THE EASTWOOD–SINGER GAUGE IN EINSTEIN SPACES

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Abstract

Electrodynamics in curved space-time can be studied in the Eastwood–Singer gauge, which has the advantage of respecting the invariance under conformal rescalings of the Maxwell equations. Such a construction is here studied in Einstein spaces, for which the Ricci tensor is proportional to the metric. The classical field equations for the potential are then equivalent to first solving a scalar wave equation with cosmological constant, and then solving a vector wave equation where the inhomogeneous term is obtained from the gradient of the solution of the scalar wave equation. The Eastwood–Singer condition leads to a field equation on the potential which is preserved under gauge transformations provided that the scalar function therein obeys a fourth-order equation where the highest-order term is the wave operator composed with itself. The second-order scalar equation is here solved in de Sitter space-time, and also the fourth-order equation in a particular case, and these solutions are found to admit an exponential decay at large time provided that square-integrability for positive time is required. Last, the vector wave equation in the Eastwood–Singer gauge is solved explicitly when the potential is taken to depend only on the time variable.
I. INTRODUCTION

Even before attempting a functional-integral quantization of gauge theories, the analysis of hyperbolic classical field equations supplemented by gauge-fixing conditions is quite important. For example, in the case of Maxwell’s electrodynamics, the action functional
\[-\frac{1}{4} \int F_{ab} F^{ab} \sqrt{-g} d^4x\]
leads to the field equation
\[P_a^b A_b = 0, \quad (1.1)\]
having defined the operator
\[P_a^b \equiv -\delta_a^b \Box + R_a^b + \nabla_a \nabla^b, \quad (1.2)\]
where \(\nabla\) is the Levi–Civita connection with associated Ricci tensor \(R_a^b\). Equation (1.1) should be supplemented by a gauge-fixing condition \([1]\), i.e. an equation having general form
\[\Phi(A) = 0, \quad (1.3)\]
\(\Phi\) being a functional defined on the space of gauge connection one-forms \(A_b dx^b\). In particular, the Lorenz-type choice \([2]\)
\[\Phi(A) = \Phi_L(A) \equiv \nabla^b A_b, \quad (1.4)\]
reduces Eq. (1.1) to
\[(-\delta_a^b \Box + R_a^b) A_b = 0, \quad (1.5)\]
which is indeed very convenient, being the covariant vector wave equation in curved space-time. However, the Lorenz functional has an inhomogeneous transformation law under conformal rescalings \(\hat{g}_{ab} = \Omega^2 g_{ab}\) of the space-time metric, i.e. \([3]\)
\[\hat{\nabla}^b A_b = \Omega^{-2} \left( \nabla^b A_b + 2 Y^b A_b \right), \quad (1.6)\]
having denoted by \(Y_a\) the logarithmic derivative of the conformal factor, i.e. \(Y_a \equiv \nabla_a \log \Omega = \nabla_a \Omega / \Omega\). There is nothing bad or undesirable in this property, but relatively few people know that, in Ref. \([3]\), a solution was found to the problem of achieving both field equations and gauge-fixing condition of the conformally invariant type. For this purpose, the idea is to add lower order curvature terms to a differential operator which, by itself, is not conformally invariant. In the electromagnetic case, one therefore considers the tensor
\[S^{ab} \equiv -2 R^{ab} + \frac{2}{3} R g^{ab}, \quad (1.7)\]
and the third-order differential operator 3

\[ D^a \equiv \nabla_b \left( \nabla^b \nabla^a + S^{ab} \right) \]  

(1.8)

acting on one-forms. From the point of view of spectral geometry and fiber-bundle theory, the analysis of these higher-order operators is indeed well motivated and rather natural 4, 5. The gauge-fixing condition imposed is

\[ D^b A_b = 0, \]  

(1.9)

and it is conformally invariant because 3

\[ \hat{D}^b A_b = \Omega^{-4} \left[ D^b A_b + 2 Y^b \nabla^a (\nabla_a A_b - \nabla_b A_a) \right], \]  

