\(\tau\)-FUNCTION EVALUATION OF GAP PROBABILITIES IN ORTHOGONAL AND SYMPLECTIC MATRIX ENSEMBLES

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It has recently been emphasized that all known exact evaluations of gap probabilities for classical unitary matrix ensembles are in fact \(\tau\)-functions for certain Painlevé systems. We show that all exact evaluations of gap probabilities for classical orthogonal matrix ensembles, either known or derivable from the existing literature, are likewise \(\tau\)-functions for certain Painlevé systems. In the case of symplectic matrix ensembles all exact evaluations, either known or derivable from the existing literature, are identified as the mean of two \(\tau\)-functions, both of which correspond to Hamiltonians satisfying the same differential equation, differing only in the boundary condition. Furthermore the product of these two \(\tau\)-functions gives the gap probability in the corresponding unitary symmetry case, while one of those \(\tau\)-functions is the gap probability in the corresponding orthogonal symmetry case.

1 Introduction

An ensemble of \(N \times N\) random matrices \(X\) with joint probability density of the matrix elements proportional to
\[
\exp \left( \sum_{j=1}^{\infty} a_j \text{Tr}(X^j) \right) =: \prod_{j=1}^{N} g(x_j),
\]
x\(_j\) denoting the eigenvalues, is invariant under similarity transforms \(X \mapsto A^{-1}XA\). In particular, if \(X\) is an Hermitian matrix with real, complex and quaternion real elements, labelled by the parameter \(\beta\) taking the values \(\beta = 1, 2\) and \(4\) respectively, then the subgroups of unitary matrices which conserve this feature of \(X\) under similarity transformations are the orthogonal \((\beta = 1)\), unitary \((\beta = 2)\) and unitary symplectic matrices \((\beta = 4)\). For this reason the ensemble is said to have an orthogonal \((\beta = 1)\), unitary \((\beta = 2)\) or symplectic symmetry \((\beta = 4)\). The eigenvalue probability density function (PDF) for these ensembles has the explicit form
\[
\frac{1}{C} \prod_{j=1}^{N} g(x_j) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta,
\]

The function \(g(x)\) in (1.1) and (1.2) is referred to as a weight function. In the cases \(\beta = 1\) and \(\beta = 2\) the weight functions
\[
g(x) = g_1(x) = \begin{cases} 
  e^{-x^2}/2, & \text{Gaussian} \\
  x^{(a-1)/2} e^{-x^2} (x > 0), & \text{Laguerre} \\
  (1 - x)^{(a-1)/2}(1 + x)^{(b-1)/2} (-1 < x < 1), & \text{Jacobi} \\
  (1 + x^2)^{-(a+1)/2}, & \text{Cauchy}
\end{cases}
\]
and
\[
g(x) = g_2(x) = \begin{cases} 
  e^{-x^2}, & \text{Gaussian} \\
  x^a e^{-x} (x > 0), & \text{Laguerre} \\
  (1 - x)^a(1 + x)^b (-1 < x < 1), & \text{Jacobi} \\
  (1 + x^2)^{-a}, & \text{Cauchy}
\end{cases}
\]
are said to define classical matrix ensembles with an orthogonal and unitary symmetry respectively, or simply classical orthogonal and unitary ensembles (a similar definition applies in the symplectic case — see e.g. [3]). We recall (see e.g. the introduction of [12]) that the Cauchy ensemble includes as a special case the PDF

$$\frac{1}{C} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta,$$  \hspace{1cm} (1.5)$$

which specifies the Dyson circular ensembles. The two ensembles are related by the stereographic projection

$$e^{i\theta_j} = \frac{1 - ix_j}{1 + ix_j}. \hspace{1cm} (1.6)$$

In particular, changing variables in (1.5) according to (1.6) gives a PDF of the form (1.2) with $g(x)$ a Cauchy weight function, which in the cases $\beta = 1$ and 2 is specified by (1.3) and (1.4) with $\alpha = N$.

Our interest is in a special property of the probability $E_\beta(0; I; g(x); N)$ of having no eigenvalues in the interval $I$ when the eigenvalue PDF is specified by (1.2) in the case that $g(x)$ is classical. The probability is specified as a multiple integral by

$$E_\beta(0; I; g(x); N) = \frac{1}{C} \prod_{j=1}^{N} \int_{I_0 \setminus I} dx_j \ g(x_j) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta, \hspace{1cm} (1.7)$$

where $I_0$ is the interval of support of $g(x)$. The special property is that for $g(x)$ classical $E_\beta$ admits Painlevé transcendent evaluations for certain $I$ (the evaluations are in some cases restricted also to certain scaled limits). We will focus on a structural aspect of these formulas, by showing that in the orthogonal case all known Painlevé transcendent evaluations can be identified as $\tau$-functions for Hamiltonians associated with the Painlevé functions, and in the symplectic case as the mean of two $\tau$-functions.

Our work builds on the recently emphasized [13, 7] fact that all gap probabilities for classical unitary ensembles that have been characterized as the solution of a single differential equation, are in fact $\tau$-functions for certain Painlevé systems. Such characterizations of the gap probability for classical unitary matrix ensembles are known when the gap consists of a single interval including an end-point of the support [27, 2, 17, 31, 7], or a double interval symmetrically placed about the origin again including the end-points of the support or the origin (applicable to even weight functions only) [27, 32, 31]. In the special case of the gap probability for scaled, infinite GUE matrices in the bulk, the identification as a $\tau$-function for a Painlevé system was made by Okamoto and quoted in the original paper of Jimbo et al. [20, pg. 152] deriving the Painlevé evaluation. For the more general problem of characterizing the gap probabilities in the case of multiple excluded intervals, the fact that the probability is the $\tau$-function for certain integrable systems associated with monodromy preserving deformations of linear differential equations with rational coefficients was a main theme of [20], and then generalized to a more general setting (but not the most general case of interest in random matrix theory) by Palmer [25]. Harnad and Its [18] have recently discussed the work of Palmer from a Riemann-Hilbert problem perspective. Identifications of the gap probabilities in the case of multiple excluded intervals as $\tau$-functions in the Sato theory is a theme of the work of Adler, van Moerbeke and collaborators (see e.g. [3]).

The situation with the exact evaluation of gap probabilities for matrix ensembles with an orthogonal symmetry is immediately different due to the restricted number of evaluations in terms of Painlevé transcendents presently known [30, 10, 11]. In the orthogonal case, the exact evaluations can be catalogued into two distinct mathematical structures — the finite $N$ ensembles and their scaling limit for which the $\tau$-function identification is immediate, and the infinite Gaussian and Laguerre ensembles scaled at the soft and hard edges respectively in which the known Painlevé transcendent evaluations reduce to a $\tau$-function...
Laguerre: $e^{-x^2}$

Jacobi: $(1 - x^2)^\alpha$

classical ensemble

Table 1: Even classical weights, their transformed form, and the corresponding classical unitary ensemble with $N/2$ eigenvalues.

after some calculation. In the symplectic case all known exact evaluations result from a formula relating the gap probability in the symplectic case to that in the orthogonal and unitary cases. Further special features of the exact evaluations in the orthogonal and unitary cases then allows the exact evaluations in the symplectic case to be identified as the mean of two $\tau$-functions, both of which correspond to Hamiltonians satisfying the same differential equation, differing only in the boundary condition.

