Transport of a Luttinger liquid in the presence of a time dependent impurity

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Abstract

We show that the macroscopic current and charge can be formulated as a Quantum Mechanical zero mode problem. We find that the current is given by the velocity operator of a particle restricted to move around a circle. As an explicit example we investigate a Luttinger liquid of length $L$ which is perturbed by a time dependent impurity. Using the statistical mechanics of zero modes we computed the non-equilibrium current. In particular we show that in the low temperature limit, $L_T/L > 1$, the zero mode method introduced here becomes essential for computing the current.
I. INTRODUCTION

In a two dimensional electron gas, the presence of an external potential gives rise to a one-dimensional conducting channel named quantum wire \( [1] \). Quantum wires are characterized by a quantized conductance of \( 2e^2/h \) per channel. The presence of the electron-electron interaction is characterized by the interaction parameter “\( K \)”, which affects the tunneling conductance \( [2,3] \). In quantum Hall experiments, the inter-edge tunneling is also characterized by the interaction parameter “\( K \)” \( [4] \). The analysis of the transport phenomena requires the understanding of the macroscopic quantities such as charge and current at zero momentum. We will show within the Hamiltonian formalism that the quantized zero mode operators can be used to compute the macroscopic current in the presence of interaction and time dependent scatterers. As an explicit example we will compute the non-equilibrium current for a Luttinger liquid, perturbed by a time dependent backscattering impurity. Using the interaction picture we will find an exact expression for the zero mode coordinates. The time derivative of the zero mode will give the non-equilibrium current. We will compute the current in two limits, \( L/L_T > 1 \) - quantum wire case and \( L/L_T < 1 \) - mesoscopic case (\( L_T \) is the thermal coherent length given by \( L_T = \hbar v/(K_B T) \)). Comparing with the results given in ref. \( [2] \) for \( L \to \infty \) we find similar results for (“\( f \)” is the impurity frequency and “\( f_{DS} \)” is the equivalent frequency for the drain-source voltage) \( f = 0, f_{DS} \neq 0, \) and \( L/L_T > 1 \). For the time dependent impurity, \( f \neq 0 \) and \( f_{DS} \neq 0 \) the current is affected by the frequency \( f \) only when \( \hbar |f + f_{DS}| > K_B T \). We find an interesting result in the limit \( L_T/L > 1 \). In this limit the single particle fermionic spectrum (absent in the standard bosonization) reproduced by the zero mode becomes essential for the calculation of the current. We find for \( f \neq 0 \) and \( f_{DS} = 0 \) that the current “\( I \)” is proportional to \( ef \) and is periodic in \( eV_{GS} \), the static chemical potential, reflecting the quantization of charge.

The plan of this paper is as the follows: Chapter \( [1] \) is devoted to the method of the zero mode bosonization. In Chapter \( [1] \) we introduce the macroscopic current which is identified
with the time derivative of the macroscopic coordinate. Chapter IV contains the analysis of the thermodynamics of the Luttinger liquid. In Chapter V we compute the macroscopic current for the time dependent impurity potential. Chapter VI is restricted to the computation of the conductance for the Luttinger liquid. Finally in Chapter VII we conclude the paper.

II. THE ZERO MODE BOSONIZATION

Bosonization has become the standard method in one dimension since it allows to incorporate in a trivial way the interactions. For transport problems one must be able to add and subtract particles, such a thing is not trivial within the Bosonization method [3]. This problem has been solved by using zero modes in addition to the particle-hole bosonic variables [5–8]. Here we will show how the zero mode method can be used for computing macroscopic quantities such as current and charge. Following Ref. [5,6] we replace the spinless Fermion $\psi(x)$ by $\psi(x) = e^{iK_F x} \psi_R(x) + e^{-iK_F x} \psi_L(x)$ where $R$ and $L$ represent the right and left movers, with $K_F = \pi/2a$, $\psi_R(x) = \chi_R \tilde{\psi}_R(x)$, and $\psi_L(x) = \chi_L \tilde{\psi}_L(x)$ where $\chi_i = A_i + A_i^\dagger$, $i = R, L$ such that $\{A_i, A_j^\dagger\} = \delta_{i,j}$, $\{A_i, A_j\} = 0$ are the fermionic degrees of freedom ("x" independent) which keep the anticommutation relation between the left and right movers [8] (An alternative possibility used in Ref. [5] is to represent the $\chi_i$ in terms of the Klein’s factors). The operators $\tilde{\psi}_R(x)$ and $\tilde{\psi}_L(x)$ are expressed in terms of bosonic variables and zero modes for a chain of length $L$,

$$\tilde{\psi}_R(x) = \sqrt{\frac{K_F}{2\pi}} e^{i\alpha_R} e^{i\frac{2\pi}{L} (\hat{N}_R - \frac{1}{2}) x} e^{i\theta_R(x)}$$

$$\tilde{\psi}_L(x) = \sqrt{\frac{K_F}{2\pi}} e^{-i\alpha_L} e^{-i\frac{2\pi}{L} (\hat{N}_L - \frac{1}{2}) x} e^{-i\theta_L(x)}$$

(1)

