TIME SPLITTING COMBINED WITH EXPONENTIAL WAVE INTEGRATOR FOURIER PSEUDOSPECTRAL METHOD FOR QUANTUM ZAKHAROV SYSTEM

GENGEN ZHANG*

South China Research Center for Applied Mathematics and Interdisciplinary Studies
South China Normal University
Guangzhou 510631, China

(Communicated by Xiao-Ping Wang)

Abstract. In this paper we develop a time splitting combined with exponential wave integrator (EWI) Fourier pseudospectral (FP) method for the quantum Zakharov system (QZS), i.e. using the FP method for spatial derivatives, a time splitting technique and an EWI method for temporal derivatives in the Schrödinger-like equation and wave-type equations, respectively. The scheme is fully explicit and efficient due to fast Fourier transform. Numerical experiments for the QZS are presented to illustrate the accuracy and capability of the method, including accuracy tests, convergence of the QZS to the classical Zakharov system in the semi-classical limit, soliton-soliton collisions and pattern dynamics of the QZS in one-dimension, as well as the blow-up phenomena of QZS in two-dimension.

1. Introduction. The main concern of this paper is to develop the efficient and accurate numerical method for the quantum Zakharov system (QZS)[28, 26, 25]

\[
\begin{align*}
\tau iE_t + \Delta E - \varepsilon^2 \Delta^2 E &= NE, \\
N_{tt} - \Delta N + \varepsilon^2 \Delta^2 N &= \Delta |E|^2, \\
E(x,0) &= E_0(x), \\
N(x,0) &= N_0(x), \\
N_t(x,0) &= N_1(x),
\end{align*}
\]

which is originally introduced by Garcia et al. [19] and Haas et al. [27] as a model describing the nonlinear interaction between high-frequency quantum Langmuir waves and low-frequency quantum ionacoustic waves. Here the complex function \( E(x,t) \) is the slowly varying envelope of the rapidly oscillatory electric field, the real function \( N(x,t) \) is the deviation of the ion density from its equilibrium value. In fact, \( \varepsilon = \frac{\hbar \omega}{\kappa B T_e} \) is a parameter that measures the importance of the quantum effects in modulational instabilities and the ratio of the ion plasma and electron thermal energies, \( \kappa_B \) is the Boltzmann constant, \( T_e \) is the electron fluid temperature, \( \hbar \) is Planck’s constant divided by \( 2\pi \), and \( \omega_i \) is the ion plasma frequency. In the semiclassical limit \( \varepsilon \to 0 \), the QZS recovers the classical Zakharov system (ZS), which have been widely applied to various physical problems, such as plasma [49], the

2020 Mathematics Subject Classification. Primary: 35Q55, 65M06, 65M70; Secondary: 65M12.

Key words and phrases. Quantum Zakharov system, time splitting method, exponential wave integrator method, soliton-soliton collisions, pattern dynamics.

The author is supported by the National Natural Science Foundation of China (Grant No.11701110), the China Postdoctoral Science Foundation (Grant No.2020M682746).

* Corresponding author: Gengen Zhang.
theory of molecular chains [13], hydrodynamics [14] and so on. As compared to the classical ZS, Marklund [37] pointed out that the combined effect of a quantum correction and partial coherence may give rise to an increased modulational instability growth rate, and this result may be of significance in dense astrophysical plasmas and laboratory laser-plasma systems.

Analogous to the classical ZS, the QZS (1) satisfies the mass conservation law
\[
\int_{\mathbb{R}^d} |E(t)|^2 dx = \int_{\mathbb{R}^d} |E(0)|^2 dx,
\]
and the energy conservation law
\[
\int_{\mathbb{R}^d} \left( |\nabla E|^2 + \frac{1}{2}(|\nabla u|^2 + N^2) + \varepsilon^2 |\Delta E|^2 + \frac{\varepsilon^2}{2} |\nabla N|^2 + N|E|^2 \right) dx = \text{constant},
\]
where \(\Delta u = N_t\).

In recent years, the QZS (1) have been investigated extensively, for the well-posedness of QZS, we refer to [12, 15, 16, 17, 18, 24, 25, 32, 48] and references therein. The low-regularity global well-posedness of the adiabatic limit of the QZS and its semi-classical limit were studied in [12]. For the hyperchaos and temporal dynamics of the QZS, Misra et al. [39] established the coexistence of novel hyperchaotic attractors, whose appearance is explained by means of the analysis of Lyapunov exponent spectra as well as the Kaplan-Yorke dimension. Misra and Shukla [40] revealed that many coherent solitary patterns can be excited and saturated via the modulational instability of unstable harmonic modes, and this evolution of the solitary patterns may undergo the states of spatially partial coherence, coexistence of spatiotemporal chaos and temporal chaos. In both two and three-spatial dimensions, Haas and Shukla [27] pointed out that the quantum corrections prevent the collapse of localized Langmuir envelope fields, and the quantum terms can produce an oscillatory behavior of the width of the approximate Gaussian solutions.

In addition, the time splitting (TS) method and exponential wave integrator (EWI) method are very efficient, thus, they have been widely used in solving PDE numerically. Wang et al. [44] proposed various TS Fourier pseudospectral (FP) methods for N-coupled nonlinear Schrödinger equation. Hofstätter et al. [29] studied the convergence analysis of TS pseudospectral methods adapted for time-dependent Gross-Pitaevskii equations with additional rotation term. Bao and Dong [1] investigated a Gautschi-type exponential integrator FP method for Klein-Gordon equation. Based on TS-EWI sine pseudospectral method, Bao and Su [5] derived an uniformly accurate numerical method for the ZS with a dimensionless parameter. Liao et al. [36] formulated the TS-EWI-FP method for approximations of the coupled Schrödinger-Boussinesq system. Readers can refer to the works [2, 9, 20, 35, 43, 51] for more relevant references.

Up to now, the numerical studies of the ZS are very rich, one could reference [4, 5, 6, 7, 8, 10, 11, 21, 33, 34, 41, 46, 42] for more details. In particular, TS spectral methods have been proposed by Bao et al. [7, 5] and Jin et al. [33, 34] to solve the generalized ZS and vector ZS for multi-component plasmas. Xiao et. al. [47] proposed a conservative linearly-implicit difference scheme for the fractional modified Zakharov system with quantum correction in one-dimension, and show some dynamical phenomena. Very recently, Zhang and Su [50] developed a highly accurate conservative method for solving QZS (1). However, tracing the literature regarding the studies of QZS (1), the numerical methods proposed in the literature are limited. The main purpose of this paper is to construct a TS-EWI-FP method
for QZS (1), and show some peculiar dynamical behaviors in numerical examples. This scheme is fully explicit and highly accurate at the spectrally accurate in space and second order in time. It is very efficient due to fast Fourier transform, and one only needs to solve two independent linear algebraic equations at each time step.

