Strong Scalar QED in Inhomogeneous Electromagnetic Fields

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(Dated:)

Strong QED has attracted attention recently partly because many astrophysical phenomena have been observed to involve electromagnetic fields beyond the critical strength for electron-positron pair production and partly because terrestrial experiments will generate electromagnetic fields above or near the critical strength in the near future. In this talk we critically review QED phenomena involving strong external electromagnetic fields. Strong QED is characterized by vacuum polarization due to quantum fluctuations and pair production due to the vacuum instability. A canonical method is elaborated for pair production at zero or finite temperature by inhomogeneous electric fields. An algorithm is advanced to calculate pair production rate for electric fields acting for finite periods of time or localized in space or oscillating electric fields. Finally, strong QED is discussed in astrophysics, in particular, strange stars.

Keywords: Strong QED, Inhomogeneous Electromagnetic field, Schwinger pair production, Strange star

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I. INTRODUCTION

Recently strong QED (quantum electrodynamics) has attracted much attention not only from theoretical interest but also from astrophysical observations and terrestrial experimental tests in the near future. From a theoretical viewpoint, calculating the full nonperturbative effective action under the influence of strong external electromagnetic fields, in particular, inhomogeneous fields, is still a challenging task (for a recent review and references, see Ref. [1] and also Ref. [2]). From an experimental viewpoint, in the near future electromagnetic fields from X-ray free electron lasers from LCLS (Linac Coherent Light Source) at SLAC [3] and TESLA (TeV Energy Superconducting Linear Accelerator) at DESY [4] may attain a strength almost comparable to the critical value for electron-positron pair production, which will directly test strong QED [3]. Interestingly, astrophysical sources have been predicted and observed that can have electromagnetic fields greater than the critical strength. Neutron stars have magnetic fields ranging from $10^8$ G to $10^{15}$ G and more than one-tenth of them have magnetic fields stronger than $10^{14}$ G, the so-called magnetars (for a review and references, see Ref. [6]), at least one order greater than the critical strength. Another interesting astrophysical objects with a ultra-strong electromagnetic field are strange quark stars, hypothetical objects, which may have electric fields with one or two order higher than the critical strength [7, 8] (see also Ref. [9] for review and references).

Vacuum fluctuations due to a strong external electromagnetic field contribute nonlinear terms to the classical Maxwell theory and the electromagnetic theory thus becomes highly nonlinear. Physics in strong electromagnetic fields drastically differs from the Maxwell theory [10, 11]. The cyclotron energy of an electron in a strong magnetic field can be greater than the rest mass energy of electron, the equivalent value leading to the critical strength of magnetic field $B_c = m^2c^3/eh$ ($4.4 \times 10^{13}$ G). Similarly, in a strong electric field, virtual pairs of electrons and positrons can gain energy comparable to or greater than the rest mass energy of electron or positron. The electric field whose potential energy across the Compton wavelength is the rest mass energy of electron is the critical value $E_c = m^2c^3/eh$ ($2.2 \times 10^{15}$ V/cm). For magnetic fields greater than the critical value, nonlinear contributions to the Maxwell term make the vacuum polarized by quantum fluctuations and the vacuum polarization causes nonlinear effects such as birefringence (propagation of photons in the magnetic vacuum), which plays an important role in the physics of magnetars [10]. For strong electric fields the vacuum decays due to an imaginary part of the effective action and thus leads to Schwellinger pair production [12]. Strange stars can emit electron-positron pairs more efficiently than photons [13, 14, 15]. On the other hand, in the standard QED with the minimal interaction magnetic fields are stable up to $B = 10^{12}$ G due to the instability from the self-interaction of an electron and up to the range $B = 10^{51} - 10^{55}$ G due to the instability from magnetic monopole production at the string or Planck scale [16]. However, the Pauli interaction may open a window for pair production by a far weaker inhomogeneous magnetic field and would have astrophysical applications [17].

