Thermal Analysis of Conductive-Convective-Radiative Heat Exchangers With Temperature Dependent Thermal Conductivity

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ABSTRACT In this paper, one dimensional mathematical model of convective-convective-radiative fins is presented with thermal conductivity depending on temperature. The temperature field with insulated tip is determined for a fin in convective, conductive and radiative environments. Moreover, an intelligent soft computing paradigm named as the LeNN-WOA-NM algorithm is designed to analyze the mathematical model for the temperature field of convective-convective-radiative fins. The proposed algorithm uses function approximating ability of Legendre polynomials based on artificial neural networks (ANN’s), global search optimization ability of Whale optimization algorithm (WOA), and local search convergence of Nelder-Mead algorithm. The proposed algorithm is applied to illustrate the effect of variations in coefficients of convection, radiation heat losses, and dimensionless parameter of thermal conductivity on temperature distribution of conductive-convective and radiative fins in convective and radiative environments. The experimental data establishes the effectiveness of the design scheme when compared with techniques in the latest literature. It can be observed that accuracy of approximate temperature increases with lower values of $N_c$ and $N_r$ while decreases with increase in $\lambda$. The quality of solutions obtained by LeNN-WOA-NM algorithm are validated through performance indicators including absolute errors, MAD, TIC, and ENSE.

INDEX TERMS Conductive-convective-radiative fin, temperature-dependent thermal conductivity, temperature distribution, weighted Legendre neural networks, hybrid soft computing, whale optimization algorithm, Nelder Mead algorithm.

I. INTRODUCTION

Heat exchangers or fins are also known as extended surfaces which are commonly used as an element of heat dissipation, that improves the performance and efficiency of equipment [1]. Fins have various applications in air conditioning, energy systems equipment, chemical processes, heat exchanger, cooling systems for computer equipment and refrigeration. Extended surfaces or fins are designed in different shapes for a class of longitudinal fins with a cross section much less than one dimensional (1D) extended surfaces or length directional. In particular, temperature-dependent behavior is revealed by thermal conductivity when dramatic changes in temperature of the fins occurred. This results in a nonlinear fin problem. Another source of nonlinearity arises from radiation. For example, measurable results from an experiment reveal that heat loss due to radiation is around 15–20 percent of the total heat loss along a fin cooled by...
natural convection and radiation [2]. As a result, radiation heat transfer has a significant impact on heat exchanger performance, particularly at high temperatures [3]. Thus, similar to conduction and convection, radiation has a substantial influence on temperature distribution and is important for increasing the thermal efficiency of fins, especially for devices with a low convection heat transfer coefficient. Heat transfer in fins is related to one dimensional nonlinear problem, where heat transfer coefficient and thermal conductivity are temperature dependent. Effect of variations in thermal conductivity and heat transfer of several nonlinear models has been extensively studied in [4]–[10]. Various techniques have been designed to study the approximate temperature distribution of extended surfaces in convective-conductive nonlinear fin problems. Chiu et al. [11] and Arsalan [12] proposed Adomian decomposition method to model analytical solution in the form of power series. In addition, other numerical methods that are used to find temperature distribution of fins are homotopy perturbation method [13], [14], homotopy analysis method [15], variational iteration method [16], [17], differential transformation method [18], Galerkin’s method [19] and the series method [20]. Moitsheki [21] applied classical Lie symmetry techniques to find exact solutions of the fin problem with a power-law temperature-dependent thermal conductivity. Abbasbandy and Shivanian [22] in 2017 calculate closed-form solutions for heat transfer in a straight fin. In recent times, Sun and Li [23] studied the exact solution of the nonlinear fin problem with exponentially temperature-dependent thermal conductivity and heat transfer coefficient. [24]–[30] recently focused on the study of optimization of various nonlinear models representing physical phenomenon. Radiation, in addition to convection, is another source of heat loss. When heat loss through natural convection is comparable to heat loss from an extended (fin) surface, radiation heat loss cannot be neglected. Thus, for the devices having a low convection heat transfer coefficient, convection and radiation heat transfer coefficients play a vital role. Convection and radiation heat transfer must be used to evaluate high performances of convective, conductive, and radiative extend surfaces (fins). Meanwhile, a strong nonlinear impact on temperature is exhibited by radiation heat loss transfer. Most of above listed methods have been designed to study the thermal distribution and performance of conductive, convective and radiative extend surfaces. DTM method is developed by [31] to study convective and radiative fins with thermal conductivity depending on temperature. [32] find the series solution for convective radiative conduction equation of nonlinear fin with temperature-dependent thermal conductivity. [33] studies a radial fin of uniform thickness with convective heating at the base and convective-radiative cooling at the tip. Generalized variational iteration method is used by Miansari et al. [34] to deal with nonlinear fin problem with radiation heat loss. Atouei uses collocation method [35], Runge-Kutta method [36] and least square method [37] to analyze temperature distribution and performance of radiative-convective semi-spherical extended surfaces. Optimal linearization method (OLM) [38] was developed to find approximate solutions for temperature field in convective and radiative heat transfers. An integral equation method is introduced by Huang and Li [39] to find an analytical and approximate distribution of temperature and fin performance for convective, conductive, and radiative fin. Multiple shape fins along with longitudinal fins has been widely studied such as T-shaped fins [40], [41], 2D orthotropic convection pin fin [42], [43] and stepped fins [44], [45]. By considering, Cattaneo-Christov heat flux model Khan and Alzahrani [28], [46] studied the impact of variable thermal conductivity over a variable thick surfaces. The classical lie point symmetry method is applied by Mhlongo et al. [47] to investigate the behavior of temperature when subjected to heat flow jump and base temperature jump. [48], [49] analyzed the mathematical model of non-Fourier heat conduction on wet extended surfaces.

In recent times, the heat transfer performance of fin gained the attention of researchers due to dramatic changes in the behavior of fin with temperature variations. Thus, it becomes necessary to design a method that can easily calculate the distribution of temperature in a fin. Unlike approaches available in the literature, this paper focuses on strengthening the concept of artificial neural networks (ANN’s). ANN based meta heuristic algorithms are used to solve variety of nonlinear problems arising in fluid dynamics [50]–[54], civil engineering [55], [56], wire coating dynamics [57], thermal engineering [58], [59], biomathematics [60]–[62], financial marketing [63]–[65], fuzzy systems [66]–[68] and petroleum engineering [69]. These potential application of stochastic techniques encourage the authors to strengthen computational ability of ANN’s based on Legendre neural networks to study the temperature distribution of fins. The innovative contribution of the given study are summarized as follows:

- A mathematical model for temperature distribution of fin with thermal conductivity in the conductive, convective and radiative environment is presented.
- A novel computing paradigm is design by using function approximating ability of orthogonal Legendre polynomials with hybridization of the Whale optimization algorithm (WOA) and the Nelder-Mead algorithm (NM). Proposed methodology is named as LeNN-WOA-NM.
- Further, the design scheme is utilized to study the influence of variations in coefficient of radiation and convection.
- The results obtained by design soft computing paradigm are compared with integral method and exact solution which shows the accuracy of design algorithm with minimum absolute errors in the solutions.
- Verification and validation of the performance analysis based on statistics in terms of standard deviations, mean absolute deviations, absolute errors, Theil’s inequality coefficient, variance and error in Nash Sutcliffe efficiency for design scheme have been evaluated by executing LeNN-WOA-NM algorithm for 100 independent runs.
II. PROBLEM FORMULATION

Consider a conductive-convective-radiative fin of length $L$ and cross-sectional $A$ with a temperature-dependent thermal conductivity shown in Figure 1. It is assumed that the fin is made of isotropic solid material relatively long compared to its cross-section. Moreover, the temperature at the base of the fin in a convection environment is considered uniform. By convection-radiation, heat is dissipated on the surface of the fin in a convection environment is considered uniform. Its cross-section. Moreover, the temperature at the base of the fin is denoted by $T_a$. The fin is at rest while heat flows in a steady state. Over the entire surface of the fin, convection heat transfer coefficient $N_c$ is considered to be uniform while thermal conductivity $k(T)$ depends on temperature, which is defined as \[ k(T) = k \left[ 1 + \lambda' (T - T_a) \right], \] where ambient temperature is presented by $T_a$. When inner temperature $T$ of fins is equal to ambient temperature ($T = T_a$) then $k$ denotes thermal conductivity. Temperature change is denoted by $\lambda'$. At any cross section during flow of heat, $T$ is invariant and varies only with longitudinal directions. Therefore, the phenomena presented in Figure 1 satisfies one dimensional non-linear differential equation for heat transfer which is given by Eq (2) [22].

