UNIQUENESS OF CONVEX ANCIENT SOLUTIONS TO MEAN CURVATURE FLOW IN HIGHER DIMENSIONS

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Abstract. In this paper, we consider noncompact ancient solutions to the mean curvature flow in $\mathbb{R}^{n+1}$ ($n \geq 3$) which are strictly convex, uniformly two-convex, and noncollapsed. We prove that such an ancient solution is a rotationally symmetric translating soliton.

1. Introduction

Throughout this paper, we fix an integer $n \geq 3$. Our goal in this paper is to classify all noncompact ancient solutions to mean curvature flow in $\mathbb{R}^{n+1}$ which are convex, uniformly two-convex, and noncollapsed in the sense of Sheng and Wang [9]:

Theorem 1.1. Let $M_t$, $t \in (-\infty, 0]$, be a noncompact ancient solution of mean curvature flow in $\mathbb{R}^{n+1}$ which is strictly convex, uniformly two-convex, and noncollapsed. Then $M_t$ is a rotationally symmetric translating soliton.

If we evolve a closed, embedded, two-convex hypersurface by mean curvature flow, then it is well known that any blow-up limit is an ancient solution which is weakly convex, uniformly two-convex, and noncollapsed (see [4], Theorem 1.10, or [10],[11]). If we combine this result with Theorem 1.1, we obtain the following result:

Corollary 1.2. Consider an arbitrary closed, embedded, two-convex hypersurface in $\mathbb{R}^{n+1}$, and evolve it by mean curvature flow. At the first singular time, the only possible blow-up limits are shrinking round spheres; shrinking round cylinders; and the unique rotationally symmetric translating soliton.

In a recent paper [2], we obtained a classification of noncompact ancient solutions in $\mathbb{R}^3$ which are convex and noncollapsed. The proof of Theorem 1.1 draws on similar techniques. In Section 2, we derive asymptotic estimates for the solution in the cylindrical region. These estimates tell us that, for $-t$ large, the rescaled surface $(-t)^{-\frac{n}{2}} M_t \cap B_{5n}(0)$ is $O((-t)^{-\frac{n}{2}})$-close to a cylinder of radius $\sqrt{2(n - 1)}$. In Section 3, we combine this estimate with a barrier argument in the spirit of [11] to conclude that $\liminf_{t \to -\infty} H_{\max}(t) > 0$, where $H_{\max}(t)$ denotes the supremum of the mean curvature of $M_t$.

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In Section 4, we establish a higher-dimensional version of the Neck Improvement Theorem in \[2\]. This step requires significant modifications in the higher-dimensional setting. In order to formulate the Neck Improvement Theorem, we need a notion of \(\varepsilon\)-symmetry in higher dimensions, which generalizes the one introduced in \[2\]. We say that a point \((\bar{x}, \bar{t})\) in space-time is \(\varepsilon\)-symmetric if there exists a collection of rotation vector fields \(K = \{K_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2}\}\) (with a common axis of rotation) such that \(|K_\alpha| H \leq 10n\) at \((\bar{x}, \bar{t})\) and \(|\langle K_\alpha, \nu \rangle| H \leq \varepsilon\) in the parabolic neighborhood of \(\mathcal{P}(\bar{x}, \bar{t}, 10, 100)\). The main difference between the two-dimensional case and the higher-dimensional case is that, instead of a single rotation vector field in ambient space, we need to consider a collection of rotation vector fields which share a common axis. The statement of the Neck Improvement Theorem can be summarized as follows: if \((\bar{x}, \bar{t})\) lies on a neck and every point in a sufficiently large parabolic neighborhood of \((\bar{x}, \bar{t})\) is \(\varepsilon\)-symmetric, then the point \((\bar{x}, \bar{t})\) itself is \(\frac{\varepsilon}{2}\)-symmetric.

Let us sketch the main ideas involved in the proof of the Neck Improvement Theorem. By scaling, we may assume that \(H(\bar{x}, \bar{t}) = \sqrt{\frac{n-1}{2}}\). By assumption, we can find rotation vector fields \(\bar{K}_\alpha\) such that \(|\bar{K}_\alpha| H \leq 10n\) at \((\bar{x}, \bar{t})\) and \(|\langle \bar{K}_\alpha, \nu \rangle| H \leq \varepsilon\) in the parabolic neighborhood of \(\mathcal{P}(\bar{x}, \bar{t}, 10, 100)\). Moreover, the vector fields \(\bar{K}_\alpha\) have a common axis of rotation, which we may assume to be the \(x_{n+1}\)-axis. One key observation is that the function \(u_\alpha := \langle \bar{K}_\alpha, \nu \rangle\) satisfies the linearized equation \(\frac{\partial}{\partial t} u_\alpha = \Delta u_\alpha + |A|^2 u_\alpha\). The linearized equation on the cylinder can be analyzed using separation of variables. The upshot is that we can find coefficients \(A_{\alpha,1}, \ldots, A_{\alpha,n}, B_{\alpha,1}, \ldots, B_{\alpha,n}\) such that

\[
|\langle \bar{K}_\alpha, \nu \rangle - (A_{\alpha,1}x_1 + \ldots + A_{\alpha,n}x_n) - (A_{\alpha,1}x_1 + \ldots + A_{\alpha,n}x_n)x_{n+1}| < \varepsilon
\]

in the parabolic neighborhood \(\mathcal{P}(\bar{x}, \bar{t}, 10, 100)\). For each \(\alpha\), we are able to offset the terms involving \(A_{\alpha,i}\) and \(B_{\alpha,i}\) if we replace the rotation vector field \(\bar{K}_\alpha\) by a new rotation vector field \(\bar{K}_\alpha\) whose axis of rotation may differ from that of \(\bar{K}_\alpha\). In doing so, we need to be careful to ensure that the modified rotation vector fields \(\bar{K}_\alpha\) all share a common axis of rotation. This is a difficulty which is not present in the two-dimensional case. In order to overcome this obstacle, we exploit certain relations among the coefficients \(A_{\alpha,1}, \ldots, A_{\alpha,n}, B_{\alpha,1}, \ldots, B_{\alpha,n}\) which can be derived from the divergence theorem.

In Section 5, we iterate the Neck Improvement Theorem to conclude that any ancient solution which satisfies the assumption of Theorem 1.1 is rotationally symmetric. Finally, in Section 6, we classify ancient solutions with rotational symmetry, thereby completing the proof of Theorem 1.1.

2. Asymptotic analysis as \(t \to -\infty\)

Suppose that \(M_t, t \in (-\infty, 0]\), is a noncompact ancient solution of mean curvature flow in \(\mathbb{R}^{n+1}\) which is strictly convex, uniformly two-convex, and
noncollapsed. We consider the rescaled flow \( \bar{M}_\tau = e^{\frac{\tau}{2}} M \cdot e^{-\tau} \). The surfaces \( \bar{M}_\tau \) move with velocity \(- (H - \frac{1}{2} \langle x, \nu \rangle) \nu \). Given any sequence \( \tau_j \to -\infty \), Theorem 1.11 in [4] implies that a subsequence of the surfaces \( \bar{M}_{\tau_j} \) converges in \( C_\infty^{\infty} \) to a cylinder of radius \( \sqrt{2(n-1)} \) with axis passing through the origin. Let us denote by \( \Sigma = \{ x_1^2 + \ldots + x_n^2 = 2(n-1) \} \) the cylinder of radius \( \sqrt{2(n-1)} \) around the \( x_{n+1} \)-axis.

**Proposition 2.1.** For each \( \tau \), we have

\[
\int_{\bar{M}_\tau} e^{-\frac{|x|^2}{4}} \leq \int_{\Sigma} e^{-\frac{|x|^2}{4}}.
\]

**Proof.** A standard calibration argument shows that any convex hypersurface (or, more generally, any star-shaped hypersurface) is outward-minimizing. From this, we deduce that area(\( \bar{M}_\tau \cap B_r(p) \)) \( \leq Cr^n \) for all \( p \in \mathbb{R}^{n+1} \) and all \( r > 0 \). We next consider an arbitrary sequence \( \tau_j \to -\infty \).

After passing to a subsequence, the surfaces \( \bar{M}_{\tau_j} \) converge in \( C_\infty^{\infty} \) to a cylinder of radius \( \sqrt{2(n-1)} \) with axis passing through the origin. Consequently,

\[
\int_{\bar{M}_{\tau_j}} e^{-\frac{|x|^2}{4}} \to \int_{\Sigma} e^{-\frac{|x|^2}{4}}
\]
as \( j \to \infty \). On the other hand, the function

\[
\tau \mapsto \int_{\bar{M}_\tau} e^{-\frac{|x|^2}{4}}
\]
is monotone decreasing in \( \tau \) by work of Huisken [4]. From this, the assertion follows.

As discussed above, there exists a smooth function \( S(\tau) \) taking values in \( SO(n+1) \) such that the rotated surfaces \( \tilde{M}_\tau = S(\tau)M \) converge to the cylinder \( \Sigma \) in \( C_\infty^{\infty} \). Hence, we can find a function \( \rho(\tau) \) with the following properties:

- \( \lim_{\tau \to -\infty} \rho(\tau) = \infty \).
- \( -\rho(\tau) \leq \rho'(\tau) \leq 0 \).

The hypersurface \( \tilde{M}_\tau \) can be written as a graph of some function \( u(\cdot, \tau) \) over \( \Sigma \cap B_{2\rho(\tau)}(0) \), so that

\[
\{ x + u(x, \tau) \nu_\Sigma(x) : x \in \Sigma \cap B_{2\rho(\tau)}(0) \} \subset \tilde{M}_\tau,
\]

where \( \nu_\Sigma \) denotes the unit normal to \( \Sigma \) and \( \| u(\cdot, \tau) \|_{C^4(\Sigma \cap B_{2\rho(\tau)}(0))} \leq \rho(\tau)^{-4} \).

As in [2], it is necessary to fine tune the choice of \( S(\tau) \). Let \( \varphi \geq 0 \) be a smooth cutoff function such that \( \varphi = 1 \) on \( [-\frac{1}{2}, \frac{1}{2}] \) and \( \varphi = 0 \) outside \( [-\frac{2}{3}, \frac{2}{3}] \). By the implicit function theorem, we can choose \( S(\tau) \) in such a way that \( u(x, \tau) \) satisfies the orthogonality relations

\[
\int_{\Sigma \cap B_{\rho(\tau)}(0)} e^{-\frac{|x|^2}{4}} \langle Ax, \nu_\Sigma \rangle u(x, \tau) \varphi \left( \frac{x_{n+1}}{\rho(\tau)} \right) = 0
\]
for every matrix \( A \in \text{so}(n+1) \). In addition, we can arrange that the matrix \( A(\tau) = S'(\tau)S(\tau)^{-1} \in \text{so}(n+1) \) satisfies \( A(\tau)_{ij} = 0 \) for \( i, j \in \{1, \ldots, n\} \).

**Proposition 2.2.** There exists a constant \( L_0 \) such that for all \( L \in [L_0, \rho(\tau)] \)

\[
\int_{M_\tau \cap \{|x_n+1| \geq L\}} e^{-\frac{|x|^2}{4}} - \int_{\Sigma \cap \{|x_n+1| \geq L\}} e^{-\frac{|x|^2}{4}} \geq - \int_{\Delta_\tau \cap \{|x_n+1| = L\}} e^{-\frac{|x|^2}{4}} \langle \omega, \nu_{\text{fol}} \rangle.
\]

Here, \( \omega = (0, \ldots, 0, 1) \) denotes the vertical unit vector in \( \mathbb{R}^{n+1} \), \( \nu_{\text{fol}} \) denotes the unit normal to the shrinker foliation in \( [1] \), and \( \Delta_\tau \) denotes the region between \( \Sigma \) and \( M_\tau \).

