Modules with ⊛(⊛'or ⊛'') Condition
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Keywords. SH-submodule, SI-submodule, conditions $\mathcal{ISH}_\text{top}$-module, $\mathcal{2SH}_\text{top}$-module, $\mathcal{SI}_\text{top}$-module, ⊛, ⊛'
and ⊛'' conditions.

Abstract. In this paper we introduce and study modules with ⊛ (⊛' or ⊛'') condition. We give several properties of these types of modules and some relationships between them.

Introduction
Let M be an R-module, where R is a commutative ring with unity. Recall that a nonzero submodule N of M is called strongly hollow (briefly, SH-submodule) if whenever $L_1, L_2 \leq M$, $N \subseteq L_1 + L_2$ implies either $N \subseteq L_1$ or $N \subseteq L_2$, [1]. A submodule N of M is called strongly irreducible (SI-submodule) if whenever $L_1, L_2 \leq M$, $N \supseteq L_1 \cap L_2$ implies $N \supseteq L_1$ or $N \supseteq L_2$, [4]. The sets $\{K: K \text{ is a SH-submodule of } M\}$, $\{K: K \text{ is a proper SH-submodule of } M\}$ and $\{K: K \text{ is a nonzero proper SI-submodule of } M\}$ are denoted $\mathcal{ISH}_\text{top}(M)$, $\mathcal{2SH}_\text{top}(M)$ and $\mathcal{SI}_\text{top}(M)$ respectively,[8]. In [8] we studied and topologized these sets by setting that for any $L \leq M$

We prove that $(\mathcal{ISH}(M), \tau(M))$ is a topological space (see [8, Th.2.1.9]). Also we see that $\zeta(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.2.4.1]). This leaded us to call an R-module M a $\mathcal{ISH}_\text{top}$-module if $\zeta(M)$ is closed under finite union. Equivalently M is a $\mathcal{ISH}_\text{top}$-module if $(\mathcal{ISH}(M), \tau(M))$ is a topological space.

Beside these we see that $\zeta(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.3.2.1]). This leaded us to call an R-module M a $\mathcal{SI}_\text{top}$-module if $\zeta(M)$ is closed under a finite union. Equivalently M is a $\mathcal{SI}_\text{top}$-module if $(\mathcal{SI}(M), \tau(M))$ is a topological space.

We notice that, for any $L_1 \leq M$, $L_2 \leq M$, if $\mathcal{ISH}(L_1) = \mathcal{ISH}(L_2)$ (or $\mathcal{2SH}(L_1) = \mathcal{2SH}(L_2)$ or $\mathcal{SI}(L_1) = \mathcal{SI}(L_2)$) then it is not necessarily that $L_1 = L_2$, as the following examples show.

(1) Consider the Z-module Z, $\mathcal{ISH}(3Z) = \mathcal{ISH}(\{0\}) = \emptyset$ but $3Z \neq \{0\}$.
(2) For the Z-module $Z_{12}$, $\mathcal{2SH}(\langle 2 \rangle) = \emptyset = \mathcal{2SH}(\langle 2 \rangle)$ but $\langle 2 \rangle \neq \langle 2 \rangle$.
(3) For the Z-module $Z_{12}$, $\mathcal{SI}(\langle 6 \rangle) = \emptyset = \mathcal{SI}(\{0\})$ but $\langle 6 \rangle \neq \{0\}$.
These observations lead us to introduce the following conditions:

\[ \bigcirc : \quad V(L_1) = V(L_2) \implies L_1 = L_2, \quad \text{for each } L_1, L_2 \leq M. \]
\[ \bigcirc' : \quad V(L_1) = V(L_2) \implies L_1 = L_2, \quad \text{for each } L_1, L_2 \leq M. \]
\[ \bigcirc'' : \quad V(L_1) = V(L_2) \implies L_1 = L_2, \quad \text{for each } L_1, L_2 \leq M. \]

This paper is devoted to study modules with \( \bigcirc \) (\( \bigcirc' \), \( \bigcirc'' \) respectively). Also we shall study the behaviour \( \text{Spec}(M) \), \( \text{Spec}(M) \) and \( \text{Spec}(M) \) respectively when \( M \) satisfies \( \bigcirc \) (\( \bigcirc' \), \( \bigcirc'' \)).

\section*{S.1 Modules with the Condition \( \bigcirc \)}

We start this by the following remarks and examples.

\textbf{Remarks and Examples 1.1:}

1. The \( \mathbb{Z} \)-module \( \mathbb{Z} \) does not satisfies \( \bigcirc \) since for each \( L, N \leq \mathbb{Z}, L \neq \mathbb{Z} \),

\[ V(L) = V(N) = \emptyset \]

2. Every simple module satisfies \( \bigcirc \).

3. Let \( M, M' \) be two isomorphic \( R \)-modules. Then \( M \) satisfies \( \bigcirc \) if and only if \( M' \) satisfies \( \bigcirc \).

