Strange Work in Strange Places:
Quantum Field Theory in Curved Space

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ABSTRACT

Astronomers seem to be observing the fossilized remnants of quantum gravitational processes which took place during the epoch of *primordial inflation* that is conjectured to have occurred during the first $10^{-32}$ seconds of cosmic history. We give a non-technical description of what causes these processes and how they become preserved to survive to the current epoch. We also discuss some of the secondary effects which should result.

PACS numbers: 04.50.Kd, 95.35.+d, 98.62.-g

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1 Introduction

There are two reasons why quantum field theory in curved space might not be considered an appropriate topic for a book celebrating the 100th anniversary of general relativity:

1. Albert Einstein never accepted quantum mechanics; and

2. Applying quantum mechanics to gravity may show that general relativity is not the correct theory of gravity.

However, Einstein loved to pursue ideas to their logical conclusions, and he did important work on quantum mechanics despite his misgivings about it. So the research we will describe really is in the tradition of the great man’s thought, and it certainly concerns his theory of general relativity.

Until recently, most work on quantum field theory in curved space was motivated by events from the 1970’s. This fertile decade witnessed Stephen Hawking’s brilliant insight that quantum mechanics causes black holes to emit thermal radiation [1], and William Unruh’s demonstration that accelerated observers should experience a similar effect even in flat space [2]. These are results of the first magnitude, and still an area of active research as regards the eventual decay of black holes. However, all of the calculations of this sort which it is presently possible to perform seem to have been done. There are plenty of things we still don’t understand, but answering those questions is likely to require measurements of the effects to motivate simplifying assumptions that would facilitate more ambitious computations. Unfortunately, no such measurements exist, nor are any likely to become available in the near future because the known black hole candidates are all large (which means the radiation they emit is small) and very far away. For example, Figure 1 shows the orbits of star around the giant black hole near the center of our galaxy. From these orbits we can estimate its mass to be about 4.3 million times the mass of our own Sun. The radiation coming from a black hole of that size would have a temperature only about $10^{-14}$ degree Kelvin above absolute zero. We could not detect this faint a source of radiation even if it were produced in an Earth-bound laboratory, and it is in fact about $2.5 \times 10^{17}$ km away. There are smaller black hole candidates which have higher temperatures and are nearer, but the numbers are still impossible.
Figure 1: Stars orbiting the giant black hole near the center of our galaxy in the constellation of Sagittarius.
Recent years have seen much work on quantum field theory effects in cosmology. Unlike the situation for black holes, there are still plenty of relatively simple calculations to be made. There is also an abundant stream of data to test the results, and to motivate improvements. For example, Figure 2 displays a full sky map from the Planck satellite of the tiny temperature fluctuations which were impressed upon radiation from the Big Bang by the gravity from quantum fluctuations in matter during the first $10^{-32}$ seconds of existence. These same fluctuations served as the seeds for the aggregation of matter to form stars and galaxies. Among other things, they represent the first quantum gravitational effects which have ever been resolved. About $10^8$ bits of this data have already been recovered. The present technique for resolving these fluctuations gives us something like an x-ray of the universe because it reveals the direction in which the fluctuation is located but not its distance from us. There is a more promising method in which one measures the redshift of the 21 centimeter radiation emitted by Hydrogen gas throughout the universe. Greater redshift means further distance so we can reconstruct what is effectively a CT scan of the universe, containing vastly more information. The full development of this technique over the course of the next several decades might provide a staggering $10^{21}$ bits of data. These are the sorts of numbers which intoxicate physicists.

So topicality, simplicity and ready access to data all combine to draw attention away from quantum effects in black holes to quantum effects in cosmology. There is also the important consideration of the present authors’ expertise. As interesting as we find the work on black holes, it is not our tale to tell, whereas we have worked extensively on quantum field theory in cosmology.

Ours is not a simple story. Men of genius have laboured for centuries to extend human understanding of natural law. The magnitude of their achievement is evident from the huge barrier of background information which must be assimilated before accounts of modern physics become comprehensible. Even graduate students in our subject are dismayed by the frequency with which the dismissive phrase ”Introduction to” appears in the titles of their courses and textbooks! We have done our best to distil the essential facts and explain them in section 2. Section 3 describes the production of scalars and gravitons in cosmology, which is thought to have produced the fluctuations imaged in Figure 2. These might be termed the primary effect of cosmological expansion. Section 4 describes some of the fascinating secondary effects. In section 5 we list some of the problems which are still open.
Figure 2: Relative temperatures (cooler for blue, warmer for red) of radiation from different regions of the sky as imaged by the Planck satellite. The average temperature is $T_0 = 2.7255$ degrees on the Kelvin scale, and the fluctuations are about $3 \times 10^{-5}$ degree Kelvin.

We close with a note of caution. The general public is rightly impressed with the erudition displayed by physicists and astronomers, but one should bear in mind that they are still human, and as liable as anyone else to harbour unreasonable prejudices. In the panel discussion at a 1997 conference on the mounting evidence for the existence of black holes, Werner Israel shared a charming anecdote about this [4]:

Twenty years ago I spent a very pleasant sabbatical year at the ... Institute in ... Although this story is true, I have blanked out the names because its significance is generic – it could have happened at any of dozens of institutions at that time. Shortly after my arrival there was a coffee party, and after some warm words of welcome, the Director of the Institute remarked, “Werner is going to be with us for a year. We should all talk to him and try to cure him of these silly notions he has about the possibility of black holes.”

The Schwarzschild solution was discovered in 1916, just one year after the introduction of general relativity. And it was obvious, from the recognition of white dwarf stars in the 1920’s, that compact objects very close to black
holes exist in nature [3]. (Indeed, our Sun will end this way.) Yet learned physicists for decades rejected the possibility that slightly more massive stars might collapse to form black holes. While this attitude had softened to bemused toleration by Israel’s time, the future Nobel laureate Subrahmanyan Chandrasekhar was publicly ridiculed during the 1930’s for his demonstration that white dwarf stars of more than 1.4 solar masses cannot avoid collapse. The history of science is littered with similar cases, and it leads to dismal reflections on the human condition that no generation of scientists ever seems to rise above the reflex to persecute ideas which they find unattractive. So we enjoin the public to scepticism about unsupported opinion, no matter who espouses it. Scientists should be trusted only as far as their pronouncements are confirmed by experiment and observation, or by a provable chain of deduction from principles which have been established by experiment and observation.

2 Some Key Facts

This section is intended to give simplified explanations of a number of key physics theories and techniques, and we begin with a review of notation. Large and small numbers are represented using “scientific notation,” for example,

\[ 7.28 \times 10^{-27} \quad \text{and} \quad 3.085 \times 10^{16}. \]  

We use the MKS system of units in which mass is measured in kilograms (kg), length in meters (m), time in seconds (s) and charge in Coulombs (C). Five physical constants have standard symbols:

\[ c \approx 3.00 \times 10^8 \text{ \frac{m}{s}}, \]  
\[ G \approx 6.67 \times 10^{-11} \text{ \frac{m^3}{kg\cdot s}}, \]  
\[ \hbar \approx 1.054 \times 10^{-34} \text{ \frac{kg\cdot m^2}{s}}, \]  
\[ e \approx 1.602 \times 10^{-19} \text{ C}, \]  
\[ \epsilon_0 \approx 10^{-12} \text{ \frac{C^2\cdot s^2}{kg\cdot m^3}}. \]

Note that we use triple horizontal lines (≡) to indicate the quantity on the left “is defined to be” the quantity on the right. Scientists reserve the equals sign
for cases in which both sides of the relation have independent definitions.

In both physics and mathematics we have tried to employ concepts from high school which should be familiar to any educated reader. However, calculus was invented to give the laws of physics their simplest expression and there are some points at which it cannot be avoided. We use an over-dot to indicate differentiation with respect to time,

$$\dot{f}(t) \equiv \frac{df(t)}{dt}, \quad \ddot{f}(t) \equiv \frac{d^2f(t)}{dt^2}.$$ \hspace{1cm} (7)

Some more technical discussions have been relegated to appendices. Although we will not employ any vector or tensor calculus, we will use vectors. They are indicated by an over-arrow,

$$\vec{r} \equiv (x, y, z), \quad \vec{A} \equiv (A_x, A_y, A_z).$$ \hspace{1cm} (8)

The scalar product of two vectors is indicated with a dot,

$$\vec{A} \cdot \vec{B} \equiv A_x B_x + A_y B_y + A_z B_z.$$ \hspace{1cm} (9)

We shall also occasionally need complex numbers and their norms,

$$z = a + ib \implies |z|^2 = a^2 + b^2.$$ \hspace{1cm} (10)

An important special case is the exponential of an imaginary number,

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$ \hspace{1cm} (11)

Remember that the arguments of trigonometric functions such as those in expression (11) depend upon radians, not degrees!

### 2.1 General Relativity

Everyone who has studied physics is familiar with Isaac Newton’s three laws of motion. In the original Latin his first law reads:

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitetur statum illum mutare.
Which is to say, "Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed." Einstein's theory of general relativity describes how gravity modifies the geometrical concept of a straight line so that an object moving in a gravitational field, with no other forces, really does follow a straight line. General relativity does this by introducing a new dynamical variable known as the metric tensor field, which pervades space and time just like the electromagnetic fields which carry electric and magnetic forces and also describe the propagation of light.

The metric tensor field quantifies how observers at different points in space and time compare times, distances and the directions needed to determine if something is moving in a straight line. Popular science readers are probably aware of statements about how moving clocks seem run more slowly, and how objects seem contracted in the direction of motion, with both effects becoming large as the velocity approaches the speed of light. Those are special relativistic concepts which describe how observers at the same point, but moving with respect to one another, compare their notions of time, distance and direction. General relativity has to do with how observers at different points make the same comparison, even if they are not moving with respect to one another.

