DYNAMIC TRANSITIONS OF GENERALIZED KURAMOTO-SIVASHINSKY EQUATION

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Abstract. In this article, we study the dynamic transition for the one dimensional generalized Kuramoto-Sivashinsky equation with periodic condition. It is shown that if the value of the dispersive parameter \( \nu \) is strictly greater than \( \nu^* \), then the transition is Type-I (continuous) and the bifurcated periodic orbit is an attractor as the control parameter \( \lambda \) crosses the critical value \( \lambda_0 \). In the case where \( \nu \) is strictly less than \( \nu^* \), then the transition is Type-II (jump) and the trivial solution bifurcates to a unique unstable periodic orbit as the control parameter \( \lambda \) crosses the critical value \( \lambda_0 \). The value of \( \nu^* \) is also calculated in this paper.

1. Introduction. The one dimensional Kuramoto-Sivashinsky (KS) equation is given by

\[
\frac{\partial u}{\partial t} + \mu \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} = 0.
\]

(1)

The KS equation arises in many different physical fields, such as flame front propagation and instabilities [15, 16], reaction diffusion systems [5] and long wave on the interface between two viscous fluids [4]. Ma and Wang [6, 7, 8, 9] first gave the following classification scheme for dissipative system. It says that dynamic transitions of all dissipative systems can be classified into three categories, namely, Type-I (Continuous), Type-II (Jump) or Type-III (Random). For Type-I transition, the transition states are represented by a local attractor \( \Sigma_\lambda \), which attracts a neighborhood of the basic solution. This says that as the control parameter crosses a threshold, the transition states stay in the close neighborhood of the basic state. There are many physical systems which can undergo a continuous transition. For classical example, see [10] regarding the bifurcation and stability of the solutions of the Boussinesq equations, and the onset of the Rayleigh-Benard convection. See also [14] for Type-I transition of Benard convection of fluids on spherical shells. For Type-II transition, the transition states are represented by some local attractors.
away from the basic solution. Intuitively speaking, this type of transition corresponds to the case in which the dissipative system undergoes a more drastic change as the control parameter crosses the critical threshold. For Type-III transition, the transition states are represented by two local attractors, with one as in Type-II transition and the other as in Type-I transition. In this case the fluctuations of the basic state can be divided into two regions such that fluctuations in one of the regions lead to continuous transitions and those in the other region lead to jump transitions. For example of Type-II and Type-III transitions, the reader is referred to [11] which study the dynamic phase transitions associated with the spatial-temporal oscillation of the Belousov-Zhabotinsky reactions. See also [2] for detailed classification of dynamic transitions in the study of surface tension driven convection. The paper also contains discussion on the mechanism and formation of hexagonal structures which depend on the geometry of the domain.

In this paper we study the generalized Kuramoto-Sivashinsky (GKS) equation in one-dimensional space with periodic boundary condition given by

\[
\frac{\partial u}{\partial t} + \mu \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^3 u}{\partial x^3} + \frac{u}{\mu} \frac{\partial u}{\partial x} = 0.
\]

(2)
The constants \( \mu > 0 \) and \( \nu \neq 0 \) are two parameters where \( \nu \) is also known as a dispersive parameter. When \( \nu = 0 \), (2) reduces to the standard KS equation.

Spectral and nonlinear stabilities of periodic traveling wave solutions of a generalized Kuramoto-Sivashinsky equation have been studied [1]. Note that

\[
\frac{1}{\mu} \frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{1}{\mu} \frac{\partial^2 u}{\partial x^2} + \frac{\nu}{\mu} \frac{\partial^3 u}{\partial x^3} + \frac{1}{\mu} \frac{u}{\partial x} = 0.
\]

(3)

Now letting \( \tau = \mu t \), we can write (3) as

\[
\frac{\partial u}{\partial \tau} = - \frac{\partial^4 u}{\partial x^4} - \frac{1}{\mu} \frac{\partial^2 u}{\partial x^2} - \frac{\nu}{\mu} \frac{\partial^3 u}{\partial x^3} - \frac{1}{\mu} \frac{u}{\partial x},
\]

(4)

With \( \lambda = \frac{1}{\mu} \) and writing \( \tau \) as \( t \), we consider the GKS equation in the form

\[
\frac{\partial u}{\partial t} = - \frac{\partial^4 u}{\partial x^4} - \lambda \frac{\partial^2 u}{\partial x^2} - \nu \lambda \frac{\partial^3 u}{\partial x^3} - \lambda \frac{u}{\partial x},
\]

(5)

\[
\frac{\partial^j u}{\partial x^j}(x + 2k\pi) = \frac{\partial^j u}{\partial x^j}(x), \quad k \in \mathbb{Z}, \quad j = 0, 1, 2, 3,
\]

\[
u(x, 0) = \phi(x),
\]

and we assume that the average flow vanishes

\[
\int_0^{2\pi} u(x, t) \, dx = 0.
\]

(6)

It has been shown, see [12], that KS equation (1) undergoes Type-I transition as the control parameter \( \lambda = \frac{1}{\mu} \) crosses a critical value. Furthermore the system bifurcates from \( (u, \lambda) = (0, 1) \) to an attractor \( \Sigma_\lambda \) which is homeomorphic to \( S^1 \). However, to the best of author’s knowledge, dynamic transition of GKS (2) has not been studied. The objective of this article is therefore to study the dynamic transition of GKS using dynamic transition theory developed by Ma and Wang [12, 13]. The main result of this paper is as follows.