4. Let \( M_1, M_2 \) be \( R \)-modules. If \( M_1, M_2 \) satisfies \( \bigcirc \) condition, then \( M_1 \oplus M_2 \) may not be satisfy \( \bigcirc \), as an example: Let \( M_1 = \mathbb{Z}_3 \) as a \( \mathbb{Z} \)-module and \( M_2 = \mathbb{Z}_4 \) as a \( \mathbb{Z} \)-module. Each of \( M_1 \) and \( M_2 \) satisfies \( \bigcirc \). However \( \mathbb{Z}_3 \oplus \mathbb{Z}_4 \cong \mathbb{Z}_{12} \) and \( \mathbb{Z}_{12} \) does not satisfy \( \bigcirc \).

5. Let \( M \) be an \( R \)-module. Then \( M \) satisfies \( \bigcirc \) as \( R \)-module if and only if \( M \) satisfies \( \bigcirc \) as \( R/\text{ann} \) \( M \).

\textbf{Proposition 1.2.} Let \( M \) be an \( R \)-module such that every nonzero submodules is \( \text{SH} \). Then \( M \) satisfies \( \bigcirc \).

\textbf{Proof:} First note that \( V(<0>) = \emptyset \neq V(N) \) for each \( N \neq <0> \). Let \( L, N \leq M, L \neq (0), N \neq (0) \) such that \( V(L) = V(N) \). Since \( L \subseteq L \) and \( L \) is a \( \text{SH} \)-submodule by hypothesis, \( L \in V(L) = V(N) \). It follows that \( L \subseteq N \). Similarly, \( N \in V(N) = V(L) \) and hence \( N \subseteq L \). Thus \( L = N \).

Recall that an \( R \)-module \( M \) is called chained if the lattice of its submodules is linearly ordered by inclusion [10].

\textbf{Corollary 1.3.} Let \( M \) be a chained \( R \)-module. Then \( M \) satisfies \( \bigcirc \).

The following theorem gives a characterization of modules with the condition \( \bigcirc \).

\textbf{Theorem 1.4:}

Let \( M \) be an nonzero \( R \)-module. Then \( M \) satisfies \( \bigcirc \) if and only if every nonzero submodule of \( M \) can be represented as sum of \( \text{SH} \)-submodules.

\textbf{Proof:} (\( \Rightarrow \)) Let \( (0) \neq K \leq M \). Then \( V(K) \neq \emptyset \), since if \( V(K) = \emptyset \), then \( V(K) = V(0) \) and hence \( K = (0) \) (by \( \bigcirc \)), which is a contradiction. Set \( N = \sum_{W \in V(K)} W \) and let \( L \in V(N) \). Then \( L \subseteq V(K) \).

But for each \( W \in V(K), W \subseteq K \), so \( N \subseteq K \). Thus \( L \subseteq K \) and hence \( L \in V(N) \). Thus \( V(N) \subseteq V(K) \).

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Now, let $L \in V(K)$, then $L \subseteq N$ (by definition of N). Hence $L \in V(K)$ and so $V(N) \subseteq V(K)$ Thus $V(N) \subseteq V(K)$ and by $\odot$, $K=N$. Thus $K$ is a sum of SH-submodules.

$(\Leftarrow)$ Let $N \leq M$, $N \neq 0$. Then $N = \sum_{i \in \Lambda} T_i$, where $T_i$ is an SH-submodule of $M$, $\forall \ i \in \Lambda$ (by hypothesis).

Since for each $i \in \Lambda$, $T_i \subseteq N$, we have $T_i \in V(N)$. Hence $\bigvee_{T_i \in V(N)} \sum_{T_i \in V(N)} T_i = N = \bigvee_{S_i \in V(N)} \sum_{S_i \in V(N)} S_i$, by the previous argument. Then $N_1 = N_2$. Thus $M$ satisfies $\odot$.

**Corollary 1.5.** Let $M$ be a semisimple R-module such that every simple submodule is SH. Then $M$ satisfies $\odot$.

Recall that an R-module is called comultiplication if for each $N \leq M$, $N = M$ for some ideal $I$ of $R$. Let $N(M) = \{m \in M \mid Im = (0)\}$.

An R-module is called distributive if for each $N, L, W \leq M$, $N \cap (L + W) = (N \cap L) + (N \cap W)$.

**Lemma 1.6:** [8,Cor.1.1.9], [8,Prop.1.1.11] Let $M$ be a comultiplication (or distributive) R-module. Then every simple submodule of $M$ is a SH-submodule.

