A Remark on Smoothing Out Higher Codimension Branes

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Abstract

We discuss some issues arising in studying (linearized) gravity on non-BPS higher codimension branes in an infinite-volume bulk. In particular, such backgrounds are badly singular for codimension-3 and higher $\delta$-function-like branes with non-zero tension. As we discuss in this note, non-trivial issues arise in smoothing out such singularities. Thus, adding higher curvature terms might be necessary in this context.
I. INTRODUCTION AND SUMMARY

As was originally proposed in [1], one can reproduce four-dimensional gravity on a 3-brane in 6 or higher dimensional infinite-volume bulk if one includes an induced Einstein-Hilbert term on the brane. Gravity then is almost completely localized on the brane with only ultra-light modes penetrating into the bulk, so that gravity is four-dimensional at distance scales up to an ultra-large cross-over scale $r_c$ (beyond which gravity becomes higher dimensional), which can be larger than the present Hubble size. In particular, this is the case for codimension-2 and higher tensionless branes [1] as well as for codimension-2 non-zero tension branes [2].

A careful analysis of linearized gravity in such backgrounds requires smoothing out higher codimension singularities [1,3]. This is already the case for tensionless branes, where the background is non-singular (in fact, it is flat), but the graviton propagator is singular. In the case of non-zero tension branes the situation is even more complicated as in the case of $\delta$-function-like branes the background itself becomes singular (for phenomenologically interesting non-BPS branes on which we focus in this paper). More precisely, in the codimension-2 case the singularity is very mild as in the extra 2 dimensions the background has the form of a wedge with a deficit angle, so the singularity is a simple conical one [2]. As was discussed in [3], this singularity can be consistently smoothed out. In this case the gravity on the brane was analyzed in [2,3], where it was found that the behavior of gravity is essentially unmodified compared with the tensionless brane cases.

In codimension-3 and higher cases the singularities are more severe. The purpose of this note is to study these backgrounds using a smoothing out procedure discussed in [3]. This procedure goes as follows. Consider a codimension-$d$ $\delta$-function-like source brane in $D$-dimensional bulk. Let the bulk action simply be the $D$-dimensional Einstein-Hilbert action, while on the brane we have the induced $(D-d)$-dimensional Einstein-Hilbert term as well as the cosmological term corresponding to the brane tension. As we have already mentioned, the background in this case is singular [4]. One way to smooth out such a singularity is to replace the $(D-d)$-dimensional world-volume of the brane by its product with a $d$-dimensional ball $B_d$ of some non-zero radius $\epsilon$. As was pointed out in [3], in this case already for a tensionless brane the gravitational modes on the brane contain an infinite tower of tachyonic modes. This can be circumvented by considering a partial smoothing out where one replaces the $(D-d)$-dimensional world-volume of the brane by its product with a $(d-1)$-sphere $S^{d-1}$ of radius $\epsilon$ [3]. As was pointed out in [3], this suffices for smoothing out higher codimension singularities in the graviton propagator as in the codimension-1 case the propagator is non-singular [3]. Moreover, in the case of tensionless branes as well as in the case of a non-zero tension codimension-2 brane we then have only one tachyonic mode which is expected to be an artifact of not including non-local operators on the brane. The question we would like to address here is whether this smoothing out procedure can also cure singularities of the background itself in the case of codimension-3 and higher non-zero tension branes.

We find that there are no non-singular solutions of the aforementioned type. One possibility here could be to add higher curvature terms which might help cure these singularities. In particular, as was recently discussed in [3], higher curvature terms are expected to smooth out higher codimension singularities in the graviton propagator.
II. SETUP

The brane world model we study in this paper is described by the following action:

\[
S = \tilde{M}_P^{D-3} \int_{\Sigma} d^{D-1}x \sqrt{-\tilde{G}} \left( \tilde{R} - \tilde{\Lambda} \right) + M_P^{D-2} \int d^Dx \sqrt{-G} \ R .
\]  

(1)

