Linear Diophantine Equations, Group CSPs, and Graph Isomorphism

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Abstract

In recent years, we have seen several approaches to the graph isomorphism problem based on “generic” mathematical programming or algebraic (Gröbner basis) techniques. For most of these, lower bounds have been established. In fact, it has been shown that the pairs of non-isomorphic CFI-graphs (introduced by Cai, Fürer, and Immerman in 1992 as hard examples for the combinatorial Weisfeiler-Leman algorithm) cannot be distinguished by these mathematical algorithms. A notable exception were the algebraic algorithms over the field $\mathbb{F}_2$, for which no lower bound was known. Another, in some way even stronger, approach to graph isomorphism testing is based on solving systems of linear Diophantine equations (that is, linear equations over the integers), which is known to be possible in polynomial time. So far, no lower bounds for this approach were known.

Lower bounds for the algebraic algorithms can best be proved in the framework of proof complexity, where they can be phrased as lower bounds for algebraic proof systems such as Nullstellensatz or the (more powerful) polynomial calculus. We give new hard examples for these systems: families of pairs of non-isomorphic graphs that are hard to distinguish by polynomial calculus proofs simultaneously over all prime fields, including $\mathbb{F}_2$, as well as examples that are hard to distinguish by the systems-of-linear-Diophantine-equations approach.

In a previous paper, we observed that the CFI-graphs are closely related to what we call “group CSPs”: constraint satisfaction problems where the constraints are membership tests in some coset of a subgroup of a cartesian power of a base group ($\mathbb{Z}_2$ in the case of the classical CFI-graphs). Our new examples are also based on group CSPs (for Abelian groups), but here we extend the CSPs by a few non-group constraints to obtain even harder instances for graph isomorphism.

1 Introduction

The graph isomorphism problem is famous for its unsolved complexity status, and despite exciting recent developments in graph isomorphism testing \cite{14}, a polynomial time algorithm is not in sight. Recently, generic mathematical programming and algebraic techniques applied to graph isomorphism have received considerable attention \cite{3, 6, 13, 16, 11, 17, 19}. The basic idea is to encode the isomorphism problem for two given graphs $G$ and $H$ into a system of equalities and inequalities in variables $[v \mapsto w]$ for vertices $v \in V(G)$ and $w \in V(H)$. The intended meaning of the variable $[v \mapsto w]$ is to indicate whether $v$ is mapped to $w$ (value 1) or not (value 0). The coding details depend on the exact algorithmic framework: sometimes we use linear equalities and inequalities, sometimes we use linear and quadratic equalities, and sometimes we use additional variables such as $[v_1 \mapsto w_1, \ldots, v_\ell \mapsto w_\ell]$ indicating that $v_i$ is mapped to $w_i$ for $i = 1, \ldots, \ell$. Furthermore, we interpret the equations over different fields and rings. Then we solve or try to solve the system...
(using different methods like linear or semi-definite programming or Gröbner bases), which should tell us whether the given graphs are isomorphic, but not always does. All the polynomial time algorithms based on this paradigm either correctly detect that the graphs are non-isomorphic or give no definite answer. In the former case, we say that the algorithm distinguishes the graphs. Hence to prove that the algorithm is a complete isomorphism test we have to show that it distinguishes all pairs of non-isomorphic graphs. Not surprisingly, most of these algorithms have been proved to be incomplete. Somewhat surprisingly, despite the considerable variation of systems that have been studied, it has turned out that all these algorithms are very similar in their distinguishing power. In particular, they all fail to distinguish the non-isomorphic pairs of CFI-graphs, introduced by Cai, Fürer, and Immerman [9] to prove that the Weisfeiler-Leman (WL) algorithm, a combinatorial graph isomorphism test, is incomplete. The distinguishing power as well as the running time of all these algorithms is governed by a parameter \( \ell \), which is the degree of the polynomials considered by a Gröbner basis algorithm, the “level” in a hierarchy of linear and semidefinite programming relaxations, or the “dimension” of the WL algorithm. Proving lower bounds for any of the algorithms means proving lower bounds on the parameter \( \ell \) necessary to distinguish the input graphs. For almost all of the algorithms, the CFI-graphs yield a lower bound on \( \ell \) that is linear in the size \( n \) of the input graphs. As the running time of the algorithms is \( n^{\Theta(\ell)} \), these lower bounds not only show that the polynomial time restrictions of the algorithms are incomplete isomorphism tests, but are in fact much stronger. There are two algorithms among those considered in this context whose incompleteness had not been established. The first is based on solving systems of linear Diophantine equations (that is, linear equations over the integers), which is possible in polynomial time (see, for example, [19]). The second is based on the Gröbner basis algorithm over fields of characteristic 2. We prove lower bounds for both of these algorithms. Before we explain how these lower bounds are obtained, let us discuss both algorithms in more detail.

The systems of linear Diophantine equations for two graphs \( G, H \) are obtained as follows. We start from a standard integer linear program in the variables \( [v \mapsto w] \) whose nonnegative integral solutions are the isomorphisms between \( G \) and \( H \). As the standard LP-relaxation is fairly weak, we strengthen the system using lift-and-project methods, specifically the Sherali-Adams hierarchy [20]. The \( \ell \)-th level of the hierarchy for graph isomorphism consists of \( n^{\Theta(\ell)} \) linear equalities in the variables \( [v_1 \mapsto w_1, \ldots, v_{\ell} \mapsto w_{\ell}] \). Now instead of dropping the integrality constraints, as one typically does in combinatorial optimisation, we drop the nonnegativity constraints and are left with a system of linear Diophantine equations. Our Diophantine isomorphism test solves this system over the integers, which is possible in time \( n^{\Theta(\ell)} \), and then answers “non-isomorphic” if no solution exists. What is remarkable about this algorithm is that it distinguishes all pairs of CFI-graphs, and not only that, but also the variants of the CFI-graphs modulo \( p \) for all primes \( p \). (The CFI-graphs may be viewed as graph encodings of systems of linear equations modulo 2, and they have natural variants modulo \( p \).) As the CFI-graphs and their variants are used in all previous lower bound proofs—arguably, the CFI-construction is the only systematic construction of hard examples for graph isomorphism that is known—this explains why no lower bounds for the Diophantine isomorphism test were known.

Algebraic algorithms for graph isomorphism start from similar equations in variables \( [v \mapsto w] \) as the integer linear program, except that nonnegativity constraints are replaced by polynomial equations \( [v \mapsto w]^2 = [v \mapsto w] \) to ensure \( \{0,1\}\)-solutions. These algorithms can best be analysed by algebraic proof systems such as Nullstellensatz [5] or the (more powerful) polynomial calculus [10], which captures the power of the Gröbner basis algorithm. In this setting, non-isomorphic graphs can be efficiently distinguished if they have a refutation of low degree over some field \( \mathbb{F} \). In a previous paper [6], we established degree lower bounds for graph isomorphism in the polynomial calculus over all fields except fields of characteristic 2. These lower bounds were obtained by a
reduction from the so-called Tseitin tautologies in a version due to Buss et al. [7], for which lower bounds were known in all characteristics but 2. In this paper, the lower bounds are based on a different construction due to Alekhnovich and Razborov [1], which also provides hard instances over fields of characteristic 2. More significantly, we construct families of pairs of non-isomorphic graphs for which we can prove lower bounds for the polynomial calculus that simultaneously hold for all prime fields.

To prove the lower bounds, both for linear Diophantine equations and the polynomial calculus over all prime fields, we cannot use the CFI-instances, because they are distinguished by both algorithms (in the case of polynomial calculus: the CFI-instances modulo \( p \) are distinguished over the field \( \mathbb{F}_p \)). In [6], we established a close connection between the CFI-instances and what we call group CSPs, that is, constraint satisfaction problems where the constraints are membership tests in some coset of a cartesian power of a base group. For the “classical” CFI-instances modulo \( p \), this group is \( \mathbb{Z}_p \). We can associate a pair of non-isomorphic graphs with any instance of an unsatisfiable group CSP, but unfortunately, this generalisation still does not suffice for the lower bound proof. A crucial new idea of this paper is to enhance the group CSPs by an additional constraint of bounded size. This yields what we call an \( e \)-extended group CSP, where \( e \) is the size (number of permitted values) of the non-group constraint. We show that for every fixed \( e \), instances of \( e \)-extended group CSPs can still be translated to pairs of non-isomorphic graphs. We apply this construction to group CSPs over the group \( \mathbb{Z}_2 \times \mathbb{Z}_3 \) and use the non-group constraint to introduce a “disjunction” between the subgroups \( \mathbb{Z}_2 \times \{0\} \) and \( \{0\} \times \mathbb{Z}_3 \). The resulting pairs of non-isomorphic graphs are hard to distinguish for linear Diophantine equations and the polynomial calculus simultaneously over all prime fields.

We believe that our construction, while still rooted in the CFI-constructions, adds a genuinely new aspect and thus provides new hard graph-isomorphism instances which may be useful in other contexts as well.

Related Work

The connection between the linear programming approach to graph isomorphism and the 1-dimensional WL algorithm (a.k.a colour refinement) goes back to Tinhofer [21]. The correspondence between the levels of the Sherali-Adams hierarchy and the higher-dimensional WL was established by Atserias and Maneva [3] and independently Malkin [16] and later refined by Grohe and Otto [13]. O’Donnell et. al. [17] and Codenotti et al. [11] proved that that even the more powerful semi-definite Lasserre hierarchy fails to distinguish CFI-graphs.

A related approach of applying algebraic techniques to graph isomorphism was initiated by the authors of this paper in [6]. We proved lower bounds for the polynomial calculus over all fields of characteristic \( \neq 2 \) and also established close connections between the algebraic approach, the linear programming approach, and the WL-algorithm. For a detailed discussion of these connections, we refer the reader to [6].

2 Linear Equations for Graph Isomorphism and CSP

2.1 Preliminaries

In general, we use standard notation and terminology, but let us highlight a few points. We denote the vertex and edge set of a directed or undirected graph \( G \) by \( V(G) \) and \( E(G) \), respectively. We

\footnote{We suspect that the Diophantine isomorphism test can distinguish these graphs for all group CSPs, or at least all Abelian group CSPs, but we can only prove this for Abelian groups that are direct products of prime groups \( \mathbb{Z}_p \).}
denote the edges of an undirected graph by \( vw \) (instead of \( \{v, w\} \)) and the edges of a directed graph by \( (v, w) \). An orientation of an undirected graph \( G \) is a directed graph \( D \) such that \( V(D) = V(G) \) and \( E(D) \) contains exactly one of \( (v, w), (w, v) \) for all \( vw \in E(G) \).

If \( G \) is an undirected graph, for every set \( W \subseteq V(G) \) we let \( E(W) \) be the set of all edges incident with a vertex in \( W \) and \( \partial(W) \), the boundary of \( W \), the set of all edges incident with a vertex in \( W \) and a vertex in \( V \setminus W \). Note that \( \partial(W) = \partial(V \setminus W) = E(W) \cap E(V \setminus W) \). If \( D \) is a directed graph, for every subset \( W \subseteq V(D) \), we let \( \partial_-(W) \) be the set of all edges of \( D \) with head in \( W \) and tail in \( V \setminus W \) and \( \partial_+(D) \) the set of all edges of \( H \) with tail in \( W \) and head in \( V \setminus W \). We write \( \partial(v) \) instead of \( \partial(\{v\}) \), and similarly \( \partial_-(v), \partial_+(v) \).

Both for undirected and directed \( G \) and \( W \subseteq V(G) \), by \( G[W] \) we denote the induced subgraph of \( G \) with vertex set \( W \), and we let \( G \setminus W := G[V(G) \setminus W] \). Moreover, for \( F \subseteq E(G) \) we let \( G - F := (V(G), E(G) \setminus F) \).

The degree of a vertex \( v \) of an undirected graph is \( |\partial(v)| \), and the degree of a vertex \( v \) of a directed graph is \( |\partial_-(v)| + |\partial_+(v)| \). A directed or undirected graph is \( d \)-regular if every vertex has degree \( d \).