\[
\frac{d}{dx} \left( k(T)A \frac{dT}{dx} \right) - Ph(T - T_a) - \varepsilon \sigma P \left( T^4 - T_s^4 \right) = 0, \tag{2}
\]

where $0 < X < L$, $\varepsilon$ is surface emissivity, $h$ is convection heat transfer coefficient, Stefan-Boltzmann constant is denoted by $\rho$ and sink temperature for radiation is presented by $T_s$. Now, introducing non-dimensional variables as follows

\[
\theta = \frac{T - T_a}{T_b - T_a}, \quad x = \frac{X}{L}, \quad l = \frac{LP}{A}, \tag{3}
\]

\[
N_c = l^2 \frac{hA}{k}, \quad N_r = l^2 \frac{\varepsilon \sigma A T_s^3}{kP}, \quad \lambda = \lambda' (T_b - T_a), \tag{4}
\]

where $T_b$ and $\lambda$ is dimensionless parameter so, Eq (2) can be written as

\[
\frac{d}{dx} \left( (1 + \lambda \theta) \frac{d\theta}{dx} \right) - N_c \theta - N_r \theta^4 = 0, \quad 0 < x < 1, \tag{5}
\]

Higher order Legendre polynomials are generated by using a recursive relation given by Eq (8) [72].

\[
L_{n+1}(x) = \frac{1}{n+1} \left[ (2n+1) x L_n(x) - n L_{n-1}(x) \right], \tag{8}
\]

We consider trial solution for Eq (6) presenting non-linear fin problem with temperature dependent thermal conductivity as

\[
\theta_{approx}(x) = \sum_{n=0}^{N} \delta_n L_n (\psi_n x + \xi_n), \tag{9}
\]

where $\delta_n$, $\psi_n$ and $\xi_n$ are unknown neurons that would be determined in course of solution. Figure 2 shows structure of

![Schematic of a conductive, convective and radiative fin.](image-url)
Legendre neural networks. As Eq (9) is continuous and differentiable therefore $\theta'$ and $\theta''$ can be calculated as following

$$\theta'_{\text{approx}}(x) = \sum_{n=1}^{N} \delta_n L'_n (\psi_n x + \xi_n),$$  

(10)

$$\theta''_{\text{approx}}(x) = \sum_{n=4}^{N} \delta_n L''_n (\psi_n x + \xi_n).$$  

(11)

Plugging, $\theta$, $\theta'$ and $\theta''$ in Eq (6) will model governing differential equation of conductive, convective and radiative fins. Mathematical model in terms of input, hidden and output layers is shown in Figure 4.

A. THE WHALE OPTIMIZATION ALGORITHM

Whale optimization algorithm (WOA) is a nature-inspired stochastic optimization technique developed by Mirjalili and Lewis [73] which mimics the foraging of humpback whales. It is a global search optimizer that utilizes population search space to determine the global optimum solution for optimization problems. Likewise, other population-based meta-heuristic algorithms, WOA, start optimizing the given problem by generating random candidate solutions. It then improves the set with each iteration until a satisfaction criterion for an ending is achieved. WOA is inspired by the bubble net hunting strategy of humpback whales as shown in Figure 3.

From Figure 3(b) it can be observed that humpback whales encircles the prey by moving in spiral path and creating bubbles along the way. The mathematical model for bubble net mechanism is given by

$$\tilde{X}(t+1) = \begin{cases} \tilde{X}^*(t) - \tilde{A} \cdot D, & \text{if } p < 0.5 \\ D' \cdot \exp(b \cdot \cos(2\pi l)) + \tilde{X}^*(t), & \text{if } p \geq 0.5, \end{cases}$$

(12)

where “$p$” is a random value in $[0,1]$, $b$ is shape of logarithmic spiral and $l$ is a random number in $[-1,1]$. $X^*$ represents best solution obtained so far while $D$ and $D'$ are defined by following equations

$$D = |\tilde{C} \cdot \tilde{X}_{\text{rand}} - \tilde{X}|,$$

(13)

$$D' = |\tilde{X}^*(t) - \tilde{X}(t)|,$$

(14)

$\tilde{A}$ and $\tilde{C}$ are coefficient vectors and given as follows:

$$\tilde{A} = 2\tilde{a} - \tilde{a},$$

(15)

$$\tilde{C} = 2\tilde{r}.$$  

(16)
TABLE 2. Description of parameter settings for design algorithm.

| Algorithm | Parameters | Settings | Parameters | Settings |
|-----------|------------|----------|------------|----------|
| LeNN-WOA  | Technique  | Metaheuristic | Candidate selection | Random search |
|           | Max. Iterations | 5,000 | Function tolerance | $10^{-1.8}$ |
|           | Bounds (Lb, Ub) | [-1, 1] | Fitness Limit | $10^{-20}$ |
|           | Search agents | 70 | Other settings | Default |
| NM        | Initialization | Global best solution of WOA | X-Tolerance ‘TolX’ | 1.0E-20 |
|           | Function evaluations | 150,000 | Max. iter | 2,000 |
|           | Fitness Limit | $10^{-20}$ | Other settings | Default |

TABLE 3. Comparison between temperature distributions obtained by exact, Integral method and proposed computing technique for different values of $N_c$ with $N_r = \lambda = 0$.

| $N_c$ | Integral Method | WOA | WOA-NN | Exact Method | WOA | WOA-NN | Exact Method | WOA | WOA-NN | Exact Method | WOA | WOA-NN | Exact Method |
|-------|----------------|------|--------|-------------|------|--------|-------------|------|--------|-------------|------|--------|-------------|
| 0.5   | 0.7931         | 0.7853 | 0.6471 | 0.6531 | 0.6541 | 0.6556 | 0.6591 | 0.6591 | 0.2598 | 0.2658 | 0.2658 |
| 0.2   | 0.7952         | 0.7953 | 0.6494 | 0.6513 | 0.6511 | 0.6507 | 0.6517 | 0.6517 | 0.2607 | 0.2667 | 0.2667 |
| 0.1   | 0.8112         | 0.8112 | 0.6490 | 0.6501 | 0.6501 | 0.6484 | 0.6486 | 0.6486 | 0.2618 | 0.2678 | 0.2678 |
| 0.05  | 0.8555         | 0.8555 | 0.6474 | 0.6474 | 0.6474 | 0.6474 | 0.6474 | 0.6474 | 0.2628 | 0.2688 | 0.2688 |
| 0.025 | 0.9458         | 0.9458 | 0.6460 | 0.6460 | 0.6460 | 0.6460 | 0.6460 | 0.6460 | 0.2638 | 0.2698 | 0.2698 |
| 0.0125| 0.9630         | 0.9630 | 0.6446 | 0.6446 | 0.6446 | 0.6446 | 0.6446 | 0.6446 | 0.2648 | 0.2708 | 0.2708 |
| 0.00625| 0.9905        | 0.9905 | 0.6432 | 0.6432 | 0.6432 | 0.6432 | 0.6432 | 0.6432 | 0.2658 | 0.2718 | 0.2718 |

$r$ is a random vector between $[0,1]$ and $a$ decreases linearly from 2 to 0 with the course of iterations.

The first component of Eq (12) illustrates the encircling mechanism, whereas the second mimics the bubble-net strategy. The variable $p$ switches between these two components with an equal probability. Output $X^*$ depends on the value of $p$. WOA starts the process with a set of random solutions. At each iteration, update the position of search agents with respect to either a randomly chosen search agent or the best solution obtained so far. Working procedure of the WOA is shown through flow chart in Figure 5. Initial parameter setting for WOA is given in Table 2.
**B. NELDER-MEAD ALGORITHM**

Nelder-Mead (NM) algorithm is a direct search method known as the downhill simplex method developed by Nelder and Mead in 1965 to solve different problems without any information about the gradient [74]. NM is a single path following a local search optimizer that can find good results.
if initialized with a better initial solution. A simplex consisting of $n+1$ vertices is set up to minimize a function $f$ with $n$ dimensions [75]. NM algorithm generates a sequence of simplices by following four basic steps, named reflection, expansion, contraction, and shrink. Initially, the points ($x^1$, $x^2$, ..., $x^{n+1}$) are generated and corresponding values of objective function are evaluated.

**Sorting**: Objective values for corresponding vertices of simplex are sorted to obtain centroid ($x^0$), worst ($x^b$), next to worst ($x^{aw}$) and best ($x^b$) values in all points.