**Theorem 1.1.** The dynamic transition for the one dimensional generalized Kuramoto-Sivashinsky equation with periodic condition is
1. Type-I (continuous) and the bifurcated periodic orbit is an attractor as the control parameter $\lambda$ crosses the critical value $\lambda_0 = 1$, whenever the dispersive parameter $\nu \in (\nu^*, \infty)$.

2. Type-II (jump) and the trivial solution bifurcates to a unique unstable periodic orbit as the control parameter $\lambda$ crosses the critical value $\lambda_0 = 1$, whenever the dispersive parameter $\nu \in (-\infty, \nu^*)$.

Here $\nu^*$ is the root of

$$
\zeta(\nu) = \frac{-3}{144 + 64\nu^2} + \frac{\nu^2(1728 - 2256\nu^2)}{(1728 - 2256\nu^2)^2 + (3456 - 480\nu^2)^2}
\frac{\nu(576\nu + 240\nu^2)}{(144 - 60\nu^2)^2 + 36864\nu^2},
$$

which is $\nu^* \simeq -3.36217$. Furthermore, the bifurcated periodic solution can be expressed as

$$
u = x_1 e_1 + x_2 e_2 + o(|\lambda - 1|),
$$

$$x_1(t) = \left(\frac{\lambda - 1}{|b|}\right)^{1/2} \cos \nu \lambda t + o(|\lambda - 1|),
$$

$$x_2(t) = \left(\frac{\lambda - 1}{|b|}\right)^{1/2} \sin \nu \lambda t + o(|\lambda - 1|).$$

One crucial component of dynamic transition theory is center manifold reduction. Under the assumption (25) which is known as Principle of Exchange of Stability (PES), we reduce (8) to a center manifold, then the type of transitions for (8) at $(0, \lambda_0)$ is completely dictated by its reduced equations (33) and (34) near $\lambda = \lambda_0$. Using first-order approximation formula (See Theorem A.1.1. in [13]) we obtain the reduced equations (78) and (89). Analysis of these reduced equations give the above dynamic transition theorem, providing a precise criterion for the transition type.

The paper is organized as follows. Linear eigenvalue problem for system (8) is studied in Section 2. Center manifold reduction and its first-order approximation are addressed in Section 3. Proof of the main theorem is given in Section 4.

2. Mathematical setting and linear problem. For the mathematical setting of (5) we let

$$H_1 = \{u \in H^4(0, 2\pi) : \int_0^{2\pi} u dx = 0, \frac{\partial^j u}{\partial x^j}(x + 2k\pi) = \frac{\partial^j u}{\partial x^j}(x), \quad k \in \mathbb{Z}, \quad j = 0, 1, 2, 3\},$$

$$H = \{u \in L^2(0, 2\pi) : \int_0^{2\pi} u dx = 0\}.$$

Notice that $H_1$ and $H$ are two Hilbert spaces and $H_1 \hookrightarrow H$. We then write (5) as

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda),
$$

$$u(x, 0) = \phi(x),$$

where $L_\lambda = -A + \lambda B$, with

$$Au = \frac{\partial^4 u}{\partial x^4}, \quad \text{and} \quad Bu = -\frac{\partial^2 u}{\partial x^2} - \nu \frac{\partial^3 u}{\partial x^3}. $$
Since $A : H_1 \to H$ is a linear homeomorphism and $B_\lambda : H_1 \to H$ is a linear compact operator, we see that the operators $L_\lambda = -A + \lambda B$ are parameterized linear completely continuous fields and is continuously depending on $\lambda \in \mathbb{R}$. Also notice that $G(u, \lambda) = -\lambda u \frac{\partial}{\partial u}$ is a bounded mapping satisfying $G(u, \lambda) = o(\|u\|_{H_1})$.

Next we consider the eigenvalue problem

$$L_\lambda u = \beta(\lambda) u.$$  \hfill (9)

This eigenvalue problem is equivalent to

$$-u_{xxxx} - \lambda u_{xx} - \nu \lambda u_{xxx} = \beta(\lambda) u,$$  \hfill (10)

which has eigenvalues

$$\beta_n(\lambda) = n^2(\lambda - n^2) + i\nu \lambda n^3,$$  \hfill (11)

with corresponding eigenfunctions

$$\varphi_n(x) = \cos(nx) + i\sin(nx).$$  \hfill (12)

Also for $L_\lambda u = -u_{xxxx} - \lambda u_{xx} - \nu \lambda u_{xxx}$, its adjoint can be calculated. By writing $(L_\lambda u, \varphi) = (u, L_\lambda^* \varphi)$, we have

$$L_\lambda^* \varphi = -\varphi_{xxxx} - \lambda \varphi_{xx} + \nu \lambda \varphi_{xxx}.$$  \hfill (13)

