THE COHOMOLOGICAL EXCESS OF CERTAIN MODULI SPACES OF CURVES OF GENUS $g$

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ABSTRACT. The open set $\tilde{M}_g^{<k} \subset M_g$ parametrizes stable curves of genus $g$ having at most $k$ rational components. By the work of Looijenga, one expects that the cohomological excess of $\tilde{M}_g^{<k}$ is at most $g - 1 + k$. In this paper we show that when $k = 0$, the conjectured upper bound is sharp by showing that there is a constructible sheaf on $H_g^{<0}$ (the hyperelliptic locus) which has non-vanishing cohomology in degree $3g - 2$.

1. INTRODUCTION

This research originated from a conjecture by Looijenga which can be stated as follows:

Conjecture 1.1. The coarse moduli space $M_g$ of smooth curves of genus $g > 1$ can be covered by $g - 1$ open affine subvarieties.

Fontanari and Pascolutti [3] prove this conjecture for genus $2 \leq g \leq 5$. (The paper of Fontanari and Looijenga [2] is also relevant.) This conjecture gives bounds on the topological complexity of the moduli space. For example, it would imply the cohomological dimension of constructible sheaves on $M_g$ is at most

$$\dim M_g + (g - 2) = 4g - 5.$$ 

For local systems on $M_g$ this was established by Harer. Looijenga’s conjecture may be viewed as a generalization of Harer’s theorem.

The minimum number of open affine subsets needed to cover a variety less one is called the affine covering number. Later Roth and Vakil [12] introduced the closely related affine stratification number (asn).

The Deligne-Mumford compactification $\overline{M}_g$ is a projective variety which contains $M_g$ as the complement of a normal crossings divisor and parametrizes stable curves of genus $g$, that is, projective curves whose singularities are nodes and whose smooth locus has no components of non-negative Euler characteristic.

Graber and Vakil [7] have introduced a filtration of this variety: $\overline{M}_g^{<k}$ parametrizes stable curves of genus $g$ having at most $k$ rational components. Roth and Vakil extend Looijenga’s conjecture to the following:

Conjecture 1.2. The affine stratification number of $\overline{M}_g^{<k}$ is at most $g - 1 + k$.

Along with other topological consequences this conjecture would prove that the cohomological dimension for constructible sheaves on $\overline{M}_g^{<k}$ is at most $\dim \overline{M}_g^{<k} + g - 1 + k = 4g - 4 + n + k$. We shall denote the cohomological dimension for constructible sheaves on a variety $X$ by $\text{ccd}(X)$. It is the minimum integer $d$ such that $H^n(X, \mathcal{F}) = 0$ for any $n > d$ and any constructible sheaf $\mathcal{F}$ on $X$.

Looijenga introduced another invariant, called cohomological excess (ce).

Definition 1.3. The cohomological excess of a non-empty variety $X$, denoted $\text{ce}(X)$, is the maximum of the integers $\text{ccd}(W) - \dim W$, where $W$ runs over all the Zariski closed subsets $W \subset X$.

Looijenga’s aim was to give an upper bound for the cohomological excess of the moduli space of smooth curves $M_{g,n}$ and, more in general, for certain open subsets of $\overline{M}_{g,n}$. The expected upper bound in the case of $\overline{M}_{g,n}$ is the following:

Conjecture 1.4. The cohomological excess of $\overline{M}_{g,n}$ is at most $g - 1 + k$.

The conjecture above is consistent with Conjecture 1.2 since $\text{ce}(X) \leq \text{asn}(X)$. In this paper we show that the upperbound in the conjecture above is sharp when $k = 0$. We consider the locus $H_g^{<0} = \overline{M}_g^{<0} \cap H_g$.
of stable hyperelliptic curves in $\mathcal{M}_g^{\leq 0}$. Then there is a constructible sheaf $\mathcal{L}$ on $\mathcal{H}_g^{\leq 0}$ whose cohomological dimension is

$$3g - 2 = (g - 1) + (2g - 1) = (g - 1) - \dim \mathcal{H}_g^{\leq 0}. $$

The space $\mathcal{H}_g$ is a quotient of $\mathcal{M}_g$ by the action of the symmetric group $S_{2g+2}$. The constructible sheaf $\mathcal{L}$ on $\mathcal{H}_g^{\leq 0}$ is obtained by taking the push forward of the constant sheaf $\mathbb{C}$, under the quotient map. We prove the following result:

**Lemma 1.5.** The cohomology group $H^{3g-2}(\mathcal{H}_g^{\leq 0}, \mathcal{L})$ is non-zero, and $H^k(\mathcal{H}_g^{\leq 0}, \mathcal{L}) = 0$ for $k > 3g - 2$.

As a consequence we have:

**Theorem 1.6.** $ce(\mathcal{M}_g^{\leq 0}) \geq g - 1$

**Remark.** By Conjecture 1.4 and Lemma 1.5. The cohomology group $H^{3g-2}(\mathcal{H}_g^{\leq 0}, \mathcal{L})$ is non-zero, and $H^k(\mathcal{H}_g^{\leq 0}, \mathcal{L}) = 0$ for $k > 3g - 2$. It is not hard to see that $\dim(\mathcal{H}_g^{\leq k}) \leq g - 1 + k$, hence this result also shows that $\dim(\mathcal{H}_g^{\leq 0}) = g - 1$.

2. **Combinatorial Preliminaries**

Here we recall some definitions from graph theory, explained in greater detail in Getzler and Kapranov [6, Section 2].

**Definition 2.1.** A graph $G$ is a tuple $(F, V, \sigma)$, where

1. $F(G)$ is the set of flags of $G$;
2. $V(G)$ is a partition of $F(G)$, whose parts are called vertices.
3. $\sigma : F(G) \to F(G)$ is an involution.