(1.10)

where \( \nabla^a (\nabla_a A_b - \nabla_b A_a) = \nabla^a F_{ab} = 0 \) by virtue of the vacuum Maxwell equations imposed upon the potential \( A_b \). The authors of Ref. 3 focused on background metrics which satisfy the vacuum Einstein equations. In all dimensions greater than two, this condition is equivalent to Ricci flatness, which in turn implies the vanishing of the tensor \( S^{ab} \) in (1.7), while the operator in (1.8) reduces then to \( D^a = \square \nabla^a \), and the gauge-fixing (1.9) is implied 3 by the Lorenz condition \( \nabla^b A_b = 0 \).

Section 2 studies the Eastwood–Singer gauge in Einstein spaces, while its behavior under gauge transformations of the Maxwell potential is considered in section 3. Sections 4 and 5 solve in de Sitter space-time, under suitable assumptions, the associated second-order and fourth-order scalar equations, respectively, while section 6 solves the vector wave equation in the Eastwood–Singer gauge when the potential depends only on the time variable. Concluding remarks are presented in section 7. Our analysis is entirely classical, with space-time metric of Lorentzian signature.

II. EASTWOOD–SINGER GAUGE IN EINSTEIN SPACES

The restriction to Ricci-flat space-times, however, is not mandatory, and another interesting class of background space-times are the Einstein spaces, for which the Ricci tensor is proportional to the metric, i.e. \( R_{ab} = \Lambda g_{ab} \). They include the de Sitter and anti-de Sitter geometries. One then finds, in four space-time dimensions,

\[ S^{ab} = \frac{2}{3} \Lambda g^{ab}. \]  

(2.1)
This tensor is covariantly constant by virtue of the metric-compatible condition $\nabla g = 0$, so that the Leibniz rule and the Eastwood–Singer gauge (1.9) yield eventually

$$\Box \nabla^b A_b + S^{ab} \nabla_b A_a = 0 \Rightarrow \left( \Box + \frac{2}{3} \Lambda \right) \nabla^b A_b = 0. \quad (2.2)$$

Moreover, the field equation (1.1) takes the form

$$\left( - \Box + \Lambda \right) A_a = \nabla_a (-\nabla^b A_b). \quad (2.3)$$

To sum up, if we define $\psi \equiv \text{div} A = \nabla^b A_b$, we first have to solve a scalar wave equation with cosmological constant, i.e.

$$\left( \Box + \frac{2}{3} \Lambda \right) \psi = 0. \quad (2.4)$$

The gradient of $-\psi$ provides the inhomogeneous term in the wave equation for the Maxwell potential, i.e.

$$\left( - \Box + \Lambda \right) A_a = \nabla_a (-\psi). \quad (2.5)$$

The solutions of (2.5) can be split as

$$A_a = A_a^{(0)} + \tilde{A}_a, \quad (2.6)$$

where $A_a^{(0)}$ obeys the homogeneous wave equation in Einstein spaces, i.e.

$$\left( - \Box + \Lambda \right) A_a^{(0)} = 0, \quad (2.7)$$

while $\tilde{A}_a$ is a particular solution of Eq. (2.5). As a consistency check, the sum (2.6) should fulfill Eq. (2.5).

