A basic inequality for submanifolds in a cosymplectic space form

Jeong-Sik Kim and Jaedong Choi*

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Abstract

For submanifolds tangent to the structure vector field in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants of the submanifold, namely its sectional curvature and scalar curvature on one side; and its main extrinsic invariant, namely squared mean curvature on the other side. Some applications including inequalities between the intrinsic invariant $\delta_M$ and the squared mean curvature are given. The equality cases are also discussed. 2000 AMS Subject Classification: 53C40, 53D15.

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1 Introduction

To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the natural interests in the submanifold theory. Let $M$ be an $n$-dimensional Riemannian manifold. For each point $p \in M$, let $(\inf K) (p) = \inf \{ K(\pi) : \text{plane sections } \pi \subset T_p M \}$. Then, the well defined intrinsic invariant $\delta_M$ for a $M$ introduced by B.-Y. Chen([4]) is

$$\delta_M (p) = \tau (p) - (\inf K) (p), \quad (1)$$

where $\tau$ is the scalar curvature of $M$ (see also [6]).

In [3], Chen established the following basic inequality involving the intrinsic invariant $\delta_M$ and the squared mean curvature for $n$-dimensional submanifolds $M$ in a real space form $R (c)$ of constant sectional curvature $c$:

$$\delta_M \leq \frac{n^2 (n - 2)}{2(n - 1)} \| H \|^2 + \frac{1}{2} (n + 1) (n - 2) c. \quad (2)$$

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The above inequality is also true for anti-invariant submanifolds in complex space forms \( \tilde{M}(4c) \) as remarked in [7]. In [5], he proved a general inequality for an arbitrary submanifold of dimension greater than two in a complex space form. Applying this inequality, he showed that (2) is also valid for arbitrary submanifolds in complex hyperbolic space \( CH^m(4c) \). He also established the basic inequality for a submanifold in a complex projective space \( CP^m \).

A submanifold normal to the structure vector field \( \xi \) of a contact manifold is anti-invariant. Thus \( C \)-totally real submanifolds in a Sasakian manifold are anti-invariant, as they are normal to \( \xi \). An inequality similar to (2) for \( C \)-totally real submanifolds in a Sasakian space form \( \tilde{M}(c) \) of constant \( \varphi \)-sectional curvature \( c \) is given in [8]. In [9], for submanifolds in a Sasakian space form \( \tilde{M}(c) \) tangential to the structure vector field \( \xi \), a basic inequality along with some applications are presented.

There is another interesting class of almost contact metric manifolds, namely cosymplectic manifolds([10]). In this paper, submanifolds tangent to the structure vector field \( \xi \) in cosymplectic space forms are studied. Section 2 contains necessary details about submanifolds and cosymplectic space forms are given for further use. In section 3, for submanifolds tangent to the structure vector field \( \xi \) in cosymplectic space forms, we establish a basic inequality between the main intrinsic invariants, namely its sectional curvature function \( \tilde{K} \) and its scalar curvature function \( \tau \) of the submanifold on one side, and its main extrinsic invariant, namely its mean curvature function \( \|H\| \) on the other side. In the last section, we give some applications including inequalities between the intrinsic invariant \( \delta_M \) and the extrinsic invariant \( \|H\| \). We also discuss the equality cases.

2 Preliminaries

Let \( \tilde{M} \) be a \((2m+1)\)-dimensional almost contact manifold([2]) endowed with an almost contact structure \((\varphi, \xi, \eta)\), that is, \( \varphi \) is a \((1,1)\) tensor field, \( \xi \) is a vector field and \( \eta \) is 1-form such that \( \varphi^2 = -I + \eta \otimes \xi \) and \( \eta(\xi) = 1 \). Then, \( \varphi(\xi) = 0 \) and \( \eta \circ \varphi = 0 \).

