TOPOLOGICAL TRANSITIVE SEQUENCE OF COSINE OPERATORS ON ORLICZ SPACE

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Abstract. For a Young function \( \phi \) and a locally compact second countable group \( G \), let \( L^\phi(G) \) denote the Orlicz space on \( G \). In this article, we present a necessary and sufficient condition for the topological transitivity of a sequence of cosine operators \( \{ C_n \}_{n=1}^{\infty} := \{ \frac{1}{2}(T^n_g + S^n_g) \}_{n=1}^{\infty} \), defined on \( L^\phi(G) \). We investigate the conditions for a sequence of cosine operators to be topological mixing. Moreover, we go on to prove the similar results for the direct sum of a sequence of the cosine operators. At the last, an example of topological transitive sequence of cosine operators is given.

1. Introduction and preliminaries

A sequence of bounded linear operators \( \{ T_n \}_{n=1}^{\infty} \) acting on a Fréchet space \( X \) is said to be topological transitive if for each pair of non-empty open sets \( (U, V) \) in \( X \), there exists an \( n \in \mathbb{N} \) such that \( T_n(U) \cap V \neq \emptyset \).

A single bounded linear operator \( T \) is topological transitive whenever the sequence of its iterates, that is \( \{ T^n \}_{n=0}^{\infty} \) is topological transitive where \( T^0 \) is the identity map. Furthermore, a sequence \( \{ T_n \}_{n=1}^{\infty} \) is called hypercyclic if there is a vector \( x \in X \) whose orbit \( \{ T_nx : n = 0, 1, 2, \ldots \} \) is dense in \( X \). Such a vector is called a hypercyclic vector for that sequence. Analogously, when the sequence \( \{ T^n \}_{n=0}^{\infty} \) is dense in \( X \), we say that an operator \( T \) is hypercyclic. It is worth noting that these two notions, topological transitivity and hypercyclicity in a single case are more likely equivalent on a Fréchet space \( X \). An operator \( T \) is topologically mixing whenever for each pair of non-empty open sets \( (U, V) \) in \( X \), there exists an \( N \in \mathbb{N} \) such that \( T^n(U) \cap V \neq \emptyset \) for all \( n \geq N \). The operators of the form “identity plus a backward shift” are the example of topologically mixing operators which are also hypercyclic. An operator \( T \) on a Fréchet space \( X \) is weakly mixing if and only if \( T \oplus T \) is hypercyclic on \( X \oplus X \). Note that weakly mixing maps are topologically transitive but in the topological setting, the converse is not true. More detailed information concerning dynamics of linear operators may be found in the best interesting books.

A continuous, even and convex function \( \phi : \mathbb{R} \to \mathbb{R}^+ \cup \{0\} \) is called a Young’s function whenever \( \phi(0) = 0 \) and \( \lim_{t \to \infty} \phi(t) = \infty \). Corresponding for each Young’s function \( \phi \), there is another Young’s function \( \psi : \mathbb{R} \to \mathbb{R}^+ \cup \{0\} \) defined by \( \psi(y) := \sup \{ x | y - \phi(x) : x \geq 0 \} \), which is called complementary Young’s function of \( \phi \).

Let \( G \) be a locally compact and second countable group with the identity element \( e \). Consider a right

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invariant Haar measure $\lambda$ on $G$. Let $L^\phi(G)$ denote the set of all Borel measurable functions $f$ on $G$ such that

$$\int_G |f(x)\nu(x)| \, d\lambda(x) < +\infty,$$

for each measurable $\nu$ with

$$\Psi(\nu) := \int_G \psi(\nu) \, d\lambda \leq 1.$$

The set of all Borel functions $\nu$ on $G$ such that $\Psi(\nu) \leq 1$ will be denoted by $\Omega$. Plainly $L^\phi(G)$ is a vector space. Now we assume that the Young’s function $\phi$ vanishes only at zero. This guarantees that $L^\phi(G)$ equipped with the norm

$$\|f\|_\phi := \sup_{\Psi(\nu) \leq 1} \int_G |f(x)\nu(x)| \, d\lambda(x),$$

is a Banach space and called an Orlicz space $\Omega^\phi$.

Another equivalent norm on $L^\phi(G)$ is defined by

$$N_\phi(f) = \inf\{k > 0 : \int_G \phi\left(\frac{|f|}{k}\right) \, d\mu \leq 1\},$$

which is so-called the Luxemburg norm.

