For many observables, the most difficult part of a single logarithmic resummation is the analytical treatment of the observable’s dependence on multiple emissions. We present a general numerical method, which allows the resummation specifically of these single logarithms. Some first applications and new results for the thrust major, the oblateness and the two-jet rate in the Durham algorithm are also presented.

1 Introduction

The formation of jets in hadronic collisions is surely one of the most striking phenomena in high energy collisions. Their study allows a precise measurement of $\alpha_s$, is important for any search for new physics and is useful to test our understanding of strong interaction dynamics. Generally jet cross sections are quite large but it might be difficult to get an insight into details of the interaction mechanics. One way is provided by the study of event shapes and jet rates.

2 Resummation of event shape distributions

Event shape variables can be studied at variety of levels, either through relatively inclusive properties, such as their mean value, or more exclusively by examining integrated (or differential) distributions. The integrated distribution $\Sigma(v)$ accumulates contributions both from real and virtual emissions. In spite of the fact that both contributions are separately divergent, infrared safety of the observable ensures that their sum is finite at any order in perturbation theory. In the (more inclusive) $v \sim 1$ region the observable is sensitive only to the total energy and momentum flow, while in the (less inclusive) region $v \to 0$ the mismatch between real and virtual emissions gets emphasized, so that divergences do still cancel but large logarithmic enhanced contributions are produced, which need to be resummed at any order in perturbation theory.

*From here onwards, capital letters denote the observable, while lower case the numerical value.*
2.1 Classification of single logarithmic terms

In spite of the fact that resummation programmes are quite cumbersome it is possible to classify systematically in a simple way all leading (DL) and subleading (SL) contributions. DL contributions are due to emissions which are both soft and collinear, while SL corrections are found to be due to

- soft, large angle emissions;
- hard, collinear radiation;
- effects due to the running of $\alpha_s$;
- multiple emission effects, i.e. taking into account properly how all emissions add up together and interfere to contribute to the value of the observable.

As far as the first three items are concerned, to obtain a resummed prediction at SL level it is enough to understand how a single emission affects the observable and to exponentiate this result in a naive way. Therefore the last point turns out to be the only non trivial task in any resummation programme and it is exactly this contribution which we address here.

2.2 Standard treatment of multiple emission effects

To understand how multiple emission effects are treated usually let us consider for instance the single jet broadening $B$ (in the following $k_{ti}$ denote the secondary transverse momenta)

$$2BQ = \sum_i k_{ti}, \quad \text{with} \quad k_{ti} = |\vec{k}_{ti}|. \quad (1)$$

Resummation is achieved factorizing matrix element and the phase space (using an independent emission picture\cite{1,2} and the definition of the observable via a Mellin transform

$$\Theta(2BQ - \sum_i k_{ti}) = \int_C \frac{d\nu}{2\pi i \nu} e^{2\nu BQ} e^{-\nu k_{ti}}. \quad (2)$$

But, actually, at SL level one needs to take into account the fact that due to all secondary emissions the hard parton will take a recoil $p_t$. Therefore the $\Theta$-function needs to be replaced by

$$\Theta(2BQ - \sum_i k_{ti}) \Rightarrow \int d^2p_t \delta^2(\vec{p}_t + \sum_i \vec{k}_{ti}) \Theta(2BQ - \sum_i k_{ti} - p_t). \quad (3)$$

To factorize the $\delta$-function one needs now an additional two-dimensional Fourier transform. In the case of the thrust minor, which is defined as the radiation out of a plane, to fix kinematically the plane one needs an additional five-dimensional integral, so that generally an analytical treatment can become very involved. Furthermore, in certain cases, as for the thrust major, the oblateness and the 3-jet resolution, an analytical treatment seems to be unfeasible.

3 Numerical treatment of multiple emission effects

Since multiple emission effects have the same origin it is natural to look for a general approach to the problem. The idea is to relate multiple emission effects of the observable $V$ under study to a 'simple' reference variable $V_s$ which has the same double logarithmic structure, i.e. the same dependence on one soft-collinear emission, but which factorizes in a trivial way.

