UNIFORM DISTRIBUTION FOR THE PERTURBED EIGENFUNCTIONS OF A POINT SCATTERER ON THE THREE-DIMENSIONAL TORUS

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Abstract. We show that for a point scatterer (the Laplacian perturbed by a delta-potential) on the standard three-dimensional flat torus, almost all of the perturbed eigenfunctions are uniformly distributed in phase space.

1. Introduction

Consider a point scatterer on the standard three-dimensional flat torus \( \mathbb{T}^3 = \mathbb{R}^3/2\pi \mathbb{Z}^3 \), which is formally given by

\[
-\Delta + \alpha \delta_{x_0}
\]

where \(-\Delta\) is the associated Laplacian on \( \mathbb{T}^3 \), \( \delta_{x_0} \) is the Dirac potential at \( x_0 \) and \( \alpha \) is a coupling parameter.

We want to study quantum ergodicity of this system, which is a key question in the field of Quantum Chaos. A classical result regarding quantum ergodicity is Schnirelman’s theorem \([14, 4, 17]\), which asserts that for classically ergodic systems, the quantum counterpart is quantum ergodic, i.e. almost all eigenstates are uniformly distributed in phase space. However, our system is not classically ergodic but an intermediate system; its classical dynamics is nearly integrable.

Rigorously, a point scatterer is obtained as a self-adjoint extension of the Laplacian acting on functions vanishing near \( x_0 \). Such extensions are parametrized by \( \phi \in (-\pi, \pi] \), where \( \phi = \pi \) corresponds to the standard Laplacian \( (\alpha = 0 \text{ in (1.1)}) \). For \( \phi \neq \pi \), the eigenfunctions of the corresponding operator consist of eigenfunctions of the Laplacian which vanish at \( x_0 \) (and are not affected by the scatterer, so they are related to the unperturbed problem), and new, perturbed eigenfunctions. Since the latter are the only eigenfunctions which feel the scatterer, we have to consider only them in order to study the physics of the perturbed problem.

In a recent paper \([16]\), the author was able to prove quantum (unique) ergodicity regarding the perturbed eigenfunctions of a point scatterer on the

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standard three-dimensional flat torus, but only for observables which do not depend on the momentum. Our goal now is to prove quantum ergodicity regarding the perturbed eigenfunctions in full phase space.

We remark that a two-dimensional analogue of the theorem in [16] (i.e. quantum ergodicity in configuration space) was proved for a general two-dimensional flat torus by Rudnick and Ueberschär [12]; in a recent work by Kurlberg and Ueberschär, they prove an analogue of the theorem in the current paper for the standard two-dimensional flat torus, and for an irrational torus they show that within a density one sequence the following semiclassical measures arise: Liouville in position times a superposition of 4 Dirac measures in momentum.

The unit (co)tangent bundle of $\mathbb{T}^3$ is the compact metric space $S^*\mathbb{T}^3 \simeq \mathbb{T}^3 \times S^2$, on which we have the Liouville probability measure $\mu$ which is the normalized product of the Lebesgue measure $m$ on $\mathbb{T}^3$ and the Lebesgue measure $\sigma$ on $S^2$. Observables are smooth, zero-th order $\xi$-homogeneous functions $a(x, \xi)$ on $S^*\mathbb{T}^3$. In order to quantize them, we use the notion of pseudo-differential operators on $\mathbb{T}^3$, which will be discussed in greater detail in Chapter 3 below. As we will see, quantizing $a \in C^\infty(S^*\mathbb{T}^3)$ leads to an operator $\text{Op}(a)$, which is a bounded operator on $L^2(\mathbb{T}^3)$.

We now state the main theorem of this paper. For every $\phi \in (-\pi, \pi)$, let $\Lambda_{\phi}$ be the set of the perturbed eigenvalues of the point scatterer, with the corresponding $L^2$-normalized eigenfunctions $g_\lambda$ ($\lambda \in \Lambda_{\phi}$). We prove:

**Theorem 1.1.** Fix $\phi \in (-\pi, \pi)$. There is a subset $\Lambda_{\phi,\infty} \subseteq \Lambda_{\phi}$ of density one so that for all $a \in C^\infty(S^*\mathbb{T}^3)$,

$$\langle \text{Op}(a) g_\lambda, g_\lambda \rangle \to \int_{S^*\mathbb{T}^3} a d\mu$$

as $\lambda \to \infty$ along $\Lambda_{\phi,\infty}$.

Approximating an observable with a linear combination of the functions $Y_{l,m}(\xi) e^{ix \cdot \zeta}$, where $Y_{l,m}$ are spherical harmonics, the question reduces to an arithmetic one. The main case is $\zeta = 0$ (the other cases will follow from the proof of proposition 3.9 in [16]), for which a new arithmetic ingredient is used – an estimate due to Duke, by which he proved in [5, 6] the equidistribution of integer lattice points on a sphere of radius $n$, $n \not\equiv 0, 4, 7 (8)$, $n \to \infty$ (conjectured by Linnik, and proved independently by Golubeva and Fomenko [8, 9]). It will be combined with Siegel’s lower bound for $r_3(n)$, the number of representations of $n$ as a sum of three squares ($n \not\equiv 0, 4, 7 (8)$) [15], which was also a key ingredient in proving the analogue result in configuration space in [16].