Notice that in a Hilbert space,

$$\sigma(L_\lambda^*) = \{ \lambda \in \mathbb{C} : \lambda \in \sigma(L_\lambda) \}.$$  

Also for linear completely continuous field, the spectrum of $L_\lambda$ consists of eigenvalues. Hence if we write $L_\lambda z = \beta(\lambda) z$, with $z = x + iy$ and $\beta(\lambda) = \alpha(\lambda) + i\sigma(\lambda)$, then we have $L_\lambda^* z^* = \beta(\lambda) z^*$, with $z^* = x^* + iy^*$ and $\beta(\lambda) = \alpha(\lambda) - i\sigma(\lambda)$. To avoid using complex eigenfunctions, we then call the $x$ and $y$ as eigenfunctions for $L_\lambda$. Similarly $x^*$ and $y^*$ are eigenfunctions for $L_\lambda^*$.

For $e_1 = \sin x$, $e_2 = \cos x$, $e_1^* = \sin x$, and $e_2^* = \cos x$, we compute and obtain

$$L_\lambda e_1 = (\lambda - 1) \sin x + \nu \lambda \cos x = \alpha_1(\lambda) e_1 + \sigma_1(\lambda) e_2,$$  \hfill (14)

$$L_\lambda e_2 = (\lambda - 1) \cos x - \nu \lambda \sin x = -\sigma_1(\lambda) e_1 + \alpha_1(\lambda) e_2,$$  \hfill (15)

$$L_\lambda^* e_1^* = (\lambda - 1) \sin x - \nu \lambda \cos x = \alpha_1(\lambda) e_1^* - \sigma_1(\lambda) e_2^*,$$  \hfill (16)

$$L_\lambda^* e_2^* = (\lambda - 1) \cos x + \nu \lambda \sin x = \sigma_1(\lambda) e_1^* + \alpha_1(\lambda) e_2^*.$$  \hfill (17)

That is,

$$e_1 = \sin x, \quad e_2 = \cos x,$$  \hfill (18)

$$e_1^* = \sin x, \quad e_2^* = \cos x,$$  \hfill (19)

are eigenfunctions according to our definition and in general we have

$$e_{2n-1} = e_{2n-1}^* = \sin nx,$$  \hfill (20)

$$e_{2n} = e_{2n}^* = \cos nx, \forall n \in \mathbb{N}.$$  \hfill (21)

With these eigenfunctions, we re-index $\beta_n$ and write the eigenvalues as

$$\beta_{2n-1}(\lambda) = \alpha_n(\lambda) + i\sigma_n(\lambda),$$  \hfill (22)

$$\beta_{2n}(\lambda) = \alpha_n(\lambda) - i\sigma_n(\lambda),$$  \hfill (23)
where \( \alpha_n(\lambda) = n^2(\lambda - n^2) \) and \( \sigma_n(\lambda) = \nu \lambda n^3 \), such that

\[
L_\lambda e_{2n-1} = \alpha_n(\lambda)e_{2n-1} + \sigma_n(\lambda)e_{2n},
\]
\[
L_\lambda e_{2n} = -\sigma_n(\lambda)e_{2n-1} + \alpha_n(\lambda)e_{2n}.
\]

(24)

With these eigenvalues, we have (with \( \lambda_0 = 1 \)),

\[
\text{Re}\beta_1(\lambda) = \text{Re}\beta_2(\lambda) = \begin{cases} < 0 & \text{if } \lambda < 1, \\ = 0 & \text{if } \lambda = 1, \\ > 0 & \text{if } \lambda > 1, \end{cases} 
\]

\[
\text{Im}\beta_1(\lambda_0) = -\text{Im}\beta_2(\lambda_0) \neq 0,
\]
\[
\text{Re}\beta_j(\lambda_0) < 0, \forall j \geq 3,
\]

(25)

which is known as Principle of Exchange of Stability (PES).

3. Center manifold reduction. By the spectral theorem, see Theorem 3.4 [12], we have the following decomposition

\[
H = E_1^\lambda \oplus E_2^\lambda,
\]
\[
H_1 = E_1^\lambda \oplus E_2^\lambda,
\]
\[
E_1^\lambda = \text{span}\{e_1, e_2\} = \text{span}\{\sin x, \cos x\},
\]
\[
E_2^\lambda = \text{span}\{e_n : n \geq 3\}.
\]

(26)

Let \( u \in H \), we write

\[
u = x_1e_1 + x_2e_2 + y,
\]

(27)

where \( x_1 \in \mathbb{R}, x_2 \in \mathbb{R} \) and \( y \in E_2^\lambda \). We can then write

\[
\frac{du}{dt} = L_\lambda u + G(u, \lambda)
\]

as

\[
\frac{dx}{dt} = L_\lambda x + P_1G(x + y, \lambda),
\]

\[
\frac{dy}{dt} = L_\lambda y + P_2G(x + y, \lambda).
\]

(29)

