Modal Operators for Coequations

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Outline

I. The co-Birkhoff Theorem
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II. Deductive completeness
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III. The □ operator
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The Birkhoff variety theorem

Let $\mathbb{P} : \text{Set} \to \text{Set}$ be a polynomial functor, and $X$ an infinite set of variables.

Theorem (Birkhoff’s variety theorem (1935)). A full subcategory $\mathcal{V}$ of $\text{Set}^{\mathbb{P}}$ is closed under

- products,
- subalgebras and
- quotients (codomains of regular epis)

just in case $\mathcal{V}$ is definable by a set of equations $E$ over $X$, i.e.,

$$\mathcal{V} = \{ \langle A, \alpha \rangle \mid \langle A, \alpha \rangle \models E \}.$$
The covariety theorem

Let $\Gamma : \mathcal{E} \rightarrow \mathcal{E}$ be a functor bounded by $C \in \mathcal{E}$.

**Theorem.** A full subcategory $\mathcal{V}$ of $\mathcal{E}_\Gamma$ is closed under

- coproducts,
- images (codomains of epis) and
- (regular) subcoalgebras

just in case $\mathcal{V}$ is definable by a coequation $\varphi$ over $C$, i.e.,

$$\mathcal{V} = \{ \langle A, \alpha \rangle \mid \langle A, \alpha \rangle \models \varphi \}.$$
Coequations

A coequation over $C$ is a subobject of $UHC$, the cofree coalgebra over $C$. 
Coequations

A **coequation** over \( C \) is a subobject of \( UHC \), the cofree coalgebra over \( C \).

A coalgebra \( \langle A, \alpha \rangle \) satisfies \( \varphi \) just in case, for every homomorphism

\[
p: \langle A, \alpha \rangle \longrightarrow HC,
\]

the image of \( p \) is contained in \( \varphi \) (i.e., \( \text{Im}(p) \leq \varphi \)).

\[
U \langle A, \alpha \rangle \longrightarrow UHC
\]
Example

The cofree coalgebra $H^2$
Example

A coequation.
Example

This coalgebra satisfies $\varphi$. 
Example

Under any coloring, the elements of the coalgebra map to elements of $\varphi$. 
Example

This coalgebra doesn’t satisfy $\varphi$. 
Example

If we paint the circle red, it isn’t mapped to an element of $\mathcal{F}$. 
Coequations as predicates

Since a coequation $\varphi$ over $C$ is just a subobject of $UHC$, a coequation can be viewed as a **predicate** over $UHC$. 
Coequations as predicates

Since a coequation \( \varphi \) over \( C \) is just a subobject of \( UHC \), a coequation can be viewed as a predicate over \( UHC \). Hence, the coequations over \( C \) come with a natural structure. We can build new coequations out of old via \( \land \), \( \neg \), \( \forall \), etc.
Coequations as predicates

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$$\langle A, \alpha \rangle \text{ satisfies } \varphi \text{ just in case, for every } p: \langle A, \alpha \rangle \rightarrow HC,$$

$$\text{Im}(p) \leq \varphi.$$
Coequations as predicates

Since a coequation \( \varphi \) over \( C \) is just a subobject of \( UHC \), a coequation can be viewed as a **predicate** over \( UHC \). Coequation satisfaction can be stated in terms of predicate satisfaction.

\[
\langle A, \alpha \rangle \text{ satisfies } \varphi \text{ just in case, for every } p: \langle A, \alpha \rangle \to HC, \quad \exists a \in A (p(a) = x) \vdash \varphi(x).
\]
Birkhoff’s deduction theorem

A set of equations $E$ is **deductively closed** just in case $E$ satisfies the following:

(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) $E$ is closed under the $\mathbb{P}$-operations;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$. 
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(iv) $E$ is closed under the $P$-operations;
(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

Items (i)–(iv) ensure that $E$ is a congruence and hence uniquely determines a quotient of $FX$. 

