Enclosure method and reconstruction of a linear crack in an elastic body

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Abstract. In this paper we report a recent application of the idea of the enclosure method to an inverse problem related to a crack (inverse crack problem) in an elastic body. The problem is to extract information about the location and shape of an unknown crack from a single set of the surface displacement field and traction on the boundary of an arbitrary homogeneous elastic plate. Both anisotropic and isotropic bodies are considered. In states of both plane stress and plane strain, an extraction formula of an unknown crack is given provided: the crack is linear; one of two end points of the crack is known and located on the boundary of the body; a well-controlled surface traction is given on the boundary of the body.

1. Introduction
The enclosure method [2, 3, 4, 5] is a methodology in inverse problems for partial differential equations. The method yields a partial information about the location of unknown discontinuity which appears as discontinuity of the coefficients of a partial differential equation or a part of the boundary of the common domain of definition of solutions of the equation.

The enclosure method of a single measurement version [2] can be divided into three parts.

(i) Find a special solution of the formal adjoint of the governing equation for the background medium which is parameterized by a large parameter \( \tau \) and divides the whole space into two parts: in one part the absolute value of the solution decays as \( \tau \rightarrow \infty \); in another part the solution grows as \( \tau \rightarrow \infty \).

(ii) Construct an indicator function of independent variable \( \tau \) by multiplying the governing equation of the medium by the special solution constructed in (i), integrating over the domain of definition and extracting only the integral on the known boundary of the domain.

(iii) Study the asymptotic behaviour of the indicator function as \( \tau \rightarrow \infty \).

The aim of this expository paper is to report a recent application [6, 7] of this method to an inverse problem for the crack in an elastic body in which the governing equation becomes an elliptic system. The problem is to extract information about the location and shape of an unknown crack from a single set of the surface displacement field and traction on the boundary of the elastic plate which is in the both of anisotropic and isotropic. This is an industrial and theoretical important problem.
We show how the method yields the extraction formula of an unknown crack provided: the crack is linear; one of two end points of the crack is known and located on the boundary of the body; a well-controlled surface traction is given on the boundary of the body. The formula gives a finite number of approximate values of the support function of an unknown crack which gives the signed distances from the origin of the coordinates to the support line of the crack.

2. An inverse crack problem

By \(u = (u_i)_{i=1,2,3}\), \(\varepsilon = (\varepsilon_{ij})_{i,j=1,2,3}\) and \(\sigma = (\sigma_{ij})_{i,j=1,2,3}\) we denote the displacement vector, the strain tensor and the stress tensor, respectively. \(C = (C_{ij})_{i,j=1,2,6}\) and \(S = (S_{ij})_{i,j=1,2,6}\) are an elasticity tensor and the compliance tensor, respectively. And the following contracted notation is used: \(\varepsilon = (\varepsilon_i)_{i=1,2,6}\) and \(\sigma = (\sigma_i)_{i=1,2,6}\), where

\[
\begin{align*}
\varepsilon_1 &= \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{13}, \quad \varepsilon_6 = 2\varepsilon_{12}, \\
\sigma_1 &= \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12}.
\end{align*}
\]

The linear elasticity equations for a homogeneous material consist of the constitutive law (Hooke’s law) \(\sigma_1 = C_{ij}\varepsilon_j\), \(C_{ij} = C_{ji}\), \(\varepsilon_i = S_{ij}\sigma_j\), \(S_{ij} = S_{ji}\) \((i,j = 1,2,\cdots,6)\), the equilibrium conditions without any body forces

\[
\frac{\partial}{\partial x_j} \sigma_{ij} = 0, \quad i,j = 1,2,3
\]

and the strain-displacement relation is written as

\[
\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}), \quad u_{i,j} = \frac{\partial}{\partial x_i} u_i, \quad i,j = 1,2,3.
\]

Here and in what follows we use the summation convention.

Let \(\Omega \) be a bounded convex domain in \(\mathbb{R}^2\), representing a homogeneous anisotropic or isotropic elastic plate. For plane stress, that is, \(\sigma_3 = \sigma_4 = \sigma_5 = 0\), the \(S_{ij}\) components of interest are \(S_{ij}\), \(i,j = 1,2,6\) and \(C_{ij}\) are exchanged with \(\tilde{C}_{ij}\). For plane strain, that is, \(u_1 = u_1(x_1,x_2), u_2 = u_2(x_1,x_2), u_3 = 0\), the \(C_{ij}\) components of interest are \(C_{ij}\), \(i,j = 1,2,6\) and \(S_{ij}\) are exchanged with \(\tilde{S}_{ij}\) as follows:

\[
\tilde{C}_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}, \quad \tilde{S}_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \quad i,j = 1,2,6.
\]

Note here that \(C_{11} = C_{22}, C_{16} = C_{26} = 0, 2C_{66} = C_{11} - C_{12}\) holds in the case of an isotropic material.

