A uniqueness theorem for the AdS soliton

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The stability of physical systems depends on the existence of a state of least energy. In gravity, this is guaranteed by the positive energy theorem. For topological reasons this fails for nonsupersymmetric Kaluza-Klein compactifications, which can decay to arbitrarily negative energy. For related reasons, this also fails for the AdS soliton, a globally static, asymptotically toroidal \( \Lambda < 0 \) spacetime with negative mass. Nonetheless, arguing from the AdS/CFT correspondence, Horowitz and Myers proposed a new positive energy conjecture, which asserts that the AdS soliton is the unique state of least energy in its asymptotic class. We give a new structure theorem for static \( \Lambda < 0 \) spacetimes and use it to prove uniqueness of the AdS soliton. Our results offer significant support for the new positive energy conjecture and add to the body of rigorous results inspired by the AdS/CFT correspondence.

The positive energy theorem singles out Minkowski spacetime as the “ground state”, or spacetime of lowest mass-energy, within the class of asymptotically flat spacetimes with local energy density \( \geq 0 \) and without naked singularities. In the presence of a negative cosmological constant the appropriate ground state is anti-de Sitter spacetime. These ground states are regular, globally static, supersymmetric, and of constant curvature. Moreover, it is known that Minkowski spacetime is the unique asymptotically flat, regular, stationary vacuum spacetime. The analogous uniqueness result for the asymptotically globally adS case is proved in [3].

A simple scaling argument suggests that a ground state cannot have negative mass, for then it could be scaled to produce a state of even lesser mass. Consider, then, the surprising properties of the adS soliton, first examined by Horowitz and Myers, which is a negative mass, globally static Einstein spacetime with cosmological constant \( \Lambda < 0 \). The metric in \( n + 1 \geq 4 \) spacetime dimensions is

\[
ds^2 = -r^2 dt^2 + \frac{1}{V(r)} dr^2 + V(r) d\phi^2 + r^{n-2} \sum_{i=1}^{n-2} (dy_i)^2
\]

where \( V(r) = \frac{r^2}{\ell^2} \left( 1 - \frac{r_0^n}{r^n} \right) \), \( \ell^2 = -\frac{n(n-1)}{2\Lambda} \), and \( r_0 \) is a constant. Regularity demands that \( \phi \) be identified with period \( \beta_0 = \frac{2\pi}{\ell^2} \). The periods of the \( y_i \) are arbitrary. The soliton is “asymptotically locally anti-de Sitter” with boundary at conformal infinity (scri) foliated by spacelike \( (n-1) \)-tori. The time slices of spacetime itself, when conformally completed, are topologically the product of an \( (n-2) \)-torus and a disk (a solid torus in \( 3 + 1 \)). The soliton spacetime is neither supersymmetric nor of constant curvature, but has minimal energy under small metric perturbations. Remarkably, one cannot vary the soliton mass by simple scaling. To see why, note that rescaling the parameter \( r_0 \rightarrow kr_0 \) in [3] has the same effect as the coordinate transformation \( (t, r, y', \phi) \rightarrow (kt, k^{-1}r, ky', k\phi) \). Thus the new metric (with parameter \( kr_0 \)) is isometric to the original one, provided the conformal boundary data are chosen to agree, so they must have the same physical mass.

Horowitz and Myers found that the negative mass of the adS soliton has a natural interpretation as the Casimir energy of a non-supersymmetric gauge theory on the conformal boundary. If a non-supersymmetric version of the adS/CFT conjecture holds, as is generally hoped, then this would indicate that the soliton is the lowest energy solution with these boundary conditions. This led them to postulate a new positive energy conjecture, that the soliton is the unique lowest mass solution for all spacetimes in its asymptotic class. The validity of this conjecture is thus an important test of the non-supersymmetric version of the adS/CFT correspondence. The conjecture is all the more remarkable because the soliton topology has certain circles that are not contractible at infinity but are contractible in the bulk. This leads to the failure of spinorial methods to produce a positive energy theorem here and is linked to a known instability in Kaluza-Klein theory.

