr \\
+ \left\langle 1_{\mathfrak{fc}, T \not= \emptyset} (r) G \left( r, X^n_r, Y_r \right) - 1_{[t, T]} (r) G \left( r, X_r, Y_r \right) \right. \left. \, dA^n_r \right| \\
\text{(18)}
\end{align*}

and

\begin{align*}
dR_r^{(n)} &= \left\langle Y^n_r - Y_r, 1_{\mathfrak{fc}, T \not= \emptyset} (r) G \left( r, X_r, Y_r \right) \left( dA^n_r - dA_r \right) \right| \right. \left. \\
\text{(19)}
\end{align*}

From Lemma 2.8 with \( a = 1/2 \), it follows that

\begin{align*}
\mathbb{E} \sup_{r \in [0,T]} e^{2\lambda \left( r + A^n_r \right)} \left| Y^n_r - Y_r \right|^2 + \mathbb{E} \left( \int_0^T e^{2\lambda \left( r + A^n_r \right)} \left| Z^n_r - Z_r \right|^2 \, dr \right) \\
\leq C_a \mathbb{E} \left[ e^{2\lambda \left( T + A^n_T \right)} \left| \kappa \left( X^n_T \right) - \kappa \left( X_T \right) \right|^2 + \left( \int_0^T e^{\lambda \left( r + A^n_r \right)} dL_r^{(n)} \right)^2 \\
+ \int_0^T e^{2\lambda \left( r + A^n_r \right)} dR_r^{(n)} \right].
\end{align*}
Consequently by Lemmas 3.2, 3.3 and 3.5 below, we have
\[
\limsup_{n \to \infty} \mathbb{E} \sup_{r \in [0,T]} |Y^n_r - Y_r|^2 \leq \limsup_{n \to \infty} \mathbb{E} \sup_{r \in [0,T]} e^{2\lambda(r + A^n_r)} |Y^n_r - Y_r|^2 = 0.
\]

We now deduce
\[
|Y^{t_n,x_n}_n - Y^{t,x}_t|^2 \leq 2\mathbb{E} |Y^{t_n,x_n}_n - Y^{t,x}_t|^2 + 2\mathbb{E} |Y^{t,x}_t - Y^{t,x}_t|^2
\]
\[
\leq 2\mathbb{E} \sup_{r \in [0,T]} |Y^n_r - Y_r|^2 + 2\mathbb{E} |Y^{t,x}_t - Y^{t,x}_t|^2
\]
\[
\to 0, \quad \text{as } n \to \infty;
\]
since the last term on the right tend to 0 from the a.s. continuity of the mapping \(s \to Y^{t,x}_s\) and uniform integrability, which follows from (14). The result follows.

Since the constants \(C_1, C_2, C_3\) and \(C_4\) appearing in (13), (14), (15) and (16) are uniform w.r.t. \((t, x)\), those estimates are valid for \((X^n, A^n, Y^n, Z^n, U^n, V^n)\) for all \(n \geq 0\), with the same constants, which are independent of \(n\). This fact will be used repeatedly in the proofs below.

**Lemma 3.2:** We have
\[
\lim_{n \to \infty} \mathbb{E} \left( e^{2\lambda(T + A^n_T)} |\kappa \left(X^n_T\right) - \kappa \left(X_T\right)|^2 \right) = 0
\]

**Proof:** By Lebesgue’s dominated convergence theorem and (6) (also taking in account the boundedness (5-jj) and (13)), we have
\[
\mathbb{E} \left( e^{2\lambda(T + A^n_T)} |\kappa \left(X^n_T\right) - \kappa \left(X_T\right)|^2 \right)
\]
\[
\leq \left( \mathbb{E} e^{4\lambda(T + A^n_T)} \right)^{1/2} \left( \mathbb{E} |\kappa \left(X^n_T\right) - \kappa \left(X_T\right)|^4 \right)^{1/2}
\]
\[
\leq C_\lambda \left( \mathbb{E} |\kappa \left(X^n_T\right) - \kappa \left(X_T\right)|^4 \right)^{1/2}
\]
\[
\to 0, \quad \text{as } n \to \infty.
\]

**Lemma 3.3:** Let \(L^{(n)}\) be defined by (18). Then
\[
\int_0^T e^{\lambda(r + A^n_r)} dL^{(n)}_r \to 0
\]
in mean square, as \(n \to \infty\).

**Proof:** By (5-jj) we get
\[
\mathbb{E} \left( \int_0^T e^{\lambda(r + A^n_r)} dL^{(n)}_r \right)^2 \leq 3 \left[ \mathbb{E} (\Lambda_n) + \mathbb{E} (\Gamma_n) + \mathbb{E} (\Delta_n) \right].
\]
where

\[
\Lambda_n = \left( \int_0^T \left| \mathbf{1}_{t_n, T}^\ast (r) F \left( r, X_r^n, Y_r, Z_r \right) - \mathbf{1}_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right|^2 \, dr \right)^2,
\]

\[
\Gamma_n = \left( \int_0^T \left| G \left( r, X_r^n, Y_r \right) - G \left( r, X_r, Y_r \right) \right|^2 \, dA_r^n \right)^2,
\]

\[
\Delta_n = \left( \int_0^T \left| G \left( r, X_r, Y_r \right) \right|^2 \left| \mathbf{1}_{t_n, T}^\ast (r) - \mathbf{1}_{[t, T]} (r) \right|^2 \, dA_r^n \right)^2. \tag{20}
\]

**Step 1.** $\mathbb{E} (\Lambda_n) \to 0$:

Since

\[
\mathbf{1}_{t_n, T}^\ast (r) F \left( r, X_r^n, Y_r, Z_r \right) - \mathbf{1}_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \to 0 \quad \text{a.e.} \quad r \in [0, T],
\]

and

\[
\left| \mathbf{1}_{t_n, T}^\ast (r) F \left( r, X_r^n, Y_r, Z_r \right) - \mathbf{1}_{[t, T]} (r) F \left( r, X_r, Y_r, Z_r \right) \right|^2 \leq C \left( 1 + |Y_r|^2 + |Z_r|^2 \right),
\]

then by Lebesgue’s dominated convergence theorem $\mathbb{E} \Lambda_n \to 0$.

