Achievable Rate Region of the Zero-Forcing Precoder in a $2 \times 2$ MU-MISO Broadcast VLC Channel With Per-LED Peak Power Constraint and Dimming Control

Amit Agarwal and Saif Khan Mohammed, Senior Member, IEEE

Abstract—In this paper, we consider the $2 \times 2$ multiuser multiple-input-single-output (MU-MISO) broadcast visible light communication (VLC) channel with two light emitting diodes (LEDs) at the transmitter and a single photodiode (PD) at each of the two users. We propose an achievable rate region of the zero-forcing precoder in this $2 \times 2$ MU-MISO VLC channel under a per-LED peak and average power constraint, where the average optical power emitted from each LED is fixed for constant lighting, but is controllable (referred to as dimming control in IEEE 802.15.7 standard on VLC). We analytically characterize the proposed rate region boundary and show that it is Pareto-optimal. Further analysis reveals that the largest rate region is achieved when the fixed per-LED average optical power is half of the allowed per-LED peak optical power. We also propose a novel transceiver architecture where the channel encoder and dimming control are separated, which greatly simplifies the complexity of the transceiver. A case study of an indoor VLC channel with the proposed transceiver reveals that the achievable information rates are sensitive to the placement of the LEDs and the PDs. An interesting observation is that for a given placement of LEDs in a $5 \times 5 \times 3$ m room, even with a substantial displacement of the users from their optimum placement, reduction in the achievable rates is not significant. This observation could therefore be used to define “coverage zones” within a room where the reduction in the information rates to the two users is within an acceptable tolerance limit.

Index Terms—Multi-user, multiple-input-multiple-output, rate region, visible light communication, zero-forcing.

I. INTRODUCTION

Visible light communication (VLC) is a form of optical wireless communication (OWC) technology which can provide high speed indoor wireless data transmission using existing infrastructure for lighting. One distinctive advantage of VLC technology is that it utilizes the unused visible band of the electromagnetic spectrum and does not interfere with the existing radio frequency (RF) communication in the UHF (Ultra High Frequency) band [1], [2].

In VLC systems, it is common to use intensity modulation (IM) via light emitting diode (LEDs) for transmission of information signal and direct detection (DD) via photodiodes (PDs) for the recovery of the information signal [1]. Contrary to RF systems, in VLC systems the modulation symbols must be non-negative and real valued as information is communicated by modulating the power/ intensity of the light emitted by the optical source (LED). The modulation symbols are also constrained to be less than a pre-determined value as the intensity of the light emitted by the LED is peak constrained due to safety regulations and also due to the limited linear range of the transfer function of LEDs [1], [3]. Moreover, due to constant lighting the mean value of the modulation symbol is also fixed (i.e., non-time varying) and can be adjusted according to the users’ requirement (dimming target) [4], [5].

Due to these constraints, analysis performed for RF systems is not directly applicable to VLC systems. For example, the capacity of the RF single-input-single-output (SISO) additive white Gaussian noise (AWGN) channel is well known and it has been shown that the Gaussian input distribution is capacity achieving. For the case of the optical wireless AWGN SISO channel with IM/DD transceiver, closed form expression for the capacity is still not known, though several inner and outer bounds have been proposed [6]–[8]. However, it has been shown that the capacity achieving input distribution for the IM/DD SISO AWGN optical wireless channel is discrete [9], and has been computed numerically in [10]. Similarly, for the case of dimmable VLC IM/DD SISO channel with peak constraint, there is no closed from expression for the capacity. However following a similar approach as in [6], an upper and lower bound is presented in [11].

Recently, there has been a lot of interest in multi-user multiple-input multiple-output/single-output (MU-MIMO/MISO) VLC systems, where multiple LEDs are used for information transmission to multiple non-cooperative PDs (users) [12], [1]. Such systems have been shown to enhance the system sum rate when compared to SISO VLC systems [13], [14].
In [13], the information sum rate of MU-MIMO VLC broadcast systems has been studied under the non-negativity constraint on the signal transmitted from each LED, and also a per-LED average transmitted power constraint with no dimming control. The block diagonalization precoder in [13] is used to suppress the multi-user interference and the numerically computed achievable sum rate is shown to be sensitive to the placement of the users and the rotation of the PDs. However, they do not consider peak power constraints which is important due to eye safety regulations and also due to the requirement of limited interference to other VLC systems.

Per-LED peak and average power constraint has been considered in [14], where the sum-rate of the zero forcing (ZF) precoder is maximized in a IM/DD based MU-MIMO/MISO VLC systems. However, in many practical scenarios fairness is required and therefore maximizing the sum rate might not always be the desired operating regime. For example we would like to find the maximum possible rate such that each user gets the same rate. Such operating points can only be obtained from the rate region characterization of the MU-MIMO VLC systems. In [15], authors have proposed inner and outer bounds on the capacity region of a two user IM/DD broadcast VLC system where the transmitter has a single LED and each user has a single PD. Per-LED average and peak power constraints are considered. The authors have extended their work to more than two users in [16]. However, in both [15] and [16], the transmitter has only one LED. Furthermore, dimming control is not considered in [13]–[16].

The capacity/achievable rate region of a IM/DD based VLC broadcast channel where the transmitter has $N > 1$ LEDs and $M > 1$ users having one PD each, is still an open and challenging problem, primarily due to the non-negativity, peak and average constraints on the electrical signal input to each LED.

In this paper, we consider the smallest instance of this open problem along with dimming control, i.e., with $N = 2$ LEDs at the transmitter and $M = 2$ users (each having one PD). Dimming control is required in indoor VLC systems since the illumination should not vary with time on its own and should be controllable by the users. Therefore, in this paper, in addition to the peak and non-negativity constraints, we constrain the average optical power radiated by each LED to be fixed, i.e., non-time varying. Subsequently in this paper we refer to this system as the $2 \times 2$ MU-MISO VLC broadcast system.

The major contributions of this paper are as follows:

1) In Section III, we propose an achievable rate region for the $2 \times 2$ MU-MISO VLC broadcast system with the ZF precoder. In this section through analysis we show that the per-LED non-negativity and peak constraint restricts the information symbol vector for the two users (i.e., $(u_1, u_2)$) to lie within a parallelogram $R_{ij}$. Each achievable rate pair $(R_1, R_2)$ then corresponds to a rectangle which lies within $R_{ij}$. The rate $R_i$, $i = 1, 2$ to the $i$th user depends on the length of the rectangle along the $u_i$-axis. Due to the same average optical power constraint at each LED, these rectangles should also have their midpoint (i.e., point of intersection of the diagonals of the rectangle) at a fixed point on the diagonal of $R_{ij}$ denoted by D.\footnote{Out of the two diagonals of $R_{ij}$, we refer to the one which has one end point at the origin $(u_1, u_2) = (0,0)$.} This fixed point D on the diagonal of $R_{ij}$ is non-time varying, but can be controlled by the user depending upon the illumination requirement. This feature of the proposed system enables dimming control.

2) In Section III, We also mathematically define the proposed rate region of the ZF precoder for a fix dimming target.

3) In Section IV, we analytically characterize the boundary of the proposed rate region by deriving explicit expressions for the largest possible length along the $u_2$-axis of some rectangle inside $R_{ij}$ whose midpoint coincides with the fixed point D on the diagonal of $R_{ij}$ and whose length along the $u_1$-axis is given. Through analysis we also show that the rate region boundary is Pareto-optimal.

4) We also analyze the variation in the rate region with change in the dimming level. In depth analysis reveals that the largest rate region is achieved when the fixed point D lies at the midpoint of the diagonal of $R_{ij}$, i.e., when the fixed per-LED average optical transmit power is half of the per-LED peak optical power.

5) For practical scenarios with fairness constraints, through analysis we show that the largest achievable rate pair $(R_1, R_2)$ such that $R_2 = \alpha R_1$ is given by the unique intersection of the proposed rate region boundary with the straight line $R_2 = \alpha R_1$.

6) In Section V, from the point of view of practical implementation we also propose a novel transceiver architecture where the same channel encoder can be used irrespective of the level of dimming control.

7) Analytical results have been supported with numerical simulations in Section VI. It is observed that for a fixed placement of the two LEDs, the achievable information rates are a function of the placement of the two PDs/users. Specifically, we observe that for a given placement of the two LEDs, there exists an optimal placement of the two users which maximizes the symmetric rate. Another interesting observation is that in a $5 \times 5 \times 3$ m (height) room with the two LEDs attached to the ceiling and the two PDs placed in the horizontal plane at a height of 50 cm above the floor, even a user displacement of 60 cm from the optimal placement results in only approx. a 10 percent reduction in the symmetric rate when compared to the symmetric rate with the optimal placement of PDs.\footnote{For this study the dimming control is such that the average optical power radiated from each LED is 30 percent of the peak allowed optical power.} This allows for substantial mobility of the user terminals around their optimal placement which is specially desirable when the user terminals are mobile/portable. A practical application of the results derived in this paper could be in defining coverage zones for the PDs/users, i.e., the maximum allowable displacement of the users for a fixed desired upper limit on the percentage loss in the achievable information rates.
II. SYSTEM MODEL

We consider a 2 × 2 IM/DD MU-MISO VLC broadcast system. The transmitter of the MISO system is equipped with two LEDs and each user has a single photo-diode (PD) (see Fig. 1). The LED converts the information carrying electrical signal to an intensity modulated optical signal and the PD at each user converts the received optical signal to electrical signal. The transmitter performs beamforming of the information symbols towards the two non-cooperative users. Let \( u_1 \in U_1 \) and \( u_2 \in U_2 \) be the information symbols intended for the first and second user respectively, where \( U_1 \) and \( U_2 \) are the information symbol alphabets for user 1 and user 2 respectively. Let \( x_i \) be the optical power transmitted from the \( i \)th LED (\( i = 1, 2 \)). At any time instance, the transmitted optical power vector \( x \triangleq [x_1, x_2]^T \) is given by

\[
x = Au,
\]

where \( u \triangleq [u_1, u_2]^T \) and \( A \in \mathbb{R}^{2 \times 2} \) is the beamforming matrix. In this paper, we consider the following power constraints for our dimmable VLC system.

The instantaneous power transmitted from each LED is non-negative and is less than some maximum limit \( P_0 \) due to skin and eye safety regulations [3]. Further, such a maximum limit on the transmitted power is required also due to limited interference requirement to the neighboring VLC systems, i.e.

\[
0 \leq x_i \leq P_0, \quad i = 1, 2.
\]

Since our VLC system is dimmable we further impose a per-LED average power constraint of the type

\[
E[x_i] = \xi P_0, \quad i = 1, 2,
\]

where \( 0 \leq \xi \leq 1 \) is the dimming target [5]. For the sake of analysis, we define \( x_i' \triangleq \frac{x_i}{P_0} \), i.e., \( x_i' = x_i / P_0 \). Consequently, the normalized optical power transmitted from each LED must satisfy the following constraints given by

\[
0 \leq x_i' \leq 1 \quad \text{and} \quad E[x_i'] = \xi, \quad i = 1, 2.
\]

III. AN ACHIEVABLE RATE REGION OF THE CHANNEL IN (5)

In this section, we derive an achievable rate region for the channel in (5) using the ZF precoder. Towards this end, in Section III-A we firstly explain as to how, the transmission power constraints (per-LED peak and average) are met by the proposed precoder. Secondly, in Section III-B we explain as to how the information alphabet set (from where the information symbol vector \( u \) is chosen) is designed depending on the channel realization \( H \). We give explicit mathematical expression for the chosen information alphabet set as a function of the channel realization \( H \) and dimming target \( \xi \). Thirdly, in Section III-C we derive mathematical expressions for the instantaneous optical power transmitted from each LED. Finally, in Section III-D we propose the achievable rate region for the ZF precoder.

A. Per-LED Peak and Average Power Constraints

For the 2 × 2 MU-MISO system discussed in Section II, the ZF precoding matrix is uniquely given by \( A = P_0 H^{-1} \), i.e., \( x' = x / P_0 = Au / P_0 = P_0 H^{-1} u / P_0 = H^{-1} u \). Thus

\[
\text{Thus} \quad 0 \leq x_i' \leq 1 \quad \text{and} \quad E[x_i'] = \xi, \quad i = 1, 2.
\]

3 Since each of the two users has a single PD we will be interchangeably using user and PD in subsequent discussions.

4 We assume channel state information (CSI) at the transmitter, which can be acquired through feedback of CSI by the users (PDs) using an uplink wireless channel. The PDs can acquire downlink CSI through pilot transmissions from the LEDs at the transmitter.

5 In subsequent discussions, by “received electrical signal”, we refer to the “normalized received electrical signal”.

6 Note that \( h_{ki} \)'s are non-negative and model the overall gains of the line of sight (LOS) optical path between the \( i \)th LED and the \( k \)th user and also the responsivity of the PD of the \( k \)th user [3].

7 Note that the above noise impairments of the received signal are the main impairments that are commonly assumed in VLC systems [3].

8 In this paper, we only consider the scenario where \( H \) is invertible i.e., \( \det(H) \neq 0 \). When \( \det(H) = 0 \), the indoor 2 × 2 MU-MISO VLC broadcast channel is reduced to an effective indoor 2 × 1 VLC channel with one LED transmitter and two single PD users, for which the capacity region has been studied in [15].

9 For the scenario where \( \det(H) = 0 \), we consider the rates of the two users to be zero. If we do not restrict ourselves to the ZF precoder then non-zero rates can be simultaneously achieved for both the users [15]. Indeed, the 2 × 2 MU-MISO broadcast channel reduces to 2 × 1 MU-MISO channel, for which the capacity region is studied in [15]. However, in this paper, emphasis is on studying the rate region of the 2 × 2 MU-MISO broadcast VLC channels with the ZF precoder since this has not been studied before, and therefore we only consider the scenario where \( \det(H) \neq 0 \).
The normalized received signal vector is given by
\[ y = Hx' + n = HH^{-1}u + n = u + n. \] (6)
i.e., there is no multi-user interference (MUI). Since
\[ u = Hx' = [h_1, h_2][x'_1, x'_2]^T, \] (7)
and \( 0 \leq x'_i \leq 1, i = 1, 2 \) (see (4)) it follows that, the information signal vector \( u \) must be limited to the region
\[ R_{\perp}(H) \equiv \{ u | u = Hx', 0 \leq x'_i \leq 1, 0 \leq x'_2 \leq 1 \}. \] (8)
The region \( R_{\perp}(H) \) is a parallelogram with its two non parallel sides as \( h_1 \) and \( h_2 \) (see \( R_{\perp}(H) \) in Fig. 2). In addition to this, the diagonal of the parallelogram \( R_{\perp}(H) \) is the vector \( h_1 + h_2 \) as shown in Fig. 2.

Let \( E[u] \equiv [E[u_1], E[u_2]]^T \) be the mean information symbol vector. From (7) and (4), the mean information symbol vector is given by
\[ E[u] = HE[x'] = [h_1, h_2]E[x'_1, x'_2]^T \overset{(a)}{=} \xi(h_1 + h_2), \] (9)
where step (a) follows from (5). Therefore, the mean information symbol pair \( (E[u_1], E[u_2]) \) is a point corresponding to the tip of the vector \( E[u] = \xi(h_1 + h_2) \). Equivalently, if the distribution of the information symbol vector \( E[u] = [E[u_1], E[u_2]]^T \) is \( \xi(h_1 + h_2) \), then the mean transmit optical signal is given by
\[ E[x] = [E[x_1], E[x_2]]^T \overset{(a)}{=} P_0H^{-1}E[u] \overset{(b)}{=} P_0H^{-1}(h_1 + h_2) \overset{(c)}{=} P_0[H^{-1}\xi(h_1 + h_2)]^T \overset{(d)}{=} P_0[h_1 + h_2]^T, \] (10)
where step (a) follows from the fact that the transmit optical signal vector \( x = Au \) and \( A = P_0H^{-1} \). Step (b) follows from the fact that \( E[u] = \xi(h_1 + h_2) \) and step (c) follows from the fact that \( (h_1 + h_2) = H[11]^T \). From the above equation we have \( E[x_1] = P_0\xi \) and \( E[x_2] = P_0\xi \), i.e., this shows that the proposed precoder satisfies the per-LED average power constraint (dimming constraint).

From (8) it is clear that the vector \( (h_1 + h_2) \) is a diagonal of \( R_{\perp}(H) \) (see Fig. 2). For a given \( 0 \leq \xi \leq 1 \), the tip of the mean information symbol vector \( \xi(h_1 + h_2) \) is therefore a fixed point on the diagonal \( (h_1 + h_2) \). We denote this point by
\[ D(H, \xi) = (E[u_1], E[u_2]) \overset{(a)}{=} (\xi(h_{11} + h_{12}), \xi(h_{21} + h_{22})). \] (11)

B. Design of the Information Alphabet Sets \( U_1 \) and \( U_2 \) for a Given Channel Realization \( H \)

With the ZF precoder, the broadcast channel in (5) is reduced to two parallel SISO (single-input-single-output) optical channels between the transmitter and the two users (see (6)). Since \( u_1 \in U_1 \) and \( u_2 \in U_2 \) are independent and originate from different codebooks, it follows that \( (u_1, u_2) \in U_1 \times U_2 \).

From (8), we know that \( (u_1, u_2) \) must belong to the parallelogram \( R_{\perp}(H) \) and therefore
\[ U_1 \times U_2 \subset R_{\perp}(H). \] (12)

In general we choose \( U_1 \) to be intervals of the type \( [a_i, b_i], i = 1, 2 \) [9]. Let the length of the intervals \( U_1 \) and \( U_2 \) be \( L_1 \) and \( L_2 \) respectively, i.e., \( L_1 \equiv [b_i - a_i], i = 1, 2 \). With \( U_1 \) and \( U_2 \) as intervals, it is clear that \( U_1 \times U_2 \) must be a rectangle whose length along the \( u_1 \) axis is \( L_1 \) and that along the \( u_2 \) axis is \( L_2 \).

In this paper we assume \( u_1 \) and \( u_2 \) to be uniformly distributed in the interval \( U_1 \) and \( U_2 \) respectively [12]. Therefore, for a given \( U_1 \) and \( U_2 \), the mean information symbol pair \( (E[u_1], E[u_2]) \) will lie at the point of intersection of the two diagonal of the rectangle \( U_1 \times U_2 \). We will subsequently call this point of intersection as the “midpoint” of the rectangle \( U_1 \times U_2 \) and will denote it by \( \mathcal{C}(U_1, U_2) \).

From (11), it follows that the mean information symbol pair must exactly coincide with \( D(H, \xi) \), i.e.
\[ \mathcal{C}(U_1, U_2) = D(H, \xi) \] (13)

We have seen earlier that restricting the information symbol vector \( u \) to lie within \( R_{\perp}(H) \) guarantees that the per-LED peak power constraint is met and therefore we choose \( U_1 \) and \( U_2 \) such that the rectangle \( U_1 \times U_2 \) lies within \( R_{\perp}(H) \).