We comment that all this is very different from the two-dimensional case, where arithmetic questions about spheres are replaced by questions about circles. The two-dimensional analogue to Duke’s estimate is the theorem of Erdős and Hall [17] about the distribution of lattice points on circles; an obvious complication constructing the density one sequence $\Lambda_{\phi,\infty}$ of perturbed
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3 eigenvalues is due to the fact that the theorem of Erdős and Hall is not
deterministic, i.e. lattice points on circles are equidistributed only for an
unspecified density one set of compatible n’s.

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2. Point Scatterers on the Torus

Let $T^3 = \mathbb{R}^3 / 2\pi \mathbb{Z}^3$ be the standard flat torus. A rigorous definition of the
operator (1.1) can be found in [16], following [3, 12]. For the convenience of
the reader we give here a brief summary:

Let $D_0 = C^\infty_c (T^3 \setminus \{x_0\})$ be the domain of $C^\infty$ functions vanishing in a
neighborhood of $x_0$, and define an operator on $L^2 (T^3)$ by $-\Delta x_0 = -\Delta |D_0$.
For the adjoint of $-\Delta x_0$ we have

\[ \text{Dom} (-\Delta x_0^*) = H^2 (T^3 \setminus \{x_0\}) = \{ f \in L^2 (T^3) : \exists A \in \mathbb{C}, -\Delta f + A \delta x_0 \in L^2 (T^3) \} , \]

and the self-adjoint extensions of $-\Delta x_0$ are indexed by a parameter $\phi \in (-\pi, \pi]$; the domain of the corresponding operators $-\Delta_{\phi, x_0}$ contains the functions $f \in \text{Dom} (-\Delta_{\phi, x_0})$ such that

\[ \exists a \in \mathbb{C}, f (x) = a \left( \cos \frac{\phi}{2} \cdot - \frac{1}{4\pi |x - x_0|} + \sin \frac{\phi}{2} \right) + o (1), \quad x \to x_0 \]

and their action on $f \in \text{Dom} (-\Delta_{\phi, x_0})$ is given by

\[ -\Delta_{\phi, x_0} f = -\Delta f + A \delta x_0 = -\Delta f + a \cos \frac{\phi}{2} \delta x_0 \quad (2.1) \]

so we define a point scatterer to be one of these extended operators $-\Delta_{\phi, x_0}$.

We remark that for $\phi = \pi$ we have $\text{Dom} (-\Delta_{\pi, x_0}) = H^2 (T^3)$ and $-\Delta_{\pi, x_0} f = -\Delta f$, so this extension recovers the standard Laplacian $-\Delta_{\infty}$ on the domain

\[ H^2 (T^3) \quad (\text{which is the unique self-adjoint extension of } -\Delta_{C^\infty (T^3)}). \]

The functions $(2\pi)^{-3/2} e^{i \xi \cdot x} (\xi \in \mathbb{Z}^3)$ form an orthonormal basis of eigenfunctions of $-\Delta_{\infty}$ for $L^2 (T^3)$. The corresponding eigenvalues are the norms $|\xi|^2$ of vectors in $\mathbb{Z}^3$, i.e. the set $N_3$ of integers which are sums of three
squares, and each eigenvalue is of multiplicity $r_3 (n)$ which is the number of representations of $n = a^2 + b^2 + c^2$ with $a, b, c \in \mathbb{Z}$ integers.

For the perturbed operator (2.1) with $\phi \neq \pi$ we still have the nonzero eigenvalues from the unperturbed problem $(0 \neq \lambda \in \sigma (-\Delta_{\infty}))$, with multipli-
cities decreased by one, as well as a new set $\Lambda = \Lambda_\phi$ of eigenvalues (referred
as the perturbed eigenvalues), each appearing with multiplicity one, with the corresponding eigenfunctions being multiples of the Green’s function

\[ G_\lambda (x, x_0) = (\Delta + \lambda)^{-1} \delta x_0. \]
The main tool we use for studying the Green’s functions $G_\lambda$ is their $L^2$-expansion, which is given by

$$G_\lambda(x, x_0) = -\frac{1}{8 \pi^3} \sum_{\xi \in \mathbb{Z}^3} \frac{\exp(i \xi \cdot (x - x_0))}{|\xi|^2 - \lambda}.$$ 

We denote by

$$g_\lambda(x) := \frac{G_\lambda(x, x_0)}{\|G_\lambda\|_2}$$

the $L^2$-normalized Green’s function, or the normalized perturbed eigenfunctions of the scatterer.

