CONTINUITY OF THE DUAL HAAR MEASURE

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ABSTRACT. Given a locally compact group bundle, we show that the system of the Plancherel weights of their C*-algebras is lower semi-continuous. As a corollary, we obtain that the dual Haar system of a continuous Haar system of a locally compact abelian group bundle is also continuous.

Let $G$ be a locally compact abelian group with Haar measure $\lambda$. The dual Haar measure $\hat{\lambda}$ on the dual group $\hat{G}$ is the Haar measure on $\hat{G}$ which makes the Fourier transform defined by

$$F(f)(\chi) = \int f(\gamma)\overline{\chi(\gamma)}d\lambda(\gamma)$$

for $f$ in the space $C_c(G)$ of complex-valued continuous functions with compact support on $G$, an isometry from $L^2(G, \lambda)$ to $L^2(\hat{G}, \hat{\lambda})$. Suppose now that $p : G \to X$ is a locally compact abelian group bundle. Here, we mean that $G$ and $X$ are locally compact Hausdorff spaces, $p$ is a continuous surjection and the fibres $G_x = p^{-1}(x)$ are abelian groups. We also require that the multiplication, as a map from $G \times_X G$ to $G$, and the inverse, as a map from $G$ to $G$ are continuous. A Haar system for $G$ is a family of measures $(\lambda_x)_{x \in X}$, where for all $x \in X$, $\lambda_x$ is a Haar measure of $G_x$; it is said to be continuous if for all $f \in C_c(G)$, the function $x \mapsto \int fd\lambda_x$ is continuous. Then we can form the dual group bundle $\hat{p} : \hat{G} \to X$. As a set, $\hat{G}$ is the disjoint union of the dual groups $\hat{G}_x$. Moreover, it can be identified with the spectrum of the commutative C*-algebra $C^*(G)$, where $G$ is viewed as a locally compact groupoid with Haar system $(\lambda_x)_{x \in X}$. Hence it is endowed with a locally compact topology. It can be checked ([12, Corollary 3.4]) that $\hat{p} : \hat{G} \to X$ is indeed a locally compact abelian group bundle in the above sense. Of course, one expects that the dual Haar system $(\hat{\lambda}_x)_{x \in X}$ is continuous. This is stated as Proposition 3.6 of [12]. However it was recently pointed to the authors by Henrik Kreidler that their proof is defective. This note corrects this and gives a more general result, based on the fact that the dual Haar measure $\hat{\lambda}$ on the dual group $\hat{G}$ of a locally compact abelian group $G$, viewed as its Haar weight and defined as the canonical weight of $L^\infty(\hat{G})$ acting on $L^2(\hat{G}, \hat{\lambda})$, corresponds under the Fourier transform to the Plancherel weight of $G$, defined as the canonical weight of the von Neumann algebra $VN(G)$ of $G$ acting on $L^2(G, \lambda)$. Therefore one can consider a locally compact group bundle $p : G \to X$ where the fibres $G_x$ are no longer abelian. Our main result is Proposition 2.3 which says that the Plancherel weight of $G_x$ varies continuously in a suitable sense. This lead us to the definition of a lower semi-continuous $C_0(X)$-weight on a $C_0(X)$-C*-algebra which we illustrate by three examples.
The first section recalls the construction of the canonical weight of a left Hilbert algebra and the properties which are needed to prove the crucial Corollary 1.6. In the second section, we consider the case of a locally compact group bundle and prove our main result. This example motivates the definition of a $C_0(X)$-weight of a $C_0(X)$-$C^*$-algebra, given in the third section. The fact that the Plancherel $C_0(X)$-weight of the $C^*$-algebra of a group bundle is densely defined and lower semi-continuous gives Proposition 3.6 of [12].

