On an Alternative Approach to Gravitation.

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Summary

Several energy-momentum "tensors" of gravitational field are considered and compared in the lowest approximation. Each of them together with energy-momentum tensor of point-like particles satisfies the conservation laws when equation of motion of particles are the same as in general relativity. It is shown that in Newtonian approximation the considered tensors differ one from the other in the way their energy density is distributed between energy density of interaction (nonzero only at locations of particles) and energy density of gravitational field.

Starting from Lorentz invariance the Lagrangians for spin-2, mass-0 field are constructed. They differ only by divergences. From these Lagrangians by Belinfante-Rosenfeld procedure the energy-momentum tensors are built. Only one of them is suitable for explaining the perihelion shift. This tensor does not coincide with Weinberg’s one (directly obtainable from Einstein equation).

It is noted that phenomenological field-theoretical approach (utilizing only vertices and propagators) can lead to modification of theory in the region of strong field, where till now no observational data are available.

1 Introduction

General relativity is a complete, elegant, and self-consistent theory. Yet there is a necessity to obtain gravity by field-theoretical means starting from flat spacetime, see e.g.[1-5]. It is widely believed that on this way even dropping the general covariance requirement we naturally get general relativity. It is supposed that in the lowest nonlinear approximation this is demonstrated in detail by Thirring [2]. Yet this conclusion can not be drawn from [2], see Sec.2.

In this connection and also because the gravitational collapse is considered as the greatest crisis in physics [6] the research into possible alternative
theories acquire especial significance. It is quite natural to make the first step and to consider the simplest processes by utilizing vertices; the graviton propagator is known by analogy with electrodynamics.

In the lowest nonlinear approximation it is necessary to know only 3-graviton vertex. We assume the simplest possibility: the source of graviton is the energy-momentum tensor of two other gravitons. In higher approximations probably other vertices will be needed. Along this path one can find out what theories are possible without assuming general covariance and a priori restriction on vertices. An important step in this direction was made by Thirring [1-2]. We continue his investigation in the same approximation and restrict ourselves to point-like classical particles as sources of gravitation. Mainly we are interested in the simplest system consisting of a Newtonian center and test particle moving in its field.

In general relativity classical particles move along geodesics in Riemannian space. This is the incarnation of equivalence principle and it is more reliable than specific equation determining the gravitational field [9]. Besides, the equation of motion for particles are obtainable from variational principle without any appeal to general relativity, see equation (2) on page 181 in [6]. It is easy to verify also that eq.(11) in [2], derived in this way coincide exactly with that of general relativity. As to the equation determining gravitational field, it is possible to think that the phenomenological field-theoretical approach will lead to more complicated algorithm for getting the field. An interesting possibility in this direction was pointed out by Schwinger [10], see also Sec. 6.

It is reasonable to believe together with Einstein that for some reason or other the singular behaviour near the gravitational radius does not correspond to reality, see §15 in Pais’s book [11] and Einstein paper [12]. At present the Schwarzschild singularity is considered as fictitious by many researchers because the geometry is nonsingular there. See however the text after (2.2.6) in [13] and after eq.(9.40) in [14], where they say convincingly about physical singularity. By field-theoretical approach it is difficult to understand why in a finite system the acceleration of a freely falling particle becomes unlimited when it nears the horizon. Such a behaviour should be connected with the fact that according to [15] the gravitational energy in vacuum outside the sphere of radius R goes to $-\infty$ for $R \to r_g$. In conformity with this the energy of matter and gravitational field inside the sphere of radius $R$ goes to $+\infty$, in such way that total energy of spherical body
is equal to its mass. But if a theory predicts that the absolute value of field energy outside sphere of radius $R$ might be greater than total energy of a body then the analogy with electrodynamics suggests that the concept of external field becomes inapplicable [16]. The belief in general relativity in similar circumstances is based upon the concept of nonlocalizability of gravitational energy, see, e.g. §20.4 in [6]. What is more, general relativity does not need as a rule the gravitational energy-momentum pseudotensor.

The situation changes drastically, when we begin to construct gravity theory starting from flat spacetime. Here the gravitational energy-momentum tensor appears to be necessary to describe the 3-graviton interaction. The nonlinear correction to the motion of a test particle depends on the chosen energy-momentum tensor. The latter is build from field Lagrangian, which is not unique as one can add to it some divergence terms. This leads to different energy-momentum tensors. They can give rise to gravitational energy densities, which may have even different signs. The question of sign of energy density is of interest by itself. Provided the sign turns out to be positive, one should expect the weakening of gravitational interaction at $r \sim r_g = 2GM$ in comparison with Newtonian one in order that the gravitational energy outside the sphere of radius $r$ were much less than the mass $M$ of the center. The possibility of decreasing of interaction at small distances is suggested also by the behaviour of attraction force between two bodies supported by Weil’s strut, see, §35 [17].

In order to understand in what way the various energy-momentum tensors differ one from the other we consider the following tensors: Thirring’s [1,2], Landau-Lifshitz’s [16], Papapetrou-Weinberg’s [9] and tensor obtained from the Lagrangian given in Exercise 7.3 in [6]. The second and third tensors are representatives of general relativity, the rest are build from Lagrangians of free field of spin-2, mass-0 particles and symmetrized by the Belinfante-Rosenfeld procedure, see, §1 Ch.7 in [18].

In the considered approach it is suitable to subdivide the 3-graviton vertex into three vertices in accordance with three possibilities for choosing two gravitons out of three to form the energy-momentum tensor, the source for the third graviton, see Sec.3. So three diagrams contribute to nonlinear correction to the field. In contrast with this the energy-momentum tensor, figuring in the solution of Einstein equation by iteration procedure, is so defined that the correction to field is given simply by means of propagator, i.e. by one diagram only, see Sec.2.
The main result of the paper is this: starting from quadratic Lagrangians (differing by divergence terms) of spin-2, mass-0 particles, the energy-momentum tensors are constructed by Belinfante-Rosenfeld procedure. It turns out that only certain combination of these tensors is fitted for correct description of perihelion shift. This combination does not coincide with Papapetrou-Weinberg tensor.

The investigation of possibilities of phenomenological approach to gravitation without use of general covariance seems to us very promising. Valuable undertaking in this direction was made in [19].

2 Thirring’s energy-momentum tensor

Throughout the paper we use
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]

In Sections 2, 3, 5 \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \); in Sections 4 and 6 \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1, 1) \). In this Sec. we use Thirring’s notation [2]; both greek and latin indices run from 0 to 3.

The gravitational field is described by the symmetric tensor \( h_{\mu\nu} \), which contains spin-2 and lower spins, see, e.g. [3]. The unnecessary spins (spin-1 and one of spin-0) are excluded by Hilbert gauge:
\[ \bar{h}_{\mu\nu} \equiv (h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h)_{,\nu} = 0, \quad h = h_{,\sigma}, \quad h_{,\nu} \equiv \frac{\partial}{\partial x^\nu} h. \]

One way to obtain Thirring’s tensor is to start from general relativity Lagrangian \( \sqrt{-g}R \). If we remove terms with second derivatives of \( g_{\mu\nu} \) into divergence terms and drop the latter, we get the function \( G(x) \) in eq.(93.1) in [16]. Retaining in it only quadratic in \( h_{\mu\nu} \) terms we get
\[ G(x) = \frac{1}{4}[h_{\mu\nu,\lambda} h_{,\mu\nu} - 2h_{\mu\nu,\lambda} \lambda_{,\mu} + 2h_{\nu\mu,\lambda} h_{,\nu} - h_{,\lambda} h_{,\lambda}] \]

This is equivalent to Thirring’s Lagrangian [2]
\[ \mathcal{L} = \frac{1}{2}[\psi_{\mu\nu,\lambda} \psi_{,\mu\nu,\lambda} - 2\psi_{\mu\nu,\lambda} \psi_{,\lambda,\mu} + 2\psi_{\nu\mu,\lambda} \psi_{,\nu} - \psi_{,\lambda} \psi_{,\lambda}]. \]
Here
\[ \psi_{\mu\nu} = -h_{\mu\nu}/2f, \quad \psi = \psi_\sigma^\sigma, \quad f^2 = 8\pi G, \quad G = 6.67 \cdot 10^{-8} \text{cm}^3/\text{g} \cdot \text{sec}^2. \] (5)

Using $\psi_{\mu\nu}$ instead of $h_{\mu\nu}$ is justified because then the analogy with electrodynamics becomes more close: $\psi_{\mu\nu}$ is analogous to vector potential $A_\mu$ and has the same dimensionality, $M\sqrt{G}$ has the dimensionality of electromagnetic charge. We note that the Lagrangian (4) exactly corresponds to Schwinger’s Lagrangian [10], who uses the notation $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $2h_{\mu\nu}^{Sch} = -h_{\mu\nu}^T = -h_{\mu\nu}$.

