Cosmological perturbations in FRW model with scalar field within Hamilton-Jacobi formalism and symplectic projector method

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Abstract

The Hamilton-Jacobi analysis is applied to the dynamics of the scalar fluctuations about the Friedmann-Robertson-Walker (FRW). The gauge conditions are found from the consistency conditions. The physical degrees of freedom of the model are obtain by symplectic projector method. The role of the linearly dependent Hamiltonians and the gauge variables in Hamilton-Jacobi formalism is discussed.

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1 Introduction

The cosmological models which include the theory of a scalar field coupled to gravity play an important role during the last period. A transformation from a reparametrization-invariant system to an ordinary gauge system was applied for deparametrizing cosmological models. In the path integral approach to false vacuum decay with the effect of gravity, there is an unsolved problem, called the negative mode problem. A conjecture it was proposed that there should be no supercritical supercurvature mode. This conjecture was verified for a wide variety of tunnelling potentials [1]. For the monotonic potentials no negative modes were reported about the Hawking-Turok instanton. For a potential with a false vacuum the Hawking-Turok instanton it was shown that we obtain a negative mode for certain initial data [2]. It was shown that the cosmological perturbations in Lorentzian regime are related to the cosmic microwave background radiation and large scale structure formation [3, 4, 5]. The unconstrained reduced action corresponding to the dynamics of scalar fluctuations about the FRW background was obtained by applying Dirac’s method of singular Lagrangian systems [6, 7]. The results were applied to the negative mode problem in the description of tunnelling transitions with gravity [8]. There are several known methods in obtaining and dealing with unconstrained quadratic action in terms of the physical variables [4, 5, 9] in the theory of scalar field coupled to gravity in non-spatially flat FRW Universe but the main problem appears at the quantum level [8]. For these reasons new quantization methods as Hamilton-Jacobi method and the symplectic projector method [10, 11, 12, 13] should be applied on the theory mentioned above. By adding a surface term to the action functional the gauge invariance of the systems whose Hamilton-Jacobi equation is separable was improved [14].

Hamilton-Jacobi formalism (HJ) based on Carathéodory’s idea [15] gained a considerable importance during the last decade due to its various applications to quantization of constrained systems [16].

However, some difficulties may occur for HJ in dealing with linear dependent constraints. The main problem comes from the construction of the canonical Hamiltonian. Let us assume that the canonical Hamiltonian is a linear combination of two terms and the second one is proportional to a given field having its momentum zero. After imposing the integrability condition for that momentum we obtain a new constraint, therefore the canonical Hamiltonian is a linear combination of two constraints.
Therefore one of interesting and yet not solved question is how to deal with the total differential equations within HJ in the above mentioned case.

Another issue is related to the gauge fixing procedure within HJ formalism. Can we find inside of HJ a mechanism to obtain the gauge fixing condition?

In order to analyze the above mentioned open problems we have to apply HJ formalism to a constrained system possessing linearly dependent constraints.

For these reasons the application of HJ formalism to cosmological perturbations in FRW model with scalar field is an interesting issue.

The paper is organized as follows:

In Section 2 the model is presented. Section 3 presents briefly HJ formalism. The symplectic projector method is discussed in Section 4. In Section 5 the gauge fixing conditions of the investigated model are discussed inside HJ formalism and the true degrees of freedom are obtained within symplectic projector method. Finally, Section 6 is dedicate to our conclusions.

2 The model

The action of the system of scalar matter field coupled to gravity is given by

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{2k} - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right].
\]  (1)

Here \( k = 8\pi G \) represents the reduced Newton’s constant and the scalar potential field is denoted by \( V(\phi) \). [8]

By expanding the metric and the scalar field over an FRW type background one obtains

\[
ds^2 = a(\eta)^2 \left[ -(1 + 2A(\eta)Y)d\eta^2 + 2B(\eta)Y_i^id\eta dx^i \right.
\]

\[
+ \{ \gamma_{ij}(1 - 2\Psi(\eta)Y) + 2E(\eta)Y_{ij}dx^idx^j \},
\]

\[
\phi = \varphi(\eta) + \Phi(\eta)Y.
\]  (2)

