Coupling non-gravitational fields with simplicial spacetimes

Jonathan R McDonald\textsuperscript{1,2} and Warner A Miller\textsuperscript{1}

\textsuperscript{1} Department of Physics, Florida Atlantic University, Boca Raton, FL 33431, USA
\textsuperscript{2} Institute for Applied Mathematics, Friedrich Schiller University, 07743 Jena, Germany

E-mail: jonathan.mcdonald@uni-jena.de

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Abstract
The inclusion of source terms in discrete gravity is a long-standing problem. Providing a consistent coupling of source to the lattice in the Regge calculus (RC) yields a robust unstructured spacetime mesh applicable to both numerical relativity and quantum gravity. RC provides a particularly insightful approach to this problem with its purely geometric representation of spacetime. The simplicial building blocks of RC enable us to represent all matter and fields in a coordinate-free manner. We provide an interpretation of RC as a discrete exterior calculus framework into which non-gravitational fields naturally couple with the simplicial lattice. Using this approach we obtain a consistent mapping of the continuum action for non-gravitational fields to the Regge lattice. In this paper we apply this framework to scalar, vector and tensor fields. In particular we reconstruct the lattice action for (1) the scalar field, (2) Maxwell field tensor and (3) Dirac particles. The straightforward application of our discretization techniques to these three fields demonstrates a universal implementation of the coupling source to the lattice in RC.

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1. Non-gravitational source in simplicial spacetime

Regge calculus (RC) is a discrete coordinate-free formulation of Einstein’s geometric theory of gravitation [1]. Its simplicial spacetime directly incorporates the local Poincaré invariance of general relativity into finite domains [2]. The simplicial geometry is by now well understood [3], but how is one to couple non-gravitational sources to this lattice? We are unaware of a completely unified approach to the incorporation of matter into Regge calculus (RC) which utilizes the natural geometric structure and locally finite implementation of both the Poincaré and diffeomorphism symmetry. In this manuscript we provide such a description.
While the inherent representation of a local tangent space within each simplex provides us with an *a priori* orthogonal and holonomic frame, i.e. locally flat Minkowski metric structure with finite domain, the way in which we couple source to this structure must also utilize the discrete diffeomorphism invariance [4, 5]. It is this invariance that automatically enforces the conservation of stress–energy at each vertex in the lattice. This is equivalent to identifying the local structure on which fields must be embedded so as to be consistent with the automatic preservation of conservation of source [6]. In this manuscript we provide a unified approach to constructing non-gravitational fields in RC based on the interpretation of the Regge lattices as discrete differential forms [7, 8]. We begin in section 2 by outlining the basics of a discrete exterior calculus. We then apply this framework to the Klein–Gordon (section 3), Maxwell (section 4) and the Dirac fields (section 5).

2. Discrete differential forms

In order to accurately retain the geometric properties of fields in a discrete framework, one must embed, in a consistent way, the fields into the lattice through a correspondence between the geometry of the lattice and field variables [7, 9]. This has been the primary focus of methods in computational electromagnetism [9–11] where the focus is on demonstrating increased stability and automatic retention of internal symmetries. In these approaches to the discretization of Maxwell’s equations the electromagnetic 4-potential $A^\mu$ is constructed from chains of 1-forms that naturally emerge in the underlying triangulation [11]. The intuitive notion is that one projects a continuous $k$-form onto an algebraic $k$-chain determined by the $k$-simplexes of the lattice. In this manuscript we provide a cursory overview of this construction and its geometry. A more complete discussion of the implementation of discrete exterior calculus is available in Desbrun et al [8] or Arnold et al [9].

