HISTORIC WANDERING DOMAINS NEAR CYCLES

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Dedicated to Professor Lorenzo J. Díaz on the occasion of his 60th birthday

Abstract. We explain how to obtain non-trivial historic contractive wandering domains for a dense set of diffeomorphisms in certain type of $C^r$-Newhouse domains of homoclinic tangencies in dimension $d \geq 3$ and $r \geq 1$. In particular, this gives for the first time a contribution to Takens’ last problem in the $C^1$ topology and in dimension $d > 2$. We show how these Newhouse domains can be obtained arbitrarily close to diffeomorphisms exhibiting heterodimensional cycles (in dimension $d = 3$) or non-transverse equidimensional cycles (in any dimension $d \geq 3$) associated with periodic points with non-real complex leading multipliers.

1. Introduction

A non-trivial historic contractive wandering domain for a given map $f$ on a $C^\infty$ Riemannian compact manifold $M$ is a non-empty connected open set $D \subset M$ which satisfies the following conditions:

- $f^i(D) \cap f^j(D) = \emptyset$ for $i, j \geq 0$ with $i \neq j$,
- the union of the $\omega$-limit set for points in $D$ is not equal to a single orbit,
- the diameter of $f^i(D)$ converges to zero if $i \to \infty$,
- the orbit of any point $x$ in $D$ has historic behavior, i.e., the sequence of empirical measure $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$ does not converge in the weak* topology.

Non-trivial contractive wandering domains were early observed by Bohl and Denjoy (see [Boh16, Den32]) for $C^1$ diffeomorphisms on a circle. Following these results, similar phenomena were observed for high dimensional examples as well as one-dimensional maps in real analytic category [BGLT94, Har89, KNS17, Kni81, KM10, McS93, NS96, Sul85, BL89, Lyu89]. However, these domains are not historic in the sense of the last condition above. The existence of non-trivial historic contractive wandering domains were first studied by Colli and Vargas [CV01] for some two-dimensional example which is made up of an affine thick horseshoe with $C^2$-robust homoclinic tangencies. More recently in [KS17] (see also [BB20]) it was proved that any two-dimensional diffeomorphism in any $C^2$-Newhouse domain (open sets of $C^2$-diffeomorphisms with robust homoclinic tangencies) is contained in the closure of diffeomorphisms having non-trivial historic contractive wandering domains. In this paper, we will explain how this result could be generalized to higher dimensions and in the $C^1$ topology for certain class of Newhouse domains. This provides for the first time examples of smooth dynamical systems in dimension greater than two, as well as in the $C^1$ topology, where it is not possible to get rid of historical behavior by eliminating negligible sets of diffeomorphisms and of initial conditions. This problem was raised by Ruelle in [Rue01] and similarly by Takens in [Tak08] being nowadays known as Takens’ last problem. Also answering [BKN+20, Question 2], we give some conditions ensuring that diffeomorphisms with
certain types of heterodimensional and equidimensional cycles have historic contractive non-trivial wandering domains.

1.1. **Wandering domains for Newhouse domains in higher dimensions.** Following [BD12], we say that a $C^r$-open set $N$ of diffeomorphisms is a $C^r$-Newhouse domain if there exists a dense set $\mathcal{D}$ in $N$ such that every $g \in \mathcal{D}$ has a homoclinic tangency associated with some hyperbolic periodic saddle. Furthermore, if these homoclinic tangencies satisfy a given property $\mathcal{P}$, then we may call it a $C^r$-Newhouse domain of homoclinic tangencies satisfying $\mathcal{P}$. See Definition 2.1 for a formal definition of homoclinic tangency (and heterodimensional cycle).

The first example of a $C^r$-Newhouse domain was obtained by Newhouse [New70] in any surface for $r \geq 2$. Multidimensional $C^r$-Newhouse domains for $r \geq 2$ was constructed by Palis and Viana [PV94], Romero [Rom95] and Godchenko, Shil’nikov and Turaev in [GTS93b] (see also [GST08]). Namely, from these papers, $C^r$-Newhouse domains with $r \geq 2$ can be constructed in any manifold of dimension $d \geq 2$ arbitrarily $C^r$-close to any $C^r$-diffeomorphism having a homoclinic tangency associated with a hyperbolic periodic point $P$. This means that if $f$ is a $C^r$-diffeomorphism ($r \geq 2$) with a homoclinic tangency associated with $P$, then $f \in N$ where $N$ is a $C^r$-Newhouse domain of homoclinic tangencies. Moreover, these homoclinic tangencies are associated with periodic points satisfying similar multiplier condition as $P$. The same result was previously established by Newhouse in [New79] for surface dynamics.

One of the first examples of $C^1$-Newhouse domains was obtained in three-dimensional manifolds by Bonatti and Díaz in [BD99] associated with homoclinic tangencies to periodic points with complex eigenvalues and involving heterodimensional cycles. Later, Asaoka in [Asa08] provides $C^1$-Newhouse domains in any manifold of dimension $d \geq 3$. As Asaoka himself mentioned in [Asa09], his example is essentially the same that Simon previously provided in [Sim72]. Both examples are based on normally hyperbolic non-trivial attractors. More recently, again Bonatti and Díaz in [BD12] have constructed similar examples of $C^1$-Newhouse domains but now associated with blender-horseshoes which are more abundant objects than hyperbolic non-trivial attractors. It is unknown if $C^1$-Newhouse domains can be obtained arbitrarily close to homoclinic tangency in dimension $d \geq 3$. It is also unknown

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1Some comments on [BD99] are necessary. This paper appeared before the stabilization theory of heterodimensional cycles [BD08, BDK12]. The trick used by Bonatti and Díaz was to consider a robust heterodimensional cycle coming from [BD96] (where blenders were introduced) and additionally they assumed that this cycle is homoclinically related to another heterodimensional cycle with complex eigenvalues. They showed that, by a $C^1$-perturbation, the homoclinic classes of the involved periodic points are $C^1$-robustly linked. This provides a $C^1$-open set where densely there exist homoclinic tangencies associated with periodic points with complex multipliers. That is, they construct a $C^1$-Newhouse domain of homoclinic tangencies of periodic points with complex multipliers. At that time, they did not know if this open set corresponds with an open set of robust tangencies (i.e., where the homoclinic tangencies associated with a non-trivial hyperbolic set persists under perturbations). But now, from [BD12] we can conclude that, indeed, $C^1$-robust tangencies associated with a blender-horseshoe appear in this open set.

2We refer to [BD12] to the precise definition of $cu$ and $cs$-blender-horseshoe. Here it suffices to understand that these objects are a certain class of horseshoes in dimension $d \geq 3$. 

whether $C^1$-Newhouse domains exist for surface dynamics. Moreira’s result [Mor11] provides a strong evidence suggesting that there are no Newhouse domains in the $C^1$-topology for surface dynamics.