(30)

Inner product of (29) with \( e_1 \) and \( e_2 \) give

\[
\frac{dx_1}{dt} = [\alpha_1(\lambda)x_1 - \sigma_1(\lambda)x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + y, \lambda)e_1 \, dx,
\]

(31)

\[
\frac{dx_2}{dt} = [\sigma_1(\lambda)x_1 + \alpha_1(\lambda)x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + y, \lambda)e_2 \, dx.
\]

(32)

By the center manifold theorem, there exist a function, \( \Phi(\cdot, \lambda) : B_\epsilon(0) \cap E_1^\lambda \to E_2^\lambda \), see [3], such that the bifurcation of (31), (32) and (30) is equivalent to

\[
\frac{dx_1}{dt} = [\alpha_1(\lambda)x_1 - \sigma_1(\lambda)x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + \Phi(x), \lambda)e_1 \, dx,
\]

(33)

\[
\frac{dx_2}{dt} = [\sigma_1(\lambda)x_1 + \alpha_1(\lambda)x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + \Phi(x), \lambda)e_2 \, dx.
\]

(34)

The key is to find a good approximation of the center manifold function \( \Phi \) so that (33) and (34) with \( \Phi \) replaced by the approximation provides a complete bifurcation information.
Note that the higher order term in this case is $G(u, \lambda) = -\lambda u \frac{\partial u}{\partial x}$. Let us define

$$G(u, v, \lambda) = -\lambda u \frac{\partial v}{\partial x},$$

then $G(u, v, \lambda)$ is bilinear as

$$G(\alpha u_1 + \beta u_2, v, \lambda) = \alpha G(u_1, v, \lambda) + \beta G(u_2, v, \lambda),$$
$$G(u, \alpha v_1 + \beta v_2, \lambda) = \alpha G(u, v_1, \lambda) + \beta G(u, v_2, \lambda).$$

The approximate center manifold function $\Phi$ can then be expressed as (See Theorem A.1.1. in [13])

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(3),$$

where

$$o(k) = o(||x||^k) + O\left(|\text{Re}\beta(\lambda)||x||^k\right).$$

Functions $\Phi_1, \Phi_2, \Phi_3$ are calculated by

$$-\mathcal{L}_\lambda \Phi_1 = x_2^2 G_{11} + x_2^2 G_{22} + x_1 x_2 (G_{12} + G_{21}),$$

$$\left[(-\mathcal{L}_\lambda)^2 + 4\sigma_1^2\right](-\mathcal{L}_\lambda) \Phi_2 = 2\sigma_1^2 \left[x_1^2 - x_2^2\right] (G_{22} - G_{11}) - 2x_1 x_2 (G_{12} + G_{21}),$$

$$\left[(-\mathcal{L}_\lambda)^2 + 4\sigma_1^2\right] \Phi_3 = \sigma_1 \left[(-\mathcal{L}_\lambda)^2 \left[x_2^2 - x_1^2\right] (G_{12} + G_{21}) + 2x_1 x_2 (G_{11} - G_{22})\right],$$

where $G_{ij} = P_2 G(e_i, e_j, \lambda)$ for $1 \leq i, j \leq 2$, and $\mathcal{L}_\lambda = L_\lambda|_{E_2}: E_2 \to E_2$. Let

$$\Phi_1(x_1, x_2) = \Phi_1 = \sum_{n=3}^{\infty} y_n(x_1, x_2)e_n = \sum_{n=3}^{\infty} y_n e_n,$$

$$\Phi_2(x_1, x_2) = \Phi_2 = \sum_{n=3}^{\infty} p_n(x_1, x_2)e_n = \sum_{n=3}^{\infty} p_n e_n,$$

$$\Phi_3(x_1, x_2) = \Phi_3 = \sum_{n=3}^{\infty} q_n(x_1, x_2)e_n = \sum_{n=3}^{\infty} q_n e_n,$$

together with,

$$\int_0^{2\pi} G(e_1, e_1, \lambda)e_{2n-1} dx = \begin{cases} -\frac{\lambda}{2}\pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\int_0^{2\pi} G(e_1, e_1, \lambda)e_{2n} dx = 0 \quad \text{for all } n \geq 2,$$

$$\int_0^{2\pi} G(e_1, e_2, \lambda)e_{2n-1} dx + \int_0^{2\pi} G(e_2, e_1, \lambda)e_{2n-1} dx = 0 \quad \text{for all } n \geq 2,$$

$$\int_0^{2\pi} G(e_1, e_2, \lambda)e_{2n} dx + \int_0^{2\pi} G(e_2, e_1, \lambda)e_{2n} dx = \begin{cases} -\lambda\pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\int_0^{2\pi} G(e_2, e_2, \lambda)e_{2n-1} dx = \begin{cases} \frac{\lambda}{2}\pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases},$$

$$\int_0^{2\pi} G(e_2, e_2, \lambda)e_{2n} dx = 0 \quad \text{for all } n \geq 2.$$
the coefficients \( y_n, p_n \) and \( q_n \) can be computed. Using (41)-(46), the coefficients \( y_n \) for \( \Phi_1(x_1, x_2) \) are

