Existence and multiplicity of Homoclinic solutions for the second order Hamiltonian systems

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Abstract

In this paper we study the existence and multiplicity of homoclinic solutions for the second order Hamiltonian system
\[ \ddot{u} - L(t)u(t) + W_u(t, u) = 0, \quad \forall t \in \mathbb{R}, \]
by means of the minmax arguments in the critical point theory, where \( L(t) \) is unnecessary uniformly positively definite for all \( t \in \mathbb{R} \) and \( W_u(t, u) \) satisfies the asymptotically linear condition.

Keywords: Homoclinic solution; Second order Hamiltonian system; Linking structure; Mountain pass theorem

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1 Introduction and the main result

Consider the second order Hamiltonian systems
\[ \ddot{u} - L(t)u(t) + W_u(t, u) = 0, \quad \forall t \in \mathbb{R}, \tag{1.1} \]
where \( L \in C \left( \mathbb{R}, \mathbb{R}^{N^2} \right) \) is a symmetric matrix valued function, \( W \in C^1 \left( \mathbb{R} \times \mathbb{R}^N, \mathbb{R} \right) \). We say that a solution \( u \) of (1.1) is homoclinic (to 0) if \( u \in C^2 \left( \mathbb{R}, \mathbb{R}^N \right), u \neq 0, u(t) \to 0 \) and \( \dot{u}(t) \to 0 \) as \( |t| \to \infty \).

The existence and multiplicity of homoclinic solutions for (1.1) have been extensively investigated in many papers via the variational methods, see, e.g., [1-6, 8, 9, 12, 14-19].

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Most of them treat the superquadratic case (see [1-6, 8, 9, 12, 14-16]), while [18, 19] consider the asymptotically quadratic case and [5, 17] treat the subquadratic case. But except for [5, 14] all known results are obtained under the following assumption that $L(t)$ is uniformly positively definite for all $t \in \mathbb{R}$, that is, there exists a constant $l_0 > 0$ such that

$$\langle L(t)u, u \rangle \geq l_0 |u|^2, \quad t \in \mathbb{R}, \quad u \in \mathbb{R}^N,$$

where $\langle \cdot, \cdot \rangle$ and $|\cdot|$ are the standard inner product and the associated norm in $\mathbb{R}^N$ respectively and we will always use these notations.

In this paper, we study the homoclinic solutions of (1.1) where $L(t)$ is unnecessary uniformly positively definite for all $t \in \mathbb{R}$, and $W(t, u)$ satisfies subquadratic condition. More precisely, $L$ satisfies

(L1) The smallest eigenvalue of $L(t) \to \infty$ as $|t| \to \infty$, i.e.,

$$l(t) \equiv \inf_{|u|=1, \ u \in \mathbb{R}^N} \langle L(t)u, u \rangle \to \infty, \quad \text{as } |t| \to \infty,$$

(L2) For some $a > 0$ and $\bar{r} > 0$, one of the following is true:

(i) $L \in C^1(\mathbb{R}, \mathbb{R}^N)$ and $|L'(t)| \leq a |L(t)|$, $\forall |t| \geq \bar{r}$, or

(ii) $L \in C^2(\mathbb{R}, \mathbb{R}^N)$ and $L''(t) \leq a L(t)$, $\forall |t| \geq \bar{r}$,

where $L'(t) = (d/dt)L(t)$ and $L''(t) = (d^2/dt^2)L(t)$,

and $W(t, u)$ satisfies

(W1) $W(t, u) \geq 0$, $W(t, 0) = 0$ and $W_u(t, u) = o(|u|)$ as $u \to 0$ uniformly in $t$, $|W_a(t, u)| \leq C_W(|u|)$ for some $C_W > 0$.

In what follows it will always be assumed that (L1) is satisfied. Denote by $A$ the self-adjoint extension of the operator $-(d^2/dt^2) + L(t)$ with domain $D(A) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let $\{E(\lambda) : -\infty < \lambda < \infty\}$ and $|A|$ be the spectral resolution and the absolute value of $A$ respectively, and $|A|^{1/2}$ be the square root of $|A|$ with domain $D(|A|^{1/2})$. Set $U = I - E(0) - E(-0)$, where $I$ is the identity map on $L^2$. Then $U$ commutes with $A$, $|A|$ and $|A|^{1/2}$, and $A = U|A|$ is the polar decomposition of $A$ (see [11]). Let $E = D(|A|^{1/2})$,
and define on $E$ the inner product and norm
\[(u, v)_0 = (|A|^{1/2}u, |A|^{1/2}v)_2 + (u, v)_2,\]
\[\|u\|_0 = (u, u)_0^{1/2},\]
where $(\cdot, \cdot)_2$ denotes the inner product in $L^2$; then $E$ is a Hilbert space.

In order to learnt about the spectrum of $A$, we first need the following lemma from [5] (cf. Lemma 2.1 in [5]).

**Lemma 1.1.** Suppose that $L$ satisfies (L1), then $E$ is compactly embedded in $L^p \equiv L^p(\mathbb{R}, \mathbb{R}^N)$ for $2 \leq p \leq \infty$.

**Remark 1.2.** It is easy to see that $E$ is continuously embedded in $H^{1,2}(\mathbb{R}, \mathbb{R}^N)$ from the fact that $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $E$ and the proof of Lemma 2.1 in [5].

