I-Love-Q to the extreme

Hector O Silva© and Nicolás Yunes©

Department of Physics, eXtreme Gravity Institute, Montana State University, Bozeman, MT 59717, United States of America

E-mail: hector.okadadasilva@montana.edu

Received 6 October 2017, revised 5 November 2017
Accepted for publication 9 November 2017
Published 5 December 2017

Abstract

Certain bulk properties of neutron stars, in particular their moment of inertia, rotational quadrupole moment and tidal Love number, when properly normalized, are related to one another in a nearly equation of state independent way. The goal of this paper is to test these relations with extreme equations of state at supranuclear densities constrained to satisfy only a handful of generic, physically sensible conditions. By requiring that the equation of state be (i) barotropic and (ii) its associated speed of sound be real, we construct a piecewise function that matches a tabulated equation of state at low densities, while matching a stiff equation of state parametrized by its sound speed in the high-density region. We show that the I-Love-Q relations hold to 1 percent with this class of equations of state, even in the extreme case where the speed of sound becomes superluminal and independently of the transition density. We also find further support for the interpretation of the I-Love-Q relations as an emergent symmetry due to the nearly constant eccentricity of isodensity contours inside the star. These results reinforce the robustness of the I-Love-Q relations against our current incomplete picture of physics at supranuclear densities, while strengthening our confidence in the applicability of these relations in neutron star astrophysics.

Keywords: general relativity, neutron stars, equation of state

(Some figures may appear in colour only in the online journal)

1. Introduction

Neutron stars are ideal laboratories for probing fundamental physics. The energy density inside their inner core can be a few times above nuclear saturation density, where matter transmutes into forms that cannot be recreated in the laboratory. Their compactness creates large gravitational fields, which demand the use of relativistic gravity for their modeling. Their rotation frequency can rival that of the blades of the fastest professional kitchen blender, while
supporting the most extreme magnetic fields. These are just a few of the many striking features
that make neutron stars invaluable tools to study extreme physics.

An outstanding problem in neutron star astrophysics is the determination of the equation
of state (EoS) of cold-nuclear matter inside the core, which is a crucial ingredient in the
prediction of observables, such as the (gravitational) mass $M$ and radius $R$ [1–3]. While the
EoS in the crust region is fairly well-understood, there is a large degree of uncertainty at 1–2
times the nuclear saturation energy density $\varepsilon_n \simeq 150 \text{ MeV } \text{fm}^{-3} \simeq 2.67 \times 10^{15} \text{ g } \text{cm}^{-3}$, and
this uncertainty increases further at densities $\approx 10^{15} \text{ g } \text{cm}^{-3}$ common in the inner core. One
could imagine using precise measurements of the mass of neutron stars to observationally
determine the EoS. Unfortunately, competing EoS models predict neutron stars that fill a
large portion of the mass-radius ($M$-$R$) plane, introducing degeneracies when converting from
observed masses to constraints on the EoS.

But not all is lost on the observational front. Accurate measurements of massive $\approx 2M_\odot$
pulsars [4, 5] set a solid lower mass bound that viable EoSs must respect, thus ruling out a
number of candidate EoSs. Similarly, one can use the observed population of pulsars to place
a statistical upper limit on the maximum mass of neutron stars [6, 7], thus further restricting
the space of viable EoSs. Unfortunately, while the mass of neutron stars can sometimes be
accurately measured (see [7] for an overview), measuring the radius is currently much more
difficult [8]. Future simultaneous mass-radius measurements will hopefully tighten the $M$-$R$
relation, thus placing even stronger constraints on the EoS.

The uncertainties of the EoS also impact our ability to determine other global properties of
neutron stars, which are important in various astrophysical scenarios. The moment of inertia
determines the spin–orbit coupling correction to the rate of advance of the periastron in binary
pulsars [9, 10], an effect that may be measurable in the future with the double-pulsar system
PSR J0737-3039 [11–13]. The stellar ellipticity, the moment of inertia and the quadrupole
moment affect the modeling of x-rays emitted by hot spots on the surface of neutron stars,
as light is affected by the curvature of the exterior spacetime when it escapes [14–18]. The
quadrupole moment and the tidal Love number (associated with the tidal deformability of the
star) are also important in the modeling of gravitational waves emitted in the inspiral of binary
neutron stars, as these introduce finite-size corrections to models constructed in the point-
particle limit [19–27].

Uncertainties in the EoS can therefore introduce degeneracies that limit the amount of
information that can be extracted from observations, unless one finds approximately EoS-
independent relations between model parameters. Imagine for example that the $f$, $w$ and
$p$-mode frequencies $\omega_i$ of pulsating neutron stars are extracted from future gravitational wave
observations. Approximately EoS-independent relations between these frequencies and the
compactness $C \equiv M/R$ or ‘average density’ $M/R^3$, could then be used to extract the mass and
the radius of neutron stars from the observation of any two frequencies [28, 29]. In practice,
of course, the accuracy of this extraction is limited not only by the statistical uncertainties in
the measurement of the frequencies, but also potential systematic uncertainties in the degree
of EoS-independence of the relations themselves.

In this spirit, [30, 31] found that the moment of inertia $I$, the quadrupole moment $Q$ and
tidal Love number $\lambda$, when properly normalized, are linked together in a remarkable (nearly)
EoS-independent way (see [32, 33] for recent reviews). The analytic fits obtained by [30, 31],
connecting two of the quantities in the $I$-Love-$Q$ trio, hold to better than 1% for all EoS con-
sidered, covering a very broad spectrum of nuclear physics models. These universal relations
can be used in a number of physical scenarios. For instance, take the case of the PSR J0737-
3039 system. If a measurement of the moment of inertia of Pulsar A is made, knowledge of
its mass $M$ and spin, allows us to automatically infer its quadrupole moment $Q$. The $I$-Love-$Q$
relations can also be used to break degeneracies between various parameters in models of x-ray pulse profiles and gravitational waves. In this way, these relations reduce the size of the parameter space of the models and allow the remaining parameters to be more accurately determined [32].

