Small ball probabilities for a class of
time-changed self-similar processes

KEI KOBAYASHI

University of Tennessee
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Abstract

This paper establishes small ball probabilities for a class of time-changed processes \( X \circ E \), where \( X \) is a self-similar process and \( E \) is an independent continuous process, each with a certain small ball probability. In particular, examples of the outer process \( X \) and the time change \( E \) include an iterated fractional Brownian motion and the inverse of a general subordinator with infinite Lévy measure, respectively. The small ball probabilities of such time-changed processes show power law decay, and the rate of decay does not depend on the small deviation order of the outer process \( X \), but on the self-similarity index of \( X \).

1 Introduction

Let \( W \) be a one-dimensional standard Brownian motion and let \( E_\beta \) be the inverse of a stable subordinator \( D_\beta \) of index \( \beta \in (0,1) \), independent of \( W \). Nane [21] established that the small ball probability of the time-changed Brownian motion \( W \circ E_\beta \) is given by

\[
\mathbb{P}\left( \sup_{0 \leq t \leq 1} |W(E_\beta(t))| \leq \epsilon \right) \sim \frac{32\Gamma(\beta)\sin(\beta\pi)}{\pi^4} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} \epsilon^2 \quad \text{as} \quad \epsilon \downarrow 0,
\]

where \( \Gamma(\cdot) \) is Euler’s Gamma function and the notation \( f(x) \sim g(x) \) for two positive functions \( f \) and \( g \) means that \( \lim f(x)/g(x) = 1 \). The result is interesting since the small ball probability of \( W \circ E_\beta \) shows power law decay unlike the exponential decay observed for the original Brownian motion \( W \):

\[
-\log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |W(t)| \leq \epsilon \right) \sim \frac{\pi^2}{8} \epsilon^{-2} \quad \text{as} \quad \epsilon \downarrow 0.
\]

Moreover, the rate of decay in (1) does not depend on the stability index \( \beta \) of the underlying stable subordinator \( D_\beta \); the dependence on \( \beta \) only appears as a small deviation constant independent of \( \epsilon \).

The proof of (1) provided in [21] essentially relies on the following expression for the Laplace transform of the random variable \( E_\beta(1) \) and its asymptotic behavior along the negative real axis (see e.g. Proposition 1(a) of [5] and Theorem 1.4 of [22]):

\[
\mathbb{E}[e^{-aE_\beta(1)}] = \mathbb{E}_\beta(-a) \sim \frac{1}{a\Gamma(1-\beta)} \quad \text{as} \quad a \to \infty.
\]
Here, \( E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\beta+1)} \) is the Mittag–Leffler function with parameter \( \beta \). Nane [21] also extended the result to a time-changed process \( X \circ E_\beta \), where the outer process \( X \) is a self-similar process possessing a certain small ball probability, which particularly includes the case of a fractional Brownian motion. However, the exact small deviation constant cannot be specified unlike the situations considered in [2]; see Remark 12 for details of this point.

The main motivation to analyze such time-changed processes comes from their non-standard diffusion structures. In particular, the time-changed Brownian motion \( W \circ E_\beta \) is non-Gaussian and non-Markovian, and is widely used to model subdiffusions, where particles spread more slowly than the classical Brownian particles do. Namely, the particles represented by the time-changed Brownian motion are trapped and immobile during the constant periods of the time change \( E_\beta \). One interesting aspect of the time-changed Brownian motion is that its transition probabilities satisfy the following time-fractional generalization of the Fokker–Planck or forward Kolmogorov equation:

\[
\partial_t^\beta p(t, x) = \frac{1}{2} \partial_x^2 p(t, x), \quad t > 0, \ x \in \mathbb{R}.
\]

Here, \( \partial_t^\beta \) denotes the Caputo fractional derivative operator in time of order \( \beta \) (see e.g. [22]). The correspondence between the time-changed Brownian motion and the fractional Kolmogorov equation has been extended to those for different classes of time-changed processes and stochastic differential equations they drive; see e.g. [8, 9, 10, 13, 16, 17, 18]. The fractional Kolmogorov equations have found many applications in a wide range of scientific areas, including physics, [19, 27], finance [7, 15], hydrology [3], and biology [25].

In this paper, we establish small ball probabilities for a class of time-changed processes \( X \circ E \), where \( X \) is a self-similar process and \( E \) is a continuous process independent of \( X \), each with a certain small ball probability (Theorems 1 and 11). This largely extends the results in [21] in terms of both the outer process \( X \) and the time change \( E \). Examples of \( X \) and \( E \) that can be handled within our framework include an iterated fractional Brownian motion and the inverse of a general subordinator with infinite Lévy measure, respectively. Our strategy is to employ a version of the Tauberian theorem (Lemma 3) along with a general fact concerning a subordinator (Proposition 4), which is a different approach from what was taken in [21] to derive (1). In particular, even when \( E \) is the inverse of a stable subordinator, our method does not rely on the asymptotic expression for the Mittag–Leffler function given in (3).

