Pattern formation in Laplacian growth: Theory

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Abstract

A first-principles statistical theory is constructed for the evolution of two dimensional interfaces in Laplacian fields. The aim is to predict the pattern that the growth evolves into, whether it becomes fractal and if so the characteristics of the fractal pattern. Using a time dependent map the growing region is conformally mapped onto the unit disk and the problem is converted to the dynamics of a many-body system. The evolution is argued to be Hamiltonian, and the Hamiltonian is shown to be the conjugate function of the real potential field. Without surface effects the problem is ill-posed, but the Hamiltonian structure of the dynamics allows introduction of surface effects as a repulsive potential between the particles and the interface. This further leads to a field representation of the problem, where the field’s vacuum harbours the zeros and the poles of the conformal map as particles and antiparticles. These can be excited from the vacuum either by fluctuations or by surface effects. Creation and annihilation of particles is shown to be consistent with the formalism and lead to tip-splitting and side-branching. The Hamiltonian further allows to make use of statistical mechanical tools to analyse the statistics of the many-body system. I outline the way to convert the distribution of the particles into the morphology of the interface. In particular, I relate the particles statistics to both the distributions of the curvature and the growth probability along the physical interface. If the pattern turns fractal the latter distribution gives rise to a multifractal spectrum, which can be explicitly calculated for a given particles distribution. A ‘dilute boundary layer approximation’ is discussed, which allows explicit calculations and shows emergence of an algebraically long tail in the curvature distribution which points to the onset of fractality in the evolving pattern.
1. Introduction

Notwithstanding the abundance of phenomenological knowledge the morphologies of interfaces that grow in Laplacian fields are poorly understood theoretically. In many cases these interfaces exhibit a rich variety of convoluted patterns. Known examples are diffusion-limited aggregation (DLA), solidification of supercooled liquid, electrodeposition and growth of bacterial colonies, to mention but a few. At present there is no sufficient theoretical understanding to allow for reliable predictions of the asymptotic patterns that such growths evolve into, starting from the basic equations of motion (EOM). Existing analyses are either of an effective medium type[1] or employ renormalisation group techniques assuming similarity solutions and limit distributions. Recently two of the latter generic approaches managed to yield quite accurate values for the scaling of the growth size (or the mass) with the linear size of the growth[2][3].

A different direction to treat this problem was suggested exactly half a century ago[4]. It was proposed that two-dimensional free interfaces (i.e., in the absence of surface tension) be analysed by conformally mapping the growing region onto the unit disk and studying the evolution of the map rather than that of the interface. This idea was followed by Richardson[5] who discovered that such a map enjoys a set of independent constants of motion. Shraiman and Bensimon[6] took this issue a step further showing explicitly how the problem can be converted to a many-body system and pointing out that the free interface evolution is mathematically ill-posed. They demonstrated that the formalism breaks down after a finite time due to instabilities with respect to growth of perturbations along the interface on ever shorter length scales. Without the curbing effect of surface tension, irregularities develop into cusp singularities along the physical interface. There have been theoretical efforts to counteract this catastrophic sharpening by using small surface tension to cut off the short length scales in a renormalisable manner[7]. These, however, met with another difficulty: such an ad-hoc inclusion turns out to constitute a singular perturbation to the EOM of the system. This, in turn, means that solutions based on this approach are strongly sensitive to initial conditions in the sense that very similar initial conditions can result in completely different morphologies[8]. Yet, numerous observations show that the final morphological properties are rather robust to the details of the growth and to changes in initial conditions. A different approach has been proposed recently to prevent formation of cusps by arguing that tip-splitting reduces high surface curvatures. Implementing this idea into the EOM of the equivalent many-body system results in production of particles[9], as will be elaborated on in this paper.

A significant question in this context, that has not been addressed much, is whether the system supports a Hamiltonian or a Lyapunov function[10][11]. This point is essential to a fundamental understanding of this highly nonequilibrium
growth process and therefore to the thrust of this paper, as will become clear below. Although, as formulated, the problem is known to support a set of conserved quantities[5][9][12], it is unclear whether these can assist in finding a Hamiltonian for the system.