The canonical energy-momentum tensor following from (4), has the form
\[ T^\mu_\nu = \phi_{\mu\nu,\delta} \phi_{\mu,\nu}^{\gamma\delta} - \frac{1}{2} \phi_{\gamma\delta} \phi_{\mu,\nu}^{\nu,\mu} - \eta^{\gamma\delta} \frac{f}{L}, \]
\[ L = \frac{1}{2} [\phi_{\mu,\nu,\lambda} \phi_{\mu,\nu}^{\nu,\lambda} - \frac{1}{2} \phi_{\nu,\lambda} \phi_{\nu,\lambda}^{\nu,\mu} - 2 \phi_{\mu,\nu,\lambda} \phi_{\lambda,\mu}^{\nu,\lambda}], \] (6)

\[ \phi_{\mu\nu} \equiv \bar{\psi}_{\mu\nu} = \psi_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \psi, \quad \psi = \psi_\sigma^\sigma. \] (7)

Using $\phi_{\mu\nu}$ instead of $\psi_{\mu\nu}$ is handy as many expressions become more compact and the consequences of imposition of Hilbert gauge more clear.

The energy-momentum tensor for a static point-like mass (Newtonian center)
\[ T^\mu_\nu = M \delta(\vec{x}) \delta_{\mu0} \delta_{\nu0}. \] (8)

In linear approximation this source generate the field
\[ \phi_{\mu\nu} = -\bar{h}_{\mu\nu}/2f = \frac{fM}{4\pi |\vec{r}|} \delta_{\mu0} \delta_{\nu0}, \quad \bar{h}_{\mu\nu} = 4\phi \delta_{\mu0} \delta_{\nu0}, \] (9)

satisfying Hilbert condition (2). For one Newtonian center $\phi = -GM/r$. For several centers
\[ \phi = -G \sum_a \frac{m_a}{|\vec{r} - \vec{r}_a|}. \] (10)

In terms of
\[ h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}, \quad \bar{h} = \bar{h}_\sigma^\sigma, \] (11)

we have
\[ h_{\mu\nu} = 2\phi \delta_{\mu\nu}, \quad h = h_\sigma^\sigma = -4\phi = -\bar{h}. \] (12)
The energy density of field (9) is positive

\[ T^0_0 = \frac{1}{8\pi G} (\nabla \phi)^2. \]  

(13)

But \( T^{\gamma \delta} \) ought to be supplemented to symmetric one by the spin part:

\[ \theta^{\gamma \delta} = T^{\gamma \delta} + T^{\delta \gamma}. \]  

(14)

For Newtonian center Thirring obtains

\[ T^{s \gamma \delta} = -\frac{1}{\pi G} (\nabla \phi)^2 \delta_{\gamma 0} \delta_{\delta 0}. \]  

(15)

So in this case \( \theta^{00} \) is negative

\[ \theta^{00} = -\frac{7}{8\pi G} (\nabla \phi)^2. \]  

(16)

Turning now to conservation laws of total energy-momentum we remind first how matters stand in general relativity. There the energy-momentum tensor of point-like particles \( T^\mu_\nu \) is connected with its counterpart in special relativity \( T^\mu_\nu \) by the relation, see (33.4), (33.5) and (106.4) in [16]:

\[ \sqrt{-g} T^\mu_\nu = T^\mu_\nu = \sum_a m_a u^\mu u^\nu \frac{ds}{dt} \delta(\vec{x} - \vec{x}_a(t)), \quad u^\mu = dx^\mu / ds, \]  

(17)

g is determinant of \( g_{\mu \nu} \). In terms of \( T^\mu_\nu \) the conservation laws are (see (96.1) in [16])

\[ T^\mu_{\nu,\mu} = [T^{\mu \tau} (\eta_{\tau \nu} + h_{\tau \nu})]_{,\mu} = \frac{1}{2} h_{\mu \sigma,\nu} T^{\mu \sigma}. \]  

(18)

We shall see below that \( T^{\mu \tau} h_{\tau \nu} \) can be interpreted as (part of) interaction energy-momentum tensor.

As is known the equation of motion of particles in general relativity

\[ \frac{d^2 x^k}{ds^2} + \Gamma^k_{mj} u^m u^i = 0 \]  

(19)

is contained in conservation laws. Indeed from

\[ T^p_{\ jk \ ;j} = T^p_{\ jk} + \Gamma^p_{mj} T^p_{\ mk} + \Gamma^p_{mj} T^p_{\ jm} = 0 \]  

(20)
taking into account that from definition of $\tilde{T}^{jk}$ in (17)

$$\tilde{T}^{jk} = -\frac{1}{2}(-g)^{-\frac{1}{2}}(-g)^{j,k}(T^{jk} + (-g)^{j,k}T^{jk})$$

we get

$$\gamma_{m,j}^{jk} + \Gamma_{mj}^{k} \gamma_{mj}^{k} = 0. \tag{22}$$

This is equivalent to (19), because [9]

$$\gamma_{jm,j} = \sum_{a} \frac{dp_{m}}{dt} \delta(\vec{x} - \vec{x}(t)). \tag{23}$$

Going back to field-theoretical approach, we rewrite the equation of motion of particles (19) in the lowest approximation

$$\frac{d\mu}{ds} = \frac{d^{2}x^{\mu}}{ds^{2}} = \frac{1}{2}h_{\alpha\beta}^{\mu}u^{\alpha}u^{\beta} - h_{\alpha,\beta}^{\mu}u^{\alpha}u^{\beta}. \tag{24}$$

Just at such movement of particles the divergence of total energy-momentum ought to be zero, and inversely, from zero divergence follows eq. (24). From (23) and (24) we find

$$\gamma_{\gamma,\delta}^{\gamma} = \frac{1}{2}h_{\alpha\beta}^{\gamma}h_{\alpha,\gamma}^{\beta} - h_{\alpha,\gamma}^{\beta}T^{\alpha,\gamma}. \tag{25}$$

This agrees with (22) in considered approximation. With the same accuracy this can be rewritten as

$$(\gamma_{\gamma,\delta}^{\gamma} + \gamma_{\gamma,\delta}^{\beta})_{,\gamma} = \frac{1}{2}h_{\alpha\beta}^{\gamma}h_{\alpha,\gamma}^{\beta}. \tag{26}$$

Using linearized Einstein equation for $\varphi_{\mu\nu}$

$$\varphi_{\mu\nu,\lambda} = \frac{1}{2}h_{\mu\nu}^{,\rho}h_{\rho,\lambda}^{,\lambda} - \varphi^{,\rho,\mu,\lambda} - \varphi^{,\rho,\mu,\lambda} = fT_{\mu\nu} \quad ,$$

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}h_{\mu\nu}T \quad , \quad T = T_{\sigma}^{\sigma} \quad , \tag{27}$$

we get

$$fT_{\gamma,\delta}^{\gamma} = \frac{1}{2}h_{\alpha,\beta}^{\gamma}h_{\alpha,\beta}^{\gamma} = -\frac{1}{2}h_{\alpha,\beta}^{\delta}h_{\alpha,\beta}^{\beta} = -\frac{1}{2}h_{\alpha,\beta}^{\delta}T^{\alpha,\beta}. \tag{28}$$
Spin part of energy-momentum tensor is conserved by itself and do not contribute to conservation laws:

\[ T^\gamma_\delta \gamma = 0. \]  

(29)

So from (26) and (28) it follows that the conserved tensor contains in itself the interaction tensor [2]

\[ \int T^\gamma_\delta = T^\gamma_\alpha h_\alpha^\delta. \]  

(30)

But it is not symmetric. In order to understand the reason we have to consider the properties of \( T^\gamma_\delta \) in detail despite the fact that it does not take part in conservation laws written in the form of eqs.(26) and (28). According to known rules [2,18] we have

\[ T^j^k = -F^{jik}_{,i} - F^{kij}_{,i} - F^{ikj}_{,i}, \]  

(31)

\[ F^{jik} = \frac{\partial L}{\partial \varphi_{a \beta,i}} (\varphi^j_{,\alpha} - \varphi^j_{,\alpha} \eta^k_\beta), \quad L = \frac{1}{2} f. \]  

(32)

The antisymmetric part of (31) is contained only in the last term. For it we have

\[ -F^{ikj}_{,i} = \left( \frac{\partial L}{\partial \varphi_{a,j,i}} \right)_{,i} \varphi^j_{,a} - \left( \frac{\partial L}{\partial \varphi_{a,k,i}} \right)_{,i} \varphi^j_{,a,i} + \varphi^j_{,a,i} = \frac{\partial L}{\partial \varphi_{a,k,i}} \varphi^j_{,a,i}. \]  

(33)

The first two terms on the right hand side symmetrize the interaction tensor, the last two terms symmetrize the canonical one.

