Equitable Colorings of Planar Graphs without Short Cycles

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Abstract

An equitable coloring of a graph is a proper vertex coloring such that the sizes of every two color classes differ by at most 1. Chen, Lih, and Wu conjectured that every connected graph $G$ with maximum degree $\Delta \geq 2$ has an equitable coloring with $\Delta$ colors, except when $G$ is a complete graph or an odd cycle or $\Delta$ is odd and $G = K_{\Delta,\Delta}$. Nakprasit proved the conjecture holds for planar graphs with maximum degree at least 9. Zhu and Bu proved that the conjecture holds for every $C_3$-free planar graph with maximum degree at least 8 and for every planar graph without $C_4$ and $C_5$ with maximum degree at least 7.

In this paper, we prove that the conjecture holds for planar graphs in various settings, especially for every $C_3$-free planar graph with maximum degree at least 6 and for every planar graph without $C_4$ with maximum degree at least 7, which

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improve or generalize results on equitable coloring by Zhu and Bu. Moreover, we prove that the conjecture holds for every planar graph of girth at least 6 with maximum degree at least 5.

**Key Words:** Equitable coloring; Planar graph; Cycle; Girth

# 1 Introduction

Throughout this paper, all graphs are finite, undirected, and simple. We use $V(G)$, $|G|$, $E(G)$, $e(G)$, $\Delta(G)$, and $\delta(G)$, respectively, to denote vertex set, order, edge set, size, maximum degree, and minimum degree of a graph $G$. We write $xy \in E(G)$ if $x$ and $y$ are adjacent. The graph obtained by deleting an edge $xy$ from $G$ is denoted by $G - \{xy\}$. For any vertex $v$ in $V(G)$, let $N_G(v)$ be the set of all neighbors of $v$ in $G$. The *degree* of $v$, denoted by $d_G(v)$, is equal to $|N_G(v)|$. We use $d(v)$ instead of $d_G(v)$ if no confusion arises. For disjoint subsets $U$ and $W$ of $V(G)$, the number of edges with one end in $U$ and another in $W$ is denoted by $e(U,W)$. We use $G[U]$ to denote the subgraph of $G$ induced by $U$.

An *equitable* $k$-coloring of a graph is a proper vertex $k$-coloring such that the sizes of every two color classes differ by at most 1. We say that $G$ is equitably $k$-colorable if $G$ has an equitable $k$-coloring.

It is known [2] that determining if a planar graph with maximum degree 4 is 3-colorable is NP-complete. For a given $n$-vertex planar graph $G$ with maximum degree 4, let $G'$ be a graph obtained from $G$ by adding $2n$ isolated vertices. Then $G$ is 3-colorable if and only if $G'$ is equitably 3-colorable. Thus, finding the minimum number of colors need to color a graph equitably even for a planar graph is an NP-complete problem.

Hajnal and Szemerédi [4] settled a conjecture of Erdős by proving that every graph $G$ with maximum degree at most $\Delta$ has an equitable $k$-coloring for every $k \geq 1 + \Delta$. In its ‘complementary’ form this result concerns decompositions of a sufficiently dense graph into cliques of equal size. This result is now known as Hajnal
and Szemerédi Theorem. Later, Kierstead and Kostochka [6] gave a simpler proof of Hajnal and Szemerédi Theorem in the direct form of equitable coloring. The bound of the Hajnal-Szemerédi theorem is sharp, but it can be improved for some important classes of graphs. In fact, Chen, Lih, and Wu [11] put forth the following conjecture.

**Conjecture 1.** Every connected graph $G$ with maximum degree $\Delta \geq 2$ has an equitable coloring with $\Delta$ colors, except when $G$ is a complete graph or an odd cycle or $\Delta$ is odd and $G = K_{\Delta, \Delta}$.

Lih and Wu [8] proved the conjecture for bipartite graphs. Meyer [9] proved that every forest with maximum degree $\Delta$ has an equitable $k$-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$ colors. This result implies conjecture holds for forests. The bound of Meyer is attained at the complete bipartite $K_{1,m}$: in every proper coloring of $K_{1,m}$, the center vertex forms a color class, and hence the remaining vertices need at least $m/2$ colors. Yap and Zhang [13] proved that the conjecture holds for outerplanar graphs. Later Kostochka [5] extended the result for outerplanar graphs by proving that every outerplanar graph with maximum degree $\Delta$ has an equitable $k$-coloring for each $k \geq 1 + \lceil \Delta/2 \rceil$. Again this bound is sharp.

