A Unified Analysis Method for Online Optimization in Normed Vector Space

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Abstract

This paper studies online optimization from a high-level unified theoretical perspective. We not only generalize both Optimistic-DA and Optimistic-MD in normed vector space, but also unify their analysis methods for dynamic regret. Regret bounds are the tightest possible due to the introduction of $\phi$-convex. As instantiations, regret bounds of normalized exponentiated subgradient and greedy/lazy projection are better than the currently known optimal results. By replacing losses of online game with monotone operators, and extending the definition of regret, namely ‘regret’, we extend online convex optimization to online monotone optimization.

Index terms—optimistic online learning, online convex optimization, online monotone optimization, dynamic regret, static regret, generalized Bregman divergence, $\phi$-convex

1 Introduction

The online convex optimization problem introduced by Zinkevich (2003) can be regarded as a repeated game between the learner and the adversary (environment). At round $t$, the learner chooses a map $x_t$ from a hypothesis class $C$ as prediction, and the adversary feeds back a convex loss function $\varphi_t$, then the learner suffers an instantaneous loss $\varphi_t(x_t)$. The goal is to minimize

$$\text{regret}(z_1, z_2, \cdots, z_T) := \sum_{t=1}^{T} \varphi_t(x_t) - \sum_{t=1}^{T} \varphi_t(z_t),$$

where $z_t \in C$ represents an arbitrary reference strategy in round $t$, and $T$ is the number of rounds. Equation (1) is called the dynamic regret. For static regret, which often appears in the literature (e.g., Cesa-Bianchi and Lugosi, 2006; Shalev-Shwartz, 2012; McMahan, 2017; Hazan, 2019; Orabona, 2019), simply let $z_t \equiv z$. Although dynamic regret has attracted widespread attention recently (e.g., Hall and Willett, 2013; Jadabaie et al., 2015; Mokhtari et al., 2016; Zhang et al., 2018; Campolongo and Orabona, 2021; Kalhan et al., 2021), it still lacks systematic research.

Generally, strategies for online optimization fall into two major families, Mirror Descent (MD) and Follow The Regularized Leader (FTRL). MD was introduced by Beck and Teboulle (2003), and a special form of MD can be traced back to Nemirovskiï and Yudin (1983). FTRL was introduced by Abernethy et al. (2008), and its core ideas can be traced back to Shalev-Shwartz and Singer (2006). FTRL with surrogate linearized losses is also called Dual Averaging (DA Nesterov, 2009). An algorithm is optimistic, if the prediction of the impending loss is added to the update rule. The Optimistic-MD was proposed by Chiang et al. (2012), and extended by Rakhlin and Sridhara (2013), who also proposed the Optimistic-FTRL. Both DA and MD can
be unified via the Unified Mirror Descent (UMD) according to Juditsky et al. (2019). However, the level of abstraction and generality of UMD is not enough.

For strategies based on surrogate linearization losses, online convex optimization naturally extends to online monotone optimization. This idea was first proposed by Gemp and Mahadevan (2016) based on earlier research. After digging deeper, we found that the novelty of online monotone optimization is to allow the online game to abandon the concept of loss function. Therefore, we argue that the concept of online monotone optimization needs to be rigorously reformulated.

This paper addresses the online optimization problem from a high-level unified theoretical perspective. We not only generalize both Optimistic-DA and Optimistic-MD in normed vector space, but also unify their analysis methods for dynamic regret. We focus on optimistic online learning since optimism is at the hub. For a non-optimistic version, it suffices to set the estimated term to be null, and for learning with delay, it suffices to modify the estimated term to delete the unobserved loss subgradients (Flaspohler et al., 2021). Dynamic regret is chosen as the performance metric due to its generality. For static regret, it suffices to fix the reference strategies to be constant over time. Normed vector space is chosen over \( \mathbb{R}^n \) because it shows more essential details. The contributions of this paper are as follows.

- We present a unified analysis method for online optimization in normed vector space using dynamic regret as the performance metric. Our analysis is based on two relaxation strategies, namely S-I and S-II (Optimistic-MD), which are obtained through the relaxation of S, a two-parameter variant strategy covering Optimistic-DA. The analysis process relies on the generalized cosine rule and \( \phi \)-convex, both of which depend on the generalized Bregman divergence.

- The regret bounds are the tightest possible. Our analysis shows that the last term of all upper bounds is an extra subtraction term, and \( \phi \)-convex further tightens the upper bounds. Instantiations in Section 6 show that the regret bounds for normalized exponentiated subgradient and greedy/lazy projection are better than the currently known optimal results.

- All strategies are not only suitable for online convex optimization, but also for online monotone optimization. We formalize the online monotone optimization problem, and propose the definition of regret of the generalized version of regret, which allows the absence of losses in online game. This is natural and mathematically rigorous.

## 2 Preliminaries

Let \( E \) be a normed vector space over \( \mathbb{R} \) and let \( E^* \) be the dual space of \( E \). We denote by \( \| \cdot \|_E \) the norm of \( E \), and denote by \( \| \cdot \|_{E^*} \) the norm of \( E^* \). Without causing ambiguity, the subscript of the norm is usually omitted, for example, the space in which the element is located is known. We denote by \( \langle \cdot, \cdot \rangle \) the scalar product for the duality \( E^* \) and \( E \). Usually the elements with superscript * are points in \( E^* \).

We use superscript ★ to denote the Fenchel conjugate of a function. Throughout this paper, we introduce \( Q_\rho (x) = \frac{1}{2} x^2 + \chi_{[-\rho, \rho]} (x) \), where \( \forall \rho \in (0, +\infty) \) and \( \forall x \in \mathbb{R} \), and its Fenchel conjugate is \( Q^*_\rho (x) = \frac{1}{2} x^2 - \frac{1}{2} (|x| - \rho)_+^2 \), where \( \forall x \in \mathbb{R}, x_+ := \max \{ x, 0 \} \).

See Appendix A for basis of convex analysis.

Next, we present definitions of generalized Bregman divergence and subdifferential with symmetrical beauty.
**Definition 1** (Generalized Bregman Divergence). The generalized Bregman divergence w.r.t. a proper function $\varphi$ is defined as

$$B_{\varphi} (x, y^*) := \varphi (x) + \varphi^* (y^*) - \langle y^*, x \rangle, \quad \forall (x, y^*) \in E \times E^*.$$  

**Definition 2** (Subdifferential). The subdifferential of a proper function $\varphi$ at $x$ is

$$\partial \varphi (x) := \{ x^* \in E^* \mid B_{\varphi} (x, x^*) = 0 \}.$$  

Any element in $\partial \varphi (x)$ is a subgradient of $\varphi$ at $x$, denoted by $x^\varphi$.

**Remark 3.** This remark illustrates the motivation for the above two definitions. The subdifferential of $\varphi$ at $x$ is usually defined as follows (Section 2.4 of Zălinescu, 2002),

$$\{ x^* \in E^* \mid \varphi (y) - \varphi (x) \geq \langle x^*, y - x \rangle, \forall y \in E \},$$

which is equivalent to $\{ x^* \in E^* \mid \varphi^* (x^*) + \varphi (x) = \langle x^*, x \rangle \}$ according to the definition of Fenchel conjugate. Generally, the definition of Bregman divergence is associated with a continuously differentiable function. Definition 1 of Joulani et al. (2020) extends the definition of Bregman divergence by replacing continuously differentiable with directionally differentiable. We first consider utilizing subdifferentiable to define Bregman divergence, and then we notice that $\forall y^* \in \partial \varphi (y)$,

$$\varphi (x) - \varphi (y) - \langle y^*, x - y \rangle = \varphi (x) + \varphi^* (y^*) - \langle y^*, x \rangle,$$

which implies that a symmetrical aesthetic expression can be used to extend the definition of Bregman divergence and equivalently describe the definition of subdifferential.

Similar to Lemma 3.1 of Chen and Teboulle (1993), we have the following generalized cosine rule, which plays a key role in the derivation of regret upper bounds.

**Lemma 4** (Generalized Cosine Rule). If $(y, y^*) \in \text{gra} \, \partial \varphi$, then

$$B_{\varphi} (x, y^*) + B_{\varphi} (y, z^*) - B_{\varphi} (x, z^*) = \langle z^* - y^*, x - y \rangle.$$  

**Remark 5.** We call Lemma 4 the generalized cosine rule, because if $E$ is a Hilbert space and $\varphi = \frac{1}{2} \| \cdot \|^2$. Lemma 4 is instantiated as the ordinary cosine rule. Indeed, under the above settings, $B_{\frac{1}{2} \| \cdot \|^2} (x, y^*) = \frac{1}{2} \| x - y^* \|^2$, and $y^* = y$ since $\partial \left( \frac{1}{2} \| \cdot \|^2 \right)$ is the identity map (Proposition 3.6 of Chidume, 2009). Thus, we have

$$\| x - y \|^2 + \| y - z^* \|^2 - \| x - z^* \|^2 = 2 \langle z^* - y, x - y \rangle.$$  

The following lemma plays a pivot role in the equivalent transformation of update rules.

**Lemma 6.** If $\varphi$ is proper, convex and lower semicontinuous, then

$$x^* \in \partial \varphi (x) \iff x \in \partial \varphi^* (x^*),$$

and

$$\partial \varphi (x) = \arg \max_{x^* \in E^*} \langle x^*, x \rangle - \varphi^* (x^*), \quad \partial \varphi^* (x^*) = \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x).$$

Now we define a generalized version of strongly convex — $\phi$-convex.

**Definition 7** ($\phi$-Convex). A function $\varphi$ is $\phi$-convex if

$$B_{\varphi} (x, y^*) \geq \phi (\| x - y \|), \quad \forall x \in E, \quad \forall (y, y^*) \in \text{gra} \, \partial \varphi,$$

where $\phi$ is convex and $\phi \geq 0, \phi (0) = 0$.

**Lemma 8.** Let $\alpha$ be an affine function. If $\varphi$ is $\phi$-convex, then $\varphi + \alpha$ is $\phi$-convex.
3 Online Convex Optimization

Let $E$ be a normed vector space over $\mathbb{R}$ and let $C \neq \emptyset$ be a closed convex subset of $E$. The online convex optimization problem can be formalized as follows. At round $t$,

- the player chooses $x_t \in C$ according to some algorithm,
- the adversary (environment) feeds back a loss function $\varphi_t$ with $\text{dom} \partial \varphi_t \subseteq C$.

Although we did not specify that $\varphi_t$ is convex, according to Lemma 39, $\varphi_t + \chi_C$ is convex and lower semicontinuous, where $\chi_C(x) = 0$ iff $x \in C$. We choose the dynamic regret (Equation 1) as the performance metric. For static regret, it suffices to fix the reference strategies to be constant over time.

4 Strategies

In this section, we derive two relaxation variant forms, namely $S$-I and $S$-II, by relaxing the strategy $S$, a two-parameter variant covering Optimistic-DA. The inclusion relationship between strategies in this section is as follows,

$$S$-I $\supset S$ $\supset \begin{cases} S | \theta_t = 1 & = \text{Optimistic-DA}, \\ S | \eta_t = 1 & \subset S$-II $= \text{Optimistic-MD}. \end{cases}$$

Next, we elaborate on their relationship. We start with the definition of $S$.

The strategy $S$ can be formalized as the following two-parameter update rule,

$$x_t \in \arg\min_{x \in E} \left( \sum_{i=1}^{t-1} \theta_i x^*_i + \theta_t \tilde{x}^*_t, x \right) + \frac{1}{\eta_t} B_\psi (x, a_\psi) = \partial \psi^* \left( a_\psi - \eta_t \sum_{i=1}^{t-1} \theta_i x^*_i - \eta_t \theta_t \tilde{x}^*_t \right),$$

where $\tilde{x}^*_t \in E^*$ is the estimated linear loss function corresponding to $x^*_t$, $\eta_t > 0$ is the cumulative parameter since it acts on the cumulative quantity $\sum_{i=1}^{t-1} \theta_i x^*_i$, $\theta_t > 0$ is the instantaneous parameter since it acts on the instantaneous quantity $x^*_t$, $\psi : C \rightarrow \mathbb{R}$ is $\phi$-convex, $(a_\psi, a_\psi^\phi) \in \text{gra} \partial \psi$, and the equality follows from Lemma 6. The equivalent two-step iterative form can be rearranged as follows,

$$\tilde{x}^*_t = a_\psi - \eta_t \sum_{i=1}^{t-1} \theta_i x^*_i, \quad x_t \in \partial \psi^* \left( \tilde{x}^*_t - \eta_t \theta_t \tilde{x}^*_t \right),$$

where $\tilde{x}^*_t$ is the intermediate variable. The corresponding function form of $S$ can be written as $\left( x_t, \tilde{x}^*_t \right) = S (x^*_t, x^*_{t-1}; \eta_t, \theta_t)$, $\tilde{x}^*_1 = a_\psi$.