Defining a dynamical theory such a general relativity consists of specifying three things:

1. What the dynamical variable is;

2. How the dynamical variable affects the rest of physics; and

3. How the rest of physics affects the dynamical variable.

Because we must avoid complicated mathematics it is best to proceed by analogy with electromagnetism which should be familiar from introductory physics. The dynamical variables of electromagnetism are the electric and magnetic fields, and they affect physics through the forces they exert on charged particles. These fields are sourced by charges and currents through Maxwell’s equations. We have just seen that the dynamical variable of general relativity is the metric tensor field, and that it affects the rest of physics by defining the way times, lengths and directions are compared at different points in space and time. It turns out that the source for the metric tensor field is the energy and momentum per unit volume, and the stresses (non-gravitational force per unit area) at each point in space and time. In the
elegant mathematics of general relativity these sources are assembled into a single quantity known as the stress-energy tensor. In general relativity the stress-energy tensor sources the metric tensor field through the Einstein equations.

Figure 3: The Laser Interferometer Gravitational Wave Observatory (LIGO) located at Livingston, Louisiana in the United States. Another American LIGO detector exists at Hanford, Washington. These detectors work by using laser light bounced back and forth inside a 4 kilometer vacuum pipe. When a gravitational wave passes through the pipe it induces small distortions which can be detected by precise measurements of interference patterns produced by the light.

No sketch of this brevity can do justice to the scope of general relativity but our narrative does require a comment on the triune nature of general relativity and of all other known force laws. For reasons which are not fully understood, the dynamical variables of all force fields break up into three parts:
1. A gauge part which can be fixed arbitrarily and amounts, for gravity, to the choice of coordinate system;

2. A constrained part that is determined by the source, which is the stress-energy tensor for general relativity; and

3. A radiation part which can exist without any source and is generically produced by making sources undergo acceleration.

Humans see using electromagnetic radiation; we produce it by causing charges to accelerate. We have not yet directly detected gravitational radiation but there is strong indirect evidence for its existence because we can observe the effect of energy being carried away by gravitational radiation from pairs of neutron stars which are closely orbiting around one another. (Recall that even uniform circular motion entails centripetal acceleration.) Figure 3 shows one of the devices which are attempting to directly detect gravitational waves. In mid 2015 this device will resume operation at a greater sensitivity than ever before, and hopes are high that it will observe the bursts of gravitational radiation from mergers between compact objects such as neutron stars and black holes.

2.2 Cosmology

As we stated above, the metric tensor field defines how observers at different points in space and time compare times, distances and directions. If an observer’s time and position vector are $(t, \vec{x})$, then the metric tensor field tells him what is the infinitesimal distance to a nearby observer whose time and position vector are $(t + dt, \vec{x} + d\vec{x})$. With no gravitational fields present this relation is,

$$ds^2 = -c^2 dt^2 + d\vec{x} \cdot d\vec{x}. \quad (12)$$

Expression (12) encompasses all of special relativity, and represents the simplest geometry of general relativity. This way in which an earlier theory — in this case, special relativity — is nested within a more advanced theory — in this case, general relativity — is known as a Correspondence Limit. All new theories must reduce to older theories in the appropriate correspondence limit.

The next most complicated geometry is that of cosmology,

$$ds^2 = -c^2 dt^2 + a^2(t) d\vec{x} \cdot d\vec{x}. \quad (13)$$
It differs from the geometry of flat space only by the function of time $a^2(t)$ which multiplies the space terms. When a physicist says that the universe is expanding, what he means is that the scale factor $a(t)$ is growing. It is called “the scale factor” because equation says that one must multiply coordinate separations by the scale factor to convert them to physical distance. We know this expansion is occurring because we can see what it does to the light emitted (or absorbed) in known atomic transitions whose wave length is fixed by the energy of the transition according to physical laws which we assume to be the same throughout the universe. If this light was emitted at time $t$ with wavelength $\lambda$ then the wavelength we measure at the current time $t_0$ is,

$$\lambda_0 = \frac{a(t_0)}{a(t)} \times \lambda \equiv (1 + z)\lambda . \quad (14)$$

Equation (14) assumes that the emitter is not moving with respect to us; if it is moving there is also a Doppler shift (like the police use to catch speeders!) due to the motion. The factor $z$ in equation (14) is known as the (cosmological) redshift. We can recognize atomic spectral lines which have been redshifted by more than a factor of 7, and we can recognize thermal radiation which has experienced a redshift of over 1000.

Physicists quantify the expansion of the universe through time derivatives of the scale factor. The first derivative of its logarithm is known as the Hubble parameter,

$$H(t) \equiv \frac{d \ln[a(t)]}{dt} = \frac{\dot{a}(t)}{a(t)} . \quad (15)$$

Many people have seen the lovely pictures produced by the Hubble Space Telescope, for example, Figure 4 However, the actual reason American tax payers provided a billion US dollars to build and operate the telescope was to enable astronomers to measure the current value of the Hubble parameter to an accuracy of 10%. Today measurements of the cosmic microwave background radiation such as Figure 2 can do an even better job, $H_0 = \left(67.74 \pm 0.23\right) \frac{\text{km}}{\text{Mpc} \cdot \text{s}} = \left(2.195 \pm 0.007\right) \times 10^{-18} \text{ Hz} . \quad (16)$

The Hubble parameter was not always so small. The equations of general relativity predict a relation, known as the ΛCDM model, which seems to be valid to redshifts as large as $z \sim 10^{22}$,

$$H = H_0 \sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + 1 - \Omega_m - \Omega_r} , \quad (17)$$
Figure 4: Hubble Space Telescope image of the Whirlpool Galaxy (M51), a giant spiral galaxy located about $2.2 \times 10^{23}$ m from us.
where observations give the following values [6],

$$\Omega_r = \left( 9.29 \pm 0.050 \right) \times 10^{-5}, \quad \Omega_m = 0.315 \pm 0.017. \quad (18)$$

The relative abundances of the lightest nuclear isotopes preserve a fossilized record of conditions at $z \sim 10^9$, during which the Hubble parameter was 17 orders of magnitude larger than its current value. So the picture to keep in mind is a universe which was once much smaller, much hotter and much more rapidly expanding than it is now.

The second derivative of the scale factor is combined with lower derivatives to form a dimensionless quantity known as the deceleration parameter,

$$q(t) \equiv -\frac{\ddot{a}}{a^2}. \quad (19)$$

There is a funny story about the minus sign in expression (19), which also illustrates the fallibility of scientists, the crucial importance of checking beliefs with experiment and observation, and just how fast things are changing in this particular field. The story begins with modern physicists’ frustration at Benjamin Franklin’s unfortunate choice for the signs of electrical charge more than 250 years ago. In those years people were just learning about electricity, and they had discovered that electricity can be produced by rubbing glass with silk and also by rubbing amber with fur; du Fay termed them “vitreous” and “resinous” electricity, respectively. Franklin had the brilliant insight that these were not two different sorts electricity but rather that one represented an excess of charge and the other was a relative absence of the same kind of charge. He was quite right about that, but he unfortunately chose to label vitreous electricity as “positive” charge and resinous electricity as “negative” charge, whereas we now understand that rubbing glass with silk carries away negatively charged electrons and rubbing amber with fur adds electrons. Franklin’s sign choice means that electrons — which are the basis of our electrical power industry — carry negative charge. One consequence is that electrical engineers must remember that a wire carrying positive charge to the right actually consists of electrons moving to the left. The cartoon in Figure 5 indicates how irritating they find this!

Physicists hate negative-valued parameters because they inhibit us from telling the general tendency of things just by glancing at an equation. Cosmologists were determined not to introduce another negative-valued parameter into physics. Because we were certain that gravity must be slowing down
Figure 5: Cartoon about the frustration electrical engineers feel with Franklin’s sign choice.
the expansion — the same way it slows a rock which is thrown into the air — we inserted the minus sign into expression (19) to keep the parameter positive. But the universe does as it pleases, without regard to human prejudice, and its expansion has actually been accelerating for almost 6 billion years. So all our cleverness has resulted in a new negative-valued parameter $q_0$, 

$$q_0 = -0.54 \pm 0.01.$$  

Sometimes you just can’t win! This was only shown in 1998, and the leaders of the two teams who made the measurement were awarded the 2011 Nobel Prize in physics. Explaining why the universe is accelerating is regarded by many as the greatest problem confronting fundamental theory, and it didn’t exist 20 years ago.

Popular science readers have probably seen accounts of how superstring theory will provide the ultimate explanation of all natural phenomena. It is sobering to quote the words of a leading string theorist when he was told the result about cosmic acceleration:

I’m sure the data is wrong because string theory predicts a negative cosmological constant. And if it’s right, I’m going to stop doing physics.

Superstring theory does predict that the final state of the universe cannot be one of accelerated expansion, but the observations showing that our 13.8 billion year old universe is accelerating did turn out to be right. The reaction of string theorists was to introduce a huge number of free parameters (estimates involve numbers like $10^{500}$) so that it can be made to produce a long phase of accelerated expansion before the ultimate decay into deceleration. It can also be made to predict anything else. Scientists have for generations argued that the hallmark of a scientific theory is its ability to make predictions which can be checked. The present version of superstring theory does not meet this criterion but some thinkers have argued that, rather than abandon string theory, we must change the definition of “science.” We leave the reader to form his own opinion about this but we again urge scepticism about pronouncements which are not supported by observation and experiment, or by a rigorous chain of deduction from principles which have been established by observation and experiment.

The same ΛCDM model which led to relation (17) for the Hubble param-
Figure 6: Sketch of how the deceleration parameter is thought to have changed throughout cosmic history. The parameter $N$ is the number of e-foldings since the end of primordial inflation. If the scale factor at the end of inflation is $a_i$ then $N \equiv \ln[a(t)/a_i]$.  

eter also predicts that the deceleration at redshift $z$ is,

$$q = \frac{\Omega_r(1+z)^4 + \frac{1}{2}\Omega_m(1+z)^3 - 1 + \Omega_m + \Omega_r}{\Omega_r(1+z)^4 + \Omega_r(1+z)^3 + 1 - \Omega_m - \Omega_r}.$$  \hspace{1cm} (21)

Even though the universe has been accelerating for almost half of its life, we can see back to early times during which it was decelerating. Relation (21) implies that the cosmological redshift was only about $z \sim 0.63$ then, which is well within our ability to observe. This phase of deceleration is important in explaining today’s universe because acceleration works against the usual tendency of gravity to pull things together. The earlier phase of deceleration is why matter was able to collapse into stars, then into galaxies and finally into the galactic clusters we can observe today. After the onset of acceleration the formation of larger structures stopped and now the universe is being pulled apart. Of the approximately 100 billion galaxies we can now observe it has been calculated that we will eventually only be able to see the handful which are gravitationally bound to our own galaxy.

