SINGULAR COMPACTNESS AND DEFINABILITY FOR Σ-COTORSION AND GORENSTEIN MODULES

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Abstract. We introduce a general version of singular compactness theorem which makes it possible to show that being a Σ-cotorsion module is a property of the complete theory of the module. As an application of the powerful tools developed along the way, we give a new description of Gorenstein flat modules which implies that, regardless of the ring, the class of all Gorenstein flat modules forms the left-hand class of a perfect cotorsion pair. We also prove the dual result for Gorenstein injective modules.

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Introduction

The aim of the paper is to establish new structural and approximation results about two types of homologically defined (and at least in the first case very well known) classes of modules:

(1) Gorenstein flat and Gorenstein injective modules and
(2) Σ-cotorsion modules.

What these seemingly distant classes of modules have in common is the rather non-obvious fact that one can learn deep facts about their structure using infinite combinatorics and set-theoretically flavored homological tools. This is despite the fact that the statements of the main results (Theorems 2.3, 3.11 and 4.6 and their corollaries) are of purely module-theoretic and homological nature, and can be explained without any set theory. It is their proofs where infinite combinatorics plays crucial role, and the key ingredient brought by this paper is a new version of

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Shelah’s singular compactness theorem for direct systems which do not necessarily consist of monomorphisms.

Gorenstein homological algebra, which is a version of relative homological algebra with roots on one hand in commutative algebra (and especially the celebrated Auslander–Buchsbaum formula) and on the other hand in modular representation theory of finite groups, has been developed for almost half a century; an interested reader may find a more detailed overview in the introduction of [15]. Our main result here is that, for any ring, the classes of Gorenstein injective and Gorenstein flat modules sit in complete cotorsion pairs, the class of Gorenstein injective modules is enveloping and the class of Gorenstein flat modules is covering. In particular, any ring is GF-closed in the sense of [11].

This contribution is perhaps best explained in the context of the new impetus which Gorenstein homological algebra recently got from the study of abelian model structures [22] and which allowed to import homotopical theoretic techniques. Since it was not known in general whether the standard classes of Gorenstein flat or injective modules had good approximation properties, Bravo, Hovey and Gillespie [14] were led to introduce a modification of the definitions of these classes, to ensure the existence of the required approximations in this way. Our results can thus be summarized as that this change was not necessary: the classes of Gorenstein flat and injective modules have good approximation properties and induce abelian model structure on their own for every ring, regardless of how daunting the ring is. The model structures of [14] can then be recovered as a localization (Bousfield localization at the level of model categories or triangulated localization at the level of their homotopy categories) of the model structures arising from the standard classes.

The class of Σ-cotorsion modules, on the other hand, was studied [8, 29, 31] in an attempt to generalize model theoretic methods for modules to arbitrary additive finitely accessible category (in the terminology of [1]; they are also known under the term locally finitely presented additive categories [16]). Every finitely accessible additive category is equivalent to the category \(\text{FL}(R)\) of flat modules over a ring \(R\) (possibly non-unital, but with enough idempotents) and admits a natural (pure) exact structure inherited from \(\text{Mod-}R\). Moreover, as a consequence of the solution to the Flat Cover Conjecture [12], this exact structure has enough injective objects, which are precisely the flat and cotorsion \(R\)-modules. The main theme of [30, 32] is that there are even enough indecomposable flat cotorsion modules in order to cogenerate \(\text{FL}\), so that one can go on and define the Ziegler spectrum for \(\text{FL}\) (at least as a set, the topology still has not been defined in general at the time of writing this paper).

A Σ-cotorsion module is one whose every direct sum of copies is cotorsion. Thus Σ-cotorsion modules generalize classical Σ-pure-injective modules, which are well behaved and characterized by chain conditions on definable subgroups.

Our main result here is that Σ-cotorsion modules are also characterized by a version of chain conditions, but these are way more complicated. As a consequence, if \(C\) is a Σ-cotorsion module, then any module in the smallest definable class (= first-order axiomatizable and closed under direct sums and summands) containing \(C\) is Σ-cotorsion as well. This, in particular, shows that being Σ-cotorsion is not a property of a particular module, but rather of its first-order theory in the language of modules over a given ring. The notion of Σ-cotorsion module is therefore one where homological algebra, model theory and infinite combinatorics meet each other in a fascinating way.
As already mentioned, our results are based on a collection of methods involving homological algebra and infinite combinatorics (stationarity and Mittag-Leffler condition, almost-freeness, singular compactness), which have been thoroughly studied by several authors in the last two decades. We use these to treat the following general questions for a class $\mathcal{B} \subseteq \text{Mod-}R$:

(a) Given a module $M$ such that $\text{Ext}^1_R(M, \mathcal{B}) = 0$, when can we write $M = \varinjlim M_i$, where $\text{Ext}^1_R(M_i, \mathcal{B}) = 0$ and all the $M_i$ are $\kappa$-presented for some fixed cardinal $\kappa$, independent of $M$?

(b) Conversely, suppose that $M = \varinjlim M_i$, where $\text{Ext}^1_R(M_i, \mathcal{B}) = 0$. When can we conclude that $\text{Ext}^1_R(M, \mathcal{B}) = 0$?

Our general strategy to understand classes of modules of the form $\text{Ker} \text{Ext}^1_R(\cdot, \mathcal{B})$ is first to give a positive answer to (a), using non-trivial closure properties of $\mathcal{B}$ (e.g. under direct limits or direct sums). In the best cases we can reach $\kappa = \aleph_0$, which reduces our questions to countably presented modules, which are in general very well understood. Then we use (b) for a possibly larger class $\mathcal{B}' \supseteq \mathcal{B}$.

The paper is organized as follows. We first collect the essential tools of infinite combinatorics in homological algebra in Section 1 including the novel Theorems 1.8 and 1.9.

In section Section 2 we prove Theorem 2.3, which says that $\Sigma$-cotorsionness is a property of a first-order theory, and study the corresponding intricate chain conditions (Definition 2.5) which generalize previously known special cases for countable rings [31, Theorem 12] and non-discrete valuation domains [8, Theorem 3.8].

In Section 3, we introduce a new class of projectively coresolved Gorenstein flat modules, which turns out to be a part of a complete hereditary cotorsion pair (Theorems 3.4 and 3.9). This allows us to prove that also Gorenstein flat modules are a part of a complete cotorsion pair (Theorem 3.11 and Corollary 3.12) and to define two new Quillen equivalent abelian model structures.

In Section 4, we prove that also Gorenstein injectives sit in a complete cotorsion pair (Theorem 4.6) and can be used to define an abelian model structure, whose existence was previously known only for particular cases of rings [40, Theorem 7.12].

Finally, the last Sections 5 and 6 are devoted to giving a proof for our new version of singular compactness (which we use in the form of Lemma 6.1). The paper is concluded by appendix which describes standard but somewhat technical and not trivial operations on direct systems, which we constantly use.

1. Relative projectivity, injectivity and general tools

Unless stated otherwise, by a module, we mean a right $R$-module where $R$ is an associative unital ring. We denote the class of all modules by Mod-$R$ and the class of all left $R$-modules by $R$-Mod. In fact, our techniques work equally well for modules over small preadditive categories (in the sense of [39, Appendix B]) or, equivalently, unitary $R$-modules $M$ (i.e. satisfying $MR = M$) over (associative) rings with enough idempotents (see [33, §1.3] for precise definitions).

A cotorsion pair is a pair $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ of classes of modules such that $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = \mathcal{B}^\perp$. Cotorsion pairs are most useful if they are complete, i.e. for each $M \in \text{Mod-}R$ there exist short exact sequences

$$0 \longrightarrow B^M \longrightarrow A^M \overset{p}{\longrightarrow} M \longrightarrow 0$$

and

$$0 \longrightarrow M \overset{i}{\longrightarrow} B_M \longrightarrow A_M \longrightarrow 0$$

with $A^M, A_M \in \mathcal{A}$ and $B^M, B_M \in \mathcal{B}$. The map $p$ is called a special $\mathcal{A}$-precover of $M$ while $i$ is called a special $\mathcal{B}$-preenvelope of $M$. A key observation [27, Theorem 6.11(b)] is that any cotorsion pair generated by a set $\mathcal{S}$ of modules, i.e. such
that \( \mathcal{B} = S^\perp \), is complete. In practice, naturally arising cotorsion pairs are usually proved to have this property.

A cotorsion pair \( \mathcal{C} = (\mathcal{A}, \mathcal{B}) \) is hereditary if, \( \operatorname{Ext}^n_R(A, B) = 0 \) for each \( A \in \mathcal{A}, \ B \in \mathcal{B} \) and \( n \geq 1 \). Equivalently, one might require that \( \mathcal{A} \) be closed under kernels of epimorphisms, or that \( \mathcal{B} \) be closed under cokernels of monomorphisms, \([27, \text{Lemma } 5.24]\).

We say that \( \mathcal{D} \subseteq \text{Mod-}R \) is a definable class if \( \mathcal{D} \) is closed under products, direct limits and pure submodules. For a class \( \mathcal{C} \subseteq \text{Mod-}R \), we denote by \( \operatorname{Cogen}(\mathcal{C}) \) the class of modules cogenerated by \( \mathcal{C} \), i.e. the class of all submodules of products of modules from \( \mathcal{C} \). Analogously, we denote by \( \operatorname{Cogen}_n(\mathcal{C}) \) the class of all pure submodules of products of modules from \( \mathcal{C} \), and by \( \mathcal{C} \) the definable closure of the class \( \mathcal{C} \), i.e. the smallest definable class containing \( \mathcal{C} \). Furthermore, we use the notations \( \mathcal{C}^+ = \bigcap_{C \in \mathcal{C}} \ker \operatorname{Ext}^1_R(C, -) \) and \( ^+\mathcal{C} = \bigcap_{C \in \mathcal{C}} \ker \operatorname{Ext}^1_R(\mathcal{C}, -) \). If \( \mathcal{C} = \{C\} \), we write just \( \operatorname{Cogen}(C), \operatorname{Cogen}_n(C), C, C^+ \) or \( ^+C \), respectively.

Every definable class is closed under pure-epimorphic images by \([12, \text{Theorem } 3.4.8]\), and the definable closure \( \overline{\mathcal{C}} \) of a class \( \mathcal{C} \) can be constructed, for instance, by closing \( \mathcal{C} \) under products, then under pure submodules and finally under pure-epimorphic images. Note also that, for any definable class \( \mathcal{D} \), there is by \([12, \text{Corollary } 5.3.52]\) a pure-injective module \( C \in \mathcal{D} \) such that \( \operatorname{Cogen}(C) = \mathcal{D} \). The module \( C \) is called an elementary cogenerator of the definable class \( \mathcal{D} \).

Further, given a right (left, resp.) \( R \)-module \( M \), \( M^\perp \) stands for the character module of \( M \), i.e. the left (right, resp.) \( R \)-module \( \operatorname{Hom}_R(M, \mathbb{Q}/\mathbb{Z}) \). Recall that if \( \mathcal{C} \) is closed under products and direct limits and \( M \in \mathcal{C} \), then \( M^\perp \in \mathcal{C} \) (cf. \([15, \text{Lemma } 5.3]\)), this is because \( M^\perp \) is elementarily equivalent to \( M \), so it purely embeds into an ultrapower of \( M \), and hence is a summand there).

Finally, for a regular uncountable cardinal \( \lambda \), we call a directed system \( \mathcal{M} = (M_i, f_{ji}; M_i \to M_j \mid i \leq j \in I) \) of modules \( \lambda \)-continuous, provided that the poset \( (I, \leq) \) has suprema of all chains of length \( < \lambda \) and, for any such chain \( J \subseteq I \), we have \( \operatorname{sup} J = \lim_{\alpha \in J} M_{\alpha} \). It is easy to see that \( \mathcal{M} \) is then \( \lambda \)-directed. A well-directed system \( (M_{\alpha}, f_{\beta\alpha}; M_{\alpha} \to M_{\beta} \mid \alpha \leq \beta < \sigma) \) is called a filtration of a module \( M \) if all the maps in the system are inclusions, \( M_0 = 0, M_\sigma = \lim_{\alpha < \sigma} M_\alpha \) for each limit ordinal \( \theta < \sigma \) and \( M = \lim_{\alpha < \sigma} M_\alpha \).

The following definition contains notions which are fundamental in this paper.

**Definition 1.1.** Let \( R \) be a ring and \( M, N \in \text{Mod-}R \). We say that a homomorphism \( f: M \to N \) is \( \mathcal{C} \)-injective if \( \operatorname{Hom}_R(f, C) \) is surjective for all \( C \in \mathcal{C} \). Moreover, for an uncountable regular cardinal \( \lambda \), we say that a module \( M \) is almost \((\mathcal{C}, \lambda)\)-projective, if \( M \) is the direct limit of a \( \lambda \)-continuous directed system consisting of \( < \lambda \)-presented modules from \( \overline{\mathcal{C}} \). If moreover all the colimit maps are \( \mathcal{C} \)-injective, then we call the module \( M \) \((\mathcal{C}, \lambda)\)-projective.

**Remark.** If \( M \) is \( < \lambda \)-presented, then (almost) \((\mathcal{C}, \lambda)\)-projectivity of \( M \) amounts to \( M \in \overline{\mathcal{C}} \).

Let us start with an easy observation.

**Lemma 1.2.** If \( f: M \to N \) is a \( \mathcal{C} \)-injective map and \( D \in \operatorname{Cogen}(\mathcal{C}) \) a module with \( \operatorname{Ext}^1_R(\operatorname{Coker}(f), D) = 0 \), then \( f \) is \( D \)-injective.

**Proof.** By our assumptions, \( \ker(f) \subseteq \ker(h) \) for any \( h \in \operatorname{Hom}_R(M, C) \) where \( C \in \mathcal{C} \). It immediately follows that the same holds for any \( h \in \operatorname{Hom}_R(M, D) \) since \( D \in \operatorname{Cogen}(\mathcal{C}) \). From the hypothesis on \( \operatorname{Coker}(f) \), we see that the inclusion \( \operatorname{Im}(f) \subseteq N \) is \( D \)-injective, hence \( f \) is \( D \)-injective as well. \( \square \)
We are interested in when \( \text{Ext}^1_R(\lim M_i, C) = 0 \) holds true for a direct system of modules \((M_i \mid i \in I)\). The following lemma gives us a tool to handle this situation in a special case.

**Lemma 1.3.** Let \( R \) be a ring, \( C \) be a module, and \((M_i, f_{ji} \mid i < j \in I)\) be a directed system of modules such that \( \text{Ext}^1_R(M_i, C) = 0 \) for each \( i \in I \). Then the following are equivalent:

1. \( \text{Ext}^1_R(\lim M_i, C) = 0 \).
2. For each family \((g_{ji} \colon M_i \to C \mid i < j)\) of morphisms such that
   \[ g_{ki} = g_{ji} + g_{kj}f_{ji} \]
   for each \( i, j, k \in I \) with \( i < j < k \),
   there is a family \((g_i \colon M_i \to C \mid i \in I)\) of morphisms such that
   \[ g_i = g_{ji} + g_{j}f_{ji} \]
   for each \( i, j \in I \) with \( i < j \).

**Proof.** It is well known that there is the following exact sequence for \( \lim M_i \):

\[
\cdots \xrightarrow{\delta_3} \bigoplus_{i_0 < i_1 < i_2} M_{i_0i_1i_2} \xrightarrow{\delta_0} \bigoplus_{i_0 < i_1} M_{i_0i_1} \xrightarrow{\delta} \bigoplus_{i_0 \in I} M_{i_0} \to \lim M_i \to 0
\]

where \( M_{i_0i_1\ldots i_n} = M_{i_0} \) for all \( i_0 < i_1 < \cdots < i_n \) in \( I \) and

\[
(\delta_0 \mid M_{ij})(x) = (x, -f_{ji}(x)) \in M_i \times M_j
\]

\[
(\delta_1 \mid M_{ijk})(x) = (x, -x, -f_{ji}(x)) \in M_k \times M_{ij} \times M_{jk}
\]

If we apply the functor \( \text{Hom}_R(\_, C) \) to that long exact sequence, we get in general a complex. Since \( \text{Ext}^1_R(\bigoplus M_{i_0}, C) = 0 \), we conclude that \( \text{Ext}^1_R(\lim M_i, C) = 0 \) if and only if this complex is exact at \( \text{Hom}_R(\bigoplus_{i_0 < c_1} M_{i_0i_1}, C) \). However, we have:

\[
\text{Hom}_R(\delta^0, C)((g_i)_i) = (g_i - g_jf_{ji})_{i < j}
\]

\[
\text{Hom}_R(\delta^1, C)((g_{ji})_{i < j}) = (g_{ki} - g_{ji} - g_{kj}f_{ji})_{i < j < k}
\]

Hence, the exactness condition translates precisely to the condition (2) of the statement. \( \square \)

As a fruitful corollary, we obtain the following generalization of what is referred to as the Eklof Lemma in [27, Lemma 6.2].

**Lemma 1.4.** Let \( R \) be a ring. Let \( C \subseteq \text{Mod-}R, \sigma \) be a limit ordinal and \( M = (M_\alpha, f_{\beta\alpha} : M_\alpha \to M_\beta \mid \alpha < \beta \leq \sigma) \) be a continuous well-ordered direct system of modules such that \( M_\alpha \in \mathcal{C} \) and \( f_{\alpha+1, \alpha} \) is \( \mathcal{C} \)-injective for all \( \alpha < \sigma \). Then \( M_\sigma \in \mathcal{C} \).

**Proof.** Pick any \( C \in \mathcal{C} \) and suppose that, as in the condition (2) of Lemma [23] we have a collection of maps \((g_{\beta\alpha})_{\alpha < \beta}\) with \( g_{\beta\alpha} : M_\alpha \to C \) and \( g_{\gamma\alpha} = g_{\beta\alpha} + g_{\gamma\beta}f_{\beta\alpha} \) whenever \( \alpha < \beta < \gamma < \sigma \). We need to find \( g_\alpha : M_\alpha \to C \) such that

\[ g_\alpha = g_{\beta\alpha} + g_{\gamma\beta}f_{\beta\alpha} \]

for each \( \alpha < \beta < \sigma \).

By substracting \( g_{\gamma\alpha} = g_{\beta\alpha} + g_{\gamma\beta}f_{\beta\alpha} \), this can be equivalently reformulated to

\[ g_\alpha - g_{\gamma\alpha} = (g_{\beta\alpha} - g_{\gamma\beta})f_{\beta\alpha} \]

for each \( \alpha < \beta < \gamma < \sigma \).

