Optical metrics and birefringence of anisotropic media

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Abstract

The material tensor of linear response in electrodynamics is constructed out of products of two symmetric second rank tensor fields which in the approximation of geometrical optics and for uniaxial symmetry reduce to “optical” metrics, describing the phenomenon of birefringence. This representation is interpreted in the context of an underlying internal geometrical structure according to which the symmetric tensor fields are vectorial elements of an associated two-dimensional space.

Key words: Anisotropic media, geometrical optics, optical metric, birefringence

1 Introduction

Electrodynamics in media is a broadly investigated subject with a variety of different facets [1, 2, 3]. As far as the propagation of waves is concerned, the approximation of geometrical optics has played a major role. Within this approach Maxwell’s equations are transformed into a set of algebraic equations for the wave vector. In a relativistic context (see [4, 5, 6, 7]), a specific feature

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of the wave propagation in media is the appearance of an “optical” metric, introduced by Gordon [4]. With respect to this quantity light in isotropic media propagates as in vacuum. Light rays follow geodesics [5] and the wave vector is a null vector of this metric. The optical metric is characterized by the refraction index of the medium which is composed of the coefficients of dielectricity and magnetic permeability. The latter quantities, in turn, enter the material tensor that relates the excitation (induction) tensor to the field strength tensor. For the motivation of this paper it is essential that this happens in a manner which allows one to represent the fourth rank material tensor as a combination of products of the second rank optical metric. Written in terms of the optical metric, the material tensor for an isotropic medium has the same structure as the “material” tensor for the vacuum, only that the spacetime metric within the latter is replaced by the optical metric [8]. Here we ask to what extent this type of representation can be generalized to anisotropic media. This is anything but straightforward since light propagation in anisotropic media is accompanied by the phenomenon of birefringence. A hint here is the fact that in the simplest anisotropic case, the case of uniaxial symmetry, birefringence can be described by two Lorentzian optical metrics [7] (under more general circumstances they have to be replaced by Finsler metrics). Given the mentioned representation of the material tensor in the isotropic case, this raises the question whether there exists a bi-metric representation of this tensor in anisotropic media. While the problems of birefringence and bi-metricity have been considered for non-linear electrodynamics (birefringence as a non-linear phenomenon) [9, 10], we are not aware of an explicit representation of the material tensor in terms of two optical metrics in the linear case, describing the so-called ordinary and extraordinary waves in uniaxial media. We shall demonstrate that the material tensor can indeed be written as a combination of products of those two optical metrics which characterize the light cone structure for uniaxial symmetry. A construction of the material tensor out of two symmetric second rank tensor fields is even possible in general anisotropic media, but the interpretation of these quantities as optical metrics is restricted to the uniaxial case. We also show that the transition between different representations of the material tensor in terms of different pairs of second rank tensor fields is governed by invariance properties in an associated two-dimensional internal vector space.

The paper is organized as follows. In section 2 we consider Maxwell’s equations and the general constitutive laws which mediate between the tensors of excitation (induction) and field strength. We decompose the material tensor with respect to contribution parallel and orthogonal to the four velocity of the medium. We recall the role of an optical metric in isotropic media and provide different representations of the material tensor in terms of this metric. Section 3 is a collection of basic relations of geometrical optics. Section 4, which is the main section of the paper, is devoted to the construction of two symmetric second rank tensor field in terms of which a representation of the material tensor is obtained. Novel geometric features of this representation are pointed out. In the approximation of geometrical optics Maxwell’s equations reduce to a fourth order algebraic equation for the wavevector, the “extended” Fresnel equation.
(cf. [11]). For the case of uniaxial symmetry this equation is shown to factorize where the two symmetric tensor fields reduce to optical metrics for ordinary and extraordinary wave propagation. A summary of the paper is given in section 6.

2 Maxwell’s equations and constitutive laws

2.1 General theory

The covariant, source free Maxwell equations in continuous electromagnetic media are

\[ \nabla_k F^{*ik} = 0 \quad \text{and} \quad \nabla_k H^{ik} = 0, \]  

(1)

where

\[ F^{*ik} = \frac{1}{2\sqrt{-g}} \epsilon^{ikls} F_{ls} \]  

(2)

is the dual to the field strength tensor \( F^{ik} \) and \( H^{ik} \) is the induction (or excitation) tensor. Both \( F^{ik} \) and \( H^{ik} \) are antisymmetric, i.e., \( F^{ik} = -F^{ki} \) and \( H^{ik} = -H^{ki} \). Here, \( \nabla_k \) denotes the covariant derivative with respect to the spacetime metric \( g^{ik} \) (with determinant \( g \) and signature + − − −) and \( \frac{1}{\sqrt{-g}} \epsilon^{ikls} \) is the Levi-Civita (pseudo-) tensor where \( \epsilon^{ikls} \) is completely antisymmetric with \( \epsilon^{0123} = -\epsilon_{0123} = 1 \).

The set of equations (1) has to be completed by constitutive relations, linking the electromagnetic inductions to the field strengths. Restricting ourselves to linear dielectric, permeable media, the most general structure for the corresponding constitutive laws is

\[ H^{ik} = C^{ikmn} F_{mn}, \]  

(3)

where \( C^{ikmn} \) is the material tensor which characterizes the electromagnetic properties of the medium in the linear response approximation. This tensor has the symmetries

\[ C^{ikmn} = C^{mnik} = -C^{kmin} = -C^{iknm}. \]  

(4)

Consequently, it has 21 independent components. (See, however, [11] for an approach without the symmetry \( C^{ikmn} = C^{mnik} \)). Assuming that the medium is characterized by a four-velocity field \( U^i \), normalized by \( U^i U_i = 1 \), we may define the vectors \( D^i, H^i, E^i, \) and \( B^i \) as [12]

\[ D^i = H^{ik} U_k, \quad H^i = H^{*ik} U_k, \quad E^i = F^{ik} U_k, \quad B^i = F^{*ik} U_k. \]  

(5)
These vectors are orthogonal to the four-velocity vector $U^i$,
\[ D^i U_i = 0 = E^i U_i \quad \text{and} \quad H^i U_i = 0 = B^i U_i. \quad (6) \]

The field strength tensor can be decomposed according to
\[ F_{mn} = \delta_{mn}^{pq} E_p U_q - \eta_{mnk} B^k, \quad (7) \]
where $\delta_{mn}$ is the generalized 4-indices $\delta$–Kronecker tensor
\[ \delta_{mn}^{ik} = \delta_{m}^{i} \delta_{n}^{k} - \delta_{k}^{i} \delta_{n}^{m} \]
and $\eta_{mnk}$ is an antisymmetric tensor orthogonal to $\mathbf{U}^i$, defined as
\[ \eta_{mnk} \equiv \sqrt{-g} \epsilon_{mnls} U^s, \quad \eta_{ikl} \equiv \frac{1}{\sqrt{-g}} \epsilon^{ikls} U^s. \quad (8) \]

A useful identity is
\[ -\eta_{ikp} \eta_{mnp} = \delta_{mn}^{i} \Delta_k - \Delta_m \Delta^i_k, \quad (9) \]
where
\[ \delta_{mn}^{ikl} = \delta_{m}^{i} \delta_{n}^{k} \delta_{l}^{s} + \delta_{m}^{k} \delta_{n}^{i} \delta_{l}^{s} + \delta_{m}^{i} \delta_{n}^{k} \delta_{l}^{s} - \delta_{k}^{i} \delta_{n}^{m} \delta_{l}^{s} - \delta_{k}^{i} \delta_{m}^{n} \delta_{l}^{s} - \delta_{k}^{i} \delta_{n}^{m} \delta_{l}^{s} \]
and
\[ \Delta^{ik} = g^{ik} - U^i U^k. \quad (10) \]