The outline of this paper is organized as follows. In Section 2, for one-dimensional QZS, we propose EWI-FP method based on Gautschi-type quadrature for solving the wave-type equation, then we present the TS-FP discretization of Schrödinger-like equation. Enlightened by TS-EWI-FP method for one-dimensional problem, we provide TS-EWI-FP method with more details for two-dimensional QZS in Section 3. In Section 4, we investigate the accuracy of TS-EWI-FP method for QZS with solitary-wave solutions, and show some complex dynamical behaviors. Finally, some conclusions are drawn in Section 5.

2. A TS-EWI-FP method. In this section, we propose an efficient and accurate numerical method for QZS in one dimension \((d = 1)\), i.e. the FP method for spatial derivatives, the TS-EWI method by using a TS technique and an EWI method for temporal derivatives in the Schrödinger-like equation and wave-type equation, respectively. Similar to most works for the simulation of the classical ZS [6, 7, 4, 5, 10, 33], the whole space problem (1) will be truncated into a bounded domain \(\Omega = [a, b]\) with periodic boundary conditions:

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t E + \partial_x^2 E - \varepsilon^2 \partial_x^4 E - NE = 0, \\
\partial_x^2 N - \partial_x^4 N + \varepsilon^2 \partial_x^6 N - \partial_x^8 (|E|^2) = 0, \\
E(x, 0) = E_0(x), \quad N(x, 0) = N_0(x), \quad \partial_t N(x, 0) = N_1(x), \\
x \in (a, b), \quad t > 0,
\end{array} \right.
\end{align*}
\]

(2)

2.1. An EWI-FP method for the wave-type equation. Let \(\tau > 0\) be the time step size, and the time steps \(t_n := n\tau\) for \(n = 0, 1, 2, \cdots\). Choose a mesh size \(h := (b - a)/M\) with \(M\) being an even positive integer, and the grid points \(x_j := a + jh\) for \(j = 0, 1, \cdots, M\). Define the index sets

\[
\mathcal{T}_M = \left\{ j | j = -\frac{M}{2}, -\frac{M}{2} + 1, \cdots, \frac{M}{2} - 1 \right\}, \quad \mathcal{T}_M^0 = \left\{ j | -\frac{M}{2}, -\frac{M}{2} + 1, \cdots, \frac{M}{2} \right\}
\]

and

\[
X_M := \{ v = (v_0, v_1, \cdots, v_M)^T \ | v_0 = v_M \} \subseteq \mathbb{C}^{M+1} \text{ with } \|v\|_h^2 = h \sum_{j=0}^{M-1} |v_j|^2,
\]

\[
Y_M := \text{span} \{ \Phi_l(x) = e^{i\mu_l(x-a)} | x \in \Omega \}
\]

with

\[
\mu_l = \frac{2\pi l}{b - a}, \quad l \in \mathcal{T}_M.
\]

Let \(\mathcal{P}_M : H^1(\Omega) \to Y_M\) be the standard \(L^2\) -projection operator, and let \(\mathcal{I}_M : X_M \to Y_M\) be the standard interpolation operator as

\[
(\mathcal{P}_M \psi)(x) = \sum_{l=-M/2}^{M/2-1} \hat{\psi}_l \Phi_l(x), \quad (\mathcal{I}_M \phi)(x) = \sum_{l=-M/2}^{M/2-1} \hat{\phi}_l \Phi_l(x), \quad x \in \Omega,
\]

where \(\hat{\psi}_l\) and \(\hat{\phi}_l\) are the Fourier and discrete Fourier transform coefficients of the function \(\psi(x)\) and vector \(\phi\) (with \(\phi_j = \phi(x_j)\) for \(j \in \mathcal{T}_M^0\) when involved), respectively, defined as

\[
\hat{\psi}_l = \frac{1}{b-a} \int_a^b \psi(x)e^{-i\mu_l(x-a)}dx, \quad \hat{\phi}_l = \frac{1}{M} \sum_{j=0}^{M-1} \phi_j e^{-i\mu_l(x_j-a)}, \quad l \in \mathcal{T}_M.
\]

(3)
Obviously, we have
\[ \| (\mathcal{L}_M \phi) (x) \|_{L^2}^2 = (b - a) \sum_{l=-M/2}^{M/2-1} |\tilde{\phi}_l|^2, \quad \int_a^b (\mathcal{L}_M \phi) (x) dx = (b - a) \tilde{\phi}_0. \]

