A homogeneous brane-world universe

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Abstract. A homogeneous, Kantowski-Sachs type, bouncing brane-world universe is presented. The bulk has a positive cosmological constant and the Killing algebra so(1, 3) ⊕ so(3). The totality of the source terms of the effective Einstein equation combine to a solid with different radial and tangential pressures.

1. Introduction

The 5-dimensional Einstein equation imposed on a bulk containing a single brane located at $y = 0$ (generalized Randall-Sundrum type II model [1]), with the energy-momentum tensor $T_{AB}$ (obeying $T_{AB}n^B = 0$, where $n^A = \delta^A_y$ is the unit normal to the brane) is

$$\tilde{G}_{AB} = \tilde{\kappa}^2 \left[ -\tilde{\Lambda} g_{AB} + \delta(y) \{ -\lambda g_{AB} + T_{AB} \} \right].$$

Here $\tilde{\kappa}^2 = 8\pi/\tilde{M}_p^3$ is the 5-dimensional coupling constant ($\tilde{M}_p$ being the 5-dimensional Planck mass) and $\lambda$ is the brane tension. Due to the Gauss-Codazzi equations and the junction conditions across the $Z_2$-symmetrically embedded brane, the effective Einstein equation on the brane emerges [2]:

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \kappa^4 S_{\mu\nu} - \mathcal{E}_{\mu\nu},$$

where $\kappa^2 = 8\pi/M_p^2$ is the 4-dimensional coupling constant and $g_{AB} = \bar{g}_{AB} - n_A n_B$ is the induced metric. The 5-dimensional and 4-dimensional energy scales and cosmological constants are related to each other and the brane tension via

$$\tilde{\kappa}^4 \lambda = 6\kappa^2,$$

$$2\Lambda = \tilde{\kappa}^2 \Lambda + \kappa^2 \lambda.$$

Here $S_{\mu\nu}$ are local quadratic energy-momentum corrections, given by

$$S_{\mu\nu} = \frac{1}{12} T_{\alpha}^{\alpha} T_{\mu\nu} - \frac{1}{4} T_{\mu\alpha} T_{\alpha}^{\nu} + \frac{1}{24} g_{\mu\nu} \left[ 3T_{\alpha\beta} T^{\alpha\beta} - (T_{\alpha}^{\alpha})^2 \right].$$

and $\mathcal{E}_{\mu\nu}$ is the $y \to 0$ limit of the $\mathcal{E}_{AB} = \tilde{C}_{ACBD} n^C n^D$ projection of the bulk Weyl tensor.

Various solutions of this new scenario have already been considered, the most recent of them being the Gödel brane [3]. Due to the cosmological implications,
likely to be experimentally tested, the Friedmann branes were widely employed [4], [5], [6]. The results of [7] and [8] suggested that all bulk solutions with Friedmann branes should be 5-dimensional Schwarzschild-anti de Sitter (SAdS). However a class of brane-world solutions generalizing the Einstein static universe was found [9], which disobey this rule. The simplest of them, the generalized Einstein static brane with curvature index $\epsilon = 0$ is embedded in a flat bulk in [9]. The solutions given in [9] bear in common a negative energy density $\rho_E = -\lambda$ of the perfect fluid source, however the effective energy density (computed with respect to the unit tangent vector along the fluid flow lines) from all contributions to the source (including the 4-dimensional cosmological constant) is

$$\kappa^2 \rho_{\text{eff}} = \tilde{G}_{AB} u^A u^B = G_{\mu\nu} u^\mu u^\nu = \tilde{\Lambda} \kappa^2,$$

positive for any bulk cosmological constant $\tilde{\Lambda} > 0$.

In this paper we present (among other metrics with peculiar signatures) a related homogeneous brane-world solution, obtained by applying the complex transformation $t \to i\chi$, $\chi \to it$, (5) to the solutions found in [9]. From a 4-dimensional point of view the brane in this solution is a solid with spherical and homogeneous symmetries and constant tensions. The notation follows closely that of Ref. [9].

