Chiral Symmetry of SYM theory in hyperbolic space at finite temperature

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Abstract

We study a holographic gauge theory living in the AdS$_4$ space-time at finite temperature. The gravity dual is obtained as a solution of the type IIB superstring theory with two free parameters, which correspond to four dimensional (4D) cosmological constant ($\lambda$) and the dark radiation ($C$) respectively. The theory studied here is in confining and chiral symmetry broken phase for $\lambda < 0$ and small $C$. When $C$ is increased, the transition to the deconfinement phase has been observed at a finite value of $C/|\lambda|$. It is shown here that the chiral symmetry is still broken for a finite range of $C/|\lambda|$ in the deconfinement phase. In other words, the chiral phase transition occurs at a larger value of $C/|\lambda|$ than the one of the deconfinement transition. So there is a parameter range of a new deconfinement phase with broken chiral symmetry. In order to study the properties of this phase, we performed a holographic analysis for the meson mass-spectrum and other quantities in terms of the probe D7 brane. The results of this analysis are compared with a linear sigma model. Furthermore, the entanglement entropy is examined to search for a sign of the chiral phase transition. Several comments are given for these analyses.

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1 Introduction

The holographic approach \[1, 2, 3\] is expected to be applicable also to the supersymmetric Yang Mills (SYM) theory in curved space-time as well as in the flat Minkowski space-time. It would be interesting to make clear the properties of SYM theory in the curved space-time and in cosmologically developing universe. In this direction, some approaches have been extended to the field theory in the Friedmann-Robertson-Walker (FRW) type space-time \[4, 5, 6\]. In this case, the 4D cosmological constant (\(\lambda\)) can be introduced as a free parameter in obtaining the 5D sector of the 10D supergravity solution. The dynamical properties of the 4D SYM theory on the boundary are then characterized by the sign of \(\lambda\). Through the holographic approach, it has been found that the SYM theory is in the confinement (deconfinement) phase for negative (positive) \(\lambda\) \[4, 5, 6\]. This implies that the dynamical properties of the SYM fields are largely influenced by the geometry of the background space-time.

Furthermore, in this approach, one more free parameter (\(C\)) can be introduced as an integration constant in the solution of the supergravity. At first, this term has been added as the “dark radiation” to the 5D supergravity solution in the context of the brane world \[7, 8\]. Since it is defined in the 5D space-time, then its meaning in the 4D theory was mysterious. In this context, afterward, it could be interpreted as the projection of the 5D Weyl term \[9, 10\]. On the other hand, from the viewpoint of holography, it has been found for \(\lambda = 0\) that \(C\) corresponds to the thermal radiation of the SYM fields at a finite temperature \[11\], then the system is in the deconfinement phase. Actually, the 5D metric in this case can be rewritten to the AdS\(_5\)-Schwarzschild form, where \(C(>0)\) corresponds to the black-hole mass in this metric.

Then, it is easy to image that the above two parameters, \(C\) and the negative \(\lambda\), compete with each other to realize the opposite phase of the theory, namely the confinement and the deconfinement respectively. In fact, we find a phase transition at the point where these two opposite effects are balanced \[11, 12, 13, 14, 15\]. As a result, the SYM theory is in the deconfinement phase for \(b_0 > r_0\). Here the density of dark radiation \(C\) and the magnitude of \(|\lambda|\) are denoted by using \(b_0\) and \(r_0\), which are shown in the formula \(2.5\) of the next section. By using these parameters, the phase diagram of the SYM theory in the FRW space-time is obtained as in the Fig. \[\text{Fig.}\]

In the deconfinement phase, the temperature \(T\), which is given by the Hawking temperature of the 5D metric, appears. Then the critical temperature \((T_c)\) of confinement deconfinement transition is given as \(T_c = 0\), which corresponds to the critical line \(r_0 = b_0\) in the Fig. \[\text{Fig.}\]. In the region \(b_0 > r_0\), the temperature, which depends on \(r_0\), monotonically increases with \(b_0\).

