Requiem for the ideal clock

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Abstract

The problem of time remains an unresolved issue in all known physical descriptions of the Universe. One aspect of this problem is the conspicuous absence of time in the Wheeler-Dewitt equation, which is the analogue of the Schrodinger equation for the Universal wavefunction. Page and Wootters famously addressed this problem by providing a mechanism for effectively introducing time evolution into this timeless cosmological picture. Their method, which is sometimes called the conditional probability interpretation (CPI), requires the identification of an internal clock system that is meant to keep time for the remainder of the Universe. Most investigations into this idea employ the idealized limit of a non-interacting clock system, the so-called ideal clock. However, by allowing for interactions, we have found the counter-intuitive result that a non-interacting clock is not necessarily the optimal choice, even if it is ideal. In particular, the uncertainty that is associated with the physical measurement of an atomic clock is found to decrease monotonically as the interactions grow stronger. This observation, which is reinforced by a previous study using a semi-classical clock, paves the way to an independent argument that is based on the energy conservation of any isolated system. Our conclusion is that ideal clocks must be prohibited from the CPI when recovering cosmological time evolution. Interactions are necessary for describing time evolution as a strict matter of prin-
ciple. Lastly, we also consider the implications of this result for the experience of time in the evolution of the Universe.
1 Introduction

1.1 Background and Motivations

The ‘problem of time’ and its various components have been described *ad nauseam* within the vast collection of literature on the subject. As the second law of thermodynamics is a rare physical principle providing a direction from the past to the future, it is usually the first facet of such discussions but is closely followed by a second. This would be the lack of not only a direction but a common description of time in the two fundamental theories — quantum theory and general relativity — which respectively view time as an external parameter and an abstract spacetime dimension. This is an important disagreement to settle before one even contemplates broaching the daunting subject of quantum gravity. But, even with these issues aside, a subtle point is often excluded from such discussions: There is a distinction between the usual parametrizations of time as it appears in mathematical expressions and the emergent phenomenon of time evolution as it is understood through our life-long experiences. Most of our trouble in describing time actually lies within the purview of the latter. Whether it is the absolute time of the quantum world or the abstract dimension of classical relativity, our restricted movement in time remains an unresolved puzzle.

This lack of an explanation for our passage through time is brought to the fore by a third facet of the problem: the timelessness of the Universe. Wheeler and De Witt introduced this notion in the form of a mathematical statement [1], since named after them,

\[ \hat{H} |\psi\rangle = 0. \]  

(1)
Here, $\hat{H}$ represents a quantum analogue of the Hamiltonian constraint of general relativity (although the equation should really be viewed as semi-classical) and $|\psi\rangle$ represents the total state of the Universe. This somewhat ad hoc but generally accepted equation enforces a total energy of zero for the Universe and, as a consequence, imposes an entirely static description on $|\psi\rangle$. Yet, even if the mathematics is sound, the imposed timelessness on the state of the Universe is at odds with our experiences from within. The question is then the same as before, only more so: Why do we experience a directed evolution in time?

This paradoxical situation was taken up by Page and Wootters, who managed to resolve it into a workable theory which is indeed capable of describing evolution [2] (see also [3]). The premise is to divide the entire state $|\psi\rangle$ into two strongly entangled subsystems: a ‘clock’ $C$ and the remainder of the Universe $R$. (The entanglement is necessarily quantum.) The evolution of $R$ is then to be described in terms of a measurement of one of the clock’s variables. To further clarify, at no point is time measured directly, as there is no appearance of time in the conventional sense. Rather, an eigenvalue of $C$, such as the location of its center of mass $\vec{x}_{CM}$, is used to provide an effective time variable. The evolution of $\vec{x}_{CM}$ would then be accessible to $R$ because of its mutual entanglement with $C$. A more detailed description is provided in Appendix A.

The Page–Wootters’ approach has met with some amount of resistance; most notably, Kuchar’s concerns that their clock could not describe a succession of time measurements and, therefore, no description of evolution would be possible [4]. These concerns have since been been countered by Dolby [5]
(and independently by Giovannetti et al. [6], also see [7, 8]), who furthered the Page–Wootters’ treatment while renaming it as the conditional probability interpretation (CPI). Dolby showed that the CPI is consistent by adopting an integration variable to ‘sync’ $C$ and $R$ and thus play the role of an abstract time parameter. This integration is basically the same as tracing out the clock system, a procedure which is favored by many others.

