Full Length Research Paper

Triple Shehu transform and its properties with applications

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Received 14 January, 2021; Accepted 19 April, 2021

In the current paper, the concept of one-dimensional Shehu Transform have been generalized into three-dimensional Shehu Transform namely, Triple Shehu Transform (TRHT). Further, some main properties, several theorems and properties related to the TRHT have been established. Triple Shehu transform was used in solving fractional partial differential equations, with the fractional derivative described in Caputo sense. The proposed scheme finds the solution without any discretization, transformation or restrictive assumptions. Several examples are given to check the reliability and efficiency of the proposed technique.

Key words: Caputo fractional derivative, exponential order, Triple Shehu transforms, partial derivative, uniqueness.

INTRODUCTION

Integral transforms is one of the most easy and effective methods for solving problems arising in Mathematical Physics, Applied Mathematics and Engineering Science which are defined by differential equations, difference equations and integral equations. The main idea in the application of the method is to transform the unknown function of some variable to a different function of a complex variable. With this, the associated differential equation can be directly reduced to either a differential equation of lower dimension or an algebraic equation in the new variable. There are several forms of integral transforms such as Laplace transform (Papoulis, 1957, Debnath and Bhatta, 2014, Rehman et al., 2014 and Dhunde et al., 2013), Sumudu transform (Kilicman and Gadain, 2010; Mahdy et al., 2015 and Mahdy et al., 2015a), Eltayeb and Kilicman, 2010, and Mechee and Naemah, 2020. Aboodh transform (Aboodh, 2013), Elzaki transform (Elzaki, 2011), Variational homotopy perturbation method (Mahdy et al., 2015b), Alternative variational iteration method (Mtawal et al., 2020) and one form may be obtained from the other by a transformation of the coordinates and the functions. Recently, in 2019 Maitama and Zhao introduced a new type of integral transform as a generalization of both Laplace transform and Sumudu transform for solving differential equations in the time domain, and provided some theorems on this transform. The study was further reinforced by Issa and Mensah (2020), Alfaqeih and Misirli (2020), Aggarwal et al. (2019), Mahdy and Mtawal (2016), Mtawal and Alkaleeli (2020). In this paper, we extend and generalized

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some results of Thakur et al. (2018), Abdon (2013) and Alfaqeih (2019); in particular, we extend one-dimensional Shehu transform into three-dimensional Shehu transform, and provide some examples to show the effectiveness of our results.

**PRELIMINARIES**

Here, we recall the definitions of Shehu and Double Shehu transform.

**Definition 1**

The Shehu Transform (\( \mathcal{H} \)) (Maitama and Zhao, 2019) is defined over the set of the functions

\[
B = \left\{ f(t): \exists M, \mu_1, \mu_2 > 0, \left| f(t) \right| < M e^{-\left| t \right|}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}
\]

by the following formula

\[
\mathcal{H}(f(t)) = F(s, u) = \int_0^{\infty} e^{-\left(\frac{s}{u}\right)} f(x) dx.
\]

And the inverse Shehu transform is defined by

\[
\mathcal{H}^{-1}(F(s, u)) = f(t), t \geq 0,
\]

\[
= \frac{1}{2\pi i} \int_{\alpha - i \infty}^{\alpha + i \infty} e^{\left(\frac{u}{s}\right)} F(s, u) ds.
\]

**Definition 2**

A real function \( f(t), t > 0 \), is considered to be in the space \( C_m, m \in R \), if there exists a real number \( \alpha > m \), so that \( f(t) = t^\alpha g(t) \), where \( g(t) \in [0, \infty) \), and it is said to be in the space \( C_m, m \in N \). (Podlubny, 1999; He, 2014).

**Definition 3**

The left-sided Riemann–Liouville fractional integral of order \( \alpha \geq 0 \), of a function \( f \in C_\alpha, \sigma \geq -1 \), (Podlubny, 1999; He, 2014) is defined as:

\[
J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t, \alpha > 0.
\]

Here \( \Gamma(.) \) is the gamma function.

