A Stabilized Normal Form Algorithm for Generic Systems of Polynomial Equations

Simon Telen, Marc Van Barel

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Abstract

We propose a numerical linear algebra based method to find the multiplication operators of the quotient ring \( \mathbb{C}[x]/I \) associated to a zero-dimensional ideal \( I \) generated by \( n \) \( \mathbb{C} \)-polynomials in \( n \) variables. We assume that the polynomials are generic in the sense that the number of solutions in \( \mathbb{C}^n \) equals the Bézout number. The main contribution of this paper is an automated choice of basis for \( \mathbb{C}[x]/I \), which is crucial for the feasibility of normal form methods in finite precision arithmetic. This choice is based on numerical linear algebra techniques and governed by the numerical properties of the given generators of \( I \).

1 Introduction

Consider the following problem. Given \( n \) polynomials \( f_1, \ldots, f_n \in k[x_1, \ldots, x_n] \) with \( k \) an algebraically closed field, find all the points \( x \in k^n \) where they all vanish: \( f_1(x) = \ldots = f_n(x) = 0 \). Here, we will work over the complex numbers \( k = \mathbb{C} \). The ring of all polynomials in the \( n \) variables \( x_1, \ldots, x_n \) with coefficients in \( \mathbb{C} \) is denoted by \( \mathbb{C}[x_1, \ldots, x_n] \). For short, we will denote \( x = (x_1, \ldots, x_n) \) and an element \( f \in \mathbb{C}[x] \) can be written as

\[
f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha
\]

where we used the short notation \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). The support \( S(f) \) of \( f \) is defined as

\[
S(f) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : c_\alpha \neq 0\}.
\]

A set of \( n \) polynomials \( \{f_1, \ldots, f_n\} \subset \mathbb{C}[x] \) defines a square ideal

\[
I = \langle f_1, \ldots, f_n \rangle = \{g_1 f_1 + \ldots + g_n f_n : g_1, \ldots, g_n \in \mathbb{C}[x]\} \subset \mathbb{C}[x].
\]
The affine variety associated to $I$ is
\[ \mathbb{V}(I) = \{ x \in \mathbb{C}^n : f(x) = 0, \forall f \in I \} = \{ x \in \mathbb{C}^n : f_1(x) = \ldots = f_n(x) = 0 \}. \]

In this paper, we assume that the variety $\mathbb{V}(I)$ consists of finitely many points \{\(z_1, \ldots, z_N\)\} \(\subset \mathbb{C}^n\). Such a variety is called $0$-dimensional.

A well known result in algebraic geometry states that the quotient $k[x_1, \ldots, x_n]/I$ with $I \subset k[x_1, \ldots, x_n]$ a 0-dimensional ideal and $k$ an algebraically closed field is isomorphic as a $k$-algebra to a finite dimensional $k$-vectorspace $V$ with multiplication defined by a pairwise commuting set of $n$ square matrices over $k$. This set of matrices corresponds to a set of generators of $k[x_1, \ldots, x_n]/I$ and the size of each matrix is equal to the number of points in $\mathbb{V}(I) \subset k^n$, counting multiplicities. Once the (generating) multiplication matrices are known in some basis, we can answer several questions about the variety $\mathbb{V}(I)$. For example, we can retrieve the solutions of the system by computing their eigenstructure and we can evaluate any polynomial on $\mathbb{V}(I)$. Our goal is to compute the multiplication matrices in a numerically stable way for square ideals satisfying some genericity assumptions.

There are many approaches to the problem of solving systems of polynomial equations. The different methods are often subdivided in homotopy methods, subdivision methods and algebraic methods. Homotopy continuation uses Newton iteration to track solution paths, starting from a simple initial system and gradually transforming it into the target system. These ideas have led to highly successful solvers [1, 16]. However, performing some numerical experiments one observes that for large systems some solutions might be lost along the way. The continuation gives up on certain paths when, for example, they seem to be diverging to infinity or they enter an ill-conditioned region. Normal form algorithms belong to the category of algebraic methods. The earliest versions of these algorithms use Groebner bases [5, 6] and doing so they make an implicit choice of basis for $\mathbb{C}[x]/I$. It turns out that these methods are numerically unstable and infeasible for large systems of equations (high degree, many variables). More recent algorithms are based on border bases [13, 15, 14]. Essentially, they fix a basis $\mathcal{O}$ for $\mathbb{C}[x]/I$ and construct the multiplication matrices of the coordinate functions by calculating the normal forms of $x_1 \cdot \mathcal{O}$, \ldots, $x_n \cdot \mathcal{O}$ with respect to $\mathcal{O}$. Border bases are a generalization of Groebner bases and they can be used to enhance the numerical stability of normal form algorithms. However, there are no algorithms that make a choice of $\mathcal{O}$ based on the conditioning of the normal form computation problem. This is mentioned as an open problem in [14]. In this paper we present such an algorithm that makes an automatic choice of $\mathcal{O}$, which does not necessarily correspond to a Groebner basis, nor to a border basis.

In the following section we discuss our genericity assumptions and some properties of the systems that satisfy them. Section 3 briefly reviews the multiplica-
tion maps in $\mathbb{C}[x]/I$ and their properties. In Section 4 we introduce the so called Macaulay matrices. Section 5 presents the algorithm and in the final section we present some numerical experiments.

