CODING AND TILING OF JULIA SETS FOR SUBHYPERBOLIC RATIONAL MAPS

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Abstract. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a subhyperbolic rational map of degree $d$. We construct a set of coding maps $\text{Cod}(f) = \{\pi_r : \Sigma \to J\}_r$ of the Julia set $J$ by geometric coding trees, where the parameter $r$ ranges over mappings from a certain tree to the Riemann sphere. Using the universal covering space $\phi : \tilde{S} \to S$ for the corresponding orbifold, we lift the inverse of $f$ to an iterated function system $I = (g_i)_{i=1,2,\ldots,d}$. For the purpose of studying the structure of $\text{Cod}(f)$, we generalize Kenyon and Lagarias-Wang’s results: If the attractor $K$ of $I$ has positive measure, then $K$ tiles $\phi^{-1}(J)$, and the multiplicity of $\pi_r$ is well-defined. Moreover, we see that the equivalence relation induced by $\pi_r$ is described by a finite directed graph, and give a necessary and sufficient condition for two coding maps $\pi_r$ and $\pi_r'$ to be equal.

1. Introduction

The method of symbolic dynamics is prevalent in the study of dynamical systems. Particularly, to investigate an attractor (or repeller) of a dynamical system, one often uses symbolic dynamics to code the attractor. In the present paper, we study coding maps of Julia sets for rational maps. We emphasize that we treat not only individual coding maps but totalities of coding maps.

We say a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the Riemann sphere to itself is subhyperbolic if each critical point is either preperiodic or attracted to an attracting cycle. We will construct a set of coding maps $\text{Cod}(f) = \{\pi_r\}_r$ from the full shift to the Julia set $J$ by using ‘geometric coding trees.’ Geometric coding tree technique is developed by Przytycki and his coauthors for general holomorphic maps ([19], [20], [21]). See also [16], §1.16. The parameter $r$ ranges over radials, which are mappings from a certain topological tree to the Riemann sphere. One goal of considering $\text{Cod}(f)$ is to understand the combinatorics of $f$. In view of the construction of geometric coding trees, the structure of $\text{Cod}(f)$ reflects the combinatorics of $f$.

We know that coding is effectively used in the study of dynamics of interval maps (see for example [3], [18], [4]). Coding often works in the parameter space as well as in the dynamical space. Recall that the dynamical space is just an interval $I$ with natural partition, that is, $I$ is divided into several subintervals by the turning points. This partition gives a natural coding of $I$, and we obtain a nice invariant called the kneading sequence, which is defined as the symbol sequences corresponding to the forward orbits of the turning points. For example, in a certain family of real polynomial maps the kneading sequences almost completely classify these maps up to topological conjugacy. In particular, for the quadratic family $x \mapsto ax(x - 1) \ (0 < a < 4)$, we have the monotonicity of the kneading sequence (and the topological entropy). Roughly speaking, the natural coding parametrizes the bifurcation of the quadratic family. However, in a larger family of complex
rational maps, coding does not seem to work well in the parameter space. The main reason of this difficulty is the absence of natural partitions. We do not have a nice invariant like the kneading sequence. In such a situation, thus it is less important to consider individual coding maps. This is why we treat totalities of coding maps.

A complete description of Cod($f$) is quite difficult except a few cases including $f(z) = z^2$, $f(z) = z^2 - 2$, etc. It is unfortunate that even the case $f(z) = z^d$ with $d \geq 3$ and the Cantor set case (e.g. $f(z) = z^2 - 3$) are complicated. Thus we will try to find tools to manage Cod($f$), keeping in mind the following natural and naive problems: (1) What is the canonical coding? (2) Are there any good structures on Cod($f$)? The present paper does not completely solve these problem, but gives several fundamental facts which will be a help to approach these problems. Our main results are concerned with the multiplicities of coding maps and the equivalence relations on the space of symbol sequences.

In our setting, $f^{-1}$ can be lifted by the ‘universal covering’ $\phi : \tilde{S} \to S = \mathcal{C} - AP$, where AP is the set of attracting periodic points. Let $d$ be the degree of $f$. For $i = 1, 2, \ldots, d$, there exists a holomorphic contraction $g_i : \tilde{S} \to \tilde{S}$ depending on $r$ such that the diagram

$$
\begin{array}{ccc}
\tilde{S} - f^{-1}(AP) & \leftarrow g_i & \tilde{S} \\
\phi & & \phi \\
S - f^{-1}(AP) & \rightarrow & S
\end{array}
$$

commutes. Thus there exists a compact set $K$ such that $K = \bigcup_{i=1}^{d} g_i(K)$ by Hutchinson [7]. We call $K$ the Julia tile with respect to $r$. If $r$ is ‘suitable’, then the coding map $\pi_r$ is onto, and so $\phi(K) = J$.

The situation above is analogous to that of self-affine tiling. We recall self-affine tiling briefly. Let $A$ be an $n \times n$ expanding integral matrix, and let $d = |\det A|$. Since $AZ^n \subset \mathbb{Z}^n$, its projection $f : \mathbb{T}^n \to \mathbb{T}^n$ is well-defined, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the $n$-torus. A lift of $f$ has the form $x \mapsto Ax + a, a \in \mathbb{Z}^n$. Choosing $a_1, a_2, \ldots, a_d \in \mathbb{Z}^n$, we obtain $d$ contractions $g_i : x \mapsto A^{-1}(x - a_i)$ by which we have the commuting diagram

$$
\begin{array}{ccc}
\mathbb{R}^n & \leftarrow g_i & \mathbb{R}^n \\
\phi & & \phi \\
\mathbb{T}^n & \rightarrow & \mathbb{T}^n
\end{array}
$$

for $i = 1, 2, \ldots, d$, and the self-affine set $K$ with $K = \bigcup_{i=1}^{d} g_i(K)$. Kenyon [11] showed that if $K$ has positive Lebesgue measure, then $K$ tiles $\mathbb{R}^n$ (i.e. there exists a set of translations $T \subset \mathbb{Z}^n$ such that (1) $\mathbb{R}^n = \bigcup_{t \in T} (K + t)$, and (2) the Lebesgue measure of $(K + t) \cap (K + t')$ vanishes for distinct $t, t' \in T$). See for more details [11] and [13].

We generalize Kenyon and Lagarias-Wang’s results and obtain:

**Tiling Theorem** (Theorem 22). Let $\mu$ be the equilibrium state for $f : J \to J$ with constant weight, and $\tilde{\mu}$ the lift of $\mu$ to $\tilde{S}$. If $\tilde{\mu}(K)$ is positive, then $K$ tiles $\tilde{J} = \phi^{-1}(J)$, that is, there exists a set $T$ of deck transformations of $\phi : \tilde{S} \to \tilde{S}$ such that (1) $\tilde{J} = \bigcup_{t \in T} t(K)$, and (2) $\tilde{\mu}(t(K) \cap t'(K)) = 0$ for distinct $t, t' \in T$. 


Multiplicity Theorem (Theorem 28). If \( \tilde{\mu}(K) \) is positive, then there exists \( n > 0 \) depending on \( r \) such that \( \pi_r \) is almost \( n \)-to-one, that is, \( \#\pi^{-1}_r(\omega) = n \) for \( m \)-almost all \( \omega \in \Sigma \), where \( m \) is the Bernoulli measure with identically distributed weight. The number \( n = n_r \) is said to be the multiplicity of \( \pi_r \).

For a given coding map \( \pi_r \), it is difficult to calculate its multiplicity in general. The most primitive way to do that is directly seeing the equivalence relation induced by \( \pi_r \) on \( \Sigma \). We will show that the equivalence relation is described by a finite graph, which is a version of Fried’s result ([5], Lemma 1). See also [10].

Finite Graph Theorem (Theorem 47). We can construct a directed graph with vertex set \( V \), edge set \( E \), and weight \( \alpha : E \to \{1, 2, \ldots, d\}^2 \) such that \( \pi_r(\omega_1 \omega_2 \cdots) = \pi_r(\omega'_1 \omega'_2 \cdots) \) if and only if there exists a sequence \( e_1, e_2, \ldots \in E \) with \( \alpha(e_i) = (\omega_i, \omega'_i) \) and (the terminal vertex of \( e_i \)) = (the initial vertex of \( e_{i+1} \)) for \( i = 1, 2, \ldots \).

We expect that any subhyperbolic rational maps have the canonical coding maps. The word ‘canonical’ has vagueness, but it become clearer by the notion of multiplicity. If there exists the simplest nontrivial coding map, we may consider it canonical. We will look at several examples of subhyperbolic rational maps later, and will find out that each of them has a nontrivial coding map which is apparently the simplest. These coding maps have the features (1) the multiplicities are equal to one, (2) the connecting sets \( E = \bigcup_{i \neq j} (g_i(K) \cap g_j(K)) \cup \bigcup_{i} \{x \in g_i(K) | \#g_i^{-1}(x) \geq 2\} \) are small, and (3) the Julia tiles \( K \) have simple shapes. While the second and the third features are still vague, the first one mathematically makes sense. Thus our conjecture is that any subhyperbolic rational maps have a coding maps with multiplicity one.

Another conjecture is that \( \text{Cod}(f) \) has some structure. For example, we expect that there exists a natural action on \( \text{Cod}(f) \) by which we can control the diversity of multiplicities of \( \pi_r \).

Structure Theorem (Theorem 35). \( \pi_r = \pi_{r'} \) if and only if \( r \) and \( r' \) are freely homotopic mod \( N_r \).

This theorem says that
\[
\text{Cod}(f) \approx \bigcup_N \{r \mid N_r = N\}/(\text{freely homotopic mod } N),
\]
where \( N_r \) denotes the maximal invariant subgroup with respect to \( r \) of the fundamental group of \( \mathbb{C} - \{\text{postcritical points}\} \) we will define later. Let \( \mathcal{A}(f) \) be the monoid of rational maps commutative to \( f \). Then \( \mathcal{A}(f) \) acts on \( \text{Cod}(f) \) by \( (R, \pi) \mapsto R \circ \pi \), and the multiplicity of \( R \circ \pi \) is equal to the degree of \( R \) times the multiplicity of \( \pi \) (Proposition 39).

The organization of the present paper is as follows. After giving the definitions of coding maps and Julia tiles in Section 2 we consider several examples of subhyperbolic rational maps in Section 3. In one example, the Lévy Dragon appears as a Julia tile. Section 4 supplies the definitions of invariant subgroups and the equilibrium states, and shows some basic facts. We prove Tiling Theorem and Multiplicity Theorem in Section 5. In Section 6, we discuss the structure of...
Cod($f$), prove Structure Theorem, and show a couple of examples. Finite Graph Theorem is proved in Section 4. In Section 5, we see that several results hold for non-subhyperbolic rational maps.

2. Definition

In this section, we give the construction of coding maps after Przytycki. We obtain Julia tiles by lifting the coding maps to the universal covering spaces. Let $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a rational map of degree $d$.

**Definition 1.** We say that
$$C = C_f = \{ \text{critical points of } f \} = \{ c \mid f \text{ is not locally homeomorphic at } c \}$$
is the critical set of $f$. The postcritical set is defined by
$$P = P_f = \{ f^n(c) \mid c \in C, n > 0 \}.$$Let $AP$ denote the set of attracting periodic points of $f$. We write
$$S' = \hat{\mathbb{C}} - P \quad \text{and} \quad S = \hat{\mathbb{C}} - AP.$$

**Definition 2.** Let
$$Q = Q_d = \bigcup_{i=1}^{d} [0, 1]_i / (0_i \sim 0_j)$$
be the topological tree made of $d$ copies of the unit interval with all the origins identified. We say that a continuous map $r : Q \rightarrow S'$ is a radial if $r(0) = f \circ r(1_i)$ for $i = 1, 2, \ldots, d$. We say that $\bar{x} = r(0)$ is the basepoint of $r$. A radial is considered as a $d$-tuple of curves $(l_i : [0, 1] \rightarrow S')_i$ with the same initial point $\bar{x}$ such that $f \circ l_i(1) = \bar{x}$. A radial $r$ is said to be proper if $r(1_i) \neq r(1_j)$ whenever $i \neq j$. We write the set of radials with basepoint $\bar{x}$ by
$$\text{Rad}(f, \bar{x}).$$Set
$$L = L(f, \bar{x}) = \{ l : [0, 1] \rightarrow S' \mid l \text{ is continuous and } l(0) = \bar{x} \},$$
$$\Lambda(f, \bar{x}) = \{ l \in L(f, \bar{x}) \mid l(1) \in f^{-1}(\bar{x}) \}.$$Then $\text{Rad}(f, \bar{x}) = \Lambda(f, \bar{x})^d$.

For a curve $l \in L$ and $x \in f^{-k}(\bar{x})$, we define a curve $F_x(l) : [0, 1] \rightarrow S$ as the lift of $l$ by $f^k$ with initial point $x$, that is, $f^k \circ F_x(l) = l$ and $F_x(l)(0) = x$. Since $l$ does not pass through $P$, the curve $F_x(l)$ is uniquely defined.

**Notation 3.** Let $(\sigma, \Sigma)$ be the one-sided fullshift of $d$ symbols. Namely,
$$\Sigma = \{ 1, 2, \ldots, d \}^N$$is the set of one-sided infinite sequences of $\{ 1, 2, \ldots, d \}$, and $\sigma : \Sigma \rightarrow \Sigma$ is the shift map ($\sigma(w_1 w_2 \cdots) = w_2 \cdots$).

