\[ \rho^0 - \omega \] mixing in the presence of a weak magnetic field

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We calculate the momentum dependence of the \( \rho^0 - \omega \) mixing amplitude in vacuum with vector nucleon-nucleon interaction in presence of a constant homogeneous weak magnetic field background. The mixing amplitude is generated by the nucleon-nucleon (\( NN \)) interaction and thus driven by the neutron-proton mass difference along with a constant magnetic field. We find a significant effect of magnetic field on the mixing amplitude. We also calculate the Charge symmetry violating (CSV) \( NN \) potential induced by the magnetic field dependent mixing amplitude. The presence of the magnetic field influences the \( NN \) potential substantially which can have important consequences in highly magnetized astrophysical compact objects, such as magnetars. The most important observation of this work is that the mixing amplitude is non-zero, leading to positive contribute to the CSV potential if the proton and neutron masses are taken to be equal.

PACS numbers: 12.38.Mh, 13.75.Cs, 21.30.Fe, 21.65.Cd
Keywords: mixing amplitude, CSV \( NN \) potential, magnetic field

I. INTRODUCTION

Recent years have witnessed significant progress in understanding the properties of strongly interacting nuclear matter in presence of a magnetic background \cite{1}. Such studies draw their motivation both from heavy-ion collision experiments and the physics of neutron stars. Magnetic field with the strength of \( eB \sim (m^2_e - 15m^2_e) \) can be achieved in the laboratory in non-central heavy-ion collisions at RHIC and LHC \cite{2,3}. On the other hand, a similar environment can be expected in the interior of magnetars \cite{4,5,6}. Several novel properties of the strongly interacting matter under extreme conditions have been studied like chiral magnetic effect \cite{7,8,9}, magnetic catalysis \cite{10}, inverse magnetic catalysis \cite{11}, phase structure of QCD \cite{12}, superconductivity of vacuum \cite{13,14}, properties of mesons \cite{15,16,17,18,19}, photon polarization \cite{20,21}, dilepton production \cite{22,23,24,25,26}, and many more.

Another phenomenologically important quantity to study concerns the charge symmetry of nuclear matter and its violation. Experimentally, charge symmetry violation (CSV) can be observed in a charge-conjugate system such as the difference between \( pp \) and \( nn \) scattering length in the \( 1S_0 \) state with the experimental value \( \Delta a_{\text{CSV}} = a^N_{pp} - a^N_{nn} = 1.6 \pm 0.6 \text{ fm}/c \) \cite{27,28}. Such a non-Coulombic interaction can also contribute to the binding energy difference of the light mirror nuclei which is known as the Nolen-Schiffer (NS) anomaly \cite{29,30}. The CSV effect has been incorporated into the neutron-proton form-factor, the hadronic \( \tau \) decay contribution \cite{31}, decay of the \( \Psi \rightarrow (J/\Psi)\pi^0 \), hadronic vacuum correction to \( g - 2 \) \cite{32}, pion form factor \cite{33}, and isospin asymmetric nuclear matter \cite{34,35}. At the level of QCD, CSV occurs via the small mass difference between up and down quarks and via electromagnetic interaction of quarks \cite{36}. Consequently, charge symmetry is violated at the hadronic level because of the neutron-proton mass difference. The major contribution to CSV is the isospin mixing of vector mesons, mainly \( \rho^0 - \omega \) mixing \cite{37,38}, in single boson exchange model of the two nucleon force. Other examples of the mesons mixing are \( \pi - \eta \) and \( \pi - \eta' \) mixing \cite{39,40}.

The mixing of vector mesons at the \( \rho^0 - \omega \) pole is quite different from its sign and magnitude in the space-like region which is pertinent to the construction of the CSV \( NN \) potential. Goldman, Henderson, and Thomas \cite{41} find that the \( NN \) potential has a node at around 0.9 fm implying that the potential changes sign. Similar results were...
reported using several different theoretical approaches including mixing via $q\bar{q}$ loop driven by the $u-d$ quark mass difference \[52, 53\], and via $NN$ loop using the small neutron-proton mass difference \[54\]. Soon after their study it was argued in Ref. \[53\] that the strong momentum dependent mixing amplitude must vanish at the transition from time-like to space-like region. Moreover, QCD sum-rule \[53\], calculation also gives a large momentum dependence of the coupling. Since the $NN$ potential involves the space-like region, the long range $NN$ potential is strongly suppressed by the momentum dependent of $\rho - \omega$ mixing amplitude. As argued in Ref. \[50\], the off-shell dependence of $\rho^0 - \omega$ mixing is not sufficient to determine the CSV potential. In contrast to the momentum dependent mixing amplitude, the "mixed propagator" field theory approach \[57–59\] would restore the conventional role of the $\rho^0 - \omega$ mixing.

