AN ON-AVERAGE MAEDA-TYPE CONJECTURE IN THE LEVEL ASPECT

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Abstract. We present a conjecture on the average number of Galois orbits of newforms when fixing the weight and varying the level. This conjecture implies, for instance, that the central $L$-values (resp. $L$-derivatives) are nonzero for $100\%$ of even weight prime level newforms with root number $+1$ (resp. $-1$).

1. Introduction

Let $S_k(N)$ (resp. $S_{k}^{\text{new}}(N)$) be the space of weight $k$ elliptic cusp (resp. new) forms of level $\Gamma_0(N)$. For $S$ a Hecke-stable subspace of $S_k(N)$ such that the set of normalized eigenforms is closed under the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denote by $\text{Orb}(S)$ the set of Galois orbits of normalized eigenforms in $S$.

Maeda’s conjecture states that for any even $k$ such that $S_k(1) \neq 0$, $\#\text{Orb}(S_k(1)) = 1$, and in fact that $T_2$ (or any $T_p$) acts irreducibly (over $\mathbb{Q}$) on $S_k(1)$ with Galois group of type $S_n$. More generally, let $\text{Sq}_r$ denote the set of squarefree positive integers with exactly $r$ prime factors. Then for $N \in \text{Sq}_r$, $S_k(N)$ has $2^r$ Atkin–Lehner eigenspaces. A generalization of Maeda’s conjecture [Tsa14] states that, for fixed $N \in \text{Sq}_r$, one has $\#\text{Orb}(S_k^{\text{new}}(N)) = 2^r$ for all $k$ sufficiently large. It follows from trace formula methods that $\#\text{Orb}(S_k^{\text{new}}(N)) \geq 2^r$ for all sufficiently large $k$, and this can be made effective [Mar18].

On the other hand, if we fix $k = 2$ and vary $N \in \text{Sq}_r$, one expects a strict inequality $\#\text{Orb}(S_2(N)) > 2^r$ infinitely often due to the existence of sufficiently many elliptic curves (or abelian surfaces of GL(2) type) of squarefree level. However, we predict the following on-average analogue of Maeda’s conjecture in the level aspect.

Conjecture 1.1. Let $k \geq 2$ be even. Then the average number of Galois orbits of $S_k^{\text{new}}(N)$ over all $N \in \text{Sq}_r$ is $2^r$, i.e.,

$$
\lim_{X \to \infty} \frac{\sum_{N \in \text{Sq}_r(X)} \#\text{Orb}(S_k^{\text{new}}(N))}{\#\text{Sq}_r(X)} = 2^r,
$$

where $\text{Sq}_r(X) = \{N \in \text{Sq}_r : N \leq X\}$. In fact, for a fixed prime $p$, $T_p$ acts irreducibly on each of the $2^r$ Atkin–Lehner eigenspaces of $S_k^{\text{new}}(N)$ for $100\%$ of $N \in \text{Sq}_r$ coprime to $p$.

In particular, this asserts that each Atkin–Lehner eigenspace is spanned by a single Galois orbit $100\%$ of the time, analogous to the usual Maeda conjecture.

One application of Maeda’s conjecture is to non-vanishing of central $L$-values for newforms of full level [CF99] (see also [KZ81, Corollary 2]). The above conjecture similarly has applications to non-vanishing $L$-values and derivatives. We recall that if a newform

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$f \in S_k(N)$ has root number $-1$, then the central $L$-value $L(k, f)$ is forced to vanish by the functional equation.

**Theorem 1.2.** Let $k \geq 2$ be even, and assume Conjecture 1.1. Let $\mathcal{F}_k^I$ denote the collection of newforms in $\bigcup_N \text{prime} S_k(N)$, partially ordered by level. Then

1. for $100\%$ of $f \in \mathcal{F}_k^I$ with root number $+1$, we have $L(k, f) \neq 0$; and
2. for $100\%$ of $f \in \mathcal{F}_k^I$ with root number $-1$, we have $L'(k, f) \neq 0$.

Let us expand a bit on the context of these statements.

First, Maeda’s conjecture for full level was formulated on the basis of computational data and recent work verifies Maeda’s conjecture for $k \leq 14000$ ([GM12], [Ben]). See [Tsa14] and [DPT] for evidence for the analogue for squarefree level as well as a possible generalization to non-squarefree level. Our conjecture is based on both heuristics in Section 2 and data in Section 3.

