RADIATION STRUCTURES ON ISOTROPIC HYPERSURFACES

G. BURDET and M. PERRIN

Abstract

The geometrical structures (in the sense of E. Cartan) are analyzed which underlie the gravitational radiation phenomenon. Among the results are:
- the introduction of the adapted frame bundle to a congruence of isotropic hypersurfaces in a Lorentzian manifold,
- the description of the reduced frame bundle which admits a unique radiation connection induced from the amiant space-time one,
- the determination of the automorphisms of an integrable radiation structure,
- the repercussions of the geometry on the shape of the stress-energy tensor in the Einstein’s field equations.

Mathematics Subject Classification (1991): 53C10, 53B15, 83C05, 83C35.
1. Introduction

In all the standard books of differential geometry the theory of Riemannian submanifolds of a Riemannian manifold benefits by a large display having been a field of investigation as old as the differential geometry itself. In what concerns the pseudo-Riemannian case authors are usually more discreet and the isotropic degenerate case is a priori discarded in general. However in spite (or in reason) of their pathological nature, isotropic submanifolds often appear in physics of the Lorentzian space-time. At least one can think to the following problems:

i) It was early recognized (Mach, Dirac 1938 [1a]) that every cosmological model should be formulated from data on the past light-cone of the observers, which has led for instance Dirac [1b] to propose the front form of relativistic dynamics, but also to study causality and Cauchy development problems using a time parameter whose level surfaces are isotropic hypersurfaces in space-time [2]. By the way let us note that the tools to analyse the formation of singularities or the (semi) global existence of solutions of Einstein’s equations are not available yet (for a recent reference see [3]).

In the seventies the same ideas underlie the attempts of particles interaction modelisations through the infinite momentum frame, the light-cone quantization and the extreme parton model techniques.

ii) The system of Einstein’s equations is of hyperbolic type and as such its characteristics are hypersurfaces across which solutions might suffer discontinuities of their derivatives. Then the radiation concept is formalized as these characteristic isotropic hypersurfaces and the bicharacteristic rays along which disturbances are propagated. Physically these properties have led to the gravitational wave concept in vogue during the sixties but next forsaked because of the non-convincing results of the miscellaneous attempts of detection which followed Weber’s experience. But new elements in favour of existence of gravitational waves were deduced from the observations of the binary pulsar in 1974 [4], generating a revival of interest for this subject, and, at the present time, several gravitational wave detection experiments are in preparation as well as in U.S.A. (cf. the ”Laser Interferometer Gravitational Wave Observatory” or LIGO project) as in several European countries (cf the italo-french project VIRGO).

iii) In the sixties the fascinating aspect of unexpected features in Einstein’s theory came to development namely the existence of singularities in solutions. An ”outside“ observer does not perceive the singularity itself but an event horizon. Conversely light cannot escape the singularity, it becomes infinitely redshifted as the horizon is approached. Then, modulo some reservations it has been shown that horizons are isotropic hypersurfaces [5].

Hence theoretical physicists have widely used isotropic hypersurfaces with their degenerate metric induced by the embedding into the Lorentzian space-time. Now if it is clear that on a manifold with a degenerate metric there is not a unique related affine connection, concerning isotropic hypersurfaces, due
to the presence of the ambiant Levi-Cività connection, the opinions of authors are divergent (see for instance [6]).

In the fifties, by taking back the marvellous geometrical Cartan’s ideas and essentially under the impulsion of Erhesman [7] and Chern [8] the notion of connection in fibre bundles and the theory of $G$-structures have been developed. But these techniques have not been used in the previous referred theoretical physics works and consequently the purpose of this paper is to show that a clear description of connections associated to isotropic hypersurfaces can be given by using the just above mentioned techniques. Here we do not want to take into account for the presence of caustics and focal points and we shall restrict ourselves to isotropic hypersurfaces which are codimension-one differentiable submanifolds of space-time. The paper is organized as follows:

- in Sect.2 the bundle of adapted frames over an isotropic hypersurface is described.
- In Sect.3 the non-unicity in the bundle of adapted frames of the connection induced by the Levi-Cività connection of the ambiant space-time is pointed out and it is shown that a unique induced radiation connection can be defined over a particular subbundle of the bundle of adapted frames: the so defined radiation structure.
- Sect.4 is devoted to the descriptions of generalized radiation connections in radiation structures.
- In Sect.5 the (infinitesimal) automorphisms of the above introduced radiation structures are described.
- A closer inspection of the so-introduced geometrical structures reveals important constraints on the physical content of the right hand side of the Einstein’s field equations. This point is commented in the conclusion.

2. Bundle of Adapted Frames over a congruence of isotropic hypersurfaces

Let $(V_{n,1}, g)$ denote the Lorentz space-time i.e. a $(n + 1)$-dimensional smooth manifold endowed with an indefinite metric tensor $g$ with signature $n - 1$. In $(V_{n,1}, g)$ we want to consider $n$-dimensional isotropic hypersurfaces $V_n$ with a one-dimensional foliation induced by an isotropic vector field $\xi$, each leaf of which being tangent to an isotropic geodesic: a bicharacteristic ray. In fact the direction of $\xi$ corresponds to the characteristic rays of the isotropic hypersurface $V_n$. Therefore, instead of the vector field $\xi$, we have to consider the line field $[\xi]$ which is the span of the vector field $\xi$ i.e. $[\xi] = \{\lambda \xi, \lambda \in \mathbb{R}\}$, so we have just to suppose that $\xi$ is both isotropic ($g(\xi, \xi) = 0$) and tangent to a congruence of unparametrized geodesics.

At each point $x$ of $V_n$ the cone of isotropic directions of $x$ has a first order contact with $V_n$, but the tangent space $T_x V_n$ does not contain any timelike vector, the future light cone of $x$ being entirely on one side of $V_n$, and the past light cone of $x$ being entirely on the other side. Moreover as an
isotropic hypersurface of the Lorentzian ambiant space-time, \( V_n \) inherits of a "degenerate" metric \( \beta \), the kernel of which is generated by the line field \([\xi]\).

Let \( G\ell(V_{n,1}) \) be the principal fibre bundle of linear frames on \( V_{n,1} \). The presence of the symmetric metric tensor \( g \) leads to the reduction of \( G\ell(V_{n,1}) \) to the bundle of orthonormal frames \( O(V_{n,1}) \), the structural group of which being \( O(n,1) := \{ a \in G\ell(n+1,\mathbb{R}) | ^t aSa = S \} \), where \( t \) denotes the transposition between rows and columns, and \( S \) is chosen as the following \((n + 1) \times (n + 1)\) symmetric matrix

\[
S = \begin{pmatrix}
0 & 0 & -1 \\
0 & \mathbb{I}_{n-1} & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]  
(2.1)

Moreover let us take into account for the presence of the given isotropic line field \([\xi]\) a representative element of which being written as \(^t(0 \ 0 \ 1)\), and look for the subgroup \( G \) of \( O(n,1) \) which keeps the isotropic direction fixed. It is easy to see that \( G \) is realized by the matrices of the following form :

\[
\begin{pmatrix}
a^{-1} & 0 & 0 \\
^tRU & R & 0 \\
\frac{1}{2}aU^2 & aU & a
\end{pmatrix}
\]  
(2.2)

where \( R \in O(n-1) \) i.e. \(^tRR = \mathbb{I}_{n-1}\), \( a \in \mathbb{R}^n \) and \( U \) is a \((n - 1)\) dimensional row \((U^2 \) denoting the scalar \(^t[UU]\)).