In this paper I first introduce the EOM of the interface. I show that the interface follows Hamiltonian dynamics and that the Hamiltonian is directly related to the physical two dimensional potential. I then derive the EOM of the equivalent many-body system, whose particles are the zeros and poles of the map. Using the existence of a Hamiltonian formulation, surface effects can be described as a field that either repels particles from the surface or gives rise to particles production. Both approaches overcome the ill-posedness of the problem and extend its range of validity to infinite time. The Hamiltonian also allows to analyse the statistical mechanics of the many-body system and in particular the spatial distribution of the particles inside the unit disk. I show how to extract information on the morphology of the interface from the distribution of poles and zeros. This includes a calculation of the moments of the growth probability distribution along the interface and the distribution of the curvature. I demonstrate the calculation for the case when the particles form a dilute gas near the unit circle and show that the distribution of the curvature along the interface develops an algebraic tail, implying non-negligible probabilities of formation of particularly sharp protrusions. This tail points to the onset of fractality and self-similarity, a feature that is not assumed a-priori. The third moment of the growth probability along the interface is calculated explicitly in terms of the distribution of the particles. This moment gives the fractal dimension of the pattern.

2. The basic problem and the interface’s EOM

The fundamental problem of Laplacian growth can be formulated as follows. Consider a Jordan curve, $\gamma(s)$, embedded in two dimensions which represents the physical interface. This curve is parametrised by $0 \leq s < 2\pi$, and is fixed at a given value of the potential field (electrostatic potential for electrodeposition or concentration for diffusion controlled processes). A higher potential is assigned to a circular boundary whose radius, $R$, is very large compared to the growth size. The potential field, $\Phi$, outside the area that is enclosed by $\gamma$ satisfies Laplace’s Eq.

$$\nabla^2 \Phi = 0 \ .$$ (2.1)

The interface is assumed to grow at a rate that is proportional to the local gradient of the field which is normal to the interface[13]

$$v_n = -\nabla \Phi \cdot \hat{n} \ .$$ (2.2)
This rate is assumed to be sufficiently slow so that at any time $t$ the Laplacian field can be considered to be static. Denoting by $\zeta = x + iy$ the physical (complex) plane, we now conformally map at each instant of time, $t$, the curve onto the unit circle (UC) in a mathematical $z$ plane via $\zeta = F(z,t)$. The time-dependent interface is recovered from the map by $\gamma(s,t) = \lim_{z \to e^{is}} F(z,t)$. The field gradient along the curve is $-\nabla \Phi(\zeta) = -\left[ \frac{\partial \Phi(\zeta)}{\partial \zeta} \right]^* = -i/(zF')^*$, where * stands for complex conjugate and the prime indicates derivative with respect to $z$. Using the fact that $z = e^{is} = 1/z^*$ on the interface, Shraiman and Bensimon[6] derived the EOM for $\gamma$:

$$\partial_t \gamma(s,t) = -i \partial_s \gamma(s,t) \left[ |\partial_s \gamma(s,t)|^{-2} + ig(s) \right].$$ (2.3)

The first term within the square brackets on the r.h.s. of (2.3) represents the normal growth rate as constituted by relation (2.2). The second term, however, is added by hand (but it is uniquely determined) and represents a tangential velocity, or ‘sliding’, of a point along the interface. Denoting the entire square brackets on the r.h.s. of Eq. (2.3) as the limit of an analytic function $G(z,t)$ (whose explicit form is unique and explicitly determinable - see below), the EOM for $F(z,t)$ becomes

$$\dot{F} = zF'G.$$ (2.4)

For reasons to become clear below let us also write down the logarithmic derivative of this equation with respect to $z$:

$$\frac{d}{dt} \ln F' = \frac{1}{F'} \frac{d}{dz} (zF'G).$$ (2.5)

