The pseudorelativistic Hartree equation with a general nonlinearity: existence, non existence and variational identities

DIMITRI MUGNAI∗
Dipartimento di Matematica e Informatica
Università di Perugia
Via Vanvitelli 1, 06123 Perugia - Italy
tel. +39 075 5855043, fax. +39 075 5855024,
e-mail: mugnai@dmi.unipg.it

Abstract
We prove several existence and non existence results of solitary waves for a class of nonlinear pseudo–relativistic Hartree equations with general nonlinearities. We use variational methods and some new variational identities involving the half Laplacian.

Keywords: pseudo–relativistic Hartree equation, positive potential
2000AMS Subject Classification: 35Q55, 35A15, 35J20, 35Q40, 35Q85

1 Introduction, motivations and main result
In [19], Fröhlich e Lenzmann studied the Schrödinger equation with a Hartree nonlinearity

\[ i\psi_t = \sqrt{-\Delta + m^2} \psi - \left(\frac{1}{|x|} \ast |\psi|^2\right) \psi \quad \text{in } \mathbb{R}^3 \quad (1.1) \]

as a description of pseudorelativistic boson stars (see [17] for the rigorous derivation of the model). Here \( \psi(t, x) \) is a complex–valued wave field (a one-particle wave function), the symbol \( \ast \) stands for the usual convolution in \( \mathbb{R}^3 \), and \( 1/|x| \) is the Newtonian gravitational potential (after setting all physical constants equal to 1). The operator \( \sqrt{-\Delta + m^2} \), which coincides with the so–called half Laplacian \( (-\Delta)^{1/2} \) when \( m = 0 \), describes the kinetic and rest energy of a relativistic particle of mass \( m \geq 1 \), and can be defined in several ways. For example, we

∗Research supported by the Miur PRIN Variational methods and nonlinear PDEs

†The kinetic energy of a fermion with mass \( m > 0 \) is described by the pseudo–differential operator \( \sqrt{-\Delta + m^2} - m \).
can associate to $\sqrt{-\Delta + m^2}$ its symbol $\sqrt{k^2 + m^2}$ in the following way: for any $f \in H^1(\mathbb{R}^N)$ with Fourier transform $\mathcal{F}f$, we define

$$\mathcal{F}(\sqrt{-\Delta + m^2}f)(k) = \sqrt{|k|^2 + m^2} \mathcal{F}f(k),$$

which is actually well defined in $H^{1/2}(\mathbb{R}^N)$, see [20] for a complete description of this method.

We will follow another approach to define $\sqrt{-\Delta + m^2}$, extending to the whole Euclidean space the “Dirichlet to Neumann” procedure (see, for example, [9]), which consists in realizing the nonlocal operator $\sqrt{-\Delta + m^2}$ in $\mathbb{R}^3$ through a local problem in $\mathbb{R}^4_+ = \mathbb{R}^3 \times (0, \infty)$. In view of our general statements, we will present this procedure in every dimension $N \geq 2$: for any function $u \in \mathcal{S}(\mathbb{R}^N)$ there exists a unique function $v \in \mathcal{S}(\mathbb{R}^N)$ such that

$$\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
v(x, 0) = u(x) & \text{on } \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\} \simeq \mathbb{R}^N,
\end{cases} \quad (1.2)$$

i.e. $v$ is the generalized harmonic extension of $u$ in $\mathbb{R}^{N+1}_+$. Here, and in the following, we shall denote by $x$ a generic point of $\mathbb{R}^N$. Now, consider the operator $T$ defined as

$$Tu(x) = -\frac{\partial v}{\partial x_{N+1}}(x, 0). \quad (1.3)$$

It is readily seen that the system

$$\begin{cases}
-\Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
w(x, 0) = -\frac{\partial v}{\partial x_{N+1}}(x, 0) = Tu(x) & \text{on } \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\} \simeq \mathbb{R}^N,
\end{cases}$$

admits the unique solution $w(x, y) = -\frac{\partial v}{\partial x_{N+1}}(x, y)$, and thus by (1.2) we immediately have

$$T(Tu(x)) = -\frac{\partial w}{\partial x_{N+1}}(x, 0) = \frac{\partial^2 v}{\partial x_{N+1}^2}(x, 0) = (-\Delta v + m^2 v)(x, 0).$$

In conclusion, $T^2 = -\Delta + m^2$, i.e. the operator $T$ mapping the Dirichlet datum $u$ to the Neumann datum $-\frac{\partial v}{\partial x_{N+1}}(\cdot, 0)$ is a square root of the generalized Laplacian $-\Delta + m^2$ in $\mathbb{R}^{N+1}_+$.

Going back to (1.1), such an equation does not take into account the presence of outer influences or mutual interactions among particles, so that it seems natural to include the presence of a potential, considering

$$i\psi_t = \sqrt{-\Delta + m^2} \psi - \lambda (W * |\psi|^2) \psi + F'(\psi) \quad \text{in } \mathbb{R}^N, \quad (1.4)$$

where $\lambda \in \mathbb{R}$ and $F: \mathbb{R} \to \mathbb{R}$ is a $C^1$ potential having some good invariant (common in Abelian Gauge Theories, see [6], [23], [24]): typically, one requires some conditions of the form $F(e^{i\theta}u) = F(u)$ and $F'(e^{i\theta}u) = e^{i\theta}F'(u)$ for any function $u$ and any $\theta \in \mathbb{R}$, which is obviously satisfied by linear combinations.
of power–like nonlinearities. Finally, \( W \) is a radially symmetric weight function which generalizes in \( \mathbb{R}^N \) the Newtonian potential \( 1/|x| \) in \( \mathbb{R}^3 \). In particular, we will also assume that \( W \) can be decomposed as sum of a bounded function plus another function having suitable integrability (see Theorem 1.1 for the precise assumptions).

We are interested in solitary wave solutions of (1.4) of the form
\[
\psi(x, t) = e^{-i\omega t}u(x), \quad \text{where} \quad \omega \in \mathbb{R} \text{ and } u : \mathbb{R}^N \rightarrow \mathbb{R};
\]
thus, since the Fourier transform acts only on the \( x \)–variables, it is readily seen that \( u \) solves
\[
\sqrt{-\Delta + m^2} u - \omega u - \lambda (W * u^2) u + F'(u) = 0 \quad \text{in} \quad \mathbb{R}^N.
\]
Actually, there is no reason to discard an \( x \)–dependence in the potential \( F = F(x, s) \), so that we will actually consider the following nonlinear Hartree equation with potential:
\[
\sqrt{-\Delta + m^2} u - \omega u - \lambda (W * u^2) u + F_s(x, u) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.5}
\]
Let us remark that in \( \mathbb{R}^3 \) with \( W(x) = 1/|x| \) such an equation is equivalent to the following system of pseudorelativistic Schrödinger–Poisson type:
\[
\sqrt{-\Delta} u + m^2 u - \omega u - \lambda u \phi + F_s(x, u) = 0 \quad \text{in} \quad \mathbb{R}^3, \\
-\Delta \phi = 4\pi u^2 \quad \text{in} \quad \mathbb{R}^3, \tag{1.6}
\]
where the second equation represents the repulsive character of the Coulomb force (the attractive case is described by the equation \( \Delta \phi = 4\pi u^2 \), see [29]).

We also note that this very last system is the pseudorelativistic version of the classical Schrödinger–Poisson system, or Hartree–Fock equation, which has been object of a large interest in the last decade, also for its applications in modelling molecules and crystals, and we only quote [3], [4], [10], [11], [12], [13], [15], [16], [22], [25], [28] and the papers cited therein as a brief references list.

When \( N = 3 \), the case \( \lambda = 1 \) and \( F \equiv 0 \) was studied in [21] (for existence of spherically symmetric solutions), in [15] (for stability of standing wave solutions of (1.1)), in [19] (for instability of standing wave solutions of (1.1) when \( m = 0 \)).

The fact that the natural domain for the governing operator in (1.5) is \( H^{1/2}(\mathbb{R}^N) \), forces to decrease the range of natural exponents in the nonlinearities which could appear in the equation, according to the Sobolev Embedding Theorem; more precisely, if \( N = 3 \), the critical exponent is 3; hence, all subcritical superlinearities in the equation may have a growth between 1 and 2. In fact, equation (1.5) was considered in [14] when \( \lambda = 1 \), \( \omega < m \) and \( m > 0 \), with
\[
F(x, s) = F(s) = -\frac{|s|^p}{p}, \quad p \in \left( 2, \frac{2N}{N-1} \right). \tag{1.7}
\]
Let us remark that the prototype potential defined in (1.7) is not positive, while for physical reasons a potential suitable to model physical phenomena
Pseudorelativistic Hartree equation: existence and nonexistence

should be nonnegative: indeed, the fact that $F$ is nonnegative implies that the potential energy density of a solution of equation (1.5) has more nonnegative contributions, as one can see from the expression of the Lagrangian in equation (2.24) below. Another reason to consider positive potential is that, if we consider the classical autonomous electrostatic case $-\Delta u + F'(u) = 0$, calling “rest mass” of the particle $u$ the quantity

$$\int F(u) \, dx$$

(see [7]), the fact that $F$ is positive implies that the systems under consideration has - a priori - positive mass, which is, of course, relevant from a physical viewpoint.

Therefore, in this paper, as far as existence is concerned, we shall consider equation (1.5) under the assumptions that $F$ is nonnegative and $m > 0$.

Using the approach with the operator $T$ introduced in (1.3), we rewrite equation (1.5) as the following system:

$$\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+,
-\partial v / \partial x_{N+1} = \omega v + \lambda (W * v^2) v - F_s(x, v) & \text{on } \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\} \simeq \mathbb{R}^N.
\end{cases}
$$

(1.8)

As usual, for physical reasons, we look for solutions having finite energy, i.e. $v \in H^1(\mathbb{R}^{N+1}_+)$, $H^1(\mathbb{R}^{N+1}_+)$ being the usual Sobolev space endowed with the scalar product

$$\langle v, w \rangle_{H^1} := \int_{\mathbb{R}^{N+1}_+} (Dv \cdot Dw + m^2 vw) \, dx \, dx_{N+1}$$

and norm $\|v\| = (\int |Dv|^2 + m^2 \int v^2)^{1/2}$.

**Definition 1.** A function $v \in H^1(\mathbb{R}^{N+1}_+)$ is said to be a (weak) solution of problem (1.8) iff for every $w \in H^1(\mathbb{R}^{N+1}_+)$ there holds

$$\int_{\mathbb{R}^{N+1}_+} (Dv \cdot Dw + m^2 vw) \, dx \, dx_{N+1} = \int_{\mathbb{R}^N} \left[ \omega v + \lambda (W * v^2) v - F_s(x, v) \right] w \, dx.
$$

(1.9)

**Remark 1.1.** We remark that, whenever a solution $v \in H^1(\mathbb{R}^{N+1}_+)$ of (1.8) is given, then

$$\int_{\mathbb{R}^N} F_s(x, v) w \, dx \in \mathbb{R} \quad \text{for all } w \in H^1(\mathbb{R}^{N+1}_+)$$

without assuming any growth condition on $F$.