Considerations of this sort lead one to wonder how the very large scale universe got to be as uniform as it seems to be. In view of the smoothing out process which is even now being caused by the current phase of cosmic expansion, one might suspect that another, much earlier phase of accelerated expansion could do the job. This epoch is known as primordial inflation and it is conjectured to have occurred during the first $10^{-32}$ seconds of cosmic history. There are very strong reasons for taking this idea seriously. One example is the cosmic microwave background radiation whose tiny temperature fluctuations are imaged in Figure 2. This radiation is left over from a time about 380,000 years into cosmic history, when the universe was slightly over 1000 times smaller than it is today, and hence over 1000 times hotter. At that instant the universe finally became cool enough for electrons and protons to form neutral Hydrogen, at which point light could propagate almost freely. We are seeing the highly redshifted glow from that hot plasma. It is in almost perfect thermal equilibrium — the fluctuations shown Figure 2 are only about one part in $10^5$, which is far, far better equilibrium than the air of any normal room. As everyone knows who has watched ice melt in a glass of water, it takes time for things to reach the same temperature after they are put in contact. Without primordial inflation the light we are seeing would have come from about 3000 regions which are so distant from one another that not even light could have travelled between them during the brief time the universe had existed by then. Primordial inflation avoids the problem by
causing all of the sky we can now observe to have come from a region which
was so small that it had plenty of time to reach equilibrium.

Figure 6 shows how we think the deceleration parameter behaved over
cosmic history. In order to avoid compressing all early events into a single
point, the scale of time has been replaced by $N$, the number of e-foldings
since the end of inflation. If primordial inflation ended at time $t_i$ then the
value of $N$ at any time $t$ is,

\[ N = \ln \left( \frac{a(t)}{a(t_i)} \right). \] (22)

Significant events on the graph are:

- **Primordial Inflation** during which the deceleration parameter was very
close to $q = -1$ and the Hubble parameter may have been 56 orders of
magnitude larger than it is today. Note that this period is not described
by the $\Lambda$CDM model (17) and (21).

- **Radiation Domination** during which the $\Omega_r (1 + z)^4$ term was much
larger than any other term in equations (17) and (21). Hence the
deceleration parameter was close to $q = +1$ and the universe was so
hot that thermal motion made all matter move at nearly the speed of
light.

- **Big Bang Nucleosynthesis** during which neutrons and protons fell out of
thermal equilibrium and the seven lightest nuclear isotopes were formed
(the heavier ones formed much later during supernovae).

- **Matter-Radiation Equality** when the $\Omega_r (1 + z)^4$ and $\Omega_m (1 + z)^3$ terms
in equation (17) became equal.

- **Recombination** during which electrons and protons formed neutral Hy-
drogen and light was able to propagate freely.

2.3 Quantum Mechanics

Quantum mechanics is not a dynamical theory in the sense of having its own
dynamical variable, force law and source equation. It is rather a procedure,
known as “quantization”, which can be applied to any physical theory. So
one should not imagine that there are special, quantum field equations which
are different from those of classical field theory\textsuperscript{[1]} It means precisely the same thing to solve these equations quantum mechanically as it does classically: one expresses the dynamical variable at any time in terms of its values — and those of all other dynamical variables — at some initial time.

As a very simple example, consider the dynamical system comprised of a single point particle of mass \( m \) moving freely in one dimension whose position at time \( t \) is \( x(t) \). For that system the initial value solution is,

\[
x(t) = x_0 + \frac{p_0}{m} t ,
\]

where \( x_0 \) is the initial position and \( p_0 \) is the initial momentum. These two initial values are called \textit{conjugate pairs} and every dynamical variable has them. The solution \textsuperscript{(23)} is valid in both classical physics and in quantum physics; the only difference is what \( x_0 \) and \( p_0 \) mean. In classical physics they are just numbers — for example, \( x_0 \) might be five meters and \( p_0 \) might be minus three kilogram-meters per second. Quantization consists of making the initial values of conjugate pairs into random numbers whose probability distributions are tied to one another.

Educated readers are familiar with the concept of quantities which are distributed randomly such as human intelligence. Figure\textsuperscript{7} shows the famous “bell curve”. The probability that a randomly selected human’s IQ score \( i \) lies between and two values \( i_1 \) and \( i_2 \) is given by the area under the curve between \( i_1 \) and \( i_2 \). What quantum mechanics does is to make every conjugate pair, like \( x_0 \) and \( p_0 \) above, into random variables whose probability densities are inversely related so that when one variable is very well known, the other is poorly known. To understand precisely how this works requires a little integral calculus which is explained in Appendix A. The important thing for our discussion is that, if the value of \( x_0 \) is known to an uncertainty of \( \pm \Delta x \), and the value of \( p_0 \) is known to an uncertainty of \( \pm \Delta p \), then the product of the two uncertainties obeys a famous inequality known as the Uncertainty Principle,

\[
\Delta x \times \Delta p \geq \hbar .
\]

The constant \( \hbar \approx 1.054 \times 10^{-34} \) kg\(\cdot\)m\(^2\)/s is very small when measured in terms of normal units, which is why we perceive no quantum effects in everyday life. For example, most people would consider that a measurement of

\footnote{Note that we use the term “classical” in contradistinction to “quantum”, without regard to whether or not special or general relativistic effects are included.}
Figure 7: The normal probability curve which describes human IQ scores and many other randomly distributed quantities. The height of the curve for any score gives the relative probability for a randomly chosen human to achieve that score. The percentage figures on the x-axis give the fraction of the total population whose scores fall within this range.

Some object’s position was exceptionally accurate if it determined to within $\Delta x_0 = 10^{-10}$ m, but this would still allow the same object’s momentum to be measured to within $\Delta p_0 \approx 10^{-24}$ kg–m/s, which also seems wonderfully accurate. But the fact that we cannot make both $\Delta x_0$ and $\Delta p_0$ zero has terrific importance. One simple consequence is the stability of atoms against the collapse of their electrons onto the central nucleus, which is energetically favoured in classical physics.

Consider the simplest atom, Hydrogen, whose energy is,

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 |x|}. \quad (25)$$

In classical physics we can make the energy arbitrarily negative by taking the electron-proton separation $x$ be close to zero while keeping the kinetic energy smaller in magnitude. This means it is energetically favorable for a classical Hydrogen atom to collapse and emit a burst of electromagnetic radiation. The Uncertainty Principle is what prevents that, because forcing $x$ to take any particular value (such as zero) to within a small uncertainty $\Delta x$, causes the uncertainty on the momentum to become large, which results in a very
large positive kinetic energy. This is easy to see from expression (25) if we make the energy as negative as possible by assuming that the average values of $x$ and $p$ are both zero, so that only their uncertainties contribute,

$$E \rightarrow \frac{\Delta p^2}{2m} - \frac{e^2}{4\pi \varepsilon_0 \Delta x}. \quad (26)$$

Now use the Uncertainty Principle to eliminate $\Delta p$ and make a few simple algebraic rearrangements to derive a lower bound on the energy,

$$E \geq \frac{\hbar^2}{2m \Delta x^2} - \frac{e^2}{4\pi \varepsilon_0 \Delta x} = \frac{1}{2m} \left( \frac{\hbar}{\Delta x} - \alpha mc \right)^2 - \frac{1}{2} \alpha^2 mc^2 \geq -\frac{1}{2} \alpha^2 mc^2. \quad (27)$$

The symbol $\alpha \equiv e^2/(4\pi \varepsilon_0 \hbar c) \approx 1/137$ is known as the fine structure constant and we will encounter it again. The rest mass energy of the electron is usually expressed in million electron-volts or MeV, $mc^2 \approx 0.511$ MeV. Hence our bound (27) implies that the energy of a quantum mechanical Hydrogen atom cannot drop below the well known value,

$$E \geq -\frac{1}{2} \alpha^2 mc^2 \approx \frac{1}{2} \left( \frac{1}{137} \right)^2 \times 511000 \text{ eV} \approx -13.6 \text{ eV}. \quad (28)$$

The agreement would not have been so perfect had we been more precise about defining what is meant by $\Delta x$, and about the atom living in three spatial dimensions, but none of this would change the fact that the Uncertainty Principle stabilizes atoms against collapse.

### 2.4 Quantum Field Theory

A field is a dynamical variable which depends upon time $t$ as well as space $\vec{x}$. Familiar examples from introductory physics are the electrodynamic scalar and vector potentials: $\Phi(t, \vec{x})$ and $\vec{A}(t, \vec{x})$. Classical fields have initial value solutions, just like (23), except that there are conjugate pairs for each space point $\vec{x}$. A quantum field obeys the same equation as its classical counterpart, the only difference is that the conjugate pairs of its initial values are random numbers which obey the Uncertainty Principle (24).

The effect of quantization explains the otherwise puzzling statements popular science readers may have seen that general relativity provides a classical theory of gravitation which agrees with all known observations, however, quantizing this perfect theory results in nonsense. That seems contradictory
until one realizes that the wonderful classical agreement comes from solutions for which both partners of almost all conjugate pairs have been set to zero. In classical physics they are just numbers which can be fixed as we choose, but the Uncertainty Principle of quantum mechanics means there must be a minimum amount of disturbance in each conjugate pair. It is the cumulative effect of having each of the infinite number of conjugate pairs a little disturbed from zero which makes quantum general relativity so problematic. It is possible that we are not doing the computation correctly (more on that in the next subsection) and that nonlinear effects resolve the apparent difficulties. But it is easy to understand how problems might arise from the addition of an infinite number of small contributions to the energy and momentum which sources gravitation.