**III. BEHAVIOR OF THE EASTWOOD–SINGER CONDITION UNDER GAUGE TRANSFORMATIONS**

In classical gauge theory with space-time metric of Lorentzian signature, the gauge-fixing (also called “supplementary”) condition leads to a convenient form of the field equation for the potential. For example, for classical electrodynamics in the Lorenz gauge (1.4), the wave equation reduces to Eq. (1.5) as we said. However, while the Maxwell Lagrangian $-\frac{1}{4} F_{ab} F^{ab}$ is invariant under gauge transformations

$$^f A_b \equiv A_b + \nabla_b f, \quad (3.1)$$
where \( f \) is a freely specifiable function of class \( C^1 \), the Lorenz gauge (1.4) (as well as any other admissible gauge) is not invariant under (3.1). Nevertheless, to achieve the desired wave equation on \( A_b \), it is rather important to make sure that both \( A_b \) and the gauge-transformed potential \( f A_b \) obey the same gauge-fixing condition, i.e.

\[
\Phi(A) = 0, \quad \Phi(fA) = 0. \tag{3.2}
\]

A more general situation, here not considered for simplicity, is instead the case when only the gauge-transformed potential obeys the gauge-fixing condition, i.e.

\[
\Phi(A) \neq 0, \quad \Phi(fA) = 0. \tag{3.3}
\]

From the point of view of constrained Hamiltonian systems, the gauge-fixing equations can be viewed as constraint equations to be preserved following Dirac’s method [7]. This turns a theory with first-class constraints into a second-class set [7].

The counterpart of (3.2) for pure gravity is the well known problem of imposing the de Donder gauge on metric perturbations, and then requiring its invariance under infinitesimal diffeomorphisms. One then finds [8] that the covector occurring in the transformation of metric perturbations under infinitesimal diffeomorphisms should obey a vector wave equation (but with opposite sign of the Ricci tensor with respect to Eq. (2.5)).

On imposing the equations (3.2), one arrives at a proper subset of the original set of gauge transformations (3.1), where \( f \) is no longer freely specifiable, but obeys a differential equation whose form depends on the choice of gauge-fixing functional \( \Phi(A) \). In general, \( \Phi(A) \) changes under gauge transformations in a way encoded in a differential operator of first, second or higher order.

In our paper, by virtue of (1.9), this implies that the function \( f \) should obey the fourth-order equation

\[
D^b \nabla_b f = 0, \tag{3.4}
\]

which, in Einstein spaces, reads as

\[
\left( \Box^2 + \frac{2}{3} \Lambda \Box \right) f = 0. \tag{3.5}
\]

Since the fourth-order operator in Eq. (3.5) is obtained from the composition

\[
\Box \cdot \left( \Box + \frac{2}{3} \Lambda \right),
\]
the general solution can be decomposed as

\[ f = f_0 + f_1, \quad (3.6) \]

where \( f_0 \) belongs to the kernel of the scalar wave operator \( \Box + \frac{2}{3} \Lambda \) occurring in Eq. (2.4), while \( f_1 \) is a particular solution of Eq. (3.5).

We might also exploit a Hodge decomposition of the potential according to

\[ A_b = A_{\perp b} + \nabla_b \phi, \quad (3.7) \]

where \( A_{\perp b} \) is the transverse part obeying

\[ \nabla^b A_{\perp b} = 0, \]

while \( \psi \equiv \nabla^b A_b = \Box \phi \). This would lead to the equations

\[ (-\Box + \Lambda) A_{\perp b} = 0, \quad \left( \Box + \frac{2}{3} \Lambda \right) \psi = 0, \]

\[ \phi = \Box^{-1} \psi, \quad A_b = A_{\perp b} + \nabla_b (\Box^{-1} \psi), \]

which however are not substantially easier to solve, since our space-time is de Sitter rather than Minkowski. We thus go on with fourth-order equations hereafter. Note also that, if we were studying the quantum theory via Euclidean functional integrals, our higher-order differential operators would be elliptic operators whose one-loop contribution is obtained from their functional determinant. We are instead studying the gauge-fixed equations of the classical theory, e.g. Eqs. (1.5) or (2.3), which are hyperbolic equations. Moreover, it would be misleading to consider the analogy between Eq. (3.4) and the equation for zero-modes of the ghost operator of the quantum theory, since Eq. (3.4) is a classical hyperbolic equation, and no consideration of eigenfunctions belonging to zero-eigenvalues is involved therein.