The map needs in general to satisfy several constraints[9]: First, we want the topology of the boundary far away to remain unchanged under the map, which enforces $\lim_{z \to \infty} F \sim z$. Second, the map must have no branch cuts. These two conditions are satisfied by the following general form

$$F' = A(t) \prod_{n=1}^{N} \frac{z - Z_n(t)}{z - P_n(t)},$$ (2.6)

where the time dependence appears in the scaling factor, $A$, and the locations of the zeros and poles of $F'$. The first constraint imposes ‘charge neutrality’, i.e., the number of poles should equal the number of zeros. The second constraint imposes a ‘dipolar neutrality’, namely, $\sum_n Z_n = \sum_n P_n$. It is possible to show that this form, and with a proper choice of the number $N$, enables to describe any initial simply connected cuspless curve that is allowed by this process. Therefore, this form is quite general and not merely a small class of maps[14]. Integrating (2.6) we have

$$\zeta = F(z,t) = A(t) \left[ 1 + \sum_{n=1}^{N} R_n \ln(z - P_n) \right].$$ (2.7)
The quantities $R_n$ are the residues of the product in (2.6) at $P_n$ (without the prefactor $A$), and

$$Q_n = 2 \prod_{m=1}^{N} \frac{(1/Z_n - P_m^*)(Z_n - P_m)}{(1/Z_n - Z_m^*)(Z_n - Z_{m'})} \quad m' \neq n$$

$$G = G_0 + \sum_{n=1}^{N} \frac{Q_n}{z - Z_n}$$

$$G_0 = \sum_{m=1}^{N} \frac{Q_m}{2Z_m} + \prod_{m=1}^{N} \frac{P_m}{Z_m}.$$  

By contour-integrating around the location of the poles and zeros in (2.4) and (2.5) one obtains their EOM:

$$-A^2(t) \dot{Z}_n = Z_n \left\{ G_0 + \sum_{m'} \frac{Q_n + Q_{m'}}{Z_n - Z_{m'}} \right\} + Q_n \left\{ 1 - \sum_{m} \frac{Z_n}{Z_n - P_m} \right\} \equiv f^{(Z)}_n(\{Z\}; \{P\})$$

$$-A^2(t) \dot{P}_n = P_n \left\{ G_0 + \sum_{m} \frac{Q_m}{P_n - Z_m} \right\} \equiv f^{(P)}_n(\{Z\}; \{P\}).$$  

(2.8)

The behaviour of the particles under these these equations was discussed in detail by Blumenfeld and Ball[9]and will not be repeated here. From (2.4), and using the requirement that there are no branch cuts, one finds that the quantities $A(t)R_n(t)$ are constants of the motion that are independent of each other and which are determined only by initial conditions. These constants are directly related to those found by Richardson[5]and mineev[12]. The time evolution of $A(t)$ is straightforwardly found from (2.4)

$$\dot{A}(t) = A(t)G_0.$$  

(2.9)

This relation shows how this prefactor depends on the distribution of the particles inside the unit disk. This rescaling factor is related directly to the fractal dimension of the pattern, if it becomes fractal, as will be discussed in section 5.

3. The Hamiltonian structure

Whenever a set of dynamical equations appears on the scene the first pertinent question is whether it supports a Lyapunov function or a Hamiltonian. If it does this gives access to a powerful bag of tools. I argue that the present process is indeed Hamiltonian and further relate it to the actual field $\Phi$. The first hint that the process is Hamiltonian comes from the physical EOM (2.3), when only normal growth is considered:

$$\partial_t \gamma(s, t) = -i \partial_s \gamma(s, t) \frac{1}{|\partial_s \gamma(s, t)|^2} = -i \frac{\delta_s}{\delta \gamma^*}.$$  

(3.1)
This expression and its complex conjugate are equivalent to Hamilton’s equations. This relation suggests that the (rescaled) length of the actual growth, \( s \), may play a role of a Lyapunov functional, while \( \gamma \) plays the role of a field, whose real and imaginary parts are the canonical variables. Although Eq. (3.1) is not formally correct this appealing interpretation begs the question whether the EOM for the map, \( F \), also supports such a structure. It is not difficult to show[15] that this is indeed so: Multiply the complex conjugate of (2.3) by \( \partial_s \gamma \) and rewrite its imaginary part in terms of the map[4]

\[
\text{Im} \left\{ F_\ast F_\ast \right\} = 1.
\]