From [5], under the above assumption (L1) on $L$ and by Lemma 1.1, we know that $A$ possesses a compact resolvent and the spectrum $\sigma(A)$ consists of only eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$, with a corresponding eigenfunctions $(e_n)(Ae_n = \lambda_ne_n)$, forming an orthogonal basis in $L^2$. Let $n^- = \#\{i|\lambda_i < 0\}$, $n^0 = \#\{i|\lambda_i = 0\}$, and $\bar{n} = n^- + n^0$. Set $E^- = \text{span}\{e_1, \ldots, e_{n^-}\}$, $E^0 = \text{span}\{e_{n^-+1}, \ldots, e_\bar{n}\} = \ker A$ and $E^+ = \text{span}\{e_{\bar{n}+1}, \ldots\}$. Then one has the orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$ with respect to the inner product $(\cdot, \cdot)_0$ on $E$. Now we introduce on $E$ the following inner product and norm:
\[(u, v) = (|A|^{1/2}u, |A|^{1/2}v)_2 + (u^0, v^0)_2,\]
\[\|u\| = (u, u)^{1/2},\]
where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E = E^- \oplus E^0 \oplus E^+$. Clearly the norms $\| \cdot \|$ and $\| \cdot \|_0$ are equivalent (cf. [5]). From now on $\| \cdot \|$ will be used.

**Remark 1.3.** Note that the decomposition $E = E^- \oplus E^0 \oplus E^+$ is also orthogonal with respect to both $(\cdot, \cdot)$ and $(\cdot, \cdot)_2$.

**Remark 1.4.** Since the norms $\| \cdot \|$ and $\| \cdot \|_0$ on $E$ are equivalent, then by Lemma 1.1 for any $2 \leq p \leq \infty$, there exists $\beta_p > 0$ such that
\[|u|_p \leq \beta_p\|u\|, \quad \forall u \in E,\] (1.2)
where $| \cdot |_p$ is the norm on $L^p$. 

3
For later use, let
\[ a(u, v) = (|A|^{1/2} U u, |A|^{1/2} v)_2, \quad \forall u, v \in E \]  
be the quadratic form associated with \( A \). For any \( u \in D(A) \) and \( v \in E \), we have
\[ a(u, v) = \int \langle \dot{u}, \dot{v} \rangle + \langle L(t)u, v \rangle \, dt \]  
and (1.4) holds for all \( u, v \in E \) since \( D(A) \) is dense in \( E \). Moreover, by definition
\[ a(u, u) = ((P^+ - P^-)u, u) = \|u^+\|^2 - \|u^-\|^2 \]  
for all \( u = u^- + u^0 + u^+ \in E \), where \( P^\pm : E \to E^\pm \) are the orthogonal projections with respect to the inner product \((\cdot, \cdot)\).

We further make the following assumptions on \( W \):

(W2) \( W_u(t, u) = M(t)u + w_u(t, u) \) with \( M \) a bounded, continuous symmetric \( N \times N \) matrix-valued function and \( w_u(t, u) = o(|u|) \) as \( |u| \to \infty \), \( \forall t \in \mathbb{R} \);

(W3) \( m_0 := \inf_{t \in \mathbb{R}} \left[ \inf_{|u|=1, \; u \in \mathbb{R}^N} \langle M(t)u, u \rangle \right] > \inf (\sigma(A) \cap (0, \infty)) \);

(W4) \( 0 \notin \sigma_p(A - M) \), where \( \sigma_p(A - M) \) is the point spectrum of \( A - M \), \( M \) is the operator defined on \( L^2 \) by
\[ (Mu)(t) := M(t)u(t), \quad t \in \mathbb{R}, \; u \in L^2. \]

From the above spectral result of the operator \( A \), the set \( \sigma(A) \cap (0, m_0) \) consists of only eigenvalues of finite multiplicity, where \( m_0 \) is defined in (W3). Let \( \ell \) denote the number of eigenvalues (counted with multiplicity) lying in \( (0, m_0) \).

Then we have our main result:

**Theorem 1.5.** Suppose that (L1), (L2) and (W1)–(W4) are satisfied. Then (1.1) has at least one nontrivial homoclinic solution. If in addition \( W(t, u) \) is even in \( u \), then (1.1) has at least \( \ell \) pairs of nontrivial homoclinic solutions.
Remark 1.6. There are functions $L$ and $W$ which satisfy the conditions in our Theorem 1.5 but do not satisfy the corresponding conditions in [1-6, 8, 9, 12, 14-19]. For example, let
\[
L(t) = \begin{cases} (e^{t^2} - 2)I_N, & |t| \leq 1/\sqrt{e} \\ (\ln t^2)I_N, & |t| > 1/\sqrt{e}, \end{cases}
\]
\[
W(t, u) = \frac{1}{2} (e^{-t^2} + a) |u|^2 \left(1 - \frac{1}{\ln (e + |u|)}\right).
\]
Simple computation shows that $M(t) = (e^{-t^2} + a) I_N$ in (W2) and we can choose suitable $a > \inf(\sigma(A) \cap (0, \infty))$ such that (W4) holds due to the special spectral result of $A$ above.

2 Variational setting and proof of the main result

In order to establish a variational setting for the problem (1.1), we further need the following lemma which can be found in [5].