Here \( M_P \) is the (reduced) \( D \)-dimensional Planck mass, while \( \tilde{M}_P \) is the (reduced) \((D-1)\)-dimensional Planck mass; \( \Sigma \) is a source brane, whose geometry is given by the product \( R^{D-d-1,1} \times S_{\epsilon}^{d-1} \), where \( R^{D-d-1,1} \) is the \((D-d)\)-dimensional Minkowski space, and \( S_{\epsilon}^{d-1} \) is a \((d-1)\)-sphere of radius \( \epsilon \) (in the following we will assume that \( d \geq 3 \)). The quantity \( \tilde{M}_P^{D-3} \tilde{\Lambda} \) plays the role of the tension of the brane \( \Sigma \). Also,

\[
\tilde{G}_{mn} \equiv \delta_m^M \delta_n^N G_{MN} \bigg|_{\Sigma} ,
\]

(2)

where \( x^m \) are the \((D-1)\) coordinates along the brane (the \( D \)-dimensional coordinates are given by \( x^M = (x^m, r) \), where \( r \geq 0 \) is a non-compact radial coordinate transverse to the brane, and the signature of the \( D \)-dimensional metric is \((-+, \ldots , +))\); finally, the \((D-1)\)-dimensional Ricci scalar \( \tilde{R} \) is constructed from the \((D-1)\)-dimensional metric \( \tilde{G}_{\mu\nu} \). In the following we will use the notation \( x^i = (x^\alpha, r) \), where \( x^\alpha \) are the \((d-1)\) angular coordinates on the sphere. Moreover, the metric for the coordinates \( x^i \) will be (conformally) flat:

\[
\delta_{ij} \ dx^i dx^j = dr^2 + r^2 \gamma_{\alpha\beta} \ dx^\alpha dx^\beta ,
\]

(3)

where \( \gamma_{\alpha\beta} \) is the metric on a unit \((d-1)\)-sphere. Also, we will denote the \((D-d)\) Minkowski coordinates on \( R^{D-d-1,1} \) via \( x^\mu \) (note that \( x^m = (x^\mu, x^\alpha) \)).

The equations of motion read

\[
R_{MN} - \frac{1}{2} G_{MN} R + \sqrt{-\tilde{G}} \tilde{G}^{M}{}_{M} \delta_{N}{}^{n} \left[ \tilde{R}_{mn} - \frac{1}{2} \tilde{G}_{mn} \left( \tilde{R} - \tilde{\Lambda} \right) \right] \tilde{L} \delta(r - \epsilon) = 0 ,
\]

(4)

where

\[
\tilde{L} \equiv \tilde{M}_P^{D-3} / M_P^{D-2} .
\]

(5)

To solve these equations, let us use the following ansatz for the background metric:

\[
ds^2 = \exp(2A) \eta_{\mu\nu} \ dx^\mu dx^\nu + \exp(2B) \ \delta_{ij} \ dx^i dx^j ,
\]

(6)

where \( A \) and \( B \) are functions of \( r \) but are independent of \( x^\mu \) and \( x^\alpha \) (that is, we are looking for solutions that are radially symmetric in the extra dimensions). We then have (here prime denotes derivative w.r.t. \( r \)):

\[
\tilde{R}_{\mu\nu} = 0 ,
\]

(7)

\[
\tilde{R}_{\alpha\beta} = \lambda \ \tilde{G}_{\alpha\beta} ,
\]

(8)

\[
\tilde{R} = (d - 1) \lambda ,
\]

(9)

\[
R_{\mu\nu} = -\eta_{\mu\nu} e^{2(A-B)} \left[ A'' + (d - 1) \frac{1}{r} A' + (D - d)(A')^2 + (d - 2) A'B' \right] ,
\]

(10)
\[ R_{rr} = -(d - 1) \left[ B'' + \frac{1}{r} B' \right] + (D - d) \left[ A'B' - (A')^2 - A'' \right], \]  
\[ R_{\alpha\beta} = - r^2 \gamma_{\alpha\beta} \left[ B'' + (2d - 3) \frac{1}{r} B' + (d - 2)(B')^2 + (D - d) A'B' + (D - d) \frac{1}{r} A' \right], \]  
\[ R = - e^{-2B} \left[ 2(d - 1) B'' + 2(d - 1)^2 \frac{1}{r} B' + 2(D - d) A'' + 2(D - d)(d - 1) \frac{1}{r} A' + (d - 1)(d - 2)(B')^2 + (D - d)(D - d + 1)(A')^2 + 2(D - d)(d - 2) A'B' \right], \]  
where \[ \lambda \equiv \frac{d - 2}{e^2} e^{-2B(\epsilon)}. \]  

