THE GRAVITATIONAL CONTENT OF LORENTZIAN COMPLEX STRUCTURES

C. N. Ragiadakos
Pedagogical Institute
Mesogion 396, Agia Paraskevi, TK 15341, Greece
email: crag@pi-schools.gr

ABSTRACT
The definition of a positive energy is investigated in a renormalizable
4-dimensional generally covariant model, which depends on the lorentzian
complex structure and not the metric of spacetime. The gravitational content
of the lorentzian complex structures is revealed by identifying the spacetime
with special 4-dimensional surfaces of the $G_{2,2}$ Grassmannian manifold. The
lorentzian complex structure is found to be a codimension-4 CR structure and
its classification is studied using the Chern-Moser and Cartan methods. The
spacetime metric is found to be a Fefferman-like metric of this codimension-4
CR structure. The open CR manifolds "hanging" from the points of the $U(2)$
characteristic boundary of the $SU(2, 2)$ classical domain belong into
representations of the Poincaré group and are related to the particle spectrum
of the model.
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1 INTRODUCTION

After the recent failure of ATLAS and CMS experiments to find minimal supersymmetry effects and (large) higher spacetime dimensions, doubts on the physical relevance of the superstring model start to appear. Despite these experimental difficulties, the proponents of the string model do not give up, because they think that it is the unique quantum mechanically self-consistent model, which includes gravity. In fact, it is not unique! Long time ago I wrote down and I quantized the following simple 4-dimensional Yang-Mills-like action, which depends on the lorentzian complex structure and not the metric of the spacetime.

\[ I_G = \int d^4z \sqrt{-g} g^{\alpha\beta} F_{\alpha\beta} F_{\alpha\beta} + c. c. = 2 \int d^4z F_{0\alpha} F_{0\beta} + c. c. \]

\( F_{j\alpha\beta} = \partial_{j} A_{\alpha\beta} - \partial_{\alpha} A_{j\beta} - \gamma f_{jk} A_{ia} A_{kb} \) \hspace{1cm} (1.1)

Lorentzian complex structures have been introduced by Flaherty in order to study spacetimes with two geodetic and shear free congruences. Using the ordinary null tetrad \((\ell_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu})\), the lorentzian metric \(g_{\mu\nu}\) and the complex structure \(J_{\nu}^{\mu}\) take the form

\[ g_{\mu\nu} = \ell_{\mu} n_{\nu} + n_{\mu} \ell_{\nu} - m_{\mu} \bar{m}_{\nu} + \bar{m}_{\mu} m_{\nu} \]
\[ J_{\nu}^{\mu} = i(\ell_{\mu} n_{\nu} - n_{\mu} \ell_{\nu} - m_{\mu} \bar{m}_{\nu} + \bar{m}_{\mu} m_{\nu}) \]

\[ (\ell^{\mu} m_{\nu} - \ell_{\mu} m^{\nu})(\partial_{\mu} \ell_{\nu}) = 0 \hspace{1cm} (\ell^{\mu} m_{\nu} - \ell_{\mu} m^{\nu})(\partial_{\mu} m_{\nu}) = 0 \]
\[ (n^{\mu} m_{\nu} - n_{\mu} m^{\nu})(\partial_{\mu} n_{\nu}) = 0 \hspace{1cm} (n^{\mu} m_{\nu} - n_{\mu} m^{\nu})(\partial_{\mu} m_{\nu}) = 0 \]

That is when the spin coefficients \(\kappa, \sigma, \lambda, \nu\) vanish, which implies that the real vectors \(\ell^{\mu}\) and \(n^{\mu}\) define geodetic and shear free congruences. Then Frobenius theorem states that there are four independent complex functions \((z^{\alpha}, \bar{z}^{\tilde{\alpha}})\), \(\alpha = 0, 1\), such that

\[ dz^{\alpha} = f_{\alpha} \ell_{\mu} dx^{\mu} + h_{\alpha} m_{\mu} dx^{\mu} \hspace{1cm} d\bar{z}^{\tilde{\alpha}} = f_{\tilde{\alpha}} n_{\mu} dx^{\mu} + h_{\tilde{\alpha}} \bar{m}_{\mu} dx^{\mu} \]
\[ \ell = \ell_{\alpha} dz^{\alpha} \hspace{1cm} m = m_{\alpha} dz^{\alpha} \]
\[ n = n_{\tilde{\alpha}} d\bar{z}^{\tilde{\alpha}} \hspace{1cm} m = m_{\tilde{\alpha}} d\bar{z}^{\tilde{\alpha}} \]

These four functions are the structure coordinates of the (integrable) complex structure. In the present case of lorentzian spacetimes the coordinates \(z^{\alpha}\)
are not complex conjugate of \(z^a\), because \(J^\alpha_\mu\) is no longer a real tensor, like the ordinary complex structures. We always have \(z^a = f^a(z^b)\), while the condition \(dz^0 \wedge d\bar{z}^0 \wedge dz^0 \wedge dz^1 \neq 0\) has to be imposed.

Using the null tetrad, the generally covariant form of model action takes the form

\[
I_G = \int d^4x \sqrt{-g} \left\{ (\ell^\mu m^\rho F_{\mu \rho}) (n^\nu m^\sigma F_{\nu \sigma}) + (\ell^\mu \overline{m}^\rho F_{\mu \rho}) (n^\nu \overline{m}^\sigma F_{\nu \sigma}) \right\}
\]

(1.5)

\[
F_{\mu \nu} = \partial_\mu A_{\nu} - \partial_\nu A_{\mu} - \gamma f_{jik} A_{\mu} A_{k\nu}
\]

with the following term of Lagrange multipliers

\[
I_C = \int d^4x \sqrt{-g} \left\{ \phi_0 (\ell^\mu m^\nu - \ell^\nu m^\mu) (\partial_\mu \ell_\nu) + \phi_1 (\ell^\mu m^\nu - \ell^\nu m^\mu) (\partial_\mu m_\nu) + \phi_0 (n^\mu m^\nu - n^\nu m^\mu) (\partial_\mu n_\nu) + \phi_1 (n^\mu m^\nu - n^\nu m^\mu) (\partial_\mu \overline{m}_\nu) + c.c. \right\}
\]

(1.6)

which impose the integrability conditions of the lorentzian complex structure. These terms are essential, because they assure the metric independence of the action leading to its renormalizability\[18\]. In brief, this model is a conventional lagrangian generally covariant model which is renormalizable because of its increased symmetry.