There is, however, another concern which presents a stumbling block for the CPI; the so-called clock ambiguity [10]. To elaborate, along with the requisite condition of strong entanglement, a ‘good’ clock in the CPI should satisfy two other requirements (see, e.g., [11]). The first is that the clock should be able to serve as an effective measuring device, meaning that it can access a sufficient amount of distinguishable states. The second is that the clock $C$ be weakly interacting with the remainder $R$, as interactions would naturally threaten the degree of their entanglement and also blur the delineation of the two systems. The latter condition for the clock is often extended to the limit of zero interactions, leading to the notion of an ‘ideal’ clock system. The essence of the clock ambiguity problem is then the existence of a large (and possibly infinite) number of choices for good clocks, so that any such description of $R$’s dynamics is somewhat arbitrary.

Not too long ago, Marletto and Vedral resolved this ambiguity by arguing that it is natural to limit considerations to ideal clocks and any choice of ideal clock is related to any other by a unitary transformation [11]. On the other hand, one might argue — as we recently did [12] — that interactions are an inevitable consequence of a realistic Universe and, as such, cannot be

\footnote{For a contrary opinion regarding Dolby’s resolution, see [9].}
dismissed out of hand or even taken to zero as a limiting case. Our previous investigation in [12] considered a semi-classical clock; the coherent-state description of a damped harmonic oscillator.\footnote{Our working assumption in both [12] and the current analysis is that the size of the clock system is small enough in comparison to its complement $R$ for the interactions to have a negligible effect on the latter. For a different approach, see [13].} Following a procedure that was motivated in part by [14], we found that the ideal-clock limit was not the optimal choice as far as it concerns minimizing uncertainty in the clock readings. As it happens, this uncertainty decreases monotonically as the damping grows stronger. And so, given the previously discussed importance of having relatively weak interactions, the optimal choice for a damping parameter is small but finite, and it depends inversely on the running time of the clock. It is implicit in this conclusion that the clock can only run efficiently for a finite duration before a ‘resetting’ is required.\footnote{See Subsection 1.2 for our actual meaning of ‘resetting’.} Otherwise, one could simply impose the double scaling limit of infinitesimally weak damping and an infinitely long running time. This time limit is important in what follows.

In order to advance our investigation into the use of interacting clocks, we sought out a system with a truly quantum description. Atomic clocks, as first suggested in practice by Rabi [15], fit quite naturally into this picture given that they are subjected to decohering interactions. As will be made clear later in the paper, decohering atomic clocks are not much different than damped, coherent oscillators with regard to large uncertainties in the clock readings being correlated with weak interactions. What is different, however, is the option of an infinitely long running time. This choice is ruled out by fiat in the current scenario, meaning that the double scaling limit is no longer
in play.

In spite of the small sample size for clock systems, we will further assert that the incorporation of interactions is a generic requirement for the CPI. This argument is based upon exposing the properties of completely isolated systems. In a similar manner to the above treatment of $|\psi\rangle$, the conserved total energy of an isolated system would restrict any description of its time evolution to the absolute time parameter which is prescribed by the Schrödinger equation. So that, in spite of previous claims to the contrary, ideal clocks can only provide a static, non-evolving description which is as timeless as the Universe in the Wheeler–DeWitt equation.

1.2 ‘Disclaimer’

Before proceeding, let us briefly comment on the perspective of the current paper and its authors. As will be argued in an upcoming discourse [16], it is ‘the problem with time’ that is the real problem and not time itself. Nevertheless, we would argue that, irrespective of any problem with time, the basic premise of the Wheeler–DeWitt equation — that the Universe is inherently timeless — must be correct even though the equation itself may well be flawed. This stands to reason given that the Universe is a closed system; meaning that, as it is employed here, the Wheeler–DeWitt equation (as well as the CPI by extension), should be regarded as a metaphorical or toy-model description of a more realistic and intricate picture. As such, we would then also argue that the subsequent discussion is relevant regardless of one’s personal stand on either the alleged problem or the equation in question. With this as our current mindset, any discussion regarding the
interpretation of time and the problems thereof will be kept to a minimum (however, see [16]).