Then it becomes possible to study the properties of the finite temperature SYM theory in the deconfinement phase in the FRW space-time, where the three space is hyperbolic one. As mentioned above, for the case of \(\lambda = 0\) and finite \(b_0\), the bulk solution is reduced to the well known AdS\(_5\)-Schwarzschild. So the SYM theory in its
special limit of this background has been studied already, see for example [16]. In this case, the two critical temperatures of confinement-deconfinement ($T_c$) and the chiral symmetry restoration ($T_\chi$) transitions are the same, namely $T_c = T_\chi = 0$. On the other hand, we show here that the value of $T_\chi$ shifts from $T_c = 0$. Namely we find $0 = T_c < T_\chi$ in the case of finite $\lambda(<0)$, and then the critical line is given by $b_0 = 0.76r_0$. It is shown in the Fig. 1 in the parameter space of $b_0$ and $r_0$. Then there exists a phase, where quark and gluons are not confined but the chiral symmetry is not restored before the system cools down to the confinement phase.

This implies a non-trivial thermal property of the SYM theory in the FRW space-time for negative $\lambda$. In order to show and understand the details of this point, we have examined the embedding of the D7 probe brane in the background. For each embedded D7 brane, we have examined also the spectrum of the Nambu-Goldstone boson and a massive meson modes in the region of the newly found deconfinement phase where the chiral symmetry is still broken. Then we compare our results with some theoretical consequences obtained from some typical phenomenological models for the chiral symmetry breaking of QCD in order to make clear the dynamical properties implied by our holographic model.

The outline of this paper is as follows. In the next section, a five dimensional (5D) space-time is given by a solution of supergravity as the dual of SYM theory in the AdS$_4$ background. In Sec.3, the spontaneous chiral symmetry breaking is studied by embedding the D7 probe brane, and then the spectra of two scalar mesons are examined. Since one of them corresponds to the Nambu-Goldstone boson, then the Gell-Mann-Oakes-Lenner relation and pion decay constant are also examined. In Sec.4,
the mass relations of the two scalars are discussed in terms of the sigma model, then we suggest a modified sigma model which implied by our holographic analysis. In Sec. 5, we searched a sign of the chiral transition in the entanglement entropy. We could find a change of the behavior of the entanglement entropy, but it could not lead to the transition point. The summary and discussions are given in the final section.

2 Gravity dual

The holographic dual to the large $N$ gauge theory embedded in a space-time with dark energy and dark radiation is solved by the gravity on the following form of the metric

$$ds^2_{10} = \frac{r^2}{R^2} (-\bar{n}^2 dt^2 + \bar{A}^2 a_0^2(t) \gamma_{ij}(x) dx^i dx^j) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2,$$

(2.1)

where

$$\gamma_{ij}(x) = \delta_{ij} \left( 1 + k \frac{\bar{r}^2}{4 \bar{r}_0^2} \right)^{-2}, \quad \bar{r}^2 = \sum_{i=1}^{3} (x^i)^2,$$

(2.2)

and $k = \pm 1$, or 0. The arbitrary scale parameter $\bar{r}_0$ of three space is set hereafter as $\bar{r}_0 = 1$. The solution is obtained from 10D supergravity of type IIB theory [11, 12, 13, 14].

The factors $\bar{A}$ and $\bar{n}$ are obtained by introducing two free parameters as mentioned below. Here, we use the following form of solution,

$$\bar{A} = \left( 1 + \left( \frac{r_0}{r} \right)^2 \right)^2 + \left( \frac{b_0}{r} \right)^4 \right)^{1/2},$$

(2.3)

$$\bar{n} = \frac{\left( 1 + \left( \frac{r_0}{r} \right)^2 \right)^2 - \left( \frac{b_0}{r} \right)^4}{A},$$

(2.4)

$$r_0 = \sqrt{|\lambda|} R^2/2, \quad b_0 = R \bar{c}_0, \quad \bar{c}_0 = CR^3/(4a_0^4),$$

(2.5)

where the dark radiation $C$ is introduced as an integral constant in solving the equation of motion (??). This solution expresses the case of negative $\lambda$. Here, the ”dark energy” (or cosmological term) $\lambda(t)$ is introduced by the following equation,

$$\left( \frac{a_t}{a_0} \right)^2 + \frac{k}{a_0^2} = \lambda$$

(2.6)

Although it is possible to consider a time dependent $\lambda$ as in [11], we set it here as a constant $\lambda$ for simplicity. In the following, our discussion would be concentrated to the case of negative constant $\lambda$ and we assume small time derivative of $a_0(t)$. We should notice the following fact that the solution $a_0 = 1/\sqrt{|\lambda|} = \text{constant}$ is actually allowed for negative constant $\lambda$ and $k = -1$.  