The fixed points of $\sigma$ are called leaves and the set of all leaves is denoted by $L(G)$. The orbits of size 2 of $\sigma$ are called edges and the set of all edges is denoted by $E(G)$. Let

$$F(v) = \{ f \in F(G) | f \in v \}$$

be the set of flags incident on the vertex $v \in V(G)$.

A graph $G$ has a geometric realization $|G|$, which is the one-dimensional cell complex with 1-cells $[0, 1] \times F(G)$, modulo the identifications $0 \times f \sim 0 \times f'$ if $f, f' \in F(v)$, and $1 \times f \sim 1 \times \sigma(f)$.

For example, the geometric realization of the graph

$$G = (\{1, \ldots, 9\}, \{1, 4, 6, 8\}, \{2, 3, 5, 7, 9\}, (45)(67)(89))$$

is shown in Figure 1.

![Figure 1. The geometric realization of G](image)

Let $b_i(G) = \dim H_i(G, \mathbb{C})$, for $i = 0, 1$. We only consider connected graphs, that is graphs $G$ with $b_0(G) = 1$. If $G$ is connected, the following equality holds:

$$b_1(G) = |V(G)| - |E(G)| + 1.$$ 

A tree is a graph $T$ with $b_0(T) = 1$ and $b_1(T) = 0$, that is $|T|$ is connected and simply connected.

We consider graphs along with labeling $g : V(G) \to \mathbb{Z}_{\geq 0}$ of the vertices. The number $g(v)$ is the genus of the vertex $v$. The genus of the graph $G$ is

$$g(G) = \sum_{v \in V(G)} g(v) + b_1(G).$$
A graph $G$ is **stable** if its vertices satisfy the inequality:

$$2g(v) - 2 + |F(v)| > 0.$$  

A graph of genus $g$ and $n$ leaves will be said to be of type $(g, n)$. For example, in Figure 1 if both the vertices of the graph have genus 0, then the graph is stable of type $(2, 3)$.

The **stabilization** of a labelled graph $G$ is constructed by deleting vertices $v$ of $G$ of genus 0 containing one or two flags.

A numbering of leaves is a bijection $L(G) \to \{n\}$, where $\{n\} = \{1, \ldots, n\}$. A graph with a numbering of its leaves is called a numbered graph.

Two numbered graphs are isomorphic if there is an isomorphism of the underlying graphs that preserves the genus of vertices and the numbering of leaves. Let $\Gamma(g, n)$ be the isomorphism classes of stable graphs of type $(g, n)$. $\Gamma(g, n)$ is finite, as it has been proved in [6, Lemma 2.16].

Let $M_{g,n}$ be the (coarse) moduli space of smooth genus $g$ algebraic curves over $\mathbb{C}$ with $n$ marked points, and let $\overline{M}_{g,n}$ be its Deligne-Mumford compactification, the moduli space of stable curves of arithmetic genus $g$ with $n$ marked points.

To each stable curve of genus $g$ with $n$ marked points we associate a stable graph of type $(g, n)$, called the dual graph: the vertices of the dual graph correspond to the irreducible components of the curve, each labelled by the geometric genus of the corresponding component, the edges of the graph correspond to nodes, and the leaves correspond to the marked points.

We have a stratification of $\overline{M}_{g,n}$ corresponding to isomorphism classes of the dual graphs in $\Gamma(g, n)$, as explained in [6,14]. For an isomorphism class in $\Gamma(g, n)$ choose a representative $G$. Let

$$M_G = \{ [C] \in \overline{M}_{g,n} \mid \text{dual graph of } C \text{ is isomorphic to } G \}.$$  

$M_G$ is open in its closure $\overline{M}_G$, and the set of subvarieties $\{M_G\}$ as $G$ varies over isomorphism classes of graphs in $\Gamma(g, n)$ stratifies $\overline{M}_{g,n}$.

There is a partial order $\prec$ on $\Gamma(g, n)$, $[G] \prec [G']$ if $G'$ can be obtained from a graph isomorphic to $G$ by contracting a subset of the edges and relabelling the vertices (genus of a vertex of $G'$ is the genus of the sub-graph which is the pre-image of the vertex, see [6, Section 2]). Then $[G] \prec [G']$ if an only if $M_G \prec M_{G'}$.

### 3. Hyper-elliptic Locus

A hyperelliptic curve of genus $g$ is a smooth algebraic curve which admits a degree 2 map to $\mathbb{P}^1$ ramified over $2g + 2$ points. The locus $H_g$ is the subvariety of $M_g$ parametrizing hyperelliptic curves of genus $g$ and $\overline{H}_g$ is its closure in $\overline{M}_g$. There is an isomorphism:

$$\overline{H}_g \cong \overline{M}_{0,2g+2}/S_{2g+2}$$

Here, $S_n$ is the symmetric group on $n$ letters and it acts on $\overline{M}_{0,n}$ by permuting the marked points.

Let us recall how the isomorphism is obtained. Let $\mathcal{H}ur_{g,d}$ be the Hurwitz space parametrizing degree $d$ simply branched covers of $\mathbb{P}^1$ of genus $g$. By simple branching we mean each fiber has at least $d - 1$ points. Let $\mathcal{H}ur_{g,d}$ be its compactification by admissible covers, as in [9, Section 4]. By Riemann-Hurwitz there are $r = 2g + 2d - 2$ points over which ramification occurs. We have two maps; $\phi : \mathcal{H}ur_{g,d} \to \overline{M}_{0,r}/S_r$ by remembering only the points in $\mathbb{P}^1$ over which branching occurs, and $\psi : \mathcal{H}ur_{g,d} \to \overline{M}_g$. The admissible cover may not be a stable curve, but we can stabilize it to obtain a genus $g$ curve and thus obtain $\psi$.

