Quantifying the Conceptual Error in Dimensionality Reduction

Tom Hanika¹,² and Johannes Hirth¹,²

¹ Knowledge & Data Engineering Group, University of Kassel, Germany
² Interdisciplinary Research Center for Information System Design
University of Kassel, Germany
tom.hanika@cs.uni-kassel.de, hirth@cs.uni-kassel.de

Abstract Dimension reduction of data sets is a standard problem in the realm of machine learning and knowledge reasoning. They affect patterns in and dependencies on data dimensions and ultimately influence any decision-making processes. Therefore, a wide variety of reduction procedures are in use, each pursuing different objectives. A so far not considered criterion is the conceptual continuity of the reduction mapping, i.e., the preservation of the conceptual structure with respect to the original data set. Based on the notion scale-measure from formal concept analysis we present in this work a) the theoretical foundations to detect and quantify conceptual errors in data scalings; b) an experimental investigation of our approach on eleven data sets that were respectively treated with a variant of non-negative matrix factorization.

Keywords: Formal Concept Analysis; Dimension Reduction; Conceptual Measurement; Data Scaling

1 Introduction

The analysis of large and complex data is presently a challenge for many data driven research fields. This is especially true when using sophisticated analysis and learning methods, since their computational complexity usually grows at least superlinearly with the problem size. One aspect of largeness and complexity is the explicit data dimension, e.g., number of features, of a data set. Therefore, a variety of methods have been developed to reduce exactly this data dimension to a computable size, such as principal component analysis, singular value decomposition, or factor analysis [11]. What all these methods have in common is that they are based on the principle of data scaling [8].

A particularly challenging task is to apply Boolean factor analysis (BFA) [15], as the distinct feature values are restricted to either 0 (false) or 1 (true). For example, given the binary data set matrix \( K \), the application of a BFA yields two

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binary data matrices $S,H$ of lower dimension, such that $S \cdot H$ approximates $K$ with respect to a previously selected norm $\| \cdot \|$. The factor $S$ can be considered as a lower dimensional representation of $K$, i.e., a scaling of $K$. The connection between the scaling features of $S$ and the original data features of $K$ is represented by $H$. The quality of an approximation, and therewith the quality of a scale $S$, is usually scored through the Frobenius norm of $K - S \cdot H$, or other functions [3, 9], such as the Residual Sum of Squares or, in the binary setting, the Hamming distance. These scoring functions give a good impression of the extent to which the linear operator $K$ is approximated by $S \cdot H$, yet, they are incapable to detect the deviation of internal incidence structure of $S$ with respect to $K$, which we want to call conceptual scaling error.

A well defined formalism from Formal Concept Analysis (FCA) [6] to analyze the resulting inconsistencies in (binary) data scaling is scale-measures [5, 7]. In this work, we build up on this notion and introduce a comprehensive framework for quantification of quantifying the conceptual errors of scales. The so introduced mathematical tools are capable of determining how many conceptual errors arise from a particular scaling $S$ and pinpoint which concepts are falsely introduced or lost. For this we overcome the potential exponential computational demands of computing complete conceptual structures by employing previous results on the scale-measures decision problem [8]. We motivate our results with accompanying examples and support our results with an experiment on eleven data sets.

2 Scales and Data

FCA Recap In the field of Formal concept analysis (FCA) [6, 14] the task for data scaling, and in particular feature scaling, is considered a fundamental step for data analysis. Hence, data scaling is part of the foundations of FCA [6] and it is frequently investigated within FCA [5, 7].

The basic data structure of FCA is the formal context, see our running example $\mathbb{K}_W$ Figure 1 (top). That is a triple $(G, M, I)$ with non-empty and finite set $G$ (called objects), finite set $M$ (called attributes) and a binary relation $I \subseteq G \times M$ (called incidence). The tuple $(g,m) \in I$. Any context $S = (H, N, J)$ with $H \subseteq G, N \subseteq M$ and $J = I \cap (H \times N)$ we call induced sub-context of $K$, and denote this relation by $\leq$, i.e., $S \leq K$.

The incidence relation $I \subseteq G \times M$ gives rise to the natural Galois connection between $P(G)$ and $P(M)$, which is the pair of operators $\cdot^\prime: \mathcal{P}(G) \to \mathcal{P}(M)$, $A \mapsto A^\prime = \{m \in M \mid \forall a \in A: (a,m) \in I\}$, and $\cdot^\prime: \mathcal{P}(M) \to \mathcal{P}(G)$, $B \mapsto B^\prime = \{g \in G \mid \forall b \in B: (g,b) \in I\}$, each called derivation. Using these operator, a formal concept is a pair $(A,B) \in \mathcal{P}(G) \times \mathcal{P}(M)$ with $A^\prime = B$ and $A = B^\prime$, where $A$ and $B$ are called extent and intent, respectively.

This restriction is in general not necessary. Since data sets can be considered finite throughout this work and since this assumption allows a clearer representation of the results within this work, it was made.

Both operators are traditionally denoted by the same symbol $\cdot^\prime$. 