Let $W$ be the set of words (i.e. finite sequences) of $d$ symbols, and $W_k$ the set of words of length $k$:
$$W = \bigcup_{k=1}^{\infty} W_k, \quad W_k = \{ 1, 2, \ldots, d \}^k.$$For $w \in W$, we write
$$\Sigma(w) = \{ w \omega_1 \omega_2 \cdots \mid \omega_1 \omega_2 \cdots \in \Sigma \} \subset \Sigma.$$
Construction of coding maps. Suppose $f$ is subhyperbolic. For a radial $r = (l_i)_i$, we inductively define curves $l_w : [0, 1] \to S'$ and points $x_w$ for $w \in W$. First we set $x_i = l_i(1)$ for $i \in W_1 = \{1, 2, \ldots, d\}$. If $l_w$ and $x_w$ are determined for $w \in W_k$, we set $l_{iw} = l_i F_w(l_w)$ and $x_{iw} = l_{iw}(1)$ for $i \in W_1$. By the expandingness of $f$, we have 

\[ l_w = \lim_{k \to \infty} l_{w_1 w_2 \cdots w_k}, \quad x_w = \lim_{k \to \infty} x_{w_1 w_2 \cdots w_k} \quad \text{for} \quad \omega = \omega_1 \omega_2 \cdots \in \Sigma. \]

Note that $l_w$ does not converge, but by a suitable change of parametrization, the limit $l_w$ is well-defined and unique up to parametrization. Clearly, $x_w = l_w(1)$ for $\omega \in \Sigma$. Since $x_w$ is an accumulation point of $f^{-k}(\bar{x})$, $k = 1, 2, \ldots$, the point belongs to the Julia set (for example, see [17], 4.7). It is easily seen that $f(x_w) = x_{\pi w}$ and the mapping $\omega \mapsto x_w$ is continuous. We denote this mapping by $\pi = \pi_r : \Sigma \to J$, and call it the coding map of $J$ for $r$. We write

\[ \text{Cod}(f) = \{ \pi_r \mid r \text{ is a radial} \}. \]

**Remark 4.**

- $l_u \cdot F_{x_w}(l_w)$ is equal to $l_{uw}$ up to parametrization for $u \in W$ and $w \in W$ (or $w \in \Sigma$).
- $f \pi(\Sigma) = \pi(\Sigma)$.
- If $r$ is proper, then $\pi_r : \Sigma \to J$ is onto. However, the converse is not always true.
- The image of $\pi$ is either a perfect set or a singleton. Indeed, suppose $\pi(\Sigma)$ has a isolated point $p$. Since $\pi^{-1}(p)$ is open, there exists $k$ such that $\sigma^k \pi^{-1}(p) = \Sigma$. Therefore $\pi(\Sigma) = f^k(p)$.
- If $\pi(\Sigma)$ is perfect, then $\# f^{-k}(x) \cap \pi(\Sigma) \to \infty$ as $k \to \infty$ for any $x \in \pi(\Sigma)$. Indeed, it is sufficient to show that there exists $k$ such that $\# f^{-k}(x) \cap \pi(\Sigma) \geq 2$. Assume that $f^{-k}(x) \cap \pi(\Sigma) = \{y_k\}$ for any $k$. There exists $k_0$ such that $y_k \notin C$ for $k \geq k_0$. Take $\omega \in \Sigma$ with $\pi(\omega) = y_{k_0}$. Then $\pi(w(\omega)) = y_{k_0 + m}$ for any $w \in W_m$. This means that $F_{x_w}(l_w)$'s are the same for $w \in W_m$. Thus $x_w$'s are the same for $w \in W_m$. Consequently, the accumulation points of $\{x_w\}_{w \in W_m}$ ($m \to \infty$) consist of one point.

From now on, we suppose that $f$ is subhyperbolic.

**Definition 5.** A function $\rho : S \to \mathbb{N}$ is called a ramification function for $f$ if $\rho(x) = 1$ for $x \notin P$ and $\rho(f(x))$ is a multiple of $\deg_x f \cdot \rho(x)$. The minimal ramification function is called the canonical ramification function and denoted by $\rho_f$.

For a ramification function $\rho$, we have a universal covering $\phi : \tilde{S} \to S$ for the orbifold $(S, \rho)$ (i.e. $\tilde{S}$ is a connected and simply connected Riemann surface and $\phi$ is the holomorphic branched covering such that the local degree $\deg_x \phi$ is $\rho(\phi(\bar{x}))$ for every $\bar{x} \in \tilde{S}$). See [17], Appendix E.

The universal covering $\phi : \tilde{S} \to S$ is constructed as follows. Let

\[ G = \pi_1(S', \bar{x}) \]

be the fundamental group of $S'$. For

\[ \gamma \in \Gamma = \Gamma(f, \bar{x}) = \{ \gamma \in L(f, \bar{x}) \mid \gamma(0) = \gamma(1) \}, \]

we denote the homotopy class for $\gamma$ by $[\gamma] \in G$. Let $AP = \{a_1, a_2, \ldots, a_p\}$ and $P - AP = \{b_1, b_2, \ldots\}$. Choose simple closed curves $A_1, A_2, \ldots, A_p, B_1, B_2, \cdots \in \Gamma$ so that $A_i$ separates $a_i$ from the other points of $AP$, and $B_j$ separates $b_j$ from the other points of $AP$. Then $G$ is generated by $[A_i], [B_j]$ ($i = 1, 2, \ldots, p$, $j = 1, 2, \ldots$).
Let $N = N^\rho$ be the normal subgroup of $G$ generated by $[B_j^{\rho(b_j)}], b_j \in P - \text{AP}$. Then $[A_k^i] \notin N, k \in \mathbb{Z}$ and $[B_j^i] \notin N, 0 < |k| < \rho(b_j)$ in our case. The quotient group $G^\rho = G/N$ is called the fundamental group of the orbifold $(S, \rho)$. Let

$$\tilde{L} = \{ l : [0, 1] \to S \mid l \text{ is continuous, } l(0) = \tilde{x} \text{ and } l(t) \in S' \text{ for } 0 \leq t < 1 \}.$$ 

We set

$$\tilde{S} = \tilde{L} / \sim_N,$$

where $l \sim_N l'$ if $l(1) = l'(1)$ and there exists a closed curve $\gamma \in \Gamma$ obtained by perturbing $l'^{-1}$ near $l(1)$ such that $[\gamma] \in N$. We obtain a branched covering $\phi : \tilde{S} \to S$ defined by $\phi([l]) = l(1)$. The surface $\tilde{S}$ is considered as either $\{|z| < 1\}$ or $\mathbb{C}$.

The covering $\phi : \tilde{S} - \phi^{-1}(P) \to S'$ corresponds to the normal subgroup $N \subset G$. Take $\tilde{x} \in \phi^{-1}(\tilde{x})$. Since $(f \circ \phi)_* \pi_1(\tilde{S} - \phi^{-1}(P), z) \supseteq N$, for $z \in \phi^{-1}((\tilde{x})$ there exists a holomorphic covering $g : \tilde{S} - \phi^{-1}(P) \to \tilde{S} - \phi^{-1}(P)$ such that

$$f \circ \phi \circ g = \phi$$

and $g(\tilde{x}) = z$. We extend $g$ to a holomorphic branched covering $\tilde{g} : \tilde{S} \to \tilde{S}$. We say that $g$ is the contraction associated with $z$ (with respect to the basepoint $\tilde{x}$).

By retaking a bigger ramification function if necessary, we may assume $\tilde{S} = \{|z| < 1\}$. Thus $g$ is contracting with respect to the Poincaré metric. Hence $f$ is uniformly expanding near $J$ with respect to the projection metric, that is, there exists $c > 1$ such that $\|Df_z\| > c$ for $z$ in some neighborhood of $J$. See [17], §19.

**Definition 6.** Fix $\tilde{x} \in \phi^{-1}(\tilde{x})$. For $l \in \tilde{L}$, we denote by $\tilde{l}$ the lift of $l$ to $\tilde{S}$ with $\tilde{l}(0) = \tilde{x}$. Let $r = (\tilde{l}_i)$ be a radial. We have the contractions $g_1, g_2, \ldots, g_d : \tilde{S} \to \tilde{S}$ associated with $\tilde{l}_i$. Since $g_i$’s are contracting, we have the attractor $K$ of the iterated function system $\mathcal{I} = (g_1, g_2, \ldots, g_d)$ (i.e. $K \subset \tilde{S}$ is the unique nonempty compact set with $K = \bigcup_{i=1}^{d} g_i(K)$). See [17]. We use the notation

$$g_w = g_{w_1} \circ g_{w_2} \circ \cdots \circ g_{w_k}$$

for $w = w_1 w_2 \cdots w_k \in W$. It is known that a surjective coding map

$$\hat{\pi} : \Sigma \to K$$

is defined by $\hat{\pi}(\omega_1 \omega_2 \cdots) = \lim_{k \to \infty} g_{w_1 w_2 \cdots w_k}(z)$, which is independent of $z \in \tilde{S}$.

**Proposition 7.** $\phi \circ \hat{\pi} = \pi$.

**Proof.** We show that $\tilde{l}_w(1) = g_w(\tilde{x})$ for any $w \in W$ by induction. Assume that $\tilde{l}_w(1) = g_w(\tilde{x})$ for $w \in W_k$. Then $\phi \circ g_w(\tilde{x}) = x_w$. By (2.1), $f^k \circ \phi \circ g_w \circ \tilde{l}_i = \tilde{l}_i$. Hence $F_{x_w}(l_i) = \phi \circ g_w \circ \tilde{l}_i$. Therefore $l_{w_1} = l_{w_1} \cdot (\phi \circ g_w \circ \tilde{l}_i)$ lifts to $\tilde{l}_{w_1} \cdot (g_w \circ \tilde{l}_i)$. Hence $\tilde{l}_{w_1}(1) = g_w \circ \tilde{l}_i(1) = g_{w_1}(\tilde{x})$.

From $\hat{\pi}(w_1 w_2 \cdots) = \lim_{k \to \infty} g_{w_1 w_2 \cdots w_k}(\tilde{x})$ and $g_w(\tilde{x}) = \tilde{l}_w(1)$, it follows that $\phi \circ \hat{\pi}(w_1 w_2 \cdots) = \lim_{k \to \infty} x_{w_1 w_2 \cdots w_k} = \pi(w_1 w_2 \cdots)$. \hfill \Box

**Remark 8.** Let $[\rho]_N = (\tilde{l}_i(1)) \in (\phi^{-1}f^{-1}(\tilde{x}))^d$.

Since the iterated function system $\mathcal{I}$ is determined by $(\tilde{l}_i(1))$, so is the coding map $\pi_r$. 

3. **Examples**

3.1. Let \( f(z) = z^d \). Then \( C = P = AP = \{0, \infty\} \), \( S = \mathbb{C} - \{0\} \), and the Julia set \( J \) is the unit circle \( \{|z| = 1\} \). Since \( P - AP = \emptyset \), we have a unique ramification function \( \rho(x) = 1 \), and so \( N = N^\rho \) is trivial. Thus \( \hat{S} \) is Euclidean (i.e. \( \hat{S} = \mathbb{C} \)). We take an universal covering \( \phi = z \mapsto e^{-2\pi iz} : \mathbb{C} \to \mathbb{C} - \{0\} \). Fix basepoints \( \bar{x} = 1 \in \mathbb{C} - \{0\} \) and \( \hat{x} = 0 \in \hat{C} \). Then \( \phi^{-1} f^{-1}(\hat{x}) = \frac{1}{2}\mathbb{Z} \). The contraction associated with \( n/d \in \frac{1}{2}\mathbb{Z} \) is \( z \mapsto z/d + n/d \). The coding maps \( \pi_r \) such that \([r]_N = (n_1/d, n_2/d, \ldots, n_d/d)\) with \( \{n_1, n_2, \ldots, n_d\} = \{0, 1, \ldots, d - 1\} \mod d \), for which the attractors \( K_r \) are intervals of length one, are considered canonical.

Consider the case \( d = 2 \). If \([r]_N = (n_1/2, n_2/2)\) with \( n_1 \leq n_2 \), then \( K_r \) is the closed interval \([n_1, n_2]\), and \( \pi_r \) is almost \((n_2-n_1)\)-to-one map provided \( n_2 - n_1 > 0 \). Since \( K_r \) is a closed interval, \( \phi^1(J_r) = \mathbb{R} \) whenever \( n_2 - n_1 \) is positive.

Consider the case \( d = 3 \). Let \( r \) be a radial with \([r]_N = (n_1/3, n_2/3, n_3/3) = (km_1/3, km_2/3, km_3/3) \), where \( k \) is the greatest common divisor of \( n_1, n_2, \) and \( n_3 \). Kenyon [12] proved that \( \mu(K_r) \), the 1-dimensional Lebesgue measure of \( K_r \) is positive if and only if \( m_1 + m_2 + m_3 = 0 \mod 3 \), and that \( \mu(K_r) = k \) and \( K_r \) tiles \( \phi^{-1}(J_r) = \mathbb{R} \). Remark that \( K_r \) is not necessarily an interval. Related topics are discussed by Lagarias and Wang [15].

3.2. Let \( f(z) = z^2 - 2 \). Then \( C = \{0, \infty\} \), \( P = \{-2, 2, \infty\} \), \( AP = \{\infty\} \), \( S = \mathbb{C} \), and the Julia set \( J \) is the interval \([-2, 2]\). The canonical ramification function is \( \rho(z) = 2 \) if \( z = -2 \), \( \rho(z) = 1 \) otherwise. Note that \( \hat{S} \) is Euclidean, and take a universal branched covering \( \phi = z \mapsto 2\cos \pi z : \mathbb{C} \to \mathbb{C} \). Then \( \phi^{-1}(J_r) \) is the real axis and \( f \circ \phi(z) = \phi(\pm 2z + 2n), n \in \mathbb{Z} \). Fix basepoints \( \bar{x} = 0 \) and \( \hat{x} = 1/2 \). Then \( f^{-1}(\hat{x}) = \{\pm \sqrt{2}\} \), \( \phi^{-1}(\sqrt{2}) = \{1/4 + 2n, 7/4 + 2n \mid n \in \mathbb{Z}\} \) and \( \phi^{-1}(\sqrt{2}) = \{3/4 + 2n, 5/4 + 2n \mid n \in \mathbb{Z}\} \). The contraction associated with \( \pm 1/4 + n \in \phi^{-1} f^{-1}(\hat{x}) \) is \( z \mapsto \pm z/2 + n \). If \([r]_N = (k_1, k_2) \in (\phi^{-1} f^{-1}(\hat{x}))^2 \), then \( K_r \) is the closed interval \([m_1, m_2]\), where

\[
m_1 = \min\{2n_1, 2n_2\}, m_2 = \max\{2n_1, 2n_2\} \text{ if } k_1 = 1/4 + n_1, k_2 = 1/4 + n_2,
\]

\[
m_1 = \min\{2n_1, 2n_2 - n_1\}, m_2 = \max\{2n_1, 2n_2 - n_1\}
\]

\[
\text{if } \{k_1, k_2\} = \{1/4 + n_1, -1/4 + n_2\},
\]

\[
m_1 = \min\{4n_1 - 2n_2, 4n_2 - 2n_1\}/3, m_2 = \max\{4n_1 - 2n_2, 4n_2 - 2n_1\}/3
\]

\[
\text{if } k_1 = -1/4 + n_1, k_2 = -1/4 + n_2.
\]

The coding maps \( \pi_r \) such that \([r]_N = (k_1, k_2) \) with \( \{k_1, k_2\} = \{1/4 + n_1, -1/4 + n_2\} \) \(|\{3n_1 - n_2\} = 1\), for which the attractors \( K_r \) are intervals of length one, are considered canonical.