The fermionic fields given in Eq.1 obey antiperiodic boundary conditions. $\theta_R(x)$ and $\theta_L(x)$ represent the particle-hole excitation with no zero mode which obey $\theta_R/L(x + L) = \theta_R/L(x)$.
θ_{R/L}(x). α_R, \hat{N}_R and α_L, \hat{N}_L are the zero modes which obey the commutation rules: $[\alpha_R, -\hat{N}_R] = [\alpha_L, \hat{N}_L] = i$. The operators $\hat{N}_R$ and $\hat{N}_L$ measure the number of fermions, $\hat{N}_R|N_R; \theta_R\rangle = N_R|N_R; \theta_R\rangle$, $\hat{N}_L|N_L; \theta_L\rangle = N_L|N_L; \theta_L\rangle$ with the eigen values $N_R$ and $N_L$. $\alpha_R$ and $\alpha_L$ play the role of the coordinate of a particle on a circle, $0 \leq \alpha_{R/L} \leq 2\pi$, and $N_R$, $N_L$ are the canonical conjugate momentum for a rotator. As a result $e^{\mp i\alpha_R}$ and $e^{\pm i\alpha_L}$ act as a creation (annihilation) operator of particles, $e^{\mp i\alpha_R}|N_R; \theta_R\rangle = |N_R \pm 1; \theta_R\rangle$, $e^{\pm i\alpha_L}|N_L; \theta_L\rangle = |N_L \mp 1; \theta_L\rangle$.

III. THE MACROSCOPIC CURRENT

Next we will introduce the macroscopic currents. We will use the method used in quantum mechanics where one replaces the “momentum” operator with the “velocity” one. In each channel we add $N_R$ and $N_L$ charges. using the balistic time $f_B^{-1} = L/v_F$ we introduce two currents:

$$I_R = -\frac{eF}{L} N_R; \quad I_L = \frac{eF}{L} N_L; \quad \tilde{I} = \bar{I} = I_R + I_L = \frac{eF}{L} (N_L - N_R).$$

(2)

Using the fact that the coordinator operator $\alpha_R$ is conjugate to the momentum operator $\hat{N}_R$ and $\alpha_L$ is conjugate to $\hat{N}_L$ allows to define the velocity operators, $\dot{\alpha}_R$ and $\dot{\alpha}_L$, according to the rules of quantum mechanics. Using the Heisenberg equation of motion, we will identify the “velocity” operator with the current one. We start by defining the chiral currents $I_L$, $I_R$, and $I = I_R + I_L$:

$$I_R = \frac{e}{2\pi} \frac{d}{dt} \alpha_{R,H}(t); \quad I_L = \frac{e}{2\pi} \frac{d}{dt} \alpha_{L,H}(t); \quad I = \frac{e}{2\pi} \frac{d}{dt} \alpha_H(t).$$

(3)

In Eq.(3), $\alpha_{R,H}(t)$, $\alpha_{L,H}(t)$, and $\alpha_H(t)$ are the Heisenberg operators:

$$\alpha_{L/R,H}(t) = e^{iHt/\hbar} \alpha_{L/R} e^{-iHt/\hbar}; \quad \hat{N}_{L/R,H}(t) = \hat{N}_{L/R}$$

(4)

with $\alpha = \alpha_L + \alpha_R$ being the coordinate, $\hat{J} = \hat{N}_L - \hat{N}_R$ the current, $\hat{Q} = \hat{N}_L + \hat{N}_R$ the charge, and $[\alpha, \hat{J}] = i2$. 


Using the case of non-interacting zero mode hamiltonian allows to show the identity between the definitions in Eqs. (2) and (3). We consider the hamiltonian:

\[ H_{L/R}^{(n=0)} = \frac{\pi v_F\hbar}{L} \hat{N}_{L/R} \]

\[ H_0^{(n=0)} = H_L^{(n=0)} + H_R^{(n=0)} = \frac{\pi v_F\hbar}{2L} \left( \hat{J}_2 + \hat{Q}_2 \right). \tag{5} \]

The Heisenberg equation of motion gives:

\[ I_{L/R} = \frac{e}{2\pi} \frac{d\alpha_{L/R,H}}{dt} = \frac{e}{i\hbar} [\alpha_{L/R,H}, H_{L/R}^{(n=0)}] = \pm \frac{ev}{L} \hat{N}_{L/R} \tag{6} \]

with

\[ \hat{N}_{L/R,H}(t) = \hat{N}_{L/R}; \quad \hat{J}_H(t) = \hat{J} \]

\[ I = \frac{e}{2\pi} \frac{d\alpha_H}{dt} = \frac{e}{i\hbar} [\alpha_H, H_0^{(n=0)}] = \frac{ev}{L} \hat{J}_H. \tag{7} \]

Using an eigenstate with a fixed number of fermions, \(|N_R, \theta_R\rangle \otimes |N_L, \theta_L\rangle\), allows to establish that the expectation values \langle I_L \rangle, \langle I_R \rangle, and \langle I \rangle given in Eqs.(3) and Eqs.(7) are identical with the definition given in Eq.(2).