It should be further noted that the notion of a ‘resetting’ time, which was introduced in [12] and motivated in analogy to ordinary timepieces, is not meant to imply that some outside agent is needed to formally reinitiate the timing procedure. The time of resetting rather means that for which the perturbative formalism breaks down and, then, either the clock can no longer effectively serve its purpose or a more sophisticated treatment is required. And, because of the above viewpoint, we are in no way suggesting that temporal evolution would, at this point, come to a crashing halt in the physical Universe. Nonetheless, a finite duration for the Universe, if it is isolated, is not unreasonable insofar as it would eventually have to stop evolving on account of the second law of thermodynamics or its accelerated expansion (or both).

It is important to keep in mind that the time \( t \) of the atomic clock is not, itself, the time which is “seen” by the remainder of the Universe. This \( t \) plays the same role as, for example, Dolby’s aforementioned abstract time [5] or, in other words, it is simply an integration variable. In this version of the Page–Wootters method, the actual time parameter would rather be the value of some observable property of the clock system. Provided that the clock system \( C \) and its complement, the remainder \( R \), are maximally entangled, the states of \( R \) would necessarily be correlated to the eigenstates of the relevant operator and, thus, with its eigenvalues as well. Meaning that the clock operator in question need not be the Wheeler–DeWitt Hamiltonian (with the \( R \) states traced out) as it is in more standard versions of the
Page–Wootters framework. See Appendix A for further clarification on this methodology. For the case of an atomic clock, in particular, R’s perceived time would be related to the inverse of the clock’s resonant frequency. Note, though, that our current interest is with the efficiency of the clock rather than the actual clock readings.

One final comment: For the discussion on isolated systems in Section 4, the arguments apply just as well to classical (sub)systems as they do to quantum ones.

1.3 Contents

The remainder of the paper is laid out as follows. The next section briefly describes the atomic-clock procedure and its applicability to the CPI. Section 3 reports on the effects of decoherence and identifies the optimal atomic clock from the CPI perspective. Section 4 presents a general argument for our claim that interactions are a necessary feature in any consistent description of time evolution. Section 5 provides a brief summary, and some additional details about the maths are included in four appendices.

2 The atomic clock

In 1945, Rabi presented the first practical approach for obtaining a time measurement using atomic frequencies [15]. (See, e.g., [17] for a textbook account.) The method provides a time measurement by counting the cycles of an electromagnetic oscillator and dividing by the oscillator frequency \( \omega \). A standardized unit of time can be defined by setting the frequency \( \omega \) to the
transition frequency of an electron in a particular atom. We will adopt the notation \( \omega_{21} = E_2 - E_1 \) for the transition frequency, where \( E_{1,2} \) are the ground and excited state respectively and \( \hbar \) has been set to unity here and throughout. In order to ensure that \( \omega \) is as close to the desired transition frequency as possible, the atoms in question are set in the ground state and then exposed to the oscillator. By modulating \( \omega \), one will change the probability of finding the exposed electrons in the excited state and can then plot this probability \( P_{ex} \) versus \( \omega \). The maximum value for \( P_{ex} \) corresponds to the resonant frequency, \( \theta \equiv \omega - \omega_{21} = 0 \), and the full width at half maximum (FWHM) of the plot measures \( \delta \omega \), the uncertainty in \( \omega \) (and, consequently, that of the time measurements). One finds that \( \delta \omega \propto \lambda \), where \( \lambda \) is the amplitude of the oscillating wave.

Improvements to the Rabi method were later made by Ramsey [18]. That author showed that exposing the atoms to the oscillator for two short times (or pulses) \( \tau \), separated by a longer non-interaction time \( T \), would reduce the uncertainty of the measurements [18]. When the value of the frequency is sufficiently close to resonance, \( \theta \ll \lambda \), its uncertainty rather goes as \( \delta \omega = \frac{\pi}{T} \), where \( T \) has become known as the Ramsey time. This suggests that taking the limit \( T \to \infty \) would minimize the uncertainty. However, \( T \) must indeed be finite as the Ramsey process still requires a period of exposure \( \tau \) to take place immediately after \( T \).

The atomic clock could, of course, be made arbitrarily accurate by setting the system to the resonance case. Assuming, for the sake of argument, that such accuracy could be achieved at least as a matter of principle, one ends up with a description of time that cannot be distinguished from the absolute
time parameter already appearing in the Schrödinger equation. Ultimately, we will claim that the limiting case of a non-interacting clock leads to a time description which is similarly indistinguishable from absolute time, as the CPI then fails to account for our passage through time. To prove this, the atomic clock will be allowed to interact with the rest of the Universe through the inclusion of decoherence effects.