\footnote{Actually, the single jet broadening, taken here for illustration, is a non-global observable, so that an independent emission picture mistreats some of the single logs}
For an observable given by the sum of \( n \) contributions, i.e. sensitive to all emissions, the natural choice of the simple variable becomes

\[
V = \sum_i v(k_i) \quad \Rightarrow \quad V_s = \max[v(k_1), \ldots, v(k_n)].
\] (4)

By construction \( V_s \) is sensitive only to the 'largest' emission and this definition also ensures that \( V_s \) factorizes in a simple way (i.e. needs no Mellin transforms)

\[
\Theta(V_s - \max[v(k_1), \ldots, v(k_n)]) = \prod_i \Theta(V_s - v(k_i)).
\] (5)

As a consequence the resummation of \( V_s \) becomes straightforward. For instance for the case of \( e^+ e^- \to 2 \text{ jets} \) the resummed distribution becomes (the case \( e^+ e^- \to 3 \text{ jets} \) is also trivial)

\[
\Sigma_s(v_s) = e^{-R_s(v_s)} \quad \mathcal{R}_s(v_s) = C_F \int \frac{Q^2}{\pi k_t^2} \frac{d^2 k_t}{2\pi} \alpha_s(k_t) \int_{-\ln Q^2/k_t^{3/4}} \ln \frac{Q_s^{3/4}}{k_t^{3/4}} d\eta \Theta(v(k) - v_s).
\] (6)

The aim is now to exploit the 'simple', known distribution \( \Sigma_s \) to compute the real observable \( D \).

The two differential distributions \( D = d\Sigma/d\ln v \) are related by a simple convolution

\[
\frac{D(v)}{v} = \int \frac{dv_s}{v_s} D_s(v_s)P(v|v_s),
\] (7)

where \( P(v|v_s) \) denotes the conditional probability of having a value of \( v \) for \( V \) given a value of \( v_s \) for \( V_s \). The two distributions have the same double logarithmic structure so that \( v \sim v_s \) and one can expand \( D_s(v_s) \sim e^{-\mathcal{R}_s(v_s)} \) around the value of \( v_s \)

\[
D_s(v_s) = \mathcal{SL} D_s(v)e^{-R'\ln \frac{v}{v_s}},
\] (8)

where \( R' \equiv -dR/d\ln v \) is a single logarithmic function. Combing eq. (7) and (8) one obtains

\[
D(v) = \mathcal{SL} D_s(v)\mathcal{F}(R'),
\] (9)

with

\[
\mathcal{F}(R') = \int \frac{dx}{x} e^{-R'\ln x}p(x, R'), \quad x = v/v_s, \quad p(x, R') = v P(v/v_s).
\] (10)

At SL level the same relation holds then for the integrated distribution

\[
\Sigma(v) = \Sigma_s(v)\mathcal{F}(R').
\] (11)

This expression shows explicitly that all non trivial multiple effects can be factored out and embodied in \( \mathcal{F} \). The aim is then to compute \( \mathcal{F} \) i.e. \( p(x, R') \) in a general way.

### 3.1 Numerical implementation: the procedure to get \( \mathcal{F} \)

To compute the function \( \mathcal{F} \) numerically one writes a Monte Carlo[4] which starts from an arbitrary Born configuration with an arbitrary number of hard emitting legs and

- 0 generates the first 'largest' emission with \( v(k_1) = v_s \) (i.e. one starts fixing \( V_s = v_s \));
- \( I \) generates an additional soft-collinear (SC) emission (according to the phase space \( R' \)) which satisfies \( v(k_i) < v(k_{i-1}) \);
- \( II \) if \( v(k_i) < \epsilon \ll 1 \), where \( \epsilon \) defines the precision of the procedure, the emission (and all subsequent ones) is too small to affect the value of the observable, therefore the procedure is stopped and the momenta generated are passed to a standard routine which computes the value of \( V \); otherwise one sets \( i \to i + 1 \) and goes back to step \( I \).
4 Recollection of known results and new results

As a first check of the procedure we compared some Monte Carlo results with results from the literature for the function $F$ for the thrust $T_\mu$ and the total and wide broadening $T_\nu$ and for the thrust minor $T_M$. Comparisons show that our numerical results turn out to be interchangeable with analytical ones.

We then applied the procedure to obtain some new predictions for thrust major and the oblateness, which have so far defied analytical resummation and to the two-jet rate in the Durham algorithm, for which only a subset of the single logs had up to now been calculated.

We first used the Monte Carlo procedure to compute the function $F$ for these variables, combining this with $\Sigma_s$ we get the first full resummed prediction at SL level which we then match with second order exact results.

5 Conclusions

The study of event shapes and jet rates has proved to be a powerful tool to test QCD and to measure $\alpha_s$. Many variables have been studied in the last years, but never in a general way. This work is a first step towards a complete numerical resummation of any $n$-jet observable in any hard process. What has been done so far is to implement multiple emission effects generally, to test the procedure and to obtain some new predictions. What remains to be done is to make the procedure completely automatic, i.e. userfriendly and to exploit the method to obtain a bunch of new predictions especially in hadron-hadron collisions, which will be the main object of most experimental analyses in the near future.

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