APPLICATIONS OF TRANSCENDENTAL NUMBER THEORY TO DECISION PROBLEMS FOR HYPERGEOMETRIC SEQUENCES

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Abstract. A rational-valued sequence is hypergeometric if it satisfies a first-order linear recurrence relation with polynomial coefficients. In this note we discuss two decision problems, the membership and threshold problems, for hypergeometric sequences. The former problem asks whether a chosen target is in the orbit of a given sequence, whilst the latter asks whether every term in a sequence is bounded from below by a given value.

We establish decidability results for restricted variants of these two decision problems with an approach via transcendental number theory. Our contributions include the following: the membership and threshold problems are both decidable for the class of rational-valued hypergeometric sequences with Gaussian integer parameters.

1. Introduction

1.1. Motivation. The membership and threshold problems for recurrence sequences pertain to fundamental aspects of automated verification, the analysis of algorithms, and computational modelling. Indeed, they frequently appear in discussions on topics such as weighted automated, loop termination and Markov processes. The membership problem asks whether a chosen target is in the orbit of (commonly, is reached by) a given sequence. Meanwhile, the threshold problem asks whether every term in a real-valued sequence is bounded from below by a given value (commonly, the threshold).

Arguably, the most famous such membership problem is the Skolem Problem for linear recurrence sequences with constant coefficients. Skolem asks whether a given recurrence sequence vanishes at some point; that is to say, is zero a term in the sequence? The decidability of Skolem is a long-standing open problem; in fact, decidability of Skolem is only known for linear recurrence sequences of order at most four [14, 24].

In this paper we restrict our focus to the membership and threshold problems for hypergeometric sequences (the Hypergeometric Membership and Hypergeometric Threshold Problems, respectively). Here a hypergeometric sequence is a rational-valued first-order linear recurrence sequence with polynomial coefficients; that is to say, a sequence \( \langle u_n \rangle \) of rationals that satisfies a relation of the form

\[
 p(n)u_{n+1} = cq(n)u_n
\]

where \( p, q \in \mathbb{Q}[x] \) and \( c \in \mathbb{Q} \).

For the avoidance of doubt, the input tuple \( (\langle u_n \rangle, t) \) for both the Hypergeometric Membership and Hypergeometric Threshold Problems is a
hypergeometric sequence \( (u_n) \) and a non-zero rational \( t \). The former problem is to determine whether there is an \( n \in \mathbb{N}_0 \) such that \( u_n = t \). The latter problem is to determine whether \( u_n \geq t \) for each \( n \in \mathbb{N}_0 \). Both of the decision problems are open.

1.2. Contributions. In this paper we present decidability results for classes of hypergeometric sequences. These classes place restrictions on the polynomial coefficients \( p, q \in \mathbb{Q}[n] \) in (1). For a hypergeometric sequence that satisfies (1), we call the roots of the associated polynomial coefficients the parameters of the sequence.

Our main contributions are as follows:

1. The membership and threshold problems for the class of hypergeometric sequences with Gaussian integer parameters are both decidable (Proposition 3.1). We also establish decidability for these decision problems when the sequence parameters are drawn from the integers of any imaginary quadratic number field (Theorem 3.4).

2. For a class \( \mathcal{C} \) of hypergeometric sequences whose parameters obey certain algebraic properties, the membership and threshold problems are decidable if Schanuel’s conjecture is true (Theorem 4.5).

Recall that Schanuel’s conjecture is a grand unifying prediction in transcendental number theory that subsumes several principal results (cf. [11, 3, 26]). We delay a formal statement of the conjecture (to the Preliminaries) and definition of the class \( \mathcal{C} \) (to Section 4).

1.3. Related works. Given a non-degenerate hypergeometric sequence \( (u_n) \) (definition in the Preliminaries), we shall observe (Proposition 2.3) that there is a single difficult instance of both the membership and threshold problems. That is to say, for all but at most one value of \( t \), the instance \(( (u_n) , t )\) of the membership (resp. threshold) problem can be decided by elementary means. The problem instances when \( t \) is the critical value are both open. The object of this paper is to show that for restricted classes of hypergeometric sequences we can employ techniques from transcendental number theory to settle restricted variants of the hypergeometric membership and threshold problems.

Previous works by Kenison et al. [8] and Nosan et al. [17] established respective reductions from the membership and threshold problems for hypergeometric sequences to determining equality between gamma products, the Gamma Product Equality Problem (GPEP). Here a gamma product is an expression of the form \( \Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_\ell) \) where \( \Gamma \) denotes the gamma function, \( \Gamma(z) = \int_0^\infty x^{z-1} e^{-x} \, dx \) for \( z \in \mathbb{C} \) with \( \text{Re}(z) > 0 \). It is possible to analytically extend the domain of \( \Gamma \) to the whole complex plane minus the non-positive integers where the function has simple poles.

In [8], Kenison et al. establish decidability of the Hypergeometric Threshold Problem subject to a strengthening of the Period Conjecture (see [10, Conjecture 1]) posed by Kontsevich and Zagier. Previous works [8, 17] also consider the restriction of the Hypergeometric Membership Problem to the class of sequences with rational parameters. Deciding GPEP under this restriction reduces to determining all multiplicative relations of the Gamma function for given inputs (for full details see [17, Theorem 5]). The latter
problem is the subject of an older conjecture due to Rohrlich [11, 26]. The Rohrlich Conjecture predicts that any multiplicative relation of the form $\prod_{j=1}^{n}(2\pi)^{-1/2}\Gamma(\alpha_j) = \theta$ for appropriate $\alpha_j \in \mathbb{Q}$ and real-algebraic $\theta$ can be derived from the standard functional identities for the gamma function (the translation, reflection, and multiplicative properties discussed in the Preliminaries).

1.4. Comparisons to related works. Suppose that $(u_n)_n$ is a hypergeometric sequence that satisfies recurrence (1). Without loss of generality, we associate a unique pair of polynomial coefficients $p, q \in \mathbb{Q}[X]$ to $(u_n)_n$ by assuming that both polynomials are of lowest degree and that $q$ is monic. We call $r(n) := cq(n)/p(n)$ the shift quotient of the sequence $(u_n)_n$.