We have a confession to make: there is not any such thing as a point particle. In reality the wave functions of quantum mechanical particles are just one linearized (and typically nonrelativistic limit of a) solution of a quantum field. So there is a quantum field which describes each kind of particle: electrons, quarks, neutrinos, etc. A good way of categorizing these fields is by the spins of the particles they describe. Fundamental theory is subject to powerful constraints which limit the possibilities to just these:

- **Spin 0 (scalar) fields** like the Higgs scalar which was recently discovered at the Large Hadron Collider;
- **Spin \( \frac{1}{2} \) fermions** like the electron;
- **Spin 1 (vector) fields** like electromagnetism or the weak and strong nuclear forces;
- **Spin \( \frac{3}{2} \) gravitinos** which occur in a conjectured extension of general relativity known as supergravity; and
- **Spin 2 (tensor) fields** of which the metric of general relativity is the only possibility.

Particles also have mass \( m \), which is a characteristic of the type of particle, as well as energy \( E \) and momentum \( \vec{p} \) which change depending upon how the particle moves. An important relation (due to Einstein) exists between these three quantities in flat space (that is, with \( a(t) = 1 \),

\[
E = \sqrt{m^2 c^4 + \vec{p} \cdot \vec{p} c^2}.
\] (29)
In the next section we will see how the Einstein relation (29) changes in an expanding universe.

We reiterate that all particles are really excitations of quantum fields, and hence probability waves. This has been tested past the point of any dispute. For example, electrons have been made to show double slit interference patterns, just like light waves. Another important relation (this one due to Louis DeBroglie) connects the wave length $\lambda$ to the magnitude $p$ of the momentum,

$$\lambda = \frac{\hbar}{p} = \frac{\hbar c}{\sqrt{E^2 - m^2 c^4}} = \frac{\hbar}{\sqrt{2mK + \frac{K^2}{c^2}}}.$$  

(30)

The final form on the right is expressed in terms of the kinetic energy $K \equiv E - mc^2$, and it explains why we don’t normally recognize the wave nature of electrons or other particles with nonzero mass. Suppose the electron has the amount of kinetic energy it is pretty much guaranteed to have from thermal motion at room temperature, which is $K \approx 6 \times 10^{-21}$ Joules. Substituting this into expression (30) gives a wave length of only $\lambda \approx 10^{-9}$ m which is about 500 times smaller than visible light. We can directly perceive the wave nature of massless particles such as the photons which constitute electromagnetic radiation, and gravitons which constitute gravitational radiation. However, quantum mechanics offers a surprise for these radiation fields in that they really consist of discrete quanta. According to relations (29,30) light or gravity waves of wave length $\lambda$ corresponds to discrete particles of energy $E = \hbar c/\lambda$. You can have this amount of energy in the radiation field, or any multiple of this amount, but you could not have 1.5 of this energy, or $\sqrt{2}$ times this energy. This discrete nature of light was demonstrated long ago in the photo-electric effect, which is what powers solar cells. The quantization of gravitational radiation has not yet been demonstrated, and the gravitational interaction is too weak for individual gravitons to be seen at the LIGO detector of Figure 3. However, we will learn in the next section that cosmological measurements can establish the quantization of gravitational radiation. With a bit of luck, they might even establish the existence of classical gravitational radiation before LIGO detects it!

The spooky effects of quantum field theory all derive from the undoubted fact that each conjugate pair in a quantum field is a little bit disturbed. This little bit of disturbance causes even empty space to behave as if it was filled with classical particles, of all different wave lengths $\lambda = \hbar/p$, which exist for
a time $\Delta t$ governed by the *energy-time uncertainty principle*

$$\Delta t \leq \frac{\hbar}{E} = \frac{\hbar}{\sqrt{m^2c^4 + \hbar^2c^2k^2}},$$

(31)

where $k \equiv 2\pi/\lambda$ is called the wave number of the particle. The particles which obey relation (31) are known as virtual particles and, once their existence is accepted, one can understand quantum field theory effects just by using classical physics. Of course that had to be true because quantum fields obey exactly the same equations as classical fields do; the only difference is that each of their conjugate pairs must be a little nonzero, which is precisely what virtual particles represent.

Figure 8: Feynman diagram representing the lowest order process contributing to the scattering of two photons.

As an example, consider the scattering of light by light. Many readers learned in high school physics that two light rays pass right through one another in empty space, with no effect at all. With virtual particles that is no longer quite true. Figure 8 is a Feynman diagram which represents two quantized light particles (known as photons) coming in from the right. Then they are absorbed by a virtual electron and anti-electron (known as a positron) which exist for a brief period of time given by relation (31). When the virtual pair’s short lifetime is up, the two photons are re-emitted to the left, but with slightly different momenta than they originally had.

All quantum field theory effects can be understood in this way. And it should be obvious that whatever strengthens the lifetime $\Delta t$ — and also the rate at which virtual particles are emitted — will make quantum field theory effects become stronger. We will see in the next section that the expansion of
the universe does this. However, even in flat space one can see from relation \(31\) that making the mass smaller increases \(\Delta t\) at fixed \(k\). That is why the strongest quantum field theoretic effects come from the lightest particles. One can also see that making the wave length long at fixed \(m\) increases \(\Delta t\).

2.5 Perturbation Theory

Physicists are very good at math, but most of the problems in general relativity and quantum field theory are harder than we can solve. What we do instead is to develop approximate solutions in the form of power series expansions around the answers to simpler problems whose exact solution is known. Educated readers are familiar with the basic technique from computing square roots using the expansion,

\[
\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \ldots = 1 + \frac{x}{2} - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} \left(-\frac{x}{2}\right)^n. \tag{32}
\]

(The double factorial symbol means to skip every other number, for example, \(7!! = 7 \cdot 5 \cdot 3 \cdot 1 = 105\).) For example, we know that the square root of 4 is exactly 2, and the square root of 9 is exactly 3, but we don’t have exact results for the numbers in between. Suppose we want the square root of 5. We can get a series expansion by first writing 5 as 4 + 1, then factoring out the 4 and using expression \(32\) with \(x = \frac{1}{4}\),

\[
\sqrt{5} = \sqrt{4 + 1} = 2 \sqrt{1 + \frac{1}{4}} = 2 + \frac{1}{4} - 2 \sum_{n=2}^{\infty} \frac{(2n-3)!!}{n!} \left(-\frac{1}{8}\right)^n, \tag{33}
\]

\[
= 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \frac{7}{131072} - \ldots \tag{34}
\]

With a scientific calculator one sees that \(\sqrt{5} \approx 2.23607\). This six digit accuracy is achieved by just the five terms shown in \(34\).

Physicists who study quantum field theory employ a similar approximation technique known as the loop expansion. The first term is called tree order, the second term is called one loop, the third term is called two loop, and so on. It is sometimes difficult even getting the tree order result, and computing loop corrections becomes prohibitively difficult very fast, so computations of real things are never done to more than 4-5 orders. This disturbs some people but it is well to recall that physicists can only avoid error by constantly
checking their predictions against experiment and observation, and there are important practical limitations on how accurately these measurements can be performed. One of the best measured quantities is the gyromagnetic ratio of the electron (the ratio of its actual magnetic dipole moment to the classical prediction) which is accurately known to 12 digits. That corresponds to 4-5 orders in the loop expansion, and there is simply no point to pushing the theoretical prediction any farther until the measurement can be done with greater accuracy.

Every quantum field theory has a *loop counting parameter* which quantifies the loop expansion. For quantum electrodynamics (QED) this parameter is \( \alpha/2\pi \approx 1/861 \), where \( \alpha \equiv e^2/(4\pi\epsilon_0\hbar c) \) is the fine structure constant we saw in equation (27). A typical prediction of QED takes the form,

\[
\text{(QED Prediction)} = \left( \text{Tree Result} \right) \left\{ 1 + A \frac{\alpha}{2\pi} + B \left( \frac{\alpha}{2\pi} \right)^2 + C \left( \frac{\alpha}{2\pi} \right)^3 + \ldots \right\}.
\]

(35)

The “Tree Result” could also involve \( \alpha \). The coefficients \( A, B \) and \( C \) can involve other parameters of the problem but they are assumed to be of order one, so one can see that the 4-loop correction \( (\alpha/2\pi)^4 \approx 1.8 \times 10^{-12} \) really does give 12-digit accuracy, which is the best we can currently measure.

As one might expect, the loop counting parameter of quantum gravity involves Newton’s constant of universal gravitation, \( G \). With the other factors it takes the form \( \hbar G \omega^2/c^5 \), where \( \omega \) is the characteristic frequency of whatever process is being studied. For cosmological particle production that frequency is the Hubble parameter, \( H(t) \). From expression (16) one sees that the current effect is hopelessly too small to resolve,

\[
\frac{\hbar G H^2_0}{c^5} \approx 10^{-123}.
\]

(36)

However, the Hubble parameter \( H_i \) of primordial inflation might be as much as 56 orders of magnitude larger. The best current upper bound is [6],

\[
\frac{\hbar G H^2_i}{c^5} < 4 \times 10^{-11}.
\]

(37)

This number is interesting because it is says that quantum gravitational effects are small during primordial inflation, but within our ability to measure. The fact that the number is small means we don’t need to worry about more than the first two terms.
3 Primary Effects

Quantum gravitational effects from primordial inflation are observable for three reasons:

1. The quantum gravitational loop counting parameter associated with the rate of inflationary expansion \(37\) is small enough that we can compute reliably with quantum general relativity, but large enough that sensitive measurements can resolve effects which involve one or two powers of it;

2. Accelerated expansion allows significant numbers of virtual gravitons and the simplest type of massless scalars to emerge from empty space and persist forever; and

3. The fluctuations these particles induce become fossilized so that they can survive to be observed at late times.

We have already discussed the first point. In this section we will give a simple way of understanding the second and third points. Then we discuss the primordial power spectra which represent the primary effects.