Writing \( F = \omega + i\chi \), with \( \omega \) and \( \chi \) real functions, this equation asserts that the Jacobian of the transformation from the coordinates \( t - s \) to the coordinates \( \omega - \chi \) is unity, \( \partial (\omega, \chi) / \partial (t, s) = 1 \). It is then straightforward to show that when \( z \to e^{is} \)

\[
\dot{\omega} = \frac{\partial s}{\partial \chi}; \quad \dot{\chi} = -\frac{\partial s}{\partial \omega} \quad \text{and} \quad \dot{F} = -i \frac{\partial s}{\partial F_\ast}.
\]  

(3.2)

Since \( s = \text{Im}\{\ln z\} \) Eq. (3.2) generalises to

\[
\dot{F} = -i \frac{\partial}{\partial F_\ast} \text{Im}\{\ln z\} = -i \frac{\partial}{\partial F_\ast} \text{Im}\{\Phi\},
\]  

(3.3)

where \( \Phi \) is the complex potential in the physical plane. Thus the trajectories of the system follow the stream lines, which are the conjugate of the equipotential lines in the physical plane.

Yet another way to obtain a similar structure is as follows: Define the complex function \( \Psi \equiv F' + iz \). By manipulating Eq. (2.5)[16] one obtains

\[
\frac{\partial \Psi}{\partial t} = -i \frac{\partial H_0}{\partial \Psi^*},
\]  

(3.4)

where \( H_0 \equiv zF'G \), which appears on the r.h.s. of (2.5) is complex and therefore does not enjoy a convenient translation into Hamilton’s equations.

The main point regarding the above arguments is not as much the exact form of the Hamiltonian but rather that the p.d.e. that governs the interface’s evolution indeed follows Hamiltonian dynamics. In other words, given an initial value of the Hamiltonian (the ‘energy’) the system then follows a trajectory that keeps this value constant. Since the many-body formulation is an equivalent description of the growth process it follows that the system of poles and zeros must also keep this quantity constant and hence the latter also supports a Hamiltonian structure. To find the many-body Hamiltonian one inverts relation (2.7) to express \( \ln z \) in terms of \( F \) and then \( \text{Im}\{\ln z\} = F(F, F^*) \). I will not pursue this direction further here but rather argue that the very existence of a Hamiltonian already paves the way to much progress.
4. Surface effects

Turning to consider surface effects, it has already been mentioned that without surface tension (or capillary forces) cusps form along the interface due to instability of small corrugations[17]. But real growth processes clearly do not admit cusps. In the cases that concern us here this is because the system has to expend a macroscopic surface energy as the curvature increases. What does the procedure of cusp formation correspond to in the many-body system? A local protrusion (the incipient cusp) is caused, in the many-body system, by a zero approaching the UC. Thus prohibition of high curvatures naturally corresponds to keeping that zero from approaching the UC too closely and hence to an effective repulsive potential between the particles and the interface defined by the UC. It should be stressed that only the existence of a Hamiltonian makes it possible to use the term ‘repulsive potential’ with any proper meaning.

There are different ways to incorporate this idea into the theory: One is by introducing a surface potential term in the many-body Hamiltonian[16]. Although this may sound somewhat difficult since we don’t know the exact Hamiltonian of the many-body system, one can nevertheless insert such a term in the p.d.e. \( H = H_0 + V \), and derive the modified EOM for the particles. The choice of the surface potential term determines the nature of the growth to a large extent. The stronger the repulsion, the smoother the resulting interfaces. An example of a possible simple repulsive potential term is

\[
V = \sigma \lim_{z \to \epsilon \imath s} \ln [K(\{Z\}, \{P\})] ,
\]

where \( K \) is the (complex) curvature in terms of the locations of the zeros and the poles[9]

\[
K(s, \{Z\}, \{P\}) = \lim_{z \to \epsilon \imath s} |F'|^{-1} \left\{ 1 + \sum_{n=1}^{N} \left\{ \frac{Z_n}{z - Z_n} - \frac{P_n}{z - P_n} \right\} \right\} ,
\]

whose real part is the physical curvature. This particular surface potential is simple in that it contributes a constant repelling term in the EOM of the particles, (2.8). It has recently been shown that a term that diverges as a particle approaches the UC would do better to describe the physics[16].