Here \( \gamma_{ij} \) represents the three-dimensional metric on the constant curvature space sections, \( a \) and \( \phi \) denote the background field values and \( A, \Psi, \Phi, B \) and \( E \) are small perturbations. In addition, \( Y \) represents a normalized function of 3-dimensional Laplacian, \( \Delta Y = -k^2Y \), and vertical line denotes the covariant derivative with respect to \( \gamma_{ij} \).
The background fields $a$ and $\phi$ are subjected to the following equations

\begin{align*}
\mathcal{H}^2 - \mathcal{H}' + K &= \frac{k}{\phi} \phi^2, \\
2\mathcal{H}' + \mathcal{H}^2 + K &= \frac{k}{2} \left( -\phi'^2 + 2a^2V(\phi) \right) \\
\phi'' + 2\mathcal{H}\phi' + a^2 \frac{\delta V}{\delta \phi} &= 0.
\end{align*} \tag{3}

In (3) the prime denotes a derivative with respect to conformal time $\eta$, $\mathcal{H} = \frac{a'}{a}$ and $K$ denotes the curvature parameter which takes the values 1, 0, -1 for closed, flat and open universes, respectively.

The total action containing only the second order terms becomes

\[ S = S^{(0)} + S^{(2)}, \tag{4} \]

where $S^{(0)}$ represents the action of the background solution and $S^{(2)}$ is quadratic in perturbations. The corresponding Lagrangian is as follows

\[ \mathcal{L} = \frac{a^2}{2k} \left[ -6\Psi'^2 + 2(k^2 - 3K)\Psi^2 \right. \\
- \left. \left\{ (a^2 \frac{\delta^2 V}{\delta \phi^2} + k^2)\Phi^2 + 6\phi'\Phi' \Phi \right\} -2(\mathcal{H}' + 2\mathcal{H}^2 - K)A^2 \right]. \tag{5} \]

3 Hamilton-Jacobi formalism

HJ formalism presented in this section is based on Carathéodory’s idea of equivalent Lagrangians [15]. This approach can be considered an alternative method of quantization of constrained systems and it was subjected under an intense debate during the last decade [16, 17, 18, 19, 20, 21, 22, 23]. The starting point of this method is a singular Lagrangian $\mathcal{L}$, therefore the corresponding Hessian matrix is singular. In this case a set of primary constraints appears naturally. Instead of working with one Hamiltonian, in this method we use the initial canonical Hamiltonian $H_0$ and all primary constraints $H'_\alpha$. Namely, the ”Hamiltonians” are

\[ H'_\alpha = p_\alpha + H_\alpha(t_\beta, q_\alpha, p_\alpha), \tag{6} \]
where $\alpha, \beta = n - r + 1, \cdots, na = 1, \cdots, n - r$ and the canonical one

$$H_0 = p_a w_a + \dot{q}_\mu p_\mu \mid_{p_\nu = H_\nu} - L(t, q_i, \dot{q}_i, \dot{q}_a = w_a), \quad \nu = 0, n - r + 1, \cdots, n. \quad (7)$$

Using (6) and (7) a set of total differential equations is obtained

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \quad dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \quad dp_\mu = -\frac{\partial H'_\alpha}{\partial t_\mu} dt_\alpha, \quad \mu = 1, \cdots, r, \quad (8)$$

together with HJ function $z$ is defined by

$$dz = \left( -H_\alpha + p_\alpha \frac{\partial H'_\alpha}{\partial p_\alpha} \right) dt_\alpha, \quad (9)$$

where $t_\alpha$ are gauge variables $[15]$. The next step is to investigate the integrability of the system $[18]$. On the surface of constraints the system of differential equations is integrable if and only if

$$[H'_\alpha, H'_\beta] = 0. \quad (10)$$

The difficulties appear in this formalism when the Hamiltonians are not in the form (6). In this case the physical significance from HJ point of view is lost. Therefore, we have to make a canonical transformation to be able to recover the physical significance. Finding a suitable canonical transformation for a given constrained system is not an easy task in general. The surface terms play an important role in finding the integrability conditions in HJ formalism.

If (10) is not fulfilled, then another set of "Hamiltonians" arises and we subject them to the integrability conditions. The process ends when no new "Hamiltonian" appears.

### 4 Symplectic Projector Method

Let us assume that a system admits only second constraints

$$\phi^m(\zeta^M) = 0, \quad (11)$$

where $\zeta^M = (x^a, p^a)$, $M = 1, 2, \cdots, 2N$ are the coordinates.