**Proof.** Analogous to [2], Proposition 2.2.

**Proposition 2.3.** There exists a constant \( L_0 \) such that

\[
\int_{\Sigma \cap \{|x_n+1| \leq L\}} e^{-\frac{|x|^2}{4}} |\nabla^\Sigma u(x, \tau)|^2 \leq C \int_{\Sigma \cap \{|x_n+1| \leq \frac{L}{2}\}} e^{-\frac{|x|^2}{4}} u(x, \tau)^2
\]

and

\[
\int_{\Sigma \cap \{|x_n+1| \leq L\}} e^{-\frac{|x|^2}{4}} u(x, \tau)^2 \leq CL^{-2} \int_{\Sigma \cap \{|x_n+1| \leq \frac{L}{2}\}} e^{-\frac{|x|^2}{4}} u(x, \tau)^2
\]

for all \( L \in [L_0, \rho(\tau)] \).

**Proof.** Lemma 4.11 in [1] implies that \( |\langle \omega, \nu_{\text{fol}} \rangle| \leq CL^{-1} |x_1^2 + \ldots + x_n^2 - 2(n-1)| \) for each point \( x \in \Delta_\tau \cap \{|x_n+1| = L\} \). This gives

\[
\int_{\Delta_\tau \cap \{|x_n+1| = L\}} e^{-\frac{x^2}{4}} |\langle \omega, \nu_{\text{fol}} \rangle| \leq CL^{-1} \int_{\Delta_\tau \cap \{|x_n+1| = L\}} e^{-\frac{x^2}{4}} |x_1^2 + \ldots + x_n^2 - 2(n-1)|
\]

\[
\leq CL^{-1} \int_{\Sigma \cap \{|x_n+1| = L\}} e^{-\frac{|x|^2}{4}} u^2,
\]

hence

\[
\int_{\tilde{M}_\tau \cap \{|x_n+1| \geq L\}} e^{-\frac{|x|^2}{4}} - \int_{\Sigma \cap \{|x_n+1| \geq L\}} e^{-\frac{|x|^2}{4}} \geq -CL^{-1} \int_{\Sigma \cap \{|x_n+1| = L\}} e^{-\frac{|x|^2}{4}} u^2
\]

by Proposition 2.2. We next observe that

\[
\int_{\tilde{M}_\tau \cap \{|x_n+1| \leq L\}} e^{-\frac{|x|^2}{4}} - \int_{\Sigma \cap \{|x_n+1| \leq L\}} e^{-\frac{|x|^2}{4}} = \int_{-L}^{L} \left( \int_{S^{n-1}} e^{-\frac{(|\nabla^\Sigma u|^2 + u)^2}{4}} \sqrt{2(n-1) + u}^{n-2}
\]

\[
\cdot \sqrt{2(n-1) + u} \left( 1 + \left( \frac{\partial u}{\partial z} \right)^2 + |\nabla S^{n-1} u|^2 \right.
\]

\[
- e^{-\frac{n-1}{2}} \sqrt{2(n-1)^{n-1}} \right) dz.
\]
By assumption, the height function $u$ satisfies $|u| + |\frac{\partial u}{\partial x}| + |\nabla^{S_{n-1}} u| \leq o(1)$ for $|x_{n+1}| \leq L$. From this, we deduce that

\[
\int_{\overline{M}_r \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} - \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} 
\geq \int_{-L}^{L} \left( \int_{S^{n-1}} e^{-\frac{|z|^2}{4}} \left[ e^{-\frac{(n-1)u^2}{4}} (\sqrt{2(n-1)} + u)^{n-1} - e^{-\frac{4}{n-1}} \sqrt{2(n-1)}^{n-1} + \frac{1}{C} |\nabla^{\Sigma} u|^2 \right] \right) dz
\geq \int_{-L}^{L} \int_{0}^{2\pi} e^{-\frac{4\pi}{n-1}} \left[ -Cu^2 + \frac{1}{C} |\nabla^{\Sigma} u|^2 \right] d\theta dz
\]

where $C > 0$ is a large constant that depends only on $n$. Putting these facts together, we conclude that

\[
\int_{\overline{M}_r \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} - \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} \geq \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} \left[ -Cu^2 + \frac{1}{C} |\nabla^{\Sigma} u|^2 \right] - CL^{-1} \int_{\Sigma \cap \{|x_{n+1}| = L\}} e^{-\frac{|z|^2}{4}} u^2.
\]

Using Proposition 2.2, we obtain

\[
\int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} |\nabla^{\Sigma} u|^2 \leq C \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} u^2 + CL^{-1} \int_{\Sigma \cap \{|x_{n+1}| = L\}} e^{-\frac{|z|^2}{4}} u^2.
\]

On the other hand, using the divergence theorem, we obtain

\[
L \int_{\Sigma \cap \{|x_{n+1}| = L\}} e^{-\frac{|z|^2}{4}} u^2 = \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} \text{div}_{\Sigma} (e^{-\frac{|z|^2}{4}} u^2 x^{\text{tan}})
= \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} \left( u^2 - \frac{1}{2} x^2_{n+1} u^2 + 2u \langle x^{\text{tan}}, \nabla^{\Sigma} u \rangle \right)
\leq \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} \left( u^2 - \frac{1}{4} x^2_{n+1} u^2 + 4 |\nabla^{\Sigma} u|^2 \right),
\]

and consequently

\[
L^2 \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} u^2 + L \int_{\Sigma \cap \{|x_{n+1}| = L\}} e^{-\frac{|z|^2}{4}} u^2
\leq C \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|z|^2}{4}} |\nabla^{\Sigma} u|^2 + CL^2 \int_{\Sigma \cap \{|x_{n+1}| \leq \frac{L}{2}\}} e^{-\frac{|z|^2}{4}} u^2.
\]
To summarize, we have shown that
\[
\int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|x|^2}{4}} |\nabla^\Sigma u|^2 \leq C \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|x|^2}{4} + \frac{1}{4} u^2}.
\]
If \( L \) is sufficiently large, this gives
\[
\int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|x|^2}{4}} |\nabla^\Sigma u|^2 \leq C \int_{\Sigma \cap \{|x_{n+1}| \leq \frac{L}{2}\}} e^{-\frac{|x|^2}{4} + \frac{1}{4} u^2}.
\]
This proves the first statement. Using the inequality
\[
0 \leq \int_{\Sigma \cap \{|x_{n+1}| \leq L\}} e^{-\frac{|x|^2}{4}} \left( u^2 - \frac{1}{4} x_{n+1}^2 u^2 + \frac{1}{4} |\nabla^\Sigma u|^2 \right),
\]
the second statement follows.

Let us denote by \( H \) the space of all functions \( f \) on \( \Sigma \) such that
\[
\| f \|_{H}^2 = \int_{\Sigma} e^{-\frac{|x|^2}{4}} f^2 < \infty.
\]
We define an operator \( L \) on the cylinder \( \Sigma \) by
\[
Lf = \Delta_{\Sigma}^n f - \frac{1}{2} \langle x^\tan, \nabla^\Sigma f \rangle + f.
\]
This can be rewritten as
\[
Lf = \frac{\partial^2}{\partial z^2} f + \frac{1}{2(n-1)} \Delta_{S^{n-1}} f - \frac{1}{2} z \frac{\partial}{\partial z} f + f.
\]
Let \( Y_m \) be a basis of eigenfunctions of \( \Delta_{S^{n-1}} \), and let \( \lambda_m \) denote the corresponding eigenfunctions. Note that \( \lambda_0 = 0, \lambda_1 = \ldots = \lambda_n = n - 1, \) and \( \lambda_{n+1} = 2n \). Moreover, \( Y_0 = 1, Y_1 = x_1, \ldots, Y_n = x_n \), up to scaling. The eigenfunctions of \( L \) are of the form \( H_l(\frac{z}{2}) Y_m \), where \( H_l \) denotes the Hermite polynomial of degree \( l \). The corresponding eigenvalues are given by \( 1 - \frac{l}{2} - \frac{\lambda_m}{2(n-1)} \). Thus, there are \( n + 2 \) eigenfunctions that correspond to positive eigenvalues of \( L \), and these are given by \( 1, z, x_1, \ldots, x_n \), up to scaling. The span of these eigenfunctions will be denoted by \( H_+ \). Moreover, there are \( n + 1 \) eigenfunctions of \( L \) with eigenvalue 0, and these are given by \( z^2 - 2, x_1 z, \ldots, x_n z \), up to scaling. The span of these eigenfunctions will be denoted by \( H_0 \). The span of all remaining eigenfunctions will be denoted by \( H_- \). With this understood, we have
\[
\langle Lf, f \rangle_{H} \geq \frac{1}{2} \| f \|_{H}^2 \quad \text{for } f \in H_+,
\]
\[
\langle Lf, f \rangle_{H} = 0 \quad \text{for } f \in H_0,
\]
\[
\langle Lf, f \rangle_{H} \leq -\frac{1}{n-1} \| f \|_{H}^2 \quad \text{for } f \in H_-.
\]
As in Lemma 2.4 in [2], we can show that the function \( u(x, \tau) \) satisfies
\[
\frac{\partial}{\partial \tau} u = Lu + E(A(\tau)x, \nu_\Sigma),
\]
where \( E \) is an error term satisfying \( |E| \leq O(\rho(\tau)^{-1}) (|u| + |\nabla_\Sigma u| + |A(\tau)|) \).

We next define \( \hat{u}(x, \tau) = u(x, \tau) \varphi\left(\frac{x_n+1}{\rho(\tau)}\right) \). The function \( \hat{u}(x, \tau) \) satisfies
\[
\frac{\partial}{\partial \tau} \hat{u} = L\hat{u} + \hat{E} + \langle A(\tau)x, \nu_\Sigma \rangle \varphi\left(\frac{x_n+1}{\rho(\tau)}\right),
\]
where \( \hat{E} \) is an error term satisfying \( \|\hat{E}\|_\mathcal{H} \leq O(\rho(\tau)^{-1}) (\|\hat{u}\|_\mathcal{H} + |A(\tau)|) \) (cf. [2], Lemma 2.5).

**Lemma 2.4.** We have \( |A(\tau)| \leq O(\rho(\tau)^{-1}) \|\hat{u}\|_\mathcal{H} \) and
\[
\left\| \frac{\partial}{\partial \tau} \hat{u} - L\hat{u} \right\|_\mathcal{H} \leq O(\rho(\tau)^{-1}) \|\hat{u}\|_\mathcal{H}.
\]

**Proof.** Analogous to [2], Lemma 2.6.

We now define
\[
U_+(\tau) := \|P_+ \hat{u}(\cdot, \tau)\|^2_\mathcal{H},
\]
\[
U_0(\tau) := \|P_0 \hat{u}(\cdot, \tau)\|^2_\mathcal{H},
\]
\[
U_-(\tau) := \|P_- \hat{u}(\cdot, \tau)\|^2_\mathcal{H},
\]
where \( P_+, P_0, P_- \) denote the orthogonal projections to \( \mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_- \), respectively. Then
\[
\frac{d}{d\tau} U_+(\tau) \geq U_+(\tau) - O(\rho(\tau)^{-1}) (U_+(\tau) + U_0(\tau) + U_-(\tau)),
\]
\[
\left| \frac{d}{d\tau} U_0(\tau) \right| \leq O(\rho(\tau)^{-1}) (U_+(\tau) + U_0(\tau) + U_-(\tau)),
\]
\[
\frac{d}{d\tau} U_-(\tau) \leq -\frac{2}{n-1} U_-(\tau) + O(\rho(\tau)^{-1}) (U_+(\tau) + U_0(\tau) + U_-(\tau)).
\]

Clearly, \( U_+(\tau) + U_0(\tau) + U_-(\tau) = \|\hat{u}\|^2_\mathcal{H} \to 0 \) as \( \tau \to -\infty \).

**Lemma 2.5.** We have \( U_0(\tau) + U_-(\tau) \leq o(1) U_+(\tau) \).

**Proof.** The ODE lemma of Merle and Zaag (cf. Lemma 5.4 in [1] or Lemma A.1 in [3]) implies that either \( U_0(\tau) + U_-(\tau) \leq o(1) U_+(\tau) \) or \( U_+(\tau) + U_-(\tau) \leq o(1) U_0(\tau) \).