Recall that an instance of the constraint satisfaction problem (CSP) is a triple \((X, D, \mathcal{C})\), where \( X \) is a finite set of variables, \( D \) a finite domain, and \( \mathcal{C} \) a set of constraints of the form \((x, R)\), where \( x \in X^k \) and \( R \subseteq D^k \), for some \( k \geq 0 \). An assignment \( \varphi : X \to D \) satisfies the constraint if \( \varphi(x) \in R \). The arity of the constraint is \( k \), and the arity of the instance \((X, D, \mathcal{C})\) is the maximum arity of its constraints. When the domain is clear from the context we specify CSPs by the set \( \mathcal{C} \) of their constraints and let the variables be given implicitly. In this case, we refer to the set of variables of \( \mathcal{C} \) by \( \text{Var}(\mathcal{C}) \) and to the domain by \( \text{Dom}(\mathcal{C}) \).

### 2.2 Equations for Graph Isomorphism and CSP

Given two graphs \( G, H \) we introduce for \( \ell \geq 1 \) a system of linear equations \( L_{\text{iso}}^\ell(G, H) \). These systems form a hierarchy \( L_{\text{iso}}^1 \subset L_{\text{iso}}^2 \subset \cdots \) and are equivalent to the Sherali-Adams hierarchy of relaxations for a natural linear programming formulation of the graph isomorphism problem, see [6] for a more detailed discussion of encodings. The variables of \( L_{\text{iso}}^\ell(G, H) \) are \([\pi]\) for sets \( \pi \subseteq V(G) \times V(H) \) of size \(|\pi| \leq \ell \). We interpret these sets as partial mappings from \( V(G) \) to \( V(H) \). For sets \( \pi \) that do not correspond to partial mappings the system will have an equation \([\pi] = 0\), and thus we can ignore such \( \pi \). To emphasise the partial-mapping view, we write \([v_1 \mapsto w_1, \ldots, v_m \mapsto w_m]\) instead of \([[\{ (v_1, w_1), \ldots, (v_m, w_m) \}]]\). We also write \([\pi, v \mapsto w]\) instead of \([\pi \cup \{ (v, w) \}]\).

We say that \( \pi \) is a partial isomorphism from \( G \) to \( H \) if it is an injective partial mapping that additionally preserves adjacencies, that is \( vw \in E(G) \iff \pi(v)\pi(w) \in E(H) \). If \( G \) and \( H \) are coloured graphs, partial isomorphisms are also required to preserve colours. We let \( L_{\text{iso}}^\ell(G, H) \) be the following system of linear equations:

\[
\sum_{w \in V(H)} [\pi, v \mapsto w] = [\pi] \quad \text{for all } \pi \subseteq V(G) \times V(H) \text{ of size } |\pi| \leq \ell - 1 \text{ and all } v \in V(G),
\]

\[
\sum_{w \in V(H)} [\pi, v \mapsto w] = [\pi] \quad \text{for all } \pi \subseteq V(G) \times V(H) \text{ of size } |\pi| \leq \ell - 1 \text{ and all } v \in V(G),
\]

\[
[\emptyset] = 1,
\]

\[
[\pi] = 0 \quad \text{for all } \pi \subseteq V(G) \times V(H) \text{ of size } |\pi| \leq \ell \text{ such that } \pi \text{ is not a partial isomorphism.}
\]

Note that for all \( \ell \geq 2 \) the system \( L_{\text{iso}}^\ell(G, H) \) has a nonnegative integral solution if and only if the graphs are isomorphic.
In the same way we define for a CSP $C$ and $\ell \geq 1$ the system of linear equations $L^\ell_{\text{csp}}(C)$. The variables of our system are $[\psi]$ for sets $\psi \subseteq \text{Var}(C) \times \text{Dom}(C)$ of size $|\psi| \leq \ell$. We interpret sets as partial mappings from $\text{Var}(C)$ to $\text{Dom}(C)$, which are intended to be partial solutions, that is, partial mappings that satisfy all constraints whose variables are in the domain of $\psi$. We denote the domain of $\psi$ by $\text{dom}(\psi)$ and also use notations like $[x_1 \mapsto \gamma_1, \ldots, x_m \mapsto \gamma_m]$ or $[x \mapsto \gamma]$ or $[\psi, x \mapsto \gamma]$. We let $L^\ell_{\text{csp}}(C)$ be the following system of linear equations:

$$
\sum_{\gamma \in D} [\psi, x \mapsto \gamma] = [\psi] \quad \text{for all } \psi \subseteq \text{Var}(C) \times \text{Dom}(C) \text{ of size } |\psi| \leq \ell - 1 \text{ and all } x \in \text{Var}(C),
$$

(2.E)

$$
[\emptyset] = 1,
$$

(2.F)

$$
[\psi] = 0 \quad \text{for all } \psi \subseteq \text{Var}(C) \times \text{Dom}(C) \text{ of size } |\psi| \leq \ell \text{ such that } \psi \text{ is not a partial solution.}
$$

(2.G)

If $C$ is a $k$-ary CSP-instance and $\ell \geq k$, then the system $L^\ell_{\text{csp}}(C)$ has a nonnegative integral solution if and only if $C$ is satisfiable. We are interested in the (not necessarily nonnegative) integral solutions of $L_{\text{iso}}$ and $L_{\text{csp}}$. As one step towards this we also consider a certain type of rational solutions: for an integer $p$, a $p$-solution of a system of linear equations is a satisfying assignment over $\mathbb{Z}$ which has a value of the form $p^z$ for $z \in \mathbb{Z}$. The next lemma states a criterion for the existence of integral solutions.

**Lemma 2.1.** Let $L$ be a system of linear equations over $\mathbb{Z}$, and let $p, q \in \mathbb{Z}$ be co-prime. If $L$ has a $p$-solution and a $q$-solution, then it has an integral solution.

**Proof.** Suppose that $L$ is of the form $\{ x \mid Mx = b \}$ for a matrix $M \in \mathbb{Z}^{m \times n}$ and a vector $b \in \mathbb{Z}^m$. Let $x, y \in \mathbb{Q}^n$ be solution vectors over $\{0\} \cup \{p^z \mid z \in \mathbb{Z}\}$ and $\{0\} \cup \{q^z \mid z \in \mathbb{Z}\}$, respectively. If one of these solutions is already integral there is nothing to prove. Otherwise, let $z \geq 1$ be maximal such that $x$ contains a value of the form $p^{-z}$ or $y$ contains a value of the form $q^{-z}$. Note that $p^z x, q^z y \in \mathbb{Z}^n$. Because $p^z$ and $q^z$ are relatively prime, there are integers $\alpha, \beta \in \mathbb{Z}$ such that $\alpha p^z + \beta q^z = 1$. Now we have

$$
M \cdot (\alpha p^z x + \beta q^z y) = \alpha p^z Mx + \beta q^z My = \alpha p^z b + \beta q^z b = (\alpha p^z + \beta q^z)b = b.
$$

(2.H)

Hence, $\alpha p^z x + \beta q^z y$ is an integral solution for $L$. \qed

### 2.3 Tseitin Tautologies

For a directed graph $H$, and Abelian group $\Gamma$, and a mapping $\sigma : V(H) \rightarrow \Gamma$, the $\Gamma$-Tseitin tautology $C^{H,\Gamma,\sigma}$ is the following CSP with domain $\Gamma$ and variables $x_e$ for $e \in E(H)$. For every $v \in V(H)$ of degree $k$, the CSP $C^{H,\Gamma,\sigma}$ has a $k$-ary constraint $C^{H,\Gamma,\sigma}(v)$ defined by the equation

$$
\sum_{e \in \partial_+(v)} x_e - \sum_{e \in \partial_-(v)} x_e = \sigma(v)
$$

(2.I)

over $\Gamma$. It is easy to see that if $\sum_{v \in V(H)} \sigma(v) \neq 0$, then $C^{H,\Gamma,\sigma}$ has no solution. For $\Gamma = \mathbb{Z}_2$, the $\Gamma$-Tseitin tautologies are the classical Tseitin tautologies [22]. The graph $H$ is typically a $k$-regular expander graph (see Appendix A).

The $\mathbb{Z}_p$-Tseitin tautologies were defined in [1] and have been used to prove degree lower bounds for polynomial calculus over all fields whose characteristic $q$ contains a primitive $p$th root of unity. In [1], this lower bound was extended to all fields of characteristic $q \neq p$, even for the more restricted variant of Boolean $\mathbb{Z}_p$-Tseitin tautologies $B^{H,\mathbb{Z}_p,\sigma}$, which have the same variables and constraints as the $\mathbb{Z}_p$-Tseitin tautologies, but the smaller domain $\{0, 1\}$.

5
3 Extended Group CSPs

We recall the notion of group CSPs introduced in [3], but restrict ourselves to finite Abelian groups \( \Gamma \) (written additively). An instance of a \( \Gamma \)-CSP has domain \( \Gamma \) and constraints of the form \( ((x_1, \ldots, x_k), \Delta + \gamma) \), where \( \Delta \leq \Gamma^k \) is a subgroup and \( \Delta + \gamma \) a coset for some \( \gamma \in \Gamma^k \). For example, the \( \Gamma \)-Tseitin tautologies \( \mathcal{C}_{H,\Gamma}^{\Delta+\gamma} \) are \( \Gamma \)-CSPs (but not their Boolean versions).

With each constraint \( C = ((x_1, \ldots, x_k), \Delta + \gamma) \), we associate the homogeneous constraint \( \tilde{C} = ((x_1, \ldots, x_k), \Delta) \). For an instance \( \mathcal{C} \), we let \( \tilde{\mathcal{C}} = \{ \tilde{C} \mid C \in \mathcal{C} \} \). For every group CSP \( \mathcal{C} \) we define two graphs \( G(\mathcal{C}), \tilde{G}(\mathcal{C}) \) that are isomorphic if and only if \( \mathcal{C} \) is satisfiable. Let \( \mathcal{C} \) be a \( \Gamma \)-CSP. We construct a coloured graph \( G(\mathcal{C}) \) as follows.

- For every variable \( x \in \text{Var}(\mathcal{C}) \) we take vertices \( \gamma(x) \) for all \( \gamma \in \Gamma \). We colour all these vertices with a fresh colour \( L(x) \).
- For every constraint \( C = ((x_1, \ldots, x_k), \Delta + \gamma) \in \mathcal{C} \) we add vertices \( \beta(C) \) for all \( \beta \in \Delta + \gamma \). We colour all these vertices with a fresh colour \( L(C) \). If \( \beta = (\beta_1, \ldots, \beta_k) \), we add an edge \( \{\beta(C), \beta_i(x)\} \) for all \( i \in [k] \). We colour this edge with colour \( M(i) \).

We let \( \tilde{G}(\mathcal{C}) \) be the graph \( G(\tilde{\mathcal{C}}) \) where for all constraints \( C \in \mathcal{C} \) we identify the two colours \( L(C) \) and \( L(\tilde{C}) \). We call \( G(\mathcal{C}) \) and \( \tilde{G}(\mathcal{C}) \) the CFI-graphs over \( \mathcal{C} \). The CFI-graphs over the \( \mathbb{Z}_2 \)-Tseitin tautologies \( \mathcal{C}_{H,\mathbb{Z}_2}^{\Delta+\gamma} \) are just the “standard” CFI-graphs, going back to Cai, F"urer, and Immerman [9]. These graphs have been intensively studied and applied in the finite-model-theory literature (and elsewhere).

It was show in [3] Lemma 2.1] that the CFI-graphs over \( \mathcal{C} \) are isomorphic if and only if \( \mathcal{C} \) is satisfiable. The next lemma additionally shows \( p \)-solutions can be transferred from the corresponding \( \mathcal{L}_{\text{csp}} \) to \( \mathcal{L}_{\text{iso}} \).

**Lemma 3.1.** Let \( \Gamma \) be an Abelian group and \( \mathcal{C} \) a \( \Gamma \)-CSP of arity \( k \).

(a) \( \mathcal{C} \) is satisfiable if and only if \( G(\mathcal{C}) \) and \( \tilde{G}(\mathcal{C}) \) are isomorphic.

(b) If \( \mathcal{L}_{\text{csp}}^k(\mathcal{C}) \) has a \( p \)-solution, then so does \( \mathcal{L}_{\text{iso}}^k(G(\mathcal{C}), \tilde{G}(\mathcal{C})) \).