**Corollary 1.7.** Let $M$ be a semisimple comultiplication (or distributive). Then $M$ satisfies $\odot$.

**Proof:** It follows by Lemma 1.6 and Cor. 1.5.

**Remarks 1.8:**

1. The condition “every simple submodule is a SH-submodule” is necessary in Cor. 1.5, as for example: It is clear that the vector space $\mathbb{R}^2$ over $\mathbb{R}$ is semisimple, but $N_1 = \mathbb{R}(1,0)$ is a simple submodule of $\mathbb{R}^2$ and it is not SH. However $\mathbb{R}^2$ does not satisfy $\odot$ since $V(N_1) = V(N_2) = \phi$, where $N_2 = \mathbb{R}(0,1)$.

2. The converse of Cor. 1.5 is not true in general, for example: consider the $Z$-module $Z_4$, $Z_4$ satisfies $\odot$ and every nonzero submodule of $Z_4$ is SH. However $Z_4$ is not semisimple.

Before giving the next result, we give the following lemma.

**Lemma 1.9.** Let $M$ be an R-module. Then $\text{SH} \text{Spec}(M)$ is a $T_1$-space if and only if every SH-submodule is minimal SH in $\text{SH} \text{Spec}(M)$

**Proof:** $(\Rightarrow)$ If $\text{Spec}(M)$ is $T_1$. Suppose $\text{Spec}(M) = \phi$. Then nothing to prove. Let $\text{Spec}(M) \neq \phi$. Since $\text{Spec}(M)$ is $T_1$, then for any $N \in \text{Spec}(M)$, $N$ is closed; that is $\{N\} = V(L)$ for some $L \leq M$. If $N$ is not minimal SH, then there exists $K \in \text{Spec}(M)$ and $K \subseteq N$. Hence $K \subseteq L$; that is $K \in V(L) = \{N\}$ and $K = N$ which is a contradiction. Therefore $N$ is a minimal SH-submodule.

$(\Leftarrow)$ Let $N \in \text{Spec}(M)$. Then $N$ is SH and so $N \in V(N)$. Assume there exists $L \leq M$, $L \neq M$ such that $L \in V(N)$. It follows that $L$ is SH and $L \subseteq N$. Hence $N$ is not a minimal SH-submodule which contradicts the hypothesis.

**Theorem 1.10.** Let $M$ be a comultiplication R-module. Then $\text{Spec}(M)$ is $T_1$ and $M$ satisfies $\odot$ if and only if $M$ is semisimple and every SH-submodule is minimal SH.
Proof: \((\Rightarrow)\) Since \(\text{Spec}(M)\) is \(T_1\), then by lemma 1.9, every SH-submodule is minimal SH. But M satisfies \(\odot\), so by Th.1.4, every submodule is a sum of SH-submodules. On the other hand, M is comultiplication and \(\text{Spec}(M)\) is \(T_1\) imply \(S(M) = \text{Spec}(M)\) by [8, Cor.2.2.19] where \(S(M) = \) set of all simple submodules of M. Thus every submodule of M is a sum of simple submodules; that is M is semisimple.

\((\Leftarrow)\) It follows by Cor.1.7 and Lemma 1.9.

Recall that an R-module M is called multiplication if for each \(N \leq M\), there exists \(I \leq R\) such that \(N = IM\), [3].

Lemma 1.11. Let M be a faithful finitely generated multiplication over a comultiplication ring and let \((0) \neq N \leq M\). Then N is a minimal SH-submodule of M if and only if N is simple.

Proof: \((\Rightarrow)\) Since M is multiplication, \(N = IM\) for some ideal I of R. Then by [8, Prop.1.2.1] I is aSH ideal of R. We claim that I is a simple ideal of R. If I is not simple, then there exists a simple ideal J of R such that \(J \subsetneq I\) since R is comultiplication. Moreover by Lemmal 1.6, J is a SH ideal of R and so by [8, Prop.1.2.1] JM is a SH-submodule. But M is a faithful finitely generated multiplication, so by [5,Th.3.1] J \(\subsetneq IM\). Thus N is not a minimal SH-submodule of M, which is a contradiction. Thus I is a simple ideal of R and so N is a simple submodule of M.

\((\Leftarrow)\) Let N be a simple submodule of M. Since M is multiplication then \(N = IM\) for some ideal I of R. It is easy to check that I is a simple ideal of R. Then by Lemma 1.6, I is a SH ideal and so [8, Prop.1.2.1], N is a SH-submodule. Thus N is a minimal SH-submodule of M.

By using Lemma 1.11, we have the following immediate result.

Corollary 1.12. Let R be a comultiplication ring and let \(J \leq R\). Then J is a minimal SH ideal if and only if J is a simple (minimal) ideal.

Now we have the following:

Theorem 1.13. Let M be a faithful finitely generated multiplication over a comultiplication ring R. Then \(\text{Spec}(M)\) is \(T_1\) and M satisfies \(\odot\) if and only if M is semisimple and every SH-submodule of M is minimal SH.