It may further be noted that in asymmetric nuclear matter $\rho^0 - \omega$ mixing plays an important role in determining the symmetry energy which in turn affects the EOS of neutron star. It has been argued in Ref. \[60\] that $\rho^0 - \omega$ mixing has an important effect on the symmetry energy. In fact the symmetry energy is softened both at sub- and super-saturation densities. It is also to be noted that the change in symmetry energy modifies the equation of state (EOS) of nuclear matter. Since the mixing depends both on the magnetic field and the density of the nuclear medium, there $B$-dependent mixing in vacuum and intend to extend this calculation in nuclear matter in near future. $\rho^0 - \omega$ mixing in magnetic field might also affect the cooling of neutron star via neutrino emission through $NN \to NN\gamma\gamma$ where $NN$ cross section will be different because of the $B$-dependent $\rho^0 - \omega$ mixing. In addition to that, the medium masses of $\rho$ and $\omega$ will also be affected in magnetic field due to $\rho^0 - \omega$ mixing \[53\].

To explore the possible momentum dependence of the $\rho^0 - \omega$ mixing amplitude in the presence of a weak external magnetic field, we revisit the problem of $\rho^0 - \omega$ mixing in vacuum. The mixing amplitude is generated by $NN$ loop and led by the neutron-proton mass difference along with a background magnetic field. The effect of external magnetic field on fermionic propagators is taken into account using Schwinger propagator \[61\]. In the present calculation, assuming that the magnetic field strength is weak i.e., $eB \ll m^2/\omega$, compatible with the strength observed in the interior of magnetars. In the presence of a magnetic field, the momentum dependence of $\rho^0 - \omega$ mixing amplitude is modified, and it will affect the CSV $NN$ potential. Moreover, to examine the magnetic field dependent contribution, we also perform calculations with equal nucleon masses in vacuum.

The paper is organized as follows. In Sec. II, we discuss the formalism required for the explicit calculation of the momentum dependent $\rho^0 - \omega$ mixing amplitude in presence of a weak magnetic field. In Sec. III, we use the magnetic field dependent mixing amplitude to determine the CSV $NN$ potential and discuss the numerical results. Finally in Sec. IV we conclude with a brief summary and discussions. Some details of the calculations are provided in the Appendix.

II. $\rho^0 - \omega$ MESON MIXING AMPLITUDE

In the one-boson exchange (OBE) models, the $NN$ interaction is mediated by the exchange of several mesons. For the purpose of this calculation, we are interested in the mixing between the neutral isovector $\rho^0$ meson and the isoscalar $\omega$ meson. The vector meson nucleon interaction Lagrangian corresponding to $\rho^0 - \omega$ mixing that we use is the following:

$$\mathcal{L}_{\rho NN} = g_{\rho} \bar{\Psi} \gamma_{\mu} \Phi^{\mu} \Psi,$$

$$\mathcal{L}_{\rho NN} = g_{\rho} \bar{\Psi} \left[ \gamma_{\nu} + \frac{C_{\rho}}{2M} \sigma_{\mu\nu} \partial^{\mu} \right] \tau \cdot \Phi^{\nu} \Psi,$$

where $\Psi$ and $\Phi$ are the nucleon and meson fields, respectively. From the above interaction Lagrangian one can find the vertex factors $\Gamma_{\rho}^{\nu} = g_{\rho} \gamma^{\nu}$ and $\Gamma_{\rho}^{\nu} = g_{\rho} [\gamma^{\nu} + \frac{C_{\rho}}{2M} \sigma^{\nu\lambda} q_{\lambda}]$. In this paper we use the coupling constants determined by the Bonn group \[32\]. The appropriate Bonn couplings are $g_{\rho}^2/4\pi = 10.6$, $g_{\omega}^2/4\pi = 0.41$ and $C_{\rho} = f_{\rho}/g_{\rho} = 6.1$. In the present calculation, $NN\omega$ tensor coupling is not included for its negligible contribution.