\[
y_{2n-1}(x_1, x_2) = \begin{cases} \frac{-\lambda}{\sigma_2^2(\lambda) + \sigma_2^2(\lambda)} \left[ \frac{1}{2} \sigma_2(\lambda) (x_2^2 - x_1^2) \right] - \sigma_2(\lambda) x_1 x_2 & \text{if } n = 2 \\ 0 & \text{if } n \geq 3, \end{cases}
\]

and

\[
y_{2n}(x_1, x_2) = \begin{cases} \frac{-\lambda}{\sigma_2^2(\lambda) + \sigma_2^2(\lambda)} \left[ \frac{1}{2} \sigma_2(\lambda) (x_2^2 - x_1^2) + \sigma_2(\lambda) x_1 x_2 \right] & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}
\]

Hence the nonzero terms for \( \Phi_1(x_1, x_2) \) are \( y_3(x_1, x_2) \) and \( y_4(x_1, x_2) \). Similarly, using (41)-(46), the coefficients \( p_n(x_1, x_2) \) for \( \Phi_2(x_1, x_2) \) are

\[
p_{2n-1}(x_1, x_2) = \begin{cases} \frac{2}{A_{12} + B_{12}} \lambda \sigma_2^2(\lambda) \left[ A_{12}(x_1^2 - x_2^2) - 2B_{12} x_1 x_2 \right] & \text{if } n = 2 \\ 0 & \text{if } n \geq 3, \end{cases}
\]

and

\[
p_{2n}(x_1, x_2) = \begin{cases} \frac{2}{A_{12} + B_{12}} \lambda \sigma_2^2(\lambda) \left[ B_{12}(x_1^2 - x_2^2) + 2A_{12} x_1 x_2 \right] & \text{if } n = 2 \\ 0 & \text{if } n \geq 3. \end{cases}
\]

Hence the nonzero terms for \( \Phi_2(x_1, x_2) \) are \( p_3(x_1, x_2) \) and \( p_4(x_1, x_2) \). To obtain the coefficients \( q_n \) for \( \Phi_3(x_1, x_2) \), the following need to be computed.

\[
\int_0^{2\pi} e_3 e_{2n-1} \, dx = \begin{cases} \pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}
\]

\[
\int_0^{2\pi} e_3 e_{2n} \, dx = 0 \quad \text{for all } n \geq 2,
\]

\[
\int_0^{2\pi} e_4 e_{2n-1} \, dx = 0 \quad \text{for all } n \geq 2,
\]

\[
\int_0^{2\pi} e_4 e_{2n} \, dx = \begin{cases} \pi & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}
\]

Using (51)-(54), the coefficients \( q_n \) for \( \Phi_3(x_1, x_2) \) are

\[
q_{2n-1}(x_1, x_2) = \frac{\lambda \sigma_1(\lambda)}{A_{22}^2 + B_{22}^2} \{(x_2^2 - x_1^2)[B_{22} \alpha_2(\lambda) - A_{22} \sigma_2(\lambda)] + 2x_1 x_2[A_{22} \alpha_2(\lambda) + B_{22} \sigma_2(\lambda)]\},
\]

if \( n = 2 \) and zero for \( n \geq 3 \), and

\[
q_{2n}(x_1, x_2) = \frac{\lambda \sigma_1(\lambda)}{A_{22}^2 + B_{22}^2} \{(x_2^2 - x_1^2)[A_{22} \alpha_2(\lambda) + B_{22} \sigma_2(\lambda)] + 2x_1 x_2[A_{22} \sigma_2(\lambda) - B_{22} \alpha_2(\lambda)]\},
\]

if \( n = 2 \) and zero for \( n \geq 3 \).

Hence the nonzero terms for \( \Phi_3(x_1, x_2) \) are \( q_3(x_1, x_2) \) and \( q_4(x_1, x_2) \).

Let

\[
(G(e_i, e_j, \lambda), e_k) = \int_0^{2\pi} -\lambda e_i \frac{\partial e_j}{\partial x} e_k \, dx,
\]

we then compute \((G(x_1 e_1 + x_2 e_2 + y, \lambda), e_i)\) for \( i = 1, 2 \).
With
\[ \int_0^{2\pi} -\lambda e_1 \frac{\partial e_1}{\partial x} e_1 \, dx = 0, \]  
(58)
\[ \int_0^{2\pi} -\lambda e_2 \frac{\partial e_2}{\partial x} e_1 \, dx = 0, \]  
(59)
\[ \int_0^{2\pi} -\lambda e_2 \frac{\partial e_1}{\partial x} e_1 \, dx = 0, \]  
(60)
\[ \int_0^{2\pi} -\lambda e_2 \frac{\partial e_2}{\partial x} e_1 \, dx = 0, \]  
(61)
we obtain
\[ (G(x_1 e_1 + x_2 e_2 + y, \lambda), e_1) \]
\[ = \int_0^{2\pi} -\lambda x_1 e_1 \frac{\partial y}{\partial x} e_1 \, dx + \int_0^{2\pi} -\lambda x_2 e_2 \frac{\partial y}{\partial x} e_1 \, dx \]
\[ + \int_0^{2\pi} -\lambda y x_1 \frac{\partial e_1}{\partial x} e_1 \, dx + \int_0^{2\pi} -\lambda y x_2 \frac{\partial e_2}{\partial x} e_1 \, dx + \int_0^{2\pi} -\lambda y \frac{\partial e_1}{\partial x} e_1 \, dx. \]  
(62)