The governing system of equations for plane elasticity is

\[
A(\partial_{\bar{x}}) u = 0, \quad (2.1)
\]

where \(u = (u_1,u_2)^T\), the superscript \(T\) denotes matrix transposition and the operator \(A(\partial_{\bar{x}}) = A(\partial/\partial x_1, \partial/\partial x_2)\) has the symbol

\[
A(\xi_1,\xi_2) = \begin{pmatrix}
C_{11}\xi_1^2 + C_{66}\xi_2^2 + (C_{16} + C_{16})\xi_1\xi_2 & C_{16}\xi_1^2 + C_{26}\xi_2^2 + (C_{12} + C_{66})\xi_1\xi_2 \\
C_{16}\xi_1^2 + C_{26}\xi_2^2 + (C_{12} + C_{66})\xi_1\xi_2 & C_{66}\xi_1^2 + C_{22}\xi_2^2 + (C_{26} + C_{26})\xi_1\xi_2
\end{pmatrix}.
\]
Let $\nu = (\nu_1, \nu_2)^T$ be the unit outward normal to $\partial \Omega$. The operator $T(\partial_x) = T(\partial/\partial x_1, \partial/\partial x_2)$ has the symbol

$$T(\xi_1, \xi_2) = \begin{pmatrix}
(C_{11}\xi_1 + C_{16}\xi_2)\nu_1 + (C_{16}\xi_1 + C_{66}\xi_2)\nu_2 & (C_{16}\xi_1 + C_{12}\xi_2)\nu_1 + (C_{66}\xi_1 + C_{26}\xi_2)\nu_2 \\
(C_{16}\xi_1 + C_{66}\xi_2)\nu_1 + (C_{12}\xi_1 + C_{26}\xi_2)\nu_2 & (C_{66}\xi_1 + C_{26}\xi_2)\nu_1 + (C_{26}\xi_1 + C_{22}\xi_2)\nu_2
\end{pmatrix},$$

and is called the boundary stress operator.

Next, we define the internal energy density by

$$E(u, u) = \frac{1}{2}\sigma^T \varepsilon = \frac{1}{2}\varepsilon^T C\varepsilon = \frac{1}{2}\sigma^T S\sigma.$$

For the $E(u, u)$ to be positive we must have

$$\varepsilon^T C\varepsilon > 0 \quad \text{and} \quad \sigma^T S\sigma > 0. \quad (2.2)$$

We denote by $\gamma$, which is the straight line segment $PQ$ for any $P \in \partial \Omega$ and $Q \in \Omega$ the crack in $\Omega$. Let $Q'$ be a point of intersection of an extension of crack $\gamma$ and $\partial \Omega$. $PQ'$ divides $\Omega$ into two open convex sets $\Omega_+$ and $\Omega_-$. For any $u$ defined in $\Omega \setminus \gamma$ we denote by $u_{\pm}$ the restriction of $u$ onto $\Omega_{\pm}$.

Let $\Gamma_D$ be an arbitrary nonempty open subset of a connected component of $\partial \Omega \setminus \{P, Q'\}$ with $\Gamma_D \subset \partial \Omega_-$ and $P \notin \Gamma_D$. Given $g \in L^2(\partial \Omega \setminus \Gamma_D)$ let $u$ be a weak solution of the problem

$$\begin{cases}
Au = 0 \quad \text{in} \quad \Omega \setminus \gamma, \\
T u_{\pm} = 0 \quad \text{on} \quad \gamma_{\pm}, \\
u = 0 \quad \text{on} \quad \Gamma_D, \quad T u = g \quad \text{on} \quad \partial \Omega \setminus \Gamma_D.
\end{cases} \quad (*)$$

Note that we take $\nu = \nu_{\pm}$ on $\gamma_{\pm}$ that is the unit outward normal relative to $\Omega_{\pm}$. On the crack $\gamma$ the traction-free condition is imposed. Under the condition (2.2) one knows the existence and uniqueness of $u$ in a suitable function space.

In [6, 7] we considered the following problem and gave an answer by using the enclosure method.

