An analytic treatment of the Gibbs-Pareto behavior in wealth distribution

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Abstract

We develop a general framework, based on Boltzmann transport theory, to analyze the distribution of wealth in societies. Within this framework we derive the distribution function of wealth by using a two-party trading model for the poor people while for the rich people a new model is proposed where interaction with wealthy entities (huge reservoir) is relevant. At equilibrium, the interaction with wealthy entities gives a power-law (Pareto-like) behavior in the wealth distribution while the two-party interaction gives a Boltzmann-Gibbs distribution.

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1 INTRODUCTION

Inequality in the distribution of wealth in the population of a nation has provoked a lot of studies. It is important for both economists and physicists to understand the root cause on this inequality: whether stochasticity or a loaded dice is the main culprit for such a lop-sided distribution. While it has been empirically observed by Pareto \cite{Pareto} that the higher wealth group distribution has a power-law tail with exponent varying between 2 and 3,
the lower wealth group distribution is exponential or Boltzmann-Gibb’s like [2, 3]. The Boltzmann-Gibb’s law has been shown to be obtainable when trading between two people, in the absence of any savings, is totally random [4, 5, 6]. The constant finite savings case has been studied earlier numerically by Chakraborti and Chakrabarti [1] and later analytically by us [5]. As regards the fat tail in the wealth distribution, several researchers have obtained Pareto-like behavior using approaches such as random savings [7, 8], inelastic scattering [9], generalized Lotka Volterra dynamics [10], analogy with directed polymers in random media [11], and three parameter based trade-investment model [12].

In this paper, we try to identify the processes that lead to the wealth distribution in societies. Our model involves two types of trading processes – tiny and gross. The tiny process involves trading between two individuals while the gross one involves trading between an individual and the gross-system. The philosophy is that small wealth distribution is governed by two-party trading while the large wealth distribution involves big players interacting with the gross-system. The poor are mainly involved in trading with other poor individuals. Whereas the big players mainly interact with large entities/organizations such as government(s), markets of nations, etc. These large entities/organizations are treated as making up the gross-system in our model. The gross-system is thus a huge reservoir of wealth. Hence, our model invokes the tiny channel at small wealths while at large wealths the gross channel gets turned on.

2 GENERAL FRAMEWORK

We will now develop a formalism similar to Boltzmann transport theory so as to obtain the distribution function $f(y, \dot{y}, t)$ for wealth $y$, net income $\dot{y}$ (or total income after consumption) as a function of time $t$. Similar to
Boltzmann’s postulate we also postulate a dynamic law of the form

\[
\frac{\partial f}{\partial t} = \left\{ \frac{\partial f}{\partial t} \right\}_{\text{ext source}} + \left\{ \frac{\partial f}{\partial t} \right\}_{\text{interaction}}.
\] (1)

The first term on the right hand side (RHS) describes the evolution due to external income sources only, while the second one represents contribution from entirely internal interactions.

2.1 Model for tiny-trading

Individuals, possessing wealth smaller than a cutoff wealth \( y_c \), engage in two-party trading where two individuals 1 and 2 put forth a fraction of their wealth \((1 - \lambda_t) y_1\) and \((1 - \lambda_t) y_2\) respectively [with \( 0 \leq \lambda_t < 1 \)]. Then the total money \((1 - \lambda_t)(y_1 + y_2)\) is randomly distributed between the two. The total money between the two is conserved in the two-party trading process. We assume that probability of trading by individuals having certain money is proportional to the number of individuals with that money.

Since in a two-party trading there is no external income source, then \( \left\{ \frac{\partial f}{\partial t} \right\}_{\text{ext source}} = 0 \). The second term on the RHS in Eq. (1) can be obtained as follows in terms of a balance equation.

\[
\frac{\partial f}{\partial t} = \left\{ \frac{\partial f}{\partial t} \right\}_{\text{interaction}} = \text{gains} - \text{losses}.
\] (2)

In Eq. (2) the two terms on the RHS can be expressed in terms of transition rates \( r(y_1, y_2; y_1', y_2') \) for a pair of persons to go from moneys \( y_1, y_2 \) to moneys \( y_1', y_2' \) respectively. Then, we have on assuming that the distribution function is only a function of wealth and time,

\[
\frac{\partial f(y_1, t)}{\partial t} = \int r(y_1', y_2', y_1, y_2) f(y_1', t) f(y_2', t) dy_2 dy_1' dy_2' - \int r(y_1, y_2, y_1', y_2') f(y_1, t) f(y_2, t) dy_2 dy_1 dy_2'.
\] (3)

