The relation of thermal fluctuation and information-entropy of One-dimensional Rindler Oscillator

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Within the framework of thermo-field-dynamics(TFD), the information-entropies associated with the measurements of position and momentum for one-dimensional Rindler oscillator are derived, and the connection between its information-entropy and thermal fluctuation is obtained. A conclusion is drawn that the thermal fluctuation leads to the loss of information.

Key Words: Rindler space-time, information-entropy, generalized uncertainty relation

PACS number(s): 04.90.+e, 05.40+j, 03.65.-w

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I. INTRODUCTION

As a joint-realm of information theory and statistical physics, information-entropy has received a great deal of investigations\textsuperscript{1−3}. Generally, for a thermal state, the thermal fluctuation results in losses of information and increases of the uncertainty and information-entropy. Therefore our interest here concerns what is the connection between the thermal fluctuation and information-entropies.

The generalized uncertainty relation\textsuperscript{4−8} is widely discussed. But in these papers the “generalized uncertainty relations” have different meanings. In our previous paper\textsuperscript{4}, the generalized uncertainty relation of one-dimensional Rindler oscillator, which is related to the thermal nature of the quantum state, was derived. In fact, we presented a thermal modification to the uncertainty principle, it is the acceleration that induces the thermal effect (Unruh effect). As is well know, a Minkowski vacuum is equivalent to a thermal bath\textsuperscript{9,10} for a Rindler observer. For a Rindler uniformly accelerated observer, there are not only general quantum fluctuations but also thermal fluctuations related to his acceleration. On the other hand, the authors\textsuperscript{6,7} presented a gravitationally-induced modification to the momentum and position uncertainty relation which comes from a definition of momentum and position operators that incorporates certain dynamics relevant at high energies and differs from the usual definition.

In this paper, we present the information-entropies associated with the
measurements of position and momentum for one-dimensional Rindler oscillator in the coordinate representation, and derive a relation between information-entropies and fluctuations. Compared with other methods such as Wigner function method\textsuperscript{11,12}, our method introduced in this paper is different from others. Our strategy is as follows. First we calculate the information-entropies in the coordinate representation. Comparing the result with the generalized uncertainty relation, we derive the relation of the thermal fluctuation and the information-entropies. From the results of this paper, we find that the thermal fluctuation and quantum fluctuation are separated.

II. RINDLER AND MINKOWSKI SPACE-TIME

The coordinates in Rindler space-time can be obtained from the Minkowski coordinates $T$, $X$ under the transformations

\begin{align*}
T &= g^{-1} e^{g \xi} \sinh g \eta \\
X &= g^{-1} e^{g \xi} \cosh g \eta & \text{for region R} (1)
\end{align*}

and

\begin{align*}
T &= -g^{-1} e^\tilde{g} \xi \sinh g \tilde{\eta} \\
X &= -g^{-1} e^\tilde{g} \xi \cosh g \tilde{\eta} & \text{for region L} (2)
\end{align*}
where \( g = \text{constant} > 0 \), the Rindler coordinates \((\eta, \xi)\) and \((\tilde{\eta}, \tilde{\xi})\) cover the space-time regions \( R \) and \( L \) respectively. Both regions \( R \) and \( L \) are quadrants of Minkowski space-time as shown in Fig. 1. The region \( L \) is called the mirror space-time region of \( R \). With the method of standard Rindler quantization\(^{13}\), we can obtain two groups of annihilation and creation operators \((b, b^\dagger)\) and \((\tilde{b}, \tilde{b}^\dagger)\) corresponding to the Rindler modes in the regions \( R \) and \( L \), respectively, and the vacuum state defined by these two groups of annihilation and creation operators is \(|0\rangle^R\) in region \( R \) and \(|\tilde{0}\rangle^R\) in region \( L \), respectively. The modes that are corresponding to these two groups of annihilation and creation operators are complete in the region \( R \) and \( L \) respectively, but they are not complete in the whole Minkowski space-time region. Here the position is denoted by \( \xi \) in region \( R \), and by \( \tilde{\xi} \) in region \( L \), while the momentum is denoted by \( p_R \) in region \( R \), and by \( \tilde{p}_R \) in region \( L \).