The results to be established in this paper show that the small ball probability of a certain time-changed process \( X \circ E \) has power law decay whose rate depends on the self-similarity index of the outer process \( X \), but not on the small deviation order of \( X \). In a particular case of a time-changed Brownian motion \( W \circ E \) with the time change \( E \) being the inverse of a general subordinator with infinite Lévy measure, the dependence on \( E \) is reflected on the associated small deviation constant. We will specify that constant when the underlying subordinator is a Gamma subordinator or a tempered stable subordinator; these specific time changes have been recently investigated to analyze anomalous diffusions observed in various natural phenomena (see e.g. [12]). This will allow us to examine how the small ball probabilities for the important subclasses of time-changed processes vary according to the choice of the parameters defining the underlying subordinators. In particular, our result with the time change being the inverse of a tempered stable subordinator recovers (1) as an immediate corollary; see Remark 10 for details.
2 Small ball probabilities for time-changed Brownian motions

Let $E$ be a stochastic process in $\mathbb{R}^1$ with continuous, nondecreasing paths starting at 0. One way to construct such a process is through a subordinator. Namely, let $D$ be a subordinator with Laplace exponent $\psi$ and infinite Lévy measure $\nu$; i.e. $D$ is a one-dimensional nondecreasing Lévy process with càdlàg paths starting at 0 with Laplace transform

$$
\mathbb{E}[e^{-sD(t)}] = e^{-t\psi(s)}, \quad \text{where } \psi(s) = bs + \int_0^\infty (1 - e^{-sx}) \nu(dx), \quad s > 0,
$$

with $b \geq 0$ and $\int_0^\infty (x \wedge 1) \nu(dx) < \infty$. The assumption that the Lévy measure is infinite (i.e. $\nu(0, \infty) = \infty$) implies that $\psi$ is an increasing function with $\lim_{s \to \infty} \psi(s) = \infty$ and $D$ has strictly increasing paths with infinitely many jumps (see e.g. Theorem 21.3 of [24]). Let $E$ be the inverse or first hitting time process of $D$; i.e.

$$
E(t) := \inf\{u > 0; D(u) > t\}, \quad t \geq 0.
$$

Since $D$ has strictly increasing paths, the process $E$, called an inverse subordinator, has continuous, nondecreasing paths starting at 0 (see e.g. Lemma 2.7 of [13]). It is known that $E$ generally does not have independent or stationary increments (see Section 3 of [17]), which implies that even if $X$ is a Gaussian or Lévy process independent of $E$, the time-changed process $X \circ E$ no longer has the same structure. Hence, existing results on small ball probabilities of Gaussian or Lévy processes cannot be directly applied to find the small ball probability of $X \circ E$.

A stochastic process $X$ in $\mathbb{R}^1$ is called a self-similar process of index $H > 0$ if for every $a > 0$, $(X(at))_{t \geq 0} \overset{d}{=} (a^H X(t))_{t \geq 0}$. Important examples of self-similar processes include fractional Brownian motions, iterated fractional Brownian motions, and stable Lévy processes. Brief definitions of these processes will be provided in examples in Section 3.

The following theorem largely extends Theorem 2.1 of [21] to the case when the time change is given by the inverse of a non-stable subordinator.

**Theorem 1.** Let $E$ be the inverse of a subordinator $D$ with infinite Lévy measure $\nu$, independent of a one-dimensional standard Brownian motion $W$. Then for all $T > 0$ at which $\nu$ has no mass (i.e. $\nu(\{T\}) = 0$),

$$
\mathbb{P}\left(\sup_{0 \leq t \leq T} |W(E(t))| \leq \epsilon\right) \sim \frac{32}{\pi^3} \nu(T, \infty) \sum_{k=1}^\infty \frac{(-1)^k}{(2k - 1)^3} \epsilon^2 \quad \text{as } \epsilon \downarrow 0.
$$

This is interpreted as $\mathbb{P}(\sup_{0 \leq t \leq T} |W(E(t))| \leq \epsilon) = o(\epsilon^2)$ if $\nu(T, \infty) = 0$.

**Remark 2.** 1) If $\nu(T, \infty) > 0$, then the small ball probability of the time-changed Brownian motion $W \circ E$ has a power law decay. Moreover, the rate of decay of the small ball probability does not depend on the choice of the inverse subordinator $E$; the dependence on $E$ is reflected only on the constant $\nu(T, \infty)$.

2) In the degenerate case when $E(t) = t$, clearly $\nu \equiv 0$ and hence the small ball probability becomes $o(\epsilon^2)$. This is because the small ball probability for the Brownian motion $W$ (without a time change) has an exponential decay as in (2).

3) If $E = E_\beta$ is the inverse of a $\beta$-stable subordinator $D_\beta$, then (5) immediately recovers (1). Indeed, using the explicit form of the Lévy measure of $D_\beta$ (see e.g. Example
1.3.18 of [1]), we observe that

\[ \nu(T, \infty) = \int_T^{\infty} \frac{\beta}{\Gamma(1 - \beta)} x^{-1-\beta} \, dx = \frac{T^{-\beta}}{\Gamma(1 - \beta)}. \]

When \( T = 1 \), the last expression coincides with \( \Gamma(\beta) \sin(\beta \pi) / \pi \) due to Euler’s reflection formula; consequently, the expression (5) takes the specific form given in (1).

The proof of Theorem 1 requires some auxiliary facts to be established first.

