MORITA EQUIVALENCE OF DEFORMATIONS OF KLEINIAN SINGULARITIES

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Abstract. In this note we classify all Morita equivalent pairs of (classical) generalized Weyl algebras for generic values of the parameters, thus positively settling a 30 year old question posed by T. Hodges. We also prove a similar result for noncommutative deformations of arbitrary Kleinian singularities provided the corresponding parameters are very generic.

1. Introduction

Recall that classical generalized Weyl algebras introduced by V. Bavula, also considered by T. Hodges under the name of noncommutative deformations of type $A$ singularities [H2], are defined as follows. Let $k$ be a field and let $v \in k[h]$. Then the corresponding generalized Weyl algebra $A(v)$ is defined as a quotient of $k\langle x, y, h \rangle$ by the following relations

$$[h, x] = x, \quad [h, y] = -y, \quad xy = v(h), \quad yx = v(h - 1).$$

Generalized Weyl algebras have been extensively studied in the literature. The isomorphism problem for generalized Weyl algebras (over $\mathbb{C}$) was solved by Bavula and Jordan [BJ]. Namely, they proved that $A(v) \cong A(v')$ if and only if $v'(h) = av(bh + c)$ with $a, b \in \mathbb{C}^*, c \in \mathbb{C}$. It is not surprising that the Morita equivalence problem turned out to be more difficult.

From now on, we say that $v \in \mathbb{C}[h]$ is generic if $v$ has no multiple roots and the differences between its distinct roots are not integers. It is well-known that the global dimension of $A(v)$ equals 1 precisely for generic values of $v$.

It was observed by Hodges [H2] that (for a generic $v$) $A(v)$ is Morita equivalent to $A(v')$ if the roots of $v$ and $v'$ differ by integers. Earlier, it was also proved by Hodges [H1] that $U(\mathfrak{sl}_2)/(\Delta - \lambda), U(\mathfrak{sl}_2)/(\Delta - \lambda')$ are Morita equivalent if and only if $\lambda' \in \pm \lambda + \mathbb{Z}$ (here $\lambda \notin \mathbb{Z}$). Recall that the algebras $U(\mathfrak{sl}_2)/(\Delta - \lambda)$ (maximal primitive quotients of $U(\mathfrak{sl}_2)$) correspond to $A(v)$ with quadratic $v$. In other words, if $\deg(v) = 2$, then $A(v), A(v')$ are Morita equivalent only when they obviously are. This led Hodges [H2] to pose the following question.

Question 1 (Hodges). Let $A(v)$ and $A(v')$ be Morita equivalent, with $v$ generic. Then are the algebras $A(v), A(v')$ obviously Morita equivalent?

The main goal of this paper is to give an affirmative answer to the above question. Before stating it, recall that given an associative ring $R$, its Picard
group Pic(\(R\)) is defined as the group of isomorphism classes of invertible \(R\)-bimodules. Given an automorphism \(f \in \text{Aut}(R)\) denote by \(R_f\) the \(R\)-bimodule which is \(R\) as a left module, and on which the right action is given by \(f\). Then \(R_f \in \text{Pic}(R)\) and this way we may view \(\text{Aut}(R)\) as a subgroup of \(\text{Pic}(R)\). Our main result is as follows.

**Theorem 1.1.** Let \(v = \prod_{i=1}^n (h - t_i) \in \mathbb{C}[h]\) be generic. Then \(\text{Pic}(A(v)) = \text{Aut}(A(v))\). Let \(B\) be a domain Morita equivalent to \(A(v)\). Then either \(\text{Pic}(B)/\text{Aut}(B)\) is uncountable, so in particular \(B\) is not isomorphic to any \(A(v')\), or

\[ B \cong A(v'), \quad v'(h) = \prod_{i=1}^n (h - t_i + d_i) \]

for some \(d_i \in \mathbb{Z}\).

A similar result is also obtained in the upcoming paper \[CEEF2\].

A few comments about the above theorem are in order. The case of linear \(v\) corresponds to the first Weyl algebra \(A(v) \cong A_1(\mathbb{C})\), in which case the above result is due to Stafford \[S\]; in fact, he showed that the result holds in arbitrary characteristic. In particular, it was Stafford who emphasized the importance of the invariant \(\text{Pic}(R)/\text{Aut}(R)\) to distinguish between non-isomorphic Morita equivalent domains. The case of quadratic \(v\) was proved in \[CEEF1\].

Our proof crucially relies on two nontrivial results in the literature: the first being the \(\text{Aut}(A(v))\)-equivariant description of isomorphism classes of ideals in \(A(v)\) in terms of certain quiver varieties due to F. Eshmatov \[E\] and Berest–Chalykh–Eshmatov \[BCE\]; and the second one being the transitivity of the action of \(\text{Aut}(A(v))\) on those varieties proved in \[CEET\].

Noncommutative deformations of all Kleinian singularities, i.e. the algebras \(O^\lambda\) (which incorporate generalized Weyl algebras \(A(v)\)), were introduced by Crawley-Boevey and Holland \[CBH\]. We are able to show an analogous result for Morita equivalences between these algebras provided that the parameter \(\lambda\) is very generic (its entries are algebraically independent over \(\overline{\mathbb{Q}}\)), see Theorem 2.4.

2. **Noncommutative deformations of general Kleinian singularities**

We start by briefly recalling the definition of deformed preprojective algebras introduced by Crawley-Boevey and Holland \[CBH\]. Let \(Q\) be an arbitrary finite quiver with the set of vertices \(I\), and let \(\overline{Q}\) be its double. For each \(i \in I\), denote by \(e_i\) the corresponding idempotent. Let \(\lambda \in \mathbb{C}^I\). Then the deformed preprojective algebra \(\Pi^\lambda(Q)\) with parameter \(\lambda\) is defined as the quotient of the path algebra of \(\overline{Q}\) by the following relations:

\[ \sum_{a \in Q} [a, a^*] = \sum_{i \in I} \lambda_i e_i. \]
Next, we need to recall the reflection functors (for \( \lambda \neq 0 \))

\[
E_i : \Pi^\lambda(Q)\text{-Mod} \to \Pi^{r_i(\lambda)}(Q)\text{-Mod}
\]
satisfying \( E_i^2 = \text{Id} \), introduced by Crawley-Boevey and Holland [CBH]. First, they define the symmetrized Ringel form \((\cdot, \cdot) : \mathbb{Z}^I \times \mathbb{Z}^I \to \mathbb{Z} \), via:

\[
(\alpha, \beta) = (\alpha, \beta) + (\beta, \alpha), \quad \text{where} \quad (\alpha, \beta) := \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in Q} \alpha_{t(a)} \beta_{h(a)}.
\]

