FOLD COBORIDMS AND A POINCARÉ-HOPF TYPE THEOREM FOR THE SIGNATURE

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Abstract. We give complete geometric invariants of cobordisms of framed fold maps. These invariants are of two types. We take the immersion of the fold singular set into the target manifold together with information about non-triviality of the normal bundle of the singular set in the source manifold. These invariants were introduced in the author’s earlier works. Secondly we take the induced stable partial framing on the source manifold whose cobordisms were studied in general by Koschorke. We show that these invariants describe completely the cobordism groups of framed fold maps into $\mathbb{R}^n$. Then we are looking for dependencies between these invariants and we study fold maps of $4k$-dimensional manifolds into $\mathbb{R}^2$. We construct special fold maps which are representatives of the fold cobordism classes and we also compute the cobordism groups. We obtain a Poincaré-Hopf type formula, which connects local data of the singularities of a fold map of an oriented $4k$-dimensional manifold $M$ to the signature of $M$. We also study the unoriented case analogously and prove a similar formula about the twisting of the normal bundle of the fold singular set.

1. Introduction

Let $n \geq 1$ and $q \geq 0$. A smooth map $f$ of an $(n+q)$-dimensional manifold $M$ into $\mathbb{R}^n$ is called a fold map if it can be written as

$$f(x_1, \ldots, x_{n+q}) = \left(x_1, \ldots, x_{n-1}, \sum_{i=n}^{n+q} \pm x_i^2\right)$$

in local charts around each critical point $p \in M$ and critical value $f(p)$. Fold maps are natural generalizations of Morse functions, and play a basic role in the theory of singular maps. For example, it is always possible to deform any map of an at least 2 dimensional closed orientable manifold with even Euler characteristic into $\mathbb{R}^2$ to obtain a fold map, see [El70, Kå00, Le65, Mi84].

Given a fold map $f : M^{2+q} \to \mathbb{R}^2$, the singular set of $f$, i.e. the set of points in $M$ where the rank of the differential of $f$ is less than 2, is a 1-dimensional submanifold of $M$. A fold map, which we always presume being in generic position, restricted to its singular set is a generic immersion. In the case of $q$ odd, Chess [Ch80] found a relation between the number of double points of this immersion, the twisting of the normal bundle of the singular set in $M$ and certain Stiefel-Whitney classes. Namely, let $k \geq 1$ and

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$f : M^{2k+1} \to \mathbb{R}^2$ be a fold map of a closed orientable manifold. Then

$$t(f) + \tau(f) \equiv \begin{cases} 
0 \mod 2 & \text{if } k \text{ is odd,} \\
 w_{2}^{w_{2k-1}}[M^{2k+1}] \mod 2 & \text{if } k \text{ is even.}
\end{cases}$$

Here $w_i$ denotes the $i$th Stiefel-Whitney class, $t(f)$ denotes the number of double points of the immersed singular set of $f$ in $\mathbb{R}^2$ and $\tau(f)$ measures how non-trivial the normal bundle of the singular set of $f$ in $M$ is, for the precise definitions, see Section 3.2 of the present paper.

In [Ch80] this result is called a Poincaré-Hopf type theorem because it relates some topological property of a $(4k+1)$-dimensional manifold $M$ to the local behavior of its map into $\mathbb{R}^2$ (which of course corresponds to some local behavior of some vector fields on $M$).

In the present paper, we look for this type of results in the case of $4k$-dimensional manifolds. We obtain

**Theorem 1.1.** Let $k \geq 1$ and $f : M^{4k} \to \mathbb{R}^2$ be a fold map of a closed oriented $4k$-dimensional manifold. Then

$$t(f) + \tau(f) \equiv \frac{\sigma(M^{4k})}{2} \mod 2.$$

Here $\sigma(M^{4k})$ denotes the signature of the closed oriented $4k$-dimensional manifold $M^{4k}$. The invariants $t$ and $\tau$ are again the number of double points of the immersed singular set and the twisting of the normal bundle of the singular set of a fold map.

By [Le65] the manifold $M^{4k}$ has a fold map into $\mathbb{R}^2$ if and only if the Euler characteristic of $M^{4k}$ is even. A corollary of Theorem 1.1 gives the condition for fold maps when the signature of $M^{4k}$ is divisible by 4.

**Corollary 1.2.** Let $k \geq 1$ and $M^{4k}$ be a closed oriented $4k$-dimensional manifold. Then the following are equivalent.

(i) The signature $\sigma(M^{4k})$ is divisible by 4.

(ii) There is a fold map $f : M^{4k} \to \mathbb{R}^2$ such that $t(f) \equiv \tau(f) \mod 2$.

Each of these implies that for any fold map $f : M^{4k} \to \mathbb{R}^2$ we have $t(f) \equiv \tau(f) \mod 2$.

In the non-orientable case, we have

**Theorem 1.3.** Let $k \geq 1$ and $f : M^{4k} \to \mathbb{R}^2$ be a fold map of a closed (possibly non-orientable) $4k$-dimensional manifold. Then

$$\tau^1(f) + \tau^2(f) \equiv 0 \mod 2.$$

Again, for the precise definitions of the invariants $\tau^1$ and $\tau^2$, which measure the twisting of the normal bundle of the singular set of a fold map, see Section 3.2 As we mentioned, the relation between the Euler characteristic of the source manifold of a map and the singularities of the map is quite well-known. However, the relation between the signatures of oriented source manifolds and their singular maps is still much less understood. Existing results in [ES03, Oz02, SY06] establish formulas about the signatures of oriented manifolds and their singular maps but for only 4-dimensional source manifolds. In [TY06] formulas about non-oriented 4-manifolds and their singularities are obtained.

We obtain Theorems 1.1 and 1.3 by considering cobordisms of fold maps (see Definition 3.1). It is well-known that two closed manifolds are cobordant if and only if the corresponding Stiefel-Whitney numbers (and Pontryagin numbers in the oriented case) coincide. We have the notion of a *cobordism of singular maps* [Sz79, RS98], and if there are
two given maps, one can ask about easily applicable procedures, namely checking cobordism invariants, to decide whether the two fold maps are cobordant. Fold maps restricted to their singular sets are immersions, and we introduced and used geometric cobordism invariants of fold maps in [Ka08, Ka09a, Ka09b] in terms of these immersions with prescribed normal bundles which describe the image of the fold singular set in the target manifold together with more detailed informations about the “twisting” of the tubular neighborhood of the singular set in the source manifold (we summarize these results in Section 3.2). More precisely, the fold singularities

\[
(x_1, \ldots, x_{n+q}) \mapsto \left( x_1, \ldots, x_{n-1}, \sum_{i=n}^{n+\lambda-1} x_i^2 + \sum_{i=n+\lambda}^{n+q} x_i^2 \right),
\]

where \(0 \leq \lambda \leq (q + 1)/2\), form an \((n-1)\)-parameter family of the index \(\lambda\) Morse singularities

\[
(x_n, \ldots, x_{n+q}) \mapsto \sum_{i=n}^{n+\lambda-1} x_i^2 + \sum_{i=n+\lambda}^{n+q} x_i^2.
\]

Roughly speaking this Morse singularity is a map \(\varphi: \mathbb{R}^{q+1} \to \mathbb{R}\) and has a symmetry group \(G_\lambda\) acting on \(\mathbb{R}^{q+1}\) and \(\mathbb{R}\), respectively. The singular locus \(S_\lambda\) of a fold map \(f: M \to \mathbb{R}^n\) consisting of fold singular points with \(\lambda\) many “-” signs is an \((n-1)\)-dimensional submanifold of \(M\) and the normal bundle of the immersion \(f|_{S_\lambda}: S_\lambda \to \mathbb{R}^n\) can be induced from the line bundle \(l^{n}_\lambda \to BG_\lambda\) which we get by taking the action of \(G_\lambda\) on the target \(\mathbb{R}\) of the Morse singularity \(\varphi\). By assigning this immersion to the fold map and by looking at cobordisms, we get a homomorphism \(\xi_\lambda\) from the cobordism group of fold maps into the cobordism group \(\text{Imm}(l^{n}_\lambda, n)\),

which denotes the cobordism group of codimension 1 immersions into \(\mathbb{R}^n\) whose normal bundles are induced from \(l^{n}_\lambda\). These homomorphisms \(\xi_\lambda\) are the geometric cobordism invariants which we introduced in [Ka08, Ka09a, Ka09b].

In this paper, we define further invariants which describe the cobordism class of the source manifold together with its mapping away from the singular set as well. Namely, for a given fold map, we take the stably framed cobordism class of its source manifold (see Section 4), which notion was studied in general by Koschorke [Ko81]. More precisely, on the source manifold of a fold map \(f: M^{n+q} \to \mathbb{R}^n\) with oriented singular set we obtain a stable (partial) framing simply by pulling back the parallelization of the tangent space \(T\mathbb{R}^n\) by the modified differential \(df + \alpha: TM \oplus \varepsilon^1 \to T\mathbb{R}^n\), where \(\alpha: \varepsilon^1 \to T\mathbb{R}^n\) is just a homomorphism having full rank near the singular set of \(f\) and consequently \(df + \alpha\) is an epimorphism, see Section 2.2 and also [An04, Sae92]. Looking at the cobordisms of these stable framings, we obtain our homomorphism \(\sigma_{n,q}\) which maps the cobordism class of the fold map \(f\) to the cobordism class of the stably framed source manifold \(M^{n+q}\).

Finally, by using a result of Ando [An04, Theorem 0.1] about the existence of fold maps, we show that our invariants give complete cobordism invariants of (framed) fold maps (see Definition 3.2, Theorem 5.1, Corollary 5.2).

Namely, for \(n > 0, q \geq 0\), let us denote by \(\mathcal{I}_{n,q}\) the homomorphism

\[
(\sigma_{n,q}, \xi_1, \ldots, \xi_{[(q+1)/2]}),
\]
which maps the cobordism class of a (framed) fold map into the direct sum of the cobordism
group of stably framed manifolds and the groups $\text{Imm}(l^1_\lambda, n)$ for $1 \leq \lambda \leq (q + 1)/2$ as
described above.

Then we obtain the following result.

**Theorem 1.4.** The homomorphism $\Im_{n,q}$ is injective.

It is important to note that Theorems 1.1 and 1.3 and also the result of [Ch80] show
dependencies between the cobordism invariants $\xi_0, \ldots, \xi_{\lfloor (q + 1)/2 \rfloor}$ and $\sigma_{n,q}$.

We prove Theorem 1.4 by applying the h-principle of Ando [An04] in a short and
simple way. We have the analogous results when we consider orientations on the manifolds
and on the cobordisms as well. When we arrive to applications and computations, what
simplifies everything is that the line bundles $l^1_\lambda$ are in fact trivial bundles in almost all of
the cases. For example, we obtain

**Theorem 1.5.** For $k \geq 1$, the cobordism group of fold maps of oriented $4k$-dimensional
manifolds into $\mathbb{R}^2$ is isomorphic to

$$\Omega_{4k}^{2|X} \oplus \mathbb{Z}_{2}^{4k-2},$$

where $\Omega_{4k}^{2|X}$ is the cobordism group of closed oriented $4k$-dimensional manifolds with even
Euler characteristic.

For the unoriented case, we have

**Theorem 1.6.** For $k \geq 1$, the cobordism group of fold maps of $4k$-dimensional manifolds
into $\mathbb{R}^2$ is isomorphic to

$$\mathfrak{N}_{4k}^{2|X} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{6}^{6k-3},$$

where $\mathfrak{N}_{4k}^{2|X}$ is the cobordism group of closed unoriented $4k$-dimensional manifolds with even Euler characteristic.

To prove our Theorems 1.1 and 1.3 we also construct representatives of the generators
of these cobordism groups of fold maps. In the case of Theorem 1.5, the direct summand $\mathbb{Z}_2^{4k-2}$ is generated by the classes of such fold maps which are Morse function bundles
over immersed 1-dimensional manifolds in $\mathbb{R}^2$. In [Ka08] we introduced these Morse
function bundles to detect direct summands of the cobordism groups of fold maps into
$\mathbb{R}^n$. We recall these results in a detailed form in Section 6 in the special case of $n = 2$.

To find representatives of generators of the direct summand $\Omega_{4k}^{2|X}$, we need two things.
At first, we apply [AK80, Theorem 3] to get fiber bundles over $S^2$ which we map then
into $\mathbb{R}^2$ by constructing some fold maps. The invariants $t, \tau$ and the signature of the
source manifold are typically zero for these maps. Secondly, we construct a specific fold
map $f_C: \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2$, which we use to get more non-zero values for $\tau$ and the
signature.

In the case of Theorem 1.6, the direct summand $\mathbb{Z}_2 \oplus \mathbb{Z}_{6}^{6k-3}$ is generated by the classes
of Morse function bundles and the map $f_C$ similarly to the oriented case. The direct
summand $\mathfrak{N}_{4k}^{2|X}$ is generated by classes of fold maps of fiber bundles over $S^2$ similarly to
the oriented case but now we apply results of [Br69] to get these bundles.

So having invariants which encode geometric information, computing the cobordism
groups, constructing representatives of the generators of the cobordism groups, and checking
the values of the invariants on them lead to formulas about geometric properties —
in this paper we implement this concept in the case of oriented $4k$-dimensional manifolds.
(k \geq 1) and their fold maps into the plane. In this way, we obtain Theorem 1.1. For the unoriented case, we similarly obtain Theorem 1.3.

Other results about cobordisms of fold and singular maps can be found for example in \cite{An06, An08, EST07, Ik04, IS03, Ka05, Ka07, KT12, RS98, Sad09, Sad12, Sae06, SST10, Sz79, Sz08, ST19}.

The paper is organized as follows. In Sections 2, 3 and 4 we give and study the basic definitions, in Sections 5 and 7 we state our main results, in Section 6 we recall necessary results about Morse function bundles and in Section 8 we prove our results.

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Notations. In this paper the symbol \( \sqcup \) denotes the disjoint union. For any number \( x \) the symbol \( \lfloor x \rfloor \) denotes the greatest integer \( i \) such that \( i \leq x \). For an integer \( k \geq 0 \) the symbol \( \mathbb{Z}^k \) denotes the trivial \( k \)-dimensional vector bundle over the space \( X \). For a pair of spaces \((X, A)\) and a vector bundle \( \xi \) over \( X \) the symbol \( \xi|_A \) denotes the bundle induced by the inclusion \( A \subset X \). For a smooth manifold \( X \) the symbol \( TX \) denotes its tangent space. The symbol \( \pi_n^k(X) \) (resp. \( \pi_n^k \)) denotes the \( n \)-th stable homotopy group of the space \( X \) (resp. spheres). The symbol \( \text{Imm}(\eta^k,n) \) denotes the cobordism group of codimension \( k \) immersions into \( \mathbb{R}^n \) whose normal bundles are induced from the vector bundle \( \eta^k \) (this group is isomorphic to \( \pi_n^k(T\eta^k) \), where \( T\eta^k \) is the Thom space of \( \eta^k \), see \cite{We65}). The symbol \( \Omega_m \) (resp. \( \Omega_m^\text{or} \)) denotes the cobordism group of oriented (resp. unoriented) \( m \)-dimensional manifolds. If \( W \) is a manifold with boundary \( X \), then the tangent space of the submanifold \( X \) is a codimension 1 subbundle of the restriction \( TW|_X \). Any tangent vector \( v \in TW|_p \), where \( p \in X \), such that \( v \) and \( TX|_p \) generate \( TW|_p \) is called a normal section of \( X \) at \( p \). All manifolds are of class \( C^\infty \).

2. Preliminaries

2.1. Maps with fold singularities. Let \( n \geq 1 \) and \( q \geq 0 \). Throughout this section let \( Q \) and \( N \) be smooth manifolds of dimensions \( n+q \) and \( n \) respectively. Let \( p \in Q \) be a singular point of a smooth map \( f: Q \to N \). The smooth map \( f \) has a fold singularity of index \( \lambda \) at the singular point \( p \) if we can write \( f \) in some local coordinates around \( p \) and \( f(p) \) in the form

\[
f(x_1, \ldots, x_{n+q}) = \left( x_1, \ldots, x_{n-1}, \sum_{i=n}^{n+\lambda-1} -x_i^2 + \sum_{i=n+\lambda}^{n+q} x_i^2 \right)
\]

for some \( 0 \leq \lambda \leq q+1 \) (the index \( \lambda \) is well-defined if we consider that \( \lambda \) and \( q+1-\lambda \) represent the same absolute index).

A fold singularity is a definite fold singularity if \( \lambda = 0 \) or \( \lambda = q+1 \) and it is an indefinite fold singularity of index \( \lambda \) if \( 1 \leq \lambda \leq q \).

Let \( S_\lambda(f) \subset Q \) denote the set of fold singularities of index \( \lambda \) of \( f \). Note that \( S_\lambda(f) = S_{q+1-\lambda}(f) \). Let \( S_f \) denote the set \( \bigcup_\lambda S_\lambda(f) \) of all the fold singularities of \( f \). Then \( S_f \) is a smooth \((n-1)\)-dimensional submanifold of \( Q \). If \( f: Q \to N \) is a generic fold map, then the restriction of \( f \) to \( S_f \) is a generic codimension one immersion into the target manifold \( N \). Each connected component of the manifold \( S_f \) has its own index \( \lambda \) if we consider that \( \lambda \) and \( q+1-\lambda \) represent the same index.
Since every fold map is in general position after a small perturbation, and we study maps under the equivalence relation cobordism (see Definition 3.1), in this paper we can restrict ourselves to studying fold maps which are in general position. Without mentioning we presume that a fold map $f$ is generic.

For a fold map $f: Q \to N$ and for an index $0 \leq \lambda \leq \lfloor q/2 \rfloor$ the normal bundle of the codimension one immersion $f|_{S_\lambda(f)}: S_\lambda(f) \to N$ has a canonical orientation or framing (i.e. trivialization) by identifying the fold singularities of index $\lambda$ with the fold germ $(x_1, \ldots, x_{n+q}) \mapsto (x_1, \ldots, x_{n-1}, \sum_{i=n}^{n+\lambda-1} -x_i^2 + \sum_{i=n+\lambda}^{n+q} x_i^2)$.

**Definition 2.1** (Framed fold map). We say that a fold map $f: Q \to N$ is framed if

1. the normal bundle of the codimension 1 immersion $f|_{S_f}: S_f \to N$ is oriented so that for each index $0 \leq \lambda \leq \lfloor q/2 \rfloor$ the orientation of the normal bundle of the immersion $f|_{S_\lambda(f)}$ coincides with the canonical orientation and
2. for odd $q$ the normal bundle of the immersion $f|_{S_{(q+1)/2}(f)}$ is orientable and it is oriented.

**Remark 2.2.** (i) If we have a framed fold map $f$ into an oriented manifold $N$, then since the fold singular set $S_f$ is immersed into $N$, the manifold $S_f$ has an induced orientation given by this framing and the orientation of the target manifold $N$.

(ii) A fold map $f: Q \to N$ can be framed if and only if the cokernel bundle of the differential $df: TQ \to f^*TN$ restricted to $TQ|_{S_f}$ is a trivial line bundle.

