Scalar curvatures in almost Hermitian geometry and some applications

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Abstract On an almost Hermitian manifold, there are two Hermitian scalar curvatures associated with a canonical Hermitian connection. In this paper, two explicit formulas on these two scalar curvatures are obtained in terms of the Riemannian scalar curvature, norms of the components of the covariant derivative of the fundamental 2-form with respect to the Levi-Civita connection, and the codifferential of the Lee form. Then we use them to get characterization results of the Kähler metric, the balanced metric, the locally conformal Kähler metric or the $k$-Gauduchon metric. As corollaries, we show partial results related to a problem given by Lejmi and Upmeier (2020) and a conjecture by Angella et al. (2018).

Keywords J-scalar curvature, canonical Hermitian connection, Hermitian scalar curvature, the first Chern form, balanced metric, $k$-Gauduchon metric

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1 Introduction

In Kähler geometry, the complex structure is parallel with respect to the Levi-Civita connection. Hence on a Kähler manifold, the complex geometry is very compatible with the underlying Riemannian geometry. Yau [43] proved that the Kodaira dimension of a compact Kähler manifold with positive total scalar curvature must be $-\infty$. For a Hermitian non-Kähler manifold, the complex structure is not parallel with respect to the Levi-Civita connection. One usually chooses the Chern connection instead of the Levi-Civita connection and hence uses the Chern scalar curvature. Chiose et al. [10] successfully extended Yau’s result to the non-Kähler case and showed that a Moishezon manifold is uniruled if and only if it admits a balanced metric of positive total Chern scalar curvature. Recently, Yang [42] proved that a compact complex manifold $M$ admits a Hermitian metric with positive (resp. negative) Chern scalar curvature if and only if its canonical (resp. anti-canonical) bundle is not pseudo-effective.

The purpose of this paper is to study various scalar curvatures on an almost Hermitian manifold. We first recall some definitions given by Gauduchon [18]. Assume that $(M, J, h)$ is an almost Hermitian manifold of real dimension $2n$. Let $F$ be the fundamental 2-form associated with $h$ and $J$, and $\nabla$ be

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Theorem 1.1. Let \( (M, J, h) \) be an almost Hermitian manifold of real dimension \( 2n \). Then

\[
\begin{align*}
s_1(t) &= \frac{s}{2} \left( \frac{5}{12} |(dF)^-|^2 + \frac{1}{16} |N^0|^2 + \frac{1}{4} |(dF)^0_+|^2 \right) \\
&\quad + \left( \frac{1}{4(n-1)} \cdot \frac{t-1}{2} \right) |\alpha_F|^2 + \frac{t-2}{2} \delta \alpha_F,
\end{align*}
\]

and

\[
\begin{align*}
s_2(t) &= \frac{s}{2} \left( \frac{1}{12} |(dF)^-|^2 + \frac{1}{32} |N^0|^2 - \frac{t^2-2t}{4} |(dF)^0_+|^2 \right) \\
&\quad - \left( \frac{t^2-2t}{4(n-1)} + \frac{(t+1)^2}{8} \right) |\alpha_F|^2 - \frac{t+1}{2} \delta \alpha_F.
\end{align*}
\]

Here, \((dF)^-\) is the sum of \((3,0)\) and \((0,3)\) components of \(dF\), \((dF)^0_+\) is the primitive part of \((dF)^+\) which is the sum of \((2,1)\) and \((1,2)\) components of \(dF\), \(N^0 = N - bN\) where \(N\) is the Nijenhuis tensor of \(J\) and \(bN\) is its skew-symmetric part, and \(\alpha_F = J\delta F\) where \(\delta\) is the codifferential operator, i.e., the formal adjoint of \(d\) in the metric \(h\).

Gray and Hervella [22] introduced that in a natural way, there are precisely sixteen classes of almost Hermitian manifolds of real dimension \(2n \geq 6\). In fact, the conditions of each class are defined by letting some elements of the set \(\{(dF)^-, N^0, (dF)^0_+, \alpha_F\}\) equal zero [18]. For example, \(K\) is the class of Kähler manifolds \((dF)^- = N^0 = (dF)^0_+ = \alpha_F = 0\), i.e., \(\nabla F = 0\); \(W_3\) is the class of Hermitian semi-Kähler (i.e., balanced [30]) manifolds \((dF)^- = N^0 = \alpha_F = 0\); \(W_4\) is the class of locally conformal Kähler manifolds \((dF)^- = N^0 = (dF)^0_+ = 0\); \(W_2 \oplus W_3\) is the class \((dF)^- = \alpha_F = 0\); \(W_3 \oplus W_4\) is the class of \((dF)^- = 0\); \(W_1 \oplus W_4\) is the class \(N^0 = (dF)^0_+ = 0\); \(W_2 \oplus W_3 \oplus W_4\) is the class \((dF)^- = 0\). These classes will be mentioned in the following.

We can now give some applications of Theorem 1.1.

Theorem 1.2. Let \( (M, J, h) \) be a compact almost Hermitian manifold of real dimension \( 2n \geq 6 \). If \((M, J, h) \in W_2 \oplus W_3 \oplus W_4\) and \(t \geq 1 - \frac{1}{2(n-1)}\), then

\[
\int_M (2s_1(t) - s)dv \geq 0.
\]

If \((M, J, h) \in W_1 \oplus W_4\) and \(t \leq 1 - \frac{1}{2(n-1)}\), then

\[
\int_M (2s_1(t) - s)dv \leq 0.
\]
In both cases, the equality holds if and only if $(M, J, h)$ is a Kähler manifold, or $t = 1 - \frac{1}{2(n-1)}$ and it is a locally conformal Kähler manifold.

Recently, Lejmi and Upmeier presented a problem in [28, Remark 3.3]: are there any compact almost Hermitian manifolds of real dimension $2n \geq 6$ with $2s_1(1) = s$ such that they are non-Kähler? Theorem 1.2 implies the following non-existence result.

**Corollary 1.3** (See Corollary 5.3). Let $(M, J, h)$ be a compact almost Hermitian manifold of real dimension $2n \geq 6$. If it belongs to the class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and satisfies $2s_1(1) = s$, then it is a Kähler manifold.

Very recently, Grama and Oliveira [19] used our Theorem 1.1 to calculate the Hermitian scalar curvatures on generalized flag manifolds and thus obtained examples of compact non-Kähler almost Hermitian manifolds with $2s_1(1) = s$.

**Theorem 1.4** (See Theorem 5.5). Let $(M, J, h)$ be a compact almost Hermitian manifold of real dimension $2n \geq 6$.

1. If $(M, J, h) \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ and $t \in \mathbb{R} \setminus (-3 - 2\sqrt{3}, -3 + 2\sqrt{3})$, then
   \[
   \int_M (s_1(t) - s_2(t))dv \geq 0.
   \]
   The equality holds if and only if $(M, J, h)$ is a Kähler manifold, or $t = 1$ and it is a balanced manifold.

2. If $(M, J, h) \in \mathcal{W}_1 \oplus \mathcal{W}_4$ and $t \in [-1, \frac{1}{3}]$, then
   \[
   \int_M (s_1(t) - s_2(t))dv \leq 0.
   \]
   The equality holds if and only if $(M, J, h)$ is a Kähler manifold.

Recently, the concept of “Kähler-like” was introduced and studied by Yang and Zheng first in [41] for the Levi-Civita connection and the Chern connection and later extended by Angella et al. [2] to other canonical Hermitian connections. As a corollary of Theorem 1.4(1), we give the partial result in [2, Conjecture 2].

**Corollary 1.5** (See Corollary 5.7). Let $(M, J, h)$ be a compact Hermitian manifold of real dimension $2n \geq 6$. If it is Kähler-like for a canonical Hermitian connection $\mathcal{D}^t$ when $t \in \mathbb{R} \setminus \{(-3 - 2\sqrt{3}, -3 + 2\sqrt{3}) \cup \{1\}\}$, then it is a Kähler manifold.

For $t = -1$, we have the following special result.

**Theorem 1.6** (See Theorem 5.8). Let $(M, J, h)$ be a compact Hermitian manifold of real dimension $2n \geq 6$. Then
\[
\int_M (s_1(-1) - s_2(-1))dv = \int_M (|dF|^2 - |\alpha_F|^2)dv.
\]
Moreover, if $h$ is a Gauduchon metric (i.e., $\delta\alpha_F = 0$) and $s_1(-1) = s_2(-1)$, then $h$ is also a $k$-Gauduchon metric [14] for $k = 1, 2, \ldots, n - 2$.

