Non-Abelian Born-Infeld theory without the square root

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Abstract

A non-Abelian Born-Infeld theory is presented. The square root structure that characterizes the Dirac-Born-Infeld (DBI) action does not appear. The procedure is based on an Abelian theory proposed by Erwin Schrödinger that, as he showed, is equivalent to Born-Infeld theory. We briefly mention other possible similar proposals. Our results could be of interest in connection with string theory and possible extensions of well known physical results in the usual Born-Infeld Abelian case.

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Seventy years ago Erwin Schrödinger wrote a paper entitled Contributions to Born’s New Theory of the Electromagnetic Field [1]. As is known and he himself pointed out the classical (Dirac-Born-Infeld-DBI) Born’s theory [2, 3, 4, 5, 6] can be constructed by means of the two vectors $B$ and $E$, the magnetic induction and the electric field-strength respectively. The partial derivatives of the Lagrangian with respect to the components of $B$ and $E$ define a second pair of vectors correspondingly $H$, the magnetic field and $-D$, the electric displacement. It was already shown by Born that one can choose four different ways to write a Lagrange function in terms of one of the magnetic vectors with one of the electric vectors. For each of these theories, the Lagrangians have essentially the same structure.

Schrödinger proposed a theory whose structure is entirely different from the above mentioned. He used two complex combinations of $B$, $E$, $H$ and $D$ as independent variables

$$\Omega = B - iD, \quad \Sigma = E + iH,$$

and constructed a Lagrangian in such a way that the complex conjugate of one of these variables is identical with the partial derivative of the Lagrangian with respect to the other one. This he called the condition of conjugateness. The Lagrangian is

$$\mathcal{L} = \frac{\Omega^2 - \Sigma^2}{\Omega \cdot \Sigma}. \quad (2)$$

In this Lagrangian the square root structure typical of the Dirac-Born-Infeld action has disappeared. The Lagrangian results rational and homogeneous of the zeroth degree.

Schrödinger showed that the classical treatment of the Lagrangian (2) is entirely equivalent to Born’s theory (DBI), this will be shown below. He then writes consequently it can not provide us with any new insight, which could not, virtually, be derived from Born’s original treatment as well. He recognized that for practical calculations, however, the imaginary vectors structure will hardly be useful. Quoting, once more Schrödinger yet for certain theoretical considerations of a general kind I am inclined to consider the present treatment as the standard form on account of its extremal simplicity, the Lagrangian being simply the ratio of the two invariants, whereas in Maxwell’s theory it was equal to one of them.

In this work we will present a generalization of Schrödinger’s Lagrangian (2) to a non-Abelian gauge theory. The complexification of the fields will provide us with a direct clue...
to find the non-Abelian framework based on the Abelian Schrödinger’s representation. As already recognized by Schrödinger himself, also our non-Abelian complex formalism provides a structure hardly to handle with for actual calculations. We will not attempt, in this work, to make calculations with our complex non-Abelian gauge theory. The work, however, is motivated by the fact that in the framework of string theory, the possibility to define a non-Abelian generalization of the standard Dirac-Born-Infeld bosonic and/or supersymmetric actions has been extensively explored beginning in 1990. If this theory could be constructed, it should represent the world-volume U(N) gauge theory that arises when one has N coincident type II Dp branes. The symmetrized trace prescription proposed by Tseytlin seems to be correct up to terms of the order $F^4$, but if fails at higher orders. Also in the supersymmetric set up certain terms cannot be expressed in terms of symmetrized traces.

At this stage we want only to show a consistent and relatively straightforward way to generalize Schrödinger’s representation of the DBI action to non-Abelian gauge field theories. It will be, as in the Abelian case, the ratio of two invariants and does not have the usual form of a square root. The formal structure is simple and provides us with a different starting point to investigate another non-Abelian generalization of the Dirac-Born-Infeld action. We will make a brief comment on other possible ways to generalize Schrödinger’s construction.

We will begin by reviewing the main aspects of Schrödinger’s proposal. His Lagrangian is then written in terms of the usual tensorial formulation in Electrodynamics. Next we present our non-Abelian proposal, for which, as in the Abelian case the corresponding condition of conjugateness will allow us to identify complex fields and reduce them to the number of the usual physical fields of the corresponding non-Abelian gauge theory. We conclude with a few remarks.