2 Generic total degree systems

We say that a polynomial $f \in \mathbb{C}[x]\{0\}$ is of degree $d$ if

$$\max_{\alpha \in S(f)} |\alpha| = d,$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We denote $\deg(f) = d$. Accordingly, we say that a square polynomial system in $n$ variables given by $\{f_1, \ldots, f_n\}$ is of degree $(d_1, \ldots, d_n)$ if $\deg(f_i) = d_i, i = 1, \ldots, n$. A polynomial $f \in \mathbb{C}[x]\{0\}$ is called homogeneous of degree $d$ if $|\alpha| = d, \forall \alpha \in S(f)$.

Consider the projective $n$-space $\mathbb{P}^n = (\mathbb{C}^{n+1}\{0\})/\sim,$

where $(a_0, \ldots, a_n) \sim (b_0, \ldots, b_n)$ iff $a_i = \lambda b_i, i = 0, \ldots, n, \lambda \in \mathbb{C}\{0\}$. We can interpret $\mathbb{P}^n$ as the union of $n+1$ copies of $\mathbb{C}^n$, each of them given by putting one of the coordinates equal to 1. We will also think of $\mathbb{P}^n$ as the union of $\mathbb{C}^n$ corresponding to $x_0 = 1$ and the set $\{x_0 = 0\}$, called the hyperplane at infinity. For more on projective space, see [5]. Note that the equation $f = 0$ with $f \in \mathbb{C}[x_0, \ldots, x_n]$ is well defined over $\mathbb{P}^n$ if and only if $f$ is homogeneous. Starting from a polynomial $f \in \mathbb{C}[x]$ in $n$ variables of degree $d$, we can obtain a homogeneous polynomial $f^h \in \mathbb{C}[x_0, \ldots, x_n]$, called the homogenization of $f$ as

$$f^h = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The following theorem was proved by Étienne Bézout for the intersection of algebraic plane curves in $\mathbb{P}^2$. The generalization is often referred to as Bézout’s theorem.

**Theorem 1** (Bézout). A system of $n$ homogeneous equations of degree $(d_1, \ldots, d_n)$ in $n+1$ variables with a finite number of solutions in $\mathbb{P}^n$ has exactly $d_1 \cdots d_n$ solutions in $\mathbb{P}^n$, counting multiplicities.

**Proof.** The theorem is a corollary of Theorem 7.7 in [10].

It is not difficult to show that for almost all systems with degree $(d_1, \ldots, d_n)$, all $d_1 \cdots d_n$ solutions lie in the overlapping part of the affine charts of $\mathbb{P}^n$ [6]. Hence, if the $n$ homogeneous equations in $n+1$ variables of Theorem 1 are the homogenizations of $n$ affine equations $f_1 = \ldots = f_n = 0$ in $n$ variables, all of the $d_1 \cdots d_n$
solutions correspond to points in $\mathbb{C}^n \subset \mathbb{P}^n$.

The kind of systems that we consider in this paper are the ones that satisfy the assumption of Bézout’s theorem. Namely, we assume that the homogenized equations $f^h_1 = \ldots = f^h_n = 0$ have a finite number of solutions in $\mathbb{P}^n$. We denote $\mathcal{I} = \langle f^h_1, \ldots, f^h_n \rangle$ and $\mathbb{V}(\mathcal{I}) = \{ x \in \mathbb{P}^n : f^h_1(x) = \ldots = f^h_n(x) = 0 \}$. Furthermore, we assume that none of the solutions lie on the hyperplane at infinity. Note that this last assumption is not really restrictive: a random linear change of projective coordinates will move all of the solutions away from the hyperplane $\{ x_0 = 0 \}$ with probability 1.

3 Multiplication in $\mathbb{C}[x]/I$

In this section we briefly review the $\mathbb{C}$-algebra structure of the quotient ring $\mathbb{C}[x]/I$ and the properties of multiplication in this ring. For an extensive treatment one can consult [5, 6, 15]. Consider the following equivalence relation on $\mathbb{C}[x]$: $f \sim g \iff f - g \in I$.

Now, every polynomial $f \in \mathbb{C}[x]$ defines a residue class $[f] = f + I$ with respect to $\sim$. We call the polynomial $f$ a representative of the residue class $[f]$. The set of all such residue classes is the quotient ring $\mathbb{C}[x]/I$. Note that $[0] = I$. One can check that the scalar multiplication and addition operations

$$\alpha[f] = [\alpha f], \quad [f] + [g] = [f + g]$$

with $\alpha \in \mathbb{C}$ and $f, g \in \mathbb{C}[x]$ are well defined. This implies that $\mathbb{C}[x]/I$ is a vector space. Moreover, to show that $\mathbb{C}[x]/I$ is a $\mathbb{C}$-algebra, it can be checked that multiplication $[f] \cdot [g] = [fg]$ is well defined. The following theorem allows us to describe these operations on $\mathbb{C}[x]/I$ using linear algebra.

**Theorem 2.** For a zero-dimensional ideal $I$, the dimension of $\mathbb{C}[x]/I$ as a vector space is equal to the number of points in $\mathbb{V}(I) \subset \mathbb{C}^n$, counting multiplicities.

**Proof.** For the proof of this theorem we refer to [6].

We now consider the map $m_f : \mathbb{C}[x]/I \rightarrow \mathbb{C}[x]/I$ given by

$$m_f([g]) = [f] \cdot [g] = [fg], \forall g \in \mathbb{C}[x].$$

This map is linear, so once we choose a basis $\mathcal{O}$ for $\mathbb{C}[x]/I$, it can be represented by an $N \times N$ matrix, where $N$ is the number of solutions (counting multiplicities).
Under our genericity assumptions, $N$ is the Bézout number: $N = \prod_{i=1}^{n} d_i$. Once we have fixed a basis of $\mathbb{C}[x]/I$, we will no longer make a distinction between the map $m_f$ and its matrix representation. The matrix representing multiplication by $f$ is called a multiplication matrix of $f$. Its eigenstructure has the following remarkable properties.