Then \( G \) can be written as a semi-direct product \((\mathbb{R} \otimes O(n-1)) \otimes \mathbb{R}^{n-1}\). Let us note that the homogeneous space \( O(n,1)/G \) is diffeomorphic to the \((n - 1)\)-sphere, and is known as the \((n - 1)\)-dimensional M"obius space in conformal geometry \([9]\).

Hence one is led to consider the reduction of \( O(V_{n,1}) \) to a principal fibre bundle with \( G \) as structure group : it will be denoted by \( G(V_{n,1}) \) and called the frame bundle adapted to the triplet \((V_{n,1}, g, [\xi])\).

Let \( (e_0e_1 \ldots e_{n-1}e_n) := (e_0 \leq e_n) \) denote a moving frame where \( e_0 \) is isotropic, \( e \) is a collection of \((n - 1)\) space-like vectors and \( e_n = \xi \) at each point of \( V_{n,1} \) such that :

\[
\begin{align*}
g(e_0, e_0) &= g(e_n, e_n) = 0 \\
g(e_A, e_B) &= \delta_{AB} \quad \forall A, B \in [1, n - 1] \\
g(e_0, e_A) &= g(e_n, e_A) = 0 \\
g(e_0, e_n) &= -1
\end{align*}
\]  
(2.3)

The right action of an element \((a, R, U) \in G\) on a moving frame is given by :
\[(e_0 \leq e_n) \mapsto (a^{-1}e_0 + eR^tU + \frac{1}{2}aU^2e_n \quad eR + aUe_n \quad ae_n) \quad (2.4)\]

and the corresponding action on a dual coframe is written as

\[
\theta = \begin{pmatrix} \theta^0 \\ \theta^1 \\ \vdots \\ \theta^n \end{pmatrix} \mapsto \begin{pmatrix} a\theta^0 \\ -a^tU\theta^0 + t\bar{R}\theta \\ a^{-1}\theta^n - U^tR\bar{\theta} + \frac{1}{2}aU^2\theta^0 \end{pmatrix}.
\quad (2.5)
\]

Obviously \(g = t^t \theta S \theta\) is kept invariant under this action.

Now let \(V_n\) be a \(n\)-dimensional hypersurface of \(V_{n,1}\) and denote \(i : V_n \to V_{n,1}\) the inclusion map. Then \(G(V_{n,1})\) will be called the frame bundle of \(V_{n,1}\) adapted to the hypersurface \(V_n\) if, at each point of \(V_n\), \((e_0 \leq e_n)\) is a frame of \(V_n\).

Indeed let us consider the bundle \(G(V_n)\) induced by \(i\) from \(G(V_{n,1})\), it is a principal \(G\)-bundle over \(V_n\) with a homomorphism also denoted by \(i : G(V_n) \to G(V_{n,1})\) which induces \(i : V_n \to V_{n,1}\) and corresponds to the identity automorphism of \(G\). Then, in each point of \(V_n\), an element of \(G(V_n)\) is a moving frame of \(V_n\). On \(V_n\) we have \(\theta^0 = 0\) and the action of \(G\) on a coframe is given by

\[
\begin{pmatrix} \bar{\theta} \\ \theta_n \end{pmatrix} \mapsto \begin{pmatrix} t\bar{R}\bar{\theta} \\ a^{-1}\theta^n - U^tR\bar{\theta} \end{pmatrix}.
\quad (2.6)
\]

From (2.4) and (2.6) we see that, at each point of \(V_n\), \([\xi] = [e_n]\) and \(\beta = t^t \bar{\theta} \bar{\theta}\) are left invariant under the action of \(G\). So it is now clear why \(G(V_{n,1})\) has been called the frame bundle adapted to the geometric structure induced by \(i\) over \(V_n\) i.e. the structure consisting in the smooth symmetric 2-covariant tensor field \(\beta = i^*g\) which is degenerate, its kernel being spanned by \(\xi\). Now let us show that \(G(V_n)\) is really the right frame bundle to consider over \((V_n, \beta, \xi)\) i.e. that \(G(V_n)\) is a reduction of \(G\ell(V_n)\) the bundle of linear frames on \(V_n\). First let us introduce the so-called degenerate orthogonal groups for \(p \leq q\) as follows

\[
O^{q-p}(p) := \{g \in G\ell(q, \mathbb{R}) | gS^p(q)^t g = S^p(q)\}
\quad (2.7)
\]

where \(S^p(q)\) denotes a 2-contravariant symmetric tensor, degenerate of order \(q - p\), and

\[
O_{q-p}(p) := \{g \in G\ell(q, \mathbb{R}) | t^t gS_p(q)g = S_p(q)\}
\quad (2.8)
\]

where \(S_p(q)\) denotes a 2-covariant symmetric tensor degenerate of order \(q - p\). For \(p = q\), \(O^0(p) = O_0(p)\) and they are both isomorphic to \(O(p)\) the usual orthogonal group.

Let \(G\ell(V_n)\) be the bundle of linear frames over \(V_n\), it is a principal fibre bundle with structural group \(G\ell(\dim V_n, \mathbb{R}) = G\ell(n, \mathbb{R})\). Then \(\mathbb{R}^n\) is the
standard fibre of the tangent bundle $T(V_n)$ associated with $Gℓ(V_n)$. Since any element $r ∈ Gℓ(V_n)$ over $x ∈ V_n$ can be considered as a one-to-one linear mapping of $\mathbb{R}^n$ onto $T_x(V_n)$ $y → ry = Y$, at each point of $V_n$ it is possible to associate with the metric $\beta$ a bilinear form $(\cdot , \cdot )_\beta$ on $\mathbb{R}^n$ defined by

$$(y, y')_\beta = (r^{-1}Y, r^{-1}Y') = \beta(Y, Y').$$

This bilinear form can be written

$$(y, y')_\beta = ^t y S_{n-1}(n)y$$

where $S_{n-1}(n)$ is the $n × n$ matrix which represents $\beta$ and $y ∈ \mathbb{R}^n$ is written as a column with $n$ elements, $^t y$ denoting the corresponding transposed row.

