THE EQUIVARIANT COHOMOLOGY RINGS OF REGULAR NILPOTENT HESSENBerg VARIETIES IN LIE TYPE A : RESEARCH ANNOUNCEMENT

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Dedicated to the memory of Samuel Gitler (1933-2014)

Abstract. Let $n$ be a fixed positive integer and $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ a Hessenberg function. The main result of this manuscript is to give a systematic method for producing an explicit presentation by generators and relations of the equivariant and ordinary cohomology rings (with $\mathbb{Q}$ coefficients) of any regular nilpotent Hessenberg variety $\text{Hess}(h)$ in type A. Specifically, we give an explicit algorithm, depending only on the Hessenberg function $h$, which produces the $n$ defining relations $\{f_{h(j), j}\}_{j=1}^n$ in the equivariant cohomology ring. Our result generalizes known results: for the case $h = (2, 3, 4, \ldots, n, n)$, which corresponds to the Peterson variety $\text{Pet}_n$, we recover the presentation of $H^*_S(\text{Pet}_n)$ given previously by Fukukawa, Harada, and Masuda. Moreover, in the case $h = (n, n, \ldots, n)$, for which the corresponding regular nilpotent Hessenberg variety is the full flag variety $\text{Flags}(\mathbb{C}^n)$, we can explicitly relate the generators of our ideal with those in the usual Borel presentation of the cohomology ring of $\text{Flags}(\mathbb{C}^n)$. The proof of our main theorem includes an argument that the restriction homomorphism $H^*_T(\text{Flags}(\mathbb{C}^n)) \to H^*_S(\text{Hess}(h))$ is surjective. In this research announcement, we briefly recount the context and state our results; we also give a sketch of our proofs and conclude with a brief discussion of open questions. A manuscript containing more details and full proofs is forthcoming.

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Introduction

This paper is a research announcement and is a contribution to the volume dedicated to the illustrious career of Samuel Gitler. A manuscript containing full details is in preparation [1].

Hessenberg varieties (in type A) are subvarieties of the full flag variety $\text{Flags}(\mathbb{C}^n)$ of nested sequences of subspaces in $\mathbb{C}^n$. Their geometry and (equivariant) topology have been studied extensively since the late 1980s [6, 7, 8]. This subject lies at the intersection of, and makes connections between, many research areas such as: geometric representation theory [26, 14], combinatorics [12, 23], and algebraic geometry and topology [5, 20]. Hessenberg varieties also arise in the study of the quantum cohomology of the flag variety [22, 25].

The (equivariant) cohomology rings of Hessenberg varieties has been actively studied in recent years. For instance, Brion and Carrell showed an isomorphism between the equivariant cohomology ring of a regular nilpotent Hessenberg variety with the affine coordinate ring of a certain affine curve [5]. In the special case of Peterson varieties $\text{Pet}_n$ (in type A), the second author and Tymoczko provided an explicit set of generators for $H^*_S(\text{Pet}_n)$ and also proved a Schubert-calculus-type “Monk formula”, thus giving a

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presentation of $H_2^*(Pet_n)$ via generators and relations [10]. Using this Monk formula, Bayegan and the second author derived a “Giambelli formula” [3] for $H_2^*(Pet_n)$ which then yields a simplification of the original presentation given in [10]. Drellich has generalized the results in [10] and [3] to Peterson varieties in all Lie types [10]. In another direction, descriptions of the equivariant cohomology rings of Springer varieties and regular nilpotent Hessenberg varieties in type A have been studied by Dewitt and the second author [9], the third author [18], the first and third authors [2], and Bayegan and the second author [3]. However, it has been an open question to give a general and systematic description of the equivariant cohomology rings of all regular nilpotent Hessenberg varieties [19, Introduction, page 2], to which our results provide an answer (in Lie type A).

Finally, we mention that, as a stepping stone to our main result, we can additionally prove a fact (cf. Section 3) which seems to be well-known by experts but for which we did not find an explicit proof in the literature: namely, that the natural restriction homomorphism $H_1^*(\text{Flags}(\mathbb{C}^n)) \rightarrow H_1^*(\text{Hess}(h))$ is surjective when Hess(h) is a regular nilpotent Hessenberg variety (of type A).

