A Class of Even Walks and Divergence of High Moments of Large Wigner Random Matrices*†‡

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Abstract
We consider the Wigner ensemble of $n \times n$ random matrices $\hat{A}^{(n)}$ with truncated to the interval $(-U_n, U_n)$ elements and study the moments $M_{2s}^{(n)} = E \text{Tr} (\hat{A}^{(n)})^{2s}$ by using their representation as the sums over the set of weighted even closed walks $W_{2s}$.

We construct a subset $W_{2s}' \subset W_{2s}$ such that the corresponding sub-sum diverges in the limit $n, s \to \infty$, $s_n = O(n^{2/3})$ for any truncation of the form $U_n = 2n^{1/6+\epsilon}$ with $\epsilon > 0$, provided the probability distribution of the matrix elements $a_{ij}$ belongs to a class of distributions with the twelfth moment unbounded. This result allows us to put forward a conjecture that the existence of $E |a_{ij}|^{12}$ is necessary for the existence of the universal upper bound of the sequence $M_{2s}^{(n)}$ as $n \to \infty$ and eventually for the edge spectral universality in Wigner ensembles or random matrices.

Running title: Even Walks and Divergence of Moments

1 Wigner ensembles of random matrices

The Wigner ensemble of random matrices is given by the family of real symmetric (or Hermitian) random matrices $\{A^{(n)}\}$ with the matrix elements

$$A_{ij}^{(n)} = \frac{1}{\sqrt{n}} a_{ij}, \quad i, j = 1, \ldots, n,$$

where $\{a_{ij}, 1 \leq i \leq j\}$ are real jointly independent random variables of the same law such that

$$E a_{ij} = 0, \quad \text{and} \quad E |a_{ij}|^2 = v^2.$$  

We denote $V_{2k} = E |a_{ij}|^{2k}$ for $k \geq 2$ and assume that the probability distribution of random variables $a_{ij}$ is symmetric.

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We study the moments of the Wigner random matrices that are obtained from (1.1) by the standard truncation procedure. Namely, let us consider the truncated variables
\[ \hat{a}_{ij}^{(n)} = \begin{cases} a_{ij}, & \text{if } |a_{ij}| < U_n, \\ 0, & \text{if } |a_{ij}| \geq U_n \end{cases} \]
and determine matrices
\[ \hat{A}_{ij}^{(n)} = \frac{1}{\sqrt{n}} \hat{a}_{ij}^{(n)}, \quad i, j = 1, \ldots, n. \]
We will refer to the family \( \{ \hat{A}_n \} \) as to the ensemble of truncated Wigner random matrices. In the case of Hermitian matrices we assume that \( a_{ij} = b^{(1)}_{ij} + ib^{(2)}_{ij} \), where \( \{ b^{(l)}_{ij}, l = 1, 2 \} \) are i.i.d. random variables such that (1.2) holds and consider truncation of \( b^{(l)}_{ij} \).

Since the pioneering works of E. Wigner [9], one extensively uses the natural representation of the averaged moments of (1.1) as the sum over a family of closed paths of \( 2s \) steps. More precisely, one can write equality
\[ \hat{M}_{2s}^{(n)} = \frac{1}{n^s} \sum_{I_{2s} \in \mathcal{I}_{2s}^{(n)}} \Pi(I_{2s}), \]
where \( I_{2s} = (i_0, i_1, \ldots, i_{2s-1}, i_0) \), \( \mathcal{I}_{2s}^{(n)} \) is the set of all even closed trajectories \( I_{2s} \) over the set \( \{1, \ldots, n\} \) and \( \Pi(I_{2s}) \) is the weight given by the mathematical expectation of the product of \( a_{ij} \)'s that corresponds to \( I_{2s} \),
\[ \Pi(I_{2s}) = E \left( \hat{a}^{(n)}_{i_0, i_1} \cdots \hat{a}^{(n)}_{i_{2s-1}, i_0} \right). \]
The trajectory \( I_{2s} \) can be viewed as an \( n \)-realization of an even closed walk \( w_{2s} \in \mathcal{W}_{2s} \) such that in the corresponding multigraph \( g_{2s} = g(w_{2s}) \) each couple of vertices is joined by even number \( 2l \) of edges, \( l \geq 0 \) and the total number of edges is \( 2s \). For the rigorous definitions of trajectories, walks and their graphs see [3].