Given the tremendous potential of the I-Love-Q relations, one is led to wonder whether they are valid even when only a minimal number of assumptions on the EoS and on how neutron stars are modeled is made; if so, we can then expect them to be valid for any physically reasonable EoS. This leads to the question of whether one can construct a reasonable set of minimal assumptions that leads to a family of physically reasonable EoSs. Studies along this line were pioneered in the mid-70s by Rhoades and Ruffini [34] (see also [35] and [36] for a review) with the goal of obtaining theoretical upper limits on the mass of neutron stars, important for connecting massive neutron stars to solar mass black holes and allowing us to distinguish between these compact objects in x-ray binary observations [37, 38]. This question was partially addressed by [39] who focused on the I-Love relation (and other nearly EoS-independent relations, such as that between the binding energy and the compactness and between the moment of inertia and compactness). The goal of this paper is to complete this picture by investigating the full set of I-Love-Q relations with a physically-reasonable family of EoSs constructed with minimal assumptions. As an intermediate step, we also obtain upper bounds on the moment of inertia and quadrupole moment valid for slowly-rotating neutron stars.

We find that the I-Love-Q relations continue to hold when using such a family of EoSs as well as when using tabulated (‘realist’) EoSs. We follow Rhoades and Ruffini [34] and model the EoS as a piecewise function that reduces to a given tabulated EoS at low densities and transitions to a stiff EoS with speed of sound \( v_s \in \mathbb{R}_{>0} \) at a transition density \( \bar{\epsilon} = \eta \rho_n \), with \( \eta \in \mathbb{R}_{>0} \). Our main findings are summarized in figure 1, which shows that the I-Love-Q relations are robust to changes to the EoS parameter \( \eta \) for fixed \( v_s = 1 \) (for a similar plot where we fix \( \eta = 1 \) and vary \( v_s \), see figure 6). Since these EoSs extremize the bulk stellar properties, our results strongly suggest the following:

| The I-Love-Q relations are valid to approximately 1% accuracy for any physically reasonable EoS provided that |
|---|
| (i) the stars do not rotate maximally, |
| (ii) magnetic fields are not extremely strong, |
| (iii) and general relativity is valid. |

Strong magnetic fields, of the size expected in magnetars, would contribute to the elliptical deformation of the neutron star, thus strongly modifying its quadrupole moment [40]. Extremely rapid rotation, of magnitude comparable to that of the fastest millisecond pulsars, will also deteriorate the relations [41, 42]. Although theories outside of general relativity have so far been shown to also contain I-Love-Q relations, albeit modified from the general relativity ones, we cannot discard the possibility a priori of exotic theories where these relations do not hold.

The remainder of this paper explains how these results are obtained in detail. In section 2, we introduce the EoS used in [34] and the physical requirements we require it satisfy. In section 3, we describe how we construct slowly-rotating and tidally deformed neutron stars. In section 4, we present some novel upper bounds on the reduced rotational quadrupole moment \( Q \) and the moment of inertia \( I \). Finally, in section 5, we confront the I-Love-Q relations against our numerical integrations and close in section 6 by summarizing our main findings and discussing some possible future research. All throughout, we work in geometric units where \( c = 1 = G \).

3
2. The equation of state

In order to obtain a theoretical upper bound on the maximum mass $M_{\text{max}}$ for neutron stars, Rhoades and Ruffini [34] assumed that the star consists of two layers:

(i) an outer layer whose contribution to the total (gravitational) mass $M$ is determined by some known EoS, and

(ii) an inner core whose contribution to the total mass must be extremized by performing a variational calculation.

The two layers are matched near the nuclear saturation density, above which the EoS is poorly understood. This calculation is made with the following additional assumptions:

(a) the structure of a neutron stars is determined by the general relativistic stellar structure equations, i.e. the Tolman–Oppenheimer–Volkoff (TOV) equations [43];

(b) the energy density is positive ($\varepsilon > 0$) and related to pressure by a barotropic (single parameter) EoS, i.e. $p = p(\varepsilon)$;

(c) $\frac{dp}{d\varepsilon} \geq 0$, such that the sound speed $v_s^2(\equiv \frac{dp}{d\varepsilon})$ is positive and matter is stable against microscopic collapse;

(d) $\nu_s \leq 1$, i.e. fluid perturbations are causal.

Rhoades and Ruffini [34] showed that the EoS for the inner core that maximizes the total mass of a static, spherically symmetric neutron star is that with $v_s = 1$, which can be written analytically as $p = p_0 + (\varepsilon - \varepsilon_0)$, where $p_0$ and $\varepsilon_0$ are the pressure and energy densities at
the transition between the two layers respectively. They found that \(M_{\text{max}}\) is around \(\approx 3.2 M_{\odot}\), although this bound does depend on the values of \(\varepsilon_0\) and the EoS used in the low-density region \([36, 37, 44]\).