The $\rho^0 - \omega$ mixing amplitude is generated because of the difference between proton and neutron loop contribution as shown in Fig[4]

$$\Pi_{\rho\omega}^{\nu\mu}(q^2) = \Pi_{\rho\omega}^{\nu\mu}(p^2) - \Pi_{\rho\omega}^{\nu\mu}(n^2),$$

where $p(n)$ stands for proton (neutron). The polarization tensor of $\rho^0 - \omega$ mixing due to $NN$ excitations is calculated using standard Feynman rules and is given by

$$i\Pi_{\rho\omega}^{\mu\nu}(N^2)(q^2) = \int \frac{dk}{(2\pi)^4} \text{Tr} \left[ \Gamma_{\omega}^{\nu}(q) S_N(k)(\bar{\tau}_{\rho}^{\nu}(-q)S_N(k+q)) \right].$$
where subscript N denotes either p (proton) or n (neutron). The Feynman propagator for the neutron is

$$S_n(k) = \frac{k + m_n}{k^2 - m_n^2}$$

(5)

To include the effect of a constant background magnetic field, we use Schwinger’s proper time method [61]. Without any loss of generality, we assume the magnetic field $B$ to be along the $z$ direction. As we are interested in the weak field regime, i.e., $eB \ll \frac{m_p^2}{\omega}$, the magnetic field dependent proton propagator can be written as a power series in $eB$, that up to order $(eB)^2$ read as

$$S_p(k) = S^{(0)}(k) + S^{(1)}(k) + S^{(2)}(k)$$

(6)

where

$$S^{(0)}(k) = \frac{k + m_p}{k^2 - m_p^2}$$

(7)

$$S^{(1)}(k) = eB \frac{i\gamma_2(\gamma \cdot k) + m_p}{(k^2 - m_p^2)^2}$$

(8)

$$S^{(2)}(k) = (eB)^2 \frac{-2k_\perp^2}{(k^2 - m_p^2)^2} \left[ \frac{k + m_p}{k^2 - m_p^2} - \frac{\gamma \cdot k_\perp}{k^2 - m_p^2} \right]$$

(9)

We decompose the metric tensor into two parts $g^{\mu\nu} = g^{\mu\nu}_{||} - g^{\mu\nu}_{\perp}$, where $g^{\mu\nu}_{||} = \text{diag}(1, 0, 0, -1)$ and $g^{\mu\nu}_{\perp} = \text{diag}(0, 1, 1, 0)$. Also, we use $k_\perp^2 = k_2^2 - k_3^2$ and $k_\parallel^2 = k_1^2 + k_2^2$.

The magnetic field independent vacuum contribution to the self-energy is

$$i\Pi^{\mu\nu(N)}(q^2) = \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Gamma^{\mu}_{\omega}(q) S_N(k) \Gamma^{\nu}_{\omega}(-q) S_N(k + q) \right]$$

$$= g_{\omega\nu} \int \frac{d^4k}{(2\pi)^4} T^{\mu\nu}(k, k + q) \frac{1}{(k^2 - m_N^2 + i\epsilon)((k + q)^2 - m_N^2 + i\epsilon)}$$

(10)

where

$$T^{\mu\nu}(k, k + q) = \left( 2k^\mu k^\nu + k^\mu q^\nu + k^\nu q^\mu - g^{\mu\nu}(k^2 + k \cdot q - m_N^2) + \frac{C_{\rho M}}{2M^2} m_N (g^{\mu\nu} q^2 - q^\mu q^\nu) \right)$$

(11)

After the momentum integration, one may write the field free polarization tensor as

$$\Pi^{\mu\nu(N)}_{\rho\omega;(\text{vac})}(q^2) = (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) \Pi^{(N)}_{\rho\omega;(\text{vac})}(q^2),$$

(12)

where

$$\Pi^{(N)}_{\rho\omega;(\text{vac})}(q^2) = -\frac{g_{\rho\omega}}{4\pi^2} q^2 \int_0^1 dx \left[ 2x(1-x) + \frac{C_{\rho M}}{2M^2} m_N \right] \left( \frac{1}{\epsilon} - \gamma_E - \ln(\frac{\Delta}{\mu^2}) \right),$$