According to (2.8) the bilinear form $(\cdot , \cdot )_\beta$ is invariant under the action of $O_1(n - 1)$. By choosing

$$S_{n-1}(n) = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$$

it is easy to verify that $O_1(n - 1)$ is a semi-direct product $(O(n - 1) ⊗ \mathbb{R}) ⊗ \mathbb{R}^{n-1}$ in which we recognize the group $G$ previously realized as a subgroup of $Gℓ(n + 1, \mathbb{R})$ by the matrices (2.3). Here $G$ is realized as a subgroup of $Gℓ(n, \mathbb{R})$ by matrices of the form

$$\begin{pmatrix} R & 0 \\ aU & a \end{pmatrix}.$$

The invariance of $(\cdot , \cdot )_\beta$ by $O_1(n - 1)$ implies that Rel.(2.10) is independent of the choice of $r$ modulo a right action of an element of $O_1(n - 1)$ as a subgroup of $Gℓ(n, \mathbb{R})$ into $Gℓ(V_n)$ i.e. it leads to a reduction of $Gℓ(V_n)$ to a $O_1(n - 1)$-structure, so we are led to the following definition :

Definition : The bundle of adapted linear frames over $[V_n, \beta, [\xi]]$, a congruence of isotropic hypersurfaces generated by a given line field $[\xi]$, is a $G$-structure (i.e. a subbundle $G(V_n) ↪ Gℓ(V_n)$) where $G$ is the degenerate orthogonal group $O_1(n - 1)$.

In some particular cases such as the $pp$-waves for instance, a stronger condition is involved, namely instead of considering $(V_{n,1}, g)$ equipped with a line field $[\xi]$ it must be endowed with a covariantly constant vector field $\xi$. This leads to introduce another reduction of $Gℓ(V_{n,1})$ to a principal $G_I$-bundle $G_I(V_{n,1})$ where $G_I$ denotes the stabilizer of $\xi$. According to (2.4) it is deduced from $G$ by setting $a = 1$ in (2.3). So the dilation is excluded and we are left with $G_I = O(n - 1) ⊗ \mathbb{R}^{n-1}$.

As previously let us introduce the bundle $G_I(V_n)$ induced from $G_I(V_{n,1})$ by the inclusion map $i$. Now $\beta$ and $\xi$ are left invariant under the action of $G_I$ (as a consequence of rel.(2.4) with $a = 1$).

Then from $\xi$ it is possible to construct a degenerate symmetric 2-contravariant tensor field $Z = \xi \otimes \xi$. For the cotangent bundle $T^*(V_n)$ a construction similar to the one above described in the case of the tangent bundle can be done.
Now at each point \( x \) of \( V_n \), let us denote by \( Y = ry \) the element of \( T_x^\ast(V_n) \) corresponding to \( y \in \mathbb{R}^n \). Then an invariant bilinear form \((,)_\xi\) is defined by

\[
(y, y')_\xi = t^y S^1(n)y' = Z(Y, Y')
\]

(2.12)

where \( S^1(n) \) is the \( n \times n \) matrix which represents \( Z \).

Following the previous choice (2.11) let us set

\[
S^1(n) = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & 1 \end{pmatrix}.
\]

(2.13)

Then owing to (2.7) it can be shown that the bilinear form \((,)_\xi\) is invariant under \( O^{n-1}(1) \) which can be written as \( O^{n-1}(1) = Gl(n-1, \mathbb{R}) \otimes \mathbb{R}^{n-1} \).

The group which leaves invariant both bilinear forms \((,)_\beta\) and \((,)_\xi\) is the intersection \( O_1(n-1) \cap O^{n-1}(1) \) i.e. the group \( G_I \) above introduced which can be realized as a subgroup of \( Gl(n, \mathbb{R}) \) by the matrices of the form

\[
\begin{pmatrix} R & 0 \\ U & 1 \end{pmatrix}.
\]

(2.14)

So in this case we are led to a \( G_I \)-structure and to the following definition.

**Definition** : The bundle of adapted linear frames over \([V_n, \beta, \xi]\), a congruence of isotropic hypersurfaces generated by a given vector field \( \xi \), is a \( G_I \)-structure where \( G_I \) is the intersection \( O_1(n-1) \cap O^{n-1}(1) \) of two degenerate orthogonal groups.

### 3. Restriction of the metric connection to the bundle of adapted linear frames over congruences of isotropic hypersurfaces. The radiation connection

In the previous section the presence of the metric connection of \((V_{n,1}, g)\) has been alluded through the properties of the vector field \( \xi \). Here we want to study the properties of the metric connection with respect to the bundles of linear adapted frames \( G(V_{n,1}) \) and \( G_I(V_{n,1}) \).

In the bundle of Lorentzian frames \( O(V_{n,1}) \) let us consider the Levi-Civit\`a connection \( \varphi \) induced by the metric connection. It takes values in the Lie algebra of \( O(n,1) \) denoted by \( \mathcal{L}(O(n,1)) \). For our purpose it is convenient to introduce a dense subset in \( O(n,1) \) defined as the set of matrices given by the product of three matrix subgroups symbolically denoted

\[
(R^{n-1}) (\mathbb{R} \otimes O(n-1)) (R^{n-1})^\ast
\]

each factor corresponding to a subgroup of \( O(n,1) \) parametrized as follows :

\[
(R^{n-1}) \equiv \left\{ \begin{pmatrix} 1 & \nabla^t \\ 0 & I_{n-1} \\ 0 & 0 \end{pmatrix}, \nabla \text{ column } \in \mathbb{R}^{n-1} \right\}
\]
\[(\hat{\mathbf{R}} \otimes O(n-1)) \equiv \left\{ \begin{pmatrix} a^{-1} & R \\ 0 & a \end{pmatrix} \right\}, \ a \in \hat{\mathbf{R}}, R \in O(n-1) \]

\[(\mathbb{R}^{n-1})^* \equiv \left\{ \begin{pmatrix} 1 \\ \frac{1}{2} U \\ U \\ 0 \end{pmatrix}, \ U \text{ row } \in \mathbb{R}^{n-1} \right\}. \]

Let us note that with \((\hat{\mathbf{R}} \otimes O(n-1)) \otimes (\mathbb{R}^{n-1})^*\) the parametrization of \(G\) used in (2.4) is recovered.

Owing to this symmetric decomposition of a dense subset in \(O(n,1)\), the Lie algebra \(\mathcal{L}(O(n,1))\) can be decomposed in the following way

\[\mathcal{L}(O(n,1)) = \mathcal{L}(\mathbb{R}^{n-1}) + \mathcal{L}(\hat{\mathbf{R}} \otimes O(n-1)) + \mathcal{L}((\mathbb{R}^{n-1})^*) ,\]

a vector space direct sum.

Hence the Livi-Civita connection \(\varphi\) can be written in the following matrix form

\[\varphi = \begin{pmatrix} -\phi_n^a & t\phi^0_a & 0 \\ t\phi^0_a & \phi^0 & 0 \\ 0 & 0 & \phi^a_n \end{pmatrix}, \quad a, b, c \in [0, n], \]

where \(\phi_n^a\) is \(\mathbb{R}\)-valued, \(\bar{\varphi} = \{\phi^A_n, A \in [1, n-1]\}\) is \(\mathcal{L}(\mathbb{R}^{n-1})\)-valued,

\(\bar{\phi} = \{\phi^n_A, A \in [1, n-1]\}\) is \(\mathcal{L}((\mathbb{R}^{n-1})^*)\)-valued and \(\phi = \{\phi^B_A, A, B \in [1, n-1]\}\) is \(\mathcal{L}(O(n-1))\)-valued i.e. such that \(\phi^B_A + \phi^A_B = 0\).

