Hamilton-Jacobi theory for one dimensional autonomous systems using parametric transformations

G. González *

Departamento de Matemáticas y Física
I.T.E.S.O.

Periférico Sur # 8585 C.P. 45090
Guadalajara, Jalisco, México.

Abstract

A necessary and sufficient condition for a parameter transformation that leaves invariant the energy of a one dimensional autonomous system is obtained. Using a parameter transformation the Hamilton-Jacobi equation is solved by a quadrature. An example of this approach is given.

Keywords: Parameter transformation, Lagrangian, Hamiltonian, Hamilton-Jacobi equation, autonomous system.

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*E-mail: gabel_glez@yahoo.com
1 Introduction

It is well known that the problem of integrating Hamilton’s canonical equations of motion, which are expressed as a system of ordinary differential equations of the first order, can be replaced by an equivalent problem of finding a complete solution of a single non linear partial differential equation of the first order, called the Hamilton-Jacobi equation [4]. If we succeed in finding a complete solution of the Hamilton-Jacobi partial differential equation, the motion of the dynamical system can be easily found directly from this complete solution by using only elementary algebraic operations [9].

The Hamilton-Jacobi equation can be considered the most elegant and powerful method known for finding the general solution to the mechanical equations of motion and gives an important physical example of the deep connection between first order partial differential equations and first order ordinary differential equations [8]. Each complete integral of the Hamilton-Jacobi equation gives rise to a family of solutions of Hamilton’s equations and is the generating function of the canonical transformation which maps the dynamical system to a trivial one with vanishing Hamiltonian.

Apart from its practical aspects, the Hamilton-Jacobi theory leads to a geometric picture of dynamics relating the dynamics to wave motion and has been the starting point for Schrödinger to state the wave equation in quantum mechanics [2].

The applicability of using the Hamilton-Jacobi procedure depends in our ability to solve a first order non linear partial differential equation which can be very complicated to solve in some cases [7]. The main purpose of this article is to show that, under certain conditions, it is possible to introduce a parameter transformation
which allow us to obtain a complete integral for the Hamilton-Jacobi equation for one dimensional autonomous systems without having to solve the partial differential equation given by the Hamilton-Jacobi method.

## 2 Parameter transformations and the Hamilton-Jacobi equation

Newton’s equation of motion for one dimensional autonomous systems can be written as the following dynamical system

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = F(x, v), \quad (1)$$

where $x$ is the position of the particle, $v$ is the velocity and $F(x, v)$ is the force divided by the mass of the particle. It is well known that the Lagrangian $L(x, v)$ for a one dimensional system always exist [3] thus we may obtain (1) from the Euler-Lagrange equation [4]

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v} \right) = \frac{\partial L}{\partial x}, \quad (2)$$

and the associated constant of motion for (1) from the Legendre transformation [4]

$$K(x, v) = v \frac{\partial L}{\partial v} - L, \quad (3)$$

the numerical value of (3) will be referred as the energy of the system. Knowing the constant of motion of (1) we can obtain the associated Hamiltonian by expressing the constant of motion in terms of the canonical variables $H(x, p) = K(x, v(x, p))$ and using this Hamiltonian we obtain the Hamilton-Jacobi equation

$$H \left( x, \frac{\partial S}{\partial x} \right) + \frac{\partial S}{\partial t} = 0, \quad (4)$$
where $S$ is known as Hamilton’s principal function [4].

Let us now consider the following parameter transformation of the form

$$
\tau = \tau(t), \quad \tilde{x} = x(t), \quad \text{(5)}
$$

which does not affect the position variable and it will be assumed that the functions $\tau(t)$ are of class $C^2$ such that $d\tau/dt > 0$. The problem is to find a parameter transformation which leaves invariant the energy of the system when subject to this type of transformation. To do this let $L(x, v)$ and $\tilde{L}(\tilde{x}, \tilde{v})$ be the Lagrangians for (1) before and after the parameter transformation respectively. Assume that $\partial^2 L/\partial v^2$ and $\partial^2 \tilde{L}/\partial \tilde{v}^2$ do not vanish or become infinite in some region $R$ of the dynamical space then we have the following