1. Background on Hessenberg varieties

In this section we briefly recall the terminology required to understand the statements of our main results; in particular we recall the definition of a regular nilpotent Hessenberg variety, denoted Hess(h), along with a natural $S^1$-action on it. In this manuscript we only discuss the Lie type A case (i.e. the $GL(n, \mathbb{C})$ case). We also record some observations regarding the $S^1$-fixed points of Hess(h), which will be important in later sections.

By the flag variety we mean the homogeneous space $GL(n, \mathbb{C})/B$ which may also be identified with
\[ \text{Flags}(\mathbb{C}^n) := \{ V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n) \mid \dim \mathbb{C}(V_i) = i \}. \]

A Hessenberg function is a function $h : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ satisfying $h(i) \geq i$ for all $1 \leq i \leq n$ and $h(i+1) \geq h(i)$ for all $1 \leq i < n$. We frequently denote a Hessenberg function by listing its values in sequence, $h = (h(1), h(2), \ldots, h(n) = n)$. Let $N : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a linear operator. The Hessenberg variety (associated to $N$ and $h$) Hess($N$, $h$) is defined as the following subvariety of $\text{Flags}(\mathbb{C}^n)$:
\[ \text{Hess}(N, h) := \{ V_\bullet \in \text{Flags}(\mathbb{C}^n) \mid NV_i \subseteq V_{h(i)} \text{ for all } i = 1, \ldots, n \} \subseteq \text{Flags}(\mathbb{C}^n). \]

If $N$ is nilpotent, we say Hess($N$, $h$) is a nilpotent Hessenberg variety, and if $N$ is a principal nilpotent operator then Hess($N$, $h$) is called a regular nilpotent Hessenberg variety. In this manuscript we restrict to the regular nilpotent case, and as such we denote Hess($N$, $h$) simply as Hess(h) where $N$ is understood to be the standard principal nilpotent operator, i.e. $N$ has one Jordan block with eigenvalue 0.

Next recall that the following standard torus
\[ T = \left\{ \begin{pmatrix} g_1 & & \\ & g_2 & \\ & & \ddots \\ & & & g_n \end{pmatrix} \mid g_i \in \mathbb{C}^* (i = 1, 2, \ldots, n) \right\} \]

acts on the flag variety $\text{Flags}(\mathbb{C}^n)$ by left multiplication. However, this $T$-action does not preserve the subvariety Hess(h) in general. This problem can be rectified by considering instead the action of the following circle subgroup $S$ of $T$, which does preserve Hess(h) ([17 Lemma 5.1]):
\[ S := \left\{ \begin{pmatrix} g & & \\ & g^2 & \\ & & \ddots \\ & & & g^n \end{pmatrix} \mid g \in \mathbb{C}^* \right\} \]

Recall that the $T$-fixed points $\text{Flags}(\mathbb{C}^n)^T$ of the flag variety $\text{Flags}(\mathbb{C}^n)$ can be identified with the permutation group $S_n$ on $n$ letters. More concretely, it is straightforward to see that the $T$-fixed points are the set
\[ \{(e_{w(1)}) \subseteq (e_{w(1)}, e_{w(2)}) \subseteq \cdots \subseteq (e_{w(1)}, e_{w(2)}, \ldots, e_{w(n)}) = \mathbb{C}^n) \mid w \in S_n \} \]
where \(e_1, e_2, \ldots, e_n\) denote the standard basis of \(\mathbb{C}^n\).

It is known that for a regular nilpotent Hessenberg variety \(\text{Hess}(h)\) we have

\[
\text{Hess}(h)^S = \text{Hess}(h) \cap (\text{Flags}(\mathbb{C}^n))^T
\]

so we may view \(\text{Hess}(h)^S\) as a subset of \(S_n\).