Most of the literature explores strategies in the form of arg min, this paper elaborates from the perspective of multivalued maps, which makes the analysis simpler.

**Remark 9.** If $\theta_t \equiv \theta$, then the strategy $S$ degenerates into Optimistic-DA, abbreviated as $S | \theta_t = 1$.

Indeed, it suffices to consider the case $\theta_t \equiv 1$, that is,

$$x_t \in \arg\min_{x \in E} \left( \sum_{i=1}^{t-1} x^*_i + \tilde{x}^*_t, x \right) + \frac{1}{\eta_t} B_\psi (x, a_\psi).$$
The type-I relaxation variant form of $S$ (named as $S$-I) is formalized as

$$
\tilde{x}_t^\psi = a^\psi - \eta_t \sum_{i=1}^{t-1} \theta_i x_i^t, \quad \tilde{x}_t \in \partial \psi^* \left( \tilde{x}_t^\psi \right), \quad \tilde{x}_1^\psi \in \partial \psi \left( \tilde{x}_1 \right).
$$

The function form of $S$-I can be written as $\left( x_t, \tilde{x}_t^\psi, \tilde{x}_t^\psi \right) = S$-I $\left( \tilde{x}_t^\psi, \tilde{x}_t^\psi, \eta_t, \theta_t \right), \tilde{x}_1^\psi = a^\psi$.

**Remark 10.** $S$-I allows $\tilde{x}_t^\psi \neq \tilde{x}_t^\psi$, which makes $S$ a special case of $S$-I. Indeed, $\tilde{x}_t^\psi$ and $\tilde{x}_t^\psi$ are both elements of $\partial \psi \left( \tilde{x}_1 \right)$ according to Lemma 6.

If $\eta_t = \eta$, then the strategy $S$ degenerates into the following form (without loss of generality, one can set $\eta_t = 1$, and denote it as $S \big|_{\eta_t = 1}$).

$$
\tilde{x}_t^\psi = \tilde{x}_t^\psi - \theta_t x_t^1, \quad x_t \in \partial \psi^* \left( x_t - \theta_t \tilde{x}_1^\psi \right).
$$

The type-II relaxation variant form of $S$ (named as $S$-II) can be formalized as

$$
\tilde{x}_t^\psi = \tilde{x}_t^\psi - \theta_t x_t^1, \quad \tilde{x}_t \in \partial \psi^* \left( \tilde{x}_t^\psi \right), \quad \tilde{x}_1^\psi \in \partial \psi \left( \tilde{x}_1 \right), \quad \tilde{x}_1^\psi = a^\psi.
$$

The function form of $S$-II can be written as $\left( x_t, \tilde{x}_t^\psi, \tilde{x}_t^\psi \right) = S$-II $\left( \tilde{x}_t^\psi, \tilde{x}_t^\psi, \theta_t \right), \tilde{x}_1^\psi = a^\psi$.

**Remark 11.** $S$-II allows $\tilde{x}_t^\psi = \tilde{x}_t^\psi$, which makes $S \big|_{\eta_t = 1}$ a special case of $S$-II. In fact, $S$-II is Optimistic-MD in normed vector space.

### 5 Regret Analysis

In this section, we use a unified analysis method to prove the regret upper bounds for $S$, $S$-I and $S$-II. We start with the generalized strategy $S$-I.

**Theorem 12** (Dynamic Regret for S-I). If $\partial \psi \left( C \right) = E^*$ or $C$ is compact, then $\forall a \in C$, $S$-I enjoys the following dynamic regret upper bound,

$$
\text{regret} \left( z_1, z_2, \ldots, z_T \right) \leq \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \left[ B_\psi \left( z_t, \tilde{x}_t^\psi \right) - B_\psi \left( z_t, a^\psi + \eta_t \frac{\tilde{x}_t^\psi - a^\psi}{\eta_t + 1} \right) \right] + \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \left[ B_\psi \left( x_{t+1}, \tilde{x}_t^\psi \right) - B_\psi \left( x_{t+1}, \tilde{x}_t^\psi \right) \right] + \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \phi^* \left( \eta_t \theta_t || x_t^1 - \tilde{x}_1^\psi || \right) - \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} B_\psi \left( x_t, \tilde{x}_t^\psi \right), \quad \forall z_t \in C,
$$

where $\left( x_{t+1}, \frac{1}{\eta_t \theta_t} \left( \tilde{x}_t^\psi - a^\psi \right) \right) \in \text{gra} \partial \frac{1}{\eta_t \theta_t} \left( \psi - a^\psi \right)$.

**Proof.** $\partial \psi \left( C \right) = E^*$ or the compactness of $C$ are sufficient conditions to guarantee that $x_t \in C$. See Appendix E for general analysis. The following proof focuses on the derivation of the dynamic regret upper bound for $S$-I.
S-I can be rearranged as follows,

\[ x_t^\psi + \eta_t \theta_t x_t^\psi = \tilde{x}_t, \quad \tilde{x}_t = a^\psi - \eta_t \sum_{i=1}^{t-1} \theta_i x_i^\psi = a^\psi + \eta_t X_t^\psi, \]

where \( X_t^\psi \) is an auxiliary variable, \( \tilde{\psi} : C \to \mathbb{R} \) is a temporarily unknown convex function.

Note that

\[ \phi (x_t) - \phi (z_t) \leq \frac{1}{\eta_t} \left( \theta_t x_t^\psi, x_t - z_t \right), \quad x_t^\psi \in \partial \phi_t (x_t), \quad (2) \]

and

\[
\begin{align*}
\langle \theta_t x_t^\psi, x_t - z_t \rangle &= \langle \theta_t x_t^\psi, x_t - X_{t+1} \rangle + \langle \theta_t x_t^\psi, x_t - X_{t+1} \rangle + \langle \theta_t x_t^\psi, x_t - X_{t+1} \rangle \\
&= - \left( X_t^\psi - X_{t+1}^\psi, z_t - X_{t+1} \right) - \frac{1}{\eta_t} \left( \tilde{x}_t - x_t^\psi, x_t - z_t \right) + \langle \theta_t x_t^\psi, x_t - X_{t+1} \rangle \\
&= B_{\tilde{\psi}} (z_t, X_t^\psi) - B_{\tilde{\psi}} (z_t, X_{t+1}^\psi) - B_{\tilde{\psi}} (X_{t+1}, X_{t+1}^\psi) + \frac{1}{\eta_t} B_{\tilde{\psi}} (X_{t+1}, \tilde{x}_t^\psi) - \frac{1}{\eta_t} B_{\tilde{\psi}} (x_t, \tilde{x}_t^\psi) \\
&\leq B_{\tilde{\psi}} (z_t, X_t^\psi) - B_{\tilde{\psi}} (z_t, X_{t+1}^\psi) - B_{\tilde{\psi}} (X_{t+1}, X_{t+1}^\psi) + \frac{1}{\eta_t} B_{\tilde{\psi}} (X_{t+1}, \tilde{x}_t^\psi) - \frac{1}{\eta_t} B_{\tilde{\psi}} (x_t, \tilde{x}_t^\psi) \\
&\quad + \frac{1}{\eta_t} \phi^* (\eta_t, \eta_t \left\| x_t^\psi - \tilde{x}_t^\psi \right\|), \quad (3)
\end{align*}
\]

where \( X_{t+1} \in \partial \tilde{\phi}^* (X_{t+1}^\psi) \), the last “=” uses the generalized cosine rule (Lemma 4), and “\( \leq \)” uses the \( \phi \)-convexity of \( \tilde{\phi} \) and the following inequality,

\[
\langle \eta_t \theta_t x_t^\psi - \eta_t \theta_t \tilde{x}_t^\psi, x_t - X_{t+1} \rangle \leq \eta_t \theta_t \left\| x_t^\psi - \tilde{x}_t^\psi \right\| \left\| x_t - X_{t+1} \right\| \\
\leq \phi \left( \left\| x_t - X_{t+1} \right\| \right) + \phi^* (\eta_t, \theta_t \left\| x_t^\psi - \tilde{x}_t^\psi \right\|).
\]

If \( \tilde{\psi} = \frac{1}{\eta_t} (\psi - a^\psi) \), then \( \tilde{\phi}^* (y_t) = \frac{1}{\eta_t} \psi^* (a^\psi + \eta_t y_t) \), according to \( X_t^\psi = \frac{1}{\eta_t} (\tilde{x}_t^\psi - a^\psi) \) and \( X_{t+1} = \frac{1}{\eta_{t+1}} (\tilde{x}_{t+1}^\psi - a^\psi) \), we have

\[
\begin{align*}
B_{\tilde{\psi}} (z_t, X_t^\psi) &= \frac{1}{\eta_t} B_{\tilde{\psi}} (z_t, \tilde{x}_t) \\
B_{\tilde{\psi}} (z_t, X_{t+1}^\psi) &= \frac{1}{\eta_t} B_{\tilde{\psi}} (z_t, a^\psi + \frac{\eta_t}{\eta_{t+1}} (\tilde{x}_{t+1}^\psi - a^\psi)) \\
B_{\tilde{\psi}} (X_{t+1}, X_{t+1}^\psi) &= \frac{1}{\eta_t} B_{\tilde{\psi}} (X_{t+1}, \tilde{x}_t^\psi) \quad (4)
\end{align*}
\]

To complete the proof, it suffices to combine Equations (2) to (4). \( \square \)

**Remark 13.** The proof of Theorem 12 provides a general analysis method for dynamic regret. The decomposition of the instantaneous dynamic regret (the first line of Equation 3) follows from Appendix A.1 of Zhao et al. (2020). In fact, Zhao et al. (2020) deduced the dynamic regret for online extra-gradient descent, which is a special case of S-II. Later in this section (Proposition 18), we show the dynamic regret for S-II.
Next, we analyze each term of the dynamic regret upper bound for $S$-I. In order to reduce the item
\[
\sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \left[ B_{\psi} \left( z_t, \bar{x}_t^\psi \right) - B_{\psi} \left( z_t, a^\psi + \frac{\eta_t}{\eta_{t+1}} \left( \bar{x}_{t+1}^\psi - a^\psi \right) \right) \right],
\]
one feasible way is to set $\theta_t \equiv \theta$ and $z_t \equiv z$, the other is to set $\eta_t \equiv \eta$ and $z_t \equiv z$. Indeed, we have the following lemma.

**Lemma 14.** If $\eta_t \geq \eta_{t+1}$, then
\[
\sum_{t=1}^{T} \frac{1}{\eta_t} \left[ B_{\psi} \left( z_t, \bar{x}_t^\psi \right) - B_{\psi} \left( z_t, a^\psi + \frac{\eta_t}{\eta_{t+1}} \left( \bar{x}_{t+1}^\psi - a^\psi \right) \right) \right] \leq \frac{1}{\eta_{T+1}} B_{\psi} \left( z, a^\psi \right).
\]
If $\theta_{t-1} \leq \theta_t$, then
\[
\sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( z_t, \bar{x}_t^\psi \right) - B_{\psi} \left( z_t, \bar{x}_{t+1}^\psi \right) \right] \leq \frac{1}{\theta_1} B_{\psi} \left( z, a^\psi \right) + \sum_{t=2}^{T} \frac{1}{\theta_t} \| \partial \psi (z_t) - \bar{x}_t^\psi \| \| z_t - z_{t-1} \|.
\]

Combining Theorem 12 and Lemma 14, we directly obtain the static regret for $S$-I.

**Corollary 15 (Static Regret for $S$-I).** If $\partial \psi \left( C \right) = E^*$ or $C$ is compact, then $\forall a \in C$, under the assumption of $\eta_t \geq \eta_{t+1}$, $S$-I $\left| \eta \equiv 1 \right|$ enjoys the following static regret upper bound,
\[
\text{regret} \left( z, z, \cdots, z \right) \leq \frac{1}{\eta_{T+1}} B_{\psi} \left( z, a^\psi \right) + \sum_{t=1}^{T} \frac{1}{\eta_t} \left[ B_{\psi} \left( X_{t+1}, \bar{x}_t^\psi \right) - B_{\psi} \left( X_{t+1}, \bar{x}_{t+1}^\psi \right) \right] + \sum_{t=1}^{T} \frac{1}{\eta_t} \phi^* \left( \eta_t \| x_t^* - \bar{x}_t^\psi \| \right) - \sum_{t=1}^{T} \frac{1}{\eta_t} B_{\psi} \left( x_t, \bar{x}_t^\psi \right), \quad \forall z \in C,
\]
where $\left( X_{t+1}, \frac{1}{\eta_{t+1}} \left( \bar{x}_{t+1}^\psi - a^\psi \right) \right) \in \text{gra} \partial \frac{1}{\eta_t} \left( \psi - a^\psi \right)$, and under the assumption of $\theta_{t-1} \leq \theta_t$, $S$-I $\left| \eta \equiv 1 \right|$ enjoys the following static regret upper bound,
\[
\text{regret} \left( z, z, \cdots, z \right) \leq \frac{1}{\theta_1} B_{\psi} \left( z, a^\psi \right) + \sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( X_{t+1}, \bar{x}_t^\psi \right) - B_{\psi} \left( X_{t+1}, \bar{x}_{t+1}^\psi \right) \right] + \sum_{t=1}^{T} \frac{1}{\theta_t} \phi^* \left( \theta_t \| x_t^* - \bar{x}_t^\psi \| \right) - \sum_{t=1}^{T} \frac{1}{\theta_t} B_{\psi} \left( x_t, \bar{x}_t^\psi \right), \quad \forall z \in C,
\]
where $\left( X_{t+1}, \bar{x}_{t+1}^\psi - a^\psi \right) \in \text{gra} \partial \left( \psi - a^\psi \right)$.