One can see that perturbed eigenvalues are the solutions to the equation

$$(2.2) \sum_{\xi \in \mathbb{Z}^3} \left\{ \frac{1}{|\xi|^2 - \lambda} - \frac{|\xi|^2}{|\xi|^4 + 1} \right\} = c_0 \tan \frac{\phi}{2}$$

with

$$c_0 = \sum_{\xi \in \mathbb{Z}^3} \frac{1}{|\xi|^4 + 1}$$

so the perturbed eigenvalues interlace between the elements of

$$\mathcal{N}_3 = \{0 = n_0 < n_1 < n_2 < \ldots \},$$

and denoting them by $\lambda_k = \lambda^k_\phi$ we can write

$$\lambda_0 < n_0 < \lambda_1 < n_1 < \lambda_2 < \cdots < n_k < \lambda_{k+1} < n_{k+1} < \ldots.$$ 

Recall that a subset $\Lambda' = \{\lambda_{j_k}\} \subseteq \Lambda$ is of density $a$ $(0 \leq a \leq 1)$ in $\Lambda$ if

$$\lim_{J \to \infty} \frac{1}{J} \# \{k \in \mathbb{N} : j_k \leq J\} = a$$

or equivalently

$$\lim_{X \to \infty} \frac{\# \{\lambda \in \Lambda' : \lambda \leq X\}}{\# \{\lambda \in \Lambda : \lambda \leq X\}} = a.$$

### 3. Pseudo-Differential Operators on the Torus

In order to quantize observables $a \in C^\infty(S^*T^3)$, one can use the special structure of $T^3$ as a compact group, to get a global definition of pseudo-differential operators on $T^3$, instead of using the theory of (Hörmander’s) pseudo-differential operators on $\mathbb{R}^3$ locally, which could be rather inconvenient. This idea goes back to Agranovich [11], and it was proved by McLean [11] that both definitions of pseudo-differential operators on the torus are equivalent. A recent monograph by Ruzhansky [13] gives a very comprehensive treatment of this subject – we present here the basic definitions of pseudo-differential operators on $T^n$ (and in particular on $T^3$) using their notations.
Let $\sigma : \mathbb{Z}^n \to \mathbb{C}$, and let $e_j$ be the standard basis elements of $\mathbb{R}^n$. Let $\Delta_\xi$ be the partial difference operator defined by

$$\Delta_\xi \sigma (\xi) = \sigma (\xi + e_j) - \sigma (\xi),$$

and for a multi-index $\alpha$ define

$$\Delta^\alpha_\xi = \Delta_\xi^{\alpha_1} \cdots \Delta_\xi^{\alpha_n}.$$

Following the notation in [13], for $m \in \mathbb{R}, 0 \leq \rho, \delta \leq 1$, define $S_{\rho,\delta}^m (\mathbb{T}^n \times \mathbb{R}^n)$ to be the set of functions $a (x, \xi)$ which are smooth in $x$ for all $\xi \in \mathbb{Z}^n$, and satisfy

$$\left| \Delta^\alpha_\xi \partial_\xi^\beta a (x, \xi) \right| \leq C_{a,\alpha,\beta,m} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}$$

for every $x \in \mathbb{T}^n$, $\alpha, \beta$ multi-indices, and $\xi \in \mathbb{Z}^n$. Here $\langle \xi \rangle = \left( 1 + |\xi|^2 \right)^{1/2}$.

For every symbol $a (x, \xi) \in S_{\rho,\delta}^m (\mathbb{T}^n \times \mathbb{R}^n)$, define a toroidal symbol $\tilde{a} : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ by $\tilde{a} = a |_{\mathbb{T}^n \times \mathbb{Z}^n}$. We clearly have $\tilde{a} (x, \xi) \in S_{\rho,\delta}^m (\mathbb{T}^n \times \mathbb{Z}^n)$. Define

$$\text{Op} (a) f (x) = \text{Op} (\tilde{a}) f (x) = \sum_{\xi \in \mathbb{Z}^n} e^{ix\cdot\xi} a (x, \xi) \hat{f} (\xi).$$

One verifies (see [13]) that the operator $\text{Op} (a) : C^\infty (\mathbb{T}^n) \to C^\infty (\mathbb{T}^n)$ is well defined and continuous. Note that for a symbol $a (x, \xi) = \sum_{|\alpha| \leq m} a_\alpha (x) \xi^\alpha$ we get that

$$\text{Op} (a) f (x) = \sum_{|\alpha| \leq m} a_\alpha (x) \sum_{\xi \in \mathbb{Z}^n} e^{ix\cdot\xi} \xi^\alpha \hat{f} (\xi)$$

$$= \sum_{|\alpha| \leq m} a_\alpha (x) \sum_{\xi \in \mathbb{Z}^n} e^{ix\cdot\xi} (-i\partial)\alpha f (\xi)$$

$$= \sum_{|\alpha| \leq m} a_\alpha (x) (-i\partial)\alpha f (x)$$

as one would expect.