In the above equation, conservation law requires that \( y_1 + y_2 = y_1' + y_2' \). Hence in the first integral we treat \( y_2 \) as redundant and integrate out with respect
to it to yield a normalization constant. Similarly in the second integral $y_2'$ is integrated out. Now for the transition rate in the first integral of the above equation we have

$$r(y_1', y_2'; y_1, y_2) \propto \frac{1}{(1 - \lambda t)(y_1' + y_2')}$$

if $\lambda ty_1' \leq y_1 \leq y_1' + (1 - \lambda t)y_2'$ and zero otherwise. On taking into account the restriction that no one can have negative money and setting $y_1' + y_2' = L$, the first integral in Eq. (3) at equilibrium is proportional to

$$
\int_y^\infty dL \int_{a(y_1, L, \lambda t)}^{b(y_1, L, \lambda t)} dy_1' \mathcal{F}(y_1', L, \lambda t),
$$

where

$$a(y_1, L, \lambda t) \equiv \max[0, \{y_1 - (1 - \lambda t)L\}/\lambda t],$$

$$b(y_1, L, \lambda t) \equiv \min[L, y_1/\lambda t],$$

and

$$\mathcal{F}(y_1', L, \lambda t) \equiv \frac{f(y_1')f(L - y_1')}{(1 - \lambda t)L}.$$ 

As regards the second integral in Eq. (3), at equilibrium we assume that the transition from $y_1$ to all other levels is proportional to $f(y_1)$. Also, since at equilibrium $\frac{\partial f(y_1,t)}{\partial t} = 0$, we obtain the distribution function to be

$$f(y) = \int_y^\infty dL \int_{a(y_1, L, \lambda t)}^{b(y_1, L, \lambda t)} dx \mathcal{F}(x, L, \lambda t).$$

The above result was obtained earlier by us using an alternate route [5].

On introducing an upper cutoff $y_c$ for the two-party trading, the contribution to the distribution function $f(y)$ from the tiny channel becomes

$$\gamma \int_y^\infty dL \int_{a(y_1, L, \lambda t)}^{b(y_1, L, \lambda t)} dx \mathcal{F}(x, L, \lambda t) \mathcal{H}(x, L, y_c).$$

(7)
In the above equation
\[ \mathcal{H}(x, L, y_c) \equiv [1 - \theta(x - y_c)][1 - \theta(L - x - y_c)], \]
with \( \theta(x) \) being the unit step function and \( \gamma = \frac{1}{\int_0^{y_c} dx f(x)} \) is a normalization constant introduced to account for the less than unity value of the probability of picking a person below \( y_c \).

### 2.2 Model for gross-trading

Next, we will analyze the contribution to the distribution function \( f(y) \) from gross-trading. An individual possessing wealth larger than a cutoff wealth \( (y_c) \) trades with a fraction \( (1 - \lambda_g) \) of his wealth \( y \) with the gross-system. The latter puts forth an equal amount of money \( (1 - \lambda_g)y \). The trading involves the total sum \( 2(1 - \lambda_g)y \) being randomly distributed between the individual and the reservoir. Thus on an average the gross-system’s wealth is conserved.

When only the gross-trading channel is operative, we have
\[
\frac{\partial f(y_1, t)}{\partial t} = \int r(y'_1; y_1)f(y'_1, t)dy'_1 - \int r(y_1; y'_1)f(y_1, t)dy'_1, \tag{8}
\]
where \( r(y'_1; y_1) \) is the transition rate from \( y'_1 \) to \( y_1 \) through interaction with the reservoir. Total money involved in trading (between individual and the gross-system) is \( 2y'_1(1 - \lambda_g) \). After interaction, the resulting money \( y_1 \) of the individual satisfies the following constraints \( \lambda_g y'_1 \leq y_1 \leq (2 - \lambda_g)y'_1 \). The first transition rate in the above equation is given by
\[
r(y'_1; y_1) \propto \frac{1}{2y'_1(1 - \lambda_g)}, \tag{9}
\]
if \( \lambda_g y'_1 \leq y_1 \leq (2 - \lambda_g)y'_1 \) and zero otherwise. As before, at equilibrium the second integral in Eq. (8) is proportional to \( f(y_1) \) when only the gross channel is operative. Then the distribution function \( f(y) \) is given by
\[
f(y) = \int_{y/(2 - \lambda_g)}^{y/\lambda_g} \frac{dx f(x)}{2x(1 - \lambda_g)}. \tag{10}
\]
Now it is interesting to note that the solution of the above equation is given by

\[ f(y) = \frac{c}{y^n}. \]