1.1.1. Newhouse domains of tangencies with non-real complex leading multipliers. First, we will consider Newhouse domains of diffeomorphisms of dimension $m \geq 3$ with homoclinic tangencies associated with periodic saddles whose multipliers $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, \gamma$ satisfy that $\lambda_1$ and $\lambda_2$ are non-real complex conjugate, that is, $\lambda_{1,2} = \lambda e^{\pm i \varphi}$ with $\lambda \in \mathbb{R}, \varphi \neq 0, \pi$, and

$$|\lambda_1| < |\lambda| < 1 < |\gamma| \quad \text{with} \quad |\lambda^2 \gamma| < 1 < |\lambda \gamma| \quad \text{for } j \neq 1, 2. \quad (1)$$

Recall that a saddle is sectional dissipative if the product of any pair of multipliers is less than one in absolute value. This implies that the unstable index (dimension of the unstable manifold) needs to be one. A periodic point satisfying (1) has also unstable index one, it is dissipative (product of all multipliers is less than one) but it is not sectional dissipative. On the other hand, the leading multipliers of a periodic point $P$ of period $n$ of a diffeomorphism $f$ are the nearest eigenvalues of $Df^n(P)$ to the unit circle. Thus, roughly speaking, we are considering Newhouse domains of homoclinic tangencies associated with dissipative but non-sectionally dissipative periodic points with non-real complex leading multipliers. Note that when we say “non-real complex multipliers” we are asking that some of the multipliers are not real (but not necessarily all).

As we have mentioned, this kind of Newhouse domains can be obtained for $r \geq 2$ arbitrarily $C^r$-close to diffeomorphism with a homoclinic tangency associated with a periodic point satisfying (1). For $r = 1$ (actually for any $r \geq 1$), let us consider a $C^1$-diffeomorphism $f$ with the following properties:

(H1) $f$ has a non-transverse equidimensional cycle associated with hyperbolic periodic points $P$ and $Q$. That is, both $P$ and $Q$ have the same unstable index, its stable and unstable invariant manifolds meet transversely and cyclically and also have at least one topologically non-transverse intersection;

(H2) $Q$ has multipliers satisfying (1);

(H3) $P$ is homoclinically related to a $\cs$-blender-horseshoe $\Gamma$.

It is not difficult to see that a $C^1$-Newhouse domain of homoclinic tangencies associated with periodic points satisfying (1) can be also obtained arbitrarily $C^1$-close to $f$ under the assumptions (H1)-(H2)-(H3). See Proposition 2.3. As a corollary, we will obtain a similar result as in [BD99] on the approximation of Newhouse domains associated with homoclinic tangencies to saddle periodic points satisfying (1) from heterodimensional cycles in dimension $d = 3$ and non-transverse equidimensional cycle in dimension $d \geq 3$ with non-real complex leading multipliers. Moreover, we will show that any diffeomorphism in a Newhouse domain of this type is contained in the closure of diffeomorphisms having historic contractive non-trivial wandering domains.
Theorem A. Let \( N \) be a \( C^r \)-Newhouse domain \((r \geq 1)\) of homoclinic tangencies associated with periodic points satisfying \((1)\). Then there is a dense set \( D \) of \( N \) such that, every \( f \in D \) has a non-trivial historic contractive wandering domain. Moreover, the set \( N \) can be obtained arbitrarily

- \( C^r \)-close to a diffeomorphism having a non-transverse equidimensional cycle satisfying \((H1)-(H2)\) for \( r \geq 2 \) and satisfying \((H1)-(H2)-(H3)\) for \( r = 1 \).
- \( C^1 \)-close to a three-dimensional diffeomorphism having a heterodimensional cycle associated with a pair of hyperbolic periodic saddles with non-real complex multipliers and where the multipliers of some of these saddles satisfy \((1)\).

The idea behind the proof of the first part of the above theorem is a reduction of the homoclinic tangency to a two-dimensional smooth normally-hyperbolic attracting invariant manifold where the restricted dynamics has a dissipative saddle. After that, we apply the result in [KS17]. This strategy is not new and was successfully applied to find other types of complex dynamics in [Rom95] and [KS06]. However, we cannot apply Romero’s result [Rom95, Thm. C], even in the three-dimensional case (see also [KS06, Thm. 4, Rem. 1, Sec. 4.1] or [GTS93b, Lem. 2]) because of the following difficulty. Recall first Romero’s result in the three-dimensional case. Let \( f \) be a \( C^r \)-diffeomorphism \((r \geq 2)\) having a homoclinic tangency associated with a periodic point \( P \) with real multipliers \( \nu, \lambda, \gamma \) such that

\[
|\nu| < |\lambda| < 1 < |\gamma| \quad \text{and} \quad J(P) = |\lambda \gamma| > 1.
\]

The case where \( P \) has complex multipliers is reduced to the above case using [PV94, Sec. 5]. According to [Rom95, Thm. C], arbitrarily \( C^r \)-close to \( f \) there exists a diffeomorphism \( g \) which has a two-dimensional normally hyperbolic attracting smooth invariant manifold \( S \) such that the two-dimensional restriction \( g|_S \) has a homoclinic tangency associated with a periodic point \( Q \) with \( J(Q) > 1 \). Since \( Q \) is not a dissipative periodic point, we cannot apply [KS17] to \( g|_S \). To work around this problem, we use the rescaling theory in [GST08] working directly with the complex multipliers instead of reducing the problem to the case of real leading multipliers as in [PV94, Rom95].

1.1.2. Historic wandering domains from Tatjer homoclinic tangencies. In dimension three, we can also obtain wandering domains from another type of Newhouse domains. Namely, we will consider Newhouse domains associated with Tatjer homoclinic tangencies. To introduce these tangencies, we need some preliminaries.

Let \( P \) be a hyperbolic saddle fixed point of a three-dimensional diffeomorphism \( f \). For simplicity of the exposition, we have chosen a fixed point, but all terminologies and concepts are valid if \( P \) is a periodic point. Suppose that \( Df(P) \) has real eigenvalues \( \lambda_s, \lambda_{cu} \) and \( \lambda_{uu} \) satisfying

\[
|\lambda_s| < 1 < |\lambda_{cu}| < |\lambda_{uu}|.
\]

Thus the tangent space at \( P \) has a dominated splitting of the form \( E^s \oplus E^{cu} \oplus E^{uu} \) given by the corresponding eigenspaces. The unstable manifold \( W^u(P) \) is tangent at \( P \) to the bundle \( E^u = E^{cu} \oplus E^{uu} \). On the other hand, according to [HPS77], the extremal bundle \( E^{uu} \) can be also integrated providing a one-dimensional manifold \( W^{uu}(P) \) called strong unstable
manifold. Moreover, this bundle can be uniquely extended to $W^u(P)$ providing a foliation $\mathcal{F}^{uu}(P)$ of this manifold by one-dimensional leaves $\ell^{uu}(Y)$ containing $Y \in W^u(P)$. We assume additionally that the center-stable bundle $E^s \oplus E^{cu}$ is also extended and integrated along the stable manifold $W^s(P)$ of $P$. Although the extended center-stable bundle is not unique, any center-stable manifold contains $W^s(P)$ and any two of these manifolds are tangent to each other at every point of $W^u(P)$. Finally, a three-dimensional diffeomorphism as above has a Tatjer homoclinic tangency associated with $P$ (which corresponds to the type I in [Tat01]) if

(T1) $W^s(P)$ and $W^u(P)$ have a quadratic tangency at $Y$ which does not belong to the strong unstable manifold $W^{uu}(P)$ of $P$,
(T2) $W^s(P)$ is tangent to the leaf $\ell^{uu}(Y)$ of $\mathcal{F}^{uu}(P)$ at $Y$,
(T3) $W^u(P)$ is transverse to any center-stable manifold at $Y$.