By introducing the Ricci coefficients, each component of the connection form can be expressed in terms of the soldering form \(\theta\) as follows

\[\varphi^a_b = \gamma^a_{cb} \theta^c \] (3.2)

\(a, b, c \in [0, n]\), the Ricci’s coefficients \(\gamma\) being related to the Christoffel’s symbols of the linear connection by

\[\gamma^a_{bc} = e^a_b e^c \{\Gamma^a_{\beta\gamma} \theta^\alpha_{\beta} - \partial_\beta \theta^a_{\gamma}\} \] (3.3)

where \(\alpha, \beta, \ldots\) are the indices of a local coordinate system.

Then at each point of \(V_{n,1}\) the covariant derivative of any vector field \(X (= X^0 e_0 + X^A e_A + X^n e_n)\) is given by:

\[\nabla X = (dX^0 + X^A \phi^0_A + X^0 \phi^0_0) \otimes e_0 + (dX^C + X^A \phi^C_A + X^0 \phi^C_0 + X^n \phi^C_n) \otimes e_C + (dX^n + X^A \phi^n_A + X^n \phi^n_0) \otimes e_n\] (3.4)
where $\phi_0^0 = -\phi_n^0$, $\phi_A^0 = \phi_A^n$, $\phi_0^n = \phi_A^n$.

In the restriction to $\mathcal{L}(G)$ the components $\phi_A^n$ (hence $\phi_n^n$) disappear. So if we consider a vector field $X$ tangent to the isotropic hypersurface $V_n$ (i.e. corresponding to $X^0 = 0$), rel. (3.4) reduces to

$$\nabla X = (dX^C + X^A\phi_A^C) \otimes e_C + (dX^n + X^A\phi_A^n + X^n\phi_n^n) \otimes e_n$$  \hfill (3.5)

where there is no component along $e_0$. Therefore for any vector field $Y$ tangent to $V_n$, $\nabla_Y X$ is tangent to $V_n$. But things do not go on the same way in what concerns one-forms (covariant vectors) because of the degeneracy of the structure. Indeed, for a one-form $f (= f_0 \theta^0 + f_A \theta^A + f_n \theta^n)$, the covariant derivative with respect to the Levi-Civita connection can be written as

$$\nabla f = (df_0 - f_A \phi_A^0 - f_0 \phi_0^0) \otimes \theta^0 + (df_C - f_A \phi_C^A - f_0 \phi_0^C - f_n \phi_n^C) \otimes \theta^C + (df_n - f_n \phi_n^n) \otimes \theta^n$$ \hfill (3.6)

By restriction to a $\mathcal{L}(G)$-valued connection and for a one-form of $V_n$ ($f_0 = 0$), one gets

$$\nabla f = -f_A \phi_A^0 \otimes \theta^0 + (df_C - f_A \phi_C^A - f_n \phi_n^C) \otimes \theta^C + (df_n - f_A \phi_A^n - f_n \phi_n^n) \otimes \theta^n$$ \hfill (3.7)

in which a $\theta^0$-component remains, showing that $\nabla_Y f \forall Y \in TV_n$, is not a 1-covariant tensor of $V_n$. Hence the following proposition is established

**Proposition 3.1**

There is no connection induced on a congruence of isotropic hypersurfaces $[V_n]$ by the reduction to $G(V_{n,1})$ of the amiant Levi-Civita connection.

To make disappear the $\theta_0$ term in (3.7) it is required that $\phi_A^0 = 0 (= \phi_A^n)$. This leads to restrict the structural group $G$ to a subgroup $G_R$ in which the invariant subgroup $R^{n-1}$ has been discarded, that is $G_R = \mathbb{R} \otimes O(n-1)$. Therefore in a $G_R(V_{n,1})$-subbundle the covariant derivatives of a contravariant and a covariant vector as given by (3.5) and (3.7) (with $\phi_A^0 = 0$) along a vector field of $V_n$ are both well defined tensors of $V_n$. So one gets :

**Proposition 3.2**

There is a unique connection induced on $[V_n]$ by the reduction to the bundle $G_R(V_{n,1})$ of the Levi-Civita connection of the amiant space-time. This connection will be called the radiation connection.

A local description of the radiation connection is given in the following section.
But here we want to underline that the structural group of any $G$-bundle can be reduced to $G_R$. This statement follows from the fact that the quotient of $G$ by $G_R$ is contractible, being topologically equivalent to $\mathbb{R}^{n-1}$.

4. Radiation connections on degenerate metric structures

Now we want to discard the ambient space-time $(V_{n,1}, g)$ and to study the degenerate metric structure $(V_n, \beta, [\xi])$ per se by keeping in mind the fibre bundles introduced in the previous section, namely $G(V_n)$ the fibre bundle corresponding to the fibre bundle of adapted frames on $V_n$, $G_R(V_n)$ which corresponds to the fibre bundle in which there is the radiation connection and also $G_I(V_n)$ which corresponds to the strict invariance of the vector field $\xi$.

Obviously every connection in $G(V_n)$, $G_R(V_n)$ or $G_I(V_n)$ determines a linear connection of $V_n$, and to keep in memory the origin of these fibre bundles we shall adopt the following definitions.

Definition: A $G$-radiation connection is a linear connection with vanishing torsion induced by a connection in $G(V_n)$. Alike for $G_R$ and $G_I$-structures respectively.

4.A. $G$-radiation connections.

**Proposition 4.1**

With respect to any $G$-radiation connection the degenerate metric $\beta$ is parallel.

**Proof**: In a moving frame, $\beta$ can be written as

$$\beta = t^\theta S \theta$$

where $S := S_{n-1}(n)$ is the $n$-dimensional matrix defined in (2.11). Then

$$\nabla_c \beta = t^\theta (t^c \gamma_c S + S \gamma_c) \theta \quad c \in [1,n]$$

where $\gamma_c = (\gamma^a_{cb})$ denotes the $n \times n$ matrix of Ricci coefficients introduced in (3.2).

Hence $\nabla_c \beta = 0$ is equivalent to $t^c \gamma_c S + S \gamma_c = 0$ in which we recognize, by definition, the Lie algebra $\mathcal{L}(O_1(n-1)) = \mathcal{L}(G)$. Q.E.D.

**Proposition 4.2**

With respect to any $G$-radiation connection the line field $[\xi]$ is such that

$$\nabla [\xi] = \chi \otimes [\xi]$$

where $\chi$ is a real one-form corresponding to the dilation component of the connection.

