2-Nilpotent Multiplier and 2-Capability of Finite 2-Generator $p$-Groups of Class Two

Farangis Johari and Azam Kaheni

Abstract. Let $p$ be a prime number. We give the explicit structure of 2-nilpotent multiplier for each finite 2-generator $p$-group of class two. Moreover, 2-capable groups in that class are characterized.

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1. Introduction and Motivation

The first classification for finite 2-generator $p$-groups of class two for an odd prime $p$ was given by Bacon and Kappe [2]. In 1999, Kappe, Vissher, and Sarmin [6] generalized the previous classification to the case $p = 2$. Later, Ahmad, Magidin, and Morse [1] discovered that these classifications were incomplete. They corrected these omissions by giving a full classification of these groups as stated in the following.

Theorem 1.1. [1, Theorem 1.1] Let $p$ be a prime and $n > 2$. Every 2-generated $p$-group of class exactly 2 corresponds to an ordered 5-tuple of integers, $(\alpha, \beta, \gamma; \rho, \sigma)$, such that:

(i) $\alpha \geq \beta \geq \gamma \geq 1$;
(ii) $\alpha + \beta + \gamma = n$;
(iii) $0 \leq \rho \leq \gamma$ and $0 \leq \sigma \leq \gamma$;

where $(\alpha, \beta, \gamma; \rho, \sigma)$ corresponds to the group presented by

$$G = \langle a, b \mid [a, b]^{p^\gamma} = [a, b, b] = [a, b, a] = 1, a^{p^\alpha} = [a, b]^{p^\rho}, b^{p^\beta} = [a, b]^{p^\sigma} \rangle.$$ 

Moreover:

FAMILY (1) If $\alpha > \beta$, then $G$ is isomorphic to:

(a) $(\alpha, \beta, \gamma; \rho, \gamma)$, when $\rho \leq \sigma$.
(b) $(\alpha, \beta, \gamma; \gamma, \sigma)$, when $0 \leq \sigma < \sigma + \alpha - \beta \leq \rho$ or $\sigma < \rho = \gamma$.
(c) $(\alpha, \beta, \gamma; \rho, \sigma)$, when $0 \leq \sigma < \rho < \min(\gamma, \sigma + \alpha - \beta)$.
FAMILY (2) If $\alpha = \beta > \gamma$, or $\alpha = \beta = \gamma$, and $p > 2$, then $G$ is isomorphic to $(\alpha, \beta, \gamma; \min(p, \sigma), \gamma)$.

FAMILY (3) If $\alpha = \beta = \gamma$, and $p = 2$, then $G$ is isomorphic to:

(a) $(\alpha, \beta, \gamma; \min(p, \sigma), \gamma)$, when $0 \leq \min(p, \sigma) < \gamma - 1$.
(b) $(\alpha, \beta, \gamma; \gamma - 1, \gamma - 1)$, when $p = \sigma = \gamma - 1$.
(c) $(\alpha, \beta, \gamma; \gamma, \gamma)$, when $\min(p, \sigma) \geq \gamma - 1$ and $\max(p, \sigma) = \gamma$.

The groups listed in 1(a)–3(c) are pairwise nonisomorphic.

The idea of computing the nonabelian tensor square and the nonabelian exterior square for these groups was started in [2,6], but these computations were based on the previous classification. Magidin and Morse [8] computed various homological functions for these groups, using the new classification in [1]. These functions include the nonabelian tensor square, the nonabelian exterior square, and the Schur multiplier. They also determined which of these groups are capable. A group $G$ is called capable if $G$ is isomorphic to the center factor of some group $H$. The following result gives necessary and sufficient conditions for the capability of 2-generator $p$-groups of class two.

Lemma 1.2. [8, Theorems 63 and 67] Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator $p$-group of class two with the presentation as in Theorem 1.1. Then the following results hold:

(a) For an odd prime number $p$, $G$ is capable if and only if $\alpha - \beta = \rho - \sigma$ and $\beta = \rho$.
(b) For $p = 2$, $G$ is capable if and only if it meets one of the following conditions:

(i) $\rho \leq \sigma, \alpha = \beta$, and $\rho = \gamma$;
(ii) $\rho > \gamma, (\alpha - \beta) = (\rho - \sigma) > \delta_{\beta \gamma}$, and $\rho = \gamma$, where $\delta_{\beta \gamma}$ is the Kronecker delta;
(iii) $\rho = \sigma = \gamma = \beta$, and $\alpha = \gamma + 1$.

Using Theorem 1.1 and Lemma 1.2, all capable 2-generator $p$-groups of class two are listed as follows:

Corollary 1.3. Let $G$ be a 2-generator $p$-group of class two. Then the following assertions hold:

(a) If $p$ is odd, then $G$ is capable if and only if $G$ is isomorphic to exactly one of the following groups.

(i) $G \cong G_1 = \langle a, b \mid [a, b]^{p^\alpha} = a^\rho b^\sigma = b^\alpha a = [a, b, b] = [a, b, a] = 1 \rangle$.

(ii) $G \cong G_2 = \langle a, b \mid [a, b]^{p^\gamma} = a^\rho b^\sigma = b^\alpha a = [a, b, b] = [a, b, a] = 1, \alpha > \gamma \rangle$.

(iii) $G \cong G_3 = \langle a, b \mid [a, b]^{p^\gamma} = a^\rho b^\sigma = b^\alpha a = [a, b, b] = [a, b, a] = 1, b^\beta = [a, b]^{p^\beta} \rangle$, where $\alpha - \beta = \gamma - \sigma$ and $0 \leq \sigma < \gamma$.

(b) If $p = 2$, then $G$ is capable if and only if $G$ is isomorphic to exactly one of the following groups.

(iv) $G \cong G_4 = \langle a, b \mid [a, b]^{2^\beta} = a^{2^\beta} = b^{2^\sigma} = [a, b, b] = [a, b, a] = 1 \rangle$.

(v) $G \cong G_5 = \langle a, b \mid [a, b]^{2^\beta} = a^{2^\beta} = b^{2^\sigma} = [a, b, b] = [a, b, a] = 1 \rangle$.