It is not seen directly from (6) and (31) that field energy-momentum tensor \( \theta^\gamma_\delta \) in (14) is symmetric. This agrees with the fact that the proof of symmetry utilizes the Euler-Lagrange equations for field which is considered as free [18]. We are interested in interacting field. So using linearized Einstein equation (27) with source, we get

\[ \left( \frac{\partial L}{\partial \varphi_{a,j,i}} \right)_{,i} \varphi^k_{,a} - \left( \frac{\partial L}{\partial \varphi_{a,k,i}} \right)_{,i} \varphi^j_{,a,i} = f (T^{aj}_{a} \varphi^k_{a} - T^{ak}_{a} \varphi^j_{a}) \]

\[ = f (T^{aj}_{a} \varphi^k_{a} - T^{ak}_{a} \varphi^j_{a}) = \frac{1}{2} (T^{ak}_{a} h^i_{a} - T^{aj}_{a} h^k_{a}). \]  

(34)
In the last equation we use the connection between $\psi_{\mu\nu}$ and $h_{\mu\nu}$, see (5). Substituting in (30) $\gamma \to j, \delta \to k$, we see that the sum of (30) and (34) is symmetric. This result retains if we start from another Lagrangian differing from Thirring's one in (6) by divergence because the linearized equation remains the same.

One should take into account however that the symmetric part of $s^T_{jk}$ can also contain terms of interaction type. So for the Lagrangian in (6) similarly to (34) we find

$$-F_{ijk,i} - F_{kij,i} = [fT - 2\varphi_{\alpha}^i,\varphi^j_{\alpha}] + f[\overline{T}^{\alpha j} \varphi_{\alpha}^k + \overline{T}^{ak} \varphi_{\alpha}^j]
+ 2(\varphi_{\alpha}^{ij} i \varphi^k_{\alpha} + \varphi^{ki,\alpha} i \varphi^j_{\alpha}) + 2\varphi^{j\alpha,i} \varphi_{a,i} - (\varphi^{\alpha j^i,\alpha} \varphi_{a,i} + \varphi^{\alpha,\alpha} \varphi_{a,i}^j)
- 2(\varphi^{j\alpha,i} \varphi_{a,i} + \varphi^{j\alpha,i} \varphi_{a,i}^j) + 2\varphi^{ki,\alpha} \varphi_{a,i}^j. \quad (35)$$

As a result we get for $s^T_{jk}$

$$s^T_{jk} = 2[(\varphi_{\alpha}^{ij} i \varphi^k_{\alpha} + \varphi^{ki,\alpha} i \varphi^j_{\alpha}) - \varphi_{\alpha}^{i\alpha,j} \varphi_{a,i} - \varphi^{j\alpha,i} \varphi_{a,i}^k - \varphi^{j\alpha,i} \varphi_{a,i}^j]
- \varphi^{j\alpha,i} \varphi_{a,i} + \varphi^{ki,\alpha} \varphi_{a,i}^j] - 2\varphi^{j\alpha,i} \varphi_{a,i} + 2fT^{\alpha j} \varphi_{a,i}^k. \quad (36)$$

Here last but one term, added to $fT_{jk}$, makes it symmetric. The last term can be rewritten in terms of $h_{\mu\nu}$ in the form, see (9) and (11), (30),

$$-T^{\alpha j} (h_{\alpha}^k - \frac{1}{2} \eta_{\alpha} \eta^k h) = -\frac{\int T_{jk}}{2} + \frac{1}{2} T^{\alpha j} h. \quad (37)$$

So the symmetrization of $\int T_{jk}$ in (30) is reduced to its replacement by $\frac{1}{2} T^{\alpha j} h$. This tensor is nonzero only where particles are present. For Newtonian centers the corresponding energy density

$$\frac{1}{2} T^{00} h = -2T^{00} \phi \quad (38)$$

is positive (contrary to our intuition and) contrary to $\int T^{00}$ in (30), see (12) and (10), where $h$ and $\phi$ are given for Newtonian centers.

We note that the use of linearized Einstein eq. (27) in the expression for $\int T_{jk}$ leads to that eq. (29) is satisfied only with considered accuracy. The presence of interaction energy-momentum tensor means the appearance of
such vertex: the energy-momentum tensor of matter together with one of gravitons serves as a source for other graviton, see Fig.1

Thirring assumes that his tensor $\theta^{\gamma\delta}$ (see (14), (6), (31)) is an analog of energy-momentum tensor figuring in the r.h.s. of Einstein equation when iteration procedure is used. In other words the nonlinear correction to field is given only by diagram (2a) in Fig.2. On this Fig. the short straight line has only conditional meaning: it represents the source of gravitons, namely the energy-momentum tensor build of two gravitons (real or virtual) shown on Fig.2 as joining the ends of this line. The graviton emerging from the middle of straight line is emitted or absorbed by this source. On diagram (2a) the energy-momentum tensor is build from gravitons of Newtonian center. On diagrams (2b) and (2c) one of the virtual gravitons of Newtonian center interact with energy-momentum tensor of two other gravitons. All three diagrams on Fig.2 correspond to one Feynman diagram obtained by contracting the short straight line to a point.

The contribution to nonlinear correction for field from diagram (2a) is easy to obtain. Indeed, from (14), (6) and (15) we have

$$\theta_{jk} = T_{jk} = \frac{GM^2}{4\pi} \left( \frac{x^j x^k}{r^6} - \frac{1}{2} \frac{\delta_{jk}}{r^4} \right), \quad j, k = 1, 2, 3. \quad (39)$$

Using now the field equation in Hilbert gauge with $\theta^{\mu\nu}$ from (16) and (39)

$$\Box \tilde{h}^{\mu\nu} = -16\pi G \theta^{\mu\nu}, \quad \Box = \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (40)$$

we find

$$\tilde{h}^{00} = -7\phi^2, \quad \tilde{h}^{ik} = -\frac{G^2 M^2}{r^4} x^i x^k, \quad \phi = -\frac{GM}{r}, \quad i, k = 1, 2, 3. \quad (41)$$

Here easily verifiable relations

$$\nabla^2 \frac{x^i x^k}{r^4} = \frac{2\delta_{ik}}{r^4} - \frac{4x^i x^k}{r^6}, \quad \nabla^2 \frac{1}{r^2} = \frac{2}{r^4} \quad (42)$$

were used. The obtained $\tilde{h}^{\mu\nu}$ satisfies Hilbert condition. Going over to $h_{\mu\nu} = \tilde{h}^{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h}$, we find the following nonlinear corrections

$$h_{00} = -4\phi^2, \quad h_{ik} = -G^2 M^2 \left( \frac{x^i x^k}{r^4} + \frac{3\delta_{ik}}{r^2} \right). \quad (43)$$
Index 2 in $h_{\mu\nu}$, indicating the order of correction in powers of $G$, is dropped for brevity.

Finally from (12) and (43) we have

$$ds^2 = g_{00} dt^2 - (1 - 2\phi + 3\phi^2) \delta_{ik} dx^i dx^k - \phi^2 \frac{x_i x_k dx^i dx^k}{r^2},$$  \hspace{1cm} (44)

$$g_{00} = 1 + 2\phi - 4\phi^2.$$  \hspace{1cm} (45)

The transition to spherical coordinates is given by the relations

$$\delta_{ik} dx^i dx^k = dr^2 + r^2 (d\theta^2 + \sin^2 \phi d\phi^2), \quad \frac{x_i x_k dx^i dx^k}{r^2} = (dr)^2.$$  

The nonlinear correction $-4\phi^2$ in $g_{00}$ in (45) is of special interest for us. The correct value necessary to explain the perihelion shift is $+2\phi^2$ (if used in equation for geodesic, see [19] and Sec. 6). The quantity $g_{00}$ is observable in principle by redshift $\omega = \omega_0 (g_{00})^{-\frac{1}{2}}$. Why the nonlinear correction in (45) turns out to be negative? It will appear later on that it is caused solely by the source (15), see eqs. (95) and (117). The sources (8) and (15) have different signs, but the corresponding fields have the same sign. The answer is simple. The correction in (45) is only a small (and negative) part. The larger and positive part goes for converting initially bare mass in Newtonian potential into a dressed one, see eq. (113). Now the negative sign of correction is clear: the mass of Newtonian center at infinitely large distance appears as $M$, but at finite distance the test particle feels a greater mass and greater attraction, because (15) is negative.

There are some technical problems on how to take into account the contributions from diagrams (2b), (2c) and Fig.1. To simplify the problem as much as possible we may look only into corrections to Newtonian potential. As it will appear in Sec.6, it seems impossible to get the right value on any assumption about the contributions from these diagrams for the case of Thirring’s tensor.

We note here that Thirring obtained from his tensor the necessary correction. Yet his result is objectionable as he used illdefined gauge

$$\Box^2 \Lambda = \frac{G^2 M^2}{4r^2},$$
see eq. (83) in [2]. Namely the source of \( \Lambda \) fall of too slowly for large \( r \) and the integral in (112), see below, diverge for large \( r' \).