In particular, we use this quantization for zero-th order positively homogeneous symbols $a (x, \xi) \in S_{0,1}^0 (\mathbb{T}^3 \times \mathbb{R}^3)$ (in the sense that there exists $r > 0$ such that $a (x, \lambda \xi) = a (x, \xi)$ for $|\xi| \geq r$ and $\lambda \geq 1$). Since $a (x, \xi)$ is a smooth $\xi$-homogeneous function of order zero (for $|\xi| \geq r$), we can identify it with a smooth function on the unit cotangent bundle of $\mathbb{T}^3$, and we write $a \in C^\infty (S^* \mathbb{T}^3)$.

One proves that in this case, $\text{Op} (a)$ extends to a bounded operator on $L^2 (\mathbb{T}^3)$. Moreover, we have the following result on the $L^2$-norm of $\text{Op} (a)$ (theorem 4.8.1 in [13]): Let $k \in \mathbb{N}$ and $k > n/2$. Let $a : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ be such that $\left| \partial_\xi^\beta a (x, \xi) \right| \leq C$ for all $(x, \xi) \in \mathbb{T}^n \times \mathbb{Z}^n$ and all $|\beta| \leq k$. Then the
operator $\text{Op}(a)$ extends to a bounded linear operator on $L^2(\mathbb{T}^n)$, and there exists a constant $C_k$ (which depends only on $k$) such that

$$\|\text{Op}(a)\|_{L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)} \leq C_k \sum_{|\alpha| \leq k} \sup_{y \in \mathbb{T}^n} \sup_{\xi \in \mathbb{Z}^n} |\partial_\nu^\alpha a(y, \xi)|^2.$$ 

Thus, for $n = 3$, $k = 2$, $a \in S_{0,1}^0(\mathbb{T}^3 \times \mathbb{R}^3)$ we have

$$\|\text{Op}(a)\|_{L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)} \leq C \sum_{|\alpha| \leq 2} \sup_{y \in \mathbb{T}^3} \sup_{\xi \in \mathbb{Z}^3} |\partial_\nu^\alpha a(y, \xi)|^2. \tag{3.1}$$

4. BOUNDS FOR THE GREEN’S FUNCTIONS AND TRUNCATION

We collect some auxiliary results proved in [16]:

We have the following lower bound for the $L^2$-norm of the Green’s function:

$$\|G_\lambda\|_2^2 \gg \lambda^{1/2 - \varepsilon}.$$ 

For $L > 0$, define the truncated Green’s function by

$$G_{\lambda,L}(x) = -\frac{1}{8\pi^3} \sum_{||\xi| - \lambda| < L} e^{i\xi \cdot (x-x_0)} \frac{e^{i\xi \cdot (x-x_0)}}{||\xi|^2 - \lambda},$$

and the $L^2$-normalized truncated Green’s function by $g_{\lambda,L} = \frac{G_{\lambda,L}}{\|G_{\lambda,L}\|_2}$.

For $L = \lambda^\delta$, $\delta > 0$, we have

$$\|g_{\lambda} - g_{\lambda,L}\|_2 \to 0$$

as $\lambda \to \infty$ and

$$\|G_{\lambda,L}\|_2 = \|G_\lambda\|_2 (1 + o(1)).$$

It is also proved that for all $f \in C^\infty(\mathbb{T}^3)$

$$|\langle f g_\lambda, g_\lambda \rangle - \langle f g_{\lambda,L}, g_{\lambda,L} \rangle| \leq 2 \|f\|_\infty \|g_\lambda - g_{\lambda,L}\|_2,$$

and a similar proof shows that for all $a \in C^\infty(S^*\mathbb{T}^3)$

$$|\langle \text{Op}(a) g_\lambda, g_\lambda \rangle - \langle \text{Op}(a) g_{\lambda,L}, g_{\lambda,L} \rangle| \leq 2 \|\text{Op}(a)\|_{L^2(\mathbb{T}^3) \to L^2(\mathbb{T}^3)} \|g_\lambda - g_{\lambda,L}\|_2. \tag{4.1}$$

5. BASIS ELEMENTS

The following lemma shows that we can approximate a smooth function on $\mathbb{T}^3 \times S^2$ by a finite linear combination of functions of the form $e^{i\xi \cdot \zeta l m}(x, \xi) = Y_{l,m}(\xi) e^{i\xi \cdot \zeta}$, where $Y_{l,m}(\xi)$ is the spherical harmonic of degree $l$ and order $m$ (normalized so that $\int_{S^2} |Y_{l,m}|^2 \, d\sigma = 1$).

**Lemma 5.1.** For all $a \in C^\infty(\mathbb{T}^3 \times S^2)$, $\varepsilon > 0$, there exist $N_1$, $N_2$ and

$$P(x, \xi) = \sum_{|\xi| \leq N_1} \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\xi,l,m} e^{i\xi \cdot \zeta l m}(x, \xi) \in C^\infty(\mathbb{T}^3 \times S^2),$$

where $c_{\xi,l,m}$ are constants.
such that for all \( x \in \mathbb{T}^3, \xi \in S^2 \) and for all multi-index \( \alpha \) such that \(|\alpha| \leq 2\), we have \(|\partial_x^\alpha (a - P)(x, \xi)| < \varepsilon\).