3.3. Let \( f(z) = -(z - 1)^2/4z \). (This map is conjugate to the map \( z \mapsto (z - 2)^2/z^2 \), which is discussed in [1], §4.3.) Then \( C = \{-1, 1\} \), \( P = \{1, 0, \infty\} \), \( AP = \emptyset \) and \( S = J = \hat{\mathbb{C}} \). The canonical ramification function is \( \rho(0) = \rho(\infty) = 4 \), \( \rho(1) = 2 \), and \( \rho(z) = 1 \) otherwise. Note that \( \hat{S} \) is Euclidean. If \( \phi : \mathbb{C} \to \hat{\mathbb{C}} \) is an elliptic function of order four with lattice \( 2\Gamma = \{2n + 2mi \mid n, m \in \mathbb{Z}\} \) such that \( \phi(iz) = \phi(z), \phi(0) = \infty, \phi(1) = 1, \phi(1+i) = 0 \), then \( \phi(\alpha z + \beta) = f(\phi(z)) \) for \( \alpha \in \{\pm 1, \pm 1 + i\}, \beta \in 2\Gamma \) and \( \phi \) is a universal covering for the orbifold \((S, \rho)\). Clearly, \( \phi^{-1}(J_r) = \mathbb{C} \).

Note
that \( \phi(z) = a\phi(2z)^2 \) satisfies the above condition, where \( \phi \) is the Weierstrass elliptic function for the lattice \( \Gamma = \{ n + mi \mid n, m \in \mathbb{Z} \} \) and \( a \) is some constant. It is easily seen that the map \( f \) is a version of Lattès’ example. (For example, see [1], §4.3 or [17], §7. Using the addition formula, we have
\[
\phi(\alpha z)^2 = -(\phi(z)^2 - g_2/4)^2/4\phi(z)^2 \quad \text{for} \quad \alpha \in \{ \pm 1 + i, \pm 1 - i \},
\]
where \( g_2 \) is a nonzero constant. The constant \( a \) is equal to \( 4/g_2 \).)

Fix basepoints \( \tilde{x} = -1 \) and \( \tilde{x} = 1/2 + i/2 \). Then \( \phi^{-1}f^{-1}(\tilde{x}) = \{ s + n + mi \mid n, m \in \mathbb{Z}, s = 1/2 \text{ or } i/2 \} \). The contraction associated with \( x \in \phi^{-1}f^{-1}(\tilde{x}) \) is
\[
\begin{align*}
    z &\mapsto (1 - i)z/2 + n + mi & \text{if } x = 1/2 + n + mi \\
    z &\mapsto (1 + i)z/2 + n + mi & \text{if } x = i/2 + n + mi \\
    z &\mapsto (-1 + i)z/2 + n + mi & \text{if } x = -1/2 + n + mi \\
    z &\mapsto (-1 - i)z/2 + n + mi & \text{if } x = -i/2 + n + mi,
\end{align*}
\]
where \( n + mi \in (1 + i)\Gamma = \{ n + mi \mid n + m \text{ is even} \} \). The attractor \( K_r \) is a compact set with integral Lebesgue measure. For example, in the case \( [r] = (-i/2 + 1 + i, -1/2 + 2) \), \( K_r \) is the triangle with vertices \( 0, 2, 1 + i \). This case is considered as canonical. In the case \( [r] = (i/2, 1/2 + 1 + i) \), \( K_r \) is the Lévy Dragon (Figure 4). Thus Tiling Theorem gives another proof of the well-known fact that the Lévy Dragon tiles \( \mathbb{R}^2 \) and has nonempty interior.

We can calculate the 2-dimensional Lebesgue measure \( \mu(K_r) \) for \( [r] = (\alpha_1/2 + \beta_1, \alpha_2/2 + \beta_2) \) \( \alpha_j \in \{ \pm 1, \pm i \}, \beta_j \in (1 + i)\Gamma \) as follows.
\[
\mu(K_r) = \begin{cases} 
2|\beta_2 - \beta_1|^2 & \text{if } \alpha_1 = \alpha_2 \\
\frac{|\beta_2 - \beta_1|^2}{2(1 - i)\alpha_2 - 2(1 - i)\alpha_1} & \text{if } (\alpha_1, \alpha_2) = (1, i), (i, 1) \\
10 & \text{if } (\alpha_1, \alpha_2) \in A_1 \times A_2 \text{ or } A_2 \times A_1 \\
25 & \text{if } (\alpha_1, \alpha_2) = (-1, -i), (-i, -1)
\end{cases}
\]
where \( A_1 = \{ 1, i \}, A_2 = \{ -1, -i \} \). The details are left to the reader.

3.4. Let \( f(z) = z^2 - 3 \). Then \( C = \{ 0, \infty \}, P = \{ -3, 6, 33, \cdots, \infty \}, \) \( AP = \{ \infty \}, S = \mathbb{C} \), and the Julia set \( J \) is a Cantor set in the real axis. The canonical ramification function is \( \rho(z) = 2 \) for \( z = -3, 6, 33, \ldots, \rho(z) = 1 \) otherwise. Thus \( \tilde{S} \) is hyperbolic (i.e. \( S = \{ |z| < 1 \} \)), and for any radial, the corresponding contractions \( g_1, g_2 \) are not invertible.

If we take radials \( r_1, r_2 \) and \( r_3 \) as in Figures 2, 3 and 4 then \( \pi_{r_1} \) is a homeomorphism, \( \pi_{r_2} \) is exactly two-to-one, and \( \pi_{r_3} \) is at most three-to-one respectively. Moreover, \( \#\pi_{r_3}^{-1}(x) = 1 \) for \( \mu \)-almost all \( x \). So, we say that \( \pi_{r_3} \) is almost one-to-one. See Section 4 for the proof. The coding map \( \pi_{r_1} \) is considered canonical.

4. FURTHER SETTING

In this section, we define an invariant subgroup and the equilibrium state (the Brolin-Lyubich measure).

Let \( f \) be a subhyperbolic rational map of degree \( d \), and \( \rho = \rho_f \) the canonical ramification function. Fix a basepoint \( \tilde{x} \).

**Definition 9.** For a subgroup \( N \subset G = \pi_1(S', \tilde{x}) \), we have a covering
\[
\phi_N : (S_N', x_N^N) \to (S', \tilde{x})
\]
Figure 1. The Lévy Dragon.

Figure 2. The radial $r_1$.

Figure 3. The radial $r_2$. 
with \( \phi_N \circ \pi_1(S'_N, x^N) = N \). If \( N^\rho \subset N \), then we can extend \( \phi_N \) to a branched covering

\[
\phi_N : (S_N, x^N) \to (S, \bar{x}),
\]

where \( S_N - \phi_N^{-1}(P) = S'_N \). For \( l \in \overline{L(f, \bar{x})} \), we denote by \( l^N \) the lift of \( l \) to \( S'_N \) (or \( S_N \)) with \( l^N(0) = x^N \).

If \( N' \) is a subgroup with \( N' \subset N \subset G \), then there exists a unique covering

\[
\psi_{N', N} : (S'_{N'}, x^{N'}) \to (S'_N, x^N)
\]

such that \( \phi_N \circ \psi_{N', N} = \phi_{N'} \). If \( N^\rho \subset N' \), then \( \psi_{N', N} \) is extended to a branched covering \( \psi_{N', N} : S_{N'} \to S_N \).

Suppose \( N \) is a normal subgroup. The group of deck transformations for \( \phi_N \) is identified with \( G/N \). For \( \gamma \in \Gamma(f, \bar{x}) \), we denote the quotient class for \( [\gamma] \) by \( [\gamma]_N \).

**Definition 10.** Let \( l \in \Lambda(f, \bar{x}) \) with terminal point \( x \). Consider a homomorphism \( f_* : \pi_1(S' - f^{-1}(P), x) \to G \) induced from \( f \) and a homomorphism \( l_*^{-1} : \pi_1(S' - f^{-1}(P), x) \to G \) induced from the inclusion and the path \( l^{-1} \). A subgroup \( N \subset G \) is said to be invariant with respect to \( l \) if \( N \subset f_* (l_*^{-1})^{-1} (N) \), or equivalently if \( F_x(\gamma) \) is a closed curve and \( [F_x(\gamma)l^{-1}] \in N \) for every \( \gamma \) with \( [\gamma] \in N \).

If \( N \) and \( N' \) are invariant with respect to \( l \), then so is the subgroup generated by \( N \) and \( N' \). If \( N \) and \( N' \) are invariant with respect to \( l \) and \( l' \) respectively, then \( N \cap N' \) is invariant both with respect to \( l \) and \( l' \). We denote by \( N_l \) the maximal invariant subgroup with respect to \( l \). A subgroup \( N \) is invariant with respect to a radial \( r = (l_i) \) if \( N \) is invariant with respect to all \( l_i, i = 1, 2, \ldots, d \). The maximal invariant subgroup \( N_r \) with respect to \( r \) is equal to \( \bigcap_{i=1}^{d} N_{l_i} \). It is evident that \( N^\rho \) is invariant with respect to every radial for any ramification function \( \rho \).

**Proposition 11.** Let \( l \in \Lambda(f, \bar{x}) \) and let \( N \subset G \) be a subgroup. If \( N \) is invariant with respect to \( l \), then there exists a covering \( g : S'_N \to S'_N - \phi_N^{-1} f^{-1}(P) \) such that \( f \circ \phi_N \circ g = \phi_N \) and \( g(x^N) = l_N(1) \). If \( N^\rho \subset N \), then \( g \) is extended to a branched covering \( g : S_N \to S_N \).

**Proof.** Let \( [\gamma] \in N \). Then \( [F_x(\gamma)l^{-1}] \in N \), and so \( lF_x(\gamma)l^{-1} \) can be lifted to a closed curve in \( S'_N - \phi_N^{-1} f^{-1}(P) \). This means \( (f \circ \phi_N) \pi_1(S'_N - \phi_N^{-1} f^{-1}(P), l_N(1)) \supset N = \phi_N \circ \pi_1(S'_N, x^N) \). Therefore we have a covering \( g : S'_N \to S'_N - \phi_N^{-1} f^{-1}(P) \) with \( f \circ \phi_N \circ g = \phi_N \) and \( g(x^N) = l_N(1) \). If \( N^\rho \subset N \), we can extend the map \( g \) on \( S_N \). \( \square \)
Remark 12. The covering $g$ above is contractive in the pullback metric $\phi_s^* \nu$, where $\nu$ is the expanding metric for $f$.

Let $r = (l_i)$ be a radial, and $N$ an invariant subgroup with respect to $r$. There exist contractions $g_1, g_2, \ldots, g_d : S_N^r \to S_N$ with $f \circ \phi_N \circ g_i = \phi_N$ and $g_i(x^N) = l_i^N(1)$. Similarly to Proposition 19,

\[(4.1) \quad l_i^N(1) = g_w(x^N).
\]

For simplicity, we assume $N^o \subset N$.

Definition 13. We have a nonempty compact set $K = K^r_N \subset S_N$ with $K = \bigcup_{i=1}^d g_i(K)$, and a coding map $\pi^r_N = \pi^r_N : \Sigma \to K$ with $\pi^N(x) = g_i\pi^N(x)$. We say $K$ is a Julia tile if $\phi_N(K) = J$.

From Proposition 14, we immediately have

\[\pi = \phi_N \circ \pi^N.\]

Proposition 14. Let $r$ be a radial. The maximal invariant subgroup $N_r$ coincides with the normal subgroup

\[N_r' = \{[\gamma] \in G \mid F_{x_w}(\gamma) \text{ is a closed curve for any } w \in W\}.\]

Proof. If $F_{x_w}(\gamma)$ is a closed curve for any $w \in W$, then $F_{x_w}(l_i F_{x_w}(\gamma) l_i^{-1}) = l_i l_{x_w} F_{x_w}(\gamma) l_i^{-1}$ is a closed curve for any $i = 1, 2, \ldots, d$ and any $w \in W$. Thus $N_r'$ is invariant with respect to $r$.

If $[\gamma] \in N_r$, then $F_{x_w}(\gamma)$ is a closed curve and $[l_i F_{x_w}(\gamma) l_i^{-1}] \in N_r$ for $i = 1, 2, \ldots, d$. Inductively, we see that $F_{x_w}(\gamma)$ is a closed curve and $[l_w F_{x_w}(\gamma) l_w^{-1}] \in N_r$ for $w \in W$. Thus $[\gamma] \in N_r'$.

Definition 15. We consider the monodromy actions $\eta_k$ of $G$ on $f^{-k}(\bar{x})$ for $k = 1, 2, \ldots$, that is, $\eta_k([\gamma])(x) = x \cdot [\gamma] = F_{x}(\gamma)(1)$. We set

\[\hat{N} = \bigcap_{k=1}^\infty \ker \eta_k = \{[\gamma] \in G \mid F_{x}(\gamma) \text{ is a closed curve for any } k \geq 1 \text{ and any } x \in f^{-k}(\bar{x})\}.
\]

The quotient group

\[\hat{G} = G/\hat{N}\]

is called the reduced fundamental group of $f$.

From Proposition 14, we immediately have

Proposition 16. If $l \in \Lambda(f, \bar{x})$, then $\hat{N}$ is invariant with respect to $l$. If $r = (l_i)$ is a proper radial, then $\hat{N} = N_r$.

Proposition 17. Let $l, l' \in \Lambda(f, \bar{x})$, and let $N$ be an invariant subgroup with respect to $l$ and $l'$. We have two coverings $g, g' : S_N^l \to S_N - \phi_N^{-1} f^{-1}(P)$ corresponding to $l, l'$ respectively by Proposition 14. Suppose there exists a deck transformation $t : S_N^l \to S_N$ for the covering $\phi_N$ such that $t(z) = z'$ and $g(z) = g'(z')$ for some $z, z' \in S_N$. Then $g' \circ t = g$.