Upon contraction, equation (9) yields the further identity
\[ -\frac{1}{2} \eta_{ikl} \eta_{klm}^{i} = -\delta_{mn}^{i} \Delta_k - \Delta_m \Delta^i_k. \quad (11) \]

### 2.2 Decomposition of the material tensor $C^{ikmn}$

We can uniquely decompose the tensor $C^{ikmn}$ as
\[ C^{ikmn} = \frac{1}{2} \left[ \varepsilon^{im} U^k U^n - \varepsilon^{im} U^k U^m + \varepsilon^{kn} U^i U^m - \varepsilon^{km} U^i U^n \right] - \frac{1}{2} \eta^{ikl} (\mu^{-1})_{ls} \eta_{mn}^{s} - \frac{1}{2} \left[ \eta^{ikl} (U^m \nu^{i n} - U^n \nu^{i m}) + \eta^{lmn} (U^i \nu^{k i} - U^k \nu^{i i}) \right]. \quad (12) \]

Here $\varepsilon^{im}$, $(\mu^{-1})_{pq}$ and $\nu^{pm}$ are defined as
\[ \varepsilon^{im} = 2C^{ikmn} U_i U_n, \]
\[ (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pik} C^{ikmn} \eta_{mnq}, \]
\[ \nu^{pm} = \eta_{pik} C^{ikmn} U_n. \quad (13) \]
The tensors $\varepsilon_{ik}$ and $(\mu^{-1})_{ik}$ are symmetric, but $\nu^k_i$ is in general non-symmetric. The dot denotes the position of the second index when lowered. These three tensors are spacelike, i.e., they are orthogonal to $U^i$,

$$\varepsilon_{ik}U^k = 0, \quad (\mu^{-1})_{ik}U^k = 0, \quad \nu^k_i U^i = 0 = \nu^k_i U_k.$$  \hfill (14)

The total trace of the $C^{ikmn}$ tensor is equal to

$$C^{ikmn}g_{im}g_{kn} = \varepsilon^k_k + (\mu^{-1})^k_k.$$  \hfill (15)

With the help of the definitions (3), (5) and the decomposition (12), we obtain the linear laws

$$D^i = \varepsilon^{im}E_m - B^i \nu^i_m, \quad \text{and} \quad H_i = (\mu^{-1})_{im}B^m + \nu^m_i E_m.$$  \hfill (16)

Obviously, the tensors $\varepsilon^{im}$ and $\mu_{pq}$ are the four-dimensional analogues of the dielectricity tensor and the magnetic permeability tensor, respectively. The tensor $\nu^m_p$ describes magneto-electric cross effects. Thus, the 21 components of $C^{ikmn}$ consist of the 6 components of $\varepsilon^{im}$, the 6 components of $(\mu^{-1})_{pq}$ and the 9 components of $\nu^m_p$. For media without cross effects the number of independent components reduces to 12.

### 2.3 Special cases

#### 2.3.1 Vacuum

The vacuum is characterized by

$$H^{ik} = F^{ik} = g_{im}g_{kn}F_{mn},$$  \hfill (17)

which corresponds to a “material” tensor

$$C_{(\text{vac})}^{ikmn} = \frac{1}{2} (g_{im}g_{kn} - g_{in}g_{km}).$$  \hfill (18)

Under this condition the quantities (13) reduce to

$$\varepsilon^{ik} = (\mu^{-1})^{ik} = \Delta^{ik}, \quad \nu^i_k = 0.$$  \hfill (19)

Alternatively, introducing the latter relations into Eq. (12), we obtain the special case (18).

#### 2.3.2 Isotropic medium

A spatially isotropic medium is characterized by

$$D^i = \varepsilon E^i \quad \text{and} \quad H_i = \frac{1}{\mu}B_i.$$  \hfill (20)
with two phenomenological scalars, the dielectricity $\varepsilon$ and the permeability $\mu$. The relations (20) are the special case of the general structure (16) for

$$
\varepsilon^{ik} = \Delta^{ik}, \quad (\mu^{-1})^{ik} = \frac{1}{\mu} \Delta^{ik}, \quad \nu_i^k = 0.
$$

(21)

For $\varepsilon$ and $\mu$ equal to unity, we recover the vacuum case (19). Introducing the expressions (21) for $\varepsilon^{ik}, (\mu^{-1})^{ik}$ and $\nu^k_i$, into (12) one obtains the material tensor

$$
C^{ikmn}_{(isotr)} = \frac{1}{2 \mu} \left[ g^{im}g^{kn} - g^{in}g^{km} \right] + \frac{1}{2} \left( \varepsilon - 1 \right) \left( g^{im}U^nU^k + g^{kn}U^mU^i - g^{km}U^iU^n \right).
$$

(22)

A convenient representation is obtained in terms of the second rank tensor

$$
g^{*ik} = \frac{1}{n^2} \left[ g^{ik} + (n^2 - 1) U^iU^k \right] \equiv \frac{1}{n^2} \Delta^{ik} + U^iU^k,
$$

(23)

where $n$ is the refractive index, defined by $n^2 = \varepsilon \mu$. The inverse tensor is

$$
g^{*km} = n^2 \left[ g_{km} + \left( \frac{1}{n^2} - 1 \right) U_kU_m \right] \equiv n^2 \Delta_{km} + U_kU_m,
$$

(24)

such that

$$
g^{*ik}g^{*km} = \delta^i_m.
$$

(25)

Obviously, we have

$$
g^{*ik}U_k = U^i, \quad g^{*ik}U_iU_k = 1, \quad g^{*km}U^m = U_k, \quad g^{*km}U^kU^m = 1,
$$

(26)

i.e., the tensors $g^{*ik}$ and $g^{*km}$ preserve the norm of the four-velocity vector. (Note, however, that generally indices are raised and lowered with the metric tensor $g_{ik}$.) Furthermore, the relation

$$
g^{*ln} = \frac{\varepsilon^{ln}}{\varepsilon n^2} + U^lU^n
$$

(27)

for the isotropic case $\varepsilon^{ik} = \varepsilon \Delta^{ik}$ is valid. With the help of both $g^{ik}$ and $g^{*ik}$ the tensor (22) can be written as

$$
C^{ikmn}_{(isotr)} = \frac{n^2}{2 \mu} \left\{ g^{im}g^{kn} - g^{in}g^{km} \right\} - \frac{n^2}{(n^2 - 1)} \left[ (g^{im} - g^{*im})(g^{kn} - g^{km}) \right.
$$

$$
- \left. (g^{in} - g^{*in})(g^{km} - g^{*km}) \right\},
$$

(28)

which will be useful for later reference. The most compact and elegant way, however, is the symmetric vacuum type form (cf. [8])

$$
C^{ikmn}_{(isotr)} = \frac{n^4}{2 \mu} \left( g^{*im}g^{*kn} - g^{*in}g^{*km} \right).
$$

(29)
In such a form the material tensor $C^{ikmn}_{(isot)}$ for the isotropic medium, up to the factor $\frac{1}{\Omega^4}$, has the structure of \(1\), with the metric $g^{ik}$ replaced by $g_{ik}$.