One could also attempt to generalize Conjecture 1.1 to non-squarefree levels. In the case of non-squarefree level, the naive expectation for the number of non-CM Galois orbits should be given by the number of possible local representation types at ramified places, which varies with both the primes $p | N$ as well as $v_p(N)$. We do not consider this here, in part because it would be difficult to generate a convincing amount of data for non-squarefree level and partly because even the correct generalization of Maeda’s conjecture to non-squarefree level is not clear (see [DPT]). Similarly, we do not consider nontrivial nebentypus or odd weights, but it is reasonable to expect an analogue of Conjecture 1.1 in these settings as well.

We also discuss the possibility of stronger statements than Conjecture 1.1. For instance, assuming a conjecture of Roberts [Rob18] on the finitude of rational newforms in large weights, we suggest that for large weights one may have an exact equality $\#\text{Orb}(S^\text{new}_k(N)) = 2^r$ for all but finitely many $N \in \text{Sq}_e$ (Conjecture 2.2), which would be an exact analogue of Tsaknis’ generalized Maeda conjecture in the level aspect. We also briefly discuss how often Hecke polynomials are irreducible or Galois groups are of type $S_n$ in Sections 2.3 and 3.4. In addition, we raise some questions related to [Rob18], [Mur99] and [KSW08] based on our data (Questions 3.1 and 3.2).

Lipnowski and Schaeffer [LS] formulated a conjecture in a similar vein as Conjecture 1.1, that the rational Hecke modules of each Atkin–Lehner eigenspace are asymptotically simple. Restricting to $N$ prime, this means that for $\varepsilon = \pm 1$ the maximal size of an orbit in the Atkin–Lehner $\varepsilon$-eigenspace $S^\text{new,\varepsilon}_k(N)$ should be asymptotic to the dimension of $S^\text{new,\varepsilon}_k(N)$, i.e.,

\begin{equation}
\lim_{N \to \infty} \frac{\max\{\#O : O \in \text{Orb}(S^\text{new,\varepsilon}_k(N))\}}{\dim S^\text{new,\varepsilon}_k(N)} = 1.
\end{equation}

While neither (1.1) nor (1.2) imply the other, (1.1) is morally stronger in that it is suggestive of (1.2) but not conversely. Namely, (1.1) implies (1.2) holds for a density 1 subsequence of primes, but (1.2) does not imply any bounds on the number of Galois orbits, even restricting to some density 1 subset of primes.

We also remark that Conjecture 1.1 and (1.2) assert very strong statements about the growth of degrees of rationality fields of newforms. For instance it is known that, as the level grows, the proportion of newforms with rationality fields of bounded degree tends to 0 [Bin17]. Moreover, there exist sequences of newforms with rationality fields of degrees
growing at least logarithmically in the level (e.g., [BM16], [BPGR]). Conjecture 1.1 and (1.2) would imply that the growth of rationality fields is generically linear in the level.

Regarding Theorem 1.2, the conclusion is expected from minimalist type conjectures (see [Bru95] for $k = 2$) and the Katz–Sarnak philosophy. The non-vanishing of central $L$-values $L(k, f)$ and $L$-derivatives $L'(k, f)$ is already known for a positive proportion of such $f$ by [Van99], [KM00] and [IS00]. In [IS00], a connection is made between large proportions of non-vanishing of central values (via a more refined density conjecture) and Landau–Siegel zeroes. This suggests a connection between the average number of Galois orbits and Landau–Siegel zeroes.

The proof of Theorem 1.2 is an immediate application of the above non-vanishing results. The obstruction to extending this to squarefree level is that one needs the existence of non-vanishing $L$-values or $L$-derivatives in 100% of the Atkin–Lehner eigenspaces. While this may very well be accessible by current analytic methods, to our knowledge it has not been considered. One similarly has applications of Conjecture 1.1 to questions such as non-vanishing of twisted $L$-values.

In [MW], we give another application of Conjecture 1.1 to zeroes of automorphic forms on definite quaternion algebras.

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2. HEURISTICS

2.1. Random polynomials. Let $k \geq 2$ be even, $N \in \text{Sq}_r$ and $n = \dim S_{k}^{\text{new}}(N) \sim \frac{(k-1)\varepsilon(N)}{12}$. Let $\varepsilon$ denote a sign pattern for $N$, i.e., a collection $(\varepsilon_p)_{p|N}$ such that $\varepsilon_p = \pm 1$ for each $p|N$. For $p|N$, let $W_p$ be the Atkin–Lehner involution at $p$ on $S_{k}^{\text{new}}(N)$. Denote by $S_{k}^{\text{new},\varepsilon}(N) = \{ f \in S_{k}^{\text{new}}(N) : W_p f = \varepsilon_p f \text{ for } p|N \}$ the Atkin–Lehner eigenspace of newforms associated to $\varepsilon$. By [Mar18], we know that $\dim S_{k}^{\text{new},\varepsilon}(N) \sim \frac{n}{\pi}$ (asymptotically as $kN \to \infty$ for $(k, N) \in 2\mathbb{Z}_{>0} \times \text{Sq}_r$), and in fact we know good error estimates.