Next, define a simple reflection for a given loop-free vertex \( i \in I \) as follows:

\[
s_i : \mathbb{Z}^I \to \mathbb{Z}^I, \quad s_i(\alpha) = \alpha - (\alpha, \epsilon_i) \epsilon_i,
\]

where \( \epsilon_i \) is the \( i \)-th coordinate vector, and the corresponding reflection \( r_i : \mathbb{C}^I \to \mathbb{C}^I \) defined with the property that \( r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha) \), explicitly

\[
r_i(\lambda)_j = \lambda_j - (\epsilon_i, \epsilon_j) \lambda_i.
\]

One now defines the Weyl group \( W \) of \( Q \) as the group of automorphisms of \( \mathbb{Z}^I \) generated by \( s_i, i \in I \). Note that \( W \) acts on \( \mathbb{C}^I \) by \( w(\lambda) \cdot \alpha = \lambda \cdot w^{-1}(\alpha) \) for all \( w \in W, \lambda \in \mathbb{C}^I, \alpha \in \mathbb{Z}^I \).

Given a left \( \Pi^\lambda \)-module \( M \), we put \( M_i = e_i M \), where recall that \( e_i \in \Pi^\lambda \) denotes the idempotent of the vertex \( i \in I \). So, \( M = \bigoplus_{i \in I} M_i \). Then \( (E_i(M))_j = M_j \) for \( j \neq i \). Also, \( (E_i(M))_i \) is a direct summand of \( \bigoplus_{a \in Q, h(a) = i} M_{t(a)} \). By composing reflection functors, for any element of the Weyl group \( w \in W \), we get the functor, which is an equivalence of categories

\[
E_w : \Pi^\lambda\text{-Mod} \to \Pi^{w(\lambda)}\text{-Mod}.
\]

Given \( \alpha \in \mathbb{N}^I \), denote by \( M_Q(\lambda, \alpha) \) the variety of isomorphism classes of \( \Pi^\lambda(Q) \)-modules of dimension \( \alpha \). Then for each \( w \in W \), the translation functor \( E_w \) induces an isomorphism of algebraic varieties

\[
R_w : M_Q(\lambda, \alpha) \cong M_Q(w(\lambda), w(\alpha)).
\]

Let \( \lambda \in Z(\mathbb{C}[\Gamma]) \), where \( \Gamma \) is a finite subgroup of \( SL_2(\mathbb{C}) \). Then to this datum Crawley-Boevey and Holland [CBH] associated an algebra \( O^\lambda \), which is a non-commutative deformation of the Kleinian singularity \( \mathbb{C}^2/\Gamma \). So, \( O^\lambda \) is a filtered algebra such that its associated graded algebra is isomorphic to \( O(\mathbb{C}^2/\Gamma) \). Recall the definition of \( O^\lambda \). At first, one defines the corresponding symplectic reflection algebra

\[
H^\lambda = (\Gamma \ltimes \mathbb{C}(x, y))/([x, y] - \lambda).
\]

Then \( O^\lambda = eH^\lambda e \), where \( e = \frac{1}{\# \Gamma} \sum_{g \in \Gamma} g \). Recall that the algebras \( H^\lambda \) admit well-known higher dimensional generalizations called symplectic reflection algebras introduced by Etingof and Ginzburg [EG].

Now let us recall how the algebras \( O^\lambda \) relate to deformed preprojective algebras. Let \( Q \) be the McKay graph of \( \Gamma \) with arbitrary orientation, thus vertices of \( Q \) correspond to simple representations \( V_i \) of \( \Gamma \) (the trivial module corresponds to vertex 0), and with an arrow \( i \to j \) if \( \text{Hom}_\Gamma(V \otimes V_i, V_j) \neq 0 \). Let \( I \) denote the set of vertices of \( Q \). Then, an important result of [CBH] asserts that \( \Pi^\lambda(Q) \) is Morita
equivalent to $H^\lambda$, and $\mathcal{O}^\lambda \cong e_0 \Pi^\lambda e_0$. Here we first write $\lambda \in \mathbb{C}^I$ as $\lambda = \sum_{i \in I} f_i \tau_i$, where $\tau_i$ is the idempotent in $Z(\mathbb{C}[\Gamma])$ corresponding to the simple $\Gamma$-module $V_i$. Now $\lambda$ has $i$-th entry $f_i \delta_i$, where $\delta_i := \dim(V_i)$. Moreover, the algebras $\Pi^\lambda$ and $\mathcal{O}^\lambda$ are Morita equivalent if $\lambda \cdot \alpha \neq 0$ for all Dynkin roots $\alpha$.

From now on, we say that $\lambda \in \mathbb{C}^I$ is generic if $\lambda \cdot \alpha \neq 0$ for any root $\alpha$. Similarly, $\lambda \in \mathbb{C}^I$ is said to be very generic if its coordinates are algebraically independent over $\mathbb{Q}$. Recall that $\mathcal{O}^\lambda$ is commutative if and only if $\lambda \cdot \delta = 0$, and $\text{gl.dim}(\mathcal{O}^\lambda) = 1$ if and only if $\lambda$ is generic [CRH, Theorem 0.4].

Let $W_{\text{ext}}$ be the group generated by the reflections $s_i$ and the diagram automorphisms of $\mathcal{Q}$. So $W_{\text{ext}}$ is the extended affine Weyl group of the affine root system of $\mathcal{Q}$. Hence given a generic $\lambda \in \mathbb{C}^I$, for any $w \in W_{\text{ext}}$, we have the corresponding equivalence of categories

$$\tilde{E}_w : \mathcal{O}^\lambda \text{-Mod} \to \mathcal{O}^{w(\lambda)} \text{-Mod}.$$ 

It is natural to pose the analogue of Hodges’ question in this more general setup. We show that the answer is yes for a very generic $\lambda$ (defined just above). Here we also need to recall that as observed by Boyarchenko [B], translations by

$$d \in \Lambda := \{ \xi \in \mathbb{Z}^I, \xi \cdot \delta = 0 \}$$

provide Morita equivalent pairs of algebras $\mathcal{O}^\lambda$ for generic $\lambda$ such that $\lambda \cdot \delta = 1$:

$$\mathcal{O}^\lambda \text{-Mod} \cong \mathcal{O}^{\lambda + d} \text{-Mod}.$$ 

We need the following result which should be well-known. Its proof easily follows along the lines of considering the standard Koszul resolution for symplectic reflection algebras, see [EG, proof of Theorem 1.8].