**Lemma 3** (A version of the Tauberian theorem). Let \( V \) be a nonnegative random variable and let \( A \) and \( \theta \) be positive constants. Then

\[ \mathbb{E}[e^{-aV}] \sim A a^{-\theta} \text{ as } a \to \infty \]

if and only if

\[ \mathbb{P}(V \leq \epsilon) \sim \frac{A}{\Gamma(\theta + 1)} \epsilon^\theta \text{ as } \epsilon \downarrow 0. \]

**Proof.** This follows from Corollary 1a and Theorem 4.3 of Chapter V of [26]. \( \square \)

**Proposition 4.** Let \( E \) be the inverse of a subordinator \( D \) with infinite Lévy measure \( \nu \). Then for all \( T > 0 \) at which \( \nu \) has no mass,

\[ \mathbb{P}(E(T) \leq \epsilon) \sim \nu(T, \infty) \epsilon \text{ as } \epsilon \downarrow 0. \]

This is interpreted as \( \mathbb{P}(E(T) \leq \epsilon) = o(\epsilon) \) if \( \nu(T, \infty) = 0 \).

**Proof.** See Appendix. \( \square \)

**Lemma 5.** Let \( E \) be the inverse of a subordinator \( D \) with Laplace exponent \( \psi \). Then for any fixed \( a > 0 \), the Laplace transform of the function \( t \mapsto \mathbb{E}[e^{-aE(t)}] \) exists and is given by

\[ \mathcal{L}_t[\mathbb{E}[e^{-aE(t)}]](s) = \frac{\psi(s)}{s} (\psi(s) + a)^{-1}, \quad s > 0. \]

**Proof.** See Appendix. \( \square \)

**Remark 6.** Lemma 5 implies that if \( E = E_\beta \) is the inverse of a \( \beta \)-stable subordinator, then for a fixed \( a > 0 \),

\[ \mathcal{L}_t[\mathbb{E}[e^{-aE_\beta(t)}]](s) = \frac{s^{\beta-1}}{s^{\beta} + a}, \quad s > 0. \]

Since the right hand side coincides with the Laplace transform of the function \( t \mapsto E_\beta(-at^\beta) \) (see e.g. [22]), we recover the well-known formula \( \mathbb{E}[e^{-aE_\beta(t)}] = E_\beta(-at^\beta) \), which is used to derive (1) in [21]. In the proof of Theorem 1, we use (8) to guarantee the use of the Fubini Theorem.

**Proof of Theorem 1.** By Theorem 1 of [6] (also see the proof of Theorem 2.1 of [21]), for all \( \epsilon > 0 \),

\[ \mathbb{P} \left( \sup_{0 \leq t \leq 1} |W(t)| \leq \epsilon \right) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp \left( -\frac{(2k-1)^2 \pi^2}{8 \epsilon^2} \right). \]
For a fixed $\epsilon > 0$, since $E$ is a continuous, nondecreasing process independent of $W$, which is self-similar with index $1/2$, a simple conditioning argument along with the use of (9) yields

$$
\mathbb{P} \left( \sup_{0 \leq t \leq T} |W(E(t))| \leq \epsilon \right) = \mathbb{E} \left[ \mathbb{P} \left( \sup_{0 \leq s \leq E(T)} |W(s)| \leq \epsilon \mid E \right) \right] = E \left[ \mathbb{P} \left( \sup_{0 \leq s \leq 1} |W(s)| \leq \frac{\epsilon}{E(T)^{1/2}} \mid E \right) \right] = f_\epsilon(T),
$$

where

$$
f_\epsilon(t) := \frac{4}{\pi} \mathbb{E} \left[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \exp \left( - \frac{(2k-1)^2 \pi^2 E(t)}{8\epsilon^2} \right) \right].
$$

We also introduce the auxiliary function

$$
g_\epsilon(t) := \frac{4}{\pi} \mathbb{E} \left[ \sum_{k=1}^{\infty} \frac{1}{2k-1} \exp \left( - \frac{(2k-1)^2 \pi^2 E(t)}{8\epsilon^2} \right) \right].
$$

Then by the Fubini Theorem for nonnegative integrands (applied to the product measure $\mathbb{P} \times$ counting measure $\times (e^{-st} dt)$ and the formula (8), the Laplace transform of the function $t \mapsto g_\epsilon(t)$ is given by

$$
\mathcal{L}_t[\{g_\epsilon\}](s) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \mathcal{L}_t \left[ \mathbb{E} \left[ \exp \left( - \frac{(2k-1)^2 \pi^2 E(t)}{8\epsilon^2} \right) \right] (s) \right]
= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\psi(s)}{2k-1} \left( \psi(s) + \frac{(2k-1)^2 \pi^2}{8\epsilon^2} \right)^{-1}
\leq \frac{4}{\pi} \frac{\psi(s)}{s} \sum_{k=1}^{\infty} \frac{8\epsilon^2}{(2k-1)^3 \pi^2} < \infty, \quad s > 0.
$$

