Chiral Splitting at Work.

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Abstract

Second-order equations of motion on a group manifold that appear in a large class of so-called chiral theories are presented. These equations are presented and explicitly solved for cases of semi-simple, finite-dimensional Lie groups.

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1 Introduction

In this paper we would like to present a solution of a seemingly complicated differential equation by means of a trick known in 2-dimensional field theory as chiral splitting \[4\]. The equations themselves arise in a quite natural way as equations of motion of a particle on a group manifold $G$. The phase space is then the cotangent bundle $T^*G$, which can be identified with $G \times \mathcal{G}$, provided there exists a non-degenerate $Ad$-invariant quadratic form on $\mathcal{G}$ (Killing form). The group $G$ is equipped with its natural left and right action on itself, but the liftings of those actions to actions on the bundle are not unique. The inequivalent liftings are labelled by group-cocycles $\theta$ described below. Choosing particular liftings (and a particular cocycle) typically requires a modification to the canonical symplectic structure on $T^*G$ in order to make it invariant under the lifted actions. The lifted actions are hamiltonian with respect to the modified symplectic form and they admit weak momentum mappings. If the Hamiltonian is taken as a sum of the squares of the momentum mappings (for the canonical lift corresponding to $\theta = 0$ that gives $H = \frac{1}{2}p^2$) there is a family of dynamics corresponding to different group cocycles.

Many details and much deeper discussion of this geometry can be found in \[2\], \[3\]. Here we would like to concentrate rather on the equations themselves.

2 The Equations

Consider a phase space $G \times \mathcal{G}$, where $G$ is a Lie group and $\mathcal{G}$ is its Lie algebra. Of course it can be identified with $TG$. We assume that there exist an $Ad$-invariant metric $K$ on $\mathcal{G}$.

Let $\theta$ be a group cocycle, i.e.

$$
\forall g_1, g_2 \in G \quad \theta(g_1 g_2) = \theta(g_1) + Ad_{g_1} \theta(g_2) \in \mathcal{G},
$$

and let $\Sigma$ be the derivative of $\theta$:

$$
\Sigma(X) := \frac{d}{dt} \theta(e^{tX})|_{t=0} \quad \forall X \in \mathcal{G}.
$$

We assume that $\theta$ is symplectic with respect to $K$, i.e.

$$
K(\Sigma(X) , Y) = -K(X , \Sigma(Y)) \quad \forall X, Y \in \mathcal{G}
$$

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and the symplectic structure on $G \times G$ is given by the 2-form:

$$\Omega = dK(p, g^{-1}dg) - K(Ad_g^{-1}\theta(g), dg^{-1}dg).$$ (4)

The moments corresponding to the right and left lifted actions are:

$$J^r := p + \theta(g^{-1}) ; \quad J^l := -Ad_g p - \theta(g).$$ (5)

For the Hamiltonian

$$H = \frac{1}{4}K(J^r, J^r) + \frac{1}{4}K(J^l, J^l)$$ (6)

one gets the following equations of motion

$$g^{-1}\dot{g} = p$$ (7)

$$\dot{p} = -\Sigma(\theta(g^{-1})) + [p, \theta(g^{-1})].$$ (8)

The equations (7),(8) are not transparent and even if one is able to write down a formal solution, it is in general not obvious what the trajectories look like etc\footnote{3}

3 Abelian case

There is one case when the equations of motion (7), (8) are particularly simple and this is when $G$ is abelian ($G \cong \mathbb{R}^n$). Then the $Ad$ representation is trivial and the condition (4) tells us that $\theta$ is a linear operator. The derivative $\Sigma$ is numerically equal to $\theta$ (although it acts on a different space), the translations in velocities are trivial and (8) reduce to\footnote{4}

$$\ddot{x} = -\theta^T \theta x.$$ (9)

Here $\theta^T$ is a transposition defined with respect to the Euclidean form on $\mathbb{R}^n$. The r.h.s of (9) is never positive. Therefore in the abelian case the motion is either oscillatory (negative eigenvectors) or free (null directions of $\theta$).

