Scalar field entanglement entropy for a small Schwarzschild black hole

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Abstract
We consider scalar field entanglement entropy generated with a black hole of (sub)Planck mass scale, thus implying the unitary evolution of gravity. The dependence on the dimension of the Hilbert space for degrees of freedom located behind the horizon is taken into account. The obtained results contain polylogarithmic terms.

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1. Introduction

Bekenstein demonstrated in 1973 [1] that the black hole (BH) entropy $S_{BH}$ is proportional to its horizon area $A$. As is well known from the ‘no-hair’ theorem, the observer can determine the mass, angular momentum and charge of the BH but no other properties, thus giving rise to the entropy origin problem. The second problem is the information-loss problem initiated by Hawking in 1975 [2], which in combination with [3] determined that the BH entropy $S_{BH} = A/4$ in Planck units. The BH appeared to generate a large amount of entropy with almost no chance to read out any information about the matter falling below its horizon, thus implying the non-unitary evolution. To date, finding a solution for both the problems is a challenging task for any theory of quantum gravity. At present, many approaches to answer them have been proposed.

In 1996, Strominger and Vafa [4] proposed to consider BH entropy with the help of string theory. Such an approach has appeared to be a fruitful one; for more information on the topic, one can read reviews such as [5] or the recent ones [6, 7] and the references therein.

Another approach to the problems is based on loop quantum gravity. It provides a count of microscopic BH microstates and therefore determines its entropy. One can read more on the topic in [8–10] and in recent papers such as [11, 12]. In [13], the authors applying a similar approach conclude that the BH radiation spectrum should become less entropic as it evaporates. It leads to the possibility of information recovery from the BH due to the increasing role of quantum effects in the evolution of a small BH. Such a conclusion supports the concept of information preservation with a BH that was reconsidered first by Hawking in 2005 [14] and may interfere with our own results presented in [15].
The estimation of BH entropy with the help of the quantum tunneling approach is proposed in such papers as [16].

In spite of the variety of methods to calculate $S_{\text{BH}}$, it is widely accepted that the entropy should be generated with the BH event horizon. As the horizon separates the whole spacetime into accessible and non-observable regions, any distant observer should trace out all the degrees of freedom localized inside the BH. Therefore, the horizon can serve not only as an entropy generator, but as a depository for the degrees of freedom giving rise to it. Such an approach was proposed by Srednicki [17]; for the review see [18]. Investigation of the BH horizon as the depository results in the holographic principle [19]; among all the publications on the topic, we would like to mention [20, 21], where the authors use holography to calculate the entanglement entropy from AdS/CFT.

In [22], energy and entropy divergences arising in ’t Hooft’s brick wall model [23] are considered in the framework of the uncertainty principle. The authors raise the question of similarity between the entanglement and the statistical definitions of BH entropy. In [24], higher order corrections within the brick wall formalism for arbitrary spin have been found.

As the horizon separates the space into observable and non-observable ones, it is logical to assume that $S_{\text{BH}}$ originates from entanglement. Such a viewpoint is interesting since then there is no information loss at all and gravity obeys unitary evolution [14]. For the other approaches resulting in the unitary nature of the BH evolution, one can read [25] and the recent review [26]. The situation has much in common with the restricted access to some code represented in the Schmidt basis: being able to read some part of the code only, one concludes at the non-zeroth entropy. But reducing the inaccessible part of the code reduces the entropy; a similar behavior should be observed for the decreasing horizon area. Such a process (i.e. accessing a part of the code only) cannot be viewed as one that generates entropy.

In this paper, we apply the approach presented in [15] to a BH of (sub)Planck mass scale. We count the entanglement entropy of a scalar field separated with the BH’s horizon into two parts, and compare it to $S_{\text{BH}}$. We realize that the case of the Planck scale BH is a bit speculative. As is mentioned in [2], such small BHs cannot be considered as some classical background metric for any quantum field. In such a case, one is expected to apply quantum gravity. But, as such a theory has not been built till now, we try to take a look beyond.

In the author’s opinion, the (sub)Planck mass scale BHs seem to be of great interest in the light of modern heavy ion collision experiments. In the case of appearance during LHC experiments, small BHs will witness other dimensions via their spectrum radiation characteristics, and therefore, we hope our investigation might be useful in this sphere. Also, the results we derived may be helpful in the analysis of quark–gluon bag models.