The rest of this paper is organized as follows. In Section 2, we first give a brief description of almost Hermitian geometry. Then the formula of $s - s_J$ on an almost Hermitian manifold is obtained. The explicit formulas of $s_1(0) - \frac{1}{3}s$ and $s_2(0) - \frac{1}{3}s$ are also obtained in Section 3. In Section 4, Theorem 1.1 is proved by using the structure equations of connections $\mathcal{D}^0$ and $\mathcal{D}^1$. In Section 5, some applications of Theorem 1.1 are discussed.

For convenience, we fix the index ranges $1 \leq i, j, k, \ldots \leq n$ and $1 \leq A, B, C, \ldots \leq 2n$. We use the Einstein summation convention, i.e., the repeated indices are summed over. The space of the smooth tangent vector fields on $M$ is denoted by $\Gamma(TM)$.

## 2 $J$-scalar curvature of an almost Hermitian manifold

In this section, we first give a brief description of almost Hermitian geometry. In particular, we review the decomposition of the covariant derivative of the fundamental 2-form with respect to the Levi-Civita
connection. We then introduce an important formula on the difference of the Riemannian scalar curvature and the J-scalar curvature. Some applications are also discussed.

Let \((M, J, h)\) be an almost Hermitian manifold of complex dimension \(n\). \(J\) is an almost complex structure which is compatible with the Riemannian metric \(h = \langle \cdot, \cdot \rangle\). The fundamental 2-form \(F\) associated with \(h\) and \(J\) is defined by \(F(X, Y) = h(JX, Y)\) for all \(X, Y \in \Gamma(TM)\) and the volume form of the metric \(h\) is \(dv = \frac{F^n}{n!}\).

Another important differential form of \((M, J, h)\) is the Lee form \(\alpha_F = J\delta F\), where \(\delta = - * d*\) is the codifferential operator, i.e., the formal adjoint of \(d\) in the metric \(h\). It can be determined by the formula

\[
dF = (dF)_0 + \frac{1}{n-1} \alpha_F \wedge F, \tag{2.1}
\]

where \((dF)_0\) is the primitive part of \(dF\). Denote by \((dF)^+\) the sum of (2, 1) and (1, 2) components of \(dF\) and by \((dF)^-\) the sum of (3, 0) and (0, 3) components of \(dF\). The formula (2.1) implies

\[
(dF)^+ = (dF)_0^+ + \frac{1}{n-1} \alpha_F \wedge F, \tag{2.2}
\]

where \((dF)_0^+\) is the primitive part of \((dF)^+\).

The Nijenhuis tensor \(N\) of \(J\) is a (1, 2)-type tensor defined by

\[
N(X, Y) = [X, Y] + J[X, Y] + J[X, JY] - [JX, JY] \tag{2.3}
\]

for all \(X, Y \in \Gamma(TM)\). It is well known that \(N \equiv 0\) if and only if \(J\) is integrable [31]. \(N\) can be viewed as a (0, 3)-tensor: \(N(X, Y, Z) = (X, N(Y, Z))\). Denote by \(bN\) its skew-symmetric part:

\[
bN(X, Y, Z) = \frac{1}{3} \left( N(X, Y, Z) + N(Y, Z, X) + N(Z, X, Y) \right).
\]

Define \(N^0 = N - bN\). Clearly, \(bN^0 = 0\).

Denote by \(\nabla\) the Levi-Civita connection of the metric \(h\). We have \(\langle \nabla_X F \rangle(Y, Z) = \langle (\nabla_X J)Y, Z \rangle\). We also use the notation \(\langle \nabla F \rangle(X, Y, Z) = \langle \nabla_X F \rangle(Y, Z)\). According to [18, Proposition 1], \(\nabla F\) can be decomposed as

\[
\langle \nabla F \rangle(Y, Z) = \frac{1}{3} (dF)^-(X, Y, Z) - \frac{1}{2} N^0(X, Y, Z) + \frac{1}{2} (dF)^+(X, Y, Z) - \frac{1}{2} (dF)^+(X, JY, JZ) \tag{2.4}
\]

for all the vector fields \(X, Y\) and \(Z\).

The metric \(h\) induces a natural inner product, also denoted by \(\langle \cdot, \cdot \rangle\), on the bundle \(\Lambda^k M\) of all the real \(k\)-forms and on the bundle \(TM \otimes \Lambda^k M\) of all the \(TM\)-valued \(k\)-forms. We use the norm conventions as in [17,18]. Then we see that by (2.1) and (2.2),

\[
|dF|^2 = \frac{|\alpha_F|^2}{n-1} + |(dF)^+_0|^2 + |(dF)^-_0|^2, \tag{2.5}
\]

and by (2.4),

\[
|\nabla F|^2 = \frac{|\alpha_F|^2}{n-1} + |(dF)^+_0|^2 + \frac{1}{4} |N^0|^2 + \frac{1}{3} |(dF)^-_0|^2. \tag{2.6}
\]

The Riemannian curvature tensor \(R\) of the metric \(h\) is defined by

\[
R(X, Y, Z, W) = \langle \nabla_Z \nabla_W Y - \nabla_W \nabla_Z Y - \nabla_{[Z, W]} Y, X \rangle
\]

for all the vector fields \(X, Y, Z\) and \(W\). It induces a curvature operator \(\mathfrak{R}: \Lambda^2 TM \to \Lambda^2 TM\) defined by

\[
\langle \mathfrak{R}(X \wedge Y), Z \wedge W \rangle = R(X, Y, Z, W).
\]

It can also be viewed as an endomorphism of \(\Lambda^2 M\) via \(h\).
Let \(\{e_1, e_2, \ldots, e_{2n}\}\) be a local orthonormal frame field of the metric \(h\). Its dual coframe field is denoted by \(\{\omega^1, \omega^2, \ldots, \omega^{2n}\}\). Then the Ricci tensor \(\text{Ric}\) of \(h\) is defined by \(\text{Ric}(X, Y) = R(e_A, X, e_A, Y)\) and the scalar curvature \(s\) of \(h\) is defined by \(s = \text{Ric}(e_A, e_A)\). The curvature operator \(\mathcal{R}\) of \(\Lambda^2 M\) is given by \(\mathcal{R}(\omega^A \wedge \omega^B) = \frac{1}{2} R_{ABCD} \omega^C \wedge \omega^D\), where \(R_{ABCD} = R(e_A, e_B, e_C, e_D)\).

On the other hand, on an almost Hermitian manifold \((M, J, h)\), there is a \(J\)-twisted version of the Ricci tensor called the \(J\)-Ricci tensor (also called the \(*\)-Ricci tensor in some literature) and denoted by \(\text{Ric}_J\). It is defined by \(\text{Ric}_J(X, Y) = R(e_A, X, J e_A, JY)\). In general, \(\text{Ric}_J \neq \text{Ric}\) and \(\text{Ric}_J(X, Y) \neq \text{Ric}_J(Y, X)\). However, we have \(\text{Ric}_J(X, Y) = \text{Ric}_J(JY, JX)\). Hence, we can introduce a 2-form \(\rho_J\), called the \(J\)-Ricci form as follows:

\[
\rho_J(X, Y) = -\text{Ric}_J(X, JY).
\]

Then we define the \(J\)-scalar curvature \(s_J\) as \(s_J = \text{Ric}_J(e_A, e_A)\). By a direct calculation, we have

\[
\rho_J = \mathcal{R}(F) \quad \text{and} \quad s_J = 2(\mathcal{R}(F), F).
\]

Let \(\Delta_d = d\delta + \delta d\) be the Hodge Laplacian associated with the metric \(h\). Applying the Bochner-Weitzenböck formula to the fundamental 2-form \(F\), we obtain (see [17])

\[
\Delta_d F = \nabla^* \nabla F + \frac{2(n - 1)}{n(2n - 1)} s \cdot F - 2\mathfrak{W}(F) + \frac{n - 2}{n - 1} (\text{Ric}_0(J\cdot, \cdot) - \text{Ric}_0(\cdot, J\cdot)),
\]

where \(\nabla^*\) is the formal adjoint of the Levi-Civita connection \(\nabla\) in the metric \(h\), \(\text{Ric}_0\) is the trace-free part of \(\text{Ric}\), and \(\mathfrak{W}\) is the Weyl curvature operator.