**Reflection**: In this step, reflection point $x^r$ is determined by Eq (17).

$$x^r = x^0 + \alpha (x^0 - x^b).$$

### TABLE 4. Comparison between temperature distribution for PROBLEM II, III and IV obtained by proposed algorithm for variations in $\lambda$ with fixed value of coefficient of convective heat loss ($N_c = 1$).

| $x$ | $N_c$ | $\lambda = 1.0$ | Integral Method | Exact | LeSNN-WOA-SSM | $\lambda = 2.0$ | Integral Method | Exact | LeSNN-WOA-SSM | $\lambda = 3.0$ | Integral Method | Exact | LeSNN-WOA-SSM |
|-----|-------|-----------------|-----------------|-------|-----------------|-----------------|-----------------|-------|-----------------|-----------------|-----------------|-------|-----------------|
| 1   | 0.000 | 0.208           | 0.170           | 0.7704 | 0.7742           | 0.7782           | 0.8167          | 0.8165 | 0.8165          | 0.8165          | 0.8165          | 0.8165 |
| 2   | 0.659 | 0.457           | 0.657           | 0.7170 | 0.7234           | 0.7255           | 0.7656          | 0.7706 | 0.7706          | 0.7706          | 0.7706          | 0.7706 |
| 3   | 0.541 | 0.023           | 0.8223          | 0.8759 | 0.4694           | 0.4694           | 0.7241          | 0.7553 | 0.7553          | 0.7553          | 0.7553          | 0.7553 |
| 0.3 | 1     | 0.7303          | 0.7316          | 0.7316 | 0.7342           | 0.7342           | 0.8328          | 0.8329 | 0.8329          | 0.8329          | 0.8329          | 0.8329 |
| 2   | 0.6823 | 0.6860         | 0.6860          | 0.7464 | 0.7464           | 0.7464           | 0.7901          | 0.7907 | 0.7907          | 0.7907          | 0.7907          | 0.7907 |
| 3   | 0.6458 | 0.6524         | 0.6524          | 0.7138 | 0.7156           | 0.7156           | 0.7567          | 0.7580 | 0.7580          | 0.7580          | 0.7580          | 0.7580 |
| 0.6 | 1     | 0.8093          | 0.8085          | 0.8085 | 0.8063           | 0.8063           | 0.8181          | 0.8222 | 0.8222          | 0.8222          | 0.8222          | 0.8222 |
| 2   | 0.7788 | 0.7790         | 0.7790          | 0.7810 | 0.8211           | 0.8211           | 0.8509          | 0.8514 | 0.8514          | 0.8514          | 0.8514          | 0.8514 |
| 3   | 0.7464 | 0.7564         | 0.7564          | 0.7595 | 0.7594           | 0.7594           | 0.8258          | 0.8271 | 0.8271          | 0.8271          | 0.8271          | 0.8271 |

### TABLE 5. Maximum and minimum absolute errors (AE) in solutions of design scheme for cases of Problem I.

| $x$ | $N_c = 0.5$ | Maximum AE | Minimum AE | $N_c = 1.0$ | Maximum AE | Minimum AE | $N_c = 2.0$ | Maximum AE | Minimum AE | $N_c = 4.0$ | Maximum AE | Minimum AE |
|-----|-------------|-------------|------------|-------------|------------|------------|-------------|------------|------------|-------------|------------|------------|
| 0.0 | 1.30E-07    | 1.20E-11    | 3.50E-06   | 1.50E-13    | 4.23E-07   | 1.64E-11   | 1.09E-04    | 1.52E-10   | 1.30E-07    | 1.20E-11    | 3.50E-06   |
| 0.1 | 1.60E-07    | 4.32E-11    | 1.48E-07   | 2.93E-13    | 2.32E-06   | 9.64E-11   | 4.00E-05    | 1.19E-09   | 1.60E-07    | 4.32E-11    | 1.48E-07   |
| 0.2 | 1.90E-07    | 1.70E-11    | 1.42E-06   | 5.05E-13    | 2.53E-09   | 1.07E-11   | 8.14E-05    | 1.79E-10   | 1.90E-07    | 1.70E-11    | 1.42E-06   |
| 0.3 | 1.11E-04    | 1.70E-11    | 1.36E-06   | 4.04E-14    | 1.50E-06   | 2.22E-11   | 1.81E-05    | 7.63E-10   | 1.11E-04    | 1.70E-11    | 1.36E-06   |
| 0.4 | 6.28E-08    | 1.84E-11    | 3.30E-07   | 7.05E-13    | 7.43E-07   | 3.58E-14   | 5.39E-06    | 6.84E-11   | 6.28E-08    | 1.84E-11    | 3.30E-07   |
| 0.5 | 1.71E-07    | 3.91E-14    | 1.35E-06   | 2.65E-13    | 1.70E-07   | 2.80E-11   | 4.33E-05    | 7.64E-10   | 1.71E-07    | 3.91E-14    | 1.35E-06   |
| 0.6 | 8.84E-04    | 1.58E-11    | 4.64E-07   | 2.38E-13    | 1.74E-06   | 5.05E-12   | 3.78E-05    | 2.33E-10   | 8.84E-04    | 1.58E-11    | 4.64E-07   |
| 0.7 | 1.71E-09    | 1.67E-11    | 8.59E-07   | 1.19E-12    | 8.44E-07   | 1.77E-11   | 8.10E-06    | 4.99E-10   | 1.71E-09    | 1.67E-11    | 8.59E-07   |
| 0.8 | 1.54E-07    | 7.41E-13    | 3.41E-07   | 5.07E-14    | 5.34E-07   | 1.34E-11   | 2.83E-05    | 6.87E-10   | 1.54E-07    | 7.41E-13    | 3.41E-07   |
| 0.9 | 1.74E-07    | 3.44E-12    | 6.01E-09   | 2.24E-12    | 3.64E-06   | 5.21E-11   | 5.48E-05    | 1.19E-09   | 1.74E-07    | 3.44E-12    | 6.01E-09   |
| 1.0 | 1.60E-07    | 8.13E-12    | 9.79E-07   | 3.83E-13    | 1.62E-06   | 8.55E-12   | 2.84E-05    | 1.01E-10   | 1.60E-07    | 8.13E-12    | 9.79E-07   |
TABLE 6. Maximum and minimum absolute errors (AE) in solutions of design scheme for cases of Problem II.

| $\lambda$ | 1.0 | 2.0 | 3.0 |
|-----------|-----|-----|-----|
| $x$       | Max | Min | Max | Min | Max | Min |
| 0         | 9.23E-06 | 7.23E-11 | 1.49E-05 | 3.77E-11 | 8.97E-06 | 2.22E-10 |
| 0.1       | 3.59E-05 | 6.23E-10 | 1.07E-05 | 4.45E-10 | 7.37E-05 | 2.21E-09 |
| 0.2       | 5.54E-06 | 2.34E-10 | 1.41E-05 | 2.52E-10 | 2.64E-05 | 7.38E-10 |
| 0.3       | 1.56E-05 | 4.69E-10 | 1.35E-06 | 1.86E-10 | 2.44E-05 | 1.22E-09 |
| 0.4       | 3.28E-05 | 4.95E-12 | 3.23E-06 | 8.06E-11 | 7.25E-06 | 9.07E-11 |
| 0.5       | 1.61E-06 | 6.11E-10 | 1.18E-05 | 2.68E-10 | 2.20E-05 | 1.40E-09 |
| 0.6       | 2.84E-05 | 1.89E-10 | 7.75E-06 | 5.70E-13 | 2.94E-07 | 1.89E-10 |
| 0.7       | 4.39E-05 | 5.24E-10 | 4.68E-09 | 2.55E-10 | 1.57E-05 | 1.05E-09 |
| 0.8       | 6.05E-07 | 8.23E-10 | 1.05E-05 | 5.89E-11 | 3.79E-08 | 8.89E-10 |
| 0.9       | 1.28E-04 | 1.54E-09 | 1.55E-05 | 3.37E-10 | 6.64E-06 | 2.04E-09 |
| 1         | 3.42E-05 | 1.52E-10 | 1.17E-05 | 3.98E-11 | 1.18E-06 | 1.89E-10 |

TABLE 7. Maximum and minimum absolute errors (AE) in solutions of design scheme for cases of Problem III.

| $\lambda$ | 1.0 | 2.0 | 3.0 |
|-----------|-----|-----|-----|
| $x$       | Max | Min | Max | Min | Max | Min |
| 0         | 6.21E-05 | 4.23E-11 | 7.48E-06 | 4.13E-13 | 3.22E-06 | 1.53E-11 |
| 0.1       | 3.10E-04 | 7.68E-10 | 1.18E-05 | 1.69E-11 | 3.89E-06 | 2.38E-10 |
| 0.2       | 3.04E-11 | 1.31E-09 | 1.08E-05 | 7.55E-11 | 1.92E-06 | 3.01E-10 |
| 0.3       | 1.64E-04 | 1.64E-13 | 1.53E-07 | 1.44E-12 | 8.59E-08 | 1.39E-11 |
| 0.4       | 8.15E-05 | 8.40E-10 | 7.24E-06 | 6.50E-11 | 1.61E-06 | 2.02E-10 |
| 0.5       | 1.63E-05 | 4.45E-12 | 1.78E-05 | 1.03E-13 | 1.02E-06 | 2.47E-11 |
| 0.6       | 1.67E-06 | 6.88E-10 | 1.03E-05 | 9.89E-11 | 2.83E-09 | 1.62E-10 |
| 0.7       | 7.32E-05 | 8.13E-12 | 1.64E-07 | 9.54E-13 | 8.29E-07 | 2.94E-11 |
| 0.8       | 4.81E-05 | 7.33E-10 | 2.42E-05 | 2.00E-10 | 3.99E-07 | 2.39E-10 |
| 0.9       | 2.93E-04 | 2.92E-10 | 3.93E-05 | 1.07E-10 | 3.80E-07 | 7.10E-11 |
| 1         | 1.18E-04 | 1.15E-11 | 3.97E-05 | 5.45E-12 | 2.10E-11 | 2.32E-12 |