Next we consider the Luttinger liquid zero mode hamiltonian and show that the current operator is changed such that \(v\) is replaced by \(v_J = vK\).

\[ H_0^{(n=0)} = \frac{\pi v\hbar}{2L} \left[ \frac{K}{K} \hat{J}_2 + \hat{Q}_2 \right], \quad K \leq 1. \tag{8} \]

Here the Heisenberg equation of motion gives the result obtained in Ref. [5],

\[ I = \frac{e}{2\pi} \frac{d\alpha_H}{dt} = \frac{e}{i\hbar} [\alpha_H, H_0^{(n=0)}] = \frac{evK}{L} \hat{J}_H. \tag{9} \]

For the remaining part we will compute the macroscopic zero mode current \( I = [e/(2\pi)]d\alpha_H/dt \) for a Luttinger liquid in the presence of a time dependent backscattering impurity potential \( H_1(t) \) localized at \( x = 0 \).
\[ H = H_0 + H_1(t) \quad (10) \]

\[ H_0 = H_0^{(n=0)} + H_0^{(n\neq0)} \quad (11) \]

\[ H_0^{(n=0)} = \frac{\pi \hbar}{L} \left[ \frac{K}{2} \hat{J}^2 + \frac{1}{2K} \hat{Q}^2 \right] \quad (12) \]

\[ H_0^{(n\neq0)} = \int_0^L \frac{dx}{2\pi} \hbar \left[ \frac{K}{2} (\partial_x \phi)^2 + \frac{1}{2K} (\partial_x \theta)^2 \right] \quad (13) \]

\[ H_1(t) = \frac{\lambda}{2} [\chi_R \chi_L e^{-i\alpha} e^{-i\theta(x=0)} e^{-i\omega t} + \text{h.c.}], \quad t > 0. \quad (14) \]

In obtaining Eqs. (10)-(14) we have used the bosonic fields \( \theta \) and \( \phi \): \( \theta(x) = \theta_R + \theta_L \), \( \phi(x) = \theta_L - \theta_R \). Similarly for zero modes we define the charge and current operators: \( \hat{Q} = \hat{N}_R + \hat{N}_L \), \( \hat{J} = \hat{N}_L - \hat{N}_R \), \( \alpha = \alpha_L + \alpha_R \), \( [\alpha, \hat{J}] = 2i \). The operators \( \hat{Q} \) and \( \hat{J} \) have the eigenvalues \( Q = N_R + N_L \) and \( J = N_L - N_R \). \( H_0 \) represents the Luttinger liquid model for a wire of length \( L \) characterized by the repulsive interaction \( K \leq 1 \) and Fermi velocity \( v \). \( H_1(t) \) represents a time dependent backscattering impurity driven at a frequency \( \omega = 2\pi f \) and momentum \( q \sim 2K_F \). (Microscopically one obtains such a term when we couple a driven optical phonon field with a frequency \( \omega = \omega(q) \) and momentum \( q \sim 2K_F \) to the electrons. It is assumed that the coupling occurs only in a restricted region in space \( d \ll L \) which justifies to approximate the problem by a localized impurity.) The effect of the time dependent impurity is similar to an oscillating atom which induces a charge density wave, \( \psi_R \psi_L \Delta_{CDW} + \text{h.c.}, \Delta_{CDW} \sim \frac{\lambda}{2} e^{-i\omega t} \). In the absence of any other source a charge density wave will depend on the relative phase between the field and the electrons.

The macroscopic current for the hamiltonian \( H \) given in Eqs. (11) to (14) is obtained from the Heisenberg equation of motion:

\[ I_H(t) = \frac{e}{2\pi} \frac{\alpha_H}{dt} = \frac{e}{i\hbar} [\alpha_H, H] = \frac{evK}{L} \hat{J}_H(t). \quad (15) \]

From Eq. (15) we observe that the current is determined by the Heisenberg representation,
\[ \hat{J}_H(t) = \exp \left( \frac{i}{\hbar} Ht \right) \hat{J} \exp \left( -\frac{i}{\hbar} Ht \right) \]  \hspace{1cm} (16)

For convenience we will use the interaction picture, where

\[ \alpha_I(t) = \exp \left( \frac{i}{\hbar} H_0^{(n=0)} t \right) \alpha(0) \exp \left( -\frac{i}{\hbar} H_0^{(n=0)} t \right); \]

\[ \hat{J}_I(t) = \exp \left( \frac{i}{\hbar} H_0^{(n=0)} t \right) \hat{J}(0) \exp \left( -\frac{i}{\hbar} H_0^{(n=0)} t \right) \equiv \hat{J} \]  \hspace{1cm} (17)

and

\[ \alpha_H(t) = U^\dagger(t,0) \alpha_I(t) U(t,0); \quad \hat{J}_H(t) = U^\dagger(t,0) \hat{J}_I(t) U(t,0) \]  \hspace{1cm} (18)

where \( U(t,0) \) is the unitary evolution operator,

\[ U(t,0) = T \exp \left[ -\frac{i}{\hbar} \int_0^t dt_1 H_I(t_1) \right]; \quad H_I(t) = \exp \left( \frac{i}{\hbar} H_0 t \right) H_1(t) \exp \left( -\frac{i}{\hbar} H_0 t \right). \]  \hspace{1cm} (19)