In case of $d = 2$ and $r = 2g + 2$, $\phi$ is an isomorphism, and $\psi$ an embedding onto $\overline{H}_g$, giving a very explicit description of $\overline{H}_g$. Let us denote by $\pi : \overline{M}_{0,2g+2} \to \overline{H}_g$, the quotient map to $\overline{M}_{0,2g+2}/S_{2g+2}$ followed by the isomorphism.

#### 3.1. Strata

Associated to a curve $C$, such that $[C] \in \overline{H}_g$, are two combinatorial objects. The first is the dual graph of $C$, which is a stable graph of type $(g, 0)$. The second is the dual graph of a curve $D$ such that $[D] \in \pi^{-1}([C])$, which is a stable graph of type $(0, 2g + 2)$. Of course if $[D'] \in \pi^{-1}([C])$ is another pre-image then $[D'] = \sigma[D]$ for some permutation $\sigma \in S_{2g+2}$. Hence, the dual graph of $D'$ is the same as the dual graph of $D$ but with a renumbering of the leaves.

In fact, if we take the quotient $\Gamma(0, 2g + 2)/S_{2g+2}$, then the graphs correspond to the strata of $\overline{M}_{0,2g+2}/S_{2g+2}$. These are the graphs that we obtain if we forget the numbering of the leaves. The following algorithm describes how to obtain the dual graph of $\pi([D])$ from the dual graph of $D$. This defines a function $\Gamma(0, 2g + 2) \to \Gamma(g, 0)$ which as
described factors through \( \Gamma(0, 2g + 2)/S_{2g+2} \), and in fact \( \Gamma(0, 2g + 2)/S_{2g+2} \to \Gamma(g, 0) \) is injective as will be clear from the algorithm. But before giving the algorithm we have the following definitions:

Let \( T = (F, V, \sigma) \) be a stable graph of type \((0, 2k)\). The parity \( p : F(T) \to \mathbb{Z}/2 \) is a function satisfying \( p \circ \sigma = p \), so that both flags of an edge have same parity. (We say that a flag is even or odd according to whether its parity is \( 0 \) or \( 1 \); in drawing graphs, the even edges will be dashed.) The leaves of \( T \) are odd. The parity of an edge \( e \) is determined as follows: deleting \( e \) produces two connected graphs \( G_1 \) and \( G_2 \), both of which have either an even or an odd number of leaves, since the total number of leaves must be \( 2k \); the edge \( e \) is even or odd accordingly.

**Definition 3.1.** The **ramification number** \( \rho(v) \) of a vertex \( v \in V(T) \) is the number of its flags that are odd

\[
\rho(v) = |\{ f \in F(v) \mid p(f) = 1 \}|
\]

**Algorithm 3.2.** Given a tree \( T \) corresponding to a curve \( C \) in \( \overline{M}_{0,2g+2} \), the dual graph of \( \pi([C]) \in \overline{M}_g \) is the stabilization of the graph \( G \) defined as follows:

- There are two edges in \( G \) for each even edge of \( T \), and one edge in \( G \) for each odd edge of \( T \). (The leaves of \( T \) do not contribute flags to \( G \).)
- A vertex \( v \) of \( T \) contributes a single vertex to \( G \), of genus \( (\rho(v) - 2)/2 \), unless \( \rho(v) = 0 \), in which case it contributes two vertices of genus 0.

![Figure 2. Illustrations of the Algorithm](image)

Figure 2 illustrates the algorithm. The left-most graphs correspond to dual graphs of curves in \( \overline{M}_{0,2g+2} \) (since all vertices are genus 0 there is no need to label them). The middle graphs are the dual graphs of the admissible covers and the rightmost graphs are the graphs of the images in \( \overline{H}_g \). Now let us see why the above algorithm works. Most of this is in fact explained in [8, Section 3-G].

Let \( C \) be a curve in \( \overline{M}_{0,2g+2} \) and \( f : C' \to C \) be the admissible double covering. Assume that \( C \) has 2 irreducible components \( C_1 \) and \( C_2 \); the more general case is only notationally more complicated. If the node connecting the components is an odd node, then both components have an odd number of marked points. Also \( f : f^{-1}(C_i) \to C_i \) are actual branched double covers, so by Riemann-Hurwitz there must be even number of branch points and by definition of admissible cover, there must be ramification over the node. This shows that there is ramification over the odd nodes, whereas similar reasoning shows that the even nodes have two pre-images in the admissible cover.

It is easy to see that each \( f^{-1}(C_i) \) is smooth and has Euler characteristic \( 4 - \rho(C_i) \), where \( \rho \) is the branching number. The rest is self-explanatory.

### 3.2. A Filtration

Graber and Vakil [7] have defined a filtration of \( \overline{M}_g \) by open subsets \( \overline{M}^{\leq k}_g \) corresponding to stable genus \( g \) curves with at most \( k \) rational components (irreducible components of geometric genus 0). This filtration induces a filtration \( \overline{H}^{\leq k}_g \) on \( \overline{H}_g \), and hence, on \( \overline{M}_{0,2g+2} \) through the map \( \pi : \overline{M}_{0,2g+2} \to \overline{H}_g \). Let us determine which strata of \( \overline{M}_{0,2g+2} \) are in \( \overline{M}^{(k)}_{0,2g+2} := \pi^{-1} \left( \overline{H}^{\leq k}_g \right) \).

For a tree \( T \) of type \((0, 2g + 2)\), let the **edge-valence** of a vertex \( v \in V(T) \) be the number \( \nu(v) = |F(v) \setminus L(T)| \), that is the number of edges of that vertex.

Call a vertex \( v \in V(T) \) **internal** if \( \nu(v) > 1 \); otherwise it is **external**.