**Step 2.** $\mathbb{E} (\Gamma_n) \to 0$:

We have $\Gamma_n \to 0$, $\mathbb{P}$-a.s., because

\[
\Gamma_n = \left( \int_0^T \left| G \left( r, X_r^n, Y_r \right) - G \left( r, X_r, Y_r \right) \right|^2 \, dA_r^n \right)^2 \leq \left( A_T^n \right)^2 \sup_{r \in [0, T]} \left| G \left( r, X_r^n, Y_r \right) - G \left( r, X_r, Y_r \right) \right|^4.
\]

Since for all $q > 1$

\[
\mathbb{E} \Gamma_n^q \leq C \mathbb{E} \left[ \left( 1 + |Y|_{L^q}^4 \right) |A_T^n|^{2q} \right] \leq C_1 \left( 1 + \mathbb{E} |Y|_{L^q}^{8q} + \mathbb{E} |A_T^n|^{4q} \right) \leq C_2,
\]

then the sequence of random variables $\Gamma_n$ is uniformly integrable and therefore $\mathbb{E} (\Gamma_n) \to 0$.

**Step 3.** $\mathbb{E} (\Delta_n) \to 0$:

We have

\[
\Delta_n = \left( \int_0^T \left| G \left( r, X_r, Y_r \right) \right|^2 \left| \mathbf{1}_{t_n, T}^\ast (r) - \mathbf{1}_{[t, T]} (r) \right|^2 \, dA_r^n \right)^2 \leq \left( \sup_{r \in [0, T]} \left| G \left( r, X_r, Y_r \right) \right|^4 \right) \left( \int_0^T \left| \mathbf{1}_{t_n, T}^\ast (r) - \mathbf{1}_{[t, T]} (r) \right|^2 \, dA_r^n \right)^2.
\]
\[= \left( \sup_{r \in [0,T]} |G(r, X_r, Y_r)|^4 \right) |A^n_{t_n} - A^n_t|^2 \]
\[\to 0, \quad P - \text{a.s.,} \]

where we have used (7) on the last line. Moreover for \( q > 1, \)

\[
\mathbb{E} \Delta_n^q \leq \mathbb{E} \left[ \sup_{r \in \dagger 0, \dagger T} |G(r, X_r, Y_r)|^{4q} |A^n_{t_n} - A^n_t|^{2q} \right] \\
\leq C \left( \mathbb{E} \sup_{r \in [0,T]} |G(r, X_r, Y_r)|^{8q} + \mathbb{E} \sup_{r \in \dagger 0, \dagger T} |A^n_r|^{4q} \right) \\
\leq C_1
\]

Consequently, by uniformly integrability, we conclude that \( \mathbb{E} (\Delta_n) \to 0. \)

Consider \( N \in \mathbb{N}, N > T \) and the partition \( \pi_N : 0 = r_0 < r_1 < \ldots < r_i < \ldots < r_N = T \) with \( r_i = \frac{iT}{N} \). We denote \( \lfloor r \rfloor N = \max \{ r_i : r_i \leq r \} = \left[ \frac{rN}{T} \right] \), where \( \lfloor x \rfloor \) is the integer part of \( x \). Given a continuous stochastic process \( (H_t)_{t \in [0,T]} \), we define

\[ H^N_t = \sum_{i=0}^{N-1} H_{t_i} 1_{[r_i, r_{i+1})} (r) + H_T 1_{[T]} (r) = H_{\lfloor r \rfloor N}. \]

**Lemma 3.4:** Let \( 1 < q < 2 \). There exists a positive constant \( C \) independent of \( (t,x) \), \( (t_n, x_n) \in [0,T] \times \mathcal{D} \) and \( N \in \mathbb{N} \) such that

\[
\limsup_{n \to \infty} \mathbb{E} \left( \int_0^T |Y^n_r - Y^{n,N}_r|^q (dA^n_r + dA_r) \right) \\
\leq \frac{C}{N^{q/2}} + C \left[ \mathbb{E} \max_{\lfloor 1 \rfloor N} (A_{r_i} - A_{r_{i-1}})^{2q/(2-q)} \right]^{(2-q)/4}. \]

**Proof:** Since

\[
Y^n_{s,N} + \int_{[s,N]} (U^n_t \, dr + V^n_t \, dA^n_t) = Y^n_s + \int_{[s,N]} 1_{\dagger t_n, \dagger T} (r) F (r, X^n_r, Y^n_r, Z^n_r) \, dr, \\
+ \int_{[s,N]} 1_{\dagger t_n, \dagger T} (r) G (r, X^n_r, Y^n_r) \, dA^n_r \\
- \int_{[s,N]} \{Z^n_r, dB_r\}, \forall \ s \in [0,T],
\]

then

\[
|Y^n_{s,N} - Y^n_s|^q \leq \frac{C}{N^{q/2}} \left[ \int_{[s,N]} \left( |U^n_r|^2 + |F (r, X^n_r, Y^n_r, Z^n_r)|^2 \right) \, dr \right]^{q/2} \\
+ C \left( A^n_s - A^n_{[s,N]} \right)^{q/2} \left[ \int_{[s,N]} \left( |V^n_r|^2 + |G (r, X^n_r, Y^n_r)|^2 \right) \, dA^n_r \right]^{q/2}
\]
Hence
\[ \mathbb{E} \left( \int_0^T |Y_r^n - Y_r^{n,N}|^q (dA_r^n + dA_r) \right) \leq \alpha_{n,N} + \beta_{n,N} + \gamma_{n,N}. \]

We have first
\[
\alpha_{n,N} = \frac{C}{N^{q/2}} \mathbb{E} \left[ \int_0^T \left( \int_{[s|N]} \left( |U_r^n|^2 + |F(r, X_r^n, Y_r^n, Z_r^n)|^2 \right) dr \right)^{q/2} (dA_s^n + dA_s) \right],
\]
\[
\leq \frac{C}{N^{q/2}} \mathbb{E} \left[ (A_T^n + A_T) \left( \int_0^T \left( |U_r^n|^2 + |F(r, X_r^n, Y_r^n, Z_r^n)|^2 \right) dr \right)^{q/2} \right],
\]
\[
\leq \frac{C}{N^{q/2}} \left[ \mathbb{E} (A_T^n + A_T)^{\frac{2}{2-q}} \left( \mathbb{E} \int_0^T |U_r^n|^2 dr + \mathbb{E} \int_0^T |F(r, X_r^n, Y_r^n, Z_r^n)|^2 dr \right)^{\frac{q}{2}} \right].
\]