**Lemma 2.1.** Let $\lambda \in Z(\mathbb{C}[\Gamma])$ be very generic. Then the natural map

$$Z(\mathbb{C}[\Gamma]) \to H^\lambda/[H^\lambda, H^\lambda]$$

is surjective with kernel $\mathbb{C}\lambda$.

The next result is a direct corollary of using the Hattori–Stallings traces just as was originally done by Hodges in [H2]. This technique was also used by Berest, Etingof, and Ginzburg to classify Morita equivalent pairs of rational Cherednik algebras of $S_n$ for generic parameters [BEG]. Our result is a generalization of a result by Richard–Solotar [RS, Theorem 2.6.2] to noncommutative deformations of arbitrary Kleinian singularities.

Given $\lambda \in \mathbb{C}^I$, define $\lambda := (\lambda_i, i \neq 0) \in \mathbb{C}^{|I|-1}$. Recall that given a finitely generated projective $A$-module $P$, its Hattori–Stallings trace is defined as the image in $A/[A, A] = H_0(A)$ of the trace of an idempotent corresponding to $P$. Thus we have the trace homomorphism

$$tr = tr_A : K_0(A) \to H_0(A).$$

**Proposition 2.2.** Let $H^\lambda, H^{\lambda'}$ be Morita equivalent, with $\lambda \cdot \delta = \lambda' \cdot \delta = 1$. Then there exist $A \in GL_{n-1}(\mathbb{Z})$ and $d \in \mathbb{Z}^{n-1}$ such that $\lambda' = A\lambda + d$. 


Proof. Recall that $\tau_i \in Z(\mathbb{C}[\Gamma])$ denotes the primitive central idempotent corresponding to $V_i$, and $\lambda = \sum_{i \in I} f_i \tau_i, \lambda' = \sum_{i \in I} f_i' \tau_i$. Moreover, the vectors $\lambda, \lambda' \in \mathbb{C}^I$ have $i$-th coordinate $f_i \delta_i, f_i' \delta_i$ respectively. So by our assumption $\sum_i f_i \delta_i^2 = \sum_i f_i' \delta_i^2 = 1$.

Let $F : H^\lambda \text{-Mod} \to H^{\lambda'} \text{-Mod}$ be an equivalence of categories, then $F$ preserves the Hattori-Stallings traces. Also, as $F$ must descend to an equivalence $\mathcal{O}^\lambda \text{-Mod} \to \mathcal{O}^{\lambda'} \text{-Mod}$, so $F$ must preserve the rank, where we define: $\text{rank}_\lambda(M) := \text{rank}_{\mathcal{O}_\lambda}(eM)$. It is well-known that $K_0(H^\lambda) = K_0(\mathbb{C}[\Gamma])$ has a basis $\{[P_i], i \in I\}$, where $P_i = H^\lambda \otimes_{\mathbb{C}[\Gamma]} V_i$ [BEG]. Also, $\text{rank}([P_i]) = \delta_i$ and $\text{tr}([P_i]) = \tau_i / \delta_i$.

Denote by $\bar{K}(H^\lambda)$ the kernel of the rank homomorphism $\text{rank} : K_0(H^\lambda) \to \mathbb{Z}$. Then $\{[P_i] - \delta_i[P_0] =: \varepsilon_i, i > 0\}$ is a $\mathbb{Z}$-basis of $\bar{K}(H^\lambda)$. Since $\sum_{i \in I} f_i \tau_i = 0$ in $H^\lambda/[H^\lambda, H^\lambda]$ and $\tau_i = \text{tr}(\delta_i \varepsilon_i) + \delta_i^2 \tau_0, \ i > 0$,

we conclude that (setting $\text{tr}_\lambda := \text{tr}_{H^\lambda}$ and $\text{tr}_{\lambda'} := \text{tr}_{H^{\lambda'}}$)

$$\text{tr}_{\lambda}^{-1}(e_0) = -\sum_{i > 0} \delta_i f_i \varepsilon_i, \quad \text{tr}_{\lambda'}^{-1}(e_0) = -\sum_{i > 0} \delta_i f_i' \varepsilon_i.$$ 

Since $\text{rank}(F(H^\lambda e_0)) = 1$, the desired result follows from the following commutative diagram

\[
\begin{array}{ccc}
\bar{K}_0(H^\lambda) & \xrightarrow{\text{tr}_\lambda} & \bar{K}_0(H^{\lambda'}) \\
\downarrow & & \downarrow \\
H_0(H^\lambda) & \xrightarrow{H_0(F)} & H_0(H^{\lambda'})
\end{array}
\]

We next recall the following simple result. It should be well-known, but we give a proof here for the sake of completeness.

Lemma 2.3. Let $R$ be an affine root system spanning an $\mathbb{R}$-vector space $V$. Let $T \in GL(V)$ such that there is a set $S \subset R$ containing all imaginary roots and finitely many other roots, such that $T(R \setminus S), T^{-1}(R \setminus S) \subset R$. Then $T$ is an element of the extended affine Weyl group of $R$.

Proof. Let $\delta$ denote the minimal imaginary positive root of $R$. Let $R'$ denote the finite root system whose affinization is $\tilde{R}$. So, $R = (R' \times \mathbb{Z} \delta) \sqcup \mathbb{Z} \delta$. Thus for all but finitely many $\alpha \in R \setminus \mathbb{Z} \delta$ we have $T(\alpha) \in R$. Recall that for given roots $\alpha, \beta$, if $m\alpha + \beta \in R$ for infinitely many $m$, then $\alpha$ must be a multiple of $\delta$. This implies $T(\delta) = \pm \delta$. Thus, without loss of generality $T(\delta) = \delta$. Then $\tilde{T} : V/\mathbb{R} \delta \to V/\mathbb{R} \delta$ is a linear transformation preserving the finite root system $\tilde{R} \cong \tilde{R'}$. Now, it easily follows that $\tilde{T}$ belongs to the subgroup generated by the Weyl group and diagram automorphisms of $\tilde{R}$. Hence $T$ belongs to the extended affine Weyl group of $R$. □
Theorem 2.4. Let $O^\lambda$ and $O^{\lambda'}$ be Morita equivalent noncommutative deformations of a Kleinian singularity. If $\lambda$ is very generic then $\lambda' = tw(\lambda)$ for some $w \in W_{\text{ext}}$ and $t \in \mathbb{C}^*$.