The physical content of the model has been studied in my previous works\[17\],\[19\],\[20\]. In this work we focus on the mathematical methods which could be used for the definition of a positive energy and the related classification of the lorentzian complex structures. In section II, I specify the general class of metrics, which are compatible with a lorentzian complex structure. I have not yet found an appropriate definition of the energy quantity of the model. A formal definition of the energy may be undertaken using the existence of a coordinate system, where the ordinary contracted derivative of the Einstein tensor vanishes. But this coordinate system has to be related to the geodetic coordinates of \(\ell^\mu\) and \(n^\mu\) which transform according to a Poincaré group. This defined energy, suggested by the great success of the Einstein general relativity, must be proved to be positive.

The Poincaré group naturally emerges if the spacetime is described as a surface of \(G_{2,2}\) in section III and a codimension-4 CR manifold in section IV, where the classification of lorentzian complex structures is undertaken. In section V, I find that the spacetime class of metrics compatible with a lorentzian complex structure are Fefferman-like metrics of the corresponding codimension-4 CR structure. This direct relation of the "particles" of the model with the formidable mathematical machinery of bounded domains, their CR-boundaries and the natural emergence of the Poincaré group may permit us to define the energy quantity.
2 AN ENERGY DEFINITION FOR THE MODEL

The integrable complex structure is determined by a null tetrad up to the following transformations

\[ \ell'_\mu = \Lambda \ell_\mu, \quad n'_\mu = \frac{1}{N} n_\mu \]
\[ m'_\mu = M m_\mu, \quad \nu'_\mu = \frac{1}{M} \nu_\mu \]  
(2.1)

Under these transformations the Newman-Penrose spin coefficients transform as follows

\[ \alpha' = \frac{1}{M} \alpha + \frac{M}{4M} (\pi - \tau) + \frac{1}{4M} \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \beta' = \frac{1}{M} \beta + \frac{M}{4M} (\tau + \pi) + \frac{1}{4M} \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \gamma' = \frac{1}{M} \gamma + \frac{M}{4M} (\mu - \nu) + \frac{1}{4M} \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \epsilon' = \frac{1}{N} \epsilon + \frac{M}{4M} (\rho - \sigma) + \frac{1}{4N} D \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \mu' = \frac{1}{N} (\mu + \overline{\nu}) + \frac{1}{2N} (\mu - \overline{\nu}) + \frac{1}{4N} \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \rho' = \frac{1}{N} (\rho + \overline{\sigma}) + \frac{1}{2N} (\rho - \overline{\sigma}) - \frac{1}{4N} D \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \tau' = \frac{1}{N} (\tau + \overline{\pi}) + \frac{1}{2N} (\tau - \overline{\pi}) - \frac{1}{4N} D \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \pi' = \frac{1}{N} (\pi + \overline{\pi}) + \frac{1}{2N} (\pi - \overline{\pi}) + \frac{1}{4N} \ln \frac{\Lambda}{\sqrt{M}} \]
\[ \kappa' = \frac{1}{N M} \kappa, \quad \sigma' = \frac{M}{N M} \sigma, \quad \nu' = \frac{N}{N M} \nu, \quad \lambda' = \frac{N}{N M} \lambda \]  
(2.2)

We see that the following relations

\[ \rho' - \overline{\rho}' = \frac{N}{M} (\rho - \overline{\rho}) \]
\[ \mu' - \overline{\mu}' = \frac{N}{M} (\mu - \overline{\mu}) \]
\[ \tau' + \overline{\tau}' = \frac{N}{M} (\tau + \overline{\tau}) \]  
(2.3)

establish the corresponding quantities as relative invariants of the complex structure, analogous to the Levi forms of the CR structure[9].

Not all the metrics admit a lorentzian complex structure. In that case, the (non-conformally flat) metric uniquely determines the lorentzian complex structure through the integrability conditions

\[ \Psi_{ABCD} o^A o^B o^C o^D = 0 = \Psi_{ABCD} \ell^A \ell^B \ell^C \ell^D \]  
(2.4)

where \( o^A \) and \( \ell^A \) is the spinor dyad of the integrable null tetrad and \( \Psi_{ABCD} \) is the Weyl tensor in the spinor notation. Namely, they are principal null directions of the Weyl spinor \( \Psi_{ABCD} \). But the inverse is not true. The class \( J[g_{\mu\nu}] \) of metrics, which are compatible with a lorentzian complex structure, is determined by the following general form

\[ g'_{\mu\nu} = \phi^2 g_{\mu\nu} + \psi^2 (\ell_\mu n_\nu + n_\mu \ell_\nu) \]
\[ g'^{\mu\nu} = \frac{1}{\phi^2} g^{\mu\nu} - \frac{\psi^2}{\phi^2 + \psi^2} (\ell^{\mu} n^{\nu} + n^{\mu} \ell^{\nu}) \]  
(2.5)

5
This is a generalization of the conformal (Weyl) class of metrics and a Yamabe-like problem may be posed with one more scalar quantity constant, besides the scalar curvature. I do not actually see any physical relevance of this mathematical problem. Instead I have already pointed out\[17,19\] that the derivation of Einstein’s gravity implies that the energy of the model should be defined using the Einstein tensor of a metric from the class $J[g_{\mu\nu}]$. The assumed in General Relativity dominant energy condition must be proven in the present model. I want to point out that the existence of a positive conserved quantity seems to be essential for the quantum stability of the model.