In [14], Zhang and Yap essentially proved the conjecture holds for planar graphs with maximum degree at least 13. Later Nakprasit [10] extended the result to all planar graphs with maximum degree at least 9.

Other studies focused on planar graphs without some restricted cycles. Li and Bu [7] proved that the conjecture holds for every planar graph without $C_4$ and $C_6$ with maximum degree at least 6. Zhu and Bu [15] proved that the conjecture holds for every $C_3$-free planar graph with maximum degree at least 8 and for every planar graph without $C_4$ and $C_5$ with maximum degree at least 7. Tan [11] proved that the conjecture holds for every planar graph without $C_4$ with maximum degree at least 7. Unfortunately the proof contains some flaws.

In this paper, we prove that each graph $G$ in various settings has an equitably $m$-colorable such that $m \leq \Delta$. Especially we prove that the conjecture holds for planar graphs in various settings, especially for every $C_3$-free planar graph with maximum...
degree at least 6 and for every planar graph without $C_4$ with maximum degree at
least 7, which improve or generalize results on equitable coloring by Zhu and Bu [15].
Moreover, we prove that the conjecture holds for every planar graph of girth at least
6 with maximum degree at least 5.

2 Preliminaries

Many proofs in this paper involve edge-minimal planar graph that is not equitably
$m$-colorable. The minimality is on inclusion, that is, any spanning subgraph with
fewer edges is equitably $m$-colorable. In this section, we describe some properties
of such graph that appear recurrently in later arguments. The following fact about
planar graphs in general is well-known and can be found in standard texts about
graph theory such as [12].

Lemma 1. Every planar graph $G$ of order $n$ and girth $g$ has $e(G) \leq (g/(g-2))(n-2)$. Especially, a $C_3$-free planar graph $G$ has $e(G) \leq 2n - 4$ and $\delta(G) \leq 3$.

Let $G$ be an edge-minimal $C_3$-free planar graph that is not equitably $m$-colorable
with $|G| = mt$, where $t$ is an integer. As $G$ is planar and without $C_3$, a graph $G$ has
an edge $xy$ where $d(x) = \delta \leq 3$. By edge-minimality of $G$, the graph $G - \{xy\}$ has
an equitable $m$-coloring $\phi$ having color classes $V'_1, V'_2, \ldots, V'_m$. It suffices to consider
only the case that $x, y \in V'_1$. Choose $x \in V'_1$ such that $x$ has degree $\delta$ and order
$V'_1, V'_2, \ldots, V'_i$ in a way that $N(x) \subset V'_1 \cup V'_2 \cup \cdots \cup V'_i$. Define $V_1 = V'_1 - \{x\}$ and
$V_i = V'_i$ for $1 \leq i \leq m$.

We define $\mathcal{R}$ recursively. Let $V_i \in \mathcal{R}$ and $V_j \in \mathcal{R}$ if there exists a vertex in $V_j$
which has no neighbors in $V_i$ for some $V_i \in \mathcal{R}$. Let $r = |\mathcal{R}|$. Let $A$ and $B$ denote
$\bigcup_{V_i \in \mathcal{R}} V_i$ and $V(G) - A$, respectively. Furthermore, we let $A'$ denote $A \cup \{x\}$ and $B'$
denote $B - \{x\}$. From definitions of $\mathcal{R}$ and $B$, $e(V_i, \{u\}) \geq 1$ for each $V_i \in \mathcal{R}$ and
$u \in B$. Consequently $e(A, B) \geq r[(m - r)t + 1]$ and $e(A', B') \geq r(m - r)t$.

Suppose that there is $k$ such that $k \geq \delta + 1$ and $V_k \in \mathcal{R}$. By definition of $\mathcal{R}$,
there exist $u_1 \in V_{i_1}, u_2 \in V_{i_2}, \ldots, u_s \in V_{i_s}, u_{i_{s+1}} \in V_{i_{s+1}} = V_k$ such that
$e(V_1, \{u_1\}) = \ldots = e(V_k, \{u_k\}) = \ldots = e(V_s, \{u_s\}) = \ldots = e(V_{i_{s+1}}, \{u_{i_{s+1}}\}) =$
\( e(V_1, \{u_2\}) = \cdots = e(V_s, \{u_{s+1}\}) = 0 \). Letting \( W_1 = V_1 \cup \{u_1\}, W_i = (V_i \cup \{u_2\}) - \{u_1\}, \ldots , W_i = (V_i \cup \{u_{s+1}\}) - \{u_s\}, \) and \( W_k = (V_k \cup \{x\}) - \{u_{s+1}\} \), otherwise \( W_i = V_i \), we get an equitable \( m \)-coloring of \( G \). This contradicts to the fact that \( G \) is a counterexample.