(13)

where $\Delta = m_N^2 - x(1-x)q^2$, $\mu$ is an arbitrary renormalization scale. $\gamma_E$ is the Euler-Mascheroni constant and $\epsilon = 2 - \frac{d}{2}$ contains the singularity, which diverges as $d \to 4$. Since the individual self-energy contribution of proton...
and neutron diverges, the singularity can be removed by the difference between proton and neutron loop contribution and we obtain the magnetic field independent mixing amplitude as

$$\Pi_{\mu\nu(\text{vac})}(q^2) = \Pi_{\mu\nu(\text{vac})}^{(p)}(q^2) - \Pi_{\mu\nu(\text{vac})}^{(n)}(q^2)$$

$$= \frac{g_{\rho\omega} q^2}{4\pi^2} \int_0^1 dx \left(2x(1-x) + \frac{C_F}{2} \ln \frac{m_p^2 - x(1-x)q^2}{m_n^2 - x(1-x)q^2} \right) \tag{14}$$

It can clearly be seen that if we do not distinguish between the proton and neutron mass, the mixing amplitude vanishes. In absence of magnetic field, the CSV NN potential in vacuum does not exist for $m_p = m_n$.

We now discuss the magnetic field dependent $\rho^0 - \omega$ mixing amplitude. In this paper, we are mainly concerned with the $B$- dependent mixing amplitude up to $\mathcal{O}((eB)^2)$ which is reasonable in the weak field regime. The first order contribution of magnetic field to $\rho^0 - \omega$ mixing is (as explicitly shown in the Appendix A)

$$i\Pi_{\mu\nu(\text{vac})}^{(1)(p)}(q^2) = \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \Gamma_\nu^\rho(q) S_p^{(0)}(k) \Gamma_\sigma^\mu(q) S_p^{(1)}(k) + \Gamma_\nu^\rho(q) S_p^{(1)}(k) \Gamma_\sigma^\mu(q) S_p^{(0)}(k) \right]$$

$$i\Pi_{\mu\nu(\text{vac})}^{(1)(n)}(q^2) = -8\pi C_F \frac{m_p e B g_{\rho\omega}}{2M} \int \frac{d^4 k}{(2\pi)^4} \epsilon^\lambda\rho\sigma k_\lambda q_\rho b_\mu u_\sigma \frac{1}{(k^2 - m_p^2)((k + q)^2 - m_p^2)^2}$$

$$= 0 \tag{15}$$

Hence, the linear order contribution of order $eB$ vanishes.

The second order contribution of magnetic field in $\rho^0 - \omega$ mixing is given by (see Appendix B for details)

$$i\Pi_{\mu\nu(\text{vac})}^{(2)(p)}(q^2) = \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \Gamma_\nu^\rho(q) \Gamma_\sigma^\mu(q) S_p^{(2)}(k) + \Gamma_\nu^\rho(q) S_p^{(0)}(k) \Gamma_\sigma^\mu(q) S_p^{(1)}(k) + \Gamma_\nu^\rho(q) S_p^{(1)}(k) \Gamma_\sigma^\mu(q) S_p^{(0)}(k) \right]$$

$$\Pi_{\mu\nu(\text{vac})}^{(2)(p)}(q^2) = (eB)^2 \frac{g_{\rho\omega} g_{\mu\nu}}{\pi^2} \int_0^1 dx \left[ x^3 \left[ \frac{1}{\Delta} + \frac{x(1-x)q^2 + x(4x-1)q^2_\perp + 2m_p^2}{3\Delta^2} + \frac{2x^2(x(1-x))q^2 + 2m_p^2|q^2_\perp|}{3\Delta^3} \right] \right.$$\n
$$+ x^2 \left[ \frac{1}{\Delta} - \frac{x(1-x)q^2_\perp}{4\Delta^2} \right] - x(1-x) \left[ \frac{1}{2\Delta} + \frac{2x(1-x)q^2_\perp - m_p C_F q^2_\perp (xq^2_\perp - (x+1)|q^2_\perp|)}{4\Delta^2} \right] \right] \tag{16}$$

It is clearly seen that the contribution of the magnetic field dependent mixing amplitude is finite; i.e., no divergences appear in the weak field limit. The correction term that is quadratic in field strength. In absence of weak external magnetic field, the total contribution to the mixing amplitude can be written as

$$\Pi_{\rho\omega}^1(q^2) = \Pi_{\rho\omega(\text{vac})}(q^2) + \Pi_{\rho\omega(\text{vac})}^{(2)}(q^2) \tag{17}$$