Strictly speaking, the Rhoades–Ruffini approach is only valid for static stars. Numerical results supporting the idea that the EoS proposed in \([34]\) also yields the maximum mass of rapidly uniformly rotating neutron stars was reported in \([45]\). In the same vein as the variational calculations of \([34]\), but considering slowly-rotating neutron stars (in the sense made precise in section 3), Sabbadini and Hartle \([46]\) showed that the moment of inertia is maximized when the inner layer has constant central density \(\varepsilon_c \geq \varepsilon_0\). Note however, that an incompressible fluid has the unrealistic property of having an infinite speed of sound. Koranda et al \([47]\) (see also \([48]\) and references therein) showed that the maximum rotation frequency is obtained for an EoS, where \(v_i = 1\) in the inner core, while the outer layer is maximally soft, i.e. \(p = 0\) for \(\varepsilon \geq \varepsilon_0\). Assuming a known EoS in the outer layer has the effect of decreasing this frequency by a few percent. More recently, \([49, 50]\) provided numerical evidence that the tidal Love number is maximized by the EoS of Rhoades and Ruffini \([34]\). The astute reader will notice that a study on the maximum quadrupole moment that can be supported by a neutron star is currently missing.

To be conservative in our study of the extremal bulk properties of neutron stars, we here consider the following extreme EoS (xEoS) family

\[
p(\varepsilon) = \begin{cases} 
p_{\text{ms}}(\varepsilon), & \varepsilon < \varepsilon_0 \\
p_{\text{ms}}(\varepsilon_0) + v_s^2(\varepsilon - \varepsilon_0), & \varepsilon \geq \varepsilon_0.
\end{cases}
\]

where \(\varepsilon_0\) is a matching density that separates the neutron star interior into two regions: one where matter has a speed of sound \(v_i \in \mathbb{R}_{>0}\) (when the energy density is less than \(\varepsilon_0\)) and another one, \(p_{\text{ms}}(\varepsilon)\), where matter is described by the EoS MS \([51]\) (see also \([47]\)). We thus follow \([49]\) and assume a relatively stiff EoS for the known, outer region of the star. We choose the matching energy density \(\varepsilon_0\) to be a multiple of the nuclear saturation density (taken as \(\varepsilon_n = 2.7 \times 10^{14} \text{ g cm}^{-3}\)):

\[
\varepsilon_0 = \eta \varepsilon_n,
\]

where \(\eta \in \mathbb{R}_{>0}\). In the limit \(\eta \to \infty\), the xEoS reduces to \(p_{\text{ms}}(\varepsilon)\), while as \(\eta \to 0\), the EoS is given by \(p = p_{\text{ms}}(\varepsilon_0) + v_s^2(\varepsilon - \varepsilon_0)\). Our choice of EoS differs from that of \([49]\) in that we allow for \(v_i \neq 1\) so that we can explore possible further deviations from universality in the I-Love-Q relations.

The speed of sound controls the stiffness of the xEoS, and thus, its range warrants some further comments. On general grounds, an upper bound of \(v_s^2 = 1/3\) can be obtained for systems displaying conformal symmetry, which have zero trace of the energy-momentum tensor. Calculations of the speed of sound in strongly interacting relativistic systems (which are not conformal), have shown consistently that \(v_s^2 < 1/3\) (see \([52]\) for further details). Interestingly, Bedaque and Steiner \([52]\) found that the observations of \(\approx 2M_{\odot}\) neutron stars \([4, 5]\) are in tension with the upper bound \(v_s^2 = 1/3\), assuming an EoS of the form of equation (1) with \(v_s^2 = 1/3\) and with the region below twice nuclear saturation density described by a non-relativistic model for hadronic interactions subject to constrains coming from nuclear physics experiments. More recently, Alsing et al \([7]\) found strong statistical evidence that the maximum speed of sound has a lower bound of \(v_s \gtrsim 2/5\) by analyzing the neutron star mass distribution.

\[1\] There is some persistent confusion in the literature regarding the acronym used for this EOS. The MS EoS that supports neutron stars with maximum masses of \(\approx 2.7 M_{\odot}\) has been called both MS0 and MS1. To avoid confusing the reader further, we will refer to this EoS as MS only.
But what about the upper bound of $v_s$? Formally, $\sqrt{dp/d\varepsilon}$ is the phase velocity of sound waves in the neutron star fluid. In non-dispersive fluids, this number coincides with the group velocity, which is required to be $<1$ by causality. Neutron star interiors, however, are expected to be dispersive (see e.g. [45] and references therein), and one could then have a violation of $v_s < 1$. Nonetheless, van Oeveren and Friedman [49] (based on [53]) have shown recently that causality actually does imply $dp/d\varepsilon \leq 1$ for a two parameter EoS, $p = p(\varepsilon, s)$ where $s$ is the entropy per baryon for stable relativistic fluids. For more complicated multi-parameter EoSs (e.g. including different particle species), $dp/d\varepsilon \leq 1$ follows from local stability assuming $v_s < 1$. Given all of this, we will here mostly assume that $v_s \leq 1$, but we will consider violations of this conditions just to see if the I-Love-Q relations continue to hold even then.

To confirm that the xEoS maximizes the values of the bulk properties of neutron stars, we also consider a few other less stiff choices of outer EoSs (in comparison with the MS EoS) in the $\varepsilon < \varepsilon_0$ region. In particular, we will consider the following outer EoSs: MPA1 [54], BSk21 [55] and SLy4 [56], in decreasing order of stiffness. All of these support $2M_\odot$ neutron stars as shown in the left panel of figure 2.