## 3 Geometrical optics

### 3.1 Basic relations

We assume the field strength tensor to be given in terms of a four potential $A_m$ by

$$F_{mn} = \nabla_m A_n - \nabla_n A_m.$$  \hfill (30)

Geometrical optics is based on the leading-order approximation in the eikonal derivatives of

$$A_l = a_l e^{i\phi}, \quad k_m \equiv \nabla_m \Phi,$$  \hfill (31)

where $\Phi$ is the phase, $a_l$ is a slowly varying amplitude and $k_m$ is a wave four-vector. In this approximation (for complex quantities always the real part has to be taken)

$$F_{mn} = i (k_m A_n - k_n A_m)$$  \hfill (32)

and Maxwell’s equations reduce to

$$k_l C^{lmn} (k_m A_n - k_n A_m) = 0.$$  \hfill (33)

### 3.2 Vacuum

In the vacuum case Eq. (33) takes the form

$$k^i (g^{ln} k_l A_n) - A^i (g^{lm} k_l k_m) = 0.$$  \hfill (34)

Contraction with $U_i$ yields

$$(U_i k^i) (g^{ln} k_l A_n) = (U_i A^i) (g^{lm} k_l k_m).$$  \hfill (35)

For the Landau gauge $U_i A^i = 0$ and $U_i k^i \neq 0$ it follows that

$$g^{ln} k_l A_n = 0.$$  \hfill (36)

While this is identical to the Lorentz gauge, it is not imposed as a separate condition in the present context but it is a consequence of the condition $U_i A^i = 0$. With (35) the field equation (34) admits a nontrivial solution for the potential $A^i$ if and only if

$$g^{lm} k_l k_m = 0,$$  \hfill (37)

i.e., the wave four vector $k^i$ has to be a null vector with respect to the metric $g^{ik}$. Moreover, the curves to which $k^a$ is a tangent vector are geodesics \(1\).
3.3 Isotropic medium

With the help of the optical metric (23) and with (29) the Maxwell equations for an isotropic medium can be written in the compact form

\[ \frac{n^4}{2\mu} \left( g^{*im} k_m g^{*lm} k_l A_n - g^{*in} A_n g^{*lm} k_l k_m \right) = 0. \] (38)

Proceeding as in the vacuum case, projection along \( U_i \) leads to

\[ (U_i k^i) \left( g^{*lm} k_l A_n \right) = (U_i A^i) \left( g^{*lm} k_l k_m \right). \] (39)

For \( U_i A^i = 0 \) (Landau gauge) and \( U_i k^i \neq 0 \) we get

\[ g^{*lm} k_l A_n = 0, \] (40)

which, because of \( U_i A^i = 0 \) coincides with (36) and the Maxwell equations simplify to

\[ g^{*lm} A_n g^{*lm} k_l k_m = 0. \] (41)

This equation admits a nontrivial solution for the potential, if and only if

\[ g^{*lm} k_l k_m = 0, \] (42)

i.e. if the wave four-vector \( k_m \) is a null vector with respect to \( g^{*lm} \), which justifies the name optical metric \( g^{*ik} \) for this quantity. It was proven by Pham Mau Quan (see also [14]) that the corresponding light curves are null geodesics of the metric \( g^{*ik} \).

4 Anisotropic media

4.1 Representation of the material tensor

In the isotropic case it turned out to be possible to find a metric \( g^{*ik} \) in terms of which the matter tensor has the quasi-vacuum representation (29). Moreover, \( g^{*ik} \) is an optical metric. With respect to \( g^{*ik} \) the wave vector is a null vector. Both for the representation of the matter tensor and for the light propagation in geometrical optics the transition from vacuum to an isotropic medium consists in replacing the spacetime metric by the optical metric \( g^{*ik} \). It is natural to ask, to what extent these features may be generalized also to anisotropic media. A useful hint here is the circumstance that for anisotropic media of the simplest kind, the uniaxial crystal, there exist two optical metrics which account for the phenomenon of birefringence (see, e.g., [7]). This seems to suggest a bi-metric representation of the matter tensor in such media. The construction of \( C^{iklm} \) out of two different symmetric second rank tensor fields is the subject of the present section. While these tensor fields cannot be expected to be optical metrics in the general case, for an uniaxial symmetry, however, a decomposition
of \( C^{ikmn} \) in terms of two optical metrics will indeed be possible. With respect to each of them light propagation proceeds as in vacuum in a similar sense as described by (42) for isotropic media. To the best of our knowledge, this problem has not been addressed in the literature so far. Our strategy will be as follows. In a first step we simply assume the existence of two symmetric tensor fields as construction elements of the material tensor. This choice is not unique and we have to single out a suitable pair of symmetric tensors. This will be done with the help of a geometric interpretation according to which these tensors can be regarded as vectors in a two-dimensional space. The next step will then be the explicit construction of these tensor fields and, finally, the demonstration that they reduce to two different optical metrics in the appropriate limit.

This strategy implies that we consider media for which the tensor \( C^{ikmn} \) in (12) is reducible to the form

\[
C^{ikmn} = \frac{1}{2\hat{\mu}} \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} \left( g^{im(\alpha)} g^{kn(\beta)} - g^{in(\alpha)} g^{km(\beta)} \right),
\]

where \( g^{im(\alpha)} \) and \( g^{im(b)} \) are a pair of (for the moment unknown) symmetric, non-degenerate tensor fields. The indices \( \alpha \) and \( \beta \) take the values \( a \) and \( b \) and the sum is over all possible combinations. By construction, the expression (43) satisfies the symmetry conditions (4). For a given tensor \( C^{ikmn} \) with generally 21 independent components the representation (43) may be regarded as a set of algebraic equations to determine the 10 components of \( g^{im(a)} \), the 10 components of \( g^{im(b)} \) and, additionally, one of the three quantities \( G_{(a)}(a)/\hat{\mu}, G_{(b)}(b)/\hat{\mu} \), and \( G_{(a)}(b)/\hat{\mu} = G_{(b)}(a)/\hat{\mu} \) (we assume \( G_{(\alpha)(\beta)} \) to be symmetric). Two of these quantities remain arbitrary, the corresponding freedom will be used later on. The representation (43) is a direct generalization of (18) and (29) for the cases of vacuum and isotropic media, respectively. Since we require for the isotropic limit \( g^{im(a)} = g^{im(b)} = g^{*im} \) the decomposition (29) with \( \hat{\mu} = \frac{1}{n^2} \) to hold, we obtain

\[
\sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} = 1 \tag{44}
\]

as an additional condition. The choice of the symmetric tensors \( g^{im(a)} \) and \( g^{im(b)} \) in the decomposition (43) is not unique. A first guess could be (cf. [10], (D.1.80))

\[
\begin{align*}
C^{ikmn}_{(1,2)} &= \frac{1}{4\hat{\mu}} \left( g^{im(1)} g^{kn(2)} - g^{in(1)} g^{km(2)} + g^{im(2)} g^{kn(1)} - g^{in(2)} g^{km(1)} \right),
\end{align*}
\]

which corresponds to

\[
G_{(1)(1)} = G_{(2)(2)} = 0, \quad G_{(1)(2)} = G_{(2)(1)} = \frac{1}{2}, \quad \det(G_{(1,2)}) = -\frac{1}{4} \tag{46}
\]
where \( \det(G_{(1,2)}) \) is the determinant \( G_{(1)(1)}G_{(2)(2)} - G_{(1)(2)}G_{(2)(1)} \) of the matrix \( G_{(\alpha)(\beta)} \) for this case. So far we will not use, however a specific choice but further consider the general representation (43). As any symmetric tensor, the quantities \( g^{ik(\alpha)} \) can be decomposed with respect to their components parallel and orthogonal to the four-velocity \( U^i \),

\[
g^{ik(\alpha)} = B^{(\alpha)}U^iU^k + \tilde{D}^{i(\alpha)}U^k + \tilde{k}^{k(\alpha)}U^i + \tilde{S}^{ik(\alpha)},
\]

where

\[
B^{(\alpha)} = g^{ik(\alpha)}U_iU_k, \quad \tilde{D}^{i(\alpha)} = \Delta_i^p g^{ik(\alpha)}U_k, \quad \tilde{S}^{pq(\alpha)} = \Delta_i^p g^{ik(\alpha)}\Delta_i^q.
\]