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\[\text{\textsuperscript{4}}\text{Both operators are traditionally denoted by the same symbol $\cdot^\prime$.}\]
The two possible compositions of the derivation operators lead to two closure operators \( \cdot'': \mathcal{P}(G) \rightarrow \mathcal{P}(G) \) and \( \cdot'': \mathcal{P}(M) \rightarrow \mathcal{P}(M) \), and in turn to two closure spaces \( \text{Ext}(\mathbb{K}) := (G,'') \) and \( \text{Int}(\mathbb{K}) := (M,'') \). Both closure systems are represented in the (concept) lattice \( \mathcal{B}(\mathbb{K}) = (\mathcal{B}(\mathbb{K}), \leq) \), where the set of all formal concepts is denoted by \( \mathcal{B}(\mathbb{K}) := \{(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M) \mid A' = B \land B' = A\} \) the order relation is \( (A, B) \leq (C, D) :\Rightarrow A \subseteq C \). An example drawing of such a lattice is depicted in Figure 1 (bottom).

### 2.1 Scales-Measures

The basis for quantifying the conceptual error of a data scaling is the continuity of the scaling map with respect to the resulting closure spaces. We say a map map \( f : G_1 \rightarrow G_2 \) is **continuous** with respect to the closure spaces \((G_1, c_1)\) and \((G_2, c_2)\) if and only if for all \( A \in \mathcal{P}(G_2) \) we have \( c_1(f^{-1}(A)) \subseteq f^{-1}(c_2(A)) \). That is, a map is continuous iff the preimage of closed sets is closed. Within FCA this notion is captured by the following definition.

**Definition 1 (Scale-Measure (cf. Definition 91, [6])).** Let \( \mathbb{K} = (G, M, I) \) and \( \mathbb{S} = (G_\mathbb{S}, M_\mathbb{S}, I_\mathbb{S}) \) be a formal contexts. The map \( \sigma : G \rightarrow G_\mathbb{S} \) is called an \( \mathbb{S} \)-measure of \( \mathbb{K} \) into the scale \( \mathbb{S} \) iff the preimage \( \sigma^{-1}(A) := \{g \in G \mid \sigma(g) \in A\} \) of every extent \( A \in \text{Ext}(\mathbb{S}) \) is an extent of \( \mathbb{K} \).
| S | W | LW | PLants | Animals | Land Plants | Water Plants | Land Animal | Water Animal | Mammal |
|---|---|---|---|---|---|---|---|---|---|
| dog | × | × | × | | | | | | |
| leech | × | × | | | | | | | |
| corn | × | × | | | | | | | |
| bream | × | × | | | | | | | |
| water | × | × | × | | | | | | |
| weeds | × | × | | | | | | | |
| bean | × | × | | | | | | | |
| frog | × | × | × | | | | | | |
| reed | × | × | × | | | | | | |

Figure 2. A scale context (top), its concept lattice (bottom right) for which id_G is a scale-measure of the context in Figure 1 and the reflected extents σ^{-1}[Ext(S)] (bottom left) indicated as in gray.

The (scaling-)map σ can be understood as an interpretation of the objects from K using the attribute (features) of S. Hence, we denote σ^{-1}[Ext(S)] := \bigcup_{A \in Ext(S)} σ^{-1}(A) as the set of extents that are reflected by the scale context S. In Figure 2 we depicted an example scale-context S, where the attributes of the scale-context are constructed by conjunctions of attributes from K_W, as seen in Figure 1. This scaling is based on the original object set G and we observe that S reflects twelve out of the nineteen concepts from \mathcal{B}(K_W).

For any two scale-measures (σ, S), (ψ, T) we say [8] that (σ, S) is finer than (ψ, T), iff ψ^{-1}(T) ⊆ σ^{-1}[Ext(S)]. Dually we say then that (σ, S) is coarser than (ψ, T). From both relation an equivalence relation ∼ arises naturally. The set of all possible scale-measures for some K, denoted by \mathcal{G}(K) := \{(σ, S) | σ is a S-measure of K\}, is therefore ordered. Furthermore, it is known [8] that factorizing \mathcal{G}(K) by ∼ leads to a lattice ordered structure \mathcal{G}(K) = (\mathcal{G}(K)/∼, ≤), called the scale-hierarchy of K. This hierarchy is isomorphic to the set of all sub-closure systems.
of \( \text{Ext}(\mathcal{K}) \), i.e., \{ \mathcal{Q} \subseteq \text{Ext}(\mathcal{K}) \mid \mathcal{Q} \) is a Closure System on \( \mathcal{G} \}, ordered by set inclusion \( \subseteq \).