This particularly implies that $g_\epsilon(t) < \infty$ for (Lebesgue) almost every $t > 0$, but by the monotonicity of the function $g_\epsilon$, we must have $g_\epsilon(t) < \infty$ for all $t > 0$. Therefore, due to the Fubini Theorem, the expectation and summation in the definition of $f_\epsilon(t)$ are interchangeable. Thus,

$$
\frac{1}{\epsilon^2} f_\epsilon(T) = \frac{1}{\epsilon^2} \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \mathbb{E} \left[ \exp \left( - \frac{(2k-1)^2 \pi^2 E(T)}{8\epsilon^2} \right) \right]
= \frac{32}{\pi^3} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} \varphi_T \left( \frac{(2k-1)^2 \pi^2}{8\epsilon^2} \right),
$$

where $\varphi_T(a) := a \mathbb{E}[e^{-aE(T)}]$ for $a > 0$. By (7) along with Lemma 3, it follows that $\varphi_T(a) \to \nu(T, \infty)$ as $a \to \infty$. Therefore, letting $\epsilon \downarrow 0$ in (10) and using the dominated convergence theorem (which is allowed since $\sum_{k=1}^{\infty} 1/(2k-1)^3 < \infty$), we obtain (5). \qed

**Remark 7.** In the proof of (1) provided in [21], where the time change is given by the inverse of a stable subordinator, the asymptotic facts about the Mittag–Leffler function
play a significant role (see equations (2.4) and (2.5) of that paper); they are employed to
 Guarantee the use of the Fubini theorem and the dominated convergence theorem. For
the inverse of a general non-stable subordinator, however, the quantity \( E[e^{-aE(t)}] \) cannot
be represented via a special function like the Mittag–Leffler function. To overcome this
difficulty, the proof provided above employs the explicit form of the Laplace transform
of \( t \mapsto E[e^{-aE(t)}] \) (Lemma 5) as well as a version of the Tauberian theorem (Lemma 3)
along with a general result concerning subordinators (Proposition 4).

Now we turn our attention to examples of time changes \( E \) which are not considered
in [21] but can be handled by Theorem 1. This will entail small ball probabilities for
some of the important time-changed Brownian motions representing anomalous diffu-
sions observed in various fields of science.

Let us introduce the upper incomplete Gamma function \( \Gamma(z,x) \) defined by
\[
\Gamma(z,x) = \int_x^{\infty} e^{-u} u^{z-1} du.
\]
Obviously \( \Gamma(z,0) \) coincides with the Gamma function \( \Gamma(z) \). Note that for \( x > 0 \), the
integral defining \( \Gamma(z,x) \) is finite even when \( z \leq 0 \). In particular, for \( \beta \in (0, 1) \) and \( x > 0 \),
a simple application of integration by parts yields
\[
\Gamma(-\beta, x) = \frac{x^{-\beta} e^{-x} - \Gamma(1-\beta, x)}{\beta}.
\]
\[(11)\]

Example 8 (An inverse Gamma subordinator as a time change). Let \( E \) be the inverse
of a Gamma subordinator \( D \) with parameters \( c, b > 0 \); i.e., the Laplace exponent of \( D \)
in (4) is given by \( \psi(s) = c \log(1 + s/b) \). Then for all \( T > 0 \), using the explicit form of
the Lévy measure (see e.g. Example 1.3.22 of [1]), we obtain
\[
\nu(T, \infty) = \int_T^{\infty} cx^{-\beta}e^{-bx} dx = c \int_0^{\infty} t^{-1} e^{-t} dt = c \Gamma(0, bT).
\]
Hence, (5) with \( \nu(T, \infty) \) replaced by \( c \Gamma(0, bT) \) yields the small ball probability of the
time-changed Brownian motion.

Example 9 (An inverse tempered stable subordinator as a time change). Let \( D \) be a
tempered stable subordinator with stability index \( \beta \in (0, 1) \) and tempering function
\( q(x) \), which implies that the Lévy measure of \( D \) takes the form
\[
\nu(dx) = x^{-\beta-1} q(x) dx \quad \text{with} \quad q(x) = \int_0^{\infty} e^{-\lambda x} \mu(d\lambda),
\]
where \( \mu \) is a finite measure on \((0, \infty)\); see [23] for details. By the Fubini theorem,
\[
\nu(T, \infty) = \int_0^{\infty} \int_T^{\infty} x^{-\beta}e^{-\lambda x} dx \mu(d\lambda) = \int_0^{\infty} x^{\lambda} \Gamma(-\beta, \lambda T) \mu(d\lambda).
\]
Note that \( \Gamma(-\beta, \lambda) \) has an alternative expression given by (11).

Remark 10. Suppose that the tempering function \( q(x) \) in (12) is given by the simple
exponential tilting \( q(x) = \beta e^{-\lambda x}/\Gamma(1-\beta) \), where \( \lambda > 0 \) is a fixed constant. Then the
Laplace exponent in (4) takes the form \( \psi(s) = (s + \lambda)^\beta - \lambda^\beta \), and using (11), one can write the constant \( \nu(1, \infty) \) in (13) as

\[
\nu(T, \infty) = \frac{e^{-\lambda T} T^{-\beta} - \lambda^\beta \Gamma(1 - \beta, \lambda T)}{\Gamma(1 - \beta)}.
\]

Letting \( \lambda \downarrow 0 \) yields \( \nu(T, \infty) = T^{-\beta}/\Gamma(1 - \beta) \), which coincides with the constant found in (6) for the inverse stable subordinator; this makes sense since a tempered stable subordinator with the tempering factor \( \lambda \) set to be 0 is merely a stable subordinator.