If $P$ has stable index equals two, the above definition applies to $f^{-1}$.

Similarly to the results obtained in [GTS93a, GST08] strange attractors, normally hyperbolic attracting smooth invariant circles and hyperbolic sinks are also obtained by unfolding a Tatjer homoclinic tangency under the following extra assumptions [Tat01, GGT07]. The first extra assumption is the dissipativeness: the homoclinic tangency is associated with a saddle periodic point $P$ whose multipliers are $\lambda_s$, $\lambda_c$ and $\lambda_u$ with

$$|\lambda_s| < 1 < |\lambda_{ul}| \quad |\lambda_u| < |\lambda_c| < |\lambda_{ul}| \quad \text{and} \quad |\lambda_s\lambda_c\lambda_u| < 1 \quad (\text{dissipativeness}).$$

Recall a periodic point is said to be sectional dissipative when the absolute value of the product of any pair of multipliers is less than one. Conversely, the second extra assumption required is the non-sectional dissipativeness of $P$: either

(Case A) $|\lambda_c| < 1$, \quad $|\lambda_c\lambda_u| > 1$
(Case B) $|\lambda_c| > 1$.

For short we will say that $P$ is dissipative but non-sectional dissipative periodic point when both above extra assumptions are satisfied.

On the other hand, observe that the conditions (T1) and (T3) are generic. This means that for an arbitrarily small perturbation one can always assume that a homoclinic tangency under the assumption (T2) is, in fact, a Tatjer tangency (of type I). Although (T1) is a codimension one condition, we must observe that the required tangency in (T2) is a condition of codimension $3 - \dim[T_YW^u(P) + T_Y\ell^{uu}(Y)] = 2$.

For more details about tangencies of large codimension see also [BR17, BR21a, BP20].

**Theorem B.** Let $f$ be a three-dimensional $C^r$-diffeomorphism ($r \geq 2$) with a Tatjer homoclinic tangency associated with a dissipative but non-sectional dissipative periodic point. Then, $C^r$-arbitrarily close to $f$, there are a $C^r$-Newhouse domain $N$ (associated with sectional dissipative periodic points) and a dense subset $\mathcal{D}$ of $N$ such that, every $f \in \mathcal{D}$ has a non-trivial historic contractive wandering domain.
The following result completes the case \( r = 1 \). But to achieve this, we need to introduce a special type of Newhouse domains. Namely, we will deal with a Newhouse domain \( N \) satisfying that there exists a dense set \( \mathcal{D} \) of \( N \) such that a map in \( \mathcal{D} \) displays a Tatjer homoclinic tangency associated with dissipative but non-sectional dissipative periodic point. Observe that because of the extra degeneration on the codimension of the homoclinic tangency, the existence of a Newhouse domain associated with Tatjer tangencies is a non-trivial problem. However, as we will explain in §2.3, following essentially the strategy of [KNS17], we construct such Newhouse domains near certain non-transverse equidimensional cycles. Finally, we will show that this open class of diffeomorphisms also is in the closure of maps having non-trivial historic contractive wandering domains.

**Theorem C.** Let \( N \) be a \( C^r \)-Newhouse domain of Tatjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points with \( r \geq 1 \). Then, there is a dense set \( \mathcal{D} \) of \( N \) such that, every \( f \in \mathcal{D} \) has a non-trivial historic contractive wandering domain.

Moreover, this type of \( C^r \)-Newhouse domains can be obtained arbitrarily \( C^1 \)-close to a \( C^r \)-diffeomorphism having heterodimensional cycle associated with periodic points with non-real complex multipliers such that at least one of them is dissipative but non-sectional dissipative.

As we explain in Section 2.3, the proof of the main result in [KNS17] has a gap. The additional assumption on the multipliers that appears in the second part of the above theorem solves this gap. Actually, the conclusion of Theorem C is stronger than [KNS17, Thm. 1.1] since the non-trivial contracting wandering domains obtained are also historic.

1.2. Attracting circles, strange attractors, sinks and non-trivial homoclinic classes. From [GST08], it also follows the coexistence of infinitely many normally hyperbolic attracting invariant smooth circles (and sinks) for a residual subset of diffeomorphisms in a \( C^r \)-Newhouse domain associated with saddle periodic points satisfying (1). This result is only proved in [GST08] in the case \( r \geq 2 \) but the case \( r = 1 \) also holds since \( C^\infty \)-diffeomorphisms with homoclinic tangencies are \( C^1 \)-dense in a \( C^1 \)-Newhouse domain and the attracting circles (and sinks) are \( C^1 \)-robust (they are normally hyperbolic). From [Tat01, GGT07], the same results are obtained for \( C^r \)-Newhouse domains (\( r \geq 1 \)) of Tatjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points.

On the other hand, attracting compact invariant sets having a dense orbit with at least one positive Lyapunov exponent obtained from Hénon-like maps, the so-called Hénon-like strange attractors, are non-hyperbolic. This lack of hyperbolicity prevents stability under perturbations, and thus, the classical arguments (see [PT93]) to provide coexistence of infinitely many of such attractors do not work. This difficulty was overcome by Colli [Col98] and Leal [Lea08]. From these papers, it follows that, in a \( C^r \)-Newhouse domain (\( r \geq 1 \)) associated with homoclinic tangencies of sectional dissipative periodic points, there exists a dense set of diffeomorphisms exhibiting the coexistence of infinitely many non-hyperbolic strange attractors (see §2.4). Once again, this result can be translated to the \( C^r \)-Newhouse domains considering in Theorems A and C. This is because the main tool behind the proof of these theorems is a reduction of the dynamics to a two-dimensional attracting smooth invariant
manifold where the restriction of the diffeomorphism has a homoclinic tangency associated with a dissipative periodic point. Then, one can apply [KNS17] as well as [Col98].

Notice that Hénon-like strange attractors are, in fact, non-trivial attracting homoclinic classes. Recall that a homoclinic class is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the hyperbolic periodic orbit. By attracting we understand that there exists an open neighborhood $V$ of the homoclinic class such that the forward image of the closure of $V$ is strictly inside of $V$. And, we say that the homoclinic class is non-trivial if it is not reduced to a sink or repeller. Although, as mentioned, Hénon-like strange attractors are not stable under perturbation, non-trivial attracting homoclinic classes are $C^1$-robust. This observation allows us to get the following result:

**Theorem D.** Let $N$ be a $C^r$-Newhouse domain $(r \geq 1)$ of one of the following types:

(i) homoclinic tangencies associated with sectional dissipative periodic points,
(ii) homoclinic tangencies associated with periodic points satisfying (1),
(iii) Tatjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points.