At high SNR \( (P_0/\sigma >> 1) \), uniformly distributed information symbol is near capacity achieving [10].
Further, to satisfy the per-LED average power constraint, since the midpoint of the rectangle $U_1 \times U_2$ is at $D(H, \xi)$ (whose coordinates in the $u_1$-$u_2$ plane are given in (11)) and the length of the intervals $U_1$ and $U_2$ are $L_1$ and $L_2$ respectively, the information alphabet sets $U_1$ and $U_2$ depend on the channel and are given by the intervals

$$U_1 = [\xi(h_{11} + h_{12}) - (L_1/2), \, \xi(h_{11} + h_{12}) + (L_1/2)]$$

(14)

$$U_2 = [\xi(h_{21} + h_{22}) - (L_2/2), \, \xi(h_{21} + h_{22}) + (L_2/2)]$$

(15)

where $L_1$ and $L_2$ are such that $U_1 \times U_2 \subset R_{//}(H)$.

C. The Instantaneous Transmit Optical Power Levels for a Choosen $U_1, U_2$

Let $\text{Rect}(L_1, L_2, D(H, \xi))$ denote the unique rectangle having its midpoint at $D(H, \xi)$ and whose length along the $u_1$ axis is $L_1$ and that along the $u_2$ axis is $L_2$ (see Fig. 2). With the information symbols restricted to the alphabet sets

$$U_1 = [\xi(h_{11} + h_{12}) - (L_1/2), \, \xi(h_{11} + h_{12}) + (L_1/2)]$$

and

$$U_2 = [\xi(h_{21} + h_{22}) - (L_2/2), \, \xi(h_{21} + h_{22}) + (L_2/2)]$$

respectively, the transmitted optical power vector would lie in the set

$$P(H, U_1 \times U_2) \triangleq \{(x_1, x_2) = P_0 \mathbf{H}^{-1} \mathbf{u} \mid \mathbf{u} \in U_1 \times U_2\}.$$  

(16)

The instantaneous optical transmit powers in terms of the channel $H$ and the information symbols $u_1$ and $u_2$ are given by $x = [x_1 \, x_2]^T = P_0 \mathbf{H}^{-1} [u_1 \, u_2]^T$. It follows that

$$x_1 = \frac{P_0}{h_{12}h_{22} - h_{11}h_{21}} (h_{22}u_1 - h_{12}u_2)$$

$$x_2 = \frac{P_0}{h_{11}h_{22} - h_{12}h_{21}} (h_{11}u_2 - h_{21}u_1)$$

(17)

Since $u_1 \in U_1$ and $u_2 \in U_2$ and $U_i$, $i = 1, 2$ are intervals (given in (14) and (15)), the instantaneous optical transmit power from each LED would lie in some range given by

$$x_1^{\min} \leq x_1 \leq x_1^{\max}, \quad x_2^{\min} \leq x_2 \leq x_2^{\max},$$

(18)

where

$$x_1^{\min} \triangleq P_0 \left[ \frac{\xi + h_{22}L_1 + h_{12}L_2}{2(h_{11}h_{22} - h_{12}h_{21})} \right]$$

$$x_1^{\max} \triangleq P_0 \left[ \frac{\xi - h_{22}L_1 + h_{12}L_2}{2(h_{11}h_{22} - h_{12}h_{21})} \right]$$

$$x_2^{\min} \triangleq P_0 \left[ \frac{\xi + h_{11}L_2 + h_{21}L_1}{2(h_{11}h_{22} - h_{12}h_{21})} \right]$$

$$x_2^{\max} \triangleq P_0 \left[ \frac{\xi - h_{11}L_2 + h_{21}L_1}{2(h_{11}h_{22} - h_{12}h_{21})} \right].$$

(19)

We call $x_1^{\max}$ and $x_i^{\min}$ as the maximum and minimum levels of the instantaneous transmit optical power from the $i$th LED.$^{13}$ Through the following example we illustrate as to how the intervals $U_1$ and $U_2$ can be selected in order that the proposed precoder results in optical transmit signals which satisfy the per-LED peak and average power constraints.

**Example 1:** For a given dimming target $\xi = 0.3$, per-LED peak power constraint $P_0$ and channel matrix $H_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}$ the corresponding parallelogram $R_{//}(H_1)$ is shown in Fig. 3. The point $D(H = H_1, \xi = 0.3)$ coincides with the tip of the vector

$$\mathbf{u} = \begin{bmatrix} 0.3 & 0.3 \end{bmatrix}.$$  

(20)

Therefore the co-ordinates of the point $D(H_1, \xi)$ is $(1.5, 1.2)$. This point is shown in Fig. 3 marked with a star. We choose a rectangle ABCD in the $u_1$-$u_2$ plane, whose horizontal length (length along the $u_1$-axis) is $L_1 = 2$ and the vertical length (length along the $u_2$ axis) is $L_2 = 0.4$ and whose midpoint is at $(1.5,1.2)$. Hence in our notation $ABC D = \text{Rect}(2, 0.4, (1.5, 1.2))$ which can be seen to lie completely inside $R_{//}(H_1)$, which guarantees that the transmit optical signals satisfy the per-LED peak power constraint (see Footnote 10). This corresponds to choosing $U_1 = [0.5, 2.5]$ and $U_2 = [1, 1.4]$ which matches exactly with $U_1$ and $U_2$ calculated from (14) and (15). The mean information vector is then given by $\mathbf{E}[\mathbf{u}] = [\mathbf{E}[u_1] \mathbf{E}[u_2]]^T = [(0.5 + 2.5)/2 (1 + 1.4)/2]^T = [1.5 1.2]^T = D(H_1, \xi = 0.3)$ (see (20) above), i.e., which guarantees that the per-LED average power constraint is satisfied (see the discussion before (10)). From Fig. 3 we see that $U_1 \times U_2 = ABC D = \text{Rect}(2, 0.4, (1.5, 1.2))$ lies entirely inside $R_{//}(H)$ and therefore it follows that the transmit optical signal satisfies the per-LED peak power constraint (see the discussion in Footnote 10).

For the example here, in Fig. 4 we have plotted the set $A_1 B_1 C_1 D_1 = P(H_1, U_1 \times U_2 = ABC D)$ (normalized

$^{13}$In this paper, LED 1 refers to the LED whose channel vector has a higher inclination angle (from the $u_1$ axis) than the inclination angle of the channel vector of the other LED. Due to this convention, det$(H) = h_{11}h_{22} - h_{12}h_{21}$ will always be negative and the equations for the minimum and maximum levels given in (19) will hold. The fact that this convention results in negative det$(H)$ is explained in (28) of Section IV. However, irrespective of the used convention, the maximum transmit optical power level will always be greater than the minimum transmit optical power level.
by \( P_0 \), from where it is observed that for the channel realization \( \mathbf{H}_1 \) the set of transmit power vectors \( P(\mathbf{H}_1, \mathbf{u}_1 \times \mathbf{u}_2 = ABCD) \) (normalized by \( P_0 \)) lies completely inside the square region \((0 \leq x_1' \leq 1, 0 \leq x_2' \leq 1)\), i.e., the transmit optical power vector \([x_1, x_2]^T\) satisfies the per-LED peak power constraint \(0 \leq x_1 \leq P_0 \) and \(0 \leq x_2 \leq P_0 \). From (19) the minimum and maximum instantaneous normalized transmit optical power levels are given by

\[
\begin{align*}
\frac{x_1^\text{min}}{P_0} &= \left[ 0.3 + \frac{1 \times 2 + 4 \times 0.4}{2} \right] = 0.1364, \\
\frac{x_1^\text{max}}{P_0} &= \left[ 0.3 - \frac{1 \times 2 + 4 \times 0.4}{2} \right] = 0.4636, \\
\frac{x_2^\text{min}}{P_0} &= \left[ 0.3 + \frac{1 \times 0.4 + 3 \times 2}{2} \right] = 0.0091, \\
\frac{x_2^\text{max}}{P_0} &= \left[ 0.3 - \frac{1 \times 0.4 + 3 \times 2}{2} \right] = 0.5909,
\end{align*}
\]

which matches with the normalized minimum and maximum instantaneous power levels as shown in Fig. 4.

Another possible valid information alphabet set is given by the rectangle \( MNOP = \text{Rect}(L_1 = 0.6, L_2 = 1.2, D(\mathbf{H}_1, \xi = 0.3) = (1.5, 1.2)) \) as shown in Fig. 3. The midpoint of \( MNOP \) is the same as that of \( ABCD \) and it also lies completely inside \( R_{ij}(\mathbf{H}_1) \). However, the intervals \( \mathbf{u}_1 = [1.2, 1.8] \) and \( \mathbf{u}_2 = [0.6, 1.8] \) of \( MNOP \) are different from that of \( ABCD \). Due to different information alphabet sets, the set of instantaneous transmit power vectors \( P(\mathbf{H}_1, MNOP) \) for \( \mathbf{u}_1 \times \mathbf{u}_2 = MNOP \) is different from \( P(\mathbf{H}_1, ABCD) \). The set \( P(\mathbf{H}_1, MNOP) \) (normalized by \( P_0 \)) is shown in Fig. 4 (see \( M_1 N_1 O_1 P_1 \)).

Example 2: Let us consider the same dimming target \( \xi = 0.3 \), same per-LED peak power constraint \( P_0 \) as in example 1. However, let \( \mathbf{H} = \mathbf{H}_2 = \begin{bmatrix} 3/4 \ 3/5 \ -1/5 \ -1/4 \end{bmatrix} \). Note that \( \mathbf{H}_2 \neq \mathbf{H}_1 \) (channel realization considered in example 1) and therefore \( R_{ij}(\mathbf{H}_2) \) and \( R_{ij}(\mathbf{H}_1) \) are different, as can be observed in Fig. 5. Further, from Fig. 5 we also observe that the rectangle \( ABCD \) of example 1 (shown as a dotted rectangle in Fig. 5) which has its midpoint at \( \mathbf{D}(\mathbf{H}_1, \xi) \) and which lies completely inside \( R_{ij}(\mathbf{H}_1) \) cannot be chosen as a possible information alphabet set when the channel realization changes to \( \mathbf{H} = \mathbf{H}_2 \). This is because, as can be seen in Fig. 5, the rectangle \( ABCD \) is no longer inside \( R_{ij}(\mathbf{H}_2) \) and its midpoint does not coincide with \( \mathbf{D}(\mathbf{H}_2, \xi) \).

Further, it is also observed that even if we translate the rectangle \( ABCD \) to \( PQRS \) so as to shift its midpoint to \( \mathbf{D}(\mathbf{H}_2, \xi) \), the new rectangle \( PQRS \) does not lie within \( R_{ij}(\mathbf{H}_2) \) and is therefore not a valid information alphabet set. Hence, with changing channel realization \( \mathbf{H} \), the same information alphabet set cannot be used as it might no longer satisfy the per-LED peak and average-power constraint. Therefore, the choice of information alphabet set depends on the channel realization. In Fig. 6 we have plotted the set of instantaneous transmit optical power vectors \( P(\mathbf{H}_2, \mathbf{u}_1 \times \mathbf{u}_2 = PQRS) \) (normalized by \( P_0 \)
which is observed to be not completely inside the square region $0 \leq x_2^i \leq 1, 0 \leq x_1^i \leq 1$, which confirms that the information alphabet set $PQRS$ is not valid for $H = H_1$ as it results in transmit optical vectors which do not satisfy the per-LED power constraint.

From the above discussions we have the following remark.

Remark 1: If we choose the information symbol alphabets $U_1$ and $U_2$ such that $U_1 \times U_2 = \text{Rect}(L_1, L_2, D(H, \xi)) \subset R_{i/H}(H)$ and if the information symbols are uniformly distributed in $U_i, i = 1, 2$, then the transmit optical vector $x = P_0 H^{-1} u$ satisfies both the per-LED peak and average power constraints.

D. The Proposed Achievable Rate Region

The ZF precoder transforms the broadcast channel into two parallel SISO channels $y_i = u_i + n_i, u_i \in U_i, i = 1, 2$. Let $R_1$ and $R_2$ denote the information rates achieved on these SISO channels with $u_i$ distributed uniformly in $U_i$. Any given $U_i$ and $U_2$ satisfying the conditions in (12) and (13) would satisfy the optical power constraints in (4) and would therefore correspond to an achievable rate pair for the broadcast channel in (5). Since a rectangle in the $u_1$-$u_2$ plane corresponds to a unique $U_1$ and $U_2$ and vice versa, it follows that any rectangle lying inside the parallelogram $R_{i/H}(H)$ and having its midpoint at $D(H, \xi)$ will correspond to an achievable rate pair. In this paper, for the broadcast channel in (5), we therefore propose an achievable rate region which consists of rate pairs corresponding to such rectangles (one such rectangle is shown in Fig. 2). We define our proposed rate region more precisely in the following. Towards this end, we first formally define the achievable rate of a SISO AWGN optical channel, where the transmitted information symbol is constrained to lie in an interval.

Result 1: [From [6], [10]] The achievable information rate of a SISO channel $y = u + n$ (where $u \sim Unif[a, b]$ and $n \sim \mathcal{N}(0, \sigma^2/P_0)$) depends on the interval $[a, b]$ only through its length $L = |b - a|$, and is given by the function

$$C(L = |b - a|, P_0/\sigma) \triangleq I(u; y),$$

(21)

here $Unif[a, b]$ denote the uniform distribution in the interval $[a, b]$ and $I(u; y)$ is the mutual information between $u$ and $y$.

Result 2: [From [6], [10]] The function $C(L, P_0/\sigma)$ is continuous with respect to $L$ and increases monotonically with increasing $L$ for a fixed $P_0/\sigma$.

Since, $\text{Rect}(L_1, L_2, D(H, \xi))$ denote the unique rectangle having its midpoint $D(H, \xi)$ and whose length along the $u_1$ axis is $L_1$ and that along the $u_2$ axis is $L_2$ (see Fig. 2) Any such rectangle $\text{Rect}(L_1, L_2, D(H, \xi)) \subset R_{i/H}(H)$ will correspond to an achievable rate pair given by

$$(R_1, R_2) \triangleq (C(L_1, P_0/\sigma), C(L_2, P_0/\sigma))$$

(22)

For a given $(H, P_0/\sigma, \xi)$ the proposed achievable rate region for the ZF precoder is given by

$$R_{ZF}(H, P_0/\sigma, \xi) \triangleq \bigcup_{(L_1, L_2) \in S(H)} \{C(L_1, P_0/\sigma), C(L_2, P_0/\sigma)\},$$

(23)

where $S(H) \triangleq \{(L_1 \geq 0, L_2 \geq 0) \mid \exists \text{Rect}(L_1, L_2, D(H, \xi)) \subset R_{i/H}(H)\}$ is the set of all feasible $(L_1, L_2)$ pairs for which $\text{Rect}(L_1, L_2, D(H, \xi))$ lies completely within $R_{i/H}(H)$. With change in the channel realization $H, R_{i/H}(H)$ and $D(H, \xi)$ changes, and therefore the set of all such feasible $(L_1, L_2)$ pairs will also change, which will result in a different rate region, i.e., the proposed rate region $R_{ZF}(H, P_0/\sigma, \xi)$ changes with changing channel realization $H$. As an example, $(L_1 = 2, L_2 = 0.4)$ is a feasible $(L_1, L_2)$ pair when $H = H_1$ in example 1. With $H = H_1, R_{i/H}(H_1)$ is depicted by the dotted parallelogram in Fig. 5, and the rectangle $\text{Rect}(L_1 = 2, L_2 = 0.4, D(H_1, \xi) = 0.3))$ is depicted by the dotted rectangle $ABCD$. The rate pair corresponding to this rectangle is $(R_1, R_2) = (C(L_1 = 2, P_0/\sigma), C(L_2 = 0.4, P_0/\sigma))$. When the channel realization changes to $H = H_2$ example 2, the parallelogram $R_{i/H}(H)$ changes from $R_{i/H}(H_1)$ to $R_{i/H}(H_2)$ and $D(H_1, \xi) = 0.3$ also changes from $D(H_1, \xi) = 0.3$ to $D(H_2, \xi) = 0.3$ (see Fig. 5). However, the $(L_1, L_2)$ pair $(L_1 = 2, L_2 = 0.4)$ is no more feasible when $H = H_2$ since the rectangle $\text{Rect}(L_1 = 2, L_2 = 0.4, D(H = H_2, \xi = 0.3))$ (shown in the figure as $PQRS$) is not completely within the new $R_{i/H}(H_2)$. Hence the rate pair $(C(L_1 = 2, P_0/\sigma), C(L_2 = 0.4, P_0/\sigma))$ which was achievable with $H = H_1$ is no more achievable when the channel realization changes to $H = H_2$. Hence, it is clear that the proposed rate region $R_{ZF}(H, P_0/\sigma, \xi)$ depends on the channel realization $H$.

From (22) and (23) we have the following definition of the achievable rate pair

**Definition 1:** $(a, b) \in R_{ZF}(H, P_0/\sigma, \xi) \iff \exists$ a unique $(L_a \geq 0, L_b \geq 0)$ s.t. rectangle $\text{Rect}(L_a, L_b, D(H, \xi)) \subset R_{i/H}(H)$ and $(a, b) = (C(L_a, P_0/\sigma), C(L_b, P_0/\sigma))$.

Uniqueness follows from the fact that $C(x, P_0/\sigma)$ is a monotonic and continuous function of $x$ (see Result 2) and hence the $(L_a, L_b)$ pair satisfying $(a, b) = (C(L_a, P_0/\sigma), C(L_b, P_0/\sigma))$ will be unique.

IV. Characterizing the Boundary of the Rate Region $R_{ZF}(H, P_0/\sigma, \xi)$

In this section, we completely characterize the boundary of the rate region, $R_{ZF}(H, P_0/\sigma, \xi)$, for a fixed $(H, P_0/\sigma, \xi)$. Towards this end, for each information rate $R_1$ achievable by the first user, we find the corresponding maximum possible information rate $R_2$ achievable by the second user. Each pair of $R_1$ and its corresponding maximum possible $R_2$ is therefore a point on the boundary of the proposed rate region. By increasing $R_1$ from 0 to its maximum possible value, all such $(R_1, R_2)$ pairs characterize the boundary of the rate region.

From (23), we know that any achievable rate pair $(R_1, R_2)$ in the proposed rate region $R_{ZF}(H, P_0/\sigma, \xi)$ corresponds to some rectangle $\text{Rect}(L_1, L_2, D(H, \xi))$. The rate to the $i$th user, i.e., $R_i = C(L_i, P_0/\sigma), i = 1, 2$ depends only on the length of this rectangle along the $u_i$-axis. Since the $C(L, P_0/\sigma)$ function is monotonic and continuous in its first argument, each value of $R_i$ corresponds to a unique $L_i$ and vice versa. Therefore, towards characterizing the boundary of $R_{ZF}(H, P_0/\sigma, \xi)$, we note that for a given $R_1$, i.e., for a given length $L_1$ along the $u_1$-axis, we would like to find the largest possible $R_2$, i.e., the largest
possible $L_2$ such that the rectangle $\text{Rect}(L_1, L_2, D(H, \xi))$ lies entirely inside $R_{i/1}(H)$. Hence, we can characterize the boundary of $R_{ZF}(H, P_0/\sigma, \xi)$ simply by varying $L_1 = x$ from 0 to its maximum possible value (denoted by $L_1^{\text{max}}(\xi)$), and for each value of $L_1$, $x \in [0, L_1^{\text{max}}(\xi)]$ we find the largest possible $L_2 = L_2^2(x)$ which gives us a corresponding rate pair $(R_1, R_2) = (C(L_1 = x, P_0/\sigma), C(L_2 = L_2^2(x), P_0/\sigma))$ on the boundary of the rate region $R_{ZF}(H, P_0/\sigma, \xi)$.