**Proof**: The covariant derivative corresponding to a linear connection in a moving frame verifies
\[ \nabla_{e_a} e_n = \gamma_{an}^b e_b. \] (4.4)

For a $G$-radiation connection $\phi^B_n = 0$ so that $\gamma^B_{an} = 0 \ \forall B \in [1, n-1]$. Hence

\[ \nabla_{e_a} e_n = \gamma^n_{an} e_n. \] (4.5)

So, in a moving frame in which $\xi = e_n$, it can be set $\gamma^n_{an} = \chi_a$ and then

\[ \nabla_a \xi = \chi_a \xi. \] (4.6)

**Proposition 4.3**

With respect to a $G$-radiation connection

i) the line field $[\xi]$ is geodesic $\nabla_\xi \xi = \chi(\xi) \xi$

ii) the expansion of $[\xi]$ does not vanish, $\text{div}\ \xi = \chi(\xi)$.

The proof is obvious from (4.5) and let us note that $\text{div}\ \xi = \chi_n$.

**Proposition 4.4**

The structure equations and Bianchi’s identities of a $G$-radiation connection are given by

\[
\begin{align*}
(a) \quad 0 &= \overline{\omega} = d\theta + \phi \wedge \overline{\theta} \\
(b) \quad 0 &= \overline{\omega}^n = d\theta^n + \phi \wedge \overline{\theta} + \phi^n_{an} \wedge \theta^n \\
(c) \quad \Phi &= d\phi + \phi \wedge \phi \wedge \overline{\theta} + \phi^n_n \wedge \theta^n \\
(d) \quad \Phi &= d\phi + \phi \wedge (\phi - \Phi_{n-1}) \\
(e) \quad \Phi^n &= d\phi^n \\
(a') \quad \Phi \wedge \chi = 0 \\
(b') \quad \Phi \wedge \chi + \Phi^n \wedge \theta^n = 0 \\
(c') \quad D\Phi = 0 \\
(d') \quad D\Phi = 0 \\
(e') \quad D\Phi^n = 0
\end{align*}
\] (4.7)

*Proof:* By using the same conventions as in (3.1) any $G$-radiation connection can be written under the matrix form

\[
\begin{pmatrix}
\phi \\
\overline{\theta} \\
\phi^n
\end{pmatrix}
\] (4.8)

from which the torsion and curvature 2-forms are easily deduced and the Bianchi’s identities follow.

If $\{\overline{\theta}, \theta_n\} := \theta$ denote the soldering form of $G(V_n)$ every component of the connection can be decomposed by using Ricci’s coefficients as previously (see (3.2)). The antisymmetry of the $\mathcal{L}(O(n-1))$-valued component $\phi$ leads to :

\[ \gamma^A_{CB} = -\gamma^B_{CA} \quad \text{and} \quad \gamma^A_{nB} = -\gamma^B_{nA} \] (4.9)

Let us now study if among all existing $G$-radiation connections one of them is a privileged one, in other words if the miracle of the Riemannian geometry occurs again.
Proposition 4.5

All $G$-radiation connections have a common orthogonal component.

Proof: To compare the $\mathcal{L}(O(n-1))$-components of two $G$-radiation connections $\Gamma$ and $\Gamma'$ let us set $\Delta \phi := \phi' - \phi = (\gamma' - \gamma) \theta := (\Delta \gamma) \theta$.

Then from (4.7a) one can deduced

\[
\begin{align*}
(a) \quad & \Delta \gamma^{A}_{[BC]} = 0 & \quad & \Delta \gamma^{A}_{nB} = 0 \\
(b) \quad & \Delta \gamma^{n}_{[AB]} = 0 & \quad & \Delta \gamma^{n}_{[nB]} = 0
\end{align*}
\]

and from (4.7b)

\[
\begin{align*}
(a) \quad & \Delta \phi^{n}_{[AB]} = 0 & \quad & \Delta \gamma^{n}_{[AB]} = 0 \\
(b) \quad & \Delta \phi^{n}_{[nB]} = 0 & \quad & \Delta \gamma^{n}_{[nB]} = 0
\end{align*}
\]

Then (4.7a) together with the antisymmetry properties (4.9) leads to $\Delta \gamma^A_{BC} = 0 \quad \forall A, B, C$. As $\Delta \gamma^A_{nB} = 0$ it can be deduced $\Delta \phi^A_B = 0 \quad \forall A$ and that is to say all $G$-radiation connections have the same $\mathcal{L}(O(n-1))$-component.

In what concerns the other components it remains

\[
\begin{align*}
(a) \quad & \Delta \phi^{n}_{[AB]} = \Delta \gamma^{n}_{[AB]} \theta^A \\
(b) \quad & \Delta \phi^{n}_{[nB]} = \Delta \gamma^{n}_{[nB]} \theta^A + \Delta \gamma^{n}_{nn} \theta^n.
\end{align*}
\]

Proposition 4.6

On the degenerate structure $(V_n, \beta, [\xi])$ the $G$-radiation connections are in correspondence with the symmetric 2-contravariant tensors on $V_n$.

Proof: It is a consequence of (4.11).

Let us give the corresponding local expressions. In terms of its Ricci’s coefficients and in a local coordinate system the Christoffel’s symbols of a $G$-radiation connection are given by

\[
\Gamma_{\beta \gamma}^\alpha = e^\alpha_a \left\{ \partial_\beta \theta^a_\gamma + \theta^b_\beta \theta^c_\gamma \gamma_{bc} \right\}.
\]

By performing a standard algebra one gets

\[
\Gamma_{\beta \gamma}^\alpha = (e \otimes e)^{\alpha \rho} B_{\rho, \beta \gamma} + e^\alpha_n \Gamma^\rho_{\beta \gamma} \theta^n
\]

where $B_{\rho, \beta \gamma}$ is recognized as a Koszul’s term defined by

\[
\begin{align*}
B_{\rho, \beta \gamma} &= (\overline{\theta}_{(\beta \partial_\gamma)} \overline{\theta}_{\rho}) + (\overline{\theta}_\rho \overline{\partial}(\beta \gamma)) - (\overline{\theta}_{(\beta \partial_\rho)} \overline{\theta}_\gamma) \equiv \partial_\rho \beta_{\beta \gamma} - \frac{1}{2} \partial_\rho \beta_{\beta \gamma}.
\end{align*}
\]

In the expression (4.14) a two-covariant tensor $e \otimes e$ is arising, it is degenerate its kernel being spanned by $\theta^n$. This leads to locally introduce a quasi inverse $\beta_f$ of $\beta$ (i.e. a two-covariant tensor $\beta_f$ such that its contraction with $\beta \beta_f$ gives $\beta_f$) associated to the one-form $f$ such that $f(\xi) = 1$ and here
identified with \( \theta^n \). More exactly \( \beta_f \) is associated to the choice of a projective one-form \([f]\) such that \([f](\xi) = 1\), in such a way one can write

\[
\beta \beta_f = 1 - [f] \otimes [\xi].
\]  
(4.16)

It is then easy to verify that the general expression which verifies (4.14) can be written under the following form:

\[
\Gamma_{\beta \gamma}^\alpha = \beta_f^{\alpha \rho} B_{\rho, \beta \gamma} + \xi^\alpha Z_{\beta \gamma}
\]  
(4.17)

where \( Z_{\beta \gamma} \) is an arbitrary 2-contravariant symmetric tensor (cf. proposition 4.5).