Proof. Without loss of generality we may assume that $\lambda \cdot \delta = \lambda' \cdot \delta = 1$. By Proposition 2.2 we have that $\lambda' = \lambda \Lambda + d$ for some $\Lambda \in GL_n(\mathbb{Z})$ and $d \in \Lambda$ (see (1)), where $n = |I|$. Now, recall that translations by $d \in \Lambda$ induce Morita equivalences and belong to $W_{\text{ext}}$. Thus, without loss of generality we may assume that $\lambda' = \Lambda \lambda$. It follows that there exists a finitely generated ring $S \subset \mathbb{C}$, containing $\lambda_0, \ldots, \lambda_{n-1}$, such that $O^{\lambda'}, O^\lambda$ are Morita equivalent over $S$. Denote by $T : \mathbb{R}^n \to \mathbb{R}^n$ the linear transformation $T(v) = Av$. By Lemma 2.2, we may assume without loss of generality that for infinitely many non-imaginary roots $\alpha$, $T(\alpha)$ is not a root. It follows that there exists a homomorphism $\chi : S \to \bar{\mathbb{Q}}$ and a non-imaginary root $\alpha$, so that $\chi(\lambda) \cdot \alpha = 0$, while $\chi(\lambda') \cdot \beta \neq 0$ for any root $\beta$. So, $O^{\chi(\lambda)}, O^{\chi(\lambda')}$ are (still) Morita equivalent, while $\text{gl.dim}(O^{\chi(\lambda)}) \neq \text{gl.dim}(O^{\chi(\lambda')})$, a contradiction. \(\square\)

3. Characteristic $p \gg 0$ approach

The goal of this section is to prove Theorem 3.3 below, which plays a key role in proving our main result. Its proof is based on the reduction mod $p \gg 0$ method. To state the result, we need to introduce the following subgroup of $\text{Aut}(A(v))$. Let $k$ be a field, $v \in k[h]$. Let $G$ be the subgroup of $\text{Aut}(A(v))$ generated by $\exp(\lambda \text{ad } x^n), \exp(\lambda \text{ad } y^n)$ for $\lambda \in k, n \geq 1$, and the diagonal automorphisms $\phi_t, t \in k^*$, defined as

$$\phi_t(x) = tx, \quad \phi_t(y) = t^{-1}y, \quad \phi_t(h) = h.$$ 

Notice that the definition of $G$ makes sense when $\text{char } k > 0$ also. Following [BJ], we call a polynomial $v \in k[h]$ reflexive if $v(h) = (-1)^d v(a - h)$ for some $a \in k, d = \text{deg}(h)$. If $v$ is reflexive, then we have an involution $\Omega \in \text{Aut}(A(v))$ defined as follows:

$$\Omega(x) = y, \quad \Omega(y) = (-1)^d x, \quad \Omega(h) = 1 + a - h.$$ 

It was proved by Bavula and Jordan [BJ, Theorem 3.29] that if $v$ is reflexive, then $\text{Aut}(A(v))$ is generated by $G$ and $\Omega$, otherwise $\text{Aut}(A(v)) = G$. Recall that $\Omega$ normalizes $\Gamma$.

For $v(h) = \prod_i(h - c_i) \in \mathbb{C}[h]$, let $S \subset \mathbb{C}$ be a finitely generated subring of $\mathbb{C}$ containing all roots of $v$. Given a base change $S \to k$ to a characteristic $p$ field $k$, we set $v^{[p]}(h) := \prod_i(h - (c_i^p - c_i)) \in k[h]$.

We also need to recall the commutative deformations of type A Kleinian singularities: the commutative Poisson $k$-algebra $B(v)$ generated by $x, y, h$ with the relation $xy = v(h)$ and the following Poisson brackets

$$\{h, x\} = x, \quad \{h, y\} = -y, \quad \{x, y\} = v'(h).$$ 

We need the following characteristic $p \gg 0$ analogue of the Poisson version of the aforementioned theorem of Bavula–Jordan on automorphisms of $A(v)$.
(The original proof in [BJ] also goes through directly for this analogue.) If $v$ is reflexive, then $B(v)$ admits the Poisson automorphism $\Omega$ defined just as for $A(v)$. The subgroup $W \leq \text{Aut}(B(v))$ of Poisson automorphisms is defined similarly to the subgroup $G \leq \text{Aut}(A(v))$. Now one has:

**Theorem 3.1.** Given a natural number $d$ there exists $N \in \mathbb{N}$, such that for all primes $p > N$, the following holds. Let $k$ be an algebraically closed field of characteristic $p$, and let $v \in k[h]$. Let $\phi : B(v) \to B(v)$ be a Poisson $k$-algebra automorphism, such that $\deg(\phi(x)), \deg(\phi(y)), \deg(v) \leq d$. Then $\phi$ belongs to the subgroup generated by $\Omega$ and $W$ if $v$ is reflexive, otherwise $\phi \in G$.

Recall that the center of $A(v)$ is trivial for $\text{char } k = 0$, while in the case of $\text{char } k = p$ it is large and is described as follows [BC]. The center $Z(A(v))$ is generated by $x^p, y^p, h^p - h$ subject to the relation

$$x^p y^p = \prod_i (h^p - h - (c_i^p - c_i)),$$

where $v(h) = \prod_i (h - c_i)$.