In particular, if $N$ is an invariant normal subgroup, then for the two coverings $g, g' : S_N^l \to S_N - \phi_N^{-1} f^{-1}(P)$ corresponding to $l, l'$, there exists a deck transformation $t$ with $g' \circ t = g$. 

Proof. Write \( y = g(z) = g'(z') \) and \( x = \phi_N(z) = \phi_N(z') \). Consider the induced homomorphisms

\[
\begin{align*}
g_* &: \quad \pi_1(S'_N, z) \rightarrow \pi_1(S'_N - \phi_N^{-1}f^{-1}(P), y), \\
g'_* &: \quad \pi_1(S'_N, z') \rightarrow \pi_1(S'_N - \phi_N^{-1}f^{-1}(P), y), \\
\phi_{N*} &: \quad \pi_1(S'_N, z) \rightarrow \pi_1(S', x), \\
\phi_{N*}' &: \quad \pi_1(S'_N, z') \rightarrow \pi_1(S', x), \\
(f \circ \phi_N)_* &: \quad \pi_1(S'_N - \phi_N^{-1}f^{-1}(P), y) \rightarrow \pi_1(S', x).
\end{align*}
\]

The existence of \( t \) implies \( \phi_{N*}\pi_1(S'_N, z) = \phi_{N*}'\pi_1(S'_N, z') \). By (2.7) and the injectivity of \((f \circ \phi)_* \), we have \( g_*\pi_1(S'_N, z) = g'_*\pi_1(S'_N, z') \). Therefore there exists a unique homeomorphism \( t' : S'_N \rightarrow S_N \) such that \( g' \circ t = g \) and \( t'(z) = z' \). By the uniqueness of \( t \), we have \( t = t' \). \( \square \)

Definition 18. A Borel measure \( \mu \) defined as follows is called the Brolin-Lyubich measure for \( f \). Let

\[
\mu_k = d^{-k} \sum_{y \in f^{-k}(z)} \delta_y,
\]

where \( \delta_y \) is the point mass at \( y \). We define \( \mu \) as the weak* limit of \( \mu_k \), that is, \( \int h \, d\mu = \lim_{k \to \infty} \int h \, d\mu_k \) for any continuous function \( h \). It is known that the limit \( \mu \) exists for any rational map of the Riemann sphere and \( \mu \) is the unique equilibrium state, hence \( f \) is strongly mixing with respect to \( \mu \) (see [15]). In our case, since \( f \) is expanding, the direct proof is not difficult. (We can also construct \( \mu \) via a Markov partition. Cf. [2, 8, 9, 6].)

Proposition 19. The measure \( \mu \) is characterized as the unique Borel probability measure satisfying

\[
\int h(x) \, d\mu(x) = d^{-1} \int \sum_{y \in f^{-1}(x)} h(y) \, d\mu(x)
\]

for any continuous function \( h \).

Proof. The operator \( \mu \mapsto L(\mu) \) defined by

\[
\int h(x) \, dL(\mu)(x) = d^{-1} \int \sum_{y \in f^{-1}(x)} h(y) \, d\mu(x)
\]

is a contraction of the space of Borel probability measures having compact supports in \( S \) with metric \( D(\mu, \mu') = \sup_h |\int h \, d\mu - \int h \, d\mu'| \), where the supremum runs over all functions \( h \) with Lipschitz constant (which is taken with respect to the expanding metric on \( S \)) less than or equal to one. Indeed, if the Lipschitz constant of \( h(x) \) is less than \( \alpha \), then the Lipschitz constant of \( d^{-1} \sum_{y \in f^{-1}(x)} h(y) \) is less than \( \alpha c \), where \( c \) is the contraction constant (see the end of Definition 6). \( \square \)

Remark 20. Lyubich [15] proved that (1.2) holds for any rational map \( f \) and any Borel function \( h(x) \).

Let \( N \) be a subgroup of \( G \). A Borel measure \( \mu_N \) on \( S'_N \) (or \( S_N \)) is defined as the lift of \( \mu \), that is, for a small Borel set \( E \subset S'_N \) for which the restriction \( \phi|E \) is injective, we set \( \mu_N(E) = \mu(\phi(E)) \). We write

\[
J_N = \phi_N^{-1}(J).
\]

It is easily seen that \( \mu \) (or \( \mu_N \)) is supported on \( J \) (or \( J_N \)).
Hence we have

**Proposition 21.**

1. \( \mu \) is an invariant measure (i.e. \( \mu(E) = \mu(f^{-1}(E)) \) for any Borel set \( E \subset S \)).
2. \( f \) is strongly mixing with respect to \( \mu \) (i.e. \( \lim_{k \to \infty} \mu(A \cap f^{-k}(B)) = \mu(A)\mu(B) \) for Borel sets \( A, B \subset S \)).
3. For a Borel set \( E \subset S \), \( d \cdot \mu(E) = \mu(f(E)) \) provided \( f : E \to f(E) \) is injective.
4. Let \( 1 \leq i \leq d \). For a Borel set \( E \subset S_N', \ d \cdot \mu_N(g_i(E)) = \mu_N(E) \) provided \( g_i : E \to g_i(E) \) is injective. In general, \( g_i : E \to g_i(E) \) is at most one-to-one, then \( d \cdot \mu_N(g_i(E)) \leq \mu_N(E) \leq nd \cdot \mu_N(g_i(E)) \).

**Proof.** We need to show only 3. 4 is an immediate consequence of 3.

Suppose \( E \) is so small that there exists an open set \( U \supset E \) with \( f : U \to f(U) \) injective. Then for any continuous function \( h \) with support included in \( U \), we have \( \int h d\mu = d^{-1} \cdot \int h \circ (f|U)^{-1} d\mu \) by Proposition 19. Thus \( \mu(E) = d^{-1} \mu(f(E)) \). In general, divide \( E \) into small parts. \( \square \)

5. **Julia tiling**

In this section, we prove Tiling Theorem and Multiplicity Theorem. Moreover, we give several necessary and sufficient conditions for suitability of a coding map.

Let \( f \) be a subhyperbolic rational map of degree \( d \). Let \( r \) be a radial with basepoint \( \bar{x} \), and \( N \) an invariant normal subgroup with respect to \( r \) such that \( N^{\rho r} \subset N \). Write \( K = K_N^r \).

**Theorem 22.** If \( \pi(\Sigma) = J \), then there exists a subset \( T \) of the group of deck transformations \( G/N \) such that \( \mu_N(t(K) \cap t'(K)) = 0 \) for \( t \neq t' \) and \( \bigcup_{t \in T} t(K) = J_N \).

**Proof.** By 4 of Proposition 21 and \( K = \bigcup_{i=1}^{d} g_i(K) \), we have

\[
\mu_N(K) \leq \sum_{i=1}^{d} \mu_N(g_i(K)) \leq \mu_N(K).
\]

Thus

\[
\mu_N(K) = \sum_{i=1}^{d} \mu_N(g_i(K)) = d \cdot \mu_N(g_i(K)), \quad i = 1, 2, \ldots, d,
\]

(5.1)

\[
\sum_{i \neq j} \mu_N(g_i(K) \cap g_j(K)) = 0, \quad \text{and} \quad \sum_{i=1}^{d} \mu_N(\{x \in K \mid \#(g_i^{-1}(x) \cap K) \geq 2\}) = 0.
\]

(5.2)

There exists a subset \( U \subset K \) open in the relative topology of \( J_N \). Indeed, \( \phi_N(K) = \phi_N(\pi_N(\Sigma)) = \pi(\Sigma) = J \) implies \( J_N = \bigcup_{t \in T} t(K) \). Since \( G/N \) is countable, \( K \) has a nonempty interior in the relative topology by the Baire category theorem. Take \( \omega = \omega_1 \omega_2 \cdots \in \Sigma \) with \( \pi_N(\omega) \in U \). If \( k_0 \) is sufficiently large, we have \( g_w(K) \subset U \) for \( w = \omega_1 \omega_2 \cdots \omega_{k_0} \in W_{k_0} \). Note that \( \pi_N(w^\infty) \in U \). Therefore for any \( x \in J_N \), there exists \( k \geq 0 \) such that \( g_w^k(x) \in U \). Consequently, \( \bigcup_{k=0}^{\infty} g_w^{-k}(K) = J_N \).
For $\mu_N$-almost all $x \in J_N$, there uniquely exists $y_x \in K$ such that $g_u(y_x) = g_w^k(x)$ for some $k \geq 0$ and $u \in W_{kk_0}$. Indeed, let

$$E_k = \bigcup_{u \neq w \in W_k} (g_u(K) \cap g_w(K)) \cup \bigcup_{w \in W_k} \{x \in K \mid \#(g_w^{-1}(x) \cap K) \geq 2\}$$

and $E' = \bigcup_{w \in W_k} g_w^{-1}(\bigcup_{k > 0} E_k)$. Each $g_w : K \to K$ is at most finite-to-one. Thus $\mu_N(E') = 0$ by (4) of Proposition 21 and (5.2). Let $x \in J_N - E'$ and let $k \geq 0$ be the minimal integer such that $g_w(x) \in K$. Since $g_w^k(x) \notin E_{kk_0}$, there exists a unique $u \in W_{kk_0}$ such that $g_u^k(x) \in g_u(K)$ and $\#g_u^{-1}(g_u(x)) \cap K = 1$. If $k' > k$ and $g_u'(y_x) = g_u(y_x)$, then necessarily $u' = w^{k'-k}u$, and so $g_u^{-1}(g_u^k(x)) \cap K = g_u^{-1}(g_u^{k'}(x)) \cap K$.

By Proposition 17 there exists a unique deck transformation $t_x \in G/N$ such that $g_u^k \circ t_x = g_u$ and $t_x(y_x) = x$. Let $T = \{t_x \mid x \in J_N - E'\}$. Then $\bigcup_{t \in T} t(K)$ includes $J_N - E'$. Since $\bigcup_{t \in T} t(K)$ is closed, it is equal to $J_N$. By the uniqueness of $y_x$, we can see that $(t_x(K) \cap t_x(K)) - E' = \emptyset$ whenever $t_x \neq t_x'$.

**Corollary 23.** If $\pi(\Sigma) = J$, then $K$ has nonempty interior in the relative topology of $J_N$.

**Proof.** We have already shown this statement in the beginning of the second paragraph of the proof of Theorem 22. □

**Corollary 24.** If $\pi(\Sigma) = J$, then $d \cdot \mu_N(g_i(A)) = \mu_N(A), 1 \leq i \leq d$ for any Borel set $A \subset K$.

**Proof.** Let $A \subset K$ be a Borel set. Let $E = E_1$ be the set defined in the proof of Theorem 22. Then $g_i : A - g_i^{-1}(E) \to g_i(A) - E$ is one-to-one. Therefore $\mu_N(g_i(A)) = \mu_N(g_i(A) - E) = d^{-1} \mu_N(A - g_i^{-1}(E)) = d^{-1} \mu_N(A)$ by (4) of Proposition 21. □

**Definition 25.** We denote by $m$ the identically distributed Bernoulli measure on $\Sigma$. Namely, $m(\Sigma(w)) = d^{-k}$ for every $w \in W_k$.

**Proposition 26.** If $\pi(\Sigma) = J$, then the conditional measure $\mu_N|K$ is the invariant measure for the iterated function system $\mathcal{I}$, that is,

$$\mu_N(A \cap K) = d^{-1} \sum_{i=1}^{d} \mu_N(g_i^{-1}(A) \cap K)$$

for any Borel set $A$. In particular, we have $\mu_N|K = \pi_{N*}m$ (i.e. $\mu_N(A)/\mu_N(K) = m(\pi_{N*}^{-1}(A))$ for any Borel set $A \subset K$).

**Proof.** (5.3) is an immediate consequence of Corollary 22. It is easy to see that $\pi_{N*}m$ is also the invariant measure for $\mathcal{I}$ with weight $(1/d, 1/d, \ldots, 1/d)$. The uniqueness of the invariant measure (22) implies $\mu_N|K = \pi_{N*}m$. □

**Theorem 27.** The following are equivalent.

1. $\pi(\Sigma) = J$.
2. For distinct words $w, w' \in W$, $[w, l_{w', w}^{-1}]$ is nontrivial, and $\pi(\Sigma) \notin P$.
3. For distinct words $w, w' \in W$, $[w, l_{w', w}^{-1}] \notin N_r$.
4. $\lim_{k \to \infty} \#(g_u(x^{N_k}) \mid w \in W_k) = 1/k$ for $N = \{1\}$, and $\pi(\Sigma) \notin P$.
5. $\lim_{k \to \infty} \#(g_u(x^{N_k}) \mid w \in W_k) = 1/k$ for $N = N_r$.
6. For any $x \in J$, $\pi^{-1}(x)$ is finite.
Proof. The implications $\mathbf{1} \Rightarrow \mathbf{5}$ and $\mathbf{6} \Rightarrow \mathbf{4}$ are trivial.

$\mathbf{2} \Rightarrow \mathbf{1}, \mathbf{3} \Rightarrow \mathbf{2}, \mathbf{4}$ or $\mathbf{6}$ implies $\# \{w(x^N) \mid w \in W_k \} = d^k$. Recall $\mathbf{4.1}$.

$\mathbf{1} \Rightarrow \mathbf{2}$. If $\{l_w^1\} \in N$ for some $w, w' \in W_n$, then $g_w = g_{w'}$. Thus $\# \{g_w(x^N) \mid u \in W_{kn} \} \leq (d^n - 1)^k$.

$\mathbf{5} \Rightarrow \mathbf{1}$. Since $f \pi(\Sigma) = \pi(\Sigma)$, we have $\mu(\pi(\Sigma)) = 1$ by the ergodicity. Thus $\mathbf{1}$ follows from the compactness of $\pi(\Sigma)$.

$\mathbf{6} \Rightarrow \mathbf{3}$. Since $\{l_w\}$ is bounded for $w \in W$, it follows from $\mathbf{3}$ that there exists $n > 0$ such that

\begin{equation}
\# \{w \mid x_w = x_w' \} \leq n
\end{equation}

for any $w \in W$.

