ON A GEOMETRIC DESCRIPTION OF $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$, AND A $p$-ADIC AVATAR OF $\widetilde{GT}$

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ABSTRACT. We develop a $p$-adic version of the so-called Grothendieck-Teichmüller theory (which studies $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ by means of its action on profinite braid groups or mapping class groups). For every place $v$ of $\mathbb{Q}$, we give some geometrоро-combinatorial descriptions of the local Galois group $\text{Gal}(\mathbb{Q}_v/\mathbb{Q}_v)$ inside $\text{Gal}(\mathbb{Q}/\mathbb{Q})$. We also show that $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$ is the automorphism group of an appropriate $\pi_1$-functor in $p$-adic geometry.

Classification: 11R32, 14H30, 14G22, 14G20, 20F28, 20F36.
1. Introduction

1.1. Profinite groups which are the absolute Galois group $G_k = Gal(\bar{k}/k)$ of some field $k$ have been the object of extensive study, especially in the case where $k$ is a number field. In that case, $G_k$ has more structure: it comes equipped with a constellation of closed subgroups $G_{k_v} = Gal(\bar{k}_v/k_v)$ attached to the places $v$ of $k$, and fitting together in an arithmetically relevant way.

The problem then arises to describe these local Galois groups $G_{k_v}$ in $G_k$.

1.2. A purely group-theoretic approach to this problem has been found by Artin (for archimedean $v$) and Neukirch (for non-archimedean $v$): namely, the subgroups $G_{k_v} \subset G_k$ for $v|k$ real are exactly the subgroups of order two; the subgroups $G_{k_v}$ for $v$ non-archimedean are exactly the closed subgroups which have, abstractly, the algebro-topological structure of the absolute Galois group of a local field\(^1\) and are maximal for this property, cf. [27, § XII.1]. This is however more a characterization than a description of the local Galois groups - rather, of the set of local Galois groups $G_{k_v}$ attached to a fixed place $v|k$ of $k$.

1.3. In this paper, we examine the problem from a completely different viewpoint, aiming at a geometric solution (with combinatorial flavour). Our approach is inspired by Grothendieck’s leitmotiv of studying $G_k$ via its outer action on the “geometric” algebraic fundamental group $\pi_{alg}^1(\bar{X}_k)$ of smooth geometrically connected algebraic varieties $X$ defined over $k$. In fact, the problem of describing the local Galois groups in $G_k$ has been raised explicitly by Grothendieck in the context of geometric actions on fundamental groups and his dream of anabelian geometry ([11], note 4).\(^2\)

1.4. In the simplest non-trivial case ($k = \mathbb{Q}$ and $X = \mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}$), it turns out that the outer action of $G_\mathbb{Q}$ on the profinite group $\pi_{alg}^1(X_\mathbb{Q})$ is faithful (Belyi). An embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ being fixed, $\pi_{alg}^1(X_\mathbb{Q})$ is the profinite completion of the usual ‘transcendental’ fundamental group $\pi_{top}^1(X^{an})$ of the complex-analytic manifold $X^{an}$ (a discrete free group of rank two).

\(^1\)which is “known”, cf. [27, § VII.5].

\(^2\)we became aware of these lines on having another look at [11] just before completing this work: “Parmi les points cruciaux de ce dictionnaire [anabélien], je prévois [...] une description des sous-groupes d’inertie de $\Gamma [ = G_\mathbb{Q}]$, par où s’amorce le passage de la caractéristique zéro à la caractéristique $p > 0$, et à l’anneau absolu $\mathbb{Z}$.”
One can then recover $G_{\mathbb{R}}$ inside $G_{\mathbb{Q}}$ as the intersection of $G_{\mathbb{Q}}$ and $Out\,\pi_1^{\text{top}}(X^{an})$ in $Out\,\pi_1^{\text{alg}}(X_{\overline{\mathbb{Q}}})$, (cf. 3.3.1, and 3.3.2 for a more general statement).

1.5. We shall give a similar description of the local Galois group $G_{\mathbb{Q}_{p}}$, embedded into $G_{\mathbb{Q}}$ via a fixed embedding $\mathbb{Q} \hookrightarrow \mathbb{C}_{p} = \widehat{\mathbb{Q}}_{p}$. The required ingredient is a fundamental group for rigid-analytic “$p$-adic manifolds” playing the role of $\pi_1^{\text{top}}$. In particular, the profinite completion of such a fundamental group should coincide with the algebraic fundamental group $\pi_1^{\text{alg}}$, in the case of an algebraic $p$-adic manifold.

Such a theory has been set up in [1] (for completely different purposes), and will be outlined below (§4). Let us just say here that its corner-stone is the notion of tempered etale covering, which generalizes in a minimal way both finite etale coverings and infinite topological coverings.

The corresponding tempered fundamental groups $\pi_1^{\text{temp}}$ encapsulate combinatorial information about the bad reduction of all finite etale coverings of the base. Their topology is a little complicated (for instance, the tempered fundamental group of $\mathbb{P}^1_{\mathbb{C}_{p}} \setminus \{0, 1, \infty\}$ is complete but not locally compact, hence is neither discrete nor profinite); but in dimension one, they have a suggestive combinatorial description as inverse limit of a sequence of fundamental groups of certain finite graphs of groups, which could be considered as non-archimedean analogues of Grothendieck’s ‘dessins d’enfants’.

1.6. For any smooth geometrically connected algebraic variety $X$ over $\mathbb{Q}_{p}$, there is a canonical outer action of $G_{\mathbb{Q}_{p}}$ on $\pi_1^{\text{temp}}(X^{an})$, where $X^{an}$ denotes the rigid-analytic manifold attached to $X_{\mathbb{C}_{p}}$, cf. prop. 5.1.1.

Coming back to the case of $X = \mathbb{P}^1_{\mathbb{Q}_{p}} \setminus \{0, 1, \infty\}$, one version of our main result is that one can recover $G_{\mathbb{Q}_{p}}$ inside $G_{\mathbb{Q}}$ as the intersection of $G_{\mathbb{Q}}$ and $Out\,\pi_1^{\text{temp}}(X^{an})$ in $Out\,\pi_1^{\text{alg}}(X_{\overline{\mathbb{Q}}})$, cf. thm. 7.2.1 (a more general statement is given in 7.2.3). We also study the structure of $Out\,\pi_1^{\text{temp}}(X^{an})$ for algebraic curves $X$. We show that this group is, like $\pi_1^{\text{temp}}(X^{an})$ itself, the inverse limit of a sequence of finitely generated discrete groups (6.1.4).

1.7. According to the philosophy of the Grothendieck-Teichmüller theory, $\mathbb{P}^1_{\mathbb{Q}_{p}} \setminus \{0, 1, \infty\}$ should be seen as the special case $n = 4$ of the moduli space $\mathcal{M}_{0,r}$ of curves of genus 0 with $r$ ordered marked points, and one should study as well the Galois action on the geometric fundamental group of these moduli spaces (and also in higher genus). The main player here is Drinfeld’s

\footnote{following the referee’s advice, we have changed our previous terminology ‘temperate’ to ‘tempered’}
Grothendieck-Teichmüller group $\hat{G}T$, a kind of “geometric upper bound” for $G_{\mathbb{Q}}$.

We introduce a closed subgroup $\hat{G}T_p$ of the profinite group $\hat{G}T$, the local Grothendieck-Teichmüller group at $p$, defined in terms of the rigid-analytic manifolds attached to the moduli spaces $\mathcal{M}_{0,r}$ over $\mathbb{C}_p$ (8.6.3). This group is a kind of “geometric upper bound” for $G_{\mathbb{Q}_p}$: we prove that $G_{\mathbb{Q}_p}$ is the intersection of $G_{\mathbb{Q}}$ and $\hat{G}T_p$ in $\hat{G}T$ (thm. 8.7.1).

This sheds some new light on the longstanding problem of how close is $\hat{G}T$ to $G_{\mathbb{Q}}$ (cf. 8.7.2).

1.8. Another attempt to describe the absolute Galois group of a number field $k$ as the full automorphism group of some geometric structure consists in looking at sufficiently many $k$-varieties at a time and at the geometric fundamental group as a functor. Geometric algebraic fundamental groups give rise to a functor $\pi_{k}^{alg}$ from the category $\mathcal{V}_k$ of smooth geometrically connected varieties over a number field $k$ to the category $\mathcal{T}$ of topological groups up to inner automorphisms. Pop has shown that $G_k = \text{Aut} \pi_{k}^{alg}$ ([29], unpublished).

On the other hand, for any $p$-adic place $v$ of $k$, there is a functor $\pi_{\mathbb{C}_p}^{temp}$ : $\mathcal{V}_k \rightarrow \mathcal{T}$ given by the tempered fundamental groups of associated $p$-adic manifolds. Our final result is that $G_{k_v} = \text{Aut} \pi_{\mathbb{C}_p}^{temp}$ (thm. 9.2.2).

Thus in some sense, the arithmetic of finite extensions of $\mathbb{Q}_p$ is embodied in analytic geometry over $\mathbb{C}_p$. 
2. GEOMETRIC FUNDAMENTAL GROUPS AND GALOIS ACTIONS

2.1. Let $X$ be a smooth geometrically connected algebraic variety over $k$ endowed with a geometric point $x$. Grothendieck’s algebraic fundamental group $\pi_{1}^{alg}(X, x)$ is the profinite group which classifies all finite etale (pointed) coverings of $(X, x)$:

$$\{\text{finite etale coverings of } X\} \sim \{\text{finite } \pi_{1}^{alg}(X, x)\text{-sets}\}.$$  

This group depends on $x$ only up to inner automorphism.

Let $\bar{k}$ be a separable closure of $k$, and let $G_k = Gal(\bar{k}/k)$ stand for the absolute Galois group. The group $\pi_{1}^{alg}(X_{\bar{k}}, x)$ is sometimes called the geometric fundamental group of $X$ (pointed at $x$). There is a functorial exact sequence

$$\{1\} \rightarrow \pi_{1}^{alg}(X_{\bar{k}}, x) \rightarrow \pi_{1}^{alg}(X, x) \rightarrow G_k \rightarrow \{1\},$$

which splits canonically if $x$ comes from a $k$-rational point of $X$. Whence a Galois action

$$G_k \xrightarrow{\rho} Out \pi_{1}^{alg}(X_{\bar{k}}),$$

which lifts to an action

$$G_k \xrightarrow{\rho_x} Aut \pi_{1}^{alg}(X_{\bar{k}}, x)$$

if $x$ comes from a $k$-rational point of $X$ (here $Out = Aut/Inn$ denotes as usual the group of outer automorphisms (in $\rho$, we drop the base point from the notation since it is irrelevant).

2.2. We assume henceforth that $char \ k = 0$. Under this assumption, it is known that the profinite groups $\pi_{1}^{alg}(X_{\bar{k}}, x)$ are finitely generated, which implies that $Aut \pi_{1}^{alg}(X_{\bar{k}}, x)$ and $Out \pi_{1}^{alg}(X_{\bar{k}})$ are also finitely generated (hence metrizable\(^\text{4}\)) profinite groups. More precisely, there are only finitely many open normal subgroups of $\pi_{1}^{alg}(X_{\bar{k}}, x)$ of index dividing\(^\text{4}\) $n$, hence their intersection $U_n$ is a characteristic\(^\text{6}\) open subgroup of $\pi_{1}^{alg}(X_{\bar{k}}, x)$ (hence is preserved by $G_k$). It follows that $U_n = \pi_{1}^{alg}(X_n, x_n)$ for a well-defined finite Galois etale (pointed) covering $(X_n, x_n) \rightarrow (X_{\bar{k}}, x)$ with $X_n$ geometrically connected; in fact, the tower of characteristic subgroups $U_n$ is

\(^4\)recall that a profinite group is metrizable if and only if it has countably many open subgroups, or, equivalently, if it is a countable inverse limit of finite groups, \([\text{32}, \S \text{4.1.3}]\) \([\text{35}, \S \text{IX.2.8}]\), or else, if and only if it is a closed subgroup of a quotient of the profinite free group on two generators \(F_2\) \([\text{32}, \S \text{4.1.6}]\) (a profinite variant of the fact that any countable discrete group is a subgroup of a quotient of \(F_2\)); because of these equivalences, one sometimes says ‘separable’ instead of ‘metrizable’

\(^6\)i.e. stable under every automorphism of the profinite group $\pi_{1}^{alg}(X_{\bar{k}}, x)$
the tower of geometric fundamental groups of a ‘tower’ of (pointed) finite Galois etale coverings

\[ \ldots \to X_{n'} \to X_n \to \ldots \to X_{\bar{k}}, \quad n|n'. \]

We denote by \( \Gamma_n = \pi_{1,\text{alg}}(X_{\bar{k}}, x)/U_n \) the Galois group of the covering \( X_n/X_{\bar{k}} \).

