Effect of soil mass motion on nonlinear forced vibrations of beams on elastic foundation

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Abstract. Based on the nonlinear equation of motion of the beam on the Winkler foundation with the consideration of finite-depth soil mass motion, the nonlinear forced vibrations of the beam were investigated. Applying the Galerkin method and the method of multiple scales, the frequency-response equation and the second-order approximate solution of the primary resonance of the beam were obtained. Then, by means of the numerical calculation and parameter analysis, the effects of parameters about the soil mass motion on the frequency-response curves of the beam on the Winkler foundation were explored, such as foundation depth, soil mass, and Winkler foundation stiffness. The results show that the effect of the soil mass motion on the nonlinear forced vibration of the beam on the Winkler foundation is significant. When the effect of soil mass motion is introduced into the equation of motion of the beam, the range and the softening behavior of primary harmonic resonance of the system are reduced.

1. Introduction

The Winkler foundation model is widely used in engineering practice and can be used to analyze the static and dynamic responses of soil-structure interaction problems [1, 2, 3]. Compared with other elastic foundation models, the Winkler model is the most concise one, in which the elastic foundation is divided into a series of independent linear springs. The subgrade reaction of the Winkler foundation can be expressed as a linear function of the displacement of the beam [4, 5].

Based on the theory of beam on the Winkler foundation, various studies about the linear and nonlinear dynamic responses of soil-structure interaction systems have been published. Thambiratnam and Zhuge [6] analyzed the free vibration of beams on elastic foundation. Applying the nonlinear Winkler model, Li et al. [7] proposed a simplified model of equivalent calculated pile length for the soil-pile-structure interaction analysis. Considering the dynamic interaction between the pipe pile and the outer and inner soil, Liu et al. [8] used the Winkler model to study the vertical vibration of pipe pile groups. Based on the recognition of the energy transfer between the foundation and its supporting structure during the vibration process, it has received increasing attention that the study on the effect of finite-depth soil mass motion on the dynamic response of beams on the Winkler foundation. Jaiswal and Iyengar [9] studied the dynamic response of an infinite beam on a finite depth foundation subjected to a moving force. Considering the effect of soil-structure interaction on the nonlinear dynamic response of the beam and using the expression of subgrade reaction obtained from the equation of motion of Winkler foundation, Ma et al. [10] established a nonlinear dynamic model of the
beam on Winkler foundation with the consideration of finite-depth soil mass motion, and analyzed the linear and nonlinear free vibration of beams on elastic foundation. Therefore, in order to study the nonlinear dynamic response of beams on the Winkler foundation, it is necessary to explore the nonlinear forced vibration of beams on the Winkler foundation with the consideration of finite-depth soil mass motion.

In this paper, based on the nonlinear equation of motion of beams on the Winkler foundation with the soil mass motion, by taking the primary resonance response of the beam as a calculation object, the effects of soil mass motion on the nonlinear forced vibrations of beams on the Winkler foundation are investigated. Using the Galerkin method and the method of multiple scales, the frequency-response equation of the beam is determined. Then, the numerical results of the primary resonance of the beam on the Winkler foundation are obtained. Meanwhile, the effects of the depth, soil mass, elastic modulus, and Winkler parameter of foundation on the primary resonance of the beam are explored.

2. Nonlinear dynamic model

As shown in Fig. 1, this study takes a finite-length beam on elastic foundation subjected to simple harmonic lateral excitation as the research object, in which \( L, b \) and \( h \) are the length, width, and height of beam. \( H \) is the depth of foundation. A Cartesian coordinate system \( O-xy \) is established with the left end of the beam as the origin \( O \) and the axis of the beam as the \( x \)-direction. \( p(x,t) = P \cos(\Omega t) \) is the harmonic excitation, where \( P \) and \( \Omega \) denote the amplitude and frequency of the excitation, respectively. \( u(x,t) \) and \( v(x,t) \) denote the in-plane displacements of the beam along the \( x \) and \( y \) directions.

![Figure 1. Beam on elastic foundation](image)

Referring to Ref. [10], the dimensionless equation of motion of the beam on Winkler foundation with the consideration of finite-depth soil mass motion

\[
v'' + v \left[ 1 + \frac{\rho_s H}{m} \int_0^1 \phi \, dy \right] + \frac{v''}{h} \left[ c + c_s \int_0^1 (1 - \phi) \, dy \right] + k_f v = \frac{P}{m} \cos(\Omega t) - \frac{h}{2} \rho_s H \left[ \nu'(\nu' - k_f \nu) \right] \cos(\Omega t),
\]

where \( \rho_s \) is the soil mass per unit depth per unit length of the beam; \( m \) is the beam mass per unit length of the beam; \( \phi \) is the attenuation function of foundation displacement along the \( y \)-direction; \( c \) is the viscous damping coefficient per unit length of the beam; \( c_s \) is the viscous damping coefficient for the foundation per unit depth per unit length of the beam; \( k_f \) is the modulus of subgrade reaction of Winkler foundation; the dot indicates the derivative with respect to \( t \); the prime indicates the derivative with respect to \( x \).