**Definition 2.3** (Oriented fold map). A fold map $f: Q \to N$ is oriented if there is a chosen consistent orientation of every fiber at their regular points, i.e., if the kernel of the differential of the restriction $f|_R$ is an oriented bundle, where $R$ denotes the set of regular points of $f$.

For example, a fold map between oriented manifolds is naturally oriented. In this paper, we deal with fold maps into Euclidean spaces. For such a map an orientation is equivalent to an orientation of its source manifold (we fix orientations for Euclidean spaces).

Note that there exist oriented fold maps $f: Q^{n+q} \to \mathbb{R}^n$ with odd $q$, which cannot be framed in the sense of Definition 2.1 (for example for $n = 3$ and $q = 1$, see [Sae03]).

2.2. Existence of framed fold maps. We will study and recall results about the relation between

- existence of continuous fiberwise epimorphisms $TQ \oplus \varepsilon_Q^1 \to T\mathbb{R}^n$,
- having $n$ linearly independent continuous sections in $TQ \oplus \varepsilon_Q^1$ and
- existence of framed fold maps $Q \to \mathbb{R}^n$.

Fix the standard Riemannian metric on $\mathbb{R}^n$. When there is a Riemannian metric on $Q$, we always consider the Riemannian metric on $TQ \oplus \varepsilon_Q^1$ by defining $\varepsilon_Q^1$ to be perpendicular to $TQ$. At first observe that if there is a given Riemannian metric on $Q$, then having a fiberwise epimorphism $TQ \oplus \varepsilon_Q^1 \to T\mathbb{R}^n$ is equivalent to having $n$ linearly independent sections in $TQ \oplus \varepsilon_Q^1$.

Following [Sae92] Lemma 3.1 and [An04] Lemma 3.1, given a framed fold map $g: Q \to \mathbb{R}^n$, we will construct a fiberwise epimorphism $\varphi: TQ \oplus \varepsilon_Q^1 \to T\mathbb{R}^n$, which will depend on the given framed fold map $g$, a chosen Riemannian metric $g$ on $Q$ and a chosen $r > 0$, where this $r$ also depends on $g$ and $g$. 
So if \( g: Q \to \mathbb{R}^n \) is a framed fold map and \( g \) is a Riemannian metric on \( Q \), then let \( r = r(g, g) > 0 \) be the radius of a compact tubular neighborhood \( N_r(S_g) \) of the singular set \( S_g \) in \( Q \). If we have these \( g, g \) and \( r \), we define

\[
\varphi(g, g, r): TQ \oplus \varepsilon^1_Q \to T\mathbb{R}^n
\]

in the following way.

Consider the differential \( dg \) of \( g \) as a homomorphism \( TQ \to g^*T\mathbb{R}^n \) and take the commutative diagram

\[
\begin{array}{ccc}
TQ|_{S_g} & \xrightarrow{i} & TQ \\
\downarrow & & \downarrow \\
S_g & \xrightarrow{i} & Q
\end{array}
\]

Moreover we have a fiberwise monomorphism

\[
g: TQ|_{S_g} \to g^*T\mathbb{R}^n |_{S_g}
\]

where \( i: S_g \to Q \) is the embedding of the singular set. Then \( dg \circ i_* \) maps \( TQ|_{S_g} \) into a codimension 1 subbundle of \( g^*T\mathbb{R}^n |_{S_g} \). Having the pulled-back standard Riemannian metric on \( g^*T\mathbb{R}^n \) we take the orthogonal complement \((\text{im} \, dg \circ i_*)^\perp \) in \( g^*T\mathbb{R}^n |_{S_g} \). This is the cokernel bundle of \( dg \) over \( S_g \), which is a trivial bundle \( \varepsilon^1_{S_g} \) since \( g \) is a framed fold map. We denote this cokernel bundle by \( \theta_g \to S_g \). Of course \( \theta_g \subset g^*T\mathbb{R}^n \) and \( \theta_g \cong \varepsilon^1_{S_g} \). Moreover we have a fiberwise monomorphism

\[
g_*|_{S_g}: \theta_g \to T\mathbb{R}^n
\]

of this trivial bundle since it is a subbundle of \( g^*T\mathbb{R}^n \), and \( g^*T\mathbb{R}^n \) is sent by the pull-back homomorphism \( g_*: g^*T\mathbb{R}^n \to T\mathbb{R}^n \) fiberwise isomorphically into \( T\mathbb{R}^n \). Then over \( S_g \) we have a fiberwise epimorphism

\[
dg|_{S_g} + \text{id}_{\theta_g}: TQ|_{S_g} \oplus \theta_g \to g^*T\mathbb{R}^n |_{S_g},
\]

which we denote by \( \tilde{\varphi} \). Of course \( \tilde{\varphi} \) depends on the Riemannian metric on \( \mathbb{R}^n \) but that is fixed so \( \tilde{\varphi} \) depends really only on the map \( g \). We just have to extend this \( \tilde{\varphi} \) somehow over the entire \( TQ \oplus \varepsilon^1_Q \) and then to compose with \( g_* \) to get the claimed \( \varphi \).

So take a Riemannian metric \( g \) on \( Q \) and take \( r = r(g, g) \). Let us map a point in \( N_r(S_g) \) to the closest point in \( S_g \), this defines a map \( p: N_r(S_g) \to S_g \). Then take the commutative diagram

\[
\begin{array}{ccc}
p^*\left(g^*T\mathbb{R}^n |_{S_g}\right) & \xrightarrow{p_*} & g^*T\mathbb{R}^n |_{S_g} \\
\downarrow & & \downarrow \\
N_r(S_g) & \xrightarrow{p} & S_g
\end{array}
\]

where \( p^*\left(g^*T\mathbb{R}^n |_{S_g}\right) \) is canonically isomorphic to \( g^*T\mathbb{R}^n |_{N_r(S_g)} \) so that this isomorphism is the identity over \( S_g \) since \( T\mathbb{R}^n \) is a trivial bundle with the standard trivialization. Then, by the isomorphism and the diagram, the pull-back by \( p \) of our trivial cokernel bundle \( \theta_g \) in \( g^*T\mathbb{R}^n |_{S_g} \) yields an \( \varepsilon^1_{N_r(S_g)} \) subbundle in \( g^*T\mathbb{R}^n |_{N_r(S_g)} \).

Denote the bundle projection \( \varepsilon^1_{N_r(S_g)} \to N_r(S_g) \) by \(\pi\). Then extend \( \tilde{\varphi} \) over \( N_r(S_g) \) to get a fiberwise epimorphism \( \tilde{\varphi}: TQ|_{N_r(S_g)} \oplus \varepsilon^1_{N_r(S_g)} \to g^*T\mathbb{R}^n \) by the formula

\[
(v, w) \mapsto \left(\varphi(g, g, r)\right)(v) + \alpha(\pi(w))w
\]

for \( v \in TQ|_{N_r(S_g)} \) and \( w \in \varepsilon^1_{N_r(S_g)} \subset g^*T\mathbb{R}^n |_{N_r(S_g)} \), where \(\alpha: N_r(S_g) \to [0, 1] \) is equal to \( r \) minus the distance from \( S_g \) multiplied by \( 1/r \). Observe that \(\alpha(\pi(w)) = 0 \) if \( w \) is over the boundary of \( N_r(S_g) \). We suppose that \(\alpha \) is smooth near \( S_g \).
Hence $\tilde{\varphi}$ extends continuously over the entire $TQ \oplus \varepsilon^1_Q$ by applying only $dg$ over $Q - N_r(S_g)$. So we obtain the fiberwise epimorphism
\[ \varphi(g, q, r) = g_* \circ \tilde{\varphi}(g, q, r). \]

Obviously $\varphi(g, q, r)$ covers the map $g$.

Conversely, if we have any fiberwise epimorphism $\varphi: TQ \oplus \varepsilon^1_Q \to T\mathbb{R}^n$, then there is a framed fold map $g: Q \to \mathbb{R}^n$. More precisely, we have the following.

**Theorem 2.4** (Theorem 3.2 in [An04]). Let $n \geq 2$ and $q \geq 0$. Assume there is a fiberwise epimorphism $\psi: TQ^{n+q} \oplus \varepsilon^1_Q \to T\mathbb{R}^n$ over a continuous map $g: Q^{n+q} \to \mathbb{R}^n$. Then there is a framed fold map $f: Q^{n+q} \to \mathbb{R}^n$ homotopic to $g$.

**Remark 2.5.** If the bundle $TQ \oplus \varepsilon^1_Q$ has $n$ linearly independent sections $e_1, \ldots, e_n$ and there is a given Riemannian metric $g$ on $Q$, then mapping the orthogonal complement of the subspace spanned by the sections to $0$ and mapping the sections to the standard framing of $T\mathbb{R}^n$ we get a fiberwise epimorphism $\psi: TQ \oplus \varepsilon^1_Q \to T\mathbb{R}^n$ over some map $g: Q \to \mathbb{R}^n$. By Theorem 2.4 from $\psi$ we get a framed fold map $f: Q \to \mathbb{R}^n$ homotopic to $g$. This framed fold map $f$ also gives a fiberwise epimorphism $\varphi(f, g, r): TQ \oplus \varepsilon^1_Q \to T\mathbb{R}^n$ for some $r > 0$ and hence $n$ linearly independent sections $e'_1, \ldots, e'_n$ in $TQ \oplus \varepsilon^1_Q$. As the proof of [An04] Theorem 3.2 shows, we get $f$ by constructing from $\psi$ an appropriate section $s$ of $Q$ into the 2-jet space and by applying [An04] Theorem 2.1 which gives the homotopy from $g$ to $f$. This homotopy is the result of a homotopy in the formal 2-jet space, which gives a homotopy of fiberwise epimorphisms from $\psi$ to $\varphi(f, g, r)$ as well, which gives a homotopy from $e_1, \ldots, e_n$ to $e'_1, \ldots, e'_n$.

We will use an easy modification of the previous statement for the “relative case” as follows. As usual, for a smooth map $f: Q \to N$ and $x \in Q$, $y \in N$, we express a 2-jet in $J^2_{x,y}(Q, N)$ as a pair $(a, b)$ where $a \in \text{Hom}(T_xQ, T_yN)$, $b \in \text{Hom}(S^2(T_xQ), T_yN)$ and $S^2(T_xQ)$ denotes the 2-fold symmetric product of $T_xQ$, see [An04] pages 32–33.

**Theorem 2.6.** Let $k \geq 2$ and $m \geq k$. Let $g$ be a Riemannian metric on the $m$-dimensional manifold $W$ and let $C$ be a closed subset of $W$. Let $f: W \to \mathbb{R}^k$ be a continuous map such that the restriction of $f$ to a neighborhood $U$ of $C$ is smooth and has only definite fold singularities. Let $\psi: TW \oplus \varepsilon^1_W \to T\mathbb{R}^k$ be a fiberwise epimorphism over $f$, and suppose that $\psi$ is equal to $\varphi(f, g, r)$ for some $r > 0$ over $U$. Then, there exists a framed fold map $g: W \to \mathbb{R}^k$, which coincides with $f$ on $C$.

**Proof.** Let $\xi$ be the kernel of $\psi$. Then in the same way as in the proof of [An04] Theorem 3.2 we obtain a $(k - 1)$-dimensional manifold $V$ in $W$ such that $\xi|_p \subset T_pW$ exactly at the points $p \in V$. It follows that for the singular set $S$ of $f|_U$ we have $S \subset V$. Also we have that the rank of $\psi|_{T_pW}$ is equal to $k$ for $p \in W - V$ and it is equal to $k - 1$ for $p \in V$. Clearly $\psi|_{TW}$ induces a homomorphism $\Psi: TW \to f^*(T\mathbb{R}^k)$ by pulling back by
This extends to the entire \( \xi \) metric group of the definite fold singularity is the full orthogonal group. Let us denote this extension by \( \beta \).

Observe that the cokernel bundle of \( \Psi \) in \( f^*(T\mathbb{R}^k)|_V \) at the points of \( V \) is the trivial bundle: the sequence

\[
0 \rightarrow \xi|_V \rightarrow TW|_V \xrightarrow{\Psi|_V} f^*(T\mathbb{R}^k)|_V \rightarrow \text{coker } \Psi|_V \rightarrow 0
\]

is obviously exact and then by the Whitney product formula we have

\[
w_1(\text{coker } \Psi|_V) + w_1(TW|_V) - w_1(\xi|_V) = w_1(f^*(T\mathbb{R}^k)|_V).
\]

This implies that

\[w_1(\text{coker } \Psi|_V) = w_1(\varepsilon^1_V),\]

because

\[w_1(f^*(T\mathbb{R}^k)|_V) + w_1(\xi|_V) - w_1(TW|_V) = w_1(\varepsilon^1_V)\]

again by the Whitney product formula since the sequence

\[
0 \rightarrow \xi|_V \rightarrow TW|_V \oplus \varepsilon^1_V \rightarrow f^*(T\mathbb{R}^k)|_V \rightarrow 0
\]

is exact. So by (2.3) the bundle coker \( \Psi|_V \) is trivial. Now we want to define a non-singular symmetric map \( \beta: \xi|_V \otimes \xi|_V \rightarrow \varepsilon^1_V \) whose target \( \varepsilon^1_V \subset f^*(T\mathbb{R}^k)|_V \) is just the cokernel bundle of \( \Psi \) at the points of \( V \). For the singular set \( S \subset V \) we have already such a map: the positive definite symmetric form given by the definite fold singularities of \( f|_U \). We denote this by

\[
\beta: \xi|_S \otimes \xi|_S \rightarrow \varepsilon^1_S.
\]

This extends to the entire \( \xi|_V \otimes \xi|_V \) as a non-singular symmetric map because the symmetry group of the definite fold singularity is the full orthogonal group. Let us denote this extension by \( \tilde{\beta} \).

So if we define the section \( s: W \rightarrow \Omega^{m-k+1,0}(W,\mathbb{R}^k) \), where \( \Omega^{m-k+1,0}(W,\mathbb{R}^k) \) denotes the union of regular and fold jets in the 2-jet space \( J^2(W,\mathbb{R}^k) \), as

\[
s(p) = \left( p, f(p), \Psi_p, \tilde{\beta}_p \right),
\]

then by applying [An04] Theorem 2.1 we get the statement.

\[
\square
\]

3. Cobordism of maps

3.1. Cobordism of framed and oriented maps.

**Definition 3.1** (Cobordism). Two fold maps \( f_i: Q_i \rightarrow \mathbb{R}^n, \ i = 0, 1 \), of closed \((n+q)\)-dimensional manifolds \( Q_i^{n+q} \) are cobordant if there exists a fold map

\[
F: X \rightarrow \mathbb{R}^n \times [0,1]
\]

of a compact \((n+q+1)\)-dimensional manifold \( X \) such that

(i) \( \partial X = Q_0 \amalg Q_1 \) and

(ii) \( F|_{Q_0 \times [0,\varepsilon]} = f_0 \times \text{id}_{[0,\varepsilon]} \) and \( F|_{Q_1 \times (1-\varepsilon,1]} = f_1 \times \text{id}_{(1-\varepsilon,1]} \), where \( Q_0 \times [0,\varepsilon) \) and \( Q_1 \times (1-\varepsilon,1] \) are small collar neighborhoods of \( \partial X \) with the identifications \( Q_0 = Q_0 \times \{0\} \) and \( Q_1 = Q_1 \times \{1\} \).
We call the map $F$ a \textit{cobordism} between $f_0$ and $f_1$.

When the fold maps $f_i : Q_i \to \mathbb{R}^n$, $i = 0, 1$, are oriented, we say that they are \textit{oriented cobordant} (or shortly \textit{cobordant} if it is clear from the context) if they are cobordant in the above sense via an oriented fold map $F$, such that the orientations are compatible on the boundary of $X$.

This clearly defines an equivalence relation on the set of fold maps of closed $(n + q)$-dimensional manifolds into $\mathbb{R}^n$. The equivalence classes are called \textit{cobordism classes}. We denote the set of cobordism classes of fold maps by $\text{Cob}(n, q)$. For the oriented version an upper index "$O$" applies and we write $\text{Cob}^O(n, q)$. By taking disjoint union of maps, the sets $\text{Cob}(n, q)$ and $\text{Cob}^O(n, q)$ become groups as one can see easily.

\textbf{Definition 3.2} (Framed cobordism). Two framed fold maps $f_i : Q_i \to \mathbb{R}^n$, $i = 0, 1$, of closed $(n + q)$-dimensional manifolds $Q_i$ are \textit{framed cobordant} if they are cobordant in the sense of Definition 3.1 by a framed fold map $F : X \to \mathbb{R}^n \times [0, 1]$ such that the orientation of the normal bundle of the immersion $F|_{S_F} : S_F \to \mathbb{R}^n \times [0, 1]$ restricted to $\partial X \cap S_F$ coincides with that of the immersions $f_i|_{S_{f_i}} : S_{f_i} \to \mathbb{R}^n \times \{i\}$, $i = 0, 1$. Similarly two oriented framed fold maps are \textit{oriented framed cobordant} if they are framed cobordant so that this cobordism is also an oriented cobordism.

We denote the corresponding framed cobordism groups by $\text{Cob}_{fr}(n, q)$ and $\text{Cob}_{fr}^O(n, q)$. For $q$ even the group $\text{Cob}_{fr}(n, q)$ is naturally isomorphic to $\text{Cob}(n, q)$ and the group $\text{Cob}_{fr}^O(n, q)$ is naturally isomorphic to $\text{Cob}^O(n, q)$ by the natural forgetful map from the framed cobordism group to the unframed one.

\textbf{3.2. Cobordism invariants of fold maps.} We introduced and used geometric invariants of cobordisms of fold maps \cite{Ka08} Section 2, namely the homomorphisms

$$
\xi_\lambda : \text{Cob}(n, q) \to \text{Imm} \left( \varepsilon^{1}_{B(O(\lambda) \times O(q+1-\lambda))}, n \right)
$$

for $0 \leq \lambda < (q+1)/2$ and

$$
\xi_{(q+1)/2} : \text{Cob}(n, q) \to \text{Imm}(l^1, n)
$$

for $q$ odd and $\lambda = (q+1)/2$, where $l^1$ is an appropriate line bundle. These homomorphisms are defined as follows. Restricting the fold map $f : Q^{n+q} \to N^n$ to a tubular neighborhood of its index $\lambda$ fold singular set $S_\lambda(f)$ we get a bundle over $S_\lambda(f)$ with fiber the mapping

$$
\varphi : (x_1, \ldots, x_{q+1}) \mapsto \sum_{i=1}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{q+1} x_i^2.
$$

By \cite{Ja78, Wa80} the structure group of this bundle can be reduced to a maximal compact subgroup, namely to the group $O(\lambda) \times O(q+1-\lambda)$ in the case of $0 \leq \lambda < (q+1)/2$ and to the group generated by the group $O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right)$ and the transformation

$$
T = \begin{pmatrix}
0 & I_{(q+1)/2} \\
I_{(q+1)/2} & 0
\end{pmatrix}
$$

in the case of $q$ odd and $\lambda = (q+1)/2$, see, for example, \cite{Sae92}. We denote this latter group by $\left\langle O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T \right\rangle$.

