Some eigenstates for a model associated with solutions of tetrahedron equation.

II. A bit of algebraization

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Abstract

This paper adds two observations to the work solv-int/9701016 where some eigenstates for a model based on tetrahedron equation have been constructed. The first observation is that there exists a more “algebraic” construction of one-particle states, resembling the 1 + 1-dimensional algebraic Bethe ansatz. The second observation is that the strings introduced in solv-int/9701016 are symmetries of a transfer matrix, rather than just eigenstates.

Introduction

This work is a continuation of the work [1] where some eigenstates were introduced for a model based on the solutions to the tetrahedron equations described in paper [2]. It contains two separate observations on two sorts of eigenstates introduced in [1]: particle-like states (plane waves and their superpositions), and string-like states. Those observations are explained in Sections I and II respectively. The reasons for doing this rather technical work are explained in Section III.

Let us give some definitions and remarks. We will depict the operators graphically in such a way that each operator will have some number of “incoming edges” and the same number of “outgoing edges” (or “links”). To each edge corresponds its own copy of a two-dimensional complex linear space, and to several edges of the same (incoming or outgoing) kind together corresponds the tensor product of their spaces. Each of the mentioned two-dimensional spaces has a basis of a 0-particle and 1-particle vectors.
For any, maybe infinite, collection of edges, we will define this collection’s 0-particle vector, or vacuum, as the tensor product of 0-particle vectors throughout the collection (in this paper, the meaning of infinite tensor products will be always clear). Further, we will identify a 1-particle vector in an edges with its tensor product with the 0-particle vectors in all the collection’s other edges and define a collection’s 1-particle vector as a formal sum over all its edges of the corresponding 1-particle vectors, with any complex coefficients. Then, we can define in an obvious way the 2-particle, 3-particle etc. states.

According to the above, an operator acts from the tensor product of “incoming” spaces to the tensor product of “outgoing”, i.e. different, spaces. Still, sometimes we will assume that all the edges along one straight line represent the same two-dimensional space. This is convenient e.g. when we write out the tetrahedron equation, as in formula (1) below, and this will never lead to confusion.

1 Algebraization of one-particle-state construction

In this section, we will present a one-parameter family of “creation operators”. When applied to the “vacuum”, these operators produce one-particle states—plane waves, described already in the paper [1]. As we will see, the very construction of these operators presupposes that they act on vectors which don’t differ much from the “vacuum”. We will not try to make this statement more exact here. Instead, in this paper we will assume from the beginning that the domain of definition of those operators consists of only one-dimensional space generated by vacuum, leaving the extension of that domain for further work.

1.1 Description of transfer matrices from which the creation operators are constructed

Creation operators will be transfer matrices on a kagome lattice with some special boundary conditions. Graphically, such a transfer matrix is depicted in Figure 1. We are considering the eigenstates of transfer matrix made up of “hedgehogs”, as in work [1]. Naturally, a kagome transfer matrix must be such that it would be possible to bring the hedgehogs through it using the tetrahedron equation.

The present work, as well as [1], is based only on simple solutions of that equation.

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that can be found in [2]. For those solutions, the following form of equation holds:

\[ S_{01,12}^0, 12^0, 02^0, 12^0, 03^0, 13^0, 02^0, 12^0, 03^0, 23^0, 01^0, 02^0, 12^0, 03^0, 13^0, 23^0, 01^0, 02^0, 12^0, 03^0, 13^0, 23^0, (\varphi_0, \varphi_1, \varphi_2) \]

\[ = S_{12,23}^1, 13^1, 23^1, 12^1, 03^1, 23^1, 12^1, 03^1, 23^1, 01^1, 02^1, 12^1, 03^1, 13^1, 23^1, 01^1, 02^1, 12^1, 03^1, 13^1, 23^1, (\varphi_1, \varphi_2, \varphi_3). \]  

(1)

Here a number 0, 1, 2, or 3 is attached to a plane, that is, to a face of the tetrahedron. An operator \( S_{ij,ik,jk} \) acts in the tensor product of three linear spaces corresponding to the lines—intersections of those planes.

Assuming that parameters \( \varphi_1, \varphi_2, \varphi_3 \) belong to the “hedgehogs” \( S_{12,13,23} \) and are given, there remains one free parameter \( \varphi_0 \), where the number 0 is attached to the plane of the kagome lattice.