For the wave-type equation in (2), we discretize it in space by the FP method and in time by an EWI method which has been widely used for discretizing oscillatory PDEs [1, 3, 30] and second order ODEs [22, 23, 31]. Specifically, find \( E_M := E_M(x,t) \) and \( N_M := N_M(x,t) \), i.e.
\[ E_M(x,t) = \sum_{l=-M/2}^{M/2-1} \tilde{E}_l(t) \Phi_l(x), \quad N_M(x,t) = \sum_{l=-M/2}^{M/2-1} \tilde{N}_l(t) \Phi_l(x) \] such that
\[ \partial_t N_M - \partial_{xx} N_M + \varepsilon^2 \partial_{xxxx} N_M = \mathcal{P}_M (\partial_{xx} |E_M|^2). \] (5)
Substituting (4) into (5) and noticing the orthogonality of the basis functions in \( X_M \), for \( t = t_n + w(n \geq 0) \) and \( w \in \mathbb{R} \), we obtain
\[ \frac{d^2}{dw^2} \tilde{N}_l (t_n + w) + \xi_i^2 \tilde{N}_l (t_n + w) + \mu_i^2 \tilde{G}_l^n (w) = 0, \] (6)
where \( \xi_i = \sqrt{\mu_i^2 + \varepsilon^2 \mu_i^4}, \tilde{G}_l^n (w) = (|E_M|^2)_i (t_n + w) \).
Using the variation-of-constants formula [22, 23], the general solution of above second-order ODE (6) is
\[ \tilde{N}_l (t_n + w) = \cos (\xi_i w) \tilde{N}_l (t_n) + \frac{\sin (\xi_i w)}{\xi_i} \tilde{N}_l' (t_n) \]
\[ - \frac{\mu_i}{\sqrt{1 + \varepsilon^2 \mu_i^2}} \int_0^w \tilde{G}_l^n (s) \sin (\xi_i (w - s)) ds, \quad n \geq 0. \] (7)
Differentiating (7) with respect to \( w \), we have
\[ \tilde{N}_l' (t_n + w) = - \xi_i \sin (\xi_i w) \tilde{N}_l (t_n) + \cos (\xi_i w) \tilde{N}_l' (t_n) \]
\[ - \mu_i^2 \int_0^w \tilde{G}_l^n (s) \cos (\xi_i (w - s)) ds, \quad n \geq 0. \] (8)
For \( n \geq 1 \), in (7), choosing \( w = \tau \) and \( w = -\tau \) respectively, then summing the corresponding equations together, we obtain the following recursion relationship
\[ \tilde{N}_l (t_{n+1}) = 2 \cos (\xi_i w) \tilde{N}_l (t_n) - \tilde{N}_l (t_{n-1}) \]
\[ - \frac{\mu_i}{\sqrt{1 + \varepsilon^2 \mu_i^2}} \int_0^\tau \tilde{G}_l^n (s + \tilde{G}_l^n (-s)) \sin (\xi_i (\tau - s)) ds. \] (9)
Similarly, we get
\[ \tilde{N}_l' (t_{n+1}) = \tilde{N}_l' (t_{n-1}) - 2 \xi_i \sin (\xi_i w) \tilde{N}_l (t_n) \]
\[ - \mu_i^2 \int_0^\tau \tilde{G}_l^n (s) + \tilde{G}_l^n (-s) \cos (\xi_i (\tau - s)) ds. \] (10)
In order to design an explicit scheme, we adopt the Cautschi-type quadrature [3, 22, 23, 30, 31, 45] to approximate the integrals in (9) and (10).
Let $N^n_M(x)$ and $\tilde{N}^n_M(x)$ be the approximations of $N(x, t_n)$ and $\partial_t N(x, t_n)$, respectively. Then the details of EWI-FP method for the wave-type equation is organized as follows:

$$
N^n_M(x) = \sum_{l=-M/2}^{M/2-1} \tilde{N}^n_l \Phi_l(x), \quad \tilde{N}^n_M(x) = \sum_{l=-M/2}^{M/2-1} \tilde{N}^n_l \Phi_l(x),
$$

where, for $n \geq 1$

$$
\tilde{N}^{n+1}_l = 2 \cos(\xi_l t) \tilde{N}^n_l - \tilde{N}^{n-1}_l - \frac{2}{1+\epsilon^2 \mu^2 l^2} \tilde{G}_l^n(0)(1 - \cos(\xi_l t)),
$$
$$
\tilde{N}^{n+1}_l = \tilde{N}^{n-1}_l - 2 \xi_l \sin(\xi_l t) \tilde{N}^n_l - \frac{2|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \tilde{G}_l^n(0) \sin(\xi_l t).
$$

When $n = 0$, from the initial conditions (2), we have

$$
\tilde{E}_l^0 = (E_0)_l, \quad \tilde{N}^0_l = (N_0)_l, \quad \tilde{N}_l^0 = (N_1)_l.
$$

In order to evaluate the first step value $\tilde{N}^1_l$ and $\tilde{N}^1_l$, setting $w = t$ in (7) and (8), we get

$$
\tilde{N}_l(t_1) = \cos(\xi_l t) \tilde{N}_l(t_0) + \frac{\sin(\xi_l t)}{4l^2} \tilde{N}^0_l(t_0) - \frac{|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \int_0^t \tilde{G}_l^n(s) \sin(\xi_l (t - s)) ds,
$$
$$
\tilde{N}_l^{t_1} = -\xi_l \sin(\xi_l t) \tilde{N}_l(t_0) + \cos(\xi_l t) \tilde{N}_l^{t_0} - \frac{|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \tilde{G}_l^n(0) \sin(\xi_l t).
$$

Then, based on (11) and (13), approximating integrals via the Cautschi-type quadrature $[3, 22, 23, 30, 31, 45]$, we have

$$
\tilde{N}^1_l = \cos(\xi_l t) \tilde{N}^0_l + (\tau - \frac{\xi_l^2 \tau^3}{6}) \tilde{N}^0_l - \frac{1}{1+\epsilon^2 \mu^2 l^2} \tilde{G}_l^n(0)(1 - \cos(\xi_l t)),
$$
$$
\tilde{N}^1_l = -\xi_l \sin(\xi_l t) \tilde{N}^0_l + \cos(\xi_l t) \tilde{N}^0_l - \frac{|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \tilde{G}_l^n(0) \sin(\xi_l t),
$$

where we use the relation $\frac{\sin(\xi_l t)}{4l^2} = \tau - \frac{\xi_l^2 \tau^3}{6} + O(\tau^5)$.

In fact, the above procedure is not suitable for practice computing due to the difficulty in evaluating the integrals defining the Fourier transform coefficients in (11)-(14) via the integration given in (3). Here, by approximating the integrals in (11)-(14) by the numerical quadrature on the grids $\{x_j\}_{j=0}^{M-1}$, we present an efficient implementation by choosing interpolation stated in (3) rather than the projection (integration).

Let $N^n_j$ and $\tilde{N}^n_j$ be the approximations of $N(x_j, t_n)$ and $\partial_t N(x_j, t_n)$, respectively. Denote

$$
N^n_j = \sum_{l=-M/2}^{M/2-1} \tilde{N}^n_l \Phi_l(x_j), \quad \tilde{N}^n_j = \sum_{l=-M/2}^{M/2-1} \tilde{N}^n_l \Phi_l(x_j),
$$