Let $V$ be an open set including $\pi(\Sigma)$. Take an open set $V'$ such that $\pi(\Sigma) \subset V'$ and $\forall \pi \subset V$. Then there exists $k_0 > 0$ such that if $k > k_0$, then $x_w \in V'$ for every $w \in W_k$. Hence $d^k/n \leq \# \{x_w \mid w \in W_k \} \cap V' \leq \# f^{-1}(\bar{x}) \cap V'$.

Therefore $1/n \leq \mu(V)$, and so $1/n \leq \mu(\pi(\Sigma))$.

$\mathbf{3} \Rightarrow \mathbf{2}$. Since $\mathbf{3} \Rightarrow \mathbf{1}, \mathbf{3}$ implies $\pi(\Sigma) \not\subseteq P$.

$\mathbf{7} \Rightarrow \mathbf{3}$. Suppose there exist distinct $w, w' \in W$ such that $[l_w l_w^{-1}] \in N_r$. Set $\Omega = \{w \in \Sigma \mid \sigma^k w \in \Sigma(w) \text{ for infinitely many } k\}$.

Then $m(\Omega) = 1$. For any $w = u_1 w_2 u_3 \cdots \in \Omega$, $\pi^{-1}(\pi(\omega))$ includes an uncountable set $\{u_1 w_2 w_3 \cdots \mid u_i = w \text{ or } w'\}$. Indeed, note that if $[l_w l_w^{-1}]$, $[l_w l_w^{-1}] \in N_r$, then $[l_w l_w^{-1}] \in N_r$. Thus $x_{u_1 w_2 w_3 \cdots} = x_{u_1 w_2 w_3 \cdots}$ if $u_i = w$ or $w'$. Hence $\pi(\{u_1 w_2 w_3 \cdots \mid u_i = w \text{ or } w'\}) = \{\pi(w)\}$.

$\mathbf{2} \Rightarrow \mathbf{3}$. Suppose $\pi(\Sigma) \not\subseteq P$ and there exist distinct $w, w' \in W$ such that $[l_w l_w^{-1}] \in N_r$. If we take a large $k_0$, then $F_{x_w}(l_w l_w^{-1})$ is such small for $u \in W_k, k \geq k_0$ that $F_{x_w}(l_w l_w^{-1})$ is either homotopically trivial or winding around some point of $P$ several times in $S'$. Since $\pi(\Sigma) \not\subseteq P$, there exists a word $u$ such that $F_{x_u}(l_w l_w^{-1})$ is homotopically trivial. Then $[l_w l_w^{-1}]$ is trivial.

$\mathbf{6} \Rightarrow \mathbf{3}$. Suppose there exist distinct $w, w' \in W$ such that $[l_w l_w^{-1}] \in N_r$. Then $g_w = g_{w'}$ by $\mathbf{4.1}$. From $\mathbf{5.2}$, we have $\mu_N(g_w(K)) = 0$, and hence $\mu_N(K) = 0$ by $\mathbf{5.1}$. Therefore $\mu(\pi(\Sigma)) = 0$.

$\mathbf{3} \Rightarrow \mathbf{6}$. Let $p \in J$. Since $[l_w]$ is bounded for $w \in \Sigma$, $L(p) = \{w \mid \omega \in \pi^{-1}(p)\}$ is divided into a bounded number of homotopy classes mod $N_r$, that is,

$L(p) = \bigcup_{i=1}^{m(p)} (l_w, l_w^{-1} \in L(p)) \iff \{l_w l_w^{-1} \in N_r \}$

with $m(p) \leq M$ for some $M > 0$ independent of $p$.

Set

$A(p, k) = \{w \in W_k \mid \Sigma(w) \cap \pi^{-1}(p) \neq \emptyset\}$

$= \{w \in W_k \mid \text{there exists } \omega \in \pi^{-1}f^k(p) \text{ such that } \pi(\omega) = p\}$,

$B(p, k, \omega) = \{x_w \mid w \in W_k, \pi(\omega) = p\}$.

Then $\# B(p, k, \omega)$ is equal to or less than $\deg_{\omega, p} f^k$, the local degree of $f^k$ at $p$. Indeed, if $\pi(\omega) = f^k(p)$, then we have exactly $\deg_{\omega, p} f^k$ lifts of $l_w$ by $f^k$ with terminal point $p$. Thus there exists $b > 0$ independent of $p, k$ and $\omega$ such that $\# B(p, k, \omega) < b$. 

(7) $m(\{\omega \in \Sigma \mid \pi^{-1}(\pi(\omega)) \text{ is at most countable}\}) > 0$.

(8) $\mu(\pi(\Sigma)) > 0$.
It is easily seen that $B(p, k, \omega) = B(p, k, \omega')$ whenever $[l_{\omega}l_{\omega'}^{-1}] \in N_r$. Therefore choosing $\omega_i$ so that $l_{\omega_i} \in L(f^k(p))$, we have

\[
\#A(p, k) \leq n \cdot \# \{x \in A(p, k) \}
\]

\[
= n \cdot \# \bigcup_{\omega \in \pi^{-1}f^k(p)} B(p, k, \omega)
\]

\[
= n \cdot \# \bigcup_{i=1}^{m(f^k(p))} B(p, k, \omega_i)
\]

\[
\leq nMb.
\]

for any $k > 0$, where $n$ is defined in (5.5). Hence

\[
\pi^{-1}(p) = \bigcap_{k=1}^{\infty} \bigcup_{w \in A(p, k)} \Sigma(w).
\]

**Theorem 28.** There exists an integer $n \geq 0$ such that for any invariant subgroup $N \subset G$ with respect to $r$,

1. $\#\pi^{-1}(p) = \#\phi_N^{-1}(p) \cap K = n$ for $\mu$-almost all $p \in J$,
2. $\#\pi^{-1}(p) \geq \#\phi_N^{-1}(p) \cap K \geq n$ for any $p \in J - P$, and
3. $\mu_N(\phi_N^{-1}(A) \cap K) = n\mu(A)$ for any Borel set $A \subset \hat{G}$. In particular, $\mu_N(K) = n$.

**Proof.** First remark that $\pi = \phi_N \circ \pi_N$ and $\phi_N = \psi_{N,N_r} \circ \phi_{N_r}$ imply $\#\pi^{-1}(p) \geq \#\phi_N^{-1}(p) \cap K^N \geq \#\phi_N^{-1}(p) \cap K^{N_r}$ for every $p \in J$. Let $E_k \subset S_N$ be the set as in the proof of Theorem 27. Then $D = \bigcup_{k=1}^{\infty} E_k$ is a null set. If $x \in K^{N_r} - D$, then $\#(\pi_N^{-1}(x)) = 1$. Therefore $\#\pi^{-1}(p) = \#\phi_N^{-1}(p) \cap K^{N_r}$ for every $p \in J - \phi_{N_r}(D)$.

If $\pi(\Sigma) \neq J$, then $\mu(\pi(\Sigma)) = 0$ by Theorem 27, so the conditions are satisfied for $n = 0$.

Suppose $\pi(\Sigma) = J$. We show that $h(x) = \#\phi_N^{-1}(x) \cap K$ is a Borel function. To this end, let $A_{k,\epsilon}$ be the set of $x \in J$ such that for some $z_i \in K, i = 1, 2, \ldots, k$, $\phi_N(z_i) = x, 1 \leq i \leq k$ and the distance between $z_i$ and $z_j$ is equal to or bigger than $\epsilon$ for $0 \leq i \neq j \leq k$. Then $A_{k,\epsilon}$ is closed, and so $\{x \mid h(x) \geq k\} = \bigcup_{k=1}^{\infty} A_{k-1,1/n}$ is a Borel set for $k \geq 2$.

For a Borel set $E \subset J$, using $h(x) = d^{-1} \sum_{y \in f^{-1}(x)} h(y)$ and $\int_E h(x) d\mu(x) = \int_{f^{-1}(E)} h \circ f(y) d\mu(y)$ (substitute $h$ of (4.2) by $1_{f^{-1}(E)} \cdot h \circ f$), we have

\[
\int_E h(x) d\mu(x) = \int_{f^{-1}(E)} h(y) d\mu(y).
\]

Hence for any $E$ and any $k > 0$, we have $\int_E h(x) d\mu(x) = \int_{f^{-1}(E)} h(y) d\mu(y)$, which converges to $\mu(E) \int J h(x) d\mu(x)$ as $k \to \infty$ by the strong mixing condition. Therefore $h(x)$ is constant almost everywhere. Thus (1) is verified. By the definition of $\mu_N$, we see that $\mu_N(\phi_N^{-1}(A) \cap K) = n\mu(A)$ for any Borel set $A$.

If $z_i, z_i' \in K, i = 1, 2, \ldots$ satisfy $\lim_{i \to \infty} z_i = \lim_{i \to \infty} z_i' = z$, $z_i \neq z_i'$, and $\phi_N(z_i) = \phi_N(z_i')$, then $\phi_N(z) \in P$. Since $\{x \mid \#\phi_N^{-1}(x) \cap K = n\}$ is dense in $J$, we have $\#\phi_N^{-1}(x) \cap K \geq n$ for every $x \in J - P$. In general, $\#\phi_N^{-1}(x) \cap K \geq \max\{n - \max_{y \in \phi_N^{-1}(x)} \deg_{\phi_N} + 1, 1\}$ for every $x \in J$.

**Definition 29.** We call the number $n$ above the multiplicity of $\pi_r$ and denote by $n_r$ or $\nu(\pi_r)$.

**Corollary 30.** If $\pi(\Sigma) = J$, then $\mu = \pi_r(m)$. 
Definition 32. Let $N \subset G$ be a subgroup. We say that $l$ and $l' \in \Lambda(f, \bar{x})$ are homotopic modulo $N$ with basepoint held fixed if $l(1) = l'(1)$ and $[l, l^{-1}] \in N$. We denote by $\Lambda_N(f, \bar{x})$ the set of homotopy classes modulo $N$ with basepoint $\bar{x}$ held fixed. Then $\Lambda_N(f, \bar{x})$ is identified with $\phi_N^{-1}f^{-1}(\bar{x})$, and with $\{g : S_N' \to S_N' | f o \phi_N o g = \phi_N\}$ as well. The natural projection from $\Lambda(f, \bar{x})$ to $\Lambda_N(f, \bar{x})$ is denoted by $l \mapsto [l]_N$.

We say that two radials $r = (l_i)$ and $r' = (l'_i) \in \text{Rad}(f, \bar{x}) = \Lambda(f, \bar{x})^d$ are homotopic modulo $N$ with basepoint held fixed if $[l_i]_N = [l'_i]_N$ for $i = 1, 2, \ldots, d$. We write $\text{Rad}_N(f, \bar{x}) = \Lambda_N(f, \bar{x})^d$ and $[r]_N = ([l_i]_N)_i$.

Definition 33. We say that two radials $r \in \text{Rad}(f, \bar{x})$ and $r' \in \text{Rad}(f, \bar{x}')$ are freely homotopic if there exists a homotopy $H : Q \times [0, 1] \to S'$ between $r$ and $r'$ such that $H(1, t) : Q \to S'$ is a radial for every $0 \leq t \leq 1$.

Let $Y = \{\gamma : [0, 1] \to S' | \gamma(0) = \bar{x}, \gamma(1) = \bar{x}'\} \subset L$.

We define an operation of $\gamma \in Y$ from $\Lambda(f, \bar{x})$ to $\Lambda(f, \bar{x}')$ by $\gamma \cdot l = \gamma^{-1}F_{\bar{x}}(\gamma)$.

For a subgroup $N \subset G$, the operation $\gamma : \Lambda_N(f, \bar{x}) \to \Lambda_N(f, \bar{x}')$ naturally descends to $\gamma : \Lambda_N(f, \bar{x}) \to \Lambda_{\gamma^{-1}(N)}(f, \bar{x}')$. Suppose $N'$ is invariant with respect to $l$ and $\gamma, \gamma' \in Y$ satisfy $[\gamma \gamma'^{-1}] \in N'$. Then for $N \supset N'$, $\gamma \cdot [l]_N = \gamma' \cdot [l]_N$. In particular, in the case $\bar{x} = \bar{x}'$, the operation $\gamma \cdot l = \gamma^{-1}F_{\bar{x}}(\gamma)$ is well-defined for every $[\gamma]_{\hat{N}} \in \hat{G} = G/\hat{N}$. Identifying $\hat{G}$ with the group of deck transformations of $\phi_N : S_{\hat{N}} \to S$, we write the action $t : \Lambda_N(f, \bar{x}) \to \Lambda_{tN}(f, \bar{x})$ by $t \cdot g_t = tg_t t^{-1}$.
for $t \in \hat{G}$, where we use the identification $\Lambda_{N}(f, \bar{x}) = \{ g : S_{N} \to S_{N} \mid f \circ \phi_{N} \circ g = \phi_{N} \}$. Clearly, the operation of $\gamma \in Y$ from $\text{Rad}(f, \bar{x})$ to $\text{Rad}(f, \bar{x}')$ is defined diagonally. From (6.1)

\[(6.2) \quad [\gamma]_{N} : \{ [r]_{N} \mid r \in \text{Rad}(f, \bar{x}) \} \to \{ [r]_{N} \mid r \in \text{Rad}(f, \bar{x}) \}\]

is well-defined for every $[\gamma]_{N} \in \hat{G}$.

Two radials $r \in \text{Rad}(f, \bar{x})$ and $r' \in \text{Rad}(f, \bar{x}')$ are freely homotopic if and only if there exists $\gamma \in Y$ such that $\gamma \cdot [r]_{e} = [r']_{e'}$, where $e$ and $e'$ are the trivial subgroups. Thus we have a generalization of Definition 33.

**Definition 34.** Let $N \subset G$ be a subgroup. We say that $r$ and $r'$ are **freely homotopic modulo** $N$ if there exists $\gamma \in Y$ such that $\gamma : [r]_{N} = [r']_{\gamma^{-1}(N)}$.

**Theorem 35.** Let $r = (l_{i}) \in \text{Rad}(f, \bar{x})$ and $r' = (l'_{i}) \in \text{Rad}(f, \bar{x}')$. Then $\pi_{r} = \pi_{r'}$ if and only if $r$ and $r'$ are freely homotopic modulo $N_{r}$.