**Proposition 3.3.** Let \( C \) be a curve in \( \overline{M}_{0,2g+2} \) with corresponding dual graph \( G \). Then the image \( \pi([C]) \in \overline{H}_g \) has a rational component if and only if \( G \) has an internal vertex \( v \) with \( \rho(v) \leq 2 \). Furthermore, the number of rational components of \( \pi([C]) \) is given by

\[
2|\{v \in V(G) \mid \rho(v) = 0\}| + |\{v \in V(G) \mid v \text{ internal and } \rho(v) = 2\}|
\]
Proof. From Algorithm 3.2, it is clear that the only vertices of $G$ that contribute a genus 0 vertex to the dual graph of the admissible cover are those vertices $v \in V(G)$ with $\rho(v)=0$ or $\rho(v)=2$.

If $\rho(v)=0$, then $v$ meets no leaves, so it is internal. Hence in the admissible cover it lifts up to 2 vertices of genus 0 each of which is connected to at least 3 edges, and hence survives stabilization. Hence $v$ contributes 2 vertices of genus 0 to the stabilization of the admissible cover.

On the other hand if $\rho(v)=2$, then the vertex lifts up to one vertex of genus 0 in the admissible cover. If $\nu(v)=1$, then $v$ meets two leaves and an even edge. But then the vertex corresponding to it in the admissible cover has just 2 edges on it, and disappears after stabilization. If $\nu(v)>1$, then the corresponding vertex in the admissible cover meets at least 3 edges and survives stabilization. $\square$

The proposition above tells us exactly which curves belong to $\overline{M}_{0,2g+2}^{(k)}$. The following bound will be useful later. For a stable pointed curve $C$, let $\delta(C)$ denote the number of nodes of $C$.

**Proposition 3.4.** For a curve $C$ in $\overline{M}_{0,2g+2}$, if $[C] \in \overline{M}_{0,2g+2}^{(k)}$ then $\delta(C) \leq g + k - 1$.

**Proof.** For a stable curve of genus $g$ which has $r$ irreducible components, $\delta$ nodes and the geometric genera of the irreducible components are $g_i$, we have the following equality:

$$g = \sum_{i=1}^{r} (g_i - 1) + \delta + 1$$

This yields $\delta = (g - 1) - \sum_{i=1}^{r} (g_i - 1)$. Hence, if the curve has at most $k$ rational components, then $\delta \leq g + k - 1$.

By Algorithm 3.2 for a curve $D$ in $\overline{M}_{0,2g+2}$, $\pi([D])$ has at least as many nodes as $D$. $\square$

3.3. **A Constructible Sheaf.** Consider the constant sheaf $\mathbb{C}$ on $\overline{M}_{0,2g+2}$. Let $\mathcal{L} := \pi_* \mathbb{C}$. (If the action of $S_{2g+2}$ on $\overline{M}_{0,2g+2}$ were free, $\mathcal{L}$ would be a local system, but in any case it is a constructible sheaf.)

Consider the restriction of $\mathcal{L}$ on $\overline{H}^g_{\leq 0}$ and denote it by $\mathcal{L}$ as well. Then,

$$H^*(\overline{H}^g_{\leq 0}, \mathcal{L}) \cong H^*(\overline{M}_{0,2g+2}^{(0)}, \mathbb{C})$$

By Poincaré Duality, Lemma 1.3 is a corollary of the following lemma, proved in Section 4.

**Lemma 3.5.** The cohomology group $H^{k}_{g}(\overline{M}_{0,2g+2}^{(0)})$ is non-zero, and $H^{k}_{g}(\overline{M}_{0,2g+2}^{(0)}) = 0$ for $k < g$.

4. **COHOMOLOGICAL COMPUTATIONS**

4.1. **A Spectral Sequence.** The spectral sequence we describe here is “dual” to the spectral sequence of Deligne [1] section 1.4] for mixed Hodge theory of smooth quasi-projective varieties (see Getzler [4] section 3.7).

Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$, and $D$, a simple normal crossings divisor. By that we mean $D = D_1 \cup \ldots \cup D_N$, where each $D_i$ is a co-dimension 1 smooth sub-variety and all intersections of $D_i$ are transverse. Let

$$X = X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} = \emptyset$$

be the following filtration on $X$: $X_1 = D$, and

$$X_k = \bigcup_{|I|=k} \bigcap_{i \in I} D_i$$

Let $X_k^c = X_k \setminus X_{k+1}$. Then we have $H^*(X_k, X_{k+1}) \cong H^*(X_k^c)$, where $H^*$ denotes cohomology and $H^*_c$ compactly supported cohomology with complex coefficients.

Consider the spectral sequence associated to this filtration on $X$. We have

$$E_1^{p,q} = H^{p+q}(X_{-p}, X_{-p+1}) = H^{p+q}_c(X_{-p})$$

and the differential $d_1$ is given by the composition of maps

$$E_1^{p,q} \xrightarrow{d_1} E_1^{p+1,q}$$

where

$$H^{p+q}(X_{-p}, X_{-p+1}) \xrightarrow{\delta} H^{p+q+1}(X_{-p-1}, X_{-p})$$

and

$$E_1^{p,q} = H^{p+q}(X_{-p}, X_{-p+1}) = H^{p+q}_c(X_{-p})$$
where $i$ and $\delta$ are the maps in the long exact sequence of a pair $W \subset Z$ as follows:
\[
\cdots \to H^{i-1}(W) \xrightarrow{\delta} H^i(Z, W) \xrightarrow{i} H^i(Z) \to \cdots
\]

