Fermion Numbers 1/2 of Sphalerons from Spectral Mirror Symmetry

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Abstract

Motivated by the pioneering work of Jackiw and Rebbi, we explore the properties of fermionic Hamiltonians under discrete transformations in the background of the $SU(2)_L$ S sphaleron of the electroweak standard model. We find that CP is not a symmetry of the system along the noncontractible loop of field configurations that connects topologically distinct vacua and includes the sphaleron. By augmenting the CP transformation with an additional operation, we observe that the Dirac Hamiltonian is odd under the new transformation precisely at the sphaleron, and this ensures the mirror symmetry of the spectrum. This symmetry also indicates that the zero mode is self-conjugate. As a consistency check, we show that the fermionic zero mode discovered by Ringwald in the sphaleron background is invariant under the new transformation. This symmetry is broken elsewhere along the loop. For the vacua, this symmetry is ensured by the usual CP invariance. The fermion numbers $\frac{1}{2}$ of the sphaleron then follow immediately from the reasonings presented by Jackiw and Rebbi or equivalently from the spectral deficiency $\frac{1}{2}$ of the Dirac sea. The relevance of this analysis to other solutions is also discussed.

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1 Introduction

In their seminal paper on the subject of charge fractionalization, Jackiw and Rebbi [1] studied the Dirac equation in classical bosonic backgrounds for a number of field theories. In their analysis, they first showed that the Dirac equations admit normalizable, static zero-energy solutions. From there, they went on to prove that a non-degenerate, c-number zero mode implied that the background bosonic configuration carried half-integer fermion numbers. A key condition in their analysis was fermion number conjugation invariance of system, which implied spectral mirror symmetry in the classical bosonic background, along with fermion number conjugation invariance of the fermionic zero mode. Since then, charge fractionalization has been thoroughly studied and has found a wide range of applications in different areas of physics, such as particle physics [2–10], cosmology [11–14], condensed matter physics [15–18] and polymer physics [19–21].

The above argument was made for solitons. Nevertheless, it can be extended to other solutions as well. Another class of solutions that can be found in certain field theories are sphalerons, which are saddle-point solutions in field configuration space [22, 23]. In this context, configuration space is the infinite-dimensional function space in which all finite energy, static field configurations exist. To prove the existence of sphaleron solutions in this space,
one can perform an infinite-dimensional version of the minimax procedure used by Lyusternik and Snirelman [24]. The method to do this is to first form a noncontractible loop (more generally, a noncontractible n-sphere, depending on the problem at hand) in field configuration space and find the configuration along the loop that maximizes the field energy functional. Then one shrinks this loop to minimize the maximal energy [25]. If the configuration that is singled out satisfies the field equations of motion, then a sphaleron has been constructed.

An important member of this class of solutions is the “S” sphaleron [25] of the electroweak standard model. Its importance stems from the role that it is believed to play in the early Universe, including the generation of the matter-antimatter asymmetry of the Universe [26–28]. Following the discovery of this solution in hadronic models [29, 30], the sphaleron was rediscovered [22] in \( SU(2)_L \) theory and its properties and implications for cosmology were detailed in [26]. There, the baryon and lepton numbers of the sphaleron were calculated and both were shown to be \( \frac{1}{2} \) [26]. This was done by integrating the temporal component of the Chern-Simons current over 3-space and obtaining the resultant Chern-Simons charge for the sphaleron configuration, which is just its baryon or lepton number.

At this point, it is worth mentioning that sphalerons differ from solitons in an important way: whereas solitons are static, stable, finite-energy solutions in real time (\( t^2 > 0 \)), sphalerons are static, unstable, finite-energy solutions in real time and therefore quickly decay to the vacuum configuration [31]. To understand this better, note that a single map from spatial infinity (with spherical compactification), \( S^2_\infty \), to the Higgs vacuum manifold three-sphere, \( S^3_{\text{Higgs}} \), is topologically trivial, i.e.

\[
\pi_0 \left( \text{Maps} \left( S^2_\infty \to S^3_{\text{Higgs}} \right) \right) \cong \pi_2(S^3) \cong I,
\]

(1.1)

where \( \left[ \text{Maps} \left( S^2_\infty \to S^3_{\text{Higgs}} \right) \right] \), as defined in [23], is the topological equivalent of field configuration space, \( \pi_m(S^n) \) classifies the homotopy groups of spheres, \( I \) is the identity map and \( \cong \) signifies an isomorphism. This means that any single mapping can be continuously deformed to the identity map. In order to achieve nontrivial topology, we can instead consider a one-parameter family of maps from \( S^2_\infty \) to \( S^3_{\text{Higgs}} \). If the map begins and ends at the vacuum configuration and is appropriately defined [22, 25], then the resulting loop in field configuration space will be noncontractible, i.e.
where the loop belongs to the first homotopy sector of the above map.