3. Slowly-rotating and tidally deformed neutron stars

We construct families of slowly-rotating neutron stars solutions using the perturbative approach introduced by Hartle and Thorne [57, 58], using the xEoS described in section 2. In this approach, rotation is taken as a small perturbation upon a static spherically symmetric stellar background configuration—a solution of the TOV equations [43]. The perturbative parameter is $\xi \equiv \Omega/\Omega^*$, where $\Omega$ is the rotation frequency of the star and $\Omega^*$ is the (Newtonian) mass-shedding frequency $\Omega^* \equiv \sqrt{M^*/R^*_c}$, where in turn $M$ and $R_c$ are the mass and areal radius of the non-rotating model. In realistic astrophysical scenarios, the slow-rotation approximation is sufficiently accurate. Even for the fastest spinning neutron star observed, PSR J1748-2446ad [59], for which $\Omega \simeq 4500$ Hz, then $\xi \simeq 0.1$ [60] if we assume it has a mass and radius of $M = 1.4M_\odot$ and $R = 10$ km.

From these families of slowly-rotating neutron stars, we extract their moment of inertia $I$, the (rotational) quadrupole moment $Q$ and the Love number. To do so, we begin by integrating the Hartle–Thorne system of equations at zeroth, first and second perturbative order in $\xi$, in the form used in [61] after correcting some misprints pointed out in [62]. To test the accuracy of our code, we compare our results to the tables in [62] finding excellent agreement, and we test our implementation of the xEoS by reproducing the results of Kalogera and Baym [37] in the non-rotating limit. From these numerical solutions, we can then easily extract the moment of inertia and the quadrupole moment as described, e.g. in [31].

When present in a binary, a neutron star is tidally deformed by the gravitational field of its companion. This deformation is predominantly encoded in the $l = 2$ (electric-type) tidal Love number $\lambda$ [21, 63–66]. We calculate this quantity using the formalism of the tidal deformations of static neutron star (solutions of the TOV equations), as presented in [67]. We validate our numerical calculations through comparisons with the results of [31].

Figure 2 shows the mass-radius ($M$-$R$) curves for a family of neutron stars parametrized by central density along any curve, using a variety of EoSs. The left panel shows the $M$-$R$ relation using a few different realistic EoSs that span the entire stellar interior. The right panel shows the same relation but using xEoS for fixed $v_s = 1$ but different choices of the transition density, as parameterized by $\eta$. Whenever the central energy density is such that $\varepsilon_c > \varepsilon_0$ (for a given value of $\eta$) the star has a core region described by a maximally stiff fluid. As a result, neutron
stars then have masses and radii that are substantially different from that of a star with same central energy density but described entirely by the realistic EoS.

4. Upper bounds on the moment of inertia and on the quadrupole moment

Before studying the I-Love-Q relations, let us discuss upper bounds on the moment of inertia and the quadrupole moment. Although similar bounds for the moment of inertia had already been studied for an incompressible fluid core in [46, 68, 69], ours is, to our knowledge, the first study of its kind that refers to the quadrupole moment. Throughout this section, we assume that $v_s = 1$ and only study the sensitivity of the upper bounds on the transition density $\varepsilon_0$.

In what follows, we will study the behavior of the dimensionless $\bar{I}$, $\bar{\lambda}$, and $\bar{Q}$, so let us discuss here how these quantities are defined. The dimensionless moment of inertia $I \equiv I/M^3_*$, the Love number $\bar{\lambda} \equiv \lambda/M^5_*$, and the quadrupole moment $Q \equiv -Q/(M^2_\chi)$, where $\chi \equiv S/M^2_*$, $S$ is the spin angular momentum and $Q$ is the quadrupole moment. As defined here then, $I$ and $Q$ are independent of the rotation frequency of the neutron star in the slow-rotation approximation [30, 31].

We start by verifying that the xEoS does indeed maximize $I$, $Q$ and $\bar{\lambda}$. Indeed, figure 3 shows that the values of each of these quantities is larger than those obtained when using the realistic EoS in our catalog. In each panel, we fix $\eta = 1$ in the xEoS. If we were to increase $\eta$, the curve would move downwards until, unsurprisingly, it would overlap the MS EoS curve. On the other hand, if we were to decrease $\eta$, the curve would move upwards, because then it would begin to resemble an isothermal fluid sphere (i.e. a polytope in the $n \to \infty$ limit), where the universality deteriorates. This behavior can be observed in figure 4, where we construct
Figure 3. The dimensionless quantities $\bar{I}$, $\bar{Q}$ and $\bar{\lambda}$ as a function of the gravitational mass $M_\ast$. For a given mass $M_\ast$, the xEoS gives an upper bound for each of these quantities.

Figure 4. The dimensionless quantities $I$, $Q$ and $\bar{\lambda}$ (solid, dashed and dotted lines respectively) as a function of $\eta$. We consider a family of solutions with fixed gravitational mass $M_\ast = 1.4M_\odot$. In the limit $\eta \to \infty$ the curves converge to the result of using only EoS MS.
families of constant $M_*=1.4\, M_\odot$ stars using the xEoS for different values of $\eta$. In all cases, $I$, $Q$ and $\lambda$ increase as $\eta \to 0$, while they converge to the constant values ($I \simeq 17.4$, $Q \simeq 8.10$ and $\lambda \simeq 1.37 \times 10^3$) as $\eta \to \infty$.

Besides examining the dimensionless $I$, $Q$ and $\lambda$, it is also interesting to consider the regular, dimensionful, moment of inertia and quadrupole moment. Figure 5 (upper panels) shows the maximum moment of inertia $I_{\text{max,45}} \equiv I_{\text{max}}/(10^{45} \, \text{g cm}^2)$ as a function of $\eta$. As in the case of realistic EoSs, we find that the maximum moment of inertia occurs for a neutron star with mass $M \lesssim M_{\text{max}} [60]$. This seems to be in contradiction with the behavior of $I = I(M_*)$ shown in figure 3; in reality it is not because $I$ is normalized by $M_\odot^{-3}$, which explains the increase of $I$ at lower masses. For large $\eta$, $I_{\text{max,45}}$ converges to $\simeq 4.9$, the maximum moment of inertia we would find using the MS EoS. In the other limit, $\eta \to 0$, the only scaling parameter is the transition density $\varepsilon_0$ which has units of length$^{-2}$. Since the moment of inertia has units of length$^3$, we find it scales as $\varepsilon_0^{-3/2}$. More precisely,

![Figure 5](image-url)
I_{\text{max},45} \simeq 14.7 \times \left(\frac{\varepsilon_0}{\varepsilon_n}\right)^{-3/2}, \quad \eta \lesssim 1.25 \quad (3)

which is shown by the solid line in the top-left panel of figure 5. This same reasoning has been used to explain the low \( \eta \) behavior of the maximum mass, radius and tidal Love number in [47, 49].