**Definition 4**

If \( f \in C^n_m, n \in N \cup \{0\} \). The left Caputo fractional derivative of \( f \) in the Caputo sense (Podlubny, 1999; He, 2014) is defined as follows:

\[
D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad n-1 < \alpha \leq n, \\ D^\alpha f(t), \quad \alpha = n. \end{cases}
\]

**Definition 5**

The Mittag–Leffler function \( E_{\alpha, \beta} \) (Kilbas et al., 2004) are defined as

\[
E_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha + 1)}, \quad \alpha \in R, \quad \text{Re}(\alpha) > 0,
\]

\[
E_{\alpha, \beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in R, \quad \text{Re}(\alpha), \text{Re} (\beta) > 0.
\]

These functions are generalization of the exponential function. Some special cases of the Mittag-Leffler function are as follows:

\[
E_1(t) = e^t, \quad E_{1,1}(t) = E_0(t).
\]

**Theorem 1**

If \( \alpha > 0, a \in R \) and \( |a| < \left(\frac{s}{u}\right)^{\alpha} \), (Khalouta and Kadem, 2019), then

\[
\mathcal{H}^{-1} \left[ \left(\frac{s}{u}\right)^{\alpha - \beta} \right] = t^{\beta-1} E_{\alpha, \beta} (-at^{\alpha}).
\]

**Definition 6**

The single Shehu Transform (\( \mathcal{H} \)) of a function \( f(x, y, z) \)
with respect to the variables \( x, y \) and \( z \) respectively (Maitama and Zhao, 2019) are defined by:

\[
H_i \left( f \left( x, y, z \right) \right) = \int_0^\infty e^{-\frac{mx}{y}} f \left( x, y, z \right) dx,
\]

\[
H_j \left( f \left( x, y, z \right) \right) = \int_0^\infty e^{-\frac{n}{v}} f \left( x, y, z \right) dy,
\]

\[
H_k \left( f \left( x, y, z \right) \right) = \int_0^\infty e^{-\frac{rz}{k}} f \left( x, y, z \right) dz.
\]

**Definition 7**

The double Shehu Transform \( \mathcal{H} \) of a function \( f(x,y) \) (Alfaqeh and Misirli, 2020) over the set of the functions\( A = \left\{ f(x,y) : \exists M, \mu_i, \mu_j > 0, \left| f(x,y) \right| < M e^{-\frac{\mu_i}{x}} \text{ for } (x,y) \in \mathbb{R}^2, j = 1, 2 \right\} \)

by the following formula

\[
\mathcal{H} \left( f(x,y) \right) = e^{-\frac{\mu_1}{x}} \int_0^\infty e^{-\frac{\mu_2}{y}} f(x,y) dx dy,
\]

and the inverse double Shehu transform is defined by

\[
\mathcal{H}^{-1}(\mathcal{H}(f(x,y))) = f(x,y)
\]

\[
= \frac{1}{2\pi i} \int_{\mu=\infty} e^{\mu z} \left( \frac{1}{2\pi i} \int_{\rho=\infty} e^{\rho y} e^{\frac{z}{y}} H_2(f(x,y)) \right) dy\]

**Definition 8**

The double Shehu Transform \( \mathcal{H} \) of a function \( f(x,y,z) \) with respect to \( xy, xz \) and \( yz \) respectively (Alfaqeh and Misirli, 2020), are defined by:

\[
\mathcal{H}_y \left( f \left( x, y, z \right) \right) = \int_0^\infty \int_0^\infty e^{-\frac{ax}{w}} f \left( x, y, z \right) dx dy,
\]

\[
\mathcal{H}_x \left( f \left( x, y, z \right) \right) = \int_0^\infty \int_0^\infty e^{-\frac{ay}{v}} f \left( x, y, z \right) dx dz,
\]

\[
\mathcal{H}_z \left( f \left( x, y, z \right) \right) = \int_0^\infty \int_0^\infty e^{-\frac{az}{w}} f \left( x, y, z \right) dy dz.
\]

**RESULTS**

Here, we introduce the definition of Triple Shehu transform and Triple Shehu transform of partial and fractional derivatives which are used further in this paper; moreover, we apply Triple Shehu transform for some basic functions.