The Hypergeometric Membership Problem. In [17], Nosan et al. prove that the Hypergeometric Membership Problem with rational parameters is decidable; their $p$-adic argument considers the prime divisors of the terms $u_n$ and the target $t$. Briefly, their strategy (for non-degenerate cases) shows that for sufficiently large $n \in \mathbb{N}$, there is a prime divisor $p$ of $u_n$ that cannot also be a prime divisor of $t$. This is sufficient to reduce such instances of the membership problem to an exhaustive search for $t$ amongst the initial terms of the hypergeometric sequence.

On the one hand, the work by Nosan et al. can settle decidability of the membership problem for examples of the form

$$u_{n+1} = \frac{(n+1)(n+7/9)(n+5/9)}{(n+11/9)(n+8/9)(n+2/9)}u_n,$$

which we cannot handle herein. A heuristic argument as to why such examples are beyond our approach is given in Section 5. On the other hand, we establish decidability results for certain classes of hypergeometric sequences whose parameters are not necessarily rationals such as

$$v_{n+1} = \frac{n^4 - 4n^3 + 9n^2 - 10n + 5}{\Phi_{12}(n-4)}v_n$$

and

$$w_{n+1} = \frac{n^2 - 4n + 5}{n^2 - 4n + 13}w_n$$

(here $\Phi_{12}(n)$ is the 12th cyclotomic polynomial).

During the preparation of this note, there has been rapid progress on the decidability the Hypergeometric Membership Problem with algebraic parameters (building on the $p$-adic methods of [17]). Work in preparation [9], establishes decidability results for the Hypergeometric Membership Problem in two directions. First, those authors settle decidability when the polynomial coefficients $p$ and $q$ of (1) have distinct splitting fields. This result directs us towards difficult cases (such as those discussed in Proposition 3.1 and Theorem 3.4) where $p$ and $q$ have the same splitting field. Second, those authors give an alternative proof for the decidability of the Hypergeometric Membership Problem with Gaussian integer parameters (Proposition 3.1 below) as well as results for related parameter families.

The Hypergeometric Threshold Problem. One of the main strengths of the approach in this paper is the consideration of the Hypergeometric Threshold Problem; by comparison, the $p$-adic techniques employed by Nosan et al. [17] are only applicable to the Hypergeometric Membership Problem. Note,
for example, that are our approach establishes decidability results for the threshold problem to sequences such as \( \langle v_n \rangle_n \) and \( \langle w_n \rangle_n \) above.

In [8], Kenison et al. consider the related Hypergeometric Inequality Problem as follows: given two hypergeometric sequences \( \langle s_n \rangle_n \) and \( \langle t_n \rangle_n \), determine whether \( s_n \geq t_n \) for each \( n \in \mathbb{N}_0 \). Those authors gave a Turing reduction from the Hypergeometric Inequality Problem to the Hypergeometric Threshold Problem. As a consequence of Proposition 3.1 herein, the decidability of the Hypergeometric Inequality Problem with Gaussian integer parameters is decidable (the aforementioned reduction reduces the problem to deciding the Hypergeometric Threshold Problem with Gaussian integer parameters).

1.5. Background. There is a large corpus of research into infinite product identities related to the gamma function (see, for example, [5, 1, 7]). This is particularly relevant to our approach, via transcendence theory, to resolving equality testing problems. An illustrative example is given by Borwein et al. [4, pages 4–6]. Those authors give the following identity,

\[
\prod_{k=2}^{\infty} \frac{k^5 - 1}{k^5 + 1} = \frac{2 \cdot \Gamma(-\omega_{10}) \Gamma(-\omega_{10}^2) \Gamma(-\omega_{10}^3) \Gamma(\omega_{10}^4)}{5 \cdot \Gamma(\omega_{10}) \Gamma(-\omega_{10}^2) \Gamma(\omega_{10}^3) \Gamma(-\omega_{10}^4)}
\]

where \( \omega_{10} = e^{2\pi i/10} \). It is unknown whether the quotient on the right-hand side of the preceding line is algebraic.

Identities for (short) gamma products and quotients are also the subject of much research interest (a non-exhaustive list includes [22, 25, 13, 16, 5, 18]). The identity

\[
\frac{\Gamma(1/14) \Gamma(9/14) \Gamma(11/14)}{\Gamma(3/14) \Gamma(5/14) \Gamma(13/14)} = 2,
\]

is an application of the functional properties of the gamma function (see Subsection 2.2), is taken from [5]. This example illustrates that even when exact values of the gamma function at rational points are not known (indeed it is not known even whether \( \Gamma(1/14) \) is algebraic or transcendental), there is a wealth of interest in the multiplicative relations between said values.

1.6. Structure. This paper is structured as follows. In the next section we gather together relevant preliminary material. In Section 3, we establish decidability of the Hypergeometric Membership and Threshold Problems with Gaussian integer parameters (Proposition 3.1). We then push the techniques further and establish decidability of the Hypergeometric Membership and Threshold Problems with parameters drawn from the integers of any imaginary quadratic number field (Theorem 3.4). In Section 4, we first define a class of hypergeometric sequence \( \mathcal{C} \). Then we show that subject to the truth of Schanuel’s conjecture, the Hypergeometric Membership and Threshold Problems for sequences in \( \mathcal{C} \) are both decidable (Theorem 4.5). Finally, we summarise our work and make suggestions for future avenues of research in the conclusion (Section 5).

2. Preliminaries

2.1. Shift quotients and infinite products. From the definition of a hypergeometric sequence, it is clear that we have a product formulation of the
nth term \( u_n = u_0 \cdot \prod_{k=0}^n r(k) \). Thus the membership and threshold problems reduce to analysing the sequence of partial products \( \prod_{k=0}^n r(k) \). In the sequel, we shall assume that \( r(n) \neq 0 \) for each \( n \in \mathbb{N} \). Let us sketch a high-level overview of our approach. Given an instance \((u_n)_n, t)\) of the Membership (resp. Threshold) Problem we want to show there is an effectively computable upper bound \( N \) that depends only on the sequence and target (resp. threshold) such that \( u_n = t \) only if \( n < N \). Thus we reduce the Membership (resp. Threshold) Problem to an exhaustive search that asks whether \( t \in \{u_0, u_1, \ldots, u_{N-1}\} \).

The following straightforward lemma appears in previous works [8, 17].

**Lemma 2.1.** Consider the class of hypergeometric sequences \((u_n)_n\) shift quotients \( r(n) \) either diverge to \( \pm \infty \) or converge to a limit \( \ell \) with \( |\ell| \neq 1 \). For this class of sequences, the Hypergeometric Membership and Threshold Problems are both decidable.