The piecewise-flat simplicial lattice of RC is an image of a curvilinear simplicial complex on a continuous manifold under an appropriate isomorphism. We use this map to project the $k$-forms onto the simplicial complex and identify its projections into the lattice. We start by explicitly constructing such simplicial forms. If $\omega$ is a $k$-form, then we define the lattice representation of $\omega$ by

$$\Pi(\omega, \sigma^k) := \langle \omega | \sigma^k \rangle = \int_{\sigma^k} \omega,$$

where $\sigma^k$ is a $k$-dimensional simplex. This ‘projection inner-product’ maps each continuum form into its corresponding piecewise-flat simplicial form. The key feature of the discrete exterior calculus approach that is captured by equation (1) is that any physical field contributes scalar weights to the simplicial differential forms naturally supplied by the lattice. This greatly simplifies any calculations on the simplicial net as one need only to work with $k$-simplexes and their associated scalar value. This closely mimics the fundamental approach of RC to gravitation in which geometric observables are defined via weighting of appropriate $k$-simplexes with scalar weights [12, 13].

In addition to the discrete differential forms one must introduce a Hodge operator to map simplicial forms to dual forms in the lattice. The construction of such an operator is not unique [11] as one can arbitrarily construct a consistent mapping based on circumcenters, incenters, barycenters, etc. Despite the ambiguity in constructing a dual lattice, there are distinct advantages to utilizing the circumcentric dual. It is this dual that appears to be most natural in RC [3–5, 12, 14–16] because of its orthogonality to the simplicial lattice. This orthogonality provides a factorization of $d$-volumes associated with any given $k$-simplex into orthogonal components which lie in either the simplicial or dual lattice. This factorization of
the homology (Delaunay lattice, $\sigma$) and the co-homology (Voronoi lattice, $\star \sigma$) was essential in defining the Einstein–Hilbert action in RC [13] and the vertex-based scalar curvature [18]. However, one may choose to define the dual in any consistent manner.

Moreover, to ensure that the dual is a well-defined [17] lattice we restrict ourselves to Delaunay simplicial lattices and their circumcentric Voronoi duals. This restriction is a natural assumption since we wish to require that any local topology constructed from the hybrid $d$-volumes remains Hausdorff. The Hodge dual in a simplicial $d$-dimensional geometry thus maps $k$-forms from $k$-simplexes to $(d-k)$-forms on the $(d-k)$-faces of the circumcentric Voronoi dual. Explicitly this can be defined as [11]

$$
\frac{1}{|\sigma^k|} \left( \langle \alpha \mid \sigma^k \rangle \right) = \text{Sign} (\sigma^k) \frac{1}{|\star \sigma^k|} \left( \langle \star \alpha \mid \star \sigma^k \rangle \right),
$$

(2)

where $|\sigma^k|$ is the volume of the $k$-simplex $\sigma^k$ and where one includes an appropriate sign based on whether the $k$-simplex is space-like or time-like. This allows us to explicitly define the right-hand side of equation (2) as a mapping from $\star \alpha$ to elements $\star \sigma^k$ of the dual lattice

$$
\Pi (\star \alpha) := \langle \star \alpha \mid \star \sigma^k \rangle = \int \star \alpha.
$$

(3)

The assignment of $p$-forms $\omega \in \Lambda^p$ to either the simplicial or the dual lattice is somewhat arbitrary. For the sake of clarity we examine the geometric meaning of those elements of $\Lambda^k$ to identify such an assignment. In this spirit, a 1-form is equivalent to a covector, and a general $k$-form is given as the totally anti-symmetric product of $k$ 1-forms. The identification of 1-forms with covectors indicates that the dual of a $(d-1)$-form is identified as a vector on a manifold. We thus make the identification that forms on the circumcentric dual represent the standard $k$-forms while the elements of the simplicial lattice represent the dual space. For further clarity, we identify simplicial forms as $k$-forms in $\Lambda^k$ and dual forms, i.e. forms represented on the dual lattice, as $k$-forms in $\Lambda^k$.

With the definition of the dual forms we can introduce an inner-product of two discrete forms on the manifold:

$$
(\omega, \eta) := \int \langle \omega \mid \eta \rangle = \int \omega \wedge \star \eta \quad \text{(continuum)}
$$

$$
(\omega, \eta) := \sum_{\sigma^k} \langle \omega \wedge \star \eta \rangle_{\sigma^k} = \sum_{\sigma^k} \langle \omega \mid \eta \rangle V^{(d)}_{\sigma^k} = \sum_{\sigma^k} \omega (\sigma^k) \eta (\sigma^k) V^{(d)}_{\sigma^k} \quad \text{(lattice)}.
$$

