Higher Order Moments of the Matter Distribution
in Scale–Free Cosmological Simulations
with Large Dynamic Range

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Abstract. We calculate reduced moments $\bar{\xi}_q$ of the matter density fluctuations, up to order $q = 5$, from counts in cells produced by Particle–Mesh numerical simulations with scale–free Gaussian initial conditions. We use power–law spectra $P(k) \propto k^n$ with indices $n = -3, -2, -1, 0, 1$. Due to the supposed absence of characteristic times or scales in our models, all quantities are expected to depend on a single scaling variable. For each model, the moments at all times can be expressed in terms of the variance $\bar{\xi}_2$, alone. We look for agreement with the hierarchical scaling ansatz, according to which $\bar{\xi}_q \propto \bar{\xi}_2^{q-1}$. For $n \leq -2$ models we find strong deviations from the hierarchy, which are mostly due to the presence of boundary problems in the simulations. A small, residual signal of deviation from the hierarchical scaling is however also found in $n \geq -1$ models. For the first time, due to our large dynamic range and careful checks of scaling and shot–noise effects, we are able to detect evolution away from the perturbation theory result.

Subject headings: Galaxies: formation, clustering – large–scale structure of the Universe.
1. Introduction

A fundamental problem in the analysis of the matter distribution in the universe is to choose simple statistical tools able to provide the most compact information on both the initial conditions and the subsequent evolution of density fluctuations. In this sense, the study of count probabilities has proved a useful statistical technique. They can be used to follow the action of gravity during the mildly nonlinear as well as fully nonlinear phases of structure formation. The count probability approach, dating back to the early work of Hubble (1934), has been recently applied to a number of galaxy samples: Efstathiou et al. (1990) calculated the variance of IRAS–selected galaxies in the QDOT sample for roughly cubical cells of various sizes, while Loveday et al. (1992) performed the same analysis in the Stromlo–APM redshift survey; Saunders et al. (1991) computed the skewness of density fluctuations, after smoothing the QDOT galaxy distribution by a Gaussian filter. A statistical analysis of the CfA and SSRS optical galaxy samples in terms of moments of counts in cells has been recently performed by Gaztañaga (1992; see also Gaztañaga & Yokoyama 1993). A more recent analysis of this type, up to the fifth connected moment, has been performed by Bouchet et al. (1993) on the 1.2 Jy IRAS Galaxy Redshift Survey (see also Bouchet, Davis, & Strauss 1992). Compared to connected correlation functions of order $q$, $\xi_q(x_1, \ldots, x_q)$, reduced moments (or *cumulants*) of the same order, $\bar{\xi}_q$, of the fractional density fluctuation enhance the signal–to–noise ratio, though at the expense of reducing the amount of geometrical information. One has the following connection between the above quantities

$$ \bar{\xi}_q(R) \equiv \int d^3x_1 \ldots d^3x_q W_R(x_1) \ldots W_R(x_q) \xi_q(x_1, \ldots, x_q), \quad (1) $$

where $W_R(x)$ defines a suitable filter over a volume of size $R$. These relations allow one to connect the results on moments of galaxy counts in cells with the large amount of available data on galaxy correlation functions. Actually, moments of counts in cells can be related to the $\xi_q$ only after shot–noise subtraction
(see the following Section). Early observations of higher order (i.e. \( q > 2 \)) correlation functions established the validity of the so-called hierarchical scaling ansatz according to which correlations of order \( q \) can be expressed as suitable sums of products of \( q - 1 \) two-point functions (Groth & Peebles 1977; Fry & Peebles 1978; Sharp, Bonometto, & Lucchin 1984). If the two-point function scales with distance as a power-law, \( \xi(r) \propto r^{-\gamma} \), or if the filtering radius is larger than the typical correlation length, a related hierarchy holds for the reduced moments,

\[
\overline{\xi}_q(R) = S_q \overline{\xi}_2^{q-1}(R), \quad q > 2,
\]