IV. THE SCALAR WAVE EQUATION IN DE SITTER SPACE-TIME

The equations (2.4), (2.5) and (3.5) cannot be solved without choosing a particular Einstein space. For this purpose, we focus on four-dimensional de Sitter space-time in the expanding-universe coordinates, in which the line element reads as

\[ ds^2 = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2), \quad (4.1) \]
where the coordinates \((t, x, y, z)\) can take all values from \(-\infty\) to \(+\infty\). This implies that

\[
\Box \psi = \left( -\frac{\partial^2}{\partial t^2} - 3H \frac{\partial}{\partial t} + e^{-2Ht} \Delta \right) \psi, \tag{4.2}
\]

with \(\Delta\) equal to minus the flat Laplacian in Cartesian coordinates, i.e. (our sign convention leads to a positive-definite leading symbol for the Laplacian)

\[
\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{4.3}
\]

The scalar wave equation (2.4) can be now solved by the factorization ansatz

\[
\psi = A(x, y, z)\chi(t), \tag{4.4}
\]

with the functions \(A\) and \(\chi\) satisfying the equations

\[
e^{2Ht} \left( \frac{\ddot{\chi}}{\chi} + 3H \frac{\dot{\chi}}{\chi} - \frac{2}{3} \Lambda \right) = \left( \frac{\Delta A}{A} \right) = k, \tag{4.5}
\]

having denoted by \(k\) a positive constant. Interestingly, the second-order ordinary differential equation for \(\chi\) can be solved exactly, bearing in mind that \(\Lambda = 3H^2\), in the form (up to a multiplicative parameter depending on \(H\) and \(k\))

\[
\chi(t) = e^{-\frac{2}{3}Ht} \left[ C_1 \Gamma \left( 1 - \frac{\sqrt{17}}{2} \right) I_{-\frac{\sqrt{17}}{2}} \left( \frac{\sqrt{k}}{H} e^{-Ht} \right) + i\sqrt{\frac{\sqrt{17}}{2}} C_2 \Gamma \left( 1 + \frac{\sqrt{17}}{2} \right) I_{\frac{\sqrt{17}}{2}} \left( \frac{\sqrt{k}}{H} e^{-Ht} \right) \right], \tag{4.6}
\]

where \(C_1\) and \(C_2\) are constants and \(I_\nu\) is the modified Bessel function of first kind and order \(\nu\). The occurrence of both \(I_{-\frac{\sqrt{17}}{2}}\) and \(I_{\frac{\sqrt{17}}{2}}\) in (4.6) means that the general solution contains infinitely many terms which decay exponentially at large \(t\) as well as infinitely many terms which blow up exponentially at large \(t\). The requirement of square-integrable solutions as \(t\) belongs to the positive half-line picks out \(I_{-\frac{\sqrt{17}}{2}}\left( \frac{\sqrt{k}}{H} e^{-Ht} \right)\) in (4.6), and enforces the choice \(C_1 = 0\). Moreover, the function \(A\) occurring in (4.4) can be expressed in the form

\[
A(x, y, z) = \alpha(x)\beta(y)\gamma(z), \tag{4.7}
\]

with \(\alpha, \beta, \gamma\) obeying the second-order equations

\[
\left[ \frac{d^2}{dx^2} - \lambda_1 \right] \alpha = 0, \quad \left[ \frac{d^2}{dy^2} - \lambda_2 \right] \beta = 0, \quad \left[ \frac{d^2}{dz^2} - \lambda_3 \right] \gamma = 0, \tag{4.8}
\]
subject to
\[ \lambda_1 + \lambda_2 + \lambda_3 = k. \]  
(4.9)

The various choices of sign for \( \lambda_1, \lambda_2, \lambda_3 \) give rise to eight different combinations, which should all satisfy the constraint (4.10). The solutions of Eqs. (4.9) will be oscillatory for negative \( \lambda_i \) and exponentially growing for positive \( \lambda_i \), i.e.

\[ \alpha(x) = D_1 \cos(\sqrt{|\lambda_1|} x) + D_2 \sin(\sqrt{|\lambda_1|} x), \]  
(4.10)

or

\[ \alpha(x) = \tilde{D}_1 \cosh(\sqrt{\lambda_1} x) + \tilde{D}_2 \sinh(\sqrt{\lambda_1} x), \]  
(4.11)

and entirely analogous formulae for \( \beta(y) \) and \( \gamma(z) \).