As support for the new positive energy conjecture, we will give a uniqueness theorem for the adS soliton, singling it out as the only suitable ground state in the class of spacetimes with similar asymptotics. Our theorem is similar in spirit to [2], but relates to asymptotically “locally” adS spacetimes with Ricci flat conformal boundary. The proof is based on the fact that the soliton indeed shares one important geometric property with known ground states: its universal cover admits a foliation by totally geodesic null surfaces rules by complete, achronal null geodesics called null lines. In other words, the spacetime admits non-focusing plane waves. A similar idea underlies the approach to mass positivity in [2], a significant difference being that the null lines we will construct here do not approach scri. Our construction of the null surfaces relies on results in [3].
Our arguments will be of necessity terse. Herein we give the flavour of the proof; details will appear in [14]. Our results considerably generalize a uniqueness result of [13]. Of related interest is a uniqueness theorem of Anderson for asymptotically hyperbolic Einstein metrics on 4-dimensional Riemannian manifolds [13].

Following the formalism of Chruściel and Simon [6], we consider static spacetimes of the form

\[ M^{n+1} = \mathbb{R} \times \Sigma, \quad g = -N^2 dt^2 + h, \]

where \((\Sigma, h, N)\) is conformally compactifiable. Thus we assume that \((\Sigma, h, N)\) is the interior of a compact manifold with boundary \(\bar{\Sigma} = \Sigma \cup \partial \Sigma\) such that \((\Sigma, h, N)\) extends to a smooth function \(\bar{N}\) on \(\bar{\Sigma}\), with \(\bar{N} = 0\) and \(d\bar{N} \neq 0\) along \(\partial \Sigma\) and \((\Sigma, h, N)\) satisfies condition (C). This ba-

\[ \Delta = \nabla^2, \quad \nabla \text{ is the covariant derivative on } (\Sigma, h, N) \]

\[ \tilde{\Delta} \tilde{N} = \frac{2\Lambda}{n-1} + n\tilde{W}, \]

where \(\tilde{W} := \tilde{h}^{ab}\tilde{\nabla}_a \tilde{\nabla}_b \tilde{N} = N^{-2}\tilde{h}^{ab}\nabla_a N \nabla_b N\).

Let us now consider how condition (C) relates to the sign of the mass. As we are using the conformal approach, the Ashtekar-Magnon [13] mass expression involving the electric part of the Weyl tensor is especially convenient. Consider a conformally compactifiable static spacetime \((\Sigma, h, N)\), and view \(\Sigma\) as the slice \(t = 0\) passing through a point \(p \in \Sigma\) lifts to a unique future directed null geodesic \(\gamma\) passing through \(p\). Moreover, if \(\gamma\) is a length minimizing geodesic segment in \((\Sigma, \tilde{h})\) then it lifts to an achronal null geodesic segment \(\eta\). Thus, a line (inextendible geodesic, length minimizing on each segment) in \((\Sigma, \tilde{h})\) lifts to a null line (achronal inextendible null geodesic) in \((M, g)\). This ba-

\[ \tilde{h} = \tilde{W}^{-1} dx^2 + h_{AB}(x, x^C)dx^A dx^B, \]

where \(h_{AB}\) is the induced metric on \(x = \text{const}\) surfaces. Let \(T^a\) denote the future-timelike unit normal to \(\Sigma\) in the rescaled spacetime metric, and on \(\Sigma\), let \(n^a = -\tilde{W} \partial_x\) be the outward pointing unit normal field to the slices \(x = \text{const}\) near the conformal boundary. The Weyl mass is then given, up to a positive constant, by \(\int_{\Sigma} \mu dA\), where \(dA\) is the volume element on \(\partial \Sigma\). The mass aspect \(\mu\), up to a positive constant, is given by
\[
\mu = \lim_{x \to 0} \frac{\tilde{E}_{ac}T^aT^c}{x^{n-2}} = \lim_{x \to 0} \frac{\tilde{C}_{abcd}n^bn^dT^aT^c}{x^{n-2}},
\]

where \(\tilde{C}_{abcd}\) is the Weyl tensor (and \(\tilde{E}_{ac}\) its electric part) of the conformal spacetime metric \(\tilde{g} = N^{-2}g\).