\footnote{3}{Note however that for $\theta = 0$ one recovers the motion along geodesics ($\dot{p} = 0$).
\footnote{4}{We write $x$ instead of $g$ as it is an element of a vector space here.}
4 Semi-simple case

On a semi-simple, finite-dimensional Lie group $G$ the cocycle $\theta$ is necessarily of the form:

$$\theta(g) = \text{Ad}_g \mu - \mu; \quad \mu \in G$$  \hfill (10)

Then from (2) one has:

$$\Sigma(X) = \text{ad}_X \mu \quad \forall X \in G$$  \hfill (11)

and the equations of motion take the form:

$$g^{-1} \dot{g} = p; \quad \dot{p} = [p + \mu, \text{Ad}_g^{-1} \mu - \mu].$$  \hfill (12)

Solving these equations seems difficult, because the 'brute force' methods are likely to give the solution in form of ordered exponents, which are hardly transparent. Nevertheless it turns out that there is a simple formula for the solution of (12) for a starting point $(g_0, p_0)$, given by:

$$g(t) = \exp\{-\mu t\} g_0 \exp\{(p_0 + \mu + \text{Ad}_{g_0} \mu)t\} \exp\{-\mu t\}.$$  \hfill (13)

Deriving (13) relies heavily on parametrizing as much of the phase space as possible by 'chiral momenta' (5), which satisfy simple, time independent linear equations:

$$\dot{J}^l = \Sigma(J^l); \quad \dot{J}^r = -\Sigma(J^r).$$  \hfill (14)

Let us assume the following Ansatz for the solution (5):

$$g(t) = u_l(t) u_r(t).$$  \hfill (15)

Then using (7) and the cocycle property (4) we can write

$$J^r(g, p) = u_l^{-1} \dot{u}_r + \theta(u_l^{-1}) + \text{Ad}_{u_l^{-1}}(u_l^{-1} \dot{u}_l + \theta(u_l^{-1}))$$  \hfill (16)

and

$$- J^l(g, p) = \dot{u}_l u_l^{-1} + \theta(u_l) + \text{Ad}_{u_l}(\dot{u}_r u_r^{-1} + \theta(u_r)).$$  \hfill (17)

5 In fact this means considering the model as a symplectic reduction of a larger system by suitable constrains. The details of this geometric interpretation can be found in a forthcoming paper.
Introducing new variables

\[ \xi_l := u_l^{-1} \dot{u}_l + \theta(u_l^{-1}) \]  

and

\[ \xi_r := \dot{u}_r u_r^{-1} + \theta(u_r) \]  

we can write

\[ J_r = Ad_{u_r}^{-1}(\xi_l + \xi_r - 2\theta(u_r)) \]  

\[ - J^l = Ad_{u_l}(\xi_l + \xi_r - 2\theta(u_l^{-1})) \]  

A straightforward calculation shows that each of the equations (14) is equivalent to:

\[ (\xi_l + \xi_r) + \left( \Sigma - \frac{1}{2} ad_{\xi_l+\xi_r}(\xi_l - \xi_r) \right) = 0. \]  

This resembles the zero-curvature equation (the Gauss law constraint) of chiral two-dimensional field theory [4]. The set of solutions to (22) is much too large for our purposes. We shall restrict \( \xi_l \) and \( \xi_r \) (simplifying the equations at the same time) by using an obvious gauge invariance of (13). The transformation \( u_r \rightarrow hu_r, u_l \rightarrow u_l h^{-1} \) with \( h \) arbitrarily time-dependent, leaves (3) and (22) invariant, while (18), (19) change as follows:

\[ (\xi_l + \xi_r) \rightarrow Ad_h(\xi_l + \xi_r) + 2\theta(h) \]  

\[ (\xi_l - \xi_r) \rightarrow Ad_h(\xi_l - \xi_r) - \dot{h} h^{-1} \]  

From the above equations one can see that the difference \( \xi_l - \xi_r \) can always be transformed to be zero. Moreover, such a 'gauge' remains invariant under time-independent transformations.

Then the equation (22) tells us that \( \xi = \xi_l = \xi_r \) is time independent. Also, using time independent gauge transformations, one can always make \( \xi \) belong to the kernel of \( \Sigma \).