For simplicity, Eq. (1) can be re-written as

\[
v + \frac{P}{\alpha} \cos(\Omega t) - \frac{h}{2\alpha} \left( \nu'' + \nu'^{'} \right) = \frac{P}{m} \cos(\Omega t) + G_{21}(v,v) + G_{22}(v,v) + G_{23}(v,v) + G_{3}(v,v,v),
\]
where $\beta = -\frac{H}{m}$, $\alpha = 1 + \beta \int_0^1 \rho \mathrm{d}y$, $\mu = \frac{c + c_i \int_0^1 (1 - \varphi) \mathrm{d}y}{\alpha}$; $L_v = \frac{1}{\alpha} \nu'' + \frac{k_f}{\alpha} \nu$ is the linear operator; $G_{21} = -\frac{h(1-\alpha)}{2\alpha} (\nu \psi')'$, $G_{22} = \frac{hk_f}{2\alpha} (\nu \psi')'$, and $G_{23} = -\frac{hc_i}{\alpha} \int_0^1 (1 - \varphi) \mathrm{d}y (\nu \psi')'$ are the quadratic nonlinear operators; $G_i = -\frac{1}{\alpha} \left[ \nu' (\nu \psi')' \right]'$ is the cubic nonlinear operator.

3. Primary resonance ($\Omega = \omega_n$)

In this study, the displacement of the beam $v(x,t)$ can be expanded as

$$v(x,t; \varepsilon) = \sum_{i=0}^{i} \varepsilon^i v_i(x, T_0, T_2) + \cdots,$$

where $\varepsilon$ is a small bookkeeping parameter, $T_i (i = 0, 1, 2)$, $\partial / \partial t = D_0 + \varepsilon^3 D_2 + \cdots$, $D_i = \partial / \partial T_i$.

In order to study the primary resonance of the beam, the detuning parameter $\sigma$ is introduced. Then, the following relationship between the excitation frequency and natural frequency of elastic foundation: $\Omega = \omega_n + \varepsilon^3 \sigma$, where $\omega_n$ is the $n$th order frequency of beam on elastic foundation. To balance the term effects of the nonlinearities [11], we rescale the viscous damping term $\mu$ as $\varepsilon^2 \mu$ and the excitation amplitude $P$ as $\varepsilon^3 P$. Substituting Eq. (3) into (2) and equaling coefficients of like powers of $\varepsilon$, we can obtain

Order $\varepsilon^0$:

$$D_0 v_0 + L V_0 = 0;$$

Order $\varepsilon^2$:

$$D_0 v_2 + L V_2 = G_{21} \left( v_j, v_i \right) + G_{22} \left( v_j, v_i \right);$$

Order $\varepsilon^3$:

$$D_0^3 v_3 + L V_3 = \frac{P}{\alpha} \cos(\Omega T_0) - \mu D_0 v_3 - 2D_0 D_2 v_1 + G_{21} \left( v_j, v_i \right) + G_{22} \left( v_j, v_i \right) + G_3 \left( v_j, v_i, v_i \right),$$

Following Ma et al. [10], the case of boundary conditions, clamped-free beam, is chosen as the representative example. Therefore, the boundary conditions can be written as

$$v_i (0, T_0, T_2) = 0; \quad v_i' (0, T_0, T_2) = 0; \quad v_i'' (1, T_0, T_2) = 0; \quad v_i''' (1, T_0, T_2) = 0;$$

For the primary resonance of the $n$th order mode, the general solution of Eq. (4) is

$$v_i (x, T_0, T_2) = A_i (T_2) e^{i \omega_n x} \phi_i (x) + cc,$$

where $A_i$ is the amplitude; $\phi_i (x)$ is the $n$th order mode shape; $i = \sqrt{-1}$; cc stands for the complex conjugate of the preceding terms.