**Proof.** Let us first assume that \( r(n) \) diverges to \( \pm \infty \). In this case it is easily seen that, for each \( t \in \mathbb{Q} \), there exists a computable \( N \in \mathbb{N} \) such that if \( n \geq N \) then \( |u_n| = |u_0 \cdot \prod_{k=0}^n r(k)| > |t| \). Thus the Membership Problem in such instances reduces to an exhaustive search that asks whether \( t \in \{u_0, u_1, \ldots, u_{N-1}\} \). Mutatis mutandis, decidability is similarly established for instances of the Membership Problem where \( r(n) \) converges to a limit \( \ell \) with \( |\ell| \neq 1 \).

Similar elementary reasoning proves that the Hypergeometric Threshold Problem for the aforementioned class of sequences is also decidable. \( \square \)

Thus, for both of the decision problems, a single challenging case remains: the case where the shift quotient \( r(n) \in \mathbb{Q}(n) \) of the sequence converges to \( \pm 1 \) as \( n \to \infty \). We say an infinite product \( \prod_{k=0}^{\infty} r(k) \) converges if the sequence of partial products converges to a finite non-zero limit (otherwise the product is said to diverge). Recall the following classical theorem [27, 5].

**Theorem 2.2.** Consider the rational function

\[
r(k) := \frac{c(k + \alpha_1) \cdots (k + \alpha_m)}{(k + \beta_1) \cdots (k + \beta_{m'})}
\]

where we suppose that each \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_{m'} \) is a complex number that is neither zero nor a negative integer. The infinite product \( \prod_{k=0}^{\infty} r(k) \) converges to a finite non-zero limit only if \( c = 1, m = m' \), and \( \sum_j \alpha_j = \sum_j \beta_j \). Further, the value of the limit is given by

\[
\prod_{k=0}^{\infty} r(k) = \prod_{j=1}^{m} \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.
\]

Following the assumptions in Theorem 2.2, it is useful to introduce the following terminology for shift quotients. We call a shift quotient \( r(k) \) (as above) harmonious if \( r(k) \) satisfies the assumptions \( c = 1, m = m' \), and \( \sum_j \alpha_j = \sum_j \beta_j \). From Theorem 2.2 it is immediately apparent that a hypergeometric sequence \((u_n)_n\) with shift quotient \( r \) converges to a finite non-zero limit only if \( r \) is harmonious.
Nosan et al. [17] prove a specialisation of the following result. We include a proof (which is all but identical to the proof of Proposition 2 in [17]) here for the sake of completeness.

**Proposition 2.3.** Let \((u_n)_n\) be a hypergeometric sequence whose shift quotient is given by a ratio of two polynomials with real coefficients. For such sequences, the Hypergeometric Membership and Threshold Problems are both Turing-reducible to the following decision problem. Given \(d \in \mathbb{N}, \alpha_1, \ldots, \alpha_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}\) (the roots of some \(P(x) \in \mathbb{R}[x]\)), and \(\beta_1, \ldots, \beta_d \in \mathbb{C} \setminus \mathbb{Z}_{<0}\) (the roots of some \(Q(x) \in \mathbb{R}[x]\)), determine whether

\[
\frac{\Gamma(\beta_1) \cdots \Gamma(\beta_d)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_d)} = t
\]

for \(t \in \mathbb{Q} \setminus 0\).

**Proof.** From Lemma 2.1, we need only consider cases where the associated shift quotient \(r(k)\) converges to ±1 and, by Theorem 2.2, we can assume without loss of generality that \(r(k)\) is harmonious. We treat the case that the sequence of partial products \((\prod_{k=0}^n r(k))_n\) is eventually strictly increasing. The case where the sequence of partial products is eventually strictly decreasing follows *mutatis mutandis*.

Consider an instance \(((u_n)_n, t)\) of the Hypergeometric Membership (Threshold) Problem with \(r(k)\) as above. Let \(\tau := \prod_{k=0}^n r(k)\). Assume that the sequence of partial products \((\prod_{k=0}^n r(k))_n\) is eventually strictly increasing. Then there exists a computable \(N \in \mathbb{N}\) such that \(u_n < \tau\) for each \(n \geq N\). There are two subcases to consider. First, if \(\tau < t\) then it is clear that \(u_n < t\) for each \(n \geq N\) and so decidability in this instance reduces to an exhaustive search that asks whether \(t \in \{u_0, u_1, \ldots, u_{N-1}\}\). Second, if \(\tau > t\) then there exists an \(N_1 \in \mathbb{N}\) such that \(u_n > t\) for each \(n \geq N_1\). Thus, again, decidability in this instance reduces to an exhaustive search.

All that remains is to decide whether \(\tau \leq t\). It is clear that, by computing \(\tau\) to sufficient precision, the problem of determining whether \(\tau < t\) or \(\tau > t\) is recursively enumerable. Thus we need only test whether the equality \(\tau = t\) holds. By Theorem 2.2, we know that \(\tau = \prod_{j=1}^m \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}\), from which we deduce the desired result.

For the sake of brevity, we omit the argument for reduction from the Hypergeometric Threshold Problem. The argument is near identical to the reasoning displayed in the reduction from the Hypergeometric Membership Problem. \(\square\)

### 2.2. The gamma function

The gamma function has been studied by luminaries such as Euler, Gauss, Legendre, and Weierstrass (amongst many others). We briefly recall standard results for the gamma function. Further details and historical accounts are given in a number of sources (cf. [27, 2]).

The standard relations for the gamma function give the functional identities: the recurrence or translation property \(\Gamma(z + 1) = z\Gamma(z)\) for \(z \notin \mathbb{Z};\) the reflection property \(\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)\) for \(z \notin \mathbb{Z};\) and the Gauss multiplication formula

\[
\prod_{k=0}^{n-1} \Gamma(z + k/n) = (2\pi)^{(n-1)/2} n^{1/2 - nz} \Gamma(nz)
\]

where \(\Gamma(z)\) is the gamma function of \(z\).
is valid so long as none of the \( n \) functions encounter a pole. (The case \( n = 2 \) is more commonly known as Legendre’s duplication formula.)

The next lemma is well-known. We include a proof for the sake of completeness.