3
3 Chiral phase transition at finite temperature

3.1 D7 brane embedding

Here, we study the chiral condensate and the $q - \bar{q}$ meson spectrum of the boundary theory by embedding the probe D7 brane for the flavor quarks. The D7-brane action is given by the Dirac-Born-Infeld (DBI) and the Chern-Simons (CS) terms as follows,

$$S_{D7} = -\tau_7 \int d^8 \xi e^{-\phi} \sqrt{-\det (g_{ab} + 2\pi\alpha' F_{ab})} + T_7 \int \sum_i \left( e^{2\pi\alpha' F_{(2)} \wedge c_{(a_1...a_i)}} \right)_{0...7},$$

$$g_{ab} \equiv \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}, \quad c_{(a_1...a_i)} \equiv \partial_{a_1} X^{\mu_1} \ldots \partial_{a_i} X^{\mu_i} C_{\mu_1...\mu_i},$$

where $\tau_7$ is the brane tension. The DBI action involves the induced metric $g_{ab}$ and the $U(1)$ world volume field strength $F_{(2)} = dA_{(1)}$.

The solution given in the previous section is obtained for the case of constant dilaton. So it does not play any role in the present case. For simplicity, we consider the dual theory on the boundary at $r = \infty$. Then the induced metric for the above D7 brane is obtained as follows. Consider the above background (2.1) and rewrite it as follows

$$ds_{10}^2 = \frac{r^2}{R^2} ds^2_{(4)} + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2,$$

$$ds^2_{(4)} = (\bar{n}^2 dt^2 + \bar{A}^2 a_0(t)^2 \gamma_{ij}(x) dx^i dx^j),$$

$$r^2 = \rho^2 + (X^8)^2 + (X^9)^2.$$

Then, the induced metric of the D7 brane is obtained as

$$ds^2 = \frac{r^2}{R^2} ds^2_{(4)} + \frac{R^2}{r^2} \left( 1 + w'^2 \right) d\rho^2 + \rho^2 d\Omega_3^2,$$

where the profile of the D7 brane is taken as $(X^8, X^9) = (w(\rho), 0)$ and $w' = \partial_\rho w$, then

$$r^2 = \rho^2 + w^2.$$

In the present case, there is no R-R field, so the action is given as

$$S_{D7} = -\tau_7 \Omega_3 \int d^4 x a_0^3(t) \gamma^3(x) \int d\rho \rho^3 \bar{n} \bar{A}^3 \sqrt{1 + w'^2(\rho)},$$
where $\Omega_3$ denotes the volume of $S^3$ of the D7’s world volume.

From this action, the equation of motion for $w$ is obtained as

$$w'' + \left(\frac{3}{\rho} + \frac{\rho + \rho w'}{r} \partial_r (\log(\bar{n}\bar{A}^3))\right) w'(1 + w'^2) - \frac{w}{r} (1 + w'^2)^2 \partial_r (\log(\bar{n}\bar{A}^3)) = 0. \quad (3.9)$$

The constant $w \neq 0$ is not the solution of this equation, so the supersymmetry is broken except for the case of trivial solution $w = 0$.

### 3.2 Embedded solutions and chiral symmetry breaking

In the confinement region, the chiral symmetry is spontaneously broken as shown in \[11\]. The phase transition of the present model to the deconfinement occurs when the density of the dark radiation increases and exceeds a critical point, which is given by $b_0 = r_0$ (See the Fig.1). From this point, the Hawking temperature, $T_H(=\frac{\sqrt{b_0}}{\pi R \sqrt{1 - (r_0/b_0)^2}})$, appears for $b_0 > r_0$. At this stage, when the temperature appears, the chiral symmetry is usually restored. In the present case, however, we could observe the chiral symmetry restoration after transferring into the deconfinement phase at a finite $T_H$, which depends on $r_0$, as shown in the Fig.1 above. These facts are explained below through the numerical analysis.

All solutions of the above equation (3.9) have the following asymptotic form

$$w(\rho) = m_q + \frac{c + 4m_q r_0^2 \log(\rho)}{\rho^2} + \cdots, \quad (3.10)$$

at large $\rho$. In the second term of the right hand side of (3.10), the term proportional to $\log(\rho)$ arises from the loop corrections of the SYM theory since the conformal symmetry would be broken due to the existence of the cosmological constant in the present case \[4, 5, 6\]. We could show however that this term is proportional to the quark mass $m_q$, which is given by the asymptotic value of $w(\infty)$, the first term. In order to see the spontaneous chiral symmetry breaking, it is enough to see chiral condensate $\langle \bar{\Psi} \Psi \rangle = c = \rho^2w_{\rho \to \infty}$. Then the analysis is simply performed for $m_q = 0$.