We can construct such \( g_\alpha \) by transfinite induction on \( \alpha < \sigma \). The initial morphism \( g_0 \) can be chosen arbitrarily. If \( \beta = \alpha + 1 \) is a successor ordinal, we take \( g_\beta \) as a lift of \( g_\alpha - g_{\alpha\beta} : M_\alpha \to C \) over \( f_{\beta\alpha} : M_\alpha \to M_\beta \), using the \( \mathcal{C} \)-injectivity of \( f_{\beta\alpha} \).

Finally, if \( \alpha \) is a limit ordinal, we use the continuity of the direct system and the fact that \((g_\alpha - g_{\delta\alpha} \mid \delta < \alpha)\) is a cocone of the direct system \((M_\delta \mid \delta < \alpha)\), and take \( g_\alpha : M_\alpha \to C \) as the colimit map corresponding to this cocone. \( \square \)
Remark. It follows from the lemma above that, given an uncountable regular cardinal $\kappa$, a $\kappa$-presented module which is $(C, \kappa)$-projective belongs to $\mathcal{C}$. Indeed, if $M = (M_i, f_{ji} : M_i \to M_j \mid i < j \in I)$ is any $\kappa$-continuous directed system of $\kappa$-presented modules whose direct limit is $M$, it has a continuous well-ordered subchain $M' = (M_{\alpha} \mid \alpha < \kappa)$ with the same direct limit. If $M$ witnesses the $(C, \kappa)$-projectivity of $M$, so does $M'$ and we can use the lemma.

If, on the other hand, $\kappa$ is singular, we have Lemma 6.1 instead. We will use the lemma here, but postpone its fairly technical proof to Sections 5 and 6 for the sake of better readability.

Recall from [3] that, for $Q \subseteq R$-Mod, a module $M$ is $Q$-Mittag-Leffler, if the canonical morphism $\rho: M \otimes_R \prod_{i \in I} Q_i \to \prod_{i \in I}(M \otimes_R Q_i)$ is injective for any subset $\{Q_i \mid i \in I\}$ of $Q$.

Further, given a module $M$ and $C \subseteq \text{Mod-}R$, we say that $M$ is $C$-stationary provided that for some (equivalently any) directed system $F = (F_i, f_{ji} : F_i \to F_j \mid i < j \in I)$ consisting of finitely presented modules with $\lim_{\to} F = M$, the corresponding inverse system $\text{Hom}_R(F, C)$ of abelian groups satisfies the Mittag-Leffler condition for each $C \in C$. This means that, for any $C \in C$ and $i, j \in I$, there exists $j \in I$, $j \geq i$, such that, for all $g \in \text{Hom}_R(F_j, C)$, we have $gf_{ji} \in \text{Im}((\text{Hom}_R(f_{ki}, C))$ for any $i \leq k \in I$. Moreover, if we denote by $f_i : F_i \to M$ the canonical colimit map, we say that $M$ is strict $C$-stationary if we have even $gf_{ji} \in \text{Im}((\text{Hom}_R(f_j, C))$. These two concepts coincide for $C$ (locally) pure-injective. See [3] Section 2] for this result and other ones relating the notions of (strict) stationary and Mittag-Leffler module.

One can use the following lemma to present a large $C$-stationary module as the direct limit of a $\lambda$-continuous directed system consisting of small $C$-stationary modules. Compare it with [35] Theorem 2.6]. Here we denote, for a set $I$ and a cardinal number $\lambda$, by $[I]^{< \lambda}$ the set of all subsets of $I$ of cardinality $< \lambda$.

**Lemma 1.5.** Let $C$ be a $\text{module}$, and $M$ be a $C$-stationary module. Then for each uncountable regular cardinal $\lambda$, there exists a $\lambda$-continuous directed system $\mathcal{L}$ consisting of $< \lambda$-presented $C$-stationary modules such that $M = \lim_{\to} \mathcal{L}$. Moreover, for every $L$ from the system $\mathcal{L}$ and a pure-injective module $D \in \mathcal{C}$, the canonical colimit map $L \to M$ is $D$-injective.

**Proof.** Consider a direct system $F = (F_i, f_{ji} : F_i \to F_j \mid i < j \in I)$ consisting of finitely presented modules such that $M = \lim_{\to} F_i$, and denote by $f_i : F_i \to M$ the canonical colimit maps. We can w.l.o.g. assume that $(I, \leq)$ does not have the largest element and, using [31] Corollary 2.10, Theorem 2.11], that $C$ is an elementary cogenerator of $\mathcal{C}$. Thus each pure-injective $D \in \mathcal{C}$ is a direct summand in a product of copies of $C$.

Since $M$ is strict $C$-stationary, we can define a map $\sigma : I \to I$ so that for each $i \in I$, any cardinal $\kappa$ and each $g \in \text{Hom}_R(F_{\sigma(i)}, C^{\kappa})$, we have $gf_{\sigma(i)} \in \text{Im}((\text{Hom}_R(f_i, C^{\kappa}))$. Further, we fix a map $\delta : I^2 \to I$ satisfying $i, j \leq \delta(i, j)$. For each $X \in [I]^{< \lambda}$, we construct the set $\tilde{X}$ as the union of a chain of the sets $X_1 = X$, $X_{2n} = X_{2n-1} \cup \delta(X_{2n-1}, i)$, and $X_{2n+1} = X_{2n} \cup \sigma(X_{2n})$ for $1 \leq n < \omega$.

Then $\tilde{X}$ is a directed subposet of $(I, \leq)$ closed under $\sigma$ and $\delta$. Moreover, $|\tilde{X}| < \lambda$ since $\lambda$ is uncountable. We put $\mathcal{L} = (\lim_{\to} F_{|X} \mid X \in [I]^{< \lambda})$ where we take the canonical colimit factorization maps as morphisms. It is easy to see that $\mathcal{L}$ is $\lambda$-continuous and that it consists of $< \lambda$-presented modules.

Pick any $L = \lim_{\to} F_{|X} \in \mathcal{L}$ and, for all $i \in \tilde{X}$, denote by $g_i : F_i \to L$ the colimit maps. Finally, let $g : L \to M$ be the canonical factorization, so that $f_i = gg_i$. 


By the construction, $L$ is strict $C$-stationary. Let $D \in C$ be arbitrary pure-injective and let $(-)^*$ denote the functor $\text{Hom}_R(-, D)$. It remains to show that the map $g^*: \text{End}(g) \to \text{im}(g)^*$ is surjective. Let us write $g^*: \text{End}(g) \to \text{im}(g)^*$.

First notice that for each $h: L \to D$, we have $\text{Ker}(h) \subseteq \text{Ker}(g)$: indeed, if $y \not\in \text{Ker}(h)$, then $h g_i(x) \neq 0$ for some $i \in \bar{X}$ where $g_i(x) = y$. However, by the construction, $h g_i \in \text{Im}(\text{Hom}_R(f_i, D))$, and so $f_i(x) = g g_i(x) \neq 0$ as well. Hence $y \not\in \text{Ker}(g)$. It follows that $\theta$ is an isomorphism and it suffices to prove the surjectivity of $\theta$.

To this end, consider the short exact sequence $0 \to \text{Im}(g) \to M \to \text{Coker}(g) \to 0$ as the direct limit of a direct system $\mathcal{D}$ of short exact sequences $0 \to \text{Im}(f_i) \to M \to \text{Coker}(f_i) \to 0$, where $i$ runs through $\bar{X}$. Since $D$ is pure-injective, we have

$$\lim \text{Ext}_R^1(\text{Coker}(f_i), D) \cong \text{Ext}_R^1(\text{Coker}(g), D)$$

by a classic result of Auslander (see e.g. [27, Lemma 6.28]).

Applying the functor $(-)^*$ to the direct system $\mathcal{D}$, we obtain for each $i \in \bar{X}$ the following commutative diagram with exact rows where $\epsilon_i: \text{Im}(f_i) \to \text{Im}(g)$ denotes the inclusion:

$$
\begin{array}{c}
\begin{array}{ccc}
M^* & \xrightarrow{\theta} & (\text{Im}(g))^* \\
\| & & \downarrow \epsilon_i^* \\
M^* & \xrightarrow{\epsilon_i^* \theta} & (\text{Im}(f_i))^* \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Ext}_R^1(\text{Coker}(g), D) \\
\downarrow \\
\text{Ext}_R^1(\text{Coker}(f_i), D).
\end{array}
\end{array}
$$

We will prove that $\eta$ is the zero map. By the previous paragraph, this amounts to showing that $\epsilon_i \eta = 0$ for all $i \in \bar{X}$. However, $\epsilon_i \eta = \eta_i \epsilon_i^*$, and $\text{Im}(\epsilon_i^*) \subseteq \text{Im}(\epsilon_i^* \theta) = \text{Ker}(\eta_i)$, by the construction of $L$. Hence $\eta_i \epsilon_i^* = 0$ for each $i \in \bar{X}$ and $\theta$ is an epimorphism.

**Remark.** Notice that the same proof would apply if we allow the $L$ from the statement of Lemma [15] to be the direct limit of a directed subsystem of $L$.

**Definition 1.6.** Let $S = (M_i, f_{ij}: M_i \to M_j | i < j \in I)$ be a directed system of modules. For each $\eta$ regular uncountable, we denote by $S^\eta$ the directed system of modules consisting of direct limits of directed subsystems of $S$ of cardinality $< \eta$ and canonical factorization maps between them. So if $g_{kk}: N_k \to N_l$ is a morphism in $S^\eta$, then $\text{im}(g_{kk}) \subseteq N_k = \lim_{\to} \mathcal{D}_k$, $N_l = \lim_{\to} \mathcal{D}_l$ where $\mathcal{D}_k, \mathcal{D}_l$ are directed subsystems of $S$ of cardinality $< \eta$ and $\mathcal{D}_k \subseteq \mathcal{D}_l$.

**Remark.** The latter definition is often used in conjunction with Observation [16].

The typical use is as follows. If $S$ is $\aleph_1$-continuous and $\mathcal{D} \in S^\eta$, then $\mathcal{D}$ is often not $\aleph_1$-continuous itself. Observation [16] allows us to find an $\aleph_1$-continuous direct system $\mathcal{D}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq S$ and $\lim_{\to} \mathcal{D} = \lim_{\to} \mathcal{D}'$. Thus, for instance, if $S$ witnesses almost $(C, \aleph_1)$-projectivity, so does $\mathcal{D}'$.

In the lemma below, we use the notion of a filter-closed class from [16]. Note that a class is filter-closed, for instance, provided that it is closed under direct products and either direct limits of monomorphisms, or pure submodules. On the other hand, any filter-closed class is closed under taking arbitrary direct products and direct sums.

**Lemma 1.7.** Let $B$ be a filter-closed class of modules and $M \in +B$ an almost $(B, \aleph_1)$-projective module. Then $M$ is $(\text{Cogen}_*(B), \lambda)$-projective for any regular uncountable cardinal $\lambda$. In particular, $M \in +\text{Cogen}_*(B)$.

**Proof.** Put $\mathcal{D} = \text{Cogen}_*(B)$ and $\mathcal{A} = +\mathcal{D}$. Note that $\mathcal{D}$ is necessarily filter-closed. Let $\mathcal{T}$ be a directed system witnessing that $M$ is almost $(B, \aleph_1)$-projective. Let $\kappa$
be the least infinite cardinal such that $M$ is $\kappa$-presented. We proceed by induction on $\kappa$.

For $\kappa = \aleph_0$, we have to show only that $M \in \mathcal{A}$. However, this follows immediately from [45] Proposition 4.3.

Now, let $\kappa$ be uncountable. By induction on $\aleph_0 < \lambda \leq \kappa$, $\lambda$ regular, we prove that there exists a directed system $C_\lambda$ consisting of modules from $\mathcal{A}$ and witnessing that $M$ is $\mathcal{B}$-projective. For $\lambda = \aleph_1$, [45] Proposition 4.3 gives us that $\mathcal{T}$ witnesses almost $(\mathcal{D}, \aleph_1)$-projectivity of $M$, which in fact means none other than that $\mathcal{T}$ consists of modules from $\mathcal{A}$. Using [46] Lemma 2.3, we obtain a directed subsystem $C_{\aleph_0}$ of $\mathcal{T}$ witnessing $(\mathcal{B}, \aleph_1)$-projectivity of $M$.

Let $\lambda > \aleph_1$. Consider the system $\mathcal{T}^\lambda$ (see Definition 1.4). By Observation 1.3 the modules from $\mathcal{T}^\lambda$ are almost $(\mathcal{B}, \aleph_1)$-projective. As before, [46] Lemma 2.3 provides us with a $\lambda$-continuous directed subsystem $\mathcal{C}$ of $\mathcal{T}^\lambda$ such that $\lim_{\mathcal{C}} \mathcal{C} = M$ and all the colimit maps from modules in $\mathcal{C}$ to $M$ are $\mathcal{B}$-injective. We are going to show that $\mathcal{C}$ contains a $\lambda$-continuous directed subsystem consisting of modules from $\mathcal{A}$. In fact, it is enough to show that modules from $\mathcal{A}$ occur cofinally in the directed system $\mathcal{C}$. Indeed, suppose we are given a well-ordered subsystem $\mathcal{S}$ of $\mathcal{C}$ of cardinality $< \lambda$ consisting of modules from $\mathcal{A}$. Since all the connecting maps in $\mathcal{S}$ are $\mathcal{B}$-injective, it follows from Lemma 1.4 that $\lim_{\mathcal{S}} \mathcal{S}$ belongs to $\mathcal{B}$, and hence to $\mathcal{A}$ by the inductive hypothesis for $\kappa$ (we have $\lambda \leq \kappa$ and $\lim_{\mathcal{S}} \mathcal{S}$ is $< \lambda$-presented).

We distinguish the following three cases.

Case 1: $\lambda = \eta^+$ for $\eta$ regular. Consider the system $\mathcal{C}_\eta^\lambda$ which comes from the application of the construction in Definition 1.4 to the system $\mathcal{C}_\eta$, which witnesses the $(\mathcal{B}, \eta)$-projectivity of $M$. Notice in particular that each direct subsystem $\mathcal{S} \subseteq \mathcal{C}_\eta$ of cardinality $\leq \eta$ either has a supremum in $\mathcal{C}_\eta$ (if the cardinality of $\mathcal{S}$ is $< \eta$) or has a cofinal well-ordered subsystem $\mathcal{S}'$ which is indexed by $\eta$. In the latter case, we can replace $\mathcal{S}$ by $\mathcal{S}'$ and assume w.l.o.g. that $\mathcal{S}$ is continuous (since $\mathcal{C}_\eta$ is $\eta$-continuous). As above, it follows from Lemma 1.4 and the inductive hypothesis that all elements of $\mathcal{C}_\eta^\lambda$ belong to $\mathcal{A}$.

Finally, we can, by Construction A.2, intersect the system $\mathcal{C}$ above with the system $\mathcal{C}_\eta^\lambda$. The resulting $\lambda$-continuous common subsystem $\mathcal{C}_\lambda$ has all the properties which we require.

Case 2: $\lambda = \eta^+$ for $\eta$ singular. Set $\mathcal{M}_\lambda = \mathcal{C}$ and, for all regular uncountable $\theta < \eta$, let us denote by $\mathcal{M}_\theta$ the system $\mathcal{C}_\theta$. The set $\{\mathcal{M}_\theta \mid \aleph_0 < \theta = \text{cf}(\theta) \leq \lambda\}$ has cardinality $\leq \eta$ which allows us to use Construction A.2 to intersect all the systems into one which we denote by $\mathcal{C}_\lambda$. Then objects of $\mathcal{C}_\lambda$ are in $\mathcal{B}$ by Lemma 6.1 (see also the remark below Lemma 1.4, since each object in $\mathcal{C}_\lambda$ is an $\eta$-presented module which is almost $(\mathcal{D}, \theta)$-projective for all regular uncountable $\theta < \eta$). Thus, the objects in $\mathcal{C}_\lambda$ are also in $\mathcal{A}$ by the inductive hypothesis and the maps in $\mathcal{C}_\lambda$ are $\mathcal{B}$-injective since such are the maps in $\mathcal{C}$.

Case 3: $\lambda$ is a limit regular cardinal. We can denote $\mathcal{C} = (M_i, f_{ij}: M_i \to M_j \mid i < j \in I)$, and for each $i \in I$, let $f_i: M_i \to M$ be the canonical colimit map. We construct an increasing countable subposet of $(I, \leq)$ as follows:

Pick any $i_0 \in I$. Assume that $i_n$ is defined and that $M_{i_n}$ is $\eta$-presented for $\aleph_0 < \eta < \lambda$. We pick a module $N_{i_n}$ from the system $\mathcal{C}_{\eta^+}$ and a factorization $g_{i_n}: M_{i_n} \to N_{i_n}$ of $f_{i_n}$ through the $\mathcal{B}$-injective canonical colimit map $h_{i_n}: N_{i_n} \to M$. Subsequently, we pick $i_{n+1} \in I$, $i_n < i_{n+1}$, and $g'_{i_n}: N_{i_n} \to M_{i_{n+1}}$ in such a way that $h_{i_n} = f_{i_{n+1}}g'_{i_n}$ and $f_{i_{n+1}i_n} = g'_{i_n}g_{i_n}$.

The module $M_{i_\omega} = \lim_{i < \omega} M_{i_n}$ is a $< \lambda$-presented member of the system $\mathcal{C}$. On the other hand $M_{i_\omega}$ is the direct limit of the directed system
Lemma 1.5, so each $u \in C$ since $K$ system $\mathbb{K}$ consists of modules which are projective.

By [45, Proposition 4.3], we obtain an $\aleph_1$-injective connecting maps. It follows that $M_k$ belongs to $\mathcal{B}$ by Lemma 1.4, hence $M_k \oplus A$ by the inductive hypothesis.

Since $i_0 \in I$ was arbitrary, we have shown that modules from $\mathcal{A}$ occur in the system $C$ cofinally, whence there is a subsystem $C_\lambda$ of $C$ consisting of modules from $\mathcal{A}$ witnessing ($\mathcal{B}, \lambda$)-projectivity of $M$.

We have finished the construction of the systems $C_\lambda$. Notice that each module in $C_\lambda$ is almost ($\mathcal{B}, \aleph_1$)-projective since $C_\lambda \subseteq T^\lambda$. It also belongs to $\mathcal{B}$ and so it is even ($\mathcal{B}, \aleph_1$)-projective by [46] Lemma 2.3. Further, since any morphism $f: M_i \to M_j$ in $C_\lambda$ is $\mathcal{B}$-injective and $M_j \in \mathcal{B}$, the cokernel of $f$ belongs to $\mathcal{B}$. Moreover, if we apply Construction A.1 to $f: M_i \to M_j$ and the direct systems $\mathcal{M}$ and $\mathcal{N}$ which witness that $M_i$ and $M_j$ are ($\mathcal{B}, \aleph_1$)-projective, respectively, one directly checks that the maps $u_k$ from the conclusion are $\mathcal{B}$-injective. Hence $\text{Coker}(f)$ is an almost ($\mathcal{B}, \aleph_1$)-projective $\lambda$-presented module in $\mathcal{B}$ and, by the induction hypothesis, $\text{Coker}(f) \in \mathcal{A} = \mathcal{D}$. By Lemma 1.2 the morphisms in $C_\lambda$ are even $\mathcal{D}$-injective.