Next, by writing \( y = \Phi(x_1, x_2) \) we have
\[ (G(e_i, \Phi, \lambda), e_j) = \int_0^{2\pi} -\lambda e_i \frac{\partial \Phi}{\partial x} e_j \, dx, \]  
(63)
\[ (G(\Phi, e_i, \lambda), e_j) = \int_0^{2\pi} -\lambda \frac{\partial e_i}{\partial x} e_j \, dx, \]  
for \( i = 1, 2 \).

To avoid fourth-order term in later calculations, we drop the last term \( \int_0^{2\pi} -\lambda y \frac{\partial e_1}{\partial x} e_1 \, dx \) in (62). We then obtain the approximate equation of (31) as
\[ \frac{dx_1}{dt} = [\sigma_1(\lambda) x_1 - \sigma_1(\lambda) x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + \Phi, \lambda) e_1 \, dx \]
\[ = \sigma_1(\lambda) x_1 - \sigma_1(\lambda) x_2 + \frac{x_1}{\pi} \left[ (G(e_1, \Phi, \lambda), e_1) + (G(\Phi, e_1, \lambda), e_1) \right] \]
\[ + \frac{x_2}{\pi} \left[ (G(e_2, \Phi, \lambda), e_1) + (G(\Phi, e_2, \lambda), e_1) \right]. \]  
(64)

Similarly, we have
\[ \frac{dx_2}{dt} = [\sigma_1(\lambda) x_1 + \sigma_1(\lambda) x_2] + \frac{1}{\pi} \int_0^{2\pi} G(x + \Phi, \lambda) e_2 \, dx \]
\[ = \sigma_1(\lambda) x_1 + \sigma_1(\lambda) x_2 + \frac{x_1}{\pi} \left[ (G(e_1, \Phi, \lambda), e_2) + (G(\Phi, e_1, \lambda), e_2) \right] \]
\[ + \frac{x_2}{\pi} \left[ (G(e_2, \Phi, \lambda), e_2) + (G(\Phi, e_2, \lambda), e_2) \right]. \]  
(65)

From (47)-(50) and (55)-(56), we see that \( \Phi \) can be written as
\[ \Phi(x_1, x_2) = \varphi_3(x_1, x_2) e_3 + \varphi_4(x_1, x_2) e_4, \]  
(66)
where
\[ \varphi_3(x_1, x_2) = y_3(x_1, x_2) + p_3(x_1, x_2) + q_3(x_1, x_2), \]
\[ \varphi_4(x_1, x_2) = y_4(x_1, x_2) + p_4(x_1, x_2) + q_4(x_1, x_2). \]  
(67)

With these notations, we have
\[ (G(e_1, \Phi, \lambda), e_1) = \varphi_3(G(e_1, e_3, \lambda), e_1) + \varphi_4(G(e_1, e_4, \lambda), e_1), \]  
(68)
For (65), we need the following information

\[ (G(\Phi, e_1, \lambda), e_1) = \varphi_3(G(e_3, e_1, \lambda), e_1) + \varphi_4(G(e_4, e_1, \lambda), e_1), \]  

\[ (G(\Phi, e_2, \lambda), e_1) = \varphi_3(G(e_2, e_2, \lambda), e_1) + \varphi_4(G(e_2, e_4, \lambda), e_1), \]  

\[ (G(\Phi, e_2, \lambda), e_1) = \varphi_3(G(e_3, e_2, \lambda), e_1) + \varphi_4(G(e_4, e_2, \lambda), e_1). \]  

Since

\[ (G(e_1, e_3, \lambda), e_1) = \lambda \pi, \]  

\[ (G(e_1, e_4, \lambda), e_1) = 0, \]  

\[ (G(e_4, e_1, \lambda), e_1) = 0, \]  

\[ (G(e_2, e_3, \lambda), e_1) = 0, \]  

\[ (G(e_2, e_4, \lambda), e_1) = \pi \lambda, \]  

\[ (G(e_4, e_2, \lambda), e_1) = -\frac{\lambda}{2} \pi, \]  

we have from (72)-(75) that

\[ (G(e_1, \Phi, \lambda), e_1) = \lambda \varphi_3 \pi, \]  

\[ (G(\Phi, e_1, \lambda), e_1) = -\frac{\lambda \varphi_3}{2} \pi, \]  

\[ (G(e_2, \Phi, \lambda), e_1) = \lambda \varphi_4 \pi, \]  

\[ (G(\Phi, e_2, \lambda), e_1) = -\frac{\lambda \varphi_4}{2} \pi. \]  