QED describes the interaction between charged particles and photons. The success of QED is based on the perturbation theory in the weak-field limit. However, QED has not been completely understood yet in the opposite case of strong electromagnetic fields partly because the full nonperturbative QED action is not known except for some exactly solved cases [1, 2]. Historically, the
effective action of an electron in a constant electromagnetic field was obtained by Heisenberg and Euler [18], and also by Weisskopf [19]. Using the proper time method, Schwinger found the one-loop effective action for a spin-1/2 fermion with charge $q$ and mass $m$ in a constant electromagnetic field [12]

$$\mathcal{L}_{\text{eff}} = -\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-ms^2}}{s^3} \times \left[ (qs)^2 \frac{\text{Re} \cosh(qsX)}{\text{Im} \cosh(qsX)} - 1 - \frac{2}{3}(qs)^2 \mathcal{F} \right], \quad (1)$$

where $X = \left[2(\mathcal{F} + i\mathcal{G})\right]^{1/2} = \mathcal{X} + i\mathcal{Y}$.

Here, $\mathcal{F}$ is the negative of the Maxwell term, $-\mathcal{L}_{\text{Maxwell}},$

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\mathbf{B}^2 - \mathbf{E}^2), \quad (3)$$

and $\mathcal{G}$ is another Lorentz invariant tensor

$$\mathcal{G} = \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = \mathbf{E} : \mathbf{B}, \quad (4)$$

where $\tilde{F}_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual field tensor. The one-loop effective action was also obtained in Ref. [20].

The one-loop effective action has two important aspects. First, in the weak-field limit the nonlinear contribution to the real part

$$\text{Re}\mathcal{L}^{(1)} = \frac{2}{45mc^2} \left(\frac{q^2}{4\pi\hbar}\right)^2 \left(\frac{\hbar}{mc}\right)^3 \left(4\mathcal{F}^2 + 7\mathcal{G}^2\right), \quad (5)$$

makes the vacuum polarized by quantum fluctuations. In a pure strong magnetic field the leading term becomes

$$\mathcal{L}^{(1)} = \frac{(qB)^2}{24\pi^2} \ln \left(\frac{2qB}{mc}\right). \quad (6)$$

The ratio $\mathcal{L}^{(1)}/\mathcal{L}_{\text{Maxwell}} = -(q^2/12\pi^2) \ln(qB/m^2)$ is the one-loop QED $\beta$-function related with the renormalization group [11]. Second, in spinor QED in a pure strong electric field pairs are produced at the rate per unit time and unit volume

$$w_{\text{fermion}} = 2\text{Im}\mathcal{L}_{\text{eff}} = \frac{2}{(2\pi)^3} \sum_{n=1}^\infty \left(\frac{qE}{n}\right)^2 e^{-\frac{q^2n^2}{4\pi^2}}, \quad (7)$$

and in scalar QED pairs at the rate

$$w_{\text{boson}} = 2\text{Im}\mathcal{L}_{\text{eff}} = \frac{2}{(2\pi)^3} \sum_{n=1}^\infty (-1)^{n+1} \left(\frac{qE}{n}\right)^2 e^{-\frac{q^2n^2}{4\pi^2}}, \quad (8)$$

where $q$ is the charge of the electron.

Here, the factor of 2 is the spin multiplicity. The scattering amplitude of the ingoing vacuum to the outgoing vacuum decays according to Eqs. (7) or (8), leading to Schwinger pair production. One way to understand Schwinger pair production is to compute the imaginary part of the effective action. Another way is to find the vacuum solution of the field equation and then calculate the number of pairs produced by the field. The tunneling interpretation is that charged pairs in the Dirac sea can tunnel quantum mechanically through the potential barrier lowered by a uniform electric field. Pair production is an efficient mechanism for energy extraction from objects with strong electromagnetic fields.