Every scale-measure \((\mathcal{S}, \sigma) \in \mathcal{S}(\mathcal{K})\) does allow for a canonical representation \([8]\), i.e., \((\sigma, \mathcal{S}) \sim (\text{id}, \mathcal{K}_{\sigma^{-1}(\text{Ext}(\mathcal{S}))})\). This representation, however, very often eludes human explanation to some degree. This issue can be remedied through a related approach called \textit{logical scaling} \([12]\) that is a representation of the scale-context attributes by conjunction, disjunction, and negation of the original attributes, formally \( M_\mathcal{S} \subseteq L(M, \{ \land, \lor, \neg \}) \). Such a representation does always exist:

**Proposition 1** (CNF of Scale-measures (cf. Proposition 23, [8])). Let \( \mathcal{K} \) be a context, \((\sigma, \mathcal{S}) \in \mathcal{S}(\mathcal{K}) \). Then the scale-measure \((\psi, \mathcal{T}) \in \mathcal{S}(\mathcal{K})\) given by

\[
\psi = \text{id}_G \quad \text{and} \quad \mathcal{T} = \bigcup_{A \in \sigma^{-1}(\text{Ext}(\mathcal{S}))} (G, \{ \land A \}, I_\phi)
\]

is equivalent to \((\sigma, \mathcal{S})\) and is called conjunctive normal form of \((\sigma, \mathcal{S})\).

3 Conceptual Errors in Data Scaling

Scaling and factorizations procedures are essential to almost all data science (DS) and machine learning (ML) approaches. For example, relational data, such as a formal context \( \mathcal{K} \), is often scaled to a lower (attribute-) dimensional representation \( \mathcal{S} \). Such scaling procedures, e.g., \textit{principle component analysis}, \textit{latent semantic analysis}, \textit{non-negative matrix factorization}, however, do almost always not account for the conceptual structure of the original data \( \mathcal{K} \). Hence, in order to comprehend the results of DS/ML procedures, it is crucial to investigate to what extent and which information is lost during the scaling process. In the following we want to introduce a first approach to quantify and treat error in data scalings through the notion of scale-measures. For this we need the notion of \textit{context apposition} \([6]\). Given two formal contexts \( \mathcal{K}_1 := (G_1, M_1, I_1) \), \( \mathcal{K}_2 := (G_2, M_2, I_2) \) with \( G_1 = G_2 \) and \( M_1 \cap M_2 = \emptyset \), then \( \mathcal{K}_1 \upharpoonright \mathcal{K}_2 := (G, M_1 \cup M_2, I_1 \cup I_2) \). If \( M_1 \cap M_2 \neq \emptyset \) the apposition is constructed disjoint union of the attribute sets.

**Proposition 2.** Let \( \mathcal{K}, \mathcal{S} \) be formal contexts, \( \sigma : G_\mathcal{K} \to G_\mathcal{S} \) a map and let \( \mathcal{A} = \mathcal{K} \upharpoonright (G_\mathcal{K}, M_3, I_3) \) with \( I_3 = \{ (g, \sigma(g)) \mid g \in G_\mathcal{K} \} \circ I_3 \), then \((\sigma, \mathcal{S})\) and \((\text{id}_{G_\mathcal{K}}, \mathcal{K})\) are scale-measures of \( \mathcal{A} \).

**Proof.** It is to show that \( \text{id}_{G_\mathcal{K}}^{-1}(\text{Ext}(\mathcal{K})) \) and \( \sigma^{-1}(\text{Ext}(\mathcal{S})) \) are subsets of \( \text{Ext}(\mathcal{A}) \). The set \( \text{Ext}(\mathcal{A}) \) is the smallest intersection closed set containing \( \text{Ext}(\mathcal{K}) \) and \( \text{Ext}(G_\mathcal{K}, M_3, I_3) \). Thus \( \text{id}_{G_\mathcal{K}}^{-1}(\text{Ext}(\mathcal{K})) \subseteq \text{Ext}(\mathcal{A}) \). Let \( C \in \text{Ext}(\mathcal{S}) \), then there exists a representation \( C = D_\mathcal{S} \circ D_\mathcal{K} \), with \( D \subseteq M_\mathcal{S} \). The derivation \( D_\mathcal{K} \) is an extent in \( \text{Ext}(G_\mathcal{K}, M_3, I_3) \) and is equal to \( \sigma^{-1}(D_\mathcal{S}) \), since \( I_\sigma = \{ (g, \sigma(g)) \mid g \in G_\mathcal{S} \} \circ I_3 \). Thus, \( C = \sigma^{-1}(C) \in \text{Ext}(G_\mathcal{K}, M_3, I_3) \) and therefore we find \( \sigma^{-1}(C) \in \text{Ext}(G_\mathcal{K}, M_3, I_3) \).

The extent closure system of the in Proposition 2 constructed context \( \mathcal{A} \), is equal to the join of \( \text{Ext}(\mathcal{K}) \) and \( \sigma^{-1}(\text{Ext}(\mathcal{S})) \) in the closure system of all closure systems on \( G \) (cf. Proposition 13 [8]). Hence, \( \text{Ext}(\mathcal{A}) \) is the smallest closure system on \( G \), for which \( \text{Ext}(\mathcal{K}) \) and \( \sigma^{-1}(\text{Ext}(\mathcal{S})) \) are contained.
In the above setting one can consider \( K \) and \( S \) as \emph{consistent scalings} of \( K \). Based on this the question for representing and quantifying \emph{inconsistencies} arises.