Then, there is a residual set $R$ of $N$ such that, every $f \in R$ exhibits the coexistence of infinitely many (independent) non-trivial attracting homoclinic classes.

In the topology $C^1$, it has been known for sometime [BD03] that the coexistence of infinitely many (pairwise disjoint) non-trivial attracting homoclinic classes are locally generic. In fact, recently, it has been also proved [BCF18, Thm. 5] that for $C^1$-generic diffeomorphisms these homoclinic classes could be taken with entropy uniformly large. To conclude, we want to remark that, also in the $C^1$-topology, we obtain the following:

**Corollary I.** Arbitrarily $C^1$-close to a three-dimensional $C^r$-diffeomorphism $f$ having a heterodimensional cycle associated with periodic points with non-real complex multipliers where at least one of them is dissipative but non-sectional dissipative, there exists a locally residual set of diffeomorphisms exhibiting the coexistence of infinitely many (independent) non-trivial attracting homoclinic classes.

The proof of this result is just applied Theorem D to the $C^r$-Newhouse domain of Tatjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points obtained from Theorem C arbitrarily $C^1$-close to $f$.

2. Proof of the theorems

Before proving the main theorems, let us provide the formal definition of the main two dynamical configurations in this paper:

**Definition 2.1.** A diffeomorphism $f$ has a

(1) homoclinic tangency associated with a transitive hyperbolic set $\Lambda$ if there is a pair of points $x, y \in \Lambda$ such that $W^s(x)$ and $W^u(y)$ has a non-transverse intersection. The tangency is said to be $C^1$-robust if there is a $C^1$-neighborhood $U$ of $f$ such that any $g \in U$ has a homoclinic tangency associated with the continuation $\Lambda_g$ of $\Lambda$ for $g$. 


(2) heterodimensional cycle associated with transitive hyperbolic sets $\Lambda$ and $\Sigma$ if these sets have different unstable indices and their invariant manifolds meet cyclically, that is,

$$W^u(\Lambda) \cap W^u(\Sigma) \neq \emptyset \quad \text{and} \quad W^s(\Lambda) \cap W^s(\Sigma) \neq \emptyset.$$  

The heterodimensional cycle is said to be $C^1$-robust if there is a $C^1$-neighborhood $U$ of $f$ such that any $g \in U$ has a heterodimensional cycle associated with the continuations $\Lambda_\varepsilon$ and $\Sigma_\varepsilon$ of $\Lambda$ and $\Sigma$ for $f$.

2.1. Proof of Theorem A. Let us assume that $N$ is a $C^r$-Newhouse domain of homoclinic tangencies associated with periodic points satisfying (1) with $r \geq 1$. Recall, according to [GST08], a homoclinic tangency is said to be simple if the tangency is quadratic, of codimension one and, in the case that the dimension $m \geq 3$, any extended unstable manifold is transverse to the leaf of the strong stable foliation which passes through the tangency point. Thus, since these properties are generic, by an arbitrarily small $C^s$-perturbation with $r \geq 1$, we obtain that maps $f$ with a simple homoclinic tangency associated with a periodic point $Q$ satisfying (1) can be obtained densely in the $C^r$-Newhouse domain $N$. Moreover, we can assume that $f$ is in fact $C^k$ with $k > r$.

We need to consider a two-parameters unfolding $f_\varepsilon$ of $f = f_0$ with $\varepsilon = (\mu, \varphi)$ where $\mu$ is the parameter that controls the splitting of the tangency and $\varphi$ is the value for which the argument of the complex multiplier of $Q$ is perturbed. As usual, $T_0 = T_0(\varepsilon)$ denotes the local map. In this case, this map corresponds to $f_\varepsilon^n$, where $\varepsilon$ is the period of $Q$ and it is defined on a neighborhood $W$ of $Q$. By $T_1 = T_1(\varepsilon)$ we denote the map $f_\varepsilon^n$ from a neighborhood $\Pi^{-}$ of a tangent point $Y^- \in W^u_{\text{loc}}(Q, f_0) \cap W$ of $f_0$ to a neighborhood $\Pi^{+}$ of $Y^+ = f_0^n(Y^-) \in W^s_{\text{loc}}(Q, f_0) \cap W$. Then, for $n$ large enough, one defines the first return map $T_n = T_1 \circ T^n_0$ on a subset $\sigma_n = T^{-n}_0(\Pi^{-}) \cap \Pi^{+}$ of $\Pi^{+}$ where $\sigma_n \rightarrow W^u_{\text{loc}}(Q)$ as $n \rightarrow \infty$. According to [GST08, Lemma 1 and 3] we have the following result:

**Lemma 2.2.** There exists a sequence of open set $\Delta_n$ of parameters converging to $\varepsilon = 0$ such that for these values the map $T_n$ has a two-dimensional normally hyperbolic attracting invariant $C^k$-manifold $M_n$ in $\sigma_n$ which, after a $C^k$-smooth transformation of coordinates on $\sigma_n$, the restriction of the map is given by

$$\bar{x} = y, \quad \bar{y} = M - Bx - y^2 - R_n(xy + o(1)).$$  

The rescaled parameters $M$, $B$ and $R_n$ are functions of $\varepsilon \in \Delta_n$ such that $R_n$ converges to zero as $n \rightarrow \infty$ and $M$ and $B$ run over asymptotically large regions which, as $n \rightarrow \infty$, cover all finite values. Namely,

$$M \sim \gamma^{2n}(\mu + O(\gamma^{-n} + \lambda^n)), \quad B \sim (\lambda \gamma)^n \cos(n\varphi + o(1)) \quad \text{and} \quad R_n \sim \frac{2J_1}{B}(\lambda^2 \gamma)^n$$

where $J_1 \neq 0$ is the Jacobian of the global map $T_1$ calculated at the homoclinic point $Y^-$ for $\varepsilon = 0$. The $o(1)$-terms tend to zero as $n \rightarrow \infty$ along with all the derivatives up to the order $k$ with respect to the coordinates and up to the order $k - 2$ with respect to the rescaled parameters $M$ and $B$.

The dynamics of the generalized Hénon map

$$\bar{x} = y, \quad \bar{y} = M - Bx - y^2 - R_nxy$$  

(3)
Figure 1. Bifurcation diagram near the Bogdanov-Takens point BT\(_n\) in the cases \(R_n > 0\). The case \(R_n < 0\) is similarly changing the stability of the periodic points. The curves SN\(_n\) \(\setminus\) \{BT\(_n\)\} and H\(_n\) correspond to saddle-node and Hopf bifurcations. The curves T\(-\)\(_n\) and T\(_+\)\(_n\) are curves of homoclinic tangencies associated with a dissipative fixed point.

was studied in [GG00, GG04, GKM05] (see also [GGT07]). For small \(R_n\), the map (3) has, on the parameter plane \((M, B)\), a bifurcation point

\[
\text{BT}_n : \quad M = \frac{-1 - R_n}{(1 + R_n/2)^2}, \quad B = 1 + \frac{R_n}{1 + R_n/2}. \tag{4}
\]

At this point, (3) has a fixed point with a pair of eigenvalues equal to +1. As it was shown in [GKM05] (see also [GGT07, Tat01]), the Generalized Hénon family unfolds generically a Bogdanov-Takens bifurcation at BT\(_n\). Figure 1 is showed the local picture of this bifurcation (c.f. [BRS96]).

