SU($N$) coherent states and irreducible Schwinger bosons

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Abstract
We exploit the SU($N$) irreducible Schwinger boson to construct SU($N$) coherent states. This construction of an SU($N$) coherent state is analogous to the construction of the simplest Heisenberg–Weyl coherent states. The coherent states belonging to irreducible representations of SU($N$) are labeled by the eigenvalues of ($N$ − 1) SU($N$) Casimir operators and characterized by ($N$ − 1) complex orthonormal vectors describing the SU($N$) group manifold.

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1. Introduction

The concept of coherent states was introduced by Schrödinger [1] in the context of a harmonic oscillator. These harmonic oscillator coherent states, also called canonical coherent states, have been widely used in physics [2–6]. The next important coherent states are spin coherent states or SU(2) coherent states which are associated with angular momentum or the SU(2) group. Like canonical coherent states, they too have found wide applications in different branches of physics such as quantum optics, statistical mechanics, nuclear physics and condensed matter physics [2, 5–8]. It is known that these spin coherent states can also be constructed using harmonic oscillators by exploiting either the Holstein–Primakov or the Schwinger boson representation of the SU(2) Lie algebra [9–11]. This harmonic oscillator formulation of spin coherent states is appealing because of its simplicity as this construction is analogous to the simplest and oldest canonical coherent state construction. Further, unlike the standard construction of coherent states [6], this method does not require any knowledge of group representations or group elements and their actions on a particular weight vector, leading to many technical simplifications (see section 2). In fact, these SU(2) coherent states in terms of harmonic oscillators have been implicitly contained in the seminal work of Schwinger [9].
way back in 1952. The aim of this work is to show that the above simple, uniform and explicit
construction of canonical and spin or SU(2) coherent states can also be easily extended to
all higher SU(N) groups. This is in contrast to the standard construction of SU(N) coherent
states (i.e. by applying the SU(N) group elements on a particular weight vector) which is
known to become more and more tedious as N increases [12, 13]. In the past, this problem
has led to various different approaches [12–18] to construct SU(3) and SU(N) coherent
states. In [12], SU(N) coherent states belonging to SU(N) symmetric representations are
constructed by using fixed-order polynomials of complex N-plets. In [13], a very special
characterization of SU(N) group elements is exploited to construct SU(N) coherent states. In
[14–16], the Schwinger boson representation of SU(N) Lie algebra is used to construct SU(N)
coherent states. In [17], the Schwinger oscillator representation of SU(3) coherent states is
analyzed to discuss its relationship with the standard harmonic oscillator coherent states. In
[18], SU(3) coherent states are constructed using a special parametrization of SU(3) group
elements.

This work exploits the SU(N) irreducible Schwinger bosons [19, 20] to construct SU(N)
coherent states belonging to an arbitrary irreducible representation of SU(N). By definition,
the SU(N) irreducible Schwinger bosons creation operators carry all the symmetries of
SU(N) irreducible representations (see sections 3 and 4). As a result all SU(N) irreducible
representation states are monomials (not polynomials) of SU(N) irreducible Schwinger boson
creation operators acting on the corresponding vacuum states. Therefore, as in the case of
canonical and spin coherent states, they are the natural candidates for constructing the SU(N)
coherent states. These coherent states are defined on the SU(N) group manifold characterized
by an orthonormal set of \( (N - 1) \) complex SU(N) vectors. In addition, the SU(N) coherent
states are labeled by the integer eigenvalues of \( (N - 1) \) Casimir operators which are the \( (N - 1) \)
types of Schwinger boson number operators (see sections 3 and 4).

The organization of the paper is as follows. We start with a very brief discussion on a
harmonic oscillator or canonical coherent states because the SU(N) coherent state construction
in this work is analogous to this simplest construction. In section 2, we illustrate this
similarity by constructing SU(2) coherent states in terms of a doublet of harmonic oscillators
or equivalently SU(2) Schwinger bosons transforming as the fundamental representation of
SU(2). As mentioned earlier, this construction has been exploited by Schwinger [9] to compute
SU(2) re-coupling coefficients. At the end of section 2 (section 3), the simplifications obtained
by this Schwinger boson approach to coherent states over the standard approach of applying
SU(2) (SU(3)) group elements on a particular weight vector are highlighted. In sections 3 and
4, we further extend the above SU(2) construction to SU(3) and SU(N), respectively. These
SU(3) and then SU(N) extensions of SU(2) coherent states are again trivial as they correspond
to:

- Replacing an SU(2) Schwinger boson doublet by \( (N - 1) \) SU(N) irreducible Schwinger
  bosons N-plets,
- Replacing an SU(2) group manifold (i.e. a doublet of complex numbers) by an SU(N)
  group manifold (i.e. \( N - 1 \) N-plets of complex numbers).

Section 3 on SU(3) is added to make the transition from SU(2) (section 2) to SU(N)
(section 4) easy.

In the simplest example of the Heisenberg–Weyl group, the Lie algebra contains three
generators. It is defined in terms of creation annihilation operators \( (a, a^\dagger) \) satisfying

\[
[a, a^\dagger] = \mathcal{I}, \quad [a, \mathcal{I}] = 0, \quad [a^\dagger, \mathcal{I}] = 0. \tag{1}
\]
This algebra has only one infinite-dimensional unitary irreducible representation. The states within this representation are the occupation number states \(|n⟩= (a^†)^n \sqrt{n!} |0⟩\) with \(n = 0, 1, 2, \ldots\).