In case \( U^i \) is lightlike with respect to one or both tensors \( g^{im(\alpha)} \), the corresponding coefficients \( B^{(\alpha)} \) vanish. We shall assume, however, that, analogously to the vacuum and isotropic media cases, the four velocity \( U^i \) is timelike with respect to both tensors \( g^{im(\alpha)} \). Under this condition the factors \( B^{(\alpha)} \) in front of \( U^iU^k \) in (47) are different from zero and can be absorbed into the coefficients \( G^{(\alpha)(\beta)} \). The decomposition (47) then reduces to

\[
g^{ik(\alpha)} = U^iU^k + D^{i(\alpha)}U^k + D^{k(\alpha)}U^i + S^{ik(\alpha)},
\]

with quantities \( D^{p(\alpha)} \) and \( S^{pq(\alpha)} \) redefined accordingly.

The magneto-electric cross terms \( \nu_{m-p} \) in (13) are determined by the coefficients \( D^{p(\alpha)} \) via

\[
\nu_{m-p} = \frac{1}{2\mu} \sum_{(\alpha)(\beta)} G^{(\alpha)(\beta)}\eta_{pi}k \left( D^{k(\beta)}g^{im(\alpha)} - D^{i(\alpha)}g^{km(\beta)} \right).
\]

Obviously, a sufficient condition for excluding magneto-electric cross effects is \( D^{p(\alpha)} = 0 \). (Note that even in the presence of cross effects one of the \( D^{p(\alpha)} \) can be transformed to zero by assuming \( U^i \) to be an eigenvector of the corresponding \( g^{ik(\alpha)} \).) In the following we shall restrict ourselves to \( D^{p(\alpha)} = 0 \) and use

\[
g^{ik(\alpha)} = U^iU^k + S^{ik(\alpha)} \quad \Rightarrow \quad U_i g^{im(\alpha)} = U^m.
\]

Applying the structure (51) to (13), the spatial tensors \( \varepsilon^{im} \) and \( (\mu^{-1})_{pq} \) in (13) can be written as

\[
\varepsilon^{im} = \frac{1}{\mu} \left\{ \left[ G^{(a)(a)} + G^{(a)(b)} \right] S^{im(\alpha)} + \left[ G^{(a)(b)} + G^{(b)(b)} \right] S^{im(\beta)} \right\},
\]

and
\[
(\mu^{-1})_{pq} = -\frac{1}{\mu} \eta^p_{ik} \eta^q_{mn} \left[ G(a)(a) S^{im(a)} S^{kn(a)} + G(b)(b) S^{im(b)} S^{kn(b)} + 2 G(a)(b) S^{im(a)} S^{kn(b)} \right], \quad (53)
\]

respectively. In a next step we make use of the remaining two degrees of freedom, mentioned in the discussion following equation (43). From (52) it is obvious that there exists a special configuration, for which \(\varepsilon^{im}\) depends only on one of the spatially projected (cf. (49)) metrics, say on \(S^{im(a)}\). This configuration, which we denote by \((a) = (A)\) and \((b) = (B)\), is characterized by the two relations
\[
G(A)(A) + G(B)(A) = 1, \quad G(A)(B) + G(B)(B) = 0 \quad (54)
\]

which obviously satisfy (44). Introducing the abbreviation \(\gamma \equiv G(A)(B) = G(B)(A) = -\text{det}(G(A,B))\), where the latter quantity denotes the determinant \(G(A)(A) G(B)(B) - G(A)(B) G(B)(A)\) of the matrix \(G(A,B)\) for this configuration, the material tensor (43) takes the form
\[
C_{ijkmn}^{(A,B)} = \frac{1}{2\mu} \left\{ \left[ g^{im(A)} g^{kn(A)} - g^{im(A)} g^{km(A)} \right] 
- \gamma \left[ (g^{im(A)} - g^{im(B)})(g^{kn(A)} - g^{kn(B)}) 
- (g^{in(A)} - g^{in(B)})(g^{km(A)} - g^{km(B)}) \right] \right\}, \quad (55)
\]

which generalizes the representation (28) for the isotropic case. The factor \(\gamma\) generalizes the (positive) expression \(\frac{n^2}{n^2-1}\) in (28) and has to be positive as well (cf. subsection 4.2 below). The quantities (52) and (53) specify to
\[
\varepsilon^{im} = \frac{1}{\mu} S^{im(A)}, \quad (56)
\]

and
\[
(\mu^{-1})_{pq} = -\frac{1}{\mu} \eta^p_{ik} \eta^q_{mn} \left[ S^{im(A)} S^{kn(A)} 
- \gamma \left( S^{im(A)} - S^{im(B)} \right) \left( S^{kn(A)} - S^{kn(B)} \right) \right], \quad (57)
\]

respectively.

It is evident from (51) that (56) implies
\[
\varepsilon^{im} = \frac{1}{\mu} \left[ g^{im(A)} - U^i U^m \right], \quad (58)
\]

independent of the value of the parameter \(\gamma\). This parameter contributes to the tensor \((\mu^{-1})_{pq}\) only. In turn, (58) provides us with an explicit expression for \(g^{im(A)}\) in terms of the tensor \(\varepsilon^{im}\), which in the isotropic limit reduces to the optical metric (27).
4.2 Geometric implications

The change between the explicit representations \((45)\) to \((55)\) of the material tensor corresponds to a transformation \((\gamma > 0)\)

\[
g^{ik(1)} = g^{ik(A)} + \sqrt{\gamma} \left(g^{ik(A)} - g^{ik(B)}\right) \tag{59}
\]

and

\[
g^{ik(2)} = g^{ik(A)} - \sqrt{\gamma} \left(g^{ik(A)} - g^{ik(B)}\right) \tag{60}
\]

or, inversely,

\[
g^{ik(A)} = \frac{1}{2} \left(g^{ik(1)} + g^{ik(2)}\right) \tag{61}
\]

and

\[
g^{ik(B)} = g^{ik(A)} + \frac{1}{2} \sqrt{\gamma} \left(g^{ik(2)} - g^{ik(1)}\right). \tag{62}
\]

Transformations between different sets of symmetric tensors in the representation \((43)\) can be regarded as transformations in a two-dimensional space with a metric \(G^{(\alpha)(\beta)}\), which is symmetric and non-degenerate, i.e.,

\[
G^{(\alpha)(\beta)} = G^{(\beta)(\alpha)}, \quad G \equiv \det(G^{(\alpha)(\beta)}) \neq 0. \tag{63}
\]

This geometric interpretation requires a Kronecker delta \(\delta^{(\alpha)}_{(\beta)}\) to exist together with a contravariant tensor \(G^{[\mu][\nu]}\), satisfying \(G^{[\mu][\nu]} G^{(\nu)(\beta)} = \delta^{(\mu)}_{(\beta)}\). Furthermore, in such a space there exists an anti-symmetric Levi-Civita (pseudo-) tensor

\[
E^{(\alpha)(\beta)} = \sqrt{|G|} \epsilon^{(\alpha)(\beta)}, \quad \epsilon^{(\alpha)(\beta)} = -\epsilon^{(\beta)(\alpha)}, \quad \epsilon^{(1)(2)} = 1. \tag{64}
\]

