A note on four-point functions of conformal operators in $N = 4$
Super-Yang Mills.

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Abstract

We find that the first-order correction to the free-field result for the four-point function of the conformal operator $\text{Tr}(\phi^i \phi^j)$ is nonvanishing and survives in the limit $N_c \to \infty$.

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I. Introduction and conclusion

Since the conjecture by Maldacena that gauged supergravity on an anti-de Sitter background is dual to the large $N_c$, large $g^2_{YM}N_c$ limit of conformal field theories, many efforts have been made to test the conjecture as well as to extract predictions about the behaviour of the conformal field theories in this limit. In the last class falls the recent conjecture by Intriligator \cite{1} that the S-duality of type IIB supergravity on $\text{AdS}_5 \times S_5$ constrains the correlation functions of conformal operators $\mathcal{O}_i$ in short representations of the conformal group of the dual CFT: $d = 4 \ N = 4$ super-Yang-Mills. In the limit of small $g_{YM}$ and large $g^2_{YM}N_c$ they must obey the selection rule

$$\langle \prod_{i=1}^{n} \mathcal{O}^{(q_i)}_i(x_i) \rangle = 0 \text{ unless } \sum_i q_i = 0 \quad (1)$$

The $q_i$ are the charges of a $U(1)_Y$ bonus symmetry of the correlation functions, present in the double limit $g^2_{YM}, N \to \infty$. This $U(1)_Y$ is the maximal compact subgroup of the $SL(2, \mathbb{Z})$ S-duality group.

An attempt to prove the validity of this selection rule using harmonic $N = 4$ superspace in the last section of the paper \cite{1} found the suspicious result that this symmetry should hold for all values of $g^2_{YM}$ and $N_c$. This would in turn imply that all correlation functions are given by their free field values. In this brief note we merely want to point out that it is very easy to construct a counter-example. This is perhaps a little superfluous as it was already pointed out in the literature that $n \geq 4$ correlation functions receive $1/g^2_{YM}N_c$ corrections from the AdS point of view \cite{2}. This should be enough evidence to preclude that the answer is free field theory. That $N = 4$ SYM is interacting is also evident from the instanton calculations in \cite{3}. These corrections are subleading in $N_c$, however, and conceivably the theory could still simplify in the limit $N_c \to \infty$. The calculation presented below, that the four-point function of the $\text{Tr} \phi^a \phi^b$ conformal operator receives planar diagram corrections at order $g^2_{YM}N_c$ corroborates the AdS result and strengthens the instanton calculations from the weak coupling point of view: $N = 4$ SYM at large $N_c$ is interacting.

II. Computation of the correlator $\langle (\text{Tr} \phi^4) \rangle$

We will perform our calculation in $N = 1$ superspace. In $N = 0$ components the conformal operator $\text{Tr} \phi^a \phi^b$ transforms in the $20'$ representation of $R$-symmetry group $SU(4)_R \simeq SO(6)$. The $\phi^a$ are the six real scalar fields of $N = 4$ SYM transforming in the adjoint of the gauge group $SU(N_c)$; $a$ is an $SO(6)$ vector index.
In $N = 1$ superfields these operators belong to the three supermultiplets $\text{Tr}\Phi_i\Phi_j$, $\text{Tr}\Phi_i e^V \bar{\Phi}^j$, $\text{Tr}\Phi^i\bar{\Phi}^j$ transforming in the $6$, $8$ and $\bar{6}$ of the manifest $SU(3)$ flavor symmetry of $N = 4$ SYM in $N = 1$ language. The $e^V$ in $\text{Tr}\Phi_i e^V \bar{\Phi}^j$ is necessary for gauge invariance of the operator.\footnote{One may of course covariantize the antichiral field $\bar{\Phi}$ to a “vector” representation $\bar{\Phi} \rightarrow e^V \bar{\Phi}$, in which case the operators look more symmetric.}

The $N = 1$ Lagrangian is

$$\mathcal{L} = -\int d^2 \theta \frac{1}{2} \text{Tr} W^a W_a - \int d^2 \theta \text{Tr} \Phi_1 [\Phi_2, \Phi_3] + \int d^2 \bar{\theta} \text{Tr} \Phi^1 [\Phi^2, \Phi^3] + \int d^2 \theta d^2 \bar{\theta} \text{Tr} \Phi^i e^V \bar{\Phi}^i,$$

and we include a source term for the conformal operators

$$\mathcal{L}_{\text{source}} = \int d^2 \theta J^{ij} \text{Tr} \Phi_i \Phi_j + \int d^2 \bar{\theta} \bar{J}^{ij} \text{Tr} \Phi^i \bar{\Phi}^j.$$

The field strength equals $W_a = \bar{D}^2 e^{-V} D_a e^V$ in terms of the prepotential $V$. All fields are in the adjoint representation of the gauge group $SU(N_c)$ and we have chosen antihermitian generators, i.e. $\text{Tr} T^A T^B = -\delta^{AB} C_2(\text{Ad}) = -\delta^{AB} N_c$. The full Lagrangian, including the source term, $\mathcal{L} + \mathcal{L}_{\text{source}}$, is multiplied by the inverse of the coupling constant squared $1/g^2_{YM} N_c$. This corresponds to the normalization of the conformal operator $O_{ij} = \frac{1}{g^2_{YM} N_c} \text{Tr} \Phi_i \Phi_j$ where the free-field result is $g^2_{YM} N_c$ independent.

It will suffice for our purposes to compute the single correlator $\langle O_{11} O_{22} \bar{O}_{11} \bar{O}_{22} \rangle$ and show that its first-order correction is nonzero. The connected free-field result for this correlator, given by figure 1, vanishes as all the propagators are diagonal in the flavor group $SU(3)$.