with constant coefficients \( S_q \). The hierarchical scaling of Eq. (2) can be given a theoretical justification in two different regimes. Starting from Gaussian density fluctuations, perturbation theory shows that the action of gravity, already in the mildly nonlinear regime, predicts the above hierarchical structure (Peebles 1980; Fry 1984b; Goroff et al. 1986; Bernardeau 1992). The hierarchical scaling also represents a self-consistent solution of the BBGKY equations in the fully relaxed, highly nonlinear regime (Davis & Peebles 1977; Fry 1984a; Hamilton 1988). The validity of this ansatz has been successfully tested in numerical simulations by a number of authors: Coles & Frenk (1991); Bouchet & Hernquist (1992, hereafter BH); Weinberg & Cole (1992, hereafter WC); Lahav et al. (1993, hereafter LIIS); Fry, Melott, & Shandarin (1993). Most of these works, however, deal with the skewness, \( \overline{\xi}_3 \), vs. the variance, \( \sigma^2 \equiv \overline{\xi}_2 \), relation (see also Silk & Juszkiewicz 1991). Indeed, much recent work has been devoted to a detailed analysis of the skewness ratio \( S_3 \equiv \overline{\xi}_3(R)/\overline{\xi}_2^2(R) \). In particular, Juszkiewicz & Bouchet (1992) and Juszkiewicz, Bouchet, & Colombi (1993) used a second order perturbation theory in Eulerian coordinates to compute the dependence of \( S_3 \) on the type of window function as well as on the spectral index \( n \) for Gaussian density fluctuations with scale-free spectra \( P(k) \propto k^n \). Bouchet et al. (1992) used second order perturbation expansion in Lagrangian space to evaluate the dependence of \( S_3 \) on the density parameter \( \Omega \).
The relevance of the primordial skewness of density fluctuations in determining both the dynamical evolution and the present texture of the matter distribution has been discussed by Moscardini et al. (1991), Messina et al. (1992) and WC. The quantity $S_3$ has been calculated by Coles et al. (1993) for the mass and galaxy distribution in $N$–body simulations of skewed Cold Dark Matter (CDM) models and shown that this quantity can be used as a powerful test to discriminate among various statistical distributions of primordial fluctuations.

From the theoretical point of view, even if we accept that the mass distribution follows the hierarchical law on scales affected by nonlinear evolution, one still has to ask whether such a scaling is stable against the nonlinear (and possibly non–local) biasing process leading to the galaxy distribution. This issue has been partially solved by Fry & Gaztañaga (1993), who showed that the hierarchical scaling is indeed preserved by a rather general type of bias in the limit of small fluctuations. Conversely, it might be that the observed scaling of higher order correlation functions is entirely due to the bias mechanism, instead of reflecting the true statistical properties of the underlying matter distribution. Actually, Politzer & Wise (1984), for the Gaussian case, and Matarrese, Lucchin, & Bonometto (1986), for the non–Gaussian one, argued that Eq.(2) is also implied on intermediate and large scales, if the biasing mechanism requires a high–density threshold, while further terms deriving from the Kirkwood expansion (e.g. Peebles 1980) also appear on small scales. On the other hand, biasing with moderate threshold may lead to the hierarchical form (Melott & Fry 1986). Any possible contamination of the hierarchical scaling law due to redshift–space distortions has been shown to be negligible by a number of authors (Bouchet et al. 1992; LIIS; Coles et al. 1993; Juszkiewicz, Bouchet, & Colombi 1993).

In this paper we test the validity of the hierarchical scaling law of Eq.(2) for the third (skewness), fourth (kurtosis) and fifth connected moments of the
density contrast in $N$–body simulations of the gravitational evolution of scale–free Gaussian models with spectral index $n = -3, -2, -1, 0, 1$. We also study the dependence of $S_q$ (for $q = 3, 4, 5$) on the primordial spectral index $n$. Moments of counts in cells have already been computed in numerical simulations for some of these models. Bouchet & Hernquist (1992), besides considering CDM and Hot Dark Matter models, run a tree code with white–noise (i.e. $n = 0$) initial conditions. In the frame of a comparison of Particle–Mesh simulations with both Gaussian and non–Gaussian initial conditions, WC calculate $S_3$ for $n = -2, -1, 0$ initial spectra. Finally, LIIS analyze counts in cells in tree code simulations for various models, including $n = -1, 0, 1$ scale–free models. Our analysis, besides considering a wider ensemble of power–law models, covers a much larger dynamic range both in time and resolution. In Table I we show the range in the $rms$ fluctuation $\sigma$ and spectra studied for our simulation and those cited in this paragraph. It appears that ours is the only study to date with sufficient dynamic range and control of boundary conditions to reliably detect evolution away from the perturbation theory result. We also have looked at a variety of pure power–law models, so that such dependence can be detected.

The plan of the paper is as follows. In Section 2 we give the theoretical background. Section 3 presents the numerical simulations and the cautions used in the following analysis, while Section 4 discusses our results on the analysis of the moments of counts in cells. Conclusions are summarized in Section 5.