The point here is that the decomposition \((43)\) is invariant with respect to transformations in this associated space. In order to demonstrate this, we consider linear transformations

\[
g^{mn(\alpha)'} = t^{(\alpha)'}_{(\alpha)} g^{mn(\alpha)}, \quad G^{(\alpha)'}(\beta) = t^{(\alpha)'}_{(\alpha)} t^{(\beta)}_{(\beta)'} G^{(\alpha)(\beta)}, \tag{65}
\]

and require that the scalar product \(G^{(\alpha)(\beta)} g^{mn(\alpha)} g^{ik(\beta)}\) remains invariant, i.e.,

\[
G^{(\alpha)'}(\beta) g^{im(\alpha)'} g^{ln(\beta)'} = G^{(\alpha)(\beta)} g^{im(\alpha)} g^{ln(\beta)}. \tag{66}
\]
Contraction with $U_i U_m$ yields
\[ \sum_{(\alpha)'(\beta)'} G_{(\alpha)'(\beta)'} g^{ln(\beta)'} = \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} g^{ln(\beta)}. \] (67)

Identifying now $(\alpha)'$ and $(\beta)'$ with $(A)$ and $(B)$, respectively, as well as $(\alpha)$ and $(\beta)$ with (1) and (2), respectively, upon using (54) and (46) we reproduce the transformation (61). Similarly, the invariance
\[ E_{(\alpha)'(\beta)'} g^{im(\alpha)'} g^{ln(\beta)'} = E_{(\alpha)(\beta)} g^{im(\alpha)'} g^{ln(\beta)}. \] (68)
via
\[ \sum_{(\alpha)'(\beta)'} E_{(\alpha)'(\beta)'} g^{ln(\beta)'} = \sum_{(\alpha)(\beta)} E_{(\alpha)(\beta)} g^{ln(\beta)}. \] (69)
provides us with the second transformation (62). As a consistency check, we also realize that
\[ G' = \frac{\partial g}{\partial g'} G = 4\gamma G, \] (70)
which is the correct relations between the determinants $G' \equiv \det(G_{(A,B)})$ and $G \equiv \det(G_{(1,2)})$. Here, $g$ in (70) stands symbolically for the set $g^{ik(1)}$, $g^{ik(2)}$, and $g'$ stands for the set $g^{ik(A)}$, $g^{ik(B)}$.

The relations (59) - (70) constitute an underlying geometric structure for the representation of the material tensor in terms of second rank symmetric tensor fields. The coefficients $G_{(\alpha)(\beta)}$ play the role of a metric in the associated two-dimensional space, the tensors $g^{ik(\alpha)}$ are vectorial objects in this space.

4.3 Maxwell’s equations

With the general expression (43) Maxwell’s equations are
\[ C^{lmn} k_l k_m A_n = 0 \]
\[ \Rightarrow \frac{1}{2\mu} \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} \left( g^{im(\alpha)} g^{ln(\beta)} - g^{in(\alpha)} g^{lm(\beta)} \right) k_l k_m A_n = 0. \] (71)

Contracting these equations with $U_i$ similarly as for the previous isotropic case, we obtain
\[ U^m k_m \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} g^{ln(\beta)} k_l A_n = U^n A_n \sum_{(\alpha)(\beta)} g^{lm(\beta)} k_l k_m. \] (72)
With the Landau-gauge $U^m A_n = 0$ and $U^m k_n \neq 0$, the relation generalizing (40) is

$$g^{ln}_{(\text{eff})} k_l A_n = 0, \quad g^{ln}_{(\text{eff})} = \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} g^{ln(\beta)}.$$

(73)

The structure of this relation suggests to identify $g^{ln}_{(\text{eff})}$ with one of the symmetric tensors $g^{ln(\alpha)}$, let’s say $g^{ln(a)}$, out of which the material tensor is constructed. This is equivalent to

$$g^{ln}_{(\text{eff})} = g^{ln(a)} = \left[ G^{ln(a)} + G^{ln(b)} \right] g^{ln(b)}$$

$$\Rightarrow G^{ln(a)} + G^{ln(b)} = 1, \quad G^{ln(b)} + G^{ln(b)} = 0.$$

(74)

It is remarkable, that this choice again results in the previously derived set of coefficients, characterized by (54). The identification $g^{ln}_{(\text{eff})} = g^{ln(a)}$ in (74) is an alternative way to single out this option which we again denote by $(A)$ and $(B)$. In the present context we have the additional relation

$$g^{ln(A)} k_l A_n = 0,$$

(75)

which has the structure of a generalized Lorentz gauge. We emphasize again (see the remarks following (54)) that this is not imposed as a condition here but that it follows from the Landau gauge $U^m A_n = 0$.

4.4 Constructing $g^{kn(A)}$ and $g^{kn(B)}$

4.4.1 General relations

While the tensor $g^{ik(A)}$ is given in terms of $\varepsilon^{ik}$ by (55), it is less straightforward to obtain a corresponding expression for $g^{ik(B)}$. Following (7) we consider the eigenvectors $X_i^{(1)}, X_i^{(2)},$ and $X_i^{(3)}$ of the tensor $\varepsilon^{ik}$. These eigenvectors have the properties

$$g_{ik} X_i^{(a)} X_i^{(b)} = -\delta_{(a)(b)}, \quad g_{ik} X_i^{(a)} U^k = 0,$$

(66)

$$X_i^{(1)} X_i^{(1)} + X_i^{(2)} X_i^{(2)} + X_i^{(3)} X_i^{(3)} = -g^{ik} + U^i U^k,$$

(77)

$$\eta_{ik} X_i^{(a)} X_i^{(b)} = \varepsilon_{(a)(b)(c)} X_i^{(c)},$$

(78)

where $(a), (b),... = (1), (2), (3)$ are tetrad indices and the double index $(c)$ denotes a sum. Relation (78) is a consequence of the fact that the scalar function $\eta_{ik} X_i^{(a)} X_i^{(b)} X_i^{(c)}$ is totally antisymmetric with respect to indices $(a), (b), (c)$ and
has to be proportional to the Levi-Civita symbol $\varepsilon_{(a)(b)(c)}$. The dielectric permeability can be decomposed according to

$$\varepsilon^{ik} = - \sum_{(a)} \varepsilon_{(a)} X^i_{(a)} X^k_{(a)}, \quad \varepsilon^k_k = \varepsilon_{(1)} + \varepsilon_{(2)} + \varepsilon_{(3)},$$

(79)

where the terms $\varepsilon_{(a)}$ denote the (positive) eigenvalues, corresponding to the eigenvector $X^i_{(a)}$. Since the tensor $\varepsilon^{ik}$ is orthogonal to the four-velocity vector $U^i$, the corresponding eigenvalue $\varepsilon_{(0)}$ is equal to zero, and the velocity does not appear in this decomposition.

The tetrad components $k_{(a)}$ and $A_{(b)}$ of the wave four-vector and of the electromagnetic potential four-vector, respectively, are

$$k_{(a)} \equiv k_i X^i_{(a)}, \quad A_{(b)} \equiv A_i X^i_{(a)}$$

(80)

and we may write

$$k^i = U^i (U^j k_j) - k_{(a)} X^i_{(a)}, \quad A^i = - A_{(a)} X^i_{(a)},$$

(81)

The tensor fields $g^{im(\alpha)}$ are describable in terms of $U^i$ and $X^i_{(a)}$ according to (cf. (51))

$$g^{im(\alpha)} = U^i U^m + S^{(a)(b)}_{(\alpha)} X^i_{(a)} X^m_{(b)},$$

(82)

where

$$S^{(a)(b)}_{(\alpha)} = S^{m n}_{(\alpha)} X^m_{(a)} X^n_{(b)},$$

(83)

and where we again have neglected the magneto-electric cross effects. In the following we use the first and second equations (13) to construct the tensors $g^{im(A)}$ and $g^{im(B)}$.