A Young’s function $\phi$ is said to satisfy condition $\Delta_2$-regular if there is a constant $k > 0$ such that $\phi(2t) \leq k\phi(t)$ for large values of $t$ when $\lambda(G) < \infty$. In case $\lambda(G) = \infty$, $\phi(2t) \leq k\phi(t)$ for each $t > 0$. Some examples of such Young’s functions may be found in [2, Example 2.8]. If $\phi$ is $\Delta_2$-regular, then the space $C_c(G)$ of all continuous functions on $G$ with compact support is dense in $L^\phi(G)$, and the dual space $(L^\phi(G), \| \cdot \|_\phi)$ is $(L^\phi(G), N_\phi(\cdot))$. For further information the interested reader is referred to [11].

It is well known that the hypercyclic phenomenon is occurred only on infinite-dimensional and separable spaces [3, 8]. For this reason, we assume that $G$ is second countable and $\phi(x) = 0$ if and only if $x = 0$ [11, p. 87, Theorem 1]. Throughout this paper, the Banach space of all essentially bounded and measurable functions on $G$ is denoted by $L^\infty(G)$ and $N(f, r)$ denotes a neighborhood of $f \in L^\phi(G)$ with radius $r > 0$. A bounded continuous function $w : G \to (0, \infty)$ is called a weight. For $g \in G$, let $\delta_g$ be the unit point mass at $g$. Given a weight $w$ on $G$ and $g \in G$, a weighted translation $T_{g, w} : L^\phi(G) \to L^\phi(G)$ is defined by

$$T_{g, w}(f) := w \cdot f \ast \delta_g, \quad f \in L^\phi(G)$$

where $f \ast \delta_g$ is the following convolution

$$f \ast \delta_g(t) := \int_G f(tx^{-1}) \, d\delta_g(x) = f(tg^{-1}), \quad t \in G.$$

Indeed it is the right translation of $f$ by $g^{-1}$. Moreover, it is easy to check that $f \ast \delta_g \in L^\phi(G)$ whenever $f \in L^\phi(G)$. Recall that an element $g \in G$ is called a torsion element if it is of finite order. An element $g \in G$ is called periodic if the closed subgroup $G(g)$ generated by $g$ is compact. Further, an element in $G$ is aperiodic if it is not periodic. Equivalently, $g \in G$ is an aperiodic element, if and only if for any compact subset $K \subset G$, there exists an $N \in \mathbb{N}$ such that $K \cap Kg^{-n} = \emptyset$ for $n > N$ [6, Lemma 2.1]. It is worth noting that a weighted translation $T_{g, w}$ cannot be hypercyclic whenever $\|w\|_\infty \leq 1$ or $g$ is a torsion
element [6, 2]. The hypercyclic weighted translation on locally compact groups have been characterized by C. Chen [6] in details. In addition, he has studied the hypercyclicity of weighted convolution operators in [7]. Moreover, the hypercyclic weighted translations on Orlicz spaces $L^\phi(G)$ has been studied in [2].

In the case $w^{-1} := \frac{1}{w} \in L^\infty(G)$, the weighted translation operator $T_{g,w}$ is invertible and its inverse is $T_{g^{-1}, w^{-1} \delta_g}$ which will be denoted by $S_{g,w}$ throughout this paper. For each $n \in \mathbb{Z}$, the cosine operator $C_n : L^\phi(G) \rightarrow L^\phi(G)$ is defined by

$$C_n := \frac{1}{2}(T_{n,g,w} + S_{n,g,w}).$$

The study of cosine operator on Banach spaces is originally due to the work done in [4] by Bonilla and Miana. They gave sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous cosine operator function. Afterwards, T. Kalmes in [9] characterized the hypercyclicity of cosine operator functions on $L^p(\Omega)$ ($\Omega$ is open subset of $\mathbb{R}^d$) generated by second order partial differential operators. He also showed that the hypercyclicity and weakly mixing of these type of operators are equivalent.

Furthermore, a necessary and sufficient condition for the topological transitivity of the cosine operator $C_n$ on $L^p(G)$ has been already studied in [5]. Nevertheless, in this paper, we are going to generalize that condition to Orlicz space $L^\phi(G)$ on which the topological transitivity of the cosine operator $C_n$ stays still in force.

2. Main Result

In this section we present our main results with some immediate consequences. We begin with the following theorem which give a necessary and sufficient condition on weight so that cosine operator is topological transitive.

**Theorem 2.1.** Let $g \in G$ be an aperiodic element of $G$ and let $\phi$ be a $\Delta_2$-regular Young’s function. Let $w, w^{-1} \in L^\infty(G)$. If $C_n := \frac{1}{2}(T_{n,g,w} + S_{n,g,w})$ is a cosine operator on $L^\phi(G)$, then the following statements are equivalent.