In absence of magnetic field, we obtain the mixing amplitude at the on-shell $\omega$ and $\rho$ meson point $\Pi_{\rho\omega}(m_\omega^2) = -4314$ MeV$^2$ and $\Pi_{\rho\omega}(m_\rho^2) = -4152$ MeV$^2$ respectively, which compares well with the experimental values [49]. In Fig. 2(a) we have shown the variation of the mixing amplitude at the point $(q^2 = m_\omega^2)$ with weak external magnetic field. We have used the condition that the strength of the external field is much lower than the square of the vector meson mass, i.e., $eB \ll m_p^2/\omega$. In both the meson mass, we have observed that the mixing amplitude, $\Pi_{\mu\nu}(q^2 = m_\rho^2/\omega)$ decreases with the increase of external magnetic field strength. In presence of background magnetic field, the mixing amplitude is non-zero, even in the limit $m_p = m_n$ as shown in Fig. 2(b). It is seen that, taking the limit $(m_p = m_n)$, the mixing amplitude vanishes at $eB = 0$ and hence, we see a decreasing behavior of mixing amplitude with increasing $eB$.

The momentum dependence of the $\rho^0 - \omega$ mixing amplitude is displayed in Fig. 3 at a different magnetic field strength. In absence of $eB$, the mixing amplitude has a node at exactly $q^2 = 0$ [53, 54], and, consequently, there is a change of sign of the mixing amplitude. Fig. 3(a) displays the mixing amplitude which is diminished with increasing values of $eB$ at same values of $q^2$. It is also clearly noticed that the value of $\Pi_{\rho\omega}$ decreases with the increase of $q^2$ at fixed values of background magnetic field. Similar behavior can be observed in Fig. 3(b) where $eB$ is varied keeping $q^2$ fixed. The effect of magnetic field on the mixing amplitude is greater in the time-like region than the space-like region. It is clearly visible that the node is shifted towards the space-like region in presence of magnetic field.
III. CHARGE SYMMETRY VIOLATING POTENTIAL

Now we evaluate the CSV $NN$ potential induced by the $\rho^0 - \omega$ mixing in presence of an external weak magnetic field. The momentum space CSV potential due to $\rho^0 - \omega$ mixing is given by \[54, 62\]:

$$V_{NN}^{\rho\omega}(q) = -\frac{g_{\rho\omega} \Pi_{\rho\omega}(q)}{(q^2 + m_{\rho}^2)(q^2 + m_{\omega}^2)}$$

(18)

Here, we neglected the contribution due to the external legs. Because of the extended structure of hadrons, one needs to incorporate meson-nucleon vertex correction which would be sufficient to take into account the inner structure of the hadrons. In our analysis, form factors are introduced by parameterizing the point coupling as \[32\]:

$$g_i \rightarrow g_i \left( \frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 + q^2} \right)$$

(19)

The cutoff parameter $\Lambda_i$ can be related directly to the hadron size and the numerical values for the cutoffs ($\Lambda_i$) are determined from the fit of the empirical $NN$ data \[32\].

To convert the CSV potential to configuration space, we make use of the identity

$$\frac{1}{(q^2 + m_{\rho}^2)(q^2 + m_{\omega}^2)} = \frac{1}{m_{\rho}^2 - m_{\omega}^2} \left( \frac{1}{q^2 + m_{\rho}^2} - \frac{1}{q^2 + m_{\omega}^2} \right),$$

(20)
potential can be obtained analytically \[54\] but in case of non-zero $eB$ amplitude. We see that there is a stronger suppression of the NN factor. Magnetic field dependent mixing amplitude leads to a clear enhancement of the magnetic field dependent CSV potential for two special cases: (a) $B \perp r$ and (b) $B \parallel r$. In Fig. 5 we present the role of the off-shell contribution of $\rho^0 - \omega$ mixing in the CSV NN potential. The contribution of the background magnetic field to the NN potential is clearly shown in both the graphs. We see that the $B$-independent CSV potential have a node around 0.9 fm \[51, 54\] with form factors. As the magnetic field is turned on, the occurrence of the node in the potential is around at 0.35 fm (at $eB = 0.05 \text{ GeV}^2$). We also notice that a non-zero $B$-dependent $\rho^0 - \omega$ contribution to the NN interaction is found to be much larger than without $B$-dependent mixing amplitude. It is also interesting to examine the CSV potential in presence of weak field regime at $m_p = m_n$, as is shown in the inset of Fig. 5. It is seen that the effect of magnetic field on the NN potential is found to be always positive in space-like region, and consequently, there is no node in the NN potential which leads to a significant effect on CSV.