**Theorem 1** A necessary and sufficient condition for a parameter transformation to leave invariant the energy of a one dimensional autonomous system is that

$$
\frac{\partial^2 L}{\partial v^2} \frac{dv}{dt} = \frac{\partial^2 \tilde{L}}{\partial \tilde{v}^2} \frac{d\tilde{v}}{d\tau}.
$$

**Proof 1** The Euler-Lagrange equation for $L(x, v)$ may be written as

$$
\frac{\partial^2 L}{\partial x \partial v} v + \frac{\partial^2 L}{\partial v^2} \frac{dv}{dt} = \frac{\partial L}{\partial x},
$$

similarly the Euler-Lagrange equation for $\tilde{L}(\tilde{x}, \tilde{v})$ is given by

$$
\frac{\partial^2 \tilde{L}}{\partial \tilde{x} \partial \tilde{v}} \tilde{v} + \frac{\partial^2 \tilde{L}}{\partial \tilde{v}^2} \frac{d\tilde{v}}{d\tau} = \frac{\partial \tilde{L}}{\partial \tilde{x}},
$$

by assumption of the theorem we have

$$
\frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial v} v = \frac{\partial \tilde{L}}{\partial \tilde{x}} - \frac{\partial^2 \tilde{L}}{\partial \tilde{x} \partial \tilde{v}} \tilde{v}.
$$
therefore
\[ \frac{\partial}{\partial x} \left( L - v \frac{\partial L}{\partial v} \right) = \frac{\partial}{\partial \tilde{x}} \left( \tilde{L} - \tilde{v} \frac{\partial \tilde{L}}{\partial \tilde{v}} \right), \]
which implies
\[ \frac{\partial K(x, v)}{\partial x} = \frac{\partial \tilde{K}(\tilde{x}, \tilde{v})}{\partial \tilde{x}}. \]
Taking the total time derivative of \( \tilde{K}(\tilde{x}, \tilde{v}) \) we obtain
\[ \frac{d\tilde{K}}{dt} = \frac{\partial \tilde{K}}{\partial \tilde{x}} \frac{d\tilde{x}}{dt} + \frac{\partial \tilde{K}}{\partial \tilde{v}} \frac{d\tilde{v}}{dt}, \]
using the fact that \( \frac{\partial \tilde{K}}{\partial \tilde{v}} = (\partial^2 \tilde{L}/\partial \tilde{v}^2) \tilde{v} \) then
\[ \frac{d\tilde{K}}{dt} = \frac{d\tilde{x}}{dt} \frac{d\tau}{dt} \left( \frac{\partial \tilde{K}}{\partial \tilde{x}} + \frac{\partial^2 \tilde{L}}{\partial \tilde{v}^2} \frac{d\tilde{v}}{d\tau} \right) \]
but \( \frac{dx}{dt} = (d\tilde{x}/d\tau)(d\tau/dt) \), \( \frac{\partial K}{\partial x} = \frac{\partial \tilde{K}}{\partial \tilde{x}} \) and \( \partial^2 L/\partial v^2 (dv/dt) = \partial^2 \tilde{L}/\partial \tilde{v}^2 (d\tilde{v}/d\tau) \) then
\[ \frac{dK}{dt} = \frac{d\tilde{K}}{d\tilde{\tau}} \]
which implies that \( K(x, v) = \tilde{K}(\tilde{x}, \tilde{v}) \) and completes the proof of the theorem.

The main advantage of using this type of parameter transformation is that neither the position or the energy of the system are affected, using this fact and taking into account that we are dealing with an autonomous system we can express the solution to the Hamilton-Jacobi equation as [4]
\[ S = W(x) - Et, \quad \tilde{S} = \tilde{W}(x) - E \tau, \quad (6) \]
where \( S \) and \( \tilde{S} \) are Hamilton’s principal functions for (1) before and after the parameter transformation respectively and \( W \) and \( \tilde{W} \) represents the solution to a nonlinear first order partial differential equation given by [4]
\[ H \left( x, \frac{dW}{dx} \right) = E, \quad \tilde{H} \left( x, \frac{d\tilde{W}}{dx} \right) = E, \quad (7) \]
which is usually known as Hamilton’s characteristic function. If we know how the
generalized linear momentum transforms under (5) and recalling that
\[ \frac{dW}{dx} = p, \quad \frac{d\tilde{W}}{dx} = \tilde{p}, \] (8)
we may express Hamilton’s principal function for (1) as
\[ S = \int p(x, \tilde{p}(x, E)) \, dx - Et. \] (9)