Since \((A_s^n)_{s \geq 0}\) and \((A_s)_{s \geq 0}\) are increasing stochastic processes,
\[
\beta_{n,N} = C \mathbb{E} \int_0^T \left( (A_s^n - A_{s|N]}^n)^{\frac{q}{2}} \left( \int_{[s|N]} \left( |V_r^n|^2 + |G(r, X_r^n, Y_r^n)|^2 \right) dA_r^n \right)^{\frac{q}{2}} \right) (dA_s^n + dA_s),
\]
\[
\leq C \mathbb{E} \left[ \left( \int_0^T |V_r^n|^2 + |G(r, X_r^n, Y_r^n)|^2 \right) dA_r^n \right]^{\frac{q}{2}} \sum_{i=1}^N \int_{r_{i-1}}^{r_i} (A_s^n - A_{s|N]}^n)^{\frac{q}{2}} (dA_s^n + dA_s) \right],
\]
\[
\leq C \left[ \mathbb{E} \left( \sum_{i=1}^N (A_{r_i}^n - A_{r_{i-1}}^n)^{q/2} (A_{r_i}^n + A_{r_i}^n - A_{r_{i-1}}^n) \right) \right]^{2/(2-q)} \left( 2-q \right)/2.
\]

Since by \((5-j)\)
\[
\lim_{n \to \infty} \mathbb{E} \sup_{r \in [0,T]} |A_r^n - A_r|^p = 0, \quad \text{for all } p > 0,
\]
and
\[
\mathbb{E} \sup_{r \in [0,T]} |A_r|^p + \sup_{n \in \mathbb{N}} \left( \mathbb{E} \sup_{r \in [0,T]} |A_r^n|^p \right) < \infty, \quad \text{for all } p > 0,
\]
we infer that for all \(N \in \mathbb{N}\)
\[
\limsup_{n \to \infty} \beta_{n,N} \leq C \left[ \mathbb{E} \left( \sum_{i=1}^N (A_{r_i} - A_{r_{i-1}})^{q/2} (A_{r_i} - A_{r_{i-1}}) \right) \right]^{2/(2-q)} \left( 2-q \right)/2
\]
\[
\leq C \left[ \mathbb{E} \left( \max_{i=1,N} (A_{r_i} - A_{r_{i-1}})^{q/2} A_T \right) \right]^{2/(2-q)} \left( 2-q \right)/2
\]
\[
\leq C_1 \left[ \mathbb{E} \max_{i=1,N} (A_{r_i} - A_{r_{i-1}})^{2q/(2-q)} \right]^{(2-q)/4}.
\]
We finally consider

\[ \gamma_{n,N} = C \mathbb{E} \int_0^T \left| \int_{[t,s]} \langle Z^n_r, dB_r \rangle \right|^q (dA^n_s + dA_s) \]

\[ = C \mathbb{E} \sum_{i=1}^N \int_{r_{i-1}}^{r_i} \left| \int_{[t,s]} \langle Z^n_r, dB_r \rangle \right|^q (dA^n_s + dA_s) \]

\[ \leq C \sum_{i=1}^N \mathbb{E} \left[ \sup_{s \in [r_{i-1}, r_i]} \left| \int_{r_{i-1}}^{r_i} \langle Z^n_r, dB_r \rangle \right|^q \left( A^n_{r_i} - A^n_{r_{i-1}} + A_{r_i} - A_{r_{i-1}} \right) \right] \]

\[ \leq C \sum_{i=1}^N \left[ \mathbb{E} \left[ \sup_{s \in [r_{i-1}, r_i]} \left| \int_{r_{i-1}}^{r_i} \langle Z^n_r, dB_r \rangle \right|^2 \right]^{q/2} \left[ \mathbb{E} \left( A^n_{r_i} - A^n_{r_{i-1}} + A_{r_i} - A_{r_{i-1}} \right)^2 \right]^{2-q/2} \right] \]

\[ \leq C \sum_{i=1}^N \left( \mathbb{E} \int_{r_{i-1}}^{r_i} |Z^n_r|^2 \, dr \right)^{q/2} \left[ \mathbb{E} \left( A^n_{r_i} - A^n_{r_{i-1}} + A_{r_i} - A_{r_{i-1}} \right)^2 \right]^{2-q/2} \]

From the above and the following Hölder’s inequality, which is valid for all nonnegative real numbers \( a_i, b_i, 1 \leq i \leq N \), and for \( 1 < q < 2 \),

\[ \sum_{i=1}^N a_i^{q/2} b_i^{(2-q)/2} \leq \left( \sum_{i=1}^N a_i \right)^{q/2} \left( \sum_{i=1}^N b_i \right)^{(2-q)/2}, \]

we deduce that

\[ \gamma_{n,N} \leq C_2 \left[ \sum_{i=1}^N \mathbb{E} \left( A^n_{r_i} - A^n_{r_{i-1}} + A_{r_i} - A_{r_{i-1}} \right)^2 \right]^{(2-q)/2}. \]

Hence for all \( N \in \mathbb{N} \)

\[ \limsup_{n \to \infty} \gamma_{n,N} \leq C \left[ \sum_{i=1}^N \mathbb{E} \left( A_{r_i} - A_{r_{i-1}} \right)^{2/(2-q)} \right]^{(2-q)/2} \]

\[ \leq C \left[ \mathbb{E} \left( \max_{i=1,N} \left( A_{r_i} - A_{r_{i-1}} \right) \right)^q \sum_{i=1}^N \left( A_{r_i} - A_{r_{i-1}} \right) \right]^{(2-q)/2} \]

\[ \leq C_1 \left[ \mathbb{E} \max_{i=1,N} \left( A_{r_i} - A_{r_{i-1}} \right)^{2q/(2-q)} \right]^{(2-q)/4}. \]

The result follows.

\[ \square \]

**Lemma 3.5:** Let \( R^{(n)} \) be defined by (19). Then

\[ \limsup_{n \to \infty} \mathbb{E} \int_0^T e^{2\lambda (r + A^n_r)} \, dR_r^{(n)} = 0. \]
Corollary 3.7: Of Remarks 2.2 and 2.7, uniform integrability properties allowing to pass to the limit in Lemmas 3.2 and 3.3 yield the same continuity result as in Theorem 3.1.