We now explain how to rule out the second case. If \( U_+(\tau) + U_-(\tau) \leq o(1) U_0(\tau) \), then \( \frac{\hat{u}(\cdot, \tau)}{\|\hat{u}(\cdot, \tau)\|_\mathcal{H}} \) converges with respect to \( \cdot \) to the subspace \( \mathcal{H}_0 = \text{span}\{z^2 - 2, x_1 z, \ldots, x_n z\} \). The orthogonality relations above imply that \( \hat{u}(\cdot, \tau) \) is orthogonal to \( \langle Ax, \nu_\Sigma \rangle \) for each \( A \in \text{so}(n+1) \). In other words, \( \hat{u}(\cdot, \tau) \) is orthogonal to \( x_1 z, \ldots, x_n z \). Therefore, \( \frac{\hat{u}(\cdot, \tau)}{\|\hat{u}(\cdot, \tau)\|_\mathcal{H}} \) converges to a non-zero multiple of \( z^2 - 2 \).
Let $\Omega_\tau$ denote the region enclosed by $\tilde{M}_\tau$, and let $\mathcal{A}(z, \tau)$ denote the area of the intersection $\Omega_\tau \cap \{x_{n+1} = z\}$. By the Brunn-Minkowski inequality, the function $z \mapsto \mathcal{A}(z, \tau)^{\frac{1}{n}}$ is concave. Since $\tilde{M}_\tau$ is noncompact, it follows that the function $z \mapsto \mathcal{A}(z, \tau)^{\frac{1}{n}}$ is monotone.

Note that $\mathcal{A}(z, \tau) = \frac{1}{n} \int_{S^{n-1}} (\sqrt{2(n-1) + u(\cdot, \tau)})^n$ for $|z| \leq \rho(\tau)$. Consequently, the function $z \mapsto \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] d\theta$ is monotone. In particular, we either have

\[
\int_{-3}^{-1} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz 
\leq \int_{-1}^{1} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz 
\leq \int_{1}^{3} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz
\]

or

\[
\int_{-3}^{-1} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz 
\geq \int_{-1}^{1} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz 
\geq \int_{1}^{3} \left( \int_{S^{n-1}} [(\sqrt{2(n-1) + u(\cdot, \tau)})^n - \sqrt{2(n-1)^n}] \right) dz.
\]

However, this leads to a contradiction since $\sup |u(\cdot, \tau)| \to 0$ and $\hat{u}(\cdot, \tau)/|u(\cdot, \tau)|_H$ converges to a non-zero multiple of $z^2 - 2$. This is a contradiction. The proof of Lemma 2.5 is now complete.

**Lemma 2.6.** For each $\varepsilon > 0$, we have $\|u(\cdot, \tau)\|_{C^4(S^{n-1} \times [-10n, 10n])} \leq o(e^{(1-\varepsilon)\tau})$ and $|A(\tau)| \leq o(e^{(1-\varepsilon)\tau}).$

**Proof.** Lemma 2.5 gives $U_0(\tau) + U_-(\tau) \leq o(1)U_+(\tau)$. Substituting this back into the ODE for $U_+(\tau)$ gives

\[
\frac{d}{d\tau}U_+(\tau) \geq U_+(\tau) - o(1)U_+(\tau).
\]

Consequently, for every $\varepsilon > 0$, we have $U_+(\tau) \leq o(e^{(1-\varepsilon)\tau})$. Using the estimate $U_0(\tau) + U_-(\tau) \leq o(1)U_+(\tau)$, we obtain

\[
\|u\|_H^2 = U_+(\tau) + U_0(\tau) + U_-(\tau) \leq o(e^{(1-\varepsilon)\tau}).
\]

This implies $|A(\tau)| \leq o(1)\|u\|_H \leq o(e^{(1-\varepsilon)\tau})$. Moreover, standard interpolation inequalities imply $\|u(\cdot, \tau)\|_{C^4(S^{n-1} \times [-10n, 10n])} \leq o(e^{(1-\varepsilon)\tau})$. 

Consequently, the limit \( \lim_{\tau \to -\infty} S(\tau) \) exists. Without loss of generality, we may assume that \( \lim_{\tau \to -\infty} S(\tau) = \text{id} \). Then \( |S(\tau) - \text{id}| \leq o(e^{\frac{(1-\varepsilon)^\tau}{\tau}}) \).

**Lemma 2.7.** We have

\[
\sup_{M_\tau \cap \{ |x_{n+1}| \leq 2 \}} |x_1^2 + \ldots + x_n^2 - 2(n-1)| \leq e^{\frac{\tau}{\sqrt{\tau}}}
\]

if \( -\tau \) is sufficiently large.

**Proof.** By the previous lemma,

\[
\sup_{x \in M_\tau \cap B_{x_0}(0)} |x_1^2 + \ldots + x_n^2 - 2(n-1)| \leq o(e^{\frac{(1-\varepsilon)^\tau}{\tau}}).
\]

In view of the convexity of \( \bar{M}_\tau \), it follows that

\[
\sup_{\bar{M}_\tau \cap \{ |x_{n+1}| \leq 2 \}} (x_1^2 + \ldots + x_n^2) \leq 2(n-1) - e^{\frac{\tau}{\sqrt{\tau}}}
\]

if \( -\tau \) is sufficiently large. Let

\[
\Sigma_a = \{ (x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \ldots + x_n^2 = u_a(-x_{n+1})^2, -a \leq x_{n+1} \leq 0 \}
\]

be the self-similar shrinker constructed in [1]. By Lemma 4.4 in [1], \( u_a(2) \leq \sqrt{2(n-1) - a^{-2}} \). Since \( M_\tau \) converges to \( \Sigma \) in \( C^\infty_\text{loc} \), the surface \( \bar{M}_\tau \cap \{ x_{n+1} \leq -2 \} \) encloses the surface \( \Sigma_a \cap \{ x_{n+1} \leq -2 \} \) if \( -\tau \) is sufficiently large (depending on \( a \)). On the other hand, the estimate \( \inf_{x \in M_\tau \cap B_{x_0}(0)} (x_1^2 + \ldots + x_n^2) \geq 2(n-1) - o(e^{\frac{(1-\varepsilon)^\tau}{\tau}}) \) guarantees that the boundary \( \bar{M}_\tau \cap \{ x_{n+1} = -2 \} \) encloses the boundary \( \Sigma_a \cap \{ x_{n+1} = -2 \} \) provided that \( -\tau \) is sufficiently large and \( a \leq e^{-\frac{(1-\varepsilon)^\tau}{\tau}} \). By the maximum principle, the surface \( \bar{M}_\tau \cap \{ x_{n+1} \leq -2 \} \) encloses \( \Sigma_a \cap \{ x_{n+1} \leq -2 \} \) whenever \( -\tau \) is sufficiently large and \( a \leq e^{-\frac{(1-\varepsilon)^\tau}{\tau}} \).

By Theorem 8.2 in [1], \( u_a(y) \geq \sqrt{2(n-1)(1-a^{-2}y^2)} \). This gives

\[
\inf_{\bar{M}_\tau \cap \{ |x_{n+1}| \leq -2 \}} (x_1^2 + \ldots + x_n^2) \geq 2(n-1) - e^{\frac{\tau}{\sqrt{\tau}}}
\]

if \( -\tau \) is sufficiently large. An analogous argument gives

\[
\inf_{\bar{M}_\tau \cap \{ 2 \leq x_{n+1} \leq e^{-\frac{\tau}{\sqrt{\tau}}} \}} (x_1^2 + \ldots + x_n^2) \geq 2(n-1) - e^{\frac{\tau}{\sqrt{\tau}}}
\]

if \( -\tau \) is sufficiently large. Putting these facts together, we conclude that

\[
\inf_{\bar{M}_\tau \cap \{ |x_{n+1}| \leq e^{-\frac{\tau}{\sqrt{\tau}}} \}} (x_1^2 + \ldots + x_n^2) \geq 2(n-1) - e^{\frac{\tau}{\sqrt{\tau}}}
\]

if \( -\tau \) is sufficiently large. This completes the proof of Lemma 2.7.

**Lemma 2.8.** Let \( \varepsilon_0 > 0 \) be given. If \( -\tau \) is sufficiently large (depending on \( \varepsilon_0 \)), then every point in \( \bar{M}_\tau \cap \{ |x_{n+1}| \leq \frac{1}{2} e^{-\frac{\tau}{\sqrt{\tau}}} \} \) lies at the center of an \( \varepsilon_0 \)-neck.
Proof. This is a consequence of Lemma 2.7.

Proposition 2.9. We have
\[ \sup_{x \in \tilde{M}_\tau \cap B_{r_0}(0)} \left| x_1^2 + \ldots + x_n^2 - 2(n - 1) \right| \leq O(e^\frac{\tau}{1000}). \]

Proof. We repeat the argument above, this time with \( \rho(\tau) = e^{-\frac{\tau}{1000}} \). As above, we consider the rotated surfaces \( \tilde{M}_\tau = S(\tau)M_\tau \), where \( S(\tau) \) is a function taking values in \( SO(n + 1) \). We write each surface \( M_\tau \) as a graph over the cylinder, i.e.
\[ \{ x + u(x, \tau) : x \in \Sigma \cap B_{2e^{-\frac{\tau}{1000}}}(0) \} \subset \tilde{M}_\tau, \]
where \( \|u(\cdot, \tau)\|_{C^4(\Sigma \cap B_{2e^{-\frac{\tau}{1000}}}(0))} \leq O(e^\frac{\tau}{1000}) \). As above, the matrices \( S(\tau) \) are chosen so that the orthogonality relations
\[ \int_{\Sigma \cap B_{e^{-\frac{\tau}{1000}}}(0)} e^{-\frac{|x|^2}{4}} \langle Ax, \nu_S \rangle u(x, \tau) \varphi(e^{\frac{\tau}{1000}} x_{n+1}) = 0 \]
are satisfied for all \( A \in so(n+1) \). Then the function \( \hat{u}(x, \tau) = u(x, \tau) \varphi(e^{\frac{\tau}{1000}} x_{n+1}) \) satisfies
\[ \left\| \frac{\partial}{\partial \tau} \hat{u} - \tilde{L} \hat{u} \right\|_H \leq O(e^{\frac{\tau}{1000}}) \| \hat{u} \|_H. \]

Hence, if we define
\[ U_+(\tau) := \| P_+ \hat{u}(\cdot, \tau) \|_H^2, \]
\[ U_0(\tau) := \| P_0 \hat{u}(\cdot, \tau) \|_H^2, \]
\[ U_-(\tau) := \| P_- \hat{u}(\cdot, \tau) \|_H^2, \]
then
\[ \frac{d}{d\tau} U_+(\tau) \geq U_+(\tau) - O(e^{\frac{\tau}{1000}}) (U_+(\tau) + U_0(\tau) + U_- (\tau)), \]
\[ \left| \frac{d}{d\tau} U_0(\tau) \right| \leq O(e^{\frac{\tau}{1000}}) (U_+(\tau) + U_0(\tau) + U_- (\tau)), \]
\[ \frac{d}{d\tau} U_-(\tau) \leq -U_-(\tau) + O(e^{\frac{\tau}{1000}}) (U_+(\tau) + U_0(\tau) + U_- (\tau)). \]

As above, the ODE lemma of Merle and Zaag (cf. Lemma 5.4 in [1]) gives \( U_0(\tau) + U_-(\tau) \leq o(1)U_+(\tau) \). This gives
\[ \frac{d}{d\tau} U_+(\tau) \geq U_+(\tau) - O(e^{\frac{\tau}{1000}}) U_+(\tau), \]
hence \( U_+(\tau) \leq O(e^\tau) \). Thus, \( U_0(\tau) + U_-(\tau) \leq o(1)U_+(\tau) \leq O(e^\tau) \). Consequently, \( \| \hat{u} \|_H \leq O(e^\frac{\tau}{1000}) \). By Lemma 2.4, \( |A(\tau)| \leq O(e^\frac{\tau}{2}) \). Since \( \lim_{\tau \to -\infty} S(\tau) = id \), we obtain \( |S(\tau) - id| \leq O(e^\frac{\tau}{2}) \). Finally, \( u \) satisfies an equation of the form \( \frac{\partial}{\partial \tau} u = \tilde{L}u + \langle A(\tau)x, \nu_S \rangle \), where \( \tilde{L} \) is an elliptic operator of second order whose coefficients depend on \( u, \nabla u, \nabla^2 u, \) and \( A(\tau) \). As \( \tau \to -\infty \), the coefficients of \( \tilde{L} \) converge smoothly to the corresponding coefficients of
L. Hence, standard interior estimates for parabolic equations imply that
\[ \|u(\cdot, \tau)\|_{C^4(S^{n-1} \times [-10n, 10n])} \leq O(e^{\tau}). \]
Combining this estimate with the estimate \[ |S(\tau) - \text{id}| \leq O(e^{\tau}), \]
we conclude that
\[ \sup_{x \in M_t \cap B_{5n}(0)} |x_1^2 + \ldots + x_n^2 - 2(n - 1)| \leq O(e^{\tau}). \]

3. LOWER BOUND FOR \( H_{\text{max}}(t) \) AS \( t \to -\infty \)

Let \( M_t, t \in (-\infty, 0] \), be a noncompact ancient solution of mean curvature
flow in \( \mathbb{R}^{n+1} \) which is strictly convex, uniformly two-convex, and noncolapsed. Let \( H_{\text{max}}(t) \) be the supremum of the mean curvature of \( M_t \).