**Theorem 3.** Let $I$ be a zero-dimensional ideal in $\mathbb{C}[x]$ and let $m_f$ be the multiplication matrix of $f \in \mathbb{C}[x]$ with respect to a given basis $O = \{[b_1], \ldots, [b_N]\}$ of $\mathbb{C}[x]/I$. Then

$$\det(m_f - \lambda I) = (-1)^{N} \prod_{z \in \mathbb{V}(I)} (\lambda - f(z))^\mu(z)$$

where $N = \dim \mathbb{C}[x]/I$, $I$ is the identity matrix of size $N \times N$ and $\mu(z)$ is the multiplicity of the root $z$. Also, the row vector

$$[b_1(z) \ldots b_N(z)]$$

spans the left eigenspace of the eigenvalue $f(z)$ for all $z \in \mathbb{V}(I)^\dagger$.

**Proof.** For the proof, we refer the reader to [6, Chapter 4].

Theorem 3 implies that if we want to compute the coordinates of the solutions $z_1, \ldots, z_N$, we can construct the multiplication matrices $m_{x_1}, \ldots, m_{x_N}$ corresponding to the coordinate functions and compute their eigenvalues. Another possibility is to use the eigenvectors [15, 6]. Note that, according to Theorem 3, the left eigenvectors do not depend on the choice of $f$. In fact, neither do the right ones. By their definition, it is not difficult to see that the multiplication maps must commute. They form a family of commuting matrices, so they must share common eigenspaces [15]. We note here that when the set of eigenvectors spans $\mathbb{C}^N$ (that is, when all solutions are simple), the matrices $m_{x_1}, \ldots, m_{x_n}$ are simultaneously diagonalizable. We will give an example of the construction of the multiplication matrices of the coordinate functions in Section 5. To work out this example, we will need the notion of a normal form.

**Definition 1** (Normal form). Let $O = \{[b_1], \ldots, [b_N]\}$ be a basis for $\mathbb{C}[x]/I$. Given any polynomial $g \in \mathbb{C}[x]$, let

$$[g] = a_1[b_1] + \ldots + a_N[b_N] = [a_1b_1 + \ldots + a_Nb_N], \quad a_i \in \mathbb{C}$$

be the unique representation of $[g]$ in the basis $O$. We say that $a_1b_1 + \ldots + a_Nb_N$ is the normal form of $g$ w.r.t. $O$. We denote $\overline{g}^O = a_1b_1 + \ldots + a_Nb_N$.

Note that for any $g \in \mathbb{C}[x]$, if the basis elements $[b_i]$ are given by monomials: $[b_i] = [x^{\alpha_i}]$, we have that $S(\overline{g}^O) \subset \{\alpha_1, \ldots, \alpha_N\}$. In general $S(\overline{g}^O) \subset \bigcup_{i=1}^{N} S(b_i)$.

$^\dagger$Note that in general $\#\mathbb{V}(I) \leq N$ where equality only holds if all solutions are simple.
4 Macaulay matrices

A Macaulay matrix associated to the set of polynomials \( \{f_1, \ldots, f_n\} \subset C[x] \) is a matrix over \( C \) in which each column corresponds to a monomial \( x^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0} \). Furthermore, such a Macaulay matrix has \( n \) block rows, each of which corresponds to one of the polynomials in the set. The \( j \)-th row of the \( i \)-th block row is the vector representation of a polynomial \( x^{\beta_{ij}} f_i \in I, \beta_{ij} \in \mathbb{Z}^n_{\geq 0} \) in the basis \( \{x^\alpha\} \) of monomials corresponding to the columns. For example, denote \( R = C[x] \) and for an ideal \( J \subset R \), we denote by \( J_{\leq t} \) the elements in \( J \) of degree \( \leq t \). Let \( d_i = \deg(f_i) \). For \( t \geq \max_i d_i \), consider the linear map

\[
\bigoplus_{i=1}^n R_{\leq t-d_i} \longrightarrow I_{\leq t},
\]

\[
(a_1, \ldots, a_n) \longrightarrow a_1 f_1 + \cdots + a_n f_n.
\]

The transpose of the matrix representation of this map with respect to the standard monomial basis of \( R_{\leq t} \) is a Macaulay matrix. We will call such a Macaulay matrix a dense Macaulay matrix. We clarify this and introduce some notation by means of an example.

**Example 1.** Let \( I = \langle f_1, f_2 \rangle \subset C[x_1, x_2] \) be generated by \( f_1 = a + bx_1 + cx_2 \) and \( f_2 = d + ex_1 + fx_2 + gx_1^2 + hx_1 x_2 + jx_2^2 \) with \( a, \ldots, j \in C \). It is clear that \( I_{\leq 2} \) is a subset of \( R_{\leq 2} = C[x_1, x_2]_{\leq 2} \) which is spanned as a \( C \)-vector space by \( 1, x_1, x_2, x_1^2, x_1 x_2, x_2^2 \). Using this basis to represent elements of \( I_{\leq 2} \) and \( R_{\leq 1} \oplus R_{\leq 0} = \text{span}(1, x_1, x_2) \oplus \text{span}(1) \) we get the transpose of the matrix

\[
M = \begin{bmatrix}
1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \\
f_1 & a & b & c \\
x_1 f_1 & a & b & c \\
x_2 f_1 & a & b & c \\
f_2 & d & e & f & g & h & j
\end{bmatrix}
\]

for the matrix representation of

\[
R_{\leq 1} \oplus R_{\leq 0} \longrightarrow I_{\leq 2},
\]

\[
(a_1, a_2) \longrightarrow a_1 f_1 + a_2 f_2.
\]

