Limit Cycles of Piecewise Smooth Differential Equations on Two Dimensional Torus

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Abstract In this paper we study the limit cycles of some classes of piecewise smooth vector fields defined in the two dimensional torus. The piecewise smooth vector fields that we consider are composed by linear, Ricatti with constant coefficients and perturbations of these one, which are given in (3). Considering these piecewise smooth vector fields we characterize the global dynamics, studying the upper bound of number of limit cycles, the existence of non-trivial recurrence and a continuum of periodic orbits. We also present a family of piecewise smooth vector fields that posses a finite number of fold points and, for this family we prove that for any 2\(k\) number of limit cycles there exists a piecewise smooth vector fields in this family that presents \(k\) number of limit cycles and prove that some classes of piecewise smooth vector fields presents a non-trivial recurrence or a continuum of periodic orbits.

Keywords Piecewise smooth differential equations · Limit cycles · Global dynamics in torus

Mathematics Subject Classification Primary 34A36 · 34C07 · 34C23 · 34C60
1 Introduction

In this paper we consider piecewise smooth differential equations of the form

\[
\dot{x} = \begin{cases} 
X^+(x) & \text{if } x \in \mathbb{T}^+, \\
X^-(x) & \text{if } x \in \mathbb{T}^- 
\end{cases}
\]  

(1)

where the two dimensional torus \( \mathbb{T} \) is decomposed as \( \mathbb{T} = \mathbb{T}^+ \cup \mathbb{T}^- \), \( \mathbb{T}^+ \) denotes the upper half of torus and \( \mathbb{T}^- \) the bottom half of torus. We also denote by \( \Sigma = \mathbb{T}^+ \cap \mathbb{T}^- \) a smooth curve breaking \( \mathbb{T} \) into two connected components and \( X^+, X^- \) are smooth vector fields on \( \mathbb{T} \).

The dynamics over \( \Sigma \) is defined to satisfy the Filippov’s convention (see [6]). For simplicity, a differential equation like (1) will be denoted by \((X^+, X^-)\) and referred as vector field (1).

The theory of piecewise smooth vector fields (PSVF) has been developing very fast in the last years, mainly due to its strong relation with branches of applied sciences as also this branch of research is in the boundary between mathematics, physics and engineering, see [8] and [15] for a recent survey on this subject, where models from control theory are discussed. This kind of systems are described by piecewise systems of differential equations, so that we have a smooth system defined in regions of the phase portrait. The common frontier between the regions that separate the smooth vector fields is called switching manifold (or discontinuity manifold).

The study of piecewise smooth dynamical systems defined on torus is not new, but has been restricted to the case of discrete dynamical systems. There are a large number of results for piecewise maps [1,2,4] and [16], but there is a lack of theoretical results for the case of piecewise dynamical systems where the flow is the solution of a piecewise differential equation.

The investigation of the number and stability of limit cycles of the relevant class of vector fields is one of the most relevant problems in the classical qualitative theory of dynamical systems. This study started with [13] and [11]. In this paper, the main objective is to start this investigation considering piecewise smooth vector fields given in two zones where one of them is linear in torus, Ricatti with constant coefficients, some families of perturbations of then given by applications (3) and a family of PSVF presenting a finite number of fold points, see (6).

This paper is divide as follows: in Sect. 2 we formalize some basic concepts on PSVF, as the first return map in this scenario and present some techniques that we use in the proof of the main results. In Sect. 3 the main problems are presented, in Sect. 4 we prove these results presented and in Sect. 5 we finalize the paper presenting some numerical examples of PSVF with the maximum number of limit cycles.

2 Basic Theory

2.1 Filippov’s Convention

First of all, in this work the model of construction of the two dimensional torus, denoted by \( \mathbb{T} \) that we consider is provided by the following equivalent relation in \( \mathbb{Q} = [0, 1] \times [0, 1] \subset \mathbb{R}^2 \):

\[
(x, y) \sim (z, w) \iff x - z \in \mathbb{Z}, y - w \in \mathbb{Z}.
\]

(2)

Consider \( \Sigma_1 = \{(x, y) \in \mathbb{Q}; y = 0\} \) and \( \Sigma_2 = \{(x, y) \in \mathbb{Q}; y = \frac{1}{2}\} \). We denote by \( h_1(x, y) = y \) and \( h_2(x, y) = y - \frac{1}{2} \), in this way we can write \( \Sigma_1 = h_1^{-1}(0) \) and

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\(\Sigma_2 = h_2^{-1}(0)\). Clearly the switching manifolds \(\Sigma_1\) and \(\Sigma_2\) are the separating boundaries of the regions \(\mathbb{T}^- = \{(x, y) \in \mathbb{T}; 0 \leq y \leq \frac{1}{2}\}\) and \(\mathbb{T}^+ = \{(x, y) \in \mathbb{T}; \frac{1}{2} \leq y \leq 1\}\).

Designate by \(\mathcal{X}'\) the space of \(C^r\)-vector fields on \(\mathbb{T}\) endowed with the \(C^r\)-topology with \(r = \infty\) or \(r \geq 1\) large enough for our purposes. Call \(\Omega'\) the space of vector fields \(X : \mathbb{T} \to \mathbb{T}\) such that

\[
X(x, y) = \begin{cases} 
X^+(x, y), & \text{for } (x, y) \in \mathbb{T}^+,
X^-(x, y), & \text{for } (x, y) \in \mathbb{T}^-,
\end{cases}
\]

where \(X^+ = (X^+_1, X^+_2)\) and \(X^- = (X^-_1, X^-_2)\) are in \(\mathcal{X}'\). Let \(h \in \{h_1, h_2\}\) and \(\Sigma \in \{\Sigma_1, \Sigma_2\}\). We denote \(X^\pm h(p) = (X^\pm(p), \nabla h(p))\) and \((X^\pm)^n h(p) = (X^\pm(p), \nabla(X^\pm)^{n-1} h(p))\) the Lie’s derivatives, where \(\langle \cdot, \cdot \rangle\) denote the canonical inner product. We may consider \(\Omega' = \mathcal{X}' \times \mathcal{X}'\) endowed with the product topology and denote any element in \(\Omega'\) by \(X = (X^+, X^-)\), which we will accept to be multivalued in points of \(\Sigma\). The basic results of differential equations, in this context, were stated by Filippov in [6]. Related theories can be found in [5, 10, 15] and references therein.

On \(\Sigma \in \{\Sigma_1, \Sigma_2\}\) we generically distinguish three regions: crossing region: \(\Sigma^c = \{p \in \Sigma; X^+_2(p)X^-_2(p) > 0\}\), stable sliding region: \(\Sigma^s = \{p \in \Sigma; X^+_2(p) < 0, X^-_2(p) > 0\}\) and unstable sliding region: \(\Sigma^u = \{p \in \Sigma; X^+_2(p) > 0, X^-_2(p) < 0\}\).

When \(q \in \Sigma^s\), following the Filippov’s convention (see [6], chapter 2, page 52), the sliding vector field associated to \(X \in \Omega'\) is the vector field \(\hat{X}^s\) tangent to \(\Sigma^s\), expressed in coordinates \((q, \delta)\) as

\[
\hat{X}^s(q, \delta) = \frac{1}{(X^- h - X^+ h)(q)}((X^- h)X^+ - (X^+ h)X^-)(q, \delta)
= \frac{1}{(X^-_2 - X^+_2)(q)}((X^+_2 - X^-_2)(q, \delta)),
\]

where \(\delta \in [0, 1/2]\) and which, by a time rescaling, is topologically equivalent to the normalized sliding vector field

\[
X^s(q) = (X^+_1 - X^-_1)(q).
\]

The points \(q \in \Sigma\) such that \(X^s(q) = 0\) are called pseudo equilibrium of \(X\) and the points \(p \in \Sigma\) such that \(X^+ h(p)X^- h(p) = 0\) are called tangential singularities of \(X\) (i.e., the trajectory through \(p\) is tangent to \(\Sigma\)). We say that \(q \in \Sigma\) is a regular point if \(q \in \Sigma^c\) or \(q \in \Sigma^s\) and \(X^s(q) \neq 0\).