2. Moments of Counts in Cells

The quantities $\xi_q(R)$ defined above represent the reduced moments of the density fluctuation $\delta_R(x) \equiv \bar{\rho}_R(x)/\bar{\rho} - 1$ (here $\bar{\rho}_R$ is the density smoothed over the scale $R$ and $\bar{\rho}$ its average); these can be derived from the moment generating function $\mathcal{M}(s) \equiv \int d\bar{\rho}_R P(\bar{\rho}_R) \exp(is\bar{\rho}_R/\bar{\rho})$, where $P(\bar{\rho}_R)$ gives the probability density of the continuous variable $\bar{\rho}_R(x)$. One has

$$\ln \mathcal{M}(s) = is + \sum_{q=2}^{\infty} \frac{(is)^q}{q!} \xi_q(R). \quad (3)$$
Given the probability density or the moment generating function, one can easily generate discrete count–probabilities \( P_m \) (e.g. Peebles 1980). In fact, the count \( P_m \) can be understood as the probability that, in a realization of the stochastic process \( \varrho_R \), \( m \) objects are found in a randomly placed cell of volume \( V = Nv \), where \( v \) is the specific volume \( 1/\varrho \) and \( N \) is the expected number of objects in that volume. Consider then \( \varrho_R \) as giving the mean density of an ensemble of local Poisson count distributions with mean \( Nv\varrho_R \), whose moment generating function is \( \exp[Nv\varrho_R(e^{is} - 1)] \). Note that the Poisson model does not necessarily provide a good representation of discreteness effects (see e.g. Coles & Frenk 1991; Borgani et al. 1993); in particular, it is likely to fail when the expected number of objects \( N \) in the cell volume is smaller than unity, i.e., when \( V \ll v \).

Averaging over the \( \varrho_R \) ensemble produces the moment generating function for the discrete process as

\[
M_{\text{dis}}(s) = \int d\varrho_R \ P(\varrho_R) \ \exp[Nv\varrho_R(e^{is} - 1)] = M[-i(e^{is} - 1)N] : \quad (4)
\]

the moment generating function of discrete counts, \( M_{\text{dis}}(s) \), is obtained from the continuous one, \( M(s) \), by the replacement \( is \rightarrow (e^{is} - 1)N \). By inverse Fourier transforming \( M_{\text{dis}}(s) \) one gets \( P_{\text{dis}}(\varrho_R) = \sum_{m=0}^{\infty} \delta(\varrho_R - m\varrho)P_m \), where \( \delta \) is the Dirac delta–function and the counts \( P_m \) are defined through a Poisson transform of \( P(\varrho_R) \),

\[
P_m = \int d\varrho_R \ P(\varrho_R) \ \frac{(Nv\varrho_R)^m}{m!} e^{-Nv\varrho_R}. \quad (5)
\]

One can then define the normalized central moments of the counts in cells of volume \( V \) as \( \mu_q \equiv \langle ((m - \overline{m})/\overline{m})^q \rangle \), where

\[
\langle m^q \rangle = \sum_{m=1}^{\infty} m^qP_m = (-i)^q \frac{d^q M_{\text{dis}}(s)}{ds^q} \bigg|_{s=0}, \quad (6)
\]

and \( \overline{m} \equiv \sum_{m=1}^{\infty} mP_m = N \). For small \( N \), at fixed moments of \( \varrho_R/\varrho \), the count distribution is shot–noise dominated and \( M_{\text{dis}}(s) \) reduces to the moment generating function of a Poisson process with mean \( N \): if the cell volume \( V \) is
too small, statistical fluctuations dominate the realization and one is unable to get any faithful statistical information on $\mathcal{P}(\varrho_R)$ from the counts. This can be seen by writing $\mathcal{P}(\varrho_R)$ in Eq.(4) through its Fourier transform and expanding in reduced moments $\xi_q(R)$,

$$M_{\text{dis}}(s) = \int dy \exp[y(e^{is} - 1)] \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{it(N-y)} \exp\left[\sum_{q=2}^{\infty} \frac{(itN)^q}{q!} \xi_q(R)\right], \quad (7)$$

which for small $N$ and fixed $\xi_q(R)$ yields

$$M_{\text{dis}}(s) \approx \int dy \exp[y(e^{is} - 1)] \delta(y - N) = \exp[N(e^{is} - 1)]. \quad (8)$$

When dealing with the $N \to 0$ limit above, one should, however, consider the volume dependence of the connected moments. Actually, if the hierarchical scaling of Eq.(2) holds and $\sigma^2 \propto N^{-\gamma/3}$ as $N \to 0$, shot–noise dominates for small filtering scales, provided that the effective spectral index $n_{\text{eff}} \equiv \gamma - 3$ is smaller than zero. Note that on small scales $n_{\text{eff}} \neq n$.

Conversely, shot–noise may even dominate for large cell sizes, $N \to \infty$, where $\gamma \to n + 3$, if the spectral index $n$ is larger than zero. Except for these cases, one generally expects that the discrete counts $NP_m$ reduce to the original continuous distribution of $\varrho_R$, for large $N$ and $m$ and fixed $m/N$, as a property of the Poisson transform. In fact, using the asymptotic representation of the Poisson counts,

$$\frac{(NV\varrho_R)^m}{m!} e^{-NV\varrho_R} \sim \frac{1}{\sqrt{2\pi NV\varrho_R}} \exp\left[-\frac{(m - NV\varrho_R)^2}{2NV\varrho_R}\right], \quad (9)$$

in the integrand of Eq.(5), and taking the limit for $N \to \infty$ one gets $NP_m \sim \mathcal{P}(\varrho_R)$, with $\varrho_R = \varrho m/N$.