1. The Plancherel weight of a locally compact group

We recall first some elements of Tomita-Takesaki’s theory, using the standard notation from [16]. Given a left Hilbert algebra $A$ where the product, the involution and the scalar product are respectively denoted by $ab, a^*,$ and $(a|b)$, we denote by $H$ the Hilbert space completion of $A$ and by $\pi : A \rightarrow \mathcal{L}(H)$ the left representation. We denote by $M = \pi(A)^{\prime\prime}$ the left von Neumann algebra of $A$. We denote by $S$ the closure of the involution $a \mapsto a^*$ and by $F$ its adjoint. The domain of $S$ [resp. $F$] is denoted by $D^S$ [resp. $D^F$] and one writes $\xi^S = S\xi$ [resp. $\eta^S = F\eta$] for $\xi \in D^S$ [resp. for $\eta \in D^F$]. An element $\eta \in H$ is called right bounded if there exists a bounded operator $\pi'(\eta)$ on $H$ such that $\pi'(\eta)a = \pi(a)\eta$ for all $a \in A$. One then writes $\xi\eta = \pi'(\eta)\xi$ for all $\xi \in H$. The set of right bounded elements is denoted by $B'$. One shows that $A' = B' \cap D^F$ with involution $\xi^F$ is a right Hilbert algebra in the same Hilbert space $H$. An element $\xi \in H$ is called left bounded if there exists a bounded operator $\pi(\xi)$ on $H$ such that $\pi(\xi)\eta = \pi'(\eta)\xi$ for all $\eta \in A'$. One then writes $\xi\eta = \pi(\xi)\eta$ for all $\eta \in H$. If $\xi$ is left bounded and $\eta$ is right bounded, the notation is consistent: $\pi(\xi)\eta = \xi\eta = \pi'(\eta)\xi$. The set of left bounded elements is denoted by $B$. We shall need the following well-known lemmas.

Lemma 1.1. Let $\mathcal{B}$ be the set of left bounded elements of $H$. Then,

(i) $\mathcal{B}$ is a linear subspace of $H$ containing $A$.

(ii) $\pi(\mathcal{B})$ is contained in the left von Neumann algebra $M$.

(iii) $\mathcal{B}$ is stable under $M$. More precisely, for $T \in M$ and $\xi \in \mathcal{B}$, $\pi(T\xi) = T\pi(\xi)$.

Proof. See [2, Section 2].

Lemma 1.2. Let $\xi, \xi' \in \mathcal{B}$ such that $\pi(\xi') = \pi(\xi)^*$. Then $\xi$ belongs to $D^S$ and $\xi' = \xi$. 


Proof. For all \( \eta_1, \eta_2 \in \mathcal{A}' \), one has:

\[
(\xi | \eta_1 \eta_2^\flat) = (\xi | \pi'(\eta_2^\flat)\eta_1) = (\xi | \pi'(\eta_2)^*\eta_1)
= (\pi'(\eta_2)\xi | \eta_1) = (\pi(\xi)\eta_2 | \eta_1) = (\pi(\xi)^*\eta_2^\flat | \eta_1)
= (\eta_2 | \pi'(\eta_1)\xi^I)
= (\pi'(\eta_1)^*\eta_2 | \xi^I)
= (\pi'(\eta_1^I)\eta_2 | \xi^I)
= (\eta_2^I \eta_1^I | \xi^I)
\]

According to [16, Lemma 3.4] and the comments which follow, this suffices to conclude. \( \Box \)

**Definition 1.3.** The canonical weight \( \tau \) of the left Hilbert algebra \( \mathcal{A} \) is the map \( \tau: \mathcal{M}_+ \rightarrow [0, \infty] \) defined by

\[
\tau(T) = \begin{cases} 
\|\xi\|^2 & \text{if } \exists \xi \in \mathcal{B} : T = \pi(\xi)^*\pi(\xi) \\
\infty & \text{otherwise}
\end{cases}
\]

**Lemma 1.4.** [2, Théorème 2.11] The above canonical weight \( \tau \) is well-defined and is a faithful, semi-finite, \( \sigma \)-weakly lower semi-continuous weight on \( \mathcal{M}_+ \).

The following lemma is certainly well-known but I did not find a reference for it.

**Lemma 1.5.** For all \( \xi \in \mathcal{B} \), one has

\[
\tau(\pi(\xi)\pi(\xi)^*) = \begin{cases} 
\|\xi^I\|^2 & \text{if } \xi \in \mathcal{D}^I \\
\infty & \text{otherwise}
\end{cases}
\]

**Proof.** Suppose that \( \xi \in \mathcal{D}^I \). Then \( \xi \) belongs to the full Hilbert algebra \( \mathcal{A}'' = \mathcal{B} \cap \mathcal{D}^I \) of \( \mathcal{A} \). Therefore, \( \xi^I \) is also left bounded and \( \pi(\xi)^* = \pi(\xi^I) \). Thus,

\[
\tau(\pi(\xi)\pi(\xi)^*) = \tau(\pi(\xi^I)^*\pi(\xi^I)) = \|\xi^I\|^2
\]