The definition of symplectic projector is the following (for more details see $[10, 11, 12, 13]$ and the references therein)
\[ \Lambda^{MN} = \delta^{MN} - J^{ML} \frac{\delta \phi_m}{\delta \zeta^L} \Delta^{-1}_{mn} \frac{\delta \phi_n}{\delta \zeta^N} , \]  
\tag{12} 

where \( \Delta^{-1}_{mn} \) represents the inverse of the following matrix

\[ \Delta_{mn} = \{ \phi_m, \phi_n \} \]  
\tag{13} 

and \( J^{MN} \) denotes the symplectic two form. Thus, the action of the symplectic projector defined by (12) is to project \( \zeta^M \) onto local variables on the constraint surface defined below

\[ \zeta^* = \Lambda^{MN} \zeta^N. \]  
\tag{14} 

5 The model and the gauge fixing conditions within HJ formalism

From the Lagrangian density (5) we obtain the canonical momenta as

\[ \Pi_\Psi = \frac{6a^2 \sqrt{\gamma}}{k} (\Psi' + \frac{k}{2} \psi' \Phi - H A) \]
\[ \Pi_\Phi = a^2 \sqrt{\gamma} (\Phi' - \phi' A) , \]
\[ \Pi_A = 0. \]  
\tag{15} 
\tag{16} 

From (16) we conclude that \( A \) is a gauge variable and that \( H'_1 = \Pi_A \) represents a "Hamiltonian". By using (15) and (16) the canonical Hamiltonian becomes

\[ H_C = -\frac{k}{12a^2 \sqrt{\gamma}} \Pi_\Psi^2 + \frac{1}{2a^2 \sqrt{\gamma}} \Pi_\Phi^2 + \frac{k}{2} \phi' \Pi_\Phi \Phi \]
\[ + a^2 \sqrt{\gamma} [ -\frac{k^2 - 3K}{k} \Psi^2 + \frac{1}{2} \left( a^2 \frac{\delta^2 V}{\delta \phi} - \frac{k \phi'^2}{2} \right) \Phi^2 ] \]
\[ + A \{ \phi' \Pi_\Phi - H \Pi_\Psi + a^2 \sqrt{\gamma} \left[ (a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H}) \Phi + \frac{2(k^2 - 3K)}{k} \Psi \right] \} \]  
\tag{17} 

therefore

\[ H'_0 = p_0 + H_C. \]  
\tag{18}
The next step in HJ formalism is to obtain the total differential equations by using (16) and (18). In our case we obtain the following set of total differential equations

\[ d\Psi = -\frac{k\Pi d\tau}{6a^2\sqrt{\gamma}} + \frac{k}{2}\phi'\Phi d\tau - A\mathcal{H}d\tau, \quad (19) \]

\[ d\Phi = \frac{(\Pi\phi}{a^2\sqrt{\gamma}} + A\phi')d\tau, \quad (20) \]

\[ d\Pi\Psi = 2a^2\sqrt{\gamma}\frac{k^2}{k} - \frac{3K}{k} k\Psi d\tau - \frac{2a^2\sqrt{\gamma}A(k^2 - 3K)}{k} d\tau, \quad (21) \]

\[ d\Pi\Phi = -\frac{\Pi\phi}{a^2\sqrt{\gamma}} d\tau - Aa^2\sqrt{\gamma}(a^2\delta V\delta\phi + 3\phi'\mathcal{H})d\tau \]

\[-a^2\left(\frac{\delta^2 V}{\delta\phi\delta\phi} - \frac{3}{2}k\phi'^2 + k^2\right)\Phi d\tau. \quad (22) \]

Taking into account (17) and the consistency condition

\[ d\Pi_A = 0, \quad (23) \]

we get a new ”Hamiltonian” denoted by \( H_2 \). Namely, the form of \( H_2 \) is given by

\[ H_2 = \phi'\Pi - \mathcal{H}\Pi + a^2\sqrt{\gamma}[(a^2\delta V\delta\phi + 3\phi'\mathcal{H})\Phi + \frac{2(k^2 - 3K)}{k}\Psi]. \quad (24) \]

In order to close the chain the variation of \( H_2 \) must be zero, otherwise a new constraint will appear. By using (19), (20), (21) and (22) we may find after some tedious calculations that if

\[ A - \Psi = 0, \quad \Pi\Psi = 0, \quad (25) \]

then \( dH_2 = 0 \) provided that \( A \) is given as a function of background fields, \( \Phi \) and \( \Pi\Phi \).