In this paper we are concerned with problem (1.8) in presence of a generally nontrivial potential $F$, always neglected in the previous papers, except for [14]; hence, in order to cover the trivial case $F \equiv 0$, for the main existence result (see Theorem 1.1) we shall use the following general superlinear and subcritical assumptions, which, however, can be relaxed for the other existence and for the non existence results, see Propositions 1.1, 1.2 and 1.3 and Theorem 1.3 below:
Pseudorelativistic Hartree equation: existence and nonexistence

5

\( F_1 \) \( F : \mathbb{R}^N \times \mathbb{R} \to [0, \infty) \) is such that the derivative \( F_s : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, \( F(x, s) = F(|x|, s) \) for a.e. \( x \in \mathbb{R}^N \) and for every \( s \in \mathbb{R} \), and \( F(x, 0) = F_s(x, 0) = 0 \) for a.e. \( x \in \mathbb{R}^N \);

\( F_2 \) \( \exists C_1, C_2 > 0 \) and \( 2 < \ell < p < 2N/(N-1) \) such that

\[ |F_s(x, s)| \leq C_1 |s|^\ell - 1 + C_2 |s|^{p-1} \text{ for a.e. } x \in \mathbb{R}^N \text{ and every } s \in \mathbb{R}; \]

\( F_3 \) \( \exists k \geq 2 \) such that \( 0 \leq sF_s(x, s) \leq kF(x, s) \) for a.e. \( x \in \mathbb{R}^N \) and every \( s \in \mathbb{R} \);

**Remark 1.2.** Condition \( F_3 \) is a kind of reversed Ambrosetti–Rabinowitz condition, already used, for instance, in [5], [24] and [25].

**Remark 1.3.** As it will be clear from the existence proof, the requirement that \( F \) depends radially on the space variable is a technical assumption which lets us reduce the problem to a radial setting and use some compactness properties in the associated Sobolev space.

**Remark 1.4.** From \( F_1 \) we immediately get that \( F \) has an absolute minimum point for \( s = 0 \) and for a.e. \( x \in \mathbb{R}^N \), so that problem (1.8) always admits the trivial solution \( u = 0 \). Moreover, by direct integration of \( F_2 \) we get

\[ 0 \leq F(x, s) \leq \frac{C_1}{\ell} |s|^\ell + \frac{C_2}{p} |s|^p \quad (1.10) \]

for every \( s \in \mathbb{R} \) and a.e. \( x \in \mathbb{R}^N \), so that the potential \( F \) is superquadratic at 0 and subcritical at infinity, as in the case of (1.7). For this reason, in \( F_3 \) we exclude the case \( k < 2 \), since by simple calculations we would get \( F \equiv 0 \), already considered in the related literature.

**Remark 1.5.** Our assumptions include the case \( F \equiv 0 \), but even the more intriguing cases in which \( F(x, s) = 0 \) for \( x \) belonging to a proper subset of \( \mathbb{R}^N \) - a ball (so that the potential \( F \) is active only in an exterior domain), an annulus, ... - or for some values of \( s \) - for instance when \( s \) is large or small. All these situations are completely new, and, to our best knowledge, this paper is the first one to consider (1.8) with a potential, even possibly vanishing somewhere. In particular, we remark that no control from below is assumed on the potential \( F \), as done, for instance, in [6] for a Klein–Gordon–Maxwell system with positive potential.

Since \( u = 0 \) is a solution of problem (1.8), we are interested in nontrivial solutions. But, before giving our first existence result, we make precise the assumptions on the weight potential \( W \), which will be assumed from now on:

\[ W) : W : \mathbb{R}^N \to [0, \infty) \text{ is such that } W(x) = W(|x|), W = W_1 + W_2, \text{ where } W_1 \in L^r(\mathbb{R}^N) \text{ for some } r \in (N/2, \infty), \text{ and } W_2 \in L^\infty(\mathbb{R}^N). \]
Remark 1.6. If \( N = 3 \) we can take the Newton potential \( W(x) = 1/|x| \), which is bounded at infinity and in \( L^r \) near the origin for \( r < N \), so that all our results cover the physical problem (1.6). However, if we require that solutions have a constant \( L^2 \) norm equal to \( M \) (like in [21]), it turns out that the Newton potential is critical, in the sense that if \( W(x) = 1/|x| \), then a radial, real–valued, nonnegative ground state solitary wave in \( H^{1/2}(\mathbb{R}^3) \) does exist only if \( M < M_c \), \( M_c \) being the Chandrasekhar limit mass for boson stars modelled by (1.1) (and in such a case the solution is found via a minimization process). Therefore, not fixing the \( L^2 \) norm of solutions leaves us more freedom in the quest of solutions.

Moreover, another classical potential we can treat when \( N = 3 \) is any Yukawa type two body interaction, that is
\[
W(x) = e^{-\mu |x|}/|x|, \quad \mu \geq 0.
\]

Remark 1.7. In contrast to [14] and to the behaviour of the Newton or Yukava potential, we do not require that \( W(x) \to 0 \) as \( |x| \to \infty \), so that we are allowed to consider a wider class of kernels.

We are ready for our main existence result.

Theorem 1.1. Assume that \( F \) satisfies \( F_1 \), \( F_2 \) and \( F_3 \) with \( k \leq 4 \) and that \( W \) satisfies \( W \). Then for any \( \lambda > 0 \) and \( \omega < m \) there exists a nontrivial solution \( v \in H^1(\mathbb{R}^{N+1}) \) of problem (1.8) which is radially symmetric in the first \( N \) variables. If, in addition, \( F(x,s) \geq F(x,|s|) \) for a.e. \( x \in \mathbb{R}^N \) and all \( s \in \mathbb{R} \), then \( v \) is strictly positive in \( \mathbb{R}^{N+1} \).

The assumption \( \lambda > 0 \) in Theorem 1.1 is not restrictive. Indeed, just under a more general version of assumption \( F_3 \) above, problem (1.8) admits only the trivial solution when \( \lambda \leq 0 \), independently of any possible symmetry:

Proposition 1.1. Suppose that \( F \) satisfies \( F_4 \) for a.e. \( x \in \mathbb{R}^N \) and all \( s \in \mathbb{R} \), and let \( v \in H^1(\mathbb{R}^{N+1}) \) be a solution of problem (1.8) with \( \lambda \leq 0 \) and \( \omega < m \). Then \( v = 0 \).

For the more general non existence results in \( \mathbb{R}^3 \) we assume that \( F \) satisfies a weaker form of \( F_1 \), i.e. we require that \( F \) is independent of the \( x \)-variable and needs not be nonnegative:

\[
F_1' \quad F : \mathbb{R}^3 \to \mathbb{R} \text{ is of class } C^1 \text{ with } F(0) = F'(0) = 0.
\]

The non existence results we shall prove are consequences of the following variational identity involving the generalized half Laplacian, which we consider of independent interest:

Theorem 1.2. Assume \( F_1' \); if \( v \in H^{1/2}(\mathbb{R}^3) \) is a solution of (1.6) such that
\[
\int_{\mathbb{R}^3} F(v) \, dx \in \mathbb{R},
\]
then
\[
0 = -\int_{\mathbb{R}^4} |Dv|^2 \, dX - 2m^2 \int_{\mathbb{R}^4} v^2 \, dX + \frac{3}{2} \omega \int_{\mathbb{R}^3} v^2 \, dx + \frac{5}{4} \lambda \int_{\mathbb{R}^3} v^2 \phi \, dx - 3 \int_{\mathbb{R}^3} F(v) \, dx.
\]

(1.11)
As aforesaid, by exploiting (1.11) we are able to prove

**Theorem 1.3.** Assume $F_1'$; let $v \in H^{1/2}(\mathbb{R}^3)$ be a solution of (1.6) such that

$$
\int_{\mathbb{R}^3} F(v) \, dx \in \mathbb{R}.
$$

If

$$
\omega \leq 0, \lambda \leq 0 \text{ and } F'(s)s \leq 3F(s) \quad \forall \ s \in \mathbb{R}, \quad (1.12)
$$

or

$$
\omega \leq 0, \lambda \leq 0 \text{ and } 3F(s) \leq F'(s)s \quad \forall \ s \in \mathbb{R}, \quad (1.13)
$$

or

$$
\lambda \leq 0, \omega \in \left[0, m\sqrt{\frac{2}{9}}\right] \text{ and } F(s) \geq 0 \quad \forall \ s \in \mathbb{R}, \quad (1.14)
$$

or

$$
\lambda > 0, \omega \in (0, m) \text{ and } 0 \leq F'(s)s \leq 2F(s) \quad \forall \ s \in \mathbb{R}, \quad (1.15)
$$

then $v \equiv 0$.

As a straightforward application, we have the following

**Corollary 1.1.** If $v \in H^{1/2}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ solves (1.6), then $v \equiv 0$ provided that

- $F(s) = \frac{|s|^p}{p}$ and
  - $\omega \leq 0, \lambda \leq 0, \text{ and } p > 0$, or
  - $\lambda \leq 0, \omega \in \left[0, m\sqrt{\frac{2}{9}}\right] \text{ and } p > 0$, or
  - $\lambda > 0, \omega \in (0, m), p \in (0, 2]$
- $F(s) = -\frac{|s|^p}{p}$, $\omega \leq 0, \lambda \leq 0$ and $p \in \left(0, \frac{3}{2}\right] \cup [3, \infty)$.

**Remark 1.8.** In condition (1.14), contrary to the assumptions of Proposition 1.1, no sign assumption is made on $F'$.

The Corollary above shows that the existence result proved in [14] for $\lambda > 0$, $\omega < m$ and $F(s) = -|s|^p/p$ with $2 < p < 2N/(N-1)$ is somehow optimal, since in the supercritical case only the trivial solution shows up, independently of any possible symmetry.

On the other hand, such a Corollary also guarantees that our requirement of working with a superquadratic $F$ when $\lambda > 0$ and $\omega < m$ is not only a technical one, since if $F$ is subquadratic or quadratic ($p = 2$), then only the trivial solution can exist.

Actually, Theorem 1.3 is a special case of a more general result, which is the following:

**Theorem 1.4.** Assume $F_1'$; let $v \in H^{1/2}(\mathbb{R}^3)$ be a solution of (1.6) such that

$$
\int_{\mathbb{R}^3} F(v) \, dx \in \mathbb{R}.
$$

If

$$
\omega \leq 0, \lambda \leq 0 \text{ and } \exists \rho \leq 1 \text{ s.t. } \rho F'(s)s \leq 3F(s) \quad \forall \ s \in \mathbb{R}, \quad (1.16)
$$

or
Proposition 1.2. Assume or on hold true (see [1, Theorems 7.58 and 7.57]): then \( v \) of \( F \) the corresponding ones in (1.12) and (1.13), since no assumptions on the sign or \( F \) is ever made.

We remark that the conditions in (1.16) and (1.17) are not consequences of the corresponding ones in (1.12) and (1.13), since no assumptions on the sign of \( F \) or \( F' \) is ever made.