Suppose now that the left handside is finite. Then there exists \( \eta \in \mathcal{B} \) such that \( \pi(\xi)\pi(\xi)^* = \pi(\eta)^*\pi(\eta) \). Let \( \pi(\xi) = U|\pi(\xi)| \) [resp. \( \pi(\eta) = V|\pi(\eta)| \)] be the polar decomposition of \( \pi(\xi) \) [resp. \( \pi(\eta) \)]. All these operators belong to the left von Neumann algebra \( \mathcal{M} \). Our assumption implies that \( |\pi(\eta)| = U|\pi(\xi)|U^* \). Therefore,

\[
(U^*V^*\pi(\eta))^* = \pi(\eta)^*VU = |\pi(\eta)|V^*VU = |\pi(\eta)|U = U|\pi(\xi)| = \pi(\xi)
\]

Moreover, according to Lemma 1.1, \( U^*V^*\pi(\eta) = \pi(U^*V^*\eta) \) and \( \zeta = U^*V^*\eta \) is left bounded. Since \( \pi(\xi) = \pi(\xi)^* \) we deduce from Lemma 1.2 that \( \xi \in \mathcal{D}^I \). \( \Box \)
We consider now the left Hilbert algebra associated with the left regular representation of a locally compact group $G$ endowed with a left Haar measure $\lambda$. Using the framework of [14, pages 56-58], we choose the left Hilbert algebra $A = C_c(G)$, where the product is the usual convolution product, the involution, denoted by $f^*$ instead of $f^\#$, is $f^*(\gamma) = f(\gamma^{-1})$ and the scalar product is

$$(f | g) = \int f\overline{g}d\lambda^{-1}.$$ 

The left representation is denoted by $L$ instead of $\pi$. It acts on the Hilbert space $H = L^2(G, \lambda^{-1})$ by $L(f)g = f * g$ for $f, g \in C_c(G)$. The left von Neumann algebra $M$ is the group von Neumann algebra $VN(G)$. The canonical weight $\tau$ of this left Hilbert algebra is also called the Plancherel weight of the group $G$ ([15, 17]). We also refer the reader to [6, Section 2] for a deep study of its properties. It satisfies

$$\tau(L(f^* * f)) = \|f\|^2 = \int |f|^2d\lambda^{-1} = (f^* * f)(e)$$

for $f \in C_c(G)$. When $G$ is abelian, we can use the Fourier transform to compute the Plancherel weight (see [17, VII.3]). The canonical weight $\hat{\tau}$ of the commutative Hilbert algebra $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$ is the integral with respect to the dual Haar measure $\hat{\lambda}$ on $L^\infty(\hat{G})$. This weight is called the Haar weight of the dual group $\hat{G}$. The Fourier transform $\mathcal{F}$ implements an isomorphism between the full left Hilbert algebra of the regular representation of $G$ and the Hilbert algebra $L^2(\hat{G}, \hat{\lambda}) \cap L^\infty(\hat{G})$. Therefore, $\tau(T) = \hat{\tau}((\mathcal{F} \circ T \circ \mathcal{F}^{-1})$ for $T \in VN(G)$. One has in particular

$$\tau(L(a)) = \int \mathcal{G}(a)d\hat{\lambda},$$

for $a \in C^*(G)_+$ where $\mathcal{G} : C^*(G) \to C_0(\hat{G})$ is the Gelfand transform.

Let us express Lemma 1.5 in the case of the left regular representation of a locally compact group $G$ with a left Haar measure $\lambda$.

**Corollary 1.6.** For all left bounded element $\xi$ of $L^2(G, \lambda^{-1})$, one has

$$\tau(L(\xi)L(\xi)^*) = \int |\xi|^2d\lambda.$$

**Remark 1.7.** One can give a more direct proof of this result since the equality $L(\xi)^*\eta = \xi^* \eta$ can be established by usual integration techniques and [5, Lemme 3.1]. However, its natural framework is Tomita-Takesaki’s theory.

