Proof of a conjecture on the determinant of the walk matrix of rooted product with a path

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\section*{ABSTRACT}

The walk matrix of an \( n \)-vertex graph \( G \) with adjacency matrix \( A \), denoted by \( W(G) \), is \([e, Ae, \ldots, A^{n-1}e]\), where \( e \) is the all-ones vector. Let \( G \circ P_m \) be the rooted product of \( G \) and a rooted path \( P_m \) (taking an endvertex as the root), i.e. \( G \circ P_m \) is a graph obtained from \( G \) and \( n \) copies of \( P_m \) by identifying each vertex of \( G \) with an endvertex of a copy of \( P_m \). Mao et al. [A new method for constructing graphs determined by their generalized spectrum. Linear Algebra Appl. 2015;477:112–127.] and Mao and Wang [Generalized spectral characterization of rooted product graphs. Linear Multilinear Algebra. 2022. DOI:10.1080/03081087.2022.2098226.] proved that, for \( m = 2 \) and \( m \in \{3, 4\} \),

\[
\det W(G \circ P_m) = \pm a_0 m \left( \det W(G) \right)^m,
\]

where \( a_0 \) is the constant term of the characteristic polynomial of \( G \). Furthermore, in the same paper, Mao and Wang conjectured that the formula holds for any \( m \geq 2 \). In this paper, we verify this conjecture using the technique of Chebyshev polynomials.

\section{1. Introduction}

Let \( G \) be a simple graph with vertex set \([1, \ldots, n]\). The \textit{adjacency matrix} of \( G \) is the \( n \times n \) symmetric matrix \( A = (a_{ij}) \), where \( a_{ij} = 1 \) if \( i \) and \( j \) are adjacent; \( a_{ij} = 0 \) otherwise. We use \( \phi(G; x) \) to denote the characteristic polynomial of \( G \), i.e. \( \phi(G; x) = \det(xI - A(G)) \). The zeroes of \( \phi(G; x) \) are called the eigenvalues (spectrum) of \( G \).

For a graph \( G \), the \textit{walk matrix} of \( G \) is

\[
W = W(G) := [e, Ae, \ldots, A^{n-1}e],
\]

where \( e \) is the all-ones vector. Note that walk matrices are clearly integral but are usually not symmetric. Compared to general integral matrices, walk matrices of graphs have
some special properties. For example, the determinant of any walk matrix of order \( n \) is always a multiple of \( 2^\left\lfloor \frac{n}{2} \right\rfloor \); see [1,2]. This kind of matrix has attracted increasing attention in recent years as many interesting properties of graphs are closely related to the corresponding walk matrices. A typical result is a theorem of Wang [3] which says that any graph \( G \) with \( 2^{-\left\lfloor \frac{n}{2} \right\rfloor} \det W(G) \) odd and square-free is uniquely determined, up to isomorphism, by its generalized spectrum (DGS for short). Here, the generalized spectrum of a graph \( G \) means the spectrum of \( G \) together with that of its complement \( \overline{G} \). For more recent studies involving walk matrices, we refer to [2,4,5].

In [6], in order to construct DGS-graphs, the authors considered the rooted product graph \( G \circ P_2 \) and proved the following theorem. Throughout this paper, we shall fix a graph \( G \) and use \( a_0 \) to denote the constant term of the characteristic polynomial of \( G \).

**Theorem 1.1 ([6]):** If \( G \) is a graph, then
\[
\det W(G \circ P_2) = \pm a_0(\det W(G))^2.
\]

Recently, Mao and Wang [7] extended the above formula to \( G \circ P_m \) for \( m = 3, 4 \). Precisely, they proved:

**Theorem 1.2 ([7]):** If \( G \) is a graph and \( m = 3 \) or \( 4 \), then
\[
\det W(G \circ P_m) = \pm a_0^\left\lfloor \frac{m}{2} \right\rfloor (\det W(G))^m.
\]

In the same paper, the authors proposed the following natural conjecture, which (if true) unifies and extends the above two theorems.

**Conjecture 1.1 ([7]):** If \( G \) is a graph and \( m \geq 2 \), then
\[
\det W(G \circ P_m) = \pm a_0^\left\lfloor \frac{m}{2} \right\rfloor (\det W(G))^m.
\]

The main aim of this paper is to confirm this conjecture.