In the next two Sections we shall see how energy-momentum "tensors" of general relativity differ from Thirring's tensor.

3  Landau-Lifshitz pseudotensor of energy-momentum

This pseudotensor in the sense and approximation considered here is a tensor. In the lowest approximation with help of relation
\[
\sqrt{-g}g^{ik} \approx (1 + \frac{\hbar}{2})(\eta^{ik} - h^{ik}) \approx \eta^{ik} - \bar{h}^{ik}.
\]
we get from eq. (96.9) in [16]
\[
t^{ik} = \frac{1}{16\pi G} \left[ \bar{h}^{ik,l} \bar{h}_{lm,n} - \bar{h}^{ii,l} \bar{h}^{km,m} - \bar{h}^{km,p} \bar{h}_{mp} - \bar{h}^{im,p} \bar{h}_{mp} + \eta^{ik} \left( \frac{1}{2} \bar{h}_{mn,p} \bar{h}^{pm,n} + \frac{1}{8} h^{m} h_{m} - \frac{1}{4} \bar{h}_{pq,m} \bar{h}^{pq,m} \right) \right].
\]
(47)

Comparison with canonical tensor (6) shows that it is connected with \( t^{ik} \) by the relation
\[
t^{ik} = \bar{T}^{ik} + F^{ik}, \quad \bar{h}^{ik} = -2f \phi_{ik},
\]
\[
F^{ik} = \frac{1}{16\pi G} \left[ \bar{h}^{ik,l} \bar{h}_{lm,n} - \bar{h}^{ii,l} \bar{h}^{km,m} - \bar{h}^{km,p} \bar{h}_{mp} - \bar{h}^{im,p} \bar{h}_{mp} + \eta^{ik} \left( \frac{1}{2} \bar{h}_{mn,p} \bar{h}^{pm,n} + \frac{1}{8} h^{m} h_{m} - \frac{1}{4} \bar{h}_{pq,m} \bar{h}^{pq,m} \right) \right].
\]
(48)

From (14) we see that now in place of \( \bar{T}^{ik} \) stands \( F^{ik} \). But \( \bar{T}^{ik} \) was a conserved quantity, see.(29). So \( F^{ik} \) should rather play the role of interaction energy-momentum tensor. Indeed, taking into account that in considered approximation \( \bar{h}^{ik} \) satisfies the linearized Einstein equation
\[
\bar{h}_{np,j} - \bar{h}_{jp,n} + \eta_{np} \bar{h}_{qr,pr} = -16\pi G \bar{T}_{np},
\]
we find
\[
F^{jk} = \bar{h}^{kn,i} \bar{T}_{ni} = \bar{h}^{kn,i} \bar{T}_{ni} - \frac{1}{2} h^{i} \bar{T}_{i}.
\]
(50)
Now we check that conservation laws \[16\]
\[
\frac{\partial}{\partial x^k} \left( (-g)[\mathcal{T}_{ik} + i_{ik}] \right) = 0
\]  
are fulfilled. From (48), (28) and (50) we have
\[
i_{ik,i} = -\frac{1}{2} h^{iq,k} \mathcal{T}_{iq} + h^{kn,i} \mathcal{T}_{ni} - \frac{1}{2} h^{i} \mathcal{T}_{i}. \tag{52}
\]
For matter energy-momentum tensor from (17) we get
\[
(-g)^{\mathcal{T}_{ik}} = \sqrt{-g} \mathcal{T}_{ik} \approx (1 + \frac{h}{2}) \mathcal{T}_{ik}, \quad -g \approx 1 + h. \tag{53}
\]
From here
\[
\left((-g)^{\mathcal{T}_{ik}}\right)_{,i} = \mathcal{T}_{ik,i} + \frac{1}{2} h_{i} \mathcal{T}_{i}, \tag{54}
\]
the terms of order \(h^2\) being dropped. Now it follows from (25), (52) and (54) that (51) is fulfilled. As seen from (53) here too there is a tensor, which is nonzero only where particles are located. Surprisingly it coincides with Thirring’s interaction tensor, see (37) and text below it.

Although \(-g^{\mathcal{T}_{ik}} + i_{ik}\) differs from Thirring’s \(\mathcal{T}_{ik} + i_{ik} + \theta_{ik}\), for Newtonian centers they coincide. Now we turn to Newtonian approximation. According to Problem 1 in §106 in \[16\] the energy density of gravitational field in this approximation is given by \(T^{00}\) in (13), (10). But there is also energy density of interaction \(\mu \phi\), where \(\mu\) is density of particles. Using Poisson equation \(\nabla^2 \phi = 4\pi G \mu\) and ignoring the problems connected with point-like nature of particles, we can write (utilizing integration by parts)
\[
\int \mu \phi dV = -\int \frac{1}{4\pi G} (\nabla \phi)^2 dV, \tag{55}
\]
The density in the integrand on the r.h.s. contains now not only the energy density of interaction, but also the proper energy density of particle’s self-field. The density on the l.h.s. is nonzero only at particle locations, the density on the r.h.s. is nonzero where the field is nonzero. The integration by parts deprive us the possibility to retain the previous physical meaning of integrand. If nevertheless we do this, then adding to (13) the energy
density in the r.h.s. of (55) we get the effective gravitational energy density in Newtonian approximation
\[
-\frac{1}{8\pi G} (\nabla \phi)^2. \tag{56}
\]
To bring this in agreement with \( t^{00} \) we ought, according to a foot-note in [16], take into account the contribution from \( (-g) T^{p\ 00} \). Let us do it. For \( t^{00} \) we have
\[
t^{00} = -\frac{7}{8\pi G} (\nabla \phi)^2, \tag{57}
\]
where \( \phi \) is the potential of Newtonian centers. Now
\[
(-g) T^{p\ 00} = \sqrt{-g} T^{00} \approx T^{00}(1 + \frac{h}{2}) \approx T^{00} - 2\phi\mu, \tag{58}
\]
see (12). The sought for agreement will be reached only after we rewrite a la Thirring [2] \( T^{00} \) in terms of observables. From (17)
\[
T^{00} = \sum_a m_a \frac{dx_a^0}{ds} \delta(\vec{x} - \vec{x}_a(t)). \tag{59}
\]
In the presence of gravitational field
\[
ds^2 = g_{00} dt^2 (1 - v^2). \tag{60}
\]
Here \( v^2 \) is physical velocity, see §88 in [16]. Hence
\[
\frac{dx^0}{ds} = \frac{1}{\sqrt{g_{00}(1 - v^2)}} \approx \frac{1}{\sqrt{1 - v^2}} = \frac{h_{00}}{2}, \quad v^2 \ll 1. \tag{61}
\]
Thus after going over to the observable velocity we obtain the term
\[
-\frac{h_{00}}{2} \mu = -\phi\mu. \tag{62}
\]
detached from \( T^{00} \). Equation (12) was used to get the r.h.s.. Together with corresponding term in (58) this leads to
\[
-3\mu \phi \Rightarrow \frac{3}{4\pi G} (\nabla \phi)^2, \tag{63}
\]
where arrow corresponds to going over in (55) from integrand on the l.h.s.
to the integrand on r.h.s. Now the sum of (57) and (63) gives the expected
(56).

The consideration of Newtonian approximation makes the following point
of view very enticing: The energy density of an isolated point-like particle
should be positive; Hilbert gauge exclude the unnecessary spins and then pos-
sitivity seems quite natural, because the presence of virtual gravitons should
not make the energy density negative. The attraction is described by inter-
action energy density and so the latter must be negative. Neither Thirring
tensor nor Landau-Lifshitz one satisfies this requirement. The MTW tensor
does. Unfortunately I failed to fit this idea into existing approach to
gravitation, see Sec.6.