## 2. The Plancherel weight of a group bundle

We consider now the case of a locally compact group bundle $p : G \to X$. We assume that the groups $G_x$ have a left Haar measure $\lambda^x$ such that $x \mapsto \lambda^x$ is continuous. This is the particular case of a locally compact groupoid with Haar system where the range and source maps coincide. Thus, we can construct its $C^*$-algebra $C^*(G)$ as usual. If the groups $G_x$ are abelian, it is a commutative $C^*$-algebra. Then the Gelfand transform identifies it with $C_0(\hat{G})$. The space $\hat{G}$
is the total space of the bundle \( \hat{\rho} : \hat{G} \to X \), where the fibre \( \hat{G}_x \) is the dual group of \( G_x \). For \( a \in C^*(G)_+ \) and \( x \in X \), we define

\[
T(a)(x) \overset{\text{def}}{=} \int \mathcal{G}(a)d\hat{\lambda} = \int \mathcal{G}_x(a_x)d\hat{\lambda}_x = \tau_x \circ L_x(a_x),
\]

where \( L_x \) is the left regular representation of \( G_x \) and \( \tau_x \) is the Plancherel weight of \( G_x \). The last expression is defined when the groups are non-abelian and we turn now to this case.

**Lemma 2.1.** Let \( G \to X \) be a locally compact group bundle with Haar system \( \lambda = (\lambda^x)_{x \in X} \). Then,

(i) \( x \mapsto C^*(G_x) \) is an upper semi-continuous field of \( C^* \)-algebras;

(ii) its sectional algebra is \( C^*(G) \).

**Proof.** See for example [10, Section 5]. \( \square \)

Equivalently, this lemma says that \( C^*(G) \) is a \( C_0(X) \)-\( C^* \)-algebra. We shall view an element \( a \) of \( C^*(G) \) as a continuous field \( x \mapsto a_x \), where \( a_x \in C^*(G_x) \).

For \( a \in C^*(G)_+ \), we define as in the commutative case the function

\[
T(a) : x \mapsto \tau_x(L_x(a_x)),
\]

where \( L_x \) is the left regular representation of \( C^*(G_x) \) and \( \tau_x \) is the Plancherel weight of \( G_x \). Our main result will be that the function \( x \mapsto T(a)(x) \) is lower semi-continuous. We consider now weights on \( C^* \)-algebras rather than von Neumann algebras and refer the reader to [1] for the main definitions and results.

**Lemma 2.2.** Let \( \varphi : A_+ \to [0, \infty] \) be a weight on a \( C^* \)-algebra \( A \). Assume that \((e_i)_{i \in I}\) is an approximate identity for \( A \) such that \( \|e_i\| \leq 1 \) for all \( i \in I \). Let \( a \in A_+ \). Then,

(i) \( \varphi(a^{1/2}e_ia^{1/2}) \leq \varphi(a) \);

(ii) if \( \varphi \) is lower semi-continuous, then \( \varphi(a) = \sup_i \varphi(a^{1/2}e_ia^{1/2}) \).

**Proof.** For (i), we have \( a^{1/2}e_ia^{1/2} \leq \|e_i\|a \leq a \). Since \( \varphi \) is increasing, we obtain the desired inequality. For (ii), since \( a^{1/2}e_ia^{1/2} \) converges to \( a \), we have

\[
\varphi(a) \leq \lim \inf \varphi(a^{1/2}e_ia^{1/2}) \leq \sup \varphi(a^{1/2}e_ia^{1/2}) \leq \varphi(a)
\]

\( \square \)

**Proposition 2.3.** Let \( G \to X \) be a locally compact group bundle with Haar system \( \lambda = (\lambda^x)_{x \in X} \). Given \( a \in C^*(G)_+ \), the function

\[
T(a) : x \mapsto \tau_x(L_x(a_x))
\]

is lower semi-continuous.