Then an EWI-FP method for wave equation reads as follows:

$$
\tilde{N}^{n+1}_l = 2 \cos(\xi_l t) \tilde{N}^n_l - \tilde{N}^{n-1}_l - \frac{2}{1+\epsilon^2 \mu^2 l^2} \tilde{G}_l^n(0)(1 - \cos(\xi_l t)),
$$
$$
\tilde{N}^{n+1}_l = \tilde{N}^{n-1}_l - 2 \xi_l \sin(\xi_l t) \tilde{N}^n_l - \frac{2|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \tilde{G}_l^n(0) \sin(\xi_l t), \quad n \geq 1,
$$
$$
\tilde{E}_l^0 = (E_0)_l, \quad \tilde{N}^0_l = (N_0)_l, \quad \tilde{N}_l^0 = (N_1)_l,
$$
$$
\tilde{N}_l^0 = \cos(\xi_l t) \tilde{N}_l^0 + (\tau - \frac{\xi_l^2 \tau^3}{6}) \tilde{N}_l^0 - \frac{1}{1+\epsilon^2 \mu^2 l^2} \tilde{G}_l^n(0)(1 - \cos(\xi_l t)),
$$
$$
\tilde{N}_l^1 = -\xi_l \sin(\xi_l t) \tilde{N}_l^0 + \cos(\xi_l t) \tilde{N}_l^0 - \frac{|\mu_l|}{\sqrt{1+\epsilon^2 \mu^2 l^2}} \tilde{G}_l^n(0) \sin(\xi_l t).
$$
2.2. A TS-FP method for Schrödinger-like equation. For the Schrödinger-like equation in (2), we adopt the TS-FP method, which has been widely used in the literature [29, 44, 43]. The basic idea of the TS method for the nonlinear equations is to decompose a problem into linear and nonlinear subproblems on each time step. Considering the Strang splitting scheme for the Schrödinger-like equation, from time \( t = t_k \) to \( t = t_{k+1} \) by writing \( t = t_k + s \), we can split it in three splitting steps. One solves first

\[
i \partial_t E(x, t_k + s) + \partial^2 E(x, t_k + s) - \varepsilon^2 \partial^4 E(x, t_k + s) = 0, \tag{16}
\]

for a half time step \( \frac{\tau}{2} \), followed by solving the equation

\[
i \partial_t E(x, t_k + s) - N(x, t_k + s) E(x, t_k + s) = 0, \tag{17}
\]

for time step \( \tau \), and followed by integrating (16) for another \( \frac{\tau}{2} \).

For the linear subproblem (16), we discretize it in space by FP method as follows

\[
E_l(t) = \tilde{E}_l(t_{n}) e^{-i\xi_l^2 t (t - t_n)}, \quad t \in [t_n, t_{n+1}]. \tag{18}
\]

For the nonlinear subproblem (17), we have

\[
i \frac{d}{dt} \frac{E(x, t)}{E(x, t)} = N(x, t) dt, \quad t \in [t_n, t_{n+1}].
\]

Integrating the above equation from \( t_n \) to \( t_{n+1} \), and then approximating the integral on \([t_n, t_{n+1}]\) via trapezoidal rule, we obtain

\[
E(x, t_{n+1}) \approx E(x, t_{n}) e^{-i\tau N(x, t_{n}) / 2}. \tag{19}
\]

Based on (18) and (19), from \( t_n \) to \( t_{n+1} \) with \( n = 0, 1, \cdots \), the TS-FP method for solving the Schrödinger-like equation reads:

\[
E_j^n = \sum_{l=-M/2}^{M/2-1} E_{l}^{n} e^{-i\xi_l^2 \tau / 2} e^{i\mu_l (x_j - a)},
\]

\[
E_{j*}^n = E_j^* e^{-i\tau \frac{\xi_j^2}{2}},
\]

\[
E_{j}^{n+1} = \sum_{l=-M/2}^{M/2-1} (E_{l*}) e^{-i\xi_l^2 \tau / 2} e^{i\mu_l (x_j - a)}, \tag{20}
\]

where \( N_j^{n+1} \) is evaluated by (15), i.e.

\[
N_j^{n+1} = \sum_{l=-M/2}^{M/2-1} \tilde{N}_l^{n+1} e^{i\mu_l (x_j - a)}.
\]

**Theorem 2.1.** The discretizations (20) for Schrödinger-like equation posses the following property:

\[
h \sum_{j=0}^{M-1} |E_j^{n+1}|^2 = h \sum_{j=0}^{M-1} |E_j^n|^2, \quad n \geq 0.
\]

**Proof.** Using the interpolation properties and the third formula of (20), it holds that

\[
\tilde{E}_l^{n+1} = (E_{l*}) e^{-i\xi_l^2 \tau / 2}. \tag{21}
\]
Noticing the Parseval’s identity

\[ \sum_{l=-M/2}^{M/2-1} |\tilde{E}_n^l|^2 = \frac{1}{M} \sum_{j=0}^{M-1} |E_j^n|^2. \]

Thus, it follows from (20) and (21) that

\[

t = \frac{M}{2} - 1 \sum_{j=0}^{M-1} |E_j^n|^2 = h \sum_{j=0}^{M-1} \left| E_j^n e^{-0.5 \tau (N_j^n + N_{j+1}^n)} \right|^2
\]

\[
= \frac{M}{2} - 1 \sum_{j=0}^{M-1} \left| \tilde{E}_n^{l+1} e^{-0.5 \tau} \right|^2 = h \sum_{j=0}^{M-1} \left| E_j^n e^{-0.5 \tau} \right|^2
\]

and the proof is completed. □

**Remark 1.** It is worth pointing out that the QZS (2) is non-linearly coupled, while the proposed scheme (15)-(20) is decoupled and solving the non-linear algebraic equations is successfully avoided, which makes the computational efficiency greatly improved.

### 3. Extended to two-dimensional QZS.

We consider the two-dimensional QZS in a finite domain \( \Omega = (a, b) \times (c, d) \) with periodic boundary conditions. Denote \( h_1 := (b-a)/M, h_2 := (d-c)/K \) with \( M \) and \( K \) are even positive integers, and the grid points \( (x_j, y_k, t_n) := (a + j h_1, c + k h_2, n \tau) \) for \( 0 \leq j \leq M, 0 \leq k \leq K \) and \( n \geq 0 \).