Einstein used the Levi-Civita connection to equate the covariantly conserved tensor $E^{\mu\nu}$ with the matter energy-momentum tensor

$$E^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = 8\pi k T^{\mu\nu} \quad (2.6)$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the matter fields. In the present model, general covariance does not permit the definition of an energy-momentum tensor. Therefore we use the existence of a coordinate system such that

$$\partial_\mu (\sqrt{-g} E^{\mu\nu}) = 0 \quad (2.7)$$

to define the conserved quantity

$$\mathcal{E}(g_{\mu\nu}) = \int_{\mathcal{A}} \sqrt{-g} E^{\mu\nu} dS_\mu \quad (2.8)$$

in the precise coordinate system. Recall that singularities are not permitted in Quantum Field Theory. Therefore we have to consider only regular spacetime metrics.

The considered quantity $\mathcal{E}(g_{\mu\nu})$ depends on the metric $g_{\mu\nu}$ and it does not characterize the complex structure, therefore it cannot be the energy definition of the lorentzian complex structure. I think that the energy of a complex structure is properly defined by the following minimum

$$E[J^\nu] = \min_{g_{\mu\nu} \in J[g_{\mu\nu}]} \mathcal{E}(g_{\mu\nu}) \quad (2.9)$$

where the minimum is taken over all the class $J[g_{\mu\nu}]$ of metrics. Apparently the mathematical conjecture is that such a minimum exists, which is not at all evident! That is, in the present model, the positive energy condition must be proved!

This conserved quantity depends only on the moduli parameters of the complex structure. Minkowski spacetime determines the vacuum sector of the model because $E[J^\nu] = 0$, for complex structures compatible with the Minkowski metric. From the 2-dimensional solitonic models\[3\], we know that the minima of the energy characterize the solitons. Assuming that $E[J^\nu]$ is a smooth function of the moduli parameters, we can always expand it around a minimum.

$$E[J^\nu] \simeq E + \sum_q \xi_q \pi_q a_q \quad (2.10)$$
where $E$ and $\varepsilon_q$ are positive parameters. These variables and $a_q$ are moduli parameters of the complex structure. $E$ is defined to be the energy of the soliton characterized by the minimum and $\varepsilon_q$ are the energies of the excitation modes.

This formal procedure implies the Einstein equations if the following points are mathematically clarified: 1) The positivity of $\mathcal{E}(g_{\mu\nu})$ for at least a subclass of $J[g_{\mu\nu}]$. 2) The precise coordinate system satisfying (2.7) properly transforms under the Poincaré group found in my previous work[19] and which will be outlined in the next section. 3) The proof will explicitly single out the precise Einstein metric from all the other induced[19] metrics on the manifold.

3 THE $G_{2,2}$ DESCRIPTION OF THE LORENTZIAN COMPLEX STRUCTURES

It is trivial to show from [14] that the structure coordinates $(z^\alpha, z^{\tilde{\alpha}})$, $\alpha = 0, 1$ satisfy the relations

$$dz^0 \wedge dz^1 \wedge d\bar{z}^0 \wedge d\bar{z}^1 = 0$$

$$dz^\tilde{0} \wedge dz^\tilde{1} \wedge d\bar{z}^\tilde{0} \wedge d\bar{z}^\tilde{1} = 0$$

$$dz^\tilde{0} \wedge dz^\tilde{1} \wedge d\bar{z}^\tilde{0} \wedge d\bar{z}^\tilde{1} = 0$$

that is, there are two real functions $\Psi_{11}$, $\Psi_{22}$ and a complex one $\Psi_{12}$, such

$$\Psi_{11}(z^\alpha, z^{\tilde{\alpha}}) = 0 \quad , \quad \Psi_{12}(z^\alpha, z^{\tilde{\alpha}}) = 0 \quad , \quad \Psi_{22}(z^\alpha, z^{\tilde{\alpha}}) = 0$$

(3.2)

This surface may be considered as the characteristic boundary of a domain which is holomorphically equivalent to a bounded domain in $\mathbb{C}^4$, through the positive definite condition of the following $2 \times 2$ matrix

$$\Psi = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12} & \Psi_{22} \end{pmatrix} > 0$$

(3.3)

which occurs when $\Psi_{11} + \Psi_{22} > 0$ and $\det \Psi > 0$. Notice that the boundary conditions $\Psi_{11} + \Psi_{22} = 0$ and $\det \Psi = 0$ imply the above four relations which determine the lorentzian complex structure.

The mathematical study of this kind of problems is performed after their projective formulation. For this purpose I consider the rank-2 $4 \times 2$ matrices $X^{\mu i}$ with every column being a point of an algebraic surface $K_i(X^{\mu i})$ of the $CP^3$ projective space. Then I consider that the $2 \times 2$ matrix $\Psi$ has the form

$$\Psi = X^\dagger EX - \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix} > 0$$

(3.4)
where $E$ is an $SU(2,2)$ invariant $4 \times 4$ matrix and $G_{ij} = G_{ij}(X^{mi}, X^{mj})$ are homogeneous functions. This projective form emerged from the consideration of lorentzian complex structures asymptotically compatible with the Minkowski metric and the Penrose observation that a geodetic and shear free congruence of Minkowski spacetime can be described by a null twistor satisfying an algebraic condition. In the simple case $G_{ij} = 0$ it is a first kind Siegel domain for

$$E = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

(3.5)

which is holomorphic to the $SU(2,2)$ invariant bounded classical domain given by

$$E = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(3.6)

That is the form osculates the surface with the Shilov boundary of the $SU(2,2)$ invariant classical domain.