**Definition 9**

Let \( f \) be a continuous function of three variables; then, the Triple Shehu transform (TRST) of \( (x, y, z) \) is defined by

\[
H^{(1)}_o \left( f \left( x, y, z \right) \right) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{ax}{w}} e^{-\frac{by}{v}} e^{-\frac{cz}{k}} f \left( x, y, z \right) dx dy dz.
\]

Where \( x, y, z \geq 0 \) and \( s, q, r, u, v \) and \( k \), are Shehu variables, provided the integral exists.

Also, the inverse Triple Shehu transform is defined by

\[
H^{(2)}_o \left( F(s,q,r),(u,v,k) \right) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{ax}{w}} e^{-\frac{by}{v}} e^{-\frac{cz}{k}} H_2(f(x,y,z)) dx dy dz.
\]

**Existence and uniqueness of TRST**

Here, we debate the existence and uniqueness of the Triple Shehu transform and prove it.

**Definition 10**

A function \( f(x,y,z) \) is said to be of exponential order \( a > 0, b > 0, c > 0, \) as \( x, y, z \to \infty \) if there are positive constants \( m, x, y \) and \( z \) (Alfaqeh, 2019) such that

\[
\left| f(x,y,z) \right| \leq M e^{(ax+by+cz)} \text{ for all } x > X, y > Y, z > Z,
\]

and, we write

\[
f(x,y,z) = O \left( e^{(ax+by+cz)} \right) \text{ as } x, y, z \to \infty.
\]

Or, equivalently,

\[
\sup_{x, y, z > 0} \left( \left| f(x,y,z) \right| \right) < \infty.
\]
Theorem 2

Let \( f(x, y, z) \) be a continuous function on the interval 
\((0, X), (0, Y), (0, Z) \) and of exponential order \( e^{(ax + by + cz)} \). 
Then the Triple Shehu transform of \( f(x, y, z) \) exists

\[
\forall s > a u, \quad q > b v, \quad r > c k.
\]

Proof

Let \( f(x, y, z) \) be of exponential order \( e^{(ax + by + cz)} \) such that

\[
\left| f(x, y, z) \right| \leq M e^{(ax + by + cz)} \quad \text{for all} \quad x > X, \ y > Y, \ z > Z.
\]

Then, we have

\[
\left| H_{xyz}^3 (f(x, y, z)) \right| = \left| \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} e^{(ax + by + cz)} f(x, y, z) \, dx \, dy \, dz \right|
\]

\[
\leq M \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} e^{(ax + by + cz)} \left| f(x, y, z) \right| \, dx \, dy \, dz
\]

\[
\leq M \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} e^{(ax + by + cz)} \, dx \, dy \, dz
\]

\[
= M \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} \left( e^{(ax + by + cz)} \right) \, dx \, dy \, dz
\]

\[
= M \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} \left( e^{(ax + by + cz)} \right) \, dx \, dy \, dz
\]

\[
= M \int_{0}^{X} \int_{0}^{Y} \int_{0}^{Z} \left( e^{(ax + by + cz)} \right) \, dx \, dy \, dz
\]

Thus, the proof is complete.

In the next theorem, we show that \( f(x, y, z) \) can be uniquely obtained from \( F[(s, q, r), (u, v, k)] \).