Similar to Section 2, we first introduce the corresponding EWI-FP method for two-dimensional wave-type equation. Let \( N_{jk}^n \) and \( \tilde{N}_{jk}^n \) be the approximations of \( N(x_j, y_k, t_n) \) and \( N_t(x_j, y_k, t_n) \), respectively. Denote

\[
N_{jk}^n = \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \tilde{N}_{lm}^n e^{i \mu l (x_j - a)} e^{i \nu m (y_k - c)},
\]

\[
\tilde{N}_{jk}^n = \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \tilde{N}_{lm}^n e^{i \mu l (x_j - a)} e^{i \nu m (y_k - c)}.
\]
Theorem 3.1. The discretizations TS-FP (23) for two-dimensional Schrödinger-like equation possesses the following properties:

\[ Q^n+1 = Q^n, \quad n \geq 0, \]

where \( Q^n = h_1 h_2 \sum_{j=0}^{M-1} \sum_{k=0}^{K-1} |E^n_{jk}|^2. \)

Proof. Using the interpolation properties and the third formula of (23), it holds that

\[ \sum_{l=-M/2}^{K/2} \sum_{m=-K/2}^{M/2-1} |\tilde{E}^n_{lm}|^2 = \frac{1}{MK} \sum_{j=0}^{N-1} \sum_{k=0}^{K-1} |E^n_{jk}|^2. \]

Notice the Parseval’s identity

\[ \sum_{l=-M/2}^{K/2} \sum_{m=-K/2}^{M/2-1} |\tilde{E}^n_{lm}|^2 = \frac{1}{MK} \sum_{j=0}^{N-1} \sum_{k=0}^{K-1} |E^n_{jk}|^2. \]
Thus, it follows from (23) and (24) that

\[ Q^{n+1} = MKh_1 h_2 \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \left| \tilde{E}_{lm}^{n+1} \right|^2 \]

\[ = MKh_1 h_2 \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \left| (\tilde{E}^{**})_{lm} e^{-i\eta_m^2 \tau/2} \right|^2 \]

\[ = MKh_1 h_2 \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \left| (\tilde{E}^{**})_{lm} \right|^2 = h_1 h_2 \sum_{j=0}^{M-1} \sum_{k=0}^{K-1} |E_{jk}^*|^2 \]

\[ = h_1 h_2 \sum_{j=0}^{N-1} \sum_{k=0}^{K-1} \left| E_{jk}^* e^{-0.5i\tau(N_{jk}^n + N_{jk}^{n+1})} \right|^2 = h_1 h_2 \sum_{j=0}^{M-1} \sum_{k=0}^{K-1} |E_{jk}^*|^2 \]

\[ = MKh_1 h_2 \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \left| \tilde{E}_{lm}^{n} e^{-0.5i\eta_m^2 \tau} \right|^2 \]

\[ = MKh_1 h_2 \sum_{l=-M/2}^{M/2-1} \sum_{m=-K/2}^{K/2-1} \left| \tilde{E}_{lm}^{n} e^{-0.5i\eta_m^2 \tau} \right|^2 \]

and the proof is completed. \(\square\)

4. **Numerical examples.** Some numerical results are presented in this section. We present numerical examples including accuracy tests, solitary-wave collisions and pattern dynamics in one-dimensional QZS, as well as the blow up phenomena of two-dimensional QZS, to demonstrate the efficiency and accuracy of the explicit TS-EWI-FP method for QZS.

![Figure 1](attachment:conv_exp_mn.png)

**Figure 1.** Convergence of \(E\) (left) and \(N\) (right) between the QZS and classical ZS in Example 4.1.
Table 1. Spatial errors of the scheme at $T = 1$ for Example 4.1 with different $\varepsilon, \tau = 10^{-5}$.

| $\varepsilon$ | $h = 1$ | $1/2$ | $1/2^2$ | $1/2^3$ |
|---------------|---------|-------|---------|---------|
| $\frac{1}{2\pi}$ | 1.41e-2 | 6.83e-5 | 3.78e-9 | 4.77e-12 |
| $\frac{1}{\pi}$ | 4.27e-2 | 1.34e-4 | 4.80e-9 | 5.02e-12 |
| $\frac{1}{2\pi}$ | 5.01e-2 | 2.49e-4 | 4.94e-9 | 5.22e-12 |
| $\frac{1}{\pi}$ | 5.06e-2 | 2.60e-4 | 7.75e-9 | 5.25e-12 |
| $\frac{1}{2\pi}$ | 5.06e-2 | 2.61e-4 | 8.20e-9 | 5.25e-12 |
| $\frac{1}{\pi}$ | 5.06e-2 | 2.61e-4 | 8.20e-9 | 5.25e-12 |

4.1. **Accuracy test.** Let $E_{h,\tau}$ and $N_{h,\tau}$ be the numerical solution of QZS by using the TS-EWI-FP method with mesh size $h$ and time step $\tau$. To quantify the numerical methods, we define the error functions as

$$e_\varepsilon = \| E(\cdot, T) - E_{h,\tau}(\cdot, T) \|_{L^\infty}, \quad n_\varepsilon = \| N(\cdot, T) - N_{h,\tau}(\cdot, T) \|_{L^\infty}. \quad (25)$$

**Example 4.1.** Consider the QZS (2) with initial conditions

$$E_0(x) = i\sqrt{1.5\text{sech}(x)}e^{x/4},$$
$$N_0(x) = -2\text{sech}^2(x), \quad N_1(x) = -2\text{sech}^2(x)\tanh(x). \quad (26)$$

We solve the problem on the interval $[-32, 32]$. Table 1 shows the spatial errors of the TS-EWI-FP method at $T = 1$ under different $h$ and $\varepsilon$ with a very small time step $\tau = 10^{-5}$ such that the discretization error in time is negligible, and the reference solution is taken to be the numerical solution when sufficiently small spatial step $h = \frac{1}{2\pi}$ and time step $\tau = 10^{-5}$ are used. Table 2 display the temporal errors of the TS-EWI-FP method at $T = 1$ under different $\tau$ and $\varepsilon$ with a fixed mesh size $h = \frac{1}{2\pi}$, and the reference solution is taken to be the numerical solution when sufficiently small spatial step $h = \frac{1}{\pi}$ and time step $\tau = \frac{1}{2400}$ are used. It can be clearly observed that for solitary-wave solution, TS-EWI-FP method is really spectrally accurate in space and second-order in time.

Next, for the Example 4.1, we apply the TS-EWI-FP method to study numerically convergence rate of the QZS to its limiting model, i.e. the classical ZS ($\varepsilon = 0$). In order to do so, we choose the same initial and boundary conditions (26).

For fixed $h = \frac{1}{16}$, $\tau = \frac{1}{100}$, Figure 1 shows the errors between the solutions of the QZS and the corresponding classical ZS, which are obtained numerically by the
Table 2. Temporal errors of the scheme at $T = 1$ for Example 4.1 with different $\varepsilon$, $h = 1/2^4$.

| $\varepsilon$ | $\tau_0 = 1/20$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ |
|---------------|-----------------|-------------|-------------|-------------|-------------|-------------|
| $\varepsilon = 1/2^2$ | 1.85e-3 | 3.80e-4 | 7.31e-5 | 1.38e-5 | 3.55e-6 | 8.03e-7 |
| rate | - | 2.28 | 2.38 | 2.41 | 1.96 | 2.14 |
| $\varepsilon = 1/2^3$ | 6.81e-4 | 1.69e-4 | 4.22e-5 | 1.05e-5 | 2.63e-6 | 6.52e-7 |
| rate | - | 2.01 | 2.00 | 2.00 | 2.00 | 2.01 |
| $\varepsilon = 1/2^4$ | 7.56e-4 | 1.85e-4 | 4.61e-5 | 1.15e-5 | 2.87e-6 | 7.13e-7 |
| rate | - | 2.03 | 2.01 | 2.00 | 2.00 | 2.01 |

same TS-EWI-FP method at $T = 5$. It indicates that the solution of the QZS in Example 4.1 converges to the classical ZS quadratically with respect to $\varepsilon$, and it is consistent with the convergence rates of the exact result in [18].