\textbf{Definition 2 (Conceptual Scaling Error).} Let \( K, S \) be formal contexts and \( \sigma : G_K \rightarrow G_S \), then the conceptual scaling error of \( (\sigma, S) \) with respect to \( K \) is the set \( \mathcal{E}^K_{\sigma,S} := \sigma^{-1}[\text{Ext}(S)] \setminus \text{Ext}(K) \).

The conceptual scaling error \( \mathcal{E}^K_{\sigma,S} \) consists of all pre-images of closed object sets in \( S \) that are not closed in the context \( K \), i.e., the object sets that contradict the scale-measure criterion. Hence, \( \mathcal{E}^K_{\sigma,S} = \emptyset \) iff \( (\sigma, S) \in \mathcal{G}(K) \).

In the following, we denote by \( \sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)} := \sigma^{-1}[\text{Ext}(S)] \cap \text{Ext}(K) \) the set of consistently reflected closed object sets of \( S \) by \( \sigma \). This set can be represented as the intersection of two closure systems and is thereby a closure system as well. Using this notation together with the canonical representation Proposition 1 we can find the following statement.

\textbf{Corollary 1.} For \( K, S \) and \( \sigma : G_K \rightarrow G_S \), there exists a scale-measure \( (\psi, T) \in \mathcal{G}(K) \) with \( \psi^{-1}(\text{Ext}(T)) = \sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)} \).

The conceptual scaling error \( \mathcal{E}^K_{\sigma,S} \) does not constitute a closure system on \( G \), since it lacks the top element \( G \). Moreover, the meet of elements \( A, D \in \mathcal{E}^K_{\sigma,S} \) can be closed in \( K \) and thus \( A \land D \notin \mathcal{E}^K_{\sigma,S} \).

To pinpoint the cause of the conceptual scaling inconsistencies we may investigate the scale’s attributes using the following proposition.

\textbf{Proposition 3 (Deciding Scale-Measures (cf. Proposition 20, [8])).} Let \( K \) and \( S \) be two formal contexts and \( \sigma : G_K \rightarrow G_S \), then TFAE:

\begin{enumerate}
  \item \( \sigma \) is a \( S \)-measure of \( K \)
  \item \( \sigma^{-1}(m^{l_S}) \in \text{Ext}(K) \) for all \( m \in M_S \)
\end{enumerate}

Based on this result, we can decide if \( (\sigma, S) \) is a scale-measure of \( K \) solely based on the attribute extents of \( S \). In turn this enables us to determine the particular attributes \( n \) that cause conceptual scaling errors, i.e. \( \sigma^{-1}(n^{l_S}) \notin \text{Ext}(K) \). We call the set of all these attributes the \emph{attribute scaling error}.

\textbf{Corollary 2.} For formal contexts \( K, S \) and map \( \sigma : G_K \rightarrow G_S \) let the set \( O = \{ m \in M_S \mid \sigma^{-1}(m^{l_S}) \in \text{Ext}(K) \} \). Then \( (\sigma, (G_S, O, I_S \cap G_S \times O)) \) is a scale-measure of \( K \).

\textbf{Proof.} Follows directly from applying Proposition 3.

The thus constructed scale-measure does not necessarily reflect all extents in \( \sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)} \). For this, consider the example \( K = (\{1,2,3\}, \{1,2,3\}, =) \) with \( S = (\{1,2,3\}, \{1,2,3\}, \neq) \) and the map \( \text{id}_{\{1,2,3\}} \). The error set is equal to \( \mathcal{E}^K_{\sigma,S} = (\{1,2,3\}) \). Hence, none of the scale-attributes \( M_S = \{1,2,3\} \) fulfills the scale-measure property. By omitting the whole set of attributes \( M_S \), we result in the context \( (G, \{\}, \{\}) \) whose set of extents is equal to \( \{G\} \). The set \( \sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)} \) however is equal to \( \{\}, \{1\}, \{2\}, \{3\}, \{1,2,3\} \).
3.1 Representation and Structure of Conceptual Scaling Errors