We have therefore examined the numerical solutions for $m_q = 0$ at various points in the parameter space $b_0 - r_0$ in order to find the transition points. Three typical solutions are shown in the left figure of Fig.2. They are classified as Minkowski type (M-type) ($w(0) > r_H = \sqrt{b_0^2 - r_0^2}$), Black-hole type (B-type), which ends on $r_H$ at $\rho = \rho_{\text{min}}$, and the trivial solution $w = 0$, which is always the solution of Eq.(3.9).

Here we performed the numerical analyses for fixed $r_0 = 1$ by varying $b_0$. In this case, the region of $b_0$ is separated to the following three one.

* For $b_0 > 1.31$, we find only the trivial solution for $m_q = 0$. Then the chiral symmetry is restored.
Fig. 2: **Left;** The solutions of \( w(\rho) \) with \( m_q = 0 \) for \( b_0 = 1.3r_0, \mu = 1/R = 1.0, \) and \( r_0 = 1.0. \) The solution (c) represents the trivial one, \( w = 0. \) The circle denotes the horizon \( r_H = \sqrt{b_0^2 - r_0^2} = 0.83. \)** **Right;** The \( c - m_q \) relations of embedded solutions for \( b_0 = 1.2, 1.3, \) and 1.4 are shown with the same other parameters of the left figure. For the case of \( b_0 = 1.3, \) the embedded solutions at the three points, (a), (b), and (c), are shown in the left figure.

- For \( 1.28 < b_0 < 1.31, \) there are three types of solution mentioned above.
- For \( b_0 < 1.28, \) there are M-type and the trivial solutions.

The distributions of the general solutions including these three types of solutions are shown in the right of the Fig. 2 for \( b_0 = 1.2, 1.3, \) and 1.4. Then one might consider that the chiral symmetry may be broken in the region \( b_0 < 1.31, \) however we must compare the free energies in order to see which solution is favored for the given \( b_0 \) when plural solutions for the same \( m_q \) exist.

The free energy for each solution is obtained by substituting the solution \( w(\rho) \) into the Wick rotated Euclidean D7 action (3.8). Here the normalized free energy,

\[
E = \int d\rho \rho^3 \bar{n} A^3 \sqrt{1 + w'^2(\rho)}, \tag{3.11}
\]

is evaluated, and then the values subtracted the one for the trivial solution are shown in the Fig. 3.

This figure shows the followings.

- The free energy of B-type solution is always larger than the other two. Then this type of solution can not be realized.
- The value of \( E_{M-type} - E_{trivial} \) crosses zero at \( b_0 = 1.31. \) This implies that a transition from M-type solution (\( \langle \bar{\Psi} \Psi \rangle = c > 0 \)) to the trivial solution (\( \langle \bar{\Psi} \Psi \rangle = 0 \))
occurs at this point. This point is therefore the chiral phase transition point. The order parameter $\langle \bar{\Psi} \Psi \rangle$ has a gap at this transition point.

The transition line $b_0 = 1.31 \ r_0$ is shown in the Fig. As a result, we find two critical lines (a) and (b) in the parameter plane $(b_0, r_0)$. The line (a) represents the transition point from confinement to deconfinement and (b) does represent the critical line from chiral symmetry breaking phase to the restoring phase. This result implies the fact that the density of the dark radiation necessary for the restoration of the chiral symmetry is larger than the one needed for realizing the deconfinement phase.

Here we remember that the role of the dark radiation is to screen the force needed for the confinement. The same kind of force would be necessary for the spontaneous mass generation of massless quarks. The above result, that the chiral transition needs larger value of $C$ than the case of the confinement-deconfinement transition, implies that the range of the force necessary to break the chiral symmetry is shorter than the one for the confinement.