We conclude our list of results with two existence statements which can be proved without condition \( F_3 \), but only assuming the positivity and the subcriticality of \( F \); in this case, \( \lambda \) appears as a Lagrange multiplier:

**Proposition 1.3.** Assume \( F_1 \), \( F_2 \) and suppose that \( \omega < m \). Then there exists \( \lambda \in \mathbb{R} \) such that the associated problem (1.8) admits a nontrivial solution \( u \in H^1_0(\mathbb{R}^{N+1}_+) \). If in addition, \( 2F(x,s) \leq F_s(x,s)s \) for all \( s \in \mathbb{R} \) and a.e. \( x \in \mathbb{R}^N \), then \( \lambda > 0 \).

**Proposition 1.4.** Assume \( F_1 \), \( F_2 \) and suppose that \( \omega < m \). Moreover, suppose that \( F(x,s) = F(x,-s) \) for all \( s \in \mathbb{R} \) and a.e. \( x \in \mathbb{R}^N \). Then there exists a sequence \( (\lambda_n)_n \in \mathbb{R} \) such that the associated problems (1.8) with \( \lambda = \lambda_n \) admit a couple of nontrivial solutions \( \pm u_n \in H^1_0(\mathbb{R}^{N+1}_+) \). If in addition, \( 2F(x,s) \leq F_s(x,s)s \) for all \( s \in \mathbb{R} \) and a.e. \( x \in \mathbb{R}^N \), then \( \lambda_n > 0 \) for any \( n \in \mathbb{N} \).

From a physical viewpoint, positive solutions are the most interesting ones, and in fact we can prove

**Proposition 1.5.** If \( \lambda \leq 0 \) or \( F(x,s) \geq F(x,|s|) \) for a.e. \( x \in \mathbb{R}^N \) and all \( s \in \mathbb{R} \), the solution found by Proposition 1.2 is strictly positive in \( \mathbb{R}^{N+1}_+ \).

**Remark 1.9.** If only \( F_1 \) and \( F_2 \) are in force, the assumption \( \lambda \leq 0 \) does not contradict Proposition 1.4 above.

## 2 Proofs of the existence theorems

Let us start making precise some notations:

- \((x,x_{N+1})\) a point of \( \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0, \infty) \),
- \(\|v\|\) the norm of \( v \) in \( H^1(\mathbb{R}^{N+1}_+) \),
- \(\|v\|_q\) the norm of \( v \) in \( L^q(\mathbb{R}^{N+1}_+) \),
- \(\|v\|_q\) the norm of (the trace of) \( v \) in \( L^q(\mathbb{R}^N) \).

Let us recall that any \( v \in H^1(\mathbb{R}^{N+1}_+) \), \( N \geq 2 \), admits trace (still denoted by \( v \) for simplicity) on \( \partial \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \{0\} \simeq \mathbb{R}^N \), and that the following embeddings hold true (see [1] Theorems 7.58 and 7.57):

\[
H^1(\mathbb{R}^{N+1}_+) \hookrightarrow W^{1,q}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N),
\]
where $\chi = 1 - \frac{N+1}{q} + \frac{N}{q} \in [0,1)$ and $q \leq \sigma \leq Nq/(N - \chi q)$. Let us note that 
$q = 2N/(N - 1 + 2\chi) \geq 2$ for any $\chi \leq 1/2$ and that $Nq/(N - \chi q) = 2N/(N - 1)$ for any $\chi$; in particular
\[ H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^2(\mathbb{R}^N) \quad \text{and} \quad H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^{2N/(N-1)}(\mathbb{R}^N). \]
In addition, 
\[ \frac{4N}{N+2} < \frac{Nq}{N - \chi q} \quad \text{for any } \chi \geq 0, \]
and
\[ \frac{4N}{N+2} \geq q \]
as soon as $\chi \geq 1 - N/4$. In particular, by interpolation, we get that any $v \in H^1(\mathbb{R}^{N+1}_+)$ is such that $v^2 \in (L^2(\mathbb{R}^N))^\prime = L^{2q}('\mathbb{R}^N)$.
Moreover, if $v \in C^\infty(\mathbb{R}^{N+1}_+)$, we have
\[
\int_{\mathbb{R}^N} |v(x,0)|^q dx = - \int_{\mathbb{R}^N} dx \int_0^\infty \frac{\partial}{\partial x_{N+1}} |v(x,x_{N+1})|^q dx_{N+1} = - q \int_{\mathbb{R}^{N+1}} |v(x,x_{N+1})|^{q-2} v(x,x_{N+1}) \frac{\partial v}{\partial x_{N+1}} (x,x_{N+1}) dx dx_{N+1},
\]
and by the Hölder inequality
\[
|v(x,0)|_q \leq q^{1/q} \|v\|_{2q-2}^{1-1/q} \|v_{x_{N+1}}\|_{2}^{1/q} \leq q^{1/q} \|v\|_{2q-2}^{1-1/q} \|Dv\|_{2}^{1/q}.
\]
By interpolation, the Sobolev inequality and by density, we get that
\[
|v(x,0)|_q \leq c_q \|v\| \quad \text{for any } v \in H^1(\mathbb{R}^{N+1}_+),
\]
provided that $2 \leq 2q - 2 \leq (N + 1)/(N - 1)$, that is $2 \leq q \leq 2N/(N - 1)$.
Applying the Young inequality to (2.19) we also get
\[
|v(x,0)|_q^q \leq \frac{q^2}{4} \int_{\mathbb{R}^{N+1}_+} |v|^{2q-2} dx dx_{N+1} + \frac{1}{\varepsilon} \int_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x_{N+1}} \right|^2 dx dx_{N+1}
\]
for any $\varepsilon > 0$, and in particular, when $q = 2$, we have
\[
|v(x,0)|_2^2 \leq \varepsilon \int_{\mathbb{R}^{N+1}_+} |v|^2 dx dx_{N+1} + \frac{1}{\varepsilon} \int_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x_{N+1}} \right|^2 dx dx_{N+1} \quad \text{for any } \varepsilon > 0,
\]
Now, write
\[
\int_{\mathbb{R}^N} (W * v^2) v^2(x) dx = \int_{\mathbb{R}^N} (W_1 * v^2) v^2(x) dx + \int_{\mathbb{R}^N} (W_2 * v^2) v^2(x) dx.
\]
By Hölder’s inequality, we can estimate the right hand side of the previous inequality by
\[
|v|_2^2 |W_1 * v^2|_{q'} + |W_2|_{\infty} |v|_2^2.
\]
Now apply Young’s inequality for convolutions choosing \( q \) so that \( 1/q' = 1/r + 1/q - 1 \), that is \( q = 2r/(2r - 1) \), estimating with

\[
|v|^2_v |W_1|_r |v^2|_q + |W_2|_\infty |v|^2_v;
\]

note that, since \( r > N/2 \), we have \( 1 < q < N/N - 1 \). Finally, by the interpolation and the Sobolev inequalities, we get that there exists \( C = C(W) > 0 \) such that

\[
\int_{\mathbb{R}^N} (W \ast v^2) v^2(x) \, dx \leq C \|v\|^4 \quad \text{for any} \quad v \in H^1(\mathbb{R}^{N+1}). \tag{2.23}
\]

In view of the previous remarks, by exploiting the radial symmetry of \( W \), the proof of the following result is straightforward:

**Proposition 2.1.** A function \( v \in H^1(\mathbb{R}^{N+1}) \) is a solution of system \([1.8]\) iff \( v \) is a critical point of the \( C^1 \) functional \( J : H^1(\mathbb{R}^{N+1}) \to \mathbb{R} \) defined as

\[
J(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}} (|Dv|^2 + m^2 v^2) \, dx dx_{N+1} - \int_{\mathbb{R}^N} \left[ \frac{\omega}{2} v^2 + \lambda W(v^2)v^2 - F(x, v) \right] \, dx.
\tag{2.24}
\]

Due to the lack of compactness of the rotation group in \( \mathbb{R}^N \), we will look for critical points of \( J \) constrained on the space of functions which are radially symmetric in the first \( N \) variables, that is

\[
H^1_1(\mathbb{R}^{N+1}) = \left\{ v \in H^1(\mathbb{R}^{N+1}) : v(Mx, x_{N+1}) = v(x, x_{N+1}) \text{ for any} \quad M \in O(N) \right\};
\]

here \( O(N) \) denotes the orthogonal group in \( \mathbb{R}^N \). Since the problem under consideration is invariant by rotation around the \( x_{N+1} \)-axis, if \( v \in H^1_1(\mathbb{R}^{N+1}) \) is a critical point of \( J \) constrained on \( H^1_1(\mathbb{R}^{N+1}) \), then \( v \) is also a critical point of \( J \) on the whole of \( H^1(\mathbb{R}^{N+1}) \) by the Principle of Symmetric Criticality of Palais, see [26]. Hence, we will look for critical points of \( J \) constrained on \( H^1_1(\mathbb{R}^{N+1}) \). In particular, we want to show that under the assumptions of Theorem [1.1] the functional \( J \) constrained on \( H^1_1(\mathbb{R}^{N+1}) \) satisfies the hypothesis of the Mountain Pass Theorem (see [2]).

First, we prove that \( J \_{H^1(\mathbb{R}^{N+1})} \) has a strict minimum point in \( v = 0 \). Indeed, taking \( \varepsilon = m \) in (2.22), since \( F \geq 0 \), using (2.23), we have

\[
J(v) \geq \left( \frac{1}{2} - \frac{\omega}{2m} \right) \int_{\mathbb{R}^{N+1}} |Dv|^2 \, dx dx_{N+1} + \left( \frac{m^2}{2} - \frac{m\omega}{2} \right) \int_{\mathbb{R}^{N+1}} v^2 \, dx dy - C \|v\|^4
\]

for any \( v \in H^1_1(\mathbb{R}^{N+1}) \), and the claim follows, since \( \omega < m \).

Moreover, if \( v \in H^1_1(\mathbb{R}^{N+1}) \) is such that \( v(x, 0) \neq 0 \), taken \( t > 0 \) we have

\[
J(tv) = \frac{t^2}{2} \int_{\mathbb{R}^{N+1}} (|Dv|^2 + m^2 v^2) \, dx dx_{N+1}
\]

\[
- \int_{\mathbb{R}^N} \left[ \frac{\omega t^2}{2} v^2 + \lambda t^4 (W \ast v^2)v^2 - F(x, tv) \right] \, dx
\]

\[
\leq At^2 - Bt^4 + Ct^\ell + Dt^p
\]
by (1.10), where $A, B, C, D$ are positive constants. Now, if $p < 4$, we have that $A t^2 - B t^3 + C t^4 + D t^5 \to -\infty$ as $t \to \infty$. Note that the requirement $p < 4$ is automatically satisfied, since $p < \frac{2N}{N-1} < 4$ for every $N \geq 2$.

We also need

**Lemma 2.1.** $J$ satisfies the Palais–Smale condition on $H^1_\nu(\mathbb{R}^{N+1})$, i.e., every sequence $(v_n)_n$ in $H^1_\nu(\mathbb{R}^{N+1})$ such that $(J(v_n))_n$ is bounded and $J'(u_n) \to 0$ in $(H^1_\nu(\mathbb{R}^{N+1}))'$ as $n \to \infty$, admits a convergent subsequence.