The Minkowski vacuum is defined by general annihilation and creation operators \((a, a^\dagger)\), \( a |0\rangle^M = 0 \). Because of the different selecting of modes, we have another group of Minkowski annihilation and creation operators \((d, d^\dagger)\) and \((\tilde{d}, \tilde{d}^\dagger)\). The relations between them and the Rindler annihilation operators satisfy the Bogoliubov transformations

\[
\begin{align*}
    d &\equiv R(\theta) b R^\dagger(\theta) \\
    \tilde{d} &\equiv R(\theta) \tilde{b} R^\dagger(\theta)
\end{align*}
\]

where \([d, d^\dagger] = [\tilde{d}, \tilde{d}^\dagger] = 1\). The unitary transformation (called thermal
transformation) is

\[ R(\theta) = \exp \left\{ -\theta (\beta) \left( \hat{b}\hat{b}^\dagger - \hat{b}^\dagger\hat{b} \right) \right\} \]  \hspace{1cm} (4)

where

\[ \tanh [\theta (\beta)] = \exp \left( -\frac{\beta\hbar\omega}{2} \right) \]  \hspace{1cm} (5)

\[ \beta = \frac{1}{K_B T} , \quad K_B \text{ is the Boltzmann constant and } T \text{ is temperature here.} \]

The vacuum state defined by \((d, d^\dagger)\) and \((\tilde{d}, \tilde{d}^\dagger)\) is equivalent to the Minkowski vacuum state \(|0\rangle_M\)

\[ d |0\rangle_M = \tilde{d} |0\rangle_M = 0 \]  \hspace{1cm} (6)

For one-dimensional Rindler oscillator, we construct position and momentum \((x, p)\) from \((d, d^\dagger)\) and their tilde conjugate quantities \((\tilde{x}, \tilde{p})\) from \((\tilde{d}, \tilde{d}^\dagger)\) as follows:

\[ x = \sqrt{\frac{\hbar}{2m\omega}} (d + d^\dagger) \]

\[ p = -i\sqrt{\frac{m\omega\hbar}{2}} (d - d^\dagger) \]  \hspace{1cm} (7)

and

\[ \tilde{x} = \sqrt{\frac{\hbar}{2m\omega}} (\tilde{d} + \tilde{d}^\dagger) \]
\[
\hat{p} = -i \sqrt{\frac{m \omega \hbar}{2}} (\hat{d} - \hat{d}^\dagger)
\]  

(8)

The relation between Rindler vacuum and Minkowski vacuum is

\[
|0\rangle_M = R(\theta) |0, \bar{0}\rangle_R,
\]

(9)

Where \(|0, \bar{0}\rangle_R\) is a direct product of the Rindler vacuum state in region R and L, and \(R(\theta)\) describes the effect of a thermal bath in which a quantum harmonic oscillator immerses. From Eq.(9), we can say loosely that a thermalizing operator \(R(\theta)\) heats the ground state of a zero-temperature harmonic oscillator (Rindler vacuum) into a thermal state with a finite temperature for a Rindler uniformly accelerating observer. Note that any operator in region L commutes with any tilde operator in region R for bosons in this paper. Consequently any Minkowski vacuum expectation for the Rindler observer coincides with its canonical ensemble average in statistical mechanics.

**III. INFORMATION-ENTROPY OF ONE-DIMENSIONAL RINDLER OSCILLATOR**

In this section, we will derive the reduced probability densities, and discuss the information-entropies associated with the measurements of position and momentum for one-dimensional Rindler oscillator. Comparing the generalized uncertainty relation of one-dimensional Rindler oscillator, we present the relation of information-entropies with quantum and thermal fluctuations. In this paper, we use the information-entropy given by Shannon.$^{14}$
\[ S_A[\Psi] = - \sum_\alpha \langle \alpha | \Psi \rangle^2 \ln \langle \alpha | \Psi \rangle^2 \quad (10) \]

where \( \{ |\alpha\rangle \} \) is the set of eigenstates of \( A \).