We can conclude that the optimal range of cell sizes $R \equiv V^{1/3}$ depends on the spectral slope $n$: in a numerical simulation such as ours, with mean interparticle distance $\ell$, one should require $R > \text{Max}[\ell, \ell \sigma^{-2/3}(\ell)]$, for every $n$; for $n > 0$, however, one should also require $R < \ell \sigma^{-2/3}(R)$. 8
From Eqs.(3), (4) and (6) above one can explicitly find the required relations among the moments of the counts and the reduced moments of the continuous variable \( \delta_R \). Inverting these relations, one finally obtains the cumulants \( \xi_q \) as a function of the moments of counts in cells \( \mu_n \), up to order \( q \). We have, in particular,

\[
\begin{align*}
\xi_2 &= \mu_2 - \frac{1}{N}, \\
\xi_3 &= \mu_3 - 3 \frac{\mu_2}{N} + \frac{2}{N^2}, \\
\xi_4 &= \mu_4 - 6 \frac{\mu_3}{N} - 3 \mu_2^2 + 11 \frac{\mu_2}{N^2} - \frac{6}{N^3}, \\
\xi_5 &= \mu_5 - 10 \frac{\mu_4}{N} - (10 \mu_2 - \frac{35}{N^2}) \mu_3 + 30 \frac{\mu_2}{N} - 50 \frac{\mu_2}{N^3} + \frac{24}{N^4}.
\end{align*}
\]

In what follows we shall also consider the ratios \( S_3 \equiv \xi_3/\xi_2^2 \), \( S_4 \equiv \xi_4/\xi_2^3 \) and \( S_5 \equiv \xi_5/\xi_2^4 \), in order to test whether the hierarchical scaling relations apply, i.e., whether these ratios are scale–independent, i.e. independent of the variance.

Theoretical predictions for the value of \( S_3 \) have been obtained using second order perturbation theory. In order to get a consistent prediction for \( S_4 \) and \( S_5 \) one respectively needs third and fourth order perturbative calculations. Goroff et al. (1986) have computed these ratios for initially Gaussian perturbations in standard CDM, by a clever summation of tree diagrams. Filtering the density field by a Gaussian window they obtain the values \( S_3 \approx 3, S_4 \approx 16 \) and \( S_5 \sim 100 \) on large scales, where the spectral slope tends to the Zel’dovich value \( n = 1 \). For scale–free Gaussian initial perturbations in an Einstein–de Sitter model, Juszkiewicz, Bouchet, & Colombi (1993) find the relation

\[
S_3^{(p)} = \frac{34}{7} - (n + 3), \quad -3 \leq n < 1,
\]

using a spherical top–hat filter, while for \( n = 1 \) the perturbative prediction formally diverges. In the latter case, taking into account that numerical simulations cannot reproduce the initial spectrum above the Nyquist frequency,
they find $S_3^{(p)}(n = 1) = 1.9$. We shall compare these perturbative estimates with our numerical results. The use of a sharp cubic filter instead of a spherical one is not expected to introduce big changes in the $S_3$ vs. $n$ relation. Actually, we have numerically verified that the two filters give essentially equivalent results, provided all quantities are compared at equal smoothing volumes (see also, LIIS).

Perturbation theory implies an expansion of a series. As is well known in basic physics, the series contains higher and higher order powers of the perturbed quantity. It can only be expected to converge to the correct result if this quantity is small. Indeed, perturbation theory is going to fail as $\sigma$ gets of order unity, simply because the gravitational field becomes arbitrarily large around regions of orbit mixing. Thus perturbation theory ought to give better results for small $\sigma$, and we can use that to compare with our procedures. The value of numerical simulations such as these is that we can investigate the nonlinear regime.

We believe that, for the first time, due to our large dynamic range and careful checks of scaling, we are able to detect the evolution away from the perturbation theory result.

3. Numerical Simulations

The simulations studied herein are numerical models of the clustering of collisionless matter in an expanding universe. We wish to investigate the above scale–invariant behavior in the case of Gaussian initial conditions. In order to implement this, we use an $\Omega = 1$ universe, as to choose otherwise would introduce a preferred scale or time. We use pure power–law initial perturbation spectra $P(k) \propto k^n$ with $n = -3, -2, -1, 0, 1$.