3 The decohering atomic clock

In order to analyze the effects of decoherence on the atomic clock, we will be incorporating the dynamics of the Lindblad equation [20]. The basic idea is to allow the clock system to decohere during the Ramsey interval $T$, as the time $\tau$ of the oscillator pulse is taken to be small enough to ignore the effects of decoherence during these brief periods of exposure. A significant portion of our method follows an approach that was sketched out by Weinberg [19].

As outlined in Appendix B, the first step is to derive the evolution operator for the Ramsey setup when decoherence is included. The next step is to use this operator to calculate the probability of finding the system in the excited state $P_{ex}(t)$ after both pulses and the Ramsey time have transpired. As explained in Appendix C, this process leads to

$$P_{ex}(2\tau + T) = \frac{4\lambda^2}{\Omega^2} \sin^2(\Omega\tau) \left[ 2 \cos(\Omega\tau) + \frac{\theta^2}{2\Omega^2} \sin^2(\Omega\tau) \right]$$

$$+ e^{-\alpha T} \left( 2 \cos^2(\Omega\tau) \cos((\theta - \beta)T) 
- \frac{\theta}{2\Omega^2} \sin^2(\Omega\tau) \cos((\theta - \beta)T) 
- \frac{2\theta}{4\Omega} \sin(2\Omega\tau) \sin((\theta - \beta)T) \right),$$

(2)
where $\alpha$ and $\beta$ are the real and imaginary parts of the ‘decoherence factor’ 
$\gamma$ (i.e., $\gamma$ is one of the eigenvalues of the non-unitary portion of the Linblad equation) and 
$\Omega = \sqrt{\lambda^2 + \frac{\theta^2}{4}}$ is known as the Rabi frequency.

Following Ramsey, we will fix the pulse time $\tau$ by maximizing the probability for the idealized case of 
$\theta = T = 0$ [18]. Making this choice and setting $P_{ex}(2\tau + T) = 1$, one finds that $\tau = \frac{\pi}{4\lambda}$. Then, with the substitution of $\tau$ and the assumption that $\theta \ll \lambda$ (i.e., the system is close to resonance), 
eq (2) reduces down to

$$P_{ex}(2\tau + T) = \frac{1}{2} \left[ 1 - e^{-\alpha T} \cos((\theta - \beta)T) \right].$$

This expression closely resembles one from [19], where it was applied in a different context.

The uncertainty in $\omega$ for the current case — again calculated as the FWHM from the plot of $P_{ex} \times \omega$ — is found to be $\delta = \frac{\pi}{T}$, exactly the same as before. This would suggest that the inclusion of decoherence has no bearing on the precision of the measurements. However, this is not true because the maximum outcome for the probability has definitely diminished. Put into more physical terms, the decoherence of the system reduces its ability to function as a quantum clock. More rigorously, a constrained minimization of the uncertainty in the probability $P_{ex}(2\tau + T)$ leads to the following relation,

$$\alpha T \sim \mathcal{O}(1),$$

as elaborated on in Appendix D.

The above outcome indicates that the desire for a long Ramsey time $T$
(which must anyways be finite) for the purposes of minimizing the uncertainty must be balanced against the (similarly finite) effects of decoherence. This
is really just another way of justifying the previously stipulated condition of a weakly interacting clock, which translates into $\alpha T \lesssim \mathcal{O}(1)$.

And so, in attempting to impose the ideal-clock limit of $\alpha \to 0$, one is stymied by both the condition of a finite $T$ and the proclivity for more accurate measurements. Our conclusion is that the ideal limit of an atomic clock is neither a tenable nor an optimal choice.

4 What can be said about ideal clocks

The story that the above result seems to be telling is one where interactions are a necessary feature in the framework of the CPI. The current objective is to both generalize and strengthen our conclusions about atomic clocks (and, previously, coherent states [12]). This will be accomplished with an independent, qualitative argument.