There are two ways to drop the term $B_{\psi} \left( X_{t+1}, \bar{x}_t^\psi \right) - B_{\psi} \left( X_{t+1}, \bar{x}_{t+1}^\psi \right)$. One is to set $\bar{x}_t^\psi = \bar{x}_t^\psi$, which forces $S$-I back to $S$. The regret for $S$ is formalized into the following two corollaries. The other is to set $\psi$ the squared norm on Hilbert space. See Section 6.2 for analysis details.

**Corollary 16 (Dynamic Regret for $S$).** If $\partial \psi \left( C \right) = E^*$ or $C$ is compact, then $\forall a \in C$, $S$ enjoys the following dynamic regret upper bound,
\[
\text{regret} \left( z_1, z_2, \cdots, z_T \right) \leq \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \left[ B_{\psi} \left( z_t, \bar{x}_t^\psi \right) - B_{\psi} \left( z_t, a^\psi + \frac{\eta_t}{\eta_{t+1}} \left( \bar{x}_{t+1}^\psi - a^\psi \right) \right) \right] + \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} \phi^* \left( \eta_t \theta_t \| x_t^* - \bar{x}_t^\psi \| \right) - \sum_{t=1}^{T} \frac{1}{\eta_t \theta_t} B_{\psi} \left( x_t, \bar{x}_t^\psi \right), \quad \forall z_t \in C.
\]
Corollary 17 (Static Regret for $S$). If $\partial \psi (C) = E^*$ or $C$ is compact, then $\forall a \in C$, under the assumption of $\eta_t \geq \eta_{t+1}$, $S|_{\eta_t=1}$ (which is Optimistic-DA) enjoys the following static regret upper bound,

$$\text{regret} (z, z \cdots, z) \leq \frac{1}{\eta_{t+1}} B_\psi (z, a^\psi) + \sum_{t=1}^{T} \frac{1}{\eta_t} \phi^* (\eta_t \|x_t^* - \tilde{x}_t^*\|) - \sum_{t=1}^{T} \frac{1}{\eta_t} B_\psi (x_t, \tilde{x}_t) , \ \forall z \in C,$$

and under the assumption of $\theta_{t-1} \leq \theta_t$, $S|_{\eta_t=1}$ enjoys the following static regret upper bound,

$$\text{regret} (z, z \cdots, z) \leq \frac{1}{\theta_1} B_\psi (z, a^\psi) + \sum_{t=1}^{T} \frac{1}{\theta_t} \phi^* (\theta_t \|x_t^* - \tilde{x}_t^*\|) - \sum_{t=1}^{T} \frac{1}{\theta_t} B_\psi (x_t, \tilde{x}_t) , \ \forall z \in C.$$

Note that the term $\phi^* (\eta_t \|x_t^* - \tilde{x}_t^*\|)$ contains the function $\phi$. Compared with the strongly-convex, $\phi$-convex allows the regret upper bounds to be more finely controlled. For example, the upper bounds of using $Q_\rho$-convex are tighter than that of using 1-strongly-convex, since $Q_\rho^*$ has an extra subtraction term, where $\rho = \sup_{x,y \in C} \|x - y\|$. For the term $B_\psi (x_t, \tilde{x}_t^*)$, one can relax regret bounds via the $\phi$-convexity of $\psi$, that is,

$$B_\psi (x_t, \tilde{x}_t^*) \geq \phi (\|x_t - \tilde{x}_t\|) .$$

Now we show the dynamic regret for $S$-II, which is the general form of Appendix A.1 of Zhao et al. (2020).

Proposition 18 (Dynamic Regret for $S$-II). If $\partial \psi (C) = E^*$ or $C$ is compact, then $\forall a \in C$, $S$-II (which is Optimistic-MD) enjoys the following dynamic regret upper bound,

$$\text{regret} (z_1, z_2, \cdots, z_T) \leq \sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_\psi (z_t, \tilde{x}_t) - B_\psi (z_t, \tilde{x}_t^*) \right] + \sum_{t=1}^{T} \frac{1}{\theta_t} \phi^* (\theta_t \|x_t^* - \tilde{x}_t^*\|) - \sum_{t=1}^{T} \frac{1}{\theta_t} B_\psi (x_t, \tilde{x}_t) , \ \forall z_t \in C.$$

Proof. This proof is similar to the proof of Theorem 12. Note that $S$-II can be rearranged as follows,

$$x_t^* + \theta_t \tilde{x}_t^* = \tilde{x}_t^* , \ \tilde{x}_t^* = \tilde{x}_{t-1}^* - \theta_{t-1} x_{t-1}^* ,$$

and

$$\varphi_t (x_t) - \varphi_t (z_t) \leq \frac{1}{\theta_t} \left( \theta_t x_t^* - x_t - z_t \right) , \ \ x_t^* \in \partial \varphi_t (x_t) , \ \ \ \ \ \ (5)$$

where

$$\left\langle \theta_t x_t^* - x_t - z_t \right\rangle = \left\langle \theta_t x_t^* - \tilde{x}_{t+1}^* - z_t \right\rangle + \left\langle \theta_t \tilde{x}_t^* - x_t - \tilde{x}_{t+1}^* \right\rangle + \left\langle \theta_t x_t^* - \theta_t \tilde{x}_t^* - x_t - \tilde{x}_{t+1}^* \right\rangle$$

$$= - \left( \tilde{x}_t^* - \tilde{x}_{t+1}^* ; z_t - \tilde{x}_{t+1}^* \right) - \left( \tilde{x}_t^* - \tilde{x}_{t+1}^* ; x_t - \tilde{x}_{t+1}^* \right) - \left( \tilde{x}_t^* - \tilde{x}_{t+1}^* ; x_t - \tilde{x}_{t+1}^* \right)$$

$$\leq B_\psi (z_t, \tilde{x}_t^*) - B_\psi (z_t, \tilde{x}_{t+1}^*) - B_\psi (x_t, \tilde{x}_t^*) - B_\psi (\tilde{x}_{t+1}, \tilde{x}_t^*) + \phi^* (\theta_t \|x_t^* - \tilde{x}_t^*\|) . \ \ \ \ \ \ (6)$$

To complete the proof, it suffices to combine Equations (5) and (6). \qed
Note that

\[
\sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( \hat{z}_t, \tilde{x}_t^\psi \right) - B_{\psi} \left( \hat{z}_t, \hat{x}_t^\psi \right) \right] = \sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( \hat{z}_t, \tilde{x}_t^\psi \right) - B_{\psi} \left( \hat{z}_t, \tilde{x}_{t+1}^\psi \right) \right] \\
+ \sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( \hat{z}_t, \tilde{x}_{t+1}^\psi \right) - B_{\psi} \left( \hat{z}_t, \hat{x}_{t+1}^\psi \right) \right].
\]

This decomposition makes the bound of Proposition 18 extremely similar to that of Theorem 12. The method of reducing the term \( \sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_{\psi} \left( \hat{z}_t, \tilde{x}_t^\psi \right) - B_{\psi} \left( \hat{z}_t, \tilde{x}_{t+1}^\psi \right) \right] \) is covered by Lemma 14.

For the term \( B_{\psi} \left( \hat{z}_t, \tilde{x}_{t+1}^\psi \right) - B_{\psi} \left( \hat{z}_t, \hat{x}_{t+1}^\psi \right) \), there are two feasible ways to drop it. One is to set \( \tilde{x}_{t+1}^\psi = \hat{x}_{t+1}^\psi \), which forces \( S\text{-II} \) back to \( S_{\|y_t\|_1} \). The other is to set \( \psi \) be the squared norm on Hilbert space. See Section 6.2 for analysis details.

Next, we investigate the effect of introducing auxiliary strategies in addition to primary strategies on regret bounds. This is a general extension of Appendix B of Flaspohler et al. (2021).

Note that \( \{x_t\}_{t\geq 1} \) is determined by the estimated sequence \( \{\hat{x}_t^\psi\}_{t\geq 1} \), which is arbitrary. Without changing the primary strategy sequence \( \{x_t\}_{t\geq 1} \), we introduce an auxiliary strategy sequence \( \{y_t\}_{t\geq 1} \), which is determined by \( \{\hat{y}_t^\psi\}_{t\geq 1} \). The instantaneous dynamic regret can be decomposed as follows,

\[
\varphi_t (x_t) - \varphi_t (z_t) = \left( x_t^\psi, x_t - z_t \right) + \left( \hat{x}_t^\psi, y_t - z_t \right), \quad x_t^\psi \in \partial \varphi_t (x_t).
\]

The regret bound for the auxiliary term is simply replacing \( x_t, \tilde{x}_t^\psi \) and \( \hat{x}_t^\psi \) with \( y_t, \hat{y}_t^\psi \) and \( \tilde{y}_t^\psi \) respectively. Therefore, it suffices to obtain the upper bound for the drift term. Indeed, we have the following proposition.

**Proposition 19.** If an auxiliary strategy sequence \( \{y_t\}_{t\geq 1} \) determined by \( \{\hat{y}_t^\psi\}_{t\geq 1} \) is introduced in addition to the primary strategy sequence \( \{x_t\}_{t\geq 1} \), then the term \( \phi^* (\xi \|x_t^\psi - \hat{x}_t^\psi\|) \) in regret upper bounds is replaced by \( \Phi_{\xi} (x_t^\psi, \hat{x}_t^\psi) \), and \( x_t, \tilde{x}_t^\psi \) and \( \hat{x}_t^\psi \) in remaining terms are replaced by \( y_t, \hat{y}_t^\psi \) and \( \tilde{y}_t^\psi \) respectively, where

\[
\Phi_{\xi} (x^\psi, \hat{x}^\psi) = \phi^* (\xi \|x^\psi - \hat{x}^\psi\|) + \inf_{y > 0} \left( \frac{1}{\gamma} \phi^* (\gamma \xi \|x^\psi\|) + \frac{1}{\gamma} \phi^* (\xi \|\hat{x}^\psi - \hat{y}^\psi\|) \right).
\]

If \( \phi = Q_\rho \), and \( \hat{y}^\psi = \lambda \hat{x}^\psi + (1 - \lambda) x^* \), where \( \lambda = \min \left\{ \frac{\|x^*\|}{\|x^* - \hat{x}^\psi\|}, 1 \right\} \), then

\[
\Phi_{\xi} (x^\psi, \hat{x}^\psi) = Q_\rho^* (\xi \min \{\|x^\psi - \hat{x}^\psi\|, \|x^\psi\|\} + \xi \|x^\psi\| \min \{\xi (\|x^\psi - \hat{x}^\psi\| - \|x^\psi\|)_+, \rho\}) \\
\leq \xi^2 Q_\rho^* (\|x^\psi - \hat{x}^\psi\|) - \frac{1}{2} \xi \min \{\|x^\psi - \hat{x}^\psi\|, \|x^\psi\|\} - \rho_2^2.
\]

**Remark 20.** \( Q_\rho \)-convex tightens the regret upper bound more finely. The Huber penalty in Flaspohler et al. (2021) corresponds to the \( Q_\infty \)-convex case, that is, the last subtraction term in the second line of Equation (7) vanishes.

### 6 Instantiations

All the above analyses are based on the abstract description of the \( \phi \)-convexity of \( \psi \). In this section, we instantiate \( \psi \) in two forms, the negative entropy and the squared norm.
6.1 \( \psi \) is Negative Entropy

In this part, we set \( E = (\mathbb{R}^{n+1}, \|\cdot\|_1) \), and then \( E^* = (\mathbb{R}^{n+1}, \|\cdot\|_\infty) \).

**Lemma 21.** Set \( \psi \) be the negative entropy, that is,

\[
\psi (w) = \langle w, \ln w \rangle + \chi_{\mathbb{R}^n} (w),
\]

where the probability simplex \( \Delta^n := \{ w \in E \mid w \geq 0, \|w\|_1 = 1 \} \) is a compact subset of \( E \). Then \( \psi \) is \( Q_2 \)-convex, and the strategy \( S \) can be instantiated as the following Optimistic Normalized Exponentiated Subgradient (ONES),

\[
w_t = \mathcal{N} \left( a \circ e^{-\theta_t \sum_{i=1}^{t-1} \ell_i} \right),
\]

where \( \mathcal{N} \) is the normalization operator, \( \circ \) denotes the Hadamard product, \( \ell_i \in E^* \) is the loss vector, \( \ell_i \) is the estimated vector corresponding to \( \ell_t \).