**Lemma 2.1** ([5, Lemma 2.3]). If $L$ satisfies (L1) and (L2), then $D(A)$ is continuously embedded in $H^{2,2}(\mathbb{R}, \mathbb{R}^N)$ and consequently, we have
\[
|u(t)| \to 0 \text{ and } \dot{u}(t) \to 0 \text{ as } |t| \to \infty, \forall u \in D(A).
\]

For any fixed $b > 0$, let $k$ be the number of eigenvalues of the operator $A$ (counted with multiplicity) lying in $[-b, b]$. Denote by $f_i$ ($1 \leq i \leq k$) the corresponding eigenfunctions and set
\[
L^b = \text{span}\{f_1, \ldots, f_k\},
\]
then we have the orthogonal decomposition
\[
L^2 = L^b \oplus L^{b+}, \quad u = u^b + u^{b+}.
\]
where $L^{b+}$ is the orthogonal complement of $L^b$ in $L^2$.

Correspondingly, $E$ has the decomposition
\[
E = E^b \oplus E^{b+} \text{ with } E^b = L^b \text{ and } E^{b+} = E \cap L^{b+},
\]
orthogonal with respect to both the inner products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. Then we have the following lemma which will be used.
Lemma 2.2. For any fixed $b > 0$, let $E = E^b - \oplus E^b +$ as above, then

$$b|u|^2 \leq \|u\|^2 \quad \text{for all } u \in E^b +,$$

where $| \cdot |_2$ is the norm on $L^2$.

Proof. It is obvious from the definition of the norm $\| \cdot \|$ on $E$ and the distribution of the eigenvalues of $A$. \qed

By virtue of the quadratic form in (1.3), we define a functional $\Phi$ on $E$ by

$$\Phi(u) = \frac{1}{2} a(u, v) - \Psi(u)$$

$$= \frac{1}{2} \int_{\mathbb{R}} (|\dot{u}|^2 + \langle L(t)u, u \rangle) dt - \Psi(u)$$

$$= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \Psi(u) \quad \text{where } \Psi(u) = \int_{\mathbb{R}} W(t, u) dt \quad (2.1)$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. By (W1) and Lemma 1.1, $\Phi$ and $\Psi$ are well defined. Furthermore, we have

Proposition 2.3. Let (L1), (L2) and (W1) be satisfied. Then $\Psi \in C^1(E, \mathbb{R})$, and hence $\Phi \in C^1(E, \mathbb{R})$. Moreover,

$$\Psi'(u)v = \int_{\mathbb{R}} \langle W_u(t, u), v \rangle dt \quad (2.2)$$

$$\Phi'(u)v = (u^+, v^+) - (u^-, v^-) - \Psi'(u)v$$

$$= (u^+, v^+) - (u^-, v^-) - \int_{\mathbb{R}} \langle W_u(t, u), v \rangle dt \quad (2.3)$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$ and $v = v^- + v^0 + v^+ \in E = E^- \oplus E^0 \oplus E^+$, and critical points of $\Phi$ on $E$ are homoclinic solutions of (1.1).

Proof. We first verify (2.2) by definition. Let $u \in E$. Using (W1), by the mean value
theorem and the H"older inequality, we have

\[
\left| \int_{|t|>T} (W(t, u + v) - W(t, u) - \langle W_u(t, u), v \rangle) dt \right| \\
\leq C \left( \int_{|t|>T} (|u| + |v|)^2 dt \right)^{1/2} |v|_2 \\
\leq C\beta_2 \left( \int_{|t|>T} (|u| + |v|)^2 dt \right)^{1/2} \|v\|, \forall T > 0, \forall v \in E,
\] (2.4)

where \( C \) is a constant and the last inequality holds by (1.2). In view of Lemma 1.1, for any \( \varepsilon > 0 \), there is a \( \delta_1 > 0 \) and \( T_\varepsilon > 0 \) such that

\[
C\beta_2 \left( \int_{|t|>T_\varepsilon} (|u| + |v|)^2 dt \right)^{1/2} \leq \varepsilon/2,
\] (2.5)

for all \( v \in E, \|v\| \leq \delta_1 \).

From Remark 1.2, \( u \in H^{1,2}(\mathbb{R}, \mathbb{R}^N) \). Define \( \Psi_T : E \to \mathbb{R} \) by

\[
\Psi_T(u) := \int_{-T}^T W(t, u) dt, \forall u \in E.
\]

It is known (see, e.g., [13]) that \( \Psi_T \in C^1(H^{1,2}([-T, T], \mathbb{R}^N), \mathbb{R}) \) for any \( T > 0 \). Therefore, for the \( \varepsilon \) and \( T_\varepsilon \) given above, by Remark 1.2 there is a \( \delta_2 = \delta_2(\varepsilon, T_\varepsilon, u) \) such that

\[
\left| \int_{-T_\varepsilon}^{T_\varepsilon} (W(t, u + v) - W(t, u) - \langle W_u(t, u), v \rangle) dt \right| \leq \frac{\varepsilon}{2}\|v\|,
\] (2.6)

for all \( v \in E, \|v\| \leq \delta_2 \).