Let us start by reconsidering the Wheeler–DeWitt equation. Its timelessness can be attributed to the Universe having a total net energy of zero, as per the right-hand side of eq. (1). The precise value of the energy, however, is really besides the point. Any (strictly) constant value for the energy would imply that the dynamics of the Universe are frozen, rendering time a meaningless concept. Let us now consider the more familiar case of an isolated (sub-)system as it would be described in a textbook on quantum mechanics. The dynamics of this system are similarly frozen, yet we attribute it with a time parameter all the same; namely that of the time-independent Schrödinger equation. Where did this time come from? There are only two possibilities: the system’s notion of time was put in by hand or it was inher-
ited from a larger, ancestral system. But can the idealized clock system of
the CPI answer this same question about the the origin of its notion of time?
The CPI ‘rulebook’ does not permit us to put in time by hand and neither
can the clock inherit its time from an ancestor, as the only one available
is the timeless Universe à la Wheeler and DeWitt. And so, with no notion
of time available and no opportunity to interact with its environment, the
ideal clock cannot possibly evolve relative to another system or component
thereof. In short, the idealized clock could never serve as a timepiece for
another system any more or less than the Universe as a whole could.

A sequence of states could still be described for these timeless, isolated
systems as illustrated in Marletto and Vedral’s treatment [11]. Each suc-
cessive state of \( R \) is identified with a time measurement of \( C \) and a history
is produced. However, as pointed out by those same authors, this picture
provides neither a flow of time nor an arrow of time — both of these concepts
should be viewed as fictitious within this timeless framework. There is sim-
ply no motivation for moving from one state to the next, and no provision
for a sense of movement through time without also assuming an absolute,
external time along with an imposed direction. The ideal-clock scenario then
precludes the possibility of a clock which itself can experience time or can
provide a measurement of time for an external agent.

Taking our lead from the above argument and including interactions as
a matter of principle, we arrive at a very different result. The requirement
of an open system for \( C \) immediately allows for a clock with a sense of
evolving in time and, likewise, for its complement \( R \). The resulting time
evolution includes a description of not just the history of states for \( C \) and \( R \)
but also an arrow in time thanks to the non-reversible effects of decoherence and/or damping. We thus have a way of reconciling our experience of passing through time with the timeless state of the Universe.

5 Conclusions

Our investigation into atomic-clock systems showed that the optimal choice of clock requires a compromise between fending off the effects of decohering interactions and maximizing the accuracy of the clock. This reinforced a previous result on coherent-state clocks and led to a new view on the description of time within the framework of the CPI. The restriction to the ideal-clock (non-interacting) limit prohibits any description of motion through time; there can only be a static series of states with no motivation for any movement between them. The inclusion of interactions, however, resolves this issue as the interacting clock system can evolve through its relation with the complementary system. Elevating this framework to ‘reality’ (or, rather, some simplistic description thereof), one would translate this evolution into a passage through time, rather than the inclusion of an abstract, absolute time dimension which sits ‘outside’ of our experience. Our conclusion is that the use of an ideal clock in the CPI not only fails at being the optimal choice in practice but also represents a misleading assumption in principle; interactions must be included as a strict rule.

This motivation to use only interacting clocks within the CPI does, however, reintroduce the clock ambiguity as the ideal-clock limit can no longer be called upon to resolve the issue. As the inclusion of interactions appears
to come along with a free arrow of time, there could well be a solution to the
clock ambiguity which utilizes a preference for clocks obeying the second law,
rather than appealing to the redundancy of ideal clocks. This possibility is
currently under investigation \[16\].

**Acknowledgments**

The research of AJMM received support from an NRF Incentive Funding
Grant 85353 and NRF Competitive Programme Grant 93595. KLHB is sup-
ported by an NRF bursary through Competitive Programme Grant 93595
and a Henderson Scholarship from Rhodes University. This work is based on
the research also supported in part by the National Research Foundation of
South Africa (Grant Numbers: 111616).

A Evolution according to Page and Wootters

Section \[1.1\] outlined the Page–Wootters method of recovering time. The
method is based on the timeless description of the Universe as a pure state
\(\left| \psi \right\rangle\), which is governed by the Hamiltonian \(\hat{H}\) in the Wheeler–DeWitt equa-
tion. Here, we describe the method in more detail along with an explanation
of the role of the abstract variable \(t\) which is discussed in Section \[1.2\].