Consider a Hecke operator $T_p (p \nmid N)$ acting on $S_{k}^{\text{new}}(N)$ or some $S_{k}^{\text{new},\varepsilon}(N)$. Then each Galois orbit $O$ of newforms in these spaces corresponds to a factor $g_{O,p}(x)$ of the characteristic polynomial $c_{T_p}(x) \in \mathbb{Z}[x]$ of $T_p$. More precisely, we must have $g_{O,p}(x) = m_{O,p}(x)^j$ for some irreducible polynomial $m_{O,p}(x) \in \mathbb{Z}[x]$, and the roots of $g_{O,p}(x)$ are in one-to-one correspondence with the $T_p$-eigenvalues of the newforms in $O$.

Thus a random model for the number and degrees of factors of $c_{T_p}(x)$ will provide a simple heuristic for an upper bound on the number and sizes of Galois orbits of $S_{k}^{\text{new}}(N)$. In fact, the main result of [KSW08] implies that for a given Galois orbit $O$, $g_{O,p}(x) = m_{O,p}(x)$ for 100% of primes $p$. Hence we may use a model for the factorization of $c_{T_p}(x)$ for an arbitrary $p \nmid N$ to model the Galois orbits of $S_{k}^{\text{new}}(N)$.

We recall two heuristic principles on random polynomials, which are quite robust to the model being considered.

(RP1) In the absence of simple reasons for nontrivial factors, the probability that a well-behaved random polynomial in $\mathbb{Z}[x]$ is irreducible tends to 1 (typically quickly) as the size of the polynomial grows.
(RP2) Asymptotically, the probability that a well-behaved random polynomial in $\mathbb{Z}[x]$ is reducible over $\mathbb{Q}$ is proportional to the probability that it has a linear factor over $\mathbb{Q}$.

We do not attempt to define the notions of “well-behaved” or the “size” of the polynomial—indeed we use them in a somewhat vague sense here—but just remark that by size we have in mind some combination of size of the coefficients and the degree of the polynomial. The principle (RP1) has been long studied, and there are many results in this direction. See, e.g., [BBB+18] for a recent study of (RP2), which the authors term universality.

Apart from the decomposition of $S_k^\text{new}(N)$ into Atkin–Lehner eigenspaces, there are no obvious reasons why the characteristic polynomials of $T_p$’s acting on $S_k^\text{new}(N)$ should factor for $p \nmid N$. Thus (RP1) suggests that, for $p \nmid N$, $T_p$ acts irreducibly on each Atkin–Lehner eigenspace 100% of the time. This is our first heuristic why Conjecture 1.1 should be true.

To be more precise, we recall a simple model for Hecke polynomials on Atkin–Lehner eigenspaces recently proposed by Roberts [Rob18].

Consider the collection $\mathcal{P}_d(t)$ of degree $d$ monic polynomials in $\mathbb{Z}[x]$ all roots real size at most $t$. It follows from [AP14a, Theorem 4.1] that $|\mathcal{P}_d(t)|$ is approximately

$$R_d(t) := (2t)^{d(d+1)/2} \prod_{j=1}^{d} \frac{(j - 1)!^2}{(2j - 1)!} = 2^d t^{d(d+1)/2} \prod_{j=1}^{d-1} \left( \frac{j}{2j + 1} \right)^{d-j}$$

when $d + t$ is large (cf. [AP14b, Theorem 3.1]). Thus the probability that a random polynomial in $\mathcal{P}_d(t)$ has a factor of degree $e \leq \frac{d}{2}$ is approximately

$$P_{d,e}(t) = \frac{R_e(t)R_{d-e}(t)}{2^d R_d(t)}$$

$$= \frac{1}{2^\delta t^{e(d-e)}} \prod_{j=1}^{e-1} \left( \frac{2j + 1}{j} \right)^{j} \prod_{j=e}^{d-e} \left( \frac{2j + 1}{j} \right)^{j} \prod_{j=d-e}^{d-1} \left( \frac{2j + 1}{j} \right)^{d-j},$$

where $\delta = 1$ if $d = 2e$ and 0 otherwise.

Note (for $d > 2$)

$$P_{d,1}(t) = \frac{1}{td-1} \prod_{j=1}^{d-1} \frac{2j + 1}{j}$$

and

$$P_{d,e}(t) < \left( \frac{3}{t} \right)^{e(d-e)}.$$ 

Consequently, we see that both $P_{d,e}(t) \to 0$ and $P_{d,1}(t) \gg \sum_{e=2}^{\lfloor d/2 \rfloor} P_{d,e}(t)$ as $d + t \to \infty$, provided $t > 3$. In fact, by refining the above bound, this analysis works for fixed $t > 2$ and $d \to \infty$. Thus (RP1) and (RP2) hold for this model with fixed $t > 2$ and $d \to \infty$.