**Proof.** Proposition 2.10 of [11] gives the existence of an approximate unit \((e_i)\) of \( C^*(G) \) where each \( e_i \) is a finite sum of elements of the form \( f \ast f^* \) with \( f \in C_c(G) \); moreover, according to its construction, \( \|e_i\|_I \) tends to 1, where \( \|f\|_I = \max(\sup_x \int |f|d\lambda^x, \sup_x \int |f|d\lambda_x) \) and \( \lambda_x = (\lambda^x)^{-1} \). Replacing \( e_i \) by \((1/\|e_i\|_I)e_i\), we can assume that \( \|e_i\|_I = 1 \) for all \( i \in I \). We then have \( \|e_i\| \leq \|e_i\|_I \leq 1 \). For
all \(x \in X\), the image \((e_i(x))\) of \((e_i)\) in \(C^*(G_x)\) satisfies the same assumptions. Since the Plancherel weight \(\tau_x\) is \(\sigma\)-weakly lower semi-continuous, we have

\[
\tau_x(L_x(a_x)) = \sup_i \tau_x(L_x(a_x^{1/2}e_i(x)a_x^{1/2})).
\]

We will show that the function \(x \mapsto \tau_x(L_x(a_x^{1/2}e_i(x)a_x^{1/2}))\) is lower semi-continuous. This will imply that the function \(x \mapsto \tau_x(L_x(a_x))\) is lower semi-continuous as lower upper bound of a family of lower semi-continuous functions. It suffices to show that \(x \mapsto \tau_x(L_x(a_x^{1/2}*(f*f)_x*a_x^{1/2}))\) is lower semi-continuous for all \(f \in C_c(G)\).

Let us fix \(x \in X\). According to Lemma 1.1, \(f_x\) and \(g_x = L_x(a_x^{1/2})f_x\) are left bounded elements of \(L^2(G_x, \lambda_x)\). Thus we have according to Lemma 1.5 :

\[
\tau_x(L_x(a_x^{1/2}*(f*f)_x*a_x^{1/2})) = \tau_x(L_x(g_x)L_x(g_x)^*) = \int |g_x(\gamma)|^2d\lambda^x(\gamma).
\]

Note that \(g : x \mapsto g_x\) is an element of the \(C_0(X)\)-Hilbert module \(L^2(G, \lambda^{-1})\) (see [7, Section 2]). If the groups \(G_x\) are unimodular, then

\[
\int |g(\gamma)|^2d\lambda^x(\gamma) = \int |g(\gamma)|^2d\lambda_x(\gamma)
\]

depends continuously on \(x\). If not, we first observe that for all \(\rho \in C_c(G)\), \(\rho g\) belongs to \(L^2(G, \lambda^{-1})\). Let \(D : G \to \mathbb{R}^+_1\) be the modular function of the group bundle \(G\). It is a continuous homomorphism such that \(\lambda^x = D\lambda_x\) for all \(x \in X\) (see [8, Lemma 2.4]). There exists an increasing sequence \((D_n)\) of continuous non-negative functions with compact support which converges pointwise to \(D\). Then, for all \(x \in X\), \(\int D_n(\gamma)|g(\gamma)|^2d\lambda_x(\gamma)\) is an increasing sequence which converges to

\[
\int D(\gamma)|g(\gamma)|^2d\lambda_x(\gamma) = \int |g(\gamma)|^2d\lambda^x(\gamma).
\]

Therefore, the function \(x \mapsto \int |g(\gamma)|^2d\lambda^x(\gamma)\) is lower semi-continuous as a limit of an increasing sequence of continuous functions. \(\square\)

### 3. \(C_0(X)\)-WEIGHTS ON \(C_0(X)\)-C*-ALGEBRAS

In this section, given a topological space \(X\), \(LSC(X)_+\) denotes the convex cone of lower semi-continuous functions \(f : X \to [0, \infty]\).

**Definition 3.1.** Let \(X\) be a locally compact Hausdorff space. A \(C_0(X)\)-weight on a \(C_0(X)\)-C*-algebra \(A\) is a map

\[
\Phi : A_+ \to LSC(X)_+
\]

such that

(i) \(\Phi(a + b) = \Phi(a) + \Phi(b)\) for all \(a, b \in A_+\);

(ii) \(\Phi(ha) = h\Phi(a)\) for all \(h \in C_0(X)_+\) and \(a \in A_+\).

It is called lower semi-continuous if \(a_n \to a\) implies \(\Phi(a) \leq \liminf \Phi(a_n)\) and densely defined if its domain, defined as

\[
P = \{a \in A_+ : \Phi(a) \text{ is finite and continuous}\}
\]

is dense in \(A_+\).
In order to include the first example, we need to modify slightly this definition. Given a topological space \( X \), we denote by \( C_b(X) \) the space of complex-valued bounded continuous functions on \( X \). We say that a C*-algebra \( A \) is a \( C_b(X) \)-algebra if it is endowed with a nondegenerate morphism of \( C_b(X) \) into the centre of the multiplier algebra of \( A \). Then we define a \( C_b(X) \)-weight by replacing \( C_0(X) \) by \( C_b(X) \) in the above definition.

