Fréchet cardinals

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Abstract

An infinite cardinal \( \lambda \) is called Fréchet if the Fréchet filter on \( \lambda \) extends to a countably complete ultrafilter. We examine the relationship between Fréchet cardinals and strongly compact cardinals under a hypothesis called the Ultrapower Axiom.

1 Introduction

An infinite cardinal \( \lambda \) is called Fréchet if the Fréchet filter on \( \lambda \) extends to a countably complete ultrafilter. In this paper, we examine the relationship between Fréchet cardinals and strongly compact cardinals. Obviously if \( \kappa \) is strongly compact, then every cardinal \( \lambda \) with \( \text{cf}(\lambda) \geq \kappa \) is Fréchet. The converse is not provable in ZFC. Our focus is proving strong converses to this fact under an assumption called the Ultrapower Axiom (UA), which serves as a regularity property for countably complete ultrafilters. Our main theorem is the following:

**Theorem 1.1 (UA)**. If \( \delta \) is a Fréchet successor cardinal or a Fréchet inaccessible cardinal then some \( \kappa \leq \delta \) is \( \delta \)-strongly compact.

This theorem is a key step in the proof of the equivalence of strong compactness and supercompactness assuming UA ([1]). The situation for Fréchet singular cardinals and weakly inaccessible cardinals is not quite as clear and is wrapped up in the analysis of isolated cardinals, defined below.

The other main result of this paper uses the analysis of Fréchet cardinals to improve the main result of [2] by removing its cardinal arithmetic hypothesis.

**Theorem 1.2 (UA)**. The Mitchell order wellorders the class of generalized normal ultrafilters.

We will have to cite a number of results from the author’s thesis and from the papers [2], [3], and [1].

2 Uniform ultrafilters

**Definition 2.1.** An ultrafilter \( U \) on a set \( X \) is Fréchet uniform if for all \( A \in U, |A| = |X| \). A cardinal is Fréchet if it carries a countably complete Fréchet uniform ultrafilter.
Apart from Fréchet uniformity, there is another definition of uniformity that is often used. These two notions coincide for ultrafilters on regular cardinals but diverge everywhere else.

**Definition 2.2.** An ultrafilter \( U \) on an ordinal \( \alpha \) is *tail uniform* (or just *uniform*) if \( \alpha \setminus \beta \in U \) for all \( \beta < \alpha \). An ordinal is *tail uniform* (or just *uniform*) if it carries a countably complete uniform ultrafilter.

The basic relationship between uniform ordinals and Fréchet cardinals is quite simple:

**Lemma 2.3.** An ordinal is uniform if and only if its cofinality is Fréchet.

**Definition 2.4.** For any ordinal \( \alpha \), \( \text{Un}_\alpha \) denotes the set of uniform countably complete ultrafilters on \( \alpha \), \( \text{Un}_{<\alpha} = \bigcup_{\beta<\alpha} \text{Un}_\beta \), \( \text{Un}_{\leq\alpha} = \bigcup_{\beta\leq\alpha} \text{Un}_\beta \), and \( \text{Un} = \bigcup_{\alpha\in\text{Ord}} \text{Un}_\alpha \).

Thus \( \alpha \) is uniform if and only if \( \text{Un}_\alpha \neq \emptyset \). Note that if \( \alpha \) is a successor ordinal, then \( \alpha \) is uniform since there is a uniform principal ultrafilter on \( \alpha \).

**Definition 2.5.** For any \( U \in \text{Un} \), \( \text{sp}(U) \) denotes the unique ordinal \( \alpha \) such that \( U \in \text{Un}_\alpha \).

### 3 The Fréchet successor operation

**Definition 3.1.** For any ordinal \( \gamma \), \( \gamma^\sigma \) denotes the least Fréchet cardinal strictly greater than \( \gamma \).

The following conjecture drives our analysis:

**Conjecture 3.2 (UA).** Suppose \( \gamma \) is an ordinal and \( \lambda = \gamma^\sigma \). Either \( \gamma^\sigma = \gamma^+ \) or \( \gamma^\sigma \) is measurable.

We will verify this conjecture assuming UA + GCH in Theorem 5.1, and we will also prove various approximations to it assuming UA alone. But we begin with a related ZFC fact:

**Lemma 3.3.** For any ordinal \( \gamma \), either \( \gamma^\sigma = \gamma^+ \) or \( \gamma^\sigma \) is a limit cardinal.

This is an immediate consequence of the following lemma.

**Lemma 3.4.** Suppose \( \lambda \) is a cardinal and \( \lambda^+ \) is Fréchet. Either \( \lambda \) is Fréchet or \( \lambda \) is singular and all sufficiently large regular cardinals below \( \lambda \) are Fréchet.

**Proof.** Fix a countably complete Fréchet uniform ultrafilter \( U \) on \( \lambda^+ \).

Assume first that \( \lambda \) is regular. By a theorem of Prikry [4], \( U \) is \( \lambda \)-decomposable, and therefore \( \lambda \) is Fréchet as desired.

Assume instead that \( \lambda \) is singular. Let \( \iota = \text{cf}^M\lambda^+(\sup j_U[\lambda^+]) \). By a theorem of Ketonen [5], every set of ordinals \( X \) such that \( |X| \leq \lambda^+ \) is contained in a set of ordinals \( X' \in M_U \) such that \( |X'|^{M_U} \leq \iota \).

**Case 1.** \( \iota < \sup j_U[\lambda] \)
Fix $\gamma < \lambda$ such that $\iota < j_U(\gamma)$. We claim that any regular $\delta$ with $\gamma \leq \delta < \lambda$ is Fréchet. Note that $\text{cf}^{M_U}(\sup j_U[\delta]) \leq \iota < j_U(\gamma) < \sup j_U[\delta]$ so $\sup j_U[\delta]$ is a singular ordinal in $M_U$. On the other hand by elementarity, $j_U(\delta)$ is a regular cardinal in $M_U$. It follows that $\sup j_U[\delta] < j_U(\delta)$. Therefore $j_U$ is discontinuous at $\delta$, and it follows that $\delta$ is Fréchet.

**Case 2. $\iota \geq \sup j_U[\lambda]$**

Note that $\iota < j_U(\lambda)$ and $\text{cf}(\iota) = \lambda^+$. Therefore there is a tail uniform (but not Fréchet uniform) ultrafilter $W$ on the ordinal $\iota$. Let $Z = \{ X \subseteq \lambda : j_U(X) \cap \iota \in W \}$. We claim $Z$ is a Fréchet uniform countably complete ultrafilter on $\lambda$. Suppose $X \in Z$, and we will show $|X| = \lambda$. Since $X \subseteq \lambda$ and $j_U(X) \cap \iota \in W$, we have that $j_U(X) \cap \iota$ is cofinal in $\iota$. Since $\iota$ is regular in $M_U$, $|j_U(X) \cap \iota|^{M_U} = \iota \geq \sup j_U[\lambda]$. It follows easily that $|X| \geq \lambda$ as desired.

**Proof of Lemma 3.3.** Suppose $\gamma^\sigma$ is not a limit cardinal. Then $\gamma^\sigma = \lambda^+$ for some cardinal $\lambda$. By Lemma 3.4, $\lambda$ is either Fréchet or else a limit of Fréchet cardinals. Suppose towards a contradiction that $\gamma < \lambda$. Then there is a Fréchet cardinal in the interval $(\gamma, \lambda]$. This contradicts that $\gamma^\sigma = \lambda^+$.

We conclude this section by pointing out a consequence of Lemma 3.4 for $\omega_1$-strongly compact cardinals, a notion due to Bagaria-Magidor [6]:

**Corollary 3.5.** Let $\kappa$ be the least $\omega_1$-strongly compact cardinal. Then $\kappa$ carries a countably complete Fréchet uniform ultrafilter.

**Question 3.6.** Let $\kappa$ be the least $\omega_1$-strongly compact cardinal. Does $\kappa$ carry $2^{\omega_1}$ countably complete Fréchet uniform ultrafilters?

## 4 Approximating ultrafilters

In this section we exposit two lemmas that allow us to approximate ultrafilters by smaller ultrafilters.

The first lemma is due to the author, but has likely been discovered by others before him.

**Lemma 4.1.** Suppose $U$ is an ultrafilter on a set $X$ and $Y \subseteq j_U(A)$ is a set. Then there is an ultrafilter $D$ on $A^Y$ and an elementary embedding $k : M_D \rightarrow M_U$ with $j_U = k \circ j_D$ and $Y \subseteq \text{ran}(k)$.

**Proof.** Choose for each $y \in Y$ a function $f_y : X \rightarrow A$ such that $y = [f_y]_U$. Let $g : X \rightarrow A^Y$ be defined by $g(x)(y) = f_y(x)$. One calculates that

$$[g]_U(j_U(y)) = j_U(g)([id]_U)(j_U(y)) = j_U(f_y)([id]_U) = [f_y]_U$$

so letting $D = f_\iota(U)$ and $k : M_D \rightarrow M_U$ be the factor embedding, $A \subseteq \text{ran}(k)$ since $[g]_U$ and $j_U[Y]$ are contained ran($k$).

**Corollary 4.2.** Suppose $U$ is an ultrafilter and $\gamma$ is an ordinal. Then there is an ultrafilter $D$ on $2^\gamma$ and an elementary embedding $k : M_D \rightarrow M_U$ with $j_U = k \circ j_D$ and $\gamma \subseteq \text{ran}(k)$.
The second lemma, due to Silver, is much more interesting. To put it in context, we first spell out a correspondence between partitions modulo an ultrafilter and definability over an ultrapower that is implicit in Silver’s proof.

**Definition 4.3.** Suppose \( P \) is a partition of a set \( X \) and \( A \) is a subset of \( X \). Then the restriction of \( P \) to \( A \) is the partition \( P \upharpoonright A \) defined by

\[
P \upharpoonright A = \{ A \cap S : S \in P \text{ and } A \cap S \neq \emptyset \}\]

**Definition 4.4.** Suppose \( U \) is an ultrafilter on a set \( X \). Let \( \mathbb{Q}_U \) be the preorder on the collection of partitions of \( X \) defined by setting \( P \leq Q \) if there exists some \( A \in U \) such that \( Q \upharpoonright A \) refines \( P \upharpoonright A \). Let \( \mathbb{Q}_U^* \) be the quotient partial order.

**Definition 4.5.** Suppose \( U \) is an ultrafilter. Let \( \mathbb{P}_U \) be the preorder on \( M_U \) defined by setting \( x \leq y \) if \( x \) is definable over \( M \) from \( y \) and parameters in \( j_U[V] \). Let \( \mathbb{P}_U^* \) be the quotient partial order.