Our approach has much in common with the others presented in [27–29]. However, our approach has some differences. We estimate entropy via the volume of the BH and angular momentum, while in [27, 28], shell volume near the horizon and momentum are utilized. As for [29], it is based on the thermal atmosphere surrounding the horizon.

This paper is organized as follows. In section 2, we present a general idea of our model. Section 3 discusses the model itself. Entropy estimation and its analysis are presented in section 4. Discussion and conclusions are presented in section 5.

2. Basic concepts

Throughout the paper, Planck units were used.

We consider a Schwarzschild BH of mass $M$ and consequently of radius $r = 2M$ and some scalar field surrounding it. The field is supposed to be in some pure state $|\Phi\rangle$ in the Kruskal frame of reference (FR) and to have no influence on the background metric (quasiclassical...
FRs are connected via the Bogolubov transformations \[2, 30\], with the particles being radiated with it. The creation and annihilation boson operators in both FRs are connected via the Bogolubov transformations \[2, 30\],

\[
a^\dagger = \frac{1}{\sqrt{1 - \zeta^2}} b_{\text{out}}^\dagger - \frac{\zeta}{\sqrt{1 - \zeta^2}} b_{\text{in}}, \quad a = \frac{1}{\sqrt{1 - \zeta^2}} b_{\text{out}} - \frac{\zeta}{\sqrt{1 - \zeta^2}} b_{\text{in}},
\]

where \(a, a^\dagger\) are the annihilation and creation operators in the Kruskal FR, \(b_{\text{in(out)}}, b_{\text{in(out)}}^\dagger\) are the annihilation and creation operators in the accelerated FR inside (outside) the horizon and \(\zeta\) is defined as

\[
\zeta = \exp(-4\pi M\omega),
\]

where \(\omega\) is the energy of the field quanta generated at the BH’s event horizon under the Unruh effect.

The Kruskal field \(|\Phi\rangle\) will be detected with the observer from the accelerated FR in the state

\[
|\Phi\rangle = \sqrt{\frac{1 - \zeta^2}{1 - \zeta^{2N}}} \sum_{n=0}^{N-1} \zeta^{2n} |n\rangle_{\text{in}} |n\rangle_{\text{out}}, \quad a|\Phi\rangle = 0,
\]

(2)

where \(N = N_{\text{in(out)}}\) is the dimension of the in- (out-) Hilbert subspaces \[30\]. Here, the in- and out-components (denoted by the corresponding subscripts) describe the parts of the field under and above the horizon. We emphasize that now we are working with a single mode of the scalar field only. Later, we will integrate over all \(\omega\) possible to take into account all the modes.

As one can see, (2) is exactly the Schmidt decomposition \[31, 32\], and hence one obtains density matrices of the in- and out-components

\[
\rho_{\text{in}} = \text{Tr}_{\text{out}} |\Phi\rangle \langle \Phi| = \frac{1 - \zeta^2}{1 - \zeta^{2N}} \sum_{n=0}^{N-1} \zeta^{2n} |n\rangle_{\text{in}} \langle n|,
\]

\[
\rho_{\text{out}} = \text{Tr}_{\text{in}} |\Phi\rangle \langle \Phi| = \frac{1 - \zeta^2}{1 - \zeta^{2N}} \sum_{n=0}^{N-1} \zeta^{2n} |n\rangle_{\text{out}} \langle n|.
\]

As we see, different observers handle different density matrices. Although the Kruskal observer detects the pure state \(|\Phi\rangle\), the accelerated one, because of having access to the out-component of the field (i.e. to the outgoing radiation) only, detects a mixture with entropy

\[
\sigma(N, \zeta) = -\text{Tr} \rho_{\text{out}} \ln \rho_{\text{out}} = -\frac{1 - \zeta^2}{1 - \zeta^{2N}} \sum_{n=0}^{N-1} \zeta^{2n} \ln \left(\frac{1 - \zeta^2}{1 - \zeta^{2N}} \zeta^{2n}\right)
\]

\[
= -\ln \frac{1 - \zeta^2}{1 - \zeta^{2N}} - \left(\frac{\zeta^2}{1 - \zeta^2} - N \frac{\zeta^{2N}}{1 - \zeta^{2N}}\right) \ln \zeta^2.
\]