**Lemma 2.4.** Suppose that \( \rho, w \in \mathbb{C} \) are complex numbers such that both \( \rho + w, \rho - w \) lie in the domain of the gamma function and \( w \) is neither an integer nor a half-integer. Up to multiplication by an integer, we have the following equalities

\[
\Gamma(\rho + w)\Gamma(\rho - w) = \begin{cases} 
\frac{2\pi e^{\pi i w}}{w(1 - e^{2\pi i w})} & \text{if } \rho \text{ is an integer, or} \\
\frac{2\pi e^{\pi i w}}{e^{2\pi i w} + 1} & \text{if } \rho \text{ is a half-integer.}
\end{cases}
\]

**Proof.** Up to multiplication by an integer, we have the following equalities

\[
\Gamma(\rho + w)\Gamma(\rho - w) = \begin{cases} 
\Gamma(w)\Gamma(-w) & \text{if } \rho \text{ is an integer, or} \\
\Gamma(1/2 + w)\Gamma(1/2 - w) & \text{if } \rho \text{ is a half-integer.}
\end{cases}
\]

Consider the first of the two cases above. The reflection and recurrence formula leads to

\[
\Gamma(w)\Gamma(-w) = \frac{\Gamma(w)\Gamma(1 - w)}{-w} = \frac{\pi}{-w\sin(\pi w)} = -\frac{2\pi i}{w(e^{\pi i w} - e^{-\pi i w})}.
\]

For the second case, we employ the cosine variant of Euler’s reflection formula to obtain

\[
\Gamma(1/2 + w)\Gamma(1/2 - w) = \frac{\pi}{\cos(\pi w)} = \frac{2\pi i}{e^{\pi i w} + e^{-\pi i w}}.
\]

The equalities in the statement of the lemma quickly follow. \( \square \)

We will make reference to a straightforward corollary of **Lemma 2.4** in the work that follows.

**Corollary 2.5.** Let \( \rho \) be an integer or half-integer. Subject to our previous assumptions and up to multiplication by an integer, we have the following equalities

\[
\Gamma(\rho + bi)\Gamma(\rho - bi) = \begin{cases} 
\frac{2\pi i e^{\pi b}}{b(1 - e^{2\pi b})} & \text{if } \rho \text{ is an integer, or} \\
\frac{2\pi i e^{\pi b}}{1 + e^{2\pi b}} & \text{if } \rho \text{ is a half-integer.}
\end{cases}
\]

2.3. **Number fields.** We recall standard results for quadratic fields below (cf. [23, Chapter 3]). A number field \( K \) is quadratic if \([K : \mathbb{Q}] = 2\). A field \( K \) is quadratic if and only if there is a square-free integer \( d \) such that \( K = \mathbb{Q}(\sqrt{d}) \). Further, a quadratic field \( \mathbb{Q}(\sqrt{d}) \) is imaginary if \( d < 0 \).

**Theorem 2.6.** Suppose that \( d \in \mathbb{Z} \) is square-free. Then the algebraic integers of \( \mathbb{Q}(\sqrt{d}) \) are given by \( \mathbb{Z}(\sqrt{d}) \) if \( d \not\equiv 1 \pmod{4} \) or \( \mathbb{Z}(1/2 + \sqrt{d}/2) \) if \( d \equiv 1 \pmod{4} \).

We include the following standard lemma for ease of reference.

**Lemma 2.7.** Let \( L/\mathbb{Q} \) be a finite Galois extension and suppose that \( \tilde{P} \in L(X_1, \ldots, X_m) \) is a polynomial such that \( \tilde{P}(s_1, \ldots, s_m) = 0 \) with \( (s_1, \ldots, s_m) \in \mathbb{C}^m \). Then there is a polynomial \( Q \in \mathbb{Q}(X_1, \ldots, X_m) \) such that \( Q(s_1, \ldots, s_m) = 0 \).
Proof. Let \( \hat{P} = \sum_{(t_1, \ldots, t_m)} c(t_1, \ldots, t_m) X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m} \) and for each \( \sigma \in G \) (the Galois group of \( L/Q \)) let \( \sigma(\hat{P}) = \sum_{(t_1, \ldots, t_m)} \sigma(c(t_1, \ldots, t_m)) X_1^{t_1} X_2^{t_2} \cdots X_m^{t_m} \).

Let \( Q = N_{L/Q}(\hat{P}) := \prod_{\sigma \in G} \sigma(\hat{P}) \). It is clear that each of the coefficients of the polynomial \( Q \) is rational since the coefficients are invariant under the action of the group \( G \). Further,

\[
Q(s_1, \ldots, s_m) = \hat{P}(s_1, \ldots, s_m) \prod_{\sigma \in G \setminus \{e\}} \sigma(\hat{P})(s_1, \ldots, s_m) = 0,
\]

as desired. \( \square \)

2.4. **transcendental number theory.** The transcendence degree of a field extension is a measure of the size of the extension. In fact, for finitely generated extensions of \( L/Q \) (such those that we consider), the transcendence degree indicates the largest cardinality of an algebraically independent subset of \( L \) over \( Q \). For a field extension \( L/Q \), a subset \( \{\xi_1, \ldots, \xi_n\} \subset L \) is algebraically independent over \( Q \) if for each polynomial \( P(X_1, \ldots, X_n) \in Q[X_1, \ldots, X_n] \) we have that \( P(\xi_1, \ldots, \xi_n) = 0 \) only if \( P \) is identically zero.

It is useful to recall the Gelfond–Schneider Theorem that establishes the transcendentality of \( \alpha^\beta \) for algebraic numbers \( \alpha \) and \( \beta \) except for the cases where \( \alpha = 0, 1 \) or \( \beta \) is rational.

Schanuel’s conjecture is a unifying prediction in transcendental number theory. If Schanuel’s conjecture is true, then it generalises several of the principal results in transcendental number theory such as: the Gelfond–Schneider Theorem, the Lindemann–Weierstrass Theorem, and Baker’s theorem (cf. [11, 3, 26]). The conjecture predicts that given \( \xi_1, \ldots, \xi_n \) rationally linearly independent complex numbers, then from the set of elements \( \xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n} \) one can pick a subset of size at least \( n \) that is algebraically independent over \( Q \).

**Conjecture 2.8** (Schanuel). Suppose that \( \xi_1, \ldots, \xi_n \) are complex numbers that are linearly independent over the rationals \( Q \). Then the transcendence degree of the field extension \( Q(\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}) \) over \( Q \) is at least \( n \).

