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Coefficient Estimates and Fekete–Szegö Functional Inequalities for a Certain Subclass of Analytic and Bi-Univalent Functions

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Abstract: The present paper introduces a new class of bi-univalent functions defined on a symmetric domain using Gegenbauer polynomials. For functions in this class, we have derived the estimates of the Taylor–Maclaurin coefficients, \( |a_2| \) and \( |a_3| \), and the Fekete-Szegö functional. Several new results follow upon specializing the parameters involved in our main results.

Keywords: Gegenbauer polynomials; bi-univalent functions; analytic functions; Fekete–Szegö problem

MSC: 30C45

1. Definitions and Preliminaries

Let \( A \) denote the class of all analytic functions \( f \) defined in the open unit disk \( U = \{ \xi \in \mathbb{C} : |\xi| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). Thus, each \( f \in A \) has a Taylor–Maclaurin series expansion of the form

\[
f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in U).
\] (1)

Further, let \( S \) denote the class of all functions \( f \in A \) which are univalent in \( U \). Let the functions \( f \) and \( g \) be analytic in \( U \). We say that the function \( f \) is subordinate to \( g \), written as \( f \prec g \), if there exists a Schwarz function \( w \), which is analytic in \( U \) with \( w(0) = 0 \) and \( |w(\xi)| < 1 \) (\( \xi \in U \)) such that

\[
f(\xi) = g(w(\xi)).
\]

In addition, if the function \( g \) is univalent in \( U \), then the following equivalence holds

\[
f(\xi) \prec g(\xi) \quad \text{if and only if} \quad f(0) = g(0)
\]

and

\[
f(U) \subset g(U).
\]

It is well known that every function \( f \in S \) has an inverse \( f^{-1} \), defined by

\[
f^{-1}(f(\xi)) = \xi \quad (\xi \in U)
\]

and

\[
f^{-1}(f(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq \frac{1}{4})
\]
where
\[
    f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.
\]  
(2)

A function is said to be bi-univalent in \( \mathbb{U} \) if both \( f(\xi) \) and \( f^{-1}(\xi) \) are univalent in \( \mathbb{U} \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1). Example of functions in the class \( \Sigma \) are
\[
\frac{\xi}{1-\xi}, \quad \log \frac{1}{1-\xi}, \quad \log \sqrt{\frac{1+\xi}{1-\xi}}.
\]

However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( \mathbb{U} \) such as
\[
\frac{2\xi - \xi^2}{2} \quad \text{and} \quad \frac{\xi}{1-\xi^2}
\]
are also not members of \( \Sigma \).

Lewin [1] investigated the bi-univalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [2] conjectured that \( |a_2| < \sqrt{2} \). Netanyahu [3], on the other hand, showed that \( \max_{f \in \Sigma} |a_2| = 4/3 \).

The coefficient estimate problem for each of the Taylor–Maclaurin coefficients \( |a_n| \) \( (n \geq 3; n \in \mathbb{N}) \) is presumably still an open problem.

Similar to the familiar subclasses \( S^*(\xi) \) and \( K(\xi) \) of starlike and convex functions of order \( \xi(0 \leq \xi < 1) \), respectively, Taha [4] introduced certain subclasses of the bi-univalent function class \( \Sigma \), \( S^*_\Sigma(\xi) \) and \( K_\Sigma(\xi) \) of bi-starlike functions and of bi-convex functions of order \( \xi(0 < \xi \leq 1) \), respectively. For each of the function classes \( S^*_\Sigma(\xi) \) and \( K_\Sigma(\xi) \), they found non-sharp estimates on the first two Taylor–Maclaurin coefficients. For some intriguing examples of functions and characterization of the class \( \Sigma \), see [5–17].

In 1933, Fekete and Szegö [18] obtained a sharp bound of the functional \( \eta a_2^2 - a_3 \), with real \( \eta \) \( (0 \leq \eta \leq 1) \) for a univalent function \( f \). Since then, the problem of finding the sharp bounds for this functional of any compact family of functions \( f \in A \) with any complex \( \eta \) is known as the classical Fekete–Szegö problem or inequality.

