Proof of the Matrix-valued Pólya Positivstellensatz via pure states

Li Gao and Colin Tan

Abstract. Let $\Sigma$ denote the linear form $x_1 + \cdots + x_n$. By a classical Positivstellensatz of Pólya, if a real form $f$ is strictly positive on the standard simplex, then $\Sigma^m f$ has strictly positive coefficients for some nonnegative integer $m$. Pólya’s Positivstellensatz generalizes to a square symmetric matrix $B$ of forms having fixed degree. Namely, if such a matrix $B$ is positive definite when evaluated at each point on the standard simplex, then the entrywise product $\Sigma^m \cdot B$ strictly has positive definite coefficients for some nonnegative integer $m$. We give an algebraic proof of this Positivstellensatz for matrices of forms using the technique of pure states and a criterion of Goodearl-Handelman.

1. Introduction and historical review

Fix a positive integer $n$, and let $\Sigma$ denote the linear form $x_1 + \cdots + x_n$ in $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$. Pólya proved that if a form $f \in \mathbb{R}[x]$ (i.e. homogeneous polynomial) is strictly positive on the standard codimension-1 simplex $\Delta_n := \{x \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0, \Sigma(x) = 1\}$, then $\Sigma^m f$ has strictly positive coefficients for some nonnegative integer $m \in \mathbb{N}$ [10] (reproduced in [7, pp. 57–60]).

Powers-Reznick gave an upper bound on the least $m$ for which $\Sigma^m f$ has strictly positive coefficients [11]. For $f$ of fixed degree, their bound is an explicit continuous function of the coefficient vector of $f$. As essentially observed by Robinson [14, pp. 28–33], such a bound on $m$ implies a generalization of Pólya’s Positivstellensatz to matrices of forms with corresponding bounds. Indeed, Scherer-Hol explicated Robinson’s method and stated this matrix-valued generalization of Pólya’s Positivstellensatz as [15, Theorem 3]. Without the explicit bounds, the Matrix-valued Pólya Positivstellensatz is:

**Theorem 1.1.** Let $B$ be a square symmetric matrix, whose entries are forms in $\mathbb{R}[x]$ of fixed degree. If $B(x)$ is positive definite for all $x \in \Delta_n$, then the entrywise product $\Sigma^m \cdot B$ strictly has positive definite coefficients for some nonnegative integer $m$.

2010 Mathematics Subject Classification. Primary 12D99, 15B48, 52A07; secondary 14P99, 26C99.

Key words and phrases. form, matrix, positive definite, pure state.
Here, given a square symmetric matrix $B$ whose entries are forms in $\mathbb{R}[x]$ of some fixed degree $d \in \mathbb{N}$, we say that $B$ strictly has positive definite coefficients if $B = \sum_{|\alpha| = d} P_\alpha x^\alpha$ where every coefficient $P_\alpha$ of degree-$d$ monomials $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ is a positive definite matrix of scalars (for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with length $|\alpha| := \alpha_1 + \cdots + \alpha_n = d$). By the entrywise product $\Sigma^m \cdot B$, we mean the matrix whose $(i,j)$th entry is $\Sigma^m f_{ij}$, where $B = (f_{ij})$. Note that this theorem specializes to Pólya’s Positivstellensatz when $B$ is a $1 \times 1$ matrix.

Other than Robinson’s approach, Theorem 1.1 is also a special case of a hermitian Positivstellensatz of Quillen [13] (where the hermitian form equals zero when evaluated on distinct basis elements). Both the explicit bounds of Powers-Reznick required for Robinson’s approach as well as Quillen’s proof uses methods from analysis. What is new in this paper is the alternative algebraic proof of Theorem 1.1 that we give below.

Our approach is based on a recent proof of the Representation Theorem by Burgdorf-Scheiderer-Schweighofer that uses techniques from convex geometry [4]. The Representation Theorem is a fundamental result in real algebraic geometry. In various versions, this result was rediscovered and proved by Stone, Krivine, Kadison, Dubois and other mathematicians (see [8] or [9] for a statement and see [12] Section 5.6 for historical remarks). In fact, Pólya’s Positivstellensatz can be deduced from the Representation Theorem, as first observed by Wörmann [17] (see also [1]).

However, the Representation Theorem alone appears inadequate to prove the Matrix-valued Pólya’s Positivstellensatz. Our main observation is that the arguments of Burgdorf-Scheiderer-Schweighofer in their proof of the Representation Theorem holds in greater generality. Briefly, we reduce Theorem 1.1 to the Goodearl-Handelman Criterion [5, Lemma 6.4] (see Lemma 2.1 below) and characterize the pure states of a concrete preordered abelian group admitting an order unit (see Proposition 4.4) to complete the proof.