Theorem 3

Let \( F_1[(s, q, r), (u, v, k)] \) and \( F_2[(s, q, r), (u, v, k)] \) be the Shehu transform of the continuous functions \( f_1(x, y, z) \) and \( f_2(x, y, z) \) defined for \( x, y, z \geq 0 \) respectively. If

\[
F_1[(s, q, r), (u, v, k)] = F_2[(s, q, r), (u, v, k)],
\]

then

\[
f_1(x, y, z) = f_2(x, y, z).
\]

Proof

If we presume \( \alpha, \beta, \gamma \) to be sufficiently large, then since

\[
f(x, y, z) = \frac{1}{2 \pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i(qs + ur + vs + uk)}}{(s + a'u^2)(q + b'v^2)(r + k'c^2)} \, ds \, dv \, dk.
\]

We deduce that

\[
f(x, y, z) = \frac{1}{2 \pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i(qs + ur + vs + uk)}}{(s + a'u^2)(q + b'v^2)(r + k'c^2)} \, ds \, dv \, dk;
\]

\[
= f(x, y, z).
\]

This proves the uniqueness of the TRHT.

TRHT of some elementary functions

1. If \( f(x, y, z) = A, x, y, z > 0 \). Then \( H_{xyz}^3 (A) = A \left( \frac{uvk}{sqr} \right) \).

2. If \( f(x, y, z) = x, y, z \). Then \( H_{xyz}^3 (x, y, z) = \left( \frac{uvk}{sqr} \right) \).

3. If \( f(x, y, z) = e^{(ax + by + cz)} \). Then

\[
H_{xyz}^3 (e^{(ax + by + cz)}) = \frac{ukv}{(s - au)(q - bv)(r - kc)}.
\]

4. If \( f(x, y, z) = e^{(ax + by + cz)} \). Then

\[
H_{xyz}^3 (e^{(ax + by + cz)}) = \frac{ukv}{(x - au)(q - bv)(r - kc)}.
\]

Consequently,

\[
H_{xyz}^3 (\cos(ax + by + cz)) = \frac{ukv}{(s^2 + a^2u^2)(q^2 + b^2v^2)(r^2 + k^2c^2)};
\]

and,

\[
H_{xyz}^3 (\sin(ax + by + cz)) = \frac{ukv}{(s^2 + a^2u^2)(q^2 + b^2v^2)(r^2 + k^2c^2)}.
\]

If \( f(x, y, z) = f_1(x)f_2(y)f_3(z) \), then
\[ H_{yz}^3 \left( f(x,y,z) \right) = \frac{1}{ab^2c} F \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right) \]

Then for \( a,b,c > 0 \), we have

\[ H_{yz}^3 \left( f(a x, b y, c z) \right) = \frac{1}{a b^2 c} F \left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right) \]

**Proof**

We have

\[ H_{yz}^3 \left( f(a x, b y, c z) \right) = \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right)} f(a x, b y, c z) dx dy dz. \]

Let \( t = ax, w = by, l = cz \).

Then

\[ H_{yz}^3 \left( f(t,w,l) \right) = \frac{1}{a b^2 c} \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{t}{a}, \frac{w}{b}, \frac{l}{c} \right)} f(t,w,l) dt dw dl \]

**First shifting property**

Let \( f(x,y,z) \) be a functions such that

\[ H_{yz}^3 \left( f(x,y,z) \right) = F \left[ \left( s, q, r, (u,v,k) \right) \right] \]

Then for real constants \( a,b,c \), we have

\[ H_{yz}^3 \left( e^{(a x+b y+c z)} f(x,y,z) \right) = F \left[ \left( s-a u, q-b v, r-c k, (u,v,k) \right) \right] \]

**Proof**

We have, by definition

\[ H_{yz}^3 \left( e^{(a x+b y+c z)} f(x,y,z) \right) = \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right)} e^{(a x+b y+c z)} f(x,y,z) dx dy dz \]

Let \( f(x,y,z) \) be a functions such that

\[ H_{yz}^3 \left( f(x,y,z) \right) = F \left[ \left( s, q, r, (u,v,k) \right) \right] \]

Then for any constants, \( \alpha, \beta \), we have

\[ H_{yz}^3 \left( \alpha f(x,y,z) + \beta g(x,y,z) \right) = \alpha H_{yz}^3 \left( f(x,y,z) \right) + \beta H_{yz}^3 \left( g(x,y,z) \right). \]