4.4.2 The tensor $g^{im(A)}$

While $g^{im(A)}$ is already known to be given by (58), we briefly sketch how it is alternatively derived in the tetrad formalism. Together with (82), the first equation (13) becomes

$$2 C^{ikmn} U_k U_n = - \varepsilon_{(a)} \delta^{(a)(b)} X^i_{(a)} X^m_{(b)}.$$

(84)

With $\Delta^{ik} = - \delta^{(a)(b)} X^i_{(a)} X^k_{(b)}$ it takes the form

$$\frac{1}{\mu} X^i_{(a)} X^m_{(b)} \sum_{(\alpha)(\beta)} G^{(\alpha)(\beta)} S^{(a)(b)}_{(\alpha)} = - \varepsilon_{(a)} \delta^{(a)(b)} X^i_{(a)} X^m_{(b)}.$$

(85)
Contraction with the tetrad vectors yields
\[ \sum_{(\alpha)(\beta)} G_{(\alpha)(\beta)} S_{(\alpha)(b)}^{(a)(b)} = - \hat{\mu} \varepsilon_{(a)} \delta^{(a)(b)}. \] (86)

With the coefficients (54) we obtain
\[ S_{(A)}^{(a)(b)} = - \hat{\mu} \varepsilon_{(a)} \delta^{(a)(b)}, \] (87)
such that the first of the tensors is
\[ g_{im}^{(A)} = U^i U^m + \hat{\mu} \varepsilon^{im}. \] (88)

This reproduces (58). For the choice \( \hat{\mu} = \frac{1}{\mu^{(1)}} \) and for \( \varepsilon^{ik} = \varepsilon^{\Delta ik} \) we recover the optical metric (27) for the isotropic medium.

### 4.4.3 The tensor \( g^{im(B)} \)

Applying the tetrad representation (22) to the second equation (13) yields
\[ - \frac{1}{2} \eta_{\mu \nu} C^{ikmn} \eta_{mnq} = - (\mu^{-1})_{(a)(b)} X^a_{\mu} X^b_{\nu}, \] (89)

where \( - (\mu^{-1})_{(a)(b)} \) are the tetrad components of the magnetic permeability tensor. With (78) it transforms into
\[ G_{(\alpha)(\beta)} S_{(a)}^{(c)(d)} S_{(b)}^{(e)(f)} \epsilon_{(a)(c)(e)} \epsilon_{(b)(d)(f)} = 2 \hat{\mu} (\mu^{-1})_{(a)(b)}. \] (90)

Here \( \epsilon_{(a)(c)(e)} \) is the Levi-Civita symbol in three-dimensional space, tetrad indices being regarded as Euclidean. The equations (90) form a system of 6 quadratic equations for the 6 components of \( S^{(a)(b)}_{(B)} \). Let us restrict ourselves to the case of a diagonal \( (\mu^{-1})_{(a)(b)} \) tensor, i.e., to
\[ (\mu^{-1})_{(a)(b)} = \frac{1}{\mu_{(a)}} \delta_{(a)(b)}, \quad \mu_{(a)} > 0. \] (91)

This ansatz allows us to search for an explicit solution of the basic equations (90) for which the \( S_{(B)}^{(a)(b)} \) are diagonal, i.e.,
\[ S_{(B)}^{(a)(b)} = 0, \text{ if } (a) \neq (b). \] (92)

Using the structures (54) of \( G_{(\alpha)(\beta)} \) and (91) of \( (\mu^{-1})_{(a)(b)} \), we obtain from (90) the system of equations
\[ S_{(B)}^{(2)(2)} S_{(B)}^{(3)(3)} + S_{(B)}^{(2)(2)} \hat{\mu} \varepsilon_{(3)} + S_{(B)}^{(3)(3)} \hat{\mu} \varepsilon_{(2)} + (1 - \gamma^{-1}) \hat{\mu}^2 \varepsilon_{(2)} \varepsilon_{(3)} = - \frac{\hat{\mu}}{\gamma \mu_{(1)}}, \] (93)
\[
S^{(1)(1)}_B + S^{(3)(3)}_B + S^{(1)(1)}_B \tilde{\mu} \varepsilon^{(1)} + S^{(3)(3)}_B \tilde{\mu} \varepsilon^{(3)} + (1 - \gamma^{-1}) \tilde{\mu}^2 \varepsilon^{(1)} \varepsilon^{(3)} = -\frac{\tilde{\mu}}{\gamma \mu^{(2)}}, \tag{94}
\]

\[
S^{(1)(1)}_B + S^{(2)(2)}_B + S^{(1)(1)}_B \tilde{\mu} \varepsilon^{(2)} + S^{(2)(2)}_B \tilde{\mu} \varepsilon^{(1)} + (1 - \gamma^{-1}) \tilde{\mu}^2 \varepsilon^{(1)} \varepsilon^{(2)} = -\frac{\tilde{\mu}}{\gamma \mu^{(3)}}. \tag{95}
\]

After the substitution

\[
S^{(a)(b)}_B = -\tilde{\mu} \varepsilon^{(a)} \delta^{(a)(b)} + Y^{(a)(b)}, \tag{96}
\]

where the first term on the right-hand side coincides with \( S^{(a)(b)}_A \) from (87), the system (93)-(95) takes the form

\[
Y^{(2)(2)} Y^{(3)(3)} = \tilde{\mu}^2 \varepsilon^{(2)} \varepsilon^{(3)} M^{(1)}, \tag{97}
\]

\[
Y^{(1)(1)} Y^{(3)(3)} = \tilde{\mu}^2 \varepsilon^{(1)} \varepsilon^{(3)} M^{(2)}, \tag{98}
\]

\[
Y^{(1)(1)} Y^{(2)(2)} = \tilde{\mu}^2 \varepsilon^{(1)} \varepsilon^{(2)} M^{(3)}, \tag{99}
\]

where

\[
M^{(a)} \equiv 1 - \frac{\varepsilon^{(a)}}{\mu^{(a)} \tilde{\mu} \varepsilon^{(1)} \varepsilon^{(2)} \varepsilon^{(3)}}. \tag{100}
\]

In the isotropic limit with \( \varepsilon^{(a)} = \varepsilon, \mu^{(a)} = \mu \) and \( \tilde{\mu} = 1/(\varepsilon n^2) \), the \( M^{(a)} \) quantities vanish identically. Solving the symmetric system (97)-(99) leads to the following solutions of the system (93)-(95):

\[
S^{(1)(1)}_B = -\tilde{\mu} \varepsilon^{(1)} \left[ 1 \pm \sqrt{\frac{M^{(2)} M^{(3)}}{\gamma M^{(1)}}} \right], \tag{101}
\]

\[
S^{(2)(2)}_B = -\tilde{\mu} \varepsilon^{(2)} \left[ 1 \pm \sqrt{\frac{M^{(1)} M^{(3)}}{\gamma M^{(2)}}} \right], \tag{102}
\]

\[
S^{(3)(3)}_B = -\tilde{\mu} \varepsilon^{(3)} \left[ 1 \pm \sqrt{\frac{M^{(1)} M^{(2)}}{\gamma M^{(3)}}} \right]. \tag{103}
\]