The $\varphi$-bundle over $S_\lambda(f)$ has a “source” bundle and also a “target” bundle part over $S_\lambda(f)$. This results from the source $\mathbb{R}^{q+1}$ and the target $\mathbb{R}$ of $\varphi$ since the structure group of $\varphi$ acts on these. So we have a $(q+1)$-dimensional vector bundle and also a 1-dimensional vector bundle over $S_\lambda(f)$ and $\varphi$ maps fiberwise between them.
It follows that this 1-dimensional vector bundle over $S_\lambda(f)$, where $0 \leq \lambda \leq (q+1)/2$, can be induced from the trivial line bundle

$$\varepsilon^1 \rightarrow B(O(\lambda) \times O(q+1-\lambda))$$

for $\lambda \neq (q+1)/2$ and from an appropriate line bundle

$$l^1 \rightarrow B\left(O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\right)$$

for $q$ odd and $\lambda = (q+1)/2$.

Now, restricting the fold map $f$ to its fold singular set $S_\lambda(f)$ of index $\lambda$ we get an immersion and the homomorphisms $\xi_\lambda$ and $\xi_{(q+1)/2}$ map the cobordism class $[f]$ to the cobordism class of this immersion with normal bundle induced from the line bundle $\varepsilon^1 \rightarrow B(O(\lambda) \times O(q+1-\lambda))$ or $l^1 \rightarrow B\left(O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\right)$, respectively.

### 3.2.1. Invariants of oriented fold maps.

For oriented fold maps, we have the analogous statements but we have to consider the subgroup of the automorphisms of $\varphi: \mathbb{R}^n+q \rightarrow \mathbb{R}$ whose elements act preserving the orientations of $\mathbb{R}^n+q$ and $\mathbb{R}$ simultaneously or reversing the orientations simultaneously. That is, we consider the subgroup $S(O(\lambda) \times O(q+1-\lambda))$ of orientation preserving transformations of the group $O(\lambda) \times O(q+1-\lambda)$ and the trivial line bundle $\varepsilon^1 \rightarrow BS(O(\lambda) \times O(q+1-\lambda))$ in the case of $0 \leq \lambda < (q+1)/2$, and the appropriate subgroup denoted by $S\langle O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\rangle$ of the group $\langle O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\rangle$ and the corresponding line bundle $l^1 \rightarrow BS\langle O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\rangle$ in the case of $q$ odd and $\lambda = (q+1)/2$. So in the case of oriented fold maps, we have the homomorphisms

$$\xi^O_\lambda: \text{Cob}^O(n, q) \rightarrow \text{Imm}\left(\varepsilon^1_{BS(O(\lambda) \times O(q+1-\lambda))}, n\right)$$

for $0 \leq \lambda < (q+1)/2$ and

$$\xi^O_{(q+1)/2}: \text{Cob}^O(n, q) \rightarrow \text{Imm}(l^1, n)$$

for $q$ odd and $\lambda = (q+1)/2$, just like in the case of non-oriented fold maps. We used these homomorphisms in [Ka08b] [Ka08] to describe cobordisms of fold maps.

### 3.2.2. Invariants of framed fold maps.

For $0 \leq \lambda < (q+1)/2$, clearly there is the similar framed cobordism invariant

$$\xi_\lambda: \text{Cob}_{fr}(n, q) \rightarrow \text{Imm}\left(\varepsilon^1_{B(O(\lambda) \times O(q+1-\lambda))}, n\right).$$

For framed fold maps when $q$ is odd and $\lambda = (q+1)/2$ we have to consider only that largest subgroup of $\langle O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right), T\rangle$ whose elements act trivially on the target $\mathbb{R}$ of $\varphi$. This is $O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right)$ because $T$ acts as multiplication by $-1$ on the target $\mathbb{R}$ of $\varphi$. So we have the homomorphism

$$\xi_{(q+1)/2}: \text{Cob}_{fr}(n, q) \rightarrow \text{Imm}\left(\varepsilon^1_{B(O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right))}, n\right).$$

It is easy to get the analogous homomorphisms for oriented framed fold maps as well.

Since the group $\text{Imm}(\varepsilon^1_X, n)$, where $X$ is a space, is isomorphic to $\pi^s_{n-1} \oplus \pi^s_{n-1}(X)$, all these homomorphisms $\xi_\lambda$, where $0 \leq \lambda \leq (q+1)/2$, map from a framed cobordism group
into a direct sum of stable homotopy groups. The first coordinate map of $\xi_\lambda$ mapping to $
pi_{n-1}^s$ will be denoted by

$$t_\lambda : \text{Cob}_{fr}(n, q) \to \npi_{n-1}^s$$

and the second will be

$$\tau_\lambda : \text{Cob}_{fr}(n, q) \to \npi_{n-1}^s(BG),$$

for the suitable subgroup $G$ of the orthogonal group $O(q + 1)$.

When $n = 2$, the sum for $\lambda \geq 0$ of these homomorphisms will be used, i.e. take

$$\sum_{0 \leq \lambda \leq (q+1)/2} t_\lambda$$

and denote it by $t$ and take

$$\sum_{0 \leq \lambda \leq (q+1)/2} \tau_\lambda$$

and denote it by $\tau$.

So we have two homomorphisms

$$t : \text{Cob}_{fr}(2, q) \to \mathbb{Z}_2,$$

because $\pi_1^s = \mathbb{Z}_2$ and

$$\tau : \text{Cob}_{fr}(2, q) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

because $\pi_1^s(BG) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for any $1 \leq \lambda \leq (q + 1)/2$ and $G$ at hand, and in the case of $\lambda = 0$ we consider $\pi_1^s(B(O(0) \times O(q + 1))) = \{0\} \oplus \mathbb{Z}_2$, which we consider as a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We will denote the two components of $\tau$ by $\tau^1$ and $\tau^2$.

The analogous homomorphisms for the case of oriented framed maps are denoted the same way. In that case we have

$$t : \text{Cob}_{fr}^O(2, q) \to \mathbb{Z}_2,$$

and

$$\tau : \text{Cob}_{fr}^O(2, q) \to \mathbb{Z}_2$$

because then $\pi_1^s(BG) = \mathbb{Z}_2$ for any $1 \leq \lambda \leq (q + 1)/2$ and $G$ at hand, and in the case of $\lambda = 0$ we have $\pi_1^s(BS(O(0) \times O(q + 1))) = \{0\}$, which we consider as a subgroup of $\mathbb{Z}_2$.

To simplify the notation, if $f$ is a fold map, we will refer to $t([f])$ and $\tau([f])$ as $t(f)$ and $\tau(f)$, respectively. When $n = 2$, the value $t(f)$ is just the number of double points mod 2 of the immersion into $\mathbb{R}^2$ of the singular set of the generic fold map $f$.

Remark 3.3. Clearly in the case of non-framed (but possibly oriented or non-oriented) fold maps into $\mathbb{R}^2$ of even codimension $q$ we have the analogous homomorphisms $t$ and $\tau$ because such fold maps and their cobordisms can be naturally framed.

We will consider non-framed oriented fold maps to $\mathbb{R}^2$ of odd codimension $q$ as well.

In that case we also have the analogous homomorphisms $t$ and $\tau$ mapping into $\mathbb{Z}_2$ since for $t_\lambda$ and $\tau_\lambda$, where $0 \leq \lambda < (q + 1)/2$, everything works the same way as in the framed case. Moreover an easy computation shows that the target of the homomorphism

$$\xi_{(q+1)/2}^O : \text{Cob}^O(2, q) \to \text{Imm}(\tilde{l}^1, 2)$$

is also $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, which corresponds to the number of double points mod 2 and the twisting of the normal bundle of the the index $(q + 1)/2$ fold singularities in the source manifold, just like in the case of framed fold maps.
4. Cobordism of manifolds with stable framings

4.1. Stably framed manifolds and their cobordisms.

Definition 4.1 (Stably \((n-1)\)-framed manifolds). For \(n > 0, q \geq 0\) an \((n+q)\)-dimensional manifold \(Q\) is stably \((n-1)\)-framed if the vector bundle \(TQ \oplus \varepsilon^1_Q\) has \(n\) sections that are linearly independent at every point of \(Q\) (shortly, we say that the vector bundle \(TQ \oplus \varepsilon^1_Q\) has \(n\) independent sections).

Definition 4.2 (Stably \((n-1)\)-framed cobordism). Let \(Q_i\) be closed (oriented) stably \((n-1)\)-framed \((n+q)\)-dimensional manifolds, i.e., the vector bundles \(TQ_i \oplus \varepsilon^1_{Q_i}\) have \(n\) independent sections \(e^1_i, \ldots, e^n_i\) \((i = 0, 1)\). We say that the manifolds \(Q_0\) and \(Q_1\) are stably (oriented) \((n-1)\)-framed cobordant if

(i) they are (oriented) cobordant in the usual sense by a compact (oriented) \((n+q+1)\)-dimensional manifold \(W\),
(ii) the vector bundle \(TW \oplus \varepsilon^1_W\) has \(n+1\) independent sections \(f^1, \ldots, f^{n+1}\),
(iii) the sections \(f^j\), \(j = 1, \ldots, n\), restricted to the boundary \(Q_0 \sqcup Q_1\) of \(W\) coincide with the sections \(e^j_i\) \((j = 1, \ldots, n\) and \(i = 0, 1)\).
(iv) the section \(f^{n+1}\) restricted to the boundary part \(Q_0\) (resp. \(Q_1\)) of \(W\) coincides with an inward (resp. outward) normal section of \(\partial W\).

We denote the set of stably \((n-1)\)-framed cobordism classes of closed stably \((n-1)\)-framed \((n+q)\)-dimensional manifolds by \(C_{n+q}(n)\) (and by \(C^O_{n+q}(n)\) in the oriented case) which is an abelian group with the disjoint union as operation.

We obtain homomorphisms

\[
\sigma_{n,q} : \text{Cob}_{fr}(n, q) \to C_{n+q}(n)
\]

and

\[
\sigma^O_{n,q} : \text{Cob}^O_{fr}(n, q) \to C^O_{n+q}(n),
\]

which map a cobordism class of a framed fold map \(g : Q^{n+q} \to \mathbb{R}^n\) to the stably \((n-1)\)-framed cobordism class of the source manifold \(Q^{n+q}\) precisely as follows.

In Section 2.2 we constructed a fiberwise epimorphism

\[
\varphi(g, q, r) : TQ \oplus \varepsilon^1_Q \to T\mathbb{R}^n
\]

from a given framed fold map \(g : Q^{n+q} \to \mathbb{R}^n\). We will define the cobordism group of fiberwise epimorphisms of this form which we will denote by \(\mathcal{E}(n, q)\) and give a homomorphism from the group \(\text{Cob}_{fr}(n, q)\) to \(\mathcal{E}(n, q)\). Then we will define a homomorphism from \(\mathcal{E}(n, q)\) to the group \(C_{n+q}(n)\). The composition

\[
\text{Cob}_{fr}(n, q) \to \mathcal{E}(n, q) \to C_{n+q}(n)
\]

of these two homomorphisms will be \(\sigma_{n,q}\).

Definition 4.3. Let \(Q_0\) and \(Q_1\) be closed \((n+q)\)-dimensional manifolds. Let

\[
\psi_i : TQ_i \oplus \varepsilon^1_{Q_i} \to T\mathbb{R}^n,
\]

\(i = 0, 1\), be fiberwise epimorphisms. We say that \(\psi_0\) and \(\psi_1\) are cobordant if

(i) there is a compact \((n+q+1)\)-dimensional manifold \(W\) such that \(\partial W = Q_0 \sqcup Q_1\),
(ii) there is a fiberwise epimorphism \(\Psi : TW \oplus \varepsilon^1_W \to T(\mathbb{R}^n \times [0, 1])\) and
(iii) for \(i = 0, 1\) the bundle homomorphism \(\Psi\) restricted to \(TQ_i \oplus \varepsilon^1_{Q_i}\) maps into \(T(\mathbb{R}^n \times \{i\})\) and it is equal to \(\psi_i\).
The naturally resulting cobordism group is denoted by \( \mathcal{E}(n,q) \).

We have the following (we use the notations of Section 2.2).

**Lemma 4.4.** Let \( g: Q^{n+q} \rightarrow \mathbb{R}^n \) be a framed fold map. Let \( \varphi \) be a Riemannian metric on \( Q \) and let \( r = r(g, \varphi) \).

(i) If \( 0 < r' < r \), then \( \varphi(g, \varphi, r') \) is cobordant to \( \varphi(g, \varphi, r) \).

(ii) If \( \varphi' \) is another Riemannian metric on \( Q \) and \( r' = r'(g, \varphi') \), then there is a positive \( r'' < \min(r, r') \) such that \( \varphi(g, \varphi, r'') \) and \( \varphi(g, \varphi', r'') \) are cobordant.

(iii) If \( (Q', \varphi') \) is another Riemannian manifold and \( g': Q' \rightarrow \mathbb{R}^n \) is another framed fold map such that \( g \) and \( g' \) are framed cobordant, then there is a positive \( r'' < \min(r(g, \varphi), r'(g', \varphi')) \)

such that \( \varphi(g, \varphi, r'') \) and \( \varphi(g', \varphi', r'') \) are cobordant.

**Proof.** We get (i) by continuously modifying along \( Q \times \{t\} \), \( t \in [0,1] \), the radius \( r \) of the tubular neighborhood \( N_r(S_g) \) until we get \( r' \) in the manifold \( Q \times [0,1] \) equipped with the Riemannian metric equal to the direct sum of \( g \) and the standard metric on \( [0,1] \). We also modify the function \( \alpha: N_r(S_g) \rightarrow [0,1] \) such that it stays smooth near \( S_g \). The proof of (ii) and (iii) is an easy exercise in constructing Riemannian metrics on \( Q \times [0,1] \) in the case of (ii) and on a cobordism \( W \) in the case of (iii).

This lemma implies that the construction in Section 2.2 induces a homomorphism from the cobordism group of framed fold maps \( \text{Cob}_{fr}(n,q) \) to the cobordism group \( \mathcal{E}(n,q) \).

Now, we want a homomorphism from \( \mathcal{E}(n,q) \) to the cobordism group of stably \((n-1)\)-framed manifolds. The kernel of a fiberwise epimorphism \( \varphi: TQ \oplus \varepsilon_Q^1 \rightarrow T\mathbb{R}^n \) has an orthogonal complement if there is a given Riemannian metric on \( Q \). Then this orthogonal complement has \( n \) frames since it is mapped isomorphically onto \( T\mathbb{R}^n \). So we have the stably \((n-1)\)-framed manifold \( Q \) and the image of the cobordism class \([\varphi]\) is defined to be the stably framed cobordism class of \( Q \). But again we have to prove that these \( n \) frames do not depend on the Riemannian metric on \( Q \) up to cobordism.

**Lemma 4.5.** We have the following.

(i) If \( q_1 \) and \( q_2 \) are Riemannian metrics on \( Q \), then the two stably \((n-1)\)-framed manifolds obtained from the fiberwise epimorphism \( TQ \oplus \varepsilon_Q^1 \rightarrow T\mathbb{R}^n \) depending on \( q_1 \) and \( q_2 \) are cobordant.

(ii) If the fiberwise epimorphisms \( TQ_i \oplus \varepsilon_Q^1 \rightarrow T\mathbb{R}^n \), \( i = 0,1 \), are cobordant, then for some Riemannian metrics on \( Q_0 \) and \( Q_1 \) the obtained stably \((n-1)\)-framed manifolds \( Q_0 \) and \( Q_1 \) are cobordant.

**Proof.** The proof is an easy exercise in extending Riemannian metrics on \( Q \times [0,1] \) in the case of (i) and constructing Riemannian metrics on a cobordism \( W \) in the case of (ii).

All these arguments imply that we have the well-defined homomorphisms

\[ \text{Cob}_{fr}(n,q) \rightarrow \mathcal{E}(n,q) \quad \text{and} \quad \mathcal{E}(n,q) \rightarrow C_{n+q}(n). \]

**Definition 4.6.** We denote by \( \sigma_{n,q} \) the composition

\[ \text{Cob}_{fr}(n,q) \rightarrow \mathcal{E}(n,q) \rightarrow C_{n+q}(n). \]

Analogously, in the oriented case we have the well-defined homomorphism

\[ \sigma_{n,q}^O: \text{Cob}_{fr}^O(n,q) \rightarrow C_{n+q}^O(n). \]
Proposition 4.7. The homomorphisms $\sigma_{n,q}$ and $\sigma_{n,q}^O$ are surjective.

Proof. Let us take a cobordism class $\omega$ in $C_{n+q}(n)$ represented by a stably $(n-1)$-framed manifold $Q$. Then by Remark 2.5 there is a framed fold map $f: Q \to \mathbb{R}^n$ such that the stable $(n-1)$-framing given by $f$ is homotopic to the stable $(n-1)$-framing which was given originally on $Q$. This homotopy yields a cobordism between the two stable $(n-1)$-framings. Hence $\sigma_{n,q}$ maps the cobordism class of the framed fold map $f$ to the cobordism class $\omega$. The proof for $\sigma_{n,q}^O$ is similar. □

In fact [Ko81] deals extensively with the groups $C_{n+q}(n)$ and $C_{n+q}^O(n)$, which are naturally isomorphic through stabilization to cobordism groups denoted in [Ko81] by $\mathcal{N}_{n+q}(n-1,n-1)$ and $\Omega_{n+q}(n-1,n-1)$, respectively, and Koschorke computes them for low $n$ (see [Ko81, Theorem 6.6, Proposition 7.17 and Theorems 12.1, 12.8, 19.39, 19.40, 19.41]).

Definition 4.8. Let $\mathcal{N}_m^2$ and $\Omega_m^2$ denote the kernels of the homomorphism $w_m: \mathcal{N}_m \to \mathbb{Z}_2$

and

$w_m: \Omega_m \to \mathbb{Z}_2$,

respectively, where for a closed $m$-dimensional manifold $M^m$, $w_m([M^m])$ is the Stiefel-Whitney number for the $m$th Stiefel-Whitney class of $M^m$. In other words $\mathcal{N}_m^2$ (resp. $\Omega_m^2$) is the cobordism group (resp. oriented cobordism group) of manifolds of even Euler characteristic.

For example, the group $\Omega_4^2$ is isomorphic to $\mathbb{Z}$ and it is generated by the class $[\mathbb{C}P^2 \# \mathbb{C}P^2]$. By [Ko81] Proposition 7.17 and Theorems 12.8, 19.39, we have the following.

Proposition 4.9. For $k \geq 1$, we have

(i) $C_{k+1}(2) \cong \mathbb{Z}_2 \oplus \mathcal{N}_k^{2}$,

(ii) $C_{4k}^O(2) \cong \Omega_{4k}^2$ and the isomorphism is induced by the forgetful map $C_{4k}^O(2) \to \Omega_{4k}$.