(3)

where the relation

\[
\sum_{n=0}^{N-1} \zeta^{2n} = \frac{1}{2 \ln \zeta} \frac{2}{\zeta} \sum_{n=0}^{N-1} \zeta^{2na} \bigg|_{\zeta = 1} = \frac{(1 - \zeta^{2N}) \zeta^2 - N(1 - \zeta^2) \zeta^{2N}}{(1 - \zeta^2)^2}
\]

has been used.
As one can see from (3), $\sigma(N, \zeta)$ depends on two parameters: $N$ and $\zeta$. The main problem here is to estimate the value of $N$. It is easy in the asymptotic of a large BH, as one can use the limit $N \to \infty$ then; such an asymptotic is popular in the literature. In [15], it was applied too. However, here we consider small BHs with mass $M \leq 1$ and therefore have to take into account the finiteness of $N$. The direct estimation of the magnitude of $N$ may be done via the calculation of the field energy. However, it leads to the integral which can be solved approximately only, and therefore is not discussed here. Anyway, we expect $N \gg 1$ since otherwise the scalar field will influence the BH and thus violate the quasiclassical approach. Such an assumption seems to be reasonable and to have no contradictions with the model. For a BH with (sub)Planck mass $M \leq 1$ and for the small rest mass of scalar field quanta, $N$ must be large enough to encode all the degrees of freedom.

3. Model construction

Expression (3) is written for some mode with fixed parameters of the radiated field component. Model construction requires correct contribution estimation of all the modes to the entropy, that is of the system symmetry and of the energy spectrum determined with $\omega$.

Due to the spherical symmetry, we must take into account the contributions from all the angular momenta $l$ and their projections $-l \leq \mu \leq l$ possible. The range on $l$ is well defined and can be written in the following form:

$$0 \leq \sqrt{l(l+1)} \leq \sqrt{L(L+1)} = rp = 2M\sqrt{\omega^2 - m^2},$$

where $p$ is the momentum of the field quantum radiated away.

In such a case, the entropy from (3) should be multiplied with

$$\sum_{l=0}^{L} \sum_{\mu=-l}^{\mu=l} 1 = 4M^2(\omega^2 - m^2) + \sqrt{16M^2(\omega^2 - m^2)} + 1 + \frac{1}{2}.$$ (4)

But, taking into account angular degrees of freedom is not enough. In (3), $\sigma(N, \zeta)$ is defined for the fixed $\omega$ only. As we want to estimate the contribution from all the modes, we should integrate over all the $\omega$ possible. Therefore, we write down the following integral for the radiation entropy $S(N, M, m)$:

$$S(N, M, m) = \frac{V}{(2\pi)^3} \int_{m}^{M} \sum_{l=0}^{L} \sum_{\mu=-l}^{\mu=l} \sigma(N, \zeta) \, d\omega,$$

where $V = 4\pi r^3/3 = 2^5\pi M^3/3$ is the BH volume confined within the horizon and $m$ is the rest mass of the radiated quanta. The upper integral bound is equal to $M$ here because the energy of the field quanta cannot exceed the BH mass. Substituting (3) and (4) into the integral, we obtain for the radiation entropy

$$S(N, M, m) = \frac{M}{6\pi^3} \int_{m}^{M} \sigma(N, \zeta) \left[ 1 + 8M^2(\omega^2 - m^2) + \sqrt{1 + 16M^2(\omega^2 - m^2)} \right] \, d\omega,$$ (5)

where $S_{BH} = 4\pi M^2$ is the Bekenstein–Hawking entropy and $\zeta$ is defined in (1).

4. Entropy estimation

Before we proceed, let us make some estimations of $m$. In the case of being not equal to 0, $m$ should be of elementary particles’ mass order, i.e.

$$m = 0 \quad \text{or} \quad 10^{-23} \leq m \leq 10^{-18} \quad \Rightarrow \quad m \gtrapprox 0.$$ (6)
where $10^{-23}$ is of the order of the electron mass $m_e$ and $10^{-18}$ is of the order of the $Z^0$ boson mass $m_{Z^0}$. So it should be taken into account that $m$ is a small number.