2. Statement of the main theorem

In this section we state the main result of this paper. We first recall some notation and terminology. Let \(E_i\) denote the subbundle of the trivial vector bundle \(\text{Flags}(\mathbb{C}^n) \times \mathbb{C}^n\) over \(\text{Flags}(\mathbb{C}^n)\) whose fiber at a flag \(V_i\) is just \(V_i\). We denote the \(T\)-equivariant first Chern class of the line bundle \(E_i/E_{i-1}\) by \(\tilde{\tau}_i \in H^2_T(\text{Flags}(\mathbb{C}^n))\). Let \(C_i\) denote the one dimensional representation of \(T\) through the map \(T \to \mathbb{C}^*\) given by \(\text{diag}(g_1, \ldots, g_n) \mapsto g_i\). In addition we denote the first Chern class of the line bundle \(ET \times_T C_i\) over \(BT\) by \(t_i \in H^2(BT)\). It is well-known that the \(t_1, \ldots, t_n\) generate \(H^*(BT)\) as a ring and are algebraically independent, so we may identify \(H^*(BT)\) with the polynomial ring \(\mathbb{Q}[t_1, \ldots, t_n]\) as rings. Furthermore, it is known that \(H^*_T(\text{Flags}(\mathbb{C}^n))\) is generated as a ring by the elements \(\tilde{\tau}_1, \ldots, \tilde{\tau}_n, t_1, \ldots, t_n\). Indeed, by sending \(x_i\) to \(\tilde{\tau}_i\) and the \(t_i\) to \(t_i\) we obtain the following isomorphism:

\[
H^*_T(\text{Flags}(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n, t_1, \ldots, t_n]/(e_i(x_1, \ldots, x_n) - e_i(t_1, \ldots, t_n) \mid 1 \leq i \leq n).
\]

Here the \(e_i\) denote the degree-\(i\) elementary symmetric polynomials in the relevant variables. In particular, since the odd cohomology of the flag variety \(\text{Flags}(\mathbb{C}^n)\) vanishes, we additionally obtain the following:

\[
H^*(\text{Flags}(\mathbb{C}^n)) \cong \mathbb{Q}[x_1, \ldots, x_n]/(e_i(x_1, \ldots, x_n) \mid 1 \leq i \leq n).
\]

As mentioned in Section 1 in this manuscript we focus on a particular circle subgroup \(S\) of the usual maximal torus \(T\). For this subgroup \(S\), we denote the first Chern class of the line bundle \(ES \times_S \mathbb{C}\) over \(BS\) by \(t \in H^2(\text{BS})\), where by \(\mathbb{C}\) we mean the standard one-dimensional representation of \(S\) through the map \(S \to \mathbb{C}^*\) given by \(\text{diag}(g, g^2, \ldots, g^n) \mapsto g\). Analogous to the identification \(H^*(BT) \cong \mathbb{Q}[t_1, \ldots, t_n]\), we may also identify \(H^*(BS)\) with \(\mathbb{Q}[t]\) as rings.

Consider the restriction homomorphism

\[
H^*_T(\text{Flags}(\mathbb{C}^n)) \to H^*_S(\text{Hess}(h)).
\]

Let \(\tau_i\) denote the image of \(\tilde{\tau}_i\) under \((2.1)\). We next analyze some algebraic relations satisfied by the \(\tau_i\). For this purpose, we now introduce some polynomials \(f_{i,j}(x_1, \ldots, x_n, t) \in \mathbb{Q}[x_1, \ldots, x_n, t]\).

First we define

\[
p_i := \sum_{k=1}^{i} (x_k - kt) \quad (1 \leq i \leq n).
\]

For convenience we also set \(p_0 := 0\) by definition. Let \((i, j)\) be a pair of natural numbers satisfying \(n \geq i \geq j \geq 1\). These polynomials should be visualized as being associated to the \((i, j)\)-th spot in an \(n \times n\) matrix. Note that by assumption on the indices, we only define the \(f_{i,j}\) for entries in the lower-triangular part of the matrix, i.e., the part at or below the diagonal. The definition of the \(f_{i,j}\) is inductive, beginning with the case when \(i = j, i.e.,\) the two indices are equal. In this case we make the following definition:

\[
f_{j,j} := p_j \quad (1 \leq j \leq n).
\]

Now we proceed inductively for the rest of the \(f_{i,j}\) as follows: for \((i, j)\) with \(n \geq i > j \geq 1\) we define:

\[
f_{i,j} := f_{i-1,j-1} + (x_j - x_i - t)f_{i-1,j}.
\]

Again for convenience we define \(f_{i,0} := 0\) for any \(*\).

To make the discussion more concrete, we present an explicit example.