Recall that given an associative flat $\mathbb{Z}$-algebra $R$ and a prime number $p$, the center $Z(R/pR)$ of its reduction mod $p$ acquires a natural Poisson bracket (see for example [BK, Section 5.2]), which we refer to as the reduction mod $p$ bracket, defined as follows. Given $a, b \in Z(R/pR)$, let $z, w \in R$ be their respective lifts. Then the Poisson bracket $\{a, b\}$ is defined to be

$$\frac{1}{p} [z, w] \mod R \in Z(R/pR).$$

So, given a base change $S \to k$ to a characteristic $p$ field $k$, where $S \subset \mathbb{C}$ is a finitely generated ring containing all roots of $v$, then $Z(A(v) \otimes_S k)$ is equipped with the reduction mod $p$ Poisson bracket as defined above and we get an isomorphism

$$Z(A(v) \otimes_S k) \cong B(v^p),$$

preserving the Poisson brackets with the minus sign (see [BK, Section 5.2] for the case of the Weyl algebra; the general case is similar).

**Remark 3.2.** It follows from the realization of $\mathcal{O}^\lambda$ (for $\lambda \cdot \delta = 1$) as a quantum Hamiltonian reduction [Hol] that in characteristic $p \gg 0$ we have that

$$Z(\mathcal{O}^\lambda \mod p) \cong \mathcal{O}^{\lambda[p]},$$

where $\lambda[p] = \lambda^p - \lambda i$. So, $\lambda[p] \cdot \delta = 0$. Recall that $\mathcal{O}^\lambda$ is commutative if and only if $\lambda \cdot \delta = 0$. In particular, if $\mathcal{O}^{\lambda_1}, \mathcal{O}^{\lambda_2}$ are Morita equivalent, then we have an isomorphism of Poisson algebras $\mathcal{O}^{\lambda[p]} \cong \mathcal{O}^\lambda[p]$ for $p \gg 0$.

**Theorem 3.3.** Let $v \in \mathbb{C}[h]$ be arbitrary. Then $\text{Aut}(A(v))$ is a normal subgroup of $\text{Pic}(A(v))$. Moreover, if $v$ is reflexive, or the roots of $v$ are algebraically independent over $\mathbb{Q}$, then $\text{Pic}(A(v)) = \text{Aut}(A(v))$. 


Proof. Recall that given an automorphism \( f \) of an algebra \( A \), by \((A)_f \) we denote the \( A \)-bimodule which is \( A \) as a left module and on which the right action is by \( f \). Let \( M \in \text{Pic}(A(v)) \). The idea is to show that \( M \text{Aut}(A(v))M^{-1} \mod p \subset \text{Aut}(A(v)) \) for all \( p \gg 0 \). Recall that given a bimodule \( N \) over \( A(\bar{v}) \), where \( \bar{v} \in k[h] \) and \( k \) is a characteristic \( p \) field, the support of \( N \) (denoted by \( \text{Supp}(N) \)) is the closed subvariety of \( \text{Spec}(Z(A(\bar{v})) \otimes Z(A(\bar{v}))) \) corresponding to the annihilator of \( N \) in \( Z(A(\bar{v})) \otimes Z(A(\bar{v})) \). A standard argument shows that for \( N \in \text{Pic}(A(\bar{v})) \), its support must be the graph of an automorphism \( \phi \in \text{Aut}(Z(A(v))) \), and \( N \) is uniquely determined by its support.

At first, we consider the case when \( v \) is not reflexive, so \( \text{Aut}(A(v)) = G \). Let \( N' \) be a bimodule corresponding to an element \( f \in G \), so \( N' = (A(v))_f \). Let \( M \in \text{Pic}(A(v)) \) and put \( N = MN'M^{-1} \in \text{Pic}(A(v)) \). We need to show that \( N \cong A \) as a left module. Let \( S \subset C \) be a large enough finitely generated ring containing the roots of \( v \), such that all the above modules are defined over it. We first verify that for any base change \( S \to k \) to an algebraically closed field of characteristic \( p \), we have

\[
N_k = N \otimes_S k \cong A(\bar{v}),
\]

where \( \bar{v} \) denotes the image of \( v \) in \( k[h] \). Indeed, let \( \phi \in \text{Aut}(Z(A(\bar{v}))) \) be the automorphism whose graph is the support of \( M_k \). Recall that \( Z(A(\bar{v})) \cong B(v^{[p]}) \).

So, we may view \( \phi \) as a Poisson automorphism of \( B(v^{[p]}) \). We also see that \( f|_{Z(A(\bar{v}))} = \tilde{f} \in W \). By using Theorem 3.1, we conclude that \( \phi \) belongs to the subgroup generated by \( \Omega \) and \( W \) (and \( \phi \in W \) if \( v^{[p]} \) is not reflexive). Since \( \Omega \) normalizes \( W \), it follows that \( \phi \tilde{f} \phi^{-1} \in W \). So, \( \text{Supp}(N_k) \) is the graph of an element of \( W \). Since \( G|_{Z(A(v))} = W \), we conclude that there exists \( \psi \in G \) so that \( N_k \cong A(\bar{v}) \psi \). Thus, we see that \( N_k \cong A(\bar{v}) \) as left modules.

Since \( N \) as a left module has rank 1, we may assume that \( N \) is a left ideal in \( A(v) \). Moreover, we may assume that \( N \) is a submodule of \( A(v) \) over \( S \). Let us equip \( N \) with the induced filtration from \( A(v) \). Then \( N \) is a principal ideal if and only if \( \text{gr}(N) \) is a principal ideal in \( \text{gr}(A(v)) \). Now, since we have proved that \( N_k \cong A(\bar{v}) \), it follows that \( \text{gr}(N)_k \) is a principal ideal in \( \text{gr}(A(\bar{v})) \). Hence, \( \text{gr}(N) \) is a principal ideal and we are done.

Now suppose \( v \) is reflexive, then so is \( v^{[p]} \). Then any Poisson automorphism of \( B(v^{[p]}) = Z(A(\bar{v})) \) that belongs to the subgroup generated by \( \Omega \) and \( W \) can be lifted to an automorphism of \( A(\bar{v}) \), and the rest of the proof is identical to the case of a non-reflexive \( v \).

Finally, suppose \( v \) satisfies one of the assumptions in the lemma. Then it follows from the above considerations that \( M_k \in \text{Aut}(A(\bar{v})) \) for infinitely many \( k \) with \( \text{char } k \gg 0 \). Then as explained above, it follows that \( M \) belongs to the image of \( \text{Aut}(A(v)) \) in the Picard group. \( \square \)

Remark 3.4. The usage of the reduction mod \( p \gg 0 \) technique for statements of the form \( \text{Pic}(R) = \text{Aut}(R) \), where \( R \) is a filtered quantization of a Poisson algebra is quite typical: it was used by Stafford for the Weyl algebra \([S]\). It is also worth mentioning that a recent work of C. Dodd \([D]\) showed that \( \text{Pic}(A_n(\mathbb{C})) = \ldots \)
Aut(\(A_n(\mathbb{C})\)), settling a conjecture by Belov-Kanel and Kontsevich, where the reduction mod \(p \gg 0\) approach is at the heart of everything.