**Remark 3.2.** The above definition of a \( C_0(X) \)-weight on a \( C_0(X) \)-C*-algebra is motivated by the examples below; it is different from the definition of a C*-valued weight given in [9].

**Lemma 3.3.** The domain \( P \) of a \( C_b(X) \)-weight or of \( C_0(X) \)-weight is hereditary: if \( 0 \leq b \leq a \) and \( a \in P \), then \( b \in P \).

**Proof.** See [4, Lemme 4.4.2.i]. \( \square \)

**Example 3.4.** The canonical center-valued trace (cf [3, Section 3.4]). Let \( A \) be a C*-algebra and let \( X = \text{Prim}(A) \) its primitive ideal space endowed with the Jacobson topology. Through the Dauns-Hofmann theorem, we view \( A \) as a \( C_b(X) \)-algebra. Given \( a \in A_+ \), according to [13, Proposition 4.4.9], one can define \( \Phi(a)(P) = \text{Trace}(\pi(a)) \) where \( P \) is a primitive ideal and \( \pi \) is any irreducible representation admitting \( P \) as kernel, and \( \Phi(a) \) is lower semi-continuous on \( X \). Condition (i) of the definition results from the additivity of the trace and condition (ii) from the irreducibility of \( \pi \). Thus \( \Phi : A_+ \to LSC(X)_+ \) is a \( C_b(X) \)-weight. Since the usual Trace is \( \sigma \)-weakly lower semi-continuous, \( \Phi \) is lower semi-continuous. By definition, \( \Phi \) is densely defined if and only if \( A \) is a continuous-trace C*-algebra.

**Example 3.5.** The Plancherel \( C_0(X) \)-weight of a group bundle. This is the example described in the previous section:

**Theorem 3.6.** Let \( p : G \to X \) be a locally compact group bundle endowed with a continuous Haar system. Then the map

\[
\mathcal{T} : C^*(G)_+ \to LSC(X)_+
\]

such that \( \mathcal{T}(a)(x) = \tau_x(L_x(a_x)) \) for \( a \in C^*(G)_+ \) and where \( \tau_x \) is the Plancherel weight of \( G_x \) and \( L_x \) the left regular representation of \( G_x \) is a densely defined lower semi-continuous \( C_0(X) \)-weight, which we call the Plancherel \( C_0(X) \)-weight of the group bundle.

**Proof.** The main point was to show that the range of this map is contained in \( LSC(X)_+ \), which was done in Proposition 2.3. The conditions (i) and (ii) of the definition are clear. It is densely defined since its domain contains the elements \( f^* * f \) where \( f \in C_c(G) \), whose linear span is dense in \( C^*(G) \). The lower semi-continuity of \( \tau_x \circ L_x \) for all \( x \in X \) gives the lower semi-continuity of \( \mathcal{T} \). \( \square \)

**Example 3.7.** The commutative case. Let \( \pi : Y \to X \) be a continuous, open and surjective map, where \( Y \) and \( X \) are locally compact Hausdorff spaces. For \( x \in X \), let \( Y_x = \pi^{-1}(x) \) be the fibre over \( x \). Endowed with the fundamental family
of continuous sections $C_c(Y)$, $x \mapsto C_0(Y_x)$ is a continuous field of $C^*$-algebras. Its $C^*$-algebra of continuous sections is identified to $C_0(Y)$. Thus $C_0(Y)$ is a $C_0(X)$-$C^*$-algebra. The following proposition gives a description of the densely defined lower semi-continuous $C_0(X)$-weights of $C_0(Y)$.

**Proposition 3.8.** In the above situation, we have natural one-to-one correspondence between

(i) the continuous $\pi$-systems of measures, by which we mean families of Radon measures $\alpha = (\alpha_x)_{x \in X}$ on $Y$, where $\alpha_x$ is supported on $\pi^{-1}(x)$, and such that for all $f \in C_c(Y)$, the function $x \mapsto \int f \, d\alpha_x$ is continuous;

(ii) the densely defined and lower semi-continuous $C_0(X)$-weights of $C_0(Y)$.