4 Papapetrou- Weinberg energy-momentum
tensor.

Einstein equation can be recast in such a way that gravitational energy -
momentum "tensor" can be easily identified in coordinate system that goes
over to Lorentzian at large distances from gravitating bodies [9]. In the
lowest approximation this tensor have the form, see eq. (7.6.14) in [9])

\[
t_{\mu\nu} = \frac{1}{8\pi G} \left[ -\frac{1}{2} h_{\mu\nu} R^{(1)} + \frac{1}{2} \eta_{\mu\nu} h^{\rho\sigma} R_{\rho\sigma}^{(1)} + R_{\mu\nu}^{(2)} - \frac{1}{2} \eta_{\mu\nu} R^{(2)} \right], \quad R^{(1)} = R^{(1,2)}_{\mu}.
\]

(64)

In this Section we use Weinberg notation:

\[
\eta_{\mu\nu} = \text{diag}(-1,1,1,1), \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} \equiv h^{W}_{\mu\nu} = -h^{Thir}_{\mu\nu} = -h^{LL}_{\mu\nu}.
\]

(65)

Greek indices run from 0 to 3, latin ones from 1 to 3. \( R^{(1,2)}_{\mu\nu} \) is Ricci tensor in
the first and second approximation in powers of \( h_{\mu\nu} \). The indices are raised
and lowered with \( \eta_{\mu\nu} \). In terms of \( h_{\mu\nu} \) we get

\[
R^{(1)}_{\mu\nu} = \frac{1}{2} (\bar{h}_{\mu\nu,\sigma} - \bar{h}_{\sigma,\mu,\nu} - \bar{h}_{\sigma,\nu,\mu}) - \frac{1}{4} \eta_{\mu\nu} \bar{h}_{,\sigma} \sigma, \quad R^{(1)} = -\bar{h}^{\mu\nu}_{,\mu\sigma} - \frac{1}{2} \bar{h}_{,\sigma} \sigma,
\]

(66)

\[
R^{(2)}_{\mu\nu} = \frac{1}{2} \bar{h}^{\lambda\nu}_{,\mu\kappa} [\bar{h}_{\mu\nu,\kappa\lambda} + \bar{h}_{\kappa\nu,\mu\lambda} - \bar{h}_{\lambda\nu,\mu\kappa} - \bar{h}_{\mu\kappa,\nu\lambda}] - \frac{1}{4} (\bar{h}^{\lambda}_{,\mu} \bar{h}_{,\kappa\lambda} + \bar{h}^{\nu}_{,\kappa} \bar{h}_{,\mu\nu})
\]

15
\[
\begin{align*}
+ \frac{1}{4} \bar{h}(\bar{h}_{\mu\nu} + \bar{h}_{\mu\nu,\lambda} - \bar{h}_{\mu,\lambda} - \bar{h}_{\nu,\mu}) - \frac{1}{4}(\bar{h}^{\nu}_{\mu,\nu} \bar{h}_{\nu,\nu} + \bar{h}^{\nu}_{\mu,\nu} \bar{h}_{\mu,\nu}) + \\
\bar{h}^{\sigma}(\frac{1}{2} \bar{h}_{\mu,\sigma} + \frac{1}{4} \bar{h}_{\mu,\sigma,\nu} + \frac{1}{2} \bar{h}^{\nu}_{\sigma,\nu} (\bar{h}^{\sigma}_{\mu,\nu} + \bar{h}^{\sigma}_{\mu,\nu}) \\
- \bar{h}_{\mu,\nu} (\bar{h}_{\mu,\nu,\lambda} - \bar{h}_{\mu,\nu} \bar{h}_{\sigma,\nu} - \bar{h}_{\nu,\mu} \bar{h}_{\sigma,\nu}) + \frac{1}{8} \bar{h}_{\mu,\nu} \bar{h}_{\mu,\nu} \\
+ \eta_{\mu\nu} [\frac{1}{4} \bar{h}^{\nu}_{\mu,\nu} - \frac{1}{8} \bar{h}_{\mu,\nu} - \frac{1}{4} \bar{h}^{\nu}_{\sigma,\nu} \bar{h}^{\sigma}_{\mu,\nu} - \frac{1}{8} \bar{h}^{\nu}_{\sigma,\nu} \bar{h}^{\sigma}_{\mu,\nu}],
\end{align*}
\]

(67)

\[
R^{(2)} = \bar{h}^{\nu}_{\mu,\nu} (\bar{h}_{\nu,\nu,\lambda} - \frac{1}{2} \bar{h}_{\nu,\nu,\sigma}) + \bar{h}^{\nu}_{\sigma,\nu} \bar{h}^{\sigma}_{\nu,\nu} - \frac{1}{2} \bar{h}^{\nu}_{\sigma,\nu} \bar{h}^{\sigma}
+ \frac{1}{2} \bar{h}_{\nu,\nu} \bar{h}_{\nu,\nu,\lambda} - \frac{3}{4} \bar{h}_{\nu,\nu} \bar{h}_{\nu,\nu,\lambda} - \frac{1}{2} \bar{h}^{\nu}_{\lambda,\nu,\lambda} + \frac{1}{8} \bar{h}_{\nu,\nu} \bar{h}^{\nu}_{\lambda,\nu}. (68)
\]

For Newtonian center from (9-12) we obtain

\[
\bar{h}_{\mu,\nu} = -\bar{h}^{T}_{\mu,\nu} = -4\phi \delta_{\mu\nu} \delta_{\nu\nu}, \quad h_{\mu,\nu} = -h^{T}_{\mu,\nu} = -2\phi \delta_{\mu\nu}, \quad h = h^{W} = h^{T} = -4\phi = -\bar{h}. (69)
\]

Nonzero components of \( t_{\mu\nu} \) are

\[
t_{00} = -\frac{3}{8\pi G} (\nabla \phi)^{2} = \frac{3GM^{2}}{8\pi r^{4}}, \quad t_{ik} = \frac{GM^{2}}{8\pi r^{6}} (7\delta_{ik} r^{2} - 14x^{i} x^{k}). (70)
\]

In Hilbert gauge from equation

\[
\nabla^{2} \bar{h}_{\mu,\nu} = -16\pi G t_{\mu,\nu} (71)
\]

we find, cf. with (40), (42),

\[
\bar{h}_{00} = \frac{3G^{2}M^{2}}{r^{2}} = 3\phi^{2}, \quad \bar{h}_{ik} = -7G^{2}M^{2} \frac{x^{i} x^{k}}{r^{4}}. (72)
\]

It is easy to check that (72) satisfies the Hilbert condition (2). In terms of \( h_{\mu,\nu} \) we have

\[
h_{00} = -2\phi^{2}, \quad h_{ik} = G^{2}M^{2} \left( \frac{5\delta_{ik}}{r^{2}} - 7 \frac{x^{i} x^{k}}{r^{4}} \right). (73)
\]

In the expressions (71-73) \( h_{\mu,\nu} \) is nonlinear correction.

On the other hand in harmonic coordinates the Schwarzschild solution has the form [9]

\[
-d\tau^{2} = \frac{1 + \phi}{1 - \phi} dt^{2} + (1 - \phi)^{2} (d\vec{x})^{2} + \frac{1 - \phi}{1 + \phi} \phi^{2} \frac{x^{i} x^{k}}{r^{2}} dx^{i} dx^{k}. (74)
\]
So in the considered approximation this gives
\[ g_{00} = -(1 + 2\phi + 2\phi^2), \tag{75} \]
\[ g_{ik} = 1 - 2\phi\delta_{ik} + \phi^2 (\delta_{ik} + \frac{x_i x_k}{r^2}). \tag{76} \]
From (69) we have \( h^{(1)}_{00} = -2\phi \), from (73) \( h^{(2)}_{00} = -2\phi^2 \), and there is agreement with (75). As to the nonlinear correction for \( g_{ik} \), in (73) it differs from the one in (76) by a gauge. Really, subtracting from \( h_{ik} \) in (73) the nonlinear part of (76), we find
\[ G^2 M^2 \left( \frac{4\delta_{ik}}{r^2} - \frac{8x_i x_k}{r^4} \right) = 2G^2 M^2 (\Lambda_{i,k} + \Lambda_{k,i}), \quad \Lambda_i = \frac{x_i}{r^2}, \tag{77} \]
i.e. a gauge.
Going back to \( t_{00} \) in (70), we note that this density is negative and does not coincide with any density of other tensors. At the same time the equation of motion of particles is contained in the conservation laws of total energy-momentum tensor. We shall check it in considered approximation. For gravitational part the calculations give
\[ t^{\mu\kappa} = -h^{\nu}_{\sigma,\nu} \mathcal{T}^{\mu\sigma} + \frac{1}{2} h_{\sigma} \mathcal{T}^{\mu\sigma} - \frac{1}{2} h^{\rho\sigma,\mu} \mathcal{T}_{\rho\sigma} - h^{\nu\lambda} \mathcal{T}^{\mu,\nu,\lambda}. \tag{78} \]
The energy-momentum tensor for particles, figuring in conservation laws, has a rather complicated form by construction [9]
\[ \tau^{\mu\kappa} = \eta^{\mu\sigma} \eta^{\nu\kappa} g_{\sigma\alpha} g_{\tau\beta} \mathcal{T}^{\alpha\beta} \approx \mathcal{T}^{\mu\kappa} + h^{\mu}_{\alpha,\kappa} \mathcal{T}^{\alpha\kappa} + h^{\kappa}_{\alpha} \mathcal{T}^{\alpha\mu}. \tag{79} \]
From here with the considered accuracy
\[ \tau^{\mu\kappa} = \mathcal{T}^{\mu\kappa} + h^{\mu}_{\alpha,\kappa} \mathcal{T}^{\alpha\kappa} + h^{\kappa}_{\alpha} \mathcal{T}^{\alpha\mu} + h^{\kappa}_{\alpha,\kappa} \mathcal{T}^{\alpha\mu}. \tag{80} \]
So
\[ (\tau^{\mu\kappa} + t^{\mu\kappa})_{,\kappa} = \mathcal{T}^{\mu\kappa} - \frac{1}{2} h^{\rho\sigma,\mu} \mathcal{T}_{\rho\sigma} + h^{\mu\alpha,\kappa} \mathcal{T}_{\alpha\kappa} + \frac{1}{2} h^{\kappa}_{,\sigma} \mathcal{T}^{\mu\sigma}. \tag{81} \]
Further from (53) we get
\[ \mathcal{T}^{\mu\kappa} \approx \mathcal{T}^{\mu\kappa} - \frac{1}{2} h^{\kappa}_{,\kappa} \mathcal{T}^{\mu\kappa}. \tag{82} \]
Taking into account (25) we see that the r.h.s. of (81) is zero.

