We present results for the interaction of two kaons at maximal isospin. The calculation is based on 2+1+1 flavour gauge configurations generated by the ETM Collaboration (ETMC) featuring pion masses ranging from about 230 MeV to 450 MeV at three values of the lattice spacing. The elastic scattering length $a_{I=1}^0$ is calculated at several values of the bare strange quark and light quark masses. We find $M_{K^0a_0} = -0.397(11)(^{+8}_{-6})$ as the result of a chiral and continuum extrapolation to the physical point. This number is compared to other lattice results.
1. Introduction

We investigated the scattering of 2 $K^+$-mesons in the maximum isospin channel of $I_3 = 1$. Due to the fact of $SU(3)$-symmetry breaking by the, compared to the light quark, heavy strange quark, $\chi$-PT often suffers from sizeable corrections. Lattice QCD offers a non-perturbative way to access this interaction and helps in understanding the dynamics of the underlying strong interaction at low energies. We focus on the elastic scattering length of this system which has not been determined experimentally. The analysis of the $K^+K^-$-system proceeds very similarly to the one of $\pi\pi$-scattering at $I_3 = 2$ detailed in ref. [1]. Our study is the first to take into account three values of the lattice spacing.

2. Elastic Scattering in Lüscher’s formalism

The scattering length $a_0$ for the scattering of two mesons is defined by the low energy limit

$$
\lim_{q \to 0} q \cot(\delta_0(q)) = -\frac{1}{a_0},
$$

where $q$ is the momentum transfer in the center of mass frame and $\delta_0$ the phaseshift of the outgoing wave function. In a series of papers, [4]- [5], Lüscher related the scattering length to the energy shift $\delta E$ of a system of two particles [5] in a finite volume. This energy shift is the deviation of the total energy of the interacting two particle system, $E_{KK}$, from its expected value in the absence of the interaction between the two particles, $2E_K$. For the case of two $K^+$-mesons at maximal isospin in the center of mass frame it reads:

$$
\delta E_{KK}^{I_3=1} = E_{KK} - 2E_K = -\frac{4\pi a_0}{M_K L^3} \left[ 1 + c_1 \frac{a_0}{L} + c_2 \left( \frac{a_0}{L} \right)^2 \right] + O(L^{-6}),
$$

with the spatial lattice extent $L$ and the kaon mass $M_K$, where the kaons are at rest.

3. Lattice Action and Operators

We work with Wilson twisted mass Lattice QCD (tmLQCD) at maximal twist introduced in ref. [6]. This guarantees automatic $O(a)$ improvement of all physical quantities of interest as shown in ref. [7]. The gauge configurations have 2+1+1 dynamical quark flavours and were generated by the ETMC. In total 3 different lattice spacings in the range from 0.619 fm to 0.885 fm and a pion mass range from 230 MeV to 450 MeV build a sound basis for continuum and chiral extrapolations. In this analysis we use twisted mass light valence quarks at maximal twist and Osterwalder-Seiler (OS) strange valence quarks on a Wilson twisted mass fermion-sea. This mixed action approach enables tuning of the valence strange quark mass to its physical value without having to regenerate gauge configurations. The cost of this is introducing small unitarity breaking effects. The Dirac operator for the light sector (used as valence and sea Dirac operator) reads:

$$
D_\ell = D_W + m_0 + i\mu_\ell \gamma_5 \tau^3,
$$

where $D_W$ is the Wilson Dirac operator. For the strange sector, which is purely valence, the twisted mass Dirac operator yields:

$$
D_s = \frac{1}{2} \left[ D_W + m_0 + i\mu_\ell \gamma_5 \tau^3 \right],
$$

with $\mu_\ell = m_0 + c_1 \frac{a_0}{L} + c_2 \frac{a_0}{L^2}$. The factor $\tau^3$ with values $\tau^3 = 0$ and $\tau^3 = 1$ for light and strange quarks, respectively, is inserted into the Dirac operator. The twisted mass parameter $a_0$ is adjusted such that the strange quark mass $m_s$ at physical $\tau_0$ extrapolated to the $a_s = 0$ limit matches the physical strange quark mass $m_s^\text{phys}$. The lattice spacing $L$ is related to the physical distance $r$ via $L = \sqrt{4\pi a_0 / m_0}$.
with the Wilson Dirac operator $D_W$ and the Pauli matrices $\tau^i, i = 1,2,3$ acting in flavour space. The parameter $\mu_\ell$ denotes the twisted mass $\pm \mu_\ell$ for the spinor $\chi_\ell$ on which it acts ($\chi_\ell = (u,d)^T$).