**Proof.** Let $(v_n)_n$ in $H^1_\nu(\mathbb{R}^{N+1})$ be as in the statement above. We first prove that $(v_n)_n$ is bounded. Indeed, by assumption, there exist $A, B > 0$ such that

$$4J(v_n) - J'(v_n)(v_n) \leq A + B\|v_n\|.$$ 

On the other hand,

$$4J(v_n) - J'(v_n)(v_n) = \int_{\mathbb{R}^{N+1}} \left[ |Dv_n|^2 + m^2 v_n^2 \right] dx - \omega \int_{\mathbb{R}^N} v_n^2 dx$$

$$+ \int_{\mathbb{R}^N} \left[ 4F(x, v_n) - F_s(x, v_n) v_n \right] dx$$

$$\geq \left( 1 - \frac{\omega}{m} \right) \int_{\mathbb{R}^{N+1}} |Dv_n|^2 dx - m(m - \omega) \int_{\mathbb{R}^{N+1}} v_n^2 dx$$

by (2.22) applied again with $\varepsilon = m$ and by $F_3)$. In conclusion, there exist $C > 0$ such that

$$C\|v_n\|^2 \leq A + B\|v_n\|,$$

and thus

$$(v_n)_n$$

is bounded in $H^1_\nu(\mathbb{R}^{N+1})$, \hspace{1cm} (2.25)

as claimed, and so we can assume that $v_n \rightharpoonup v$ in $H^1_\nu(\mathbb{R}^{N+1})$.

By (2.20), (denoting as usual the trace of a function by the function itself) we get that $(v_n)_n$ is bounded also in $L^q(\mathbb{R}^N)$ for any $q \in \left(2, \frac{2N}{N-1}\right)$, so that we may assume, without loss of generality, that $v_n \rightharpoonup v$ in $L^q(\mathbb{R}^N)$. However, by the compact embedding of $H^1_\nu(\mathbb{R}^{N+1})$ in $L^q(\mathbb{R}^{N+1})$ for any $q \in \left(\frac{2N}{N+1}, \frac{2N}{N-1}\right)$, we actually have that

$$v_n \rightharpoonup v$$

in $L^q(\mathbb{R}^N)$ for any $q \in \left(\frac{2N}{N-1}, \frac{2N}{N+1}\right)$, \hspace{1cm} (2.26)

Up to subsequence, we can also assume that $v_n \to v$ a.e. in $\mathbb{R}^{N+1}$.

We will now show that $v_n \to v$ in $H^1_\nu(\mathbb{R}^{N+1})$. First, $F_s(x, v_n) \to F_s(x, v)$ a.e., since $F_s$ is a Carathéodory function by $F_1$, and by $F_2$

$$|F_s(x, v_n)(v_n - v)| \leq C_1 |v_n|^{|\ell|} |v_n - v| + C_2 |v_n|^{|p-1|} |v_n - v|.$$ 

For instance,

$$\int_{\mathbb{R}^N} |v_n|^{|\ell|} |v_n - v| dx \leq |v_n|^{|\ell|} |v_n - v|_{\ell} \to 0 \quad \text{as } n \to \infty$$
Requiring $2\sigma$ where $1 + \frac{1}{r'} = \frac{1}{r} + \frac{1}{s}$, Analogously, we have

$$|F_\sigma(x,v)(v_n - v)| \leq C_1|v|^{r-1}|v_n - v| + C_2|v|^{s-1}|v_n - v| \to 0 \quad \text{as } n \to \infty,$$

and so Lebesgue’s Theorem gives

$$\int_{\mathbb{R}^N} |F_\sigma(x,v_n) - F_\sigma(x,v)|(v_n - v) \, dx \to 0 \quad \text{as } n \to \infty. \quad (2.27)$$

Moreover, for any $q \in \left(2, \frac{2N}{N-1}\right)$, so that $q' \in \left(\frac{2N}{N-1}, 2\right)$, we get

$$\int_{\mathbb{R}^N} (W \ast v_n^2) v_n (v_n - v) \, dx \leq |v_n - v_q| \left(\int_{\mathbb{R}^N} |v_n(z)|^{q'} \left(\int_{\mathbb{R}^N} W(x-z) v_n^2(x) \, dx\right) \, dz\right)^{1/q'} \quad (2.28)$$

First, Minkowski’s inequality implies that for any $\varphi \in [1, \infty]$ there holds

$$|W \ast v_n^2|_{\varphi} \leq |W_1 \ast v_n^2|_{\varphi} + |W_2 \ast v_n^2|_{\varphi}. \quad (2.29)$$

Now, by Hölder’s inequality and Young’s inequality for convolutions, for any $t \in (1, \infty)$ we have

$$\int_{\mathbb{R}^N} |v_n|^{q'} \left(\int_{\mathbb{R}^N} W_1(x-z) v_n^2(x) \, dx\right)^{q'/2} \, dz \leq |v_n|^{q'/2}_{q'/2} |W_1 \ast v_n^2|_{q'/2} \leq |v_n|^{q'/2}_{q'/2} |W_1|^{3/2}, \quad (2.30)$$

where $1 + \frac{1}{q'} = \frac{1}{r} + \frac{1}{s}$. Choosing $t$ such that $q't = 2\sigma$, that is $\sigma = \frac{3q'r}{2q'r+r-s}$, we finally have

$$|v_n(W_1 \ast v_n^2)|_{q'} \leq |v_n|^{3}_{3/2}|W_1|_r. \quad (2.31)$$

Requiring $2\sigma \in \left[2, \frac{2N}{N-1}\right]$ implies

$$\frac{2r}{r+2} \leq q' \leq \frac{2Nr}{rN-3r+2N}.$$  

Since $r > N/2$, we have

$$\frac{2N}{N+4} < \frac{2r}{r+2} < 2 \quad \text{and} \quad \frac{2N}{N+1} < \frac{2Nr}{rN-3r+2N} < \frac{2N}{N-3};$$

finally have the nontrivial possibility $q' \in \left(\frac{2N}{N+1}, 2\right)$, that is $q \in \left(2, \frac{2N}{N-1}\right)$, with obvious meaning if $N \leq 3$.

In this way (2.28), (2.31) and (2.25) imply the existence of $C > 0$ such that

$$\int_{\mathbb{R}^N} |v_n|^{q'} \left(\int_{\mathbb{R}^N} W_1(x-z) v_n^2(x) \, dx\right)^{q'/2} \, dz \leq C \quad \text{for every } n \in \mathbb{N}. \quad (2.32)$$
On the other hand, proceeding as above, we have

\[ |v_n(W_2 * v_n^2)_{q'}| \leq |v_n|_{q't} |v_n|_{2\sigma} W_2, \]

where \( t \) can be chosen so that \( q't = 2\sigma \in \left[ 2, \frac{2N}{N-1} \right] \), obtaining, as above,

\[ |v_n(W_2 * v_n^2)_{q'}| \leq |v_n|_{q't} |v_n|_{2\sigma} W_2 \leq C \quad (2.33) \]

for some \( C > 0 \) and all \( n \in \mathbb{N} \).

In conclusion, from (2.29) we get

\[
\left| \int_{\mathbb{R}^N} (W * v_n^2) v_n(v_n - v) \, dx \right| \leq C |v_n - v|_q \tag{2.34}
\]

for some \( C > 0 \) and all \( n \in \mathbb{N} \).

Reasoning as above, one can also prove that

\[
\left| \int_{\mathbb{R}^N} (W * v^2) v(v_n - v) \, dx \right| \leq C |v_n - v|_q \tag{2.35}
\]

for some \( C > 0 \) and all \( n \in \mathbb{N} \).

We also need to prove that \( v \) is a critical point of \( J \) on \( H^1_0(\mathbb{R}_+^{N+1}) \). To this purpose, start from \( J'(v_n)(w) \to 0 \) as \( n \to \infty \) for every \( w \in H^1_0(\mathbb{R}_+^{N+1}) \), i.e.

\[
\langle v_n, w \rangle_{H^1} - \int_{\mathbb{R}^N} [\omega v_n w + \lambda W * v_n^2 w - F_s(x, v_n) w] \, dx \to 0.
\]

Thus, by the weak convergence of \( v_n \) to \( v \), it is enough to prove that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} [\lambda(W * v^2) v_n w - F_s(x, v_n) w] \, dx = \int_{\mathbb{R}^N} [\lambda(W * v^2) v w - F_s(x, v) w] \, dx.
\]

First, by \( F_2 \) we have

\[
\left| \int_{\mathbb{R}^N} F_s(x, v_n) w \, dx \right| \leq \int_{\mathbb{R}^N} [C_1 |v_n|_{q-1} + C_2 |v_n|_{p-1}] |w| \, dx \quad \text{(by Hölder’s inequality)}
\]

\[
\leq C_1 |v_n|_{q-1} |w|_{q} + C_2 |v_n|_{p-1} |w|_{p} \to C_1 |v|_{q-1} |w|_{q} + C_2 |v|_{p-1} |w|_{p}.
\]

Hence, by the Generalized Dominated Convergence Theorem

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} F_s(x, v_n) w \, dx = \int_{\mathbb{R}^N} F_s(x, v) w \, dx.
\]

Then,

\[
\left| \int_{\mathbb{R}^N} [(W * v_n^2) v_n w - (W * v^2) v w] \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^N} (W * v_n^2) (v_n - v) w \, dx + \int_{\mathbb{R}^N} (W * (v_n^2 - v^2)) w \, dx \right|,
\]
and starting like in (2.30), we can estimate the previous identity with
\[(W_1 \sigma + W_2 \sigma_\infty) \left(|v_n| - v|_{2\sigma}|v_n|_{2\sigma} + |v_n^2 - v^2|v|_{2\sigma}|w|_{2\sigma},\right)\]
where \(2\sigma \in \left(2, \frac{2N}{N-1}\right)\). Hence, proceeding as done to obtain (2.31) and (2.32), the last quantity goes to 0 as \(n \to \infty\) and thus we can conclude that \(v\) is a critical point of \(J\) on \(H_1^1(\mathbb{R}^{N+1})\).

Finally, we now know that \(J'(v_n)(v_n - v) = [J'(v_n) - J'(v)](v_n - v) \to 0\) as \(n \to \infty\), i.e.
\[
||v_n - v||^2 - \omega \int_{\mathbb{R}^N} (v_n - v)^2 \, dx - \lambda \int_{\mathbb{R}^N} (v_n(W * v_n^2) - v(W * v^2)) (v_n - v) \, dx \\
+ \int_{\mathbb{R}^N} [F_s(x,v_n) - F_s(x,v)](v_n - v) \, dx \to 0 \quad \text{as} \ n \to \infty.
\]
By (2.22) applied again with \(\varepsilon = m\), from (2.27), (2.34), (2.35) and (2.26), we have
\[
o(1) = J'(v_n)(v_n - v) \\
\geq \left(1 - \frac{\omega}{m}\right) \int_{\mathbb{R}^{N+1}} |D(v_n - v)|^2 \, dx \int_{\mathbb{R}^{N+1}} + m(m - \omega) \int_{\mathbb{R}^{N+1}} (v_n - v)^2 \, dx \, dx_{N+1}
\]
where \(o(1) \to 0\) as \(n \to \infty\). Since \(\omega < m\), we finally get that \(v_n \to v\) in \(H_1^1(\mathbb{R}^{N+1})\), as claimed. \(\square\)

At this point, the existence of a nontrivial critical point \(v\) for \(J\) is proved by applying the Mountain Pass Theorem, and so the first part of Theorem 1.1 holds true.