2 Orthogonal matrix ensembles

2.1 Finite $N$ ensembles

It has been shown in [10] that for the classical weights [13], having the additional property of being even (which is the case for the Gaussian, symmetric Jacobi ($a = b$) and Cauchy weights),

$$E_1(0; (-s, s); g_1(x); N) = E_2(0; (0, s^2); x^{-1/2}g_2(x^{1/2}); N/2)$$

where on the RHS $x > 0$, and it is assumed $N$ is even. Now a unitary ensemble with weight $x^{-1/2}g_2(x^{1/2})$, in which $g_2(x)$ is an even classical weight, is equal to another unitary ensemble with a classical weight, after a suitable change of variables as detailed in Table 1. Hence it follows that

$$E_2(0; (0, s^2); x^{-1/2}g_2(x^{1/2}); N/2) = \begin{cases} E_2(0; (0, s^2); x^{-1/2}e^{-x}; N/2), & \text{Gaussian} \\ E_2(0; (-1, 2s^2 - 1); (1 + x)^{-1/2}(1 - x)^a; N/2), & \text{symmetric Jacobi} \\ E_2(0; (-1, (s^2 - 1)/(s^2 + 1)); (1 + x)^{-1/2}(1 - x)^a-N+1/2; N/2), & \text{Cauchy} \end{cases}$$

This substituted in (2.1) gives $E_1(0; (-s, s); g_1(x); N)$ for the even classical orthogonal ensembles in terms of $E_2$ for certain classical unitary ensembles. The latter furthermore have the gap free interval including an end-point of the support of the weight function. In such a case, we can deduce from the existing literature that $E_2$, and consequently $E_1$, is a $\tau$-function for an appropriate Painlevé system.

Consider first $E_2(0; (0, s); x^a e^{-x}; N)$, specifying the probability that there are no eigenvalues in the interval $(0, s)$ of the Laguerre unitary ensemble. Following [13] and [19] introduce the Hamiltonian $H_V$ associated with the Painlevé V equation by

$$tH_V = q(q - 1)^2p^2 - \left\{(v_2 - v_1)(q - 1)^2 - 2(v_1 + v_2)q(q - 1) + tq\right\}p + (v_3 - v_1)(v_4 - v_1)(q - 1)$$

where the parameters $v_1, \ldots, v_4$ are constrained by

$$v_1 + v_2 + v_3 + v_4 = 0.$$

The relationship of (2.3) to $P_V$ can be seen by eliminating $p$ in the Hamilton equations

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}.$$
One finds that $q$ satisfies the equation

$$y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{1}{t} y' + \frac{(y-1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{t} + \delta \frac{y(y+1)}{y-1} \tag{2.6}$$

with

$$\alpha = \frac{1}{2} (v_2 - v_1)^2, \quad \beta = -\frac{1}{2} (v_2 - v_1)^2, \quad \gamma = 2v_1 + 2v_2 - 1, \quad \delta = -\frac{1}{2} \tag{2.7}$$

This is the general $P_V$ equation with $\delta = -\frac{1}{2}$ (recall that the general $P_V$ equation with $\delta \neq 0$ can be reduced to the case with $\delta = -\frac{1}{2}$ by the mapping $t \mapsto \sqrt{-2\delta} t$). Now introduce the auxiliary Hamiltonian

$$\sigma_V = t H_V + (v_3 - v_1)(v_4 - v_1). \tag{2.8}$$

Of course with $t H_V$ replaced by $\sigma_V$ in (2.5), the Hamilton equations remain unchanged so $\sigma_V/t$ is also a Hamiltonian for the same $P_V$ system. The quantity $\sigma_V$ satisfies the second order, second degree differential equation

$$(t \sigma'')^2 - [\sigma - t \sigma' + 2(\sigma')^2 + (v_0 + v_1 + v_2 + v_3) \sigma']^2 + 4(v_0 + \sigma')(v_1 + \sigma')(v_2 + \sigma')(v_3 + \sigma') = 0, \tag{2.9}$$

with

$$v_0 = 0, \quad v_1 = v_2 - v_1, \quad v_2 = v_3 - v_1, \quad v_3 = v_4 - v_1 \tag{2.10}$$

(because (2.9) is symmetrical in $\{v_k\}$ any permutation of these values is also valid). Conversely, each solution (with $\sigma'' \neq 0$) of (2.9) leads to a solution of the system (2.7) (2.3).

The $\tau$-function associated with the Hamiltonian $\sigma_V/t$ is specified by

$$\sigma_V = t \frac{d}{dt} \log \tau_{\sigma_V}(t). \tag{2.11}$$

But from the work of Tracy and Widom [27] we know that

$$t \frac{d}{dt} \log E_2(0; (0, t); x^a e^{-x}; N) \tag{2.12}$$

satisfies (2.9) with

$$v_0 = 0, \quad v_1 = 0, \quad v_2 = N + a, \quad v_3 = N,$$

subject to the boundary condition

$$\sigma(t) \sim \frac{\Gamma(N + a + 1)}{t^{a+1}} \quad \text{as} \quad t \to 0^+. \tag{2.13}$$

Consequently, after equating (2.11) and (2.12), and normalizing $\tau_{\sigma_V}$ so that $\tau_{\sigma_V}(0) = 1$, we have

$$E_2(0; (0, t); x^a e^{-x}; N) = \tau_{\sigma_V}(t) \bigg|_{v_0 = 0, v_1 = 0, v_2 = N + a, v_3 = N} \tag{2.14}$$

With $a = -1/2$ we see that this corresponds to the Gaussian case of (2.3). Recalling (2.4) then gives the sought $\tau$-function formula for the Gaussian orthogonal ensemble,

$$E_1(0; (-s, s); e^{-x^2/2}; N) = \tau_{\sigma_V}(s^2) \bigg|_{v_0 = 0, v_1 = 0, v_2 = N - 1/2, v_3 = N} \tag{2.15}$$

Consider next $E_2(0; (-1, s); (1-x)^a(1+x)^b; N)$, specifying the probability that there are no eigenvalues in the interval $(-1, s)$ of the Jacobi unitary ensemble. According to (2.3) this is relevant to both the
symmetric Jacobi and Cauchy cases. Introduce the Hamiltonian $H_{VI}$ associated with the Painlevé VI equation by \[22\]

$$t(t-1)H_{VI} = q(q-1)(q-t)p^2 - \{\chi_0(q-1)(q-t) + \chi_1(q-t) + (\theta - 1)q(q-1)\}p + \chi(q-t) \tag{2.15}$$

where

$$\chi = \frac{1}{4}(\chi_0 + \chi_1 + \theta - 1)^2 - \frac{1}{4}\chi_\infty^2.$$  

Eliminating $p$ from the corresponding Hamilton equations \((\ref{2.13})\) shows that $q$ satisfies the P$_{VI}$ equation

$$y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) y'$$

$$+ \frac{y(y-1)(y-t)}{t^2(t-1)^2} (\alpha + \beta \frac{t}{y^2} + \gamma \frac{t-1}{(y-1)^2} + \delta \frac{t(t-1)}{(y-t)^2})$$

with

$$\alpha = \frac{1}{2}\chi_\infty^2, \quad \beta = -\frac{1}{2}\chi_0^2, \quad \gamma = \frac{1}{2}\chi_1^2, \quad \delta = \frac{1}{2}(1-\theta^2).$$