**Lemma 4.6.** Suppose \( U \) is an ultrafilter on a set \( X \). Then \( \mathbb{P}_U^* \cong \mathbb{Q}_U^* \).

*Proof.* It suffices to define an order-embedding \( \Phi : \mathbb{P}_U \to \mathbb{Q}_U \) that is essentially surjective in the sense that for any \( x \in \mathbb{P}_U \) there is some \( P \in \mathbb{Q}_U \) such that \( x \) and \( \Phi(P) \) are equivalent in \( \mathbb{P}_U \).

For \( P \in \mathbb{Q}_U \), let \( \Phi(P) \) be the unique \( S \in j_U(P) \) such that \([id]_U \in S\). We claim that \( \Phi : \mathbb{P}_U \to \mathbb{Q}_U \) is order-preserving and essentially surjective.

Suppose \( P, Q \in \mathbb{Q}_U \) and \( P \leq Q \). Fix \( A \in U \) such that \( Q \upharpoonright A \) refines \( P \upharpoonright A \). Then \( \Phi(P) \) is definable in \( M_U \) from the parameters \( \Phi(Q), j_U(P), j_U(A) \) as the unique \( S \in j_U(P) \) such that \( \Phi(Q) \cap j_U(A) \subseteq S \cap j_U(A) \).

Conversely suppose \( \Phi(P) = j_U(f)(\Phi(Q)) \) for some \( f : Q \to P \). Let \( A \subseteq X \) consist of those \( x \in X \) such that \( x \in f(S) \) where \( S \) is the unique element of \( Q \) with \( x \in S \). Then \( A \in U \) since \([id]_U \in j_U(f)(S) \) where \( S = \Phi(Q) \) is the unique \( S \in j_U(Q) \) such that \([id]_U \in S \). Moreover for any \( S \in Q, S \cap A \subseteq f(S) \cap A \), so \( Q \upharpoonright A \) refines \( P \upharpoonright A \).

We conclude by showing that \( \Phi \) is essentially surjective. Fix \( x \in \mathbb{P}_U \). In other words, \( x \in M_U \), so \( x = j_U(f)([id]_U) \) for some \( f : X \to V \). Let

\[
P = \{ f^{-1}([y]) : y \in \text{ran}(f) \}\]

Then \( \Phi(P) \) is interdefinable with \( x \) over \( M_U \) using parameters in \( j_U[V] \): \( \Phi(P) \) is the unique \( S \in j_U(P) \) such that \( x \in j_U(f)[S] \); and since \( j_U(f)\Phi(P)\) = \( \{x\}, x = \bigcup j_U(f)\Phi(P)\). \( \Box \)

**Definition 4.7.** Suppose \( U \) is an ultrafilter on \( X \) and \( \lambda \) is a cardinal. Then \( U \) is \( \lambda \)-indecomposable if every partition of \( X \) into \( \lambda \) pieces is \( U \)-equivalent to a partition of \( X \) into fewer than \( \lambda \) pieces.

In other words, \( U \) is \( \lambda \)-indecomposable if and only if there is no Fréchet uniform ultrafilter \( W \) on \( \lambda \) with \( W \leq_{\text{RK}} U \).

**Theorem 4.8** (Silver). Suppose \( \delta \) is a regular cardinal and \( U \) is an ultrafilter on \( X \) that is \( \lambda \)-indecomposable for all \( \lambda \in [\delta, 2^\delta] \). Then there is an ultrafilter \( D \) on some \( \gamma < \delta \) and an elementary embedding \( k : M_D \to M_U \) such that \( j_U = k \circ j_D \) and \( j_U(2^\delta)^+ \subseteq \text{ran}(k) \).
Proof. If $\kappa$ is a cardinal, we call a set $P$ a $\leq \kappa$-partition of $X$ if $P$ is a partition of $X$ such that $|P| \leq \kappa$.

We begin by proving the existence of a maximal $\leq 2^\delta$-partition $P$ of $X$ in the order $\mathcal{Q}_U$. Thus we will find a partition $P$ of $X$ such that for any refinement $Q$ of $P$, there is some $A \in U$ such that $P \upharpoonright A$ refines $Q \upharpoonright A$.

Suppose there is no such $P$. We construct by recursion a sequence $\langle P_\alpha : \alpha \leq \delta \rangle$ of $\leq 2^\delta$-partitions of $X$. Let $P_0 = \{\delta\}$. If $\alpha < \delta$ and $P_\alpha$ has been defined, then let $P_{\alpha+1}$ be a $\leq 2^\delta$-partition of $X$ witnessing that $P_\alpha$ is not maximal: thus $P_{\alpha+1}$ refines $P_\alpha$ but for any $A \in U$, $P_\alpha \upharpoonright A$ does not refine $P_{\alpha+1} \upharpoonright A$; since $P_{\alpha+1}$ refines $P_\alpha$, this is the same as saying $P_\alpha \upharpoonright A \neq P_{\alpha+1} \upharpoonright A$ for all $A \in U$. If $\gamma$ a nonzero limit ordinal and $P_\alpha$ is defined for all $\alpha < \gamma$, let $P_\gamma$ be the least common refinement of the $P_\alpha$ for $\alpha < \gamma$.

Since $U$ is $\lambda$-indecomposable for all $\lambda \in [\delta, 2^\delta]$, there is some $A \in U$ such that $|P_\delta \upharpoonright A| < \delta$. For $\alpha \leq \delta$, let $P'_\alpha = P_\alpha \upharpoonright A$. We claim that the $P'_\alpha$ are eventually constant. Since $P'_\delta$ is the least common refinement of the $P'_\alpha$ and $\delta$ is regular, it is not hard to show there is some $\alpha < \delta$ with the property that for each $S \in P'_\alpha$, there is a unique $S' \subseteq S$ with $S' \subseteq S$. Then since $\bigcup P'_\alpha = \bigcup P'_\delta = A$, we must in fact have $S' = S$. So $P'_\alpha = P'_\delta$, which implies $P'_\beta = P'_\alpha$ for all $\beta \geq \alpha$ as claimed.

This is a contradiction since by our choice of $P_{\alpha+1}$, $P_{\alpha+1} \upharpoonright A \neq P_\alpha \upharpoonright A$ for all $A \in U$. Therefore our assumption was false, and there is a maximal $\leq 2^\delta$-partition $P$ of $X$. By the indecomposability of $U$, we may assume $|P| < \delta$.

Let $D = \{Q \subseteq P : \bigcup Q \in U\}$. Let $k : M_D \rightarrow M_U$ be the factor embedding. By the maximality of $P$ among $\leq 2^\delta$-partitions and the correspondence of Lemma 4.6, $j_U(2^\delta) \subseteq \text{ran}(k)$. Assume towards a contradiction that there is some $e < j_U(2^\delta)$ with $e \notin \text{ran}(k)$. Then the ultrafilter derived from $j_U$ using $e$ witnesses that $U$ is $(2^\delta)^+$-indecomposable. By a theorem of Prikry [4], letting $\lambda = \text{cf}(2^\delta)$, $U$ is $\lambda$-indecomposable. But $\lambda \in [\delta, 2^\delta]$, which is a contradiction. \hfill \square

## 5 A weakening of Conjecture 3.2

In this section we prove:

**Theorem 5.1 (UA).** Suppose $\delta$ is a regular cardinal that is not Fréchet. Either $\delta^\sigma \leq 2^\delta$ or $\delta^\sigma$ is measurable.

For the proof, we introduce some definitions we will use throughout this paper.

**Definition 5.2.** Suppose $U$ is a countably complete ultrafilter and $W' \in (\text{Un}_\gamma)_{M_U}$ for some ordinal $\gamma'$. Then $U^-(W') = \{X \subseteq \gamma : j_U(X) \cap \gamma' \in W'\}$ where $\gamma$ is the least ordinal such that $j_U(\gamma) \geq \gamma'$.

We briefly recall the definition of the Ketonen order.

**Definition 5.3.** For $U, W \in \text{Un}$, we set $W <_E U$ if setting $\alpha = [\text{id}]_U$, there is some $W' \in \text{Un}_{\leq \alpha}^{M_U}$ with $W = U^-(W')$.

Proofs of the following theorems appear in [7].
Theorem 5.4. The Ketonen order is a strict wellfounded partial order of $U$. 

Theorem 5.5. The following are equivalent:

(1) The Ketonen order is linear

(2) The Ultrapower Axiom holds.

Definition 5.6. If $\lambda$ is a Fréchet cardinal, then $U_\lambda$ denotes the $<_E$-least Fréchet uniform countably complete ultrafilter on $\lambda$.

The internal relation is a minor variant of the generalized Mitchell order on countably complete ultrafilters.

Definition 5.7. The internal relation $\sqsubseteq$ is defined on countably complete ultrafilters $U, W$ by setting $U \sqsubseteq W$ if $j_U \upharpoonright M_W$ is an internal ultrapower embedding of $M_W$.

We will use the following lemma from [8]:

Definition 5.8. Suppose $U, W \in U$. Then $t_U(W)$ denotes the $<_E$-least ultrafilter $W' \in U_{M_U}$ such that $U^-(W') = W$. The function $t_U : U \to U_{M_U}$ is called the translation function associated to $U$.

Note that for any $U, W \in U$, $U^-(j_U(W)) = W$ and so $t_U(W) \leq_E j_U(W)$. Equality holds if and only if $U \sqsubseteq W$.

Lemma 5.9 (UA). For any $U, W \in U$, $U \sqsubseteq W$ if and only if $t_U(W) = j_U(W)$. 

The following simple lemma will be refined in Theorem 8.8 using a much harder argument:

Lemma 5.10 (UA). Suppose $\gamma$ is an ordinal and $\lambda = \gamma^n$. Then for all $D \in U_{<\lambda}$, $D \sqsubseteq U_\lambda$.

Proof. Let $U = U_\lambda$. It is not hard to see that $U$ is the $<_E$-least countably complete uniform ultrafilter that is not isomorphic to an ultrafilter in $U_{<\gamma}$. Therefore to prove the lemma, it suffices to show that for all $D \in U_{<\gamma}$, $D \sqsubseteq U$. (This is not really necessary, but it simplifies notation.)