Now, assume also that \(F(x,s) \geq F(x,|s|)\). Recall that for the critical point found above, the critical level \(\beta = J(v)\) is defined as
\[
\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),
\]
where \(\Gamma = \{\gamma \in C([0,1], H^1_1(\mathbb{R}^{N+1})) : \gamma(0) = 0, J(\gamma(1)) < 0\}\). Now, the map \(t \mapsto J(tv)\) is increasing and has a strict maximum point at \(t = 1\); taken \(\tau > 0\) such that \(J(\tau v) < 0\), define \(\tilde{\gamma}(t) = \tau t|v|\), so that \(\tilde{\gamma} \in \Gamma\), by the additional assumption on \(F\); moreover, the map \(t \mapsto J(t|v|)\) has a unique maximum point at \(t = 1\). Thus, it is readily seen that \(J(t\tau v) \geq J(\tilde{\gamma}(t))\), so that
\[
\beta = \max_{t \in [0,1]} J(t v) \geq \max_{t \in [0,1]} J(\tilde{\gamma}(t)) \geq \beta.
\]
Hence, also \(t \mapsto t|v|\) is a path giving the same critical level. If, by contradiction, \(|v|\) were not a critical point for \(J\), we could define a path \(\tilde{\gamma} \in \Gamma\), obtained deforming \(\tilde{\gamma}\) via the gradient flow, in such a way that \(\max_{t \in [0,1]} J(\tilde{\gamma}(t)) < \beta\), contradicting the definition of \(\beta\) itself. In conclusion, also \(|v|\) is a critical point for \(J\).
Finally, if there exists \((\bar{x}, \bar{x}_{N+1}) \in \mathbb{R}_+^{N+1}\) such that \(v((\bar{x}, \bar{x}_{N+1}) = 0\), by the maximum principle for the equation in \(\mathbb{R}_+^{N+1}\), we get \(x_{N+1} = 0\) and by the boundary condition we have \(\frac{\partial u}{\partial n}(\bar{x}, 0) = 0\), in contradiction with Hopf’s Lemma. Hence \(v > 0\) and Theorem 1.1 is completely proved.

Remark 2.1. In [14] the proof of the Palais–Smale condition for the related problem is given without using the compact embedding of \([30]\), but exploiting the assumption that \(W(x) \to 0\) as \(|x| \to \infty\), which here we do not require.

We now give the Proof of Proposition 1.2. Since \(\omega < m\), the quantity

\[
\frac{1}{2} \int_{\mathbb{R}_+^{N+1}} (|Dv|^2 + m^2 v^2) \, dx dy - \int_{\mathbb{R}^N} \frac{\omega}{2} v^2 dx
\]

defines a norm \(\|v\|^2_2\) in \(H^1_r(\mathbb{R}_+^{N+1})\) which is equivalent to the usual one. Indeed, applying (2.22) with \(\varepsilon = m\), we get

\[
\|v\|_2^2 \geq \left(1 - \frac{\omega}{m}\right) \int_{\mathbb{R}_+^{N+1}} |Dv|^2 dx dy + (m^2 - \omega m) \int_{\mathbb{R}_+^{N+1}} v^2 dx dy \geq C\|v\|^2
\]

for any \(v \in H^1_r(\mathbb{R}_+^{N+1})\).

Now, let us set

\[
M = \left\{ v \in H^1_r(\mathbb{R}_+^{N+1}) : \int_{\mathbb{R}^N} v^2 (W \ast v^2) \, dx = 1 \right\} = \mathcal{G}^{-1}(0),
\]

where \(\mathcal{G}(v) = \int_{\mathbb{R}^N} v^2 (W(x) \ast v^2) \, dx - 1\).

It is easy to see that \(M\) is a non empty differentiable manifold of codimension 1. Indeed, for any \(v \in H^1_r(\mathbb{R}_+^{N+1})\) such that \(v(x, 0) \neq 0\), consider the map \(\psi : [0, \infty) \to \mathbb{R}\) defined as

\[
\psi(t) = t \int_{\mathbb{R}^N} v^2 (W \ast v^2) \, dx. \tag{2.37}
\]

It is clear that \(\psi\) is a strictly increasing function such that \(\psi(0) = 0\) and \(\psi(t) \to \infty\) as \(t \to \infty\), and thus \(M\) is non empty. Moreover, if \(v \in M\) and we assume that \(\mathcal{G}'(v)(w) = 0\), i.e. (using the symmetry of \(W\))

\[
\int_{\mathbb{R}^N} (W \ast v^2) vw \, dx = 0 \quad \forall w \in H^1_r(\mathbb{R}_+^{N+1}),
\]

we get in particular

\[
\int_{\mathbb{R}^N} v^2 (W \ast v^2) \, dx = 0,
\]

a contradiction with the fact that \(v \in M\).
Now, since \( F \geq 0 \), we immediately see that the \( C^1 \) functional \( I : H^1_+ (\mathbb{R}^{N+1}) \to \mathbb{R} \) defined as
\[
I(v) = \frac{1}{2} ||v||^2 + \int_{\mathbb{R}^N} F(x,v) \, dx
\]
is bounded below. Let \((v_n)_n \subset M\) be a minimizing sequence for \( I \) on \( M \). It is readily seen that \((v_n)_n\) is bounded, so that we may assume that \( v_n \rightharpoonup v \) in \( H^1_+ (\mathbb{R}^{N+1}) \), \( v_n \to v \) in \( L^q (\mathbb{R}^N) \) for every \( q \in \left( 2, \frac{2N}{N-2} \right) \) and a.e. in \( \mathbb{R}^N \). Moreover, by Ekeland’s Variational Principle (see for example [32, Theorem 8.5]) we can also assume that \( I'_M (v_n) \to 0 \), i.e. there exists a sequence \((\mu_n)_n\) in \( \mathbb{R} \) such that
\[
I'(v_n) (w) - \mu_n \int_{\mathbb{R}^N} (W * v_n^2) v_n w \, dx \to 0 \quad (2.38)
\]
as \( n \to \infty \) for every \( w \in M \), and hence for any \( w \in H^1_+ (\mathbb{R}^{N+1}) \). Since \( v_n \in M \) for any \( n \geq 1 \) and \((v_n)_n\) is bounded, from \( I'(v_n) (v_n) - \mu_n \to 0 \), i.e.
\[
\|v_n\|^2 - \int_{\mathbb{R}^N} F_s (x,v_n,v_n) \, dx - \mu_n \to 0,
\]
we get that also \( \int_{\mathbb{R}^N} F_s (x,v_n,v_n) \, dx - \mu_n \) is bounded. By \( F_2 \) we have
\[
\left| \int_{\mathbb{R}^N} F_s (x,v_n,v_n) \, dx \right| \leq C_1 \int_{\mathbb{R}^N} |v_n|^p \, dx + C_2 \int_{\mathbb{R}^N} |v_n|^q \, dx \leq C
\]
for some universal constant \( C > 0 \), since \((v_n)_n\) is bounded. In conclusion, also \((\mu_n)_n\) is bounded. Hence, we can suppose that there exists \( \lambda \in \mathbb{R} \) such that \( \mu_n \to \lambda \) as \( n \to \infty \). Proceeding as in the proof of Lemma [2.1], we can now show that, up to subsequences, \( v_n \rightharpoonup v \) in \( H^1_+ (\mathbb{R}^{N+1}) \). Moreover, \( M \) being closed in \( H^1_+ (\mathbb{R}^{N+1}) \), we get that \( v \in M \) is a nontrivial minimum point of \( I \) in \( M \). Passing to the limit in (2.38), we get
\[
I'(v)(w) - \lambda \int_{\mathbb{R}^N} (W * v^2) v w \, dx = 0
\]
for all \( w \in M \) - hence for all \( w \in H^1_+ (\mathbb{R}^{N+1}) \) - i.e. \( v \) solves (1.8) with \( \lambda \) given as a Lagrange multiplier.

If in addition \( 2F(x,s) \leq F_s (x,s)s \) for all \( s \in \mathbb{R} \) and a.e. \( x \in \mathbb{R}^N \), then
\[
\lambda \int_{\mathbb{R}^N} (W * v^2) v^2 \, dx = I'(v)(v) = \|v\|^2 + \int_{\mathbb{R}^N} F_s (x,v) v \, dx
\]
\[
= 2I(v) + \int_{\mathbb{R}^N} [F_s (x,v) v - 2F(x,v)] \, dx \geq 2I(v) > 0,
\]
and so \( \lambda > 0 \).

**Proof of Proposition 1.3** It is an application of the following multiplicity theorem based on the Krasnoselskii genus, see [31, Theorem 5.7]:
Theorem 2.1. Suppose that $I$ is an even $C^1$ functional on a complete symmetric $C^{1,1}$ manifold $M$ contained in a Banach space $B$, and suppose that $I$ satisfies $(PS)$ and is bounded from below. Let

$$\tilde{\gamma}(M) = \sup \{ \gamma(K) : K \subset M \text{ is compact and symmetric} \} \leq \infty.$$ 

Then $I$ admits at least $\tilde{\gamma}(M)$ pairs of critical points on $M$.

Here $\gamma(A)$ denotes the Krasnoselskii genus of a symmetric set $A$, defined as

$$\gamma(\emptyset) = 0, \text{ and when } A \neq \emptyset, \gamma(A) = \inf \{ m \in \mathbb{N} : \exists h \in C^0(A, \mathbb{R}^m \setminus \{0\}) \text{ odd}, \infty \text{ if } \{ m \in \mathbb{N} : \exists h \in C^0(A, \mathbb{R}^m \setminus \{0\}) \text{ odd} \} = \emptyset.$$ 

In particular $\gamma(A) = \infty$ for any symmetric set containing 0; see [27] or [31] for an introduction to the genus and some related results and applications.

In our case, we proceed as in the previous proof, restricting $I$ on the symmetric manifold $M$. Imitating the steps above, one can see that $I$ satisfies the $(PS)$ condition on $M$, while, applying Theorem 2.1, the existence part of the Proposition follows from the following Lemma, whose proof is given in the Appendix:

Lemma 2.2. $\tilde{\gamma}(M) = \infty$.

The final statement in Proposition 1.3 is exactly as in the previous proof. □

We conclude this section with the

Proof of Proposition 1.4. If $\lambda \leq 0$, it is enough to repeat the final part of the proof of Proposition 1.2 replacing the functional $I$ by the functional

$$\mathcal{I}(v) = \frac{1}{2} \|v\|_{2}^{2} + \int_{\mathbb{R}^N} F(x, v^+) \, dx,$$

where $v^+ = \max\{v, 0\}$, so that the minimum point solves

$$\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+,
-rac{\partial v}{\partial x_{N+1}} = \omega v + \lambda (W * v^2) v - F(x, v^+) & \text{on } \mathbb{R}^N \times \{0\} = \partial \mathbb{R}^{N+1}_+.
\end{cases}$$

Using $v^-$ as test function, we find

$$-\|v^+\|_{2}^{2} = -\lambda \int_{\mathbb{R}^N} (W * v^2) \,(v^-)^2 \, dx - \int_{\mathbb{R}^N} F_s(x, v^+) v^- \, dx$$

$$= -\lambda \int_{\mathbb{R}^N} (W * v^2) \,(v^-)^2 \, dx \geq 0,$$

so that $v = v^+ \geq 0$.