This proposed characterization of the rate region boundary therefore reduces the problem of finding rate pairs on the boundary of the proposed rate region to the geometrical problem of finding rectangles of a given/fixed length along the $u_1$-axis having the largest possible length along the $u_2$-axis such that the rectangles lie completely inside $R_{i/1}(H)$ and have their midpoint at $D(H, \xi)$.

For a given $(L_1 = x, L_2 = L_2^2(x))$ the corresponding information rate pair lies on the boundary of the proposed rate region $R_{ZF}(H, P_0/\sigma, \xi)$. We denote this information rate pair by $(R_{1d}(x, P_0/\sigma, \xi), R_{2d}(x, P_0/\sigma, \xi))$. From (22), this information rate pair is given by

$$
R_{1d}(x, P_0/\sigma, \xi) \triangleq C(L = x, P_0/\sigma).
$$

This then completely characterizes the boundary of the rate region $R_{ZF}(H, P_0/\sigma, \xi)$, which is given by

$$
R_{ZF}(H, P_0/\sigma, \xi) = \bigcup_{0 \leq x \leq L_1^{\text{max}}(\xi)} (R_{1d}(x, P_0/\sigma, \xi), R_{2d}(x, P_0/\sigma, \xi)).
$$

It is noted that the analysis done in this paper is applicable to any placement of the users and the LEDs. Subsequently, we follow the following convention that, by LED 1 we shall refer to the LED whose channel vector has a higher inclination angle (from the $u_1$-axis) than the inclination angle of the channel vector of the other LED.

Let the inclination of the vector $h_1$ and $h_2$ from the $u_1$ axis be $\theta_1$ and $\theta_2$ respectively (see Fig. 2). From our definition of LED 1 and LED 2 (see the above paragraph), it follows that $\theta_1 > \theta_2$. Therefore it follows that $\tan \theta_1 > \tan \theta_2$. Since $\tan \theta_1 = h_1/h_{11}$, $\tan \theta_2 = h_2/h_{12}$.

Hence, $\tan \theta_1 > \tan \theta_2$ implies that $h_1/h_{11} - h_2/h_{12} > 0$,

$$
h_1h_{12} - h_2h_{11} < 0, \quad \text{i.e.} \quad \det(H) < 0
$$

In the following proposition, we first compute the maximum value of $L_1$ and subsequently we derive the maximum value of $L_2$ for each value of $L_1$.

**Proposition 1:** The largest possible value of $L_1$ (i.e., length of the interval $L_1$) such that there exists a rectangle $\text{Rect}(L_1, L_2, D(H, \xi))$ ($L_2 \geq 0$) which lies completely inside the parallelogram $R_{i/1}(H)$, is given by

$$
L_1^{\text{max}}(\xi) \triangleq \max_{0 \leq x \leq L_1^{\text{max}}(\xi)} \frac{-2\det(H)\max(h_{11}, h_{12})}{\text{det}(H)}, \quad 0 \leq \xi \leq 1/2
$$

$$
\frac{-2(1 - \xi)\text{det}(H)\max(h_{11}, h_{12})}{\text{det}(H)}, \quad 1/2 \leq \xi \leq 1.
$$

**Proof:** See Appendix A. ■

It is clear from (29) in Proposition 1 that $L_1^{\text{max}}(\xi)$ is a continuous function of $\xi$ and $L_1^{\text{max}}(\xi) = L_1^{\text{max}}(1 - \xi)$.

**Remark 2:** The function $L_1^{\text{max}}(\xi)$ is a continuous function of $\xi$ and is symmetric about $\xi = 1/2$, i.e.

$$
L_1^{\text{max}}(\xi) = L_1^{\text{max}}(1 - \xi), \quad 0 \leq \xi \leq 1
$$

From (29) it is clear that since $\text{det}(H) < 0$ (see (28)) $L_1^{\text{max}}(\xi)$ is linearly increasing for $0 \leq \xi \leq 1/2$ and is linearly decreasing for $1/2 \leq \xi \leq 1$. Hence $L_1^{\text{max}}(\xi)$ has a unique maximum at $\xi = 1/2$.

**Remark 3:** The function $L_1^{\text{max}}(\xi)$ has its unique maximum at $\xi = 1/2$, i.e.

$$
\arg \max_{0 \leq \xi \leq 1} L_1^{\text{max}}(\xi) = 1/2
$$

**Proposition 2:** For a given $L_1 = x \in [0, L_1^{\text{max}}(\xi)]$, the largest possible $L_2 \geq 0$ such that there exists a rectangle $\text{Rect}(x, L_2, D(H, \xi)) \subset R_{i/1}(H)$, is given by

$$
L_2^{\text{max}}(\xi, x) \triangleq \max_{0 \leq x \leq L_1^{\text{max}}(\xi)} \frac{2\min(L_{2\text{up}}(x), L_{2\text{down}}(x))}{\text{det}(H)},
$$

$$
= 2\min(L_{2\text{up}}(x), L_{2\text{down}}(x)),
$$

where $L_{2\text{up}}(x)$ is given by

$$
\text{Case I: } 0 \leq \xi \leq \frac{h_{11} + h_{12}}{h_{11}}
$$

$$
L_{2\text{up}}(x) = \left\{ \begin{array}{ll}
\frac{-\xi\det(H) - \xi h_{21}}{h_{11}}, & 0 \leq x \leq L_1^{\text{max}}(\xi)
\end{array} \right.
$$
Using Lemma 3 along with the definition of the rate region boundary in (26) we get the following result.

**Result 3:** The proposed rate region boundary \( R_{ZF}^B(d)(H, P_0/\sigma, \xi) \) is symmetric about \( \xi = 1/2 \), i.e.

\[
R_{ZF}^B(d)(H, P_0/\sigma, \xi) = R_{ZF}^B(d)(H, P_0/\sigma, (1 - \xi)), \quad \forall \xi \in [0, 1].
\]  

(38)

The following theorem shows that for \( 0 \leq \xi \leq 1 \), the largest rate region is achieved when \( \xi = 1/2 \).

**Theorem 1:** For a fixed \( \xi \in [0, 1] \),

\[
R_{ZF}(H, P_0/\sigma, \xi) \subseteq R_{ZF}(H, P_0/\sigma, 1/2).
\]  

(39)

**Proof:** See Appendix D.

The proposed rate region boundary \( R_{ZF}^B(d)(H, P_0/\sigma, \xi) \) can be used to compute many practical operating points. Consider a case where we are interested in finding the largest achievable rate pair \((R_1, R_2)\) such that \( R_2 = \alpha R_1 \). This operating point could make sense, if for example the average data throughput requested by user 2 is \( \alpha \) times that of the throughput requested by user 1.

Moreover, for a given \( \alpha > 0 \) and \( P_0/\sigma \), the maximum achievable rate pair of the form \((r, \alpha r)\) is given by \((R_{ZF}^a(\alpha), \alpha R_{ZF}^a(\xi))\) where \( R_{ZF}^a(\xi) \) is defined as

\[
R_{ZF}^a(\xi) = \max_{(r, \alpha r) \in R_{ZF}^B(d)} r |r, \alpha r|_{H, P_0/\sigma, \xi}.
\]  

(40)

**Theorem 2:** \( R_{ZF}^a(\xi) \) is unique and \((R_{ZF}^a(\alpha), \alpha R_{ZF}^a(\xi))\) lies on the boundary \( R_{ZF}^B(d)(H, P_0/\sigma, \xi) \).

**Proof:** See Appendix E.

**Remark 4:** From the proof in Appendix E it is clear that Theorem 2 is non-trivial as it depends on the monotonicity and continuity of \( L_2^\xi(x) \), which is shown in Lemma 1. If Lemma 1 were not true, Theorem 2 would not hold.

**Result 4:** Using Theorem 2 and (38) of Result 3 it follows that for a given \( \alpha > 0 \), \( R_{ZF}^a(\xi) \) is symmetric about \( \xi = 1/2 \), i.e.

\[
R_{ZF}^a(\xi) = R_{ZF}^a(1 - \xi), \quad \forall \alpha > 0, \xi \in [0, 1].
\]  

(41)

**Corollary 2.1:** From the geometrical interpretation of Theorem 2 it follows that \((R_{ZF}^a(\xi), \alpha R_{ZF}^a(\xi))\) lies on the intersection of the straight line \( R_2 = \alpha R_1 \) and the rate region boundary \( R_{ZF}^B(d)(H, P_0/\sigma, \xi) \). Further, from the Pareto-optimality of the proposed rate region boundary, it follows that there is only a unique point of intersection between the line \( R_2 = \alpha R_1 \) and \( R_{ZF}^B(d)(H, P_0/\sigma, \xi) \).

Many other interesting operating points can also be found using the proposed characterization of the rate region boundary. For example, given that the first and the second user request rates \( r_1 \) and \( r_2 \) respectively, one could be interested in the operating rate pair \((R_{op}^1, R_{op}^2)\) given by

\[
(R_{op}^1, R_{op}^2) = \arg \min_{(R_1, R_2) \in R_{ZF}(H, P_0/\sigma, \xi)} |R_1 - r_1| + |R_2 - r_2|.
\]  

(42)

In Appendix F we show that if \((r_1, r_2, \xi) \in R_{ZF}(H, P_0/\sigma, \xi)\) then \((R_{op}^1, R_{op}^2) = (r_1, r_2)\) or else \((R_{op}^1, R_{op}^2)\) lies on the boundary of the proposed rate region, i.e., \((R_{op}^1, R_{op}^2) \in R_{ZF}^B(d)(H, P_0/\sigma, \xi)\).
rate region boundary in Section IV, we then propose a simple algorithm to compute this operating rate pair. Instead of performing a brute-force search over the entire rate region, the proposed algorithm first checks if \((r_1, r_2) \in R_{ZF}^{BD}(H, P_0/\sigma, \xi)\) and if not then it searches over all rate pairs \((R_1, R_2)\) only on the boundary of the proposed rate region. The proposed characterization of all rate pairs on the boundary helps us to simplify this search.

### A. Maximum Symmetric Rate \(R^\text{sym} (\xi)\)

Note that for the special case of \(\alpha = 1, R^\text{m, max} (\xi)\) is nothing but the maximum achievable symmetric rate which we shall denote by

\[
R^\text{sym} (\xi) \triangleq R^\text{m, max} (\xi). \tag{43}
\]

From Theorem 2 it is clear that the maximum symmetric rate is nothing but the largest rate \(R\) such that the rate pair \((R, R)\) lies on the boundary \(R_{ZF}^{BD}(H, P_0/\sigma, \xi)\). From the characterization of the boundary points in (26), it follows that there must exist \((x, L^\xi (x))\) for some \(0 \leq x \leq L^\xi_{\text{max}} (\xi)\) such that

\[
R = C(x, P_0/\sigma), \quad \text{and} \quad R = C(L^\xi (x), P_0/\sigma) \tag{44}
\]

and therefore

\[
x = L^\xi (x) \tag{45}
\]

since from Result 2 we know that \(C(L, P_0/\sigma)\) is a continuous and monotonic function. From (22) it follows that there exists a rectangle Rect\((x, L^\xi (x), D(H, \xi)) \subset R_1/\{H\}\) corresponding to the rate pair \((R, R)\) where \(x\) satisfies (45).

Since \(x = L^\xi (x)\) it follows that this rectangle is a square. Further, from the definition of \(L^\xi (x)\) in (32) it follows that this is the largest sized square whose midpoint is at \(D(H, \xi)\) and has side length \(x\).

Hence, the maximum achievable symmetric rate corresponds to the largest sized square which is completely inside \(R_1/\{H\}\) and has its midpoint at \(D(H, \xi)\).

It is noted that the proof of all theorems in this paper is based on the proposed rate region boundary characterization and does not rely on explicit mathematical expression for the rate function \(C(., .)\).

### V. A NOVEL TRANSCIEVER ARCHITECTURE

In this section we propose a novel transceiver architecture for the practical implementation of the proposed \(2 \times 2\) MU-MISO VLC system to achieve any rate pair \((R_1, R_2) \in R_{ZF}^{BD}(H, P_0/\sigma, \xi)\) (see Section IV), under a per-LED peak power constraint of \(P_0\) and a controllable dimming target.

In Fig. 7, we have shown the block diagram of both the transmitter and the receiver. The block diagram in Fig. 7(a) depicts the transmitter, the block diagram in Fig. 7(b) depicts the controller for the transmitter which we call as Tx controller and the block diagram in Fig. 7(c) depicts the receiver. The working of this transceiver is as follows.

Consider a scenario where the rate requested by User 1 and User 2 are \(R^{\text{gt}}_1\) bpcu and \(R^{\text{gt}}_2\) bpcu respectively and to satisfy the lighting requirement inside the room the required dimming target is \(\xi\). We call this rate pair \((R^{\text{gt}}_1, R^{\text{gt}}_2)\), as the target rate pair of the system. The Tx controller first checks if this target rate pair lies in the proposed achievable rate region \(R_{ZF}^{BD}(H, P_0/\sigma, \xi)\) (see Section IV). If the target rate pair lies inside the proposed achievable rate region, (i.e., \((R^{\text{gt}}_1, R^{\text{gt}}_2) \in R_{ZF}^{BD}(H, P_0/\sigma, \xi)\)), then the Tx controller flags 1, otherwise it flags 0 (see status output of the Tx controller in Fig. 7(b)). If this flag is 1, then the Tx controller provides \(L_1\) and \(L_2\), the lengths of the intervals \(U_1\) and \(U_2\). From (23) we know that since \((R^{\text{gt}}_1, R^{\text{gt}}_2) \in R_{ZF}^{BD}(H, P_0/\sigma, \xi)\), there must exist some \((L_1, L_2)\) such that \(R^{\text{gt}}_1 = C(L_1, P_0/\sigma)\) and \(R^{\text{gt}}_2 = C(L_2, P_0/\sigma)\). From Result 2 we also know that for a given \(P_0/\sigma\), \(C(x, P_0/\sigma)\) is a monotonic function of \(x\), and therefore there exists a corresponding inverse function \(C^{-1}(R, P_0/\sigma)\) such that \(C^{-1}(C(L, P_0/\sigma), P_0/\sigma) = L\) and \(C(C^{-1}(R, P_0/\sigma), P_0/\sigma) = R\). It then follows \(L = C^{-1}(R^{\text{gt}}_i, P_0/\sigma), \quad i = 1, 2\) see Fig. 7(c). In the Tx controller we also have a block which outputs the mean information symbol vector \(\xi(h_1 + h_2) = [\xi(h_{11} + h_{12})\xi(h_{21} + h_{22})]^T\) (defined in (9)).

Further, in Fig. 7(a) the information bits for user 1 and user 2 are coded separately using independent codebooks each having i.i.d. codeword symbols which are uniformly distributed...
TABLE I

| PD area | 1 cm² |
|----------|-------|
| Receiver Field of View (FOV) | 60 [deg.] |
| Refractive index of a lens at the PD | 1.5 |
| Semi-angle at half power | 70 [deg.] |

in $[-1/2, 1/2]$. The codeword symbols for user 1 and user 2 are denoted by $u_1'$ and $u_2'$ respectively (note that $u_1$ and $u_2$ are the information symbols for User 1 and User 2 respectively). From Section III, we know that the information symbols for the $i$th user must be uniformly distributed in the interval $\mathcal{U}_i$, i.e., $u_i \in \mathcal{U}_i = [\xi(h_{1i} + h_{2i}) - L_i/2, \xi(h_{1i} + h_{2i}) + L_i/2]$ (since the horizontal length of the rectangle corresponding to the rate pair $(R_1, R_2)$ is $L_1$, the vertical length of this rectangle is $L_2$ and its midpoint is $D(H, \xi)$). Therefore, starting with the codeword symbol $u_i'$ we can get the information symbol $u_i$ by

$$u_i = L_i u_i' + \xi(h_{1i} + h_{2i}), \quad i = 1, 2.$$  \hspace{1cm} (46)

This is also shown in Fig. 7(a). The information vector $[u_1, u_2]^T$ is then precoded with $H^{-1}$ and scaled by $P_0$ to give the transmit signal vector $[x_1, x_2]^T$. It is noted that the proposed transmitter architecture in Fig. 7(a) allows us to use the same channel encoder/codebook irrespective of the dimming target $\xi$. This is because the effect of the dimming control is only in shifting the mean of the information symbols $(u_1, u_2)$ (see the adders in Fig. 7(a)).

At the receiver after performing the operations shown in Fig. 7(c), we obtain the received vector as given by (5).

VI. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we present numerical results in support of the results reported in previous sections. For all numerical results we consider an indoor office room environment where the room is $5 \text{ m} \times 5 \text{ m}$ and its height is $3 \text{ m}$. The two LEDs are attached to the ceiling and the two PDs (users) are placed at a height of $50 \text{ cm}$ from the floor of the room. The two LEDs and the PDs lie in a plane perpendicular to the floor of the plane. The LEDs are placed $60 \text{ cm}$ apart and the ratio $P_0/P_1$ is fixed to $70 \text{ dB}$. The channel gains are modeled for an indoor line of sight (LOS) channel. The other parameters used for simulation are given in Table I. All these parameters and the channel model are taken from prior work [3], [14], [17], [18].

In Fig. 8, for a LED separation of $0.6 \text{ m}$ and PD (user) separation of $4 \text{ m}$ such that the placement of both the LEDs and the PDs is symmetric, we plotted the proposed rate region boundary $R_{12}^{BD}(H, P_0, \sigma, \xi)$ (see 26), for $\xi = 0.1, 0.2, 0.3, 0.4, 0.5, 0.7$.

For a given $\xi$, it is observed that the boundary is indeed Pareto-optimal as is stated in Lemma 2. We also observe that as $\xi$ increases from $\xi = 0.1$ to $\xi = 0.5$, the rate region expands and then it shrinks with further increase in $\xi$ from $\xi = 0.5$ onwards to $\xi = 1$. We have also observed that rate region boundary is same for both $\xi = 0.3$ and $\xi = 1 - 0.3 = 0.7$ as is stated in Result 3 (see the dotted line and the solid line marked with circle in Fig. 8). It is also observed that $\xi = 1/2$ gives us the largest rate region as is stated in Theorem 1. The expansion/shrinking of the rate region with changing $\xi$ is explained in the following.