**Theorem 1.3:** Conjecture 1.1 is true.

The overall strategy of the proof is the same as in [7] which is based on explicit computations of eigenvalues and eigenvectors. Indeed, most derivations in [7] for \( m = 3, 4 \) can be easily extended to any positive \( m \) except for some resultant-related computations. A new finding of this paper is that most of the computations involved are closely related to Chebyshev polynomials of the second kind. The final proof of Theorem 1.3 is given in Section 4 and, as a direct consequence of Theorem 1.3, the paper concludes with a method to construct large DGS-graphs from small ones.

It is worth mentioning that the Chebyshev polynomials have been successfully used to study the walk matrices of two other kinds of graphs; see [8,9].
2. Eigenvalues and eigenvectors of $G \circ P_m$

We always regard the endvertex of $P_m$ as its root vertex. For an $n$-vertex labelled graph $G$, the *rooted product graph* $G \circ P_m$, is a graph obtained from $G$ and $n$ copies of $P_m$ by identifying the root vertex of the $i$th copy of $P_m$ with the $i$th vertex of $G$ for $i = 1, 2, \ldots, n$; see Figure 1 for an illustration. This is a special case of rooted product graphs $G \circ H$ introduced by Godsil-McKay [10] and Schwenk [11].

**Definition 2.1:** Let $A = (a_{ij})$ be an $m \times n$ matrix and $B$ a $p \times q$ matrix. The *Kronecker product* $A \otimes B$ is the $pm \timesqn$ block matrix:

$$
A \otimes B = \begin{bmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}.
$$

By appropriately labelling the vertices in $G \circ P_m$, the adjacency matrix $A(G \circ P_m)$ has a nice structure.

**Observation 2.1:** $A(G \circ P_m) = A(P_m) \otimes I_n + D_1 \otimes A(G)$, where $I_n$ is the identity matrix of order $n$ and $D_1$ is the diagonal matrix $\text{diag}(1, 0, \ldots, 0)$ of order $m$.

The following lemma is a special case of a decomposition of $\phi(G \circ H; x)$ derived by Schwenk [11] (see also [10,12]).

**Lemma 2.1:** $\phi(G \circ P_m; x) = (\phi(P_{m-1}; x))^n\phi(G; \frac{\phi(P_m; x)}{\phi(P_{m-1}; x)}) = \prod_{i=1}^n(\phi(P_m; x) - \lambda_i\phi(P_{m-1}; x))$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $G$.

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**Figure 1.** Graph $G$ (left) and the rooted product graph $G \circ P_4$ (right).
Let $U_n(x)$ be the $n$-th Chebyshev polynomial of the second kind, defined by

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$ 

It is known that $U_n(x)$ satisfies the three-term recurrence relation: $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ and the initial conditions: $U_0(x) = 1$ and $U_1(x) = 2x$. Define $S_n(x) = U_n(x)/2$. Then $S_n(x)$ is a monic polynomial with integral coefficients and is referred to as the renormalized Chebyshev polynomial [13]. It is well known that $\phi(P_m; x) = S_m(x)$.

**Definition 2.2:** Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $G$ and $\xi_1, \ldots, \xi_n$ be the corresponding normalized eigenvector. We use $\mu_i^{(j)} (j \in \{1, 2, \ldots, m\})$ to denote all the zeroes of $S_m(x) - \lambda_i S_{m-1}(x)$ for $i \in \{1, 2, \ldots, n\}$ and write

$$\eta_i^{(j)} = \frac{1}{S_{m-1}(\mu_i^{(j)})} \begin{bmatrix} S_{m-1}(\mu_i^{(j)}) \\ S_{m-2}(\mu_i^{(j)}) \\ \vdots \\ S_0(\mu_i^{(j)}) \end{bmatrix} \otimes \xi_i.$$ 

It should be pointed out that $S_{m-1}(\mu_i^{(j)})$ is never zero: see Corollary 3.1 in Section 3.