**Example 1.** Suppose \(n = 4\). Then the \(f_{i,j}\) have the following form.

\[
\begin{align*}
f_{i,i} &= p_i \quad (1 \leq i \leq 4) \\
f_{2,1} &= (x_1 - x_2 - t)p_1 \\
f_{3,2} &= (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2 \\
f_{4,3} &= (x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2 + (x_3 - x_4 - t)p_3 \\
f_{3,1} &= (x_1 - x_3 - t)(x_1 - x_2 - t)p_1
\end{align*}
\]
\( f_{4,2} = (x_1 - x_3 - t)(x_1 - x_2 - t)p_1 + (x_2 - x_4 - t)\{(x_1 - x_2 - t)p_1 + (x_2 - x_3 - t)p_2\} \\
\( f_{4,1} = (x_1 - x_4 - t)(x_1 - x_3 - t)(x_1 - x_2 - t)p_1 \)

We can describe and visualize the inductive algorithm determining the \( f_{i,j} \) as follows. First, place in the lower-triangular entries of a \( 4 \times 4 \) matrix the following polynomials in the variables \( x_i \) and \( t \) (recall the \( p_i \) are by definition linear polynomials in these variables):

\[
\begin{array}{c|c|c|c}
   & x_1 - t & p_1(x_1 - x_2 - t) & x_2 - 2t \\
   p_1 \prod_{i=2}^{3} (x_1 - x_i - t) & p_2(x_2 - x_3 - t) & x_3 - 3t \\
   p_1 \prod_{i=2}^{4} (x_1 - x_i - t) & \{p_1(x_1 - x_2 - t) + p_2(x_2 - x_3 - t)\} \times (x_2 - x_4 - t) & p_3(x_3 - x_4 - t) & x_4 - 4t \\
\end{array}
\]

For the following description we introduce some terminology. For \( 0 < k \leq n - 1 \), we refer to the lower-triangular matrix entries in the \((i, j)\)-th spots where \( i - j = k \) as the \( k\)-th lower diagonal. (Equivalently, the \( k\)-th lower diagonal is the “usual” diagonal of the lower-left \((n - k) \times (n - k)\) submatrix.) The polynomials in each matrix entry above are determined inductively, starting on the main diagonal:

1. First place the linear polynomial \( x_i - it \) in the \( i\)-th entry along the diagonal.
2. Proceed inductively as follows. Temporarily denote by \( \Delta_{a,b} \) the entry in the \((a, b)\)-th box, assuming it has already been defined. Suppose \((i, j)\) satisfies \( i > j \). Define

\[
\Delta_{i,j} := \left( \sum_{t=1}^{j} \Delta_{i-j+t-1,t} \right) (x_j - x_i - t).
\]

In words, this means the following. Suppose \( k = i - j > 0 \). Then \( \Delta_{i,j} \) is the product of \( (x_j - x_i - t) \) with the sum of the entries in the boxes which are in the “diagonal immediately above the \((i,j)\) box” (i.e. the boxes which are in the \((k - 1)\)-st lower diagonal), but we omit any boxes to the right of the \((i,j)\) box (i.e. in columns \( j + 1 \) or higher). With this notation in place, the polynomials \( f_{i,j} \) are obtained by taking the sum of the entries in the \((i,j)\)-th box and any boxes “to its left” in the same lower diagonal. More precisely,

\[
(2.5) \quad f_{i,j} = \sum_{k=1}^{j} \Delta_{i-j+k,k}.
\]

The same algorithm works for general \( n \).

We are now ready to state our main result.

**Theorem 2.1.** Let \( n \) be a positive integer and \( h : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\} \) a Hessenberg function. Let \( \text{Hess}(h) \subset \mathcal{F}_n \) denote the corresponding regular nilpotent Hessenberg variety equipped with the circle \( S^1 \)-action described above. Then the restriction map

\[
H^*_T(\mathcal{F}_n) \rightarrow H^*_T(\text{Hess}(h))
\]

is surjective. Moreover, there is an isomorphism of \( \mathbb{Q}[t] \)-algebras

\[
H^*_T(\text{Hess}(h)) \cong \mathbb{Q}[x_1, \ldots, x_n, t]/I(h)
\]

sending \( x_i \) to \( \tau_i \) and \( t \) to \( t \) and we identify \( H^*(BS) = \mathbb{Q}[t] \). Here the ideal \( I(h) \) is defined by

\[
(2.6) \quad I(h) := (f_{h(j),j} \mid 1 \leq j \leq n).
\]