For a given $\xi$, the points on the rate region boundary correspond to rectangles in the $u_1 - u_2$ plane having their midpoints at $D(H, \xi)$, i.e., on the diagonal $(h_1 + h_2)$ and at a distance of $\xi||h_1 + h_2||$ from the origin. As $\xi$ increases, the midpoint of the rectangles move away from the origin and towards the interior of the parallelogram $R_{12}^{BD}(H)$. This allows us to fit bigger rectangles and hence the rate region expands. As $\xi$ is increased beyond $\xi = 0.5$ the midpoint of the rectangles moves towards the origin and hence the rate region shrinks.

In Fig. 9, for a fixed user separation of $s = 4 \text{ m}$, an LED separation of $d = 60 \text{ cm}$ and symmetric placement of LEDs and PDs, we plot the maximum achievable symmetric rate
optimal location for the two users from the origin. The displacement vector of the two users from the origin is around 20% as compared to when the displacement is 0 cm to 40 cm.

We next study the variation in the maximum symmetric rate when the two users (PDs) are moved along a line parallel to the ceiling (at a height of 50 cm above the floor) while the two LEDs are stationary and fixed to the ceiling with a fixed separation of 60 cm between them and the dimming target is also fixed to the ceiling (at a height of 50 cm above the floor) while the two LEDs are co-planar. Further, the two LEDs and the two PDs are co-planar. In Fig. 10, we plot the symmetric rate on the vertical axis as a function of the displacement of the two users from the origin (origin is the point of intersection of the perpendicular bisector of the line joining the LEDs with the line joining the two users, see Fig. 1). In Fig. 10 a positive displacement implies that the user PD is located on the right side of the origin and vice versa (see Fig. 1).

It is observed that the maximum symmetric rate is almost zero if the displacement of both the users is same, i.e., the two users are almost co-located. In the figure this is represented by the dark black region. This is expected since in that case the channels to the users is also the same and hence the performance of the ZF precoder degrades. From the figure we observe that starting with both the users at the origin, as user 2 moves towards the right and User 1 moves towards left the maximum symmetric rate increases sharply (in the figure the colour changes from dark black to light black to gray to white as the displacement vector moves from $(0, 0)$ to $(-1.2, 1.2)$). This happens because as the users move away from each other, their channels become distinct i.e., the angles between the vectors $h_1$ and $h_2$ increases and hence the area of $R^\text{sym} / (H)$ increases. This result in an increase in the largest sized square that can be fit into $R^\text{sym} / (H)$ with center at $D(H, \xi)$. This then implies that the maximum symmetric rate would also increase (see the discussion in Section IV-A for the correspondence between the largest square and the maximum symmetric rate). With further increase in the separation between the two users the angular separation between the channel vectors does not increase as sharply as before. At the same time, due to increased path loss from the LEDs to the users, the area of $R^\text{sym} / (H)$ starts decreasing which results in the decrease in the maximum symmetric rate. This can be seen in the figure, as the colour changes back from white to gray, as we move from the displacement vector $(-1.2, 1.2)$ to $(-2.5, 2.5)$. This shows that the maximum symmetric rate is dependent on the location of the users and therefore there is an optimal location\(^{20}\) for both the users which results in the highest symmetric rate. In Fig. 10 the optimum location is $(-1.2, 1.2)$, or $(1.2, -1.2)$.

Next in Fig. 11, for a fixed LED separation of 60 cm we plot the percentage loss in the maximum symmetric rate $R^\text{sym} / (\xi)$ (w.r.t. the symmetric rate at the optimum location) with the users’ displacement from their optimum location for two different values of $\xi = 0.1, 0.3$.

It is observed that the percentage loss increases with increasing displacement of the PDs from their optimal location. Further, the increase in the percentage loss is small when the displacement is small as compared to when the displacement is large. For example, with $\xi = 0.3$, the percentage loss increases only by 6% as the displacement increases from 0 cm to 40 cm. However with a further increase in displacement from 40 cm to

\(^{19}\)Displacement is nothing but the distance of the user from the origin (see Fig. 1.

\(^{20}\)By the optimum user location we mean the displacement vector of the users at which we get maximum $R^\text{sym} / (\xi)$.
80 cm, the percentage loss increases sharply from 6% to 30%. A similar behavior is also observed with $\xi = 0.1$, though for a given displacement the loss is greater when $\xi = 0.1$ as compared to when $\xi = 0.3$. A practical application of this study could be in defining coverage zones\(^{21}\) for the PDs, i.e., the maximum allowable displacement for a fixed desired upper limit on the percentage loss. For example, in the current setup with $\xi = 0.3$, for a 20% upper limit on the percentage loss, the maximum allowable displacement is roughly 70 cm. It therefore appears that indoor VLC systems allow for a lot of flexibility in the movement of the user terminals without significant loss in the information rate.

VII. CONCLUSION

We have proposed an achievable rate region for the $2 \times 2$ MU-MISO broadcast VLC channel under per-LED peak power constraint and dimming control. The boundary of the proposed rate region has been analytically characterized. We propose a novel transceiver architecture to implement such systems. Interestingly, the design of encoder/codebook is independent of the dimming target, which reduces the complexity of the transceiver. Work done in this paper reveals that, in an indoor setting, the two users have enough mobility around their optimal placement without sacrificing their information rates. Our work can also be applied to a 2-D setting, where the users are allowed to move in a plane rather than being restricted to a line.

APPENDIX A
PROOF OF PROPOSITION 1

Proof: Under the condition in (28), to find $L_{1}^{\max}(\xi)$ we need to consider three scenarios that cover all geometrically possible parallelograms $R_{1}//(H)$: (a) $(h_{11} < h_{12}$ and $h_{21} > h_{22}$); (b) $(h_{11} < h_{12}$ and $h_{21} < h_{22}$); and (c) $(h_{11} < h_{12}$ and $h_{21} > h_{22}$).

For a given dimming target, $\xi$, let $L_{3}$ denote the length of the longest line segment parallel to the $u_{1}$-axis lying completely inside $R_{1}//(H)$ and whose midpoint coincides with the point $D(H, \xi)$ ($D(H, \xi)$ is defined in (11)). For any rectangle $\text{Rect}(L_{1}, L_{2}, D(H, \xi)) \subset R_{1}//(H)$, its side along the $u_{1}$ axis is a line segment inside $R_{1}//(H)$. From the definition of $L_{3}$, it follows that $L_{1} \leq L_{3}$ for any rectangle $\text{Rect}(L_{1}, L_{2}, D(H, \xi)) \subset R_{1}//(H)$. Additionally, the longest line segment of length $L_{3}$ corresponds to a rectangle $\text{Rect}(L_{3}, L_{2} = 0, D(H, \xi)) \subset R_{1}//(H)$.

\[^{21}\text{In Fig. 11 we have a 100% loss in information rate when the two users are located at the same location, i.e., when they have moved 1.2 m towards each other from their optimum position. This is because, when the two users are at the same location we have det$(H) = 0$ for which no communication is possible with the ZF precoder. In general, if we do not restrict ourselves to the ZF precoder, then for the det$(H) = 0$ scenario the $2 \times 2$ MU-MISO VLC broadcast channels channel reduces to a $2 \times 1$ MU broadcast VLC channel for which from [15] we know that non-zero rates can be achieved for both the users, and therefore the loss will not be 100%. It is possible to generalize the proposed precoder to consider the det$(H) = 0$ scenario, which might improve the coverage. We however do not consider this generalization in this paper.}\]
Since $L_i^{\text{max}}(\xi)$ is the length of the line segment parallel to the $u_1$-axis having its midpoint at point $P$ and lying completely inside $R_{i/1}(H)$, it follows that
\[
L_i^{\text{max}}(\xi) = 2 \min(PP_1, PP_2),
\] (51)
where both the line segments $PP_1$ and $PP_2$ are parallel to the $u_1$-axis. Further, $P_1$ lies on the line $OA$ whereas $P_2$ lies on the line $OC$ as shown in Fig. 12. Next, we evaluate $PP_1$ and $PP_2$. To this end, from Fig. 12, we compute the length of the line segment $PP_1$ as follows
\[
PP_1 = PP_3 - P_1P_3
\]
\[
\begin{align*}
& (a) \quad \xi(h_{11} + h_{12}) - OP_3/\tan \theta_1 \\
& (b) \quad \xi(h_{11} + h_{12}) - \xi(h_{21} + h_{22})h_{11}/h_{21} \\
& = \xi(h_{12}h_{21} - h_{11}h_{22})/h_{21} = -\xi \det(H)/h_{21},
\end{align*}
\] (52)
where step (a) follows from the fact that, $PP_3$ is equal to the coordinate of the point $D(H, \xi)$ along the $u_1$-axis and therefore from (11), we have $PP_3 = \xi(h_{11} + h_{12})$. In step (a) we have also used the fact that since $OP_3P_3$ is a right angle triangle having $\angle OP_3P_3 = \theta_1$. Hence, it follows that $P_1P_3 = OP_3/\tan \theta_1$. Step (b) also follows from two facts. Firstly, $OP_3$ is equal to the co-ordinate of the point $D(H, \xi)$ along the $u_2$-axis and therefore from (11), we have that $OP_3 = \xi(h_{21} + h_{22})$ and secondly, from (27), we know that $\tan \theta_1 = h_{21}/h_{11}$. Similarly from Fig. 12, we calculate the length of $PP_2$ as follows
\[
PP_2 = PP_3 - P_1P_3
\]
\[
\begin{align*}
& (a) \quad OP_3/\tan \theta_2 - \xi(h_{11} + h_{12}) \\
& (b) \quad \xi(h_{21} + h_{22})h_{12}/h_{22} - \xi(h_{11} + h_{12}) \\
& = \xi(h_{12}h_{21} - h_{11}h_{22})/h_{22} = -\xi \det(H)/h_{22},
\end{align*}
\] (53)
where step (a) follows from the fact that, $PP_3$ is equal to the co-ordinate of the point $D(H, \xi)$ along the $u_1$-axis and therefore from (11), we have that $PP_3 = \xi(h_{11} + h_{12})$. In step (a) we have also used the fact that $OP_3P_3$ is a right angle triangle having $\angle OP_3P_3 = \theta_2$. Hence, it follows that $P_1P_3 = OP_3/\tan \theta_2$. Step (b) follows from two facts. Firstly, $OP_3$ is equal to the co-ordinate of the point $D(H, \xi)$ along the $u_2$-axis and therefore from (11), we have that $OP_3 = \xi(h_{21} + h_{22})$ and secondly, $\tan \theta_2 = h_{22}/h_{12}$.

Using (52) and (53) in (51) we see that when $0 \leq \xi \leq h_{22}/(h_{21} + h_{22}) = \min(h_{21}, h_{22})/(h_{21} + h_{22})$ (since $h_{21} > h_{22}$ in scenario (a)), we have
\[
L_i^{\text{max}}(\xi) = -2\xi \det(H) \min \left( \frac{1}{h_{21}}, \frac{1}{h_{22}} \right) = -2\xi \det(H) \max(1/h_{21}, 1/h_{22}).
\] (54)

**Computation of $L_i^{\text{max}}(\xi)$ when $Q = D(H, \xi) \in \text{Region 2}$:**

Point $Q = D(H, \xi)$ lies in Region 2 if and only if
\[
OT \leq OQ \leq OM,
\] (55)
where $M$ is the point of intersection of the line segment $AA'$ and the diagonal $OB$ (see Fig. 12). In the following we firstly show that $D(H, \xi) \in \text{Region 2}$ if and only if
\[
\frac{h_{22}}{h_{21} + h_{22}} \leq \xi \leq \frac{h_{21}}{h_{21} + h_{22}}.
\] (56)
Towards this end, we firstly derive an expression for $OM$. From the right angle triangle $OM_1M$ in Fig. 12, we know that $OM = MM_1/\sin \gamma$ and since $MM_1 = h_{21}$, $\sin \gamma = (h_{21} + h_{22})/OB$. We have
\[
OM = \frac{h_{21}OB}{h_{21} + h_{22}}.
\] (57)
Since the point $Q$ is nothing but the point $D(H, \xi)$, from (11), we have $OQ = OB$ and from (49), it follows that $OT = h_{22}/(h_{21} + h_{22})$. In (55), we substitute $OQ$ by $\xi OB$, $OM$ by the R.H.S in (57) and $OT$ by $h_{22}/(h_{21} + h_{22})$ to get
\[
\frac{h_{22}OB}{h_{21} + h_{22}} \leq \xi \leq \frac{h_{21}OB}{h_{21} + h_{22}}.
\] (58)
For all such values of the dimming target, $\xi$, satisfying (56), it follows that $D(H, \xi) \in \text{Region 2}$. Next, we evaluate $L_i^{\text{max}}(\xi)$ when $h_{22}/(h_{21} + h_{22}) \leq \xi \leq h_{21}/(h_{21} + h_{22})$. For this scenario, from Fig. 12 we see that
\[
L_i^{\text{max}}(\xi) = 2 \min(QQ_1, QQ_2),
\] (59)
where construction of $QQ_1$ and $QQ_2$ is similar to the construction of $PP_1$ and $PP_2$ (see Fig. 12), except the fact that $QQ_2$ lies on $CB$ instead of $OC$. Next, we evaluate $QQ_1$ and $QQ_2$. To this end, using the similar steps as for the evaluation of $PP_1$ (see (52)) we have,
\[
QQ_1 = QQ_3 - Q_1Q_3 = -\xi \det(H)/h_{21}.
\] (60)
However, evaluation of $QQ_2$ is not the same as evaluation of $PP_2$, as the point $Q_2$ lies on the line segment $CB$ whereas the point $P_2$ lies on the line $OC$. Towards this end, using Fig. 12, we evaluate $QQ_2$ as follows
\[
QQ_2 = Q_3Q_2 - Q_3Q
\]
\[
= Q_3Q_4 + Q_4Q_2 - Q_3Q
\]
\[
= h_{12} + \frac{CQ_4}{\tan \theta_1} - \xi(h_{11} + h_{12})
\]
\[
= h_{12} + \frac{\xi(h_{21} + h_{22}) - h_{22} - \xi(h_{11} + h_{12})}{h_{21}/h_{11}}
\]
\[
= -\frac{\det(H)(1 - \xi)}{h_{21}}.
\] (61)
Using (60) and (61) in (59), we see that when $h_{22}/(h_{21} + h_{22}) \leq \xi \leq h_{21}/(h_{21} + h_{22})$, i.e. $\min(h_{21}, h_{22})/(h_{21} + h_{22}) \leq \xi \leq \max(h_{21}, h_{22})/(h_{21} + h_{22})$ (since $h_{21} > h_{22}$ in scenario (a)), we have
\[
L_i^{\text{max}}(\xi) = -2\frac{\det(H)}{h_{21}} \min \left( \xi, \frac{1 - \xi}{h_{21}} \right)
\] (62)
where step (a) follows from the fact that \( h_{21} = \max(h_{21}, h_{22}) \), since for scenario (a), we know that \( h_{21} > h_{22} \).

**Computation of \( L_1^{\max}(\xi) \) when \( S = D(H, \xi) \in \text{Region 3} \):**

Point \( S = D(H, \xi) \) lies in Region 3 = \( AA'B \) if and only if \( OM \leq OS \leq OB \). Using (57) \( \frac{OM}{OB} = \frac{h_{21}}{h_{21} + h_{22}} \) and the fact that \( OS = \xi OB \) (from (11)), we have

\[
h_{21}/(h_{21} + h_{22}) \leq \xi \leq 1.
\]

Next, we evaluate \( L_1^{\max}(\xi) \) when \( h_{21}/(h_{21} + h_{22}) \leq \xi \leq 1 \). From Fig. 12 it is clear that when \( S = D(H, \xi) \in \text{Region 3} \) then \( L_1^{\max}(\xi) \) is given by

\[
L_1^{\max}(\xi) = 2 \min(\xi, 1-\xi),
\]

where \( S_1 \) and \( S_2 \) are the intersections of the straight line parallel to the \( u_1 \) axis passing through \( S \), with the line segment \( AB \) and \( CB \) respectively. Next, we evaluate \( SS_1 \) and \( SS_2 \). To this end, using similar steps as for the evaluation of \( QQ_2 \) (see (61)) we have,

\[
SS_2 = S_3S_2 - S_3S_1 = S_3S_4 + S_1S_2 - S_3S_1 = -\frac{\det(H)(1-\xi)}{h_{21}}
\]

However, evaluation of \( SS_1 \) is not same as evaluation of \( QQ_1 \), as the point \( S_1 \) lies on the line segment \( AB \) whereas the point \( Q_1 \) lies on the line \( OA \). Towards the evaluation of \( SS_1 \), using Fig. 12, we have

\[
SS_1 = SS_3 - S_1S_3
\]

\[
= \xi(h_{11} + h_{12}) - (S_3A_1 + A_1S_1)
\]

\[
= \xi(h_{11} + h_{12}) - \left(h_{11} + \frac{AA_1}{\tan \theta_2}\right)
\]

\[
= \xi(h_{11} + h_{12}) - \left(h_{11} + \frac{\xi(h_{21} + h_{22}) - h_{21}}{(h_{21}/h_{22})}\right)
\]

\[
= -\frac{\det(H)(1-\xi)}{h_{22}}
\]

Using (66) and (67) in (65), we see that when \( h_{21}/(h_{21} + h_{22}) \leq \xi \leq 1 \), i.e. \( \max(h_{21}, h_{22})/(h_{21} + h_{22}) \leq \xi \leq 1 \) (since \( h_{21} > h_{22} \) in scenario (a)), we have

\[
L_1^{\max}(\xi) = -2 \det(H)(1-\xi) \min \left(\frac{1}{h_{21}}, \frac{1}{h_{22}}\right),
\]

where step (a) follows from the fact that \( h_{21} = \max(h_{21}, h_{22}) \), since for scenario (a), we know that \( h_{21} > h_{22} \).

Therefore for the scenario (a), we have the expression of \( L_1^{\max}(\xi) \) as follows

\[
L_1^{\max}(\xi) = \begin{cases} 
-2\xi \det(H) \max(h_{21}, h_{22}) & 0 \leq \xi \leq \eta_1 \lesssim \min(h_{21}, h_{22})/(h_{21} + h_{22}) \\
-2\det(H) \min(\xi, 1-\xi) \max(h_{21}, h_{22}) & \eta_1 \leq \xi \leq \eta_2 \lesssim \max(h_{21}, h_{22})/(h_{21} + h_{22}) \\
-2(1-\xi) \det(H) \max(h_{21}, h_{22}) & \eta_2 \leq \xi \leq 1.
\end{cases}
\]

Since

\[
\eta_1 \lesssim \min(h_{21}, h_{22})/(h_{21} + h_{22}) \leq 1/2
\]

and

\[
\eta_2 \lesssim \max(h_{21}, h_{22})/(h_{21} + h_{22}) \geq 1/2,
\]

where the above two inequalities follows from the simple mathematical manipulations, and therefore the expression in (69) can be further simplified. To this end, we consider two cases based on the values of \( \xi \).