4. Classification of ideals via quiver varieties

Let \(\mathcal{O}^\lambda\) be a noncommutative deformation of a Kleinian singularity, with \(\lambda\) generic (so \(\text{gl.dim}(\mathcal{O}^\lambda) = 1\)). Then it was proved by Baranovsky, Ginzburg, and Kuznetsov [BGK] that there is a bijection between the set of isomorphism classes of indecomposable projective \(\mathcal{O}^\lambda\)-modules and a countable disjoint union of certain quiver varieties.

In the case of deformations of type \(A\) Kleinian singularities \(A(v)\), automorphism groups \(\text{Aut}(A(v))\) are very big, so it is interesting to consider the natural action of \(\text{Aut}(A(v))\) on the set of isomorphism classes of indecomposable projective \(\mathcal{O}^\lambda\)-modules. An explicit and \(\text{Aut}(A(v))\)-equivariant version of the aforementioned theorem of Baranovsky–Ginzburg–Kuznetsov was proved by Eshmatov [E] and Berest, Chalykh, and Eshmatov [BCE] (building on earlier works on ideals of the Weyl algebra and the Calogero–Moser spaces by Berest, Chalykh, and Wilson).

In this section we recall the bijection constructed in [BCE] and observe that it is compatible with reflection functors (Lemma 4.1), which plays a crucial role in our proof on Theorem 1.1.

From now on, let \(Q\) denote the quiver corresponding to the extended Dynkin diagram \(\tilde{A}_m\), which is a cycle with \(m\) vertices. (This corresponds to the cyclic group \(\Gamma\) of size \(m\).) Let \(Q_\infty\) denote the quiver obtained from \(Q\) by attaching an additional vertex, labeled \(\infty\), with an arrow to the vertex \(0\).

Let \(\lambda = (\lambda_0, \ldots, \lambda_m-1) \in \mathbb{C}^m\) be a regular (generic) vector. Also let \(\alpha \in \mathbb{Z}^{m+1}\) be a positive root for \(Q_\infty\) with \(\alpha_\infty = 1\), and write \(\alpha = (1, \eta)\). Set \(\lambda_\alpha := (-\lambda \cdot \eta, \lambda) \in \mathbb{C}^{m+1}\) and \(B := e_\infty \Pi^{\lambda_\alpha}(Q_\infty) e_\infty\). It is known that \(\Pi^\lambda(Q) = \Pi^{\lambda_\alpha}(Q_\infty)/(e_\infty)\) [BCE, Lemma 6].

It is well-known that a generalized Weyl algebra \(A(v)\) can be identified with \(\mathcal{O}^\lambda = e_0 \Pi^\lambda(Q) e_0\) for an appropriate \(\lambda\). Indeed, let \(\lambda = (\lambda_1, \ldots, \lambda_n-1) \in \mathbb{C}^{n-1}\), then define \(a = (a_1, \ldots, a_n)\) as follows: \(a_n = 0\), and

\[
a_i = (n - i + \lambda_i)/n, \quad i < n.
\]

Put \(v = \prod_{i=1}^n (h - a_i)\). Then \(A(v) \cong \mathcal{O}^\lambda\) [M, Lemma 7.1].

In the above setting, Berest–Chalykh–Eshmatov considered the functor

\[ F : \Pi^{\lambda_\alpha}(Q_\infty)\text{-Mod} \to \Pi^\lambda(Q)\text{-Mod} \]

defined as follows

\[ F(M) := \text{Hom}_{\Pi^{\lambda_\alpha}(Q_\infty)}(\Pi^{\lambda_\alpha}(Q_\infty)/(e_\infty), \Pi^{\lambda_\alpha}(Q_\infty) \otimes_B M e_\infty). \]

Recall that \(M_{Q_\infty}(\lambda_\alpha, \alpha)\) denotes the quiver variety of isomorphism classes of \(\Pi^{\lambda_\alpha}(Q_\infty)\)-modules with dimension vector \(\alpha\). From now on we denote \(M_{Q_\infty}(\lambda_\alpha, \alpha)\) by \(M_{Q_\infty}(\lambda, \alpha)\) to simplify the notation. Then as was proved in [BCE], the functor \(F\) restricts on \(M_{Q_\infty}(\lambda_\alpha, \alpha)\) to an injective function with values in indecomposable projective modules over \(\Pi^\lambda(Q)\). Thus, we have the following inclusion
\[ F_\lambda : \bigsqcup_{\alpha} M_{Q_\infty}(\lambda, \alpha) \hookrightarrow \{\text{isoclasses of indec. projective } \Pi^\lambda(Q)\text{-modules}\}. \]

In order to extend \( F_\lambda \) to a bijection, we need to consider for each vertex \( i \) of \( Q \) an analog of \( Q_\infty \) defined by adding to \( Q \) the vertex \( \infty \) with an arrow to the vertex \( i \). Specifically, given \( i \in \mathbb{Z}/m\mathbb{Z} \), let \( \lambda^i \in \mathbb{C}^m \) be obtained from \( \lambda \) by rotating its coordinates by \( i \), so \((\lambda^i)_j = \lambda_{i+j}, j \in \mathbb{Z}/m\mathbb{Z} \). Since cyclic rotations belong to diagram automorphisms of \( Q \), we get that \( \Pi^\lambda(Q) \cong \Pi^\lambda(Q) \), which induces the bijection

\{\text{isoclasses of indec. proj. } \Pi^\lambda(Q)\text{-mod}\} \cong \{\text{isoclasses of indec. proj. } \Pi^\lambda(Q)\text{-mod}\}.

Thus, as was shown in \([E, \text{Theorem } 5]\), by combining the above bijection with functions \( F_\lambda \) above, we obtain the following bijection onto the set of isomorphism classes of indecomposable projective \( \Pi^\lambda(Q)\)-modules

\[ F_\lambda : \bigsqcup_{i \in \mathbb{Z}/m\mathbb{Z}} \bigsqcup_{\alpha} M_{Q_\infty}(\lambda^i, \alpha) \cong \{\text{isoclasses of indec. projective } \Pi^\lambda(Q)\text{-modules}\}. \]

Next, we need to check that \( F \) commutes with the reflection functors. This is the subject of the following lemma.