If $F(x, s) \geq F(x, |s|)$ one can immediately see that $I(v) \geq I(|v|)$, so that $|v|$ is again a minimum point for $I$.

The strong maximum principle implies $v > 0$ in both cases. □
3 Variational identities and proof of the non existence results

We start this section with the first non existence result, whose proof is very easy and can be obtained without additional new tools. In particular, here we don’t need the new variational identities for the half Laplacian (see Lemma 3.4 and equation (1.3)), which will be developed below in order to prove the more general non existence results.

Proof of Proposition 1.1 Taking $v$ as test function in (1.9), if $\omega \leq 0$ we have

$$0 = \int_{\mathbb{R}^{N+1}_+} (|Dv|^2 + m^2 v^2) \, dx \, dx' - \omega \int_{\mathbb{R}^N} v^2 \, dx - \lambda \int_{\mathbb{R}^N} (W \ast v^2) v^2 \, dx + \int_{\mathbb{R}^N} F_s(x,v) \, dx \geq \int_{\mathbb{R}^{N+1}_+} (|Dv|^2 + m^2 v^2) \, dx \, dx' ,$$

and the thesis follows. If $\omega \in (0, m)$, applying (2.22) with $\varepsilon = m$, we find

$$0 = \int_{\mathbb{R}^{N+1}_+} (|Dv|^2 + m^2 v^2) \, dx \, dx' - \omega \int_{\mathbb{R}^N} v^2 \, dx - \lambda \int_{\mathbb{R}^N} (W \ast v^2) v^2 \, dx + \int_{\mathbb{R}^N} F_s(x,v) \, dx \geq \left(1 - \frac{\omega}{m}\right) \int_{\mathbb{R}^{N+1}_+} |Dv|^2 \, dx \, dx' + (m^2 - \omega m) \int_{\mathbb{R}^{N+1}_+} v^2 \, dx \, dx' \geq C \|v\|^2$$

for some $C > 0$, and again the claim is proved.

Now, we show by some variational identities that the existence results of the previous section (in particular Theorem 1.1) are, in some sense, optimal, provided that $F = F(s)$.

Let $v \in H^1(\mathbb{R}^{N+1}_+)$ be a solution of (1.8) with $F = F(s)$. By reasoning as in [13, Theorem 3.2 and Proposition 3.9], we can show that

- $v \in L^\infty(\mathbb{R}^{N+1}_+)$,
- $v \in L^p(\mathbb{R}^N)$, $\forall p \in [2, \infty]$,
- $v \in C^{0,\alpha}(\mathbb{R}^{N+1}_+) \cap W^{1,q}(\mathbb{R}^N \times (0,R))$ for any $q \in [2, \infty)$ and all $R > 0$,
- if $F$ is of class $C^{0,\alpha}(\mathbb{R})$, then $v \in C^{1,\alpha}(\mathbb{R}^{N+1}_+) \cap C^2(\mathbb{R}^{N+1}_+)$ and is a classical solution of (1.8).

Let us set $X = (x,x_{N+1})$ with $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. Moreover, define

$$\Delta_R = \left\{ X \in \mathbb{R}^N \times [0,\infty) : |X|^2 \leq R \right\},$$

$$S^+_R = \left\{ X \in \mathbb{R}^{N+1}_+ : |X| = R \text{ and } x_{N+1} > 0 \right\}$$
Pseudorelativistic Hartree equation: existence and non existence

and

\[ b_r = \left\{ X \in \Delta_R : x_{N+1} = 0 \right\}. \]

Finally, for shortness, we write \( v_i \) in place of \( v_{x_i} \).

We state the following result for regular functions, like solutions of (1.8) if \( F \) is of class \( C^{0,\alpha} \), with obvious generalization for Sobolev functions in \( H^2_{loc}(\mathbb{R}^{N+1}_+) \cap H^1(\mathbb{R}^{N+1}_+) \), since functions in this space admit traces on every manifold appearing in the calculations below.

**Lemma 3.1.** For any \( v \in C^{1,\alpha}(\mathbb{R}^{N+1}_+) \cap C^2(\mathbb{R}^{N+1}) \) and for every \( R > 0 \) there holds

\[
\int_{\Delta_R} -\Delta v \cdot Dv \, dX = \frac{1 - N}{2} \int_{\Delta_R} |Dv|^2 \, dX + \int_{b_R} v_{N+1} Dv \cdot x \, dx \\
+ \int_{S^+_R} \left[ \frac{R}{2} |Dv|^2 - \frac{1}{R} |Dv \cdot X|^2 \right] \, d\sigma,
\]

\[
(3.39)
\]

\[
\int_{\Delta_R} g(v) X \cdot Dv \, dX = -(N + 1) \int_{\Delta_R} G(v) \, dX + R \int_{S^+_R} G(v) \, dx.
\]

Here \( g : \mathbb{R} \to \mathbb{R} \) is any continuous function and \( G(s) = \int_0^s g(t) \, dt \).

**Proof.** Fix \( i \in \{1, \ldots, N + 1\} \). Denoting by \( \nu \) the outward unit vector to \( \partial \Delta_R \), we have

\[ \nu(X) = \begin{cases} \frac{X}{|X|} & \text{if } x_{N+1} > 0, \\ (0, \ldots, 0, -1) & \text{if } x_{N+1} = 0, \end{cases} \]

so that

\[ X \cdot \nu(X) = \begin{cases} |X| & \text{if } x_{N+1} > 0, \\ 0 & \text{if } x_{N+1} = 0. \end{cases} \]

For any \( i, j \in \{1, \ldots, N + 1\} \), by Green’s formula we have

\[
\int_{\Delta_R} v_i v_j x_j \, dX = \frac{1}{2} \int_{\partial \Delta_R} (v_i)^2 x_j \nu_j \, d\sigma - \frac{1}{2} \int_{\Delta_R} (v_i)^2 \, dX.
\]

Hence, denoting by \( \delta_{ij} \) the usual Kronecker symbol, we have

\[
- \int_{\Delta_R} v_i v_j x_j \, dX = - \int_{\partial \Delta_R} v_i v_j x_j \nu_i \, d\sigma + \int_{\Delta_R} [\delta_{ij} v_i v_j + v_i v_j x_j] \, dX \\
= \int_{\partial \Delta_R} \left[ \frac{1}{2} (v_i)^2 x_j \nu_j - v_i v_j x_j \nu_i \right] \, d\sigma \\
+ \int_{\Delta_R} \left[ \delta_{ij} v_i v_j - \frac{1}{2} (v_i)^2 \right] \, dX.
\]

Summing up over \( i, j \in \{1, \ldots, N + 1\} \), by (3.41), we get (3.39).
In order to prove (3.40), observe that for any \( i \in \{1, \ldots, N+1\} \) we have
\[
\int_{\Delta_R} g(v) v_i x_i dX = \int_{\partial \Delta_R} G(v) x_i v_i d\sigma - \int_{\Delta_R} G(v) dX.
\]
Summing up, using (3.41), we obtain (3.40).

We now focus on the 3-dimensional case, i.e. on problem (1.6).

Lemma 3.2. If \( v \in L^3(\mathbb{R}^3) \) and \( \phi(x) = \frac{1}{|x|^2} * v^2 \), then
\[
\int_{b_R} v \phi Dv \cdot x dx = -\frac{3}{2} \int_{b_R} v^2 \phi dx + \frac{1}{16\pi} \int_{b_R} |D\phi|^2 dx
+ \int_{\partial b_R} \left[ \frac{R}{2} v^2 \phi - \frac{1}{8\pi} \left( \frac{R}{2} |D\phi|^2 - \frac{1}{R} |x \cdot D\phi|^2 \right) \right] d\tau.
\]  
(3.42)

Proof. Let us note that \( \phi(x) \in H^2_{loc}(\mathbb{R}^3) \cap D^1(\mathbb{R}^3) \) is a solution of
\[
-\Delta \phi = 4\pi v^2 \text{ in } \mathbb{R}^3,
\]  
(3.43)

where \( D^1(\mathbb{R}^3) = C^\infty_0(\mathbb{R}^3) \), with \( ||\phi||^2 = \int_{\mathbb{R}^3} |D\phi|^2 \) (see also [16]).

Operating as we did to prove Lemma 3.1 replacing \( \Delta_R \) with \( b_R \), we can prove that
\[
\int_{b_R} v \phi x \cdot Dv dx = -\frac{1}{2} \int_{b_R} v^2 x \cdot D\phi dx - \frac{3}{2} \int_{b_R} v^2 \phi dx + \frac{R}{2} \int_{\partial b_R} v^2 \phi d\tau
\]  
(3.44)

and
\[
-\int_{b_R} \Delta \phi x \cdot D\phi dx = -\frac{1}{2} \int_{b_R} |D\phi|^2 dx + \int_{\partial b_R} \left[ \frac{R}{2} |D\phi|^2 - \frac{1}{R} |x \cdot D\phi|^2 \right] d\tau,
\]  
(3.45)

see also [15] Lemma 3.1.

On the other hand, from (3.43) we have
\[
4\pi \int_{b_R} v^2 x \cdot D\phi dx = -\int_{b_R} \Delta x \cdot D\phi dx.
\]  
(3.46)

Starting from (3.44), using (3.46) and (3.45), the claim follows.