Thus, in case of \( C_3 \)-free planar graph, we assume \( \mathcal{R} \subseteq \{V_1, V_2, \ldots , V_\delta \} \) where \( \delta \leq 3 \) is the minimum degree of non-isolated vertices.

We summarize our observations here.

**Observation 2.** If \( G \) is an edge-minimal \( C_3 \)-free planar graph that is not equitably \( m \)-colorable with order \( mt \), where \( t \) is an integer, then we may assume

(i) \( \mathcal{R} \subseteq \{V_1, V_2, \ldots , V_\delta \} \) where \( \delta \leq 3 \) is the minimum degree of non-isolated vertices;

(ii) \( e(u, V_i) \geq 1 \) for each \( u \in B \) and \( V_i \in \mathcal{R} \);

(iii) \( e(A, B) \geq r[(m - r)t + 1] \) and \( e(A', B') \geq r(m - r)t \).

### 3 Results on \( C_3 \)-free Planar Graphs

First, we introduce some useful tools and notation that will be used later.

**Theorem 3.** \( \exists \) (Grötzsch, 1959) If \( G \) is a \( C_3 \)-free planar graph, then \( G \) is 3-colorable.

**Lemma 4.** Let \( m \) be a fixed integer with \( m \geq 1 \). Suppose that any \( C_3 \)-free planar graph of order \( mt \) with maximum degree at most \( \Delta \) is equitably \( m \)-colorable for any integer \( t \geq k \). Then any \( C_3 \)-free planar graph with order at least \( kt \) and maximum degree at most \( \Delta \) is also equitably \( m \)-colorable.

**Proof.** Suppose that any \( C_3 \)-free planar graph of order \( mt \) with maximum degree at most \( \Delta \) is equitably \( m \)-colorable for any integer \( t \geq k \). Consider a \( C_3 \)-free planar graph \( G \) of order \( mt + r \) where \( 1 \leq r \leq m - 1 \) and \( t \geq k \). If \( r = m - 1 \) or \( m - 2 \), then \( G \cup K_{m-r} \) is equitably \( m \)-colorable by hypothesis. Thus also is \( G \). Consider \( r \leq m - 3 \). Let \( x \) be a vertex with minimum degree \( d \). We assume that \( G - \{x\} \) is equitably \( m \)-colorable to use induction on \( r \). Thus the coloring of \( G - \{x\} \) has \( r + 1 \)
color classes with size \( t - 1 \). Since there are at most \( d \) forbidden colors for \( x \) where \( d \leq 3 \), we can add \( x \) to a color class of size \( t - 1 \) to form an equitable \( m \)-coloring of \( G \). This completes the proof \( \Box \)

**Lemma 5.** \cite{1} If \( G \) is a graph with maximum degree \( \Delta \geq |G|/2 \), then \( G \) is equitably \( \Delta \)-colorable.

**Observation 6.** By Lemmas 4 and 5, for proving that the conjecture holds for \( C_3 \)-free planar graphs it suffices to prove only \( C_3 \)-free planar graphs of order \( \Delta t \) where \( t \geq 3 \) is a positive integer.

**Lemma 7.** \cite{14} Let \( G \) be a graph of order \( mt \) with chromatic number \( \chi \) such that \( \chi \leq m \), where \( t \) is an integer. If \( e(G) \leq (m - 1)t \), then \( G \) is equitably \( m \)-colorable.

**Lemma 8.** Suppose \( G \) is a \( C_3 \)-free planar graph with \( \Delta(G) = \Delta \). If \( G \) has an independent \( s \)-set \( V' \) and there exists \( U \subseteq V(G) - V' \) such that \( |U| > s(1 + \Delta)/2 \) and \( e(u, V') \geq 1 \) for all \( u \in U \), then \( U \) contains two nonadjacent vertices \( \alpha \) and \( \beta \) which are adjacent to exactly one and the same vertex \( \gamma \in V' \).

**Proof.** Let \( U_1 \) consist of vertices in \( U \) with exactly one neighbor in \( V' \). If \( r = |U_1| \), then \( r + 2(|U| - r) \leq \Delta s \) which implies \( r \geq 2|U| - \Delta s > s \). Consequently, \( V' \) contains a vertex \( \gamma \) which has at least two neighbors in \( U_1 \). Since \( G \) is \( C_3 \)-free, this two neighbors are not adjacent. Thus \( U_1 \) contains two nonadjacent vertices \( \alpha \) and \( \beta \) which are adjacent to exactly one and the same vertex \( \gamma \in V' \). \( \Box \)

**Lemma 9.** \cite{10} If a graph \( G \) has an independent \( s \)-set \( V' \) and there exists \( U \subseteq V(G) - V' \) such that \( e(u, V') \geq 1 \) for all \( u \in U \), and \( e(G[U]) + e(V', U) < 2|U| - s \), then \( U \) contains two nonadjacent vertices \( \alpha \) and \( \beta \) which are adjacent to exactly one and the same vertex \( \gamma \in V' \).