In paper [3], it is proved that if there exists \( \delta_0 > 0 \) such that \( E |a_{ij}|^{12+\delta_0} < \infty \), then there exists a truncation \( U_n = n^{1/6-\varepsilon}, \varepsilon > 0 \) such that
\[ P \left\{ A^{(n)} \neq \hat{A}^{(n)} \text{ infinitely often} \right\} = 0 \]
and that the moments (1.3) are bounded from above
\[ \limsup_{n,s_n \to \infty} \frac{1}{4v^2s_n} \hat{M}_{2s_n}^{(n)} \leq \mathcal{L}(\vartheta), \quad s_n = \lfloor \vartheta n^{2/3} \rfloor, \quad \vartheta > 0 \]
where \( \mathcal{L}(\vartheta) \) is a universal constant that does not depend on the particular values of \( V_{2k}, 1 \leq k \leq 6 \). Relation (1.4) is obtained by the standard application to random matrices the Borel-Cantelli lemma, and the bound (1.5) is proved with the help of a completed and improved version of the method developed by Ya. Sinai and A. Soshnikov [5, 6, 7] to study the sum (1.3).

In the present note we consider the inverse situation. We prove that if the probability distribution of \( a_{ij} \) is such that \( E |a_{ij}|^{12} \) does not exist, then for any truncation
Theorem 1. Let the probability distribution of \( a_{ij} \) \((1.2)\) has a symmetric density \( f(x) = F'(x) \) of the form \((1.7)\). Then for any choice of \( \epsilon > 0 \) in \( U_n = 2n^{1/6+\epsilon} \), the high moments of the corresponding Wigner ensemble of truncated matrices diverge; namely,

\[
\limsup_{n \to \infty} \frac{1}{(4n^2)^{s_n}} M_{2s_n}^{(n)} = +\infty,
\]

where \( s_n = \lfloor \vartheta n^{2/3} \rfloor \) with any \( \vartheta > 0 \).

Let us complete this section by the following comments. By itself, Theorem 1 does not imply the necessity of the twelfth moment for the bound \((1.5)\). However, one can see that in many cases the truncation \( \bar{a}_{ij}^{(n)} \) combined with the reasoning based on the Borel-Cantelli lemma leads to the optimal conditions for the moments of \( a_{ij} \) (see the papers [2] and [4] for the necessary and sufficient conditions for the semicircle law to be valid; see [1] for the convergence of the spectral norm).

Therefore in view of Theorem 1, it is natural to put forward a hypothesis that the bound \( \mathbb{E}|a_{ij}|^{12} < +\infty \) represents the necessary condition for the existence of the upper bound \((1.5)\). At the end of the present paper, we discuss the optimal conditions on \( a_{ij} \) in asymptotic regimes different from that of \((1.5)\).

Finally, let us note that the bound \((1.5)\) and its proof play the central role in the demonstration of the universality of the edge spectral distribution of large Wigner random matrices [7]. Thus one can relate our results with the necessary and sufficient conditions of the spectral universality in the spectral theory of random matrices.