Contracting (2.8) by \(F\) implies

\[
(\Delta_d F, F) - (\nabla^* \nabla F, F) = \frac{2(n - 1)}{2n - 1} s - 2(\mathfrak{W}(F), F).
\]

For the fundamental 2-form \(F\), \(\langle \nabla^* \nabla F, F \rangle = |\nabla F|^2\), \(\langle d\delta F, F \rangle = |\alpha_F|^2 + \delta \alpha_F\) and \(\langle \delta d F, F \rangle = |dF|^2 + \delta \alpha_F\) (see [17]). Then it follows that

\[
|dF|^2 + |\alpha_F|^2 + 2\delta \alpha_F - |\nabla F|^2 = \frac{2(n - 1)}{2n - 1} s - 2(\mathfrak{W}(F), F).
\]

By the decomposition of the Riemannian curvature tensor [3], we have

\[
2(\mathfrak{W}(F), F) = s_J - \frac{1}{(2n - 1)} s.
\]

This is an important identity in [13].

Inserting (2.10), (2.5) and (2.6) into (2.9) and then simplifying it, finally we obtain

\[
s - s_J = \frac{2}{3} |dF| - \frac{1}{4} |N^0|^2 - |\alpha_F|^2 + 2\delta \alpha_F.
\]

When \((M, J, h)\) is conformally flat, i.e., \(\mathfrak{W} = 0\), (2.10) implies

\[
s - s_J = \frac{2(n - 1)}{2n - 1} s,
\]

and hence (2.11) implies

\[
s = \frac{2n - 1}{2(n - 1)} \left( \frac{2}{3} |dF| - \frac{1}{4} |N^0|^2 + |\alpha_F|^2 + 2\delta \alpha_F \right).
\]

We use the formula (2.11) to get the following result.
Theorem 2.1. Let \((M, J, h)\) be a compact almost Hermitian manifold of real dimension \(2n \geq 6\). If \((M, J, h) \in W_1 \oplus W_3 \oplus W_4\), then
\[
\int_M (s - s_J) dv \geq 0.
\]
The equality holds if and only if \((M, J, h)\) is a balanced manifold.

Proof. If \((M, J, h) \in W_1 \oplus W_3 \oplus W_4\), then \(N^0 = 0\). Hence in this case, the formula (2.11) says
\[
s - s_J = \frac{2}{3} (|dF|^{-2} + |\alpha_F|^2 + 2\delta \alpha_F),
\]
which implies
\[
\int_M (s - s_J) dv = \frac{2}{3} \int_M |dF|^{-2} dv + \int_M |\alpha_F|^2 dv \geq 0.
\]
The above equality holds if and only if \((dF)^- = \alpha_F = 0\). Hence \((M, J, h) \in W_3\), i.e., it is a balanced manifold.

Remark 2.2. Recently, Yang and Zheng [41] proved that a compact Hermitian manifold \((M, J, h)\) is balanced if its Riemannian curvature tensor \(R\) satisfies the Gray-Kähler-like condition [20]. This result can be deduced from Theorem 2.1 as the Gray-Kähler-like condition implies \(s = s_J\).

By the same method as in Theorem 2.1, we obtain the following theorem.

Theorem 2.3. Let \((M, J, h)\) be an almost Hermitian manifold of real dimension \(2n \geq 6\). If \((M, J, h) \in W_2 \oplus W_3\), then \(s - s_J \leq 0\), and the equality holds if and only if it is a balanced manifold.

Proof. If \((M, J, h) \in W_2 \oplus W_3\), then \((dF)^- = \alpha_F = 0\). Hence, by (2.11) we have
\[
s - s_J = -\frac{1}{4} |N^0|^2 \leq 0.
\]
The above equality holds if and only if \(N^0 = 0\). Hence \((M, J, h) \in W_3\), i.e., it is a balanced manifold.

Remark 2.4. Note that \((dF)^- = (dF)^+ = 0\) on almost Hermitian surfaces. In this case, we have similar results from (2.11) as in Theorems 2.1 and 2.3. For a compact Hermitian surface \((M, J, h)\), i.e., \((M, J, h) \in W_4\), we get \(\int_M (s - s_J) dv \geq 0\), and the equality holds if and only if it is a Kähler surface [38]. For an almost Kähler surface \((M, J, h)\), i.e., \((M, J, h) \in W_2\), we get \(s - s_J \leq 0\), and the equality holds if and only if it is a Kähler surface.

Remark 2.5. We give some interesting observations. It follows from Theorem 2.1 and the formula (2.12) that a compact real hyperbolic manifold of dimension \(2n \geq 4\) does not admit a compatible \(W_2 \oplus W_3 \oplus W_4\)-structure, and in particular, does not admit a compatible complex structure, since a hyperbolic metric is conformally flat with \(s < 0\) (see [17,23]). Theorem 2.3 and the formula (2.12) imply that the standard round metric on the 6-sphere \(S^6\) does not admit a compatible \(W_2 \oplus W_3\)-structure since it is conformally flat with \(s > 0\) (see [23]). By the same reason, the standard induced metric on the Hopf manifold \(S^1 \times S^{2n-1}\) does not admit a compatible \(W_2 \oplus W_3\)-structure.

3 Scalar curvatures of the Lichnerowicz connection

In this section, we show explicit formulas of two Hermitian scalar curvatures of the Lichnerowicz connection on an almost Hermitian manifold. Preparing for the next section, we also present the structure equations of the Lichnerowicz connection.

Let \((M^{2n}, J, h)\) be an almost Hermitian manifold and \(\nabla\) be the Levi-Civita connection of the metric \(h\). The Lichnerowicz connection \(D^0\) is defined by
\[
D^0_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y
\]
and its curvature tensor $K^0$ by
\[ K^0(X, Y, Z, W) = \langle D^0_Z D^0_W Y - D^0_W D^0_Z Y - D^0_{[Z,W]} Y, X \rangle \]
for all $X, Y, Z, W \in \Gamma(TM)$. $K^0$ is related to the Riemannian curvature tensor $R$ as follows \[12,21\]:
\[ K^0(X, Y, Z, W) = \frac{1}{2} R(X, Y, Z, W) + \frac{1}{2} R(J X, J Y, Z, W) \]
\[ + \frac{1}{4} \langle (\nabla Z J) X, (\nabla W J) Y \rangle - \frac{1}{4} \langle (\nabla W J) X, (\nabla Z J) Y \rangle. \] (3.1)

By contracting $K^0$, we can define various Ricci forms and Hermitian scalar curvatures as in the Hermitian case \[16,29\]. Here, we only consider two Hermitian scalar curvatures, denoted by $s_1(0)$ and $s_2(0)$ as follows. Choose a local $J$-adapted $h$-orthonormal frame field $\{e_A\}_{A=1,2,\ldots,2n} = \{e_i, e_{n+1} = J e_i\}_{i=1,2,\ldots,n}$ and hence a unitary frame field $\{u_i = \frac{1}{\sqrt{2}}(e_i - \sqrt{-1}e_{n+1})\}_{i=1,2,\ldots,n}$. Set $u_i = \overline{u}_i$.

Two Hermitian scalar curvatures $s_1(0)$ and $s_2(0)$ of the Lichnerowicz connection are defined, respectively, by
\[ s_1(0) = K^0(u_i, u_i, u_j, u_j) \quad \text{and} \quad s_2(0) = K^0(u_i, u_j, u_i, u_j). \]

By (3.1) and (2.7), we have
\[ s_1(0) = R(u_i, u_i, u_j, u_j) + \frac{1}{4} \langle (\nabla u_j) u_i, (\nabla u_j) u_i \rangle - \frac{1}{4} \langle (\nabla u_j) u_i, (\nabla u_j) u_i \rangle \]
\[ = \frac{1}{2} s_j + \frac{1}{8} \langle (\nabla e_n) J e_A, (\nabla J e_n) J e_A \rangle. \] (3.2)

Since the repeated indices are summed over and we change $e_A$ to $J e_A$, it follows that
\[ \langle (\nabla e_n) J e_A, (\nabla J e_n) J e_A \rangle = -\langle (\nabla e_n) F e_A, (\nabla J e_n) J e_A \rangle = -\langle (\nabla e_n) F (e_A, e_C) (\nabla J e_n) F (e_A, e_C) \rangle. \]

Then the decomposition (2.4) of $\nabla F$ implies
\[ \langle (\nabla e_n) J e_A, (\nabla J e_n) J e_A \rangle = -2 |(dF)^+|^2 + \frac{2}{3} |(dF)^-|^2 + \frac{1}{2} |N^0|^2. \] (3.3)