TABLE 8. Maximum and minimum absolute errors (AE) in solutions of design scheme for cases of Problem IV.

| $\lambda$ | 1.0 | 2.0 | 3.0 |
|-----------|-----|-----|-----|
| $x$       | Max | Min | Max | Min | Max | Min |
| 0         | 3.95E-04 | 2.15E-09 | 2.07E-06 | 1.70E-10 | 6.39E-05 | 6.83E-11 |
| 0.1       | 6.96E-05 | 1.79E-08 | 5.50E-06 | 1.90E-09 | 3.67E-05 | 4.04E-10 |
| 0.2       | 2.35E-04 | 4.92E-09 | 1.23E-06 | 1.95E-09 | 7.37E-05 | 2.42E-11 |
| 0.3       | 1.06E-04 | 6.48E-09 | 1.40E-06 | 3.12E-06 | 6.53E-06 | 2.08E-10 |
| 0.4       | 1.74E-07 | 2.44E-10 | 4.41E-06 | 8.83E-10 | 2.03E-05 | 8.77E-12 |
| 0.5       | 1.71E-04 | 5.23E-09 | 8.71E-07 | 3.54E-06 | 7.48E-05 | 1.18E-10 |
| 0.6       | 1.92E-04 | 1.59E-09 | 1.70E-06 | 6.54E-10 | 5.45E-05 | 9.71E-11 |
| 0.7       | 5.34E-05 | 2.10E-06 | 6.00E-06 | 3.30E-10 | 2.35E-07 | 4.11E-10 |
| 0.8       | 6.25E-05 | 6.61E-09 | 1.89E-07 | 1.16E-09 | 7.23E-05 | 2.05E-10 |
| 0.9       | 3.61E-04 | 7.90E-09 | 1.19E-05 | 2.59E-10 | 1.27E-04 | 1.95E-10 |
| 1         | 1.30E-04 | 5.51E-10 | 2.16E-06 | 4.11E-12 | 9.09E-05 | 1.23E-11 |

where $\alpha$ is reflection coefficient. If $f(x^1) \leq f(x^r) < f(x_{n+1})$ then iteration is terminated and “$x^r$” is accepted.

Expansion: The expansion point $x^e$ is computed by using the equation given below

$$x^e = x^0 + \gamma (x^r - x^0), \quad (18)$$

If $f(x^r) \leq f(x^e)$ then $x^e$ would be accepted and iteration will be stopped.

Contraction: If objective value at $x^r$ is strictly greater then objective value at $x^{mv}$ then this steps contraction is applied.

a) If $f(x^r) < f(x^h)$ then the outside contraction is applied by using Eq. (19).

$$x^{oc} = x^0 + \beta (x^r - x^0), \quad (19)$$

$\beta$ is an expansion coefficient

b) If $f(x^r) > f(x^h)$ then the inside contraction is applied by using Eq. (20).

$$x^{ic} = x^0 + \beta (x^h - x^0), \quad (20)$$
TABLE 9. Comparison through performance indices for each case of problem I obtained during 100 independent execution of proposed technique.

| Cases   | Min    | Mean   | Median  | Mode   | Std.   | Var.   |
|---------|--------|--------|---------|--------|--------|--------|
| $N_c = 0.5$ | 1.53E-11 | 3.94E-06 | 1.08E-07 | 1.53E-11 | 2.31E-05 | 5.36E-10 |
| $N_c = 1.0$ | 5.36E-13 | 3.47E-06 | 9.79E-08 | 5.36E-13 | 1.38E-05 | 1.91E-10 |
| $N_c = 2.0$ | 2.68E-12 | 3.01E-05 | 4.79E-07 | 2.68E-12 | 2.10E-04 | 4.40E-08 |
| $N_c = 4.0$ | 5.19E-10 | 1.10E-04 | 4.97E-06 | 5.19E-10 | 6.01E-04 | 3.61E-07 |
| Fit     | $N_c = 0.5$ | 1.83E-05 | 5.40E-05 | 2.79E-05 | 1.83E-05 | 1.10E-04 | 1.21E-08 |
|         | $N_c = 1.0$ | 2.46E-05 | 5.30E-05 | 2.95E-05 | 2.46E-05 | 8.63E-05 | 7.44E-09 |
|         | $N_c = 2.0$ | 2.42E-05 | 1.06E-04 | 3.00E-05 | 2.42E-05 | 3.49E-04 | 1.22E-07 |
|         | $N_c = 4.0$ | 1.90E-05 | 2.50E-04 | 4.55E-05 | 1.90E-05 | 9.72E-04 | 9.45E-07 |
| MAD     | $N_c = 0.5$ | 6.09E-06 | 1.73E-05 | 9.22E-06 | 6.09E-06 | 3.28E-05 | 1.07E-09 |
|         | $N_c = 1.0$ | 8.83E-06 | 1.97E-05 | 1.07E-05 | 8.83E-06 | 3.13E-05 | 9.75E-10 |
|         | $N_c = 2.0$ | 9.73E-06 | 4.34E-05 | 1.34E-05 | 9.73E-06 | 1.32E-04 | 1.74E-08 |
|         | $N_c = 4.0$ | 1.15E-05 | 1.27E-04 | 2.53E-05 | 1.15E-05 | 4.76E-04 | 2.26E-07 |
| TIC     | $N_c = 0.5$ | 8.04E-07 | 3.57E-05 | 1.88E-06 | 8.04E-07 | 2.67E-04 | 7.12E-08 |
|         | $N_c = 1.0$ | 5.07E-07 | 8.50E-06 | 7.27E-07 | 5.07E-07 | 5.00E-05 | 2.50E-09 |
|         | $N_c = 2.0$ | 2.09E-07 | 4.72E-05 | 3.21E-07 | 2.09E-07 | 3.74E-04 | 1.40E-07 |
| ENSE    | $N_c = 0.5$ | 7.12E-08 | 1.98E-04 | 4.09E-07 | 7.12E-08 | 1.75E-03 | 3.08E-06 |

Shrinkage: It is a final step and the result is calculated by Eq. (21).

\[ x^i = x^i + \delta \left( x^b - x^i \right) \]  \hspace{1cm} (21)

where \( \delta \) is shrink coefficient. The resulting simplex generated by NM algorithm for succeeding iterations can be written as \( X = x^i, i = 1, 2, 3, \ldots, n+1 \). Parameter setting for Nelder-Mead algorithm are given in Table 2.

IV. LeNN-WOA-NM ALGORITHM

The steps for the proposed hybridized algorithm are summarized as:

**Step 1: Initialization:** Randomly generates an initial population using Eq (9), with number of parameters equal to number of neurons in LeNN's structure. Parameters setting to initialize WOA is demonstrated in Table 2.
Step 2: Fitness evaluation: Calculate the fitness value for each individual of candidate space by using Eq (22).

Step 3: Termination criteria: Terminate the process of fitness evaluation, if any of the following termination criteria is achieved.

• When maximum number of predefined iterations is achieved.

• When fitness value $\varepsilon \leq 10^{-25}$

Step 4: Ranking: Rank the individuals of the population on the basis of values of the fitness function $\varepsilon$.

Step 5: Storage: Store the values of weights and fitness function.

Step 6: Initialization of NM: Nelder-Mead algorithm is used for further speedy fine tuning of the results, starting with global best values of $\delta_n, \psi_n$, and $\xi_n$ obtained by WOA. Parameters setting for NM algorithm is shown in Table 2.

Step 7: Refinement: NM algorithm uses MATLAB built-in function “FMINSEARCH” to update the weights and evaluate the fitness function using Eq (22). The execution of the process stops when predefined stopping criteria is attained.

Step 8: Storage: Store the refined best values of $\zeta_n, \psi_n$ and $\delta_n$ along with fitness. The procedure in executed for 100 independent runs to obtain large set of statistical data.
LeNN-WOA-NM algorithm has a simple structure and easy to implement. WOA updates the position of individual using global search ability and bubble net strategy of humpback whales while NM algorithm further complements its local convergence. Since, Legendre polynomials are orthogonal on \([-1, 1]\) so the experimental analysis shows that proposed algorithm converges to best solutions for number of real-world problems by training the weights from the interval \([-1, 1]\). It has been noticed that convergence of design scheme is slightly affected by increasing the domain.