So far, we apprehended $\mathcal{E}_{\sigma,S}^K$ as the set of erroneous preimages. However, the conceptual scaling error may be represented as a part of a scale-measure: i) The first approach is to analyze the extent structure of $\sigma^{-1}\left[\text{Ext}(S)\right]$. This leads to a scale-measure $(\sigma,S)$ of the apposition $S | K$, according to Proposition 2. The conceptual scaling error $\mathcal{E}_{\sigma,S}^K$ is a subset of the reflected extents of $(\sigma,S)$. ii) The second approach is based on our result in Corollary 1. The conceptual scaling error $\mathcal{E}_{\sigma,S}^K$ cannot be represented as scale-measure of $K$. However, since $\mathcal{E}_{\sigma,S}^K \subseteq \sigma^{-1}\left[\text{Ext}(S)\right]$ there is a scale-measure of $S$ that reflects $\mathcal{E}_{\sigma,S}^K$ (right, Figure 3). Such a scale-measure can be computed using the canonical representation of scale-measures as highlighted by $\phi(\mathcal{E}_{\sigma,S}^K)$ in Figure 3. Since the scale-hierarchy is join-pseudocomplemented [8], we can compute a smaller representation of $\sigma^{-1}\left[\text{Ext}(S)\right]|_{\text{Ext}(K)}$ and $\mathcal{E}_{\sigma,S}^K$. In detail, for any $\sigma^{-1}\left[\text{Ext}(S)\right]|_{\text{Ext}(K)}$ there exists a least element in $\Sigma(S)$ whose join with $\sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)}$ yields $\sigma^{-1}[\text{Ext}(S)]$. Due to its smaller size, the so computed join-complement can be more human comprehensible than $\mathcal{E}_{\sigma,S}^K$. iii) The third option is based on splitting the scale context according to its consistent attributes, see Corollary 2. Both split elements are then considered as scale-measures of $S$. This results in two smaller, potentially more comprehensible, concept lattices. Additionally, all discussed scale-measures can be given in conjunctive normalform.

Figure 3. The conceptual scaling error and the consistent part of $(\sigma,S)$ in $\Sigma(K)$ (left). The right represents both parts as scale-measures of $S$. 


3.2 Computational Tractability

The first thing to note, with respect to the computational tractability, is that the size of the concept lattice of \( S \), as proposed in i) (above) is larger compared to the split approaches, as proposed in ii) and iii). This difference results in order dimensions for the split elements that are bound by the order dimension of \( S \) (cf. Proposition 24, [8]). The approach in ii) splits the scale \( S \) according to the conceptual scaling error \( E_{\sigma,K,S} \), a potentially exponentially sized problem with respect to \( S \). The consecutive computation of the join-complement involves computing all meet-irreducibles in \( \sigma - 1 [\text{Ext}(S)] \), another computationally expensive task. In contrast, approach iii) splits \( S \) based on consistent attributes and is takes therefore polynomial time in the size of \( S \). However, as shown in the example after Corollary 2, approach iii) may lead to less accurate representations.

Attributes of \( K \)

| cleared land (CL), draft (Dr), dung (Dv), education (Ed), eggs (Eg), feathers (Fe), fiber (Fi), guarding (G), guiding (Gu), hunting (Hu), lawn mowing (LM), leather (Le), measure (Me), milk (Mi), meat (Mo), narcotic detection (ND), ornamental (O), pack (Pa), pest control (PC), pets (Pe), plowing (Pl), policing (Ps), racing (Ra), rescuing (Re), research (Re), service (Se), show (Sh), skin (Sk), sport (Sp), therapy (Th), truffle harvesting (TH), vellum (V), weed control (WC), working (W) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|--- |
Table 1. Quantifying the Conceptual Scaling Error for approximations $K_{\approx} = S \cdot H$ of data sets $K$ by Binary Matrix Factorization. Cells with '-' where not computed due to computational intractability. Density (D), Attribute Error (AE), Conceptual Scaling Error (CE), Hemming distance between $I_K$ and $I_{S \cdot H}$ relative to $|G| \cdot |M|$ (H%), Frobenius measure between $I_K$ and $I_{S \cdot H}$.

| Content K | Approximated Context $K_{\approx} = S \cdot H$ | Respective Scale $S$ |
|-----------|-----------------------------------------------|-------------------|
|           | $|G|$ | $|M|$ | D | (28) | Frobenius | $H\%$ | (28) | AE | CE | $|M|$ | D | (28) | AE | CE |
| Diagnosis | 120  | 17  | 0.471 | 88 | 13.04 | 8.3 | 26 | 6 | 7 | 10.250 | 6 | 0 | 0 |
| Hayes-Roth | 132  | 18  | 0.218 | 215 | 16.40 | 11.3 | 33 | 8 | 26 | 10.350 | 12 | 3 | 8 |
| Domestic  | 41   | 55  | 0.158 | 292 | 8.49  | 3.2 | 148 | 14 | 68 | 10.183 | 34 | 6 | 15 |
| Zoo       | 101  | 43  | 0.365 | 4579 | 15.52 | 5.5 | 442 | 13 | 347 | 70.315 | 25 | 2 | 3 |
| Chess     | 346  | 683 | 0.491 | 229585 | 22473 | 22473 | 22473 | 22473 | 22473 | 22473 | 22473 | 22473 | 22473 |
| Mushroom  | 8124 | 119 | 0.395 | 4579 | 15.52 | 5.5 | 442 | 13 | 347 | 70.315 | 25 | 2 | 3 |
| HIV-1PC   | 6590 | 162 | 0.055 | 115615 | 22138 | 4.6 | 303 | 32 | 229 | 100.154 | 330 | 12 | 236 |
| Plant-Habits | 34781 | 68 | 0.127 | 322.16 | 4.4 | 68 | - | - | 80.128 | 256 | 8 | 255 |
| Airbnb-Berlin | 22552 | 145 | 0.007 | 130.29 | 0.5 | - | 0 | 0 | 120.657 | 8 | 1 | 1 |
| UFC-Fights | 5144 | 1915 | 0.001 | 101.43 | 0.1 | 2 | 0 | 44 | 44.032 | 2 | 44 | 1 |
| Recipes   | 175206 | 58 | 0.057 | 492.8 | 2.9 | 7 | - | - | 50.230 | 256 | 4 | 187 |