In the static setting, the mass aspect \(\mu\) can be directly related to the geometry of \((\Sigma, \tilde{h})\). One finds, using the field equations, that, up to a positive constant, \(\mu = -\partial \tilde{R}/\partial x^{n-2}\big|_{x=0}\). (Chruściel and Simon had previously identified, in the 3 + 1 dimensional case, \(-\partial \tilde{R}/\partial x|_{x=0}\) as the mass aspect.)

Let \(\tilde{H}\) denote the mean curvature function of the slices \(x = \text{const}\) with respect to the outward normal \(n^a\); along each such slice, \(\tilde{H} = \nabla_ao^a = \text{the trace of the second fundamental form} = \text{the sum of the principal curvatures.}\)

Using the field equations and the Gauss equation, one can show that \(\tilde{H}\) is related to \(\mu\) by

\[(n-2)\sqrt{W}\tilde{H} = -\frac{x^{n-1}}{2(n-1)}\mu + O(x^n)\]

when the conformal boundary has Ricci flat induced metric. In order to establish Equation (11), we must carry out a Fefferman-Graham type boundary analysis, for the metric (6) relevant to our situation, and subject to the field equations (5) and (6). This analysis implies \(\tilde{R}|_{x=0} = 0\) and \(\partial^2 \tilde{R}/\partial x|_{x=0} = 0\) for \(1 \leq \ell \leq n-3\).

For Ricci flat conformal boundary, as is the case for the soliton, equation (11) implies that if the mass aspect \(\mu\) is (pointwise) negative, the level surfaces \(\tilde{N} = c\) near conformal infinity are outwardly mean convex in \((\tilde{\Sigma}, \tilde{h})\), i.e., have strictly positive mean curvature (and hence the sum of the principal curvatures is positive). In other words, if the mass aspect is negative (as it is for the adS soliton) then condition (C) holds in the mean. However, our proof of uniqueness of the adS soliton requires not just mean convexity, but (weak) convexity near infinity, and so we need to impose the following condition. We say that \((\Sigma, h, N)\) satisfies condition (S) provided the principal curvatures of the level surfaces \(\tilde{N} = c\) near infinity are either all non-negative or all non-positive. This condition holds trivially in the adS soliton, since all but one of the principal curvatures vanish. It also holds in the Kottler spacetimes, regardless of the sign of the mass.

**Theorem 2:** Consider a static spacetime as in (3) such that (i) \((\Sigma, h, N)\) is conformally compactifiable, (ii) the static vacuum field equations hold, and (iii) condition (S) holds. Suppose in addition, that

(a) The boundary geometry of \((\tilde{\Sigma}, \tilde{h})\) is the same as that of (1), i.e., \(\partial \tilde{\Sigma} = T^{n-2} \times S^1, \tilde{h}|_{\partial \tilde{\Sigma}} = \delta \phi^2 + \sum_{i=1}^{n-2} (dy^i)^2\), with the same periods for \(\phi\) and the \(y^i\).

(b) The mass aspect \(\mu\) of \((\Sigma, h, N)\) is pointwise negative.

(c) Given the inclusion map \(i : \partial \tilde{\Sigma} \to \tilde{\Sigma}\), the kernel of the induced homomorphism of fundamental groups, \(i_* : \Pi_1(\partial \tilde{\Sigma}) \to \Pi_1(\tilde{\Sigma})\), is generated by the \(S^1\) factor.