As already mentioned in the introduction Schrödinger’s proposal begins by postulating the Lagrangian (2) (the singular case $\Omega \cdot \Sigma = 0$ is discussed in [1]). The complex combinations (1) are considered as independent variables but such that their complex conjugates denoted by $\ast$ fulfill

$$\Omega^\ast = \frac{\partial L}{\partial \Sigma} = -\frac{2\Sigma}{\Omega \cdot \Sigma} - \frac{\Omega^2 - \Sigma^2}{(\Omega \cdot \Sigma)^2} \Omega,$$

$$\Sigma^\ast = \frac{\partial L}{\partial \Omega} = \frac{2\Omega}{\Omega \cdot \Sigma} - \frac{\Omega^2 - \Sigma^2}{(\Omega \cdot \Sigma)^2} \Sigma,$$

where $\Omega$ and $\Sigma$ are complex fields.
this he called the condition of conjugateness.

Schrödinger then remarks that to get the field-equations corresponding to (2) one should not pay attention to the relation (1), but actually consider $\Omega$ and $\Sigma$ as fundamental variables. He then, assumes (as in Born’s theory and, as well known, in Maxwell’s theory) that the six complex vector $\Omega$ and $\Sigma$ is the four dimensional curl of a potential four-vector, and consequently only its four components are to be varied independently. This is equivalent to assume that the field equations are

$$\nabla \times \Sigma + \frac{\partial \Omega}{\partial t} = 0, \quad \nabla \cdot \Omega = 0.$$  \hspace{1cm} (4)

Then, using the conjugateness condition one can obtain by variation in the usual way

$$\nabla \times \Sigma^* + \frac{\partial \Omega^*}{\partial t} = 0, \quad \nabla \cdot \Omega^* = 0.$$ \hspace{1cm} (5)

It is also shown, using (3), that $\mathcal{L}$ becomes purely imaginary and is also equal to

$$\mathcal{L} = -\frac{\Omega^* \Sigma^* - \Sigma \Omega^*}{\Omega^* \cdot \Sigma^*}.$$ \hspace{1cm} (6)

The stress energy momentum tensor is calculated and it is proved that there always exists a Lorentz frame in which all the four composing three vectors are parallel in a certain world point. Further simplification is obtained by making use of the fact that the six components of $\Omega$ and $\Sigma$ can be multiplied by a factor $e^{i\gamma}$, $\gamma$ real. It is called the $\gamma$-transformation. It does not interfere with the conjugateness condition, for in (3) the right-hand sides take the factor $e^{-i\gamma}$. The numerical values of the Lagrangian (2) remain unmodified as well as those of the stress-energy-momentum tensor components. The application to (4) and (5) with $\gamma = const.$ produces another solution, though with the same energy, momentum and stress densities as before in every world point.

A consequence of (3) is

$$\Omega^* \cdot \Sigma + \Sigma^* \cdot \Omega = 0.$$ \hspace{1cm} (7)

Making use of the above mentioned Lorentz transformation one can make all components vanish except, say, $\Omega_1$ and $\Sigma_1$ and choose $\gamma$ so as to make $\Omega_1$ real. Through the relation (7) one can write

$$\Omega_1 (\Sigma_1 + \Sigma_1^*) = 0,$$ \hspace{1cm} (8)

$\Sigma_1$ results imaginary and can be put as

$$\Sigma_1 = iA\Omega_1,$$ \hspace{1cm} (9)
where $A$ is a real constant. By substitution in (3) it is easily seen that the only allowed expressions for $\Omega_1$ and $\Sigma_1$ are

\[ \Omega_1 = \frac{\sqrt{1-A^2}}{A}, \quad \Sigma_1 = i\sqrt{1-A^2}, \quad (10) \]

$A$ takes values from $-1$ to $+1$ and the positive sign of the square root should be taken. This is called the "standard field". It is purely magnetic field with permeability $A^{-1}$, a purely electric field with dielectric constant $A^{-1}$ can be obtained by a $\gamma-$ transformation. This standard field does not require then a further $\gamma$-transformation, but only a Lorentz transformation would be necessary to obtain the most general field. The Lagrangian for the standard case results then in

\[ \mathcal{L} = -i\frac{1+A^2}{A}. \quad (11) \]