Lemma 3.3. If \( v \in H^1(\mathbb{R}^3) \), there exists a sequence \( R_n \to \infty \) such that
\[
\int_{\partial b_{R_n}} \left[ \frac{R_n}{2} v^2 \phi - \frac{1}{8\pi} \left( \frac{R_n}{2} |D\phi|^2 - \frac{1}{R_n} |x \cdot D\phi|^2 \right) \right] d\tau \to 0
\]  
as \( n \to \infty \).
Proof. We follow the lines of [8]. First, let us note that on $\partial b_{R_n}$ we have
$$\frac{1}{R_n}|x \cdot D\phi|^2 \leq \frac{1}{R_n}|x|^2|D\phi|^2 = R_n|D\phi|^2 \in L^1(\mathbb{R}^3).$$
Moreover, $v^2 \in L^{6/5}(\mathbb{R}^3)$ by interpolation, and $\phi \in L^6(\mathbb{R}^3)$ by the Sobolev inequality: hence $v^2\phi \in L^1(\mathbb{R}^3)$. Thus it is enough to prove that, if $f \in L^1(\mathbb{R}^3)$, then there exists a sequence $R_n \to \infty$ such that
$$R_n \int_{\partial b_{R_n}} |f| d\tau \to 0.$$
Assume this is not the case, so that there exists $\varepsilon, R_0 > 0$ such that
$$R \int_{\partial b_R} |f| d\tau \geq \varepsilon \text{ for every } R \geq R_0.$$ 

Then
$$\infty > \int_{\mathbb{R}^3} |f| dx = \int_{0}^{\infty} dR \int_{\partial b_R} |f| d\tau \geq \int_{R_0}^{\infty} \frac{\varepsilon}{R} dR = \infty,$$
and a contradiction arises. \hfill \Box

Lemma 3.4. If $v \in H^1(\mathbb{R}^3)$, then
$$\int_{\mathbb{R}^3} v \left( \frac{1}{|x|} * v^2 \right) Dv \cdot x dx = -\frac{5}{4} \int_{\mathbb{R}^3} v^2 \left( \frac{1}{|x|} * v^2 \right) dx.$$

Proof. From Lemma 3.2, applying Lemma 3.3, we find
$$\int_{\mathbb{R}^3} v\phi Dv \cdot x dx = -\frac{3}{2} \int_{\mathbb{R}^3} v^3 \phi dx + \frac{1}{16\pi} \int_{\mathbb{R}^3} |D\phi|^2 dx.$$ 

On the other hand, $\phi$ being a solution in $D^1(\mathbb{R}^3)$ of (3.43), we get
$$\int_{\mathbb{R}^3} |D\phi|^2 dx = 4\pi \int_{\mathbb{R}^3} v^2 \phi dx.$$ 

Substituting, we get the claim. \hfill \Box

We are now ready to prove the variational identity (1.11) for the generalized half Laplacian, which we believe to be quite useful in studying system (1.6):

Proof of Theorem 1.2. Multiply the first equation in (1.6) by $X \cdot Dv$ and integrate on $\Delta_R$. Applying Lemma 3.1 with $g(s) = s$ we get
$$0 = -\int_{\Delta_R} |Dv|^2 dX + \int_{S_R^+} \left[ \frac{R}{2} |Dv|^2 - \frac{1}{R} |Dv \cdot X|^2 \right] d\sigma + \int_{b_R} v_4 Dv \cdot x dx - 2m^2 \int_{\Delta_R} v^2 dX + \frac{m^2 R}{2} \int_{S_R^+} v^2 d\sigma. \hspace{1cm} (3.48)$$ 

By the boundary condition in (1.8) we have
$$\int_{b_R} v_4 Dv \cdot x dx = -\int_{b_R} \left[ \omega v + \lambda \left( \frac{1}{|x|} * v^2 \right) v - F'(v) \right] Dv \cdot x dx.$$
Operating as in the proof of Lemma 3.2 (see also [15, Lemma 3.1]), setting \( \phi = \left( \frac{1}{|x|} * v^2 \right) \), we can prove that

\[
\int_{bR} [\omega v + \lambda \phi v - F'(v)] \, Dv \cdot x \, dx
\]

\[
= \frac{\lambda}{16\pi} \int_{bR} |D\phi|^2 \, dx - \frac{3}{2} \int_{bR} (\omega + \lambda \phi) v^2 \, dx + 3 \int_{bR} F(v) \, dx \tag{3.49}
\]

\[
+ \int_{\partial bR} \left[ \frac{R}{2} (\omega + \lambda \phi) v^2 \phi - \frac{\lambda}{8\pi} \left( \frac{R}{2} |D\phi|^2 - \frac{1}{R} |x \cdot D\phi|^2 \right) - RF(v) \right] \, d\tau.
\]

Substituting (3.49) into (3.48), we obtain

\[
0 = - \int_{\Delta R} |DV|^2 dX + \int_{S_R^+} \left[ \frac{R}{2} |DV|^2 - \frac{1}{R} |DV \cdot X|^2 \right] \, d\sigma
\]

\[
- \frac{\lambda}{16\pi} \int_{R^3} |D\phi|^2 \, dx + \frac{3}{2} \int_{R^3} (\omega + \lambda \phi) v^2 \, dx - 3 \int_{bR} F(v) \, dx
\]

\[
- \int_{\partial bR} \left[ \frac{R}{2} (\omega + \lambda \phi) v^2 \phi - \frac{\lambda}{8\pi} \left( \frac{R}{2} |D\phi|^2 - \frac{1}{R} |x \cdot D\phi|^2 \right) - RF(v) \right] \, d\tau
\]

\[
- 2m^2 \int_{\Delta R} v^2 dX + \frac{m^2 R}{2} \int_{S_R^+} v^2 \, dx.
\]

As in the proof of Lemma 3.3 we can find a sequence \( R_n \to \infty \) such that the integrals over \( \partial bR_n \) and over \( S_R^+ \) go to 0 as \( n \to \infty \). In this way (3.50) gives

\[
0 = - \int_{R^3} |DV|^2 dX - 2m^2 \int_{R^3} v^2 dX
\]

\[
- \frac{\lambda}{16\pi} \int_{R^3} |D\phi|^2 \, dx + \frac{3}{2} \int_{R^3} (\omega + \lambda \phi) v^2 \, dx - 3 \int_{R^3} F(v) \, dx.
\]

By substituting in the equation above the term \( \int |D\phi|^2 \) taken from (3.47), we finally get (1.11). \( \square \)

We are now ready for the

**Proof of Theorem 1.3** Since \( v \) is a solution of (1.8), we have

\[
\int_{R^3} \left( |DV|^2 + m^2 v^2 \right) \, dX = \int_{R^3} [\omega v + \lambda \phi v - F'(v)] v \, dx. \tag{3.51}
\]

Now we isolate \( \int_{R^3} |DV|^2 \, dX \) in (3.51) and substituting in (1.11), we get

\[
0 = -m^2 \int_{R^3} v^2 \, dX + \frac{\omega}{2} \int_{R^3} v^2 \, dx + \frac{1}{4} \lambda \int_{R^3} v^2 \phi \, dx
\]

\[
- \int_{R^3} \left[ 3F(v) - F'(v) v \right] \, dx.
\]
Thus, if (1.12) holds, we get $v \equiv 0$.

On the other hand, if we isolate $\int_{\mathbb{R}^{+}} v^2 dX$ in (3.51) and we substitute in (1.11), we get

$$0 = \int_{\mathbb{R}^{+}} |Dv|^2 dX - \frac{\omega}{2} \int_{\mathbb{R}^3} v^2 dx - \frac{3}{4} \lambda \int_{\mathbb{R}^3} v^2 \phi dx$$

$$+ \int_{\mathbb{R}^3} [2F'(v)v - 3F(v)] dx.$$  

Thus, if (1.13) holds, again we get $v \equiv 0$.

Now, if $\omega > 0$ and $\lambda \leq 0$, starting from (1.11), using (2.22), we have

$$0 \leq \left(-1 + \frac{3\omega}{2}\mu\right) \int_{\mathbb{R}^{+}} |Dv|^2 dX + \left(\frac{3\omega \mu}{2} - 2m^2\right) \int_{\mathbb{R}^3} v^2 dX$$

$$+ \frac{5}{4} \lambda \int_{\mathbb{R}^3} v^2 \phi dx - 3 \int_{\mathbb{R}^3} F(v) dx.$$  

Choosing $\mu \in \left[\frac{3\omega}{2}, \frac{4m^2}{3\omega}\right]$, by (1.14) we obtain again $v \equiv 0$. If $\omega = 0$ the conclusion is easier.

The last statement needs a longer proof and the following estimate, which also establishes a non vanishing property for nontrivial solutions of (1.6), and for whose proof we direct the reader to the Appendix:

**Lemma 3.5.** If $v \in H^1(\mathbb{R}^{N+1})$ solves (1.8) with $\lambda > 0$, $\omega \in (0, m)$ and $F_s(x, s) s \geq 0$ for every $s \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$, then

$$\|v\|^2 \leq \frac{m}{m - \omega} \lambda \int_{\mathbb{R}^N} (W * v^2)v^2 dx$$  

(3.52)

and there exists $C = C(m, \omega) > 0$ such that

$$\|v\|^2 \geq \frac{C}{\lambda}$$  

(3.53)

for any nontrivial solution of (1.6).

Let us start noting that for any solution $v$ and for any $\rho > 0$ we have

$$0 = (1.11) + \rho J'(v)v$$

$$= (\rho - 1) \int_{\mathbb{R}^3} |Dv|^2 dX + (\rho - 2)m^2 \int_{\mathbb{R}^3} v^2 dX + \left(\frac{3}{2} - \rho\right) \omega \int_{\mathbb{R}^3} v^2 dx$$

$$+ \left(\frac{5}{4} - \rho\right) \lambda \int_{\mathbb{R}^3} v^2 \phi dx + \int_{\mathbb{R}^3} [\rho F'(v)v - 3F(v)] dx.$$

(3.54)
Now, take any $\rho < 3/2$ and start from (3.54); applying (2.22) with $\varepsilon > 0$, we obtain
\begin{align*}
0 & \leq \left[ \rho - 1 + \varepsilon \omega \left( \frac{3}{2} - \rho \right) \right] \int_{\mathbb{R}^{N+1}_+} |Dv|^2 dx dN+1 \\
& + m^2 \left[ \rho - 2 + \frac{\omega}{\varepsilon m^2} \left( \frac{3}{2} - \rho \right) \right] \int_{\mathbb{R}^{N+1}_+} v^2 dx dN+1 \\
& + \left( \frac{5}{4} - \rho \right) \lambda \int_{\mathbb{R}^N} v^2 \phi dx + \int_{\mathbb{R}^3} [\rho F'(v)v - 3F(v)] dx.
\end{align*}
(3.55)

By choosing $\varepsilon$ small enough, we can suppose that
$$\rho - 1 + \varepsilon \omega \left( \frac{3}{2} - \rho \right) \leq \rho - 2 + \frac{\omega}{\varepsilon m^2} \left( \frac{3}{2} - \rho \right),$$
so that (3.55) becomes
\begin{align*}
0 & \leq \left[ \rho - 2 + \frac{\omega}{\varepsilon m^2} \left( \frac{3}{2} - \rho \right) \right] \|v\|^2 + \left( \frac{5}{4} - \rho \right) \lambda \int_{\mathbb{R}^N} v^2 \phi dx + \int_{\mathbb{R}^3} [\rho F'(v)v - 3F(v)] dx.
\end{align*}

By applying (3.52) with $W(x) = 1/|x|$ and $N = 3$, we find
\begin{align*}
0 & \leq \left\{ \left[ \rho - 2 + \frac{\omega}{\varepsilon m^2} \left( \frac{3}{2} - \rho \right) \right] \frac{m}{m - \omega} + \left( \frac{5}{4} - \rho \right) \right\} \lambda \int_{\mathbb{R}^N} v^2 \phi dx \\
& + \int_{\mathbb{R}^3} [\rho F'(v)v - 3F(v)] dx.
\end{align*}
(3.56)

After some calculations, we find that the coefficient of $\int v^2 \phi$ in the inequality above is non positive iff
$$\frac{\omega}{m} \left( \frac{1}{\varepsilon m} - 1 \right) \geq \frac{3\omega}{2\varepsilon m^2} - \frac{5\omega}{4m} - \frac{3}{4}. $$

If $\varepsilon < 1/m$ is small enough, it is clear that both sides of the previous inequality are positive, so that we are allowed to choose
$$\rho \geq \left( 6 - 5\varepsilon m - \frac{3\varepsilon m^2}{\omega} \right) \frac{1}{4(1 - \varepsilon m)}. $$

Let us remark that the function $g(\varepsilon) = (6 - 5\varepsilon m - 3\varepsilon m^2/\omega)[4(1 - \varepsilon m)]^{-1}$ is strictly decreasing in $(0, 1/m)$, and that $g(0^+) = 3/2$, so that a possible choice on $\rho < 3/2$ is given for $\varepsilon < 1/m$.