Therefore fix $D \in U_{<\gamma}$, and we will show $D \sqsubseteq U$. Let $U' = t_D(U)$. It suffices by Lemma 5.9 to show that $U' = j_D(U)$. Since $U' \leq_E j_D(U)$, we just need to show that $U' \models D$. Assume towards a contradiction that $U' <_E j_D(U)$. In $M_D$, $j_D(U)$ is the $<_E$-least countably complete uniform ultrafilter that is not isomorphic to an ultrafilter in $U_{<\gamma}$. Therefore $U'$ is isomorphic to some $U'' \in U_{<\gamma}$. It follows that there is some $Z \in U_{<\gamma}$ such that $j_Z = j_{U''} \circ j_D = j_{M_D} \circ j_{D'}$. (That is, $Z$ is isomorphic to the sum of $D$ with $U'$.)

Since $D' = U'$, there is an elementary embedding $k : M_D \to M_{U''}$ such that $k \circ j_D = j_{U''} \circ j_D$, defined by $k(j_U)(\text{id}|U)) = j_{U''}(j_D(f))([\text{id}]_{U''})$. It follows that $U \leq_{\text{RK}} Z$. But a Fréchet uniform ultrafilter on $\lambda$ cannot lie below an ultrafilter on $\gamma < \lambda$ in the Rudin-Keisler order. (Fix a function $f : \gamma \to \lambda$ such that $U = f_*(Z)$; then $f[\gamma] \in U$, contradicting that $U$ is Fréchet uniform.) This is a contradiction, so the assumption that $U' <_E j_D(U)$ was false, completing the proof.
Lemma 6.3. Suppose \( j : V \rightarrow M \) is a countable embedding. Suppose \( \lambda = \delta^n \) and assume \( 2^{\delta} < \lambda \). Suppose \( U \) is a countably complete ultrafilter such that \( D \subseteq U \) for all \( D \in \mathcal{U}_{\lambda} \). Then \( U \) is \( \lambda \)-complete.

Proof. There are no Fréchet cardinals in the interval \( [\delta, 2^{\delta}] \), so in particular \( U \) is \( \lambda \)-indecomposable for all cardinals \( \lambda \in [\delta, 2^{\delta}] \). Theorem 4.11 yields some \( D \in \mathcal{U}_{\lambda} \) such that \( j_D \upharpoonright 2^{\delta} = j_U \upharpoonright 2^{\delta} \). Since \( D \subseteq U, j_U \upharpoonright 2^{\delta} = j_D \upharpoonright 2^{\delta} \in M_U \). Thus \( U \) is \( 2^{\delta} \)-supercompact.

As a consequence of the Kunen inconsistency theorem [8], either \( j \) is a \( \kappa \)-supercompact, or \( j \) is discontinuous at every regular cardinal in \( [\delta, 2^{\delta}] \) or \( U \) is \( 2^{\delta} \)-complete. The former cannot hold since there are no Fréchet cardinals in the interval \( [\delta, 2^{\delta}] \). Therefore \( U \) is \( 2^{\delta} \)-complete. Let \( \kappa \) be the completeness of \( U \). Since \( \kappa \) is measurable, \( \kappa \) is Fréchet, and so since \( \kappa > \delta, \kappa \geq \delta^n = \lambda \). Thus \( U \) is \( \lambda \)-complete.

Proof of Lemma 6.1. Let \( \lambda = \delta^n \) and assume that \( \lambda > 2^{\delta} \). Let \( U = U_\lambda \). By Lemma 5.10, \( D \subseteq U \) for all \( D \in \mathcal{U}_{\lambda} \). By Lemma 5.11, \( U \) is \( \lambda \)-complete. Therefore \( U \) is a \( \lambda \)-complete ultrafilter on \( \lambda \), so \( \lambda \) is measurable. □

6 Nonisolation lemmas

Lemma 6.1 (UA). Suppose \( \lambda \) is a limit cardinal. Suppose there is some \( W \in \mathcal{U} \) such that \( j_W \) is discontinuous at \( \lambda \) and \( U_\lambda \subseteq W \). Then \( \lambda \) is a limit of Fréchet cardinals.

For the proof, we need another fact about the internal relation, a sort of dual to Lemma 5.9.

Definition 6.2. Suppose \( U \in \mathcal{U}_{\lambda} \) and \( W \) is a countably complete ultrafilter. Then \( s_W(U) = \{ X \subseteq \lambda : X \in M_U \text{ and } j_W[X] \in U \} \) where \( \lambda = \sup j_W[\lambda] \).

Lemma 6.3. If \( U, W \in \mathcal{U} \) then \( j_{M_W} = j_U \upharpoonright M_W \).

Corollary 6.4. If \( U, W \in \mathcal{U} \), then \( U \subseteq W \) if and only if \( s_W(U) \in M_W \).

We also need a standard fact about covering in ultrapowers.

Lemma 6.5. Suppose \( j : V \rightarrow M \) is an ultrapower embedding. Suppose \( \lambda \) is a cardinal and \( \lambda' \) is an \( M \)-cardinal. Assume there is a set \( X \subseteq M \) with \( |X| = \lambda \) such that \( j[X] \subseteq X' \) for some \( X' \in M \) such that \( |X'| \leq \lambda' \). Then any set \( A \subseteq M \) with \( |A| = \lambda \) is contained in a set \( A' \in M \) such that \( |A'| \leq \lambda' \).

Proof. Fix \( a \in M \) such that every element of \( M \) is of the form \( j(f)(a) \) for some function \( f \). Choose functions \( \{ f_x : x \in X \} \) such that \( A = \{ j(f_x)(a) : x \in X \} \). Let \( \{ g_x : x \in j(X) \} = j(\langle f_x : x \in X \rangle) \). Let \( A' = \{ g_x(a) : x \in X' \} \). Easily \( A' \in M \), \( |A'| \leq |X'| \), and \( A \subseteq A' \), as desired. □

Proof of Lemma 6.1. Let \( U = U_\lambda \). Let \( \lambda' = \sup j_W[\lambda] \) and let \( U' = s_W(U) \).

Case 1. \( U' \) is Fréchet uniform in \( M_W \).
In this case \( U' \) witnesses that \( \lambda' \) is Fréchet in \( M_W \). Since \( j_W \) is discontinuous at \( \lambda \), a simple reflection argument implies that \( \lambda \) is a limit of Fréchet cardinals.

**Case 2.** \( U' \) is not Fréchet uniform in \( M_W \).

Fix \( X' \in U' \) and \( \gamma < \lambda \) such that \( |X'|^{M_W} < j_W(\gamma) \). Let \( X = j_W^{-1}[X'] \). Then \( X \in U \), so since \( U \) is Fréchet uniform, \( |X| = \lambda \). But \( j_W[X] \subseteq X' \). It follows from Lemma 6.5 that every set \( A \subseteq M_W \) with \( |A| \leq \lambda \) is covered by a set \( A' \in M_W \) with \( |A'|^{M_W} \leq \lambda' \).

It follows that \( j_W \) is discontinuous at every regular cardinal in the interval \([\gamma, \lambda]\). (This is a standard consequence of \((\gamma, \lambda)\)-regularity, which is what we have established. If \( \delta \in [\gamma, \lambda] \) is a regular cardinal, then \( cf^{M_W}(sup j_W[\delta]) < j_W(\gamma) \) by the covering property of \( M_W \), so \( j_W(\delta) \), being regular in \( M_W \), is not equal to \( sup j_W[\delta] \).) Therefore \( \lambda \) is a limit of Fréchet cardinals.

As a corollary of the proof we also have the following fact:

**Lemma 6.6 (UA).** Suppose \( \lambda \) is a Fréchet limit cardinal and for some countably complete \( W \), \( U_\lambda \sqsubseteq W \) and \( W \not\sqsubseteq U_\lambda \). Then \( \lambda \) is a limit of Fréchet cardinals.

**Proof.** Let \( U = U_\lambda \). We may assume that \( j_W \) is continuous at \( \lambda \).

**Claim 1.** \( t_W(U) \) is not Fréchet uniform in \( M_W \).

**Proof.** Assume towards a contradiction that \( t_W(U) \) is Fréchet uniform in \( M_W \). By the definition of \( t_W(U) \), \( t_W(U) \leq_{M_W} j_W(U) \). By elementarity \( j_W(U) \) is the \( \leq_{M_W} \)-least Fréchet uniform ultrafilter on \( j_W(\lambda) = sup j_W[\lambda] \) in \( M_W \). Since \( sup j_W[\lambda] \leq sp(t_W(U)) \), this implies \( j_W(U) \leq_{M_W} t_W(U) \). It follows that \( t_W(U) = j_W(U) \). But by Lemma 5.9, \( W \sqsubseteq U \), contrary to hypothesis. 

Therefore there is some \( X' \in t_W(U) \) such that \( |X'|^{M_W} < j_W(\gamma) \) for some \( \gamma < \lambda \). We now proceed as in Lemma 6.1 to show all sufficiently large regular cardinals below \( \lambda \) are Fréchet.

\[ \Box \]

### 7 The strong compactness of \( \kappa_{U_\delta} \)

The main theorem of this section is the key to the supercompactness analysis of [1].

**Theorem 7.1 (UA).** If \( \delta \) is a Fréchet successor cardinal or a Fréchet inaccessible cardinal, then \( \kappa_{U_\delta} \) is \( \delta \)-strongly compact.

The theorem requires a sequence of lemmas. The first is related to the phenomenon of commuting ultrapowers first discovered by Kunen.

**Lemma 7.2 (UA).** Suppose \( U \) and \( W \) are countably complete ultrafilters. The following are equivalent:

1. \( U \sqsubseteq W \) and \( W \sqsubseteq U \).
2. \( j_U(j_W) = j_W \upharpoonright M_U \).

\[ \Box \]
(3) $jw(jv) = jv \upharpoonright MW$. 

The proof appears in [8].

We also need a consequence of Corollary 4.2.

Lemma 7.3 (UA). Suppose $\kappa$ is a strong limit cardinal and $W$ is a countably complete ultrafilter. The following are equivalent:

(1) $jw[\kappa] \subseteq \kappa$ and for all $D \in \text{Un}_{<\kappa}$, $D \subseteq W$.

(2) $W$ is $\kappa$-complete.

Proof. We will prove (1) implies (2).

We claim $jw[\alpha] \in MW$ for all $\alpha < \kappa$. Let $\alpha' = \text{sup} jw[\alpha]$, and note that $\alpha' < \kappa$, so $2^{\alpha'} < \kappa$. By Corollary 4.2, there is a countably complete ultrafilter $D$ on $2^{\alpha'}$ such that $jD[\alpha] = jw[\alpha]$. By assumption $D \subseteq W$, so $jD[\alpha] \in MW$. Hence $jw[\alpha] \in MW$.