IV. SUMMARY AND CONCLUSION

In the present paper, we have investigated the momentum dependence of $\rho^0 - \omega$ mixing amplitude as well as the role of momentum dependence of $\rho^0 - \omega$ mixing amplitude in CSV NN potential in the presence of an external magnetic field for the first time. The $\rho^0 - \omega$ mixing was assumed to be generated by the NN loops and hence driven by the neutron-proton mass difference along with a constant magnetic field. We have restricted ourselves to the weak
FIG. 5: (Color online) The off-shell contribution from $\rho^0 - \omega$ mixing to the CSV potential as a function of $NN$ separation. Both cases without magnetic field ($eB = 0$) and with magnetic field ($eB = 0.05$ GeV$^2$) are shown. Left panel: $B || r$. Right panel $B \perp r$. The CSV potential for $m_p = m - n$ is shown in the inset.

field limit, where the external field satisfies $eB \ll m_p^2/\omega$ and used the Schwinger’s proper-time method to describe the fermionic propagator. The effect of the background magnetic field appears as a correction to the momentum dependence of $\rho^0 - \omega$ mixing amplitude, which is relevant to study the properties of magnetars and magnetized hadronic medium relativistic heavy-ion collisions. Although in the weak field limit, the first correction is quadratic in the field. One has to also take into account the linear order corrected fermionic propagator in $B$. We find that the presence of the magnetic field modifies the mixing amplitude. It is seen that the mixing amplitude decreases with the increase of the strength of the magnetic field at the on-shell meson mixing point. This happens even if the Hamiltonian preserves the isospin symmetry, i.e., $m_p = m_n$. It is important to note that the change in the sign of the momentum dependence of $\rho^0 - \omega$ mixing amplitude is shifted towards the space-like region for non-zero $eB$ in contrast to the result found in the absence of magnetic field. Furthermore, the $NN$ potential generated by the off-shell dependence of $\rho^0 - \omega$ mixing is evaluated numerically. We have found that a node in the $NN$ potential occurs at $r \sim 0.35$ fm for $eB = 0.05$ GeV$^2$. Interestingly, we also find that the effect of the magnetic field to the $NN$ potential is always positive in the space-like region if we assume that each of the nucleon masses are taken to be equal. Moreover, one needs to extend this calculation in the dense medium to study the changes in various properties of magnetars.

Appendix A: Calculation of $\Pi^{\mu\nu(p)}_{\rho\omega(vac)}$

We have

$$i\Pi^{\mu\nu(p)}_{\rho\omega(vac)} = \int \frac{d^4k}{(2\pi)^4} eB g_\rho g_\omega \left[ \frac{T^{\mu\nu 1}}{(k - m^2_p)((k + q)^2 - m^2_p)^2} + \frac{T^{\mu\nu 2}}{(k - m^2_p)^2((k + q)^2 - m^2_p)} \right].$$

(23)

where

$$T^{\mu\nu 1} = \text{Tr}[\gamma^\mu(k + m_p)(\gamma^\nu - \frac{C_p}{2M}i\sigma^{\alpha\lambda}q_\lambda)i\gamma_1 \gamma_2 (\gamma \cdot (k + q))_|| + m_p)],$$

$$T^{\mu\nu 2} = \text{Tr}[\gamma^\mu \gamma_1 \gamma_2 (\gamma \cdot k)_|| + m_p)(\gamma^\nu - \frac{C_p}{2M}i\sigma^{\alpha\lambda}q_\lambda)((k + q)_|| + m_p)]$$

(24)

We use $i\gamma_1 \gamma_2 = -\gamma^5 b\gamma$, with $u^\mu = (1, 0, 0, 0)$ and $b^\mu = (0, 0, 0, 1)$. Using that

$$T^{\mu\nu 1}_{\rho\nu} = -4i\frac{C_p}{2M} m_p e^{\alpha\lambda\rho\sigma} k_\alpha q_\lambda b_\rho u_\sigma,$$

$$T^{\mu\nu 1}_{\rho\nu} = -4i\frac{C_p}{2M} m_p e^{\alpha\lambda\rho\sigma} (k + q)_\alpha q_\lambda b_\rho u_\sigma$$