Orthogonal polynomials have been studied extensively as early as they were discovered by Legendre in 1784 [19]. In the mathematical treatment of model problems, orthogonal polynomials often arise to find solutions of ordinary differential equations under certain conditions imposed by the model.

The importance of orthogonal polynomials for contemporary mathematics and a wide range of their applications in physics and engineering is beyond any doubt. It is well-known that these polynomials play an essential role in problems of the approximation theory. They occur in the theory of differential and integral equations and mathematical statistics. Their applications in quantum mechanics, scattering theory, automatic control, signal analysis, and axially symmetric potential theory are also known [20,21].

Very recently, Amourah et al. [22] considered the Gegenbauer polynomials, whose generating function \( H_\alpha(x, \xi) \) is given by
\[
H_\alpha(x, \xi) = \frac{1}{(1 - 2x\xi + \xi^2)^\alpha},
\]  
(3)
where \( x \in [-1, 1] \) and \( \xi \in \mathbb{U} \). For a fixed \( x \), the function \( H_\alpha \) is analytic in \( \mathbb{U} \), so it can be expanded in a Taylor series as
\[
H_\alpha(x, \xi) = \sum_{n=0}^{\infty} C^\alpha_n(x)\xi^n,
\]  
(4)
where \( C^\alpha_n(x) \) is a Gegenbauer polynomial of degree \( n \).
Obviously, $H_\alpha$ generates nothing when $\alpha = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

\[ H_0(x, \xi) = 1 - \log(1 - 2x\xi + \xi^2) = \sum_{n=0}^{\infty} C_n^\alpha(x)\xi^n \] (5)

for $\alpha = 0$. Moreover, it is worth to mention that a normalization of $\alpha$ to be greater than $-1/2$ is desirable $[21,23]$. The following recurrence relations can also define Gegenbauer polynomials:

\[ C_n^\alpha(x) = \frac{1}{n[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_n^\alpha(x)]}, \] (6)

with the initial values

\[ C_0^\alpha(x) = 1, \quad C_1^\alpha(x) = 2ax \quad \text{and} \quad C_2^\alpha(x) = 2a(1+\alpha)x^2 - a. \] (7)

Many researchers have recently explored bi-univalent functions associated with orthogonal polynomials, refs. $[24-28]$ to mention a few. For a Gegenbauer polynomial, as far as we know, there is little work associated with bi-univalent functions in the literature.

Motivated essentially by the work of Amourah et al. $[22,29,30]$, we introduce here a new subclass of bi-univalent functions subordinate to Gegenbauer polynomials and obtain bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete–Szegö functional problems for functions in this new class.

2. Coefficient Bounds of the Class $\mathcal{S}_C(x, \lambda, \mu, \delta)$

**Definition 1.** Let $\alpha$ be a nonzero real constant $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 0$, $\zeta = \frac{2\lambda+\mu}{2\lambda+1}$. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{S}_C(x, \lambda, \mu, \delta)$ if the following subordinations are satisfied:

\[ (1 - \lambda) \left( \frac{f(\xi)}{\xi} \right)^\mu + \lambda f''(\xi) \left( \frac{f'(\xi)}{\xi} \right)^{\mu-1} + \zeta \delta z f'''(\xi) \prec H_\alpha(x, \xi) \] (8)

and

\[ (1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} + \zeta \delta w g''(w) \prec H_\alpha(x, w), \] (9)

where $x \in (\frac{1}{2}, 1]$, $\zeta = \frac{2\lambda+\mu}{2\lambda+1}$, the function $g(w) = f^{-1}(w)$ is defined by (2) and $H_\alpha$ is the generating function of the Gegenbauer polynomial given by (3).

First, we give the coefficient estimates for the class $\mathcal{S}_C(x, \lambda, \mu, \delta)$ given in Definition 1.