ONES\( |_{\ell_t=1} \) can be rearranged as

\[
\tilde{w}_t = \mathcal{N} \left( a \circ e^{-\theta_t \sum_{i=1}^{t-1} \ell_i} \right), \quad a \in \Delta^n,
\]

\[
w_t = \mathcal{N} \left( \tilde{w}_t \circ e^{-\theta_t \ell_t} \right),
\]

and its function form is \( (w_t, \tilde{w}_t) = \text{ONES} \left( \ell_t, \ell_{t-1}; \eta_t, 1 \right) \). ONES\( |_{\eta_t=1} \) can be rearranged as

\[
\tilde{w}_t = \mathcal{N} \left( \tilde{w}_{t-1} \circ e^{-\theta_t \ell_{t-1}} \right), \quad a \in \Delta^n,
\]

\[
w_t = \mathcal{N} \left( \tilde{w}_t \circ e^{-\theta_t \ell_t} \right),
\]

and its corresponding function form is \( (w_t, \tilde{w}_t) = \text{ONES} \left( \ell_t, \ell_{t-1}; 1, \theta_t \right) \).

**Remark 22.** It is worth mentioning that ONES has two parameters, and ONES\( |_{\eta_t=1} \) has a plethora of different names, such as Optimistic Hedge, Optimistic Exponential Weights, etc.

The static regret for ONES is formalized into the following corollary.

**Corollary 23 (Static Regret for ONES).** Under the assumption of \( \eta_t \geq \eta_t+1 \), ONES\( |_{\theta_t=1} \) enjoys the following static regret upper bound,

\[
\text{regret} (u, u, \cdots, u) \leq \frac{1}{\eta_{T+1}} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \frac{1}{\eta_t} Q_2^* \left( \eta_t, \|\ell_t - \tilde{\ell}_t\|_{\infty} \right) - \sum_{t=1}^{T} \frac{1}{\eta_t} \left( w_t, \ln \frac{w_t}{\tilde{w}_t} \right), \quad (8)
\]

or the corresponding version with auxiliary strategies,

\[
\text{regret} (u, u, \cdots, u) \leq \frac{1}{\eta_{T+1}} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \frac{1}{\eta_t} \Phi_{\eta_t} \left( \ell_t, \tilde{\ell}_t \right) - \sum_{t=1}^{T} \frac{1}{\eta_t} \left( v_t, \ln \frac{v_t}{\tilde{v}_t} \right), \quad (9)
\]

where \( \forall u \in \Delta^n, (v_t, \tilde{v}_t) = \text{ONES} \left( \ell_t, \ell_{t-1}; \eta_t, 1 \right), \tilde{\ell}_t = \lambda \tilde{\ell}_t + (1 - \lambda) \ell_t, \lambda = \min \left\{ \frac{\|\ell_t\|_\infty}{\|\ell_t - \tilde{\ell}_t\|_{\infty}}, 1 \right\} \), and \( \Phi \) follows from Equation (7) with \( \rho = 2 \).
Under the assumption of \( \theta_{t-1} \leq \theta_t \), ONES\( \big|_{\eta_t=1} \) enjoys the following static regret upper bound,

\[
\text{regret} (u, u, \ldots, u) \leq \frac{1}{\theta_1} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \frac{1}{\theta_t} Q^*_2 \left( \theta_t, \|\ell_t - \tilde{\ell}_t\|_\infty \right) - \sum_{t=1}^{T} \frac{1}{\theta_t} \left( w_t, \ln \frac{w_t}{\tilde{w}_t} \right),
\] (10)
or the corresponding version with auxiliary strategies,

\[
\text{regret} (u, u, \ldots, u) \leq \frac{1}{\theta_1} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \frac{1}{\theta_t} \Phi_{\theta_t} \left( \ell_t, \tilde{\ell}_t \right) - \sum_{t=1}^{T} \frac{1}{\theta_t} \left( v_t, \ln \frac{v_t}{\tilde{v}_t} \right),
\] (11)

where \( \forall u \in \Delta^n \), \((v_t, \tilde{v}_t) = \text{ONES} \left( \tilde{\ell}_t, \ell_{t-1}; 1, \theta_t \right) \). \( \tilde{\ell}_t = \lambda \tilde{\ell}_t + (1 - \lambda) \ell_t, \lambda = \min \left\{ \frac{\|\ell_t\|}{\|\ell_t - \tilde{\ell}_t\|_\infty}, 1 \right\} \), and \( \Phi \) follows from Equation (7) with \( \rho = 2 \).

**Remark 24.** According to Lemma 43,

\[
\left\langle w_t, \ln \frac{w_t}{\tilde{w}_t} \right\rangle \geq \frac{1}{2} \|w_t - \tilde{w}_t\|^2_t,
\]

then Equation (8) (as well as Equation 10) can be relaxed as

\[
\frac{1}{\eta_{t+1}} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \frac{\eta_t}{2} \|\ell_t - \tilde{\ell}_t\|^2_\infty - \sum_{t=1}^{T} \frac{1}{2\eta_t} \left( \eta_t \|\ell_t - \tilde{\ell}_t\|_\infty - 2 \right)^2 + \sum_{t=1}^{T} \frac{1}{\eta_t} \|w_t - \tilde{w}_t\|^2_t.
\]

Without the first subtraction term, this bound is a refined version of Theorem 19 of Syrgkanis et al. (2015), without all the subtraction terms, this bound is directly implied by Theorem 7.28 of Orabona (2019).

By dropping the last term \( \left\langle v_t, \ln \frac{v_t}{\tilde{v}_t} \right\rangle \), Equation (9) (as well as Equation 11) can be relaxed as

\[
\frac{1}{\eta_{t+1}} \left( u, \ln \frac{u}{a} \right) + \sum_{t=1}^{T} \eta_t Q^*_\ell \left( \|\ell_t - \tilde{\ell}_t\|_\infty \right) - \frac{1}{2} \sum_{t=1}^{T} \left( \eta_t \min \{ \|\ell_t - \tilde{\ell}_t\|_\infty, \|\ell_t\|_\infty \} - 2 \right)^2 + \sum_{t=1}^{T} \frac{1}{\eta_t} \|w_t - \tilde{w}_t\|^2_t.
\]

Without the last subtraction term, this bound is directly implied by Theorem 3 of Flasopoulos et al. (2021).

### 6.2 \( \psi \) is Squared Norm

In this part, we set \( E \) be a Hilbert space over \( \mathbb{R} \).

**Lemma 25.** Set \( \psi \) be the squared norm on \( C \), that is,

\[
\psi (x) = \frac{1}{2} \|x\|^2 + \chi_C (x),
\]

and let \( \rho = \sup_{x, y \in C} \|x - y\| \). Then \( \psi \) is \( Q_\rho \)-convex, \( \partial \psi (C) = E \), and S-1 can be instantiated as the following Optimistic Lazy Projection (OLP).

\[
\tilde{x}_t = PC \left( a - \eta_t \sum_{i=1}^{t-1} \theta_i x_i^t \right), \quad x_t = PC \left( \tilde{x}_t - \eta_t \theta_i \tilde{x}_i \right),
\]
S-II can be instantiated as the following Optimistic Greedy Projection (OGP),

\[
\begin{align*}
\hat{x}_t &= P_C (\hat{x}_{t-1} - \theta_{t-1} x^*_t), \\
x_t &= P_C (\hat{x}_t - \theta_t \hat{x}_t),
\end{align*}
\]

where \( P_C \) represents the projection onto the closed convex subset \( C \).

The function form of OLP can be written as \( (x_t, \hat{x}_t) = \text{OLP} (\hat{x}_t, x^*_{t-1}; \eta_t, \theta_t) \), and the function form of OGP can be written as \( (x_t, \hat{x}_t) = \text{OGP} (\hat{x}_t, x^*_{t-1}; \theta_t) \).

**Remark 26.** It is worth mentioning that the lazy projection here has two parameters, while the usual lazy projection has only one parameter. McMahan (2017) proved that the lazy projection algorithm is just DA and the greedy one is just MD. Here we proved that the two-parameter strategy OLP is just S-I. It should be emphasized that all projection algorithms must be in Hilbert space, and both S-I and S-II run in normed vector space. The instantiations require the extra constraint that \( \partial \varphi \) is the identity map.

**Corollary 27** (Dynamic Regret for OLP). Under the assumption of \( \theta_{t-1} \leq \theta_t \), \( \text{OLP}_{\eta_t \equiv 1} \) enjoys the following dynamic regret upper bound,

\[
\text{regret} (z_1, z_2, \ldots, z_T) \leq \frac{1}{2 \theta_1} \| z_1 - a \|^2 + \sum_{t=2}^{T} \frac{1}{\theta_t} \| z_t - a + \sum_{i=1}^{t-1} \theta_i x^*_i \| \| z_t - z_{t-1} \| \\
+ \sum_{t=1}^{T} \frac{1}{\theta_t} Q^*_p (\theta_t, \| x^*_t - \hat{x}_t \|) - \sum_{t=1}^{T} \frac{1}{2 \theta_t} \| x_t - \hat{x}_t \|^2, \quad \forall z_t \in C,
\]

or the corresponding version with auxiliary strategies,

\[
\text{regret} (z_1, z_2, \ldots, z_T) \leq \frac{1}{2 \theta_1} \| z_1 - a \|^2 + \sum_{t=2}^{T} \frac{1}{\theta_t} \| z_t - a + \sum_{i=1}^{t-1} \theta_i x^*_i \| \| z_t - z_{t-1} \| \\
+ \sum_{t=1}^{T} \frac{1}{\theta_t} \Phi_{\theta_t} (x^*_t, \hat{x}_t) - \sum_{t=1}^{T} \frac{1}{2 \theta_t} \| y_t - \bar{y}_t \|^2, \quad \forall z_t \in C,
\]

where \( (y_t, \bar{y}_t) = \text{OLP} (\bar{y}_t, x^*_{t-1}; 1, \theta_t), \bar{y}_t = \lambda \hat{x}_t + (1 - \lambda) x^*_t, \lambda = \min \left\{ \frac{\| x^*_t \|}{\| x^*_t - \hat{x}_t \|}, 1 \right\} \), and \( \Phi \) follows from Equation (7).

**Corollary 28** (Static Regret for OLP). Under the assumption of \( \eta_t \geq \eta_{t+1} \), \( \text{OLP}_{\theta_t \equiv 1} \) enjoys the following static regret upper bound,

\[
\text{regret} (z, z, \ldots, z) \leq \frac{1}{2 \eta_{T+1}} \| z - a \|^2 + \sum_{t=1}^{T} \frac{1}{\eta_t} Q^*_p (\eta_t, \| x^*_t - \hat{x}_t \|) - \sum_{t=1}^{T} \frac{1}{2 \eta_t} \| x_t - \hat{x}_t \|^2,
\]

or the corresponding version with auxiliary strategies,

\[
\text{regret} (z, z, \ldots, z) \leq \frac{1}{2 \eta_{T+1}} \| z - a \|^2 + \sum_{t=1}^{T} \frac{1}{\eta_t} \Phi_{\eta_t} (x^*_t, \hat{x}_t) - \sum_{t=1}^{T} \frac{1}{2 \eta_t} \| y_t - \bar{y}_t \|^2,
\]

where \( \forall z \in C, (y_t, \bar{y}_t) = \text{OLP} (\bar{y}_t, x^*_{t-1}; \eta_t, 1), \bar{y}_t = \lambda \hat{x}_t + (1 - \lambda) x^*_t, \lambda = \min \left\{ \frac{\| x^*_t \|}{\| x^*_t - \hat{x}_t \|}, 1 \right\} \), and \( \Phi \) follows from Equation (7).
Under the assumption of $\theta_{t-1} \leq \theta_t$, OLP$_{\theta_t \equiv 1}$ enjoys the following static regret upper bound,

$$\text{regret}(z, z, \cdots, z) \leq \frac{1}{2\theta_1} \|z - a\|^2 + \sum_{t=1}^{T} \frac{1}{\theta_t} Q^{\ast}_{\rho}(\theta_t, \|x^{\ast}_t - \bar{x}_t\|) - \frac{1}{2\theta_t} \|x_t - \bar{x}_t\|^2,$$

or the corresponding version with auxiliary strategies,

$$\text{regret}(z, z, \cdots, z) \leq \frac{1}{2\theta_1} \|z - a\|^2 + \sum_{t=1}^{T} \frac{1}{\theta_t} \Phi_{\theta_t}(x^{\ast}_t, \bar{x}_t) - \frac{1}{2\theta_t} \|y_t - \bar{y}_t\|^2,$$

where $\forall z \in C, (y_t, \bar{y}_t) = \text{OLP}(\tilde{y}_t, x^{\ast}_{t-1}; 1, \theta_t)$, $\tilde{y}_t = \lambda \bar{x}_t + (1 - \lambda) x^{\ast}_t$, $\lambda = \min \left\{ \frac{\|x^{\ast}_t\|}{\|x^{\ast}_t - \bar{x}_t\|}, 1 \right\}$, and $\Phi$ follows from Equation (7).