A tangential singularity \(q \in \Sigma\) of \(X^+\) is a fold point of \(X^+\) if \(X^+ h(q) = 0\) but \((X^+)^2 h(q) \neq 0\), visible tangency if \((X^+)^2 h(q) > 0\) and invisible tangency if \((X^+)^2 h(q) < 0\).

**Definition 1** The flow \(\phi_X\) of \(X \in \Omega'\) is obtained by the concatenation of flows of \(X^+, X^-\) and \(X^s\), denoted by \(\phi_{X^+}, \phi_{X^-}\) and \(\phi_{X^s}\), respectively.

In the following we define a fold-regular point.

**Definition 2** Let \(X = (X^+, X^-) \in \Omega'\), we say that \(p \in \Sigma\) is a fold-regular point of \(X\) if \(p\) is a fold point of \(X^+\) and \(X^-(p)\) is transversal to \(\Sigma\) at \(p\).

### 2.2 Extended Chebyshev Systems

We say that a ordered set of complex–valued functions \(\mathcal{F} = (g_0, g_1, \ldots, g_k)\) defined on a proper real interval \(I\) is an Extended Chebyshev system or ET–system on \(I\) if and only if any
nontrivial linear combination of functions in $\mathcal{F}$ has at most $k$ zeros counting multiplicities. We say that $\mathcal{F}$ is an Extended Complete Chebyshev system or an ECT-system on $I$ if and only if for any $s$, $0 \leq s \leq k$, we get that $(g_0, g_1, \ldots, g_s)$ is an ET-system. For more details, see the book of Berezin and Zhidkov [3], and the book of Karlin and Studden [7].

In order to prove that $\mathcal{F}$ is a ECT-system on $I$ it is necessary and sufficient to show that $W(g_0, g_1, \ldots, g_s)(t) \neq 0$ on $I$ for $0 \leq s \leq k$. Here $W(g_0, g_1, \ldots, g_s)(t)$ denotes the Wronskians of the functions $(g_0, g_1, \ldots, g_s)$ with respect to $t$.

In [9] the authors proved that for a family of $n+1$ linearly independent analytical functions where at least one of that possess constant sign in its domain, there exists a linear combination of these functions having at least $n$ simple zeros. Precisely, they proved the following result:

**Theorem B** Let $\mathcal{F} = [g_0, g_1, \ldots, g_n]$ be an ordered set of real $C^\infty$ functions on $(a, b)$ such that there exists $\xi \in (a, b)$ with $W(g_0, g_1, \ldots, g_{n-1})(\xi) = W_{n-1}(\xi) \neq 0$. Then next properties hold:

(a) If $W_n(\xi) \neq 0$ then for each configuration of $m \leq n$ zeros, taking into account their multiplicity, there exists $f \in \text{Span}(\mathcal{F})$ with this configuration of zeros.

(b) If $W_n(\xi) = 0$ and $W'_{n}(\xi) \neq 0$ then for each configuration of $m \leq n + 1$ zeros, taking into account their multiplicity, there exists $f \in \text{Span}(\mathcal{F})$ with this configuration of zeros.

### 3 Main Results For PSVF in the Two Dimensional Torus

One of the mains objective of this paper is study the linear and Ricatti (with constant coefficients) vector fields in $\mathbb{T}$, that we denote by

$$X^\omega_L(x, y) = (a^\omega y + b^\omega, c^\omega y + d^\omega),$$

$$X^\omega_R(x, y) = (1, e^\omega + f^\omega y + g^\omega y^2),$$

respectively, where $a^\omega, b^\omega, c^\omega, d^\omega, e^\omega, f^\omega, g^\omega \in \mathbb{R}$ and $\omega = +$ or $\omega = -$, depending if the vector fields is defined in $\mathbb{T}^+$ or $\mathbb{T}^-$. The special case of $X^\omega_L$ where $a^\omega = c^\omega = 0$ in $X^\omega_L$ will be denoted by $X^\omega_C$ (constant vector field).

In the following, we perturb these PSVF considering the applications defined in $\mathbb{T}$:

$$F_1(x, y) = (-x + x^2, 0), \quad F_2(x, y) = (\eta_1 y + \eta_2 y^2, 0), \quad F_3(x, y) = (\cos(2\pi x), 0),$$

(3)

where $\eta_1, \eta_2 \in \mathbb{R}$ and small. We denote the piecewise smooth vector field $X_{LL}$ composed by linear vector fields in each half torus, $X_{LR}$ composed by a linear vector fields in $\mathbb{T}^-$ and Ricatti vector fields on $\mathbb{T}^+$ and $X_{RR}$ composed by the Ricatti vector fields with constant coefficients in each half torus. Considering the PSVF $X_{LL}, X_{LR}$ and $X_{RR}$ we perform the following perturbations:

$$X_{LL2+} = X_{LL} + (F_2, \overrightarrow{0}), \quad X_{RR1+} = X_{RR} + \varepsilon(F_1, \overrightarrow{0}), \quad X_{RR2+} = X_{RR} + (F_2, \overrightarrow{0})$$

$$X_{RR3+} = X_{RR} + \varepsilon(F_3, \overrightarrow{0}), \quad X_{LR1-} = X_{LR} + \varepsilon(\overrightarrow{0}, F_1), \quad X_{LR2+} = X_{LR} + (F_2, \overrightarrow{0})$$

$$X_{LR2-} = X_{LR} + (\overrightarrow{0}, F_2), \quad X_{LR3-} = X_{LR} + \varepsilon(\overrightarrow{0}, F_3),$$

(4)

where $\overrightarrow{0}$ denotes the null vector fields $(0, 0)$ in $\mathbb{T}$.

**Remark 1** We only consider the perturbation of $X_{LL}$ and $X_{RR}$ in $\mathbb{T}^+$, however by symmetry, we have the same results that we obtained if we consider perturbations in $\mathbb{T}^-$. 

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For each of the families presented in (4) we consider the following subfamilies

\[ \Omega^1_{L^\omega} = \{ X_{L^\omega}; c^\omega > 0, d^\omega > -\frac{c^\omega}{2} \}, \quad \Omega^2_{L^\omega} = \{ X_{L^\omega}; c^\omega < 0, -1 < \frac{c^\omega}{(c^\omega + 2d^\omega)} < 0 \}, \]

\[ \Omega^3_{L^\omega} = \{ X_{L^\omega}; c^\omega < 0, -1 < \frac{c^\omega}{d^\omega} < 0 \}, \quad \Omega^4_{L^\omega} = \{ X_{L^\omega}; c^\omega > 0, d^\omega > 0 \}, \]

\[ \Omega^5_{L^\omega} = \{ X_{L^\omega}; a^\omega > 0, b^\omega > 0 \}, \quad \Omega^6_{L^\omega} = \{ X_{L^\omega}; a^\omega < 0, -1 < \frac{a^\omega}{(2b^\omega)} < 0 \}, \]

\[ \Omega^1_{R^\omega} = \left\{ X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left( \frac{f^\omega}{2} \right)^2, \tan^{-1} (\theta^\omega_1) > \tan^{-1} (\theta^\omega_2) \right\}, \]

\[ \Omega^2_{R^\omega} = \left\{ X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left( \frac{f^\omega}{2} \right)^2, \tan^{-1} (\theta^\omega_2) > \sec^{-1} (\theta^\omega_3) \right\}, \]

\[ \Omega^3_{R^\omega} = \left\{ X_{R^\omega}; f^\omega > 0, e^\omega g^\omega > \left( \frac{f^\omega}{2} \right)^2, \tan^{-1} (\theta^\omega_1) > \sec^{-1} (\theta^\omega_3) \right\}, \]

where \( \theta^\omega_1 = \frac{f^\omega + 2e^\omega}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}}, \theta^\omega_2 = \frac{f^\omega + g^\omega}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}} \) and \( \theta^\omega_3 = \frac{2\sqrt{e^\omega g^\omega}}{\sqrt{4e^\omega g^\omega - (f^\omega)^2}} \).