(vi) $G \cong G_6 = \langle a, b \mid [a, b]^{2^\gamma} = a^{2^\alpha} = b^{2^\alpha} = [a, b, b] = [a, b, a] = 1, \alpha > \gamma \rangle$. 
(vii) $G \cong G_7 = \langle a, b \mid [a, b]^{2^7} = a^{2^\alpha} = [a, b, b] = [a, b, a] = 1, b^{2^\beta} = [a, b]^{2^\gamma} \rangle$, where $\alpha - \beta > \delta \beta \gamma, \alpha - \beta = \gamma - \sigma, \alpha > \beta, \gamma > \sigma,$ and $\delta_{ij}$ is the Kronecker delta.

Burns and Ellis [4] generalized the notion of capability for groups to c-capability. A group $G$ is called c-capable if $G \cong H/Z_c(H)$ for some group $H$, where $Z_c(H)$ is the c-th term of the upper central series of $H$. Clearly, each $(c + 1)$-capable group is c-capable. The converse of this statement is not true in general. A counterexample was given by Burns and Ellis [4]. Despite of this flaw example, someone try to find groups in which two concepts of capability and c-capability are equivalent for them. Finitely generated abelian groups, extra special $p$-groups, and also generalized extra special $p$-groups are some such groups (for more details, see [4, 11, 12]). Recently, Monfared, Kayvanfar, and Johari [10] showed that “capability” and “2-capability” coincide for finite 2-generator 2-groups of class two. More precisely, they gave a classification of all 2-capable 2-generator 2-groups of class two and proved that for these groups each capable group is 2-capable. Furthermore, they computed the 2-nilpotent multiplier of 2-capable 2-generator 2-groups of class two.

The main result of the present paper is to classify all 2-capable finite 2-generator $p$-groups of class two. At first, we determine the structure of 2-nilpotent multipliers for all capable 2-generator $p$-groups of class two when $p$ is odd. Then, for these $p$-groups, we show that each capable group is 2-capable. Finally, we compute 2-nilpotent multipliers for all noncapable 2-generator $p$-groups of class two for an arbitrary prime number $p$, to clarify the structure of 2-nilpotent multiplier for each 2-generator $p$-group of class two.

Now, we state our main results.

The following result describes 2-nilpotent multipliers for all capable finite 2-generator $p$-groups of class two for $p > 2$.

**Theorem A.** Let $p$ be an odd prime number and $G$ be a capable finite 2-generator $p$-group of class two with the presentation as in Corollary 1.3(a). Then

$$M^{(2)}(G) \cong \begin{cases} Z_p^{(5)} & \text{if } G \cong G_1, \\ Z_p^{(2)} \oplus Z_p^{(3)} & \text{if } G \cong G_2, \\ Z_p^{(5)} \oplus Z_p^{(3)} \oplus Z_p^{(3)} & \text{if } G \cong G_3, \end{cases}$$

where $Z_r^{(t)}$ denotes the direct sum of $t$ copies of $Z_r$, in which $Z_r$ is the cyclic group of order $r$.

The next result gives the exact structure of 2-capable finite 2-generator $p$-groups of class two.

**Theorem B.** Let $G$ be a finite 2-generator $p$-group of class two for odd prime $p$. Then $G$ is 2-capable if and only if $G$ is isomorphic to exactly one of the $p$-groups $G_1, G_2,$ or $G_3$.

The following result is immediately obtained from Corollary 1.3, Theorem B, and [10, Corollary 4.4].
Corollary C. Let $G$ be a finite 2-generator $p$-group of class two. Then $G$ is 2-capable if and only if $G$ is capable.

Recall that the epicenter of a group $G$, $Z^*(G)$, is the smallest central subgroup of $G$ with capable quotient, which is defined in [3]. In particular, $G$ is capable if and only if $Z^*(G) = 1$. Following [4,9], the 2-epicenter of $G$, $Z^*_2(G)$, is the smallest subgroup of $Z_2(G)$ with 2-capable quotient. In fact, $G$ is 2-capable if and only if $Z^*_2(G)$ is trivial. Moreover, $Z^*(G) \subseteq Z^*_2(G)$. (For more information, see [4, Lemma 2.1]).

Here, we prove that the epicenter and the 2-epicenter for a finite 2-generator $p$-group of class two coincide.

Theorem D. Let $G$ be a finite 2-generator $p$-group of class two. Then $Z^*(G) = Z^*_2(G)$.

In the sequel, we give the exact structure of 2-nilpotent multipliers for all capable 2-generator $p$-groups of class two in Sects. 2 and 4. Moreover, the 2-capability of such groups is discussed in Sect. 3.

2. 2-Nilpotent Multipliers of Some Capable $p$-Groups

Let $G$ be a group presented as $F/R$ to be the quotient group of a free group $F$ by a normal subgroup $R$. From [7], the Baer invariant of a group $G$ with respect to the variety of nilpotent groups of class at most 2 is called the 2-nilpotent multiplier of $G$, $\mathcal{M}^{(2)}(G)$, and defined as follows:

$$\mathcal{M}^{(2)}(G) \cong (R \cap \gamma_3(F))/[R,F,F],$$

in which $\gamma_3(F) = [F,F,F]$. It is well known that the 2-nilpotent multiplier of $G$ is abelian and independent of the choice of its free presentation (see [7]).

We need the notion of basic commutator for computing the 2-nilpotent multipliers of groups.

Definition 2.1. Let $X$ be an arbitrary subset of a free group and select an arbitrary total order for $X$. The basic commutators on $X$, their weight $wt$, and the ordering among them are defined as follows:

(i) The elements of $X$ are basic commutators of weight one, ordered according to the total order previously chosen.

(ii) Having defined the basic commutators of weight less than $n$, a basic commutator of weight $n$ is $d = [s,k]$, where:

(a) $s$ and $k$ are basic commutators and $wt(s) + wt(k) = n$, and

(b) $s > k$, and if $s = [s_1,s_2]$, then $k \geq s_2$.