**Proof.** For $r'$, the notation $x_{w}$, $l'_{w}$, $l_{w}$ and $F'_{x}(\cdot)$ are defined in a trivial way.

Suppose $\pi = \pi_{r} = \pi_{r'}$. For $\omega \in \Sigma$, write $\gamma_{\omega} = l_{w}^{-1}l'_{w}^{-1}$. Remark that

\[(6.3) \quad F_{x_{w}}(\gamma_{\omega}) = F_{x_{w}}(l_{w})F'_{x_{w}}(l'_{w})^{-1}\]

for $w \in W$ and $\omega \in \Sigma$ with $\pi(\omega) \notin P$, and

\[(6.4) \quad l_{w}F_{x_{w}}(l_{w})F'_{x_{w}}(l'_{w})^{-1}l'_{w}^{-1} = \gamma_{w}\omega\]

for $w \in W$ and $\omega \in \Sigma$. Since the curve $F_{x_{w}}(l_{w})F'_{x_{w}}(l'_{w})^{-1}$ joins $x_{w}$ and $x_{w}'$, the curve $F_{x_{w}}(l_{w})F'_{x_{w}}(l'_{w})^{-1}F_{x_{w}}(l_{w})F_{x_{w}}(l_{w})^{-1}$ is closed for any $w \in W$ and $\omega, \omega' \in \Sigma$. It follows from this that $[\gamma_{w}\gamma_{\omega}^{-1}] \in N_{r}$ provided $\pi(\omega), \pi(\omega') \notin P$.

First we assume $\pi(\Sigma) \notin P$. By (6.3) and (6.4), $\gamma_{\omega}^{-1}l_{w}F_{x_{w}}(\gamma_{\omega})$ and $\gamma_{\omega}^{-1}l_{w}l'_{w}$ are homotopic in $S'$ with endpoints held fixed for $i = 1, 2, \ldots, d$ whenever $\pi(\omega) \notin P$. Hence, since $[\gamma_{\omega}^{-1}l_{w}] \in \gamma^{-1}N_{r}$, we have $\gamma_{\omega} : [r]_{N} = [r']_{\gamma^{-1}(N_{r})}$. If $\pi(\Sigma) \subset P$, then $\pi(\Sigma)$ consists of one fixed point (see Remark 3), say $p$. Note that $f$ is one-to-one near $p$. Therefore if we modify $\gamma_{\omega}$ near $p$ into $\gamma_{\omega}'$, avoiding $P$, then $F_{x_{w}}(\gamma_{\omega}')$ coincides with $F_{x_{w}}(l_{w})F'_{x_{w}}(l'_{w})^{-1}$ except near $p$. Hence we use the same discussion as above to conclude that $r$ and $r'$ are freely homotopic modulo $N_{r}$.

Suppose $r$ and $r'$ are freely homotopic modulo $N_{r}$. Then there exists $\gamma \in Y$ such that $[\gamma_{w}l_{w}F_{x_{w}}(\gamma_{w})^{-1}l'_{w}] \in N_{r}$ for $i = 1, 2, \ldots, d$. Let $x' \in \phi_{N}^{-1}(\bar{x}')$ be the terminal point of the lift $\gamma_{N}^{r}$ of $\gamma$ to $S_{N}$ whose initial point is $x^{N}$. We have the contractions $g_{l_{1}}, g_{l_{2}}, \ldots, g_{l_{d}} : S_{N} \to S_{N}$ corresponding to $r'$ with respect to the basepoints $\bar{x}'$ and $x'$. Let $l'_{i}$ denote the lift of $l'_{w}$ to $S_{N}$, whose initial point is $x'$, and $l_{N}^{r}, l_{N}^{r'}$ the lift of $lF_{x_{w}}(\gamma)$ to $S_{N}$, whose initial point is $x^{N}$. Then the terminal points of $l_{N}^{r}, l_{N}^{r'}$ are $g_{l}(x^{N}), g_{l}'(x')$ respectively. By assumption, the terminal point of $\gamma_{N}^{r}l_{N}^{r}$ coincides with that of $l_{N}^{r}l_{N}^{r}$. Thus $\gamma_{N}^{r}l_{N}^{r}$ joins $g_{l}(x^{N})$ and $g_{l}'(x')$. Since $\gamma_{0}^{N}l_{N}^{r}$ is a lift of $\gamma$ by $f \circ \phi_{N}$, we have $g_{l}(\gamma_{N}^{r}) = \gamma_{N}^{r}l_{N}^{r} = g_{l}'(\gamma_{N}^{r})$. Hence $g_{l} = g_{l}'$.

Consequently, $\pi_{r}^{N} = \pi_{r'}^{N}$, and so $\pi_{r} = \phi_{N} \circ \pi_{r}^{N} = \phi_{N} \circ \pi_{r'}^{N} = \pi_{r'}$. \qed

**Corollary 36.**

\[\text{Cod}(f) \approx \{ [r]_{N_{r}} \mid r \in \text{Rad}(f, \bar{x}) \}/\hat{G},\]

where the action of $\hat{G}$ is defined in (6.2).
Corollary 37. If $\pi_r = \pi_{r'}$, then $N_r$ and $N_{r'}$ are conjugate (i.e. $N_{r'} = \gamma_r^{-1}(N_r)$ for some $\gamma \in Y$).

Example 38.

(1) $f(z) = z^d$. Since $N_{p/2}$ is trivial, the fundamental group $G_{p/2}$ is equal to $G = \mathbb{Z}$. For any curve $l \in \Lambda(f, \bar{x})$, the maximal invariant subgroup $N_l$ is trivial. Thus $\hat{G} = G_{p/2}$ and $\text{Cod}(f) \approx (\phi^{-1}f^{-1}(\bar{x}))^d/\hat{G} = \mathbb{Z}^d/\mathbb{Z}$, where $n/d \in \phi^{-1}f^{-1}(\bar{x})$ is identified with $n \in \mathbb{Z}$, and $\mathbb{Z}$ acts on $\mathbb{Z}^d$ by

\[ n \cdot (n_1, n_2, \ldots, n_d) = (n_1 + (d-1)n, n_2 + (d-1)n, \ldots, n_d + (d-1)n). \]

Thus

\[ \text{Cod}(f) \approx \{[n_1, n_2, \ldots, n_d] : n_i \in \mathbb{Z}, 0 \leq n_i \leq d - 2 \}, \]

where $[n_1, \ldots]$ denotes the equivalence class including $(n_1, \ldots)$.

(2) $f(z) = z^2 - 2$. The fundamental group $G_{p/2}$ is equal to $\text{Iso}(2\mathbb{Z}) = \{x \mapsto ax + 2n \mid a = \pm 1, n \in \mathbb{Z}\}$. For $l \in \Lambda(f, \bar{x})$, let $l_\infty = \phi(x)$ be the fixed point of $f$ such that $g(x) = x$ for the contraction $g$ with respect to $l$. If $l_\infty = -1$, then $N_l = N_{p/2}$; if $l_\infty = 2$, then $N_l/N_{p/2} = \{\text{id}, x \mapsto -x + 4n\} \subset \text{Iso}(2\mathbb{Z})$ for some $n \in \mathbb{Z}$. Therefore if $\pi_r(\Sigma) \neq \{2\}$, $N_r = N_{p/2}$; otherwise, $N_r/N_{p/2} = \{\text{id}, x \mapsto -x + 4n\}$. In the latter case, the branched covering $\phi_N : S_{N_r} \to S$ is given by $\phi_N(z) = \phi(\sqrt{z}) = 2 \cos 2\pi \sqrt{z}$. We have $\hat{G} = G_{p/2}$.

By Corollary 36,

\[ \text{Cod}(f) \approx \{2\} \cup \{[r]_{N_{p/2}} \in \text{Rad}_{N_{p/2}}(f, \bar{x}) \mid \pi_r(\Sigma) \neq \{2\}\}/\hat{G}, \]

where $2 : \Sigma \to J$ is the coding map with image $\{2\}$. Let us identify $\text{Rad}_{N_{p/2}}(f, \bar{x}) = (\phi^{-1}f^{-1}(\bar{x}))^2$ with $\{\pm 1\} \times \mathbb{Z}$ by $\pm 1/4 + n \leftrightarrow (\pm 1, n)$. The action of $\text{Iso}(2\mathbb{Z})$ on $\{\{\pm 1\} \times \mathbb{Z}\}$ is given by

\[ (x \mapsto ax + 2n) \cdot ((\pm 1, n_1), (\pm 2, n_2)) = ((\pm 1, an_1 + (2 - \epsilon_1)n), (\pm 2, an_2 + (2 - \epsilon_2)n)). \]

It is easily seen that $\pi_r(\Sigma) = \{2\}$ if and only if $[r]_{N_{p/2}} = (b_1, b_2)$, $b_1, b_2 \in \{(1, n), (-1, 3n)\}$ for some $n \in \mathbb{Z}$. Thus

\[ \text{Cod}(f) \approx \{\{1, 0\}\} \cup \{\{(\pm 1, 0), (\pm 2, n) : \epsilon \in \{\pm 1\}, n \in \mathbb{N}\} \cup \{\{(-1, 1), (\epsilon, m) : \epsilon \in \{\pm 1\}, m \in \mathbb{Z}\}. \]

In general, consider $f_d(z) = 2T_d(z/2)$, where $T_d$ is the Chebyshev polynomial of degree $d$ (i.e. $f_d \circ \phi(z) = \phi(\pm dz + 2n)$). Then $f_2(z) = z^2 - 2$. For the basepoint $\bar{x} = 0$, $\phi^{-1}f_{d}^{-1}(\bar{x}) = \{\pm 1/2 + 2n\}/d \subset \mathbb{Z}$. Similarly to the above,

\[ \text{Cod}(f_d) \approx \{2\} \cup \{-2\} \cup \{[r]_{N_{p/2}} \in \text{Rad}_{N_{p/2}}(f_d, \bar{x}) \mid \pi_r(\Sigma) \neq \{2\}, \{-2\}\}/\hat{G}, \]

where the term $\{-2\}$ appears if $d$ is odd. Letting $\pm 1/2 + 2n/d$ correspond to $\pm (1, n) \in \{\pm 1\} \times \mathbb{Z}$, the action of $\hat{G} = \text{Iso}(2\mathbb{Z})$ on $\text{Rad}_{N_{p/2}}(f, \bar{x}) \approx \{\pm 1\} \times \mathbb{Z}$ is given by

\[ (x \mapsto ax + 2n) \cdot ((\epsilon_i, n_i)) = ((\epsilon_i, an_i + (d - \epsilon_i)n)). \]
(3) $f(z) = -(z - 1)^2/4z$. The group $G^\circ$ is equal to
\[ \text{Iso}^+(2\Gamma) = \{ z \mapsto az + 2b \mid a = \pm 1, \pm i, \ b \in \Gamma \}. \]
Similarly to the above, if $\pi_r(\Sigma) \neq \{ \infty \}$, $N_r = N^\circ$; otherwise $N_r/N^\circ = (z \mapsto iz + 2b)$ for some $b \in \Gamma'$. Hence $\hat{G} = G^\circ$. We have
\[ \text{Cod}(f) \approx \{ \infty \} \cup \{ [r]_{N_r} \in \text{Rad}_\Sigma(f, x) \mid \pi_r(\Sigma) \neq \{ \infty \} \}/\hat{G}, \]
where $\infty$ denotes the coding map with image $\{ \infty \}$. Letting $\alpha/2 + (1 + i)\beta$ ($\alpha \in A = \{ \pm 1, \pm i \}, \beta \in \Gamma$) correspond to $(\alpha, \beta) \in A \times \Gamma$, we have the action of $\text{Iso}^+(2\Gamma)$ on $(A \times \Gamma)^2$ by
\[
(z \mapsto az + 2b) \cdot ((\alpha_1, \beta_1), (\alpha_2, \beta_2)) = ((\alpha_1, a\beta_1 + (2 - (1 - i)\alpha_1)b), (\alpha_2, a\beta_2 + (2 - (1 - i)\alpha_2)b)).
\]
Thus
\[
\text{Cod}(f) = \{(1,0), (0,1)\} \cup \{ (\alpha_1, (\alpha_2, \beta)) : \alpha_i \in A, \beta \in \Gamma_+ \}
\cup \{ (\alpha_1, 1 + i) : (\alpha_2, \beta) : \alpha_1 \in \{-1, -i\}, \alpha_2 \in A, \beta \in \Gamma \},
\]
where $\Gamma_+ = \{ n + mi \in \Gamma \mid n > 0, m \geq 0 \}$.

Let $\mathcal{A}(f) = \{ R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid R$ is rational, $\deg R \geq 1, R \circ f = f \circ R \}$. Then $\mathcal{A}(f)$ forms a monoid under composition of maps.

**Proposition 39.** The monoid $\mathcal{A}(f)$ acts on $\text{Cod}(f) = \{ \pi_r \}$ by $\pi \mapsto R \circ \pi$. Moreover, $\text{mul}(R \circ \pi) = \deg R \cdot \text{mul}(\pi)$ for $\pi \in \text{Cod}(f)$.

**Proof.** For $R \in \mathcal{A}(f)$, $R(J) = J = R^{-1}(J)$. Indeed, if $x$ belongs to the Fatou set of $f$, then so does $R(x)$ (see for example [1], Theorem 4.2.9). Thus $J_f = J_R$ if $\deg R \geq 2$. If $\deg R = 1$, we have $J \subset R(J)$, so $R^{-1}(J) \subset J$. On the other hand, $R(J) \subset J \subset R^{-1}(J)$ since $R^{-1} \in \mathcal{A}(f)$.

Since $R : J \to J$ is $(\deg R)$-to-one except for finite points, $\text{mul}(R \circ \pi) = \deg R \cdot \text{mul}(\pi)$. For a radial $r$, we can assume that $r$ does not intersect $R^{-1}(P_f)$. It is easily seen that $R \circ \pi_r = \pi_{R \circ r}$. Thus $R \circ \pi_r \in \text{Cod}(f)$. \hfill \Box

**Proposition 40.** Every $R \in \mathcal{A}(f)$ of degree $D = \deg R$ bigger than one is a subhyperbolic rational map with $P_f = P_R$ and $\rho_f = \rho_R$. Moreover, exceptional points of $f$ coincide with those of $R$.