Another approach is to make a deeper use of the fact that the particles move in fact in a field. The field can have a vacuum that can accommodate particles and antiparticles (zeros and poles). ‘Exciting’ the vacuum (say, by fluctuations) can then effect creation and annihilation of zeros and poles, a mechanism that allows for tip-splitting and side-branching[9]. But before considering such a farfetched interpretation we need to convince ourselves that such a picture is consistent with
the present formalism and that it does not contradict any of the basic premises. To this end consider again the derivative of the map (Eq. (2.6)) and rewrite it at the initial moment (say, $t_0$) in the, seemingly redundant, form

$$F'(z,t_0) = A(t_0) \prod_{n=1}^{N} \frac{z - Z_n(t_0)}{z - P_n(t_0)} \prod_{k=1}^{\infty} \frac{z - \Gamma_k(t_0)}{z - \Gamma_k(t_0)}.$$  (4.3)

The second product is unity at $t_0$ and at any time thereafter under the free interface evolution because all its terms simply cancel out. It is straightforward to see that, under the free-growth EOM, at any time $t > t_0$ $F'$ will retain this form with only the position of the particles in the first product and the value of $A$ changing. Consider now what happens when a zero, $Z_i$, and a pole, $P_j$, collide. At the instant of collision the particles occupy the same location and therefore their corresponding terms, $(z - Z_i)/(z - P_j)$, cancel out in the first product on the r.h.s. of (4.3). This is exactly an annihilation event. Such an event conserves the balance between the numbers of zeros and poles, so that charge neutrality is not violated. It also does not violate the dipolar neutrality, as can be immediately verified. Turning to production events, the second product on the r.h.s. of (4.3) can now be interpreted as a ‘vacuum’ of pairs of zeros and poles that do not manifest unless they are ‘excited’. One can envisage two routes for this to occur: i) A fluctuation, of whatever origin, can virtually separate such a pair; ii) A particle that moves with a high kinetic energy can knock the pair apart. Particles with high velocities are usually zeros that are close to the unit circle and therefore give rise to locally high curvatures.

So the form (2.6) is also naturally suited for creation. Immediately after a zero-pole pair has been excited their locations are very close and it is quite straightforward to verify from the EOM that such a close pair interact repulsively, pushing apart and hence maintaining their identities. Differently expressed, once created the particles are stable. Thus a pair, say at $\Gamma_j$, can be excited into two individual particles at $Z_j = \Gamma_j + \delta_1$ and $P_j = \Gamma_j + \delta_2$. Under production, as under annihilation, the basic constraints need to be satisfied. Namely, an excitation is in pairs for charge neutrality, and dipolar neutrality is satisfied by imposing a relation between the locations of the excited particles (I should mention that excitation by field fluctuations has to occur in quartets rather than in pairs due to the dipolar constraint). In a particular implementation of this idea Blumenfeld and Ball [9] proposed that excitation of a new pair is triggered by the proximity of a zero, say $Z_n$, to the unit circle (scenario ii above). Such proximity leads to a high curvature in front of $Z_n$ at $s = \arg\{Z_n\}$ and the new zero-pole pair is excited once the local curvature reaches a threshold value. To satisfy dipolar neutrality the new pole occupies the location of $Z_n$ prior to the production event, while the other two zeros are equidistantly and oppositely situated around the pole. It turns out that the orientation of the zeros is not limited by the constraints on the system and needs to be imposed following
another criterion. Blumenfeld and Ball introduced an energetic criterion along the following lines: Recognising that surface energy increases with increasing curvature, the location of the particles needs to minimise the local curvature at $s$. As it happens, this minimum corresponds to placing the two zeros in the azimuthal direction about the location of the newly born pole. In the physical plane, such an event constitutes exactly tip-splitting. Placing the zeros along a radial ray equidistantly from the pole maximises the local energy and corresponds to side-branching in the physical plane. Since the system would rather minimise its local energy, these results suggest that tip-splitting may have an energetic, rather than only stochastic, origin. This particular procedure is not universal and other systems may follow other criteria for particles production. It should be emphasised that it is quite plausible that fluctuations of the field superimose on this mechanism and stochastically induce tip-splitting and side-branching. Further studies in this direction are currently being carried out and will not be elaborated on here.