Now, if $\kappa$ is regular, $C_\kappa$ contains a well-ordered directed subsystem $S$ such that $\varinjlim S = M$. Using Lemma 1.3 we deduce that $M \in \mathcal{A}$. If, on the other hand, $\kappa$ is singular, we use that $\mathcal{D}$ is filter-closed and apply Lemma 6.1 to show that $M \in \mathcal{A}$. Finally, we can use [46] Lemma 2.3 once more to pick, for each $\lambda \leq \kappa$ regular uncountable, from the system $C_\lambda$ a subsystem witnessing that $M$ is ($\mathcal{D}, \lambda$)-projective.

Our first theorem constitutes a partial converse of the remark after Lemma 1.4.

**Theorem 1.8.** Let $\mathcal{B} \subseteq \text{Mod-}R$ be a class closed under direct limits and products. Then each module from $\mathcal{B}^\lambda$ is ($\mathcal{B}, \lambda$)-projective for any $\lambda$ regular uncountable. Consequently, $\mathcal{B}^\lambda = \mathcal{B}^\mathcal{B}$.

**Proof.** By the assumption on $\mathcal{B}$ (see [45] Lemma 5.3)), an elementary cogenerator $C$ of $\mathcal{B}$ is contained in $\mathcal{B}$. In particular, $\mathcal{B} = \text{Cogen}_1(\mathcal{B})$. Our result will follow from Lemma 1.7 once we show that each $M \in \mathcal{B}^\lambda$ is almost ($\mathcal{B}, \aleph_1$)-projective.

To this end, note that $M$ is strict $C$-stationary by [45] Lemma 4.2. If we fix a short exact sequence

$$0 \longrightarrow N \overset{f}{\longrightarrow} P \longrightarrow M \longrightarrow 0$$

with $P$ projective, then, since $(C(I))^{cc} \in \mathcal{B}$ for all sets $I$ and since $P$ is strict $C$-stationary because it is projective, $N$ is strict $C$-stationary as well by [45] Lemma 4.4.

Let $\mathcal{L}$ be an $\aleph_1$-continuous directed system provided by Lemma 1.5 for $N$, and let $S$ be the directed system consisting of all countably generated direct summands of $P$ and inclusions. By Construction A.1 we obtain an $\aleph_1$-continuous directed system $\mathcal{K}$, consisting of the cokernels of the morphisms $u_k$ from the construction, such that $\varprojlim \mathcal{K} = M$. Notice that modules from $\mathcal{K}$ are in $\mathcal{B}^\lambda$. Indeed, $f$ is $\mathcal{B}$-injective since $C \in \mathcal{B}$, and each colimit map from the directed system $\mathcal{L}$ is $\mathcal{B}$-injective by Lemma 1.5 so each $u_k$ is $\mathcal{B}$-injective and the claim follows. An application of Lemma 1.5 to $M$ provides us, on the other hand, with an $\aleph_1$-continuous directed system $\mathcal{K}'$ whose all objects are $\mathcal{B}$-stationary. If we intersect the two systems using Construction A.2 we obtain an $\aleph_1$-continuous directed system consisting of countably presented modules which are $\mathcal{B}$-stationary and in $\mathcal{B}^\lambda$, hence also in $\mathcal{B}$ by [45] Proposition 4.3]. This direct system witnesses that $M$ is almost ($\mathcal{B}, \aleph_1$)-projective. □
We can also deduce the following crucial result which generalizes [27, Theorem 8.17]. For the first time, it appeared in an unpublished manuscript [47].

**Theorem 1.9.** Let $\theta \leq \kappa$ be uncountable cardinals, $\theta$ regular. Let $C, M$ be modules and $(M_\alpha, f_{\beta\alpha} \mid \alpha < \beta \leq \theta)$ be a $\theta$-continuous directed system such that $M = M_\theta$ and all $M_\alpha$ are $\theta$-generated modules for $\alpha < \theta$. Suppose that $\text{Ext}^1_R(M_\alpha, C^{(\kappa)}) = 0$ for all $\alpha < \theta$. Then the following conditions are equivalent:

1. $\text{Ext}^1_R(M, C^{(\kappa)}) = 0$.

2. There is a closed unbounded subset $X \subseteq \theta$ such that $f_{\beta\alpha}$ is $C^{(\kappa)}$-injective for all $\alpha, \beta < \theta$.

**Remark.** For the implication $(2) \implies (1)$, we do not need the assumption that the modules $M_\alpha$ are $\theta$-generated for $\alpha < \theta$.

**Proof.** $(2) \implies (1)$. We can w.l.o.g. assume that $f_{\beta\alpha}$ is $C^{(\kappa)}$-injective for each $\alpha < \beta < \theta$ and use Lemma [1.3].

$(1) \implies (2)$. Put $D = C^{(\kappa)}$. Possibly restricting ourselves just to indices in some closed unbounded subset of $\theta$, we can always assume that whenever $f_{\beta\alpha}$ is not $D$-injective for some $\alpha < \beta$, then already $f_{\alpha+1,\alpha}$ was not $D$-injective.

Suppose now for contradiction that the set

$$E = \{ \alpha < \theta \mid f_{\alpha+1,\alpha} \text{ is not } D\text{-injective} \}$$

is stationary in $\theta$; that is, it intersects every closed unbounded subset of $\theta$. Fix $h_\alpha \in \text{Hom}_R(M_\alpha, D)$ such that $h_\alpha$ does not factorize through $f_{\alpha+1,\alpha}$ for each $\alpha \in E$. Put $h_\alpha = 0$ for $\alpha \in \theta \setminus E$.

We will inductively construct homomorphisms $g_{\beta\alpha} : M_\alpha \to D^{(\beta)}$ for all $\alpha < \beta \leq \theta$ such that

(a) $g_{\alpha+1,\alpha}$ is the composition of $h_\alpha$ with the $\alpha$th canonical inclusion $D \to D^{(\alpha+1)}$ (in particular, $\text{Im}(g_{\alpha+1,\alpha}) \cap D^{(\alpha)} = \{0\}$), and

(b) $g_{\gamma\alpha} = g_{\beta\alpha} + g_{\gamma\beta}f_{\beta\alpha}$ for all $\alpha < \beta < \gamma < \theta$.

For $\beta = 1$, we just put $g_{10} = h_0$. Suppose we have constructed $g_{\beta\alpha}$ for all $\alpha < \beta < \gamma$ for some $\gamma \leq \theta$. If $\gamma = \delta + 1$ for some $\delta$, put $g_{\gamma\alpha} = h_\delta$ and $g_{\gamma\alpha} = g_{\delta\alpha} + h_\delta f_{\delta\alpha}$ for each $\alpha < \delta$. If $\gamma$ is a limit ordinal, define $g_{\gamma\alpha}$ as the maps $g_{\gamma\alpha}$ given by Lemma [1.3] condition (2). It is straightforward to check that the maps defined in this way satisfy the required conditions.

Put

$$X = \{ \lambda < \theta \mid \text{Im } g_{\lambda\alpha} \subseteq D^{(\lambda)} \text{ for each } \alpha < \lambda \}$$

It is easy to check that $X$ is closed unbounded in $\theta$. Hence, the set $X'$ consisting of the limit ordinals in $X$ is closed unbounded too, and there is some $\lambda \in X' \cap E$.

Denote by $\pi$ the $\lambda$th canonical projection $D^{(\theta)} \to D$. First we show that $\pi g_{\lambda\alpha} = 0$. Choose an arbitrary $x \in M_\lambda$. Since the $M_\lambda = \lim_{\mu<\lambda} M_\mu$ by continuity of the direct system, there is $\alpha < \lambda$ and $y \in M_\alpha$ such that $x = f_{\lambda\alpha}(y)$. We have the equality:

$$\pi g_{\lambda\alpha}(y) = \pi g_{\lambda\alpha}(y) + \pi g_{\theta\lambda}f_{\lambda\alpha}(y)$$

But $\pi g_{\lambda\alpha}(y) = 0$ since $\lambda \in X$ and $\pi g_{\lambda\alpha}(y) = 0$ by definition of $g_{\lambda\alpha}$. Hence $0 = \pi g_{\theta\lambda}f_{\lambda\alpha}(y) = \pi g_{\lambda\theta}(x)$. The claim follows since $x \in M_\lambda$ was arbitrary.

On the other hand, we know that $g_{\lambda\lambda} = g_{\lambda+1,\lambda} + g_{\theta,\lambda+1}f_{\lambda+1,\lambda}$. Composing this with $\pi$, we get:

$$0 = \pi g_{\lambda\lambda} = \pi g_{\lambda+1,\lambda} + \pi g_{\theta,\lambda+1}f_{\lambda+1,\lambda} = h_\lambda + \pi g_{\theta,\lambda+1}f_{\lambda+1,\lambda}$$

But this implies that $h_\lambda$ factorizes through $f_{\lambda+1,\lambda}$, a contradiction to the choice of $h_\lambda$ for $\lambda \in E$.

Hence, $E$ is not stationary. Therefore, we can choose a closed unbounded subset $X \subseteq \theta$ such that $X \cap E = \varnothing$ and (2) follows. □
2. \( \Sigma \)-COTORSION MODULES AND \( C \)-STATIONARITY

Let \( \mathcal{FL} \) denote the class of all flat \( R \)-modules and \( \mathcal{EC} = \mathcal{FL}^\perp \). The modules in \( \mathcal{EC} \) are called (Enochs) cotorsion modules. They generalize pure-injective modules and have been studied from that perspective in the series of papers \([28, 30, 32]\), especially regarding their direct sum decomposition properties. Among cotorsion modules, one may specialize to \( \Sigma \)-cotorsion modules, i.e. those whose every sum of copies is cotorsion. This is an intriguing class of modules at the boundary of homological algebra, model theory and set theory. These modules are far more complicated than \( \Sigma \)-pure-injective ones and have been studied in \([8, 29, 31]\).

In this section, we prove that being \( \Sigma \)-cotorsion is a property of the first-order theory of a module rather than the individual module alone, and give an analysis of the resulting theory, extending jointly the descriptions in \([31, \text{Theorem 12}] \) (for theory of a module rather than the individual module alone, and give an analysis complicated than \( \Sigma \)-pure-injective ones and have been studied in \([8, 29, 31]\).)

Proposition 2.2. Let \( \kappa \) be a nonzero cardinal, \( F, C \in \text{Mod-} R \) with \( F \) \( \kappa \)-presented. Assume that \( F \) is almost \( (C^{(\kappa)}, \lambda) \)-projective for each \( \lambda \) regular uncountable. Then \( F \) is \( (\text{Cogen}_{\kappa}, C, \lambda) \)-projective for each regular \( \lambda \) \( \geq \aleph_0 \), and so \( F \in \perp \text{Cogen}_{\kappa}, C \).

Proof. Put \( C = \{C^{(\kappa)}\} \) and \( B = \text{Cogen}_{\kappa}, C \). We work by induction on \( \kappa \). If \( \kappa \) is finite, our assumption says that \( \text{Ext}_{\kappa}^{1}(F, C) = 0 \) and \( F \) is finitely presented. Consequently, \( F \in \perp B \).

If \( \kappa = \aleph_0 \), we have \( \text{Ext}_{\kappa}^{1}(F, C^{(\aleph_0)}) = 0 \) and our result follows from \([16, \text{Proposition 2.7}] \). Note that for \( F \) countably presented, the \((B, \lambda)\)-projectivity amounts to \( F \in \perp B \).

Let \( \lambda > \aleph_0 \) and, for each regular uncountable \( \lambda \), let \( S_{\lambda} \) denote a directed system of modules witnessing that \( F \) is almost \((C, \lambda)\)-projective. Notice that \( S_{\aleph_1} \), even witnesses that \( F \) is almost \((B, \aleph_1)\)-projective by \([16, \text{Proposition 2.7}] \). Furthermore, using Lemma 2.1 with \( \nu = \aleph_0 \), we can w.l.o.g. assume that, if \( \lambda \) is a successor cardinal, all the modules in \( S_{\lambda} \) are almost \((C, \eta)\)-projective for all regular uncountable \( \eta \). Hence they are \((B, \eta)\)-projective for all regular uncountable \( \eta \) by the inductive hypothesis, provided they are \( < \kappa \)-presented. In particular, \( S_{\lambda} \) witnesses almost \((B, \lambda)\)-projectivity of \( F \) for all infinite successor cardinals \( \lambda \leq \kappa \).

To conclude our proof, we will show that \( F \in \perp B \) and use Lemma 2.1. We remind the reader that \( F \in \perp C \) since we assume that \( F \) is almost \((C^{(\kappa)}, \kappa^+)\)-projective. We discuss several cases depending on \( \kappa \):

Case 1: If \( \kappa \) is singular, we get \( F \in \perp B \) by Lemma 2.1.

Case 2: Suppose that \( \kappa = \mu^+ \) is successor cardinal. Then the system \( S_{\kappa} \) can be taken w.l.o.g. well-ordered, so \( S_{\kappa} = (F_{\alpha}, f_{\beta\alpha} : F_{\alpha} \to F_{\beta} \mid \alpha < \beta < \kappa) \) where
each $F_\alpha$ is $< \kappa$-presented. As was explained above, we can assume that $S_\kappa$ consists of modules which are $(\mathcal{B}, \lambda)$-projective for all regular uncountable cardinals $\lambda$ (in particular $F_\alpha \in \mathcal{B}$). Finally, by applying Theorem \[\text{[13]}\] we can also assume that $S_\kappa$ witnesses $(\mathcal{C}, \kappa)$-projectivity.

We fix any $\lambda > \aleph_0$ regular and, for each $\alpha < \kappa$, let $T_\alpha$ denote a system witnessing that $F_\alpha$ is $(\mathcal{B}, \lambda)$-projective. If we put $M = T_\alpha$ and $N = T_{\alpha + 1}$, Construction \[\text{[A.1]}\] will provide us with the system $K$ which is easily seen to witness almost $(\mathcal{C}, \lambda)$-projectivity of $\text{Coker}(f_{\alpha + 1, \alpha})$. Just use the properties of $M$ and $N$ together with the $C$-injectivity of $f_{\alpha + 1, \alpha}$. Since $\lambda$ was arbitrary, we can use the inductive hypothesis to deduce that $\text{Coker}(f_{\alpha + 1, \alpha}) \in \mathcal{B}$. Subsequently, the map $f_{\alpha + 1, \alpha}$ is $\mathcal{B}$-injective by Lemma \[\text{[12]}\] for every $\alpha < \kappa$, and Lemma \[\text{[13]}\] yields $F \in \mathcal{B}$.

Case 3: Let $\kappa$ be a weakly inaccessible (i.e. uncountable, regular and limit) cardinal. As in the previous case, $S_\kappa$ can be taken well-ordered, of the form $S_\kappa = (F_\alpha, f_{\beta\alpha} : F_\alpha \to F_\beta \mid \alpha < \beta < \kappa)$ where each $F_\alpha$ is $< \kappa$-presented. Possibly by intersecting $S_\kappa$ with $S^*_\kappa$, we can also w.l.o.g. assume that $S_\kappa$ consists of almost $(\mathcal{B}, \kappa)$-projective modules.

We next show that $S_\kappa$ contains a cofinal subsystem consisting of modules which are $(\mathcal{B}, \lambda)$-projective for each regular uncountable $\lambda$. To this end, suppose that $M_0$ is an arbitrary module from $S_\kappa$ (i.e. $M_0 = F_\alpha$ for some $\alpha < \kappa$). Then $M_0$ is $\kappa_0$-presented for an infinite cardinal $\kappa_0 < \kappa$. Let $R_0$ be the directed system obtained by intersecting $S_\kappa$ with the systems $S^\lambda_\kappa$ where $\lambda$ runs through the uncountable regular cardinals $\leq \kappa_0^\kappa$ (see Construction \[\text{[A.2]}\]). We continue recursively: once $M_n$, $\kappa_n$ and $R_n$ are defined, we find a $\kappa_{n + 1}$-presented module $M_{n + 1} \in R_n$ above $M_n$ where $\kappa_n < \kappa_{n + 1} < \kappa$; finally, we let $R_{\kappa + 1}$ be the intersection of $R_n$ with the systems $S^\lambda_\kappa$ where $\kappa_n < \lambda = \text{cf}(\lambda) \leq \kappa_{n + 1}^\kappa$. Then $M = \lim_{\kappa_n < \omega} M_n$ belongs to $R_\kappa$ for each $n < \omega$, hence $M$ is almost $(\mathcal{C}, \lambda)$-projective for each $\lambda$ regular uncountable. It follows from the inductive hypothesis that $M$ is $(\mathcal{B}, \lambda)$-projective for each $\lambda > \aleph_0$ regular.

To summarize so far, we can w.l.o.g. assume that $F_\alpha$ is $(\mathcal{B}, \lambda)$-projective (for each $\lambda > \aleph_0$ regular) whenever $\alpha$ is a non-limit ordinal. Put $F_\kappa = F$ and suppose, for the sake of contradiction, that there exists a limit ordinal $\delta < \kappa$ such that $F_\delta$ is not $(\mathcal{B}, \eta)$-projective for some regular uncountable $\eta$. Take the least such $\delta$.

Similarly to Case 2, we fix a regular cardinal $\lambda > \aleph_0$ and, for each $\alpha < \delta$, denote by $T_\alpha$ a system witnessing that $F_\alpha$ is $(\mathcal{B}, \lambda)$-projective. If we put $M = T_\alpha$ and $N = T_{\alpha + 1}$, Construction \[\text{[A.1]}\] will provide us with the system $K$ which witnesses almost $(\mathcal{C}, \lambda)$-projectivity of $\text{Coker}(f_{\alpha + 1, \alpha})$. As before, we can use the inductive hypothesis to deduce that $\text{Coker}(f_{\alpha + 1, \alpha}) \in \mathcal{B}$, so that the map $f_{\alpha + 1, \alpha}$ is $\mathcal{B}$-injective by Lemma \[\text{[12]}\] for every $\alpha < \delta$, and Lemma \[\text{[13]}\] yields $F_\delta \in \mathcal{B}$. Lemma \[\text{[17]}\] then gives us the desired contradiction.

\[\square\]

Remark. Going through the proof, we can see that, for $\kappa$ singular, the same conclusion holds assuming only that $F$ is almost $(C(\kappa), \lambda)$-projective for all regular uncountable $\lambda < \kappa$.

Now we can prove the model-theoretic nature of the $\Sigma$-cotorsion property.