Proof: \((\Rightarrow)\) By Lemma 1.9, every SH-submodule of M is minimal SH and by Lemma 1.11, every minimal SH-submodule is a simple submodule. But by Th.1.4, every submodule of M is a sum of SH-submodules. Thus every submodule of M is a sum of simple submodules. Therefore M is semisimple.

\((\Leftarrow)\) Since M is semisimple, every submodule is a sum of simple submodule. But by Lemma 1.11, every simple submodule is minimal SH. Thus every submodule of M is a sum of SH-submodules and so that by Th.1.4, M satisfies \(\odot\).

On the other hand, since every SH-submodule of M is minimal SH, then by Lemma 1.9, \(\text{Spec}(M)\) is a \(T_1\)-space.

Next we have the following:

Theorem 1.14. Let M be a faithful finitely generated multiplication R-module. Then M satisfies \(\odot\) if and only if R satisfies \(\odot\).

Proof: \((\Rightarrow)\) Let I and J be ideal of R such that \(\text{V}(I) = \text{V}(J)\). We claim that \(\text{V}(IM) = \text{V}(JM)\). To see this, let \(K \in \text{V}(IM)\) Then K is a SH-submodule and \(K \subseteq IM\). But by [8,Prop.1.2.1], there exists a SH ideal T of R such that \(K = TM\). Thus \(TM \subseteq IM\) and so by [5,Th.3.1], \(T \subseteq I\). It follows that \(T \in \text{V}(IM)\) that is \(T \subseteq J\) which implies \(K=TM \subseteq JM\). Thus \(K \in \text{V}(JM)\) and hence \(\text{V}(IM) \subseteq \text{V}(JM)\).
Similarly $\mathcal{V}(IM) \subseteq \mathcal{V}(JM)$ Therefore $\mathcal{V}(IM) \subseteq \mathcal{V}(JM)$ and so $IM = JM$ and then by [5, Th.3.1], $I = J$. Thus $R$ satisfies $\ominus$. ($\Rightarrow$) The proof is Similarly.

Remark 1.15. The condition (M is faithful) in Th.1.14 cannot be dropped, as for example. The $Z$-module $Z_8$ is finitely generated multiplication but not faithful. However $Z_8$ satisfies $\ominus$, but $Z$ does not satisfy $\ominus$.

Corollary 1.16. Let $M$ be a finitely generated $R$-module. Then the following statements are equivalent:

1. $M$ satisfies $\ominus$ as $R$-module.
2. $M$ satisfies $\ominus$ as $R$-$\ominus$-module.
3. $R$ satisfies $\ominus$.

where $R = R/\text{ann } M$.

Recall that a submodule $N$ of an $R$-module is called second if for each ideal $I$ of $R$, either $IK = K$ or $IK = (0)$, [13].

To give our next result, first we introduce the following Lemma.

Lemma 1.17. Let $M$ be an $R$-module such that every $SH$-submodule is second. Then $\mathcal{V}(N) = \mathcal{V}(0 : I) = \mathcal{V}(N + (0 : I)) = \mathcal{V}(N : I)$ For any $N \subseteq M$, $I \subseteq R$.

Proof: By [8, Prop.2.1.9], $\mathcal{V}(N + (0 : I)) \subseteq \mathcal{V}(N : I)$; hence $\mathcal{V}(N + (0 : I)) = \mathcal{V}(N : I)$. Now, let $K \subseteq \mathcal{V}(N)$ or $K \subseteq \mathcal{V}(0 : I)$. Then $IK \subseteq N$ and $K$ is a $SH$-submodule. Hence by hypothesis $K$ is second, so that either $IK = K$ or $IK = (0)$. It follows that either $K \subseteq N$ or $IK = (0)$.

Recall that a submodule $N$ of an $R$-module $M$ is called copure if for each $I \subseteq R$, $N + (0 : I) = (N : I)$, [12]

Theorem 1.18. Let $M$ be an $R$-module with $\ominus$ condition such that every $SH$-submodule is second. Then every submodule of $M$ is copure.

Proof: Let $N \subseteq M$. By Lemma 1.17 $\mathcal{V}(N + (0 : I)) = \mathcal{V}(N : I)$ for any $I \subseteq R$. Hence by condition $\ominus$, $N + (0 : I) = (N : I)$ for any $I \subseteq R$; that is $N$ is copure.

Proposition 1.19. Let $M$ be an $R$-module which satisfies $\ominus$. Then

1. $M$ is Noetherian (Artinian) if and only if $\text{Spec}(M)$ satisfies a.c.c (d.c.c) on closed set.
2. $M$ is Noetherian (Artinian) if and only if $\text{Spec}(M)$ satisfies d.c.c (a.c.c) on open sets.