Let $n_\epsilon = \dim S_k^{\text{new,}\epsilon}(N) \approx \frac{(k-1)\varphi(N)}{12 \pi^2}$. Then a crude model for the characteristic polynomial of $T_p$ acting on $S_k^{\text{new,}\epsilon}(N)$ is a random polynomial in $\mathcal{P}_{n_\epsilon}(2p^{(k-1)/2})$. One obvious defect is that, even for fixed $N$ and $\epsilon$, the probability of reducibility decreases rapidly as $p \to \infty$.

For odd $N$, we will just use a model for the factorization of $c_T^2$ as a model for the Galois orbits. In light of both [KSW08] and observed data (see Table 6), the factorization
of $c_T$ does seem to be a very good model for the sizes of Galois orbits. Further, as in [Rob18], we may view $P_{nc,e}(2^{k+1})$ as a rough model for the probability that $S_k^{\text{new},e}(N)$ has a Galois orbit of size $e$ (or a collection of smaller Galois orbits whose sizes sum to $e$), even when $N$ is even.

This model is not very accurate—as pointed out in [Rob18], it severely underpredicts the actual number of factorizations of $c_T$'s. In fact it suggests all but finitely many Atkin–Lehner eigenspaces consist of a single Galois orbit as $N \to \infty$ along $\text{Sq}_r$, which should not be true at least in weight 2 (cf. Section 2.2 and Question 3.1). We will partially address this by also considering arithmetic statistics in Section 2.2 below. However, the model at least suggests the following principles which we believe in accordance with the data and general expectations about randomness. In the following statements, we consider $r \geq 1$ fixed, $N \in \text{Sq}_r$, $k \geq 2$ even, $\varepsilon$ a sign pattern for $N$ and $p \nmid N$.

- (wt) Given $N$, $\varepsilon$ and $p$, the probability that $T_p$ acts reducibly on $S_k^{\text{new},\varepsilon}(N)$ decreases rapidly as $k$ becomes large.
- (lev) Given $\varepsilon$ and $p$, the probability that $T_p$ acts reducibly on $S_k^{\text{new},\varepsilon}(N)$ decreases rapidly as $N$ grows coprime to $p$.
- (lin) If $k+N$ is large, the probability that $T_p$ acts reducibly on $S_k^{\text{new},\varepsilon}(N)$ is roughly equal to the probability that $T_p$ acting on $S_k^{\text{new},\varepsilon}(N)$ has a rational eigenvalue.

Note (wt) and (lev) follow from (RP1) since increasing $k+N$ increases the dimensions of the Atkin–Lehner subspaces. Similarly, (lin) follows from (RP2). Specifically, (lev) suggests Conjecture 1.1.

Before we discuss other heuristics and data, we discuss the relation of the above heuristics with the following recent conjecture of [Rob18] restricted to our setting of squarefree level. (In particular, we do not need to account for quadratic twist classes or CM forms.) By a rational newform, we mean a newform in $S_k(N)$ with rational Fourier coefficients, i.e., a newform whose Galois orbit has size 1. Let $F_k$ be the set of newforms which lie in some $S_k(N)$ with $N$ squarefree.

**Conjecture 2.1** (Roberts). Fix $k \geq 6$. There are only finitely many rational newforms in $F_k$. Further, if $k \geq 52$, there are no rational newforms in $F_k$.

Roberts’ support for his conjecture comes from his rough heuristic model together with an apparent lack of motivic sources for rational newforms in higher weight and computations in weights $k \leq 50$ and levels $N \leq C_k$. Here the bound $C_k$ depends on $k$—e.g., $C_6 = 1000$, $C_{10} = 450$, $C_{20} = 150$, $C_{30} = 100$ and $C_{40} = 30$.

Of course when $k = 2$, rational newforms correspond to isogeny classes of elliptic curves, so we expect infinitely many rational newforms (cf. Section 2.2). For $k = 4$, Roberts remarks that it is unclear if there should be infinitely many rational newforms or not, and that this is related to the existence of suitable Calabi–Yau threefolds.

Let $\omega(N)$ denote the number of prime divisors of $N$. Then (lin) suggests the following more speculative conjecture, which implies something much stronger than Conjecture 1.1 if Roberts’ conjecture is true for $k$.

**Conjecture 2.2.** Fix $k$. Suppose there are only finitely many rational newforms in $F_k$. Then $\# \text{Orb}(S_k^{\text{new}}(N)) = 2^{\omega(N)}$ for all but finitely many squarefree $N$. 


