Verifying the existence of maximum likelihood estimates for generalized linear models

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Abstract

A fundamental problem with nonlinear models is that maximum likelihood estimates are not guaranteed to exist. Though nonexistence is a well known problem in the binary choice literature, it presents significant challenges for other models as well and is not as well understood in more general settings. These challenges are only magnified for models that feature many fixed effects and other high-dimensional parameters. We address the current ambiguity surrounding this topic by studying the conditions that govern the existence of estimates for (pseudo-)maximum likelihood estimators used to estimate a wide class of generalized linear models (GLMs). We show that some, but not all, of these GLM estimators can still deliver consistent estimates of at least some of the linear parameters when these conditions fail to hold. We also demonstrate how to verify these conditions in models with high-dimensional parameters, such as panel data models with multiple levels of fixed effects.

JEL Classification Codes: C13, C18, C23, C25

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1 Introduction

Estimators based on count data models are widely used in applied economic research (Cameron and Trivedi, 2013; Winkelmann, 2008). In particular, Poisson regression has exploded in popularity since the publication of Santos Silva and Tenreyro (2006). Given this widespread and long-standing popularity, it is genuinely surprising that economists have only recently become aware that count data models are not guaranteed to have maximum likelihood (ML) solutions. More precisely, Santos Silva and Tenreyro (2010) show that the first-order conditions that maximize the likelihood of Poisson models might not have a solution if regressors are perfectly collinear over the subsample where the dependent variable is nonzero. Beyond this observation, however, Santos Silva and Tenreyro (2010) caution that “it is not possible to provide a sharp criterion determining the existence” of Poisson ML estimates. Moreover, although nonexistence is a well-known issue in binary choice models, it seemingly remains unknown if similar issues could arise in other nonbinary choice models besides Poisson, and the connections between these various cases remain unknown as well.

In this paper, we resolve several key aspects of this ambiguity. We show that nonexistence of ML estimates is a potential problem for a broad class of generalized linear models (GLMs), including Poisson, binary choice, as well as several other models. We also clarify that this problem continues to be salient for pseudo-maximum likelihood (PML) estimators of these models and, furthermore, that some common PML estimators are affected by nonexistence in ways that cannot be remedied without changing the estimator. For cases in which simpler remedies are possible, we discuss computational methods for detecting and resolving nonexistence and propose a novel algorithm that works well even in settings that require a complex array of high-dimensional covariates, such as panel data models with multiple levels of fixed effects.

We derive our main results in part by drawing on a largely uncredited contribution by Verbeek (1989), who established necessary and sufficient conditions governing the existence of ML estimates for a broad class of GLMs. Using Verbeek’s earlier results as our starting point, we show that for many GLMs, even when the ML estimates can nominally be said to “not exist”, at least

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1Poisson pseudo-maximum likelihood (PML) estimators have emerged as a workhorse approach for studying health outcomes (Manning and Mullahy, 2001), patent citations (Figueiredo et al., 2015), trade (de Bromhead et al., 2019), migration (Bertoli et al., 2019), commuting (Brinkman and Lin, 2019), auctions (Bajari and Hortacsu, 2003), finance (Cohn et al., 2021), and many other economic applications with nonnegative dependent variables and/or with data assumed to be generated by a constant-elasticity model.

2As of this writing, both printings of Verbeek (1989, 1992) together have only nine citations listed on Google Scholar. As the editor’s note appended to Verbeek (1992) explains, Albert Verbeek’s motivation for writing this paper was identical to that of Santos Silva and Tenreyro (2010): frustration over standard algorithms either failing to converge or converging to nonsensical estimates.
some of the linear parameters can usually be consistently estimated. We also add new results for PML estimation approaches that have only become popular in the years following Verbeek (1989) and that turn out not to share these useful properties. For example, the log-link gamma PML estimator sometimes recommended in fields such as international trade and health care economics has very different conditions governing nonexistence than Poisson and suffers from more dire consequences when it occurs.

In addition to discussing how to detect such a problem, we also provide guidance on what can be done about it. At the moment, this is another area in need of clarity. Even for binary choice models, where nonexistence is well-known as the so-called “separation” problem, textbooks that mention the topic generally stop short of suggesting remedies (Zorn, 2005; Eck and Geyer, 2021). The binary choice literature has filled this gap primarily by presenting a choice between two main ways of solving the problem, each with its limitations. On the one hand, the most common approach is to drop a regressor from the model (Zorn, 2005; Allison, 2008; Rainey, 2016). This is also the only approach that has been discussed in the context of nonbinary choice models (Santos Silva and Tenreyro, 2010; Larch et al., 2019). On the other hand, dropping a regressor has implications for the estimation and identification of the other parameters, and often it is not obvious which regressor is the “right” one to drop. Thus, the leading alternative recommended for binary choice settings is to assume the data have been drawn randomly from a known prior distribution (Heinze and Schumper, 2002; Gelman et al., 2008). These methods could be adapted to nonbinary choice settings as well. However, they still necessarily involve altering the model in a way that affects estimation. Furthermore, they are not currently compatible with models that include high-dimensional fixed effects, which are widely used in the international trade literature (Head and Mayer, 2014; Yotov et al., 2016) and are becoming increasingly popular in applied work in general.