4.2. Soliton-soliton collisions.

Example 4.2. The initial data are chosen as (cf. [50])

$$E_0(x) = i \sum_{j=1}^{2} \sqrt{2 \left(1 - v_j^2\right)} \text{sech} \left(x - x_j\right) e^{i v_j (x - x_j)/2},$$

$$N_0(x) = -2 \sum_{j=1}^{2} \text{sech}^2 \left(x - x_j\right), \quad N_1(x) = -4 \sum_{j=1}^{2} v_j \text{sech}^2 \left(x - x_j\right) \tanh \left(x - x_j\right),$$

(27)
which represents two solitary waves located at the initial positions $x = x_1$ and $x = x_2$, respectively, moving to the right or left depending on the sign of the propagation velocity $v_j (j = 1, 2)$. We test the following cases:

(i). $x_1 = -x_2 = -30$, $v_1 = -v_2 = 1/2$;
(ii). $x_1 = -x_2 = -30$, $v_1 = 3/4$, $v_2 = -1/2$;
(iii). $x_1 = -x_2 = -5$, $v_1 = 3/4$, $v_2 = -1/2$.

All computations are performed with $h = 200/1024$ and $\tau = 1/200$ on the interval $\Omega = [-100, 100]$. Figures 2-4 show the interaction of two solitary waves for the classical ZS ($\varepsilon = 0$) and QZS ($\varepsilon \neq 0$) under cases (i)-(iii), respectively. It can be clearly seen that all the soliton-soliton collisions are not elastic. First, we consider the collisions of two initially well-separated solitons move towards each other at the same propagation velocity (cf. case (i) in Figure 2), they collide and fuse into a static pulse with the strengthened amplitude and the narrower width. At the same time, two amplitude-weakened solitons with propagation velocities changed and some small radiation are generated during the collision. When the propagation velocities are different (cf. case (ii) in Figure 3), the static soliton is replaced by a moving solitary wave. Next, when the two solitons are not initially well-separated (cf. case (iii) in Figure 4), the dynamics is much more complicated. Unlike the cases (i) and (ii), the soliton remains as a soliton with constant density and propagation velocity after collision, and there is a periodic perturbation on the position of some
localized pulse. From Figures 2-4, we can draw the following observations. (1) The soliton-soliton collisions for the QZS differs from the classical ZS, and the quantum effect makes the chaos more obvious after collision. The larger the quantum effect is, the more obvious the chaos is after colliding. (2) There are small outgoing waves emitting before collision.

4.3. Pattern dynamics.

**Example 4.3.** The initial conditions are selected as (cf. [40])

\[ E_0(x) = E_0 (1 + \beta \cos(kx)), \]
\[ N_0(x) = -\sqrt{2} E_0 k \beta \cos(kx), \quad N_1(x) = 0, \]

(28)

where \( E_0 = (k/\sqrt{2})(1+\varepsilon^2 k^2) \) represents the amplitude of the pump Langmuir wave, \( 0 < k < \sqrt{2} E_0 \) and \( \beta \) is a small perturbation constant.

The QZS with the initial conditions (28) was simulated using TS-EWI-FP method with time step \( \tau = \frac{1}{200} \) and mesh size \( h = \frac{1}{8} \) on the interval \( \Omega = [-100, 100] \). Take \( T = 200, k = 0.7, \beta = 0.001 \). Figure 5 displays that many solitary pattern trains appear from the master mode and unstable harmonic modes. At the same time, solitons being strong after a long interval of time, and the central stationary soliton disappears right after collision that sets it into a few oscillations. It can be observed that the QZS is more unstable than in the classical ZS. For example, when \( \varepsilon = 1/8, \)]
Figure 4. Inelastic collision between two solitons in Example 4.2 under case (iii).

Figure 5 shows that two solitary patterns excited adjacent to the centre collide and fuse into another pattern with strengthened amplitude and narrower width, coexistent with master modes. Therefore the QZS is in the coexistence of spatiotemporal chaos and temporal chaos. The numerical results compare very well with the results presented in [40].

4.4. Two dimension case. We also apply the TS-EWI-FP method to investigate the accuracy and the blow-up phenomena of two-dimensional QZS.

Example 4.4. The initial conditions are selected as

\[ E_0(x, y) = \cos^2 \frac{\pi x}{8} \cos^2 \frac{\pi y}{8}, \quad N_0(x, y) = 0, \quad N_1(x, y) = 0, \]  

We confine this problem on a periodical cell \( \Omega = [-8, 8] \times [-8, 8] \). Since the exact solution is not known, we take the numerical solution obtained by the proposed TS-EWI-FP method with \( h_1 = h_2 = \frac{1}{32} \), \( \tau = \frac{1}{6400} \) as the reference solution. Table 3 shows the temporal errors of the TS-EWI-FP method at \( T = 1 \) under different \( \tau \) and \( \epsilon \) with a small spatial step \( h_1 = h_2 = \frac{1}{32} \). From Table 3, we observe that TS-EWI-FP method is really second-order in time.
Next, for the Example 4.4, we apply the TS-EWI-FP method to study numerically convergence rate of the QZS to its limiting model, i.e. the classical ZS ($\varepsilon = 0$). In order to do so, we choose the same initial and boundary conditions (29).