S.2 Modules with the Condition $\ominus'$

In this section, we study modules that satisfy $\ominus'$. Some properties of these modules are analogous to that of modules with the condition $\ominus$.

As we mention in the introduction, a module with condition $\ominus'$ if it satisfies the condition $\ominus'$, where

$\ominus'$: if for each $L$, $N \subseteq M$, $\mathcal{V}(L) = \mathcal{V}(N)$ implies $L = N$
Remarks and Examples 2.1

(1) Every simple module satisfies $\ominus'$. 

(2) If every proper nonzero submodule of $M$ is SH, then $M$ satisfies $\ominus'$ for each proper nonzero submodules of $M$. 

(3) The $Z$-module $Z_4$ does not satisfy $\ominus'$ since $\frac{Z}{Z_4}$ is maximal but $\frac{Z}{Z_4} \neq \phi$. 

(4) If $M_1$ and $M_2$ are $R$-modules such that $M_1 \cong M_2$ then $M_1$ satisfies $\ominus'$ if and only if $M_2$ satisfies $\ominus'$. 

(5) Let $M$ be an $R$-module. Then $M$ satisfies $\ominus'$ as $R$-module if and only if $M$ satisfies $\ominus'$ as $R/\text{ann} M$-module. 

Theorem 2.2. Let $M$ be a nonzero $R$-module. Then $M$ satisfies $\ominus'$ if and only if every proper submodule of $M$ is an intersection of SH-submodules.

Proof: $(\Rightarrow)$ Let $K \leq M$. Then $\frac{V(K)}{V(N)} = \phi$, because if $\frac{V(K)}{V(N)} = \phi = \frac{V(M)}{V(K)}$, and hence by $\ominus'$, $K = M$ which is a contradiction. Put $N = \bigcap_{W \in \frac{V(K)}{V(N)}} W$. Let $L \leq M$. Then $L$ is SH and $L \supseteq N$. 

Thus $\frac{V(N)}{V(K)} \leq \frac{V(K)}{V(N)}$. (1) 

Now let $L \leq \frac{V(K)}{V(N)}$. Since $N = \bigcap_{W \in \frac{V(K)}{V(N)}} W$, then $L \supseteq N$; that is $L \leq \frac{V(K)}{V(N)}$. 

Hence $\frac{V(K)}{V(N)} = \phi = \frac{V(M)}{V(K)}$ (2) 

Thus by (1) and (2), $\frac{V(N)}{V(K)} \leq \frac{V(K)}{V(N)}$ and by $\ominus'$, $K = N = \bigcap_{W \in \frac{V(K)}{V(N)}} W$, i.e. $K$ is an intersection of SH-submodules.

$(\Leftarrow)$ Let $N \leq M$. Then $N = \bigcap_{W \in \frac{V(K)}{V(N)}} W$, where $W \in \frac{V(K)}{V(N)}$. It follows that for each $I \in \Lambda$, $N_i \supseteq N$ and so that $N_i \leq \frac{V(N)}{V(K)}$. 

Thus $\bigcap_{W \in \frac{V(K)}{V(N)}} W = \bigcap_{N_i \leq \frac{V(N)}{V(K)}} L_i$. 

Assume $\frac{V(L)}{V(N)} = \frac{V(N)}{V(K)}$. It follows $N = L$. 

Recall that an R-module $M$ is called cosemisimple if every submodule of $M$ is an intersection of maximal submodules, [2].

Proposition 2.3. Let $M$ be a cosemisimple R-module. If $\text{Max}(M) \subseteq \frac{V(M)}{V(N)}$ then $M$ satisfies $\ominus'$, where $\text{Max}(M)$ is the set of all maximal submodules in $M$.

Proof: Let $N \leq M$. Since $M$ is cosemisimple, then $N = \bigcap_{W \in \frac{V(K)}{V(N)}} W$, where $W \in \text{Max}(M)$. Hence by hypothesis, $W$ is a SH-submodule, and this implies that $N$ is an intersection of SH-submodules. 

Then by Theorem 2.2, $M$ satisfies $\ominus'$. 

Remark 2.4. The condition “$\text{Max}(M) \subseteq \frac{V(M)}{V(N)}$” is necessary in Prop.2.3, as for example:

$Z_30$ as a $Z$-module does not satisfy $\ominus'$ since $\frac{V(Z_30)}{V(2)} = \phi$ but $\frac{Z_30}{V(2)} \neq Z_30$. But $Z_30$ is cosemisimple, also $\text{Max}(Z_30) \subseteq \frac{V(M)}{V(N)}$, since $\frac{Z_30}{V(2)}$ is a maximal submodule but not SH. 