Remark 3.6: If \( D = \mathbb{R}^d \) (\( A^t \equiv 0 \), there is no \( G \) nor \( \psi = 0 \)), then, under the conditions of Remarks 2.2 and 2.7, uniform integrability properties allowing to pass to the limit in Lemmas 3.2 and 3.3 yield the same continuity result as in Theorem 3.1.

Theorem 3.1 yields

Corollary 3.7: Corollary 14 from [5] holds true.

and in the particular case \( \varphi = \psi \equiv 0 \)
Corollary 3.8: Proposition 4.1 from [11] holds true.

4. Infinite horizon BSDEs: continuity

Let us consider the forward–backward problem (3) and (8) on the interval $[0, \infty)$ with $f, g, F$ and $G$ independent of time argument, $\kappa = 0$ and $\varphi = \psi \equiv 0$, $u_0 = 0$, that is:

- the forward reflected SDE starting from $x$ at $t = 0$:
  
  $\begin{align*}
  (j) & \quad X^x_s \in \overline{D} \text{ for all } s \geq 0, \\
  (jj) & \quad 0 = A^x_0 \leq A^x_s \leq A^x_u \text{ for all } 0 \leq s \leq u, \\
  (jjj) & \quad X^x_s + \int_0^s \nabla \phi \left( X^x_r \right) \, dA^x_r = x + \int_0^s f \left( X^x_r \right) \, dr \\
  & \quad + \int_0^s g \left( X^x_r \right) \, dB_r, \quad \forall \, s \geq 0, \\
  (jv) & \quad A^x_s = \int_0^s 1_{Bd(D)} \left( X^x_r \right) \, dA^x_r, \quad \forall \, s \geq 0.
  \end{align*}$

and the BSDE on $[0, \infty)$ with the final data $0$:

$$Y^x_s = \int_s^\infty F \left( X^x_r, Y^x_r, Z^x_r \right) \, dr + \int_s^\infty G \left( X^x_r, Y^x_r \right) \, dA^x_r - \int_s^\infty Z^x_r \, dB_r, \quad s \geq 0. \tag{21}$$

Denote $(X^x_s, A^x_s, Y^{x;n}_s, Z^{x;n}_s) = (X^{0,x}_s, A^{0,x}_s, Y^{0,x}_s, Z^{0,x}_s), n \in \mathbb{N}$, the solution of the forward–backward problem (3) and (8) on the time interval $[0, n]$ with $(Y^{x;n}_s, Z^{x;n}_s) = 0$, for $s > n$; hence

$$Y^{x;n}_s = \int_s^n F \left( X^x_r, Y^{x;n}_r, Z^{x;n}_r \right) \, dr + \int_s^n G \left( X^x_r, Y^{x;n}_r \right) \, dA^x_r - \int_s^n Z^{x;n}_r \, dB_r, \quad s \in [0, n]. \tag{22}$$

From Theorem 3.1, the mapping

$$x \mapsto Y^{x;n}_0 : \overline{D} \to \mathbb{R}^m \text{ is continuous.} \tag{23}$$

Estimates on the approximating Equation (22) and the continuity result (23) yield:

**Theorem 4.1:** Under the assumptions (H.3) and $\max \left\{ (\mu_F + \ell^2_F), \mu_G \right\} \leq \lambda < 0$ there exists a unique pair $(Y^x, Z^x) \in S^0_m[0, T] \times \Lambda^0_{m \times k}(0, T)$ solving the BSDE (21) in the following sense:

$$\begin{align*}
(j) & \quad Y^x_s = Y^x_T + \int_s^T F \left( X^x_r, Y^x_r, Z^x_r \right) \, dr + \int_s^T G \left( X^x_r, Y^x_r \right) \, dA^x_r - \int_s^T Z^x_r \, dB_r, \\
(jj) & \quad \mathbb{E} \sup_{r \geq 0} e^{2\lambda (r + A^x_r)} \left| Y^x_r \right|^2 + \mathbb{E} \int_0^\infty e^{2\lambda (r + A^x_r)} \left| Z^x_r \right|^2 \, dr < \infty, \\
(jjj) & \quad \lim_{N \to \infty} \mathbb{E} \sup_{r \geq N} e^{2\lambda (r + A^x_r)} \left| Y^x_r \right|^2 = 0.
\end{align*} \tag{24}$$

Moreover the mapping

$$x \mapsto u(x) = Y^x_0 : \overline{D} \to \mathbb{R}^m \text{ is continuous.} \tag{25}$$
Proof: The existence and uniqueness of the solution to (24) was proved by Pardoux and Zhang in [11], Theorem 2.1 (the result is also given in [10], Section 5.6.1). Proving here the continuity property (25) we obtain, once again, the existence of the solution; the uniqueness is an easy consequence of Lemma 2.8 via the assumptions of the Theorem.

Using (H.3) we also deduce by Lemma 2.8 with \( a = 1/2 \) (or directly from (12)) that for \( 0 \leq s \leq n \):

\[
\mathbb{E} \left[ \sup_{r \in I_s, n} e^{2\lambda (r + A^x_r)} \left| Y_r^{x,n} \right|^2 \right] + \mathbb{E} \int_s^n \left( e^{2\lambda (r + A^x_r)} \left| Z_r^{x,n} \right|^2 \right) dr \\
\leq C \mathbb{E} \left[ e^{2\lambda (s + A^x_s)} \left| Y_s^{x,n} \right|^2 \right] + \left( \int_s^n e^{\lambda (r + A^x_r)} \left| G(X_r^x, 0) \right| dr \right)^2 \\
\leq C' \mathbb{E} \left[ \left( \int_s^n e^{\lambda (r + A^x_r)} (dr + dA^x_r) \right)^2 \right] \\
\leq \frac{C'}{\lambda} e^{2\lambda (s + A^x_s)} \\
\leq \frac{C'}{\lambda} e^{2\lambda s}.
\]

Since \((Y_s^{x,n}, Z_s^{x,n}) = 0\), for \( s > n \) we infer that for all \( s \geq 0 \) and \( n \in \mathbb{N} \),

\[
\mathbb{E} \sup_{r \geq s} e^{2\lambda (r + A^x_r)} \left| Y_r^{x,n} \right|^2 + \mathbb{E} \int_s^\infty e^{2\lambda (r + A^x_r)} \left| Z_r^{x,n} \right|^2 dr \leq \frac{C}{\lambda} e^{2\lambda s}.
\]