**Lemma 2.2:** For any $i \in \{1, 2, \ldots, n\}$, each zero of $S_m(x) - \lambda_i S_{m-1}(x)$ is simple, i.e. $\mu_i^{(j_1)} \neq \mu_i^{(j_2)}$ for any distinct $j_1$ and $j_2$ in $\{1, 2, \ldots, m\}$.

**Proof:** The zeroes of the renormalized Chebyshev polynomials $S_m(x)$ and $S_{m-1}(x)$ are $\{a_k = 2 \cos \frac{kr}{m+1}: 1 \leq k \leq m\}$ and $\{b_k = 2 \cos \frac{k\pi}{m}: 1 \leq k \leq m - 1\}$. Note that $a_1 > b_1 > a_2 > b_2 > \cdots > a_{m-1} > b_{m-1} > a_m$. Moreover, as each zero $a_k$ is a simple zero, we find that the sequence $S_m(+\infty), S_m(b_1), S_m(b_2), \ldots, S_m(b_{m-1}), S_m(-\infty)$ must have alternating signs. Write $f(x) = S_m(x) - \lambda_i S_{m-1}(x)$. Since $S_{m-1}(b_k) = 0$ for $k = 1, 2, \ldots, m - 1$ and $m - 1 = \deg S_{m-1}(x) < \deg S_m(x) = m$, the signs of $S_m(x)$ and $f(x)$ are the same for each $x \in \{b_1, \ldots, b_{m-1}\} \cup [+\infty, -\infty]$. This means that the sequence $f(+\infty), f(b_1), f(b_2), \ldots, f(b_{m-1}), f(-\infty)$ also has alternating signs. By the Intermediate Value Theorem for continuous functions, $f(x)$ has at least one zero in each of the $m$ intervals: $(-\infty, b_{m-1}), (b_{m-1}, b_{m-2}), \ldots, (b_2, b_1)$ and $(b_1, +\infty)$. Since $f(x)$ is a polynomial of degree $m$, we conclude that all zeroes of $f(x)$ are simple, as desired. 

**Lemma 2.3:** Let $\tilde{A}$ denote the adjacency matrix of $G \circ P_m$. Then $\tilde{A}\eta_i^{(j)} = \mu_i^{(j)}\eta_i^{(j)}$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.
Proof: We fix $i$ and $j$ and write $s_k = S_k(\mu_i^{(j)})$ ($k = 0, 1, \ldots, m - 1$) for simplicity. By Observation 2.1 and some basic properties of the Kronecker product, we obtain

$$\tilde{A}_{\eta_i}^{(j)} = \frac{1}{s_{m-1}}(A(P_m) \otimes I_n + D_1 \otimes A(G))((s_m, s_{m-1}, \ldots, s_0) \otimes \xi_i)$$

$$= \frac{1}{s_{m-1}} \left( A(P_m) \begin{bmatrix} s_m \\ s_{m-1} \\ \vdots \\ s_0 \end{bmatrix} \otimes \xi_i + \begin{bmatrix} \lambda_i s_m \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes (\lambda_i \xi_i) \right)$$

$$= \frac{1}{s_{m-1}} \left( A(P_m) \begin{bmatrix} s_{m-2} + \lambda_i s_m \\ s_{m-3} + s_{m-1} \\ \vdots \\ s_0 + s_2 \\ s_1 \end{bmatrix} \otimes \xi_i \right).$$

(2)

By Definition 2.2, we see that $\lambda_i s_{m-1} = s_m$. Noting that $S_1(x) = xS_0(x)$ and $S_k(x) + S_{k+2}(x) = xS_{k+1}(x)$ for any $k \geq 0$, we obtain

$$\begin{bmatrix} s_{m-2} + \lambda_i s_m \\ s_{m-3} + s_{m-1} \\ \vdots \\ s_0 + s_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} s_m \\ s_{m-1} \\ \vdots \\ s_0 \\ s_1 \end{bmatrix} = \mu_i^{(j)}.$$

(3)

Now, Equation (2) becomes

$$\tilde{A}_{\eta_i}^{(j)} = \frac{1}{s_{m-1}} \mu_i^{(j)} \begin{bmatrix} s_m \\ s_{m-1} \\ \vdots \\ s_0 \\ s_1 \end{bmatrix} \otimes \xi_i = \mu_i^{(j)} \eta_i^{(j)}.$$

3. Resultants of Chebyshev-related polynomials

Definition 3.1: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$. The resultant of $f(x)$ and $g(x)$, denoted by $\text{Res}_x(f(x), g(x))$,
or simply $\text{Res}(f(x), g(x))$, is defined to be

$$a_n^m b_m^n \prod_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_i - \beta_j),$$

where $\alpha_i$'s and $\beta_j$'s are the zeroes (in complex field $\mathbb{C}$) of $f(x)$ and $g(x)$, respectively.