Using the following spinorial form of the rank-2 matrix $X^{mj}$ in its unbounded realization

$$X^{mj} = \begin{pmatrix} \lambda^{Aj} \\ -i\epsilon_{A'B'} \lambda^{Bj} \end{pmatrix}$$

(3.7)

and the null tetrad

$$L^a = \frac{1}{\sqrt{2}} \lambda^{A1} \lambda B1 \sigma^a_{A'B'} , \quad N^a = \frac{1}{\sqrt{2}} \lambda^{A2} \lambda B2 \sigma^a_{A'B'} , \quad M^a = \frac{1}{\sqrt{2}} \lambda^{A2} \lambda B1 \sigma^a_{A'B'}$$

$$\epsilon_{AB} \lambda^{A1} \lambda^{B2} = 1$$

the above relations take the form

$$\Psi_{11} = 2\sqrt{2} y^a L_a - G_{11}(Y^{mi}, Y^{n1})$$

$$\Psi_{12} = 2\sqrt{2} y^a M_a - G_{12}(Y^{mi}, Y^{n2})$$

$$\Psi_{22} = 2\sqrt{2} y^a N_a - G_{22}(Y^{mi}, Y^{n2})$$

(3.9)

where $y^a$ is the imaginary part of $r^a = x^a + iy^a$ defined by the relation $r^a_{A'B'} = r^a_{\sigma^a_{A'B'}}$ and $\sigma^a_{A'B'}$ being the identity and the three Pauli matrices. The surface satisfies the relation

$$y^a = \frac{1}{2\sqrt{2}} [G_{22} N^a + G_{11} L^a - G_{12} M^a - G_{11} M^a - G_{12} L^a - G_{12} M^a]$$

(3.10)

which combined with the computation of $\lambda^{A1}$ as functions of $r^a$, using the Kerr conditions $K_i(X^{mi})$, permit us to compute $y^a = y^a(x)$ as functions of the real part of $r^a$.

Notice that this surface does not generally belong into the Siegel domain, because $y^0$ and

$$y^a y^b \eta_{ab} = \frac{1}{8} [G_{22} G_{11} - G_{12} G_{12}]$$

(3.11)
are not always positive. But the regular surfaces (with an upper bound) can always be brought inside the Siegel domain (and its holomorphic bounded classical domain) with an holomorphic complex time translation.

\[
\begin{pmatrix}
\lambda'^{Aj} \\
\omega'^{j}_{B'}
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
dI & I
\end{pmatrix}\begin{pmatrix}
\lambda^{Aj} \\
\omega^{j}_{B'}
\end{pmatrix}
\] (3.12)

In the case of an asymptotically flat space time, its point at infinity, which is on the Shilov boundary, remains intact. Therefore we can always assume that the bounded domain (3.4) is always inside the \( SU(2, 2) \) invariant classical domain.

A typical example is the case \( X^1 \Gamma X = 0 \) with

\[
\Gamma = \begin{pmatrix}
-2b & I \\
I & 0
\end{pmatrix}
\] (3.13)

and the Kerr functions \( K_1 = X^{11} + X^{31} = 0 \) and \( K_2 = X^{02} + X^{22} = 0 \). Then using the following definition of the structure coordinates

\[
z^0 = \frac{X^{21}}{X^{01}} , \quad z^1 = \frac{X^{11}}{X^{01}} , \quad \bar{z}^0 = i\frac{X^{32}}{X^{12}} , \quad \bar{z}^1 = -\frac{X^{02}}{X^{12}}
\] (3.14)

we easily find the relations

\[
\Psi_{11} = i(z^0 - \bar{z}^0) - 2z^1\bar{z}^1 - 2b(1 + z^1\bar{z}^1)
\]

\[
\Psi_{12} = z^1(1 - iz^0 + 2b) - \bar{z}^1(1 + iz^0 + 2b)
\] (3.15)

\[
\Psi_{22} = i(z^0 - \bar{z}^0) - 2z^1\bar{z}^1 - 2b(1 + z^1\bar{z}^1)
\]

which will be described below. We find \( y^a = (b, 0, 0, 0) \) which does not induce any gravity, because this lorentzian complex structure is compatible with the Minkowski metric.

The asymptotically flat lorentzian complex structures \( (X^{11}EX^1 = 0) \), \( X^{12}EX^2 = 0 \) belong into irreducible representations of the \( SU(2, 2) \) group which is broken down exactly \([19]\) to its \( Poincaré \times dilation \) subgroup by the infinity point on the Shilov boundary.

The real part of \( r^a \) determine a characteristic set of coordinates because it properly transforms under the Poincaré group. In the case of asymptotically flat lorentzian complex structures there are two other coordinate systems, which properly transform under the Poincaré group too. These are the geodetic coordinates of \( \ell^\mu \) and \( n^\mu \) which have the general forms

\[
x_{(+), A'}^A = \frac{i_w^1}{\lambda^{C1}_A} \omega^{rA}_C + r\lambda^{A}_A \lambda^{1}_A , \quad \forall r \quad and \quad \lambda^{C1}_A \omega^{rA}_C \neq 0
\]

\[
x_{(-), A'}^A = \frac{i_w^2}{\lambda^{C2}_A} \omega^{sA}_C + s\lambda^{A}_A \lambda^{2}_A , \quad \forall s \quad and \quad \lambda^{C2}_A \omega^{sA}_C \neq 0
\] (3.16)

These characteristic coordinate systems should be related to the definition of the energy of a lorentzian complex structure, for this quantity to be a component of a four-momentum.
4 CLASSIFICATION OF LORENTZIAN COMPLEX STRUCTURES

Flaherty worked with the complex structure preserving connection $\gamma^a_{bc}$ with the following non vanishing components

$$\gamma^a_{\beta\gamma} = g^{a\tilde{a}} \partial_\beta g_{\gamma\tilde{a}}, \quad \gamma_{\beta\gamma} = g^{a\tilde{a}} \partial_\beta g_{\alpha\tilde{a}}$$