**Corollary 29 (Dynamic Regret for OGP).** Under the assumption of $\theta_{t-1} \leq \theta_t$, OGP enjoys the following dynamic regret upper bound,

$$\text{regret}(z_1, z_2, \cdots, z_T) \leq \frac{1}{2\theta_1} \|z_1 - a\|^2 + \sum_{t=2}^{T} \frac{\rho}{\theta_t} \|z_t - z_{t-1}\|^2$$

$$+ \sum_{t=1}^{T} \frac{1}{\theta_t} Q^{\ast}_{\rho}(\theta_t, \|x^{\ast}_t - \bar{x}_t\|) - \frac{1}{2\theta_t} \|x_t - \bar{x}_t\|^2, \quad \forall z_t \in C,$$

or the corresponding version with auxiliary strategies,

$$\text{regret}(z_1, z_2, \cdots, z_T) \leq \frac{1}{2\theta_1} \|z_1 - a\|^2 + \sum_{t=2}^{T} \frac{\rho}{\theta_t} \|z_t - z_{t-1}\|^2$$

$$+ \sum_{t=1}^{T} \frac{1}{\theta_t} \Phi_{\theta_t}(x^{\ast}_t, \bar{x}_t) - \frac{1}{2\theta_t} \|y_t - \bar{y}_t\|^2, \quad \forall z_t \in C,$$

where $(y_t, \bar{y}_t) = \text{OGP}(\tilde{y}_t, x^{\ast}_{t-1}; 1, \theta_t)$, $\tilde{y}_t = \lambda \bar{x}_t + (1 - \lambda) x^{\ast}_t$, $\lambda = \min \left\{ \frac{\|x^{\ast}_t\|}{\|x^{\ast}_t - \bar{x}_t\|}, 1 \right\}$, and $\Phi$ follows from Equation (7).

**Corollary 30 (Static Regret for OGP).** Under the assumption of $\theta_{t-1} \leq \theta_t$, OGP enjoys the following static regret upper bound,

$$\text{regret}(z, z, \cdots, z) \leq \frac{1}{2\theta_1} \|z - a\|^2 + \sum_{t=1}^{T} \frac{1}{\theta_t} Q^{\ast}_{\rho}(\theta_t, \|x^{\ast}_t - \bar{x}_t\|) - \frac{1}{2\theta_t} \|x_t - \bar{x}_t\|^2,$$

or the corresponding version with auxiliary strategies,

$$\text{regret}(z, z, \cdots, z) \leq \frac{1}{2\theta_1} \|z - a\|^2 + \sum_{t=1}^{T} \frac{1}{\theta_t} \Phi_{\theta_t}(x^{\ast}_t, \bar{x}_t) - \frac{1}{2\theta_t} \|y_t - \bar{y}_t\|^2,$$

where $\forall z \in C, (y_t, \bar{y}_t) = \text{OGP}(\tilde{y}_t, x^{\ast}_{t-1}; 1, \theta_t)$, $\tilde{y}_t = \lambda \bar{x}_t + (1 - \lambda) x^{\ast}_t$, $\lambda = \min \left\{ \frac{\|x^{\ast}_t\|}{\|x^{\ast}_t - \bar{x}_t\|}, 1 \right\}$, and $\Phi$ follows from Equation (7).

Appendix K provides the proof of Corollary 27. Corollary 29 can be proved in the same way. Corollaries 28 and 30 can be obtained by setting $z_t \equiv z$ directly.

**Remark 31.** The regret bounds of the above four corollaries are better than the currently known optimal results. This is due to the fact that the last term of all upper bounds is the extra subtraction term and the more refined choice of $Q^{\ast}_{\rho}$-convexity.
7 Online Monotone Optimization

In this section, we argue monotonicity rather than convexity is the natural boundary for $S$-I and $S$-II. By allowing the absence of loss functions, we extend the scope of application of all update rules to online monotone optimization. Although Gemp and Mahadevan (2016) proposed the idea of online monotone optimization, we emphasize that the concept of online monotone optimization needs to be rigorously reformulated. See Romano et al. (1993) for the potential theory of monotone multivalued operators. Part of the relevant basis is compiled in Appendix L.

Let $E$ be a normed vector space over $\mathbb{R}$ and let $C \neq \emptyset$ be a closed convex subset of $E$. The online monotone optimization problem can be formalized as follows. At round $t$,

the player chooses $x_t \in C$ according to some algorithm,

the adversary (environment) feeds back a monotone operator $M_t : C \rightrightarrows E^*$,

where “\rightrightarrows” emphasizes that the map is multivalued. We choose the following generalized dynamic regret as the performance metric.

**Definition 32 (regret).** The instantaneous generalized dynamic regret of strategy $x_t$, relative to the reference strategy $z_t$, in round $t$ is defined as

$$\text{regret}^n(z_t) := \inf \int_{\gamma_n(z_t, x_t)} \langle M_t \alpha, \, d\alpha \rangle, \quad \forall n \in \mathbb{N},$$

where the infimum operator traverses all possible decompositions of $\gamma_n(z_t, x_t) = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \cdots + \frac{1}{\alpha_n}$, $\forall \alpha_i \in C$, $\alpha_i$ represents the map $\lambda \mapsto \alpha + \lambda (x - \alpha)$, $\lambda \in [0, 1]$, and if $n \in \mathbb{N}_*$, $\exists \mu \in \mathbb{R}$ such that $\forall x, y \in C$, $\langle M_t x, y - x \rangle \geq \mu$.

**Remark 33.** The online convex optimization problem is manifested as alternate actions of multivalued maps $\partial \varphi : C \rightrightarrows E^*$ and $\partial \psi^* : E^* \rightrightarrows C$. Correspondingly, the multivalued maps for online monotone optimization are $M_t : C \rightrightarrows E^*$ and $\partial \psi^* : E^* \rightrightarrows C$. Even if $\partial \psi^*$ is replaced by a monotone operator, we can always choose a conservative monotone operator such that its potential is $\psi^*$. In brief, online monotone optimization is to replace $\partial \varphi$ in online convex optimization with a monotone operator $M_t$. The generalized dynamic regret is a series of performance metrics determined by $n \in \mathbb{N}$, where $n$ represents the maximum meta-algorithm nesting level that the online algorithm can accommodate, and obviously, regret$^n z_t \leq$ regret$^{n-1} z_t$, where $z_t$ represents the reference strategy in round $t$.

Next, we analyze the rationality of online monotone optimization.

Firstly, Definition 32 is well-defined since for any fixed $n \in \mathbb{N}_*$,

$$\int_{\gamma_n(z_t, x_t)} \langle M_t \alpha, \, d\alpha \rangle = \left( \int_{\frac{1}{z_t \alpha_1}} \right. + \left. \int_{\frac{1}{\alpha_1 \alpha_2}} + \cdots + \int_{\frac{1}{\alpha_n x_t}} \right) \langle M_t \alpha, \, d\alpha \rangle \geq (n + 1) \mu.$$

Secondly, the definition of generalized dynamic regret extends the original definition of dynamic regret.

On one hand, if the feedback is a loss function (we treat the restricted loss function $\varphi_t + \chi_C$ as the loss function, and for simplicity, we still denote it as $\varphi_t$), then according to Lemma 55, the potential of $M_t = \partial \varphi$ can be chosen as $\varphi_t$, and

$$\varphi_t(x_t) - \varphi_t(z_t) = \int_{\gamma(z_t, x_t)} \langle \partial \varphi_t(\alpha), \, d\alpha \rangle = \text{regret}^n z_t,$$

where $\gamma(z_t, x_t)$ denotes an arbitrary finite-length oriented polyline in $C$ from $z_t$ to $x_t$, and the integral is path-independent since $\partial \varphi$ is conservative.
On the other hand, if $M_t$ is a conservative monotone operator, and its potential is $\varphi_t$, then by Definition 53, we have

$$\text{regret}^n z_t = \int_{\gamma_n(z_t, x_t)} \langle M_t \alpha, d\alpha \rangle = \varphi_t (x_t) - \varphi_t (z_t),$$

and $\varphi_t$ is convex and lower semicontinuous according to Lemma 54. Typically in this case the superscript $n$ can be omitted.

**Remark 34.** If $M_t$ is a non-conservative monotone operator, the integral is path-dependent, and its potential does not exist, that is, the loss function cannot be defined. Online monotone optimization allows situations where the loss function cannot be defined. This point of view is different from Gemp and Mahadevan (2016); Gemp and Mahadevan (2017); Gemp (2019), who proposed the idea of online monotone optimization, but did not to abandon the loss function. In essence, the online game emphasizes the interaction between the two parties and weakens the restrictions on the feedback of the adversary.

**Remark 35.** For online convex optimization problem, we can regard the loss function as the potential and the regret as the potential difference. This perspective is guaranteed by strict mathematics.

Thirdly, surrogate linear losses are compatible with the settings of online monotone optimization.

For online convex optimization problem, it suffices to study the corresponding online surrogate linear optimization problem. Note that $\partial \varphi_t$ plays the bridging role in transforming the online convex optimization problem into the corresponding online surrogate linear optimization problem. Since $\partial \varphi_t$ is monotone, we replace $\partial \varphi_t$ with a monotone operator $M_t : C \Rightarrow E^*$. According to the definition of integral (Definition 48), $\forall n \in \mathbb{N}$, we have

$$\text{regret}^n z_t \leq \int_{x_t} \langle M_t \alpha, d\alpha \rangle \leq \langle x_t^*, x_t - z_t \rangle, \quad x_t^* \in \partial \varphi_t (x_t),$$

which is an extension of Equation (12).

Finally, the superscript $n$ represents the maximum meta-algorithm nesting level that the online algorithm can accommodate.

In online meta-learning, the outermost meta-algorithm maintains a group of experts $\{e_i\}_{i \in I}$, and tracks the best one through the combination of expert advice using weight $w_t$. Note that $\forall n \in \mathbb{N}_+$, the instantaneous generalized dynamic regret can be decomposed as the following recursive inequality,

$$\text{regret}^n z_t \leq \int_{x_t(j)} \langle M_t \alpha, d\alpha \rangle + \inf \int_{\gamma_{n-1}(z_t, x_t(j))} \langle M_t \alpha, d\alpha \rangle \leq \langle \ell_t, w_t - 1 \rangle + \text{regret}^{n-1} z_t,$$

where $1_j$ is the one-hot vector corresponding to the expert $e_j$, $\ell_t \in \langle M_t \overline{x}_t, x_t \rangle$, $\overline{x}_t = \langle w_t, x_t \rangle$, $x_t = \{x_t(i)\}_{i \in I}$, $x_t(i)$ represents the suggestion of the expert $e_i$.

If $n - 1 > 0$, then expert $e_j$ can still be a meta-algorithm, which causes the inequality to continue downward recursion.
For example, \( \forall n \in \mathbb{N}_+ \), regret\(^n\) can accommodate one meta-algorithm nesting, that is,
\[
\text{regret}^n z_t \leq \left( \int_{x_t(j)x_t(j)} + \int_{z_t x_t(j)} \right) \langle M_t \alpha, \, d\alpha \rangle \leq \langle \ell_t, \, w_t - 1_j \rangle + \langle x_t (j)^*, \, x_t (j) - z_t \rangle, \tag{13}
\]
where \( x_t (j)^* \in M_t x_t (j) \). However, regret\(^0\) \( z_t \) has no corresponding decomposition, which implies that regret\(^0\) cannot accommodate any meta-algorithms.

Corresponding to Equation (13), in online convex optimization with meta-algorithm, the instantaneous dynamic regret is usually decomposed as the following form (Zhang et al., 2018),
\[
\varphi_t (\bar{x}_t) - \varphi_t (z_t) = \varphi_t (\langle w_t, x_t \rangle) - \varphi_t \left( \langle 1_j, x_t \rangle \right) + \varphi_t (x_t (j)) - \varphi_t (z_t) \\
\leq \langle \ell_t, \, w_t - 1_j \rangle + \langle x_t (j)^*, \, x_t (j) - z_t \rangle,
\]
where \( \ell_t \in \langle \partial \varphi_t (\bar{x}_t), x_t \rangle \), and \( x_t (j)^* \in \partial \varphi_t (x_t (j)) \).

8 Conclusions and Future Work

In this paper, we present a unified analysis method for tighter dynamic regret upper bounds of \( S-I \) and \( S-II \) in normed vector space, and extend online convex optimization to online monotone optimization, which expand the application scope of \( S-I \) and \( S-II \). Our analysis is systematic and mathematically rigorous.