Finally, we remark that the converse of Theorem 1.1 is true. This is because, for a symmetric matrix whose entries are forms of fixed degree, strictly having positive definite coefficients is a certificate for being pointwise positive definite on $\Delta_n$. In this sense, Theorem 1.1 can be regarded as a Positivstellensatz for symmetric matrices $B$ of forms having fixed degree, relative to $\Delta_n$.
We are grateful for Prof. CheeWhye Chin’s encouragement to pursue this line of research. We thank Prof. Dr. Markus Schweighofer for pointing out Robinson’s approach and the reference to [14]. Prof. Dr. Claus Scheiderer also gave many wonderful insights on states and Positivstellensatze. Professor Wing-Keung To explained how Theorem 1.1 is a special case of Quillen’s Positivstellensatz. Last but not least, we convey heartfelt appreciation to Professor Zhihua Yang for his advice to pay attention to reviewing the literature.

2. The Goodearl-Handelman Criterion

We recall some preliminaries on order units and pure states, leading to a statement of the Goodearl-Handelman Criterion [5, Lemma 4.1] (see Lemma 2.1 below). We follow closely the modern exposition of this criterion as given by Burgdorf-Scheiderer-Schweighofer [4].

Let $G$ be an abelian group, written additively, and let $M \subseteq G$ be a submonoid, i.e. a subset containing 0 and closed under addition. Associated to $M$ is a preorder (i.e. reflexive and transitive binary relation) on $G$ defined by $g' \leq_M g$ whenever $g - g'$ lies in $M$. An element $u \in M$ is an order unit of $(G, M)$ if, for each $g \in G$, there exists an integer $k \in \mathbb{Z}$ such that $k + f \in G$, i.e. if $M + \mathbb{Z}u = G$.

Suppose that $(G, M)$ has an order unit $u \in M$. A state of $(G, M, u)$ is an additive map $\varphi : G \rightarrow \mathbb{R}$ to the reals such that $\varphi|_M \geq 0$ and $\varphi(u) = 1$. In particular, each state of $(G, M, u)$ is monotone, in the sense that $g' \leq_M g$ implies $\varphi(g') \leq \varphi(g)$ (where $\mathbb{R}$ is ordered linearly as usual) for all $g', g \in G$. We regard the set of states, denoted by $S(G, M, u)$, as a subset of the product vector space $\mathbb{R}^G = \prod_G \mathbb{R}$, via the injection $\varphi \mapsto (\varphi(g))_{g \in G} : S(G, M, u) \hookrightarrow \mathbb{R}^G$. As such $S(G, M, u)$ is compact and convex. A state $\varphi$ of $(G, M, u)$ is pure if, for any two states $\varphi_1, \varphi_2$ of $(G, M, u)$, the equation $2\varphi = \varphi_1 + \varphi_2$ implies $\varphi = \varphi_1 = \varphi_2$. In other words, the pure states are the extreme points of $S(G, M, u) \hookrightarrow \mathbb{R}^G$. It follows more generally that, if a pure state $\varphi$ is a proper convex combination of two states $\varphi_1$ and $\varphi_2$ (i.e. $\varphi = c\varphi_1 + (1 - c)\varphi_2$ for some $0 < c < 1$), then $\varphi = \varphi_1 = \varphi_2$.

The following version of the Goodearl-Handelman Criterion was stated by Burgdorf-Scheiderer-Schweighofer in [4, Theorem 2.5].
Lemma 2.1 (Goodearl-Handelman Criterion). Let $G$ be an abelian group, let $M \subseteq G$ be a submonoid and suppose that $u \in M$ is an order unit of $(G, M)$. For each $g \in G$, if $\varphi(g) > 0$ for all pure states $\varphi$ of $(G, M, u)$, then $kg \in M$ for some positive integer $k$.

The stronger requirement that $\leq_M$ is a partial order, or equivalently that $M \cap (-M) = \{0\}$, was assumed by Goodearl-Handelman in their original statement of [5, Lemma 4.1]. However this assumption is not necessary, as noted by Burgdorf-Scheiderer-Schweighofer, leading to their formulation of the Goodearl-Handelman Criterion as restated in Lemma 2.1 above.