For the one-dimensional oscillator in Rindler space-time region \( R \), its Hamiltonian is

\[ H = \frac{1}{2m} p_R^2 + \frac{1}{2} m \omega^2 \xi^2 = \left( b^\dagger b + \frac{1}{2} \right) \hbar \omega \quad (11) \]

The wave function of ground state in the coordinate representation is

\[ \langle \xi | 0 \rangle_R = \left( \frac{m \omega}{\pi \hbar} \right)^{\frac{1}{4}} \exp \left\{ - \frac{m \omega}{2 \hbar} \xi^2 \right\} \quad (12) \]

where, \( p_R = -i \hbar \frac{d}{d\xi} \equiv -i \hbar \partial_\xi \), \( m \) is the mass, \( \omega \) is the angular frequency, and

\[ b = \frac{1}{\sqrt{2m\hbar w}} (ip_R + m\omega \xi) \]
\[ b^\dagger = \frac{1}{\sqrt{2m\hbar w}} (-ip_R + m\omega \xi) \quad (13) \]

are the corresponding annihilation and creation operators in the Rindler region \( R \), respectively. Using the tilde rules in Thermal field dynamics, we introduce

\[ \tilde{H} = \frac{1}{2m} \tilde{p}_R^2 + \frac{1}{2} m \omega^2 \tilde{\xi}^2 = \left( \tilde{b}^\dagger \tilde{b} + \frac{1}{2} \right) \hbar \omega \quad (14) \]

in Rindler region \( L \). Substituting Eq.(13) into Eq.(4), one has
\[ R(\theta) = \exp \left\{ \frac{i}{\hbar} \left( \xi \tilde{p}_R - \tilde{\xi} p_R \right) \right\} \]  \hspace{1cm} (15)

with \( \theta \equiv \theta (\beta) \). From Appendix B.4 in Ref.15, the last formula can be written as

\[ R(\theta) = \exp \left\{ -\tanh (\theta) \tilde{\xi} \partial_{\tilde{\xi}} \right\} \exp \left\{ \ln \cosh (\theta) \right\} \left( \xi \partial_{\xi} - \tilde{\xi} \partial_{\tilde{\xi}} \right) \exp \left\{ -\tanh (\theta) \xi \partial_{\tilde{\xi}} \right\} . \]  \hspace{1cm} (16)

Using the following operator properties

\[ e^{C\partial_y} f (y) = f (y + C) \]  \hspace{1cm} (17)

and

\[ e^{C_y \partial_y} f (y) = f (ye^C) , \]  \hspace{1cm} (18)

one can gave the wave function of Minkowski vacuum in Rindler coordinate representation

\[
\begin{align*}
\langle \tilde{\xi}, \xi | 0 \rangle_M &= R(\theta) \left( \frac{m\omega}{\pi \hbar} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m\omega}{2\hbar} \left( \xi^2 + \tilde{\xi}^2 \right) \right\} \\
&= \left( \frac{m\omega}{\pi \hbar} \right)^{\frac{1}{2}} \exp \left\{ -\frac{m\omega}{2\hbar} \left( \xi \cosh (\theta) - \tilde{\xi} \sinh (\theta) \right)^2 \right. \\
& \quad \left. + \left( \xi \cosh (\theta) - \tilde{\xi} \sinh (\theta) \right)^2 \right\}. \hspace{1cm} (19)
\end{align*}
\]
We can see that when $\beta \to \infty$, from Eq.(4) and Eq.(5) we have $\theta (\beta) \to 0$, $R(\theta) \to 1$, so $\langle \bar{\xi}, \xi \mid 0 \rangle_M$ is reduced to $\langle \bar{\xi}, \xi \mid 0, \bar{0} \rangle_R$.

The matrix element of density operator in position presentation for one-dimensional Rindler oscillator is

$$
\rho_{\xi',\xi} = \int_{-\infty}^{+\infty} \langle \xi', \bar{\xi} \mid 0 (\beta) \rangle \langle 0 (\beta) \mid \bar{\xi}, \xi \rangle \, d\bar{\xi}
$$

$$
= \frac{m\omega}{\pi\hbar} \int_{-\infty}^{+\infty} \exp \left\{ -\frac{m\omega}{2\hbar} \left[ \left( \xi' \cosh (\theta) - \bar{\xi} \sinh (\theta) \right)^2 + \left( \xi \cosh (\theta) - \bar{\xi}' \sinh (\theta) \right)^2 \right] \right\}
$$

$$
\cdot \exp \left\{ -\frac{m\omega}{2\hbar} \left[ \left( \xi \cosh (\theta) - \bar{\xi} \sinh (\theta) \right)^2 + \left( \bar{\xi} \cosh (\theta) - \xi \sinh (\theta) \right)^2 \right] \right\} \, d\bar{\xi}
$$