The equations of motion then read:
\[ (D - d) \left[ \frac{1}{2}(D - d - 1)(A')^2 + (d - 1) \frac{1}{r} A' + (d - 1) A'B' \right] + \]  
\[ (d - 1)(d - 2) \left[ \frac{1}{2}(B')^2 + \frac{1}{r} B' \right] = 0, \]  
\[ (D - d) \left[ A'' + \frac{1}{2}(D - d + 1)(A')^2 + (d - 2) \frac{1}{r} A' + (d - 3) A'B' \right] + \]  
\[ (d - 2) \left[ B'' + \frac{1}{2}(d - 3)(B')^2 + (d - 2) \frac{1}{r} B' \right] + \frac{1}{2} e^B \left[ \bar{\Lambda} - (d - 3) \lambda \right] \bar{L} \delta(r - \epsilon) = 0, \]  
\[ (D - d - 1) \left[ A'' + \frac{1}{2}(D - d)(A')^2 + (d - 1) \frac{1}{r} A' + (d - 2) A'B' \right] + \]  
\[ (d - 1) \left[ B'' + \frac{1}{2}(d - 2)(B')^2 + (d - 1) \frac{1}{r} B' \right] + \frac{1}{2} e^B \left[ \bar{\Lambda} - (d - 1) \lambda \right] \bar{L} \delta(r - \epsilon) = 0. \]  

Here the third equation is the \( (\mu\nu) \) equation, the second equation is the \( (\alpha\beta) \) equation, while the first equation is the \( (rr) \) equation. Note that the latter equation does not contain second derivatives of \( A \) and \( B \). The solution for \( B' \) is given by (we have chosen the plus root, which corresponds to solutions with infinite-volume extra space):
\[ B' = - \frac{1}{r} - \frac{D - d}{d - 2} A' + \sqrt{\frac{1}{r^2} + \frac{1}{\kappa^2} (A')^2}, \]  
where we have introduced the notation \[ \frac{1}{\kappa^2} \equiv \frac{(D - d)(D - 2)}{(d - 1)(d - 2)^2}. \]  

III. THE BACKGROUND

We can solve the above equations of motion as follows. First, consider the difference of \( (15) \) and \( (16) \):
\begin{align*}
(D - d) \left[ 2A'B' - (A')^2 - A'' + \frac{1}{r} A' \right] + \\
(d - 2) \left[ (B')^2 - B'' + \frac{1}{r} B' \right] - \frac{1}{2} e^B \left[ \Lambda - (d - 3) \lambda \right] \tilde{L} \delta(r - \epsilon) = 0 . \tag{20}
\end{align*}

Plugging (18) into this equation we obtain the following equation:

\begin{align*}
\frac{A'[rA'' + A]}{\sqrt{\kappa^2 + r^2(A')^2}} + \frac{(d - 2) (A')^2}{\kappa} + \frac{\kappa}{2(d - 2)} e^B \left[ \Lambda - (d - 3) \lambda \right] \tilde{L} \delta(r - \epsilon) = 0 . \tag{21}
\end{align*}

This equation can then be integrated. Thus, let

\begin{equation}
Q \equiv \frac{1}{\kappa} r A' . \tag{22}
\end{equation}

Then we have

\begin{align*}
\frac{QQ'}{\sqrt{1 + Q^2}} + \frac{(d - 2) 1}{r} Q^2 + \frac{\epsilon}{2(d - 2)} e^B \left[ \Lambda - (d - 3) \lambda \right] \tilde{L} \delta(r - \epsilon) = 0 . \tag{23}
\end{align*}

Here we are interested in non-singular solutions such that \( A \) and \( B \) are constant for \( r < \epsilon \), and asymptote to some finite values as \( r \to \infty \). The corresponding solution for \( Q(r) \) is given by (here \( \theta(x) \) is the Heavyside step-function):

\begin{equation}
Q(r) = \frac{2f(r)}{1 - f^2(r)} \theta(r - \epsilon) , \tag{24}
\end{equation}

where

\begin{equation}
f(r) \equiv \left( \frac{r_*}{r} \right)^{d - 2} , \tag{25}
\end{equation}

and \( r_* \) is the integration constant. Note that due to the discontinuity at \( r = \epsilon \) we have the following matching condition:

\begin{align*}
(d - 2) \frac{2f^2(\epsilon)}{1 - f^2(\epsilon)} + \frac{\epsilon \tilde{L}}{2} e^{B(\epsilon)} \left[ \Lambda - (d - 3) \lambda \right] = 0 . \tag{26}
\end{align*}

Note that, as it should be, this matching condition is the same as the one that follows from equation (20).