To obtain \( n \) one then solves the equation

\[ (2 - \lambda_g)^n - \lambda_g^n = 2n(1 - \lambda_g), \]

and obtains \( n = 1, 2 \). Only \( n = 2 \) is a realistic solution because it gives a finite cumulative probability. It is of interest to note that the solution is \textit{independent of} \( \lambda_g \). Also, clearly the distribution function makes sense only for \( y > 0 \). On taking into account an upper cutoff \( y_c \), the contribution to the distribution function \( f(y) \) from the gross channel is

\[ \int_{y/(2-\lambda_g)}^{y/\lambda_g} dx f(x) \frac{dx f(x)}{2x(1 - \lambda_g)} \theta(x - y_c). \]

(12)

2.3 Hybrid model

Here an individual possessing wealth larger than a cutoff wealth \( y_c \) does trading with the gross-system, while individuals possessing wealth smaller than \( y_c \) engage in two-party tiny-trading. Hence from Eqs. (7) and (12), the distribution function is obtained to be

\[
\begin{align*}
\gamma \int_y^\infty dL \int_{\alpha(y,L,\lambda_t)}^{\beta(y,L,\lambda_t)} dx F(x,L,\lambda_t) \mathcal{H}(x,L,y_c) \\
+ \int_{y/(2-\lambda_g)}^{y/\lambda_g} dx f(x) \frac{dx f(x)}{2x(1 - \lambda_g)} \theta(x - y_c).
\end{align*}
\]

(13)

Now, it must be pointed out that when the savings \( \lambda_t = 0, \lambda_g \neq 0 \), and \( y \to 0 \), Eq. (13) yields (up to a proportionality constant) the following same result as the purely tiny-trading case without an upper cutoff \([5]\):

\[ f'(y) \propto -f(y)f(0). \]

(14)

In obtaining the above equation we again assumed that the function \( f(y) \) and its first and second derivatives are well behaved. Then the solution for small \( y \) is given by

\[ f(y) \propto f(0) e^{\exp[-yf(0)]}. \]

(15)
3 RESULTS AND DISCUSSION

The distribution function \( f(y) \) can be obtained by solving the nonlinear integral Eq. (13). To this end, we simplify Eq. (13) for computational purposes as follows:

\[
f(y) = \gamma G(y, \lambda_t, y_c) \int_{2y_c}^{y} dL \int_{a(y, L, \lambda_t)}^{b(y, L, \lambda_t)} dx \mathcal{F}(x, L, \lambda_t) \mathcal{H}(x, L, y_c) + [1 - \theta(y - y_{as})] \int_{y/(2 - \lambda_g)}^{y/\lambda_g} dx f(x) \frac{dx}{2x(1 - \lambda_g)} \theta(x - y_c) + \theta(y - y_{as}) f(y_{as}) \frac{y_{as}^2}{y^2},
\]

where \( G(y, \lambda_t, y_c) \equiv 1 - \theta[y - (2 - \lambda_t)y_c] \) and \( y > y_{as} \) gives the asymptotic behavior \( f(y) \propto 1/y^2 \). In our calculations, we have taken \( y_{as} \) to be at least \( 20y_c \) and obtained \( f(y) \) for all \( y \) less than 2000 times the average wealth per person \( y_{av} \). We solved Eq. (16) iteratively by choosing a trial function, substituting it on the RHS and obtaining a new trial function and successively substituting the new trial functions over and over again on the RHS until convergence is achieved. The criterion for convergence was that the difference between the new trial function \( f_n \) and the previous trial function \( f_p \) satisfies the accuracy test \( \sum_i |f_n(y_i) - f_p(y_i)| / \sum_i f_p(y_i) \leq 0.002 \) [13].