Thus, the second symmetric tensor is

\[
g^{im(B)} = U^i U^m + \tilde{\mu} \varepsilon^{im} + \tilde{\mu} \sqrt{M^{(1)} M^{(2)} M^{(3)}} \sum_{(a)=(1)} \frac{\varepsilon^{(a)} X^i_{(a)} X^m_{(a)}}{M^{(a)}}. \tag{104}
\]

The first two terms on the right-hand side coincide with \( g^{im(A)} \) from (88). In the isotropic limit we have \( g^{im(B)} = g^{im(A)} = g^{im} \). The parameter \( \gamma \) remains undetermined. Using the explicit expressions (88) and (104) in (55) it is obvious that \( C^{ikmn} \) is independent of \( \gamma \). A proper choice of \( \gamma \) will be essential, however, to make \( g^{im(B)} \) an optical metric in special cases to be discussed below.
4.5 The characteristic equation

To obtain the solution of Maxwell’s equations \(C_{ikmn}k_kk_mA_n = 0\) with \([55]\) in the approximation of geometrical optics, we project these equations onto the directions \(X_i(a)\),

\[
\gamma \left( S_{(A)}^{(a)(b)} - S_{(B)}^{(a)(b)} \right) \left[ k_{(b)}(k, A_{(b)}) + A_{(b)}(H_{(A)} - H_{(B)}) \right] - S_{(A)}^{(a)(b)} A_{(b)} H_{(A)} = 0. \quad (105)
\]

Here we introduced the following abbreviations:

\[
(k, A)_{(B)} \equiv g^{im(B)}k_iA_m, \quad H^{(a)} \equiv g^{im(a)}k_i k_n. \quad (106)
\]

Explicitly we obtain

\[
(k, A)_{(B)} = \mp \hat{\mu} \sqrt{\frac{M_{(1)}M_{(2)}M_{(3)}}{\gamma}} \sum_{(c) = (1)}^{(3)} \frac{\varepsilon_{(c)}k_{(c)}A_{(c)}}{M_{(c)}}, \quad (107)
\]

\[
H^{(A)} = (k_i U^i)^2 - \hat{\mu} \sum_{(c) = (1)}^{(3)} \varepsilon_{(c)} k_{(c)}^2, \quad (108)
\]

\[
H^{(A)} - H^{(B)} = \pm \hat{\mu} \sqrt{\frac{M_{(1)}M_{(2)}M_{(3)}}{\gamma}} \sum_{(c) = (1)}^{(3)} \frac{\varepsilon_{(c)}k_{(c)}^2}{M_{(c)}}. \quad (109)
\]

The equations \([105]\) may then be written as

\[
\delta^{(a)(b)} \left\{ \mp \hat{\mu} \sqrt{\frac{M_{(1)}M_{(2)}M_{(3)}}{M_{(a)}}} A_{(b)} \sum_{(c) = (1)}^{(3)} \frac{\varepsilon_{(c)}k_{(c)}^2}{M_{(c)}} - k_{(b)} \sum_{(c) = (1)}^{(3)} \frac{\varepsilon_{(c)}k_{(c)}A_{(c)}}{M_{(c)}} \right\} + A_{(b)} H^{(A)} \right\} = 0, \quad (110)
\]

or, in components,

\[
\left( \frac{H^{(A)}}{\hat{\mu}M_{(2)}M_{(3)}} + \frac{\varepsilon_{(2)}k_{(2)}^2}{M_{(2)}} + \frac{\varepsilon_{(3)}k_{(3)}^2}{M_{(3)}} \right) A_{(1)} - \frac{\varepsilon_{(2)}k_{(1)}k_{(2)}}{M_{(2)}} A_{(2)} - \frac{\varepsilon_{(3)}k_{(1)}k_{(3)}}{M_{(3)}} A_{(3)} = 0, \quad (111)
\]

\[
- \frac{\varepsilon_{(1)}k_{(1)}k_{(2)}}{M_{(1)}} A_{(1)} + \left( \frac{H^{(A)}}{\hat{\mu}M_{(1)}M_{(3)}} + \frac{\varepsilon_{(1)}k_{(1)}^2}{M_{(1)}} + \frac{\varepsilon_{(3)}k_{(3)}^2}{M_{(3)}} \right) A_{(2)} - \frac{\varepsilon_{(3)}k_{(2)}k_{(3)}}{M_{(3)}} A_{(3)} = 0, \quad (112)
\]

18
\[
\begin{align*}
- \frac{\varepsilon(1)k(1)k(3)}{M(1)} A(1) - \frac{\varepsilon(2)k(2)k(3)}{M(2)} A(2) \\
+ \left( \frac{H^{(A)}}{\mu M(1)M(2)} + \frac{\varepsilon(1)k^2(1)}{M(1)} + \frac{\varepsilon(2)k^2(2)}{M(2)} \right) A(3) &= 0. \quad (113)
\end{align*}
\]

In addition, we have to consider Eq. (75), which in terms of the tetrad components reads

\[
k(1)\varepsilon(1)A(1) + k(2)\varepsilon(2)A(2) + k(3)\varepsilon(3)A(3) = 0. \quad (114)
\]

Formally, we have four equations to find three amplitude coefficients \(A(a)\). However, one can check directly, that one of the set of equations (113) - (115), say (113), happens to be a linear combination of two others, (111) and (112), and of (114). Thus, we can find one of the quantities, say \(k(3)\varepsilon(3)A(3)\), from (114) and insert it into (111) and (112). This provides us with a system of two equations for \(A(1)\) and \(A(2)\):

\[
\begin{align*}
\left( \frac{H^{(E)}}{\mu M(2)M(3)} + \frac{\varepsilon(2)k^2(2)}{M(2)} + \frac{\varepsilon(1)k^2(1) + \varepsilon(3)k^2(3)}{M(3)} \right) A(1) \\
- \left( \frac{1}{M(2)} - \frac{1}{M(3)} \right) \varepsilon(2)k(1)k(2)A(2) &= 0, \quad (115)
\end{align*}
\]

\[
\begin{align*}
\left( \frac{1}{M(1)} - \frac{1}{M(3)} \right) \varepsilon(1)k(1)k(2)A(1) \\
- \left( \frac{H^{(E)}}{\mu M(1)M(3)} + \frac{\varepsilon(1)k^2(1)}{M(1)} + \frac{\varepsilon(2)k^2(2) + \varepsilon(3)k^2(3)}{M(3)} \right) A(2) &= 0. \quad (116)
\end{align*}
\]

The solution of the system (115), (116) requires the Cramer determinant to be zero. The resulting characteristic equation is

\[
\pm \sqrt{\gamma M(1)M(2)M(3)} \left( \varepsilon(1)k^2(1) + \varepsilon(2)k^2(2) + \varepsilon(3)k^2(3) \right) \left( \frac{H^{(A)} - H^{(B)}}{\mu} \right) \\
+ \left( \frac{H^{(A)}}{\mu} \right) \left[ (M(2) + M(3))\varepsilon(1)k^2(1) + (M(1) + M(3))\varepsilon(2)k^2(2) \\
+ (M(1) + M(2))\varepsilon(3)k^2(3) + \left( \frac{H^{(A)}}{\mu} \right) \right] = 0. \quad (117)
\]

This is a fourth order equation for the wave vector. It corresponds to the extended Fresnel equation in \[11\] (Equations (D.2.23) and (D.2.44)). In the isotropic limit \(M_1 = M_2 = M_3 = 0\) we have the double solution

\[
\left( H^{(A)} \right)^2 = \left( H^{(B)} \right)^2 = (g^{ij}k_i k_j)^2 = 0, \quad (118)
\]
which coincides with (42). For \( n^2 = 1 \) we recover, of course, the vacuum case (37).