Prior to present the theorem we denote the following numbers

\[ \Delta_{LL} = \frac{1}{2(c_e-c^-)^2} \left( c^- \left( c^+ (a^- c^- + a^+ c^-) + 2c^- \log \left( c^+ \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+ c^+ - a^+ d^+) \right) \right. \]

\[ + 2(c^+)^2 \log \left( \frac{c^+}{2d^+} + 1 \right) (b^+ c^+ - a^+ d^+), \]

\[ \Delta_{LR} = \frac{1}{2(c^-)^2} \left( a^- c^- + (4(c^-)^2)(-\tan^{-1} (\theta^\omega_1) + \tan^{-1} (\theta^\omega_2))/\sqrt{-(f^\omega)^2 + 4e^\omega g^\omega} + 2b^- c^- \log(1 + c^-/(2d^-)) - 2a^- d^- \log(1 + c^-/(2d^-)) \right), \]

\[ \Delta_{RR} = \sec^{-1} (\theta^\omega_3) - \tan^{-1} (\theta^\omega_2) + \tan^{-1} (\theta^\omega_2) - \sec^{-1} (\theta^\omega_3), \]

\[ \Delta_{LL^2+} = \frac{1}{8(c_e)^3} \left( 8 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (d^+ (d^+ \eta_2 - c^+ (a^+ + \eta_1)) + b^+ (c^+)^2) + c^+ (4c^+ (a^+ + c^+ + \eta_1) + \eta_2 (3c^+ - 4d^+)) \right), \]

\[ \Delta_{RR^2+} = \frac{1}{2(g^+)^2 \sqrt{4e^\omega g^\omega - (f^\omega)^2} \sqrt{4e^\omega g^\omega + (f^\omega)^2}} \left( \tan^{-1} (\theta^\omega_2) (4e^\omega \eta_2 g^+ - 2(f^\omega)^2 \eta_2 + 2f^+ \eta_1 g^+ - 4(g^+)^2) + 2 \tan^{-1} (\theta^\omega_1) (-g^+ (2e^\omega \eta_2 + f^+ \eta_1) + (f^\omega)^2 \eta_2 + 2(g^+)^2) + \sqrt{4e^\omega g^\omega - (f^\omega)^2} \log \left( g^+ (e^+ + f^+ + g^+) \right) (\eta_1 g^+ - f^+ \eta_2) \right) + \log \left( 1 + (\theta^\omega_2)^2 \right) (f^+ \eta_2 - \eta_1 g^+) - f^+ \eta_2 \log(4) + \eta_1 g^+ \log(4) + \eta_2 g^+) + 4(g^+)^2 \sqrt{4e^\omega g^\omega - (f^\omega)^2} \left( \tan^{-1} (\theta^\omega_2) - \sec^{-1} (\theta^\omega_3) \right) \right) \]
\[ \Delta_{LR2-} = \frac{1}{2(c^+)^2(g^-)^2 \sqrt{4e^g - (f^-)^2}} \left( \sqrt{4e^g - (f^-)^2} \left( 2(g^-)^2 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) \right) \right) \]
\[= \frac{1}{2(c^+)^2(g^-)^2 \sqrt{4e^g - (f^-)^2}} \left( \sqrt{4e^g - (f^-)^2} \left( 2(g^-)^2 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) \right) \right) \]
\[= 2c^+ \eta_1 g^- \log \left( \frac{1}{\sqrt{1 + (\theta_2^-)^2}} \right) - c^+ \sqrt{e^{-g}} \eta_2 g^- \sqrt{\frac{(f^-)^2}{e^{-g}}} \]
\[= 2c^+ \eta_1 g^- \log \left( \frac{1}{\sqrt{1 + (\theta_2^-)^2}} \right) - c^+ \sqrt{e^{-g}} \eta_2 g^- \sqrt{\frac{(f^-)^2}{e^{-g}}} \]
\[= 2c^+ f^- \eta_2 \log \left( \frac{\sqrt{4e^g - (f^-)^2}}{2\sqrt{e^{-g}}} \right) + 2c^+ f^- \eta_2 \log \left( \frac{1}{\sqrt{1 + (\theta_2^+)^2}} \right) \]
\[+ c^+ f^- \eta_2 - c^+ \eta_1 g^- \log(4) + c^+ \eta_2 g^- \right) + \left( 2(c^+)^2 \left( -g^- (2e^{-\eta_2} + f^- \eta_1) \right) \right) \]
\[+ (f^-)^2 \eta_2 + 2(g^-)^2 \right) \left( \tan^{-1}(\theta_2^-) - \sec^{-1}(\theta_3^-) \right) \]

\[ \Delta_{LR2+} = \frac{1}{8(c^+)^3 \left( (f^-)^2 - 4e^{-g} \right)} \left( (f^-)^2 - 4e^{-g} \right) \left( 8 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) \right) \]
\[\left( c^+ \left( b^+ c^+ - d^+ (a^+ + \eta_1) \right) + (d^+)^2 \eta_2 + c^+ (4c^+ a^+ + \eta_1) + \eta_2 (3c^+ - 4d^+) \right) \]
\[+ 16(c^+)^3 \sqrt{e^{-g} - (f^-)^2} \left( \sec^{-1}(\theta_3^-) - \tan^{-1}(\theta_2^-) \right) \]

In Theorem 1 we prove that these subfamilies correspond the piecewise smooth vector fields where the first return map \( P : \Sigma_1 \rightarrow \Sigma_1 \) is defined.

**Theorem 1** Consider the PSVF given in (5).

(a) If \( \Delta_{LL} \) is a rational number then \( X_{LL} \) presenting a continuum of periodic orbits and if \( \Delta_{LL} \) is not a rational number then all trajectories of \( X_{LL} \) are dense.

(b) If \( \Delta_{LR} \) is a rational number then \( X_{LR} \) presenting a continuum of periodic orbits and if \( \Delta_{LR} \) is not a rational number then all trajectories of \( X_{LR} \) are dense.

(c) If \( \Delta_{RR} \) is a rational number then \( X_{RR} \) presenting a continuum of periodic orbits and if \( \Delta_{RR} \) is not a rational number then all trajectories of \( X_{RR} \) are dense.

Considering the perturbations \( F_{i}^{0} \), we get

(d) If \( \Delta_{LL2+} \) is a rational number then \( X_{LL2+} \) presenting a continuum of periodic orbits and if \( \Delta_{LL2+} \) is not a rational number then all trajectories of \( X_{LL2+} \) are dense.

(e) If \( \epsilon > 0 \) then the maximum number of limit cycles for \( X_{RR1+} \) is two. Besides then this upper bound is reached.

(f) If \( \Delta_{RR2+} \) is a rational number then \( X_{RR2+} \) presenting a continuum of periodic orbits and if \( \Delta_{RR2+} \) is not a rational number then all trajectories of \( X_{RR2+} \) are dense.

(g) The maximum number of limit cycles of \( X_{RR3+} \) is two and this upper bound is reached.

(h) The maximum number of limit cycles of \( X_{LR1-} \) is two and this upper bound is reached if \( X_{LR1-} \in \left[ \Sigma_{L+}^1 \cup \Sigma_{L+}^2 \right] \cap \Omega_{R-}^2, \epsilon < 0, a^+ c^+ > 0, \) and \( b^+ c^+ > a^+ d^+ \).

(i) If \( \Delta_{LR2-} \) is a rational number then \( X_{LR2-} \) presenting a continuum of periodic orbits and if \( \Delta_{LR2-} \) is not a rational number then all trajectories of \( X_{LR2-} \) are dense.