Combining (3.2) and (3.3), we get
\[ s_1(0) = \frac{1}{2} s_j + \frac{1}{4} |(dF)^+|^2 - \frac{1}{12} |(dF)^-|^2 - \frac{1}{16} |N^0|^2. \] (3.4)

For $s_2(0)$, by the similar method, we have
\[ s_2(0) = R(u_i, u_j, u_i, u_j) + \frac{1}{4} \langle (\nabla u_j) u_i, (\nabla u_j) u_i \rangle - \frac{1}{4} \langle (\nabla u_j) u_i, (\nabla u_j) u_i \rangle. \] (3.5)

By the first Bianchi identity, it follows that
\[ R(u_i, u_j, u_i, u_j) = R(u_i, u_i, u_j, u_j) + R(u_i, u_j, u_i, u_j) \]
\[ = \frac{1}{2} R(u_i, u_i, u_j, u_j) + \frac{1}{4} (4 R(u_i, u_j, u_i, u_j) + 2 R(u_i, u_i, u_j, u_j)) \]
\[ = \frac{1}{4} s_j + \frac{1}{8}. \] (3.6)

Note that $\alpha_F(X) = -\delta F(J X) = \langle (\nabla e_A) F e_A, J X \rangle = \langle (\nabla e_A) J e_A, J X \rangle$. Then
\[ \langle (\nabla u_j) u_i, (\nabla u_j) u_i \rangle = \frac{1}{2} \langle (\nabla e_A) J e_A, (\nabla J e_n) F (e_A, e_B) \rangle = \frac{1}{2} |\alpha_F|^2. \] (3.7)

For
\[ |dF|^2 = \frac{1}{6} dF(e_A, e_B, e_C) dF(e_A, e_B, e_C) \]
Theorem 3.1. Let \( (M, J, h) \) be an almost Hermitian manifold of real dimension \( 2n \). Then

\[
\begin{align*}
\sqrt{s_1(0)} &= \frac{s}{2} - \frac{5}{12} |(dF)^{-1}|^2 + \frac{1}{16} |N^0|^2 - \frac{1}{4} |(dF)^0|^2 + \frac{3 - 2n}{4(n - 1)} |\alpha F|^2 - \delta \alpha F, \\
sv &= \frac{s}{2} - \frac{1}{12} |(dF)^{-1}|^2 + \frac{1}{32} |N^0|^2 - \frac{1}{8} |\alpha F|^2 - \frac{1}{2} \delta \alpha F.
\end{align*}
\]

Combining (2.11), (3.4) and (3.9) yields two identities in the following theorem.

Remark 3.2. The integral formula of (3.11) was given in [7]. This formula is very useful in characterizing different types of almost Hermitian structures, in particular, for those with vanishing first Chern class [7,33].

Note that for other canonical Hermitian connections mentioned in Section 1, the corresponding curvature tensors are extremely complicated, not so simple as in the formula (3.1). In the next section, we should exploit the moving frame method, which turns out to be very effective in our study of curvatures

\[
\omega = \omega_A^B \wedge \omega_B + \Omega_A^B, \quad \text{where } \omega_A^B + \omega_B^A = 0, \quad \text{and } \Omega_A^B = \frac{1}{2} R_{ABCD} \omega_C \wedge \omega_D.
\]

Set

\[
J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},
\]

where \( I_n \) is the \( n \times n \) identity matrix. Then we have the following decompositions of \( \omega \) and \( \Omega \):

\[
\begin{align*}
\omega &= \frac{1}{2} (\omega - J_0 \omega J_0) + \frac{1}{2} (\omega + J_0 \omega J_0), \\
\Omega &= \frac{1}{2} (\Omega - J_0 \Omega J_0) + \frac{1}{2} (\Omega + J_0 \Omega J_0).
\end{align*}
\]

In fact, \( \frac{1}{2} (\omega - J_0 \omega J_0) \) is the connection form matrix of the Lichnerowicz connection [26].

We denote the unitary coframe field by \( \theta^i = \frac{1}{\sqrt{2}} (\omega^i + \sqrt{-1} \omega^{n+i}) \), and set

\[
\begin{align*}
\varphi^i_j &= \frac{1}{2} (\omega^i_j + \omega^{n+i}_{n+j}) + \frac{\sqrt{-1}}{2} (\omega^{n+i}_{j} - \omega^i_{n+j}), \\
\mu^i_j &= \frac{1}{2} (\omega^i_j - \omega^{n+i}_{n+j}) + \frac{\sqrt{-1}}{2} (\omega^{n+i}_{j} + \omega^i_{n+j}).
\end{align*}
\]
Clearly, $\varphi_j^i + \overline{\varphi_i^j} = 0$ and $\mu_j^i + \overline{\mu_i^j} = 0$. Then the structure equations of the Lichnerowicz connection are

\begin{align}
d\theta^i &= -\varphi_j^i \wedge \theta^j + \tau^i, \\
d\varphi_j^i &= -\varphi_k^i \wedge \varphi_j^k + \Phi_j^i, \\
\end{align}

where $\tau^i$'s are the torsion forms defined by $\tau^i = -\mu_j^i \wedge \overline{\theta^j}$ and $\Phi_j^i$'s are the curvature forms:

$$\Phi_j^i = -\mu_j^k \wedge \mu_k^i + \frac{1}{2} (\Omega_j^i + \Omega_j^{i+1}) + \frac{\sqrt{-1}}{2} (\Omega_j^{i+1} - \Omega_j^{i+2}).$$

Hence, by the skew-symmetric property of $Ω$ and $μ$, the first Chern form $ρ_1(0)$ associated with the Lichnerowicz connection is

$$ρ_1(0) = \sqrt{-1} \text{tr}(Φ) = -\sqrt{-1} μ_j^i \wedge μ_i^j - Ω_j^{i+2},$$

where two forms $-\sqrt{-1} μ_j^i \wedge μ_i^j$ and $-Ω_j^{i+2}$ are globally defined. In fact, these two forms can be rewritten in a more familiar version as follows [33]:

$$-Ω_j^{i+2} = Re(F) = ρ_j$$

and

$$-\sqrt{-1} μ_j^i \wedge μ_i^j(X, Y) = \frac{1}{4} (J(∇_X J)e_A, (∇_Y J)e_A),$$

where $X, Y \in Γ(TM)$. Hence, if $(M, J, h)$ is a Hermitian manifold, i.e., $∇_JX = J∇_X J$, then $-\sqrt{-1} μ_j^i \wedge μ_i^j$ is a non-negative $(1, 1)$-form.

As a classical example, we consider the 6-sphere $S^6$. For the standard round metric, the associated fundamental 2-form $F$ of a compatible almost complex structure $J$ satisfies $Re(F) = F$. If $J$ is integrable, then by (3.15), $ρ_1(0)$ is a closed positive 2-form. It is a contradiction since the second Betti number of $S^6$ is zero. Moreover, we can get the following well-known result.

**Proposition 3.3** (See [6, 27, 29, 32, 33, 36]). The 6-sphere $S^6$ does not admit a complex structure compatible with any metric in a small neighborhood of the standard round metric.

### 4 Curvatures of the canonical Hermitian connections

In this section, by using the structure equations of the Lichnerowicz connection $D^0$ and the Chern connection $D^1$ on an almost Hermitian manifold, we obtain the curvature formulas of canonical Hermitian connections $D^t$ for all $t$. Then we show the explicit formulas of two Hermitian scalar curvatures $s_1(t)$ and $s_2(t)$ in terms of the Riemannian scalar curvature $s$, norms of decompositions of the covariant derivative $∇F$ of the fundamental 2-form $F$ with respect to the Levi-Civita connection $∇$, and the codifferential $δ_{αF}$ of the Lee form $α_F$.

Given an almost Hermitian manifold $(M, J, h)$, we use a local unitary frame field $\{u_i\}$ and its dual coframe field $\{\theta^i\}$ as in Section 3. Locally, let $ψ = (ψ_j^i)$ be the matrix of the Chern connection form. Then the structure equations of the Chern connection are

\begin{align}
dθ^i &= -ψ_j^i \wedge θ^j + T^i, \\
dψ_j^i &= -ψ_k^i \wedge ψ_j^k + Ψ_j^i, \\
\end{align}

where $T^i$'s are the torsion forms and $Ψ_j^i$'s are the curvature forms. As $\{u_i\}$ is a unitary frame, $ψ_j^i + ψ_j^i = 0$. As $T^i$ has no $(1, 1)$-component, it can be written as

$$T^i = \frac{1}{2} T_{jk}^i θ^j \wedge θ^k + \frac{1}{2} T_{jk}^i \overline{θ^j} \wedge \overline{θ^k},$$
where $T^{i}_{jk} + T^{i}_{kj} = 0$ and $T^{i}_{jk} + T^{i}_{kj} = 0$. Hence the torsion tensor $T^{D^1}$ of the Chern connection $D^1$, defined by

$$T^{D^1}(X,Y) = D^1_X Y - D^1_Y X - [X,Y],$$

can be expressed by $T^{D^1} = u_i \otimes T^i + u_i \otimes T^i$.