Our own suggested remedy, which only involves dropping the separated observations, is gen-

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3Our recommendation to drop separated observations is ostensibly similar to Allison (2008)’s suggestion to “do nothing”, as doing nothing could result in approximately valid estimates and inferences for at least some of the model parameters. However, in general, doing nothing is likely to result in—at best—lack of numerical convergence, and—at worst—convergence to incorrect values and to overstating of statistical significance. In the words of Geyer (2018), “no one knows how much applied statistics is garbage because of this”. Also, some software packages drop separated observations by default (e.g., Stata’s `probit` command), but they generally are not adept at detecting these observations, nor do they usually provide theoretical justification for this practice in their documentation. Our companion website offers examples and discussion.

4This popularity is only likely to increase in the near future thanks to a series of computational innovations that have made models with multiple levels of fixed effects more feasible to compute (see Figueiredo et al., 2015; Larch et al., 2019; Stammann, 2018; Bergé, 2018; Correia et al., 2020) as well as a growing literature on bias corrections for incidental parameter bias (see Arellano and Hahn, 2007; Fernández-Val and Weidner, 2016).
erally very simple to implement through our algorithm and does not have any of these limitations. This is mainly because of an insight advanced independently by Verbeek (1989), Geyer (1990), and Clarkson and Jennrich (1991): any GLM suffering from separation can be nested within a “compactified” GLM where the conditional mean of each observation is allowed to go to its boundary values. The (pseudo-)likelihood function always has a maximum somewhere in the compactified parameter space; thus, we can transform the problem of nonexistence to one of possible corner solutions. More importantly, observations with a conditional mean at the boundary in the more compactified model are effectively perfectly predicted observations. These observations offer no information about the parameters with interior solutions and, as we will show, can be quickly detected even for very complex models. Dropping these observations then results in a standard (non-compactified) version of the model that is assured to produce the same model fit as the compactified version, as well as the same point estimates and inferences of the parameters with interior solutions. We also show that the estimates are consistent and that correct inference requires only careful attention to which of the regressors are involved in separation. The resulting output, on the whole, is no different than what one would observe with a perfectly collinear regressor, and the problems of interpretation and inference turn out to be very similar as well.

Exactly who first discovered that nonexistence could occur for Poisson regression and other nonbinary choice GLMs is open to interpretation. The literature starts with Haberman (1973, 1974)’s derivation of a necessary and sufficient condition for the existence of estimates for log-linear frequency table models (including Poisson frequency tables). However, it was known at the time that this condition was difficult to verify for higher-dimensional tables (see Albert and Anderson, 1984), a still-unsettled problem we indirectly solve in this paper. Soon thereafter, Wedderburn (1976) independently derived a sufficient but not necessary condition for the existence of estimates across a wide class of GLMs. His result can be shown to be equivalent to the later result from Santos Silva and Tenreyro (2010) for the Poisson model. Silvapulle (1981) and Albert and Anderson (1984) are then credited with demonstrating the concept of “separation” for binary choice models, a term that we will adapt to describe nonexistence in other models as well. The latter paper also conjectures that their analysis may generalize to the class of log-linear frequency table models considered by Haberman (1973, 1974). A few years later, Silvapulle and Burridge (1986) showed how linear programming methods may be used to detect separation in binary choice models. They state, but do not prove, that this linear programming problem is also equivalent to the condition for the existence of estimates for the class of models considered by Haberman (1973, 1974). To our knowledge, Verbeek (1989) was the first to derive an explicit, unifying link between
these earlier results for binary choice models and the more general GLM setting, though Geyer (1990) and Clarkson and Jennrich (1991) each independently derived similar results for related classes of models shortly thereafter.\(^5\)

We add to this earlier literature in three main ways. First, by considering an expanded set of estimation approaches, we offer a more detailed treatment of how the separation problem varies across GLMs and estimators thereof. For example, a significantly stricter set of conditions governs the existence of estimates for gamma PML and inverse Gaussian PML than for Poisson, logit, and probit—a result that raises concerns about applications of the former estimators to settings where zero outcomes are common, such as health care cost analysis and international trade. Importantly, these are precisely the settings where these estimators have come into common usage; see, e.g., Manning and Mullahy (2001); Egger and Staub (2015).\(^6\) Second, we clarify that at least some of the linear parameters can be consistently estimated in the presence of separation as well as how to obtain valid asymptotic inferences—though, again, it is important to note these results do not extend to all of the estimators we consider.\(^7\)