The coherent states of the Heisenberg–Weyl group are defined over a complex manifold as
\[|z⟩|_∞ = \exp(za^†) |0⟩ = \sum_{n=0}^{∞} F_n(z) |n⟩.\] (2)

In (2) the subscript \([∞]\) on the coherent states is the irreducible representation index. It implies that these coherent states are defined over the infinite-dimensional irreducible representation of the group. The sum in (2) runs over all the basis vectors \(|n⟩\) belonging to this infinite-dimensional representation. The coefficients
\[F_n(z) = \frac{z^n}{\sqrt{n!}},\] (3)
are the coherent state expansion coefficients which are the analytic functions of the group manifold coordinate \(z\). The resolution of the identity property of the coherent state (2) follows from the group transformation property. Let us define the operator:
\[O|_∞ \equiv \int \exp(-|z|^2) \, dz \, d\bar{z} \, |z⟩|_∞ \langle z|.|_∞.\] Under the Heisenberg–Weyl group element \(g_{hw} \equiv \exp(iα + wa - \bar{w}a^†)\):
\[|z⟩|_∞ \rightarrow e^{iα + zw - \bar{w}w} |z - \bar{w}||_∞.\] It is trivial to see that the operator \(O|_∞\) defined above is invariant under \(g_{hw}\). Therefore, by the Schur’s lemma it is proportional to a unity operator.

The purpose of this work is to generalize (2) and (3) for the Heisenberg–Weyl group to SU(\(N\)) for arbitrary \(N\) (see (10), (12) for SU(2); (35), (37) for SU(3); and (55), (57) for SU(\(N\))). We start with SU(2) construction [9, 10] first.

2. SU(2) coherent states

The Heisenberg–Weyl coherent state construction can be readily generalized to the simplest compact group SU(2) by utilizing the Schwinger representation of the SU(2) Lie algebra: \([Ja, Jb] = iϵabcJc\). We define [9]
\[J^a = \frac{1}{2}a^†_α(σ^a)^α_β a^β.\] (4)

In (4), \(σ^a\) with \(a = 1, 2, 3\) denoting the three Pauli matrices. The doublet of harmonic oscillator creation and annihilation operators \(a^a\) and \(a^†_a\) or equivalently Schwinger bosons in (4) satisfy the simple bosonic commutation relations \([a^a, a^†_β] = δ^α_β\) with \(α, β = 1, 2, 3\). The vacuum state \(|0⟩\) of these two oscillators will be denoted by \(|0⟩\). Under SU(2) transformations, the Schwinger boson creation operators transform as fundamental representations:
\[a^α_α \rightarrow a^β_β \left(\frac{\exp(iθ^a σ^α)}{2}\right)_α^β.\] (5)

The defining equations (4) imply that the SU(2) Casimir operator is simply the total number operator:
\[C = 2 \sum_{α=1}^{3} a^†_α a^α \equiv a^† \cdot a.\] (6)

The eigenvalues of \(C\) will be denoted by \(n\). The various states in the irreducible representation \(n(= 2j)\) are
\[|α(α_2 \cdots α_3)|_n \equiv a^†_{α_1} a^†_{α_2} \cdots a^†_{α_n} |0⟩.\] (7)
The corresponding SU(2) Young tableau is shown in figure 1. Note that the state in (7) is invariant under all $n!$ permutations of the SU(2) indices $\alpha_1, \alpha_2, \ldots, \alpha_n$. This is because all SU(2) creation operators on the right-hand side of (7) commute amongst themselves. In other words, the SU(2) Schwinger boson creation operators carry the symmetries of the SU(2) Young tableau\(^1\) which is shown in figure 1. Therefore, the $(n+1)$ states in (7) belong to SU(2) irreducible representation with the total angular momentum \(j = n/2\).

The SU(2) group manifold $S^3$ can also be described by a doublet of complex numbers $(z_1, z_2)$ of unit magnitude:

$$|z| = |z_1|^2 + |z_2|^2 = 1.$$  

This is because any SU(2) matrix $U_2$ can be written as

$$U_2 = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

with $U_2 U_2^\dagger = U_2^\dagger U_2 = 1$ and $|U_2| = 1$. At this stage one can trivially combine the SU(2) irreducible states in (7) and the SU(2) group manifold coordinates in (8) to construct the generating function of the SU(2) coherent states:

$$|z\rangle \equiv |z_1, z_2\rangle = \exp(z \cdot a^\dagger) |0\rangle = \sum_{n=0}^{\infty} \frac{(z \cdot a^\dagger)^n}{n!} |0\rangle = \sum_{n=0}^{\infty} |z_n\rangle.$$  

Above, $z \cdot a^\dagger \equiv \sum \alpha_1 a_1^\dagger + \sum \alpha_2 a_2^\dagger$ and $|z_n\rangle$ is the coherent state in the SU(2) representation $j = n/2$:

$$|z\rangle_n = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = 1}^{2} F^{\alpha_1 \alpha_2 \ldots \alpha_n} (z) a^\dagger_{\alpha_1} a^\dagger_{\alpha_2} \ldots \alpha_n |0\rangle = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_n = 1}^{2} F^{\alpha_1 \alpha_2 \ldots \alpha_n} (z) |\alpha_1 \alpha_2 \ldots \alpha_n=2\rangle_{[j]}.$$  

Like in the Heisenberg–Weyl case (3), the SU(2) coherent state structure functions in the irreducible representation $j = \frac{n}{2}$ are

$$F^{\alpha_1 \alpha_2 \ldots \alpha_n} (z^1, z^2) \equiv |n\rangle z^{\alpha_1} z^{\alpha_2} \ldots z^{\alpha_n}.$$  

Note that they are the analytic functions of group manifold coordinates. The resolution of the identity property again follows from the group transformation laws. The coherent state structure in (10) and the SU(2) transformations (5) imply that under the group transformations $|z^1, z^2\rangle_{[n]} \rightarrow |z^1', z^2\rangle_{[n]}$, where $(z^1, z^2)$ are the SU(2) rotated coherent state coordinates:

$$z^\alpha = \left( \exp i \left( \frac{\alpha \beta}{2} \right) \right)^\alpha \beta n.$$  

\(^1\) This obvious symmetry argument will not be true for higher SU(N) (sections 3 and 4) leading to the definition of SU(N) irreducible Schwinger bosons. In terms of SU(N) irreducible Schwinger bosons, the SU(N) irreducible states will be monomials like (7).
Therefore, under the SU(2) transformations the coherent states \(|z\rangle \equiv |z_1, z_2\rangle\) transform amongst themselves on \(S^3\) as constraint (8) remains invariant under (13). Again, we define the operator
\[
\mathcal{O}_a = \int d\mu(z)(|z\rangle_{\{a\}} |z\rangle) = \int d^2z \Delta(z) \delta(|z|^2 - 1) |z\rangle_{\{a\}} |z\rangle.
\]
(14)
The operator \(\mathcal{O}_a\) is invariant under all SU(2) transformations of the coherent states \(|z\rangle_{\{a\}}\).