**Proof.** Recall that $p$ is called an exceptional point of $f$ if $\# \bigcup_{k=0}^\infty f^{-k}(p) < \infty$. In fact, then $p$ is a critical point of degree $d = \deg f$, $f^2(p) = p$, $\bigcup_{k=0}^\infty f^{-k}(p) = \{ p, f(p) \}$, and $f(p)$ is also exceptional.

First we show that if $x = R^k(y)$ and $y$ is not an exceptional point of $f$, then there exist $m > 0$ and $z \in f^{-m}(x)$ such that $\deg_x(R^k)$ is a divisor of $\deg_x(f^m)$. Take $m > 0$ so that $\# f^{-m}(y) > 2D^k - 2$. Since $\# C_{R^k} \leq 2D^k - 2$, there exists $w \in f^{-m}(y)$ such that $\deg_w(R^k) = 1$. Setting $z = R^k(w)$, we have $\deg_w(R^k) \deg_w(f^m) = \deg_w(R^k f^m) = \deg_w(f^m R^k) = \deg_w(f^m) \deg_w(R^k) = \deg_w(f^m)$.

Second we show that if $y$ is an exceptional point of $f$, then $y$ is an exceptional point of $R$. Suppose $y$ is an exceptional point of $f$. By commutativity, $R(y), R^2(y), \ldots$ are exceptional points of $f$. Since a rational map has at most two exceptional points, $R^2(y) = R(y)$ and $f^{-2}(R(y)) = \{ R(y) \}$. Assume that $R(y)$ is not an exceptional point of $R$. Then $X_k = R^{-2k}(R(y)) - \{ R(y) \}$ is nonempty for $k \in \mathbb{N}$. From $R^{2k} \circ f^2(X_k) = R(y)$, we have $f^2(X_k) \subset X_k$. Hence $X_k$ contains a
periodic cycle of \( f \). This contradicts the fact that the Fatou set includes at most finite periodic cycles.

From the claims above, we deduce that \( P_f = P_R \) and \( \rho_f = \rho_R \). \( \square \)

**Corollary 41.** If \( f \) has exactly one critical point \( c \) with \( \# \{ f^k(c) \mid k \in \mathbb{N} \} = \infty \), then \( \mathcal{A}(f) \) is generated by \( f \).

**Proof.** Let \( R \in \mathcal{A}(f) \). We show that \( R \circ f(c) = f^k(c) \) for some \( k \in \mathbb{N} \). To this end, we assume that \( R \circ f(c) \notin \{ f^k(c) \mid k \in \mathbb{N} \} \). Since \( f(c) \in P, R \circ f(c) \in P \). Thus \( R \circ f(c) \) is eventually periodic under the iteration of \( f \). Therefore there exist \( m, n \in \mathbb{N} \) such that \( f^m \circ R \circ f(c) = f^{m+kn} \circ R \circ f(c) = R \circ f^{m+kn+1}(c) \) for every \( k \in \mathbb{N} \). This implies a contradiction that \( \# R^{-1}(f^m \circ R \circ f(c)) = \infty \).

From \( R \circ f(c) = f^k(c) \), we have \( R(f^n(c)) = f^{n+k-1}(c) \). Hence \( R = f^{k-1} \) by the identity theorem. \( \square \)

**Corollary 42.** Let \( \rho = \rho_f \) be the canonical ramification function, and \( g : S_{N_f} \to S_{N_f} \) a contraction with \( f \circ \phi_{N_f} \circ g = \phi_{N_f} \). Then \( \mathcal{A}(f) \) is the set of rational maps \( R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that there exists a branched covering \( g' : S_{N_f} \to S_{N_f} \) with \( R \circ \phi_{N_f} \circ g' = \phi_{N_f} \) and \( g \circ g' = g' \circ g \circ t \) for some deck transformation \( t \).

**Proof.** If \( R \in \mathcal{A}(f) \), then there exists a branched covering \( g' : S_{N_f} \to S_{N_f} \) with \( R \circ \phi_{N_f} \circ g' = \phi_{N_f} \) by Proposition 40. From \( f \circ R \circ \phi_{N_f} \circ g' = \phi_{N_f} \) and \( f \circ R \circ \phi_{N_f} \circ g \circ g' = R \circ f \circ \phi_{N_f} \circ g \circ g' = \phi_{N_f} \), it follows that \( g \circ g' = g' \circ g \circ t \) for some \( t \) by Proposition 47.

Conversely, suppose there exists a branched covering \( g' : S_{N_f} \to S_{N_f} \) with \( R \circ \phi_{N_f} \circ g = \phi_{N_f} \) and \( g \circ g' = g' \circ g \circ t \) for some deck transformation \( t \). Then \( R \circ f \circ \phi_{N_f} \circ g \circ g' = \phi_{N_f} \) and \( f \circ R \circ \phi_{N_f} \circ g' \circ g \circ t = \phi_{N_f} \). Hence

\[
R \circ f(x) = f \circ f \circ \phi_{N_f} \circ g \circ g'((\phi_{N_f} \circ g \circ g')^{-1}(x)) = \phi_{N_f}((\phi_{N_f} \circ g' \circ g \circ t)^{-1}(x)) = f \circ R(x)
\]

for any \( x \in S \). \( \square \)

**Definition 43.** A coding map \( \pi \in \text{Cod}(f) \) is said to be prime if there are no \( R \in \mathcal{A}(f) \) and no \( \pi' \in \text{Cod}(f) \) such that \( \pi = R \circ \pi' \) and \( \text{deg} \, R \geq 2 \). A radial \( r \) is said to be prime if \( \pi_r \) is prime.

We write

\[
\text{Cod}'(f) = \{ \pi \in \text{Cod}(f) \mid \pi \text{ is prime} \} \approx \text{Cod}(f) / \mathcal{A}(f)
\]

\[
\mathcal{M} = \{ \text{mul}(\pi) \mid \pi \in \text{Cod}(f), \, \mathcal{M}' \}
\]

**Corollary 44.** (1) \( 0 \in \mathcal{M}' \), (2) \( \mathcal{M}' \setminus \{ 0 \} \neq \emptyset \), (3) \( \mathcal{M} = \{ \text{deg} \, R \cdot n \mid R \in \mathcal{A}(f), \, n \in \mathcal{M} \} \). In particular, \( \mathcal{M} \supset \{ d^k \cdot n \mid n \in \mathcal{M}', k = 0, 1, 2, \ldots \} \) since \( f^k \in \mathcal{A}(f) \).

**Example 45.** We calculate \( \mathcal{M}' \), \( \mathcal{M} \) and \( \text{Cod}'(f) \) for a couple of examples in Section 4.

\[
(1) \, f(z) = z^d. \quad \text{The monoid } \mathcal{A}(f) = \{ e^{2\pi im/d}z^k \mid k \in \mathbb{Z}, m = 0, 1, \ldots, d-1 \} \text{ is identified with } \{(m, k) \mid \text{the product is given by } (m, k)(m', k') = (m + km', kk') \}. \text{ The action of } \mathcal{A}(f) \text{ on } \text{Cod}(f) = \mathbb{Z}^d / \mathbb{Z} \text{ is given by}
\]

\[
(m, k) \cdot [n_1, n_2, \ldots, n_d] = [kn_1 + m, kn_2 + m, \ldots, kn_d + m].
\]

Hence

\[
\text{Cod}(f) / \mathcal{A}(f) \approx \{ [0, n_2, \ldots, n_d] \mid n_2, \ldots, n_d \text{ are mutually prime} \}.
\]
Since $1 \in M'$, we have $M = \{0\} \cup \mathbb{N}$.

In the case $d = 2$, $\text{Cod}(f)$ is identified with $\mathbb{Z}$ by $n_2 - n_1 \leftrightarrow [n_1, n_2]$, and $\text{mul}([n_1, n_2]) = n_2 - n_1$. We can see that $\text{Cod}'(f)$ is identified with $\{[0, 0], [0, 1], [0, -1]\}$. Thus $M' = \{0, 1\}$.

In the case $d = 3$, $\text{Cod}'(f) \approx \{[0, n_2, n_3] | n_2, n_3 \text{ are mutually prime}\}$ is infinite, and $M' = \{0, 1\}$.

In the case $d = 4$, $M' \supset \{0, 1, 2\}$ (see [3], Example 3.1).

(2) $f(z) = z^2 - 2$. Then $\text{Cod}(f) \approx (\{+, -\} \times \mathbb{Z})/\text{Iso}(\mathbb{Z})$ and $A(f) = \{f_d | d = 1, 2, \ldots\} \approx \mathbb{N}$, where $f_d(z) = z^2 - 2$. The action of $A(f)$ on $\text{Cod}(f)$ is given by $n \cdot [\{n_1, n_2\}, (c_1, c_2)] = [(c_1, n c_1), (c_2, n c_2)]$. Hence

$\text{Cod}'(f) \approx \{[+, 0], (0, 0), [0, 1], (0, 0), (0, 0), (0, 0), 0\} \cup \{([0, 0], (e_2, 1)) : e_1 \in \{+, -\} \cup \{(-, 1), (-, 1, 1), (0, 0), 0\} \}

Thus $M = \{0, 1, 2, 3, 4\} | n = 0, 1, \ldots\} \cup \{0\} \cup \mathbb{N}$.

(3) $f(z) = -(z - 1)^2/2$. The monoid $A(f) = \{R_b | b \in \Gamma\} \approx \Gamma$, where $R_b$ is the rational map with $R_b(z) = b$. The action of $A(f)$ on $\text{Cod}(f)$ is given by $b \cdot [(a, b, \beta), (a, b, \beta)] = [(a, b, \beta), (a, b, \beta)]$. Hence

$\text{Cod}'(f) \approx \{[0, 1], (0, 0)\} \cup \{[0, 1, (0, 1, 1 + i)], (1, 1), (0, 1, 1 + i)\} \cup \{([0, 1, 1), (1, 0, 1)) : (a, 0, 1) \in A^2 - \{(-1, 1), (0, 1, 1)\}\}

Thus $M' = \{0, 1, 4, 5\} | n = 0, 1, \ldots\} \cup \{0\} \cup \mathbb{N}$.

(4) $f(z) = z^2 - 3$. Then $A(f)$ is generated by $f$. In this case, the complete solution has not been obtained. We will show that $1, 2 \in M'$ in Example 49 (we conjecture $M' = \{1, 2\}$). Consequently, $0, 2^k \in M, k \in \mathbb{N}$.

We pose the following problem:

**Problem.** For a given $f$, determine $M'$ and $M$.

It is easily seen that $M = \{0\} \cup \mathbb{N}$ if $f$ is postcritically finite polynomial map. In general case, however, it is unknown whether $1 \in M$ or not.

### 7. Equivalence relations on the word space

Let $r = (l_i)$ and $r' = (l'_i)$ be radial words with basepoints $\bar{x}$ and $\bar{x}'$ respectively, and $N \subset G$ a subgroup. For $r'$, the notation $x_w', l'_w, l'_\omega$ and $F'_\omega(\cdot)$ are defined in a trivial way. Let $M$ be a real number bigger than $\sup_{\omega \in \Sigma} |l_w| + \sup_{\omega \in \Sigma} |l'_w|$, and let

$Y' = \{\gamma \in Y : |\gamma| < M\}, \quad Y'_N = Y'/\sim_N,$

where $\gamma \sim_N \gamma'$ if $[\gamma \gamma^{-1}] \in N$. It is clear that $Y'_N$ is a finite set if $N^{\rho_i} \subset N$. The equivalence class of $\gamma$ is denoted by $[\gamma]_N$.

**Proposition 46.** For $\omega = \omega_1 \omega_2 \cdots, \omega' = \omega_1 \omega'_2 \cdots \in \Sigma$, $\pi_r(\omega) = \pi_r(\omega')$ if and only if there exist curves $\gamma_0, \gamma_1, \ldots \in Y'$ such that

\[
[l_{\omega_k} F_{x_{\omega_k}}(\gamma_k)]^{l_{\omega_k}}_{\omega_k}^{-1} [\gamma_k^{-1}] \text{ is trivial}
\]
for \( k = 1, 2, \ldots \). Moreover, \( \text{(6.1)} \) can be replaced with

\[
[\omega_k F_{x_{\omega_k}}(\gamma_k)l_{\omega_k}^{-1}l_{\gamma_{k-1}}^{-1}] \in N_r.
\]

**Proof.** Suppose \( p = \pi_r(\omega) = \pi_r(\omega') \). We take \( \gamma_k = l_{\sigma_k}\omega_l\omega_{k-1}^{-1} \). Then \( F_{x_{\omega_k}}(\gamma_k) = F_{x_{\omega_k}}(l_{\sigma_k}\omega_l\omega_{k-1}^{-1}) \). Thus if \( p \notin P \), then \( \gamma_0, \gamma_1, \ldots \) satisfy the condition above. If \( p \in P \), then modify each \( \gamma_k \) in a small neighborhood of \( f^k(p) \) so that \( \text{(6.1)} \) holds.