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(iv) $E$ is closed under the $\mathbb{P}$-operations;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

Item (v) ensures that $E$ is a stable $\mathbb{P}$-algebra, i.e., closed under substitutions.
Birkhoff’s deduction theorem

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(i) $x = x \in E$;

(ii) $t_1 = t_2 \in E \Rightarrow t_2 = t_1 \in E$;

(iii) $t_1 = t_2 \in E$ and $t_2 = t_3 \in E \Rightarrow t_1 = t_3 \in E$;

(iv) $E$ is closed under the $\mathbb{P}$-operations;

(v) $t_1 = t_2 \in E \Rightarrow t_1[t/x] = t_2[t/x] \in E$.

**Theorem (Birkhoff completeness theorem).**

$E = Th_{Eq}(V)$ for some class $V$ iff $E$ is deductively closed.
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem).

\[ E = Th_{\text{Eq}}(V) \text{ for some class } V \text{ iff } E \text{ is deductively closed.} \]

The duals of the closure conditions yield two modal operators in the coalgebraic setting.
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem). $E = Th_{\text{Eq}}(V)$ for some class $V$ iff $E$ is deductively closed.

The duals of the closure conditions yield two modal operators in the coalgebraic setting.

- Taking the least congruence generated by $E$ corresponds to taking the largest subcoalgebra of $\varphi$. 
Dualizing the completeness theorem

Theorem (Birkhoff completeness theorem). 
\[ E = \text{Th}_{\text{Eq}}(V) \text{ for some class } V \text{ iff } E \text{ is deductively closed.} \]

The duals of the closure conditions yield two modal operators in the coalgebraic setting.

- Taking the least congruence generated by \( E \) corresponds to taking the largest subcoalgebra of \( \varphi \).
- Closing \( E \) under substitutions corresponds to taking the largest invariant coequation contained in \( \varphi \).
Dualizing the completeness theorem

The duals of the closure conditions yield two modal operators in the coalgebraic setting.

- Taking the least congruence generated by $E$ corresponds to taking the largest subcoalgebra of $\varphi$.
- Closing $E$ under substitutions corresponds to taking the largest invariant coequation contained in $\varphi$.

Theorem (Invariance theorem). $\varphi$ is a generating coequation just in case $\varphi$ is an invariant subcoalgebra of $HC$. 
Theories/Generating coequations

A set of equations $E$ is the **equational theory** for some class $V$ of algebras iff

- $V \models E$;
- If $V \models E'$, then $E' \subseteq E$. 
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- $V \models E$;
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A coequation $\varphi$ is the **generating coequation** for some class $V$ of coalgebras iff

- $V \models \varphi$;
- If $V \models \psi$, then $\varphi \vdash \psi$. 

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**Theories/Generating coequations**
Theories/Generating coequations

A coequation \( \varphi \) is the **generating coequation** for some class \( V \) of coalgebras iff

- \( V \models \varphi \);
- If \( V \models \psi \), then \( \varphi \vdash \psi \).

A generating coequation gives a measure of the “coequational commitment” of \( V \).
Invariant coequations

Let $\varphi \subseteq UHC$. We say that $\varphi$ is invariant just in case, for every “repainting”

$$p: UHC \rightarrow C,$$ 

equivalently, every homomorphism $\tilde{p}: HC \rightarrow HC$, we have

$$\exists \tilde{p}\varphi \leq \varphi.$$
Invariant coequations

Let $\varphi \subseteq UHC$. We say that $\varphi$ is invariant just in case, for every “repainting”

$$p : UHC \longrightarrow C,$$

equivalently, every homomorphism $\tilde{p} : HC \rightarrow HC$, we have

$$\exists c \in UHC (\tilde{p}(c) = x \land \varphi(c)) \vdash \varphi(x).$$
Invariant coequations

Let $\varphi \subseteq UHC$. We say that $\varphi$ is **invariant** just in case, for every “repainting”

$$p:UHC \rightarrow C,$$

equivalently, every homomorphism $\tilde{p}:HC \rightarrow HC$, we have

$$\exists c \in UHC (\tilde{p}(c) = x \land \varphi(c)) \vdash \varphi(x).$$

In other words, however we repaint $HC$, the elements of $\varphi$ are again (under this new coloring) elements of $\varphi$. 
Example (cont.)