In Fig. 1, using a log-log plot we depict the distribution function \( f(y) \) for the constant savings case \( \lambda_t = \lambda_g = 0.5 \) with the average money per person \( y_{av} \) being set to unity and with the values of the wealth cutoff \( y_c = 3, 5, 10 \). As expected, for larger values of \( y_c \), the Pareto-like \( 1/y^2 \) behavior sets in later. The transition to purely gross-trading occurs at \( (2 - \lambda_t)y_c \), while below \( \lambda_g y_c \) it is purely two-party tiny-trading. Thus the transition from purely tiny-trading to purely gross-trading occurs in Fig. 1 over a region of width \( y_c \). However, all the tails merge irrespective of the cutoff. At smaller values of \( y \) the behavior of \( f(y) \), depicted in the inset, is similar to the purely two-party trading model studied earlier (see Ref. [5]). The curves in the inset appear to be close because here the trading is two-party and is governed by the same
Figure 1: Plot of the wealth distribution function for savings $\lambda_t = \lambda_g = 0.5$ and various wealth cutoff values $y_c = 3, 5, 10$. The average money per person $y_{av}$ is set to unity. The dotted lines are guides to the eye.
Figure 2: Wealth distribution $f(y)$ at average wealth $y_{av} = 1$, wealth cutoff $y_c = 5$, and various values of savings $\lambda_t = \lambda_g = 0.1, 0.5, 0.8$. 
savings. Next, in Fig. 2 we plot \( f(y) \) with the cutoff \( y_c = 5, y_{av} = 1 \), and for values of savings fraction \( \lambda_t = \lambda_g = \lambda = 0.1, 0.5, 0.8 \). Here the power-law behavior \( (1/y^2) \) takes over for \( y > (2 - \lambda)y_c \) and hence at lower savings it sets in later. In the power-law region the curves merge together. As shown in the inset of Fig. 2, at smaller values of \( y \) the \( f(y) \)s become zero with the higher peaked curves (corresponding to larger \( \lambda s \)) approaching zero faster similar to the case of the purely two-party trading model in our earlier work [5]. Here the transition from purely tiny- to purely gross-trading at higher \( \lambda \) is sharper because the transition occurs over a region of width \( 2(1 - \lambda)y_c \). Lastly, in Fig. 3, we show the distribution function \( f(y) \) for the zero savings case in the tiny-channel \( (\lambda_t = 0) \) and for various savings \( \lambda_g = 0.2, 0.5, 0.9 \) in
the gross-channel with $y_{av} = 1$ and $y_c = 5$. The distribution, as expected, decays exponentially (or Boltzmann-Gibbs-like) for small values of $y$ and has power-law ($1/y^2$) behavior at large values. The curves merge in the Pareto-like region and, in fact, $f(y) \approx 0.1/y^2$ in all the three figures at large values of $y$. In Fig. 3 too, for reasons mentioned earlier, the transition is sharper at larger values of $\lambda_g$. Fig. 3 takes into account the fact that, in societies, the rich tend to have higher savings fraction ($\lambda$) compared to the poor. Actually, if the savings fraction were to increase gradually with wealth, one can expect a more gradual change in the transition region of the distribution rather than the sharp local maxima (around $y \approx 6.5$) shown by the $\lambda_g = 0.9$ curve.

In all the figures anomalous looking kinks/shoulders appear at the crossover between the Boltzmann-Gibbs-like and the Pareto-like regimes. This is due to the sharp cut-off at $y_c$ that we introduced using a step function. However, a kink seems to be generic in these kind of distributions in real populations (as borne out by the empirical data in Fig. 9 of Ref. [2]) indicating that two different dynamics may be operative in the two regimes. Different societies have the onset of Pareto-like behavior at different wealths which is indicative that the cut-off has to be obtained empirically based on various factors like the social structure, welfare policies, type of markets, form of government, etc.

In Japan the wealth/income distribution vanishes at zero wealth/income and then rises to a maximum (see Ref. [2]). In US the distribution seems to be a maximum at zero wealth/income (see Ref. [2]). Both these aspects can be covered in our model as the poor in general are known to save very little. If their savings are zero, one gets the Boltzmann-Gibbs behavior at the poor end. On the other hand, if the savings are small one gets a maximum close to zero and the distribution vanishes at zero wealth.

It would be interesting to deduce the savings pattern from the wealth distribution. While it has been observed that the rich tend to save more than the poor, how gradually the savings change as wealth increases can
perhaps be inferred from the change in slope. However, as explained below, the middle region (involving the middle-class) has been modeled quite crudely by us and needs to be refined before a serious connection with savings pattern can be attempted.