4 Chiral transition and Nambu-Goldstone (NG) bosons

Here we study the spectra of mesons constructed by the quark and anti-quark. They are given by solving the equations of motion for the fluctuation of the fields on the D7
brane embedded as in the previous section. In the phase of chiral symmetry breaking, generally, we could find the Nambu-Goldstone bosons and Gell-Mann-Oakes-Lenner relation for massless and small mass quarks respectively. In the phase of region B, quarks are deconfined since the linear potential between quark and anti-quark is lost, however we expect the existence of the Nambu-Goldstone (NG) boson and a massive scalar due to the spontaneous chiral symmetry breaking.

The spectra of these mesons are obtained by solving the equations of motion of the fields on the D7 brane. Consider the fluctuations of the scalar fields which are defined as,

\[ X^9 = \tilde{\phi}^9, \quad X^8 = w(\rho) + \phi^8. \]

The fields, \( \tilde{\phi}^9 \) and \( \phi^8 \) are corresponding to the Nambu-Goldstone boson and a massive scalar. We notice here this phenomenon represents also the breaking of a global \( U(1) \) symmetry at the same time.

The wave functions are given in the following factorized form,

\[ \tilde{\phi}^k = \varphi^k(t, x^i)\phi_l^k(\rho)\mathcal{Y}_l(S^3), \quad (k = 9, 8) \]

where \( \mathcal{Y}_l(S^3) \) denotes the spherical harmonic function on three dimensional sphere with the angular momentum \( l \). We derive linearized field equations for these scalars from the D7 action, then we study the linearized field equations for \( \phi_l^9(\rho) \) and \( \phi_l^8(\rho) \).

We get the linearized field equations of \( \phi_l^9(\rho) \) and \( \phi_l^8(\rho) \) for \( w \neq 0 \) as follows

\[ \partial_\rho^2 \phi_l^9 + \frac{1}{L_0} \partial_\rho(L_0) \partial_\rho \phi_l^9 + (1 + w^2) \left[ \left( \frac{R}{r} \right)^4 \frac{m_l^2}{n^2} - \frac{l(l + 2)}{\rho^2} - 2K(1) \right] \phi_l^9 = 0, \tag{4.1} \]

\[ L_0 = \rho^3 n A^3 \frac{1}{\sqrt{1 + w^2}}, \quad K(1) = \frac{1}{n A^2} \partial_\rho (n A^3) \tag{4.2} \]

and

\[ \partial_\rho^2 \phi_l^8 + \frac{1}{L_1} \partial_\rho(L_1) \partial_\rho \phi_l^8 + (1 + w^2) \left[ \left( \frac{R}{r} \right)^4 \frac{m_l^2}{n^2} - \frac{l(l + 2)}{\rho^2} - 2(1 + w^2)(K(1) + 2w^2 K(2)) \right] \phi_l^8 = -2 \frac{1}{L_1} \partial_\rho (L_0 w' K(1)) \phi_l^8 \tag{4.3} \]

\[ L_1 = \frac{L_0}{1 + w^2}, \quad K(2) = \frac{1}{n A^3} \partial_\rho^2 (n A^3). \tag{4.4} \]

Here notice the following points.

i) At first, for the 4D part of the wave-function, \( \varphi^k(x^\mu) \), we assumed the following eigenvalue equation,

\[ \Box_4 \varphi^k(x^\mu) = \frac{1}{\sqrt{g_4}} \partial_\mu \sqrt{g_4} g^\mu\nu \partial_\nu \varphi^k(x^\mu) = m_k^2 \varphi^k(x^\mu) \]
where $\tilde{g}_4 = -\det\tilde{g}_{\mu\nu}$ and

$$
\tilde{g}_{\mu\nu}dx^\mu dx^\nu = (-dt^2 + a_0^2(t)\gamma_{ij}(x)dx^i dx^j) .
$$

(4.6)

ii) The above operator $\Box_4$ comes from the eight dimensional Laplacian $\Box_8$ on the induced metric (3.6) on the D7 brane. In fact $\Box_8$ is expanded as

$$
\Box_8 = \frac{1}{\sqrt{g^{(8)}}} \partial_a g^{ab} \sqrt{g^{(8)}} \partial_b ,
$$

(4.7)

$$
= \frac{1}{\sqrt{g^{(8)}}} \partial_\mu g^{\mu\nu} \sqrt{g^{(8)}} \partial_\nu + \cdots ,
$$

(4.8)

$$
\sqrt{g^{(8)}} = \sqrt{g^{(4)}} \rho^3 \sqrt{1 + w^2} , \quad \sqrt{g^{(4)}} = (a_0 \gamma A)^3 n ,
$$

(4.9)

where $a, b = 0 \sim 7$, $\mu, \nu = 0 \sim 3$ and the elliptics denotes the derivative terms with respect to the other coordinates, $\rho$ and the one of $S^3$.