For fixed $h_1 = h_2 = \frac{1}{2\pi}$, $\tau = \frac{1}{400}$, Figure 6 shows the errors between the solutions of the QZS and the corresponding classical ZS, which are obtained numerically by the same TS-EWI-FP method at $T = 5$. It indicates that the solution of the QZS
Table 3. Temporal errors of the scheme at $T = 1$ for Example 4.4 with different $\varepsilon$, $h = 1/2^4$.

| $\varepsilon$ | $\tau_0 = 1/20$ | $\tau_0/2$ | $\tau_0/2^2$ | $\tau_0/2^3$ | $\tau_0/2^4$ | $\tau_0/2^5$ |
|---------------|------------------|-------------|---------------|---------------|---------------|---------------|
| $\frac{1}{2\pi}$ | 3.23e-2 | 8.06e-3 | 2.01e-3 | 5.02e-4 | 1.25e-4 | 3.11e-5 | 2.00 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.01 |
| $\frac{1}{3\pi}$ | 2.01e-2 | 5.03e-3 | 1.26e-3 | 3.14e-4 | 7.83e-5 | 1.94e-5 | 2.01 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.01 |
| $\frac{1}{4\pi}$ | 1.92e-2 | 4.80e-3 | 1.20e-3 | 3.00e-4 | 7.49e-5 | 1.86e-5 | 2.01 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.01 |
| $\frac{1}{5\pi}$ | 1.92e-2 | 4.79e-3 | 1.20e-3 | 2.99e-4 | 7.47e-5 | 1.85e-5 | 2.01 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.01 |
| $\frac{1}{6\pi}$ | 1.92e-2 | 4.79e-3 | 1.20e-3 | 2.99e-4 | 7.47e-5 | 1.85e-5 | 2.01 |
| rate | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.01 |

Example 4.5. Consider a two-dimensional problem
\begin{align*}
iE_t + \Delta E - \varepsilon^2 \Delta^2 E + NE &= 0, \\
N_{tt} - \Delta N + \varepsilon^2 \Delta^2 N + \mu \Delta |E|^2 &= 0 \tag{30}
\end{align*}

with the initial conditions
\begin{align*}
E_0(x,y) = \frac{1}{\sqrt{\pi}} e^{-x^2+y^2}, \\
N_0(x,y) = \nu |E_0(x,y)|^2, \\
N_1(x,y) &= 0. \tag{31}
\end{align*}

When $\varepsilon = 0$, mathematically, the solution of the classical ZS may be blow-up in finite time [38], and this problem have been utilized in [6, 34] to demonstrate their numerical behavior. Moreover, taking quantum effects into account, we apply the TS-EWI-FP method to investigate the blow-up phenomena of the QZS (30)-(31).
We solve the QZS (30)-(31) on the rectangle $\Omega = [-8, 8] \times [-8, 8]$ with mesh size $h_1 = h_2 = \frac{1}{2^4}$ and time step $\tau = \frac{1}{1000}$. When $\mu = \nu = 20$ and $\varepsilon = \frac{1}{2^5}$, Figure 7 shows the surface plots of electron density $|E|^2$ and ion density fluctuation $N$ at different times. One can see that a singularity indeed starts to form as time evolves.

5. **Conclusion.** In this paper, we formulate an efficient and accurate TS-EWI-FP method for computing QZS (1). The scheme is fully explicit and efficient due to
the fast Fourier transform. This scheme is spectral order in space and second-order in time. Numerical examples have been performed to show the numerical accuracy. In addition, some applications, such as convergence of the QZS to the classical ZS in the semi-classical limit, soliton-soliton collisions, pattern dynamics of one-dimensional QZS and the blow-up phenomena of two-dimensional QZS, have also been simulated to show the capability of the TS-EWI-FP method for the QZS.

**Acknowledgments.** The author would like to specially thank Professor Weizhu Bao for his valuable suggestions and comments.

**REFERENCES**

[1] W. Bao and X. Dong, Analysis and comparison of numerical methods for Klein-Gordon equation in nonrelativistic limit regime, *Numer. Math.*, 120 (2012), 189–229.

[2] W. Bao, X. Dong and X. Zhao, An exponential wave integrator sine pseudospectral method for the Klein-Gordon-Zakharov system, *SIAM J. Sci. Comput.*, 35 (2013), A2903–A2927.

[3] W. Bao, X. Dong and X. Zhao, Uniformly accurate multiscale time integrators for highly oscillatory second order differential equations, *J. Math. Study*, 47 (2014), 111–150.

[4] W. Bao and C. Su, Uniform error bounds of a finite difference method for the Zakharov system in the subsonic limit regime via an asymptotic consistent formulation, *Multiscale Model. Simul.*, 15 (2017), 977–1002.

[5] W. Bao and C. Su, A uniformly and optically accurate method for the Zakharov system in the subsonic limit regime, *SIAM J. Sci. Comput.*, 40 (2018), A929–A953.

[6] W. Bao and F. Sun, Efficient and stable numerical methods for the generalized and vector Zakharov system, *SIAM J. Sci. Comput.*, 26 (2005), 1057–1088.

[7] W. Bao, F. Sun and G. W. Wei, Numerical methods for the generalized Zakharov system, *J. Comput. Phys.*, 190 (2003), 201–228.

[8] W. Bao and X. Zhao, A uniformly accurate multiscale time integrator spectral method for the Klein-Gordon-Zakharov system in the high-plasma-frequency limit regime, *J. Comput. Phys.*, 327 (2016), 270–293.

[9] W. Bao and X. Zhao, Comparison of numerical methods for the nonlinear Klein-Gordon equation in the nonrelativistic limit regime, *J. Comput. Phys.*, 398 (2019), 108886, 30 pp.

[10] Y. Cai and Y. Yuan, Uniform error estimates of the conservative finite difference method for the Zakharov system in the subsonic limit regime, *Math. Comp.*, 87 (2018), 1191–1225.

[11] Q. Chang, B. Guo and H. Jiang, Finite difference method for generalized Zakharov equations, *Math. Comput.*, 64 (1995), 537–553, S7–S11.

[12] B. J. Choi, Global well-posedness of the adiabatic limit of quantum Zakharov system in 1D, preprint, (2019), arXiv:1906.10807v2.

[13] A. S. Davydov, Solitons in molecular systems, *Phys. Scr.*, 20 (1979), 387–394.

[14] L. M. Degtyarev, V. G. Nakhankov and L. I. Rudakov, Dynamics of the formation and interaction of Langmuir solitons and strong turbulence, *Sov. Phys. JETP*, 40 (1974), 264–268.

[15] Y. Fang, H. Shih and K. Wang, Local well-posedness for the quantum Zakharov system in one spatial dimension, *J. Hyperbolic Differ. Equ.*, 14 (2017), 157–192.

[16] Y. Fang, J. Segata and T. Wu, On the standing waves of quantum Zakharov system, *J. Math. Anal. Appl.*, 458 (2018), 1427–1448.