In this talk, within the framework of canonical quantum field theory we critically review the Schwinger mechanism at zero or finite temperature in inhomogeneous electric fields motivated by terrestrial experiments or astrophysics. Exact solutions of the Klein-Gordon or Dirac equation minimally coupled to inhomogeneous electromagnetic fields are known only for a few models. The Sauter-type electric field that extends for a finite region or lasts for a finite period is the most well-known model [21]. However, one cannot find, in general, solutions for arbitrary electromagnetic fields, so he or she has to employ some approximation schemes. It is known that pair production by a constant electric field that extend over a finite region has a finite size effect and differs from that by a constant field [22]. Also the pair production rate by a Sauter-type electric field obtained by the worldline instanton method depends on the characteristic scale in a nontrivial way [23, 24]. Applying the phase-integral method [25] to find the WKB instanton action for the field equation with an electric and/or magnetic field in a fixed direction, the pair production rate is obtained for the Sauter-type electric field either in space or time with or without a constant magnetic field [26, 27]. Further, a perturbative method is advanced to calculate the WKB instanton action for pair production by any analytical electric field and is then applied to strange stars to calculate the production rate of electron-positron pairs. The thermal effect on the pair production is also studied [28].

The organization of this talk is as follows. In Sec. II, we critically review the Schwinger mechanism and then thermal effects on pair production. In Secs. III and IV, we apply the WKB instanton action method to inhomogeneous electric fields that act on for a finite period or extend for a finite region or oscillate. In Sec. IV, we apply strong QED to calculate the pair production rate from strange stars.

II. CANONICAL METHOD FOR PAIR PRODUCTION

In real physical systems electric fields are either confined to finite regions or turned on for finite periods of time. For such electric fields it is a nontrivial task to calculate the pair production rate. Instead of applying the proper time method or path integral method, we employ an approximation scheme such as the WKB approximation and phase integral in canonical quantum field theory. For the sake of convenience we consider only scalar QED, but the formalism here can be directly applied to spinor
A. Schwinger Pair Production

The Klein-Gordon equation for a charged boson with \( q (q > 0) \) and \( m \) takes the form (in units with \( \hbar = c = 1 \) and with metric signature \((+,−,−,−)\))

\[
[n^{μν}(∂_μ + iqA_μ)(∂_ν + iqA_ν) + m^2]Φ(x,t) = 0.
\]

Hereafter we further restrict our study to time-dependent electric fields along the \( z \) direction with gauge potentials of the form \( A_z(t) = -E_0g(t) \) for any analytic function \( g(t) \). Then the Fourier mode, \( Φ(x,t) = e^{ik·x}φ_k(t) \), satisfies

\[
\left[ \frac{∂^2}{∂t^2} + m^2 + k_z^2 + (k_z + qE_0g(t))^2 \right]φ_k(t) = 0.
\]

The solution can be used to quantize the position operators as

\[
\hat{a}_k(t) = \varphi_k(t)\hat{a}_k(t) + \varphi_k^*(t)\hat{b}_k(t),
\quad \hat{b}_k(t) = \varphi_k(t)\hat{b}_k(t) + \varphi_k^*(t)\hat{a}_k(t),
\]

and the momentum operators as

\[
π_k(t) = \varphi_k^*(t)a_k^+(t) + \varphi_k(t)b_k(t),
\quad π_k^*(t) = \varphi_k^*(t)b_k^+(t) + \varphi_k(t)a_k(t).
\]

On the other hand, in quantum mechanics, Eq. \( 10 \) is a one-dimensional scattering problem with inverted potential. The positive (asymptotic) solution \( φ_{k,\text{in}} \) at one asymptotic region \( t = -∞ \) defines the (asymptotic) ingoing vacuum and another positive (asymptotic) solution \( φ_{k,\text{out}} \) at the other region \( t = +∞ \) defines the (asymptotic) outgoing vacuum. As an incident solution from \( t = +∞ \) is partially transmitted over the barrier to \( t = -∞ \) and partially reflected by the barrier back to \( t = +∞ \), the ingoing solution is related with the outgoing solution as

\[
φ_{k,\text{in}} = μ_kφ_{k,\text{out}} + ν_kφ_{k,\text{out}}^*.
\]

That is, the ingoing positive frequency solution is mixed both with the outgoing positive solution and with the outgoing negative solution, which is the origin of particle production by an external field \( 30 \). As the Wronskian

\[
ψ_2(t)φ_k(t) - φ_k(t)ψ_2^*(t) = i,
\]

is constant, the coefficients satisfy the relation

\[
|μ_k|^2 - |ν_k|^2 = 1.
\]