We express the Nijenhuis tensor $N$ in terms of the torsion form of the Chern connection. For $[X,Y] = D^1_X Y - D^1_Y X - T^{D^1}(X,Y)$, $N$ can be rewritten as

$$N(X,Y) = -2T^{D^1}(X,Y) - 2JT^{D^1}(JX,Y).$$

Hence if we set $N_{ij} = \langle N(u_i, u_j), u_k \rangle$, then

$$N_{ij} = -2\langle T^{D^1}(u_i, u_j) + JT^{D^1}(J u_i, u_j), u_k \rangle = -4T^k_{ij},$$

and we can write

$$N = \frac{1}{2}N_{ij} u_k \otimes \theta^i \wedge \theta^j + \frac{1}{2}N_{ik} u_k \otimes \theta^i \wedge \theta^i.$$

We now express $dF$ in terms of the torsion of the Chern connection. By the structure equation (4.1), we have

$$dF = \sqrt{-1}(T^i \wedge \bar{\theta}^i - \theta^i \wedge T^i).$$

Hence,

$$\alpha F = J \delta F = -J * d\frac{F^{n-1}}{(n-1)!} = T^i_{j} \theta^j + \bar{T}^i_{j} \bar{\theta}^j,$$

$$dF^+ = \frac{\sqrt{-1}}{2}(T^i_{jk} \theta^j \wedge \theta^k \wedge \bar{\theta}^i - T^{ik}_{jk} \bar{\theta}^j \wedge \bar{\theta}^k \wedge \theta^i)$$

and

$$dF^- = \frac{\sqrt{-1}}{2}(T^i_{jk} \bar{\theta}^j \wedge \bar{\theta}^k \wedge \theta^i - T^{ik}_{jk} \theta^j \wedge \theta^k \wedge \bar{\theta}^i).$$

By the above expressions, a direct calculation shows the following formulas on norms:

$$|T^{D^1}|^2 = T^i_{jk} \bar{T}^i_{jk} + T^{ik}_{jk} \bar{T}^{ik}_{jk},$$

$$|\alpha F|^2 = 2T^i_{jk} \bar{T}^i_{jk},$$

and

$$|(dF^+)|^2 = T^i_{jk} \bar{T}^i_{jk}.$$

Combining the structure equations (3.12) and (4.1), we have

$$(\varphi^i_j - \psi^i_j) \wedge \theta^i + T^i_j - \tau^i = 0.\quad (4.7)$$

Since $\varphi^i_j + \bar{\varphi}^i_j = 0$, $\psi^i_j + \bar{\psi}^i_j = 0$ and $\mu^i_j + \bar{\mu}^i_j = 0$, by comparing the types of the forms in (4.7), we get (see [26])

$$\varphi^i_j - \psi^i_j = \frac{1}{2}T^i_{jk} \theta^k - \frac{1}{2}\overline{T^i_{jk}} \bar{\theta}^k,$$

$$\mu^i_j = -\frac{1}{2}T^i_{jk} \bar{\theta}^k + \frac{1}{2}(T^i_{jk} + T^i_{kj}) \theta^k.$$

Set $\gamma = (\varphi^i_j) = (\varphi^i_j - \psi^i_j)$. Locally let $\psi(t) = (\psi^i_j(t))$ be the connection form matrix of the canonical Hermitian connection $D^1$. Then $\psi(0) = \varphi - \tau \gamma$. In particular, $\psi(0) = \varphi$ and $\psi(1) = \psi$.

The structure equations of the canonical Hermitian connection $D^1$ are

$$d\theta^i = -\psi^i_j(t) \wedge \theta^j + T^i_j(t),$$

$$d\psi^i_j(t) = -\psi^i_k(t) \wedge \psi^k_j(t) + \Psi^i_j(t).\quad (4.10)$$
where \( T^i(t)'s \) are the torsion forms and \( \Psi^i_j(t)'s \) are the curvature forms. In particular, \( T^i(0) = \tau^i \), \( \Psi^i_j(0) = \Phi^i_j \), \( T^i(1) = T^i \) and \( \Psi^i_j(1) = \Psi^i_j \). Then by (4.10), we have

\[
\Psi^i_j(t) = \Phi^i_j - t(d\tau^i_j + \psi^i_k \wedge \gamma^k_j + \psi^k_i \wedge \gamma_j^k) + (t^2 - 2t)\gamma^i_k \wedge \gamma_j^k. \tag{4.11}
\]

Now by (4.11), the first Chern form \( \rho_1(t) \) associated with the canonical Hermitian connection \( D^t \) is

\[
\rho_1(t) = \sqrt{-1} \text{tr}(\Psi(t)) = \rho_1(0) - \sqrt{-1}td\gamma^i_j. \tag{4.12}
\]

Then by (4.8) and (4.3),

\[
\rho_1(t) = \rho_1(0) - \sqrt{-1} \frac{t}{2}(dT^i_j \theta^k - T^i_j \theta^k) = \rho_1(0) + \frac{t}{2}d\delta F.
\]

The above formula is also obtained by using the different method in [12, 18].

Proposition 4.1. Let \((M, J, h)\) be a compact Hermitian manifold and \( c_1^{\text{AC}}(M) \) be the first Aeppli-Chern class of the anti-canonical bundle \( K_M^* \). Then \( \rho^{(1)}(t) \) represents \( c_1^{\text{AC}}(M) \) in \( H^{1,1}(M; \mathbb{R}) \). More precisely,

\[
\rho^{(1)}(t) = \rho_1(1) + \frac{t}{2}d\delta F + (\partial \bar{\partial}^* F + \bar{\partial} \partial^* F).
\]

Proof. For a Hermitian manifold, by (4.12) and (4.13) we have

\[
\rho^{(1)}(t) = \rho_1(t) = \frac{1}{2} \left( \rho_1(0) + \frac{t}{2}d\delta F + J(\rho_1(0) + \frac{t}{2}d\delta F) \right)
\]

\[
= \frac{1}{2} \left( \rho_1(1) + \frac{t-1}{2}d\delta F + \rho_1(1) + \frac{t-1}{2}Jd\delta F \right)
\]

\[
= \rho_1(1) + \frac{t-1}{4}(d\delta F + Jd\delta F).
\]

Since on a compact Hermitian manifold \( d = \partial + \bar{\partial} \) and \( \delta = -*d* = \partial^* + \bar{\partial}^* \), where \( \partial^* = -*\partial \ast \) and \( \bar{\partial}^* = -*\bar{\partial} \ast \), respectively, are formal adjoints of \( \partial \) and \( \bar{\partial} \) in the metric \( h \), we have

\[
d\delta F + Jd\delta F = 2(\partial \bar{\partial}^* F + \bar{\partial} \partial^* F).
\]

The result now follows. \( \square \)

As in Section 3, we define the curvature tensor \( K^t \) of the canonical Hermitian connection \( D^t \) as follows:

\[
K^t(X, Y, Z, W) = \langle D^t_X D^t_Y Z - D^t_X D^t_Y Z, D^t_{[Z,W]} Y, X \rangle,
\]

where \( X, Y, Z, W \in \Gamma(TM) \). Hence the curvature form

\[
\Psi^i_j(t) = \frac{1}{2} K^t_{ijkl} \theta^k \wedge \theta^l + K^t_{ijkl} \bar{\theta}^k \wedge \bar{\theta}^l + \frac{1}{2} K^t_{ijkl} \bar{\theta}^k \wedge \bar{\theta}^l,
\]

where \( K^t_{ijkl} = K^t(u_i, u_j, u_k, u_l) \) and the others are similar. They satisfy \( K^t_{ijkl} + K^t_{ijlk} = 0 \) and \( K^t_{ijkl} + K^t_{ijlk} = 0 \).
Proposition 4.2. Let \((M, J, h)\) be an almost Hermitian manifold. Then the components of the curvature tensor \(K^i\) of the canonical Hermitian connection \(D^i\) are given by

\[
K^i_{ijkl} = R_{ijkl} + \frac{1}{4} T^i_{jk}(T^k_{pl} + T^k_{lp} - T^k_{pj}) - \frac{1}{4} T^i_{ij}(T^k_{lp} + T^k_{pj} - T^k_{jk})
- \frac{1}{2}(T^k_{jk} - T^k_{lj}) + \frac{1}{4} T^k_{lp} + T^k_{pj} - T^k_{jk},
\]

where \(T^i_{jk}\) and \(T^i_{jl}\) are defined by

\[
dT^i_{jk} + T^p_{jk}T^i_p - T^p_{k(j}T^i_{p)} = T_{jk,l}\theta^l + T_{jk,l}\overline{\theta}.
\]

Proof. We start by the formula (4.11). We first deal with the term \(\Phi^i_j\) in (4.11), which has the expression (3.14).