To obtain a sphaleron solution, one usually starts with an ansatz. In the full $SU(2)_L \times U(1)_Y$ theory, the presence of the $U(1)$ field downgrades the spherically symmetric “hedgehog” ansatz for the $SU(2)_L$ sphaleron [22, 32] to an axially symmetric ansatz [33, 34] for the sphaleron configuration. Later on, a specific sphaleron solution for non-vanishing mixing angle was presented in [35]. Following this, a number of other sphaleron solutions were introduced both in electroweak theory and in other field theories as well [23, 25, 36–43].

The next step in the study of electroweak sphalerons was to add fermions to the theory. For the $SU(2)$ theory, a normalizable zero-energy solution of the Dirac equation was shown to exist precisely at the sphaleron [44, 45]. Later on, in the level-crossing picture for the $SU(2)_L$ theory, the change in fermionic eigenvalues from one vacuum to a neighboring vacuum, through the sphaleron, was numerically determined [46]. There, it was shown that as one traverses the path beginning at one vacuum, passing through the sphaleron and ending at a neighboring vacuum, a single negative eigenvalue of the Dirac Hamiltonian arises from the Dirac sea, crosses the zero energy level precisely at the sphaleron and enters the positive energy continuum at the next vacuum$^1$. This numerical study thus reconfirmed the existence of a zero energy bound state in the sphaleron background. A similar numerical analysis [48] has also been done in the background of the S sphaleron for when the fermions are non-degenerate in mass. The splitting of the fermion masses forces one to consider an axially symmetric ansatz for the fermionic fields. There, the authors showed that based on the results obtained using an “almost spherically symmetric ansatz”, i.e., when the mass difference is small, the hedgehog ansatz for fermions is a good approximation for studying fermionic level-crossing in the background of the sphaleron. A similar conclusion for the sphaleron’s properties was also reached in [26] for when the bosonic ansatz is axially symmetric.

An important observation that can be made in the results of [46] and [48] is that, when the change in the spectrum of the fermionic Hamiltonian is monitored as the non-contractible loop (NCL) connecting neighboring vacua

$^1$This non-vanishing spectral flow can be understood in terms of the Atiyah-Singer index theorem [47], which relates the analytic index of the Dirac operator (the parameter $\mu$ of the NCL) to the topological charge of the NCL.
through the sphaleron is traversed, the fermionic spectrum for the bound states is symmetric about $E = 0$ at the sphaleron and fails to be so as one travels toward the vacua in either direction along the NCL. If the whole fermionic spectrum, including the continua, has mirror symmetry, this would suggest, based on the results of [1], that the fermion numbers of the sphaleron are $\frac{1}{2}$.

The main goal of this paper is to present a rederivation of the half-integer fermion numbers of $SU(2)_L$ sphalerons by adopting an approach that is based on discrete symmetries. To do this, we find the transformation operator, which includes CP transformations, under which the Dirac Hamiltonian is odd. Hence we show that the entire spectrum of the Dirac Hamiltonian has mirror symmetry in the presence of the sphaleron. We then use the results presented by [1] to argue that the presence of the zero mode mandates half-integer fermion numbers for the sphaleron. In addition to presenting an alternative route to the calculation of the sphaleron’s fermion numbers than the standard one used in [26], and confirming the numerical results of [46, 48], we believe this approach has a number of other advantages, which will be further elaborated in Section 4.

The outline of this paper is as follows: In Section 2, we briefly review the bosonic sector of electroweak theory and the sphaleron ansatz of $SU(2)_L$ Yang-Mills-Higgs (YMH) theory in the limit of vanishing weak mixing angle. In Section 3, we analyze the behavior of the Dirac Hamiltonian operator under a CP transformation. For fermions in the background of $SU(2)_L$ sphalerons, we do this for all configurations along the NCL. Bearing in mind the nontrivial topology of the NCL, we then augment CP to arrive at a suitable choice for the fermion number conjugation operator. Finally, we perform a consistency check on the zero mode discovered by Ringwald [45] in the sphaleron background. In Section 4, we summarize our results and present an outlook.

