Efficient Algorithm for Perturbative Calculation of Multiloop Feynman Integrals

Boris Kastening\textsuperscript{1}\textsuperscript{*} and Hagen Kleinert\textsuperscript{2}\textsuperscript{†}

\textsuperscript{1}Institut f"ur Theoretische Physik, Universit"at Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
\textsuperscript{2}Institut f"ur Theoretische Physik, Freie Universit"at Berlin, Arnimallee 14, 14195 Berlin, Germany

We present an efficient algorithm for calculating multiloop Feynman integrals perturbatively.

1. Recently, a new method has been proposed to calculate Feynman integrals of multiloop diagrams perturbatively \cite{1}. Together with the solution procedure for graphical recursion relations developed in Ref. \cite{2}, this should ultimately lead to the completely automatized computer generation of perturbation expansions of field theories up to high orders. Such expansions are needed in all strongly coupled fluctuating field systems, for example those describing the critical phenomena close to second-order phase transitions \cite{3}. So far, expansions have been limited to seven loops only \cite{4,5}, which are barely sufficient to yield critical exponents \cite{6} with an accuracy comparable to experimental data \cite{7}.

In this note, we would like to show how the expansions proposed in \cite{1} can be performed most efficiently, such that they can be carried out on a computer to high orders in a limited computer time.

2. A basic Feynman integral with \(L\) loops, \(n\) internal momenta, and \(E\) external momenta \(k_1, \ldots, k_E\) has the form

\[
I_{a_k}^D = \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_L}{(2\pi)^D} \prod_{k=1}^{n} \frac{1}{(1 + \kappa q_k^2)^a_k}, \quad n \geq L, \tag{1}
\]

with some powers \(a_k\), where \(q_k\) are the momenta carried by the lines, and the integrations run over all loop momenta \(p_i\). The line momenta \(q_k\) are linear combinations of the loop momenta \(p_i\) and the external momenta \(k_j\).

For simplicity, we have set all masses equal to unity. A Feynman integral with different non-zero masses can be reduced to (1) by an appropriate rescaling of the line momenta \(q_k\). Following Ref. \cite{1}, we view the integral (1) as a special case \(I_{a_k}^D = I_{a_k}^D(1)\) of the function

\[
I_{a_k}(\kappa) = \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_L}{(2\pi)^D} \prod_{k=1}^{n} \frac{e^{a_k(\kappa-1)}q_k^2}{(1 + \kappa q_k^2)^a_k}, \tag{2}
\]

to be calculated perturbatively via a Taylor series expansions in powers of \(\kappa\).

It is the purpose of this note to point out that the simplest way to derive such an expansion is by rewriting each generalized propagator in the Schwinger parametric form

\[
e^{a(\kappa-1)q^2/(1 + \kappa q^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty dt t^{a-1} e^{-t|a(\kappa-1)-\kappa|q^2}. \tag{3}
\]

Then the integrals (2) take the form

\[
I_{a_k}(\kappa) = \frac{1}{\Gamma(a_1)} \cdots \frac{1}{\Gamma(a_L)} \int_0^\infty dt_1 t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_n t_n^{a_n-1} e^{-t_n} \times \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_L}{(2\pi)^D} \exp \left\{ - \sum_{k=1}^{n} [a_k(1-\kappa) + \kappa t_k] q_k^2 \right\}, \tag{4}
\]

Collecting the \(L\) loop momenta \(p_i\) and the \(E\) external momenta \(k_j\) in single vector symbols

\[
p = (p_1, \ldots, p_L), \quad k = (k_1, \ldots, k_E), \tag{5,6}
\]

we rewrite

\[\text{Email: boris@thphys.uni-heidelberg.de}\]
\[\text{Email: kleinert@physik.fu-berlin.de, URL: http://www.physik.fu-berlin.de/~kleinert}\]
and complete the squares to

\[ \sum_{k=1}^{n} [a_k(1-\kappa) + \kappa t_k] q_k^2 = \frac{1}{2} p^T M p + p^T M' k + \frac{1}{2} k^T M'' k \]

and complete the squares to

\[ \sum_{k=1}^{n} [a_k(1-\kappa) + \kappa t_k] q_k^2 = \frac{1}{2} (p + M^{-1} M')^T M (p + M^{-1} M') + \frac{1}{2} k^T \left( M'' - M'^T M' \right) k, \]

with symmetric matrices \( M \) and \( M'' \). After a shift \( p \rightarrow p - M^{-1} M' k \) of integration variables, the \( p_k \) integrations become Gaussian, and we obtain

\[
I^D_{a_k}(\kappa) = \frac{1}{\Gamma(a_1) \cdots \Gamma(a_L)} \int_0^\infty dt_1 t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_L t_L^{a_L-1} e^{-t_L} \left( \frac{1}{(2\pi)^D} \int \frac{d^D p}{(2\pi)^D} e^{-\frac{1}{2} k^T (M'' - M'^T M') k} \right) \]

where the matrices \( M, M', \) and \( M'' \) depend on \( \kappa \) and the \( t_k \) through linear combinations

\[ c_k(\kappa, t_k) \equiv a_k(1-\kappa) + \kappa t_k. \]

Although the entries of the matrix \( M \) depend on the routing of the loop momenta through the different lines, the determinant of \( M \) is invariant under changes of the routing, except for trivial relabelings of the \( a_k \).