The integral (5) cannot be calculated directly due to the strong integrand dependence on the integral boundaries. The situation is complicated with the exact entropy dependence on $N$. In [15], the dependence on $N$ had been neglected because of the large BH mass, but here, we cannot use the same trick.

To estimate the entropy, at first we decompose $\sigma(N, \zeta)$ from (3) into series

$$\sigma(N, \zeta) = \sum_{n=1}^{N} \zeta^{2n} \left( \frac{1}{m} - 2 \ln \zeta \right) - N \zeta^{2N} \left( \frac{1}{N} - 2 \ln \zeta \right) + O[\zeta^{2(n+1)}].$$

(7)

Due to (1), $\zeta$ exponentially depends on $\omega$, and thus such a decomposition is good at the higher integral bound $\omega = M$. But one can argue that at the lower integral bound $\omega = m$, such a series expansion may fail: due to the smallness of $m$, which follows from (6), $\zeta$ will not differ from unity a lot. However, the neglected terms in the expansion are of the order of $\zeta^{2(N+1)}$, so here we must take into account not the mass $m$ itself but the product $Nm$ in the exponent. As we discussed at the end of section 2, the number $N$ is expected to be large since otherwise the scalar field will influence the BH, thus violating the quasiclassical approach. So we conclude that the decomposition (7) is applicable at the whole range $m \leq \omega \leq M$ except in the case $m = 0$.

The second step is the decomposition of the square root term in the integrand from (5). Expanding it into a series with respect to $\omega$, one meets the problem at the upper bound of the integral since $M \leq 1$, so we use the following trick. As

$$\forall \omega \in [m, M] \quad 1 + 16M^2\omega^2 > 16M^2m^2 \quad \Rightarrow \quad (1 + 4M\omega)^2 > 8M\omega + 16M^2m^2,$$

which allows us to rewrite the square root from (5) in the following form:

$$\sqrt{1 + 16M^2(\omega^2 - m^2)} = (1 + 4M\omega) \sqrt{1 - \frac{8M\omega + 16M^2m^2}{(1 + 4M\omega)^2}}$$

$$= (1 + 4M\omega) \left(1 - \frac{4M\omega + 8M^2m^2}{(1 + 4M\omega)^2} + O\left[\frac{(M\omega + 2M^2m^2)^2}{(1 + 4M\omega)^4}\right]\right)$$

$$\approx 4M\omega + \frac{1 - 8M^2m^2}{1 + 4M\omega}.$$ (8)

Estimating the error for (8), one can note that the expression in the square brackets increases with decreasing $\omega$. As a result, the error of the decomposition applied will be of the order of $O[M^2m^2(1 + 2m)^2(1 + 4Mm)^{-4}]$, and thus is small due to (6).

Substituting (7) and (8) into (5), we obtain

$$\frac{S(N, M, m)}{S_{BH}} \approx \frac{1}{24\pi^3} \left( \sum_{n=1}^{N} \alpha_n - N\alpha_N \right) \Bigg|_{\omega = M}^{\omega = m},$$

(9)

where

$$\alpha_n = \zeta^{2n} \left[ \frac{1 + 4M\omega + 8M^2(2\omega^2 - m^2)}{\pi n^2} + \frac{1 + 6M\omega + 8M^2(2\omega^2 - m^2)}{4\pi^2 n^3} + \frac{3 + 16M\omega}{2\pi^3 n^4} \right]$$

$$+ (1 - 8M^2m^2)(2\pi - 1/n)e^{2\pi n} \text{Ei}[-2\pi n(1 + 4M\omega)],$$

(10)

where $\text{Ei}(x) = \int_{-\infty}^{x} e^{-t}/t \, dt$. 

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Expressions (9) and (10) give an approximate estimation for the scalar field entanglement entropy for a small BH.

From (9), it is hard to achieve the power law since in any order the corresponding terms will vanish after substituting the integral boundaries. We have no explanation of this fact except that for the BH of the (sub)Planck mass scale, the radiation spectrum should change to take into account quantum gravity effects. A quite similar conclusion was made in [13] also with the help of loop quantum gravity.