Finally, we introduce a simple-but-powerful method for detecting separation in models with a large number of fixed effects, a conceptually nontrivial task that would ordinarily require solving a high-dimensional linear programming problem. Because our algorithm relies on repeated iteration of a least-squares-with-equality-constraints regression, it can take advantage of the recent innovations of Correia (2017), who shows how to solve high-dimensional least-squares problems in nearly linear time. To our knowledge, the only other method that has been suggested for detecting separation in large ML settings is that of Eck and Geyer (2021). Their algorithm works by iteratively solving for the null eigenvectors of the information matrix, whereas ours avoids large matrix operations altogether, making it substantially more scalable. Our method also has the advantage of being simple to program, and our proof of its effectiveness requires only knowledge of elementary least-squares algebra.

The rest of the paper proceeds as follows. Section 2 formally establishes the problem of separation in GLMs, including its sufficient and necessary conditions. Section 3 discusses how to address

\(^5\)The results by Geyer (1990) apply to the class of linear exponential families, while the work of Clarkson and Jennrich (1991) applies to linear parameter models (also known as “models with a linear part”; see Stirling, 1984).

\(^6\)Manning and Mullahy (2001) leave aside the issue of zero outcomes in their paper, but they indicate that gamma PML is generally a good model for health care cost data and also remark that “there is ostensibly nothing in the above analysis that would preclude applications to data where realizations of \(y\) are either positive or zero, as is common in many health economics applications.” Our own findings indicate that zeroes do pose a distinct problem for gamma PML estimation that must be carefully taken into account.

\(^7\)Gourieroux et al. (1984, Appx 1.1) and Fahrmeir and Kaufmann (1985) (Sec. 2.2) both assume in their proofs of consistency that the solutions for the linear parameters are interior. We present a proof that relies on a suitable reparameterization of the separated model such that the results of Gourieroux et al. (1984) apply directly.
separation in setups with and without fixed effects. Section 4 concludes. Further details are available in the Appendix, including additional proofs and results of interest. We have also created a website devoted to the separation problem, which provides numerous examples illustrating the methods and principles described in this paper.

2 Nonexistence in generalized linear models

The class of GLM-based estimators we consider is defined by the maximization of the following log (pseudo-)likelihood objective function, corresponding to distributions of the exponential family:

$$l(\beta) = \sum_i l_i(\beta) = \sum_i \left[ \alpha_i(\varphi) y_i \theta_i - \alpha_i(\varphi) b(\theta_i) + c(y_i, \varphi) \right], \quad (1)$$

For brevity, we will generally refer to this objective function as the “likelihood”, though we will use “pseudo-likelihood” when strictly discussing PML estimators. The individual term \(l_i\) will be the “likelihood contribution” or “pseudo-likelihood contribution”. \(y_i \geq 0\) is an outcome variable, \(x_i\) is a set of \(M\) regressors \((x_1, x_2, \ldots, x_M)\), and \(\beta \in \mathbb{R}^M\) is an \(M \times 1\) vector of parameters to be estimated. The function \(\alpha_i(\varphi) > 0\) is usually of the form \(w_i/\varphi\), where \(w_i\) is a known weight, and \(\varphi\) is a potentially unknown scale or dispersion parameter.\(^8\) \(\theta_i = \theta(x_i\beta; \nu)\) is the canonical location parameter, which links the linear predictor of a given observation \(x_i\beta\) to its likelihood contribution \(l_i\) and its conditional mean \(\mu_i \equiv E[y_i|x_i] = b'(\theta_i)\). Note that \(\theta(x_i\beta; \nu)\) is continuous, strictly increasing, and twice differentiable in \(x_i\beta\) and that \(b(\theta_i)\) is continuous, increasing, and convex in \(\theta_i\). Notably, these last few restrictions together ensure that the quantities \(\theta_i\), \(x_i\beta\), and \(\mu_i\) are each increasing with respect to one another and that \(l(\beta)\) is continuous in \(\beta\). We further assume that \(\lim_{x_i\beta \to -\infty} \mu_i = 0\) to rule out the simple linear model, which always has a solution.\(^9\) \(\nu\) is an additional dispersion parameter that allows us to also consider negative binomial models (see Table 1). Lastly, \(c(\cdot)\) is a known real-valued function that depends on the specific GLM.

The first-order condition for the \(m\)-th individual parameter, \(\beta_m\), follows from the GLM score function:

$$s(\beta_m) = \sum_i s_i(\beta_m) = \sum_i \alpha_i(\varphi) \left[ y_i - b'(\theta_i) \right] \theta'(x_i\beta; \cdot) x_{mi} = 0 \quad \forall m. \quad (2)$$

Examples of models conforming to this framework notably include binary choice models, count

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\(^8\) \(\varphi\) is not associated with the problem of separation and will henceforth be treated as known. The results we document apply to models with unknown scaling factors without loss of generality. Table 1 gives examples. For more information on this class of models, see P. McCullagh (1989) Section 2.2.2.