In fact, the annihilation and creation operators at two asymptotic regions are related through Bogoliubov transformations

\[
\hat{a}_{k,\text{in}} = \mu_k\hat{a}_{k,\text{out}} - \nu_k\hat{b}_{k,\text{out}}^*,
\quad \hat{b}_{k,\text{in}} = \mu_k\hat{b}_{k,\text{out}} - \nu_k\hat{a}_{k,\text{out}}^*.
\]

The inverse Bogoliubov transformations are

\[
\hat{a}_{k,\text{out}} = \mu_k\hat{a}_{k,\text{in}} + \nu_k\hat{b}_{k,\text{in}}^*,
\quad \hat{b}_{k,\text{out}} = \mu_k\hat{b}_{k,\text{in}} + \nu_k\hat{a}_{k,\text{in}}^*.
\]

Therefore, the outgoing vacuum contains the ingoing particles/antiparticles as \( 31 \)

\[
\langle 0,\text{out} \mid \sum_k \hat{a}_{k,\text{in}}\hat{a}_{k,\text{in}}^* \rangle = \sum_k |ν_k|^2,
\quad \langle 0,\text{out} \mid \sum_k \hat{b}_{k,\text{in}}\hat{b}_{k,\text{in}}^* \rangle = \sum_k |ν_k|^2,
\]

and, conversely, the ingoing vacuum evolves into outgoing particle/antiparticle states as

\[
\langle 0,\text{in} \mid \sum_k \hat{a}_{k,\text{out}}\hat{a}_{k,\text{out}}^* \rangle = \sum_k |ν_k|^2,
\quad \langle 0,\text{in} \mid \sum_k \hat{b}_{k,\text{out}}\hat{b}_{k,\text{out}}^* \rangle = \sum_k |ν_k|^2.
\]

B. Hamiltonian Approach

Pair production by time-dependent electric fields can also be described by the Hamiltonian formalism. The Hamiltonian formalism is particularly appropriate for studying thermal effects because the density operator should satisfy the Liouville-von Neumann equation with respect to the Hamiltonian itself. The complex scalar field has the Hamiltonian

\[
H = \int [dk \left[ π_k^*π_k + ω_k^2(t)φ_k^*φ_k \right]],
\]

where \( [dk] = d^3k/(2\pi)^3 \) and

\[
ω_k^2(t) = m^2 + k_z^2 + (k_z + qE_0g(t))^2.
\]

The field operators are quantized as

\[
\hat{φ}(t,x) = \int [dk \left[ φ_k(t)\hat{a}_k(t) + φ_k^*(t)\hat{b}_k(t) \right]] e^{ik·x},
\]

\[
\hat{φ}^*(t,x) = \int [dk \left[ φ_k(t)\hat{b}_k(t) + φ_k^*(t)\hat{a}_k(t) \right]] e^{-ik·x}
\]

and the momentum operators as

\[
\hat{π}(t,x) = \int [dk \left[ φ_k^*(t)\hat{a}_k(t) + φ_k(t)\hat{b}_k(t) \right]] e^{-ik·x},
\]

\[
\hat{π}^*(t,x) = \int [dk \left[ φ_k^*(t)\hat{b}_k(t) + φ_k(t)\hat{a}_k(t) \right]] e^{ik·x}.
\]

In the Hamiltonian approach both quantum states and the density operator can be found simultaneously using the operators that satisfy the Liouville-von Neumann equation \( 32 \)

\[
\frac{∂ρ(t)}{∂t} + [\hat{H}(t), \hat{H}(t)] = 0.
\]
In fact, there are the time-dependent annihilation and creation operators satisfying Eq. (24) for particles
\[
\hat{a}_k(t) = i[\varphi_k(t) \hat{\pi}_k - \varphi_k^*(t) \hat{\phi}_k],
\]
and for antiparticles
\[
\hat{b}_k(t) = -i[\varphi_k(t) \hat{\pi}_k - \varphi_k^*(t) \hat{\phi}_k],
\]
where
\[
\hat{b}_k(t) = 1 \hat{b}_k(t)|0; t\rangle = \hat{a}_k^\dagger(t)|0; t\rangle = 0 \quad (\text{for any } k),
\]
and multi-particle and antiparticle states by
\[
|n_k, \cdots ; n_k, \cdots ; t\rangle = \frac{\hat{a}_{n_k1}^\dagger(t)}{\sqrt{n_k1!}} \cdots \frac{\hat{a}_{n_k2}^\dagger(t)}{\sqrt{n_k2!}} |0; t\rangle.
\]