As some numerical evidence for this conjecture, Roberts observed that in his calculations for $k \geq 6$ that there were only four squarefree levels where an Atkin–Lehner eigenspace has multiple Galois orbits with no orbits of size 1, which is in line with \((\text{lin})\).

Moreover, in light of the second part of Roberts’ conjecture, the above heuristics suggest that for $k$ sufficiently large, each Atkin–Lehner eigenspace may only be a single Galois orbit for arbitrary squarefree $N$. Put another way, it is possible that the generalized Maeda conjecture for squarefree level is true with a uniform bound on the weight: there exists some absolute $k_0$ such that for any $r \geq 0$, \(#\text{Orb}(S^\text{new}_k(N)) = 2^{\omega(N)}\) for all $k \geq k_0$ and all squarefree $N$. (The conjecture in \([\text{Tsa14}]\) merely asserts that this should be true if we allow $k_0$ to depend upon $N$.)

### 2.2. Elliptic curves.

As mentioned before, random polynomial models as above seem too crude to accurately predict the frequency of small Galois orbits. One perspective is that the existence of small Galois orbits is due to the existence of suitable motives, which seem to be hard to model without a deep understanding of arithmetic. However, we can supplement the random polynomial model above with heuristics from arithmetic geometry.

In particular, based on the principle \((\text{wt})\), we expect that Conjecture 1.1 should be true if it is true in weight 2. Our data below corroborate this idea, by indicating the average number of Galois orbits converges to $2^r$ faster the higher the weight is (see Section 3.1). Moreover, by the principle \((\text{lin})\), we expect that Conjecture 1.1 should be true if, as $N \to \infty$ in $S_{\text{qr}}$, 0% of weight 2 newforms are rational.

By dimension formulas, we know that the number of weight 2 newforms of level $N \leq X$, $N \in S_{\text{qr}}$, grows at least as fast as $\frac{X^2}{\log X}$ (up to a constant, this is the asymptotic for $r = 1$). However the number of isogeny classes of elliptic curves of arbitrary conductor $\leq X$ is $O(X^{1+\epsilon})$ [DK00]. (Heuristics of Watkins [Wat08] suggest it is actually $O(X^{5/6})$.) Consequently, 0% of weight 2 newforms along levels in $S_{\text{qr}}$ are rational. This gives arithmetic support for our belief in Conjecture 1.1.

In fact, generalizing work of Serre, Binder [Bin17] showed that, for any weight $k$ and fixed degree $A$, 0% of weight $k$ newforms of levels $N$ have a rationality field of degree $\leq A$ for any sequence of $N \to \infty$ with $\omega(N)$ bounded.

### 2.3. Galois groups.

We have discussed heuristics for Conjecture 1.1, which is an on-average analogue of two aspects of Maeda’s conjecture in the level aspect: the number of Galois orbits and the irreducibility of the action of $T_p$. The remaining part of Maeda’s conjecture is that the rationality fields of newforms are of type $S_n$. Since random polynomials tend to have Galois groups $S_n$, it is reasonable to expect that Conjecture 1.1 also holds with the added statement that $c_{T_p}$ has Galois group $S_n$.

There are examples of newforms with rationality fields whose Galois group is not a full symmetric group, even when the whole Atkin–Lehner space has only a single Galois orbit (see Section 3.2). Thus at most we can hope for statements of the form: for almost all newforms in a given sequence, the rationality field has Galois group $S_n$. Analogous to questions about finiteness of number of rational newforms, one can ask if “almost all” here can be interpreted to mean “all but finitely many”, i.e., given a sequence of Galois orbits of newforms (in distinct quadratic twist classes), are the Galois groups of type $S_n$ for all but finitely many orbits? Unfortunately, we do not have precise enough heuristics or abundant enough data to speculate about this.
Table 1. Average numbers $A_{k,1}(X)$ of Galois orbits for $r = 1$

| $k$ | $X = 250$ | 500 | 1000 | 2500 | 5000 | 10000 |
|-----|-----------|-----|------|------|------|-------|
| 2   | 2.038     | 2.484 | 2.679 | 2.684 | 2.577 | 2.483 |
| 4   | 2.057     | 2.042 | 2.030 | 2.016 |       |       |
| 6   | 1.981     | 2.000 | 2.000 |       |       |       |
| 8   | 2.000     |       |       |       |       |       |
| 10  | 1.981     |       |       |       |       |       |
| 12  | 1.943     |       |       |       |       |       |