It is known that every semisimple module is cosemisimple. Hence we have:

$$\frac{V(K)}{V(N)} \leq \frac{V(M)}{V(N)}$$
Corollary 2.5. Every semisimple module $M$ with $\text{Max}(M) \subseteq \text{Spec}(M)$ satisfies $\odot'$. However for a ring $R$, we have:

Theorem 2.6. Every semisimple ring $R$ satisfies $\odot'$.

Proof: Let $I, J \subseteq R$ such that $V(I) = V(J)$. Since $R$ is semisimple, $I, J$ are direct summands of $R$. Hence $I = \langle e \rangle$, $J = \langle f \rangle$ for some idempotent elements $e, f \in R$. It follows that $V(\langle e \rangle) = V(R) = \emptyset$, $V(\langle f \rangle) = V(R) = \emptyset$, hence $V(\langle e \rangle) \cap V(\langle 1 - e \rangle) = \emptyset$, $V(\langle f \rangle) \cap V(\langle 1 - f \rangle) = \emptyset$. Then $V(\langle e \rangle) \cap V(\langle 1 - e \rangle) = \emptyset$, $V(\langle f \rangle) \cap V(\langle 1 - f \rangle) = \emptyset$ (since $V(\langle e \rangle) = V(\langle f \rangle)$), which imply that $V(\langle e \rangle + \langle 1 - e \rangle) = V(R)$ and $V(\langle f \rangle + \langle 1 - f \rangle) = V(R)$. But $R$ is comultiplication, since $R$ is semisimple. Then by [8,Prop 2.5.14] $\langle e \rangle + \langle 1 - f \rangle = R$ and $\langle f \rangle + (1 - e) = R$. Now to prove $\langle e \rangle = \langle f \rangle$. Let $x \in \langle f \rangle$, then $x = cf$ for some $c \in R$ and $x = cf = r_1 e + r_2 (1 - f)$ for some $r_1, r_2 \in R$. It follows that $c^2 = cef + r_2 (1 - f)$, hence $x = cf = cef$. Thus $x \in \langle e \rangle$, and so $\langle f \rangle \subseteq \langle e \rangle$. Similary $\langle e \rangle \subseteq \langle f \rangle$. Thus $I = \langle e \rangle = \langle f \rangle = J$ and $R$ satisfies $\odot'$.

To give the next result we need the following lemmas. First compare the following Lemma with Lemma 1.9.

Lemma 2.7. Let $M$ be an $R$-module. Then $\text{Spec}(M)$ is a $T_1$-space if and only if every proper $SH$-submodule is maximal $SH$ in $\text{Spec}(M)$.

Proof: It is analogous to the proof of lemma 1.9, so is omitted.

Recall that a topological space $(X, \tau)$ is called cofinite if the only closed subsets of $X$ are finite sets or $X$. Equivalently $\tau = \{U: U \subseteq X \text{ and } X - U \text{ is a finite set}\} \cup \{\emptyset\}$.

Lemma 2.8. Let $M$ be an $R$-module. Then $\text{Spec}(M)$ is a cofinite topological space if and only if every proper $SH$-submodule is maximal $SH$ in $\text{Spec}(M)$ and for any $N \subseteq M$, either $V(N) = \text{Spec}(M)$ or $V(N)$ is a finite set.

Proof: $(\Rightarrow)$ Since $\text{Spec}(M)$ is a cofinite topological space, then $\text{Spec}(M)$ is a $T_1$-space. Hence by Lemma 2.7, every proper $SH$-submodule is a maximal $SH$-submodule. Moreover for any $N \subseteq M$ it is clear that either $V(N) = \text{Spec}(M)$ or $V(N)$ is a finite set.

$(\Leftarrow)$ For any $N \subseteq M$, $\chi(N) = \text{Spec}(M)$ or $V(N)$ is finite.

Either finite or $\emptyset$. Thus $\text{Spec}(M)$ is cofinite.

Compare the following Lemma with Lemma 1.11.

Lemma 2.9. Let $M$ be a faithful generated multiplication over a comultiplication top ring $R$, if $N \not\cong M$, then $N$ is a maximal $SH$-submodule of $M$ if and only if $N$ is a maximal submodule of $M$.

Proof: $(\Rightarrow)$ Let $N \not\cong M$ and $N$ is a maximal $SH$-submodule. Then by [8,Prop.1.2.1] there exists a $SH$-ideal $I$ of $R$ such that $N = IM$. Suppose $N$ is not a maximal submodule, so there exists a maximal submodule $W$ of $M$ such that $W \supseteq N$, since $M$ is multiplication. Also $W = JM$ for some maximal ideal $J$ of $R$, because $M$ is multiplication. But by [8,Cor.2.5.6] $J$ is a $SH$-ideal and hence by [8,Prop.1.2.1], $W$ is a $SH$-submodule. But $N$ is a maximal $SH$-submodule, so $N = W$; that is $N$ is a maximal submodule.