3.1 Inflationary Particle Production

We learned in section 2.4 that any particle is really a probability wave. It can have an arbitrary length \(\lambda\), which fixes its energy and momentum by relations \(29\) and \(30\) in flat space. In the cosmological geometry \(13\) a fixed wave length \(\lambda\) corresponds to a physical length of \(a(t) \times \lambda\), so it should not be surprising that the energy of a particle with wave length \(\lambda = 2\pi/k\) is,

\[
E(t, k) = \sqrt{m^2c^4 + \left(\frac{\hbar ck}{a(t)}\right)^2}.
\]

(38)

One interesting consequence is that the expansion of the universe lowers the energy of a particle.

Another thing we learned in section 2.4 is that quantum field theoretic effects can be understood as the classical response to virtual particles which emerge from empty space at time \(t\) and persist until time \(t + \Delta t\). In flat space the persistence time \(\Delta t\) is given by relation \(31\), but the alert reader
will note a problem in that the cosmological energy (38) changes with time! What time should we evaluate it at? Stating the answer precisely requires some integral calculus which is explained in Appendix B. However, it should not seem surprising that the answer involves a sort of average over all of the changing energies between \( t \) and \( t + \Delta t \).

Because \( E(t, k) \) decreases as the universe expands, the persistence time \( \Delta t \) is increased by cosmological expansion. As long as the particle’s mass \( m \) is nonzero its energy (38) cannot be smaller than \( mc^2 \), so \( \Delta t \) remains finite. However, massless particles have \( E(t, k) = \hbar ck/a(t) \), and it can be that the energy goes to zero so rapidly that the persistence time \( \Delta t \) becomes infinite. In Appendix B we demonstrate that the condition for this is inflation (which means negative deceleration parameter \( q(t) \)) with \( ck < -q(t)H(t)a(t) \).

The combination \( c/H(t)a(t) \) is known as the Hubble radius, and it represents roughly the physical distance that a local observer can see. So the wavelengths we are discussing are of cosmological scale. That is very important because physicists know that there is something wrong with our understanding of quantum general relativity at scales smaller than the Planck length, \( \sqrt{\hbar G/c^3} \sim 10^{-35} \) m. Using quantum general relativity at these scales or smaller would not be reliable. On the other hand, using it at much larger scales should be right. (This is yet another one of those correspondence limits which are so important to physics!) Fortunately for us, the smallest Hubble radius for primordial inflation which is consistent with the observations of Figure 2 is about \( c/Hi \sim 10^{-30} \) m. That is very small, but still a hundred thousand times larger than the Planck length, so we should be able to trust quantum general relativity.

Just because long wavelength massless virtual particles persist forever during inflation does not mean they necessary mediate significant quantum field theoretic effects. One must still check that there is no suppression of the rate at which these particles emerge from empty space. In flat space this rate is simply a constant but, as with many things, it can become time dependent in the cosmological geometry (13). A simple way to analyze the problem is by changing the time coordinate from \( t \) to the conformal time \( \eta \) which obeys the differential equation \( dt = a(t)d\eta \). In terms of conformal time the cosmological geometry (13) is the same as the flat space geometry (12) up to an overall multiplicative constant,

\[
-c^2 dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} = a^2 \left[ -c^2 d\eta^2 + d\vec{x} \cdot d\vec{x} \right].
\]
Almost all massless particles possess a symmetry known as *conformal invariance* which states that they are the same in conformal coordinates as in flat space. This means that the rate at which conformally invariant particles emerge from empty space per unit conformal time is the same constant $\Gamma_{\text{flat}}$ as it is in flat space. It follows that the emission rate per unit physical time falls off with $a(t)$,

$$
\frac{dN}{dt} = \frac{d\eta}{dt} \times \frac{dN}{d\eta} = \frac{1}{a(t)} \times \Gamma_{\text{flat}}.
$$

(40)

Therefore, any long wavelength conformally invariant virtual particle which emerges from empty space during inflation can persist forever, but very few emerge. There are just two sorts of massless particles which are not conformally invariant, and so avoid the reduced emission rate of expression (40). They are the simplest kind of spin zero particles, and the particles associated with quantized gravitational radiation, which are known as gravitons.

### 3.2 How Inflationary Perturbations Fossilize

Both gravitons and massless minimally coupled scalars of wave number $k$ have the same wave function $u(t,k)$. It obeys the same equation as a point mass attached to the same sort of spring whose force law $F = -m\omega^2x$ is familiar from introductory physics. The details are explained in Appendix C but the key point is that the mass and characteristic frequencies of this oscillator are time dependent,

$$
m(t) = \sqrt{\frac{32\pi G\hbar}{c}} \times a^3(t) \quad \omega(t,k) = \frac{ck}{a(t)}.
$$

(41)

For inflation the frequency $\omega(t,k)$ falls off so rapidly that the particle stops oscillating because the force acting on it gets so small. However, the fact that the mass $m(t)$ grows more rapidly that $\omega^2(t,k)$ falls off means that the system freezes in with a substantial amount of potential energy. One can count the number $N(t,k)$ of particles produced by simply considering this potential energy to be $N(t,k)$ times the energy $\hbar ck/a(t)$ of a single particle.

Explicit computations are not possible for general $q(t)$ but Figure 6 shows it should be a good approximation to set $q_i = -1$, which means $H(t)$ is some constant $H_i$. These approximations reveal a staggering amount of particle production,

$$
q_i = -1 \implies N(t,k) = \left[ \frac{H_i a(t)}{2ck} \right]^2.
$$

(42)
Note that the particle number is small initially when $ck \gg H(t)a(t)$, it becomes of order one at $t_k$, the time of first horizon crossing when $ck = H(t_k)a(t_k)$, and it grows explosively for late times. Under the same approximation one can also show that the amplitude of $u(t,k)$ approaches the constant,

$$q_i = -1 \implies \lim_{t \to \infty} |u(t,k)| = \frac{H_i}{ck} \left[ \frac{8\pi\hbar G}{c^3 k^2} \right]^\frac{1}{4}$$

The fact that $u(t,k)$ approaches a constant is how perturbations from gravitons and the simplest kind of massless scalars become fossilized so that they can be seen at much later times.

The process of fossilization does not depend upon the assumption we made in expressions (42-43) that $q(t) = -1$. To see this, consider the time derivative of the product of $H(t)$ and $a(t)$ for general $q(t)$,

$$\frac{d}{dt} [H(t)a(t)] = \frac{d}{dt} [\dot{a}(t)] = \ddot{a}(t) = -\left(\frac{a\ddot{a}}{\dot{a}^2}\right) a = -q(t)H^2(t)a(t).$$

(44)

From expression (44), we see that $H(t)a(t)$ increases during inflation ($q(t) < 0$) and decreases during deceleration ($q(t) > 0$). The wave numbers $k = \frac{2\pi}{\lambda}$ of interest for us pass through three stages of evolution over the course of the cosmological history depicted in Figure 6:

1. At the beginning of primordial inflation they obey $ck \gg H(t)a(t)$. While this is true the wave function $u(t,k)$ oscillates rapidly and its magnitude falls off like $1/a(t)$. During this phase the occupation number $N(t,k)$ is very small.

2. First horizon crossing $ck = H(t_k)a(t_k)$ occurs at some time $t_k$ about 50 e-foldings before the end of primordial inflation. After this point the characteristic frequency $\omega(t,k) = ck/a(t)$ becomes negligible with respect to the Hubble parameter $H(t)$ and the wave function $u(t,k)$ approaches a constant. After first horizon crossing the occupation number $N(t,k)$ grows rapidly.

3. Second horizon crossing $ck = H(T_k)a(T_k)$ occurs at some time $T_k$ after the end of primordial inflation and before the onset of the current phase of cosmic acceleration. At this point the wave function $u(t,k)$ begins to change with time again, although according to the altered conditions which prevail at late times.

29
Of course there are large \( k \) wave numbers which never experienced first horizon crossing. We can measure them at current times but they are not very interesting because they did not become highly populated during primordial inflation, nor did they ever fossilize. There are also some small \( k \) wave numbers which have passed through first horizon crossing but never experienced second horizon crossing. These wave numbers are still fossilized and their physical wave lengths are so large that the limited portion of the universe we can see does not allow us to perceive their spatial variation. People who think they know things without doing calculations or measurements claim that these wave numbers can have no effect at all today but it seems safer to conclude that whatever effect these modes have must seem spatially (but perhaps not temporally) constant to us. There may also be some very small \( k \) wave numbers which were fossilized when primordial inflation began and are still fossilized today.

### 3.3 The Primordial Power Spectra

There are two primary effects: the ensemble of gravitons which are produced during inflation and the gravitational response to inflation-enhanced quantum fluctuations in whatever matter fields provided the source for primordial inflation. The first effect is weaker than the second but conceptually simpler. It was originally described by the brilliant Russian cosmologist Alexei Starobinsky in 1979 [7]. This effect is usually represented by a quantity known as the \textit{tensor power spectrum} \( \Delta^2_h(k) \) which can be expressed in terms of the wave function \( u(t,k) \),

\[
\Delta^2_h(k) = \frac{k^3}{2\pi^2} \times 2 \times \sqrt{\frac{32\pi \hbar G}{c^3}} \times \left| u(t,k) \right|^2 \left\{ 1 + \left( \text{Loop Corrections} \right) \right\}, \tag{45}
\]

\[
\approx \frac{16\hbar G H^2(t_k)}{\pi c^5} \left\{ 1 + \# \frac{\hbar G H^2}{c^5} + \ldots \right\}. \tag{46}
\]

Because \( u(t,k) \) in expression (45) is evaluated long after first horizon crossing and long before second horizon crossing, the wave function is at its constant, fossilized value. The approximation (46) was obtained by substituting the \( q(t) = -1 \) result (43) into expression (45). At this stage no one is sure how to compute loop corrections (debates about this sometimes degenerate into shouting matches!) although these corrections should be suppressed by the quantum gravitational loop counting parameter \( \delta_5 \). One of the
many things which is not clear is when the slightly time dependent Hubble parameter should be evaluated.