**Theorem 1.** Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{S}_C(x, \lambda, \mu, \delta)$. Then,

\[ |a_2| \leq \frac{2|a|\sqrt{2|a| \x}}{\sqrt{|\lambda(\lambda + \mu + 2\zeta\delta)^2 - 2|a(1 + \alpha)(\lambda + \mu + 2\zeta\delta)^2 - ((2\lambda + \mu)(1 + \mu) - 12\zeta\delta)a^2\lambda^2| \lambda^2 x^2|}}, \]

and

\[ |a_3| \leq \frac{4a^2\lambda^2}{(\lambda + \mu + 2\zeta\delta)^2} + \frac{2|a|\sqrt{2|a| \x}}{2\lambda + \mu + 6\zeta\delta}. \]

**Proof.** Let $f \in \mathcal{S}_C(x, \lambda, \mu, \delta)$. From Definition 1, for some analytic functions $w, v$ such that $w(0) = v(0) = 0$ and $|w(\xi)| < 1, |v(w)| < 1$ for all $\xi, w \in \mathbb{U}$, then we can write

\[ (1 - \lambda) \left( \frac{f(\xi)}{\xi} \right)^\mu + \lambda f''(\xi) \left( \frac{f'(\xi)}{\xi} \right)^{\mu-1} + \zeta \delta \xi f'''(\xi) = H_\alpha(x, w(\xi)) \] (10)
and
\[(1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu - 1} + \zeta \delta wg''(w) = H_a(x, v(w)), \quad (11)\]

From Equalities (10) and (11), we obtain that
\[f'(\xi) = 1 + C_1^e(x)c_1\xi + \left[C_1^e(x)c_2 + C_2^e(x)c_1^2\right]\xi^2 + \cdots \quad (12)\]
and
\[g'(w) = 1 + C_1^e(x)d_1w + \left[C_1^e(x)d_2 + C_2^e(x)d_1^2\right]w^2 + \cdots . \quad (13)\]

It is fairly well known that if
\[|w(\xi)| = |c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots | < 1, \quad (\xi \in \mathbb{U})\]
and
\[|v(w)| = |d_1w + d_2w^2 + d_3w^3 + \cdots | < 1, \quad (w \in \mathbb{U}),\]
then
\[|c_j| \leq 1 \quad \text{and} \quad |d_j| \leq 1 \quad \text{for all} \quad j \in \mathbb{N}, \quad (14)\]
see [31].

Thus, upon comparing the corresponding coefficients in (12) and (13), we have
\[\lambda + \mu + 2\zeta\delta)a_2 = C_1^e(x)c_1, \quad (15)\]
\[(2\lambda + \mu) \left[\left(\frac{\mu - 1}{2}\right)a_2^2 + \left(1 + \frac{6\delta}{2\lambda + 1}\right)a_3\right] = C_1^e(x)c_2 + C_2^e(x)c_1^2, \quad (16)\]
\[\lambda + \mu + 2\zeta\delta)a_2 = C_1^e(x)d_1, \quad (17)\]
and
\[(2\lambda + \mu) \left[\left(\frac{\mu + 3}{2} + \frac{12\delta}{2\lambda + 1}\right)a_2^2 - \left(1 + \frac{6\delta}{2\lambda + 1}\right)a_3\right] = C_1^e(x)d_2 + C_2^e(x)d_1^2. \quad (18)\]

It follows from (15) and (17) that
\[c_1 = -d_1 \quad (19)\]
and
\[2(\lambda + \mu + 2\zeta\delta)^2a_2^2 = |C_1^e(x)|^2 \left(c_1^2 + d_1^2\right). \quad (20)\]

If we add (16) and (18), we get
\[(2\lambda + \mu) \left(1 + \mu + \frac{12\delta}{2\lambda + 1}\right)a_2^2 = C_1^e(x)(c_2 + d_2) + C_2^e(x)(c_1^2 + d_1^2). \quad (21)\]

Substituting the value of \((c_1^2 + d_1^2)\) from (20) in the right hand side of (21), we deduce that
\[\left[(2\lambda + \mu)(1 + \mu) + 12\zeta\delta - \frac{2C_2^e(x)(\lambda + \mu + 2\zeta\delta)^2}{|C_1^e(x)|^2}\right]a_2^2 = C_1^e(x)(c_2 + d_2). \quad (22)\]