The matrix \( M \) is clearly a Macaulay matrix. By the support of \( M \), we mean the set of exponent vectors

\[
S(M) = \{ \alpha \in \mathbb{Z}^n_{\geq 0} : x^\alpha \text{ corresponds to a column of } M \}.
\]

To describe the row content of \( M \), we define the sets

\[
\Sigma_i(M) = \{ \beta_{ij} \in \mathbb{Z}^n_{\geq 0} : x^{\beta_{ij}} f_i \text{ corresponds to a row of the } i\text{-th block row of } M \}.
\]


The set $\Sigma_i$ is also called the set of shifts of $f_i$. Note that, given the polynomials $f_i$, $M$ is defined up to row and column permutations by $S(M)$ and $\Sigma_i(M)$, $1 \leq i \leq n$ and in order to be feasible, these sets must satisfy

$$S(x^{\beta_{ij}} f_i) \subset S(M), \forall \beta_{ij} \in \Sigma_i(M), 1 \leq i \leq n.$$ 

In this example, we have $S(M) = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2)\}$, $\Sigma_1(M) = \{(0,0), (1,0), (0,1)\}$, $\Sigma_2(M) = \{(0,0)\}$.

A Macaulay matrix of this type has a natural homogeneous interpretation. We show this by continuing the previous example.

**Example 2.** Homogenizing the equations we get $f_1^h = ax_0 + bx_1 + cx_2$ and $f_2^h = dx_0^2 + ex_0x_1 + fx_0x_2 + gx_1^2 + hx_1x_2 + jx_2^2$, where the superscript $h$ indicates the homogenization and it should not be confused with the coefficient $h \in C$ of the monomial $x_1x_2$ in $f_2$. We denote $T = \langle f_1^h, f_2^h \rangle \subset C[x_0, x_1, x_2]$. Now, one can verify that $M$ is also the Macaulay matrix of $\{f_1^h, f_2^h\}$ with

$$S^h(M) = \{(2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2)\} \subset \mathbb{Z}^3,$$

$\Sigma_1^h(M) = \{(1,0,0), (0,1,0), (0,0,1)\}$, $\Sigma_2^h(M) = \{(0,0,0)\}$. It is clear how this can be generalized to any dense Macaulay matrix: if the associated map has image in $I_{\leq t}$, homogenize the exponent vectors in $S(M)$ to degree $t$ in $\mathbb{Z}^{n+1}_{\geq 0}$, and the exponent vectors in $\Sigma_i(M)$ to degree $t - d_i$. The associated linear map is given as follows. Denoting $R = C[x_0, x_1, \ldots, x_n]$ and the degree $t$ part of a graded $R$-module $M$ by $M_t$ (the grading is induced by the standard grading on $R$), $M$ represents the map

$$\bigoplus_{i=1}^n R_{t-d_i} \longrightarrow T_t,$$

$$(a_1, \ldots, a_n) \longrightarrow a_1 f_1^h + \cdots + a_n f_n^h.$$

In the generic case, this map is surjective.

Macaulay matrices are used to give determinantal formulations of resultants [6] and to solve systems of polynomial equations [6, 8, 3]. They form a natural first step in reformulating the root finding problem as a linear algebra problem. The following theorem is straightforward [8].

**Theorem 4.** Let $S(M) = \{\alpha_1, \ldots, \alpha_t\}$ be the support of a Macaulay matrix $M$ of $\{f_1, \ldots, f_n\}$, where $\alpha_i$ corresponds to the $i$-th column of $M$. Let $I = \langle f_1, \ldots, f_n \rangle$. The point $z \in \mathbb{C}^n$ satisfies $z \in \mathbb{V}(I)$ if and only if the vector

$$v(z) = (x^{\alpha_1}(z), \ldots, x^{\alpha_t}(z))^T$$

satisfies $Mv(z) = 0$. 

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It is clear that Theorem 4 generalizes to the projective interpretation of $M$.

**Theorem 5.** Let $S^h(M) = \{\alpha^h_1, \ldots, \alpha^h_l\}$ be the support of a (homogeneously interpreted) Macaulay matrix $M$ of $\{f^h_1, \ldots, f^h_n\}$, where $\alpha^h_i$ corresponds to the $i$-th column of $M$. Denote $I = \langle f^h_1, \ldots, f^h_n \rangle$. The point $z^h \in \mathbb{P}^n$ satisfies $z^h \in \mathbb{V}(I)$ if and only if the point

$$v(z^h) = (x^{\alpha^h_1}(z^h), \ldots, x^{\alpha^h_l}(z^h))^\top,$$

viewed as a point in $\mathbb{P}^{d-1}$, satisfies $Mv(z^h) = 0$ (note that here $x = (x_0, x_1, \ldots, x_n)$ is short for an $n+1$-tuple). This condition is well defined, since $v(\lambda z^h) = \lambda t v(z^h)$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $t = |\alpha^h_i|$.