## 2 Bosonic Fields

Consider the bosonic sector of the well-established electroweak Lagrangian

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - \lambda (\Phi^\dagger \Phi - \eta^2)^2, \quad (2.1)$$
where the $U(1)$ field strength tensor is given by
\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu},
\] (2.2)
the $SU(2)$ field strength tensor is given by
\[
G_{\mu\nu}^{a} = \partial_{\mu} B_{\nu}^{a} - \partial_{\nu} B_{\mu}^{a} + g e^{abc} B_{\mu}^{b} B_{\nu}^{c},
\] (2.3)
and the covariant derivative of the Higgs field is
\[
D_{\mu} \Phi = \left( \partial_{\mu} - ig \frac{\tau^a}{2} B_{\mu}^{a} - ig' Y A_{\mu} \right) \Phi.
\] (2.4)
The non-vanishing vacuum expectation value (VEV) of the Higgs field
\[
\langle \Phi \rangle = \eta \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\] (2.5)
spontaneously breaks the gauge symmetry such that
\[
SU(2)_L \times U(1)_Y \xrightarrow{SSB} U(1)_{EM}.
\] (2.6)
This ensures that the Maxwell field remains massless, while the masses of the Higgs and remaining gauge bosons are given by
\[
M_{W} = \frac{1}{\sqrt{2}} g \eta, \quad M_{Z} = \frac{1}{\sqrt{2}} \sqrt{g^2 + g'^2} \eta, \quad M_{H} = 2 \eta \sqrt{\lambda}.
\] (2.7)
Finally, the weak mixing angle $\theta_w$ and electric charge $e$ are determined by
\[
\tan \theta_{w} = \frac{g'}{g}, \quad e = g \sin \theta_{w}.
\] (2.8)

\section{SU(2)$_L$ Sphaleron}
In the limit of vanishing mixing angle, the $U(1)$ field decouples and this allows for a spherically symmetric ansatz for the gauge and Higgs fields of the NCL. To this end, consider the following map:
\[
U : S^1 \wedge S^2 \sim S^3 \to SU(2), \quad (\mu, \theta, \phi) \mapsto U(\mu, \theta, \phi),
\] (2.9)
where $\wedge$ is the smash product\(^2\) and $\mu$ is the loop parameter. The map $U$ needs to be topologically nontrivial (of winding number 1 in this case) with the appropriate boundary conditions. A suitable representation is $[22, 25]$

$$U (\mu, \theta, \phi) = -iy^1 \tau_1 - iy^2 \tau_2 - iy^3 \tau_3 + y^4 I_2,$$  \hspace{1cm} (2.10)

where

$$
\begin{pmatrix}
  y^1 \\
  y^2 \\
  y^3 \\
  y^4 \\
\end{pmatrix}
= 
\begin{pmatrix}
  -\sin \mu \sin \theta \sin \phi \\
  -\sin \mu \sin \theta \cos \phi \\
  \sin \mu \cos \mu (\cos \theta - 1) \\
  \cos^2 \mu + \sin^2 \mu \cos \theta \\
\end{pmatrix},
\hspace{1cm} (2.11)
$$

and $\tau^i$, the generators in weak isospace, are the usual Pauli matrices. Using the above map, the ansatz\(^3\) for the static gauge and Higgs fields of the $SU(2)_L$ sphaleron barrier becomes $[22]$

$$B (\mu, r, \theta, \phi) = -\frac{f (r)}{g} dU (\mu, \theta, \phi) U^{-1} (\mu, \theta, \phi),$$

$$\Phi (\mu, r, \theta, \phi) = \eta h (r) U (\mu, \theta, \phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \eta \begin{pmatrix} e^{-i\mu \cos \phi} \\ 0 \end{pmatrix} (1 - h (r)) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$ \hspace{1cm} (2.12)

where the radial functions have the following boundary conditions:

$$\lim_{r \to 0} f (r) = 0, \hspace{1cm} \lim_{r \to \infty} f (r) = 1,$$

$$\lim_{r \to 0} h (r) = 0, \hspace{1cm} \lim_{r \to \infty} h (r) = 1.$$ \hspace{1cm} (2.13)

Some remarks are in order. The field $B$ is the $SU(2)$-valued one-form

$$B (\mu, r, \theta, \phi) = B_r dr + B_\theta d\theta + B_\phi d\phi = B_i dx^i,$$ \hspace{1cm} (2.14)

for which we impose the polar gauge condition $B_r = 0 [22]$. We assume that in the polar gauge there exists a limiting field

$$\Phi^\infty (\theta, \phi) \equiv \lim_{r \to \infty} \Phi (r, \theta, \phi),$$ \hspace{1cm} (2.15)

such that $|\Phi^\infty| = 1$ and

$$\Phi^\infty (\theta = 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ \hspace{1cm} (2.16)

Observe that, of the gauge and Higgs fields configurations of the NCL, only the sphaleron ($\mu = \frac{\pi}{2}$) and the vacuum ($\mu = 0$) are parity eigenstates.

