HODGE NUMBERS OF ARBITRARY SECTIONS FROM LINEAR
SECTIONS

HERBERT CLEMENS

To Enrico Arbarello, my first graduate student and dear life-long friend.

Abstract. Let $|L|$ be the total space of the inverse of a very ample line
bundle $\pi : L^{-1} \to B$ over a projective manifold $B$. Any section of $L^{-1} \to B$
is isomorphic to $B$ and the Hodge numbers of any proper smooth multisection
are determined by the degree $d$ of that multi-section as are the Hodge numbers
of any smooth complete intersection of multi-sections of degrees $(d_1, \ldots, d_r)$.
In this paper recursive formulae are given for those Hodge numbers in terms
of the integers $\{d_1, \ldots, d_r\}$ and the Hodge numbers of the linear sections
$B, B\cdot B, \ldots, B^\dim B$.

The recursion proceeds by induction on dimension and degree. Its proof relies
on the theory of asymptotic mixed Hodge structures.

An interesting corollary is that the Lefschetz hyperplane property is weak-
ened by one degree in this setting. That is, relative vanishing does not reach
the middle degree of the hyperplane section but only to degree one less than
the middle degree.

As an application, in an Appendix, we calculate closed formulae for all
Hodge numbers of all smooth complete intersections for the case $\dim B = 3$.

Contents

1. Introduction 2
2. Computing Hodge numbers by induction on dimension and degree 3
   2.1. Case One: Projective submanifolds of the total space of a negative
        line bundle on a projective manifold 3
   2.2. Case Two: Smooth hypersurface sections of a projective manifold 4
   2.3. Recursive formula 5
3. Common setting for Cases One and Two 6
   3.1. Small resolution of linear family over the $t$-disk 6
   3.2. Topological decomposition of $V_d$ 6
   3.3. The governing exact diagram of morphisms of mixed Hodge structures 7
   3.4. Mixed Hodge structures 8
4. The computation 9
   4.1. Duality and the Lefschetz hyperplane theorem 9
   4.2. Accommodating the exceptional locus of the small resolution 10
   4.3. Ampleness, the ‘hard’ Lefschetz theorem and the circle bundle $T$ 10
   4.4. The monodromy operator 11
   4.5. Breaking down the governing diagram 12
   4.6. The theorem 13

Date: November 4, 2022.
Appendix A. Hodge numbers of proper complete intersections in $Y = |L|$ for a very ample line bundle $L^{-1}$ on a complex projective threefold $B$

A.1. Hodge numbers of curves $(V_{d_0} \cap V_{d_1} \cap V_{d_2}) \subseteq |L| \subseteq Q$
A.2. Hodge numbers of surfaces $(V_{d_0} \cap V_{d_2}) \subseteq |L| \subseteq Q$
A.3. Hodge numbers of the threefolds $V_d \subseteq |L| \subseteq Q$

References

1. Introduction

Long ago the author was among those who worked out the asymptotic mixed Hodge theory for a one-parameter degeneration of Kähler manifolds in the complex analytic setting [1]. Quite recently, as part of a collaboration with string theorists, I was called upon to compute the Hodge numbers of certain elliptically fibered Calabi-Yau threefolds and fourfolds in settings where standard computational tools are less accessible. Even though it is well established that such formulae 'can be computed,' I found it annoying that I could not find in the literature closed formulae for Hodge numbers of complete intersections of a fixed complex projective manifold of dimension $n + 1$ whose Hodge numbers were given, or more generally of complete intersections of multi-sections of an ample line bundle over a complex projective manifold of dimension $n$ whose Hodge numbers were given, even for low values of $n$. The purpose of this paper is to lay out the recursive algorithm that allows the derivation of such formulas in all dimensions and to derive such formulas for $n \leq 3$.

Although the computations involved may require some patience to follow, the principle by which they are derived is quite simple, any complete intersection $V$ specializes linearly to the transverse union of two complete intersections $V' \cup V''$ of lower degree unless the degree was one to start with. Then asymptotic mixed Hodge theory allows one to read off the Hodge numbers of the original complete intersection from those of the two components $V'$ and $V''$ of the specialization, the intersection $V' \cap V''$ of those two components, and finally the intersection $V \cap V' \cap V''$ of $V$ with the intersection of those two components.

The computational method is by complete induction on degrees and dimension, using asymptotic mixed Hodge theory 'in reverse.' Using modern mathematical software it is presumably a simple matter to write a closed formula for the Hodge numbers of all complete intersections for any fixed $n$. In the Appendix to this paper we provide a roadmap for such a program by working out the formulas for all proper complete intersections in the case in which $n \leq 3$.

'In reverse' simply means that since the dimensions of the graded pieces of the Hodge filtration of the asymptotic mixed Hodge structure are the same as those for the Hodge structure of the nearby smooth fiber (see [2], §13), and the latter can be more easily computed by computing the former, namely for a degeneration that breaks the nearby fiber into simpler pieces in the limit as described above. This strategy is certainly not a new one, going way back to the 19th-century technique of computing the genus of a smooth plane curve of degree $d$ by degenerating it into a union of $d$ projective lines meeting transversally, or inductively, degenerating it into a union of two transversely intersecting smooth curves of degrees $d_1$ and $d_2$ respectively where $d_1 + d_2 = d$. Our setting in Case One below is projective...
submanifolds of the total space of $Y = |L|$ of the inverse of a very ample line bundle $\pi : L^{-1} \to B$ over a projective manifold $B$. The zero scheme of any section of $L^{-1} \to B$ is isomorphic to $B$ and the Hodge numbers of any proper smooth multisection are determined by the degree $d$ of that multi-section as are the Hodge numbers of any smooth complete intersection of multi-sections of degrees $(d_1, \ldots, d_r)$. In this paper recursive formulae are given for computing those Hodge numbers in terms of the integers $\{d_1, \ldots, d_r\}$ and the Hodge numbers of the linear sections. The same method works when $Y$ is a project manifold (Case Two below) although in that case, the derivation of Hodge numbers of complete intersections is well known.

However we believe the case where $Y$ is the total space of a negative line bundle on a projective manifold (Case One below) is new. We include both cases in the combined exposition below, since it requires no additional work and the explicit formulae in the case in which $Y$ is compact are needed in the derivation in which $Y$ is non-compact. Also, it seems that these formulas, even in the $Y$-compact case, do not explicitly appear in the literature. In short each induction step is always established as described above, namely by using that, if a smooth proper divisor $V_d$ specializes linearly to a sum of smooth divisors $V_{d_1} + V_{d_2}$ where $d = d_1 + d_2$, allowing us to derive the Hodge numbers $h^{p,q}(V_d)$ as simple sums of certain Hodge numbers of $V_{d_1}, V_{d_2}, V_{d_1} \cap V_{d_2}$, and $V_d \cap V_{d_1} \cap V_{d_2}$. A corollary in Case One of the derivation is that the Lefschetz hyperplane property is weakened by one degree in the Case One setting. That is, relative vanishing does not reach the middle degree of the hypersurface section but only to degree one less than the middle degree.

2. Computing Hodge numbers by induction on dimension and degree

2.1. Case One: Projective submanifolds of the total space of a negative line bundle on a projective manifold. Let $B \subseteq \mathbb{P}^M = \mathbb{P} \left( H^0 \left( L^{-1} \right) \right)$ be a smooth complex projective manifold of dimension $n$ imbedded by the complete linear system of a very ample line bundle $L/B$. We assume that $B$ does not contain the point $[1, 0, \ldots, 0] \in \mathbb{P}^M$. Then the zero-schemes of homogeneous forms

$$F(y) \in \text{Sym}^d \left( H^0 \left( L \right) \right)$$

be a line bundle such that $L^{-1} = \mathcal{O}_B (N)$ is very ample so that its complete linear system gives an imbedding

$$B \to \mathbb{P} \left( H^0 \left( L^{-1} \right) \right) = \mathbb{P}^M,$$

We denote homogeneous coordinates

$$\mathbb{P}_{[y_0, \ldots, y_M]} = \mathbb{P}^M$$

and let $|\mathcal{O}_{\mathbb{P}^M} (-1)|$ denote the total space of the line bundle $\mathcal{O}_{\mathbb{P}^M} (-1)$ and form the fibered product

$$\begin{align*}
Y & \to |\mathcal{O}_{\mathbb{P}^M} (-1)| \\
\downarrow & \\
B & \to \mathbb{P}^M.
\end{align*}$$
Definition 1. We call a homogeneous form
\[ F(y) \in \text{Sym}^d \left( H^0(L) \right) = \text{Sym}^d \left( \mathbb{C}[y_0, \ldots, y_M] \right) \]
monic if \( F([1, 0, \ldots, 0]) \neq 0 \), that is, the coefficient of \( y_0^d \) in \( F \in \mathbb{C}[y_0, \ldots, y_M] \) is not zero.

If \( F(y) \) is monic, then we have 'affine' coordinate \( \vartheta = y_0 \in H^0 \left( \mathcal{O}_{\mathbb{P} \left( H^0(L^{-1}) \right)}(1) \right) \)
and we can write
\[ F(y) = \sum_{j=0}^d f_j(y_1, \ldots, y_M) \cdot \vartheta^{d-j} \]
where \( f_j \) is homogeneous of degree \( j \). Also
\[ (F) := \{ y : F(y) = 0 \} \subseteq Y \]
is proper over \( B \) and forms a \( d \)-sheeted branched cover of \( B \). Given monic \( F_d(y) \) and \( G_d(y) \) we can form a linear family\n\[ \{ t \cdot F_d(y) + G_d(y) = 0 \} \subseteq \Delta \times Y, \]
so that all of its forms for \( |t| < \varepsilon \) are monic and so its fibers over the \( t \)-disk \( \Delta \) are proper. Thus 'linear variation' and 'degeneration of projective varieties' makes sense in this context. For monic \( F_1(y) \in H^0 \left( \mathcal{O}_{\mathbb{P}M}(1) \right) \) and \( \{ F_1(y) = 0 \} \cap Y =: V_1 \)
is isomorphic to \( B \), \( V_1 \cdot V_1 \) is a divisor in the linear system associated with \( L^{-1} \), etc., and finally the cardinality of \( V_1^{\dim B} \) is the degree of \( B \subseteq \mathbb{P}^M \).