**Inverse Crack Problem** Assume that $P$ is known and $Q$ is unknown. Find $Q$ from the boundary data $u|_{\partial \Omega \setminus \Gamma_D}$, $g$ and $T u$ on $\Gamma_D$ for a single weak solution $u$ of $(*)$.

In order to describe the result we introduce two important functions and a condition on the direction. We identify the set of all unit vectors in $\mathbb{R}^2$ with the unit circle $S^1$.

First define the support function of $\gamma$ by the formula

$$h_\gamma(\omega) = \sup_{x \in \gamma} x \cdot \omega, \quad \omega \in \mathbb{R}^2 \setminus \{0\}.$$

**DEFINITION 1** It is well known that there exists a direction $\vartheta \in S^1$ such that, for all $x \in \Omega$ $(x - P) \cdot \vartheta > 0$. Let $\omega \in \mathbb{R}^2 \setminus \{0\}$ satisfy $\omega \cdot \vartheta > 0$. We say that $\omega$ is regular with respect to $\gamma$ if the intersection of the line $x \cdot \omega = h_\gamma(\omega)$ with $\gamma$ is just given by the tip $Q$. 
It is clear that given \( \omega \) with \( \omega \cdot \vartheta > 0 \), is regular with respect to \( \gamma \) if and only if \( \omega \) is not parallel to the unit normal of \( \gamma \).

In the enclosure method, the following complex exponential solution of (2.1) in \( \mathbb{R}^2 \) plays the central role.

In the case of isotropic body, let \( \omega \in S^1 \) and take \( \omega^\perp \in S^1 \) perpendicular to \( \omega \) satisfying \( \det (\omega^\perp, \omega) > 0 \). For \( \tau > 0 \) we define a function \( v = (\omega + i\omega^\perp) e^{\tau \omega (\omega + i\omega^\perp)} \).

In the case of anisotropic body, let \( \omega = \omega^{(1)} + \text{Re}\{\lambda\} \omega^{(2)} \) for given two linearly independent real vectors \( \omega^{(1)} = (\omega_1^{(1)}, \omega_2^{(1)})^T, \omega^{(2)} = (\omega_1^{(2)}, \omega_2^{(2)})^T \in S^1 \). For \( \tau > 0 \) we define a special solution \( v \) of the equation (2.1) in \( \mathbb{R}^2 \) : 
\[
v = a_\lambda e^{\tau \omega (\omega^{(1)} + i\omega^{(2)})}.
\]

Here \( \lambda \) is a complex number having a positive imaginary part and a solution of the equation
\[
S_{11}(\omega_2^{(1)} + \lambda \omega_2^{(2)})^4 - 2S_{16}(\omega_1^{(1)} + \lambda \omega_1^{(2)})(\omega_2^{(1)} + \lambda \omega_2^{(2)})^3 \\
+ (2S_{12} + S_{66})(\omega_1^{(1)} + \lambda \omega_1^{(2)})^2(\omega_2^{(1)} + \lambda \omega_2^{(2)})^2 \\
- 2S_{26}(\omega_1^{(1)} + \lambda \omega_1^{(2)})^3(\omega_2^{(1)} + \lambda \omega_2^{(2)}) + S_{22}(\omega_1^{(1)} + \lambda \omega_1^{(2)})^4 = 0
\]

and \( a_\lambda = (p_\lambda, q_\lambda)^T \) with
\[
p_\lambda = S_{11}(\omega_2^{(1)} + \lambda \omega_2^{(2)})^2 + S_{12}(\omega_1^{(1)} + \lambda \omega_1^{(2)})^2 - S_{16}(\omega_1^{(1)} + \lambda \omega_1^{(2)})(\omega_2^{(1)} + \lambda \omega_2^{(2)}),
\]
\[
q_\lambda = S_{12}(\omega_1^{(1)} + \lambda \omega_1^{(2)})(\omega_2^{(1)} + \lambda \omega_2^{(2)}) + \frac{S_{22}(\omega_1^{(1)} + \lambda \omega_1^{(2)})^3}{\omega_1^{(1)} + \lambda \omega_1^{(2)}} - S_{26}(\omega_1^{(1)} + \lambda \omega_1^{(2)})^2.
\]

**REMARK 1** When \( \omega^{(1)} = (1, 0)^T \) and \( \omega^{(2)} = (0, 1)^T \), \( \lambda \) satisfies the equation (2.3) if and only if the \( \lambda \) is a Stroh eigenvalue of \( C \) (cf. [11]).