2 Proof of Theorem 1

Let us briefly explain the structure of the elements \( w_{2s} ' \in \mathcal{W}_{2s} ' \) of \((1.6)\) which is based on a fairly simple principle. We assume that each walk \( w_{2s} ' \) consists of two parts; during the first half-part it creates a vertex \( \zeta_0 \) of high self-intersection degree, where
edges of multiplicity 10 meet each other. On its second part, the walk $w'_2s$ enters $\zeta_0$ and then performs a sequence of "there-and-back" trips along the multiple edges. The vanishingly small proportion of the half-part trajectories with $\zeta_0$ of this kind is compensated by the large number of choices where to go by these there-and-back steps form the second half-part. The cardinality of trajectories of this structure is such that it is impossible to get a finite upper bound for the sub-sum (1.6) with $U_n = 2n^{1/6+\epsilon}$, $\epsilon > 0$.

2.1 Construction of the walks $\mathcal{W}_{2s}'$

Let us split the time interval $[0, 2s]$ into two parts, the $X$-part containing $2s' + 2$ steps and the $Y$-part containing $2L$ steps with obvious equality $2s' + 2 + 2L = 2s$. The chronological order is as follows: the first $2s' + 1$ steps belong to the $X'$-part; these are followed by $2L$ steps of the $Y$-part and the final step $[2s - 1, 2s]$ is attributed again to the $X$-part. We determine in a natural way the sub-walks that correspond to the $X$- and $Y$-parts of $w_{2s}$ and denote them by $w^{(X)}_{2s}$ and $w^{(Y)}_{2s}$, respectively. We also consider the first $2s'$ steps as the $X'$-part of the walk (see figure 1).

![Figure 1: Partition of $[0, 2s]$ into $X'$- and $Y$-parts and Catalan structures](image)

2.1.1 Catalan structures on $X'$-part

Regarding the $X'$-part, one can create a Catalan structure by attributing to the $2s'$ steps $s'$ signs " + " and $s'$ signs " − " [8]. Following [5], we will say that each step labelled by + is marked, the remaining steps being the non-marked ones. The Catalan structure is in one-to-one correspondence with the set $T_{s'}$ of plane rooted trees $T$ constructed with the help of $s'$ edges or equivalently, with the set of the Dyck paths [8]. This correspondence can be determined by the lexicographic (or in other words, by the chronological) run over the element $T \in T_{s'}$ that produces also a Dyck path. In this representation, the signs + and − correspond to the ascending and the descending steps of the Dyck path, respectively.

It is known that the cardinality of $T_{s'}$ is given by the Catalan number,

$$|T_{s'}| = C(s') = \frac{(2s')}{s'!(s'+1)!}. \quad (2.1)$$

In our construction, we are mainly related with a subset $\tilde{T}_{s'}$ of Catalan structures on $X'$-part such that the majority of vertices of the corresponding trees have the degree 6 with 5 children edges. To describe this subset, we introduce the number $s'' = \lfloor s'/5 \rfloor$ and consider the set of all Catalan trees $T_{s''}$. Given $T \in T_{s''}$, we start
the lexicographic run over $T$ and after each descending step — we add the sequence $(+, -, +, -, +, +, +, -)$. Then we get a Dyck path of $10s''$ steps. We denote by $T_{s'} \{ T_{s''} \}$ the set of all Dyck paths obtained by this procedure. To get the subset $\tilde{T}_{s'}$, we consider the elements of $T_{s'} \{ T_{s''} \}$ added by all possible Catalan structures on the remaining $2s' - 10s''$ steps.

The final specification is that we are going to use the subset of trees $\tilde{T}_{s'}^{(d_0)} \subset T_{s''}$ that have vertices with the number of children not greater than a given value $d_0 > 2$.

The following estimate from below

$$|\tilde{T}_{s'}^{(d_0)}| \geq \left( 1 - (2s'' + 1)e^{-(d_0-2)\ln(4/3)} \right) C(s'')$$  \hspace{1cm} (2.2)

can be proved, in particular, by using the recurrence relations for $C(s')$ (see [3]).

