Conditional and unconditional information inequalities: an algebraic example

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Abstract

We provide a simple example showing that some conditional information inequalities (even in a weak form) cannot be derived from unconditional inequalities.

1 Introduction

Three conditional information inequalities were considered in [2, 4, 8]:

\begin{align*}
(1) & \quad I(A:B|C) = I(A:B) = 0 \implies I(C:D) \leq I(C:D|A) + I(C:D|B) \\
(2) & \quad I(A:B|C) = I(B:D|C) = 0 \implies I(C:D) \leq I(C:D|A) + I(C:D|B) + I(A:B) \\
(3) & \quad I(A:B|C) = H(C|AB) = 0 \implies I(C:D) \leq I(C:D|A) + I(C:D|B) + I(A:B)
\end{align*}

Here $A, B, C, D$ are random variables (with a finite range), $H(U|V)$ stands for conditional Shannon entropy of $U$ given $V$, and $I(U:V|W)$ is the amount of mutual information in $U$ and $V$ when $W$ is known.

Question. Can these conditional inequalities be derived from their unconditional versions?

For example, inequality (1) could be a consequence of an unconditional inequality (with “Lagrange multipliers”)

$$I(C:D) \leq I(C:D|A) + I(C:D|B) + \kappa[I(A:B|C) + I(A:B)]$$

if the latter were true for some $\kappa > 0$. This is not the case, as shown in [8] (for all three inequalities (1-3)). In this note we consider a statement that can be considered as a conditional inequality that is weaker than both (1) and (3).\footnote{Authors are supported by ANR NAFTI grant 008-01}

\begin{align*}
I(A:B|C) = I(A:B|D) = H(C|AB) = I(C:D|A) = I(C:D|B) = I(A:B) = 0 \implies I(C:D) = 0
\end{align*}

We give a simple algebraic example that shows that even this weak statement is essentially conditional, therefore providing an alternative simple proof of the conditional nature of inequalities (1) and (3).

Theorem. There is no constant $\kappa$ such that

$$I(C:D) \leq \kappa[I(A:B|C) + I(A:B|D) + H(C|AB) + I(C:D|A) + I(C:D|B) + I(A:B)] \quad (*)$$

for all random variables $A, B, C, D$.\footnote{The condition $I(A:B|D) = 0$ is not needed to derive (1) and (3). We added it to make the statement weaker (and the result below stronger).}
2 Example

Consider a quadruple \( \langle A, B, C, D \rangle \) of geometric objects on the affine plane over the finite field \( F_q \):

- First choose a random non-vertical line \( C \) defined by the equation \( y = c_0 + c_1 x \) (the coefficients \( c_0 \) and \( c_1 \) are independent random elements of the field);
- then pick uniformly at random a parabola \( D \) in the set of all non-degenerate parabolas \( y = d_0 + d_1 x + d_2 x^2 \) (where \( d_0, d_1, d_2 \in F_q, d_2 \neq 0 \)) that intersect \( C \) at exactly two points;
- and call two intersection points \( A \) and \( B \) in a random order.

One can easily compute both sides of the inequality (\( \ast \)) for these \( A, B, C, D \), and get the following inequality:

\[
1 - \log_2 \left( \frac{q}{q - 1} \right) \leq 4\kappa \log_2 \left( \frac{q}{q - 1} \right)
\]

This leads to a contradiction: the left-hand side is \( 1 - O(\frac{1}{q}) \) and the right-hand side is \( O(\frac{1}{q}) \) for large \( q \).

Informal explanations: The mutual information between the line and the parabola is approximately 1 bit because \( C \) and \( D \) intersect at two distinct points iff the corresponding equation discriminant is a non-zero quadratic residue, which happens almost half of the time. Other information quantities vanish since the involved random geometrical objects are almost independent.

It is in fact easy to check that they are all equal to \( \log_2 \left( \frac{q}{q - 1} \right) \). For instance, given \( A \) there are \( q \) equiprobable lines \( C \). Given \( A \) and \( D \), there are \( q - 1 \) equiprobable lines since now the tangent to \( D \) at \( A \) is excluded \( (A \neq B) \). Hence \( I(C;D|A) = \log_2(q) - \log_2(q - 1) \). All other computations are similar.

3 Historical comments and motivation

To every quadruple of discrete random variables corresponds an entropy vector consisting of the fifteen entropies of all possible non-empty subsets of variables (in some fixed order). A vector is called entropic if it is the entropy vector of some tuple of random variables. The set of all entropic vectors has a very complex structure (e.g., it is not even closed). However, the closure of the set of all entropic vectors is easier to study: it is a convex cone. This closure is called the cone of asymptotically entropic vectors.

Some bounds for the cone of asymptotically entropic vectors are simple and well-known. For instance, these vectors satisfy Shannon inequalities, i.e., all non-negative linear combinations of the basic inequality

\[
H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z) \geq 0.
\]

This inequality means that the conditional mutual information \( I(X;Y|Z) \) is non-negative.

N. Pippenger asked in 1986 a general question [1]: does there exist any linear inequality for entropies other than Shannon inequalities? The first inequality of this kind was conditional [2] and appeared in 1997, followed by a similar unconditional one [3] a year later. Quite many (actually, infinitely many) other inequalities have been found since then. In fact, the set of all valid information inequalities cannot be reduced to any finite number of “basic” inequalities: F. Matuš proved that the cone of asymptotically entropic vectors for quadruple of random variables is not polyhedral [5].

Conditional equalities (1-3) express other subtle properties of entropic vectors. These inequalities imply conditional independence inference rules (which can not be deduced from Shannon’s inequalities). These rules were extensively studied by several authors, see a survey by M. Studený in [6].

In [7] it was asked if any conditional information inequality (in particular (1)) can be obtained as a direct consequence of some unconditional inequality. The negative answer was given in [8]. In the present paper we give a simplified proof of this result, in a slightly stronger form for (1) and (3).

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