We list some basic properties of resultants for convenience.

**Lemma 3.1:** Let $f(x) = a_n x^n + \cdots + a_0 = a_n \prod_{i=1}^n (x - \alpha_i)$ and $g(x) = b_m x^m + \cdots + b_0 = b_m \prod_{j=1}^m (x - \beta_j)$. Then the followings hold:

(i) $\text{Res}(f(x), g(x)) = a_n^m b_m^n \prod_{i=1}^n g(\alpha_i) = (-1)^m b_m^m \prod_{j=1}^m f(\beta_j)$;
(ii) If $m < n$ then $\text{Res}(f(x) + tg(x), g(x)) = \text{Res}(f(x), g(x))$ for any $t \in \mathbb{C}$;
(iii) $\text{Res}(f(tx), g(tx)) = t^{mn} \text{Res}(f(x), g(x))$ for any $t \in \mathbb{C} \setminus \{0\}$.

We need the following result due to Dilcher and Stolarsky [14]; see [15,16] for different proofs.

**Lemma 3.2 ([14]):** For any integer $m \geq 1$,

$$\text{Res}(U_m(x), U_{m-1}(x)) = (-1)^{\frac{m(m-1)}{2}} 2^{m(m-1)}.$$

**Corollary 3.1:** For any integers $m, n \geq 1$ and $i \in \{1, \ldots, n\}$,

$$\prod_{j=1}^m S_{m-1}(\mu_i^{(j)}) = (-1)^{\frac{m(m-1)}{2}}.$$

**Proof:** Recall that $S_{m-1}(x) = U_{m-1}(x/2)$ is a monic polynomial and so is $S_m(x) - \lambda_i S_{m-1}(x)$. Since each $\mu_i^{(j)}$ is a zero of $S_m(x) - \lambda_i S_{m-1}(x)$, we obtain, by Lemma 3.1,

$$\prod_{j=1}^m S_{m-1}(\mu_i^{(j)}) = \text{Res}(S_m(x) - \lambda_i S_{m-1}(x), S_{m-1}(x))$$

$$= \text{Res}(S_m(x), S_{m-1}(x))$$

$$= \left(\frac{1}{2}\right)^{\frac{m(m-1)}{2}} \text{Res}(U_m(x), U_{m-1}(x))$$

$$= (-1)^{\frac{m(m-1)}{2}}.$$

**Lemma 3.3 ([17]):** For any integer $m \geq 1$ and any complex number $t$,

$$\text{Res} \left( U_m(x) + t U_{m-1}(x), \sum_{k=0}^{m-1} U_k(x) \right) = (-1)^{\frac{m(m-1)}{2}} t^{\left\lfloor \frac{m}{2} \right\rfloor} 2^{m(m-1)}.$$
Proof: The proof of this lemma is due to Terrence Tao [17]. Since $U_m(x) + tU_{m-1}(x)$ is of degree $m$ and $\sum_{k=0}^{m-1} U_k(x)$ is of degree $m-1$ with leading coefficient $2^{m-1}$, the resultant factors as

$$(-1)^{m(m-1)} 2^{m(m-1)} \prod_{j=1}^{m-1} (U_m(\beta_j) + tU_{m-1}(\beta_j))$$

where $\beta_1, \ldots, \beta_{m-1}$ are zeroes of $\sum_{k=0}^{m-1} U_k(x)$.