Now we turn to Newtonian approximation. Terms of interaction tensor are contained in both $\tau^{\mu\nu}$ (three last terms in the r.h.s. of (79)) and in $t_{\mu\nu}$. From (64), (66-68), using (49), which preserve its form in the notation of this Section, we find the following terms of interaction tensor contained in $t_{\mu\nu}$:

$$-\frac{1}{2}h_{\mu\nu}T - \eta_{\mu\nu}(\bar{h}^{\rho\sigma}T_{\rho\sigma} - \frac{1}{4}\bar{h}T) - \frac{1}{2}\bar{h}T_{\mu\nu}.$$  

From here and the first equation in (70) we get in Newtonian approximation

$$t_{00} = -\frac{3}{8\pi G}(\nabla\phi)^2 - 6\phi T_{00}. \quad (83)$$

Here $\phi$ is the same as in (10). From (79) and (53) we get in this approximation

$$\tau^{00} = (-g)^{-\frac{1}{2}}T^{00} + 2h^{0}_{0}T^{00} \approx T^{00}(1 - \frac{1}{2}\bar{h}) - 2h_{00}T^{00} = T^{00} + 6\phi T^{00}. \quad (84)$$

Thus in Newtonian approximation the interaction terms in the sum of (83) and (84) are cancelled out. The agreement with Newtonian approximation (56) is achieved in the same way as for Landau-Lifshitz tensor: $T^{00}$ on the r.h.s. of (84) detaches term (62), which is equivalent (in accordance with (63)) to $\frac{1}{4\pi G}(\nabla\phi)^2$. Together with the first term on the r.h.s. of (83) this gives (56).

Weinberg shows in detail that his energy-momentum tensor has all required characteristics. But this tensor does not help us to find energy-momentum tensor of two gravitons as represented by straight line on diagrams of Fig.2. By construction Weinberg’s tensor gives the gravitational field only via diagram (2a). The field-theretical description tell us that test particle is not quite passive. It does not simply follows the command ”move along geodesic” but itself takes part in the creation of field in which it moves, see Fig.1 and Fig.(2b, 2c). From this viewpoint one can expect that e.g. photon and graviton scatter differently on Newtonian center as only in the latter case all three diagrams of Fig.2 contribute.
In this Section it is handy for us to use again Thirring’s notation. Up to divergence terms the Lagrangian (6) may be rewritten as

\[ L = \frac{1}{2} \phi_{\mu\nu,\lambda} \phi^{\mu\nu,\lambda} - \frac{1}{4} \phi_{\lambda} \phi^{\lambda} - \phi_{\mu\nu,\mu} \phi^{\phi^{\lambda^\nu,\lambda}}. \]  

(85)

The corresponding canonical energy-momentum tensor

\[ f_{\mathcal{T} \ jk} = \frac{\partial L}{\partial \phi_{\mu\nu, j}} \phi^{\mu\nu, k} - \eta_{jk} \mathcal{L}, \]  

(86)

acquires the form

\[ f_{\mathcal{T} \ jk} = \tilde{T}_{jk} - \frac{1}{2} \eta_{jk} \tilde{T}, \quad \tilde{T}_{jk} = \phi^{\mu\nu,k} \phi_{\mu\nu}^{\ j} - \frac{1}{2} \phi^{k,\ j} - 2 \phi_{\jmath}^{\nu,k} \phi_{\nu\sigma}^{\ j}, \tilde{T} = -T = 2\mathcal{L}. \]  

(87)

Spin part is given by (31-32) with substitution \( L \rightarrow \mathcal{L} \). We dwell on differences from Thirring’s tensor. In symmetric in \( jk \) tensor

\[ F^{ijk} + F^{kij} = (\varphi^{\alpha,i,j} - \varphi^{\alpha,j,i}) \varphi_{\alpha}^{\ k} + (\varphi^{\alpha,i,k} - \varphi^{\alpha,k,i}) \varphi_{\alpha}^{\ j} \]

\[ -2 \varphi^{i,j,k} + (2 \varphi^{\alpha,j,i} \eta^{k,i} - \varphi^{\alpha,k,i} - \varphi^{\alpha,j,k}) \varphi_{\alpha}^{\ i} + \varphi^{\alpha} \varphi^{\ j} + \varphi^{\ j} \varphi^{\ i} \]  

(88)

there is no derivatives over \( x^i \). This means that in divergence \( F^{ijk} + F^{kij} \), there are no terms of interaction tensor. In antisymmetric in \( jk \) tensor

\[ F^{ijk} = (\varphi^{\alpha,k,i} - \varphi^{\alpha,i,j}) \varphi_{\alpha}^{\ j} - \varphi^{\alpha} \varphi_{\alpha}^{\ j} + (\varphi^{\alpha,i,j} - \varphi^{\alpha,j,i}) \varphi_{\alpha}^{\ k} + \varphi^{\ j} \varphi^{\ i} \]  

(89)

such terms are present. Hence, using linearized Einstein equation (27) in the expression for \( F^{ijk}, \mathcal{T} \), we get

\[ -F^{ijk}, = f(\mathcal{T}^{\ j\alpha} \varphi_{\ k}^{\ \alpha} - \mathcal{T}^{\ k\alpha} \varphi_{\ j}^{\ \alpha}) + \varphi^{\alpha} \varphi_{\alpha}^{\ j} - \varphi^{\alpha} \varphi_{\alpha}^{\ k}. \]  

(90)

Terms with \( f \) together with (30) give symmetric interaction tensor in accordance with (34). Other two terms on the r.h.s. of (90) supplement \( f_{\mathcal{T} \ jk} \) to symmetric one, see (87). So we get

\[ \theta^{jk} = f_{\mathcal{T} \ jk} + s_{\mathcal{T} \ jk} = \phi^{\mu\nu,k} \phi_{\mu\nu}^{\ j} - \frac{1}{2} \phi^{k,\ j} + \varphi^{\ j} \varphi_{\alpha}^{\ k} + \varphi^{\mu\nu,\mu} \varphi_{\mu\nu}^{\ j} \]

\[ -\varphi^{\alpha,i,j} \varphi_{\alpha}^{\ k,i} + (\varphi^{\alpha,j,i} + \varphi^{\alpha,k,i}) \varphi_{\alpha}^{\ j} + \varphi^{\ j} \varphi_{\alpha}^{\ k} + \]  

\[ 2 \varphi^{\alpha} \varphi^{\ j} \varphi_{\alpha}^{\ k} + (\varphi^{\ j} \varphi_{\alpha}^{\ k} + \varphi^{\alpha} \varphi^{\ j} \varphi_{\alpha}^{\ k}) \varphi_{\alpha}^{\ j} + \]

\[ 2 \eta^{k} (\varphi^{\alpha,j,i}, \varphi_{\alpha}^{\ j}, \varphi^{\ j} \varphi_{\alpha}^{\ k} + \varphi^{\ j} \varphi_{\alpha}^{\ k} \varphi_{\alpha}^{\ j} + \varphi^{\ j} \varphi_{\alpha}^{\ k} \varphi^{\ j} \varphi_{\alpha}^{\ k}) \]  

(91)
From (91) and (30), using the relation between $\varphi_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ in (9), and taking into account (34), we find

$$\theta^{jk} + \text{int} \ T^{jk} = \frac{1}{32\pi G} \left[ (\bar{h}^\mu_{\nu,k} \bar{h}_{\mu\nu} - \frac{1}{2} \bar{h}^k_{,\nu} \bar{h}^j_{,\mu} - (\bar{h}^j_{,\sigma,\nu} \bar{h}_{k,\mu} + \bar{h}^k_{,\sigma,\nu} \bar{h}_{j,\mu}) \right. $$

$$ - (\bar{h}^\alpha_{i,j} \bar{h}_{\alpha,i} + \bar{h}^{\alpha,k} \bar{h}_{\alpha,k} + \bar{h}^{\alpha,j} \bar{h}_{\alpha,j} \bar{h}^i_{,\alpha} + 2 \bar{h}^i_{,\sigma,\alpha} \bar{h}_{j,\mu} + 2 \bar{h}^i_{,\sigma,\nu} \bar{h}^j_{,\mu} - 2 \bar{h}^i_{,\sigma,\nu} \bar{h}^j_{,\mu} \bar{h}^k_{,\alpha}) \right]$$

$$ + \frac{1}{2} (\mathcal{T}^{\alpha \nu} \bar{h}_{\alpha,\nu} + \mathcal{T}^{\nu \alpha} \bar{h}_{\nu,\alpha}), \quad (92)$$

It is easy to verify that total energy-momentum tensor consisting of (17) and (92) is conserved.