By the Kunen inconsistency theorem there cannot be a $j : V \rightarrow M$ such that $j[\lambda] \in M$ for some $\lambda$ with $j[\lambda] \subseteq \lambda$ and $\text{crt}(j) < \lambda$. Taking $\lambda = \kappa$ and $j = jw$, it follows that $W$ is $\kappa$-complete.

This lemma has the following corollary.

Lemma 7.4 (UA). Suppose $U$ and $W$ are countably complete ultrafilters such that $\text{Un}_{<\kappa_U} \subseteq W$ and $\text{Un}_{<\kappa_W} \subseteq U$. Either $U \not\subseteq W$ or $W \not\subseteq U$.

Proof. Suppose not. Then $U \subseteq W$ and $W \subseteq U$. By Lemma 7.2, $jU(jw) = jw \upharpoonright MU$ and $jw(jv) = jv \upharpoonright MW$. In particular, $jU(\kappa_W) = \kappa_W$ and $jw(\kappa_U) = \kappa_U$. Therefore $\kappa_U \neq \kappa_W$, and we may assume by symmetry that $\kappa_U < \kappa_W$. But then $jw[\kappa_W] \subseteq \kappa_W$ and $\text{Un}_{<\kappa_W} \subseteq U$. So $U$ is $\kappa_W$-complete by Lemma 7.3. But this contradicts that $\kappa_U < \kappa_W$.

Lemma 7.5 (UA). Suppose $W$ is a countably complete ultrafilter and $\nu$ is an isolated cardinal such that $U_\nu \subseteq W$ and for all $D \in \text{Un}_{<\nu}$, $D \subseteq W$. Then $W$ is $\nu^+$-complete.

Proof. Suppose towards a contradiction $\kappa_W \leq \nu$. Then $\text{Un}_{<\kappa_W} \subseteq U_\nu$ by Lemma 5.10. By assumption $\text{Un}_{<\kappa_U} \subseteq W$. Therefore by Lemma 7.4, $W \not\subseteq U_\nu$. But by Lemma 6.6, it follows that $\nu$ is not isolated, and this is a contradiction.

Theorem 7.6 (UA). Suppose $\lambda$ is a Fréchet cardinal that is either regular or isolated. For any $\gamma$ with $\kappa_{U_\lambda} \leq \gamma < \lambda$, either $\gamma^\sigma = \gamma^+$ or $\gamma^\sigma = \lambda$.

Proof. Let $\nu = \gamma^\sigma$. Assume $\nu < \lambda$. Assume towards a contradiction that $\nu \neq \gamma^+$. By Lemma 3.3, $\nu$ is a limit cardinal. So $\nu$ is isolated. Since $\lambda$ is either regular or isolated, we have $U_\nu \subseteq U_\lambda$ and for all $D \in \text{Un}_{<\nu}$, $D \subseteq U_\lambda$. But by Lemma 7.5, it follows that $U_\lambda$ is $\nu^+$-complete, contradicting that $\nu \geq \gamma \geq \kappa_{U_\lambda}$.

For the next theorem we need a fact that is essentially due to Ketonen [5].

Lemma 7.7 (UA). Suppose $\delta$ is a regular Fréchet cardinal and every regular cardinal in the interval $[\kappa_{U_\delta}, \delta]$ is Fréchet. Then $\kappa_{U_\delta}$ is $\delta$-strongly compact.
Proof. Let $U = U_\delta$. Since $\sup j_U[\delta]$ is not tail uniform in $M_U$, $\cf^{M_U}(\sup j_U[\delta])$ is not Fréchet in $M_U$. Therefore we have $\cf^{M_U}(\sup j_U[\delta]) < j_U(\kappa_U)$, since otherwise $\cf^{M_U}(\sup j_U[\delta])$ is an $M_U$-regular cardinal in the interval $j_U(\kappa_U, \delta]$ and hence is Fréchet by elementarity. This implies that every set of ordinals of size $\delta$ in $M_U$ is covered by a set of ordinals in $M_U$ of size less than $j_U(\kappa_U)$. Hence $j_U : V \to M_U$ witnesses that $\kappa_U$ is $\delta$-strongly compact.

Theorem 7.8 (UA). Suppose $\delta$ is Fréchet cardinal. Assume $\delta$ is either a successor cardinal or a regular limit of Fréchet cardinals. Then $\kappa_{U_\delta}$ is $\delta$-strongly compact.

Proof. Fix $\gamma$ with $\kappa_{U_\lambda} \leq \gamma^+ < \delta$. We claim $\gamma^\sigma < \delta$. If $\delta$ is a limit of Fréchet cardinals this is immediate. Suppose instead that $\delta = \lambda^+$. By Lemma 3.4, either $\lambda$ is Fréchet or $\lambda$ is a limit of Fréchet cardinals. Therefore since $\gamma < \lambda$, $\gamma^\sigma \leq \lambda < \delta$, as claimed.

It follows that every successor cardinal in the interval $[\kappa_{U_\delta}, \delta]$ is Fréchet. But then by Prikry’s theorem [4], every regular cardinal in the interval $[\kappa_{U_\delta}, \delta]$ is Fréchet. By Lemma 7.7, it follows that $\kappa_{U_\delta}$ is $\delta$-strongly compact.

Proof of Theorem 7.1. By Theorem 7.8, we have reduced to the case that $\delta$ is a strongly inaccessible isolated cardinal. By Theorem 5.1, $\delta$ is measurable. The proof will be complete if we show $\kappa_{U_\delta} = \delta$, since $\delta$ is certainly $\delta$-strongly compact. Let $W$ be a normal ultrafilter on $\delta$. We claim $U_\delta = W$. Otherwise $U_\delta <_E W$, and so since $W$ is normal, $U_\delta < W$. This implies $\delta$ is a limit of measurable cardinals, which contradicts that $\delta$ is isolated.

8 Ultrafilters on an isolated cardinal

In this section we enact a fairly complete analysis of the ultrafilters that lie on an isolated cardinal. For the rest of the section, $\lambda$ will denote a fixed isolated cardinal and $U$ will denote $U_\lambda$.

We start with a characterization of $U$.

Theorem 8.1 (UA). The ultrafilter $U$ is the unique countably complete uniform ultrafilter on $\lambda$ such that $[\text{id}]_U$ is a generator of $j_U$.

Proof. Suppose towards a contradiction $U'$ is the $<_E$-least countably complete uniform ultrafilter on $\lambda$ such that $[\text{id}]_{U'}$ is a generator of $j_{U'}$ and $U' \neq U$. Since $[\text{id}]_{U'}$ is a generator of $U'$, $U'$ is Fréchet uniform. Therefore by the definition of $U$, $U <_E U'$.

Let $(i', i') : (M_U, M_{U'}) \to N$ be a comparison of $(j_U, j_{U'})$. Then $i'(\cdot[j_U])$ is a generator of $i' \circ j_{U'} = i \circ j_U$. Since $i(\cdot[j_U]) < i'(\cdot[j_U])$, it follows that $i'(\cdot[j_U])$ is an $i'(\cdot[j_U])$-generator of $i \circ j_U$ and hence $i'(\cdot[j_U])$ is a generator of $i : M_U \to N$.

Let $W$ be the ultrafilter derived from $i$ using $i'(\cdot[j_U])$. Then $W$ is Fréchet uniform. Moreover since $i'(\cdot[j_U]) \supseteq i'(\cdot[j_U][\lambda]) = \sup i \circ j_U[\lambda]$, $\text{sp}(W) \supseteq \sup j_U[\lambda]$.

If $\text{sp}(W) < j_U(\lambda)$, then an easy reflection argument yields that $\lambda$ is a limit of Fréchet cardinals, contradicting that $\lambda$ is isolated. So $\text{sp}(W) = j_U(\lambda)$. Therefore in $M_U$, $W$ is a countably complete uniform ultrafilter on $j_U(\lambda)$ such that $[\text{id}]_{j_U}^M$ is a generator of $j_{U'}$. Moreover $U^-(W) = U'$ so $W \neq j_U(U)$. It follows that $j_U(U') \leq_{M_U} W$.

But $W = t_U(U')$. Since $t_U(U') \leq_{M_U} j_U(U')$, it follows that $t_U(U') = j_U(U')$. This implies $U \sqsubset U'$. But this contradicts Lemma 6.1.
We then analyze the generators of $U^n$.

**Definition 8.2.** We define finite sets of ordinals $p_n$ by induction setting $p_0 = \emptyset$ and

$$p_{n+1} = j_U(p_n) \cup \{ j_U(j_U^n)([id]_U) \}$$

**Lemma 8.3 (UA).** $j_U(j_U^n)([id]_U)$ is a $j_U(p_n)$-generator of $j_U^{n+1}$.

For the proof we need an abstract version of the Dodd-Jensen lemma:

**Proposition 8.4.** Suppose $M$ and $N$ are inner models and $j : M \to N$ and $k : M \to N$ are elementary embeddings such that $j$ is definable from parameters over $M$. Then $j(\alpha) \leq k(\alpha)$ for any ordinal $\alpha$.

This yields the following fact:

**Lemma 8.5 (UA).** Suppose $Z$ is a countably complete ultrafilter and $h : M_U \to M_Z$ is an internal ultrapower embedding with $j_Z = h \circ j_U$. Then $h([id]_U)$ is the least generator of $M_Z$ above $\sup j_Z[\lambda]$.  

**Proof.** Let $\xi$ be the least generator of $j_Z$ above $\sup j_Z[\lambda]$, so $h([id]_U) \leq \xi$. Let $U'$ be the ultrafilter derived from $Z$ using $\xi$ and let $k : M_{U'} \to M_Z$ be the canonical factor embedding. By Theorem 8.1, $U' = U$. By Proposition 8.4, $\xi = k([id]_U)$ for some ordinal $\alpha$. Thus $h([id]_U) = \xi$, as desired.

**Proof of Lemma 8.3.** Suppose not. Let $p \subset j_U(j_U^n)([id]_U)$ be the least parameter such that

$$j_U(j_U^n)([id]_U) \in H^{M_{U^{n+1}}}_U(j_U^{n+1}[V] \cup j_U(p_n) \cup p)$$

**Case 1.** $p \prod j_U^{n+1}[\lambda]$.

Note that $s_U^n(U)$ is the ultrafilter derived from $j_U \upharpoonright H_U^n$ using $j_U(j_U^n)([id]_U)$. It follows that $s_U^n(U)$ is isomorphic over $M_{U^n}$ to the uniform $M_{U^n}$-ultrafilter $U'$ derived from $j_U \upharpoonright H_U^n$ using $p$. In other words, for some ordinal $\delta < \sup j_U^{n+1}[\lambda]$, there is some $f : [\delta]^{<\omega} \to j_U^{n+1}[\lambda]$ with $f : M_{U^n}$ such that $X \in s_U^n(U)$ if and only if $f^{-1}[X] \subset U'$.