V. SOLUTION OF THE FOURTH-ORDER SCALAR EQUATION

As far as the fourth-order scalar wave equation (3.5) is concerned, its form in four-dimensional de Sitter space-time with metric (4.1) reads as (bearing again in mind that \( H^2 = \frac{\Lambda}{3} \))

\[
\left[ \frac{\partial^4}{\partial t^4} + 6H \frac{\partial^3}{\partial t^3} + (7H^2 - 2e^{-2Ht} \triangle) \frac{\partial^2}{\partial t^2} - 2H(e^{-2Ht} \triangle + 3H^2) \frac{\partial}{\partial t} + 4H^2 e^{-2Ht} \triangle + e^{-4Ht} \triangle^2 \right] f = 0. \]  
(5.1)

With the notation of Eq. (3.4), the hard part of the analysis consists in finding \( f_1 \), for which a factorized ansatz is not as helpful as for Eq. (2.4). However, we notice that Eq. (5.1) admits a particular exact solution depending on \( t \) only, because it then reduces to

\[
\left[ \frac{d^4}{dt^4} + 6H \frac{d^3}{dt^3} + 7H^2 \frac{d^2}{dt^2} - 6H^3 \frac{d}{dt} \right] f = 0. \]  
(5.2)

This means we are here dealing with a proper subset of the general set of scalar functions occurring in the gauge freedom described by Eq. (3.1), by restricting \( f \) to depend on \( t \) only. The constant-coefficient equation (5.2) is solved by the exponential \( e^{\alpha t} \), where the real parameter \( \alpha \) solves the algebraic equation

\[ \alpha(\alpha^3 + 6H \alpha^2 + 7H^2 \alpha - 6H^3) = 0, \]  
(5.3)
whose four roots are

\[ \alpha_1 = 0, \quad \alpha_2 = -3H, \quad \alpha_3 = \frac{1}{2}(3 + \sqrt{17})H, \quad \alpha_4 = \frac{1}{2}(3 - \sqrt{17})H. \]  

Equation (5.2) is therefore solved by

\[ f(t) = \tilde{C}_1 + \tilde{C}_2 e^{-3Ht} + \tilde{C}_3 e^{-\frac{1}{2}(3+\sqrt{17})Ht} + \tilde{C}_4 e^{-\frac{1}{2}(3-\sqrt{17})Ht}. \]  

In this case, the requirement that \( f \) should be square-integrable for \( t \) lying on the positive half-line enforces the choice \( \tilde{C}_1 = \tilde{C}_4 = 0 \). When we allow for a non-vanishing spatial gradient of \( f \), we may expect infinitely many terms which decay exponentially as \( t \to \infty \) and are weighted by square-integrable functions of \( (x, y, z) \). Their evaluation goes beyond our present capabilities.

Indeed, Eq. (3.5) might be split into the pair of second-order equations

\[ \left( \Box + \frac{2}{3} \Lambda \right) f = v, \]  
\[ \Box v = 0, \]  

but these are second-order equations with variable coefficients, whose analysis is not necessarily more powerful than the single fourth-order equation (5.1).