**Proof**

Using definition of TRST, we obtain

\[ H_{yz}^3 \left( \alpha f(x,y,z) + \beta g(x,y,z) \right) = \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right)} \left( \alpha f(x,y,z) + \beta g(x,y,z) \right) dx dy dz \]

\[ = \alpha \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right)} f(x,y,z) dx dy dz + \beta \int_0^1 \int_0^1 \int_0^1 e^{\left( \frac{x}{a}, \frac{y}{b}, \frac{z}{c} \right)} g(x,y,z) dx dy dz \]

\[ = \alpha H_{yz}^3 \left( f(x,y,z) \right) + \beta H_{yz}^3 \left( g(x,y,z) \right). \]

**Change of scale property**

Let \( f(x,y,z) \) be a functions such that
TRST of derivative of a function of three variables

1) The TRST of mixed derivative of a function of three variables is given by:

\[
H_{nm} (\frac{\partial^2 f(x,y,z)}{\partial x \partial y}) = \left(\frac{su}{v}k\right)\left(\frac{\partial^2 f(x,y,0)}{\partial x \partial y}\right) + \left(\frac{sv}{u}a\right)\left(\frac{\partial^2 f(0,y,z)}{\partial x \partial y}\right) + \left(\frac{su}{v}k\right)\left(\frac{\partial^2 f(x,0,z)}{\partial x \partial y}\right) + \left(\frac{sv}{u}a\right)\left(\frac{\partial^2 f(0,0,0)}{\partial x \partial y}\right).
\]

2) The TRST of nth partial derivative of a function of three variables is given by

\[
H_{mn} (\frac{\partial^n f(x,y,z)}{\partial x^n}) = \left(\frac{u}{v}\right)^n \int f((s,q,r),(u,v,k)) - \sum_{m=0}^{n-1} \left(\frac{u}{v}\right)^m H_{mn} (\frac{\partial^m f(0,y,z)}{\partial x^m})\left(\frac{\partial f(x,y,z)}{\partial x^n}\right).
\]

3) The TRST of the partial fractional Caputo derivatives of a function of three variables is given by:

\[
H_{mn} (\frac{\partial^\alpha f(x,y,z)}{\partial x^\alpha}) = \left(\frac{u}{v}\right)^\alpha \int f((s,q,r),(u,v,k)) - \sum_{m=0}^{\alpha-1} \left(\frac{u}{v}\right)^m H_{mn} (\frac{\partial^m f(0,y,z)}{\partial x^m})\left(\frac{\partial f(x,y,z)}{\partial x^\alpha}\right).
\]

Multiplying by \(x^n\), \(y^m\), \(z^p\)

Let \(f(x,y,z)\) be a functions such that

\[
H^3_{xyz} \left( f(x,y,z) \right) = F[(s,q,r),(u,v,k)]
\]

Then

\[
H^3_{xyz} \left( x^n y^m z^p f(x,y,z) \right) = \frac{\partial^{n+m+p}}{\partial x^n \partial y^m \partial z^p} F[(s,q,r),(u,v,k)].
\]

Proof

We have

\[
H^3_{xyz} (f(x,y,z)) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{x+y+z}{u}} f(x,y,z) dx dy dz
\]

Therefore,

\[
(-1)^{m+n+p} \frac{\partial^{m+n+p}}{\partial x^m \partial y^n \partial z^p} F[(s,q,r),(u,v,k)]
\]

In the same way, integrating Equation (16) with respect to \(y\), \(x\), we can get the required result.