In particular, the set $X = f([\delta]^{<\omega})$ is such that $X \in s_U^n(U)$ and $X^{M_{U^n}} \leq \delta$. Let $X = j_U^n[X]$. Then $X \in U$, so $X \subset \lambda$. But by Lemma 6.5, this implies every set $A \subset N_{U^n}$ with $|A| \leq \lambda$ is covered by a set $A' \subset M_{U^n}$ with $|A'|^{M_{U^n}} \leq \delta$. As in Lemma 6.1, it follows that $j_U^n$ is discontinuous at all sufficiently large regular cardinals below $\lambda$, which contradicts that $\lambda$ is isolated.

**Case 2.** $p \not\prod j_U^{n+1}[\lambda]$.

Since $p$ is a set of generators of $j_U^{n+1}$, it follows that there is a set of parameters of $j_U^{n+1}$ in the interval $[\sup j_U^{n+1}[\lambda], j_U(j_U^n)([id]_U))$. But this contradicts Lemma 8.5 with $Z = U^{n+1}$ and $h \circ j_U(j_U^n)$.

**Corollary 8.6 (UA).** $p_n$ is the least parameter $p$ with $M_{U^n} = H^{M_{U^n}}(j_U^n[V] \cup p)$.  

**Definition 8.7.** If $W$ is a countably complete ultrafilter on $\lambda$, then $p_W$ denotes the least parameter $p$ with $M_W = H^{M_W}(j_W[V] \cup p \cup q)$ for some $q \prod j_W[\lambda]$.  

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Thus $p_W$ is the piece of the Dodd parameter of $W$ contained in the interval $[\sup j_W[\lambda], j_W(\lambda))$.

**Theorem 8.8 (UA).** Suppose $W$ is a countably complete ultrafilter on $\lambda$. Let $n = |p_W|$ and let $\lambda_* = \sup j_U^n[\lambda]$. There is some $D \in \text{Un}^{M_{U^n}}_{<\lambda_*}$ with $j_D^{M_{U^n}} \circ j_{U^n} = j_W$ and $j_D^{M_{U^n}}(p_n) = p_W$.

**Proof.** We may assume by induction that the theorem is true for all $W' <_E W$. We may also assume without loss of generality that $W$ is $\lambda$-decomposable. It follows that $U \not\subseteq W$ by Lemma 6.1.

Let $(h, i) : (M_U, M_W) \to N$ be the canonical comparison of $(j_U, j_W)$. Let $p = i(p_W)$ and let $\xi = [\text{id}]_U$.

**Claim 1.** $h(\xi) = \min p$.

**Proof.** Suppose not. Then by Lemma 8.5, $h(\xi) < \min p$.

**Subclaim 1.** $h(\xi)$ is a $p$-generator of $h \circ j_U$.

**Proof.** We have that $h = j_W^{M_U}$ where $W' = t_U(W)$. Since $U \not\subseteq W$, we must have $t_U(W) <_E j_U(W)$. It is easy to check that $p = (p_W)^{M_U}$ where the parameter is defined at $j_U(\lambda)$ rather than at $\lambda$. Therefore by our induction hypothesis and elementarity, there is some $n < \omega$ such that $h = k \circ j_U(j_{U^n})$ where $k : M_{U^n}^{M_{U^n}} \to N$ is an elementary embedding such that $k(j_U(p_n)) = p$. By Lemma 8.3, $j_U(j_{U^n})(\xi)$ is a $j_U(p_n)$-generator of $j_{U^n+1} = j_U(j_{U^n}) \circ j_U$. Therefore $k(j_U(j_{U^n})(\xi))$ is a $k(j_U(p_n))$-generator of $k \circ j_U(j_{U^n}) \circ j_U$. Replacing like terms, $h(\xi)$ is a $p$-generator of $h \circ j_U$. \qed

It follows that $h(\xi)$ is a generator of the embedding $i : M_W \to N$. Let $Z$ be the ultrafilter derived from $i$ using $h(\xi)$. Then $\text{sp}(Z) \geq \sup j_W[\lambda]$ since $h(\xi) \geq \sup i \circ j_W[\lambda]$. On the other hand since $h(\xi) < \min p$, $\text{sp}(Z) \leq \min p < j_W(\lambda)$. But in $M_W$, $Z$ is a Fréchet uniform ultrafilter, since $[\text{id}]_Z$ is a generator of $j_Z^{M_W}$. This contradicts that by isolation $j_W(\lambda) = (\gamma)^{M_W}$ for some $\gamma < \sup j_W[\lambda]$. This contradiction proves Claim 1. \qed

By the definition of a canonical comparison, it follows that $i$ is the identity, $N = M_W$, and $h : M_U \to M_W$ is an internal ultrapower embedding. Moreover, as in the proof of Claim 1, our induction hypothesis implies that for some $n < \omega$,\[ h = j_D^{M_{U^{n+1}}} \circ j_{U^n}(j_U) \]

where $D \in \text{Un}_{<\lambda_*}$ for $\lambda_* = \sup j_U^n[j_U(\lambda)]$ and $j_D^{M_{U^{n+1}}} (j_U(p_n)) = p \setminus (h(\xi) + 1)$. Since $j_{U^{n+1}}(\lambda) = (\gamma)^{M_{U^{n+1}}}$, there is in fact some $D' \in \text{Un}_{\sup j_{U^{n+1}}[\lambda]}$ with $D' \equiv D$ in $M_{U^{n+1}}$. Thus replacing $D$ with $D'$, we may assume $D \in \text{Un}_{\sup j_{U^{n+1}}[\lambda]}$.

Since $p_{n+1} = j_U(p_n) \cup j_U(j_{U^n})(\xi)$, we have $j_D^{M_{U^{n+1}}}(p_{n+1}) = p \setminus (h(\xi) + 1) \cup \{h(\xi)\} = p$, with the final equality following from Claim 1.

Putting everything together, we have shown that there is some $D \in \text{Un}_{\sup j_{U^{n+1}}[\lambda]}$ such that $j_W = j_D^{M_{U^{n+1}}} \circ j_{U^{n+1}}$ and such that $j_D^{M_{U^{n+1}}}(p_{n+1}) = p_W$. This proves the theorem. \qed

We will use the following consequence of this theorem, which improves Lemma 5.10.
Corollary 8.9 (UA). Suppose $D$ is a countably complete $M_U$-ultrafilter on an ordinal below $\sup j_U[\lambda]$. Then $D \in M_U$.

Proof. Let $W$ be a countably complete ultrafilter on $\lambda$ such that $j_W = j_D^{M_U} \circ j_U$. An easy calculation shows that $p_W = \{j_D^{M_U}([\text{id}])\}$. Therefore by Theorem 8.8, there is an internal ultrapower embedding $k : M_U \to M_W$ such that $j_W = k \circ j_U$ and $k([\text{id}]) = j_D^{M_U}([\text{id}])$. It follows that $k = j_D^{M_U}$. But then $j_D^{M_U}$ is an internal ultrapower embedding, so $D \in M_U$. □

9 The continuum function below an isolated cardinal

In this section we prove a theorem that shows Theorem 5.1 is optimal in a sense:

Theorem 9.1 (UA). Suppose $\lambda$ is an isolated cardinal and $\gamma < \lambda$ is Fréchet. Then $2^\gamma < \lambda$.

We need a theorem from [1]:

Theorem 9.2 (UA). If $W$ is a countably complete ultrafilter and $\lambda$ is a cardinal such that $U_{<\lambda} \subseteq W$, then for all Fréchet cardinals $\gamma < \lambda$, $W$ is $\gamma$-supercompact.

Proof. Suppose $\delta$ is the least cardinal such that $M_W$ is not closed under $\delta$-sequences. If $\lambda < \delta$, we are done, so assume $\delta < \lambda$. We must show that no cardinal in the interval $[\delta, \lambda]$ is Fréchet.

Claim 1. $\delta$ is not Fréchet.

Proof. Assume towards a contradiction that $\delta$ is Fréchet. Note that $\delta$ is a regular cardinal. We claim $\delta$ is not isolated. This is because if $\delta$ were isolated, then since $U_{<\delta} \subseteq W$, by Lemma 7.5, $W$ would be $\delta^+$-complete, contradicting that $M_W$ is not closed under $\delta$-sequences. Thus $\delta$ is not isolated.

By Theorem 7.8, it follows that $\text{crt}(j_\delta)$ is $\delta$-strongly compact, so by the main theorem of [1], $M_\delta$ is closed under $<\delta$-sequences and has the tight covering property at $\delta$. Now $j_\delta(M_W) \subseteq M_W$ and $j_\delta(M_\delta) = M_{j_\delta(W)}$ is closed under $< j_\delta(\delta)$-sequences inside of $M_\delta$. In particular, since $\delta < j_\delta(\delta)$,

$$ [\text{Ord}]^{\delta} \cap M_\delta \subseteq M_W $$

(1)

Fix a set of ordinals $S$ of cardinality $\delta$. By the tight covering property at $\delta$, there is some $A \in M_\delta$ containing $j_\delta[S]$ such that $|A|^{M_\delta} = \delta$. Let $A' = A \cap j_\delta(S)$, so that $j_\delta^{-1}[A'] = S$. By (1), $A' \in M_W$. Since $U_\delta \subseteq W$, $j_\delta \upharpoonright M_W$ is definable over $M_W$. Therefore $j_\delta^{-1}[A'] \in M_W$, so $S \in M_W$. It follows that $M_W$ is closed under $\delta$-sequences, contradicting the definition of $\delta$. □

To show no cardinal in the interval $[\delta, \lambda]$ is Fréchet, it now suffices to show that $\delta^\sigma \geq \lambda$ so $\lambda$ is a strong limit cardinal. Since $\delta$ is regular but not Fréchet, $\delta^\sigma > \delta^+$. Let $\nu = \delta^\sigma$. Assume towards a contradiction that $\nu < \lambda$. Then $U_\nu \subseteq W$ and $U_{<\nu} \subseteq W$. Therefore $W$ is $\nu^+$-complete, contradicting that $M_W$ is not closed under $\nu$-sequences. □

We need the following lemma whose proof requires ideas from [3]:

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Lemma 9.3 (UA). Suppose $\lambda$ is a limit of Fréchet cardinals. Then $\lambda$ is a strong limit cardinal.