(iii) The basic commutators of weight $n$ follow those of weight less than $n$. The basic commutators of weight $n$ are ordered among themselves in any total order, but the most common used total order is lexicographic order, that is if $[b_1,a_1]$ and $[b_2,a_2]$ are basic commutators of weight $n$, then $[b_1,a_1] < [b_2,a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$. 
Let $G$ be a finite 2-generator $p$-group of class two for $p > 2$ with the free presentation $F/R$ such that $F$ is a free group generated by $\{a, b\}$ with a normal subgroup $R$. Then $\gamma_3(F) \subseteq R$, because of nilpotency of $G$. Thus,

$$\mathcal{M}^{(2)}(G) \cong \frac{R \cap \gamma_3(F)}{[R, F, F]} \cong \frac{\gamma_3(F)/\gamma_5(F)}{[R, F, F]/\gamma_5(F)},$$

where $\gamma_5(F)$ is the 5-th term of the lower central series of $F$. It is known that $\gamma_3(F)/\gamma_5(F)$ is the free abelian group with the basis of all basic commutators of weights 3 and 4 on $\{a, b\}$. By considering $a > b$, we have

$$\gamma_3(F)/\gamma_5(F) = \langle [a, b, a] \gamma_5(F), [a, b, b] \gamma_5(F), [a, b, b, a] \gamma_5(F), [a, b, a] \gamma_5(F), [a, b, b] \gamma_5(F), [a, b, a] \gamma_5(F) \rangle.$$

Therefore, one can gain the structure of $\mathcal{M}^{(2)}(G)$ by providing a suitable basis for $[R, F, F]/\gamma_5(F)$.

Now, we determine the structure of 2-nilpotent multipliers of $p$-groups in Corollary 1.3, whenever $p$ is odd.

**Lemma 2.2.** Let $G$ be the group $G_1$ presented in Corollary 1.3(a)(i). Then

$$\mathcal{M}^{(2)}(G) \cong \mathbb{Z}_{p^5}^{(5)}.$$

**Proof.** Let $F/R$ be a free presentation for $G_1$ such that $F$ is a free group on $\{a, b\}$ and $R = \langle a^{p^5}, b^{p^5}, [a, b]^{p^5}, [a, b, b], [a, b, a] \rangle^F$. Hence,

$$\frac{[R, F, F]}{\gamma_5(F)} = \langle [a^{p^5}, f_1, f_2], [b^{p^5}, f_3, f_4], [[a, b]^{p^5}, f_5, f_6] \mid f_i \in F, 1 \leq i \leq 6 \rangle^{[R, F, F]} / \gamma_5(F).$$

Using [11, Lemma 3.3], we obtain

$$[a^{p^5}, f_1, f_2]^{f} \equiv [a, f_1, f_2]^{p^5} [a, f_1, f_2]^{(5^5)} [a, f_1, f_2, f]^{p^5} \pmod{\gamma_5(F)},$$

$$[b^{p^5}, f_3, f_4]^{f} \equiv [b, f_3, f_4]^{p^5} [b, f_3, f_4]^{(5^5)} [b, f_3, f_4, f]^{p^5} \pmod{\gamma_5(F)},$$

and

$$[[a, b]^{p^5}, f_5, f_6]^{f} \equiv [a, b, f_5, f_6]^{p^5} \pmod{\gamma_5(F)}$$

for all $f_i, f \in F$. Since $p > 2$, $[a, f_1, f_2]^{p^5} \gamma_5(F) \in [R, F, F]/\gamma_5(F)$. Therefore,

$$A = \{ [a, b]^{p^5} \gamma_5(F), [a, b, b]^{p^5} \gamma_5(F), [a, b, b]^{p^5} \gamma_5(F), [a, b, a]^{p^5} \gamma_5(F) \}$$

generates $[R, F, F]/\gamma_5(F)$. Since $\gamma_3(F)/\gamma_5(F)$ is a free abelian group, it is easy to see that $A$ is linearly independent, and so $A$ is a basis for $[R, F, F]/\gamma_5(F)$. Thus

$$[R, F, F] \equiv \langle [a, b]^{p^5}, [a, b, a]^{p^5}, [a, b, b]^{p^5}, [a, b, a]^{p^5} \rangle \pmod{\gamma_5(F)}.$$

Since $\gamma_3(F)/\gamma_5(F) \cong \mathbb{Z}^{(5)}$ and $[R, F, F]/\gamma_5(F) \cong (p^5 \mathbb{Z})^5$, we get $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}_{p^5}^{(5)}$. This proof is complete.

**Lemma 2.3.** Let $G$ be the group $G_2$ presented in Corollary 1.3(a)(ii). Then

$$\mathcal{M}^{(2)}(G) \cong \mathbb{Z}_{p^2}^{(2)} \oplus \mathbb{Z}_{p^3}^{(3)}.$$
Proof. Let $F/R$ be a free presentation for $G_2$ such that $F$ is a free group on $\{a, b\}$ and $R = \langle [a, b]^p, a^p, b^p, [a, b, b], [a, b, a], \alpha > \gamma \rangle^F$. By some commutator computations similarly to the proof of Lemma 2.2, one can reach that

$$[R, F, F] \equiv \langle [a, b, b]^p, [a, a]^p, [a, b, b]^p, [a, b, b, a]^p, [a, b, a]^p, [a, b, a]^p \rangle (\text{mod} \gamma_5(F)).$$

Thus $\mathcal{M}^{(2)}(G_2) \cong \frac{\gamma_3(F)}{\gamma_5(F)} \cong \Z_{p^\alpha} \oplus \Z_{p^\gamma}$, as desired. $\square$

**Lemma 2.4.** Let $G$ be the group $G_3$ presented in Corollary 1.3(a)(iii). Then $\mathcal{M}^{(2)}(G) \cong \Z_{p^\alpha} \oplus \Z_{p^\beta} \oplus \Z_{p^\gamma}$.