Substituting Eq. (8) into (5), we can obtain

$$D_0 v_2 + L V_2 = \left[ 1 - \frac{\omega_n^2 (\alpha - 1)}{k_f} \right] (A_i e^{2i \omega_n x} + A_i \bar{A}_i) \times G_2 \left( \phi_i, \phi_i \right) + cc,$$

where the bar denotes the complex conjugate. The solution of Eq. (9) can be written as

$$v_2 (x, T_0, T_2) = \left[ 1 - \frac{\omega_n^2 (\alpha - 1)}{k_f} \right] A_i e^{2i \omega_n x} \psi_{in} (x) + \left[ 1 - \frac{\omega_n^2 (\alpha - 1)}{k_f} \right] A_i \bar{A}_i \psi_{2n} (x) + cc,$$

where $\psi_{in} (x)$ and $\psi_{2n} (x)$ are the second-order configuration functions. Using the method of undetermined coefficients, the solutions of $\psi_{in} (x)$ and $\psi_{2n} (x)$ can be obtained [10].

Substituting Eqs. (8) and (10) into Eq. (6), the solvability condition can be obtained.
2a_i T\alpha A_n + \mu_n A_n - \Gamma_n A^2_n - \frac{P_n}{2a} e^{i\tau_i} = 0 \tag{11}

where \( p_n = \int_0^1 \mu \phi_n(x)dx \); \( 2\mu_n = \int_0^1 \phi_n^2(x)dx \);

\[
\Gamma_n = \frac{-\omega_n^2 (\alpha-1)}{k_f} \left[ 1 - \frac{\omega_n^2 (\alpha-1)}{k_f} \right] \times \left[ 4\int_0^1 G_{22} (\Psi_n, \phi_n) \phi_n dx + \sum_{i=1}^3 \int_0^1 G_{22} (\phi_i, \Psi_m) \phi_i dx \right] + \left[ 1 - \frac{\omega_n^2 (\alpha-1)}{k_f} \right] \times \sum_{i=1}^3 \int_0^1 G_{22} (\phi_i, \Psi_m) + G_{22} (\Psi_m, \phi_i) \phi_i dx + 3\int_0^1 G_1 (\phi_i, \phi_i, \phi_i) \phi_i dx \right]. \tag{12}
\]

We express \( A_n \) in the polar form

\[
A_n = \frac{1}{2} a_n (T_2) e^{i(\varphi_n)} \tag{13},
\]

where \( a_n \) and \( \beta_n \) represent the amplitude and phase, respectively. Substituting Eq. (13) into Eq. (11) and separating real and imaginary parts, we obtain

\[
\frac{p_n}{2a} \cos \gamma_n + \hat{a}_n \omega_n + \frac{\Gamma_n a_n^2}{8} = 0 \tag{14};
\]

\[
\frac{p_n}{2a} \sin \gamma_n + \hat{a}_n \omega_n + \frac{1}{2} \mu_n a_n \omega_n = 0 \tag{15},
\]

where \( \gamma_n = \sigma T_2 - \beta_n \). Setting \( \hat{a}_n = 0 \) and \( \gamma_n = 0 \) in Eqs. (14) and (15), the frequency-response equation can be obtained

\[
\sigma = -\frac{\Gamma_n a_n^2}{8} \left[ \left( \frac{p_n}{2a\omega_n a_n} \right)^2 - \mu_n^2 \right] \tag{16}.\]

Meanwhile, substituting Eqs. (8) and (10) into Eq. (3), the second-order approximation of the displacement \( v(x,t) \) can be written as

\[
v(x,t) = \phi_n(x) a_n \cos(\Omega t - \gamma_n) + \frac{1}{2} \left[ 1 - \frac{\omega_n^2 (\alpha-1)}{k_f} \right] a_n^2 \left[ \cos(2\Omega t - 2\gamma_n)\psi_n(x) + \psi_n(x) \right] + \cdots, \tag{17}
\]

where \( \varphi \) has been set equal to 1.

4. Numerical calculation

In order to study the numerical results, Table 1 gives the dimensional parameters and material properties of the beam and elastic foundation [10]. Referring to Ref. [12] and Table 1, the dimensional Winkler foundation parameter \( k_f = 6.235 \times 10^9 \text{N/m}^2 \) can be calculated. In this section, the effects of parameter \( H \), \( \rho_s \), \( E_s \), and \( k_f \) on the primary resonance of the beam are explored.