Table 2. Average numbers $A_{k,2}(X)$ of Galois orbits for $r = 2$

| $k$ | $X = 250$ | 500 | 1000 | 2500 | 5000 | 10000 |
|-----|-----------|-----|------|------|------|-------|
| 2   | 3.243     | 4.386 | 5.292 | 5.615 | 5.608 | 5.442 |
| 4   | 4.405     | 4.352 | 4.250 | 4.135 |       |       |
| 6   | 4.108     | 4.069 | 4.042 |       |       |       |
| 8   | 3.973     | 3.986 |       |       |       |       |
| 10  | 4.013     |       |       |       |       |       |
| 12  | 4.000     |       |       |       |       |       |

Table 3. Average numbers $A_{k,3}(X)$ of Galois orbits for $r = 3$

| $k$ | $X = 250$ | 500 | 1000 | 2500 | 5000 | 10000 |
|-----|-----------|-----|------|------|------|-------|
| 2   | 3.708     | 5.885 | 8.652 | 11.34 | 12.30 | 12.29 |
| 4   | 7.500     | 8.902 | 9.237 | 8.701 |       |       |
| 6   | 8.167     | 8.557 | 8.348 |       |       |       |
| 8   | 8.292     | 8.197 |       |       |       |       |
| 10  | 8.083     |       |       |       |       |       |
| 12  | 7.958     |       |       |       |       |       |

3. Data

3.1. LMFDB data. First we present some numerical evidence for Conjecture 1.1 using data from LMFDB [LMFDB]. LMFDB contains data for the newforms in $S^\text{new}_k(N)$ whenever $Nk^2 \leq 40000$. Using this data, we computed the average number $A_{k,r}(X)$ of Galois orbits over the spaces $S^\text{new}_k(N)$ where $N \in \mathbb{Q}$ with $N < X$ for several values of $X \leq 10000$, $2 \leq k \leq 12$ and $1 \leq r \leq 3$. The data are summarized in Tables 1 to 3, corresponding respectively to $r = 1$, 2 and 3. Blank spaces in the tables denote the situations where there is not sufficient data in LMFDB to compute the averages.

Our first remark about the data is that, for fixed $k, r$ the rough shape of the graph $A_{k,r}(X)$ as a function of $X$ initially increases with $N$ (corresponding to the range where some Atkin–Lehner spaces are 0) and then is essentially decreasing. In accordance with our heuristics, it appears that $A_{k,r}(X)$ tends to $2^r$ faster the larger $k$ is and the smaller $r$ is, as the dimensions of the Atkin–Lehner eigenspaces are larger in these situations.

In particular, the data for the case $k = 2$ and $r = 3$ are not sufficient to make it numerically apparent whether the average tends to $2^3$. However, the data on the whole seems to be in support of Conjecture 1.1, and also the notion that Conjecture 1.1 should be
true if it is true for \( k = 2 \). In addition, it seems reasonable to expect that, for given \( k \), the distribution of sizes of Galois orbits along a sequence of Atkin–Lehner eigenspaces depends primarily on dimension of the Atkin–Lehner eigenspaces and not to any significant amount on the number \( r \) of prime factors of \( N \). Thus, at least to our mind, we can be confident about Conjecture 1.1 if we are in the special case of \( k = 2 \) and \( r = 1 \), which is what we focus on below.

3.2. **Data for \( S_2(N) \), \( N \) prime.** To gather more numerical evidence for Conjecture 1.1, we computed \( A_{2,1}(X) \) for \( X \leq 60000 \). These calculations were carried out in parallel using Sage on the University of Oklahoma’s supercomputing facilities (OSCER) over the course of several weeks. See Fig. 1 for a graph of \( A_{2,1}(X) \), which appears to be eventually decreasing to 2, as conjectured. We remark that \( A_{2,1}(60000) \approx 2.3016 \) (compare with Table 1). The apparent slow rate of convergence is expected in light of the well-known numerical phenomenon that the proportion of weight 2 newforms accounted for by elliptic curves tends to 0 quite slowly.

One consequence of Conjecture 1.1 would be that, in 100% of prime levels, \( S_2(N) \) has exactly 2 Galois orbits. (For \( N > 59 \), \( S_2(N) \) has at least 2 orbits.) In fact, provided the number of Galois orbits does not grow too fast along any subsequence, this is equivalent to the \( k = 2, r = 1 \) case of Conjecture 1.1. Table 4 summarizes how often we get exactly 2 (or 3, or 4, etc) Galois orbits in weight 2 in certain ranges. These numerics suggest that indeed there are exactly 2 Galois orbits 100% of the time. We remark that for prime \( N < 60000 \), the maximum number of Galois orbits is 10.