Therefore, we obtain the solution for the Hamilton-Jacobi equation by a quadrature.
This way of solving the Hamilton-Jacobi equation is useful to solve complicated
problems as it would be shown in the following example.

3 Example

Consider a relativistic particle of mass at rest \( m \) under the action of a constant
force \( \lambda > 0 \) and immersed in a medium that exerts some type of friction which is
proportional to the square of the velocity. The classical equation of motion for this
system is given by
\[ m \frac{dv}{dt} = (\lambda - \gamma v^2)(1 - \frac{v^2}{c^2})^{3/2}, \] (10)
where \( \gamma \) is a positive real parameter and \( c \) represents the speed of light. Writing
(10) at first order of approximation in \( \frac{v^2}{c^2} \) we have
\[ m \frac{dv}{dt} = (\lambda - \gamma v^2)(1 - \alpha^2 v^2), \] (11)
where \( \alpha^2 = \frac{3}{2x^2} \). The Lagrangian associated to this system is [5]
\[ L(x, v) = \frac{mv}{2\alpha} \tanh^{-1}(\alpha v) e^{-2x(\lambda \alpha^2 - \gamma)/m} - \frac{m\lambda}{2(\lambda \alpha^2 - \gamma)} \left( e^{2x(\lambda \alpha^2 - \gamma)/m} - 1 \right), \] (12)
and the Hamiltonian of this system is given by [5]

\[ H(x, p) = \frac{m}{2} e^{-2x(\lambda \alpha^2 - \gamma)/m} \sum_{n=0}^{\infty} \alpha^n \nu^{2n+2}(x, p) + \frac{m \lambda}{2(\lambda \alpha^2 - \gamma)} \left( e^{-2x(\lambda \alpha^2 - \gamma)/m} - 1 \right), \]  

(13)

where \( \nu^{2n+2}(x, p) \) is given by

\[ \nu^{2n+2}(x, p) = \left( \frac{p}{m} e^{-2\gamma x/m} \frac{2n + 1}{(n + 1)!} \left( \frac{2 \lambda x}{m} \right)^n \right)^{(2n+2)/(2n+1)}. \]  

(14)

The Hamiltonian (13) is valid for the case \(|\alpha \nu| < 1\) and has physical meaning only when \( p > 0 \) and \( x > 0 \). Once knowing the Hamiltonian we can obtain the Hamilton-Jacobi equation for the system

\[ \frac{m}{2} e^{-2x(\lambda \alpha^2 - \gamma)/m} \sum_{n=0}^{\infty} \alpha^n \left( \frac{\partial S}{\partial x} e^{-2\gamma x/m} \frac{2n + 1}{m(n + 1)!} \left( \frac{2 \lambda x}{m} \right)^n \right)^{(2n+2)/(2n+1)} + \frac{m \lambda}{2(\lambda \alpha^2 - \gamma)} \left( e^{-2x(\lambda \alpha^2 - \gamma)/m} - 1 \right) + \frac{\partial S}{\partial t} = 0, \]  

(15)

therefore one has to solve (15) to obtain Hamilton’s principal function, which at first sight may seem like a formidable task, but it can be done applying the approach described in the last section.