(4.1)

where the metric is written in the structure coordinate system $(z^\alpha, z^{\tilde{a}})$, $\alpha = 0, 1$

$$ds^2 = 2g_{a\tilde{b}} dz^a dz^{\tilde{b}}$$

(4.2)

He showed that if the torsion of this connection $T^c_{ab} = \gamma^c_{ba} - \gamma^c_{ab}$ vanishes, the complex structure is kaehlerian, $d(J_{\mu\nu} dx^\mu \wedge dx^\nu)$, and the vectors of the null tetrad are hypersurface orthogonal. This means that the complex structure is trivial and apparently compatible with the Minkowski metric. But the inverse is not valid. There are non-trivial complex structures (with non-vanishing torsion) which are also compatible with the Minkowski spacetime. Hence we cannot use this torsion to describe the gravitational content of the complex structure. But we may use all the invariant tensors (torsion, curvature and their covariant derivatives) created by this connection to classify the lorentzian complex structure.

The four real conditions (3.2) imply that the spacetime, which admits an integrable lorentzian complex structure, is a CR manifold with codimension four. Following the ordinary procedure we can find the corresponding four real forms. It is convenient to use the notation $\partial f = \frac{\partial f}{\partial z^a} dz^a$ and $\partial f = \frac{\partial f}{\partial z^{\tilde{a}}} dz^{\tilde{a}}$. Then we find

$$\ell = 2i\partial \Psi_{11} = i(\partial - \tilde{\partial})\Psi_{11} = -2i\tilde{\partial}\Psi_{11}$$

$$n = 2i\tilde{\partial}\Psi_{22} = i(\tilde{\partial} - \partial)\Psi_{22} = -2i\partial\Psi_{22}$$

$$m_1 = i(\partial + \tilde{\partial} - \tilde{\partial})\Psi_{12} = 2i\tilde{\partial}\Psi_{12}$$

$$m_2 = i(\partial + \tilde{\partial} - \partial)\Psi_{12} = 2i\partial\Psi_{12}$$

(4.3)

These forms restricted on the manifold are real, because of $d\Psi_{ij} = 0$ and the special dependence of each function on the structure coordinates $(z^\alpha, z^{\tilde{a}})$. The relations become simpler if we use the complex form

$$m = m_1 + im_2 = 2i\tilde{\partial}\Psi_{12} = -2i\partial\Psi_{12} = i(\partial - \tilde{\partial})\Psi_{12}$$

(4.4)

Notice that these forms coincide with the null tetrad up to a multiplicative factor. The general CR transformation is actually restricted to a factor, because the dimension of the manifold coincides with its codimension.
A general null tetrad has the following differential forms

\[
d\ell = (\varepsilon + \overline{\varepsilon})n \wedge \ell + (\overline{\tau} - \alpha - \beta)m \wedge \ell + (\rho - \overline{\rho})m \wedge \overline{m} - \kappa n \wedge \overline{m}
\]

\[
dn = -((\gamma + \overline{\gamma})\ell \wedge n + (\alpha + \overline{\beta} - \pi)n \wedge m + (\mu - \overline{\mu})m \wedge \overline{m} - \lambda \ell \wedge \overline{m} - \sigma n \wedge \overline{m})
\]

\[
dm = (\gamma - \overline{\gamma})\ell \wedge m + (\varepsilon - \overline{\varepsilon} - \rho)n \wedge m + (\alpha - \beta)m \wedge \overline{m} - (\rho - \overline{\rho})m \wedge \overline{m} - \kappa n \wedge \overline{m}
\]

(4.5)

It is integrable if \(\kappa = \sigma = \lambda = \nu = 0\). Then the transformations of the spin coefficients (2.2) imply that the vanishing or not of the quantities \((\rho - \overline{\rho})\), \((\mu - \overline{\mu})\), \((\tau + \overline{\tau})\) are relative invariants of the lorentzian complex structure. If these quantities vanish, the complex structure is kaehlerian, and the vectors of the null tetrad are hypersurface orthogonal. That is the complex structure is trivial and apparently compatible with the Minkowski metric.

The classification of the lorentzian complex structures may be approached using the CR structure techniques. In the next two subsections I will outline the Chern-Moser normal form and the \(SU(2, 2)\) Cartan connection methods.

4.1 The Chern-Moser normal form method

It has already pointed out[10],[1] that the explicit conditions \(\Psi_{11}(z^\alpha, z^\beta) = 0\), \(\Psi_{22}(z^{\tilde{\alpha}}, z^{\tilde{\beta}}) = 0\) and the corresponding holomorphic transformations \(z^\alpha = f^\alpha(z^\alpha)\) and \(z^{\tilde{\alpha}} = f^{\tilde{\alpha}}(z^{\tilde{\alpha}})\) which preserve the lorentzian complex structure, are exactly those of the 3-dimensional CR structures[5]. Therefore we may use the Moser procedure for the classification of the lorentzian complex structures. For each hypersurface type CR structure we consider the following Moser expansions

\[
U = z^1 \overline{z}^1 + \sum_{k \geq 2, j \geq 2} N_{jk}(u)(z^1)(\overline{z}^1)^k
\]

\[
N_{22} = N_{32} = N_{33} = 0
\]

\[
V = z^1 \overline{z}^1 + \sum_{k \geq 2, j \geq 2} \tilde{N}_{jk}(v)(\tilde{z}^1)(\overline{\tilde{z}}^1)^k
\]

(4.6)

where \(z^0 = u + iU\), \(z^{\tilde{0}} = v + iV\) and the functions \(N_{jk}(u)\), \(\tilde{N}_{jk}(v)\) characterize the lorentzian complex structure. By their construction these functions belong into representations of the isotropy subgroup of \(SU(1, 2)\) symmetry group of the hyperquadric. Notice that the corresponding Moser chains are determined by \(n^0 \frac{\partial}{\partial n^0}\) and \(\ell^0 \frac{\partial}{\partial \ell^0}\) respectively and they should be related to \(x^a\) geodetic coordinates.