**Proof.** For the first statement we repeat the argument from [3], showing that for every satisfying assignment \( \varphi \) of \( \mathcal{C} \) there is an isomorphism \( \pi_\varphi \) between \( G(\mathcal{C}) \) and \( \tilde{G}(\mathcal{C}) \) and for every isomorphism \( \pi_\varphi \) between \( G(\mathcal{C}) \) and \( \tilde{G}(\mathcal{C}) \) there is a satisfying assignment \( \varphi_\pi \) of the \( \mathcal{C} \). Let \( G = (V,E) := G(\mathcal{C}) \) and \( \tilde{G} = (\tilde{V}, \tilde{E}) := \tilde{G}(\mathcal{C}) \). Let \( \varphi : \mathcal{X} \to \Gamma \) be a satisfying assignment for \( \mathcal{C} \). We define a mapping \( \pi_\varphi : V \to \tilde{V} \) as follows:

- For every \( x \in \mathcal{X} \) and \( \gamma \in \Gamma \) we let \( \pi_\varphi(\gamma(x)) := (\gamma - \varphi(x))^{(x)} \).
- For every \( C = (x_1, \ldots, x_k, \Delta + \gamma) \in \mathcal{C} \) and every \( \beta = (\beta_1, \ldots, \beta_k) \in \Delta + \gamma \) we let
  \[ \pi_\varphi(\beta(C)) := (\beta_1 - \varphi(x_1), \ldots, \beta_k - \varphi(x_k))^{(C)} \].

To see that this is well defined, note that \( \varphi(x) := (\varphi(x_1), \ldots, \varphi(x_k)) \in \Delta + \gamma \), because \( \varphi \) satisfies the constraint \( C \). Thus
  \[ \beta - \varphi(x) = (\beta_1 - \varphi(x_1), \ldots, \beta_k - \varphi(x_k)) \in \Delta \].
It is easy to see that the mapping $\pi_\varphi$ is bijective. To see that it is an isomorphism, consider, for some constraint $C = ((x_1, \ldots, x_k), \Delta + \gamma) \in \mathcal{C}$ and some $i \in [k]$, a vertex $\beta(C)$, where $\beta = (\beta_1, \ldots, \beta_k) \in \Delta + \gamma$, and a vertex $\gamma(x_i)$, where $\gamma \in \Gamma$. Then
\[
\{\beta(C), \gamma(x_i)\} \in E \iff \beta_i = \gamma \iff \beta_i - \varphi(x_i) = \gamma - \varphi(x_i)
\]
\[
\iff \{\pi_\varphi(\beta(C)), \pi_\varphi(\gamma(x_i))\} \in \widetilde{E}.
\]
To prove the backward direction, suppose that $\pi$ is an isomorphism from $G$ to $\tilde{G}$. We define an assignment $\varphi_\pi : \mathcal{X} \to \Gamma$ by
\[
\varphi_\pi(x)^{(x)} = \pi^{-1}(0^{(x)}).
\]
(Here $0^{(x)}$ denotes the $x$-copy of the unit element $0 \in \Gamma$ in the graph $\tilde{G}$.) To see that $\varphi_\pi$ is a satisfying assignment, consider a constraint $C = (x_1, \ldots, x_k, \Delta + \gamma) \in \mathcal{C}$. Let $\beta = (\beta_1, \ldots, \beta_k)$ with $\beta_i = \varphi_\pi(x_i)$. We need to prove that $\beta \in \Delta + \gamma$. We have $\pi(\beta_i^{(x_i)}) = 0^{(x_i)}$. As $0 = (0, \ldots, 0) \in \Delta$, the vertex $0^{(C)} \in \widetilde{V}$ has edges to all vertices $\pi(\beta_i^{(x_i)})$. Thus the vertex $\pi^{-1}(0^{(C)})$ has colour $L(C) = L(C)$ and edges to the vertices $\beta_i^{(x_i)}$. This implies that $\pi^{-1}(0^{(C)}) = \alpha^{(C)}$ for some $\alpha \in \Delta + \gamma$ and $\alpha = (\beta_1, \ldots, \beta_k) = \beta$. This concludes the proof of [a]

For the translation [b] of satisfying assignments from $L^{\ell}_{csp}(G)$ to $L^{\ell}_{iso}$, let $\Phi_{csp} : \text{Var}(L^{\ell}_{csp}(G)) \to \mathbb{Q}$ be a $p$-solution. We define an assignment $\Phi_{iso} : \text{Var}(L^{\ell}_{iso}(G^{(C)}, G^{(C)})) \to \mathbb{Q}$ as follows. We let $\Phi_{iso}([0]) = 1$. Let $\pi = \{(v_1, w_1), \ldots, (v_m, w_m)\}$. If $\pi$ is not a partial isomorphism we let $\Phi_{iso}([\pi]) = 0$. Otherwise, all $(v_i, w_i)$ are of the form $(\gamma(x), \gamma'(x))$ or $(\alpha(C), (\beta(C))$. We use
\[
\psi_\pi := \{(x, \gamma - \gamma') | (\gamma(x), \gamma'(x)) \in \pi\}
\]
\[
\cup \{(x, \alpha_j - \beta_j) | ((\alpha_1, \ldots, \alpha_{k'}), (\beta_1, \ldots, \beta_{k'})) \in \pi
\]
for some $C = ((x_1, \ldots, x_{k'}), \Delta + \gamma) \in \mathcal{C}$ and $1 \leq j \leq k'$
\[
\text{(3.C)}
\]
and set $\Phi_{iso}([\pi]) := \Phi_{csp}([\psi_\pi])$, noting that $|\psi_\pi| \leq k|\pi| \leq k\ell$. It is clear that if $\Phi_{csp}$ takes values from $\{0\} \cup \{p^z | z \in \mathbb{Z}\}$ for some prime $p$, then so does $\Phi_{iso}$. We have to check that this assignment satisfies the equations [2.A]–[2.D] from $\Phi_{iso}$. First note that [2.C] and [2.D] are satisfied by definition. We show that $\Phi_{iso}$ satisfies all equations of the form [2.A] \sum_{w \in V(H)} [\pi, v \mapsto w] = [\pi]$ for some $v$, the argument for [2.B] is symmetric. First suppose that $v = \gamma(x)$, it follows that
\[
\sum_{w \in V(H)} \Phi_{iso}([\pi, \gamma(x) \mapsto w]) = \sum_{\gamma' \in \Gamma} \Phi_{iso}([\pi, \gamma(x) \mapsto \gamma'(x)])
\]
\[
= \sum_{\gamma' \in \Gamma} \Phi_{csp}([\psi_\pi, x \mapsto \gamma']) = \Phi_{csp}([\psi_\pi]) = \Phi_{iso}([\pi]), \text{ (3.E)}
\]
where [3.E] follows from [2.E]. Now suppose that $v = \beta(C)$ for some $\beta = (\beta_1, \ldots, \beta_{k'})$ and $C = ((x_1, \ldots, x_{k'}), \Delta + \gamma)$. Similar as above we have
\[
\sum_{w \in V(H)} \Phi_{iso}([\pi, \beta(C) \mapsto w]) = \sum_{\alpha \in \Gamma^{k'}} \Phi_{iso}([\pi, \beta(C) \mapsto \alpha(C)])
\]
\[
= \sum_{\alpha \in \Gamma^{k'}} \Phi_{csp}([\psi_\pi, x_1 \mapsto \alpha_1, \ldots, x_{k'} \mapsto \alpha_{k'}]) = \Phi_{csp}([\psi_\pi]) = \Phi_{iso}([\pi]), \text{ (3.H)}
\]
This concludes the proof of [b]
Now we extend group CSPs by small non-group constraints and provide a similar graph encoding for them as for group CSPs. Extended group CSPs will later play a crucial role in the lower bound arguments for the Diophantine equations (Section 4) and the polynomial calculus (Section 6). Let $\Gamma$ be a finite Abelian group. An $e$-extended $\Gamma$-CSP has a constraint set $C^* = C \cup \{C_{arb}\}$, where $C$ defines a $\Gamma$-CSP, and $C_{arb} = (x, R_{arb})$ is an additional constraint with $|R_{arb}| \leq e$. For $\gamma \in R_{arb}$ we let $C_{\gamma} := C \cup \{(x, \{\gamma\})\}$ be the CSP obtained by fixing the variables in the constraint $C_{arb}$. Observe that $C_{\gamma}$ is a $\Gamma$-CSP, for every $\gamma \in \Gamma^k$, even if $C^*$ is not. Furthermore, $C \cup \{C_{arb}\}$ is satisfiable if and only if there exists an $\gamma \in R_{arb}$ such that $C_{\gamma}$ is satisfiable. We now show how to encode extended group CSPs into instances of graphs isomorphism, that is, we prove an analogue of Lemma 3.1 for extended group CSPs. Let us start by reviewing a well-known “or-construction” for graph isomorphism (see [15]). For mutually disjoint graphs $G_1, \ldots, G_{\ell}$, we let $G_1 \uplus \ldots \uplus G_{\ell}$ be the disjoint union of the $G_i$ (we also write $\biguplus_{i=1}^{\ell} G_i$), and we let $\langle G_1, \ldots, G_{\ell}\rangle$ be the graph obtained from the disjoint union $G_1 \uplus \ldots \uplus G_{\ell}$ by adding fresh vertices $v_1, \ldots, v_\ell$ and edges from $v_i$ to all vertices in $V(G_1)$ and from $v_i$ to all vertices in $V(G_{i-1}) \cup V(G_i)$ for all $i \geq 2$. Thus $\langle G_1, \ldots, G_{\ell}\rangle$ encodes an ordered sequence of the graphs $G_i$ and it is not hard to show that two sequence graphs $\langle G_1, \ldots, G_{\ell}\rangle$ and $\langle H_1, \ldots, H_{\ell}\rangle$ are isomorphic if and only if all pairs $G_i, H_i$ are isomorphic.

**Definition 3.2.** Let $(G^0_i, G^1_i)_{i \in [\ell]}$ be a sequence of pairs of graphs. We define the graph pair $(G^0, G^1) = \bigvee_{i \in [\ell]} (G^0_i, G^1_i)$ as follows. For $j \in \{0, 1\}$ let

$$G^j = \bigcup \left\{ \langle G^a_{i_1}, \ldots, G^a_{i_\ell} \rangle \mid \sum_{i=1}^{\ell} a_i \equiv j \pmod{2} \right\}.$$  

(3.1)

**Lemma 3.3.** Let $(G^0, G^1) = \bigvee_{i \in [\ell]} (G^0_i, G^1_i)$. Then $G^0$ and $G^1$ are isomorphic if and only if there exists an $i$ such that $G^0_i, G^1_i$ are isomorphic.

**Proof.** Suppose that $G^0$ and $G^1$ are isomorphic. As both graphs consist of $2^{\ell-1}$ connected components, every isomorphism is a combination of isomorphisms between the sequence graphs and hence between the components of the corresponding sequence graphs. As all pairs of sequence graphs from $G^0$ and $G^1$ differ in at least one component, it follows that some pair $G^0_i, G^1_i$ has to be isomorphic. For the other direction suppose that $G^0_i$ and $G^1_i$ are isomorphic. There is a bijection between the sequence graphs of $G^0$ and $G^1$ such that $\langle G^a_{i_1}, \ldots, G^a_{i_\ell} \rangle$ is matches with $\langle G^b_{i_1}, \ldots, G^b_{i_\ell} \rangle$ where $b_j = 1 - a_j$ and $b_j = a_j$ for $j \neq i$. By combining the isomorphism between $G^0_i$ and $G^1_i$ with automorphisms on $G^a_{j}$ for $j \neq i$ it follows that all pairs of sequence graphs and hence $G^0$ and $G^1$ are isomorphic.

**Lemma 3.4.** Suppose that $C^* = C \cup \{(x, R_{arb})\}$ is an $e$-extended group CSP of arity $k$ and let $(G^0_{C_{arb}}, G^1_{C_{arb}}) := \bigvee_{\gamma \in R_{arb}} (G(C_{arb}), \tilde{G}(C_{arb}))$.