Sketch. Let $\kappa$ be the supremum of the isolated cardinals below $\lambda$. If $\kappa = \lambda$ then it is easy to see that $\lambda$ is a limit of measurable cardinals so $\lambda$ is a strong limit cardinal. Otherwise by Theorem 7.6, every successor cardinal in the interval $[\kappa, \lambda]$ is Fréchet. Applying Theorem 9.2 to $U_\delta$ for successor cardinals $\delta \in [\kappa, \lambda]$, it follows that $\kappa$ is $\delta$-supercompact for all $\delta < \lambda$. Now by the main theorem of [3], GCH holds for every cardinal in the interval $[\kappa, \lambda]$. So $\lambda$ is a strong limit cardinal.

Proof of Theorem 9.1. Assume towards a contradiction that $\lambda$ is the least isolated cardinal such that for some Fréchet cardinal $\gamma < \lambda$, $2^\gamma \geq \lambda$. Since $U_\lambda$ satisfies the hypotheses of Theorem 9.2, $U_\lambda$ is $\gamma$-supercompact. In particular, $P(\gamma) \subseteq M_{U_\lambda}$, and therefore $(2^\gamma)_{M_{U_\lambda}} \geq 2^\gamma \geq \lambda$.

Claim 1. $\lambda < j_{U_\lambda}(\kappa_{U_\lambda})$.

Proof. Since $j_{U_\lambda}(\kappa_{U_\lambda})$ is inaccessible in $M_{U_\lambda}$ and $(2^\gamma)_{M_{U_\lambda}} \geq \lambda$, it suffices to show $\gamma < j_{U_\lambda}(\kappa_{U_\lambda})$. If $j_{U_\lambda}(\kappa_{U_\lambda}) < \gamma$, then $j_{U_\lambda}$ witnesses that $\kappa_{U_\lambda}$ is a huge cardinal. In particular, some $\kappa < \kappa_{U_\lambda}$ is $\kappa_{U_\lambda}$-supercompact. A standard theorem on the propagation of supercompactness (see [10]) implies $\kappa$ is $\gamma$-supercompact.

By the Kunen inconsistency theorem, there is an inaccessible cardinal $\delta \leq \gamma$ such that $j_{U_\lambda}(\delta) > \gamma$. By elementarity, in $M_{U_\lambda}$, $\kappa$ is $j_{U_\lambda}(\delta)$-supercompact. By the results of [3] applied in $M_{U_\lambda}$, $(2^\gamma)_{M_{U_\lambda}} = \gamma^+$, which contradicts the fact that $(2^\gamma)_{M_{U_\lambda}} \geq \lambda$.

By Corollary 8.9, it follows that $U_\lambda \cap M_{U_\lambda} \in M_{U_\lambda}$. In particular, $\lambda$ is Fréchet in $M_{U_\lambda}$. Similarly $\gamma$ is Fréchet in $M_{U_\lambda}$. Of course $\lambda$ is a limit cardinal in $M_{U_\lambda}$. In fact, $\lambda$ is isolated in $M_{U_\lambda}$. Assume not. Then $\lambda$ is a limit of Fréchet cardinals in $M_{U_\lambda}$. Therefore by Lemma 9.3, $\lambda$ is a strong limit cardinal. But this contradicts that $(2^\gamma)_{M_{U_\lambda}} \geq \lambda$.

Therefore $M_{U_\lambda}$ satisfies that $\lambda$ is an isolated cardinal and there is a Fréchet cardinal $\gamma < \lambda$ such that $2^\gamma \geq \lambda$. But by elementarity, in $M_{U_\lambda}$, $j_{U_\lambda}(\lambda)$ is the least cardinal with this property. It follows that $j_{U_\lambda}(\lambda) = \lambda$, which contradicts that $\lambda < \sup j_{U_\lambda}(\kappa_{U_\lambda})$.

Putting Theorem 5.1 and Theorem 9.1 together, and using Theorem 7.6, we have a rough picture of the continuum function below a nonmeasurable isolated cardinal:

Proposition 9.4 (UA). Suppose $\lambda$ is a nonmeasurable isolated cardinal. Let

$$\delta = \sup \{ \gamma^+ : \gamma < \lambda \text{ is Fréchet} \}$$

be the strict supremum of the Fréchet cardinals below $\lambda$. Then $2^{<\delta} < \lambda \leq 2^\delta$.

10 The Mitchell order without GCH

In this section, we apply the machinery of this paper to improve the main result of [2]. We first make some general remarks to explain the presentation of the result we have chosen here.

We begin by stating some folklore facts about canonical representatives for the Rudin-Keisler equivalence classes of countably complete ultrafilters.
Definition 10.1. A countably complete ultrafilter $U$ is *seed-minimal* if $|\text{id}|_U$ is the least ordinal $\xi$ such that $M_U = H^{M_U}(j_U[V] \cup \{\xi\})$.

We include a combinatorial reformulation without proof.

Lemma 10.2. A countably complete ultrafilter $U$ on $\lambda$ is seed-minimal if and only if no regressive function on $\lambda$ is one-to-one on a set in $U$. □

The following is immediate:

Lemma 10.3. Every countably complete ultrafilter is isomorphic to a unique seed-minimal ultrafilter. □

Definition 10.4. Suppose $A$ is a set. An ultrafilter $U$ on $P(A)$ is *fine* if for all $a \in A$, for $U$-almost all $\sigma \in P(A)$, $a \in \sigma$. A fine ultrafilter $U$ on $P(A)$ is *normal* if all choice functions $f : P(A) \to A$ are constant on set in $U$.

A nonprincipal ultrafilter is a *generalized normal ultrafilter* if it is seed-minimal and isomorphic to a normal fine ultrafilter on $P(A)$ for some set $A$.

We include a combinatorial reformulation of the notion of a generalized normal ultrafilter.

Definition 10.5. An ultrafilter $U$ on a cardinal $\lambda$ is *weakly normal* if every regressive function on $\lambda$ takes fewer than $\lambda$ values on a set in $U$.

Theorem 10.6. If $U$ is an ultrafilter on a cardinal $\lambda$, the following are equivalent:

(1) $U$ is weakly normal and $j_U[\lambda] \in M_U$.

(2) $U$ is a generalized normal ultrafilter. □

The proof of this theorem essentially appears in [2]. It is due to Solovay when $\lambda$ is regular and to the author when $\lambda$ is singular.

The main theorem of this section is the following:

Theorem 10.7 (UA). The Mitchell order is linear on generalized normal ultrafilters.

As a corollary one can extract various corollaries about the linearity of the Mitchell order on normal fine ultrafilters; we omit these facts here since this sort of thing appears in [2].

Our proof of Theorem 10.7 requires proving a stronger theorem:

Theorem 10.8 (UA). Suppose $U$ is a generalized normal ultrafilter and $D$ is a countably complete uniform ultrafilter with $D <_E U$. Then $D \triangleleft U$.

In [2], we proved the same theorem using a different hypothesis:

Theorem 10.9. Suppose $\lambda$ is a cardinal such that $2^{<\lambda} = \lambda$. Suppose $U$ is a generalized normal ultrafilter on $\lambda$ and $D$ is a countably complete uniform ultrafilter with $D <_E U$. Then $D \triangleleft U$. □

This will be used in the proof of Theorem 10.8.

We also need some more facts from the general theory of the internal relation.
Lemma 10.10 (UA). Suppose $U$ is a nonprincipal countably complete ultrafilter. Let $D$ be the $<_E$-least countably complete uniform ultrafilter such that $D \not\subseteq U$. Then for any countably complete ultrafilter $W$, if $W \subseteq U$, then $W \subseteq D$.

Proof. We may assume without loss of generality that $U \in \mathcal{U}$. Moreover it suffices to show that for any $W \in \mathcal{U}$, if $W \subseteq U$ then $W \subseteq D$. So fix such a $W$. To show that $W \subseteq D$, it suffices to show that $t_W(D) = j_W(D)$, and for this it suffices to show that $j_W(D) \leq t_W(D)$. For this it is enough to show that in $M_U$, $t_W(D) \not\subseteq j_W(U)$.

Assume towards a contradiction that $M_U$ satisfies that $t_W(D) \subseteq j_W(U)$. In other words, $j_M(U) \cap j_W(U)$ restricts to an internal ultrapower embedding of $M_U$. But note that $j_M(U) \cap j_W(U)$ restricts to an internal ultrapower embedding of $M_U$. It follows easily that $j_D$ restricts to an internal ultrapower embedding of $M_U$, contradicting the fact that $D \not\subseteq U$. □

The following key fact about internal ultrapowers is proved in [11].

Theorem 10.11 (UA). Suppose $j_0 : V \to M_0$ and $j_1 : V \to M_1$ are ultrapower embeddings and $(i_0, i_1) : (M_0, M_1) \to N$ is their canonical comparison. Then for any ultrapower embedding $h : N \to N'$, the following are equivalent:

(1) $h$ is an internal ultrapower embedding of $N$.

(2) $h$ is definable over both $M_0$ and $M_1$. □

In fact, we will only use a weak corollary of this theorem:

Corollary 10.12 (UA). Suppose $D, U, W \in \mathcal{U}$ and $W \subseteq D, U$. Then in $M_U$, $s_U(W) \subseteq t_U(D)$.

We also use the following theorems from [3].

Proposition 10.13 (UA). Suppose $\lambda$ is a cardinal and $U$ is a countably complete ultrafilter such that $M_U$ is closed under $\lambda$-sequences. Then for any countably complete ultrafilter $W$ on an ordinal $\gamma < \lambda$, $W \subseteq M_U$. □

Theorem 10.14 (UA). Suppose $\lambda$ is a cardinal and $\lambda^+$ carries a countably complete uniform ultrafilter. Then $2^{<\lambda} = \lambda$. □

We could make do without using the following theorem, proved in Section 12, but it will be convenient:

Theorem 10.15 (UA). For any cardinal $\lambda$, $|\mathcal{U}_{\leq \lambda}| \leq (2^\lambda)^+$.
a singular strong limit cardinal by Theorem 10.14, and hence $2^\gamma = \gamma^+$ by Solovay’s theorem [12].

Let $D$ be the $<_E$-least ultrafilter such that $D \not \subseteq U$. Thus $D \leq E U$. By Proposition 10.13, $D$ is a uniform ultrafilter on $\lambda$.