The spinors $\chi_\ell$ (and $\bar{\chi}_\ell$) are connected to their physical counterparts, $\psi_\ell$ ($\bar{\psi}_\ell$), via a chiral rotation,

$$
\bar{\psi}_\ell = \bar{\chi}_\ell \exp \left(i \gamma_5 \tau_3 \frac{\omega}{2} \right), \quad \psi_\ell = \exp \left(i \gamma_5 \tau_3 \frac{\omega}{2} \right) \chi_\ell,
$$

around the twist angle $\omega$. Working at maximal twist means $\omega = \pi/2$. For the OS strange valence quarks we use the operator:

$$
D_s = D_W + m_0 + i\mu_\ell \gamma_5. \tag{3.3}
$$

Further details of the formulation of the OS action can be found in ref. [9]. The gauge configurations have been generated with the Iwasaki gauge action of ref. [10]. The ensembles cover 11 values of $a\mu_\ell$ at 3 different lattice spacings $a$. We calculated strange quark propagators for 3 different strange quark mass parameters $a\mu_\ell$ per lattice spacing. Tab. 2 states the corresponding values of $a\mu_\ell$ for every $\beta$. The one and two particle operators, in the physical basis denoted by $K(t)$ and $\phi_{KK}(t)$,

$$
K(t) = \sum_\chi \bar{\psi}_\ell(\vec{x},t)\gamma_5 \psi_\ell(\vec{x},t), \quad \phi_{KK}(t) = K(t)K(t),
$$

respectively, lead to the correlation functions of the single kaon $C_K(t)$ and the two kaon system $C_{KK}(t)$. They are defined as follows:

$$
C_K(t) = \langle K(t)K(0) \rangle, \quad C_{KK}(t) = \langle \phi_{KK}(t)\phi_{KK}^\dagger(0) \rangle.
$$

We extract the ground state energy of each correlation function at times large enough such that excited states have decayed sufficiently. Because we work at zero total momentum this approach yields the lattice kaon mass $M_K$ from $C_K$ and the total energy $E_{KK}$ of the $K^+K^+$-system from $C_{KK}$. Unfortunately, the spectral decomposition of $C_{KK}$ gets distorted by terms constant in Euclidean time, preventing the naive extraction of $E_{KK}$, as detailed in ref. [11]. Following ref. [11] we use a

| ensemble | $\beta$ | $a\mu_\ell$ | $a\mu_\sigma$ | $a\mu_\delta$ | $(L/a)^T$ | $N_{\text{conf}}$ |
|-----------|--------|-------------|--------------|--------------|-------------|-----------|
| A30.32    | 1.90   | 0.0030      | 0.150        | 0.190        | $32^3 \times 64$ | 280       |
| A40.24    | 1.90   | 0.0040      | 0.150        | 0.190        | $24^3 \times 48$ | 404       |
| A40.32    | 1.90   | 0.0040      | 0.150        | 0.190        | $24^3 \times 64$ | 250       |
| A60.24    | 1.90   | 0.0060      | 0.150        | 0.190        | $24^3 \times 48$ | 314       |
| A80.24    | 1.90   | 0.0080      | 0.150        | 0.190        | $24^3 \times 48$ | 306       |
| B35.32    | 1.95   | 0.0035      | 0.135        | 0.170        | $32^3 \times 64$ | 250       |
| B55.32    | 1.95   | 0.0055      | 0.135        | 0.170        | $32^3 \times 64$ | 311       |
| B85.24    | 1.95   | 0.0085      | 0.135        | 0.170        | $32^3 \times 64$ | 296       |
| D30.48    | 2.10   | 0.0030      | 0.120        | 0.1385       | $48^3 \times 96$ | 369       |
| D45.32sc  | 2.10   | 0.0045      | 0.0937       | 0.1077       | $32^3 \times 64$ | 301       |

Table 1: The gauge ensembles used in this study. The labelling of the ensembles follows the notation in Ref. [8]. In addition to the relevant input parameters we give the lattice volume and the number of evaluated configurations, $N_{\text{conf}}$. 