For each degree \( d \) and monic \( F_d \) we write
\[ V_d := \{ F_d = 0 \} \subseteq |L| \]
and if \( d = d_1 + d_2 \), we can form the linear degeneration
\[ \{ t \cdot F_d(y) - F_{d_1}(y) \cdot F_{d_2}(y) = 0 \} \subseteq \mathbb{C} \times |L|. \]

Our goal will be to explicitly derive the Hodge numbers of
\[ V_{d_1} \cdot \ldots \cdot V_{d_r} \]
for \( r < n \) from the Hodge numbers of \( B \). Of course if \( r = n \), the zero-th Hodge number is
\[ d_1 \cdot \ldots \cdot d_n \cdot \deg B. \]

2.2. Case Two: Smooth hypersurface sections of a projective manifold.
An alternative situation is that in which \( Y \) is a projective manifold of dimension \( n+1 \) and \( B \) is a smooth hyperplane section. Here the 'monic' condition is irrelevant but the same recursive formula given in Theorem 2 below applies. In this case one uses the same reasoning as in Case One to derive explicit recursive formulas for the Hodge numbers of complete intersections of any projective manifold from the Hodge numbers of the manifold \( Y \) and the (middle) Hodge numbers of a smooth hyperplane section. In this case Theorem 2 below applies to the setting
\[ \{ t \cdot F_d(y) - F_{d_1}(y) \cdot F_{d_2}(y) = 0 \} \subseteq \mathbb{C} \times Y \subseteq \mathbb{C} \times \mathbb{P}^M \]
where \( B \) is a smooth hyperplane section of \( Y \), \( \mathcal{O}_B(N) = \mathcal{O}_{\mathbb{P}^M}(1)|_B \), and \( F_d \) is any homogeneous form of degree defining a smooth hypersurface of \( Y \).
2.3. **Recursive formula.** In either Case One or Case Two, let

\[ V_{d_1} \cdot \ldots \cdot V_{d_r} \]

denote the transverse intersection of \( r \) smooth proper hypersurfaces of \( Y \) of degrees \( d_1, \ldots, d_r \) respectively. The purpose of note is to prove the following:

**Theorem 2.** In either Case One or Case Two above, given the Hodge numbers of \( V \) on dimension and degree using only the following formulae:

\[ V, V^1, \ldots, V^n, \]

the Hodge numbers of \( V_{d_1} \cdot \ldots \cdot V_{d_r} \) can be computed recursively by complete induction on dimension and degree using only the following formulae:

Let \( V_d := V_{d_1} + V_{d_2} \) be as in either (2.1) or (2.2) above, and let \( V_{d_2} \) denote the blow-up of \( V_{d_2} \) blown up along the submanifold \( V_d \cap V_{d_1} \cap V_{d_2} \).

For \( p + q = k > n + 1 \)

\[ h^{p,q}(V_d) = h^{p,q}(V_1), \]

for \( k = p + q = n + 1 \)

\[ h^{p,q}(V_d) = h^{p,q}(\ker (H^{n+1}(V_{d_1}) \oplus H^{n+1}(V_{d_2}) \to H^{n+1}(V_{d_1} \cap V_{d_2}))) - h^{p,q}(V_{d_1} \cap V_{d_2}) = \begin{cases} \dim(\ker (H^{p,q}(V_{d_1}) \oplus H^{p,q}(V_{d_2}) \to H^{p,q}(V_{d_1} \cap V_{d_2}))) \\ + h^{p-1,q-1}(V_{d_1} \cap V_{d_2}) - h^{p,q}(V_{d_1} \cap V_{d_2}) \end{cases}, \]

and for \( p + q = k = n \)

\[ h^{p,q}(V_d) = \begin{cases} h^{p,q}_{prim}(V_{d_1} \cap V_{d_2}) + h^{p,q}_{prim}(V_{d_2}) + h^{p-1,q-1}(V_d \cap V_{d_1} \cap V_{d_2}) \\ + h^{p,q}_{prim}(V_{d_1} \cap V_{d_2}) \end{cases}. \]

**Remark 3.** The complete induction in the above theorem in Case One requires application of either Case One or Case Two with prior intersections playing the role of \( Y \) at various steps.

As an example of the application of Case Two of the formulae in Theorem 2 we calculate the Hodge numbers of the quintic threefold \( X_5 \subseteq \mathbb{P}^4 = Y \) by degenerating it linearly

\[ \{ t \cdot G_5 + G_3, G_2 = 0 \} \subseteq \mathbb{C} \times \mathbb{P}^4 \]

into the union of a cubic threefold \( X_3 \) and a quadric threefold \( X_2 \), taking care that the intersection \( X_5 \cap X_3 \cap X_2 \) is transverse. Rewriting the equation of the family as

\[ \begin{bmatrix} t & G_2 \\ G_3 & G_5 \end{bmatrix} = 0 \subseteq \mathbb{C} \times \mathbb{P}^4 \]

we must do a small resolution over the curve \( C := \{ t = 0 \} \cap X_5 \cap X_3 \cap X_2 \) where all four entries of the above matrix vanish. This has the effect of blowing up \( C \) in either \( X_2 \) or \( X_3 \) and while modifying the degeneration to normal crossing form without base locus as considered in [1]. Noting that \( X_2 \cap X_3 \) is a K3-surface, Theorem 2 and the table

| \( h^{3,0}_{prim} \) | \( X_3 \) | \( X_2 \) | \( X_3 \cap X_2 \) | \( X_5 \cap X_3 \cap X_2 \) |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 19 |
allow us to compute conclude that

\[ h^{3,0}(X_5) = h^{3,0}(X_3) + h^{3,0}(X_2) + h^{2,0}(X_3 \cap X_2) = 1 \]
\[ h^{2,1}(X_5) = \left( h^{2,1}(X_3) + h^{2,1}(X_2) + h^{2,0}(X_3 \cap X_2) + h^{1,0}(X_5 \cap X_3 \cap X_2) \right) = 101. \]

3. Common setting for Cases One and Two

3.1. Small resolution of linear family over the t-disk. We begin by rewriting (2.1) or (2.2) in the form

\[
\begin{pmatrix} t & F_{d_1} \\ F_{d_2} & F_d \end{pmatrix} = 0
\]

from which one sees that the total space of the family in \( \mathbb{C} \times \mathbb{P}^M \) has a nodal locus on the set

\[ \{ t = F_{d_1} = F_{d_2} = F_d = 0 \} \]

Using either ratios of rows or ratios of columns in the above matrix gives two small resolutions of the total space of the family and deposits the exceptional locus in either \( V_{d_1} \) or \( V_{d_2} \). We choose \( V_{d_2} \) which is thereby blown up along the codimension-two submanifold \( \{ F_{d_1} = F_d = 0 \} \). We denote the blown up \( V_{d_2} \) as \( \tilde{V}_{d_2} \). We have Hodge subspaces

\[ H^{j,k}(\tilde{V}_{d_2}) = H^{j,k}(V_{d_2}) \oplus H^{j-1,k-1}(V_{d_1} \cap V_{d_2} \cap V_d) \oplus (1,1) \]

where \((1,1)\) denotes the so-called Hodge-Tate mixed Hodge structure (dimension 1, weight 2, pure type \((1,1)\)). Furthermore \( V_{d_1} \cap \tilde{V}_{d_2} = V_{d_1} \cap V_{d_2} \).

A detailed description of this normal-crossing degeneration and its asymptotic Hodge theory is given in \( \text{[1]} \).

3.2. Topological decomposition of \( V_d \). \( V_d \) is constructed topologically by first removing a small regular open neighborhood \( U_{V_{d_1} \cap V_{d_2}} \) of \( V_{d_1} \cap V_{d_2} \) from \( V_{d_1} \cup \tilde{V}_{d_2} \), giving respectively \( V_{d_1}' \) as a deformation retraction of \( V_{d_1} \setminus (V_{d_1} \cap V_{d_2}) \) and \( \tilde{V}_{d_2}' \) as a deformation retraction of \( \tilde{V}_{d_2} \setminus (V_{d_1} \cap V_{d_2}) \). Then \( \partial V_{d_1}' \cong \partial \tilde{V}_{d_2}' =: T \) where \( T \) is a circle bundle over \((V_{d_1} \cap V_{d_2})\) and the isomorphism reverses orientation. \( V_d \) is then given topologically by pasting \( V_{d_1}' \) to \( \tilde{V}_{d_2}' \) by identifying corresponding points of their common boundary. There is a natural 'contraction' mapping

\[(3.1) \quad \rho : V_d \to V_{d_1} \cup \tilde{V}_{d_2} \]

obtained by the inclusion of \( V_d \) into a regular neighborhood of \( V_{d_1} \cup \tilde{V}_{d_2} \) in the (smooth) total space of the family. These constructions are described in detail in §5-7 of \( \text{[1]} \) where the topology of the horizontally and vertically exact diagram...
is also explained as are the hypercohomology groups \( H^\ast \) characterized by the fact that their addition completes the cohomology group mappings

\[
\rho^\ast : V_d H^\ast (V_d \cup \tilde{V}_d) \to H^\ast (V_d)
\]

to a long exact cohomology sequence. The exact cohomology diagram \( \text{(3.2)} \) can be thought of as intertwining horizontal Mayer-Vietoris exact sequences with a vertical mapping cone exact sequence and vertical ‘log’ exact sequences derived from residue maps via the Gysin isomorphism.

3.3. The governing exact diagram of morphisms of mixed Hodge structures. We now apply the theory of mixed Hodge structures to the topological set-up described in the previous Subsection, in particular to the diagram \( \text{(3.2)} \). For readers unfamiliar with mixed Hodge theory, a gentle introduction can be found in \([1]\), a more in-depth treatment in \([5]\) and finally the full theory in \([3]\).

The cohomology mappings \( \text{(3.2)} \) associated to the mapping cone of \( \rho \) in \( \text{(3.1)} \) in the case of the the topological decomposition of \( H^\ast \) give rise to morphisms of mixed Hodge structures explained in Chapter 3 of \([1]\). Our strategy will be to deduce the Hodge numbers

\[
H^p (V_d; \mathbb{C}) = \sum_{p+q=n} H^{p,q} (V_d)
\]

from the Hodge numbers of all the other cohomology groups in \( \text{(3.2)} \), all of which involve only \( H^\ast (V_d') \) with \( d' < d \) and/or \( \dim V_d' < \dim V_d \). This and closed expressions for \( \sum_{k=1}^n k^m \) with \( m \leq n + 1 \) then allow us to recursively build closed formulas for Hodge numbers of proper complete intersections of \( Y \) for \( \dim Y \leq \).
We carry out the recursive algorithm in detail for \( n \leq 3 \) in the Appendix to this paper. It seems to the author that, following the paradigm in the Appendix, computer programs that make the corresponding calculations for arbitrary (fixed) \( n \) ought not to be too difficult to design.