This paper only focuses on the online optimization problem of the learner suffering from the hitting cost. We leave the smoothing online optimization problem (in which the learner suffers both a hitting cost and a switching cost) to future research. Monotonic optimization is a natural extension from convex optimization to non-convex optimization. Exploring other non-convex optimization frameworks that are different from online monotone optimization is left to future work. We also hope that the conclusions of this article encourage the research of adaptive algorithms with tighter dynamic regret upper bounds.
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A  Basis of Convex Analysis

We use "gra" to represent the graph of a map (function), and "epi" to represent the epigraph of a map (function).

Definitions 36 to 38 and Lemma 40 are compiled from Section 1.4 of Brezis (2011).

**Definition 36 (Convex Function).** A proper function $\varphi$ is convex if $\text{epi } \varphi$ is a convex subset of $E \times \mathbb{R}$, or equivalently, $\text{dom } \varphi := \varphi^{-1} \mathbb{R}$ is a convex subset of $E$, and

\[ \varphi (\lambda x + (1 - \lambda) y) \leq \lambda \varphi (x) + (1 - \lambda) \varphi (y), \quad \forall x, y \in \text{dom } \varphi, \quad \forall \lambda \in [0, 1]. \]

**Definition 37 (Lower Semicontinuous).** A proper function $\varphi$ is lower semicontinuous if any one of the following three equivalent conditions holds:

(a) $\forall \lambda \in \mathbb{R}, \varphi^{-1} (\lambda, +\infty)$ is open.
(b) $\lim \inf_{x \to x_0} \varphi (x) \geq \varphi (x_0), \forall x_0 \in \text{dom } \varphi.$
(c) $\varphi$ is closed.

**Definition 38 (Fenchel Conjugate).** Let $\varphi$ be a proper function. $\varphi^* (x^*) := \sup_{E} x^* - \varphi$ is the Fenchel conjugate function of $\varphi$, and $\varphi^*$ is convex and lower semicontinuous.

**Lemma 39.** Let $C \neq \emptyset$ be a closed convex subset of $E$. If $\text{dom } \varphi = \text{dom } \partial \varphi = C$, then $\varphi$ is convex and lower semicontinuous.

**Proof.** Note that $\varphi (y) \geq \langle x^*, y - x \rangle + \varphi (x), \quad \forall x \in C, \quad \forall x^* \in \partial \varphi (x), \quad \forall y \in E,$

which implies that

$\varphi (y) \geq \sup_{x \in C, x^* \in \partial \varphi (x)} \langle x^*, y - x \rangle + \varphi (x), \quad \forall x \in C, \quad \forall y \in E,$

therefore we obtain $\varphi = f + \chi_C$, where

\[ f (y) = \sup_{x \in C, x^* \in \partial \varphi (x)} \ell_{x,x^*} (y), \quad \text{and} \quad \ell_{x,x^*} (y) = \langle x^*, y - x \rangle + \varphi (x). \]

Note that

\[ \text{epi } f = \bigcap_{x \in C, x^* \in \partial \varphi (x)} \text{epi } \ell_{x,x^*}, \]

$\text{epi } \varphi = (C \times \mathbb{R}^+) \cap \text{epi } f$ is closed and convex. According to Definitions 36 and 37, we conclude that $\varphi$ is convex and lower semicontinuous. 

**Lemma 40 (Fenchel-Moreau).** If $\varphi$ is proper, convex and lower semicontinuous, then $\varphi^{**} = \varphi$, where $\varphi^{**} (x) := \sup_{E} x - \varphi^*.$

B  Proof of Lemma 4

**Proof.**

\[ B_{\varphi} (x, y^*) + B_{\varphi} (y, z^*) - B_{\varphi} (x, z^*) \]

\[ = (\varphi (x) + \varphi^* (y^*) - \langle y^*, x \rangle) + (\varphi (y) + \varphi^* (z^*) - \langle z^*, y \rangle) - (\varphi (x) + \varphi^* (z^*) - \langle z^*, x \rangle) \]

\[ = \varphi^* (y^*) + \varphi (y) - \langle z^*, y \rangle + \langle z^* - y^*, x \rangle \]

\[ = \langle z^* - y^*, x - y \rangle, \]

where the last "follows from $\varphi^* (y^*) + \varphi (y) = \langle y^*, y \rangle$ since $(y, y^*) \in \text{gra } \partial \varphi.$"
C  Proof of Lemma 6

Proof. Since $\varphi^{**} = \varphi$ by Lemma 40, we have

$$x^* \in \partial \varphi (x) \iff x, x^* \text{ satisfies the equation } \varphi^* (x^*) + \varphi (x) = \langle x^*, x \rangle \iff x \in \partial \varphi^* (x^*).$$

Given $x^*$. On one hand, if $\partial \varphi^* (x^*) \neq \emptyset$, then $\forall x \in \partial \varphi^* (x^*)$, $\varphi^* (x^*) = \langle x^*, x \rangle - \varphi (x)$ holds, that is, the equation $\langle x^*, x \rangle - \varphi (x) = \sup_{x \in E} \langle x, x \rangle - \varphi (x)$ holds, thus, $x \in \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x)$; If $\partial \varphi^* (x^*) = \emptyset$, then $\forall x \in E, \varphi^* (x) = \langle x^*, x \rangle - \varphi (x)$ does not hold, that is, the equation $\langle x^*, x \rangle - \varphi (x) = \sup_{x \in E} \langle x, x \rangle - \varphi (x)$ does not hold, thus $\arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x) = \emptyset$. In summary, $\partial \varphi^* (x^*) \subset \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x)$.

On the other hand, if $\arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x) \neq \emptyset$, then $\forall x \in \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x)$, $(x^*, x) - \varphi (x) = \sup_{x \in E} \langle x, x \rangle - \varphi (x)$ holds, that is, the equation $\varphi^* (x^*) = \langle x^*, x \rangle - \varphi (x)$ holds, thus, $x \in \partial \varphi^* (x^*)$; If $\arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x) = \emptyset$, then $\forall x \in E, \langle x, x \rangle - \varphi (x) = \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x)$ does not hold, that is, the equation $\varphi^* (x^*) = \langle x^*, x \rangle - \varphi (x)$ does not hold, thus $\partial \varphi^* (x^*) = \emptyset$. In summary, $\arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x) \subset \partial \varphi^* (x^*)$.

Therefore, we have $\partial \varphi^* (x^*) = \arg \max_{x \in E} \langle x^*, x \rangle - \varphi (x)$.

Similarly, we have $\partial \varphi (x) = \arg \max_{x \in E^*} \langle x^*, x \rangle - \varphi^* (x^*)$.\hfill \qed

D  Proof of Lemma 8

Proof. Let $\alpha = \ell + c$, where $\ell \in E^*$ and $c \in \mathbb{R}$. Let $(y, y^{\ell + c}) \in \text{gra } \partial (\varphi + \alpha)$, we have $y^{\ell + c} \in \partial (\varphi + \alpha) y = \partial \varphi (y) + \ell$, i.e., $(y, y^{\ell + c} - \ell) \in \text{gra } \partial \varphi$, then $\forall x \in E$,

$$B_{\varphi+\alpha} (x, y^{\ell + c}) = (\varphi + \alpha) x + (\varphi + \alpha) y^{\ell + c} - \langle y^{\ell + c}, x \rangle = \varphi (x) + \varphi^* (y^{\ell + c} - \ell) - \langle y^{\ell + c} - \ell, x \rangle = B_{\varphi} (x, y^{\ell + c} - \ell) \geq \phi (\|x - y\|).$$

\hfill \qed

E  Supplementary Proof of Theorem 12

The supplementary proof of Theorem 12 relies on the following lemma.

Lemma 41 (Bolzano-Weierstrass. Theorem 6.21 of Muscat, 2014). In a metric space, a subset $C$ is compact iff every sequence in $C$ has a subsequence that converges in $C$.

Proof. It suffices to prove that, under sufficient conditions that $\partial \psi (C) = E^*$ or $C$ is compact, $x \in C$ can be guaranteed by the following general expression included in the update rule,

$$x \in \arg \min_{x \in E} \psi (x^* - x) = \partial \psi^* (x^*), \quad \forall x^* \in E^*. \quad (14)$$

(a) If $\partial \psi (C) = E^*$, then Equation (14) is equivalent to $x^* \in \partial \psi (x)$ according to Lemma 6. Since $\text{dom } \partial \psi = C$, we have that $x \in C$.

(b) If $C$ is compact, according to Lemma 39, $f = \psi - x^*$ is convex and lower semicontinuous on $C$, thich implies that $\{ f^{-1} (\alpha, +\infty) \mid \alpha \in \mathbb{R} \}$ is an open cover of $C$, then $\exists m \in \mathbb{N}$, such that $\{ f^{-1} (\alpha_i, +\infty) \}_{i=1}^m$ covers $C$, that is, $f (C) > \min_i \alpha_i$. Let $\beta = \inf f (C)$, then $\exists y_n \in C$, such that

$$\beta \leq f (y_n) < \beta + \frac{1}{n},$$

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according to Lemma 41, there exists a subsequence $y_{n_k} \to y_0 \in C$, which implies that
\[ \beta \leftarrow f(y_{n_k}) \to f(y_0), \]
thus $f(y_0) = \beta$, which leading to $x = y_0 \in C$.

\[ \square \]

**F  Proof of Lemma 14**

**Proof.** Note that
\[
\sum_{t=1}^{T} \frac{1}{\eta_t} \left[ B_\psi \left( z, \tilde{x}_t^{\psi} \right) - B_\psi \left( z, a^{\psi} + \frac{\eta_t}{\eta_{t+1}} (\tilde{x}_{t+1}^{\psi} - a^{\psi}) \right) \right]
\]
\[
\leq \frac{1}{\eta_1} B_\psi \left( z, \tilde{x}_1^{\psi} \right) + \sum_{t=1}^{T} \left[ \frac{1}{\eta_{t+1}} B_\psi \left( z, \tilde{x}_{t+1}^{\psi} \right) - \frac{1}{\eta_t} B_\psi \left( z, a^{\psi} + \frac{\eta_t}{\eta_{t+1}} (\tilde{x}_{t+1}^{\psi} - a^{\psi}) \right) \right]
\]
\[
\leq \frac{1}{\eta_1} B_\psi \left( z, \tilde{x}_1^{\psi} \right) + \sum_{t=1}^{T} \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) B_\psi \left( z, a^{\psi} \right) = \frac{1}{\eta_{T+1}} B_\psi \left( z, a^{\psi} \right),
\]
where the second \( \leq \) follows from the convexity of $B_\psi (z, \cdot)$. Let $\mu_t = \frac{\eta_{t+1}}{\eta_t} \leq 1$, 
\[
\frac{1}{\eta_{t+1}} B_\psi \left( z, \tilde{x}_{t+1}^{\psi} \right) - \frac{1}{\eta_t} B_\psi \left( z, a^{\psi} + \frac{\eta_t}{\eta_{t+1}} (\tilde{x}_{t+1}^{\psi} - a^{\psi}) \right) \leq \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) B_\psi \left( z, a^{\psi} \right)
\]
is equivalent to
\[
B_\psi \left( z, \tilde{x}_{t+1}^{\psi} \right) \leq \mu_t B_\psi \left( z, a^{\psi} + \frac{1}{\mu_t} (\tilde{x}_{t+1}^{\psi} - a^{\psi}) \right) + (1 - \mu_t) B_\psi \left( z, a^{\psi} \right).
\]
For the second half of Lemma 14,
\[
\sum_{t=1}^{T} \frac{1}{\theta_t} \left[ B_\psi \left( z_t, \tilde{x}_t^{\psi} \right) - B_\psi \left( z_t, \tilde{x}_{t+1}^{\psi} \right) \right]
\]
\[
\leq \frac{1}{\theta_1} B_\psi \left( z_1, \tilde{x}_1^{\psi} \right) - \frac{1}{\theta_T} B_\psi \left( z_T, \tilde{x}_T^{\psi} \right) + \sum_{t=2}^{T} \frac{1}{\theta_t} \left[ B_\psi \left( z_t, \tilde{x}_t^{\psi} \right) - B_\psi \left( z_{t-1}, \tilde{x}_t^{\psi} \right) \right]
\]
\[
\leq \frac{1}{\theta_1} B_\psi \left( z, a^{\psi} \right) + \sum_{t=2}^{T} \frac{1}{\theta_t} \left( \partial_\psi (z_t) - \tilde{x}_t^{\psi} - z_t - z_{t-1} \right),
\]
where the last \( \leq \) follows from the convexity of $\psi$.

\[ \square \]

**G  Proof of Proposition 19**

This is an extension of Appendix B of Flasphol et al. (2021).