By using the above expansion and the approximation to neglect the time derivative of $a_0(t)$, we arrive at the above equations (4.1) and (4.3), which are used to find mass spectra of mesons according to [6].

### 4.1 Numerical Results in the phase A and B

We calculate equations for the modes of $\phi^8$ (massive scalar) and $\phi^9$ (Nambu-Goldstone boson) and obtain $m_8(m_s)$ and $m_9(m_{NG})$ which are shown in the Fig.4 for small quark mass. In the spectra of $\phi^9$, we find the massless NG boson for $m_q = 0$, and then the mass of this mode, denoted by $m_{NG}$, behaves as $m_{NG}^2 \propto m_q$ for the small quark mass $m_q$ as expected as the Gell-Mann-Oakes-Renner (GOR) relation [17].

The GOR relation is shown in the Fig.4 for several $b_0$. They should be expressed as follows,

$$
m_{NG}^2 = \frac{2m_q \langle \bar{\Psi}\Psi \rangle}{f_\pi^2} ,
$$

(4.10)

where $f_\pi$ denotes the pion decay constant. Since $\langle \bar{\Psi}\Psi \rangle$ is given for each $b_0$ through the solution of the D7 embedding profile, then we could obtain the pion decay constant from the relation (4.10). The numerical estimation of $f_\pi$ is shown in the Fig.5.

We find the following fact. The value of $f_\pi$ is almost constant within the confinement region, $b_0 < r_0$, then it increases with $b_0$ in the deconfinement region. This seems to be reasonable since the decay channel may increase in the deconfinement phase.
Fig. 4: **Left:** The quark mass ($m_q$) dependence of $m_{NG}^2$ (Nambu-Goldstone boson ($\phi^0$)). **Right:** The $m_s^2$ (red line) for massive scalar ($\phi^8$) is shown by the solid lines. The dotted lines are the prediction from the sigma model given below. The line (1) denotes the one for (4.13) and (2) for (4.18).

Fig. 5: $b_0$ dependence of the chiral condensate $<\bar{\Psi}\Psi>$ and the pion decay constant $f_\pi$. 
4.2 Comparison with the linear $\sigma$ model

In principle, it would be possible to derive an effective theory of mesons from the D7 brane action as a functional of $\phi^8$ and $\phi^9$ instead of $\pi$ and $\sigma$. Then the correspondence of the parameters of the D7 action to the one of the usual sigma model will be obtained. However, there are various higher order terms of the meson fields, and then we will find a very complicated effective action. Then this calculation will be postponed as a future work.

Here we instead compare our results given above with the usual linear sigma model in order to find a consistent sigma model by adding some improvement if it is necessary. Usually, the Lagrangian density of a linear sigma model at the mean field level is given by

$$L = \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda_1}{4}(\sigma^2 + \pi^2)^2 + h\sigma, \quad (4.11)$$

where $\mu$, $\lambda$ and $h$ are the parameters while $\sigma$ and $\pi$ are the mean fields. The last term proportional to $h$, which plays a role of the quark mass term, breaks $U(1)$ chiral symmetry explicitly. For the small value of $h$, the vacuum is determined from the stationary conditions $\frac{\partial L}{\partial \sigma} = \frac{\partial L}{\partial \pi} = 0$, which lead us to

$$(\sigma, \pi) = \left(f_\pi + \frac{h}{2\mu^2}, 0\right) \quad (4.12)$$

with $f_\pi \equiv \sqrt{\frac{\mu^2}{\lambda_1}}$. From this vacuum, one obtains the mass spectra such that

$$M^2_\pi(h) = \frac{h}{f_\pi}, \quad M^2_\sigma(h) = 2\mu^2 + \frac{3h}{f_\pi} = 2\mu^2 + 3M^2_\pi(h) \quad (4.13)$$

When we compare these results with those obtained from the holographic method, we should restrict the region of the parameter $b_0$ to the confinement phase, namely to $b_0 < 1$ for $r_0 = 1$. Hereafter, we consider the case of $r_0 = 1$. The reason is that the above $\sigma$-model should be considered for the confinement phase since the dynamical quarks are not included, and then the Lagrangian is written only by mesons.