Notice that the above ‘tower’ is canonical and functorial in \( X \), and the same is true for the corresponding ‘tower’ of finite groups

\[ \ldots \to \Gamma_{n'} \to \Gamma_n \to \ldots \to \{1\}, \quad n|n'. \]

Moreover, its formation is compatible with extension of algebraically closed fields \( \bar{k} \hookrightarrow \bar{k}' \). One then has (cf. \[22, \S 3.4. \text{ex.6}] for the case of \( \text{Aut} \))

\[ \pi_{1,\text{alg}}(X_{\bar{k}}, x) = \lim_{\leftarrow} \Gamma_n, \]

\[ \text{Aut} \pi_{1,\text{alg}}(X_{\bar{k}}, x) = \lim_{\leftarrow} \text{Aut} \Gamma_n, \]

\[ \text{Out} \pi_{1,\text{alg}}(X_{\bar{k}}) = \lim_{\leftarrow} \text{Out} \Gamma_n. \]

The topology of the group of (outer) automorphisms defined by the inverse limit is called the topology of congruence subgroups; thus, the group of (outer) automorphisms of \( \pi_{1,\text{alg}}(X_{\bar{k}}, x) \) is closed for the (profinite) topology of congruence subgroups.

2.2.1. Observation. For any normal closed subgroup \( U \) of \( \pi_{1,\text{alg}}(X_{\bar{k}}) \) contained in \( U_n \), the quotient \( U_n/U \) is a characteristic subgroup of \( \pi_{1,\text{alg}}(X_{\bar{k}})/U \).

Indeed, by definition, the homomorphisms \( \pi_{1,\text{alg}}(X_{\bar{k}}) \to \pi_{1,\text{alg}}(X_{\bar{k}})/U \to \Gamma_n \) induce bijections between the sets of finite quotients of order dividing \( n \) of each of these groups.

In particular, the kernel of \( \Gamma_{n'} \to \Gamma_n \) is characteristic, for any multiple \( n' \) of \( n \).

2.3. In this paper, ‘curve’ is an abbreviation for ‘smooth geometrically connected algebraic variety of dimension one’. A curve \( X \) over \( k \) is called hyperbolic or anabelian if its geometric fundamental group is non-abelian. If \( X \) is affine, this just means that \( X_{\bar{k}} \) is not isomorphic to the projective line minus one or two points.

The following result will be of constant use throughout the present paper:

2.3.1. Theorem (Belyi, Matsumoto[21]). Assume that \( k \) is a number field. Then the Galois action

\[ G_k \overset{\rho}{\to} \text{Out} \pi_{1,\text{alg}}(X_{\bar{k}}) \]

is faithful for any hyperbolic affine curve \( X \).

\[ \square \]

\[ ^7 \text{I am endebted to one of the referees for this observation} \]
This relies in turn on Belyi’s theorem (used several times in the sequel) according to which any curve defined over \( k \) admits a rational function which is ramified only above 0, 1, \( \infty \).

3. \( G_Q, G_R \) AND GEOMETRIC FUNDAMENTAL GROUPS

3.1. Now \( k \) is a number field, and we fix an embedding \( \iota : \bar{k} \hookrightarrow \mathbb{C} \). For any smooth geometrically connected algebraic \( k \)-variety \( X \) with a geometric point \( x \), one has canonical isomorphisms

\[
\pi_1^{alg}(X_{\bar{k}}, x) = \pi_1^{alg}(X_\mathbb{C}, x) = \pi_1^{top}(X_\mathbb{C}^{an}, x),
\]

the latter group being the profinite completion of the usual topological fundamental group \( \pi_1^{top}(X_\mathbb{C}^{an}, x) \) which classifies topological (= étale) coverings of the complex-analytic manifold attached to \( X_\mathbb{C} \).

If \( \iota(k) \subset \mathbb{R} \), there is a Galois action

\[
G_R = \mathbb{Z}/2\mathbb{Z} \xrightarrow{\rho_{\infty}} \text{Out} \ \pi_1^{top}(X_\mathbb{C}^{an}, x),
\]

which lifts to an action

\[
G_R \xrightarrow{\rho_{\infty}, x} \text{Aut} \ \pi_1^{top}(X_\mathbb{C}^{an}, x)
\]

if \( x \) comes from a real point of \( X \). This action is compatible with the (outer) action of \( G_K \) on the profinite completion \( \hat{\pi}_1^{alg}(X_{\bar{k}}, x) \).

3.2. Assume that \( X \) is an affine curve. Then \( \pi_1^{top}(X_\mathbb{C}^{an}, x) \) is a free group of finite rank, which implies that it is residually finite, i.e. embeds into its profinite completion. It follows that the following natural homomorphisms are injective:

\[
\pi_1^{top}(X_\mathbb{C}^{an}, x) \hookrightarrow \pi_1^{alg}(X_{\bar{k}}, x),
\]

\[
\text{Aut} \ \pi_1^{top}(X_\mathbb{C}^{an}, x) \hookrightarrow \text{Aut} \ \pi_1^{alg}(X_{\bar{k}}, x),
\]

\[
\text{Out} \ \pi_1^{top}(X_\mathbb{C}^{an}) \hookrightarrow \text{Out} \ \pi_1^{alg}(X_{\bar{k}}).
\]

For the injectivity of the third homomorphism, one uses in addition the following

3.2.1. Lemma. A free group \( F \) of finite rank \( > 1 \) is its own normalizer in its profinite completion \( \hat{F} \).

For lack of reference\(^8\), we indicate a proof. Let \( a \) belong to the normalizer of \( F \) in \( \hat{F} \), and let \( x_1, \ldots, x_r \) be a basis of \( F \). A classical result of Stebe says that \( F \) is conjugacy-separated: if \( x, y \in F \) are conjugate in every finite

\(^8\)not necessarily finite

\(^9\)after submission of this paper, the reference [9, thm. 2] was pointed out to the author by A. Tamagawa and one of the referees
quotient of $F$, they are conjugate in $F$ [20, prop. 4.9]. In particular, there are elements $a_i \in F$ such that $a x_i a^{-1} = a_i x_i a_i^{-1}$. Hence $a_i^{-1} a$ belongs to the centralizer of $x_i$ in $\hat{F}$, which is $x_i^Z$. Therefore, there are elements $z_i \in \mathbb{Z}$ such that $a = a_i x_i z_i$. For $i = 1, 2$, this implies $x_1^{z_1} x_2^{z_2} \in F$, whence $z_1, z_2 \in \mathbb{Z}$, and $a \in F$. □

We denote by $\text{Out} \pi_{1}^{\text{top}}(X_{\text{an}})$ the completion of $\text{Out} \pi_{1}^{\text{top}}(X_{\text{an}})$ with respect to the topology of congruence subgroups, i.e. the closure of $\text{Out} \pi_{1}^{\text{top}}(X_{\text{an}})$ in $\text{Out} \pi_{1}^{\text{alg}}(\overline{X}_{k})$:

$$\text{Out} \pi_{1}^{\text{top}}(X_{\text{an}}) = \varprojlim \text{Im}[\text{Out} a_{1}^{\text{top}}(X_{\text{an}}) \to \text{Out} \Gamma_{n}]$$

(this limit is clearly a quotient of the profinite completion of $\text{Out} \pi_{1}^{\text{top}}(X_{\text{an}})$ and embeds into $\varprojlim \text{Out} \Gamma_{n} = \text{Out} \pi_{1}^{\text{alg}}(\overline{X}_{k})$).

Since we shall have to compare repeatedly the (outer) automorphism group of a topological group with the (outer) automorphism group of its profinite completion, we include here the

3.2.2. Sorite. a) Let $\phi : G \to H$ be a surjective homomorphism of topological groups, and let $\hat{\phi} : \hat{G} \to \hat{H}$ be the induced homomorphism of their profinite completions.

Let $\tau$ be an automorphism of $G$, and let $\hat{\tau}$ be the corresponding automorphism of $\hat{G}$. If $\hat{\tau}$ induces an automorphism of $\hat{H}$ (via $\hat{\phi}$), then $\tau$ induces an automorphism of $H/ \ker(H \to \hat{H})$.

b) Let $F$ be an open subgroup of finite index of a topological group $G$. We assume that $G$ embeds into its profinite completion $\hat{G}$. Let $\tau$ be as before an automorphism of $G$. If $\hat{\tau}$ preserves the image $\overline{F}$ of $F$ in $\hat{G}$, then $\tau$ preserves $F$.

Proof. a) Replacing $G$ and $H$ by $G/ \ker(G \to \hat{G})$ and $H/ \ker(H \to \hat{H})$ respectively, one may assume that they embed into their profinite completion. Let $\text{Aut}(G, \ker \phi)$ (resp. $\text{Aut}(\hat{G}, \ker \hat{\phi})$) denote the group of automorphisms of $G$ (resp. $\hat{G}$) which preserve the kernel of $\phi$ (resp. $\hat{\phi}$). The sorite follows from the equality

$$\text{Aut}(G, \ker \phi) = \text{Aut}(\hat{G}, \ker \hat{\phi}) \cap \text{Aut} G,$$

b) One has $F = F \cap G$, whence

$$\text{Aut}(G, F) = \text{Aut}(\hat{G}, \overline{F}) \cap \text{Aut} G.$$  

\footnote{10} the profinite completion of a topological group is the inverse limit of its finite quotients (i.e. quotients by open subgroups of finite index)

\footnote{11} in the category of topological groups, of course
3.3. For any hyperbolic affine curve $X$ over a number field $k$ (and a fixed complex embedding $i$ of $k$), we have encountered two closed subgroups of $Out\,\pi_1^{alg}(X_k): \overline{Out\,\pi_1^{top}(X^an)}$ and $G_k$. We shall study their intersection, starting with the case of $P^1 \setminus \{0, 1, \infty\}$.

3.3.1. Theorem. If $X = P^1_k \setminus \{0, 1, \infty\}$, then

$$Out\,\pi_1^{top}(X^an) \cap G_k = \overline{Out\,\pi_1^{top}(X^an)} \cap G_k = \begin{cases} G_R & \text{if } i(k) \subset \mathbb{R} \\ \{1\} & \text{otherwise}. \end{cases}$$

Proof. We already know that $G_R \subset Out\,\pi_1^{top}(X^an) \cap G_k$ if $i(k) \subset \mathbb{R}$. It remains to show that $\overline{Out\,\pi_1^{top}(X^an)} \cap G_k \subset G_R$ (from which it follows that the intersection is trivial if $i(k) \not\subset \mathbb{R}$). We may and shall assume that $k = \mathbb{Q}$.

Since $\pi_1^{top}(X^an)$ is a free group of rank two, $Out\,\pi_1^{top}(X^an)$ is faithfully represented on the abelianization $(\pi_1^{top}(X^an))^{ab} \cong \mathbb{Z}^2$ according to a classical theorem of Nielsen (cf. [20, §1.4.5]). Hence $Out\,\pi_1^{top}(X^an) \cong GL_2(\mathbb{Z})$ and $\overline{Out\,\pi_1^{top}(X^an)}$ is a quotient of the profinite completion $\overline{GL_2(\mathbb{Z})}$ and commutes with the subgroup $\{\pm id\} \subset GL_2(\mathbb{Z}) \cong Out\,\pi_1^{top}(X^an)$. This subgroup lies in $Out\,\pi_1^{top}(X^an) \cap G_Q$, and coincides with $G_R$. It follows that $\overline{Out\,\pi_1^{top}(X^an)} \cap G_Q$ commutes with $G_R$. But it is well-known that $G_R$ is its own centralizer in $G_Q$ (cf. [27, §12.1.4]).

For more general affine hyperbolic $k$-curves $X$, we have:

3.3.2. Theorem. Assume that some finite etale covering of $X_k$ admits a non-constant rational function which omits at least three values. Assume also that $i(k) \subset \mathbb{R}$. Then

$$Out\,\pi_1^{top}(X^an) \cap G_k = G_R.$$

A fortiori, if the geometric point $x$ is $k$-rational, then the subgroup of $G_k$ which stabilizes $\pi_1^{top}(X^an, x) \subset \pi_1^{alg}(X_k, x)$ is $G_R$.

(Note that the assertion about ‘Aut’ is less precise than the statement about ‘Out’.)
Proof. From the assumption, there is a finite extension $k'$ of $k$ in $C$, a finite etale covering $Y \to X_{k'}$ and a dominant morphism

$$Y \xrightarrow{\psi} Z = \mathbb{P}^1_{k'} \setminus \{0, 1, \infty\}.$$  

(We may also assume that $X$, $Y$, $Z$ are endowed with compatible $k'$-rational geometric points $x$, $y$, $z$ respectively). The result will follow from 3.3.1 by the following dévissage:

3.3.3. Lemma. The statement

$$(*)_{X,k} \quad \text{Out} \, \pi^\text{top}_1(X^{an}) \cap G_k \subset G_R$$

satisfies

a) $(*)_{X,k'} \Rightarrow (*)_{X,k}$ if $k'$ is a finite extension of $k$ in $C$,

b) $(*)_{Z,k} \Rightarrow (*)_{Y,k}$ if there is a non-constant morphism $f : Y \to Z$,

c) $(*)_{Y,k} \Rightarrow (*)_{X,k}$ if $Y$ is a finite étale covering of $X$,

provided $\iota(k) \subset \mathbb{R}$.