The subset we need $\tilde{T}_{s'}^{(d_0)}$ is constructed with the elements of $T_{s'} \{ T_{s''}^{(d_0)} \}$ completed on the remaining $2s' - 10s''$ steps by all possible Catalan structures. We will say that the edges of $T = \tilde{T}_{s'}^{(d_0)}$ represent the principal edges of the corresponding element of $\tilde{T}_{s'}^{(d_0)}$, while the others are the supplementary ones.

Using (2.1) with $s'$ and taking into account inequalities

$$\sqrt{2\pi k} \left( \frac{k}{e} \right)^k \leq k! \leq e\sqrt{2\pi k} \left( \frac{k}{e} \right)^k,$$  \hspace{1cm} (2.3)

we can write that in the limit of large $s'$,

$$|\tilde{T}_{s'}| \geq \frac{(2s'!)!}{s'^!(s'' + 1)!} \geq \frac{4^{s'/5+3}}{e^2 \sqrt{\pi}} \cdot \frac{1}{(s' + 1)^{3/2}}.$$

Then we can deduce from (2.2) the estimate

$$|\tilde{T}_{s'}^{(d_0)}| \geq \left( 1 - s''e^{-(d_0-2)\ln(4/3)} \right) \frac{4^{s'/5+3}}{2e^2 \sqrt{\pi}} \cdot \frac{1}{(s' + 1)^{3/2}}. \hspace{1cm} (2.4)$$

### 2.1.2 Self-intersections on $X'$-part

Regarding the $X'$-part of $w_{2s}$, we will say that the graph $\tilde{g}(w_{2s}^{(X)})$ with simple non-oriented edges represents the frame of the walk $w_{2s}$ that we denote by $F(w_{2s})$. Using the Catalan structure on the $X'$-part, we construct $w_{2s}^{(X)}$ in the following way.

At the zero instant of time, we put the walk at the root vertex $w(0) = \rho$. Here and below we simply say that $\{w_{2s}(t), 0 \leq t \leq 2s\}$ represent the vertices of the graph $g(w_{2s})$ instead of more rigorous but cumbersome expression.

We start the run over the Catalan structure $T$; if the step $[t, t+1]$ is marked, then the walk either creates a new vertex $w(t+1)$ or arrives at one of the already existent vertices. We say that the edge $(w(t), w(t+1))$ is marked. If the step $[t, t+1]$ is such that $w(t+1)$ joins an existing vertex $\beta$, then we say that $t+1$ is the instant of the self-intersection of the walk [5]. The total number of arrivals at $\beta$ at the marked instants of time is called the self-intersection degree of $\beta$; we denote this number by $\kappa(\beta)$. If $\kappa(\beta) = 2$, then we say that $\beta$ is the vertex of simple self-intersection.

If the instant of time $[t, t+1]$ is non-marked, then the walk performs the step along the marked edge $(\beta, \gamma)$ attached to $w(t) = \beta$ such that $(\beta, \gamma)$ is passed odd number of times during the interval $[0, t]$. This marked edge is uniquely determined.
Let us consider a particular tree \( T_{s'} \in \tilde{T}_s^{(d_0)} \) together with its predecessor \( T_{s''} \in T_{s''} \) and choose \( D \) edges \( e_1', \ldots, e_D' \) among the principal edges of this \( T_{s'} \). We want these edges to be such that for any couple \( e_i', e_j' \), the distance between their vertices in \( T_{s''} \) is not less than 3. It is easy to see that the number of possible choices is estimated from below by

\[
\frac{1}{D!} s'' \left( s'' - 2d_{0}^3 \right) \left( s'' - 3d_{0}^3 \right) \cdots \left( s'' - (D - 1)d_{0}^3 \right) \geq \frac{1}{D!} \left( s'' - Dd_{0}^3 \right)^D. \tag{2.5}
\]