Informally and visually, we may describe the ideal \( I(h) \) defined in (2.6) as follows. As in Example above, we may visualize the \( f_{i,j} \) as being associated to the lower-triangular \((i,j)\)-th entry in an \( n \times n \) matrix, as
follows:
\[
\begin{pmatrix}
  f_{1,1} & 0 & \cdots & \cdots & 0 \\
  f_{2,1} & f_{2,2} & 0 & \cdots & \\
  f_{3,1} & f_{3,2} & f_{3,3} & \ddots & \\
  \vdots & \vdots & \vdots & \ddots & \\
  f_{n,1} & f_{n,2} & \cdots & \cdots & f_{n,n}
\end{pmatrix}
\]

Similarly, any Hessenberg function \( h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) determines a subspace of the vector space \( M(n \times n, \mathbb{C}) \) of matrices as follows: an \((i, j)\)-th entry is required to be 0 if \( i > h(j) \). If we represent a Hessenberg function \( h \) by listing its values \( (h(1), h(2), \ldots, h(n)) \), then the Hessenberg subspace can be described in words as follows: the first column (starting from the left) is allowed \( h(1) \) non-zero entries (starting from the top), the second column is allowed \( h(2) \) non-zero entries, and so on. For example, if \( h = (3, 3, 4, 5, 7, 7, 7) \) then the Hessenberg subspace is
\[
\begin{pmatrix}
  \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
  \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
  0 & 0 & \ast & \ast & \ast & \ast & \ast \\
  0 & 0 & 0 & \ast & \ast & \ast & \ast \\
  0 & 0 & 0 & 0 & \ast & \ast & \ast \\
  0 & 0 & 0 & 0 & 0 & \ast & \ast \\
\end{pmatrix} \subseteq M(7 \times 7, \mathbb{C}).
\]

Then the ideal \( I(h) \) can be described as being “generated by the \( f_{i,j} \) in the boxes at the bottom of each column in the Hessenberg space”. For instance, in the \( h = (3, 3, 4, 5, 7, 7, 7) \) example above, the generators are \( \{f_{3,1}, f_{3,2}, f_{4,3}, f_{5,4}, f_{7,5}, f_{7,6}, f_{7,7}\} \).

Our main result generalizes previous known results.

**Remark 1.** Consider the special case \( h = (2, 3, \ldots, n, n) \). In this case the corresponding regular nilpotent Hessenberg variety has been well-studied and it is called a Peterson variety \( Pet_n \) (of type \( A \)). Our result above is a generalization of the result in [11] which gives a presentation of \( H^*_S(Pet_n) \). Indeed, for \( 1 \leq j \leq n - 1 \), we obtain from (2.4) and (2.2) that
\[
f_{j+1,j} = f_{j,j-1} + (x_j - x_{j+1} - t)f_{j,j} = f_{j,j-1} + (-p_j - p_{j+1} - 2t)p_j
\]
and since \( f_{n,n} = p_n \) we have
\[
H^*_S(Pet_n) \cong \mathbb{Q}[x_1, \ldots, x_n, t]/(f_{j,j-1} + (-p_j - p_{j+1} - 2t)p_j, p_n \mid 1 \leq j \leq n - 1)
\]
\[
= \mathbb{Q}[p_1, \ldots, p_{n-1}, t]/((-p_j - p_{j+1} - 2t)p_j, p_n \mid 1 \leq j \leq n - 1)
\]
which agrees with [11]. (Note that we take by convention \( p_0 = p_n = 0 \).)

The main theorem above also immediately yields a computation of the ordinary cohomology ring. Indeed, since the odd degree cohomology groups of \( Hess(h) \) vanish [29], by setting \( t = 0 \) we obtain the ordinary cohomology. Let \( f_{i,j} := f_{i,j}(x, t = 0) \) denote the polynomials in the variables \( x_i \) obtained by setting \( t = 0 \). A computation then shows that
\[
\hat{f}_{i,j} = \sum_{k=1}^{j} x_k \prod_{\ell=j+1}^{i} (x_k - x_{\ell}).
\]
(For the case \( i = j \) we take by convention \( \prod_{\ell=j+1}^{i} (x_k - x_{\ell}) = 1 \).) We have the following.