**Case (a):** \( 0 \leq \xi \leq 1/2 \)

For this case we know that \( \xi \leq (1-\xi) \) and hence

\[
\min(\xi, 1-\xi) = \xi
\]

Since from (70) and (71), we know that \( \eta_1 \leq 1/2 \) and \( \eta_2 \geq 1/2 \) and hence, for \( 0 \leq \xi \leq 1/2 \) from (69) and (72) we have

\[
L_1^{\max}(\xi) = -2\xi \det(H) \max(h_{21}, h_{22})
\]

**Case (b):** \( 1/2 \leq \xi \leq 1 \)

For this case we know that \( \xi \geq (1-\xi) \) and hence

\[
\min(\xi, 1-\xi) = (1-\xi)
\]

Since from (70) and (71), we know that \( \eta_1 \leq 1/2 \) and \( \eta_2 \geq 1/2 \) and hence, for \( 1/2 \leq \xi \leq 1 \) from (69) and (74) we have

\[
L_1^{\max}(\xi) = -2(1-\xi) \det(H) \max(h_{21}, h_{22})
\]

Therefore from (73) and (75) the final expression of \( L_1^{\max}(\xi) \) for scenario (a) is as follows

\[
L_1^{\max}(\xi) = \begin{cases} 
-2\xi \det(H) \max(h_{21}, h_{22}) & 0 \leq \xi \leq 1/2 \\
-2(1-\xi) \det(H) \max(h_{21}, h_{22}) & 1/2 \leq \xi \leq 1
\end{cases}
\]

Using similar arguments as for scenario (a), we evaluate \( L_1^{\max}(\xi) \) for **Scenario (b):** \( (h_{21} \leq h_{22}) \); and for **Scenario (c):** \( (h_{12} \leq h_{11} \text{ and } h_{21} > h_{22}) \) as follows.

To this end, we first partition \( B_i/(H) \) into three regions as shown in Fig. 13. Next, we denote \( D(H, \xi) \) by the point \( P \) if \( D(H, \xi) \) lies in Region 1, by the point \( Q \) if \( D(H, \xi) \) lies in Region 2 and by the point \( S \) if \( D(H, \xi) \) lies in Region 3.
Using (57) and (49) we can also show that the point $D(H, \xi)$ lies in Region 1 if and only if $0 \leq \xi \leq \frac{\min(h_{21}, h_{22})}{h_{21} + h_{22}}$, $D(H, \xi)$ lies in Region 2 if $\frac{\min(h_{21}, h_{22})}{h_{21} + h_{22}} \leq \xi \leq \frac{\max(h_{21}, h_{22})}{h_{21} + h_{22}}$, and it lies in Region 3 if $\frac{\max(h_{21}, h_{22})}{h_{21} + h_{22}} \leq \xi \leq 1$. Next, we evaluate $L_{1}^{\text{max}}(\xi)$ when $D(H, \xi)$ lies in Region $i$, $i = 1, 2, 3$.

Following similar steps as for scenario (a), from Fig. 13 it follows that when $D(H, \xi) \in$ Region 1, i.e., when $0 \leq \xi \leq \frac{\min(h_{21}, h_{22})}{h_{21} + h_{22}}$,

$$L_{1}^{\text{max}}(\xi) = \frac{-2\xi \det(H)}{\max(h_{21}, h_{22})}.$$  

Similarly, using Fig. 13 it can be shown that when $D(H, \xi) \in$ Region 2, i.e., when $\frac{\min(h_{21}, h_{22})}{h_{21} + h_{22}} \leq \xi \leq \frac{\max(h_{21}, h_{22})}{h_{21} + h_{22}}$,

$$L_{1}^{\text{max}}(\xi) = -2 \det(H) \frac{\min(\xi, 1 - \xi)}{\max(h_{21}, h_{22})}.$$  

Further, using Fig. 13 it can be shown that when $D(H, \xi) \in$ Region 3, i.e., when $\frac{\max(h_{21}, h_{22})}{h_{21} + h_{22}} \leq \xi \leq 1$,

$$L_{1}^{\text{max}}(\xi) = \frac{-2(1 - \xi) \det(H)}{\max(h_{21}, h_{22})}.$$  

Following the steps used to arrive at (76) from (69) in scenario (a), for scenario (b) and (c) also we get the same final expression for $L_{1}^{\text{max}}(\xi)$ as in (76). This completes the proof. ■

### Appendix B

**Proof of Proposition 2**

To this end, for a fixed $\xi$ and given $L_{1} = x$, we construct all such possible rectangles $\text{Rect}(L_{1} = x, L_{2}, D(H, \xi)) \subseteq R_{i}/(H)$ and among them we choose the rectangle having the maximum possible vertical length.

To get this rectangle we first construct a horizontal line segment of the length $x$ parallel to the $u_{1}$-axis such that its midpoint coincides with the point $D(H, \xi)$. We denote this line segment by $\text{LINE}(x, D(H, \xi))$, i.e.

$$\text{LINE}(x, D(H, \xi)) \triangleq \{v = (v_{1}, v_{2}) \in \mathbb{R}^{2} | v_{1} \in S_{1}, v_{2} \in S_{2}\},$$

where $S_{1} \triangleq \{v_{1} \in \mathbb{R} | |v_{1} - \xi(h_{11} + h_{12})| \leq (x/2)\}$ and $S_{2} \triangleq \{v_{2} \in \mathbb{R} | v_{2} = \xi(h_{21} + h_{22})\}$. Since $x \leq L_{1}^{\text{max}}(\xi)$, from the definition of $L_{1}^{\text{max}}(\xi)$ it follows that $\text{LINE}(x, D(H, \xi))$ lies completely inside $R_{i}/(H)$.

Given any horizontal line segment of length $x$ parallel to the $u_{1}$-axis, any rectangle inside $R_{i}/(H)$ having this line segment as one of its side can be constructed in two possible ways, either by extending it vertically downwards or extending it vertically upwards.\(^{21}\) Subsequently, we shall refer to these construction methods as “downward extension” and “upward extension”.

Using this for a given $L_{2} > 0$ we can construct a rectangle by extending the line segment $\text{LINE}(x, D(H, \xi))$ vertically upwards. We denote this rectangle by

$$\text{Rect}^{\text{up}}(x, L_{2}, D(H, \xi)) \triangleq \text{Rect}(x, L_{2}, D(H, \xi)) + C_{0},$$

where $C_{0} \triangleq (0, L_{2}/2)$. Let $L_{1}^{\text{up}, \xi}(x)$ denote the largest possible vertical length of all such rectangles which lie completely inside $R_{i}/(H)$ and are constructed by the upward extension of the line segment $\text{LINE}(x, D(H, \xi))$, i.e.

$$L_{1}^{\text{up}, \xi}(x) \triangleq \max_{\text{Rect}^{\text{up}}(x, L_{2}, D(H, \xi)) \subseteq R_{i}/(H)} L_{2}.$$  

Similarly, we construct any rectangle of vertical length $L_{2} \geq 0$ by extending the line segment $\text{LINE}(x, D(H, \xi))$ vertically downwards. We denote this rectangle by

$$\text{Rect}^{\text{down}}(x, L_{2}, D(H, \xi)) \triangleq \text{Rect}(x, L_{2}, D(H, \xi)) + C_{1},$$

where $C_{1} \triangleq (0, -L_{2}/2)$. Let $L_{1}^{\text{down}, \xi}(x)$ denote the largest possible vertical length of all such rectangles which lie completely inside $R_{i}/(H)$ and are constructed using the “downward extension” method, i.e.

$$L_{1}^{\text{down}, \xi}(x) \triangleq \max_{\text{Rect}^{\text{down}}(x, L_{2}, D(H, \xi)) \subseteq R_{i}/(H)} L_{2}.$$  

$L_{2}^{\xi}(x)$ is the maximum possible vertical length of any rectangle lying completely inside $R_{i}/(H)$ with horizontal side length equal to $x$ and having its mid point at $D(H, \xi)$, which is the midpoint of the line segment $\text{LINE}(x, D(H, \xi))$. Equivalently,

\(^{21}\)Once we fix $\xi$, location of the point $D(H, \xi)$ gets fixed (see (11)).

\(^{22}\)By extending a horizontal line segment vertically downwards/upwards, we mean that we create a rectangle by drawing two vertical lines from the end points of this horizontal line segment in the downward/upward direction and then connecting the other two end points of these two vertical lines to form a rectangle.
such a maximal rectangle must be symmetric about the line segment $LINE(x, D(H, \xi))$. Since such a maximal rectangle $Rect(x, L^\xi_2(x), D(H, \xi))$ lies inside $R_{ij}(H)$, from (83) and (85) it follows that

$$Rect(x, L^\xi_2(x), D(H, \xi)) \subseteq S_3 \cup S_4,$$

where $S_3 \triangleq Rect^{up}(x, L^{up, \xi}_2(x), D(H, \xi))$ and $S_4 \triangleq Rect^{down}(x, L^{down, \xi}_2(x), D(H, \xi))$ and $S_3 \cap S_4 = LINE(x, D(H, \xi))$. Further since the maximal rectangle $Rect(x, L^\xi_2(x), D(H, \xi))$ has the maximum possible vertical length and is symmetric about $LINE(x, D(H, \xi))$, it follows that $L^\xi_2(x)/2 = L^{up, \xi}_2(x)$ if $L^{up, \xi}_2(x) \leq L^{down, \xi}_2$, and $L^\xi_2(x)/2 = L^{down, \xi}_2(x)$ if $L^{down, \xi}_2(x) \leq L^{up, \xi}_2(x)$, i.e.

$$L^\xi_2(x) = 2 \min(L^{down, \xi}_2(x), L^{up, \xi}_2(x)).$$

We next derive expressions for $L^{up, \xi}_2(x)$ and $L^{down, \xi}_2(x)$ for $0 \leq x \leq L^{max}(\xi)$ for a fixed $0 \leq \xi \leq 1$. We firstly consider scenario (a) ($h_{11} < h_{12}$ and $h_{21} > h_{22}$).

### A. Computation of $L^{up, \xi}_2(x)$ for Scenario (a)

Towards this end, we divide $R_{ij}(H)$ into two regions, Region $i, i = 1, 2$, namely Region 1 = $OAA_1$ and Region 2 = $A_1CB$ (see Fig. 14). Note that in Fig. 14, the straight line $AA_1A_2$ is parallel to the $u_2$-axis and $A_2$ is the point of intersection of this line segment with the side $OC$ of $R_{ij}(H)$. Next, we evaluate expressions for $L^{up, \xi}_2(x)$ depending upon the region where $D(H, \xi)$ lies. In Fig. 14, we denote $D(H, \xi)$ by the point $P$ if $D(H, \xi)$ lies in Region 1 and by the point $Q/Q'$ if $D(H, \xi)$ lies in Region 2.

![Fig. 14. Evaluation of $L^{up, \xi}_2(x)$ for Scenario (a) ($h_{11} < h_{12}$ and $h_{21} > h_{22}$).](image)

Computation of $L^{up, \xi}_2(x)$ when $D(H, \xi) = P \in Region 1$: The point $D(H, \xi) = P \in Region 1 = OAA_1$ if

$$0 \leq OP \leq OT,$$  \hspace{1cm} (87)

where $T$ is the point of intersection of the line segment $AA_2$ with the diagonal $OB$ (see Fig. 14). Next, we evaluate expression for $OT$. Towards this end, from the similarity of the triangles $OTA_2$ and $OBB_1$ it follows that $\frac{OT}{TH} = \frac{OA}{OB}$. Further, from Fig. 14, it follows that $OA_2 = h_{11}$ and $OB_1 = h_{11} + h_{12}$ and therefore we have,

$$OT = \frac{h_{11}}{h_{11} + h_{12}} \cdot OB.$$  \hspace{1cm} (88)

Since the point $P$ is nothing but the point $D(H, \xi)$, from (11) we have $OP = \xi OB$. Therefore, using (88) and $OP = \xi OB$ in (87) we have, $D(H, \xi) \in Region 1$ if $0 \leq \xi \leq \frac{h_{11}}{h_{11} + h_{12}}$.

When $0 \leq \xi \leq \frac{h_{11} - h_{12}}{h_{11} + h_{12}}$, from (83) it follows that for evaluating $L^{up, \xi}_2(x)$, we need to construct rectangles $Rect^{up}(x, L_2, D(H, \xi))$ using the “upward extension” of the line segment $LINE(x, D(H, \xi) = P) = EF$ as shown in Fig. 14. $L^{up, \xi}_2(x)$ is then the largest possible vertical length of all such rectangles which lie inside $R_{ij}(H)$. From Fig. 14, it is clear that during the upward extension of the line $EF$, with increasing vertical length $L_2$ of the constructed rectangle $Rect^{up}(x, L_2, D(H, \xi))$, the vertically upward line from $E$ will be the first to move out of $R_{ij}(H)$ when compared to the vertical line from $F$. Hence it follows that in Fig. 14, for $x = EF$ and $0 \leq \xi \leq \frac{h_{11} - h_{12}}{h_{11} + h_{12}}$, we have $L^{up, \xi}_2(x) = EH$. To evaluate $EH$, we firstly denote the $(u_1, u_2)$ coordinates of the point $E$ by $(u_1^E, u_2^E)$. From the definition of $LINE(x, D(H, \xi))$ and (11) it is clear that

$$u_1^E = \xi(h_{11} + h_{12}) - x/2, \quad u_2^E = \xi(h_{21} + h_{22}).$$  \hspace{1cm} (89)

When $0 \leq \xi \leq \frac{h_{11} - h_{12}}{h_{11} + h_{12}}$ and $0 \leq x \leq L^{max}(\xi)$, using Fig. 14, $EH$ is computed as follows

$$L^{up, \xi}_2(x) = EH = E_1H - E_2E = u_1^E \tan \theta_1 - u_2^E$$

$$= \left(\xi(h_{11} + h_{12}) - \frac{x}{2}\right) \frac{h_{21}}{h_{11}} - \xi(h_{21} + h_{22})$$

$$= -\xi(\det(H) - \frac{x}{2}) \frac{h_{21}}{h_{11}}$$  \hspace{1cm} (90)

where $E_1$ is the point of intersection of the extension of the line segment $EH$ and the $u_1$-axis (see Fig. 14). Step (a) follows from (89) and (27).

Computation of $L^{up, \xi}_2(x)$ when $D(H, \xi) = Q \in Region 2$: Point $D(H, \xi) = Q$ lies in Region 2 = $A_1CB$ if and only if $OT \leq OQ \leq OB$. Since $Q$ denote the point $D(H, \xi)$, from (11) we have, $OQ = \xi OB$ and from (88) we have $OT = \frac{h_{11} + h_{12}}{h_{11} + h_{12}}$. Hence, it follows that $D(H, \xi)$ lies in Region 2 if $\frac{h_{11} + h_{12}}{h_{11} + h_{12}} \leq \xi \leq 1$. We next evaluate $L^{up, \xi}_2(x)$ when $\xi$ lies in this interval.

From (83), it follows that for evaluating $L^{up, \xi}_2(x)$, we need to construct rectangles $Rect^{up}(x, L_2, Q = D(H, \xi))$ using the “upward extension” of the line segment $LINE(x, Q =
$D(\mathbf{H}, \xi)$ as shown in Fig. 14. $L_{up,\xi}^2(x)$ is then the largest possible vertical length of all such rectangles which lie inside $R_{ij}(\mathbf{H})$. From Fig. 14, it is clear that during the upward extension of the line segment $LINE(x, Q^i = D(\mathbf{H}, \xi))$, the upper left vertex of the constructed rectangle having the largest vertical length will either intersect with the side $OA$ of $R_{ij}(\mathbf{H})$ or with the side $AB$ of $R_{ij}(\mathbf{H})$ (see rectangles $E'F'G'H'$ and $E''F''G''H''$ in Fig. 14). The upper left vertex intersects with the side $OA$ if and only if the lower left vertex of the constructed rectangle, i.e., the leftmost point of the line segment $LINE(x, D(\mathbf{H}, \xi))$ (see $E'$ in Fig. 14) lies inside Region 1, i.e.,

$$u_1^D(\mathbf{H}, \xi) - x/2 \leq h_{11}, \quad \text{i.e.}$$

$$2\xi h_{12} - 2(1 - \xi)h_{11} \leq x, \quad (91)$$

where $u_1$ coordinate of the point $D(\mathbf{H}, \xi)$ is denoted by $u_1^D(\mathbf{H}, \xi)$. From (11), we know that $u_1^D(\mathbf{H}, \xi) = \xi(h_{11} + h_{12})$. On the other hand, the upper left vertex intersects with the side $AB$ of $R_{ij}(\mathbf{H})$ if and only if the lower left vertex of the constructed rectangle, i.e., the leftmost point of the line segment $LINE(x, Q^i = D(\mathbf{H}, \xi))$ (see $E''$ in Fig. 14) lies inside Region 2, i.e.,

$$u_2^D(\mathbf{H}, \xi) - x/2 \geq h_{11}, \quad \text{i.e.}$$

$$x \leq 2\xi h_{12} - 2(1 - \xi)h_{11}. \quad (92)$$

From the above, we know that when $x$ satisfies (91), i.e., $2\xi h_{12} - 2(1 - \xi)h_{11} \leq x$, the lower left vertex of the constructed rectangle lies in Region 1 and the upper left vertex lies on the side $OA$. Hence, we have

$$L_{up,\xi}^2(x) = E'H' = E''H'' - E'E''$$

$$= -\xi \det(\mathbf{H}) - \frac{x}{h_{12}} h_{21} \quad (93)$$

Similarly, when $x$ satisfies (92), i.e., $2\xi h_{12} - 2(1 - \xi)h_{11} \geq x$, the lower left vertex of the constructed rectangle lies in Region 2 and the upper left vertex lies on the side $AB$. Hence, we have

$$L_{up,\xi}^2(x) = E''H''$$

$$= E''_1H'' - E''_2H''$$

$$= E''_1E_2 + E_2H'' - E''_1H''$$

$$= h_{21} + AE_2 \tan \theta_2 - E''_1E''$$

$$= h_{21} + (u_1^{E''} - h_{11}) \tan \theta_2 - u_2^{E''} \quad (a)$$

$$= h_{21} + \left(\xi(h_{11} + h_{12}) - \frac{x}{2} - h_{11}\right) \frac{h_{22}}{h_{12}} - \xi(h_{21} + h_{22})$$

$$= \frac{-1 - \xi \det(\mathbf{H}) - \frac{x}{2} h_{22}}{h_{12}} \quad (b)$$

where $E_2$ is the point of intersection of the line $E''H''$ with $AA'$ and $(u_1^{E''}, u_2^{E''})$ are the $(u_1, u_2)$ coordinates of the point $E''$. Step (a) follows from right angle triangle $AE_2H''$. Step (b) follows from the fact that, $AE_2 = u_1^{E''} - h_{11}$ and $E_1E'' = u_2^{E''}$. In step (c) the expression for $u_1^{E''}$ and $u_2^{E''}$ follows from the definition of $LINE(x, D(\mathbf{H}, \xi))$ in (81) and (11) and the value of $\tan \theta_2$ follows from (27). Therefore, when $D(\mathbf{H}, \xi) \in$ Region 2, (i.e., $\pi_{h_{11} + h_{12}} < \xi < 1$) and $(0 \leq x \leq 2\xi h_{12} - 2(1 - \xi)h_{11}$), we have

$$L_{up,\xi}^2(x) = \frac{-(1 - \xi) \det(\mathbf{H}) - \frac{x}{2} h_{22}}{h_{12}} \quad (94)$$

Therefore, in scenario (a) ($h_{11} < h_{12}$ and $h_{21} > h_{22}$), from (90), (93) and (94) we finally have

$$L_{up,\xi}^2(x) =$$

$$\begin{cases}
-\xi \det(\mathbf{H}) - \frac{x}{2} h_{21} & 0 \leq \xi \leq \mu_1 \text{ and } 0 \leq x \leq L_{1,\text{max}}^\text{up}(\xi) \\
-\xi \det(\mathbf{H}) - \frac{x}{2} h_{22} & \mu_1 \leq \xi \leq \frac{h_{11}}{h_{11} + h_{12}} \text{ and } 0 \leq x \leq \eta_3(\xi) \\
-\xi \det(\mathbf{H}) - \frac{x}{2} h_{21} & \mu_1 \leq \xi \leq 1 \text{ and } \eta_3(\xi) \leq x \leq L_{1,\text{max}}^\text{up}(\xi)
\end{cases} \quad (95)$$

B. Computation of $L_{down,\xi}^2(x)$ for Scenario (a)

Evaluation of $L_{down,\xi}^2(x)$ is similar to that of $L_{up,\xi}^2(x)$. In Fig. 15, we partition the parallelogram $R_{ij}(\mathbf{H})$ into two regions, namely, Region 1 $= CC_{1}B$ and Region 2 $= OAC_{1}C$. Next, we evaluate the expression for $L_{down,\xi}^2(x)$ depending upon the region where $D(\mathbf{H}, \xi)$ lies. In Fig. 15, we denote $D(\mathbf{H}, \xi)$ by the point $P$ if $D(\mathbf{H}, \xi)$ lies in Region 1, by point $Q$ or $Q'$ if $D(\mathbf{H}, \xi)$ lies in Region 2.