[17] Y. Fang and K. Nakanishi, Global well-posedness and scattering for the quantum Zakharov system in $L^2$, *Proc. Amer. Math. Soc.*, 6 (2019), 21–32.

[18] Y. Fang, H. Kuo, H. Shih and K. Wang, Semi-classical limit for the quantum Zakharov system, *Taiwan. J. Math.*, 23 (2019), 925–949.

[19] L. G. Garcia, F. Haas, L. P. L. de Oliveira and J. Goedert, Modified Zakharov equations for plasmas with a quantum correction, *Phys. Plasmas*, 12 (2005), 012302.

[20] L. Gauckler, On a splitting method for the Zakharov system, *Numer. Math.*, 139 (2018), 349–379.

[21] R. T. Glassey, Approximate solutions to the Zakharov equations via finite differences, *J. Comput. Phys.*, 100 (1992), 377–383.

[22] V. Grimm, On error bounds for the Gautschi-type exponential integrator applied to oscillatory second-order differential equations, *Numer. Math.*, 100 (2005), 71–89.
[23] V. Grimm, A note on the Gautschi-type method for oscillatory second-order differential equations, Numer. Math., 102 (2005), 61–66.
[24] Y. Guo, J. Zhang and B. Guo, Global well-posedness and the classical limit of the solution for the quantum Zakharov system, Z. Angew. Math. Phys., 64 (2013), 53–68.
[25] B. Guo, Z. Gan, L. Kong and J. Zhang, The Zakharov System and its Soliton Solutions, Science Press, Beijing, 2016.
[26] F. Haas, Variational approach for the quantum Zakharov system, Phys. Plasmas, 14 (2007), 042309.
[27] F. Haas and P. K. Shukla, Quantum and classical dynamics of Langmuir wave packets, Phys. Rev. E, 79 (2009), 066402.
[28] F. Haas, Quantum Plasmas: An Hydrodynamic Approach, Springer Series on Atomic, Optical, and Plasma Physics, 65, Springer, New York, 2011.
[29] H. Hofstätter, O. Koch and M. Thalhammer, Convergence analysis of high-order time-splitting pseudo-spectral methods for rotational Gross-Pitaevskii equations, Numer. Math., 127 (2014), 315–364.
[30] M. Hochbruck and A. Ostermann, Exponential integrators, Acta Numer., 19 (2010), 209–286.
[31] M. Hochbruck and C. H. Lubich, A Gautschi-type method for oscillatory second-order differential equations, Numer. Math., 83 (1999), 403–426.
[32] J.-C. Jiang, C. K. Lin and S. Shao, On one dimensional quantum Zakharov system, Discrete Contin. Dyn. Syst., 36 (2016), 5445–5475.
[33] S. Jin, P. A. Markowich and C. Zheng, Numerical simulation of a generalized Zakharov system, J. Comput. Phys., 201 (2004), 376–395.
[34] S. Jin and C. Zheng, A Time-splitting spectral method for the generalized Zakharov system in multi-dimensions, J. Sci. Comput., 26 (2006), 127–149.
[35] X. Li and L. Zhang, Error estimates of a trigonometric integrator sine pseudo-spectral method for the extended Fisher-Kolmogorov equation, Appl. Numer. Math., 131 (2018), 39–53.
[36] F. Liao, L. Zhang and S. Wang, Time-splitting combined with exponential wave integrator fourier pseudospectral method for Schrödinger-Boussinesq system, Commun. Nonlinear Sci. Numer. Simulat., 55 (2018), 93–104.
[37] M. Marklund, Classical and quantum kinetics of the Zakharov system, Phys. Plasmas, 12 (2005), 082110, 5 pp.
[38] V. Masselin, A result of the blow-up rate for the Zakharov system in dimension 3, SIAM J. Math. Anal., 33 (2001), 440–447.
[39] A. P. Misra, D. Ghosh and A. R. Chowdhury, A novel hyperchaos in the quantum Zakharov system for plasmas, Phys. Lett. A, 372 (2008), 1469–1476.
[40] A. P. Misra and P. K. Shukla, Pattern dynamics and spatiotemporal chaos in the quantum Zakharov equations, Phys. Rev. E, 79 (2009), 056401.
[41] G. C. Papanicolaou, C. Sulem, P. L. Sulem and X. P. Wang, Singular solutions of the Zakharov equations for Langmuir turbulence, Phys. Fluids B, 3 (1991), 969–980.
[42] C. Su and X. Zhao, A uniformly first-order accurate method for Klein-Gordon-Zakharov system in simultaneous high-plasma-frequency and subsonic limit regime, J. Comput. Phys., 428 (2021), 110064, 22 pp.
[43] A. Taleei and M. Dehghan, Time-splitting pseudo-spectral domain decomposition method for the soliton solutions of the one- and multi-dimensional nonlinear Schrödinger equations, Comput. Phys. Commun., 185 (2014), 1515–1528.
[44] S. Wang, T. Wang and L. Zhang, Numerical computations for N-coupled nonlinear Schrödinger equations by split step spectral methods, Appl. Math. Comput., 222 (2013), 438–452.
[45] Y. Wang and X. Zhao, Symmetric high order Gautschi-type exponential wave integrators pseudospectral method for the nonlinear Klein-Gordon equation in the nonrelativistic limit regime, Int. J. Numer. Anal. Mod., 15 (2018), 405–427.
[46] Y. Xia, Y. Xu and C. Shu, Local discontinuous Galerkin methods for the generalized Zakharov system, J. Comput. Phys., 229 (2010), 1238–1259.
[47] A. Xiao, C. Wang and J. Wang, Conservative linearly-implicit difference scheme for a class of modified Zakharov systems with high-order space fractional quantum correction, Appl. Numer. Math., 146 (2019), 379–399.
[48] S. Yao, J. Sun and T. Wu, Stationary quantum Zakharov systems involving a higher competing perturbation, Electron. J. Differential Equations, 2020 (2020), 18 pp.
[49] V. E. Zakharov, Collapse of langmuir waves, Sov. Phys. JETP, 35 (1972), 908–914.
[50] G. Zhang and C. Su, A conservative linearly-implicit compact difference scheme for the quantum Zakharov system, *J. Sci. Comput.*, **87** (2021), 71.

[51] X. Zhao, On error estimates of an exponential wave integrator sine pseudospectral method for the Klein-Gordon-Zakharov system, *Numer. Meth. Part. D. E.*, **32** (2016), 266–291.

Received for publication October 2020.

*E-mail address: zhanggen036@163.com*