Conversely, suppose there exist curves \( \gamma_0, \gamma_1, \ldots \) satisfying the condition \( \text{(6.2)} \). We write \( (\omega)_k = \omega_1\omega_2 \cdots \omega_k \). Taking the product of

\[
[l_{(\omega)_k} F_{x_{(\omega)_k}}(\gamma_k)l_{(\omega)_k}^{-1}l_{(\omega)_k}^{-1} F_{x_{(\omega)_k}}(\gamma_{k-1})^{-1}l_{(\omega)_k}^{-1}l_{(\omega)_k}^{-1}] = [l_{(\omega)_k} F_{x_{(\omega)_k}}(\gamma_k)l_{(\omega)_k}^{-1}l_{(\omega)_k}^{-1}l_{(\omega)_k}^{-1}] \in N_r
\]

from \( k = n \) to 1, we have \( [l_{(\omega)_n} F_{x_{(\omega)_n}}(\gamma_n)l_{(\omega)_n}^{-1}l_{(\omega)_n}^{-1}] \in N_r \). Thus the distance between \( x_{(\omega)_n} \) and \( x'_{(\omega)_n} \) is less than \( |F_{x_{(\omega)_n}}(\gamma_n)| \leq e^{-n}M \). \( \square \)

**Theorem 47.** There exists a weighted directed graph \((V, E, \alpha)\):

- The vertex set \( V \) is finite.
- The edge set \( E \) is finite. Each edge \( e \in E \) has its initial vertex \( e^- \in V \) and its terminal vertex \( e^+ \in V \). (We do not assume that \( e^-_k = e^+_1 \) and \( e^-_0 = e^+_1 \) imply \( e_0 = e_1 \).)
- The weight function \( \alpha : E \to \{1, 2, \ldots, d\}^2 \), such that for \( \omega = \omega_1\omega_2 \cdots \omega_k \in \Sigma \),

\[
\pi_r(\omega) = \pi_r'(\omega') \iff \text{there exist } e_1, e_2, \ldots \in E \text{ such that } e_i^+ = e_{i+1}^- \text{ and } \alpha(e_i) = (\omega_i, \omega_i')
\]

**Proof.** Let \( V \) be the maximal subset of \( Y'_N \) such that for any \( [\gamma]_N \in Y'_N \), there exists \( [\gamma']_N \in V \) with \( [\gamma]_N = [l_{i} F_{x_{i}}(\gamma')l_{i-1}^{-1}]_N \) for some \( i \in \{1, 2, \ldots, d\} \). Set

\[
E = \{(\gamma]_N, [\gamma']_N, i, j) \in V^2 \times \{1, 2, \ldots, d\}^2 \mid [\gamma]_N = [l_i F_{x_i}(\gamma')l_{i-1}^{-1}]_N \}
\]

For \( e = ([\gamma]_N, [\gamma']_N, i, j) \in E \), define \( e^- = [\gamma]_N, e^+ = [\gamma']_N \), and \( \alpha(e) = (i, j) \).

Suppose there exist \( e_1, e_2, \ldots \in E \) such that \( e_i^+ = e_{i+1}^- \) and \( \alpha(e_i) = (\omega_i, \omega_i') \). Since there exist curves \( \gamma_k, k = 0, 1, \ldots \) such that \( [\gamma_k]_N = e_{k+1}^- \), we have \( \pi_r(\omega) = \pi_r'(\omega') \) by proposition 46. Conversely, suppose \( \pi_r(\omega) = \pi_r'(\omega') \). By Proposition 46 there exists \( [\gamma_0]_N, [\gamma_1]_N, \ldots \in Y' \) such that \( [l_{\omega_k} F_{x_{\omega_k}}(\gamma_k)l_{\omega_k}^{-1}]_N = [\gamma_{k-1}]_N \) for \( k = 1, 2, \ldots \). Then \( [\gamma_k]_N \in V \), and so \( e_k = ([\gamma_{k-1}]_N, [\gamma_k]_N, \omega_k, \omega_k') \), \( k = 1, 2, \ldots \) satisfy the condition. \( \square \)

**Corollary 48.** Let \( V \) be the vertex set constructed in Theorem 47. If \( \pi(\Sigma) = J \), then

\[
\#\pi^{-1}(x) \leq \max_w \#\{w' \mid x_w = x_{w'}\} \cdot \max_{p \in C \cap U, k \geq 1} \deg_p j^k \cdot \sqrt{2\#V}
\]

for any \( x \in J \).

**Proof.** An immediate consequence of \( \text{(5.3)} \). \( \square \)

It is easily seen that if \( \pi(\Sigma) = J \), then the Julia set \( J \) is topologically identified with the quotient space \( \Sigma/\sim_r \), where the equivalence relation \( \sim_r \) is defined by \( \omega \sim_r \omega' \iff \pi_r(\omega) = \pi_r(\omega') \). Considering \( r = r' \), we obtain from Theorem 47 an algorithm to calculate \( \sim_r \) provided \( Y' \) is determined.
Example 49. Let \( f(z) = z^2 - 3 \). In order to obtain \( \sim_r \) for the radials \( r = r_1, r_2, r_3 \) in Section 3.4, we use Theorem 17. Let us set generators \([B_1],[B_2],\ldots \) of \( G \) as shown in Figure 24. Then \([B^2_i],[B_iB^2_i] \in \hat{N}, k = 1, 2, \ldots \) Take a simply connected open domains \( U_1 \subset U_2 \) such that \( B_1 \subset U_1, B_2 \subset U_2, [-3,3] \subset U_1, [-6,6] \subset U_2, C \subset C - U_1, [33, \infty) \subset C - U_2 \), and \( f^{-1}(U_1) \subset U_i \). For the radial \( r = r_j \) above, we take \( r \) so that the image of \( r \) is included in \( U_i \) \((i = 1 \text{ if } j = 1, 2, i = 2 \text{ if } j = 3)\) without loss of generality. For any \( k \), there exists \( n \) such that \( F_2(B^k_i) \subset U_2 \) for each \( x \in f^{-n}(\hat{x}) \). Therefore, for \( r = r_1, r_2 \), the set \( V \) in Theorem 17 is included \( \{1_{\hat{N}}, [B_1]_{\hat{N}} \} \), where \( 1_{\hat{N}} = \hat{N} \) is the unit element. If \( r = r_3 \), \( V \) is included in

\[
\{1_{\hat{N}}, [B_1]_{\hat{N}}, [B_2]_{\hat{N}}, [B_1B_2]_{\hat{N}}, [B_1B_2B_1]_{\hat{N}}, [B_2B_1B_2]_{\hat{N}},
\[(B_1B_2)^2]_{\hat{N}}, (B_2B_1)^2]_{\hat{N}} \},
\]

the subgroup generated by \([B_1]_{\hat{N}} \) and \([B_2]_{\hat{N}} \).

(1) \( r = r_1 \). We have

\[
\begin{align*}
e & \leftarrow (11) e, \quad (22) e \\
B_1 & \leftarrow (12) e, \quad (21) e
\end{align*}
\]

where \( e \) denotes the trivial loop. (For example, \( "B_1 \leftarrow (11) e, (22) e" \) indicates that \( l_1F_{x_1}(B_1)l_2^{-1} \) and \( l_2F_{x_2}(B_1)l_1^{-1} \) are trivial.) Thus

\[
V = \{1_{\hat{N}}\}, \quad E = \{(1_{\hat{N}},1_{\hat{N}},1,1), (1_{\hat{N}},1_{\hat{N}},2,2)\}.
\]

It follows from this that the equivalence relation \( \sim_r \) is trivial, that is, \( \omega \sim_r \omega' \) if and only if \( \omega = \omega' \). Consequently, \( \pi_r \) is bijective.

(2) \( r = r_2 \). We have

\[
\begin{align*}
e & \leftarrow (11) e, \quad (22) e \\
B_1 & \leftarrow (12) \ B_1^{-1}, \quad (21) B_1
\end{align*}
\]

("(11) \ B_1^{-1}, (22) B_1" indicates that \( l_1F_{x_1}(B_1)l_2^{-1} \) is homotopic to \( B_1^{-1} \) and \( l_2F_{x_2}(B_1)l_1^{-1} \) is homotopic to \( B_1 \).) Thus

\[
V = \{1_{\hat{N}}, [B_1]_{\hat{N}}\}, \\
E = \{(1_{\hat{N}},1_{\hat{N}},1,1), (1_{\hat{N}},1_{\hat{N}},2,2), ([B_1]_{\hat{N}}, [B_1]_{\hat{N}},1,2), ([B_1]_{\hat{N}}, [B_1]_{\hat{N}},1,2)\}.
\]

It follows from this that

\[
\omega = \omega_1\omega_2\cdots \sim_r \omega' = \omega_1'\omega_2'\cdots \iff \omega = \omega' \text{ or } \omega_k \neq \omega'_k, k = 1, 2, \ldots
\]

(for example, \( 111 \cdots \sim_r 222 \cdots \) and \( 1212 \cdots \sim_r 2121 \cdots \)). Consequently, \( \pi_r \) is exactly two-to-one.

(3) \( r = r_3 \). We have

\[
\begin{align*}
e & \leftarrow (11) e, \quad (22) e \\
B_1 & \leftarrow (12) B_2, \quad (21) B_2^{-1} \\
B_2 & \leftarrow (11) e, \quad (22) B_2^{-1}B_1B_2 \\
B_1B_2 & \leftarrow (12) B_1B_2, \quad (21) B_1B_2^{-1} \\
B_2B_1B_2 & \leftarrow (12) B_2B_1B_2, \quad (21) B_2B_1B_2^{-1}
\end{align*}
\]
etc. Thus
\[
V = \{1, [B_2]_N, [B_1, B_2]_N, [B_2 B_1]_N, [B_2 B_1 B_2]_N\},
\]
\[
E = \{(1, 1, 1, 1, 1, 1), (1, 1, 1, 2, 2),
(1, [B_2]_N, [B_2]_N, 1), ([B_2 B_1]_N, [B_2 B_1]_N, 1, 2),
([B_2 B_1]_N, [B_2 B_1]_N, 2, 1), ([B_2 B_1]_N, [B_2 B_1]_N, 2, 1),
([B_2 B_1]_N, [B_2 B_1]_N, 1, 2), ([B_2 B_1]_N, [B_2 B_1]_N, 2, 1),
([B_2 B_1]_N, [B_2 B_1]_N, 2, 1), ([B_2 B_1]_N, [B_2 B_1]_N, 2, 2)\}.
\]

By Corollary 45, \(\pi_r\) is at most three-to-one. The graph \((V, E, \alpha)\) is diagrammatically shown as

\[
\begin{array}{c}
11, 22 \circlearrowleft 1 \rightarrow 11 \quad B_2 \downarrow 21 \equiv 12 \quad B_2 B_1 \circlearrowright 21 \\
12 \circlearrowleft B_1 B_2 \downarrow 22 \equiv 12 \quad B_2 B_1 B_2
\end{array}
\]

We can see that if \(\omega \in \Sigma\) contains a word 12121 infinitely many times, then \(\pi_r^{-1} \Pi_r(\omega) = \{\omega\}\). Consequently, the multiplicity of \(\pi_r\) is equal to one.

![Figure 5. Generators of G.](image)

8. Non-subhyperbolic case

A couple of our results are true for non-subhyperbolic rational maps. In fact, if \(f\) is geometrically finite (i.e. \(J \cap P\) is finite), then almost all of our results are applicable, but it is possible that \(J \not\subset S\). Thus the Julia tile \(K\) might be noncompact. The details are left to the reader.

For general rational maps, we need some restriction. Let \(f : \hat{C} \rightarrow \hat{C}\) be a rational map. If \(S' = \hat{C} - P\) has a connected component \(U\) such that \(f^{-1}(U) \subset U\), then we have a radial \(r\) in \(U\), but \(x_\omega\) does not converge to a point in general. We define

\[
\Pi(\omega) = \bigcup_{k=1}^{\infty} F_{x_{\omega_1 \omega_2 \cdots \omega_k}}(U)
\]

for \(\omega = \omega_1 \omega_2 \cdots \in \Sigma\). Then we have

**Proposition 50.**  (1) \(f(\Pi(\omega)) = \Pi(\sigma \omega)\), (2) \(\Pi(\omega) \subset J\), (3) \(\omega \mapsto \Pi(\omega)\) is upper semicontinuous, that is, if a sequence \(\omega^1, \omega^2, \ldots\) in \(\Sigma\) converges to \(\omega\), then \(\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \Pi(\omega^k) \subset \Pi(\omega)\) and (4) \(\Pi(\omega)\) is connected.
Proof. (1) is immediately deduced from the definition.

(2): Suppose a point \( x \in \Pi(\omega) \) belongs to the Fatou set. It is easily seen that \( f^n(x) \) converges to neither a (super)attracting cycle nor a parabolic cycle. Thus \( f^n(x) \in \Pi(\sigma^n\omega) \) for some \( n \) is contained in either a Siegel disc or a Herman ring. This contradicts the fact that \( f : f^{-1}(U) \to U \) is expanding in the Poincaré metric on \( U \).

(3): Set \( \omega = \omega_1 \omega_2 \cdots \) and \( \omega^k = \omega_1^k \omega_2^k \cdots \). For any \( n \), there exists \( m_0 \) such that \( \omega^n \in \Sigma(\omega_1 \omega_2 \cdots \omega_n) \) whenever \( m \geq m_0 \). Since
\[
\Pi(\omega^n) \subset \bigcup_{\omega' \in \Sigma} F_{x_1 \omega_2 \cdots \omega_n} (l_{\omega'}) = \bigcup_{\omega' \in \Sigma} F_{x_1 \omega_2 \cdots \omega_n} (l_{\omega'})
\]
for \( m \geq m_0 \), we have \( \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \Pi(\omega^k) \subset \bigcup_{\omega' \in \Sigma} F_{x_1 \omega_2 \cdots \omega_n} (l_{\omega'}) \) for every \( n \).

(4): \( \Pi(\omega) \) is connected since \( \bigcup_{\omega' \in \Sigma} F_{x_1 \omega_2 \cdots \omega_n} (l_{\omega'}) \) is connected.

From (2) and (4) immediately,

Corollary 51. If \( J \) is totally disconnected, then \( \Pi(\omega) \) is a singleton for every \( \omega \in \Sigma \).

Remark 52. In general, by Przytycki’s result [20], \( \Pi(\omega) \) is a singleton for every \( \omega \in \Sigma \) except for \( \omega \) in a “thin” set.

Theorem 53. The multiplicity of \( \Pi \) is well-defined, that is, there exists \( n_\tau \in \mathbb{N} \cup \{ \infty \} \) such that \( \#\{ \omega \in \Sigma \mid x \in \Pi(\omega) \} = n_\tau \) for \( \mu \)-almost all \( x \in J \).

Proof. It is sufficient to show that the function \( h(x) = \#\{ \omega \in \Sigma \mid x \in \Pi(\omega) \} \) is Borel. Let \( A_{k,\epsilon} \) be the set of \( x \in J \) such that for some \( \omega^i \in K, i = 1,2,\ldots,k \), \( x \in \Pi(\omega^i), 1 \leq i \leq k \) and the distance between \( \omega^i \) and \( \omega^j \) is equal to or bigger than \( \epsilon \) for \( 0 \leq i \neq j \leq k \), where we consider an arbitrary compatible distance function on \( \Sigma \). Then \( A_{k,\epsilon} \) is closed since \( \omega \mapsto \Pi(\omega) \) is upper semicontinuous. Thus \( \{ h(x) \geq k \} \) is Borel.

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