Proof. Let $F/R$ be a free presentation for $G_3$ such that $F$ is a free group on $\{a, b\}$ and $R = \langle [a, b]^p, a^p, b^p, [a, b, b], [a, b, a], [b^\beta, [a, b]^p] \rangle^F$. One can easily check that $[R, F, F]$ is generated by the following set

$$\{[a, b, f_1, f_2]^p, [a, f_3, f_4]^p, [a, b, f_5, f_6]^p, [a, b, a, f_1, f_2]^p, [a, b, a, f_3, f_4]^p, [a, b, a, f_5, f_6]^p \}$$

Similarly to the proof of Lemma 2.2, $[R, F, F]/\gamma_5(F)$ is generated by the elements $[a, b, a]^p \gamma_5(F), [a, b, b]^p \gamma_5(F), [a, b, b]^p \gamma_5(F), [a, b, b]^p \gamma_5(F)$, and $[a, b, a]^p \gamma_5(F)$. Therefore, the following set

$$\{[a, b, b]^p, [a, b, b]^p, [a, b, b]^p, [a, b, b]^p, [a, b, b]^p, [a, b, b]^p \}$$

is a basis for the group $[R, F, F]$ in modulo $\gamma_5(F)$. By easy calculations, the set $\{[a, b, b], [a, b, a], [a, b, b], [a, b, a], [a, b, b]^p \gamma_5(F), [a, b, a]^p \gamma_5(F)\}$ is also a basis for the free abelian group $\gamma_3(F)/\gamma_5(F)$. Thus,

$$\mathcal{M}^{(2)}(G_3) \cong \frac{\gamma_3(F)}{\gamma_5(F)} \cong \frac{[a, b, b]^p}{[a, b, b]^p} \oplus \frac{[a, b, a]^p}{[a, b, a]^p} \oplus \frac{[a, b, b]}{[a, b, b]^p} \oplus \frac{[a, b, a]}{[a, b, a]^p} \oplus \frac{[a, b, a]}{[a, b, a]^p} \cong \Z_{p^\alpha} \oplus \Z_{p^\beta} \oplus \Z_{p^\gamma},$$

as required. $\square$

Now, we are ready to prove Theorem A.

**Proof of Theorem A.** The result is obtained by Lemmas 2.2, 2.3, and 2.4. $\square$

3. 2-Capability of Some $p$-Groups

This section is devoted to determine all 2-capable finite 2-generator $p$-groups of class two for odd prime $p$.

The following results are useful for determining the 2-capability of groups.

**Lemma 3.1.** ([9, Theorem 4.4] and [4, Lemma 2.1(vii)]) Let $N$ be a normal subgroup of a group $G$ contained in $Z_2(G)$. Then

(i) $N \subseteq Z_2(G)$ if and only if the natural map $\mathcal{M}^{(2)}(G) \rightarrow \mathcal{M}^{(2)}(G/N)$ is a monomorphism.

(ii) The sequence $\mathcal{M}^{(2)}(G) \rightarrow \mathcal{M}^{(2)}(G/N) \rightarrow N \cap \gamma_3(G) \rightarrow 1$ is exact.
An immediate result of Lemma 3.1 is as follows:

**Corollary 3.2.** Let $G$ be a group of nilpotency class two and $N$ be a normal subgroup of $G$. Then $N \trianglelefteq Z^2_2(G)$ if and only if $\mathcal{M}^{(2)}(G) \cong \mathcal{M}^{(2)}(G/N)$.

Note that, the 2-capability implies that the capability. For this, we just need to discuss the 2-capability of groups in Corollary 1.3(a).

**Theorem 3.3.** Let $G$ be a $p$-group such that $G \cong G_1$ or $G \cong G_2$ with the presentations as in Corollary 1.3. Then $G$ is 2-capable.

*Proof.* Assume that $G \cong G_1$. From [4, Theorem 1.3], $G_1/G'_1$ is 2-capable, and so $Z^2_2(G_1) \subseteq G'_1 = \langle [a, b] \rangle$. Let $x$ be a nontrivial element of $Z^2_2(G_1)$. If $\langle x \rangle = \langle [a, b]^\gamma \rangle$ with $\gcd(p, r) = 1$, then $\langle x \rangle = G'$. Using [11, Theorem 2.3] and Lemma 2.2, $\mathcal{M}^{(2)}(G_1/G'_1) \cong \mathbb{Z}^{(2)}_p$ and $\mathcal{M}^{(2)}(G_1) \cong \mathbb{Z}^{(5)}_p$. Therefore, we will have a contradiction by Lemma 3.1(i). Now, let $\gcd(p, r) \neq 1$. Then $r = p^s t$ for some $s$ with $1 \leq s < \alpha$ and $\gcd(p, t) = 1$. Hence, $\langle x \rangle = \langle [a, b]^\gamma \rangle$. We denote the image of $y \in G_1$ in $G_1/\langle x \rangle$ by $\tilde{y}$. Thus

$$G_1/\langle x \rangle = \langle \tilde{a}, \tilde{b} \mid \tilde{a}^\alpha = \tilde{b}^\alpha = [\tilde{a}, \tilde{b}, \tilde{a}] = [\tilde{a}, \tilde{b}, \tilde{b}] = 1, \alpha > s \rangle \cong G_2.$$

By Lemmas 2.2 and 2.3, $|\mathcal{M}^{(2)}(G_1)| > |\mathcal{M}^{(2)}(G_1/\langle x \rangle)|$, which is a contradiction, using Lemma 3.1(i). So, $Z^2_2(G_1) = 1$, and the result follows in this case. Now, let $G \cong G_2$. By a similar way, we get $Z^2_2(G_2) = 1$, as desirable. \hfill $\square$

The following lemma is an essential key in the proof of Theorem 3.5.

**Lemma 3.4.** Let $G$ be the group $G_3$ presented in Corollary 1.3(iii). Assume that $1 \neq d = a^{i'p^{\beta+k_1}} b^{j'p^{\beta+k_2}}$ and $gcd(p, i') = gcd(p, j') = 1$, where $k_1$ and $k_2$ are integers. Then the following results hold:

(i) $k_1 = k_2 = k$, $d = a^{i'p^{\beta+k}} b^{j'p^{\beta+k}}$, and $k < \alpha - \beta$.

(ii) $|d| = p^{\alpha - \beta - k}$.

