ON ORBIT CLOSURES FOR INFINITE TYPE QUIVERS

CALIN CHINDRIS

Abstract. For the Kronecker quiver, Zwara [7, Theorem 1] has found an example of a representation whose orbit closure is neither unibranch nor Cohen-Macaulay. In this note, we explain how to extend this example to all infinite type quivers without oriented cycles.

In this short note, we use quiver exceptional sequences to reduce, in a “hom-controlled” manner, the list of all representation-infinite quivers without oriented cycles to just $\tilde{A}_2$. This observation combined with Zwara’s example from [7, Theorem 1] yields:

Theorem 0.1. Let $Q$ be a representation-infinite, connected quiver without oriented cycles. Then, there exists a representation $W$ whose orbit closure is neither unibranch nor Cohen-Macaulay.

In [6], Zwara showed that the orbit closures of representations of Dynkin quivers are always unibranch. Hence, we deduce that the Dynkin quivers are precisely the ones with the property that all orbit closures are unibranch.

In what follows, we first review some background material from quiver theory and then prove Theorem 0.1. Throughout this note, we work over an algebraically closed field $k$ of characteristic zero.

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver, where $Q_0$ is the set of vertices, $Q_1$ is the set of arrows and $t, h : Q_1 \to Q_0$ assign to each arrow $a \in Q_1$ its tail $ta$ and head $ha$, respectively. A representation $V$ of $Q$ over $k$ is a family of finite dimensional $k$-vector spaces $\{V(x) | x \in Q_0\}$ together with a family $\{V(a) : V(ta) \to V(ha) | a \in Q_1\}$ of $k$-linear maps. If $V$ is a representation of $Q$, we define its dimension vector $d_V$ by $d_V(x) = \dim_k V(x)$ for every $x \in Q_0$. Thus the dimension vectors of representations of $Q$ lie in $\Gamma = \mathbb{Z}^{Q_0}$, the set of all integer-valued functions on $Q_0$.

Given two representations $V$ and $W$ of $Q$, we define a morphism $\varphi : V \to W$ to be a collection of linear maps $\{\varphi(x) : V(x) \to W(x) | x \in Q_0\}$ such that for every arrow $a \in Q_1$, we have $\varphi(ha)V(a) = W(a)\varphi(ta)$. We denote by $\text{Hom}_Q(V, W)$ the $k$-vector space of all morphisms from $V$ to $W$. Let $W$ and $V$ be two representations of $Q$. We say that $V$ is a subrepresentation of $W$ if $V(x)$ is a subspace of $W(x)$ for all vertices $x \in Q_0$ and $V(a)$ is the restriction of $W(a)$ to $V(ta)$ for all arrows $a \in Q_1$. In this way, we obtain the abelian category $\text{Rep}(Q)$ of all quiver representations of $Q$.

A representation $W$ is said to be a Schur representation if $\text{End}_Q(W) \cong \mathbb{C}$. The dimension vector of a Schur representation is called a Schur root.

From now on, we assume that our quivers are without oriented cycles. For two quiver representations $V$ and $W$, consider Ringel’s canonical exact sequence [5]:

$$0 \to \text{Hom}_Q(V, W) \to \bigoplus_{x \in Q_0} \text{Hom}_k(V(x), W(x)) \to \bigoplus_{a \in Q_1} \text{Hom}_k(V(ta), W(ha)),$$

Date: August 24, 2007.

2000 Mathematics Subject Classification. Primary 16G20; Secondary 05E15.

Key words and phrases. Exceptional sequences, orbit closure, unibranch, quivers.
where \( d^W_\alpha((\varphi(x))_{x \in Q_0}) = (\varphi(ha)V(a) - W(a)\varphi(ta))_{a \in Q_1} \) and \( \text{Ext}^1_Q(V, W) = \text{coker}(d^W_\alpha). \)

If \( \alpha, \beta \) are two elements of \( \Gamma \), we define the Euler inner product by

\[
\langle \alpha, \beta \rangle_Q = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).
\]

(When no confusion arises, we drop the subscript \( Q \).)

It follows from (1) and (2) that

\[
\langle d^W_\alpha, d^W_\beta \rangle = \dim_k \text{Hom}_Q(V, W) - \dim_k \text{Ext}^1_Q(V, W).
\]

A dimension vector \( \beta \) is called a real Schur root if there exists a representation \( W \in \text{Rep}(Q, \beta) \) such that \( \text{End}_Q(W) \simeq k \) and \( \text{Ext}^1_Q(W, W) = 0 \) (we call such a representation exceptional). Note that if \( \beta \) is a real Schur root then there exists a unique, up to isomorphism, exceptional \( \beta \)-dimensional representation.

For \( \alpha \) and \( \beta \) two dimension vectors, consider the generic ext and hom:

\[
\text{ext}_Q(\alpha, \beta) := \min\{\dim_k \text{Ext}^1_Q(V, W) \mid (V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)\},
\]

and

\[
\text{hom}_Q(\alpha, \beta) := \min\{\dim_k \text{Hom}_Q(V, W) \mid (V, W) \in \text{Rep}(Q, \alpha) \times \text{Rep}(Q, \beta)\}.
\]

Given two dimension vectors \( \alpha \) and \( \beta \), we write \( \alpha \perp \beta \) provided that \( \text{ext}_Q(\alpha, \beta) = \text{hom}_Q(\alpha, \beta) = 0 \).