C. Pair Production at Finite Temperature

The finite temperature QED effective action was calculated in a constant magnetic field [34], a constant electromagnetic field [33], and at finite density [36]. The Schwinger proper-time method was used to derive the effective action in a constant electromagnetic field, which exhibits the Schwinger mechanism at high temperature [37]. However, depending on the formalism employed to calculate the effective action, pairs are either produced [38, 39] or not produced [40]. The QED effective action from the imaginary-time formalism has nonzero imaginary part at two-loop [41]. In this paper we follow the real-time formalism in Ref. [29] to obtain pair production at finite temperature at one-loop.

As \(\hat{a}_k(t), \hat{a}_k^\dagger, \hat{b}_k(t), \hat{b}_k^\dagger\) satisfy Eq. (24), the density operator for particles can be found as [33]
\[
\hat{\rho}_{nk}(t) = \frac{1}{Z_k} \exp\left[-\beta E_k \left(\hat{a}_k^\dagger(t) \hat{a}_k(t) + \frac{1}{2}\right)\right],
\]
with \(\beta = 1/(kT)\) is the inverse temperature, and for antiparticles as
\[
\hat{\rho}_{nk}(t) = \frac{1}{Z_k} \exp\left[-\beta E_k \left(\hat{b}_k^\dagger(t) \hat{b}_k(t) + \frac{1}{2}\right)\right].
\]

Then the time-dependent vacuum, an exact state of the time-dependent annihilation and creation operators as
\[
\hat{a}_k(t) 0; t) = \hat{b}_k(t) 0; t) = 0 \quad (\text{for any } k),
\]
and multi-particle and antiparticle states by

Here, the Bogoliubov coefficients are
\[
\mu_k(\infty) = \langle \varphi_k^*(\infty) \varphi_k^\dagger(\infty) - \varphi_k^*(\infty) \varphi_k^\dagger(\infty)\rangle,
\]
\[
\nu_k(\infty) = \langle \varphi_k^*(\infty) \varphi_k^\dagger(\infty) - \varphi_k^*(\infty) \varphi_k^\dagger(\infty)\rangle.
\]
When there is a uniform magnetic field \(B\) in addition to \(E(t)\), each mode of the scalar field obeys the equation
\[
\ddot{\varphi}_{nk}(t) + \omega_{nk}^2 \varphi_{nk}(t) = 0,
\]
where
\[
\omega_{nk}^2 = k_z^2 + qE_0g(t) + B(2n + 1) + m^2.
\]

III. TIME-DEPENDENT ELECTRIC FIELDS

Pair production by localized electric fields in time or space significantly differs from that by the constant electric field due to a duration or a size effect [22, 23, 24, 26, 27]. In this section we exploit an analytical method to calculate the Schwinger pair production rate by an electric field acting for a finite period of time or an oscillating electric field. This case is characterized by a homogeneous time-dependent electric field \(E(t)\) with the maximum strength \(E_0\) and the time scale \(T\) defined as
\[
T = \frac{1}{2E_0} \int_0^\infty E(t)dt.
\]
The pair production rate is determined by two dimensionless parameters
\[
\epsilon = \frac{m}{qE_0 T}, \quad \delta = \frac{qE_0}{\pi m^2}.
\]
Pair production is allowed for any \(\epsilon\) but is strongly suppressed for \(\epsilon \gg 1\).