(i) $(C_n)_{n \in \mathbb{N}_0}$ is topological transitive.

(ii) For each non-empty compact subset $K \subset G$ with $\lambda(K) > 0$, there exist sequences of Borel sets $(E_k), (E_k^+)$ and $(E_k^-)$ in $K$, and a sequence $(n_k)$ of positive numbers such that for $E_k = E_k^+ \cup E_k^-$, we have

$$\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{K \setminus E_k} |\nu(x)| \, d\lambda(x) = 0.$$
\[ \lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_k} \varphi_{n_k}(x) |\nu(x g^{n_k})| \, d\lambda(x) = 0, \]
\[ \lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_k^-} \varphi_{2n_k}(x) |\nu(x g^{2n_k})| \, d\lambda(x) = 0, \]
\[ \lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_k^-} \tilde{\varphi}_{n_k}(x) |\nu(x g^{n_k})| \, d\lambda(x) = 0. \]

Proof. (i) ⇒ (ii). In spite of being different underlying spaces, the approach of the proof is followed like as done in [5]. Let \( K \) be a compact subset of \( G \) such that \( \lambda(K) > 0 \). Since \( g \in G \) is an aperiodic element, there exists \( N \in \mathbb{N} \) such that \( K \cap K g^{\pm n} = \emptyset \) for \( n > N \) [6, Lemma 2.1]. Denote the characteristic function of \( K \) defined on \( G \) by \( \chi_K \). Clearly \( \chi_K \in L^\phi(G) \). Take \( U = N(\chi_K, \epsilon^2) \) and \( V = N(-\chi_K, \epsilon^2) \) in the definition of the topological transitive for the sequence \( (C_n)_{n \in \mathbb{N}_0} \). Then for each \( \epsilon \in (0, 1) \), there exist \( f \in L^\phi(G) \) and \( m \in \mathbb{N} \), such that

\[ \|f - \chi_K\|_\phi < \epsilon^2 \quad \text{and} \quad \|C_m f + \chi_K\|_\phi < \epsilon^2. \]

Hence, we can write that

\[ \|\text{Re}(f) - \chi_K\|_\phi < \epsilon^2 \quad \text{and} \quad \|\text{Re}(C_m f) + \chi_K\|_\phi = \|C_m \text{Re}(f) + \chi_K\|_\phi < \epsilon^2, \]

where \( \text{Re}(f) \) is the real part of the complex valued function \( f \). Since the maps \( \text{Re} : L^\phi(G, \mathbb{C}) \to L^\phi(G, \mathbb{R}) \) and \( f \mapsto f^+ := \max\{0, f\} \) from \( L^\phi(G, \mathbb{R}) \) to \( L^\phi(G, \mathbb{R}) \) are continuous and also commute with both \( T_{g,w} \) and \( S_{g,w} \), hence without loss of generality we may assume that the function \( f \) is positive.

Therefore, for any Borel subset \( F \subset G \), we have

\[
\|C_m f^+ \chi_F\|_\phi \leq \|(C_m f)^+\|_\phi = \|(C_m f + \chi_K - \chi_K)^+\|_\phi \\
\leq \|(C_m f + \chi_K)^+\|_\phi + \|(-\chi_K)^+\|_\phi \\
= \|(C_m f + \chi_K)^+\|_\phi \leq \|C_m f + \chi_K\|_\phi < \epsilon^2. 
\]

Set \( A = \{x \in K : |f(x) - 1| > \epsilon\} \). Then

\[
\epsilon^2 > \|f - \chi_K\|_\phi = \sup_{\nu \in \Omega} \int_{J_G} |f(x) - \chi_K(x)| |\nu(x)| \, d\lambda(x) \\
\geq \sup_{\nu \in \Omega} \int_{K} |f(x) - 1| |\nu(x)| \, d\lambda(x) \\
> \sup_{\nu \in \Omega} \int_{A} \epsilon |\nu(x)| \, d\lambda(x). 
\]

Therefore, we have

\[
\sup_{\nu \in \Omega} \int_{A} |\nu(x)| \, d\lambda(x) < \epsilon.
\]

Set \( B_m = \{x \in K : |C_m f(x) + 1| \geq \epsilon\} \). Then, by the similar argument, we get

\[
\sup_{\nu \in \Omega} \int_{B_m} |\nu(x)| \, d\lambda(x) < \epsilon.
\]
Now, let \( E_m := \{ x \in K : |f(x) - 1| < \epsilon \} \cap \{ x \in K : |C_m f(x) + 1| < \epsilon \} \). Then, for \( x \in E_m \), we get \( f(x) > 1 - \epsilon > 0 \) and \( C_m f(x) < \epsilon - 1 < 0 \). Also,

\[
\sup_{\nu \in \Omega} \int_{E_m} |\nu(x)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_{A \cup B_m} |\nu(x)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_A |\nu(x)| \, d\lambda(x) + \sup_{\nu \in \Omega} \int_{B_m} |\nu(x)| \, d\lambda(x) < \epsilon + \epsilon = 2\epsilon.
\]