Computation of $L_{down,\xi}^2(x)$ when $D(\mathbf{H}, \xi) \in$ Region 1: The point $D(\mathbf{H}, \xi) = P \in$ Region 1 $= CC_{1}B$ iff

$$OT \leq OP \leq OB,$$
where \( T \) is the point of intersection of the straight line \( CC_1 \) with the diagonal \( OB \) (see Fig. 15). Note that the straight line \( CC_1 \) is parallel to the \( u_2 \)-axis (see Fig. 15). Next, we derive an expression for \( OT \). Towards this end, from the similarity of the triangles \( OTC_3 \) and \( OBB_1 \) it follows that

\[
OT = \frac{h_{12}}{h_{11}+h_{12}} OB. \tag{97}
\]

Since \( P = D(H, \xi) \), from (11) we have \( OP = \xi OB \). Therefore, using (97) and \( OP = \xi OB \) in (96) we have, \( D(H, \xi) \in \) Region 1 if

\[
\frac{h_{12}}{h_{11}+h_{12}} \leq \xi \leq 1.
\]

When \( \frac{h_{11}+h_{12}}{h_{12}} \leq \xi \leq 1 \), from (85), it follows that for evaluating \( L_{2, \text{down}}(\xi) \), we need to construct rectangles \( \text{Rect}_{\text{down}}(x, L_2, \ldots, D(H, \xi)) \) using the “downward extension” of the line segment \( \text{LINE}(x, D(H, \xi) = P) = EF \) as shown in Fig. 15. \( L_{2, \text{down}}(\xi) \) is then the largest possible vertical length of all such rectangles which lie inside \( R_{j//H} \). From Fig. 15, it is clear that during the downward extension of the line \( EF \), with increasing vertical length \( L_2 \) of the constructed rectangle \( \text{Rect}_{\text{down}}(x, L_2, D(H, \xi)) \), the vertically downward line from \( F \) will be the first to move out of \( R_{j//H} \) when compared to the vertical line from \( E \). Hence, it follows that in Fig. 15, for \( x = EF \) and \( \frac{h_{12}}{h_{11}+h_{12}} \leq \xi \leq 1 \), we have \( L_{2, \text{down}}(\xi) = FJ \). Let us denote the \((u_1, u_2)\) coordinates of the point \( F \) by \((u^F_1, u^F_2)\). From the definition of \( \text{LINE}(x, P = D(H, \xi)) \) in (81) and from (11) we have

\[
\begin{align*}
\frac{h_{11}}{h_{11}+h_{12}} \leq \xi \leq 1 \quad \text{and} \quad 0 \leq x \leq L_1^\text{max}(\xi) \text{, using Fig. 15, } FJ \text{ is computed as follows:}
\end{align*}
\]

\[
L_{2, \text{down}}(\xi) = FJ
\]

\[
= F_1 F - F_1 J
\]

\[
= F_1 F - F_1 C_2 - C_2 J
\]

\[
= F_1 F - h_{22} - C_2 \tan \theta_1
\]

\[
= u^F_1 - h_{22} - (u^F_1 - h_{12}) \tan \theta_1
\]

\[
= \left( h_{21} + h_{12} \right) \frac{h_{21}}{h_{11}} - \left( h_{11} + h_{12} \right) \frac{x}{2} - \frac{h_{12}}{h_{11}}
\]

\[
= -(1-\xi) \text{det}(H) - \frac{x}{2} h_{21} - \frac{h_{12}}{h_{11}}, \tag{99}
\]

where \( F_1 \) is the point of intersection of the extension of the line \( FJ \) with the \( u_1 \)-axis (see Fig. 15). Step (a) follows from the right angle triangle \( CC_3J \). Step (b) follows from the fact that, \( CC_2 = u^F_1 - h_{12} \) and \( F_1 F = u^F_2 \). In step (c) we use the expressions for \( u^F_1 \) and \( u^F_2 \) from (98) and the value of \( \tan \theta_1 \) from (27).

**Computation of \( L_{2, \text{down}}(\xi) \) when \( Q = D(H, \xi) \) in Region 2:** We know from Fig. 15, that \( Q = D(H, \xi) \in \) Region 2 = \( OAC_1C \) if

\[
0 \leq OQ \leq OT. \tag{100}
\]

Since \( Q \) is nothing but \( D(H, \xi) \), from (11) we have \( OQ = \xi OB \). Further, from (97) we have \( OT = \frac{h_{12}}{h_{11}+h_{12}} OB \). Therefore, it follows that \( D(H, \xi) \in \) Region 2 if \( \xi \leq h_{12}/(h_{11} + h_{12}) \). Next, we evaluate \( L_{2, \text{down}}(\xi) \) when \( \xi \) lies in this interval.

From (85), it follows that for evaluating \( L_{2, \text{down}}(\xi) \), we need to construct rectangles \( \text{Rect}_{\text{down}}(x, L_2, D(H, \xi)) \) using the “downward extension” of the line segment \( \text{LINE}(x, D(H, \xi)) \) as shown in Fig. 15. \( L_{2, \text{down}}(\xi) \) is then the largest possible vertical length of all such rectangles which lie inside \( R_{j//H} \). From Fig. 15, it is clear that during the downward extension of the line segment \( \text{LINE}(x, D(H, \xi)) \), the lower right vertex of the constructed rectangle having the largest vertical length will either intersect with the side \( OC \) of \( R_{j//H} \) or with the side \( CB \) of \( R_{j//H} \) (see rectangles \( E'F'J'K' \) and \( E''F''J''K'' \) in Fig. 15). The lower right vertex intersects with the side \( CB \) if and only if the upper right vertex of the constructed rectangle, i.e., the rightmost point of the line segment \( \text{LINE}(x, D(H, \xi)) \) (see \( F'' \) in Fig. 15) lies inside Region 1, i.e.

\[
\begin{align*}
u_1^D(H, \xi) + x/2 & \geq h_{12}, \text{ i.e.} \\
u_1^D(H, \xi) + x/2 & \leq 2(1-\xi)h_{12} - 2\xi h_{11}, \tag{101}
\end{align*}
\]

where \( u_1^D(H, \xi) \) denote the \( u_1 \)-coordinate of the point \( D(H, \xi) \). From (11) we know that \( u_1^D(H, \xi) = (h_{11} + h_{12}) \). On the other hand, the lower right vertex of the constructed rectangle intersects with the side \( OC \) of \( R_{j//H} \) if and only if the upper right vertex of the constructed rectangle, i.e., the rightmost point of the line segment \( \text{LINE}(x, D(H, \xi)) \) (see \( F'' \) in Fig. 15) lies inside Region 2, i.e.

\[
\begin{align*}
u_1^D(H, \xi) + x/2 & \leq h_{12}, \text{ i.e.} \\
u_1^D(H, \xi) + x/2 & \leq 2(1-\xi)h_{12} - 2\xi h_{11}. \tag{102}
\end{align*}
\]

From the above, we know that when \( x \leq 2(1-\xi)h_{12} - 2\xi h_{11} \), the upper right vertex of the constructed rectangle lies in Region 2 and the lower right vertex lies on the side \( OC \). Hence, we have \( L_{2, \text{down}}(\xi) \) \( = F'' \). Towards this end, we firstly denote the \((u_1, u_2)\) coordinates of the point \( F'' \) by \((u^{F''}_1, u^{F''}_2)\). From the definition of \( \text{LINE}(x, D(H, \xi)) \) in (81) and from (11) we have

\[
\begin{align*}
u_1^{F''} = \xi(h_{11} + h_{12}) + \frac{x}{2}, \quad u_2^{F''} = \xi(h_{21} + h_{22}). \tag{103}
\end{align*}
\]

Next, for \( 0 \leq \xi \leq h_{12}/(h_{11} + h_{12}) \) and \( 0 \leq x \leq 2(1-\xi)h_{12} - 2\xi h_{11} \), we evaluate expression for \( L_{2, \text{down}}(\xi) \) \( = F'' \) as follows.

\[
L_{2, \text{down}}(\xi) = F'' \tag{104}
\]
where $F''_u$ is the point of intersection of the line $F''_rJ''_r$ extended downward with the $u_1$-axis (see Fig. 15). Step (a) follows from the two facts. Firstly, from the fact that $F''_rF''_u$ is the $u_2$ coordinate $F''_r$, and secondly from the right angle triangle $OF''_rJ''_r$, we have $\tan \theta_2 = \frac{F''_rJ''_r}{u_1'}$, i.e. $F''_rJ''_r = u_1' \tan \theta_2$. Step (b) follows from (103) and (27).

On the other hand, when $x \geq 2(1-\xi)h_{12} - 2\xi h_{11}$, the upper right vertex of the constructed rectangle lies in Region 1 and the lower right vertex lies on the side $CB$. Hence, we have $L_{2\downarrow,\xi}^{down}(x) = F''_rJ''_r$. Towards this end, we firstly denote the $(u_1, u_2)$ coordinates of the point $F''_r$ by $(u_1', u_2')$. From the definition of $\text{LINE}(x, D(H, \xi))$ in (81) and from (11) we have

$$u_1' = \xi(h_{11} + h_{12}) + \frac{x}{2}, \quad u_2' = \xi(h_{21} + h_{22}).$$  

(105)

The steps involved in the evaluation of $F''_rJ''_r$ is exactly the same as for the evaluation of $F_J$ in (99). Hence, from Fig. 15 we have

$$L_{2\downarrow,\xi}^{down}(x) = F''_rJ''_r = (F''_rF''_u - F''_rJ''_r) = \frac{1}{2}(1-\xi)\det(H) - \frac{x}{2}h_{21},$$

(106)

Therefore, in scenario (a) ($h_{11} < h_{12}$ and $h_{21} > h_{22}$), from (99), (104) and (106) we finally have

$$L_{2\downarrow,\xi}^{down}(x) = \begin{cases} \frac{-1}{2}(1-\xi)\det(H) - \frac{x}{2}h_{21}, & \mu_2 \leq \xi \leq 1 \quad \text{and} \quad 0 \leq x \leq L_1^{\max}(\xi) \\ -\xi\det(H) - \frac{x}{2}h_{22}, & 0 \leq \xi \leq \mu_2 \quad \text{and} \quad 0 \leq x \leq \eta_1(\xi) \\ -\xi\det(H) - \frac{x}{2}h_{21}, & 0 \leq \xi \leq \mu_2 \quad \text{and} \quad \eta_1(\xi) \leq x \leq L_1^{\max}(\xi) \end{cases},$$

(107)

where $\mu_2 = \frac{h_{12}}{h_{11} + h_{12}}$ and $\eta_1(\xi) = 2(1-\xi)h_{12} - 2\xi h_{11}$. In the following, we derive expressions for $L_{2\uparrow,\xi}^{up}(x)$ and $L_{2\downarrow,\xi}^{down}(x)$ for scenario (b) ($h_{21} \leq h_{22}$); and (c) ($h_{12} \leq h_{11}$ and $h_{21} > h_{22}$).

Similarly, for scenarios (b) and (c) also, by using the upward and downward extension methods we derive the expressions for $L_{2\uparrow,\xi}^{up}(x)$ and $L_{2\downarrow,\xi}^{down}(x)$ respectively by constructing rectangles which lie inside $R_J(H)$ and have the maximum possible vertical lengths for a given horizontal length. It turns out that the expression for $L_{2\uparrow,\xi}^{up}(x)$ is exactly the same as that for scenario (a).

**APPENDIX C**

**PROOF OF LEMMA 3**

**Proof:** To prove (37) we consider its R.H.S. $L_2^{1-\xi}(x)$. From (32) the R.H.S. is given by

$$L_2^{(1-\xi)}(x) = \min(L_2^{up,1-\xi}(x), L_2^{down,1-\xi}(x)).$$  

(108)

where the expression for $L_2^{up,1-\xi}(x)$ is given by,

**Case I:** For $0 \leq (1-\xi) \leq \frac{h_{11}}{h_{11} + h_{12}}$, i.e. $\frac{h_{12}}{h_{11} + h_{12}} \leq \xi \leq 1$ from (33) we have

$$L_2^{up,1-\xi}(x) = \begin{cases} \left(-1-\xi\right)\det(H) - \frac{x}{2}h_{21}, & 0 \leq x \leq L_1^{\max}(1-\xi) \\ \eta_1(\xi) \leq x \leq L_1^{\max}(\xi) \\ \eta_1(\xi) \leq x \leq L_1^{\max}(\xi) \end{cases}.$$  

(109)

where step (a) follows from (30) and step (b) follows from (36). Case II: For $\frac{h_{11}}{h_{11} + h_{12}} \leq (1-\xi) \leq 1$, i.e. $0 \leq \xi \leq \frac{h_{12}}{h_{11} + h_{12}}$, from (34) we have

$$L_2^{up,1-\xi}(x) = \begin{cases} \left(-1-\xi\right)\det(H) - \frac{x}{2}h_{21}, & 0 \leq x \leq \eta_1(1-\xi) \\ -\xi\det(H) - \frac{x}{2}h_{22}, & 0 \leq x \eta_1(\xi) \leq \frac{h_{12}}{h_{11} + h_{12}} \leq \xi \leq L_1^{\max}(1-\xi) \end{cases}.$$  

(110)

From (111) we also have

$$L_2^{up,\xi}(x) = \max(L_2^{up,1-\xi}(x), L_2^{down,1-\xi}(x)) = 2\min(L_2^{up,1-\xi}(x), L_2^{down,1-\xi}(x)) = 2\min(L_2^{up,\xi}(x), L_2^{down,\xi}(x)) = L_2^{\xi}(x) = L.H.S.,$$  

(113)

where step (a) follows from (32). This therefore completes the proof.

**APPENDIX D**

**PROOF OF THEOREM 1**

To prove this theorem we need the following Lemma.

**Lemma 4:** For a fixed $x \in [0, L_1^{\max}(1/2)]$, the function $L_2^{\xi}(x)$ attains its maximum at $\xi = 1/2$, i.e.

$$L_2^{\xi}(x) \leq L_2^{1/2}(x) \forall \xi \in [f(x), 1/2]$$  

(114)
where for any $0 \leq x \leq L_1^{\max}(1/2)$, $f(x)$ is the unique value such that

$$L_1^{\max}(f(x)) = x \quad \text{and} \quad f(x) \leq 1/2. \quad (115)$$

**Proof:** To prove Lemma 4 we consider two cases (a) $h_{12} \leq h_{11}$; and (b) $h_{12} \geq h_{11}$. From Lemma 3, we know that for a fixed $x \in [0, L_1^{\max}(1/2)]$, $L_2^x(x)$ is symmetric about $\xi = 1/2$, hence we consider $\xi$ only in the range $[0,1/2]$. The proof of Lemma 4 is as follows

**Case(a) $h_{12} \leq h_{11}$:**

$$h_{12} \leq h_{11}, \quad \text{i.e.} \quad 1/2 \leq \frac{h_{11}}{h_{11} + h_{12}}. \quad (116)$$

Since $\frac{h_{11}}{h_{11} + h_{12}} \geq 1/2$ and $\xi \in [0,1/2]$, we have $\xi \leq \frac{h_{11}}{h_{11} + h_{12}}$. Therefore from (33) we have

$$L_{2}^{up,\xi}(x) = -\xi \text{det}(H) - \frac{\xi}{h_{11}}, \quad 0 \leq x \leq L_1^{\max}(1/2). \quad (117)$$

From the above equation it is clear that $L_{2}^{up,\xi}(x)$ is an increasing function of $\xi \in [0,1/2]$ and therefore for any $\xi \in [0,1/2]$ we have

$$\xi \leq 1/2 \Rightarrow L_{2}^{up,\xi}(x) \leq L_{2}^{up,1/2}(x). \quad (118)$$

From (112) we know that $L_{2}^{up,\xi}(x) = L_{2}^{down,(1-\xi)}(x)$, and therefore for $\xi = 1/2$

$$L_{2}^{up,1/2}(x) = L_{2}^{down,1/2}(x). \quad (119)$$

Using (119) in (32) we have

$$L_{2}^{up,1/2}(x) = L_{2}^{down,1/2}(x) = L_2^{\xi=1/2}(x)/2 \quad \text{i.e.} \quad \min(L_{2}^{up,\xi}(x), L_{2}^{down,\xi}(x)) \leq L_2^{\xi=1/2}(x) \quad \text{i.e.} \quad L_2^\xi(x) \leq L_2^{\xi=1/2}(x). \quad (121)$$

Therefore for case(a) ($h_{12} \leq h_{11}$) and $\xi \in [0,1/2]$ finally we have

$$L_2^\xi(x) \leq L_2^{\xi=1/2}(x), \quad 0 \leq x \leq L_1^{\max}(1/2). \quad (122)$$

**Case(b) $h_{11} \leq h_{12}$:**

$$h_{11} \leq h_{12}, \quad \text{i.e.} \quad 1/2 \leq \frac{h_{12}}{h_{11} + h_{12}}. \quad (123)$$