The first-order corrections are given by Feynman diagrams of the form given in figure 2. We may ignore self-energy corrections to the propagator as in $N = 4$ there is no wave-function renormalization at one-loop. In principle one should also include all the connected but not 1PI diagrams. However, these graphs vanish by virtue of the tracelessness of the $SU(N_c)$ structure constants. Moreover, the couplings to the vector multiplet are diagonal in flavors and therefore the graphs in figures 2b, 2c and 2d again vanish trivially. Thus the sole graph in figure 3a remains to be evaluated.

The effective action for the sources $J^{ij}$ and $\bar{J}_{ij}$ which is responsible for this diagram is given by

$$S_{\text{eff}} = -\frac{16 g^2_{YM} N_c}{C_2(\text{Ad})} f_{abc} f^{abc} \int \prod_{i=1}^6 d^8 z_i \delta^{8 \delta_{13}} D^2 \bar{D}^2 \delta^8_{16} D^2 \bar{D}^2 \delta^8_{35} \delta^8_{56} D^2 \bar{D}^2 \delta^8_{26} \delta^8_{24} D^2 \bar{D}^2 \delta^8_{45} \times J^{11}(z_1) J^{22}(z_2) \bar{J}_{11}(z_3) \bar{J}_{22}(z_4).$$

\footnote{One may of course covariantize the antichiral field $\bar{\Phi}$ to a “vector” representation $\bar{\Phi} \rightarrow e^V \bar{\Phi}$, in which case the operators look more symmetric.}
We perform the $D$-algebra such that only terms that have the structure $J \bar{J} D^2 J \bar{J}$ contribute to the scalar components of $O$ and keep only these terms. We find

$$S_{\text{eff}} = 16 g_Y^2 N_c (N_c^2 - 1) \int \prod_{i=1}^{6} d^4 z_i \frac{\delta_{13}^8 \delta_{16}^8 \delta_{35}^8 \delta_{56}^8 \delta_{24}^8 \delta_{45}^8}{\Box \Box \Box \Box \Box \Box} \times J^{11}(z_1) D^2 J^{22}(z_2) J_{11}(z_3) D^2 J_{22}(z_4) + \ldots$$

The Grassmann integrations may be performed and as usual the final expression is local in Grassmann space. For $N_c \gg 1$, $g_Y^2 N_c$ fixed, the result translates into the nonvanishing first-order correction to the correlation function

$$\langle O_{11}(z_1) O_{22}(z_2) \bar{O}^{11}(z_3) \bar{O}^{22}(z_4) \rangle|_{\theta = 0} = 16 g_Y^2 N_c (N_c^2 - 1) \int \prod_{i=1}^{6} d^4 x_i \frac{1}{x_{1a}^2 x_{2a}^2 x_{3a}^2 x_{4a}^2} \frac{1}{x_{13a}^2 x_{16a}^2 x_{35a}^2 x_{56a}^2} \frac{1}{x_{24a}^2 x_{24a}^2} \frac{1}{x_{2}^2 x_{3}^2 x_{4}^2} \frac{1}{x_{13}^2 x_{16}^2 x_{35}^2 x_{56}^2} \frac{1}{x_{24}^2 x_{24}^2} \times J^{11}(z_1) D^2 J^{22}(z_2) J_{11}(z_3) D^2 J_{22}(z_4) + \ldots$$

where $2F_1(1, 1; 2; 1 - y) = - \ln(y)/(1 - y)$ is a special case of the hypergeometric function. Eq.(6) is the expected result for a four-point correlator of conformal operators [4].

From eq.(6) one sees that the component expression for the correlator of $O_{ij}|_{\theta = 0} = \text{Tr} \varphi_i \varphi_j$ ($\varphi_i$ is a complex combination of the six $\phi^a$) receives only a contribution from the exchange of auxiliary fields. Integrating these out in the component form of the
action (3) yields the effective interaction

\[ S^{(N=0)\text{int}} = \ldots + \text{Tr}[\phi_i, \bar{\phi}^j][\phi_j, \bar{\phi}^i] - 2\text{Tr}[\phi_i, \phi_j][\bar{\phi}^i, \bar{\phi}^j] \]  \hspace{1cm} (7)

This interaction is responsible for the diagram in figure 3b. One sees immediately that its contribution is exactly that of eq. (6).

As a last comment we should emphasize that because the free-field result is vanishing, the conclusion that $N = 4$ SYM at large $N_c$ is interacting is robust. The corrections cannot be compensated by redefining or re-normalizing the operators.

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3a. 3b.

Figure 3: Sole contribution to the correlator $\langle \mathcal{O}_{11} \mathcal{O}_{22} \bar{\mathcal{O}}^{11} \bar{\mathcal{O}}^{22} \rangle$ in superfields and components

References

[1] K. Intriligator, “Bonus Symmetries of $N = 4$ Super-Yang-Mills Correlation Functions via AdS duality”, hep-th/9811047.

[2] T. Banks, M. Green, J. High Energy Phys. 05 (1998) 002, hep-th/9804170.

[3] M. Bianchi, M. Green, S. Kovacs, G. Rossi, J. High Energy Phys. 08 (1998) 013, hep-th/9807033.
   N. Dorey, T. Hollowood, V. Khoze, M. Mattis and S. Vandoren, hep-th/9810243.

[4] P. Di Francesco, P. Mathieu, D. Sénéchal, *Conformal Field Theory*, Springer Verlag (1997).