4.2. The singular set of the stably framed source manifold. It is important to know what $\sigma_{n,q}$ does if we watch it through the isomorphism stated in (i) of Proposition 4.9. The next statement explains in geometric terms the homomorphism $\chi''$ defined in [Ko81] page 130 (12.5)], which plays an important role in the theory of framed bordisms. Following [Ko81] Definition 2.1 and pages 26–27 we will denote by $\Omega_1(point; \varepsilon^1)$, where $\varepsilon^1$ is the trivial line bundle over a point, the first normal bordism group of the point with trivial coefficients. This group is isomorphic to the stable homotopy group $\pi_1^s$, see [Ko81] equation (2.2) on page 27]. Recall that $\pi_1^s \cong \mathbb{Z}_2$ and hence $\Omega_1(point; \varepsilon^1) \cong \mathbb{Z}_2$. The generator is represented by a circle embedded in a high dimensional Euclidean space with trivialized normal bundle such that the trivialization is twisted once as we move along the circle.
Proposition 4.10. Let \( q \geq 0 \). Under the isomorphism \( \mathcal{C}_{2+q}(2) \cong \mathbb{Z}_2 \oplus \mathfrak{N}_{2+q}^2 \chi \) obtained in [Ko81] Theorem 12.8, the homomorphism

\[
\sigma_{2,q} : \text{Cob}_{fr}(2,q) \to \mathbb{Z}_2 \oplus \mathfrak{N}_{2+q}^2 \chi
\]

is the map

\[
[g : M^{2+q} \to \mathbb{R}^2] \mapsto ([g|_{S_\eta}], [M^{2+q}]),
\]

where \([M]\) is the cobordism class of \( M \) and \([g|_{S_\eta}] \in \mathbb{Z}_2\) is the cobordism class of the immersion \( g|_{S_\eta} \) of the fold singular set of \( g \) into \( \mathbb{R}^2 \).

Proof. By definition the homomorphism \( \sigma_{2,q} \) maps a fold cobordism class \([g : M^{2+q} \to \mathbb{R}^2]\) to the cobordism class of the stably 1-framed source manifold \([M]\) in \( \mathcal{C}_{2+q}(2) \). By [Ko81] Proposition 7.17 the group \( \mathcal{C}_{2+q}(2) \) is isomorphic to the cobordism group \( \mathfrak{N}_{2+q}(1,1) \), which is by definition the cobordism group of closed \((2+q)\)-dimensional manifolds admitting one non-zero vector field. If the closed manifold \( N \) represents a class in \( \mathfrak{N}_{2+q}(1,1) \), then the tangent space \( TN \) has one non-zero vector field and then \( TN \oplus \varepsilon_N^1 \) has two linearly independent vector fields: the second vector field is the natural framing of the bundle \( \varepsilon^1_N \) added to \( TN \). This is called stabilization and yields a representative of a class in \( \mathcal{C}_{2+q}(2) \). This is the natural stabilizing isomorphism

\[
\text{St} : \mathfrak{N}_{2+q}(1,1) \to \mathcal{C}_{2+q}(2)
\]

established in [Ko81].

Then [Ko81] Theorem 12.8 says that \( \mathfrak{N}_{2+q}(1,1) \) is isomorphic to \( \mathbb{Z}_2 \oplus \mathfrak{N}_{2+q}^2 \chi \) via a map

\[
(\chi'', f) : \mathfrak{N}_{2+q}(1,1) \to \mathbb{Z}_2 \oplus \mathfrak{N}_{2+q}^2 \chi,
\]

where \( f \) is just the forgetful map which forgets all the framings. The map \( \chi'' \) is defined as follows (see [Ko81] page 130 (12.5)). Take a representative \( N^{2+q} \) of a class \( x \in \mathfrak{N}_{2+q}(1,1) \), then \( TN = \varepsilon_N^1 \oplus \eta_N \) for some \((q+1)\)-dimensional vector bundle \( \eta_N \). Let \( S \subset N \) be the zero set of a generic smooth section of \( \eta_N \). Then \( S \) is a closed smooth 1-dimensional manifold and since \( TS \oplus \eta_S = TN|_{S} = \varepsilon_N^1 \oplus \eta_S \) over \( S \), adding an appropriate bundle \( \eta^1 \) over \( S \) such that \( \eta_S \oplus \eta^1 \) is a trivial bundle, we have a given stable parallelization of \( TS \). Hence \( S \) represents an element in the bordism group \( \Omega_1(\text{point}; \varepsilon^1) \cong \mathbb{Z}_2 \), see [Ko81] Definition 2.1 and the following explanation on page 27]. This element is the value \( \chi''(x) \).

Notice that the isomorphism \( \mathcal{C}_{2+q}(2) \cong \mathbb{Z}_2 \oplus \mathfrak{N}_{2+q}^2 \chi \) mentioned in the statement of the proposition is just \( (\chi'', f) \circ \text{St}^{-1} \).

Now, we want to compute

\[
(\chi'', f) \circ \text{St}^{-1} \circ \sigma_{2,q}(\langle g \rangle)
\]

of a class \([g : M^{2+q} \to \mathbb{R}^2] \in \text{Cob}_{fr}(2,q)\). The class \([g : M^{2+q} \to \mathbb{R}^2] \in \text{Cob}_{fr}(2,q)\) is mapped by \( \sigma_{2,q} \) to the class of the stably 1-framed source manifold \([M]\) in \( \mathcal{C}_{2+q}(2) \). So we have

\[
TM \oplus \varepsilon^1_M \cong \eta_M \oplus \varepsilon^2_M
\]

for some \( \eta_M \). Then we get \( \text{St}^{-1}([M]) \in \mathfrak{N}_{2+q}(1,1) \) by the following process: since \([M]\) is in the image of the stabilizing isomorphism, the manifold \( M \) is stably 1-framed cobordant by some \( W \) to a manifold \( M' \) such that

1. the bundle \( TM' \oplus \varepsilon^1_M \) has two linearly independent vector fields,
2. the second vector field of this framing coincides with the natural framing of the bundle \( \varepsilon^1_M \).
We can delete this second vector field together with the summand $\varepsilon_{1M}^\prime$. This means we obtain a representative $M'$ of the class $\text{St}^{-1}([M])$ such that $TM'$ has one non-zero vector field. To get the value

$$(\chi'', f)(\text{St}^{-1}([M]))$$

in the group $\mathbb{Z}_2 \oplus \mathfrak{H}_{2+q}$ at first observe that $f \circ \text{St}^{-1}([M])$ is just the cobordism class $[M] \in \mathfrak{H}_{2+q}$. Now, we are going to look for $\chi'' \circ \text{St}^{-1}([M]) = \chi''([M'])$. Computing $\chi''([M'])$ goes as above. Namely, $TM' = \varepsilon_{1M}^\prime \oplus \eta_{M'}$ for some bundle $\eta_{M'}$ and we can find the stably parallelized 1-dimensional manifold $S' \subset M'$ by taking the zero set of a generic section of $\eta_{M'}$ over $M'$. Of course we could find the same $S' \subset M'$ in $TM' \oplus \varepsilon_{1M}^\prime = (\varepsilon_{1M}^\prime \oplus \eta_{M'}) \oplus \varepsilon_{1M}^\prime$; instead of $TM'$ as well if we do not delete that second vector field specified in (2). Besides these, we can find an $\eta_{M'}$ in the entire $TW \oplus \varepsilon_{1W}$ which restricts to our $\eta_{M'}$ in $TM' \oplus \varepsilon_{1M}^\prime$ and to $\eta_{M}$ in $TM \oplus \varepsilon_{1M}^\prime$. And hence we find a stably parallelized 2-dimensional manifold $\tilde{S} \subset W$ as the zero set of a generic section of $\eta_{W}$ in $TW \oplus \varepsilon_{1W}$. Summarizing, we obtain that the zero set $S \subset M$ of a generic section of $\eta_{M}$ over the original manifold $M$, where we had $TM \oplus \varepsilon_{1M}^\prime \cong \eta_{M} \oplus \varepsilon_{1M}^\prime$, is cobordant to $S'$ by the surface $\tilde{S}$ in the sense of the group $\Omega_1(\text{point}; \varepsilon_{1W})$.

As a result we get that if we look for the value $\chi'' \circ \text{St}^{-1} \circ \sigma_{2,q} \left( [g : M^{2+q} \to \mathbb{R}^2] \right)$, then it is enough to consider the stably 1-framed manifold $M$ as a representative in $\mathcal{C}_{2+q}(2)$ and to find the zero set $S \subset M$ of a generic section of $\eta_{M}$ where $TM \oplus \varepsilon_{1M}^\prime \cong \eta_{M} \oplus \varepsilon_{1M}^\prime$, as we obtain from the map $g$. Then this $S$ and its stable parallelization give the same element in $\Omega_1(\text{point}; \varepsilon_{1W})$ as $\chi'' \circ \text{St}^{-1} \circ \sigma_{2,q}([g])$.

So having $TM \oplus \varepsilon_{1M}^\prime \cong \varepsilon_{2M}^2 \oplus \eta_{M}$ obtained from a fold map $g : M^{2+q} \to \mathbb{R}^2$, we have to understand the geometric meaning of this stable parallelized $S$. We got the bundle $\eta_{M}$ as the kernel of the fiberwise epimorphism

$$\tilde{\varphi}_{g; q, r} : TM \oplus \varepsilon_{1M} \to g^*T\mathbb{R}^2$$

taken from the fold map $g$ by applying (2.2) in Section 2.2. So the bundle $\eta_{M}$ is completely contained in $TM$ exactly at the singular points of $g$. If we project fiberwisely the unit vector of $\varepsilon_{1M}^\prime$ to $\eta_{M}$ perpendicularly by the Riemannian metric used in Section 2.2 (note that the direction $\varepsilon_{1M}^\prime$ is perpendicular to $TM$), then we get a continuous section $s$ of $\eta_{M}$, which is zero exactly at the singular set $S_g$ and smooth near $S_g$. Approximate this continuous section $s$ by a smooth section of $\eta_{M}$, which coincides with $s$ near the singular set $S_g$.

Then the singular set $S_g$ is a smooth 1-dimensional submanifold of $M$, obviously $TS_g$ and $\eta_{M}|_{S_g}$ are subbundles of $TM|_{S_g}$ and we have

$$\eta_{M}|_{S_g} \oplus TS_g = TM|_{S_g},$$

where the second isomorphism is being induced by $\tilde{\varphi}$. This equation gives the stable framing of $TS_g$ after adding $\eta_{M}|_{S_g}$ to both sides. Restricted to $S_g$ the homomorphism $\tilde{\varphi}$ has the form

$$\eta_{M}|_{S_g} \oplus TS_g \oplus \varepsilon_{1S_g} \to g^*T\mathbb{R}^2,$$

where

$$(u, v, w) \mapsto dg(u, v) + w, \quad (u, v, w) \in \eta_{M}|_{S_g} \oplus TS_g \oplus \varepsilon_{1S_g}.$$

The isomorphism in (4.1) can be made explicit: if it is

$$\iota: \eta_{M}|_{S_g} \oplus TS_g \oplus \varepsilon_{1S_g} \to \eta_{M}|_{S_g} \oplus g^*T\mathbb{R}^2|_{S_g},$$
then
\[ \iota(u, v, w) = (u, \varphi_{g,v,w}(0, v, w)) = (u, dg(0, v) + w). \]

Furthermore, since \( dg|_{S_g} \) maps \( \eta_M|_{S_g} \) to 0 and \( g|_{S_g} : TS_g \to g^*T\mathbb{R}^2 \) coincides with
\[ v \mapsto dg(0, v), \quad v \in TS_g, \]
we have that the map \( TS_g \oplus \varepsilon^k_{S_g} \to g^*T\mathbb{R}^2 \), \((v, w) \mapsto dg(0, v) + w\) is the same as the fiberwise isomorphism
\[ g|_{S_g} + \nu_g : TS_g \oplus \varepsilon^1_{S_g} \to g^*T\mathbb{R}^2 \]
induced by the immersion \( g|_{S_g} : S_g \to \mathbb{R}^2 \) and its trivial normal bundle \( \nu_g \).

Then for some \( k \in \mathbb{N} \) we have the diagram
\[
\begin{array}{cc}
\eta_M|_{S_g} \oplus TS_g \oplus \varepsilon^k_{S_g} & \xrightarrow{\iota} & \eta_M|_{S_g} \oplus g^*T\mathbb{R}^2|_{S_g} \\
\eta_M|_{S_g} \oplus TS_g \oplus \varepsilon^1_{S_g} & \xrightarrow{\kappa} & \varepsilon^k_{S_g} \oplus g^*T\mathbb{R}^2|_{S_g}
\end{array}
\]
\[
\begin{array}{cc}
\varepsilon^1_{S_g} \oplus TS_g \oplus \varepsilon^1_{S_g} & \xrightarrow{\kappa} & \varepsilon^1_{S_g} \oplus g^*T\mathbb{R}^2|_{S_g} \\
\varepsilon^1_{S_g} \oplus TS_g \oplus \varepsilon^k_{S_g} & \xrightarrow{\iota} & \eta_M|_{S_g} \oplus g^*T\mathbb{R}^2|_{S_g}
\end{array}
\]
where the top vertical downward arrows are the identity isomorphisms on the direct summands \( TS_g \oplus \varepsilon^1_{S_g} \) and \( g^*T\mathbb{R}^2|_{S_g} \), respectively. The homomorphism \( \kappa \) is defined as
\[ \kappa(u, v, w) = (u, dg(0, v) + w) \text{ for } (u, v, w) \in \varepsilon^k_{S_g} \oplus TS_g \oplus \varepsilon^1_{S_g}. \]

The bottom vertical upward arrows are inclusions. These imply that the diagram is commutative.

Thus \( g|_{S_g} + \nu_g \) is stably equivalent to the isomorphism giving the stable framing of \( TS_g \) through \( \iota \) in (4.1).

Hence
\[ \chi'' \circ \text{St}^{-1} \circ \sigma_{2,q} \left( [g : M^{2+q} \to \mathbb{R}^2] \right) = [g|_{S_g}] \]
in the group \( \mathbb{Z}_2 \).

\[ \square \]

5. THE COMPLETE INVARIANTS OF COBORDISMS OF FRAMED FOLD MAPS

Recall that the group \( \text{Imm}(\varepsilon_X, n) \), where \( X \) is a topological space, is identified with the group \( \pi^a_{n-1} \oplus \pi^a_{n-1}(X) \). Throughout this section \( \lambda \) denotes non-negative integers referring to the absolute indices of the fold singularities.

Denote by \( \mathcal{S}_{n,q}^O \) the homomorphism
\[
\left( \sigma_{n,q}, \xi^O_1, \ldots, \xi^O_{\lfloor(q+1)/2\rfloor} \right)
\]
and by \( \mathcal{S}_{n,q} \) the homomorphism
\[
\left( \sigma_{n,q}, \xi_1, \ldots, \xi_{\lfloor(q+1)/2\rfloor} \right).
\]

**Theorem 5.1.** Let \( n \geq 1, q \geq 0 \). Then, the homomorphisms
\[
\mathcal{S}_{n,q} : \text{Cob}_{fr}(n, q) \to \mathcal{C}_{n+q}(n) \oplus \bigoplus_{1 \leq \lambda \leq \lfloor(q+1)/2\rfloor} \pi^a_{n-1} \oplus \pi^a_{n-1}(B(O(\lambda) \times O(q + 1 - \lambda)))
\]
and
\[
\mathcal{S}_{n,q}^O : \text{Cob}_{fr}^O(n, q) \to \mathcal{C}_{n+q}^O(n) \oplus \bigoplus_{1 \leq \lambda \leq \lfloor(q+1)/2\rfloor} \pi^a_{n-1} \oplus \pi^a_{n-1}(BS(O(\lambda) \times O(q + 1 - \lambda)))
\]
are injective. Hence two framed fold maps \( f_i : Q_i^{n+q} \to \mathbb{R}^n, i = 0, 1, \) are framed cobordant if and only if
\[
\mathcal{S}_{n,q}([f_0]) = \mathcal{S}_{n,q}([f_1])
\]
and two oriented framed fold maps \( f_i : Q_i^{n+q} \to \mathbb{R}^n, i = 0, 1, \) are oriented framed cobordant if and only if
\[
\mathcal{S}^O_{n,q}([f_0]) = \mathcal{S}^O_{n,q}([f_1]) .
\]

The proof will be given in Section [5]. For \( q \) even, since fold maps with even codimension can be framed naturally, in the statement of Theorem 5.1 the group \( \text{Cob}^O_{fr}(n, q) \) can be replaced by \( \text{Cob}^O(n, q) \) and \( \text{Cob}_{fr}(n, q) \) can be replaced by \( \text{Cob}(n, q) \). For \( q \) odd, there is a forgetful map \( \text{Cob}^O(n, q) \to \text{Cob}^O(n, q) \), which is obviously surjective if \( n \leq 2 \).

**Corollary 5.2.** For \( k \geq 0 \), the homomorphism \( \mathcal{S}^{(O)}_{n,2k} \) gives a complete invariant of the (oriented) cobordism group \( \text{Cob}^{(O)}(n, 2k) \) of fold maps.

**Remark 5.3.** By Corollary 5.2 the homomorphisms
\[
\mathcal{S}^O_{n,0} : \text{Cob}^O(n, 0) \to \mathcal{C}^O_n(n) \quad \text{and} \quad \mathcal{S}_{n,0} : \text{Cob}(n, 0) \to \mathcal{C}_n(n)
\]
are injective, and by Proposition 4.7 they are surjective as well. Hence, we have
\[
\text{Cob}^O(n, 0) \cong \mathcal{C}^O_n(n) \quad \text{and} \quad \text{Cob}(n, 0) \cong \mathcal{C}_n(n).
\]

Since the group \( \mathcal{C}^O_n(n) \) is isomorphic to \( \pi^n_\ast \), we obtain another argument for the isomorphism \( \text{Cob}^O(n, 0) \cong \pi^n_\ast \) (for the original proof, see Ando [An02a, An06]).

**Corollary 5.4.** (1) The cobordism group \( \text{Cob}(2, 0) \) of fold maps from unoriented surfaces into \( \mathbb{R}^2 \) is isomorphic to \( \mathbb{Z}_2 \). A fold map from an unoriented surface to \( \mathbb{R}^2 \) is null-cobordant if and only if its singular set is immersed into \( \mathbb{R}^2 \) with an even number of double points.

(2) The cobordism group \( \text{Cob}(3, 0) \) of fold maps from unoriented 3-manifolds into \( \mathbb{R}^3 \) is isomorphic to \( \mathbb{Z}_2 \). A fold map from an unoriented 3-manifold to \( \mathbb{R}^3 \) is null-cobordant if and only if the immersion of its singular set into \( \mathbb{R}^3 \) is null-cobordant.