For this case, in order to prove $L_2^\xi(x) \leq L_2^{\xi=1/2}(x)$, we further consider two different cases on the basis of the values of $x \in [0, L_1^{\max}(1/2)]$ (a) $x \in [0, h_{12} - h_{11}]$; and (b) $x \in [h_{12} - h_{11}, L_1^{\max}(1/2)]$

(a) $x \in [0, h_{12} - h_{11}]$

From (35) we have

$$\eta_1(\xi) = 2(1 - \xi)h_{12} - 2\xi h_{11}$$

$$= 2h_{12} - 2\xi(h_{11} + h_{12}). \quad (124)$$

From the above equation it is clear that $\eta_1(\xi)$ is monotonically decreasing with $0 \leq \xi \leq 1/2$ and therefore we have

$$\eta_1(1/2) \leq \eta_1(\xi) \leq \eta_1(0)$$

$$h_{12} - h_{11} \leq \eta_1(\xi) \leq 2h_{12} \quad (125)$$

and since we know that $x \in [0, h_{12} - h_{11}]$, hence for any value of $\xi \in [0,1/2]$, $x$ will always be less than $\eta_1(\xi)$, i.e. $x \leq \eta_1(\xi)$. Therefore from (35) we have

$$L_2^{down,\xi}(x) = -\frac{\xi \text{det}(H) - \frac{\xi}{h_{11}}}{h_{12}}. \quad (126)$$

From the above equation it is clear that for a fixed $x \in [0, h_{12} - h_{11}]$, $L_2^{down,\xi}(x)$ is a monotonically increasing function of $\xi \in [0,1/2]$. Therefore for any $\xi \in [0,1/2]$ we have

$$L_2^{down,\xi}(x) \leq L_2^{down,1/2}(x) = L_2^{\xi=1/2}(x)/2, \quad \text{i.e.} \quad \min(L_{2}^{up,\xi}(x), L_{2}^{down,\xi}(x)) \leq L_2^{\xi=1/2}(x)/2, \quad \text{i.e.} \quad 2 \min(L_{2}^{up,\xi}(x), L_{2}^{down,\xi}(x)) \leq L_2^{\xi=1/2}(x) \quad \text{i.e.} \quad L_2^\xi(x) \leq L_2^{\xi=1/2}(x). \quad (127)$$

Therefore for case(b.I) finally we have

$$L_2^\xi(x) \leq L_2^{\xi=1/2}(x), \quad x \in [0, h_{12} - h_{11}]. \quad (128)$$

case (b.II) $x \in [h_{12} - h_{11}, L_1^{\max}(1/2)]$

From (34) we have

$$\eta_3(\xi) = 2\xi h_{12} - 2(1 - \xi)h_{11}$$

$$= 2\xi(h_{12} + h_{11}) - 2h_{11} \quad (129)$$

It is clear from the above equation that $\eta_3(\xi)$ is monotonically increasing with $\xi$ and hence for $\xi \in [0,1/2]$ we have

$$\eta_3(0) \leq \eta_3(\xi) \leq \eta_3(1/2), \quad \text{i.e.} \quad -2h_{11} \leq \eta_3(\xi) \leq h_{12} - h_{11}.$$ 

Since $x \in [h_{12} - h_{11}, L_1^{\max}(1/2)]$ we have

$$\eta_3(\xi) \leq h_{12} - h_{11} \leq x, \quad \text{i.e.} \quad \eta_3(\xi) \leq x. \quad (130)$$

Therefore for case(b.II), from (33) and (34) we have

$$L_2^{up,\xi}(x) = -\frac{\xi \text{det}(H) - \frac{\xi}{h_{11}}}{h_{11}}, \quad h_{12} - h_{11} \leq x \leq L_1^{\max}(1/2) \quad (131)$$

It is clear that $L_2^{up,\xi}(x)$ is monotonically increasing with $\xi$ and hence using the similar argument as for (121) we can show that

$$L_2^\xi(x) \leq L_2^{\xi=1/2}(x). \quad (132)$$

---

25Uniqueness follows from the fact that $L_1^{\max}(\xi)$ is continuous, increases linearly when $\xi \in [0,1/2]$, has a unique maximum at $\xi = 1/2$, and $L_1^{\max}(\xi) = L_1^{\max}(1 - \xi)$. 
Therefore using (128) and (132)\textsuperscript{26} for case (b) \((h_{12} \geq h_{11})\) we have
\[
L_2^0(x) \leq L_2^{\xi-1/2}(x), \quad 0 \leq x \leq L_1^{\text{max}}(1/2).
\]
(133)

Therefore finally from (133) and (121) and Lemma 3 we have for any \(\xi \in [0, 1]\)
\[
L_2^0(x) \leq L_2^{\xi-1/2}(x).
\]
This completes the proof of Lemma 4.

\textit{Proof:} Next using this Lemma we prove Theorem 1.

Towards this end, we consider an arbitrary \(\xi \in [0, 1/2]\), for which we show that \(R_{ZF}(H, P_0/\sigma, \xi) \subseteq R_{ZF}(H, P_0/\sigma, 1/2)\). For \(\xi \in [1/2, 1]\), the proof is similar due to the symmetricity of the \(L_2^0(x)\) and \(L_1^{\text{max}}(\xi)\) functions (see Remark 2 and Lemma 3).

For a given \(\xi \in [0, 1/2]\) let \((R_1, R_2) \in R_{ZF}(H, P_0/\sigma, \xi)\). From (22), we know that there exists a \(\text{Rect}(L_1 \geq 0, L_0 \geq 0, D(H, \xi)) \subset R_{ZF}(H)\) which corresponds to this rate pair \((R_1, R_2)\). Further from proposition 2, it follows that there exists
\[
L_2^0(L_1) \geq L_2.\]
(134)

From Lemma (4) we know that for any a given \(\xi \in [0, 1/2]\) and \(0 \leq L_1 \leq L_1^{\text{max}}(\xi)\), there exists
\[
L_2^{\xi-1/2}(L_1) \geq L_2^0(L_1).
\]
(135)

From (135) and (134) we get
\[
L_2^{\xi-1/2}(L_1) \geq L_2.\]
(136)

We know that for \(\xi = 1/2\) there exists a rectangle \(\text{Rect}(L_1, L_2^{\xi-1/2}(L_1), D(H, 1/2)) \subset R_{ZF}(H)\). From (136) it follows that there exists a rectangle \(\text{Rect}(L_1, L_2, D(H, 1/2)) \subset R_{ZF}(H)\) and hence \(\text{Rect}(L_1, L_2, D(H, 1/2)) \subset R_{ZF}(H)\). The rate pair corresponding to the rectangle \(\text{Rect}(L_1, L_2, D(H, 1/2))\) is \((R_1, R_2)\) and therefore \((R_1, R_2) \in R_{ZF}(H, P_0/\sigma, 1/2)\). 

\text{APPENDIX E}

\text{PROOF OF THEOREM 2}

\textit{Proof:} The proof of Theorem 2 is as follows. Let \((a, a)\) be any arbitrary rate pair of the form \((r, r)\) lying strictly inside the rate region \(R_{ZF}(H, P_0/\sigma, \xi)\) and which does not lie on the boundary \(R_{ZF}^{\partial}(H, P_0/\sigma, \xi)\) (see the point \(P\) in Fig. 16).

We then show that there exists the unique rate pair \((a^*, a^*)\) which lies on the boundary \(R_{ZF}^{\partial}(H, P_0/\sigma, \xi)\) such that \(a^* > a\). This shows that the rate pair \((\alpha R_{\text{max}}(\xi), \alpha^* R_{\text{max}}(\xi)) = (a^*, a^*)\) is the unique rate pair of the form \((r, r)\) which lies on the boundary.

Since, the rate pair \((a, a) \in R_{ZF}(H, P_0/\sigma, \xi), \) from (22) such a pair \((a, a)\) will correspond to the rectangle \(\text{Rect}(y, y, D(H, \xi))\) inside the parallelogram \(R_{ZF}(H)\), where \(0 \leq y < L_1^{\text{max}}(\xi)\) such that
\[
a = C(y, P_0/\sigma), \quad \text{and} \quad y < t
\]
(137)

\(26\)In (128), \(x \in [0, h_{12} - h_{11}], \) whereas in (132), \(x \in [h_{12} - h_{11}, L_1^{\text{max}}(1/2)], \) both of these cases we have the same result and for the union of both of these cases \(x \in [0, L_1^{\text{max}}(1/2)].\)
Hence \( f(x) \) is a monotonically increasing function of \( x \). Further, since \( C(x, P_0/\sigma) \) and \( L^L_2(x) \) are continuous functions, it follows that \( f(x) \) is also continuous. It is clear that
\[
f(y) \overset{(a)}{=} \alpha C(y, P_0/\sigma) - C(L^L_2(y), P_0/\sigma) \overset{(b)}{=} \alpha a - a_1 < 0, \tag{145}
\]
where step (a) follows from (144), step (b) follows from (137) and (139), and step (c) follows from (140). Similarly
\[
f(t) \overset{(a)}{=} \alpha C(t, P_0/\sigma) - C(L^L_2(t), P_0/\sigma) = \alpha C(t, P_0/\sigma) - C(z, P_0/\sigma) = \alpha C(t, P_0/\sigma) - \alpha a
\]
\[
= \alpha C(t, P_0/\sigma) - a
\]
\[
\overset{(d)}{>} 0 \tag{146}
\]
where step (a) follows from (144), step (b) follows from the fact that \( L^L_2(t) = z \) (see (141)). Step (c) follows from (138). Step (d) follows from (143).

Further, since \( f(y) < 0 \), \( y \in [0, L^\text{max}_1(\xi)] \) and \( f(x), x \in [0, L^\text{max}_1(\xi)] \) is monotonically increasing in \( x \), it follows that \( f(x) < 0 \). Similarly, \( f(x = L^\text{max}_1(\xi)) > 0 \) since \( f(t) > 0 \) and \( L^\text{max}_1(\xi) \geq t \). Since, \( f(x) \) is a monotonically increasing and continuous in \( [0, L^\text{max}_1(\xi)] \), and \( f(0) < 0 \), it follows that there exists a unique \( x^* \in [0, L^\text{max}_1(\xi)] \) such that \( f(x^*) = 0 \) [19]. The uniqueness follows from the monotonicity of \( f(x) \). That is, from (144) we have
\[
o C(x^*, P_0/\sigma) = C(L^L_2(x^*), P_0/\sigma). \tag{147}
\]
Let \( a^* = C(x^*, P_0/\sigma) \) and therefore from (147) it follows that \( \alpha a^* = C(L^L_2(x^*), P_0/\sigma) \). From (26), it is clear that the rate pair \((a^*, \alpha a^*) \in R^R_{ZF}(H, P_0/\sigma, \xi)\). Uniqueness of such a rate pair follows from the uniqueness of \( x^* \). Further since \( f(y) < 0 = f(x^*) \) (see (145) and \( f(x) \) is monotonically increasing, it follows that
\[
x^* > y. \tag{148}
\]
From Result (2) we know that \( C(x, P_0/\sigma) \) is monotonically increasing in \( x \) and therefore form (148) \( a^* = C(x^*, P_0/\sigma) > C(y, P_0/\sigma) = a \). Therefore we have shown that the unique rate pair \((a^*, \alpha a^*) \) lies on the boundary \( R^R_{ZF}(H, P_0/\sigma, \xi) \) and \( a^* > a \) for any arbitrary choice of \( a \), where \((a, \alpha a)\) lies strictly inside \( R_{ZF}(H, P_0/\sigma, \xi) \). As shown in Fig. 16, the point \((a^*, \alpha a^*)\) lies on the line \( R_2 = \alpha R_1 \) and also on the boundary \( R^R_{ZF}(H, P_0/\sigma, \xi) \). This therefore completes the proof. \( \blacksquare \)

**APPENDIX F**

**Definition 2:** Let \((r_1, r_2)\) be the rate requested by User 1 and User 2, respectively. Consider the operating rate pair \((R^R_{1p}, R^R_{2p})\) such that
\[
(R^R_{1p}, R^R_{2p}) = \arg \min_{(R_1, R_2) \in R_{ZF}(H, P_0/\sigma, \xi)} \mathcal{J}((R_1, R_2), (r_1, r_2)), \tag{149}
\]
where, the objective function \( \mathcal{J}((R_1, R_2), (r_1, r_2)) \overset{(a)}{=} |R_1 - r_1| + |R_2 - r_2| \).

In the next theorem we show that if \((r_1, r_2)\) lies inside the proposed rate region \( R_{ZF}(H, P_0/\sigma, \xi) \) then the operating point \((R^R_{1p}, R^R_{2p}) = (r_1, r_2)\), and otherwise it lies on the boundary of the rate region, i.e., \( R^R_{ZF}(H, P_0/\sigma, \xi) \).

**Theorem 3:** The solution to (149) is given by
\[
(R^R_{1p}, R^R_{2p}) = \begin{cases} (r_1, r_2), & \text{if } (r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi) \vspace{1mm} \notag \\
\arg \min \limits_{(r_1, r_2) \in R^R_{ZF}(H, P_0/\sigma, \xi)} \mathcal{J}((R_1, R_2), (r_1, r_2)), & \text{otherwise.} \end{cases} \tag{150}
\]

**Proof:** See Appendix F-A.

In (150) we need to check whether \((r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\) or not. The following lemma gives us a method to make this check efficiently using the \( L^L_2(\xi) \) and the \( L^L_1(x) \) functions defined in Propositions 1 and 2 respectively.

**Lemma 5:** The rate pair
\[
(r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi) \iff r_1 \leq C(L^L_1(\xi), P_0/\sigma) \& r_2 \leq C(L^L_2(C^{-1}(r_1, P_0/\sigma)), P_0/\sigma) \tag{151}
\]
where for a given \( P_0/\sigma \), the inverse function of \( C(x, P_0/\sigma) \) is denoted by \( C^{-1}(y, P_0/\sigma) \).

**Proof:** From (29) we know that \( L^L_1(\xi) \) is the maximum possible value of \( L_1 \) such that \( \exists \text{Rect}(L_1, L_2 \geq 0, D(H, \xi)) \subseteq R_{ZF}(H) \). Hence, the maximum rate achievable by User 1 is the rate corresponding to the length \( L^L_1(\xi) \), and therefore, for any rate \((r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi) \), \( r_1 \) must be less than \( C(L^L_1(\xi), P_0/\sigma) \), i.e.,
\[
(r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi) \Rightarrow r_1 \leq C(L^L_1(\xi), P_0/\sigma) \tag{152}
\]
Since \( C(x, P_0/\sigma) \) is continuous and monotonically increasing in \( x \), from (152) it follows that there must exist a unique \( x \in [0, L^L_1(\xi)] \) such that \( r_1 = C(x, P_0/\sigma) \), and therefore
\[
C^{-1}(r_1, P_0/\sigma) = x. \tag{153}
\]
Further, from our definition of rate region boundary in (26) the rate pair \((r_1, r_2^*) \overset{(a)}{=} (C(x, P_0/\sigma), C(L^L_2(x), P_0/\sigma)) \) lies on the boundary. Also, from the definition of the boundary (see Para I of Section IV) we know that \( r_2^* = C(L^L_2(x), P_0/\sigma) \) is the largest possible rate of User 2 when the rate of User 1 is \( r_1 = C(x, P_0/\sigma) \). Hence, using the above fact we have
\[
(r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi) \Rightarrow r_2 \leq r_2^* = C(L^L_2(x), P_0/\sigma) \tag{154}
\]
where step (a) follows from (153). From the above two equations we have shown the if part of Lemma 5. We can prove the only if part of Lemma 5 by using Definition 1, Para 1 of

\( ^{27} \)The inverse exists since \( C(x, P_0/\sigma) \) is continuous and monotonically increasing with \( x \).
Section IV, (26) and following similar line of arguments as used to prove the if part. This therefore completes the proof.

Using Theorem 3 and Lemma 5, in the following algorithm we present a procedure for computing the operating rate pair \( (R_{1}^{op}, R_{2}^{op}) \).

### A. Proof of Theorem 3

For the requested rate pair \( (r_{1}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \), the operating point \( (R_{1}^{op}, R_{2}^{op}) \) is nothing but \( (r_{1}, r_{2}) \), because in this scenario \( J((R_{1}^{op}, R_{2}^{op}), (r_{1}, r_{2})) = J((R_{1}, R_{2}), (r_{1}, r_{2})) = 0 \), i.e.

\[
(r_{1}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \Rightarrow (R_{1}^{op}, R_{2}^{op}) = (r_{1}, r_{2}) \tag{155}
\]

If \( (r_{1}, r_{2}) \notin R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \), we will show that \( (R_{1}^{op}, R_{2}^{op}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \). The proof is by contradiction.

#### Algorithm 1: Operating Rate Pair \( (R_{1}^{op}, R_{2}^{op}) \) for Requested User Rate Pair \( (r_{1}, r_{2}) \).

**Input:** \( (r_{1}, r_{2}) \) : Requested rate pair, \( \xi : \) dimming target

**Output:** \( (R_{1}^{op}, R_{2}^{op}) \) : Operating rate pair

**Initialize:** \( (R_{1}^{op}, R_{2}^{op}) = (0, 0) \), \( \text{min}_\text{val} = |r_{1}| + |r_{2}| \)

1. if \( r_{1} \leq C(L_{1}^{max}(\xi), P_{0} / \sigma) \) \& \( r_{2} \leq C(L_{2}^{max}(\xi), P_{0} / \sigma, \xi) \) then
2. \( (R_{1}^{op}, R_{2}^{op}) = (r_{1}, r_{2}); \)
3. else
4. for \( x = 0 \) to \( L_{1}^{max}(\xi) \) do
5. \( R_{1} = C(x, P_{0} / \sigma); \)
6. \( R_{2} = C(L_{2}(x), P_{0} / \sigma); \)
7. \( J((R_{1}, R_{2}), (r_{1}, r_{2})) = |R_{1} - r_{1}| + |R_{2} - r_{2}|; \)
8. if \( J((R_{1}, R_{2}), (r_{1}, r_{2})) \leq \text{min}_\text{val} \) then
9. \( \text{min}_\text{val} = J((R_{1}, R_{2}), (r_{1}, r_{2})); \)
10. \( (R_{1}^{op}, R_{2}^{op}) = (R_{1}, R_{2}); \)
11. end if
12. end for
13. end if

Let us assume that \( (R_{1}^{op}, R_{2}^{op}) = (R'_{1}, R'_{2}) \) where \( (R'_{1}, R'_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) and \( (R'_{1}, R'_{2}) \notin R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \). We will then show that there will exist a rate pair \( (R_{1}^{*}, R_{2}^{*}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) such that \( J((R'_{1}, R'_{2}), (r_{1}, r_{2})) < J((R_{1}^{*}, R_{2}^{*}), (r_{1}, r_{2})) \) which contradicts the fact that \( (R_{1}^{op}, R_{2}^{op}) = (R'_{1}, R'_{2}) \).

Towards this end, we consider the rate pair \( (R'_{1}, r_{2}) \). The rate pair \( (R_{1}^{*}, r_{2}) \) can either lie inside the rate region \( R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) or outside the rate region (see the point \( P \) in Figs. 17 and 18).