Combining (2.4), (2.5) with (2.6) and taking \( \delta = \min\{\delta_1, \delta_2\} \), then we obtain

\[
\left| \int_{\mathbb{R}} (W(t, u + v) - W(t, u) - \langle W_u(t, u), v \rangle) dt \right| \leq \varepsilon\|v\|
\]

for all \( v \in E, \|v\| \leq \delta \). Thus (2.2) follows immediately by the definition of Fréchet derivatives. Due to the form of \( \Phi \) in (2.1), (2.3) also holds.
We then prove that $\Psi'$ is continuous. Suppose $u_n \to u_0$ in $E$ and hence $u_n \to u_0$ in $L^\infty$ by Lemma 1.1. Note that

$$\sup_{\|v\|=1} \|(\Psi'(u_n) - \Psi'(u_0))v\| = \sup_{\|v\|=1} \left| \int \langle W_u(t, u_n) - W_u(t, u_0), v \rangle dt \right|$$

$$\leq \sup_{\|v\|=1} \left( \int |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} |v|_2$$

$$\leq \beta_2 \left( \int |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2}$$

(2.7)

where $\beta_2$ is the constant in (1.2).

Note that, by Lemma 1.1, $(u_n)$ is bounded in $L^2$ since $u_n \to u_0$ in $E$, i.e., there exists a constant $M_0 > 0$ such that $|u_n|_2 \leq M_0$, $\forall n \in \mathbb{N}$. By (W1), for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|W_u(t, u_n)| \leq \frac{\varepsilon}{2(M_0 + |u_n|_2)} |u|, \forall u \in \mathbb{R}^N, |u| \leq \eta.$$  

(2.8)

Due to $u_0 \in H^{1,2}(\mathbb{R}, \mathbb{R}^N)$ and $u_n \to u_0$ in $L^\infty$, there exist $T_\varepsilon > 0$ and $N_1 \in \mathbb{N}$ such that for all $n > N_1$ and $|t| \geq T_\varepsilon$, it holds that

$$|W_u(t, u_n(t))| \leq \frac{\varepsilon}{2(M_0 + |u_n|_2)} |u_n(t)|,$$

$$|W_u(t, u_0(t))| \leq \frac{\varepsilon}{2(M_0 + |u_0|_2)} |u_0(t)|.$$  

(2.9)

Observe also that $(u_n)$ is bounded in $L^\infty$, then by (W1) and Lebesgue’s Dominated Convergence Theorem,

$$\left( \int_{-T_\varepsilon}^{T_\varepsilon} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} \to 0, \text{ as } n \to \infty.$$  

Then there exists $N_2 \in \mathbb{N}$ such that for all $n > N_2$,

$$\left( \int_{-T_\varepsilon}^{T_\varepsilon} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} \leq \varepsilon/2$$
Combining this with (2.9) and taking \( N = \max \{ N_1, N_2 \} \), we have

\[
\left( \int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} \\
\leq \left( \int_{-T_e}^{T_e} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} + \left( \int_{|t| > T_e} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(M_0 + |u_0|_2)}(|u_n|_2 + |u_0|_2) \leq \varepsilon
\]

for all \( n > N \). This shows that

\[
\left( \int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u_0)|^2 dt \right)^{1/2} \to 0, \quad n \to \infty.
\]

Thus the continuity of \( \Psi' \) follows immediately by (2.7). Consequently, the form of \( \Phi \) yields \( \Phi \in C^1(E, \mathbb{R}) \).

Finally, we show that critical points of \( \Phi \) on \( E \) are homoclinic solutions of (1.1). Note first that, by means of a standard argument, (1.3)—(1.5) and (2.3) imply that critical points of \( \Phi \) belong to \( C^2(\mathbb{R}, \mathbb{R}^N) \) and satisfy (1.1). Now for any critical point \( u \) of \( \Phi \) on \( E \), by (W1) and Lemma 1.1, one has

\[
|Au|^2 = \int_{\mathbb{R}} |W_u(t, u)|^2 dt \\
\leq C_W^2 |u|^2 < \infty.
\]

where \( C_W \) is the constant in (W1). Thus \( u \in D(A) \) and \( u \) is a homoclinic solution of (1.1) by Lemma 2.1. The proof is completed.

We will make use of minimax arguments to prove our main result and first state two results of this type from Rabinowitz [13] and Ghoussoub [10] here. One is the following linking theorem:
Theorem 2.4 ([13, Theorem 5.3]). Let $E$ be a real Banach space with $E = V \oplus X$, where $V$ is finite dimensional. Suppose $\Phi \in C^1(E, \mathbb{R})$, satisfies $(PS)$-condition, and

$(\Phi_1)$ there are constants $\rho, \alpha > 0$ such that $\Phi |_{\partial B_\rho \cap X} \geq \alpha$, and

$(\Phi_2)$ there is an $e \in \partial B_\rho \cap X$ and $R > \rho$ such that if $Q \equiv (B_R \cap V) \oplus \{re \mid 0 < r < R\}$, then $\Phi |_{\partial Q} \leq 0$.

where $B_r$ is an open ball in $E$ of radius $r$ centered at 0.

Then $\Phi$ possess a critical value $c \geq \alpha$ which can be characterized as

$$c \equiv \inf_{h \in \Gamma} \max_{u \in Q} \Phi(h(u)),$$

where

$$\Gamma = \{h \in C(Q, E) \mid h = \text{id} \text{ on } \partial Q\}.$$

The other one is the $\mathbb{Z}_2$-symmetric Mountain Pass Theorem:

Theorem 2.5 ([10, Corollary 7.22]). Let $\Phi$ be an even $C^1$-functional satisfying $(PS)$ on $X = Y \oplus Z$ where $\dim(Y) = k < \infty$. Assume $\Phi(0) = 0$ as well as the following conditions:

(1) There is $\rho > 0$ and $\alpha \geq 0$ such that $\inf \Phi(S_\rho(Z)) \geq \alpha$.