**Proof.** The instantaneous dynamic regret can be decomposed as the following form,
\[
\nu_t (x_t) - \nu_t (z_t) \leq \left( x_t^*, x_t - z_t \right) = \left( x_t^*, x_t - y_t \right) + \left( x_t^*, y_t - z_t \right), \quad x_t^* \in \partial \nu_t (x_t).
\]
The regret bound for the auxiliary term is simply replacing \( x_i, \bar{x}_i^y \) and \( \bar{x}_i^y \) with \( y_i, \bar{y}_i^y \) and \( \bar{y}_i^y \) respectively. For the drift term, note that \( f_i(y) \) is \( \phi \)-convex according to Lemma 8, then

\[
\begin{align*}
(f_i(x_i) + \langle \eta_i \theta_i \bar{y}_i, x_i \rangle) &- (f_i(y_i) + \langle \eta_i \theta_i \bar{y}_i, y_i \rangle) \geq \phi (\| x_i - y_i \|), \\
(f_i(y_i) + \langle \eta_i \theta_i \bar{x}_i, y_i \rangle) &- (f_i(x_i) + \langle \eta_i \theta_i \bar{x}_i, x_i \rangle) \geq \phi (\| x_i - y_i \|).
\end{align*}
\]

Adding the above two inequalities and utilizing the Fenchel-Young inequality, we have

\[
\phi (\| x_i - y_i \|) \leq \phi^* (\langle \eta_i \theta_i \bar{x}_i^* - \bar{y}_i^* \rangle).
\]

Then \( \forall \gamma > 0, \)

\[
\langle x_i^*, x_i - y_i \rangle \leq \frac{1}{\gamma} \phi^* (\gamma \| x_i^* \|) + \frac{1}{\gamma} \phi (\| x_i - y_i \|) \leq \frac{1}{\gamma} \phi^* (\gamma \| x_i^* \|) + \frac{1}{\gamma} \phi^* (\| x_i^* - \bar{y}_i^* \|).
\]

Thus, the term \( \phi^* (\langle \xi \| x_i^* - \bar{y}_i^* \rangle) \) in regret upper bounds is replaced by \( \Phi_\xi (x_i^*, \bar{x}_i^*) \), where

\[
\Phi_\xi (x^*, \bar{x}^*) = \phi^* (\langle \xi \| x^* - \bar{y}^* \rangle) + \inf_{\gamma > 0} \left( \frac{1}{\gamma} \phi^* (\gamma \xi \| x^* \|) + \frac{1}{\gamma} \phi^* (\xi \| \bar{x}^* - \bar{y}^* \|) \right).
\]

If \( \phi = Q_\rho, \) then

\[
\Phi_\xi (x^*, \bar{x}^*) = Q_\rho^* (\xi \| x^* - \bar{y}^* \|) + \inf_{\gamma > 0} \left( \frac{1}{\gamma} Q_\rho^* (\gamma \xi \| x^* \|) + \frac{1}{\gamma} Q_\rho^* (\xi \| \bar{x}^* - \bar{y}^* \|) \right).
\]

Let \( f(\gamma) = \frac{1}{\gamma} Q_\rho^* (\gamma \xi \| x^* \|) + \frac{1}{\gamma} Q_\rho^* (\xi \| \bar{x}^* - \bar{y}^* \|). \) If \( \xi \| \bar{x}^* - \bar{y}^* \| > \rho, \) then

\[
\inf_{\gamma > 0} f(\gamma) = \min \left\{ \inf_{\gamma \| x^* \| > \rho} f(\gamma), \inf_{\gamma \| x^* \| \leq \rho} f(\gamma) \right\} = \rho \xi \| x^* \|
\]

since

\[
\inf_{\gamma \| x^* \| > \rho} f(\gamma) = \rho \xi \| x^* \| + \inf_{\gamma \| x^* \| > \rho} \frac{1}{\gamma} (\xi \| \bar{x}^* - \bar{y}^* \| - \rho) = \rho \xi \| x^* \|,
\]

\[
\inf_{\gamma > 0, \gamma \| x^* \| \leq \rho} f(\gamma) = \inf_{\gamma > 0, \gamma \| x^* \| \leq \rho} \left( \frac{\gamma}{2} (\xi \| x^* \|)^2 + \frac{\rho}{2\gamma} (\xi \| \bar{x}^* - \bar{y}^* \| - \rho) \right) \geq \inf_{\gamma > 0, \gamma \| x^* \| \leq \rho} \left( \frac{\gamma}{2} (\xi \| x^* \|)^2 + \frac{\rho^2}{2\gamma} \right) \geq \rho \xi \| x^* \|.
\]

If \( \xi \| \bar{x}^* - \bar{y}^* \| \leq \rho, \) then

\[
\inf_{\gamma > 0} f(\gamma) = \min \left\{ \inf_{\gamma > 0, \xi \| x^* \| \leq \rho} f(\gamma), \inf_{\gamma \| x^* \| > \rho} f(\gamma) \right\} = \xi^2 \| x^* \| \| \bar{x}^* - \bar{y}^* \|
\]

since

\[
\inf_{\gamma \| x^* \| \leq \rho} f(\gamma) = \inf_{\gamma > 0, \xi \| x^* \| \leq \rho} \left( \frac{\gamma}{2} (\xi \| x^* \|)^2 + \frac{1}{2\gamma} (\xi \| \bar{x}^* - \bar{y}^* \|)^2 \right) = \xi^2 \| x^* \| \| \bar{x}^* - \bar{y}^* \|,
\]

\[
\inf_{\gamma \| x^* \| > \rho} f(\gamma) = \rho \xi \| x^* \| + \inf_{\gamma \| x^* \| > \rho} \frac{1}{2\gamma} (\xi^2 \| \bar{x}^* - \bar{y}^* \|^2 - \rho^2) = \frac{1}{2\rho} \xi \| x^* \| \xi^2 \| \bar{x}^* - \bar{y}^* \|^2 \geq \xi^2 \| x^* \| \| \bar{x}^* - \bar{y}^* \|. \]
In summary,

\[ \inf_{y \geq 0} \left( \frac{1}{\gamma} Q_{\rho}^* (\gamma \xi \| x^* \|) + \frac{1}{\gamma} Q_{\rho}^* (\xi \| \tilde{x}^* - \tilde{y}^* \|) \right) = \xi \| x^* \| \min \{ \xi \| \tilde{x}^* - \tilde{y}^* \|, \rho \}. \]

If \( \tilde{y}^* = \lambda \tilde{x}^* + (1 - \lambda) x^* \), \( \lambda = \min \left\{ \frac{\| x^* \|}{\| \tilde{x}^* - x^* \|}, 1 \right\} \), then

\[ \Phi_\xi (x^*, \tilde{x}^*) = Q_{\rho}^* (\xi \| x^* - \tilde{y}^* \|) + \xi \| x^* \| \min \{ \xi \| \tilde{x}^* - \tilde{y}^* \|, \rho \} \]
\[ = Q_{\rho}^* (\xi \min \{\| x^* - \tilde{x}^* \|, \| x^* \|\}) + \xi \| x^* \| \min \{ \xi \| x^* - \tilde{x}^* \| - \| x^* \|, \rho \} \]
\[ \leq Q_{\rho}^* (\xi \min \{\| x^* - \tilde{x}^* \|, \| x^* \|\}) + \xi^2 \| x^* \| \left( \| x^* - \tilde{x}^* \| - \| x^* \| \right)^2 \]
\[ = \xi^2 Q_{x^*}^* (\| x^* - \tilde{x}^* \|) - \frac{1}{2} (\xi \min \{\| x^* - \tilde{x}^* \|, \| x^* \|\} - \rho)^2. \]

\( \Box \)

**H Proof of Lemma 21**

The proof of Lemma 21 relies on the following lemmas.

**Lemma 42** (Heine-Borel, Theorem 5.5 of Gamelin and Greene, 1999). Let \( S \) be a subset of metric space \( \mathbb{R}^n \), then \( S \) is compact iff \( S \) is closed and bounded.

**Lemma 43** (Example 2.5 of Shalev-Shwartz, 2012). \( \psi (w) = \langle w, \ln w \rangle + \chi_{\Delta^n} (w) \) is 1-strongly-convex w.r.t \( \| \cdot \| \) over the probability simplex \( \Delta^n \).

**Lemma 44** (Table 2.1 of Shalev-Shwartz, 2012). \( \psi (w) = \langle w, \ln w \rangle + \chi_{\Delta^n} (w) \) and \( \psi^* (w^*) = \ln \left( 1, e^{w^*} \right) \) are a pair of Fenchel conjugate functions, where \( 1 \) represents the all-ones vector in \( \mathbb{R}^{n+1} \).

**Proof:** Note that \( \Delta^n \) is a bounded closed subset of \( \mathbb{R}^{n+1} \) with metric \( \| \cdot \|_1 \). According to Lemma 42, \( \Delta^n \) is a compact subset.

Lemma 43 is equivalent to

\[ \psi (w) - \psi (u) \geq \langle u^\psi, w - u \rangle + \frac{1}{2} \| w - u \|_1^2, \quad \forall u, w \in \Delta^n, \quad \forall u^\psi \in \partial \psi (u). \]

Remove the restriction on \( w \) by adding \( \chi_{\Delta^n} (w) \) on both sides of the above inequality, and note that \( \| w - u \|_1 \leq \| w \|_1 + \| u \|_1 = 2 \) for all \( w \in \Delta^n \), we obtain that

\[ B_{\psi} (w, u^\psi) \geq \frac{1}{2} \| w - u \|_1^2 + \chi_{[-2,2]} (\| w - u \|_1), \quad \forall u \in \Delta^n, \quad \forall u^\psi \in \partial \psi (u), \]

which implies that \( \psi \) is \( Q_2 \)-convex.

According to Lemma 44, we have that

\[ \partial \psi (w) \ni 1 + \ln w, \quad \forall w \in \Delta^n, \quad w > 0, \]
\[ \partial \psi^* (w^*) = N e^{w^*}. \]
Choose \( \partial \varphi = 1 + \ln \), then the strategy \( S \) is instantiated as
\[
w_t = \mathcal{N} \left( a \circ e^{-\eta} \left( \sum_{i=1}^{t-1} \theta_i u_i + \theta_t \hat{w}_t \right) \right).
\]
To complete the proof, it suffices to replace \( w_t^* \) and \( \hat{w}_t^* \) with \( \ell_t \) and \( \hat{\ell}_t \) respectively. \( \square \)

## I Proof of Corollary 23

**Proof.** \( \forall u, w \in \Delta^n, w^\psi = 1 + \ln w \) since \( \partial \varphi = 1 + \ln \). By Lemma 44,
\[
B_{\psi} (u, w^\psi) = \langle u, \ln u \rangle + \ln \left( 1, e^{w^\psi} \right) - \langle u, w^\psi \rangle = \left( u, \ln \frac{u}{w} \right).
\]
Note that \( \Delta^n \) is compact and \( \psi \) is \( Q_2 \)-convex. To complete the proof, we just apply Corollary 17. \( \square \)

## J Proof of Lemma 25

The proof of Lemma 25 relies on the following lemma.

**Lemma 45** (Theorem 5.2 of Brezis, 2011). Let \( C \neq \emptyset \) be a closed convex subset of Hilbert space \( E \). \( \forall x \in E, \exists! x_0 \in C, \) such that \( \| x - x_0 \| = \inf \| x - C \|. \) \( x_0 \) is called the projection of \( x \) onto \( C \) and is denoted by \( x_0 = P_C (x) \). Moreover, \( \langle x - x_0, C - x_0 \rangle \leq 0 \).