Let \( g \) be a compatible Riemannian metric with \((\varphi, \xi, \eta)\), that is, \( g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \) or equivalently, \( g(X, \varphi Y) = -g(\varphi X, Y) \) and \( g(X, \xi) = \eta(X) \) for all \( X, Y \in TM \). Then, \( \tilde{M} \) becomes an almost contact metric manifold equipped with an almost contact metric structure \((\varphi, \xi, \eta, g)\). An almost contact metric manifold is cosymplectic([2]) if \( \nabla_X \varphi = 0 \), where \( \nabla \) is the Levi-Civita connection of the Riemannian metric \( g \). From the formula \( \nabla_X \varphi = 0 \) it follows that \( \nabla_X \xi = 0 \).

A plane section \( \sigma \) in \( T_p\tilde{M} \) of an almost contact metric manifold \( \tilde{M} \) is called a \( \varphi \)-section if \( \sigma \perp \xi \) and \( \varphi(\sigma) = \sigma \). \( \tilde{M} \) is of constant \( \varphi \)-sectional curvature if the sectional curvature \( \tilde{K}(\sigma) \) does not depend on the choice of the \( \varphi \)-section \( \sigma \) of \( T_p\tilde{M} \) and the choice of a point \( p \in \tilde{M} \). A cosymplectic manifold \( \tilde{M} \) is of constant \( \varphi \)-sectional curvature...
curvature $c$ if and only if its curvature tensor $\tilde{R}$ is of the form (10)

$$4\tilde{R}(X,Y,Z,W) = c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)$$

$$+ g(X,\varphi W)g(Y,\varphi Z) - g(X,\varphi Z)g(Y,\varphi W)$$

$$- 2g(X,\varphi Y)g(Z,\varphi \eta)g(W,\varphi \eta)$$

$$+ g(X,\varphi Y)g(Z,\varphi \eta)g(W,\varphi \eta) - g(Y,\varphi Z)g(X,\varphi \eta)g(W,\varphi \eta)$$

$$+ g(Y,\varphi W)g(X,\varphi \eta)g(Z,\varphi \eta) - g(Y,\varphi W)g(X,\varphi \eta)g(Z,\varphi \eta)\}.$$  (3)

Let $M$ be an $(n+1)$-dimensional submanifold of a manifold $\tilde{M}$ equipped with a Riemannian metric $g$. The Gauss and Weingarten formulae are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N$$

for all $X, Y \in TM$ and $N \in T^\perp M$, where $\tilde{\nabla}$, $\nabla$ and $\nabla^\perp$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, $M$ and the normal bundle $T^\perp M$ of $M$ respectively, and $h$ is the second fundamental form related to the shape operator $A$ by $g(h(X,Y),N) = g(A_N X,Y)$.

Let $\{e_1,\ldots,e_{n+1}\}$ be an orthonormal basis of the tangent space $T_p M$. The mean curvature vector $H(p)$ at $p \in M$ is

$$H(p) \equiv \frac{1}{n+1} \sum_{i=1}^{n+1} h(e_i,e_i).$$  (4)

The submanifold $M$ is totally geodesic in $\tilde{M}$ if $h = 0$, and minimal if $H = 0$. We put

$$h_{ij}^r = g(h(e_i,e_j),e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n+1} g(h(e_i,e_j),h(e_i,e_j)).$$

3 A basic inequality

Let $M$ be a submanifold of an almost contact metric manifold. For $X \in TM$, let

$$\varphi X = PX + FX, \quad PX \in TM, \quad FX \in T^\perp M.$$ 

Thus, $P$ is an endomorphism of the tangent bundle of $M$ and satisfies

$$g(X,PY) = -g(PX,Y), \quad X,Y \in TM.$$ 

For a plane section $\pi \subset T_p M$ at a point $p \in M$,

$$\alpha(\pi) = g(e_1,Pe_2)^2 \quad \text{and} \quad \beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2$$

are real numbers in the closed unit interval $[0,1]$, which are independent of the choice of the orthonormal basis $\{e_1,e_2\}$ of $\pi$.