Therefore,
\[
[j^a, \mathcal{O}_a] = 0, \quad \forall \ a = 1, 2, 3
\]
(15)
where \(j^a\) is defined in (4). Schur’s lemma implies that \(\mathcal{O}_a\) is proportional to the identity operator. Before generalizing (10) to SU\((N)\), it is illustrative to briefly mention the standard group theoretical coherent state construction procedure [6]. We characterize the SU(2) group elements \(U\) by the Euler angles, i.e. \(U(\theta, \phi, \psi) \equiv \exp{-i\theta J_3}\exp{-i\phi J_2}\exp{-i\psi J_1}\). The ranges of these Euler angles are
\[
0 \leq \theta \leq \pi \quad \text{and} \quad 0 < \phi, \psi \leq 2\pi.
\]
The SU(2) coherent states are constructed as
\[
|\theta, \phi, \psi\rangle = U(\theta, \phi, \psi) |j, j\rangle = \sum_{m=-j}^{+j} C_m(\theta, \phi, \psi) |j, m\rangle,
\]
(16)
The SU(2) coherent state structure functions or the coefficients \(C_m(\theta, \phi, \psi)\) are given by
\[
C_m(\theta, \phi, \psi) = \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} e^{-i(m\phi+j\psi)} \left(\sin \frac{\theta}{2}\right)^{j-m} \left(\cos \frac{\theta}{2}\right)^{j+m}.
\]
(17)
The coherent state constructed in (10) is directly related to the conventional coherent state in (16). The coherent state in (10) written in the occupation number basis is
\[
|z_1, z_2\rangle_{n=2j} = \sum_{n_1, n_2} \delta_{n_1+n_2, n} \frac{(z_1)^{n_1}(z_2)^{n_2}}{\sqrt{n_1!n_2!}} |n_1, n_2\rangle.
\]
(18)
Now changing this \(|n_1, n_2\rangle\) basis to the \(|j, m\rangle\) basis by defining \(n_1 = j - m\) and \(n_2 = j + m\) and also by defining
\[
z_1 = e^{i\phi/2}e^{-i\psi/2} \sin \frac{\theta}{2}, \quad z_2 = e^{-i\phi/2}e^{-i\psi/2} \cos \frac{\theta}{2}
\]
we write (18) as
\[
|z_1, z_2\rangle_j = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{+j} \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} e^{-i(m\phi+j\psi)} \left(\sin \frac{\theta}{2}\right)^{j-m} \left(\cos \frac{\theta}{2}\right)^{j+m} |j, m\rangle
\]
(19)
The additional factor \(\frac{1}{\sqrt{2j+1}}\) is because of the normalization \(j(z_1, z_2|z_1, z_2)j = \frac{1}{2j+1}\). This relation shows the equivalence between the standard group theoretical construction of the coherent state (16) and the corresponding Schwinger boson construction.

We note that the generalization of the standard SU(2) construction (16) to the higher SU\((N)\) group is difficult [12]. This is because we need to know all the SU\((N)\) representations, Euler angles and the group elements to implement this procedure. On the other hand, the coherent states in (10) (also see (36) and (56) for SU(3) and SU\((N)\), respectively) are a straightforward generalization of the Heisenberg–Weyl coherent states in (2) and bypass all
the problems mentioned above. In fact, even for \( N = 3 \) the computation of coherent state structure functions analogous to (17) is extremely tedious and involved (see section 3 for more details). As expected, these complications grow with \( N \). Motivated by this, our aim in this work is to further extend the simple SU(2) coherent state construction (10) to SU(\( N \)) with arbitrary \( N \). As we will see the only new input required for this purpose is the replacement of SU(2) Schwinger bosons by SU(\( N \)) irreducible Schwinger bosons [19, 20]. We first deal with the SU(3) group in detail.

3. SU(3) coherent states

We start with a brief review of SU(3) irreducible Schwinger bosons [19, 20] and then construct SU(3) coherent states.

3.1. The irreducible Schwinger boson representations of SU(3)

As the rank of an SU(3) group is 2 it has two fundamental triplet representations. We take them to be two independent harmonic oscillator triplets or equivalently Schwinger boson and denote them by \( a_1^\dagger \) and \( a_2^\dagger \), respectively. The SU(3) generators in terms of these Schwinger bosons are

\[
Q^a = a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger, \quad a = 1, 2, \ldots, 8.
\]

(20)

In (20), \( \lambda^a \)'s are the Gell–Mann matrices. Under SU(3) transformations both types of Schwinger boson creation operators transform as fundamental representations

\[
a_1^\dagger[i] \rightarrow a_1^\dagger[i] \left( \exp i \theta^a \frac{\lambda^a}{2} \right)^\beta, \quad i = 1, 2.
\]

(21)

The defining equations (20) immediately imply that the two SU(3) Casimirs commuting with all the generators are the two total number operators:

\[
C[1] \equiv N_1 \equiv a_1^\dagger a_1^\dagger, \quad C_2 \equiv N_2 \equiv a_2^\dagger a_2^\dagger.
\]

(22)

It is obvious that \([C[i], Q^a] = 0, \forall i = 1, 2, a = 1, 2, \ldots, 8\), as each \( Q^a \) contains one creation and one annihilation operator of either type \( i = 1 \) or \( i = 2 \). Their eigenvalues are denoted by \( n_1 \) and \( n_2 \), respectively. The corresponding irreducible representation with \( n_1 \geq n_2 \) is denoted\(^2\) by \([n_1, n_2]\). The associated SU(3) Young tableau is shown in figure 2.