3. Multiplicative Law governing pure states of certain modules

As in the previous section, let $G$ be an abelian group, let $M \subseteq G$ be a commutative submonoid and suppose that $u \in M$ is an order unit of $(G, M)$. We shall show that if $G$ has the additional structure of a module over a (commutative unital) ring $A$ such that $M$ is closed under the action of some archimedean subsemiring of $A$, then every pure state of $(G, M, u)$ satisfies a certain multiplicative law (see Proposition 3.1 below).

This result is essentially due to Burgdorf-Scheiderer-Schweighofer and follows verbatim from [4, p.123]. There they discussed the case where $G$ is contained in $A$, so that $G$ is an ideal of $A$. Nonetheless, their proof holds more generally for any $A$-module $G$. For the convenience of the reader, we shall reproduce their proof below.

Let $A$ be a ring (all rings are commutative with unit), and let $S \subseteq A$ be a subsemiring, i.e. a subset containing 0, 1 and closed under addition and multiplication. Recall that $S \subseteq A$ is said to be archimedean if, for each $a \in A$, there exists an integer $k \in \mathbb{Z}$ such that $k + a \in S$, i.e. if $S + \mathbb{Z} = A$. In other words, a subsemiring $S \subseteq A$ is archimedean if and only if 1 is an order unit of $(A, S)$ (c.f. Section 2), where the multiplicative structure is forgotten.

Now suppose that the abelian group $G$ is equipped with an $A$-action. So $G$ is an $A$-module. We say that a submonoid $M \subseteq G$ is a subsemimodule over $S$ (or $S$-subsemimodule, for short) if it is closed under the action restricted to $S$, i.e. if $SM \subseteq M$. The following lemma is essentially [4, Proposition 4.1], with the condition that $G$ be contained in $A$ removed. As mentioned in loc. cit.,
precedents of this result can be found in the work of Bonsall-Lindenstrauss-
Phelps [2, Theorem 10], Krivine [9, Theorem 15] and Handelman [6, Proposition
1.2].

**Proposition 3.1** (Multiplicative Law). Let $G$ be a module over a ring $A$, and let
$M \subseteq G$ be a subsemimodule over some archimedean subsemiring of $A$. Suppose that
$u \in M$ is an order unit of $(G, M)$. Then each pure state $\varphi$ of $(G, M, u)$ satisfies

$$
\varphi(\mathbf{a}g) = \varphi(\mathbf{a}u)\varphi(g) \quad \text{for all } \mathbf{a} \in A, g \in G.
$$

As a preparation for the proof of this lemma, we require the following obser-
vations. Let $A, G, M, u$ be as in Proposition 3.1. Given a map $\varphi : G \to \mathbb{R}$ to the
reals, we associate to each $a \in A$ satisfying $\varphi(au) \neq 0$ a map $\varphi_a : G \to \mathbb{R}$ given by

$$
\varphi_a(g) := \frac{\varphi(\mathbf{a}g)}{\varphi(\mathbf{a}u)} \quad (g \in G).
$$

Let $S \subseteq A$ be the archimedean subsemiring that acts on $M$. So $M$ is a $S$-
subsemimodule of $G$. The reader can verify that if $\varphi$ is a state of $(G, M, u)$
and $s \in S$ satisfies $\varphi(su) > 0$, then $\varphi_s$ is also a state of $(G, M, u)$. Furthermore,
if $\varphi$ is a state and $s_1, s_2 \in S$ satisfy $\varphi(s_1 u), \varphi(s_2 u) > 0$, so that $s_1 + s_2 \in S$
and $\varphi((s_1 + s_2)u) > 0$, then $\varphi_{s_1 + s_2}$ is a proper convex combination of
the states $\varphi_{s_1}$ and $\varphi_{s_2}$:

$$
\varphi(s_1 u)\varphi_{s_1} + \varphi(s_2 u)\varphi_{s_2} = \varphi((s_1 + s_2)u)\varphi_{s_1 + s_2}.
$$

**Proof of Proposition 3.1** Let $S \subseteq A$ be the archimedean subsemiring that acts
on $M$ and let $\varphi$ be a given pure state of $(G, M, u)$. Since $A = S + \mathbb{Z}$ and $G =
M + Zu$, it suffices to show that (1) holds whenever $a \in S$ and $g \in M$.

Let $a \in S$ and $g \in M$ be given. Then $au \in M$ since $M$ is closed under the $S$-
action and contains the order unit $u$. Hence $\varphi(au) \geq 0$. We split the discussion
into two cases: either $\varphi(au) = 0$ or $\varphi(au) > 0$.