**Proposition 3.1.** For each \( t \), \( H_{\text{max}}(t) < \infty \).

**Proof.** Let us fix a time \( t \) and a small number \( \varepsilon \). It follows from Proposition 3.1 in [5] that every point in \( M_t \) which lies outside some large compact set must lie at the center of an \( \varepsilon \)-neck. Hence, if \( H_{\text{max}}(t) = \infty \), then the surface \( M_t \) contains a sequence of \( \varepsilon \)-necks with radii converging to 0, but
this cannot happen in a convex hypersurface.

**Corollary 3.2.** The function \( H_{\text{max}}(t) \) is continuous and monotone increasing in \( t \).

**Proof.** It follows from work of Haslhofer and Kleiner [4], [5] that \[ \left| \frac{\partial}{\partial t} H \right| \leq CH^3 \]
for some uniform constant \( C \). This implies that \( H_{\text{max}}(t) \) is continuous in \( t \). In particular, \( H_{\text{max}}(t) \) is uniformly bounded from above on every compact time interval. Consequently, \( H_{\text{max}}(t) \) is monotone increasing in \( t \) by Hamilton’s Harnack inequality [3].

**Proposition 3.3.** We have \( \lim \inf_{t \to -\infty} H_{\text{max}}(t) > 0 \).

**Proof.** The results in the previous section imply that
\[ \sup_{x \in (-t)^{-\frac{1}{2}} (M_t \cap B_{5n(-t)^{\frac{1}{2}}}(0))} |x_1^2 + \ldots + x_n^2 - 2(n - 1)| \leq O((-t)^{-\frac{1}{2}}). \]
Since \( M_t \) has exactly one end, we can assume without loss of generality that \( M_t \cap \{x_{n+1} \geq 0\} \) is noncompact and \( M_t \cap \{x_{n+1} \leq 0\} \) is compact. There exists a large constant \( K \) with the following property: if \(-t\) is sufficiently large, then the cross-section
\[ (-t)^{-\frac{1}{2}} (M_t \cap \{x_{n+1} = -2(-t)^{\frac{1}{2}}\}) \]
lies outside the sphere
\[ \{x_1^2 + \ldots + x_n^2 = (\sqrt{2(n - 1)} - K(-t)^{-\frac{1}{2}})^2, x_{n+1} = -2\}. \]
We now recall the self-similar shrinkers constructed in [1]. For \( a > 0 \) large, there exists a self-similar shrinker

\[
\Sigma_a = \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \ldots + x_n^2 = u_a(-x_{n+1})^2, \ -a \leq x_{n+1} \leq 0 \}
\]
satisfying \( H = \frac{1}{2} \langle x, \nu \rangle \). Consequently, the hypersurfaces

\[
\Sigma_{a,t} := (-t)^{\frac{3}{2}} \Sigma_a + (0, \ldots, 0, Ka^2)
\]

\[
= \{(x_1, \ldots, x_n, x_{n+1}) : x_1^2 + \ldots + x_n^2 = (-t)u_a((-x_{n+1} + Ka^2)(-t)^{-\frac{1}{2}})^2, \ K a^2 - a(-t)^{\frac{3}{2}} \leq x_{n+1} \leq K a^2 \}
\]
evolve by mean curvature flow.

As in [2], we can use the hypersurfaces \( \Sigma_{a,t} \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \} \) as barriers. In the limit as \( t \to -\infty \), the rescaled surfaces \((-t)^{-\frac{1}{2}} M_t \) converge in \( C^\infty_{\text{loc}} \) to the cylinder \( \{ x_1^2 + \ldots + x_n^2 = 2(n-1) \} \). Furthermore, the rescaled surfaces \((-t)^{-\frac{1}{2}} (\Sigma_{a,t} \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \}) \) converge to \( \Sigma_a \cap \{ x_{n+1} \leq -2 \} \) as \( t \to -\infty \). Consequently, \( \Sigma_{a,t} \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \} \) lies inside \( M_t \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \} \) if \( -t \) is sufficiently large (depending on \( a \)).

By our choice of \( K \), the cross-section

\[
(-t)^{-\frac{1}{2}} (M_t \cap \{ x_{n+1} = -2(-t)^{\frac{3}{2}} \})
\]
lies outside the sphere

\[
\{ x_1^2 + \ldots + x_n^2 = (\sqrt{2(n-1)} - K(-t)^{-\frac{1}{2}})^2, x_{n+1} = -2 \}.
\]

Moreover, the cross-section

\[
(-t)^{-\frac{1}{2}} (\Sigma_{a,t} \cap \{ x_{n+1} = -2(-t)^{\frac{3}{2}} \})
\]
is a sphere

\[
\{ x_1^2 + \ldots + x_n^2 = u_a(2 + Ka^2(-t)^{-\frac{3}{2}})^2, x_{n+1} = -2 \}.
\]

Using Lemma 4.4 in [1], we obtain \( u_a(2) \leq \sqrt{2(n-1)} \) and \( u_a(2) - u_a(1) \leq -a^{-2} \) if \( a \) is sufficiently large. Since the function \( u_a \) is concave, we obtain

\[
u_a(2 + Ka^2(-t)^{-\frac{3}{2}}) \leq u_a(2) + Ka^2(-t)^{-\frac{3}{2}} (u_a(2) - u_a(1)) \leq \sqrt{2(n-1)} - K(-t)^{-\frac{1}{2}}
\]

for \( -t \geq 4K^2a^2 \). Consequently, the cross-section \( \Sigma_{a,t} \cap \{ x_{n+1} = -2(-t)^{\frac{3}{2}} \} \) lies inside the cross-section \( M_t \cap \{ x_{n+1} = -2(-t)^{\frac{3}{2}} \} \) whenever \( -t \geq 4K^2a^2 \) and \( a \) is sufficiently large. By the maximum principle, the hypersurface \( \Sigma_{a,t} \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \} \) lies inside the hypersurface \( M_t \cap \{ x_{n+1} \leq -2(-t)^{\frac{3}{2}} \} \) whenever \( -t \geq 4K^2a^2 \) and \( a \) is sufficiently large. For \( -t = 4K^2a^2 \), the tip of \( \Sigma_{a,t} \) has distance \( a(-t)^{\frac{3}{2}} - Ka^2 = Ka^2 = -\frac{1}{4K} \) from the origin. Consequently, the intersection \( M_t \cap \{ x_1 = \ldots = x_n = 0, x_{n+1} \leq \frac{1}{4K} \} \) is non-empty if \( -t \) is sufficiently large. In particular, \( \limsup_{t \to -\infty} H_{\max}(t) > 0 \). Since \( H_{\max}(t) \) is monotone increasing in \( t \), it follows that \( \liminf_{t \to -\infty} H_{\max}(t) > 0 \).
4. THE NECK IMPROVEMENT THEOREM

Definition 4.1. Let \( K = \{ K_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) be a collection of vector fields in \( \mathbb{R}^{n+1} \). We say that \( K \) is a normalized set of rotation vector fields if there exists an orthonormal basis \( \{ J_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) of \( \text{so}(n) \subset \text{so}(n+1) \), a matrix \( S \in O(n+1) \) and a point \( q \in \mathbb{R}^{n+1} \) such that

\[
K_\alpha(x) = SJ_\alpha S^{-1}(x - q).
\]

Note that we do not require that the origin lies on the axis of rotation.

Lemma 4.2. We can find a large constant \( C \) and small constant \( \varepsilon_0 > 0 \) with the following property. Let \( M \) be a hypersurface in \( \mathbb{R}^{n+1} \), and assume that, after suitable rescaling, \( M \) is \( \varepsilon_0 \)-close (in the \( C^4 \)-norm) to a cylinder \( S^1 \times [-5, 5] \) of radius 1. Suppose that \( K^{(1)} = \{ K_\alpha^{(1)} : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) and \( K^{(2)} = \{ K_\alpha^{(2)} : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) are two normalized sets of rotation vector fields with the following properties:

- \( \max_\alpha |K_\alpha^{(1)}| H \leq 10n \) and \( \max_\alpha |K_\alpha^{(2)}| H \leq 10n \) at the point \( \bar{x} \).
- \( \max_\alpha |\langle K_\alpha^{(1)}, \nu \rangle| H \leq \varepsilon \) and \( \max_\alpha |\langle K_\alpha^{(2)}, \nu \rangle| H \leq \varepsilon \) in a geodesic ball around \( \bar{x} \) in \( M \) of radius \( H(\bar{x})^{-1} \).

Then

\[
\inf_{\omega \in O(n+1)} \sup_{B_{100H(\bar{x})^{-1}}(\bar{x})} \max_\alpha \left| K_\alpha^{(1)} - \sum_{\beta=1}^{\frac{n(n-1)}{2}} \omega_{\alpha\beta} K_\beta^{(2)} \right| H(\bar{x}) \leq C\varepsilon.
\]

Proof. Analogous to [2].

Definition 4.3. Let \( M_t \) be a solution of mean curvature flow. We say that a point \( (\bar{x}, \bar{t}) \) is \( \varepsilon \)-symmetric if there exists a normalized set of rotation vector fields \( K = \{ K_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) such that \( \max_\alpha |K_\alpha| H \leq 10n \) at the point \( (\bar{x}, \bar{t}) \) and \( \max_\alpha |\langle K_\alpha, \nu \rangle| H \leq \varepsilon \) in the parabolic neighborhood \( \mathcal{P}(\bar{x}, \bar{t}, 10, 100) \)[1]

Note that the condition that \( \max_\alpha |K_\alpha| H \leq 10n \) at the point \( (\bar{x}, \bar{t}) \) ensures that the distance of the point \( \bar{x} \) from the axis of rotation of \( K = \{ K_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) is at most \( C(n) H(\bar{x}, \bar{t})^{-1} \).

Theorem 4.4 (Neck Improvement Theorem). There exists a large constant \( L \) and a small constant \( \varepsilon_1 \) with the following property. Suppose that \( M_t \) is a solution of mean curvature flow. Moreover, suppose that \((\bar{x}, \bar{t})\) is a point in space-time with the property that every point in \( \mathcal{P}(\bar{x}, \bar{t}, L, L^2) \) lies at the center of an \( \varepsilon_1 \)-neck and is \( \varepsilon \)-symmetric, where \( \varepsilon \leq \varepsilon_1 \). Then \( (\bar{x}, \bar{t}) \) is \( \frac{\varepsilon}{2} \)-symmetric.