Next, we solve for \( A \) and \( B \):

\begin{align*}
A(r) &= A(\epsilon) , \quad r \leq \epsilon , \tag{27} \\
A(r) &= A_\infty - \frac{\kappa}{d - 2} \ln \left( \frac{1 + f(r)}{1 - f(r)} \right) , \quad r > \epsilon , \tag{28} \\
B(r) &= B(\epsilon) , \quad r \leq \epsilon , \tag{29} \\
B(r) &= B_\infty - \frac{D - d}{d - 2} (A(r) - A_\infty) + \frac{1}{d - 2} \ln \left( 1 - f^2(r) \right) , \quad r > \epsilon , \tag{30}
\end{align*}

where \( A_\infty \) and \( B_\infty \) are the asymptotic values of \( A(r) \) respectively \( B(r) \) as \( r \to \infty \).
To complete our task here, we must check that (17) is also satisfied. In fact, it is more convenient to check the difference of (17) and (15):

\[
(D - d - 1)A'' - (d - 1)\frac{1}{r}A' - (D - 2)A'B' + \\
(d - 1) \left[ B'' + \frac{1}{r}B' \right] + \frac{1}{2} e^B \left[ \Lambda - (d - 1)\lambda \right] \tilde{L} \delta(r - \epsilon) = 0 .
\]

By direct examination it is not difficult to check that this equation is indeed satisfied subject to the following matching condition:

\[
(d - 1) \frac{2f^2(\epsilon)}{1 - f^2(\epsilon)} - \frac{D - 2}{d - 2} \frac{2\kappa f(\epsilon)}{1 - f^2(\epsilon)} + \frac{\epsilon\tilde{L}}{2} e^{B(\epsilon)} \left[ \Lambda - (d - 1)\lambda \right] = 0 .
\]

We can rewrite the matching conditions (26) and (32) as follows:

\[
(d - 2) \frac{2f^2(\epsilon)}{1 - f^2(\epsilon)} + \frac{\epsilon\tilde{L}}{2} e^{B(\epsilon)} \left[ \Lambda - (d - 3)\lambda \right] = 0 ,
\]

\[
(D - 2) \frac{2\kappa f(\epsilon)}{1 - f^2(\epsilon)} + \frac{\epsilon\tilde{L}}{2} e^{B(\epsilon)} \left[ \Lambda + (d - 1)\lambda \right] = 0 .
\]

Let us study possible solutions to these matching conditions.

Thus, let us see if there are non-singular solutions with \( r^* < \epsilon \) for which \( f(\epsilon) < 1 \). Then these matching conditions can only be satisfied if \( \tilde{\Lambda} < 0 \). On the other hand, the parameter \( \lambda \) is positive. Let

\[
\tilde{\Lambda} \equiv -\gamma \lambda ,
\]

where \( \gamma \) is a positive parameter. Then we have:

\[
f(\epsilon) = \rho \frac{\gamma + d - 3}{\gamma - d + 1} ,
\]

where

\[
\rho \equiv \kappa \frac{D - 2}{d - 2} = \sqrt{(d - 1)(D - 2)} \cdot \frac{D - d}{D} .
\]

Note that we must have \( \gamma > d - 1 \). On the other hand, note that for \( d \geq 3 \) we always have \( \rho > \sqrt{2} \). This then implies that there is no solution with \( f(\epsilon) < 1 \), that is, there is no solution with \( r^* < \epsilon \). In other words, there are no non-singular solutions of this type. That is, there are no solutions with non-zero brane tension (for vanishing brane tension smooth solutions were discussed in [3]).

Let us end with the following remark. Note that in the action (1) we included not only the \((D - d)\)-dimensional Einstein-Hilbert term for the Minkowski part \( R^{D-d-1,1} \) but also for the sphere part. It is not difficult to see that dropping (or rescaling) the latter does not change any conclusions.
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