**Proof.** For (1) we have \( \text{Cob}(2, 0) \cong \mathcal{C}_2(2) \cong \mathbb{Z}_2 \oplus \mathfrak{N}_2^2 \) because \( \mathfrak{N}_2^2 = 0 \). Proposition 4.10 gives the second part of the statement. For (2) we have \( \text{Cob}(3, 0) \cong \mathcal{C}_3(3) \cong \mathbb{Z}_2 \oplus \mathfrak{N}_3^{2} \) by [Ko81] Theorem 12.8 and then \( \text{Cob}(3, 0) \cong \mathbb{Z}_2 \) because \( \mathfrak{N}_3^2 = 0 \). Proposition 4.10 gives the second part of the statement. \( \square \)

**Corollary 5.5.** By [Ko81] Proposition 7.17 and Theorem 19.40 we know that for \( k \geq 1 \) we have \( \mathcal{C}^O_{4k-1}(2) \cong \Omega_{4k-1} \). The image of \( \mathfrak{S}^O_{2,4k-3} \) is a subgroup of \( \Omega_{4k-1} \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^{2k-1} \) and when \( k = 1 \) this means that \( \text{Cob}^O_{fr}(2, 1) \subset \mathbb{Z}_2 \odot \mathbb{Z}_2 \). It is easy to construct two framed fold maps \( f_{1,2} : M_{1,2} \to \mathbb{R}^2 \) on 3-manifolds such that the index 1 fold singular set of \( f_1 \) is immersed with one double point into \( \mathbb{R}^2 \), the index 1 fold singular set of \( f_2 \) is immersed without double points into \( \mathbb{R}^2 \), the twisting \( \tau_1([f_1]) = 0 \) and the twisting \( \tau_1([f_2]) = 1 \), see, for example (3-1) and (1)-(4) in [Ka09b, page 328] with the role \( N^n = \mathbb{R}^2 \). Then the cobordism invariants \( t_1 \) and \( \tau_1 \) distinguish between \([f_1]\) and \([f_2]\), hence \( \text{Cob}^O_{fr}(2, 1) \cong \mathbb{Z}_2 + \mathbb{Z}_2 \). The forgetful map \( \text{Cob}^O_{fr}(2, 1) \to \text{Cob}^O(2, 1) \) is clearly surjective and even in \( \text{Cob}^O(2, 1) \) the invariants \( t_1 \) and \( \tau_1 \) distinguish between \([f_1]\) and \([f_2]\), so \( \text{Cob}^O(2, 1) \cong \mathbb{Z}_2 + \mathbb{Z}_2 \). For other proofs of this fact not using h-principle, see [Ka07] and [Ka09b] Theorem 2.9.
6. Morse function bundles over immersions

In this section, we recall some results of [Ka08] for the convenience of the reader. These results will be used in Sections 8.2, 8.3 and 8.4.

6.1. Cobordism classes of Morse function bundles. For \( q \geq 2 \) and \( 1 \leq j < (q+1)/2 \) we construct fold maps \( \varphi_{j,q} \) of some \((2+q)\)-dimensional manifolds into \( \mathbb{R}^2 \), where the \( \varphi_{j,q} \) will also depend on some other parameters. The cobordism classes of these fold maps \( \varphi_{j,q} \) will serve as generators of an important direct summand of the cobordism group \( \text{Cob}^O(2, q) \). We will construct similar maps in the unoriented case as well.

For \( q \geq 2 \) and \( 1 \leq j < (q+1)/2 \), let \( h_j : S^{q+1} \to \mathbb{R} \) be a Morse function of the \((q+1)\)-dimensional sphere onto the closed interval \([-\varepsilon, \varepsilon]\) with four critical points \( a, b, c, d \in S^{q+1} \) of index \( 0, j-1, j, q+1 \), respectively, such that \( h_j(a) = -\varepsilon, h_j(b) = -\varepsilon/2, h_j(c) = 0 \) and \( h_j(d) = \varepsilon \). Recall the following result from [Ka08].

**Lemma 6.1** (Lemma 3.2 [Ka08]). There exists an identification of the Morse function \( h_j \) around its critical point of index \( j \) with the fold germ

\[
\gamma(x_1, \ldots, x_{q+1}) = (-x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_{q+1}^2),
\]

such that under this identification

1. any automorphism in the automorphism group \( O(1) \times O(q) \) (in the case of \( j = 1 \))
2. any automorphism in the automorphism group \( O(1, j-1) \times O(q+1-j) \) (in the case of \( j > 1 \))

of the fold germ \( \gamma \) can be extended to an automorphism of the Morse function \( h_j \).

Following [Ka08] Section 3 in the special case of \( n = 2 \), we define the group homomorphisms

\[
\alpha_1 : \text{Imm} \left( \varepsilon_{BS(O(1) \times O(q))}^1, 2 \right) \to \text{Cob}^O(2, q)
\]

and

\[
\alpha_j : \text{Imm} \left( \varepsilon_{BS(O(1,j-1) \times O(q+1-j))}^1, 2 \right) \to \text{Cob}^O(2, q)
\]

for \( 2 \leq j < (q+1)/2 \) as follows.

We first define \( \alpha_1 \). Let \([m : M^1 \to \mathbb{R}^2]\) be an element of \( \text{Imm} \left( \varepsilon_{BS(O(1) \times O(q))}^1, 2 \right) \). Then the normal bundle of the immersion \( m \) is induced from the trivial line bundle \( \varepsilon_{BS(O(1) \times O(q))}^1 \) by a map

\[
\mu : M^1 \to BS(O(1) \times O(q)).
\]

By Lemma 6.1 the symmetries in \( S(O(1) \times O(q)) \) extend to symmetries of the Morse function \( h_1 \). Hence the inducing map \( \mu \) yields a bundle with fiber the Morse function \( h_1 \) and base space the 1-dimensional manifold \( M^1 \). The Morse function \( h_1 \) has the source \( S^{q+1} \) and the target \([-\varepsilon, \varepsilon]\) so this \( h_1 \)-bundle over \( M^1 \) consists of an \( S^{q+1} \)-bundle over \( M^1 \), an \([-\varepsilon, \varepsilon]\)-bundle over \( M^1 \) and a map \( \beta \) between the total spaces of these two latter bundles. The map \( \beta \) fiberwise can be identified with \( h_1 \). We denote this \( S^{q+1} \)-bundle over \( M^1 \) by \( p_q : Q^{2+q}_{1,q,m} \to M^1 \), this \([-\varepsilon, \varepsilon]\)-bundle over \( M^1 \) by \( p_1 : J \to M^1 \) and then \( \beta \) maps \( Q^{2+q}_{1,q,m} \) to \( J \) (and as we said the restriction of \( \beta \) to a fiber \( S^{q+1} \) can be identified with \( h_1 : S^{q+1} \to [-\varepsilon, \varepsilon] \)). So we have the commutative diagram

\[
\begin{pmatrix}
1 & 0 \\
0 & M
\end{pmatrix}
\]

where \( M \) is an element of the group \( O(k) \).

---

1 Let \( O(1,k) \) denote the subgroup of the orthogonal group \( O(k+1) \) whose elements are of the form

\[
\begin{pmatrix}
1 & 0 \\
0 & M
\end{pmatrix}
\]

where \( M \) is an element of the group \( O(k) \).
where $\beta$ maps fiberwise.

Obviously, since $M^1$ is immersed into $\mathbb{R}^2$ (by $m$), the total space of the normal bundle of $m$ is also immersed into $\mathbb{R}^2$ (by $\nu$, say). Identify $J$ with the normal bundle of $m$ (in fact this normal bundle is a line bundle having an $[-\varepsilon, \varepsilon]$ subbundle and this and $J$ are the same $[-\varepsilon, \varepsilon]$ bundle over $M^1$, i.e. both of them are induced by $\mu$) and compose $\beta$ with $\nu$ to get the fold map $\nu \circ \beta$, which we denote by $\varphi_{1,q,m} : Q^{2+q}_{1,q,m} \to \mathbb{R}^2$.

Now let $\alpha_1([m])$ be the fold cobordism class of $\varphi_{1,q,m}$. This definition depends only on the immersion cobordism class of $m$ as one can see easily by doing the analogous constructions for cobordisms.

The definition of the group homomorphism $\alpha_j$ is similar: for an element $[m']$ in $\text{Imm} \left( \varepsilon_{BS(O(1,j-1) \times O(q+1-j))}^1, 2 \right)$ we define the fold map $\varphi_{j,q,m'} : Q^{2+q}_{j,q,m'} \to \mathbb{R}^2$ and its cobordism class $\alpha_j([m'])$ for $j > 1$ in a completely analogous way.

Now for formal reasons let $O(1,0)$ denote just the group $O(1)$, which is not a very good notation but we will be using it. For convenience, we extend each $\alpha_j$ to the other $i \neq j$ summands of the group

$$\bigoplus_{1 \leq i < (q+1)/2} \text{Imm} \left( \varepsilon_{BS(O(1,i-1) \times O(q+1-i))}^1, 2 \right)$$

as the identically zero homomorphism.

6.2. Direct summands of fold cobordism groups. We have the following statement (see also [Ka08, Remark 3.3]).

**Proposition 6.2.** The homomorphism $\sum_{1 \leq j < (q+1)/2} \alpha_j$ is an isomorphism onto a direct summand of $\text{Cob}^O(2,q)$. The group $\text{Cob}^O(2,q)$ contains $\mathbb{Z}_2^{1+q}$ as a direct summand.

**Proof.** At first, notice that for all $2 \leq j < (q+1)/2$, since we consider immersions into the plane, the group

$$\text{Imm} \left( \varepsilon_{BS(O(1,j-1) \times O(q+1-j))}^1, 2 \right)$$

is the same as the group

$$\text{Imm} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right),$$

so for all $j$ the homomorphisms $\alpha_j$ are in fact homomorphisms of type

$$\alpha_j : \text{Imm} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \to \text{Cob}^O(2,q).$$

As before, for convenience, we extend each $\alpha_j$ to the other $i \neq j$ summands of the group

$$\bigoplus_{1 \leq i < (q+1)/2} \text{Imm} \left( \varepsilon_{BS(O(i) \times O(q+1-i))}^1, 2 \right)$$

as the identically zero homomorphism.
Take the composition
\[
\bigoplus_{1 \leq j < (q+1)/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \to \text{Cob}^O(2, q) \to \\
\bigoplus_{1 \leq j < (q+1)/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right),
\]
where the first arrow is the homomorphism \( \sum_{1 \leq j < (q+1)/2} \alpha_j \) and the second arrow is the homomorphism \((\xi_1^O, \ldots, \xi_{[q/2]}^O)\).

If we show that this composition
\[
(\xi_1^O, \ldots, \xi_{[q/2]}^O) \circ \sum_{1 \leq j < (q+1)/2} \alpha_j
\]
is an isomorphism, then the statement of the proposition follows.

For each \( 1 \leq j < (q+1)/2 \), we fix a basis \{[i_j], [e_j]\} of the domain of \( \alpha_j \), which is
\[
\text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \cong \pi_1 \oplus \pi_1 (BS(O(j) \times O(q+1-j))) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]
Let \( i_j : S^1 \to \mathbb{R}^2 \) be an immersion with 1 double point and with trivial normal bundle induced from the bundle \( \varepsilon_{BS(O(j) \times O(q+1-j))}^1 \) by a constant map \( S^1 \to BS(O(j) \times O(q+1-j)) \).

Hence \([i_j]\) represents \((1, 0)\) in this \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Let \( e_j : S^1 \to \mathbb{R}^2 \) be an immersion without any multiple points and with the normal bundle induced from the bundle \( \varepsilon_{BS(O(j) \times O(q+1-j))}^1 \) by a map \( S^1 \to BS(O(j) \times O(q+1-j)) \) which represents the non-trivial element in
\[
\mathbb{Z}_2 \cong \pi_1 (BS(O(j) \times O(q+1-j))) = \pi_{N+1} (S^NBS(O(j) \times O(q+1-j))) \cong \mathbb{Z}
\]
\[
H_{N+1} (S^NBS(O(j) \times O(q+1-j)); \mathbb{Z}) = H_1 (BS(O(j) \times O(q+1-j)); \mathbb{Z}),
\]
where “\( S^N \)” denotes the \( N \)th suspension for a large \( N \). Then the index \( j \) indefinite fold singular set of \( \varphi_{j,q,e_j} \) is a circle whose normal bundle in \( O_{2q}^{j,q,e_j} \) has the gluing transformation which is orientation reversing on both of the \( O(j)- \) and \( O(q+1-j)- \) invariant subspaces, as one can see easily. Then \([e_j]\) represents \((0, 1)\) in \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

For a \( 1 \leq j' < (q+1)/2 \), what are \( \xi_{j'}^O \circ \alpha_j ([i_j]) \) and \( \xi_{j'}^O \circ \alpha_j ([e_j]) \)? We know that \( \alpha_j ([i_j]) = [\varphi_{j,q,i_j}] \) and \( \alpha_j ([e_j]) = [\varphi_{j,q,e_j}] \). The fold maps \( \varphi_{j,q,i_j} \) and \( \varphi_{j,q,e_j} \) have fold singularities of absolute indices 0, \( j - 1 \), and 1. Besides the fact that \( \xi_{j'}^O ([\varphi_{j,q,i_j}]) = [i_j] \) and \( \xi_{j'}^O ([\varphi_{j,q,e_j}]) = [e_j] \) always hold we have that \( \xi_{j'}^O ([\varphi_{j,q,i_j}]) \) or \( \xi_{j'}^O ([\varphi_{j,q,e_j}]) \) can be non-zero only if \( j' = j - 1 \). This shows that the matrix of
\[
(\xi_1^O, \ldots, \xi_{[q/2]}^O) \circ \sum_{1 \leq j < (q+1)/2} \alpha_j
\]
is an upper triangular matrix with 1s along the diagonal. This finishes the proof. \( \square \)

**Remark 6.3.** A little more information is that we have
\[
\xi_1^O ([\varphi_{2,q,i_2}]) = [i_1], \xi_2^O ([\varphi_{3,q,i_3}]) = [i_2], \ldots, \xi_{[q/2]-1}^O ([\varphi_{[q/2],q,i_{[q/2]-1}}]) = [i_{[q/2]-1}].
\]

To have the similar result for the \([e_j]\)s we have to see how the symmetry of the Morse function \( h_j \) which acts non-trivially on the critical point of index \( j \) acts on the critical point of index \( j - 1 \) for \( j - 1 \geq 1 \). By [Ka08, Proof of Lemma 3.2] we have that for \( 2 \leq j \leq [q/2] \),
\[
\xi_{j-1}^O ([\varphi_{j,q,e_j}]) = [e_{j-1}].
\]
For the unoriented case let
\[ \bar{\alpha}_j : \text{Imm} \left( \varepsilon_{B(O(j) \times O(q+1-j))}^1, 2 \right) \to \text{Cob}(2, q), \]
\[ 1 \leq j < (q+1)/2, \]
be the homomorphisms like the \( \alpha_j \). Note that
\[ \text{Imm} \left( \varepsilon_{B(O(j) \times O(q+1-j))}^1, 2 \right) \cong \pi^1_0 \oplus \pi^1_1 \left( B(O(j) \times O(q+1-j)) \right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2. \]

Similarly to the previous arguments, we obtain

**Proposition 6.4.** The homomorphism \( \sum_{1 \leq j < (q+1)/2} \bar{\alpha}_j \) is an isomorphism onto a direct summand of \( \text{Cob}(2, q) \). The group \( \text{Cob}(2, q) \) contains \( \mathbb{Z}_2^{q+2} \) as a direct summand.

**Proof.** The proof is completely analogous to the proof of Proposition 6.2.

For each \( 1 \leq j < (q+1)/2 \), we take the standard basis \( \{i_j, e_j^1, e_j^2\} \) of the domain \( \mathbb{Z}_2^q \) of \( \bar{\alpha}_j \). The immersions \( i_j, e_j^1 \) and \( e_j^2 \) are defined as follows. Each of them maps \( S^1 \) into \( \mathbb{R}^2 \), the immersion \( i_j \) has one double point, the immersions \( e_j^1 \) and \( e_j^2 \) have no multiple points.

The normal bundle of each of them is induced from the bundle \( \varepsilon_{B(O(j) \times O(q+1-j))}^1 \). The normal bundle of \( i_j \) is induced by the constant map \( S^1 \to B(O(j) \times O(q+1-j)) \), the normal bundle of \( e_j^1 \) by a map \( S^1 \to B(O(j) \times O(q+1-j)) \) which represents the element \((1, 0)\) in \( \pi^1_0 \left( B(O(j) \times O(q+1-j)) \right) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), i.e. twists the \( O(j) \) component but not the \( O(q+1-j) \) component, and the normal bundle of \( e_j^2 \) by a map \( S^1 \to B(O(j) \times O(q+1-j)) \) which represents the element \((0, 1)\) in \( \pi^1_1 \left( B(O(j) \times O(q+1-j)) \right) \) so it does not twist the \( O(q+1-j) \) component.

Then we construct the fold maps \( \tilde{\varphi}_{j,q,m} : \tilde{Q}^{2+q}_{j,q,m} \to \mathbb{R}^2 \) just like in the oriented case, where \( m \) runs over the elements of \( \{i_j, e_j^1, e_j^2 : 1 \leq j < (q+1)/2\} \).

Then we show that the matrix of the homomorphism
\[ (\xi_1, \ldots, \xi_{[q/2]}) \circ \sum_{1 \leq j < (q+1)/2} \bar{\alpha}_j \]
is non-singular (over the field \( \mathbb{Z}_2 \)). Details are left to the reader. \( \square \)

7. Fold maps into the plane and a Poincaré-Hopf type formula for the signature

7.1. Results about oriented fold maps. Let \( n = 2, q \geq 0 \) and let \( 0 \leq \lambda \leq (q+1)/2 \).

Recall from Section 3.2 that the group
\[ \pi^{q+1}_0 \oplus \pi^{q+1}_1 \left( B\left( O(\lambda) \times O(q+1-\lambda) \right) \right) \]
for \( 0 \leq \lambda < (q+1)/2 \) and the group
\[ \text{Imm}(\tilde{T}_1, 2) \]
for the case of \( q \) odd and \( \lambda = (q+1)/2 \), which are the targets of the homomorphisms \( \xi^O_\lambda \), are isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) if \( \lambda \geq 1 \) and to \( \mathbb{Z}_2 \oplus \{0\} \) if \( \lambda = 0 \). Recall that the first component of the homomorphism \( \xi^O_\lambda \), where \( 0 \leq \lambda \leq (q+1)/2 \), i.e. the homomorphism
\[ t_\lambda : \text{Cob}^O(2, q) \to \mathbb{Z}_2 \]
maps a fold cobordism class \( [f] \) to the cobordism class of the immersion
\[ f|_{S_\lambda(f)} : S_\lambda(f) \to \mathbb{R}^2 \]
of the 1-dimensional manifold $S_{\lambda}(f)$. The cobordism class of this immersion is an element\(^2\) in $\mathbb{Z}_2$. Simplifying the notation we often refer to $t_{\lambda}([f])$ as $t_{\lambda}(f)$. It is just the mod 2 number of double points of the $f$-image in $\mathbb{R}^2$ of the index $\lambda$ fold singular set of the generic fold map $f$.