Furthermore, the auxiliary Hamiltonian

$$h_{VI} = t(t-1)H_{VI} + (b_1b_3 + b_1b_4 + b_3b_4)t - \frac{1}{2} \sum_{1 \leq j < k \leq 4} b_jb_k, \tag{2.16}$$

where

$$b_1 = \frac{1}{2}(\chi_0 + \chi_1), \quad b_2 = \frac{1}{2}(\chi_0 - \chi_1), \quad b_3 = \frac{1}{2}(\theta - 1 + \chi_\infty), \quad b_4 = \frac{1}{2}(\theta - 1 - \chi_\infty),$$

satisfies the differential equation

$$h_{VI}'(t(1-t)h_{VI}')^2 + \{h_{VI}'[2h_{VI} - (2t-1)h_{VI}'] + b_1b_2b_3b_4\}^2 = \prod_{k=1}^{4}(h_{VI}' + b_k^2) \tag{2.17}$$

and conversely, each solution of \((\ref{2.17})\) such that $h_{VI}' \neq 0$ leads to a solution of the corresponding Hamilton equations. Now, we know from the work of Haine and Semengue \[17\], and Borodin and Deift \[7\], that

$$\sigma(t) := t(t-1)\frac{d}{dt} \log E_2(0; (-1, -1 + 2t); (1-x)^a(1+x)^b; N) - b_1b_2t + \frac{1}{2}(b_1b_2 + b_3b_4)$$

with

$$b_1 = b_2 = N + \frac{a+b}{2}, \quad b_3 = \frac{a+b}{2}, \quad b_4 = \frac{a-b}{2}$$

satisfies \((\ref{2.17})\). Comparing with \((\ref{2.16})\) we see that with this choice of parameters

$$t(t-1)\frac{d}{dt} \log E_2(0; (-1, -1 + 2t); (1-x)^a(1+x)^b; N) = h_{VI} + b_1b_2t - \frac{1}{2}(b_1b_2 + b_3b_4). \tag{2.18}$$

Thus, denoting the RHS of \((\ref{2.18})\) by $\tilde{h}_{VI}$, we see from \((\ref{2.16})\) that $\tilde{h}_{VI}/t(t-1)$ is a Hamiltonian for the P$_{VI}$ system, and defining the corresponding $\tau$-function by

$$\tilde{h}_{VI} = t(t-1)\frac{d}{dt} \log \tau_{\tilde{h}_{VI}}(t)$$

we have that

$$E_2(0; (-1, -1 + 2t); (1-x)^a(1+x)^b; N) = \tau_{\tilde{h}_{VI}}(t) \bigg|_{b_1=b_2=N+(a+b)/2}^{b_3=(a+b)/2, b_4=(a-b)/2}. \tag{2.19}$$

Recalling \((\ref{2.2})\) and \((\ref{2.1})\) then gives the sought $\tau$-function formulas for the gap probabilities in the Jacobi orthogonal and Cauchy orthogonal ensembles,

$$E_1(0; (-s, s); (1-x^2)^{(a-1)/2}; N) = \tau_{h_{VI}}(s^2) \bigg|_{b_1=b_2=N+(a-1)/2}^{b_3=(a-1)/2, b_4=(a+1)/2}. \tag{2.20}$$

$$E_1(0; (-s, s); (1+x^2)^{(a+1)/2}; N) = \tau_{h_{VI}}\left(\frac{s^2}{s'^2+1}\right) \bigg|_{b_1=b_2=a/2}^{b_3=(a-N)/2, b_4=(a-N+1)/2}. \tag{2.21}$$
2.2 Bulk scaling limit

Let us consider now the \( N \to \infty \) bulk scaling limit of an orthogonal ensemble, and the quantity \( E_1^{\text{bulk}}(0; 2s) \) specifying the probability that there are no eigenvalues in an interval of length \( 2s \) with the mean spacing between eigenvalues equal to unity. By an appropriate scaling, each of the probabilities in (2.14), (2.20) and (2.21) tends to \( E_1^{\text{bulk}}(0; 2s) \). For example, in the Gaussian case the required scaling is \( s \mapsto \pi s/\sqrt{2N} \) and so

\[
E_1^{\text{bulk}}(0; 2s) = \lim_{N \to \infty} E_1(0; (-\frac{\pi s}{\sqrt{2N}}, \frac{\pi s}{\sqrt{2N}}); e^{-x^2/2}; N).
\]

This scaling applied to (2.14) is known to lead to the result [10]

\[
E_1^{\text{bulk}}(0; 2s) = \exp \left( -\int_0^{\pi s^2} \sigma_B(t) \left|_{a=-1/2} \frac{dt}{t} \right. \right)
\]

where \( \sigma_B(t) \) satisfies the equation

\[
(t\sigma_B'^2)^2 + \sigma_B'((4\sigma_B - 1) - a^2(\sigma_B')^2 = 0
\]

subject to the boundary condition

\[
\sigma_B(t) \sim \frac{1}{4} \left[ J_{a+1}(\sqrt{t}) - J_{a-1}(\sqrt{t}) \right] \sim \frac{t^{1+a}}{2^{2+a} \Gamma(1+a) \Gamma(2+a)}.
\]

In fact the expression (2.22) is precisely the \( \tau \)-function for a particular P_{III} system. To see this, following Okamoto [24], introduce the Hamiltonian

\[
th = q^2p^2 - (q^2 + v_1q - t)p + \frac{1}{2}(v_1 + v_2)q.
\]

Substituting this form of \( H \) in the Hamilton equations (2.5) and eliminating \( p \) shows that \( y(s) = q(t)/s \), \( t = s^2 \), satisfies the general Painlevé III equation (Painlevé III' in the notation of [24])

\[
\frac{d^2y}{ds^2} = \frac{1}{y} \left( \frac{dy}{ds} \right)^2 - \frac{1}{s} \frac{dy}{ds} + \frac{1}{s} (\alpha y^2 + \beta) + \gamma y^3 + \delta y
\]

with

\[
\alpha = -4v_2, \quad \beta = 4(v_1 + 1), \quad \gamma = 4, \quad \delta = -4.
\]

It is shown in [24] that the auxiliary quantity

\[
h = th + \frac{1}{4}v_1^2 - \frac{1}{2}t
\]

satisfies the equation

\[
(th'')^2 + v_1v_2h' - (4h')^2 + (h - th) - \frac{1}{4}(v_1^2 + v_2^2) = 0,
\]

and conversely all solutions of this equation (assuming \( h'' \neq 0 \)) lead to the P_{III'} system. It is a simple exercise to verify from the fact that \( h \) satisfies (2.26), the result that

\[
\sigma_{\text{III'}}(t) := -(th) \left|_{t=t/4} \frac{v_1}{4}(v_1 - v_2) + \frac{t}{4} \right.
\]

satisfies the equation

\[
(t\sigma_{\text{III'}}'^2)^2 - v_1v_2(\sigma_{\text{III'}}')^2 + \sigma_{\text{III'}}'((4\sigma_{\text{III'}} - 1) - (\sigma_{\text{III'}} - t\sigma_{\text{III'}}') - \frac{1}{4}(v_1 - v_2)^2 = 0.
\]
We note from (2.27) that $-\sigma_{III}(t)/t$ is a Hamiltonian for the $P_{III}'$ system, so we can introduce the corresponding $\tau$-function by

$$\sigma_{III}'(t) = -t \frac{d}{dt} \log \tau_{III}(t).$$  \hspace{2cm} (2.29)