By keeping the facts that Haar measure \( \lambda \) is right invariant, \( T_{g,w}^m f^+ \) and \( S_{g,w}^m f^+ \) are positive in the mind, with the aid of (1) we get that

\[
2\epsilon^2 > \|2(C_m f^+) \chi_{E_m} \|_{\phi} = \| (T_{g,w}^m f^+ + S_{g,w}^m f^+) \chi_{E_m} \|_{\phi} \geq \| T_{g,w}^m f^+ \chi_{E_m} \|_{\phi}
\]

\[
= \sup_{\nu \in \Omega} \int_{E_m} |T_{g,w}^m f^+(x)||\nu(x)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_{E_m} |w(x)w(xg^{-1})\ldots w(xg^{-m+1})f^+(xg^{-m})||\nu(x)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_{E_m} w(xg^m)w(xg^{m-1})\ldots w(xg)f^+(x)||\nu(xg^m)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_{E_m} \varphi_m(x)f^+(x)||\nu(xg^m)| \, d\lambda(x) > \sup_{\nu \in \Omega} \int_{E_m} (1 - \epsilon)\varphi_m(x)||\nu(xg^m)| \, d\lambda(x).
\]

Therefore,

\[
\sup_{\nu \in \Omega} \int_{E_m} \varphi_m(x)||\nu(xg^m)| \, d\lambda(x) < \frac{2\epsilon^2}{1 - \epsilon}.
\]

By the similar argument, we get

\[
2\epsilon^2 > \|(S_{g,w}^m f^+) \chi_{E_m g^m} \|_{\phi} > (1 - \epsilon) \sup_{\nu \in \Omega} \int_{E_m} \varphi_m(x)||\nu(xg^m)| \, d\lambda(x)
\]

and thus

\[
\sup_{\nu \in \Omega} \int_{E_m} \varphi_m(x)||\nu(xg^m)| \, d\lambda(x) < \frac{2\epsilon^2}{1 - \epsilon}.
\]

Hence, the first part of Condition (ii) holds as \( \epsilon \) is arbitrary.

Now, let \( E^-_m = \{ x \in E_m : T_{g,w}^m f(x) < \epsilon - 1 \} \) and \( E^+_m = E_m \setminus E^-_m \). Then, for \( x \in E^+_m \), we have

\[
\epsilon - 1 > C_m f(x) = \frac{1}{2} T_{g,w}^m f(x) + \frac{1}{2} S_{g,w}^m f(x) \geq \frac{1}{2}(\epsilon - 1) + \frac{1}{2} S_{g,w}^m f(x)
\]

and therefore

\[
S_{g,w}^m f(x) < \epsilon - 1, \quad x \in E^+_m.
\]

Now, consider the following

\[
(1 - \epsilon) \sup_{\nu \in \Omega} \int_{E^+_m} \varphi_{2m}(x)||\nu(xg^{2m})| \, d\lambda(x)
\]

\[
= \sup_{\nu \in \Omega} \int_{E^+_m} |w(xg^{2m})w(xg^{2m-1})w(xg^{2m-2})\ldots w(xg)||S_{g,w}^m f^-(x)||\nu(xg^{2m})| \, d\lambda(x)
\]
In the last inequality, from the fact $K \cap K g^{\pm 2m} = \emptyset$, has been already used. Therefore, we get

$$\sup_{\nu \in \Omega} \int_{E_{m}^{+}} |\varphi_{2m}(x)| \nu(xg^{2m}) |d\lambda(x) < \frac{4\epsilon^2}{(1 - \epsilon)}.$$ 

In similar lines, we also have

$$\sup_{\nu \in \Omega} \int_{E_{m}^{+}} \varphi_{2m}(x) |\nu(xg^{2m})| d\lambda(x) < \frac{4\epsilon^2}{(1 - \epsilon)}.$$ 

Since $\epsilon$ is arbitrary, last two condition of (ii) part also fulfilled.