(iii) $G/\langle d \rangle \cong \langle a_1, b_1 \mid a_1^\alpha = b_1^\alpha = [a_1, b_1]^\gamma = [a_1, b_1, a_1] = 1, b_1^\beta = [a_1, b_1]^\sigma, a_1^{p^{\beta+k}} = b_1^{p^{\beta+k}}, \beta + k < \alpha \rangle$.

(iv) $\mathcal{M}^{(2)}(G/\langle d \rangle) \cong \mathbb{Z}_{p^{\beta+k}} \oplus \mathbb{Z}_{p^{\gamma}} \oplus \mathbb{Z}_{p^\sigma}$.

*Proof.* Consider the image of $y \in G$ in $G/\langle d \rangle$ by $\tilde{y}$. As a result,

$$G/\langle d \rangle = \langle \tilde{a}, \tilde{b} \mid \tilde{a}^\alpha = \tilde{b}^\alpha = [\tilde{a}, \tilde{b}]^\gamma = 1, \tilde{b}^\beta = [\tilde{a}, \tilde{b}]^\sigma, \tilde{a}^{i'p^{\beta+k_1}} \rangle = 1 \neq \tilde{b}^{j'p^{\beta+k_2}} = [\tilde{a}, \tilde{b}, \tilde{a}] = 1.$$

Since $\alpha - \beta = \gamma - \sigma$ and $|a^{i'p^{\beta+k_1}}| = |b^{j'p^{\beta+k_2}}|$, we obtain $k_1 = k_2$. Put $k_1 = k_2 = k$. Thus

$$1 \neq d = a^{i'p^{\beta+k}} b^{j'p^{\beta+k}} \quad \text{and} \quad gcd(p, i') = gcd(p, j') = 1.$$

If $\beta + k \geq \alpha$, then $G/\langle d \rangle \cong G$, and so $d = 1$, which is a contradiction. Thus, $\beta + k < \alpha$. The case (i) is obtained. Now, we want to compute the order of $d$. Using [8, Section 5.1, p. 28], $Z(G) = \langle a^\gamma, [a, b], b^\alpha \rangle$, and hence, $\langle a^{i'p^{\beta+k}}, b^{j'p^{\beta+k}} \rangle \subseteq Z(G)$. Since $\gamma - \sigma = \alpha - \beta > k$ and $|b| = |a| = p^\alpha$, we get
\(|a^i p^{\beta + k}| = p^{\alpha - \beta - k} = |b^j p^{\beta + k}| = |d| = p^{\alpha - \beta - k} \neq 1\). The case (ii) is obtained.

Without loss of generality, taking \(i' = j' = 1\), using [1, Proposition 3.1], we have

\[G/d \cong H = \langle a_1, b_1 \mid a_1 p^\alpha = b_1 p^\alpha = [a_1, b_1] p^{\gamma} = 1, b_1^p = [a_1, b_1] p^\alpha, a_1^p = b_1^{-p^{\beta + k}}, [a_1, b_1, a_1] = 1, \beta + k < \alpha \rangle.\]

By [3, Proposition 1.1], we get \(a_1^{p^{\beta + k}} \in Z^*(H) \subseteq Z^2*(H)\). We denote the image of \(y \in H\) in \(H/G/Z\) by \(\bar{y}\). As a result,

\[H_1 = H/\langle a_1^{p^{\beta + k}} \rangle = \langle \bar{a_1}, \bar{b_1} \mid \bar{a_1} p^{\beta + k} = [\bar{a_1}, \bar{b_1}] p^{\alpha + k} = 1, \bar{b_1} p^\alpha = [\bar{a_1}, \bar{b_1}] p^\alpha, [\bar{a_1}, \bar{b_1}, \bar{a_1}] = 1 \rangle.\]

Now, Lemma 2.4 and Corollary 3.2 imply that

\[\mathcal{M}(2)(G/d) \cong \mathcal{M}(2)(H) \cong \mathcal{M}(2)(H_1) \cong \mathbb{Z}_{p^{\beta + k}} \oplus \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\mu} ; \]

and the case (iv) is proved.

**Theorem 3.5.** Let \(G\) be the group \(G_3\) presented in Corollary 1.3(iii). Then \(G\) is 2-capable.

**Proof.** Put \(M = \langle a^\sigma, b^\beta \rangle\). Using [8, Section 5.1, p. 28], \(Z(G_3) = \langle a^\gamma, [a, b], b^\sigma \rangle\), and so \(M \trianglelefteq G_3\). Consider the image of \(y \in G_3\) in \(G_3/M\) by \(\bar{y}\). As a result,

\[G_3/M = \langle \bar{a}, \bar{b} \mid \bar{a} p^\beta = \bar{b} p^\alpha = [\bar{a}, \bar{b}, \bar{a}] = [\bar{a}, \bar{b}, \bar{b}] = 1, \beta > \sigma \rangle \cong G_2.\]

From Theorem 3.3, \(G_3/M\) is 2-capable and hence \(Z^2_2(G_3) \subseteq M\). We claim that \(Z^2_2(G_3) = 1\). By the way of contradiction, assume that \(1 \neq d \in Z^2_2(G_3)\) is an arbitrary element. Then \(d = a^{i p^\beta} b^{j p^\beta}\) such that \(i\) and \(j\) are integers. Suppose that \(i = i' p^{k_1}, j = j' p^{k_2}\), and \(gcd(p, i') = gcd(p, j') = 1\), where \(k_1\) and \(k_2\) are integers. Lemmas 2.4 and 3.4 imply that \(M^2(G_3/d) \cong \mathbb{Z}_{p^{\beta + k}} \oplus \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\mu} \) and \(\mathcal{M}(2)(G_3) \cong \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta} \oplus \mathbb{Z}_{p^\mu} \). Hence, \(|\mathcal{M}(2)(G_3)| > |\mathcal{M}(2)(G_3/d)|\), and we will have \(d \notin Z^2_2(G_2)\), by Lemma 3.1(i). This contradiction completes the proof.

**Proof of Theorem B.** Let \(G\) be 2-capable. Then \(G\) is capable, and so Corollary 1.3(a) implies that \(G\) is isomorphic to one of the groups \(G_1, G_2, \) or \(G_3\). The converse holds by Theorems 3.3 and 3.5.