**Theorem 2.3.** If $C$ is a $\Sigma$-cotorsion module, then every module from the definable closure of $\{C\}$ is $(\Sigma)$-cotorsion.

**Proof.** Let $F$ be an arbitrary flat module. Then $F$ is a $\lambda$-continuous direct limit of $< \lambda$-presented flat modules, hence it is almost $(C(\kappa), \lambda)$-projective for any $\kappa$ and regular $\lambda > \aleph_0$. By Proposition \[\text{[2.2]}\] we have $F \in \mathcal{B}$. Since the class $\mathcal{FL}$ of all flat modules is resolving, we get even $\mathcal{FL} \subseteq \mathcal{C}$ (every $M \in \mathcal{C}$ is a pure-epimorphic image of a module from $\text{Cogen}_\kappa(C)$).

\[\square\]
Corollary 2.4. Let $\kappa = |R| + \aleph_0$, $F$ be a flat $\Sigma$-cotorsion module and $\mu$ a cardinal. Then any pure submodule of $F^{(\mu)}$ splits. In particular, $F$ is a direct sum of $\kappa$-presented modules with local endomorphism rings.

Proof. Let $P$ be a pure submodule in $F^{(\mu)}$. Using Theorem 2.3, we get that $P$ is cotorsion. Consequently $P$ splits in $F^{(\mu)}$ since $F^{(\mu)}/P$ is flat. The rest follows from [1, Theorem 1.1]. \hfill $\square$

The point in the following rather technical recursive definition is that $\mathcal{S}_C(M)$ holds if and only if a certain set of first-order sentences, determined by the module $M$, holds in $C$. We can look at $\mathcal{S}_C$ as a sort of ‘hereditary’ $C$-stationarity.

Definition 2.5. For $M \in \text{Mod-R}$, we denote by $\text{pres}(M)$ the least cardinal $\theta$ such that $M$ is $\theta$-presented. For a module $C$, we recursively define a property $\mathcal{S}_C$ of a module $M$ by stating that $\mathcal{S}_C(M)$ holds if and only if

1. $M$ is $C$-stationary and $\text{pres}(M) \leq \aleph_0$, or
2. $\theta = \text{pres}(M)$ is uncountable regular and there exists a $\theta$-continuous well-ordered directed system $\mathcal{M} = (M_\alpha, f_{\beta\alpha} : M_\alpha \to M_\beta \mid \alpha < \beta < \theta)$ such that
   a. $M_0 = 0$ and $\text{pres}(M_\alpha) < \theta$ for each $\alpha < \theta$,
   b. $\lim\mathcal{M}$ $M$,
   c. $\mathcal{S}_C(\text{Coker}(f_{\beta\alpha}))$ holds for each $\alpha < \beta < \theta$, or
3. $\theta = \text{pres}(M)$ is singular and, for each infinite successor cardinal $\lambda < \theta$,
   a. $M$ is the direct limit of a $\lambda$-continuous directed system consisting of $< \lambda$-presented modules satisfying $\mathcal{S}_C$.

Remark. Notice that $\mathcal{S}_C$ implies $\mathcal{S}_J$ whenever $J$ lies in the definable closure of $C$ since $C$-stationarity yields $J$-stationarity in this case. Moreover, it follows by [35, Proposition 2.2] that $\mathcal{S}_C(M)$ implies that $M$ is $C$-stationary for any $M \in \text{Mod-R}$.

We have the following interesting characterization of pure-projectivity.

Proposition 2.6. Let $C$ be an elementary cogenerator of $\text{Mod-R}$. Then $\mathcal{S}_C(M)$ holds if and only if $M$ is pure-projective.

Proof. First, notice that $C$-stationary means Mittag-Leffler. The if part easily follows from the fact that every pure-projective module decomposes as a direct sum of countably presented pure-projective (equivalently countably presented Mittag-Leffler) modules by [2, Theorem 26.1].

The only-if part goes by induction on $\kappa = \text{pres}(M)$. For $\kappa \leq \aleph_0$, $C$-stationarity of $M$ amounts to $M$ being pure-projective.

If $\kappa$ is regular uncountable, we consider the system $\mathcal{M}$ from the definition of $\mathcal{S}_C$. By the remark above, $M$ is $C$-stationary. From Lemma 1.5, we get a $\kappa$-continuous directed system $\mathcal{L}$ consisting of $< \kappa$-presented pure submodules of $M$ and inclusions such that $\lim\mathcal{L} = M$. Intersecting $\mathcal{M}$ and $\mathcal{L}$ together by Construction A.2, we obtain a filtration $(M_\alpha \mid \alpha < \kappa)$ consisting of $< \kappa$-presented pure submodules of $M$ such that $\mathcal{S}_C(M_{\alpha+1}/M_\alpha)$ for each $\alpha < \kappa$. However, by the inductive hypothesis, $M_{\alpha+1}/M_\alpha$ is pure-projective, whence the filtration splits and $M \cong \bigoplus_{\alpha < \kappa} M_{\alpha+1}/M_\alpha$.

If $\kappa$ is singular, we use (again) the definition of $\mathcal{S}_C$ together with Lemma 1.5 to obtain, for each $\lambda < \kappa$ infinite successor cardinal, a $\lambda$-continuous directed system $\mathcal{N}_{\lambda}$ consisting of $< \lambda$-presented pure submodules of $M$ satisfying $\mathcal{S}_C$. Using the inductive hypothesis, we see that the system $\mathcal{N}$, in fact, consists of (direct sums of countably presented) pure-projective modules. From the classical Shelah’s singular compactness theorem [15, §2 II, Theorems 3.1 and 4.1], we conclude that $M$ is as well a direct sum of countably presented pure-projective modules. \hfill $\square$
In the statement of the proposition below, $\text{PE}(C)$ denotes the pure-injective envelope of $C$.

**Proposition 2.7.** Let $\kappa$ be an infinite cardinal and $M, C \in \text{Mod}-R$ where $M$ is $\kappa$-presented. Consider the following conditions:

1. $M$ is $(\text{Cogen}_\kappa(C), \lambda)$-projective for all regular $\lambda > \aleph_0$;
2. $M$ is almost $(C(\kappa^+), \lambda)$-projective for all regular $\lambda > \aleph_0$;
3. $\mathfrak{S}_C(M)$ holds.

Then (1) $\iff$ (2) $\implies$ (3). If $M$ is almost $(\text{PE}(C), \aleph_1)$-projective, then (3) $\implies$ (1).

**Proof.** The implication (1) $\implies$ (2) is trivial. The converse one is exactly Proposition 2.2.

(1) $\implies$ (3). By induction on $\kappa$. If $\kappa = \aleph_0$, it follows by [45, Lemma 4.2]. Let $\kappa$ be uncountable. Using Lemma 2.1 and the inductive hypothesis, we can assume that, for all uncountable cardinal successors $\lambda \leq \kappa$, there is a system $\mathcal{S}_\lambda$ consisting of modules satisfying $\mathfrak{S}_C$ such that $\mathcal{S}_\lambda$ witnesses $(\text{Cogen}_\lambda(C), \lambda)$-projectivity of $M$.

If $\kappa$ is singular, we obtain $\mathfrak{S}_C(M)$ by the very definition.

Assume therefore that $\kappa$ is uncountable regular. As in the proof of Proposition 2.2, we can w.l.o.g. suppose that $\mathcal{S}_\kappa = (F_\alpha, f_{\beta \alpha} : F_\alpha \to F_\beta \mid \alpha < \beta < \kappa)$ and, using Construction $\Delta.1$ we deduce that $\text{Coker}(f_{\beta \alpha})$ is almost $(\text{Cogen}_\lambda(C), \lambda)$-projective for each regular $\lambda > \aleph_0$ and ordinals $\alpha < \beta < \kappa$. Hence $\mathfrak{S}_C(\text{Coker}(f_{\beta \alpha}))$ holds by the inductive hypothesis, and subsequently $\mathfrak{S}_C(M)$ holds as well.

(3) $\implies$ (1). Put $J = \text{PE}(C)$. Then $\text{Cogen}_\kappa(C) \subseteq \text{Cogen}_\kappa(J)$, and $\mathfrak{S}_J(M)$ holds by Remark. By induction on $\text{pres}(M)$, we prove that $\text{Ext}^1_R(M, C) = 0$ whenever $\mathfrak{S}_J(M)$ and there is a directed system $T$ witnessing that $M$ is almost $(J, \aleph_1)$-projective. If $M$ is countably presented, we can use [45, Proposition 4.3].

Otherwise, since $M$ is $J$-stationary by Remark, we can use Lemma $\Gamma.3$ to obtain a system $\mathcal{S}$, consisting of $J$-stationary modules, such that $\mathcal{S}$ witnesses the $(J, \aleph_1)$-projectivity of $M$ (if $\mathcal{S} \not\subseteq J$, just intersect $\mathcal{S}$ and $T$ using Construction $\Delta.2$). By [45, Proposition 4.3], $\mathcal{S}$ witnesses also that $M$ is almost $(\text{Cogen}_\kappa(J), \aleph_1)$-projective. Furthermore, using the pure-injectivity of $J$ (so that $J$ is closed under direct limits by [24, Theorem 6.19]), it follows from the remark after Lemma $\Gamma.3$ that the system $\mathcal{S}^\rho$ from Definition $\Delta.3$ witnesses the $(J, \lambda)$-projectivity of $M$ for each regular $\lambda > \aleph_0$.

If $\text{pres}(M)$ is singular, we can w.l.o.g. assume that $\mathcal{S}^\rho$ consists of modules satisfying $\mathfrak{S}_C$ for all successor cardinals $\aleph_0 < \lambda < \text{pres}(M)$; just use Construction $\Delta.2$ and $\mathfrak{S}_C(M)$. Since each $N \in S^\lambda$ also satisfies $\mathfrak{S}_J$ and, thanks to Observation $\Delta.3$ is almost $(J, \aleph_1)$-projective, we have $\text{Ext}^1_R(N, C) = 0$ by the inductive hypothesis and, consequently, $S$ witnesses almost $(\text{Cogen}_\kappa(C), \lambda)$-projectivity of $M$.

Whence we can use Lemma $\Gamma.4$ to conclude that $\text{Ext}^1_R(M, C) = 0$.

Now let $\theta = \text{pres}(M)$ be regular uncountable. Take $\mathcal{M}$ from the definition of $\mathfrak{S}_C(M)$ and intersect it with $S^\theta$ using Construction $\Delta.2$ (we can assume that $M_0 = 0$ is in the resulting system). Let $\alpha < \beta < \theta$ be arbitrary. Then $\text{Coker}(f_{\beta \alpha}) \in J$, since $f_{\beta \alpha}$ is $J$-injective and $M_\beta \in J$. We apply Construction $\Delta.1$ with $\alpha = \aleph_1$ to build, from $\mathcal{S}$, an $\aleph_1$-continuous directed system $T'$ consisting of countably presented modules from $J'$ such that $\text{lim}_{\alpha \in J'} \mathcal{M} = \text{Coker}(f_{\beta \alpha})$. Since this shows that $\text{Coker}(f_{\beta \alpha})$ is almost $(J, \aleph_1)$-projective, and also $\mathfrak{S}_J(f_{\beta \alpha})$ by the choice of $M$, we may use the inductive hypothesis to obtain that $\text{Coker}(f_{\beta \alpha}) \in \text{Cogen}_\kappa(C)$. In particular $M_\beta \in \text{Cogen}_\kappa(C)$ (which is the case $\alpha = 0$). Lemma $\Gamma.2$ gives that $f_{\beta \alpha}$ is $\text{Cogen}_\kappa(C)$-injective. Since $\alpha, \beta$ were chosen arbitrarily, we conclude that $\text{Ext}^1_R(M, C) = 0$ using Lemma $\Gamma.4$. 


To summarize, starting with an almost \((J, \aleph_1)\)-projective module \(M\) satisfying (3), we have shown that \(M \in \perp B\) and is almost \((B, \aleph_1)\)-projective for \(B = \text{Cogen}_\ast(C)\). Then (1) follows by Lemma 1.7 □

**Corollary 2.8.** Let \(R\) be a ring, \(C\) be an \(R\)-module and \(\kappa = |R| + \aleph_0\). Then \(C\) is \(\Sigma\)-cotorsion if and only if \(C^{(\kappa)}\) is cotorsion, if and only if \(\mathcal{E}_C(M)\) holds for each \((\kappa\text{-presented})\) flat \(R\)-module \(M\).

*Proof.* Apply Proposition 2.7 for each \(\kappa\text{-presented} flat module \(M\), and use the fact that, under our assumptions, each flat module has a filtration with consecutive factors flat and \(\kappa\text{-presented}. Notice also that each flat module is almost \((\text{PE}(C), \aleph_1)\)-projective since it is the direct limit of an \(\aleph_1\)-continuous directed system consisting of flat modules. □

**Example 2.9.** The implication \((3) \implies (2)\) in Proposition 2.7 does not hold without the additional assumption. To see this, let \(R\) be an \(\aleph_0\)-noetherian ring and \(C\) be a \(\Sigma\)-pure injective \(R\)-module which is not injective (e.g. \(R = \mathbb{Z}\) and \(C\) is nonzero finite). By Baer injectivity test, there exists a (cyclic) countably presented module \(M\) such that \(\text{Ext}_R^1(M, C) \neq 0\). However, \(\mathcal{E}_C(M)\) follows from the \(\Sigma\)-pure injectivity of \(C\), cf. [45, Lemma 5.1]. Thus (3) holds and (2) does not hold true.

**Example 2.10.** The condition \((2)\) in Proposition 2.7 does not imply \(M \in \perp \overline{C}\). Indeed, let \(R\) be a ring which is not right coherent and \(C\) an injective cogenerator of \(\text{Mod-}R\). Then \(C\) contains a module \(D\) which is not absolutely pure. Let \(M\) be a finitely presented module such that \(\text{Ext}_R^1(M, D) \neq 0\), whence \(M \notin \perp \overline{C}\). However, we see that the condition \((2)\) from Proposition 2.7 holds true since \(C^{(\kappa)}\) is absolutely pure for any \(\kappa\) (and so \(\text{Ext}_R^1(M, C^{(\kappa)}) = 0\)).

### 3. Projectively coresolved Gorenstein flat modules

In the next two sections we study implications of our set-theoretic tools to Gorenstein homological algebra. The highlight of this section is the fact that, for any ring, Gorenstein flat covers always exist. We also construct new model structures on the category of modules over any ring which refines the Gorenstein AC-projective model structure from [14] (in that the model structure from [14] is a Bousfield localization of ours).

By a *projectively coresolved Gorenstein flat module* or a \(\text{PGF}\)-module for short, we mean a syzygy module in an acyclic complex

\[
\cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow \cdots
\]

(\#) consisting of projective modules which remains exact after tensoring by arbitrary injective left \(R\)-module. We denote the class of all such modules by \(\mathcal{PGF}\). For a comparison, recall that a module is *Gorenstein projective* if it is a syzygy in an acyclic complex (\#) consisting of projective modules which remains exact after applying the functor \(\text{Hom}_R(\cdot, P)\) for arbitrary projective module \(P\).

Finally, a module is *Gorenstein flat* if it is a syzygy in an acyclic complex (\#) consisting of flat modules which remains exact after tensoring by arbitrary injective left \(R\)-module. We denote the classes of Gorenstein projective and Gorenstein flat modules by \(\mathcal{GP}\), \(\mathcal{GF}\) respectively.

We are going to apply results from the two previous sections to prove a general statement which yields \(\mathcal{PGF} \subseteq \perp \overline{R}\). Consequently, each module from \(\mathcal{PGF}\) is Gorenstein projective. On the other hand, each Gorenstein AC-projective module in the sense of Bravo, Gillespie and Hovey (cf. [14]) is a \(\text{PGF}\)-module. Moreover, if \(R\) is left coherent, then \(\mathcal{PGF}\) is precisely the class of all Gorenstein AC-projective
modules since $\overline{K_R}$ coincides with the class of all level (or equivalently flat) right $R$-modules in this case.

The next lemma is folklore.

**Lemma 3.1.** Let $P^\bullet$ be a complex. If we sum up all its shifts, we get a 1-periodic complex which induces an exact sequence $0 \to H \to M \to \bigoplus_{n \in \mathbb{Z}} P^n \to M \to 0$ such that $H \cong \bigoplus_{n \in \mathbb{Z}} H^n(P^\bullet)$ and $M \cong \bigoplus_{n \in \mathbb{Z}} P^n / B^n(P^\bullet)$ where $B^n(P^\bullet)$ denotes the $n$th coboundary module of $P^\bullet$ and $H^n(P^\bullet)$ the $n$th cohomology module.

In particular, if $K \in \mathcal{PGF}$, then $K$ is a direct summand in a module $M$ such that $M \cong P/M$ with $P$ projective and $\text{Tor}_1^R(M, I) = 0$ for each injective left $R$-module $I$.

We believe that the following statement could be of independent interest. Combined with the lemma above, it generalizes [14, Theorem A.6].

**Proposition 3.2.** Let $I$ be a left $R$-module and

$$M \xrightarrow{f} P \to M \to 0$$

an exact sequence of modules with $P \in \mathcal{IF}$. Assume that $f \otimes_R I^\theta$ is injective for all cardinals $\theta$. Then $f$ is $\overline{I^\theta}$-injective; in particular, $M \in \mathcal{IF}$.

**Proof.** Put $C = I^\varsigma$. By [14, Lemma 4.2], $P$ is (strict) $C$-stationary, equivalently $I$-Mittag-Leffler (cf. [34, Corollary 2.6]). By our assumption on $f \otimes_R I^\theta$, $M$ is $I$-Mittag-Leffler and $C$-stationary as well.

Let $\kappa$ denote the least infinite cardinal such that $P$ is $\kappa$-presented. Notice that $M$ is $\kappa$-presented as well. We are going to prove by induction on $\kappa$ that $M \in \mathcal{IF}(C)$. For $\kappa = \aleph_0$, it holds: $f$ is $C$-injective (by our assumption for $\theta = 1$ and the well-known relations between $\otimes$ and Hom) which yields $\text{Ext}_1^R(P, C) = 0$ since $\text{Ext}_1^R(P, C) = 0$, and it remains to use [14, Proposition 4.3].

Now assume that $\kappa$ is regular uncountable. Using Lemma [14, for $\lambda = \kappa$, we obtain a $\kappa$-continuous directed system $\mathcal{L} = \mathcal{L}_\kappa$ consisting of $C$-stationary $\kappa$-presented modules such that $\lim \mathcal{L} = M$. Passing to a cofinal subsystem, we can w.l.o.g. assume that $\mathcal{L}$ is well-ordered by $\kappa$. We have $\mathcal{L} = \{ (L_\alpha, f_{\beta\alpha} \mid \alpha < \beta < \kappa) \}$. For each $\alpha < \kappa$, let $f_{\alpha}: L_\alpha \to M$ be the canonical colimit map.