One of our heuristics for Conjecture 1.1 uses the idea (lin), that most of the time an Atkin–Lehner eigenspace has multiple Galois orbits, the multiple orbits are accounted for by the existence of a rational newform. In Table 5, we summarize the number of small Galois orbits in various ranges, and observe that indeed most of the time there is a small Galois orbit, it is of size 1. In fact, there are no orbits of “moderate” size: for
Table 4. Counts of number of Galois orbits for $S_2(N)$ with $N$ prime

| $N$ | number of orbits | $2$ | $3$ | $4$ | $5$ | $6$ | $7+$ | % with $2$ orbits |
|-----|------------------|-----|-----|-----|-----|-----|-----|------------------|
| $0 < N < 10000$ | $777$ | $331$ | $67$ | $25$ | $9$ | $6$ | $63.2\%$ |
| $10000 < N < 20000$ | $786$ | $193$ | $39$ | $9$ | $5$ | $1$ | $76.1\%$ |
| $20000 < N < 30000$ | $769$ | $176$ | $31$ | $5$ | $1$ | $1$ | $78.2\%$ |
| $30000 < N < 40000$ | $768$ | $158$ | $23$ | $7$ | $2$ | $0$ | $80.2\%$ |
| $40000 < N < 50000$ | $750$ | $162$ | $14$ | $3$ | $1$ | $0$ | $80.6\%$ |
| $50000 < N < 60000$ | $765$ | $121$ | $28$ | $6$ | $4$ | $0$ | $82.8\%$ |

Table 5. Counts of small Galois orbits for $S_2(N)$ with $N$ prime

| $N$ | size of orbits | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
|-----|----------------|-----|-----|-----|-----|-----|-----|-----|
| $0 < N < 10000$ | $329$ | $212$ | $76$ | $28$ | $20$ | $11$ | $18$ |
| $10000 < N < 20000$ | $200$ | $104$ | $16$ | $3$ | $0$ | $0$ | $0$ |
| $20000 < N < 30000$ | $176$ | $80$ | $5$ | $2$ | $1$ | $0$ | $0$ |
| $30000 < N < 40000$ | $171$ | $56$ | $5$ | $1$ | $0$ | $0$ | $0$ |
| $40000 < N < 50000$ | $140$ | $56$ | $7$ | $0$ | $0$ | $0$ | $0$ |
| $50000 < N < 60000$ | $152$ | $57$ | $2$ | $0$ | $0$ | $0$ | $0$ |

10000 $< N < 60000$, there are no Galois orbits of size $6 \leq d \leq 300$. This suggests the following question:

**Question 3.1.** Fix $k, r, d$. Are there infinitely many Galois orbits of size $d$ in the union of spaces $S_k^{\text{new}}(N)$ with $N \in S_0$?

Note that for $d = 1$, this is just asking about the infinitude of rational newforms, which is the topic of Roberts’ conjecture discussed above. In fact, Roberts’ conjecture together with Conjecture 2.2 would imply that for $k$ large, the answer is negative for all $r, d$. On the other hand, when $k = 2$ and $d$ is small, we expect this question has a positive answer. So the most novel case of this question is when $k$ is small but $d$ is not. At least for $k = 2$, $r = 1$ and $d$ sufficiently large, our data suggest the answer may be no.

### 3.3. The method

Now we describe our method to compute Galois orbits. For an odd prime $N$, let $B = B_N$ be the definite quaternion algebra of discriminant $N$. Then we computed the Brandt matrix $T_2^B$ for a maximal order $O_B$ of $B$, which acts on the space of $M$ quaternionic modular forms associated to $O_B$. This space of quaternionic modular forms is Hecke isomorphic to $M_2(N) \cong \mathbb{C}E_{2,N} \oplus S_2(N)$, where $E_{2,N}$ is the normalized weight $2$ level $N$ holomorphic Eisenstein series. The Eisenstein eigenvalue of $T_2^B$ is $3$, and thus the eigenvalues of $T_2^B$ acting on $B_N$ are $3$ together with the eigenvalues of $T_2$ acting on $S_2(N)$. We compute the characteristic polynomial $c_{T_2^B}(x) = (x - 3)c_{T_2}(x)$.

If $c_{T_2}(x)$ has no repeated factors, then the number of Galois orbits in $S_2(N)$ is simply the number of irreducible factors of $c_{T_2}(x)$. If $c_{T_2}(x)$ has repeated factors, then we repeat the above calculation with successive $T_p$’s until this method succeeds. We performed these calculations in parallel by treating different $N$ on different cores. Most of the calculation time is spent computing the Brandt matrices $T_p^B$ and their characteristic polynomials,
Table 6. Frequency that $p$ is the smallest prime such that $T_p$ acts on $S_2(N)$ with no repeated eigenvalues ($N < 60000$ prime)

| $p$ | frequency |
|-----|-----------|
| 2   | 5815      |
| 3   | 158       |
| 5   | 42        |
| 7   | 14        |
| 11  | 15        |
| 13  | 2         |
| 17  | 3         |
| 19  | 4         |
| 23  | 3         |
| 47  | 1         |

and the computational complexity increases both with $N$ and with $p$. For $N$ close to 60000, this calculation for a single $T_p$ can take over 24 hours of wall time.