Let us now consider the relation between \( I_{\text{max}} \) and the maximum mass \( M_{\text{max}} \) and its radius \( R_{\text{max}} \). For realistic EoSs, these quantities have been numerically found to be related by

\[
I_{\text{max},45} \simeq k \Xi_{\text{max}}, \quad \Xi_{\text{max}} \equiv \left(\frac{M_{\text{max}}}{M_{\odot}}\right) \left(\frac{R_{\text{max}}}{10 \text{ km}}\right)^2,
\]

(4)

where \( k \approx 0.97 \) [70]. Using the xEoS, we find a similar relationship, but with \( k = 1.14 \), as shown in the top-right panel of figure 5.

The maximum quadrupole moment obeys relations similar to those for the moment of inertia, although a note of warning is necessary. The quadrupole moment \( Q \), unlike the moment of inertia \( I \), scales with the expansion parameter \( \xi \) as \( Q = \xi^2 Q^* \), where \( Q^* \) is the quadrupole moment of a star rotating at the mass-shedding frequency \( \Omega^* \). Since by definition \( 0 \leq \xi \leq 1 \), the most conservative upper bound on the quadrupole moment would seem to be \( Q < Q^* \), but this is incorrect because the \( Q = \xi^2 Q^* \) scaling is only valid in the slow-rotation approximation. Comparing the calculation of \( Q \) using the Hartle–Thorne formalism to full numerical integrations using the RNS code [72], Berti et al [62] found that the slow-rotation and exact \( Q \) disagree by 10% for stellar models with \( \xi \approx 0.2 \) when \( M = 1.4 M_{\odot} \) and by 20% for the maximum mass configuration (for a given EoS). These errors grow with increasing \( \xi \), with the Hartle–Thorne calculation systematically overestimating the values of \( Q \).

Akin to the case of \( I_{\text{max}} \), we find that the maximum quadrupole moment \( [Q_{\text{max},45}^* \equiv Q_{\text{max}}^* / (10^{45} \text{ g cm}^2)] \) satisfies

\[
Q_{\text{max},45}^* \equiv \xi^2 Q_{\text{max},45}^* \simeq 0.31 \xi^2 \times \left(\frac{\varepsilon_0}{\varepsilon_n}\right)^{-3/2}, \quad \eta \lesssim 1.25 \quad (5)
\]

and

\[
Q_{\text{max},45}^* = \xi^2 Q_{\text{max},45}^* \simeq 3.98 \xi^2 \Xi_{\text{max}}, \quad (6)
\]

as shown in the bottom panels of figure 5. We expect these relations to hold for neutron stars spinning up to \( \xi \lesssim 0.1 \), based on [62]. To extend the applicability of equations (5) and (6) to \( \xi \gtrsim 0.1 \), a study considering rapidly rotating neutron stars would be required [73], but this is outside the scope of this paper.

5. Validity of the I-Love-Q relations for the matched equation of state

5.1. Universal relations and the matched equation of state

Let us now test the I-Love-Q relations using the xEoS of equation (1). In previous work, Yagi and Yunes [30, 31] showed that the I-Love-Q relations can be fitted to the function

2 An improved fit can be obtained by also adding the compactness \( C_{\text{max}} \equiv (M_{\text{max}} / M_{\odot}) / (R_{\text{max}} / \text{km}) \) in the fitting function (see e.g. equation (11) in [71]).
\[
\ln y_i = \sum_{k=0}^{4} c_k (\ln x_i)^k, 
\]

where \(y_i, x_i\) are a pair of variables from the \(I, Q, \lambda\) trio and \(c_k\) are numerical (fitting) constants (see table 1 in [32]). This fit was carried out using a very large sample of tabulated (realistic) EoSs, but of course nobody has yet considered the \(I\)-\(Love\)-\(Q\) relations for the xEoS of equation (1), as we do next in this section.

Figure 1 already showed that the \(I\)-\(Love\)-\(Q\) relations are satisfied for the xEoS of equation (1), using a variety of transition energy densities \(\varepsilon_0\) parameterized by \(\eta\) with a fixed sound speed \(v_s = 1\). Section 3 taught us that the lower the value of \(\eta\), the larger the values of \(\bar{I}, \bar{Q}\) and \(\bar{\lambda}\) (see figure 4). As shown in the bottom panels of figure 1, the non-universality of the \(I\)-\(Love\)-\(Q\) relations (as quantified by the relative fractional error from the fits) remain always below \(\pm 1\%\) for the range of \(\eta\) we considered.

Let us now repeat this study but holding the transition density fixed by setting \(\eta = 1\) and varying the sound speed \(v_s\). Figure 6 shows the \(I\)-\(Love\)-\(Q\) relations for \(v_s \in (1/3, 5/3)\), thus including examples of (admittedly unphysical) superluminal values. Surprisingly, even in this extremal (and unphysical) situation, the \(I\)-\(Love\)-\(Q\) relations stand firm and once again the relative fractional errors with respect to the fitting functions remain below \(\pm 1\%\).