By Theorem [14.3], we obtain a system $S = S_\kappa$ witnessing that $P$ is $(\mathcal{O}, \kappa)$-projective; as before, we can assume that it is well-ordered, so $S = \{ (S_\alpha, g_{\beta\alpha} \mid \alpha < \beta < \kappa) \}$. Again, let $g_{\alpha}: S_\alpha \to P$ denote the canonical colimit map for each $\alpha < \kappa$. Possibly dropping some indices, Construction [A.1] provides us with a $\kappa$-continuous directed system $\mathcal{U} = \{ (u_\alpha: L_\alpha \to S_\alpha, f_{\beta\alpha} \mid \alpha < \beta < \kappa) \}$ of morphisms with $\lim \mathcal{U} = f$. We also get the well-ordered directed system $\mathcal{K} = \{ (\text{Coker}(u_\alpha), h_{\beta\alpha} \mid \alpha < \beta < \kappa) \}$ with canonically defined morphisms. It follows that $M = \lim \mathcal{K}$.

If we apply Construction [A.2] to $\mathcal{L}$ and $\mathcal{K}$, we can assume, by possibly passing to a closed and unbounded subset of $\kappa$, that $L_\alpha = \text{Coker}(u_\alpha)$ and that the canonical colimit map $\text{Coker}(u_\alpha) \to M$ equals $f_{\alpha}$ as well as that $h_{\beta\alpha} = f_{\beta\alpha}$ for all $\alpha < \beta < \kappa$.

Let $\theta$ be a cardinal and put $D = I^\theta$. We claim that $u_\alpha \otimes_R I^\theta$ is injective for every $\alpha < \kappa$, or equivalently: $u_\alpha$ is $D$-injective. To see this, notice that $f$ is $D$-injective by the hypothesis and $f_{\alpha}$ is $D$-injective by the statement of Lemma [14.3] for each $\alpha < \kappa$ (note that $D \in \mathcal{C}$). The claim follows from the identity $f_{\alpha} = g_{\alpha}u_\alpha$.

Since $S_\alpha \in \mathcal{IF}(C)$, we may use the inductive hypothesis to deduce that $L_\alpha \in \mathcal{IF}(\text{Cogen}_\kappa(C))$ for each $\alpha < \kappa$. Considering the following diagram with exact rows and columns.
we see that \( \text{Coker}(g_{\alpha+1,\alpha}) \in \mathcal{C} \) holds, by the property of \( \mathcal{S} \), and that \( \bar{u} \otimes_R \bar{t} \) is injective for any \( \theta \): indeed, just apply the functor \(- \otimes_R \bar{t}\) on the diagram and use the \( 3 \times 3 \) lemma. Once more using the inductive hypothesis, we infer that \( \text{Coker}(g_{\alpha+1,\alpha}) \in \mathcal{C} \) for each \( \alpha < \kappa \). By Lemma 1.2 \( f_{\alpha+1,\alpha} \) is \( \text{Cogen}_*(C) \)-injective, and Lemma 1.3 yields \( M \in \mathcal{C} \).

Now let \( \kappa \) be singular. For each \( \lambda < \kappa \) regular uncountable, there are systems \( \mathcal{L}_\lambda, \mathcal{S}_\lambda \) such that \( \mathcal{S}_\lambda \) witnesses that \( P \) is \((\mathcal{C}, \lambda)\)-injective and \( \mathcal{L}_\lambda \) is provided by Lemma 1.3. By a similar argument as in the regular step, combining Construction \( \Lambda_1 \) and \( \Lambda_2 \) we can obtain a \( \lambda \)-continuous directed system \( \mathcal{U}_\lambda = (u_j : L_j \to S_j \mid j \in J) \) of morphisms with \( \lim_{\mathcal{U}} = f \), \( L_j \in \mathcal{L}_\lambda, S_j \in \mathcal{S}_\lambda \), and such that the induced directed system \( (\text{Coker}(u_j) \mid j \in J) \) of modules is identical with \( \mathcal{L}_\lambda \).

As in the regular step, we observe that \( u_j \otimes_R \bar{t} \) is injective for each \( \theta \) and \( j \in J \). By the inductive hypothesis, we get that \( L_j \in \mathcal{C} \) for all \( j \in J \). Using Lemma 2 with \( D = \text{Cogen}_*(C) \) and \( \nu = \aleph_0 \), we obtain \( M \in \mathcal{C} \). This finishes the induction.

Finally, we show that \( f \) is \( \mathcal{C} \)-injective. Note that \( \mathcal{C} \) is just the closure of \( \text{Cogen}_*(C) \) under pure-epimorphic images. Let \( h : D \to E \) be a pure-epimorphism with \( D \in \text{Cogen}_*(C) \). We have to check that every morphism \( m : M \to E \) factorizes through \( f \). However, \( \text{Ker}(h) \in \text{Cogen}_*(C) \) yields the existence of \( n : M \to D \) such that \( hn = m \). Since \( M \in \mathcal{C} \) and \( f \) is \( \mathcal{C} \)-injective, \( f \) is \( \text{Cogen}_*(C) \)-injective by Lemma 1.2. We can thus factorize the morphism \( n \) through \( f \).

The composition of the resulting map \( r : P \to D \) with \( h \) is the desired factorization. \( \square \)

**Example 3.3.** Let \( 0 \to M \xrightarrow{f} P \to M \to 0 \) be a short exact sequence, \( R \) right coherent and \( I = R \). The assumptions of Proposition 1.2 hold whenever \( P \) is an \( fp \)-projective module, i.e. \( P \in \mathcal{C} \), where \( \mathcal{C} \) is none other than the class of absolutely pure (i.e. \( fp \)-injective) modules. As a result, we get that \( M \) is \( fp \)-projective as well.

Using the reduction from Lemma 3.1, we obtain a generalization of [25 Theorem 3.6]: over a right coherent ring, every syzygy module in an acyclic complex of \( fp \)-projective modules is \( fp \)-projective. By the dual reasoning to the one in the proof of [4 Theorem 4.3], we get the equality \( \text{dw}(fpProj) = \text{dg}(fpProj) \) over a right coherent ring.

We are now ready to prove that

**Theorem 3.4.** Every projectively coresolved Gorenstein flat module belongs to the class \( \mathcal{C} \), in particular it is Gorenstein projective.

**Proof.** Let \( K \in \text{PGF} \) be arbitrary. By Lemma 3.1 \( K \) is a direct summand of a module \( M \) such that \( M \cong P/M \) with \( P \) projective and \( \text{Tor}_R^n(M, I) = 0 \) for any
injective left $R$-module $I$. We use Proposition 3.2 for $I = (R_R)^c$ and $f : M \rightarrow P$ the inclusion.

Recall that, over any ring $R$, we have $\mathcal{FL} \subseteq \mathcal{FR}$. We sum up what we achieved in the corollary below. Note that $(R_R)^c$ is the class of all absolutely pure left $R$-modules if and only if $R$ is left coherent, if and only if $\mathcal{FR}$ coincides with the class of all flat modules.

**Corollary 3.5.** Let $K$ be a right $R$-module. Then the following conditions are equivalent:

1. $K$ is projectively coresolved Gorenstein flat;
2. $K$ is a syzygy in a long exact sequence of projective modules which stays exact after applying the functor $\text{Hom}_R(\cdot, F)$ for any $F \in \text{Mod-}R$ from the definable closure of \{R\};
3. $K$ is a syzygy in a long exact sequence of projective modules which stays exact after applying the functor $- \otimes_R I$ for any $I \in \text{R-Mod}$ from the definable closure of \{(R_R)^c\}.

We can prove more than just $\mathcal{PGF} \subseteq \mathcal{FR}$. We will see in Theorem 3.9 that $\mathcal{PGF}$ forms the left-hand class of a hereditary cotorsion pair which is generated by a set, hence complete.

To do so, we will need to construct a filtration of an exact sequence of the form $M \xrightarrow{f} P \xrightarrow{g} M \rightarrow 0$ by exact sequences of the same form. Ignoring the exactness for the moment, we will view the sequences as representations of the quiver

$$Q: \quad \begin{array}{c} p \rightarrow \rightarrow m \\ f \downarrow g \end{array}$$

in the category $\text{Mod-}R$ which are bound by the relation $gf = 0$. The category of such representations is equivalent to the category of left modules over the path algebra $R_1 = R^{\text{op}}Q/(gf)$ (see [5, Proposition III.1.7], whose proof is valid also in our situation).

**Remark.** The ring $R_1$ is isomorphic to the subring of the full matrix ring $M_3(R^{\text{op}})$ formed by matrices of the form

$$\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix}.$$ 

This isomorphism can be obtained from the left action $R_1 \rightarrow \text{End}_R(M)$ on the faithful left $R_1$-module corresponding to the representation

$$R \oplus R \xrightarrow{(0 1)} R.$$ 

Let us denote by $\mathcal{RE}$ the full subcategory of $\text{R-Mod}$ formed by all modules corresponding to the representations

$$P \xrightarrow{g} M \quad (\dagger)$$

with $M \xrightarrow{f} P \xrightarrow{g} M \rightarrow 0$ right exact.

---

1One can alternatively use [13, Theorem A.6] to prove Corollary 3.5.
Lemma 3.6. If $M, P$ are $\kappa$-presented in $\text{Mod-R}$ and $U \in R_1\text{-Mod}$ corresponds to a representation of the form $[1]$, then $U$ is $\kappa$-presented in $R_1\text{-Mod}$. The class $\mathcal{RE}$ is closed in $R_1\text{-Mod}$ under cokernels and extensions.

Proof. The first assertion follows from the fact that $R_1$ is free of finite rank as a right $R$-module. The closure under cokernels comes from the fact that the cokernel of a map between right exact sequences is right exact. For the closure under extensions, we consider the elements of $R_1\text{-Mod}$ as cochain complexes of right $R$-modules concentrated in degrees $-1, 0, 1$ and use the long exact sequence of cohomologies. \hfill \Box

Now, given an infinite cardinal $\nu$, we say that a ring $R$ is right $\nu$-coherent if each $\nu$-generated right ideal of $R$ is $\nu$-presented. If every right ideal of $R$ is $\nu$-generated, we say that $R$ is right $\nu$-noetherian.

Proposition 3.7. Let $\nu$ be an infinite cardinal, $W$ an injective cogenerator in $\text{Mod-R}$ and $I$ a left $R$-module. Let moreover $\mathcal{D}$ be a definable subclass of $\text{Mod-R}$ such that $I^\circ, W \in \mathcal{D}$. Assume that $\mathcal{E} : M \rightarrow P \rightarrow M \rightarrow 0$ is an exact sequence with $P \in \mathcal{D}$ and $f \otimes_R P^\theta$ injective for any cardinal $\theta$. Let moreover

1. $R$ be a right $\nu$-noetherian ring, or
2. $R$ be a right $\nu$-coherent ring, $f$ be a monomorphism and $P$ be projective, or
3. $f \otimes_R P^\theta$ be injective for all cardinals $\theta$ (in this case, set $\nu = \aleph_0$).

Then there exists a filtration $\mathfrak{F} = (\mathcal{E}_\alpha : M_\alpha \rightarrow P_\alpha \rightarrow M_\alpha \rightarrow 0 \mid \alpha \leq \sigma)$ of $\mathcal{E}$ where $P_{\alpha+1}/P_\alpha \in \mathcal{D}$ and $M_{\alpha+1}/M_\alpha$ is $\nu$-presented with $M_{\alpha+1}/M_\alpha \in \mathcal{D}^\sigma$ for each $\alpha < \sigma$.

Proof. Let $\kappa$ be the least infinite cardinal such that $P$ (and hence also $M$) is $\kappa$-presented. For the sake of nontriviality, let us assume that $\kappa > \nu$. We follow the lines of the proof of Proposition $[32]$ proving by induction on $\kappa$ the existence of the filtration $\mathfrak{F}$. The only difference is the choice of the systems $\mathcal{L}_\lambda, \mathcal{S}_\lambda$.

Using Theorem $[13]$, we let $\mathcal{S}_\lambda$ be a system witnessing $(\mathcal{D}, \lambda)$-projectivity of $P$. Since $W \in \mathcal{D}$, we can w.l.o.g. assume that $\mathcal{S}_\lambda$ consists of submodules of $P$ and inclusions.

It is easy to observe that, if (1) or (2) holds, every $\eta$-generated submodule of $M$ is $\eta$-presented whenever $\eta \geq \nu$. Put $C = I^\circ$ in this case, and $C = I^\circ \oplus W$ if (3) holds true. By Proposition $[32]$ we know that $M \in \mathcal{C}$, thus we can consider a system $\mathcal{L}_\lambda$ witnessing that $M$ is $(C, \lambda)$-projective (using Theorem $[13]$ again). For $\lambda > \nu$, we can w.l.o.g. assume that the elements of $\mathcal{L}_\lambda$ are, in fact, $< \lambda$-presented submodules of $M$. Indeed, in cases (1) and (2) we can intersect $\mathcal{L}_\lambda$ with the system of all $< \lambda$ generated submodules of $M$ using Construction $[32]$ while in case (3) we use that $W \in \mathcal{C}$.

Furthermore, we can, as in the proof of Proposition $[32]$, without loss of generality express the morphism $f : M \rightarrow P$ as the direct limit a $\lambda$-continuous directed system $U_\lambda = (u_j : L_j \rightarrow S_j \mid j \in J)$ of morphisms with $\lim U_\lambda = f$, $L_j \in \mathcal{L}_\lambda$, $S_j \in \mathcal{S}_\lambda$, and such that the induced directed system $(\text{Coker}(u_j) \mid j \in J)$ of modules is identical with $\mathcal{L}_\lambda$. If $U \in R_1\text{-Mod}$ corresponds to the representation as in $[1]$, we have in particular expressed $U$ as a $\lambda$-continuous direct limit of left $R_1$-submodules, which are all contained in $\mathcal{RE}$.

If $\kappa = \lambda$ is regular, we obtain a filtration with $< \kappa$-presented consecutive factors, which we refine by applying the inductive hypothesis to these factors, to get the desired $\mathfrak{F}$. Note that the consecutive factors in $\mathfrak{F}$ are also contained in $\mathcal{RE}$ by Lemma $[34]$.

Finally, if $\kappa$ is singular, we apply the classical singular compactness theorem to $U \in R_1\text{-Mod}$, cf. $[27]$ Theorem 7.29, to get $\mathfrak{F}$. \hfill \Box
Note that, over a right coherent ring $R$, the condition (3) in the statement of Proposition 3.7 is equivalent to the injectivity of the map $f$.

Now we discuss some basic closure properties of the classes of ordinary and of projectively coresolved Gorenstein flat modules.

**Lemma 3.8.** Let $B \in \mathcal{PGF}$ and $C \in \mathcal{GF}$. Then $A \in \mathcal{GF}$ provided that it fits into one of the following two short exact sequences:

1. $0 \to C \to A \to B \to 0$.
2. $0 \to A \to B \to C \to 0$.

Moreover, if $C \in \mathcal{PGF}$, then $A$ is a PGF-module as well. Finally, $\mathcal{PGF}$ is closed under transfinite extensions.

**Proof.** The proof mimics the one of [30 Theorem 2.5]. We give just a brief sketch. Let us denote by $i$ the monomorphism $C \to A$ and by $f$ the epimorphism $A \to B$ in the short exact sequence from (1). There are short exact sequences $0 \to B \to P \to B' \to 0$ and $0 \to C \to F \to C' \to 0$ with $P$ projective, $F$ flat, $B' \in \mathcal{PGF}$ and $C'$ Gorenstein flat. We define a morphism $m_1 : A \to F$ as a factorization of $g$ through $i$; this is possible, since $FL \subseteq \mathcal{PGF}^\perp$ by Theorem 3.3. Further, we let $m_2 = hf$ and let $m : A \to F \oplus P$ be determined by $m_1$ and $m_2$.

Then $m$ is a monomorphism, $\text{Coker}(m)$ is an extension of $C'$ by $B'$, and we can repeat the process to obtain an acyclic complex of flat modules where all syzygies, including $A$, belong to $\text{Ker Tor}_n^R(-, I)$ for all injective left $R$-modules $I$ and $n > 0$ (see also [11 Lemma 2.4(2)]).

To prove the alternative (2), we take the pushout of $h$ and the epimorphism $B \to C$ from (2). In the resulting short exact sequence $0 \to A \to P \to H \to 0$, we see, by (1), that $H \in \mathcal{GF}$ since it is an extension of $C$ by $B'$. By [11 Lemma 2.4(3)], it follows that $A$ is Gorenstein flat.

The proof of the moreover clause is analogous. In this case, we have $F$ projective and $C' \in \mathcal{PGF}$.

Finally, the construction used in the proof of the alternative (1) for $C \in \mathcal{PGF}$ can be iterated to show that a transfinite extension of PGF-modules is again a syzygy in an acyclic complex of projective modules. The rest follows from Eklof lemma. □

Recall that a class $W \subseteq \text{Mod-}R$ is called *thick* provided that it is closed under direct summands, extensions, and taking kernels of epimorphisms and cokernels of monomorphisms.

**Theorem 3.9.** $\mathcal{PGF} = (\mathcal{PGF}, \mathcal{PGF}^\perp)$ is a complete hereditary cotorsion pair with $\mathcal{PGF}^\perp$ thick. If $R$ is right $\aleph_0$-coherent, then $\mathcal{PGF}$ is of countable type and each module from $\mathcal{PGF}$ is filtered by countably presented PGF-modules.

Moreover, $R$ is right perfect if and only if $\mathcal{PGF}$ coincides with the class $\mathcal{GF}$ of all Gorenstein flat modules.

**Proof.** Let $(A,B)$ be the cotorsion pair generated by a representative set $A_0$ of $\nu$-presented modules from $\mathcal{PGF}$; here $\nu$ is as in Proposition 3.7(2). Then $A$ consists precisely of direct summands of $A_0$-filtered modules by [27 Corollary 6.14]. Using this, Proposition 3.7, the Eklof Lemma and Lemma 3.1, we see that $\mathcal{PGF} \subseteq A$. Since $R_\nu \in \mathcal{PF}$ and $\mathcal{PF}$ is closed under taking transfinite extensions by Lemma 3.3 and direct summands by [36 Proposition 1.4], we get $A = \mathcal{PGF}$. The completeness of $\mathcal{PGF}$ follows by [27 Corollary 6.14 and Theorem 6.11]. The thickness of $\mathcal{PGF}^\perp$ stems from the definition of a PGF-module. The statement about right $\aleph_0$-coherent rings follows from [27 Theorem 7.13].