Theorem 5 implies that every point $z^h \in \mathbb{V}(I) \subset \mathbb{P}^n$ generates a direction $v(z^h)$ in the nullspace of $M$. We will now present a way to construct the dense Macaulay matrix such that its null space is spanned by these directions. In the Macaulay matrix $M$ with support $S(M) = \{\alpha \in \mathbb{Z}^n_{\geq 0} : |\alpha| \leq t\}$, the number of columns is

$$\#S(M) = \binom{t+n}{n}.$$

Consider the shifts

$$\Sigma_i(M) = \{\beta \in \mathbb{Z}^n_{\geq 0} : |\beta| \leq t - d_i\}.$$

It is clear that the resulting matrix $M$ is the dense Macaulay matrix of degree $t$.

**Theorem 6.** Under our genericity assumptions, for $M$ constructed as above with $t \geq \sum_{i=1}^n d_i - n$, we have $\dim \text{null}(M) = N$. Equivalently, for these values of $t$: $\#S(M) - N = \text{rank}(M)$.

**Proof.** This result was known by Macaulay [12]. The degree $t = \sum_{i=1}^n d_i - n$ is called the degree of regularity in [3, 8].

**Theorem 7.** For $t \geq \sum_{i=1}^n d_i - (n - 1)$ we have that

$$\binom{t-1+n}{n} \geq N.$$

**Proof.** The number $\binom{t-1+n}{n}$ is the number of monomials of degree at most $t - 1 = \sum_{i=1}^n d_i - n$. The number $N = \prod_{i=1}^n d_i$ is the number of monomials in the set $\{\alpha \in \mathbb{Z}^n_{\geq 0} : \alpha_i \leq d_i - 1, i = 1, \ldots, n\}$. The highest degree monomial in this set has degree $\sum_{i=1}^n d_i - n$.

It will become clear later that the properties of $M$ given in Theorem 6 and Theorem 7 are exactly the properties we need in our algorithm. We also want $M$ to be as small as possible to reduce memory use and computational effort. We therefore set $t = \sum_{i=1}^n d_i - (n - 1)$.  

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5 Normal form computation using the Macaulay matrix

In this section, we assume that the ideal $I = \langle f_1, \ldots, f_n \rangle$ is generic in the sense that the number of points in $\mathbb{V}(I) \subset \mathbb{C}^n$ is equal to the Bézout number and all solutions have multiplicity one.

5.1 An example

We introduce the ideas of our algorithm by a simple example. Consider the ideal $I = \langle f_1, f_2 \rangle \subset \mathbb{C}[x_1, x_2]$ given by $f_1(x_1, x_2) = x_1^2 + x_2^2 - 2 = 0, f_2(x_1, x_2) = 3x_1^2 - x_2^2 - 2 = 0$. We will use linear combinations of $f_1, x_1f_1, x_2f_1, f_2, x_1f_2, x_2f_2$ to find the normal forms. The variety $\mathbb{V}(I) = \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$ is 0-dimensional and the system satisfies the genericity assumptions. A possible basis for $\mathbb{C}[x_1, x_2]/I$ is $\mathcal{O} = \{[1], [x_1], [x_2], [x_1x_2]\}$. We construct the dense Macaulay matrix $M$ of degree $t = \sum_{i=1}^d d_i - (n - 1) = 3$ as presented in Section 4, ordering the columns such that these monomials correspond to the last four columns:

$$
M = \begin{pmatrix}
    x_1^2 & x_2^2 & x_1^2y & x_1x_2 & x_2 & 1 & x_1 & x_2 & x_1x_2 \\
    f_1 & 1 & 1 & 1 & -2 & -2 \\
x_1f_1 & 1 & 1 & 1 & -2 & -2 \\
x_2f_1 & 3 & -1 & 3 & -1 & -2 & -2 \\
f_2 & 1 & 1 & 1 & -2 & -2 \\
x_1f_2 & 3 & -1 & 3 & -1 & -2 & -2 \\
x_2f_2 & 1 & 1 & 1 & -2 & -2 & -2
\end{pmatrix}.
$$

To construct the multiplication maps $m_{x_1}$ and $m_{x_2}$ with respect to $\mathcal{O}$, we need to calculate the normal forms of $x_1^2, x_1^2x_2, x_2^2, x_1x_2^2$ in $\mathcal{O}$. All of these monomials appear in the left block column of $M$. Inverting this column block and applying it from the left to $M$ gives

$$
\tilde{M} = \begin{pmatrix}
    x_1^2 & x_2^2 & x_1^2y & x_1x_2 & x_2 & 1 & x_1 & x_2 & x_1x_2 \\
    x_1^2 - 1 & 1 & 1 & -1 & -1 \\
x_1^2 - 1 & 1 & 1 & -1 & -1 \\
x_1^2x_2 - x_1 & 1 & 1 & -1 & -1 \\
x_1x_2^2 - x_1 & 1 & 1 & -1 & -1 \\
x_2^2 - x_2 & 1 & 1 & -1 & -1
\end{pmatrix}.
$$

Note that the left block was square because of the properties of the dense Macaulay matrix. The rows of $\tilde{M}$ are linear combinations of the rows of $M$, so they represent polynomials in $I$. Hence, for example, $[x_1^2 - 1] = [0]$ modulo $I$ and the normal form
of $x_1^2$ is 1. Using the information in $\tilde{M}$ we can construct $m_{x_1}$ and $m_{x_2}$. This gives

$$m_{x_1} = \begin{bmatrix} [x_1] & [x_1][x_2] & [x_1][x_1] & [x_1][x_1][x_2] \\ [1] & 0 & 1 & 0 \\ [x_1] & 1 & 0 & 0 \\ [x_2] & 0 & 0 & 0 \\ [x_1][x_2] & 0 & 0 & 1 \end{bmatrix},$$

$$m_{x_2} = \begin{bmatrix} [x_2] & [x_2][x_1] & [x_2][x_2] & [x_2][x_1][x_2] \\ [1] & 0 & 0 & 1 \\ [x_1] & 0 & 0 & 0 \\ [x_2] & 1 & 0 & 0 \\ [x_1][x_2] & 0 & 1 & 0 \end{bmatrix}.$$