Combining Eqs. (16), (17), and (18), we obtain the Heisenberg representation of the current \( I_H(t) \),

\[ I_H(t) = \frac{eV}{L} U^\dagger(t,0) \hat{J} U(t,0) \]  \hspace{1cm} (20)

In order to compute the current we have to perform the statistical average of Eq.(20). In particular we will be interested in computing the current \( I_H(t) \) in the presence of an external source of voltage \( \tilde{V}_{DS} \). This requires to add to the zero mode hamiltonian a source term of the form \( e\tilde{V}_{DS} \hat{J} \). As a result, the zero mode hamiltonian \( H_0^{(n=0)} \) is replaced by \( H_0^{(n=0)} + e\tilde{V}_{DS} \hat{J} \), causing a change in the current operator \( I_H(t) \rightarrow I_H(t) + (e^2/\hbar)\tilde{V}_{DS} \). This simple procedure has been questioned in Ref. [9] where the external voltage source connected to the wire is screened. As a result it has been argued that the screened voltage source must be used and not the applied voltage \( \tilde{V}_{DS} \). In this paper we will adopt a thermodynamic point of view. We will replace the external voltage \( \tilde{V}_{DS} \) by two chemical potentials \( \mu_L \) and \( \mu_R \). As a result no external source will be added to the zero mode Hamiltonian \( H_0^{(n=0)} \). Therefore the Heisenberg equation of motion for the current will be given by Eq.(20). The only change
will occur at the level of the thermodynamic expectation values which will depend on the chemical potentials in the reservoirs. (The chemical potentials in the reservoirs are different from the ones in the wires. This difference reflects the screening phenomena.) In this paper we will define the conductance with respect to the chemical potential in the reservoir. Following Ref. [10] we will consider that the thermal reservoir has two chemical potentials, \( \mu_R \) and \( \mu_L \). We will choose \( (\mu_L + \mu_R)/2 \equiv eV_{GS} \) and \( \mu_L - \mu_R \equiv eV_{DS} \). The presence of the chemical potential has no effects on the equation of motion. Only the statistical average will be affected. The reservoir will be characterized by the partition function,

\[
Z(n\neq0) = Tr\, e^{-\beta H_0^{(n\neq0)}}
\]

\[
Z(n=0) = Tr\left[ e^{-\beta H_0^{(n=0)}} e^{-\beta(\mu_R \hat{N}_R + \mu_L \hat{N}_L)} \right]
\]

In order to compute the non-equilibrium current caused by \( H_1(t) \), we will compute the statistical average using the partition function \( Z \). We introduce the notation:

\[
\rho(H_0) = \rho\left(H_0^{(n\neq0)}\right) \rho\left(H_0^{(n=0)}\right) \equiv \langle \cdots \rangle
\]

where

\[
\rho\left(H_0^{(n\neq0)}\right) = \left[Z(n\neq0)\right]^{-1} e^{-\beta H_0^{(n\neq0)}} \equiv \langle \cdots \rangle_{n\neq0}; \tag{23}
\]

\[
\rho\left(H_0^{(n=0)}\right) = \left[Z(n=0)\right]^{-1} e^{-\beta H_0^{(n=0)}} e^{-\beta(\mu_R \hat{N}_R + \mu_L \hat{N}_L)} \equiv \langle \cdots \rangle_{n=0}. \tag{24}
\]

**IV. THERMODYNAMICS OF THE ZERO MODE LUTTINGER LIQUID**

We will compute the zero mode partition function \( Z^{(n=0)} \equiv Z^{(n=0)}(\mu_R, \mu_L) \). Using the partition function we will compute the zero mode expectation values, \( \langle \hat{N}_R \rangle_{n=0} \), \( \langle \hat{N}_L \rangle_{n=0} \), \( \langle \hat{N}_R^2 \rangle_{n=0} \), and \( \langle \hat{N}_L^2 \rangle_{n=0} \). These result will be used to compute the current in the presence of the time dependent perturbation \( H(t_1) \) given by Eq. (14). We will consider two cases,
$L/L_T \gg 1$ and $L/L_T \leq 1$.

**a)** The $L/L_T \gg 1$ case. In this limit the sum over $N_R$ and $N_L$ in Eq. (22) is replaced by a Gaussian integral. As a result we obtain:

$$
\langle \hat{N}_R \rangle_{n=0} \rightarrow \frac{\mu_R}{\hbar} \frac{L}{Kv}, \quad \langle \hat{N}_L \rangle_{n=0} \rightarrow \frac{\mu_L}{\hbar} \frac{L}{Kv},
$$

$$
\langle \hat{J} \rangle_{n=0} \rightarrow \frac{eV_{DS}}{\hbar} \frac{L}{Kv}, \quad \langle \hat{Q} \rangle_{n=0} \rightarrow 2 \frac{eV_{GS}}{\hbar} \frac{L}{Kv}
$$

and

$$
\langle \hat{N}_R^2 \rangle_{n=0} - \langle \hat{N}_R \rangle_{n=0}^2 = \langle \hat{N}_L^2 \rangle_{n=0} - \langle \hat{N}_L \rangle_{n=0}^2 \rightarrow \frac{1}{2\pi K} \left( \frac{L}{L_T} \right)
$$

$$
\langle \hat{J}^2 \rangle_{n=0} - \langle \hat{J} \rangle_{n=0}^2 = \langle \hat{Q}^2 \rangle_{n=0} - \langle \hat{Q} \rangle_{n=0}^2 \rightarrow \frac{1}{\pi K} \left( \frac{L}{L_T} \right).
$$