The mass relation between $M^2_\pi$ and $M^2_\sigma$ given by (4.13) however definitely deviates from our holographic results in the region of $b_0 < 1$. Rather it gives a good fit with the one given near the transition point of chiral symmetry, namely at $b_0 = 1.3$. So it might be better to modify the above $\sigma$-model in order to get a better correspondence to our analysis in the region of $b_0 < 1$.

Then, we show an example of a possible improvement of (4.11) to more favourite form. This is performed by adding the next order term of $(\sigma^2 + \pi^2)$ as follows,

$$L = \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda_1}{4}(\sigma^2 + \pi^2)^2 - \frac{\lambda_2}{6}(\sigma^2 + \pi^2)^3 + h\sigma. \quad (4.14)$$
This type of improvement is justified when we suppose that the improvement is based on the derivation of the effective $\sigma$-model by expanding the D7 probe brane in terms of the $\pi$ and $\sigma$ fields. In general, this expansion provides many terms of higher powers of the fields $\pi$ and $\sigma$, and then the form will be very complicated. In this sense, we should say that the above form is the simplest extended version. The vacuum values of the fields are obtained as above by the following setting

$$(\sigma, \pi) = (\sigma_0, 0), \quad \text{(4.15)}$$

where $\sigma_0$ is the real solution of

$$h = (-\mu^2 + \lambda_1 \sigma_0^2 + \lambda_2 \sigma_0^4)\sigma_0. \quad \text{(4.16)}$$

Then we have the masses for $\pi$ and $\sigma$ as,

$$M_\pi^2 = -\mu^2 + \lambda_1 \sigma_0^2 + \lambda_2 \sigma_0^4 = \frac{h}{\sigma_0}, \quad M_\sigma^2 = -\mu^2 + 3\lambda_1 \sigma_0^2 + 5\lambda_2 \sigma_0^4. \quad \text{(4.17)}$$

From these, we have the following relation

$$M_\sigma^2 = 2M_\pi^2 + \mu^2, \quad \text{(4.18)}$$

by setting the following relation for the parameters

$$\lambda_2 = -\frac{\lambda_1}{3\sigma_0^2}. \quad \text{(4.19)}$$

This relation is a tuning to get (4.18), which provides a better fitting to our holographic results as shown in the right of the Fig. 4

### 5 Entanglement Entropy near the transition region

Here we study the entanglement entropy ($S_{EE}$) near the transition region to find a sign of the phase transition if it existed. $S_{EE}$ is given by calculating the minimum area of the surface $A$ whose boundary $\partial A$ is set at the boundary of the bulk. As given in [14, 15] the holographic entanglement entropy is given by

$$S_{EE} = \frac{S_{\text{area}}}{4G_N}, \quad \text{(5.1)}$$

where $S_{\text{area}}$ denotes the minimal surface whose boundary is defined by $\partial A$ and the surface is extended into the bulk. On this formula, the details are seen in [15].

Here we see the regularized finite part $\tilde{S}_{finite}$. This quantity contains two contributions from the curvature $r_0$ and the dark radiation $b_0$. In order to see how the dark radiation affects on the entropy, we consider here the following quantity,

$$S_{finite} \equiv \tilde{S}_{finite} - \tilde{S}_{finite}|_{b_0=0}, \quad \text{(5.2)}$$
Fig. 6: The left figure is the relation between $S_{\text{finite}}/V$ and $b_0$ for $r_0 = R = 1$ and $a_0 = 0.5$. For large $b_0$ region and small $b_0$ region, $S_{\text{finite}}/V$ can be fitted by $S_{\text{finite}}/V = 26.1T_H^3$ (blue line) and $S_{\text{finite}}/V = 0.97b_0^4$ (red line) respectively. The right figure shows the relation at small $b_0$ region. There is no specific phenomenon at the chiral transition point ($b_0 = 1.37$).

by subtracting the $S_{\text{finite}}|_{b_0=0}$ from $S_{\text{finite}}$.

Fig.6 shows the relation between $S_{\text{finite}}/V$ and $b_0$. Here, $V$ is the volume of the sphere with radius $p = p_0$ in $FRW_4$ space,

$$V = 4\pi a_0^3 \int_0^{p_0} \gamma^3 p^2 dp = 2\pi a_0^3 \left( \frac{4p_0(4 + p_0^2)}{(p_0^2 - 4)^2} + \log \frac{2 - p_0}{2 + p_0} \right). \quad (5.3)$$

From the results shown in the Fig.6, we can observe the followings.