2. Homogeneous universe

As in [9] we introduce the notation

$$\tilde{\kappa}^2 \tilde{\Lambda} = 3 \epsilon \Gamma^2, \quad \Gamma > 0.$$ (6)

Here $\epsilon = 0, \pm 1$ carries the sign of the bulk cosmological constant. The one-parameter class of solutions found in [9] is:

$$\Gamma^2 ds_E^2 = -F^2(y; \epsilon) dt^2 + d\chi^2 + \mathcal{H}_E^2(\chi; \epsilon) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + dy^2, (7)$$

with the metric functions

$$F(y; \epsilon) = \begin{cases} A \cos \left( \sqrt{2} y \right) + B \sin \left( \sqrt{2} y \right) & \epsilon = 1, \\ A + \sqrt{2} By & \epsilon = 0, \\ A \cosh \left( \sqrt{2} y \right) + B \sinh \left( \sqrt{2} y \right) & \epsilon = -1. \end{cases}$$ (8)

and

$$\mathcal{H}_E(\chi; \epsilon) = \begin{cases} \cos \chi & \epsilon = 1, \\ \chi & \epsilon = 0, \\ \sinh \chi & \epsilon = -1. \end{cases}$$ (9)

(For $\epsilon = 1$ we have changed the function sin into cos. This being a simple translation in the coordinate $\chi$, it does not change the solution. This new form of the metric is required in order to obtain the new solutions presented below.)

By the complex transformation (5) the line element becomes

$$\Gamma^2 ds^2 = -dt^2 + F^2(y; \epsilon) d\chi^2 + \mathcal{H}^2(t; \epsilon) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + dy^2,$$ (10)

with the metric function $\mathcal{H}(t; \epsilon)$ given by

$$\mathcal{H}(t; \epsilon) = \begin{cases} \cosh t & \epsilon = 1, \\ it & \epsilon = 0, \\ isin t & \epsilon = -1. \end{cases}$$ (11)
Alternatively, we can define $\mathcal{H}$ by relations very similar to the ones given in [9]:
\[
(\partial_t \mathcal{H})^2 = \epsilon \mathcal{H}^2 - 1 , \quad \partial^2_{\chi} \mathcal{H} = \epsilon \mathcal{H} .
\] (12)
The new metrics solve the Einstein equations (1) in the bulk. In the cases $\epsilon = 0, -1$ the metric has the signature $(- + - - +)$ and we do not study further these solutions.

The $\epsilon = 1$ case gives the new one-parameter solution:
\[
\Gamma^2 d\tilde{s}^2 = -dt^2 + A \cos \left( \sqrt{2} y \right) + B \sin \left( \sqrt{2} y \right) \right]^2 d\chi^2 + \cosh^2 t \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + dy^2 .
\] (13)
The symmetries of this metric form the Killing algebra $so(1,3) \oplus so(3)$, as shown in the Appendix.

The projected part of the bulk Weyl tensor characterizing the metric (13) is
\[
\mathcal{E}_{AB} = \tilde{C}_{ACBD} n^C n^D = \frac{1}{2} \Gamma^2 (u_{A,U} B + 3 e_{A} e_{B} - l_{AB}) ,
\] (14)
where $u^A = \Gamma \delta^A_0$ is the unit 4-velocity along $\partial/\partial t$, $e^A = \Gamma^{-1} \delta^A_1$ is the homogeneous Killing vector of the metric (13) and $l_{AB} = g_{AB} + u_{A} u_{B} - e_{A} e_{B}$.