**Corollary 2.2.** Let the notation be as above. There is a ring isomorphism
\[
H^*(Hess(h)) \cong \mathbb{Q}[x_1, \ldots, x_n]/\overline{I}(h)
\]
where \( \overline{I}(h) := (\hat{f}_{h(j),j} \mid 1 \leq j \leq n) \).
Remark 2. Consider the special case $h = (n, n, \ldots, n)$. In this case the condition in (1.1) is vacuous and the associated regular nilpotent Hessenberg variety is the full flag variety $\mathcal{F}lags(\mathbb{C}^n)$. In this case we can explicitly relate the generators $f_{h(j) = n, j}$ of our ideal $I(h) = I(n, n, \ldots, n)$ with the power sums $p_r(x) = p_r(x_1, \ldots, x_n) := \sum_{k=1}^n x_k^r$, thus relating our presentation with the usual Borel presentation, see e.g. [13]. More explicitly, for $r$ be an integer, $1 \leq r \leq n$, define

$$q_r(x) = q_r(x_1, \ldots, x_n) := \sum_{k=1}^{n+1-r} x_k \prod_{\ell=n+2-r}^n (x_k - x_\ell).$$

Note that by definition $q_r(x) = \hat{f}_{n,n+1-r}$ so these are the generators of $I(n, n, \ldots, n)$. The polynomials $q_r(x)$ and the power sums $p_r(x)$ can then be shown to satisfy the relations

$$(2.7) \quad q_r(x) = \sum_{i=0}^{r-1} (-1)^i e_i(x_{n+2-r}, \ldots, x_n) p_{r-i}(x).$$

Remark 3. In the usual Borel presentation of $H^*(\mathcal{F}lags(\mathbb{C}^n))$, the ideal $I$ of relations is taken to be generated by the elementary symmetric polynomials. The power sums $p_r$ generate this ideal $I$ when we consider the cohomology with $\mathbb{Q}$ coefficients, but this is not true with $\mathbb{Z}$ coefficients. Thus our main Theorem 2.1 does not hold with $\mathbb{Z}$ coefficients in the case when $h = (n, n, \ldots, n)$, suggesting that there is some subtlety in the relationship between the choice of coefficients and the choice of generators of the ideal $I(h)$.

### 3. Sketch of the proof of the main theorem

We take a moment to sketch the outline of the proof of the main result (Theorem 2.1) above. As a first step, we show that the elements $\tau_i$ satisfy the relations $f_{h(j) = n, j} = f_{h(j) = \tau_1, \ldots, \tau_n, t} = 0$. The main technique of this part of the proof is (equivariant) localization, i.e. the injection

$$(3.1) \quad H^*_S(\text{Hess}(h)) \rightarrow H^*_S(\text{Hess}(h)^S).$$

Specifically, we show that the restriction $f_{h(j) = n, j}(w)$ of each $f_{h(j)}$ to an $S$-fixed point $w \in \text{Hess}(h)^S$ is equal to 0; by the injectivity of (3.1), this then implies that $f_{h(j) = n, j} = 0$ as desired. This part of the argument is rather long and requires a technical inductive argument based on a particular choice of total ordering on $\text{Hess}(h)^S$ which refines a certain natural partial order on Hessenberg functions. Once we show $f_{h(j) = n, j} = 0$ for all $j$, we obtain a well-defined ring homomorphism which sends $x_i$ to $\tau_i$ and $t$ to $t$:

$$(3.2) \quad \varphi_h : \mathbb{Q}[x_1, \ldots, x_n, t]/(f_{h(j), j} \mid 1 \leq j \leq n) \rightarrow H^*_S(\text{Hess}(h)).$$

We then show that the two sides of (3.2) have identical Hilbert series. This part of the argument is rather straightforward, following the techniques used in e.g. [11].

The next key step in our proof of Theorem 2.1 relies on the following two key ideas: firstly, we use our knowledge of the special case where the Hessenberg function $h$ is $h = (n, n, \ldots, n)$, for which the associated regular nilpotent Hessenberg variety is the full flag variety $\mathcal{F}lags(\mathbb{C}^n)$, and secondly, we consider localizations of the rings in question with respect to $R := \mathbb{Q}[t] \setminus \{0\}$. For the following, for $h = (n, n, \ldots, n)$ we let $\mathcal{H} := \text{Hess}(h = (n, n, \ldots, n)) = \mathcal{F}lags(\mathbb{C}^n)$ denote the full flag variety and let $I$ denote the associated ideal $I(n, n, \ldots, n)$. In this case we know that the map $\varphi := \varphi(n, n, \ldots, n)$ is surjective since the Chern classes $\tau_i$ are known to generate the cohomology ring of $\mathcal{F}lags(\mathbb{C}^n)$. Since the Hilbert series of both sides are identical, we then know that $\varphi$ is an isomorphism.