The monomial states constructed out of the two SU(3) fundamental Schwinger bosons

\[
|\alpha_1 \alpha_2 \cdots \alpha_n_1 : \beta_1 \beta_2 \cdots \beta_n_2 \rangle \equiv (a_1^\dagger a_1^\dagger \cdots \alpha_{n_1}^\dagger [1]) (a_2^\dagger a_2^\dagger \cdots \beta_{n_2}^\dagger [2]) |0\rangle
\]

(23)

\(^2\) Note that \( n_1 \) and \( n_2 \) are the numbers of two SU(3) fundamental triplet representations. The anti-triplet (i.e. 3\(^*\)) representation is the \([n_1 = 1, n_2 = 1]\) representation.

Figure 2. SU(3) Young table in the \([n_1, n_2]\) representation. The monomial state (31) in terms of SU(3) irreducible Schwinger bosons and not (23) carries all the symmetries of this SU(3) Young tableau.
are eigenstates of $\mathcal{C}[1]$ and $\mathcal{C}[2]$ with eigenvalues $n_1$ and $n_2$, respectively. However, unlike the SU(2) case, where the monomial states (7) are SU(2) irreducible, the corresponding monomial states (23) are SU(3) reducible and do not form the SU(3) irreducible representation $[n_1, n_2]$. Like in the SU(2) case, the monomial state (23) carries the horizontal symmetries of SU(3) Young tableaux as the Schwinger bosons creation operators on the right-hand side commute amongst themselves. However, the vertical antisymmetry needs to be imposed to get SU(3) irreducibility. We achieve this by imposing the following constraint [20, 21]:

$$\hat{L}_{12} \equiv a_1^\dagger [1] \cdot a_2^\dagger [2] = 0$$

(24)
on the monomial states (23). As an example, the simple $[n_1 = 1, n_2 = 1]$ irreducible representation of SU(3) corresponds to the anti-triplet $3^*$ and the corresponding states are given by

$$[\alpha; \beta]_{[n_1 = 1, n_2 = 1]} = - [\beta; \alpha]_{[n_1 = 1, n_2 = 1]} \equiv (a_\alpha^\dagger [1]a_\beta^\dagger [2] - a_\beta^\dagger [1]a_\alpha^\dagger [2])|0\rangle.$$  

(25)

The three states in (25) trivially satisfy constraint (24) as they are antisymmetric. Therefore, they belong to the SU(3) irreducible representation $3^*$. In recent works [19, 20], we have defined SU(3) irreducible Schwinger bosons $A^i[i]$ with $i = 1, 2$ as

$$A_\alpha^i [1] = a_\alpha^i [1]$$

(26)

$$A_\alpha^i [2] = a_\alpha^i [2] - \frac{1}{N_1 - N_2 + 2} (a_\alpha^i [2] \cdot a [1])a_\alpha^i [1].$$

(27)

The SU(3) irreducible Schwinger bosons in (26) are constructed such that [19, 20]

$$[a^i [1] \cdot a [2]], A^a [1] \simeq 0$$

$$[a^i [1] \cdot a [2]], A^a [2] \simeq 0$$

(28)

where ‘$\simeq 0$’ means that the commutators are weakly zero. In other words, the above commutators annihilate all SU(3) irreducible states satisfying (24). The irreducible Schwinger bosons further satisfy

$$[A_\alpha^i [1], A_\beta^j [1]] = 0, \quad [A_\alpha^i [2], A_\beta^j [2]] = 0.$$ 

(29)

The defining equations (26) imply that their transformation properties are exactly the same as (21):

$$A_\alpha^i [i] \rightarrow A_\beta^i [\hat{A}] (\exp i\theta^a \frac{\lambda^a}{2})^i, \quad i = 1, 2.$$ 

(30)

We now consider the most general monomial state constructed out of SU(3) irreducible Schwinger bosons:

$$|\alpha_1 \alpha_2 \cdots \alpha_{n_1}; \beta_1 \beta_2 \cdots \beta_{n_2} \rangle_{[n_1, n_2]} \equiv \left(A_{\beta_1}^2 [2]A_{\beta_2}^2 [2] \cdots A_{\beta_{n_2}}^2 [2]\right)\left(A_{\alpha_1}^1 [1]A_{\alpha_2}^1 [1] \cdots A_{\alpha_{n_1}}^1 [1]\right)|0\rangle.$$ 

(31)

This monomial state directly creates the SU(3) irreducible state corresponding to the SU(3) Young tableau with $n_1$ and $n_2$ boxes in the first and second rows, respectively. This is because the inbuilt constraint (24) ensures the vertical antisymmetry and the commutators (29) ensure the horizontal symmetries of SU(3) Young tableau. It is easy to check that the antisymmetric $3^*$ states in (25) are simply created by applying $A_\beta^2 [2]A_\alpha^1 [1]$ on the vacuum:

$$A_\beta^2 [2]A_\alpha^1 [1]|0\rangle = \frac{1}{2} (a_\alpha^1 [1]a_\beta^2 [2] - a_\beta^2 [1]a_\alpha^1 [2])|0\rangle \equiv \frac{1}{2} [\alpha; \beta]_{[n_1 = 1, n_2 = 1]}.$$
To appreciate this further we explicitly construct the simplest mixed octet representation with \(n_1 = 2\) and \(n_2 = 1\) in (31):

\[
|\alpha_1\alpha_2; \beta\rangle_{|n_1=2, n_2=1} = A^\dagger_2[1]A^\dagger_2[1]|A^\dagger_1[1]|0\rangle = \frac{1}{2} \left\{ (a^\dagger_2[2]a^\dagger_1[1] - a^\dagger_2[1]a^\dagger_1[2])a^\dagger_2[1] \\
+ (a^\dagger_2[2]a^\dagger_1[1] - a^\dagger_2[1]a^\dagger_1[2])a^\dagger_1[1] \right\} |0\rangle.
\]  