3 Extensions

This section establishes small ball probabilities for a large class of time-changed self-similar processes which includes the time-changed Brownian motions discussed in the previous section.

Let \( X = (X(t))_{t \geq 0} \) be a self-similar process starting at 0 and extend \( X \) for \( t < 0 \) using an independent copy; i.e. let \( X' \) be an independent copy of \( X \) and set \( X(t) := X'( -t) \) for \( t < 0 \). We call the so-defined process \( X = (X(t))_{t \in \mathbb{R}} \) a two-sided process.

Let \( E = (E(t))_{t \geq 0} \) be an independent continuous process starting at 0, independent of \( X \), such that \( \mathbb{P}(\sup_{0 \leq t \leq T} |E(t)| \leq \epsilon) \approx \epsilon^\sigma \) as \( \epsilon \downarrow 0 \).

The proof employs an idea presented in the proof of Theorem 1 of [2].

**Theorem 11.** Let \( X \) be a two-sided self-similar process starting at 0 of index \( H > 0 \) such that

\[
-\log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon \right) \approx e^{-\tau} \quad \text{as} \quad \epsilon \downarrow 0
\]

for some \( \tau > 0 \). Let \( E \) be a continuous process starting at 0, independent of \( X \), such that

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} |E(t)| \leq \epsilon \right) \approx \epsilon^\sigma \quad \text{as} \quad \epsilon \downarrow 0
\]

for some \( T > 0 \) and \( \sigma > 0 \). Then

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} |X(E(t))| \leq \epsilon \right) \approx \epsilon^{\sigma/H} \quad \text{as} \quad \epsilon \downarrow 0.
\]

**Proof.** For any \( \theta > 0 \), assumption (15) is equivalent to

\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} |E(t)|^{1/\theta} \leq \epsilon \right) \approx \epsilon^{\theta \sigma} \quad \text{as} \quad \epsilon \downarrow 0.
\]

which, by the weak order analogue of Lemma 3 (see the discussion given in Chapter V of [26]) with \( V = \sup_{0 \leq t \leq T} |E(t)|^{1/\theta} \), implies that

\[
\mathbb{E}[e^{-a \sup_{0 \leq t \leq T} |E(t)|^{1/\theta}}] \approx a^{-\theta \sigma} \quad \text{as} \quad a \to \infty.
\]

This is equivalent to

\[
\mathbb{E}[e^{-a \sup_{0 \leq t \leq s, t \leq T} |E(t) - E(s)|^{1/\theta}}] \approx a^{-\theta \sigma} \quad \text{as} \quad a \to \infty
\]

7
due to the inequalities
\[
\frac{1}{2} \sup_{0 \leq s,t \leq T} |E(t) - E(s)| \leq \sup_{0 \leq t \leq T} |E(t)| = \sup_{0 \leq t \leq T} |E(t) - E(0)| \leq \sup_{0 \leq s,t \leq T} |E(t) - E(s)|.
\]
Now, by assumption (14), there exist constants \(c_1, c_2, \epsilon_0 > 0\) such that for all \(\epsilon \in (0, \epsilon_0]\),
\[
e^{-c_1\epsilon^{-\tau}} \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon \right) \leq e^{-c_2\epsilon^{-\tau}}.
\]
Setting \(c_3 := e^{-c_1\epsilon_0^{-\tau}}\) and \(c_4 := e^{c_2\epsilon_0^{-\tau}}\), we observe that for all \(\epsilon > 0\),
\[
\text{(18)} \quad c_3\epsilon^{-c_1\epsilon^{-\tau}} \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon \right) \leq c_4\epsilon^{-c_2\epsilon^{-\tau}}.
\]
Let \(N := \inf_{0 \leq t \leq T} E(t)\) and \(M := \max_{0 \leq t \leq T} E(t)\). The assumption that \(E(0) = 0\) implies that \(N \leq 0\) and \(M \geq 0\). For \(\epsilon > 0\), using continuity of \(E\), independence between \((X(t))_{t \geq 0}\) and \((X(t))_{t < 0}\), independence between \(X\) and \(E\), and the self-similarity of \(X\), we observe that
\[
\mathbb{P}\left( \sup_{0 \leq t \leq T} |X(E(t))| \leq \epsilon \right) = \mathbb{P}\left( \sup_{N \leq s \leq 0} |X(s)| \leq \epsilon, \sup_{0 \leq s \leq M} |X(s)| \leq \epsilon \right)
\[
= \mathbb{E}\left[ \mathbb{P}\left( \sup_{N \leq s \leq 0} |X(s)| \leq \epsilon | E \right) \mathbb{P}\left( \sup_{0 \leq s \leq M} |X(s)| \leq \epsilon | E \right) \right]
\[
= \mathbb{E}\left[ \mathbb{P}\left( \sup_{0 \leq s \leq 1} |X(s)| \leq \frac{\epsilon}{(-N)^\nu H} | E \right) \mathbb{P}\left( \sup_{0 \leq s \leq 1} |X(s)| \leq \frac{\epsilon}{M^\nu H} | E \right) \right].
\]
By the upper bound in (18) and the elementary inequality \((x + y)^p \leq d(p)(x^p + y^p), x, y \geq 0\), with \(p = \tau H\), it follows that
\[
e^{-\sigma/H} \mathbb{P}\left( \sup_{0 \leq t \leq T} |X(E(t))| \leq \epsilon \right) \leq c_2^2 e^{-\sigma/H} \mathbb{E}\left[ e^{-c_2\epsilon^{-\tau}\{(N)^\nu H + M^\nu H}} \right]
\[
\leq c_2^2 e^{-\sigma/H} \mathbb{E}\left[ e^{-c_2\epsilon^{-\tau}(M-N)^\nu H} \right]
\[
= c_2^2 e^{-\sigma/H} \mathbb{E}\left[ e^{-c_2\epsilon^{-\tau} \sup_{0 \leq s,t \leq T} |E(t)-E(s)|^{\nu H}} \right]
\[
= \frac{c_2^2}{(\bar{c}_2)^{\theta/\nu}} \varphi_{T,\theta,\sigma}(\bar{c}_2\epsilon^{-\tau}),
\]
where \(\bar{c}_2 := c_2/d(\tau H)\), \(\theta := 1/(\tau H)\) and
\[
\varphi_{T,\theta,\sigma}(a) := a^{\theta/\nu} \mathbb{E}[e^{-a \sup_{0 \leq s,t \leq T} |E(t) - E(s)|^{1/\theta}}].
\]
Now, (17) implies that \(\limsup_{a \to \infty} \varphi_{T,\theta,\sigma}(a) < \infty\), and hence, the desired upper bound follows. The lower bound is obtained in a similar manner. \(\square\)