In order to derive the desired expansion of \( I^D_{a_k}(\kappa) \) in powers of \( \kappa \), we expand the integrand on the right hand side of (1) in powers of \( \kappa \), whose coefficients are polynomials in the parameters \( t_i, (i = 1, \ldots, L) \). The \( t_i \)-integrals can then all be performed using the formula

\[ \int_0^\infty dt e^{-t} = \Gamma(\gamma + 1). \]

For diagrams without external momenta, appearing in the perturbation expansions for the ground state of quantum field theories, (1) simplifies to

\[
I^D_{a_k}(\kappa) = \frac{(2\pi)^{-LD/2}}{\Gamma(a_1) \cdots \Gamma(a_L)} \int_0^\infty dt_1 t_1^{a_1-1} e^{-t_1} \cdots \int_0^\infty dt_L t_L^{a_L-1} e^{-t_L} \left( \frac{1}{(\det M)^{D/2}} \right). \]

More general Feynman integrals than those in Eq. (1) may contain loop momenta \( p_k \) in the numerator of the integrand. These can be calculated with a simple extension of the above technique, by introducing “source terms” \( \Sigma_{i=1}^L \cdot p_i \) into the exponents of (2) and (3), and appropriately differentiate the resulting \( \kappa \)-expansion with respect to \( \kappa \), which are set equal to zero at the end.

3. As a first example, take the exactly solvable one-loop integral

\[
I^D_{a}(\kappa) = \int \frac{d^D p}{(2\pi)^D} \frac{1}{(1 + \kappa p^2)^a} = \frac{\Gamma(a - D/2)}{(4\pi)^{D/2} \Gamma(a)}. \]

Its \( \kappa \)-generalized version can be expressed in terms of a confluent hypergeometric function,

\[
I^D_{a}(\kappa) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{a(\kappa - 1)p^2}}{(1 + \kappa p^2)^a} = \left( \frac{\Gamma(D/2)}{(4\pi \kappa)^{D/2}} \right) \Psi\left( \frac{D}{2}, 1 + \frac{D}{2} - a; \frac{a(1 - \kappa)}{\kappa} \right) - a - \frac{a(1 - \kappa)}{\kappa} \]

\[
= \frac{1}{(4\pi \kappa)^{D/2}} \left[ \frac{\Gamma(a - D/2)}{\Gamma(a)} \right] 1_F^1 \left( \frac{D}{2}, 1 + \frac{D}{2} - a; \frac{a(1 - \kappa)}{\kappa} \right) + \frac{\Gamma(D/2 - a)}{\Gamma(D/2)} (1 - \kappa)^{a-D/2} \right] 1_F^1 \left( a, 1 + a - \frac{D}{2} ; \frac{a(1 - \kappa)}{\kappa} \right) \]

with

\[ 1_F^1(a; \beta; z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(\beta)_k k!}, \quad (a)_s \equiv \frac{\Gamma(a + s)}{\Gamma(a)} = \prod_{r=0}^{s-1} (a + r) \] (Pochhammer's symbol).
In Ref. [1] this was calculated perturbatively via a Wick expansion. Here we use our general formula (12) for vacuum integrals. The number of loops is $L = 1$, and we identify

$$q_1 = p, \quad a_1 = a, \quad c_1 = a(1 - \kappa) + \kappa t, \quad M = 2(c_1), \quad \text{det} M = 2c_1.$$ (16)

Expanding $(\text{det} M)^{-D/2}$ in powers of $\kappa$, and performing the resulting integrals over $t$ in Eq. (12), we find directly the perturbation expansion for the loop integrals (13) in any dimension $D$:

$$I^D_{a_k}(\kappa) = \frac{1}{(4\pi a)^{D/2}} \left[ 1 + \frac{D (2 + D) \kappa^2}{8a} - \frac{D (2 + D) (4 + D) \kappa^3}{24a^2} + \frac{(2 + a) D (2 + D) (4 + D) (6 + D) \kappa^4}{128a^3} \right. + \left. \frac{(6 + 5a) D (2 + D) (4 + D) (6 + D) (8 + D) \kappa^5}{960a^4} \right] + \mathcal{O}(\kappa^6).$$ (17)

The expansion can easily extended any desired order. It agrees, of course, with what we would obtain from the exact expression (14) via a large-argument expansion of the confluent hypergeometric function.