If \( n, l \in \mathbb{N} \) and \( s \in [0, n] \), then

\[
Y_s^{x,n+l} - Y_s^{x,n} = Y_n^{x,n+l} + \int_s^n dK_r - \int_s^n \left( Z_r^{x,n+l} - Z_r^{x,n} \right) dB_r,
\]

where

\[
dK_r = \left[ F(X_r^x, Y_r^{x,n+l}, Z_r^{x,n+l}) - F(X_r^x, Y_r^{x,n}, Z_r^{x,n}) \right] dr \\
- \left[ G(X_r^x, Y_r^{x,n+l}) - G(X_r^x, Y_r^{x,n}) \right] dA^x_r.
\]

By the assumption (H.3) we have

\[
\left\{ Y_r^{x,n+l} - Y_r^{x,n}, dK_r \right\} \leq \mu_F \left| Y_r^{x,n+l} - Y_r^{x,n} \right|^2 dr + \ell_F \left| Y_r^{x,n+l} - Y_r^{x,n} \right| Z_r^{x,n+l} - Z_r^{x,n} \right| dr \\
+ \mu_G \left| Y_r^{x,n+l} - Y_r^{x,n} \right|^2 dA^x_r \\
\leq \frac{1}{4} \left| Z_r^{x,n+l} - Z_r^{x,n} \right|^2 dr + \left| Y_r^{x,n+l} - Y_r^{x,n} \right|^2 \lambda (dr + dA^x_r).
\]
Therefore by Lemma 2.8 (with $a = 1/2$) and (26) we get

$$
\mathbb{E} \left[ \sup_{r \in [0,n]} e^{2\lambda (r + A^x_r)} \left| Y_r^{x;n} - Y_r^n \right|^2 \right] + \mathbb{E} \int_0^n e^{2\lambda (r + A^x_r)} \left| Z_r^{x;n} - Z_r^n \right|^2 \, dr \\
\leq C \mathbb{E} \left[ e^{2\lambda n} \left| Y_r^{x;n} \right|^2 \right] \\
\leq \frac{C}{|\lambda|} e^{2\lambda n}.
$$

(28)

Hence

$$
\mathbb{E} \sup_{r \geq 0} e^{2\lambda (r + A^x_r)} \left| Y_r^{x;n+l} - Y_r^n \right|^2 + \mathbb{E} \int_0^\infty e^{2\lambda (r + A^x_r)} \left| Z_r^{x;n+l} - Z_r^n \right|^2 \, dr \leq \frac{C}{|\lambda|} e^{2\lambda n}
$$

and consequently there exists $(Y_s^x, Z_s^x)_{s \geq 0}$ a pair of progressively measurable stochastic process, $(Y_s^x)_{s \geq 0}$ having continuous trajectories, such that for all $s \geq 0$

$$
\mathbb{E} \sup_{r \geq s} e^{2\lambda (r + A^x_r)} \left| Y_r^{x} - Y_r^n \right|^2 + \mathbb{E} \int_s^\infty e^{2\lambda (r + A^x_r)} \left| Z_r^{x} - Z_r^n \right|^2 \, dr < \frac{C}{|\lambda|} e^{2\lambda s}
$$

and

$$
\mathbb{E} \sup_{r \geq 0} e^{2\lambda (r + A^x_r)} \left| Y_r^{x} - Y_r^n \right|^2 + \mathbb{E} \int_0^\infty e^{2\lambda (r + A^x_r)} \left| Z_r^{x} - Z_r^n \right|^2 \, dr \\
\leq \frac{C}{|\lambda|} e^{2\lambda n} \to 0, \quad \text{as } n \to \infty.
$$

Since for all $0 \leq T \leq n$:

$$
Y_s^{x;n} = Y_T^{x;n} + \int_s^T F \left( X_r^x, Y_r^{x;n}, Z_r^{x;n} \right) \, dr + \int_s^T G \left( X_r^x, Y_r^{x;n} \right) \, dA_r^x - \int_s^T Z_r^{x;n} \, dB_r, \quad s \in [0, n],
$$

then passing to limit as $n \to \infty$ (possibly along a subsequence) we obtain that $(Y_s^x, Z_s^x)_{s \geq 0}$ is a solution of (24).

Let $y, x \in \overline{D}$. Since

$$
\left| Y_0^y - Y_0^x \right| \leq \left| Y_0^y - Y_0^{y;n} \right| + \left| Y_0^{y;n} - Y_0^{x;n} \right| + \left| Y_0^{x;n} - Y_0^x \right| \\
\leq 2 \sqrt{C} e^{\lambda n} + \left| Y_0^{y;n} - Y_0^{x;n} \right|, \quad \text{for all } n \in \mathbb{N}.
$$

and $\lambda < 0$, the continuity property (25) follows from (23).

We finally deduce

**Corollary 4.2:** Theorem 5.1 in [11] holds true.


5. Viscosity solutions

5.1. Parabolic PDEs

We recall some results on the viscosity solutions of the PVI (2) from [4,5,8,10]. At the same time, we formulate the definition of the notion of viscosity solution of our system of equations.

We assume that the hypotheses (H.1), . . . , (H.5) are satisfied and we let the dimension of the Brownian motion be \( k = d \).

Denote \( \mathbb{S}^d \) the set of symmetric matrices from \( \mathbb{R}^{d \times d} \).

Let \( h : [0, T] \times \overline{D} \to \mathbb{R} \) be a continuous function.

A triple \((p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d\) is a parabolic super-jet to \( h \), at \((t, x) \in [0, T] \times \overline{D}, \) if for all \((s, x') \in [0, T] \times \overline{D}, \)

\[
\begin{align*}
    h(s, x') & \leq h(t, x) + p(s-t) + \langle q, x' - x \rangle + \frac{1}{2} \langle X(x' - x), x' - x \rangle \\
    & + o(|s-t| + |x' - x|^2). \\
\end{align*}
\]

The set of parabolic super-jets at \((t, x)\) is denoted by \( \mathcal{P}_h^{2,+} \); the set of parabolic sub-jets is defined by \( \mathcal{P}_h^{2,-} \).