40.0) [40.0 42.0], ii) Hayes-Roth iii) Zoo iv) Mushroom v) HIV-1ProteaseCleavage [13] and vi) Plant-Habits four kaggle\(^6\) data sets vii) Top-Chess-Players with rating,rank, games, birth_year ordinally scaled, viii) neighborhood data from the Airbnb-Berlin data sets, ix) A_fighter and B_fighter from the UFC-Fights data sets and x) Recipes [10]. The eleventh data set is generated from the Wikipedia list of Domesticated Animals.\(^7\) This data set is also used for a qualitative analysis.

We summarized all data sets in Table 1, first and second major column. As dimension reduction method, we employ the binary matrix factorization [15] of the Nimfa [16] framework. Their algorithm is an adaption of the non-negative matrix factorization (NMF). In addition to the regular NMF a penalty and a thresholding function are applied to binarize the output. For any given data set $K$ two binary factors $S, H$ with $K \approx S \cdot H$ are computed.

The BMF factorization algorithm takes several parameters, such as convergence $\lambda_w, \lambda_h$, which we left at their default value of 1.1. We increased the maximum number of iterations to 500 to ensure convergence and conducted ten runs, of which took the best fit. The target number of attribute (features) in $|M_S|$ was set approximately to $\sqrt{|M_K|}$ to receive a data dimension reduction of one magnitude. We depicted the results, in particular the quality of the factorizations, in Table 1 (major column three and four) where $K_{\approx} = S \cdot H$ is the BMF approximation of $K$. Our investigation considers standard measures, such as Frobenius norm ($Frob$) and relative Hamming distance ($H\%$), as well as the proposed conceptual scaling error ($CE$) and attribute scaling error ($AE$). For the large data sets, i.e., the last four in Table 1, we omitted computing the number of concepts due to its computational intractability, indicated by '-'. Therefore we were not

\(^6\) https://www.kaggle.com/, i) https://www.kaggle.com/odartey/top-chess-players and https://www.fide.com/, ii) https://www.kaggle.com/brittabettendorf/berlin-airbnb-data/, iii) https://www.kaggle.com/rajeevuru/ufadata

\(^7\) https://en.wikipedia.org/w/index.php?title=List_of_domesticated_animals, 25.02.2020
able to compute the conceptual scaling errors of the approximate data sets \( \mathbb{K}_\infty \). However, the conceptual scaling error of the related scales \( \mathbb{S} \) is independent of the computational tractability of CE of \( \mathbb{K}_\infty \). We observe that the values for Frob and for H% differ vastly among the different data sets. For Example H% varies from 0.1 to 11.3. We find that for all data sets \(|\mathbb{B}(\mathbb{K})|\) is substantially larger than \(|\mathbb{B}(\mathbb{K}_\infty)|\), independently of the values of Frob and H%. Hence, BMF leads to a considerable loss of concepts. When comparing the conceptual and attribute scaling error to Frob and H%, we observe that the novel conceptual errors capture different aspects than the classical matrix norm error. For example, Domestic and Chess have similar values for H%, however, their error values with respect to attributes and concepts differ significantly. In detail, the ratio of \(|CE|/|\mathbb{B}(\mathbb{K}_\infty)|\) is 0.98 for Chess and 0.46 for Domestic, and the ratio for \(|AE|/|M_\infty|\) is 0.36 for Chess and 0.25 for Domestic. While we do not know the number of concepts for Airbnb-Berlin, we do know that conceptual scaling error of the related \( \mathbb{K}_\infty \) is 0 due to AE being 0 and Proposition 3. The factorization of the UFC-Fights produced an empty context \( \mathbb{K}_\infty \). Therefore, all attribute derivations in \( \mathbb{K}_\infty \) are the empty set, whose pre-image is an extent of \( \mathbb{K} \), hence, AE is 0. We suspect that BMF is unable to cope with data sets that exhibit a very low density. It is noteworthy that we cannot elude this conclusion from the value of the Frob and H%. By investigating the binary factor \( \mathbb{S} \) using the conceptual scaling error and the attribute error, we are able to detect the occurrence of this phenomenon. In detail, we see that 44 out of 44 attributes are inconsistent. We can take from our investigation that low H% and Frob values do not guaranty good factorizations with respect to preserving the conceptual structure of the original data set. In contrast, we claim that the proposed scaling errors are capable of capturing such error to some extent. On a final note, we may point out that the conceptual scaling errors enable a quantifiable comparison of a scaling \( \mathbb{S} \) to the original data set \( \mathbb{K} \), despite different dimensionality.