### 3.4. Mixed Hodge structures.

For all the mixed Hodge structures we will consider, \( W \) will denote the weight filtration and \( F \) will denote the Hodge filtration. Taking to account that the rank-one Hodge-Tate structure \((1,1)\) has weight 2, all weight and Hodge filtration of terms are just the standard ones of the cohomology groups represented, except for \( H(X) \). The right-hand vertical exact sequence

\[
\cdots \rightarrow d N H^k (V_d \cap V_d) \rightarrow \cdots \rightarrow H^k \rightarrow H^k (T) \rightarrow \Res \rightarrow H^k (V_d \cap V_d) \rightarrow \cdots
\]

is the standard exact sequence for the circle bundle \( T/V_d \). \( A \) (\( T \)) is quasi-isomorphic to a mapping cone of

\[
(d N \cdot) \colon H^{-2} (V_d \cap V_d) \otimes (1,1) \rightarrow H (V_d \cap V_d)
\]

in the abelian category of mixed Hodge structures and that therefore endows \( H^k (T) \) with its mixed Hodge structure

\[
W_k (H^k (T)) = \image (\rho^*)
\]

\[
\frac{W_k (H^k (T))}{W_k (H^k (T))} = \ker ((d N \cdot) : H^{k-1} (V_d \cap V_d) \otimes (1,1) \rightarrow H^k (V_d \cap V_d))
\]

Since \( N \) is ample

\[
\ker ((d N \cdot) : H^{k-1} (V_d \cap V_d) \otimes (1,1) \rightarrow H^{k+1} (V_d \cap V_d)) = 0
\]

if \( k < n \) and

\[
\ker ((d N \cdot) : H^{k-2} (V_d \cap V_d) \otimes (1,1) \rightarrow H^k (V_d \cap V_d)) = 0
\]

if \( k \geq n \). So

\[
(3.3) \quad \begin{cases}
H^k (T) = W_k (H^k (T)) & \text{if } k < n \\
H^k (T) = \frac{W_k (H^k (T))}{W_k (H^k (T))} & \text{if } k \geq n.
\end{cases}
\]

Furthermore

\[
(3.4) \quad \frac{H^k (V_d \cap V_d)}{H^k (V_d \cap V_d)} \cong \begin{cases}
H^{k-2} (V_d \cap V_d) \otimes (1,1) & \text{if } k \leq n - 1 \\
H^{k+1} (V_d \cap V_d) & \text{if } k \geq n.
\end{cases}
\]

Also \( A' (T) \) is quasi-isomorphic to a mapping cone of

\[
A' (V_d) \rightarrow A' (\log (V_d \cap V_d)) \oplus A' (\log (V_d \cap V_d))
\]

inducing the same mixed Hodge structure.

The cohomology of \( V_d \cup \bar{V}_d \) and its mixed Hodge structure are given by the isomorphism

\[
H^k (V_d \cup \bar{V}_d) \cong \mathbb{H}^k \left( A' (V_d) \oplus A' (\bar{V}_d) \rightarrow A' (V_d \cap \bar{V}_d) \right)
\]

with weight filtration

\[
\frac{W_{k-1} (H^k (V_d \cup \bar{V}_d))}{H^k (V_d \cup \bar{V}_d)} = \ker \left( \frac{H^k (V_d) \oplus H^k (\bar{V}_d) \rightarrow H^k (V_d \cap \bar{V}_d)}{H^k (V_d \cup \bar{V}_d)} \right).
\]
The mixed Hodge structure on $H^* (V_d)$, called the asymptotic mixed Hodge structure, is explained in [1]. It is given by the isomorphism

$$H^i (V_d) \cong H^i \left( A_{V_{d_1}} (\log (V_{d_1} \cap V_{d_2})) \oplus A_{V_{d_2}} (\log (V_{d_1} \cap V_{d_2}) \to A^1 (T)) \right).$$

Its weight filtration is given by

$$W_{k-1} (H^k (V_d)) = \text{image} (W_{k-1} (H^{k-1} (T)))$$

$$W_k (H^k (V_d)) = \text{image} \left( H^k \left( V_{d_1} \cup V_{d_2} \right) \right)$$

$$W_{k+1} (H^k (V_d)) = H^k (V_d).$$

The mapping $\rho^*$ in (3.2) is the mapping $\mu$ in diagram (3.6) of [1].

The middle vertical sequence is the standard ‘log’ exact sequence of mixed Hodge structures, and the left-side horizontal maps are morphisms of mixed Hodge structures.

4. The computation

4.1. Duality and the Lefschetz hyperplane theorem. The exact duality of standard exact sequences permit the definitions

$$\begin{array}{ccc}
H^k (V_{d_1}) & \Downarrow \quad H^{2n-k} (V_{d_1}) \\
H^k (V_{d_1} \cap V_{d_2}) & \Downarrow \quad H^{2n-k-2} (V_{d_1} \cap V_{d_2}) \\
H^{k+1} (V_{d_1}, (V_{d_1} \cap V_{d_2})) & \Downarrow \quad H^{2n-(k+1)} (V_{d_1} - (V_{d_1} \cap V_{d_2})) \\
H^{k+1} (V_{d_1}) & \Downarrow \quad H^{2n-(k+1)} (V_{d_1})
\end{array}$$

\[ \rho^* \]

\[ \text{Push-forward} \]

Since $H^k (V_{d_1}, (V_{d_1} \cap V_{d_2})) = 0$ for $k < n - 1$ and by duality and ampleness

$$H^k (V_{d_1} - (V_{d_1} \cap V_{d_2})) = 0$$

for $k > n$. Also for $k > n$ we have

$$\left( H^{2n-2-k} (V_{d_1} \cap V_{d_2}) \right)^{\vee} \cong H^k (V_{d_1} \cap V_{d_2}) \cong H^k (V_{d_1}) \cong \left( H^{2n-k} (V_{d_1}) \right)^{\vee}$$

so that the push-forward map

$$H^k (V_{d_1} \cap V_{d_2}) \to H^{k+2} (V_{d_1})$$

is an isomorphism for $k > n - 1$ and surjective for $k = n - 1$. Then the commutative diagram

$$\begin{array}{ccc}
H^k (V_{d_1} \cap V_{d_2}) & \Downarrow N \quad H^{k+2} (V_{d_1}) \\
H^{k+2} (V_{d_1} \cap V_{d_2}) & \Downarrow N \quad H^{k+4} (V_{d_1})
\end{array}$$

implies that

$$\begin{array}{ccc}
H^k \text{prim} (V_{d_1} \cap V_{d_2}) & \to H^{k+2} \text{prim} (V_{d_1}) \\
H^{k+2} \text{prim} (V_{d_1} \cap V_{d_2}) & \to H^{k+4} \text{prim} (V_{d_1})
\end{array}$$

induced by push-forward is an isomorphism for $k > n - 1$ and surjective for $k = n - 1$. In fact for $k = n - 2$ the cokernel of (4.2) is $H^n \text{prim} (V_{d})$ because of the isomorphisms...
indicated in the diagram

\[
\begin{align*}
H^{n-4} (V_{d_1} \cap V_{d_2}) & \rightarrow H^{n-2} (V_{d_1}) \\
\downarrow^{N} & \quad \downarrow^{N} \\
H^{n-2} (V_{d_1} \cap V_{d_2}) & \rightarrow H^{n} (V_{d_1}) \\
\downarrow^{N} & \quad \downarrow^{N} \\
H^{n} (V_{d_1} \cap V_{d_2}) & \rightarrow H^{n+2} (V_{d_1}).
\end{align*}
\]

(4.3)

The above analysis is identical if the roles of \(d_1\) and \(d_2\) are reversed.

4.2. Accommodating the exceptional locus of the small resolution. The morphism of exact sequences of mixed Hodge structures

\[
\begin{align*}
\ldots & \rightarrow H^{k-2} (V_{d_1} \cap V_{d_2}) & \leftrightarrow & H^{k-2} (V_{d_1} \cap V_{d_2}) & \rightarrow & H^{k} (V_{d_2}) & \rightarrow & H^{k} (\tilde{V}_{d_2}) \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
H^{k} (V_{d_2}) & \rightarrow & H^{k} (\tilde{V}_{d_2}) & \rightarrow & H^{k} (V_{d_2} - (V_{d_1} \cap V_{d_2})) & \rightarrow & H^{k} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) & \rightarrow & H^{k+1} (V_{d_2}) \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
H^{k} (V_{d_2} - (V_{d_1} \cap V_{d_2})) & \rightarrow & H^{k} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) & \rightarrow & H^{k} (V_{d_1} \cap V_{d_2}) & \rightarrow & H^{k+1} (V_{d_2}) \\
\downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow & \quad & \downarrow \\
\ldots & \rightarrow & \ldots
\end{align*}
\]

\[
\begin{align*}
\ldots & \rightarrow H^{k} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) = H^{k} (V_{d_2} - (V_{d_1} \cap V_{d_2})) \oplus H^{k-2} (V_{d_2} \cap V_{d_1} \cap V_{d_2}) \oplus (1, 1) \\
\end{align*}
\]

yields that for all \(k\)

\[
H^{k} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) = H^{k} (V_{d_2} - (V_{d_1} \cap V_{d_2})) \oplus H^{k-2} (V_{d_2} \cap V_{d_1} \cap V_{d_2}) \oplus (1, 1).
\]

4.3. Ampleness, the ‘hard’ Lefschetz theorem and the circle bundle \(T\).

As we have seen in Subsection 3.4 in [3,2] the right-hand vertical exact sequence for the circle bundle \(T/ (V_{d_1} \cap V_{d_2})\) where \(N = c_1 (O_B (1))\) yields

\[
H^{k} (T) = \begin{cases}
\text{coker} \left( H^{k-2} (V_{d_1} \cap V_{d_2}) \times H^{k} (V_{d_1} \cap V_{d_2}) \right) + \\
\text{ker} \left( H^{k-1} (V_{d_1} \cap V_{d_2}) \times H^{k-1} (V_{d_1} \cap V_{d_2}) \right) \times H^{k+1} (V_{d_1} \cap V_{d_2})
\end{cases}
\]

Since \(N\) is ample, it polarizes cohomology so that this formula reduces to

\[
H^{k} (T) = H^{k} \text{prim} (V_{d_1} \cap V_{d_2}) \quad \text{if } k \leq \text{dim} (V_{d_1} \cap V_{d_2}) = n - 1
\]

\[
H^{k} (T) = H^{k} \text{prim} (V_{d_1} \cap V_{d_2}) \oplus (1, 1) \quad \text{if } k \geq \text{dim} (V_{d_1} \cap V_{d_2}) + 1 = n.
\]

By (4.1.3) the long exact cohomology sequence

\[
\begin{align*}
& \ldots \rightarrow H^{n-1} (T) \rightarrow H^{n} (V_{d_1}) \rightarrow H^{n} (V_{d_1} - (V_{d_1} \cap V_{d_2})) \oplus H^{n} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) \\
& \rightarrow H^{n} (T) \rightarrow H^{n+1} (V_{d_1}) \rightarrow H^{n+1} (V_{d_1} - (V_{d_1} \cap V_{d_2})) \oplus H^{n+1} (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})) \\
& \rightarrow H^{n+1} (T) \rightarrow H^{n+2} (V_{d_1}) \rightarrow \ldots
\end{align*}
\]
The monodromy operator. We have the exact sequence of mixed Hodge structures

\[ 0 \rightarrow \text{coker} \left( H^k(V_d) \oplus H^k(\hat{V}_d) \rightarrow H^k(V_d \cap \hat{V}_d) \right) \rightarrow H^{k+1}(V_d \cup \hat{V}_d) \rightarrow \text{ker} \left( H^{k+1}(V_d) \oplus H^{k+1}(\hat{V}_d) \rightarrow H^{k+1}(V_d \cap \hat{V}_d) \right) \rightarrow 0. \]