5. Noise and statistical analysis

The foregoing assumed mostly a deterministic evolution. It is known\cite{18} that the dynamical EOM of the p.d.e. leads to a chaotic growth in the sense that if one starts from very close initial conditions, one ends up very quickly with different deterministic structure. This implies an efficient spread of the solutions in phase space which, combined with the good fortune of having a Hamiltonian dynamics, ensures the existence of a Gibbs measure. Namely, one expects to be able to construct a partition function

$$Z = \int e^{-\beta H} \mathcal{D}R\ ,$$

(5.1)

where $\mathcal{D}R \equiv \prod_{n=1}^{N} d^2 Z_n d^2 P_n$ is a an infinitesimal volume in phase space and the Hamiltonian is that of the many-body system. The quantity $\beta$ reflects the ‘noise’ in the system and is obtained as a Lagrange multiplier by imposing an average ‘energy’ constraint on the distribution. Expectation values of quantities such as the moments of the curvature and moments of the field gradient, $|\nabla \Phi|$, can now be found via

$$\langle X \rangle \equiv E\{X\} = \frac{1}{Z} \int X \ e^{-\beta H} \mathcal{D}R\ .$$

(5.2)

Since, at present, the explicit form of $H$ is unknown, this approach is not easy to implement.

Alternatively, we can construct a master equation for the evolution of the distribution of the particles, $\mathcal{N}(\{Z\}, \{P\})$, using Liouville’s theorem

$$\frac{\partial \mathcal{N}}{\partial t} + \sum_{n=1}^{N} f_n^{(Z)} \frac{\partial \mathcal{N}}{\partial Z_n} + f_n^{(P)} \frac{\partial \mathcal{N}}{\partial P_n} = \Gamma\ ,$$

(5.3)
where $\Gamma$ represents collisions and noise. To use Liouville’s theorem let me confine myself here to systems with conserved number of singularities (no particles production). An extension to a nonconserved number of particles is not difficult once a self-consistent renormalisation is introduced and will be discussed elsewhere. Expecting a steady state distribution after rescaling the growth by $A(t)$ we can discard the explicit time derivative. A solution of this equation yields everything there is to know about the interface’s statistics. Unfortunately, as common in statistical mechanics, a general exact solution is impossible. Rather than trying to solve this equation in some approximation, let me demonstrate how such a solution can be converted into information about the physical interface.

First, the distribution of the curvature, $P_K$, can be derived from $N$ using the relation

$$P_K = \int N(\{Z\}, \{P\}) \delta \left\{ K_1 - \frac{1}{|F'|} \left[ 1 + \text{Re} \sum_{n=1}^{N} \left( \frac{1}{1-Z_n e^{-is}} - \frac{1}{1-P_n e^{-is}} \right) \right] \right\} d\mathcal{R} ,$$

(5.4)

where $K_1$ is the measurable curvature along the interface and $\delta$ denotes Dirac’s delta-function. More generally, the distribution of any morphology-related quantity, $X$, $P(X)$, that is expressible in terms of the locations of the particles can be found in this manner. As an explicit example of using the statistics, I now turn to calculate the values of the moments of the growth probability distribution along the interface. This calculation has been recently carried out by Blumenfeld and Ball[9]and is only briefly reviewed here. These moments play a central role in pattern formation and growth, mostly because they were shown to lead to an asymptotically stable multifractal function that is independent of initial conditions or details of the growth.