On the brane ($y = 0$) the metric (13) induces the 4-metric:
\[
\Gamma^2 ds^2 = -dt^2 + A^2 d\chi^2 + \cosh^2 t \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) .
\] (15)

As the continuity of the metric across the brane requires the same value of the constant $A$ on both sides of the brane, the constant $A$ can be absorbed into the coordinate $\chi$. The brane has spherical symmetry but it is also homogeneous, as is the Kantowski-Sachs metric [11] of ordinary 4-dimensional general relativity. Having the scale-factors $1/\Gamma$ and $\cosh t/\Gamma$, it is different from the Kantowski-Sachs brane cosmologies presented in [11]. The metric (15) has positive curvature $R = 6 \Gamma^2$, therefore it is not contained in the Bianchi classification [12]. However, as shown in the Appendix, it has even more symmetries. The remaining three Killing symmetries are boost-like. The complete algebra of the Killing vectors on the brane is $so(1,3) \oplus \mathbb{R}$. The scale-factor $\cosh t$ implies a bouncing character of this brane-universe in the time-parameter $t$.

The extrinsic curvature of any hypersurface $y = \text{const.}$ in the metric (13) has only one nonvanishing component
\[
K_{\chi\chi} = \frac{1}{\Gamma} F(y) \partial_y F(y) .
\] (16)
In consequence the jump in the extrinsic curvature across the brane (at $y = 0$) is
\[
[K_{\mu\nu}] = \sqrt{2}[B] (e_{\mu} e_{\nu}) .
\] (17)
A $Z_2$-symmetric bulk solution is given by Eq. (13) with $y$ replaced by $|y|$. Then $[B]$ is replaced by $2B$ in Eq. (17).

The relation $[K_{\mu\nu}] = \bar{\kappa}^2 \left( \left[ T_{\mu\nu} + \frac{1}{2} (\lambda - T) g_{\mu\nu} \right] \right)$ gives the brane source
\[
T_{\mu\nu} = \rho u_{\mu} u_{\nu} + \lambda e_{\mu} e_{\nu} - \rho l_{\mu\nu} ,
\]
\[
\bar{\kappa}^2 (\rho + \lambda) = 2 \sqrt{2} B \Gamma .
\] (18)
The energy-momentum tensor describes a perfect fluid with negative energy density $\rho = -\lambda$ only in the case $B = 0$. This condition would imply no jump in the extrinsic curvature, therefore the matter on the brane being generated as a response to the brane tension $\lambda$. In the generic case the energy-momentum tensor (18) represents a
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As \( \lambda > 0 \), the possibility of a positive energy density is left open, in contrast with the solutions presented in [9]. Then the radial principal pressure is positive, while the tangential ones are negative.

The quadratic source term [4] gives

\[
\tilde{\kappa}^4 S_{\mu \nu} = \frac{\kappa^2 \lambda}{2} g_{\mu \nu} - \kappa^2 T_{\mu \nu},
\]

and the totality of the source terms in the effective 4d Einstein equation combine to

\[
G_{\mu \nu} = \Gamma^2 (u_A u_B - 3 e_A e_B - l_{AB}).
\]

(In the derivation we have employed Eqs. (3) and (6) several times.)

3. Concluding remarks

We have presented a homogeneous brane-world universe obtained from the Einstein brane-universe by applying the complex transformation [3]. This universe has a bouncing character. Its source given by Eq. (18) is difficult to interpret in terms of ordinary matter. However, from the classical general relativistic point of view, on the right hand side of Eq. (20) we have a generalized solid with energy density \( \kappa^2 \rho_{eff} = \Gamma^2 \), radial tension \( \kappa^2 p_{rad}^{eff} = -3 \Gamma^2 \) and tangential tension \( \kappa^2 p_{tan}^{eff} = -\Gamma^2 \).

This is one of the rare solutions of the Einstein field equations with symmetry group \( G_7 \), from among which only the Einstein static universe and some special plane waves were known previously [13].

It is worth to note, that the Einstein static brane presented in [9], with \( \rho^E = -\lambda < 0 \) and \( \tilde{\kappa}^2 (\rho^E + p^E) = -2\sqrt{2B} \Gamma \), can be interpreted from the general relativistic point of view as having effective energy density \( \kappa^2 p_{eff}^{E} = 3\epsilon \Gamma^2 \) and isotropic tension \( \kappa^2 p_{eff}^{\epsilon} = -\epsilon \Gamma^2 \), a perfect solid for \( \epsilon = 1 \).