The term proportional to the logarithm of $S_{BH}$ may be obtained in the following way. As one can see, the leading-order term from (10) is proportional to $\zeta^2 n^{-n}$. Neglecting the higher powers of $n$, we obtain from (9) and (10) that

$$\frac{S(N, M, m)}{S_{BH}} \propto \left( \sum_{n=1}^{N} \frac{\zeta^{2n}}{n} - \zeta^{2N} \right)_{\omega=M} \Bigg|_{\omega=m},$$

which after setting $N \to \infty$ and applying (1) transforms to

$$\frac{S(\infty, M, m)}{S_{BH}} \propto \ln \frac{1 - e^{-8\pi M^2}}{1 - e^{-8\pi Mm}}.$$  \(11\)

Taking into account (6), one can note that $Mm \ll 1$ and then easily extract the term proportional to $\ln M$ from (11). However, one should keep in mind that the entanglement entropy is measured in $S_{BH}$ units already. So the derived term is not the logarithm correction in its common sense, but the one proportional to it.

As we see from (9), the terms containing higher powers of $1/n$ transform to polylogarithms of orders 2, 3 and 4. So we have obtained the other corrections to the scalar entanglement entropy.

Finally, we would like to give an upper bound for the scalar entanglement entropy. As one can note from (5), the integrand is non-negative for any values of $\omega$ and $N$. Therefore, from (9), it follows that the entropy takes maximum values at its boundary:

$$\frac{S(N, M, m)}{S_{BH}} \leq \frac{S(N \to \infty, 1, 0)}{4\pi} \approx 1.462 \times 10^{-3}.$$  

Therefore, the scalar field entanglement entropy cannot be responsible for all the entropy generation: its contribution is less than 1%.

5. Discussion

Summing up, we estimated the radiation entropy of the scalar field generated with the horizon of a BH with mass $M \leq 1$. This paper complements [15], where the case $M > 1$ was considered. It is based on similar principles.

In this paper, we considered the influence of the dimension number $N$ of the in(out)side Hilbert subspace with respect to the BH horizon. $N$ is usually taken to be infinite for simplicity, but here we could not do so due to the smallness of the BH. Estimation of the magnitude of $N$ deserves further research.

The results demonstrate no area law dependence. We suppose this as a consequence of quantum gravity effects. The term proportional to the logarithm of the BH area is obtained. It is not the logarithm correction term in the common sense: it is the product of the logarithm and $S_{BH}$ itself. The other correction terms contain polylogarithms of orders from 2 to 4.

The upper bound of the scalar entanglement entropy does not exceed 1% of $S_{BH}$. Comparing the entanglement entropy upper bound to [15], we note that its contribution is almost the same. Such a result follows from the fact that for the upper bound estimation on the
entropy, similar assumptions were used (i.e. \( N \to \infty \)). Taking the finiteness of \( N \) will reduce the scalar entropy contribution to \( S_{\text{BH}} \) even more, but will not change the result significantly. So we conclude that the scalar field entanglement entropy does not dominate in \( S_{\text{BH}} \).

The smallness of the scalar field contribution to BH entropy might be a consequence of our restrictions to the scalar field only. BH entropy is detected via the particles radiated away. So the degrees of freedom encoded with other quantum numbers should play a more significant role. Here, the analogy with some register may be observed. One-symbol language provides linear growth of the number of possible states of the register, while even the binary language provides the exponential one. In such a case, other quantum numbers such as spin and its projections are expected to increase the entanglement entropy contribution significantly enough. Such a supposition needs further investigation.

The contribution of the scalar entanglement entropy appeared to be small in comparison to the results obtained in [17, 18, 33]. Such a discrepancy should have been expected due to the differences in the models and assumptions considered.

Our approach has much in common with the one presented in [27, 28] where the upper bound on the entropy is derived. Compared to the papers, here we present the analytical expression for the entropy for a small BH, taking into account its dependence on \( N \).

As is well known, some of the LHC detectors are designed to explore ‘new physics’; it implies looking for additional dimensions. In case there are such, the BH might appear during the collisions and therefore might be detected via its Hawking radiation. This paper might be helpful in searching for possible BH generation during collisions. Despite the fact that here we considered the case of a \( 3 + 1 \)-dimensional BH, which cannot be observed on the LHC, this paper may shed some light on the topic since such a BH is expected to be small: its mass cannot exceed 1 due to energy restrictions. Also, the presented results can be helpful for the analysis of a quark–gluon bag model or similar ones.

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