First we consider the system (2) with the functions \( \varphi, \psi : \mathbb{R}^m \to ]-\infty, +\infty] \) 'decoupled' in the sense that \( \varphi(u_1, \ldots, u_m) = \varphi_1(u_1) + \cdots + \varphi_m(u_m) \) and \( \psi(u_1, \ldots, u_m) = \psi_1(u_1) + \cdots + \psi_m(u_m) \), where \( \varphi_i, \psi_i : \mathbb{R} \to ]-\infty, +\infty] \) are l.s.c. convex functions; hence \( \partial \varphi(u_1, \ldots, u_m) = \partial \varphi_1(u_1) \times \cdots \times \partial \varphi_m(u_m) \) and similarly for \( \partial \psi \).

We also assume that \( F_i \), the \( i \)th coordinate of \( F \), depends only on the \( i \)th row of the matrix \( Z \).

Consider the system

\[
\begin{align*}
    (a) \quad & \frac{\partial u_i(t,x)}{\partial t} + \mathcal{L}_i u_i(t,x) + F_i(t,x,u(t,x), \nabla u_i(t,x))^* g(t,x) \in \partial \varphi_i(u_i(t,x)), \\
    & t \in (0,T), \ x \in D, \ i \in \overline{1, m}, \\
    (b) \quad & -\frac{\partial u_i(t,x)}{\partial n} + G_i(t,x,u(t,x)) \in \partial \psi_i(u_i(t,x)), \\
    & t \in (0,T), \ x \in B_d(\overline{D}), \ i \in \overline{1, m}, \\
    (c) \quad & u(T,x) = \kappa(x), \ x \in \overline{D},
\end{align*}
\]

where the operator \( \mathcal{L}_i \) has been defined in (1).

Define \( \Phi_i, \Gamma_i : [0, T] \times \overline{D} \times \mathbb{R}^m \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}, \ i \in \overline{1, m}, \) to be the functions:

\[
\begin{align*}
    \Phi_i(t,x,y,q,X) &= \frac{1}{2} \text{Tr}(g^*g(t,x)X) + \langle q, f(t,x) \rangle + F_i(t,x,y,q^*g(t,x)) \ \\
    \Gamma_i(t,x,y,q) &= -\langle \nabla \varphi(x), q \rangle + G_i(t,x,y).
\end{align*}
\]

If \( u = (u_1, \ldots, u_m)^* : [0, T] \times \overline{D} \to \mathbb{R}^m \), then for each \( i \in \overline{1, m} \) we have

\[
\begin{align*}
    \Phi_i(t,x,u(t,x), \nabla u_i(t,x), D^2 u_i(t,x)) &= \mathcal{L}_i u_i(t,x) + F_i(t,x,u(t,x), \nabla u_i(t,x))^* g(t,x), \\
    \Gamma_i(t,x,u(t,x), \nabla u_i(t,x)) &= -\frac{\partial u_i(t,x)}{\partial n} + G_i(t,x,u(t,x)).
\end{align*}
\]
We shall use the notations \( a \wedge b \overset{\text{def}}{=} \min \{ a, b \} \) and \( a \lor b \overset{\text{def}}{=} \max \{ a, b \} \).

We now formulate four known results about the function \( u(t,x) \) defined by (17) being the solution of a system of generalized semilinear parabolic PDEs. In the first statement, \( \varphi = \psi \equiv 0 \).

**Theorem 5.1 (Pardoux, Zhang [11]: Theorem 4.3; Pardoux, Răşcanu [10]: Theorem 5.43):** Consider the parabolic system (30) with \( \varphi = \psi \equiv 0 \). Then the continuous function \( u: [0, T] \times \overline{D} \to \mathbb{R}^m \) defined by (17) is the unique viscosity solution of the parabolic partial differential system (30) in the sense that the three following properties hold:

1. \( u(T,x) = \kappa(x) \), for all \( x \in \overline{D} \);
2. \( u \) is a viscosity sub-solution that is, for any \( i \in \overline{1,m} \):
   - for any \( (t,x) \in (0,T) \times \overline{D} \), any \( (p,q,X) \in \mathcal{P}^{2,+} u_i(t,x) \),
     
     \[ p + \Phi_i (t,x,u(t,x),q,X) \geq 0, \]
   - for any \( (t,x) \in (0,T) \times \text{Bd} (\overline{D}) \), any \( (p,q,X) \in \mathcal{P}^{2,+} u_i(t,x) \),
     
     \[ [ p + \Phi_i (t,x,u(t,x),q,X) ] \lor \Gamma_i (t,x,u(t,x),q) \geq 0; \]
3. \( u \) is a viscosity super-solution that is, for any \( i \in \overline{1,m} \):
   - for any \( (t,x) \in (0,T) \times \overline{D} \), any \( (p,q,X) \in \mathcal{P}^{2,-} u_i(t,x) \),
     
     \[ p + \Phi_i (t,x,u(t,x),q,X) \leq 0, \]
   - for any \( (t,x) \in (0,T) \times \text{Bd} (\overline{D}) \), any \( (p,q,X) \in \mathcal{P}^{2,-} u_i(t,x) \),
     
     \[ [ p + \Phi_i (t,x,u(t,x),q,X) ] \land \Gamma_i (t,x,u(t,x),q) \leq 0. \]

In the second statement, \( \varphi \) and \( \psi \) are decoupled in the sense explained above in this section.

**Theorem 5.2 (Mărculescu, Răşcanu [5]: Theorem 5; Pardoux, Răşcanu [10]: Theorem 5.81):** The continuous function \( u: [0, T] \times \overline{D} \to \mathbb{R}^m \) defined by (17) is the unique viscosity solution of the parabolic differential system (30) on \( \overline{D} \) in the sense that the three following properties hold:

1. \( u(T,x) = \kappa(x), \forall x \in \overline{D}, \)
2. \( u(t,x) \in \text{Dom}(\varphi), \forall (t,x) \in (0,T) \times \overline{D}, \)
3. \( u(t,x) \in \text{Dom}(\psi), \forall (t,x) \in (0,T) \times \text{Bd}(\overline{D}); \)

4. \( u \) is a viscosity sub-solution that is, for any \( i \in \overline{1,m} \):
   - for any \( (t,x) \in (0,T) \times \overline{D} \), any \( (p,q,X) \in \mathcal{P}^{2,+} u_i(t,x) \),
     
     \[ p + \Phi_i (t,x,u(t,x),q,X) \geq (\varphi_i)'_-(u_i(t,x)), \]