\(^2\)For a definition, see [25].  
\(^3\)It can be shown that, even when the ansatz is not manifestly spherically symmetric, it can always be transformed to one that is [32, 49].
2.2 Chern-Simons Charge and Baryon Number

A topologically significant number that can be assigned to any gauge field configuration is the Chern-Simons number. The $SU(2)$ contribution to the Chern-Simons current is

$$K_\mu = \frac{g^2}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{Tr} \left( G^{\nu\rho} B^\sigma + \frac{2}{3} ig B^\nu B^\rho B^\sigma \right),$$  \hspace{1cm} (2.17)

where

$$G_{\nu\rho} = \frac{1}{2} \tau^a G^a_{\nu\rho}, \quad B_\nu = \frac{1}{2} \tau^a B^a_\nu.$$  \hspace{1cm} (2.18)

The divergence of $K_\mu$ is non-vanishing. Physically, this represents the fact that the fermionic currents in the standard model contain an Abelian anomaly \cite{50}. The Chern-Simons charge is defined as

$$Q_{CS} = \int d^3 r K^0.$$  \hspace{1cm} (2.19)

When calculated in the correct gauge, namely one in which the integral of $\vec{K}.d\hat{S}$ over the surface of a sphere $S$ at spatial infinity vanishes \cite{26}, the Chern-Simons charge of the field configuration is equal to its baryon (lepton) number $Q_B$ ($Q_L$) \cite{26}. For the $SU(2)_L$ sphaleron of YMH theory, if one starts from a vacuum configuration with $Q_B$ set to zero and traverses the NCL through the sphaleron, one finds that the sphaleron will have $Q_B = \frac{1}{2}$ \cite{26}, while the neighboring vacuum will have $Q_{CS} = 1$. Furthermore, as was noted in \cite{26} and explicitly calculated in \cite{51}, even when $\theta_w \neq 0$, since the electric and magnetic fields are perpendicular at the sphaleron, the $U(1)$ field does not contribute to the baryon number. Therefore, for the axially symmetric ansatz \cite{52} of the $SU(2)_L \times U(1)_Y$ sphaleron, once again $Q_B = \frac{1}{2}$. In the next section, we present an alternative derivation of the fermion numbers of the $S$ sphaleron based on discrete symmetries reflected in the spectral mirror symmetry of the Dirac Hamiltonian.

3 Fermionic Symmetries

In this section we study the behavior of the Dirac Hamiltonian under discrete transformations including C and P in a sphaleron background. To motivate this, we first perform an analogous calculation for a simpler model involving
fermions in the background of a topologically nontrivial configuration. For
the sphaleron, when the weak mixing angle goes to zero, we perform our
analysis for arbitrary loop parameter $\mu$. For the $SU(2)_L \times U(1)_Y$ sphaleron,
only the sphaleron ansatz has been constructed and not the full barrier.
This restricts our analysis to the sphaleron when $\theta_w = 0$. Nevertheless, this
strategy can be readily extended to the full barrier once it is constructed.

3.1 MacKenzie-Wilczek Model

Let us first briefly consider a 1+1 dimensional theory of effectively massive
fermions interacting nonlinearly with a pseudoscalar field. The Lagrangian
for this theory is given by [3]

$$
\mathcal{L} = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m e^{i \phi(x)} \gamma^5 \right) \psi,
$$

with the topologically nontrivial background field given by

$$
\phi(x) = \mu \frac{x}{|x|}, \quad \mu \in (0, \pi).
$$

This model, which was extensively studied by MacKenzie and Wilczek in [3],
is in fact a chirally rotated, infinitely thin version of the one studied in [1].
Here, we are interested in the behavior of the Dirac Hamiltonian operator
under fermion number conjugation. The Hamiltonian operator is

$$
\hat{H} = -i \gamma^0 \gamma^j \partial_j + m \gamma^0 e^{i \phi(x)} \gamma^5
$$

and the choices of gamma matrices and charge conjugation operator are [3]

$$
\gamma^0 = \sigma^1, \quad \gamma^1 = i \sigma^3, \quad \gamma^5 = \gamma^0 \gamma^1 = \sigma^2, \quad \psi^C(x) = \sigma^1 \psi^*(x).
$$