**Proof.** Let \( \varepsilon > 0 \). Expanding \( a \) to its Fourier series in the variable \( x \), for every \( \xi \in S^2 \) we have

\[
a(x, \xi) = \sum_{\zeta \in \mathbb{Z}^3} a_\zeta(\xi) e^{ix \cdot \zeta}
\]

where \( a_\zeta(\xi) = \frac{1}{2\pi^3} \int_{\mathbb{T}^3} a(x, \xi) e^{-ix \cdot \zeta} \, dx \), with convergence in the sense of \( L^2(\mathbb{T}^3) \). Moreover, since \( a \in C^\infty(\mathbb{T}^3 \times S^2) \), integration by parts yields that \( a_\zeta(\xi) \ll |\zeta|^{-k} \) for all \( k \) (where the implied constant is independent of \( \xi \)), so the series in (5.1) is uniformly convergent in \( \mathbb{T}^3 \times S^2 \). Since uniform convergence implies \( L^2 \) convergence to the same (equivalence class of) function, and since for every \( \xi \) both sides of (5.1) are continuous functions on \( \mathbb{T}^3 \), we conclude that the series uniformly converges in \( \mathbb{T}^3 \times S^2 \) to \( a(x, \xi) \). We also get that for every multi-index \( \alpha \), we have

\[
\partial_x^\alpha a(x, \xi) = \sum_{\zeta \in \mathbb{Z}^3} a_\zeta(\xi) \partial_x^\alpha e^{ix \cdot \zeta}
\]

and the series in (5.2) is uniformly convergent in \( \mathbb{T}^3 \times S^2 \) to \( \partial_x^\alpha a(x, \xi) \), so we can find \( N_1 \) such that for all \( x, \xi \) and for all multi-index \( \alpha \) such that \(|\alpha| \leq 2\) we have

\[
\left| \partial_x^\alpha \left(a(x, \xi) - \sum_{|\zeta| \leq N_1} a_\zeta(\xi) e^{ix \cdot \zeta} \right) \right| < \frac{\varepsilon}{2}.
\]

For every \( \zeta \), we have the spherical harmonics expansion:

\[
a_\zeta(\xi) = \sum_{l=0}^\infty \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi)
\]

where \( c_{\zeta,l,m} = \int_{S^2} a_\zeta(\xi) \overline{Y_{l,m}(\xi)} \, d\sigma \), with convergence in the sense of \( L^2(S^2) \). Since \( a(x, \xi) \in C^\infty(\mathbb{T}^3 \times S^2) \), we easily see that for all \( \zeta \) we have \( a_\zeta(\xi) \in C^\infty(S^2) \), and hence

\[
\sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \ll l^{-k}
\]

for all \( k \) (see [2], for example), so the sum in (5.3) is uniformly convergent in \( S^2 \) for all \( \zeta \), and again, it must converge to \( a_\zeta(\xi) \). We conclude that there exists \( N_2 \) such that for all \( |\zeta| \leq N_1 \) and for all \( \xi \) we have

\[
\left| a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right| < \frac{\varepsilon}{2N_1^3}.
\]
and if we denote

\[ P(x, \xi) = \sum_{|\zeta| \leq N_1} \sum_{1 \leq l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} e_{\zeta,l,m}(x, \xi) \]

we get that for all \( x, \xi \) and for all multi-index \( \alpha \) such that \( |\alpha| \leq 2 \) we have

\[
|\partial_\xi^\alpha (a - P)(x, \xi)| \leq \left| \partial_\xi^\alpha \left( P(x, \xi) - \sum_{|\zeta| \leq N_1} a_\zeta(\xi) e^{ix\zeta} \right) \right| + \frac{\epsilon}{2}
\]

\[
= |\partial_\xi^\alpha \left( \sum_{|\zeta| \leq N_1} \left( a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right) e^{ix\zeta} \right) | + \frac{\epsilon}{2}
\]

\[
\leq N_1^2 \sum_{|\zeta| \leq N_1} \left| a_\zeta(\xi) - \sum_{l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} Y_{l,m}(\xi) \right| + \frac{\epsilon}{2} < \epsilon.
\]

We can think of \( e_{\zeta,l,m}(x, \xi) = Y_{l,m}(\xi) e^{ix\zeta} \) as a zero-th order positively homogeneous symbol in \( S_{0,1}^0(\mathbb{T}^3 \times \mathbb{R}^3) \), by extending \( Y_{l,m}(\xi), (l,m) \neq (0,0) \) homogeneously to the domain \( |\xi| \geq 1 \), and arbitrarily to the domain \( |\xi| < 1 \); the function \( Y_{0,0}(\xi) = \frac{1}{2} \sqrt{\frac{1}{\pi}} \) extends to \( \xi \in \mathbb{R}^3 \) in an obvious way. In order to prove our main theorem, we will now see that it suffices to show a simpler version of it on the functions \( e_{\zeta,l,m}(x, \xi) \).