**Proof of Theorem D.** If \(G\) is capable, then \(Z^*(G) = Z^2_2(G) = 1\), using Theorem B. Otherwise, since \(G/Z^*(G)\) is nilpotent of class at most 2, Corollary C and [4, Theorem 1.3] imply that \(G/Z^*(G)\) is 2-capable, and so \(Z^2_2(G)/Z^*(G) \subseteq Z^*_2(G/Z^*(G)) = 1\). Hence, \(Z^2_2(G) \subseteq Z^*(G)\) as required.

4. 2-Nilpotent Multipliers of Some Noncapable \(p\)-Groups

In this section, we intend to determine the structures of 2-nilpotent multipliers for noncapable finite 2-generator \(p\)-groups of class two.
A criterion for detecting a capable group is the notion of the exterior center. The exterior center of a group $G$, $Z^\wedge(G)$, is defined in [5] as follows:

$$Z^\wedge(G) = \{g \in G \mid g \wedge h = 1_{G\wedge G} \text{ for all } h \in G\},$$

where $\wedge$ denotes the operator of the nonabelian exterior square. It is shown [5] that $Z^*(G) = Z^\wedge(G)$. It implies that $G$ is capable if and only if $Z^\wedge(G) = 1$.

**Lemma 4.1.** Let $G = G_p(\alpha, \beta, \gamma; \rho, \sigma)$ be a 2-generator $p$-group of class two presented as in Theorem 1.1. If $k > \max(\rho, \sigma)$, then

$$a^{p^k} \wedge b = a \wedge b^{p^k} = (a \wedge b)^{p^k(p^k+1)/2}.$$  

**Proof.** We claim that $a^{p^k} \wedge b = (a \wedge b)^{p^k(p^k+1)/2}$. For this, we have

$$a^{p^k} \wedge b = \prod_{i=0}^{p^k-1} (a^i(a \wedge b)) = \prod_{i=0}^{p^k-1} (a \wedge a^i b)$$

$$= \prod_{i=0}^{p^k-1} (a \wedge [a^i, b]) = \prod_{i=0}^{p^k-1} (a \wedge [a^i, b])(a \wedge b)$$

$$= (a \wedge b)^{1+\cdots+p^k} \prod_{i=1}^{p^k-1} (a \wedge [a, b])^i$$

$$= (a \wedge b)^{p^k(p^k+1)/2} \prod_{i=1}^{p^k-1} (a \wedge [a, b])^i$$

$$= (a \wedge b)^{p^k(p^k+1)/2} (a \wedge [a, b])^{1+\cdots+p^k-1}$$

$$= (a \wedge b)^{p^k(p^k+1)/2} (a \wedge [a, b])^{p^k(p^k-1)/2}$$

$$= (a \wedge b)^{p^k(p^k+1)/2} (a \wedge [a, b])^{p^k(p^k-1)/2}$$

and so

$$a^{p^k} \wedge b = (a \wedge b)^{p^k(p^k+1)/2}. \quad (4.1)$$

Similarly,

$$a \wedge b^{p^k} = (b^{p^k} \wedge a)^{-1} = (b \wedge a)^{p^k(p^k+1)/2}, \quad (4.2)$$

as desirable. □

Note that, in Lemma 4.1, the condition $k > \max(\rho, \sigma)$ may be replaced by $k \geq \max(\rho, \sigma)$, whenever $p$ is an odd prime number.

Using Theorem 1.1 and Lemma 1.2, all noncapable finite 2-generator $p$-groups of class two are described as follows:
Corollary 4.2. Let $G$ be a finite 2-generator $p$-group of class two with the presentation as in Theorem 1.1. Then the following results hold:

If $p$ is an odd prime number, then $G$ is noncapable if and only if $G$ is isomorphic to exactly one of the following groups.

1. $K_1 = \langle a, b \mid [a, b]^{p\gamma} = [a, b, b] = [a, b, a] = 1 = b^{p\delta}, a^{p\alpha} = [a, b]^{p\beta} \rangle$, where $\alpha > \beta \geq \gamma$ and $0 \leq \rho < \gamma$.

2. $K_2 = \langle a, b \mid a^{p\alpha} = b^{p\beta} = [a, b]^{p\gamma} = [a, b, b] = [a, b, a] = 1 \rangle$, where $\alpha > \beta \geq \gamma$.

3. $K_3 = \langle a, b \mid a^{p\alpha} = [a, b]^{p\gamma} = [a, b, b] = [a, b, a] = 1, b^{p\delta} = [a, b]^{p\alpha} \rangle$, where $\alpha > \beta \geq \gamma$.

4. $K_4 = \langle a, b \mid [a, b]^{p\gamma} = [a, b] = [a, b, a] = 1, a^{p\alpha} = [a, b]^{p\beta}, b^{p\delta} = [a, b]^{p\alpha} \rangle$, where $\alpha > \beta \geq \gamma$ and $0 \leq \rho < \min(\gamma, \sigma + \alpha - \beta)$.

5. $K_5 = \langle a, b \mid [a, b]^{p\gamma} = [a, b, b] = [a, b, a] = 1 = b^{p\alpha}, a^{p\alpha} = [a, b]^{p\beta} \rangle$, where $\alpha > \gamma > \rho$.

If $p = 2$, then $G$ is noncapable if and only if $G$ is isomorphic to exactly one of the following groups.

6. $K_6 = \langle a, b \mid [a, b]^{2\gamma} = [a, b] = [a, b, a] = 1, \ a^{2\alpha} = [a, b]^{2\sigma}, b^{2\delta} = 1 \rangle$, where $\alpha > \beta \geq \gamma$ and $0 \leq \rho < \gamma$.

7. $K_7 = \langle a, b \mid [a, b]^{2\gamma} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = 1 = b^{2\delta} \rangle$, where $\alpha > \beta > \gamma$.

8. $K_8 = \langle a, b \mid [a, b]^{2\beta} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = 1 = b^{2\delta} \rangle$, where $\alpha > \beta + 1$.

9. $K_9 = \langle a, b \mid [a, b]^{2\gamma} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = 1, b^{2\delta} = [a, b]^{2\sigma} \rangle$, where $0 \leq \sigma < \sigma + \alpha - \beta < \gamma$.