The only-if part of the moreover clause is trivial. The other implication follows from the fact that $\mathcal{PGF} \cap \mathcal{PF}^\perp$ is precisely the class $P_0$ of all projective modules.
which stems from the thickness of $\mathcal{PGF}^\perp \supseteq \mathcal{P}_0$. Note that flat modules belong to $\mathcal{PGF}^\perp$ by Theorem 3.4. □

Example 3.10. Let $R$ be an Artin algebra. Then the cotorsion pair $\mathcal{PGF}$ is of countable type. Moreover, it is of finite type (i.e. generated by finite dimensional PGF-modules) if and only if $R$ is virtually Gorenstein, cf. [10, Theorem 5]. Of course, in this case $\mathcal{PGF} = \mathcal{GF} = \mathcal{GP}$.

Remark. Let $R$ be a right $\aleph_0$-coherent ring. It is well known (cf. [48, Lemma 3.4]) that each countably presented Gorenstein projective $R$-module is (projectively coresolved) Gorenstein flat. We have thus identified the class $\mathcal{PGF}$ as the subclass of Gorenstein projective modules filtered by countably presented Gorenstein projective modules.

The question whether Gorenstein projective modules are necessarily (projectively coresolved) Gorenstein flat remains open. However, we have manifested that, to answer this question in positive over $R$ right $\aleph_0$-coherent, it is necessary to show that each Gorenstein projective module is filtered by countably presented Gorenstein projective modules.

Our results suggest that the notion of a projectively coresolved Gorenstein flat module could serve as an alternative definition of a Gorenstein projective module over any ring. In fact, Theorem 3.9 can be viewed as a Gorenstein analogue of the Kaplansky theorem on the decomposition of projective modules.

Finally, if it happens that $\mathcal{GP} \subseteq \mathcal{GF}$, then necessarily $\mathcal{GP} = \mathcal{PGF}$. However, it is not clear whether $\mathcal{GF} \cap \mathcal{GP} \subsetneq \mathcal{PGF}^\perp$ since we do not know if $FL \cap \mathcal{GP} = \mathcal{P}_0$ holds true, cf. [7, Question 2.8].

The rest of this section is devoted to clarifying the relation between $\mathcal{PGF}$-modules and Gorenstein flat modules. We obtain a description of the class $\mathcal{GF}$ which implies that it is always closed under extensions, regardless of the ring $R$ (hence every ring is GF-closed in the terminology of [11]).

Theorem 3.11. Let $M$ be a module. Then the following conditions are equivalent:

1. $M$ is Gorenstein flat.
2. There is a short exact sequence
   \[ 0 \to K \to L \to M \to 0 \]
   with $K \in FL$ and $L \in \mathcal{PGF}$ which remains exact after applying the functor $\text{Hom}_R(-, C)$ for any (flat) cotorsion module $C$.
3. $\text{Ext}_R^1(M, C) = 0$ for all cotorsion modules $C \in \mathcal{PGF}^\perp$.
4. There is a short exact sequence
   \[ 0 \to M \to F \to N \to 0 \]
   with $F$ flat and $N \in \mathcal{PGF}$.

In particular, we get $\mathcal{GF} \cap \mathcal{PGF}^\perp = FL$.

Proof. (1) ⇒ (2). The module $M$ is a syzygy in an exact complex $F^\bullet$ consisting of flat modules which remains exact after applying the functor $- \otimes_R I$ for any $I \in R\text{-Mod}$ injective. Consider the complete cotorsion pair $(C(\mathcal{P}_0), C(\mathcal{P}_0)^\perp)$ in $\text{Ch}(R)$, where $C(\mathcal{P}_0)$ denotes the class of all complexes of projective modules, cf. [13, Theorem 4.5]. Let $0 \to G^\bullet \to P^\bullet \to F^\bullet \to 0$ be a special $C(\mathcal{P}_0)$-precover of $F^\bullet$, i.e. $P^\bullet \in C(\mathcal{P}_0)$ and $G^\bullet \in C(\mathcal{P}_0)^\perp$. It follows that the complex $G^\bullet$ is exact, whence $P^\bullet$ is exact as well. For each $i \in \mathbb{Z}$, we obtain the following commutative diagram of modules with exact rows and columns.
whence the cotorsion envelope where modules and $G$ is a pure monomorphism and $F$ is injective (see also [41, Theorem 8.6]). As $F^*$ and $G^*$ remain exact after applying $- \otimes_R I$ for an injective $I \in R$-Mod, the same holds for $F^*$, which yields $L^i \in \mathcal{P}GF$. Finally, $f^i$ is $C$-injective for any $C$ cotorsion since $g^i$ and $h^i$ are (we use that $K^{i+1}$ and $F^i$ are flat).

$(2) \implies (3)$. Let $0 \to K \xrightarrow{f} L \to M \to 0$ be a short exact sequence with $K$ flat and $L \in \mathcal{P}GF$ such that $f$ is $\mathcal{E}C \cap \mathcal{FL}$-injective. The result will follow, once we show that any morphism $g: K \to C$ with $C$ cotorsion (from $\mathcal{PF}^*$) factorizes through $f$. Let us form a flat cover $\pi: F \to C$. Then there exists a map $h: K \to F$ such that $\pi h = g$ since $K$ is flat and Ker($\pi$) cotorsion. The flat module $F$ is also cotorsion, whence we can factorize $h$ through $f$. The composition of the resulting map with $\pi$ is the desired factorization of $g$.

$(3) \implies (4)$. First, we show that $M$ is a pure-epimorphic image of a PGF-module. We start with a special $\mathcal{P}GF$-precover $\pi: P \to M$ of the module $M$. Put $K = \text{Ker}(\pi)$ and let us consider the following pushout diagram

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & \to & M^i & \to & F^i & \to & M^{i+1} & \to & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & L^i & \to & P^i & \to & L^{i+1} & \to & 0 \\
& & f^i & & h^i & & \uparrow & & \uparrow \\
0 & \to & K^i & \xrightarrow{g^i} & G^i & \to & K^{i+1} & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & & & 0 & & 0 & & 0
\end{array}
\]

where $h^i$ is pure since $F^i$ is flat. It follows that $G^*$ consists of flat modules and as such it is a direct limit of complexes from $C(P_0)$; see [41 Lemma 8.4]. On the other hand $G^* \in C(P_0)^\perp$, and so each morphism from a finitely presented complex to $G^*$ is null-homotopic. In particular, $G^*$ is pure exact and $K^i$ is flat (see also [41 Theorem 8.6]). As $F^*$ and $G^*$ remain exact after applying $- \otimes_R I$ for an injective $I \in R$-Mod, the same holds for $F^*$, which yields $L^i \in \mathcal{P}GF$. Finally, $f^i$ is $C$-injective for any $C$ cotorsion since $g^i$ and $h^i$ are (we use that $K^{i+1}$ and $F^i$ are flat).

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\[
\begin{array}{ccccccccc}
0 & 0 & G \\
\uparrow & & \uparrow \\
G & \xrightarrow{G} & G \\
\uparrow & & \uparrow \\
0 & \to & CE(K) & \to & H & \to & M & \to & 0 \\
\varepsilon & & \uparrow & & \uparrow \\
0 & \to & K & \xrightarrow{\varepsilon} & P & \xrightarrow{\pi} & M & \to & 0
\end{array}
\]

where $CE(K)$ denotes the cotorsion envelope of $K$. Since $\mathcal{P}GF^\perp$ contains all flat modules and $G$ is flat, we have $CE(K) \in \mathcal{P}GF^\perp$. By (3), the middle row splits whence the cotorsion envelope $K \hookrightarrow CE(K)$ factorizes through $\varepsilon$. In particular, $\varepsilon$ is a pure monomorphism and $\pi$ is a pure epimorphism.

Now, we form a special $\mathcal{P}GF^\perp$-preenvelope of $M$: $0 \to M \to F \to N \to 0$. Then $F \in \mathcal{P}GF^\perp$ and, at the same time, $\text{Ext}_R(F, C) = 0$ for any cotorsion module
C ∈ PGF⊥. By the preceding paragraph, F is a pure-epimorphic image of a PGF-module. In particular, any special PGF-precover \( \rho : Q \to F \) is a pure epimorphism. Since \( Q \in PGF^\perp \cap PGF \), it is projective, and so \( F \) is flat.

The implication (4) \( \implies \) (1) follows at once from [11, Lemma 2.4].

Finally, the inclusion \( GF \cap PGF^\perp \supseteq FL \) is a consequence of Theorem 3.4, and the equivalent condition (4) yields \( GF \cap PGF^\perp \subsetneq FL \).

The following corollary provides a generalization of [26, Section 3].

**Corollary 3.12.** There is a hereditary cotorsion pair \( \mathcal{S} \mathcal{F} = (GF, EC \cap PGF^\perp) \) generated by a set of modules of cardinality at most \( |R| + \aleph_0 \). The kernel of \( GF \) equals \( FL \cap EC \). The class \( GF \) is closed under direct limits, and so it is a covering class (in the sense of [27, Definition 5.5]).

**Proof.** Let us denote \( B = EC \cap PGF^\perp \). We have \( GF = \perp B \) by Theorem 3.11(3). On the other hand, \( B = GF^\perp \) since \( PGF \cup FL \subseteq GF \). So \( GF \) is a cotorsion pair. In fact, it is the supremum of \( (FL, EC) \) and \( PGF^\perp \) in the big lattice of cotorsion pairs, using the convention from [27, Chapter 12]. As such, it is generated by the union of two sets generating these cotorsion pairs, which means by a representative set of Gorenstein flat modules of cardinality at most \( \nu = |R| + \aleph_0 \). The description of the kernel \( GF \cap GF^\perp \) of \( GF \) stems from Theorem 3.11(4).

In particular, we have shown that the class \( GF \) is closed under extensions, and so it is closed under direct limits as well by [49, Lemma 3.1]. Finally, \( GF \) is covering by the well-known result due to Enochs, see e.g. [27, Corollary 5.32].

Now we discuss the interpretation of our results from the perspective of stable model structures and the corresponding homotopy categories. To that end, recall that a triple \( H = (Q, W, R) \) of classes of modules is called a Hovey triple if \( (Q \cap W, R) \) and \( (Q, W \cap R) \) are complete cotorsion pairs and the class \( W \) is thick. Such triples were introduced in [37] and their basic properties are summarized for instance in [6, §1]. In particular, if the two cotorsion pairs associated with a Hovey triple are hereditary, it defines a stable model category [38], whose homotopy category is simply the stable category \( T \) of the Frobenius exact category \( Q \cap R \) modulo the class \( Q \cap W \cap R \) of its projective-injective objects. It is well-known that \( T \) is a triangulated category and it encodes the corresponding relative homological algebra on \( \text{Mod-}R \).

The results of this section provide us, over any ring, with two previously unknown Hovey triples: \( (PGF, PGF^\perp, \text{Mod-}R) \) and \( (GF, PGF^\perp, EC) \). Since the middle classes of the triples coincide, the two stable model structures on \( \text{Mod-}R \) are Quillen equivalent in the sense of [38]. In particular, their homotopy categories are equivalent triangulated categories.

As mentioned at the beginning of the section, the class \( PGF \) contains the class of Gorenstein AC-projective modules from [14] and, if \( R \) is left coherent, the two classes coincide. From the point of model structures, the Gorenstein AC-projective model structure from [14, Theorem 8.5] is a Bousfield localization of the one given by \( (PGF, PGF^\perp, \text{Mod-}R) \) (cf. [6, §§1.4 and 1.5]), and the localization is trivial if \( R \) is left coherent. If \( R \) is not left coherent, our model structures are potentially finer (if the Bousfield localization is proper). Unfortunately, we are presently not aware of any particular example of a non-coherent ring where the two model structures differ.

We also do not know whether \( GF \) is always closed under taking pure-epimorphic images, equivalently, whether \( GF \) is precisely the class of pure-epimorphic images
of PGF-modules. Neither we know whether the implication \((2) \implies (3)\) in the following result can be reversed, see also [17, Problem 4.12].

**Proposition 3.13.** Consider the following conditions:

1. \(GF\) is a definable class of modules.
2. \(GF\) is closed under products.
3. \(R\) is left coherent.
4. \(GF\) is closed under pure-epimorphic images and pure submodules.

Then \((1) \iff (2) \implies (3) \implies (4)\).

**Proof.** \((1) \implies (2)\) is trivial. The converse will follow once we prove \((2) \implies (4)\).

\((2) \implies (3)\). Let \(F\) be a flat module and \(\kappa\) a cardinal. Then \(F^\kappa \in GF\) by \((2)\). Since \(F^\kappa\) belongs also to \(PGF^\perp\), it is flat by Theorem 3.11(4). It follows that \(R\) is left coherent.

The implication \((3) \implies (4)\) is well-known, [36, Theorems 3.6 and 2.6]. \(\square\)

**Example 3.14.** [20, Section 4, Ex.(1)] The ring

\[
R = \begin{pmatrix} Q & 0 & 0 \\ Q & Q & 0 \\ R & R & Q \end{pmatrix} / \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & 0 & 0 \end{pmatrix}
\]

is (left and) right perfect and not left coherent. Moreover, the class \(GF\) coincides with the class of all projective modules. It follows that the implication \((4) \implies (3)\) in Proposition 3.13 does not hold for this \(R\).

4. **Gorenstein injective modules**

In the whole section, \(I_0\) denotes the class of all injective modules. Recall that a complex \(I^•\) of injective modules is **totally acyclic** if it is exact and it remains exact after the application of \(\text{Hom}_R(E, -)\) with \(E\) arbitrary injective. We write \(C\text{tac}(I_0)\) for the class of all totally acyclic complexes of injective modules. A module \(M\) is called **Gorenstein injective** if it is a syzygy in a totally acyclic complex of injective modules. We denote by \(GI\) the class of all Gorenstein injective modules. In this section, we show that, over any ring, the class \(GI\) forms the right-hand class of a perfect hereditary cotorsion pair \(G\mathcal{E} = (W, GI)\) with \(W\) thick.

We start with general observations.

**Lemma 4.1.** Let \(C\) be a class of modules which is thick and closed under filtrations (i.e. \(M \in C\) whenever \(M\) has a filtration with consecutive factors in \(C\)). Then \(C\) is closed under direct limits.

**Proof.** The proof of [25, Proposition 3.1] applies. Although there \(C\) is assumed to be a thick left-hand side of a cotorsion pair, the proof only uses the fact that \(C\) is closed under transfinite extensions by the Eklof Lemma [27, Lemma 6.2]. \(\square\)

**Lemma 4.2.** Let \(C\) be a class of modules such that \(A = \perp C\) is thick, and let \(\kappa \geq |R|\) be an infinite cardinal. Assume that \(M\) is an almost \((C, \kappa^+)\)-projective module.

Then \(A\) is closed under direct limits and \(M\) possesses a filtration \(M = (M_\alpha | \alpha \leq \sigma)\) where \(M_{\alpha+1}/M_\alpha\) is a \(\kappa\)-presented module from \(A\) for each \(\alpha < \sigma\).

**Proof.** First, note that \(A\) is closed under direct limits by Lemma 4.1 in particular, \(M \in A\). Let \(S\) be a system witnessing that \(M\) is almost \((C, \kappa^+)\)-projective. Since \(\kappa \geq |R| + \aleph_0\), we can w.l.o.g. assume that \(S\) consists of submodules of \(M\) and inclusions (use Construction A.2). Suppose that the module \(M\) is \(\mu\)-presented. We proceed by transfinite induction on \(\mu\). If \(\mu \leq \kappa\), there is nothing to prove, so assume \(\mu > \kappa\).
Let \( \mu \) be a regular cardinal. Using the notation from Definition 1.6, we see that the system \( S^\mu \) consists of \( \mu \)-presented submodules of \( M \) which belong to \( A \). Since \( M \in A \) and \( A \) is thick, we can choose from \( S^\mu \) a filtration \( \mathcal{T} = (N_\alpha \mid \alpha < \mu) \) of \( M \) with consecutive factors \( \mu \)-presented and belonging to \( A \).

By the definition of \( S^\mu \) and the thickness of \( A \), \( \mathcal{T} \) consists of \( (C_\kappa, \kappa^+)\)-projective modules (use Observation A.3). Thus we can use Construction A.1 for each \( \alpha < \mu \) to obtain a system \( K_\alpha \) witnessing that \( N_{\alpha+1}/N_\alpha \) is almost \( (C_\kappa, \kappa^+)\)-projective. By the inductive hypothesis, \( N_{\alpha+1}/N_\alpha \) possesses a filtration with consecutive factors \( \kappa \)-presented modules from \( A \) for each \( \alpha < \mu \). We use these filtrations to refine \( \mathcal{T} \) into the desired filtration \( \mathcal{M} \).

If \( \mu \) is singular, we see that \( M \) is almost \( (C, \lambda)\)-projective for all regular \( \kappa < \lambda < \mu \) using that \( A \) is closed under direct limits. The inductive hypothesis and [27, Theorem 7.29] (singular compactness) give us the filtration \( \mathcal{M} \).

**Lemma 4.3.** Put \( \lambda = |R| + \aleph_0 \) and let \( \kappa \) be an infinite cardinal such that \( \kappa = \kappa^\lambda \). Assume that \( A \in \mathcal{A} = \perp C \) is a pure-injective module. Then \( A \) is almost \( (C, \kappa^+)\)-projective provided that \( A \) is closed under direct limits (of monomorphisms).

**Proof.** We use [27, Lemma 10.5] to obtain, for any \( X \subseteq A \) of cardinality \( \leq \kappa \), a \( \kappa \)-presented submodule \( C \) of \( A \) such that \( X \subseteq C \) with the property that each system of cardinality \( \leq \lambda \) consisting of \( R \)-linear equations with parameters from \( C \) has a solution in \( C \) provided that it has a solution in \( A \). In particular, \( C \) is pure in \( A \), and it follows from [19, Theorem V.1.2] that \( C \) is pure-injective (here, we use that \( A \) is pure-injective). Thus \( C \) is a direct summand of \( A \) which yields \( C \in A \).

By the preceding paragraph, we know that \( A \) is the directed union of a system consisting of \( \kappa \)-presented direct summands of \( A \). If \( A \) is closed under direct unions, we can enlarge this system to a one witnessing that \( A \) is almost \( (C, \kappa^+)\)-projective. \( \square \)

**Lemma 4.4.** The class \( \perp GI \) is thick. Subsequently, for any infinite cardinal \( \kappa \) such that \( \kappa^{\lambda} = \kappa \), each pure-injective module in \( \perp GI \) possesses a filtration with consecutive factors \( \kappa \)-presented modules from \( \perp GI \). In particular, this is the case of any injective module in \( \text{Mod-}R \).

**Proof.** The first, well known, part is straightforward from the definition of a Gorenstein injective module. The second part follows from the two preceding lemmas. \( \square \)

Now we can prove that the class of totally acyclic complexes of injective modules forms the right-hand side of a cotorsion pair generated by a set in the category \( \text{Ch}(R) \) of complexes of right \( R \)-modules.