We must show $D = U$. Let $D' = t_U(D)$.

**Claim 1.** $D'$ is $\lambda^+$-complete in $M_U$.

**Proof.** We first show that for any $W \in \text{Un}^M_U$, $M_U$ satisfies that $W \subseteq D'$. To see this, note that such an ultrafilter $W$ satisfies $W \subseteq U$ since $j_W \upharpoonright M_U = j^M_U$ by the closure of $M_U$ under $\lambda$-sequences. It follows from Corollary 10.12 that $M_U$ satisfies $s_U(W) \subseteq D'$, but this is equivalent to the fact that $M_U$ satisfies $W \supseteq D'$.

Let $U'$ be the generalized normal ultrafilter on $\gamma$ derived from $U$. Then $U' \in M_U$ by Proposition 10.13. By an argument due to Solovay, it follows that in $M_U$, every set $A \subseteq P(\gamma)$ is in the ultrapower of a countably complete ultrafilter on $\gamma$. Thus in $M_U$, $2^{2\gamma} = |\text{Un}_{\leq \gamma}| = (2\gamma)^+$ by Theorem 10.15.

Note that $(\lambda^\sigma)^M_U > \lambda^+$: otherwise $(\lambda^\sigma)^M_U = \lambda^+$, so applying Theorem 10.14 in $M_U$, we have $(2^{<\lambda})^M_U = \lambda$, and so since $P(\lambda) \subseteq M_U$, in actuality $2^{<\lambda} = \lambda$, contrary to assumption.

Therefore by Lemma 3.4, $M_U$ satisfies that $\lambda^\sigma$ is isolated. We can apply Theorem 5.1 at the Fréchet cardinal $\gamma$ to conclude that $M_U$ satisfies $2^{\gamma} < \lambda^\sigma$ and hence $(2\gamma)^+ < \lambda^\sigma$. Since $2^\gamma \geq \lambda^+$ and $P(\gamma) \subseteq M_U$, $M_U$ satisfies that $2^\gamma \geq \lambda^+$. Thus $M_U$ satisfies

$$2^{(\lambda^+)} < 2^{2\gamma} = (2\gamma)^+ < \lambda^\sigma$$

Now in $M_U$, $\lambda^+$ is a non-Fréchet regular cardinal and $2^{(\lambda^+)} < \lambda^\sigma = (\lambda^+)\sigma$. Therefore by Lemma 5.11, it follows that $D'$ is $\lambda^\sigma$-complete in $M_U$, which proves the claim. \hfill \Box

Given the claim, we finish the proof as follows. Since $\lambda$ is regular, the weak normality of $U$ implies that $[\text{id}]_U = \sup j_U[\lambda]$. Since $D'$ is $\lambda^+$-complete, $j^M_U \circ j_U[\lambda]$ is continuous at $\sup j_U[\lambda]$. Thus

$$j^M_{D'}([\text{id}]_U) = \sup j^M_{D'} \circ j_U[\lambda] = \sup j^M_D \circ j_D[\lambda] \leq j^M_D([\text{id}]_D)$$

The final inequality comes from the fact that $\sup j_D[\lambda] \leq [\text{id}]_D$; this holds because $D$ is a uniform ultrafilter on $\lambda$. Thus $U \leq E D$, and hence $U = D$, as desired. \hfill \Box

### 11 Strongly tall cardinals and set-likeness

There is a natural question left open by the results of [1] that we resolve here:

**Theorem 11.1 (UA).** Suppose $\kappa$ is the least ordinal such that for all ordinals $\alpha$, there is some ultrapower embedding $j : V \rightarrow M$ such that $j(\kappa) > \alpha$. Then $\kappa$ is supercompact. \hfill \Box

What we show here is the following:

**Proposition 11.2 (UA).** Suppose $\kappa$ is the least ordinal such that for all ordinals $\alpha$, there is some ultrapower embedding $j : V \rightarrow M$ such that $j(\kappa) > \alpha$. Then $\kappa$ is $\omega_1$-strongly compact.
Clearly $\kappa$ is then the least $\omega_1$-strongly compact cardinal. It follows that $\kappa$ is supercompact by the following theorem from [1]:

**Theorem 11.3 (UA).** The least $\omega_1$-strongly compact cardinal is supercompact. □

For the proof of Proposition 11.2, we use the following fact:

**Lemma 11.4 (UA).** Suppose $\xi$ is an ordinal and $W$ is the $<_E$-least uniform countably complete ultrafilter $Z$ such that $j_Z(\xi) \neq \xi$. Then for any countably complete ultrafilter $U$ such that $j_U(\xi) = \xi$, $U \subset W$.

**Proof.** By Lemma 5.9, it suffices to show that in $M_U$, $j_U(W) \leq_E t_U(W)$. For this it is enough to show that $j_M^{M_U}(j_U(\xi)) \neq j_U(\xi)$, since in $M_U$, $j_U(W)$ is the $<_E$-least countably complete ultrafilter $Z$ such that $j_Z(\xi) \neq \xi$. This is a consequence of the following calculation:

$$j_M^{M_U}(j_U(\xi)) = j_{t_U(W)}(j_U(\xi)) \geq j_W(\xi) > \xi = j_U(\xi)$$ □

**Proof of Proposition 11.2.** We show that for any ordinal $\alpha \geq \kappa$, $\alpha^\kappa = \alpha^+$. Let $\lambda = \alpha^\#$. Let $U = U_\lambda$. Let $\xi$ be the least fixed point of $j_U$ such that $\xi > \text{crt}(j_U)$. Let $W$ be the $<_E$-least uniform countably complete ultrafilter such that $j_U(W) \neq \xi$. By Lemma 11.4, $U \subset W$.

We claim $W \not\subset U$. Suppose towards a contradiction that $W \subset U$. Then by Lemma 7.2, $j_W(j_U \upharpoonright \text{Ord}) = j_U \upharpoonright \text{Ord}$. This is a contradiction since $\xi$ is $\Delta_1$-definable from $j_U \upharpoonright \text{Ord}$ without parameters, yet $j_W(\xi) \neq \xi$.

Thus $U \subset W$ and $W \not\subset U$. By Lemma 6.6, it follows that $\lambda$ is not an isolated cardinal. Thus $\lambda = \alpha^+$, as desired. □

We connect this up with some ideas from [8].

**Definition 11.5.** A **pointed ultrapower** is a pair $(M, \alpha)$ where $M$ is an ultrapower of $V$ and $\alpha$ is an ordinal. The class of pointed ultrapowers is denoted $\mathcal{P}$. If $\mathcal{M} = (M, \alpha)$ is a pointed ultrapower, then $\alpha_{\mathcal{M}}$ denotes $\alpha$.

We will sometimes abuse notation by writing $\mathcal{M}$ when we really mean the ultrapower $M$ such that $\mathcal{M} = (M, \alpha_{\mathcal{M}})$.

**Definition 11.6.** The **completed seed order** is defined on $\mathcal{M}_0, \mathcal{M}_1 \in \mathcal{P}$ by setting $\mathcal{M}_0 \preceq_S \mathcal{M}_1$ if there is a pair of elementary embeddings $(i_0, i_1): (\mathcal{M}_0, \mathcal{M}_1) \rightarrow N$ such that $i_1$ is an internal ultrapower embedding of $M_1$ and $i_0(\alpha_{\mathcal{M}_0}) < i_1(\alpha_{\mathcal{M}_1})$.

Not every pointed ultrapower must have a rank in the completed seed order, but for those that do, we use the following notation:

**Definition 11.7.** For any $\mathcal{M} \in \mathcal{P}$, $|\mathcal{M}|_\infty$ denotes the rank of $\mathcal{M}$ in the completed seed order if it exists.

Here we will only consider the completed seed order assuming UA, in which case we have the following fact from [8]:

**Lemma 11.8 (UA).** Suppose $\mathcal{M}_0, \mathcal{M}_1 \in \mathcal{P}$. Then the following are equivalent:
(1) $M_0 <_S M_1$.

(2) There exist internal ultrapower embeddings $(i_0, i_1): (M_0, M_1) \rightarrow N$ such that $i_0(\alpha_{M_0}) < i_1(\alpha_{M_1})$.

We have the following fact about $\mathcal{P}$:

Lemma 11.9. Suppose $N$ is an ultrapower. Then $\mathcal{P}^N \subseteq \mathcal{P}$ and $<_S \upharpoonright \mathcal{P}^N$ is a suborder of $<_S$.

Corollary 11.10. For any ultrapower $N$ and $M \in \mathcal{P}^N$, $|M|^N_\infty \leq |M|_\infty$.

With UA, we can prove sharper results:

Lemma 11.11 (UA). Suppose $N$ is an ultrapower. Then $\mathcal{P}^N \subseteq \mathcal{P}$ and $<_S \upharpoonright \mathcal{P}^N$ is equal to $<_S \upharpoonright \mathcal{P}^N$. Moreover, for any $M \in \mathcal{P}$, there is some $M' \in \mathcal{P}^N$ such that $M \equiv_S M'$.

Corollary 11.12 (UA). For any ultrapower $N$ and $M \in \mathcal{P}^N$, $|M|^N_\infty = |M|_\infty$.

Lemma 11.13 (UA). Suppose $\alpha$ is an ordinal. Then the following are equivalent:

(1) $\alpha = |(V, \xi)|_\infty$ for some ordinal $\xi$.

(2) $\alpha$ is fixed by all ultrapower embeddings.

Proof. Assume (1). Let $i: V \rightarrow N$ be an ultrapower embedding. Then $i(\alpha) = |(N, i(\xi))|^N_\infty$. By Lemma 11.9, $|(N, i(\xi))|^N_\infty \leq |(N, i(\xi))|_\infty$. It is easy to see that every predecessor of $(N, i(\xi))$ in the completed seed order is a predecessor of $(V, \xi)$ in the completed seed order. Therefore $|(N, i(\xi))|_\infty \leq |(V, \xi)|_\infty = \alpha$. Putting everything together $i(\alpha) \leq \alpha$, so $i(\alpha) = \alpha$. Since $i$ was an arbitrary ultrapower embedding, (2) holds.

It is to prove (2) implies (1) that we need UA. Fix an ultrapower embedding $j: V \rightarrow M$ such that for some $\xi'$, $\alpha$ is the rank of $(M, \xi')$ in the completed seed order. Then by Lemma 11.11, in $M$, $\alpha$ is the rank of $(M, \xi')$ in the completed seed order. But $\alpha = j(\alpha)$ since $\alpha$ is fixed by all ultrapower embeddings. It follows by the elementary of $j: V \rightarrow M$ that there is some $\xi$ such that $\alpha$ is the rank of $(V, \xi)$ in the completed seed order, as claimed.