(a) $G^0_{C_{arb}}$ and $G^1_{C_{arb}}$ are isomorphic if and only if $C^*$ is satisfiable.

(b) The size of $G^0_{C_{arb}}$ and $G^1_{C_{arb}}$ is bounded by $O(2^e |C^*|)$

(c) If $L^k_{\text{iso}}(C_{\gamma})$ has a p-solution for some $\gamma \in R_{arb}$, then $L^k_{\text{iso}}(G^0_{C_{arb}}, G^1_{C_{arb}})$ has a p-solution.

**Proof.** For $C^* = C \cup \{(x, R_{arb})\}$ consider the $\Gamma$-CSPs $C_{\gamma}$ as defined above and let $G^0_{\gamma} := G(C_{arb})$ and $G^1_{\gamma} := \tilde{G}(C_{arb})$ be the corresponding CFI-graphs, which are of size $O(|C_{arb}|)$. By definition, $G^0_{\gamma}$ and $G^1_{\gamma}$ have size $O(2^e |C^*|)$ and by Lemma 3.3 they are isomorphic if and only if $G^0_{\gamma}$ and $G^1_{\gamma}$ are isomorphic for some $\gamma \in \Gamma$. As this holds if and only if the corresponding $C_{\gamma}$ is satisfiable, it follows that $G^0_{\gamma}$ and $G^1_{\gamma}$ are isomorphic if and only if $C^*$ is satisfiable.

For (c) suppose that $L^k_{\text{iso}}(C_{\gamma})$ has a p-solution. From Lemma 3.1(b) it follows that there is a p-solution for $L^k_{\text{iso}}(G^0_{C_{arb}}, G^1_{C_{arb}})$ and $L^k_{\text{iso}}(G^1_{C_{arb}}, G^0_{C_{arb}})$. We can fix a bijection between the sequence graphs
in $G^0_c$, and $G^1_c$, such that every pair of sequence graphs differs only in component $\gamma$. To define the $p$-solution, we first set $[\pi] = 0$ if $\pi$ is not a partial isomorphism between the corresponding components of the sequence graphs that are matched by the bijection. Otherwise, let $\pi$ be a mapping between components $G^0_i$ and $G^1_i$ of two matched sequence graph. If $G^0_i = G^1_i$, then we set $[\pi] = 1$ if it is a subset of the identity mapping and $[\pi] = 0$, else. If $G^0_i \neq G^1_i$, one of both graphs is a copy of $G^0_{\hat{C}_\gamma}$ and the other is a copy of $G^1_{\hat{C}_\gamma}$. In this case we let $[\pi]$ as defined by the corresponding $p$-solution for $L^0_{iso}(G^0_{\hat{C}_\gamma}, G^1_{\hat{C}_\gamma})$ or $L^1_{iso}(G^1_{\hat{C}_\gamma}, G^0_{\hat{C}_\gamma})$. \hfill $\square$

4 Lower Bounds for Linear Diophantine Equations

In this section we prove one of our main results, the lower bound for the \textit{Linear-Diophantine-Equations algorithm} for graph isomorphism testing. Recall that the algorithm works as follows. The input consists of two graphs $G, G'$, in addition we have a parameter $\ell \geq 1$. The algorithm computes the system $L^\ell_{iso}(G, G')$ of linear equations with integer coefficient (see (2.A)–(2.D)) and solves it over the integers (for example by using the polynomial time algorithm described in [19]). If the system has no solution, the algorithm answers “not isomorphic”. If it has a solution, the algorithm answers “possibly isomorphic”. The running time of the algorithm is $n^{O(\ell)}$, where $n$ is the number of vertices of the input graphs. The algorithm is clearly sound, that is, always gives a correct answer. To show that it is complete (for some $\ell$), we would have to prove that for all pairs $G, G'$ of non-isomorphic input graphs the system $L^\ell_{iso}(G, G')$ has no integral solution. Our theorem shows that this fails in rather strong sense: for $\ell = o(n)$, there are non-isomorphic input graphs for which the system does have an integral solution.

\textbf{Theorem 4.1.} For every $\ell \geq 1$ there are non-isomorphic 3-regular graphs $G, \tilde{G}$ of size $|G| = |\tilde{G}| = O(\ell)$ such that $L^\ell_{iso}(G, \tilde{G})$ has an integral solution.

The rest of this section is devoted to a proof of this theorem. Let $E$ be a family of 2-connected 3-regular expander graphs. For the necessary definitions and the existence of such a family we refer the reader to Appendix A. The only consequence of the expansion property that we use is stated in the following lemma, which is proved in the appendix (as Corollary A.4).

\textbf{Lemma 4.2.} There is constant $c > 0$ such that for every $G \in E$ and every set $X \subseteq E(G)$ there is a set $\hat{X} \supseteq X$ of size $|\hat{X}| \leq c|X|$ such that $E(G) \setminus \hat{X}$ is either empty or the edge set of a 2-connected subgraph of $G$.

For the rest of this section, we fix a graph $G \in E$ and let $V := V(G)$, $E := E(G)$, $n := |V|$, and $m := |E|$. Note that $m = (3/2)n$, because $G$ is 3-regular. We let

$$\ell := \left\lfloor \frac{m - 1}{3c} \right\rfloor.$$

For every $X \subseteq E$, let $K_X = (W_X, Z_X)$ be the subgraph of $G$ with edge set $Z_X := E \setminus \hat{X}$ (with $\hat{X}$ from Lemma 4.2) and vertex set $W_X$ consisting of all vertices incident with an edge in $Z_X$. Then $K_X$ is either empty or 2-connected. If $|X| \leq \ell$, then

$$|Z_X| \geq m - c \cdot \ell > \frac{2}{3} m$$

and thus

$$|W_X| > \frac{2}{3} n,$$  \hfill (4.A)
because a graph of maximum degree 3 with more than \((2/3)m\) edges has more than \((2/3)(2/3)m = \(2/3\)n\) vertices.

We say that a set \(X \subseteq E\) is closed if every edge \(e \in E \setminus X\) is contained in a cycle \(Z \subseteq E \setminus X\). Here, for simplicity, we identify a cycle with its edge set. Note that the intersection of two closed sets is closed. Hence for every set \(X \subseteq E\) the set

\[
\text{cl}(X) := \bigcap_{Y \subseteq X, Y \text{ closed}} Y,
\]

which we call the closure of \(X\), is closed, and in fact the unique inclusionwise minimal closed set that contains \(X\). Observe that \(x \notin \text{cl}(X)\) if and only if there is a cycle \(Z \in E \setminus X\) such that \(x \in Z\). The forward direction of this equivalence is immediate from the definition of closed sets, and for the backward direction, note that if \(Z\) is a cycle then \(Y = E \setminus Z\) is a closed set.

We note that the operator \(\text{cl} : 2^E \to 2^E\) is a closure operator in the sense of matroid theory (see \([18\text{, Section 1.4}]\)), that is,

- \(X \subseteq \text{cl}(X) = \text{cl}^{\text{cl}}(X)\) for all \(X \subseteq E\),
- \(X \subseteq Y\) implies \(\text{cl}(X) \subseteq \text{cl}(Y)\) for all \(X, Y \subseteq E\),
- \(y \in \text{cl}(X \cup \{x\}) \setminus \text{cl}(X)\) implies \(x \in \text{cl}(X \cup \{y\})\) for all \(X \subseteq E\) and \(x, y \in E\).

To verify the last property, known as the exchange property, suppose that \(y \in \text{cl}(X \cup \{x\}) \setminus \text{cl}(X)\) and \(x \notin \text{cl}(X \cup \{y\})\). Then there is a cycle \(Z_x \subseteq E \setminus (X \cup \{y\})\) with \(x \in Z_x\). As \(y \notin \text{cl}(X)\), there is a cycle \(Z_y \subseteq E \setminus X\) with \(y \in Z_y\). As \(y \in \text{cl}(X \cup \{x\})\), we have \(x \in Z_y\). But then \((Z_y \cup Z_x) \setminus \{x\}\) contains a cycle through \(y\). This cycle has an empty intersection with \(X \cup \{x\}\), which contradicts \(y \in \text{cl}(X \cup \{x\})\).

Let us denote the matroid by \(M\). The independent sets of \(M\) are the sets \(I \subseteq E\) such that \(x \notin \text{cl}(I \setminus \{x\})\) for all \(x \in I\) (see \([18\text{, Theorem 1.4.4}]\)). A basis for \(M\) is an inclusionwise maximal independent set, and a basis for a set \(X \subseteq E\) is an inclusionwise maximal independent set \(I \subseteq X\). All bases of a set \(X\) have the same cardinality, the rank \(\text{rk}(X)\). The rank of the matroid \(M\) is \(\text{rk}(E)\).

Now let \(Z\) be the edge set of a 2-connected subgraph of \(G\). Then \(E \setminus Z\) is closed. Thus for every set \(X\) we have \(\text{cl}(X) \subseteq \hat{X}\) for the set \(\hat{X}\) of Lemma \([1\text{, Lemma 2}]\) and it follows from the lemma that

\[
|\text{cl}(X)| \leq c \cdot |X|.
\]

This implies that the rank of the matroid \(M\) is at least \(\lceil m/c \rceil\), because if \(B\) is a basis of \(M\) then \(\text{cl}(B) = E\).

We let \(H\) be an arbitrary orientation of \(G\) and \(\hat{E} := E(H)\). We let \(\Gamma\) be a finite Abelian group, \(\sigma : V \to \Gamma\). We consider the Tseitin tautology \(C := C^{H,\Gamma,\sigma}\) (see Section \([2\text{, Section 2.3}]\)). Recall that the set of variables of \(C\) is

\[
\text{Var}(C) = \{x_e \mid e \in \hat{E}\},
\]

and for each \(v \in V\) the CSP \(C\) has a constraint \(C(v)\) expressed by the following equation in the group \(\Gamma\):

\[
\sum_{e \in \partial^+_v(v)} x_e - \sum_{e \in \partial^-_v(v)} x_e = \sigma(v).
\]

(4.B)

It will be convenient for us to think of an undirected edge \(vw \in E\), its orientation \((v, w)\) or \((w, v) \in \hat{E}\), and the variable \(x_{(v,w)}\) or \(x_{(w,v)}\) as the same object, that is, identify the sets \(E\) and \(\hat{E}\).
and \(\text{Var}(\mathcal{C})\). Generically, we denote the set by \(E\), subsets by \(X,Y,Z\), and elements by \(x,y,z\), but sometimes, we still denote the elements by \(vw, (v,w)\), or \(x_{(v,w)}\) to indicate which role of an element we are thinking of at the moment.

For every subset \(W \subseteq V\) we let \(\sigma(W) := \sum_{w \in W} \sigma(w)\), and we let \(C(W)\) be the constraint

\[
\sum_{e \in \partial_+(W)} x_e - \sum_{e \in \partial_-(W)} x_e = \sigma(W). \tag{4.C}
\]

The constraints \(C(W)\) are not contained in \(\mathcal{C}\), but they are implied by the constraints \(C(v)\) of \(\mathcal{C}\), because equation (4.C) is just the sum of the equations (4.B) for \(v \in W\). Thus every solution to \(\mathcal{C}\) satisfies all constraints \(C(W)\). However, it is not the case that every partial solution \(\psi\) satisfies all constraints \(C(W)\) with \(\partial(W) \subseteq \text{dom}(\psi)\), despite the fact that \(\partial(W)\) is the set of all variables appearing in the constraint \(C(W)\).

For \(k \geq 0\), we call \(\psi \in \text{Var}(\mathcal{C}) \times \Gamma\) \(k\)-consistent if it is a partial mapping and for all \(W \subseteq V\) of size \(|W| \leq k\), if \(\partial(W) \subseteq \text{dom}(\psi)\) then \(\psi\) satisfies the constraint \(C(W)\). Note that \(\psi\) is a partial solution if and only if it is 1-consistent.

**Lemma 4.3.** Let \(X \subseteq E\) and \(\psi : X \to \Gamma\). Then \(\psi\) is \(k\)-consistent if and only if the constraint \(C(W)\) is satisfied for every \(W \subseteq V\) such that \(|W| \leq k\) and \(W\) is the vertex set of a connected component of the graph \(G = (V,E \setminus X)\).