For Newtonian centers from (92) and (9), (12) we have

$$\theta^{00} + \text{int} \ T^{00} = \frac{1}{8\pi G} (\nabla \phi)^2 + 2\mu \phi, \quad \mu = T^{00}. \quad (94)$$

For this system the MTW Lagrangian (93) coincide with Thirring’s one. The same is true for canonical energy-momentum tensors, see (6) and (86-87), but spin parts are different. It follows from (91) and (87) that for Newtonian centers MTW spin part contributes only to interaction tensor, while Thirring’s spin part contributes also to pure field part, see (15).

We note now that in Hilbert gauge for static case (for Newtonian centers) the linearized Einstein equation can be written as

$$\nabla^2 \bar{h}_{\mu\nu} = -\nabla^2 \bar{T}_{\mu\nu} = -16\pi G \bar{T}_{\mu\nu}, \quad (95)$$

see (116) below. As seen from (87) for this system $\bar{T}_{00} = 0$, i.e. there is no contribution to $h_{00}$ from diagrams of Fig.2.

Comparing MTW and Thirring tensors in Newtonian approximation we see that addition of divergence terms to Lagrangian leads to the change in subdivision of energy density between purely field part and interaction part.

### 6 On Schwinger’s explanation of perihelion shift

Schwiger gave an elementary explanation of perihelion precession [10]. His method is of interest in many respects. The offered algorithm for calculating
gravitational field is more in line with field-theoretical method than finding field via differential equation in geometrical approach. In linear approximation both methods naturally agree. In higher approximation Schwinger’s gravitational field, defined as a coefficient at $\delta T^{\mu \nu}$ in variation of amplitude

$$\delta W(T) = \int d^4 x \delta T^{\mu \nu} h_{\mu \nu}^{Sch}, \quad \delta T^{\mu \nu}, \nu = 0, \quad h_{\mu \nu} = 2 h_{\mu \nu}^{Sch}, \quad (96)$$

seems does not coincide with definition of $h_{\mu \nu}$ in (1), because the nonlinear correction to $\phi$ is $\frac{1}{2} \phi^2$, but not $\phi^2$, see the text below eq.(108). In the lowest approximation

$$W(T) = 4\pi G \int d^4 x d^4 x' T^{\mu \nu}(x) D_{\mu \rho \sigma}(x - x') T^{\rho \sigma}(x'), \quad D_{\mu \rho \sigma} = P_{\mu \rho \sigma} D_+(x - x'), (97)$$

$$D_+(x) = \frac{i}{(2\pi)^3} \int \frac{d^3 p}{2 p^0} e^{i(p \cdot x - p^0 |t|)},$$

$$P_{\mu \rho \sigma} = \frac{1}{2} (\eta_{\mu \rho} \eta_{\nu \sigma} + \eta_{\mu \sigma} \eta_{\nu \rho} - \eta_{\mu \nu} \eta_{\rho \sigma}). \quad (98)$$

The amplitude $W$ is related to the energy of system $E$ by the expression

$$W(T) = - \int dt E(t). \quad (99)$$

From the total energy $E$ Schwinger singles out the interaction energy $E^{int}$. Nonlinear correction to the interaction energy of slowly moving planet with the Sun is given then by

$$^{(2)} E^{int} = -GM \int \frac{d^3 x}{|\vec{x}|} t^{00 \text{int}}, \quad (100)$$

where $t^{00 \text{int}}$ is taken in Newtonian approximation

$$t^{00 \text{int}} = -\frac{GMm}{4\pi} \nabla \frac{1}{|\vec{x}|} \cdot \nabla \frac{1}{|\vec{x} - \vec{R}|}. \quad (101)$$

So the result is

$$^{(2)} E^{int} = \frac{m}{2} \phi^2(x), \quad \phi(x) = -\frac{GM}{R}. \quad (102)$$

Eq.(100) follows from eq.(4.34) in Ch.2 in [10] if one assumes that energy-momentum tensor of gravitational field is given by 1/8 of the expression (15),
in which $\phi$ is that of (10), i.e. the potential of Sun-planet system. The Sun’s matter energy-momentum tensor is represented by (8). It is clear that (100) is given by contribution from diagrams (2b) and (2c).

As shown in Sec.3 the energy density of gravitational field may be considered as positive. In that case the expression for $E_{\text{int}}^{(2)}$ in (102) changes sign, but then there is also a contribution from interaction energy density

$$
\mu \phi = -\frac{GMm}{R}[\delta(\vec{x}) + \delta(\vec{x} - \vec{R})].
$$

Using this instead of $t^{00\text{int}}$ in (100) and dropping the contribution from $\delta(\vec{x})$ (self-interaction is included in renormalization of Sun’s mass) we get

$$
\frac{E_{\text{int,loc}}^{(2)}}{E} = \frac{G^2 M^2 m}{R^2} = m\phi^2.
$$

This together with (102), taken with reversed sign, restores Schwinger’s result. We see that in this way the explanation of perihelion shift is achieved without the concept of negative energy density of gravitational field.

So the nonlinear correction to the potential is taken into account by substitution

$$
\phi \rightarrow \phi^{eff} = \phi(1 + \frac{1}{2}\phi).
$$

Therefore the Newtonian attraction is decreased by this correction. Qualitatively this result is quite understandable: the correction term describes the interaction of the Sun with negative interaction energy and this corresponds to repulsion. The effective potential (105) produces the acceleration

$$
-\frac{d}{dr}\phi^{eff} = -\phi'(1 + \phi) = -\frac{GM}{r^2}(1 - \frac{GM}{r}).
$$

In general relativity the expression for acceleration of particle at rest depends upon the chosen coordinates. In Schwarzschild coordinates we have

$$
\sqrt{F_iF^i} = \frac{GM}{R^2\sqrt{1 - \frac{r_g}{r}}}, \quad r_g = 2GM,
$$

see e.g. eqs. (A 61), (A 63) in Appendix in [13]. In harmonic coordinates $r = R - \frac{r_g}{2}$ the expression takes the form

$$
\sqrt{F_iF^i} = \frac{GM}{r^2(1 + \frac{r_g}{2r})\sqrt{1 - (\frac{r_g}{2r})^2}} = \frac{GM}{r^2}(1 - \frac{GM}{r} + \cdots),
$$
and does not contradict (106). We note also that in this coordinates \( g_{00} \) is given by (75), i.e. the Newtonian potential \( \phi \) is substituted by \( \phi(1 + \phi) \). The difference with (105) is caused by the fact that Schwinger obtained the correction to \( \phi \) for use in Lagrangian method, cf. \$106 in [16] and [21], not for correcting \( g_{00} \). The relation is simple. Let us denote by \( \frac{1}{2} \alpha \) and \( \beta \) the coefficients at \( \phi^2 \) in corrections to \( \phi \) in \( g_{00} \) and in Schwinger method. Then from eq. below (106.15) in [16] we have for particle at rest

\[
L \propto \frac{ds}{dt} = \sqrt{1 + h_{00}} \approx 1 + \phi + \left( \frac{\alpha}{2} - \frac{1}{2} \right) \phi^2,
\]
where

\[
h_{00} = 1 + 2\phi + \alpha\phi^2.
\]
So \( \beta = \frac{\alpha}{2} - \frac{1}{2} \).