Note that the first and the third column of $m_{x_1}$ are trivial and so are the first and the second column of $m_{x_2}$. The other columns can be read off $\tilde{M}$ directly. The eigenvalues of $m_{x_i}$ coincide with the $i$-th coordinates of the points in $\mathbb{V}(I)$. When choosing the basis $O$, we must take into account that $O$ cannot contain monomials of degree $t$ (3 in this example). Otherwise, multiplying with $x_1$ or $x_2$ gives a monomial that is not in $S(M)$. Secondly, it must be such that the resulting system is solvable. In the generic case, there is always such a choice. We consider the Macaulay matrix $M$ of degree $t = \sum_{i=1}^n d_i - (n-1)$. By $S(M)_t$ we denote the monomials in $S(M)$ of degree $t = \sum_{i=1}^n d_i - n + 1$ and by $S(M)_{<t}$ the remaining monomials. We order the columns of the Macaulay matrix in such a way that

$$M = [M_b \ M_i \ B]$$

where $M_b$ are the columns corresponding to $S(M)_t$, $B$ contains the columns corresponding to $O$ and $M_i$ corresponds to $S(M)_{<t} \setminus O$. When the polynomials $f_1, \ldots, f_n$ are generic, the set of monomials

$$O = \{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} : 0 \leq \alpha_i \leq d_i - 1, 1 \leq i \leq n\} \subset S(M)_{\leq t}$$

(2)

is a basis for $\mathbb{C}[x]/I$ [6]. This means that every monomial in $S(M) \setminus O$ has a unique normal form in $O$. In other words, there is a polynomial of the form

$$g_\alpha = x^\alpha - \sum_{b \in O} c_{\alpha b} b \in I$$

with $c_{\alpha b} \in \mathbb{C}$, for each $\alpha \in S(M) \setminus O$. Also, it follows from Property (iii) in [4, Chapter 1, p.46] and from our assumptions that for $t \geq \sum_{i=1}^n d_i - (n-1)$, the rows of $M$ span $I_{\leq t}$ linearly. Now, since $g_\alpha \in I_{\leq t}$ and the rows of $M$ span $I_{\leq t}$, every $g_\alpha$ is a linear combination of the rows of $M$. The number of polynomials $g_\alpha$ is $r = \text{rank}(M)$,
so we can apply a square matrix to the left of $M$ to transform $M$ into

$$
\begin{bmatrix}
1 & -c_{\alpha_1 b_1} & \cdots & -c_{\alpha_1 b_N} \\
1 & -c_{\alpha_2 b_1} & \cdots & -c_{\alpha_2 b_N} \\
\ddots & \vdots & \ddots & \vdots \\
1 & -c_{\alpha_r b_1} & \cdots & -c_{\alpha_r b_N} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}.
$$

The block row of zeros is introduced by the syzygies in the rows of $M^2$. This proves the following theorem.

**Theorem 8.** The matrix $M_b$ is of full column rank under our assumptions, and there is at least one possible choice of $O$ for which $[M_b \ M_i]$ is of rank $r$.

However, there are many more choices for $O$ than the ‘block basis’ from (2). From a numerical point of view, it turns out this is crucial to find the normal forms with high accuracy. The idea is simple: we choose $O$ in such a way that $[M_b \ M_i]$ is ‘as invertible as possible’, i.e., it has a small condition number.

### 5.1.1 Algorithm

The above reasoning leads to Algorithm 1 for the generic dense case. We briefly go through the different steps of the algorithm.

- **Step 2** is obvious. In step 3, we re-arrange the columns of $M$ such that $M_b$ contains the columns corresponding to $S(M)_t$ and $M_*$ contains all of the other columns. The order within the block columns is of no importance. We represented this in Algorithm 1 by a column permutation matrix $P_c$. At this point, we do not split $M_*$ into $M_i$ and $B$ as before. The actual choice of basis is made in step 6.

- **Step 4** is essential. We perform a QR factorization with optimal column pivoting to the full lower right block. That is, we do not factorize $\hat{M}_*$, but we factorize $\hat{M}_* P_i$, where $P_i$ is a column permutation matrix. The column permutation is such that it heuristically selects the ‘linearly most independent’

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2 This occurs only for $n \geq 3$, not in the example given here.
Algorithm 1 Multiplication maps of a dense system

1: procedure MultMatrices\( (f_1, \ldots, f_n) \)

2: \( M \leftarrow \) dense Macaulay matrix of degree \( \sum_{i=1}^{n} d_i - (n - 1) \)

3: \( M \leftarrow [M_b \ M_\ast] = MP_c \)

4: \( M_b = Q_b R_b \)

5: \( M \leftarrow Q_b^\ast M = \begin{bmatrix} \hat{R}_b & Z \\ 0 & \hat{M}_\ast \end{bmatrix} \)

6: \( \tilde{M}_i P_i = Q_i R_i \)

7: \( M \leftarrow M \begin{bmatrix} I & 0 \\ 0 & P_i \end{bmatrix} \)

8: \( M \leftarrow \begin{bmatrix} I & Q_i^\ast \\ Q_i & \hat{M}_i \end{bmatrix} M = \begin{bmatrix} \hat{R}_b & ZP_i \\ 0 & \hat{R}_i \end{bmatrix} = \begin{bmatrix} \hat{R}_b & \hat{Z}_1 & \hat{Z}_2 \\ 0 & \hat{R}_i & \hat{Z}_3 \end{bmatrix} \)