**Proof.** According to Lemma 45, \( \forall x \in E, \exists! x_0 = P_C (x) \in C, \) such that
\[
\frac{1}{2} \| x - y \|^2 + \chi_C (y) \geq \frac{1}{2} \| x - x_0 \|^2, \quad \forall y \in E,
\]
rearrange the above formula, we have
\[
\frac{1}{2} \| y \|^2 + \chi_C (y) - \frac{1}{2} \| x_0 \|^2 \geq \langle x, y - x_0 \rangle, \quad \forall y \in E,
\]
thus, \( x \in \partial \psi (x_0) \), which shows that \( \partial \psi (C) = E \).
Moreover, we have a stronger result that \( P_C = \partial \psi^* \), or equivalently, \( x_0 \) is the unique image of \( x \) over \( \partial \psi^* \). Suppose by contradiction that there is some \( x'_0 \neq x_0 \) such that \( x'_0 = \partial \psi^* (x) \) (and obviously \( x'_0 \in C \)), by Lemma 6, it follows that \( x \in \partial \psi (x'_0) \), that is,
\[
\frac{1}{2} \| y \|^2 + \chi_C (y) - \frac{1}{2} \| x'_0 \|^2 \geq \langle x, y - x'_0 \rangle, \quad \forall y \in E,
\]
then \( \forall y \in C, \| x - y \| \geq \| x - x'_0 \| \), which implies that \( x'_0 = P_C (x) = x_0 \), which is a contradiction.
Now we prove that \( \psi \) is \( Q_\rho \)-convex. Note that
\[
\psi^* (x^*) = \sup_{x \in E} \langle x^*, x \rangle - \frac{1}{2} \| x^* \|^2 - \chi_C (x) = \frac{1}{2} \| x^* \|^2 + \sup_{x \in C} \langle x^*, x \rangle - \frac{1}{2} \| x \|^2 - \frac{1}{2} \| x^* \|^2
\]
\[
= \frac{1}{2} \| x^* \|^2 - \frac{1}{2} \inf_{x \in C} \| x^* - x \|^2 = \frac{1}{2} \| x^* \|^2 - \frac{1}{2} \| x^* - P_C (x^*) \|^2.
\]
\[\forall z \in C, \forall z^\psi \in \partial \psi (z), \text{ we have that } z = P_C (z^\psi), \text{ and according to the cosine rule,} \]
\[
\| z^\psi - y \|^2 = \| z^\psi - z \|^2 + \| y - z \|^2 - 2 \langle z^\psi - z, y - z \rangle.
\]
Let \( y \in C \), then by Lemma 45,
\[
\|z^\phi - y\|^2 \geq \|z^\phi - z\|^2 + \|y - z\|^2,
\]
rearrange the above inequality, and note that \( \chi_C (y) \geq \chi_\rho (\|y - z\|) \),
\[
\frac{1}{2} \|y\|^2 + \chi_C (y) - \frac{1}{2} \|z\|^2 - \langle z^\phi, y - z \rangle \geq \frac{1}{2} \|y - z\|^2 + \chi_\rho (\|y - z\|),
\]
where
\[
- \frac{1}{2} \|z\|^2 + \langle z^\phi, z \rangle = - \frac{1}{2} \|z^\phi\|^2 + \frac{1}{2} \|z^\phi - P_C (z^\phi)\|^2 = \psi^* (z^\phi).
\]
Therefore, \( B_\phi (y, z^\phi) \geq Q_\rho (\|y - z\|) \).
Finally, choose \( \partial \psi \) as the identity map, since \( \forall x \in C, x \in \partial \psi (x) \), and according to \( P_C = \partial \psi^* \) and the uniqueness of the projection, S-I and S-II are respectively instantiated into the following forms,
\[
\tilde{x}_t = P_C \left( a - \eta_t \sum_{i=1}^{t-1} \theta_i x^*_i \right),
\]
\[
x_t = P_C \left( \tilde{x}_t - \eta_t \theta_t \tilde{x}_t \right),
\]
and
\[
\tilde{x}_t = P_C \left( \tilde{x}_{t-1} - \theta_{t-1} x^*_{t-1} \right),
\]
\[
x_t = P_C \left( \tilde{x}_t - \theta_t \tilde{x}_t \right).
\]

\[\square\]

**K Proof of Corollary 27**

**Proof.** Since \( \partial \psi \) is the identity map, \( \tilde{x}_t = \bar{x}_t = x_t = P_C (x^*_t) \). Note that
\[
B_\phi (z, x) = \frac{1}{2} \|z - x\|^2, \quad \forall x, z \in C.
\]
According to Lemma 45,
\[
B_\phi \left( X_{t+1}, \tilde{x}_t \right) - B_\phi \left( X_{t+1}, x_t \right) = \left( \tilde{x}_t^\phi - \tilde{x}_t, x_t - X_{t+1} \right) = \left( P_C \left( \tilde{x}_t^\phi \right) - \tilde{x}_t, P_C \left( \tilde{x}_t^\phi \right) - X_{t+1} \right) \leq 0.
\]
To complete the proof, it suffices to substitute the above settings together with Lemma 14 into Theorem 12. \[\square\]

**L Basis of Monotone Operators**

All definitions and lemmas in this appendix are compiled from Romano et al. (1993), with some minor modifications.

**Definition 46 (Monotone and Cyclic Monotone).** Let \( C \subset E, C \neq \emptyset, M : C \rightrightarrows E^* \), \( n \in \mathbb{N} \) and \( n \geq 2 \). \( M \) is \( n \)-cyclically monotone if \( \forall (x_i, x^*_i) \in \text{gra} M, i = 1, 2, \cdots, n \),
\[
\sum_{i=1}^{n} \left( x^*_i, x_{i+1} - x_i \right) \leq 0, \quad x_{n+1} = x_1.
\]
\( M \) is monotone iff \( M \) is 2-cyclically monotone; \( M \) is cyclically monotone iff \( \forall n \geq 2 \), \( M \) is \( n \)-cyclically monotone.
Remark 47. The definition of n-cyclically monotone is equivalent to the following statement, \( \forall (x_i, x_i^*) \in \text{gra} \, M, \, i = 1, 2, \ldots, n, \)
\[
\sum_{i=1}^{n} \langle x_i^*, x_i - x_{i-1} \rangle \geq 0, \quad x_0 \equiv x_n,
\]
which is to rearrange the subscripts in original definition in reverse order.

**Definition 48 (Integral).** \( C \neq \emptyset \) is a subset of \( E, \, M : C \Rightarrow E^* \) is monotone, \( \overline{xy} : t \in [0, 1] \mapsto x + t \, (y - x) \in C. \) The integral of \( M \) along \( \overline{xy} \) is defined as
\[
\int_{\overline{xy}} \langle M, \, da \rangle := \int_{0}^{1} \langle M \, (x + t \, (y - x)), \, y - x \rangle \, dt, \quad \text{and} \quad \int_{\overline{xy}+\overline{yz}} = \int_{\overline{xy}} + \int_{\overline{yz}}.
\]

Remark 49. The integral is well-defined since \( m(t) = \langle M \, (x + t \, (y - x)), \, y - x \rangle \) is a non-decreasing function on \([0, 1] \) and every non-decreasing function on \([0, 1] \) is integrable (Theorem 33.1 of Ross, 2013).

**Lemma 50.** Let \( M : E \Rightarrow E^* \) be a monotone operator, and let \( \Gamma \subset \text{dom} \, M \) be a certain finite-length oriented polyline from \( x_0 \) to \( x \). Then
\[
\int_{\Gamma} \langle M, \, da \rangle = \sup_{n, x_i, x_j^*} \sum_{i=0}^{n} \langle x_i^*, x_{i+1} - x_i \rangle, \quad x_{n+1} \equiv x,
\]
where the supremum operator traverses all possible decompositions of \( \Gamma = \overline{x_0x_1} + \overline{x_1x_2} + \cdots + \overline{x_nx}, \quad n \in \mathbb{N}^+. \)

**Definition 51 (Conservative Operator).** A monotone operator \( M \) is said to be conservative if
\[
\int_{\Gamma} \langle M, \, da \rangle = 0,
\]
for every closed polyline \( \Gamma \subset \text{dom} \, M \) with finite length.

**Lemma 52.** Let \( M : E \Rightarrow E^* \) be a multivalued map, then
1. \( M \) is cyclically monotone \( \implies M \) is conservative;
2. \( M \) is conservative and \( \text{dom} \, M \) is convex \( \implies M \) is cyclically monotone.

**Definition 53 (Potential).** Let \( C \neq \emptyset \) be a closed convex subset of \( E, \) and let \( M : C \Rightarrow E^* \) be a conservative operator. \( \varphi \) is a potential of \( M \) if
\[
\varphi(x) - \varphi(x_0) = \int_{\gamma(x_0, x)} \langle M, \, da \rangle, \quad \forall x, x_0 \in C, \quad \text{and} \quad \varphi(y) = +\infty, \quad \forall y \notin C,
\]
where \( \gamma(x_0, x) \) denotes an arbitrary finite-length oriented polyline in \( E \) from \( x_0 \) to \( x. \)

**Lemma 54.** Let \( C \neq \emptyset \) be a closed convex subset of \( E, \) and let \( M : C \Rightarrow E^* \) be a conservative operator. If \( \varphi \) is a potential of \( M, \) then \( \varphi \) is convex and lower semicontinuous.

**Proof.** \( \forall x, x_0 \in C, \) let \( \gamma(x_0, x) \) be an arbitrary finite-length oriented polyline in \( C \) from \( x_0 \) to \( x. \)
According to Lemma 50, we have
\[
\varphi(x) - \varphi(x_0) = \int_{\gamma(x_0, x)} \langle M, \, da \rangle = \sup_{n, x_i, x_i^*} \left( \sum_{i=0}^{n-1} \langle x_i^*, x_i - x_{i-1} \rangle + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \right).
\]
The subsequent proof is similar to that of Lemma 39. Indeed, for each fixed tuple \((n, x_i, x_i')\), the function \(x \mapsto \langle x_n', x - x_n \rangle + \sum_{i=0}^{n-1} \langle x_i', x_{i+1} - x_i \rangle\) is linear (and thus convex and lower semicontinuous). It follows that the superior envelope of these functions is convex and lower semicontinuous. Note that \(C\) is closed and convex, therefore we obtain that \(\varphi\) is convex and lower semicontinuous.

Lemma 55. Let \(C \neq \emptyset\) be a convex subset of \(E\). If \(\text{dom } \varphi = \text{dom } \partial \varphi = C\), then \(\partial \varphi\) is conservative, and the potential of \(\partial \varphi\) can be chosen as \(\varphi\), i.e.,

\[
\varphi (x) - \varphi (z) = \int_{\gamma(z,x)} \langle \partial \varphi (a), \, da \rangle, \quad \forall x, z \in C,
\]

where \(\gamma (z,x)\) denotes an arbitrary finite-length oriented polyline in \(C\) from \(z\) to \(x\).

Proof. The proof of Lemma 55 is derived from the following steps,

\[
dom \varphi = \text{dom } \partial \varphi = C \implies step \, 1 \implies \partial \varphi \colon C \hookrightarrow E^* \text{ is cyclically monotone}
\]

\[
\iff \partial \varphi \colon C \hookrightarrow E^* \text{ is conservative}
\]

\[
\implies step \, 3 \implies \text{the potential of } \partial \varphi \text{ can be chosen as } \varphi.
\]

The proof of step 1 is as follows. \(\forall n \geq 2, \forall (x_i, x_i') \in \text{gra } \partial \varphi, i = 1, 2, \cdots, n, x_{n+1} \equiv x_1\),

\[
\varphi (x_{i+1}) - \varphi (x_i) \geq \langle x_i', x_{i+1} - x_i \rangle,
\]

thus,

\[
\sum_{i=1}^{n} \langle x_i', x_{i+1} - x_i \rangle \leq \sum_{i=1}^{n} \varphi (x_{i+1}) - \varphi (x_i) = 0.
\]

Step 2 holds directly from Lemma 52.

The proof of step 3 relies on the following lemmas.

Lemma 56 (Lebesgue. Theorem 4.4 of Komornik, 2016). Every non-increasing or non-decreasing function \(F \colon [a, b] \to \mathbb{R}\) is a.e. differentiable.

Lemma 57 (Lebesgue-Vitali. Theorem 6.5 of Komornik, 2016). Let \(f \colon [a, b] \to \mathbb{R}\). If \(F \colon [a, b] \to \mathbb{R}\) is absolutely continuous, has bounded variation, and \(F' = f\) a.e. Then \(f\) is integrable and

\[
\int_{a}^{b} f (x) \, dx = F (b) - F (a).
\]

Let

\[
F (\lambda) = \varphi (z + \lambda (x - z)), \quad f (\lambda) = \langle \partial \varphi (z + \lambda (x - z)), x - z \rangle, \quad \lambda \in [0, 1], \quad \forall x, z \in C,
\]

then \(F\) is convex on \([0, 1]\), that is, \(\exists \tau \in [0, 1]\), such that \(F\) is non-increasing on \([0, \tau]\) and non-decreasing on \([\tau, 1]\). Thus, \(F\) has bounded variation, and \(F' = \partial F\) a.e. according to Lemma 56. Since \(\forall \lambda, \mu \in [0, 1]\),

\[
F (\mu) - F (\lambda) = \varphi (z + \mu (x - z)) - \varphi (z + \lambda (x - z)) \geq \langle \partial \varphi (z + \lambda (x - z)) - \partial \varphi (z + \mu (x - z)), x - z \rangle (\mu - \lambda) = f (\lambda) (\mu - \lambda)
\]

we have \(f (\lambda) \subset \partial F (\lambda)\), and then \(F' = f\) a.e.
Note that
\[ |F(\lambda) - F(\mu)| \leq \max \{|f(\lambda)|, |f(\mu)| \} |\lambda - \mu| \leq \max \{|\sup f(0)|, |\inf f(1)| \} |\lambda - \mu|, \]

\( F \) is Lipschitz continuous, then \( F \) is absolutely continuous. According to Lemma 57, and note that \( \partial \varphi \) is conservative, we have
\[ \int_{y(x)} \langle \partial \varphi(\alpha), \, d\alpha \rangle = \int_{x} \langle \partial \varphi(\alpha), \, d\alpha \rangle = \int_{0}^{1} f(\lambda) \, d\lambda = F(1) - F(0) = \varphi(x) - \varphi(z), \]

which completes the proof. \( \Box \)