$$
= \left[ \frac{m\omega}{\pi\hbar \left( \cosh^2 (\theta) + \sinh^2 (\theta) \right)} \right] \exp \left\{ -\frac{m\omega}{2\hbar} \left( \cosh^2 (\theta) + \sinh^2 (\theta) \right) \left( \xi'^2 + \xi^2 \right) \right\}
$$

$$
\cdot \exp \left\{ -\frac{m\omega}{2\hbar} \left[ \cosh^2 (\theta) + \sinh^2 (\theta) \right] \right\}
$$

$$
= \frac{m\omega}{\pi\hbar \left( \cosh^2 (\theta) + \sinh^2 (\theta) \right)} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh^2 (\theta)} \xi^2 \right\}
$$

$$
= \frac{m\omega}{\pi\hbar \cosh (2\theta)} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh (2\theta)} \xi^2 \right\}
$$

Taking $\xi = \xi'$ in Eq.(20), one has the probability density of position

$$
\rho_{\xi,\xi} = \frac{m\omega}{\pi\hbar \left( \cosh^2 (\theta) + \sinh^2 (\theta) \right)} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh^2 (\theta)} \xi^2 \right\}
$$

$$
\cdot \exp \left\{ -\frac{m\omega}{2\hbar} \left( \cosh^2 (\theta) + \sinh^2 (\theta) \right) \left( \xi'^2 + \xi^2 \right) \right\}
$$

$$
= \frac{m\omega}{\pi\hbar \cosh (2\theta)} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh (2\theta)} \xi^2 \right\}
$$

(21)

Similarly, we have the reduced probability density of momentum for one-dimensional Rindler oscillator

$$
\rho_{p_R,p_R} = \int_{-\infty}^{+\infty} \frac{1}{2\pi\hbar} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh (2\theta)} \xi^2 \right\} \rho_{\xi',\xi} \, d\xi \, d\xi'
$$

8
\[
\sqrt{\frac{1}{m\omega\pi\hbar \cosh (2\theta)}} \exp \left\{ -\frac{1}{m\omega\hbar} \cdot \frac{1}{\cosh (2\theta)} p_R^2 \right\} \tag{22}
\]

So, we can calculate the information-entropy with the measurements of position and momentum, respectively

\[
s_\xi = -\int_{-\infty}^{+\infty} \rho_\xi \ln (\rho_\xi) \, d\xi
= -\int_{-\infty}^{+\infty} \sqrt{\frac{m\omega}{\pi\hbar \cosh (2\theta)}} \exp \left\{ -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh (2\theta)} \xi^2 \right\}
\left[ \left( -\frac{m\omega}{\hbar} \cdot \frac{1}{\cosh (2\theta)} \xi^2 \right) + \ln \left( \sqrt{\frac{m\omega}{\pi\hbar \cosh (2\theta)}} \right) \right] \, d\xi
= \frac{1}{2} \left[ 1 + \ln \pi + \ln (\cosh (2\theta)) + \ln \left( \frac{\hbar}{m\omega} \right) \right] \tag{23}
\]

and

\[
s_{pR} = -\int_{-\infty}^{+\infty} \rho_{pR,pR} \ln (\rho_{pR,pR}) \, dp_R
= -\int_{-\infty}^{+\infty} \sqrt{\frac{1}{m\omega\pi\hbar \cosh (2\theta)}} \exp \left\{ -\frac{1}{m\omega\hbar} \cdot \frac{1}{\cosh (2\theta)} p_R^2 \right\}
\left[ \left( -\frac{1}{m\omega\hbar} \cdot \frac{1}{\cosh (2\theta)} p_R^2 \right) + \ln \left( \sqrt{\frac{1}{m\omega\pi\hbar \cosh (2\theta)}} \right) \right] \, dp_R
= \frac{1}{2} \left[ 1 + \ln \pi + \ln (\cosh (2\theta)) + \ln (m\omega\hbar) \right] \tag{24}
\]

Deutsch and Partovi\textsuperscript{1,2} discussed the sum of entropies associated with the measurements of a generic non-commutative pair of observable (A,B) in a normalized state |\Psi\rangle

\[
U [A, B : \psi] = S_A [\psi] + S_B [\psi] \tag{25}
\]
which cannot be made arbitrarily small but has an irreducible lower bound
independent of the choice of $|\Psi\rangle$.