Proof. a) $\text{Out} \, \pi^\text{top}_1(X^{an}) \cap G_k$ is finite by $(*)_{X,k'}$, and contains $G_R$ because $\iota(k) \subset \mathbb{R}$). We conclude by the following

3.3.4. Sublemma. (Artin, cf. [27, § 12.1.7]) If $H$ is a closed subgroup of $G_k$ containing $G_R$ as a subgroup of finite index, then $H = G_R$. □

For b), it is more convenient to deal with $\text{Aut}$ instead of $\text{Out}$; we pass from the latter to the former using the following consequence of 2.3.1 (replacing the assumption that $\iota(k) \subset \mathbb{R}$ by the assumption that $x$ is $k$-rational, as we may using a) and its trivial converse):

3.3.5. Sublemma. The natural homomorphism

$$H_X := (\text{Inn} \, \pi^\text{alg}_1(X_k, x).\text{Aut} \, \pi^\text{top}_1(X^{an}, x)) \cap G_k \to \text{Out} \, \pi^\text{top}_1(X^{an}) \cap G_k$$

is an isomorphism (the first intersection is taken in $\text{Aut} \, \pi^\text{alg}_1(X_k, x)$, the second one in $\text{Out} \, \pi^\text{alg}_1(X_k)$). □

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12after submission of this paper, A. Tamagawa (and one of the referees) pointed to me that this result can also be deduced from [23, thm. 1.1, rmk. 2.1], even without the assumption that some finite étale covering of $X_k$ admits a non-constant rational function which omits at least three values: indeed, let $\Gamma_{g,r}$ be the relevant mapping class group, viewed as a subgroup of index two in the group $\text{Out}^* \pi^\text{top}_1(X^{an})$ of outer automorphisms preserving the conjugacy class of the local monodromy at each puncture, and let $\Gamma_{g,r}$. The assertion follows from two facts: a) $G_k \setminus \{1\}$ does not intersect (the closure of) the image of $\Gamma_{g,r}$ in $\text{Out}^\text{alg}_1(X_k)$ [23, thm. 1.1]; b) $\text{Out}^\text{top}_1(X^{an}) \cap G_k \subset \text{Out}^* \pi^\text{top}_1(X^{an})$
b) The dominant morphism $f$ induces a homomorphism
\[
\pi_1^{\text{top}}(Y^{\text{an}}, y) \xrightarrow{f_*} \pi_1^{\text{top}}(Z^{\text{an}}, z)
\]
whose image is of finite index. We set, for short,
\[
G = \pi_1^{\text{top}}(Z^{\text{an}}, z), \quad F = f_*\pi_1^{\text{top}}(Y^{\text{an}}, y),
\]
and denote by $\text{Aut}(\hat{G}, F)$ the group of automorphisms of $\hat{G}$ preserving $F$. This homomorphism $f_*$ induces in turn a homomorphism
\[
H_Y \to (\text{Inn}\hat{G}.\text{Aut}(\hat{G}, F)) \cap G_k
\]
which is injective (because of the intersection with $G_k$ in both terms). By 3.3.4, it thus suffices to show that $(\text{Inn}\hat{G}.\text{Aut}(\hat{G}, F)) \cap G_k$ is contained in $H_Z = (\text{Inn}\hat{G}.\text{Aut} G) \cap G_k$. We shall prove the stronger statement
\[
\text{Aut}(\hat{G}, F) \subset \text{Aut} G.
\]
To this aim, it will be convenient to replace $F$ by a subgroup $F'$ which is still stable under $\text{Aut}(\hat{G}, F)$ but has the advantage of being normal (and of finite index) in $G$. We take
\[
F' = \bigcap_{g, \gamma} \text{Inn}(\gamma(g))(F) = \bigcap_{g, \gamma} \gamma(\text{Inn}(g)F) = \bigcap_{\gamma} \gamma(\bigcap_{g} \text{Inn}(g)F),
\]
where the intersection runs over $g \in G, \gamma \in \text{Aut}(\hat{G}, F)$. This is clearly a normal subgroup of $G$, stable under $\text{Aut}(\hat{G}, F)$. To see that it is of finite index, note that each $\gamma(\bigcap_{g} \text{Inn}(g)F)$ is a subgroup of the same (finite) index in $G$, and there are only finitely many such subgroups since $\hat{G}$ is of finitely type (cf. [34, §4.1.2]).

It thus suffices to show that $\text{Aut}(\hat{G}, F') \subset \text{Aut} G$, or equivalently, that the subgroup $F''$ of $\hat{G}$ generated by elements of the form $\gamma(g), g \in G, \gamma \in \text{Aut}(\hat{G}, F')$ is nothing but $G$. Note that $F'$ is normal in $F''$. There are exact sequences $\{1\} \to F' \to G \to \Gamma \to \{1\}, \quad \{1\} \to \hat{F}' \to \hat{G} \to \Gamma \to \{1\}$ with $\Gamma$ finite, whence an exact sequence
\[
\{1\} \to F'' \cap \hat{F}' \to F'' \to \Gamma \to \{1\}.
\]
But $F'$ is normal in the subgroup $F'' \cap \hat{F}'$ of its profinite completion $\hat{F}'$. Since $F'$ is free of rank $> 1$, we conclude from 3.2.1 that $F'' \cap \hat{F}' = F' \subset G$, hence $G = F''$ since $G$ surjects onto $\hat{G}/\hat{F}' = \Gamma$.

c) By the previous step, we may replace $Y/X$ by a bigger finite etale covering, and thus assume that $\pi_1^{\text{alg}}(Y_k, y)$ is a characteristic subgroup of
\( \pi_{1}^{alg}(X_{k}, x) \) with finite quotient group \( \Gamma \) (the same property then holds for the topological fundamental groups). The canonical homomorphism

\[
\text{Aut } \pi_{1}^{alg}(X_{k}, x) \to \text{Aut } \pi_{1}^{alg}(Y_{k}, y)
\]

has finite kernel, and induces a homomorphism

\[
\text{Out } \pi_{1}^{alg}(X_{k}) \to (\text{Out } \pi_{1}^{alg}(Y_{k}))/\Gamma
\]

which also has finite kernel. Hence \( \text{Out } \pi_{1}^{top}(X_{k}) \cap G_{k} \) is finite if \( \text{Out } \pi_{1}^{top}(Y_{k}) \cap G_{k} \) is, and one concludes by 3.3.4.

**Remarks.** 1) Some heuristic arguments of Baire type on moduli spaces for curves with marked points seem to indicate that not every affine hyperbolic curve satisfies the assumption of 3.3.2 (it would be nice to have a complete proof).

2) The assumption of 3.3.2 is fulfilled for \( X = \text{an elliptic curve } E \text{ minus one point} \). Indeed, by translation, one may assume that this point is the origin \( O \). Then \( E \text{ minus } E[2] \) (the 2-torsion points) is an abelian étale covering of \( E \setminus \{ O \} \), and the quotient of \( E \setminus E[2] \) by the involution \( P \mapsto -P \) is isomorphic to \( \mathbb{P}^{1} \setminus \{ 0, 1, \infty, \lambda \} \).

On the other hand, the argument of 3.3.1 applies directly to \( X = E \setminus \{ O \} \) since \( \pi_{1}^{top}(X^{an}) \cong F_{2} \). Note however that whereas \( \text{Out } \pi_{1}^{top}(X^{an}) \) is faithfully represented on the abelianization \( \pi_{1}^{top}(X^{an})^{ab} = H_{1}(X^{an}, \mathbb{Z}) \), it does not amount to the same to take the intersection with the image of \( G_{k} \) in \( \text{Out } \pi_{1}^{alg}(X_{k}) \) or in \( \text{Out } (\pi_{1}^{alg}(X_{k}))^{ab} = GL(H_{1}^{et}(X_{k}, \mathbb{Z})) \) (the latter is much bigger in general).

3) The statements of 3.3.1 and 3.3.2 do not change if one replaces \( \iota \) by its complex conjugate; hence it is enough to fix an archimedean place of \( \bar{k} \) (instead of \( \iota \)).

### 4. The Tempered Fundamental Group of a \( p \)-adic Manifold

4.1. We fix a prime number \( p \) and denote by \( C_{p} \) as usual the completion of a fixed algebraic closure \( \bar{Q}_{p} \) of \( Q_{p} \). Let \( K \) be a complete subfield of \( C_{p} \), and let \( \bar{K} \) be its algebraic closure in \( C_{p} \). We notice that \( G_{K} = \text{Gal}(\bar{K}/K) \) is a metrizable profinite group (and even a finitely generated profinite group if \( K = C_{p} \) or a finite extension of \( Q_{p} \)).

Among the several approaches to analytic geometry over \( K \), we have found Berkovich’s one \([2]\) most convenient for a discussion of unramified coverings and fundamental groups; indeed, Berkovich’s spaces are locally ringed spaces in the usual sense, as opposed to Tate’s rigid spaces which are Grothendieck’s sites (Berkovich’s spaces contain ‘more points’ than rigid...
spaces, but unlike the passage from classical algebraic varieties to schemes, all Berkovich non-classical points are closed).

In the sequel, by (analytic) \( K \)-manifold - or \( p \)-adic manifold if \( K = \mathbb{C}_p \) - we shall mean a smooth paracompact strictly \( K \)-analytic space in the sense of Berkovich [3]. Any smooth algebraic \( K \)-variety \( X \) gives rise to an analytic \( K \)-manifold \( X^{an} \), its analytification.

According to Berkovich [4], \( K \)-manifolds are locally compact, locally arcwise connected, and locally contractible, hence subject to the usual theory of universal coverings and topological fundamental groups.

By geometric point or base point of a \( K \)-manifold, we mean a point defined over some algebraically closed complete extension of \( K \).

4.2. In the sequel, we shall have to consider étale coverings of possibly infinite degree. Let us recall a definition [15] which applies both to the complex and to the non-archimedean situations: \( Y \to X \) is an étale covering (resp. a topological covering) if, locally on the base \( X \), it is a disjoint union of finite étale covering maps (resp. isomorphisms).

For complex manifolds, any étale covering gives rise to a topological covering, and conversely.

On the other hand, for non-archimedean \( K \)-manifolds, topological coverings give rise to étale coverings [15, §2.6], but not conversely: for instance, the ‘Kummer covering’

\[
P_1^1 \setminus \{0, \infty\} \to P_1^1 \setminus \{0, \infty\}, \ z \mapsto z^n, \ n > 1
\]

is a finite étale covering, but not a local homeomorphism of analytic \( K \)-manifolds (in fact the cardinality of the fiber, drops to 1 at some non-classical points; in the more traditional rigid viewpoint, this is because there is no admissible open cover over which the Kummer covering splits).

Besides the topological fundamental group \( \pi_1^{top}(X, x) \), which classifies as usual the topological coverings of the pointed connected \( K \)-manifold \((X, x)\), there is the étale fundamental group \( \pi_1^{et}(X, x) \), introduced by de Jong [15], which classifies the (possibly infinite) étale coverings of \( X \). In fact \( \pi_1^{top}(X, x) \) is a discrete quotient of the topological group \( \pi_1^{et}(X, x) \).

But neither group is a close analogue of the fundamental group of a complex manifold:

- \( \pi_1^{top}(X, x) \) is “too small”: for instance, \( \pi_1^{top}(P_1^1 \setminus \{0, 1, \infty\}, x) = \{1\} \).
- \( \pi_1^{et}(X, x) \) is “too big”: for instance, \( \pi_1^{et}(P_1^1_{\mathbb{C}_p}, x) \) is non-trivial (in fact, it is a huge non-abelian group [15, §7]).

\[\text{14}^\text{th in [3], we used this term in a less restrictive sense, allowing a non-empty boundary}\]
4.3. In order to remedy this, we have introduced in [1] an intermediate fundamental group $\pi_1^{\text{temp}}(X, x)$, the \textit{tempered fundamental group}, which classifies all tempered (pointed) coverings of $(X, x)$:

$$\{ \coprod \text{tempered coverings of } X \} \sim \{ \text{discrete } \pi_1^{\text{temp}}(X, x)-\text{sets} \}.$$ 

By definition, an etale covering $Y \to X$ is said to be \textit{tempered} if there is a commutative diagram of etale coverings

\[
\begin{array}{ccc}
\mathcal{Z} & \leftarrow & \mathcal{T} \\
& \nwarrow & \searrow \\
& \mathcal{Y} & \leftarrow \\
& \mathcal{X} & \leftarrow \\
\end{array}
\]

where $\mathcal{Z} \to \mathcal{T}$ is a (possibly infinite) topological covering, and $\mathcal{T} \to X$ is a finite etale covering.

In some sense, this is the ‘minimal’ theory of etale coverings which \textit{takes into account both the topological coverings and the finite etale coverings}. We refer to [1, \S 1, 2] for a discussion of these coverings and the precise definition of $\pi_1^{\text{temp}}(X, x)$ as a separated prodiscrete topological group. This topological group depends on the base point $x$ only up to inner automorphism [1, \S 1.4.4].

Here, we shall content ourselves with the following useful ‘criterion’:

\[4.3.1. \text{Proposition.} \] Let $g : Y \to X$ be a Galois etale covering with (discrete) group $G$. If $G$ is torsion-free (resp. virtually torsion-free), then $g$ is a topological covering (resp. a tempered covering). \hfill \square

\[\text{Remarks.} \] 1) Because of this proposition, infinite covering maps are often easier to handle than finite ones, when the point is to check whether they are local homeomorphisms; this may justify the detour to infinite coverings even if one is primarily interested in finite ones.

2) We mention in passing that there is a theory of tempered fundamental groups based at tangential base points, in dimension one [1, \S 2.2].

4.4. Let us consider the case when $X = X^{\text{an}}$ is the analytification of a geometrically connected smooth algebraic $K$-variety.

\[4.4.1. \text{Proposition.} \] $\pi_1^{\text{temp}}(X^{\text{an}}, x)$ is a countable inverse limit of discrete finitely generated groups; it particular, it is a polish group.