### 4.6 The uniaxial case

Obviously, \( H^{(A)} = 0 \) is also a solution of (117) if only two of the three quantities \( M_1, M_2 \) and \( M_3 \) are zero. Let’s assume here \( M_2 = M_3 = 0 \). For the uniaxial configuration

\[
\varepsilon_2 = \varepsilon_3 = \varepsilon, \quad \varepsilon_1 = \varepsilon (1 - \xi), \quad \mu_{(a)} = \mu,
\]

where \( 0 < \xi < 1 \), we have

\[
M_2 = M_3 = 0 \implies \hat{\mu} = \frac{1}{n^2 \varepsilon (1 - \xi)} \implies M_1 = \xi.
\]

For the expression multiplying \( H^{(A)}/\hat{\mu} \) in the brackets in (117) we may write

\[
(M_2 + M_3) \varepsilon_1 k^2_1 + (M_1 + M_3) \varepsilon_2 k^2_2 + (M_1 + M_2) \varepsilon_3 k^2_3 = \xi \varepsilon (k^2_2 + k^2_3).
\]

On the other hand, from (109),

\[
\frac{H^{(A)} - H^{(B)}}{\hat{\mu}} = \pm \sqrt{\frac{M_1}{\gamma}} \varepsilon (k^2_2 + k^2_3).
\]

With the help of the last relation the characteristic equation reduces to

\[
\frac{H^{(A)}}{\hat{\mu}} \left[ \frac{H^{(B)}}{\hat{\mu}} + \varepsilon (k^2_2 + k^2_3) \left( \xi \pm \sqrt{\frac{\xi}{\gamma}} \right) \right] = 0.
\]

For the choice

\[
\gamma = \frac{1}{\xi},
\]

and with the minus sign equations (123) factorizes into

\[
H^{(A)} H^{(B)} = (g^{mn(A)} k_m k_n) (g^{ij(B)} k_i k_j) = 0
\]

with solutions \( g^{mn(A)} k_m k_n = 0 \) and \( g^{ij(B)} k_i k_j = 0 \). Both the symmetric tensor fields \( g^{mn(A)} \) and \( g^{ij(B)} \) are optical metrics in such a case, i.e., there are two light cones describing the phenomenon of birefringence. The first optical metric
The solution $g^{mn(A)}k_mk_n = 0$ corresponds to a dispersion relation

$$\omega^2 = \frac{1}{n^2} \left( k_{(1)}^2 + \frac{k_{(2)}^2 + k_{(3)}^2}{1 - \xi} \right),$$

(127)

where $\omega \equiv k_mU_m$. This characterizes the so-called extraordinary wave. With the refractive index three-vector $n_{(a)}$, defined by

$$k_{(a)} = \omega n_{(a)},$$

(128)

it can be described as a Fresnel ellipsoid

$$\frac{n_{(1)}^2}{n^2} + \frac{n_{(2)}^2}{n^2 (1 - \xi)} + \frac{n_{(3)}^2}{n^2 (1 - \xi)} = 1.$$  

(129)

In the expression (104) for $g^{mn(B)}$ the anisotropic contributions exactly cancel for $\gamma = \xi^{-1}$ and we recover

$$g^{mn(B)} = U^m U^n + \frac{1}{n^2} \Delta^{mn},$$

(130)

which coincides with the optical metric $g^{mn}$ in (23). Here, the dispersion relation is

$$\omega^2 = \frac{1}{n^2} \left( k_{(1)}^2 + k_{(2)}^2 + k_{(3)}^2 \right),$$

(131)

which describes ordinary wave propagation.

Consequently, we have obtained an explicit representation for the material tensor $C^{ijkmn}$ with the optical metrics (126) and (130), with $\hat{\mu}$ from (124) and with $\gamma = \xi^{-1}$ according to (124). This explicit representation within the context of the geometric background revealed in (59) - (70) is the main result of this paper. While the general form of the decomposition remains true in the general case, the interpretation of $g^{mn(A)}$ and $g^{mn(B)}$ as optical metrics is valid only for uniaxial symmetry. Of course, the relations (59) - (62) between the optical metrics $g^{mn(A)}$ and $g^{mn(B)}$ and $g^{mn(1)}$ and $g^{mn(2)}$ hold in this special case as well. This means, a representation (105) is possible although $g^{mn(1)}$ and $g^{mn(2)}$ do not have an obvious physical meaning.
4.7 The energy of an electromagnetic wave

Finally, we briefly discuss some features of the electromagnetic energy-momentum tensor in this approach. The Minkowski tensor of the electromagnetic field reads

\[ T_{ij} \equiv \frac{1}{4} g_{ij} F_{mn} H^{mn} - F_{im} H_{jn} g^{mn} \]  

(132)

Taking into account Maxwell’s equations and the representation of the field strength tensor (notice that for the square of the field strength the complex conjugate has to be taken), one realizes that the scalar \( F_{mn} H^{mn} \) vanishes and the energy-momentum tensor takes the simple form

\[ T_{ij} = -2 k_i C_{j}^{lmn} A_l k_m A_n. \]  

(133)

In terms of the optical metrics and with \( (75) \) and \( (54) \) the energy scalar is equal to

\[ W \equiv T_{ij} U^i U^j = \frac{\omega^2}{\mu} g^{ln(A)} A_l A_n = -\omega^2 \varepsilon^{ln} A_l A_n. \]  

(134)

With \( (76), (79) \) and \( (81) \) one checks that this quantity is positive. Incidentally, the energy density of the electromagnetic field does not depend on whether it is calculated from the Minkowski or from the Abraham tensor [15]. Remarkably, it is only the extraordinary optical metric \( g^{ln(A)} \) from \( (126) \) which enters the expression \( (134) \). The ordinary optical metric \( g^{ln(B)} \) does not contribute here.

Generally, the Minkowski tensor is not symmetric. In the isotropic case, however, there exists a generalized symmetry (cf. [11], (E.3.50)). With \( (133), (29), (25) \) and \( (40) \) it takes the form

\[ g_{ia} T^i = \frac{\eta^4}{\mu} k_b k_a g^{skn} A_k A_n = g_{ia} T^i \]  

(135)

in the present context. A corresponding relation is not found in the anisotropic case.

5 Discussion

We have shown that the material tensor of an anisotropic medium with uniaxial symmetry can be constructed out of those two optical metrics which describe ordinary and extraordinary light propagation. This generalizes the corresponding representation of the material tensor in isotropic media in terms of Gordon’s optical metric. Furthermore, we have clarified the internal geometrical structure which underlies the decomposition of the material tensor into a combination of products of two symmetric second rank tensor fields. The latter quantities can be regarded as vectors in an associated two dimensional space where the coefficients of the decomposition play the role of a metric in this space. While we believe this type of bi-metric representation in connection with the interpretation in terms of an internal metric structure to be new, some already known results
are reproduced here in a different context as well. Generally, the wave propagation in anisotropic media is governed by a quartic equation for the wave vector, called (extended) Fresnel equation \[11\]. This equation is known to factorize into two quadratic equations in vacuum. A similar feature is true for isotropic media. Here, the factorizing implies a double solution for the wave vector as a null vector with respect to the optical metric. For an anisotropic medium with uniaxial symmetry a factorization is obtained as well. Instead of the isotropic double solution there are two solutions in terms of two optical metrics now, one of them describing ordinary, the other one extraordinary wave propagation. All these features follow as special cases of our approach. Furthermore, we recover a (generalized) symmetry property of Minkowski’s energy-momentum tensor for isotropic media.

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