**b)** The $L/L_T \leq 1$ case. This represents the low temperature limit where single particle excitations are reproduced by the zero mode theory. For this case standard bosonization is not applicable since the fermionic distribution function is absent. The partition function $Z^{(n=0)}(\mu_R, \mu_L)$ is given by:

$$
Z^{(n=0)}(\mu_R, \mu_L) = \sum_{N_L=-\infty}^{\infty} W^{N_R^2} V_L^{N_L} \sum_{N_R=-\infty}^{\infty} W^{N_R^2} V_R^{N_R(1+rN_L)}
$$

where

$$
W = \exp \left[ -\frac{\pi L_T}{2L} \left( K + K^{-1} \right) \right]; \quad V_L = \exp \left( \frac{\mu_L}{K_B T} \right), \quad V_R = \exp \left( \frac{\mu_R}{K_B T} \right);
$$

$$
r = \frac{\pi v \hbar}{L \mu_R} \left( K^{-1} - K \right).
$$

For $K = 1$, $r = 0$ and the Luttinger term $N_L N_R$ is absent in Eq. (27). As a result, $Z^{(n=0)}(\mu_R, \mu_L) = Z^{(n=0)}(\mu_R) Z^{(n=0)}(\mu_L)$. Using the Jacobi identity we find:
\[ Z^{(n=0)}(\mu_{L/R}) = \prod_{n=1}^{\infty} \left[ \frac{(1 + W^{2n-1}V_{L/R}) (1 + W^{2n-1}V_{L/R}^{-1})}{(1 + W^{2n-1})^2} \right]. \quad (31) \]

Using Eq. (31) we find
\[
\langle \hat{N}_{L/R} \rangle_{n=0} = k_B T \frac{\partial \ln Z^{(n=0)}(\mu_{L/R})}{\partial \mu_{L/R}} = \sum_{n=1}^{\infty} \left[ 1 + \exp(\beta(\epsilon_n - \mu_{L/R})) \right]^{-1} \quad (32)
\]

where \( \epsilon_n = (2\pi/L) v \hbar(n - 1/2) \) is the single particle spectrum. Similarly we obtain:
\[
\langle \hat{N}_{L/R}^2 \rangle_{n=0} - \langle \hat{N}_{L/R} \rangle_{n=0}^2 = k_B T \frac{\partial \langle \hat{N}_{L/R} \rangle_{n=0}}{\partial \mu_{L/R}} = \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \left( \frac{\epsilon_n - \mu_{L/R}}{2k_B T^*} \right). \quad (33)
\]

Next we consider the Luttinger case where \( K < 1 \) and \( r \neq 0 \) in Eq. (27). For Eq. (27) we perform first the sum over \( N_R \). As a result we find:
\[
\sum_{N_R=-\infty}^{\infty} W_{LR}^{N_R} V_{LR}^{N_R} V_{LR}^{N_R} V_{LR}^{N_L} \sim V_{LR}^{P(N_L,r)}. \quad (34)
\]

\( P(N_L,r) \) is a polynomial in \( N_L \) which can be obtained in the limit \( L/L_T \ll 1 \). We find
\[
P(N_L,r) \sim \frac{\mu_R L}{\pi v \hbar} \frac{2}{K + K^{-1}} \left[ rN_L + \frac{1}{2}(rN_L)^2 \right]. \quad (35)
\]

Substituting Eq. (34) into Eq. (27) gives:
\[
\langle \hat{N}_{L/R} \rangle_{n=0} = \sum_{n=1}^{\infty} \left[ 1 + \exp \left( \frac{\hat{\epsilon}_n - \mu_{L/R}}{k_B T^*} \right) \right]^{-1} \quad (36)
\]

where \( T^* \) and \( \hat{\epsilon}_n \) are given by:
\[
T^* \equiv T[1 + g(K)]^{-1}; \quad g(K) \equiv 2 \left( \frac{K^{-1} - K}{K^{-1} + K} \right),
\]
\[
\hat{\epsilon}_n \equiv \epsilon_n \left( \frac{K^{-1} + K}{2} \right) \frac{1 - 2 \left( \frac{K^{-1} - K}{K^{-1} + K} \right)}{1 + 2 \left( \frac{K^{-1} - K}{K^{-1} + K} \right)} , \quad (37)
\]

and
\[
\langle \hat{N}_{L/R}^2 \rangle_{n=0} - \langle \hat{N}_{L/R} \rangle_{n=0}^2 = \frac{1}{8} \sum_{n=1}^{\infty} \cosh^{-2} \left( \frac{\epsilon_n - \mu_{L/R}}{2k_B T^*} \right) \quad (38)
\]