(4)

Here, $V^{(d)} (\sigma^k)$ is the hybrid between the simplicial $k$-form and its dual. The circumcentric dual gives a particularly simple form for these volumes:

$$
V^{(d)}_{\sigma^k} = \frac{1}{|\sigma^k|} | \sigma^k | \cdot | \star \sigma^k |,
$$

(5)

resulting from the inherent orthogonality between the two lattices. We should also distinguish between the functional inner-product, $\langle \cdot, \cdot \rangle$, and the spacetime inner-product, $\langle \cdot \mid \cdot \rangle$, in that the latter acts locally on the simplicial skeleton of the manifold while the former tells us how to 'integrate' discrete forms over the entire manifold.

The discrete exterior derivative maps a $k$-simplicial (dual) form mapped on $\sigma^k$ ($\star \sigma^{d-k}$) into a $(k+1)$-simplicial (dual) form. This implies a discrete sum over the $k+1$ simplexes (dual cells) incident on $\sigma^k$ ($\star \sigma^{d-k}$). To clarify this we rely on the exterior co-derivative which maps a $k$-simplicial (dual) form into a $(k-1)$-simplicial (dual) form. The exterior derivative and co-derivative are related through the inner product as usual:

$$
\langle d \alpha \mid \sigma^k \rangle = \frac{1}{|\sigma^k|} \langle \alpha \mid \delta \sigma^k \rangle \quad \text{and} \quad \langle d \alpha \mid \star \sigma^k \rangle = \frac{1}{|\star \sigma^k|} \langle \alpha \mid \delta (\star \sigma^k) \rangle.
$$

(6)
where $\delta := \ast d \ast$. We thus have a discrete representation of Stoke’s theorem, which leads us to the discrete (co-)boundary of a (co-)boundary principle

$$d d \alpha \equiv 0 \quad \text{and} \quad \delta \delta \alpha \equiv 0.$$  

(7)

This provides the necessary de Rham cohomology with the spaces $\ast \Lambda^k$ and $\Lambda^k$ defined on the simplicial lattice and its dual, respectively.

The discrete representation of differential exterior calculus presented here is by virtue of the projection operator, structurally equivalent to continuum exterior calculus though defined on a piecewise-flat manifold. Using insight from the continuum Kirchhoff-like conservation of stress–energy [19, 20], we can explicitly show how this framework is already expressed in RC [18]. This provides a robust framework for coupling non-gravitational fields to simplicial spacetimes. In the next three sections we provide a construction of the scalar field action for scalar, vector and spinor fields. In particular we construct the electromagnetic field and the Dirac field action for simplicial spacetimes with four dimensions.

3. Discrete scalar fields

As an illustration of this construction we derive the lattice action for a scalar field and show its equivalence to its traditional finite difference representation [21]. Here we provide a direct calculation of the discrete action from the continuum action through an implementation of the discrete exterior forms of section 2. The scalar field action in the continuum is given by

$$S[\phi, \bar{\phi}] = \int d^4 x \frac{1}{2} \left[ \partial^\mu \phi \partial_\mu \bar{\phi} - m^2 \bar{\phi} \phi \right] = \frac{1}{2} \left( d \phi, d \bar{\phi} \right) - m^2 \left( \phi, \bar{\phi} \right).$$  

(8)

The differential form expression for the scalar field action contains both 1-forms and 0-forms which we must embed in the lattice. We first identify whether these are simplicial forms or dual forms. The stress–energy corresponding to the propagation of the field is directed along an edge of the lattice [5]. To maintain an appropriate identification of stress–energy with the field, we express flux of the scalar field as directed along edges of the simplicial lattice. In other words, we express each 1-form in the kinetic term of the action as projected onto the edges of the lattice. Likewise, each 0-form in the mass term is projected onto the vertices of the simplicial spacetime:

$$\Pi(\phi) = \langle \phi | v \rangle = \phi(v),$$  

$$\Pi(d \phi) = (d \phi | L) = \sum_{v \subset L} \frac{1}{|L|} \langle \phi | v \rangle = \frac{\phi(v + L) - \phi_v}{|L|}.$$  