Using this function, we define a mathematical indicator which can be calculated from the boundary data.

**DEFINITION 2** Let \( u \) be a weak solution of (\*). Define
\[
I_\omega(\tau, t) = e^{-\tau t} \left\{ \int_{\partial \Omega \setminus \Gamma_D} v \cdot g \, ds + \int_{\Gamma_D} v \cdot T u \, ds - \int_{\partial \Omega \cap \Gamma_D} u \cdot T v \, ds \right\}
\]
for \( \tau > 0 \) and \( t \in \mathbb{R} \).

3. An extraction formula

Our main result is the extraction formula of the value of the support function of \( \gamma \) from the boundary data in Inverse Crack Problem. However the surface traction \( g \) is not arbitrary.

**DEFINITION 3** We say that the surface traction \( g \) is well controlled if the condition
\[
\int_{\partial \Omega \setminus \Gamma} F(x_1, x_2) k \cdot g \, ds \neq 0
\]
holds for some \( k = (k_0, k_1, k_2)^T \neq 0 \), where \( F(x_1, x_2) k = (k_1 + k_0 x_2, k_2 - k_0 x_1)^T \).

The concept of well-controlled traction does not necessarily depend on the unknown crack. For example, any uniform pressure force field is well controlled. See [6] for other examples of well-controlled tractions which are independent of the unknown crack.

Now we state our main result.
THEOREM ([6, 7]) Assume that $\partial \Omega \setminus (\{P\} \cup \Gamma_D)$ is $C^2$. Let $g \in C^1(\partial \Omega \setminus (\{P\} \cup \Gamma_D))$ and be well controlled. Assume that $\omega$ is regular with respect to $\gamma$. The formula

$$h_\gamma(\omega) = \lim_{\tau \to \infty} \frac{1}{\tau} \log |I_\omega(\tau, 0)|,$$

is valid. Moreover we have; if $t \geq h_\gamma(\omega)$, then $\lim_{\tau \to \infty} |I_\omega(\tau, t)| = 0$; if $t < h_\gamma(\omega)$, then $\lim_{\tau \to \infty} |I_\omega(\tau, t)| = \infty$.

Note here that it is not sure whether $\omega$ is regular with respect to $\gamma$ or not. However, we can specify the crack tip $Q$ by Theorem as follows.

- Step 1. Since $\Omega$ is convex, the normal direction $\omega_1$ to a tangent line to $\partial \Omega$ at $P$ is regular with respect to $\gamma$, see Figure 1. In the case that $\partial \Omega$ is nonsmooth at $P$ as the tangent line we choose one of lines such that the intersection of the line with $\partial \Omega$ is only $P$.

- Step 2. By Theorem one can obtain the value of $h_\gamma(\omega_1)$ from the indicator function $I_{\omega_1}$ which is calculated by the boundary data. Then we can draw a line $x \cdot \omega_1 = h_\gamma(\omega_1)$, see Figure 2, which passes through $Q$.

- Step 3. We denote $\partial \Omega \cap \{x \cdot \omega_1 = h_\gamma(\omega_1)\}$ by $P_1, P_2$ and take a direction $\omega_2$ between $PP_1$ and $PP_2$. Then, $\omega_2$ is also regular with respect to $\gamma$, see Figure 3.

- Step 4. From the values $h_\gamma(\omega_1)$ and $h_\gamma(\omega_2)$ given by Theorem one can extract the location of the tip $Q$ of $\gamma$ as the intersection point of the two lines $x \cdot \omega_1 = h_\gamma(\omega_1)$ and $x \cdot \omega_2 = h_\gamma(\omega_2)$, see Figure 4.

4. A convergent series expansion of the displacement field near a crack tip

The formula in Theorem is a direct corollary of the following key lemma.

LEMMA If $\omega$ is regular with respect to $\gamma$, then there exist positive constants $\tilde{\lambda} = \tilde{\lambda}(\omega, \gamma)$ and $M = M(\omega, \gamma)$ such that $\lim_{\tau \to \infty} \tau^\tilde{\lambda} |I_\omega(\tau, h_\gamma(\omega))| = M$.

The proof of this Lemma is based on a complete asymptotic expansion of the indicator function as $\tau \to \infty$. The expansion is a consequence of the representation formula of the indicator function

$$I_\omega(\tau, t) = e^{-\tau t} \left( \int_{\gamma_+} u_+ \cdot T v_+ \, ds + \int_{\gamma_-} u_- \cdot T v_- \, ds \right)$$

(4.1)
and a convergent series expansion of the displacement field near the tip of crack.