**Remark 12.** 1) The rate of decay of the small ball probability of \(X \circ E\) in (16) does not depend on \(\tau\) appearing in (14); the information of \(\tau\) is reflected on the constant \(\bar{c}_2\) introduced in the proof.

2) Unlike Theorem 4 of [2], a simple modification of the above proof does not lead to a similar result concerning strong deviation orders (i.e. a result with \(\approx\) replaced by \(\sim\)). Indeed, if we assume (instead of (14)) that
\[
- \log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon \right) \sim k\epsilon^{-\tau} \quad \text{as} \quad \epsilon \downarrow 0
\]
for some $k > 0$, then for any $\delta \in (0, 1)$, we can find constants $c_3$ and $c_4$ such that
\[
c_3 e^{-k(1+\delta)\epsilon^{-\tau}} \leq \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X(t)| \leq \epsilon \right) \leq c_4 e^{-k(1-\delta)\epsilon^{-\tau}}
\]
for all $\epsilon > 0$. This leads to
\[
(19) \quad \frac{c_3^2}{(\tilde{c}_1)_{\theta\sigma} \varphi_{T,\theta,\sigma}(\epsilon_1 \epsilon^{-\tau})} \leq e^{-\sigma/\beta} \mathbb{P}\left( \sup_{0 \leq t \leq T} |X(E(t))| \leq \epsilon \right) \leq \frac{c_4^2}{(\tilde{c}_2)_{\theta\sigma} \varphi_{T,\theta,\sigma}(\epsilon_2 \epsilon^{-\tau})},
\]
where $\tilde{c}_1 := k(1+\delta)/d(\tau H)$, $\tilde{c}_2 := k(1-\delta)/d(\tau H)$, and $\varphi_{T,\theta,\sigma}(a)$ is as in the above proof. Now, if we further assume a strong deviation condition for the time change $E$, then $\varphi_{T,\theta,\sigma}(a)$ approaches a constant as $a \to \infty$; however, since the constants $c_3$ and $c_4$ depend on $\delta$ and do not generally approach the same value as $\delta \to 0$, a strong result for the small ball probability for $X \circ E$ does not follow from (19). Note that this issue does not occur in the proof of Theorem 4 of [2] since the logarithmic deviation is discussed in that theorem. On the other hand, in Theorem 1, the explicit formula for the small ball probability of the Brownian motion (valid for each fixed $\epsilon > 0$) allowed us to establish a strong deviation result.

We now consider some specific outer processes $X$ that can be handled within the setting of Theorem 11. Some of the examples below show that Theorem 11 indeed generalizes Theorem 2.3 of [21].

Well-known examples of self-similar processes which have logarithmic small deviation orders include a fractional Brownian motion and a symmetric stable Lévy process. Namely, if $W_H$ denotes a fractional Brownian motion with Hurst index $H \in (0, 1)$, i.e. $W_H$ is a zero mean Gaussian process with covariance function $\mathbb{E}[W_H(s)W_H(t)] = (s^{2H} + t^{2H} - |s-t|^{2H})/2$, then $W_H$ is a self-similar process of index $H$ with small deviation order given by
\[
-\log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |W_H(t)| \leq \epsilon \right) \sim c_H \epsilon^{-1/H} \quad \text{as} \quad \epsilon \downarrow 0,
\]
where $c_H$ is a positive constant depending on $H$. An explicit representation of the small deviation constant $c_H$ is found in [14].

On the other hand, if $S_\alpha$ is a symmetric stable Lévy process of stability index $\alpha \in (0, 2]$, i.e. $S_\alpha$ is a Lévy process with characteristic function $\mathbb{E}[e^{iuS_\alpha(t)}] = e^{-|u|^\alpha}$ for some positive constant $\kappa$ (see e.g. [1, 24]), then $S_\alpha$ is a self-similar process of index $H = 1/\alpha$ and
\[
-\log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |S_\alpha(t)| \leq \epsilon \right) \sim \lambda_\alpha \epsilon^{-\alpha} \quad \text{as} \quad \epsilon \downarrow 0,
\]
where $\lambda_\alpha > 0$ is some constant; see [20] for details.