**Proof.** Let \(W_0 \subseteq V\) such that \(|W_0| \leq k\) and \(\partial(W_0) \subseteq X\). Note that for every connected component \(W\) of \(G\), if \(W \cap W_0 \neq \emptyset\) then \(W \subseteq W_0\), because otherwise there are \(w \in W \cap W_0\) and \(w' \in W \setminus W_0\) such that \(ww' \in E \setminus X\), which contradicts the assumption \(\partial(W_0) \subseteq X\). Thus \(W\) is the union of vertex sets \(W_1, \ldots, W_p\) of connected components of \(G \setminus X\). But then \(\partial_+(W) = \bigcup_{i=1}^p \partial_+(W_i)\) and \(\partial_-(W) = \bigcup_{i=1}^p \partial_-(W_i)\), and the equation (4.C) is just the sum of the corresponding equations for the \(W_i\), which are satisfied by the assumption of the lemma.

**Lemma 4.4.** Let \(X \subseteq E\) such that \(\text{rk}(X) \leq \ell\), and let \(\psi : X \to \Gamma\) be \(n/3\)-consistent. Then \(\psi\) is \(2n/3\)-consistent.

**Proof.** Let \(Y\) be a basis for \(X\). Choose \(\hat{Y}\) according to Lemma 4.2 and note that \(Y \subseteq X \subseteq \text{cl}(Y) \subseteq \hat{Y}\). Let \(W_Y\) be the vertex set of the 2-connected graph \(K_Y\) with edge set \(Z_Y = E \setminus \hat{Y}\). Then \(|W_Y| > 2n/3\) by (4.A). Suppose for contradiction that \(X\) is not \(2n/3\)-consistent. Then there is a set \(W \subseteq V\) such that \(|W| \leq 2n/3\) and \(\partial(W) \subseteq X\) and \(\psi\) does not satisfy \(C(W)\). By Lemma 4.3 we may assume that \(W\) is the vertex set of a connected component of \(G - X\). As \(\psi\) is \(n/3\)-consistent, we have \(|W| > n/3\). Hence \(W \cap W_Y \neq \emptyset\). As \(K_Y\) is a connected subgraph of \(G - X\), it follows that \(W_Y \subseteq W\). Hence \(|W| > 2n/3\), which is a contradiction.

We call \(\psi\) robustly consistent if it is \(n/3\)-consistent.

**Lemma 4.5.** Let \(X \subseteq E\) such that \(\text{rk}(X) \leq \ell\), and let \(\psi : X \to \Gamma\) be robustly consistent. Then there is a unique \(\hat{\psi} : \text{cl}(X) \to \Gamma\) such that \(\psi \subseteq \hat{\psi}\) and \(\hat{\psi}\) is robustly consistent.

**Proof.** It suffices to prove that we can uniquely extend \(\psi\) to a domain \(X \cup \{x\}\) for an \(x \in \text{cl}(X) \setminus X\). So let us pick such an \(x\). Let \((W,Z)\) be the connected component of \(G - X\) that contains \(x\). As \(x\) is not contained in a cycle in \(G - X\) (otherwise it would not be in \(\text{cl}(X)\)), the edge \(x\) is a bridge (that is, separating edge) of the graph \((W,Z)\). Let \((W_1,Z_1)\) and \((W_2,Z_2)\) be the two connected components of \((W,Z - \{x\})\). Without loss of generality we assume that \(|W_1| \leq |W_2|\). Then \(|W_1| \leq n/2\), and in order to be robustly satisfiable, the mapping \(\phi\) we shall define must satisfy the constraint \(C(W_1)\).
As \( \partial(W_1) \subseteq X \cup \{x\} \) and the values \( \psi'(x') = \psi(x') \) are fixed for all \( x' \in X \), there is a unique \( \gamma \in \Gamma \) such that \( \psi' := \psi \cup \{(x, \gamma)\} \) satisfies the constraint \( C(W_1) \).

If \( |W_2| \leq n/3 \), then \( |W| = |W_1| + |W_2| \leq 2n/3 \), and as \( \psi \) is robustly consistent, it satisfies the constraint \( C(W) \). This implies that \( \psi' \) satisfies the constraint \( C(W_2) \).

All other connected components \( (W', Z') \) of \( G - (X \cup \{x\}) \) are also connected components of \( G - X \). Thus \( \psi' \) satisfies the constraint \( C(W') \) if and only \( \psi \) does, and this implies that \( \psi' \) is robustly consistent.

**Lemma 4.6.** Let \( X \subseteq E \) such that \( \text{rk}(X) \leq \ell - 1 \), and let \( \psi : X \rightarrow \Gamma \) be robustly consistent. Let \( x \in E \setminus \text{cl}(X) \). Then every \( \psi' : X \cup \{x\} \rightarrow \Gamma \) such that \( \psi \subseteq \psi' \) is robustly consistent.

**Proof.** By Lemma 4.5, we may assume without loss of generality that \( X \) is closed. Let \( \gamma \in \Gamma \) and \( \psi' := \psi \cup \{(x, \gamma)\} \). Let \( (W, Z) \) be a connected component of the graph \( G - X \) that contains \( x \). Then \( (W, Z - \{x\}) \) is connected, because \( x \not\in X = \text{cl}(X) \). Note that \( \psi' \) satisfies \( C(W) \) because \( \psi \) does and \( \partial(W) \subseteq X \).

All other connected components \( (W', Z') \) of \( G - (X \cup \{x\}) \) are also connected components of \( G - X \). Thus \( \psi' \) satisfies the constraint \( C(W') \) if and only \( \psi \) does, and this implies that \( \psi' \) is robustly consistent.

**Corollary 4.7.** Let \( Y \) be an independent set of the matroid \( \mathcal{M} \) of size \( |Y| \leq \ell \). Then every mapping \( \psi : Y \rightarrow \Gamma \) is robustly consistent.

We are now ready to turn to the linear program \( L := L^\ell_{\exp}(\mathcal{C}) \). Recall that the variables of \( L \) are \([\psi]\) for \( \psi \subseteq \text{Var}(\mathcal{C}) \times \Gamma \) of size at most \( \ell \). We define an assignments \( \Psi : \text{Var}(L) \rightarrow \mathbb{Q} \) by

\[
\Psi([\psi]) := \begin{cases} \frac{1}{|\Gamma|^r} & \text{if } \psi \text{ is robustly consistent and } \text{rk}(\text{dom}(\psi)) = r, \\ 0 & \text{if } \psi \text{ is not robustly consistent.} \end{cases} \tag{4.D}
\]

**Lemma 4.8.** \( \Psi \) is a solution to \( L \).

**Proof.** As all robustly consistent \( \psi \) are partial solutions, \( \Psi \) satisfies the equations \( \{2.G\} \).

The empty mapping is robustly consistent, because \( G \) is connected and thus the only component of \( (V, E \setminus \text{dom}(\emptyset)) = G \) contains more than \((1/3)n\) vertices. As \( \text{rk}(\emptyset) = 0 \), we have \( \Psi([\emptyset]) = 1 \), and thus \( \Psi \) satisfies \( \{2.T\} \).

To see that \( \Psi \) satisfies the equations \( \{2.E\} \), let \( \psi \subseteq \text{Var}(\mathcal{C}) \times \Gamma \) of size \( |\psi| \leq \ell - 1 \) and \( x \in E \). We have to prove that

\[
\sum_{\gamma \in \Gamma} \Psi([\psi \cup \{(x, \gamma)\}]) = \Psi([\psi]). \tag{4.E}
\]

Let \( X := \text{dom}(\psi) \) and \( r := \text{rk}(X) \). If \( \psi \) is not robustly consistent then neither is \( \psi \cup \{(x, \gamma)\} \) for any \( \gamma \), and both sides of equation \( \{4.E\} \) are zero. Suppose that \( \psi \) is robustly consistent. Then

\[
\Psi([\psi]) = \frac{1}{|\Gamma|^r}.
\]

Suppose first that \( x \in \text{cl}(X) \). Then by Lemma 4.5, there is a unique \( \gamma_x \in \Gamma \) such that \( \psi' := \psi \cup \{(x, \gamma_x)\} \) is robustly consistent. As \( x \in \text{cl}(X \cup \{x\}) \), we have \( \text{rk}(X \cup \{x\}) = r \) and thus \( \Psi([\psi']) = \frac{1}{|\Gamma|^r} \). Hence

\[
\sum_{\gamma \in \Gamma} \Psi([\psi \cup \{(x, \gamma)\}]) = \Psi([\psi']) = \frac{1}{|\Gamma|^r} = \Psi([\psi]).
\]
Suppose next that \( x \notin \text{cl}(X) \). Then by Lemma 4.6 for all \( \gamma \in \Gamma \) the mapping \( \psi \cup \{(x, \gamma)\} \) is robustly consistent. Moreover, \( \text{rk}(X \cup \{x\}) = r + 1 \) and thus \( \Psi([\psi \cup \{(x, \gamma)\}]) = \frac{1}{|\Gamma|^{r+1}} \). Hence

\[
\sum_{\gamma \in \Gamma} \Psi([\psi \cup \{(x, \gamma)\}]) = |\Gamma| \cdot \frac{1}{|\Gamma|^{r+1}} = \frac{1}{|\Gamma|^r} = \Psi([\psi]).
\]

Thus \( \Psi \) satisfies (4.E) and hence all equations (2.E).

Recall that \( p \)-solution for a system of linear equations is a rational solution that only takes values \( p^2 \) for integers \( z \). A \( p \)-group is a group of order \( p^k \) for a nonnegative integer \( k \).

**Corollary 4.9.** Let \( p \) be a prime and \( \Delta \leq \Gamma \) be a \( p \)-group. Suppose that \( \sigma(v) \in \Delta \) for all \( v \in V(H) \). Then \( L_{csp}^{\ell}(C^{H,\Gamma,\sigma}) \) has a \( p \)-solution.

**Proof.** We note that every solution \( \Psi \) to \( L_{csp}^{\ell}(C^{H,\Delta,\sigma}) \) can be extended to a solution \( \Psi' \) to \( L_{csp}^{\ell}(C) \) by letting \( \Psi'([\psi]) := \Psi([\psi]) \) for all \( \psi \subseteq \text{Var}(C) \times \Delta \) and \( \Psi'([\psi]) := 0 \) for all \( \psi \not\subseteq \text{Var}(C) \times \Delta \). Thus we can apply the previous lemma to the group \( \Delta \) and the CSP \( C^{H,\Delta,\sigma} \).

**Proof of Theorem 4.1** We let \( \Gamma \) be the group \( \mathbb{Z}_2 \times \mathbb{Z}_3 \). Let \( \Delta_2 \) be the subgroup \( \mathbb{Z}_2 \times \{0\} \) and \( \Delta_3 \) the subgroup \( \{0\} \times \mathbb{Z}_3 \). Moreover, let \( \iota_2 := (1,0) \) and \( \iota_3 := (0,1) \). We continue to work with the same graph \( G \) and orientation \( H \) of \( G \) as before. We choose an arbitrary \( v^* \in V \). We let \( \sigma_2, \sigma_3 : V \to \Gamma \) by \( \sigma_2(v) := \sigma_3(v) := (0,0) \) for all \( v \in V \setminus \{v^*\} \) and \( \sigma_p(v^*) := \iota_p \).

For \( p = 2, 3 \), we let \( \Psi_p \) be the \( p \)-solution to \( L_{csp}^{\ell}(C^{H,\Gamma,\sigma_p}) \) obtained from Corollary 4.9 applied to the \( p \)-group \( \Delta \) and \( \sigma = \sigma_p \).