Inserting the expression for $\phi(x)$ given by Eq.(3.2), the charge-conjugated Hamiltonian operator in this representation becomes

$$
\hat{H}^C \equiv C \hat{H} C^{-1} = i \gamma^0 \gamma^j \partial_j + m \left( \cos 2\mu \gamma^0 - i \frac{|x|}{x} \sin 2\mu \gamma^1 \right) e^{i \phi(x)} \gamma^5.
$$

This implies that precisely at $\mu = \frac{\pi}{2}$ the Dirac Hamiltonian is odd under
charge conjugation, i.e.

$$
\hat{H}^C \left( x, t, \mu = \frac{\pi}{2} \right) = -\hat{H} \left( x, t, \mu = \frac{\pi}{2} \right).
$$
This in turn implies that for every eigenstate with energy $E$ there is one with energy $-E$, and hence the spectrum has mirror symmetry. An important observation to make in the analysis of [3] is that, as the nontrivial classical configuration adiabatically forms from the trivial one, a bound state separates from the positive continuum at $\mu = 0$, crosses $E = 0$ at $\mu = \frac{\pi}{2}$ and joins the Dirac sea at $\mu = \pi$. Meanwhile, the spectral deficiency $\mathcal{D}$ in the positive continuum caused by the bound state starts to replenish, while deficiency starts to build up in the Dirac sea. At $\mu = \frac{\pi}{2}$ the fermionic bound state is at $E = 0$, and the spectral deficiencies in both continua are [5]

$$\mathcal{D} = \frac{\mu}{\pi}. \quad (3.7)$$

Therefore, at $\mu = \frac{\pi}{2}$, the quantum field theoretic expectation value of the fermion number operator is [3]

$$|\langle N \rangle| = \frac{1}{2}. \quad (3.8)$$

This number can be interpreted as the fermion number of the bosonic configuration.

### 3.2 CP Transformation along Noncontractible Loop

Motivated by the above results, let us now extend this analysis to the $SU(2)_L$ sphaleron. Consider the Dirac Hamiltonian operator of the electroweak theory at $\theta_w = 0$ [25]

$$\hat{\mathcal{H}} = -i \gamma^0 \gamma^j D_j + k \gamma^0 \left( \Phi_M^\dagger P_L + \Phi_M P_R \right), \quad (3.9)$$

where the matrix $\Phi_M$ contains the scalar fields of the Higgs doublet and its charge-conjugated doublet and is given by

$$\Phi_M = \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}, \quad (3.10)$$

and the projection operators are defined as

$$P_L = \frac{1}{2} \left( 1 - \gamma^5 \right), \quad P_R = \frac{1}{2} \left( 1 + \gamma^5 \right). \quad (3.11)$$
We now use the ansatz given in Eq. (2.12) to construct $\Phi_M$ shown in Eq. (3.10) and insert it into Eq. (3.9) to obtain the expression for $\hat{\mathcal{H}}$ along the NCL. The final expression for $\hat{\mathcal{H}}$ is shown in the appendix.

We use the following choice of Weyl representation for our gamma matrices

$$
\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.
$$  \tag{3.12}

In this representation, charge conjugation acts nontrivially on scalars and spinors, both of which transform in the fundamental representation of $SU(2)$, as

$$
\Phi(\vec{x}, t) \xrightarrow{C} i\tau^2 \Phi^*(\vec{x}, t), \quad \Psi(\vec{x}, t) \xrightarrow{C} i\tau^2 \gamma^0 \Psi^*(\vec{x}, t),
$$  \tag{3.13}

while under the combined transformation of C and P,

$$
\Phi(\vec{x}, t) \xrightarrow{CP} i\tau^2 \Phi^*(-\vec{x}, t), \quad \Psi(\vec{x}, t) \xrightarrow{CP} i\tau^2 \gamma^0 \gamma^2 \gamma^0 \Psi^*(-\vec{x}, t).
$$  \tag{3.14}