(9) (10)

This yields the standard finite differencing term for a scalar field on the lattice. Using this coefficient for the 1-form directed along $L$, we obtain the discrete action

$$S[\phi, \bar{\phi}] = \frac{1}{2} \left[ \sum_{L} (d \phi | d \bar{\phi}) V_{L}^{(4)} - m^2 \sum_{v} \phi \bar{\phi} V_{v}^{(4)} \right]$$

$$= \sum_{L} \frac{1}{2} \frac{\phi(v + L) - \phi_v}{|L|} \cdot \frac{\bar{\phi}(v + L) - \bar{\phi}_v}{|L|} \cdot \frac{1}{2} |L| V_{L}^{(4)} - m^2 \sum_{v} \phi(v) \bar{\phi}(v) V_{v}^{(4)}.$$  

(11)

Here we assume that $v$ is the base vertex for the oriented edge $L$. This coincides with the action suggested in [21] with the 4-volumes given by the edge-based simplicial-circumcentric dual hybrid volumes. The stress–energy tensor of the scalar field is a doubly projected tensor along the edge $L$ with conservation given by a Kirchhoff-like conservation law [5]. This can
be explicitly and simply calculated using the orthogonal decomposition of the hybrid cells. The stress–energy tensor associated with a given edge for the complex scalar field is

\[ T_{LL'} := \delta S / \delta L[\phi, \bar{\phi}] = -\frac{1}{8} \left[ \frac{(\phi(v + L) - \phi_v)(\bar{\phi}(v + L) + \bar{\phi}_v)}{L^2} - \sum_{v \subset L} m^2 \frac{\phi(v)\bar{\phi}(v)}{2} \right] V_L^* \]

with conservation of source given by

\[ \sum_{L \supset v} T_{LL} = \frac{1}{8} \sum_{L \supset v} \left[ \frac{(\phi(v + L) - \phi_v)(\bar{\phi}(v + L) - \bar{\phi}_v)}{L^2} - \sum_{v' \subset L} m^2 \frac{\phi(v')\bar{\phi}(v')}{2} \right] \approx 0. \]

This provides us with a direct and minimal coupling of the complex scalar field with the Regge lattice. It also provides us with a Kirchhoff-like expression for the conservation of scalar field stress–energy. This scalar field application provides a literal interpretation of the Kirchhoff-like conservation law at each vertex, \( v \), as a flow of field along the edges of the simplicial lattice meeting at \( v \). One can see explicitly in equation (13) the change in the field along each edge from one vertex to the other. In the next section (section 4) we extend the simplicial exterior calculus approach to Maxwell’s equations where the field is vectorial, and the correspondence with a vertex-based conservation law is not as transparent as the scalar field example done here, instead we must rely wholly on the framework set up in section 2.

4. Simplicial electromagnetic fields

The electromagnetic field is defined by the connection 1-form, \( A^\mu \), and the action is given by the square of the curvature 2-form, \( F^{\mu \nu} \):

\[ S[A] = -\frac{1}{2} \langle F, F \rangle = -\frac{1}{2} \int d^4x \ F \wedge * F = -\frac{1}{2} \int d^4x \ dA \wedge * dA. \]  

We now restrict ourselves for the rest of the manuscript to four spacetime dimensions in which we must represent the 2-form \( dA \) on a 2-simplex. The connection 1-form \( A \) is defined on the edges of the simplicial lattice. The curvature 2-form is therefore determined entirely by the values of \( A \) on its boundary:

\[ \langle dA | \sigma^3 \rangle = \frac{1}{|\sigma^2|} \langle A | \delta \sigma^2 \rangle = \sum_{L \subset \sigma^2} \langle A | L \rangle = \frac{|L|}{|\sigma^2|} \sum_{L \subset \sigma^2} A_L. \]

Here we utilize the fact that we integrate over the entire boundary which introduces a factor of \( |L| \) into the contribution from each edge.