### 4.1 Qualitative Analysis

The domestic data set includes forty-one animals as objects and fifty-five purposes for their domestication as attributes, such as pets, hunting, meat, etc. The resulting \( \mathbb{K} \) has a total of 2255 incidences and the corresponding concept lattice has 292 formal concepts. We applied the BMF algorithm as before, which terminated after 69 iterations with the scale depicted in Figure 4. The incidence of \( \mathbb{K}_\infty := \mathbb{S} \cdot H \) has seventy-three wrong incidences, i.e., wrongfully present or absent pairs, which results in H% of 3.2. The corresponding concept lattice of \( \mathbb{K}_\infty \) has 148 concepts, which is nearly half of \( \mathbb{B}(\mathbb{K}) \). Furthermore, out of these 148 concepts there are only 80 correct, i.e., in \( \text{Ext}(\mathbb{K}) \). This results in a conceptual scaling error of 68, which is in particular interesting in the light of the apparently low H% error.

To pinpoint the particular errors, we employ i)-iii) from Section 3.1. The result of the first approach is visualized in Figure 5 and displays the concept lattice of \( \sigma^{-1}[\text{Ext}(\mathbb{S})] \) in which the elements of \( \mathcal{C}^\mathbb{K}_{\sigma,\mathbb{S}} \) are highlighted in red. First, we
notice in the lattice diagram that the inconsistent extents $E_{\sigma_S}$ are primarily in the upper part. Seven out of fifteen are derivations from attribute combinations of 9, 8, and 5. This indicates that the factorization was especially inaccurate for those attributes. The attribute extents of 6, 4, and 2 are in $E_{\sigma_S}$, however, many of their combinations with other attributes result in extents of $\text{Ext}(K)$.

The resulting lattices of applying approach ii) are depicted in Figure 6, the consistent lattice of $\sigma^{-1}[\text{Ext}(S)]|_{\text{Ext}(K)}$ on the left and its join-complement on the right. The consistent part has nineteen concepts, all depicted attributes can considered in conjunctive normalform. The join-complement consists of twenty-two concepts of which the incorrect ones are marked in red.

Based on this representation, we can see that twenty out of the forty-one objects have no associated attributes. These include objects like lama, alpaca or barbary dove, which we have also indirectly identified by i) as derivations of 5, 8, 9. Furthermore, we see that thirteen out of the fifty-five attributes of $K$ are not present in any conjunctive attributes. These attributes include domestication purposes like tusk, fur, or hair. Out of our expertise we suppose that these could form a meaningful cluster in the specific data realm. In the join-complement, we can identify the attributes 5, 8, 9 as being highly inconsistently scaled, as already observed in the paragraph above.

The third approach results in a scale $S_O$ of four consistent attributes and a scale $\hat{S}_O$ of six non-consistent attributes. The scale $S_O$ has seven concepts and the scale $\hat{S}_O$ has twenty-two. We may note that the concept lattice of $S_O$ is identical to the join-complement of the previous approach. In general this is not the case, one cannot even assume isomorphy. While the concept lattice of $S_O$
does miss some of the consistent extents of $\sigma^{-1}[\Ext(S)]|_{\Ext(K)}$, we claim that the combination of $S_O$ and $S_{\hat{O}}$ still provides a good overview of the factorization shortcomings.

5 Related Work

To cope with large data sets, a multitude of methods was introduced to reduce the dimensionality. One such method is the factorization of a data set $K$ into two factors (scales) $S, H$ whose product $S \cdot H$ estimates $K$. For binary data sets, a related problem is known as the discrete basis problem [11]: For a given $k \in \mathbb{N}$, compute two binary factors $S \in \mathbb{B}^{n \times k}$ and $H \in \mathbb{B}^{k \times m}$ for which $\|K - SH\|$ is minimal. This problem is known to be NP-hard. This hardness result lead to the development of several approximation algorithms [1, 15]. For example, one approach uses formal concepts as attributes of $S$ [1] and objects of $H$. It is shown that a solution to the discrete basis problem can be given in terms of this representation. However, since the initial problem is still NP-hard, this approach is computationally intractable for large data sets.

The BMF approach [15] adapts a non-negative matrix factorization by using a penalty and thresholding function. This algorithm optimizes two initial matrices to minimize the error $\|K - SH\|$. This procedure does compute an approximation to the solution of the discrete basis problem and has lower computational cost. An additional drawback of the BMF algorithm is that it may introduce closed
Figure 7. The concept lattice of all valid (left) and invalid (right) attributes of the Domestic scale-measure. Extents in the lattice drawing of the invalid attributes that are not extents in the Domesticated Animals context are marked in red. $A=\{\text{society finch, silkmoth, fancy mouse, mink, fancy rat, striped skunk, Guppy, canary, silver fox}\}$ $T3=\{\text{fuegian dog, lama, sheep, ferret, pig, alpaca, society finch, goat, silkmoth, fancy mouse, koi, guinea pig, rabbit, hedgehog, mink, fancy rat, striped skunk, goldfish, barbary dove, Guppy, canary, pigeon, silver fox, cat, gayal}\}$ $T2=\{\text{dromedary, muscovy duck, bactrian camel, goose, turkey, hedgehog, duck, guineafowl}\}$

objects sets in $\mathbb{S} \cdot \mathbb{H}$ that are not closed in the data set $K$. The main idea of the conceptual scaling error is the efficient computation and quantification of said closed sets.