(2) There exists $R > \rho$ and a subspace $F$ of $X$ containing $Y$ such that $\dim(F) = n > k$ and $\sup \Phi(S_R(F)) \leq 0$.

There exists then critical values $c_i$ ($1 \leq i \leq n - k$) for $\Phi$ such that

(a) $0 \leq \alpha \leq c_1 \leq \cdots \leq c_{n-k}$.

(b) $\Phi$ has at least $n - k$ distinct pairs of non-trivial critical points.

In order to prove our main result by virtue of the above theorems, we need to investigate the $(PS)$-condition and the linking structure with respect to the functional. We will divide it into two parts and follow partially the ideas of the paper [7] to give the proofs of some lemmas in the two parts as follows.
Part I. The \((PS)\)-condition

we will discuss the \((PS)\)-condition in this part.

**Lemma 2.6.** Suppose that \((W1)\), \((W2)\) and \((W4)\) are satisfied, then any \((PS)\)-sequence is bounded.

**Proof.** Let \((u_n) \subset E\) be a \((PS)\)-sequence, i.e., there exists a constant \(C_0 > 0\) such that

\[
|\Phi(u_n)| \leq C_0 \quad \text{and} \quad \Phi'(u_n) \to 0. \tag{2.10}
\]

Arguing indirectly we assume that, up to a subsequence, \(\|u_n\| \to \infty\) and set \(v_n = u_n/\|u_n\|\). Then \(\|v_n\| = 1\). By Lemma 1.1 passing to a subsequence if necessary, \(v_n \rightharpoonup v\) in \(E\) and \(v_n \to v\) in \(L^p\) for all \(2 \leq p \leq \infty\). Then \(v_n\) is bounded in \(L^\infty\). Since, by \((W1)\) and \((W2)\), \(|w_u(t, u)| \leq C_w |u|\) for some \(C_w > 0\), \(w_u(t, u) = o(|u|)\) as \(|u| \to \infty\), \(\forall t \in \mathbb{R}\) and \(|u_n(t)| \to \infty\) if \(v(t) \neq 0\), then it follows, by Lebesgue’s Dominated Convergence Theorem, that

\[
\int_\mathbb{R} \frac{\langle W_u(t, u_n(t)), \varphi(t) \rangle}{\|u_n\|} dt = \int_\mathbb{R} \langle M(t)v_n(t), \varphi(t) \rangle dt + \int_{u_n(t) \neq 0} \frac{\langle w_u(t, u_n(t)), \varphi(t) \rangle}{\|u_n\|} v_n(t) dt \\
\to \int_\mathbb{R} \langle M(t)v(t), \varphi(t) \rangle dt \quad \text{as} \quad n \to \infty
\]

for all \(\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)\).

By (2.3), we have

\[
\frac{\Phi'(u_n)\varphi}{\|u_n\|} = (v_n^+, \varphi) - (v_n^-, \varphi) - \int_\mathbb{R} \frac{\langle W_u(t, u_n(t)), \varphi(t) \rangle}{\|u_n\|} dt
\]

for all \(\varphi \in C_0^\infty(\mathbb{R}, \mathbb{R}^N)\). From this we deduce, using (2.10), that

\[
(-d^2/dt^2 + L(t))v(t) = M(t)v(t),
\]

i.e.,

\[
(A - M)v = 0. \tag{2.11}
\]

We claim that \(v \neq 0\). Arguing by contradiction we assume that \(v = 0\). Choose \(b > 0\) in Lemma 2.2 such that \(C_w b < 1\), where \(C_W\) is the constant in \((W1)\). Since \(E^{b^-} \subset E\)}
in Lemma 2.2 is of finite-dimension, then the compactness of the orthogonal projection $P^b : E \to E^b \subset E$ implies $v_n^b \to v^b = 0$ in $E$.

It follows from (2.3) that

$$\frac{\Phi'(u_n)((u_n^b)^+ - (u_n^b)^-)}{\|u_n\|^2} = \|v_n^b\|^2 - \int_{u_n(t) \neq 0} \frac{W_n(t, u_n), (v_n^b)^+ - (v_n^b)^-}{|u_n|} |v_n| dt,$$

then

$$\|v_n^b\|^2 = \int_{u_n(t) \neq 0} \frac{W_n(t, u_n), (v_n^b)^+ - (v_n^b)^-}{|u_n|} |v_n| dt + \frac{\Phi'(u_n)((u_n^b)^+ - (u_n^b)^-)}{\|u_n\|^2},$$

$$\leq C_W \int_{\mathbb{R}} |(v_n^b)^+ - (v_n^b)^-| |v_n| dt + \frac{\|\Phi'(u_n)\|}{\|u_n\|},$$

$$\leq \frac{C_W}{2} \left( \int_{\mathbb{R}} |(v_n^b)^+ + (v_n^b)^-|^2 dt + \int_{\mathbb{R}} |(v_n^b)^+ - (v_n^b)^-|^2 dt \right)$$

$$+ \frac{C_W}{2} \int_{\mathbb{R}} |v_n^b|^2 dt + \frac{\|\Phi'(u_n)\|}{\|u_n\|},$$

$$= C_W \|v_n^b\|_2^2 + \frac{C_W}{2} \|v_n^b\|_2^2 + \frac{\|\Phi'(u_n)\|}{\|u_n\|},$$

$$\leq \frac{C_W}{b} \|v_n^b\|^2 + \frac{C_W}{2} \|v_n^b\|^2 + \frac{\|\Phi'(u_n)\|}{\|u_n\|},$$

where $| \cdot |_2$ is the norm on $L^2$ and $(\cdot)^+, (\cdot)^-$ are the respective components with respect to the orthogonal decomposition in Remark 1.3. The last inequality follows by Lemma 2.2.