The standard description involves the division of \(\left| \psi \right\rangle\) into the clock \(C\)
and the rest \(R\). This partitioning is accompanied by the identification of
the Hamiltonians \(\hat{H}_C = \text{Tr}_R \hat{H}\) and \(\hat{H}_R = \text{Tr}_C \hat{H}\), which govern \(C\) and \(R\)
respectively.
Interaction effects (governed by $\hat{H}_I$) between $C$ and $R$ complete the Hamiltonian, which can be written as $\hat{H} = \hat{H}_C \otimes 1 + 1 \otimes \hat{H}_R + \hat{H}_I$. Under the Page–Wootters method, these interaction effects are considered vanishingly weak and so $\hat{H}_I$ can be ignored. This leads to the approximate relation $\hat{H} \approx \hat{H}_C \otimes 1 + 1 \otimes \hat{H}_R$ and, because $\hat{H} |\psi\rangle = 0$ for physical states, it follows that

$$\hat{H}_C \approx -\hat{H}_R$$

is true when acting on physical states.

The last requirement for the Page–Wootters method is that $C$ and $R$ be in a maximally entangled state,

$$|\psi\rangle = \sum_j \alpha_j |\psi_C \rangle_j |\psi_R \rangle_j ,$$

where $|\psi_{C,R}\rangle$ are states for $C$ and $R$ respectively, a subscript of $j$ indicates a basis state and $\alpha_j$ represents numerical coefficients. In this way, the evolution of $C$ can be ‘transferred’ to $R$,

$$|\psi_j\rangle = c_j \left(e^{-i\hat{H}_C p} |\psi_C \rangle_j \right) |\psi_R \rangle_j = c_j e^{-i(\hat{H}_C - \hat{H}_R)p} \langle \psi_C |_j |\psi_R \rangle_j$$

$$\approx c_j |\psi_C \rangle_j \left(e^{i\hat{H}_R p} |\psi_R \rangle_j \right) ,$$

where $p$ refers to the eigenvalues of the conjugate to $\hat{H}_C$; in other words, $p$ is the emergent time parameter for $R$. Note that we have set $\hbar = 1$, here and throughout.

Up to this point, the standard description of the Page–Wootters method is sufficient. But, in order to analyze the case of the atomic clock, we use a variant that was inspired by Dolby [14]. What is now needed is some observable property of $C$ (but not necessarily $p$) to act as the time parameter.
for $R$. Let us denote this property by $x$. (In [12], $x$ was literally a position variable. For the atomic clock, $x$ would be related to the inverse of the resonant frequency.) Let us further denote the conjugate to the operator that measures $x$ as $\hat{\Phi}_x^C$. Then the condition of maximal entanglement is enough to ensure that there is a “mirror” operator acting on states of $R$, $\hat{\Phi}_x^R$, for which

$$\hat{\Phi}_x^C \approx -\hat{\Phi}_x^R$$

(8)
is true when acting on physical states and for some suitable choice of bases. Meaning that $\hat{\Phi}_{x,R}^C$ can (and do) play the role of effective Hamiltonians.

Let us now consider how the abstract time parameter $t$ fits in and understand why it does not require a physical interpretation. If $t$ is the ‘conventional’ time parameter for the clock operator, there should be some semiclassical relation $x = x(t)$. Then the probability for the clock to be in a state for which $x = x'$ can be expressed as an integral over the probability that $x = x'$ when conditioned on $t = t'$. In other words,

$$P_C(x') = \int_{-\infty}^{\infty} dt' \left| \langle x | \psi_C(t') \rangle \right|^2 .$$

(9)

And so $t$ is merely an integration variable as advertised. Moreover, because of the condition of maximal entanglement, the relationship

$$P_R(x') \approx P_C(x')$$

(10)
immediately follows.

**B Determining the evolution operator**

Here, Ramsey’s evolution matrix [15] is generalized to allow for decoherence.
Let us consider a two-level system at time $t = 0$ in its energy basis, 
\[ |\psi(0)\rangle = c_1 |1\rangle + c_2 |2\rangle. \]
If the external potential oscillates according to 
\[ V(t) = \lambda e^{i\omega t} + \lambda e^{-i\omega t} \]
with $\lambda$ real, then the state at a later time $t$ is 
\[ |\psi(t)\rangle = \left( \cos(\Omega t) - \frac{i\theta}{2\Omega} \sin(\Omega t) \right) \psi_{i\theta t/2} |1\rangle + \frac{e^{-i\theta t/2}}{\Omega} \sin(\Omega t) |2\rangle \]
and its density matrix is
\[ \rho(t) = |\psi(t)\rangle \langle \psi(t)| = \begin{bmatrix}
1 - \frac{\gamma^2}{\Omega^2} a^2 & \frac{i\gamma e^{i\theta t}}{\Omega} \left(b - \frac{i\theta}{2\Omega} a\right) \\
-i\gamma e^{-i\theta t} a \left(b + \frac{i\theta}{2\Omega} a\right) & \frac{\gamma^2}{\Omega^2} a^2
\end{bmatrix}, \tag{11}
\]
where $a = \sin(\Omega t)$, $b = \cos(\Omega t)$ and $\Omega = \sqrt{\lambda^2 + \frac{\theta^2}{4}}$. Alternatively, the evolution can be described by 
\[ \rho(t) = U(t, 0) \rho(0) U(t, 0)^\dagger. \]