Furthermore we have the following local relations:

\[
\chi_\alpha = -\xi^\lambda \nabla_\alpha f_\lambda = -\xi^\lambda (\partial_\alpha f_\lambda - Z_{\alpha \lambda})
\]  
(4.18)

\[
\nabla_{[\gamma} \chi_{\beta]} = \frac{1}{2} \xi^\alpha R^\lambda_{\alpha \beta \gamma} f_\lambda
\]  
(4.19)

where the \( R^\lambda_{\alpha \beta \gamma} \)’s denote the local components of the Riemannian curvature tensor which can be written

\[
R^\lambda_{\alpha \beta \gamma} = R^\lambda_{\alpha \beta \gamma} + 2\xi^\lambda \left\{ (\partial_{[\gamma} + \chi_{[\gamma}) Z_{\beta] \alpha} + \Gamma_{\alpha \beta [\gamma} Z_{\gamma] \sigma} \right\}
\]

where \( R^\lambda_{\alpha \beta \gamma} \) denotes the contribution corresponding to the Koszul’s term (i.e. which corresponds to \( Z = 0 \) in (4.17)).

4.B. \( G_R \)-radiation connections.

As suggested by results of subsection 4.A. let us briefly discuss the linear connections defined from \( \mathcal{L}(G_R) \)-valued connections we shall name \( G_R \)-radiation connections.

All the calculations of subsection 4.A can be performed again after having set \( \phi = 0 \) i.e. \( \gamma^m_{\alpha B} = 0 \). Then Propositions 4.1-2-3-5 remain unchanged and structure equations and Bianchi’s identities are deduced directly from (4.7). Therefore Prop 4.6 is modified and replaced by the following one.

**Proposition 4.7**

*On the degenerate structure \((V_n, \beta, [\xi])\) the \( G_R \)-radiation connections are in correspondence with the one-forms on \( V_n \).*

Indeed from (4.12b) the difference between the dilation components of two \( G_R \)-radiation connections is given by \( \Delta \phi^n = \Delta \gamma^n_{mn} \theta^n \), which shows that they are not related. Q.E.D.

For the Christoffel’s symbols, a direct calculation from (4.13) leads to:

\[
\Gamma_{\beta \gamma}^\alpha = \beta_f^{\alpha \rho} B_{\rho, \beta \gamma} + \xi^\alpha \left\{ \partial_{(\beta} f_{\gamma)} + \chi_{(\beta} f_{\gamma)} \right\}.
\]  
(4.20)
on \((V_n, \beta, [\xi])\) the choice of a quasi-inverse \(\beta_f\) of \(\beta\) associated with a one-form \(f\) does not fully determine a \(G_R\)-radiation connection, a supplementary one-form \(\chi\) is needed.

Now we can go back to the unique induced radiation connection defined in proposition (3.2). Indeed the local expression of the radiation connection over an isotropic hypersurface is also given by (4.20) in which the one-form \(\chi\) corresponds to the pull-back of the dilatation component of the ambient Levi-Civit\`{a} connection. While we are on the subject let us mention that the three references [5a,c,d] deal with particular choices of \(G\)-radiation connections (for \(n = 3\)) which therefore cannot be identified with the unique induced radiation connection.

4.C. \(G_I\)-radiation connections.

As mentioned in sect.2, to require the invariability of the vector field \(\xi\) leads to restrict the group \(G\) to its subgroup \(G_I\) and then to consider \(G_I\)-radiation connections. They always satisfy proposition 4.1 but propositions 4.2 and 4.3 are replaced by the following one.

**Proposition 4.8**

With respect to any \(G_I\)-radiation connection \(\xi\) is covariantly constant \(\nabla \xi = 0\), hence geodesic \(\nabla \xi \xi = 0\) and divergence free \(\text{div} \xi = 0\).

**Proof**: It is trivially deduced from the proof of proposition 4.2 by setting \(\gamma^a_{an} = 0\ \forall a \in \{1, n\}\) i.e. by accounting for \(\phi^a_n = 0\).

Structure equations and Bianchi’s identities are deduced from (4.7). Proposition 4.5 is still valid for \(G_I\)-radiation connections. Now Rel.(4.11b) being replaced by \(\Delta \gamma^a_{nB} = 0\) this leads to rewrite Prop.4.6 as follows.

**Proposition 4.9**

On the degenerate structure \((V_n, \beta, [\xi])\) the \(G_I\)-radiation connections are in correspondence with the degenerate symmetric 2-contravariant tensors on \(V_n\) the kernel of which is generated by \(\xi\).

5. Automorphisms of radiation structures

Among the elements of \(\text{Dif}(V_n)\) we have to select those which preserve the degenerate structure \((\beta, [\xi])\) that is the transformations of \(V_n\) which leave \(\beta\) and \([\xi]\) invariant in the sense that they transform a representative element of \([\xi]\) into another one. Obviously such a transformation \(f\) maps each adapted frame at an arbitrary point \(x \in V_n\) into an adapted frame at \(f(x) \in V_n\), that is to say that there exists an induced transformation \(\tilde{f}\) of \(G\ell(V_n)\) which maps \(G(V_n)\) into itself.

However due to the absence of a privileged \(G\)-radiation connection in one-to-one correspondence with \((\beta, [\xi])\), in order to promote \(f\) or \(\tilde{f}\) to the status of radiation transformation we have to prescribe that they preserve the chosen \(G\)-radiation connection (affine transformation).
Definition: A transformation \( f \) of \( V_n \) is a radiation transformation if the induced transformation \( \tilde{f} \) of \( \text{Gl}(V_n) \) maps \( G(V_n) \) endowed with a chosen radiation connection into itself.

Such a definition is fully adapted to define a notion of equivalence between two \( G_R \) (or \( G_I \))-radiation connections.

Definition: Two \( G_R \) (respectively \( G_I \))-structures are said equivalent if the two considered \( G_R \) (respectively \( G_I \))-radiation connections are subordinate to the same \( G \)-connection.

In fact two different \( G_R \) (respectively \( G_I \))-structures are associated to two distinct embeddings \( h \) and \( h' \) of \( G_R(V_n) \) (respectively \( G_I(V_n) \)) into \( G(V_n) \). Then on each fibre of the bundle \( G(V_n) \) these two embeddings \( h \) and \( h' \) define two \( G_R \)-orbits (respectively \( G_I \)-orbits) such that \( h'(p) = h(p)\lambda(x) \) for any \( p \in G_R(V_n) \) (respectively \( G_I(V_n) \)) which projects onto \( x \in V_n \), \( \lambda(x) \) being identified with an element of \( G \) which can be written under the form

\[
\begin{pmatrix} 1 & 1 \\ n & \rho(x) \\ 0 & 1 \end{pmatrix} = \lambda_R(x) \quad \text{(respectively \( \begin{pmatrix} 1 & 1 \\ n & \rho(x) \\ 0 & 1 \end{pmatrix} = \lambda_I(x) \))}
\]

where \( \rho \) is a mapping of \( V_n \) into the subgroup \( \hat{R} \subset G \) and \( \rho \) is a mapping of \( V_n \) into the subgroup \( R^{n-1} \subset G \).