**Notation.** Let \( q_{m, \Delta, t} \) denote the maximum number not exceeding \( 2mt - 4 \) such that each \( C_3 \)-free planar graph of order \( mt \), where \( t \) is an integer, is equitably \( m \)-colorable if it has maximum degree at most \( \Delta \) and size at most \( q_{m, \Delta, t} \).
The next Lemma is similar to that in [10] except that we use $V_1$ instead of $V_1'$ which is erratum. Nevertheless later arguments in [10] stand correct.

**Lemma 10.** Let $G$ be an edge-minimal $C_3$-free planar graph that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. If $e(G) \leq (r+1)(m-r)t - t + 2 + q_{r,\Delta,t}$, then $B$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1$.

**Proof.** If $e(G[A']) \leq q_{r,\Delta,t}$, then $G[A']$ is equitably $r$-colorable. Consequently, $G$ is equitably $m$-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta,t} + 1$. By Observation 2, $e(A'-V_1', B') \geq (r-1)(m-r)t$. Note that $e(G[B]) = e(G[B'])$, $e(V_1, B) = e(V_1', B') + 1$. So $e(G[B]) + e(V_1, B) = e(G[B']) + e(V_1', B') + 1 = e(G) - e(G[A']) - e(A'-V_1', B') + 1 < 2mt - 2rt - t + 3 = 2|B| - |V_1|$. By Lemma 9, $B$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1$.

**Lemma 11.** Let $G$ be an edge-minimal $C_3$-free planar graph that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. If $B$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m-r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 4$.

**Proof.** Suppose $e(G) \leq r(m-r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 3$. If $e(G[A']) \leq q_{r,\Delta,t}$, then $G[A']$ is equitably $r$-colorable. Consequently, $G$ is equitably $m$-colorable. So we suppose $e(G[A']) \geq q_{r,\Delta,t} + 1$. This with Observation 2 implies $e(G[A']) + e(A, B') \geq q_{r,\Delta,t} + 1 + r(m-r)t$. Note that $e(G[A']) + e(A, B') = e(G[A]) + e(A, B)$. Let $A_1 = (A - \{\gamma\}) \cup \{\alpha, \beta\}$ and $B_1 = (B \cup \{\gamma\}) - \{\alpha, \beta\}$. Then $e(G[A_1]) + e(A_1, B_1) \geq e(G[A]) + e(A, B) - \Delta + 2 \geq q_{r,\Delta,\gamma} + 1 + r(m-r)t - \Delta + 2$. So $e(G[B_1]) = e(G) - e(G[A_1]) + e(A_1, B_1) \leq q_{m-r,\Delta}, \gamma$ which implies $G[B_1]$ is equitably $(m-r)$-colorable. Combining with $(V_1 - \{\gamma\}) \cup \{\alpha, \beta\}, V_2, \ldots, V_r$, we have $G$ equitably $m$-colorable which is a contradiction. 

**Corollary 12.** Let $G$ be an edge-minimal $C_3$-free planar graph that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. 


Then \( e(G) \geq r(m - r)t + q_{r,\Delta,t} + q_{m-r,\Delta,t} - \Delta + 4 \) if one of the following conditions are satisfied:

(i) \((m - r)t + 1 > (t - 1)(1 + \Delta)/2;\)

(ii) \(e(G) \leq (r + 1)(m - r)t - t + 2 + q_{r,\Delta,t}.\)

**Proof.** This is a direct consequence of Lemmas 8, 10, and 11.

Now we are ready to work on \( C_3 \)-free planar graphs.