The simulations are done with a Particle–Mesh (PM) code (Hockney & Eastwood 1981) with $128^3$ particles in as many cells. In this paper we use 10 runs (two of each spectral index) out of an ensemble of 50 generated for
other systematic studies (Melott & Shandarin 1993). The PM code used in these studies is highly optimized, using a staggered mesh scheme, and has twice the dynamical resolution of any other PM code with which it has been compared [Melott 1986; Melott, Weinberg, & Gott 1988 (hereafter MWG); Weinberg 1993a,b]. Thus the studies shown here are roughly equivalent in dynamic range to usual PM runs with $128^3$ particles on a $256^3$ mesh except that we have less shot–noise and collisionality. Having a relatively large number of particles has the advantage of good mass resolution and the ability to impress initial perturbations right up to the particle Nyquist frequency $k_{Ny} = 64$. Other methods such as P$^3$M and tree codes have not yet been able to run with $128^3$ particles, which is relatively easy with PM, as can be seen by our large ensemble of such runs. More details about the particular simulations used here can be seen in Melott & Shandarin (1993).

Having stressed some advantages of our simulations, we would now like to discuss some of the precautions needed in using them, particularly for studies of scale–free processes. Resolution is one problem. Resolution has been stressed as an advantage of codes in which short–range forces are calculated accurately between point masses. In reality there are a number of different but related kinds of resolution: (a) mass resolution, essentially the reciprocal of the number of particles; (b) force resolution, essentially how accurately the force law tracks $1/r^2$ at small separations; (c) spectral resolution, equivalent to the minimum of the number of particles or Fourier analysis cells per unit length, whichever is smaller; (d) minimum scale on which two–body relaxation becomes important. PM methods are superior for all except (b), in which other methods work better. We first consider this limitation.

The growth rate of various modes in linear theory was studied in MWG for this PM code. The growth rates for PM codes are usually described as being unacceptable for $k > 0.25 k_{Ny}$, or equivalently for wavelength less than 8 cells. MWG found an equivalent performance for $0.5 k_{Ny}$ to that found in
usual PM codes at 0.25 \(k_{Ny}\). This results from the staggered mesh scheme; it has since been confirmed in other comparisons (Weinberg 1993a,b) and the PM code used here has been slightly improved since that corresponding to a further 30\% resolution increase, \(\text{i.e.}\) giving results at \(\lambda = 3\) cells equivalent to usual PM codes at \(\lambda = 8\) cells.

We still must take account of limited force resolution based on our grid scheme. The advantage of the methods used here is that we can test for the adequacy of our precautions by observing the results at various stages. In pure power–law models, properties of the distribution should depend only on \(\sigma = (\delta \rho / \rho)_{rms}\), assuming the use of the identical initial power–law smoothing windows. But \(\sigma\) increases with time, and the agreement on different scales at different times with the same \(\sigma\) is a strong consistency test. A similar strategy has already been used to find previously unknown effects from the absence of waves larger than the simulation volume (Kauffmann & Melott 1992; Gramann 1992; Melott & Shandarin 1993).

We first describe the stages of our simulations and then the restrictions we applied. Our simulations were started by using the Zel’dovich (1970) approximation, as first utilized by Klypin & Shandarin (1983). It is well–known that this approximation is inaccurate beyond the time of nonlinearity [although better than other approximations studied, with appropriate filtering; see Coles, Melott, & Shandarin (1993)] so the initial amplitude is well below unity at the Nyquist frequency.

The simulations were stopped at \(k_{n\ell} = k_{Ny}, 0.5 k_{Ny}, 0.25 k_{Ny}, \ldots, 2 k_f\) where \(k_f = 2\pi / L\) is the fundamental mode of the box; \(k_{n\ell}\) is defined by

\[
\langle \left(\frac{\delta \rho}{\rho}\right)^2 \rangle_{k_{n\ell}} \equiv \int_0^{k_{n\ell}} P(k) d^3k \equiv 1.
\]

In this study we have an available range of \(k_{n\ell}\) from 2 \(k_f\) to 64 \(k_f\). Everything in the simulations should scale as \(1/k_{n\ell}\), in the absence of boundary or resolution
problems. We make the following restrictions on what scales will be studied. We will show results based on counts in cells, in boxes of various sizes, at various stages of nonlinearity. All results for a given spectral index will be plotted together as a function of \( \sigma = (\delta \varrho / \varrho)_{\text{rms}} \) and reveal any problems.

(a) The stage with \( k_{n\ell} = k_{Ny} \) will not be used to study nonlinear effects since the code is known not to perform well at this frequency. We will begin with the stage \( k_{n\ell} = 0.5 \ k_{Ny} = 32 \ k_f \) in this study, except that we use the earlier stage to help establish the linear limit.

(b) At each stage we will not present results for one pixel of density, since this depends primarily on \( k_{Ny} \). The smallest will be a cube of 8 such cells. We expect this will be acceptable since our code performs well at wavelength of 4 cells, and collapse of \( \lambda = 4 \) perturbations will initially give rise to condensations of diameter 2. We will confirm this later.