Consider the commutative diagram

\[ H^k(V_d \cap \hat{V}_d) \rightarrow H^{k+1}(V_d \cup \hat{V}_d) \]

\[ \downarrow \rho^* \]

\[ H^k(T) \rightarrow H^{k+1}(V_d) \]

for \( k \geq n \) from (3.2). In that case by (3.3) the weight of \( H^k(T) \) is \( k + 1 \) and so the weight of its image in \( H^k(V_d) \) is also \( n + 1 \) implying that all of the cohomology of \( H^k(V_d) \) has weight greater than or equal to \( n + 1 \). Since

\[ H^{k+1}(V_d - (V_d \cap V_{\hat{d}})) \oplus H^{k+1}(\hat{V}_d - (V_d \cap \hat{V}_d)) = H^{k-1}(V_d \cap V_{\hat{d}} \cap V_{\hat{d}}) \odot (1, 1) \]

is of pure weight \( k + 1 \) as is

\[ H^k(T) = H^{k-1}_{prim}(V_d \cap V_{\hat{d}}) \odot (1, 1), \]

\( H^{k+1}(V_d) \) must have pure weight \( k + 1 \) for all \( k > n \). However by \([\Pi]\), the unipotent monodromy operator

\[ M : H^k(V_d) \rightarrow H^k(V_d) \]

is non-trivial if and only if \( \frac{W_{k+1}H^k(V_d)}{W_k(\mu^*H^k(V_d))} \) is non-zero in which case

\[ \log M : \frac{W_{k+1}H^k(V_d)}{W_k(\mu^*H^k(V_d))} \rightarrow \frac{W_{k-1}(H^k(V_d))}{W_{k-2}(H^k(V_d))} \]

is an isomorphism. Therefore (4.7) is the identity map for \( k > n \). Since \( M \) preserves the intersection pairing (4.7) is therefore the identity for all \( k \neq n \). Since the mapping \( \rho^* \) in (3.2) is the mapping \( \mu \), horizontal exactness at \( \mu \) in diagram (3.6) of \([\Pi]\) then implies that the map

\[ \rho^* : H^{k+1}((V_d \cup \hat{V}_d)) \rightarrow H^{k+1}(V_d) \]
is surjective for all \( k \neq n \).

4.5. **Breaking down the governing diagram.** We next state and prove a series of six Lemmas that will establish the 'building blocks' for the recursive algorithm that is the main goal of this paper.

From (4.4), the Lefschetz theorems and the Strictness Lemma for mixed Hodge structures we conclude the following.

**Lemma 4.** In (3.2) the mapping

\[
H^k_{prim}(V_{d_1} \cap V_{d_2}) \xrightarrow{\rho^*} H^k(T)
\]

is an isomorphism if \( k \leq n - 1 \), and the mapping

\[
H^k(T) \xrightarrow{Residue} H^{k-1}_{prim}(V_{d_1} \cap V_{d_2}) \otimes (1, 1)
\]

is an isomorphism if \( k \geq n \).

**Proof.** As mentioned above, since \( N \) is ample, (4.4) follows from the Hard Lefschetz Theorem that implies that the exterior product with the \((1, 1)\)-class \( N \) is injective up to the middle dimension \( n - 1 \) and surjective after that. \( \square \)

**Lemma 5.** The mapping cone cohomology \( \mathbb{H}^{+1} (\rho^*) \) is isomorphic to \( H^{-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \) in such a way that the horizontal sequence

\[
\ldots \rightarrow \mathbb{H}^{+1} (\rho^*) \rightarrow H^{-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \oplus H^{-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \rightarrow H^{-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \rightarrow \ldots
\]

in (3.2) in which it sits is the tautological one, that is, the first map above is the diagonal map and the second is the subtraction map.

**Proof.** The assertion follows immediately from the fact that \( V_1 \cup \tilde{V}_2 \) has the same homotopy type as the pasting of \( V_d \) to the complex disk bundle over \( V_{d_1} \cap V_{d_2} \) with boundary \( T \) to \( T \subseteq V_d \) by identifying corresponding points on \( T \). \( \square \)

**Lemma 6.** For \( k > n \),

\[
H^k (V_{d_1} - (V_{d_1} \cap V_{d_2})) = 0
\]

\[
H^k \left( \tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2}) \right) \cong H^{k-2} (V_d \cap V_{d_1} \cap V_{d_2}) \otimes (1, 1)
\]

and the residue mapping

\[
H^k \left( \tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2}) \right) \rightarrow H^{k-1} (V_{d_1} \cap V_{d_2})
\]

is zero. Therefore for \( k \geq n + 1 \) the sequence in (3.2)

\[
0 \rightarrow H^k (T) \rightarrow H^{k+1} (V_d) \rightarrow H^{k-1} (V_d \cap V_{d_1} \cap V_{d_2}) \otimes (1, 1) \rightarrow 0
\]

is exact. Therefore

\[
H^{k+1} (V_d) \cong H^{k+1} (V_1).
\]

**Proof.** Since \( (V_{d_1} \cap V_{d_2}) \) is ample as a divisor in \( V_j \)

\[
H^k (V_{d_1} - (V_{d_1} \cap V_{d_2})) = 0 = H^k (\tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2}))
\]

for \( k > n \). Since the center of the blow-up lies on \((V_{d_1} \cap V_{d_2}), \tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2})\) has the same homotopy type as \( V_{d_2} - (V_{d_1} \cap V_{d_2}) \).

\[
H^k \left( \tilde{V}_{d_2} - (V_{d_1} \cap V_{d_2}) \right) \cong H^{k-2} (V_d \cap V_{d_1} \cap V_{d_2}) \otimes (1, 1).
\]
For the last assertion, by the Lefschetz Hyperplane Theorem and duality

\[ H^{n-1}(V_d \cap V_{d_1} \cap V_{d_2}) \cong H^{2(n-2)-(k-1)}(V_d \cap V_{d_2}) \]
\[ \cong H^{2(n-2)-(k-1)}(V_d) \cong H^{k+3}(V_d) \cong H^{k-3}(V_1) \]

and

\[ H^k(T) \cong H^{k-1}_\text{prim}(V_d \cap V_{d_2}) \cong H^{2(n-1)-(k-1)}_\text{prim}(V_d \cap V_{d_2}) \]
\[ \cong H^{2(n-1)-(k-1)}(V_d) \cong H^{2(n-1)-(k-1)}_\text{prim}(V_1) \cong H^{k+1}_\text{prim}(V_1). \]

Lemma 7. The isomorphism \( \frac{H^n(T)}{W_n(H^n(T))} \rightarrow H^{n-1}_\text{prim}(V_d \cap V_{d_2}) \otimes (1, 1) \) of Hodge structures of weight \( n + 1 \) induces an additional isomorphism

\[ H^{n-1}_\text{prim}(V_d \cap V_{d_2}) \otimes (1, 1) \cong \frac{H^n(T)}{W_n(H^n(T))} \cong \frac{H^n(V_d)}{W_n(H^n(V_d))}. \]

Proof. By Subsection 4.4, \( \frac{H^n(T)}{W_n(H^n(T))} = 0 \) unless \( k = n \). By Lemma 5, the subtraction map

\[ H^{-1}(V_d \cap V_{d_2}) \otimes (1, 1) \oplus H^{-1}(V_d \cap V_{d_2}) \otimes (1, 1) \rightarrow H^{-1}(V_d \cap \tilde{V}_{d_2}) \otimes (1, 1) \]

in (3.2) is surjective. We first show that \( \gamma \in H^n(T) = H^{n-1}_\text{prim}(V_d \cap V_{d_2}) \) is the difference of residues of some

\[ (\Gamma_1, \Gamma_2) \in H^n(A_{V_{d_1}}(\log(V_d \cap V_{d_2}))) \oplus H^n(A_{\tilde{V}_{d_2}}(\log(V_d \cap V_{d_2}))). \]

By Lemma 6, we have exact

\[ H^n(A_{V_{d_1}}(\log(V_d \cap V_{d_2}))) \rightarrow H^{n-1}(V_d \cap V_{d_2}) \rightarrow H^{n+1}(V_d) \rightarrow 0 \]

and the mapping

\[ H^{n-1}(V_d) \xrightarrow{\cdot N} H^{n+1}(V_d) \]

is a bijection. Also by (4.2)

\[ \frac{H^n(V_d)}{H^{n-1}_\text{prim}(V_d)} \]

is the kernel of

\[ H^n(A_{V_{d_1}}(\log(V_d \cap V_{d_2}))) \rightarrow H^{n-1}(V_d \cap V_{d_2}) \]

and by (4.2)

\[ \frac{H^{n+1}(V_d)}{H^{n+1}_\text{prim}(V_d \cap V_{d_2})} \]

is the cokernel of

\[ H^{n-1}(V_d \cap V_{d_2}) \rightarrow H^{n+1}(V_d) \]

Therefore \( H^{n-1}_\text{prim}(V_d \cap V_{d_2}) \) is the image of \( H^n(A_{V_{d_1}}(\log(V_d \cap V_{d_2}))) \) under the residue map.

The argument for \( H^n(A_{\tilde{V}_{d_2}}(\log(V_d \cap V_{d_2}))) \) is completely analogous except that \( -N \) replaces \( N \) reflecting the reversal of the first Chern class of the normal bundle to \((V_{d_1} \cap V_{d_2})\).
Given non-zero \( \gamma \in H^n (T) = H^{n-1}_{prim} (V_d_1 \cap V_d_2) \) one can choose \( (\Gamma_1, \Gamma_2) \in H^{k-1} (V_d_1 \cap V_d_2) \otimes (1, 1) \oplus H^{k-1} (V_d_1 \cap V_d_2) \otimes (1, 1) \) such that Residue \( (\Gamma_1, \Gamma_2) \) maps to \( (\gamma, \gamma) \in H^{n-1}_{prim} (V_d_1 \cap V_d_2) \), it is the image of some non-zero
\[
(4.11) \quad \vartheta \in H^k (V_d) .
\]
The difference of two choices of \( \vartheta \) must go to zero in \( \mathbb{H}^{k+1} (\rho^*) \) since the mapping from \( \mathbb{H}^{k+1} (\rho^*) \) to \( H^{k-1} (V_d_1 \cap V_d_2) \otimes (1, 1) \oplus H^{k-1} (V_d_1 \cap V_d_2) \otimes (1, 1) \) is the diagonal mapping. So \( \gamma \) determines a well-defined non-zero element of \( \frac{H^k (V_d)}{W_k (H^k (V_d))} \). Surjectivity derives from \( \square \) where it is shown that any non-zero element of \( \frac{H^k (V_d)}{W_k (H^k (V_d))} \) must have non-trivial local monodromy and therefore give an element \( (4.10) \) with non-zero but cancelling residues.