Assuming that the system starts in its ground state, 
\[ \rho(0) = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \]
one find an evolution matrix of the form
\[ \begin{bmatrix}
\rho(t)_{11} & \rho(t)_{12} \\
\rho(t)_{21} & \rho(t)_{22}
\end{bmatrix} = \begin{bmatrix}
A & B \\
-B^* & A^*
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
A^* & -B \\
B^* & A
\end{bmatrix}. \tag{12}
\]
where
\[ |A|^2 = b^2 + \frac{\theta^2}{2\Omega^2} a^2, \]
\[ |B|^2 = \frac{\lambda^2}{\Omega^2} a^2, \]
\[ -AB = \frac{i\lambda e^{i\theta t}}{\Omega} \left(b - \frac{i\theta}{2\Omega} a\right), \]
\[ -A^* B^* = -\frac{i\lambda e^{-i\theta t}}{\Omega} \left(b + \frac{i\theta}{2\Omega} a\right). \tag{13}\]

Since $A = b - \frac{i\theta}{2\Omega} a$ is true up to a phase, the evolution matrix can be resolved into
\[ U(t, 0) = \begin{bmatrix}
b - \frac{i\theta}{2\Omega} a & -\frac{i\lambda e^{i\theta t}}{\omega} a \\
-i\frac{\lambda e^{-i\theta t}}{\omega} a & b + \frac{i\theta}{2\Omega} a
\end{bmatrix}. \tag{14}
\]
The non-decohering limit of this outcome agrees with the final form of Ramsey’s result \[18\].
Calculating the probability

The following is a calculation of the probability of finding the system in its excited state after the Ramsey procedure has been completed.

The first pulse (or exposure zone) changes the system from the ground state according to
\[
\rho(\tau) = U(\tau,0) \rho(0) U^\dagger(\tau,0) .
\]
To include decoherence during the Ramsey time period \(T\), the following Lindblad equation \([20]\) is applied \([19]\):
\[
\begin{aligned}
\dot{\rho} &= -i[H, \rho(t)] + \sum_\alpha \left[ \hat{L}_\alpha \rho(t) \hat{L}_\alpha^\dagger - \frac{1}{2} \hat{L}_\alpha^\dagger \hat{L}_\alpha \rho(t) - \frac{1}{2} \rho(t) \hat{L}_\alpha^\dagger \hat{L}_\alpha \right]. \\
\end{aligned}
\] (15)

The effect of this evolution on the system is
\[
\rho(t)_{mn} \propto e^{-i(E_m - E_n)t - \gamma_{mn}t} = e^{-i(E_m - E_n)t - \gamma_{nm}t},
\] (16)
where \(m, n\) label the eigenstates of the operators on the right-hand side of eq. (15) and \(\gamma_{mn} = \alpha + i\beta = (\alpha - i\beta)^* = \gamma^*_n\) represents the decoherent part of their eigenvalues (the state labels on \(\alpha\) and \(\beta\) are implied).

It can be seen that the diagonal terms of \(\rho(\tau)\) are insensitive to the decoherence. On the other hand, the effect on the off-diagonal terms is evident from
\[
\rho(\tau + T) = \left[ b^2 + \frac{\theta}{4\Omega^2} a^2 + \frac{i\lambda e^{i\theta} e^{(-i\omega_{21} - \gamma)T}}{\Omega} a (b - i\frac{\theta}{2\Omega} a) \right],
\] (17)
where \(a\) and \(b\) have been defined after eq. \([11]\).