We recall the following lemma from ([3]).
Lemma 3.1 If \(a_1, \ldots, a_{n+1}, a\) are \(n+2\) \((n \geq 1)\) real numbers such that
\[
\left(\sum_{i=1}^{n+1} a_i\right)^2 = n \left(\sum_{i=1}^{n+1} a_i^2 + a\right),
\]
then \(2a_1a_2 \geq a\), with equality holding if and only if \(a_1 + a_2 = a_3 = \cdots = a_{n+1}\).

Now, we prove the following

Theorem 3.2 Let \(M\) be an \((n+1)\)-dimensional \((n \geq 2)\) submanifold isometrically
immersed in a \((2m+1)\)-dimensional cosymplectic space form \(\tilde{M}(c)\) such that the
structure vector field \(\xi\) is tangential to \(M\). Then, for each point \(p \in M\) and each
plane section \(\pi \subset T_p M\), we have
\[
\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n} \|H\|^2 + \frac{c}{8} \left(3 \|P\|^2 - 6 \alpha(\pi) + 2 \beta(\pi) + (n+1)(n-2)\right). \tag{5}
\]

The equality in (5) holds at \(p \in M\) if and only if there exists an orthonormal basis
\(\{e_1, \ldots, e_{n+1}\}\) of \(T_p M\) and an orthonormal basis \(\{e_{n+2}, \ldots, e_{2m+1}\}\) of \(T_p^\perp M\) such
that (a) \(\pi = \text{Span} \{e_1, e_2\}\) and (b) the forms of shape operators \(A_r \equiv A_{e_r}\), \(r = n+2, \ldots, 2m+1\), become
\[
A_{n+2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda + \mu) I_{n-1} \end{pmatrix}, \tag{6}
\]
\[
A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n+3, \ldots, 2m+1. \tag{7}
\]

**Proof.** In view of the Gauss equation and (3), the scalar curvature and the
mean curvature of \(M\) are related by
\[
2\tau = \frac{c}{4} \left(3 \|P\|^2 + n(n-1)\right) + (n+1)^2 \|H\|^2 - \|h\|^2, \tag{8}
\]
where \(\|P\|^2\) is given by
\[
\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2
\]
for any local orthonormal basis \(\{e_1, e_2, \ldots, e_{n+1}\}\) for \(T_p M\). We introduce
\[
\rho = 2\tau - \frac{(n+1)^2(n-1)}{n} \|H\|^2 - \frac{c}{4} \left(3 \|P\|^2 + n(n-1)\right). \tag{9}
\]
From (8) and (9), we get
\[
(n+1)^2 \|H\|^2 = n(\|h\|^2 + \rho). \tag{10}
\]
Let \( p \) be a point of \( M \) and let \( \pi \subset T_p M \) be a plane section at \( p \). We choose an orthonormal basis \( \{e_1, e_2, \ldots, e_{n+1}\} \) for \( T_p M \) and \( \{e_{n+2}, \ldots, e_{2m+1}\} \) for the normal space \( T_p^\perp M \) at \( p \) such that \( \pi = \text{Span} \{e_1, e_2\} \) and the mean curvature vector \( H(p) \) is parallel to \( e_{n+2} \), then from (10) we get

\[
\left( \sum_{i=1}^{n+1} h_{ii}^{n+2} \right)^2 = n \left( \sum_{i=1}^{n+1} (h_{ii}^{n+2})^2 + \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \rho \right). \tag{11}
\]

Using Lemma 3.1, from (11) we obtain

\[
h_{11}^{n+2} h_{22}^{n+2} \geq \frac{1}{2} \left\{ \sum_{i \neq j} (h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \rho \right\}. \tag{12}
\]

From the Gauss equation and (3), we also have

\[
K(\pi) = \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + \frac{1}{2} \rho + h_{11}^{n+2} h_{22}^{n+2} - \left( \sum_{r=n+3}^{2m+1} (h_{11}^r h_{22}^r - (h_{12}^r)^2) \right). \tag{13}
\]