Comparing the case \( K \neq 1 \) to \( K = 1 \) we find that the form is preserved given that we replace \( \epsilon_n \) by \( \hat{\epsilon}_n \) and the temperature \( T \) by \( T^* \).
V. COMPUTATION OF THE MACROSCOPIC CURRENT IN THE PRESENCE OF A WEAK IMPURITY POTENTIAL

Our starting point will be Eq. (20) with $U(t,0)$ controlled by the impurity hamiltonian $H_1(t)$. For the static impurity case, standard RG calculation shows that for $K < 1$ the stable fixed point of $\lambda$ is $\lambda^* = \infty$. This means that the perturbation theory around the unstable fixed point $\lambda^* = 0$ must break for length scale $\ell > \ell_0$, where $\ell_0$ satisfies $\hat{\lambda}(\ell_0) \simeq \hat{\lambda}\ell_0^{1-K} \simeq 1$. In the remaining part we will assume that for a finite system of length $L = \ell_0 a$, $\hat{\lambda} \ll 1$, and $\hat{\lambda}(\ell_0) < 1$. Therefore for this range of parameters we can expand $U(t,0)$ in powers of $\hat{\lambda} \equiv \lambda/\Lambda$ and find

$$\langle \langle I_H(t) \rangle \rangle = e^{LvK} \langle \hat{J} \rangle_{n=0} + e^{LvK} \langle \hat{I} \rangle_{n=0} + e^{LvK} \langle \hat{J} \rangle_{n=0} + \cdots. \quad (39)$$

In Eq. (39), $c_t$ stays for the Keldysh contour: $0 \rightarrow t, t \rightarrow 0$, and $0 \rightarrow -i\beta$. The term $\langle \cdots \rangle_{n=0}$ represents the expectation values with respects to the zero mode Hamiltonian $H_0^{(n=0)}$. The second term vanishes since it contains the operators $e^{\pm i\alpha}$ (The expectation value for the terms of the forms $\langle N_R | e^{\pm i\alpha|R} | N_R \rangle$ and $\langle N_L | e^{\pm i\alpha|L} | N_L \rangle$ are zero). The third term decouples into two expectation values, $\langle \cdots \rangle_{n \neq 0}$ (the bosonic part) and $\langle \cdots \rangle_{n=0}$ (the zero mode part) which is not zero, since the zero mode coordinate $\alpha$ cancels. As a result we find:

$$\langle \langle I_H(t) \rangle \rangle = \frac{e}{L} vK \langle \hat{J} \rangle_{n=0} - e \left( \frac{\lambda}{2\hbar} \right)^2 \frac{2vK}{L} \int_0^t dt_1 e^{i(\frac{4\pi}{L}vK)t_1}. \int_0^t ds (e^{i[\theta_I(s)-\theta_I(0)]})_{n \neq 0} e^{-i\frac{4\pi}{L}vKs} \cos \left[ \left( \frac{2\pi}{L} vK \hat{J} + \omega \right) s \right] \langle \hat{J} \rangle_{n=0}$$

$$t \rightarrow \infty \frac{e}{L} vK \langle \hat{J} \rangle_{n=0} - \frac{e}{2(4\pi)} \left( \frac{\lambda}{\hbar} \right)^2 \int_0^t ds F(s) \sin \left( \frac{4\pi}{L} vKs \right) \cos \left[ \left( \frac{2\pi}{L} vK \hat{J} + \omega \right) s \right] \langle \hat{J} \rangle_{n=0}. \quad (40)$$

In Eq. (40) we observe that the current is determined by the zero mode expectation value $\langle \cdots \rangle_{n=0}$ and separately by the bosonic part $F(s) \equiv \langle e^{i[\theta_I(s)-\theta_I(0)]}\rangle_{n \neq 0}$, where $e^{i\theta_I(t)} = \cdots$
\( e^{\pm i\theta(0)} e^{-\frac{\pi}{4} H_0^{(n \neq 0)}} t \) In order to evaluate the current in Eq.(41) we have to compute the zero mode part, \( \langle \hat{J} \rangle_{n=0} \) and \( \langle \cos[2\pi(\frac{nK}{L}\hat{J}+f)s] \hat{J} \rangle_{n=0} \).

a) We consider first the quantum wire limit \( L/L_T \ll 1 \) where \( L_T = \hbar v \beta = \hbar \kappa \beta / T \) is the thermal length. Using the results given in Eqs.(25) and (26) we obtain:

\[
\langle \hat{J} \rangle_{n=0} \rightarrow \frac{eV_{DS}}{h} \frac{L}{Kv} \quad \frac{eV_{DS}}{h} \overset{\text{def}}{=} f_{DS},
\]

\[
\langle \cos \left[ 2\pi \left( \frac{nK}{L} \hat{J} + f \right) s \right] \hat{J} \rangle_{n=0} \rightarrow f_{DS} \left( \frac{L}{Kv} \right) \cos[2\pi(f_{DS}+f)s] \] (41)