(j) The maximum number of limit cycles of \( X_{LR3-} \) is two and this upper bound is reached.
(l) If $\Delta_{LR2^+}$ is a rational number then $X_{LR2^+}$ presenting a continuum of periodic orbits and if $\Delta_{LR2^+}$ is not a rational number then all trajectories of $X_{LR2^+}$ are dense.

In the following, we consider a model of a PSVF in $\mathbb{T}$ that present a finite number of fold-regular points in $\Sigma$ given by $X_{Ck} = (X_C, X_k)$ where

$$X_k(x, y) = (\alpha, \beta \cos(2k\pi x)), \quad (6)$$

is defined in $\mathbb{T}^-$ and $X_C(x, y) = (b^+, d^+)$ is defined in $\mathbb{T}^+$, where $b^+, d^+ \in \mathbb{R}$, $k$ is a positive integer and $\alpha, \beta \in \mathbb{R}$. In this case there exists a choice of parameters of $X_{Ck}$ such that $X_{Ck}$ presents a finite number of limit cycles, given in function of $k$. This result is precisely stated in the following

**Theorem 2** The piecewise smooth vector field $X_{Ck}$ presents at most $k$ limit cycles and this upper bound is reached, for every $k \geq 1$.

**Remark 2** Note that the vector field in the family $X_{Ck}$ can have no limit cycles. In this case, there are sliding regions over the switching manifold and $X_{Ck}$ may present a chaotic behavior, see for instance [17].

### 4 Proof of Main Results

#### 4.1 Preliminary Results

Before to prove the main results of this paper we need some auxiliary results. The next lemma provides the expression of the first return map for $X_{LL}$, $X_{RR}$, $X_{LR}$ and its perturbations.

**Lemma 1** Consider the PSVF given in (5) and the applications given in (3).

(a) If $X_{LL} \in \left[ \Omega^1_{L^+} \cup \Omega^2_{L^+} \right] \cap \left[ \Omega^1_{L^-} \cup \Omega^2_{L^-} \right]$ then the first return map $P_{LL} : \Sigma_1 \rightarrow \Sigma_1$ is well defined and is given by $P_{LL}(x_0) = x_0 + \Delta_{LL}$.

(b) If $X_{LR} \in \left[ \Omega^1_{L^-} \cup \Omega^2_{L^+} \right] \cap \Omega^1_{R^+}$ then the first return map $P_{LR} : \Sigma_1 \rightarrow \Sigma_1$ is well defined and is given by $P_{LR}(x_0) = x_0 + \Delta_{LR}$.

(c) If $X_{RR} \in \left[ \Omega^1_{R^+} \cap \Omega^2_{R^-} \right]$ then the first return map $P_{RR} : \Sigma_1 \rightarrow \Sigma_1$ is well defined and is given by $P_{RR}(x_0) = x_0 + \Delta_{RR}$.

(d) If $X_{LL2^+} \in \left[ \Omega^1_{L^-} \cup \Omega^2_{L^-} \right] \cap \left[ \Omega^1_{L^+} \cup \Omega^2_{L^+} \right]$ then the first return map $P_{LL2^+} : \Sigma_1 \rightarrow \Sigma_1$ is well defined and is given by $P_{LL2^+}(x_0) = x_0 + \Delta_{LL2^+}$.

(e) If $X_{RR1^+} \in \left[ \Omega^2_{R^-} \cap \Omega^3_{R^-} \right]$ and $\varepsilon$ is a small positive number then the first return map $P_{RR1^+} : \Sigma_1 \rightarrow \Sigma_1$ is well defined and is given by

$$P_{RR1^+}(x_0) = \frac{1}{2\sqrt{(\varepsilon - 4\varepsilon)}} \left( \frac{(4\varepsilon - (\varepsilon - 4\varepsilon)^2) \tan^{-1} (\theta^+_1) - \sec^{-1} (\theta^-_3)}{\sqrt{4e^+g^+ - (f^+)^2}} \right)$$

$$- \tan^{-1} \left( \frac{1}{\sqrt{4\varepsilon - (f^+)^2 - 4e^-g^-}} \sqrt{\varepsilon \left( (2x_0 - 1) \left( -((f^-)^2 + 4e^-g^-)) \right) + 4\sqrt{4e^-g^- - (f^-)^2} \tan^{-1} (\theta^-_3) - 4\sqrt{4e^-g^- - (f^-)^2} \sec^{-1} (\theta^-_3) \right) \right).$$

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(f) If $X_{RR2+} \in \left[ \Omega^1_{R-} \cap \Omega^2_{R+} \right]$ then the first return map $P_{RR2+} : \Sigma_1 \to \Sigma_1$ is well defined and is given by $P_{RR2+}(x_0) = x_0 + \Delta_{RR2+}$.

(g) If $X_{RR3+} \in \left[ \Omega^2_{R-} \cap \Omega^2_{R+} \right]$ then the first return map $P_{RR3+} : \Sigma_1 \to \Sigma_1$ is well defined and is given by

$$P_{RR3+}(x_0) = -\frac{1}{\pi} \tan^{-1} \left( \frac{\sqrt{\varepsilon + 1}}{\sqrt{1 - \varepsilon}} \tan \left( \frac{2\pi \sqrt{1 - \varepsilon^2} (\tan^{-1}(\theta^+_2) - \tan^{-1}(\theta^+_1))}{\sqrt{4e^+g^+ - (f^+)^2}} \right) \right)$$

$$- \tan^{-1} \left( \frac{(\varepsilon - 1)}{\sqrt{1 - \varepsilon^2}} \tan \left( \frac{\pi}{(f^-)^2 2e^-g^- - (f^-)^2} \right) \right)$$

$$\tan^{-1}(\theta^-_2) - 2\sqrt{4e^-g^- - (f^-)^2 \sec^{-1}(\theta^-_3)} + 4e^-g^-x_0 - (f^-)^2 x_0 \right) \right) \right).$$

(h) If $X_{LR1-} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-}$ and $\varepsilon$ is a small negative number then the first return map $P_{LR1-} : \Sigma_1 \to \Sigma_1$ is well defined and is given by

$$P_{LR1-}(x_0) = \frac{1}{2} \left( \frac{2 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+c^+ - a^+d^+) + c^+ (a^+ + c^+)}{(c^+)^2} \right)$$

$$\tanh \left( \frac{\sqrt{4 - \varepsilon \sqrt{1 - \varepsilon^2}} \tan^{-1}(\theta^+_2) - \sec^{-1}(\theta^+_3)}{\sqrt{4e^+g^+ - (f^+)^2}} \right) + \tan^{-1} \left( \frac{(2x_0 - 1) \sqrt{\varepsilon}}{\sqrt{\varepsilon - 4}} \right) \right) \right).$$

(i) If $X_{LR2-} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-}$ then the first return map $P_{LR2-} : \Sigma_1 \to \Sigma_1$ is well defined and is given by $P_{LR2-}(x_0) = x_0 + \Delta_{LR2-}$.

(j) If $X_{LR3-} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-}$ then the first return map $P_{LR3-} : \Sigma_1 \to \Sigma_1$ is well defined and is given by

$$P_{LR3-}(x_0) = \frac{1}{2 \pi (c^+)^2} \left( \frac{2 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+c^+ - a^+d^+) + a^+c^+}{(c^+)^2} \right)$$

$$+ 2(c^+)^2 \tan^{-1} \left( \frac{\sqrt{\varepsilon + 1}}{\sqrt{1 - \varepsilon}} \tan \left( \frac{2\pi \sqrt{1 - \varepsilon^2} (\tan^{-1}(\theta^-_2) - \sec^{-1}(\theta^-_3))}{\sqrt{4e^-g^- - (f^-)^2}} \right) \right)$$

$$- \tan^{-1} \left( \frac{(\varepsilon - 1) \tan(\pi x_0)}{\sqrt{1 - \varepsilon^2}} \right) \right) \right).$$

(l) If $X_{LR2+} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-}$ then the first return map $P_{LR2+} : \Sigma_1 \to \Sigma_1$ is well defined and is given by $P_{LR2+}(x_0) = x_0 + \Delta_{LR2+}$.