For \(n = m = p = 1\), we get

\[
H^3_{xyz} \left( x y z f(x,y,z) \right) = -(uvk) \frac{\partial}{\partial s} \frac{\partial}{\partial q} \frac{\partial}{\partial r} H^3_{xyz} \left( f(x,y,z) \right).
\]

Recall that the Heaviside unit step function

\[U (x-a,y-b,z-c) \] (Thakur et al., 2018) is defined by

\[U (x-a,y-b,z-c) = \begin{cases} 1 & x > a, y > b, z > c, \\ 0 & \text{Otherwise}. \end{cases} \]

Theorem 4

Let \(f(x,y,z)\) be a functions such that

\[H^3_{xyz} \left( f(x,y,z) \right) = F[(s,q,r),(u,v,k)] \]

Then for a constants \(a,b,c\) we have

\[H^3_{xyz} \left( f(x-a,y-b,z-c)\right) U(x-a,y-b,z-c) = e^{-\frac{(x+y+z)}{u}} F[(s,q,r),(u,v,k)]. \]

Where \(U(x,y,z)\) is the Heaviside unit step.

Proof

Using definition of TRST, we get

\[
H^3_{xyz} (f(x-a,y-b,z-c)U(x-a,y-b,z-c)) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{x+y+z}{u}} f(x,a,y,b,z-c) dx dy dz.
\]

\[= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{(x+y+z+a+b+c)}{u}} F[(t+w+l),u,v,k] dt dw dl.
\]
\[ e^{-\left(\frac{sa}{u} + \frac{qb}{v} + \frac{rc}{k}\right)} F\left[(s,q,r),(u,v,k)\right]. \]

**Convolution Theorem for the Triple Shehu Transform**

The convolution of the functions \( f(x,y,z), g(x,y,z) \) is denoted by \((f ** g)(x,y,z)\) and defined by

\[(f ** g)(x,y,z) = \int_0^\infty \int_0^\infty f(x-t_i,y-t_j,z-t_k) g(t_i,t_j,t_k) dt_i dt_j dt_k, \]

\[= \int_0^\infty \int_0^\infty g(x-t_i,y-t_j,z-t_k) f(t_i,t_j,t_k) dt_i dt_j dt_k. \]

**Theorem 5**

Let \( f(x,y,z), g(x,y,z) \) be of exponential order, such that

\[F[(s,q,r),(u,v,k)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{\frac{m_1}{u} + \frac{m_2}{v} + \frac{m_3}{k}} f(x,y,z) dx dy dz, \]

is converge, and in addition if

\[G[(s,q,r),(u,v,k)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{\frac{n_1}{u} + \frac{n_2}{v} + \frac{n_3}{k}} g(x,y,z) dx dy dz, \]

is absolutely converge, then

\[H_{\psi}^1[(f ** g)(x,y,z)] = H_{\psi}^1[f(x,y,z)]H_{\psi}^1[g(x,y,z)]. \]

**Proof**

We have

\[H_{\psi}^1[(f ** g)(x,y,z)] = \int_0^\infty \int_0^\infty \int_0^\infty e^{\frac{m_1}{u} + \frac{m_2}{v} + \frac{m_3}{k}} f(x-t_i,y-t_j,z-t_k) g(t_i,t_j,t_k) dt_i dt_j dt_k \]

\[= \int_0^\infty \int_0^\infty \int_0^\infty e^{\frac{n_1}{u} + \frac{n_2}{v} + \frac{n_3}{k}} f(x-t_i,y-t_j,z-t_k) g(t_i,t_j,t_k) dt_i dt_j dt_k \]

Using the Heaviside unit step function, we obtained

\[= \int_0^\infty \int_0^\infty \int_0^\infty e^{\frac{m_1}{u} + \frac{m_2}{v} + \frac{m_3}{k}} f(x-t_i,y-t_j,z-t_k) U(x-t_i,y-t_j,z-t_k) dt_i dt_j dt_k \]

\[= H_{\psi}^1[f(x,y,z)]H_{\psi}^1[g(x,y,z)]. \]

**APPLICATIONS**

Here, the Triple Shehu Transform is illustrated by studying the following examples.