(32)

The expression in (32) is first antisymmetrized amongst the column indices \(\alpha_1\) and \(\beta\), then symmetrized amongst the row indices \(\alpha_1\) and \(\alpha_2\). Therefore, it has the symmetries of SU(3) Young tableaux shown in figure 2 with \(n_1 = 2\) and \(n_2 = 1\). This simplest but non-trivial example illustrates the usefulness of the procedure to compute SU(3) representations in terms of SU(3) irreducible Schwinger bosons. In fact, it has been shown in [19, 20] that the simple monomial (31) has all the symmetries of the \([n_1, n_2]\) irreducible representation of SU(3) corresponding to the Young tableau in figure 2. We now exploit this simple fact to further extend the definition of the Heisenberg–Weyl (2) and SU(2) coherent states (10) to the SU(3) group.

3.2. Construction of SU(3) coherent states

Similar to the SU(2) cases (8) and (9), the eight-dimensional SU(3) group manifold can be characterized by two complex triplets: \(z_\alpha[1]\) and \(z_\alpha[2]\) (\(\alpha = 1, 2, 3\)) which satisfy the orthonormality constraints

\[
z[1] \cdot z[1] = 1 = z[2] \cdot z[2], \quad z[1] \cdot z[2] = 0.
\]  

(33)

This is because any SU(3) matrix \(U_3\) can be written as

\[
U_3 = \begin{pmatrix}
z[1] & z[2] & z[3] \\
z[2] & z[2] & z[3] \\
z[3] & z[2] & z[3]
\end{pmatrix}.
\]  

(34)

In (34), \(z[3] = (z[1] \land z[2]), = e_{ijk}z[i]z[j]z[k]\), where \(e_{ijk}\) is the completely antisymmetric tensor. Like the first two columns, the third column is also normalized because of the constraints (33): \(z[3] \cdot z[3] = z[1] \cdot z[1] z[2] \cdot z[2] - z[1] \cdot z[2] z[2] \cdot z[1] = 1\). Thus, the SU(3) matrix satisfies \(U_3^\dagger U_3 = U_3^\dagger U_3 = 1\) and det\(U_3\) = \(|U_3|\) = 1 due to the orthonormality constraints amongst the complex triplets \(z[1]\), \(z[2]\) and \(z[3]\).

We define the SU(3) coherent states generating function as

\[
|z[1], z[2]\rangle \equiv \exp(z[2] \cdot A^\dagger[2]) \exp(z[1] \cdot A^\dagger[1])|0\rangle
\]  

(35)

Note that this construction is an SU(3) extension of the SU(2) coherent state generating function (10). We can project the SU(3) coherent state in the representation \([n_1, n_2]\) by considering the corresponding term in the generating function (35):

\[
|z[1], z[2]\rangle_{|n_1, n_2} = \frac{(z[2] \cdot A^\dagger[2])^{n_2} (z[1] \cdot A^\dagger[1])^{n_1}}{n_2! \ n_1!} |0\rangle
\]

\[
= \sum_{\alpha_1 \ldots \alpha_{n_1}=1}^{3} \sum_{\beta_1 \ldots \beta_{n_2}=1}^{3} F^{\alpha_1 \ldots \alpha_{n_1} ; \beta_1 \ldots \beta_{n_2}} (z[1], z[2]) \\
\times \langle \alpha_1 \alpha_2 \ldots \alpha_{n_1} ; \beta_1 \beta_2 \ldots \beta_{n_2} \rangle_{|n_1, n_2}. \tag{36}
\]

In (36), the SU(3) coherent state structure functions

\[
F^{\alpha_1 \ldots \alpha_{n_1} ; \beta_1 \ldots \beta_{n_2}} (z[1], z[2]) = \frac{1}{n_1! n_2!} z[1]^{\alpha_1} z[1]^{\alpha_2} \ldots z[1]^{\alpha_{n_1}} z[2]^{\beta_1} z[2]^{\beta_2} \ldots z[2]^{\beta_{n_2}}. \tag{37}
\]
are the analytic functions of SU(3) group manifold coordinates. Like in the SU(2) case, the resolution of the identity property follows from the group transformation laws. Using the SU(3) transformations (30), we find that the SU(3) coherent states transform as

\[ z'[1] = \left( \exp i \left( \theta^a \frac{\lambda^a}{2} \right) \right)^a \beta z[2], \]

Again, like in the SU(2) case, \( z[1] \) and \( z[2] \) transform like SU(3) fundamental triplet representations and the orthonormality conditions (33) remain invariant under the SU(3) transformations. In other words, the coherent state (56) defined at a point \( (z[1], z[2]) \) transform to the coherent state at \( (z'[1], z'[2]) \) on the SU(3) group manifold. The operator \( O_{[n_1, n_2]} \):

\[ O_{[n_1, n_2]} = \int d\mu(z) \langle [z[1], z[2]]_{[n_1, n_2]} | [n_1, n_2] | [z[1], z[2]] \rangle \quad (39) \]

with the SU(3) Haar measure

\[ \int d\mu(z) = \int d^2z[1] d^2z[2] \prod_{1 \leq \alpha \leq 2} \delta(z[\alpha], z[\beta]) - \delta_{\alpha, \beta} \]

is invariant under all SU(3) transformations (21):

\[ [Q^a, O_{[n_1, n_2]}] = 0, \quad \forall a = 1, 2, \ldots, 8. \quad (40) \]

Therefore, by Schur’s lemma, \( O_{[n_1, n_2]} \) is proportional to the identity operator.