**Proof** The flow $\phi_X(t)$ where $X$ is one of the vector fields $X_{L0}, X_{R0}, X_{L20}, X_{R10}, X_{R20}, X_{R30}$ passing through $p = (x_0, y_0)$ when $t = 0$ is given, respectively, by

$$\phi_{X_{L0}}(t) = \left( \frac{1}{c^{\omega}} \right)^2 \left( -a^{\omega}c^{\omega}a^{\omega}t + a^{\omega}d^{\omega}e^{\omega t} + a^{\omega}c^{\omega}y^{0}e^{\omega t} - a^{\omega}c^{\omega}y^{0} - a^{\omega}d^{\omega} \right)$$

$$+ b^{\omega}(c^{\omega})^2 t + (c^{\omega})^2 x_0, \frac{d^{\omega}e^{\omega t} + c^{\omega}y^{0}e^{\omega t} - d^{\omega}}{c^{\omega}} \right),$$

$$\phi_{X}(t) = \left( t + x_0, \frac{1}{2 g^{\omega}} \left( \sqrt{4e^{\omega} g^{\omega} - (f^{\omega})^2} \tan \left( \frac{1}{2 t} \sqrt{4e^{\omega} g^{\omega} - (f^{\omega})^2} \right) \right) \right).$$
\[
\phi_{X_{LL}}(t) = \left( \frac{e^{\omega t}(x_0 y_0 + d^{\omega}) - d^{\omega}}{g^{\omega} - (f^{\omega})^2} \right) + 2 e^{\omega t}(x_0 y_0 + d^{\omega})(c^{\omega}(d^{\omega} + \eta_1) - 2d^{\omega} \eta_2) - 2d^{\omega} c^{\omega} d^{\omega} + 2b^{\omega}(c^{\omega})^2 t + 2(c^{\omega})^3 x_0 - 2(c^{\omega})^2 d^{\omega} \eta_1 t - (c^{\omega})^2 \eta_2 y_0^2 - 2(c^{\omega})^2 \eta_1 y_0 + 2c^{\omega}(d^{\omega})^2 \eta_2 t - 2c^{\omega} d^{\omega} \eta_1 + \eta_2 e^{\omega t}(c^{\omega} y_0 + (d^{\omega})^2 + 2c^{\omega} d^{\omega} \eta_2 y_0 + 3(d^{\omega})^2 \eta_2) \),
\]
\[
\phi_{X_{R1}}(t) = \left( \frac{1}{2\varepsilon} \sqrt{(\varepsilon - 4)e} \tan \left( \frac{1}{2} \sqrt{\varepsilon - 4} \right) + \tan^{-1} \left( \frac{2x_0 - 1}{\sqrt{\varepsilon - 4}} \right) \right),
\]
\[
\phi_{X_{R2}}(t) = \left( \frac{1}{2g^{\omega}} \left( - (\eta_1 g^{\omega} - f^{\omega})^2 \right) - \frac{2 \log \left( \frac{1}{2} \sqrt{g^{\omega} - (f^{\omega})^2} \right) + \tan^{-1} \left( \frac{g^{\omega} + 2g^{\omega} y_0}{\sqrt{g^{\omega} - (f^{\omega})^2}} \right) \right) + \log \left( 1 - \frac{(f^{\omega} + 2g^{\omega} y_0)^2}{(f^{\omega})^2 - 4g^{\omega} g^{\omega}} \right)
\]
\[
\phi_{X_{R3}}(t) = \left( \frac{1}{\pi} \tan^{-1} \left( \frac{1}{\pi t} \sqrt{\varepsilon^2 - 1} + \frac{(\varepsilon - 1) \tan^{-1} \left( \frac{\pi}{\sqrt{\varepsilon^2 - 1}} \right)}{\sqrt{\varepsilon^2 - 1}} \right) \right).
\]

In the following we detail the proof for \( X_{LL} \). In this case considering the flow \( \phi_{X_{LL}}(t) = (x_1(t), y_1(t)) \) starting in \( p = (x_0, 0) \in \Sigma_1 \), the smallest positive time \( t_1(p) \) such that \( \phi_{X_{LL}}(t_1(p)) \in \Sigma_2 \) is given by

\[
t_1(p) = \frac{\log \left( \frac{c^-}{2d^-} + 1 \right)}{c^-}.
\]
In this way, we obtain the half first return map $P_L^- : \Sigma_1 \to \Sigma_2$ given by $P_L^-(x_0, 0) = \phi_{X_{L^-}}(t_1(x_0, 0)) = (x_1, 1/2)$.

Considering now the flow $\phi_{X_{L^+}}(t) = (x_2(t), y_2(t))$ and the initial condition $p_1 = (x_1, 1/2)$ the smallest positive time such that $\phi_{X_{L^+}}(t, p_1) \in \Sigma_2$ is given by

$$t_2(p_1) = \frac{\log \left( \frac{e^+ + 2d^+}{e^+} + 1 \right)}{e^+},$$

that provides the upper half first return map $P_L^+ : \Sigma_2 \to \Sigma_1$ given by $P_L^+(x_1, 1/2) = \phi_{X_{L^+}}(t_2(x_1, 1/2)) = (x_2, 0)$. Note that a sufficient condition for $t_1(p)$ and $t_2(p_1)$ are the smallest positive times is that $X_{LL} \subseteq [\Omega_1^1 \cup \Omega_2^2] \cap [\Omega_1^1 \cup \Omega_2^2]$.

Finally, the first return map $P_{LL} : \Sigma_1 \to \Sigma_1$ is given by the composition $P_{LL}(x_0) = (P_L^+ \circ P_L^-)(x_0) = x_0 + \Delta_{LL}$.

Working similarly to the previous case we obtain the domains of definition of the first return map and the respective expressions for the other cases.

4.2 Proof of Theorem 1

Now we are able to perform the proof of Theorem 1. Consider the first return map $P_X$ for each PSVF $X$ treated in this paper. Now we define the displacement map

$$d_X(x_0) = P_X(x_0) - x_0.$$ 

The limit cycles of $X$ are given by simple zeros of $d_X$. The Lemma 1 provide us the first return map for each case. In this way, the proof of items (a), (b), (c), (d), (f), (i) and (l) follows directly because in each one of these cases the first return map is given by $P_X(x_0) = x_0 + \Delta_X$ where $\Delta_X$ is a real number given in terms of coefficients of $X$. Therefore, the iterated of $P_X$ is given by $P_X^k(x_0) = x_0 + k\Delta_X$, or equivalently the dynamics of displacement map is given by $d_X^k(x_0) = k\Delta_X$, where $k$ is an integer number. Considering the equivalent relation (2) that defines the two dimensional torus, we have that $d_X^k(x_0)$ return to $x_0$ if and only if exist $k_0$ an integer such that $k_0\Delta_X$ is an integer number or equivalently $\Delta_X$ is a rational number. Otherwise if $\Delta_X$ is not a rational number then the trajectory passing through $X_0$ never closes. Besides than in the subset of definition $\Omega_X$ given in (5) the PSVF does not present a critical point, by the Poincaré-Bendixson theorem we conclude that all trajectories are dense in torus and the proof follows for these cases.

In the following we detail the prove of items (e) and (h).