(25)
Therefore, we can write the linear order contribution of magnetic field in the $\rho^0 - \omega$ mixing amplitude

\[
\begin{align*}
\pi^{1(\rho)}_{\mu,\rho\omega(vac)}(q^2) &= -8iC_\rho \frac{m_p e B g_\omega g_\rho}{2M} \int \frac{d^4k}{(2\pi)^4} e^{\alpha\lambda\rho\omega} k_\alpha q_\lambda b_{p\sigma} u_{\sigma} \int_0^1 \frac{k^2}{(k^2 - m_p^2)((k + q)^2 - m_r^2)} \\
&= -8iC_\rho \frac{m_p e B g_\omega g_\rho}{2M} \int_0^1 dx 2x \int \frac{d^4k}{(2\pi)^4} e^{\alpha\lambda\rho\omega} q_\lambda b_{p\sigma} u_{\sigma} \frac{(k - xq)_\sigma}{k^2 - \Delta^3} \\
&= 0
\end{align*}
\]

(26)

Here, the integration involving linear terms in $k$ is zero and $e^{\alpha\lambda\rho\omega} q_\lambda q_\lambda = 0$ due to the antisymmetric properties of Levi-Civita tensor.

**Appendix B: Calculation of $\pi^{2(p)}_{\mu,\rho\omega(vac)}$**

We have

\[
\begin{align*}
\pi^{2(p)}_{\rho\omega(vac)} &= \int \frac{d^4k}{(2\pi)^4} (eB)^2 g_\omega g_\rho \left[ T_1^{\mu\nu} - \frac{2k^2}{(k^2 - m_p^2)((k + q)^2 - m_r^2)} T_2^{\mu\nu} + \frac{2(k + q)^2}{(k^2 - m_p^2)((k + q)^2 - m_r^2)} T_3^{\mu\nu} \right] \\
&= \frac{1}{(k^2 - m_p^2)(k^2 - m_r^2)} T_1^{\mu\nu}
\end{align*}
\]

(27)

where

\[
T_1^{\mu\nu} = \text{Tr}[\gamma^\mu (k + m_p - \frac{\gamma \cdot k}{k^2}) (k^2 - m_r^2)](\gamma^\nu - \frac{C_\rho}{2M} i\sigma^{\lambda\rho\omega} q_\lambda)(k + q + m_p),
\]

(28)

\[
T_2^{\mu\nu} = \text{Tr}[\gamma^\mu (k + m_p)(\gamma^\nu - \frac{C_\rho}{2M} i\sigma^{\lambda\rho\omega} q_\lambda)](k + q + m_p),
\]

(29)

where $p = k + q$. Now, we replace $k \leftrightarrow k + q$ and we find

\[
T_2^{\mu\nu} = \text{Tr}[\gamma^\mu (k + m_p)(\gamma^\nu - \frac{C_\rho}{2M} i\sigma^{\lambda\rho\omega} q_\lambda)(k + q + m_p)],
\]

(30)

and

\[
T_3^{\mu\nu} = \text{Tr}[\gamma^\mu i\gamma_1 \gamma_2 (\gamma \cdot k + m_p)(\gamma^\nu - \frac{C_\rho}{2M} i\sigma^{\lambda\rho\omega} q_\lambda) i\gamma_1 \gamma_2 (\gamma \cdot (k + q) + m_p)].
\]

(31)

The contribution of the magnetic field comes from the $O((eB)^2)$ terms:

\[
\begin{align*}
\pi^{2(p)}_{\mu,\rho\omega(vac)} &= \int \frac{d^4k}{(2\pi)^4} (eB)^2 g_\omega g_\rho \left[ 32 \left( \frac{k^2 + k \cdot q - 2m_r^2)}{(k^2 - m_r^2)^3((k + q)^2 - m_r^2)} + \frac{k^2 + k \cdot q}{(k^2 - m_p^2)((k + q)^2 - m_r^2)} \right) \\
&\quad + \frac{8k \cdot (k + q) + 4m_p \frac{C_\rho}{2M}(k \cdot q - k \cdot q^2)}{(k^2 - m_p^2)((k + q)^2 - m_r^2)} \right]
\end{align*}
\]

(32)
Using the standard procedure of Feynman parametrization and evaluation of the momentum integral and Eq. 32 reduce to Eq. 16.
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