It is assumed that the discrete forms are valued on the appropriate \( k \)-simplex such that we need only to work with the scalar values of the coefficients. Making the substitutions into the source-free Maxwell Lagrangian, the simplicial electromagnetic action becomes

\[ S[A] = -\frac{1}{4} \sum_{\sigma^2} \left( \sum_{L \subset \sigma^2} \frac{|L|}{|\sigma^2|} A_L \right)^2 \left( 2 \frac{\sigma^2 \wedge * \sigma^2}{d(d-1)} \right). \]

Introducing source into the action requires the 1-form current \( j \), whose dual is the 3-form current density, \( *j \), defined on the 3-volumes dual to an edge. The former determines the
direction of propagation of the current while the latter determines the volume pierced by the flow of the current. This introduces the interaction in the action

\[ (A, j) = \sum_L (j_L, A_L) V_L^{(d)} \]  \hspace{1cm} (17)

While gauge invariance of the potential 1-form under the transformation \( A + d\Lambda \) is automatically satisfied in the source-free terms of the action, invariance of the action should give indications of the conservation of source based on the transformations of the source terms. Under the gauge transformation, the source terms of the action give

\[ (A + d\Lambda, j) = (A, j) + \sum_L (A_{v+L} - \Lambda_v) \cdot j_L \]

\[ = (A, j) - \sum_v (\Lambda_v) \sum_{L \ni v} j_L. \]  \hspace{1cm} (18)

Under arbitrary gauge transformations, the invariance of the action implies

\[ \sum_{L \ni v} j_L = \sum_{*L \ni A(\text{cv})} *j_L = 0, \]  \hspace{1cm} (19)

where the first sum gives the Kirchhoff-like conservation of the current 1-form, while the second sum is based on the more typical representation of the divergence of the current density 3-form. This result was first shown in [22] where the projection of the 4-potential is equivalent to the construction here.

This construction follows a path to that of the Sorkin construction [22]. However, by using only discrete differential forms we express the electromagnetic field in the coordinate-free language determined by scalar weightings of simplicial elements. We thus avoid defining tangent-space-valued tensors in any strict sense so as to mirror the way gravitational variables are handled in RC.

While the action (equation (18)) is similar in character, it is explicitly distinct from the Sorkin approach to electromagnetism on the spacetime mesh [22]. Sorkin’s approach to the problem utilizes the full tangent space associated with a \( d \)-simplex to determine the affine components of the Maxwell tensor locally. The method presented here extends Sorkin’s original embedding of \( A^\mu \) into the lattice to incorporate the more recent understandings of the dual lattice structure inherent in RC [4, 5, 12–16, 18]. Using the dual lattice, one already finds that the spacetime curvature automatically decomposes into hinge-based curvature tensors proportional to the scalar curvature. In constructing the action, these hinge-curvatures are weighted by the hybrid volumes described in section 2 [13] as the determiners of the local measure on the lattice. This is a result of the projection of the continuum curvature 2-form onto the two-dimensional polygonal faces of the dual lattice, the carriers of information of spacetime curvature. Moreover we have shown [18] how this construction leads to alternative descriptions of curvature at localized sites in the lattice. The simplicial electromagnetic action outlined above follows in the same spirit as curvature’s manifestation in RC. The Maxwell tensor is locally distributed on the 2-forms of the simplicial net and weighted by the corresponding hybrid volumes. This allows us the flexibility of stringing together chains of simplicial 2-forms to define locally simple Maxwell curvature in the simplicial lattice, as was shown explicitly for spacetime curvature in RC [18].

The discrete differential form approach to defining a field on a simplicial lattice for the two cases we have examined illustrate the construction and its usefulness for non-gravitational fields in RC. We have been able to construct analogous actions directly from the continuum action via projection of the continuum fields onto the corresponding elements of the lattice.
The scalar field, in particular, provides an identical match to previous literature on scalar fields in RC. The action for the electromagnetic field differs from previous descriptions as we no longer require an explicit tangent-spaced-valued tensor to be constructed interior to each simplex. This compares well with computational electrodynamic applications for flat spacetimes. In the next section we extend this formalism and derive an action for fermionic fields that is consistent with the fundamental coordinate-free structure of and the flow of stress–energy in RC.