VI. THE VECTOR WAVE EQUATION (2.5) IN DE SITTER SPACE-TIME

When the line element (4.1) is exploited, the wave operator has the following simple action on the temporal and spatial components of the electromagnetic potential:

\[ \Box A_t = \left[ -\frac{\partial^2}{\partial t^2} - 3H \frac{\partial}{\partial t} + e^{-2Ht} \Delta \right] A_t, \]  
\[ \Box A_k = \left[ -\frac{\partial^2}{\partial t^2} - H \frac{\partial}{\partial t} + 2H^2 + e^{-2Ht} \Delta \right] A_k - 2H \frac{\partial}{\partial x^k} A_t, \]  

where \( x^1 = x, \ x^2 = y, \ x^3 = z \). We can now write down explicitly the vector wave equation (2.5) in the Eastwood–Singer gauge, bearing in mind that \( \Lambda = 3H^2 \) therein. The associated homogeneous equations (i.e. with vanishing right-hand side), under the assumption that \( A_b = A_b(t) \), can be solved by the ansatz (multiplicative constants are omitted for simplicity)

\[ A_t = e^{\alpha t}, \quad A_k = e^{\beta t}, \]  

where
leading to the algebraic equations

$$\alpha^2 + 3H\alpha + 3H^2 = 0, \quad (6.4)$$

$$\beta^2 + H\beta + H^2 = 0. \quad (6.5)$$

Hence we find

$$A_t = e^{-\frac{3}{2}Ht} \left[ U_1 \cos \left( \frac{\sqrt{3}}{2}Ht \right) + U_2 \sin \left( \frac{\sqrt{3}}{2}Ht \right) \right], \quad (6.6)$$

$$A_k = e^{-\frac{4}{2}Ht} \left[ V_1 \cos \left( \frac{\sqrt{3}}{2}Ht \right) + V_2 \sin \left( \frac{\sqrt{3}}{2}Ht \right) \right], \quad (6.7)$$

where $U_1, U_2, V_1, V_2$ are constant coefficients. For consistency, the function $\psi$ on the right-hand side of (2.5) can only depend on $t$ as well, and therefore the inhomogeneous equation for $A_t$ reads as

$$\left[ \frac{d^2}{dt^2} + 3H \frac{d}{dt} + 3H^2 \right] A_t = -\frac{d\psi}{dt}, \quad (6.8)$$

which is solved by

$$A_t = -\int G(t, t') \frac{d\psi}{dt'} dt', \quad (6.9)$$

having denoted by $G(t, t')$ the Green function of the second-order operator

$$P_t \equiv \frac{d^2}{dt^2} + 3H \frac{d}{dt} + 3H^2. \quad (6.10)$$

We therefore find $G(t, t')$ as the solution, for all $t \neq t'$, of the equation

$$P_t G(t, t') = 0, \quad (6.11)$$

subject to the continuity condition

$$\lim_{t \to t^+} G(t, t') = \lim_{t \to t^-} G(t, t'), \quad (6.12)$$

jointly with the jump condition

$$\lim_{t \to t^+} \frac{\partial G}{\partial t} - \lim_{t \to t^-} \frac{\partial G}{\partial t} = 1, \quad (6.13)$$

and the ‘boundary conditions’

$$B_1 G \equiv \alpha_{11} G(t_0, t') + \alpha_{12} \dot{G}(t_0, t') = 0, \quad (6.14)$$

$$B_2 G \equiv \alpha_{21} G(t_1, t') + \alpha_{22} \dot{G}(t_1, t') = 0. \quad (6.15)$$
If we denote by $s_1(t)$ and $s_2(t)$ the solutions of the problems

\begin{align}
  s_1 &= u : \quad P_1 u(t) = 0, \quad B_1 u(t_0) = 0, \quad (6.16) \\
  s_2 &= u : \quad P_2 u(t) = 0, \quad B_2 u(t_1) = 0, \quad (6.17)
\end{align}

we can write

\begin{align}
  G(t, t') &= A(t') s_1(t) \forall t \in ]t_0, t'[, \quad (6.18) \\
  G(t, t') &= B(t') s_2(t) \forall t \in ]t', t_1[. \quad (6.19)
\end{align}

