Variable formation plasticity matrices of a three-dimensional body when implementing a step loading procedure

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Abstract. Nowadays, the task of creating and improving computational algorithms for determining the stress-strain state (SSS) of technosphere objects and systems, taking into account the physically non-linear stage of operation of the material of structures and their elements, is quite relevant. A variational approach to the formation of a plasticity matrix at the loading step for a three-dimensional body is presented on the deformation theory of plasticity. A variant of obtaining the plasticity matrix using the operations of differentiating the covariant components of the stress tensor with respect to the covariant components of the strain tensor is considered. As an alternative to it, an option has been developed for the formation of a plasticity matrix at the loading step, based on the hypothesis of proportionality of the components of the deviator of stress increments to the components of the deviator of strain increments for three-dimensional bodies. The use of such option of forming a plasticity matrix at the loading step in software systems can significantly simplify the programming process and make it more understandable to potential users of computing systems in the design and reconstruction of technosphere objects and systems.

1. Introduction
When designing modern technosphere objects and systems, including the agro-industrial complex, the requirements for reducing the material intensity of their structural elements and the most complete use of the strength resources of the material used come to the fore. In connection with this circumstance, the task of creating and improving computational algorithms for determining the SSS of technosphere objects and systems does not lose its relevance, taking into account the physically non-linear stage of operation of the material of structures and their elements. The assumption of a limited development of plastic deformations allows us to increase the values of the payloads transferred to the designed technosphere objects, contributing to the most rational use of structural materials and their elements.

The modern development of computer technology and the computer technologies are created on the basis of its application provide the unconditional priority of numerical methods for analyzing the SSS of technosphere objects and systems, the leading position among which is the finite element method (FEM) [1-11].
Using numerical methods to study the processes of deformation of technosphere objects in a physically nonlinear formulation implement a step-by-step method of loading structures and their objects [12-20]. In this case, it becomes necessary to form a plasticity matrix at the loading step, establishing a connection between the components of the stress increment tensor and the components of the strain increment tensor.

In this article, which is based on the deformation theory of plasticity, a variational approach to the formation of a plasticity matrix at the loading step for a three-dimensional body is presented.

2. Materials and methods

2.1. The plasticity matrix of a three-dimensional body at the loading step, which is based on the calculation of the partial derivatives of the covariant components of the stress tensor with respect to the covariant components of the strain tensor

The increment of the covariant components of the stress tensor for a three-dimensional body can be expressed through the increments of the covariant components of the strain tensor by the following dependencies

\[
\Delta \sigma_{mn} = \frac{\partial \sigma_{mn}}{\partial \varepsilon_{11}} \Delta \varepsilon_{11} + \frac{\partial \sigma_{mn}}{\partial \varepsilon_{12}} \Delta \varepsilon_{12} + \frac{\partial \sigma_{mn}}{\partial \varepsilon_{13}} \Delta \varepsilon_{13} + \frac{\partial \sigma_{mn}}{\partial \varepsilon_{22}} \Delta \varepsilon_{22} + \frac{\partial \sigma_{mn}}{\partial \varepsilon_{23}} \Delta \varepsilon_{23} + \frac{\partial \sigma_{mn}}{\partial \varepsilon_{33}} \Delta \varepsilon_{33},
\]

where the indices \(m\) and \(n\) successively take the values 1–3.

To calculate the partial derivatives \(\frac{\partial \sigma_{mn}}{\partial \varepsilon_{11}}\) included in (1), it is necessary to first obtain the expressions for the covariant components of the strain tensor. Based on the second hypothesis of the theory of small elastoplastic deformations [21, 22], it is possible to establish the relationships between the covariant components of the deviators stress \(S_{mn}\) and strain \(E_{mn}\)

\[
S_{mn} = \frac{2}{3} \sigma_i E_{mn},
\]

where \(\sigma_i = \left(1.5 S_{mn} S_{mn} \right)^{1/2}\) and \(E_{mn} = \left(2/3 E_{mn} E_{mn} \right)^{1/2}\) are the intensities of stresses and strains.