3.4. Irreducibility of Hecke polynomials. In most cases, $T_2$ acts on $S_2(N)$ with no repeated eigenvalues. Even when $T_2$ does not, we typically do not have to try many $T_p$’s to find one that does. Table 6 shows for how many primes $N < 60000$ a given $p$ is minimal such that $T_p$ has no repeated eigenvalues.

More generally, given $p$ we can ask how many prime $N \neq p$ are there such that $c_{T_p}$ has no repeated roots for $S_2(N)$? Note that if we have multiple rational newforms occurring in $S_2(N)$, the naive probability that two given such newforms $f_1$ and $f_2$ have the same $a_p$ is approximately $\frac{1}{4\sqrt{p}}$ by Deligne’s bound. Since we expect that there are infinitely many prime levels $N$ where $S_2(N)$ has 2 rational newforms (e.g., coming from Neumann–Setzer elliptic curves [Set75]) we expect that $c_{T_p}$ has repeated roots for infinitely many $N$.

To avoid this situation, let us restrict to the case where $S_2(N)$ has exactly 2 Galois orbits. Out of the 4615 prime levels $N < 60000$ such that $S_2(N)$ has exactly 2 Galois orbits, there is only one level $N$ such that $T_2$ has repeated eigenvalues on $S_2(N)$, namely $N = 251$. Here $T_2$ acts irreducibly on the 17-dimensional root number +1 subspace of $S_2(251)$ but acts reducibly on the 4-dimensional root number −1 subspace. (Incidentally, a newform in the latter space has rationality field with Galois group $D_8$.) Based on the rarity of repeated eigenvalues of $T_2$, we guess that there may be no other such $N$.

This is related to a question studied in [Mur99] and [KSW08]: given a non-CM newform $f \in S_k(N)$ with rationality field $K$, how often is $\mathbb{Q}(a_p(f))$ a proper subfield of $K$? Restricted to our setting of squarefree level and trivial nebentypus, [Mur99, Conjecture 3.4] asserts that this happens for infinitely many $p$ exactly in the following cases: $k = 2$ and $K$ is quadratic, cubic or quartic with a quadratic subfield. See also [VII17] for the case $k = [K : \mathbb{Q}] = 2$. In the cases where we expect $\mathbb{Q}(a_p(f)) = K$ for all but finitely many $p$, we can ask if something stronger is true.

Question 3.2. Given $k$ and squarefree $N$, let $f \in S_k(N)$ be a non-CM newform with rationality field $K$. Suppose $k \geq 4$ or $[K : \mathbb{Q}] \geq 5$. Is $\mathbb{Q}(a_p(f)) = K$ for all $p \nmid N$?

Again, the answer may depend on $k$ and $K$, and possibly $N$, but it is not entirely unreasonable to think this question may have a positive answer if some combination $k$, $K$ and $\text{Gal}(K/\mathbb{Q})$ are sufficiently large. At least when $N = 1$, there is some evidence towards a positive answer (e.g., [VX18], [Ben]). Note that an affirmative answer to Question 3.2 under appropriate conditions would imply that $T_p$ acts irreducibly on each Galois orbit of $S_k(N)$ for all but finitely many squarefree $N$ coprime to $p$.

4. Proof of Theorem 1.2

Fix $k \geq 2$ even. For a prime level $N$, the Atkin–Lehner eigenspaces of $S_k^{\text{new}}(N)$ are simply the spaces with fixed root number $\pm 1$. 
Let $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$. It follows from an algebraicity result of Shimura [Shi76, Theorem 1] that $L(k, f^\sigma) \neq 0$ if and only if $L(k, f) \neq 0$. Thanks to the extension of the Gross–Zagier formula by Zhang, we also know that $L'(k, f^\sigma) \neq 0$ if and only if $L'(k, f) \neq 0$ [Zha97, Corollary 0.3.5].

Consequently to show Theorem 1.2, it suffices to know that for $N$ sufficiently large, there exist $f \in S_k(N)$ with $L(k, f) \neq 0$ (so $f$ necessarily has root number $+1$) and $g \in S_k(N)$ with $L'(k, g) \neq 0$ and $g$ having root number $-1$. This follows, e.g., from the works [Van99], [KM00] and [IS00] mentioned in the introduction.

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