We now build a 2-extended \( \Gamma \)-CSP \( C^* \) that is essentially the disjunction between \( C^{H,\Gamma,\sigma_2} \) and \( C^{H,\Gamma,\sigma_3} \). Note that \( C^{H,\Gamma,\sigma_2} \) and \( C^{H,\Gamma,\sigma_3} \) both have the constraints (4.B) with \( \sigma(v) = (0,0) \) for all \( v \in V \setminus \{v^*\} \); they only differ in the constraints for \( v^* \). To define \( C^* \) we add a new variable \( x^* \) and replace the constraints (4.B) for \( v^* \) by

\[
\sum_{e \in \partial_+(v)} x_e - \sum_{e \in \partial_-(v)} x_e = x^*;
\]

which still is a \( \Gamma \)-constraint. Now we add the unary constraint \( \{(x^*,\{\iota_2,\iota_3\})\} \), which is not a group constraint. \( C^* \) is the resulting 2-extended \( \Gamma \)-CSP. For \( p = 2, 3 \), we let \( C_p^* := C_{\iota_p}^* = C^* \cup \{(x^*,\{\iota_p\})\} \).

Furthermore, we let \( \Psi_p^*: \text{Var}(L_{csp}^{\ell}(C^*)) \to \mathbb{Q} \) be the assignment defined by

\[
\Psi_p^*([\psi]) := \begin{cases} 
0 & \text{if } \psi \text{ is not a partial mapping}, \\
0 & \text{if } p = 2 \text{ and } (x^*,\iota_3) \in \psi \text{ or } p = 3 \text{ and } (x^*,\iota_2) \in \psi, \\
\Psi_p([\psi \setminus \{(x^*,\iota_p)\}]) & \text{otherwise}. 
\end{cases}
\]

It is easy to see that \( \Psi_p^* \) is a \( p \)-solution to \( L_{csp}^{\ell}(C_p^*) \).

Note that all constraints of \( C \) are ternary, because the graph \( G \) is 3-regular. Let \( \ell' := \lfloor \ell/3 \rfloor \) and

\[
(G,\tilde{G}) := \bigvee_{p \in \{2,3\}} (G(C_p^*),\tilde{G}(C_p^*)).
\]

By Lemma 3.3(a), the graphs \( G \) and \( \tilde{G} \) are non-isomorphic, and by Lemma 3.2(c), the system \( L_{iso}^{\ell}(G,\tilde{G}) \) has a \( p \)-solution. Thus by Lemma 2.1 the system has an integral solution.
5 Polynomial Calculus

We now turn to an algebraic approach and encode instances of the isomorphism problem by systems of polynomial equations, which we may interpret over any field. Then we try to derive the non-solvability of the system using algebraic reasoning. Again, we obtain an algorithm that is sound, but not necessarily complete. The algorithm is parameterized by the degree $\ell$ of the polynomials that we see during the derivation, and its running time $n^{O(\ell)}$. We shall prove a lower bound by exhibiting non-isomorphic graphs that requiring degree $\ell = \Omega(n)$. The proper framework for phrasing these results is propositional proof complexity.

**Polynomial Calculus (PC)** \cite{10} is a proof system to prove that a given system of (multivariate) polynomial equations $P$ over a field $F$ has no $\{0,1\}$-solution. We always normalise polynomial equations to the form $p=0$ and just write $p$ to denote the equation $p=0$. Consequently, we view $P$ as a set of polynomials. Polynomials are derived line by line according to the following derivation rules (for polynomials $p \in P$, polynomials $f,g$, variables $x$ and field elements $a,b$):

\[
\begin{align*}
\overline{p}', & \quad \frac{x^2-x}{p'}', \\
\frac{f}{x'}', & \quad \frac{g}{ag+bf}'.
\end{align*}
\]

The axioms of the systems are all $p \in P$ and $x^2-x$ for all variables $x$. A PC refutation of $P$ is a derivation of 1 (the contradiction $1=0$). The degree of a PC derivation is the maximal degree of every polynomial in the derivation. If an instance $P$ is unsatisfiable and has a refutation of degree $d$, then it can be found in time $n^{O(d)}$ by a bounded degree variant of the Gröbner basis algorithm \cite{10}. To solve a combinatorial problem by this algebraic approach, one first encodes the instance into a set of low degree polynomials $P$ and then tries to find a PC refutation of degree $d$ over some field $F$. If such a refutation is found, we know that the instance is unsatisfiable and the algorithm rejects. Otherwise, the algorithm outputs “possibly satisfiable”. As for the Diophantine isomorphism test and other related approaches such as linear and semi-definite programming hierarchies this algorithm is sound but not necessarily complete. It can be shown, however, that completeness is achieved for $d = n+1$ over any field (where $n$ is the total number of variables in $P$).

For polynomials $f_1,\ldots,f_\ell$ and $g$ over $F$ we write $\{f_1,\ldots,f_\ell\} \models g$ if $g$ follows semantically from $f_1,\ldots,f_\ell$, that is, for every $\{0,1\}$-assignment $I$ it holds that $I(f_1) = 0, \ldots, I(f_\ell) = 0$ implies $I(g) = 0$. By $\{f_1,\ldots,f_\ell\} \models g$ we denote that there is a PC derivation of $g$ from $f_1,\ldots,f_\ell$ over $F$ and use $\{f_1,\ldots,f_\ell\} \vdash_p^d g$ if there is a refutation of degree at most $d$. For prime fields $\mathbb{F}_p$ we abbreviate $\models_{\mathbb{F}_p}, \models_{\mathbb{F}_p}^d$ by $\models_p, \models_p^d$. The following theorems will be useful for us.

**Theorem 5.1 (Derivational completeness (Theorem 5.2 in \cite{8})).** Let $f_1,\ldots,f_\ell$ and $g$ be polynomials in $n$ variables and $p$ a prime. Then

\[
\{f_1,\ldots,f_\ell\} \models_p g \iff \{f_1,\ldots,f_\ell\} \models_p g \iff \{f_1,\ldots,f_\ell\} \models_{\mathbb{F}_p}^{n+1} g. \tag{5.A}
\]

**Theorem 5.2 (Cut-elimination (Theorem 5.1 (2) in \cite{8})).** Let $F := \{f_i(x,y)\}$ be a set of polynomials $f_i(x,y)$ in variables $x_1,\ldots,x_\ell, y$ and $p$ a prime number. Let $F_0 := \{f_i(x,0)\}$, $F_1 := \{f_i(x,1)\}$ and $g$ be a polynomial. Then

\[
F_0 \vdash_p^d g \land F_1 \vdash_p^d g \Longrightarrow F \vdash_p^{d+1} g.
\]

To compare the power of the polynomial calculus for different systems of polynomials, we use *low degree reductions* \cite{7}. Fix a field $\mathbb{F}$ and let $P$ and $Q$ be two sets of polynomials in the variables $\mathcal{X}$ and $\mathcal{Y}$, respectively. A degree-$(d_1,d_2)$ *reduction* from $P$ to $Q$ is a set of degree-$d_1$ polynomials $\{f_y | y \in \mathcal{Y}\}$ in the variables $\mathcal{X}$ such that

\[
\text{for all } q(y_1,\ldots,y_\ell) \in Q \text{ and } \forall \overset{d_2}{\mathbb{F}} q(f_{y_1},\ldots,f_{y_\ell})
\]
\[ P \vdash_{d_1} \frac{d_2}{2} f_y^2 - f_y \quad \text{for all } y \in \mathcal{Y}. \tag{5.C} \]

**Lemma 5.3 (Lemma 1 in [7]).** If there is a degree-\((d_1, d_2)\) reduction from \(P\) to \(Q\) and \(Q\) has a PC refutation of degree \(d\), then \(P\) has a PC refutation of degree \(\max(d_2, d \cdot d_1)\).

### 5.1 Polynomial Encodings for Isomorphism and Constraint Satisfaction

Following [6], for graphs \(G, H\), we define a system of (multivariate) polynomials \(P_{iso}(G, H)\) in variables \([v \mapsto w], v \in V(G), w \in V(H)\). A \(\{0,1\}\)-solution to the system is intended to describe an isomorphism \(\iota\) from \(G\) to \(H\), where \([v \mapsto w] \mapsto 1\) if \(\iota(v) = w\) and \([v \mapsto w] \mapsto 0\) otherwise. The system \(P_{iso}(G, H)\) consists of the following linear and quadratic polynomials:

\[
-1 + \sum_{v \in V(G)} [v \mapsto w] \quad \text{for all } w \in V(H) \quad \text{\text{(5.D)}}
\]

\[
-1 + \sum_{w \in V(H)} [v \mapsto w] \quad \text{for all } v \in V(G) \quad \text{\text{(5.E)}}
\]

\[
[v \mapsto w] \cdot [v' \mapsto w'] \quad \text{for all } v, v' \in V(G), w, w' \in V(H) \quad \text{\text{(5.F)}}
\]

such that \(\{(v, w), (v', w')\}\) is no partial isomorphism.

Similarly, for every CSP \(C\), we define a system of (multivariate) polynomials \(P_{csp}(C)\) in variables \([x \mapsto \gamma]\) for \(x \in \text{Var}(C), \gamma \in \text{Dom}(C)\). A \(\{0,1\}\)-solution to the system is intended to describe a solution \(\alpha\), where \([x \mapsto \gamma] \mapsto 1\) if \(\alpha(x) = \gamma\) and \([x \mapsto \gamma] \mapsto 0\) otherwise. The system \(P_{csp}(C)\) consists of the following linear and quadratic polynomials:

\[
-1 + \sum_{\gamma \in D} [x_i \mapsto \gamma] \quad \text{for all } x_i \quad \text{\text{(5.G)}}
\]

\[
[x_i \mapsto \gamma] : [x_i \mapsto \gamma'] \quad \text{for all } x_i \text{ and } \gamma \neq \gamma' \quad \text{\text{(5.H)}}
\]

\[
\prod_{i=1}^{k} [x_i \mapsto \gamma_i] \quad \text{for all constraints } (x_1, \ldots, x_k), R \in C \quad \text{\text{(5.I)}}
\]

and all \((\gamma_1, \ldots, \gamma_k) \notin R\).

Again, \(P_{csp}(C)\) has a \(\{0,1\}\)-solution over some field \(\mathbb{F}\) if and only if \(C\) is satisfiable. Note the similarities between the polynomial systems \(P_{iso}(G, H)\) and \(P_{csp}(C)\) and the linear systems \(L_{iso}(G, H)\) and \(L_{csp}(C)\). One formal correspondence is the following. Suppose that \(L_{iso}(G, H) / L_{csp}(C)\) viewed as a system of linear congruencies modulo some prime \(p\) has no solution, then \(P_{iso}(G, H) / P_{csp}(C)\) has a degree \(\ell\) refutation over \(\mathbb{F}_p\). Thus, the algebraic approach is stronger than solving the linear equations modulo some prime \(p\), in that it is able to reject more unsatisfiable instances in time \(n^{O(\ell)}\). On the other hand, solving the linear system over the integers is also more powerful than solving the system modulo some prime \(p\) (because there might be a solution over \(\mathbb{Z}_p\) even though the system has no solution over \(\mathbb{Z}\)). In fact, the Diophantine isomorphism test that solves \(L_{iso}(G, H) / L_{csp}(C)\) over \(\mathbb{Z}\) is incomparable in its strength with the algebraic approach of finding a polynomial calculus refutation of degree \(\ell\).

### 6 Lower Bounds for Polynomial Calculus

In this section we prove the following lower bound, which implies that there are non-isomorphic graphs that cannot be distinguished in polynomial time by algebraic reasoning over any prime field.

**Theorem 6.1.** For every \(\ell \geq 1\) there are non-isomorphic graphs \(G, \tilde{G}\) of size \(|G| = |\tilde{G}| = O(\ell)|\) such that every polynomial calculus refutation of \(P_{iso}(G, \tilde{G})\) over some prime field \(\mathbb{F}_p\) has degree \(\Omega(\ell)\).
One main ingredient in our proof is the framework of Alekhnovich and Razborov \[1\] for proving degree lower bounds. They consider Boolean CSPs defined over an expander graph, where the variables correspond to the edges of the graph and where every constraint is defined over the edges that are incident to the same vertex \(v\). To show that such CSPs are hard to refute in polynomial calculus, they introduced the concept of immunity. A constraint \(C\) has high immunity over a field \(\mathbb{F}\), if it has no non-trivial low degree consequences, that is, if from \(P_C \models \beta\ g\) it follows that either \(g \equiv 1\) or the degree of \(g\) is large (linear in the number of variables in \(C\)). The main result of \[1\] is that if \(C\) is a Boolean CSP defined over an expander graph and every constraint \(C \in C\) is immune over \(\mathbb{F}\), then \(P_{\text{csp}}(C)\) requires polynomial calculus refutations of linear degree over \(\mathbb{F}\). As one example consider the Boolean \(\mathbb{Z}_p\)-Tseitin tautologies (see Section 2.3) where for every vertex \(v\) there is a constraint stating that the sum of ingoing minus outgoing edge variables is congruent to 0 modulo \(p\). Such parity constraints were shown to have high immunity over fields of characteristic \(\neq p\) \[1\], but they have low immunity over \(\mathbb{F}_p\) because the constraint \(2.1\) viewed as a linear equation over \(\mathbb{F}_p\) follows semantically over \(\mathbb{F}_p\). By the immunity argument this implies the following lower bound.