4. As a nontrivial example, take the integral of the watermelon diagram treated in [1] only in $D = 2$ dimensions:

$$I^D = \int \frac{d^D p_1}{(2\pi)^D} \frac{d^D p_2}{(2\pi)^D} \frac{d^D p_3}{(2\pi)^D} \frac{1}{1 + p_1^2} \frac{1}{1 + p_2^2} \frac{1}{1 + (p_1 + p_2 + p_3)^2}.$$ (18)

This integral has the powers

$$a_1 = a_2 = a_3 = a_4 = 1,$$ (19)

and we identify the line momenta as

$$q_1 = p_1, \quad q_2 = p_2, \quad q_3 = p_3, \quad q_4 = p_1 + p_2 + p_3.$$ (20)

such that the matrix $M$ is

$$M = 2 \begin{pmatrix} a_1 + a_4 & a_4 & a_4 & a_4 \\ a_4 & a_2 + a_4 & a_4 & a_4 \\ a_4 & a_4 & a_3 + a_4 & a_4 \\ a_4 & a_4 & a_4 & a_4 \end{pmatrix},$$ (21)

$$\text{det} M = 8(a_1 a_2 a_3 + a_1 a_2 a_4 + a_1 a_3 a_4 + a_2 a_3 a_4).$$ (22)

For the function $I^D_{a_k}(\kappa)$, we then obtain in any dimension $D$

$$I^D(\kappa) = \frac{1}{2^{4D} \pi^{3D/2}} \times \left[ 1 + \frac{9D (2 + D) \kappa^2}{32} - \frac{9D (2 + D) (4 + D) \kappa^3}{128} + \frac{3D (2 + D) (1048 + 522D + 81D^2) \kappa^4}{4096} \right. + \left. \frac{9D (2 + D) (4 + D) (2576 + 918D + 117D^2) \kappa^5}{40960} \right. + \left. \frac{D (2 + D) (564864 + 397744D + 110916D^2 + 15228D^3 + 891D^4) \kappa^6}{65536} \right. + \left. \frac{3D (2 + D) (4 + D) (29651840 + 15696528D + 3452148D^2 + 391068D^3 + 19683D^4) \kappa^7}{9175040} \right. + \left. \frac{3D (2 + D) (4 + D) (1419854080 + 843338336D + 212508840D^2 + 29562300D^3 + 2344950D^4 + 85779D^5) \kappa^8}{83886080} \right] + \mathcal{O}(\kappa^9).$$ (23)

For $D = 1$ this reduces to
\[
I^1(\kappa) = \frac{1}{2^{43/2}} \left[ 1 + \frac{27\kappa^2}{32} - \frac{135\kappa^3}{128} + \frac{1485\kappa^4}{4096} - \frac{97497\kappa^5}{8192} + \frac{3268929\kappa^6}{65536} - \frac{63271629\kappa^7}{262144} + \frac{22569248565\kappa^8}{16777216} + \mathcal{O}(\kappa^9) \right],
\]

and for \( D = 2 \) to

\[
I^2(\kappa) = \frac{1}{2^8\pi^3} \left[ 1 + \frac{9\kappa^2}{4} - \frac{27\kappa^3}{32} + \frac{453\kappa^4}{32} - \frac{1647\kappa^5}{64} + \frac{15157\kappa^6}{128} - \frac{157293\kappa^7}{512} + \frac{3720699\kappa^8}{16777216} + \mathcal{O}(\kappa^9) \right],
\]

thus extending easily the expansions in [1].

For \( D = 3 \), the expansion reads

\[
I^3(\kappa) = \frac{1}{2^{12}\pi^{9/2}} \left[ 1 + \frac{135\kappa^2}{32} - \frac{945\kappa^3}{128} + \frac{150435\kappa^4}{4096} - \frac{1206387\kappa^5}{8192} + \frac{48595005\kappa^6}{65536} - \frac{1079675235\kappa^7}{262144} + \frac{432899207685\kappa^8}{16777216} + \mathcal{O}(\kappa^9) \right].
\]

5. Having developed the tools for finding perturbation expansions of Feynman integrals, it remains to study the large-order behavior, and to find suitable methods for the resummation of the expansions with high accuracy. Together with the automatized generation of the Feynman diagrams of Ref. [2], this will open the way for an “industrial production” of high-loop expansions for critical exponents.

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