Moreover, computations using (13), (14) and (22) yield
\[|a_2| \leq \frac{2|a|x \sqrt{2|a|x}}{\sqrt{\mu(\lambda + \mu + 2\zeta\delta)^2 - 2|a(1 + a)(\lambda + \mu + 2\zeta\delta)^2 - ((2\lambda + \mu)(1 + \mu) - 12\zeta\delta)a^2|x^2}}.\]
Moreover, if we subtract (18) from (16), we obtain

\[
2(2\lambda + \mu) \left( 1 + \frac{6\delta}{2\lambda + 1} \right) a_3 - 2(2\lambda + \mu) \left( 1 + \frac{6\delta}{2\lambda + 1} \right) a_2^2 = C_1^a(x) (c_2 - d_2) + C_2^a(x) \left( c_1^2 - d_1^2 \right).
\]

(23)

Then, in view of (7) and (20), (23) becomes

\[
a_3 - a_2^2 = \frac{C_1^a(x)}{2(2\lambda + \mu + 2\delta)} (c_2 - d_2).
\]

Thus, applying (7), we conclude that

\[
|a_3| \leq \frac{4a^2\lambda^2}{(\lambda + \mu + 2\delta)^2} + \frac{2|ax|}{2\lambda + \mu + 6\delta}.
\]

This completes the proof of Theorem 1. □

Making use of the values of $a_2^2$ and $a_3$, we prove the following Fekete–Szegö inequality for functions in the class $\mathcal{S}_\mathbb{K}(x, \lambda, \mu, \delta)$.

**Theorem 2.** Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{S}_\mathbb{K}(x, \lambda, \mu, \delta)$. Then,

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2|ax|}{2\lambda + \mu + 6\delta}, & |\eta - 1| \leq M \\
\frac{8|ax|^2}{|1 - \eta|}, & |\eta - 1| \geq M,
\end{cases}
\]

where

\[
M = \frac{a(\lambda + \mu + 2\delta) - 2([a(1 + \alpha)(\lambda + \mu + 2\delta) - (2\lambda + \mu)(1 + \mu) - 12\delta)a^2] x^2)}{4(2\lambda + \mu + 6\delta)a^2 x^2}
\]

**Proof.** From (22) and (23)

\[
a_3 - \eta a_2^2 = a_2^2 + \frac{C_1^a(x)}{2(2\lambda + \mu + 6\delta)} (c_2 - d_2) - \eta a_2^2
\]

\[
= (1 - \eta) a_2^2 + \frac{C_1^a(x)}{2(2\lambda + \mu + 6\delta)} (c_2 - d_2)
\]

\[
= (1 - \eta) \left[ \left( \frac{C_1^a(x)}{2(2\lambda + \mu + 6\delta)} \right)^2 (c_2 + d_2) \right]
\]

\[
+ \frac{C_1^a(x)}{2(2\lambda + \mu + 6\delta)} (c_2 - d_2)
\]

\[
= C_1^a(x) \left[ \left( h(\eta) + \frac{1}{2(2\lambda + \mu + 6\delta)} \right) c_2 + \left( h(\eta) - \frac{1}{2(2\lambda + \mu + 6\delta)} \right) d_2 \right],
\]

where

\[
h(\eta) = \frac{[C_1^a(x)]^2 (1 - \eta)}{[(2\lambda + \mu)(1 + \mu) - 12\delta] [C_1^a(x)]^2 - 2(\lambda + \mu + 2\delta)^2 C_2^a(x)}
\]

Then, in view of (7), we conclude that

\[
|a_3 - \eta a_2^2| \leq \begin{cases} 
\frac{2|ax|}{2\lambda + \mu + 6\delta}, & 0 \leq |h(\eta)| \leq \frac{1}{2(2\lambda + \mu + 6\delta)} \\
\frac{8|ax|^2}{|1 - \eta|}, & |h(\eta)| \geq \frac{1}{2(2\lambda + \mu + 6\delta)}.
\end{cases}
\]

This completes the proof of Theorem 2. □
3. Corollaries and Consequences

In this section, we apply our main results to deduce each of the following new corollaries and consequences.