The second component of the homomorphism $\xi_{\lambda}^O$ for $0 \leq \lambda \leq (q + 1)/2$ is

$$\tau: \text{Cob}^O(2,q) \to \mathbb{Z}_2,$$

which maps a fold cobordism class $[f]$ to the sum $\tau_{\lambda}([f]) = \sum_{r} \tau_{\lambda,r}([f]) \mod 2$, where $\tau_{\lambda,r}([f])$ is 0 if

- the $S(O(\lambda) \times O(q + 1 - \lambda))$ bundle is trivial in the case of $0 \leq \lambda < (q + 1)/2$ and
- the $S\left(O\left(\frac{q+1}{2}\right) \times O\left(\frac{q+1}{2}\right),T\right)$ bundle is trivial in the case of $q$ odd and $\lambda = (q + 1)/2$ over the $r$th component of the 1-dimensional manifold $S_{\lambda}(f)$, and 1 otherwise\(^3\), cf. Remark 3.3.

Remark 7.1. In other words, when $q$ is even, $\tau_{\lambda}([f])$ is equal to the first Stiefel-Whitney number $\langle w_1(\delta_{\lambda}(f)), [S_{\lambda}(f)] \rangle$ of the determinant bundle $\delta_{\lambda}(f)$ of the $O(\lambda)$ bundle obtained by the projection $S(O(\lambda) \times O(q + 1 - \lambda)) \to O(\lambda)$ over $S_{\lambda}(f)$.

We refer to $\tau_{\lambda}([f])$ as $\tau_{\lambda}(f)$ and say it is the twisting of the index $\lambda$ fold germs over $S_{\lambda}(f)$. Now, take the homomorphisms

$$t: \text{Cob}^O(2,q) \to \mathbb{Z}_2,$$

$$t = \sum_{0 \leq \lambda \leq (q + 1)/2} t_{\lambda}
$$

and

$$\tau: \text{Cob}^O(2,q) \to \mathbb{Z}_2,$$

$$\tau = \sum_{0 \leq \lambda \leq (q + 1)/2} \tau_{\lambda}.$$

Some results of [Ch80] can be reformulated as follows.

**Theorem 7.2 (Chess [Ch80]).** Let $f: M^{2k+1} \to \mathbb{R}^2$ be a fold map of a closed orientable manifold. Then

$$t(f) + \tau(f) \equiv \begin{cases} 0 \mod 2 & \text{if } k \text{ is odd,} \\ w_2w_{2k-1}[M^{2k+1}] \mod 2 & \text{if } k \text{ is even.} \end{cases}$$

We are looking for a similar result if the dimension of the source manifold of a fold map is divisible by 4. To achieve this, we will compute some related cobordism groups of fold maps.

Framed fold maps with even codimension are naturally identified with fold maps. Hence when $n = 2$, $q \geq 0$ and $q = 2q'$ is even, Theorem 5.1 says that $\mathbb{Z}_{2,q}^O$ is an injective homomorphism from the group $\text{Cob}^O(2,q)$ to the group

$$\mathbb{C}^O_{2+q}(2) \oplus \pi_1^O \oplus \pi_1^O \left( BS\left(O(1) \times O(q)\right) \oplus \cdots \oplus \pi_1^O \oplus \pi_1^O \left( BS\left(O\left(\frac{q}{2}\right) \times O\left(\frac{q}{2} + 1\right)\right)\right).$$

This large direct sum is actually isomorphic to $\mathbb{C}^O_{2+q}(2) \oplus \mathbb{Z}_2^q$. Then we have the following.

\(^2\)If the immersion $f|_{S_{\lambda}(f)}$ is in general position, then its cobordism class is equal to the number of its double points modulo 2. Since our fold maps are in general position, $f|_{S_{\lambda}(f)}$ is also in general position.

\(^3\)This implies that $\tau_0: \text{Cob}^O(2,q) \to \mathbb{Z}_2$ is always the zero homomorphism.
Theorem 7.3. For \( k \geq 1 \), the group \( \text{Cob}^O(2, 4k - 2) \) is isomorphic to \( \Omega_{4k}^{2|X} \oplus \mathbb{Z}_2^{4k-2} \). An isomorphism is given by the map

\[
[f: M^{4k} \rightarrow \mathbb{R}^2] \mapsto \left( [M^{4k}], t_1(f), \tau_1(f), \ldots, t_{2k-1}(f), \tau_{2k-1}(f) \right).
\]

For example, the group \( \text{Cob}^O(2, 2) \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), and an isomorphism is given by the homomorphism

\[
[f: M^4 \rightarrow \mathbb{R}^2] \mapsto \left( \sigma(M^4)/2, [f|_{S_0(f)}], [f|_{S_1(f)}] \right).
\]

The next proposition will be important for us. We will prove later.

Proposition 7.4. For \( k \geq 1 \) there is an oriented fold map \( f: \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \rightarrow \mathbb{R}^2 \) such that \( t(f) \equiv 0 \mod 2 \) and \( \tau(f) \equiv 1 \mod 2 \).

Proof. This is the same statement as Proposition 8.4, see the proof there. \( \square \)

We obtain the following Poincaré-Hopf type formula for the signature.

Theorem 7.5. Let \( k \geq 1 \) and \( f: M^{4k} \rightarrow \mathbb{R}^2 \) be a fold map of a closed oriented \( 4k \)-dimensional manifold. Then

\[
\frac{\sigma(M^{4k})}{2} \equiv t(f) + \tau(f) \mod 2.
\]

Corollary 7.6. An isomorphism \( \text{Cob}^O(2, 2) \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2^2 \) is also given by the map

\[
[f: M^4 \rightarrow \mathbb{R}^2] \mapsto \left( \sigma(M^4)/2, [f|_{S_0(f)}], [f|_{S_1(f)}] \right).
\]

Proof. This follows from Theorems 7.3 and 7.5. \( \square \)

7.2. Results about non-oriented fold maps. Now, for the unoriented case take the homomorphisms

\[
t_\lambda: \text{Cob}(2, 4k - 2) \rightarrow \mathbb{Z}_2
\]

and

\[
\tau_\lambda: \text{Cob}(2, 4k - 2) \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2,
\]

where \( 0 \leq \lambda \leq 2k - 1 \). Define \( \tilde{\tau}_\lambda \) as the product \( \tau_1^1 \tau_2^\lambda \) taken in the field \( \mathbb{Z}_2 \) of the two components of \( \tau_\lambda \) and define

\[
\tilde{\tau} = \sum_{0 \leq \lambda \leq 2k - 1} \tilde{\tau}_\lambda.
\]

It follows easily that for \( 0 \leq \lambda \leq 2k - 1 \) we have the commutative diagrams

\[
\begin{array}{ccc}
\text{Cob}^O(2, 4k - 2) & \overset{t}{\longrightarrow} & \text{Cob}(2, 4k - 2) \\
\downarrow t_\lambda & & \downarrow t_\lambda \\
\mathbb{Z}_2 & & \mathbb{Z}_2
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Cob}^O(2, 4k - 2) & \overset{\tau}{\longrightarrow} & \text{Cob}(2, 4k - 2) \\
\downarrow \tau_\lambda & & \downarrow \tilde{\tau}_\lambda \\
\mathbb{Z}_2 & & \mathbb{Z}_2
\end{array}
\]
where the horizontal arrows denoted by \( \iota \) are the natural forgetful homomorphisms. Of course \( \tilde{\tau}_\lambda \) is not necessarily a homomorphism. But the commutative diagram above says that \( \tilde{\tau}_\lambda \) restricted to the image \( \iota (\text{Cob}^O(2, 4k-2)) \) is a homomorphism.

Notice that for \( 0 \leq \lambda \leq 2k-1 \) the composition

\[
\text{Cob}^O(2, 4k-2) \longrightarrow \text{Cob}(2, 4k-2) \xrightarrow{\tau_1^\lambda + \tau_2^\lambda} \mathbb{Z}_2
\]

is identically zero. Here the first arrow is the forgetful homomorphism and \( \tau_i^\lambda \), \( i = 1, 2 \), denote the two components of \( \tau_\lambda \). We obtain results in the unoriented case analogously to Theorems 7.3 and 7.5.

**Theorem 7.7.** For \( k \geq 1 \), the cobordism group \( \text{Cob}(2, 4k-2) \) is isomorphic to \( \mathbb{N}_{4k}^1 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{6k-3}} \).

**Theorem 7.8.** Let \( k \geq 1 \) and \( f: M^{4k} \rightarrow \mathbb{R}^2 \) be a fold map of a closed (possibly unoriented) \( 4k \)-dimensional manifold. Then

\[
\tau_1(f) \equiv \tau_2(f) \mod 2.
\]

This statement involves no number of double points and strictly speaking no topological properties of the manifold \( M \), it relates only the twistings \( \tau_1 \) and \( \tau_2 \) to each other.

8. **Proof of the results**

8.1. **Proof for the cobordism invariants.** At first, we prove Theorem 5.1.

**Proof of Theorem 5.1.** Let \( f: Q^{n+q} \rightarrow \mathbb{R}^n \) be a framed fold map. We show that if

\[
\Im([f]) = (\sigma_{n,q}([f]), \xi_1([f]), \ldots, \xi_{[(q+1)/2]}([f]))
\]

is zero, then \([f] \in \text{Cob}_{fr}(n, q)\) is also zero. Take the map

\[
f \times \text{id}_{[0, 3\varepsilon]}: Q \times [0, 3\varepsilon] \rightarrow \mathbb{R}^n \times [0, 3\varepsilon]
\]

for some small \( \varepsilon > 0 \).

Recall that

\[
\varphi: \mathbb{R}^{q+1} \rightarrow \mathbb{R},
\]

\[
\varphi(x_1, \ldots, x_{q+1}) = \sum_{i=1}^{\lambda} -x_i^2 + \sum_{i=\lambda+1}^{q+1} x_i^2
\]

is the fold singularity of index \( \lambda \). By assumption

\[
\xi_\lambda([f]) \in \text{Imm} \left( \mathbb{Z}^1_{B(O(\lambda) \times O(q+1-\lambda))} \right)_n
\]

is zero for \( 1 \leq \lambda \leq (q+1)/2 \) so we can glue these given null-cobordisms of the \( \varphi \)-bundles of \( f \) to the map

\[
f \times \text{id}_{[0, 3\varepsilon]} \text{ “at” } Q \times \{3\varepsilon\}.
\]

Such a null-cobordism is a \( \varphi \)-bundle over an \( n \)-dimensional manifold \( \Sigma_\lambda \) whose boundary \( \partial \Sigma_\lambda \) is the fold singular set \( S_\lambda(f) \). In other words, there is an \( \mathbb{R}^{q+1} \) bundle over \( \Sigma_\lambda \) with the total space \( O_\lambda \), and an \( \mathbb{R} \) bundle over \( \Sigma_\lambda \) with the total space \( R_\lambda \) and there is a map \( \Phi_\lambda: O_\lambda \rightarrow R_\lambda \) which maps fiberwise as \( \varphi \). Also \( \Sigma_\lambda \) is immersed
into $\mathbb{R}^n \times [0, 1)$ so that this immersion restricted to the boundary $S_{\lambda}(f)$ is the immersion $f|_{S_{\lambda}(f)}$ into $\mathbb{R}^n \times \{0\}$. So we have a commutative diagram

$$
\begin{array}{cccc}
DN(S_{\lambda}(f)) & \subset & DO_{\lambda} & \Phi_{\lambda}|DO_{\lambda} \\
\downarrow & & \downarrow & \downarrow \\
N(S_{\lambda}(f)) & \subset & O_{\lambda} & \Phi_{\lambda} \\
\downarrow & & \downarrow & \downarrow \\
S_{\lambda}(f) & \subset & \Sigma_{\lambda} & g \rightarrow \mathbb{R}^n \times [0, 1)
\end{array}
$$

where a lot of arrows are inclusions as it is denoted, $g$ is the immersion of $\Sigma_{\lambda}$ into $\mathbb{R}^n \times [0, 1)$ and $N(S_{\lambda}(f))$ is an open tubular neighborhood of $S_{\lambda}(f)$. The prefix “$D$” denotes the corresponding closed disk bundles. The composition $\Phi_{\lambda} \circ i$ is the given $\varphi$-bundle of $f$. The composition $g \circ j$ is the immersion $f|_{S_{\lambda}(f)}$.

Moreover the line bundle $R_{\lambda}$ over $\Sigma_{\lambda}$, which is a trivial line bundle since we are working with framed fold maps, is the normal bundle of the immersion $g: \Sigma_{\lambda} \rightarrow \mathbb{R}^n \times [0, 1)$.

So attaching the null-cobordisms to $f \times \text{id}_{[0, 3\varepsilon)}$ we obtain a framed fold map

$$
\tilde{F}: V \rightarrow \mathbb{R}^n \times [0, 1),
$$

where $V = Q \times [0, 3\varepsilon) \cup_{1 \leq \lambda \leq (q+1)/2} O_{\lambda}$ is a non-compact $(n + q + 1)$-dimensional manifold with boundary $Q$.

We also take the disk bundle $DO_{\lambda}$ of $O_{\lambda}$ and restrict the map $\Phi_{\lambda}$ to it. In this way, we obtain the restricted map $F: W \rightarrow \mathbb{R}^n \times [0, 1)$, where

1. $W \subset V$ and $W$ is a compact $(n + q + 1)$-dimensional manifold obtained from attaching to $Q \times [0, 2\varepsilon)$ the spaces $DN(S_{\lambda}(f)) \times [2\varepsilon, 3\varepsilon]$ and then the (restricted) domains $DO_{\lambda}$ of the null-cobordisms $\Phi_{\lambda}$ of the index $\geq 1$ fold singularity bundles of $f$,

2. the boundary of $W$ is equal to $Q \amalg P$, where the closed $(n + q)$-dimensional manifold $P$ is diffeomorphic to the union of

   - $Q - \bigcup_{1 \leq \lambda \leq (q+1)/2} DN(S_{\lambda}(f))$ and
   - $\bigcup_{1 \leq \lambda \leq (q+1)/2} SO_{\lambda}$, where the prefix “$S$” denotes the corresponding sphere bundles,

3. $F|_{Q \times [0, \varepsilon]} = f \times \text{id}_{[0, \varepsilon)}$, where $Q \times [0, \varepsilon)$ is a small collar neighborhood of $Q$ ($\subset \partial W$) in $W$ with the identification $Q = Q \times \{0\}$ and

4. $F$ is a restriction of $\tilde{F}$ and $\tilde{F}$ is a fold map with only definite fold singularities into $\mathbb{R}^n \times (0, 1)$ near $P$ and

5. the definite fold singularities of $F$ are of the form $S_0(f) \times [0, 2\varepsilon]$ in $Q \times [0, 2\varepsilon]$ and mapped by

   $$(f \times \text{id}_{[0, 2\varepsilon]})|_{S_0(f) \times [0, 2\varepsilon]}$$

see Figure[1].

Since $F|_{V}$ is a framed fold map, fixing a Riemannian metric $g$ on $W$ we obtain a

$$
\varphi(F, g, r): TW \oplus \varepsilon_W^1 \rightarrow T(\mathbb{R}^n \times [0, 1))
$$
fiberwise epimorphism for some small $r > 0$ as in Section 2.2. This gives a stable $n$-framing on $W$. We want this stably $n$-framed $W$ to be a cobordism between $Q$ and $P$ in the sense of Definition 4.2 but while the framing of $W$ restricts to the framing of $Q$ as Definition 4.2 requires, we do not get immediately a stable $(n-1)$-framing of $P$. This is because the framing of $W$ gives only $n+1$ linearly independent sections of $T_P \oplus \epsilon^2_P$ such that we do not know whether one of these sections is a normal section of $P$. This problem occurs because $P$ is not mapped by $F$ into some $\mathbb{R}^n \times \{t\}$, where $t \in \mathbb{R}$, while $Q$ is. But since we have $\pi_k \left( V_{n+1} (\mathbb{R}^{n+q+2}), V_n (\mathbb{R}^{n+q+1}) \right) = 0$ for the relative homotopy groups of Stiefel varieties for $k \leq n + q$ (this follows from [Hn93, Chapter 8.11.]), there is a homotopy of these $n+1$ linearly independent sections of $TP \oplus \epsilon^2_P$ through linearly independent sections such that at the end we obtain $n+1$ linearly independent sections of $TP \oplus \epsilon^2_P$, with the last section being parallel to the last $\epsilon^1_P$ summand. Deleting this we get $n$ linearly independent sections of $TP \oplus \epsilon^2_P$, and $W$ becomes a cobordism between $Q$ and $P$ in the sense of Definition 4.2. Suppose this homotopy happens over $P \times [0,1]$. We identify our $P$ with $P \times \{0\}$. Then attach this homotopy to $W$ along $P$. Hence $Q$ and $P \times \{1\}$ are stably $(n-1)$-framed cobordant by $W$ and the attached homotopy $P \times [0,1]$.

It follows that since $\sigma_{n,q}(\{f\})$ is zero, the stably $(n-1)$-framed manifold $P \times \{1\}$ is also zero in the cobordism group $C_{n+q}(n)$. So by gluing a stably $(n-1)$-framed null-cobordism of $P \times \{1\}$ to $W \cup (P \times [0,1])$ along $P \times \{1\}$, we obtain a compact $(n+q+1)$-dimensional manifold $X$ with boundary $Q$ such that the bundle $TX \oplus \epsilon^1_X$ has an $(n+1)$-framing which coincides with the stable $n$-framing of $W$ over $W$.

Since $\mathbb{R}^n \times [0,1]$ is contractible, we can extend the map $F$ to a continuous map $G: X \to \mathbb{R}^n \times [0,1]$.

Observe that if we introduce a Riemannian metric on $X$ extending the given Riemannian metric on $W$, then the $(n+1)$-framing of the bundle $TX \oplus \epsilon^1_X$ gives a fiberwise epimorphism

$$H: TX \oplus \epsilon^1_X \to T(\mathbb{R}^n \times [0,1])$$
covering the continuous map $G$ by mapping the $n + 1$ frames to the standard bases of $T(\mathbb{R}^n \times [0, 1])$ at any points of $X$. This $H$ coincides with $\varphi(F, g, r)$ over $W$ because we got $H$ by the framing determined by $\varphi(F, g, r)$.

Hence by Theorem 2.6 we see that there is a framed fold map $G': X \to \mathbb{R}^n \times [0, 1]$ which coincides with $F$ on a closed subset of $W$ which contains some $Q \times \{t\}$ for some $t \in [0, 2\varepsilon]$. So the framed fold map $f$ is framed null-cobordant. The oriented case is proved in a similar way.

8.2. Proof for the cobordism group of oriented fold maps into the plane. In order to prove Theorem 7.3 (and later Theorem 7.7 in the unoriented case), we need the following lemma.

**Lemma 8.1.** Let $A, B, C$ be abelian groups and let $\iota_B: B \to A \oplus B$ be the standard inclusion. Let $\psi: A \oplus B \to C$ be a surjective homomorphism and let $\pi_B: A \oplus B \to B$ be a homomorphism such that $\pi_B \circ \iota_B$ is an isomorphism. Suppose $(\psi, \pi_B): A \oplus B \to C \oplus B$ is injective.