**Lemma 13.** (i) \( q_{1,\Delta,t} = 0. \) (ii) \( q_{2,\Delta,t} \geq 3 \) for \( t \geq 3. \) (iii) \( q_{3,\Delta,t} \geq 2t. \)

**Proof.** (i) and (ii) are obvious. (iii) is the result of Theorem 3 and Lemma 7.

**Lemma 14.** \( q_{4,\Delta,t} \geq \min\{q_{3,\Delta,t} + 3t + 3 - \Delta, 4t - \Delta + 9\} \) for \( \Delta \geq 5 \) and \( t \geq 3. \)

**Proof.** Consider \( \Delta \geq 5 \) and \( t \geq 3. \) Suppose \( G' \) is a \( C_3 \)-free planar graph with maximum degree at most \( \Delta \) and \( e(G') \leq \min\{q_{3,\Delta,t} + 3t + 3 - \Delta, 4t - \Delta + 9\} \) but \( G' \) is not equitably 4-colorable. Let \( G \subseteq G' \) be an edge-minimal graph that is not equitably 4-colorable. From Table 1 \( e(G) > e(G'). \) This contradiction completes the proof.

| \( r \) | lower bounds on size | Reasons |
|---|---|---|
| 3 | \( q_{3,\Delta,t} + 3t + 3 - \Delta \) or \( q_{3,\Delta,t} + 3t + 2 \) | Corollary 12(ii), Lemma 13 |
| 2 | \( 4t - \Delta + 9 \) or \( 5t + 5 \) | Corollary 12(ii), Lemma 13 |
| 1 | \( q_{3,\Delta,t} + 3t + 3 - \Delta \) or \( 5t + 2 \) | Corollary 12(ii), Lemma 13 |

Table 1: Lower bounds on size of \( G \) in the proof of Lemma 14.

**Lemma 15.** \( q_{5,\Delta,t} \geq \min\{q_{3,\Delta,t} + 6t + 6 - \Delta, q_{4,\Delta,t} + 4t + 3 - \Delta, 7t + 2\} \) for \( \Delta \geq 5 \) and \( t \geq 3. \)

**Proof.** Use Table 2 for an argument similar to the proof of Lemma 14.
| r  | lower bounds on size                                      | Reasons                   |
|----|----------------------------------------------------------|---------------------------|
| 3  | \(q_{3,\Delta,t} + 6t + 6 - \Delta\) or \(q_{3,\Delta,t} + 7t + 2\) | Corollary 12(ii), Lemma 13 |
| 2  | \(q_{3,\Delta,t} + 6t + 6 - \Delta\) or \(8t + 5\)         | Corollary 12(ii), Lemma 13 |
| 1  | \(q_{4,\Delta,t} + 4t + 3 - \Delta\)                     | Corollary 12(i), Lemma 13  |

Table 2: Lower bounds on size of \(G\) in the proof of Lemma 15

Corollary 16. (1) \(q_{4,6,t}\) is at least \(5t - 3\) and \(4t + 3\) for \(t\) at least 3 and 6, respectively.
(2) \(q_{4,7,t}\) is at least \(5t - 4\) and \(4t + 2\) for \(t\) at least 3 and 6, respectively.
(3) \(q_{5,6,t}\) is at least \(9t - 6\) and \(8t\) for \(t\) at least 3 and 6, respectively.
(4) \(q_{5,7,t}\) is at least \(9t - 8\) and \(8t - 2\) for \(t\) at least 3 and 6, respectively.

Proof. The results can be calculated directly from Lemmas 13 to 15.

Corollary 17. Each \(C_3\)-free planar graph \(G\) with maximum degree at most 7 and \(|G| \geq 18\) has an equitable 6-coloring. Moreover, each \(C_3\)-free planar graph \(G\) with maximum degree 6 has an equitable 6-coloring.

Proof. Let \(G\) be an edge-minimal \(C_3\)-free planar graph that is not equitably \(\Delta\)-colorable with \(|G| = 6t\), where \(t\) is an integer at least 3, and maximum degree at most 7.

Consider the case \(r = 3\). By Corollaries 12(ii) and 16, \(e(G) > \min\{2q_{3,\Delta,t} + 9t + 3 - \Delta, q_{3,\Delta,t} + 11t + 2\} \geq 13t - 4 \geq 12t - 4\).

Consider the case \(r = 2\). By Corollary 12(i), \(e(G) > q_{4,\Delta,t} + 8t + 6 - \Delta\). It follows from Corollary 16 that \(e(G) > \min\{13t - 5, 12t + 1\} \geq 12t - 4\) for \(t \geq 3\).

Consider the case \(r = 1\). We have \(e(B', V_1) \geq 5t\) by Observation 2. But \(y\) has at most \(\Delta - 1\) neighbors in \(B'\) because \(xy \in E(G)\), so \((t - 1)\Delta - 1 \geq e(B', V_1)\).