5. The simplicial Dirac field

Before we derive our version of the Dirac action in RC it is important to note some of the key elements that go into such a construction. These elements are pulled together from considerations of (1) the Dirac field itself and from (2) the simplicial lattice geometry. The first component is the need for an orthogonal frame for the construction of a Dirac spinor. This is fundamentally related to the need to provide local representations of the Dirac $\gamma$-matrices. These matrices form their own basis in the Clifford algebra of spacetime. Therefore, one must be able to construct an isomorphism between the representations of the geometry of the spacetime (the basis 1-forms) and the $\gamma$-matrices [23]. In RC this is quite straightforward since we are provided with finite representations of spacetime whose basis 1-forms can be used to construct an orthonormal and holonomic frame. However, this cannot be taken too literally. As with the other fields, the Dirac field is a spacetime-valued field. Moreover, in light of the conservation of stress–energy as derived in [5], it is known that the flow of stress–energy is directed on the edges of the simplicial lattice. This should give some indications that the flow of field (or particles), as the carrier of stress–energy, is also directed along the edges of the simplicial lattice. In the construction given below we follow this path by defining the Dirac field on the simplicial geometry.

The standard construction of fermionic fields uses an action given by

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} \gamma^a e^a_\mu \nabla_\mu \psi - m \bar{\psi} \psi,$$

(20)

where $e^a_\mu = (e_\mu^a)^{-1}$ is the co-tetrad connecting a holonomic frame defining the covariant derivative to the orthonormal frame representation of $\gamma^a$. Before we assign a simplicial form to the fields directly we note that the field $\bar{\psi}$ is explicitly based in a representation of the double-cover $SL(2, \mathbb{C})$ and not in a representation of $SO(3,1)$. In order to make the connection with our framework more clear we write the Dirac action in a more suggestive form:

$$S[\psi, \bar{\psi}] = i \bar{\psi} (\gamma^a e_a, D) \psi - m(\bar{\psi}, \psi),$$

(21)

where $D$ is the covariant exterior differential. Here we view the first term as an inner product between operators for purely illustrative purposes. One could also expand this first term into two by including the fields $\bar{\psi}$ and $\bar{\psi}$ in the inner-product with $D$ acting on $\bar{\psi}$ in one term and $D$ acting on $\bar{\psi}$ in the second. To keep the notation compact we will assume the first term in the action to imply

$$i \bar{\psi} (\gamma^a e_a, D) \psi = i(\bar{\psi} \gamma^a e_a, D \psi) - i(\bar{\psi} \gamma^a e_a, D \bar{\psi}).$$

(22)

In this representation of the action it is evident how one must embed the field into the lattice. The co-tetrad is naturally the simplicial 1-form since the tetrad is a dual 1-form (defining the tangent space interior to a simplex). Moreover, the covariant exterior derivative maps a 0-form to a 1-form on the simplicial lattice. The 0-form nature of the field suggests that
each vertex, $v$, of the simplicial lattice is the natural placeholder for the fields, i.e. without specifically assigning a basis or tangent space to represent the field. Rather we are free to choose any simplex meeting at $v$ to determine the tangent space for the field. Once a tangent space is chosen, the vielbein explicitly determines the spinor basis and matrix representation of the Clifford algebra of the $\gamma^a$ in the standard way [23, 24]. The term $\gamma^ae_a$ is then given by the projection $[L] \cdot \Pi(\gamma) = \gamma_L$ which assigns a Dirac matrix representation, in some appropriate basis, to the edges of the lattice. Since each term in the action is a Lorentz scalar, we can make the choice of basis independently for each term in the discretized action:

$$S[\psi, \bar{\psi}] = \sum_L (i(\bar{\psi}_L \gamma_L | D_L \psi) + \text{c.c.}) V^*_L + \sum_v m \bar{\psi}(v) \psi(v) V^*_v. \quad (23)$$