In both figures 1 and 6, we focused on neutron stars with \(M \gtrsim 1M_\odot\), which falls in the range of the lowest neutron star masses observed [1], with the lowest mass precisely measured so far having \(M = 1.174 \pm 0.004M_\odot\) [74]. For very low masses \(M \approx 0.2M_\odot\), it is known that the EoS-universality breaks down [31] (see also [75]). However, the existence of neutron stars with such low-masses is at odds with supernova studies which estimate a minimum mass in the 1.15–1.20\(M_\odot\) range (see section 3.2 of [1]). Low-mass neutron stars could be formed in speculative scenarios involving the fragmentation of rapidly rotating proto-neutron stars [76]. It is however sensible to assume \(M \gtrsim 1M_\odot\) as a lower bound, above which we have shown that the \(I\)-\(Love\)-\(Q\) relations hold.
We recall that the xEoS is: (i) devised by assuming only a small number of physically sensible requirements on the properties of matter at supranuclear densities and on how stars are constructed and (ii) as a consequence, it extremizes\(^3\) the values of \(\bar{I}, \bar{Q}\) and \(\bar{\lambda}\). The results summarized in figures 1 and 6, thus provide strong support for the validity of the \(I\)-\(Q\) relations for any sensible EoS that might be used to model neutron stars in general relativity and for any possible value attainable by these quantities.

5.2. Why \(I\)-\(Love\)-\(Q\) so much?

We have seen that the \(I\)-\(Love\)-\(Q\) relations are robust under rather extreme variations of the parameters in the xEoS, but why is this so? While the explanation for the origin of the \(I\)-\(Love\)-\(Q\) remains elusive, Yagi \textit{et al} [77] (see also [78]) suggested that the (nearly) EoS-independence of these relations could be a result of an emergent symmetry due to an approximate self-similarity of the ellipsoidal isodensity contours within the star. This symmetry emerges when we flow from Newtonian, low-compactness (\(M_*/R_* \ll 1\)) stars towards relativistic, high-compactness (\(M_*/R_* \simeq 0.1\)) neutron stars. In the former case, numerical integration has revealed a breakdown of both the self-similarity of isodensity contours and of the \(I\)-\(Q\) relations [77]. On the other hand, for neutron stars, [77] found that in the region \(r \in (0.50, 0.95)R_*\) (which contributes the most to the calculation of the moment of inertia and the quadrupole moment) the eccentricity \(e\) of isodensity contours is approximately constant, changing at most by \(\simeq 10\%\). For comparison, this change can exceed \(\simeq 100\%\) in Newtonian configurations.

\(^3\)More precisely, it results in the largest values of \(I, Q\) and \(\lambda\) for a neutron star with fixed mass \(M_*\) in comparison to the other EoSs in our catalog (see figure 3).
Let us then study whether this self-similarity of the ellipticity of isodensity layers continues to hold with the $x$EoS. Figure 7 shows the ellipticity profiles, calculated following [58], for two neutron star models: one with $I = 7$ ($M \simeq 2.4M_\odot$ using the MS EoS) and another one with $I = 17$ ($M \simeq 1.4M_\odot$ using the MS EoS). Each panel shows the eccentricity profiles (normalized to the spin parameter $\chi$) for four pairs of values of $(\eta, v_s^2)$. In all cases, we see that $e/\chi$ changes very mildly in the region $r \in (0.50, 0.95)R_*$, just as [77] found in the case of realistic EoSs. In fact, in some cases, such as when $I = 17$ and $(\eta, v_s^2) = (3/4, 1)$, the eccentricity profile becomes almost flat, except near $r \gtrsim 0.95R_*$. These results strengthen the case that the approximately constant ellipticity of isodensity levels could be an explanation for the I-Love-Q relations.

6. Conclusions and outlook

The I-Love-Q relations, provide an unique way of breaking the EoS-degeneracy by establishing EoS-independent relations between the moment of inertia, the quadrupole moment and the tidal Love number. This feature, makes them powerful tools for breaking degeneracies in various astrophysical situations, as in pulse profile modeling of rotating neutron stars [79] with immediate impact to current (such as NICER [80, 81]) and future observatories (such as eXTP [82]). In the gravitational wave astronomy arena, the I-Love-Q relations can break degeneracies in the measurement of the tidal deformability in merger events involving neutron stars. Due to their potential usefulness, it is natural to ask whether the I-Love-Q relations remain valid when we demand only a limited, physically sound, number of requirement on the EoS above nuclear saturation density. We have shown that the I-Love-Q relations are surprisingly robust against an agnostic EoS model, putting them on firm ground for astrophysical applications, despite EoS uncertainties.

Our work can be extended in a number of directions. The most immediate direction would be to consider rapidly rotating neutron stars [41, 42] and to explore the validity of the three-hair relations [83, 84] using the $x$EoS. Alternatively, one could consider higher-order calculations within the Hartle–Thorne perturbative scheme. Rotating neutron star models at third-order [85] and fourth-order [84] in the slow-rotation expansion have indeed been obtained in the past.

Another interesting avenue for future research is to investigate the maximum value of the bulk properties of neutron stars in modified theories of gravity (see [86–88] for reviews). One of the major difficulties in using neutron stars as tests of general relativity is the degeneracy between our ignorance on the EoS and modifications to Einstein’s gravity [89, 90]. If we assume the same $x$EoS to obtain upper bounds on various bulk properties of neutron stars, either on a gravity-by-gravity theory case or in a parametrized way [89], we can potentially use future observations to signal the presence of new physics. Such upper bounds (notably on the maximum mass) were obtained in the past (see [36] for a review on early works in this direction). However, these works studied theories that are of little interest today (exceptions include Einstein-dilaton-Gauss–Bonnet gravity [91] and scalar-tensor gravity [92, 93]). It would be interesting to revisit this problem using more modern alternatives to general relativity.