Since the filtration is finite the spectral sequence converges to $H^{p+q}(X)$. Moreover, the vector spaces $E^{p,q}_1$ carry mixed Hodge structures and the differential is a map of mixed Hodge structures. The spectral sequence converges in the $E_2$ page and $E^{p,q}_2 = E^{p,q}_{\infty}$. To apply this to our situation note that $X = \overline{M}_{0,m}$ is a smooth projective complex variety and $D = \overline{M}_{0,m} \setminus M_{0,m}$ is a simple normal crossings divisor. The set of (isomorphism classes of) trees $\Gamma(0,m)$ can be partitioned into $\Gamma(0,m) = \Gamma_0(0,n) \sqcup \ldots \sqcup \Gamma_{m-3}(0,m)$, where trees in $\Gamma_k(0,m)$ have $k$ edges; then
\[
X_k = \bigcup_{[T] \in \Gamma_k(0,m)} \overline{MT} \quad \text{and} \quad X_k^\circ = \bigcup_{[T] \in \Gamma_k(0,m)} M_T.
\]

As above we have a spectral sequence in the category of mixed Hodge structures, with
\[
(4.1) \quad mE^{p,q}_1 = \bigoplus_{[T] \in \Gamma_p(0,m)} H^{p,q}(M_T)
\]

This spectral sequence tells us how to compute $H^*(\overline{M}_{0,m})$ from the knowledge of $H^*(M_{0,1})$ for all $l \leq m$.

The Hodge structure of $H^i(M_{0,1})$ is pure of weight $2i$ (Getzler [4]). Hence $H^i_c(M_{0,1})$ has a pure Hodge structure of weight $2i - 2(l - 3)$ and $mE^{p,q}_1$ carries a pure Hodge structure of weight $2q - 2(m - 3)$. Since the cohomology $H^i(\overline{M}_{0,m})$ carries a pure Hodge structure of weight $i$, and the odd cohomology is trivial, we conclude that $H^{2i}(\overline{M}_{0,m}) \cong mE^{2i,m-3+i}_1$ and $mE^{p,q}_2 = 0$ if $q - p \neq 2(m - 3)$. So we get a resolution of $H^{2k}(\overline{M}_{0,m})$ as follows
\[
0 \to H^{2k}(\overline{M}_{0,m}) \to \bigoplus_{[T] \in \Gamma_{m-3-k}(0,m)} H^{2k}_c(M_T) \to \cdots \to H^{m-3+k}_c(M_{0,m}) \to 0
\]

Taking duals and setting $j = m - 3 - k$, we get
\[
(4.2) \quad 0 \to H_j(M_{0,m}) \to \bigoplus_{[T] \in \Gamma_j(0,m)} H_{j-1}(M_T) \to \cdots \to \bigoplus_{[T] \in \Gamma_j(0,m)} H_0(M_T) \to H_{2j}(\overline{M}_{0,m}) \to 0
\]

### 4.2. Truncation

Now $\overline{M}_{0,2g+2}^{(0)} \subset \overline{M}_{0,2g+2}$, so if we set $\partial \overline{M}_{0,2g+2}^{(0)} = \overline{M}_{0,2g+2} \setminus \overline{M}_{0,2g+2}^{(0)}$, then
\[
H^*_{\text{tr}}(\overline{M}_{0,2g+2}^{(0)}) \cong H^*_{\text{tr}}(\overline{M}_{0,2g+2} \setminus \partial \overline{M}_{0,2g+2}^{(0)})
\]

Let $C$ be a stable genus zero curve and $T$ its dual graph. From Proposition 3.3 we have $[C] \in \overline{M}_{0,2g+2}^{(0)}$ if and only if $\rho(v) > 2$ for all internal vertices $v \in V(T)$, or, since $\rho(v)$ is even (by Riemann-Hurwitz), $\rho(v) \geq 4$. Let us call trees satisfying this condition good trees and denote the set of isomorphism classes of good trees of type $(0,2k)^0$ by $\Gamma(0,2k)^0$; then
\[
\overline{M}_{0,2g+2}^{(0)} = \bigsqcup_{[T] \in \Gamma(0,2g+2)^0} M_T.
\]

Let $\Gamma_k(0,2g + 2)^0$ be the isomorphism classes of good trees with $k$ edges, so that
\[
\Gamma(0,2g + 2)^0 = \Gamma_0(0,2g + 2)^0 \sqcup \ldots \sqcup \Gamma_{g-1}(0,2g + 2)^0
\]

The filtration on $\overline{M}_{0,2g+2}^{(0)}$ gives a filtration on the singular co-chains of the pair $(\overline{M}_{0,2g+2}^{(0)} \setminus \partial \overline{M}_{0,2g+2}^{(0)})$, and we have the associated spectral sequence:
\[
gF^{p,q}_1 = \bigoplus_{[T] \in \Gamma_p(0,2g+2)^0} H^{p+q}_c(M_T)
\]

The differential here is the same as the differential of the previous spectral sequence.

We have the bounds
\[
gF^{p,q}_1 = 0 \text{ unless } 1 - g \leq p \leq 0 \text{ and } 2g - 1 \leq q \leq 4g - 2.
\]

To see this, first note that by Proposition 3.4 a good graph can have at most $g - 1$ edges. This gives the bounds on $p$. Further when $T$ has $r$ edges, $M_T$ is an affine variety of dimension $2g - 1 - r$. Hence, the compactly supported cohomology of $M_T$ is non-trivial in degrees $2g - 1 - r$, through $4g - 2 - 2r$. This shows the bound on $q$. 


Again the spectral sequence converges and \( H^k_c(\overline{M}^{(0)}_{0,2g+2}) \cong \bigoplus g \mathbb{F}^{-s,k+s} \). As before, this is a spectral sequence in the category of mixed Hodge structures.

From the above bounds (4.3), it is clear that
\[
H_2^g(\overline{M}^{(0)}_{0,2g+2}) \cong g \mathbb{F}^{-g+1,2g-1} \cong g \mathbb{F}^{-g+1,2g-1}.
\]

Also \( g \mathbb{F}_2^{-g+1,2g-1} \) has a pure Hodge structure of weight 0.