The main result of Ref. [25] is that in the weak-field limit \((E < E_c)\) the mean number of boson pairs for each mode \(k\) per unit time and unit volume
\[
N_k = e^{-S_k}
\]
is determined by the WKB instanton action of Eq. (10)
\[
S_k = i \oint \sqrt{(k_z^2 + qE_0 g(t))^2 + m^2 + k_z^2} dt,
\]
where the integral is taken outside the contour in the complex plane of time. In Ref. [12] the production rate in scalar QED is defined as twice the imaginary part of the effective action
\[
w_k = 2 \ln(1 + e^{-S_k}) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-ns_k}.
\]
The production rate [10] is even valid for very strong electric fields \((E \gg E_c)\) or \((S_k \ll 1)\).
A few comments are in order. Taking into account spin statistics, the mean number of boson pairs with spin $s$ is given by \[ N^\text{boson}_k = e^{w_k^\text{boson}} - 1 = (1 + e^{-w_k^\text{boson}})^{2s+1} - 1, \] while the mean number of fermion pairs with spin $s$ \[ N^\text{fermion}_k = 1 - e^{-w_k^\text{fermion}} = 1 - (1 - e^{-w_k^\text{fermion}})^{2s+1}. \] where $w_k^\text{boson/fermion}$ is twice the imaginary part of the effective action for bosons and fermions, respectively, and $2s+1$ is the spin multiplicity. The mean numbers without the spin multiplicity is also obtained in Ref. \[43\]. The mean numbers \[41\] and \[42\] hold even for very strong electric fields. The first term in Eq. \[41\] is the amplification factor for boson production. In the weak-field limit ($S_k > 1$), the mean number of boson or fermion pairs is approximately given by \[ N^\text{boson/fermion}_k \approx (2s + 1)e^{-S_k^\text{boson/fermion}}. \] Another interesting point is the relation between the WKB instanton actions for bosons and fermions. For a Sauter electric field it is shown in Ref. \[28\] that the WKB instanton actions for bosons and fermions, respectively, and $2s+1$ is the spin multiplicity. Therefore, the WKB instanton action works for both scalar and spinor QED.

Now we develop an algorithm to compute the instanton action systematically. Introducing another variable \[ \zeta = g(t), \] we rewrite Eq. \[39\] as \[ S_k = i(qE_0) \oint \sqrt{\left(1 + \frac{k_z}{qE_0} \right)^2 + \frac{m^2 + k^2_\perp}{(qE_0)^2} \zeta d\zeta}, \] and expand the square root in an inverse power series \[ \sqrt{\left(1 + \frac{k_z}{qE_0} \right)^2 + \frac{m^2 + k^2_\perp}{(qE_0)^2}} = \sum_{n=0}^{\infty} C_n \zeta^n, \] and the function $1/g'(\zeta)$ in a power series \[ \frac{1}{g'(\zeta)} = \sum_{n=0}^{\infty} D_n \zeta^n. \]

Then the sum of negative residues of simple poles leads to the WKB instanton action \[ S_k = 2\pi (qE_0) \sum_{n=0}^{\infty} C_{n+2} D_n. \] The first few terms of $C_n$ are \[ C_0 = 1, \quad C_1 = \alpha_1, \quad C_2 = \frac{\alpha_2}{2}, \] \[ C_3 = -\frac{\alpha_1 \alpha_2}{2}, \quad C_4 = \frac{\alpha_1^2 \alpha_2}{2} - \frac{\alpha_2^2}{8}, \] where \[ \alpha_1 = \frac{k_z}{qE_0}, \quad \alpha_2 = \frac{m^2 + k^2_\perp}{(qE_0)^2}. \] The coefficients $D_n$ are determined by the profile of $g(t)$. For specific models, we consider a Sauter-type electric field $E(t) = E_0 \sech^2(t/T)$ and an oscillating electric field $E(t) = E_0 \cos(t/T)$.