Fortunately, these zeroes can be located explicitly using the usual trigonometric addition and subtraction identities. Telescoping the trigonometric identity

$$\sin k\theta = \frac{\cos (k - \frac{1}{2}) \theta - \cos (k + \frac{1}{2}) \theta}{2 \sin \frac{\theta}{2}}$$

we conclude that

$$\sum_{k=0}^{m-1} U_k(\cos \theta) = \frac{1}{\sin \theta} \sum_{k=1}^{m} \sin k\theta = \frac{\cos \frac{\theta}{2} - \cos \left(m + \frac{1}{2}\right) \theta}{2 \sin \theta \sin \frac{\theta}{2}} = \frac{\sin \frac{m+1}{2} \theta \sin \frac{m}{2} \theta}{2 \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2}}$$

and so the $m - 1 = \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m-1}{2} \rfloor$ zeroes of $\sum_{k=0}^{m-1} U_k(x)$ take the form $\cos \frac{2\pi j}{m+1}$ for $1 \leq j \leq \lfloor \frac{m}{2} \rfloor$ and $\cos \frac{2\pi j}{m}$ for $1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor$.

Since the first class $\cos \frac{2\pi j}{m+1}$ of zeroes are also zeroes of $U_m(x)$, and the second class $\cos \frac{2\pi j}{m}$ are zeroes of $U_{m-1}(x)$, the resultant therefore simplifies to

$$(-1)^{m(m-1)} 2^{m(m-1)} t^{\lfloor \frac{m}{2} \rfloor} \prod_{1 \leq j \leq \lfloor \frac{m}{2} \rfloor} U_{m-1} \left( \cos \frac{2\pi j}{m+1} \right) \prod_{1 \leq j \leq \lfloor \frac{m-1}{2} \rfloor} U_m \left( \cos \frac{2\pi j}{m} \right).$$

But

$$U_{m-1} \left( \cos \frac{2\pi j}{m+1} \right) = \frac{\sin \frac{2\pi mj}{m+1}}{\sin \frac{2\pi j}{m+1}} = -1$$

and similarly

$$U_m \left( \cos \frac{2\pi j}{m} \right) = \frac{\sin \frac{2\pi (m+1)j}{m}}{\sin \frac{2\pi j}{m}} = +1$$

and the lemma then follows after counting up the signs. ■

Using a similar argument as in the proof of Corollary 3.1, we obtain the following

**Corollary 3.2:** For any integers $m, n \geq 1$ and $i \in \{1, \ldots, n\}$,

$$\prod_{j=1}^{m} \sum_{k=0}^{m-1} S_k(\mu_i^{(j)}) = (-1)^{\frac{m(m-1)}{2}} (-\lambda_i) \lfloor \frac{m}{2} \rfloor.$$
4. Proof of Theorem 1.3

Lemma 4.1 ([6]): Let \( \lambda_i \) be the eigenvalues of \( G \) with normalized eigenvector \( \xi_i \) for \( i = 1, 2, \ldots, n \). Then

\[
\det W(G) = \pm \prod_{1 \leq i_1 < i_2 \leq n} (\lambda_{i_2} - \lambda_{i_1}) \prod_{1 \leq i \leq n} (e_n^T \xi_i).
\]

Definition 4.1: Let \( S(x) = \sum_{k=0}^{m-1} S_k(x) \).

The following equality is straightforward.

Lemma 4.2: \( e_n^T \eta_i^{(j)} = \frac{S(\mu_i^{(j)})}{S_{m-1}(\mu_i^{(j)})} e_n^T \xi_i \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

Proof of Theorem 1.3: Let \( \tilde{A} = A(G \circ P_m) \) and let \( 0 \leq k \leq mn - 1 \). By Lemmas 2.3 and 4.2, we have

\[
e_n^T \tilde{A}^k [\eta_1^{(1)}, \ldots, \eta_n^{(1)}, \ldots; \eta_1^{(m)}, \ldots, \eta_n^{(m)}] =
\begin{bmatrix}
(\mu_1^{(1)})^k \\
(\mu_2^{(1)})^k \\
\vdots \\
(\mu_n^{(1)})^k
\end{bmatrix}
\begin{bmatrix}
S(\mu_1^{(1)}) \\
S_{m-1}(\mu_1^{(1)}) e_n^T \xi_1 \\
S(\mu_2^{(1)}) \\
S_{m-1}(\mu_2^{(1)}) e_n^T \xi_2 \\
\vdots \\
S(\mu_n^{(1)}) \\
S_{m-1}(\mu_n^{(1)}) e_n^T \xi_n
\end{bmatrix}.
\]