1.2 Boundary conditions for creation operators

It will take some effort to describe the boundary conditions that we are going to impose on kagome transfer matrices of Subsection 1.1 to obtain out of them creation operators. The problem is that we are considering an infinite in all plane directions kagome lattice. So, first, let us draw in Figure 2 the lattice viewed from above (here, we have deformed the lattice a bit in comparison with the paper [1]). Then let us draw a dashed line \( AB \) and cut off for a while the part of the lattice lying to the left of that line (it will be explained in Subsection 1.3 that really there is much arbitrariness in choosing the line \( AB \), but let it be for now as in Figure 2). For the rest of transfer matrix, let us define the boundary condition along \( AB \) as follows. Consider all the lattice edges intersecting \( AB \). They are incoming edges for the remaining part of transfer matrix. To define the boundary conditions, we must
Figure 2
indicate some vector $\Sigma_{AB}$ in the tensor product of corresponding spaces. Let us assume that $\Sigma_{AB}$ is a 1-particle vector as defined in the Introduction, whose exact form is to be determined.

Consider the band—the part of transfer matrix lying between the lines $AB$ and $CD$. This band represents an operator acting from the space corresponding to its incoming edges into the space corresponding to its outgoing edges, where the incoming edges are those intersecting $AB$ and those pointing from behind the kagome lattice plane into the vertices situated within the band, while the outgoing edges are those intersecting $CD$ and those pointing from the vertices situated within the band at the reader. The latter vertices are marked $\odot$ in Figure 2.

Let us require that our band operator—let us call it $\mathcal{B}$—transform the tensor product $\Sigma_{AB} \otimes \Omega_\varnothing$, where $\Omega_\varnothing$ is the vacuum for the set of edges pointing into the "$\odot$" vertices, into the following sum:

$$\mathcal{B} \Sigma_{AB} \otimes \Omega_\varnothing = \kappa \Sigma_{CD} \otimes \Omega'_\varnothing + \Omega_{CD} \otimes \Psi_\varnothing,$$

where $\kappa$ is a number; $\Sigma_{CD}$ is the vector similar to $\Sigma_{AB}$, but corresponding to the edges situated one lattice period to the right, i.e. intersecting $CD$; $\Omega'_\varnothing$ is the vacuum for edges pointing from the "$\odot$" vertices to the reader (who can thus identify $\Omega'_\varnothing$ with $\Omega_\varnothing$ if desired); $\Omega_{CD}$ is the vacuum for the set of vectors intersecting $CD$; $\Psi_\varnothing$ is some vector lying in the same tensor product of spaces as $\Omega'_\varnothing$. It is remarkable that, for any $\varphi_0$, relation (2) can be satisfied with a proper choice of $\Sigma_{AB}$. The vector $\Psi_\varnothing$ will then be a 1-particle vector. Some details of calculations concerning relation (2) are explained in Subsection 1.3, while here we are going to use this relation.

The next band, lying to the right of $CD$, has vector $\kappa \Sigma_{CD}$ as its incoming vector. Thus, the whole situation is repeated up to the factor $\kappa$. On the other hand, we could have cut the lattice, instead of the line $AB$, along some other line lying, e.g., $n$ lattice periods to the left. In that case, we should have taken for the incoming vector the vector $\Sigma_{AB}$ shifted by $n$ periods to the left and multiplied by $\kappa^{-n}$. Letting $n \to \infty$, we get a $\Psi_\varnothing$-like vector in every band of the sort depicted in Figure 2. Summing up all those vectors, we get a 1-particle state of the same kind as in paper [1]. Those states are now parameterized by the parameter $\varphi_0$.

In fact, one more boundary condition must be imposed at the “right infinity” of the lattice. This is explained in the end of Subsection 1.3.

### 1.3 Some technical details

Instead of the straight line $AB$ in Figure 2, we could use any (connected) curve $l$ intersecting each straight line of the kagome lattice exactly one time in such way that a boundary condition is given in the tensor product corresponding to the edges that intersect $l$. Assuming that a 1-particle vector is given as the boundary condition,
we will require that after any deformation of \( l \) such that it passes through one of the lattice vertices, the boundary condition remain to be 1-particle.