The formulas for the covariant and contravariant components of the strain and stress deviators for a three-dimensional body have the following form [23]

\[
E_{mn} = \varepsilon_{mn} - (1/3) I_1(\varepsilon) g_{mn}, \quad E^{mn} = \varepsilon^{mn} - (1/3) I_1(\varepsilon) g^{mn},
\]

\[
S_{mn} = \sigma_{mn} - (1/3) I_1(\sigma) g_{mn}, \quad S^{mn} = \sigma^{mn} - (1/3) I_1(\sigma) g^{mn},
\]

where \(g_{mn}, g^{mn}\) is the co- and contravariant components of the metric tensor; \(I_1(\varepsilon) = g_{mn} \varepsilon_{mn} = g_{mn} \varepsilon^{mn}\), \(I_1(\sigma) = g_{mn} \sigma_{mn} = g_{mn} \sigma^{mn}\) are the first invariants of strain and stress tensors.

Taking into account the first hypothesis of the theory of small elastoplastic deformations and taking into account (3), from relations (2) we can obtain the following expressions for the covariant components of the stress tensor

\[
\sigma_{mn} = \frac{2}{3} \varepsilon_i \varepsilon_{mn} - \frac{1}{3} I_1(\varepsilon) g_{mn} \left( \frac{2}{3} \varepsilon_i - K \right),
\]

where \(K = E/(1-2\nu)\) [21, 22].

Applying to (4) the operation of differentiation with respect to the covariant components of the strain tensor [21, 22], we can write the following relation
\[
\frac{\partial \sigma_{mn}}{\partial e_{nn}} = \frac{2}{3} \left( \frac{\partial (\sigma_{i}/e_i)}{\partial e_{nn}}, e_{nn} + \frac{\sigma_{i}}{e_i} \right) - \frac{1}{3} \left( \frac{\partial (I_1(\varepsilon)g_{mn})}{\partial e_{nn}} \left( \frac{2}{3} \frac{\sigma_{i}}{e_i} - K \right) + I_1(\varepsilon)g_{mn} \right) \frac{2}{3} \frac{\partial (\sigma_{i}/e_i)}{\partial e_{mm}}. \tag{5}
\]

We obtain the partial derivatives \(\partial (\sigma_{i}/e_i)/\partial e_{nn}\) and \(\partial (I_1(\varepsilon)g_{mn})/\partial e_{mm}\) included in (5). The first of these partial derivatives can be represented as

\[
\frac{\partial (\sigma_{i}/e_i)}{\partial e_{nn}} = \frac{\partial (\sigma_{i}/e_i)}{\partial e_{i}} \frac{\partial e_{i}}{\partial e_{nn}} = \left( \frac{\partial \sigma_{i}}{\partial e_{i}} - \frac{\sigma_{i}}{e_i} \right) e_{i} e_{nn} = (E_K - E_s) \frac{1}{e_i} \frac{\partial e_{i}}{\partial e_{nn}}, \tag{6}
\]

where \(E_K = \partial \sigma_i/\partial e_i\); \(E_s = \sigma_i/e_i\) is the tangent and secant modulus of the deformation diagram.

The partial derivative \(\partial e_i/\partial e_{nn}\) included in (6) with allowance for (2) can be obtained from the following expression

\[
\frac{\partial e_i}{\partial e_{nn}} = \frac{E_{nn}}{3} (at \ m = n); \quad \frac{\partial e_i}{\partial e_{mm}} = \frac{2E_{mm}}{3} (at \ m \neq n); \quad \frac{\partial e_i}{\partial e_{mm}} = \frac{E_{mm}}{3} (at \ m = n); \quad \frac{\partial e_i}{\partial e_{mm}} = \frac{2E_{mm}}{3} (at \ m \neq n).
\]

We consider the calculation of the partial derivatives of \(\partial E_{mn}/\partial e_{mm}\) by the example of the partial derivative of \(\partial E_{11}/\partial e_{11}\)

\[
\partial E_{11}/\partial e_{11} = \left( e_{11} - \frac{1}{3} I_1(\varepsilon)g_{11} \right) / \partial e_{11} =
\]

\[
= \left( g_{11}g_{11}e_{11} + 2g_{11}g_{12}e_{12} + 2g_{11}g_{13}e_{13} + g_{12}g_{12}e_{12} + 2g_{12}g_{13}e_{23} + g_{13}g_{13}e_{23} - \frac{1}{3} I_1(\varepsilon)g_{11} \right) / \partial e_{11} =
\]

\[
= g_{11}g_{11} + 2g_{11}g_{12} \left( \frac{\partial g_{11}}{\partial e_{11}} \cdot g_{12} \cdot g_{11} + 2 \frac{\partial g_{11}}{\partial e_{11}} \cdot g_{12} \right) + 2g_{12} \left( \frac{\partial g_{11}}{\partial e_{11}} \cdot g_{11} + \frac{\partial g_{12}}{\partial e_{11}} \cdot g_{11} \right) +
\]