4.1. Effect of \( H \)

Fig. 2 shows the frequency-response curves of the primary resonance of the beam on elastic foundation with different depth \( H \), where the solid and dashed lines denote the stable and unstable solutions, respectively. Moreover, the frequency-response curve of the primary resonance of the beam without the effect of soil mass motion also be given in Fig. 2. When the effect of soil mass motion is introduced into the dynamic model of the beam, the frequency-response curves of the beam change significantly. With the increase of depth \( H \), the stiffness of the system increases, the unstable region and amplitude of the primary resonance decrease. In general, with the increase of the foundation depth, the nonlinear characteristics of the primary resonance of the beam on an elastic foundation are significantly weakened. The soil mass motion plays a hardening effect on the dynamic characteristics of the beam.
Table 1. Dimensional parameters and material properties of the elastic foundation and beam

| Parameter | Physical meaning                                      | Value       | Unit     |
|-----------|-------------------------------------------------------|-------------|----------|
| $L$       | Length of the beam                                    | 6.096       | m        |
| $b$       | Width of the beam                                     | 0.610       | m        |
| $h$       | Height of the beam                                    | 0.305       | m        |
| $\rho$    | Density of the beam                                   | $2.403 \times 10^3$ | kg/m$^3$ |
| $E$       | Young’s modulus of the beam                           | $2.482 \times 10^4$ | MPa     |
| $\nu$     | Poisson ratio of the beam                             | 0.25        | --       |
| $c$       | Damping coefficient per unit length of the beam       | 180         | N·s/m$^2$|
| $H$       | Depth of the elastic foundation                       | 10.00       | m        |
| $\rho_s$  | Foundation mass per unit depth per unit length of the beam | $1.037 \times 10^2$ | kg/m$^2$ |
| $E_s$     | Young’s modulus of the elastic foundation             | 23.94       | MPa      |
| $\nu_s$   | Poisson ratio of the elastic foundation               | 0.20        | --       |
| $c_s$     | Damping coefficient for the foundation per unit depth per unit length of the beam | 18          | N·s/m$^3$|

4.2. Effect of $\rho_s$

Fig. 3 shows the effect of the parameter $\rho_s$ on the primary resonance of the beam on the Winker foundation. Under the same conditions, with the increase of soil mass, the amplitude of the primary resonance of the beam increases, while the softening behavior of the primary resonance decreases. Compared with the numerical results shown in Fig. 2, we can get a consistent conclusion that the soil mass motion has a significant inhibiting effect on the nonlinear characteristics of the primary resonance of the beam. Therefore, in the engineering practice, the nonlinear dynamic characteristics of the supporting structure can be improved by improving the soil density in the limited depth range.

4.3. Effect of $E_s$

Fig. 4 shows the effect of elastic modulus $E_s$ of foundation on the frequency-response curves of the primary resonance of the beam on elastic foundation. Obviously, with the increase of elastic modulus $E_s$, the amplitude and the softening behavior of the primary resonance of the beam on elastic...
foundation decrease. Compared with the frequency-response curves of the beam on elastic foundation with different parameters $E_s$, it can be found that the effect of elastic modulus of foundation on the primary resonance of the beam is weak if only changing the value of $E_s$. Moreover, when the effect of soil mass motion is introduced into the dynamic model of the soil-structure interaction system, the effect of the elastic modulus of the foundation on the nonlinear dynamic response of the beam is significantly weakened.

![Figure 3. Frequency-response curves of beam on elastic foundation with different $\rho_s$.](image)

![Figure 4. Frequency-response curves of beam on elastic foundation with different $E_s$.](image)

4.4. Effect of $k_f$

To provide direct results, we need to analyze the effect of the modulus of the subgrade reaction $k_f$ on the primary resonance of the beam. Fig. 5 shows the frequency-response curves of the beam on the Winkler foundation with different $k_f$. With the increase $k_f$, the amplitude and the softening behavior of the primary resonance of the beam significantly decrease. Obviously, the Winkler foundation parameter $k_f$ plays a hardening role in the primary resonance of the beam.
5. Conclusions

Based on the nonlinear dynamic model of the beam on the Winkler foundation with consideration of the effect of the soil mass motion of finite depth, the effects of soil mass motion on the nonlinear forced vibration of the beam are investigated. Using the Galerkin method and the method of multiple scales, the frequency-response equation and the second-order approximate solution of displacement of the beam are obtained. Then, the numerical results and discussions are presented to explore the effect of foundation depth, soil mass, stiffness and Winkler foundation parameter on the primary resonance of the beam.

The numerical results show that dynamic characteristics of the beam are significantly changed when the soil mass motion is introduced into the dynamic model of the beam on the Winkler foundation. Moreover, with the increase of the foundation depth, the amplitude of the primary resonance of the beam decreases. However, the amplitude would increase with the increase of the soil mass. In addition, the soil mass motion significantly reduces the softening behavior of the nonlinear forced vibration of the beam on the Winkler foundation.

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