Its profinite completion is canonically isomorphic to $\pi_1^{\text{alg}}(X, x)$. \hfill \square

\[\text{Remark.} \] \text{recall that a discrete group is said to be virtually torsion-free is it admits a torsion-free subgroup of finite index}
(the proof of the second assertion relies on the Gabber-Lütkebohmert version of Riemann’s existence theorem [19]).

Let us recall that, according to Bourbaki [6, § IX.6.1], a topological group is polish if it is metrizable, complete, and countable at infinity (it might not be locally compact). These form a nice category of topological groups to work with: let \( \{1\} \to N \to G \to H \to \{1\} \) be a sequence of homomorphisms of topological groups which is exact in the abstract sense. Assume that \( N \) is closed in \( G \) and \( G/N \) [6, § IX.2.8 prop. 12; § IX.3.1 prop.4], and if \( H \) is also polish, then the bijective homomorphism \( G/N \to H \) is an isomorphism [6, § IX.5. ex. 28].

4.5. In the case \( K = \mathbb{C}_p \), \( \pi^\text{temp}_1(X^{an}, x) \) can be described as follows. Let us consider the ‘tower’ of finite Galois etale coverings

\[
\ldots \to X_{n'} \to X_n \to \ldots \to X, n|n', \quad \text{Gal}(X_n/X) = \Gamma_n,
\]

introduced in 2.2. Let us denote by \( \tilde{X}_n \) the universal topological covering of \( X^{an}_n \). Then \( \tilde{X}_n \) is an etale Galois covering of \( X^{an} \) [1, 2.1.2] with Galois group \( \Delta_n \) sitting in an extension

\[
\{1\} \to \pi^\text{top}_1(X^{an}_n, x_n) \to \Delta_n \to \Gamma_n \to \{1\},
\]

where the discrete group \( \pi^\text{top}_1(X^{an}_n, x_n) \) is finitely generated [1, § 1.1.3]. When \( n \) increases (by divisibility), these exact sequences form an inverse system, which is canonical and functorial in \( X \).

One has

\[
\pi^\text{temp}_1(X^{an}, x) = \lim_{\leftarrow} \Delta_n
\]

(surjective transition maps), and there is a canonical morphism commutative diagram

\[
\begin{array}{cccccc}
\{1\} & \to & \pi^\text{temp}_1(X^{an}_n) & \to & \pi^\text{temp}_1(X^{an}) & \to & \Gamma_n & \to & \{1\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\{1\} & \to & \pi^\text{top}_1(X^{an}_n) & \to & \Delta_n & \to & \Gamma_n & \to & \{1\}
\end{array}
\]

with surjective vertical maps, cf. [1, § 2.1.5., 2.1.8].

On the other hand, in dimension one, \( \pi^\text{top}_1(X^{an}_n) \) has a simple combinatorial description as the fundamental group of the (dual) graph of incidence of the semistable reduction of \( X_n \) in characteristic \( p \), cf. [15, § 5.3.], in particular it is a free group (notably, \( X^{an}_n \) is simply connected if it has good or tree-like reduction). It follows that \( \Delta_n \) is virtually free (i.e. admits a free subgroup of finite index), hence residually finite, which implies that \( \pi^\text{temp}_1(X^{an}, x) \) itself is residually finite, so that the following natural homomorphism is injective [1, § 2.1.6]:

\[
\pi^\text{temp}_1(X^{an}, x) \hookrightarrow \pi^\text{alg}_1(X^\bar{k}, x).
\]
Remark. By the theorem of Karass-Pietrovsky-Solitar \([16]\), each \(\Delta_n\), being virtually free and finitely generated, is the fundamental group of a finite graph of finite groups. It would be very interesting to exhibit a canonical geometric construction of such graphs of groups, which could be considered as \(p\)-adic analogues of Grothendieck’s “dessins d’enfants”. At present, such construction have been proposed only in the special case where the projective completion of \(X_n\) is a so-called Mumford curve, \(i.e.\) has a maximally degenerate reduction \([13][17]\), using the tree of \(SL_2\) over a local field\([16]\).

4.6. From this description, \(\pi_1^{temp}(X^{an})\) can be easily computed when \(X\) is a non-hyperbolic curve. For instance,

- \(\pi_1^{temp}(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, \infty\}) \cong \mathbb{Z}\),

and if \(X = E_j\) is an elliptic curve with invariant \(j\),

- \(\pi_1^{temp}(E_j^{an}) = \pi_1^{alg}(E_j) \cong \hat{\mathbb{Z}} \times \hat{\mathbb{Z}}\) if \(|j|_p \leq 1\) (and \(\Delta_n \cong (\mathbb{Z}/n\mathbb{Z})^2\)),

- \(\pi_1^{temp}(E_j^{an}) \cong \mathbb{Z} \times \hat{\mathbb{Z}}\) if \(|j|_p > 1\), \(i.e.\) if \(E_j\) has bad reduction (in this case, one has a commutative diagram

\[
\begin{array}{ccc}
\pi_1^{top}(X_n^{an}) & \rightarrow & \Delta_n \\
\downarrow \cong & \downarrow \cong & \downarrow \cong \\
n\mathbb{Z} & \rightarrow & \mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z}) \rightarrow (\mathbb{Z}/n\mathbb{Z})^2
\end{array}
\]

When \(X\) is hyperbolic, there is no such simple explicit description; in general, \(\pi_1^{temp}(X^{an}, x)\) is ‘lacunary’ in the sense that its open subgroups of finite index have many infinite discrete quotients. In fact, \(\pi_1^{temp}(X^{an}, x)\) encapsulates the combinatorial information about the reduction of all finite etale coverings of \(X\).

- For \(X = \text{the projective line minus } n \geq 3\) points, \(\pi_1^{temp}(X^{an})\) is not locally compact, and depends on the position of the missing points if \(n \geq 4\) (\([1] \ \S 2.3.12, 4.5.5\)).

4.7. The subtle lacunary properties of \(\pi_1^{temp}\), on which the sequel depends, really belong to the “profinite theory” and are lost if one passes to the maximal prime-to-\(p\) quotient or to the pro-\(p\) completion; indeed:

4.7.1. Proposition. If \(X\) is a curve with good reduction (if \(X\) is affine, we also require that there is no confluence of the points at infinity by reduction), the natural homomorphism

\[
\pi_1^{temp}(X^{an}, x) \rightarrow \pi_1^{alg}(X, x)^{(p')} \times \pi_1^{alg}(X, x)^{(p)}
\]

to the product of the maximal prime-to-\(p\) quotient and the maximal pro-\(p\) quotient of \(\pi_1^{alg}(X, x)\), is surjective.

\[16\] after submission of this paper and discussion with F. Kato, it seems that he is now able to construct such \(p\)-adic dessins d’enfants in the general case
Proof. Recall that $\Gamma_n$ is the quotient of $\pi_{1}^{alg}(X)$ by the intersection of its (finitely many) open subgroups of index dividing $n$. It follows that $|\Gamma_n|$ divides a power of $n$, and that there are surjective homomorphisms

$$\Gamma_n^{(p)} \to \Gamma_{p^{m_1}}, \quad \Gamma_{p^{m_2}} \to \Gamma_n^{(p)},$$

where $p^{m_1}$ (resp. $p^{m_2}$) is the greatest power of $p$ dividing $n$ (resp. $|\Gamma_n|$).

From the equality $\pi_{1}^{alg}(X, x) = \lim_{\leftarrow} \Gamma_n$, we derive that $\pi_{1}^{alg}(X, x)^{(p')} = \lim_{\rightarrow (p,n)=1} \Gamma_n$, $\pi_{1}^{alg}(X, x)^{(p)} = \lim_{\rightarrow m} \Gamma_{p^m}$.

Since $\pi_{1}^{temp}(X_{an}, x) = \lim_{\rightarrow} \Delta_n$, it thus suffices to show that if $n$ is prime to $p$ (resp. is a power of $p$), then $\Delta_n = \Gamma_n$, which means that $X_{an}$ is simply-connected. By construction, $X_n$ is a finite etale Galois covering with group $\Gamma_n$ of the curve with good reduction $X$, and $|\Gamma_n|$ is prime to $p$ (resp. is a power of $p$). A standard descent argument shows that $X_n \to X$ is actually defined over a complete discretely valued subfield $K \subset C_p$ with algebraically closed residue field. By Grothendieck’s specialization theorem [[10]] (resp. Raynaud’s specialization theorem [[30, thm.1’]]), $X_n$ has good reduction (resp. tree-like reduction), hence $X_{an}$ is simply-connected.

Remark. It follows from this that $\pi_{1}^{temp}(\mathbb{P}^1_{C_{p}} \setminus \{0, 1, \infty\})$ maps surjectively onto the free pro-nilpotent group of rank two (in contrast, it is well-known that $\pi_{1}^{top}(\mathbb{P}^1_{C_{p}} \setminus \{0, 1, \infty\})$ maps injectively into the free pro-nilpotent group of rank two).

5. A FUNDAMENTAL EXACT SEQUENCE

5.1. In this section, we establish an exact sequence relating the ‘arithmetic’ and ‘geometric’ tempered fundamental groups.

As above, $K$ is a complete subfield of $C_p$. For any analytic $K$-manifold $\mathcal{X}$, $\mathcal{X}_{C_p}$ has finitely many connected components [[15, § 2.14]], and $\mathcal{X}$ is said to be geometrically connected if there is just one; it amounts to the same to require that for every finite extension $L/K$, $\mathcal{X}_L$ is connected.

Let $(\mathcal{X}, x)$ be a geometrically connected $K$-manifold endowed with a geometric point. Let $L/K$ be a finite Galois extension contained in $C_p$. There is a functorial exact sequence of separated pro-discrete groups [[1, § 2.1.8]]

$$\{1\} \to \pi_{1}^{temp}(\mathcal{X}_L, x) \to \pi_{1}^{temp}(\mathcal{X}, x) \to Gal(L/K) \to \{1\},$$

which splits canonically if $x$ comes from a $K$-rational point of $X$. When the Galois extension $L/K$ varies, these exact sequences are compatible in
an obvious sense, and on passing to the limit, they provide another exact sequence of separated prodiscrete groups

\[ \{1\} \to \lim_{\leftarrow L} \pi_1^{\text{temp}}(\mathcal{X}_L, x) \to \pi_1^{\text{temp}}(\mathcal{X}, x) \to G_K \to \{1\} \]

(the surjectivity of \( \pi_1^{\text{temp}}(\mathcal{X}, x) \to G_K \) can be proved exactly as in [13, §2.12]).

5.1.1. **Proposition.** The canonical homomorphism

\[ \phi : \pi_1^{\text{temp}}(\mathcal{X}_{C_p}, x) \to \lim_{\leftarrow L} \pi_1^{\text{temp}}(\mathcal{X}_L, x) \]

is an isomorphism. Therefore there is a functorial exact sequence

\[ \{1\} \to \pi_1^{\text{temp}}(\mathcal{X}_{C_p}, x) \to \pi_1^{\text{temp}}(\mathcal{X}, x) \to G_K \to \{1\}, \]

whence a Galois action

\[ G_K \xrightarrow{\rho} \text{Out} \pi_1^{\text{temp}}(\mathcal{X}_{C_p}), \]

which lifts to an action

\[ G_K \xrightarrow{\rho_x} \text{Aut} \pi_1^{\text{temp}}(\mathcal{X}_{C_p}, x) \]

if \( x \) comes from a \( K \)-rational point of \( X \).

**Proof.** We shall use the following

5.1.2. **Lemma.** Let \( \phi : G \to H \) be a (continuous) homomorphism of separated prodiscrete groups. Let \( \phi^* \) be the induced functor (in the opposite direction) between discrete \( H \)- and \( G \)-sets. If \( \phi^* \) is essentially surjective (resp. an equivalence), then \( \phi \) is injective (resp. an isomorphism).

\[ \text{cf. [1], §1.4.9}. \]

In order to prove the proposition, it thus suffices to show that \( \phi^* \) is an equivalence. It is clearly fully faithful, and the essential surjectivity amounts to saying that any tempered covering of \( \mathcal{X}_{C_p} \) is defined over some finite extension of \( K \). This is clear for finite etale coverings, and it suffices to check that the universal topological covering \( \tilde{\mathcal{Y}}_{C_p} \) of any finite covering \( \mathcal{Y}_{C_p} \) of \( \mathcal{X}_{C_p} \) is defined over some finite extension of \( K \).

Let us first record the following easy

5.1.3. **Sublemma.** Let \( \mathcal{Y} \) be a geometrically connected \( K \)-manifold.

i) For any finite extension \( L/K \), \( \tilde{\mathcal{Y}}_L \) is a component of \( (\mathcal{Y})_L \).

ii) The connected components of \( (\mathcal{Y})_{C_p} \) are defined over some finite extension \( K'/K \) ([13, §2.14]), and then \( (\mathcal{Y}_{K'})_{C_p} \) is connected.

iii) For any finite extension \( L/K' \), \( \tilde{\mathcal{Y}}_L = (\mathcal{Y}_{K'})_L \): topological coverings of \( \mathcal{Y}_L \) come from topological coverings of \( \mathcal{Y}_{K'} \). \( \square \)
The fact that the universal topological covering of \( Y \) is defined over some finite extension of \( K \) (in fact over the extension \( K'/K \) just introduced) follows from the following lemma (applied to \( T = \tilde{Y}_{K'} \)):

5.1.4. **Lemma.** Let \( T \) be a geometrically connected and simply connected \( K \)-manifold. Then \( T_{C_p} \) is simply connected (in the sense of coverings).