The conditions (6.12) and (6.13) lead therefore to [10]

\begin{equation}
  G(t, t') = \frac{s_1(t^<) s_2(t^>)}{W(s_1, s_2; t')}, \quad (6.20)
\end{equation}

where $t^< = \min(t, t')$, $t^> = \max(t, t')$, and $W$ is the Wronskian

\begin{equation}
  W(s_1, s_2; t') = s_1(t') \dot{s}_2(t') - \dot{s}_1(t') s_2(t'). \quad (6.21)
\end{equation}

Now Abel’s formula for the Wronskian tells us that [10]

\begin{equation}
  W(s_1, s_2; t') = \text{cont.} \times e^{-s(t')}, \quad (6.22)
\end{equation}

where, bearing in mind (6.10), $s$ is a particular solution of the equation

\begin{equation}
  \frac{ds}{dt'} = 3H, \quad (6.23)
\end{equation}

i.e. $s(t') = 3Ht' + \text{constant}$. On denoting by $\kappa$ an integration constant, we arrive at the desired formula

\begin{equation}
  G(t, t') = \kappa e^{3Ht'} s_1(t^<) s_2(t^>). \quad (6.24)
\end{equation}

For example, on choosing the boundary conditions (cf. (6.14)–(6.17))

\begin{equation}
  s_1(t_0) = 0, \quad s_2(t_1) = 0, \quad (6.25)
\end{equation}

we obtain

\begin{align}
  s_1(t) &= e^{-\frac{3}{2} Ht} \left[ \cos \left( \frac{\sqrt{3}}{2} Ht \right) - \cot \left( \frac{\sqrt{3}}{2} Ht_0 \right) \sin \left( \frac{\sqrt{3}}{2} Ht \right) \right], \quad (6.26) \\
  s_2(t) &= e^{-\frac{3}{2} Ht} \left[ \cos \left( \frac{\sqrt{3}}{2} Ht \right) - \cot \left( \frac{\sqrt{3}}{2} Ht_1 \right) \sin \left( \frac{\sqrt{3}}{2} Ht \right) \right], \quad (6.27)
\end{align}
and we have only to plug (6.26) and (6.27) into (6.24) to obtain the explicit form of the Green function occurring in (6.9). Moreover, $\psi(t)$ in (6.8) is the combination of modified Bessel functions of Eq. (4.6), by virtue of (2.4).

The desired solution of Eq. (2.5) for $A_t$ is therefore given by the sum of (6.6), which solves the homogeneous equation, and (6.9), the latter being a solution of the inhomogeneous equation (6.8). Interestingly, thanks to the cosmological constant of de Sitter space-time, all time-dependent solutions of our vector wave equation (2.5) are square-integrable for $t \in (0, \infty)$.

When the potential $A_b$ has non-vanishing spatial gradient, our particular solutions remain helpful, because one can then look for exact or asymptotic solutions where the constant coefficients, e.g. $U_1, U_2, V_1, V_2$ in (6.6) and (6.7), are promoted to position-dependent functions [11].

VII. CONCLUDING REMARKS

The Eastwood–Singer gauge [3], whose physical motivations have been described in Section 1, has its deep mathematical roots in the theory of conformally covariant operators [4, 5]. However, explicit calculations in curved geometries of cosmological interest were lacking, to the best of our knowledge. In our paper we have first reduced its analysis in Einstein spaces to studying Eqs. (2.4), (2.5), (3.5), and then we have found exact solutions of the scalar wave equation (2.4) and a particular solution of the scalar equation (3.5). Moreover, the solution of the inhomogeneous vector wave equation (2.5) has been obtained when the potential depends only on the time variable. It now remains to be seen when the technique in Ref. [11] or, instead, in Ref. [12], can be used to solve Eq. (2.5) when the potential $A_b$ depends on all space-time coordinates.

All of this leaves aside quantization issues, for which we refer the reader to the work in Ref. [13].

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