   - for any \( (t,x) \in (0,T) \times \text{Bd}(\overline{D}) \), any \( (p,q,X) \in \mathcal{P}^{2,+} u_i(t,x) \),
     
     \[ [ p + \Phi_i (t,x,u(t,x),q,X) ] \lor \Gamma_i (t,x,u(t,x),q) \geq 0; \]
(b)  for any \((t, x) \in (0, T) \times \text{Bd}(\overline{D})\), any \((p, q, X) \in \mathcal{P}^{2,+} u_i(t, x),\)

\[
\begin{align*}
\text{either } & \ p + \Phi_i(t, x, u(t, x), q, X) \geq (\varphi_i)_-(u_i(t, x)), & \text{ or } \\
& \Gamma_i(t, x, u(t, x), q) \geq (\psi_i)_-(u_i(t, x));
\end{align*}
\]

(iii)  \(u\) is a viscosity super-solution that is, for any \(i \in \overline{1, m}:\)

(c)  for any \((t, x) \in (0, T) \times \text{Bd}(\overline{D})\), any \((p, q, X) \in \mathcal{P}^{2,-} u_i(t, x),\)

\[
\begin{align*}
\text{either } & \ p + \Phi_i(t, x, u(t, x), q, X) \leq (\varphi_i)_+(u_i(t, x)), & \text{ or } \\
& \Gamma_i(t, x, u(t, x), q) \leq (\psi_i)_+(u_i(t, x)).
\end{align*}
\]

In the next two statements, \(D = \mathbb{R}^d\) and there is no boundary condition, hence \(A^{t,x} \equiv 0\), and there is no \(G\) nor \(\psi\). We then add the assumptions stated in Remarks 2.2 and 2.7.

**Theorem 5.3 ((Pardoux, Răşcanu [8]: Theorem 4.11)):** Assume that \(D = \mathbb{R}^d\) (the system (30) is on \(\mathbb{R}^d\)) and that \(\varphi\) is decoupled. Then the continuous function \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m\) defined by (17) is the unique viscosity solution of the parabolic differential system (30)-(a) and (c) on \(\mathbb{R}^d\) in the sense that the three following properties hold:

(i)  \[
\begin{align*}
& \begin{cases}
  u(T, x) = \kappa(x), & \forall x \in \mathbb{R}^d, \\
  u(t, x) \in \text{Dom}(\varphi), & \forall (t, x) \in (0, T) \times \mathbb{R}^d,
\end{cases}
\end{align*}
\]

(ii)  \[
\begin{align*}
p + \Phi_i(t, x, u(t, x), q, X) \geq (\varphi_i)_-(u_i(t, x)), & \quad \text{for all } (p, q, X) \in \mathcal{P}^{2,+} u_i(t, x),
\end{align*}
\]

(iii)  \[
\begin{align*}
p + \Phi_i(t, x, u(t, x), q, X) \leq (\varphi_i)_+(u_i(t, x)), & \quad \text{for all } (p, q, X) \in \mathcal{P}^{2,-} u_i(t, x),
\end{align*}
\]

for any \(i \in \overline{1, m}\), any \((t, x) \in (0, T) \times \mathbb{R}^d\).

We note that in [5,8] the results are given for \(m = 1\), but with the same proof the results hold too for the quasi-decoupled system (30).

Consider finally the parabolic multivalued system (2) with \(D = \mathbb{R}^d\) and \(F\) independent of its last argument \(w\), that is \(F(t, x, y, w) \equiv F(t, x, y) \in \mathbb{R}^m\) for all \((t, x, y, w) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times m}\), namely we consider the system

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u(t, x)}{\partial t} + L_t u(t, x) + F(t, x, u(t, x)) \in \partial \varphi(u(t, x)), \\
  u(T, x) = \kappa(x), \quad x \in \mathbb{R}^d.
\end{array} \right. & \quad t \in (0, T), \ x \in \mathbb{R}^d,
\end{align*}
\]

\[\text{(32)}\]
Let $z \in \mathbb{R}^m$ and $\Phi_z : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{S}^d \to \mathbb{R}$

$$\Phi_z(t, x, y, q, X) = \frac{1}{2} \text{Tr} ((gg^*)(t, x)X) + \langle q, f(t, x) \rangle + \langle F(t, x, y), z \rangle$$

**Theorem 5.4 (Măciuc, Pardoux, Răşcanu, Zalignescu [4]: Theorem 6, Theorem 14):** The continuous function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m$ defined by (17) is the unique viscosity solution of the parabolic differential system (32) in the sense that the two following properties hold:

(i) $$\begin{cases} u(T, x) = \kappa(x), & \forall x \in \mathbb{R}^d, \\ u(t, x) \in \text{Dom}(\varphi), & \forall (t, x) \in (0, T) \times \mathbb{R}^d, \end{cases}$$

(ii) for any $(t, x) \in (0, T) \times \mathbb{R}^d$, any $z \in \mathbb{R}^m$, any $(p, q, X) \in \mathcal{P}^{2,+} \langle u(t, x), z \rangle$,

$$p + \Phi_z(t, x, u(t, x), q, X) \geq \varphi_-(u(t, x), z).$$

We remark that

$(r_1)$ the condition (ii) is equivalent to:

for any $(t, x) \in (0, T) \times \mathbb{R}^d$, any $z \in \mathbb{R}^m$, any $(p, q, X) \in \mathcal{P}^{2,-} \langle u(t, x), z \rangle$:

$$p + \Phi_z(t, x, u(t, x), q, X) \leq \varphi_+(u(t, x), z).$$

$(r_2)$ in one dimensional case ($m = 1$) condition (ii) means that $u$ is a sub–solution for $z > 0$ and a super–solution for $z < 0$.

### 5.2. Elliptic PDEs

Assume the hypotheses from Sections 1 and 2 are satisfied and moreover $f, g, F$ and $G$ are independent of time argument, $\kappa = 0$, $\varphi = \psi \equiv 0$, $u_0 = 0$ and $F_i$ the $i$th coordinate of $F$, depends only on the $i$th row of the matrix $Z$.

If $h : \overline{D} \to \mathbb{R}$ is a continuous function, then a pair $(q, X) \in \mathbb{R}^d \times \mathbb{S}^d$ is a elliptic super-jet to $h$, at $x \in \overline{D}$, if for all $x' \in \overline{D}$,

$$h(x') \leq h(x) + \langle q, x' - x \rangle + \frac{1}{2} \langle X(x' - x), x' - x \rangle + o(|x' - x|^2);$$

The set of elliptic super-jets at $x$ is denoted by $\mathcal{P}^{2,+}h(x)$; the set of elliptic sub-jets is defined by $\mathcal{P}^{2,-}_O h = -\mathcal{P}^{2,+}_O (-h)$.