**Proof.** Let \( h : Z \to T_{C_p} \) be a topological covering: every point \( t \in T_{C_p} \) (classical or not) has an open neighborhood \( V(t) \) such that \( h^{-1}V(t) \) is isomorphic to a disjoint sum of copies of \( V(t) \) via \( h \). By the argument of [3, p. 103], there is a finite Galois extension \( L/K \) contained in \( C_p \) and an open domain \( U_L(t) \subset T_L \) such that \( t \in (U_L(t))_{C_p} \subset V(t) \). The restriction of \( h \) above \( (U_L(t))_{C_p} \), being a split covering, admits a canonical \( L \)-structure. Saturating with respect to \( Gal(L/K) \), one finds by descent an open domain \( U(t) \subset T \) such that \( U(t)_L = \cup_{\sigma \in Gal(L/K)} U_L(t)^{\sigma} \) and such that the restriction of \( h \) above \( U(t)_{C_p} \) admits a canonical \( K \)-structure. By 5.1.3.iii), this actually defines a topological covering \( \tilde{Z}(t) \to U(t) \). These topological coverings glue together (using [3, § 1.3.3.a]) and define a topological covering of \( T \) which is a \( K \)-structure for \( h \), and which splits since \( T \) is simply connected (in the sense of coverings). \( \square \)

**Remark.** It does happen that \( (\tilde{Y})_{C_p} \) has several connected components (for a geometrically connected \( K \)-manifold \( Y \)). The simplest example is given by a twisted Tate elliptic curve \( Y \), cf. [2, § 4].

5.2. Let us assume that \( X = X^{an} \) for some geometrically connected smooth algebraic \( K \)-variety \( X \). Then, passing to the profinite completion, one gets a commutative diagram

\[
\begin{array}{cccccc}
\{1\} & \to & \tilde{\pi}_1^{\text{temp}}(X^{an}_{C_p}, x) & \to & \tilde{\pi}_1^{\text{temp}}(X^{an}, x) & \to & G_K & \to & \{1\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\tilde{\pi}_1^{\text{temp}}(X^{an}_{C_p}, x) & \to & \tilde{\pi}_1^{\text{temp}}(X^{an}, x) & \to & G_K & \to & \{1\} \\
\downarrow_{\cong} & & \downarrow_{\cong} & & \downarrow & & \\
\{1\} & \to & \pi_1^{\text{alg}}(X_{C_p}, x) & \to & \pi_1^{\text{alg}}(X, x) & \to & G_K & \to & \{1\}
\end{array}
\]

and the last row is the canonical exact sequence. In particular, the actions \( \rho \) (resp. \( \rho_x \)) of 6.1.1 are compatible with the homonymous actions defined in 2.1.

6. \( Out \pi_1^{\text{temp}} \) for a curve

6.1. In this section, we examine the group \( Out \pi_1^{\text{temp}}(X^{an}) \) when \( X \) is a smooth algebraic curve over \( C_p \). We fix a base point \( x \).
By characteristic quotient of a given topological group $G$, we mean the quotient of $G$ by some characteristic open subgroup.

6.1.1. Lemma. $\pi_1^{\text{top}}(X^{\text{an}}, x)$ is a characteristic quotient of $\pi_1^{\text{temp}}(X^{\text{an}}, x)$.

Remark. We do not know whether this holds for higher dimensional $X$, if $\pi_1^{\text{top}}(X^{\text{an}}, x)$ has torsion.

Proof. For a curve, $\pi_1^{\text{top}}(X^{\text{an}}, x)$ is a free, hence torsion-free, discrete group. By proposition 4.3.1, it can actually be described as the biggest torsion-free discrete quotient of $\pi_1^{\text{temp}}(X^{\text{an}}, x)$. This group-theoretic characterization makes it clear that the kernel of $\pi_1^{\text{temp}}(X^{\text{an}}, x) \to \pi_1^{\text{top}}(X^{\text{an}}, x)$ is characteristic (and of course open).

Let us recall from 4.5. that

$$\pi_1^{\text{temp}}(X^{\text{an}}, x) = \lim_{\Delta_n} \Delta_n$$

where $\Delta_n$ sits in an exact sequence

$$\{1\} \to \pi_1^{\text{top}}(X_n^{\text{an}}) \to \Delta_n \to \Gamma_n \to \{1\}.$$ 

6.1.2. Lemma. i) $\Delta_n$ is a characteristic quotient of $\pi_1^{\text{temp}}(X_n^{\text{an}}, x)$.

ii) If $n \neq n'$, both $\Gamma_n$ and $\Delta_n$ are characteristic quotients of $\Delta_n'$. In particular (for $n = n'$), the free group $\pi_1^{\text{top}}(X_n^{\text{an}})$ is a characteristic subgroup of $\Delta_n$.

Proof. i) Let us consider again the commutative diagram from 4.5

$$\begin{array}{c}
\{1\} \to \pi_1^{\text{temp}}(X_n^{\text{an}}) \to \pi_1^{\text{temp}}(X_n^{\text{an}}) \to \Gamma_n \to \{1\} \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\{1\} \to \pi_1^{\text{top}}(X_n^{\text{an}}) \to \Delta_n \to \Gamma_n \to \{1\}.
\end{array}$$

By definition, the kernel of $\pi_1^{\text{temp}}(X_n^{\text{an}}) = \pi_1^{\text{alg}}(X) \to \Gamma_n$ is characteristic. This implies that the kernel $\pi_1^{\text{temp}}(X_n^{\text{an}})$ of $\pi_1^{\text{temp}}(X_n^{\text{an}}) \to \Gamma_n$ is characteristic. On the other hand, the kernel of $\pi_1^{\text{temp}}(X_n^{\text{an}}) \to \pi_1^{\text{top}}(X_n^{\text{an}})$ is characteristic, by 6.1.1. It follows that the kernel of $\pi_1^{\text{temp}}(X_n^{\text{an}}) \to \Delta_n$ is characteristic.

ii): the profinite completion $\widehat{\Delta'}_n$ of $\Delta_n'$ is a quotient of $\pi_1^{\text{alg}}(X)$ which maps to $\Gamma_n$. By observation 2.2.1, $\Gamma_n$ is a characteristic quotient of $\widehat{\Delta'}_n$, hence of $\Delta_n'$; in other words, the kernel $K_n'$ of $\Delta_n' \to \Gamma_n$ is characteristic.

The kernel of $K_n' \to \pi_1^{\text{top}}(X_n^{\text{an}})$ is the image in $\Delta_n'$ of the kernel of $\pi_1^{\text{temp}}(X_n^{\text{an}}) \to \pi_1^{\text{top}}(X_n^{\text{an}})$. By the same argument as in 6.1.1 (using 4.3.1), it can be characterized as the smallest normal subgroup $N$ of $K_n'$ such that $K_n'/N$ is torsion-free; this shows that it is characteristic in $K_n'$, hence in $\Delta_n'$. It follows that $\Delta_n$ is a characteristic quotient of $\Delta_n'$.

□
6.1.3. **Corollary.**

\[ \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) = \lim_{\leftarrow} \text{Aut} \Delta_n, \quad \text{Out} \pi_1^{\text{temp}}(X^{an}) = \lim_{\leftarrow} \text{Out} \Delta_n. \]

Indeed, the previous lemma makes sense of the inverse limit \( \lim_{\leftarrow} \text{Aut} \Delta_n \) and provides a natural homomorphism \( \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) \to \lim_{\leftarrow} \text{Aut} \Delta_n \), which has an obvious inverse (note however that the transition maps need not be surjective). \( \square \)

We endow \( \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) \) and \( \text{Out} \pi_1^{\text{temp}}(X^{an}, x) \) with the inverse limit topology. With this topology, it is clear that \( \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) \) acts continuously on \( \pi_1^{\text{temp}}(X^{an}, x) = \lim_{\leftarrow} \Delta_n \).

6.1.4. **Proposition.** \( \text{Out} \pi_1^{\text{temp}}(X^{an}) \) is a countable inverse limit of discrete finitely generated groups; in particular it is a polish group (cf. 4.4).

**Proof.** Indeed, finitely generated virtually free groups (such as \( \Delta_n \)) have finitely generated automorphism group (Krstic, McCool [24]). \( \square \)

6.2. For a curve, we have seen that \( \pi_1^{\text{temp}}(X^{an}, x) = \lim_{\leftarrow} \Delta_n \) embeds into its profinite completion \( \pi_1^{\text{alg}}(X, x) = \lim_{\leftarrow} \Gamma_n \). A fortiori, \( \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) = \lim_{\leftarrow} \text{Aut} \Delta_n \) embeds into \( \text{Aut} \pi_1^{\text{alg}}(X, x) = \lim_{\leftarrow} \text{Aut} \Gamma_n \). For outer automorphisms, we also have:

6.2.1. **Proposition.** \( \text{Out} \pi_1^{\text{temp}}(X^{an}) \) embeds into \( \text{Out} \pi_1^{\text{alg}}(X) \); in particular, it is residually finite.

**Proof.** If \( X \) is not hyperbolic, \( \pi_1^{\text{temp}}(X^{an}) \) is abelian and coincides with \( \pi_1^{\text{alg}}(X) \) except in the case of an elliptic curve with bad reduction, for which \( \pi_1^{\text{temp}}(X^{an}) \cong \mathbb{Z} \times \hat{\mathbb{Z}} \); in either case, the statement is trivial.

We now assume that \( X \) is hyperbolic. The assumption of hyperbolicity ensures that the center of \( \pi_1^{\text{temp}}(X^{an}, x) \) is trivial (an immediate consequence of the fact that the center of its profinite completion \( \pi_1^{\text{alg}}(X, x) \) is trivial).

Let us assume that \( \varpi \in \pi_1^{\text{alg}}(X, x) \) induces an automorphism of \( \pi_1^{\text{temp}}(X^{an}, x) \) by conjugation. We have to show \( \varpi \in \pi_1^{\text{temp}}(X^{an}, x) \).

Let us fix \( n \), and consider the composed epimorphism

\[ \pi_1^{\text{alg}}(X, x) \twoheadrightarrow \hat{\Delta}_n \twoheadrightarrow \Gamma_n \]

(where \( \hat{\Delta}_n \) denotes the profinite completion of \( \Delta_n \), which sits in an exact sequence

\[ \{1\} \to \pi_1^{\text{top}}(X^{an}) \to \hat{\Delta}_n \to \Gamma_n \to \{1\}. \]

\( \text{even finitely presented} \)
Let $\pi_n$ be any element of $\pi_{1}^{\text{temp}}(X_{\text{an}}, x)$ which has the same image as $\varpi$ in $\Gamma_n$. The image $\varpi_n$ of $\varpi \cdot \pi^{-1}_n$ in $\hat{\Delta}_n$ actually belongs to $\pi_{1}^{\text{top}}(X_{\text{an}})$, and induces by conjugation an automorphism of the free group $\pi_{1}^{\text{top}}(X_{\text{an}})$ (cf. 6.1.1, 6.1.2). By 3.2.1, it follows that $\varpi_n \in \pi_{1}^{\text{top}}(X_{\text{an}})$, hence the image of $\varpi$ in $\hat{\Delta}_n$ actually belongs to $\Delta_n$. Therefore $\varpi \in \pi_{1}^{\text{temp}}(X_{\text{an}}, x) = \varprojlim \Delta_n$. □

6.2.2. Corollary. If $X$ is hyperbolic, $\pi_{1}^{\text{temp}}(X_{\text{an}}, x)$ is its own normalizer in its profinite completion. □

We denote by $\overline{\text{Out}} \pi_{1}^{\text{temp}}(X_{\text{an}})$ the closure of $\text{Out} \pi_{1}^{\text{temp}}(X_{\text{an}})$ in $\text{Out} \pi_{1}^{\text{alg}}(X)$:

$$\overline{\text{Out}} \pi_{1}^{\text{temp}}(X_{\text{an}}) = \varprojlim \text{Im} [\text{Out} \pi_{1}^{\text{temp}}(X_{\text{an}}) \to \text{Out} \Gamma_n]$$

(this limit is clearly a quotient of the profinite completion of $\text{Out} \pi_{1}^{\text{temp}}(X_{\text{an}})$ and embeds into $\varprojlim \text{Out} \Gamma_n = \text{Out} \pi_{1}^{\text{alg}}(X)$).

Remark. For the problems studied in the sequel (cf. 8.7.2), it would be interesting to know whether $\text{Out} \pi_{1}^{\text{temp}}(X_{\text{an}}) = \varprojlim \text{Out} \Delta_n$ is a profinite group or not, especially in the case $X = \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}$. This involves the question of finiteness of the outer automorphism groups of the finitely generated virtually free groups $\Delta_n$. In the abstract setting, a complete answer to this kind of question has been given by M. Pettet in terms of graphs of groups [28]. His criteria apply for instance to Kato’s presentation of $p$-adic triangle groups as amalgams [17], and could apply in the same way to the ‘$p$-adic dessins d’enfants’ invoked at the end of 4.5.

Another interesting open problem is whether the natural homomorphism $\text{Out} \pi_{1}^{\text{temp}}(X) \to GL_2(\hat{\mathbb{Z}})$ is surjective for $X = \mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, 1, \infty\}$ (one shows easily that the natural composed map

$$\pi_1^{\text{temp}}(X) \to \pi_1^{\text{alg}}(X) \to H_1^{\text{et}}(X, \hat{\mathbb{Z}}) \cong \hat{\mathbb{Z}}^2$$

is surjective).