3. **Decidability results for hypergeometric sequences with quadratic integer parameters**

**Proposition 3.1.** The Hypergeometric Membership and Threshold Problems with Gaussian integer parameters are both decidable.

**Proof.** By Theorem 2.2 and Proposition 2.3, we need only consider hypergeometric sequences with harmonic shift quotients. Given such a shift quotient \( r \), its roots and poles are either rational integers or appear as irrational complex conjugate pairs of the form \( a_m \pm b_m i \in \mathbb{Z}[i] \). By Corollary 2.5, each instance \((\langle \mu_n \rangle)_n, t) of the Hypergeometric Membership (resp. Threshold) Problem with Gaussian integer parameters reduces to testing an equality of the form

\[
\theta \pi^\ell \prod_{m} (\sinh(b_m \pi))^\epsilon_m = t.
\]

Here \( \theta \in Q \) is non-zero, \( \ell \in \mathbb{Z} \), each pair \( \Gamma(a_m + b_m i)\Gamma(a_m - b_m i) \) from Proposition 2.3 contributes a term \((\sinh(b_m \pi))^\epsilon_m \) in the finite product, and
\( e_m = \pm 1 \). We break the remainder of the proof into several subcases. Without loss of generality, we can assume that not all the roots and poles of \( r \) are rational integers, for otherwise testing (4) reduces to testing equality between two rational numbers.

We continue under the assumption that not all the roots and poles of \( r \) are rational integers. Let us examine the product \( \prod_m (\sinh(b_m \pi))^\varepsilon_m \). Up to multiplication by a rational, the following equalities hold

\[
\prod_m (\sinh(b_m \pi))^\varepsilon_m = \prod_m (e^{b_m \pi} - e^{-b_m \pi})^\varepsilon_m = \frac{f(e^n)}{g(e^n)}
\]

where \( f, g \in \mathbb{Q}[X] \) are non-trivial polynomials.

Observe that \( e^n = (e^{\pi i})^{-i} = (-1)^{-i} \); thus, by the Gelfond–Schneider theorem, \( e^n \) is transcendental. It follows that \( f(e^n)/g(e^n) \in \mathbb{Q} \) only if \( g \) is a rational multiple of \( f \). We divide this case into two further subcases:

- If \( \ell = 0 \), then, once again, testing whether (4) holds reduces to deciding whether two rationals are equal, and
- If \( \ell \neq 0 \), then testing whether (4) reduces to testing whether \( \pi^t \) is equal to a given rational number, which cannot hold for then \( \pi \) is necessarily algebraic.

All that remains is to consider the case where \( f(e^n)/g(e^n) \notin \mathbb{Q} \), which we again split into two subcases.

- If \( \ell = 0 \), then it is trivial to see that (4) cannot hold as the right-hand side is rational.
- If \( \ell \neq 0 \) and we assume, for a contradiction, that (4) holds, then a simple rearrangement of (4) shows that there is a non-trivial polynomial \( P \in \mathbb{Q}[X, Y] \) such that \( P(\pi, e^n) = 0 \). This contradicts Nesterenko’s theorem [15] that \( \pi \) and \( e^n \) are algebraically independent.

We have dispatched each of the subcases and conclude the desired result.

\[ \square \]

**Remark 3.2.** Subject to appropriate changes and employing Lemma 2.7, we can extend the result in Proposition 3.1 from instances of the membership and threshold problems \( (u_n, t) \) with \( u_0, t \in \mathbb{Q} \) to problem instances with \( u_0, t \in \mathbb{L}(\pi, e^n) \) where \( \mathbb{L} \) is any finite Galois extension of \( \mathbb{Q} \). This extension similarly holds for both Theorem 3.4 and Theorem 4.5.

Now consider the class of problem instances where \( u_0, t \in \mathbb{L}(\pi, e) \). At the time of writing, algebraic independence of \( \pi \) and \( e \) is currently unknown; however, if Schanuel’s conjecture is true, then it follows that \( \pi \) and \( e \) are algebraically independent. Thus we can similarly extend the results in Proposition 3.1 and Theorem 3.4 to instances where \( u_0, t \in \mathbb{L}(\pi, e) \) subject to the truth of Schanuel’s conjecture.

**Remark 3.3.** Analogous decision procedures to the proof of Proposition 3.1 also hold for other famous rings of integers: the Eisenstein, Kummer, and Kleinian integers. Recall that the Eisenstein integers are the elements of \( \mathbb{Z}[e] = \{a + b\zeta_3 : a, b \in \mathbb{Z}\} \) where \( \zeta_3 := e^{2\pi i/3} \). Similarly, the Kummer integers are the elements of \( \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \). Finally, the Kleinian integers are the elements of \( \mathbb{Z}[\mu] = \{a + b\mu : a, b \in \mathbb{Z}\} \) where \( \mu = -1/2 + \sqrt{-7}/2 \).
The claims for decidability in Remark 3.3, follow from the next theorem. We establish decidability of the membership and threshold problems for hypergeometric sequences whose parameters are drawn from the ring of integers of an imaginary quadratic number field.

**Theorem 3.4.** The Hypergeometric Membership and Threshold Problems for sequences with parameters drawn from the integers of an imaginary quadratic number field are both decidable.

**Proof.** Mutatis mutandis, the proof of Theorem 3.4 follows the approach in Proposition 3.1. For the sake of brevity, we shall indicate only the major changes to Proposition 3.1 here. Consider the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{d})$ where $-d \in \mathbb{N}$ is square-free. By Theorem 2.6, there are two cases to consider: first, when $d \equiv 1 \pmod{4}$ and second, when $d \equiv 1 \pmod{4}$.

We concentrate on the changes to the proof of Proposition 3.1 when $d \not\equiv 1 \pmod{4}$. Like before, we can use the recurrence formula to write $\Gamma(a + b\sqrt{d}) = \theta \Gamma(b\sqrt{d})$ where $\theta \in \mathbb{N}$. Thus all that remains is to evaluate products $\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d})$ of conjugate elements. By the reflection formula, we have

$$\Gamma(b\sqrt{d})\Gamma(-b\sqrt{d}) = -\frac{\pi}{b\sqrt{d} \sin(b\pi\sqrt{d})} = \frac{2\pi}{b\sqrt{-d}(e^{\pi b\sqrt{d}} - e^{-\pi b\sqrt{d}})}.$$ 

The important update in (4) is the product $\prod_{m}(\sinh(m\pi))^{e_{m}}$. In our new setting, the product takes the form $\prod_{m}(e^{\pi b\sqrt{d}} - e^{-\pi b\sqrt{d}})^{e_{m}}$. Observe that $e^{\pi \sqrt{d}}$ is transcendental (once again by Gelfond–Schneider) and that for each $-d \in \mathbb{N}$ the numbers $\pi$ and $e^{\pi \sqrt{d}}$ are algebraically independent over $\mathbb{Q}$ [15, Corollary 6]. The rest of the proof in this case follows as before.