**Lemma 8.** The image of the mapping
\[
H^{k-1} (T) \rightarrow W_{k-1} (H^k (V_d))
\]
\( (3.2) \) lies in \( W_k (H^k (V_d)) \) for all \( k \leq n \). This map is an isomorphism
\[
H^{k-1}_{prim} (V_d_1 \cap V_d_2) \rightarrow W_{k-1} (H^k (V_d))
\]
for \( k = n \). Furthermore
\[
\frac{W_n (H^n (V_d))}{W_{n-1} (H^n (V_d))} = H^n_{prim} (V_d_1) \oplus H^n_{prim} (V_d_2) \oplus H^{n-2} (V_d \cap V_d_1 \cap V_d_2) \otimes (1, 1)
\]
where
\[
H^n_{prim} (V_j) = \ker \left( H^n (V_j) \xrightarrow{\cdot N} H^{n+2} (V_j) \right) = \operatorname{coker} \left( H^{n-2} (V_j) \xrightarrow{\cdot N} H^n (V_j) \right).
\]
**Proof.** The first assertion follows from i) by weights. By Lemma 7 we have that
\[
H^{n-1}_{prim} (V_d_1 \cap V_d_2) \otimes (1, 1) \cong \frac{H^n (V_d)}{W_n (H^n (V_d))}.
\]
In \( \square \) it is shown that weights of classes supported small close to \( V_d_1 \cap V_d_2 \) have weight less than their degree. Since
\[
H^{n-1}_{prim} (V_d_1 \cap V_d_2) \otimes (1, 1) \rightarrow H^n (V_d) \rightarrow H^n (V_d - (V_d_1 \cap V_d_2)) \oplus H^n (\tilde{V}_d_2 - (V_d_1 \cap V_d_2))
\]
is an exact sequence of mixed Hodge structures and
\[
W_{k-1} \left( H^k (V_d_1 - (V_d_1 \cap V_d_2)) \oplus H^k (\tilde{V}_d_2 - (V_d_1 \cap V_d_2)) \right) = 0
\]
for all \( k, W_{n-1} (H^n (V_d)) = \operatorname{image} \left( H^{n-1}_{prim} (V_d_1 \cap V_d_2) \cong H^{n-1} (T) \rightarrow H^n (V_d) \right) \). Now use the result from \( \square \) that \( (4.8) \) is an isomorphism. Therefore
\[
\frac{W_n (H^n (V_d))}{W_{n-1} (H^n (V_d))} = W_n \left( H^n (V_d_1 - (V_d_1 \cap V_d_2)) \oplus H^n (\tilde{V}_d_2 - (V_d_1 \cap V_d_2)) \right) = W_n \left( H^n (V_d_1 - (V_d_1 \cap V_d_2)) \right) = W_n \left( H^n (V_d_2 - (V_d_1 \cap V_d_2)) \right) = H^n_{prim} (V_d_1) \oplus H^n_{prim} (V_d_2) \otimes (1, 1)
\]
and
\[
W_n \left( H^n (V_d_1 - (V_d_1 \cap V_d_2)) \right) = \operatorname{image} \left( H^n (V_d_1) \rightarrow H^n (V_d_1 - (V_d_1 \cap V_d_2)) \right) = H^n_{prim} (V_d_1) \oplus H^n_{prim} (V_d_2)
\]
by \( (4.3) \). \( \square \)
Lemma 9. If \( p + q = n + 1 \)
\[
H^{n+1}(V_d) \cong \frac{\ker \left( H^{n+1}(V_{d_1}) \oplus H^{n+1} \left( \tilde{V}_{d_2} \right) \rightarrow H^{n+1}(V_{d_1} \cap V_{d_2}) \right)}{H^{n+1}(V_{d_1} \cap V_{d_2})}
\]
and is of pure weight \( n + 1 \).

Proof. From (3.2), Subsection (4.4), and Lemmas 4-8 we have the follow exact diagram

\[
\begin{array}{ccc}
H^{n-1}(T) \cong H^{n-1}_{prim}(V_{d_1} \cap V_{d_2}) \uparrow \text{Residue} & \rightarrow & H^n(V_d) \downarrow \alpha \\
H^{n-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \rightarrow H^{n-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \downarrow \beta \\
H^n(V_{d_1} \cap V_{d_2}) \rightarrow H^{n+1}(V_{d_1} \cup \tilde{V}_{d_2}) \rightarrow H^{n+1}(V_d) \downarrow \rho^* \\
H^{n-1}_{prim}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \rightarrow H^{n+1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) \downarrow 0
\end{array}
\]

where the top right entry maps to the lower left entry to continue horizontal exactness and
\[(4.12)\]
\[
\text{coker } \alpha = \text{image } \beta = \ker \rho^* = \frac{H^{n-1}(V_{d_1} \cap V_{d_2}) \otimes (1, 1)}{H^{n-1}_{prim}(V_{d_1} \cap V_{d_2}) \otimes (1, 1)} \cong H^{n+1}(V_{d_1} \cap V_{d_2})
\]
of pure weight \( n + 1 \) is the kernel of \( \rho^* \).

The sequence of six Lemmas proved just above allow us to prove by weights and the Strictness Lemma that
\[
\text{coker } \left( H^n(V_{d_1}) \oplus H^n(\tilde{V}_{d_2}) \rightarrow H^n(V_{d_1} \cap V_{d_2}) \right) \subseteq \ker \rho^*.
\]

Therefore the composition
\[
\ker \left( H^{n+1}(V_{d_1}) \oplus H^{n+1}(\tilde{V}_{d_2}) \rightarrow H^{n+1}(V_{d_1} \cap V_{d_2}) \right) \downarrow \beta \\
H^{n+1}(V_{d_1} \cup \tilde{V}_{d_2}) \downarrow \rho^* \\
H^{n+1}(V_d)
\]
is surjective. So by (4.12)
\[
H^{n+1}(V_d) \cong \frac{\ker \left( H^{n+1}(V_{d_1}) \oplus H^{n+1}(\tilde{V}_{d_2}) \rightarrow H^{n+1}(V_{d_1} \cap V_{d_2}) \right)}{H^{n+1}(V_{d_1} \cap V_{d_2})}
\]
of pure weight \( n + 1 \).
4.6. The theorem. Finally we arrive at the Theorem that is the engine of the recursive algorithm to computing Hodge numbers of proper complete intersections as described in Case One and Case Two at the outset. In summary (4.5) becomes

$$\cdots \to H^{n-2}_{prim} (V_d \cap V_{d_2}) \to H^{n-1}_{prim} (V_d) \to H^{n-1} (V_d - (V_{d_1} \cap V_{d_2})) \oplus H^{n-1} \left(V_{d_2} - (V_{d_1} \cap V_{d_2})\right)$$

$$\to H^{n-1}_{prim} (V_d \cap V_{d_2}) \to H^n (V_d) \to H^n (V_d - (V_{d_1} \cap V_{d_2})) \oplus H^n \left(V_{d_2} - (V_{d_1} \cap V_{d_2})\right)$$

$$\to H^{n-1}_{prim} (V_d \cap V_{d_2}) \otimes (1,1) \to H^{n+1} (V_d) \to H^{n-1} (V_d \cap V_{d_1} \cap V_{d_2}) \otimes (1,1) \to 0$$

together with the short exact sequences

\[(4.14)\]

$$0 \to H^{k-2}_{prim} (V_d \cap V_{d_2}) \otimes (1,1) \to H^k (V_d) \to H^{k-2} (V_d \cap V_{d_1} \cap V_{d_2}) \otimes (1,1) \to 0$$

for \(k > n + 1\).

By (4.i) if \(p + q = n + 1\)

$$H^{n+1} (V_d) \cong \frac{\ker (H^{n+1} (V_{d_1}) \oplus H^{n+1} (V_{d_2}) \to H^{n+1} (V_{d_1} \cap V_{d_2}))}{H^{n+1} (V_{d_1} \cap V_{d_2})}.$$  

For \(k = n\) we have by (4.i) that

\[(4.15)\]

$$\frac{W_n (H^n (V_d))}{W_{n-1} (H^n (V_d))} = \frac{W_{n-1} (H^n (V_d))}{W_{n-1} (H^n (V_d))} = W_{n-1} (H^n (V_d) \cap H^n (V_{d_2}) \otimes (1,1))$$

$$\frac{W_{n+1} (H^{n+1} (V_d))}{W_n (H^n (V_d))} = H^{n-1}_{prim} (V_d \cap V_{d_2}) \otimes (1,1)$$

gives precisely the full weight filtration of the mixed Hodge structure on \(H^n (V_d)\).

Since the dimensions of the associated graded for the Hodge filtrations on the asymptotic mixed Hodge structure on \(H^n (V_d)\) are the same as those for the usual Hodge filtrations on \(H^n (V_d)\), we have the following result which is somewhat complicated to state but, as the example in the Introduction shows, very easy to use.

**Theorem 10. Case One:** For a very ample line bundle \(L^{-1}\) over a projective manifold \(B\) of complex dimension \(n\) we consider sections \(H^0 (L^{-d})\) as proper subvarieties \(V_d\) of \(|L|\) via the \(d\)-th power mapping

$$|L| \to |L^d|.$$  

Thus \(V_1 \cong B\).

In Case Two, \(Y\) is any smooth complex projective manifold and

$$V_1 = Y \cap H \quad \dim Y = n + 1$$

is a smooth hyperplane section.

In both cases let \(F_d\) denote the homogeneous form defining \(V_d\). Then the cohomology groups and the Hodge numbers of a (generic) complete intersection

$$V_{d_1} \cap V_{d_2} \cap \ldots \cap V_{d_r} =: V_{d_1} \cdot V_{d_2} \cdot \ldots \cdot V_{d_r},$$

with \(1 \leq r \leq n\) are given inductively in terms of the Hodge numbers of the sequence \(V_1, V_2^1, \ldots, V_1^n\) by the following formulas for \(d = d_1 + d_2\) where \((1,1)\) denotes the standard Hodge-Tate structure of dimension one and weight two:

i) For cohomology classes in degree \(k > n + 1\),

$$H^k (V_d) \cong H^k (V_1),$$

of pure weight \(k\),
for cohomology classes of degree \( n + 1 \)

\[
H^{n+1}(V_d) \cong \ker \left( \left( H^{n+1}(V_{d_1}) \oplus H^{n+1}(V_{d_2}) \right) \to H^{n+1}(V_{d_1} \cap V_{d_2}) \right).
\]

and for cohomology classes of degree \( n \)

\[
H^n(V_d) = \begin{cases}
\frac{W_n(H^n(V_d))}{W_{n-1}(H^n(V_d))} = H^{n-1}_{\text{prim}}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) & \text{if } p + q = k > n + 1 \\
H^n_{\text{prim}}(V_1) \oplus H^n_{\text{prim}}(V_2) \oplus H^{n-2}(V_{d_1} \cap V_{d_2}) \otimes (1, 1) & \text{if } k = p + q = n + 1
\end{cases}
\]

ii) If \( p + q = k > n + 1 \)

\[
h^{p,q}(V_d) = h^{p,q}(V_1),
\]

if \( k = p + q = n + 1 \)

\[
h^{p,q}(V_d) = h^{p,q}\left( \ker \left( H^{n+1}(V_{d_1}) \oplus H^{n+1}(V_{d_2}) \to H^{n+1}(V_{d_1} \cap V_{d_2}) \right) \right) - h^{p,q}(V_{d_1} \cap V_{d_2})
\]

\[
= \dim (\ker (H^{p,q}(V_{d_1}) \oplus H^{p,q}(V_{d_2}) \to H^{p,q}(V_{d_1} \cap V_{d_2}))) + h^{p-1,q-1}(V_{d_1} \cap V_{d_2}) - h^{p,q}(V_{d_1} \cap V_{d_2}),
\]

and if \( p + q = k = n \),

\[
h^{p,q}(V_d) = \begin{cases}
h^{p,q}_{\text{prim}}(V_{d_1} \cap V_{d_2}) + h^{p,q}(V_{d_1}) + h^{p-1,q-1}(V_{d_1} \cap V_{d_2}) + h^{p-1,q-1}(V_{d_1} \cap V_{d_2}) & \text{if } p + q = k = n
\end{cases}
\]

where the row of each summand indicates weight in the asymptotic mixed Hodge structure.