\[
+ 2g_{12} \left( \frac{\partial g_{11}}{\partial e_{11}} \cdot g_{12} + \frac{\partial g_{12}}{\partial e_{11}} \cdot g_{12} \right) + 2g_{13} \left( \frac{\partial g_{13}}{\partial e_{11}} \cdot g_{13} + \frac{\partial g_{13}}{\partial e_{11}} \cdot g_{12} \right) +
\]

\[
+ 2g_{13} \left( \frac{\partial g_{13}}{\partial e_{11}} \cdot g_{13} + \frac{\partial g_{13}}{\partial e_{11}} \cdot g_{12} \right)
\]

\[
\frac{\partial I_1(\varepsilon)}{\partial e_{11}} = g_{11} + e_{11} \frac{\partial g_{11}}{\partial e_{11}} + 2e_{12} \frac{\partial g_{12}}{\partial e_{11}} + 2e_{13} \frac{\partial g_{13}}{\partial e_{11}} + e_{22} \frac{\partial g_{22}}{\partial e_{11}} + 2e_{23} \frac{\partial g_{23}}{\partial e_{11}} + e_{33} \frac{\partial g_{33}}{\partial e_{11}}.
\tag{9}
\]

To obtain the partial derivatives of \(\partial g_{mn}/\partial e_{11}\), it is necessary to first perform the operations of transition from the contravariant components of the metric tensor to its covariant components [24]

\[
g_{mn} = G^{mn}/G,
\tag{10}
\]

where \(G^{mn}\) is the algebraic complement of the element \(g_{mn}\); \(G\) is the determinant of the metric tensor.

In view of (10), we can write the following expressions
\[
\frac{\partial g^{11}}{\partial \varepsilon_{11}} = \left( -\frac{1}{G} \right) 2G^{11}; \quad \frac{\partial g^{12}}{\partial \varepsilon_{11}} = \left( -\frac{1}{G} \right) 2G^{11}G^{12}; \quad \frac{\partial g^{13}}{\partial \varepsilon_{11}} = \left( -\frac{1}{G} \right) 2G^{11}G^{13}; \quad (11)
\]

\[
\frac{\partial g^{22}}{\partial \varepsilon_{11}} = \frac{2}{G} g^{33} - \frac{2}{G^2} G^{11} G^{22}; \quad \frac{\partial g^{23}}{\partial \varepsilon_{11}} = \frac{2}{G} g^{32} - \frac{2}{G^2} G^{11} G^{23}; \quad \frac{\partial g^{33}}{\partial \varepsilon_{11}} = \frac{2}{G} g^{22} - \frac{2}{G^2} G^{11} G^{33}.
\]

The partial derivatives \( \frac{\partial (I_1(\varepsilon) g_{mn})}{\partial \varepsilon_{mn}} \) included in (5) are calculated taking into account (9), for example

\[
\frac{\partial (I_1(\varepsilon) g_{11})}{\partial \varepsilon_{11}} = \frac{\partial (I_1(\varepsilon) g_{11})}{\partial \varepsilon_{11}} + I_1(\varepsilon) \frac{\partial g_{11}}{\partial \varepsilon_{11}} = \frac{\partial I_1(\varepsilon)}{\partial \varepsilon_{11}} + 2I_1(\varepsilon). \quad (12)
\]

Performing mathematical transformations similar to calculations (7–12), one can obtain all necessary partial derivatives included in (5). Thus, we can assume that all partial derivatives of \( \frac{\partial \sigma_{mn}}{\partial \varepsilon_{mn}} \) for (1) are defined. Relations (1) can be represented in matrix form

\[
\left\{ \Delta \sigma_{mn} \right\} = \left[ C_I \right] \left\{ \Delta \varepsilon_{mn} \right\}, \quad (13)
\]

where \( \left\{ \Delta \sigma_{16} \right\} = \left\{ \Delta \sigma_{11} \Delta \sigma_{12} \Delta \sigma_{13} \Delta \sigma_{12} \Delta \sigma_{13} \Delta \sigma_{33} \right\}; \quad \left\{ \Delta \varepsilon_{16} \right\} = \left\{ \Delta \varepsilon_{11} 2\Delta \varepsilon_{12} 2\Delta \varepsilon_{13} \Delta \varepsilon_{22} 2\Delta \varepsilon_{23} \Delta \varepsilon_{33} \right\}; \quad \left[ C_I \right] \) is the plasticity matrix at the loading step.