(c) Kauffmann & Melott (1992) found that for voids of size greater than size \( 0.25L \) self–similarity was broken in a model equivalent to our index \( n = -1 \); see also Gramann (1992) and Melott & Shandarin (1993). We therefore restrict ourselves to cubes of size \( L/8 \) or smaller. Combined with restriction (b), this leaves us with cells for scale from \( L/64 \) to \( L/8 \). We do not expect this to work with \( n \leq -2 \), and believe that for these values all so–called “N–body” simulations are at best crude. It would not be entirely a joke to say that in this case a model could never be big enough to be a fair sample of itself. Fortunately, it appears that the power spectrum of the universe turns over to \( n > 0 \) on large scales.

(d) We use counts in cells, rather than the usual Cloud–in–Cell method (Hockney \\& Eastwood 1981) to bin densities. This procedure increases shot–noise as compared to that present when the PM code calculates the gravitational potential. For this reason we do not use cell sizes with \( \sigma < 0.1 \). In practice, in our simulations this restriction eliminates cells where the shot–noise power is comparable to the fluctuation power impressed on the
simulation, which makes the subtraction doubtful. Following our discussion in Section 2, this only affects the largest cells at early times in our models with $n \geq 0$.

Taken together, these restrictions can eliminate problems and give us much more usable dynamic range than has been possible before in such a study. We verify that it works remarkably well by examining the agreement between various stages at the same $\sigma$. It suggests that scale dependence found in previous studies was probably a result of boundary problems. For $n \leq -2$ we find very poor agreement, as expected.

As an illustration of the different appearance of the models, Figure 1 shows, for the five spectra, grey–scale plots of $L/64$ thickness slices at the stage corresponding to $k_{n^\ell} = 8$.

4. Analysis and Results

In our scale–free simulations we expect that every quantity depends on a single scaling variable, namely the variance $\bar{\xi}_2$. Therefore we can plot the results of counts in cells at all times together as a function of the variance. Figure 2 shows the skewness $\bar{\xi}_3$ vs. the variance, for all models, i.e. for all values of $n$. The points shown are the average of the two runs; the corresponding dispersion is always small and will not be shown here. The dashed lines are the second order perturbative predictions [obtained from Eq.(14)], for the same value of $n$; these computations derive from using a spherical top–hat window, but we checked that changing from spherical to cubic filter produces essentially the same results. The solid line shows the two–parameter linear fit

$$\log \bar{\xi}_q = A_q + B_q \log \bar{\xi}_2$$

(for $q = 3$); the corresponding coefficients and related errors are reported in Table II. Figures 3 and 4, respectively, show the kurtosis $\bar{\xi}_4$ and the fifth moment $\bar{\xi}_5$ vs. $\bar{\xi}_2$, for all values of the initial spectral index. Perturbative predictions
are not available in these cases. The solid lines represent the results of linear fits from Eq.(16); best–fit coefficients and errors are reported in Table III and IV, for the fourth and fifth moment, respectively. Note that the \( n \leq -2 \) results are clearly not reliable, being highly affected by the finite box–size (see Fry, Melott, & Shandarin 1993). In all other cases \( (n = -1, 0, 1) \), the scaling of \( \xi_q \) \( (q = 3, 4, 5) \) with the variance is quite close to the hierarchical form, \( B_q = q-1 \). Nevertheless, we detect a residual deviation from this scaling, at more than three standard deviations for both the skewness and the kurtosis, while for the fifth moment this result appears at only one standard deviation. We consider rather unlikely that such a deviation is produced by boundary effects. Due to the large dynamic range of our simulations and to the careful treatment of resolution and shot–noise problems, we are led to trust these deviations, even though our results are closer to the hierarchy than previous works (BH and LIIS), whose simulations are more affected than ours by finite sample and resolution effects. For instance, for initial white–noise, BH find \( B_3 = 2.10\pm0.01, \) \( B_4 = 3.26\pm0.02 \) and \( B_5 = 4.44\pm0.04 \), while LIIS obtain \( B_3 = 2.08\pm0.01 \) and \( B_4 = 4.16\pm0.03 \). Note however, that, according to our previous discussion on shot–noise, the highest \( \sigma \) points appearing in BH, which come from cells with size smaller than the mean interparticle distance, are likely to be dominated by discreteness effects, which decreased their statistical reliability.