[1] See [7], pp. 189–190, for the definition of \( \mathcal{P}(\bar{x}, \bar{t}, 10, 100) \).
Proof. Without loss of generality, we assume $t = -1$ and $H(\bar{x}, -1) = \sqrt{\frac{n-1}{2}}$. We will assume throughout that $L$ is sufficiently large, and $\varepsilon_1$ is sufficiently small depending on $L$.

Step 1: Given any point $(x_0, t_0) \in \hat{P}(\bar{x}, -1, L, L^2)$, we can find a normalized set of rotation vector fields $K^{(x_0, t_0)} = \{K^{(x_0, t_0)}_{\alpha} : 1 \leq \alpha \leq \frac{n(n-1)}{2} \}$ such that $\max_\alpha |\langle K^{(x_0, t_0)}_{\alpha}, \nu \rangle| H \leq \varepsilon$ on the parabolic neighborhood $\hat{P}(x_0, t_0, 10, 100)$. Note that the axis of rotation depends on $(x_0, t_0)$. Using Lemma 4.2, we obtain

$$\inf_{\omega \in O(\frac{n(n-1)}{2})} \sup_{B_{100}(\bar{x})} \max_\alpha \left| K^{(x,-1)}_{\alpha} - \sum_{\beta=1}^{\frac{n(n-1)}{2}} \omega_{\alpha\beta} K^{(x_0, t_0)}_{\beta} \right| H(\bar{x}) \leq C(L)\varepsilon$$

for each point $(x_0, t_0) \in \hat{P}(\bar{x}, -1, L, L^2)$. Without loss of generality, we may assume

$$\sup_{B_{100}(\bar{x})} \max_\alpha |K^{(x,-1)}_{\alpha} - K^{(x_0, t_0)}_{\alpha}| H(\bar{x}) \leq C(L)\varepsilon$$

for each point $(x_0, t_0) \in \hat{P}(\bar{x}, -1, L, L^2)$. For abbreviation, we put $\bar{K} := K^{(x, -1)}$. Without loss of generality, we may assume that the axis of rotation of $\bar{K} = \{\bar{K}_{\alpha} : 1 \leq \alpha \leq \frac{n(n-1)}{2} \}$ is the $x_{n+1}$-axis; that is, $\bar{K}_{\alpha}(x) = J_{\alpha}x$ for some orthonormal basis $\{J_{\alpha} : 1 \leq \alpha \leq \frac{n(n-1)}{2} \}$ of $so(n) \subset so(n + 1)$.

Step 2: By assumption, every point in $\hat{P}(\bar{x}, 1, L, L^2)$ lies at the center of an $\varepsilon_1$-neck. Hence, we may write $M_t$ as a graph over the $x_{n+1}$-axis, so that

$$\left\{ (r(\theta, z, t, \theta, z)) : \theta \in S^{n-1}, z \in \left[ -\frac{L}{4}, \frac{L}{4} \right] \right\} \subset M_t.$$ 

Moreover, the function $r(\theta, z, t) - (-2(n-1)t)^{\frac{1}{2}}$ is bounded by $C(L)\varepsilon_1$ in the $C^{100}$-norm. A straightforward computation gives

$$\nu = \frac{1}{\sqrt{1 + r^{-2} |\nabla S^{n-1} r|^2 + \left( \frac{\partial r}{\partial z} \right)^2}} \left( \theta - r^{-1} \nabla S^{n-1} r, -\frac{\partial r}{\partial z} \right),$$

hence

$$\langle \bar{K}_{\alpha}, \nu \rangle = -\frac{\langle J_{\alpha}, \nabla S^{n-1} r \rangle}{\sqrt{1 + r^{-2} |\nabla S^{n-1} r|^2 + \left( \frac{\partial r}{\partial z} \right)^2}},$$

where $\nabla S^{n-1} r$ represents the gradient of the function $r$ with respect to the angular variables. Using the estimates $\max_\alpha |\langle \bar{K}_{\alpha}, \nu \rangle| \leq C(L)\varepsilon$ and $|\nabla S^{n-1} r| + |\frac{\partial r}{\partial z}| \leq C(L)\varepsilon_1$, we obtain $|\nabla S^{n-1} r| \leq C(L)\varepsilon$ and

$$|\langle \bar{K}_{\alpha}, \nu \rangle + \langle J_{\alpha}, \nabla S^{n-1} r \rangle| \leq C(L)\varepsilon_1 \varepsilon.$$

Moreover, the identity $\text{div } S^{n-1}(J_{\alpha}) = 0$ gives $\text{div } S^{n-1}(r(\theta, z, t) J_{\alpha}) = \langle J_{\alpha}, \nabla S^{n-1} r \rangle$, hence

$$|\langle \bar{K}_{\alpha}, \nu \rangle + \text{div } S^{n-1}(r(\theta, z, t) J_{\alpha})| \leq C(L)\varepsilon_1 \varepsilon.$$
Step 3: Let us fix an index $\alpha \in \{1, \ldots, \frac{n(n-1)}{2}\}$. For each point $(x_0, t_0) \in \hat{P}(\bar{x}, -1, L, L^2)$, the vector field $K^{(x_0, t_0)}_\alpha$ satisfies
\[
|\langle K^{(x_0, t_0)}_\alpha, \nu \rangle| \leq C\varepsilon (-t_0)^{\frac{1}{2}}
\]
on the parabolic neighborhood $\hat{P}(x_0, t_0, 10, 100)$. There exist real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ (depending on $(x_0, t_0)$) such that
\[
|a_1| + \ldots + |a_n| \leq C(L)\varepsilon,
\]
\[
|b_1| + \ldots + |b_n| \leq C(L)\varepsilon,
\]
and
\[
|\langle K^{(x_0, t_0)}_\alpha, \nu \rangle - (a_1 x_1 + \ldots + a_n x_n) - (b_1 x_1 + \ldots + b_n x_n)z| \leq C(L)\varepsilon_1 \varepsilon
\]
on the parabolic neighborhood $\hat{P}(x_0, t_0, 10, 100)$. Consequently, the function $u = \langle K^{(x_0, t_0)}_\alpha, \nu \rangle$ satisfies
\[
|u - (a_1 x_1 + \ldots + a_n x_n) - (b_1 x_1 + \ldots + b_n x_n)z| \leq C\varepsilon (-t_0)^{\frac{1}{2}} + C(L)\varepsilon_1 \varepsilon
\]
on the parabolic neighborhood $\hat{P}(x_0, t_0, 10, 100)$. The function $u = \langle K^{(x_0, t_0)}_\alpha, \nu \rangle$ satisfies the evolution equation
\[
\frac{\partial}{\partial t} u = \Delta_M u + |A|^2 u.
\]
Using the estimate $|u| \leq C(L)\varepsilon$ together with standard interior estimates for parabolic equations, we obtain $|\nabla u| + |\nabla^2 u| \leq C(L)\varepsilon$ on the parabolic neighborhood $\hat{P}(\bar{x}, -1, \frac{L}{2}, \frac{L^2}{4})$. This implies
\[
\left| \frac{\partial}{\partial t} u - \frac{\partial^2}{\partial z^2} u - \frac{1}{(-2(n-1)t)} \Delta_{S^{n-1}} u - \frac{1}{(-2t)} u \right| \leq C(L)\varepsilon_1 \varepsilon
\]
for $z \in [-\frac{L}{2}, \frac{L}{2}]$ and $t \in [-\frac{L^2}{16}, -1]$. We denote by $\tilde{u}$ the solution of the linear equation
\[
\frac{\partial}{\partial t} \tilde{u} = \frac{\partial^2}{\partial z^2} \tilde{u} + \frac{1}{(-2(n-1)t)} \Delta_{S^{n-1}} \tilde{u} + \frac{1}{(-2t)} \tilde{u}
\]
in the parabolic cylinder $\{z \in [-\frac{L}{2}, \frac{L}{2}], t \in [-\frac{L^2}{16}, -1]\}$ such that $\tilde{u} = u$ on the parabolic boundary $\{|z| = \frac{L}{2}\} \cup \{t = -\frac{L^2}{16}\}$. The maximum principle gives
\[
|u - \tilde{u}| \leq C(L)\varepsilon_1 \varepsilon
\]
in the parabolic cylinder $\{z \in [-\frac{L}{2}, \frac{L}{2}], t \in [-\frac{L^2}{16}, -1]\}$. In order to analyze the PDE for $\tilde{u}$, we perform separation of variables. For each $m$, we put
\[
v_m(z, t) = \int_{S^{n-1}} \tilde{u}(\theta, z, t) Y_m(\theta) d\theta.
\]
Then
\[
\frac{\partial}{\partial t} v_m(z, t) = \frac{\partial^2}{\partial z^2} v_m(z, t) + \frac{n - 1 - \lambda_m}{2(n-1)(-t)} v_m(z, t).
\]
Hence, the rescaled function \( \hat{v}_m(z, t) = (-t)^{\frac{n-1-\lambda_m}{2(n-1)}} v_m(z, t) \) satisfies
\[
\frac{\partial}{\partial t} \hat{v}_m(z, t) = \frac{\partial^2}{\partial z^2} \hat{v}_m(z, t).
\]

We first consider the case when \( m \geq n + 1 \), so that \( \lambda_m \geq 2n \). Using the estimate
\[
|v_m(z, t)| \leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon})(-t)^{\frac{n}{2}},
\]
we obtain
\[
|\hat{v}_m(z, t)| \leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon})(-t)^{1-\frac{\lambda_m}{2(n-1)}}
\]
in the parabolic cylinder \( \{ z \in [-\frac{L^2}{4}, \frac{L^2}{4}], t \in [-\frac{L^2}{16}, -1] \} \). Using the solution formula for the one-dimensional heat equation with Dirichlet boundary condition on the rectangle \( [-\frac{L^2}{4}, \frac{L^2}{4}] \times [-\frac{L^2}{16}, -1] \), we obtain
\[
|\hat{v}_m(z, t)| \leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) \left( \frac{L}{4} \right)^{2-\frac{\lambda_m}{n-1}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L \int_{-\frac{L^2}{16}}^{t} e^{-\frac{L^2}{100(t-s)}} (t-s)^{-\frac{3}{2}} (-s)^{1-\frac{\lambda_m}{2(n-1)}} ds
\]
\[
\leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) \left( \frac{L}{4} \right)^{2-\frac{\lambda_m}{n-1}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} \int_{-\frac{L^2}{16}}^{t} e^{-\frac{L^2}{200(t-s)}} (-s)^{1-\frac{\lambda_m}{2(n-1)}} ds
\]
\[
\leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) \left( \frac{L}{4} \right)^{2-\frac{\lambda_m}{n-1}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} \int_{-\frac{L^2}{16}}^{(1+\frac{1}{\sqrt{\lambda_m}})t} e^{-\frac{L^2}{200(t-s)}} (-s)^{1-\frac{\lambda_m}{2(n-1)}} ds
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} \int_{(1+\frac{1}{\sqrt{\lambda_m}})t}^{t} e^{-\frac{L^2}{200(t-s)}} (-s)^{1-\frac{\lambda_m}{2(n-1)}} ds
\]
\[
\leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) \left( \frac{L}{4} \right)^{2-\frac{\lambda_m}{n-1}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} \left( 1 + \frac{1}{\sqrt{\lambda_m}} \right)^{\frac{2n-1-\lambda_m}{2(n-1)}} (-t)^{2n-1-\frac{\lambda_m}{2(n-1)}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} e^{-\frac{L^2}{200(-t)}} (-t)^{2n-1-\frac{\lambda_m}{2(n-1)}}
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \). This implies
\[
|v_m(z, t)| \leq (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) \left( \frac{L^2}{16(-t)} \right)^{1-\frac{\lambda_m}{2(n-1)}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} \left( 1 + \frac{1}{\sqrt{\lambda_m}} \right)^{\frac{2n-1-\lambda_m}{2(n-1)}} (-t)^{2n-1-\frac{\lambda_m}{2(n-1)}}
\]
\[
+ (C \varepsilon + C(L)\varepsilon_{1\varepsilon}) L^{-\frac{1}{n-1}} e^{-\frac{L^2}{200(-t)}}
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \). Summation over \( m \) gives
\[
\sum_{m=n+1}^{\infty} |v_m(z, t)| \leq CL^{-\frac{1}{4}} + C(L)\varepsilon \varepsilon
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \).