The SU(3) coherent states (36) and the structure functions (37) are straightforward generalizations of the SU(2) coherent states (10) and the corresponding structure functions (12), respectively. The latter, in turn, are the SU(2) generalization of the oldest Heisenberg–Weyl or harmonic oscillator coherent states (2) and the associated structure functions (3). On the other hand, the standard group theoretical definition or equivalently the generalization of SU(2) coherent states definition (10) to of SU(3) corresponds to

\[ |\alpha_1, \alpha_2, \ldots, \alpha_8 \rangle \equiv U(\alpha_1, \alpha_2, \ldots, \alpha_8) |p, q, I, M, Y\rangle. \quad (41) \]

The SU(3) coherent states (41) involve the SU(3) group element in terms of eight Euler angles [22]:

\[ U(\alpha_1, \alpha_2, \ldots, \alpha_8) = \exp i (\lambda_3 \alpha_1) \exp i (\lambda_2 \alpha_2) \exp i (\lambda_3 \alpha_3) \]

acting on the SU(3) basis state \(|p, q, I, M, Y\rangle\). These SU(3) states are characterized by two integers \( p, q \) representing the eigenvalues of the two SU(3) Casimir operators. The three SU(3) magnetic quantum numbers \( I, M \) and \( Y \) are the quantum numbers of the canonical subgroup \( SU(2) \times U(1) \) of SU(3) [17] and their ranges are

\[ I = \frac{1}{2} (r + s), \quad Y = r - s + \frac{1}{2} (q - p), \quad 0 \leq r \leq p, \quad 0 \leq s \leq q. \]

Therefore, it is clear that the explicit construction of SU(3) coherent states and the computation of the corresponding structure functions (analogous to SU(2) results (16) and (17), respectively) are extremely tedious. In contrast, the present SU(3) definition (36) with simple structure functions (37) clearly bypasses all these problems. We now give the SU(N) generalization with arbitrary \( N \).

4. SU(N) coherent state

Like the previous section on SU(3), this section has two parts. In the first part, we briefly describe the SU(N) irreducible Schwinger bosons [20] and in the second part, we exploit it to define SU(N) coherent states.
4.1. Irreducible Schwinger boson representations of SU(N)

The rank of the SU(N) group is \((N - 1)\). Therefore, the fundamental constituents required to construct any arbitrary irrep. of SU(N) can be chosen to be \(N - 1\) independent Schwinger boson \(N\)-plets: \(a^{1}[1], a^{2}[2], a^{3}[3], \ldots, a^{N-1}[N-1]\) with \(\alpha = 1, 2, 3, \ldots, N\). The SU(N) generators in terms of these Schwinger bosons are

\[
Q^{a} = \sum_{i=1}^{N-1} a^{a}[i] \frac{\Lambda^{a}}{2} a[i], \quad a = 1, 2, \ldots, (N^2 - 1).
\]  

(42)

Above \(\Lambda^{a}\)'s are the generalization of Gell–Mann matrices for SU(N). The \(N(N-1)\) Harmonic oscillators present in (42) creates an \((N(N-1))\)-dimensional Hilbert space \(\mathcal{H}_{H\theta}^{N(N-1)}\). The SU(N) transformations are

\[
a^{a}_{\beta}[i] \rightarrow a^{a}_{\beta}[i] \left( \exp i \frac{a^{a}}{2} \right)^{\beta}_{\alpha} \quad i = 1, 2, \ldots, (N - 1).
\]  

(43)

The defining equations (42) imply that the \((N - 1)\) Casimirs associated with an SU(N) group, denoted by \(\mathcal{C}_{i}\), are the \((N - 1)\) number operators

\[
\mathcal{C}_{i} \equiv \mathcal{C}_{i} = a^{1}[i] \cdot a[i], \quad i = 1, 2, \ldots, (N - 1).
\]  

(44)

A particular SU(N) irreducible representation is labeled by their eigenvalues: \([n_{1}, n_{2}, \ldots, n_{N-1}]\). The SU(N) monomial eigenstates

\[
|\alpha_{1}^{[1]} \cdot \alpha_{2}^{[2]} \cdot \cdots \cdot \alpha_{N-1}^{[N-1]}\rangle = a^{1}_{\alpha_{1}}[1] \cdots a^{N-1}_{\alpha_{N-1}}[N-1]|0\rangle,
\]  

(45)

satisfy

\[
\mathcal{C}_{i} |\alpha_{1}^{[1]} \cdot \cdots \cdot \alpha_{N-1}^{[N-1]}\rangle = n_{i} |\alpha_{1}^{[1]} \cdots \alpha_{N-1}^{[N-1]}\rangle.
\]  

(46)

The Hilbert space spanned by vectors of type (45) and therefore satisfying the Casimir constraints (46) will be denoted by \(\mathcal{H}_{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{N-1}}\). It is clear that under SU(N) transformations \(\mathcal{H}_{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{N-1}}\) are left invariant and therefore the states (45) form a basis for the representation of SU(N). However, like in the SU(3) case (23), this basis is SU(N) reducible because of the presence of SU(N) invariant operators. These invariants can be removed by implementing the symmetries of the associated SU(N) Young tableau in figure 3 which contains \(n_{i}\) boxes in the \(i\)th row. To construct the SU(N) irrep. we take the convention that the \(i\)th row of an SU(N) Young tableau is created by \(a^{1}[1]\) for \(i = 1, 2, \ldots, N - 1\) as shown in figure 3. Since the Young tableaux is antisymmetric along each column of boxes, we impose the constraints [20, 21],

\[
\hat{L}_{ij} = a^{1}[i] \cdot a[j] \approx 0, \quad i, j = 1, 2, \ldots, (N - 1) \quad \text{and} \quad i < j,
\]  