It should be noted that the external potential \(V(t)\) continues to oscillate for the duration of \(T\). This introduces an additional phase factor \(e^{\pm i\omega T}\) in the off-diagonal terms of the evolution operator once the second pulse \(\tau\) is
applied. The equation governing this last exposure zone is given by

\[
\begin{bmatrix}
\rho(2\tau + T)_{11} & \rho(2\tau + T)_{12} \\
\rho(2\tau + T)_{21} & \rho(2\tau + T)_{22}
\end{bmatrix} = \begin{bmatrix} A & B \\
-B^* & A^* \end{bmatrix} \begin{bmatrix}
\rho(\tau + T)_{11} & \rho(\tau + T)_{12} \\
\rho(\tau + T)_{21} & \rho(\tau + T)_{22}\end{bmatrix} \begin{bmatrix} A^* & -B \\
B^* & A \end{bmatrix},
\]

(18)

where \(A\) and \(B\) have been defined in eq. (13).

The element \(\rho(2\tau + T)_{22}\) represents the probability of finding the system in its excited state \(P_{ex}(2\tau + T)\) and is determined to be

\[
P_{ex}(2\tau + T) = \rho(2\tau + T)_{22} = \frac{4\lambda^2}{\Omega^2} \sin^2(\Omega\tau) \left[ 2\cos(\Omega\tau) + \frac{\theta^2}{2\Omega^2} \sin^2(\Omega\tau) + e^{-\alpha T} \left( 2\cos^2(\Omega\tau) \cos((\theta - \beta)T) - \frac{\theta}{2\Omega^2} \sin^2(\Omega\tau) \cos((\theta - \beta)T) - \frac{2\theta}{4\Omega} \sin(2\Omega\tau) \sin((\theta - \beta)T) \right) \right],
\]

(19)

which correctly reduces to Ramsey’s result when \(\alpha = \beta = 0\).

D Taking limit of uncertainty

The goal here is to use a more rigorous method to substantiate the claim in Section 3 that \(\alpha T \sim \mathcal{O}(1)\).

To quantify the effect of decoherence on a clock measurement, we will calculate and then minimize the relative uncertainty of of \(P_{ex}(2\tau + T)\); that is, \(\frac{\delta P_{ex}}{P_{ex}}\) (with the time dependence now left implicit). To start, let us
consider the relation $\delta P_{ex} = \sqrt{\left(\frac{\partial P_{ex}}{\partial \omega}\right)^2 \delta \omega^2}$, which then gives

$$\frac{\delta P_{ex}}{P_{ex}} = \frac{\pi}{2} e^{-\alpha T} \sin(\Theta T) \left(1 + e^{-\alpha T} \cos(\Theta T)\right)^{-1},$$

(20)

where the definition $\Theta = \theta - \beta$ has been applied.

Eq. (20) makes it clear that the limit $T \to \infty$ minimizes the uncertainty. However, the restriction on a finite value for $T$ must still be in place in order to complete the procedure as prescribed by Ramsey. Given that the constraint (see Section 2) $T \sim \frac{1}{\delta \omega}$ is also in effect — which also ensures a finite $T$ barring the classical limit — the minimization procedure then amounts to solving

$$\frac{\partial}{\partial T} \left[\frac{\delta P_{ex}}{P_{ex}} - \Lambda(\delta \omega - \frac{\pi}{T})\right] = 0,$$

(21)

where $\Lambda$ is a Lagrange multiplier. With the help of eq. (20), the above expression resolves into

$$e^{-2\alpha T} \left(\Theta T^2 - 2\Lambda \cos^2(\Theta T)\right) + e^{-\alpha T} \left((\Theta T^2 - 4\Lambda) \cos(\Theta T) - \alpha T^2 \sin(\Theta T)\right) - 2\Lambda = 0.$$

(22)

Looking at $\Theta = \theta - \beta$ and knowing that $\theta$ will vanish as the resonant case is approached, we can apply the approximation $|\Theta| \approx -\beta$. We can further approximate $\beta \sim \pm \alpha$, as $\alpha$ is essentially the only dimensional scale in the problem. With these simplifications, it can now be readily checked that either of the limits $\alpha T \gg 1$ or $\alpha T \ll 1$ implies that $\Lambda = 0$. However, one cannot argue that $\Lambda$ vanishes as this choice effectively removes the finiteness condition on $T$. That leaves $\alpha T \sim \mathcal{O}(1)$ as the only viable solution.

Even though this is not an explicit calculation, the approximations still allow us to make a statement about the relationship between $\alpha$ and $T$. Specif-
ically, the fact that the two are related through a finite-valued product indicates that only one of the pair can be set independently. This is exactly why the finiteness of $T$ imposes the same condition on $\alpha$.

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