Although the coefficient \(R_n\) in (2) depends on \(B\), note that the range of values it takes is negligible when \(B\) is limited and \(n\) is large enough. Thus, the bifurcation diagram of (2) can be studied from the results described above for (3) assuming \(R_n = o(1)\) independent of \(B\). Thus, for any \(n\) large enough, there are values of the parameter \(\varepsilon \in \Delta_n\) such that the parameters \(M = M(\varepsilon)\) and \(B = B(\varepsilon)\) of \(T_n = T_n(\varepsilon)\) belong to the curves T\(-\)\(_n\) in Figure 1. Thus, \(T_n\) has at these parameters values a homoclinic tangency associated with a dissipative saddle fixed point. In other words, we can find a sequence \((\varepsilon_n)_n\) of parameters \(\varepsilon_n \in \Delta_n\) with \(\varepsilon_n \to 0\) such that \(g_n = f_{\varepsilon_n}\) (which approaches \(f_0 = f\)) has a normally hyperbolic attracting
two-dimensional invariant manifold $M_n$ for some iterated $m = m(n)$ where the restriction of $g^n_n$ to this manifold has a homoclinic tangency associated with a dissipative periodic point. Now, applying [KS17] to the restriction $g^n_n|_{M_n}$, we obtain a map arbitrarily $C^k$-close to $g_n$ with a non-trivial historic contractive wandering domain. In particular, we obtain that the set of maps with wandering domains are $C^r$-dense in $\mathcal{N}$. This completes the first part of Theorem A.

To prove the second part, consider firstly the case of a non-transverse equidimensional cycle (assumption (H1)). As it is well-known using the Inclination Lemma [PdM82, Lemma 7.1], a $C^r$-diffeomorphism having a non-transverse equidimensional cycle can be $C^r$-approximated by diffeomorphisms exhibiting homoclinic tangencies. Moreover, from assumption (H2), these homoclinic tangencies can be obtained associated with periodic points satisfying (1).

Hence, as it was mentioned in the introduction, according to [GTS93b] one can obtain a $C^r$-Newhouse domain as desired for $r \geq 2$ arbitrarily close to a diffeomorphism exhibiting homoclinic tangencies associated with periodic points satisfying (1). Next proposition proves the case $r = 1$. First, we need some definitions.

Recall that two periodic points of $f$ are homoclinically related if the invariant manifolds of their orbits intersect transversely and cyclically. To be homoclinic related defines an equivalence relation on the set of periodic points of $f$. Two saddles that are homoclinically related have the same unstable index. We can generalize the above notion for a pair of transitive hyperbolic sets $\Lambda$ and $\Sigma$ of $f$ by saying that they are homoclinically related if there is a pair of periodic points $P \in \Lambda$ and $Q \in \Sigma$ homoclinically related. Note that in a transitive hyperbolic set all pairs of periodic points are homoclinically related. Thus, the above periodic points $P$ and $Q$ are homoclinically related with any other periodic point $R \in \Lambda \cup \Sigma$.

We say that diffeomorphism $f$ has a non-transverse equidimensional cycle associated with transitive hyperbolic sets $\Lambda$ and $\Sigma$ if $\Lambda$ and $\Gamma$ are homoclinically related and have a heterodimensional tangency, i.e., there are $x \in \Lambda$ and $y \in \Sigma$ such that $W^s(x)$ and $W^u(y)$ has tangency. Moreover, a non-transverse equidimensional cycle as above is said to be $C^1$-robust if there exists a $C^1$-neighborhood $\mathcal{U}$ of $f$ such that every $g \in \mathcal{U}$ has a non-transverse equidimensional cycle associated with the continuations $\Gamma_g$ and $\Sigma_g$ of $\Lambda$ and $\Sigma$ respectively.

**Proposition 2.3.** Let $f$ be a $C^r$-diffeomorphism ($r \geq 1$) under the assumptions (H1) and (H3). Then $f$ can be $C^r$-approximated by diffeomorphisms exhibiting a $C^1$-robust non-transverse equidimensional cycle associated with the continuations of the periodic point $Q$ and the cs-blender-horseshoe $\Gamma$.

Moreover, if in addition $f$ satisfies also (H2), then a $C^r$-Newhouse domain of homoclinic tangencies associated with periodic points satisfying (1) is also obtained arbitrarily $C^r$-close to $f$.

**Proof.** By assumption (H1), $P$ and $Q$ are homoclinically related and by (H3), $P$ is homoclinically related to a cs-blender-horseshoe $\Gamma$. Thus $P$ and $Q$ are homoclinically related to any saddle of $\Gamma$ and, in particular, to the so-called reference saddle of the blender-horseshoe (see [BD12, Def. 3.9]). Now, by assumption (H1), $P$ and $Q$ have a heterodimensional tangency. Using the Inclination Lemma [PdM82, Lemma 7.1], one obtain easily a diffeomorphism $g$
arbitrarily $C^r$-close to $f$ having a tangency between the reference saddle of $\Gamma_g$ and the stable manifold of $Q_g$. Here $\Gamma_g$ and $Q_g$ are the continuations of $\Gamma$ and $Q$ for $g$ respectively. Then, according to [BD12, Lemma 4.10 and Corollary 4.11], we immediately conclude the first part of the proposition.

The second part of the proposition is just again a standard application of the Inclination Lemma. Indeed, from the first part of the lemma, we get a diffeomorphism $g$ arbitrarily $C^r$-close to $f$ with a $C^1$-robust equidimensional cycle associated with $Q_g$ and $\Gamma_g$. This provides a $C^r$-open set $N_g$ where each diffeomorphism $h \in N_g$ has a heterodimensional tangency between the invariant manifolds of $Q_h$ and a point $x \in \Gamma_h$. Since $x$ and $Q_h$ are also homoclinically related, using the Inclination Lemma, we obtain arbitrarily $C^r$-close to $h$ (in particular in $N_g$) a new diffeomorphism having a homoclinic tangency associated with the continuation of $Q$. Consequently, $N_g$ is a $C^r$-Newhouse domain ($r \geq 1$) of homoclinic tangencies arbitrarily $C^r$-close to $f$. Finally, since the continuation of $Q$ satisfies (H2), then these homoclinic tangencies are associated with periodic points satisfying (1). This completes the proof of the proposition.

Now we will prove that a $C^1$-Newhouse domain associated with periodic points satisfying (1) can be obtained arbitrarily $C^1$-close to a three-dimensional diffeomorphism exhibiting a certain type of heterodimensional cycle. Namely, a heterodimensional cycle associated with a pair of hyperbolic periodic points with non-real complex multipliers and where some of them satisfies (1). To prove this, in view of Proposition 2.3, it suffices the following result:

**Proposition 2.4.** Let $f$ be a three-dimensional $C^r$-diffeomorphism exhibiting a heterodimensional cycle associated with a pair of hyperbolic periodic points $P$ and $Q$ with non-real complex multipliers. Then $f$ can be $C^1$-approximated by $C^r$-diffeomorphisms exhibiting a non-transverse equidimensional cycle under the assumptions (H1) and (H3). Moreover, if in addiction $Q$ satisfies (1), then the non-transverse equidimensional cycle also verifies (H2).