The gravitational response to quantum fluctuations in whatever matter drove primordial inflation is usually quantified as the scalar power spectrum $\Delta^2_R(k)$. The original computation of it was made by Viatcheslav Mukhanov (now at Ludwig Maximilian University in Munich, Germany) and Gennady Chibisov (now deceased) [8]. Of course the result for it depends upon precisely what matter fields caused primordial inflation, which is a subject of intense debate. In the simplest model $\Delta^2_R(k)$ depends upon a wave function we might call $v(t,k)$ which behaves as an oscillator with the same characteristic frequency (41) as $u(t,k)$, but a much smaller mass $M(t) = \sqrt{\frac{c}{4\pi G}}[1 + q(t)]a^3(t)$. This oscillator is much lighter than $u(t,k)$ because $q(t)$ was so near $-1$ during primordial inflation. Of course being lighter means the amplitude is larger! In terms of $v(t,k)$ the scalar power spectrum takes the form,

$$\Delta^2_R(k) = \frac{k^3}{2\pi^2} \times \sqrt{\frac{4\pi \hbar G}{c^3}} \times \left| v(t,k) \right|^2_{t_k \ll t \ll T_k} \left\{ 1 + \left( \text{Loop Corrections} \right) \right\}, \quad (47)$$

$$\approx \frac{\hbar G H^2(t_k)}{\pi c^5 [1 + q(t_k)]} \left\{ 1 + \frac{\hbar G H^2}{c^5} + \ldots \right\}. \quad (48)$$

There are the same passionate debates about how to calculate loop corrections to $\Delta^2_R(k)$, and the same uncertainty about what time to evaluate the factors of $H(t)$ which appear in the loop counting parameter.

The scalar power spectrum was first resolved in 1992 by the Cosmic Background Explorer (COBE) satellite [9], for which George Smoot and John Mather shared the 2006 Nobel Prize. This first detection measured only a small range of the largest interesting wavelengths $\lambda = 2\pi/k$. Since then there have been two major satellite probes and scores of balloon-bourne detectors which have resolved wavelengths more than a thousand times smaller than those detected by COBE. The result is well fit by a function of the form [6],

$$\Delta^2_R(k) = A_s \left( \frac{k}{k_0} \right)^{n_s - 1}, \quad (49)$$

where the amplitude $A_s$ and scalar spectral index $n_s$ are,

$$A_s = \left( 2.142 \pm 0.049 \right) \times 10^{-9} \quad , \quad n_s = 0.9667 \pm 0.0040. \quad (50)$$
The pivot $k_0 = 0.050 \text{ Mpc}^{-1}$ corresponds to a wavelength of about $\lambda_0 = 3.9 \times 10^{24} \text{ m}$. The fact that the scalar spectral index $n_s$ is so close to one means that different wavelengths have almost the same amplitude. By comparison with the theoretical prediction expression (48) we can infer that the inflationary Hubble parameter $H(t)$ and deceleration parameter $q(t)$ were nearly constant during the approximately ten e-foldings over which the measured wavelengths experienced first horizon crossing. This is thought to have occurred about 50 e-foldings before the end of primordial inflation.

Figure 9: The BICEP2 detector located at the South Pole.

The tensor power spectrum has not yet been resolved. This is presumably due to the fact that the inflationary deceleration factor $q(t)$ is so close to $-1$, which makes the scalar power spectrum much larger than the tensor power spectrum. In March of 2014 the people who run the BICEP2 detector (depicted in Figure 9) announced that they had succeeded in resolving $\Delta^2(k)$ with huge statistical significance [10]. However, it turns out that they were mostly detecting the light from dust near our galaxy [11, 12]. The team did under-estimate the contamination from dust, and they certainly over-publicised their findings, but it’s hard to fault them much more because they did manage to achieve a huge increase in sensitivity, and the people who
possessed the best dust map (paid for by European taxpayers) had refused to make it public. When the dust map was finally released and it became apparent that their signal could be entirely due to dust, the BICEP2 team admitted their mistake and actually collaborated with their critics to produce the best current upper limit for the strength of the tensor power spectrum. The whole exercise has been a wonderful stimulant for making even more sensitive measurements in regions which are now known to be relatively free of dust.

No one knows when — or if — the tensor power spectrum will be resolved but there are many ground-based and balloon-bourne detectors which will report data over the next few years, and hopes are high. Resolving the tensor power spectrum is crucially important because it will tell us the scale of primordial inflation. Resolving $\Delta^2 \phi(k)$ over many wavelengths will also allow us to test an entire class of models for the matter theory which caused primordial inflation.

4 Secondary Effects

As we saw from equation (42), accelerated expansion leads to the explosive production of the simplest kind of massless scalars and of gravitons. This is what causes the scalar and tensor power spectra (47) and (45). However, it is just about inconceivable that these particles do nothing after having been produced. Normal particles interact with one another, and with other types of particles. Indeed, a particle could never be detected if it failed to interact with anything.

We call these interactions secondary effects. Because gravity is a weak interaction — even at the scales of primordial inflation — scalar secondary effects are typically stronger than those from gravitons. However, graviton effects are more generic because there are many, many different scalar theories — including the possibility there are none of the simplest kind of scalars — whereas there is only one theory of gravity and we know it applies.

Quantum field theory is a very tough subject, even for people who spend their lives studying physics. Some of the work described in this section required derivations of over 80 pages in the highly condensed language of theoretical physics! We shall therefore confine the discussion of this section to the barest sketch of what was found, why it happens and what it means, but we will cite the primary literature and mention the nationalities of the
authors. For each class of secondary effect (those caused by scalars and those caused by gravitons) we will follow the hierarchy of listing secondary effects on scalars, fermions, vector particles and gravitons.

4.1 Effects Mediated by Scalars

The best studied model is when the simplest kind of scalar field is given the simplest kind of interaction and studied without gravitational interactions in the simplest inflationary geometry. For this simple model the inflationary production of scalars increases the scalar field strength. That causes the following secondary effects:

- The energy density of the scalar field increases, just as it would classically if the scalar field strength were increased [13, 14].
- Scalar particles develop an increasing mass, just as they would classically when the scalar field strength increases [15, 16].

This model is the only one in which people have worked out the lowest order quantum corrections to the initial state [17] so that the system can be released at a finite time and allowed to evolve. Physicists who have studied this model include the Frenchman Tristan Brunier, Turks Emre Kahya and Vakif Onemli, and American Richard Woodard.

No one knows whether or not the newly-discovered Higgs scalar is the simplest kind of scalar, or some more complicated kind. Assuming it is the simplest kind, then it will experience explosive particle production during inflation. The effect on known fermions such as the electron has been studied. What happens is that the increasing scalar field strength causes the fermion to behave as if it had a nonzero mass — just as a classical increase in the scalar would [18, 19]. That causes the 0-point energy of each wave number of the fermion to become ever more negative [20]. If nothing else counteracts this effect then the universe would blow up in a Big Rip singularity. Meanwhile, there is no large change in the scalar mass [21], but the way the scalar contributes to the energy density of the universe becomes far more complicated and interesting than in flat space [20]. Cited authors include the Australian Leanne Duffy, German Bjorn Garbrecht, Taiwanese Shun-Pei Miao, Croatian Tomislav Prokopec, and American Richard Woodard.

The simplest kind of scalar might be charged. If one couples it to electrodynamics in the usual way then the increase of the scalar field strength
causes the photon to develop an increasing mass — just as a classical increase would [22, 23, 24, 25]. While this happens the scalar remains light [26, 27]. However, the contribution that the entire system makes to the energy density of the universe decreases [27, 28], for the same reason as a slab of dielectric is pulled inside a charged parallel plate capacitor in classical physics. One can also study how the ensemble of charged scalars produced by inflation modifies electric and magnetic forces. One dramatic effect is a rapid extinction of the force between two charges which are initially at rest with respect to one another [29]. This phenomenon of charge screening is exactly what would occur classically if there were some classical process to dump an ever-larger number of balanced, positive-negative charge pairs into the region between the two charges. Researchers include the Swiss Henri Degueldre, Croatian Tomislav Prokopec, Swede Ola Tornkvist, Greek Nikolaos Tsamis and American Richard Woodard.

Finally we come to the effects of scalars on gravitons and on the force of gravity. Although inflation certainly produces an ensemble of the simplest kind of scalars, direct computation show that they have no significant effect on the propagation of gravitons [30, 31, 32]. The reason seems to be that gravitons interact with the derivative of the scalar field strength, and though the scalar field strength builds up, its derivative does not. If that reasoning is correct then a more complicated scalar might produce some effect. For example, if the scalar had a very small mass it would still be produced efficiently during inflation, but its field strength would matter. Study of a related possibility is under way [33]. We have cited work done by the Turks Sibel Boran and Emre Kahya, Americans Katie Leonard and Richard Woodard, Korean Sohyun Park and Croatian Tomislav Prokopec.

What has not yet been worked out is the effect that scalars have on the force of gravity. There must be some effect of this type during inflation because there is one in flat space. (Another of those wonderful Correspondence Limits which physicists find so useful.) The flat space effect takes the form of a fractional correction to the force between two masses $m_1$ and $m_2$ which are separated by a distance $r$,

$$F_{\text{flat}}(r) = \frac{Gm_1m_2}{r^2} \left\{ 1 + \frac{\hbar G}{c^2 r^2} + \ldots \right\}. \quad (51)$$

The physical interpretation is that virtual scalars tend to collapse onto the sources, which makes them become stronger. Because more and more scalars
are produced during \( q(t) = -1 \) inflation there is the possibility for a growing correction of the form,

\[
F_{\text{inf}}(r) = \frac{G m_1 m_2}{r^2} \left\{ 1 + \frac{\hbar G H_i^2}{c^5} \times H_i t + \ldots \right\}. \tag{52}
\]

The formalism needed to check for such an effect has been derived \[32\] but it has not yet been used to check for the effect.