**Proposition 5.2.** Suppose that there is a subset \( \Lambda_{\phi, \infty} \subseteq \Lambda_{\phi} \) of density one so that for all \( \zeta, l, m \) with at least one of them nonzero we have

\[
\langle \text{Op}(e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} \rangle \to 0
\]

as \( \lambda \to \infty \) along \( \Lambda_{\phi, \infty} \), then theorem \( \ref{thm:main} \) follows. Here \( 0 < \delta < 1, L = \lambda^\delta \).

**Proof.** Let \( \epsilon > 0 \), and \( a(x, \xi) \in C^\infty(S^*\mathbb{T}^3) \). From lemma \( \ref{lem:approx} \) there is

\[
P(x, \xi) = \sum_{|\zeta| \leq N_1} \sum_{1 \leq l \leq N_2} \sum_{|m| \leq l} c_{\zeta,l,m} e_{\zeta,l,m}(x, \xi)
\]

such that for all \( x \in \mathbb{T}^3, \xi \in \mathbb{R}^3, |\xi| \geq r \) and for all multi-index \( \alpha \) such that \( |\alpha| \leq 2 \) we have \( |\partial_\xi^\alpha (a - P)(x, \xi)| < \epsilon \). Without loss of generality we can assume that for \( \xi \in \mathbb{Z}^3 \) such that \( |\xi| < r \) we have

\[
\partial_\xi^\alpha a(x, \xi) = \partial_\xi^\alpha P(x, \xi)
\]

for all \( x \in \mathbb{T}^3 \) and for all multi-index \( \alpha \), because for \( \lambda \) large enough

\[
\text{Op}(P) g_{\lambda,L}(x) = -\frac{1}{8\pi^3 \|G_{\lambda,L}\|_2} \sum_{|\xi|^2 - \lambda < L} e^{i(x-x_0)\cdot\xi} P(x, \xi) \frac{1}{|\xi|^2 - \lambda}
\]

does not depend on the value of \( P(x, \xi) \) for \( |\xi| < r \) (and it is easy to change \( P(x, \xi) \) in the domain \( |\xi| < r \) to get a new symbol in \( C^\infty(S^*\mathbb{T}^3) \) satisfying
so under this assumption the inequality $|\partial_{x}^{\alpha} (a - P) (x, \xi)| < \varepsilon$ holds for every $\xi \in \mathbb{Z}^{n}$. Since

$$\text{Op} (e_{0,0,0}) = \frac{1}{2} \sqrt{\frac{1}{\pi}} \text{id},$$

and since for $(\zeta, l, m) \neq (0, 0, 0)$

$$\langle \text{Op} (e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} \rangle \rightarrow 0$$

as $\lambda \rightarrow \infty$ along $\Lambda_{\phi,\infty}$, we have

$$\langle \text{Op} (P) g_{\lambda,L}, g_{\lambda,L} \rangle = \sum_{|\zeta| \leq N_{1}} \sum_{l \leq N_{2}} \sum_{|m| \leq l} c_{\zeta,l,m} \langle \text{Op} (e_{\zeta,l,m}) g_{\lambda,L}, g_{\lambda,L} \rangle$$

$$\rightarrow \frac{1}{2} \sqrt{\frac{1}{\pi}} c_{0,0,0},$$

as $\lambda \rightarrow \infty$ along $\Lambda_{\phi,\infty}$, and

$$\int_{S^{*}T^{3}} Pd\mu = \frac{1}{\text{area} (T^{3}) \text{area} (S^{2})} \sum_{|\zeta| \leq N_{1}} \sum_{l \leq N_{2}} \sum_{|m| \leq l} c_{\zeta,l,m} \left( \int_{T^{3}} e^{ix \cdot \zeta} dm \right) \left( \int_{S^{2}} Y_{l,m} (\xi) d\sigma \right)$$

$$= \frac{1}{\text{area} (T^{3}) \text{area} (S^{2})} c_{0,0,0} \left( \int_{T^{3}} 1 dm \right) \left( \int_{S^{2}} Y_{0,0} (\xi) d\sigma \right)$$

$$= \frac{1}{2} \sqrt{\frac{1}{\pi}} c_{0,0,0},$$

so

$$\langle \text{Op} (P) g_{\lambda,L}, g_{\lambda,L} \rangle \rightarrow \int_{S^{*}T^{3}} Pd\mu$$

as $\lambda \rightarrow \infty$ along $\Lambda_{\phi,\infty}$. Thus for $\Lambda_{\phi,\infty}$ large enough we have

$$\left| \langle \text{Op} (P) g_{\lambda}, g_{\lambda} \rangle - \int_{S^{*}T^{3}} Pd\mu \right| < \varepsilon + \left| \langle \text{Op} (P) g_{\lambda}, g_{\lambda} \rangle - \langle \text{Op} (P) g_{\lambda,L}, g_{\lambda,L} \rangle \right|$$

$$\leq \varepsilon + 2 \| \text{Op} (P) \|_{L^{2}(T^{3}) \rightarrow L^{2}(T^{3})} \| g_{\lambda} - g_{\lambda,L} \|_{2}$$

