Curve complexes and Garside groups

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These slides are at
http://perso.univ-rennes1.fr/bertold.wiest/CALtalk.pdf
• M. Calvez, B. Wiest, Curve graphs and Garside groups, Geometriae Dedicata

• M. Calvez, B. Wiest, Acylindrical hyperbolicity and Artin-Tits groups of spherical type, Geometriae Dedicata
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5 A bestiary of complexes
Mapping class grps $\text{Mod}(S)$
($S = \text{surface}$)

Braid groups
$B_n = \text{Mod}(D_n)$
($D_n = n$-punctured disk)

Garside grps $G$
e.g. Artin groups of spherical type
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\( \text{Mod}(S) \)  
(\( S = \text{surface} \))

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(\( D_n = n \)-punctured disk)

Garside grps \( G \)
e.g. Artin groups of spherical type

\textbf{Classical :}
- \( \text{Mod}(S) \curvearrowright \text{CC}(S) \)
- \( \text{CC} \) is \( \delta \)-hyp. \((\delta \text{ indep. of } S)\)
- For \( \varphi \in \text{Mod}(S) \),
  * \( \varphi \) finite order or reducible
    \( \implies \varphi \curvearrowright \text{elliptically} \)
  * \( \varphi \) pA \( \implies \varphi \curvearrowright \text{lox.}, \text{WPD} \)
In partic., \( \text{diam(CC)} = \infty \)

\textbf{Our contribution :}
- \( G \curvearrowright \mathcal{C}_{\text{AL}}(G) \) additional length cx.
- Conj : \( \mathcal{C}_{\text{AL}}(B_n) \overset{q.i.}{\sim} \text{CC}(D_n) \)
- \( \mathcal{C}_{\text{AL}}(G) \) is \( \delta \)-hyp. \((\delta \text{ indep. of } G)\)
- If \( G = B_n \) : \( g \) finite order or reducible \( \implies g \curvearrowright \text{ellipt.} \)
- If \( G \) any spherical type Artin :
  \( \exists g \in G \) s.t. \( g \curvearrowright \text{lox.}, \text{WPD} \)
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  - $\varphi$ finite order or reducible $\implies \varphi \curvearrowright$ elliptically
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**Our contribution:**
- $G \curvearrowright \mathcal{C}_\text{AL}(G)$ additional length cx.
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Our preferred generators of $B_n$

“Simple braids”, a.k.a. “positive permutation braids”: positive braids, any two strands crossing at most once

$\uparrow$

Permutations of $\{1, \ldots, n\}$

- **Typical example**
  
  Simple braid $x \in B_4$, permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$

- **Very special example**
  
  Half-twist $\Delta \leftrightarrow$ permutation $\begin{pmatrix} 1 & \ldots & n \\ n & \ldots & 1 \end{pmatrix}$

- **Property of $\Delta$**: “almost commutes” with all braids (and $\Delta^2$ generates $\text{Center}(B_n)$)
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Left-weighting

Example

The product $x_1 \cdot x_2$ is not left-weighted; the product $\tilde{x}_1 \cdot \tilde{x}_2$ is.

Theorem (Adjan, Thurston, Elrifai–Morton)

Every $x \in B_n$ has a unique representative of the form

$$\Delta^k \cdot x_1 \cdot \ldots \cdot x_\ell \quad (k \in \mathbb{Z})$$

with $x_i \cdot x_{i+1}$ left-weighted $\forall i$

Notation $k =$ “infimum of $x$”, $k + \ell =$ “supremum of $x$”

Remark Normal forms are described by a FSA.
The prefix-ordering

Definition  Partial ordering on $B_n$ :

$$x \preceq y :\Leftrightarrow \exists \alpha \in B_n^+, \ x \cdot \alpha = y$$

Proposition (Garside)

On $B_n^+$, the monoid of positive braids, this partial ordering is a lattice ordering : for $x, y \in B_n^+$

$$x \land y = g.c.d.(x, y) \text{ and } x \lor y = l.c.m.(x, y) \text{ exist}$$
Triangles in $Cayley(B_n)/\langle \Delta \rangle$

Picture of a triangle in $Cayley(B_n)/\langle \Delta \rangle$.
Edge paths $=$ normal form words. (They are geodesics [Charney])
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The shadowing map $\pi : \text{Cayley}(B_n) \to CC$

$D_n = \{ c_0 \}$

Look at $B_n = \text{Mod}(D_n)$, and the projection (shadowing map):$
\pi : \text{Cayley}(B_n) \to CC := CC(D_n)$
vertex $x \mapsto x.c_0$

For $x \in B_n$, the normal form $x = x_1 \cdot \ldots \cdot x_k$ represents a geodesic path in $\text{Cayley}(B_n)$ from 1 to $x$.

**Conjecture 1**

The projection of this path to $CC$
(the path $c_0 \to x_1.c_0 \to x_1x_2.c_0 \to x_1x_2x_3.c_0 \to \ldots$)
is an unparametrized quasi-geodesic in $CC$.
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For $x \in B_n$, the normal form $x = x_1 \cdot \ldots \cdot x_k$ represents a geodesic path in $\text{Cayley}(B_n)$ from 1 to $x$.

**Conjecture 1** Let us assume that this is true

The projection of this path to $CC$

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Look at triangle in $\text{Cayley}(B_n)$, edge paths = normal form words

$\exists x \in B_n^+, y \in B_n^+$

**Question** Where are the pre-images of the quasi-center?
Look at triangle in $\text{Cayley}(B_n)$, edge paths $= \text{normal form words}$

$\forall x \in B_n^+, \quad \forall y \in B_n^+$

$\text{length} = \text{length}(x \wedge y)$

Question Where are the pre-images of the quasi-center?