Let \( D \) denote the diagonal matrix in Equation (4). Let \( E^{(j)} = [\eta_1^{(j)}, \ldots, \eta_n^{(j)}] \) and

\[
M^{(j)} = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\mu_1^{(j)} & \mu_2^{(j)} & \cdots & \mu_n^{(j)} \\
\vdots & \vdots & \ddots & \vdots \\
(\mu_1^{(j)})^{mn-1} & (\mu_2^{(j)})^{mn-1} & \cdots & (\mu_n^{(j)})^{mn-1}
\end{bmatrix}_{(mn) \times n}.
\]

Then, summarizing Equation (4) for \( k \) from 0 to \( mn-1 \) leads to

\[
(W(G \circ P_m))^T [E^{(1)}, \ldots, E^{(m)}] = [M^{(1)}, \ldots, M^{(m)}] D.
\]

Claim 1: For the diagonal matrix \( D \),

\[
\det D = a_0^m \prod_{1 \leq i \leq n} (e_n^T \xi_i)^m,
\]
where $a_0 = (-1)^n \det A(G)$ is the constant term of characteristic polynomial of $G$.

By Corollary 3.1,

$$\prod_{1 \leq i \leq n} \prod_{1 \leq j \leq m} S_{m-1}(\mu_i^{(j)}) = (-1)^{\frac{m(m-1)n}{2}}.$$  

Recall that $S(x) = \sum_{k=0}^{m-1} S_k(x)$. We can rewrite Corollary 3.2 as

$$\prod_{j=1}^{m} S(\mu_i^{(j)}) = (-1)^{\frac{m(m-1)}{2}}(-\lambda_i)^{\left\lfloor \frac{m}{2} \right\rfloor}.$$  

Noting that $\prod_{i=1}^{n}(-\lambda_i) = a_0$,

$$\prod_{i=1}^{n} \prod_{j=1}^{m} S(\mu_i^{(j)}) = (-1)^{\frac{m(m-1)n}{2}} a_0^{\left\lfloor \frac{m}{2} \right\rfloor},$$

and hence Claim 1 follows by direct multiplication of all diagonal entries in $D$.

**Claim 2:** For $V = [M^{(1)}, \ldots, M^{(m)}]$,

$$\det V = \left( \prod_{i=1}^{n} \prod_{1 \leq j < j' \leq m} (\mu_{i_2}^{(j)} - \mu_{i_1}^{(j')}) \right) \left( \prod_{1 \leq i_1 < i_2 \leq n} (\lambda_{i_2} - \lambda_{i_1}) \right)^m.$$  

We use co-lexicographical order for the Cartesian product $\{1, \ldots, n\} \times \{1, \ldots, m\}$. In particular, $(i_1, j_1) < (i_2, j_2)$ (in co-lexicographical order) if either $j_1 < j_2$, or $j_1 = j_2$ and $i_1 < i_2$.

Using the familiar formula for a Vandermonde matrix, we obtain

$$\det V = \prod_{(i_1, j_1) < (i_2, j_2)} (\mu_{i_2}^{(j_2)} - \mu_{i_1}^{(j_1)})$$

$$= \left( \prod_{i=1}^{n} \prod_{1 \leq j < j' \leq m} (\mu_{i}^{(j)} - \mu_{i}^{(j')}) \right) \left( \prod_{i_1 \neq i_2, (i_1, j_1) < (i_2, j_2)} (\mu_{i_2}^{(j_2)} - \mu_{i_1}^{(j_1)}) \right). \quad (6)$$