In Eq.(41) \( f_{DS} \) is either positive or negative since it depends on the drain source voltage \( V_{DS} \). We introduce the dimensionless length “\( \ell \)”: \( \ell \overset{\text{def}}{=} \frac{L}{a} = \ell_T \) for \( |f_{DS}+f|^{-1} > \frac{L_T}{v} \) or \( \ell = \frac{V_{DS}}{|f_{DS}+f|} \ell_{f} \) for \( |f_{DS}+f|^{-1} < \frac{L_T}{v} \). Using the dimensionless length \( \ell \) defined in Eq.(41) and the function \( F(s) \sim \left( \frac{2\pi}{(v\Lambda s)^{2K}} \right) s, s < \hbar \beta \), we find:

\[
I \sim e f_{DS} \left[ 1 - \frac{\lambda^2 c_0}{4(1-K)} \ell^2(1-K) \right], \quad K < 1, \quad c_0 \approx 1 \] (42)

From Eq.(42) we observe that the current grows with the increases of temperature (decreases of \( \ell \)). The results obtained in Eq.(42) are in agreement with the one obtained in ref. [2]. We want to emphasize that according to our discussion at the beginning of this section, Eq.(42) is valid only for “\( \hat{\lambda} \)” which obeys \( \hat{\lambda} \ell^{1-K} < 1 \). (For \( \ell \rightarrow \infty \) we have to perform a duality transformation where \( \hat{\lambda} \rightarrow \hat{\ell} = 1/\hat{\lambda}, \theta \rightarrow \phi \), and \( K \rightarrow \eta = 1/K \).)

We want to mention that the formalism presented here is applicable to the “persistent current” problem [3,12,13,14]. In particular we want to emphasize that the result of the “persistent current” follows from Eq.(41) once we substitute \( eV_{DS}/h = (2\pi/L)(vK)(\Phi/\Phi_0) \), where “\( \Phi \)” is the external flux and “\( \Phi_0 \)” is the flux quantum. This correspondence follows from the fact that in the presence of an external flux “\( \Phi \)”, one obtains an equivalent problem with twisted boundary conditions. From Eq.4 it follows that the twist of the boundary condition is realized when one shift the zero mode operator \( \hat{J} \) to \( \hat{J} + 2\Phi/\Phi_0 \). As a result
the zero mode Hamiltonian $H_0^{(n=0)}$ will be shifted by, $(2\pi/L)(vK)\hbar\hat{J}\Phi/\Phi_0 + \text{Const}$. As an explicit example we consider the effect of the static impurity on the “persistent current” in the limit $L/L_T > 1$. Using the results given in Eqs.(41) and (12), we find in the limit $\Phi/\Phi_0 \to 0$

$$I = \frac{2\pi(vK)}{L} \left( \frac{\Phi}{\Phi_0} \right) \left[ 1 - \frac{\hat{\lambda}^2 c_0}{4(1-K)} \ell_T^2(1-K) \right].$$

(43)

Next we return to the time dependent impurity driven by the frequency $2\pi f$ when $f_{DS} = 0$. We compute the expectation values $\langle \hat{J} \rangle_{n=0}$, $\langle \hat{J}^2 \rangle_{n=0}$ and $\langle \cos[2\pi(vK)\hat{J} + f)s] \hat{J} \rangle_{n=0}$. We find from Eqs. (23) and (26):

$$\langle \hat{J} \rangle_{n=0} = 0, \quad \langle \hat{J}^2 \rangle_{n=0} = \frac{1}{\pi K} \left( \frac{L}{L_T} \right)$$

(44)

$$\langle \cos \left[ 2\pi \left( \frac{vK}{L} \hat{J} + f \right) s \right] \hat{J} \rangle_{n=0} = -\sin(2\pi f s)\langle \sin \left( \frac{2\pi}{L} vK \hat{J} s \right) \hat{J} \rangle_{n=0}$$

(45)

As a result we obtain:

$$I \simeq ef \left[ \frac{\hat{\lambda}^2}{4(2-K)} \left( \frac{L_T}{L} \right) \ell_T^2(1-K) \right], \quad K < 1.$$ 

(46)

Eq. (46) shows that in the quantum wire limit $L_T/L \to 0$, $I \to 0$. The current in Eq. (46) is induced by the driven field $\Delta_{CDW} \sim \frac{1}{2} e^{-i\omega t}$. The presence of this field shifts the value of $\alpha$ to $\alpha + \omega t$. As a result the current will depend on the frequency “$f$”. The current will be non-zero if $2\pi f \frac{L_T}{L} < 1$. (For high frequencies $2\pi f \frac{L_T}{L} \gg 1$ the current will vanish.) The direction of the current is determined by the initial phase of the driven field. Formally the direction of the current is determined by the total phase $\alpha(t) + \omega t = \alpha(0) + \frac{2\pi vK}{L} \hat{J} t + \omega t$. When the phase increases, the direction of the current is positive. The positive current ($\hat{J} > 0$) corresponds to $\mu_L - \mu_R > 0$. Therefore a positive current corresponds to the direction of a current which will flow when $\mu_L > \mu_R$.