(ii) $\Rightarrow$ (i). Let $U$ and $V$ be two non-empty open subsets of $L^\phi(G)$. Since $\phi$ is $\Delta_2$-regular we can choose two non-zero functions $f$ and $h$ in $C_c(G)$ such that $f \in U$ and $h \in V$. Set $K = \text{supp}(f) \cup \text{supp}(h)$, the supports of $f$ and $h$ respectively. Let $E_k \subset K$ and it satisfies condition (ii). But $g \in G$ is an aperiodic element, hence there exists $M \in \mathbb{N}$ such that $K \cap Kg^{\pm n} = \emptyset$ for all $n > M$. Subsequently, for a given $\epsilon > 0$, one can find $N \in \mathbb{N}$ such that for each $k > N$, $n_k > M$ and

$$\|h\|_\infty \cdot \sup_{\nu \in \Omega} \int_{E_k} \varphi_{n_k}(x) |\nu(xg^{n_k})| d\lambda(x) < \epsilon, \quad \|h\|_\infty \cdot \sup_{\nu \in \Omega} \int_{K \setminus E_k} |\nu(x)| d\lambda(x) < \epsilon.$$ 

Now, we have

$$\|T_{g,w}^{n_k}(h\chi_{E_k})\|_\phi = \sup_{\nu \in \Omega} \int_{G} |T_{g,w}^{n_k}(h\chi_{E_k})(x)\nu(x)| d\lambda(x) = \sup_{\nu \in \Omega} \int_{G} |w(x)w(xg^{-1}) \ldots w(xg^{-n_k+1})h(xg^{-n_k})\chi_{E_k}(xg^{-n_k})\nu(x)| d\lambda(x) = \sup_{\nu \in \Omega} \int_{G} |w(xg^{n_k})w(xg^{n_k-1}) \ldots w(xg)h(x)\chi_{E_k}(x)\nu(x)| d\lambda(x).$$
\[ \leq \|h\|_{\infty} \cdot \sup_{\nu \in \Omega} \int_{E_k} \varphi_{n_k}(x) |\nu(xg^{n_k})| \, d\lambda(x) < \epsilon. \]

Hence,
\[ \lim_{k \to \infty} \|T_{g,w}^{n_k}(h\chi_{E_k})\|_{\phi} = 0. \]
Also,
\[ \|h - h\chi_{E_k}\|_{\phi} = \sup_{\nu \in \Omega} \int_{G} |h(x) - h(x)\chi_{E_k}(x)| |\nu(x)| \, d\lambda(x) = \sup_{\nu \in \Omega} \int_{G} |h(x)| |\nu(x)| \, d\lambda(x) = \int_{K \setminus E_k} |h(X)| |\nu(x)| \, d\lambda(x) \leq \|h\|_{\infty} \cdot \int_{K \setminus E_k} |\nu(x)| \, d\lambda(x) < \epsilon. \]
By similar argument, using the conditions given in (ii) we get
\[ \lim_{k \to \infty} \|S_{g,w}^{n_k}(h\chi_{E_k})\|_{\phi} = 0, \quad \lim_{k \to \infty} \|S_{g,w}^{2n_k}(h\chi_{E_k}^{-})\|_{\phi} = \lim_{k \to \infty} \|T_{g,w}^{2n_k}(h\chi_{E_k}^{-})\|_{\phi} = 0. \]
In addition, we also have
\[ \lim_{k \to \infty} \|S_{g,w}^{n_k}(f\chi_{E_k})\|_{\phi} = \lim_{k \to \infty} \|T_{g,w}^{n_k}(f\chi_{E_k})\|_{\phi} = 0. \]
For each \( k \in \mathbb{N} \), we set
\[ v_k = f\chi_{E_k} + 2T_{g,w}^{n_k}(h\chi_{E_k}^{+}) + 2S_{g,w}^{n_k}(h\chi_{E_k}^{-}). \]
In this stage, an application of the frequently used fact i.e., \( K \cap Kg^{(m_1 - m_2)n_k} = \emptyset \) (\( m_1, m_2 \in \mathbb{Z} \) and \( m_1 \neq m_2 \)) with Minkowski inequality yield that
\[ \|v_k - f\|_{\phi} \leq \|f - f\chi_{E_k}\|_{\phi} + 2\|T_{g,w}^{n_k}(h\chi_{E_k}^{+})\|_{\phi} + 2\|S_{g,w}^{n_k}(h\chi_{E_k}^{-})\|_{\phi} \]
and
\[ \|C_{n_k}v_k - h\|_{\phi} \leq \|h - h\chi_{E_k}\|_{\phi} + \frac{1}{2}\|T_{g,w}^{m_k}(f\chi_{E_k})\|_{\phi} + \frac{1}{2}\|S_{g,w}^{n_k}(f\chi_{E_k})\|_{\phi} + \|T_{g,w}^{2n_k}(h\chi_{E_k}^{+})\|_{\phi} + \|S_{g,w}^{n_k}(h\chi_{E_k}^{-})\|_{\phi}. \]
Therefore, we have \( \lim_{k \to \infty} \|v_k - f\|_{\phi} = 0 \) and \( \lim_{k \to \infty} \|C_{n_k}v_k - h\|_{\phi} = 0 \) which gives that \( \lim_{k \to \infty} v_k = f \) and \( \lim_{k \to \infty} C_{n_k}v_k = h \). So, \( C_{n_k}(U) \cap V \neq \emptyset \) for some \( k \). Hence, the sequence \( (C_n)_{n \in \mathbb{N}_0} \) is topological transitive. \( \square \)