Now the equation (2.28) with $v_1 = v_2 = a$ is identical to (2.23), so comparison of (2.29) and (2.22) gives the $\tau$-function evaluation

$$E_{\text{bulk}}^1(0; 2s) = \tau_{III}(\pi^2 s^2) \bigg|_{v_1=v_2=-1/2}.$$

The boundary condition satisfied by $\sigma_{III}'(t)$ is the $a = -1/2$ case of (2.24),

$$\sigma_{III}'(t) \sim \frac{\sqrt{t}}{2\pi} \left[ 1 + \frac{\sin 2\sqrt{t}}{2\sqrt{t}} \right] \sim \frac{\sqrt{t}}{\pi}$$ \hspace{2cm} (2.31)

### 2.3 Cumulative distribution of the largest eigenvalue in the scaled infinite GOE

The GOE has the property that to leading order the support of the spectrum is confined to the interval $[-\sqrt{2N}, \sqrt{2N}]$. It was shown in [9] that by scaling the eigenvalues

$$\lambda \mapsto \sqrt{2N} + \frac{\lambda}{\sqrt{2N^{1/6}}},$$

so that the origin is at the right hand edge of the leading support and the eigenvalue positions then measured in units of $1/\sqrt{2N^{1/6}}$, the distribution functions describing the eigenvalues in the neighbourhood of this edge (referred to as a soft edge since the density on both sides is non-zero) are well defined.

It was shown by Tracy and Widom [30] (see [11] for a simplified derivation) that

$$E_{\text{soft}}^1(0; (s, \infty)) := F_1(s) := \lim_{N \to \infty} E_1\left(0; \left(\sqrt{2N} + \frac{s}{\sqrt{2N^{1/6}}} \right) \infty; N\right)$$

$$= e^{-\frac{1}{2} \int_{s}^{\infty} q(t) \, dt} e^{\frac{1}{2} \int_{s}^{\infty} q(t) \, dt}$$

where $q(t)$ is the solution of the non-linear equation

$$q'' = tq + 2q^3,$$ \hspace{2cm} (2.34)

subject to the boundary condition

$$q(t) \sim -\text{Ai}(t) \quad \text{as} \quad t \to \infty,$$ \hspace{2cm} (2.35)

where $\text{Ai}(t)$ denotes the Airy function. (Here we have replaced $q$ by $-q$ relative to its use in the original work; this is valid because (2.34) is unchanged by this mapping.) Since the general Painlevé II equation reads

$$q'' = tq + 2q^3 + \alpha,$$ \hspace{2cm} (2.36)

(2.34) is the special case $\alpha = 0$ of $P_{II}$. Thus (2.33) represents an explicit evaluation of the gap probability in terms of a Painlevé transcendent. It is the objective of this subsection to show that in fact (2.33) can be identified as a $\tau$-function corresponding to the Painlevé II system with $\alpha = 0$. Consequently its logarithmic derivative satisfies a single nonlinear differential equation.

Now, in the case of the probability analogous to $F_1(s)$ in the infinite, scaled Gaussian unitary ensemble (GUE), the known exact evaluation [28] allows one to immediately make an identification with a $\tau$-function
It is relevant for the purpose of identifying (2.33) to revise the theory underlying this result. Tracy and Widom [28] have derived the result

\[ E_{2 \text{soft}}^\infty(0; (s, \infty)) := F_2(s) := \lim_{N \to \infty} E_2(0; (\sqrt{2N + s}/\sqrt{2}N^{1/6}, \infty); N) = \exp \left(- \int_s^\infty R(t) \, dt \right), \tag{2.37} \]

where \( R(t) \) satisfies the second order second degree differential equation

\[ (R'')^2 + 4R'((R')^2 - tR' + R) = 0, \tag{2.38} \]

and have furthermore derived the alternative formula

\[ F_2(s) = e^{-\int_s^\infty \left(t-s\right)q^2(t) \, dt}, \tag{2.39} \]

where \( q(t) \) is the same Painlevé II transcendent as in (2.33).

To see how the evaluation (2.37) relates to a \( \tau \)-function for the Painlevé II system, we recall that in the Hamiltonian formalism of the PII equation [21], one defines a Hamiltonian \( H_{\text{II}} \) by

\[ H_{\text{II}} = -\frac{1}{2}(2q^2 - p + t)p - (\alpha + \frac{1}{2})q. \tag{2.40} \]

The canonical coordinate \( q \) and momenta \( p \) must satisfy the Hamilton equations (2.3). Elimination of the variable \( p \) between these equations shows that \( q \) satisfies the Painlevé II equation (2.36). Furthermore the Hamiltonian (2.40), regarded as a function of \( t \), satisfies the second order second degree differential equation

\[ (H_{\text{II}}')^2 + 4(H_{\text{II}}')^3 + 2H_{\text{II}}'(tH_{\text{II}}' - H_{\text{II}}) - \frac{1}{4}(\alpha + \frac{1}{2})^2 = 0, \tag{2.41} \]

referred to as the Jimbo-Miwa-Okamoto \( \sigma \)-form for \( \text{PII} \). It is also straightforward to show that \( H_{\text{II}} \) can be expressed in terms of the Painlevé II transcendent \( q \) according to

\[ H_{\text{II}} = \frac{1}{2}(q')^2 - \frac{1}{2}(q^2 + 1/2t)^2 - (\alpha + 1/2)q. \tag{2.42} \]

Finally, we recall that the \( \tau \)-function associated with the Painlevé II Hamiltonian is defined by

\[ H_{\text{II}} = d \log \tau_{\text{II}}. \tag{2.43} \]

Setting

\[ u(t; \alpha + 1/2) = -2^{1/3}H_{\text{II}}(-2^{1/3}t) \]

we see from (2.41) that \( u \) satisfies the equation

\[ (u'')^2 + 4u' \left[(u')^2 - tu' + u\right] - (\alpha + 1/2)^2 = 0. \tag{2.45} \]

Comparison of (2.45) with (2.38) shows

\[ R(t) = u(t; 0) = -2^{1/3}H_{\text{II}}(-2^{1/3}t) \bigg|_{\alpha = -1/2}. \tag{2.46} \]

In light of this identification, comparison of (2.37) and (2.43) then shows,

\[ F_2(s) = \tau_{\text{II}}(-2^{1/3}s) \bigg|_{\alpha = -1/2}. \tag{2.47} \]

The appropriate boundary condition for this \( \tau \)-function is most simply expressed in terms of \( R(t) \),

\[ R(t) \sim [\text{Ai}'(t)]^2 - t[\text{Ai}(t)]^2. \tag{2.48} \]
Thus, up to a scale factor, $F_2(s)$ is precisely the $\tau$-function associated with the Hamiltonian (2.40) for the Painlevé II system with $\alpha = -1/2$. A curious feature of (2.46), which follows from (2.42), is that $R(t)$ is naturally expressed in terms of the Painlevé II transcendent $q = q(t; -1/2)$, whereas the result (2.39) involves the Painlevé II transcendent with $\alpha = 0$. In particular, (2.37) and (2.39) give

$$R'(t) = -q^2(t; 0)$$

while (2.46), (2.40) and the first of the Hamilton equations (2.5) give

$$R'(t) = -\frac{1}{2^{1/3}} \left[ q'(t, -1/2) + q^2(t, -1/2) + \frac{t}{2} \right] \bigg|_{t \to 2^{-1/3}t}.$$  

In fact, as noted in [13], for $\epsilon = \pm 1$, it is true that [16]

$$-\epsilon 2^{1/3} q^2(-2^{-1/3}t, 0) = \frac{d}{dt} q(t, \frac{1}{2} \epsilon) - \epsilon q^2(t, \frac{1}{2} \epsilon) - \frac{1}{2} t,$$

which reconciles (2.50) with (2.49).