Consider the semi-linear elliptic partial differential system with nonlinear Robin boundary condition:

$$\begin{cases} -\mathcal{L}u_i(x) = F_i(x, u(x), (\nabla u_i(x))^* g(x)), & x \in D, \quad i \in \overline{1, m}, \\ \frac{\partial u_i}{\partial n}(x) = G_i(x, u(x)), & x \in Bd(\overline{D}), \quad i \in \overline{1, m}. \end{cases}$$

(33)
where
\[
\mathcal{L} u_i(x) = \frac{1}{2} \sum_{j,\ell=1}^d \left( g g^* \right)_{j,\ell} (t, x) \frac{\partial^2 u_i(x)}{\partial x_j \partial x_\ell} + \sum_{j=1}^d f_j(t, x) \frac{\partial u_i(x)}{\partial x_j}.
\]

Define \( \Phi_i \) and \( \Gamma_i \) as in (31).

**Theorem 5.5:** [Pardoux and Zhang [11]: Theorem 5.3] The continuous function \( x \mapsto u(x) : \overline{D} \to \mathbb{R}^m \) given by (25) is a viscosity solution of the elliptic partial differential system (33) in the sense that the two following properties hold:

1. \( u \) is a viscosity sub-solution that is, for any \( i \in \overline{1,m} \):
   a. \( \Phi_i(x, u(x), q, X) \geq 0 \), for any \( x \in \overline{D} \), any \( (q, X) \in \mathcal{P}^{2,+} u_i(x) \),
   b. \( \Phi_i(x, u(x), q, X) \lor \Gamma_i(x, u(x), q) \geq 0 \)
      for any \( x \in \partial \overline{D} \), any \( (q, X) \in \mathcal{P}^{2,+} u_i(x) \),

2. \( u \) is a viscosity super-solution that is, for any \( i \in \overline{1,m} \):
   c. \( \Phi_i(x, u(x), q, X) \leq 0 \), for any \( x \in \overline{D} \), any \( (q, X) \in \mathcal{P}^{2,-} u_i(x) \),
   d. \( \Phi_i(x, u(x), q, X) \land \Gamma_i(x, u(x), q) \leq 0 \)
      for any \( x \in \partial \overline{D} \), any \( (q, X) \in \mathcal{P}^{2,-} u_i(x) \),

6. **Erratum**

In this paper we have corrected the proofs of continuity of the function \( (t, x) \mapsto u(t, x) = Y_t^{t,x} \) from the papers [5] (Proposition 13 and Corollary 14) and [11] (Proposition 4.1 and Theorem 5.1).

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### Appendix 1

Let $\varphi : \mathbb{R}^m \to ]-\infty, +\infty]$ be a proper convex lower semicontinuous function. We denote $\text{Dom} (\varphi) = \{ y \in \mathbb{R}^m : \varphi(y) < \infty \}$; $\varphi$ is a proper function if $\text{Dom} (\varphi) \neq \emptyset$.

The subdifferential (multivalued) operator $\partial \varphi$ is defined by

$$
\partial \varphi (y) := \{ \hat{y} \in \mathbb{R}^m : \langle \hat{y}, v - y \rangle + \varphi(y) \leq \varphi(v), \quad \forall v \in \mathbb{R}^m \} ;
$$

$\partial \varphi : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a maximal monotone operator. We have

$$
\text{Dom} (\partial \varphi) \overset{\text{def}}{=} \{ y \in \mathbb{R}^m : \partial \varphi(y) \neq \emptyset \} \subset \text{Dom} (\varphi) .
$$

Recall that $\text{Dom} (\partial \varphi) = \text{Dom} (\varphi)$ and $\text{int} (\text{Dom} (\partial \varphi)) = \text{int} (\text{Dom} (\varphi))$.

For all $y \in \text{Dom} (\varphi)$ and $z \in \mathbb{R}^m$ we have

$$
\varphi^+ (y, z) \overset{\text{def}}{=} \lim_{t \searrow 0} \frac{\varphi(y + tz) - \varphi(y)}{t} \leq \lim_{t \nearrow 0} \frac{\varphi(y + tz) - \varphi(y)}{t} = \overset{\text{def}}{=} \varphi^+_+ (y, z) .
$$

$\varphi^- (y, z) = -\varphi^+_+ (y, -z)$. Moreover

$$
\hat{y} \in \partial \varphi (y) \iff \exists z \in \mathbb{R}^m, \forall \hat{y}, z \in \mathbb{R}^m, \varphi(y + tz) + \varphi(z) \leq \inf_{\hat{y}} (\varphi^- (y, z), \varphi^+_+ (y, z)) .
$$

If $m = 1$ we write $\varphi^- (y) = \varphi^- (y, 1), \varphi^+_+ (y) = \varphi^+_+ (y, 1)$ and we have

$$
\partial \varphi (y) = [\varphi^- (y), \varphi^+_+ (y)] \cap \mathbb{R} .
$$

Let $\varepsilon > 0$. The Moreau–Yosida regularization of $\varphi$ is the function $\varphi_{\varepsilon} : \mathbb{R}^m \to \mathbb{R}$

$$
\varphi_{\varepsilon} (y) \overset{\text{def}}{=} \inf \left\{ \frac{1}{2\varepsilon} |y - z|^2 + \varphi(z) : z \in \mathbb{R}^m \right\} .
$$
We mention that $\varphi_\varepsilon$ is a $C^1$ convex function and (see e.g. Pardoux and Răşcanu [10], Annex B) for all $x, y \in \mathbb{R}^m$

\begin{enumerate}[(A1)]
    \item $\varphi_\varepsilon (x) = \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon (x)|^2 + \varphi \left( x - \varepsilon \nabla \varphi_\varepsilon (x) \right)$,
    \item $\nabla \varphi_\varepsilon (x) = \partial \varphi_\varepsilon (x) \in \partial \varphi \left( x - \varepsilon \nabla \varphi_\varepsilon (x) \right)$,
    \item $|\nabla \varphi_\varepsilon (x) - \nabla \varphi_\varepsilon (y)| \leq \frac{1}{\varepsilon} |x - y|$.
\end{enumerate}