Acknowledgments

HOS thanks Thomas Sotiriou and the University of Nottingham for the support and warm hospitality during the initial stages of this work. He also thanks Davide Gerosa, for useful
advice on matplotlib [94] used to generate the figures and Leandro A Oliveira for discussions on dispersive fluids. HOS and NY thank Kent Yagi for sharing some equations of state tables for this work, validating some of our numerical calculations, profitable discussions and comments on this paper. HOS and NY acknowledge financial support through NSF CAREER grant PHY-1250636 and NASA grants NNX16AB98G and 80NSSC17M0041.

ORCID iDs

Hector O Silva https://orcid.org/0000-0002-0066-9471
Nicols Yunes https://orcid.org/0000-0001-6147-1736

References

[1] Lattimer J M 2012 Ann. Rev. Nucl. Part. Sci. 62 485-515
[2] Miller M C and Lamb F K 2016 Eur. Phys. J. A 52 63
[3] Watts A L et al 2016 Rev. Mod. Phys. 88 021001
[4] Demorest P, Pennucci T, Ransom S, Roberts M and Hessels J 2010 Nature 467 1081–3
[5] Antoniadijs J et al 2013 Science 340 6131
[6] Antoniadijs J, Tauris T M, Özel F, Barr E, Champion D J and Freire P C C 2016 arXiv:1605.01665
[7] Alsing J, Silva H O and Berti E 2017 arXiv:1709.07889
[8] Özel F and Freire P 2016 Ann. Rev. Astron. Astrophys. 54 401
[9] Barker B M and O’Connell R F 1975 Phys. Rev. D 12 329–35
[10] Damour T and Schutz B F 1988 Nuovo Cimento B 101 127–76
[11] Burgay M et al 2003 Nature 426 531–3
[12] Lyne A G et al 2004 Science 303 1153–7
[13] Lattimer J M and Schutz B F 2005 Astrophys. J. 629 979–84
[14] Cadeau C, Leahy D A and Morsink S M 2005 Astrophys. J. 618 451–62
[15] Cadeau C, Morsink S M, Leahy D and Campbell S S 2007 Astrophys. J. 654 458–69
[16] Morsink S M, Leahy D A, Cadeau C and Braga J 2007 Astrophys. J. 663 1244–51
[17] Psaltis D, Özel F and Chakrabarty D 2014 Astrophys. J. 787 136
[18] Psaltis D and Özel F 2014 Astrophys. J. 792 87
[19] Mora T and Will C M 2004 Phys. Rev. D 69 104021
Mora T and Will C M 2005 Phys. Rev. D 71 129901 (erratum)
[20] Berti E, Iyer S and Will C M 2008 Phys. Rev. D 77 024019
[21] Flanagan E E and Hinderer T 2008 Phys. Rev. D 77 021502
[22] Read J S, Markakis C, Shibata M, Uryu K, Creighton J D E and Friedman J L 2009 Phys. Rev. D 79 124033
[23] Read J S, Baiotti L, Creighton J D E, Friedman J L, Giacomazzo B, Kyutoku K, Markakis C, Rezzolla L, Shibata M and Taniguchi K 2013 Phys. Rev. D 88 044042
[24] Lackey B D, Kyotoku K, Shibata M, Brady P R and Friedman J L 2014 Phys. Rev. D 89 043009
[25] Yagi K and Yunes N 2016 Class. Quantum Grav. 33 13LT01
[26] Dietrich T, Moldenhauer N, Johnson-McDaniel B, Kyutoku K, Markakis C, Brügmann B and Tichy W 2015 Phys. Rev. D 92 124007
[27] Del Pozzo W, Li T G F, Agathos M, Van Den Broeck C and Vitale S 2013 Phys. Rev. Lett. 111 071101
[28] Andersson N and Kokkotas K D 1996 Phys. Rev. Lett. 77 4134–7
[29] Andersson N and Kokkotas K D 1998 Mon. Not. R. Astron. Soc. 299 1059–68
[30] Yagi K and Yunes N 2013 Science 341 365–8
[31] Yagi K and Yunes N 2013 Phys. Rev. D 88 024009
[32] Yagi K and Yunes N 2017 Phys. Rep. 681 1–72
[33] Doneva D D and Pappas G 2017 arXiv:1709.08046
[34] Rhoades C E Jr and Ruffini R 1974 Phys. Rev. Lett. 32 324–7
[35] Brecher K and Caporaso G 1976 Nature 259 377
[36] Hartle J B 1978 Phys. Rep. 46 201–47
[37] Kalogera V and Baym G 1996 *Astrophys. J.* 470 L61–4
[38] Fryer C L and Kalogera V 2001 *Astrophys. J.* 554 548–60
[39] Steiner A W, Lattimer J M and Brown E F 2016 *Eur. Phys. J. A* 52 18
[40] Haskell B, Ciolfi R, Pannarale F and Rezzolla L 2014 *Mon. Not. R. Astron. Soc.* 438 L71–5
[41] Doneva D D, Yazadjiev S S, Stergioulas N and Kokkotas K D 2013 *Astrophys. J.* 781 L6
[42] Chakrabarti S, Del Santo T, Gürlebeck N and Steinhoff J 2014 *Phys. Rev. Lett.* 112 201102
[43] Harrison B K, Thorne K S, Wakano M and Wheeler J A 1965 *Gravitation Theory and Gravitational Collapse* (Chicago, IL: University Chicago Press)