**Lemma 4.1.** The functor \( F \) commutes with the reflection functors \( E_i \) for all \( i \neq \infty \).

**Proof.** In what follows we denote \( \Pi^\lambda(Q_\infty) \) by \( \Pi \), and \( \Pi^\lambda_i(Q_\infty) \) by \( \Pi' \). It suffices to show that the reflection functors commute with the following functors:

\[ \Phi(-) = \text{Hom}_B(\Pi/\Pi e_\infty \Pi, -), \quad L : M \to \Pi \otimes_B M e_\infty. \]

For a \( \Pi \)-module \( M \), put \( M_i = e_i M \), so \( M = \bigoplus_i M_i \). Recall that \((E_i(M))_j = M_j \) for \( j \neq i \), while \((E_i(M))_i \) is a subspace of \( \bigoplus_{i \neq j} M_j \). We first verify that \( E_i(\Phi(M)) \subset \Phi(E_i(M)) \). Indeed, \( E_i(\Phi(M))_j \) is the annihilator of \( e_\infty \Pi e_j \) in \( M_j \).

Since \( e_\infty \Pi e_j \) belongs to the image of \( e_\infty \Pi e_j \) in \( \text{Hom}_C(M_j, M_\infty) \), we get that

\[ E_i(\Phi(M))_j \subset \Phi(E_i(M))_j, j \neq i. \]

Let

\[ v \in E_i(\Phi(M))_i \subset \bigoplus_{i \to j} \text{Ann}(e_\infty \Pi e_j). \]

Write \( v = \sum_{i \to j} v_j \). It suffices to verify that \( (e_\infty \Pi' e_i) v = 0 \). This follows from the fact that any action by a path in \( \Pi' \) ending in the vertex \( \infty \) can be written as sum of actions by an element of \( e_\infty \Pi \), so

\[ (e_\infty \Pi' e_i) v \subset \sum_{i \to j} (e_\infty \Pi e_j) v_j = 0. \]

This concludes the proof of the inclusion \( E_i(\Phi(M)) \subset \Phi(E_i(M)) \). So, replacing \( M \) by \( E_i(M) \), we get that

\[ E_i(\Phi(E_i(M))) \subset \Phi(M). \]
Now applying $E_i$ to both sides yields $\Phi(E_i(M)) \subset E_i(\Phi(M))$. Thus, $\Phi E_i = E_i \Phi$ as desired.

Next, we verify that $E_iL$ and $LE_i$ represent the same functor, $i \neq \infty$. Indeed,

$$\text{Hom}^L(E_i(L(M)), E_i(N)) = \text{Hom}^L(L(M), N) = \text{Hom}_{e_{\infty} \Pi e_{\infty}}(e_{\infty} M, e_{\infty} N).$$

On the other hand (recalling that $e_{\infty} \Pi e_{\infty} = e_{\infty} \Pi^e e_{\infty}$),

$$\text{Hom}^L(L(E_i(M)), E_i(N)) = \text{Hom}_{e_{\infty} \Pi^e e_{\infty}}(e_{\infty} M, e_{\infty} E_i(N)) = \text{Hom}_{e_{\infty} \Pi e_{\infty}}(e_{\infty} M, e_{\infty} N).$$

So, $LE_i = E_iL$ and we are done. \hfill \Box

In what follows, we denote by $W_\infty$ the subgroup of the Weyl group of $Q_\infty$ generated by the reflections $s_i$, $i \neq \infty$. Thus, by combining the bijection constructed by Berest–Chalykh–Eshmatov \cite{BCE} given above with reflection functors, we obtain that for any $w \in W_\infty$, the following diagram commutes:

$$
\begin{array}{ccc}
\bigsqcup_{\alpha} M_{Q_\infty}(\lambda_\alpha, \alpha) & \xrightarrow{F_{\lambda}} & \{\text{isoclasses of indec. projective } O^\lambda\text{-modules}\} \\
\downarrow R_w & & \downarrow R_w \\
\bigsqcup_{\alpha} M_{Q_\infty}(w(\lambda_\alpha), w(\alpha)) & \xrightarrow{F_{w(\lambda)}} & \{\text{isoclasses of indec. projective } O^{w(\lambda)}\text{-modules}\}
\end{array}
$$

5. Proof of the main theorem

Throughout this section we are assuming that the parameter $v \in \mathbb{C}[h]$ is generic. In order to prove Theorem \ref{main} we need the following result, which should be very standard. We include its proof for the reader’s convenience.

**Lemma 5.1.** Let $A$ be a Noetherian domain. Let $P$ be an indecomposable projective generator of $A\text{-Mod}$. Let $B = \text{End}(P)^{op}$. Let $X$ denote the set of isomorphism classes of indecomposable projective $A$-modules. Then the orbit of $P$ in $X$ under $\text{Pic}(A)$ equals $\text{Pic}(B)/\text{Aut}(B)$.

**Proof.** It suffices to observe that the stabilizer of $P$ in $\text{Pic}(A) = \text{Pic}(B)$ is $\text{Aut}(B)$. To show this, without loss of generality we may take $P = A$. Then an invertible $A$-bimodule $L \in \text{Pic}(A)$ stabilizes $A$ if $L \otimes_A A \cong A$ as left $A$-modules. So, $L = A$ as a left $A$-module, and the invertibility of $L$ implies that the right action of $A$ on $A$ must be given by an element of $\text{Aut}(A)$. So, the stabilizer of $A$ in $\text{Pic}(A)$ equals the image of $\text{Aut}(A)$ in $\text{Pic}(A)$. \hfill \Box

We also need the following result. Of course, eventually we show a much stronger statement.