In this section we explain how to derive the series expansion.

We denote by \( B(Q, R) \) the open disc centered at \( Q \) with radius \( R \) and set \( D_R = B(Q, R) \cap (\Omega \setminus \gamma) \). For each point \( x \) in \( D_R \) there exists a unique \((r, \theta)\) with \( 0 < r < R, -\pi < \theta < \pi \) such that

\[
x = Q - \frac{r \cos \theta}{|\omega|} \left( \cos p \omega^1 + \sin p \omega \right) + \frac{r \sin \theta}{|\omega|} \left( \sin p \omega^1 - \cos p \omega \right)
\]

where \( p \) satisfies \(-\pi < p < 0\) and is independent of \( x \). See Figure 5 for the meaning of \( p \). Note that we assume that \( \omega \) is regular with respect to \( \gamma \). We identify the point \( x \) with \((X_1, X_2) = (r \cos \theta, r \sin \theta)\) and denote \( B(Q, R) \cap \{X_2 > 0\}, B(Q, R) \cap \{X_2 < 0\} \) by \( B^+_R, B^-_R \), respectively.

Now we describe the steps to derive an expansion formula [7] of displacement field near the crack tip \( Q \). For this purpose we consider only in the coordinate system \((X_1, X_2)\) in this section. And in order to avoid some confusion we denote elements of the compliance tensor with respect to \((X_1, X_2)\) by \( \tilde{S}_{ij} \).

Here we give an essence of the derivation of an expansion formula [7] of \( u \) near \( Q \).

- **Step 1.** Choose a sufficiently small \( R \) and fix. One can find \( U \in C^\infty(D_R) \) in such a way
that
\[ \sigma_1 = \frac{\partial^2 U}{\partial X_1^2}, \quad \sigma_2 = \frac{\partial^2 U}{\partial X_2^2}, \quad \sigma_6 = -\frac{\partial^2 U}{\partial X_1 \partial X_2} \]
and \( U \in H^2(D_{sR}) \) for \( 0 < s < 1 \), \( U \) has a smooth extension of \( B^R_{sR} \) into \( B^R_{sR} \) across \( \gamma \). This \( U \) is called an Airy’s stress function. Then one knows that the equation of compatibility for the strain tensor yields the generalized biharmonic equation \([8]\)
\[ \tilde{S}_{11} \frac{\partial^4 U}{\partial X_1^4} - 2\tilde{S}_{16} \frac{\partial^4 U}{\partial X_1 \partial X_2^3} + (2\tilde{S}_{12} + \tilde{S}_{66}) \frac{\partial^4 U}{\partial X_1^2 \partial X_2^2} - 2\tilde{S}_{26} \frac{\partial^4 U}{\partial X_1^2 \partial X_2} + \tilde{S}_{22} \frac{\partial^4 U}{\partial X_1^4} = 0. \]
The corresponding characteristic equation is given by
\[ \tilde{S}_{11} \mu^4 - 2\tilde{S}_{16} \mu^3 + (2\tilde{S}_{12} + \tilde{S}_{66}) \mu^2 - 2\tilde{S}_{26} \mu + \tilde{S}_{22} = 0. \quad (4.2) \]
From (2.2) we see that the roots of (4.2) are either complex or pure imaginary pairwise conjugate \( \mu_k = \alpha_k + i\beta_k, \quad \bar{\mu}_k = \alpha_k - i\beta_k, \quad k = 1, 2, \quad \beta_k > 0. \)
In the following steps we consider only the case when \( \mu_1 \neq \mu_2 \) and see [6, 7] for the case when \( \mu_1 = \mu_2 \).