Conjecture 2 This is the correct picture;
Look at triangle in $\text{Cayley}(B_n)$, edge paths = normal form words

$x \in B_n^+$, $y \in B_n^+$

$\text{length} = \text{length}( x \wedge y )$

**Question** Where are the pre-images of the quasi-center?

**Conjecture 2** This is the correct picture; more precisely, the blue arrows are squashed in the curve complex.
Plan: construct a new space...
...where Conjectures 1 & 2 are true “by construction”

Definition: absorbable

A braid $y$ with $\inf(y) = 0$ or $\sup(y) = 0$ is absorbable if there exists a braid $x$ such that

$$\inf(x \cdot y) = \inf(x) \quad \text{and} \quad \sup(x \cdot y) = \sup(x)$$

Idea $y$ adds no length and no initial $\Delta$s to $x$.

Example $\sigma_1^k \in B_4$ is absorbable for all $k$.
(Proof: absorbed by $\sigma_3^\ell$, with $\ell \geq k$.)

Example (more surprising) $y = \sigma_2^2 \sigma_3^2 \sigma_2^2 \sigma_1 \in B_4$
absorbed by $x = \sigma_1 \sigma_2^4 \sigma_1 \sigma_2 \sigma_3$

Definition: The additional length complex $C_{AL}(B_n)$

- Take $\text{Cayley}(B_n)/\langle \Delta \rangle$
- Cone off \{ absorbable braids \}
- Transport cone everywhere in $\text{Cayley}(B_n)$ using left action
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Thm: • $C_{AL}(B_n)$ is $\delta$-hyperbolic
• Garside n.f. words are quasi-geodesics in $C_{AL}$

Proof: triangles with (edges $=$ Garside normal form words) are 2-thin.

Conjecture: $C_{AL}(B_n)$ quasi-isometric to $CC(D_n)$

• Easy: Lipschitz map $CC(D_n) \to C_{AL}(B_n)$
• Difficult: inverse map. Need to prove:
  $x \in B_n$ absorbable $\implies d_{CC}(c_0, x.c_0)$ bounded.

Theorem: $\exists x^* \in B_n$ s.t. $x^* \curvearrowright C_{AL}(B_n)$ lox., WPD

Proof: Construct $x^*$: • in left and right normal form,
• starts and ends with a single-atom-letter, and
• contains letters that are only a single atom short of being $\triangle$
Prove that $x^*$ is “contracting” in the Cayley graph.

There are analogues for all this for all irreducible Artin-Tits of spherical type.
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There are analogues for all this for all irreducible Artin-Tits of spherical type.
Application: genericity

Let $G$ be an Artin-Tits group of spherical type (e.g. $G = B_n$). Let $g \in G$ be a large random element. Then

$$\mathbb{P}(g \curvearrowright C_{AL}(G) \text{ loxodromically}) \longrightarrow 1$$

Here a “large random element” is either

- the result of a long random walk in $B_n$, or
- a random element (with uniform probability) in a large ball with center 1 in $\text{Cay}(G)$. [S.Caruso, W]

In particular, generic braids are pseudo-Anosov.

The definition of $C_{AL}$ is very naive!

Why not try adapting it to your favorite finitely gen. groups?
Application : genericity

Let $G$ be an Artin-Tits group of spherical type (e.g. $G = B_n$). Let $g \in G$ be a large random element. Then

$$P(g \curvearrowright \text{CAL}(G) \text{ loxodromically}) \longrightarrow 1$$

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Classification of Artin-Tits groups of spherical type, and their parabolic subgroups

| Type | Diagram |
|------|---------|
| $A_n$ | $n \geq 2$ |
| $B_n$ | $n \geq 2$ |
| $D_n$ | $n \geq 3$ |
| $H_3$ | |
| $H_4$ | |

Notes:
- $A_n$, $B_n$, $D_n$, $H_3$, and $H_4$ are groups of type $A$, $B$, $D$, $H_3$, and $H_4$, respectively.
- $E_6$, $E_7$, $E_8$, $F_4$, and $I_{2m}$ are other types of Artin-Tits groups.

Proposition 1.28 (Brieskorn & Saito, 1972; Deligne, 1972).

Proposition 1.29 (Brieskorn & Saito, 1972, Lemme 5.1, Théorème 7.1).

1.2.1 Subgroups paraboliques

The long the second part of this thesis we will work with specific subgroups, the parabolic subgroups, which are the algebraic analogues of the non-braided systems for braid groups.
Let $G$ be an irreducible Artin-Tits group of spherical type.

**Want**

We want $G$ to act on a $\delta$-hyperbolic complex.

(Further goal: prove $G$ is hierarchically hyperbolic, with the hierarchical structure given by parabolic subgroups.)

**Three candidate complexes!**

1. $\mathcal{C}_{COPS}(G)$ (cone off standard parabolic subgroups) = generalised arc complex $\delta$-hyperbolic ?
   - Lipschitz, *not* quasi-isom.

2. Complex of parabolic subgroups of $G$ $[C,G,G,M,W]$ (cone off normalisers of standard parabolic subgroups) = generalised curve complex $\delta$-hyperbolic ?
   - Lipschitz. Conj: quasi-isom.

3. $\mathcal{C}_{AL}(G)$ (cone off absorbables) $\delta$-hyperbolic !