The first return map for $X_{RR1^+}$ is given by item (e) of Lemma 1, so the displacement map for this case is

$$d_{X_{RR1^+}}(x_0) = \frac{1}{2} \left( \frac{\sqrt{4 - \varepsilon} \tan \left( \frac{\sqrt{4 - \varepsilon} \tan^{-1} \left( \frac{\xi_1 + (1-2x_0)\sqrt{\xi_2}}{\sqrt{4 - \varepsilon}} \right) \right) - 2x_0 + 1}{\sqrt{4e^{+}\varepsilon + (f^+)^2}} \right),$$

where $\xi_1 = \frac{\sqrt{(4 - \varepsilon)e^{\tan^{-1} \left( \frac{\xi_1^+}{\sqrt{4 - \varepsilon}} \right) - \sec^{-1} \left( \frac{\xi_2^+}{\sqrt{4 - \varepsilon}} \right)}}}{\sqrt{4e^{+}g^{+} + (f^+)^2}}$ and $\xi_2 = \frac{4\sqrt{\varepsilon} \sec^{-1} \left( \frac{\xi_1^+}{\sqrt{4 - \varepsilon}} \right) - \tan^{-1} \left( \frac{\xi_2^+}{\sqrt{4 - \varepsilon}} \right)}{\sqrt{4e^{+}g^{+} - (f^+)^2}}$ if $\varepsilon > 0$. As $X_{RR1^+} \subseteq \Omega_{R^+}^1 \cap \Omega_{R^+}^2$ then $\xi_1 > 0$ and $\xi_2 > 0$. Solving directly the equation $d_{X_{RR1^+}}(x_0) = 0$ we obtain the values

\[ \square \]
\[ x_0^\pm = \pm \csc(\xi_1) \sqrt{(4-\varepsilon)\varepsilon \sin^2(\xi_1) (4\xi_2 \cot(\xi_1) + \xi_2^2 - 4) + \xi_2 \sqrt{(4-\varepsilon)\varepsilon} + 2\varepsilon} / 4\varepsilon, \]

since \( \varepsilon > 0 \).

In fact, the radical \( R_{RR1+} = (4-\varepsilon)\varepsilon \sin^2(\xi_1) (4\xi_2 \cot(\xi_1) + \xi_2^2 - 4) \) is given in function of \( \varepsilon \) and can be written as

\[ R_{RR1+}(\varepsilon) = 16\varepsilon^2 \left( \left( \frac{\sec^{-1}(\theta_3^+) - \tan^{-1}(\theta_1^+)}{(f^+)^2 - 4e^+g^+} \right)^2 \left( \frac{4\sqrt{4e^+g^+} + (f^+)^2(\tan^{-1}(\theta_2^-) - \sec^{-1}(\theta_3^-))}{\sqrt{4e^+g^-} + (f^-)^2(\tan^{-1}(\theta_1^+) - \sec^{-1}(\theta_2^-))} - 4 \right) \right) + O(\varepsilon^{5/2}), \]

as \( X_{RR1+} \in \left[ \Omega^2_R \cap \Omega^3_{R^+} \right] \) then \( R_{RR1+} \) is positive. Therefore if \( \varepsilon > 0 \) there exists two simple zeros of \( d_{X_{RR1+}} \), or equivalently two limit cycles of \( X_{RR1+} \). In Example 1 we explicit a PSVF \( X_{RR1+} \) presenting exactly two limit cycles.

For the case \((h)\) the displacement map of \( X_{LR1-} \) is given by

\[ d_{X_{LR1-}}(x_0) = \frac{1}{2} \left( \xi_3 - 2x_0 + \frac{\tanh \left( \xi_4 + \tan^{-1} \left( \frac{(2x_0 - 1)\sqrt{\varepsilon}}{-\varepsilon - 4} \right) \right)}{\sqrt{\varepsilon}} \right), \]

where \( \xi_3 = \frac{2 \log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+c^+ - a^+d^+) + c^+ (a^+ + c^+) - \left( \frac{t}{\varepsilon + t} \right)^2 \} \) and \( \xi_4 = \frac{\sqrt{4e^+g^-} + (f^-)^2(\tan^{-1}(\theta_1^+ - \sec^{-1}(\theta_2^-))}{\sqrt{4e^+g^-} + (f^-)^2(\tan^{-1}(\theta_1^+) - \sec^{-1}(\theta_2^-))} \)

for \( X_{LR1-} \in \left[ \Omega^1_L \cup \Omega^2_{L^+} \right] \cap \Omega^2_R \) and \( \varepsilon < 0 \). Note that \( \xi_3 > 0 \). Solving directly, the equation \( d_{X_{LR1-}}(x_0) = 0 \) we obtain the values of \( x_0 \):

\[ x_0^\pm = \pm \sqrt{\frac{\varepsilon + 1}{\sqrt{\varepsilon}} \tan \left( \xi_5 + \tan^{-1} \left( \frac{(\varepsilon - 1)\tan((\xi_6 + \pi)x_0)}{\sqrt{1 - \varepsilon}} \right) \right)} - x_0, \]

These values of \( x_0 \) are real numbers since \( \varepsilon < 0 \), \( a^+c^+ > 0 \) and \( b^+c^+ > a^+d^+ \) because the radical \( R_{LR1-} = \varepsilon \left( \frac{4(\xi_3 - 1)c\coth(\xi_4)}{\varepsilon - 1} + ((\xi_3 - 2)\xi_3 + 5)\varepsilon - 16 \right) \) in terms of \( \varepsilon \) can be written as

\[ \sqrt{-\varepsilon} \left[ \frac{4\sqrt{4e^+g^-} - (f^-)^2 \left( \frac{\log \left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+c^+ - a^+d^+) + c^+ (a^+ + c^+) - \left( \frac{t}{\varepsilon + t} \right)^2 \} \right)}{\left( \frac{c^+}{c^+ + 2d^+} + 1 \right) (b^+c^+ - a^+d^+) + a^+c^+ + 16} \right] + O(\varepsilon^{3/2}), \]

so assuming that \( X_{LR1-} \in \left[ \Omega^1_L \cup \Omega^2_{L^+} \right] \cap \Omega^2_R \), \( \varepsilon < 0 \), \( a^+c^+ > 0 \) and \( b^+c^+ > a^+d^+ \) then \( R_{LR1-} > 0 \). An example of \( X_{LR1-} \) with exactly two limit cycle is presented in Example 2.

For the case \( (g) \), similarly to the previous cases, the displacement map for \( X_{RR3+} \) is given by

\[ d_{X_{RR3+}}(x_0) = -\frac{1}{\pi} \left( \tan^{-1} \left( \frac{\sqrt{\varepsilon + 1} \tan \left( \xi_5 + \tan^{-1} \left( \frac{(\varepsilon - 1)\tan((\xi_6 + \pi)x_0)}{\sqrt{1 - \varepsilon}} \right) \right)}{\sqrt{1 - \varepsilon}} \right) - x_0, \]
where \( \xi_5 = \frac{2\pi \sqrt{1-\epsilon^2 (\tan^{-1}(\theta_1^+) - \tan^{-1}(\theta_1^-))}}{\sqrt{4\epsilon^2 + (\epsilon^2 - 1)^2}} \) and \( \xi_6 = \frac{2\pi (\tan^{-1}(\theta_2^-) - \sec^{-1}(\theta_2^-))}{\sqrt{4\epsilon^2 + (\epsilon^2 - 1)^2}} \). As \( X_{RR3+} \in \left[ \Omega^2_{R-} \cap \Omega^1_{R+} \right] \) then \( \xi_5 < 0 \) and \( \xi_6 > 0 \). The solutions of equation \( d_{X_{RR3+}}(x_0) = 0 \) are given by

\[
\begin{align*}
x_0^1 &= \cos^{-1}\left( \frac{\sqrt{1-\epsilon^2 \cot(\xi_5) \sin(\xi_6) - \cos(\xi_6)}}{e} \right) - \xi_6 + k_1, \\
x_0^2 &= \cos^{-1}\left( \frac{\sqrt{1-\epsilon^2 \cot(\xi_5) \sin(\xi_6) - \cos(\xi_6)}}{e} \right) - \xi_6 + k_2,
\end{align*}
\]

where \( k_1, k_2 \) are integer numbers. In the torus, we obtain only two distinct points and the integers \( k_1 \) and \( k_2 \) are the smallest such that \( x_0^1, x_0^2 \in [0, 1] \). In Example 3 we perform a PSVF with two limit cycles.