**Corollary 1.** Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{G}_{\Sigma}^\alpha(x, \lambda, \mu) = \mathcal{C}_{\lambda}^\alpha(x, \lambda, 0)$. Then,

$$|a_2| \leq \frac{2|a|x}{\sqrt{|a(\lambda+\mu)|} - 2|a(1+a)(\lambda+\mu)|^2 - (2\lambda+\mu)(1+a^2)|x^2|},$$

$$|a_3| \leq \frac{4a^2x^2}{(\lambda+\mu)^2} + \frac{2|a|x}{2\lambda+\mu},$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|a|x}{2\lambda+\mu}, & |\eta-1| \leq N, \\ \frac{8|a|^3|1-\eta|}{|a(\lambda+\mu)|^2 - 2|a(1+a)(\lambda+\mu)|^2 - 2(2\lambda+1)a^2|x^2|}, & |\eta-1| \geq N, \end{cases}$$

where

$$N = \frac{|a(\lambda+\mu)|^2 - 2|a(1+a)(\lambda+\mu)|^2 - 2(2\lambda+1)a^2|x^2|}{4(2\lambda+1)a^2|x^2|}.$$ 

**Corollary 2.** Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{G}_{\Sigma}^\alpha(x, \lambda) = \mathcal{C}_{\lambda}^\alpha(x, \lambda, 1, 0)$. Then,

$$|a_2| \leq \frac{2|a|x}{\sqrt{|a(\lambda+1)|} - 2|a(1+a)(\lambda+1)|^2 - 2(2\lambda+1)a^2|x^2|},$$

$$|a_3| \leq \frac{4a^2x^2}{(\lambda+1)^2} + \frac{2|a|x}{2\lambda+1},$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|a|x}{2\lambda+1}, & |\eta-1| \leq N, \\ \frac{8|a|^3|1-\eta|}{|a(\lambda+1)|^2 - 2|a(1+a)(\lambda+1)|^2 - 2(2\lambda+1)a^2|x^2|}, & |\eta-1| \geq N, \end{cases}$$

where

$$N = \frac{|a(\lambda+1)|^2 - 2|a(1+a)(\lambda+1)|^2 - 2(2\lambda+1)a^2|x^2|}{4(2\lambda+1)a^2|x^2|}.$$ 

**Corollary 3.** Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{G}_{\Sigma}^\alpha(x) = \mathcal{C}_{\lambda}^\alpha(x, 1, 1, 0)$. Then,

$$|a_2| \leq \frac{2|a|x}{\sqrt{4a - 2[4a(1+a) - 6a^2]|x^2|}},$$

$$|a_3| \leq \frac{4a^2x^2}{4} + \frac{2|a|x}{3},$$

and

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2|a|x}{3}, & |\eta-1| \leq \frac{|a-2a(1+a)-3a^2|x^2|}{3a^2|x^2|}, \\ \frac{2|a|^3|1-\eta|}{|a-2a(1+a)-3a^2|x^2|}, & |\eta-1| \geq \frac{|a-2a(1+a)-3a^2|x^2|}{3a^2|x^2|}. \end{cases}$$

**Remark 1.** Special cases of Gegenbauer polynomials $C_n^\alpha(x)$ are the Chebyshev polynomials $U_n(x) = C_n^1(x)$, when $\alpha = 1$, and if $\alpha = \frac{1}{2}$, we get the Legendre polynomials $P_n(x) = C_n^{\frac{1}{2}}(x)$.

**Remark 2.** By taking $\alpha = 1$, one can deduce the above results for the various subclasses studied by Yousef et al. [26].
4. Concluding Remark

In this present investigation, we introduced and studied the coefficient problems associated with each of the new subclasses $\mathcal{S}_f^x(x)$, $\mathcal{S}_f^x(x, \lambda)$, $\mathcal{S}_f^x(x, \lambda, \mu)$ and $\mathcal{S}_f^x(x, \lambda, \mu, \delta)$ of the class of bi-univalent functions in the open unit disk $U$. We derived estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and Fekete–Szegö functional problems for functions belonging to these new subclasses.

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