We next consider the case when \( 1 \leq m \leq n \), so that \( \lambda_m = n - 1 \). In this case, the function \( v_m(z, t) \) satisfies
\[
\frac{\partial}{\partial t} v_m(z, t) = \frac{\partial^2}{\partial z^2} v_m(z, t).
\]
Moreover, given any point \((z_0, t_0) \in [-\frac{L}{4}, \frac{L}{4}] \times [-\frac{L^2}{16}, -1]\), we can find real numbers \( a_1, \ldots, a_n, b_1, \ldots, b_n \) (depending on \((x_0, t_0)\)) such that
\[
|a_1| + \ldots + |a_n| \leq C(L)\varepsilon, \\
|b_1| + \ldots + |b_n| \leq C(L)\varepsilon,
\]
and
\[
|v_m(z, t) - (a_m + b_m z)| \leq C\varepsilon(-t_0)^{\frac{3}{2}} + C(L)\varepsilon \varepsilon
\]
for \( z \in [z_0 - (-t_0)^{\frac{3}{4}}, z_0 + (-t_0)^{\frac{3}{4}}] \) and \( t \in [t_0, t_0] \). Using interior estimates for the linear heat equation, we obtain
\[
\left| \frac{\partial^2 v_m}{\partial z^2}(z, t) \right| \leq (C\varepsilon + C(L)\varepsilon \varepsilon)(-t)^{-\frac{3}{4}}
\]
in the parabolic cylinder \( \{z \in [-\frac{L}{4}, \frac{L}{4}], t \in [-\frac{L^2}{16}, -1]\} \). Using the solution formula for the one-dimensional heat equation with Dirichlet boundary condition on the rectangle \([-\frac{L}{4}, \frac{L}{4}] \times [-\frac{L^2}{16}, -1]\), we obtain
\[
\left| \frac{\partial^2 v_m}{\partial z^2}(z, t) \right| \leq (C\varepsilon + C(L)\varepsilon \varepsilon) \left(\frac{L}{4}\right)^{-1} + (C\varepsilon + C(L)\varepsilon \varepsilon) L \int_{\frac{L^2}{16}}^{t} e^{-\frac{L^2}{16} (t-s)^{-\frac{3}{4}} (-s)^{-\frac{3}{4}}} ds \\
\leq (C\varepsilon + C(L)\varepsilon \varepsilon) \left(\frac{L}{4}\right)^{-1} + (C\varepsilon + C(L)\varepsilon \varepsilon) L^{-2} \int_{\frac{L^2}{16}}^{t} (-s)^{-\frac{3}{4}} ds \\
\leq (C\varepsilon + C(L)\varepsilon \varepsilon) L^{-1}
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \). Consequently, there exist real numbers \( A_1, \ldots, A_n, B_1, \ldots, B_n \) such that
\[
|v_m(z, t) - (A_m + B_m z)| \leq C\varepsilon L^{-1} + C(L)\varepsilon \varepsilon
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \).

Finally, we consider the case \( m = 0 \). By the results in Step 2, the function \( u = \langle K_\alpha, \nu \rangle \) satisfies
\[
|u(\theta, z, t) + \text{div}^{S^{n-1}}(r(\theta, z, t) J_\theta \theta)| \leq C(L)\varepsilon \varepsilon.
\]
Integrating over $\theta \in S^{n-1}$ gives

$$\left| \int_{S^{n-1}} u(\theta, z, t) \, d\theta \right| \leq C(L)\varepsilon_1\varepsilon,$$

hence

$$|v_0(z, t)| \leq C(L)\varepsilon_1\varepsilon$$

for $z \in [-20n, 20n]$ and $t \in [-400n^2, -1]$.

**Step 4:** To summarize, for each $\alpha \in \{1, \ldots, \frac{n(n-1)}{2}\}$ we can find real numbers $A_{\alpha,1}, \ldots, A_{\alpha,n}, B_{\alpha,1}, \ldots, B_{\alpha,n}$ such that

$$|A_{\alpha,1}| + \ldots + |A_{\alpha,n}| \leq C(L)\varepsilon,$$

$$|B_{\alpha,1}| + \ldots + |B_{\alpha,n}| \leq C(L)\varepsilon,$$

and

$$|u_\alpha - (A_{\alpha,1}x_1 + \ldots + A_{\alpha,n}x_n) - (B_{\alpha,1}x_1 + \ldots + B_{\alpha,n}x_n)z| \leq CL\varepsilon \alpha,ij + C(L)\varepsilon_1\varepsilon$$

for $z \in [-20n, 20n]$ and $t \in [-400n^2, -1]$, where $u_\alpha(\theta, z, t) := \langle K_\alpha, \nu \rangle$.

**Step 5:** For each point $\theta \in S^{n-1}$, we denote by $\theta_1, \ldots, \theta_n$ the Cartesian coordinates of $\theta$. For each $i \in \{1, \ldots, n\}$, we define

$$E_i = \int_{S^{n-1}} r(\theta, 0, -1) \theta_i \, d\theta$$

and

$$F_i = \frac{1}{2} \int_{S^{n-1}} [r(\theta, 1, -1) - r(\theta, -1, -1)] \theta_i \, d\theta.$$

The inequality $|\nabla^{S^{n-1}} r| \leq C(L)\varepsilon$ implies $|E_i|, |F_i| \leq C(L)\varepsilon$ for $i \in \{1, \ldots, n\}$. By the results in Step 2, the function $u_\alpha = \langle K_\alpha, \nu \rangle$ satisfies

$$|u_\alpha(\theta, z, t) + \text{div}^{S^{n-1}} (r(\theta, z, t) J_\alpha \theta)| \leq C(L)\varepsilon_1\varepsilon.$$

A direct calculation gives

$$\text{div}^{S^{n-1}} (r(\theta, z, t) J_\alpha \theta) \theta_i = \text{div}^{S^{n-1}} (r(\theta, z, t) \theta_i J_\alpha \theta) - r(\theta, z, t) \sum_{j=1}^{n} J_{\alpha,ij} \theta_j,$$

where $J_{\alpha,ij}$ denote the components of the anti-symmetric matrix $J_\alpha$. Putting these facts together, we obtain

$$\left| u_\alpha(\theta, z, t) \theta_i + \text{div}^{S^{n-1}} (r(\theta, z, t) \theta_i J_\alpha \theta) - r(\theta, z, t) \sum_{j=1}^{n} J_{\alpha,ij} \theta_j \right| \leq C(L)\varepsilon_1\varepsilon$$

for all $i \in \{1, \ldots, n\}$. Integrating over $\theta \in S^{n-1}$ gives

$$\max_{\alpha} \left| \int_{S^{n-1}} u_\alpha(\theta, 0, -1) \theta_i \, d\theta - \sum_{j=1}^{n} J_{\alpha,ij} E_j \right| \leq C(L)\varepsilon_1\varepsilon$$
and
\[
\max_\alpha \left| \frac{1}{2} \int_{S^{n-1}} \left[ u_\alpha(\theta, 1, -1) - u_\alpha(\theta, -1, -1) \right] \theta_i d\theta - \sum_{j=1}^{n} J_{\alpha,ij} F_j \right| \leq C(L)\varepsilon_1 \varepsilon.
\]

On the other hand, using the estimate for \( u_\alpha - (A_{\alpha,1}x_1 + \ldots + A_{\alpha,n}x_n) - (B_{\alpha,1}x_1 + \ldots + B_{\alpha,n}x_n)z \) in Step 4, we obtain
\[
\max_\alpha \left| \frac{1}{2} \int_{S^{n-1}} \left[ u_\alpha(\theta, 0, -1) \right] \theta_i d\theta - c(n) A_{\alpha,i} \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon
\]
and
\[
\max_\alpha \left| \frac{1}{2} \int_{S^{n-1}} \left[ u_\alpha(\theta, 1, -1) - u_\alpha(\theta, -1, -1) \right] \theta_i d\theta - c(n) B_{\alpha,i} \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon,
\]
where \( c(n) \) is a positive constant that depends only on the dimension. Putting these facts together, we obtain
\[
\max_\alpha \left| c(n) A_{\alpha,i} - \sum_{j=1}^{n} J_{\alpha,ij} E_j \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon
\]
and
\[
\max_\alpha \left| c(n) B_{\alpha,i} - \sum_{j=1}^{n} J_{\alpha,ij} F_j \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon.
\]

Substituting this back into the estimate for \( u_\alpha - (A_{\alpha,1}x_1 + \ldots + A_{\alpha,n}x_n) - (B_{\alpha,1}x_1 + \ldots + B_{\alpha,n}x_n)z \) in Step 4, we finally conclude
\[
\max_\alpha \left| c(n) \langle \tilde{K}_\alpha, \nu \rangle - \sum_{j=1}^{n} J_{\alpha,ij} (E_jx_i + F_jx_iz) \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \).

**Step 6:** By the results in Step 4 and Step 5, we can find a normalized set of rotation vector fields \( \tilde{K} = \{ \tilde{K}_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) (with a common axis of rotation) such that
\[
\max_\alpha \left| \langle \tilde{K}_\alpha, \nu \rangle \right| \leq CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon
\]
for \( z \in [-20n, 20n] \) and \( t \in [-400n^2, -1] \). Note that the axis of rotation of the vector fields \( \tilde{K} = \{ \tilde{K}_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \) is different from the axis of rotation of the vector fields \( \tilde{K} = \{ \tilde{K}_\alpha : 1 \leq \alpha \leq \frac{n(n-1)}{2} \} \); the change in the axis of rotation is determined by the coefficients \( E_i \) and \( F_i \) defined in Step 5.
Consequently, \( (\tilde{x}, -1) \) is \((CL^{-\frac{1}{n-1}} \varepsilon + C(L)\varepsilon_1 \varepsilon)-\)symmetric. In particular, if we choose \( L \) sufficiently large and \( \varepsilon_1 \) sufficiently small (depending on \( L \)), then \( (\tilde{x}, -1) \) is \( \frac{1}{2} \)-symmetric. This completes the proof of the Neck Improvement Theorem.
5. Proof of rotational symmetry

In this section, we establish rotational symmetry. Let \( M_t, t \in (-\infty, 0] \), be a noncompact ancient solution of mean curvature flow in \( \mathbb{R}^{n+1} \) which is strictly convex, uniformly two-convex, and noncollapsed. As in [2], if \(-t\) is sufficiently large, there exists a unique point \( p_t \in M_t \) where the mean curvature attains its maximum. Moreover, this is a non-degenerate maximum in the sense that the Hessian of the mean curvature at \( p_t \) is negative definite.

Let \( \varepsilon_1 \) and \( L \) be the constants in the Neck Improvement Theorem. Recall that \( H_{\text{max}}(t) \) is uniformly bounded from below. By Proposition 3.1 in [5], we can find a large constant \( \Lambda \) such that the following holds. If \( x \) is a point on \( M_t \) such that \( |x - p_t| \geq \Lambda \), then \( x \) lies at the center of an \( \varepsilon_1 \)-neck and furthermore \( H(x, t) |x - p_t| \geq 10^6 L \).

**Proposition 5.1.** There exists a number \( T \) with the following property. If \( t \leq T, x \in M_t \) and \( |x - p_t| \geq 2^\frac{400}{n} \Lambda \), then \((x, t)\) is \( 2^{-j} \varepsilon_1 \)-symmetric.