**Example 1**

Consider the following fractional partial differential equation (Alfaqeih, 2019)

\[ D_\alpha^x \psi(x,y,z) = \frac{\partial^2 \psi(x,y,z)}{\partial x^2}, \quad 0 < \alpha \leq 1. \]

\[
\text{(17)}
\]

With

\[
\psi_x(x,y,z) = \sin(y) E_\alpha(-z^\alpha),
\psi(0,y,z) = 0,
\psi(x,0,z) = \sin(x) \sin(y).
\]

Applying Triple Shehu Transform to Equation (17), we get

\[
\bar{\psi}(s,q,r) = \left(\begin{array}{c}
s+u^2 & q^2+v^2 & \frac{r}{k} \\
\end{array}\right)^{\alpha-1} \bar{\psi}(0,y,z) - \frac{r}{k} \int_0^\infty \bar{\psi}(x,y,z) \left[\frac{x}{u}\right]^{\alpha-1} H_{\psi}^1(0,y,z) \]

Using initial conditions (18), we obtain

\[
\bar{\psi}(s,q,r) = \left(\begin{array}{c}
s+u^2 & q^2+v^2 & \frac{r}{k} \\
\end{array}\right)^{\alpha-1} \frac{u^2}{s^2+u^2} \frac{v^2}{q^2+v^2} \left(\frac{r}{k}\right)^{\alpha-1}. \]

\[
\text{(19)}
\]
Operating with the Triple Shehu inverse on both sides of Equation (19) gives
\[ \psi(x, y, z) = \sin(x) \sin(y) E_\alpha(z^\alpha). \]

**Example 2**

Consider the following fractional partial differential Equation (Alfaqeih, 2019):

\[ D_\alpha^\alpha \psi(x, y, z) = \frac{1}{5} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right), \quad 0 < \alpha \leq 1. \quad (20) \]

With

\[
\begin{align*}
\psi(x, y, 0) &= e^{x^2 + 2y}, \\
\psi(0, y, z) &= e^{2y} E_\alpha(z^\alpha), \\
\psi(x, 0, z) &= e^x E_\alpha(z^\alpha), \\
\psi(1, y, z) &= e^{1^2 + 2y} E_\alpha(z^\alpha).
\end{align*}
\]

Applying the Triple Shehu Transform to Equation (20), we get,

\[
\left( \frac{r}{k} \right)^{\alpha} \bar{\psi}(x, y, z) = \left( \frac{r}{k} \right)^{\alpha} \psi(x, y, z) = \frac{1}{5} \left[ \left( \frac{r}{k} \right)^{\alpha} \bar{\psi}(x, y, z) - \frac{q}{v} \bar{\psi}(x, 0, z) - \frac{q}{v} \bar{\psi}(0, y, z) \right].
\]

Using the conditions (21), we obtain

\[ \bar{\psi}((x, q, r), (u, v, k)) = \left( \frac{u}{s - u} \right) \left( \frac{v}{q - 2v} \right) \left( \frac{r}{k} \right)^{-1}. \quad (22) \]

Operating with the Triple Shehu inverse on both sides of Equation 22 gives

\[ \psi(x, y, z) = e^{x^2 + 2y} E_\alpha(z^\alpha). \]

**Conclusion**

In this paper, we introduced a new type of generalized integral transforms called the Triple Shehu transform, which is a generalization of single Shehu transform. Furthermore, several properties, examples and theorems of this transform were presented. To see the efficiency of the Triple Shehu transform, this transform was applied on some examples and the results show that the Triple Shehu transform method is an appropriate method for solving the fractional partial differential equations. As a new work, it will be interesting to extend known results on a Triple Laplace transform, Triple Aboodh transform, etc., to our results on a Triple Shehu transform. Finally, based on the mathematical formulations, simplicity and the findings of the proposed Triple Shehu transform, we conclude that it is highly efficient.

**CONFLICT OF INTERESTS**

The authors have not declared any conflict of interests.

**REFERENCES**

Abdon A (2013). A Note on the Triple Laplace Transform and Its Applications to Some Kind of Third-Order Differential Equation Abstract and Applied Analysis. Article ID, 769102(10).