**Case 1:** \( (R'_{1}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) : (see the point \( P \) in Fig. 17)

Since \( (R_{1}^{*}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \), from Definition 1 it follows that there exists \( (L_{1}', L_{2}) \) such that \( \text{Rect}(L_{1}', L_{2}, D(\mathbf{H}, \xi)) \subseteq \text{Rect}(L_{1}, L_{2}, \mathbf{H}) \) and

\[
r_{2} = C(L_{2}, P_{0} / \sigma). \tag{156}
\]

Further, from Lemma 5, since \( (R_{1}^{*}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) it follows that \( C(L_{2}, P_{0} / \sigma) = r_{2} \leq C(L_{2}^{(C^{-1}(R_{1}^{*}, P_{0} / \sigma))}, P_{0} / \sigma) \), since \( C(x, P_{0} / \sigma) \) is continuous and monotonically increasing in \( x \), it therefore follows that \( L_{2} \leq L_{2}^{(C^{-1}(R_{1}^{*}, P_{0} / \sigma))} \) and \( L_{2} \) lies in the range of the \( L_{2}^{(\xi)} \) function since \( L_{2}^{(\xi)} \) takes non-negative values and is continuous. Further since \( L_{2}^{(\xi)} \) is monotonically increasing it is clear that there exists a unique \( x > 0 \) such that

\[
L_{2} = L_{2}^{(\xi)(x)}. \tag{157}
\]

Hence, using (156) and (157) we have

\[
r_{2} = C(L_{2}^{(x)}, P_{0} / \sigma) \tag{158}
\]

We now define another rate pair \( (R_{1}^{*}, r_{2}) \) (point \( G \) in Fig. 17) where

\[
R_{1}^{*} \triangleq C(x, P_{0} / \sigma) \tag{159}
\]

and therefore by the characterization of \( R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi) \) in (26) it follows that

\[
(R_{1}^{*}, r_{2}) \in R_{ZF}^{d} (\mathbf{H}, P_{0} / \sigma, \xi). \tag{160}
\]

In the following we show that

\[
r_{1} > R_{1}^{*}. \tag{161}
\]
This is clear, because if \( r_1 \leq R_1^* \) then since \((R_1^*, r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\), from Lemma 6 in Appendix F-A it follows that \((r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\) which contradicts the fact that \((r_1, r_2) \notin R_{ZF}(H, P_0/\sigma, \xi)\). Since \((R_1', R_2') \in R_{ZF}(H, P_0/\sigma, \xi)\), from Definition 1 it follows that there exists \((L_1', L_2')\) such that \(\text{Rect}(L_1', L_2', D(H, \xi)) \subset R_{ZF}(H)\) and
\[
R_1' = C(L_1', P_0/\sigma), R_2' = C(L_2', P_0/\sigma)
\]
\[
(\text{a}) \quad L_1' = C^{-1}(R_1', P_0/\sigma), L_2' = C^{-1}(R_2', P_0/\sigma),
\]
where, step (a) follows from the existence of the inverse (see Footnote 27 in Lemma 5). Let,
\[
R_{2d}^b = C(L_1'(L_1', P_0/\sigma)).
\]
(163)

Then from (26) it follows that
\[
(R_1', R_2'_{2d}) \in R_{ZF}(H, P_0/\sigma, \xi)
\]
(164)

This is illustrated through Fig. 17. Since \((R_1', r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\) and \((R_1', R_2'_{2d}) \in R_{ZF}(H, P_0/\sigma, \xi)\), from the characterization of the proposed rate region boundary in Section IV it follows that
\[
r_2 \leq R_{2d}^b
\]
(165)

Here, we note that in the special case \( r_2 = R_{2d}^b \) implies that the rate pair \((R_1', r_2)\) and \((R_1', R_2'_{2d})\) are the same rate pairs and therefore \((r_1, r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\) (see (164). For this special case, we also note that since \(C(L_2, P_0/\sigma) = r_2 = R_{2d}^b = C(L_1'(L_1'), P_0/\sigma)\) we have \(L_2 = L_2'(L_1')\) (as \(C(\cdot)\) is monotonic). However, from (157) we have \(L_2 = L_2'(L_2'(x))\). Hence we have \(L_2'(L_1') = L_2 = L_2'(x)\). As \(L_2'(x)\) is monotonic it follows that \(x = L_1'\) and hence \(R_2' = C(x, P_0/\sigma) = C(L_1', P_0/\sigma) = R_1', i.e., the rate pair \((R_1', r_2)\) and \((R_1', R_2'_{2d})\) are also the same.

Hence for this special case we get that the rate pairs \((R_1', r_2), (R_1', R_2')\) and \((R_1', R_2'_{2d})\) are the same and lie on the rate region boundary \(R_{ZF}(H, P_0/\sigma, \xi)\). We then have
\[
\mathcal{J}((R_1', r_2), (r_1, r_2)) = r_1 - R_1^* \quad \text{for} \quad (161))
\]
\[
\begin{align*}
(\text{a}) & \quad r_1 - R_1^* = |r_1 - R_1'|
\end{align*}
\]
\[
(\text{b}) & \quad |r_1 - R_1'| + |r_2 - R_2'|
\]
\[
= \mathcal{J}((R_1', r_2), (r_1, r_2))
\]
(166)

where step (a) follows from the fact that for this special case \(R_1^* = R_1'\) and also that \(r_1 > R_1^*\) (see (161)). Step (b) follows from the fact that \(r_2 \notin R_2^*\) since otherwise \((R_1', R_2') = (R_1', r_2)\) and as \((R_1', r_2) \in R_{ZF}(H, P_0/\sigma, \xi)\) we would get \((R_1', R_2') \in R_{ZF}(H, P_0/\sigma, \xi)\) which contradicts the initial assumption that \((R_1', R_2')\) is not on the rate region boundary.

From (166), it is clear that we have a rate pair \((R_1^*, r_2)\) on the rate region boundary for which the value of the objective function \(\mathcal{J}((R_1^*, r_2), (r_1, r_2)) < \mathcal{J}((R_1', r_2), (r_1, r_2))\), which contradicts the fact that \((R_1^*, R_2')\) is a minimum of the objective function \(\mathcal{J}((R_1', r_2), (r_1, r_2))\) in (149).

We next consider those cases where
\[
r_2 < R_2^d.
\]
(167)

From (160), (164) and (165) we have \((R_1^*, r_2) \in R_{ZF}^{b}(H, P_0/\sigma, \xi)\) and \((R_1', R_2'_{2d}) \in R_{ZF}^{b}(H, P_0/\sigma, \xi)\), and \(r_2 \leq R_2^d\).

Since \(R_{2d} = C(L_1'(L_1'), P_0/\sigma)\) it follows that
\[
C(L_2, P_0/\sigma) = r_2 < R_2_{2d} = C(L_1'(L_1'), P_0/\sigma),
\]
\[
(\text{a}) \quad L_2 < L_2'(L_1'), P_0/\sigma,
\]
\[
(\text{b}) \quad L_2'(x) = L_2 < L_2'(L_1'),
\]
\[
(\text{c}) \quad x > L_1'
\]
\[
(\text{d}) \quad R_1' = C(x, P_0/\sigma) > C(L_1', P_0/\sigma) = R_1'
\]
\[
\Rightarrow R_1' > R_1',
\]
(168)

where, step (a) follows from the monotonicity of \(C(x, P_0/\sigma)\) function with its first argument (see Result 2). Step (b) follows from (157). Step (c) follows from the fact that \(L_2'(x)\) is a monotonically decreasing and continuous function of \(x\) (see Lemma 1). Step (d) implies from (159), (162) and Result 2.

From (161) and (168) and we have
\[
r_1 > R_1^* > R_1'.
\]
(169)

We now have a rate pair \((R_1', r_2) \in R_{ZF}^{b}(H, P_0/\sigma, \xi)\) (see (160) for which
\[
\begin{align*}
\mathcal{J}((R_1', r_2), (r_1, r_2)) = (r_2 - R_1') \quad &\quad (\text{a})
\end{align*}
\]
\[
\begin{align*}
&\quad \leq |r_1 - R_1'| + |r_2 - R_2'|
\end{align*}
\]
\[
(\text{b}) \quad \mathcal{J}((R_1', r_2), (r_1, r_2)) = (170)
\]

where step (a) follows from (169). Step (b) follows from the fact that \(r_1 - R_1' = |r_1 - R_1'|\) since \(r_1 > R_1'\) (see (169)). From (170) it is clear that we have a rate pair \((R_1', r_2)\) on the rate region boundary for which the value of the objective function \(\mathcal{J}((R_1', r_2), (r_1, r_2)) < \mathcal{J}((R_1', R_2'), (r_1, r_2))\), which contradicts the fact that \((R_1', R_2')\) is a minimum of the objective function \(\mathcal{J}((R_1', R_2'), (r_1, r_2))\) in (149).

Case II: \((R_1', r_2) \notin R_{ZF}(H, P_0/\sigma, \xi)\) (Point P in Fig. 18): From our initial assumption that \((R_1', R_2')\) lies inside the proposed rate region \(R_{ZF}(H, P_0/\sigma, \xi)\) but not on the boundary \(R_{ZF}^{b}(H, P_0/\sigma, \xi)\) we have,
\[
(R_1', R_2') \in R_{ZF}(H, P_0/\sigma, \xi) \quad \text{for} \quad (161))
\]
\[
\begin{align*}
(\text{a}) & \quad r_2 \leq C(L_1'(C^{-1}(R_1', P_0/\sigma)), P_0/\sigma)
\end{align*}
\]
\[
\begin{align*}
\Rightarrow R_2' \leq C(L_1'(C^{-1}(R_1', P_0/\sigma)), P_0/\sigma)
\end{align*}
\]
\[
(\text{b}) \quad R_2' \leq C(L_2'(L_1'), P_0/\sigma)
\]
\[
(\text{c}) \quad R_2' \leq R_{2d}^{b}
\]
(171)

where, step (a) follows from Lemma 5. Step (b) follows from the fact that \(L_1' = C^{-1}(R_1', P_0/\sigma)\) (see (162)). Step (c) follows from (163). We also have,
\[
R_2' \neq R_2^{b}
\]
(172)
As otherwise, if $R'_2 = R''_2$, then from (164) it follows that $(R'_1, R'_2) = (R'_1, R''_2) \in R''_2 (H, P_0/\sigma, \xi)$ which contradicts our initial assumption that $(R'_1, R'_2) \notin R''_2 (H, P_0/\sigma, \xi)$. Using (171) and (172) we get

$$R'_2 < R''_2.$$

(173)

Since for this case we have,

$$(R'_1, r_2) \notin R'_2 (H, P_0/\sigma, \xi), \quad (a) \quad r_2 > C(L_2^a (C^{-1} (R'_1, P_0/\sigma)), P_0/\sigma)$$

or

$$r_2 > C(L_2^a (C^{-1} (R'_1, P_0/\sigma)), P_0/\sigma)$$

(174)

Step (a) follows from Lemma 5. Step (b) follows from the fact that $R'_1 < C(L_2^a (\xi))$ since $(R'_1, R'_2) \in R'_2 (H, P_0/\sigma, \xi)$ (see lemma 5). Step (c) follows from the fact that $L_1^a = C^{-1} (R'_1, P_0/\sigma)$ (see (162)) and step (d) follows from (163). From (173) and (174) it follows that

$$R'_2 < R''_2 < r_2.$$

(175)

We then have,

$$J((R'_1, R''_2), (r_1, r_2)) = |R'_1 - r_1| + |R''_2 - r_2|$$

$$= |R'_1 - r_1| + |r_2 - R''_2|$$

$$< |R'_1 - r_1| + |r_2 - R'_2|$$

$$= |R'_1 - r_1| + |R'_2 - r_2|$$

$$= J((R'_1, R'_2), (r_1, r_2)),$$

(176)

where steps (a) and (b) follow from (175). From (176) it is clear that the rate pair $(R'_1, R''_2)$ lying on the rate region boundary has an objective function value less than the value for $(R'_1, R'_2)$ which contradicts the fact that $(R'_1, R'_2)$ is a minimum of the objective function $J((R'_1, R'_2), (r_1, r_2))$ in (149). From the analysis of Case I and Case II above it therefore follows that if $(r_1, r_2) \notin R''_2 (H, P_0/\sigma, \xi)$, then the operating pair $(R''_1, R''_2)$ in (141) must lie on the proposed boundary of the rate region, i.e.,

$$(R''_1, R''_2) = \min_{(R'_1, R'_2) \in R''_2 (H, P_0/\sigma, \xi)} |R'_1 - r_1| + |R'_2 - r_2|.$$

(177)

This along with (155) completes the proof of Theorem 3.

Lemma 6. If the rate pair $(a, b) \in R''_2 (H, P_0/\sigma, \xi)$ and if $(c, d)$ is a rate pair such that $c \leq a$ and $d \leq b$ then

$$(c, d) \in R''_2 (H, P_0/\sigma, \xi).$$

(178)

Proof: From Definition 1 we know that, $(a, b) \in R''_2 (H, P_0/\sigma, \xi) \iff \exists$ a rectangle $Rect(L_a, L_b, D (H, \xi)) \subset R''_2 (H, \sigma)$, s.t. $(a, b) = (C(L_a, P_0/\sigma), C(L_b, P_0/\sigma))$. If we show that for the rate pair $(c, d)$ there exists a rectangle $Rect(L_c, L_d, D (H, \xi)) \subset R''_2 (H, \sigma)$, s.t. $(c, d) = (C(L_c, P_0/\sigma), C(L_d, P_0/\sigma))$ then it implies that $(c, d) \in R''_2 (H, P_0/\sigma, \xi)$. Towards this end, from Result 2 where step (a) follows from (179) and step (b) follows from Result 2. Further, since it is clear that if we can fit a rectangle $Rect(L_1^a, L_2^a, D (H, \xi))$ inside $R''_2 (H, \xi)$ then we can also fit a rectangle $Rect(L_3, L_4, D (H, \xi))$ inside $R''_2 (H, \xi)$ since $L_3 \leq 1$ and $L_4 \leq 2$. Using the above fact and (180) it follows that since $L_3 \leq L_a$, $L_4 \leq L_b$, $\exists$ a rectangle $Rect(L_c, L_d, D (H, \xi)) \subset R''_2 (H, \sigma)$, s.t. $(c, d) = (C(L_c, P_0/\sigma), C(L_d, P_0/\sigma))$ and hence it implies that $(c, d) \in R''_2 (H, P_0/\sigma, \xi)$. It therefore completes the proof.

APPENDIX G

COMPARING THE PROPOSED ACHIEVABLE RATE REGION OF TWO DIFFERENT CHANNEL REALIZATIONS USING THEIR CORRESPONDING PLOTS OF $(L_1 = x, L_2 = L_2^a (x))$ PAIRS

In the following, through a numerical example we illustrate the fact that the proposed achievable rate regions depends on the channel realization and the rate region of two different channel realizations can be compared by comparing their corresponding plots of $(L_1 = x, L_2 = L_2^a (x))$ pairs even without explicitly computing the rate region boundaries.

In Figs. 19 and 20 we plot the rate region boundary and their corresponding $(L_1 = x, L_2 = L_2^a (x))$ pairs for a fixed
Fig. 20. Plot of \( (x, L_z^1(x)) \) for a fixed dimming target \( \xi = 0.5, P_0/\sigma = 75 \) dB and for two different channel realizations \( \mathbf{H}_1, \mathbf{H}_2 \).

\[ \xi = 0.5, P_0/\sigma = 75 \text{ dB and for two different channel realizations } \mathbf{H} = \mathbf{H}_1 = \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] \times 10^{-6} \text{ and } \mathbf{H} = \mathbf{H}_2 = \left[ \begin{array}{c} 3 \\ 3 \\ 1 \\ 1 \end{array} \right] \times 10^{-6} \text{ respectively. Note that we have considered the elements of the channel realization of the order } 10^{-6}. \text{ This is due to the fact that in an indoor VLC scenario (e.g., a 5 m \times 5 m \times 3 m office set-up) the channel gains are typically lies in the range of } 10^{-5} \text{ to } 10^{-6}. \]

\[ \text{It can be observed from Fig. 20 that for any } (x, L_z^1(x)) \text{ pair in the plot for } \mathbf{H} = \mathbf{H}_1, \text{ there exists a pair } (x, y) \text{ in the plot for } \mathbf{H} = \mathbf{H}_1 \text{ such that } y > L_z^1(x) \text{ (see the points A and B in Fig. 20). From this observation it follows that the proposed rate region } R_{ZF}(\mathbf{H}, P_0/\sigma, \xi) \text{ for the channel realization } \mathbf{H}_1 \text{ would be larger than the proposed rate region } R_{ZF}(\mathbf{H}, P_0/\sigma, \xi) \text{ for the channel realization } \mathbf{H}_2. \text{ This comparison is indeed true as can be verified from Fig. 19. From Fig. 19 it can also be observed that indeed the proposed achievable rate region is different for the two channel realizations } \mathbf{H}_1 \text{ and } \mathbf{H}_2. \]

References

[1] H. Elgala, R. Mesleh, and H. Haas, “Indoor optical wireless communication: Potential and state-of-the-art,” IEEE Commun. Mag., vol. 49, no. 9, pp. 56–62, Sep. 2011.
[2] S. Dimitrov and H. Hass, Principles of LED Light Communications: Towards Networked Li-Fi, Cambridge, U.K., Cambridge Univ. Press, 2015.
[3] J. M. Kahn and J. R. Barry, “Wireless infrared communications,” Proc. IEEE, vol. 85, no. 2, pp. 265–298, Feb. 1997.
[4] J. Y. Wang, J. B. Wang, M. Chen, and X. Song, “Dimming scheme analysis for pulse amplitude modulated visible light communications,” in Proc. Int. Conf. Wireless Commun. Signal Process., Oct. 2013, pp. 1–6.
[5] J. B. Wang, Q. S. Hu, J. Wang, M. Chen, and J. Y. Wang, “Tight bounds on channel capacity for dimmable visible light communications,” J. Lightwave Technol., vol. 31, no. 23, pp. 3771–3779, Dec. 2013.
[6] A. Lapi, T. S. M. Moser, and M. A. Wigger, “On the capacity of free-space optical intensity channels,” in Proc. 2008 IEEE Int. Symp. Inf. Theory, Jul. 2008, pp. 2419–2423.
[7] L. Wu, Z. Zhang, J. Dang, and H. Liu, “Capacity lower bounds of IM/DD AWGN optical wireless channels based on Fano’s inequality,” in Proc. Int. Conf. Wireless Commun. Signal Process., Oct. 2015, pp. 1–5.
[8] A. Thangaraj, G. Kramer, and G. Bcherer, “Capacity upper bounds for discrete-time amplitude-constrained AWGN channels,” in Proc. 2015 IEEE Int. Symp. Inf. Theory, Jun. 2015, pp. 2321–2325.
[9] J. G. Smith, “The information capacity of amplitude and variance constrained scalar gaussian channels,” Inf. Control, vol. 18, no. 3, pp. 203–219, 1971.