**Theorem 6.2 (Corollary 4.6 in [1]).** For every prime \(p\) there is a constant \(k_0(p)\) such that the following holds. For every \(k\), there is a directed \(k\)-regular graph \(H\) on \(n\) vertices such that for all \(p\) with \(k_0(p) \leq k\) it holds that every polynomial calculus refutation of \(P_{\text{csp}}(\mathcal{B}H,\mathbb{Z}_p,\sigma)\) over a field \(\mathbb{F}\) of characteristic \(\neq p\) requires degree \(\Omega(kn)\).

To obtain lower bounds that hold over any field we cannot apply this framework directly to \(\Gamma\text{-CSPs}, \) as for every Abelian group \(\Gamma\) the constraints always have low immunity over some prime field \(\mathbb{F}_p\). Because of this we have to use extended group CSPs and show that there is a 2-extended \((\mathbb{Z}_2 \times \mathbb{Z}_3)\)-CSP \(C\) that is at least as hard as the Boolean \(\mathbb{Z}_p\)-Tseitin tautologies for \(p \in \{2, 3\}\). This implies that \(P_{\text{csp}}(C)\) requires large polynomial calculus degree over every prime field and the same holds true for its graph encoding via the or-construction over the \(CFI\)-graphs.

In Section 3 we have already shown that solvability and \(p\)-solutions can be transferred from \((e\text{-extended})\) group CSPs to the graph isomorphism problem. We now prove the corresponding statements in the algebraic setting, showing that low degree refutations for the graph isomorphism imply low degree refutations for the system of polynomials corresponding to the underlying CSP.

**Lemma 6.3.** Let \(\mathbb{F}_p\) be a prime field, \(\Gamma\) an Abelian group, \(C\) an \(n\)-variable \(\Gamma\)-CSP of arity \(k\) and \(d_0 = (|k|! + |\Gamma|^k)^2 + 1\).

(a) There is a degree-\( (k, d_0)\) reduction from \(P_{\text{csp}}(C)\) to \(P_{\text{iso}}(G(C), \tilde{G}(C))\).

(b) There is a degree-\( (1, d_0)\) reduction from \(P_{\text{iso}}(G(C), \tilde{G}(C))\) to \(P_{\text{csp}}(C)\).

**Proof.** We define the a degree-\( (k, d_0)\) reduction from \(P_{\text{csp}}(C)\) to \(P_{\text{iso}}(G(C), \tilde{G}(C))\) as follows.

\[f_{[x \mapsto \alpha]} := [x \mapsto \alpha]\] (6.1)

\[f_{[\gamma \mapsto (\gamma - \alpha \cdot \gamma \cdot \alpha)]} := \prod_{\alpha \in \Gamma} [x_i \mapsto \alpha_i] \quad \text{for} \quad C = ((x_1, \ldots, x_k), \Delta \cup \beta)\] (6.2)

\[f_{[v \mapsto w]} := 0 \quad \text{for all other} \quad v, w\] (6.3)

It is easy to see that all polynomials \(f_{[v \mapsto w]}^2 - f_{[v \mapsto w]}\) are derivable. Let \(g\) be one of the substituted axioms \(\sum_{v \in V(G)} f_{[v \mapsto w]} - 1, \sum_{w \in V(H)} f_{[v \mapsto w]} - 1, \) or \(f_{[v \mapsto w]} f_{[v' \mapsto w']}\). Note that there is a constraint \(C \in C\) such that all variables in \(g\) are of the form \([x \mapsto \gamma]\) for some variable \(x\) occurring in \(C\). Let \(P_C \subseteq P_{\text{csp}}(C)\) be the set of polynomials for constraint \(C\). By Theorem 5.1 it suffices to show...
that \( P_C \models_p g \) as this implies that there is a degree \( k|\Gamma| + 1 \leq d_0 \) derivation of \( g \) from \( P_{\text{csp}}(C) \). Suppose that there is a \( \{0,1\} \)-assignment \( I \) that satisfies \( P_C \), we have to show that \( I(g) = 1 \). By the definition of \( P_{\text{csp}} \) it follows that there is a satisfying assignment \( \varphi \) for the constraint such that \( \varphi(x) = \gamma \) if and only if \( I([x \mapsto \gamma]) = 1 \). Let \( \pi_\varphi \) be the isomorphism between the corresponding subgraphs as defined in the proof of Lemma 3.1 and note that the definition of the substitution gives \( I(f_{[x \mapsto \gamma]}) = 1 \) if and only if \( \pi_\varphi(v) = w \). Hence, \( I(g) = 1 \) as every axiom from \( P_{\text{iso}}(G(C), \tilde{G}(C)) \) is satisfied by a \( \{0,1\} \)-assignment corresponding to an isomorphism.

For the backward direction (b) we define the degree-(1, \( d_0 \)) reduction from \( P_{\text{iso}}(G(C), \tilde{G}(C)) \) to \( P_{\text{csp}}(C) \) by \( f_{[x \mapsto \gamma]} := \{0(x) \mapsto (\gamma)(x)\} \). The argument that this is indeed a low degree reduction is similar to the one for (a). Let \( g \) be some substituted axiom, \( C \in C \) the corresponding constraint, and \( P_C \subseteq P_{\text{iso}}(G(C), \tilde{G}(C)) \) be the set of polynomials over two subgraphs of \( G(C) \) and \( \tilde{G}(C) \) that encode the constraint \( C \). Note that both subgraphs have at most \( k|\Gamma| + |\Gamma|^k \) vertices and therefore \( P_C \) contains at most \( (k|\Gamma| + |\Gamma|^k)^2 = d_0 - 1 \) variables. By Theorem 5.2 it now suffices to show that \( P_C \models_p g \). Suppose that there is a \( \{0,1\} \)-assignment \( I \) that satisfies \( P_C \). By the definition of \( P_{\text{iso}} \) it follows that there is an isomorphism \( \pi \) between the gadgets such that \( \pi(0(x)) = (\gamma)(x) \) if and only if \( I(\{0(x) \mapsto (\gamma)(x)\}) = 1 \). Let \( \varphi_\pi \) be the corresponding satisfying assignment for \( C \) (from Lemma 5.1) and note that the definition of the substitution gives \( I(f_{[x \mapsto \gamma]}) = 1 \) if and only if \( \varphi_\pi(x) = \gamma \). Hence, \( I(g) = 1 \) as every axiom from \( P_{\text{csp}}(C) \) is satisfied by a \( \{0,1\} \)-assignment corresponding to a satisfying assignment for the constraint.

The next Lemma transfers Lemma 5.4 to the algebraic setting and provides a reduction from \( e \)-extended group CSPs to graph isomorphism.

**Lemma 6.4.** Fix a prime field \( \mathbb{F}_p \). Suppose that \( C^* = C \cup \langle x, R_{\text{arb}} \rangle \) is an \( e \)-extended group CSP of arity \( k \) and let \( (G^0_{\gamma}, G^1_{\gamma}) := \bigvee_{\gamma \in R_{\text{arb}}} (G(C_{\gamma}), \tilde{G}(C_{\gamma})) \). If \( P_{\text{iso}}(G^0_{\gamma}, G^1_{\gamma}) \) has a degree-\( d \) refutation, then \( P_{\text{csp}}(C^*) \) has a refutation of degree \( O(dk|\Gamma|) \).

**Proof.** For \( \gamma \in R_{\text{arb}} \) consider the \( \Gamma \)-CSPs \( C_{\gamma} \) and let \( G^0_{\gamma} := G(C_{\gamma}) \) and \( G^1_{\gamma} := \tilde{G}(C_{\gamma}) \) be the corresponding CFI-graphs. We first apply cut-elimination to \( P_{\text{csp}}(C^*) \) and the at most \( k|\Gamma| \) variables \( \{x_i \mapsto \gamma\} \) where \( x_i \) is a variable occurring in the additional constraint \( \langle x, R_{\text{arb}} \rangle \). By Theorem 5.2 it follows that

\[
P_{\text{csp}}(C_{\gamma}) \vdash_{p}^{dk} 1 \implies P_{\text{csp}}(C^*) \vdash_{p}^{dk + k|\Gamma|} 1. \tag{6.D}
\]

By Lemma 5.3 there are degree-(\( k, d_0 \)) reductions from \( P_{\text{csp}}(C_{\gamma}) \) to \( P_{\text{iso}}(G^0_{\gamma}, G^1_{\gamma}) \) for some constant \( d_0 \). By Lemma 5.3 this implies for every \( \gamma \in R_{\text{arb}} \) and sufficiently large \( d \geq d_0 \)

\[
P_{\text{iso}}(G^0_{\gamma}, G^1_{\gamma}) \vdash_{p}^{d} 1 \implies P_{\text{csp}}(C_{\gamma}) \vdash_{p}^{dk} 1. \tag{6.E}
\]

Finally we show that there is a degree-(1, 2) reduction from \( P_{\text{iso}}(G^0_{\gamma}, G^1_{\gamma}) \) to \( P_{\text{iso}}(G^0_{\alpha}, G^1_{\alpha}) \). As mentioned in the proof of Lemma 5.3 there is a bijection \( \pi \) between the sequence graphs \( \tilde{G} \) contained in \( G^0_{\gamma} \) and \( G^1_{\gamma} \) such that every pair of sequence graphs differs only in component \( \gamma \). By fixing all other components we reduce the isomorphism test for \( G^0_{\gamma} \) and \( G^1_{\gamma} \) to testing isomorphism of one component. We denote vertices in \( G^0_{\gamma} \) and \( G^1_{\gamma} \) by \( v, \tilde{G}, G^1_{\alpha} \) referring to the vertex \( v \) in the corresponding copy of \( G^1_{\alpha} \) that is contained in the sequence \( \tilde{G} \).

\[
I_{(v, \tilde{G}, G^1_{\alpha}) \mapsto (w, \pi(\tilde{G}), G^1_{\gamma-j})} := [v \mapsto w] \quad \text{for } v \in V(G^0_{\gamma}), \ w \in V(G^1_{\gamma}), \ j \in \{0, 1\}; \tag{6.F}
\]

\[
I_{(v, \tilde{G}, G^1_{\alpha}) \mapsto (v, \pi(\tilde{G}), G^1_{\alpha-j})} := 1 \quad \text{for } v \in V(G^0_{\alpha}), \ \alpha \in R_{\text{arb}} \setminus \{\gamma\}, \ j \in \{0, 1\}. \tag{6.G}
\]
and \( f_{[v \mapsto w]} := 0 \) in all other cases. As this reduction turns every axiom of \( P_{\text{isogr}}(G^0_{csp}, G^1_{csp}) \) into a trivial polynomial or an axiom of \( P_{\text{isogr}}(G^0_{G}, G^1_{G}) \), they can be derived immediately in degree 2. By Lemma 5.3 it follows that

\[
P_{\text{isogr}}(G^0_{csp}, G^1_{csp}) \vdash^d_p 1 \implies P_{\text{isogr}}(G^0_{G}, G^1_{G}) \vdash^d_p 1 \text{ for all } \gamma \in R_{\text{arb}}.
\]  

(6.H)

The lemma follows by combining (6.I), (6.J), and (6.H).

Now we have everything in hand to prove our lower bound.