The following corollary gives a characterization of topological mixing property of cosine operators on Orlicz space \( L^\phi(G) \). Since the proof is similar to above theorem therefore we will omit the proof.

**Corollary 2.2.** Let \( g \in G \) be an aperiodic element of \( G \) and let \( \phi \) be a \( \Delta_2 \)-regular Young function. Let \( w, w^{-1} \in L^\infty(G) \). If \( C_n := \frac{1}{2}(T_{g,w}^n + S_{g,w}^n) \) is cosine operator on \( L^\phi(G) \) then the following statements are equivalent.

(i) \( (C_n)_{n \in \mathbb{N}_0} \) is topological mixing.
(ii) For each non-empty compact subset $K \subset G$ with $\lambda(K) > 0$, there exist sequences of Borel sets $(E_n)$ such that

$$\lim_{n \to \infty} \sup_{\nu \in \Omega} \int_{K \setminus E_n} |\nu(x)| \, d\lambda(x) = 0$$

and the two sequence

$$\varphi_n = \prod_{j=1}^{n} w * \delta^j_{g^{-1}} \quad \text{and} \quad \varphi_n = \left( \prod_{j=1}^{n} w * \delta^j_g \right)^{-1}$$

satisfy

$$\lim_{n \to \infty} \sup_{\nu \in \Omega} \int_{E_n} \varphi_n(x)|\nu(xg^n)| \, d\lambda(x) = 0,$$

$$\lim_{n \to \infty} \sup_{\nu \in \Omega} \int_{E_n} \varphi_n(x)|\nu(xg^n)| \, d\lambda(x) = 0.$$

We formulate the discrete version of Theorem 2.3 If $G$ is discrete group with the counting measure as its Haar measure. Then the set $E_k$ is nothing but set $K$ itself. Therefore, we have the following result.

**Corollary 2.3.** Let $g \in G$ be a non-torsion element of $G$ and let $\phi$ be a $\Delta_2$-regular Young function. Let $w, w^{-1} \in L^\infty(G)$. If $C_n := \frac{1}{2}(T_{g,w}^n + S_{g,w}^n)$ is cosine operator on $L^\phi(G)$ then the following statements are equivalent.

(i) $(C_n)_{n \in \mathbb{N}_0}$ is topological transitive.

(ii) For each non-empty finite subset $K \subset G$, there exist sequences of Borel sets $(E_k^+)$ and $(E_k^-)$ in $K$, and a sequence $(n_k)$ of positive numbers such that for $K = E_k^+ \cup E_k^-$, we have the two sequence

$$\varphi_n = \prod_{j=1}^{n} w * \delta^j_{g^{-1}} \quad \text{and} \quad \varphi_n = \left( \prod_{j=1}^{n} w * \delta^j_g \right)^{-1}$$

satisfy

$$\lim_{k \to \infty} \sup_{\nu \in \Omega} \sum_{x \in K} \varphi_{n_k}(x)|\nu(xg^{n_k})| = 0, \quad \lim_{k \to \infty} \sup_{\nu \in \Omega} \sum_{x \in K} \varphi_{n_k}(x)|\nu(xg^{n_k})| = 0,$$

$$\lim_{k \to \infty} \sup_{\nu \in \Omega} \sum_{x \in E_k^+} \varphi_{2n_k}(x)|\nu(xg^{2n_k})| = 0, \quad \lim_{k \to \infty} \sup_{\nu \in \Omega} \sum_{x \in E_k^-} \varphi_{2n_k}(x)|\nu(xg^{2n_k})| = 0.$$

Here we present a characterization of topological transitivity for a finite sequence of weighted cosine operators. We set the following notations for the sequence. For a fix $M \in \mathbb{N}$. Let $\{g_i\}_{1 \leq i \leq M}$ and $\{w_i\}_{1 \leq i \leq M}$ be the sequences of aperiodic elements of group $G$ and positive weight respectively. Then $\{T_{g_i,w_i}\}_{1 \leq i \leq M}$ is a sequence of weighted translation operators. We have the following characterization.