In the second case where $d \equiv 1 \pmod{4}$ we must additionally deal with contributions of the form $\Gamma(b/2 + b\sqrt{d}/2)\Gamma(b/2 - b\sqrt{d}/2)$. This setting introduces cases where $2 \nmid b$, which we resolve by repeated application of the recurrence formula and the cosine variant of Euler’s reflection formula. Indeed, we have

$$\Gamma(1/2 + b\sqrt{d}/2)\Gamma(1/2 - b\sqrt{d}/2) = \frac{\pi}{\cos(\pi b\sqrt{d}/2)} = \frac{2\pi}{e^{\pi b\sqrt{d}/2} + e^{-\pi b\sqrt{d}/2}},$$

and so we can construct an updated version of the product in (4). This update and analogous arguments for the transcendental properties of $e^{\pi \sqrt{d}/2}$ let us conclude decidability in this case too. □

4. **An application of Schanuel’s conjecture to decision problems for hypergeometric sequences**

We now generalise the decidability results Proposition 3.1 and Theorem 3.4. In this section we consider parameters from a wider class. Here the parameters will possess 2-fold symmetries such that the real part of the centre of rotation of each symmetry is an integer (or half-integer). The main result in this section is Theorem 4.5. Before we state and prove Theorem 4.5, we first introduce some preliminary results and notation in Subsection 4.1.
It is worth noting that our assumption on the parameters in this section is invariant under a shift \( n \mapsto n + 1 \) to the indexing. Thus we can assume, without loss of generality, that none of the parameters in our investigation is a negative integer.

4.1. **Algebraic numbers with rational real part.** Let \( C \) be the class of minimal polynomials associate with algebraic numbers that have rational real parts as discussed in [6]. Trivially, a linear polynomial with rational coefficients is always a member of \( C \) since the single root of the polynomial is rational. It is straightforward to see that an irreducible quadratic polynomial \( x^2 + a_1 x + a_0 \in \mathbb{Q}[x] \) is in \( C \) if and only if \( a_1^2 - 4a_0 < 0 \).

**Example 4.1.** The irreducible polynomial \( f(x) = x^8 - 8x^7 + 28x^6 - 56x^5 + 74x^4 - 72x^3 + 84x^2 - 88x + 41 \) has four roots with rational real part, namely \( 1 \pm i \pm \sqrt{1 - \sqrt{2}} \). The other four roots \( 1 \pm i \pm \sqrt{1 + \sqrt{2}} \) of \( f \) do not have rational real part. Note that \( f(x + 1) = x^8 + 4x^4 + 32x^2 + 4 \) is an even polynomial.

The authors of [6] achieve a complete classification of class \( C \) with the following result.

**Theorem 4.2.** Let \( F \) be a polynomial of degree at least three. Then \( F \in C \) if and only if \( F(x) = G((x - \rho)^2) \) for some \( \rho \in \mathbb{Q} \) and monic irreducible \( G \in \mathbb{Q}[X] \) that has a negative real root. In this case, \( F \) has a root with a rational real part \( \rho \).

**Corollary 4.3.** Let \( F \) be a polynomial in \( C \) of degree at least three. The roots of \( F \) that have rational real part have the same real part. Further, we have \( F(\rho - w) = F(\rho + w) \).

We partition \( C \) as follows. Let \( C_p \) be the subclass of minimal polynomials of algebraic numbers that have rational real part \( \rho \). It follows from Theorem 4.2 that \( C = \bigsqcup_{p \in \mathbb{Q}} C_p \).

We now evaluate gamma products whose inputs are given by polynomials in \( \bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n} \). It is clear from Theorem 4.2 that if \( p \in C \) and \( \deg(p) \geq 3 \), then \( \deg(p) \) is even. Further, by Corollary 4.3 roots with irrational real parts come in pairs.

**Lemma 4.4.** Suppose that \( p \in C \) and \( \deg(p) = 2d \geq 3 \). Let \( \alpha_1, \overline{\alpha}_1, \ldots, \alpha_m, \overline{\alpha}_m \) be the roots of \( p \) with rational real part \( \rho \) and let \( \alpha_{m+1}, \alpha_{m+1} - 2\omega_m, \ldots, \alpha_d, \alpha_d - 2\omega_d \) be the roots of \( p \) with irrational real parts. Here for \( k \in \{1, \ldots, m\} \) we write \( b_k := \text{Im}(\alpha_k) \) and for each \( k \in \{m+1, \ldots, d\} \) write \( \omega_k := \alpha_k - \rho \). If \( p \in \bigsqcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n} \), then (up to multiplication by an integer) the gamma product \( \prod_{\alpha \in \Gamma(\alpha) \cap \mathbb{Q}[\rho]} \) is given by

\[
\begin{align*}
\left\{ \prod_{k=1}^{m} \frac{2\pi e^{\pi b_k}}{b_k(e^{2\pi b_k} - 1)} \right\} \left\{ \prod_{k=m+1}^{d} \frac{2\pi e^{\pi \omega_k i}}{\omega_k(1-e^{2\pi \omega_k i})} \right\} & \quad \text{if } \rho \text{ is an integer}, \\
\left\{ \prod_{k=1}^{m} \frac{2\pi e^{\pi b_k}}{e^{\pi b_k} + 1} \right\} \left\{ \prod_{k=m+1}^{d} \frac{2\pi e^{\pi \omega_k i}}{e^{2\pi \omega_k i} + 1} \right\} & \quad \text{if } \rho \text{ is a half-integer}.
\end{align*}
\]

**Proof.** Taken in combination, the results of Theorem 4.2, Corollary 4.3, and Lemma 2.4 give the desired result. \( \square \)
4.2. Main result. We now prove the following generalisation of Proposition 3.1 and Theorem 3.4.

**Theorem 4.5.** Let \( \mathcal{C} \) be the class of hypergeometric sequences whose parameters have minimal polynomials in \( \bigcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n} \). If Schanuel’s conjecture is true, then we can decide the membership and threshold problems for sequences in \( \mathcal{C} \).