By (4.9), the first term \(\mu^i_p \wedge \mu^j_l\) in (3.14) has the expression

\[
\mu^i_p \wedge \mu^j_l = -\frac{1}{4} T^k_{ij}(T^k_{lp} + T^k_{pj} - T^k_{jl})\theta^l \wedge \theta^j
- \frac{1}{4} T^k_{ij}(T^k_{lp} + T^k_{pj} - T^k_{jl})\overline{\theta} \wedge \overline{\theta}
+ \frac{1}{4}(T^k_{lp}T^k_{jl} - T^k_{lp}T^k_{jl} + T^k_{lp}T^k_{jl} - T^k_{lp}T^k_{jl})\theta^l \wedge \theta^j.
\]

The second and third terms in (3.14) are Riemannian curvature components \(R(u_i, u_j, \cdot, \cdot)\), which can be written as

\[
\frac{1}{2}(\Omega^i_j + \Omega^{i+}_j) + \frac{\sqrt{-1}}{2}(\Omega^i_j - \Omega^{i+}_j) = \frac{1}{2} R_{ijkl}\theta^k \wedge \theta^l + R_{ijkl}\theta^k \wedge \overline{\theta} + \frac{1}{2} R_{ijkl}\overline{\theta} \wedge \overline{\theta}.
\]

Since \(\gamma^i_j = \varphi^i_j - \psi^i_j\), by (4.8) and (4.2), a direct calculation shows that

\[
d\gamma^i_j + \gamma^i_p \wedge \gamma^p_j + \gamma^i_j \wedge \psi^p_j = \frac{1}{2}(dT^i_{jl} + T^p_{jl}T^i_p - T^p_{jl}T^i_p - T^p_{jl}T^i_p) \wedge \theta^l
+ \frac{1}{2} T^j_{lp}T^i_p - \frac{1}{2} T^j_{lp}T^i_p - \frac{1}{2} (dT^i_{jl} + T^p_{jl}T^i_p - T^p_{jl}T^i_p) \wedge \overline{\theta}.
\]

Then by the definition (4.15) of covariant derivatives of the tensor \(T^i_{jk}\), the above identity implies

\[
d\gamma^i_j + \gamma^i_p \wedge \gamma^p_j + \gamma^i_j \wedge \psi^p_j = \frac{1}{2}(T^i_{jl,k}\theta^k + T^i_{jl,k}\overline{\theta}) \wedge \theta^l
+ \frac{1}{2} T^i_{jl,k}\theta^k - \frac{1}{2} T^i_{jl,k}\overline{\theta} - \frac{1}{2} (T^i_{jl,k}\theta^k + T^i_{jl,k}\overline{\theta}) \wedge \overline{\theta}.
\]

By (4.8), we also have

\[
\gamma^i_j \wedge \gamma^p_j = \frac{1}{4} T^i_{jk}T^p_{jl}\theta^k \wedge \theta^l + \frac{1}{4} T^i_{jk}T^p_{jl}\theta^k \wedge \overline{\theta} + \frac{1}{4} (T^p_{jk}T^i_{jl} - T^p_{jk}T^i_{jl})\theta^k \wedge \overline{\theta}.
\]

Now putting (4.16) and (4.17) into (3.14) and then putting (3.14), (4.18) and (4.19) into (4.11), finally we obtain the results.
By contracting the curvature components $K^t_{ij}n_t$ as done in Section 3, we define two Hermitian scalar curvatures $s_1(t)$ and $s_2(t)$ of the canonical Hermitian connection $D^t$:

$$s_1(t) = K^t_{ij}$$ and $$s_2(t) = K^t_{ij}.$$ 

**Theorem 4.3.** Let $(M, J, h)$ be an almost Hermitian manifold of real dimension $2n$. Then

$$s_1(t) = \frac{s}{2} - \frac{5}{12} |(dF)^-|^2 + \frac{1}{16} |N^0|^2 + \frac{t}{4} |(dF)^+_0|^2 + \left( \frac{1}{4(n-1)} + \frac{t-1}{2} \right) |\alpha_F|^2 + \frac{t-2}{2} \delta \alpha_F$$ (4.20)

and

$$s_2(t) = \frac{s}{2} - \frac{1}{12} |(dF)^-|^2 + \frac{1}{32} |N^0|^2 - \frac{t^2 - 2t}{4} |(dF)^+_0|^2$$

$$- \left( \frac{t^2 - 2t}{4(n-1)} + \frac{(t+1)^2}{8} \right) |\alpha_F|^2 - \frac{t+1}{2} \delta \alpha_F.$$ (4.21)

**Proof.** We first deal with $s_1(t)$. By (4.14), we have

$$s_1(t) = s_1(0) + \frac{t}{2} (T^t_{ij} + T^*_{ij}).$$

By (4.3), $\delta F = -J \alpha_F = -\sqrt{-1}(T^t_{ij} \theta^j - T^*_t \theta^j)$ and then using (4.1) and (4.15), we see that

$$(d\delta F, F)^t = T^t_{ij} + T^*_t = -T^t_{ij} - T^*_t.$$ (4.22)

Hence,

$$s_1(t) = s_1(0) + \frac{t}{2} (d\delta F, F) = s_1(0) + \frac{t}{2} (|\alpha_F|^2 + \delta \alpha_F).$$

From this combined with (3.10), the identity (4.20) follows.

As to $s_2(t)$, by (4.14) we have

$$s_2(t) = s_2(0) + \frac{t}{2} (T^t_{ij} + T^*_t) + \frac{t^2 - 2t}{4} (T^t_{ij} T^*_i - T^*_p T^*_p).$$

Then by (4.22) and (4.5), we have

$$s_2(t) = s_2(0) - \frac{t}{2} (|\alpha_F|^2 + \delta \alpha_F) - \frac{t^2 - 2t}{4} |(dF)^+|^2 - \frac{t^2 - 2t}{8} |\alpha_F|^2.$$ From this combined with (3.11), the identity (4.21) follows.

We now give two particular cases of Theorem 4.3. For Hermitian manifolds, since $(dF)^- = N^0 = 0$, by (2.5) we have the following corollary.

**Corollary 4.4.** Let $(M, J, h)$ be a Hermitian manifold of real dimension $2n$. Then

$$s_1(t) = \frac{s}{2} + \frac{1}{4} |dF|^2 + \frac{t-1}{2} |\alpha_F|^2 + \frac{t-2}{2} \delta \alpha_F$$ (4.23)

and

$$s_2(t) = \frac{s}{2} - \frac{t^2 - 2t}{4} |dF|^2 - \left( \frac{(t+1)^2}{8} \right) |\alpha_F|^2 - \frac{t+1}{2} \delta \alpha_F.$$ For almost Hermitian surfaces, since $(dF)^- = (dF)^+_0 = 0$, we have the following corollary.

**Corollary 4.5.** Let $(M, J, h)$ be an almost Hermitian surface. Then

$$s_1(t) = \frac{s}{2} + \frac{1}{16} |N|^2 + \frac{2t-1}{4} |\alpha_F|^2 + \frac{t-2}{2} \delta \alpha_F$$

and

$$s_2(t) = \frac{s}{2} + \frac{1}{32} |N|^2 - \frac{3t^2 - 2t + 1}{8} |\alpha_F|^2 - \frac{t+1}{2} \delta \alpha_F.$$
Remark 4.6. By Theorem 4.3, we can obtain the explicit formulas of $s_1(t)$ and $s_2(t)$ for all $t$ and for each of the 16 classes (4 classes for $n = 2$) of almost Hermitian manifolds. For Hermitian manifolds, there are many studies [1, 15, 16, 24, 25, 29] on $s_1(t)$ and $s_2(t)$ for $t = -1, 0, 1$. Recently, for almost Hermitian manifolds, $s_1(1)$ and $s_2(1)$ are also given in [28] by using the different method.