**Theorem 2.4.** Let $\{g_i\}_{1 \leq i \leq M}$ and $\{w_i\}_{1 \leq i \leq M}$ be the sequences of aperiodic elements and positive weights respectively such that $w_l, w_l^{-1} \in L^\infty(G)$. Let $C_{l,n} := \frac{1}{2}(T_{g_l,w_l}^n + S_{g_l,w_l}^n)$ be the cosine operators on $L^\phi(G)$ for $1 \leq l \leq M$, where $T_{g_l,w_l}$ is the weighted translation operator. Then the following statements are equivalent.

(i) $(C_{1,n} \oplus C_{2,n} \oplus \cdots \oplus C_{M,n})_{n \in \mathbb{N}_0}$ is topologically transitive.
(ii) For each non-empty compact subset $K \subset G$ with $\lambda(K) > 0$, there is some sequence $(n_k)$ of positive integers such that for $1 \leq l \leq M$, there exist sequences of Borel sets $(E_{l,k}^+)$, $(E_{l,k}^-)$ and $(E_{l,k}^0)$ such that for $E_{l,k} = E_{l,k}^+ \cup E_{l,k}^-$, we have

$$
\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{K \setminus E_{l,k}} |\nu(x)| \, d\lambda(x) = 0
$$

and the two sequences

$$
\varphi_{l,n_k} = \prod_{j=1}^{n_k} w_l * \delta_{g_{l,j}}^{-1} \quad \text{and} \quad \tilde{\varphi}_{l,n_k} = \left( \prod_{j=1}^{n_k} w_l * \delta_{g_{l,j}}^{-1} \right)^{-1}
$$

satisfy

$$
\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_{l,k}} \varphi_{l,n_k}(x) |\nu(xg^{n_k})| \, d\lambda(x) = 0,
$$

$$
\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_{l,k}} \tilde{\varphi}_{l,n_k}(x) |\nu(xg^{n_k})| \, d\lambda(x) = 0,
$$

$$
\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_{l,k}^+} \varphi_{l,2n_k}(x) |\nu(xg^{2n_k})| \, d\lambda(x) = 0,
$$

$$
\lim_{k \to \infty} \sup_{\nu \in \Omega} \int_{E_{l,k}^-} \tilde{\varphi}_{l,2n_k}(x) |\nu(xg^{2n_k})| \, d\lambda(x) = 0.
$$

Proof. (i) $\Rightarrow$ (ii). Let $K$ be a compact subset of $G$ such that $\lambda(K) > 0$. Since $(C_{1,n} \oplus C_{2,n} \oplus \cdots \oplus C_{N,n})_{n \in \mathbb{M}_0}$ is topological transitive, for $\epsilon \in (0, 1)$, there exist $f_l \in L^\phi(G)$ and $m \in N$ such that for $1 \leq l \leq M$, we have

$$
\|f_l - \chi_K\|_\phi < \epsilon^2 \quad \text{and} \quad \|C_{l,m}f_l + \chi_K\|_\phi < \epsilon^2.
$$

Further, to complete the proof follow the proof of part (i) $\Rightarrow$ (ii) of Theorem 2.1 to get desired conditions on weights $w_l$ for each $l$.

(ii) $\Rightarrow$ (i). Let $U_l$ and $V_l$ be non-empty open subsets of $L^\phi(G)$. Since $\phi$ is $\Delta_2$-regular we can choose two non-zero functions $f_l$ and $h_l$ in $C_c(G)$ such that $f_l \in U_l$ and $h_l \in V_l$. Set $K = \text{spt}(f_l) \cup \text{spt}(h_l)$. Let $E_{l,k} \subset K$ and it satisfies condition (ii). Now, imitate the proof of (ii) $\Rightarrow$ (i) of Theorem 2.1 to get that $C_{l,n_k}(U_l) \cap V_l \neq \emptyset$ for each $l, 1 \leq l \leq M$. \hfill $\Box$

Example 2.1. Let $G = \mathbb{Z}$. Fix an aperiodic element $g \in \mathbb{Z}$ with $g \geq 1$. Define the Young’s function $\phi(x) = (1 + |x|) \ln(1 + |x|) - |x|$, and consider the weight function

$$
w(i) = \begin{cases} 
\frac{1}{2}, & i \geq 0, \\
\frac{3}{2}, & i < 0.
\end{cases}
$$