The identity with Born’s theory (DBI) is not fully performed in Schrödinger’s work, it is mentioned that the condition of conjugateness (3) is equivalent to relations (12) (see below). This can easily be corroborated. The rest of the procedure is, according with the footnote in page 472, as follows; he refers us to Born-Infeld paper [5] and makes two corrections to misprints, there

\[ H = \frac{\partial L}{\partial B} = \frac{B - GE}{\sqrt{1 + F - G^2}}, \quad D = -\frac{\partial L}{\partial E} = \frac{E + GB}{\sqrt{1 + F - G^2}}, \quad (12) \]

with

\[ L = \sqrt{1 + F - G^2} - 1, \quad F = B^2 - E^2, \quad G = B \cdot E. \quad (13) \]

One then chooses a frame with $B||E$. Consequently $H||D||B||E$. By inserting (13) into (12) one gets

\[ H = AB, \quad E = AD, \quad (14) \]

with

\[ A = \sqrt{\frac{1 - E^2}{1 + B^2}}, \quad (15) \]

$A$ results to be the dielectric constant and also the permeability. Expressing $B^2$ in terms of $E^2$ from (15) one gets

\[ B^2 + D^2 = \frac{1 - A^2}{A^2}, \quad (16) \]

and, of course

\[ H^2 + E^2 = 1 - A^2. \quad (17) \]
These last two equations reduce to Eqs. (10) when $D$ and $E$ are abolished by a $\gamma$-transformation.

To obtain Schrödinger’s Lagrangian in tensorial notation we write explicitly the dual tensor of the electromagnetic field-strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix},$$

and

$$G^{\mu\nu} = \begin{pmatrix} 0 & -D_1 & -D_2 & -D_3 \\ D_1 & 0 & -H_3 & H_2 \\ D_2 & H_3 & 0 & -H_1 \\ D_3 & -H_2 & H_1 & 0 \end{pmatrix},$$

where $G^{\mu\nu}$ corresponds to Maxwell Electrodynamics. For a non-linear electrodynamics in general it is calculated by means of

$$G^{\mu\nu} = 2 \frac{\partial L}{\partial F^{\mu\nu}} = \frac{\partial L}{\partial I_1} 2 F^{\mu\nu} - \frac{\partial L}{\partial I_2} \tilde{F}^{\mu\nu}, \quad (18)$$

where $L$ is the Lagrangian of interest and

$$I_1 = \frac{1}{2} F^{\mu\nu} F_{\mu\nu}, \quad I_2 = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (19)$$

are the two Lorentz invariants. Equation (18) provides the constitutive equations of a theory depending of $I_1$ and $I_2$ [12].

We define now

$$\Phi^{\mu\nu} \equiv \tilde{F}^{\mu\nu} - i G^{\mu\nu}, \quad (20)$$

which results in terms of $\Omega$ and $\Sigma$ in

$$\Phi^{\mu\nu} = \begin{pmatrix} 0 & -\Omega_1 & -\Omega_2 & -\Omega_3 \\ \Omega_1 & 0 & \Sigma_3 & -\Sigma_2 \\ \Omega_2 & -\Sigma_3 & 0 & \Sigma_1 \\ \Omega_3 & \Sigma_2 & -\Sigma_1 & 0 \end{pmatrix}.$$

The Lagrangian (2) can now be written as

$$\mathcal{L} = \frac{1}{2} \Phi^{\mu\nu} \Phi_{\mu\nu} = \frac{1}{2} \Phi_{\mu\nu} \Phi_{\mu\nu} \equiv \mathcal{L} \quad (21)$$
Now, in analogy with (18), the $G$ field tensor corresponding to the Lagrangian (21) results in the following

$$
G^\alpha{}^\beta = \frac{1}{\Omega \cdot \Sigma} \begin{pmatrix}
0 & -2\Omega_1 + \mathcal{L}\Sigma_1 & -2\Omega_2 + \mathcal{L}\Sigma_2 & -2\Omega_3 + \mathcal{L}\Sigma_3 \\
2\Omega_1 - \mathcal{L}\Sigma_1 & 0 & 2\Sigma_3 + \mathcal{L}\Omega_3 & -2\Sigma_2 - \mathcal{L}\Omega_2 \\
2\Omega_2 - \mathcal{L}\Sigma_2 & -2\Sigma_3 - \mathcal{L}\Omega_3 & 0 & 2\Sigma_1 + \mathcal{L}\Omega_1 \\
2\Omega_3 - \mathcal{L}\Sigma_3 & 2\Sigma_2 + \mathcal{L}\Omega_2 & -2\Sigma_1 - \mathcal{L}\Omega_1 & 0
\end{pmatrix},
$$

(22)

where $\mathcal{L}$ is the Lagrangian defined in Eq.(2).