Moreover \( H^k_c(\overline{M}^{(0)}_{0,2g+2}) = 0 \) for \( k < g \) since \( g \mathbb{F}_1^{p,q} = 0 \) if \( q + p < g \). Hence, to complete the proof of Lemma 3.5, we just need to show \( g \mathbb{F}_2^{-g+1,2g-1} \neq 0 \).

### 4.3. A digression into Operads

Here we borrow notations and definitions from [5]. Recall that an \( S \)-module \( V \) is a sequence of chain complexes
\[
\{ V(n) \mid n \geq 0 \}
\]
together with an action of \( S_n \) on \( V(n) \).

If \( V \) is a chain complex, let \( \Sigma V \) be its shift (sometimes denoted \( V[1] \)). The gravity and hypercommutative operads [5] have as their underlying \( S \)-modules
\[
Grav(n) = \begin{cases} \Sigma^{2-n} \text{sgn}_n \otimes H_\bullet(M_0,n+1), & n \geq 2 \\ 0, & n < 2 \end{cases}
\]
and
\[
Hycomm(n) = \begin{cases} H_\bullet(M_0,n+1), & n \geq 2 \\ 0, & n < 2 \end{cases}.
\]

For an \( S \)-module \( V \), the dual \( S \)-module \( V^\vee \) is defined as
\[
V^\vee(n) = \Sigma^{n-2} \text{sgn}_n \otimes V(n)^*.
\]
The double dual \( (V^\vee)^\vee \) of an \( S \)-module is naturally isomorphic to \( V \).

Let
\[
V(n) = H_\bullet^c(M_0,n).
\]
By Poincaré duality, \( Grav(n) \cong V^\vee(n) \). So after taking duals, we have \( V \cong Grav^\vee \).

Summing the complexes
\[
0 \to H_j(M_0,m) \to \bigoplus_{[T]\in \Gamma_1(0,m)} H_{j-1}(MT) \to \ldots \to H_0(MT) \to 0
\]
of (4.2) placed in degrees \([j, 2j]\), we get an \( S \)-module
\[
\mathcal{W}(n) = \bigoplus_{[T]\in \Gamma_0(n)} H_\bullet(M_T).
\]
The cohomology of \( \mathcal{W}(n) \) is isomorphic to \( Hycomm(n) \); this is just a restatement of the Koszul duality of \( Grav \) and \( Hycomm \), since \( \mathcal{W} \) is the cobar construction for \( Grav \) (see [5]).

A diagram chase shows that the differential \( d_1 \) in the spectral sequence (4.1) is adjoint to the differential in the cobar construction for \( Grav \).

### 4.4. Proof of Lemma 3.5

As we already noted
\[
H_2^g(\overline{M}^{(0)}_{0,2g+2}) \cong g \mathbb{F}_2^{-g+1,2g-1} \cong g \mathbb{F}_2^{-g+1,2g-1},
\]
so the strategy of proof will be to show that \( d_1 : g \mathbb{F}_1^{-g+1,2g-1} \to g \mathbb{F}_1^{-g+2,2g-1} \) has a kernel. The spectral sequence \( g \mathbb{F}^{*,*} \) is a truncation of the spectral sequence \( 2g+2 \mathbb{E}^{*,*} \), as in section 4.1 and has the same differential in the first page. We identify a subspace \( V_{2g+2} \) of \( 2g+2 \mathbb{E}^{g,2g-1} \) on which the differential \( d_1 \) is non-trivial and show that the image is inside \( g \mathbb{F}_1^{-g+1,2g-1} \subset 2g+2 \mathbb{E}_1^{g,2g-1} \). First define some specific trees which will be useful in the following discussion.

For each \( l \), \( 0 \leq l \leq g \) consider the tree \( T_{l,g} \) of type \((0, 2g + 2)\) defined as follows (see Figure 4):

1. \( T_{l,g} \) has vertices \( v_0, v_1, \ldots, v_l \);
2. vertex \( v_0 \) has \( 2g - 2l + 2 \) leaves and \( v_i \) has 2 leaves for each \( i > 0 \);
(3) \( v_0 \) is connected to each \( v_i \) for \( i > 0 \) by an edge;
(4) for \( i > 0 \), the leaves of \( v_i \) are numbered \( 2i - 1, 2i \) and the leaves of \( v_0 \) are \( 2l + 1, \ldots, 2g + 2 \).

![Diagram of the tree \( T_{l,g} \)](image)

Note that the stratum \( M_{T_{l,g}} \) is isomorphic to \( M_{0,2g-l+2} \) and has dimension \( 2g - l - 1 \). One should think of \( M_{T_{l,g}} \) as the moduli space \( M_{0,2g-l+2} \) with two sets of marked points, \( l \) of them even and the rest odd. Let

\[
W_{l,g} = H^{2g-1-l}_{\epsilon}(M_{T_{l,g}}) \quad \text{for } l = 0, \ldots, g.
\]

When \( g = 4 \), and \( l = 2 \), Figure 4 shows a curve with dual graph \( T_{l,g} \), the admissible cover and its stabilization.

![Diagram of a curve with dual graph \( T_{2,4} \)](image)

The symmetric group \( S_n \) acts on \( M_{0,n+1} \) by permuting the first \( n \) marked points, and hence on \( H^\bullet(M_{0,n+1}) \). We treat \( T_{l,g} \) as a rooted tree with the leaf \( 2g + 2 \) as the root. Then

\[
\text{Aut}(T_{l,g}) \cong S_{2g-2l+1} \times (S_l \wr S_2) \subset S_{2g+1}
\]

\text{Aut}(T_{l,g}) \text{ acts on } W_{l,g} \text{ and we have an induced representation of } S_{2g+1}

\[
\text{Ind}^{S_{2g+1}}_{\text{Aut}(T_{l,g})} W_{l,g}.
\]

(This corresponds to summing over the appropriate cohomology of the strata corresponding to the rooted trees that are isomorphic to \( T_{l,g} \) after renumbering of the non-root leaves.)