Now we shall see what is in store for us if we use the diagram method. First we dwell on properties of graviton propagator (98). The polarization factor \( P_{\mu\nu\rho\sigma} \) satisfies the relations

\[
t^{\mu\nu} P_{\mu\nu\rho\sigma} = t_{\rho\sigma} - \frac{1}{2} \eta_{\rho\sigma} t \equiv \bar{t}_{\rho\sigma}, \quad P_{\mu\nu\rho\sigma} T^{\rho\sigma} = \bar{T}_{\mu\nu},
\]

\[
t^{\mu\nu} P_{\mu\nu\rho\sigma} T^{\rho\sigma} = t^{\mu\nu} T^{\mu\nu} - \frac{1}{2} T = t^{\mu\nu} \bar{T}^{\mu\nu} = \bar{T}_{\mu\nu}, \tag{109}
\]

The scalar factor \( D_+(x) \) has the representation

\[
D_+(x) = \frac{1}{4\pi} \delta_+(x^2) = \frac{i}{(2\pi)^2} \frac{1}{x^2 + i\epsilon}, \tag{110}
\]
and possesses the property

\[
\int d\tau D_+(\vec{x} - \vec{x}', \tau) = \frac{i}{(2\pi)^2} \int_0^\infty \frac{d\tau}{(\vec{x} - \vec{x}')^2 - \tau^2 + i\epsilon} = \frac{1}{4\pi |\vec{x} - \vec{x}'|}, \tag{111}
\]

For spherically symmetric body we have to deal with integrals of the kind

\[
\int d^4 x' D_+(x - x') \rho(r') = \frac{1}{4\pi} \int \frac{d^3 x'}{\sqrt{x'^2 + \vec{x}^2 - 2 \vec{x} \cdot \vec{x}'}} \rho(r') = \frac{1}{r} \int_0^r dr' r'^2 \rho(r') + \int_0^\infty dr' \rho(r'), \tag{112}
\]
By the way it is seen from here that the derivative of Newtonian potential over $r$ is determined only by the mass inside sphere of radius $r$. Assuming in (112) that $\rho = \frac{c}{r^4}$, we get

$$\frac{1}{r} \int_{\delta}^{r} dr' r'^2 \rho(r') + \int_{r}^{\infty} dr' r' \rho(r') = c \left( \frac{1}{r \delta} - \frac{1}{2r^2} \right). \quad (113)$$

Hence the divergent part at small $r'$ appears only in the term, which is absorbed by mass renormalization.

So the source (13) generate the field

$$\bar{h}_{00} = 16\pi G \int d^4 x' D_+(x - x') T_{00}(x') \implies -\phi^2. \quad (114)$$

The arrow shows that the divergent part is included in mass renormalization. The r.h.s. of (114) can be obtained also from the solution of wave equation derived from (114) by action of the operator $\partial^2 = \nabla^2 - \frac{\partial^2}{\partial t^2}$ and taking into account that

$$-\partial^2 D_+(x - x') = \delta(x - x') , \quad \nabla^2 \frac{1}{r^2} = \frac{2}{r^4}. \quad (115)$$

We note also that

$$h_{\mu\nu} = 2h^{Sch}_{\mu\nu} = 16\pi G \int d^4 x' D_+(x - x') \bar{T}_{\mu\nu}(x'). \quad (116)$$

First we try MTW energy-momentum tensor as a source of gravitational field. The canonical part of MTW tensor in Hilbert gauge has the form

$$\bar{T}^{\gamma\delta} = \frac{1}{32\pi G} \left\{ \bar{\tilde{h}}^{\mu\nu,\delta} \bar{h}_{\mu\nu,\gamma} - \frac{1}{2} \bar{\tilde{h}}^{\delta} \bar{h}_{,\gamma} + \eta^{\gamma\delta} \left[ \frac{1}{2} \bar{h}_{\mu\nu,\lambda} \bar{\bar{h}}^{\mu\nu,\lambda} - \frac{1}{4} \bar{\bar{h}}_{,\lambda} \bar{h}_{,\lambda} \right] \right\} = \bar{T}^{\gamma\delta} - \frac{1}{2} \eta^{\gamma\delta} \bar{T}. \quad (117)$$

For Newtonian centers this expression coincides with Thirring’s one, see eq.(6). As seen from (117) in this static case $\bar{T}^{00} = 0$. So there is no contribution to $h_{00}$ from this source. As noted in Sec.5 the nonlocal part of $\bar{T}^{\mu\nu}$ is zero for Newtonian centers. So assuming, as in the case of Weinberg tensor, that the contribution from diagrams of Fig.1 and (2b), (2c) should not be taken into account, we cannot obtain $g_{00}$ necessary for explaining the
perihelion shift; even a combination of MTW and Thirring tensors will not help in this case.

Thus we have to take into account all diagrams. As we know from general relativity, where the vertices are known, the extracting post-Newtonian corrections from scattering amplitudes is a cumbersome and labour consuming job [21, 22]. For this reason we assume here the Schwinger method.

Returning to Thirring tensor we find that the total correction to $\phi$ from all diagrams of Fig.1-2 is zero. Indeed starting from the energy density (56), Schwinger obtained the correction to $\phi$ as $\frac{1}{2}\phi^2$. Therefore the corresponding contribution from spin part (15) is $4\phi^3$. This correction comes from diagrams (2b) and (2c). Similarly to (103), (104) we find that the correction to $\phi$ from diagram of Fig.1 is $-2\phi^2$. Finally the diagram (2a), treated according to Schwinger method, contributes $-2\phi^2$. So we get zero instead of $\frac{1}{2}\phi^2$, necessary for use in Lagrangian method, see §106 in [16].

Now it easy to see that a linear combination of MTW and Thirring tensors with weights $\frac{1}{4}$ and $\frac{3}{4}$ gives the necessary correction $\frac{1}{2}\phi^2$. As the MTW tensor alone is unable to explain the perihelion shift, we have to give up the tempting desire to consider the gravitational energy density of an isolated point-like center as positive.

Besides the nonlinear correction (102) there is a nonrelativistic linear correction to the Sun-planet interaction energy. All nonrelativistic corrections have the form [10]

$$\frac{3p^2}{2m^2}V + \frac{1}{2m}V^2, \quad V = m\phi. \quad (118)$$

Relativistic corrections may be treated either according to Schwinger or by using (118) together with Newtonian potential energy $V$ in relativistic equation of motion in flat space, see i.g. Ch.5, §1 in [20]. Both methods give the correct result. An elaborate analysis of different aspects of planet motion in this approximation is given in [19]. Finally we remind that three other famous observational effects of general relativity are explained already by linear approximation [2,10,19].

### 7 Concluding remarks

Though general covariance was not assumed, the gauge invariances is of course retained [2,10]. For this reason the weak gravitational waves in flat space are
described as in general relativity. All considered tensors give the same energy-
momentum tensor for the plane gravitational wave. There are no a priori
reasons to believe that field-theoretical approach will give the same result
as general relativity. Similar assertion was made in [21]. Our investigation
shows that there is still much to be done to synthesize the geometrical and
field-theoretical aspects of gravitations.

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Figure captions

Fig.1: The second rank tensor formed from matter energy-momentum
tensor and graviton is a source for another graviton.

Fig.2: 3-graviton vertex. Short straight line serves only to distinguish the
roles of participating gravitons: energy-momentum tensor is formed from two
gravitons joining the straight line at its ends, this energy-momentum tensor
serves as a source for graviton emerging from the middle of the straight line.
Crosses represent external field sources.

8 References

1. W.E.Thirring, Fortschr. Physik. 7, 79 (1959).
2. W.E.Thirring, Ann. Phys. (N.Y.) 16, 96 (1961).
3. V.I.Ogievetsky and I.V.Polubarinov, Ann. Phys. (N.Y.) 35, 167 (1965).
4. S.N.Gupta, Rev. Mod. Phys. 29, 334 (1957).
5. S.S.Gerstein, A.A.Logunov, and M.A.Mestvirishvili, Dokl. Akad. Nauk,
Russia, Vol. 360, p. 332 (1998) (in Russian).
6. C.W.Misner, K.S.Thorne, J.A.Wheeler, Gravitation. San Francisco(1973).
7. E.Alvarez, Rev. Mod. Phys. 61, 561 (1989).
8. Yu.V.Graz, V.Ch. Zhukovski, and D.V.Galtsov, Classical fields, Moscow
(1991). Moscow university publishing hous (in Russian).
9. S.Weinberg, Gravitation and Cosmology, New York (1972).
10. J.Schwinger, Particles, Sources, and Fields. V.1 Addison-Wesley (1970).
11. A.Pais, The Science and the Life of Albert Einstein, Oxford (1982).
12. A. Einstein, Ann. Math. 40, 922 (1939).
13. I. D. Novikov, V. P. Frolov, *Physics of black holes*, Moscow (1986) (in Russian).
14. Venzo de Sabbata and Maurizio Gasperini, *Introduction to Gravitation*. World Scientific (1985).
15. H. Dehnen, Zeitschr. für Phys. 179, 1 Heft, 76 (1964).
16. L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, Moscow, (1973) (in Russian).
17. H. Weyl *Raum, Zeit, Materie*. Springer Verlag, Berlin (1923).
18. H. Umezawa, *Quantum Field Theory*, Amsterdam (1956).
19. H. Dehnen, H. Hönl, and K. Westpfahl, Ann. der Phys. 6, 7 Folge, Band 6, Heft 7-8, S.670 (1960).
20. A. Sommerfeld *Atombau und Spectralinien*, Band 1, Braunschweig (1951).
21. Y. Iwasaki, Prog. Theor. Phys. 46, 1587 (1971).
22. H. W. Hamber, S. Liu. Phys. Lett. B 357, 51 (1995).

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Fig. 1

Fig. 2