**Proof.** When $X$ is reduced to a point, this is a well-known result given for example in the introduction of [1]. A crucial point which we shall use again below is that a densely defined weight on $C_0(Y)$ is necessarily finite on $C_c(Y)_+$ because the linear span of its domain of definition is a dense ideal, hence it contains the minimal dense ideal, called the Pedersen ideal, which in our case is $C_r(Y)$ ([13, 5.6.3]). The proof of the general case is essentially the same. Suppose that $\alpha = (\alpha_x)_{x \in X}$ is a continuous $\pi$-system. For $f \in C_0(Y)_+$, we define $\Phi(f)(x) = \int f \, d\alpha_x$. There exists an increasing sequence $(f_n)$ in $C_c(Y)_+$ converging uniformly to $f$. Therefore, $\Phi(f) = \sup_n \Phi(f_n)$ is lower semi-continuous. It is clear that the other properties of a $C_0(X)$-weight are satisfied. Since its domain contains $C_c(Y)$, $\Phi$ is densely defined. The lower semi-continuity of the Radon measures gives the lower semi-continuity of $\Phi$. Conversely, let $\Phi$ be a densely defined and lower semi-continuous $C_0(X)$-weight on $C_0(Y)$. For all $x \in X$, we can define $\varphi_x : C_0(Y_x)_+ \to [0, \infty]$ such that $\Phi(f)(x) = \varphi_x(f|_{Y_x})$ for all $x$. Then $\varphi_x$ is a densely defined and lower semi-continuous weight on $C_0(Y_x)$. As we recalled at the beginning, there exists a unique Radon measure $\alpha_x$ such that $\phi_x(f) = \int f \, d\alpha_x$ for all $f \in C_0(Y_x)$. By assumption, the linear span of the domain of definition $P$ of $\Phi$ is dense. Since it is also an ideal, it contains the Pedersen ideal $C_r(Y)$ of $C_0(Y)$. This shows that $\alpha = (\alpha_x)_{x \in X}$ is a continuous $\pi$-system of measures. \qed

**Example 3.9.** The Haar $C_0(X)$-weight of a groupoid. This example is a particular case of the previous example. Let $G$ be a locally compact groupoid. We use the range map $r : G \to G^{(0)}$ to turn the commutative $C^*$-algebra $C_0(G)$ into a $C_0(X)$-algebra, where $X = G^{(0)}$. Then a continuous Haar system $(\lambda_x)_{x \in X}$ defines a densely defined and lower semi-continuous $C_0(X)$-weight of $C_0(G)$, which in accordance with the group case, can be called the Haar $C_0(X)$-weight of $G$. On the other hand, we cannot define a Plancherel $C_0(X)$-weight for an arbitrary groupoid $G$ since $C^*(G)$ is usually not a $C_0(X)$-algebra.

We now have all the elements to prove the Proposition 3.6 in [12].

**Corollary 3.10.** Let $p : G \to X$ a locally compact bundle of abelian groups, equipped with a continuous Haar system $\lambda = (\lambda_x)_{x \in X}$. Then the family of dual Haar measures $\hat{\lambda} = (\lambda_x)_{x \in X}$ is a continuous Haar system for $\hat{p} : \hat{G} \to X$.

**Proof.** From Theorem 3.6, we know that the Plancherel $C_0(X)$-weight $\mathcal{T}$ on $C^*(G)$ is lower semi-continuous and densely defined. Therefore $\Phi = \mathcal{T} \circ G^{-1}$, where
\( G : C^*(G) \to C_0(\hat{G}) \) is the Gelfand transform, is a lower semi-continuous and densely defined \( C_0(X) \)-weight on \( C_0(\hat{G}) \). Moreover, for all \( x \in X \), the Plancherel weight \( \tau_x \) of \( G_x \) correspond to the Haar weight of \( \hat{G}_x \), which is given by the Radon measure \( \hat{\lambda}^x \). From Proposition 3.8, the \( \hat{p} \)-system \( \hat{\lambda} = (\hat{\lambda}^x)_{x \in X} \) is continuous.

Acknowledgements. I thank Henrik Kreidler for drawing the erroneous proof of Proposition 3.6 in [12] to our attention, Michel Hilsum for discussions which led to the present proof of Corollary 3.10 and Dana Williams for sharing another proof and providing inspiring feedback and valuable comments.

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