The second factor of Equation (6) can be regrouped as

$$\prod_{1 \leq i_1 < i_2 \leq n} \left( \prod_{(i_1, j_1) < (i_2, j_2)} (\mu_{i_2}^{(j_2)} - \mu_{i_1}^{(j_1)}) \right) \left( \prod_{(i_2, j_2) < (i_1, j_1)} (\mu_{i_1}^{(j_1)} - \mu_{i_2}^{(j_2)}) \right)$$

$$= \prod_{1 \leq i_1 < i_2 \leq n} \left( \prod_{j_2=1}^{m} \prod_{j_1=1}^{m} (\mu_{i_2}^{(j_2)} - \mu_{i_1}^{(j_1)}) \right) \left( \prod_{(i_2, j_2) < (i_1, j_1)} (-1) \right). \quad (7)$$
Recall that $S_m(x) - \lambda_i S_{m-1}(x) = \prod_{j=1}^m (x - \mu_i^{(j)})$. Then

$$\prod_{j_2=1}^m \prod_{j_1=1}^m (\mu_{i_2}^{(j_2)} - \mu_{i_1}^{(j_1)}) = \prod_{j_2=1}^m \left( S_m(\mu_{i_2}^{(j_2)}) - \lambda_i S_{m-1}(\mu_{i_2}^{(j_2)}) \right)$$

$$= (\lambda_{i_2} - \lambda_{i_1})^m \prod_{j_2=1}^m S_{m-1}(\mu_{i_2}^{(j_2)}).$$

Note that for $i_1 < i_2$, the inequality $(i_2, j_2) < (i_1, j_1)$ holds if and only if $j_2 < j_1$. This means, for any fixed $i_1, i_2 \in \{1, 2, \ldots, n\}$ with $i_1 < i_2$,

$$\prod_{(i_2, j_2) < (i_1, j_1)} (-1) = (-1)^{\frac{m(m-1)}{2}}.$$

Finally, by Corollary 3.1,

$$\prod_{j_2=1}^m S_{m-1}(\mu_{i_2}^{(j_2)}) = (-1)^{\frac{m(m-1)}{2}}.$$

Now, Equation (7) reduces to

$$\left( \prod_{1 \leq i_1 < i_2 \leq n} (\lambda_{i_2} - \lambda_{i_1}) \right)^m$$

and Claim 2 holds by Equation (6).

**Claim 3:**

$$\det[E^{(1)}, \ldots, E^{(m)}] = \pm \prod_{i=1}^n \prod_{1 \leq j_2 < j_1 \leq m} \left( \mu_i^{(j_2)} - \mu_i^{(j_1)} \right).$$

Let $s_{i}^{(j_2)} = \frac{S_i(\mu_i^{(j_2)})}{S_{m-1}(\mu_i^{(j_1)})}$ and $K_{j_1,j_2}$ be the diagonal matrix:

$$K_{j_1,j_2} = \begin{bmatrix} s_{1,j_2}^{(1,j_2)} & & & \\ & s_{2,j_2}^{(2,j_2)} & & \\ & & \ddots & \\ & & & s_{n,j_2}^{(n,j_2)} \end{bmatrix}_{n \times n}.$$ (8)

Write $Q = [\xi_1, \xi_2, \ldots, \xi_n]$. Then it is routine to check the following factorization:

$$[E^{(1)}, E^{(2)}, \ldots, E^{(m)}] = \begin{bmatrix} Q & & \\ & \ddots & \\ & & Q \end{bmatrix} \begin{bmatrix} K_{1,1} & K_{1,2} & \cdots & K_{1,m} \\ K_{2,1} & K_{2,2} & \cdots & K_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ K_{m,1} & K_{m,2} & \cdots & K_{m,m} \end{bmatrix},$$ (9)

where the first factor is a block diagonal matrix with $m$ identical diagonal blocks. Since each block $K_{j_1,j_2}$ is diagonal, it is not difficult to see that, via an appropriate permutation
matrix, the block matrix \([K_{j_1,j_2}]_{m \times m}\) is similar to the following block diagonal matrix

\[
\begin{bmatrix}
L_1 \\
L_2 \\
\vdots \\
L_n
\end{bmatrix},
\]

where the \((j_1, j_2)\)th entry of each \(L_i\) is the \((i, i)\)th entry of \(K_{j_1,j_2}\). Written exactly,