1. If $\text{im}(\psi \circ \iota_B) = 0$, then $(\psi, \pi_B)$ is also surjective so $A \oplus B$ is isomorphic to $C \oplus B$.
2. If $\text{im}(\psi \circ \iota_B) = \mathbb{Z}_2$ and $(\psi, \pi_B)$ is not surjective, then $A \oplus B$ is isomorphic to $C/\text{im}(\psi \circ \iota_B) \oplus B$.

**Proof.** Let $q: C \to C/\text{im}(\psi \circ \iota_B)$ denote the quotient map and let $id_B: B \to B$ denote the identity map of $B$. We study the composition

$$A \oplus B \xrightarrow{(\psi, \pi_B)} C \oplus B \xrightarrow{q \oplus id_B} C/\text{im}(\psi \circ \iota_B) \oplus B.$$ 

We show that it is surjective. Take an element $(x, y) \in C/\text{im}(\psi \circ \iota_B) \oplus B$. There is a $c \in C$ such that $q(c) = x$ and there is an $(a, b) \in A \oplus B$ such that $\psi(a, b) = c$. Then $\pi_B(a, b) = b'$ for some $b' \in B$. Choose $b'' \in B$ such that $\pi_B \circ \iota_B(b'') = y - b'$. This means that $\pi_B(0, b'') = y - b'$. Then $\psi(a, b + b'') = \psi(a, b) + \psi(0, b'')$ and $\pi_B(a, b + b'') = \pi_B(a, b) + \pi_B(0, b'')$, so

$$q \oplus id_B \circ (\psi, \pi_B)(a, b + b'') = q \oplus id_B \left(\psi(a, b) + \psi(0, b''), \pi_B(a, b) + \pi_B(0, b'')\right) = q \oplus id_B \left(\psi(a, b) + \psi(0, b''), y\right) = \left(q \left(\psi(a, b)\right) + q \left(\psi(0, b'')\right)\right), y = (x, y)$$

because $q(\psi(0, b'')) = 0$. So $q \oplus id_B \circ (\psi, \pi_B)$ is surjective. This immediately implies (1) since then $q \oplus id_B$ is the identity map.

To see that (2) also holds note that if we have two groups $G$ and $H$, $H$ has a subgroup $\mathbb{Z}_2 \subset H$ and we have the composition

$$G \to H \to H/\mathbb{Z}_2,$$

where the first arrow, which we denote by $\alpha$, is injective but not surjective, the second arrow is the quotient map and the composition is surjective, then $G$ is isomorphic to $H/\mathbb{Z}_2$. This is because $\alpha$ has to hit all the classes $h + \mathbb{Z}_2$ and if there is a $g \in G$ which goes to $1 \in \mathbb{Z}_2 \subset H$, then for any $h \in H$ such that $h \in \text{im} \alpha$ we have that $h + 1 \in \text{im} \alpha$. So $\alpha$ would be surjective contradicting our original assumption. Hence $1 \in \mathbb{Z}_2$ is not in $\text{im} \alpha$ so the composition is injective.

---

4If $G, G', H, H'$ are abelian groups and $\alpha: G \to G'$ and $\beta: H \to H'$ are homomorphisms, then let $\alpha \oplus \beta: G \oplus H \to G' \oplus H'$ denote the homomorphism which maps $(g, h) \in G \oplus H$ to $(\alpha(g), \beta(h))$. 

Now we prove Theorem 7.3.
Proof of Theorem 7.3. Recall that by Theorem 5.1 for $k \geq 1$ the homomorphism $\mathcal{O}_{2,4k-2}$ is injective. We will apply Lemma 8.1. By Proposition 6.2 in Section 6 (see also [Ka08, Theorem 3.1 and Remark 3.3]) the group $\text{Cob}^O(2, 4k - 2)$ contains the group

$$\bigoplus_{1 \leq j \leq 2k-1} \text{Imm} \left( \varepsilon_{BS(O(j) \times O(4k-1-j))}, 2 \right)$$

as a direct summand, we denote this group by $B$, so $\text{Cob}^O(2, 4k - 2) \cong A \oplus B$ for some group $A$. With these roles we will apply Lemma 8.1. The homomorphism $(\xi^O_1, \ldots, \xi^O_{2k-1})$ will be $\pi_B$.

By Proposition 4.9 the group $\mathcal{O}^O_{4k}(2)$, which will be the group $C$, is isomorphic to the group $\Omega^O_{2k|x}$ and the forgetful map $\mathcal{C}_{4k}(2) \to \Omega^O_{2k|x}$ which forgets the framings gives an isomorphism. Since by [Le65] every closed orientable manifold of dimension $> 2$ with even Euler characteristic has a fold map into the plane, the homomorphism $\sigma^O_{2,4k-2}: \text{Cob}^O(2, 4k - 2) \to \mathcal{C}_{4k}(2)$, which will be $\psi: A \oplus B \to C$ when we apply Lemma 8.1 is surjective.

Now, let us apply Lemma 8.1. By case (1) we have, but also it is quite obvious, that to finish the proof of Theorem 7.3 it is enough to show that the source manifolds $B$ (see Section 6) represent zero in $\Omega^O_{2k|x}$. But the manifolds $Q^t_{j,4k-2,i,j}$ and $Q^t_{j,4k-2,e,j}$ are fibrations over the circle $S^1$ with fiber the $(4k - 1)$-dimensional sphere with orientation preserving linear structure group, hence they are null-cobordant.

8.3. Proof of the Poincaré-Hopf type formula. Now, we prove Theorem 7.5.

Proof of Theorem 7.5. By Theorem 7.3 the group $\text{Cob}^O(2, 4k - 2)$ is isomorphic to $\mathcal{O}^O_{4k} \oplus \mathbb{Z}_2^{2k-2}$.

We check the values

$$[f_\gamma | s_j] + \sum_{j=1}^{2k-1} \tau_j(f_\gamma)$$

and

$$\frac{\sigma(M^k_\gamma)}{2} \mod 2$$

for a system of generators $\{[f_\gamma: M^k_\gamma \to \mathbb{R}^2]\}_{\gamma \in \Gamma}$ of the cobordism group $\text{Cob}^O(2, 4k - 2)$. A system of representatives of the generators of the $\mathbb{Z}_2^{2k-2}$ summand of $\text{Cob}^O(2, 4k - 2)$ is given by the fold maps $\varphi_{j,4k-2,i,j}$ and $\varphi_{j,4k-2,e,j}$, where $[i_j]$ and $[e_j]$ are the generators of the group

$$\text{Imm} \left( \varepsilon_{BS(O(j) \times O(4k-1-j))}, 2 \right) \cong \mathbb{Z}_2,$$

$1 \leq j \leq 2k - 1$, see Section 6. Note that the source manifolds of $\varphi_{j,4k-2,i,j}$ and $\varphi_{j,4k-2,e,j}$ are oriented null-cobordant for $1 \leq j \leq 2k - 1$.

To generate the direct summand $\mathcal{O}^O_{2k|x}$ of $\text{Cob}^O(2, 4k - 2)$ as well, we construct a fold map into the plane of each element of a system of representatives of generators of the group $\mathcal{O}^O_{2k|x}$ as follows. Let us take a class $\omega \in \mathcal{O}^O_{2k|x}$. It can be written in the form

$$\omega = r[CP^{2k} \# CP^{2k}] + [Z^{4k}],$$
where \( r \geq 0 \), the manifold \( \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \) is oriented in some way and the signature \( \sigma(Z^{4k}) \) is equal to zero. By [AK80, Theorem 3], we can suppose that \( Z^{4k} \) is a fiber bundle over \( S^2 \) with a closed orientable \((4k-2)\)-dimensional manifold \( F^{4k-2} \) as fiber.

**Proposition 8.2.** There is an oriented fold map \( f_z : Z \to \mathbb{R}^2 \) such that

\[
t(f_z) \equiv 0 \mod 2
\]

and

\[
\tau(f_z) \equiv 0 \mod 2.
\]

**Proof.** Clearly there exists a Morse function \( \mu : F^{4k-2} \times [0,1] \to [0,1] \) such that \( \mu \) has no singularities near \( F^{4k-2} \times \{0\} \sqcup F^{4k-2} \times \{1\} \), \( \mu \) takes its maximum 1 on \( F^{4k-2} \times \{0\} \sqcup F^{4k-2} \times \{1\} \), and \( \mu^{-1}(1-t) = F^{4k-2} \times \{t\} \) for \( 0 \leq t \leq 1/4 \).

Let \( f_z \) be the fold map of \( F^{4k-2} \times [0,1] \times S^1 \) into \( \mathbb{R}^2 \) defined by

\[
f_z(u,s,e^{i\theta}) = (2 - \mu(u(s)))e^{i\theta}
\]

for \( u \in F^{4k-2}, s \in [0,1], \theta \in [0,2\pi] \), where we identify \( S^1 \) with the unit circle \( \{e^{i\theta} : \theta \in [0,2\pi]\} \). So we have the fold map

\[
f_z : F^{4k-2} \times [0,1] \times S^1 \to \{1 \leq |x| \leq 2\} \subset \mathbb{R}^2,
\]

which maps the boundary \( (F^{4k-2} \times \{0\}) \sqcup F^{4k-2} \times \{1\}) \times S^1 \) into the unit circle \( \{|x| = 1\} \).

Let \( D^2_+ \) and \( D^2_- \) denote the northern and southern hemispheres of \( S^2 \), respectively. According to the decomposition

\[
D^2_+ \cup D^2_- = S^2,
\]

the bundle \( F^{4k-2} \to Z^{4k} \to S^2 \) falls into two pieces, i.e. into the trivial \( F^{4k-2} \) bundles \( p_+: Z^{4k}_+ \to D^2_+ \) and \( p_-: Z^{4k}_- \to D^2_- \). Let us identify \( D^2_+ \) with the standard unit disk \( \{|x| \leq 1\} \) and \( D^2_- \) with the disk \( \{|x| \leq 2\} \) of radius 2 in \( \mathbb{R}^2 \).

We define the fold map

\[
f_z : Z^{4k} \to \mathbb{R}^2
\]

by

- \( f_z|_{Z^{4k}_+} = p_+ \),
- \( f_z|_{p_-^{-1}(|x| \leq 1)} = p_- \) and
- \( f_z|_{p_-^{-1}(|1 \leq x| \leq 2)} = \tilde{f}_z \), where \( p_-^{-1}(|1 \leq |x| \leq 2\}) = F^{4k-2} \times \{1 \leq |x| \leq 2\} \) and the annulus \( \{|1 \leq |x| \leq 2\} \) is identified with \( [0,1] \times S^1 \)

so that the resulting map \( f_z \) is a fold map.

**Lemma 8.3.** We have \( [f_z|S_{f_z}] \equiv 0 \mod 2 \) and \( \sum_{j=1}^{2k-1} \tau_j(f_z) \equiv 0 \mod 2 \).

**Proof.** It is easy to see that the \( f_z \)-image of the fold singular set of \( f_z \) consists of concentric circles, moreover each determinant bundle \( \delta_j(f_z) \), see Remark 7.1, is trivial if the set \( S_j(f_z) \) is non-empty, for \( 1 \leq j \leq 2k-1 \).

This completes the proof of Proposition 8.2. \( \Box \)

**Proposition 8.4.** Let \( k \geq 1 \). There is an oriented fold map \( f : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2 \) such that

\[
t(f) \equiv 0 \mod 2
\]

and

\[
\tau(f) \equiv 1 \mod 2.
\]
Proof. Let us orient $\mathbb{C}P^{2k}$ in the standard way. We define the fold map $f_C: \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2$ as follows. By [Le65] there is a stable map

$$g: \mathbb{C}P^{2k} \to \mathbb{R}^2$$

with only one cusp point $p \in \mathbb{C}P^{2k}$ and also we know that the index of the fold singularities around this cusp is $2k - 1$. Our plan is to “eliminate” these two cusps in the “two copies” of $\mathbb{C}P^{2k}$ in the connected sum $\mathbb{C}P^{2k} \# \mathbb{C}P^{2k}$.

Since the singular set of $g$ is connected, we can suppose that there is an embedded arc $a: [0, 1] \to \mathbb{R}^2$ such that $a(0) = g(p)$, $a(1) \in \mathbb{R}^2 - g(\mathbb{C}P^{2k})$ and $a((0, 1))$ intersects the image of the singular set of $g$ transversally and exactly at one definite fold value.

Take a small tubular neighborhood of $a([0, 1])$ in $\mathbb{R}^2$. Then the boundary $C$ of this neighborhood is a circle embedded into $\mathbb{R}^2$ which divides $g(\mathbb{C}P^{2k})$ into two regions. (Of course $C \cap g(\mathbb{C}P^{2k})$ is an embedded interval in $\mathbb{R}^2$.) One of them contains $g(p)$ and the other does not. Denote the region containing $g(p)$ by $R$, see Figure 2.

![Figure 2](image_url)

**Figure 2.** The $g$-image of $\mathbb{C}P^{2k}$ in $\mathbb{R}^2$. The thick arcs represent the $g$-image of the fold singular set going into the cusp value $g(p)$. The arc connecting $g(p) = a(0)$ and $a(1)$ intersects transversally the $g$-image of the definite fold singular set.

Then the preimage $g^{-1}(R)$ is an embedded $4k$-dimensional ball in $\mathbb{C}P^{2k}$. Moreover the $g$-preimage of $C \cap g(\mathbb{C}P^{2k})$ is the boundary of this $4k$-dimensional ball, which is an $S^{4k-1}$. And if we identify $C \cap g(\mathbb{C}P^{2k})$ with $\mathbb{R}$, then the map $g$ restricted to this $S^{4k-1}$ is a Morse function with four critical points: two definite critical points, one critical point of index $2k - 1$ and one critical point of index $2k$. This Morse function gives a handle decomposition of $S^{4k-1}$. Since the two middle critical points form a cancelling pair (because of the cusp point), the $(2k - 1)$-handle is attached to the 0-handle such that the attaching sphere is the standard $(2k - 2)$-dimensional sphere with trivial framing. Watching the Morse function “upside down” we have the same thing about the other handles.

So after identifying $g^{-1}(C \cap g(\mathbb{C}P^{2k}))$ with $S^{4k-1}$ and $C \cap g(\mathbb{C}P^{2k})$ with $\mathbb{R}$, we get the Morse function $g|_{g^{-1}(C \cap g(\mathbb{C}P^{2k}))}$, which we denote by $h$. By the previous argument, we can suppose that

$$S^{4k-1} = \{(x_1, \ldots, x_{4k})| = 1\}$$
is given in the form
\[ S^{2k-1} \times D^{2k} \cup D^{2k} \times S^{2k-1}, \]
where \( S^{2k-1} \times D^{2k} \) and \( D^{2k} \times S^{2k-1} \) are identified with the subsets \( \{ x_{2k+1}^2 + \cdots + x_{2k}^2 \leq 1/2 \} \) and \( \{ x_1^2 + \cdots + x_{2k}^2 \leq 1/2 \} \) of \( \{ \| (x_1, \ldots, x_{4k}) \| = 1 \} \), respectively. We can suppose that this decomposition coincides with the handle decomposition of \( S^{4k-1} \) given by the Morse function \( h \), \( h^{-1}(0) = \{ x_1^2 + \cdots + x_{2k}^2 = x_{2k+1}^2 + \cdots + x_{4k}^2 = 1/2 \} \), the critical point of index \( 2k-1 \) is \( (1,0,\ldots,0) \in S^{2k-1} \times \{ 0 \} \), and the critical point of index \( 2k \) is in \( \{ 0 \} \times S^{2k-1} \) with \( (2k+1) \)-st coordinate equal to 1 and other coordinates equal to 0.

Now let us take the map \( g \circ (T \circ g) : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2 \), where \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is an affine translation such that the image of \( T \circ g \) is disjoint from the image of \( g \). Since \( T \circ g : \mathbb{C}P^{2k} \to \mathbb{R}^2 \) is just a copy of the map \( g \), we also get a copy of the Morse function \( h \) applying all the previous constructions to \( T \circ g \) instead of \( g \). Roughly speaking, the Morse function \( h \) is
\[ g \vert_{T \circ g}(\mathbb{C}P^{2k}) \]
and this other Morse function is
\[ T \circ g \vert_{(T \circ g)^{-1}(T(C) \cap T \circ g(\mathbb{C}P^{2k}))}, \]
which can be naturally identified with \( h \).

Now, we want to form the connected sum \( \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \) and obtain a map
\[ f_C : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2, \]
which “coincides” with \( g \vert_{g^{-1}(\mathbb{R}^2-\mathcal{R})} \) on the first \( \mathbb{C}P^{2k} \) summand and with \( T \circ g \vert_{(T \circ g)^{-1}(\mathbb{R}^2-T(\mathcal{R}))} \) on the second \( \mathbb{C}P^{2k} \) summand. All we need is an automorphism
\[ \left( \varphi : S^{4k-1} \to S^{4k-1}, \psi : \mathbb{R} \to \mathbb{R} \right) \]
of the Morse function \( h : S^{4k-1} \to \mathbb{R} \), where both of \( \varphi \) and \( \psi \) reverse the orientation. If we have this automorphism, we can take \( g \vert_{g^{-1}(\mathbb{R}^2-\mathcal{R})} \) and \( T \circ g \vert_{(T \circ g)^{-1}(\mathbb{R}^2-T(\mathcal{R}))} \) and glue them together along the two Morse functions by this automorphism. Then we get \( f_C : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2 \), see Figure 3

We define the diffeomorphism \( \varphi : S^{4k-1} \to S^{4k-1} \) to be induced by the linear transformation
\[ (x_1, \ldots, x_{4k}) \mapsto (x_{2k+1}, \ldots, x_{4k}, x_1, \ldots, x_{2k-1}, -x_{2k}). \]
Clearly \( \varphi \) interchanges the critical points of indices \( 2k-1 \) and \( 2k \), maps the unstable and stable manifolds into the stable and unstable manifolds respectively, while reversing the orientations of the unstable manifolds.

Hence we get \( f_C : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \to \mathbb{R}^2 \), which is a stable fold map.

**Lemma 8.5.** We have \( [f_C \vert_{S_{f_C}}] \equiv 0 \mod 2 \) and \( \sum_{j=1}^{2k-1} \tau_j(f_C) \equiv 1 \mod 2. \)

**Proof.** Since we obtained the fold map \( f_C \) from two copies of the map \( g \), and the performed operations did not change the number of double points of the image of the singular sets, it is clear that \( [f_C \vert_{S_{f_C}}] \equiv 0 \mod 2. \)

We also have that \( \varphi \) interchanges the critical points of indices \( 2k-1 \) and \( 2k \), maps the unstable and stable manifolds into the stable and unstable manifolds respectively, while reversing the orientations of the unstable manifolds. Hence the twisting modulus 2 of the indefinite fold germ bundle is 1 over the component of \( S_{f_C} \) which we obtain after constructing \( f_C \) and which goes through the “connecting tube” of \( \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \). The
Figure 3. The $f_C$-image of $\mathbb{C}P^{2k} \# \mathbb{C}P^{2k}$ in $\mathbb{R}^2$ around the $f_C$-image of the “connecting tube between the two $\mathbb{C}P^{2k}$ summands”. The thick arcs represent the $f_C$-image of the fold singular set. The gluing of the “connecting tube” realizes the automorphism $(\varphi, \psi)$.