7. $G_{\mathbb{Q}}, G_{\mathbb{Q}_p}$ AND GEOMETRIC FUNDAMENTAL GROUPS

7.1. Now $k$ is a number field, and we fix an embedding $\iota_v : \bar{k} \hookrightarrow \mathbb{C}_p$. We denote by $\bar{k}_v$ the topological closure of $k$ in $\mathbb{C}_p$.

For any smooth geometrically connected algebraic $k$-variety $X$, we denote by $X^\text{an}$ the $p$-adic analytic manifold attached to $X_{\mathbb{C}_p}$.

Let us fix a $\mathbb{C}_p$-point $x$ of $X$. One has canonical isomorphisms

$$\pi_1^{\text{alg}}(X_{\bar{k}}, x) = \pi_1^{\text{alg}}(X_{\mathbb{C}_p}, x) = \pi_1^{\text{temp}}(X^\text{an}, x),$$
the latter group being the profinite completion of the tempered fundamental

group \( \pi_1^{\text{temp}}(X^{an}, x) \) which classifies tempered étale coverings of \( X^{an} \).

There is an outer action \( \rho \) of \( G_k \) on \( \pi_1^{\text{alg}}(X_k, x) \) and a compatible outer
action of \( G_{k_0} \) on \( \pi_1^{\text{temp}}(X^{an}, x) \) (6.1, 6.2).

7.2. Let us now assume that \( X \) is a hyperbolic affine curve. We have
encountered two closed subgroups of \( \text{Out} \pi_1^{\text{alg}}(X_k) : \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \) and
\( G_k \). As in 3.3, we shall study their intersection, starting with the case of
\( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). The following is a non-archimedean analogue of 3.3.1.

7.2.1. Theorem. If \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \), then
\[
\text{Out} \pi_1^{\text{temp}}(X^{an}) \cap G_k = \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \cap G_k = G_{k_0}.
\]

Proof. We already know that \( G_{k_0} \subset \text{Out} \pi_1^{\text{temp}}(X^{an}) \cap G_k \) by 5.1.1 and
5.2. It remains to show that \( \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \cap G_k \subset G_{k_0} \).

We choose a rational base point \( x \) of \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). Let \( \sigma \in G_k \) be such
that \( \rho(\sigma) \in \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \). We have to show that
\[
(\ast) \forall j \in \bar{k} \setminus \{0, 1\}, \ |j|_p \leq 1 \Rightarrow |j^\sigma|_p \leq 1.
\]

Let \( E_j \) be an elliptic curve with invariant \( j \in \bar{k} \). According to Belyi, there is
a surjective morphism \( \varphi : E_j \to \mathbb{P}^1 \) defined over \( \mathbb{Q}(j) \) which is unramified
outside \( \Sigma = \varphi^{-1}(\{0, 1, \infty\}) \).

The idea of the proof is that if the image of some \( \tau \in \text{Aut} \pi_1^{\text{temp}}(X^{an}, x) \in \text{Out} \pi_1^{\text{temp}}(X^{an}) \)
‘sufficiently close’ to \( \rho(\sigma) \), then \( \tau \) induces an isomorphism \( \pi_1^{\text{temp}}(E_j^{an}) \to \pi_1^{\text{temp}}(E_j^{an}) \), whence (\ast), since the algebraic structure
of these fundamental groups is determined by the alternative ‘good/bad’
reduction, cf. 4.6.

Let us explain the details. We fix an integer \( m \) so that the characteristic
étale covering \( X_m \) of \( X_k \) defined in 2.2 dominates \( E_j \setminus \Sigma \). This
provides a sequence of finite étale coverings with base points
\[(X_m, x_m) \to (E_j \setminus \Sigma, y) \to (X_k, x).\]

One has a commutative diagram
\[
\begin{array}{ccc}
\overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} & \xrightarrow{\rho_{\text{m}, x_m}^\sigma(\sigma)} & \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \\
\downarrow & & \downarrow \\
\overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} & \xrightarrow{\rho_{\text{p}, y^\sigma(\sigma)}} & \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \\
\downarrow & & \downarrow \\
\overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} & \xrightarrow{\rho_{\text{x}}(\sigma)} & \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} \\
\end{array}
\]

By assumption,
\[
\rho(\sigma) \in \overline{\text{Out} \pi_1^{\text{temp}}(X^{an})} = \lim_{\leftarrow} \text{Im}[\text{Out} \pi_1^{\text{temp}}(X^{an}) \to \text{Out} \Gamma_n]
\]
This implies the existence of \( \tau \in Aut \pi_1^{temp}(X^{an}, x) \), \( \varpi \in \pi_1^{alg}(X, x) \) such that \( \rho_x(\sigma) \) and \( \tau \circ ad(\varpi) \) have the same image in \( \Gamma_m \). Since \( \pi_1^{alg}(X, x) \) is open in the profinite completion \( \pi_1^{alg}(X^{an}, x) \) of \( \pi_1^{temp}(X^{an}, x) \), one has

\[
\pi_1^{alg}(X, x) = \pi_1^{temp}(X^{an}, x) \pi_1^{alg}(X, x),
\]

which allows to take \( \varpi \in \pi_1^{alg}(X, x) \). Then there is a similar commutative diagram built on \( \tau \)

\[
\begin{array}{ccc}
\pi_1^{alg}(X, x) & \to & \pi_1^{alg}(X^{an}, x) \\
\downarrow & & \downarrow \\
\pi_1^{temp}(E^a_j \setminus \Sigma, y) & \to & \pi_1^{temp}(E^a_j \setminus \Sigma^{\sigma}, y^{\sigma}) \\
\downarrow & & \downarrow \\
\pi_1^{temp}(X^a, x) & \to & \pi_1^{temp}(X^{an}, x)
\end{array}
\]

which comes, by profinite completion, from a commutative diagram

\[
\begin{array}{ccc}
\pi_1^{temp}(X^a, x) & \to & \pi_1^{temp}(X^{a, an}, x) \\
\downarrow & & \downarrow \\
\pi_1^{temp}(E^a_j \setminus \Sigma, y) & \to & \pi_1^{temp}(E^a_j \setminus \Sigma^{\sigma}, y^{\sigma}) \\
\downarrow & & \downarrow \\
\pi_1^{temp}(X^a, x) & \to & \pi_1^{temp}(X^{an}, x)
\end{array}
\]

where horizontal maps are isomorphisms (\( \tau' \) stands for the induced isomorphism).

On the other hand, the open immersion \( E^a_j \setminus \Sigma \hookrightarrow E^a_j \) induces a (strict) surjective homomorphism \( \pi_1^{temp}(E^a_j \setminus \Sigma, y) \to \pi_1^{temp}(E^a_j) \) \([1, \S 4.5.5.b]\), which is nothing but the abelianization homomorphism (this becomes clear on the profinite completions, or using 3.2.2.a). One thus has another commutative diagram

\[
\begin{array}{ccc}
\pi_1^{temp}(E^a_j \setminus \Sigma, y) & \to & \pi_1^{temp}(E^a_j \setminus \Sigma^{\sigma}, y^{\sigma}) \\
\downarrow & & \downarrow \\
\pi_1^{temp}(E^a_j) & \to & \pi_1^{temp}(E^a_j^{\sigma})
\end{array}
\]

where horizontal maps are isomorphisms.

If \( |j|_p \leq 1 \), \( E_j \) has good reduction and it follows that \( \pi_1^{temp}(E^a_j) = \pi_1^{alg}(E_j) \), hence \( \pi_1^{temp}(E^a_j^{\sigma}) = \pi_1^{alg}(E_j^{\sigma}) \), which implies in turn that \( E_j^{\sigma} \) has also good reduction at \( p \), whence \( |j^{\sigma}|_p \leq 1 \). \( \square \)

7.2.2. **Corollary.** If \( X = \mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\} \), the image of \( \text{Out} \pi_1^{temp}(X^{an}) \) in \( \text{Out} \pi_1^{alg}(X_\mathbb{Q}) \) is not dense in any open subgroup. \( \square \)
For more general affine hyperbolic $k$-curves $X$, we have the following non-archimedean analogue of 3.3.2:

7.2.3. **Theorem.** Assume that some etale covering of $X_k$ admits a non-constant rational function which omits at least three values. Then

$$\text{Out} \, \pi_{1}^{\text{temp}}(X^{\text{an}}) \cap G_k = G_{k_v}.$$ 

A fortiori, if the geometric point $x$ is $k$-rational, then the subgroup of $G_k$ which stabilizes $\pi_{1}^{\text{temp}}(X^{\text{an}}, x) \subset \pi_{1}^{\text{alg}}(X_k, x)$ is $G_{k_v}$.

**Proof.** From the assumption, there is a finite extension $k'$ of $k$ in $\mathbb{C}_p$, a finite etale covering $Y \to X_{k'}$ and a dominant morphism

$$Y \twoheadrightarrow Z = \mathbb{P}^1_{k'} \setminus \{0, 1, \infty\}.$$ 

(We may also assume that $X, Y, Z$ are endowed with compatible $k'$-rational geometric points $x, y, z$ respectively). The result will follow from 7.2.1 by the following dévissage:

7.2.4. **Lemma.** The statement

$$(*)_{X,k} \text{ Out} \, \pi_{1}^{\text{temp}}(X^{\text{an}}) \cap G_k \subset G_{k_v}$$

satisfies

a) $(*)_{X, k'} \Rightarrow (\ast)_{X, k}$ if $k'$ is a finite extension of $k$ in $\mathbb{C}_p$,

b) $(\ast)_{Z, k} \Rightarrow (\ast)_{Y, k}$ if there is a non-constant morphism $f : Y \to Z$.

c) $(\ast)_{Y, k} \Rightarrow (\ast)_{X, k}$ if $Y$ is a finite etale covering of $X$.

**Proof.** b) and c) are completely parallel to the proof of 3.3.2, replacing $\mathbb{C}$ by $\mathbb{C}_p$, $\pi_{1}^{\text{top}}$ by $\pi_{1}^{\text{temp}}$, the reference to 3.2.1 by a reference to 6.2.2, and sublemma 3.3.4 by 7.2.5.

7.2.5. **Sublemma.** (Neukirch, cf. [27, § 12.1.10]) If $H$ is a closed subgroup of $G_k$ containing $G_{k_v}$ as a subgroup of finite index, then $H = G_{k_v}$. □

As for a): let $H$ stand for $\text{Out} \, \pi_{1}^{\text{temp}}(X^{\text{an}}) \cap G_k$. It contains $G_{k_v}$. On the other hand, $\text{Out} \, \pi_{1}^{\text{temp}}(X^{\text{an}}) \cap G_{k'}$, which is nothing but $G_{k'}$, due to $(*)_{X, k'}$, is a subgroup of finite index of $H$. Therefore $H$ is closed in $G_k$ and contains $G_{k_v}$ as a subgroup of finite index. One concludes by 7.2.4. □

Remarks. 1) One version of a celebrated theorem of Mochizuki [25] says that for a hyperbolic curve $X_{k_v}$ over $k_v$, the centralizer of $G_{k_v}$ in $\text{Out} \, \pi_{1}^{\text{alg}}(X_{\mathbb{C}_p})$ is isomorphic to $\text{Aut} \, X_{k_v}$. Denoting by $X^{an}$ the $p$-adic manifold attached to $X_{\mathbb{C}_p}$, it follows that the center of $\text{Out} \, \pi_{1}^{\text{temp}}(X^{an})$ is isomorphic to the center of $\text{Aut} \, X_{k_v}$. In particular, it is trivial if $X_{k_v} = \mathbb{P}^1_{k_v} \setminus \{0, 1, \infty\}$.

2) From $G_{k_v}$, one can recover the inertia group and the Frobenius elements by a purely group-theoretic recipe [27, § 12.1.8].
3) Theorems 7.2.1 and 7.2.3 do not change if one composes \( \iota_v \) with any element of \( G_{k_v} \). Thus it is enough to fix a \( p \)-adic place \( v \) of \( \bar{k} \) (instead of \( \iota_v \)).

8. The local Grothendieck-Teichmüller group at \( p \)

8.1. In this section, we define a “\( p \)-local” version of the Grothendieck-Teichmüller group \( \hat{G}T \), which plays with respect to \( G_{\mathbb{Q}_p} \) the role that \( \hat{G}T \) plays with respect to \( G_{\mathbb{Q}} \).

For \( r \geq 4 \), let \( \mathcal{M}_{0,r} \) be the moduli space for curves of genus 0 with \( r \) ordered marked points. This smooth geometrically connected \( \mathbb{Q} \)-variety carries an action of the symmetric group \( S_r \) (permutation of the marked points). It admits a canonical smooth compactification \( \overline{\mathcal{M}}_{0,r} \) (Deligne-Mumford-Knudsen), such that the boundary \( \partial \mathcal{M}_{0,r} \) is a divisor with normal crossings; over \( \mathbb{C} \), this compactification is simply-connected \([5, \S 1.1]\), and a fortiori, \( \pi_{1,alg}^{\mathbb{Q}}(\overline{\mathcal{M}}_{0,r,\mathbb{Q}}) = \{1\} \) (from now on, we drop the implicit base point from the notation).