In the former case where $\varphi(au) = 0$, there exists $k \in \mathbb{Z}$ such that $0 \leq_M
\mathbf{g} \leq_M k u$ since $u$ is an order unit and $g$ lies in $M$. Since $a \in S$, this implies that
$0 \leq_M \mathbf{a}g \leq_M kau$. Thus, by the monotonicity and additivity of the pure state $\varphi$,

$$
0 \leq \varphi(\mathbf{a}g) \leq k\varphi(au) = 0,
$$

forcing $\varphi(\mathbf{a}g) = 0$ so that both sides of (1) equals to zero in this case.
In the latter case where \( \varphi(au) > 0 \), the archimedean property of \( S \) gives \( l \in \mathbb{Z} \) such that \( l - a \in S \). We may further choose \( l \) large enough such that \( \varphi(au) < l \).

Hence

\[
\varphi((l - a)u) = l\varphi(u) - \varphi(au) = l - \varphi(au) > 0,
\]

where the equalities follow since \( \varphi \) is a pure state. Since \( a, l - a \in S \) and \( \varphi(au), \varphi((l - a)u) > 0 \), we may apply (2) to conclude that \( \varphi_l \) is a proper convex combination of \( \varphi_a \) and \( \varphi_{l-a} \). But \( \varphi_l = \varphi \) (by direct calculation), so \( \varphi \) being an extreme point of the set of states \( S(G, M, u) \) implies that \( \varphi_a = \varphi \), which is just (1).

The following corollary of the Multiplicative Law is essentially part of [4, Lemma 4.9], again with the condition that \( G \) be contained in \( A \) removed.

**Corollary 3.2.** Let \( A, G, M, u \) be as in Proposition 3.1. For each pure state \( \varphi \) of \( (G, M, u) \), the map \( a \mapsto \varphi(au) : A \to \mathbb{R} \) is a ring homomorphism.

**Proof.** Let \( a, a' \in A \) be given. By the above Multiplicative Law (with \( g = a'y \in G \) in (1)),

\[
\varphi(aa'u) = \varphi(au)\varphi(a'u).
\]

We summarise the entire discussion in this section succinctly as follows. Given a pure state \( \varphi \) of \( (G, M, u) \), let \( \pi(\varphi) \subseteq A \) denote the kernel of the ring homomorphism \( a \mapsto \varphi(au) : A \to \mathbb{R} \). Corollary 3.2 says that \( \pi(\varphi) \) is a maximal ideal of \( A \) with residue field equal to \( \mathbb{R} \). The Multiplicative Law (Proposition 3.1) is simply the statement that the composite additive map

\[
G \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\sim} A/\pi(\varphi)
\]

is \( A \)-linear. Indeed, this composite additive map sends \( g \in G \) to the residue class \( \overline{\varphi(g)} := \varphi(g) + \pi(\varphi) \in A/\pi(\varphi) \). Multiplying this residue class by \( a \in A \) gives \( a\overline{\varphi(g)} = \overline{a\varphi(g)} = \overline{\varphi(au)} = \overline{\varphi(a)\varphi(g)} \). Thus (1) amounts to \( A \)-linearity:

\[
a\overline{\varphi(g)} = \overline{\varphi(ag)}.
\]

### 4. Proof of Theorem 1.1

Recall from Section 1 the linear form \( \Sigma := x_1 + \cdots + x_n \). The homogeneous localization of the polynomial ring \( \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \) by \( \Sigma \) is the following
subring of the field \( \mathbb{R}(x) \) of rational functions:

\[
\mathbb{R}[x](\Sigma) := \left\{ \frac{f}{\Sigma \deg f} \in \mathbb{R}(x) : f \in \mathbb{R}[x] \text{ is a form} \right\}.
\]

Let \( \mathbb{R}_+[x](\Sigma) \) be the subsemiring generated by \( \mathbb{R}_+ \) and the fractions \( x_1/\Sigma, \ldots, x_n/\Sigma \). So the elements of \( S \) are precisely the fractions \( f/\Sigma \deg f \) where \( f \) is a form all whose coefficients are nonnegative. Since \( \sum_{i=1}^n x_i/\Sigma = 1 \), it follows that \( \mathbb{R}_+[x](\Sigma) \) is an archimedean subsemiring of \( \mathbb{R}[x](\Sigma) \). This observation that \( \mathbb{R}_+[x](\Sigma) \) is archimedean is essentially due to Berr-Wörmann [1, Lemma 1], who thereby reduced Pólya’s Positivstellensatz to the Representation Theorem.