$$< C\varepsilon.$$

Using the bound in (3.1), we get that

$$\left| \langle \text{Op} (a - P) g_{\lambda}, g_{\lambda} \rangle \right|^{2} \leq \| \text{Op} (a - P) g_{\lambda} \|_{2}^{2}$$

$$\leq \| \text{Op} (a - P) \|_{L^{2}(T^{3}) \rightarrow L^{2}(T^{3})}^{2}$$

$$\leq C \sum_{|\alpha| \leq 2} \sup_{y \in T^{3}} \sup_{\xi \in \mathbb{Z}^{3}} \left| \partial_{y}^{\alpha} (a - P) (y, \xi) \right|^{2}$$

$$< C\varepsilon^{2}$$

for $\lambda \in \Lambda_{\phi,\infty}$ (we call all our constants $C$).
Proof. For \( a, n \neq 0 \) and \( n \neq 0 \), Lemma 6.1.

Lemma 6.2. For every \( a \) and \( n \neq 0 \), we have

\[ \langle \text{Op} (a) g_\lambda, g_\lambda \rangle - \int_{S^* T^3} a d\mu \leq |\langle \text{Op} (a - P) g_\lambda, g_\lambda \rangle| \\
+ |\langle \text{Op} (P) g_\lambda, g_\lambda \rangle - \int_{S^* T^3} P d\mu| \\
+ |\int_{S^* T^3} P d\mu - \int_{S^* T^3} a d\mu| \\
< C\varepsilon \]

and the theorem follows. \( \square \)

6. A Density One Set

By the theorem of Legendre and Gauss (see [10]), the Diophantine equation

\[ x_1^2 + x_2^2 + x_3^2 = n \]

has solutions in integers \( x_i (i = 1, 2, 3) \) if and only if \( n \) is not of the form \( 4^a (8k + 7) \) with \( a \in \mathbb{Z}, a \geq 0 \) and \( k \in \mathbb{Z} \), and for all \( n, r_3 (4^a n) = r_3 (n) \).

Equivalently, if we write \( n = 4^a n_1 \), with \( 4 \nmid n_1 \), then \( n \) is a sum of three squares if and only if \( n_1 \neq 7 (8) \), that is to say

\[ N_3 = \{ n \in \mathbb{N} : n = 4^a n_1, 4 \nmid n_1 \Rightarrow n_1 \neq 7 (8) \} \]

and \( r_3 (n) = r_3 (n_1) \).

For \( n \in N_3 \), write \( n = 4^a n_1 \) with \( 4 \nmid n_1 \), and define

\[ N_{good} = \{ n \in N_3 : n_1 > n^{1/2} \} \]

the set of “good” elements in \( N_3 \), and \( N_{bad} = N_3 \setminus N_{good} \) the set of “bad elements”. We show that there are very few “bad” elements in \( N_3 \):

Lemma 6.1. \# \{ \( n \in N_{bad} : n \leq X \} \leq X^{1/2} \log X \).

Proof. For \( n \in N_{bad} \), \( n \leq X \), we have \( n = 4^a n_1 \) with \( n_1 \leq n_1^{1/2} \leq X^{1/2} \) and \( a = \log_4 (n/n_1) \leq \log n/\log 4 \leq \log X \), so there are at most \( X^{1/2} \log X \) possibilities for such \( n \).

For \( \lambda \in \Lambda \), denote \( n_\lambda \) to be the closest element in \( N_3 \) to \( \lambda \) (and if there are two elements with the same distance take the smallest of them). Note that \( |n_\lambda - \lambda| \leq 1.5 \), and for \( n \neq n_\lambda \) we have \( |n - \lambda| \geq 0.5 \).

Define \( \Lambda_\infty = \{ \lambda \in \Lambda : n_\lambda \in N_{good} \} \). We show that:

Lemma 6.2. \( \Lambda_\infty \) is a density one set in \( \Lambda \).

Proof. For every \( \lambda \in \Lambda \setminus \Lambda_\infty \), we have \( n_k < \lambda < n_{k+1} \), where either \( n_\lambda = n_k \) or \( n_\lambda = n_{k+1} \), and \( n_\lambda \in N_{bad} \). Thus, for every \( n \in N_{bad} \) such that \( n \leq X + 1.5 \) there are at most two \( \lambda \in \Lambda \setminus \Lambda_\infty, \lambda \leq X \) such that \( n_\lambda = n \), so by lemma 6.1 we have

\[ \# \{ \lambda \in \Lambda \setminus \Lambda_\infty : \lambda \leq X \} \leq 2 \# \{ n \in N_{bad} : n \leq X + 1.5 \} \leq X^{1/2} \log X, \]
but
\[
\# \{ \lambda \in \Lambda : \lambda \leq X \} \geq \# \{ n \leq X : n \neq 0, 4, 7 \} \asymp X,
\]
so
\[
\frac{\# \{ \lambda \in \Lambda \setminus \Lambda_{\infty} : \lambda \leq X \}}{\# \{ \lambda \in \Lambda : \lambda \leq X \}} \ll X^{-1/2} \log X
\]
which tends to zero as \( X \to \infty \), so \( \Lambda \setminus \Lambda_{\infty} \) is a density zero set in \( \Lambda \), and therefore \( \Lambda_{\infty} \) is a density one set in \( \Lambda \). \( \square \)