Thus, we get the sum of information-entropies associated with the measure-
ments of position and momentum of one-dimensional Rindler oscillator.

$$U = s_\xi + s_{p_R} = 1 + \ln \pi + \ln (\cosh (2\theta)) + \ln \hbar$$  \hspace{1cm} (26)

where $U$ denotes the uncertainty measurement of one-dimensional Rindler
oscillator.

Now we wish to find the relation of information-entropy and fluctuation.
In our previous paper\textsuperscript{4}, according to the invariance of Bogoliubov transfor-
mation in the Thermal Field Theory, we derived the generalized uncertainty
relation of one-dimensional Rindler oscillator in Minkowski vacuum in the
coordinates representation, that is

$$\langle (\Delta p_R)^2 \rangle \langle (\Delta \xi)^2 \rangle \geq \frac{\hbar^2}{4} + \frac{\hbar^2}{4 \sinh^2 \left(\frac{\omega \hbar}{2}\right)}.$$ \hspace{1cm} (27)

For a Rindler uniformly accelerated observer, the term on the LHS( the left
hand side )of Eq.(27) describes the total fluctuations of one-dimensional
Rindler oscillator. The first term on the RHS of Eq.(27) $\langle (\Delta p)^2 \rangle \langle (\Delta x)^2 \rangle$
describes the zero-temperature fluctuation, which is a purely quantum fluc-
tuation and satisfies the general uncertainty relation. The second term on
the RHS of Eq.(27) describes a purely thermal fluctuation of one-dimensional
Rindler oscillator, which is determined by cross terms of the tilde and non-
tilde operators.

The thermal fluctuation can be written as

\[
\frac{\hbar^2}{4 \sinh^2 \left( \frac{2\omega \hbar}{2} \right)} = \frac{\hbar^2}{4} \left( \cosh^2 (2\theta) - 1 \right)
\]

(28)

Where \( \theta \) defined by Eq.(5). Comparing Eq.(26) with Eq.(28), one has

\[
\frac{\hbar^2}{4 \sinh^2 \left( \frac{2\omega \hbar}{2} \right)} = \frac{1}{4} \hbar^2 \left( \frac{e^{2(s_\xi + s_{PR})} - 1}{e^{2\pi^2 \hbar^2} - 1} \right)
\]

(29)

Hence, we derive the relation of information-entropy, quantum fluctuation and thermal fluctuation

\[
\frac{e^{2(s_\xi + s_{PR})}}{4e^{2\pi^2}} = \frac{e^{2U}}{4e^{2\pi^2}} = \frac{1}{4} \hbar^2 + \frac{\hbar^2}{4 \sinh^2 \left( \frac{2\omega \hbar}{2} \right)}
\]

(30)

IV. SUMMARY AND DISCUSSION

In Eq.(26) with \( \hbar = 1 \), when temperature \( T \to 0 \), we get

\[
S_\xi + S_{PR} = 1 + \ln \pi
\]

(31)

This result coincides with the results obtained by other methods\textsuperscript{16}.

The Eq.(30) is the key result of this paper. The term on the LHS of the Eq.(30) includes the sum of information-entropies associated with the measurements of position and momentum of one-dimensional Rindler oscillator.
The first term on the RHS of Eq.(30) $\frac{1}{4}\hbar^2$ describes zero-temperature fluctuation, which is the purely quantum fluctuation. The second term on the RHS of Eq.(30) describes the purely thermal fluctuation of one-dimensional Rindler oscillator. Thus we can make the thermal fluctuation and quantum fluctuation separated naturally. When temperature increases, $S_\xi + S_{PR}$ also increases monotonously. This result shows that the thermal fluctuation causes the loss of information.

According to Eq.(27) and (30), one have

$$\langle (\Delta p_R)^2 \rangle \langle (\Delta \xi)^2 \rangle \geq \frac{e^{2(s_\xi + s_{PR})}}{4e^2\pi^2}$$

This is the relation between the uncertainty and the information-entropy. When temperature $T \to 0$, the quantum uncertainty relation will be restored.
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