Therefore, the charge-conjugated, parity-inverted Hamiltonian becomes

$$
\hat{\mathcal{H}}^{CP}(\vec{x}, t, \mu) = \gamma^2 \gamma^0 \begin{pmatrix} -\hat{\mathcal{H}}_{22} & \hat{\mathcal{H}}_{21}^* \\ \hat{\mathcal{H}}_{12}^* & -\hat{\mathcal{H}}_{11}^* \end{pmatrix}(\vec{x}, t, \mu) \gamma^0 \gamma^2.
$$  \tag{3.15}

After inserting Eq. (A.2) into Eq. (3.15), we observe that nowhere along the NCL is $\hat{\mathcal{H}}$ odd under CP, except at the trivial vacua. However, at $\mu = \frac{\pi}{2}$, there are many cancellations and the even part reduces to

$$
\hat{\mathcal{H}}^{CP}(\vec{x}, t, \mu = \frac{\pi}{2}) + \hat{\mathcal{H}}(\vec{x}, t, \mu = \frac{\pi}{2}) = 2k\eta h(r) \gamma^0 \begin{pmatrix} \cos \theta (P_L + P_R) & -\sin \theta e^{i\phi} (P_L - P_R) \\ \sin \theta e^{-i\phi} (P_L - P_R) & \cos \theta (P_L + P_R) \end{pmatrix}.
$$  \tag{3.16}

This simply reflects the fact that the pseudoscalar spaleron configuration breaks the CP invariance of the one-generation electroweak theory that we have been considering. We now define a new transformation, $\hat{CP}$, which consists of CP and is augmented by an additional operation as follows

$$
\hat{CP} \equiv CPX,
$$  \tag{3.17}

where $X = -i\gamma^5$. By repeating the calculation leading to Eq. (3.16) for the new operation, Eq. (3.17), we see that

$$
\hat{\mathcal{H}}^{CP}(\vec{x}, t, \mu = \frac{\pi}{2}) = -\hat{\mathcal{H}}(\vec{x}, t, \mu = \frac{\pi}{2}).
$$  \tag{3.18}
The existence of a transformation under which $\hat{H}$ is odd ascertains the spectral mirror symmetry. That is, under such a transformation every eigenstate of $\hat{H}$ with energy $E$ is transformed into one with energy $-E$, the only exception being a zero energy bound state which must then be invariant under such a transformation. This can easily be verified within our model. As was mentioned in the introduction, the existence of a zero mode was shown theoretically in [44, 45] and numerically in [46]. In the next subsection we check the invariance of the zero mode discovered by Ringwald (which is the one that is relevant to our model) [45] under $\tilde{C}P$. The fermion numbers $\frac{1}{2}$ of the sphaleron then follow immediately from the reasonings presented by Jackiw and Rebbi or equivalently from the spectral deficiency $\frac{1}{2}$ of the Dirac sea. It is worth mentioning that at the trivial vacua ($\mu = 0, \pi$), $\hat{H}$ is odd under $CP$, showing that the spectrum has mirror symmetry there, reflecting the CP invariance of the one-generation $SU(2)_L$ theory. However, there are no bound states in the trivial vacua.

### 3.3 The Zero Mode

Recall that in the original analysis of Jackiw and Rebbi, the fermionic zero mode in the soliton background was fermion number self-conjugate [1]. Thus, an important consistency check on our symmetry transformation would be to operate it on the fermionic zero mode that was discovered by Ringwald at the electroweak S sphaleron [45]. To this end, consider the zero-energy solution of the Dirac equation in the sphaleron background. The ansatz for the left-handed isodoublet of the zero mode is given by [45]

$$
\Psi^i_{0,L}(\vec{x}, t) = \epsilon^{i\alpha} z(r),
$$

(3.19)

where $i = 1, 2$ is the weak isospin index, $\alpha = 1, 2$ is the spinor index and $\epsilon^{ij}$ is the Levi-Civita symbol ($\epsilon^{12} = +1$). The functional form of $z(r)$ is obtained by solving the radial part of the Dirac equation. Depending on whether the fermions are massive or massless, $z(r)$ will take on a different form [45]. For a single generation of left-handed quarks, denoted by

$$
\Psi^\alpha_{0,L} = \begin{pmatrix} u^\alpha_{0,L} \\ d^\alpha_{0,L} \end{pmatrix},
$$

(3.20)
this implies that
\[ u_{0,L}(\vec{x},t) = z(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv z(r) |\downarrow\rangle, \quad d_{0,L}(\vec{x},t) = z(r) \begin{pmatrix} -1 \\ 0 \end{pmatrix} \equiv -z(r) |\uparrow\rangle. \]