The probability that growth occurs at a point $s$ along the interface at some time $t$ is proportional to the local gradient of the field

$$p(s) = C_0 |\nabla \Phi(s, t)| = \lim_{z \to e^{is}} C_0 |F'(z, t)| ; \quad C_0 = 1/\oint_{\gamma(s,t)} |\nabla \Phi(l)| dl .$$

Therefore the quenched moments of this distribution are

$$M_q = \oint_{\gamma} |\nabla \Phi|^q dl = \frac{1}{2\pi i} \oint_{|z|=1} |F'(z)|^{1-q} \frac{dz}{z} .$$

(5.5)

Substituting from Eq. (2.6) we have

$$M_q = \frac{A(t)^{1-q}}{2\pi i} \oint \prod_{n=1}^{N} \left( \frac{z-Z_n}{z-P_n} \right)^{1-q} \prod_{n=1}^{N} \left( \frac{1-zZ_n^*}{1-zP_n^*} \right)^{1-q} \frac{dz}{z} .$$

(5.6)

The second product in the integrand, $\equiv J(z)^{(1-q)/2}$, is analytic within the unit disk while the first contains $N$ poles of order $(1-q)/2$. For $q > 1$ these poles are located
at the zeros of the map, while for $q < 1$ the poles of the integrand coincide with the
poles of the map. In both regimes a simple pole also exists at the origin. Thus it is
straightforward to evaluate this integral for odd values of $q$:

$$M_{q>1} = \frac{1}{A(t)^{2\nu}} \left\{ \left( \prod_{n=1}^{N} \frac{P_n}{Z_n} \right)^{\nu} + \frac{1}{\nu!} \sum_{n=1}^{N} \frac{d^{\nu-1}}{dz^{\nu-1}} \left[ \frac{(z - P_n)^{\nu}}{z^{\mu} - z^{\nu}} \sum_{k \neq n} \prod_{k \neq n} \left( \frac{z - P_n}{z - Z_k} \right)^{\nu} \right]_{z=Z_n} \right\}$$

(5.7)

where $\nu \equiv (q - 1)/2$ is a positive integer number. Other values of $q > 1$ can be ob-
tained either by taking into account explicitly the branch-cuts that are involved in the
calculations, or by interpolating between the integer moments. However, it is known
that the knowledge of all the odd moments of the probability density of a measure on
a finite support is sufficient to determine that probability density uniquely[19], and
therefore the present calculation suffices. Of particular interest is the third moment

$$M_3 = A(t)^{-2} \left[ \prod_{n=1}^{N} \frac{P_n}{Z_n} + \sum_{n=1}^{N} \frac{Q_n}{2Z_n} \right] = G_0/A^2(t) ,$$

(5.8)

where the last step makes use of the definition of $G_0$ in section 2. Using Eq. (2
we then obtain the exact relation between $M_3$ and $A(t)$,

$$M_3 = \dot{A}(t)/A^3(t) .$$

The third moment is directly related to the fractal dimension[20][21]by

$$D_f = \ln M_3 / \ln R_g ,$$

where $R_g$ is the linear size (the radius of gyration or the radius of the equivalent
circular capacitor) of the growth. $D_f$ can be evaluated once the distribution of the
particles is known. Moreover, if indeed the pattern turns fractal then this quantity
should asymptote to a pure number. This is another manifestation of the first-
principles nature of the present approach: It is the result that tells us whether the
morphology becomes fractal without having to assume such a solution from the out-
set.

Calculating for odd values of $q < 1$ leads to an expression similar to (5.7). Since
all these quantities depend on the locations of the particles one can find their distribu-
tion over many growth realisations via an integral similar to (5.4). An interesting
observation is the following: Comparing the expressions for the positive and negative
moments shows that these enjoy exactly the same form with the location of the zeros
interchanged with the locations of the poles. This, combined with measurements on
DLA that show a distinct difference between the negative and positive moments of
the growth probability distribution implies qualitatively different spatial distribution
of the two species of particles. A partial confirmation of this conclusion can be indeed
observed in numerical calculations where the trajectories of the poles and the zeros
display markedly different behaviours[9].
6. The dilute boundary layer approximation