We will now further discuss the motivation for using two different mechanisms to model the observed wealth distribution. The model is an approximation where the direct wealth exchange occurs between people who are in economic proximity. At the poorer end of the spectrum, the poor, who have limited economic means and avenues, come in contact with a few poor and their economic activity is modeled in terms of two-party trading. At the other end of the wealth spectrum, the rich have access to various economic avenues (e.g., markets, know-how, work force, capital, credit facilities, contacts, wealthy society, etc.) due to which they can trade with huge organizations and are thus modeled to interact with a reservoir. As regards the middle-class that is between the rich and the poor, they trade amongst themselves as well as with the poor and the reservoir. As a first step towards realizing this scenario, we included in our model only the two extreme cases of interaction. What we have not taken into account is the interaction of the middle class with the reservoir. To rectify this, in future we hope to introduce a cutoff \( y_g \) for the interaction with the reservoir such that \( y_g \) lies below the two-party trading cutoff \( y_t \). Thus, we believe that our model is a reasonable one at the poor and rich ends and is a crude approximation for the middle class. While it is true that the poor also come in market contact with wealthy organizations like the coke company, the contact is an indirect one through intermediaries. For example, the poor person deals with a richer shop-keeper selling coke who in turn deals with a richer local distributor who in turn deals with the big coke company. Lastly, we would like to add that the assumption of random distribution in two-party trading is a model studied by others as well (see Refs. \[1, 7\]). We feel that in any trading there is random fluctuation of the price around its true value. The total money put
forth for trading corresponds to the amount of random fluctuation. However
the poorer of the two puts forth less and makes the trading biased in his/her
favor. This can be justified from the fact that the poor people are constantly
looking for bargains to make ends meet.

Compared to other types of analyses involving two-party trading to ex-
plain Pareto law (see Refs. [7, 8]), our gross-trading mechanism can make
contact with the standard approach in macroeconomics as will be shown
below. Over the past, economists have developed two models, namely, the
dynastic model and the life-cycle model, to explain wealth distribution. In
the dynastic model, where bequests are vehicles of transmission of wealth
inequality, people save to improve the consumption of their descendants. On
the other hand, in the life-cycle model, where wealth of an individual is a
function of the age, people save to provide for their own consumption after re-
tirement. Both these models and their hybrid versions have had only limited
success [14]. However, one of the ingredients that goes into these models,
i.e., uninsurable shocks or stochasticity in income, has been exploited by
econophysicists with remarkable success in reproducing power-law tails.

In macroeconomics, the objective is to maximize a cumulative utility func-
tion subject to a wealth constraint [15]. Mathematically this is formulated as

\[
\max_{c_{t+i}, y_{t+i}} E_t \sum_i \beta^i u(c_{t+i}),
\]

subject to the constraint

\[
y_{t+i} = (1 + r)y_{t+i-1} + e_{t+i} - c_{t+i},
\]

where \(c_t, y_t,\) and \(e_t\) are consumption, wealth, and labor earnings respectively
at time \(t,\) \(r\) is the interest rate on wealth \(y,\) \(0 < \beta < 1\) is the time-discount
factor, \(u(c_t)\) is the concave utility function, \(E_t\) is the expectation value based
on the available information at time \(t.\) Using the method of Lagrange multi-
pliers, the conditions of optimality yield

\[
E_t[u'(c_t) - (1 + r)\beta u'(c_{t+1})] = 0,
\]
where $u'(c_t)$ is the derivative of $u(c_t)$ with respect to $c_t$. From the above equation we see that consumption at different times are related. In our work [see Eq. (19)], we introduced the stochasticity

$$y_{t+1} - y_t = \epsilon(1 - \lambda_g) y_t,$$

where $\epsilon$ is a random number with $-1 \leq \epsilon \leq 1$, which implies that

$$ry_{t+1} + e_{t+1} - c_{t+1} = \epsilon(1 - \lambda_g) y_t.$$

The above equation can be made consistent with the optimal consumption relation given by Eq. (19). Thus our results can be approached through the standard machinery in macroeconomics.

The stochasticity in wealth given by Eq. (20) implies that the spread in wealth distribution at time $t + 1$ is proportional to $y_t$ and thus wealth’s $y_{t+1} > y_t$ yield a wider spread for $y_{t+2}$ than do wealth’s $y_{t+1} < y_t$. Thus the distribution becomes more skewed to the right.

In conclusion, we introduced a new ingredient – interaction of the rich with huge entities – and obtained a Pareto-like power-law. On the other hand, the Boltzmann-Gibbs-like wealth distribution of the poorer part of the society is explained through a two-party trading mechanism. All in all, we demonstrate that stochasticity can account for the observed skewness in the wealth distribution.

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