b) Next we consider the low temperature limit $L_T/L > 1$. For this case we will make use of the results obtained in Eqs. (36) to (38). For simplicity, we will consider two problems: 1)
\( \mu_L = \mu_R = \mu \) and \( f \neq 0; \) 2) \( \mu_L - \mu_R = eV_{DS} \neq 0 \) and \( f = 0. \)

We consider first problem 1). According to Eqs. (36)-(38) we find:

\[
\langle \hat{J} \rangle_{n=0} = 0, \quad \langle \hat{Q} \rangle_{n=0} = 2 \sum_{n=1}^{\infty} \left[ 1 + \exp \frac{\hat{\epsilon}_n - \mu}{K_B T^*} \right],
\]

\[
\langle \hat{J}^2 \rangle_{n=0} = \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*}.
\] (47)

As a result we find from Eq.(41) that the current is given by:

\[
I \simeq e f \left( \frac{\tilde{f}_B}{\tilde{f}_B - f^2} \right) \hat{\lambda}^2 \ell^{2(1-K)} \left( \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*} \right).
\] (48)

where \( \tilde{f}_B = f_B K \) is the ballistic frequency and \( \ell = L/a. \) The result in Eq.(18) is obtained in the limit \( \tilde{f}_B > f. \) At resonance we introduce a broadening \( \Delta \) and find for \( |f - \tilde{f}_B| < \Delta \) the result

\[
I \simeq e f \hat{\lambda}^2 \ell^{2(1-K)} \left( \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*} \right) \hat{\lambda}^{2(1-K)} \leq 1. \)

Eqs.(17)- (19) show that the current is periodic each time the energy level \( \hat{\epsilon}_n \) coincides with the charging potential \( V_{GS} = (\mu_L + \mu_R)/(2e) = \mu/e. \) In the limit \( T \to 0 \) Eq.(18) is replaced by a Kronecker-Delta function. Tuning the length “\( L \)” such that \( \hat{\lambda}^{\ell^{1-K}} \simeq 1 \) we find,

\[
I \simeq e f \sum_{n=1}^{\infty} \delta(\hat{\epsilon}_n - eV_{GS}).
\] (50)

This result might be relevant to the surface acoustic wave experiment described in Ref. [14].

In the last part we consider problem 2), the static impurity case. We have, from Eq.(18),

\[
\langle \hat{J} \rangle_{n=0} = \sum_{n=1}^{\infty} \left[ 1 + \exp \frac{\hat{\epsilon}_n - \mu_L}{2K_B T^*} \right]^{-1} - \sum_{n=1}^{\infty} \left[ 1 + \exp \frac{\hat{\epsilon}_n - \mu_R}{2K_B T^*} \right]^{-1}.
\] (51)

We substitute in \( \mu_{L/R} = eV_{GS} \pm eV_{DS}/2 \) and find:

\[
\langle \hat{J} \rangle_{n=0} = \frac{eV_{DS}}{K_B T^*} \left( \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*} \right); \quad \mu \equiv eV_{GS}
\] (52)
As a result we find from Eq.(49) that for $\hat{\lambda}^{(1-K)} \equiv \lambda(L/a)^{1-K} < 1$, the current is given by:

$$I \simeq K \frac{e^2}{h} V_{DS} \left( \frac{L^*_T}{L} \right) \left( \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*_T} \right) \left[ 1 - \hat{\lambda}^2 \lambda^{(2(1-K))} \text{Const.} \right]$$

(53)

where $L^*_T$ is given by $L^*_T = (\bar{hv})/(K_B T^*_T)$.

### VI. THE LUTTINGER LIQUID CONDUCTANCE

Using the results given in Sections IV and V we will compute the conductance for $\hat{\lambda} = 0$ and $K \leq 1$. We find that the current is given by:

$$I = \frac{e}{L} v K \langle \hat{J} \rangle_{n=0}.$$  

(54)

Using the expectation value of $\langle \hat{J} \rangle_{n=0}$ for $\mu_L - \mu_R = eV_{DS}$ we find:

**a)** In the limit $L/L_T \to \infty$, Eq.(23) gives $\langle \hat{J} \rangle_{n=0} = (eV_{DS})/h \cdot L/(Kv)$. Substituting this value into Eq.(54) gives: $I = (e^2/h)V_{DS}$, therefore $G = e^2/h$.

**b)** In the limit $L/L_T \leq 1$, $\langle \hat{J} \rangle_{n=0}$ is given by Eq.(52). As a result the conductance is given by:

$$G = \frac{e^2}{h} K \left( \frac{L^*_T}{L} \right) \left( \frac{1}{4} \sum_{n=1}^{\infty} \cosh^{-2} \frac{\hat{\epsilon}_n - \mu}{2K_B T^*_T} \right).$$

(55)

### VII. CONCLUSION

To conclude we can say that the zero mode formalism is the proper one for computing the macroscopic current. We emphasize that the zero mode method is essential for obtaining correct results in the limit $L_T/L \geq 1$. In the limit $L_T/L \geq 1$ the current is controlled by the fermionic spectrum which is completely absent in the standard bosonization method. As an explicit demonstration of the method we have computed the Luttinger liquid current in the
presence of a time dependent impurity.
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