To gain insight into some features of the morphology let me now discuss briefly the distribution of the curvatures using a dilute boundary layer approximation for the distribution of the particles. The motivation behind this approximation is as follows: From Eq. (4.2) one can observe that the curvature is dominated by the zeros that are closest to the UC. Namely, considering a zero, \( Z_n = (1 - \rho) e^{i s_n} \) with \( \rho \ll 1 \), the curvature in front of this zero is (see Eq. (4.2))

\[
K_1(s_n) \approx C/\rho^2 ,
\]

where, to a good approximation, \( C \) is independent of either \( \rho \) or the locations of the other particles. A similar consideration for a pole near the UC shows that the local curvature is regular in \( \rho \). We can therefore neglect the contribution of poles in the following approximation. Since these are the zeros with \( \rho \ll 1 \) that dominate the curvature we consider only the zeros within a ring \( 1 - \rho_c < |z| < 1 \) and assume that their density is dilute and isotropic (the isotropy assumption can in fact pertain only to a discrete number of global arms). It can be easily shown that this corresponds to analysing the exposed parts of the physical growth. By assumption then the curvature at \( s_n \) depends mainly on the radial location of \( Z_n \), whose distribution is \( \mathcal{P}_n(\rho) \). Thus

\[
\mathcal{P}_K = \mathcal{P}_n(\rho) d\rho/dK = \text{Const.} \, K_1^{-3/2} \mathcal{P}_n \left( (C/K_1)^{1/2} \right) .
\]

In the absence of a solution to the master equation (5.3) we have no information on \( \mathcal{P}_n \). Nevertheless, we can observe that even if this distribution is well behaved (i.e., all its relevant moments are finite) the distribution of \( K_1 \) exhibits algebraically decreasing tails for high curvatures. This immediately suggests the onset of fractality, which generically originate from such tails. It is important to note that nowhere along the construction of the theory did we assume fractality or self-similarity. Yet, an algebraic tail appears which can easily generate such a structure. The moments of \( K \) are dominated by the long tail and satisfy \( \mu_q \sim K_{max}^{q-1/2} \sim a^{1/2-q} \), where \( a \) is a small cutoff lengthscale. With growth the cutoff radius reduces by a factor of \( 1/A(t) \) and therefore \( \mu_q \sim A(t)^{q-1/2} \). It is quite plausible that \( \mathcal{P}_n \) also introduces algebraic tails, directly affecting the behaviour of \( \mathcal{P}_k \). A more accurate analysis is only possible once we have a solution to Eq. (5.3) and attempts in this direction are being carried out.

Before concluding this section I should mention that it is possible, using this approximation, to also evaluate the aforementioned moments of the growth probability, \( M_q[9] \). Such an evaluation shows that these moments depend strongly on the negative moments of the distribution of zeros’ distances from the UC. If indeed the
patterns becomes multifractal, such an evaluation should yield the left hand side of the multifractal spectrum (the regime governed by the exposed parts of the growth) directly.

7. Discussion and concluding remarks

To conclude, a first-principles theory for two dimensional Laplacian growth was described. The growth was shown to be Hamiltonian with trajectories of constant energy corresponding to the stream lines in the two-dimensional physical field. This was argued to indicate that the equivalent many-body system is also Hamiltonian. Although not discussed here, an extension of this formalism to a continuous density of zeros and poles is possible and has been carried out.[16] The resulting nonlocal dynamical equations, however, currently seem too complex for an analytical treatment. Surface energy was shown to lead to a field that acts on the particles in the mathematical plane. The field concept could be incorporated either as a repulsive potential between the particles and the unit circle or by giving rise to particles production. The latter corresponds to tip-splitting and side-branching. Thus this seems a natural mechanism to prevent cusp formation along the physical interface and connect surface effects to tip-splitting. I discussed the distribution of the zeros and poles inside the unit disk and related it explicitly to the morphology of the interface. The main advantage of this approach is that the nonequilibrium growth is describable in statistical mechanics formalism, the language of equilibrium phenomena. The distribution of the curvature was analysed and an exact calculation of the moments of the growth probability distribution along the interface was discussed. To address the fundamental issue of onset of scale-invariance and self-similarity, I used an approximation which showed that the curvature distribution develops an algebraic tail, which naturally gives rise to a fractal pattern. It is this author’s belief that this approach and its possible generalisations are very promising as a framework for constructing theories for the morphology of growth processes in other dimensionalities and in fields that satisfy equations other than Laplace’s.

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