Consequently, \((t - 1)\Delta - 1 \geq 5t\). That is \(t \geq 4\) when \(\Delta \leq 7\). By Corollary 12(i), \(e(G) > q_{5,\Delta,t} + 5t - 4\). Using Corollary 16, we have \(e(G) > \min\{14t - 12, 13t - 6\}\). It follows from \(t \geq 4\) that \(e(G) > 12t - 4\).

Since we have contradiction for all cases, the counterexample is impossible. Use Lemma 4 to complete the first part of the proof.
Observation 6 implies each $C_3$-free planar graph $G$ with maximum degree 6 has an equitable 6-coloring.

Note that a graph $G$ in Corollary 17 has an equitable $m$-coloring with $m < \Delta(G)$.

**Lemma 18.** Each $C_3$-free planar graph $G$ with maximum degree at most 7 has an equitable 7-coloring.

**Proof.** Use Table 3 for an argument similar to the proof of Lemma 14.

| $r$ | lower bounds on size | Reasons   |
|-----|----------------------|-----------|
| 3   | $q_{3,\Delta,t} + 12t + q_{4,\Delta,t} + 3 - \Delta$ | Corollary 12(i), Lemma 13 |
| 2   | $q_{5,\Delta,t} + 10t + 6 - \Delta$ | Corollary 12(i), Lemma 13 |
| 1   | $q_{6,\Delta,t} + 6t + 3 - \Delta$ | Corollary 12(i), Lemma 13 |

Table 3: Lower bounds on size of $G$ in the proof of Lemma 18

Using Corollary 16 and $q_{6,\Delta,t} = 12t - 4$ from Corollary 17 we have $e(G) > 14t - 4$ for each case of $r$, which is a contradiction. Thus the counterexample is impossible. Use Observation 6 to complete the proof.

**Theorem 19.** Each $C_3$-free planar graph $G$ with maximum degree $\Delta \geq 6$ has an equitable $\Delta$-coloring.

**Proof.** Zhu and Bu [15] proved that the theorem holds for every $C_3$-free planar graph with maximum degree at least 8. Use Corollary 17 and Lemma 18 to complete the proof.

Next, we show that the conjecture holds also for a planar graph of maximum degree 5 if we restrict the girth to be at least 6.

**Corollary 20.** Each planar graph $G$ of girth at least 6 with maximum degree at most 6 and $|G| \geq 15$ has an equitable 5-coloring. Moreover, each planar graph $G$ with girth at least 6 and maximum degree $\Delta \geq 5$ has an equitable $\Delta$-coloring.
Proof. Let \( G \) be an edge-minimal planar graph of girth at least 6 that is not equitably \( \Delta \)-colorable with \( |G| = 5t \), where \( t \) is an integer at least 3, and maximum degree at most 6.

Then for \( t \geq 3 \), we have \( e(G) \leq (15/2)t - 3 \) from Lemma 11 and \( e(G) > \min\{9t - 6, 8t\} \) from Corollary 16 which leads to a contradiction. Thus the counterexample is impossible. Use Lemma 4 to complete the first part of the proof.

Observation 6 implies each planar graph \( G \) with girth at least 6 and maximum degree 5 has an equitable 5-coloring. Use Theorem 19 to complete the proof.

4 Results on Planar Graphs without \( C_4 \)

First we introduce the result by Tan [11].

Lemma 21. If a planar graph \( G \) of order \( n \) does not contain \( C_4 \), then \( e(G) \leq (15/7)n - (30/7) \) and \( \delta(G) \leq 4 \).

The proof of Lemma 21 by Tan is presented here for convenience of readers.

Proof. Let \( f \) and \( f_i \) denote the number of faces and the number of faces of length \( i \), respectively. We need only to consider the case that \( G \) is connected. A graph \( G \) cannot contain two \( C_3 \) that share the same edge since \( G \) does not contain \( C_4 \). It follows that \( 3f_3 \leq e(G) \).

Consider \( 5f - 2f_3 = 5(f_3 + f_5 + \cdots + f_n) - 2f_3 \leq 3f_3 + 5f_5 + \cdots + nf_n = \sum_{1 \leq i \leq n} if_i = 2e(G) \). Thus \( f \leq (8/15)e(G) \). Using Euler’s formula, we have \( e(G) \leq (15/7)n - (30/7) \). The result about minimum degree follows from Handshaking Lemma.

From Lemma 21, each edge-minimal counterexample graph has \( 1 \leq r \leq 4 \). The following tools in this section are quite similar to that of the previous section. Thus we omit the proofs of them.

Lemma 22. Let \( m \) be a fixed integer with \( m \geq 1 \). Suppose that any planar graph without \( C_4 \) of order \( mt \) with maximum degree at most \( \Delta \) is equitably \( m \)-colorable.
for any integer $t \geq k$. Then any planar graph without $C_4$ of order at least $kt$ and maximum degree at most $\Delta$ is also equitably $m$-colorable.