- As shown in the Fig.6, for small $b_0$ region, $S_{\text{finite}}/V$ is small and $S_{\text{finite}}/V \propto b_0^4$. The energy density of the dark radiation is also proportional to $b_0^4$, then the small deviation of $S_{\text{finite}}$ and the energy density will lead to the first law of the thermodynamics. On this point, we will discuss more in the near future.

- For large $b_0 > 1$ region, on the other hand, it increases with $b_0$ rapidly and $S_{\text{finite}}/V \propto T_H^3$ where the Hawking temperature $T_H(b_0)$ is given as

$$T_H(b_0) = \frac{\sqrt{2b_0}}{\pi R^2} \sqrt{1 - (r_0/b_0)^2}. \quad (5.4)$$

Thus, at large $b_0$ region, the entanglement entropy becomes the thermal entropy. This behavior is expected as a high temperature behaviour of the entanglement entropy $S_{\text{finite}}/V$.

- However, as far as we observe the Figure 6, there is no specific sign at the chiral transition point ($b_0 \sim 1.31$) in the behavior of the entanglement entropy.
6 Summary and Discussion

Here we have studied SYM theory in the AdS$_4$ space-time. The holographic dual is expressed by a supergravity solution which is described by two free parameters, which correspond to the negative 4D cosmological constant and the dark radiation. These two quantities work to opposite direction to realize a typical phase of the theory. The negative $\lambda$ leads the theory to the confinement phase, however, the dark radiation prevents it. Then we find confinement-deconfinement transition at their balanced point, $r_0 = b_0$, as shown previously.

Here we have pointed out that the chiral symmetry restoration does not still occur at $r_0 = b_0$ as expected in the case of usual QCD. The chiral transition is found at $b_0 = 1.31r_0$ after the deconfinement transition. So there exists a new phase, where chiral symmetry is broken but the quarks and gluons are deconfined. It is shown as the region B in the phase diagram in the Fig. 1. In this region, the Nambu-Goldstone boson is certainly observed, and then we could examine further the mass spectra of mesons made of massive quark and anti-quarks to assure the GOR relation. And then we could study the mass relation of the NG boson and the massive scalar modes as expected from the sigma model, which describes well the spontaneous chiral symmetry breaking of QCD. While the modified sigma model might be consistent with our holographic results in the confinement region A, however, we could not find a simple sigma model which is consistent in both the regions A and B.

Finally, we have examined the entanglement entropy to see the role of the dark radiation in the phase transition. To make clear the contribution of $b_0$, the deviation of the entanglement entropy given at $b_0 > 0$ from the one at $b_0 = 0$ is numerically studied. For small $b_0(< r_0)$, the deviation is small but increases like $b_0^2$, which is proportional to the energy density of the dark radiation. In the large $b_0(>> r_0)$ region, it increases rapidly and it is proportional to $b_0^3 \propto T_H^3$. This behavior is the usual thermal behavior expected in the infrared limit of the theory. However a sharp sign of the phase transition has not been observed.

Here, we would like to mention about relations between our results and those from some 4 dimensional approaches. The Polyakov-Nambu-Jona-Lasinio (PNJL) model has been often used to obtain the QCD phase diagram at finite temperature/chemical potential. In this model, in addition to chiral symmetry breaking/restoration due to the quark-anti-quark condensate which was originally developed in the Nambu-Jona-Lasinio model, the confinement-deconfinement phase transition property can be taken into account by the Polyakov loop potential. According to the model calculation, $T_r$ gets higher than $T_c$. This is the similar result with ours.

In the lattice QCD calculation, however, the result is controversial. $T_r$ gets lower than $T_c$ [18]. In [19], the authors have tried to reproduce the same result from the PNJL model by including some extra terms, but they did not succeed. As the consequence, even among the 4 dimensional approaches, there is no common picture for the QCD
phase transition properties at finite temperature. Physically speaking, in the case with $T_\chi \leq T_c$, massless baryons appear in the temperature regime $T_\chi \leq T \leq T_c$ and they would affect the thermodynamic quantities such as the equation of state.

On the other hand, the chiral phase transition in curved space has been recently discussed [20]. It is suggested that the negative curvature $R < 0$ (the negative cosmological constant) shifts the critical temperature $T_\chi$ to the increased one.

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