Remain to prove the item (j) when \( X_{LR3-} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-} \). In this case the displacement map is

\[
d_{X_{LR3-}}(x_0) = \frac{1}{\pi} \left( \tan^{-1}\left( \frac{\sqrt{e + 1} \tan(\xi_7 - \tan^{-1}\left( \frac{(e-1)\tan(\pi x_0)}{\sqrt{1-\epsilon^2}} \right))}{\sqrt{1-\epsilon}} \right) \right) + \xi_8 - x_0,
\]

where \( \xi_7 = \frac{2\pi \sqrt{1-\epsilon^2 (\tan^{-1}(\theta_2^-) - \sec^{-1}(\theta_2^-))}}{\sqrt{4\epsilon^2 + (\epsilon^2 - 1)^2}} \) and \( \xi_8 = \frac{2\log\left( e^{\pi/2} + 1 \right) (b^+ e^+ - a^+ d^+) + a^+ e^+}{2(e^+)^2} \). As \( X_{LR3-} \in \left[ \Omega^1_{L+} \cup \Omega^2_{L+} \right] \cap \Omega^2_{R-} \) then \( \xi_7 > 0 \).

Considering the change of coordinates \( z = \tan(\pi x_0) \) the map \( d_{X_{LR3-}} \) can be written

\[
d_{X_{LR3-}}(z) = \frac{C_1 z (\epsilon + 1)}{C_1 \epsilon + C_1 - z \sqrt{1-\epsilon^2}} + \frac{e + 1}{C_1 \sqrt{1-\epsilon^2} + z (e - 1)} - \frac{C_2 z}{C_2 + z} + \frac{1}{C_2 + z},
\]

where we denote by \( C_1 = \cot(\xi_7) \) and \( C_2 = \cot(\xi_8 \pi) \). Observe that \( d_{X_{LR3-}}(z) \) is given as a linear combination of the functions \( g_0(z) = \frac{1}{C_2 + z}, g_1(z) = \frac{z}{C_2 + z} \) and \( g_2(z) = \frac{1}{C_1 \sqrt{1-\epsilon^2} + z (\epsilon - 1)} \). In fact, it is sufficient to prove that the function \( g_3(z) = \frac{z}{C_1 \epsilon + C_1 - z \sqrt{1-\epsilon^2}} \) is given as a linear combination of the previous one \( g_0, g_1 \) and \( g_2 \). By a direct computation we obtain that the Wronskian \( W_3(g_0, g_1, g_2, g_3)(z) \) is zero. Therefore the set of functions \( \{g_0(z), g_1(z), g_2(z), g_3(z)\} \) is linearly dependent.

Besides than if we consider the ordered set of functions \( \mathcal{F} = \{g_0(z), g_1(z), g_2(z)\} \), the Wronskians \( W_1(z) \) and \( W_2(z) \) are given by

\[
W_1(z) = \frac{1}{(C_2 + z)^2},
\]

\[
W_2(z) = \frac{2(e - 1) \left( C_2 (e - 1) - C_1 \sqrt{1-\epsilon^2} \right)}{(C_2 + z)^3 \left( C_1 \sqrt{1-\epsilon^2} + z (\epsilon - 1) \right)^3}.
\]
So, $W_1(z) \neq 0$ and $W_2(z) \neq 0$ since $\cot(\xi_3\pi) \neq -\cot(\xi_7) \sqrt{1 + \varepsilon}$, i. e.

$$
\cot \left( \frac{2\log \left( \frac{e^{c^+} + 1}{e^{c^+} + d^+} \right) (b^+ c^- - a^+ d^+) + a^+ c^+}{2(c^+)^2} \right) 
\neq \cot \left( \frac{2\pi \sqrt{1 - \varepsilon^2} (\tan^{-1}(\phi_{k+1}) - \sec^{-1}(\phi_{k}))}{\sqrt{4e} \ v_{k} - (f^*)^2} \right) \left( \sqrt{1 + \varepsilon} \right).
$$

Therefore, by Theorem B we obtain that the upper bound of zeros of any linear combination of functions in $\mathcal{F}$ is two and besides that there exists a linear combination of $\mathcal{F}$ presenting exactly two zeros.

In this way, as the displacement map $d_{XL_{R3-}}$ is given as a specific linear combination of functions of $\mathcal{F}$ we guarantee that the upper bound of zeros of $d_{XL_{R3-}}$ is two (as a function of $z = \tan(\pi x_0)$). But if $z_1, z_2$ are the zeros of $d_{XL_{R3-}}$ then there exists $x_0 + k_1$ and $\frac{1}{2} + k_2$ real numbers where $k_1, k_2$ are integer numbers such that $\tan(\pi (x_0^1 + k_1)) = z_1$ and $\tan(\pi (x_0^2 + k_2)) = z_2$. We choose the integers $k_1$ and $k_2$ such that $x_0^1, x_0^2 \in [0, 1]$. Despite the displacement map $d_{XL_{R3-}}$ is a specific linear combination of $g_0, g_1$ and $g_2$ in Example 4 we are able to present values of the parameters such that $XL_{R3-}$ presents exactly two limit cycles.

4.3 Proof of Theorem 2

Recall that $X_{Ck}(x, y) = (X_C(x, y), X_k(x, y)$ is defined on the torus, with $X_C(x, y) = X_C^+(x, y) = (b^+, d^+)$ defined on $\Sigma^+$ and $X_k(x, y) = (\alpha, \beta \cos(2\pi k x))$ is defined on $\Sigma^-$. It is straightforward to obtain the expressions of the flows $\phi_{X_k}(t)$ and $\phi_{X_C}(t)$ of $X_k$ and $X_C^+$ respectively, passing in $p = (x_0, y_0)$ for $t = 0$, namely

$$
\phi_{X_k}(t) = \left( \alpha t + x_0, y_0 - \frac{\beta \sin(2\pi k x_0)}{2k\pi \alpha} \frac{\beta \sin(2\pi \alpha t + 2\pi k x_0)}{2k\pi \alpha} \right),
$$

$$
\phi_{X_C}(t) = \left( b^+ t + x_0, d^+ t + y_0 \right).
$$

The fly maps $P_{X_k} : \{ (x, y) \in Q; \ y = 0 \} \rightarrow \{ (x, y) \in Q; \ y = 1/2 \}$ of $\phi_{X_k}$ and $P_{X_C} : \{ (x, y) \in Q; \ y = 1/2 \} \rightarrow \{ (x, y) \in Q; \ y = 1 \}$ of $\phi_{X_C}$ are given by

$$
P_{X_k}(x_0, 0) = \left( \frac{\arcsin \left( \frac{k\pi \alpha + \beta \sin(2\pi k x_0)}{2k\pi} \right)}{2}, 1 \right),
$$

$$
P_{X_C}^+(x_0, 1/2) = \left( b^+ + 2d^+ x_0, \frac{2b}{2b} \right),
$$

where $m \in \{0, \ldots, k\}$ is such that $x_0 \in [m/k, (m + 1)/k]$. Thus the Poincaré map $P_{X_{Ck}} : \{ (x, y) \in Q; \ y = 0 \} \rightarrow \{ (x, y) \in Q; \ y = 1 \}$ is given by $P_{X_{Ck}}(x_0, 0) = (P_{X_C} \circ P_{X_k})(x_0, 0) = (P_1(x_0), 1)$, with

$$
P_1(x_0) = \frac{1}{2k\pi} \arcsin \left( \frac{\sin(2k\pi x_0) + k\alpha \pi / \beta}{\beta} \right) + \frac{b^+}{2d^+} + \frac{m}{k}.
$$

So to find limit orbit we have to find the simples zeros of the displacement map

$$
d_{XL_{k}}(x_0) = P_1(x_0) - x_0
$$
for \( x_0 \in [0, 1] \). Now we show that for every \( m = 0, \ldots, k - 1 \), there are at most one solution for (7) with \( x_0 \in (m/k, (m + 1)/k) \), thus there is at most \( k \) limit cycles for \( X_{Lk} \).