### A. Sauter-Type Electric Field

The gauge potential for $E(t) = E_0 \sech^2(t/T)$ in the $z$ direction is given by the Sauter potential \[ A_z(t) = -E_0 T \tanh \left( \frac{t}{T} \right). \] With the change of variable $\zeta = g(t) = T \tanh(t/T)$, we have the power series \[ \frac{1}{g'(t)} = \frac{1}{1 - \zeta^2} = \sum_{n=0}^{\infty} \frac{\zeta^{2n}}{T^{2n}}, \] and find $D_{2n} = 1/T^{2n}$. This means that the WKB instanton action is \[ S_k = 2\pi (qE_0 T^2) \sum_{n=0}^{\infty} \frac{C_{2n+2}}{T^{2n+2}}. \] In terms of the scaled variables and parameters \[ \lambda = \frac{k_z}{qE_0 T}, \quad \kappa = \frac{k_\perp}{m}, \quad Z = 2\pi qE_0 T^2 = \frac{2}{\delta T}, \] the leading terms of the WKB instanton action are \[ S_k = Z \epsilon^2 (1 + \kappa^2) \left[ \frac{1}{2} + \frac{\lambda^2}{2} - \frac{\epsilon^2(1 + \kappa^2)}{8} \right] + \cdots. \] In fact, the sum \[55\] can be done exactly as \[28\] \[ S_{k,1} = \frac{Z}{2} \left[ \sqrt{(1 + \lambda^2)^2 + \epsilon^2(1 + \kappa^2)} + \sqrt{(1 - \lambda)^2 + \epsilon^2(1 + \kappa^2)} - 2 \right]. \] The instanton method \[28\] gives much closer result to the exact one \[21\] than the worldline instantons \[24\].
B. Oscillating Electric Field

The oscillating electric field $E(t) = E_0 \cos(t/T)$ has many physical applications such as laser fields. The electric field from oscillating plasma due to pair production is approximately given by $E(t) = E \cos(t/T)$, where $E$ varies slowly during the oscillation period. The gauge potential $\zeta = g(t) = T \sin(t/T)$ leads to the expansion

$$\frac{1}{g(t)} = \frac{1}{\sqrt{1 - \zeta^2}} = 1 + \frac{\zeta^2}{2T^2} + \frac{3\zeta^4}{8T^4} + \frac{5\zeta^6}{16T^6} + \cdots .$$

Repeating the procedure in Sec. III A, we obtain the leading terms of the WKB instanton action

$$S_\kappa = Z \varepsilon^2 (1 + \kappa^2) \left[ \frac{1}{2} \left[ \frac{\lambda^2}{\pi} - \frac{\varepsilon^2 (1 + \kappa^2)}{16} \right] + \cdots \right].$$

IV. INHOMOGENEOUS ELECTRIC FIELD

For an inhomogeneous electric field localized in the $z$ direction, we may choose a Coulomb gauge $A_0(z) = -E_0 h(z)$, which leads to $E(z) = E_0 h'(z)$. Then the Klein-Gordon equation has the Fourier mode solution, $\Phi(x, t) = e^{ik \perp x - i\omega t} \phi_{k \perp}(z)$, given by

$$\left[ -\frac{\partial^2}{\partial z^2} + m^2 + k^2 + (\omega + qE_0 g(z))^2 \right] \phi_{k \perp}(z) = 0.$$  

The characteristic length scale $L$ is defined as

$$L = \frac{1}{2E_0} \int_{-\infty}^{\infty} E(z) dz.$$ 

As for time-dependent electric fields, two dimensionless parameters

$$\bar{\varepsilon} = \frac{m}{qE_0 L}, \quad \bar{\delta} = \frac{qE_0}{\pi m^2},$$

determine the pair production rate. Remarkably the pair production rate is again given by the WKB instanton action:

$$S_k = -i \oint \sqrt{\omega + qE_0 h(z)^2 - (m^2 + k^2)^2} dz,$$

where the integral is taken outside the contour in the complex plane of space. We again introduce the variable

$$\zeta = h(z),$$

and rewrite Eq. (64) as

$$S_k = -i(qE_0) \oint \left[ 1 + \frac{\omega}{qE_0 \zeta} \right]^2 - \frac{m^2 + k^2}{(qE_0 \zeta)^2} \frac{d\zeta}{g(\zeta)}.$$
condition is $V(-\infty) = V_q$, $V(\infty) = 0$, and $V(0) = 3V_q/4$. The Coulomb gauge potential is found \[15\]
\[
A_0(z) = \frac{\sqrt{2\pi}T}{\sinh\left[2\sqrt{\frac{\pi}{3}T(z + z_0)}\right]},
\]
and the electric field is
\[
E(z, T) = \sqrt{\frac{8\pi^3}{3}T^2 \cosh[2\sqrt{\frac{\pi}{3}T(z + z_0)}]} \sinh[2\sqrt{\frac{\pi}{3}T(z + z_0)}],
\]
whose characteristic scales are
\[
E_0 = \sqrt{\frac{8\pi^3\alpha}{3}T^2}, \quad L = \sqrt{\frac{3}{\alpha \pi} \frac{1}{2T}}.
\]
The higher (lower) the temperature is, the greater (smaller) is the maximum strength and the narrower (wider) is the width of the electric field.