Let us denote by \( \tau_1 < \ldots < \tau_D \) the instants of time that correspond to the edges \( e_j', 1 \leq j \leq n \) in \( T_{s'} \). Let us denote by \( \tilde{\tau}_i^{(j)} \), \( 1 \leq i \leq 4 \) the marked instants of time that correspond to the supplementary edges of \( T_s \) that follow after that \( e_j' \) is passed for the second time. With particular values of \( \tau_j \) pointed out, we force the walk to join the same vertex \( \zeta_0 = w(\tau_1) \) at the instants of time \( \tau_j, 2 \leq j \leq D \). Also we oblige the walk to join \( \zeta_0 \) at the instants of time \( \tilde{\tau}_i^{(j)} \). Then we get a walk that has a vertex of the self-intersection degree \( \kappa(\zeta_0) = 5D \); there are \( D \) distinct vertices \( \xi_j \) such that the edge \( (\xi_j, \zeta_0) \) is passed 10 times by \( w_{2s} \) when counted in both directions.

The last stage is to create \( \nu_2 \) simple self-intersections with the help of \( s'' - 5D \) marked steps not used before. This can be done in not more than

\[
\frac{1}{\nu_2!} \left( \frac{(s'' - 5D - 1)(s'' - 5D - 2\nu_2)}{2} \right) \cdots \left( \frac{(s'' - 5D - 2\nu_2 + 1)(s'' - 5D - 2\nu_2)}{2} \right) \geq \frac{1}{\nu_2!} \left( \frac{(s'' - 5D - 2\nu_2)}{2} \right)^{\nu_2}. \tag{2.6}
\]

ways.

The \( X' \)-part of the walk being constructed, we attribute the sign + to the step \( [2s', 2s' + 1] \); the prescription is such that \( w_{2s}(2s' + 1) = \zeta_0 \). It is clear that \( w(2s') = \rho \) and therefore the step \( [2s', 2s' + 1] \) produces the edge \( (\rho, \zeta_0) \). The last step of the \( X \)-part \( [2s - 1, 2s] \) is non-marked; it returns the walk from \( \zeta_0 \) to the root vertex \( \rho \).

### 2.1.3 The core of the walk and \( w_{2s}^{(Y)} \)

![Figure 2: The multiple edges, the core \( Q^{(i)} \) and 6 first steps of the \( Y' \)-part over the core](image)

We see that the walk \( w_{2s} \) at the instant \( t = 2s' + 1 \) enters the vertex \( \zeta_0 \) that belongs to the family of multiple edges \( Q = \{ (\xi_i, \zeta_0), \ i = 1, \ldots, D \} \), where \( \xi_i \neq \xi_j \), each edge being passed by the walk \( 2p = 10 \) times.
We refer to the set of corresponding simple non-oriented edges as to the core of the walk and denote it by $Q^{(p)} = Q^{(5)}$. The $Y$-part $w_{2s}^{(Y)}$ represents the sequence of $L$ "there-and-back" trips of the form $(\zeta_0, \xi_j, \zeta_0)$ over the core (see figure 2).

To get the appropriate estimates for the weights, we obligate the first $2D$ steps of $w_{2s}^{(Y)}$ to perform the ordered trips $(\zeta_0, \xi_1, \zeta_0)$, $(\zeta_0, \xi_2, \zeta_0), \ldots (\zeta_0, \xi_D, \zeta_0)$. As for the remaining $2L - 2D$ steps that determine the $Y'$-part of the walk, there is no restrictions on the choice to which of the vertices $\xi_i$ to go; the total number of all possible walks $w_{2L-2D}'$ is obviously given by $D^{L-D}$.

Regarding the combination of all possible sub-walks $w_{2s}^{(X)}$ and $w_{2s}^{(Y)}$ and using inequalities (2.5) and (2.6), we obtain the following estimate from below for the cardinality of $W_{2s}(d_0, D, L, \nu_2)$:

$$|W_{2s}(d_0, D, L, \nu_2)| \geq |T_{s'}^{(d_0)}| \cdot \frac{(s'' - Dd_0)!}{D!} \cdot \frac{1}{\nu_2^2} \cdot \left(\frac{(s' - 2D - 2\nu_2)^2}{5}\right)^{\nu_2} \cdot D^{L-D}, \quad (2.7)$$

where $s'' = \lfloor s'/5 \rfloor$.