For \( r = 4 \), \( \mathcal{M}_{0,4} \cong \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\} \), \( \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1_{\mathbb{Q}} \) and the action of \( S_4 \) factors through the standard action of \( S_3 \). In general, \( \mathcal{M}_{0,r} \) is isomorphic to the complement in \( (\mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\})^{r-3} \) of the partial diagonals defined by the equality of two coordinates.

Forgetting the last marked point provides an \( S_r \)-equivariant surjective morphism \( p_r : \mathcal{M}_{0,r+1} \to \mathcal{M}_{0,r} \) with fibers isomorphic to \( \mathbb{P}^1 \) minus \( r \) points.

8.2. The fundamental group \( \pi_1^{top}(\mathcal{M}_{0,r,\mathbb{C}}^{an}) \) (a mapping class group in genus zero) is generally denoted by \( \Gamma_{0,r} \), but we add a superscript \( \top \) in order to distinguish it from subsequent variants.

The fibration \( p_r \) gives rise to an exact sequence of fundamental groups

\[
\{1\} \to F_{r-1} \to \Gamma_{0,r+1}^{top} \to \Gamma_{0,r}^{top} \to \{1\}
\]

where \( F_{r-1} \cong \pi_1^{top}(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \ldots, x_r\}) \) is the discrete free group on \( r - 1 \) generators. On the other hand, according to a classical result of Nielsen-Zieschang (cf. [53]), the corresponding homomorphism

\( \Gamma_{0,r}^{top} \to \text{Out} F_{r-1} \)

induces an isomorphism

\( \Gamma_{0,r}^{top} \cong \text{Out}\,^2 \pi_1^{top}(\mathbb{P}^1_{\mathbb{C}} \setminus \{x_1, \ldots, x_r\}) \cong \ker[\text{Out} F_{r-1} \to GL_{r-1}(\mathbb{Z})] \)

(the sign \(^2\) indicates the subgroup of (outer) automorphisms which fix the conjugacy classes of the local monodromies at the points \( x_i \)). [From there, using the surjectivity of \( \text{Aut} F_n \to GL_n(\mathbb{Z}) \) \([24, \S 1.4.4]\), one derives]
the structure of \( \text{Aut } F_n \) (resp. \( \text{Out } F_n \)) for any \( n \geq 1 \): it is an extension of \( GL_n(Z) \) by an iterated extension of free groups of successive ranks 2, 3, \ldots, \( n \) (resp. 2, 3, \ldots, \( n - 1 \)].

8.3. The Grothendieck-Teichmüller group \( \hat{\text{GT}} \) introduced by Drinfel’d in connection with the theory of quantum groups is a closed subgroup of the profinite group \( \text{Aut } \hat{F}_2 \). If \( x, y \) denote fixed topological generators of \( \hat{F}_2 \), it consists of automorphisms of the form \( x \mapsto x^\lambda, y \mapsto f^{-1}y^\lambda f \) for \( \lambda \in \hat{Z}^* \) and appropriate \( f \in \hat{F}_2 \) satisfying three well-known equations I, II, III, which we do not recall here (cf. [8] [14]).

The bigger group obtained by discarding equation III (the pentagonal equation) is sometimes denoted by \( \hat{\text{GT}}^0 \).

8.4. For any \( r \geq 4 \), we set \( \Gamma_{\text{alg}}^{0,r} := \pi^{\text{alg}}_1(M_{0,r},\overline{Q}) \). Thus \( \Gamma_{\text{alg}}^{0,r} \) is identified with the profinite completion of \( \Gamma_{0,r}^{\text{top}} \) if one fixes an embedding \( \iota : \overline{Q} \hookrightarrow C \).

Following Nakamura [26], we denote by \( \text{Out}^{\sharp} \Gamma_{0,r}^{\text{alg}} \) the subgroup of outer automorphisms \( \sigma \) of \( \Gamma_{0,r}^{\text{alg}} \) which sends the conjugacy class of a generator of the inertia group at each component of \( \partial M_{0,r} \) to the conjugacy class of the same generator raised to some power \( \lambda(\sigma) \in \hat{Z}^* \) (the same \( \lambda(\sigma) \) for each component). There is an obvious homomorphism

\[
\lambda : \text{Out}^{\sharp} \Gamma_{0,r}^{\text{alg}} \rightarrow \hat{Z}^*, \quad \sigma \mapsto \lambda(\sigma).
\]

We denote by \( \hat{\text{GT}}^{(r)} := \text{Out}^{\sharp} \Gamma_{0,r}^{\text{alg}} \) the subgroup of elements which commute with the action of \( \mathfrak{S}_r \). This notation is motivated by:

8.4.1. **Theorem.** (Harbater, Schneps [12])

1) The \( \mathfrak{S}_r \)-equivariant surjective morphism \( M_{0,r+1} \xrightarrow{p_r} M_{0,r} \) induces canonical homomorphisms

\[
\text{Out}^{\sharp} \Gamma_{0,r+1}^{\text{alg}} \rightarrow \text{Out}^{\sharp} \Gamma_{0,r}^{\text{alg}}, \quad \hat{\text{GT}}^{(r+1)} \xrightarrow{q_r} \hat{\text{GT}}^{(r)}
\]

compatible with the \( \lambda \)-map.

2) For all \( r \geq 5 \), \( q_r : \hat{\text{GT}}^{(r+1)} \rightarrow \hat{\text{GT}}^{(r)} \) is an isomorphism,

3) An embedding \( \iota : \overline{Q} \hookrightarrow C \) being fixed, there is a natural commutative diagram

\[
\begin{array}{ccc}
\hat{\text{GT}}^{(5)} & \xrightarrow{q_4} & \hat{\text{GT}}^{(4)} \\
\downarrow \cong & & \downarrow \cong \\
\hat{\text{GT}} & \hookrightarrow & \hat{\text{GT}}_0
\end{array}
\]

\(^{18}\) Unlike the usage in the present paper, \( \hat{\text{GT}} \) is just a (traditional) notation and does not indicate the profinite completion of some group \( GT \); this notation helps distinguishing the profinite Grothendieck-Teichmüller group from other variants (notably the pro-unipotent one related to mixed Tate motives).
where the vertical maps are isomorphisms. □

(ι is used to identify \( \tilde{F}_2 \) with \( \Gamma^{alg}_{0.4} \) based at some tangential base point).

This result (partly based on results in [26]) provides a ‘purely algebraic’ description of \( \check{G}T \) (independent of the topology of \( \mathbb{C} \)), and at the same time, a natural interpretation of the Drinfel’d-Ihara embedding \( G_\mathbb{Q} \hookrightarrow \check{G}T \) (as an embedding of type \( \rho \), with the notation of 2.1). The question of how close is \( \check{G}T \) to \( G_\mathbb{Q} \) is a famous open problem.

8.5. In view of the results of §7, it is natural to try to replace the algebraic fundamental groups which occur in the previous theorem by tempered fundamental groups in the \( p \)-adic context.

We fix a prime number \( p \), and consider the polish groups \( \pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an}) \). It is likely that these are residually finite, but having no proof, we are led to introduce the following \( p \)-adic avatar of the mapping class group in genus zero:

\[
\Gamma_{0,r}^{temp} := \pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an})/\ker[\pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an}) \to \pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an})].
\]

Thus \( \Gamma_{0,r}^{temp} \) is, like \( \pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an}) \), a polish group, and embeds into its profinite completion, which can be identified with the \( \Gamma^{alg}_{0,r} \) if one fixes an embedding \( \iota_p : \mathbb{Q} \hookrightarrow C_p \).

Note that \( \Gamma_{0,r}^{temp} \) is our familiar group \( \pi_1^{temp}(\mathbb{P}^{1,an}_{C_p} \setminus \{0, 1, \infty\}) \).

8.5.1. Proposition. The morphism \( p_r : \mathcal{M}_{0,r+1} \to \mathcal{M}_{0,r} \) induces strict surjective homomorphisms

\[
\pi_1^{temp}(\mathcal{M}_{0,r+1,C_p}^{an}) \twoheadrightarrow \pi_1^{temp}(\mathcal{M}_{0,r,C_p}^{an}), \quad \Gamma_{0,r+1}^{temp} \twoheadrightarrow \Gamma_{0,r}^{temp}.
\]

Proof. From the description \( \mathcal{M}_{0,r,C_p} = (\mathbb{P}^1_{C_p} \setminus \{0, 1, \infty\})^{r-3} \Delta \), we deduce a factorization of \( p_r \) into a closed embedding \( j : \mathcal{M}_{0,r+1,C_p} \hookrightarrow \mathbb{A}^1_{C_p} \times \mathcal{M}_{0,r,C_p} \) followed by the second projection. This second projection induces an isomorphism on \( \pi_1^{temp} \): this follows from the fact that any finite etale covering of \( \mathbb{A}^1_{C_p} \times \mathcal{M}_{0,r,C_p} \) is of the form \( \mathbb{A}^1_{C_p} \times \mathcal{M}_1 \) for some finite etale covering \( \mathcal{M}_1 \) of \( \mathcal{M}_{0,r,C_p} \) (in the algebraic or analytic category, this does not matter thanks to [19]), and from the fact that any topological covering of \( \mathbb{A}^1_{C_p} \times \mathcal{M}_1^{an} \) is of the form \( \mathbb{A}^1_{C_p} \times \mathcal{M}_2^{an} \) for some topological covering \( \mathcal{M}_2^{an} \) of \( \mathcal{M}_1^{an} \) (since \( \pi_1^{top}(\mathbb{A}^1_{C_p}^{an}) = \{1\} \), and \( \mathbb{A}^1_{C_p} \times \mathcal{M}_1^{an} \) is homotopy equivalent to the naive cartesian product of \( \mathbb{A}^1_{C_p} \) and \( \mathcal{M}_1^{an} \) [3, pf. of 10.1]).

On the other hand, \( j \) is the inclusion of the complement of \( r \) smooth hyperplanes, and it follows from [1, §4.5.5.b] that it induces a strict surjection on \( \pi_1^{temp} \) (this result uses heavily the fact that one is dealing with polish groups). □
Remark. One has $\pi_1^{\temp}(\overline{M}_{0,r,c_p}^{an}) = \{1\}$: indeed, we have seen that any finite etale covering of $\overline{M}_{0,r,c_p}^{an}$ is trivial (in the algebraic or analytic category ([19]), and on the other hand, since $\overline{M}_{0,r}$ is Zariski open in the affine space $A^{r-3}$, $\pi_1^{\temp}(\overline{M}_{0,r,c_p}^{an}) = \pi_1^{\temp}(A_{c_p}^{r-3}) = \{1\}$ ([I] § 1.1.4).

This suggests that $\pi_1^{\temp}(\overline{M}_{0,r,c_p}^{an})$ might ‘come’ in some sense from neighborhoods of points of maximal degeneration in $\partial M_{0,r,c_p}^{an}$.

8.6. For any $r \geq 4$, we denote by $Out^p \Gamma_{0,r}^{temp}$ the subgroup of outer automorphisms $\sigma$ of $\Gamma_{0,r}^{temp}$ which send the conjugacy class of the local monodromy at each component of $\partial M_{0,r}$ to the conjugacy class of the same local monodromy raised to some power $\lambda(\sigma) \in \mathbb{Z}^*$ (the same $\lambda(\sigma)$ for each component); we refer to ([I] § 4.5.4) for the precise definition of this conjugacy class.

8.6.1. Notation. • We denote by $GT_p^{(r)} := Out^p \Gamma_{0,r}^{temp}$ the subgroup of $Out^p \Gamma_{0,r}^{temp}$ of elements which commute with the action of the symmetric group $\mathfrak{S}_r$.

• We denote by $\widehat{GT}_p^{(r)}$ the closure of the image of $GT_p^{(r)}$ in $\widehat{GT}^{(r)} \cong \widehat{GT}$.

(This notation is consistent with the notation $\widehat{GT}$ of the Grothendiek-Teichmüller group, but it should not be confused with the profinite completion $\widehat{GT}_p^{(r)}$ of $GT_p^{(r)}$; we do not know whether the natural epimorphism $\widehat{GT}_p^{(r)} \to \widehat{GT}_p^{(r)}$ is an isomorphism, but cf. 8.7.2.).

Note that $\widehat{GT}_p^{(4)}$ is a closed subgroup of $Out^p \pi_1^{\temp}(\mathcal{M}_p^{an} \setminus \{0, 1, \infty\})$.

8.6.2. Proposition. 1) The $\mathfrak{S}_r$-equivariant surjective morphism $p_r$ induces canonical injective homomorphisms

$$GT_p^{(r+1)} \xrightarrow{q_r} GT_p^{(r)}, \quad \widehat{GT}_p^{(r+1)} \xrightarrow{q_r} \widehat{GT}_p^{(r)}.$$  

2) The canonical homomorphism $GT_p^{(r)} \to \widehat{GT}_p^{(r)}$ is injective; in particular, $GT_p^{(r)}$ is residually finite.