10. $K_{10} = \langle a, b \mid [a, b]^{2\gamma} = [a, b, b] = [a, b, a] = 1, \ a^{2\alpha} = [a, b]^{2\sigma}, b^{2\delta} = [a, b]^{2\sigma} \rangle$, where $0 \leq \sigma < \rho < \min(\gamma, \sigma + \alpha - \beta)$.

11. $K_{11} = \langle a, b \mid [a, b]^{2\gamma} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = [a, b]^{2\sigma}, b^{2\delta} = 1 \rangle$, where $\alpha > \gamma > \rho$.

12. $K_{12} = \langle a, b \mid [a, b]^{2\gamma} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = [a, b]^{2\sigma}, b^{2\delta} = 1 \rangle$, where $\alpha > \rho$.

13. $K_{13} = \langle a, b \mid [a, b]^{2\beta} = [a, b, b] = [a, b, a] = 1, a^{2\beta} = [a, b]^{2\sigma}, b^{2\delta} = 1 \rangle$, where $\beta - 1 > \rho$.

14. $K_{14} = \langle a, b \mid [a, b]^{2\alpha} = [a, b, b] = [a, b, a] = 1, a^{2\alpha} = [a, b]^{2\alpha-1} = b^{2\alpha} \rangle$.

The following result gives an element in the epicenter of some noncapable finite 2-generator $p$-groups of class two.

Theorem 4.3. Let $G = K_i$ be the 2-generator $p$-group of class two presented as in Corollary 4.2. Then for $i \neq 8$, we have

(i) $b^{p\rho} \in Z^*_2(G)$, if $a^{p\alpha} = 1$;

(ii) $a^{p\beta} \in Z^*_2(G)$, if $b^{p\beta} = 1$;

(iii) $a^{p\alpha} \in Z^*_2(G)$, if $\sigma < \rho$.

Proof. Let $G = K_i$. Then $Z^*(G) = Z^*_2(G)$, by Theorem D. It is obvious that $x \in Z^*(G) = Z^*_2(G)$ if and only if $x \land a = 1 = x \land b$. Therefore, for each case, it is enough to show that three elements $b^{p\rho} \land a$, $a^{p\beta} \land b$, and $a^{p\alpha} \land b$
are trivial, respectively. Now, it is easy to achieve the desired result, using a similar method as stated in Lemma 4.1.

**Theorem 4.4.** Assume that $G$ is a noncapable 2-generator $p$-group of class two given in Corollary 4.2(1)–(5). Then the structure of 2-nilpotent multiplier of $G$ is as follows:

1. Let $G \cong K_1$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{p\alpha} \oplus \mathbb{Z}^{(3)}_{p\alpha}$.
2. Let $G \cong K_2$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{p\alpha} \oplus \mathbb{Z}^{(3)}_{p\alpha}$.
3. Let $G \cong K_3$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{p\alpha} \oplus \mathbb{Z}^{(3)}_{p\alpha}$.
4. Let $G \cong K_4$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{p\alpha} \oplus \mathbb{Z}^{(3)}_{p\alpha}$.
5. Let $G \cong K_5$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{p\alpha} \oplus \mathbb{Z}^{(3)}_{p\alpha}$.

**Proof.**

1. Let $G \cong K_1$. Then by Theorem 4.3(ii), $a^{p\alpha} \in Z^*_2(G)$. Since $G/\langle a^{p\alpha} \rangle \cong G_2$, we will have the result by Theorem A and Corollary 3.2.
2. Let $G \cong K_2$. It is not difficult to see that $G/\langle a^{p\alpha} \rangle \cong G_1$, when $\beta = \gamma$, and otherwise $G/\langle a^{p\alpha} \rangle \cong G_2$. Now, the result is in hand by Theorem 4.3(ii), Theorem A, and Corollary 3.2.
3. Let $G \cong K_3$. Theorem 4.3(i) implies that $b^{p\alpha} \in Z^*_2(G)$, and hence $\mathcal{M}^{(2)}(G) \cong \mathcal{M}^{(2)}(G/\langle b^{p\alpha} \rangle)$, by Corollary 3.2, in which

$$G/\langle b^{p\alpha} \rangle \cong \langle a_1, b_1 | a_1^{p\alpha} = b_1^{p\alpha} = [a_1, b_1]^{p^{1-\beta+\sigma}} = 1, b_1^{p\alpha} = [a_1, b_1]^{p^\alpha}, [a_1, b_1, b_1] = [a_1, b_1, a_1] = 1, 0 < \sigma < \alpha < \beta < \gamma \rangle.$$  

Since $\alpha - \beta = \alpha - \beta + \sigma - \sigma$, we get $G/\langle b^{p\alpha} \rangle$ is capable. So, $G/\langle b^{p\alpha} \rangle \cong G_3$. The result is concluded that by Theorem A.
4. Let $G \cong K_4$. From Theorem 4.3(iii), we have $a^{p\alpha} \in Z^*_2(G)$. Since,

$$G/\langle a^{p\alpha} \rangle \cong H = \langle a_1, b_1 | a_1^{p\alpha} = b_1^{p\alpha} = [a_1, b_1]^{p^{1-\beta+\sigma}} = 1, b_1^{p\alpha} = [a_1, b_1]^{p^\alpha}, [a_1, b_1, b_1] = [a_1, b_1, a_1] = 1, 0 < \sigma < \rho < \alpha < \beta < \gamma \rangle,$$

and $\rho < \sigma + \alpha - \beta$, we have $G/\langle a^{p\alpha} \rangle$ is noncapable. Theorem 4.3(ii) follows that $a_1^{p^{1-\beta+\sigma}} \in Z^*_2(H)$. So, $\mathcal{M}^{(2)}(G) \cong \mathcal{M}^{(2)}(H) \cong \mathcal{M}^{(2)}(H/\langle a_1^{p^{1-\beta+\sigma}} \rangle)$, by Corollary 3.2. Now, one can obtain the result by Theorem A, because of $H/\langle a_1^{p^{1-\beta+\sigma}} \rangle \cong G_3$.
5. Let $G \cong K_5$. Then $a^{p\alpha} \in Z^*_2(G)$, by Theorem 4.3(ii). Therefore $\mathcal{M}^{(2)}(G) \cong \mathcal{M}^{(2)}(G/\langle a^{p\alpha} \rangle) \cong \mathcal{M}^{(2)}(G_2)$, as required. 