The coequation $\phi$. 
Example (cont.)

The repainted coalgebra

The cofree coalgebra

$\varphi$ is not invariant.
Example (cont.)

The coequation $\Box \varphi$. 
The modal operator □

Let □ : Sub(\(UHC\)) \rightarrow Sub(\(UHC\)) be the comonad taking a coequation \(\varphi\) to the largest subcoalgebra \(\langle A, \alpha \rangle\) of \(HC\) such that \(A \leq \varphi\).
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the comonad taking a coequation \( \varphi \) to the largest subcoalgebra \( \langle A, \alpha \rangle \) of \( HC \) such that \( A \leq \varphi \).

As is well-known, if \( \Gamma \) preserves pullbacks of subobjects, then □ is an S4 operator.

(i) If \( \varphi \vdash \psi \) then □\( \varphi \vdash □\psi \);

(ii) □\( \varphi \vdash \varphi \);

(iii) □\( \varphi \vdash □□\varphi \);

(iv) □(\( \varphi \rightarrow \psi \)) \vdash □\( \varphi \rightarrow □\psi \);
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the comonad taking a coequation ϕ to the largest subcoalgebra ⟨A, α⟩ of HC such that A ≤ ϕ.

(i) If ϕ ⊨ ψ then □ϕ ⊨ □ψ;

(ii) □ϕ ⊨ ϕ;

(iii) □ϕ ⊨ □□ϕ;

(iv) □(ϕ → ψ) ⊨ □ϕ → □ψ;

(i) follows from functoriality.
The modal operator $\square$

Let $\square : \text{Sub}(UHC) \rightarrow \text{Sub}(UHC)$ be the comonad taking a coequation $\varphi$ to the largest subcoalgebra $\langle A, \alpha \rangle$ of $HC$ such that $A \leq \varphi$.

(i) If $\varphi \vdash \psi$ then $\square \varphi \vdash \square \psi$;
(ii) $\square \varphi \vdash \varphi$;
(iii) $\square \varphi \vdash \square \square \varphi$;
(iv) $\square (\varphi \rightarrow \psi) \vdash \square \varphi \rightarrow \square \psi$;

(ii) and (iii) are the counit and comultiplication of the comonad.
The modal operator □

Let □ : Sub(UHC) → Sub(UHC) be the comonad taking a coequation φ to the largest subcoalgebra ⟨A, α⟩ of HC such that A ≤ φ.

(i) If φ ⊩ ψ then □φ ⊩ □ψ;
(ii) □φ ⊩ φ;
(iii) □φ ⊩ □□φ;
(iv) □(φ → ψ) ⊩ □φ → □ψ;

(iv) follows from the fact that U : EΓ → E preserves finite meets.
Definition of ⨿

Let $\varphi \subseteq UHC$. Define

$$\mathcal{I}_\varphi = \{\psi \leq UHC \mid \forall p : HC \longrightarrow HC (\exists p \psi \leq \varphi)\}.$$ 

We define a functor $\Box : \text{Sub}(UHC) \rightarrow \text{Sub}(UHC)$ by

$$\Box \varphi = \bigvee \mathcal{I}_\varphi.$$ 

Then $\Box \varphi$ is the greatest invariant subobject of $UHC$ contained in $\varphi$. 
is S4

One can show that □ is an S4 operator.

(i) If φ ⊨ ψ then □φ ⊨ □ψ;
(ii) □φ ⊨ φ;
(iii) □φ ⊨ □□φ;
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One can show that □ is an S4 operator.