V. CONSTRUCTION OF FITNESS FUNCTION

Fitness function \(\epsilon\) is formulated on the basis of an unsupervised error, which is defined as the sum of mean square errors of Eq (6) and Eq (7) as

\[
\text{Minimize } \epsilon = \hat{\epsilon}_1 + \hat{\epsilon}_2,
\]

\[
\text{Minimize } \epsilon = \frac{1}{N} \sum_{m=1}^{N} \left[ d^2\theta_m \frac{d^2\theta_m}{dx^2} + \lambda \left( \frac{d\theta_m}{dx} \right)^2 - N_c \theta_m - N_r \theta_m^4 \right] + \frac{1}{2} \left( \frac{d\theta_m}{dx} (0) + (\theta(1) - 1)^2 \right),
\]

where \(N = \frac{1}{h}\) and \(h\) is a step size.
VI. PERFORMANCE MEASURES

To examine the accuracy and convergence of design scheme (LeNN-WOA-NM), in obtaining solutions for different problems of conductive-convective and radiative fins with thermal conductivity, performance measures are defined in term of fitness evaluation, mean absolute deviation.
(MAD), Theil’s inequality coefficient (TIC) and error in Nash Sutcliffe efficiency (ENSE). Mathematical formulation for these indices are given below [69].

\[ \text{MAD} = \frac{1}{n} \sum_{m=1}^{n} |\theta(x) - \theta_{\text{approx}}(x)|, \]  

(24)

\[ \text{TIC} = \frac{\sqrt{\frac{1}{n} \sum_{m=1}^{n} (\theta(x) - \theta_{\text{approx}}(x))^2}}{\left(\sqrt{\frac{1}{n} \sum_{m=1}^{n} (\theta(x))^2} + \sqrt{\frac{1}{n} \sum_{m=1}^{n} (\theta_{\text{approx}}(x))^2}\right)}, \]  

(25)

\[ \text{NSE} = \frac{\sum_{m=1}^{n} (\theta(x) - \theta_{\text{approx}}(x))^2}{\sum_{m=1}^{n} (\theta(x) - \hat{\theta}(x))^2}, \]  

\[ \hat{\theta}(x) = \frac{1}{n} \sum_{m=1}^{n} \theta(x), \]  

(26)

\[ \text{ENSE} = 1 - \text{NSE}. \]  

(27)

where, \( n \) denote a grid points.

**VII. NUMERICAL EXPERIMENTATION**

In this section, we have defined different problems to study the influence of variations in coefficients of convective heat
loss $N_c$, coefficient of radiative heat lost $N_r$ and dimensionless parameter of thermal conductivity $\lambda$ on temperature distribution of conductive-convective-radiative fins with thermal conductivity. Problems along with different cases studied in this paper are presented in the flow chart through Figure 6.
Problem I: Effect of Variations in $N_c$ on temperature distribution with no resemblance of radiation heat loss and $\lambda$.

In this problem, the proposed technique is applied to study the influence of $N_c$ on temperature distribution of fins with neglected radiation heat loss $N_r = 0$ and dimensionless parameter of thermal conductivity $\lambda = 0$. The fitness function for this problem is formulated as

$$\epsilon = \frac{1}{N} \sum_{m=1}^{N} \left( \frac{d^2 \theta_m}{dx^2} - N_c \theta_m \right)^2 + \frac{1}{2} \left( \left( \theta'(0) \right)^2 + (\theta(1)-1)^2 \right),$$  \hfill (28)

where $0 \leq x \leq 1$. Following four cases are considered.

- Case I: Eq (28) with $N_c = 0.5$.
- Case II: Eq (28) with $N_c = 1.0$.
- Case III: Eq (28) with $N_c = 2.0$.
- Case IV: Eq (28) with $N_c = 4.0$.

Problem II: Effect of Variations in dimensionless parameter of thermal conductivity $\lambda$ on temperature distribution with $N_c = N_r = 1$.

In this problem, influence of variations in $\lambda$ on temperature distribution of conductive, convective and radiative fins is considered with $N_c = N_r = 1$. Fitness function for this problem is formulated as

$$\text{Minimize} \quad \epsilon = \frac{1}{N} \sum_{m=1}^{N} \left( \frac{d^2 \theta_m}{dx^2} + \lambda \frac{d^2 \theta_m}{dx^2} + \lambda \left( \frac{d \theta_m}{dx} \right)^2 \right)^2$$

$$+ \frac{1}{2} \left( \left( \frac{d \theta}{dx}(0) \right)^2 + (\theta(1)-1)^2 \right),$$  \hfill (29)

where $0 \leq x \leq 1$, different cases for Eq (29) depending on dimensionless parameter of thermal conductivity are considered as follows.

- Case I: Eq (29) with $\lambda = 1.0$.
- Case II: Eq (29) with $\lambda = 2.0$.
- Case III: Eq (29) with $\lambda = 3.0$.

Problem III: Effect of Variations in dimensionless parameter of thermal conductivity $\lambda$ on temperature distribution with $N_c = 1$ and $N_r = 2$.

In this problem, influence of variations in dimensionless parameter of thermal conductivity on for temperature distribution of conductive, convective and radiative fin is studied with coefficient of convective heat loss ($N_c = 1$) and coefficient of radiative heat loss $N_r = 2$. Fitness function
FIGURE 19. Box plot for fitness evaluation by proposed algorithm for multiple cases of Problem I–IV.

FIGURE 20. Box plot for MAD by proposed algorithm for multiple cases of Problem I–IV.
for given problem can be written as

$$
\text{Minimize } \epsilon = \frac{1}{N} \sum_{m=1}^{N} \left( \frac{d^2 \theta_m}{dx^2} + \lambda \frac{d^2 \theta_m}{dx^2} + \lambda \left( \frac{d \theta_m}{dx} \right)^2 \right)^2 + \frac{1}{2} \left( \frac{d \theta}{dx}(0) + (\theta(1)-1)^2 \right), \quad (30)
$$

Three cases on bases of dimensionless parameter of thermal conductivity $\lambda$ are considered to study its effect on temperature distribution of fin.
FIGURE 23. Performance analysis on fitness value for different cases of Problem I–IV.

FIGURE 24. Performance analysis on MAD for different cases of Problem I–IV.
Case I: Eq (30) with $\lambda = 1.0$.
Case II: Eq (30) with $\lambda = 2.0$.
Case III: Eq (30) with $\lambda = 3.0$.

Problem IV: Effect of Variations in dimensionless parameter of thermal conductivity $\lambda$ on temperature distribution with $N_c = 1$ and $N_r = 3$. 
In this problem, effect on variations in thermal conductivity on temperature distribution of conductive, convective and radiative fins is considered with coefficients of convective heat loss $N_c = 1$ and coefficient of radiative heat loss $N_r = 3$. Fitness based error function for this problem is formulated as

$$
\text{Minimize } \epsilon = \frac{1}{N} \sum_{m=1}^{N} \left( \frac{d^2 \theta_m}{dx^2} + \lambda \theta_m \frac{d^2 \theta_m}{dx^2} + \lambda \left( \frac{d \theta_m}{dx} \right)^2 \right)^2 + \frac{1}{2} \left( \frac{d \theta_m}{dx}(0)^2 + (\theta(1)-1)^2 \right). \quad (31)
$$

Following cases on basis of changes in dimensionless parameter of thermal conductivity are considered as follows

Case I: Eq (31) with $\lambda = 1.0$.
Case II: Eq (31) with $\lambda = 2.0$.
Case III: Eq (31) with $\lambda = 3.0$.

VIII. RESULTS AND DISCUSSION

This paper has analyzed the mathematical model for temperature distribution of conductive, convective, and radiative fins with thermal conductivity. The model given by Eq (6) depends on different parameters named as the coefficient of conductive heat loss, coefficient of radiative heat loss, and dimensionless parameter of thermal conductivity. Furthermore, a new soft computing algorithm is designed to study the influence of parameters on temperature distribution of conductive-convective and radiative fins. The behavior of approximate solutions is discussed with trained neurons from [-1,1]. Results obtained by the proposed method are compared with exact solutions, and approximate solution by integration method [71], decomposition method [11], Galerkin method [70] and simplex search method [76].

To study the performance of the proposed technique by obtaining solutions to different cases of each problem, hundred independent simulations have been carried out. Figures 7-14 demonstrates the comparison of best and worst approximate solutions with exact solution along with absolute minimum errors for temperature distribution of each case of problem I, II, III, and IV, respectively. It can be observed from Figure 7 that with neglecting coefficient of radiation and thermal conductivity, temperature distribution of fin decreases and becomes strong convective with increasing coefficient of convection ($N_c = 0.5, 1, 2, 4$). Furthermore, in the presence of radiation heat loss and thermal conductivity from fin surface we consider $N_r = 1, 2, 3$ and $\lambda = 1, 2, 3$, since these non dimensionless parameters cover most of practical cases [1]. It can be viewed from Figure 9,11 and 13 that with increasing value of thermal conductivity ($\lambda$), temperature at fin tip rises and the accuracy of temperature distribution becomes higher. In addition, it can be witnessed that with increasing value of coefficient of convection, the temperature distribution of fin rises with fixed values of thermal conductivity.
and coefficient of radiation. Plots of best weights obtained by design scheme for calculating temperature distribution of each case of different problem are shown in Figures 15-18. Box plots for fitness evaluation, MAD, TIC, and ENSE are shown in Figures 19-22. It can be seen that mean values of fitness function and performance indicators lies around $10^{-4}$ to $10^{-6}$, $10^{-5}$ to $10^{-3}$, $10^{-4}$ to $10^{-5}$ and $10^{-3}$ to $10^{-5}$ respectively. The bar graphs are shown in Figures 23-26 represent minimum, mean, median, mode, standard deviation, and variance of fitness value and performance indices obtained by proposed algorithm 100 independent runs.