Corollary 11.14 (UA). The following are equivalent:

(1) For all $M \in \mathcal{P}$, $|M|_\infty$ exists.

(2) For unboundedly many ordinals $\alpha$, $\alpha$ is fixed by all ultrapower embeddings.

Proof. That (1) implies (2) is immediate. To show (2) implies (1), note that (2) implies that there are unboundedly many ordinals $\xi$ such that the rank of $(V, \xi)$ in the completed seed order is an ordinal. Therefore for all ordinals $\xi$, $(V, \xi)$ has a rank in the completed seed order, since $(V, \xi_0) <_S (V, \xi_1)$ when $\xi_0 < \xi_1$. But for any $M \in \mathcal{P}$, there is some $\xi$ such that $M <_S (V, \xi)$. Therefore $M$ has a rank in the completed seed order, i.e. $|M|_\infty$ exists.

Lemma 11.15. Exactly one of the following holds:

(1) For unboundedly many ordinals $\alpha$, $\alpha$ is fixed by all ultrapower embeddings.
(2) There is an ordinal \( \kappa \) such that for all ordinals \( \alpha \), for some ultrapower embedding \( j : V \to M \), \( j(\kappa) > \alpha \).

**Proof.** Obviously (2) implies (1) fails, so we just need to show that if (2) fails then (1) holds. Assume (2) fails. Fix an ordinal \( \xi \). We will define an ordinal \( \alpha \geq \xi \) that is fixed by all ultrapower embeddings. Let

\[
\alpha = \sup\{j(\xi) : j \text{ is an ultrapower embedding of } V\}
\]

The supremum \( \alpha \) exists since (2) fails. But for any ultrapower embedding \( i : V \to N \),

\[
i(\alpha) = \sup\{j(i(\xi)) : j \text{ is an internal ultrapower embedding of } N\}
\leq \sup\{j(\xi) : j \text{ is an ultrapower embedding of } V\}
= \alpha
\]

Therefore \( \alpha \) is fixed by all ultrapower embeddings. \( \square \)

Combining Theorem 11.1, Corollary 11.14, and Lemma 11.15, we obtain the following fact:

**Theorem 11.16 (UA).** Exactly one of the following holds:

1. For all \( M \in \mathcal{P} \), \( |M|_\infty \) exists.
2. There is a supercompact cardinal. \( \square \)

With a bit more work, one can show:

**Theorem 11.17 (UA).** Suppose \( M \) is an ultrapower and \( \alpha \) is an ordinal less than the least supercompact cardinal of \( M \). Then \( |(M, \alpha)|_\infty \) exists. \( \square \)

## 12 Counting countably complete ultrafilters

In this final section we prove a cardinal arithmetic fact whose proof requires techniques from this paper.

**Theorem 12.1 (UA).** A set \( X \) carries at most \( (2^{|X|})^+ \) countably complete ultrafilters.

Theorem 12.1 is of course a consequence of GCH since \( X \) carries at most \( 2^{|X|} \) ultrafilters, but we see no way to obtain it as a corollary of the theorems in [3]. Instead the proof given here imitates that of [3] Lemma 4.3, replacing the Mitchell order with more general concepts.

**Definition 12.2.** If \( U \) and \( U' \) are countably complete ultrafilters on ordinals \( \gamma \) and \( \gamma' \), we set \( U <_E U' \) if there is a sequence \( \langle U_\alpha : \alpha < \gamma' \rangle \) such that

1. For \( U' \)-almost all \( \alpha < \gamma' \), \( U_\alpha \) is a countably complete ultrafilter on \( \alpha \).
2. For any \( X \subseteq \gamma \), \( X \in U \) if and only if \( X \cap \alpha \in U_\alpha \) for \( U' \)-almost all \( \alpha < \gamma' \).
Definition 12.3. An ultrafilter $U$ on an ordinal $\alpha$ is tail uniform (or just uniform) if $\alpha \setminus \beta \in U$ for all $\beta < \alpha$.

Definition 12.4. For any ordinal $\alpha$, $\text{Un}_\alpha$ denotes the set of uniform countably complete ultrafilters on $\alpha$, $\text{Un}_{<\alpha} = \bigcup_{\beta<\alpha} \text{Un}_\beta$, $\text{Un}_{\leq \alpha} = \bigcup_{\beta \leq \alpha} \text{Un}_\beta$, and $\text{Un} = \bigcup_{\alpha \in \text{Ord}} \text{Un}_\alpha$.

Definition 12.5. For $U \in \text{Un}$, let $|U|_S$ denote the rank of $U$ in $<_E$ restricted to $\text{Un}$.

We use the following lemma for our main result. The proof appears in [8].

Lemma 12.6 (UA). Suppose $U$ is a nonprincipal countably complete uniform ultrafilter. Then $j_U(|U|_S) > |U|_S$.

The following bound is essentially immediate from the definition of $<_E$.

Lemma 12.7. A countably complete uniform ultrafilter on $\lambda$ has at most $\prod_{\alpha < \lambda} \text{Un}_{\leq \alpha}$ uniform $<_E$-predecessors.

Definition 12.8. An ultrafilter $U$ on a set $X$ is Fréchet uniform if for all $A \subseteq X$ with $|A| < |X|$, $X \setminus A \in U$. A cardinal $\lambda$ is Fréchet if it carries a countably complete Fréchet uniform ultrafilter.

Theorem 12.9 (UA). Suppose $\lambda$ is a Fréchet cardinal. Then for any $\alpha < \lambda$, $|\text{Un}_{\leq \alpha}| \leq 2^\lambda$.

Proof. We may assume by induction that the theorem holds below $\lambda$. If $\lambda$ is a limit of Fréchet cardinals, then the inductive hypothesis easily implies the conclusion of the theorem. Therefore assume $\lambda$ is not a limit of Fréchet cardinals.

Let $U_\lambda$ be the $<_E$-least Fréchet uniform ultrafilter on $\lambda$. Since $\lambda$ is not a limit of uniform cardinals, for every ultrafilter $W$ on a cardinal less than $\lambda$, $W \subseteq U_\lambda$.

Assume towards a contradiction that $\gamma < \lambda$ is least such that $|\text{Un}_{\leq \gamma}| > 2^\lambda$. We record that since $|\text{Un}_{\leq \gamma}| \geq \kappa_{U_\lambda}$, in fact $\gamma \geq \kappa_{U_\lambda}$.

Let $\alpha = \sup \{ |W|_S : \text{sp}(W) < \gamma \}$. We claim that for any ordinal $\xi \in [\alpha, (2^\lambda)^+]$, there is some $D$ such that $j_D(\xi) > \xi$.

To see this let $D$ be the unique countably complete uniform ultrafilter with $|D|_S = \xi$. Then $D$ is a uniform ultrafilter on $\gamma$, so $D$ is nonprincipal. Therefore by Lemma 12.6, $j_D(\xi) > \xi$ as desired.

Let $\xi$ be the least ordinal above $\alpha$ fixed by $j_{U_\lambda}$ and also by $j_D$ for all $D \in \text{Un}_{<\gamma}$. Then $\xi < (2^\lambda)^+$ since the intersection of $2^\lambda$-many $\omega$-club subsets of $(2^\lambda)^+$ is $\omega$-club. Let $W \in \text{Un}$ be the $<_E$-least ultrafilter with $j_W(\xi) > \xi$. By Lemma 12.6, $W$ is a countably complete uniform ultrafilter on $\gamma < \lambda$, so $W \subseteq U_\lambda$.

by Lemma 5.10 since $\lambda$ is not a limit of Fréchet cardinals. Moreover, since $j_{U_\lambda}(\xi) = \xi$,

$U_\lambda \subseteq W$

by Lemma 11.4.

Since $\kappa_W \leq \gamma < \lambda$,

$\text{Un}_{<\kappa_W} \subseteq U_\lambda$
by Lemma 5.10. Since $\kappa_{U_\lambda} \leq \lambda$ is a strong limit cardinal while $2^{2^{\gamma}} \geq |U_{\leq \gamma}| > 2^\lambda$, $\kappa_{U_\lambda} \leq \gamma$ and therefore

$$U_{\leq \kappa_{U_\lambda}} \subseteq W$$

by Lemma 11.4. Since $W$ and $U_\lambda$ are nonprincipal, this contradicts Lemma 7.4.

As a corollary we can prove Theorem 12.1:

**Proof of Theorem 12.1.** It clearly suffices to show that $|U_{\leq \lambda}| \leq (2^\lambda)^+$ for all cardinals $\lambda$. By induction assume that $|U_{\leq \lambda}| \leq (2^\lambda)^+$ for all cardinals $\bar{\lambda} < \lambda$.

Assume first that $\lambda$ is not Fréchet. Then any countably complete ultrafilter on $\lambda$ concentrates on a set $A \in [\lambda]^{<\lambda}$. Therefore there is a surjection from the set of pairs $(A, U)$ where $A \in [\lambda]^{<\lambda}$ and $U$ is a countably complete ultrafilter on $A$ to $U_{\leq \lambda}$. The number of such pairs is bounded by

$$[\lambda]^{<\lambda} \cdot \sup_{\bar{\lambda} < \lambda} |U_{\leq \bar{\lambda}}| \leq 2^\lambda \cdot (2^\lambda)^+$$

Thus $|U_{\leq \lambda}| \leq (2^\lambda)^+$.

Now assume that $\lambda$ is Fréchet. We claim that for any ultrafilter $U \in U_{\leq \lambda}$ has at most $2^\lambda$-many $<_E$-predecessors. By Lemma 12.7, $U$ has at most $\prod_{\alpha < \lambda} |U_{\leq \alpha}|$ predecessors. But by Theorem 12.9, for all $\alpha < \lambda$, $|U_{\leq \alpha}| \leq 2^\lambda$, so

$$\prod_{\alpha < \lambda} |U_{\leq \alpha}| \leq (2^\lambda)^\lambda = 2^\lambda$$

Thus $U$ has at most $2^\lambda$-many $<_E$-predecessors. Since $U_{\leq \lambda}$ is wellordered by $<_E$ with initial segments of length at most $2^\lambda$, $|U_{\leq \lambda}| \leq (2^\lambda)^+$.  

We remark that it is not hard to use this fact to prove that GCH holds above the least strongly compact cardinal. This proof does not yield a result that is as local as the one in [3].

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