Then the spacetime (3) determined by \((\Sigma, h, N)\) is isometric to the adS soliton (4).

Assumption (a) is a natural boundary condition. Assumption (b), together with condition (S), guarantees that condition (C) of Theorem 1 holds. Assumption (c) asserts that the generator of the \(S^1\) factor is contractible in \(\Sigma\), and moreover, that any loop in \(\partial \Sigma\) contractible in \(\Sigma\) is a multiple of the generator. As discussed in the proof, assumptions (a) and (c) together imply that \(\Pi_1(\Sigma) \approx \mathbb{Z}^{n-2}\). Were we to adopt the latter condition in lieu of assumption (c), then one could only conclude that \((\Sigma, h, N)\) is locally isometric to the adS soliton (the universal covers will be isometric, however). For further discussion of this discrete nonuniqueness relevant to the adS soliton in 3 + 1 dimensions, see [13]. We note that in 3 + 1 dimensions, assumption (b) (when \(\mu\) is constant), and the condition \(\Pi_1(\Sigma) \approx \mathbb{Z}^{n-2}\), hold automatically, cf. (3, 21).

We now sketch the proofs of Theorems 1 and 2; details will appear in [14].

**Sketch of the proof of Theorem 1.** We proceed inductively, working in \((\tilde{\Sigma}^*, \tilde{h}^*)\). If \(\Sigma^*\) is compact then Theorem 1 holds with \(k = 0\). So suppose \(\Sigma^*\) is noncompact. In this case there is a procedure for constructing a line in \((\tilde{\Sigma}^*, \tilde{h}^*)\). Fix a point \(p\) in the interior, and let \(\{p_i\}\) be a sequence of points uniformly bounded away from \(\partial \Sigma^*\), such that the distance from \(p\) to \(p_i\) tends to infinity. For each \(i\), \(p\) and \(p_i\) can be joined by a length minimizing geodesic segment \(\gamma_i\) which cannot meet the boundary (since it’s totally geodesic). In fact, by condition (C), the segments \(\gamma_i\) must be uniformly bounded away from \(\partial \Sigma^*\). Each geodesic segment \(\gamma_i\) will have a midpoint \(r_i\). Now since \(\Sigma^*\) is compact, it will have a compact fundamental domain \(D\) in \(\Sigma^*\). For each point in the covering space, there will be a covering space transformation mapping that point to a point in \(D\). We therefore apply to each geodesic segment \(\gamma_i\) a covering space transformation that maps \(r_i\) into \(D\). This produces a sequence of minimizing geodesic segments \(\sigma_i\), still uniformly bounded away from \(\partial \Sigma^*\), whose lengths are unbounded in both directions. By standard compactness results, this sequence possesses a limit curve which is a complete, length minimizing geodesic, i.e., a line, in the interior of \(\Sigma^*\).

By the relationship between Fermat and null geodesics, this line lifts to a complete null line \(\eta\) in the physical covering spacetime \((M^*, g^*)\), which is null geodesically complete and obeys the null energy condition, \(R_{ab}X^aX^b \geq 0\) for all null vectors \(X^a\). Then a null line will exist only under special circumstances. As proved in [10], \(\eta\) must be contained in a smooth achronal edgeless null hypersurface \(H\) which is totally geodesic (i.e., has vanishing
expansion and shear). In a static spacetime this has further consequences. Since \( \Sigma^* \) (viewed as the slice \( t = 0 \)) is totally geodesic, \( \mathcal{H} \) meets \( \Sigma^* \) in a totally geodesic submanifold \( W \) of codimension one in \( \Sigma^* \). But by moving \( \mathcal{H} \) invariantly under the flow generated by \( \partial/\partial t \), we see that spacetime is actually foliated by totally geodesic null hypersurfaces, and this gives rise to a foliation \( \{ W_\mu \} \) of \( \Sigma^* \) by totally geodesic hypersurfaces in \( \Sigma^* \). Moreover, it can be seen that this foliation is achieved by exponentiating out along the unit speed normal geodesics to \( W \) in the Fermat metric. The physical metric then takes the form,

\[
h^* = N^*2(u, x^A) du^2 + h_{AB}(x) dx^A dx^B.
\]