Therefore, passing to the limit as $\rho \uparrow 3/2$ in (3.56), we get
\begin{align*}
0 & \leq \frac{\omega - 3m}{4m - 4\omega} \lambda \int_{\mathbb{R}^N} v^2 \phi dx + \frac{3}{2} \int_{\mathbb{R}^3} [F'(v)v - 2F(v)] dx \leq \frac{\omega - 3m}{4m - 4\omega} \lambda \int_{\mathbb{R}^N} v^2 \phi dx
\end{align*}
by assumption on $F$ appearing in (1.15). Being the remaining coefficient a strictly negative number, we get $\int v^2 \phi = 0$, and from (3.52) also $v \equiv 0$.

Theorem 1.3 is now completely proved.
We conclude with the

**Proof of Theorem 1.4.** The first part is very similar to the proof of Theorem 1.3, which was obtained adding \((1.11)+\rho J'(v)v\) for \(\rho = 1\) and \(\rho = 2\). Now, starting from (3.54), if (1.16) holds, all the coefficients are less or equal to 0, and we get \(v \equiv 0\); if (1.17) holds, all the coefficients in (3.54) are nonnegative, and we obtain again \(v \equiv 0\).

Now assume the (1.18) holds. Since \(\omega \in \left(0, 2m\frac{(\rho - 1)(\rho - 2)}{2\rho^3 - 3}\right]\) and \(\lambda \leq 0\), starting from (3.54) and using (2.22) and the fact that \(\rho > 2\), we find

\[
0 \geq \left[\rho - 1 + \left(\frac{3}{2} - \rho\right)\frac{2\omega}{\varepsilon}\right] \int_{\mathbb{R}^4_+} |Dv|^2 dX + \left[(\rho - 2)m^2 + \left(\frac{3}{2} - \rho\right)\varepsilon\omega\right] \int_{\mathbb{R}^4_+} v^2 dX.
\]

Both the coefficients of the integrals above are nonnegative provided that

\[
\frac{2\rho - 3\omega}{\rho - 1} \leq \frac{2(\rho - 2)m^2}{(2\rho - 3)\omega},
\]

which is possible by the bound on \(\omega\).

**A Appendix**

**Proof of Lemma 2.2.** Fix \(k \in \mathbb{N}\) and take any subspace \(H_k\) of \(H^1_+(\mathbb{R}^N)\) having dimension \(k\). We first prove that \(M_k := \{v \in H_k \mid \|v\| = 1\}\) is bounded. Indeed, assume by contradiction that there exists an unbounded sequence \((v_n)\) in \(M_k\). Without loss of generality, we can assume that \(\frac{v_n}{\|v_n\|} \to v \in H_k\), where \(\|v\| = 1\).

Setting \(U_n = v_n W * v_n^2\), then

\[
\int_{\mathbb{R}^N} U_n v_n dx = \int_{\mathbb{R}^N} v_n^2 W * v_n^2 dx = 1.
\]

Dividing both sides by \(\|v_n\|^4\), we obtain

\[
0 = \lim_{n \to \infty} \int_{\mathbb{R}^N} U_n v_n dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{v_n^2}{\|v_n\|^2} W * \frac{v_n^2}{\|v_n\|^2} dx = \int_{\mathbb{R}^N} v^2 W * v^2 dx > 0
\]

since \(v\) is nontrivial, and we get a contradiction.

\(M_k\) being bounded and closed in \(H_k\), we get that \(M_k\) is a compact subset of \(H_k\).

We now show that \(\gamma(M_k) \geq k\), and being \(k\) arbitrary, we deduce that \(\tilde{\gamma}(M) = \infty\), and the lemma is proved. For this, let us consider the map \(\pi : S^{k-1} \to M\), where \(S^{k-1}\) denotes the unit sphere of \(H_k\), defined as \(\pi(u) = u_M\), \(u_M\) being the unique point of the half line \(\mathbb{R}^+ u\) intersecting \(M\); note that the definition of \(\pi\) is well posed by (2.37). Clearly \(\pi\) is an odd continuous map with \(\pi(S^{k-1}) = M_k\); thus, by [31, Proposition 5.4 (4°)], we get \(\gamma(M_k) \geq \gamma(S^{k-1})\), while \(\gamma(S^{k-1}) = k\) by [31, Proposition 5.2].

\(\square\)
Proof of Lemma 3.5 In (1.9) take \( w = v \), obtaining

\[
\|v\|^2 = \omega \int_{\mathbb{R}^N} v^2 dx + \lambda \int_{\mathbb{R}^N} (W * v^2)v^2 dx - \int_{\mathbb{R}^N} F_s(x, v) v dx
\]

\[
\leq \omega \int_{\mathbb{R}^N} v^2 dx + \lambda \int_{\mathbb{R}^N} (W * v^2)v^2 dx
\]

by the assumption on \( F_s \). Using (2.22) with \( \varepsilon = m \) we obtain

\[
\|v\|^2 \leq \frac{\omega}{m} \int_{\mathbb{R}^{N+1}} |Dv|^2 dx_{N+1} + m\omega \int_{\mathbb{R}^{N+1}} v^2 dx_{N+1} + \lambda \int_{\mathbb{R}^N} (W * v^2)v^2 dx,
\]

from which (3.52) follows.

From (3.52), by applying (2.23), we immediately get (3.53) for any solution \( v \neq 0 \).

References

[1] Adams, R.A.: Sobolev spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London, 1975.

[2] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Analysis 14, 349–381 (1973).

[3] Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger–Poisson problem. Commun. Contemp. Math 10 391–404 (2008).

[4] Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. Top. Meth. Nonlinear Anal. 11, 283–293 (1998).

[5] Benci, V., Fortunato, D.: Solitary waves in abelian gauge theories. Adv. Nonlinear Stud. 8, 327–352 (2008).

[6] Benci, V., Fortunato, D.: Spinning Q–balls for the Klein-Gordon-Maxwell equations. Comm. Math. Phys. 295 639–668 (2010).

[7] Benci, V., Fortunato, D.: Towards a Unified Field Theory for Classical Electrodynamics. Arch. Rational Mech. Anal. 173, 379–414 (2004).

[8] Berestycki, H., Lions, P.L.: Nonlinear scalar field equations, I - Existence of a ground state. Arch. Rat. Mech. Anal. 82, 313–345 (1983).

[9] Cabré, X., Tan, J.: Positive solutions of nonlinear problems involving the square root of the Laplacian. Adv. Math. 224, 2052–2093 (2010).

[10] Cancès, É., Deleurence, A., Lewin, M.: A New Approach to the Modeling of Local Defects in Crystals: The Reduced Hartree-Fock Case. Commun. Math. Phys. 281, 129–177 (2008).
Pseudorelativistic Hartree equation: existence and non existence

[11] CANCEs, É., EHRLACHER, V.: Local Defects are Always Neutral in the Thomas-Fermi-von Weizsäcker Theory of Crystals. Arch. Rational Mech. Anal. 202, 933–973 (2011).

[12] CANCEs, É., LEWIN, M.: The Dielectric Permittivity of Crystals in the Reduced Hartree–Fock Approximation. Arch. Rational Mech. Anal. 197, 139–177 (2010).

[13] CATTO, I., LE BRIS, C., LIONS, P.-L.: On some periodic Hartree–type models for crystals. Ann. Inst. H. Poincaré Anal. Non Linéaire 19, 143–190 (2002).

[14] COTI ZELATI, V., NOLASCO, M.: Existence of ground states for nonlinear, pseudorelativistic Schrödinger equations. Rend. Lincei Mat. Appl. 22, 51–72 (2011).

[15] D’APRILE, T., MUGNAI, D.: Non–existence results for the coupled Klein–Gordon–Maxwell equations. Adv. Nonlinear Stud. 4, 307–322 (2004).

[16] D’APRILE, T., MUGNAI, D.: Solitary Waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations. Proc. R. Soc. Edinb. Sect. A 134, 1–14 (2004).

[17] ELGART, A., SCHLEIN, B.: Mean field dynamics of boson stars. Comm. Pure Appl. Math. 60, 500–545 (2007).

[18] FRÖHLICH, J., JONSSON, B.L.G., LENZMANN, E.: Boson stars as solitary waves. Comm. Math. Phys. 274, 1–30 (2007).

[19] FRÖHLICH, J., LENZMANN, E.: Blowup for Nonlinear Wave Equations Describing Boson Stars. Comm. Pure Appl. Math. 60, 1691–1705 (2007).

[20] LIEB, E.H., LOSS, M.: Analysis. Graduate Studies in Mathematics 14, American Mathematical Society, 1997.

[21] LIEB, E.H., YAU, H.-T.: The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Comm. Math. Phys. 112, 147–174 (1987).

[22] MA, L., ZHAO, L.: Classification of positive solitary solutions of the nonlinear Choquard equation. Arch. Ration. Mech. Anal. 195,455–467 (2010).

[23] MUGNAI, D.: Coupled Klein–Gordon and Born–Infeld type equations: looking for solitons. R. Soc. Lond. Proc. Ser. A 460, 1519–1528 (2004).

[24] MUGNAI, D.: Solitary waves in Abelian Gauge Theories with strongly nonlinear potentials. Ann. Inst. H. Poincaré Anal. Non Linéaire 27, 1055–1071 (2010).

[25] MUGNAI, D.: The Schrödinger–Poisson system with positive potential. Comm. Partial Differential Equations 36, 1099–1117 (2011).
[26] Palais, R.S.: The principle of symmetric criticality. Comm. Math. Phys. 69, 19–30 (1979).

[27] Rabinowitz, P.H.: *Mini-max Methods in Critical Point Theory with Applications to Differential Equations*. CBMS Reg. Conf. Ser. in Math. No. 65, AMS, Providence R.I., 1986.

[28] Ruiz, D.: The Schrödinger–Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.* 237, 655–674 (2006).

[29] Ruiz Arriola, E., Soler, J.: A variational approach to the Schrödinger–Poisson system: asymptotic behaviour, breathers and stability. *J. Stat. Phys.* 103, 1069–1106 (2001).

[30] Sickel W., Skrzypczak, L.: Radial subspaces of Besov and Lizorkin-Triebel spaces: extended Strauss lemma and compactness of embeddings. *J. Fourier Anal. Appl.* 6, 639–662 (2000).

[31] Struwe, M.: *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*. Springer-Verlag, Berlin, 4th edition, 2008.

[32] Willem, M.: *Minimax Theorems*. Progr. Nonlinear Differential Equations Appl. 24. Birkhuser Boston, Inc., Boston, MA, 1996.