Thus, Eq. (3.19) can also be written as
\[ \Psi_{0,L}(\vec{x},t) = z(r) \begin{pmatrix} |\downarrow\rangle \\ -|\uparrow\rangle \end{pmatrix}. \] (3.22)

A CP transformation on Eq. (3.22) yields
\[ \Psi^{CP}_{0,L}(\vec{x},t) = i\gamma^5 \Psi_{0,L}(\vec{x},t), \] (3.23)

which shows that, as expected, the zero mode is not CP-invariant. By noting that we are performing the symmetry transformation at the sphaleron \( (\mu = \frac{\pi}{2}) \), implementing the additional factor of \(-i\gamma^5\) required by a CP transformation, we obtain
\[ \tilde{\Psi}^{CP}_{0,L}(\vec{x},t) = \Psi_{0,L}(\vec{x},t). \] (3.24)

Thus, we observe that the fermionic zero mode of Ringwald in the sphaleron background is \( \tilde{CP} \)-invariant.

4 Summary and Discussion

In this paper, we have studied the behavior of fermions under discrete transformations in sphaleron backgrounds. More specifically, we have investigated how the Dirac Hamiltonian behaves under a CP transformation in the presence of electroweak sphalerons. The solution that we chose for our analysis was the \( SU(2)_L \) sphaleron and not the \( SU(2)_L \times U(1)_Y \) sphaleron of the electroweak standard model. This is because in our analysis we used a fully parametrized sphaleron barrier and such a barrier has not been fully constructed yet for nonvanishing mixing angles. Only an axially symmetric ansatz for the sphaleron itself has been presented [35].

For the fields of the NCL passing through the S sphaleron, our calculations show that the system is not CP-invariant except at the trivial vacua. We then construct a transformation, denoted by \( \tilde{CP} \), by augmenting a CP transformation with an additional operation acting nontrivially in the Yukawa sector. We see that, for field configurations along the NCL, the Dirac Hamiltonian
is odd under $\widetilde{CP}$ precisely at the S sphaleron sitting at the top of the barrier
connecting neighboring vacua. This ensures that the spectrum has mirror
symmetry. That is, for every positive energy eigenstate there is a corre-
sponding negative energy one and the zero mode, if any, is self-conjugate. In
fact, this is the only place along the NCL that a $\widetilde{CP}$-invariant fermionic zero
mode can exist.

As an important consistency check, by performing the symmetry t rans-
formation on the fermionic zero mode discovered by Ringwald [45] in the
sphaleron background, we observe that the zero mode is indeed $\widetilde{CP}$-invariant.
This is closely analogous to the analyses of [1, 3]. There, fermion number
conjugation symmetry of the spectrum including the zero mode in the back-
ground of the classical solution was an important condition that was used in
the derivation of the half-integer fermion numbers of the background bosonic
fields. In the analyses of [1, 3], fermion number conjugation was charge conju-
gation. Our transformation operator is $\widetilde{CP}$ which reveals the spectral mirror
symmetry at the sphaleron. In this configuration, the spectral deficiency in
the Dirac sea is exactly 1/2 and one associates this to the fermion number
of the background field which is the sphaleron.

Overall, this analysis offers a number of other potential advantages. At a
basic level, it provides a useful consistency check for the numerous fractionally-
charged sphaleron ansatizes that have been discovered so far [36–43], and can
place constraints on their functional forms. Furthermore, the analyses of
[1, 3] required C-invariance, while the present analysis led to $\widetilde{CP}$-invariance.
It may be that other solutions require other novel symmetry transforma-
tions for the fermionic sector to correctly explain their fractional charges.
An important issue that our study has not addressed is what happens when
one considers three generations of fermions, where CP symmetry is violated
through the CKM and PMNS mixing matrices in the background of the even-
parity Higgs field vacuum. In any case, the approach adopted in this study
opens an avenue of inquiry that necessitates further study.

Finally, from a more practical perspective, one should bear in mind that
sphalerons are physically significant solutions in field theory. For instance,
the origin of the matter-antimatter asymmetry of the Universe is one of
the great unsolved mysteries in physics and many competing, and at times,
complementary theories are attempting to provide an explanation. Given the
central role that electroweak sphalerons play in many of these explanations,
systematically studying fermionic symmetries in their background may help
shed new light on their role in electroweak baryogenesis.