**Proof.** A hypergeometric sequence \( \langle u_n \rangle_n \) is in class \( \mathcal{C} \) if and only if the numerator and denominator of the shift quotient \( r(n) = cP(n)/Q(n) \) are given by products of irreducible factors each of which is a polynomial in \( \bigcup_{n \in \mathbb{Z}} C_n \cup C_{1/2+n} \). As noted previously, we can assume that \( r \) is harmonious and so \( c = 1 \) without loss of generality. \( p_1, \ldots, p_k \) and \( q_1, \ldots, q_k' \) be the irreducible factors of \( P \) and \( Q \), respectively.

The proof is split into two parts: a reduction to an equality testing problem and an application of Schanuel’s conjecture.

**Reduction to an equality testing problem.** From Proposition 2.3, we know that decidability of the membership and threshold problems restricted to class \( \mathcal{C} \) reduces to the problem of determining whether equality holds in

\[
\frac{\prod_{\alpha=1}^{E} \prod_{p_j(\alpha)=0} \Gamma(\alpha)}{\prod_{\beta=1}^{E} \prod_{q_j(\beta)=0} \Gamma(\beta)} = t
\]

for a given rational \( t \). Let us consider how we evaluate the gamma product \( \prod_{\alpha=1}^{E} \prod_{p_j(\alpha)=0} \Gamma(\alpha) \) in accordance with the degree \( d_j \) of the associated polynomial factor \( p_j \) (evaluating the gamma product \( \prod_{\beta=1}^{E} \prod_{q_j(\beta)=0} \Gamma(\beta) \) is similar). If \( d_j = 1 \), then \( p_j \) is a linear polynomial with a single root. Thus the aforementioned term is of the form \( \Gamma(n) = n! \) or \( \Gamma(n + 1/2) = \sqrt{\pi}n! \). If \( d_j = 2 \), then the roots of the quadratic polynomial \( p_j \) are \( a \pm bi \) where \( a \) is either an integer or a half-integer by assumption. This contribution is discussed in Corollary 2.5. We now turn to the case where \( d_j \geq 3 \). The contributions in this third case are summarised in Lemma 4.4.

For each root \( \alpha \) of either \( P \) or \( Q \), let \( b \) be either i) the imaginary part of \( \alpha \) if \( \text{Re}(\alpha) \) is an integer or a half-integer, or ii) \( b := \rho_a - \alpha \) if \( \text{Re}(\alpha) \) is irrational (here \( \rho_a \) is the rational real part of the minimal polynomial of \( \alpha \) as in Corollary 4.3). Consider the set of such numbers \( \{ b_1, \ldots, b_M \} \). We denote by \( S' := \{ s'_1, \ldots, s'_m \} \) a maximal \( \mathbb{Q} \)-linearly independent subset of \( \{ b_1, \ldots, b_M \} \) with an additional condition that \( \sqrt{\pi} \pi S' \) is also \( \mathbb{Q} \)-linearly independent (here \( \pi S' := \{ \pi s'_1, \ldots, \pi s'_m \} \)). Then for each \( k \), write \( b_k \) as a \( \mathbb{Q} \)-linear sum of elements in \( \{ s'_1, \ldots, s'_m \} \) so that

\[
b_k = \frac{x_{k1}}{y_{k1}} s'_1 + \cdots + \frac{x_{km}}{y_{km}} s'_m.
\]

We define \( s_j := s'_j/\text{lcm}(y_{j1}, y_{j2}, \ldots, y_{jM}) \) for each \( j \in \{ 1, \ldots, m \} \). Now we can write each \( b_k \) as a \( \mathbb{Z} \)-linear sum of elements in the normalised set \( S := \{ s_1, \ldots, s_m \} \).

For a given problem instance \( \langle (u_n)_n, t \rangle \), we piece together the above contributions and consider the product on the left-hand side of (5) in its entirety. Our approach is similar to the equality testing procedure for the product (4) in Proposition 3.1. By clearing denominators, we deduce that testing the
equality in (5) reduces to testing whether a certain non-trivial polynomial with rational coefficients vanishes at a given point. More specifically, we want to test whether a given polynomial \( P \in \mathbb{Q}[X_1, \ldots, X_{4m+4}] \) satisfies
\[
P\left(\sqrt{\pi}, \pi i, \pi S, \pi S_i, e^{\sqrt{\pi}}, e^{\pi i}, e^{\pi S}, e^{\pi S_i}\right) = 0.
\]
Here \( \pi S := \{\pi s_1, \ldots, \pi s_{m+1}\} \), \( e^{\pi S} := \{e^{\pi s_1}, \ldots, e^{\pi s_m}\} \), and likewise for \( e^{\pi S_i} \).

Further, we need only consider a polynomial in \( 4m + 4 \) variables as the elements \( \{b_1, \ldots, b_M\} \) for our problem instance are given by \( \mathbb{Q}\)-linear combinations of the elements of \( S \cup S_i \).

**Application of Schanuel’s conjecture.** We note that, by assumption, the elements of the set \( \{\sqrt{\pi}, \pi i, \pi S, \pi S_i\} \) are \( \mathbb{Q}\)-linearly independent. Let us apply Schanuel’s conjecture to the larger set \( \{\sqrt{\pi}, \pi i, \pi S, \pi S_i, e^{\sqrt{\pi}}, e^{\pi i}, e^{\pi S}, e^{\pi S_i}\} \) with cardinality \( 4m + 4 \). If Schanuel’s conjecture is true, then this larger set has an algebraically independent subset of size at least \( 2m + 2 \). This algebraically independent subset is necessarily \( \{\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}\} \) since the \( 2m + 2 \) elements of \( \{\sqrt{\pi}, \pi i, \pi S, \pi S_i\} \) are pairwise algebraically dependent (by construction) and \( e^{\pi i} = -1 \).