Thus, we have

\[
K(\pi) \geq \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + \frac{1}{2} \rho + \frac{1}{2} \sum_{r=n+3}^{2m+1} \sum_{i,j=1}^{n+1} (h_{ij}^r)^2 + \frac{1}{2} \sum_{r=n+3}^{2m+1} (h_{11}^r + h_{22}^r)^2, \tag{14}
\]

or

\[
K(\pi) \geq \frac{c}{4} (1 + 3\alpha(\pi) - \beta(\pi)) + \frac{1}{2} \rho, \tag{15}
\]

which in view of (\text{3.1}) yields (\text{3.3}).

If the equality in (\text{3.3}) holds, then the inequalities given by (\text{12}) and (\text{14}) become equalities. In this case, we have

\[
h_{12}^{n+2} = 0, \quad h_{11}^{n+2} = 0, \quad h_{22}^{n+2} = 0, \quad i \neq j > 2;
\]

\[
h_{12}^r = h_{22}^r = h_{ij}^r = 0, r = n + 3, \ldots, 2m + 1; \quad i, j = 3, \ldots, n + 1;
\]

\[
h_{11}^{n+3} + h_{22}^{n+3} = \cdots = h_{11}^{2m+1} + h_{22}^{2m+1} = 0. \tag{16}
\]

Furthermore, we may choose \( e_1 \) and \( e_2 \) so that \( h_{12}^{n+2} = 0 \). Moreover, by applying Lemma 3.1, we also have

\[
h_{11}^{n+2} + h_{22}^{n+2} = h_{33}^{n+2} = \cdots = h_{n+1 n+1}^{n+2} = 0. \tag{17}
\]

Thus, choosing a suitable orthonormal basis \( \{e_1, \ldots, e_{2m+1}\} \), the shape operator of \( M \) becomes of the form given by (\text{3.3}) and (\text{3.4}). The converse is straightforward.
4 Some applications

For the case $c = 0$, from (3) we have the following pinching result.

**Proposition 4.1** Let $M$ be an $(n + 1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M} (c)$ with $c = 0$ such that $\xi \in TM$. Then, we have the following

$$\delta_M \leq \frac{(n + 1)^2 (n - 1)}{2n} \|H\|^2.$$

A submanifold $M$ of an almost contact metric manifold $\tilde{M}$ with $\xi \in TM$ is called a semi-invariant submanifold ([1]) of $\tilde{M}$ if $TM = D \oplus D^\perp \oplus \{\xi\}$, where $D = TM \cap \varphi(TM)$ and $D^\perp = TM \cap \varphi(T^\perp M)$. In fact, the condition $TM = D \oplus D^\perp \oplus \{\xi\}$ implies that the endomorphism $P$ is an $f$-structure ([12]) on $M$ with $\text{rank} (P) = \text{dim} (D)$.

Now, we establish two inequalities in the following two theorem, which are analogous to that of (2).

**Theorem 4.2** Let $M$ be an $(n + 1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m + 1)$-dimensional cosymplectic space form $\tilde{M} (c)$ such that the structure vector field $\xi$ is tangential to $M$. If $c < 0$, then

$$\delta_M \leq \frac{(n + 1)^2 (n - 1)}{2n} \|H\|^2 + \frac{1}{2} (n + 1) (n - 2) \frac{c}{4}. \quad (18)$$

The equality in (18) holds if and only if $M$ is a semi-invariant submanifold with $\text{rank} (P) = 2$ and $\beta (\pi) = 0$.