Now fix a positive integer \( r \). Let \( \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) denote the abelian group of symmetric \( r \times r \) matrices with entries in \( \mathbb{R}[x](\Sigma) \). Then \( \mathbb{R}[x](\Sigma) \) acts on \( \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) by entrywise multiplication, so that \( \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) has the structure of an \( \mathbb{R}[x](\Sigma) \)-module. Each element \( G \in \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) can be written canonically as a polynomial

\[
G = \sum_{\alpha} A_{\alpha} \otimes \frac{x^\alpha}{\Sigma |\alpha|},
\]

where the coefficients \( A_{\alpha} \) are symmetric \( r \times r \) matrices with real entries, finitely many of which are nonzero. Here, recall from Section 1 that \( \alpha \) is a multi-index in \( \mathbb{N}^n \) (i.e. an \( n \)-tuple of nonnegative integers) with length \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). Say that \( G \in \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) has positive semidefinite coefficients if, when written as in (4), all the \( A_{\alpha} \)'s are positive semidefinite. Let \( \text{Sym}_r(\mathbb{R}[x](\Sigma))_+ \subseteq \text{Sym}_r(\mathbb{R}[x](\Sigma)) \) denote the submonoid of matrices having positive semidefinite coefficients. This submonoid \( \text{Sym}_r(\mathbb{R}[x](\Sigma))_+ \) is closed under the action restricted to the semiring \( \mathbb{R}_+[x](\Sigma) \), and hence is a \( \mathbb{R}_+[x](\Sigma) \)-subsemimodule of \( \text{Sym}_r(\mathbb{R}[x](\Sigma)) \). The identity \( r \times r \) matrix \( I \) has positive semidefinite coefficients, and so lies in \( \text{Sym}_r(\mathbb{R}[x](\Sigma))_+ \).

**Proposition 4.1.** The identity \( r \times r \) matrix \( I \in \text{Sym}_r(\mathbb{R}[x](\Sigma))_+ \) is an order unit of \( (\text{Sym}_r(\mathbb{R}[x](\Sigma)), \text{Sym}_r(\mathbb{R}[x](\Sigma))_+) \).

**Proof.** Let \( G = \sum_{\alpha} A_{\alpha} x^\alpha / \Sigma |\alpha| \in G \) be given. After suitable multiplication of \( \Sigma \), we obtain a nonnegative integer \( d \) such that

\[
G = \sum_{|\alpha| = d} A_{\alpha}^{(d)} x^\alpha \sum_d
\]

for some family \( \{ A_{\alpha}^{(d)} \}_{|\alpha| = d} \) of real symmetric matrices indexed by multi-indices \( \alpha \) of length \( d \). Here \( \binom{d}{\alpha} = \frac{d!}{\alpha_1! \alpha_2! \cdots \alpha_n!} \) denotes the multinomial coefficient, as usual.
Then, for an integer $N$,

$$N + G = \sum_{|\alpha| = d} (NI + A'_\alpha) \left( \frac{d}{\sum d} \right) x^\alpha.$$ 

Since there are only finitely many multi-indices $\alpha$ of length $d$, we can choose $N > 0$ sufficiently large such that all the coefficients $NI + A'_\alpha$ are positive semidefinite (or even positive definite), so that $N + G$ lies in $M$. \hfill \Box

Next, we proceed to characterize the pure states of $(\text{Sym}_r(\mathbb{R}[x](\Sigma)), \text{Sym}_r(\mathbb{R}[x](\Sigma))_+, I)$. Let $\text{Sym}_r(\mathbb{R})$ denote the $\mathbb{R}$-linear space of symmetric $r \times r$ matrices with real entries and let $\text{Sym}_r(\mathbb{R})_+ \subseteq \text{Sym}_r(\mathbb{R})$ denote the cone of positive semidefinite matrices.

**Lemma 4.2.** For each pure state $\varphi$ of $(\text{Sym}_r(\mathbb{R}[x](\Sigma)), \text{Sym}_r(\mathbb{R}[x](\Sigma))_+, I)$, the restriction

$$\varphi|_{\text{Sym}_r(\mathbb{R})} : \text{Sym}_r(\mathbb{R}) \hookrightarrow \text{Sym}_r(\mathbb{R}[x](\Sigma)) \xrightarrow{\varphi} \mathbb{R}$$

is a pure state of $(\text{Sym}_r(\mathbb{R}), \text{Sym}_r(\mathbb{R})_+, I)$.