We are now in a position to identify (2.33) with a $\tau$-function. The formula (2.50) is just the special case $a = 0$ of the identity

$$\frac{d}{dt} H_{11}(t) \bigg|_{\alpha = a - 1/2} = -2^{-1/3} \frac{d}{dt} u(-2^{-1/3}t; a) = -\frac{1}{2} \left[ q'(t, a - 1/2) + q^2(t, a - 1/2) + \frac{t}{2} \right],$$

which is derived from (2.44), (2.41) and the first of the Hamilton equations (2.5). To make use of this result we first note that the equation (2.33) can be written

$$F_1(s) = e^{-\frac{1}{2} \int_s^\infty (t-s)(q^2(t)+q'(t)) \, dt}.$$  

The identity (2.52) with $a = 1/2$ allows this in turn to be rewritten as

$$F_1(s) = \exp \left( - \int_s^\infty (t-s) \frac{d}{dt} \left[ 2^{-1/3} u(-2^{-1/3}t; 1/2) - \frac{t^2}{8} \right] dt \right)$$

$$= \exp \left( \int_s^\infty \left[ 2^{-1/3} u(-2^{-1/3}t; 1/2) - \frac{t^2}{8} \right] dt \right)$$

$$= \exp \left( - \int_s^\infty \left[ H_{11}(t) \bigg|_{\alpha = 0} + \frac{t^2}{8} \right] dt \right),$$

where the final equality follows from (2.44).

We now associate with $H$ the auxiliary Hamiltonian

$$h_{11} = H_{11} + \frac{t^2}{8}. \tag{2.55}$$

Of course, the Hamilton equations (2.5) remain valid for $H$ replaced by $h$, so $h$ is also a Hamiltonian for the same Painlevé II system. Introducing the corresponding $\tau$-function by

$$h_{11} = \frac{d}{dt} \log \tau_{h_{11}}, \tag{2.56}$$

we see from (2.50) that

$$F_1(s) = \tau_{h_{11}}(s) \bigg|_{\alpha = 0}, \tag{2.57}$$

which is our sought result. Note that $h_{11}$ satisfies (2.44) with the substitution $H_{11} = h_{11} - t^2/8$. It follows from (2.42), (2.55) and (2.39) that we seek the solution of this equation with $\alpha = 0$ and such that

$$h_{11}(t) \sim \frac{1}{2} \text{Ai}(t) + \frac{1}{2} \left\{ [\text{Ai}'(t)]^2 - t|\text{Ai}(t)|^2 \right\}.$$

Thus, we have found the Painlevé II transcendent with $\alpha = 0$. In particular, (2.37) and (2.39) give

$$R'(t) = -q^2(t; 0) \tag{2.49}$$

while (2.46), (2.40) and the first of the Hamilton equations (2.5) give

$$R'(t) = -\frac{1}{2^{1/3}} \left[ q'(t, -1/2) + q^2(t, -1/2) + \frac{t}{2} \right] \bigg|_{t \to 2^{-1/3}t} \tag{2.50}.$$
Unlike the situation with $E_{1}^\text{bulk}(0; 2s)$, in which the corresponding finite system gap probability $E_{1}(0; (-s, s); e^{-x^{2}/2}; N)$ is itself a $\tau$-function, there is no known Painlevé transcendent evaluation of the finite $N$ quantity in the definition (2.33) of $F_{1}(s)$. Nonetheless, (2.57) can be obtained as a limiting sequence of finite $N$ Painlevé transcendent evaluations, which in fact is how we were led to (2.57) in the first place [14]. The finite $N$ results are not for gap probabilities though [14]. Rather they relate to the quantity $f_{N}^{(\text{inv})}$ specifying the number of fixed point free involutions of $\{1, 2, \ldots, 2N\}$ constrained so that the length of the maximum decreasing subsequence is less than or equal to $2l$. This is specified by the generating function

$$P_{l}(t) := e^{-t^{2}/2} \sum_{N=0}^{\infty} \frac{t^{2N}}{2^{2N}N!(2N-1)!!},$$

(2.59)

which from the work of Rains [20] (see also [3]) has the integral representation

$$P_{l}(t) = e^{-t^{2}/2} \left( \frac{1}{2\pi} \right)^{l} \int_{0}^{\pi} d\theta_{1} \cdots \int_{0}^{\pi} d\theta_{l} e^{2t \sum_{j=1}^{l} \cos \theta_{j}} \prod_{j=1}^{l} |1 - z_{j}|^{2} \prod_{1 \leq j < k \leq l} |1 - z_{j}z_{k}|^{2} |z_{j} - z_{k}|^{2},$$

(2.60)

where $z_{j} := e^{i\theta_{j}}$. Although not at all obvious from the definition, it has been proved in [3] that

$$\lim_{l \rightarrow \infty} P_{l}(\frac{1}{2}(l - s(l/2)^{1/3})) = F_{1}(s).$$

(2.61)

The significance of this result from the present perspective is that we have recently shown [14] $P_{l}(t)$ to be equal to the $\tau$-function for a certain Painlevé V system which scales to the result (2.57) (the evaluation of $P_{l}(t)$ in terms of a transcendent related to Painlevé V was first given by Adler and van Moerbeke [4]).

### 2.4 Cumulative distribution of the smallest eigenvalue in the scaled infinite LOE

In the LOE, as $N \rightarrow \infty$ the spacing between the eigenvalues in the neighbourhood of the origin (referred to as the hard edge because the eigenvalue density is strictly zero for $x < 0$) is of order $1/N$. With the scaling

$$\lambda \mapsto \frac{\lambda}{4N},$$

the distribution functions describing the eigenvalues near the hard edge have well defined limits [13]. Our interest is in

$$E_{1}^{\text{hard}}(0; (0, s); (a - 1)/2) := \lim_{N \rightarrow \infty} E_{1}(0; (0, \frac{s}{4N}); x^{(a-1)/2}e^{-x^{2}/2}; N),$$

which is equal to the probability of no eigenvalues in the interval $(0, s)$ of the scaled, infinite LOE, or equivalently to the cumulative distribution of the smallest eigenvalue in the ensemble. It has been shown to have the Painlevé transcendent evaluation [1]

$$E_{1}^{\text{hard}}(0; (0, s); (a - 1)/2) = \exp \left( -\frac{1}{8} \int_{0}^{s} \left( \log \frac{s}{t} \right) q^{2}(t) \, dt \right) \exp \left( -\frac{1}{4} \int_{0}^{s} \frac{q(t)}{\sqrt{t}} \, dt \right),$$

(2.62)

where $q(t)$ satisfies the nonlinear equation

$$t(q^{2} - 1)(tq)' = q(tq)^{2} + \frac{1}{4}(t - a^{2})q + \frac{1}{4}tq^{3}(q^{2} - 2).$$

(2.63)

---

1. Since completing this work the gap probability $E_{1}(0; (s, \infty); e^{-x^{2}/2}; N)$ has been evaluated as a P$\nu$ $\tau$-function [15] and it scales to $F_{1}(s)$. 