\[
L_i = \begin{bmatrix}
\tilde{s}_{m-1}^{(i, 1)} & \tilde{s}_{m-1}^{(i, 2)} & \cdots & \tilde{s}_{m-1}^{(i, m)} \\
\tilde{s}_{m-2}^{(i, 1)} & \tilde{s}_{m-2}^{(i, 2)} & \cdots & \tilde{s}_{m-2}^{(i, m)} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{s}_0^{(i, 1)} & \tilde{s}_0^{(i, 2)} & \cdots & \tilde{s}_0^{(i, m)}
\end{bmatrix} = \begin{bmatrix}
S_{m-1}(\mu_i^{(1)}) & \cdots & S_{m-1}(\mu_i^{(m)}) \\
S_{m-2}(\mu_i^{(1)}) & \cdots & S_{m-2}(\mu_i^{(m)}) \\
\vdots & \vdots & \vdots \\
S_0(\mu_i^{(1)}) & \cdots & S_0(\mu_i^{(m)})
\end{bmatrix} \begin{bmatrix}
\frac{1}{S_{m-1}(\mu_i^{(1)})} \\
\frac{1}{S_{m-2}(\mu_i^{(1)})} \\
\vdots \\
\frac{1}{S_0(\mu_i^{(1)})}
\end{bmatrix}.
\] (10)

Since \(S_k(x)\) is a monic polynomial with degree \(k\) for each nonnegative integer \(k\), the determinant of the first factor in Equation (10) equals

\[
\det \begin{bmatrix}
(\mu_i^{(1)})^{m-1} & (\mu_i^{(2)})^{m-1} & \cdots & (\mu_i^{(m)})^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_i^{(1)} & \mu_i^{(2)} & \cdots & \mu_i^{(m)} \\
1 & 1 & \cdots & 1
\end{bmatrix} = (-1)^{\frac{m(m-1)}{2}} \prod_{1 \leq j_1 < j_2 \leq m} (\mu_i^{(j_2)} - \mu_i^{(j_1)}).
\] (11)

As \(\det Q = \pm 1\) and \(\det[K_{j_1,j_2}]_{m \times m} = \prod_{i=1}^n \det L_i\), Claim 3 follows from Equations (9)–(11) and Corollary 3.1.

By Claim 3 and Lemma 2.2, \(\det[E^{(1)}, \ldots, E^{(m)}] \neq 0\). Taking determinants for both sides of Equation (5) and using Claims 1–3,

\[
\det W(G \circ P_m) = \frac{\det[M^{(1)}, \ldots, M^{(m)}] \det D}{\det[E^{(1)}, \ldots, E^{(m)}]} = \pm a_0^{|m|} \left( \prod_{1 \leq i \leq n} (\lambda_{i_2} - \lambda_{i_1}) \right)^m \left( \prod_{1 \leq i \leq n} e_{n}^{T} \xi_{i_2} \right)^m
\]

\[
= \pm a_0^{|m|} (\det W(G))^m,
\]

where the last equality follows from Lemma 4.1.
For any even positive integer \( n \), let \( F^*_n \) be the family of all \( n \)-vertex graphs \( G \) such that \( \det W(G) = \pm 2^{\frac{n}{2}} \) and the constant term of \( \phi(G; x) \) is \( \pm 1 \). Write

\[
F^* = \bigcup_{n \text{ even}} F^*_n.
\]

As a special case of the aforementioned theorem of Wang [3], each graph in \( F^* \) is DGS.

**Lemma 4.3** ([7]): *If the constant term of \( \phi(G; x) \) is \( \pm 1 \), then so is \( \phi(G \circ P_m; x) \) for each integer \( m \geq 2 \).*

As a direct consequence of Theorem 1.3 and Lemma 4.3, we obtain the following result which was conjectured (in a slightly different form) by Mao and Wang [7, Conjecture 3.1].

**Theorem 4.1**: *If \( G \in F^* \) then for any integer \( m \geq 2 \), the graph \( G \circ P_m \in F^* \) and hence is DGS.*

Theorem 4.1 gives a simple method to construct large DGS-graphs from small ones. For example, let \( G \) be the left graph in Figure 1. It can be easily checked that \( G \in F^* \). Thus using Theorem 4.1 iteratively, we see that, for any integer sequence \( \{m_i\} \) with each \( m_i \geq 2 \), all graphs in the family

\[
G \circ P_{m_1}, (G \circ P_{m_1}) \circ P_{m_2}, ((G \circ P_{m_1}) \circ P_{m_2}) \circ P_{m_3}, \ldots
\]

are DGS.

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