Aboodh KS (2013). The new integral transform “Aboodh transform” Global Journal of Pure and Applied Mathematics 9(1):35-43.

Aggarwal S, Gupta AR, Sharma SD (2019). A new application of Shehu transform for handling Volterra integral equations of first kind. International Journal of Research in Advent Technology 7(4):439-446.

Alfaqeih S, Misirli E (2020). On Double Shehu Transform and Its Properties with Applications. International Journal of Analysis and Applications 18(3):361-395.

Alfaqeih TÖS (2019). Note on Triple Aboodh Transform and Its Application. International Journal of Engineering and Information Systems 3(3):41-50.

Debnath L, Bhatta D (2014). Integral Transforms and their applications, CRC press New York.

Dhunde RR, Bhongde NM, Dhongle PR (2013). Some remarks on the properties of double Laplace transforms. International Journal of Applied Physics and Mathematics 3(4):293-295.

Eltayeb H, Kilicman A (2010). On double Sumudu transform and double Laplace transform. Malaysian Journal of Mathematical Sciences 14(1):17-30.

Elzaki TM (2011). The new integral transform “Elzaki transform”. Global Journal of Pure and Applied Mathematics 7(1):57-64.

He JH (2014). A Tutorial Review on fractional space-time and fractional calculus. International Journal of Theoretical Physics 53(11):3698-3718.

Issa A, Mensah Y (2020). Shehu Transform: Extension to Distributions and Measures. Journal of Nonlinear Modeling and Analysis 2(4):495-503.

Khalouat A, Kadem A (2019). A New Method to Solve Fractional Differential Equations: Inverse Fractional Shehu Transform Method Applications and Applied Mathematics 14(2):926-941.

Kilbas AA, Saigo M, Saxena RK (2004). Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms and Special Functions 15(1):31-49.

Kilicman A, Gadain HE (2010). “On the applications of Laplace and Sumudu transforms,” Journal of the Franklin Institute 347(5):848-862.

Mahdy AM, Mohamed AS, Mtawa AA (2015a). Implementation of the homotopy perturbation Sumudu transform method for solving Klein-Gordon equation. Applied Mathematics 6(3):617.

Mahdy AM, Mohamed AS, Mtawa AA (2015b). Variational homotopy perturbation method for solving fractional-order Logistic differential equation. Journal of Advances in Mathematics 10(7):3632-3639.

Mahdy AMS, Mtawa AA (2016). Numerical study for the fractional optimal control problem using Sumudu transform method and Picard
method. Mitteilung Klosterneub 66(2):14-59.
Maitama S, Zhao W (2019). New integral transform: Shehu transform a
generalization of Sumudu and Laplace transform for solving
differential equations. arXiv preprint arXiv:1904.11370.
Mechee MS, Naeemah AJ (2020). A Study of Triple Sumudu Transform
for Solving Partial Differential Equations with Some Applications, Multidisciplinary European Academic Journal 2(2):1-15.
Mtawal AAH, Alkaleeli SR (2020). A new modified homotopy
perturbation method for fractional partial differential equations with
proportional delay, Journal of Advances In Mathematics 19:58-73. DOI:10.24297/jam.v19i.8876.
Mtawal AAH, Muhammed SE, Almabrok AA (2020). Application of the
alternative variational iteration method to solve delay differential
equations, International journal of Physical Sciences 15(3):112-119.
Papoulis A (1957). A new method of inversion of the Laplace transform.
Quarterly of Applied Mathematics 14(4):405-414.
Podlubny I (1999). Fractional Differential Equations. Academic Press,
New York, NY, USA.
Rehman HU, Itlikhar M, Saleem S, Younis M, Mueed A (2014). A
computational quadruple Laplace transform for the solution of partial
differential equations. Applied Mathematics 5(21):3372.
Thakur AK, Kumar A, Suryavanshi H (2018). The Triple Laplace
transforms and their properties. International Journal of Applied
Mathematics Statistical Sciences 7(4):33-44.