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| $\beta$ | 1.90 | 1.95 | 2.10 |
|---------|------|------|------|
| $a\mu_s$ | 0.0185 | 0.0160 | 0.013 |
| 0.0225 | 0.0186 | 0.015 |
| 0.0246 | 0.0210 | 0.018 |
| $a$ [fm] | 0.0885(36) | 0.0815(30) | 0.0619(18) |

Table 2: Values of the bare strange quark mass $a\mu_s$ used for the three $\beta$-values and values of the lattice spacing $a$

ratio of correlation functions

$$R(t + a/2) = \frac{C_{KK}(t) - C_{KK}(t + a)}{C_K^2(t) - C_K^2(t + a)}$$

which can be shown to behave, for large Euclidean times $t$, like

$$R(t + a/2) \propto A_R \left( \cosh (\delta E_{KK} t') + \sinh (\delta E_{KK} t') \coth (2M_K t') \right), \quad t' = t + \frac{a}{2} - \frac{T}{2}.$$  

(3.5)

Here $T$ is the total time extent of the lattice. $\delta E_{KK}$ is obtained by fitting eq. 3.5 to the lattice data of eq. 3.4. We determine $a_0$ by inserting $M_K, L$ and $\delta E_{KK}$ into eq. 2.2 and solving for $a_0$.

We apply quark field smearing in a Laplacian-Heaviside (LapH) manner to our quark fields, as proposed in ref. [12], and calculate all-to-all propagators. To reduce the computational costs we combine the LapH method with a stochastic approach (sLapH) as suggested in ref. [13], resulting in 5 (3) random vectors per light (strange) quark perambulator used in the computation. Diluting the random vectors further reduces the stochastic noise. A more detailed description of the sLapH method can be found in reference [1] and references therein.

4. Analysis Strategies

The bare quark mass parameters $a\mu_l$ and $a\mu_s$ are related to their physical counterparts via the renormalisation constant $Z_P$: $m_f = \mu_f/Z_P$. To work at the physical strange quark mass value we need to specify how to set $m_s$ in our calculation. Setting the strange quark mass in different ways serves as a consistency check.

Method A consists of fixing $a\mu_s$ to the value that reproduces the physical value of the difference

$$\Delta^2 = r_0^2(M_K^2 - M_{\pi}^2/2)$$

on each ensemble. The lattice scale is set using the Sommer Parameter $r_0 = 0.474(14)$ fm and the values for the lattice spacing determined in ref. [14]. Tab. 2 states the values employed for each lattice spacing.

Method B uses the experimental squared kaon mass to fix the strange quark mass via a fit to data for the squared kaon mass as a function of the light and strange quark masses

$$(r_0M_K)^2 = \tilde{P}_0(r_0m_l + r_0m_s) \left[ 1 + \tilde{P}_1r_0m_l + \tilde{P}_2 \left( \frac{a}{r_0} \right)^2 \right] K_{FSE}^{ESE \ell},$$

(4.2)
which includes lattice artifacts of $O(a^2)$ [14]. After the fit the continuum values of $(r_0M_K)^2$ and $r_0m_l$ from ref. [14] are used to determine the continuum value of $r_0m_l$.

With the values of $M_Ka_0$ interpolated to the strange quark mass of Method A or B we extrapolate the data for $M_Ka_0$ in $m_l$ and $a$ to the point of $m_l^{\text{phys}}$ determined in ref. [14] and the continuum. In leading order chiral perturbation theory $M_Ka_0$ depends linearly on $m_l$. Because of automatic $O(a)$-improvement we only have to consider discretisation effects of $O(a^2)$. In the continuum and the limit $\mu_l \to 0$, we have a residual value of $M_Ka_0$, stemming from the non-zero strange quark mass value, giving rise to the parameter $P_2$ in eq. 4.3. Therefore, we fit the function

$$M_Ka_0 = P_0(r_0M_K)^2 + P_1\left(\frac{a}{r_0}\right)^2 + P_2$$

(4.3)

to our data at the physical strange quark mass.