**Appendix A. Hodge numbers of proper complete intersections in \( Y = |L| \) for a very ample line bundle \( L^{-1} \) on a complex projective threefold \( B \)**

We begin with the linear table notation:

| \( V_1^1 \) | \( V_1^2 \) | \( V_1^3 \) | \( V_1 = B \) |
|---|---|---|---|
| \( h^{0,0} \) | \( l_0 \) | \( l_0 \) | \( l_0 \) |
| \( h^{1,0} \) | \( l_1 \) | \( l_2 \) | \( l_3 \) |
| \( h^{2,0} \) | \( l_3 \) | \( l_3 \) | \( l_3 \) |
| \( h^{1,1} \) | \( l_2 \) | \( l_2 \) | \( l_2 \) |
| \( h^{2,1} \) | \( l_4 \) | \( l_4 \) | \( l_4 \) |

A.1. **Hodge numbers of curves** \( (V_{d_1} \cap V_{d_1} \cap V_{d_2}) \leq |L| \leq Q \). As above let \( N = c_1(L^{-1}) \). Now the canonical bundle of \( |L| \) is

\[
p^*(K_B \otimes L^{-1})
\]

so by adjunction

\[
K_B \cong K_{V_1} = p^*(K_B \otimes L^{-1}) \otimes \mathcal{N}_{V_1||L|}.
\]

Therefore the normal bundle satisfies

\[
\mathcal{N}_{V_1||L|} = p^*L|_{V_1} = \mathcal{O}_{V_1} \otimes p^*\mathcal{O}_B (-N)
\]

\[
\mathcal{N}((V_1 \cap V_1')|_{V_1}) = \mathcal{O}_{V_1 \cap V_1'} \otimes p^*\mathcal{O}_B (N) = \mathcal{O}_{V_1 \cap V_1'} (N)
\]

\[
\mathcal{N}((V_1 \cap V''_1 \cap V''_1')|_{V_1 \cap V''_1}) = \mathcal{O}_{V_1 \cap V_1'' \cap V_1'''} (2N)
\]

\[
|V_1 \cap V_1'' \cap V_1'''| = N^3 = : l_0^{0,0}
\]
and
\[ KV_i = p^* K_B \otimes O_{V_i} \]

\[ KV_i \cap V'_i = p^* K_B \otimes O_{V_i \cap V'_i} (N) \]

\[ KV_i \cap V'_i \cap V''_i = p^* K_B \otimes O_{V_i \cap V'_i \cap V''_i} (2N) . \]

Then, abusing notation slightly,
\[
h^0 (KV_i \cap V'_i \cap V''_i) = h^0 (KV_i \cap V'_i \cap V''_i - 1) + h^0 (KV_i \cap V'_i) + (d - 1) t_0^{0,0} - 1 \\
= d^* h^0 (KV_i \cap V'_i) + \left( \sum_{k=1}^{d-1} k \right) t_0^{0,0} - (d - 1) \\
= dt_1^{1,0} + \left( \frac{d^2 - d}{2} \right) t_0^{0,0} - (d - 1) .
\]

Similarly
\[
h^0 (KV_i \cap V'_i \cap V'_1) = h^0 (KV_i \cap V'_i \cap V'_1) + h^0 (KV_i \cap V'_1 - 1 \cap V'_2) + (d_1 - 1) d_2 t_0^{0,0} - 1 \\
= d_1 h^0 (KV_i \cap V'_i \cap V'_1) + \left( \sum_{k=1}^{d_1 - 1} k \right) d_2 t_0^{0,0} - (d_1 - 1) \\
= d_1 \left( d_2 t_1^{1,0} + \left( \frac{d_2^2 - d_2}{2} \right) t_0^{0,0} - (d_2 - 1) \right) + \left( \frac{d_2^2 - d_2}{2} \right) d_2 t_0^{0,0} - (d_1 - 1) .
\]

and finally
\[
h^0 (KV_i \cap V'_i \cap V'_2) = h^0 (KV_i \cap V'_i \cap V'_2) + h^0 (KV_i \cap V'_2 - 1 \cap V'_2) + (d_0 - 1) d_1 d_2 t_0^{0,0} - 1 \\
= d_0 h^0 (KV_i \cap V'_i \cap V'_2) + \left( \sum_{k=1}^{d_0 - 1} k \right) d_1 d_2 t_0^{0,0} - (d_0 - 1) \\
= d_0 \left( d_1 d_2 t_1^{1,0} + \left( \frac{d_2^2 - d_2}{2} \right) t_0^{0,0} - (d_2 - 1) \right) + \left( \frac{d_2^2 - d_2}{2} \right) d_1 d_2 t_0^{0,0} - (d_1 - 1) .
\]

and so
\[
(A.1)
\]

\[
h^1 (V_{d_0}, V_{d_1}, V_{d_2}) = d_0 d_1 d_2 t_1^{1,0} + \left( \frac{d_0 d_1 d_2 (d_0 + d_1 + d_2 - 3)}{2} \right) t_0^{0,0} - d_0 d_1 d_2 + 1
\]

In summary we have the following table of point set and curve Hodge numbers:

| | \( V_{d_0}, V_{d_1}, V_{d_2}, V_{d_3} \) | \( V_{d_0} \cap V_{d_1} \cap V_{d_2} \subseteq |L| \subseteq Q \) |
|---|---|---|
| \( h^{0,0} \) | \( t_0^{0,0} \) | \( d_0 d_1 d_2 t_1^{1,0} d_2 d_3 \) |
| \( h^{1,0} \) | \( d_0 d_1 d_2 + \left( \frac{d_0 d_1 d_2 (d_0 + d_1 + d_2 - 3)}{2} \right) t_0^{0,0} - d_0 d_1 d_2 + 1 \)

A.2 Hodge numbers of surfaces \( \subseteq (V_{d_0} \cap V_{d_2}) \subseteq |L| \subseteq Q \). To compute the Hodge numbers of surfaces \( V_{d_0} \cap V_{d} \) we study the linear degeneration to \( V_{d_0} \cap (V_{d_1} \cup V_{d_2}) \) where \( d_1 + d_2 = d \). This time we apply Case Two of Theorem 2 with \( Y = V_{d_0} \).

The total space of the family
\[
\left\{ t F_{d_1} - F_{d_2} \cap \left( \begin{array}{l}
F_{d_1} \\
F_{d_2}
\end{array} \right) \right| F_d = 0 \right\} \subseteq \mathbb{C} \times V_{d_0}
\]

has a nodal locus at the \( t_0^{0,0} \cdot d_0 d_1 d_2 d_3 \) points
\[
V_{d_0} \cap V_{d_1} \cap V_{d_2} \cap V_{d_3}
\]

where all four entries in the above matrix are zero.
Case Two of Theorem 2 with \( Y = V_{d_0} \) and the fact that all divisors are very ample then give that for \( k = 3 \) and \( Y = V_{d_0} \) we have the surface formula

\[
\dim \ker \left( H^{2,1}(V_{d_0} \cap V_d) \right) = h^{2,1}(V_{d_0} \cap V_d) = h^{2,1}(V_{d_0} \cap V_{d_1} \cap V_{d_2}) - h^{2,1}(V_{d_0} \cap V_{d_1} \cap V_{d_2}) - h^{2,1}(V_{d_0} \cap V_{d_1} \cap V_{d_2}).
\]

Therefore by dimension

\[
h^{2,1}(V_{d_0} \cap V_d) = h^{2,1}(V_{d_0} \cap V_{d_1}) + h^{2,1}(V_{d_0} \cap V_{d_2})
\]

and so by induction for \( Y = V_{d_0} \)

\[
h^{2,1}(V_{d_0} \cap V_{d_1}) = d_1 h^{2,1}(V_{d_0} \cap V_1) = d_0 d_1 l^{1,0}.
\]

since \( h^{2,1}(V_1 \cap V_1) = l^{1,0}_2 \).

When \( p + q = 2 \), we can use Case Two of Theorem 2 \( Y = V_{d_0} \) since the formulae (A.2)

\[
\begin{align*}
\left. h^{2,0}(V_{d_0} \cap V_d) \right| & = \left\{ h^{1,0}(V_{d_0} \cap V_{d_1} \cap V_{d_2}) + h^{2,0}(V_{d_0} \cap V_{d_1}) + h^{2,0}(V_{d_0} \cap V_{d_2}) \right\} \\
\left. h^{1,1}(V_{d_0} \cap V_d) \right| & = \left\{ h^{1,1}(V_{d_0} \cap V_{d_1}) + h^{1,1}(V_{d_0} \cap V_{d_2}) + h^{1,1}(V_{d_0} \cap V_{d_1} \cap V_{d_2}) \right\}
\end{align*}
\]

allows for a complete induction starting with \( h^{p,q}(V_1 \cap V_1) = l^{p,q}_2 \). In particular

\[
h^{2,0}(V_1 \cap V_d) = \left\{ h^{1,0}(V_1 \cap V_1 \cap V_{d-1}) + h^{2,0}(V_1 \cap V_{d-1}) \right\}.
\]

Therefore if \( d > 1 \)

\[
\begin{align*}
\left. h^{2,0}(V_1 \cap V_d) \right| & = \left\{ (d - 1) l^{1,0}_1 + \left( \frac{(d-1)^2-(d-1)}{2} \right) l^{0,0}_0 - (d - 1) + 1 \right\} \\
\left. h^{2,0}(V_1 \cap V_d) \right| & = \left\{ 2d^2 - 2d + 2 \right\} l^{1,0}_1 \right. \\
\left. h^{2,0}(V_1 \cap V_d) \right| & = \left\{ \frac{(d^2-d)}{2} l^{0,0}_0 \right\} \\
\left. h^{2,0}(V_1 \cap V_d) \right| & = \left\{ \frac{d^2-d}{2} l^{0,0}_0 \right\}
\end{align*}
\]