Approximate solutions obtained by the LeNN-WOA-NM algorithm for different cases of problems I, II, III, and IV are compared with the exact solution and integral method as dictated in Tables 3 and 4. Maximum and minimum absolute errors (AE) of the proposed technique for each case of different problems are given in Tables 5-8. Minimum AE’s for case I, II, III and IV of problem I lies between $10^{-11}$ to $10^{-14}$, $10^{-12}$ to $10^{-14}$, $10^{-11}$ to $10^{-14}$ and $10^{-13}$ to $10^{-13}$ respectively. Minimum AE’s for case I, II, and III of problem II lies between $10^{-9}$ to $10^{-12}$, $10^{-10}$ to $10^{-13}$ and $10^{-9}$ to $10^{-11}$ respectively. Minimum AE’s for case I, II and III of problem III lies between $10^{-9}$ to $10^{-13}$, $10^{-10}$ to $10^{-13}$ and $10^{-10}$ to $10^{-12}$ respectively. Minimum AE’s for case I, II and III of problem IV lies between $10^{-9}$ to $10^{-10}$, $10^{-9}$ to $10^{-12}$ and $10^{-10}$ to $10^{-11}$ respectively. Statistics of fitness value, MAD, TIC, and ENSE in terms of minimum, mean, median, mode, standard deviation and variance are demonstrated through Tables 9-12. It can be seen that the minimum value of fitness function for the problem I, II, III and IV lies around $10^{-13}$, $10^{-10}$, $10^{-11}$ and $10^{-10}$ respectively. Unknown neurons in LeNN structure optimized by design algorithm for obtaining best solutions for temperature distribution of conductive, convective and radiative fins with thermal conductivity. Convergence analysis of the proposed algorithm for each case of problem I-IV is given in Table 17. Statistical results and Figures 27 demonstrates the effectiveness and accuracy of the proposed algorithm.

IX. CONCLUSION

This paper has analyzed a mathematical model for temperature distribution of fin with thermal conductivity in the conductive, convective and radiative environment. Furthermore, we have designed an intelligent soft computing paradigm named as LeNN-WOA-NM algorithm. Weighted Legendre polynomials are used to model approximate series solutions for temperature distribution under the influence of variations in thermal conductivity ($\lambda$) and coefficients of convective and radiative heat loss $N_c$ and $N_r$. We summarize our findings as follows:

- In problem I, with the increasing value of coefficient of convective heat loss, the excess of temperature is getting lower, which decreases the transfer of heat and fin becomes strong convective as shown in Figure 27(a).

- With the increasing value of dimensionless parameter $\lambda$ in thermal conductivity, the excess of temperature becomes higher, and the transfer of heat increases. Figure 27(b),(c) and (d) represent the behavior of temperature distribution for problem II, III and IV, respectively.

- Approximate solutions obtained by LeNN-WOA-NM algorithm are compared with exact solutions, and integral methods [71]. Tables 21-25 shows the accuracy of proposed technique in obtaining solutions for temperature distributions under influence of $N_c$, $N_r$ and $\lambda$.

- Minimum absolute errors in approximate solutions by design algorithm prove that LeNN-WOA-NM is efficient and accurate. Moreover, the values of performance indicators MAD, TIC and ENSE extend the worth of the designed scheme.

- Convergence of proposed algorithm has been proven by boxplots and bar graphs representing the minimum and mean values of performance indicators obtained during 100 independent runs.

- From the above-discussed figures and tables, it should be noted that the lower value of convection and radiation parameter, the higher is the accuracy of approximate solutions, while larger the value of thermal conductivity, the more accurate the approximate temperature distributions for fins.

In the future, the application of Legendre neural networks-based soft computing algorithms can be extended to solve highly nonlinear and stiff models arising in different applications of practical interest.

APPENDIX

Approximate solution obtained by proposed algorithm for problem I with $N_c = 0.5, 1.0, 2.0$ and $4.0$ are given as $x_{\text{Approx}(v)}$:

\[
\begin{align*}
&= -0.0056999 + (-0.9421679 + 1.3079032)(-0.3347157) \\
&+ \frac{5(0.6428925 + 0.00505866){v}^2}{2}(-0.3683929) \\
&+ \frac{5(1.052841 + 0.002541)(-3.1, 0.152841 + 0.00241)}{(0.70563)} \\
&+ \frac{5(-1.086182 - 0.00990)2^2(-3.0 + 1.086182 - 0.00990)^2 + 2}{8}(0.459222) \\
&+ \frac{6.3(-0.031831 + 0.06092592)^5}{8} - 0.3813831 + 0.06092592)^3(0.10915186) \\
&+ \frac{23\left(2.7693591 - 0.3620337\right)^5}{16} - 315\left(2.7693591 - 0.3620337\right)^3(0.9233865) \\
&+ \frac{42\left(-0.004245 + 0.0242191\right)^5}{16} - 0.004245 + 0.0242191)^3(-0.148039) \\
&+ \frac{64\left(5.18189 + 0.002721\right)^5}{16} - 5.18189 + 0.002721)^3(-0.49971) \\
&+ \frac{12\left(7.409142 + 0.140242\right)^5}{12} - 25.7409142 + 0.140242)^3(-0.2704175) \\
&+ \frac{12\left(8.0180 + 0.140242\right)^5}{12} - 62.0180 + 0.140242)^3(-0.2704175) \\
&+ \frac{12\left(10.0180 + 0.140242\right)^5}{12} - 100.0180 + 0.140242)^3(-0.2704175)
\end{align*}
\]
TABLE 18. Nomenclature.

| Abbreviation | Description | Abbreviation | Description |
|--------------|-------------|--------------|-------------|
| LeNN         | Legendre Neural Networks | α           | Reflection Coefficient |
| NM           | Nelder-Mead Algorithm     | β           | Expansion Coefficient |
| MAD          | Mean Absolute Deviation   | γ           | Contraction Coefficient |
| TSC          | Thiel’s inequality coefficient | δ          | Shrink Coefficient |
| NSE          | Nash Satellife Efficiency | L           | Axial length, |
| ENSF         | Error In Nash Satellife Efficiency | T㎜ | Base temperature, |
| χ            | Dimensionless constant in the thermal conductivity | ε           | Surface emissivity, |
| x            | Dimensionless axial distance measured from the tip | T[subscript]s | Sink temperature |
| X            | Axial distance measured from the fin’s tip | T[subscript]a | Ambient temperature, |
| p            | Random number | δ[subscript]f | Arbitrary constants |
| X*           | Best solution of WOA so far | A           | Cross section |
| N[subscript]c | Coefficient of convective heat loss | α[subscript]t | Expansion point in NM |
| N[subscript]r | Coefficient of radiative heat loss | A, C, F | Coefficient vectors |
| x[subscript]b | Worst value | α[subscript]b | Best value |

θ[subscript]approx (x) = \( \left( \frac{46189(0.106572a + 0.033894) + 0.709592 + 0.015608}{2} \right) (0.31448) \)

\( \theta_{approx} (x) = -0.0814582 + \left( 0.57864707x - 0.036472 \right) \left( 0.06739899 \right) \)

\( \left( \frac{46189(0.106572a + 0.033894) + 0.709592 + 0.015608}{2} \right) (0.31448) \)

\( \theta_{approx} (x) = -0.0814582 + \left( 0.57864707x - 0.036472 \right) \left( 0.06739899 \right) \)
Approximate solution obtained by proposed algorithm for problem II with $\lambda = 1.0$, 2.0 and 3.0 are given as

$\theta_{\text{approx}}(x)$

$\approx 0.269358444 + (0.98537096 + 0.821680891 - 0.4566929 + \ldots)$

$\approx \left(\frac{303.30180849 \pm 0.3045639^2}{2} \right) - \left(\frac{0.00057639}{2} \right)$

$\approx \left(\frac{50.071773 \pm 0.342294^2}{2} \right) - \left(\frac{0.329989}{2} \right)$

$\approx \left(\frac{35.033552 \pm 0.177395^2}{2} \right) - \left(\frac{0.0344}{2} \right)$

$\approx \left(\frac{63.016086 \pm 0.0195^2}{2} \right) - \left(\frac{0.0875}{2} \right)$

$\approx \left(\frac{231.0 \pm 0.030060 \pm 0.033375}{2} \right) - \left(\frac{0.01181}{2} \right)$

$\approx \left(\frac{4290.15173 \pm 0.2993^2}{2} \right) - \left(\frac{0.40698}{2} \right)$

$\approx \left(\frac{64350.01496 \pm 0.07187^2}{2} \right) - \left(\frac{0.02488}{2} \right)$

$\approx \left(\frac{12155.00461 \pm 0.27626^2}{2} \right) - \left(\frac{0.43388}{2} \right)$

$\approx \left(\frac{64189.01071 \pm 0.01347^2}{2} \right) - \left(\frac{0.23235}{2} \right)$