$(\Leftarrow)$ Let $N$ be a maximal submodule of $M$. Then $N = IM$ for some maximal ideal $I$ of $R$. Hence by [8, Cor.2.5.6], $I$ is a $SH$-ideal of $R$, hence by [8,Prop.1.2.1] $N$ is a $SH$-submodule. Thus $N$ is a maximal $SH$-submodule of $M$.

Now we can give the following theorem (compare with theorem 1.13).
Theorem 2.10. Let M be a faithful finitely generated multiplication over a comultiplication ring.

Then \( \text{Spec}(M) \) is a \( T_1 \)-space and M satisfies \( \mathcal{O}' \) if and only if M is cosemisimple and every proper SH-submodule of M is a maximal SH of M.

Proof: (\( \Rightarrow \)) By lemma 2.7, every proper SH-submodule is a maximal SH-submodule and by Lemma 2.9, every maximal SH-submodule is a maximal submodule. Thus every proper SH-submodule is a maximal submodule. Moreover by Th.2.2, every proper submodule of M is an intersection of SH-submodules. Thus every proper submodule is an intersection of maximal submodules of M; that is M is cosemisimple.

(\( \Leftarrow \)) By lemma 2.7, \( \text{Spec}(M) \) is \( T_1 \). But M is cosemiple, so every submodule is an intersection of maximal submodules. Hence by Lemma 2.9, every proper submodule is an intersection of SH-submodules. Hence by Th.2.2, M satisfies \( \mathcal{O}' \).

Compare the following result with Th.1.14.

Theorem 2.11. Let M be a finitely generated faithful multiplication \( R \)-module. Then M satisfies \( \mathcal{O}' \) if and only if \( R \) satisfies \( \mathcal{O}' \).

Proof: It is similar to the proof of Th.1.14, so is omitted.

Remark 2.12. The condition “M is faithful” is necessary in Th.2.11, as for example:

The \( Z \)-module \( Z_6 \) is a finitely generated not faithful multiplication \( Z \)-module and satisfies \( \mathcal{O}' \). But the \( Z \)-module \( Z \) does not satisfies \( \mathcal{O}' \), since \( \mathcal{V}(I) = \mathcal{V}(J) = \emptyset \) for any \( I, J \leq R \).

Corollary 2.13. Let M be a finitely generated multiplication \( R \)-module. Then the following statements are equivalent:

1. M satisfies \( \mathcal{O}' \) as \( R \)-module.
2. M satisfies \( \mathcal{O}' \) as \( R^- \)-module.
3. \( R^- \) satisfies \( \mathcal{O}' \).

where \( R^- = R/\text{ann} M \).

Recall that a proper submodule N of an \( R \)-module M is called prime if whenever \( r \in R, x \in M, rx \in N \) implies \( x \in N \) or \( r \in [N:M] \), [9].

Equivalently a proper submodule N of an \( R \)-module M is prime if for any ideal I of R and for any K \( \leq M, IK \subseteq N \) implies K \( \subseteq N \) or I \( \subseteq [N:M] \), [9].

Compare the following Lemma with Lemma 1.12.

Lemma 2.14. Let M be an \( R \)-module such that every proper SH-submodule is prime. Then for any I

\[
\mathcal{V}(N) \cup \mathcal{V}(IM) = \mathcal{V}(IM \cap N) = \mathcal{V}(IN).
\]

Proof: It is clear that \( \mathcal{V}(N) \cup \mathcal{V}(IM) \subseteq \mathcal{V}(IM \cap N) \subseteq \mathcal{V}(IN) \) . Now let \( K \in \mathcal{V}(IN) \). Then K is a SH-submodule and \( K \subseteq IN \). But by hypothesis K is prime, so either \( N \subseteq K \) or \( I \subseteq [K : M] \); that is either \( K \in \mathcal{V}(N) \) or \( K \in \mathcal{V}(IM) \). Thus \( K \in (\mathcal{V}(N) \cup \mathcal{V}(IM)) \). Hence the result is obtained.

Compare the following Theorem with Th.1.19.

Theorem 2.15. Let M be an \( R \)-module such that M satisfies \( \mathcal{O}' \) and every proper SH-submodule is prime. Then M is F-regular (i.e. every submodule of M is pure; that is \( IM \cap N = IN, \forall I \leq R \)).

Proof: It follows by Lemma 2.14 and definition of modules with the condition \( \mathcal{O}' \).

Proposition 2.16. Let M be a F-regular \( \text{top} \)-module, then every proper SH-submodule is prime.
Proof: Since $M$ is a $F$-regular, then for each $N \leq M$, $IM \cap N = IN$ for each $N \leq M$. Hence $\forall (M \cap N) = \forall (N)$. But $M$ is a top module, so

$$\forall (IM) \cup \forall (N) = \forall (IM \cap N)$$

Thus $\forall M \cap N = \forall N$ for each $N \leq M$. Hence $K \in V(IM) \cup V(N)$

Hence either $K \in V(IM)$ or $K \in V(N)$ Thus $K \subseteq IM$ or $K \subseteq N$; that is either $I \subseteq \frac{K}{M}$ or $N \subseteq K$. Therefore $K$ is prime.