We now rewrite the preceding displayed equality in terms of the (obvious) polynomial \( \hat{P} \) that absorbs the algebraically dependent inputs \( S \cup S_i \) into the coefficients. That is to say, we employ a polynomial \( \hat{P} \in L(X_1, \ldots, X_{2m+2}) \) where \( L \) is the Galois closure of the number field \( \mathbb{Q}(S, Si) \) and evaluate \( \hat{P} \) on our algebraically independent subset. It follows that the above equality holds if and only if
\[
\hat{P}(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}) = 0.
\]
By Lemma 2.7, the above holds if and only if there exists a polynomial \( Q \in \mathbb{Q}[X_1, \ldots, X_{m+2}] \) such that \( Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}) = 0 \). If Schanuel’s conjecture is true, then this equality cannot hold. This assertion follows from our preceding work. Indeed, if Schanuel’s conjecture is true, then the elements of the set \( \{\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}\} \) are algebraically independent over \( \mathbb{Q} \), from which we deduce that the equality in (5) cannot hold. The desired result follows.

**Remark 4.6.** We note that the equality test \( Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}) = 0 \) can be realised as a proposition in the first order theory of the reals with exponentiation. Macintyre and Wilkie [12] established decidability of said theory subject to the truth of Schanuel’s conjecture. As noted in previous works, careful inspection of Macintyre and Wilkie’s algorithm reveals that correctness is independent of the truth of Schanuel’s conjecture. Indeed, Schanuel’s conjecture is only used to prove termination. Thus if we apply Macintyre and Wilkie’s algorithm to determine whether the equality \( Q(\sqrt{\pi}, e^{\sqrt{\pi}}, e^{\pi S}, e^{\pi S_i}) = 0 \) holds and find the procedure terminates, then the output is certainly correct.

We note that Macintyre and Wilkie’s algorithm terminates unless the inputs constitute a counterexample to Schanuel’s conjecture. Thus, the process underlying the proof of Theorem 4.5 presents an interesting prospect.
in the sense described by Richardson in [21] (see also [20]) “A failure of the [process] to terminate would be even more interesting than [its] success.”

4.3. unnested radical and cyclotomic parameters. We draw the reader’s attention to classes of hypergeometric sequences in $\mathbb{C}$ (those that are amenable to the transcendental techniques laid out above). Our focus here is on parameters associated with unnested radicals and cyclotomic polynomials (both of which lie beyond the reach [17]); that is to say, when the shift quotient of a hypergeometric sequence has an irreducible factor of the form $n^m - a$ or $\Phi_m(n)$.

It is interesting that identities for infinite products involving cyclotomic polynomials and unnested radicals appear frequently in the literature. Indeed, for the avoidance of doubt, we do not claim the identities below are new. We direct the reader to [19, pp. 753–757] (and to further references in the latter).

Remark 4.7. We adopt the notation $\sqrt[m]{a}$ for the principal $m$th root of $a$ and $\omega_m := e^{2\pi i/m}$.

1. Note that $n^m - a \in \mathbb{Z}[n]$ is an irreducible polynomial whose roots are $\sqrt[m]{a}\omega_m^j$ for $j \in \{0, \ldots, m-1\}$. For $m$ even, $n^m - a \in \mathbb{C}_0$.

2. Let us discuss cyclotomic polynomials $\Phi_m(n)$ whose roots are the primitive $m$th roots of unity. Under the assumption that $m$ is a multiple of four, it is clear that $\Phi_m(n) \in \mathbb{C}_0$. This assumption permits us to pair together the gamma terms, indexed by $j$ and $j + m/2$, for evaluation. As an illustration to what happens without this additional assumption, observe that $\omega_{18}, \omega_{18}^5, \omega_{18}^7, \omega_{18}^{11}, \omega_{18}^{13}$ and $\omega_{18}$ are the $18$th primitive roots of unity and cannot be paired so.

4.4. Limitations of the technique. The next corollary is an immediate consequence of applying Theorem 4.5 to sequences with parameters drawn from the rational integers and the quadratic integers.

Corollary 4.8. The Hypergeometric Membership and Threshold Problems with parameters drawn from the rational integers and quadratic integers are both decidable subject to Schanuel’s conjecture.

In Remark 4.9, we show the limitation of our technique: we cannot even handle parameters drawn from a particularly well-behaved class of quartic fields, the so called biquadratic fields.

Remark 4.9. Our approach already fails at biquadratic fields (such as $\mathbb{Q}(\sqrt{5}, \sqrt{13})$ and $\mathbb{Q}(\sqrt{21}, \sqrt{33})$) because the rings of integers associated with these fields contain elements that are not amenable to our approach. Take, for example, $\theta = (5 + 3\sqrt{5} + \sqrt{13} + 3\sqrt{65})/4 \in \mathbb{Q}(\sqrt{5}, \sqrt{13})$ that satisfies $\theta^4 - 50\theta^3 - 716\theta^2 + 120\theta + 1044 = 0$. As a second example, take $\tilde{\theta} = (1 + \sqrt{21} + \sqrt{33} - \sqrt{77})/4 \in \mathbb{Q}(\sqrt{21}, \sqrt{33})$ that satisfies $\tilde{\theta}^4 - \tilde{\theta}^3 - 16\tilde{\theta}^2 + 37\tilde{\theta} - 17 = 0$. Both of these examples are taken from [28].

5. Conclusion

Our transcendence approach extends the state of the art for the membership and threshold problems for hypergeometric sequences in a novel
direction. Major obstacles to extending this method to a larger class of hypergeometric sequences are the functional properties of the gamma function. Indeed, the reflection and recurrence properties of the gamma function are indispensable for our approach. Without further functional properties or the inclusion of techniques from other mathematical disciplines, we are particularly limited to polynomial factors whose root sets have both 2-fold symmetries and centres of rotation at a point \( \rho \) in the complex plane with \( \text{Re}(\rho) \) either an integer or a half-integer.

It is not obvious that even if new functional properties were to be discovered that they would be of use when it comes to linear factors in the shift quotient. In fact, we cannot evaluate the infinite product associated with the shift quotient (2) because almost nothing is known about \( \Gamma(1/9) \). Indeed, we know very little about the transcendental properties of the gamma function. On the one hand, \( \Gamma(1/2) = \sqrt{\pi} \). On the other hand, we do not know how to express \( \Gamma \) evaluated at general rational values. Indeed, for \( s \in \{1/6, 1/4, 1/3, 2/3, 3/4 \} \) and \( n \in \mathbb{N} \), it is known that \( \Gamma(n + s) \) is a transcendental number and algebraically independent of \( \pi \) (cf. [26]). However, transcendence of the gamma function at other rational points is not known.

We close by noting that such obstacles are observed elsewhere in the literature; in particular, in relation to evaluating infinite products and gamma products identities (e.g., the product identity in (3)).

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