$\approx -0.0569247 + (0.9198956 \pm 0.4178017 \pm 0.01557818)$

$\approx \left(\frac{30.00405664 \pm 0.39807511^2}{2} \right) - \left(\frac{0.009736}{2} \right)$

$\approx \left(\frac{50.4770096 \pm 0.036763^2}{2} \right)$
Approximate solution obtained by proposed algorithm for problem III with $\lambda = 1.0, 2.0$ and $3.0$ are given as

$$
\theta_{\text{approx}}(\lambda) = \frac{0.6257893 + (-0.1167977 + 0.70370075)(-0.5616733)}{2} + \frac{3(0.29899 \cdot -0.460353)^2 - 2574(0.29899 \cdot -0.460353)^3}{-0.63660}
$$

$$
\theta_{\text{approx}}(\lambda) = \frac{0.6257893 + (-0.1167977 + 0.70370075)(-0.5616733)}{2} + \frac{3(0.29899 \cdot -0.460353)^2 - 2574(0.29899 \cdot -0.460353)^3}{-0.63660}
$$

Approximate solution obtained by proposed algorithm for problem IV with $\lambda = 1.0, 2.0$ and $3.0$ are given as

$$
\theta_{\text{approx}}(\lambda) = \frac{0.6257893 + (-0.1167977 + 0.70370075)(-0.5616733)}{2} + \frac{3(0.29899 \cdot -0.460353)^2 - 2574(0.29899 \cdot -0.460353)^3}{-0.63660}
$$

$$
\theta_{\text{approx}}(\lambda) = \frac{0.6257893 + (-0.1167977 + 0.70370075)(-0.5616733)}{2} + \frac{3(0.29899 \cdot -0.460353)^2 - 2574(0.29899 \cdot -0.460353)^3}{-0.63660}
$$

$$
\theta_{\text{approx}}(\lambda) = \frac{0.6257893 + (-0.1167977 + 0.70370075)(-0.5616733)}{2} + \frac{3(0.29899 \cdot -0.460353)^2 - 2574(0.29899 \cdot -0.460353)^3}{-0.63660}
$$
\[ \begin{aligned}
&+ \left( \frac{2510.09838 \times 0.3545}{8} - 3150.09838 \times 0.3545 \right) \left( -0.10316 \right) \\
&+ \left( \frac{4290.50656 \times 0.03962}{8} - 6903.50656 \times 0.03962 \right) \left( 0.27744 \right) \\
&+ \left( \frac{6455.81358 \times 0.3364}{8} - 12012.81358 \times 0.3364 \right) \left( 0.27103 \right) \\
&+ \left( \frac{690 \times 61358 \times 0.3364}{8} - 1260 \times 81358 \times 0.3364 \right) \left( 35 \right) \\
&+ \left( \frac{12155 \times 0.15177 \times 0.01045}{8} - 25740 \times 0.15177 \times 0.01045 \right) \left( -0.6385 \right) \\
&+ \left( \frac{+1801 \times 0.15177 \times 0.01045}{8} - 46200 \times 0.15177 \times 0.01045 \right) \left( 3 \right) \\
&+ \left( \frac{128 \times 3465 \times 0.19393 \times 0.32061}{8} - 63 \right) \left( 0.16558 \right)
\end{aligned} \]

\[ \theta_{\text{inlet}}(x) = \frac{1.7752421 \times (0.6440502 + 1.61619197) - 0.0034589}{2} - 1 \left( -0.151075 \right) \]

\[ \begin{aligned}
&+ \left( \frac{5.0 \times 0.03952 + 0.009672}{8} \times 3.0 \times 0.03952 + 0.009672 \right) \left( 0.08889 \right) \\
&+ \left( \frac{35.0 \times 0.09890 + 0.0021672}{8} - 30.0 \times 0.09890 + 0.0021672 \right) \left( 3 \right) \left( 0.08226 \right) \\
&+ \left( \frac{63.7 \times 0.19393 \times 0.01045}{8} - 70.0 \times 0.19393 \times 0.01045 \right) \left( 0.45685 \right) \\
&+ \left( \frac{15.0 \times 0.19393 \times 0.01045}{8} - 18.0 \times 0.19393 \times 0.01045 \right) \left( -0.00279 \right) \\
&+ \left( \frac{2310 \times 0.2266 \times 0.47763}{8} - 315.0 \times 0.2266 \times 0.47763 \right) \left( 0.93020 \right) \\
&+ \left( \frac{16.0 \times 0.2266 \times 0.47763}{8} - 16 \times 0.2266 \times 0.47763 \right) \left( 0.00359 \right) \\
&+ \left( \frac{12155 \times 0.005151 \times 0.97125}{8} - 25740 \times 0.005151 \times 0.97125 \right) \left( -0.01223 \right) \\
&+ \left( \frac{+1801 \times 0.005151 \times 0.97125}{8} - 46200 \times 0.005151 \times 0.97125 \right) \left( 0.0026 \right) \\
&+ \left( \frac{128 \times 3465 \times 0.19393 \times 0.32061}{8} - 63 \right) \left( -0.00882 \right)
\end{aligned} \]

References:

[1] A. Kraus, A. Aziz, and J. Welty, “Heat transfer considerations,” in *Extended Surface Heat Transfer*. New York, NY, USA: Wiley, 2002.

[2] D. W. Mueller and H. I. Abu-Mulaweh, “Prediction of the temperature in a fin cooled by natural convection and radiation,” *Appl. Thermal Eng.*, vol. 26, nos. 14–15, pp. 1662–1668, Oct. 2006.

[3] A. S. Dogonchi and Hashim, “Heat transfer by natural convection of Fe2O3-water nanofluid in an annulus between a wavy circular cylinder and a rhombus,” *Int. J. Heat Mass Transf.*, vol. 130, pp. 320–332, Mar. 2019.

[4] M. I. Khan, S. Qayyum, S. Kadry, W. A. Khan, and S. Z. Abbas, “Irreversibility analysis and heat transport in squeezing nanoliquid flow of non-Newtonian (second-grade) fluid between infinite plates with activation energy,” *Arabian J. Sci. Eng.*, vol. 45, no. 6, pp. 4939–4947, Jun. 2020.

[5] M. K. Nayak, A. A. K. Hakeem, B. Banga, M. I. Khan, M. Waqas, and O. D. Makinde, “Entropy optimized MHD 3D nanomaterial of non-Newtonian fluid: A combined approach to good absorber of solar energy and intensification of heat transport,” *Comput. Methods Programs Biomol.*, vol. 186, Apr. 2020, Art no. 105131.

[6] M. I. Khan and F. Alzahrani, “Transportation of heat through Cattaneo–Christov heat flux model in non-Newtonian fluid subject to internal resistance of particles,” *Appl. Math. Mech.*, vol. 41, no. 8, pp. 1157–1166, Aug. 2020.

[7] M. Iqaz Khan and F. Alzahrani, “Cattaneo–Christov double diffusion (CCDD) and magnetized stagnation point flow of non-Newtonian fluid with internal resistance of particles,” *Phys. Scripta*, vol. 95, no. 12, Nov. 2020, Art no. 125002.

[8] M. I. Khan, “Transportation of hybrid nanoparticles in forced convective Darcy–Forchheimer flow by a rotating disk,” *Int. Commun. Heat Mass Transf.*, vol. 122, Mar. 2021, Art no. 105177.

[9] A. Zeehsan, N. Shehzad, R. Ellahi, and S. Z. Alamri, “Convective Poiseuille flow of Al2O3–EG nanofluid in a porous wavy channel with thermal radiation,” *Neural Comput. Appl.*, vol. 30, no. 11, pp. 3371–3382, Dec. 2018.

[10] N. Shehzad, A. Zeehsan, R. Ellahi, and K. Vafai, “Connective heat transfer of nanofluid in a wavy channel: Buongiorno’s mathematical model,” *J. Mol. Liquids*, vol. 222, pp. 446–455, Oct. 2016.

[11] C.-H. Chiu and C.-K. Chen, “A decomposition method for solving the convective longitudinal fins with variable thermal conductivity,” *Int. J. Heat Mass Transf.*, vol. 45, no. 10, pp. 2067–2075, May 2002.

[12] C. Arslanturk, “A decomposition method for fin efficiency of convective straight fins with temperature-dependent thermal conductivity,” *Int. Commun. Heat Mass Transf.*, vol. 32, no. 6, pp. 831–841, May 2005.

[13] A. Rajabi, D. D. Ganji, and H. Taherian, “Application of homotopy perturbation method in nonlinear heat conduction and convection equations,” *Phys. Lett. A*, vol. 360, nos. 4–5, pp. 570–573, Jan. 2007.
S. B. Coşkun and M. T. Atay, “Determination of the fin efficiency of convective straight fins with temperature dependent thermal conductivity by using homotopy perturbation method,” Int. J. Numer. Methods Heat Fluid Flow, vol. 22, no. 2, pp. 263–272, Mar. 2012.