**Proof.** Suppose that $2\varphi|_{\text{Sym}_r(\mathbb{R})} = \overline{\varphi_1} + \overline{\varphi_2}$ where $\overline{\varphi_1}, \overline{\varphi_2}$ are states of $(\text{Sym}_r(\mathbb{R}), \text{Sym}_r(\mathbb{R})_+, I)$. For $i = 1, 2$, define the additive map $\varphi_i : \text{Sym}_r(\mathbb{R}[x](\Sigma)) \to \mathbb{R}$ given on generators by

$$(\varphi_i(A \frac{x^\alpha}{\sum |\alpha|}) = \overline{\varphi_i(A)} \varphi(I \frac{x^\alpha}{\sum |\alpha|}) \quad (A \in \text{Sym}_r(\mathbb{R}), \alpha \in \mathbb{N}^n).$$

Since $\overline{\varphi_i}$ and $\varphi$ are states in their respective spaces, it follows that $\varphi_i$ is a state of $(\text{Sym}_r(\mathbb{R}[x](\Sigma)), \text{Sym}_r(\mathbb{R}[x](\Sigma))_+, I)$. From (5),

$$(\varphi_1 + \varphi_2)(A \frac{x^\alpha}{\sum |\alpha|}) = (\overline{\varphi_1} + \overline{\varphi_2})(A) \varphi(I \frac{x^\alpha}{\sum |\alpha|}) = 2\varphi|_{\text{Sym}_r(\mathbb{R})}(A) \varphi(I \frac{x^\alpha}{\sum |\alpha|})$$

which agrees with

$$2\varphi(A \frac{x^\alpha}{\sum |\alpha|}) = 2\varphi(A) \varphi(I \frac{x^\alpha}{\sum |\alpha|}) = 2\varphi|_{\text{Sym}_r(\mathbb{R})}(A) \varphi(I \frac{x^\alpha}{\sum |\alpha|}),$$

where, in the first equality of the previous line, we used the Multiplicative Law (Proposition 3.1). Hence $2\varphi = \varphi_1 + \varphi_2$, so that $\varphi$ being a pure state implies that $\varphi = \varphi_1 = \varphi_2$. Thus, for a real symmetric matrix $A \in \text{Sym}_r(\mathbb{R})$,

$$\varphi|_{\text{Sym}_r(\mathbb{R})}(A) = \varphi(A) = \varphi_1(A) = \overline{\varphi_i(A)} \varphi(I) = \overline{\varphi_i(A)}, \quad (i = 1, 2)$$

where the last equality follows from (5). Therefore $\varphi|_{\text{Sym}_r(\mathbb{R})}$ is a pure state of $(\text{Sym}_r(\mathbb{R}[x](\Sigma)), \text{Sym}_r(\mathbb{R}[x](\Sigma))_+, I)$. \hfill \Box
We recall some standard facts about the linear space $\text{Sym}_r(\mathbb{R})$ of real symmetric matrices and the cone $\text{Sym}_r(\mathbb{R})_+ \subseteq \text{Sym}_r(\mathbb{R})$ of positive semidefinite matrices. The usual inner product $\langle A, M \rangle = \text{tr}(AM)$ on real symmetric matrices induces a pairing between $\text{Sym}_r(\mathbb{R})$ and its dual $\mathbb{R}$-linear space $\text{Sym}_r(\mathbb{R})^\vee$. Explicitly, associated to each symmetric matrix $M \in \text{Sym}_r(\mathbb{R})$ is an $\mathbb{R}$-linear functional $M^\vee : \text{Sym}_r(\mathbb{R}) \to \mathbb{R}$ given by

$$M^\vee(A) := \langle A, M \rangle \quad (A \in \text{Sym}_r(\mathbb{R})).$$

It is well-known that the cone $\text{Sym}_r(\mathbb{R})_+ \subseteq \text{Sym}_r(\mathbb{R})$ of positive semidefinite matrices is self-dual (see for e.g. [3, Example 2.24]). That is to say, the map that sends a positive semidefinite matrix $P \in \text{Sym}_r(\mathbb{R})_+$ to its associated functional $P^\vee$ is a bijection from $\text{Sym}_r(\mathbb{R})_+$ onto its dual cone

$$\text{Sym}_r(\mathbb{R})_+^\vee := \{ M^\vee \in \text{Sym}_r(\mathbb{R})^\vee : M^\vee(P) \geq 0 \text{ for all } P \in \text{Sym}_r(\mathbb{R})_+ \}.$$ 