Remark 4.7. As in the Hermitian case [16, 29], on an almost Hermitian manifold one can also define four Ricci forms of the canonical Hermitian connection $D^t$, denoted by $\rho^{(1)}(t)$, $\rho^{(2)}(t)$, $\rho^{(3)}(t)$ and $\rho^{(4)}(t)$, respectively, as follows:

$$
\rho^{(1)}(t) = -\sqrt{-1}K^{t}_{kkj} \theta^k \wedge \overline{\theta^j}, \quad \rho^{(2)}(t) = -\sqrt{-1}K^t_{jkk} \theta^k \wedge \overline{\theta^j},
$$

$$
\rho^{(3)}(t) = -\sqrt{-1}K^{t}_{kkj} \overline{\theta^k} \wedge \theta^j, \quad \rho^{(4)}(t) = -\sqrt{-1}K^t_{jkk} \overline{\theta^k} \wedge \theta^j.
$$

These Ricci forms are very useful in the research of almost Hermitian curvature flows and cohomology groups which are worthy of further study. One can refer to some related works in [8, 29, 34, 35].

5 Some applications

In this section, we show some applications of Theorem 4.3.

On a compact Hermitian manifold $(M, J, h)$ with the Chern connection, (4.23) implies the following integral formula of the Hermitian scalar curvature $s_1(1)$ and the Riemannian scalar curvature $s$:

$$
\int_M (2s_1(1) - s) dv = \frac{1}{2} \int_M |dF|^2 dv \geq 0.
$$

The equality holds if and only if $dF = 0$, i.e., $(M, J, h)$ is a Kähler manifold [16, 29].

In general, for an almost Hermitian manifold $(M, J, h)$ with the canonical Hermitian connection $D^t$, we have the following theorem.

**Theorem 5.1.** Let $(M, J, h)$ be a compact almost Hermitian manifold of real dimension $2n \geq 6$. If $(M, J, h) \in W_2 \oplus W_3 \oplus W_4$ and $t \geq 1 - \frac{1}{2(n-1)}$, then

$$
\int_M (2s_1(t) - s) dv \geq 0.
$$

If $(M, J, h) \in W_1 \oplus W_4$ and $t \leq 1 - \frac{1}{2(n-1)}$, then

$$
\int_M (2s_1(t) - s) dv \leq 0.
$$

In both cases, the equality holds if and only if $(M, J, h)$ is a Kähler manifold, or $t = 1 - \frac{1}{2(n-1)}$ and it is a locally conformal Kähler manifold.

**Proof.** If $(M, J, h) \in W_2 \oplus W_3 \oplus W_4$, then $(dF)^- = 0$. Hence by (4.20), when $t \geq 1 - \frac{1}{2(n-1)}$, we have

$$
\int_M (2s_1(t) - s) dv = \frac{1}{8} \int_M |N^0|^2 dv + \frac{1}{2} \int_M |(dF)^+_0|^2 dv + \left( t - 1 + \frac{1}{2(n-1)} \right) \int_M |\alpha_F|^2 dv \geq 0.
$$

Now if $\int_M (2s_1(t) - s) dv = 0$, then $N^0 = 0$, $(dF)^+_0 = 0$, and $t = 1 - \frac{1}{2(n-1)}$ or $t > 1 - \frac{1}{2(n-1)}$ and $\alpha_F = 0$. The result follows.

If $(M, J, h) \in W_1 \oplus W_4$, then $N^0 = 0$ and $(dF)^+_0 = 0$. Hence by (4.20), when $t \leq 1 - \frac{1}{2(n-1)}$, we have

$$
\int_M (2s_1(t) - s) dv = -\frac{5}{6} \int_M |(dF)^-|^2 dv + \left( t - 1 + \frac{1}{2(n-1)} \right) \int_M |\alpha_F|^2 dv \leq 0.
$$

Now if $\int_M (2s_1(t) - s) dv = 0$, then $(dF)^- = 0$, and $t = 1 - \frac{1}{2(n-1)}$ or $t < 1 - \frac{1}{2(n-1)}$ and $\alpha_F = 0$. The result follows. \qed
For almost Hermitian surfaces, by Corollary 4.5 we can use the similar method as in Theorem 5.1 to get the following results.

**Theorem 5.2.** Let $(M, J, h)$ be a compact almost Hermitian surface. If $t \geq \frac{1}{2}$, then

$$
\int_M (2s_1(t) - s)dv \geq 0,
$$

and the equality holds if and only if $(M, J, h)$ is a Kähler surface, or $t = \frac{1}{2}$ and it is a Hermitian surface. If $J$ is integrable and $t \leq \frac{1}{2}$, then

$$
\int_M (2s_1(t) - s)dv \leq 0,
$$

and the equality holds if and only if $(M, J, h)$ is a Kähler surface or $t = \frac{1}{2}$.

For the Chern connection $D^1$, Dabkowski and Lock [11] constructed non-compact non-Kähler Hermitian manifolds with $2s_1(1) = s$. Lejmi and Upmeier presented a problem in [28, Remark 3.3]: are there any compact almost Hermitian manifolds of real dimension $2n \geq 6$ with $2s_1(1) = s$ which are non-Kähler? Theorem 5.1 implies the following non-existence result.

**Corollary 5.3.** Let $(M, J, h)$ be a compact almost Hermitian manifold of real dimension $2n \geq 6$. If it belongs to the class $W_2 \oplus W_3 \oplus W_4$ and satisfies $2s_1(1) = s$, then it is a Kähler manifold.

**Remark 5.4.** Recently, Grama and Oliveira [19] calculated the Hermitian scalar curvatures on generalized flag manifolds by using our Theorem 4.3. They focused on providing examples of almost Hermitian manifolds satisfying $2s_1(1) = s$.

On the other hand, we consider $s_1(t) - s_2(t)$, the difference between $s_1(t)$ and $s_2(t)$ of the canonical Hermitian connection $D^t$. For a compact Hermitian manifold $(M, J, h)$ with the Chern connection $D^1$, combining two identities in Corollary 4.4 yields

$$
s_1(1) - s_2(1) = \frac{1}{2} |\alpha F|^2 + \frac{1}{2} \delta \alpha F,
$$

which implies the following well-known integral formula [16]:

$$
\int_M (s_1(1) - s_2(1))dv = \frac{1}{2} \int_M |\alpha F|^2 dv \geq 0.
$$

The equality holds if and only if $\alpha F = 0$, i.e., $(M, J, h)$ is a balanced manifold.

We now consider a compact almost Hermitian manifold $(M, J, h)$. Combining two identities in Theorem 4.3 yields

$$
s_1(t) - s_2(t) = -\frac{1}{3} |(DF)^-|^2 + \frac{1}{32} |N^0|^2 + \frac{(t - 1)^2}{4} |(DF)_0^+|^2 + c_n(t) |\alpha F|^2 + \left(t - \frac{1}{2}\right) \delta \alpha F
$$

for a constant

$$
c_n(t) = \frac{(n + 1)t^2 + (6n - 10)t + 5 - 3n}{8(n - 1)}.
$$

Hence, we get an integral formula

$$
\int_M (s_1(t) - s_2(t))dv = -\frac{1}{3} \int_M |(DF)^-|^2 dv + \frac{1}{32} \int_M |N^0|^2 dv + \frac{(t - 1)^2}{4} \int_M |(DF)_0^+|^2 dv + c_n(t) \int_M |\alpha F|^2 dv.
$$

\begin{equation}
\text{(5.1)}
\end{equation}

**Theorem 5.5.** Let $(M, J, h)$ be a compact almost Hermitian manifold of real dimension $2n \geq 6$.

1. If $(M, J, h) \in W_2 \oplus W_3 \oplus W_4$ and $t \in \mathbb{R} \setminus \{-3 - 2\sqrt{3}, -3 + 2\sqrt{3}\}$, then

$$
\int_M (s_1(t) - s_2(t))dv \geq 0.
$$
The equality holds if and only if $(M, J, h)$ is a Kähler manifold, or $t = 1$ and it is a balanced manifold.