By demanding

$$(\bar{\Phi}^{\alpha\beta})^* = -G^{\alpha\beta},$$

(23)

one gets exactly the condition of conjugateness (3). So we have really rewritten Shrödinger’s proposal and the same conclusions follow from this tensorial notation.

Taking advantage of the previous tensorial notation, we propose now for non-Abelian theories the following Lagrangian

$$
L = -\frac{1}{2} T_r(\Phi^{\alpha\beta}\Phi_{\alpha\beta}) \equiv \frac{\Pi_1}{\Pi_2},
$$

(24)

where $\Phi^{\alpha\beta} = \Phi^{\alpha\beta,a} \tau_a$, with $\tau_a$ the generators of the gauge group. As in the Abelian formulation the square root, which is so characteristic in the DBI theory and the non-Abelian generalization in 

[9], has disappeared and the Lagrangian is rational and of the zeroth degree. The procedure formally follows in the same manner as in the foregoing Abelian case. One can define a tensor

$$
G^{\alpha\beta,a} = \frac{\partial L}{\partial \Pi_1} \frac{\partial \Pi_1}{\partial \Phi^{\alpha\beta,a}} + \frac{\partial L}{\partial \Pi_2} \frac{\partial \Pi_2}{\partial \Phi^{\alpha\beta,a}},
$$

(25)

where

$$
\Phi^{\alpha\beta,a} \equiv \tilde{F}^{\alpha\beta,a} - iG^{\alpha\beta,a},
$$

(26)

with $\tilde{F}^{\alpha\beta,a}$ the dual tensor to the usual field strength tensor of the corresponding non-Abelian theory and $G^{\alpha\beta,a}$ the corresponding tensor to the Lagrangian defined only by the invariant $\Pi_1$, that is the one associated with the Yang-Mills Lagrangian under consideration and $\Pi_2$ its corresponding, so called $\theta$-term. This procedure allows us to find the constitutive equations. They can be defined by means of the use of the tensor $G^{\alpha\beta,a}$ in (25). In order to get the appropriate number of fields one needs to identify complex fields. This can be done
by imposing the condition of conjugateness in this theory, in analogy with the Abelian case
we demand
\[ (\tilde{\Phi}^{\alpha\beta,a})^* = -G^{\alpha\beta,a}. \] (27)

We know also how to calculate (25) following a similar procedure as in the Abelian formulation.

As it was assumed by Schrödinger himself in his Abelian proposal, we assume also here
that the field strength \( \Phi^{\alpha\beta,a} \) is constructed as usual, from a potential four vector and conse-
quently one gets the corresponding field equations. It is straightforward to see that, if we
construct the Lagrangian with the complex field strengths (27) it results oppositely equal to
(24). So, that \( L \) becomes purely imaginary as it happens in the Abelian formulation, Eqs.
(2,3,6).

The non-Abelian theory (24) is the most natural extension of Schrödinger’s representation
of DBI action. There is no ambiguity in ordering the product of the matrices, we take simply
the trace of the action of the non-Abelian theory under consideration and divide it by the
corresponding \( \theta - \text{term} \) which is also a trace. Being the denominator a trace, it can, by
example, easily be expanded in a series which multiplies the trace in the numerator. Other
possibilities can be considered, before taking the trace in the denominator (and numerator).
One can take the matrix product in the numerator and find the inverse matrix corresponding
to the denominator. Having to multiply these matrices, one must then give a prescription
to define the Lagrangian one would one to consider; the trace or the symmetrized trace [9]
can be used. This procedure is, however, not so simple as to take the ratio of the invariants
we are used to, \( II_1 \) and \( II_2 \) in the definition of \( L \).

In this work we have presented a classical non-Abelian proposal generalizing Born-Infeld
theory. Quantum aspects will be discussed in further work. Also, in a forthcoming paper we
will search for an expansion of the Lagrangian (24), under the condition of conjugateness (26)
in terms of the field strength and its corresponding non-Abelian gauge fields. As mentioned,
the complex fields are identified through the condition (26) and one gets the same number
of fields as in the corresponding standard Yang-Mills theory. This expansion would be a
first attempt towards a possible comparison with the results in string theory [10]. Also,
applications considering specific Lie groups will be considered to extend results that have
been studied in the Abelian case, as classical solutions that describe solitons and brane
configurations as well as physical effects related with electric fields [13] approaching limiting

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values.

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