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This equation, which is to be solved subject to the boundary condition

\[ q(t) \sim J_\alpha(\sqrt{t}) \sim \frac{1}{2^n \Gamma(1 + \alpha)t^{n/2}}, \quad (2.64) \]

is transformed \[^{[29]}\] via the substitutions

\[ t = x^2, \quad q(t) = \frac{1 + y(x)}{1 - y(x)} \quad (2.65) \]

to the PV equation \(^{[2.6]}\) for \(y(x)\) with parameters

\[ \alpha = \frac{a^2}{8}, \quad \beta = -\frac{a^2}{8}, \quad \gamma = 0, \quad \delta = -2. \quad (2.66) \]

In this subsection we will show that \((2.62)\) can be identified with a \(\tau\)-function corresponding to the PV Hamiltonian \((2.3)\).

To begin we observe that

\[ \int_0^s q(t) \sqrt{t} dt = \int_0^s (\log s - \log t) \frac{d}{dt}(\sqrt{t}q(t)) \, dt, \]

in which use is made of \((2.64)\) for its derivation. Hence we can write

\[
E_{\text{hard}}^1(0; (0, s); (a - 1)/2) = \exp \left( -\frac{1}{8} \int_0^s (\log s - \log t) \left[ q^2 + t^{-1/2}q + 2t^{1/2}q' \right] \, dt \right) \\
= \exp \left( -\frac{1}{4} \int_0^\sqrt{s} (\log s - \log t) \left[ x \frac{dq}{dx} + q + xq^2 \right] \, dx \right). \quad (2.67)
\]

But it follows from \((2.65)\) that

\[ \frac{dq}{dx} = \frac{2}{(1 - y)^2} \frac{dy}{dx}, \]

and thus

\[ x \frac{dq}{dx} + q + xq^2 = \frac{1}{(1 - y)^2} \left( 2x \frac{dy}{dx} - y^2 + 4xy + 1 \right) + x. \quad (2.68) \]

Consider now the Hamiltonian \((2.3)\). With the replacements

\[ q \mapsto y, \quad p \mapsto z, \quad t \mapsto \eta x, \quad H_V \mapsto \frac{1}{\eta} \tilde{H}_V \quad (2.69) \]

it reads

\[ x\tilde{H}_V(y, z) = y(y - 1)^2z^2 - \left\{ (v_2 - v_1)(y - 1)^2 - 2(v_1 + v_2)y(y - 1) + \eta xy \right\} z \]

\[ + (v_3 - v_1)(v_4 - v_1)(y - 1) \quad (2.70) \]

According to \((2.7)\), the remark below \((2.7)\) in parenthesis and \((2.4)\), the parameter values \((2.66)\) correspond to the Hamiltonian \((2.70)\) with

\[ \eta = 2, \quad v_1 = -v_3 = -\frac{1}{4}(a - 1), \quad v_2 = -v_4 = \frac{1}{4}(a + 1). \quad (2.71) \]

Furthermore we need to add a term \(-\frac{1}{4}(a^2 - 1)\) to the Hamiltonian in order that a well-defined limit for the auxiliary Hamiltonian specified below exists as \(x \to 0^+\). Making use of the Hamilton equations it follows that with these parameter values

\[ \frac{d}{dx}(x\tilde{H}_V) = -2yz \quad (2.72) \]

\[ x \frac{dy}{dx} = 2y(y - 1)^2z - \frac{a}{2}(y - 1)^2 + y(y - 1) - 2xy. \quad (2.73) \]
Substituting for $yz$ in (2.73) using (2.72) we see that
\[
\frac{1}{(1 - y)^2} \left[ 2x \frac{dy}{dx} - y^2 + 4xy + 1 \right] = -2 \left[ \frac{d}{dx}(x\tilde{H}_V) + \frac{a - 1}{2} \right].
\]
Substituting this in (2.68) then gives
\[
x \frac{dq}{dx} + q + xq^2 = -2 \left[ \frac{a - 1}{2} - \frac{1}{2}x + \frac{d}{dx}(x\tilde{H}_V) \right].
\]  
Finally, substituting (2.74) in (2.67) and integrating by parts we arrive at the result
\[
E_1^{\text{hard}}(0; (0, s); (a - 1)/2) = \exp \int_0^{\sqrt{s}} \left( \frac{a - 1}{2} - \frac{1}{4}x + \tilde{H}_V \right) dx.
\]  
Thus if we define the auxiliary Hamiltonian and corresponding $\tau$-function for the $P_V$ system by
\[
\tilde{h}_V = \tilde{H}_V - \frac{1}{4}x^2 + \frac{a - 1}{2}, \quad \tilde{h}_V = \frac{d}{dx} \log \tau_{h_V}
\]
we obtain the sought $\tau$-function evaluation
\[
E_1^{\text{hard}}(0; (0, s); (a - 1)/2) = \tau_{h_V}(\sqrt{s}) \bigg| \begin{array}{c}
\eta = 2 \\
v_1 = -\frac{1}{2}, v_2 = \frac{1}{2}(a - 1)
\end{array}.
\]
Note that with the parameters (2.71) it follows from (2.8) and (2.69) that
\[
x\tilde{h}_V = \sigma_V(x) - \frac{1}{4}x^2 + \frac{(a - 1)}{2}x - \frac{a(a - 1)}{4},
\]  
where $\sigma_V(x)$ satisfies (2.9) with $t \mapsto 2x$. The boundary condition for this Hamiltonian is
\[
x\tilde{h}_V(x) \sim xJ_a(x) - \frac{1}{4}x^2 \left[ J_a^2(x) - J_{a+1}(x)J_{a-1}(x) \right].
\]  
The $\tau$-function evaluation of $E_1^{\text{hard}}$ differs from those of $E_1^{\text{bulk}}$ and $E_1^{\text{soft}}$ in that no finite $N$ quantity is known which is itself a $\tau$-function, has an interpretation as a probability, and which scales to (2.77).

3 Symplectic matrix ensembles

3.1 Finite $N$ ensembles

With $N$ finite there is in fact only one symplectic matrix ensemble — the circular symplectic ensemble — for which the gap probability can be written in terms of Painlevé transcendentds using results known in the literature\footnote{Since completing this work the gap probability $E_4(0; (s, \infty); e^{-s}N)$ has been evaluated as the arithmetic mean of two $P_V$ $\tau$-functions, the Hamiltonians of which satisfy the same differential equation.}. With $E_\phi(0; (-\phi, \phi); N)$ denoting the probability that there are no eigenvalues in an interval $(-\phi, \phi)$ of the circular ensemble specified by the PDF (1.7), this is possible due to the inter-relationships between gap probabilities due to Dyson and Mehta\footnote{Since completing this work the gap probability $E_4(0; (s, \infty); e^{-s}N)$ has been evaluated as the arithmetic mean of two $P_V$ $\tau$-functions, the Hamiltonians of which satisfy the same differential equation.}

\[
E_4(0; (-\phi, \phi); N) = \frac{1}{2} \left\{ E_1(0; (-\phi, \phi); 2N) + \frac{E_2(0; (-\phi, \phi); 2N)}{E_1(0; (-\phi, \phi); 2N)} \right\},
\]  

implying the evaluation of $E_4$ from knowledge of the evaluation of $E_1$ and $E_2$.