9: \( M \leftarrow \begin{bmatrix} \hat{R}_b & \hat{Z}_1 & \hat{Z}_2 \\ 0 & \hat{R}_i & \hat{Z}_3 \end{bmatrix} \)

10: \( C \leftarrow -\begin{bmatrix} \hat{R}_b & \hat{Z}_1 \\ 0 & \hat{R}_i \end{bmatrix}^{-1} \begin{bmatrix} \hat{Z}_2 \\ \hat{Z}_3 \end{bmatrix} \)

11: \textbf{for} \( i = 1, \ldots, n \) \textbf{do}

12: \hspace{1em} Construct \( m_{x_i} \) using the normal forms in \( C \).

13: \textbf{end for}

14: \textbf{return} \( m_{x_1}, \ldots, m_{x_n} \)

15: end procedure
columns first. In step 7 we apply the corresponding permutation to the entire matrix \( M \) and in step 8 we make the entire matrix upper triangular (\( \tilde{R}_i \) is the upper non-zero block and \( \hat{R}_i \) is the square upper triangular part of \( \tilde{R}_i \)). We split the right block column into two block columns such that \( \hat{R}_i \) is square. Under our assumptions, \( \hat{R}_i \) is of full rank. Note that in the result, columns are still associated to monomials and the rows are polynomials in \( I \). With increasing row index, the support of these polynomials is contained in a shrinking subset of \( S(M) \). Note that in this step, the syzygies introduce a block row of zeros in \( M \). We drop this block row of zeros in step 9. Denoting \( r = \text{rank}(M) \), the remaining matrix \( M \) is of size \( r \times (N + r) \) by the results of Section 4.

- In step 10, we take out the left most \( r \times r \) upper triangular block and apply its inverse to the right most \( r \times N \) part with opposite sign to find the normal forms of all the monomials corresponding to the first \( r \) columns. Of course, we do not calculate the inverse, but apply backsubstitution instead. It is the condition number of this inversion that is controlled by the optimal column pivoting in step 6.

### 6 Numerical experiments

In this section, we use Algorithm 1 for some numerical experiments and compare it to Bertini [1, 2] and PHClab [9].

#### 6.1 Evaluating a polynomial function on \( \mathbb{V}(I) \)

Theorem 3 implies that we can evaluate a function \( f \in \mathbb{C}[x] \) on \( \mathbb{V}(I) \) by calculating the eigenvalues of \( m_f = f(m_{x_1}, \ldots, m_{x_n}) \). Note that this expression for \( m_f \) is well defined because of the commutativity of the \( m_{x_i} \). Algorithm 1 can be used if \( I \) satisfies the assumptions made in this paper. As a test of correctness, we have evaluated the quadric \( f(x_1, x_2) = -(x_1^2 + x_2^2) + 0.1xy + 15 \) on the variety defined by two bivariate polynomials of degree 7 and 6, shown in Figure 1. For the computed multiplication matrices, we compute

\[
\frac{\|m_{x_1}m_{x_2} - m_{x_2}m_{x_1}\|_2}{\|m_{x_1}m_{x_2}\|_2} = 5.5552 \cdot 10^{-13}.
\]

This shows that the multiplication matrices commute (up to 13 digits of accuracy).

#### 6.2 Solving generic systems

We now use the obtained multiplication maps to compute the solutions \( \mathbb{V}(I) \) of square systems of polynomial equations in the following way. We perform a simultaneous diagonalization of the identity matrix together with the \( n \) multiplication
Figure 1: Left: zero level lines in $\mathbb{R}^2$ of two bivariate polynomials of degree 7 (---) and 6 (-----) together with the real solutions (○). Right: The surface $f(x_1, x_2) = -(x_1^2 + x_2^2) + 0.1xy + 15$ and the real eigenvalues of $m_f$. maps $m_{x_i}$. For this, we use the method cpd.gevd in Tensorlab [17, 11, 7]. We compare the results (accuracy and computation time) with the homotopy solvers BertiniLab [2] and PHClab [9]. To obtain the results, we used Matlab and we generated generic polynomials $f$ in the following way. We fix a Newton polytope $P$ of $f$ and to every point in $P \cap \mathbb{Z}_{\geq 0}^n$ we assign a real number drawn from a normal distribution with $\mu = 0$ and $\sigma = 1$ (using the \texttt{randn} command in Matlab). These numbers are the coefficients of the monomials in $S(f)$. To measure the accuracy of the resulting multiplication matrices, we calculate the condition number of the matrix inverted in step 10 of Algorithm 1. The accuracy of a solution $z$ of a square system $f_1 = \ldots = f_n = 0$ is measured by the residual.

**Definition 2.** Given a square system of polynomial equations $f_1 = \ldots = f_n = 0$ with $f_1, \ldots, f_n \in \mathbb{C}[x]$ and a point $z \in \mathbb{C}^n$. The residual $r$ of $z$ is defined as

$$r_i = \frac{|f_i(z)|}{f_i,\text{abs}(z_{\text{abs}}) + 1}, \quad r = \frac{1}{n} \sum_{i=1}^{n} r_i,$$

where $| \cdot |$ denotes the absolute value, $f_{i,\text{abs}}$ is $f_i$ where the coefficients $c_{\alpha,i}$ of $f_i$ are replaced by their absolute values and $z_{\text{abs}}$ is the point in $\mathbb{C}^n$ obtained by taking the absolute values of all the components of $z$.