In particular, a real symmetric matrix $M$ is positive semidefinite if and only if $M^\vee|_{\text{Sym}_r(\mathbb{R})} \geq 0$. Furthermore, a nonzero positive semidefinite matrix $P \in \text{Sym}_r(\mathbb{R})_+$ lies on an extremal ray of $\text{Sym}_r(\mathbb{R})_+$ if and only if $P^\vee$ lies on an extremal ray of $(\text{Sym}_r(\mathbb{R})_+)^\vee$. Here a nonzero point $p$ in a cone $C \subseteq \mathbb{R}^N$ contained in some linear space is said to lie on an extremal ray if for any two points $p_1, p_2 \in C$, the equation $2p = p_1 + p_2$ implies $p_i = a_ip$ for some nonnegative real number $a_i$ (for $i = 1, 2$).

**Corollary 4.3.** For each pure state $\varphi$ of $(\text{Sym}_r(\mathbb{R}[x]_{(\Sigma)}), \text{Sym}_r(\mathbb{R}[x]_{(\Sigma)})_+, I)$, there exists a unit vector $v \in \mathbb{R}^r$ such that

$$\varphi(A) = v^t Av \quad \text{for all } A \in \text{Sym}_r(\mathbb{R}).$$

*Proof.* Let a pure state $\varphi$ of $(\text{Sym}_r(\mathbb{R}[x]_{(\Sigma)}), \text{Sym}_r(\mathbb{R}[x]_{(\Sigma)})_+, I)$ be given. By the above lemma, its restriction $\varphi|_{\text{Sym}_r(\mathbb{R})_+}$ is a pure state of $(\text{Sym}_r(\mathbb{R}), \text{Sym}_r(\mathbb{R})_+, I)$. Since $\varphi|_{\text{Sym}_r(\mathbb{R})_+} \geq 0$, there exists a positive semidefinite matrix $M$ such that $\varphi|_{\text{Sym}_r(\mathbb{R})_+} = M^\vee$ (c.f. (6)). Hence $\varphi|_{\text{Sym}_r(\mathbb{R})_+}$ being an extreme point of the set of states $S(\text{Sym}_r(\mathbb{R}), \text{Sym}_r(\mathbb{R})_+, I)$ implies that the functional $M^\vee$ lies on an extremal ray of $(\text{Sym}_r(\mathbb{R})_+)^\vee$. The self-duality of $\text{Sym}_r(\mathbb{R})_+$ then implies that $M$ itself lies on an extremal ray of $\text{Sym}_r(\mathbb{R})_+$, so that $M = vv^t$ for some vector $v \in \mathbb{R}^r$. Thus, for all real symmetric matrices $A \in \text{Sym}_r(\mathbb{R})$,

$$\varphi(A) = \varphi|_{\text{Sym}_r(\mathbb{R})_+}(A) = (vv^t)^\vee(A) = \langle A, vv^t \rangle = v^t Av.$$
In particular, since \(1 = \phi|_{\text{Sym}_r(R)_+}(I) = v^T v\), so the vector \(v = (v_i)_{i=1}^r\) has unit norm:
\[
\|v\| := \sqrt{\sum_{i=1}^r |v_i|^2} = \sqrt{v^T v} = 1.
\]

Recall from Section 1 the standard codimension-1 simplex
\[
\Delta_n = \{ x \in \mathbb{R}^n : x_1, \ldots, x_n \geq 0, \Sigma(x) = 1 \}.
\]

**Proposition 4.4.** For each pure state \(\phi\) of \((\text{Sym}_r(R[x]_\Sigma), \text{Sym}_r(R[x]_\Sigma)_+, I)\), there exists \(x \in \Delta_n\) and a unit vector \(v \in \mathbb{R}^r\) such that
\[
\phi(\sum_\alpha A_\alpha \frac{x^\alpha}{\Sigma(|\alpha|)}) = \sum_\alpha v^T A_\alpha v \frac{x^\alpha}{\Sigma(|\alpha|)} \in \mathbb{R} \quad \text{for all } \alpha \in \mathbb{N}^n, A_\alpha \in \text{Sym}_r(R).
\]