[44] Hartle J B and Sabbadin A G 1977 *Astrophys. J.* 213 831–5
[45] Friedman J L and Ipser J R 1987 *Astrophys. J.* 314 594–7
[46] Sabbadin A G and Hartle J B 1977 *Ann. Phys.* 104 95–133
[47] Koranda S, Stergioulas N and Friedman J L 1997 *Astrophys. J.* 488 799
[48] Haensel P, Lasota J P and Zdunik J L 1999 *Astron. Astrophys.* 344 151–3
[49] Van Oeveren E D and Friedman J L 2017 *Phys. Rev. D* 95 083014
[50] Moustakidis C C, Gaitanos T, Margaritis C and Lalazissis G A 2017 *Phys. Rev. C* 95 045801
Moustakidis C C, Gaitanos T, Margaritis C and Lalazissis G A 2017 *Phys. Rev. C* 95 059904 (erratum)
[51] Müller H and Serot B D 1996 *Nucl. Phys. A* 606 508–37
[52] Bedasque P and Steiner A W 2015 *Phys. Rev. Lett.* 114 031103
[53] Gerocz R P and Lindblom L 1990 *Phys. Rev. D* 41 1855
[54] Müller H, Prakash M and Ainsworth T L 1987 *Phys. Lett. B* 199 469–74
[55] Potekhin A Y, Farntina A F, Chamel N, Pearson J M and Goriely S 2013 *Astron. Astrophys.* 560 A48
[56] Douchin F and Haensel P 2001 *Astron. Astrophys.* 380 151
[57] Hartle J B 1976 *Phys. Astrophys. J.* 150 1005–29
[58] Hartle J B and Thorne K S 1969 *Astrophys. J.* 153 807
[59] Hessels J W T, Ransom S M, Stairs I H, Freire P C C, Kaspi V M and Camilo F 2006 *Science* 311 1901–4
[60] Haensel P, Potekhin A Y and Yakovlev D G 2007 *Neutron Stars 1: Equation of State and Structure* vol 326 (New York: Springer)
[61] Sumiyoshi K, Ibáñez J M and Romero J V 1999 *Astron. Astrophys. Suppl.* 134 39–52
[62] Berti E, White F, Maniopoulou A and Bruni M 2005 *Mon. Not. R. Astron. Soc.* 358 923–38
[63] Hinderer T, Lackey B D, Lang R N and Read J S 2010 *Phys. Rev. D* 82 024016
[64] Kaminen S, Prakash M and Shannon J M 2010 *Phys. Rev. D* 82 024016
[65] Kalogera V and Psaltis D 2000 *Phys. Rev. D* 61 024009
[66] Rathe C A, Özel F and Psaltis D 2016 *Phys. Rev. C* 93 032801
Rathe C A, Özel F and Psaltis D 2016 *Phys. Rev. C* 93 044905 (addendum)
[67] Bejger M and Haensel P 2002 *Astron. Astrophys.* 396 917
[68] Bejger M, Bulik T and Haensel P 2005 *Mon. Not. R. Astron. Soc.* 364 635
[69] Stergioulas N and Friedman J 1999 *Astrophys. J.* 444 306
[70] Paschalidis V and Stergioulas N 2016 arXiv:1612.03080
[71] Martínez J G, Stovall K, Freire P C C, Deneva J S, Jenet F A, McLaughlin M A, Bagchi M, Bates S D and Ridolfi A 2015 *Astrophys. J.* 812 143
[72] Silva H O, Sotani H and Berti E 2016 *Mon. Not. R. Astron. Soc.* 459 4378–88
[73] Popov S B, Blaschke D, Grigorian H and Prokhorov M E 2007 *Astrophys. Space Sci.* 308 381
[74] Yagi K, Stein L C, Pappas G, Yunes N and Apostolatos T A 2014 *Phys. Rev. D* 90 063010
[75] Stein L C, Yagi K and Yunes N 2014 *Astrophys. J.* 788 15
[76] Baubock M, Berti E, Psaltis D and Özel F 2013 *Astrophys. J.* 777 68
[77] Gentile R M, Arzoumanian Z and Okajima T 2012 *Proc. SPIE* 8443 844313
[78] Arzoumanian Z et al 2014 *Proc. SPIE* 9144 914420
[79] Zhang S N et al and (eXTP) 2016 *Proc. SPIE* 9905 99051Q
[80] Pappas G and Apostolatos T A 2014 *Phys. Rev. Lett.* 112 121101
[81] Yagi K, Kyutoku K, Pappas G, Yunes N and Apostolatos T A 2014 *Phys. Rev. D* 89 124013
[82] Benhar O, Ferrari V, Gualtieri L and Marassi S 2005 *Phys. Rev. D* 72 044028
[83] Clifton T, Ferreira P G, Padilla A and Skordas C 2012 *Phys. Rep.* 513 1–189
[84] Yunes N and Siemens X 2013 *Living Rev. Relativ.* 16 9
[85] Berti E et al 2015 *Class. Quantum Grav.* 32 243001
[89] Glampedakis K, Pappas G, Silva H O and Berti E 2015 *Phys. Rev. D* 92 024056
[90] Glampedakis K, Pappas G, Silva H O and Berti E 2016 *Phys. Rev. D* 94 044030
[91] Pani P, Berti E, Cardoso V and Read J 2011 *Phys. Rev. D* 84 104035
[92] Zaglauer H W 1992 *Astrophys. J.* 393 685–96
[93] Sotani H and Kokkotas K D 2017 *Phys. Rev. D* 95 044032
[94] Hunter J D 2007 *Comput. Sci. Eng.* 9 90–5