2.2 Trajectories and weights

It is easy to see that the graph $g_{2s} = g(w_{2s})$ of the walk $w_{2s} \in W_{2s}(d_0, D, L, \nu_2)$ we constructed has exactly $s + 1 - 5D - \nu_2 - L$ ordered vertices. Therefore given $w_{2s} \in W_{2s}'$ and assigning different values from the set $\{1, \ldots, n\}$ to the vertices of $w_{2s}$, we get the equivalence class of trajectories $C(w_{2s})$ of the cardinality

$$|C(w_{2s})| = n(n-1) \cdots (n-(s - 5D - \nu_2 - L)).$$

Taking into account the factor $n^{-s'}$, we can write that

$$\frac{1}{n^{s'}} |C(w_{2s})| = \frac{1}{n^{5D + \nu_2 + L}} \cdot \prod_{i=1}^{s - 5D - \nu_2 - L} \left(1 - \frac{i}{n}\right)$$

$$\geq \frac{1}{n^{5D + \nu_2 + L}} \cdot \exp \left\{ - \frac{(s' + 2 - \nu_2 - 5D)^2}{2n} \right\}. \quad (2.8)$$

Indeed, since we are related with the asymptotic regime $s = O(n^{2/3}), n \to \infty$, we can write that

$$\prod_{i=1}^{s - 5D - \nu_2 - L} \left(1 - \frac{i}{n}\right) = \exp \left\{ \sum_{i=1}^{s - 5D - \nu_2 - L} \ln \left(1 - \frac{i}{n}\right) \right\}$$

$$\geq \exp \left\{ - \sum_{i=1}^{s - 5D - \nu_2 - L} \frac{i}{n} \right\} \geq \exp \left\{ - \frac{(s - 5D - \nu_2 - L + 1)^2}{2n} \right\}.$$ 

Then, taking into account equality $s - L = s' + 1$, we get (2.8).

Regarding the weights of the trajectories, we observe that if $I_{2s}'$ and $I_{2s}''$ belong to the same equivalence class $C(w_{2s})$, then

$$\Pi(I_{2s}') = \Pi(I_{2s}'') = \Pi(w_{2s}).$$

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Each edge \((\alpha, \beta)\) of the frame \(F(w_{2s})\) represents a random variable \(\hat{a}_{(\alpha, \beta)}\) and these random variables are jointly independent. Each edge of the frame being passed in both directions by equal number \(k\) of times, we get the factor \(\mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^{2k}]\) without difference between the cases of Hermitian random matrices \(\hat{A}^{(n)}\) and the real symmetric \(\hat{A}^{(n)}\).

If the edge \((\alpha, \beta)\) does not belong to \(Q^{(5)}(w_{2s})\), then it produces either the factor \(\mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^2]\) or the factor \(\mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^4] \geq (\mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^2])^2\). Taking into account that \(\mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^2] \geq v^2/2\) for large values of \(n\), we can write that

\[
\prod_{(\alpha, \beta) \in F \setminus Q^{(5)}} \mathbb{E}[|\hat{a}_{(\alpha, \beta)}|^{2k_{(\alpha, \beta)}}] \geq (v^2/2)^{s-5D}.
\]

Regarding the edges of the core of \(w_{2s}\), we get the factor

\[
\prod_{i=1}^{D} \mathbb{E}[|\hat{a}_{(\xi_i, \zeta_0)}|]^{12+2l_i}, \quad \sum_{i=1}^{D} l_i = L - D,
\]

where \(l_i \geq 0\) are determined by the walk \(w_{2s}\). Elementary computations based on (1.7) show that

\[
\mathbb{E}[|\hat{a}_{(\xi_i, \zeta_0)}|]^{12+2l_i} \geq 2C_\varphi U_n^2 l_i \int_{1/2}^{1} \frac{y^{2l_i-1}}{\ln y + \ln U_n} dy \geq \frac{2C_\varphi}{\ln U_n} \tilde{U}_n^{2l_i}, \tag{2.9}
\]

where \(\tilde{U}_n = U_n/2 = n^{1/6+\epsilon}\).