Other evaluations of BMF, besides $\|K - \mathbb{S} \cdot \mathbb{H}\|$, have been considered [9]. They investigate the quality of implications in $\mathbb{S} \cdot \mathbb{H}$ for some classification task. Additionally, they use different measures [3], e.g., fidelity and descriptive loss. Other statistical approaches often find euclidean loss, Kullback-Leibler divergence, Residual Sum of Squares, adequate. All previously mentioned evaluation criteria do not account for the complete conceptual structure of the resulting data set. Moreover, they are not intrinsically able to pinpoint to the main error portions of the resulting scale data sets. Furthermore, approaches based on the computation of implications are infeasible for larger data set. An advantage of our approach is the polynomial estimation of the conceptual error in the size of $\mathbb{S}$ and $\mathbb{K}$ through the attribute scaling error.

6 Conclusion

With our work we have presented a new approach to evaluate dimension reduction methods in particular and data scaling methods in general. The proposed conceptual scaling error was derived from a natural notion of continuity, as used in scale-measures within the realm of FCA. Beyond the quantification of the conceptual error, we have succeeded in presenting a method to explicitly represent the error generated by the dimension reduction, and to visualize it with the help of conceptual lattices. For large data set we demonstrated a method for estimating the scaling error in time polynomial with respect to the data set size. In
our experiments we showed that even though a factorization using BMF terminates with apparently high accuracy, the calculated scale does reflect only about 55% consistent extents. To prevent such high conceptual errors, we envision an adaption of BMF that optimizes additionally for low conceptual errors.

References

[1] R. Belohlávek and V. Vychodil. “Discovery of optimal factors in binary data via a novel method of matrix decomposition.” In: J. Comput. Syst. Sci. 76.1 (Jan. 11, 2010), pp. 3–20.
[2] J. Czerniak and H. Zarzycki. “Application of rough sets in the presumptive diagnosis of urinary system diseases.” In: Artificial Intelligence and Security in Computing Systems. Ed. by J. Sokle and L. Drobiazgiewicz. Boston, MA: Springer US, 2003, pp. 41–51. ISBN: 978-1-4419-9226-0.
[3] S. M. Dias and N. J. Vieira. “Reducing the Size of Concept Lattices: The JBOs Approach.” In: (2010).
[4] D. Dua and C. Graff. UCI Machine Learning Repository. 2017.
[5] B. Ganter and R. Wille. “Conceptual scaling.” In: Applications of combinatorics and graph theory to the biological and social sciences. Ed. by F. Roberts. Springer-Verlag, 1989, pp. 139–167.
[6] B. Ganter and R. Wille. Formal Concept Analysis: Mathematical Foundations. Springer-Verlag, Berlin, 1999, pp. x+284.
[7] B. Ganter, J. Stahl, and R. Wille. “Conceptual measurement and many–valued contexts.” In: Classification as a tool of research. Ed. by W. Gaul and M. Schader. Amsterdam: North–Holland, 1986, pp. 169–176.
[8] T. Hanika and J. Hirth. “On the Lattice of Conceptual Measurements.” In: arXiv preprint arXiv:2012.05287 (2020).
[9] C. A. Kumar, S. M. Dias, and N. J. Vieira. “Knowledge reduction in formal contexts using non-negative matrix factorization.” In: Math. Comput. Simul. 109 (2015), pp. 46–63.
[10] S. Li. Food.com Recipes and Interactions. 2019. DOI: 10.34740/KAGGLE/DSV/783630.
[11] P. Miettinen et al. “The Discrete Basis Problem.” In: Knowledge and Data Engineering, IEEE Transactions on 20.10 (Oct. 2008), pp. 1348–1362. ISSN: 1041-4347. DOI: 10.1109/TDE.2008.53.
[12] S. Prediger and G. Stumme. “Theory-driven Logical Scaling: Conceptual Information Systems meet Description Logics.” In: Proc. KRDB’99. Ed. by E. Franconi and M. Kifer. Vol. 21. CEUR-WS.org, 1999, pp. 46–49.
[13] T. Rögnvaldsson, L. You, and D. Garwicz. “State of the art prediction of HIV-1 protease cleavage sites.” In: Bioinformatics 31.8 (2015), pp. 1204–1210.
[14] R. Wille. “Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts.” In: Ordered Sets: Proc. of the NATO Advanced Study Institute. Ed. by I. Rival. Dordrecht: Springer, 1982, pp. 445–470. ISBN: 978-94-009-7798-3.
[15] Z. Zhang et al. “Binary matrix factorization with applications.” In: 7th IEEE int. conf. on data mining (ICDM 2007). IEEE. 2007, pp. 391–400.
[16] M. Zitnik and B. Zupan. “Nimfa: A Python Library for Nonnegative Matrix Factorization.” In: JMLR 13 (2012), pp. 849–853.