(47)

on \(\mathcal{H}_{\mathcal{C}_{1}, \mathcal{C}_{2}, \ldots, \mathcal{C}_{N-1}}\). In fact, as shown in [20, 21], the \((N - 1)^{2}\) SU(N) invariant operators \(\hat{L}_{ij}(\mathcal{V})\), \(i = 1, 2, \ldots, N - 1\) form the U(N-1) Lie algebra satisfying

\[
[\hat{L}_{ij}, \hat{L}_{kl}] = \delta_{ik} \hat{L}_{jl} - \delta_{il} \hat{L}_{kj}, \quad i, j, k, l = 1, 2, \ldots, (N - 1).
\]  

(48)

Note that the set of \((N - 1)(N - 2)\) SU(N) invariant U(N-1) constraints (47) weakly commute amongst themselves:

\[
[\hat{L}_{i, j; n}, \hat{L}_{j, i; m}] = \delta_{j, i; n} \hat{L}_{i, j; n} - \delta_{i, j; m} \hat{L}_{i, j; n} = \delta_{j, i; n} \hat{L}_{i, j; m} + \delta_{i, j; m} \hat{L}_{i, j; m} \approx 0.
\]
Therefore, they form the set of the maximum number of self consistent constraints which can be imposed on \( H_{n_1, n_2, \ldots, n_{N-1}} \) to get the SU(\( N \)) irreducible representations [21].

Again following the SU(3) case, we can trivialize all the constraints in (47) in terms of SU(\( N \)) irreducible Schwinger bosons [20]. The SU(\( N \)) irreducible Schwinger boson creation operators acting on the vacuum directly create states having all the symmetries of the corresponding Young tableaux. These new irreducible Schwinger bosons \( A^i_1[i] \) for \( i = 1, 2, \ldots, N - 1 \) are related to the ordinary Schwinger bosons in (42) as follows:

\[
A^i_1[k] = a^{i_1}[k] + \sum_{r=1}^{k-1} \sum_{(i_1, \ldots, i_r) = 1}^{k-1} F^k_{i_1} F^k_{i_2} \cdots F^k_{i_r} \hat{L}_{i_1 i_2} \cdots \hat{L}_{i_r 1} a^{i_r}[1].
\]

In (49), \( k = 1, 2, \ldots, (N - 1) \) and the prime over the second summation (\( \sum' \)) implies that the ordering \( k > i_1 > i_2 > \cdots > i_r \) has to be maintained. The general form of \( F^k_i (n_1, \ldots, n_{N-1}) \) is given by

\[
F^k_i = -\frac{1}{n_i - n_k + 1 + k - i}.
\]

The defining equations (49) imply that the SU(\( N \)) irreducible Schwinger bosons also transform as SU(\( N \)) \( \mathcal{N} \)-plets:

\[
A^i_1[i] \rightarrow A^i_1[i] \left( \exp i \theta^a \frac{\lambda^a}{2} \right)^{\beta}_{\alpha}, \quad i = 1, 2, \ldots, (N - 1).
\]

Therefore, as in the SU(2) case, the Hilbert space created by the monomials of SU(\( N \)) irreducible Schwinger boson (49) creation operators is isomorphic to the space of irreducible representations of SU(\( N \)). In other words, the state

\[
| \alpha^{[1]}_1, \alpha^{[2]}_2, \ldots, \alpha^{[N-1]}_{n_{N-1}} \rangle \equiv (A^{[1]}_{n_1}[N-1] \cdots A^{[N-1]}_{n_{N-1}}[N-1]) | 0 \rangle
\]

carries all the symmetries of an SU(\( N \)) Young tableau shown in figure 3.
4.2. Construction of SU(N) coherent states

Like in SU(2) and SU(3) cases in (8) and (33), respectively, we characterize the SU(N) group manifold by \( N - 1 \) number of complex \( N \)-plets: \( \{z_\alpha[i]\}, i = 1, 2, \ldots, N - 1 \) and \( \alpha = 1, 2, \ldots, N \) following orthonormality constraints:

\[
\bar{z}[\alpha] \cdot z[\beta] = \delta_{\alpha,\beta}.
\] (53)

With the above parametrization any SU(N) matrix has the following form:

\[
U_N = \begin{pmatrix}
z_{[1]} & z_{[2]} & \cdots & z_{[N-1]}^1 & (\bar{z}_{[1]} \wedge \bar{z}_{[2]} \wedge \cdots \wedge \bar{z}_{[N-1]})_1 \\
z_{[1]} & z_{[2]} & \cdots & z_{[N-1]}^2 & (\bar{z}_{[1]} \wedge \bar{z}_{[2]} \wedge \cdots \wedge \bar{z}_{[N-1]})_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_{[1]} & z_{[2]} & \cdots & z_{[N-1]}^N & (\bar{z}_{[1]} \wedge \bar{z}_{[2]} \wedge \cdots \wedge \bar{z}_{[N-1]})_N
\end{pmatrix}.
\] (54)

At this stage we generalize (11) and (36) to define the SU(N) coherent state generating a function as

\[
|z[1], z[2], \ldots, z[N-1]\rangle = \exp(z[N-1] \cdot A^1[N-1]) \ldots \exp(z[2] \cdot A^1[2]) \times \exp(z[1] \cdot A^1[1])(0).
\] (55)