**Observation 23.** By Lemmas 5 and 22, for proving that the conjecture holds for planar graphs without $C_4$ it suffices to prove only planar graphs without $C_4$ of order $\Delta t$ where $t \geq 3$ is a positive integer.

**Lemma 24.** Suppose $G$ is a planar graph without $C_4$ with $\Delta(G) = \Delta$. If $G$ has an independent $s$-set $V'$ and there exists $U \subseteq V(G) - V'$ such that $|U| > s(2 + \Delta)/2$ and $e(u, V') \geq 1$ for all $u \in U$, then $U$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V'$.

**Notation.** Let $p_{m, \Delta, t}$ denote the maximum number not exceeding $(15/7)mt - (30/7)$ such that each planar graph without $C_4$ of order $mt$, where $t$ is an integer, is equitably $m$-colorable if it has maximum degree at most $\Delta$ and size at most $p_{m, \Delta, t}$.

**Lemma 25.** Let $G$ be an edge-minimal planar graph without $C_4$ that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. If $e(G) \leq (r + 1)(m - r)t - t + 2 + p_{r, \Delta, t}$, then $B$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1$.

**Lemma 26.** Let $G$ be an edge-minimal planar graph without $C_4$ that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. If $B$ contains two nonadjacent vertices $\alpha$ and $\beta$ which are adjacent to exactly one and the same vertex $\gamma \in V_1$, then $e(G) \geq r(m - r)t + p_{r, \Delta, t} + p_{m-r, \Delta, t} - \Delta + 4$.

**Corollary 27.** Let $G$ be an edge-minimal planar graph without $C_4$ that is not equitably $m$-colorable with order $mt$, where $t$ is an integer, and maximum degree at most $\Delta$. Then $e(G) \geq r(m - r)t + p_{r, \Delta, t} + p_{m-r, \Delta, t} - \Delta + 4$ if one of the following conditions are satisfied:

(i) $(m - r)t + 1 > (t - 1)(2 + \Delta)/2$;

(ii) $e(G) \leq (r + 1)(m - r)t - t + 2 + p_{r, \Delta, t}$. 

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Now we are ready to work on planar graphs without $C_4$.

**Lemma 28.**
(i) $p_{1,\Delta,t} = 0$. (ii) $p_{2,\Delta,t} = 2$. (iii) $p_{3,\Delta,t} \geq 6$ for $t \geq 3$. (iv) $p_{4,\Delta,t} \geq 3t$.

**Proof.** (i), (ii) and (iii) are obvious. (iv) is the result of Lemma [7].

**Lemma 29.**
$p_{5,\Delta,t} \geq \min\{p_{4,\Delta,t} + 16t + 3 - \Delta, 6t + 11 - \Delta, 7t + 2\}$ for $\Delta \geq 8$ and $t \geq 3$.

**Proof.** Use Table 4 for an argument similar to the proof of Lemma 14.

| $r$ | lower bounds on size | Reasons |
|-----|-----------------------|---------|
| 4   | $p_{4,\Delta,t} + 16t + 3 - \Delta$ or $p_{4,\Delta,t} + 4t + 2$ | Corollary 27(ii), Lemma 28 |
| 3   | $6t + 11 - \Delta$ or $7t + 8$ | Corollary 27(ii), Lemma 28 |
| 2   | $6t + 11 - \Delta$ or $8t + 4$ | Corollary 27(ii), Lemma 28 |
| 1   | $p_{4,\Delta,t} + 4t + 3 - \Delta$ or $7t + 2$ | Corollary 27(ii), Lemma 28 |

Table 4: Lower bounds on size of $G$ in the proof of Lemma 29.

**Lemma 30.**
$p_{6,\Delta,t} \geq \min\{p_{4,\Delta,t} + 8t + 5 - \Delta, 9t + 15 - \Delta, 11t + 4, p_{5,\Delta,t} + 5t + 3 - \Delta\}$ for $\Delta \geq 8$ and $t \geq 3$.