Also

\[
\left. h^{1,1}(V_1 \cap V_d) \right| = \left\{ h^{1,1}(V_1 \cap V_1) + h^{1,1}(V_1 \cap V_{d-1}) + h^{0,0}(V_1 \cap V_1 \cap V_{d-1}) \right\}.
\]
\[ h^{1,1}_{\text{prim}}(V_1 \cap V_d) = \begin{cases} (l_{2,1} - 1) l_1^{1,0} + \frac{\ldots}{2} l_0^{0,0}, & d(d - 1) + 1 \\ l_{2,1}^1 + h^{1,1}_{\text{prim}}(V_1 \cap V_{d-1}) + d(d - 1) l_0^{0,0} & \end{cases} \]

\[ h^{1,1}_{\text{prim}}(V_1 \cap V_d) = \begin{cases} (d - 1) l_1^{1,0} + (d - 1)^2 l_0^{0,0}, & (d - 1) + 1 \\ (l_{2,1} - 1) + h^{1,1}_{\text{prim}}(V_1 \cap V_{d-1}) + (d - 1)^2 l_0^{0,0} & \end{cases} \]

\[ h^{1,1}(V_1 \cap V_d) = \begin{cases} d (l_{2,1} - 1) + 2 \left( \sum_{k=1}^{d-1} k \right) l_1^{1,0} \\ + 2 \left( \sum_{k=1}^{d-1} k^2 \right) l_0^{0,0} \\ - 2 \left( \sum_{k=1}^{d-1} k \right) + (d - 1) \\ \end{cases} \]

So finally for all positive \( d \)

\[ h^{2,0}(V_1 \cap V_d) = d l_{2,0}^2 + \frac{(d^2 - d)}{2} l_1^{1,0} + \left( \frac{d^2 - 3d^2 + 2d}{6} \right) l_0^{0,0} - \frac{d^2 - 3d^2 + 2d}{2} \]

\[ h^{1,1}(V_1 \cap V_d) = d l_{2,1}^1 + (d^2 - d) l_1^{1,0} + \left( \frac{2d^2 - 3d^2 + 2d}{3} \right) l_0^{0,0} - d^2 + d. \]

Lastly we reverse the roles of \( d_0 \) and \( d \) and write

\[ h^{2,0}(V_{d_0} \cap V_d) = \begin{cases} h^{1,0}(V_{d_0} \cap V_{d-1} \cap V_1) + h^{2,0}(V_{d_0} \cap V_{V_1}) & \\ h^{2,0}(V_{d_0} \cap V_{d-1}) + h^{2,0}(V_{d_0} \cap V_{V_1}) \end{cases} \]

and so for \( d > 1 \)

\[ h^{2,0}(V_{d_0} \cap V_d) = \begin{cases} d_0 (d - 1) l_1^{1,0} + d_0 \left( \frac{d^2 - d}{2} \right) l_1^{1,0} + \left( \frac{d^2 - 3d^2 + 2d}{6} \right) l_0^{0,0} - \frac{d^2 - 3d^2 + 2d}{2} & \\ h^{2,0}(V_{d_0} \cap V_{d-1}) + d_0 l_{2,0}^2 + \left( \frac{d^2 - d}{2} \right) l_1^{1,0} + \left( \frac{d^2 - 3d^2 + 2d}{6} \right) l_0^{0,0} - \frac{d^2 - 3d^2 + 2d}{2} & \end{cases} \]

\[ h^{2,0}(V_{d_0} \cap V_d) = \begin{cases} h^{2,0}(V_{d_0} \cap V_{d-1}) + d_0 l_{2,0}^2 + \left( \frac{d^2 - d}{2} \right) l_1^{1,0} + \left( \frac{d^2 - 3d^2 + 2d}{6} \right) l_0^{0,0} - \frac{d^2 - 3d^2 + 2d}{2} & \\ + d_0 (d - 1) l_{2,0}^2 + d_0 \left( \frac{d^2 - d}{2} \right) l_1^{1,0} + \left( \frac{d^2 - 3d^2 + 2d}{6} \right) l_0^{0,0} - \frac{d^2 - 3d^2 + 2d}{2} & \end{cases} \]
\[ h^{2,0}(V_{d_0} \cap V_d) = \begin{cases} 
\left( d + d_0 \right)^2 + \left( \frac{d^2 - d_0}{2} \right) - \left( \frac{d^2 - 3d_0 + 2d_0}{6} \right) + \left( \frac{d^2 - 3d_0 + 2d_0}{2} \right) - \left( \frac{d^2 - 3d_0 + 2d_0}{2} \right) + (d - 1) \\
\end{cases} \]

\[ h^{2,0}(V_{d_0} \cap V_{d_1}) = \begin{cases} 
\left( d + d_0 d_1 \right)^2 + \left( \frac{d^2 - d_0}{2} \right) + \left( \frac{d^2 - d_0}{6} \right) + \left( \frac{d_0 d_1^2}{2} \right) - \left( \frac{d_0 d_1}{2} \right) + (d - 1) \\
\end{cases} \]

Also for \( d > 1 \)

\[ h^{1,1}(V_{d_0} \cap V_d) = \begin{cases} 
\left( d + d_0 \right)^2 + \left( \frac{d^2 - d_0}{2} \right) - \left( \frac{d^2 - 3d_0 + 2d_0}{6} \right) + \left( \frac{d^2 - 3d_0 + 2d_0}{2} \right) - \left( \frac{d^2 - 3d_0 + 2d_0}{2} \right) + (d - 1) + 1 \\
\end{cases} \]
Summarizing we now have the following table of Hodge numbers:

| $h^{1,1}$ | $(V_d \cap V_d)$ ≤ $|L|$ ≤ $Q$ |
|-----------|----------------------------------|
| $h^{0,0}$ | $d_0 : d_1 : d_2$ | 1 |
| $h^{1,0}$ | $d_0 d_1 : d_2$ | $d_0 \cdot d_1 \cdot d_2$ |
| $h^{2,0}$ | $d_0 d_1 \cdot d_2$ | $d_0 d_1 \cdot d_2 + \left( \frac{d_0^2 d_1^2 d_2}{12} \right) - d_0 d_1 \cdot l_1^{1,0}$ |
| $h^{1,1}$ | $d_0 d_1 \cdot d_2$ | $d_0 d_1 \cdot d_2 + \left( \frac{d_0^2 d_1^2 d_2}{6} \right) - \left( \frac{d_0^2 d_1^2 d_2}{2} \right)$ |

$$h^{1,1}(V_d \cap V_d) =$$

$$h^{1,1}(V_d \cap V_d) =$$

$$h^{1,1}(V_d \cap V_d) =$$

$$h^{1,1}(V_d \cap V_d) =$$

$$\frac{d_0 d_1 d_2}{12} + \left( \frac{d_0^2 d_1^2 d_2}{6} \right) - \left( \frac{d_0^2 d_1^2 d_2}{2} \right).$$
A.3. Hodge numbers of the threefolds $V_d \subseteq |L| \subseteq Q$. For $d = d_1 + d_2$ we again study the linear degeneration
\[
\{ t \cdot F_{d_1} - F_{d_2} - F_{d_2} = 0 \} \subseteq |K_{B_3}|
\]
and take a small resolution of the nodal locus of the total space
\[
\left\{ \begin{array}{c}
t \\
F_{d_1} \\
F_{d_2}
\end{array} \right| = 0 \subseteq \mathbb{C} \times |K_{B_3}|
\]
where all four entries in the above matrix are zero. We can assume the small resolution is absorbed in $V_{d_2}$ with resulting space $\tilde{V}_{d_2}$.

For $p + q = k = 5$
\[
\begin{align*}
h^{p,q}(V_d) &= h^{p-1,q-1}(V_d \cap V_{d_1} \cap V_{d_2}) + h^{p-1,q-1}(V_{d_1} \cap V_{d_2}) \\
&= h^{p,q}(V_{d_1}) + h^{p,q}(V_{d_2}) - \left\{ \begin{array}{l}1 \text{ if } p = q = 2 \\0 \text{ otherwise}\end{array}\right.
\end{align*}
\]
that is, if $p + q = k = 4$,
\[
h^{p,q}(V_d) = \begin{cases} d \cdot h^{p,q}_3 - (d - 1) & \text{if } p = q = 2 \\
d \cdot h^{p,q}_3 & \text{otherwise.} \end{cases}
\]

Again by asymptotic mixed Hodge theory as in [1] and arranging by weights in descending order, we have
\[
\begin{align*}
h^{3,0}(V_d) &= \left\{ \begin{array}{c}h^{2,0}(V_d) + h^{3,0}(V_{d_1} \cap V_{d_2}) \\
h^{3,0}(V_{d_1} \cap V_{d_2}) + h^{3,0}(V_{d_2}) \end{array} \right. \\
h^{2,1}(V_d) &= \left\{ \begin{array}{c}h^{1,1}(V_d) + h^{2,1}(V_{d_1} \cap V_{d_2}) \\
h^{2,1}(V_d) + h^{2,1}(V_{d_2}) \\
h^{1,0}(V_d \cap V_{d_1} \cap V_{d_2}) + h^{2,0}(V_{d_1} \cap V_{d_2}) \end{array} \right.
\end{align*}
\]
(A.3)