**Proof.** Let \(\phi\) be a pure state of \((\text{Sym}_r(R[x]_\Sigma), \text{Sym}_r(R[x]_\Sigma)_+, I)\). By Corollary 3.2 the map
\[
\frac{f}{\Sigma_{\deg f}} \mapsto \phi\left(\frac{f}{\Sigma_{\deg f}}\right) : \mathbb{R}[x]_\Sigma \to \mathbb{R}
\]
is a ring homomorphism. Now, every ring homomorphism from \(\mathbb{R}[x]_\Sigma\) to \(\mathbb{R}\) has the form \(\frac{f}{\Sigma_{\deg f}} \mapsto f(x)\) for some \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) with \(\Sigma(x) = x_1 + \cdots + x_n = 1\). Thus there exists an \(x \in \mathbb{R}^n\) satisfying \(\Sigma(x) = 1\) such that
\[
\phi\left(\frac{f}{\Sigma_{\deg f}}\right) = f(x) \quad \text{for all forms } f \in \mathbb{R}[x].
\]

But, for all \(i = 1, \ldots, n\), the matrix \(I_X^\Sigma\) has positive semidefinite coefficients and so lies in \(\text{Sym}_r(R[x]_\Sigma)_+\), hence \(x_i = \phi\left(\frac{I_X^\Sigma}{\Sigma(|\alpha|)}\right) \geq 0\) by the montonicity of \(\phi\). Therefore, by the definition of the standard simplex (7),
\[
\Sigma(x) = 1.
\]

Apply Corollary 4.3 to obtain a unit vector \(v \in \mathbb{R}^r\) such that \(\phi(A) = v^T A v\) for all real symmetric matrices \(A\). Hence by the Multiplicative Law (Proposition 3.1),
\[
\phi\left(\sum_\alpha A_\alpha \frac{x^\alpha}{\Sigma(|\alpha|)}\right) = \sum_\alpha \phi(A_\alpha) \phi\left(\frac{x^\alpha}{\Sigma(|\alpha|)}\right) = \sum_\alpha v^T A v x^\alpha,
\]
where we used (9) in the last equality. \(\square\)

**Proof of Theorem 1.1.** Let \(B\) be a symmetric \(r \times r\) matrix, whose entries are real forms in \(\mathbb{R}[x]\) of degree \(d\). Suppose that \(B(x)\) is positive definite for all \(x \in \Delta_n\). Since \(\Delta_n\) is compact, we may choose \(\varepsilon > 0\) small enough so that \(B(x) - \varepsilon I\) is
positive definite for all \( x \in \Delta_n \). Here \( I \) denotes the identity \( r \times r \) matrix, as above. Write

\[
B - \epsilon I \Sigma^d = \sum_{|\alpha| = d} A_\alpha x^\alpha
\]

as a polynomial whose coefficients \( A_\alpha \) are real symmetric matrices. We associate to \( B - \epsilon I \Sigma^d \) the element

\[
G := \sum_{|\alpha| = d} A_\alpha x^\alpha \Sigma^d
\]

in \( \text{Sym}_r([R[x]|_{\Sigma}) \).

We wish to apply the Goodearl-Handelman Criterion (Lemma 2.1) to show that \( G \) lies in \( \text{Sym}_r([R[x]|_{\Sigma})_+ \). For this purpose, let a pure state \( \varphi \) of \( (\text{Sym}_r([R[x]|_{\Sigma}), \ \text{Sym}_r([R[x]|_{\Sigma})_+, I) \) be given. By Proposition 4.4, there exists \( x \in \Delta_n \) and a unit vector \( v \in R^r \) such that

\[
\varphi(G) = \sum_{|\alpha| = d} v^T A_\alpha v x^\alpha = v^T (B(x) - \epsilon I) v,
\]

where the last equality follows from (11). Since \( B(x) - \epsilon I \) is positive definite, we have \( \varphi(G) > 0 \). Thus we may apply the Goodearl-Handelman Criterion to obtain a positive integer \( k \) such that \( kG \) lies in \( \text{Sym}_r([R[x]|_{\Sigma})_+ \). But \( \text{Sym}_r([R[x]|_{\Sigma})_+ \) is closed under multiplication by nonnegative scalars, hence \( G \) itself lies in \( \text{Sym}_r([R[x]|_{\Sigma})_+ \), i.e. all its coefficients \( A_\alpha \) are positive semidefinite matrices (for \( |\alpha| = d \)). Therefore, from (11),

\[
B = \epsilon I \Sigma^d + \sum_{|\alpha| = d} A_\alpha x^\alpha
\]

strictly has positive coefficients, since \( \epsilon I \Sigma^d \) strictly has positive coefficients. \( \square \)

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Li Gao, Department of Physics, Shandong University of Technology, Zhangzhou Rd.
12, Zibo 255049, China
E-mail address: gaoli2017@aliyun.com

Colin Tan, Department of Statistics & Applied Probability, National University of Singapore
E-mail address: statwc@nus.edu.sg