Note that $v_n^b \to 0$ in $L^2$ since $v_n^b \to v^b$ in $E$. Thus $\frac{C_W}{b} < 1$ and (2.10) imply $\|v_n^b\|^2 \to 0$. Then $1 = \|v_n\|^2 = \|v_n^b\|^2 + \|v_n^b\|^2 \to 0$, a contradiction.

Therefore, $v \neq 0$. Then (2.11) implies that 0 is an eigenvalue of $A - M$ which is in contradiction to (W4).

\[ \square \]

**Lemma 2.7.** Suppose that (W1), (W2) and (W4) are satisfied. Then $\Phi$ satisfies the (PS)-condition.

**Proof.** Let $(u_n) \subset E$ be an arbitrary (PS)-sequence. By Lemma 2.6 it is bounded, hence, we may assume without loss of generality that $u_n \rightharpoonup u$ in $E$ and hence $u_n^+ \rightharpoonup u^+$.
and \( u^- \to u^- \) due to \( \dim(E^-) < \infty \). By Lemma 1.1, \( u_n \to u \) and \( u_n^+ \to u^+ \) in \( L^2 \). Observe that
\[
\| u_n^+ - u_m^+ \|^2 = (\Phi'(u_n) - \Phi'(u_m))(u_n^+ - u_m^+)
+ \int_{\mathbb{R}} (W_u(t, u_n(t)) - W_u(t, u_m(t)), u_n^+ - u_m^+) dt, \quad \forall n, m \in \mathbb{N}.
\]
(2.12)

By (W1) and Hölder inequality
\[
\left| \int_{\mathbb{R}} \langle W_u(t, u_n(t)) - W_u(t, u_m(t)), u_n^+ - u_m^+ \rangle dt \right|
\leq C_W \int_{\mathbb{R}} (|u_n| + |u_m|)|u_n^+ - u_m^+| dt
\leq C_W (|u_n|_2 + |u_m|_2)|u_n^+ - u_m^+|_2 \to 0 \quad \text{as } n, m \to \infty
\]
since \( u_n \to u \) and \( u_n^+ \to u^+ \) in \( L^2 \).

Note that
\[
(\Phi'(u_n) - \Phi'(u_m))(u_n^+ - u_m^+) \to 0 \quad \text{as } n, m \to \infty
\]
since \( \Phi'(u_n) \to 0 \) and \( (u_n) \) is bounded in \( E \). Then (2.12) implies that \( (u_n^+) \) is a Cauchy sequence in \( E \). Hence \( u_n^+ \to u^+ \) in \( E \). Recall that \( \dim(E^- \oplus E^0) < \infty \), then \( u_n^- + u_n^0 \to u^- + u^0 \) in \( E \). This yields \( u_n \to u \) in \( E \) and the proof is completed.

\( \square \)

**Part II. Linking structure.**

First we have the following lemma.

**Lemma 2.8.** Let (W1) be satisfied. Then there exists \( \rho > 0 \) such that
\[
\alpha := \inf_{\partial B_\rho \cap E^+} \Phi > 0.
\]

**Proof.** By Lemma 1.1 we have
\[
|u|_\infty \to 0 \text{ as } \|u\| \to 0,
\]
(2.13)
where \( \cdot \) is the norm on \( L^\infty \). From (W1), we obtain that \( W(t, u) = o(|u|^2) \) as \( |u| \to 0 \) uniformly in \( t \). Combining this with (2.13), for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that
\[
\Psi(u) \leq \varepsilon |u|^2 \leq \varepsilon \beta_2^2 \|u\|^2, \quad \forall \|u\| \leq \delta.
\]
where $\beta_2$ is the constant in (1.2). Taking $\varepsilon = 1/(4\beta_2^2)$ and $0 < \rho < \delta$, then $\alpha := \inf \Phi(\partial B^r \cap E^+) \geq \rho^2/4 > 0$ by the form of $\Phi$ in (2.1).

Due to (W3) and the spectral result of $A$ in the previous section, we can arrange all the eigenvalues (counted with multiplicity) of $A$ in $(0, m_0)$ by $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell < m_0$ and let $e_j$ denote the corresponding eigenfunctions: $A e_j = \lambda_j e_j$ for $j = 1, \ldots, \ell$. Set $E_\ell^+ := \text{span}\{e_1, \ldots, e_\ell\}$. According to the definition of the norm on $E$, we have

$$\lambda_1 |v|^2 \leq \|v\|^2 \leq \lambda_\ell |v|^2 \quad \text{for all } v \in E_\ell^+. \quad (2.14)$$

Set $\hat{E} = E^- \ominus E^0 \oplus E_\ell^+$.

**Lemma 2.9.** Let (W1), (W2) and (W3) be satisfied and $\rho > 0$ be given by Lemma 2.8. Then there exists $R_{\hat{E}} > \rho$ such that $\Phi(u) < 0$ for all $u \in \hat{E}$ with $\|u\| \geq R_{\hat{E}}$.