**Proposition 4.5.** Let \( \lambda = |R| + \aleph_0 \) and \( \kappa \) be the least infinite cardinal such that \( \kappa^\lambda = \kappa \). There is a cotorsion pair \( \mathcal{B} = (\mathcal{B}, C_{\text{vac}}(\mathcal{I}_0)) \) in \( \text{Ch}(R) \) generated by a set of \( \kappa \)-presented complexes, hence complete. Moreover, each \( B^\bullet \in \mathcal{B} \) has a filtration with consecutive factors \( \kappa \)-presented complexes from \( \mathcal{B} \).

**Proof.** The class \( C_{\text{vac}}(\mathcal{I}_0) \) of all exact complexes of injective modules forms the right-hand class of a cotorsion pair generated by a set \( S \) consisting of \( \kappa \)-presented (in fact, even \( \lambda \)-presented) complexes by [21, Proposition 4.6]. Denote by \( T' \) a representative set of all \( \kappa \)-presented modules from \( \perp GI \). Set \( T = \{S_n(M) \mid M \in T', n \in \mathbb{Z}\} \) where \( S_n(M) \) denotes the stalk complex concentrated in degree \( n \). We claim that \( (S \cup T)^\perp = C_{\text{vac}}(\mathcal{I}_0) \).

Indeed, an exact complex \( I^\bullet \) of injective modules is totally acyclic if and only if each cocycle module \( Z^n(I^\bullet), n \in \mathbb{Z} \), is Gorenstein injective, or equivalently in \( \mathcal{I}_0^\perp \). Since \( \text{Ext}_{\text{Ch}(R)}(S_n(M), I^\bullet) \cong \text{Ext}_{\mathcal{B}}(M, Z^n(I^\bullet)) \), we get the desired conclusion by the Eklof Lemma and Lemma 4.4. The completeness of the cotorsion pair follows,
we obtain a long exact sequenceGI

It is easy to show that the cotorsion pair \( \mathcal{B} \) is hereditary and its kernel \( \mathcal{B} \cap C_{\text{tac}}(\mathcal{I}_0) \) coincides with the class of all (categorically) injective complexes. As a consequence, it follows that \( \mathcal{B} \) is also a thick class. By \( [23] \) Theorem 1.2, we can obtain a Hovey triple \((\mathcal{B}, \mathcal{Y}, \mathcal{G})\) in \( \text{Ch}(R) \), where \( \mathcal{G} \) denotes the class of all (categorically) Gorenstein injective complexes, using the following theorem for \( \text{Ch}(R) \) instead of \( \text{Mod}-R \). Recall that a cotorsion pair \( \mathcal{E} = (\mathcal{A}, \mathcal{B}) \) is perfect if \( \mathcal{A} \) is a covering class and \( \mathcal{B} \) is enveloping (see \( [27] \) Definition 5.26 for details).

**Theorem 4.6.** Let \( \mathcal{W} = \perp \mathcal{G} \mathcal{I} \). The pair \( \mathcal{E} = (\mathcal{W}, \mathcal{G} \mathcal{I}) \) is a hereditary, perfect cotorsion pair generated by a set of \( \kappa \)-presented modules where \( \kappa \) is the least infinite cardinal such that \( \kappa^{|R| + \aleph_0} = \kappa \). In particular, every module has a \( \mathcal{G} \mathcal{I} \)-envelope.

**Proof.** We are going to give two proofs. The first one using Proposition \( [17] \) and the second one entirely in the category \( \text{Mod}-R \).

For the first, set \( \mathcal{E} = (\mathcal{W}, \mathcal{W}^{\perp}) \) and let \( M \in \mathcal{W} \) be arbitrary. Then \( S_0(M) \in \mathcal{B} = \perp C_{\text{tac}}(\mathcal{I}_0) \). Thus \( S_0(M) \) possesses a filtration with consecutive factors \( \kappa \)-presented complexes in \( \mathcal{B} \) by Proposition \( [17] \). However, all these complexes are concentrated in degree 0, hence they induce a filtration of \( M \) with consecutive factors \( \kappa \)-presented modules in \( \mathcal{W} \). Since \( M \) was arbitrary, it follows from the Eklof Lemma that \( \mathcal{E} \) is generated by a representative set of \( \kappa \)-presented modules from \( \mathcal{W} \), hence \( \mathcal{E} \) is complete.

Since \( \mathcal{W} \) is thick by Lemma \( [17] \), the cotorsion pair \( \mathcal{E} \) is hereditary. Moreover, the kernel of \( \mathcal{E} \) equals \( \mathcal{I}_0 \) indeed, let \( M \in \mathcal{W}^{\perp} \cap \mathcal{W} \), then the injective envelope \( E(M) \) of \( M \) belongs to \( \mathcal{W} \), and so \( E(M)/M \in \mathcal{W} \) too by the thickness of \( \mathcal{W} \), which implies that \( M \) splits in \( E(M) \).

We want to show that \( \mathcal{E} = \mathcal{E} \mathcal{J} \). The only thing we need to check is that \( \mathcal{W}^{\perp} \subseteq \mathcal{G} \mathcal{I} \). So suppose \( G \in \mathcal{W}^{\perp} \) is arbitrary. Iteratively forming special \( \mathcal{W} \)-precovers of \( G \), we obtain a long exact sequence

\[
\cdots \rightarrow I^n \rightarrow I^{n+1} \rightarrow \cdots \rightarrow I^{-1} \rightarrow G \rightarrow 0
\]

where \( I^n \) is injective for each \( n < 0 \) and all syzygies belong to \( \mathcal{W}^{\perp} \). At the same time, all cosyzygies in any injective coresolution of \( G \) belong to \( \mathcal{W}^{\perp} \) since \( \mathcal{E} \) is hereditary. We conclude that \( G \) is a syzygy in a totally acyclic complex of injective modules, or in other words \( G \in \mathcal{G} \mathcal{I} \).

Finally, the cotorsion pair \( \mathcal{E} \mathcal{J} \) is perfect by \( [27] \) Corollary 5.32 since \( \mathcal{W} \) is closed under direct limits by Lemma \( [14] \). In particular, every module has a \( \mathcal{G} \mathcal{I} \)-envelope.

Alternatively, we can argue as follows. Let \( \mathcal{S} \) be a representative class of all \( \kappa \)-presented modules from \( \perp \mathcal{G} \mathcal{I} \) and \( \mathcal{E} = (\mathcal{X}, \mathcal{S}^{\perp}) \) be the cotorsion pair generated by \( \mathcal{S} \). By Eklof Lemma and Lemma \( [14] \), all injective modules belong to \( \mathcal{X} \). Our goal is to show that \( \mathcal{X} \) is thick.

Since \( \kappa \geq |R| \) and \( \perp \mathcal{G} \mathcal{I} \) is closed under kernels of epimorphisms, each module from \( \mathcal{S} \) has a syzygy in \( \mathcal{S} \) whence \( \mathcal{E} \) is hereditary. From \( [27] \) Theorem 7.13, we know that each module in \( \mathcal{X} \) possesses a filtration with consecutive factors (isomorphic to elements) in \( \mathcal{S} \). Using Hill Lemma \( [27] \) Theorem 7.10 (H4)], we further get, for each \( M \in \mathcal{X} \), a system \( \mathcal{M} \) of \( \kappa \)-presented submodules of \( M \) witnessing that \( M \) is almost \((\mathcal{G} \mathcal{I}, \kappa^{+})\)-projective. In fact, since \( M \in \perp \mathcal{G} \mathcal{I} \) and \( \perp \mathcal{G} \mathcal{I} \) is thick, \( \mathcal{M} \) witnesses even \((\mathcal{G} \mathcal{I}, \kappa^{+})\)-projectivity of \( M \). Having two modules \( M_1, M_2 \in \mathcal{X} \) with \( M_1 \subseteq M_2 \) and their respective systems \( \mathcal{M}_1, \mathcal{M}_2 \), we use Construction \( [14] \) to show that \( M_2/M_1 \) is almost \((\mathcal{G} \mathcal{I}, \kappa^{+})\)-projective. Consequently, \( M_2/M_1 \) belongs to \( \mathcal{X} \) by Lemma \( [12] \) and Eklof Lemma.

for instance, by \( [33] \) Proposition 2.12]. The final claim then by \( [27] \) Theorem 7.13 which holds in any finitely accessible Grothendieck category. □
We have proved that $X$ is thick and contains all injective modules. As before, we observe that $X \cap S^\perp = \mathcal{I}_0$ and finally show that $\mathcal{C} = \mathfrak{G}$ by the same argument as in the first proof.

As an immediate consequence, we obtain the Hovey triple $(\text{Mod}-R, \mathcal{W}, \mathcal{G}^{\mathcal{I}})$ in $\text{Mod}-R$. The class of Gorenstein injectives contains the class of Gorenstein AC-injectives, as defined in [14, Theorem 5.5]. Hence, similarly to the previous section, our Hovey triple refines the Gorenstein AC-injective model structure from [14 Theorem 5.5], in that the model structure from [14] is a Bousfield localization of ours.

Unfortunately, similarly as in the previous section, we do not know whether $\mathcal{W}$ is, in general, closed under pure submodules or, equivalently, pure-epimorphic images.

5. Singular step — set theoretical part

In the rest of the paper, we denote by $0$ the empty set and by $\nu$ a fixed infinite cardinal. The qualification countable is intended as having cardinality $< \aleph_1$.

Let $(I, \leq)$ be a directed (i.e. upward directed) poset with $\kappa = |I|$ singular and $\nu < \kappa$. Set $\mu = \text{cf}(\kappa)$. For each successor cardinal $\lambda$ such that $\nu < \lambda < \kappa$, we fix a set $I_\lambda \subseteq [I]^{<\lambda}$ and put $I = \bigcup \lambda I_\lambda$. We assume that for each successor cardinal $\lambda$, where $\nu < \lambda < \kappa$, the set $I_\lambda$ has the following properties:

(0) $0 \in I_\lambda$;
(1) $I_\lambda$ consists of directed subsystems of $(I, \leq)$;
(2) for each $A \in [I]^{<\lambda}$ there exists $B \in I_\lambda$ such that $A \subseteq B$;
(3) if $C \subseteq I_\lambda$ is a $\leq$-chain with $|C| < \lambda$, then $\bigcup C \in I_\lambda$.

Let us denote by $\mathcal{W}$ the set of all pairs $(A, B)$ of directed subsystems of $(I, \leq)$ such that $A \subseteq B \in [I]^{<\kappa}$. For elements $(A, B), (C, D) \in \mathcal{W}$, we shall write $(A, B) \subseteq (C, D)$ if and only if $A \subseteq C \& B \subseteq D$. This makes $\mathcal{W}$ into a poset.

Put $\mathcal{W}_0 = \{(0, A) \mid (0, A) \in \mathcal{W}\}$. In particular $(0, 0) \in \mathcal{W}_0$. For $V = (A, B) \in \mathcal{W}$, we define $|V|$ as $|B|$. For an element $V \in \mathcal{W}$ and a cardinal $\lambda$, we use the notation $[V]^{<\lambda}$ to denote the set of all $W \in \mathcal{W}$ with $W \subseteq V$ and $|W| < \lambda$. Finally, for $\mathcal{S} \subseteq \mathcal{W}$, the union $\bigcup \mathcal{S}$ is computed component-wise, i.e. if $\mathcal{S} = \{(A_j, B_j) \mid j \in J\}$, then $\bigcup \mathcal{S} = \bigcup_{j \in J} A_j \cup \bigcup_{j \in J} B_j$.

In Section 4 we are going to apply these tools in the following context: The poset $I$ will index a directed system $\mathcal{S}$ of (countably presented) modules whose direct limit is a $\kappa$-presented module of our interest. The elements in $\mathcal{W}_0$ will correspond to direct limits of ‘small’ directed subsystems of $\mathcal{S}$, while the elements from $\mathcal{W}$ will correspond to cokernels of canonical morphisms between these. Finally, the sets $\{(0, A) \mid A \in I_\lambda\}$ will encode systems witnessing almost $(\mathcal{D}, \lambda)$-projectivity (for a particular class $\mathcal{D}$), and a binary relation $\preceq$ satisfying the axioms below will be used to capture the $\mathcal{D}$-injectivity of canonical morphisms between the modules corresponding to elements from $\mathcal{W}$.

**Definition 5.1.** Consider a binary relation $\preceq$ on $\mathcal{W}$ satisfying the following axioms.

1. $\preceq$ is a partial order of $\mathcal{W}$ with the smallest element $(0, 0)$.
2. $V \preceq W \Rightarrow V \subseteq W$.
3. $(V \subseteq W \subseteq X \& V \subseteq X) \Rightarrow V \subseteq W$.
4. For every $V = (A, B) \in I^2$ with $(0, A) \preceq (0, B)$ and any successor cardinal $\lambda$ such that $\nu < \lambda \leq |B|$, there is a system $V(\lambda) \subseteq [V]^{<\lambda}$ such that:
   a. $(\forall (C, D) \in V(\lambda))(C, D) \preceq V, (0, C) \preceq (0, A), (0, D) \preceq (0, B)$;
   b. $\bigcup V(\lambda) = V$;
   c. $V(\lambda)$ is upwards directed;
   d. if $C \subseteq V(\lambda)$ is a chain and $|C| < \lambda$, then $\bigcup C \in V(\lambda)$.
Proof. Put $\lambda = |X|$. For $N \in W_0 \cap I_{\kappa^+}^2$, we define the $N$-Shelah game. It is played in turns by two players. Player I starts and chooses successively elements $X_0, X_1, \ldots$ from $W_0$ of cardinality at most $\lambda$. Player II, on each $X_n$, replies with some $N_n \in W_0 \cap I_{\kappa^+}^2$. At most $\omega$ turns are played; after the first $n + 1$ turns, we will have the following sequence:

$$X_0, N_0, X_1, N_1, \ldots, X_n, N_n.$$ 

Player II wins, if he manages to play, for each $n \in \omega$, so that $X_n \subseteq N_n$ and $N_{n-1} \subseteq N_n$ where we put $N_{-1} = N$. Otherwise, Player I immediately wins. Let $S$ denote the set of all $N \in W_0 \cap I_{\kappa^+}^2$ for which Player I possesses no winning strategy in $N$-Shelah game. We show that $(0,0) \in S$.

First, for each $K \in W_0 \cap I_{\kappa^+}^2$, we fix an $\subseteq$-increasing chain $(K^\alpha \in W_0 \mid |K^\alpha| \leq \lambda, \alpha < \lambda^+)$ such that $\bigcup_{\alpha < \lambda^+} K^\alpha = K$. Let $s$ be a strategy for Player I in $(0,0)$-Shelah game, i.e. a function that gives the first move $X_0$, and it decides what the answer should be to the play by Player II; so $X_n = s(N_0, N_1, \ldots, N_{n-1})$ for $n > 0$. We want to beat the strategy $s$. Using the properties of $I$ and $\subseteq$, we inductively construct increasing sequences $(M_n \in W_0 \cap I_{\kappa^+}^2 \mid \alpha < \lambda^+)$ and $(K^\alpha \in W_0 \cap I_{\kappa^+}^2 \mid \alpha < \lambda^+)$ in such a way that:

1. $X_0 \subseteq M_0$;
2. $M_0 \subseteq K_0$, for each $\alpha < \lambda^+$ limit;
3. $M_{\alpha+1} \supseteq M_\alpha \cup \bigcup_{\beta < \alpha} K^\beta$ for each $\alpha < \lambda^+$;
4. For each $\alpha < \lambda^+$, $s(M_\alpha, M_{\alpha+1}, \ldots, M_\alpha) \subseteq M_{\alpha+1}$, whenever $n \in \omega, \alpha_n \leq \alpha$ and $M_{\alpha_n} \leq M_{\alpha_1} \leq \cdots \leq M_{\alpha_n}$ is played by Player II according to the rules (against the strategy $s$).

Put $M = \bigcup_{\alpha < \lambda^+} M_\alpha$. We have $|M| = \lambda^+$ and $M = \bigcup_{\alpha < \lambda^+} K_\alpha \in I^2$ by (3). Considering the system $M(\lambda^+)$ given by Axiom (4) from Definition (B) it is easy to see that the set $\{\beta < \lambda^+ \mid M_\beta \in M(\lambda^+)\}$ is unbounded in $\lambda^+$. Player II is going to beat the strategy $s$, if he chooses the elements $N_n$ as the appropriate $M_\beta$ for $\beta$ from this unbounded set.

Finally, it is enough to notice that, for $X_0 = X$, it is possible to play in the $(0,0)$-Shelah game such $N_0 = N$ that $N \in S$. If not, Player I would have possessed a winning strategy in the $(0,0)$-Shelah game, a contradiction. This $N$ is the one we were looking for.

□

The main result of this section follows. Its proof employs a nontrivial enhancement of techniques coming from [39] Theorem IV.3.3]. Note that the slightly unusual (re)definition of sets $B^\alpha_n$ condenses a back-and-forth construction whose explication would just make the proof look even more technical.
Theorem 5.3. There is an increasing continuous $\succeq$-chain $\{ (0, C_\alpha) \in \mathcal{I}^2 \mid \alpha < \mu \}$ with $\bigcup_{\alpha < \mu} C_\alpha = I$.

Proof. Fix a strictly increasing continuous chain $(\nu_\alpha \mid \alpha < \mu)$ of infinite cardinals cofinal in $\kappa$ such that $\nu_0 > \mu + \nu$. For $n \in \omega$, we recursively define strictly increasing chains $(V_\alpha^n \subseteq W_0 \mid \alpha < \mu)$, where $V_\alpha^n = (0, A_{\alpha, n})$, together with (arbitrarily fixed) enumerations $A_{\alpha, n} = \{ a_{\alpha, n, \beta} \mid \beta < \nu_\alpha \}$ as follows.

First, we observe that $V_\alpha^0 = W_0$ with $\bigcup_{\nu_\alpha \leq \alpha < \mu} V_\alpha^0 \subseteq V_\alpha^0$ and such that for any $Y \subseteq W_0$ of cardinality $\nu_\alpha$ with $V_\alpha^0 \subseteq Y$, we have $V_\alpha^0 \subseteq Y$. This is possible by Lemma 5.2. Moreover, we can assume that $\bigcup_{\nu_\alpha \leq \alpha < \mu} A_{\alpha, 0} = I$.

For $n = 1$, choose $V_\alpha^1$ from $(0, V_\alpha^0)$ arbitrarily so that $V_\alpha^0 \subseteq V_\alpha^1$. Additionally, put $B_{\alpha, 1} = \{ B \mid (0, B) \in V_\alpha^0 \cup (\nu_\alpha^+, A_{\alpha, 0} \subseteq B) \}$.