### 4.2 Effects Mediated by Gravitons

Gravitons can affect themselves and other particles by scattering with them, much like the virtual electron-positron alter the motions of two photons in Figure 8. As the external particle propagates further and further through the ensemble of inflationary gravitons, one would expect it to be affected more and more provided it continues to interact effectively with inflationary gravitons. From expression (42) we see that inflationary gravitons typically have physical wave number \( H_i/c \). However, the physical wave number of the external particle redshifts like \( k/a(t) \). So finding a significant effect requires that the external particle must have some interaction with gravity in addition to its red-shifting energy \( \hbar c k/a(t) \).

The simplest kind of scalars interact with gravity only through their red-shifting energies, so they experience no significant effect \[34, 35\]. It is possible that more complicated scalars will show an effect, although this has not been checked. Work on this subject has been done by Turkish citizen Emre Kahya and American Richard Woodard.

Fermions possess a spin-spin interaction in addition to their kinetic energy, and this does not red-shift. One result is that fermions are gradually excited by inflationary gravitons until their field strengths become so large that perturbation theory is no longer valid \[36, 37, 38, 39, 40\]. It would be very interesting to see how this affects the 0-point energy density contributed by fermions. This system has been studied by Taiwanese citizen Shun-Pei Miao and American Richard Woodard.

Photons also possess a spin-spin interaction with gravitons so one would expect to find similar enhancements. This is indeed the case when detailed computations are done to quantum-correct the Maxwell equations which readers learn in beginning physics \[41\]. Inflationary gravitons make fascinating changes in electric and magnetic forces because they add momentum.
to the virtual photons which carry these forces \[42\]. For example, long range virtual photons carry only a tiny momentum, so the amount they can acquire from inflationary gravitons is large. Hence the long range electric field is strengthened \[42\]. The same effect strengthens the electric fields of dynamical photons as they propagate during inflation \[43\]. A study is under way of how this affects the 0-point energy density contributed by photons. Work on this system has been done by Croatians Drazen Glavan and Tomislav Prokopec, Americans Katie Leonard and Richard Woodard, Taiwanese Shun-Pei Miao and Chinese citizen Changlong Wang.

Of course gravitons can interact with other gravitons through their spin-spin couplings. The complexity of dealing with quantum gravity has so far prevented anyone from quantum-correcting the equations of gravity waves, but a simple approximation to these equations can be made which gives qualitatively correct results in all cases for which it can be checked. When this approximation is made for dynamical gravitons one finds that they too are excited by interaction with inflationary gravitons \[44\]. No result has yet been obtained for what inflationary gravitons do to the force of gravity, or how they affect the expansion of spacetime. The paper we have mentioned was co-authored by Americans Pedro Mora and Richard Woodard, and Greek citizen Nikolaos Tsamis.

Finally, we should note that these graviton effects are controversial, even though they seem to have sensible explanations. There are reasons for this. For example, part of the gravitational dynamical variable is arbitrary and it is sometimes difficult to be certain this arbitrary part has not contaminated a calculation to produce an incorrect result known as an gauge artefact. Some people also worry that quantum field theory computations average over portions of the wave function which have decohered, much like the famous feline of the Schrödinger Cat Paradox. In that thought-experiment one encloses a cat inside a box with a cannister of poison gas which is triggered by a completely random, nuclear decay process. If the decay occurs before the box is re-opened the gas is released and the cat dies. Averaging over a wave function which contains both the living and dead cat is a mistake because any observer who checks the box will either see a live (and pretty angry) cat or a dead cat, but not the average of both.

However, these worries can be checked, albeit with effort, and the checks never seem to satisfy the critics. The authors of this article sometimes experience a feeling of dizziness when listening to their own careful calculations being dismissed as nonsense by the very same people who consider wild and
untestable speculation with the greatest solemnity. It is difficult to avoid the feeling that some researchers are so deeply and loudly suspicious of secondary effects for no other reason than that they find the idea unattractive. The stubborn resistance to black holes noted in the Introduction is more typical than exceptional in the history of science. Far too many physicists allow pre-conceived notions to color their scientific judgements, which is why we ask the public to insist that scientists back up their opinions with observation and experiment, or with careful reasoning using principles established by observation and evidence.

But let us not leave the impression of bitter frustration. If we are right, the ever-more-precise and more abundant data will bear us out. In the meantime, vigorous criticism is one way science gradually works its way towards truth, and the motives of critics ought not to matter as long as the debate they spark is open. The authors of this article have been privileged to work on some fascinating problems at public expense. And the occasional critic has not prevented us from enjoying life, as witness Figure 10.

5 Conclusions

Quantum gravitational effects from the epoch of primordial inflation are observable for three reasons:

1. The inflationary Hubble parameter (37) is small enough for perturbation theory to be wonderfully accurate, but not so small that effects are beyond our ability to measure;

2. The accelerated expansion of primordial inflation produces huge numbers (42) of long wavelength gravitons and the simplest type of massless scalars; and

3. The wave functions (43) of these particles fossilize during primordial inflation so that they can be preserved to be observed at much later times.

These primary effects are quantified by the tensor and scalar power spectra (45) and (47). The scalar power spectrum has been resolved to 3-digit accuracy over wavelengths that differ by a factor of about a thousand. The tensor power spectrum has not yet been resolved but hopes are high that it soon will be.
Figure 10: The authors of this article are a Taiwanese-American couple who began working together in 2004 and came to enjoy each other’s company so much that they married in 2013.
Unless physics is radically different from what we believe, the primary effects must give rise to much weaker secondary effects in which inflationary scalars and gravitons alter the spacetime geometry and also change the dynamics of themselves and other particles. These secondary effects are too small to be measured now but they may become observable in several decades when the data available in 21-cm radiation has been fully analyzed.

Deriving these secondary effects has been the work of many men and women, from nations all around the world, over the course of more than a decade. And the most interesting computations remain to be done! These are the effect of gravitons on the force of gravity and on the overall geometry.

Our ability to for the first time observe quantum gravitational effects is changing fundamental theory. It has also confronted us with three great open problems:

- What caused primordial inflation?
- What is causing the current phase of inflation? and
- How do we correctly relate theoretical computations to observations?

Finding the answer to any of these problems would merit a Nobel Prize. We close with the hope that someone in the vast population of Mandarin readers who might see this article will rise to the challenge.

Acknowledgements

We are grateful for conversation and correspondence on this subject with R. H. Brandenberger, S. Deser, L. H. Ford, W. Israel, E. O. Kahya, K. E. Leonard, P. J. Mora, S. Park, T. Prokopec, A. Roura, N. C. Tsamis, C. L. Wang, S. Weinberg and H. L. Yu. This work was partially supported by Taiwan MOST grant 103-2112-M-006-001-MY3, by the Focus Group on Gravitation of the National Center for Theoretical Sciences of Taiwan, by NSF grants PHY-11-25915 and PHY-1205591, and by the Institute for Fundamental Theory at the University of Florida.
6 Appendix A: Quantum Mechanical Probability Densities

Figure 7 shows the famous “bell curve” \( \rho(i) \) whose integral gives the probability that a randomly selected human’s IQ score \( i \) will lie between any two values \( i_1 \) and \( i_2 \),

\[
\text{Prob}
\left(i_1 \leq i \leq i_2\right) = \int_{i_1}^{i_2} d i \rho(i) .
\]  

We can be precise about the relation between the probability densities of conjugate pairs for readers who understand complex numbers and integral calculus. The state of the quantized system \((23)\) is described by a single complex-valued wave function \( \psi(x_0) \) which is normalized,

\[
\int_{-\infty}^{\infty} d x_0 \left| \psi(x_0) \right|^2 = 1 .
\]  

As long as it obeys condition \((54)\), the wave function can be anything. However, once the wave function is known, it fixes the probability densities for both members of a conjugate pair as follows:

\[
\rho(x_0) \equiv \left| \psi(x_0) \right|^2 ,
\]

\[
\tilde{\rho}(p_0) \equiv \left| \int_{-\infty}^{\infty} d x_0 \exp \left[ \frac{i x_0 p_0}{\hbar} \right] \psi(x_0) \right|^2 .
\]

One can use relations \((55),(56)\) to prove the Uncertainty Principle \((24)\).

7 Appendix B: Energy-time Uncertainty Principle in Cosmology

In an expanding universe the energy \((38)\) is no longer a constant so the energy-time uncertainty principle becomes an integral,

\[
\int_t^{t+\Delta t} d t' E(t', k) \leq \hbar .
\]

The way to read expression \((57)\) is that a virtual particle of wave number \( k \), which emerges from empty space at time \( t \), can persist for a time \( \Delta t \) given
by the inequality. When \( a(t) = 1 \) the energy is constant so the integral gives just \( \Delta t \times E \), as in relation (31). Checking these sorts of correspondence limits is one important way that physicists avoid making mistakes!

As long as the mass \( m \) in expression (57) is nonzero the integral grows with \( \Delta t \) (it is always more than \( mc^2 \Delta t \)) and must eventually violate the inequality. So massive virtual particles live longer in an expanding universe, but they must eventually disappear. However, the case of massless particles is interesting,

\[
\lim_{m \to 0} \int_t^{t+\Delta t} dt' E(t', k) = \hbar c k \int_t^{t+\Delta t} \frac{dt'}{a(t')} \leq \hbar .
\]

(58)

This opens the fascinating possibility that \( a(t) \) grows so rapidly that the integral remains finite as \( \Delta t \) goes to infinity. For example, suppose the scale factor grows like a power of time,

\[
a(t) = a_1 \left( \frac{t}{t_1} \right)^b \implies H(t) = \frac{b}{t} \implies q(t) = -1 + \frac{1}{b} .
\]

(59)

Inflation corresponds to \( b > 1 \), which is also the case for which the integral in expression (58) remains finite as \( \Delta t \) goes to infinity,

\[
a(t) = a_1 \left( \frac{t}{t_1} \right)^b , \ b > 1
\]

\[
\implies \hbar c k \int_t^{t+\Delta t} \frac{dt'}{a(t')} = \frac{\hbar c k t_1}{a_1(1-b)} \left( \frac{t}{t_1} \right)^{1-b} \left| \right|_t^{t+\Delta t} \rightarrow -\frac{\hbar c k}{q(t)H(t)a(t)} .
\]

(60)

Therefore, any massless virtual particle which emerges from empty space during inflation with \( ck < -q(t)H(t)a(t) \) can persist forever!