Putting all these together, (64) can be written as

\[
\frac{dx_1}{dt} = \alpha_1(\lambda) x_1 - \sigma_1(\lambda) x_2
\]

\[
+ \left[ \frac{\lambda^2 \alpha_2(\lambda)}{4 \alpha^2_2(\lambda) + \sigma^2_2(\lambda)} + \frac{\sigma^2_1(\lambda) \lambda^2 A_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (A_{22} \sigma_2(\lambda) - B_{22} \alpha_2(\lambda))}{2 [A_{22}^2 + B_{22}^2]} \right] x_1^2
\]

\[
+ \left[ \frac{\lambda^2 \alpha_2(\lambda)}{4 \alpha^2_2(\lambda) + \sigma^2_2(\lambda)} - \frac{\sigma^2_1(\lambda) \lambda^2 B_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (A_{22} \sigma_2(\lambda) + B_{22} \alpha_2(\lambda))}{2 [A_{22}^2 + B_{22}^2]} \right] x_1 x_2
\]

\[
+ \left[ \frac{\lambda^2 \sigma_2(\lambda)}{4 \alpha^2_2(\lambda) + \sigma^2_2(\lambda)} - \frac{\sigma^2_1(\lambda) \lambda^2 A_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (A_{22} \sigma_2(\lambda) + B_{22} \alpha_2(\lambda))}{2 [A_{22}^2 + B_{22}^2]} \right] x_2^2
\]

\[+ o(|x|^3) + O(\text{Re} \beta(\lambda)||x||^3).\]

For (65), we need the following information

\[ (G(e_1, e_3, \lambda), e_2) = 0, \]  

\[ (G(e_1, e_4, \lambda), e_2) = \pi \lambda, \]  

\[ (G(e_3, e_1, \lambda), e_2) = 0, \]  

\[ (G(e_4, e_1, \lambda), e_2) = -\frac{\lambda}{2} \pi, \]  

\[ (G(e_2, e_3, \lambda), e_2) = -\pi \lambda, \]  

\[ (G(e_2, e_4, \lambda), e_2) = 0, \]  

\[ (G(e_3, e_2, \lambda), e_2) = \frac{\lambda}{2} \pi, \]  

\[ (G(e_4, e_2, \lambda), e_2) = 0. \]  

Hence, from (79)-(82), we have

\[ (G(e_1, \Phi, \lambda), e_2) = \lambda \varphi_4 \pi, \]  

\[ (G(\Phi, e_1, \lambda), e_2) = -\frac{\lambda \varphi_4}{2} \pi, \]  

\[ (G(e_2, \Phi, \lambda), e_2) = -\lambda \varphi_3 \pi, \]  

\[ (G(\Phi, e_2, \lambda), e_2) = \frac{\lambda \varphi_3}{2} \pi. \]  

Therefore, from

\[ (G(e_1, \Phi, \lambda), e_2) = \varphi_3(G(e_1, e_3, \lambda), e_2) + \varphi_4(G(e_1, e_4, \lambda), e_2), \]  

\[ (G(\Phi, e_1, \lambda), e_2) = \varphi_3(G(e_3, e_1, \lambda), e_2) + \varphi_4(G(e_4, e_1, \lambda), e_2), \]
From calculations, we have

\[ (G(e_2, \Phi, \lambda), e_2) = \varphi_3(G(e_2, e_3, \lambda), e_2) + \varphi_4(G(e_2, e_4, \lambda), e_2), \quad (87) \]

\[ (G(\Phi, e_2, \lambda), e_2) = \varphi_3(G(e_3, e_2, \lambda), e_2) + \varphi_4(G(e_4, e_2, \lambda), e_2), \quad (88) \]

we obtain the second reduced equation,

\[
\frac{dx_2}{dt} = \sigma_1(\lambda)x_1 + \alpha_1(\lambda)x_2 
\]

\[
+ \left[ \frac{-\lambda^2 \sigma_2(\lambda)}{4(a_2^2(\lambda) + \sigma_2^2(\lambda))} + \frac{\sigma_1^2(\lambda) \lambda^2 B_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (-A_{22} \alpha_3(\lambda) - B_{22} \sigma_2(\lambda))}{2[A_{22}^2 + B_{22}^2]} \right] x_1^3
\]

\[
+ \left[ \frac{\lambda^2 \sigma_2(\lambda)}{4(a_2^2(\lambda) + \sigma_2^2(\lambda))} + \frac{\sigma_1^2(\lambda) \lambda^2 A_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (-A_{22} \alpha_2(\lambda) - B_{22} \sigma_2(\lambda))}{2[A_{22}^2 + B_{22}^2]} \right] x_1^2 x_2
\]

\[
+ \left[ -\frac{\lambda^2 \sigma_2(\lambda)}{4(a_2^2(\lambda) + \sigma_2^2(\lambda))} + \frac{\sigma_1^2(\lambda) \lambda^2 B_{12}}{A_{12}^2 + B_{12}^2} + \frac{\sigma_1(\lambda) \lambda^2 (-A_{22} \alpha_2(\lambda) - B_{22} \sigma_2(\lambda))}{2[A_{22}^2 + B_{22}^2]} \right] x_2^3
\]

\[ + o(||x||^3) + O((|\text{Re}(\beta(\lambda)||x||^3)) . \]

4. Proof of the main theorem. We now give the proof of Theorem 1.1. The proof is essentially an application of Theorem 2.3.7 in [13].