Proof. The part of 1) concerning $\widehat{GT}_p^{(r)}$ follows directly from 8.4.1. From the embedding $\Gamma_{0,r}^{temp} \hookrightarrow \Gamma_{0,r}^{alg}$, we derive an embedding $Aut^p \Gamma_{0,r}^{temp} \hookrightarrow Aut^p \Gamma_{0,r}^{alg}$. The “Aut” version of 8.4.1 (cf. ([2], 3.1)) says that $p_r$ induces an injective homomorphism $Aut^p \Gamma_{0,r+1}^{temp} \hookrightarrow Aut^p \Gamma_{0,r}^{temp}$. By application of the sorite 3.2.2.a) to $G = \Gamma_{0,r+1}^{temp}$, $H = \Gamma_{0,r}^{temp}$ and $\phi = p_r$, we derive that $p_r$ induces an injective homomorphism $Aut^p \Gamma_{0,r+1}^{temp} \hookrightarrow Aut^p \Gamma_{0,r}^{temp}$. Finally,
using the surjectivity of $\Gamma_{0,r+1}^{\text{temp}} \to \Gamma_{0,r}^{\text{temp}}$ (8.5.1), we see that $p_r$ induces in turn injective homomorphisms

$$Out\, \flat \, \Gamma_{0,r+1}^{\text{temp}} \hookrightarrow Out\, \flat \, \Gamma_{0,r}^{\text{temp}}, \quad GT_p^{(r+1)} \hookrightarrow GT_p^{(r)},$$

which proves 1).

For 2), we compose the maps $q_r$ up to $r = 4$ and get an injective homomorphism $GT_p^{(r)} \to GT_p^{(4)} \hookrightarrow Out\, \Gamma_{0,4}^{\text{temp}}$. But we know (6.2.1) that $Out\, \Gamma_{0,4}^{\text{temp}}$ embeds into $Out\, \Gamma_{0,4}^{\text{alg}}$. Since the composed map $GT_p^{(r)} \to Out\, \Gamma_{0,4}^{\text{alg}}$ factors through $\widehat{GT}^{(r)}_p$, assertion 2) follows. $\square$

8.6.3. **Definition.** • We define $GT_p$ to be the intersection of the images of the $GT_p^{(r)}$ in $GT_p^{(4)}$ (for all $r \geq 4$).

• We define the local Grothendieck-Teichmüller group at $p$ to be the closure $\widehat{GT}_p$ of $GT_p$ in $\widehat{GT}_p^{(4)}$.

From 8.6.2.1), we get

8.6.4. **Corollary.** The abstract group $GT_p$ acts on the ‘tower’ of tempered mapping class groups $(\Gamma_{0,r}^{\text{temp}})^{r \geq 4}$.

On the other hand, $\widehat{GT}_p$ can be viewed as a closed subgroup of the Grothendieck-Teichmüller group $\widehat{GT}$ via the Harbater-Schneps isomorphism $\widehat{GT}_p^{(4)} \cong \widehat{GT}_0$.

8.7. We fix a $p$-adic place $v$ of $\overline{Q}$. Then the maps $\rho$ from §2 and §5 give rise to commutative squares of injective homomorphisms

$$
\begin{array}{ccc}
G_{Q_p} & \to & GT_p \\
\downarrow & & \downarrow \\
G_Q & \to & \widehat{GT}_0
\end{array}
\quad
\begin{array}{ccc}
G_{Q_p} & \to & GT_p^{(r)} \\
\downarrow & & \downarrow \\
G_Q & \to & \widehat{GT}^{(r)}
\end{array}
$$

which are compatible via the embeddings $q_r$. It turns out that these squares are cartesian:

8.7.1. **Theorem.** $G_{Q_p} = G_Q \cap GT_p = G_Q \cap \widehat{GT}_p$ in $\widehat{GT}_0$. Moreover, for any $r \geq 4$, $G_{Q_p} = G_Q \cap GT_p^{(r)} = G_Q \cap \widehat{GT}_p^{(r)}$ in $\widehat{GT}^{(r)}$.

**Proof.** This follows immediately from 7.2.1, since

$$\widehat{GT}_p^{(4)} \subset Out\, \Gamma_{1}^{\text{temp}}( \mathbb{P}_{\mathbb{C}_p}^1 \setminus \{0, 1, \infty\}). \quad \square$$

In the following corollary, we identify all $\widehat{GT}^{(r)}$ with $\widehat{GT}$ for $r \geq 5$ (according to 8.4.1).
8.7.2. Corollary. If \( G_Q = \hat{G}T \), then all these groups \( GT_p = GT_p^{(r)} = \hat{GT}_p = \hat{GT}_p^{(r)} \) would coincide with \( G_Q \) in \( \hat{GT} \), for \( r \geq 5 \).

In particular \( GT_p^{(5)} = \text{Out}_{\hat{\varphi}, \Gamma_{0,5}} \) would be a finitely generated profinite group. \( \square \)

It is tempting to try to put this property in the wrong (cf. the remark after 6.2.2)...

Remarks. 1) The analogue for \( \hat{GT}_p \) of the inertia subgroup of \( G_Q \) is the kernel of the composed map \( \hat{GT}_p \overset{\lambda}{\rightarrow} \hat{Z}^* = (\hat{Z}^{(p')})^* \times Z_p^* \rightarrow (\hat{Z}^{(p')})^* \).

2) The image of \( G_Q \) by \( \lambda \) is \( p\hat{Z} \times Z_p^* \). A natural question is whether the same holds for the image of \( \hat{GT}_p \) or even \( \hat{GT}_p^{(4)} \). After submission of this paper, A. Tamagawa has indicated to the author an argument according to which this should follow from the main results of [31], notably theorem 5.2 and proposition 5.3 of \textit{loc. cit.}

3) Since \( \lambda \) takes its values in \( \hat{Z}^* \), \( GT_p^{(r)} \) acts on the quotient of \( \Gamma_{0,r}^{\text{temp}} \) by the closed normal subgroup generated by some fixed integral powers of the local monodromies at the components of the boundary. This quotient is an orbifold fundamental group in the sense of [11], and its discrete representations correspond to certain algebraic vector bundles on \( \mathcal{M}_{0,r,c_p} \) with fuchsian integrable connection and prescribed exponents [11, \S 4.5.9], which form a \( p \)-adic analytic subvariety of the corresponding algebraic moduli space of connections. In particular, \( G_Q \) acts through a finite group on each of these \( p \)-adic analytic subvarieties.

4) As in the remark at the end of \( \S 7 \), one derives from Mochizuki’s theorem (and 8.6.2) that the center of any of the groups \( GT_p, GT_p^{(r)}, \hat{GT}_p, \hat{GT}_p^{(r)} \) is trivial.

5) In complete analogy with the above, one can introduce the archimedean avatars \( GT_{\infty}^{(r)}, \hat{GT}_{\infty}^{(r)}, GT_{\infty}, \hat{GT}_{\infty} \) (replacing \( C_p \) by \( C \) and \( \pi_1^{\text{temp}} \) by \( \pi_1^{\text{op}} \)). It turns out that all these groups coincide with \( G_R \) for any \( r \geq 4 \). This is proved by the same argument as in part i) of the proof of 3.3.1, replacing the final reference to [27, \S 12.1.4] by a reference to [18, prop. 4.ii]: \( G_R \) is its own centralizer in \( GT \).

6) Drinfel’d’s presentation of \( \hat{GT} \) uses topological generators \( x, y \) of \( \hat{F}_2 \) which are the standard local monodromies at 0 and 1 in \( \pi_1^{\text{op}}(\mathbf{P}^1_{\mathbb{C}}^{\text{an}} \setminus \{0, 1, \infty\}) \) (based at a suitable tangential base point). One could ask whether there is an analogous presentation of \( \hat{GT} \) of \( p \)-adic flavour, using local monodromies at 0 and 1 in \( \pi_1^{\text{temp}}(\mathbf{P}^1_{\mathbb{C}_p}^{\text{an}} \setminus \{0, 1, \infty\}) \). It turns out that the answer

\[ ^{19}\text{topologically generated by three elements, cf. [27, \S 7.4.1]} \]
is negative, as a consequence of the following observation: using techniques of graphs of groups, we have shown in [1, § 6.4.6] that for $p = 2$, one cannot find local monodromies $x, y, z$ at 0, 1, $\infty$ respectively such that $xyz = 1$.

9. **$G_{\mathbb{Q}_p}$ IS THE AUTOMORPHISM GROUP OF THE TEMPERED $\pi_1$-FUNCTOR**

9.1. In this section, we deal with automorphisms of fundamental groups considered as functors.

Let $\mathcal{V}_k$ be the category of all smooth geometrically connected algebraic varieties defined over $k$.

Let $\mathcal{T}$ be the category of separated topological groups up to inner automorphism: a morphism in this category is a continuous homomorphism $G \to H$ given up to conjugation by an element of $H$ (in particular, $\text{Aut}_\mathcal{T}(G) = \text{Out} G$).

In this article, we have been mainly concerned with the following functors

$$\pi^\text{alg}_k, \pi^\text{temp}_{\mathbb{C}_p} : \mathcal{V}_k \to \mathcal{T}$$

$$\pi^\text{alg}_k(X) := \pi^1_{\text{alg}}(X, x), \pi^\text{temp}_{\mathbb{C}_p}(X) := \pi^1_{\text{temp}}(X^\text{an}_{\mathbb{C}_p}, x),$$

given an embedding $k \hookrightarrow \mathbb{C}_p$ (note that in the category $\mathcal{T}$, these fundamental groups do not depend on the choice of $x$ up to canonical isomorphism; hence the above functors are uniquely defined up to unique isomorphism; similarly, $\pi^\text{temp}_{\mathbb{C}_p}$ depends only on the $p$-adic place $v$ of $k$ induced by the embedding, up to unique isomorphism).

Let $k_v$ be the topological closure of $k$ in $\mathbb{C}_p$. Functoriality of the maps $\rho$ considered in §2 and §5 may be expressed as follows: there are canonical homomorphisms

$$G_k \overset{\rho}{\to} \text{Aut} \pi^\text{alg}_k, \quad G_{k_v} \overset{\rho}{\to} \text{Aut} \pi^\text{temp}_{\mathbb{C}_p}$$

which build a commutative square if $\bar{k}$ is the closure of $k$ in $\mathbb{C}_p$:

$$
\begin{array}{ccc}
G_{k_v} & \overset{\rho}{\to} & \text{Aut} \pi^\text{temp}_{\mathbb{C}_p} \\
\downarrow & & \downarrow \\
G_k & \overset{\rho}{\to} & \text{Aut} \pi^\text{alg}_k
\end{array}
$$

9.2. The following result had been conjectured by Oda and Matsumoto [22]:

9.2.1. **Theorem.** (Pop [29], unpublished) $G_k \overset{\rho}{\to} \text{Aut} \pi^\text{alg}_k$ is an isomorphism. $\square$
(In fact, Pop proves more: $G_k$ coincides with $\text{Aut} \pi^\text{alg}_{\bar{k}}$ for any big enough full subcategory $\mathcal{W}$ of $\mathcal{V}_k$; we do not know any minimal choice for $\mathcal{W}$, but it suffices that $\mathcal{W}$ contains the open subsets of the projective spaces).

As a consequence of Pop’s theorem and theorem 7.2.1, we get

9.2.2. **Theorem.** $G_k \xrightarrow{\rho} \text{Aut} \pi^\text{temp}_{C_p}$ is an isomorphism.

**Proof.** One has a sequence of homomorphisms

$$G_k \xrightarrow{\rho} \text{Aut} \pi^\text{temp}_{C_p} \xrightarrow{\text{Out} \pi^\text{temp}_{1}(P^1_{C_p})} \text{Out} \pi^\text{alg}_{1}(P^1_{\bar{k}} \setminus \{0, 1, \infty\})$$

where the two maps on the right are injective. By 7.2.1, the intersection of $G_k$ and $\text{Out} \pi^\text{temp}_{1}(P^1_{C_p})$ in $\text{Out} \pi^\text{alg}_{1}(P^1_{\bar{k}} \setminus \{0, 1, \infty\})$ is $G_{k_v}$.

It follows that $\rho : G_{k_v} \to \text{Aut} \pi^\text{temp}_{C_p}$ admits a left inverse $\nu$. It remains to show that $\ker \nu$ is trivial.

Note that $\ker \nu = \ker[\text{Aut} \pi^\text{temp}_{C_p} \to \text{Aut} \pi^\text{alg}_{\bar{k}}]$. It follows, by 6.2.1, that the evaluation of any element of $\ker \nu$ on any curve $X$ is the identity (as outer automorphism of the fundamental group). In order to reduce the case of an arbitrary smooth geometrically connected $k$-variety $V$ to the case of a curve $X$, we choose a quasi-projective dense open subset $U \subset V$ and an embedding of $U$ in some projective space $P^N_k$. According to a theorem of Bertini-Deligne [7, 1.4], there is a dense open subset $U$ of the grassmannian of linear varieties of codimension $\dim U - 1$ in $P^N_k$ such that for all $L \in \mathcal{U}(k)$, the curve $X = L \cap U$ is geometrically connected and smooth, and the homomorphism

$$\pi^\text{top}_{1}(X^\text{an}_{C}) \to \pi^\text{top}_{1}(U^\text{an}_{C})$$

is surjective (for any fixed complex embedding $k \hookrightarrow C$). It follows that

$$\pi^\text{alg}_{1}(X_{\bar{k}}) \to \pi^\text{alg}_{1}(U_{\bar{k}})$$

is also surjective; on the other hand

$$\pi^\text{alg}_{1}(U_{\bar{k}}) \to \pi^\text{alg}_{1}(V_{\bar{k}})$$

is also surjective. Since the evaluation of any element of $\ker \nu$ on $X$ is the identity, the evaluation of any element of $\ker \nu$ on $V$ is also the identity, hence $\ker \nu = \{1\}$. \qed

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