According to [KNS17, Prop. 2.1] we can $C^1$-approximate $f$ by diffeomorphisms exhibiting a non-transverse equidimensional cycle under the assumption (H1). Actually, the first step in [KNS17, Lemma 2.2] to prove this result was to approach $f$ in the $C^1$-topology by a $C^r$-diffeomorphism $h$ having a heterodimensional cycle associated with a pair of periodic points $P'_h$ and $Q'_h$ with real multipliers which are homoclinically related to the continuations $P_h$ and $Q_h$ of $P$ and $Q$ respectively (c.f. [BD08, Thm. 2.1]). We will show that a stronger result can be obtained. Namely, we will explain that, in addition, $P'_h$ is homoclinically related to a cs-blender-horseshoe. Thus (H3) holds. This will be obtained as an immediate application of the following general lemma for the so-called co-index one heterodimensional cycles of diffeomorphisms in any dimension. First, we need to introduce some notation.

Denote by $u$-index($R$) the unstable index of a hyperbolic periodic orbit $R$. Following [BDK12, Def. 1.3], a periodic point has real central multipliers if all the leading multipliers are real and with multiplicity one.
Lemma 2.5. Let \( f \) be a \( C^r \)-diffeomorphism having a heterodimensional cycle associated with a pair of hyperbolic periodic points \( P \) and \( Q \) with \( \text{u-index}(Q) = \text{u-index}(P) + 1 \) and where at least one of them has non-real complex leading multipliers. Then, arbitrarily \( C^1 \)-close to \( f \), there is a \( C^r \)-diffeomorphism \( h \) having

1. a cs-blender-horseshoe \( \Gamma_{h}^{cs} \) and a cu-blender-horseshoe \( \Gamma_{h}^{cu} \),
2. periodic points \( P'_{h} \) and \( Q'_{h} \) with real central multipliers and homoclinically related to the continuations \( P_{h} \) and \( Q_{h} \) of \( P \) and \( Q \) respectively,

such that

3. \( h \) has a heterodimensional cycle associated with \( P'_{h} \) and \( Q'_{h} \),
4. \( P'_{h} \) and \( Q'_{h} \) are homoclinically related to \( \Gamma_{h}^{cs} \) and \( \Gamma_{h}^{cu} \) respectively.

Proof. Using the stabilization theory in [BDK12, Thm. 1 and 2], we find a \( C^r \)-diffeomorphism \( h \) arbitrarily \( C^1 \)-close to \( f \) having a \( C^1 \)-robust heterodimensional cycle between transitive hyperbolic sets \( \Lambda_{h} \) and \( \Sigma_{h} \) containing the continuation \( P_{h} \) and \( Q_{h} \) of \( P \) and \( Q \) respectively. Now, we have to go into the proof of this result to explain that, actually, \( h \) has a heterodimensional cycle associated with a pair of periodic points \( P'_{h} \) and \( Q'_{h} \) with real central multipliers belonging to \( \Lambda_{h} \) and \( \Sigma_{h} \) respectively. Moreover, \( P'_{h} \) and \( Q'_{h} \) belong to a cs-blender-horseshoe \( \Gamma_{h}^{cs} \) and cu-blender-horseshoe \( \Gamma_{h}^{cu} \) respectively. This will prove the lemma.

The key dynamical configuration is described in [BDK12, Prop. 6.1] and depicted in [BDK12, fig. 6]. In this configuration, we have a strong homoclinic intersection associated with a partially hyperbolic saddle-node/flip \( S_{g} \) whose strong stable and unstable manifolds also meet the invariant manifolds of \( P_{g} \) and \( Q_{g} \). Note that this configuration is achieved under the assumption that the initial heterodimensional cycle is non-twisted (see [BDK12, Fig. 1] for an intuitive explanation of these sort of heterodimensional cycles). However, this is not a problem as explained in detail\(^3\) in [BDK12, Sec. 7.1]. Namely, it is proved that since \( f \) has a heterodimensional cycle associated with non-real complex leading multipliers, one can always \( C^1 \)-approximate \( f \) by a \( C^r \)-diffeomorphism \( \varphi \) having a non-twisted heterodimensional cycle associated with a pair of periodic points \( P'_{\varphi} \) and \( Q'_{\varphi} \) with real leading multipliers which are homoclinically related to the continuations \( P_{\varphi} \) and \( Q_{\varphi} \) of \( P \) and \( Q \) respectively. Then, applying [BDK12, Prop. 6.1] to \( \varphi \) we get a diffeomorphism \( g \) arbitrarily \( C^r \)-close to \( \varphi \) having the dynamical configuration mentioned above. In particular, note that \( g \) has a heterodimensional cycle associated with the saddle \( P'_{g} \) and \( Q'_{g} \) homoclinically related to \( P_{g} \) and \( Q_{g} \) respectively.

We explain now following [BDK12, Sec. 3.2 and Sec. 6.1.1] how to obtain a cu-blender-horseshoe and a heterodimensional cycle as desired after an arbitrarily small \( C^r \)-perturbation of \( g \). First, observe that the dynamical configuration of \( g \) is in the assumptions of [BDK12, Thm. 3.5]. Hence, from this result, there is a \( C^r \)-diffeomorphism \( h \) arbitrarily \( C^r \)-close

\(^3\)See also [BDK12, Prop. 6.2] where the generation of non-twisted heterodimensional cycles is proved from the bi-accumulated property.
to $g$ with a heterodimensional cycle associated with the continuation $P_{h}^{\prime}$ of $P_{g}^{\prime}$ and a $cu$-blender-horseshoe $\Gamma_{h}^{cu}$ containing a hyperbolic continuation $S_{h}^{+}$ of $S_{g}$. Moreover, as observed in [BDK12, pg. 955], the dynamical configuration of $g$ implies that the saddle $S_{h}^{+}$ can be chosen homoclinically related to $Q_{h}^{\prime}$. Thus, by an arbitrarily small $C^{\prime}$-perturbation if necessary that we still denote by $h$, one can obtain a heterodimensional cycle associated with $P_{h}^{\prime}$ and $Q_{h}^{\prime}$.

Similarly, $h$ also has a $cs$-blender-horseshoe $\Gamma_{h}^{cs}$ containing a hyperbolic continuation $S_{h}^{-}$ of $S_{g}$ which can be chosen homoclinically related to $P_{h}^{\prime}$. Unfortunately, this is not notified in [BDK12] although it is rather folkloric. To justify this, observe that [BDK12, Thm. 3.5] is based in [BD08, Thm. 2.4]. The main perturbation to prove this result is depicted in [BD12, Fig. 9 and 10]. Compare such figures with [BIR16, Fig. 2 and 3]. In [BIR16, Thm. 2.3], the authors proved a similar result on the generation of robust cycles from a strong homoclinic intersection following [BD12, Sec. 4]. The generation of both $cs$-blender-horseshoe and $cu$-blender-horseshoe is proved in [BIR16, Claim 2.2 and paragraph before Claim 2.3]. These blender-horseshoes contain the hyperbolic periodic points $S_{h}^{-}$ and $S_{h}^{+}$ which are homoclinically related to $P_{h}^{\prime}$ and $P_{h}^{\prime}$ respectively as mentioned before from the dynamical configuration of $g$. This completes all properties that we need and concludes the proof.