8 Appendix C: How Inflationary Perturbations Fossilize

Both of gravitons and the simplest kind of massless scalar have the same wave function. The wave function \( u(t, k) \) for wave number \( k \) obeys the equations,

\[
\ddot{u} + 3H \dot{u} + \frac{c^2 k^2}{a^2} u = 0 , \quad \dot{u} u^* - \dot{u} u^* = \sqrt{\frac{32\pi \hbar G}{c}} \frac{i}{a^3} .
\]

(61)
Undergraduate physics majors learn to recognize this as the equation of a harmonic oscillator with a time dependent mass \( m(t) = \sqrt{\frac{c_k}{32\pi G}} a^3(t) \) and time dependent characteristic frequency \( \omega(t,k) = \frac{ck}{a(t)} \). They also learn that the quantum mechanical average value of the energy \( E(t,k) \) is,

\[
E(t,k) = \frac{1}{2} m(t) \left( |\dot{u}(t,k)|^2 + \omega^2(t,k)|u(t,k)|^2 \right).
\]

Finally, undergraduate physics majors learn how to use expression (62) to read off the number \( N(t,k) \) of particles which are present at time \( t \),

\[
E(t,k) = \hbar \omega(t,k) \left( \frac{1}{2} + N(t,k) \right).
\]

The initial factor of \( \frac{1}{2} \) in expression (63) is an example of the irreducible minimum level of excitation implied by the Uncertainty Principle.

Equation (61) cannot be solved for arbitrary deceleration parameter \( q(t) \). However, the data implies \(-1 \leq q_i \leq -0.9946\), so it should be a wonderful approximation to set \( q_i = -1 \). This means the Hubble parameter is a constant \( H_i \) and the scale factor grows exponentially \( a(t) = a_i \exp[H_i(t - t_i)] \).

Under these assumptions \( u(t,k) \) takes the form,

\[
u(t,k) = \sqrt{\frac{\hbar}{2m(t)\omega(t,k)}} \left[ 1 + \frac{iH_ia(t)}{ck} \right] \exp\left[-i\frac{k}{c} \int_{t_i}^{t} dt' \frac{dt'}{a(t')} \right]. \]

Substituting this solution (64) into the energy (62) and comparing with expression (63) is how one derives expression (42) in the main text. If we note that \( m(t)\omega(t) = \text{const} \times a^2(t) \) it is easy to show that \( u(t,k) \) approaches a constant at late times, which is how one derives expression (43) in the main text.

Before going any further it is useful to compare the results for \( u(t,k) \) with those for conformally coupled particles \( c(t,k) \). These wave functions can be solved for any scale factor \( a(t) \),

\[
c(t,k) = \sqrt{\frac{\hbar}{2m(t)\omega(t,k)}} \exp\left[-i\frac{k}{c} \int_{t_i}^{t} dt' \frac{dt'}{a(t')} \right].
\]

As it must, this explicit result shows all the features we inferred in the previous subsection on general grounds. For example, (60) resides in the complex
exponential, and expression (40) appears in the overall multiplicative factor \(1/\sqrt{m(t)\omega(t,k)} \sim 1/a(t)\). The average energy in this system turns out to be just \(\frac{1}{2}\hbar\omega(t,k)\), so the number of particles created is zero. That is only the average energy, and scales during primordial inflation are so enormous that there are still significant quantum gravitational effects, however, the late time limit of this wave function is zero,

\[
\lim_{t \to \infty} c(t,k) = 0. \tag{66}
\]

This means that nothing can survive to be observed at later times.

References

[1] S. W. Hawking, Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].

[2] W. G. Unruh, Phys. Rev. D 14, 870 (1976).

[3] A. Loeb and M. Zaldarriaga, Phys. Rev. Lett. 92, 211301 (2004) [astro-ph/0312134].

[4] W. Israel in, BLACK HOLES: THEORY AND OBSERVATION, ed. by F.W. Hehl, C. Kiefer, and R.J.K. Metzler (Springer-Verlag, Berlin, 1998), p. 487.

[5] W. Israel, Foundations of Physics 26 (1996) 595.

[6] P. A. R. Ade et al. [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO].

[7] A. A. Starobinsky, JETP Lett. 30, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. 30, 719 (1979)].

[8] V. F. Mukhanov and G. V. Chibisov, JETP Lett. 33, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. 33, 549 (1981)].

[9] G. F. Smoot, C. L. Bennett, A. Kogut, E. L. Wright, J. Aymon, N. W. Boggess, E. S. Cheng and G. De Amici et al., Astrophys. J. 396, L1 (1992).
[10] P. A. R. Ade et al. [BICEP2 Collaboration], Phys. Rev. Lett. 112, no. 24, 241101 (2014) [arXiv:1403.3985 [astro-ph.CO]].

[11] R. Adam et al. [Planck Collaboration], [arXiv:1409.5738 [astro-ph.CO]].

[12] P. A. R. Ade et al. [BICEP2 and Planck Collaborations], Phys. Rev. Lett. 114, no. 10, 101301 (2015) [arXiv:1502.00612 [astro-ph.CO]].

[13] V. K. Onemli and R. P. Woodard, Class. Quant. Grav. 19, 4607 (2002) [gr-qc/0204065].

[14] V. K. Onemli and R. P. Woodard, Phys. Rev. D 70, 107301 (2004) [gr-qc/0406098].

[15] T. Brunier, V. K. Onemli and R. P. Woodard, Class. Quant. Grav. 22, 59 (2005) [gr-qc/0408080].

[16] E. O. Kahya and V. K. Onemli, Phys. Rev. D 76, 043512 (2007) [gr-qc/0612026].

[17] E. O. Kahya, V. K. Onemli and R. P. Woodard, Phys. Rev. D 81, 023508 (2010) [arXiv:0904.4811 [gr-qc]].

[18] T. Prokopec and R. P. Woodard, JHEP 0310, 059 (2003) [astro-ph/0309593].

[19] B. Garbrecht and T. Prokopec, Phys. Rev. D 73, 064036 (2006) [gr-qc/0602011].

[20] S. P. Miao and R. P. Woodard, Phys. Rev. D 74, 044019 (2006) [gr-qc/0602110].

[21] L. D. Duffy and R. P. Woodard, Phys. Rev. D 72, 024023 (2005) [hep-ph/0505156].

[22] T. Prokopec, O. Tornkvist and R. P. Woodard, Phys. Rev. Lett. 89, 101301 (2002) [astro-ph/0205331].

[23] T. Prokopec, O. Tornkvist and R. P. Woodard, Annals Phys. 303, 251 (2003) [gr-qc/0205130].

[24] T. Prokopec and R. P. Woodard, Am. J. Phys. 72, 60 (2004) [astro-ph/0303358].
[25] T. Prokopec and R. P. Woodard, Annals Phys. 312, 1 (2004) [gr-qc/0310056].

[26] T. Prokopec, N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 24, 201 (2007) [gr-qc/0607094].

[27] T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. 323, 1324 (2008) [arXiv:0707.0847 [gr-qc]].

[28] T. Prokopec, N. C. Tsamis and R. P. Woodard, Phys. Rev. D 78, 043523 (2008) [arXiv:0802.3673 [gr-qc]].

[29] H. Degueldre and R. P. Woodard, Eur. Phys. J. C 73, no. 6, 2457 (2013) [arXiv:1303.3042 [gr-qc]].

[30] S. Park and R. P. Woodard, Phys. Rev. D 83, 084049 (2011) [arXiv:1101.5804 [gr-qc]].

[31] S. Park and R. P. Woodard, Phys. Rev. D 84, 124058 (2011) [arXiv:1109.4187 [gr-qc]].

[32] K. E. Leonard, S. Park, T. Prokopec and R. P. Woodard, Phys. Rev. D 90, 024032 (2014) [arXiv:1403.0896 [gr-qc]].

[33] S. Boran, E. O. Kahya and S. Park, Phys. Rev. D 90, no. 12, 124054 (2014) [arXiv:1409.7753 [gr-qc]].

[34] E. O. Kahya and R. P. Woodard, Phys. Rev. D 76, 124005 (2007) [arXiv:0709.0536 [gr-qc]].

[35] E. O. Kahya and R. P. Woodard, Phys. Rev. D 77, 084012 (2008) [arXiv:0710.5282 [gr-qc]].

[36] S. P. Miao and R. P. Woodard, Class. Quant. Grav. 23, 1721 (2006) [gr-qc/0511140].

[37] S. P. Miao and R. P. Woodard, Phys. Rev. D 74, 024021 (2006) [gr-qc/0603135].

[38] S. P. Miao, arXiv:0705.0767 [hep-th].

[39] S. P. Miao and R. P. Woodard, Class. Quant. Grav. 25, 145009 (2008) [arXiv:0803.2377 [gr-qc]].
[40] S. P. Miao, Phys. Rev. D 86, 104051 (2012) [arXiv:1207.5241 [gr-qc]].

[41] K. E. Leonard and R. P. Woodard, Class. Quant. Grav. 31, 015010 (2014) [arXiv:1304.7265 [gr-qc]].

[42] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, Class. Quant. Grav. 31, 175002 (2014) [arXiv:1308.3453 [gr-qc]].

[43] C. L. Wang and R. P. Woodard, arXiv:1408.1448 [gr-qc].

[44] P. J. Mora, N. C. Tsamis and R. P. Woodard, JCAP 1310, 018 (2013) [arXiv:1307.1422, arXiv:1307.1422].