A direct computation needs to find the complementary of Young’s function $\phi$, that is $\psi(x) = \exp(|x|) - |x| - 1$. Note that $\phi$ is $\Delta_2$-regular and vanishes only at zero. Choose an arbitrary $\nu \in \Omega$. Then, $\sum_{n=-\infty}^{+\infty} \psi(|\nu(n)|) \leq 1$ if and only if $\sum_{n=-\infty}^{+\infty} (\exp(|\nu(n)|) - |\nu(n)| - 1) \leq 1$. But the last is established only if $|\nu| \leq 2$. Each compact subset $K \subset \mathbb{Z}$ is a finite set, consisting of the integers $a_1 \leq a_2 \leq \cdots \leq a_m$.\hfill $\Box$
Take $E_i := \{a_1, \ldots, a_i\}$ and for each $i \geq m$, define $E_i := E_m$. Put $n_k = i$, $E_k^+ := E_i$ and $E_k^- := \emptyset$ in the statement (ii) of Theorem 2.1. In this circumstances, we have

$$
\lim_{n \to \infty} \sup_{n \in \Omega} \sum_{j \in K \setminus E_n} |\nu(j)| = 0.
$$

In addition, one may find $n_0 \in \mathbb{N}$ such that $a_1 + n_0 g \geq 0$. Hence, for each $n \geq n_0$ we have,

$$
\sup_{n \in \Omega} \sum_{i \in E_n} \phi_n(i) |\nu(i + ng)| \leq 2 \sum_{i \in E_n} \phi_n(i) \leq 2 \sum_{i \in E_n} w(i + g)w(i + 2g) \cdots w(i + ng) \leq 2 \sum_{i \in E_n} w(a_1 + g)w(a_1 + 2g) \cdots w(a_1 + ng) \leq 2 m(\frac{1}{n})n^{-n_0}w(a_1)^{|a_1|} \to 0,
$$

as $n \to \infty$. Here, $\text{card}(E_n)$ means the cardinality of the set $E_n$.

Similarly, there exists $t_0 \in \mathbb{N}$ such that $a_m - t_0 g \leq 0$ and so for each $n \geq t_0$,

$$
\sup_{n \in \Omega} \sum_{i \in E_n} \tilde{\phi}_n(i) |\nu(i + ng)| \leq 2 \sum_{i \in E_n} \tilde{\phi}_n(i) \leq 2 \sum_{i \in E_n} w^{-1}(i - g)w^{-1}(i - 2g) \cdots w^{-1}(i - ng) \leq 2 \sum_{i \in E_n} w^{-1}(a_m - g)w^{-1}(a_m - 2g) \cdots w^{-1}(a_m - t_0 g)w^{-1}(a_m - (t_0 + 1)g) \cdots w^{-1}(a_m - ng) \leq 2 m(\frac{1}{n})n^{-t_0}w^{-1}(a_m)^{|a_m|} \to 0,
$$

as $n \to \infty$. The other statements of Theorem 2.1 can be verified in this way. Therefore, by Theorem 2.1, the corresponding sequence of cosine operators to the weight $w$ and $g$, is topological transitive.

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**References**

[1] S. I. Ansari, Hypercyclic and cyclic vectors, J. Funct. Anal. 128 (1995), no. 2, 374-383.

[2] M. R. Azimi, I. Akbarbaglu, Hypercyclicity of weighted translations on Orlicz spaces, Oper. Matrices 12 (2018), no. 1, 27-37.

[3] F. Bayart, É. Matheron, Dynamics of linear operators, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2009.
[4] A. Bonilla, P. Miana, Hypercyclic and topologically mixing cosine functions on Banach spaces, Proc. Am. Math. Soc. 136 (2008) 519-528.

[5] C. Chen, Topological transitivity for cosine operator functions on groups, Topology Appl. 191 (2015), 48-57.

[6] C. Chen, C-H. Chu, Hypercyclic weighted translations on groups, Proc. Amer. Math. Soc. 139 (2011), no. 8, 2839-2846.

[7] C. Chen, C-H. Chu, Hypercyclicity of weighted convolution operators on homogeneous spaces, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2709-2718.

[8] K.-G. Grosse-Erdmann, A.P. Manguillot, Linear chaos, Universitext, Springer, London, 2011.

[9] T. Kalmes, Hypercyclicity and mixing for cosine operator functions generated by second order partial differential operators, J. Math. Anal. Appl. 365 (2010) 363-375.

[10] C. Kitai, Invariant closed sets for linear operators, Thesis (Ph.D.)University of Toronto (Canada), 1982.

[11] M. M. Rao, Z. D. Ren, Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

[12] H. N. Salas, Hypercyclic weighted shifts, Trans. Amer. Math. Soc., 347 (3) (1995) 993-1004.

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