Note that the coherent state generating function (55) contains all possible irreducible representations of SU(N). Further, the expressions for SU(N + 1) and SU(N) coherent states differ only by the last exponential factor in (55). Therefore, the present SU(N) coherent state construction is iterative in nature. Now, projecting out a specific coherent state denoted by the set of particular values of the SU(N) Casimirs, i.e. \( A^1[i] \cdot A[i] \) having the eigenvalue \( n_i \) with \( i = 1, 2, \ldots, (N - 1) \) and \( n_1 \geq n_2 \geq \cdots \geq n_{N-1} \), we get the SU(N) coherent state in the irreducible representation \( [n_1, n_2, \ldots, n_{N-1}] \):

\[
|z[1], z[2], \ldots, z[N-1]\rangle_{[n_1, n_2, \ldots, n_{N-1}]} = \frac{(z[N-1] \cdot A^1[N-1])^{n_{N-1}} \cdots (z[2] \cdot A^1[2])^{n_2} (z[1] \cdot A^1[1])^{n_1}}{n_1!n_2! \cdots n_{N-1}!} |0\rangle
\]

\[
= \sum_{a_1^{[1]}, \ldots, a_{N-1}^{[1]}=1}^{\alpha_1} \cdots \sum_{a_1^{[N-1]}, \ldots, a_{N-1}^{[N-1]}=1}^{\alpha_{N-1}} \sum_{a_1^{[1]}, \ldots, a_{N-1}^{[1]}=1}^{\alpha_1} \cdots \sum_{a_1^{[N-1]}, \ldots, a_{N-1}^{[N-1]}=1}^{\alpha_{N-1}} \times \left( z[1], z[2], \ldots, z[N-1] \right) \frac{a_1^{[1]} \cdots a_1^{[N-1]} \cdots a_1^{[N-1]} \cdots a_{N-1}^{[N-1]}}{\text{SU(N) irrep. state}}.
\] (56)

In (56), the SU(N) coherent state structure functions are given by

\[
F_{a_1^{[1]} \cdots a_{N-1}^{[N-1]} a_1^{[N-1]} \cdots a_{N-1}^{[N-1]}} = \frac{1}{n_1!n_2! \cdots n_{N-1}!} z[1]^{a_1^{[1]}} \cdots z[1]^{a_1^{[N-1]}} \cdots z[N-1]^{a_1^{[N-1]}} \cdots z[N-1]^{a_{N-1}^{[N-1]}}.
\] (57)

The states in (56) depend smoothly on the SU(N) group manifold coordinates. We now check the resolution of identity. Like in the previous SU(2) and SU(3) sections, under SU(N) transformations (51) all the \((N - 1)\) coherent state coordinates \(z[i]\) transform as \(N\)-plets:

\[
z_\alpha[i] \rightarrow z_\beta'[i] = z_\beta[i] \left( \exp i \sum_{a=1}^{N^2-1} \theta^a A^a_\alpha \right)^\beta.
\] (58)
We again define the operator $O_{[n_1, n_2, \ldots, n_{N-1}]}$ as
\[
O_{[n_1, n_2, \ldots, n_{N-1}]} \equiv \int d\mu(z) \left( \langle z[1], z[2], \ldots, z[N-1] \rangle_{[n_1, n_2, \ldots, n_{N-1}]} \right.
\times \langle z[1], z[2], \ldots, z[N-1] \rangle_{[n_1, n_2, \ldots, n_{N-1}]}.
\]
(59)

In (59), $\int d\mu(z)$ is the SU($N$) invariant Haar measure:
\[
\int d\mu(z) \equiv \prod_{\alpha=1}^{N-1} \int d^2z[\alpha] \prod_{1 \leq \alpha \leq \beta \leq N-1} \delta(z[\alpha] \cdot z^*[\beta] - \delta_{\alpha, \beta}).
\]

Under SU($N$) transformations (58), $O_{[N]}$ remains invariant. Therefore, $[O_{[a]}, O_{[n_1, n_2, \ldots, n_{N-1}]}] = 0$, $\forall a = 1, 2, \ldots, N^2 - 1$. (60)

Schur’s lemma implies
\[
O_{[n_1, n_2, \ldots, n_{N-1}]} = I_{[n_1, n_2, \ldots, n_{N-1}]}.
\]
(61)

In (61), $I_{[n_1, n_2, \ldots, n_{N-1}]}$ is proportional to the identity operator in the irreducible representation subspace. We again emphasize that the SU($N$) coherent states in (56) are the most straightforward extension of the Heisenberg–Weyl, SU(2) and SU(3) coherent states in (2), (11) and (36), respectively.

5. Conclusions

We have exploited SU($N$) irreducible Schwinger boson creation operators to construct SU($N$) coherent states. This construction is analogous to the simplest and the oldest harmonic oscillator coherent state construction. This procedure is iterative in $N$. It is also self-contained as it does not require any prior knowledge of the SU($N$) group elements and their representations. This novel SU($N$) coherent state construction can be used to compute SU($N$) Clebsch–Gordan and re-coupling coefficients. This amounts to generalizing the Schwinger method to compute these coefficients for SU(2) (see section 3 of [9] on the addition of angular momenta) to SU($N$). This is particularly interesting as these SU($N$) coupling coefficients for arbitrary $N$ are not yet known in the closed form. Work in this direction is in progress and will be reported elsewhere.

Apart from these group theoretical applications of SU($N$) coherent states constructed in this work they can also be exploited to study the SU($N$) Heisenberg model with the SU($N$) invariant Hamiltonian $[7, 23]$:
\[
\hat{H} = \sum_{x,y} J_{x,y} \sum_{a=1}^{N^2-1} Q^a(x) Q^a(y).
\]
(62)

In (62), $Q^a(x)$ and $Q^a(y)$ are the SU($N$) generators defined in (42) at lattice sites $x$ and $y$, respectively. The phase diagrams of these SU($N$) models and their large $N$ limits ($N \rightarrow \infty$) have been extensively studied [7, 23] through an SU($N$) coherent state path integral representation of (62). However, all such studies have been confined to only completely symmetric representations, i.e. $n_1 \neq 0, n_2 = n_3 = \cdots = n_{N-1} = 0$ in (46)). The present construction of SU($N$) coherent states now allows a study of the SU($N$) Heisenberg model in any arbitrary $(n_1, n_2, \ldots, n_{N-1})$ representation. It can also be exploited to construct the coherent state path integral representation of SU($N$) lattice gauge theories [24] in terms of complex $(N - 1)$ SU($N$) $N$-plets in (54).
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