**Definition 4.1.** For \( 0 \leq l \leq g \), we define the vector space \( V_{l,g} \) to be the subspace of \( W_{l,g} \) of invariants under the action of \( \text{Aut}(T_{l,g}) \)

\[
V_{l,g} = (W_{l,g})^{\text{Aut}(T_{l,g})} \cong \left( \text{Ind}^{S_{2g+1}}_{\text{Aut}(T_{l,g})} W_{l,g} \right)^{S_{2g+1}}.
\]

Recall the spectral sequences \( mE^\bullet_{-l,2g-1} \) corresponding to the cohomology of \( \overline{M}_{0,m} \) and \( gF^\bullet_{-l,2g-1} \) corresponding to the compactly supported cohomology of \( \overline{M}^{(0)}_{0,2g+2} \). Then \( V_{l,g} \subset gF^i_{-l,2g-1} \) and when \( l \leq g - 1 \), we have \( V_{l,g} \subset gF_{-l,2g-1}^{i-1} \).
We have the following diagram

\[
\begin{array}{c}
\begin{array}{c}
2g+2E^{-g,2g-1}_1 \quad \xrightarrow{d_1} \quad 2g+2E^{-g+1,2g-1}_1 \quad \xrightarrow{d_1} \quad 2g+2E^{-g+2,2g-1}_1 \\
V_{g,g} \quad \xrightarrow{d_1} \quad V_{g-1,g} \quad \xrightarrow{d_1} \quad V_{g-2,g}
\end{array}
\end{array}
\]

(4.4)

To understand the vector spaces \(V_{i,g}\), let us first analyse \(W_{i,g}\). As shown in [5], as representations of \(S_n\) we have

\[
H^{n-2}_{e}(M_{0,n+1}) \cong H_{n-2}(M_{0,n+1})
\]

(4.5)

\[
\cong \text{sgn}_n \otimes \text{Lie}(n).
\]

**Definition 4.2.** A Lie superalgebra is a \(\mathbb{Z}_2\)-graded vector space \(L = L_0 \oplus L_1\) along with a bracket \([\cdot, \cdot]\), which satisfies the following axioms: if \(a, b, c \in L\) are homogeneous elements of degree \(|a|, |b|\) and \(|c|\), then

1. \([a, b] \in L_{|a|+|b|};
2. \([a, b] = (-1)^{|a||b|}[b, a];
3. \((-1)^{|a||c|}[a, [b, c]] + (-1)^{|c||b|}[c, [a, b]] + (-1)^{|b||a|}[b, [c, a]] = 0.

A \(\mathbb{Z}\)-graded Lie algebra is defined in the same way, except that the vector space has a \(\mathbb{Z}\)-grading.

The \(S_n\)-module \(\text{Lie}(n)\) associated to the operad \(\text{Lie}\) is a submodule of the free Lie algebra with generators \(\{x_1, \ldots, x_n\}\); similarly, the \(S_n\)-module

\[
\Lambda \text{Lie}(n) = \text{sgn}_n \otimes \text{Lie}(n)[1-n]
\]

(4.6)

associated to the operad \(\Lambda \text{Lie}\) (suspension of the operad \(\text{Lie}\)) is a submodule of the free algebra with a shifted Lie bracket. Note that this is \(\mathbb{Z}\)-graded, but we can consider the underlying \(\mathbb{Z}_2\)-grading. This turns out to be a submodule of the free Lie superalgebra with generators \(\{y_1, \ldots, y_n\}\) of degree 1.

Let \(A = A_0 \sqcup A_1\) (disjoint union) be a \(\mathbb{Z}_2\)-graded set, and \(A^*\), the free monoid generated by \(A\). Denote by \(|a|\) the degree of \(a \in A^*\). Then \(\mathbb{C}(A^*)\), the \(\mathbb{C}\) vector space generated by \(A^*\) with the obvious multiplication, is called the free nonassociative algebra over \(A\). Define \([a, b] = ab - (-1)^{|a||b|}ba\). Let \(I\) be the ideal in \(\mathbb{C}(A^*)\) generated by the set

\[
\{ab + (-1)^{|a||b|}ba, (-1)^{|a||c|}[a, b, c] + (-1)^{|c||b|}[c, [a, b]] + (-1)^{|b||a|}[b, [c, a]] \mid a, b, c \in A^*\}.
\]

Then \(L_A = \mathbb{C}(A^*) / I\) with the binary operation \([\cdot, \cdot]\) is a Lie superalgebra called the free Lie superalgebra with generators \(A\).

From (4.5) and (4.6), it is clear that

\[
W_{i,g} \cong \Lambda \text{Lie}(T_{i,g}).
\]

(4.7)

In other words, \(W_{i,g}\) is spanned by free Lie superalgebra words in generators

\[
\{[a_1, a_2], [a_3, a_4], \ldots, [a_{2l-1}, a_{2l}], a_{2l+1}, \ldots, a_{2g+1}\}
\]

where \(a_i\) has degree 1, and in which each letter \(a_i\) occurs exactly once. This vector space is isomorphic to the vector space spanned by free Lie superalgebra words in generators

\[
\{b_1, \ldots, b_l, a_{2l+1}, \ldots, a_{2g+1}\}
\]

where again each generator occurs once, but now \(b_i\) has degree 0 whereas \(a_j\) has degree 1.

Let \(A\) be a \(\mathbb{Z}_2\)-graded ordered alphabet and \(A^*\) the free monoid generated by \(A\) ordered lexicographically. A word \(w\) is a Lyndon word if it is lexicographically smaller than all its cyclic rearrangements. In other words for any non-trivial factorization \(w = uv\), we have \(w < v\).