Now, we will conclude the proof of Theorem A by proving Proposition 2.4.

Proof of Proposition 2.4. Let $f$ be a three-dimensional diffeomorphism under the assumption of Proposition 2.4. Fix a $C^{\prime}$-diffeomorphism $h$ arbitrarily $C^{1}$-close to $f$ provided by Lemma 2.5. Kiriki, Soma and Nakano proved in [KNS17, Prop. 2.1] that there is a diffeomorphism $g$ arbitrarily $C^{\prime}$-close to the diffeomorphism $h$ having a non-transverse equidimensional cycle associated with the continuations $Q_{g}^{\prime}$ and $Q_{h}$ of $Q_{h}^{\prime}$ and $Q_{h}$ respectively. Thus, $h$ satisfies (H1). But, since $P_{g}^{\prime}$ is homoclinically related to the continuation $\Gamma_{h}^{cs}$ of $\Gamma_{h}^{cs}$ (which is also a $cs$-blender-horseshoe), we also have (H3). To conclude the proof note that $Q_{g}^{\prime}$ satisfies (H2) since $Q_{g}$ verifies (1).

To conclude this section we want to remark that the condition (1) on the multipliers of $Q_{g}$ is just imposed to get (H2). If this assumption is avoided, we have the following result:

**Theorem 2.6.** Let $f$ be a three-dimensional $C^{\prime}$-diffeomorphism exhibiting a heterodimensional cycle associated with a pair of hyperbolic periodic points with non-real complex multipliers. Then $f$ can be $C^{1}$-approximated by a $C^{\prime}$-diffeomorphism $g$ having simultaneously,

1. a $C^{1}$-robust homoclinic tangency,
2. a $C^{1}$-robust heterodimensional cycle, and
3. a $C^{1}$-robust non-transverse equidimensional cycle.

**Proof.** Under the assumption of this theorem, Proposition 2.4 implies that $f$ can be $C^{1}$-approximated by $C^{\prime}$-diffeomorphisms satisfying (H1) and (H3). Then, by Proposition 2.3, we can continue to $C^{1}$-approximate $f$ by a $C^{\prime}$-diffeomorphism $h$ with a $C^{1}$-robust non-transverse $\ldots$
equidimensional cycle between a saddle with non-real complex multipliers $Q_h$ and a $cs$-blender-horseshoe $\Gamma_h$. As it is well-known, by using the Inclination lemma, we can $C^r$-approximate $h$ by a diffeomorphism $g$ with a homoclinic tangency associated with a periodic point in $\Gamma_g$. Thus, according to [BD12, Thm. 4.9], there is a diffeomorphism arbitrarily $C^r$-close to $g$ having a $C^1$-robust homoclinic tangency. In particular, this diffeomorphism satisfies simultaneously (1) and (3). We will explain that also $g$ satisfies (2). To see this, observe that the first step to prove Proposition 2.4 is Lemma 2.5. This lemma provides a heterodimensional cycle which is $C^1$-robust by the presence of a blender-horseshoe. Thus $h$ and $g$ above satisfy (2). This concludes the proof. □

2.2. Proof of Theorem B. Let $f$ be a $C^r$-diffeomorphism for $r \geq 2$ with a Täuber tangency associated with a dissipative but non-sectional dissipative periodic point. By a small $C^r$-perturbation, we can assume that, $f$ is actually $C^r+1$. Now, let us explain in more detail the results obtained by Täuber in [Tat01, GGT07].

First of all, Täuber localizes in [Tat01, Prop. 3.5 and 3.7] a sequence $g_n$ of perturbations of $f$ with a $n$-periodic point $p_n$ having a Bogdanov-Takens bifurcation converging to $f$ as $n$ goes to infinity. After that, in the proof of [Tat01, Prop. 4.1 and 4.5], Täuber performs a change of variables around the $n$-periodic point $p_n$ of $g_n$ in order to calculate a manageable expression of the return map $h_n = g_n^n$. Since $p_n$ is a Bogdanov-Takens bifurcation, $h_n$ has a two-dimensional invariant center $C^r$-manifold $M_n$ which is attracting: See [Tat01, pg. 293, line 5-7] and [GGT07, Comments after Thm. 2]. Actually, in case A, the restriction of $h_n$ to this manifold is again well approached by the Generalized Hénon map (c.f [GST08, Thm. 3]). In case B, the attracting character of the center manifold $M_n$ follows from the limit return [Tat01, Pg. 299, proof of Thm 1, item 1 and 3] since the surface $y = a + bz + x^2$ is invariant and every point in $\mathbb{R}^3$ falls by one iteration of limit map into this surface. The three-dimensional limit return map has a zero eigenvalue and the study of this family can be reduced to a family of two-dimensional endomorphisms (see [PT06] reference therein). The limit map is obtained by truncating a Taylor expansion in the manageable expression of the periodic return diffeomorphism. Then, going from the limit map to the family of return maps, the zero eigenvalue becomes a real one with a small modulus. Thus, the invariant center manifold $M_n$ coming from the Bogdanov-Takens bifurcation becomes an attracting manifold providing attracting invariant smooth circle among others after bifurcation as mentioned in [Tat01, pg. 299]. Moreover, by the results of Broer et al [BRS96] (cf. [Tat01, GGT07]), near a Bogdanov-Takens bifurcation there exist homoclinic tangencies associated with a dissipative saddle periodic point into the two-dimensional invariant center manifold. Since the other direction is strong contracting, this periodic point view in three dimensions is sectional dissipative.

Similar as it did in §2.1, a three-dimensional historic wandering domain could be obtained by applying now [KS17]. Moreover, notice that, in this case, we obtain the persistence of homoclinic tangencies directly from the result of Newhouse [New79] in dimension 2. This provides a $C^r$-Newhouse domain $N$ associated with a sectional dissipative periodic points which are arbitrary $C^r$-close to $f$ and where maps with non-trivial historic contractive wandering domains are $C^r$-dense in $N$. This concludes the proof of Theorem B.
2.3. **Proof of Theorem C.** Because of Theorem B we only need to prove the existence of $C^r$-Newhouse domains of Tatjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points for $r \geq 1$.

Let us consider a three-dimensional $C^p$-diffeomorphism $f$ having a non-transverse equidimensional cycle associated with periodic points $P$ and $Q$ (assumption (H1)) for $r \geq 2$. Assume that $Q$ has complex multipliers and all the multipliers of $P$ are real. According to [KNS17, Prop. 3.1] such diffeomorphism can be $C^r$-approximated by Tatjer homoclinic tangency associated with the continuation of the periodic point $P$. Although [KNS17] deals with the case that $P$ has unstable index 2, the case of unstable index 1 also follows by simply considering $f^{-1}$. However, we cannot conclude, a priori, from this that $P$ is a dissipative but non-sectional dissipative periodic point. To do this, we need to impose an extra condition on the multipliers of $Q$. Namely, we will assume that

(H2') $Q$ has non-real multipliers and it is dissipative but non-sectional dissipative.