In order to better display possible deviations from scale–invariance we also plot the coefficients \( S_q \) as a function of \( \sigma \). In Figures 5, 6 and 7 we show \( S_3, S_4 \) and \( S_5 \), for all values of \( n \). The meaning of the solid lines and dashed ones (when present) is as before. Note that the trend of \( S_q \) with \( n \) is the same for all \( q \): \( S_q \) decreases with increasing \( n \). Although this qualitative trend is the same predicted by perturbation theory for \( q = 3 \), the value of the coefficients is only in partial agreement with it. In Tables II – IV, we report the coefficients of the one–parameter fits of \( S_q \), obtained at fixed hierarchical slope; dotted lines in Figures 5 – 7 represent these best–fit coefficients. Note that the values of \( S_3 \) are
generally different from $S_3^{(p)}$ as given by Eq.(14); however, since perturbation theory is at most consistent with mildly nonlinear evolution, we also estimated the skewness ratio by fitting only points with $\sigma \leq 0.7$: this is reported as $S_3^\star$ in Table II (we similarly define $S_4^\star$ and $S_5^\star$ in Tables III and IV). The agreement with perturbation theory is indeed improved, but we still get slightly higher values: $S_3^\star = 4.5, 3.4, 3.1, 2.0, 1.7$, instead of $S_3^{(p)} = 4.9, 3.9, 2.9, 1.9, 1.9$, for $n = -3, -2, -1, 0, 1$, respectively. In deriving the values of $S_3^\star$ above, we also used an earlier stage of the simulations corresponding to $k_{n\ell} = 64 k_f$. For their simulations with $n = 0$, BH argue that $S_3$ should tend to about 1.8 at large scales, very close to the perturbative prediction; their result is obtained from large cell sizes, corresponding to about one quarter of the computational box size, where sample effects make the resulting data less reliable.

Let us stress that agreement with second order perturbation theory is not a good test of the quality of $N$–body data, except in the linear regime. However, the behavior of such codes has already been widely tested in this regime. In the nonlinear regime, one should use $N$–body data to test the reliability of perturbation expansion results, such as the second order estimate of $S_3$ reported in Eq.(14). In this sense one can consider a success that the qualitative trend of $S_3$ with $n$ is correctly predicted by perturbation theory! It would be interesting to have similar predictions for higher order moments, to compare with our numerical results.

5. Conclusions

We have studied the properties of higher order moments of the matter distribution generated by gravity in the nonlinear regime. This was done by analyzing numerical simulations of the evolution of initially Gaussian perturbations with scale–free power–spectra (with spectral index $n = -3, -2, -1, 0, 1$) at many evolution stages and smoothing scales. Our results for $n \geq -1$ models indicate that these moments are fairly close to the hierarchical scaling ansatz, according to which connected moments $\xi_q$, of order $q$, are proportional to the
However, we detect a residual dependence, above the statistical noise, of the coefficients $S_q = \xi_q / \xi_2^{q-1}$ on the variance, i.e. on scale. In order to detect such a signal from the data we had to properly account for shot–noise, finite resolution effects and boundary problems. For models with $n \leq -2$, where the amount of large–scale power does not allow a fair representation of low frequency modes of the density field (due to the finite box size), a stronger dependence of $S_q$ on scale is found.

The skewness ratio $S_3$ is found to decrease with increasing spectral slope, $n$, i.e. with decreasing large–scale power, as correctly predicted by perturbative calculations, although our values for $S_3$ are only consistent with the theory if the small $\sigma$ limit of these quantities is considered. The same qualitative trend with $n$ is seen for the higher order coefficients $S_4$ and $S_5$.

Altogether, these results indicate that: 1) the dynamical effect of gravity is such as to generate non–Gaussian signatures on a Gaussian initial density field, already in the earliest stages of evolution and/or on large scales; 2) the hierarchical ansatz provides only an approximate description of the behavior of higher order moments of the density fluctuation field with the scale.

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Figure captions

**Figure 1.** Grey–scale plots of the projected density in slices of thickness $L/64$ at stage $k_{n\ell} = 8k_f$ in our simulations. Regions below the mean density are white; regions of density $\rho > 10$ are black; a grey scale is used in between. (a) $n = -3$; (b) $n = -2$; (c) $n = -1$; (d) $n = 0$; (e) $n = 1$.

**Figure 2.** The skewness $\bar{\xi}_3$ vs. the variance $\bar{\xi}_2$ for the different models. The solid lines represent the best–fit obtained from Eq.(16), while the dashed lines are the perturbative prediction. Different symbols refer to different stages of the simulations: $k_{n\ell} = 32k_f$, filled triangles; $k_{n\ell} = 16k_f$, asterisks; $k_{n\ell} = 8k_f$, open triangles; $k_{n\ell} = 4k_f$, open squares; $k_{n\ell} = 2k_f$, open circles. Note that circles are absent for $n = -3$; they were left out due to lack of any benefit in including them.

**Figure 3.** The kurtosis $\bar{\xi}_4$ vs. the variance $\bar{\xi}_2$ for the different models. The solid lines represent the best–fit obtained from Eq.(16). Different symbols refer to different stages of the simulations as in Figure 2.