(i) If \( \varphi \vdash \psi \) then □\( \varphi \vdash \square \psi \);
(ii) □\( \varphi \vdash \varphi \);
(iii) □\( \varphi \vdash \Box \Box \varphi \);
(iv) □(\( \varphi \rightarrow \psi \)) \vdash □\( \varphi \rightarrow \Box \psi \);

(i) - (iii) follow from the fact that □ is a comonad, as before.
is S4

One can show that \( \Box \) is an S4 operator.

(i) If \( \varphi \vdash \psi \) then \( \Box \varphi \vdash \Box \psi \);
(ii) \( \Box \varphi \vdash \varphi \);
(iii) \( \Box \varphi \vdash \Box \Box \varphi \);
(iv) \( \Box (\varphi \rightarrow \psi) \vdash \Box \varphi \rightarrow \Box \psi \);

(iv) requires an argument that the meet of two invariant co-equations is again invariant. This is not difficult.
The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$. 
The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$.

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Diamond \varphi$. 
The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$.
Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Diamond \varphi$.
Lemma. Let $[-] : \text{Sub}_\mathcal{E}(UHC) \to \text{Sub}_{\mathcal{E}_\Gamma}(HC)$ be the right adjoint to $U : \text{Sub}_{\mathcal{E}_\Gamma}(HC) \to \text{Sub}_\mathcal{E}(HC)$ (so $\Box = U \circ [-]$). Then $[\Diamond \varphi] \models \varphi$. 
The invariance theorem, revisited

Lemma. \(\langle A, \alpha \rangle \models \varphi \iff \langle A, \alpha \rangle \models \Box \varphi.\)

Lemma. \(\langle A, \alpha \rangle \models \varphi \iff \langle A, \alpha \rangle \models \mathbf{E} \varphi.\)

Lemma. \([\mathbf{E} \varphi] \models \varphi.\)

Theorem. \(\varphi\) is a generating coequation iff \(\varphi = \Box \mathbf{E} \varphi.\)
The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$.

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Diamond \varphi$.

Lemma. $[\Diamond \varphi] \models \varphi$.

Theorem. $\varphi$ is a generating coequation iff $\varphi = \Box \Diamond \varphi$.

Theorem. $\Box \Diamond \varphi \leq \Diamond \Box \varphi$, i.e., if $\varphi$ is invariant, then so is $\Box \varphi$. 
The invariance theorem, revisited

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$.

Lemma. $\langle A, \alpha \rangle \models \varphi$ iff $\langle A, \alpha \rangle \models \Box \varphi$.

Lemma. $[\Box \varphi] \models \varphi$.

Theorem. $\varphi$ is a generating coequation iff $\varphi = \Box \Box \varphi$.

Theorem. $\Box \Box \varphi \leq \Box \Box \Box \varphi$, i.e., if $\varphi$ is invariant, then so is $\Box \varphi$.

Theorem. If $\Gamma$ preserves non-empty intersections, then $\Box \Box \varphi = \Box \Box \varphi$. 
Some open questions

- Is the preservation of non-empty intersections really relevant to the conclusion that □□ = □□?
Some open questions

- Is the preservation of non-empty intersections really relevant to the conclusion that $\Box\Diamond = \Diamond\Box$?
- What is the relation between the construction of a coequation $\varphi$ and the corresponding covariety?
Some open questions

- Is the preservation of non-empty intersections really relevant to the conclusion that $\square \Box = \Box \square$?
- What is the relation between the construction of a coequation $\varphi$ and the corresponding covariety?

\[
\begin{align*}
V_{\square \varphi} &= V_{\varphi} \\
V_{\Box \varphi} &= V_{\varphi} \\
V_{\varphi \land \psi} &= V_{\varphi} \cap V_{\psi} \\
V_{\exists p \varphi} &= ? \\
V_{\neg \varphi} &= ?
\end{align*}
\]
Some open questions

- Is the preservation of non-empty intersections really relevant to the conclusion that $\Box \psi = \psi \Box$?
- What is the relation between the construction of a coequation $\varphi$ and the corresponding covariety?
- What applications do these “non-behavioral” covarieties have in computer programming semantics?