The above Moser normal forms are unique up to the isotropy group of the hyperquadric for each expansion. The transformations of the isotropy subgroup
have the form

\[
\begin{align*}
z' \equiv & \frac{\Psi^{0} z^{1} + a z^{0}}{c^{0} |1 - 2 i a z^{0} + (b - |a|^{2}) z^{1}|} \\
z' \equiv & \frac{\Psi^{0} z^{0}}{c^{0} |1 - 2 i a z^{1} + (b - |a|^{2}) z^{0}|}
\end{align*}
\]  

(4.7)

where the parameters \(a, c\) are complex and \(b\) is real. An analogous transformation ambiguity exists for the tilded structure coordinates. This freedom may be used to fix the linear terms of the \(\Psi_{12} = 0\) expansion as follows

\[
z' = z^{1} + C_{1} z^{0} + C_{2} (\overline{z})^{2} + C_{3} (\overline{z})^{2} + \ldots
\]  

(4.8)

The Moser normal forms, which determine a complex structure compatible with the Minkowski metric (without gravity content), may be found using the condition \(X^{\dagger} EX = 0\) and the two Kerr algebraic homogeneous conditions \(K_{i}(X^{\omega}) = 0\).

4.2 The Cartan connection method

The osculation \([3, 4]\) of the CR structure which describes the integrable lorentzian complex structure suggests the use of the \(U(2, 2)\) Cartan connection. The Cartan connection of a group manifold is \(\omega = g^{-1} dg\), where \(g\) is a \(4 \times 4\) matrix, which preserves the form of \(E\), \((g^{\dagger} E g = E)\) and it has the following coset space decomposition

\[
g = \left( \begin{array}{cc} I & 0 \\ -ix & I \end{array} \right) \left( \begin{array}{cc} \lambda & i\lambda b \\ 0 & (\lambda^{\dagger})^{-1} \end{array} \right)
\]  

(4.9)

where \(\lambda\) is a general complex and \(x, b\) are hermitian \(2 \times 2\) matrices. The curvature of this connection is \(\Omega = d\omega + \omega \wedge \omega = 0\). It is known that not all these \(g\) matrices determine a complex structure of the Minkowski spacetime. They do, if the matrix \(\lambda\) has the form

\[
\lambda = \left( \begin{array}{cc} k & -\bar{w} \bar{k} \\ \bar{k} & k \end{array} \right)
\]  

(4.10)

where \(k, \bar{k}\) are complex and \(w, \bar{w}\) are functions of \(x_{\alpha}^{\dagger}\) implied by the Kerr conditions \(K_{1}(w, x_{0} + x_{0}^{\dagger} w, x_{1} + x_{1}^{\dagger} w) = 0\) and \(K_{2}(\bar{w}, -x_{0}^{\dagger} \bar{w} + x_{0}^{\dagger}, -x_{1}^{\dagger} \bar{w} + x_{1}^{\dagger}) = 0\). The curvature of this lift of Minkowski spacetime continues to vanish, because \(g\) is still an element of \(U(2, 2)\).

In the general case \([3, 4]\) of a lorentzian complex structure which is not compatible with the Minkowski metric, the hermitian matrix \(x_{\alpha}^{\dagger}\) is replaced by the general complex matrix \(r_{\alpha}^{\dagger}\), which is determined by the surface defining conditions. Then \(g\) is no longer an element of \(U(2, 2)\) and the corresponding curvature does not vanish.

In the general case the Cartan connection is

\[
\omega = \left( \begin{array}{cc} e_{1} & ie_{2} \\ -ie_{0} & e_{1} \end{array} \right)
\]  

(4.11)
where \( e_0 \) and \( e_2 \) are \( 2 \times 2 \) hermitian matrices 1-forms. Identifying
\[
e_0 = \begin{pmatrix} \ell & \overline{m} \\ m & n \end{pmatrix}
\] (4.12)
the curvature \( \Omega \) satisfies the relations
\[
de_0 + e_0 \wedge e_1 - e_1^\dagger \wedge e_0 = i\Omega_{12}
\]
\[
de_1 + e_1 \wedge e_1 + e_2 \wedge e_0 = \Omega_{11}
\] (4.13)
\[
de_2 + e_1 \wedge e_2 - e_2 \wedge e_1^\dagger = -i\Omega_{21}
\]
with
\[
e_1 = \begin{pmatrix} \gamma & \nu \\ -\tau & -\gamma \end{pmatrix} \ell + \begin{pmatrix} \varepsilon & \pi \\ -\kappa & -\varepsilon \end{pmatrix} n + \begin{pmatrix} -\beta & -\mu \\ \sigma & \beta \end{pmatrix} m
\]
and
\[
e_2 = \begin{pmatrix} -\Phi_{22} & \Phi_{21} \\ \Phi_{12} & \Lambda - \Phi_{11} \end{pmatrix} \ell + \begin{pmatrix} \Lambda - \Phi_{11} & \Phi_{10} \\ \Phi_{01} & -\Phi_{00} \end{pmatrix} n + \begin{pmatrix} \Phi_{21} & -\Phi_{20} \\ -\Phi_{02} & -\Lambda - \Phi_{11} \end{pmatrix} m
\] (4.14)
and the curvature components are \( \Omega_{12} = 0 \),
\[
\Omega_{11} = \begin{pmatrix} -\Psi_2 & -\Psi_3 \\ \Psi_1 & \Psi_2 \end{pmatrix} \ell \wedge n + \begin{pmatrix} \Psi_3 & \Psi_4 \\ -\Psi_2 & -\Psi_3 \end{pmatrix} n \wedge m + \begin{pmatrix} \Psi_1 & \Psi_2 \\ -\Psi_1 & -\Psi_2 \end{pmatrix} m \wedge \overline{m}
\] (4.15)
and \( \Omega_{21} \) is directly computed.