(12)

Up to this point we have only used the null energy condition. Now using the field equations in a more explicit way, one can show \( \partial N^*/\partial u = 0 \), and hence the metric (12) is a genuine warped product. By multiplying (12) by \((N^*)^{-2}\), and showing that everything extends smoothly to the boundary, we conclude that \((\Sigma^*, h^*)\) is isometric to the Riemannian product \((\mathbb{R} \times \mathcal{W}, du^2 + d\tilde{a}^2)\), where \((\mathcal{W}, d\tilde{a}^2)\) is a complete Riemannian manifold-with-boundary. If \( \mathcal{W} \) is compact then Theorem 1 holds with \( k = 1 \). If \( \mathcal{W} \) is noncompact then one can carry out essentially the same procedure again to construct a line in \( \mathcal{W} \) lift it to a new null line spatially orthogonal to the first, and split off another \( \mathbb{R} \) factor. One can continue splitting off \( \mathbb{R} \) factors until what remains is compact.

**Sketch of the proof of Theorem 2.** Let \((\Sigma_0, h_0, N_0)\) denote the adS soliton associated with the boundary data in assumption (a), and let \((\Sigma_0, h_0, N_0)\) denote the corresponding conformally compactified soliton. Assumption (b), condition (S) and equation (11) imply that condition (C) holds, and hence we can apply Theorem 1. As shown below, \( \Pi_1(\Sigma) \approx \mathbb{Z}^{n-2} \), from which it follows that \( k = n - 2 \). Thus, the universal cover \( \check{\Sigma}^* \) splits isometrically as \( \mathbb{R}^{n-2} \times \mathcal{W} \), where \( \mathcal{W} \) is diffeomorphic to a disk, and the metric on \( \mathcal{W} \) is determined by the field equations (1) and (3). From topological censorship (20), it follows that the homomorphism \( i_* : \Pi_1(\partial \Sigma) \to \Pi_1(\Sigma) \) is onto. But \( \partial \Sigma = A \times B \), where \( A \) is the \((n-2)\) torus and \( B \) is the circle of assumption (a), whence \( \Pi_1(\partial \Sigma) \approx \Pi_1(A) \times \Pi_1(B) \).

Since by assumption (c), ker \( i_* = \Pi_1(B) \), it follows that \( i_*|_{\Pi_1(A)} : \Pi_1(A) \to \Pi_1(\Sigma) \) is an isomorphism. This implies that the covering transformations of \( \Sigma^* \) are in one-to-one correspondence, via \( i_*|_{\Pi_1(A)} \), to those of \( A^* \), the universal covering space of \( A \). Thus, \( \check{\Sigma} \approx \check{\Sigma}^*/\Pi_1(\Sigma) \approx (\mathbb{R}^{n-2} \times \mathcal{W})/\Pi_1(A) \approx (A^*/\Pi_1(A)) \times \mathcal{W} \approx A \times \mathcal{W} \), i.e., \( \check{\Sigma} \) is isometric to \( T^{n-2} \times \mathcal{W} \), where \( T^{n-2} \) is the torus in assumption (a). Because of the product structure of \( \Sigma \) and the fact that \( \tilde{N} \) depends only on \( \mathcal{W} \), the field equations (14) and (15) descend to the disk \( \mathcal{W} \), and can be solved explicitly and uniquely, subject to the appropriate boundary conditions on \( \partial \check{\Sigma} \). The result is that \( \check{\Sigma} \) is isometric to \( \Sigma_0 \) and \( \tilde{N} = \tilde{N}_0 \), from which we conclude that \((\Sigma, h, N)\) is isometric to the adS soliton \((\Sigma_0, h_0, N_0)\).

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