**Proposition 2.17.** Let $M$ be an $R$-module, if $M$ satisfies $\odot'$. Then

1. If $\text{Spec}(M)$ satisfies d.c.c (a.c.c) on closed sets, then $M$ is Noetheian (Artinian).
2. If $\text{Spec}(M)$ satisfies a.c.c (d.c.c) on open sets, then $M$ is Noetheian (Artinian).

**Proof:** It is easy so is omitted.

**S.3 Modules with the Condition $\odot''$**

In this section, we introduce modules that satisfy $\odot''$, where $\odot'' :$ for each $N, L \leq M$, $\forall (L) = \forall (N)$ implies $N = L$.

Many results about these modules are similar to that of module with $\odot$ condition. Also we give some relations modules with $\odot$ and modules with condition $\odot''$.

**Remark and Examples 3.1**

1. Every simple module $M$ does not satisfy $\odot''$, since $\forall (M) = \forall (<0>)$ but $M \neq <0>$.
2. The $\mathbb{Z}$-module $\mathbb{Z}_4$ does not satisfy $\odot''$, since $\forall (<\frac{3}{2}>) = \forall (\mathbb{Z}_4) = \{<\frac{3}{2}>\}$ but $\mathbb{Z}_4 \neq <\frac{3}{2}>$.
3. The $\mathbb{Z}$-module $\mathbb{Z}_6$ satisfies $\odot''$.

The following theorem is similar to Th.1.4

**Theorem 3.2.** Let $M$ be an nonzero $R$-module. Then $M$ satisfies $\odot''$ if and only if every proper nonzero proper submodule can be represented as sum of SI-submodules

Compare the following Lemma with Lemma 1.9.

**Lemma 3.3.** Let $M$ be an $R$-module. Then $\mathfrak{sl}(\text{Spec}(M))$ is $T_1$-space if and only if every SI-submodule is a minimal SI-submodule in $\text{Spec}(M)$

**Proof:** It is similar the proof of Lemma 1.9.

The following theorem is similar to Th.1.10.

**Theorem 3.4** Let $M$ be an $R$-module. Then $\mathfrak{sl}(\text{Spec}(M))$ is $T_1$ and $M$ satisfies $\odot''$ if and only if $M$ is semisimple and every SI-submodule is minimal SI-submodule of $M$.

The following Lemma is similar to Lemma 1.11.

**Lemma 3.5**. Let $M$ be a faithful finitely generated multiplication over comultiplication ring $R$, let $N \leq M$. If $N$ is a minimal SI-submodule, then $N$ is simple.

**Note 3.6.** The converse of Lemma 3.5 is true if $R$ is top ring.

Compare the following theorem with Th.1.13.

**Theorem 3.7.** Let $M$ be a finitely generated faithful multiplication over comultiplication ring $R$. Then $M$ semisimple and every SI-submodule is a minimal SI-submodule if and only if $\mathfrak{sl}(\text{Spec}(M))$ is a $T_1$-space and $M$ satisfies $\odot''$. 
Proof: It follows by Th.3.4, Lemma 3.5 and note 3.6. The following result is similar to Th.1.14

Theorem 3.8. Let M be a finitely generated faithful multiplication R-module. Then M satisfies $\odot^\prime$ if and only if R satisfies $\odot^\prime\prime$.

Next we give some relationships modules the condition $\odot$, and modules with condition $\odot^\prime\prime$.

Proposition 3.9. If M is a $25\text{SH}\text{top}$-module and M satisfies $\odot$ then M satisfies $\odot^\prime\prime$.

Proof: Since M satisfies $\odot$, then by Th.1.4 every nonzero submodule of M is a sum of SH-submodule. But M is a $25\text{SH}\text{top}$-module, so by [8,Prop.2.4.4] every SH-submodule is a SI-submodule. Thus every nonzero submodule is a sum of SI-submodule. Thus by Th.3.2, M satisfies $\odot^\prime\prime$.

Proposition 3.10. Let M be a $25\text{SH}\text{top}$-module and M satisfies $\odot^\prime\prime$ then M satisfies $\odot$.

Proof: Since M satisfies $\odot^\prime\prime$, every nonzero submodule of M is a sum of SI-submodule (by Th.3.2). But M is a $25\text{SH}\text{top}$, so by [8,Prop.3.2.3], every SI-submodule is a SH-submodule. Thus every submodule of M is a sum of SH-submodule. Thus M satisfies $\odot$ by Th.1.4.

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