In the present case of integrable complex structures we have \( \Phi_0 = 0 = \Phi_4 \) and the corresponding codimension-4 CR structures are classified to the following four cases

\textit{Case I} : \( \Psi_1 \neq 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 \neq 0 \)

\textit{Case II} : \( \Psi_1 \neq 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 = 0 \)

\textit{Case III} : \( \Psi_1 \neq 0 \), \( \Psi_2 = 0 \), \( \Psi_3 = 0 \)

\textit{Case D} : \( \Psi_1 = 0 \), \( \Psi_2 \neq 0 \), \( \Psi_3 = 0 \)

(4.16)

Notice that this classification is also related to the number of principal null directions that the spacetime admits through the relation (2.4).
5 A FEFFERMAN-LIKE METRIC

It is well known\[22\] that for any proper bounded domain (and its holomorphic transformations) there is a Bergman kernel

\[ K(z, w) = \sum_j \phi_j(z)\bar{\phi}_j(w) \] (5.1)

where \([\phi_j(z)]\) is a complete orthonormal system of \(L^2\) holomorphic functions. For homogeneous classical domains it can be computed. For the 4(real)-dimensional ball, the Bergman function \(K_H(z) = K_H(z, \overline{z})\) is

\[ K_H(z) = \frac{1}{\pi} \rho^{-3} \] (5.2)

where \(\rho(z, \overline{z}) > 0\) is the defining condition. In this case the behavior of the Bergman function is simple as \(z\) tends to the boundary. For the general case of a strictly pseudoconvex domain, the Bergman function has the same leading singularity, but it has a logarithmic singularity too

\[ K_H(z) = (\phi_0 + \phi_1 \rho + \phi_2 \rho^2)\rho^{-3} + \psi \log \rho + \phi \] (5.3)

where all the functions are regular on the boundary. Fefferman\[5\] has shown that these functions may be asymptotically computed using the metric

\[ ds^2_{F1} = 2 \sum_{j,k=0}^2 \frac{\partial^2 |z_0|^2 \rho}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k \] (5.4)

restricted on \(S^1 \times \text{Boundary}\), where \(S^1\) is the natural bundle \(|z_0| = 1\).

The condition \(\Psi_{11}(\overline{z^\alpha}, z^\alpha) = 0\) determines a 3-dimensional CR submanifold of the spacetime. The corresponding Fefferman metric cannot be identified with the spacetime metric because the Fefferman metric is always Petrov type N, while our spacetime metric admits at least two different principal null directions. But the Fefferman metric may be considered as an asymptotic approximation of the spacetime metric in an appropriate coordinate system. It can also be characterized among all the metrics of a 4-dimensional manifold using (among other restrictions)\[21\] the positivity of the Einstein tensor component \(E_{\mu\nu} K_\mu K_\nu\), where \(K_\mu\) is the tangent vector of \(S^1\).

Let us now proceed to define a Fefferman-like metric for the codimension-4 CR structure of the lorentzian complex structure. We will essentially osculate the Shilov boundary of the \(SU(2, 2)\) invariant classical domain. In this case the Bergman function is\[8\]

\[ K_B(Z) = \frac{1}{V} [\det(1 - Z^\dagger Z)]^{-4} \] (5.5)

where the place of the defining function \(\rho(z, \overline{z})\) takes the \(\det(1 - Z^\dagger Z)\). Hence in complete analogy to the Fefferman procedure I consider the following Kähler metric

\[ ds^2_{F4} = 2 \sum_{j,k} \frac{\partial^2 \Psi}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k \] (5.6)
where $\Psi$ is the $2 \times 2$ matrix \(\begin{bmatrix} 3 & 3 \end{bmatrix}\) which defines the spacetime as a CR manifold. A straightforward calculation gives $f_{IK} = \frac{\partial^2 (\det \Psi)}{\partial x_i \partial x_K}$

$$f_{\beta \alpha} = \Psi x_i \frac{\partial^2 \Psi}{\partial x_i \partial x_\alpha} - \Psi \frac{\partial \Psi}{\partial x_\alpha} \frac{\partial \Psi}{\partial x_i}$$

which on the characteristic boundary $\Psi = 0$ takes the form

$$ds_F^2|_M = (\ell_a n_\beta - m_a m_\beta) dz^a dz^\beta$$

which is the spacetime metric written in structure coordinates. The ambiguity factors of the null tetrads are hidden in the functions $\Psi_{ij} = 0$, which can always be multiplied with non vanishing factors, which do not affect the characteristic boundary. Notice that the null geodesics of this metric project on the Chern-Moser chains of the two hypersurface-type CR submanifolds, which is a characteristic property of the Fefferman metric. It would be interesting to see whether the present Fefferman-like metric plays the same role to the asymptotic computation of the Bergman kernel of \(\begin{bmatrix} 3 & 3 \end{bmatrix}\), like the ordinary Fefferman metric does in the case of hypersurface type CR manifolds.