The equations (9) can be written as:

\[ p + Ad_{u_l u_r} \mu + \mu = 2 Ad_{u_r}^{-1}(\xi + \mu), \]  

and the equations (18), (19) can be regarded now as simple equations of motion for \( u_r \) and \( u_l \) respectively:

\[ \dot{u}_l = -\mu u_l + u_l(\mu + \xi_l) \]  

\[ \dot{u}_r = (\xi_r + \mu) u_r - u_r \mu \]
with obvious solutions given by:

\[ u_l(t) = \exp\{-\mu t\} u_l(0) \exp\{\mu + \xi_l\} t \]  
\[ u_r(t) = \exp\{\mu + \xi_r\} t u_r(0) \exp\{-\mu t\}. \]  

(27) (28)

Now taking \( g(t) = u_l(t)u_r(t) \) gives

\[ g(t) = \exp\{-\mu t\} u_l(0) \exp\{(2\mu + \xi_l + \xi_r) t \} u_r(0) \exp\{-\mu t\} \]  

(29)

which is exactly equal to (13) because from (25) one has

\[ u_l(0) \exp\{(2\mu + \xi_l + \xi_r) t \} u_r(0) =\]

\[ = g(0) \exp\{(p(0) + \mu + Ad_{g(0)}\mu)t\}. \]  

(30)

5 Su(2) (pictures)

In this case of low dimensional group it is possible to plot some of the trajectories.

In its fundamental representation \( \text{Su}(2) \) is a group of \( 2 \times 2 \) matrices of the form:

\[ g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} ; \quad |a|^2 + |b|^2 = 1 \]  

(31)

where \( a, b \in \mathbb{C} \) and bar denotes complex conjugation. The Lie algebra \( su(2) \) is spanned by the antihermitean generators \( l_k; k = 1, 2, 3 \) proportional to Pauli matrices. The momentum is therefore given as

\[ p = p^k l_k; \quad k = 1, 2, 3. \]  

(32)

\( \text{Su}(2) \) is a simple group and therefore every cocycle is of the form (10) and without any loss of generality we can assume that

\[ \theta(g) = m(Ad_g l_3 - l_3); \]  

(33)

where \( m \in \mathbb{R} \) is an arbitrary parameter and \( l_3 \) is an anti-hermitian matrix:

\[ l_3 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}. \]  

(34)
The group is identified with $S^3$ and therefore general trajectories on it may be difficult to present. However, for trajectories starting at the group unity (or any other element of $G$ of the form $e^{it_3}$) a straightforward calculation shows that for evolution given by (13) the phase of $b$ of (31) is constant. Therefore one is left with three real parameters: $Re(a), Im(a), |b|$, and these trajectories can be presented as lying on $S^2$. The figures (1,2,3) present some of these trajectories plotted by means of the Matematica™ program.

Note that the model is in a sense 'linear', i.e. a shape of a trajectory depends only on the ratios of $p_k$ and $\mu$ and multiplying them by a constant amounts just to rescaling of the time variable $t$. The real parameter $m \in \mathbb{R}_+$ is in many respects similar to a strenght of magnetic field. In particular if initial momentum is parallell to $l_3$ then the motion is 'free' (along the big circle).

Figure 1. presents beginnings of a family of trajectories starting at the group unity (the top of an imaginary ball) for fixed $p = (0, -2, 4)$ and different values of $m$. For $m = 0$ the trajectory is a circle: As we have already pointed out if $m = 0$ the trajectories are just geodesics. As $m$ increases the circle gets more and more distorted. Then for $m = 2.11$ it closes again and then opens again etc.

Actually, it is enough to look at (13) to see that it is sufficient to have $|p + 2\mu|s = |\mu|r$ for some $s, r \in \mathbb{Z}$ to have a closed trajectory, but the period may be relatively long.

Figure 2. presents (a beginning of) a typical trajectory with initial velocity perpendicular to $\mu$. The actual values are $p = (-1, 0, 0), m = -5$ and $t = 15$.

Figure 3. presents a trajectory for $p = (-1, 2, 0)$ and $m = 4.085$. The feature of it we would like to point out are 'loops'. As the value of $m$ increases they are getting smaller and eventually vanish. At this point a trajectory has 'cusps' i.e. there are points at which the velocity drops to zero. This is very similar to a cycloidal motion of a particle in crossed electric and magnetic fields.

References

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