F. Khani and A. Aziz, “Thermal analysis of a longitudinal trapezoidal fin with temperature dependent thermal conductivity and heat transfer coefficient,” Commun. Nonlinear Sci. Numer. Simul., vol. 15, no. 3, pp. 590–601, Mar. 2010.

D. D. Ganji, A. A. Joneidi, and M. Babaelahi, “Differential transformation method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity,” Int. Commun. Heat Mass Transf., vol. 36, no. 7, pp. 757–762, Aug. 2009.

S. Abbasbandy and E. Shivanian, “New solution for the nonlinear fin problem of variable thermal conductivity, heat transfer coefficient and surface emissivity,” Therm. Sci., vol. 11, no. 5, pp. 3287–3294, Oct. 2010.

S. Kim and C.-H. Huang, “A series solution of the nonlinear fin problem with temperature-dependent thermal conductivity and heat transfer coefficient,” J. Phys. D. Appl. Phys., vol. 40, no. 9, p. 2979, 2007.

R. J. Moitsheki, T. Hayat, and M. Y. Malik, “Some exact solutions of the fin problem with a power law temperature-dependent thermal conductivity,” Nonlinear Anal. Real World Appl., vol. 11, no. 5, pp. 3287–3294, Oct. 2010.

M. Miansari, D. D. Ganji, and M. Miansari, “Application of He’s variational iteration method,” Commun. Nonlinear Sci. Numer. Simul., vol. 15, no. 3, pp. 590–601, Mar. 2010.

D. D. Ganji, G. A. Afrouzi, and R. A. Talarposhti, “Application of variational iteration method and homotopy–perturbation method for nonlinear heat diffusion and heat transfer equations,” Phys. Lett. A, vol. 368, no. 6, pp. 450–457, Sep. 2007.

S. B. Coskun and M. T. Atay, “Fin efficiency analysis of convective straight fins with temperature dependent thermal conductivity using variational iteration method,” Appl. Thermal Eng., vol. 28, nos. 17–18, pp. 2345–2352, Dec. 2008.

A. A. Joneidi, D. D. Ganji, and M. Babaeejazi, “Differential transformation method to determine fin efficiency of convective straight fins with temperature dependent thermal conductivity,” Int. Commun. Heat Mass Transf., vol. 36, no. 7, pp. 757–762, Aug. 2009.

M. G. Sobamowo, “Thermal analysis of longitudinal fin with temperature-dependent properties and internal heat generation using Galerkin’s method of weighted residual,” Appl. Thermal Eng., vol. 99, pp. 1316–1330, Apr. 2016.

S.-W. Sun and X.-F. Li, “Exact solution of a nonlinear fin problem of convective–radiative longitudinal porous fins with various profiles and multiple nonlinearities,” Int. J. Mech. Sci., vol. 136, pp. 252–263, Feb. 2018.

M. N. Bouaziz and A. Aziz, “Simple and accurate solution for convective–radiative fin with temperature dependent thermal conductivity using double optimal linearization,” Energy Convers. Manage., vol. 51, no. 12, pp. 2776–2782, Dec. 2010.

Y. Huang and X.-F. Li, “Exact and approximate solutions of convective-radiative fins with temperature-dependent thermal conductivity using integro-differential equation method,” Int. J. Heat Mass Transf., vol. 150, Apr. 2020, Art. no. 119303.
[55] W. Huang, T. Jiang, X. Zhang, N. A. Khan, and M. Sulaiman, “Analysis of beam-column designs by varying axial load with internal forces and bending rigidity using a new soft computing technique,” *Complexity*, vol. 2021, pp. 1–19, Mar. 2021.

[56] W. Waseem, M. Sulaiman, P. Kumam, M. Shoabi, M. A. Z. Raja, and S. Istan, “Investigation of singular ordinary differential equations by a neuroevolutionary approach,” *PLoS ONE*, vol. 15, no. 7, Jul. 2020, Art. no. e0235829.

[57] N. A. Khan, M. Sulaiman, P. Kumam, and A. J. Aljohani, “A new soft computing approach for studying the wire coating dynamics with Oldroyd 8-constant fluid,” *Phys. Fluids*, vol. 33, no. 3, Mar. 2021, Art. no. 036117.

[58] A. Ahmad, M. Sulaiman, A. Alhindi, and A. J. Aljohani, “Analysis of temperature profiles in longitudinal fin designs by a novel neuroevolutionary approach,” *IEEE Access*, vol. 8, pp. 113285–113308, 2020.

[59] W. Waseem, M. Sulaiman, S. Islam, P. Kumam, R. Nawaz, M. A. Z. Raja, M. Farooq, and M. Shoabi, “A study of changes in temperature profile of porous fin model using cuckoo search algorithm,” *Alexandria Eng. J.*, vol. 59, no. 1, pp. 11–24, 2020.

[60] W. Waseem, M. Sulaiman, A. Alhindi, and H. Alhakami, “A soft computing approach based on fractional order DPSO algorithm designed to solve the corneal model for eye surgery,” *IEEE Access*, vol. 8, pp. 61576–61592, 2020.

[61] Y. Zhang, J. Lin, Z. Hu, N. A. Khan, and M. Sulaiman, “Analysis of third-order nonlinear multi-singular Emden-Fowler equation by using the LeNN-WOA-NM algorithm,” *IEEE Access*, vol. 9, pp. 72111–72138, 2021.

[62] A. Ali, M. Hamraz, P. Kumam, D. M. Khan, U. Khalil, M. Sulaiman, and Z. Khan, “A k-nearest neighbours based ensemble via optimal model selection for regression,” *IEEE Access*, vol. 8, pp. 132095–132105, 2020.

[63] M. Sulaiman, M. Masihullah, Z. Hussain, S. Ahmad, W. K. Mashwani, M. A. Jan, and R. A. Khanum, “Implementation of improved grasshopper optimizer algorithm to solve economic load dispatch problems,” *Hacettepe J. Math. Stat.*, vol. 48, no. 5, pp. 1570–1589, 2019.

[64] A. H. Bukhari, M. A. Z. Raja, M. Sulaiman, S. Islam, M. Shoabi, and P. Kumam, “Fractional neuro-sequential ARFIMA-LSTM for financial market forecasting,” *IEEE Access*, vol. 8, pp. 71326–71338, 2020.

[65] M. Sulaiman, A. Ahmad, A. Khan, and S. Muhammad, “Hybridized symbiotic organism search algorithm for the optimal operation of directional overcurrent relays,” *Complexity*, vol. 2018, pp. 11–11, Jan. 2018.

[66] M. Sulaiman, I. Samiullah, A. Hamdi, and Z. Hussain, “An improved whale optimization algorithm for solving multi-objective design optimization problem of PFHE,” *J. Intell. Fuzzy Syst.*, vol. 37, no. 3, pp. 3815–3828, Oct. 2019.

[67] A. H. Bukhari, M. Sulaiman, S. Islam, M. Shoabi, P. Kumam, and M. A. Z. Rajaf, “Neuro-fuzzy modeling and prediction of summer precipitation with application to different meteorological stations,” *Alexandria Eng. J.*, vol. 59, no. 1, pp. 101–116, 2020.

[68] H. Javed, M. A. Jan, N. Tairan, W. K. Mashwani, R. A. Khanum, M. Sulaiman, H. U. Khan, and H. Shah, “On the efficiency of ensemble of constraint handling techniques in self-adaptive differential evolution,” *Mathematics*, vol. 7, no. 7, p. 635, Jul. 2019.

[69] N. A. Khan, M. Sulaiman, A. J. Aljohani, P. Kumam, and H. Alrabaiah, “Analysis of multi-phase flow through porous media for inhibition phenomena by using the LeNN-WOA-NM algorithm,” *IEEE Access*, vol. 8, pp. 196425–196458, 2020.

[70] A. Muzzio, “Approximate solution for convective fins with variable thermal conductivity,” *J. Heat Transf.*, vol. 98, no. 4, pp. 680–682, Nov. 1976.

[71] C.-N. Zhang and X.-F. Li, “Temperature distribution of convective-convective-radiative fins with temperature-dependent thermal conductivity,” *Int. Commun. Heat Mass Transf.*, vol. 117, Oct. 2020, Art. no. 104799.

[72] E. W. Weisstein, “Legendre polynomial,” *Math. Methods*, p. 43, Jun. 2015.

[73] S. Mirjalili and A. Lewis, “The whale optimization algorithm,” *Adv. Eng. Softw.*, vol. 95, pp. 51–67, May 2016.

[74] S. Singer and J. Nelder, “Nelder-mead algorithm,” *Scholarpedia*, vol. 4, no. 7, p. 2928, 2009.

[75] R. Jovanovic, S. Kais, and F. H. Alharbi, “Cuckoo search inspired hybridization of the nelder-mead simplex algorithm applied to optimization of photovoltaic cells,” 2014, arXiv:1411.0217. [Online]. Available: http://arxiv.org/abs/1411.0217

[76] B. Lia, “A simplex search method for a conductive-convective fin with variable conductivity,” *Int. J. Heat Mass Transf.*, vol. 54, nos. 23–24, pp. 5001–5009, Nov. 2011.

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