- Step 2. One can find \( \varphi_k(z_k) \) with \( z_k = X_1 + \mu_k X_2 \) \( (k = 1, 2) \) such that \( U = 2\text{Re}[\varphi_1(z_1) + \varphi_2(z_2)] \). Then \( u \) takes the form \((u_1, u_2)^T = \tilde{u} + F(X_1, X_2)k\) where
\[ \tilde{u} = \begin{bmatrix} 2\text{Re} \left[ \sum_{k=1}^2 p(\mu_k) \varphi_k(z_k) \right] \\ 2\text{Re} \left[ \sum_{k=1}^2 q(\mu_k) \varphi_k(z_k) \right] \end{bmatrix}^T \]
with a suitable \( k = (k_0, k_1, k_2)^T \) and
\[ p(\mu_k) = \tilde{S}_{11} \mu_k + \tilde{S}_{12} - \tilde{S}_{16} \mu_k, \quad q(\mu_k) = \tilde{S}_{12} \mu_k + \tilde{S}_{22} / \mu_k - \tilde{S}_{26}. \]
- Step 3. One can prove that the functions \( \varphi_1'(z_1), \varphi_2'(z_2) \) can be expanded as
\[ \varphi_1'(z_1) = \sum_{n=0}^{\infty} a_n z_1^n, \quad \varphi_2'(z_2) = \sum_{n=0}^{\infty} b_n z_2^n \]
in \( D_{R'} \) for a small \( R' > 0 \). In this step an analytic continuation argument due to Muskhelishvili [9] is essential.

Combining the steps 2 and 3, one obtains the following formula:
\[ \tilde{u}(r, \theta) = \sum_{n=0}^{\infty} A_n r^n \text{Re} \left[ \frac{i^{n-1} \psi_{1n}(\theta)}{\mu_1 - \mu_2} \right] + \sum_{n=0}^{\infty} B_n r^n \text{Re} \left[ \frac{i^{n-1} \psi_{2n}(\theta)}{\mu_1 - \mu_2} \right], \quad (4.3) \]
where \( A_n, B_n \) \( (n = 0, 1, 2, \cdots) \) are real numbers and satisfy the equations
\[ a_n = \frac{i^{n-1} \mu_2}{2(\mu_2 - \mu_1)} \left( A_n + \frac{B_n}{\mu_2} \right), \quad b_n = \frac{i^{n-1} \mu_1}{2(\mu_1 - \mu_2)} \left( A_n + \frac{B_n}{\mu_1} \right); \]
ψ_{1,n} and ψ_{2,n} are given by the formulae

\[
\psi_{1,n}(\theta) = \begin{pmatrix}
-\mu_2 p(\mu_1)(\cos \theta + \mu_1 \sin \theta) \frac{\partial}{\partial \theta} + \mu_1 p(\mu_2)(\cos \theta + \mu_2 \sin \theta) \frac{\partial}{\partial \theta}
-\mu_2 q(\mu_1)(\cos \theta + \mu_1 \sin \theta) \frac{\partial}{\partial \theta} + \mu_1 q(\mu_2)(\cos \theta + \mu_2 \sin \theta) \frac{\partial}{\partial \theta}
\end{pmatrix}
\]

\[
\psi_{2,n}(\theta) = \begin{pmatrix}
-p(\mu_1)(\cos \theta + \mu_1 \sin \theta) \frac{\partial}{\partial \theta} + p(\mu_2)(\cos \theta + \mu_2 \sin \theta) \frac{\partial}{\partial \theta}
-q(\mu_1)(\cos \theta + \mu_1 \sin \theta) \frac{\partial}{\partial \theta} + q(\mu_2)(\cos \theta + \mu_2 \sin \theta) \frac{\partial}{\partial \theta}
\end{pmatrix}
\]

The coefficients of them K_1 \equiv \sqrt{2\pi} A_1, K_2 \equiv \sqrt{2\pi} B_1 are called, in fracture mechanics [10], the stress intensity factor for mode 1 (crack opening mode), mode 2 (crack sliding mode) of fracture, respectively.

Using (4.1) and (4.3), we can compute the complete asymptotic expansion of the indicator function as τ \rightarrow \infty. We see that A_n and B_n for only n = 1, 3, 5, \ldots appear in the coefficients of the expansion and can show that if all the coefficients of the asymptotic expansion of the indicator function vanish, then A_n = B_n = 0 for all odd numbers. This condition together with (4.3) and a real analyticity of the displacement field outside the crack yields that the traction applied to the surface of the body should not be well controlled. Therefore if the surface traction is well controlled, then there exists a nonzero coefficient in the expansion of the indicator function. This is the outline of the proof of Lemma. Note that the well-controlled surface traction just ensures the existence of an odd number n with A_n + B_n \neq 0 not A_1 + B_1 \neq 0.

An application of the enclosure method of a single measurement version to an unknown polygonal cavity/inclusion in an elastic body in two dimensions remains open. The main obstruction in the application is the complexity of the behaviour of the displacement field near the vertex of the polygon [1]. However, we believe that there should be a formula similar to that of Theorem.

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