**Proof.** This follows by a repeated application of the Neck Improvement Theorem. The argument is analogous to [2].

**Theorem 5.2.** The surface \( M_t \) is rotationally symmetric for each \( t \leq T \).

**Proof.** The argument is similar to [2]. We fix a time \( \bar{t} \leq T \). For each \( j \), let \( \Omega^{(j)} \) be the set of all points \((x, t)\) in space-time satisfying \( t \leq \bar{t} \) and \( |x - p_t| \leq 2^\frac{400}{n} \Lambda \). If \( j \) is sufficiently large, then \( H(x, t) \geq n \cdot 2^{-\frac{400}{n}} \) for each point \((x, t) \in \Omega^{(j)} \). Proposition 5.1 guarantees that every point \((x, t) \in \partial \Omega^{(j)} \) is \( 2^{-j} \varepsilon_1 \)-symmetric. Consequently, given any point \((x, t) \in \partial \Omega^{(j)} \), we can find a normalized set of rotation vector fields \( \mathcal{K}^{(x, t)} = \{K^{(x, t)}_{\alpha} : 1 \leq \alpha \leq \frac{n(n-1)}{2}\} \) such that \( \max_{\alpha} |\langle K^{(x, t)}_{\alpha}, \nu \rangle| H \leq 2^{-j} \varepsilon_1 \) on \( \hat{\mathcal{P}}(x, t, 10, 100) \). Lemma 4.2 allows us to control how the axis of rotation of \( \mathcal{K}^{(x, t)} \) varies as we vary the point \((x, t)\). More precisely, if \((x_1, t_1)\) and \((x_2, t_2)\) are in \( \partial \Omega^{(j)} \) and \((x_2, t_2) \in \mathcal{P}(x_1, t_1, 1, 1)\), then

\[
\inf_{\omega \in O(n(n-1)/2)} \sup_{B_{10H(x_2, t_2)}^{-1}(x_2)} \max_{\alpha} \left| K^{(x_1, t_1)}_{\alpha} - \sum_{\beta=1}^{n(n-1)/2} \omega_{\alpha\beta} K^{(x_2, t_2)}_{\beta} \right| \leq C 2^{-j} H(x_2, t_2)^{-1}.
\]

Therefore, we can find a normalized set of rotation vector fields \( \mathcal{K}^{(j)} = \{K^{(j)}_{\alpha} : 1 \leq \alpha \leq \frac{n(n-1)}{2}\} \) such that the following holds: if \((x, t)\) is a point in \( \partial \Omega^{(j)} \) satisfying \( \bar{t} - 2^\frac{400}{n} \leq t \leq \bar{t} \), then

\[
\inf_{\omega \in O(n(n-1)/2)} \max_{\alpha} \left| K^{(j)}_{\alpha} - \sum_{\beta=1}^{n(n-1)/2} \omega_{\alpha\beta} K^{(x, t)}_{\beta} \right| \leq C 2^{-j}.
\]
at the point \((x,t)\). From this, we deduce that \(\max_{\alpha} |\langle K^{(j)}_{\alpha}, \nu \rangle| \leq C 2^{-\frac{t}{2}}\) for all points \((x,t) \in \partial \Omega^{(j)}\) satisfying \(\bar{t} - \frac{2}{400} \leq t \leq \bar{t}\). Finally, we note that \(\max_{\alpha} |\langle K^{(j)}_{\alpha}, \nu \rangle| \leq C 2^{\frac{1}{400}}\) for all points \((x,t) \in \Omega^{(j)}\) with \(t = \bar{t} - \frac{4}{400}\).

For each \(\alpha \in \{1, \ldots, \frac{n(n-1)}{2}\}\), we define a function \(f^{(j)}_{\alpha} : \Omega^{(j)} \rightarrow \mathbb{R}\) by

\[
f^{(j)}_{\alpha} := \exp(-2^{\frac{t}{400}}(\bar{t} - t)) \frac{\langle K^{(j)}_{\alpha}, \nu \rangle}{H - 2^{\frac{t}{400}}}.\]

The estimate for \(\langle K^{(j)}_{\alpha}, \nu \rangle\) implies

\[|f^{(j)}_{\alpha}(x,t)| \leq C 2^{-\frac{t}{4}}\]

for all points \((x,t) \in \partial \Omega^{(j)}\) satisfying \(\bar{t} - \frac{2}{400} \leq t \leq \bar{t}\) and for all points \((x,t) \in \Omega^{(j)}\) with \(t = \bar{t} - \frac{4}{400}\). On the other hand, the function \(f^{(j)}_{\alpha}\) satisfies the evolution equation

\[
\frac{\partial}{\partial t} f^{(j)}_{\alpha} = \Delta f^{(j)}_{\alpha} + \frac{2}{H - 2^{\frac{t}{400}}} \langle \nabla H, \nabla f^{(j)}_{\alpha} \rangle - 2^{\frac{t}{400}} \left( \frac{|A|^2}{H - 2^{\frac{t}{400}}} - 2^{\frac{t}{400}}\right) f^{(j)}_{\alpha}.\]

On the set \(\Omega^{(j)}\), we have

\[
\frac{|A|^2}{H - 2^{\frac{t}{400}}} - 2^{\frac{t}{400}} \geq \frac{1}{n} \frac{H^2}{H - 2^{\frac{t}{400}}} - 2^{\frac{t}{400}} \geq \frac{1}{n} H - 2^{\frac{t}{400}} \geq 0.
\]

By the maximum principle, we obtain

\[
\sup_{(x,t) \in \Omega^{(j)}, \bar{t} - \frac{4}{400} \leq t \leq \bar{t}} |f^{(j)}_{\alpha}(x,t)| \leq \max \left\{ \sup_{(x,t) \in \partial \Omega^{(j)}, \bar{t} - \frac{4}{400} \leq t \leq \bar{t}} |f^{(j)}_{\alpha}(x,t)|, \sup_{(x,t) \in \Omega^{(j)}, t = \bar{t} - \frac{4}{400}} |f^{(j)}_{\alpha}(x,t)| \right\} \leq C 2^{-\frac{t}{4}}
\]

for each \(\alpha \in \{1, \ldots, \frac{n(n-1)}{2}\}\). Thus, we conclude that \(\langle K^{(j)}_{\alpha}, \nu \rangle\) is rotationally symmetric for all \(t \in (-\infty, 0]\). Once we know that \(M_t\) is rotationally symmetric for \(-t\) sufficiently large, it follows from standard arguments that \(M_t\) is rotationally symmetric for all \(t \in (-\infty, 0]\).
6. Uniqueness of ancient solutions with rotational symmetry

Let \( M_t \) be an ancient solution to the mean curvature flow in \( \mathbb{R}^{n+1} \) which is strictly convex, uniformly two-convex, and noncollapsing. We may assume that \( M_t \) is symmetric with respect to the \( x_{n+1} \)-axis. Let us write \( M_t \) as a graph of a rotationally symmetric function \( f \) on \( \mathbb{R}^n \). The function \( f \) satisfies the equation

\[
 f_t = \frac{f_{rr}}{1 + f^2_r} + \frac{n-1}{r} f_r.
\]

Note that \( f(r, t) \) may not be defined for all \( r \).

Conversely, we may write the radius \( r \) as a function of \( (z, t) \), so that

\[
 f(r(z, t), t) = z.
\]

Then \( r(z, t) \) satisfies the equation

\[
 r_t = \frac{r_{zz}}{1 + r^2_z} - \frac{n-1}{r}.
\]

Since \( M_t \) is convex, we have

\[
 r > 0, \quad r_z > 0, \quad r_t < 0, \quad r_{zz} < 0.
\]

Without loss of generality, we assume that the tip of \( M_0 \) is at the origin. In other words, \( f(0,0) = 0 \) and \( r(0,0) = 0 \).

As in [2], let \( q_t = (0, \ldots, 0, f(0, t)) \) denote the tip of \( M_t \), and let \( H_{\text{tip}}(t) \) denote the mean curvature of \( M_t \) at the tip \( q_t \). By the Harnack inequality, the limit \( H := \lim_{t \to -\infty} H_{\text{tip}}(t) \) exists. Using results in Section 3, we obtain \( |q_t| \geq c(-t) \) for \(-t \) sufficiently large. This gives \( H > 0 \).

We first prove that \( f_t(r, t) \) is monotone increasing in \( t \).

**Proposition 6.1.** We have \( f_t(r, t) \geq 0 \) everywhere.

**Proof.** This is a consequence of Hamilton’s Harnack inequality for mean curvature flow [3]. See [2] for details.

We next show that \( f_t(r, t) \) is bounded from below.

**Proposition 6.2.** We have \( f_t(r, t) \geq H \) at each point in space-time. Moreover, for each \( r_0 > 0 \),

\[
 \lim_{t \to -\infty} \sup_{r \leq r_0} f_t(r, t) = H.
\]

**Proof.** Consider the flow \( M_t^{(j)} := M_{t+t_j} - q_{t_j} \) for some arbitrary sequence of times \( t_j \to -\infty \). By Theorem 1.10 in [4], the sequence \( M_t^{(j)} \) converges in \( C^\infty_{\text{loc}} \) to a smooth eternal solution, which is rotationally symmetric. At each point in time, the mean curvature at the tip of the limit solution equals \( H \). Consequently, we are in the equality case in the Harnack inequality. By [3], the limit solution must be a self-similar translator which is moving with speed \( H \). This gives

\[
 \lim_{j \to \infty} \sup_{r \leq r_0} |f_t(r, t_j) - H| = 0.
\]
for every \( r_0 > 0 \). Since \( \tilde{f}_t(r,t) \geq 0 \) by Proposition 6.1, it follows that \( f_t(r,t) \geq \mathcal{H} \) for all \( r \) and \( t \).

Using the maximum principle, we can show that \( f_t(r,t) \) is monotone increasing in \( r \).

**Proposition 6.3.** We have \( f_{tt}(r,t) \geq 0 \) everywhere.

**Proof.** Consider a time \( t_0 \) and a radius \( r_0 \) such that \( f(r_0,t_0) \) is defined. Denote by \( Q_T \) the parabolic cylinder \( Q_T = \{x_1^2 + \ldots + x_n^2 \leq r_0^2, t \in [T,t_0]\} \). It follows from the evolution equations for \( H \) and \( \langle \omega, \nu \rangle \) that the maximum \( \sup_{Q_T} H \langle \omega, \nu \rangle^{-1} \) must be attained on the parabolic boundary of \( Q_T \). This gives

\[
\sup_{x_1^2 + \ldots + x_n^2 \leq r_0^2, t=t_0} H \langle \omega, \nu \rangle^{-1} \leq \max \left\{ \sup_{x_1^2 + \ldots + x_n^2 \leq r_0^2, T \leq t \leq t_0} H \langle \omega, \nu \rangle^{-1}, \sup_{x_1^2 + \ldots + x_n^2 \leq r_0^2, t=T} H \langle \omega, \nu \rangle^{-1} \right\}.
\]

Since \( f_t(r,t) = H \langle \omega, \nu \rangle^{-1} \), it follows that

\[
\sup_{r \leq r_0} f_t(r,t_0) \leq \max \left\{ \sup_{T \leq t \leq t_0} f_t(r_0,t), \sup_{r \leq r_0} f_t(r,T) \right\}
= \max \left\{ f_t(r_0,t_0), \sup_{r \leq r_0} f_t(r,T) \right\}.
\]

Passing to the limit as \( T \to \infty \), we obtain \( \sup_{r \leq r_0} f_t(r,t_0) \leq f_t(r_0,t_0) \), as claimed.

By assumption, \( M_\ell \) is strictly convex, uniformly two-convex, and noncollapsed. Moreover, \( H_{lip}(t) \) is bounded from below by \( \mathcal{H} \). Hence, there exists a small constant \( \varepsilon_1 \in (0, \frac{1}{2m}) \) and a decreasing function \( \Lambda : (0, \varepsilon_1] \to \mathbb{R} \) such that given any \( \varepsilon \in (0, \varepsilon_1] \), if \( |\tilde{x} - q_1| \geq \Lambda(\varepsilon) \), then \( \tilde{x}, t \) is a center of \( \varepsilon \)-neck (cf. [2], Proposition 3.1). We recall three estimates from [2]. These results were stated for \( n = 2 \) in [2], but the arguments carry over directly to higher dimensions.