Under a radiation transformation \( \tilde{f} \) a \( G_R \)-structure (respectively a \( G_I \)-structure) is transformed into an equivalent one.

Definition: A vector field \( X \) on \( V_n \) is called an infinitesimal radiation transformation if the local one-parameter group of local transformations generated by \( X \) in a neighborhood of each point of \( V_n \) consists of local radiation transformations.

Proposition 5.1
With respect to an infinitesimal radiation transformation one has

i) \( L_X \beta = 0 \) where \( L_X \) denotes the Lie derivation with respect to \( X \),

ii) \( L_X \xi = [X, \xi] = k\xi \) with \( k \) = constant,

iii) \( K(X, Y) = 0 \) for all vector fields \( Y \) on \( V_n \) where \( K(X, Y) = R(X, Y) - \nabla_Y A_X \), \( A_X \) denoting the derivation defined by \( L_X - \nabla_X \), \( R \) being the curvature tensor and \( \nabla \) the covariant derivation corresponding to the chosen \( G \)-radiation connection.

This property is a direct consequence of the above definitions and iii) defines an infinitesimal affine transformation [9].

Let us denote by \( \tilde{X} \) the vector field on \( \text{Gl}(V_n) \) induced by the local group of radiation transformations prolonged to \( \text{Gl}(V_n) \).

Proposition 5.2
Corresponding to any infinitesimal radiation transformation \( X \) there is an infinitesimal automorphism \( \tilde{X} \) of \( G(V_n) \).
This proposition is a direct consequence of the definition of an infinitesimal radiation transformation.

**Proposition 5.3**

The Lie derivatives with respect to $\tilde{X}$ of the canonical one-form $\vartheta = \{\theta, \theta^n\}$ of $G\ell(V_n)$ reduced to a $G$-structure and those of the chosen $G$-radiation connection one-form $\varphi = \{\phi, \phi^n, \phi_n\}$ satisfy the following properties

$$
L_{\tilde{X}}\varphi = 0 , \quad L_{\tilde{X}}\theta^n = k\theta^n
$$

where $k$ is a constant.

Finally by comparing the right action of $G$ with those of $G_R$ ($G_I$ respectively) the following properties can be deduced.

**Proposition 5.4**

The Lie derivatives with respect to $\tilde{X}$ of the one-form $\varphi_0 = \{\phi, \phi^n\}$ of a chosen $G_R$-radiation connection and of the canonical one-form $\vartheta$ of $G\ell(V_n)$ reduced to a $G_R$-structure satisfy

$$
L_{\tilde{X}}\varphi = 0 , \quad L_{\tilde{X}}\phi^n = 0
$$

$$
L_{\tilde{X}}\theta^n = -\epsilon\theta^n
$$

where $\epsilon$ is the infinitesimal gauge transformation corresponding to the above introduced mapping $\rho$.

**Proposition 5.5**

The Lie derivatives with respect to $\tilde{X}$ of the one-form $\varphi_1 = \{\phi, \phi\}$ of a chosen $G_I$-radiation connection and of the canonical one-form $\vartheta$ of $G\ell(V_n)$ reduced to a $G_I$-structure satisfy

$$
L_{\tilde{X}}\varphi = 0 , \quad L_{\tilde{X}}\phi = -\epsilon\phi
$$

$$
L_{\tilde{X}}\theta^n = -\epsilon\theta^n
$$

where $\epsilon$ is the infinitesimal gauge transformation corresponding to the mapping $\rho$.

To illustrate the above established propositions let us treat the case of the standard radiation structure on $R^n$ in which $G(V_n)$ is diffeomorphic to the trivial bundle $R^n \times G$ endowed with the natural $G$-connection provided by the Maurer-Cartan form of $G$. The Lie algebra $\mathcal{L}(G)$ clearly contains an element of rank one and hence is of infinite type. So the general theorem on the automorphisms group of a $G$-structure cannot be applied and it cannot be claimed that it is a Lie group. Then let us perform the calculation of the infinitesimal automorphisms of this structure which firstly amounts to select among the vector fields $X$ of $V_n$ those which satisfy i) and ii) of Prop. (5.1).

In a special admissible coordinate frame system

$$(x^1, \ldots, x^n) := (\pi, x^n) \equiv (\{x^A\}, x^n(A \in [1, n - 1]))$$
in which $\theta^A_\alpha = \delta^A_\alpha$ and $e^\gamma_n = \delta^\gamma_n$ so that $\beta_{\alpha \gamma} = \theta^A_\alpha \theta^C_\gamma \delta_{AC}$ and $\xi^\gamma = \delta^\gamma_n$, the equations i) and ii) of Prop. (5.1) can be written as

$$L_X \beta_{\alpha \gamma} = 2 \nabla_\alpha (X_\gamma) = 0$$

$$L_X \xi^\gamma = -\xi^\alpha \partial_\alpha X^\gamma = k \xi^\gamma.$$  

Their general solution is given by:

$$\left\{ \begin{array}{l}
X^A = \omega^A_B (\overline{x}) x^B + a^A \\
X^n = -k x^n + f(\overline{x})
\end{array} \right. \quad (5.1)$$

where $\omega^A_B (\overline{x}) = -\omega^A_B (\overline{x})$, $\{a^A\}$ is a set of $n-1$ constants and $f(\overline{x})$ can be any function of $\overline{x}$ so that the corresponding Lie algebra is infinite dimensional. But by taking iii) of the Prop. (5.1) into account we are led to select linear (in $\overline{x}$) vector fields only.

Hence the vector fields satisfying i),ii) and iii) of Prop. (5.1) can be written as

$$\left\{ \begin{array}{l}
X^A = \omega^A_B x^B + a^A \\
X^n = k_B x^n + k x^n + k_0
\end{array} \right. \quad (5.2)$$

The Lie brackets of these vector fields generate a Lie algebra which is recognized as the Lie algebra of the inhomogeneous $G$ group isomorphic to $\mathbb{R}^n \otimes ((O(n-1) \otimes \hat{\mathbb{R}}) \otimes \mathbb{R}^{n-1})$.

Now if we recall that a $G$-structure $G(V_n)$ is said integrable if it is locally isomorphic to the standard $G$-structure on $\mathbb{R}^n$, from the above result the following proposition can be deduced immediately.

**Proposition 5.6**

The infinitesimal automorphisms of an integrable radiation structure $G(V_n)$ endowed with a radiation connection generate a Lie algebra of dimension $\frac{1}{2}(n^2 + n + 2)$ isomorphic to the Lie algebra of the inhomogeneous $G$ group namely $\mathcal{L} \left( \mathbb{R}^n \otimes \left((O(n-1) \otimes \hat{\mathbb{R}}) \otimes \mathbb{R}^{n-1}\right) \right)$.