Now we study the solutions of

\[
d^+ \arcsin \left( \frac{k\pi \alpha + \beta \sin(2\pi k x_0)}{\beta} \right) = 2x_0d^+k\pi - b^+k\pi - 2md^+\pi \tag{8}
\]

for \( x \in [0, 1] \), where \( m \in \mathbb{Z} \) is such that \( x \in (m/k, (m + 1)/k) \).

The left hand side of Eq. (8) has \( k \) branches going from \(-\infty \) to \(+\infty \), and in the right hand side we have a straight line. So the intersection of branches with the straight line produce \( k \) solutions on the interval \([0, 1]\). Without loss of generality in what follows we consider the case \( k = 1 \).

Before we conclude the analysis we discuss the tangency points of \( X_{Ck} \). For \( k = 1 \), the tangency points are \((1/4, 0)\), \((3/4, 0)\), \((1/4, 1/2)\) and \((3/4, 1/2)\). According to the signals of \( b^+, d^+, \alpha \), and \( \beta \), the segments between these points interchange between sliding (stable or unstable) and crossing.

Suppose \( b^+ \cdot \alpha < 0 \) and \( d^+ \cdot \beta < 0 \). Then the segments \([1/4, 3/4] \times \{0\}\) and \([1/4, 3/4] \times \{1/2\}\) are crossing regions. Solutions passing outside these segments cannot be limit cycles.

Note that the function \( g(x_0) = d^+ \arcsin \left( \frac{k\pi \alpha + \beta \sin(2\pi k x_0)}{\beta} \right) \) have two critical points, \( 1/4 \) and \( 3/4 \), so it is monotone on \((1/4, 3/4)\). Therefore the straight line \(2x_0d^+k\pi - b^+k\pi - 2md^+\pi \) meets the graph of \( g \) at most in one point. Thus there are at most 1 limit cycles for \( X_{Ck} \) with \( k = 1 \). It is easy to see that there are at most \( k \) limit cycles for \( X_{Ck} \). In Example 5 we provide values to the coefficients \( \alpha, \beta, b^+, d^+ \) for the \( X_{Ck} \) presents one limit cycle for \( k = 1 \).

## 5 Final Remarks and Some Examples

In the present section we exhibit explicitly values of the parameters of PSVF such that \( X_{RR1+}, X_{RR3+}, X_{LR1-} \) and \( X_{LR3-} \) present the upper bound, in each case, of the number of limit cycles.

**Example 1** If we consider the values of parameters \( e^+ = 0.466532, f^+ = 0.1, g^+ = 0.541227, e^- = -0.35481, f^- = 0.4, g^- = -0.817339 \) and \( \varepsilon = 0.02 \) the displacement map associated to \( X_{RR1+}(x, y) \) given in (4) is

\[
d_{X_{RR1+}}(x_0) = 7.05337 \tan (0.217 - \tan^{-1}(0.290888 - 0.141776 x_0)) - x_0 + 0.5.
\]

Solving the equation \( d_{X_{RR1+}}(x_0) = 0 \) we obtain the points \( x_0^1 = 0.571897 \) and \( x_0^2 = 1.97984 \) which represent the points in the torus: \( y_0^1 = 0.571897 \) and \( y_0^2 = 0.97984 \). In other words, this means that the trajectory passing through \( y_0^1 \) rotates one time before return to \( y_0^2 \).

**Example 2** With the values: \( a^+ = -4.47442, b^+ = 0, e^+ = 1, d^+ = 1, e^- = 0.0355785, f^- = \sqrt{3}, g^- = 28.1069 \) and \( \varepsilon = -0.08 \) the displacement map associated to \( X_{LR11}(x, y) \) given in (4) is

\[
d_{X_{LR1-}}(x_0) = -x_0 + 3.57071 \tanh (\tanh^{-1}(0.140028(2x_0 - 1)) + 0.28) - 0.45,
\]

and the solutions, it the torus, of \( d_{X_{LR1-}}(x_0) = 0 \) are given by the points \( x_0^1 = 0.286257 \) and \( x_0^2 = 0.763743 \).
Example 3 Putting the values of parameters $e^+ = 0.584555$, $f^+ = 0.130158$, $g^+ = 0.434921$, $e^- = 0.670355$, $f^- = \sqrt{3}$, $g^- = 1.49175$ and $\epsilon = 0.06$ the displacement map associated to $X_{RR^3}(x, y)$ given in (4) is

$$d_{X_{RR^3}}(x_0) = \tan^{-1}\left(\frac{1.06191 \tan^{-1}(0.941697 \tan(\pi x_0 + 1.4)) + 1.7}{\pi}\right) - x_0,$$

and the solutions of $d_{X_{RR^3}}(x_0) = 0$ are obtained the points $x_0^1 = 0.15119$ and $x_0^2 = 0.403176$.

Example 4 Considering the values $a^+ = 1.21371$, $b^+ = 1$, $c^+ = 1$, $d^+ = 0$, $e^- = 0.250971$, $f^- = \sqrt{3}$, $g^- = 3.98453$ and $\epsilon = 0.04$ the displacement map associated to $X_{LR^3-}(x, y)$ given in (4) is

$$d_{X_{LR^3-}}(x_0) = -x_0 + 0.31831 \tan^{-1}\left(1.04083 \tan\left(\tan^{-1}(0.960769 \tan(\pi x_0)) + 2.2\right) + 1.3\right),$$

and the solutions, in the torus, of $d_{X_{LR^3-}}(x_0) = 0$ are given by the points $x_0^1 = 0.397519$ and $x_0^2 = 0.902481$, see Fig. 1.

Example 5 Finally we provide an example with exactly $k$ limit cycles for $X_C^k$ (see Theorem 2). Given the vector field $X_k(x, y) = (\alpha, \beta \cos(2k\pi x))$ with $\alpha, \beta > 0$ and $k > 0$ an integer, we will construct a vector field $X_C(x, y) = (b^+, d^+)$ with a limit cycle. Note that for every $m = 0, \ldots, k - 1$ we have

$$P_{X_k}\left(\frac{1}{8k} + \frac{m}{k}, \frac{1}{2}\right) = \left(\frac{1}{2k\pi} \arcsin\left(\frac{k\pi \alpha}{\beta} + \frac{\sqrt{2}}{2}\right) + \frac{m}{k}, \frac{1}{2}\right),$$

so the first restriction is that $-1 < \frac{k\pi \alpha}{\beta} + \frac{\sqrt{2}}{2} < 1$. Now we fix $m = 0$ and prove that we have at least one limit cycle for $x \in [0, 1/k]$ (for every $k$).

Let $\Delta_C^k = \frac{1}{8k} - \frac{1}{2k\pi} \arcsin\left(\frac{2k\pi \alpha + \beta \sqrt{2}}{2\beta}\right)$ and consider $X_C(x, y) = (\Delta_C^k, 1/2)$, i.e., $b^+ = \Delta_C^k$ and $d^+ = 1/2$. By construction, $P_{X_C^k}(1/8k, 0) = (1/8k, 1)$, so we have a
fixed point for the Poincaré map of $X_{Ck}$; thus, a limit cycle. The derivative of the Poincaré map on this fixed point is equal to

$$P'_1(1/8k) = \left( \sqrt{\frac{\beta^2 - 2\sqrt{\pi} \alpha \beta k - 2\pi \alpha^2 k^2}{\beta^2}} \right)^{-1},$$

that is not zero under generic conditions. So this is an isolated fixed point, and the limit cycle is isolated, see Fig. 2. Thus we have exactly $k$ limit cycles.

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