Acknowledgments

I am grateful to Roy Maartens for discussions. This work was supported by the Hungarian Eötvös Fellowship.

Appendix A. Killing vectors

A set of independent Killing vectors for the metric [10] in the coordinates \( x^A = (t, \chi, \theta, \varphi, y) \) is given by

\[
\begin{align*}
K_1 &= (0, 0, 0, 1, 0), \\
K_2 &= (0, 0, -\cos \varphi, \cot \theta \sin \varphi, 0), \\
K_3 &= (0, 0, \sin \varphi, \cot \theta \cos \varphi, 0), \\
K_4 &= (-\cos \theta, 0, \partial_t (\log H) \sin \theta, 0, 0), \\
K_5 &= \left( \sin \theta \sin \varphi, 0, \partial_t (\log H) \cos \theta \sin \varphi, \partial_t (\log H) \frac{\cos \varphi}{\sin \theta}, 0 \right), \\
K_6 &= \left( \sin \theta \cos \varphi, 0, \partial_t (\log H) \cos \theta \cos \varphi, -\partial_t (\log H) \frac{\sin \varphi}{\sin \theta}, 0 \right), \\
K_7 &= (\alpha \beta)^{-1} (0, 1, 0, 0, 0),
\end{align*}
\]
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\[ K_8 = \frac{\alpha^2}{\sqrt{2} \beta} (0, -\partial_y (\log F) \sin (\beta \chi), 0, 0, \beta \cos (\beta \chi)), \]
\[ K_9 = -\frac{\alpha}{\sqrt{2} \beta} (0, \partial_y (\log F) \cos (\beta \chi), 0, 0, \beta \sin (\beta \chi)), \]  

(A.1)

where we have introduced the notations \( \alpha = \text{sgn} (\epsilon A^2 + B^2) \) and \( \beta = \sqrt{2 | \epsilon A^2 + B^2 |} \).

These Killing vectors obey
\[
[K_i, K_j] = \varepsilon_{ijk} K_k, \\
[K_{3+i}, K_{3+j}] = -\varepsilon_{ijk} K_k, \\
[K_i, K_{3+j}] = \varepsilon_{ijk} K_{3+k}, \\
[K_{6+i}, K_{6+j}] = \varepsilon_{ijk} K_{6+k}, \\
[K_{6+i}, K_j] = 0 = [K_{6+i}, K_{3+j}].
\]  

(A.2)

Thus the Killing algebra is \( \text{so}(1,3) \oplus \text{so}(3) \). From among the Killing vectors \( K_{1-7} \) are confined to the \( y = \text{const} \) sections. They span the algebra \( \text{so}(1,3) \oplus \mathbb{R} \).

The vectors \( K_{1-7} \) (without the fifth component) are the Killing vectors for the brane metric (15). Among them \( K_{1-3} \) and \( K_7 \) are spacelike and they assure the homogeneity of the constant time slices (they form the \( \text{so}(3) \oplus \mathbb{R} \) algebra). The causal character of the remaining three Killing vectors is given by
\[
\Gamma^2 g (K_4, K_4) = -1 + \sin^2 \theta \cosh^2 t, \\
\Gamma^2 g (K_5, K_5) = -1 + (1 - \sin^2 \theta \sin^2 \varphi) \cosh^2 t, \\
\Gamma^2 g (K_6, K_6) = -1 + (1 - \sin^2 \theta \cos^2 \varphi) \cosh^2 t.
\]

For \( t = 0 \) all three of them are time-like, while for other values of \( t \) the causal character depends on the actual values of \( \theta \) and \( \varphi \). The region with all three Killing vectors spacelike increases with \( |t| \).

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