Summing up, we see that the weight of the walk \(w_{2s} \in W_{2s}^2\) is bounded from below by

\[
\Pi(w_{2s}) \geq (v^2/2)^{s} \cdot \left(\frac{2^6 C_\varphi}{v^{10} \ln U_n}\right)^D \tilde{U}_{n}^{2L-2D}.
\]

Let us note that this bound based on (2.9) could be relaxed; then one gets Theorem 1 in a more general situation than that determined by (1.7). We do not discuss these generalizations here.

Now we are ready to get the estimate from below for the sum \(R_{2s}^{(n)}(1.6)\) and therefore for the moments \(\hat{M}_{2s}^{(n)}(1.3)\).

### 2.3 Estimate from below for \(R_{2s}^{(n)}(D, L)\)

Given the particular values of \(D\) and \(L\) and performing the sum over \(\nu_2\) from 0 to \(\sigma \leq s' - 5D\), we can write that

\[
R_{2s}^{(n)}(D, L) = \sum_{w_{2s} \in W_{2s}^2(d_0, D, L)} n^{-s} |C(w_{2s})| \cdot \Pi(w_{2s})
\]

\[
\geq (v^2/2)^{s} \sum_{\nu_2=0}^{\sigma} |W_{2s}^2(d_0, D, L, \nu_2)| \cdot \exp \left\{ -\frac{(s' + 2 - \nu_2 - 5D)^2}{2n} \right\}
\]

\[
\times \left(\frac{2^6 C_\varphi}{v^{10} \ln U_n}\right) \cdot \tilde{U}_{n}^{2L-2D} \cdot n^{5D+L+\nu_2}. \tag{2.10}
\]
Regarding the sum of (2.6) over $\nu_2$, we assume $\sigma = yn^{1/3}$ and get with the help of (2.2) the following inequalities
\[
\sum_{\nu_2=0}^{\sigma} \frac{1}{\nu_2!} \left( \frac{(s' - 2\nu_2 - 5D)^2}{2n} \right)^{\nu_2} \geq \exp \left\{ \frac{(s' - 2\sigma - 5D)^2}{2n} \right\} - \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{(s' - 2\sigma - 5D)^2}{2n} \right)^{\sigma+l}
\]
\[
\geq \exp \left\{ \frac{(s' - 2\sigma - 5D)^2}{2n} \right\} \left( 1 - \frac{1}{\sqrt{2\pi\sigma}} \left( \frac{e(s' - 2\sigma - 5D)^2}{2n\sigma} \right)^{\sigma} \right)
\]
\[
\geq \exp \left\{ \frac{(s' - 2\sigma - 5D)^2}{2n} \right\} \left( 1 - \left( \frac{e\theta^2}{8y} \right) \right) \geq \frac{1}{2} \exp \left\{ \frac{(s' - 2\sigma - 5D)^2}{2n} \right\}
\]
provided $s' = |s/2 - 1| \leq \delta n^{2/3}/2$ and $y > e\theta^2/4$. Using this inequality, we get from relations (2.7) and (2.10) with the help of (2.3), our main bound
\[
R_{2s}^{(n)}(D, L) \geq (v^2/2)^s |\tilde{T}_s^{(d_0)}| \cdot \frac{1}{e\sqrt{2\pi D}} \left( \frac{e(s'/5 - 1 - \delta_0^3 D)}{Dn^6} \right)^D 
\times \left( \frac{Dn^2}{2n} \right)^{-D} \left( \frac{6\epsilon C_{\varphi}}{v^{10} \ln U_n} \right)^D.
\] (2.11)