5.1 Physical Hamiltonian

To find the true degrees of freedom of the proposed model we used the symplectic projector method [10, 11, 12, 13]. The set of second class constraints to start with is as follows
\[ C_1 = \phi' \Pi + a^2 \sqrt{\gamma} \left[ \left( a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H} \right) \Phi + \frac{2(k^2 - 3K)}{k} \Psi \right] \]
\[ C_2 = \Pi_A, C_3 = \Pi_\psi, C_4 = A - \Psi. \] (26)

By using (26) we obtain the form of matrix \( \Delta \) as follows

\[
\Delta = \begin{pmatrix}
0 & 0 & 2 \frac{a^2 \sqrt{\gamma(k^2 - 3K)}}{k} & 0 \\
0 & 0 & 0 & -1 \\
-2 \frac{a^2 \sqrt{\gamma(k^2 - 3K)}}{k} & 0 & 0 & 1 \\
0 & 1 & -1 & 0
\end{pmatrix} \delta (\vec{x} - \vec{y})
\] (27)

The form of the matrix projector becomes

\[
\Lambda = \begin{pmatrix}
0 - k \frac{a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H}}{2(k^2 - 3K)} & 0 & 0 & -k \phi' \frac{2a^2 \sqrt{\gamma(k^2 - 3K)}}{k} & 0 \\
0 & 1 & 0 & k \phi' \frac{2a^2 \sqrt{\gamma(k^2 - 3K)}}{k} & 0 \\
0 - k \frac{a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H}}{2(k^2 - 3K)} & 0 & 0 & -k \phi' \frac{2a^2 \sqrt{\gamma(k^2 - 3K)}}{k} & 0 \\
0 & 0 & 0 & -k \frac{a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H}}{2(k^2 - 3K)} & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \delta (\vec{x} - \vec{y})
\] (28)

We observed that \( Tr \Lambda = 2 \), therefore we have only two true physical degrees of freedom. Let us introduce the phase space vector \( \xi \) with the following components

\[
(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5, \xi^6) = (A, \Phi, \Psi, \Pi_A, \Pi_\Phi, \Pi_\psi)
\] (29)

By using (14) we obtain

\[
\zeta_1^* = \xi^2 - \frac{k \phi'}{2a^2 \sqrt{\gamma(k^2 - 3K)}} \xi^5, \\
\zeta_2^* = \frac{k \phi'}{2a^2 \sqrt{\gamma(k^2 - 3K)}}(\xi^4 + \xi^6), \\
\zeta_3^* = -k \frac{a^2 \frac{\delta V}{\delta \phi} + 3 \phi' \mathcal{H}}{2(k^2 - 3K)} \xi^2 - \frac{k \phi'}{2a^2 \sqrt{\gamma(k^2 - 3K)}} \xi^5, \\
\zeta_4^* = 0.
\]
\[
\begin{align*}
\zeta_5^* &= -k \left( \frac{a^2 \delta \xi_{\delta} + 3 \phi \cdot \mathcal{H}}{2(k^2 - 3K)} \right) (\xi^4 + \xi^6) + \xi^5, \\
\zeta_6^* &= 0.
\end{align*}
\] (30)

We observed that
\[
\xi_1^* = \xi_3^*
\] (31)

and
\[
\zeta_5^* = -\frac{a^2 \sqrt{\gamma}}{\phi'} \left\{ \zeta_3^* \left( a^2 \frac{\delta V}{\delta \phi} + 3 \phi \cdot \mathcal{H} \right) + \frac{1}{2(k^2 - 3K)} \zeta_1^* \right\},
\] (32)

therefore only two physical variables \( \zeta_1^*, \zeta_5^* \) can be used as a starting point for the quantization of the system. As it can be seen from (31) and (32) we obtain the same degrees of freedom as in [8].

6 Conclusions

The integrability of HJ total differential equations is an open and attractive issue. In our study we obtained the gauge conditions directly from the consistency conditions within HJ formalism. This result is based on the fact that if the canonical Hamiltonian represents a sum of two terms, the second one becomes another "Hamiltonian" in HJ formalism. In other words the canonical Hamiltonian represents a case of an irregular Hamiltonian. If we denote \( H_3 = A - \Psi \) and \( H_4 = \Pi \psi \) we obtain four "Hamiltonians" in our case. As it can be seen, the obtained "Hamiltonians" are not in involution, therefore the systems corresponding to these "Hamiltonians" is not integrable. To make it integrable we work on the surface of constraints and this way leads us to the same canonical Hamiltonian from up to a constant.

The above result can be generalized for the case when the canonical Hamiltonian has the form \( H_c = H_0 + \phi_1 H_1 + \cdots \phi_n H_n \), where the fields \( \phi_1, \cdots \phi_n \) do not appear in any "Hamiltonians" \( H_1, \cdots H_n \). In this case all \( \phi_1, \cdots \phi_n \) are gauge variables and they can be fixed after imposing the integrability conditions. In order to calculate the action we have to find the linearly independent "Hamiltonians" possessing the physical significance from HJ point of view.

Since the set of four "Hamiltonians" is a second class typed in Dirac classifications the symplectic projector method was used to obtain the true
degrees of freedom of investigated model. The results were found to be in agreement with those from \[8\].

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