The next result gives the structure of 2-nilpotent multiplier of other two generator groups.

**Theorem 4.5.** Let $G$ be a noncapable 2-generator 2-group of class two given in Corollary 4.2(6)–(14).

1. Let $G \cong K_6$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{2\gamma} \oplus \mathbb{Z}^{(3)}_{2\gamma}$.
2. Let $G \cong K_7$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(2)}_{2\gamma} \oplus \mathbb{Z}^{(3)}_{2\gamma}$.
3. Let $G \cong K_8$. Then $\mathcal{M}^{(2)}(G) \cong \mathbb{Z}^{(3)}_{2\gamma-1} \oplus \mathbb{Z}^{2\gamma} \oplus \mathbb{Z}^{2\gamma+1}$. 


(4) Let \( G \cong K_9 \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^\alpha} \oplus \mathbb{Z}_{2^\beta} \oplus \mathbb{Z}_{2^9} \).

(5) Let \( G \cong K_{10} \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^{\nu - \sigma + \beta}} \oplus \mathbb{Z}_{2^9} \oplus \mathbb{Z}_{2^9} \).

(6) Let \( G \cong K_{11} \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^9} \oplus \mathbb{Z}_{2^9} \).

(7) Let \( G \cong K_{12} \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^9} \oplus \mathbb{Z}_{2^9} \).

(8) Let \( G \cong K_{13} \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^9} \).

(9) Let \( G \cong K_{14} \). Then \( \mathcal{M}(G) \cong \mathbb{Z}_{2^9} \oplus \mathbb{Z}_{2^{9-1}} \).

**Proof.** (1) Let \( G \cong K_i \), for \( i = 6, 7, 11, 12, 13, 14 \). By a similar way used in Theorem 4.4, we can observe that \( \mathcal{M}(G) \cong \mathcal{M}(G_6) \), and the result follows by [10, Theorem 3.3]. Let \( G \cong K_8 \). Since \( G/\langle a^{2^\beta} \rangle \) is 2-capable, we will have \( 1 \neq Z^*_2(G) \subseteq \langle a^{2^\beta} \rangle \). Hence, \( Z^*_2(G) = \langle a^{2^\beta + \alpha} \rangle \), and therefore \( a^{2^\alpha - 1} \in Z^*_2(G) \), because of \( \alpha > \beta + 1 \). Let \( H_0 = G \) and \( H_i = H_{i-1}/\langle a^{2^\alpha - 1} \rangle \) for all \( i \) with \( 1 \leq i \leq \alpha - \beta \). It is easy to see that \( \mathcal{M}(H_i) \cong \mathcal{M}(G) \), whenever \( \alpha - i = \beta \). Hence, if \( \alpha - 1 = \beta \), then \( \mathcal{M}(G) \cong \mathcal{M}(G/\langle a^{2^\beta} \rangle) \cong \mathcal{M}(G_4) \). Since \( \alpha > \beta \), then there exists an element \( j \) such that \( \alpha - j = \beta \). We increase \( i \) from 1 up to \( j \) and calculate \( H_i \). Note, one can check that \( \mathcal{M}(G) \cong \mathcal{M}(H_j) \cong \mathcal{M}(H_i) \cong \mathcal{M}(G_4) \) for all \( i \) with \( 0 \leq i \leq \alpha - \beta \). Hence, the result is in hand by [10, Theorem 3.3]. Now, let \( G \cong K_9 \). Then

\[
G/\langle b^{2^\alpha} \rangle \cong \langle a_1, b_1 \mid [a_1, b_1]^{2^{\nu - \sigma + \beta}} = a_1^{2^\alpha} = 1, b_1^{2^\beta} = [a, b]^{2^\sigma}, \beta > \sigma \rangle \cong G_7.
\]

Since \( b^{2^\alpha} \in Z^*(G) = Z^*_2(G) \), by Theorem 4.3, we will have \( \mathcal{M}(G) \cong \mathcal{M}(G/\langle b^{2^\alpha} \rangle) \cong \mathcal{M}(G_7) \), as desirable.

For \( G \cong K_{10} \), one can see that \( a^{2^\alpha} \in Z^*(G) = Z^*_2(G) \), by Theorem 4.3 (iii). Hence \( G/\langle a^{2^\alpha} \rangle \cong H \) in which

\[
H = \langle a_1, b_1 \mid a_1^{2^\alpha} = b_1^{2^{\nu - \sigma + \beta}}, [a_1, b_1]^{2^\sigma} = 1, b_1^{2^\beta} = [a_1, b_1]^{2^\alpha},
\]

\[
[a_1, b_1, b_1] = [a_1, b_1, a_1] = 1, 0 \leq \sigma < \rho < \sigma + \alpha - \beta, \beta < \alpha \).
\]

Since \( \rho < \sigma + \alpha - \beta \), we have \( H \) is noncapable, and by Theorem 4.3 (iii), \( a_1^{2^{\nu - \sigma + \beta}} \in Z^*(H) = Z^*_2(H) \). Therefore, \( \mathcal{M}(G) \cong \mathcal{M}(H) \cong \mathcal{M}(H/\langle a_1^{2^{\nu - \sigma + \beta}} \rangle) \), by Corollary 3.2. Note that, \( H/\langle a_1^{2^{\nu - \sigma + \beta}} \rangle \cong G_7 \) and the result follows by [10, Theorem 3.3]. \( \square \)

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Farangis Johari
Instituto de Matemática e Estatística da Universidade de São Paulo
Rua do Matão,1010
São Paulo
SPCEP 05586-090
Brazil
e-mail: farangisjohari@ime.usp.br;
farangisjohari85@gmail.com

Azam Kaheni
Department of Mathematics
University of Birjand
Birjand
Iran
e-mail: azamkaheni@birjand.ac.ir

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