**Figure 4.** The fifth connected moment $\bar{\xi}_5$ vs. the variance $\bar{\xi}_2$ for the different models. The solid lines represent the best–fit obtained from Eq.(16). Different symbols refer to different stages of the simulations as in Figure 2.

**Figure 5.** The skewness coefficient $S_3 = \bar{\xi}_3/\bar{\xi}_2^2$ vs. the $rms$ fluctuation $\sigma$ for the different models. The solid lines represent the best–fit obtained from Eq.(16), the dashed lines are the perturbative prediction, finally the dotted lines are the result of a best–fit forced to the hierarchical slope. Different symbols refer to different stages of the simulations as in Figure 2.

**Figure 6.** The kurtosis coefficient $S_4 = \bar{\xi}_4/\bar{\xi}_2^3$ vs. the $rms$ fluctuation $\sigma$ for the different models. The solid lines represent the best–fit obtained from Eq.(16), while the dotted lines are the result of a best–fit forced to the hierarchical slope. Different symbols refer to different stages of the simulations as in Figure 2.
Figure 7. The fifth moment coefficient \( S_5 = \frac{\bar{\xi}_5}{\xi_2^4} \) vs. the rms fluctuation \( \sigma \) for the different models. The solid lines represent the best–fit obtained from Eq.(16), while the dotted lines are the result of a best–fit forced to the hierarchical slope. Different symbols refer to different stages of the simulations as in Figure 2.
Table I
Comparison of Dynamic Range in Recent Similar Studies
Ratio of $\sigma_{\text{max}}$ to $\sigma_{\text{min}}$

|       | BH*   | WC**  | LIIS*** | this study |
|-------|-------|-------|---------|------------|
| $n = -3$ |       |       |         | $\sim 10$  |
| $n = -2$ |       | 1.86  |         | $\sim 20$  |
| $n = -1$ | 1.86  |       | $\sim 8$ | $\sim 64$  |
| $n = 0$  | $\sim 100$ | 1.86  | $\sim 15$ | $\sim 100$ |
| $n = +1$ |       |       | $\sim 30$ | $\sim 100$ |

* estimated from BH Figure 8.

** estimated from WC Table 2. WC did not attempt to study the evolution of these moments over a wide dynamic range.

*** numbers estimated from LIIS Figure 2. Discrepancies up to a factor two exist between results at different moments (see LIIS Table I).

Table II
Third moment coefficients

|       | $A_3$ | $B_3$ | $S_3$ | $S_3^*$ | $S_3^{(p)}$ |
|-------|-------|-------|-------|---------|-------------|
| $n = -3$ | 0.74 ± 0.01 | 2.28 ± 0.02 | 7.0   | 4.5     | 4.9         |
| $n = -2$ | 0.57 ± 0.01 | 2.07 ± 0.01 | 4.0   | 3.4     | 3.9         |
| $n = -1$ | 0.51 ± 0.01 | 2.03 ± 0.01 | 3.3   | 3.1     | 2.9         |
| $n = 0$  | 0.38 ± 0.01 | 2.04 ± 0.01 | 2.5   | 2.0     | 1.9         |
| $n = +1$ | 0.30 ± 0.01 | 2.04 ± 0.01 | 2.0   | 1.7     | 1.9‡        |

‡ see discussion in the text.
### Table III

**Fourth moment coefficients**

| $n$   | $A_4$    | $B_4$    | $S_4$ | $S_4^*$ |
|-------|----------|----------|-------|---------|
| $-3$  | $1.66 \pm 0.02$ | $3.59 \pm 0.05$ | $97.4$ | $29.5$  |
| $-2$  | $1.33 \pm 0.02$ | $3.15 \pm 0.03$ | $28.1$ | $19.1$  |
| $-1$  | $1.19 \pm 0.02$ | $3.07 \pm 0.02$ | $18.1$ | $16.0$  |
| $0$   | $0.96 \pm 0.02$ | $3.04 \pm 0.01$ | $10.0$ | $9.4$   |
| $+1$  | $0.73 \pm 0.02$ | $3.08 \pm 0.02$ | $6.2$  | $5.9$   |

### Table IV

**Fifth moment coefficients**

| $n$   | $A_5$    | $B_5$    | $S_5$ | $S_5^*$ |
|-------|----------|----------|-------|---------|
| $-3$  | $2.65 \pm 0.04$ | $4.92 \pm 0.08$ | $2091$ | $218$   |
| $-2$  | $2.15 \pm 0.04$ | $4.24 \pm 0.05$ | $319$  | $161$   |
| $-1$  | $2.00 \pm 0.03$ | $4.05 \pm 0.03$ | $147$  | $135$   |
| $0$   | $1.64 \pm 0.03$ | $4.02 \pm 0.03$ | $63$   | $86$    |
| $+1$  | $1.31 \pm 0.03$ | $4.02 \pm 0.03$ | $30$   | $48$    |