**Proof.** Use Table 5 for an argument similar to the proof of Lemma 14.

| $r$ | lower bounds on size | Reasons |
|-----|-----------------------|---------|
| 4   | $p_{4,\Delta,t} + 8t + 5 - \Delta$ or $p_{4,\Delta,t} + 9t + 2$ | Corollary 27(ii), Lemma 28 |
| 3   | $9t + 15 - \Delta$ or $11t + 8$ | Corollary 27(ii), Lemma 28 |
| 2   | $p_{4,\Delta,t} + 8t + 5 - \Delta$ or $11t + 4$ | Corollary 27(ii), Lemma 28 |
| 1   | $p_{5,\Delta,t} + 5t + 3 - \Delta$ | Corollary 27(i), Lemma 28 |

Table 5: Lower bounds on size of $G$ in the proof of Lemma 30.
Corollary 31. (1) $p_{5,8,t}$ is at least $7t - 5$ and $6t + 3$ for $t$ at least 3 and 8, respectively. (2) $p_{6,8,t}$ is at least $12t - 10$ and $9t + 7$ for $t$ at least 3 and 6, respectively. (3) $p_{7,8,t}$ is at least $18t - 15$ and $15t + 1$ for $t$ at least 3 and 6, respectively.

Proof. The results can be calculated directly from Lemmas 28 to 30.

Corollary 32. Each planar graph $G$ without $C_4$ with maximum degree at most 8 and $|G| \geq 21$ has an equitable 7-coloring. Moreover, each planar graph $G$ without $C_4$ with maximum degree 7 has an equitable 7-coloring.

Proof. Let $G$ be an edge-minimal planar graph without $C_4$ that is not equitably $\Delta$-colorable with $|G| = 7t$, where $t$ is an integer at least 3, and maximum degree at most 8.

Consider the case $r = 4$. By Corollaries 27(ii) and 31, $e(G) > \min\{p_{4,\Delta,t} + 12t + p_{3,\Delta,t} + 3 - \Delta, p_{4,\Delta,t} + 14t + 2\} \geq 15t + 1 \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 3$. By Corollaries 27(ii) and 31, $e(G) > \min\{p_{3,\Delta,t} + 12t + p_{4,\Delta,t} + 3 - \Delta, p_{3,\Delta,t} + 15t + 2\} \geq 15t + 1 \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 2$. By Corollaries 27(i) and 31, $e(G) > 10t + p_{5,\Delta,t} + 3 - \Delta \geq 15t - (30/7)$ for $t \geq 3$.

Consider the case $r = 1$. We have $e(B',V_1) \geq 6t$ by Observation 2. But $y$ has at most $\Delta - 1$ neighbors in $B'$ because $xy \in E(G)$, so $(t - 1)\Delta - 1 \geq e(B',V_1)$. Consequently, $(t - 1)\Delta - 1 \geq 5t$. That is $t \geq 4.5$ when $\Delta \leq 8$. By Corollary 27(i), $e(G) > p_{6,\Delta,t} + 6t - 5$. Using Corollary 31, we have $e(G) > \min\{18t - 15, 15t + 1\}$. It follows from $t \geq 4.5$ that $e(G) > 15t - (30/7)$.

Since we have contradiction for all cases, the counterexample is impossible. Use Lemma 22 to complete the first part of the proof.

Observation 23 implies each planar graph $G$ without $C_4$ with maximum degree 7 has an equitable 7-coloring.

Lemma 33. Each planar graph $G$ without $C_4$ with maximum degree at most 8 has an equitable 8-coloring.
| $r$ | lower bounds on size | Reasons |
|-----|-----------------------|---------|
| 4   | $p_{4,\Delta,t} + 16t + p_{4,\Delta,t} + 3 - \Delta$ or $p_{4,\Delta,t} + 19t + 2$ | Corollary 27(ii), Lemma 28 |
| 3   | $15t + p_{5,\Delta,t} + 9 - \Delta$ | Corollary 27(i), Lemma 28 |
| 2   | $p_{6,\Delta,t} + 12t + 5 - \Delta$ | Corollary 27(i), Lemma 28 |
| 1   | $p_{7,\Delta,t} + 7t + 3 - \Delta$ | Corollary 27(i), Lemma 28 |

Table 6: Lower bounds on size of $G$ in the proof of Lemma 33

**Proof.** Use Table 6 for an argument similar to the proof of Lemma 14.

Using Corollary 31 and $q_{7,\Delta,t} = 15t - 4$ from Corollary 32, we have $e(G) > (120/7)t - (30/7)$ for each case of $r$, which is a contradiction. Thus the counterexample is impossible. Use Observation 23 to complete the proof.

**Theorem 34.** Each planar graph $G$ without $C_4$ with maximum degree $\Delta \geq 7$ has an equitable $\Delta$-coloring.

**Proof.** Nakprasit [10] proved that the theorem holds for every planar graph with maximum degree at least 9. Use Corollary 32 and Lemma 33 to complete the proof.

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