**Proof.** It suffice to show that $\Phi(u) \to -\infty$ as $u \in \hat{E}$, $\|u\| \to \infty$. Arguing indirectly we assume that there exist some $c > 0$ and a sequence $(u_j) \subset \hat{E}$ with $\|u_j\| \to \infty$ such that $\Phi(u_j) \geq -c$ for all $n$. Then, setting $v_n = u_n/\|u_n\|$, we have $\|v_n\| = 1$, and we may assume without loss of generality $v_n \to v$, $v_n^- \to v^-$, $v_n^0 \to v^0$, $v_n^+ \to v^+ \in E_\ell^+$ since $\dim(\hat{E}) < \infty$.

From (2.1), we have

$$-\frac{c}{\|u_n\|^2} \leq \frac{\Phi(u_n)}{\|u_n\|^2} = \frac{1}{2}\|v_n^+\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}} W(t, u_n) dt. \quad (2.15)$$

We claim that $v^+ \neq 0$. Indeed, if not it follows from (2.15) and (W1) that $\|v_n^-\| \to 0$ and thus $v_n \to v = v^0$. Also $\int_{\mathbb{R}} \frac{W(t, u_n)}{\|u_n\|^2} dt \to 0$ and, by Lemma 1.1, $v_n \to v$ in $L^2$.

Note that by (W1) and (W2), $W(t, u) = \frac{1}{2} M(t) u \cdot u + w(t, u)$ and $|w(t, u)| \leq C_w |u|^2$ for some $C_w > 0$, $w(t, u)/|u|^2 \to 0$ as $|u| \to \infty$, $\forall t \in \mathbb{R}$. Since $|u_n(t)| \to \infty$ if $v(t) \neq 0$, we obtain

$$\int_{\mathbb{R}} \frac{|w(t, u_n)|}{\|u_n\|^2} dt = \int_{u_n(t) \neq 0} \frac{|w(t, u_n)|}{|u_n|^2} |v_n|^2 dt$$

$$\leq 2 \int_{u_n(t) \neq 0} \frac{|w(t, u_n)|}{|u_n|^2} |v_n - v|^2 dt + 2 \int_{u_n(t) \neq 0} \frac{|w(t, u_n)|}{|u_n|^2} |v|^2 dt$$

$$\leq 2C_w \int_{\mathbb{R}} |v_n - v|^2 dt + 2 \int_{u_n(t) \neq 0} \frac{|w(t, u_n)|}{|u_n|^2} |v|^2 dt$$

$$= o(1). \quad (2.16)$$
The last equality holds by $v_n \to v$ in $L^2$ and Lebesgue’s Dominated Convergence Theorem. Also, by (W3),
\[
\frac{1}{2} \int_\mathbb{R} \frac{\langle M(t)u_n, u_n \rangle}{\|u_n\|^2} dt = \frac{1}{2} \int_{u_n(t) \neq 0} \frac{\langle M(t)u_n, u_n \rangle}{\|u_n\|^2} |v_n|^2 dt \geq \frac{m_0}{2} |v_n|^2.
\] (2.17)

From (2.16), (2.17) and since \( \int_\mathbb{R} \frac{W(t,u_n)}{\|u_n\|^2} dt \to 0 \) it follows that \( |v_n|_2 \to 0 \). Due to \( \dim(\tilde{E}) < \infty \), \( 1 = \|v_n\| \to 0 \) and this contradiction implies that \( v^+ \neq 0 \). Note that (W3), (2.14) and Remark 1.3 imply that
\[
\|v^+\|^2 - \|v^-\|^2 - \int_\mathbb{R} \langle M(t)v, v \rangle dt \leq \|v^+\|^2 - \|v^-\|^2 - m_0|v|^2
\leq - (m_0 - \lambda_\ell)|v^+|^2 + \|v^-\|^2 + m_0|v^- + v^0|^2 < 0
\]

Then there is \( T > 0 \) such that
\[
\|v^+\|^2 - \|v^-\|^2 - \int_{-T}^{T} \langle M(t)v, v \rangle dt < 0.
\] (2.18)

By (2.16), we get
\[
\lim_{n \to \infty} \int_{-T}^{T} \frac{w(t,u_n)}{\|u_n\|^2} dt \to 0.
\]

Thus (2.15) and (2.18) imply that
\[
0 \leq \lim_{n \to \infty} \left( \frac{1}{2} \|v^+_n\|^2 - \frac{1}{2} \|v^-_n\|^2 - \int_{-T}^{T} \frac{W(t,u_n)}{\|u_n\|^2} dt \right)
= \frac{1}{2} \left( \|v^+\|^2 - \|v^-\|^2 - \int_{-T}^{T} \langle M(t)v, v \rangle dt \right) < 0,
\]
a contradiction.