With
\[
\zeta = -\frac{L}{\sinh\left[\frac{z + z_0}{L}\right]},
\]
the WKB instanton action becomes
\[
S_k = -i(qE_0L) \int \left(1 + \frac{\omega}{qE_0\zeta}\right)^2 - m^2 + \frac{k^2}{4} \left(qE_0\zeta\right)^2
\times \frac{d\zeta}{\sqrt{1 + \frac{\omega^2}{qE_0^2}}}.
\]
Thus, in terms of the scaled variables and parameters in Sec. IV, the leading terms of the WKB instanton action are
\[
S_k = 2\pi(qE_0L^2) \left[\frac{\omega}{qE_0L} - \frac{1}{4} \frac{\omega}{qE_0L} \frac{m^2 + \frac{k^2}{4}}{(qE_0L)^2} + \cdots\right]
= \frac{\omega}{4} \left[1 - \frac{\omega^2}{4} + \cdots\right].
\]
Then, the mean number \[42\] of electron-positron pairs per unit time and volume
\[
N_k^{\text{fermion}} = 2e^{-S_k} - e^{-2S_k}
\]
is the spectrum of emitted pairs. As $\frac{\omega}{T} = \sqrt{3\pi/\alpha} \times (\omega/T)$, hot strange stars produce more pairs of electrons and positrons than cold ones, confirming the numerical result of Ref. \[15\].

VI. CONCLUSION

In this talk, we critically reviewed the Schwinger mechanism at zero or finite temperature in inhomogeneous electric fields motivated by astrophysics or terrestrial experiments. As exact solutions of the Klein-Gordon or Dirac equation minimally coupled to inhomogeneous electromagnetic fields are known only for a few cases, for general electromagnetic fields, however, one has to rely on some approximation schemes. Inhomogeneous electric fields result in a finite size or duration effect and differs from that by a constant field \[22, 23, 24\] and an oscillating electric field. We applied the phase-integral method to find the WKB instanton action for the mode equations in inhomogeneous electromagnetic fields and then calculated the pair production rate by a Sauter-type electric field either in space or time \[26, 27, 28\] and an oscillating electric field. We also studied the thermal effect on pair production by an electric field that acts for a finite period of time \[24\]. Finally, we applied the WKB instanton action method to strange stars to calculate the electron-positron pair production rate.

The issues not treated in this talk are the effective action and the back reaction of QED at zero or finite temperature. It is a complicated task to obtain the effective action in inhomogeneous electromagnetic fields. In canonical quantum field theory, we may follow Ref. \[54\], according to which the effective action is related with the scattering amplitude as
\[
e^{iS_{\text{eff}}} = e^{i \int dt dx \mathcal{L}_{\text{eff}} = \langle 0, \text{out}|0, \text{in}\rangle}.
\]
Thus the effective action requires a complete knowledge of evolution of the ingoing vacuum to the outgoing vacuum, which may follow from the vacuum wave functional from Eqs. \[29\] and \[26\] for each mode. Another important issue is the QED back reaction problem, which is described by, for instance, the scalar QED action of the form
\[
\mathcal{L} = \phi^*(\partial_\mu + iqA_\mu)^2 \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\]
The back reaction cannot be neglected for strong electric fields because the additional electric field produced by pairs is comparable to the applied field. In fact, positive (negative) charges of produced pairs move in the same (opposite) direction of the applied electric field, so the current due to pairs induces an electric field in the opposite direction of the applied field and overshoots it until the process is reversed, which leads to the famous plasma oscillation \[47, 48, 49, 50\]. These issues will addressed in a future publication \[53\].

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