The term $+1$ in the denominator of the $r_i$ makes it clear that we are using a mixed relative and absolute criterion, to take into account the possibility that $f_{i,\text{abs}}(z_{\text{abs}}) = 0$.

We first investigate the influence of the automated choice of basis made in our algorithm. We compare it to the fixed choice of the block basis given in (2). This
is the basis that is used (implicitly) in root finding using u-resultants [6, Chapter 3]. We first check that it is not just the block basis itself that comes out of our algorithm. We generated two random dense polynomials $f_1, f_2 \in \mathbb{C}[x_1, x_2]$ of degree $d_1 = d_2 = 10$. The support of the associated dense Macaulay matrix $M$ is all monomials of degree up to $d_1 + d_2 - 1 = 19$. The basis $O$ should count 100 elements (Theorem 1). Figure 2 shows that, indeed, the choice of basis is significantly different.

![Figure 2: Left: the block basis $O$ given in (2). Right: the basis $O$ chosen by Algorithm 1. Black circles indicate the support $S(M)$ of the Macaulay matrix.](image)

We now check the accuracy of the multiplication matrices by computing the condition number of the coefficient matrix inverted in step 10. For a condition number of order $10^l$, we expect to lose $l$ accurate digits w.r.t. the machine precision. Figure 3 shows the results for bivariate systems of increasing degree up to 20. By using the QR decomposition with optimal column pivoting the condition number is controlled and it gets no larger than $\pm 10^4$. With our machine precision of order $10^{-16}$ (double precision), this means that the forward error on the multiplication matrices is of order $10^{-12}$. For the same set of generic bivariate systems of degree 1 up to 20 we also calculated the maximal residual of all of the calculated solutions. This is shown in the right part of Figure 3. One can expect that more accurate multiplication maps lead to more accurate solutions, which is confirmed by the figure. For degrees higher than 15, the solutions obtained using the block basis no longer made sense. The results are averaged out over 20 experiments. These results clearly show that a numerically justified choice of basis is crucial for the feasibility of normal form algorithms to compute multiplication matrices.

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*By degree $d$ we mean here that both polynomials $f_1$ and $f_2$ are generic of degree $d$. 

Figure 3: Left: condition number for the computation of the multiplication matrices with block basis (•) and smart choice of basis (–) for bivariate systems of increasing degree. Right: Maximal residual with the block basis (•) and the QR choice of basis (–) for the same systems.

In the following, we only work with the automated choice of basis. Some results for dense systems with more variables are shown in Figure 4. The figure shows that even for large systems, all solutions are found with a small residual. For example, in the case $n = 3$ with degree 21, there are 9261 solutions in $\mathbb{C}^3$, all found with a residual smaller than $10^{-10}$. We also note that the residual would drop to machine precision after one ‘refining’ iteration of Newton’s method.

As for the computation time, the figure shows that the method is very sensitive to the number of variables (it suffers from the ‘curse of dimensionality’). The asymptotic complexity is $O(d^{3n})$, where $d$ is the degree. Intuitively, we find the coordinates of the $d^n$ solutions as eigenvalues and the cost of the algorithm is the number of eigenvalues cubed.

We compare our method to the Matlab interfaces of the homotopy continuation packages Bertini [2] and PHCpack [9]. The results are shown in Figure 5. The figure confirms that the complexity of our method grows drastically with $n$. For $n = 2$, however, we are slightly faster for degrees at least up to 25. In all figures, the residuals of our computed solutions are slightly bigger than the ones from the homotopy methods. This is because these methods intrinsically make use of Newton-Raphson refinement. One Newton sweep over our solutions would lead to a residual of order machine precision as well, because of the quadratic convergence property. An important remark is that continuation methods do not return all solutions in all cases. The methods might give up on certain paths along the way if

\footnote{We used default settings for both solvers.}
the algorithm decides that the path seems to be diverging to infinity or if it crosses an ill-conditioned region. For $n = 2$ and degree 20, PHClab returns 398 solutions (2 solutions are lost) within slightly less than 4 seconds. For $n = 2$, degree 40, it takes 57 seconds to find 1575 out of the 1600 solutions. Using Bertini with double precision arithmetic [1], we find all solutions for $n = 2$, degree 20 within 12 seconds and 1587 out of 1600 solutions for $n = 2$, degree 40 within 350 seconds.

7 Conclusion and future work

We have presented a first normal form algorithm for zero-dimensional ideals coming from square polynomial systems that makes an automated, numerically justified choice of monomial basis for $\mathbb{C}[x]/I$ under certain genericity assumptions on $I$. Our numerical experiments show that this choice of basis makes it possible to perform the normal form computation in finite precision, while it can go terribly wrong by manually choosing a basis. Some ideas for future work are:

• Relaxing the genericity assumptions. What if the polynomials $f_1, \ldots, f_n$ are sparse?

• Solutions at infinity lead to linear dependencies in the columns of $M_b$, but it
Figure 5: Comparison of the results for PHClab (—), BertiniLab (—) and our method (—) for $n = 2, 3, 4, 5$. 
also causes the dimension of $\mathbb{C}[x]/I$ to drop. This can be incorporated in the algorithm.

- For multiple solutions of a square polynomial system, the canonical polyadic decomposition does not work. The coupling between the different coordinates can be made by using the left eigenvectors of the multiplication maps.

- The implementation is done in Matlab and a lot of computation time is spent on the construction of the Macaulay matrix $M$. We believe that an implementation in Julia, C(++), Fortran, ... could be a significant improvement.

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