(2) If $(M, J, h) \in W_1 \oplus W_4$ and $t \in [-1, \frac{1}{2}]$, then

$$\int_M (s_1(t) - s_2(t)) dv \leq 0.$$ 

The equality holds if and only if $(M, J, h)$ is a Kähler manifold.

**Proof.** (1) The equation $c_n(t) = 0$ on the variable $t$ has two solutions

$$t_{\pm}(n) = \frac{5 - 3n \pm 2\sqrt{3n^2 - 8n + 5}}{n + 1}.$$ 

Hence $c_n(t) > 0$ if and only if $t \in \mathbb{R} \setminus (t_-(n), t_+(n))$. The sequence $\{t_-(n)\}$ is monotonically decreasing and has the limit $-3 - 2\sqrt{3}$ when $n \to +\infty$, and the sequence $\{t_+(n)\}$ is monotonically increasing and has the limit $-3 + 2\sqrt{3}$ when $n \to +\infty$. Hence $c_n(t) > 0$ when $t \in \mathbb{R} \setminus (-3 - 2\sqrt{3}, -3 + 2\sqrt{3})$.

Now for $(M, J, h) \in W_2 \oplus W_3 \oplus W_4$, since $(dF)^- = 0$, the formula (5.1) implies

$$\int_M (s_1(t) - s_2(t)) dv = \frac{1}{32} \int_M |N^0|^2 dv + \frac{(t - 1)^2}{4} \int_M |(dF)_0|^2 dv + c_n(t) \int_M |\alpha_F|^2 dv$$

is non-negative. If $\int_M (s_1(t) - s_2(t)) dv = 0$, then three components of the right-hand side of the above formula are all zero. Hence the conclusion follows by the direct discussion.

(2) When $t \in [-1, \frac{1}{2}], c_n(t) < 0$. Hence for $(M, J, h) \in W_1 \oplus W_4$, since $N^0 = 0$ and $(dF)^+_0 = 0$, the formula (5.1) implies

$$\int_M (s_1(t) - s_2(t)) dv = c_n(t) \int_M |\alpha_F|^2 dv - \frac{1}{3} \int_M |(dF)^-|^2 dv \leq 0.$$ 

The equality holds if and only if $(dF)^- = \alpha_F = 0$, i.e., $(M, J, h)$ is a Kähler manifold. \hfill $\Box$

**Remark 5.6.** There are also many references such as [5, 39, 40] on the study of Hermitian manifolds with flat curvature of some canonical Hermitian connection. Recently, the concept of “Kähler-like” was introduced and studied by Yang and Zheng first in [41] for the Levi-Civita connection and the Chern connection and later extended by Angella et al. [2] to other canonical Hermitian connections. Clearly, the Kähler-like curvature condition implies $s_1(t) = s_2(t)$. Hence Theorem 5.5 is closely related to [2, Conjecture 2] which states that if a compact Hermitian manifold $(M, J, h)$ is Kähler-like for a canonical Hermitian connection $D^t$ with $t$ not equal to 1 or $-1$, then it is a Kähler manifold. As a corollary of Theorem 5.5, we give a partial result on this conjecture.

**Corollary 5.7.** Let $(M, J, h)$ be a compact Hermitian manifold of real dimension $2n \geq 6$. If it is Kähler-like for a canonical Hermitian connection $D^t$ when $t \in \mathbb{R} \setminus \{(-3 - 2\sqrt{3}, -3 + 2\sqrt{3}) \cup \{1\}\}$, then it is a Kähler manifold.

Recently, for the case $t = -1$ (i.e., the Bismut connection, now also called the Strominger connection [44]), Zhao and Zheng [44] have studied the relation between the Kähler-like condition of $D^{-1}$ and the pluriclosed metric. However, we have the following result.

**Theorem 5.8.** Let $(M, J, h)$ be a compact Hermitian manifold of real dimension $2n \geq 6$. Then

$$\int_M (s_1(-1) - s_2(-1)) dv = \int_M |(dF|^2 - |\alpha_F|^2) dv.$$ 

Moreover, if $h$ is a Gauduchon metric (i.e., $\delta \alpha_F = 0$) and $s_1(-1) = s_2(-1)$, then $h$ is also a $k$-Gauduchon metric [14], i.e., $\partial \bar{\partial} F^k \wedge F^{n-k-1} = 0$ for $k = 1, 2, \ldots, n - 2$.

**Proof.** When $t = -1$, two identities in Corollary 4.4 can be rewritten as

$$s_2(-1) = \frac{s}{2} - \frac{3}{4} |dF|^2$$

and

$$s_1(-1) = \frac{s}{2} + \frac{1}{4} |dF|^2 - |\alpha_F|^2 - \frac{3}{2} \delta \alpha_F.$$
Together with (4.22), the formula (5.5) can be rewritten as

\[ s_1(-1) - s_2(-1) = |dF|^2 - |\alpha_F|^2 - \frac{3}{2} \delta \alpha_F, \quad (5.2) \]

which implies

\[ \int_M (s_1(-1) - s_2(-1))dv = \int_M (|dF|^2 - |\alpha_F|^2)dv. \]

Next, we provide a direct calculation of \( \sqrt{-1} \partial \bar{\partial} (F^k) \wedge F^{n-k-1} \) by using the torsion of the Chern connection. First note that

\[ \sqrt{-1} \partial \bar{\partial} (F^k) \wedge F^{n-k-1} = k(k-1)\sqrt{-1} \partial F \wedge \bar{\partial} F \wedge F^{n-3} + k \sqrt{-1} \partial \bar{\partial} F \wedge F^{n-2}. \quad (5.3) \]

Formula (4.4) implies \( \sqrt{-1} \partial F \wedge \bar{\partial} F = - \frac{1}{4} T_{jk}^i T_{pq}^j \theta^i \wedge \theta^j \wedge \theta^p \wedge \overline{\theta^q} \), which implies

\[ \langle \sqrt{-1} \partial F \wedge \bar{\partial} F, F^3 \rangle = 6 T_{ki}^j T_{kj}^i - 3 T_{jk}^i T_{kj}^j. \quad (5.4) \]

By (4.4), (4.15) and (4.1), it follows that

\[ \sqrt{-1} \partial \bar{\partial} F = \frac{1}{2} (\partial (T_{jk}^i \theta^j \wedge \overline{\theta^k} \wedge \theta^p))^{(2,2)} = \frac{1}{4} (2 T_{jk}^i T_{pq}^j + T_{pq}^j T_{jk}^i) \theta^p \wedge \theta^q \wedge \overline{\theta^i} \wedge \overline{\theta^k}, \]

which implies

\[ \langle \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle = T_{jk}^i T_{ki}^j + 2 T_{ji}^j. \quad (5.5) \]

For a Hermitian manifold, by (4.5) and (4.6), we have \( |\alpha_F|^2 = 2 T_{ki}^j T_{kj}^i \) and \( |dF|^2 = T_{jk}^i T_{kj}^i \). Hence we can rewrite (5.4) as

\[ \langle \sqrt{-1} \partial F \wedge \bar{\partial} F, F^3 \rangle = 3 (|\alpha_F|^2 - |dF|^2). \quad (5.6) \]

Moreover,

\[ \langle \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle = \langle - \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle = \langle \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle. \]

Together with (4.22), the formula (5.5) can be rewritten as

\[ \langle \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle = T_{jk}^i T_{ki}^j + T_{ji}^j + T_{ji}^j = |dF|^2 - |\alpha_F|^2 - \delta \alpha_F. \quad (5.7) \]

Combining (5.3) with (5.6) and (5.7), we get

\[ \sqrt{-1} \partial \bar{\partial} (F^k) \wedge F^{n-k-1} = k(k-1) \frac{(n-3)!}{6} \langle \sqrt{-1} \partial \bar{\partial} F \wedge F^3 \rangle dv + k \frac{(n-2)!}{2} \langle \sqrt{-1} \partial \bar{\partial} F, F^2 \rangle dv \]

\[ = k \frac{(n-3)!}{6} ((n-3) (|dF|^2 - |\alpha_F|^2) - (n-2) \delta \alpha_F) dv. \]

Now if \( h \) is a Gauduchon metric (i.e., \( \delta \alpha_F = 0 \)) and \( s_1(-1) = s_2(-1) \), then (5.2) implies \( |dF|^2 - |\alpha_F|^2 = 0 \). Hence for \( k = 1, 2, \ldots, n-2 \), \( \sqrt{-1} \partial \bar{\partial} (F^k) \wedge F^{n-k-1} = 0 \).

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