As an immediate result of Lemma 2.9, we have

**Lemma 2.10.** Let (W1), (W2) be satisfied and \( \rho > 0 \) be given by Lemma 2.8. Then, letting \( e \in E^+_r \) with \( \|e\| = 1 \), there is \( R > \rho \) such that \( \sup \Phi(\partial Q) \leq 0 \) where \( Q := \{u = u_1 + re : u_1 \in E^- \oplus E^0, \|u_1\| \leq R, 0 < r < R\} \).
**Proof.** Set \( R = R_\tilde{E} \), where \( R_\tilde{E} \) is the constant in Lemma 2.9. Then, by Lemma 2.9

\[
\Phi(u) < 0, \quad \forall u \in E^- \oplus E^0 \oplus \text{span} \{e\} \subset \tilde{E}, \quad \|u\| \geq R. \tag{2.19}
\]

Observe that

\[
\partial Q = Q_1 \cup Q_2 \cup Q_3,
\]

where

- \( Q_1 := \{ u \in E^- \oplus E^0 : \|u\| \leq R \} \),
- \( Q_2 := \{ u = u_1 + Re : u_1 \in E^- \oplus E^0, \|u_1\| \leq R \} \),
- \( Q_3 := \{ u = u_1 + re : u_1 \in E^- \oplus E^0, \|u_1\| = R, \ 0 \leq r \leq R \} \).

Due to (2.19), it holds that

\[
\Phi(u) \leq 0, \quad \forall u \in Q_2 \cup Q_3.
\]

Also, in view of (W1) and the form of \( \Phi \) in (2.1), \( \Phi(u) \leq 0, \forall u \in Q_1 \). Then the proof is completed. \( \square \)

After all the above preparations, we now come to the proof of our main result.

**Proof of Theorem 1.5.** Step 1. Existence. With \( V = E^- \oplus E^0 \) and \( X = E^+ \) in Theorem 2.4, the conditions \((\Phi_1)\) and \((\Phi_2)\) there hold by Lemmas 2.8 and 2.10 respectively. \( \Phi \) satisfies the \((\text{PS})\)-condition by Lemma 2.7. Hence, \( \Phi \) has at least one critical point \( u \) with \( \Phi(u) \geq \alpha > 0 \) by Theorem 2.4. Since \( \Phi(0) = 0 \), \( u \) is a nontrivial critical point of \( \Phi \). Then (1.1) has at least one nontrivial homoclinic solution \( u \) by Proposition 2.3.

Step 2. Multiplicity. Let \( X = E \), \( Y = E^- \oplus E^0 \) and \( Z = E^+ \) in Theorem 2.5. Since \( W(t, u) \) is even in \( u \), then \( \Phi \) is even and \( \Phi(0) = 0 \) by the form of \( \Phi \) in (2.1). Lemma 2.8 shows that (1) in Theorem 2.5 holds. With \( F = \tilde{E} \) in Theorem 2.5, then Lemma 2.9 implies that (2) in Theorem 2.5 also holds. Note that \( \dim(F) - \dim(Y) = \dim(E^+_\ell) = \ell \). Therefore, \( \Phi \) has at least \( \ell \) pairs of nontrivial critical points by Theorem 2.5 and then (1.1) has at least \( \ell \) pairs of nontrivial homoclinic solutions by Proposition 2.3. \( \square \)
References

[1] F. Antonacci, Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign, Boll. Un. Mat. Ital. B (7) 10 (1996) 303–324.

[2] P.C. Carriao, O.H. Miyagaki, Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems, J. Math. Anal. Appl. 230 (1999) 157–172.

[3] C.N. Chen, S.Y. Tzeng, Existence and multiplicity results for homoclinic orbits of Hamiltonian systems, Electron. J. Differential Equations 1997 (1997) 1–19.

[4] V. Coti Zelati, P.H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, J. Amer. Math. Soc. 4 (1991) 693–727.

[5] Y.H. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995) 1095–1113.

[6] Y.H. Ding, M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, Dynam. Systems Appl. 2 (1993) 131–145.

[7] Y.H. Ding, L. Jeanjean, Homoclinic orbits for a nonperiodic Hamiltonian system, J. Differential Equations 237 (2007) 473–490.

[8] G.H. Fei, The existence of homoclinic orbits for Hamiltonian systems with the potential changing sign, Chinese Ann. Math. Ser. B 17 (1996) 403–410.

[9] P.L. Felmer, E.A. de B. Silva, Homoclinic and periodic orbits for Hamiltonian systems, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) 26 (2) (1998) 285–301.

[10] N. Ghoussoub, Duality and Perturbation Methods in Critical Point Theory, Cambridge University Press, Cambridge, 1993.

[11] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.

[12] P. Korman, A.C. Lazer, Homoclinic orbits for a class of symmetric Hamiltonian systems, Electron. J. Differential Equations 1994 (1994) 1–10.

[13] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, in: CBMS Regional Conf. Ser. in Math., vol. 65, American Mathematical Society, Providence, RI, 1986.

[14] Z.Q. Ou, C.L. Tang, Existence of homoclinic solutions for the second order Hamiltonian systems, J. Math. Anal. Appl. 291 (1) (2004) 203–213.

[15] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh Sect. A 114 (1990) 33–38.

[16] P.H. Rabinowitz, K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, Math. Z. 206 (1990) 473–499.

[17] A. Salvatore, Homoclinic orbits for a special class of nonautonomous Hamiltonian systems, in: Proceedings of the Second World Congress of Nonlinear Analysts, Part 8, (Athens, 1996), Nonlinear Anal. 30 (8) (1997) 4849–4857.
[18] S.P. Wu, J.Q. Liu, Homoclinic orbits for second order Hamiltonian system with quadratic growth, Appl. Math. J. Chinese Univ. Ser. B 10 (1995) 399–410.

[19] S.P. Wu, H.T. Yang, A note on homoclinic orbits for second order Hamiltonian system, Appl. Math. J. Chinese Univ. Ser. B 13 (1998) 251–262.