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A Dirac Hamiltonian along NCL

In this section we give the explicit functional form of the components of Eq. (3.9). As an $SU(2)$-valued $2 \times 2$ matrix, $\hat{\mathcal{H}}$ is

$$\hat{\mathcal{H}} = \begin{pmatrix} \hat{\mathcal{H}}_{11} & \hat{\mathcal{H}}_{12} \\ \hat{\mathcal{H}}_{21} & \hat{\mathcal{H}}_{22} \end{pmatrix}. \quad (A.1)$$

In the background of the gauge and Higgs fields of the NCL, Eq. (2.12), the components of $\hat{\mathcal{H}}$ are

$$\hat{\mathcal{H}}_{11} = -i \gamma^0 \gamma^j \partial_j + f(r) r \gamma^0 \gamma^3 P_L \sin \mu \cos \mu \sin^2 \theta$$
$$- f(r) r \gamma^0 \gamma^1 P_L \sin \mu \sin \theta (\cos \mu \cos \theta \cos \phi - \sin \mu \sin^2 \theta \sin \phi)$$
$$- f(r) r \gamma^0 \gamma^2 P_L \sin \mu \sin \theta (\cos \mu \cos \theta \sin \phi + \sin \mu \sin^2 \theta \cos \phi)$$
$$+ k \eta h(r) \gamma^0 \left[ e^{-i \mu \left( \frac{\cos \mu}{h(r)} + i \sin \mu \cos \theta \right) P_L} + e^{i \mu \left( \frac{\cos \mu}{h(r)} - i \sin \mu \cos \theta \right) P_R} \right], \quad (A.2a)$$

$$\hat{\mathcal{H}}_{12} = i f(r) r \gamma^0 \gamma^1 P_L e^{i(\mu + \phi)} \sin \mu$$
$$\times \left[ \cos \theta \cos \phi (\cos \mu \cos \theta - i \sin \mu) - i \sin^2 \theta \sin \phi (\cos \mu - i \sin \mu \cos \theta) \right]$$
$$+ i f(r) r \gamma^0 \gamma^2 P_L e^{i(\mu + \phi)} \sin \mu$$
$$\times \left[ \cos \theta \sin \phi (\cos \mu \cos \theta - i \sin \mu) + i \sin^2 \theta \cos \phi (\cos \mu - i \sin \mu \cos \theta) \right]$$
$$- i f(r) r \gamma^0 \gamma^3 P_L e^{i(\mu + \phi)} \sin \mu \sin \theta (\cos \mu \cos \theta - i \sin \mu)$$
$$- k \eta h(r) \gamma^0 \sin \mu \sin \theta e^{i \phi} (P_L - P_R), \quad (A.2b)$$
\[ \hat{H}_{21} = -if(r)r\gamma^0\gamma^1 P_L e^{-i(\mu+\phi)} \sin \mu \\
\times \left[ \cos \theta \cos \phi (\cos \mu \cos \theta + i \sin \mu) + i \sin^2 \theta \sin \phi (\cos \mu + i \sin \mu \cos \theta) \right] \\
- if(r)r\gamma^0\gamma^2 P_L e^{-i(\mu+\phi)} \sin \mu \\
\times \left[ \cos \theta \sin \phi (\cos \mu \cos \theta + i \sin \mu) - i \sin^2 \theta \cos \phi (\cos \mu + i \sin \mu \cos \theta) \right] \\
+ if(r)r\gamma^0\gamma^3 P_L e^{-i(\mu+\phi)} \sin \mu \sin \theta (\cos \mu \cos \theta + i \sin \mu) \\
+ k\eta h(r)\gamma^0 \sin \mu \sin \theta e^{-i\phi} (P_L - P_R), \]  
(A.2c)

\[ \hat{H}_{22} = -i\gamma^0\gamma^j \partial_j - f(r)r\gamma^0\gamma^3 P_L \sin \mu \cos \mu \sin^2 \theta \\
+ f(r)r\gamma^0\gamma^1 P_L \sin \mu \sin \theta (\cos \mu \cos \theta \cos \phi - \sin \mu \sin^2 \theta \sin \phi) \\
+ f(r)r\gamma^0\gamma^2 P_L \sin \mu \sin \theta (\cos \mu \cos \theta \sin \phi + \sin \mu \sin^2 \theta \cos \phi) \\
+ k\eta h(r)\gamma^0 \left[ e^{+i\mu} \left( \frac{\cos \mu}{h(r)} - i \sin \mu \cos \theta \right) P_L + e^{-i\mu} \left( \frac{\cos \mu}{h(r)} + i \sin \mu \cos \theta \right) P_R \right]. \]  
(A.2d)

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