Both of these examples satisfy condition (14) with $\tau = 1/H$, and they can also be handled by Theorem 2.3 of [21]. However, self-similar processes with index $H$ with $\tau \neq 1/H$ also exist as the following examples show. These processes are outside the scope of Theorem 2.3 of [21], but Theorem 11 still applies.

**Example 13** (An iterated fractional Brownian motion as an outer process). An $n$-iterated two-sided fractional Brownian motion is the process $X^{(n)}$ defined by the iteration
\[
X^{(1)}(t) := W_{H_1}(t), \quad X^{(j)}(t) := W_{H_j}(X^{(j-1)}(t)), \quad j = 2, \ldots, n,
\]
where $W_{H_1}, \ldots, W_{H_n}$ are independent two-sided fractional Brownian motions with Hurst indices $H_1, \ldots, H_n$ and small deviation constants $c_{H_1}, \ldots, c_{H_n}$, respectively. The process $X^{(n)}$ is self-similar with index $H(n) := \prod_{j=1}^n H_j$. Moreover, it is established in Section 4.2 of [2] that

$$- \log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |X^{(n)}(t)| \leq \epsilon \right) \sim c_n \epsilon^{-\tau_n} \quad \text{as} \quad \epsilon \downarrow 0,$$

where $\tau_n := 1/\sum_{i=1}^n \prod_{j=i}^n H_j$ and $c_n$ is defined iteratively by

$$c_1 := c_{H_1}, \quad c_j := (1 + \tau_{j-1}) \left[ c_{j-1} \frac{2c_{H_j}}{\tau_{j-1}} \right]^{\tau_{j-1}/(1+\tau_{j-1})}, \quad j = 2, \ldots, n.$$

Hence, condition (14) holds with $\tau = \tau_n \neq 1/H(n)$.

**Example 14** (An iterated strictly stable Lévy process as an outer process). Let $S_{\alpha_1}$ be a two-sided strictly stable Lévy process of index $\alpha_1 \in (0, 2]$. Let $S_{\alpha_2}$ be an independent strictly stable Lévy process of index $\alpha_2 \in (0, 2]$ which is not a subordinator. We call the process $X := S_{\alpha_1} \circ S_{\alpha_2}$ an *iterated strictly stable Lévy process*. It is easy to see that $X$ is self-similar with index $H = 1/(\alpha_1\alpha_2)$. Moreover, it is shown in Section 5 of [2] that

$$- \log \mathbb{P}\left( \sup_{0 \leq t \leq 1} |S_{\alpha_1}(S_{\alpha_2}(t))| \leq \epsilon \right) \sim \epsilon^{-\alpha_1\alpha_2/(1+\alpha_2)} \quad \text{as} \quad \epsilon \downarrow 0.$$

Hence, condition (14) holds with $\tau = \alpha_1\alpha_2/(1 + \alpha_2) \neq \alpha_1\alpha_2 = 1/H$.

Theorem 11 also allows us to consider time changes which are given by mixtures of independent inverse subordinators.

**Example 15** (A mixture of independent inverse subordinators as a time change). For each $j = 1, \ldots, m$, let $E_j$ be the inverse of a subordinator $D_j$ with infinite Lévy measure $\nu_j$ having no atom at $T > 0$ so that (7) holds for each $E_j$. Assume that $E_j$’s are independent and let $E := \sum_{j=1}^m c_j E_j$, where $c_j$’s are positive constants. Then it follows from Lemma 3 that

$$\mathbb{E}[e^{-aE(T)}] = \prod_{j=1}^m \mathbb{E}[e^{-ac_j E_j(T)}] \sim \left( \prod_{j=1}^m \frac{\nu_j(T, \infty)}{c_j} \right) a^{-m} \quad \text{as} \quad a \to \infty,$$

which, again by Lemma 3, is equivalent to

$$\mathbb{P}(E(T) \leq \epsilon) \sim \frac{1}{m!} \left( \prod_{j=1}^m \frac{\nu_j(T, \infty)}{c_j} \right) \epsilon^m \quad \text{as} \quad \epsilon \downarrow 0.$$

Hence, Theorem 11 applies to the time change $E$ with $\sigma = m$. Moreover, with this specific time change, it is possible to generalize Theorems 1 to obtain the small ball probability of the time-changed Brownian motion with the exact small deviation constant specified. The proof simply combines the ideas used in the proofs of Theorems 1 and 11 and hence is omitted.

Note that even if each $D_j$ is a stable subordinator, the time change $E$ defined in this example does not coincide with the inverse of a mixture of independent stable subordinators appearing in [9, 10, 18]. Indeed, in those papers, $E$ is defined to be the inverse of $D := \sum_{j=1}^m c_j D_j$, where $D_j$’s are independent stable subordinators, which implies that it has the small ball probability with $\sigma = 1$ due to Proposition 4.
Appendix

Proof of Proposition 4. By the inverse relationship between $E$ and $D$, it follows that $\mathbb{P}(E(\tau) \leq \varepsilon) = \mathbb{P}(D(\varepsilon) \geq \tau)$. Hence, we only need to verify that

$$\lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}(D(\varepsilon) \geq \tau)}{\varepsilon} = \nu(\tau, \infty) \quad \text{provided that} \quad \nu(\{\tau\}) = 0.$$  

Although this may be a well-known fact, for the sake of completeness of the discussion as well as clarification of why the assumption that $\nu(\{\tau\}) = 0$ is needed, we provide a proof below. Note that a similar argument appears in [11].