Again letting $d_1 = 1$ we use
\[
h^{2,0}(V_1 \cap V_d) = d \cdot l^{2,0}_2 + \left( \frac{d^2 - d}{2} \right) l^{1,0}_1 + \left( \frac{d^3 - 3d^2 + 2d}{6} \right) l^{p,0}_0 - \frac{d^2 - 3d + 2}{2}
\]
to obtain
\[
h^{3,0}(V_d) = \left\{ \begin{array}{c}h^{3,0}_3(V_1) + h^{3,0}_3(V_{d-1}) \\
\frac{(d - 1) \cdot l^{2,0}_2 + \left( \frac{(d - 1)^2 - (d - 1)}{2} \right) l^{1,0}_1}{d} + \left( \frac{(d - 1)^3 - 3(d - 1)^2 + 2(d - 1)}{6} \right) l^{p,0}_0 - \frac{(d - 1)^2 - 3(d - 1) + 2}{2}
\end{array} \right. \
\]
\[
+ h^{3,0}(V_{d}) + h^{3,0}(V_{d_1} \cap V_{d_2}) \}
\]
Let \( h^{3,0}(V_d) \) be defined as:

\[
h^{3,0}(V_d) = \begin{cases} 
  d \cdot t_{3,1}^{3,0} + \sum_{k=1}^{d-1} (d-1) \cdot t_{2,1}^{2,0} \\
  + \left( \frac{1}{2} \sum_{k=1}^{d-1} (d-1)^2 - \frac{1}{2} \sum_{k=1}^{d-1} (d-1) \right) t_{1,1}^{1,0} \\
  + \left( \frac{1}{6} \sum_{k=1}^{d-1} (d-1)^3 - \frac{1}{6} \sum_{k=1}^{d-1} (d-1)^2 + 2 \sum_{k=1}^{d-1} (d-1) \right) t_{0,0}^{0,0}
\end{cases}
\]

and referring in addition to

\[
h^{3,0}(V_d) = \begin{cases} 
  d \cdot t_{3,1}^{3,0} + \left( \frac{d^2 - d}{2} \right) t_{2,1}^{2,0} \\
  + \left( \frac{2(d-1)^3 + 3(d-1)^2 - (d^2 - d)}{12} \right) t_{1,1}^{1,0} \\
  + \left( \frac{d^2 - 6d^2 + 4}{24} \right) t_{0,0}^{0,0}
\end{cases}
\]

we obtain

\[
h^{1,1}(V_1 \cap V_d) = d \cdot t_{2,1}^{2,1} + (d^2 - d) t_{1,1}^{1,0} + \left( \frac{2d^3 - 3d^2 + d}{3} \right) t_{0,0}^{0,0} - d^2 + d
\]

and

\[
h^{2,1}(V_d) = \begin{cases} 
  h_{1,1}^{1,1}(V_1 \cap V_{d-1}) + \\
  h_{2,1}^{2,1}(V_1) + h_{2,1}^{2,1}(V_{d-1}) \\
  + h^{1,0}(V_d \cap V_{d-1} \cap V_1) + h^{2,0}(V_1 \cap V_{d-1})
\end{cases}
\]

and

\[
h_{2,1}^{2,1}(V_d) = \begin{cases} 
  \left( d - 1 \right) \cdot t_{2,1}^{2,1} + \left( d - 1 \right)^2 - \left( d - 1 \right) \cdot t_{1,1}^{1,0} \\
  + \left( \frac{2(d-1)^3 - 3(d-1)^2 + (d-1)}{3} \right) t_{0,0}^{0,0} - \left( d - 1 \right)^2 + (d - 1) - 1 \\
  + \left( \frac{2}{3} \right) t_{3,1}^{3,0} + h_{2,1}^{2,1}(V_{d-1}) \\
  + (d^2 - d) \cdot t_{1,1}^{1,0} + \left( \frac{(d^2 - d)(2d - 3)}{2} \right) t_{0,0}^{0,0} - (d^2 - d) + 1 \\
  + (d - 1) \cdot t_{2,1}^{2,0} + \left( \frac{(d-1)^2 - (d-1)}{2} \right) t_{1,1}^{1,0} \\
  + \left( \frac{(d-1)^3 - 3(d-1)^2 + 2(d-1)}{6} \right) t_{0,0}^{0,0} - (d-1)^2 + 3(d-1) + 2
\end{cases}
\]
Hodge numbers of arbitrary sections from linear sections

$$h_{prim}^{2,1}(V_d) = \left\{ \begin{array}{c} 
\left( l_3^{2,1} - l_3^{1,0} \right) + h_{prim}^{2,1}(V_{d-1}) + (d-1) \cdot l_2^{1,1} + (d-1) \cdot l_2^{2,0} \\
+ \left( (d-1)^2 - (d-1) \right) \cdot l_1^{1,0} + \left( (d-1)^2 + (d-1) \right) \cdot l_1^{1,0} \\
+ \left( \frac{(d-1)^2 - (d-1)}{2} \right) \cdot l_1^{0,0} \\
+ \left( \frac{4(d-1)^3 - 6(d-1)^2 + 2 (d-1)}{6} \right) \cdot l_0^{0,0} \\
+ \left( \frac{6(d-1)^2 - 3(d-1)^2 - 3(d-1)}{6} \right) \cdot l_0^{0,0} \\
+ \left( \frac{(d-1)^2 - 3(d-1)^2 + 2(d-1)}{6} \right) \cdot l_0^{0,0} \\
- \left( (d-1)^2 - (d-1) \right) - \left( (d-1)^2 + (d-1) \right) \\
\end{array} \right\} \\
- \frac{1}{2} (d-1)^2 - 3(d-1) + 2$$

$$h_{prim}^{2,1}(V_d) = \left\{ \begin{array}{c} 
\left( l_3^{2,1} - l_3^{1,0} \right) + h_{prim}^{2,1}(V_{d-1}) + (d-1) \cdot l_2^{1,1} + (d-1) \cdot l_2^{2,0} \\
+ \left( \frac{5(d-1)^2 - (d-1)}{2} \right) \cdot l_1^{1,0} \\
+ \left( \frac{11(d-1)^3 - 6(d-1)^2 + (d-1)}{6} \right) \cdot l_1^{0,0} \\
- \left( \frac{5(d-1)^2 - 3(d-1)^2 + 2(d-1)}{2} \right) \cdot l_1^{0,0} \\
\end{array} \right\}$$

$$h_{prim}^{2,1}(V_d) = \left\{ \begin{array}{c} 
\left( l_3^{2,1} - l_3^{1,0} \right) + \left( \sum_{k=1}^{d-1} k \right) \cdot l_2^{1,1} + \left( \sum_{k=1}^{d-1} k \right) \cdot l_2^{2,0} \\
+ \left( \frac{5(\sum_{k=1}^{d-1} k^2) - (\sum_{k=1}^{d-1} k)}{2} \right) \cdot l_1^{1,0} \\
+ \left( \frac{11(\sum_{k=1}^{d-1} k^3) - 6(\sum_{k=1}^{d-1} k^2) + (\sum_{k=1}^{d-1} k)}{2} \right) \cdot l_0^{0,0} \\
- \left( \frac{5(\sum_{k=1}^{d-1} k^2) - 3(\sum_{k=1}^{d-1} k^2 + 2(d-1))}{2} \right) \cdot l_0^{0,0} \\
\end{array} \right\}$$

$$h_{prim}^{2,1}(V_d) = \left\{ \begin{array}{c} 
\left( l_3^{2,1} - l_3^{1,0} \right) + \left( \frac{d^2 - d}{2} \right) \cdot l_2^{1,1} + \left( \frac{d^2 - d}{2} \right) \cdot l_2^{2,0} \\
+ \left( \frac{5\left( \frac{d^3 - 3d^2 + d}{2} \right) - \left( \frac{d^2 - d}{2} \right)}{2} \right) \cdot l_1^{1,0} \\
+ \left( \frac{11\left( \frac{d^4 - 2d^3 + d^2}{2} \right) - 6\left( \frac{d^3 - 3d^2 + d}{2} \right) + \left( \frac{d^2 - d}{2} \right)}{2} \right) \cdot l_0^{0,0} \\
- \left( \frac{5\left( \frac{2d^3 - 3d^2 + d}{2} \right) - 3\left( \frac{d^2 - d}{2} \right) + 2(d-1)}{2} \right) \cdot l_0^{0,0} \\
\end{array} \right\}$$

$$h_{prim}^{2,1}(V_d) = \left\{ \begin{array}{c} 
\left( l_3^{2,1} - l_3^{1,0} \right) + \left( \frac{d^2 - d}{2} \right) \cdot l_2^{1,1} + \left( \frac{d^2 - d}{2} \right) \cdot l_2^{2,0} \\
+ \left( \frac{5d^3 - 9d^2 + 4d}{6} \right) \cdot l_1^{1,0} \\
+ \left( \frac{11d^3 - 22d^2 + 11d^2}{2} - \frac{3d^3 - 14d^2 + 6d}{4} \right) \cdot l_0^{0,0} \\
- \left( \frac{10d^3 - 15d^2 + 5d}{6} - \frac{9d^2 - 6d}{4} + 2d - 12}{4} \right) \cdot l_0^{0,0} \\
\end{array} \right\}$$
Hodge numbers of arbitrary sections from linear sections

\[ h_{\text{prim}}^{2,1}(V_d) = \begin{cases} 
 d \left( l_3^{2,1} - l_3^{1,0} \right) + \left( \frac{d^2 - d}{2} - d_1^{1,1} \right) l_2^{1,0} + \left( \frac{d^2 - d}{2} - d_2^{2,0} \right) l_0^{0,0} \\
 + \left( \frac{5d^3 - 9d^2 + 4d}{6} \right) l_1^{1,0} \\
 + \left( \frac{11d^3 - 30d^2 + 25d^2 - 6d}{24} \right) l_0^{0,0} \\
 - \left( \frac{5d^3 - 12d^2 + 13d - 6}{6} \right) l_0^{0,0} 
\end{cases} \]

Summarizing we have the following table of Hodge numbers if \( \dim B = 3 \):

| | \( V_d \subseteq |L| \subseteq Q \) |
|---|---|
| \( h^{0,0} \) | 1 |
| \( h^{1,0} \) | \( d \cdot l_3^{0,0} \) |
| \( h^{2,0} \) | \( d \cdot l_3^{1,0} - (d - 1) \) |
| \( h^{3,0} \) | \( d \cdot l_3^{2,0} + \left( \frac{d^2 - d}{2} \right) l_2^{1,0} \) |
| | + \( \left( \frac{d^3 - 3d^2 + 2d}{6} \right) l_1^{1,0} \) |
| | + \( \left( \frac{d^3 - 3d^2 + 2d}{24} \right) l_0^{0,0} \) |
| | - \( \left( \frac{d^3 - 6d^2 + 11d - 6}{24} \right) l_0^{0,0} \) |
| \( h^{2,1} \) | \( d \left( l_3^{2,1} - l_3^{1,0} \right) + l_3^{1,0} + \left( \frac{d^2 - d}{2} - d_1^{1,1} \right) l_2^{1,0} \) |
| | + \left( \frac{2d^3 - 9d^2 + 4d}{6} \right) l_0^{0,0} \) |
| | + \left( \frac{11d^3 - 30d^2 + 25d^2 - 6d}{24} \right) l_0^{0,0} \) |
| | - \left( \frac{5d^3 - 12d^2 + 13d - 6}{6} \right) l_0^{0,0} \) |

References

[1] H. Clemens, “Degeneration of Kähler manifolds,” Duke Math. Journal, 44 (1977), 215-290.
[2] M. Levine and R Pandharipande, “Algebraic cobordism revisited.” Invent. math. 176, 63–130 (2009).
[3] C. Peters and J. Steenbrink, Mixed Hodge Structures. Berlin: Springer-Verlag (2008)
[4] A. Durfee, “A naive guide to mixed Hodge theory.” Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc. (1983), 313-320.
[5] P. Griffiths and W. Schmid, “Recent developments in Hodge theory: a discussion of techniques and results.” Proc. Internat. Colloq. on discrete subgroups of Lie groups, Oxford Univ. Press (1975).

Mathematics Dept., Ohio State University, Columbus OH 43210, USA
Email address: clemens.43@osu.edu