Regarding the latter, let $\phi$ be related to $s$ via the stereographic projection formula (1.6) with $\theta \mapsto \phi$, $x \mapsto s$. Then from the relationship between the circular ensemble and Cauchy ensemble we have

\[
E_1(0; (-\phi, \phi); 2N) = E_1(0; (-s, s); (1 + x^2)^{-(N+1)/2}; 2N)
\]
\[
E_2(0; (-\phi, \phi); 2N) = E_2(0; (-s, s); (1 + x^2)^{-2N}; 2N).
\]
Now we know from \(\text{(2.1)}\) that
\[
E_1(0; (-s, s); (1 + x^2)^{-(2N+1)/2}; 2N) = E_2(0; (0, s^2); x^{-1/2}(1 + x)^{-2N}; N)
\]
while an identity in \(\text{(10)}\) gives
\[
E_2(0; (-s, s); (1 + x^2)^{-2N}; 2N) = E_2(0; (0, s^2); x^{-1/2}(1 + x)^{-2N}; N) E_2(0; (0, s^2); x^{1/2}(1 + x)^{-2N}; N).
\]
Thus we have
\[
E_4(0; (\phi, \phi); N) = \frac{1}{2} \left\{ E_2(0; (0, s^2); x^{-1/2}(1 + x)^{-2N}; N) + E_2(0; (0, s^2); x^{1/2}(1 + x)^{-2N}; N) \right\}
\]
\[
+ E_2(0; (-1, (s^2 - 1)/(s^2 + 1)); (1 + x)^{-1/2}(1 - x)^{-1/2}; N)
\]
\[
= \frac{1}{2} \left\{ \tau_{\hat{h}_{\text{V}1}} \left( \frac{s^2}{s^2 + 1} \right) \bigg|_{b_1 = b_2 = N, b_3 = 0, b_4 = 1/2} + \tau_{\hat{h}_{\text{V}1}} \left( \frac{s^2}{s^2 + 1} \right) \bigg|_{b_1 = b_2 = N, b_3 = 0, b_4 = -1/2} \right\}
\]
where the final equality follows from \(\text{(2.19)}\). Recalling that \(\hat{h}_{\text{V}1}\) is defined as the RHS of \(\text{(2.18)}\), we see from the fact that \(\hat{h}_{\text{V}1}\) satisfies \(\text{(2.17)}\) that both cases of \(\hat{h}_{\text{V}1}\) in \(\text{(3.3)}\) satisfy the same differential equation. Comparing \(\text{(3.1)}\) and \(\text{(3.3)}\) shows the \(\tau\)-functions in the latter also give the orthogonal and unitary symmetry gap probabilities,
\[
E_1(0; (-\phi, \phi); 2N) = \tau_{\hat{h}_{\text{V}1}} \left( \frac{s^2}{s^2 + 1} \right) \bigg|_{b_1 = b_2 = N, b_3 = 0, b_4 = 1/2}
\]
(3.4)
(which is equivalent to a special case of \(\text{(2.21)}\)) and
\[
E_2(0; (-\phi, \phi); 2N) = \tau_{\hat{h}_{\text{V}1}} \left( \frac{s^2}{s^2 + 1} \right) \bigg|_{b_1 = b_2 = N, b_3 = 0, b_4 = 1/2} \tau_{\hat{h}_{\text{V}1}} \left( \frac{s^2}{s^2 + 1} \right) \bigg|_{b_1 = b_2 = N, b_3 = 0, b_4 = -1/2}.
\]
(3.5)

### 3.2 Bulk gap probability

In an obvious notation, the bulk scaled limit of \(\text{(3.1)}\) gives the formula
\[
E_1^{\text{bulk}}(0; s) = \frac{1}{2} \left\{ E_1^{\text{bulk}}(0; 2s) + E_2^{\text{bulk}}(0; 2s) \right\}.
\]
(3.6)

Using the formula \(\text{(2.22)}\) for \(E_1^{\text{bulk}}\) and a formula for \(E_2^{\text{bulk}}\) deduced from the analogue of \(\text{(3.2)}\), we have previously shown \(\text{(10)}\) that this implies the Painlevé transcendent evaluation
\[
E_4^{\text{bulk}}(0; s) = \frac{1}{2} \left\{ \exp \left( - \int_0^{(\pi s)^2} \sigma_B(t) \frac{dt}{t} \right) + \exp \left( - \int_0^{(\pi s)^2} \sigma_B(t) \frac{dt}{1/2 t} \right) \right\}
\]
where \(\sigma_B\) is specified by \(\text{(2.23)}\). Notice that the differential equation \(\text{(2.22)}\) is unchanged by \(a \mapsto -a\), so \(\sigma_B(t)\big|_{a = 1/2} = \sigma_B(t)\big|_{a = -1/2}\) and \(\sigma_B(t)\big|_{a = 1/2} = \sigma_B(t)\big|_{a = -1/2}\) differ in their characterization only by the boundary condition. The definition \(\text{(2.29)}\) of \(\tau_{\text{III}}(t)\) and the characterization of \(\sigma_{\text{III}}\) therein as the solution of \(\text{(2.28)}\) gives that \(\text{(3.7)}\) is equivalent to the \(\sigma\)-function formula
\[
E_4^{\text{bulk}}(0; s) = \frac{1}{2} \left\{ \tau_{\text{III}}(\pi^2 s^2) \bigg|_{v_1 = v_2 = -1/2} + \tau_{\text{III}}(\pi^2 s^2) \bigg|_{v_1 = v_2 = 1/2} \right\}.
\]
(3.8)

The boundary condition for \(\sigma_{\text{III}}(t)\) when \(v_1 = v_2 = -1/2\) is given by \(\text{(2.31)}\) while the corresponding condition for \(v_1 = v_2 = 1/2\) is, from \(\text{(2.24)}\),
\[
\sigma_{\text{III}}(t) \sim \sqrt{\frac{t}{2\pi}} \left[ 1 - \frac{\sin 2\sqrt{t}}{2\sqrt{t}} \right] \sim \frac{t^{3/2}}{3\pi}.
\]
(3.9)
3.3 Soft edge scaling

For the finite Gaussian ensemble the analogue of (3.1) is the coupled equations [12]

\[
E_2(0; (s, \infty); e^{-x^2}; 2N) = E_1(0; (s, \infty); e^{-x^2/2}; 2N) \left[ E_1(0; (s, \infty); e^{-x^2/2}; 2N + 1) + E_1(1; (s, \infty); e^{-x^2/2}; 2N + 1) \right] \\
+ E_1(0; (s, \infty); e^{-x^2/2}; 2N + 1) E_1(1; (s, \infty); e^{-x^2/2}; 2N) \tag{3.10}
\]

\[
E_4(0; (s, \infty); e^{-x^2}; N) = E_1(0; (s, \infty); e^{-x^2/2}; 2N) + E_1(1; (s, \infty); e^{-x^2/2}; 2N + 1) \tag{3.11}
\]

Here the only known quantity is \(E_2\). In the soft edge scaling limit the number of distinct quantities in (3.10) is reduced, and one obtains the analogue of (3.1) [12].

\[
E_4^{\text{soft}}(0; (s, \infty)) = \frac{1}{2} \left\{ E_1^{\text{soft}}(0; (s, \infty)) + \frac{E_2^{\text{soft}}(0; (s, \infty))}{E_1^{\text{soft}}(0; (s, \infty))} \right\}. \tag{3.12}
\]