**Proof.** Let \( \Phi(x, \lambda) \) be the center manifold function of (8) near \( \lambda = \lambda_0 \) and \( e_i = e_i(\lambda_0) \) for \( i = 1, 2 \). Also assume that

\[ (G(x + \Phi(x, \lambda_0), \lambda_0), e_i) = \sum_{2 \leq p + q \leq 3} a_{pq}^i x_1^p x_2^q + o(||x||^3), \quad i = 1, 2. \quad (90) \]

For (90) we introduce a number, which is called the bifurcation number and is given by

\[
b = \frac{3\pi}{4} (a_{30}^1 + a_{53}^1) + \frac{\pi}{4} (a_{12}^1 + a_{21}^1) + \frac{\pi}{2\sigma_1} (a_{10}^1 a_{02}^1 - a_{12}^0 a_{20}^1)
\]

\[+ \frac{\pi}{4\sigma_1} (a_{11}^1 a_{20}^1 + a_{10}^1 a_{21}^1 - a_{12}^0 a_{02}^1 - a_{11}^0 a_{20}^1). \quad (91) \]

Notice that in our case, we do not have second order terms, hence

\[ b = \frac{3\pi}{4} (a_{30}^1 + a_{53}^1) + \frac{\pi}{4} (a_{12}^1 + a_{21}^1). \quad (92) \]

From calculations, we have

\[
a_{30}^1 = a_{12}^1 = \pi \left\{ \frac{-3}{144 + 64\nu^2} + \frac{\nu^2(1728 - 2256\nu^2)}{(1728 - 2256\nu^2)^2 + (3456 - 480\nu^2)^2} \right. \]

\[
+ \frac{\nu(-576\nu - 240\nu^2)}{(144 - 60\nu^2)^2 + 36864\nu^2} \right\}, \quad (93) \]

\[
a_{21}^1 = a_{03}^1 = \pi \left\{ \frac{2\nu}{144 + 64\nu^2} - \frac{\nu^2(3456 - 480\nu^3)}{(144 - 60\nu^2)^2 + (3456 - 480\nu^2)^2} \right. \]

\[
+ \frac{\nu(-864 - 408\nu^2)}{(144 - 60\nu^2)^2 + 36864\nu^2} \right\}, \quad (94) \]

\[
a_{30}^2 = a_{12}^2 = \pi \left\{ \frac{-2\nu}{144 + 64\nu^2} + \frac{\nu^2(3456 - 480\nu^3)}{(144 - 60\nu^2)^2 + (3456 - 480\nu^2)^2} \right. \]

\[
+ \frac{\nu(864 + 480\nu^2)}{(144 - 60\nu^2)^2 + 36864\nu^2} \right\}. \quad (95) \]
\begin{equation}
 a_{21}^2 = a_{03}^2 = \pi \left\{ -\frac{3}{144 + 64\nu^2} + \frac{\nu^2 (1728 - 2256\nu^2)}{(1728 - 2256\nu^2)^2 + (3456 - 480\nu^3)^2} 
 + \frac{\nu (-576\nu - 240\nu^2)}{(144 - 60\nu^2)^2 + 36864\nu^2} \right\} \cdot 
 \end{equation}

From the above calculations, we notice that
\begin{equation}
 a_{30}^1 = a_{12}^1 = a_{21}^2 = a_{03}^2, 
 a_{21}^1 = a_{03}^1 = -a_{30}^2 = -a_{12}^2, 
 \end{equation}

therefore
\begin{equation}
 b = \frac{3\pi}{4} \left( a_{30}^1 + a_{03}^2 \right) + \frac{\pi}{4} \left( a_{12}^1 + a_{21}^2 \right) 
 = \frac{3\pi}{4} \left( 2a_{30}^1 \right) + \frac{\pi}{4} \left( 2a_{03}^2 \right) 
 = 2\pi a_{30}^1 
 \end{equation}

Suppose we write
\begin{equation}
 b(\nu) = 2\pi^2 \zeta(\nu). 
 \end{equation}

Numerical methods show that, \( \zeta(\nu^*) = 0 \) for \( \nu^* \approx -3.36217 \) and
\begin{equation}
 \zeta(\nu) \text{ is positive for } \nu \in (-\infty, \nu^*), 
 \end{equation}
\begin{equation}
 \zeta(\nu) \text{ is negative for } \nu \in (\nu^*, \infty). 
 \end{equation}

Hence \( b(\nu) \) is positive for \( \nu \in (-\infty, \nu^*) \) and is negative for \( \nu \in (\nu^*, \infty) \). The result then follows from Theorem 2.3.7 [13], and proof of the theorem is complete. \( \square \)

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