**Proof.** Since $c < 0$, in order to estimate $\delta_M$, we minimize $3 \|P\|^2 - 6\alpha (\pi) + 2\beta (\pi)$ in (3). For an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of $T_pM$ with $\pi = \text{span} \{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha (\pi) = \sum_{i,j=3}^{n+1} g(e_i, \varphi e_j)^2 + 2 \sum_{j=3}^{n+1} \{ (g(e_1, \varphi e_j)^2 + g(e_2, \varphi e_j)^2 \}.$$

Thus, we see that the minimum value of $3 \|P\|^2 - 6\alpha (\pi) + 2\beta (\pi)$ is zero, provided $\pi = \text{span} \{e_1, e_2\}$ is orthogonal to $\xi$ and $\text{span} \{\varphi e_j \mid j = 3, \ldots, n\}$ is orthogonal to the tangent space $T_pM$. Thus, we have (18) with equality case holding if and only if $M$ is semi-invariant such that $\text{rank} (P) = 2$ with $\beta = 0$. 
Theorem 4.3 Let $M$ be an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that $\xi \in TM$. If $c > 0$, then

$$\delta_M \leq \frac{(n+1)^2(n-1)}{2n} \|H\|^2 + \frac{1}{2}n(n+2)\frac{c}{4},$$

(19)

The equality in (19) holds if and only if $M$ is an invariant submanifold and $\beta = 1$.

**Proof.** Since $c > 0$, in order to estimate $\delta_M$, we maximize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (19). We observe that the maximum of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is attained for $\|P\|^2 = n$, $\alpha(\pi) = 0$ and $\beta(\pi) = 1$, that is, $M$ is invariant and $\xi \in \pi$. Thus, we obtain (19) with equality case if and only if $M$ is invariant with $\beta = 1$.

In last, we prove the following

Theorem 4.4 If $M$ is an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$ such that $c > 0$, $\xi \in TM$ and

$$\delta_M = \frac{(n+1)^2(n-1)}{2n} \|H\|^2 + \frac{1}{2}n(n+2)\frac{c}{4},$$

then $M$ is a totally geodesic cosymplectic space form $\tilde{M}(c)$.

**Proof.** In view of Theorem 4.3, $M$ is an odd-dimensional invariant submanifold of the cosymplectic space form $\tilde{M}(c)$. For every point $p \in M$, we can choose an orthonormal basis $\{e_1 = \xi, e_2, \ldots, e_{n+1}\}$ for $T_pM$ and $\{e_{n+2}, \ldots, e_{2m+1}\}$ for $T_p^\perp M$ such that $A_r \ (r = n+2, \ldots, 2m+1)$ take the form (19) and (20). Since $M$ is an invariant submanifold of a cosymplectic manifold, therefore it is minimal and $A_r \varphi + \varphi A_r = 0, r = n+2, \ldots, 2m+1([11])$. Thus all the shape operators take the form

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n+2, \ldots, 2m+1.$$  

(20)

Since, $A_r \varphi e_1 = 0, r = n+2, \ldots, 2m+1$, from $A_r \varphi + \varphi A_r = 0$, we get $\varphi A_r e_1 = 0$. Applying $\varphi$ to this equation, we obtain $A_r e_1 = \eta(A_r e_1) \xi = \eta(A_r e_1) e_1$; and thus $d_r = 0, r = n+2, \ldots, 2m+1$. This implies that $A_r e_2 = -c_r e_2$. Applying $\varphi$ to the both sides, in view of $A_r \varphi + \varphi A_r = 0$, we get $A_r e_2 = c_r \varphi e_2$. Since $\varphi e_2$ is orthogonal to $\xi$ and $e_2$ and $\varphi$ has maximal rank, the principal curvature $c_r$ is zero. Hence, $M$ becomes totally geodesic. As in Proposition 1.3 on page 313 of [12], it is easy to show that $M$ is a cosymplectic manifold of constant $\varphi$-sectional curvature $c$.

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Jeong-Sik Kim
Department of Mathematics Education
Sunchon National University, Sunchon 540-742, Korea
email: jskim01@hanmir.com

Jaedong Choi
Department of Mathematics
P. O. Box 335-2 Airforce Academy
Ssangsu, Namil, Chungwon, Chungbuk, 363-849, Korea
e-mail : jdong@afa.ac.kr