Twisting modulo 2 equal to 0 over the union of the other components, because each of the other components of $S_{f_C}$ in the first $\mathbb{C}P^{2k}$ summand has an identical pair in the second $\mathbb{C}P^{2k}$ summand. □

This lemma completes the proof of Proposition 8.4. □

Recall the definition of the fold maps $\varphi_{j,4k-2,i_j}$ and $\varphi_{j,4k-2,e_j}$ in Section 6.

Lemma 8.6. The fold maps $\varphi_{j,4k-2,i_j}$ and $\varphi_{j,4k-2,e_j}$, where $[i_j]$ and $[e_j]$ are the generators of the group $\text{Imm} \left( S_{BS(O(j) \times O(4k-1-j))}, 2 \right) \cong \mathbb{Z}_2$, satisfy for $1 \leq j \leq 2k-1$ the congruence $\sigma \equiv t + \tau \mod 2$ of Theorem 7.7.

Proof. Concerning the signature: The source manifolds of $\varphi_{j,4k-2,i_j}$ and $\varphi_{j,4k-2,e_j}$ are null-cobordant, since they are sphere bundles with linear structure groups. Hence

$$\sigma(Q_{j,4k-2,i_j}^{4k-2}) = \sigma(Q_{j,4k-2,e_j}^{4k-2}) = 0.$$ 

Concerning the double points: By construction, $\varphi_{j,4k-2,e_j}$ restricted to its fold singular set has 1 double point if $j = 1$ (the double point of the definite fold singular set if we perturb a little), and has no double point if $j \geq 2$. Clearly $\varphi_{j,4k-2,i_j}$ restricted to its singular set has even number of double points for each $1 \leq j \leq 2k-1$.

Concerning the twisting of the normal bundle of the singular set: The determinant bundle $\delta_l(\varphi_{j,4k-2,i_j})$ of the $O(l)$ bundle obtained by the projection

$$S(O(l) \times O(4k-1-l)) \rightarrow O(l)$$

over $S_l(\varphi_{j,4k-2,i_j})$ is trivial ($l = j-1,j$ and $l \geq 1$), the determinant bundle $\delta_l(\varphi_{1,4k-2,e_1})$ is non-trivial, and for $j \geq 2$ the determinant bundle $\delta_l(\varphi_{j,4k-2,e_j})$ is non-trivial for $l = j$ and $l = j-1$, see Remark 6.3 in Section 6.

So if we sum up the double points for the index 0,...,2k−1 fold singularities and the twistings of index 1,...,2k−1 fold singularities, we get

- zero for $\varphi_{j,4k-2,i_j}$, $1 \leq j \leq 2k-1$, because we have even number of double points and trivial twisting,
• zero for \( \varphi_{1,4k-2,e_1} \), because we have one double point (from one definite fold crossing after perturbation) and non-trivial twisting (the twisting of the index one fold singular set),

• zero for \( \varphi_{j,4k-2,e_j} \), \( 2 \leq j \leq 2k-1 \), because we have no double points and we have non-trivial twistings for both of the index \( j-1 \) and \( j \) fold singularities (and two non-trivial values are zero together).

And also all the source manifolds have zero signature. This completes the proof.

Now, for an arbitrary oriented fold map \( f: M^{4k} \to \mathbb{R}^2 \), let us write the cobordism class \([M^{4k}]\) in the form \( r[\mathbb{C}P^{2k} \# \mathbb{C}P^{2k}] + [Z^{4k}]\) as we explained before, and the cobordism class \([f]\) in the form \( r[f_C] + [f_z] + \sum_{1 \leq j \leq 2k-1} a_j[\varphi_{j,4k-2,i_j}] + b_j[\varphi_{j,4k-2,e_j}]\), where \( a_j, b_j \in \{0,1\}\). By the above, each summand satisfies the congruence \( \frac{2j}{2} \equiv t + \tau \mod 2 \) in the statement of Theorem 7.5, which completes the proof of Theorem 7.5.

8.4. Proof for the unoriented case.

**Lemma 8.7.** Let \( Z \) be a closed (possibly non-orientable) manifold which fibers over \( S^2 \). Then there exists a fold map \( f_z: Z \to \mathbb{R}^2 \) such that

\[
t(f_z) \equiv 0 \mod 2
\]

and

\[
\tau^1(f_z) \equiv \tau^2(f_z) \equiv 0 \mod 2.
\]

*Proof.* The proof is completely analogous to the proof of Proposition 8.2. Details are left to the reader.

Recall the definition of the fold maps \( \tilde{\varphi}_{j,2k-2,i_j} \), \( \tilde{\varphi}_{j,2k-2,e_j} \) and \( \tilde{\varphi}_{j,2k-2,e_j} \) in Section 6.

**Lemma 8.8.** For \( 1 \leq j \leq k-1 \) the fold maps \( \tilde{\varphi}_{j,2k-2,i_j} \), \( \tilde{\varphi}_{j,2k-2,e_j} \) and \( \tilde{\varphi}_{j,2k-2,e_j} \), where \([i_j], [e_j] \) and \([e_j^2]\) are the generators of the group \( \text{Imm}\left(\mathbb{R}_{B(O(j) \times O(2k-1-j))},2\right) \cong \mathbb{Z}_2 \), satisfy

\[
\tau^1 + \tau^2 \equiv 0 \mod 2.
\]

*Proof.* The twistings of the maps \( \tilde{\varphi}_{j,2k-2,i_j} \) are trivial.

For the map \( \tilde{\varphi}_{1,2k-2,e_1} \) we have non-trivial \( \tau^1 \)-twisting (for the index one singular set), trivial \( \tau^2 \)-twisting (again for the index one singular set) and odd number of twisting for the index zero singular set. So this map also satisfies \( \tau^1 + \tau^2 \equiv 0 \mod 2 \).

For the map \( \tilde{\varphi}_{1,2k-2,e_1} \) we have non-trivial \( \tau^2 \)-twisting, trivial \( \tau_1 \)-twisting and non-trivial twisting for the index zero singular set (three times non-trivial). So this map also satisfies \( \tau^1 + \tau^2 \equiv 0 \mod 2 \).

The other maps \( \tilde{\varphi}_{j,2k-2,e_j} \) (resp. \( \tilde{\varphi}_{j,2k-2,e_j} \)), where \( j \geq 2 \), have non-trivial \( \tau^1 \)-twisting and non-trivial \( \tau^1_{j-1} \)-twisting (resp. non-trivial \( \tau^2_{j-1} \)-twisting and non-trivial \( \tau^2_{j-1} \)-twisting) and no other twistings. So they also satisfy \( \tau^1 + \tau^2 \equiv 0 \mod 2 \).

Now we prove Theorem 7.7.

*Proof of Theorem 7.7.* We use Lemma 8.1 again but in a more sophisticated way. By Proposition 6.3 for \( k \geq 1 \), \( q = 4k - 2 \) the group \( \text{Cob}(2,q) \) contains the group

\[
\bigoplus_{1 \leq j \leq q/2} \text{Imm}\left(\mathbb{R}_{B(O(j) \times O(q+1-j))},2\right) \cong \mathbb{Z}_2^{6k-3}
\]
where \( g \) is surjective by Remark 2.5 on any stably 1-framed manifold \( M \) there is a fold map \( g \) into \( \mathbb{R}^2 \) such that the decomposition \( TM \oplus \varepsilon_1^M = \eta_M \oplus \varepsilon_2^M \) induced by \( g \) is cobordant to the given stable 1-framing of \( M \).

The homomorphism \( (\xi_1, \ldots, \xi_{q/2}) \) will play the role of \( \pi_B \).

We want to show that \( (\psi, \pi_B) \) is not only injective as Theorem 5.1 says but also surjective.

Now, suppose that \( (\psi, \pi_B) \) is not surjective. If we show that \( \text{im} (\psi \circ \iota_B) = \mathbb{Z}_2 \) and this \( \mathbb{Z}_2 \) is the last \( \mathbb{Z}_2 \) summand in \( C_{2+q}(2) \cong \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2 \), then by applying case (2) of Lemma 8.1 we get that for \( k \geq 1 \), the group \( \text{Cob}(2, 4k - 2) \) is isomorphic to

\[
\text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2^{6k-3}
\]

since then the \( \mathbb{Z}_2 \) summand of \( C_{2+q}(2) \cong \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2 \) is factored out by the quotient map \( g \) (we keep the notations of Lemma 8.1). This will lead to a contradiction as we will see later, so \( (\psi, \pi_B) \) is surjective and since it was injective, we will get the proof of the statement.

So we show that \( \text{im} (\psi \circ \iota_B) = \mathbb{Z}_2 \). By Proposition 6.1 similarly to the oriented case, the source manifolds \( \tilde{Q}_{j,q,i}^{2+q}, \tilde{Q}_{j,q}^{2+q} \) and \( \tilde{Q}_{j,q,e}^{q+1} \) of the fold maps \( \tilde{\varphi}_{j,q,i}^{2+q}, \tilde{\varphi}_{j,q,e}^{q+1} \) and \( \tilde{\varphi}_{j,q}^{q+1} \), respectively, (see Section 9) represent zero in the cobordism group \( \text{Imm}_{2+q}(\mathbb{R}^2) \). But the singular set of \( \tilde{\varphi}_{1,q,e}^{q+1} : \tilde{Q}_{1,q,e}^{q+1} \rightarrow \mathbb{R}^2 \) is immersed with exactly one double point into \( \mathbb{R}^2 \). By Proposition 6.10 this gives the non-trivial element in the direct summand \( \mathbb{Z}_2 \) of \( C_{2+q}(2) \cong \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2 \). Hence \( \text{im} (\psi \circ \iota_B) = \mathbb{Z}_2 \).

The remaining fact to show is that

\[ \text{Cob}(2, 4k - 2) \cong \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2^{6k-3} \]

cannot hold. Consider the commutative diagram

\[ \begin{array}{cccc}
\Omega_{4k}^{2+q} \oplus \mathbb{Z}_2^{4k-2} & \xrightarrow{\kappa} & \left( \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2^{6k-3} \right) & \xrightarrow{q \oplus \text{id}} & \text{Imm}_{2+q}(\mathbb{R}^2) \oplus \mathbb{Z}_2^{6k-3} \\
\mathcal{C}_{2+q}(O(2, 4k - 2)) & \xrightarrow{\tau} & \text{Cob}(2, 4k - 2) & \xrightarrow{p} & \mathcal{C}_{2+q}(2) \oplus \bigoplus_{1 \leq j \leq q/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \\
\mathbb{Z}_2 & \xrightarrow{r} & \mathcal{C}_{2+q}(2) \oplus \bigoplus_{1 \leq j \leq q/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right)
\end{array} \]

where \( \kappa \) is the natural map corresponding to the natural map

\[ \mathcal{C}_{2+q}(O(2, 4k - 2)) \oplus \bigoplus_{1 \leq j \leq q/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \rightarrow \]

\[ \mathcal{C}_{2+q}(2) \oplus \bigoplus_{1 \leq j \leq q/2} \text{Im} \left( \varepsilon_{BS(O(j) \times O(q+1-j))}^1, 2 \right) \]
and $p$ is the composition $(q \oplus \text{id}) \circ \mathcal{S}_{2,4k-2}$. Note that the arrows $\mathcal{S}^O_{2,4k-2}$ and $\mathcal{S}_{2,4k-2}$ are injective (and we know already that $\mathcal{S}^O_{2,4k-2}$ is an isomorphism).

Take the cobordism class of the map

$$f_C: \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \rightarrow \mathbb{R}^2$$

of Proposition 8.4 in $\text{Cob}^O(2, 4k - 2)$. Let us check the coordinates of $\mathcal{S}_{2,4k-2} \circ \iota([f_C])$.

Since $t(\iota([f_C])) = 0$ and $\mathbb{C}P^{2k} \# \mathbb{C}P^{2k}$ is null-cobordant in $\mathcal{N}_{4k}^{2\chi}$, we have that $\mathcal{S}_{2,4k-2} \circ \iota([f_C])$ can have non-zero coordinates only in the direct summand $\mathbb{Z}^{6k-3}_2$. Also $p(\iota([f_C]))$ can have non-zero coordinates only in the direct summand $\mathbb{Z}^{6k-3}_2$. So if $V$ denotes

$$\mathcal{S}_{2,4k-2} \circ \iota(\text{Cob}^O(2, 4k - 2)) \cap \mathbb{Z}^{6k-3}_2$$

and $W$ denotes

$$p \circ \iota(\text{Cob}^O(2, 4k - 2)) \cap \mathbb{Z}^{6k-3}_2,$$

then

$$\mathcal{S}_{2,4k-2}(\iota([f_C])) \in V$$

and

$$p(\iota([f_C])) \in W.$$

Since $\tau([f_C]) = 1$, we have $\tilde{\tau}(\iota([f_C])) = 1$. Now we want to compute $\tilde{\tau}(\iota([f_C]))$ in another way. We know that the map $\tilde{\tau}$ is a homomorphism if we restrict it to the image of $\iota$. So if we find elements $a_1, \ldots, a_l$ in $\text{Cob}^O(2, 4k - 2)$ whose $\mathcal{S}_{2,4k-2} \circ \iota$-image generates exactly the subspace $V$ of the direct summand $\mathbb{Z}^{6k-3}_2$, then writing $\iota([f_C])$ as an appropriate linear combination of $\iota(a_1), \ldots, \iota(a_l)$ we could compute $\tilde{\tau}(\iota([f_C]))$ just by taking that linear combination of the values $\tilde{\tau}(\iota(a_1)), \ldots, \tilde{\tau}(\iota(a_l))$. The classes of cobord maps $\varphi_{j,4k-2,i_j}, \varphi_{j,4k-2,e_j}$, which generate the direct summand $\mathbb{Z}^{6k-3}_2$ of $\mathcal{N}_{4k}^{2\chi} \oplus \mathbb{Z}^{6k-3}_2$ would be a natural choice for such elements $a_1, \ldots, a_l$ but the problem is that $\mathcal{S}_{2,4k-2} \circ \iota([\varphi_{1,4k-2,e_1}])$ has a non-zero coordinate in the $\mathbb{Z}^{2\chi}_2$ summand of $\mathcal{N}_{4k}^{2\chi} \oplus \mathbb{Z}^{6k-3}_2$ so the $\mathcal{S}_{2,4k-2} \circ \iota$-images of all the $[\varphi_{j,4k-2,i_j}]$ and $[\varphi_{j,4k-2,e_j}]$ do not generate exactly the subspace $V$ of the direct summand $\mathbb{Z}^{6k-3}_2$ in $\left(\mathcal{N}_{4k}^{2\chi} \oplus \mathbb{Z}^{6k-3}_2\right) \oplus \mathbb{Z}^{6k-3}_2$.

But we suppose now (in order to find a contradiction) that the arrow $p$ is an isomorphism. And since the $p \circ \iota$-images of all the $[\varphi_{j,4k-2,i_j}]$ and $[\varphi_{j,4k-2,e_j}]$ generate exactly the subspace $W$ of the direct summand $\mathbb{Z}^{6k-3}_2$, a linear combination of the $\iota([\varphi_{j,4k-2,i_j}]), \iota([\varphi_{j,4k-2,e_j}])$ has to give $\iota([f_C])$ and the same linear combination of the

$$\tilde{\tau}(\iota([\varphi_{j,4k-2,i_j}]), \tilde{\tau}(\iota([\varphi_{j,4k-2,e_j}])$$

has to give 1 (because $\tilde{\tau}(\iota([f_C])) = 1$). In this linear combination $\tilde{\tau}(\iota([\varphi_{1,4k-2,e_1}]))$ has to participate with non-zero coefficient because $\tau((\varphi_{1,4k-2,e_1})) = 1$ while the other $\tau((\varphi_{j,4k-2,e_j}))$ and $\tau((\varphi_{j,4k-2,i_j}))$ values are 0. But this leads to a contradiction because then $t([f_C]) = 1$ would hold since $t([\varphi_{1,4k-2,e_1}]) = 1$ while the other $t([\varphi_{j,4k-2,e_j}])$ and $t([\varphi_{j,4k-2,i_j}])$ values are 0. Since we know that $t([f_C]) = 0$, we have the contradiction, so the proof is finished.

\begin{lemma}
The classes

$$\mathcal{S}_{2,4k-2}([e_{j1}^1]), \mathcal{S}_{2,4k-2}([e_{j2}^2]), \mathcal{S}_{2,4k-2}([i_j])$$

for $1 \leq j \leq 2k - 1$ and $\mathcal{S}_{2,4k-2}([f_C])$ form a basis of the $\mathbb{Z}_2 \oplus \mathbb{Z}^{6k-3}_2$ direct summand of $\text{Cob}(2, 4k - 2) \cong \mathcal{N}_{4k}^{2\chi} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}^{6k-3}_2$.
\end{lemma}
Proof. This follows from the previous proof and the constructions in Section 6.

Finally, we prove Theorem 7.8.

Proof of Theorem 7.8 We just have to check that all the generators of
\[ \text{Cob}(2, 4k - 2) \cong \mathbb{N}_{4k}^{2|k} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2}^{6k-3} \]
satisfy \( \tau^1 + \tau^2 \equiv 0 \mod 2 \). The generators represented by the fold maps \( \tilde{\varphi}_{j, q, g} : \tilde{Q}_{2j} \rightarrow \mathbb{R}^2 \), where \( g \) runs over the elements of \( \{i_j, e_j^1, e_j^2 : 1 \leq j \leq q/2\} \), satisfy \( \tau^1 + \tau^2 \equiv 0 \mod 2 \) by Lemma 8.8.

By [Br69] every class in \( \mathbb{N}_{4k}^{2|k} \) fibers over \( S^2 \). Also, by Lemma 8.7 any fold map \( f_z : Z \rightarrow \mathbb{R}^2 \) where \( Z \) fibers over \( S^2 \) satisfies \( \tau^1(f_z) + \tau^2(f_z) \equiv 0 \mod 2 \).

We need one additional generator which gives us the possibility to generate the \( \mathbb{Z}_2 \) summand of \( \mathbb{N}_{4k}^{2|k} \oplus \mathbb{Z}_2 \) independently from any other summand. The map \( f_C : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \rightarrow \mathbb{R}^2 \) provided by Proposition 8.4 can serve this purpose because the classes
\[ \mathcal{J}_{2,4k-2}([f_C]), \mathcal{J}_{2,4k-2}([e_j^1]), \mathcal{J}_{2,4k-2}([e_j^2]), \mathcal{J}_{2,4k-2}([i_j]) \]
generate exactly the \( \mathbb{Z}_2 \oplus \mathbb{Z}_{2}^{6k-3} \) part of \( \text{Cob}(2, 4k - 2) \) by Lemma 8.9.

Since this map \( f_C : \mathbb{C}P^{2k} \# \mathbb{C}P^{2k} \rightarrow \mathbb{R}^2 \) also satisfies \( \tau^1(f) + \tau^2(f) \equiv 0 \mod 2 \), we get the result.

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