For a fixed real sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$, let

$$\tilde{\nu}_n(x) := \frac{\mathbb{P}(D(\varepsilon_n) \geq x)}{\varepsilon_n} \quad \text{and} \quad \tilde{\nu}(x) := \nu(x, \infty)$$

for $x > 0$. Then the proof of Theorem 1.2(i) of [4] shows that the sequence of absolutely continuous measures $\tilde{\nu}_n(x)dx$ converges vaguely to $b\delta_0(dx) + \tilde{\nu}(x)dx$, where $b \geq 0$ is the drift parameter appearing in (4) and $\delta_0$ is the Dirac measure with mass at 0. This particularly implies that for any $0 < c < d$,

$$\lim_{n \to \infty} \int_c^d \tilde{\nu}_n(x)dx = \int_c^d \tilde{\nu}(x)dx.$$

Now, assume that $\tilde{\nu}_n(T)$ does not converge to $\tilde{\nu}(T)$. Then there exist a constant $\eta > 0$ and a subsequence $\{n_k\}$ such that $|\tilde{\nu}_{n_k}(T) - \tilde{\nu}(T)| \geq \eta$ for all $k$. This implies that there exists a further subsequence $\{n_{km}\}$ such that (i) $\tilde{\nu}_{n_{km}}(T) \geq \tilde{\nu}(T) + \eta$ for all $m$ or (ii) $\tilde{\nu}_{n_{km}}(T) \leq \tilde{\nu}(T) - \eta$ for all $m$. If (i) holds, then since each $\tilde{\nu}_{n_{km}}(\cdot)$ is a decreasing function, for any $\delta \in (0, T)$,

$$\int_{T-\delta}^T \tilde{\nu}(x)dx = \lim_{m \to \infty} \int_{T-\delta}^T \tilde{\nu}_{n_{km}}(x)dx \geq \lim_{m \to \infty} \delta \tilde{\nu}_{n_{km}}(T) \geq \delta (\tilde{\nu}(T) + \eta).$$

On the other hand, the assumption that $\nu(\{\tau\}) = 0$ implies that $\tilde{\nu}(\cdot)$ is continuous at $x = T$; hence, there exists a constant $\delta \in (0, T)$ such that $\tilde{\nu}(x) - \tilde{\nu}(T) \leq \eta/2$ for all $x$ with $T - \delta \leq x \leq T$. Thus, we have

$$\int_{T-\delta}^T \tilde{\nu}(x)dx \leq \delta(\tilde{\nu}(T) + \eta/2),$$

which contradicts the estimate in (20). A similar contradiction occurs if (ii) holds. Therefore, $\tilde{\nu}_n(T)$ must converge to $\tilde{\nu}(T)$, which completes the proof. \hfill \Box

Proof of Lemma 5. For a fixed $x > 0$, since $E$ is the inverse of $D$,

$$\mathbb{P}(E(t) \leq x) = \mathbb{P}(D(x) \geq t) = 1 - \mathbb{P}(D(x) < t), \quad t > 0.$$  

Taking the Laplace transform with respect to $\tau$ on both sides, we obtain

$$\mathcal{L}_t[\mathbb{P}(E(t) \leq x)](s) = \frac{1}{s} \mathcal{L}_t[\mathbb{P}(D(x) \geq \tau)](s) = \frac{1 - \mathbb{E}[e^{-sD(x)}]}{s} = \frac{1 - e^{-x\psi(s)}}{s}, \quad s > 0,$$

where $\psi(s)$ is the Laplace transform of $\nu$. Therefore,

$$\mathcal{L}_t[\mathbb{P}(E(t) \geq x)](s) = \mathcal{L}_t[\mathbb{P}(D(x) < \tau)](s) = \frac{1}{s} \mathcal{L}_t[\mathbb{P}(D(x) < \tau)](s) = \frac{1 - e^{-x\psi(s)}}{s}, \quad s > 0.$$
where $\mathcal{L}_t[f(t)]$ and $\mathcal{L}_t[\mu(dt)]$ denote the Laplace transforms of a function $f(t)$ and a measure $\mu(dt)$, respectively. The right hand side of the above identity being differentiable with respect to $x$, so is the left hand side, and

$$\mathcal{L}_t[\mathbb{P}(E(t) \in dx)](s) = \frac{\psi(s)}{s} e^{-x\psi(s)} dx, \ s > 0.$$ 

Hence, we obtain by the Fubini theorem (for nonnegative integrands) that

$$\mathcal{L}_t[\mathbb{E}[e^{-aE(t)}]](s) = \int_0^\infty \frac{\psi(s)}{s} e^{-x(\psi(s)+a)} dx = \frac{\psi(s)}{s}(\psi(s)+a)^{-1}, \ s > 0,$$

which completes the proof.

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