In other words, the mentioned vertex is added to or withdrawn from the considered part of the lattice. Let that vertex be, e.g., such as in Figure 3. An incoming 1-particle vector for it is described by two amplitudes \( a \) and \( b \), with \( a \) corresponding to the edge 01 (see the text just after equation (1)) and \( b \) — to the edge 02. It is required that the result of transforming this incoming vector by the matrix \( S^T \) (we assume, as in work [1], that in each vertex there is a matrix of the same type as on page 96 of paper [2], but transposed) contain no three-particle part. This leads at once to the condition

\[
\frac{a}{b} = \sqrt{\tanh(\varphi_0 - \varphi_1)} \sqrt{\tanh(\varphi_0 - \varphi_2)}. \tag{3}
\]

The other ratios of the amplitudes written out in Figure 3 are simply the matrix elements of \( S^T \):

\[
\frac{c}{a} = \frac{f}{b} = \sqrt{\tanh(\varphi_0 - \varphi_1)} \sqrt{\cotanh(\varphi_0 - \varphi_2)}. \tag{4}
\]

Similar relations can be written for the kagome lattice vertices of two other kinds, that is \( \bigodot \) and \( \bigtriangledown \). It turns out that all those relations together determine the amplitudes at all lattice edges from a given one of them without contradiction. Thus, the amplitudes for the \( \bigodot \)-edges, that are incoming for the hedgehog transfer matrix, are determined correctly.

Those amplitudes give exactly its eigenstate for any fixed \( \varphi_0 \). This can be proved by a rather obvious reasoning: use the tetrahedron equation and the possibility to express the amplitudes at different edges through one another. The details are left for the reader.
There remains, however, another detail that is important: we must impose one more boundary condition, that is at the “right infinity” (see again Figure 2). In order to obtain the 1-particle vector at the \( \odot \)-edges, and no vacuum component, let us take a straight line \( C'D' \)—like \( CD \), but somewhere far to the right—and require any 1-particle vector in the space corresponding to edges that intersect \( C'D' \) be multiplied by zero, while the vacuum vector in that space be left intact. This can be interpreted as taking the scalar product of the vector at the edges intersecting \( C'D' \) and the “vacuum covector”. Then, of course, we let \( C'D' \) tend to the right infinity, so all this procedure does not change the 1-particle component of the \( \odot \)-vector, but the vacuum component vanishes.

2 Strings as symmetries

Let us introduce the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

as well as a unity matrix

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Note that the subscripts of these matrices have other meaning than the subscripts of \( S \)-matrices in equations like (1).

It follows from the explicit form of \( S \)-matrices given in paper [2] that the \( S \)-matrices commute with the operators

\[
\sigma_2 \otimes \sigma_2 \otimes \sigma_0, \quad \sigma_0 \otimes \sigma_1 \otimes \sigma_1 \quad \text{and} \quad \sigma_1 \otimes \sigma_0 \otimes \sigma_2.
\]

(5)

If now we select some set of the kagome lattice horizontal lines (to be exact, of those depicted in Figure 2 as horizontal) and consider the tensor product of matrices \( \sigma_2 \) over all vertices belonging to those lines, then the hedgehog transfer matrix \( T \) will be permutable with that product up to the fact that the lines move in the lattice plane, as explained in work [1] (the lines result from the intersection of cubic lattice faces with a plane perpendicular to a cube’s spatial diagonal, and move in that plane when the plane itself moves). This permutability follows immediately from the fact that \( S \) commutes with the first of operators (5). Similarly, it is not difficult to formulate the analogous statements for sets of oblique and vertical lines, using respectively the second and third of products (5).

Using the described symmetries of transfer matrix \( T \), we can, starting from any state vector \( \Theta_0 \) whose evolution under the action of degrees of \( T \) we can describe in this or that way (recall that \( T \) is such that its degrees are represented graphically
as “oblique layers” of the cubic lattice), build many new states $\Theta$ whose evolution we will also be able to describe.

3 Discussion

The aim of our “algebraization” is, of course, to learn to construct multi-particle states for $2 + 1$-dimensional models. Let us remind that only some special two-particle states have been constructed in paper [1], and even the superposition of two arbitrary one-particle states from that very work has not been obtained there.

In this paper, we still do not present the construction of multi-particle states. The reason for doing this work is a hope that, with the help of a proper “regularization”, the action of our creation operators can be extended from just vacuum onto one-particle, two-particle and so on (eigen) vectors.

It is clear that the superposition of two arbitrary one-particle eigenstates from work [1] cannot lie just in the 2-particle space as it is defined in Introduction. So, probably, some new terminology should be introduced to distinguish between the 2-particle space and 2-excitation states.

As for the strings providing symmetries and thus multiplying the eigenstates, it is still to be clarified whether those strings are particular cases of a family including more interesting species.

Finally, it is certainly interesting to find eigenstates for the model based on other simple solutions to the tetrahedron equation [3], and perhaps for the general model described in [1].

References

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