**Definition 0.2.** We say that \( \langle \varepsilon_1, \ldots, \varepsilon_r \rangle \) is an exceptional sequence if

1. the \( \varepsilon_i \) are real Schur roots;
2. \( \varepsilon_i \perp \varepsilon_j \) for all \( 1 \leq i < j \leq l \).

Following [4], a sequence \( \langle \varepsilon_1, \ldots, \varepsilon_r \rangle \) is called a quiver exceptional sequence if it is exceptional and \( \langle \varepsilon_j, \varepsilon_i \rangle \leq 0 \) for all \( 1 \leq i < j \leq l \).

Now, let \( \varepsilon = \langle \varepsilon_1, \ldots, \varepsilon_r \rangle \) be a quiver exceptional sequence and let \( E_i \in \text{Rep}(Q, \varepsilon_i) \) be exceptional representations. Construct a new quiver \( Q(\varepsilon) \) with vertex set \( \{1, \ldots, r\} \) and \( -\langle \varepsilon_j, \varepsilon_i \rangle \) arrows from \( j \) to \( i \). Define \( C(\varepsilon) \) to be the smallest full subcategory of \( \text{Rep}(Q) \) which contains \( E_1, \ldots, E_r \) and is closed under extensions, kernels of epimorphisms, and cokernels of monomorphisms.

For the remaining of this section, we assume that \( r \leq N - 1 \), where \( N \) is the number of vertices of \( Q \). We recall a very useful result from [4, Section 2.7] in a form that is convenient for us (see also [3]):

**Proposition 0.3.** [4] The category \( C(\varepsilon) \) is naturally equivalent to \( \text{Rep}(Q(\varepsilon)) \) with \( E_1, \ldots, E_r \) being the simple objects of \( C(\varepsilon) \). Furthermore, the inverse functor from \( \text{Rep}(Q(\varepsilon)) \) to \( C(\varepsilon) \) is a full exact embedding into \( \text{Rep}(Q) \).

Consider the linear transformation

\[
I: \mathbb{Z}^{Q(\varepsilon)_{\alpha}} \rightarrow \mathbb{Z}^{Q_0}
\]

\[
\alpha = (\alpha(1), \ldots, \alpha(r)) \rightarrow \sum_{i=1}^{r} \alpha(i)\varepsilon_i.
\]

If \( F: \text{Rep}(Q(\varepsilon)) \rightarrow \text{Rep}(Q) \) is the full exact embedding form Proposition 0.3 then \( F \) is clearly hom-controlled in the sense of Zwara [8]. Let

\[
F(\alpha): \text{Rep}(Q(\varepsilon), \alpha) \rightarrow \text{Rep}(Q, I(\alpha))
\]

be the induced morphism of varieties. Zwara [8, Theorem 2] (see also [2, Proposition 9]) showed that hom-controlled functors preserve the type of singularities of orbit closures. In particular, we have:
Proposition 0.4. Keep the same notations as above. Let $V \in \text{Rep}(Q(\varepsilon), \alpha)$. Then, $GL(\alpha)V$ is normal (or unibranch or Cohen-Macaulay) if and only if $GL(I(\alpha))F(\alpha)(V)$ has the same property.

Next, let us recall Zwara’s example from \[7\]:

Theorem 0.5. Let $\theta(2)$ be the Kronecker quiver

\[
\begin{array}{c}
1 \\
\rightarrow \rightarrow \\
2
\end{array}
\]

Label the arrows by $a$ and $b$. Consider the following representation $V \in \text{Rep}(\theta(2), (3, 3))$ defined by $V(a) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $V(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $GL(\alpha)V$ is neither unibranch nor Cohen-Macaulay.

Proof of Theorem 0.1. From \[1\] Lemma 2.1, pp. 253\], we know that any finite, connected quiver $Q$ of infinite representation type must contain a Euclidean quiver as a subquiver. Therefore, it is enough to prove the theorem for Euclidean quivers. Now, let $Q$ be a Euclidean quiver and denote by $\delta_Q$ the isotropic Schur root of $Q$.

Choose $v$ to be a vertex such that $Q \setminus v$ is a Dynkin quiver. Without loss of generality, let us assume that $v$ is a source. In this case, we take $\varepsilon_1 = \delta_Q - e_v$ and $\varepsilon_2 = e_v$. Then, $(\varepsilon_1, \varepsilon_2)$ is a quiver exceptional sequence with $\langle \varepsilon_2, \varepsilon_1 \rangle = -2$ and so the proof follows from Proposition 0.4 and Theorem 0.5.

\[\square\]

References

[1] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. Elements of the representation theory of associative algebras. Vol. 1: Techniques of representation theory, volume 65 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006.
[2] G. Bobiński and G. Zwara. Schubert varieties and representations of Dynkin quivers. Colloq. Math., 94(2):285–309, 2002.
[3] C. Chindris. Orbit semigroups and the representation type of quivers. Preprint, arXiv:0708.3413v1 [math.RT], 2007.
[4] H. Derksen and J. Weyman. The combinatorics of quiver representations. Preprint, arXiv:math.RT/0608288, 2006.
[5] C.M. Ringel. Representations of $K$-species and bimodules. J. Algebra, 41(2):269–302, 1976.
[6] G. Zwara. Degenerations of finite-dimensional modules are given by extensions. Compositio Math., 121(2):205–218, 2000.
[7] G. Zwara. An orbit closure for a representation of the Kronecker quiver with bad singularities. Colloq. Math., 97(1):81–86, 2000.
[8] G. Zwara. Unibranch orbit closures in module varieties. Ann. Sci. cole Norm. Sup., 35(6):877–895, 2002.