The following commutative diagram is crucial for the remainder of the argument.

$$\begin{array}{ccc}
R^{-1}(\mathbb{Q}[x_1, \ldots, x_n, t]/I) & \xrightarrow{R^{-1}\varphi} & R^{-1}H^*_S(\mathcal{H}) \\
\downarrow_{\text{surj}} & & \downarrow_{\text{surj}} \\
R^{-1}(\mathbb{Q}[x_1, \ldots, x_n, t]/I(h)) & \xrightarrow{R^{-1}\varphi_h} & R^{-1}H^*_S(\text{Hess}(h))
\end{array}$$

The horizontal arrows in the right-hand square are isomorphisms by the localization theorem. Since $\varphi$ is an isomorphism, so is $R^{-1}\varphi$. The rightmost and leftmost vertical arrows are easily seen to be surjective,
implying that $R^{-1}\varphi_h$ is also surjective. A comparison of Hilbert series shows that $R^{-1}\varphi_h$ is an isomorphism. Finally, to complete the proof we consider the commutative diagram

$$
\begin{array}{ccc}
\mathbb{Q}[x_1, \ldots, x_n, t]/I(h) & \xrightarrow{\varphi_h} & H^*_S(\text{Hess}(h)) \\
\downarrow\text{inj} & & \downarrow\text{inj} \\
R^{-1}\mathbb{Q}[x_1, \ldots, x_n, t]/I(h) & \xrightarrow{R^{-1}\varphi_h} & R^{-1}H^*_S(\text{Hess}(h))
\end{array}
$$

for which it is straightforward to see that the vertical arrows are injections. From this it follows that $\varphi_h$ is an injection, and once again a comparison of Hilbert series shows that $\varphi_h$ is in fact an isomorphism.

4. Open questions

We outline a sample of possible directions for future work.

- In [24], Mbirika and Tymoczko suggest a possible presentation of the cohomology rings of regular nilpotent Hessenberg varieties. Using our presentation, we can show that the Mbirika-Tymoczko ring is not isomorphic to $H^*(\text{Hess}(h))$ in the special case of Peterson varieties for $n - 1 \geq 2$, i.e. when $h(i) = i + 1, 1 \leq i < n$ and $n \geq 3$. (However, they do have the same Betti numbers.) In the case $n = 4$, we have also checked explicitly for the Hessenberg functions $h = (2, 4, 4, 4), h = (3, 3, 4, 4), h = (3, 4, 4, 4)$ that the relevant rings are not isomorphic. It would be of interest to understand the relationship between the two rings in some generality.

- In [15], the last three authors give a presentation of the (equivariant) cohomology rings of Peterson varieties for general Lie type in a pleasant uniform way, using entries in the Cartan matrix. It would be interesting to give a similar uniform description of the cohomology rings of regular nilpotent Hessenberg varieties for all Lie types.

- In the case of the Peterson variety (type A), a basis for the $S$-equivariant cohomology ring was found by the second author and Tymoczko in [16]. In the general regular nilpotent case, and following ideas of the second author and Tymoczko [17], it would be of interest to construct similar additive bases for $H^*_S(\text{Hess}(h))$. Additive bases with suitable geometric or combinatorial properties could lead to an interesting ‘Schubert calculus’ on regular nilpotent Hessenberg varieties.

- Fix a Hessenberg function $h$ and let $S : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a regular semisimple linear operator. There is a natural Weyl group action on the cohomology ring $H^*(\text{Hess}(S, h))$ of the regular semisimple Hessenberg variety corresponding to $h$ (cf. for instance [30, p. 381] and also [28]). Let $H^*(\text{Hess}(S, h))^W$ denote the ring of $W$-invariants where $W$ denotes the Weyl group. It turns out that there exists a surjective ring homomorphism $H^*(\text{Hess}(N, h)) \rightarrow H^*(\text{Hess}(S, h))^W$ which is an isomorphism in the special case of the Peterson variety. (Historically this line of thought goes back to Klyachko’s 1985 paper [21].) In an ongoing project, we are investigating properties of this ring homomorphism for general Hessenberg functions $h$.

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