Consider the following parameter transformation

\[ \tau = \int_0^t \sqrt{1 - \alpha^2 \nu^2} \, dt, \quad \tilde{x} = x, \]  

(16)

therefore equation (11) transforms into

\[ m \frac{d\tilde{v}}{d\tau} = \lambda + \tilde{\gamma} \tilde{v}^2, \]  

(17)

where \( \tilde{\gamma} = \lambda \alpha^2 - \gamma \) and the transformation equations between one set of dynamical variables \((x, v)\) to the other set of dynamical variables \((\tilde{x}, \tilde{v})\) are given by

\[ \frac{x}{\nu} = \frac{\tilde{x}}{\sqrt{1 + \alpha^2 \nu^2}}, \quad \frac{\nu}{\sqrt{1 - \alpha^2 \nu^2}} = \frac{\tilde{v}}{\sqrt{1 - \alpha^2 \nu^2}} \]  

(18)
The Lagrangian and the Hamiltonian for (17) can be obtained, and are given by [6]

\[ \tilde{L}(x, \tilde{v}) = \frac{m}{2} \tilde{v}^2 e^{-2\tilde{\gamma}x/m} - \frac{m\lambda}{2\tilde{\gamma}} (e^{-2\tilde{\gamma}x/m} - 1), \]  

(19)

\[ \tilde{H}(x, \tilde{p}) = \frac{\tilde{p}^2}{2m} e^{2\tilde{\gamma}x/m} + \frac{m\lambda}{2\tilde{\gamma}} (e^{-2\tilde{\gamma}x/m} - 1), \]  

(20)

where we have dropped the tilde for the position variable for the sake of simplicity.

Using (10), (12), (17) and (19) it is easy to convince oneself that theorem 1 is fulfill, therefore the energy is invariant under transformation (16). What we need now is to find how the generalized linear momentum transforms under (16), to do that we express \( \tilde{p} = m\tilde{v} e^{-2\tilde{\gamma}x/m} \) in terms of \( v \) using (18), which gives

\[ \tilde{p} = m\tilde{v} e^{-2\tilde{\gamma}x/m} = \frac{mve^{-2\tilde{\gamma}x/m}}{\sqrt{1 - \alpha^2v^2}} = me^{-2\tilde{\gamma}x/m} \sum_{n=0}^{\infty} \frac{(2n)!}{2^n(2n)!} \frac{\alpha^2}{2} \frac{v^{2n+1}}{v^{2n+1}}, \]  

(21)

where we have used the fact that \( |\alpha v| < 1 \) in the last step. Substituting (14) into (21) we have the way the generalized linear momentum transforms under (16)

\[ \tilde{p} = pe^{-2\lambda^2x^2/m} \sum_{n=0}^{\infty} \frac{(2n+1)(2n)!}{2^n(n+1)(n)!^2} \left( \frac{\alpha^2}{m} \right)^n = p e^{-\lambda^2x^2/m} \left( I_0(\lambda\alpha^2x/m) + I_1(\lambda\alpha^2x/m) \right), \]  

(22)

where \( I_n(z) \) is the modified Bessel function of the first kind [1]. Using the fact that

\[ \tilde{p} = e^{-\tilde{\gamma}x/m} \sqrt{2mE - \frac{m^2\lambda}{\tilde{\gamma}} (e^{-2\tilde{\gamma}x/m} - 1)}, \]  

(23)

and substituting (23) into (22) we have

\[ p(x, \tilde{p}(x, E)) = \frac{e^{\gamma x/m} \sqrt{2mE - \frac{m^2\lambda}{\alpha^2} (e^{-2(\lambda\alpha^2-\gamma)x/m} - 1)}}{I_0(\lambda\alpha^2x/m) + I_1(\lambda\alpha^2x/m)}, \]  

(24)

substituting (24) into (9) we obtain Hamilton’s principal function for \( x > 0 \)

\[ S = \int \frac{e^{\gamma x/m} \sqrt{2mE - \frac{m^2\lambda}{\alpha^2} (e^{-2(\lambda\alpha^2-\gamma)x/m} - 1)}}{I_0(\lambda\alpha^2x/m) + I_1(\lambda\alpha^2x/m)} \, dx - Et. \]  

(25)

All the expressions derived in this paper have the right limit when \( \gamma \to 0 \) and \( \alpha \to 0 \).
4 Conclusions

A necessary and sufficient condition for a parameter transformation that leaves invariant the energy of a one dimensional autonomous system was obtained. A method for solving the Hamilton-Jacobi equation using a parameter transformation was deduced. All the expressions obtained in this paper converge to the conservative case when the dissipation parameter goes to zero.
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