As a typical example, I will now compute the metric \(\begin{bmatrix} 5 & 6 \end{bmatrix}\) for the flat lorentzian complex structures generally given by the CR conditions $\Psi_{ij} = f_{ij} X^i E X^j = 0$, where $f_{ij}$ are appropriate factors such that the metric becomes Minkowski on the surface. For a change, I will use the Newman complex trajectory \(\begin{bmatrix} 2 \end{bmatrix}\) condition to specify the geodetic and shear free character of the congruences, which in the present formalism takes the form

$$X^{mj} = \begin{pmatrix} \lambda^{A_j} & \xi^{(1)}_{A'B'\lambda} B_j \\ -i\epsilon^{(1)}_{A'B'\lambda} B_1 & \lambda^{A_2} \end{pmatrix}$$

where $\xi^{(1)}$ are two generally independent trajectories. In the case of the assumption of the Kerr-Penrose conditions $K_i(X^{mj}) = 0$, the computational steps are analogous. This condition implies (ordinary) holomorphic transformations between the structure coordinates $z^0 = \tau_1 = z^0(r^a)$, $z^1 = \frac{X^1}{X^{\tau_2}} = z^1(r^a)$, $z^0 = \tau_2 = z^0(r^a)$, $z^1 = -\frac{X^{\tau_0}}{X^{\tau_2}} = z^1(r^a)$ and the complex variables $r^a$. The next step is to find factors $f_{ij}$ such that form \(\begin{bmatrix} 19 \end{bmatrix}\)

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} i(\tau^2 - r^a) L_a & i(\tau^2 - r^a) M_j \\ i(\tau^2 - r^a) M_a & i(\tau^2 - r^a) N_a \end{pmatrix}$$

where \(\begin{bmatrix} 3 & 3 \end{bmatrix}\) defines the flat null tetrad. Then the metric takes the form

$$ds_F^2 = \frac{1}{2} \sum_{a,b} \frac{\partial^2 (-\tau^2 + r^a) \tau^a}{\partial r^a \partial \tau^b} dr^a d\tau^b = \eta_{ab} dr^a dr^b$$
which apparently becomes the Minkowski metric on the surface \(g^a = \text{Im}(r^a) = 0\).

\section{ON THE U(2) AND POINCARE SYMMETRIES}

In section III we showed that a regular 4-dimensional surface can always be transferred inside the \(SU(2, 2)\) classical domain with an (holomorphic) complex time translation. Therefore we may be constrained to the regular surfaces (3.4) inside the \(SU(2, 2)\) classical domain and the flat surface on its characteristic boundary. As quantum configurations, these surfaces must belong to irreducible representation of the \(SU(2, 2)\) group.

The physically interesting asymptotically flat spacetimes, which admit a lorentzian complex structure, are equivalent\textsuperscript{19} to open surfaces with a point (the Penrose \(i^0\) point, where scri+ and scri- meet) at the Shilov boundary. This point breaks \(SU(2, 2)\) group down to its \(\text{Poincaré} \times \text{Dilation}\) subgroup, which is the isotropy group of the boundary\textsuperscript{12}.

In the case of the bounded realization of the \(SU(2, 2)\) classical domain (3.6) we represent the rank-2 matrix \(X^{mi}\) as

\[
X = \begin{pmatrix} T \\ zT \end{pmatrix}
\]

where the \(2 \times 2\) matrices \(r\) of the unbounded realization (3.7) and \(z\) are related with

\[
r = i(I + z)(I - z)^{-1}
\]

which implies that the point \(z = I\) of the characteristic boundary of the bounded realization of the homogeneous domain is mapped to the infinity of the corresponding Siegel domain. In the bounded realization, a general \(SU(2, 2)\) transformation is

\[
\begin{pmatrix} T' \\ z'T' \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} T \\ zT \end{pmatrix}
\]

\[
z' = (A_{21} + A_{22} z)(A_{11} + A_{12} z)^{-1}
\]

where the \(2 \times 2\) matrices \(A_{ij}\) must satisfy the conditions

\[
A_{11}^\dagger A_{11} - A_{21}^\dagger A_{21} = I \quad , \quad A_{11}^\dagger A_{12} - A_{21}^\dagger A_{22} = 0 \quad , \quad A_{22}^\dagger A_{22} - A_{12}^\dagger A_{12} = I
\]

\[
(6.4)
\]

The \(z = I\) stability subgroup \(P_I\) must satisfy

\[
A_{21} + A_{22} = A_{11} + A_{12}
\]

which makes the last condition of (6.4) a simple identity. This isotropy subgroup is in fact the bounded realization \(\text{Poincaré} \times \text{Dilation}\) subgroup, which becomes a linear transformation in its unbounded (Siegel domain) realization.
Hence the open surfaces "hanging" from a fixed point of the boundary belong to representations of the Poincaré group.

The characteristic boundary of the bounded $SU(2, 2)$ homogeneous domain is the $U(2)$ manifold. Under a general $U(2, 2)$ transformation the Poincaré representation of surfaces of the $z = I$ point will be transformed to the

$$U = (A_{21} + A_{22}) (A_{11} + A_{12})^{-1}$$

(6.6)

point of $U(2)$. Two $U_1$ and $U_2$ points of the boundary are always connected with the $U(2)$ group transformation $u$

$$\begin{pmatrix} T_2 \\ U_2 T_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & U_2 U_1^{-1} \end{pmatrix} \begin{pmatrix} T_1 \\ U_1 T_1 \end{pmatrix}$$

(6.7)

Therefore the corresponding isotropy subgroups are isomorphic, because $P_2 = u \cdot P_1 \cdot u^{-1}$. If in the present Quantum Field Theoretic model the vacuum is the Minkowski part of the Shilov boundary\[19] with a precise infinity point $i^0$, the $SU(2, 2)$ symmetry will be spontaneously broken to the Poincaré subgroup. The dilation group is also expected to be broken. The $U(2)$ group, which transfers the Poincaré representations of the $z = I$ infinity point to the other points of the boundary, will be broken too.

It is well known\[11] that Minkowski spacetime is topologically different than Kerr-Newman type of spacetimes, because the former can be compactified through the continuity of its geodesics at scri, while in the latter spacetimes it is obstructed by the mass. Current phenomenology indicates that Poincaré representations of the broken $U(2)$ modes of surfaces homotopic to Minkowski spacetime should be vector and scalar (Higgs) bosonic fields. The Kerr-Newman type solitonic sector should appear like the electronic multiplet of the Standard Model. I think that all these points will be clarified when an energy operator compatible with the Poincaré subgroup of $SU(2, 2)$ symmetry is found.

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