By Inclination Lemma and a $C^r$-perturbation if necessary, we can assume that $Q$ has a homoclinic tangency. Hence, according to [PV94, Sec. 5], one can $C^r$-approximate $f$ by diffeomorphisms having a homoclinic tangency associated with a periodic point $P'$ with real multipliers homoclinically related to the continuation of $Q$ and with the same local character, i.e., $P'$ is still dissipative but non-sectional dissipative. Again, by a $C^r$-perturbation, we can obtain a non-transverse equidimensional cycle associated with $Q$ and $P'$. Thus, from [KNS17] we get now a Tatjer homoclinic tangency associated with the continuation of $P'$. Summarizing,

**Lemma 2.7.** Let $f$ be a three-dimensional $C^p$-diffeomorphism ($r \geq 2$) under the assumptions (H1) and (H2'). Then $f$ can be $C^r$-approximated by diffeomorphisms displaying a Tatjer homoclinic tangencies associated with a dissipative but non-sectional dissipative periodic point.

To construct a $C^p$-Newhouse domain ($r \geq 1$) we need more extra assumptions:

(H3') $P$ is homoclinically related to a blender-horseshoe $\Gamma$.

The above assumptions (H1)-(H2')-(H3') can be rewritten as follows: either,

- $f$ satisfies (H1)-(H2)-(H3) or
- $f$ satisfies (H1), the multipliers of $Q$ are $\lambda, \gamma_1, \gamma_2$ with $|\lambda| < 1 < |\gamma|$ and $|\lambda \gamma^2| < 1$ where $\gamma_{1,2} = e^{\varphi_i} \neq 0, \pi$

and $P$ is homoclinically related to a $cu$-blender-horseshoe $\Gamma$.

Similar as indicated in the introduction, assumptions (H1)-(H2')-(H3') implies that $f$ is $C^r$-approximated by a $C^1$-robust equidimensional tangency associated with the continuation of $Q$ and $\Gamma$ (cf. [BD12, Sec. 4.3]). In other words, arbitrarily $C^r$-close to $f$, there exists a $C^1$-open set $N$ of diffeomorphisms such that any $g \in N$ has a tangency between some of the invariant manifolds of $Q$ and $\Gamma$. By a $C^r$-perturbation, we can get a smooth diffeomorphism

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See that this necessary assumption to apply the results from [Tat01, GGT07] is missing in [KNS17].
\( h \) arbitrarily close to \( g \) such that \( h \) has a non-transverse equidimensional cycle associated with \( Q \) and some periodic point \( P \in \Gamma \). Thus, we obtain a dense set \( \mathcal{D} \) in \( N \) where any \( h \in \mathcal{D} \) is smooth and satisfies \((H1)\) and \((H2')\). Then, from Lemma 2.7 we can approximate \( h \) by T atjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points. This proves the following:

**Proposition 2.8.** Arbitrarily \( C^r \)-close \((r \geq 1)\) to a \( C^r \)-diffeomorphism \( f \) satisfying \((H1)-(H2')-(H3')\), there exists a \( C^r \)-Newhouse domain \( N \) of T atjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points.

To complete the proof of Theorem C we need to show the following. Arbitrarily \( C^1 \)-close to a heterodimensional cycle associated with saddles \( P \) and \( Q \) with complex multipliers where \( Q \) satisfies \( (H2')\), one can find a \( C^r \)-Newhouse domain of T atjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points. As arguing in §2.1, heterodimensional cycles in the above assumptions can be \( C^1 \)-approximated by non-transverse equidimensional cycles satisfying assumptions \((H1)-(H2')-(H3')\). Hence, from Proposition 2.8 we get that \( f \) can be \( C^1 \)-approximated by \( C^r \)-Newhouse domains of T atjer homoclinic tangencies associated with dissipative but non-sectional dissipative periodic points.

### 2.4. Proof of Theorem D.

Let us first recall the result of Colli and Leal in [Col98] and [Lea08].

**Theorem 2.9** (Colli, Leal). Let \( f \) be a \( C^\infty \)-diffeomorphisms having a homoclinic tangency associated with a sectional dissipative periodic point. Then, there exists a \( C^\infty \)-open set \( \mathcal{U} \) containing \( f \) in its closure such that every \( g \in \mathcal{U} \) can be \( C^\infty \)-approximated by diffeomorphisms exhibit infinitely many Hénon-like strange attractors.

Now, we will consider a \( C^r \)-Newhouse domain \( N \) \((r \geq 1)\) of sectional dissipative periodic points. The set \( N^\infty = N \cap \text{Diff}^\infty(M) \) is \( C^r \)-dense in \( N \) and \( C^\infty \)-open in the set of \( C^\infty \)-diffeomorphisms \( \text{Diff}^\infty(M) \) of the manifold \( M \). Moreover, any diffeomorphism \( f \in N^\infty \) can be \( C^\infty \)-approximated by \( C^\infty \)-diffeomorphisms \( g \) having homoclinic tangencies associated with sectional dissipative periodic points. Thus, from Theorem 2.9, \( g \) can also be \( C^\infty \)-approximated by diffeomorphisms \( h \) exhibiting infinitely many Hénon-like strange attractors. Consequently, there exists a \( C^r \)-dense set \( \mathcal{D} \) in \( N \) where any \( h \in \mathcal{D} \) exhibits infinitely many Hénon-like strange attractors. In what follows, we will explain how to use this result to conclude Theorem D.

First of all, observe that as an immediate consequence of Theorem B and the proof of Theorem A, we have the following:

**Proposition 2.10.** Any \( C^r \)-Newhouse domain \((r \geq 1)\) of type (ii) or (iii) in Theorem D is also a \( C^r \)-Newhouse domain of type (i).

Thus, we can restrict our attention to the case where \( N \) is a \( C^r \)-Newhouse domain of type (i) in Theorem D, i.e., of homoclinic tangencies associated with sectional dissipative homoclinic
periodic points. Recall that, as notified in the introduction, Hénon-like strange attractors are non-trivial attracting homoclinic classes. Since non-trivial attracting homoclinic classes persist under $C^1$-perturbations, fixed a finite number $n \in \mathbb{N}$, the above result implies the existence of an open and dense set $\mathcal{O}_n$ of $\mathbb{N}$ where $n$ different (independent) homoclinic classes of this type coexists. Taking $\mathcal{R} = \cap \mathcal{O}_n$ we get the desired residual set and complete the proof.

Note that in Theorem D we cannot replace non-trivial attracting homoclinic classes by Hénon like strange attractors. A result asserting the coexistence of infinitely many strange attractors in a residual set of a Newhouse domain is not expected. However, the author of this paper proved recently in [BR21b] that such result holds in Berger domains of persistent homoclinic tangencies to sectional dissipative periodic points. That is, in an open set of parameter families that plays the same role as the notion of Newhouse domain in the free parameter case.

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