To a Lyndon word over \(A\) one can uniquely associate an element of the free Lie superalgebra generated by \(A\). This association is called the standard bracketing of a Lyndon word and is defined inductively on the length of the word. We denote the bracket of a Lyndon word \(w\) by \(B(w)\).
Suppose \( w = wv \) where \( v \) is the lexicographically smallest proper right factor of \( w \). Then \( u \) and \( v \) are both Lyndon words and \( B(w) = [B(u), B(v)] \). Let \( \mathcal{L}(A) \) be the set of all Lyndon words on the alphabet \( A \). The alphabet \( A \) can have elements in different degrees and we define the degree of \( w \in A^* \) by

\[
|w| = \sum_{i=1}^{k} |a_i|, \quad \text{if} \ w = a_1 \cdots a_k.
\]

The set

\[
\{ B(w) \mid w \in \mathcal{L}(A) \} \cup \{ [B(w), B(w)] \mid w \in \mathcal{L}(A), \text{where } |w| = 1 \}
\]

forms a basis of the free Lie superalgebra with generators \( A \) called the Lyndon basis; see for example Shtern [13] (also Reutenauer [11] Sections 4.1 and 5.1).

Clearly, the \( V_{i,g} \) are in one-to-one correspondence with the \( S_l \times S_{2g-2l+1} \) invariants of \( W_{i,g} \), which acts by permuting the letters \( \{b_1, \ldots, b_l\} \) and \( \{a_{2l+1}, \ldots, a_{2g+1}\} \) separately. This proves the first part of the following lemma.

**Lemma 4.3.** Let \( \text{Lie}_{(i,j)}[a, b] \) denote the vector space of Lie superalgebra words in the letters \( \{a, b\} \), where \( a \) has degree 1 and \( b \) has degree 0, homogeneous of degree \( i \) in \( a \) and \( j \) in \( b \). Then \( V_{i,g} \cong \text{Lie}_{(2g-2l+1, i)}[a, b] \). Furthermore, \( \dim V_{i,g} = 1 \).

**Proof.** Let us define the order \( a < b \) for the generators \( \{a, b\} \). Then the only Lyndon word with one instance of \( a \) and \( g \) instances of \( b \) is \( ab^g \); this shows that \( V_{g, g} \) has dimension 1, with basis \( B(ab^g) = \ldots [a, b], \ldots [a, b] \)

The following lemma is the main ingredient in the proof of Lemma 3.5.

**Lemma 4.4.** The differential \( d_1 : V_{g, g} \rightarrow V_{g-1, g} \) is non-zero.

**Proof.** The differential \( d_1 \) in the complex (4.2) is the adjoint of the differential of the cobar construction for the gravity operad. Hence the differential corresponds to an alternating sum of operadic compositions in the gravity operad.

Since the vector space \( V_{g, g} \) is one-dimensional with basis vector \( \omega_g = B(ab^g) \), the differential \( d_1 : V_{g, g} \rightarrow V_{g-1, g} \) is determined by its action on \( \omega_g \), which is an alternating sum of the terms obtained by replacing each instance of \( b \) by \( [a, a] \). In other words,

\[
d_1(\omega_g) = \sum_{i=1}^{g} (-1)^{i-1}[[\ldots[[a, a], b], \ldots], b].
\]

Let \( \mathcal{L}(3, g-1) \) be the set of Lyndon words in \( \{a, b\} \) with 3 instances of \( a \) and \( (g-1) \) of \( b \). We claim that for \( g \geq 2 \),

\[
d_1(\omega_g) = \begin{cases} 
2B(a^3 b^{g-1}) + (g-2)B(a^2 b a b^{g-2}) + \sum_{w \in \mathcal{L}(3, g-1) \mid w > a^2 b a b^{g-2}} n_w B(w), & \text{if } g \text{ even.} \\
(g-1)B(a^2 b a b^{g-2}) + \sum_{w \in \mathcal{L}(3, g-1) \mid w > a^2 b a b^{g-2}} n_w B(w), & \text{if } g \text{ odd.}
\end{cases}
\]

The cases \( g = 2, 3 \) are true: by the super-Jacobi identity, \( [a, [a, a]] = 0 \), hence

\[
d_1(\omega_2) = d_1[[a, [a, b]], b] - [[a, b], [a, a]] = 2[[a, [a, b]], b] = 2B(aa a b)
\]
\[
d_1(\omega_3) = d_1[[a, [a, b]], b] - [[a, b], [a, a]] = 2[[[a, [a, b]], b], [a, a]] = 2B(a a a b b)
\]

The induction now follows on combining the following results:

1. By the standard triangularity property for the Lyndon basis, if \( m < n \) are Lyndon words, and thus \( mn \) is again a Lyndon word, then

\[
[B(m), B(n)] = B(mn) + \sum_{\text{Lyndon word } w \mid w = m + n, \text{ } w > mn} n_w B(w).
\]

2. Expanding \( B(a^3 b^{g-2}) \), and applying the super-Jacobi identity, we see that if \( g > 3 \), then

\[
B(a^3 b^{g-2}) = B(a^3 b^{g-1}) + B(a^2 b a b^{g-2}) - B(a^2 b a b^{g-2} a b).
\]
From (4.8), it follows that
\[ d_1(\omega_g) = [d_1(\omega_{g-1}), b] + (-1)^{g-1}[\omega_{g-1}, [a, a]] \]
\[ = [d_1(\omega_{g-1}), b] + (-1)^g 2B(a^3b^{g-1}). \]

The lemma is proved. \( \square \)

The image of \( d_1: V_{g,g} \to V_{g-1,g} \) lies in the kernel of \( d_1: V_{g-1,g} \to V_{g-2,g} \). Since \( V_{g-1,g} \subset gF_{g-1}^{g+1,2g-1} \), this implies that \( H^2_g(M^{(0)}_{0,2g+2}) \cong gF_2^{g+1,2g-1} \) is non-trivial (see (4.4)), completing the proof of Lemma 3.5.

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