For $n > 0$ even, we choose $V_\alpha^n$ again from the set $\mathcal{I}^2_{\nu_\alpha^n}$, with $V_\alpha^n \supseteq \bigcup_{\nu_\alpha \leq \alpha < \mu} V_\alpha^n$ and such that for any $Y \subseteq W_0$ of cardinality $\nu_\alpha$ with $V_\alpha^n \subseteq Y$, we have $V_\alpha^n \subseteq Y$. Furthermore, using the assumption $\nu_0 > \mu$, we can demand that

$$A_{\alpha, n} \supseteq \{ a_{\alpha, n, \beta} \mid \gamma < \mu, \beta \leq \min\{ \nu_\alpha, \nu_\beta \} \}.$$  

Let $n > 1$ odd. By the construction, we have $A_{\alpha, n+1} \subseteq A_{\alpha, n} \subseteq \mathcal{I} \land V_{\alpha, n} \subseteq V_{\alpha, n+1}$. We also assume that the sets $B_{\alpha, n}$, $i < n$ odd, constructed in the previous odd steps are upwards directed, closed under unions of chains of cardinality $\leq \nu_\alpha$, and $\bigcup_{i < n} B_{\alpha, i} = A_{\alpha, n+1}^{-1}$.

Using Axiom (4), we can pick arbitrary $(A_{\alpha, n-2}, A_{\alpha, n}) \subseteq (A_{\alpha, n-1}, A_{\alpha, n+1}) (\alpha_1^+) \subseteq A_{\alpha, n}$. By the definition of $A_{\alpha, n}$ (see below), we have an induced chain $(0, A_{\alpha, n}) \subseteq (0, A_{\alpha, n+1}) \subseteq \cdots \subseteq (0, A_{\alpha, n-2})$ satisfying, for each odd $i < n$, $A_{\alpha, n} \subseteq B_{\alpha, n} \subseteq (A_{\alpha, n}, A_{\alpha, n+1}) (\alpha_1^+)$, where we put $A_{\alpha, n} = A_{\alpha, n}$. Set $V_\alpha^n = (0, A_{\alpha, n})$.

For each $i < n$ odd, we gradually replace the sets $B_{\alpha, n}$ by their subsets $B \in B_{\alpha, n} \subseteq A_{\alpha, n} \subseteq B \subseteq A_{\alpha, n+1}$. (Redefining $B_{\alpha, n}$ alters the set $B_{\alpha, n}$ which we replace by the subset as above, hereby we modify $B_{\alpha, n}$, and so on.) This does not harm the properties of $B_{\alpha, n}$ mentioned above. At the same time, it guarantees that we will get $(0, A_{\alpha, n}) \subseteq (0, A_{\alpha, n+2})$ for each $i \in \{ 1, 3, 5, \ldots, n \}$ in the next odd step if we define

$$B_{\alpha, n} = \{ B \mid (\exists A \in B_{\alpha, n}^{-2} (A, B) \in (A_{\alpha, n+1}, A_{\alpha, n+1}) (\alpha_1^+) \subseteq A_{\alpha, n+1} \}.$$  

Notice that $B_{\alpha, n}$ has the properties required in the next odd step too.

We claim that $S = (\bigcup_{\alpha < \mu} V_\alpha^n \mid \alpha < \mu)$ is the closed $\succeq$-chain we have been looking for. First of all, we know that $\bigcup_{\alpha < \mu} V_\alpha^n \subseteq \mathcal{I}^2_{\nu_\alpha^n}$ for all $\alpha < \mu$, using the even steps. Further, it immediately follows from the property $(\dagger)$ that $S$ is continuous. It remains to prove that $S$ is a $\succeq$-chain.

Now fix $\alpha < \mu$. For each $k \in \omega$, put $B_k = A_{\alpha, k}^2$, $A_k = \bigcup_{k \leq j < \omega} A_{\alpha, j+1}^{2k+1, 2j+1}$, $W_k = (0, B_k)$ and $V_k = (0, A_k)$. By the construction, we have the chain $V_0 \subseteq V_1 \subseteq \cdots$; indeed, for each $k \in \omega$, $V_k \subseteq W_k$ by Axiom (4), further $W_k \subseteq W_{k+1}$ by the even-step incorporation of Lemma 5.2 finally $V_k \subseteq V_{k+1}$ by Axiom (3) since $V_k \subseteq W_{k+1} \subseteq W_{k+1}$. Moreover, we have $(A_k, A_{k+1}) = (B_k, B_{k+1}) (\nu_\alpha^+)$ for each $k \leq \omega$, and $\bigcup_{k \leq \omega} A_k = \bigcup_{k \leq \omega} A_{\alpha, k}^2$.

We want to use Axiom (6) for the setting $(A_k, B_k)$. First notice that, for each $k < \omega$, the set $B_k$ belongs to $\mathcal{I}$ and we have $(0, A_k) \subseteq (0, B_k) \subseteq (A_k, A_{k+1}) \subseteq (B_k, B_{k+1}) (\nu_\alpha^+)$, and $(0, B_k) \subseteq (0, B_\omega)$ by the choice of $(0, B_0) = V_{\alpha, 0}^{2k}$. Thus $(0, A_k) \subseteq (0, B_\omega)$ for each $k < \omega$.

The relation $(A_k, B_k) \subseteq (A_{k+1}, B_{k+1})$ now follows immediately from Axiom (5) since $(A_k, A_{k+1}) \subseteq (B_k, B_{k+1})$ holds. Hence the hypotheses of Axiom (6) are satisfied, and we can conclude that $\bigcup_{\alpha < \mu} V_\alpha^n \subseteq \bigcup_{\alpha < \mu} V_{\alpha, n+1}$ for each $\alpha < \mu$. □
Remark. Analyzing the proof in detail, we see that Axiom (4) is needed only for pairs \((\lambda, |B|^+)\) of the form \((\nu^+, \nu^{++})\) or \((\nu^+, \nu_{\alpha+1})\) for \(\alpha < \mu\).

We finish this section with a lite version of the theorem above.

**Proposition 5.4.** Assume that, for all \(A \in \mathcal{I}\), we have \((0, A) \not\leq (0, B)\) whenever \((A, B) \in \mathcal{W}\). Then we obtain the same conclusion as in Theorem 5.3 without using Axioms (5) and (6), and with Axiom (4) used only for the case \(A = 0\).

**Proof.** Follow the proof of Theorem 5.3. In odd steps, instead of the subtle technical dance, do the same as in the even ones. \(\square\)

6. **Singular step — modules**

**Lemma 6.1.** Let \(R\) be a ring with enough idempotents. Let \(\kappa\) be a singular cardinal, \(M\) be a \(\kappa\)-presented module and \(\mathcal{D}\) a filter-closed class of modules. Assume that there is an infinite cardinal \(\nu\) such that, for all successor cardinals \(\nu < \lambda < \kappa\), there is a system \(S_{\lambda}\) witnessing that \(M\) is almost \((\mathcal{D}, \lambda)\)-projective. Then \(M \in \mathcal{D}\).

Furthermore, if \(S_{\lambda}\) witnesses even the \((\mathcal{D}, \lambda)\)-projectivity of \(M\) for each successor cardinal \(\nu < \lambda < \kappa\), then the same conclusion holds regardless of whether \(\mathcal{D}\) is filter-closed or not.

**Proof.** We fix a directed system \(S = (M_i, f_{ji} : M_i \to M_j \mid i \leq j \in I)\) with \(\varinjlim S = M\), and such that \((I, \leq)\) is a directed poset of cardinality \(\kappa\) and \(M_i\) is countably presented for all \(i \in I\). Moreover, using Construction \(\text{A.2}\) for \(\gamma = 1\), we can w.l.o.g. assume that, for each \(\lambda\), the system \(S_{\lambda}\) consists of direct limits of some directed subsystems of \(S\) of cardinality \(< \lambda\) and canonical colimit factorization maps between them. We define \(I_{\lambda}\) as the set of all the underlying directed posets of these subsystems of \(S\). Set \(I = \bigcup_{\nu < \text{cf}(\lambda) = \lambda < \kappa} I_{\lambda}\).

Recalling the previous section, we consider the functor \(\Phi\) from the category \((\mathcal{W}, \subseteq)\) to Mod-\(R\) sending an element \((A, B)\) to the cokernel of the canonical colimit factorization map from \(\varinjlim_{\nu \in A} M_i\) to \(\varinjlim_{\nu \in B} M_i\), an inclusion \((0, A) \subseteq (0, B)\) to this colimit factorization map, and an inclusion \((A, B) \subseteq (C, D)\) to the map \(f\) uniquely determined by the following commutative diagram:

\[
\begin{array}{cccc}
\Phi(0, C) & \longrightarrow & \Phi(0, D) & \longrightarrow & \Phi(C, D) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow f & & \\
\Phi(0, A) & \longrightarrow & \Phi(0, B) & \longrightarrow & \Phi(A, B) & \longrightarrow & 0.
\end{array}
\]

For \(V, W \in \mathcal{W}, V \subseteq W\), we define the order relation \(\leq\) by setting \(V \leq W \iff \Phi(V) \to \Phi(W)\) is \(\mathcal{D}\)-injective. Notice that for \(A \in \mathcal{I}\), one gets \(\Phi(0, A) \in \mathcal{D}\). It follows from Lemma \(1.3\) that to show \(M \in \mathcal{D}\) it is sufficient to verify the axioms from Definition 5.1 and apply Theorem 5.3.

Axioms (1) to (3) easily hold. In the Axiom (4) for \(V = (A, B)\), the modules \(\Phi(0, A), \Phi(0, B), \Phi(A, B)\) belong to \(\mathcal{D}\).

Thus we can use \(\text{[16]}\) Lemma 2.3 to find upward directed sets \(A, B \subseteq [I]^\lambda\) closed under unions of chains of length \(< \lambda\) and consisting of directed subposets of \((I, \leq)\) such that \(\bigcup A = A, \bigcup B = B\) and, for each \(C \in A \cup B\), satisfying \((0, C) \leq (0, B)\).

We use \(\text{[16]}\) Lemma 2.3 again, now for the module \(\Phi(A, B)\), to find an upward directed set \(V(\lambda) \subseteq W \cap (A \times B)\) which satisfies the hypotheses of Axiom (4).

Axiom (5) is just the \(3 \times 3\) lemma applied, for each \(D \in \mathcal{D}\), on the \(\text{Hom}_R(-, D)\)-image of the following commutative diagram with exact rows and columns.
For the Axiom (6), let $g \in \text{Hom}_R(\Phi(0, A_\nu), D)$ be arbitrary with $D \in \mathcal{D}$. Notice that $\Phi(A_k, B_k) \in \mathcal{D}$ for each $k < \omega$ since $\Phi(0, B_k) \in \mathcal{D}$ and $(0, A_k) \leq (0, B_k)$.

Using Lemma 1.4, it follows that $\Phi(A_\omega, B_\omega)$ belongs to $\mathcal{D}$, so it remains to check that $\ker(\Phi(0, A_\omega) \to \Phi(0, B_\omega)) \subseteq \ker(g)$. Assume that $x \in \Phi(0, A_\omega)$ is arbitrary such that $g(x) \neq 0$. There exists $j < \omega$ such that a preimage $y$ of $x$ can be found in $\Phi(0, A_j)$. Since $(0, A_j) \leq (0, B_\omega)$, we infer that $y \notin \ker(\Phi(0, A_\omega) \to \Phi(0, B_\omega))$, whence $x \notin \ker(\Phi(0, A_\omega) \to \Phi(0, B_\omega))$.

For the ‘furthermore’ case, we can, using Construction A.2, w.l.o.g. assume that, for each successor cardinal $\nu < \eta < \lambda$. It follows that Axiom (4) from Definition 5.1 for $A = 0$ holds in this case, which allows us to apply Proposition 5.4.

\section*{Appendix A. Manipulating with directed systems}

In the whole paper, we often use the following two constructions of directed systems of modules (or morphisms).

\textbf{Construction A.1.} Let $f : M \to N$ be a homomorphism of modules, $\lambda$ a regular uncountable cardinal, $\mathcal{M} = (M_i, f_{ji} : M_i \to M_j) \mid i < j \in I$ and $\mathcal{N} = (N_i, g_{ji} : N_j \to N_i)\mid i < j \in J$ $\lambda$-continuous directed systems of $< \lambda$-presented modules such that $\lim \mathcal{M} = M$ and $\lim \mathcal{N} = N$. Then there is a $\lambda$-continuous directed system $\mathcal{U} = (u_k : M_i \to N_{ji}(f_{ji}, g_{ji}) \mid k < l \in K)$ consisting of morphisms with domains in $\mathcal{M}$ and codomains in $\mathcal{N}$ such that $\lim \mathcal{U} = f$.

Subsequently, there is a $\lambda$-continuous directed system $\mathcal{K} = (\text{Coker}(u_k) \mid k \in K)$ consisting of $< \lambda$-presented modules (with canonically defined maps).

\textbf{Proof.} For each $i \in I$ and $j \in J$, let us denote by $f_i : M_i \to M_j$ and $g_j : N_j \to N_i$ the colimit maps, and define $f_{ji} = \text{id}_M$, and $g_{ji} = \text{id}_N$.

We define $\mathcal{U}$ as the set of all morphisms $u : M_i \to N_j$ such that $i \in I, j \in J$, $g_j u = f f_i$. For $u : M_i \to N_j, v : M_j \to N_i$, we put $u v = u$ if and only if $i \leq r, j \leq s$ and $v f_{ri} = g_{sj} u$. We easily check that $(\mathcal{U}, \leq)$ is a poset. Next, we show that it is directed.

First, fix generating sets $G = \{x_{\alpha} \mid \alpha < \mu\}$ and $H = \{y_\alpha \mid \alpha < \mu\}$ of $M_i$ and $M_\alpha$, respectively, where $\mu < \lambda$, and let $u, v \in \mathcal{U}$ be as above. We find $a \in I$ such that $i, r < a$. Since $\mathcal{N}$ is $\lambda$-continuous and $M_\alpha$ is $< \lambda$-presented, there is $b \in J, s < b$, and a morphism $w_0 : M_\alpha \to N_b$ from $\mathcal{U}$. It need not be the case that $u \leq w_0$ and $v \leq w_0$, however, for each $\alpha < \mu$, there is a $b_\alpha \geq b$ such that $g_{\alpha, b_\alpha} u(x_{\alpha}) = g_{\alpha, b_\alpha} w_0 f a_i(x_{\alpha})$ and $g_{\alpha, b_\alpha} v(y_{\alpha}) = g_{\alpha, b_\alpha} w_0 f a_r(y_{\alpha})$. Since $\mu < \lambda$ and $(J, \leq)$ is $\lambda$-directed, there is $c \in J$ such that $c \geq b_\alpha$ for each $\alpha < \mu$. It follows that $w = g_{\alpha, b_\alpha} w_0$ is in $\mathcal{U}$ and $u, v \leq w$. Subsequently, $(\mathcal{U}, \leq)$ is a directed system of morphisms. Moreover, it is $\lambda$-continuous since $\mathcal{M}$ and $\mathcal{N}$ are such.
To prove that \( \lim \mathcal{U} = f \), it is now enough to find, for arbitrary \((i, j) \in I \times J\), a morphism \( u : M_i \to N_j \) in \( \mathcal{U} \) with \( s \geq j \). This is easy (recall how we found \( u_0 \)).

The next tool allows us to merge less than \( \lambda \) directed systems which are \( \lambda \)-continuous into one. In its statement, we do not use the notation from Definition 1.6.

**Construction A.2.** Let \( M \in \text{Mod-}R \), \( \lambda \) be a regular uncountable cardinal, \( \gamma < \lambda \) and, for each \( \alpha \leq \gamma \), let \( \mathcal{M}^\alpha = (M^\alpha_{ij}, f^\alpha_{ij} : M^\alpha_i \to M^\alpha_j \mid i < j \in I_\alpha) \) be a \( \lambda \)-continuous directed system consisting of \( \lambda \)-presented modules such that \( \lim \mathcal{M}^\alpha = M \). Then the systems \( \mathcal{M}^\alpha \), \( \alpha \leq \gamma \), have a common cofinal \( \lambda \)-continuous subsystem.

More precisely: for each \( \alpha \leq \gamma \), there exists a \( \lambda \)-continuous cofinal directed subsystem \( \mathcal{N}^\alpha = (M^\alpha_i, f^\alpha_{ij} : M^\alpha_i \to M^\alpha_j \mid i < j \in J_\alpha) \) of \( \mathcal{M}^\alpha \); furthermore, for any \( \alpha, \beta \leq \gamma \), there is a bijection \( i : J_\alpha \to J_\beta \) and a directed system \( \mathcal{U} \) with \( \lim \mathcal{U} = \text{id}_M \), whose objects are isomorphisms \( u_i : M^\alpha_i \to M^\beta_{i(i)} \), \( i \in J_\alpha \), and for each \( i < j \in J_\alpha \), there is only one morphism from \( u_i \) to \( u_j \) in \( \mathcal{U} \), namely \((f^\alpha_{ij}, f^\beta_{i(i), j(i)})\).

**Proof.** We can assume that \( \gamma \) is a cardinal. The proof goes by induction on \( \gamma \). For \( \gamma = 0 \), it is trivial. Let \( \gamma = 1 \).

We use Construction A.1 with \( f = \text{id}_M \), \( \mathcal{M} = \mathcal{M}^0 \) and \( \mathcal{N} = \mathcal{M}^1 \) to obtain the system \( \mathcal{U} \) of morphisms. Using [9] Lemma 2.6, we can w.l.o.g. assume that the objects of \( \mathcal{U} \) are isomorphisms. The subsystems \( \mathcal{N}^\alpha \), \( \alpha = 0, 1 \), then consist of domains, codomains, respectively, of the isomorphisms in \( \mathcal{U} \).

By induction, we have the proof for any \( \gamma \) finite. For \( \gamma \) infinite, we use the inductive hypothesis and the following simple fact: for each \( \alpha \leq \gamma \), if \((N^\alpha_i \mid \beta < \gamma)\) is a family of \( \lambda \)-continuous cofinal directed subsystems of \( \mathcal{M}^\alpha \) such that \( N^\beta_i \supseteq N^\delta_i \) whenever \( \beta \leq \delta < \gamma \), then \( \bigcap_{\beta < \gamma} N^\beta_i \) is a \( \lambda \)-continuous cofinal directed subsystem of \( \mathcal{M}^\alpha \) as well.

We also use freely the following easy

**Observation A.3.** Let \( \lambda \) be a regular uncountable cardinal and \( \mathcal{M} \) a \( \lambda \)-continuous directed system of modules. Let \( \mathcal{K} \) be a directed subsystem of \( \mathcal{M} \). Then there is a \( \lambda \)-continuous directed subsystem \( \mathcal{K}' \) of \( \mathcal{M} \) with the same direct limit as \( \mathcal{K} \).

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