7. Proving Theorem 1.1

We are only left to prove the following proposition:

**Proposition 7.1.** Let \( \zeta, l, m \) with at least one of them nonzero, and let \( 0 < \delta < 1/28 \), \( L = \lambda^{\delta} \). Then
\[
\langle \text{Op}(e^{i \zeta \cdot l}) g_{\lambda, L}, g_{\lambda, L} \rangle \to 0
\]
as \( \lambda \to \infty \) along \( \Lambda_{\infty} \).

**Proof.** We have
\[
\langle \text{Op}(e^{i \zeta \cdot l}) G_{\lambda, L}, G_{\lambda, L} \rangle
\]
and therefore
\[
\langle \text{Op}(e^{i \zeta \cdot l}) g_{\lambda, L}, g_{\lambda, L} \rangle \ll \frac{1}{\|G_{\lambda, L}\|_2^2} \to 0
\]
as \( \lambda \to \infty \) by the proof of Proposition 3.9 in [16].

Assume now that \( \zeta = 0 \).
We have
\[
\langle \text{Op} (e_{0,l,n}) G_{\lambda,L}, G_{\lambda,L} \rangle \asymp \sum_{|\xi|^2 - \lambda < L} \frac{Y_{l,m} (\xi)}{|\xi|^2 - \lambda} \cdot \sum_{|n - \lambda| < L} \frac{W_{l,m} (n)}{(n - \lambda)^2}
\]
where
\[
W_{l,m} (n) = \sum_{|\xi|^2 = n} Y_{l,m} \left( \frac{\xi}{|\xi|} \right).
\]

If we write \( n = 4^a n_1 \) with \( 4 \nmid n_1 \), then by Duke’s estimate (see [5, 6])
\[
W_{l,m} (n) = W_{l,m} (n_1) \ll n_1^{1/28+\varepsilon} \ll n^{13/28+\varepsilon}
\]
and by Siegel’s theorem [15] \( r_3 (n) = r_3 (n_1) \gg n_1^{1/2-\varepsilon} \), so
\[
\frac{W_{l,m} (n)}{r_3 (n)} \ll n_1^{-1/28+\varepsilon}.
\]

For \( \lambda \in \Lambda_\infty \), recall that \( n_\lambda \) is the closest element in \( \mathcal{N}_3 \) to \( \lambda \) (and if there are two elements with the same distance take the smallest of them). From the definition of \( \Lambda_\infty \) we know that \( n_\lambda \in \mathcal{N}_{\text{good}} \).

Write
\[
\sum_{|n - \lambda| < L} \frac{W_{l,m} (n)}{(n - \lambda)^2} = \sum_{|n - \lambda| < L} \frac{W_{l,m} (n)}{(n - \lambda)^2} + \frac{W_{l,m} (n_\lambda)}{(n_\lambda - \lambda)^2}.
\]

Since for \( n \neq n_\lambda \) we have \(|n - \lambda| \geq 0.5:
\[
\sum_{|n - \lambda| < L} \frac{W_{l,m} (n)}{(n - \lambda)^2} \ll \sum_{|n - \lambda| < L} n^{13/28+\varepsilon} \ll \lambda^{13/28+\delta+\varepsilon}
\]
so
\[
\sum_{|n - \lambda| < L} \frac{W_{l,m} (n)}{(n - \lambda)^2} \ll \frac{\lambda^{13/28+\delta+\varepsilon}}{||G_{\lambda,L}||^2} \ll \lambda^{-1/28+\delta+2\varepsilon}
\]
which tends to zero for \( \varepsilon > 0 \) small enough, since \( \delta < \frac{1}{28} \). For the other term, writing \( n_\lambda = 4^n n_1 \) with \( 4 \nmid n_1 \), we have

\[
\frac{W_{l,m}(n_\lambda)}{(n_\lambda - \lambda)^2} \times \frac{W_{l,m}(n_\lambda)}{\|G_{\lambda,L}\|_2^2} \times \frac{W_{l,m}(n_\lambda)}{r_3(n_\lambda)} \times \sum_{n=0}^{\infty} \frac{r_3(n)}{(n-\lambda)^4} \ll \frac{W_{l,m}(n_\lambda)}{r_3(n_\lambda)} \ll n_1^{-1/28 + \varepsilon}
\]

and since \( n_\lambda \in \mathcal{N}_{\text{good}} \), we have

\[
n_1^{-1/28 + \varepsilon} \leq n_\lambda^{-1/56 + \frac{\varepsilon}{2}} \ll \lambda^{-1/56 + \frac{\varepsilon}{2}}
\]

so this term also tends to zero as as \( \lambda \to \infty \) along \( \Lambda_\infty \). \( \square \)

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