Here we take the value of the field on a given edge, $\psi_L$, to be the average field on that edge:

$$\psi_L = \frac{\psi(v + L) + \psi(v)}{2}. \quad (24)$$

The ability to freely choose an appropriate tangent space greatly simplifies the expression of the action. In particular, since each edge has multiple simplexes hinging on the edge, we can define any edge-based term of the action in the tangent space of one such simplex. This reduces the covariant derivative of the field to a flat-space differential. As with the scalar field the derivative terms reduce to finite-differencing terms and we obtain a simplified discrete action without the need for spin connections:

$$S[\psi, \bar{\psi}] = \sum_L \left( i\bar{\psi}_L \gamma_L \frac{\psi(v + L) - \psi(v)}{L} + \text{c.c.} \right) V^*_L + \sum_v m \bar{\psi}(v) \psi(v) V^*_v. \quad (25)$$

This local action utilizes the freedom to assign each flat tangent spaces to its fullest degree. This in turn reduces all derivatives and finite differences to their flat-space representations. This should simplify local calculations of the fields on the lattice. However, for non-local calculations using the Dirac field, one will necessarily introduce a spin-connection to transform one representation of the Clifford algebra to another. These are given explicitly by orthogonal transformations of the vielbein from one simplex to another. Geometrically this is viewed as transport along the edge dual to the boundary between two neighboring simplexes. A short discussion of this transformation is given in [25] where the Dirac field is represented as a the dual-vertex-based action.

### 6. A unified approach to conservation of stress–energy

The inclusion of sources into RC has been an area of active research for some years with various approaches modeled without a unified principle determining the implementation. This manuscript has provided an integrated framework based on differential forms for the embedding of fields into a simplicial lattice. The developments in computational electromagnetism using an approach called discrete exterior calculus (or discrete differential forms) together with our earlier results have led us to a formal theory for projecting smooth differential forms into a discrete pseudo-Riemannian lattice. The utility of this approach is its dependence only on the lattice structure and geometry, and its explicit lack of dependence on coordinate systems.

For scalar and fermion fields, the actions retain similarity to the more familiar finite-difference schemes on the simplicial lattice with appropriate volume weighting. For the Dirac field, we took a minimalistic approach based on the conventional quantum field theory construction of Dirac spinors. It might prove useful to also investigate a Kähler–Dirac formalism for spin-$\frac{1}{2}$ particles which explicitly uses the Clifford algebra of differential forms.
We leave this to future work. The construction of the electromagnetic field outlined in this manuscript follows the spirit of RC by embedding the fields without explicit reference to affine coordinate systems inside simplexes. The fields are directly encoded only on the edges and triangles of the lattice with dependence given by the incidence matrix of the lattice. By encoding the fields onto the simplicial skeleton we have also ensured a direct connection with the conservation of stress–energy on the lattice [5]. The Kirchhoff-like conservation principle tells us that the flow of stress–energy is directed along the edges of the lattice. This topological formulation supports the assumption that each field should be encoded onto the skeleton as we have suggested. Encoding source fields on the simplicial skeleton provides a concrete and geometric connection between the field and its conserved current, i.e. stress–energy, under diffeomorphisms.

In canonical RC there have been a variety of approaches to embedding non-gravitational fields in the simplicial lattice. To date, these approaches have been based on finite-difference schemes directly applied to the lattice or barycentric coordinate representations of field tensors. Fields have been encoded in either the simplicial skeleton or dual skeleton depending on convenience of representation. However, a universal approach toward embedding non-gravitational sources into RC with due regard to conservation of stress–energy has so far been absent from the literature. The work presented here identifies a framework for coupling source to field, in accordance with automatic conservation of source, in canonical RC. Moreover, the representations of fields and their actions do not require coordinization in the simplicial blocks. Instead we assign appropriate scalar or spin-valued weightings to elements of the simplicial skeleton through projections from the continuum representation.

In encoding the fields in the simplicial geometry based on canonical RC, we define a universal prescription for assignment of field to the lattice. This provides a foundation for carrying over these derivations to model-dependent descriptions of simplicial spacetimes. Reliance on only the canonical structure of RC without regard to specific dynamical behavior allows this approach to be universally applicable to numerical relativity or simplicial models of quantum gravity. However, appropriate modifications to the dynamics of the source fields may well be necessary to ensure consistency with application to a given model of quantum gravity. Such analysis of these foundations in the context of specific models of quantum gravity is a topic for future investigation.

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