**Proposition 5.2.** $\text{Pic}(A(v))/\text{Aut}(A(v))$ is at most countable.

**Proof.** Let $X$ be the set of isomorphism classes of indecomposable projective $A(v)$-modules. We know that the stabilizer of $A \in X$ is $\text{Aut}(A)$. We also know that there is a $G$-equivariant bijection between $X$ and the disjoint union of countably many quiver varieties, such that $G$ acts transitively on each of those quiver varieties \cite[Theorem 1.1]{CEET}. Therefore, $X/G$ is countable. Since the orbit of $A$ in $X$ is $\text{Pic}(A)/G$, we conclude that $\text{Aut}(A(v))\backslash \text{Pic}(A)/\text{Aut}(A(v))$ is at most
countable. Now, since $\text{Aut}(A(v))$ is a normal subgroup in $\text{Pic}(A(v))$ by Theorem 3.3, we obtain that $\text{Pic}(A(v))/\text{Aut}(A(v))$ is at most countable. \hfill \Box$

Finally, we show the main result of the paper. Recall that $W_{\text{ext}}$ denotes the extended affine Weyl group of $Q$, which is generated by simple reflections and diagram automorphisms of $Q$.

**Proof of Theorem 7.7** Since we are using reflection functors, it is more convenient to work with $O^\lambda \cong A(v)$. Thus, the parameter $\lambda$ is generic and $O^\lambda$ is Morita equivalent to $O^{\lambda'}$. We need to show that $O^{\lambda'} \cong O^{w(\lambda)}$ for some $w \in W_{\text{ext}}$. As before, $X$ denotes the set of isomorphism classes of indecomposable projective $O^\lambda$-modules. Given such a module $P$, we denote by $[P] \in X$ its isomorphism class. Recall that $\text{Pic}(O^\lambda)$ acts on $X$.

Now, let $P$ be an indecomposable projective $O^\lambda$-module with the property that the orbit $G \cdot [P]$ is uncountable. Put $B = \text{End}_{O^\lambda}(P)^{\text{op}}$. Then $B$ is Morita equivalent to $O^\lambda$, so we may identify $X$ with the set of isomorphism classes of indecomposable projective $B$-modules. Now the Pic($B$)-orbit of $[P]$ is uncountable, hence by Lemma 5.1, $\text{Pic}(B)/\text{Aut}(B)$ is uncountable. In particular, $B$ cannot be isomorphic to any $O^\mu$ by Proposition 5.2.

Thus, using the bijection of Berest, Chalykh, and Eshmatov, we conclude that $O^{\lambda'} \cong \text{End}_{O^\lambda}(P)^{\text{op}}$, where $[P] \in F_\lambda(M_{Q_\infty}(\mu, \alpha))$ such that $\mu$ is obtained from a cyclic permutation of $\lambda$, and $\dim M_{Q_\infty}(\mu, \alpha) = 0$. Since $\mu = w(\lambda)$, where $w$ is an element of the extended affine Weyl group of $Q$, we may assume without loss of generality that $\lambda = \mu$.

Recall that $M_{Q_\infty}(\lambda, \alpha) \neq \emptyset$ if and only if $\alpha = (1, \alpha_0, \cdots, \alpha_{n-1})$ is a positive root for $Q_\infty$, in which case by [CEET, Lemma 2.1],

$$\dim M_{Q_\infty}(\lambda, \alpha) = 2\alpha_0 - \sum (\alpha_i - \alpha_{i+1})^2 \geq 0.$$  

It was established in [CEET] that any positive root $\alpha$ for which $M_{Q_\infty}(\lambda, \alpha) \neq \emptyset$ belongs to the $W_\infty$-orbit of $(1, l, \cdots, l)$, where $2l = \dim M_{Q_\infty}(\lambda, \alpha)$. Therefore, the roots $\alpha$ for which $M_{Q_\infty}(\lambda, \alpha)$ is a zero-dimensional variety form a single $W_\infty$-orbit: that of $(1, 0, \cdots, 0) =: \theta$. The indecomposable projective $O^\lambda$-module corresponding to $M_{Q_\infty}(\lambda, \theta)$ (which is a single point) is $O^\lambda$, so $[O^\lambda] = F(M_{Q_\infty}(\lambda, \theta))$. Thus, for some $w \in W_\infty$, we have

$$[O^\lambda] = F(M_{Q_\infty}(w(\lambda), w(\theta))) = F(R_w(M_{Q_\infty}(\lambda, \theta))).$$

Now, recall the commutative diagram 2, which yields that $[P] = R_w([O^{w(\lambda)}])$ for some $w \in W$. So, $O^{\lambda'} \cong O^{w(\lambda)}$ as desired.

To show that $\text{Pic}(O^\lambda) = \text{Aut}(O^\lambda)$, we need to prove that given an invertible $O^\lambda$-bimodule $M$, then $M \cong O^\lambda$ as a left module. Since $\text{End}_{O^\lambda}(M)^{\text{op}} \cong O^\lambda$, $[M]$ must correspond to a 0-dimensional quiver variety. By the above paragraph, we know that all indecomposable projective $O^\lambda$-modules that correspond to 0-dimensional varieties $M_{Q_\infty}(\lambda, w(\theta))$ must be of the form $R_w([O^{w(\nu)}]) \cong M$ for some $w \in W_\infty$. So, $O^{w(\lambda)} \cong O^\lambda$. Now, by Bavula–Jordan’s isomorphism theorem, we may conclude that $w$ fixes $\theta$, so $M \cong O^\lambda$ and we are done. \hfill \Box
Concluding remark. It is natural to expect that Theorem 2.4 should hold for generic parameters, and \( \text{Pic}(\mathcal{O}^\lambda) = \text{Aut}(\mathcal{O}^\lambda) \). Here is why the proof of Theorem 1.1 does not work for non-type \( A \) deformations: automorphism groups of such deformations are very small (often trivial), so we are no longer able to conclude that if \( \mathcal{O}^\lambda \cong \text{End}_{\mathcal{O}^\lambda}(P) \), where \( P \) is an indecomposable projective \( \mathcal{O}^\lambda \)-module, then \( P \) must correspond to a zero-dimensional quiver variety via the bijection between indecomposable projective \( \mathcal{O}^\lambda \)-modules and quiver varieties. On the other hand, recall that P. Levy [L] has described \( \text{Aut}(\mathcal{O}^\lambda) \) for type \( D \) Kleinian singularities. The results in [L] combined with the usual reduction mod \( p \gg 0 \) technique (should) imply that \( \text{Pic}(\mathcal{O}^\lambda) = \text{Aut}(\mathcal{O}^\lambda) \) at least for generic enough \( \lambda \). We will come back to these questions elsewhere.

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