2.2. The plasticity matrix of a three-dimensional body at the loading step, which is based on the hypothesis of proportionality of the components of the deviator of stress increments to the components of the deviator of strain increments

Analyzing (7–12), it can be noted that the layout of the plasticity matrix at the loading step in the presented version is rather laborious, which undoubtedly complicates the programming procedure. Therefore, this article proposes an alternative option for the formation of a plasticity matrix at the loading step, based on the hypothesis of proportionality of the components of the deviator of stress increments to the components of the deviator of strain increments.

The increments of the covariant components of the strain tensor in this case can be expressed through the increments of the covariant components of the stress tensor as follows

\[
\Delta \varepsilon_{mn} = \frac{1.5}{E_K} \Delta \sigma_{mn} + P(\Delta \sigma) g_{mn} \Delta D,
\]

where \( P(\Delta \sigma) = \Delta \sigma_{mn} g^{mn} = \Delta \sigma_{11} g^{11} + 2\Delta \sigma_{12} g^{12} + 2\Delta \sigma_{13} g^{13} + \Delta \sigma_{22} g^{22} + 2\Delta \sigma_{23} g^{23} + \Delta \sigma_{33} g^{33}; \quad D = \left( \frac{1 - 2 \upsilon}{3E - \frac{4}{3} E_K} \right); \quad E_K = \frac{\Delta \sigma_{ij}}{\Delta \varepsilon_{ij}} \) is the tangent module of the deformation diagram.

In expanded form, relations (14) take the form

\[
\Delta \varepsilon_{11} = \left( \frac{1.5}{E_K} + g^{11} g_{11} D \right) \Delta \sigma_{11} + 2g^{12} g_{11} D \Delta \sigma_{12} + 2g^{13} g_{11} D \Delta \sigma_{13} + g^{22} g_{11} D \Delta \sigma_{22} + 2g^{23} g_{11} D \Delta \sigma_{23} +

+ g^{33} g_{11} D \Delta \sigma_{33};
\]

\[
\Delta \varepsilon_{12} = g_{12} g^{11} D \Delta \sigma_{11} + \left( \frac{1.5}{E_K} + 2g_{12} g^{12} D \right) \Delta \sigma_{12} + 2g_{12} g^{13} D \Delta \sigma_{13} + g_{12} g^{22} D \Delta \sigma_{22} + 2g_{12} g^{23} D \Delta \sigma_{23} +

+ g_{12} g^{33} D \Delta \sigma_{33};
\]

\[
\Delta \varepsilon_{13} = g_{13} g^{11} D \Delta \sigma_{11} + \left( \frac{1.5}{E_K} + 2g_{13} g^{13} D \right) \Delta \sigma_{13} + 2g_{13} g^{22} D \Delta \sigma_{22} + 2g_{13} g^{23} D \Delta \sigma_{23} +

+ g_{13} g^{33} D \Delta \sigma_{33};
\]
\[
\Delta \varepsilon_{33} = g_{333}^{11} \Delta \sigma_{11} + 2g_{333}^{12} \Delta \sigma_{12} + 2g_{333}^{13} \Delta \sigma_{13} + g_{333}^{22} \Delta \sigma_{22} + 2g_{333}^{23} \Delta \sigma_{23} + \\
+ \left( \frac{1.5}{E_k} + g_{333}^{33} D \right) \Delta \sigma_{33}.
\]

Dependencies (15) can be represented in matrix form

\[
\{ \Delta \varepsilon_{mn} \} = [B] \{ \Delta \sigma_{mn} \}.
\]

Performing the inversion operation (16), we can obtain the desired version of the plasticity matrix at the loading step

\[
\{ \Delta \sigma_{mn} \} = [C]^{-1} \{ \Delta \varepsilon_{mn} \},
\]

where \([C^{-1}] = [B]^{-1}\).

3. Discussion
Performing a comparative analysis of the obtained options for the plasticity matrix at the loading step, we can note their fundamental difference between themselves. A variant based on the application of the aforementioned hypothesis (14–17) is distinguished by its compactness and a significant simplification of the formation compared to option (1–13), based on a multi-stage procedure for calculating partial derivatives.

4. Conclusions
The use of the option of forming a plasticity matrix at the loading step \([C_I] \), based on the use of the hypothesis of proportionality of the components of the deviator of the stress increment to the components of the deviator of the strain increment, in computational algorithms, can significantly simplify the programming procedure and make it clear to potential users of software products when solving real engineering problems.

5. References
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