As noted in [11], it follows from (2.33) and (2.39) that

\[
E_4^{\text{soft}}(0; (s, \infty)) = \frac{1}{2} \left( e^{-\frac{1}{2} \int_{t=0}^{s} q(t) dt} e^{\frac{1}{2} \int_{t=0}^{s} q(t) dt} + e^{-\frac{1}{2} \int_{t=1}^{s} q(t) dt} e^{\frac{1}{2} \int_{t=1}^{s} q(t) dt} \right), \tag{3.13}
\]

where \(q(t)\) satisfies (2.34) (this result was first derived in a direct calculation [30]). The first term in (3.10) has in (2.57) been identified as a \(\tau\)-function. The second term differs from the first only in the sign of \(q(t)\). Since the differential equation (2.34) is unchanged by the replacement \(q \rightarrow -q\), we see that we can write the second term in (3.10) in a form formally identical to the first. Consequently

\[
E_4^{\text{soft}}(0; (s, \infty)) = \frac{1}{2} \left\{ \tau_{\text{H}}^{(1)}(s) + \tau_{\text{H}}^{(2)}(s) \right\} \bigg|_{s=0}, \tag{3.14}
\]

where \(h\) in \(\tau_{\text{H}}^{(1)}(s)\) is as in (2.54), while \(h\) in \(\tau_{\text{H}}^{(2)}(s)\) is characterized as the solution of the same differential equation as in (2.57), but with the boundary condition

\[
h_{\text{II}}(t) \sim -\frac{1}{2} \text{Ai}(t) + \frac{1}{2} \left\{ [\text{Ai}'(t)]^2 - t[\text{Ai}(t)]^2 \right\}, \tag{3.15}
\]

which results by substituting (2.33) without the minus sign on the RHS in (2.42) with \(a = 1/2\) and recalling (2.54) (c.f. (2.38)).

3.4 Hard edge scaling

In the case of the finite \(N\) Laguerre ensemble, the probabilities for the different symmetry classes of no eigenvalues in the interval \((0, s)\) at the hard edge of the spectrum are related by coupled equations of the form (3.10), (3.11) [12]. Consequently, in the scaled limit one obtains the analogue of (3.12) [12]

\[
E_4^{\text{hard}}(0; (0, s); \alpha + 1) = \frac{1}{2} \left\{ E_1^{\text{hard}}(0; (0, s); (\alpha - 1)/2) + \frac{E_2^{\text{hard}}(0; (0, s); \alpha)}{E_1^{\text{hard}}(0; (0, s); (\alpha - 1)/2)} \right\}, \tag{3.16}
\]

and using (2.62) and the analogous result for \(E_2^{\text{hard}}\) [23], we obtain [13]

\[
E_4^{\text{hard}}(0; (0, s); \alpha + 1) = \frac{1}{2} \left\{ \exp \left( -\frac{1}{8} \int_0^s \left( \log \frac{s}{t} \right)^2 q(t) dt \right) \exp \left( -\frac{1}{4} \int_0^s \frac{q(t)}{\sqrt{t}} dt \right) \\
+ \exp \left( -\frac{1}{8} \int_0^s \left( \log \frac{s}{t} \right)^2 q(t) dt \right) \exp \left( \frac{1}{4} \int_0^s \frac{q(t)}{\sqrt{t}} dt \right) \right\}. \tag{3.17}
\]
where \( q(t) \) satisfies (2.63). As with (3.13), the first term in (3.17) is the orthogonal ensemble result, which has been identified as a \( \tau \)-function in (2.77) above, while the second term differs from the first only in the sign of \( q(t) \). A further analogy is that (2.63), like (2.34) is unchanged by the mapping \( q \mapsto -q \), so we have

\[
E_{hard}^4(0; (0, s); a + 1) = \frac{1}{2} \left\{ \tau_{\tilde{h}V}^{(1)}(\sqrt{s}) + \tau_{\tilde{h}V}^{(2)}(\sqrt{s}) \right\}
\]

\[
\eta=2 \begin{array}{c}
u_1=-\nu_3=-\frac{4}{3}(a-1) \\
\nu_2=-\nu_4=\frac{4}{3}(a+1) 
\end{array}
\]

where \( \tilde{h}V \) in \( \tau_{\tilde{h}V}^{(1)} \) is as in (2.77), while \( \tilde{h}V \) in \( \tau_{\tilde{h}V}^{(2)} \) satisfies the same equation (recall (2.63)) but the boundary condition as determined by (2.65), (2.70), (2.76) and (2.64) is different because a minus sign must now be placed in front of (2.64). Explicitly we have for the second term

\[
 x \tilde{h}V(x) \sim \frac{1}{2} x J_a(x) - \frac{1}{4} x^2 \left[ J_a^2(x) - J_{a+1}(x) J_{a-1}(x) \right]
\]

(c.f. (2.79)). For each of the bulk, soft and hard edge cases, the \( \tau \)-functions in the evaluation of \( E_4 \) give \( E_1 \) and \( E_2 \) according to formulas analogous to (3.4) and (3.5). These are summarized in the accompanying table.

**Acknowledgement**

This work was supported by the Australian Research Council. PJF thanks A. Borodin, P. Deift and A.R. Its for stimulating discussions.

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### Gap Probabilities in Scaled Random Matrix Ensembles — Jardin d’Jimbo-Miwa-Okamoto

| Scaling Limit | Orthogonal | Unitary | Symplectic |
|---------------|------------|---------|------------|
| **Bulk**      | \( P_{III} : \quad v_1 = v_2 = \pm \frac{1}{2} \) | \( \eta = 1 \quad v_1 = v_2 = v_3 = v_4 = 0 \) | \( E_1^{bulk}(0; (-s, s)) = \tau_{III}(\pi^2 s^2)|_{-\frac{1}{2}} \) \( \quad E_2^{bulk}(0; (-s, s)) = \tau_{III}(\pi^2 s^2)|_{\frac{1}{2}} \) \( \quad E_3^{bulk}(0; (-s/2, s/2)) = \frac{1}{2} \tau_{III}(\pi^2 s^2)|_{-\frac{1}{2}} + \frac{1}{2} \tau_{III}(\pi^2 s^2)|_{\frac{1}{2}} \) |
| **Soft Edge** | \( P_{II} : \quad \alpha = 0 \) | | \( E_1^{soft}(0; (s, \infty)) = \tau_{II}^{(1)}(s) \) \( \quad E_2^{soft}(0; (s, \infty)) = \tau_{II}^{(1)}(s) \tau_{II}^{(2)}(s) \) \( \quad E_3^{soft}(0; (s, \infty)) = \frac{1}{2} \tau_{II}^{(1)}(s) + \frac{1}{2} \tau_{II}^{(2)}(s) \) |
| **Hard Edge** | \( P_{V} : \quad \eta = 2 \quad v_1 = -v_3 = -\frac{1}{4}(a-1) \quad v_2 = -v_4 = \frac{1}{4}(a+1) \) | \( P_{III} : \quad v_1 = v_2 = a \) | | \( h_1^{(1,2)}(t) \sim \pm \frac{1}{2} \text{Ai}(t) + \frac{1}{2} \left\{ |A_i(t)|^2 - t[A_i(t)]^2 \right\} \) | \( h_2^{(1,2)}(t) \sim \pm \frac{1}{2} \text{Ai}(t) + \frac{1}{2} \left\{ |A_i(t)|^2 - t[A_i(t)]^2 \right\} \) | |

For the soft edge, we have:

\[
\eta(t) \sim \frac{t}{\pi} - \frac{t^2}{\pi^2}
\]

For the hard edge, we have:

\[
\eta(t) \sim \frac{1}{4} t \left[ J_a^2(t) - J_{a+1}(t) J_{a-1}(t) \right]
\]

where \( J_a(t) \) are Bessel functions of the first kind.