Let $L = [\delta n^{2/3}/2]$. Given any positive $\epsilon$ in the truncation $U_n$, we choose $0 < \epsilon' < \epsilon$ and set
\[
D = [n^{2/3-\epsilon'}] + 1.
\] (2.12)

Using (2.4) with $d_0 = n^{\epsilon'/6}$ and taking into account that $L - D \geq (\vartheta/3)n^{2/3}$ for large values of $n$, we deduce from (2.11) the following inequalities
\[
\frac{1}{(v^2)^s} R_{2s}^{(n)}(D, L) \geq \frac{1}{8e^3\pi} \cdot \frac{1}{(\delta n^{2/3} + 1)^{3/2} n^{1/3-\epsilon'/2}} \cdot n^{\delta\epsilon' n^{2/3}}
\times \left( \frac{64\epsilon C_{\varphi}}{v^{10} (\ln n + \ln 2)n^6} \left( \frac{\vartheta}{10} - \frac{1}{n^{\epsilon'/2}} - \frac{2}{n^{2/3}} \right) \right)^{n^{2/3-\epsilon'}}
\geq C_1 \exp \left\{ n^{2/3} \left( \frac{\vartheta\epsilon'}{3} \ln n - \ln 4 \right) - n^{2/3-\epsilon'} ((6 \ln n + \ln \ln n) + C_2) - \frac{4}{3} \ln n \right\},
\]
where $C_1$ and $C_2$ are the constants that depend on $v$, $\vartheta$ and $C_{\varphi}$ only. The last expression obviously diverges as $n \to \infty$. Theorem 1 is proved.

### 3 Concluding remarks

Let us look at the main bound (2.11) and consider the principal factor
\[
\left( \frac{DU_n^2}{4n} \right)^{L-D}
\] (3.1)
that causes the divergence of $R_{2n}^{(n)}(D, L)$ for $D = O(n^{2/3-\epsilon'})$ and $L = O(n^{2/3})$, $n \to \infty$. We see that if one switches from the asymptotic regime $s_n = O(n^{2/3})$ to another one given by $s_n = O(n^{\eta})$ with $0 < \eta < 2/3$, then the truncation of the form
\[
U_n = n^{1/2 - \eta/2}
\] (3.2)
represents the critical exponent beyond that $R_{2s}(D, L)$ is divergent. This truncation guarantees (1.4) provided $E|a_{ij}|^{4/(1-\eta)+\delta} = V_{4/(1-\eta)+\delta}$ exists with $\delta > 0$. The moments (1.3) can be studied in this situation by using the combination of the method of [3] with the corresponding frame representation of the graphs of walks. The divergence of the moments in the case of $E|a_{ij}|^{4/(1-\eta)} = +\infty$ can be proved by construction of the class of walks of the type $W_{2s}'$ with the appropriately determined core $Q^{(p)}$ that gives the factors of the form (3.1).

Regarding the asymptotic regime $s_n = [\ln n]$ used in the studies of the spectral norm of $A^{(n)}$, we observe that the critical value for $U_n$ (3.2) is related with the exponent 1/2 that requires the existence of the fourth moment $E|a_{ij}|^4$. This condition is shown to be sufficient for the existence of $\limsup_{n \to \infty} ||A^{(n)}||$ obtained with the help of (1.4) and (1.5) with $s_n = O(\ln n)$, and also is proved to be the necessary one [1].

These observations reveal once more a special role played by the walks $W_{2s}'$ in the studies of high moments of Wigner random matrices. The interpolation argument with respect to the asymptotic regimes considered above also gives more support to our conjecture that the existence of the twelfth moment $E|a_{ij}|^{12}$ is the necessary and sufficient condition for the existence of the universal bound $L(\theta)$ of (1.5) as well as for the edge spectral universality of the Wigner ensembles of random matrices.

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