**Lemma 6.4.** In every \( \varepsilon_0 \)-neck, \( \varepsilon r_z = \frac{\varepsilon}{f} \leq (1 + 2\varepsilon_0)(n-1)\mathcal{H}^{-1} \).

**Lemma 6.5.** There exists a constant \( C_0 \geq 1 \) such that \( r^m |\frac{\partial r}{\partial z}| r | \leq C_0 \) holds for \( m = 1,2,3 \) at center of \( \varepsilon_0 \)-necks with \( r \geq 1 \).

**Proposition 6.6.** If \( r \geq C_1 \), then \( 0 \leq -r_{zz}(z,t) \leq C_2 r(z,t)^{-\frac{3}{2}} \).

For each \( z < 0 \), we define a real number \( T(z) \) by

\[
r(z,t) > 0 \quad \text{for} \quad t < T(z), \quad \lim_{t \to T(z)} r(z,t) = 0.
\]

The following result allows us to estimate \( r(z,t) \) in terms of \( T(z) - t \).
Corollary 6.7. We have
\[ 2(n-1) \left| \mathcal{T}(z) - t \right| \leq r(z,t)^2 \leq 2(n-1) \left| \mathcal{T}(z) - t \right| + 8C_2 [\mathcal{T}(z) - t]^\frac{1}{4} + C_1^2 \]
if \( z < 0 \) and \( r(z,t) \) is sufficiently large.

**Proof.** Let us fix a point \((\bar{z}, \bar{t})\). The inequality \((r^2 + 2(n-1)t)_t = \frac{2rrr}{r^2 + 1} < 0\) implies
\[ r(\bar{z}, \bar{t})^2 \geq 2[\mathcal{T}(\bar{z}) - \bar{t}]. \]
Moreover, if \( r \geq C_1 \), then \((r^2 + 2(n-1)t)_t = \frac{2rrr}{r^2 + 1} \geq -2C_2r^{-\frac{3}{2}}\) by Proposition 6.6. Let \( \bar{t} \leq \mathcal{T}(\bar{z}) \) be chosen so that \( r(\bar{z}, \bar{t}) = C_1 \). Then
\[ r(\bar{z}, \bar{t})^2 = C_1^2 + 2(n-1)(\bar{t} - \bar{t}) - \int_{\bar{t}}^{\bar{t}} (r(z,t)^2 + 2(n-1)t)_t dt \]
\[ \leq C_1^2 + 2(n-1)(\bar{t} - \bar{t}) + 2C_2 \int_{\bar{t}}^{\bar{t}} r(z,t) - \frac{3}{2} dt \]
\[ \leq C_1^2 + 2(n-1)(\bar{t} - \bar{t}) + 2C_2 \left[ \mathcal{T}(\bar{z}) - t \right]^{-\frac{3}{4}} dt \]
\[ \leq C_1^2 + 2(n-1)(\bar{t} - \bar{t}) - 8C_2 [\mathcal{T}(\bar{z}) - \bar{t}]^\frac{1}{4} + 8C_2 [\mathcal{T}(\bar{z}) - \bar{t}]^\frac{1}{4} \]
\[ \leq C_1^2 + 2(n-1) [\mathcal{T}(\bar{z}) - \bar{t}] + 8C_2 [\mathcal{T}(\bar{z}) - \bar{t}]^\frac{1}{4}. \]
This proves the assertion.

Lemma 6.8. Let \( \delta \) be an arbitrary positive real number. Then
\[ r(0,t)r_z(0,t) \geq (n-1)(\mathcal{H}^{-1} - \delta) \]
provided that \(-t\) is sufficiently large.

**Proof.** We may assume that \( R = r(0,t) \geq C_1 \). Then every point \((x,t)\) with \( x_{n+1} = 0 \) lies at the center of an \( \varepsilon_0\)-neck. Consequently, \(|r(z,t) - R| \leq \varepsilon_0 R\) for \(|z| \leq 2R\). Moreover, we have \( rr_z \leq (1 + 2\varepsilon_0)\mathcal{H}^{-1}\) by Lemma 6.4, and \(|(rr_z)_z| = |rr_{zz} + r^2_z| \leq C_3 R^{-\frac{3}{2}}\) for some constant \( C_3 \) by Proposition 6.6. Hence, if \( R^2 \geq 4C_3 \delta^{-1}\), then we have
\[ |r(z,t)r_z(z,t) - r(0,t)r_z(0,t)| \leq 2C_3 R^{-\frac{1}{2}} \leq \frac{\delta}{2} \]
for all \( z \in [-2R, 2R] \).
Using Corollary 6.7, we obtain
\[ r(-R,t)^2 \geq 2(n-1) [\mathcal{T}(-R) - t], \]
\[ r(-2R,t)^2 \geq 2(n-1) [\mathcal{T}(-2R) - t], \]
Moreover, if
\[ r(-2R,t)^2 \leq 2(n-1)[T(-2R)-t] + 8C_2[T(-2R)-t]\frac{1}{2} + C_1^2 \]
\[ \leq 2(n-1)[T(-2R)-t] + 8C_2r(-2R,t)^2 + C_1^2 \]
\[ \leq 2(n-1)[T(-2R)-t] + 8C_2R^2 + C_1^2. \]
From this, we deduce that
\[ r(-R,t)^2 - r(-2R,t)^2 \geq 2(n-1)[T(-R) - T(-2R)] - 8C_2R^2 - C_1^2. \]
Moreover, if \( R \) is sufficiently large, then
\[ T(-R) - T(-2R) \geq \left( H^{-1} - \frac{\delta}{2} \right) R. \]
This finally implies
\[ r(-R,t)^2 - r(-2R,t)^2 \geq 2(n-1)\left( H^{-1} - \frac{\delta}{2} \right) R, \]

hence
\[ \sup_{z \in [-2R,R]} r(z,t)r_z(z,t) \geq (n-1)\left( H^{-1} - \frac{\delta}{2} \right) \]
if \(-t\) is sufficiently large. Thus,
\[ r(0,t)r_z(0,t) \geq (n-1)\left( H^{-1} - \delta \right) \]
if \(-t\) is sufficiently large.

**Proposition 6.9.** Given \( \delta > 0 \), there exists a time \( \bar{t} \in (-\infty,0] \) (depending on \( \delta \)) such that
\[ r(z,t)r_z(z,t) \geq (n-1)\left( H^{-1} - 2\delta \right), \]
holds for all \( z \geq 0 \) and \( t \leq \bar{t} \).

**Proof.** Let \( \psi(z,t) \) denote the solution of the Dirichlet problem for the one-dimensional heat equation on the half line (see [2], Proposition 6.9). By Lemma 6.8, we can find a time \( \bar{t} \) so that \( r(0,t)r_z(0,t) \geq (n-1)(H^{-1} - \delta) \)
for \( t \leq \bar{t} \), and \( r(0,t) \geq C_1 + C_2 \) for \( z \geq 0 \) and \( t \leq \bar{t} \). Given any \( s < \bar{t} \), we define a function \( \psi^{\delta,s}(z,t) \) by
\[ \psi^{\delta,s}(z,t) = (n-1)\left( H^{-1} - 2\delta \psi(2z,t-s) \right) \]
for \( t \in (s,\bar{t}] \). We will show that \( rr_z > \psi^{\delta,s} \) for all \( z \geq 0 \) and all \( t \in (s,\bar{t}] \).

It is straightforward to verify that \( r(0,t)r_z(0,t) \geq (n-1)(H^{-1} - \delta) > \limsup_{z \to 0} \psi^{\delta,s}(z,t) \) for each \( t \in (s,\bar{t}] \); \( \liminf_{z \to \infty} r(z,t)r_z(z,t) \geq 0 > \limsup_{z \to \infty} \psi^{\delta,s}(z,t) \) for each \( t \in (s,\bar{t}] \); and \( r(z,s)r_z(z,s) \geq 0 > \limsup_{t \to s} \psi^{\delta,s}(z,t) \) for each \( z > 0 \).

On the other hand, for \( z > 0 \) and \( t \in (s,\bar{t}] \), we have, we have \( 1 + rr_z \geq 0 \), hence
\[ (rr_z)_t \geq \frac{(rr_z)_{zz}}{1 + r_z^2}. \]
Moreover,

\[
(\psi^{\delta,s})_t = \frac{1}{4} (\psi^{\delta,s})_{zz} \leq \frac{(\psi^{\delta,s})_{zz}}{1 + r_z^2}.
\]

Using the maximum principle, we conclude that \(rr_z > \psi^{\delta,s}\) for all \(z \geq 0\) and all \(t \in (s, \tilde{t}]\). Sending \(s \to -\infty\) gives the desired result.

**Corollary 6.10.** We can find a time \(T \in (-\infty, 0]\) such that \(r(z,t)^2 \geq (n-1)\mathcal{H}^{-1}z\) for all \(z \geq 0\) and \(t \leq T\). In particular, if \(t \leq T\), then the function \(f(r,t)\) is defined for all \(r \in [0, \infty)\).

**Proof.** By Proposition 6.9, we can find a time \(T \in (-\infty, 0]\) such that \(r(z,t)rr_z(z,t) \geq \frac{1}{2} (n-1)\mathcal{H}^{-1}\) for all \(z \geq 0\) and all \(t \leq T\). From this, the assertion follows easily.

**Proposition 6.11.** For each \(t \leq T\), we have \(\lim_{z \to \infty} r(z,t)rr_z(z,t) = (n-1)\mathcal{H}^{-1}\).

**Proof.** It suffices to prove that \(\liminf_{z \to \infty} r(z,t)rr_z(z,t) \geq (n-1)\mathcal{H}^{-1}\) for each \(t \leq T\). By Proposition 6.9, we know that \(\liminf_{z \to \infty} r(z,t)rr_z(z,t) \geq (n-1)(\mathcal{H}^{-1} - 2\delta)\) if \(-t\) is sufficiently large. On the other hand, Lemma 6.5 gives

\[
|rr_z| = |rr_z + rr_z| = \left| \frac{r_zr_{zz} + rr_{zzz}}{1 + r_z^2} - \frac{2rr_zr_{zz}}{(1 + r_z^2)^2} \right| \leq \frac{4C_0^2}{r^2}
\]

for \(r \geq C_1\). Using Corollary 6.10, we conclude that the quantity \(\liminf_{z \to \infty} r(z,t)rr_z(z,t)\) has the same value for all \(t \leq T\). Putting these facts together, we conclude that \(\liminf_{z \to \infty} r(z,t)rr_z(z,t) \geq (n-1)\mathcal{H}^{-1}\) for each \(t \leq T\).

**Theorem 6.12.** For each \(t \leq T\), the solution \(M_t\) is a rotationally symmetric translating soliton.

**Proof.** Since \(rr_z = \frac{f_r}{f}\), Proposition 6.11 implies

\[
\lim_{r \to \infty} \frac{f_r(r,t)}{r} = \frac{1}{n-1}\mathcal{H}
\]

for each \(t \leq T\). Using the evolution equation for \(f(r,t)\), we obtain

\[
\lim_{r \to \infty} f_t(r,t) = \lim_{r \to \infty} \frac{(n-1)f_r(r,t)}{r} = \mathcal{H}
\]

for each \(t \leq T\). Using Proposition 6.3, we conclude that \(f_t(r,t) \leq \mathcal{H}\) for all \(r \geq 0\) and all \(t \leq T\). Therefore, Proposition 6.11 gives \(f_t(r,t) = \mathcal{H}\) for all \(r \geq 0\) and all \(t \leq T\). Consequently, \(M_t\) is a translating soliton for each \(t \leq T\).

Once we know that \(M_t\) is a translating soliton for \(-t\) sufficiently large, it follows from standard arguments that \(M_t\) is a translating soliton for all \(t \in (-\infty, 0]\). This completes the proof of Theorem 6.11.
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