As a final remark let us compare the automorphisms of a degenerate structure and those of an isotropic hypersurface. Among the isometry transformations of $(V_{n,1}, g)$ we have to select the ones $F$ which do not move the isotropic rays generated by $[\xi]$ (or $\xi$ under the covariant constancy hypothesis). Then $\text{Aut} \ G(V_{n,1})$ is given by the induced transformations $\{\tilde{F}\}$ of $G\ell \ (V_{n,1})$ which map $G(V_{n,1})$ onto itself. Let us recall that $G(V_n)$ can be obtained as the pull-back of $G(V_{n,1})$ induced by the inclusion map $i : V_n \rightarrow V_{n,1}$. Hence to obtain $\text{Aut}(V_n)$ we have to select the subgroup of the automorphisms $\{\tilde{F}_0\}$ induced by the diffeomorphisms $\{F_0\}$ of $V_{n,1}$ which keep invariant the isotropic hypersurfaces.

In the above described example of a standard radiation structure embedded into the Minkowski space-time $\mathbb{R}^{n,1}$, only one translation which maps an isotropic hyperplane into another one has to be removed from the subgroup
\[ R^{n+1} \otimes \left( (O(n-1) \times \mathbb{R}) \otimes R^{n-1} \right) \subset R^{n+1} \otimes O(n,1) \]

Then the group of automorphisms of an integrable radiation structure is recovered.

6. Conclusion

In this section we restrict ourselves to the case \( n = 3 \). We have seen so far that the existence of a congruence of isotropic hypersurfaces in a Lorentzian space-time \((V_{3,1}, g)\) implies the reduction of the orthogonal frame bundle \( O(V_{3,1}) \) to the bundle \( G(V_{3,1}) \) of adapted frames. We have then to handle with \( G \)-connections, the curvature forms of which are \( \mathcal{L}(G) \)-valued. But they have also to satisfy the Bianchi’s identities leading to the disappearance of some components in the Ricci tensor, only seven of them are precisely different from zero. Consequently the physical right hand side tensor \( T \) in the Einstein’s field equations cannot be anything. It is easy to verify that for a \( G \)-connection the compatible physical tensor should be written in an adapted frame as

\[
T_G = \lambda (\theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2) + \pi \theta^0 \otimes \theta^0 + \sigma (\theta^1 \otimes \theta^2 + \theta^2 \otimes \theta^1) \\
+ \rho_1 (\theta^1 \otimes \theta^0 + \theta^0 \otimes \theta^1) + \rho_2 (\theta^2 \otimes \theta^0 + \theta^0 \otimes \theta^2) + \rho_3 (\theta^3 \otimes \theta^0 + \theta^0 \otimes \theta^3)
\]

where \( \lambda, \pi, \sigma, \rho \) are arbitrary functions on \( V_{3,1} \) a priori. We do not try to give an interpretation of \( T_G \) as a whole and directly perform the reduction to \( G_I(V_{3,1}) \) which corresponds to keep the isotropic vector field \( \xi \) covariantly constant. The disappearance of the \( R^{n-1} \)-subalgebra leads to \( R_{03} = R_{30} = 0 \). Consequently the corresponding stress-energy tensor \( T_{G_I} \) is deduced from the above expression of \( T_G \) by setting \( \lambda = 0 = \sigma \) and \( \rho_3 = \frac{1}{2} S \), \( S \) denoting the scalar curvature. Then \( T_{G_I} \) can be interpreted as the stress-energy tensor [10] of a massless particle beam with possible heat flow \{\( \rho_1, \rho_2, \rho_3 \)\} along the isotropic hypersurface, the radiation phenomena.

Finally in the case of the reduction to \( G_R(V_{3,1}) \) which makes appear the uniquely induced radiation connection, the Ricci tensor is skinny, two components being non-vanishing only. Then the expression of \( T_{G_R} \) involves the components \( \lambda \) and \( \rho_3 \) related by \( \rho_3 - \lambda = \frac{1}{2} S \), and must be interpreted as the energy tensor of the vacuum [11][12].
REFERENCES

[1] P.A.M. Dirac, a) Proc. Roy. Soc. Lond. A165, 199-208 (1938); b) Rev. Mod. Phys. 26, 392-399 (1949).
[2] R. Penrose, Null Hypersurface Initial Data for Classical Fields of Arbitrary Spin and for General Relativity, Published as Golden Oldie in Gen. Rel. Grav. 12, 225-264 (1980); H. Bondi, M. Vanderburg, A. Metzner, Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems, Proc. Roy. Soc. Lond. A269, 21 (1962); R.K. Sachs, Gravitational Waves in General Relativity. VIII. Waves in Asymptotically Flat Space-Time, Proc. Roy. Soc. Lond. A270, 103-126 (1962).
[3] P.T. Chrusciel, Semi-Global Existence and Convergence of Solutions of the Robinson-Trautman (2-dimensional Calabi) Equation, Commun. Math. Phys. 137, 289-313 (1991).
[4] J.H. Taylor and P.M. McCulloch, Ann. of the N.Y.A.S. 336, 442-446 (1980).
[5] S.W. Hawking, The Event Horizon, Lecture at the Summer School, Les Houches, C. Dewitt and B.S. Dewitt Ed. (1972).
[6] a) G. Lemmer, On Covariant Differentiation Within a Null Hypersurface, Il Nuovo Cimento 37, 1659-1672 (1965);
   b) J.B. Kammerer, Tenseur de courbure d’une hypersurface isotrope, C.R. Acad. Sc. Paris 264, 86-89 (1967);
   c) G. Dautcourt, Characteristic Hypersurfaces in General Relativity, Jour. Math. Phys. 8, 1492-1501 (1967);
   d) P. Hajicek, Exact Models of Charged Black Holes. 1. Geometry of Totally Geodesic Null Hypersurface, Commun. Math. Phys. 34, 37-52 (1973).
   e) A. Ashtekar, M. Steubel, Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity, Proc. Roy. Soc. Lond. A376, 585-607 (1981).
[7] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, in Colloque de Topologie, Bruxelles, pp. 29-55, Thone, Liège (1950).
[8] S.S. Chern, G-Structures, in Colloquium, Strasbourg, pp. 119-136, CNRS Paris (1953).
[9] S. Kobayashi, Transformation Groups in Differential Geometry, p.133, Springer Verlag Ed. (1972).
[10] C.W. Misner, K.S. Thorne, J.A. Wheeler, Gravitation, Ed. Freeman, San Francisco (1973); R.K. Sachs, H. Wu, General Relativity for Mathematicians, Springer-Verlag (1977).
[11] Ya.B. Zel’dovich, I.D. Novikov, The Structure and Evolution of the Universe, University of Chicago Press (1983).
[12] G. Burdet, M. Perrin, *Gravitational waves without gravitons.* Letters in Mathematical Physics 25, 39-45 (1992).