**Proof of Theorem 6.1.** Let be \( H \) a \( k \)-regular directed graph such that \( k \geq \max(k_0(2), k_0(3)) \) satisfies the conditions of Theorem 6.2 for \( p = 2 \) and \( p = 3 \). We choose an arbitrary vertex \( v^* \in V(H) \), let \( \sigma(v^*) := 1 \) and \( \sigma(v) := 0 \) for all \( v \in V(H) \setminus \{v^*\} \), and consider the Boolean Tseitin CSPs \( B^{H, Z_2, \sigma} \) and \( B^{H, Z_3, \sigma} \) in the variable set \( \{x_e \mid e \in E(H)\} \).

We define the unsatisfiable 2-extended \( \Gamma \)-CSP \( C^* \) for \( \Gamma = Z_2 \times Z_3 \) in variables \( \{y_e \mid e \in E(H)\} \cup \{y^*\} \) as in the proof of Theorem 4.1. That is, we let \( C^* \) be the Tseitin tautology \( C^{H, \Gamma, \sigma^*} \) for \( \sigma^* \equiv (0, 0) \) where we replace the constraints (1.3) for \( v^* \) by the \( \Gamma \)-constraint

\[
\sum_{e \in \partial_+(v)} y_e - \sum_{e \in \partial_-(v)} y_e = y^*
\]

and add the unary non-group constraint \( \{y^*, \{t_2, t_3\}\} \) for \( t_2 := (1, 0) \) and \( t_2 := (0, 1) \). Intuitively, \( C^* \) is the Tseitin tautology \( C^{H, \Gamma, \sigma^*} \) where we have \( \sigma^*(v) = (0, 0) \) for all \( v \in V(H) \setminus \{v^*\} \) and the additional constraint that either \( \sigma^*(v^*) = (0, 1) \) or \( \sigma^*(v^*) = (1, 0) \). We construct simple low degree reductions from \( P_{\text{csp}}(B^{H, Z_2, \sigma}) \) as well as from \( P_{\text{csp}}(B^{H, Z_3, \sigma}) \) to \( P_{\text{csp}}(C^*) \), which are in fact just restrictions. Fix \( p \in \{2, 3\} \). For the reduction from \( P_{\text{csp}}(B^{H, Z_p, \sigma}) \) we set for all \( (u, v) \in E(H) \)

\[
\begin{align*}
f_{y(u,v) \mapsto (0,0)} &= [x_{(u,v)} \mapsto 0], \\
f_{y(u,v) \mapsto t_p} &= [x_{(u,v)} \mapsto 1], \\
f_{y(u,v) \mapsto \gamma} &= 0, \quad \text{if } \gamma \not\in \{(0, 0), t_p\},
\end{align*}
\]

(6.I)

(6.J)

(6.K)

Furthermore, for the additional variable \( y^* \) we set

\[
\begin{align*}
f_{y^* \mapsto t_p} &= 1, \\
f_{y^* \mapsto \gamma} &= 0, \quad \text{if } \gamma \neq t_p,
\end{align*}
\]

(6.L)

(6.M)

We have to check that this substitution fulfills the requirements of low degree reductions. As every variable \( y \) from \( P_{\text{csp}}(C^*) \) is substituted by a polynomial \( f_y \) of the form \( 0, 1, [x_e \mapsto 0] \), or \([x_e \mapsto 1]\), the equations \( f_{y^*}^2 - f_y \) follow immediately. Furthermore, the additional constraint \( \{y^*, \{t_2, t_3\}\} \) is satisfied. In order to verify that all substituted axioms from vertex constraints \( C_v \) have a constant degree derivation from \( P_{\text{csp}}(B^{H, Z_p, \sigma}) \), we apply Theorem 5.1 and note that each substituted constraint \( C_v \) follows semantically from the corresponding vertex constraint \( \{0, 0\} \) in \( P_{\text{csp}}(B^{H, Z_p, \sigma}) \). As every such constraint involves at most \( k \) variables we know that the substituted equations can be derived in degree \( k+1 \). As both low degree reductions hold over every prime field, it follows by Lemma 5.3 and Theorem 6.2 that every polynomial calculus refutation of \( P_{\text{csp}}(C^*) \) over a prime field requires degree \( \Omega(\|P_{\text{csp}}(C^*)\|) \). Because \( P_{\text{csp}}(C^*) \) is a 2-extended group CSP, the lower bound for \( P_{\text{isogr}}(G, \overline{G}) \) follows from Lemma 6.4.

\[\square\]
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A Expander

We just review the bare essentials of expander graphs and refer the reader to the survey [14] for background. The expansion ratio of a graph $G$ is

$$h(G) := \min_{0 < |W| \leq |G|/2} \frac{\partial(W)}{|W|},$$

The expansion ratio of a family $\mathcal{C}$ of graphs is

$$h(\mathcal{C}) = \inf_{G \in \mathcal{C}} h(G).$$

If $\mathcal{C}$ is infinite and $h(\mathcal{C}) > 0$ we call $\mathcal{C}$ a family of expander graphs (Typically, we only use this terminology if $\mathcal{C}$ is infinite.)

Fact A.1 (Folklore). For every $d$ there exists a family of $d$-regular $d$-connected expander graphs.

Maybe the easiest way to obtain such a family is by taking random $d$-regular bipartite graphs with both parts of the same size, which asymptotically almost surely are $d$-connected [12] and have positive expansion [2].

Recall that a graph $G$ is $d$-connected if $|V(G)| > d$ and for every set $S \subseteq V(G)$ the graph $G \setminus S$ is connected. We are only interested in 2-connected graphs here. It is well known that every graph has a nice decomposition into its 2-connected components. It is convenient to state this result using tree decompositions. A tree decomposition of a graph $G$ is a pair $(T, \beta)$, where $T$ is a tree and $\beta : V(T) \to 2^{V(G)}$ such that: (i) for every $v \in V(G)$ the set of all $t \in V(T)$ such that $v \in \beta(t)$ is connected in $T$, and (ii) for every edge $vw \in E(G)$ there is a $t \in V(T)$ such that $v, w \in \beta(t)$. The adhesion of a tree decomposition $(T, \beta)$ is $\max_{t \in V(T)} |\beta(t) \cap \beta(u)|$ if $E(T) \neq \emptyset$ and 0 if $E(T) = \emptyset$.

Fact A.2 (Folklore). Every graph $G$ has a tree decomposition $(T, \beta)$ of adhesion at most 1 such that for all $t \in V(T)$, either the induced subgraph $G[\beta(t)]$ is 2-connected or $|\beta(t)| \leq 2$. 

20
We call the decomposition \((T, \beta)\) of the fact a *decomposition of \(G\) into 2-connected components.*

**Lemma A.3.** Let \(\mathcal{E}\) be a family of 3-regular 2-connected expander graphs.

Then there is constant \(c > 0\) such that for every \(G \in \mathcal{E}\) and every set \(W \subseteq V(G)\) there is a set \(\hat{W} \supseteq W\) of size \(|\hat{W}| \leq c|W|\) such that \(G \setminus \hat{W}\) is either empty or 2-connected.

**Proof.** Let \(\epsilon := \min\{1, h(\mathcal{E})\}\) and

\[ c := \frac{30}{\epsilon}. \]

Let \(n := |V(G)|\), and let \(W \subseteq V(G)\) and \(k := |W|\). Without loss of generality we may assume that

\[ \frac{30}{\epsilon}k < n; \quad (1.A) \]

otherwise we let \(\hat{W} := V(G)\).

Let \((T, \beta)\) be a tree decomposition of \(G \setminus W\) into 2-connected components. For every edge \(tu \in E(T)\), we let \(T(t, u)\) be the connected component of \(T – \{tu\}\) (the tree obtained from \(T\) by deleting the edge \(tu\)) that contains \(u\), and we let \(\gamma(t, u) := \bigcup_{s \in V(T(t, u))} \beta(s)\). We define \(T(u, t)\) and \(\gamma(u, t)\) similarly.

Now we orient every edge \(tu\) in such a way that it points to the larger of the two sets \(\gamma(t, u)\) and \(\gamma(u, t)\), breaking ties arbitrarily. Then there is a node \(s \in V(T)\) such that all edge \(st\) are oriented towards \(s\). That is, for all \(t \in N(s)\) (the set of neighbours of \(s\) in \(T\)) we have \(|\gamma(s, t)| \leq |\gamma(t, s)|\).

For every \(t \in N(s)\), we let \(\alpha(t) := \gamma(s, t) \setminus \beta(s)\). Note that

\[ |\alpha(t)| \leq \frac{|V(G) \setminus W|}{2} = n - k \quad (1.B) \]

Without loss of generality we assume that \(\alpha(t) \neq \emptyset\) for all \(t \in N(s)\).

Suppose for contradiction that \(|\beta(s)| < 3\). Let \(W' = W \cup \beta(t)\). It follows from (1.B) that there is a partition \((X, Y)\) of \(V(G) \setminus W' = \bigcup_{t \in N(s)} \alpha(t)\) such that there is no edge from \(X\) to \(Y\) in \(G\) and

\[ \frac{|V(G) \setminus W'|}{3} \leq |X| \leq |Y| \leq \frac{2|V(G) \setminus W'|}{3} \]

(both \(X\) and \(Y\) are unions of sets \(\alpha(t)\) for \(t \in N(s)\)). Then \(|X| \leq n/2\) and thus

\[ 3|W'| \geq \partial(X) \geq \epsilon|X| \geq \frac{\epsilon|V(G) \setminus W'|}{3} = \frac{\epsilon}{3}(n - |W'|). \]

This implies

\[ \frac{10}{\epsilon}(k + 2) \geq \left(\frac{9}{\epsilon} + 1\right)|W'| \geq n, \]

which contradicts (1.A). Thus \(|\beta(s)| \geq 3\), and this means that \(G[\beta(t)]\) is 2-connected.

Next, we observe that for every \(t \in N(s)\) there is at most one edge \(e = vw \in E(G)\) such that \(v \in \alpha(t)\) and \(w \in V(G) \setminus (W \cup \alpha(t))\). To see this, suppose for contradiction that there are two such edges \(v_1w_1\) and \(v_2w_2\). Then \(w_1 = w_2 =: w\) is the unique vertex in \(\beta(s) \cap \beta(t)\), and therefore \(v_1 \neq v_2\). As \(G[\beta(s)]\) is 2-connected, \(G\) has at least two neighbours in \(\beta(s)\). But then the degree of \(w\) is at least 4, which contradicts \(G\) being 3-regular.

Hence

\[ \epsilon|\alpha(t)| \leq |\partial(\alpha(t))| \leq 1 + e(\alpha(t), W), \quad (1.C) \]
where \( e(\alpha(t), W) \) is the number of edges between \( \alpha(t) \) and \( W \). Moreover, for every \( t \in N(W) \) we have \( e(\alpha(t), W) \geq 1 \), because otherwise the set \( \beta(s) \cap \beta(t) \) of size at most 1 separates \( G \), which contradicts \( G \) being 2-connected. Note that here we use the assumption \( \alpha(t) \neq \emptyset \).

As \( |\partial(W)| \leq 3k \), it follows that \( |N(s)| \leq 3k \). Then

\[
| \bigcup_{t \in N(s)} \alpha(t) | = \sum_{t \in N(s)} |\alpha(t)| \\
\leq \frac{|N(s)| + \sum_{t \in N(s)} e(\alpha(t), W)}{\epsilon} \quad \text{by (1.C)} \\
\leq \frac{|N(s)| + |\partial(W)|}{\epsilon} \\
\leq \frac{6k}{\epsilon}
\]

We let \( \widehat{W} := W \cup \bigcup_{t \in N(s)} \alpha(t) \). Then \( G \setminus \widehat{W} = G[\beta(s)] \) is 2-connected, and

\[
|\widehat{W}| \leq \left( 1 + \frac{6}{\epsilon} \right) k \leq \frac{7}{\epsilon} k \leq ck.
\]

**Corollary A.4.** Let \( \mathcal{E} \) be a family of 3-regular 2-connected expander graphs.

There is constant \( c > 0 \) such that for every \( G \in \mathcal{E} \) and every set \( X \subseteq E(G) \) there is a set \( X^* \supseteq X \) of size \( |X^*| \leq c|X| \) such that \( E(G) \setminus X^* \) is either empty or the edge set of a 2-connected subgraph of \( G \).