5. Results

![Diagrams](image)

1.a: Chiral and continuum extrapolation of $M_Ka_0$ with $m_l$ fixed via $\Delta^2$

1.b: Same as fig. 1.a, but with $m_l$ fixed via $M_K^{\text{exp}}$.

Error bands suppressed for visibility

The fit results for the methods A and B are displayed in fig. 1.a and fig. 1.b, respectively. We show the $m_l$-interpolated data of $M_Ka_0$ as a function of $(r_0M_K^2)$ and the lattice spacing (red, blue and green for rising $\beta$). In addition, the fitted functions from eq. 4.3 are shown for each value of the lattice spacing. The continuum extrapolated value of $M_Ka_0$, extrapolated to the physical value of $M_K^2$ together with the continuum version of eq. 4.3 is displayed as well. Despite different discretisation effects, the extrapolated values for $M_Ka_0$ calculated with method A and B agree well within errors. They are compiled in table 3. To investigate the different discretisation effects we made a combined fit of the data from methods A and B with common parameters $P_0$ and $P_2$ and different parameters $P_1$ and $P_1'$ for the dependence on the squared lattice spacing $a^2$. Differences in the discretisation effects should be taken into account by $P_1$ and $P_1'$. At the same time the parameters $P_0$ and $P_2$ do not change. The extrapolated value of $M_Ka_0$ is compatible with the ones from method
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|               | Method A                | Method B                | A+B combined             | $p$-value weighted |
|---------------|-------------------------|-------------------------|--------------------------|--------------------|
| $(M_{Ka_0})_{phys}$ | $-0.409(15)$           | $-0.397(11)$           | $-0.405(12)$           | $-0.397(11)(^{+0.8}_{-0.3})$ |
| $\chi^2$/dof   | 0.38                    | 0.73                    | 0.65                     | -                  |
| $p$-value      | 0.93                    | 0.67                    | 0.86                     | -                  |

Table 3: Comparison of the results for $M_{Ka_0}$ at the physical point.

A and method B, respectively. The outcome of the combined fit is shown in fig. 1.c and 1.d. The data are the same as shown in fig. 1.a and 1.b. The curves now are results from the fit of eq. 4.3 with the common parameters $P_0$ and $P_2$ and the parameters $P_1$ (fig 1.c) and $P'_1$ (fig 1.d) coming from the different lattice spacing dependence. The fit models both data sets equally well, confirming that the difference stems from discretisation effects in setting the strange quark mass. To estimate the systematic uncertainty of the Methods A and B we calculate the $p$-value weighted median of the results from A and B. The results are shown in tab. 3. The 68.54% confidence interval of the combined and weighted distribution of A and B serves as an estimate of the systematic uncertainty.

The NPLQCD collaboration also calculated $M_{Ka_0}$ in ref. [2]. The PACS-CS collaboration undertook an investigation of scattering lengths for systems of two pseudoscalar mesons in ref. [3]. Tab. 4 shows the weighted result for $M_{Ka_0}$ for our work in comparison to the calculations of NPLQCD and PACS-CS. From tab. 4 deviations beyond the statistical level of the three analyses

| Collaboration | ETMC(this work) | NPLQCD | PACS-CS |
|---------------|-----------------|--------|---------|
| $M_{Ka_0}$    | $-0.397(11)(^{+0.8}_{-0.3})$ | $-0.352(16)$ | $-0.310(10)(^{+0.32}_{-0.18})$ |

Table 4: Comparison of the results of $M_{Ka_0}$ for this work, the work of NPLQCD and the work of PACS-CS. Statistical and systematic uncertainties shown in the first and second parentheses, respectively. For the analysis by NPLQCD stated uncertainty from both uncertainties added in quadrature.
become visible. These differences might be explained with lattice artifacts not taken into account by NPLQCD and PACS-CS at the same level of control.

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