\(^9\) Also note that the linear predictor term \(x_i\beta\) is often denoted as \(\eta_i\). We keep it as \(x_i\beta\) to economize on notation.
models such as Poisson and negative binomial, and a variety of other closely related, non-GLM models such as conditional logit, multinomial logit, and the Cox proportional-hazards model. In addition, as the score vectors of many of these models can also be used to construct PML estimators for continuous data, this framework also applies to PML estimators such as Poisson, gamma, Gaussian, inverse Gaussian, and Bernoulli PML without loss of generality. Note that our interest in PML estimators represents an important deviation from Verbeek (1989) because PML estimation does not impose any restrictions on \( c(y_i, \varphi) \). As such, we can consider potential nonexistence problems in models where \( y_i = 0 \) values would otherwise be inadmissible, such as log-link gamma PML and other PML estimators with similar score functions.\(^{10}\)

On top of these general restrictions, we use two further assumptions to derive a necessary and sufficient condition for existence that holds across most of these estimators. First, we assume that the matrix of regressors \( X = x_1, x_2, \ldots, x_M \) is of full column rank. This rank assumption allows us to set aside the more widely understood case of perfectly collinear regressors, although in Section 3, we will find it useful to draw a comparison between nonexistence and perfect collinearity. Second, we assume for the moment that the individual likelihood contributions \( l_i(\beta) \) have a finite upper bound. Later on, we will consider two estimators for which \( l_i \) is not guaranteed to have a finite upper bound, gamma PML and inverse Gaussian PML. We show that the relevant criteria governing existence are not the same as when this assumption is met.

To extend and generalize the earlier result from Santos Silva and Tenreyro (2010) for Poisson models, we are now ready to prove the following proposition:

**Proposition 1 (Nonexistence)** Suppose that \( l(\beta) \) conforms to (1), the matrix of regressors \( X = x_1, x_2, \ldots, x_M \) is of full column rank, and the individual likelihood contribution \( l_i(\beta) \) has a finite upper bound. A solution for \( \beta \) that maximizes (1) will not exist if and only if there exists a linear combination of regressors \( z_i = x_i \gamma^* \) such that

\[
\begin{align*}
z_i &= 0 \quad \forall i \quad \text{s.t.} \quad 0 < y_i < \overline{y}, \quad (3) \\
z_i &\geq 0 \quad \forall i \quad \text{s.t.} \quad y_i = \overline{y}, \quad (4) \\
z_i &\leq 0 \quad \forall i \quad \text{s.t.} \quad y_i = 0, \quad (5)
\end{align*}
\]

where \( \gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_M^*) \in \mathbb{R}^M \) is a nonzero vector of the same dimension as \( \beta \) and where \( \overline{y} \) is an upper bound on \( \mu_i \) that equals 1 for binary choice models (\( \infty \) otherwise).

\(^{10}\)For more on the wide applicability of PML, see Gourieroux et al. (1984), Manning and Mullahy (2001), and Santos Silva and Tenreyro (2006).
The proof of this proposition follows Verbeek (1989), while also drawing on an earlier proof by Silvapulle (1981) specifically for binary choice models. In addition, the necessity of the condition on the boundedness of \( l(\cdot) \) function is due to Clarkson and Jennrich (1991); note that Proposition 2 later in the paper explores the implications of relaxing this assumption.

The general idea is that we want to show that if a vector \( y^* \) satisfying (3)-(5) exists, then the likelihood function \( l(\beta) \) will always be increasing if we search for a maximum in the direction associated with \( y^* \). Otherwise, if no such \( y^* \) exists, then searching in any direction from any starting point in \( \mathbb{R}^M \) under the noted conditions will cause \( l(\beta) \) to eventually decrease, such that the function must reach a maximum for some finite \( \beta^{MLE} \in \mathbb{R}^M \).

To proceed, let \( y = (y_1, y_2, \ldots, y_M) \in \mathbb{R}^M \) be an arbitrary nonzero vector of the same dimension as \( \beta \) and let \( k > 0 \) be a positive scalar. Now consider the function \( l(\beta + ky) \), which allows us to consider how the likelihood changes as we search in the same direction as \( y \) from some initial point \( \beta \). Differentiating \( l(\beta + ky) \) with respect to \( k \), we obtain

\[
\frac{dl(\beta + ky)}{dk} = \sum_i \alpha_i(\varphi) \left[ y_i - b'(\theta_i) \right] \theta'_i x_i y.
\]

Suppose there is a \( y^* \) such that \( z_i = x_i y^* \) satisfies (3)-(5). In this case, setting \( y = y^* \) the above expression becomes

\[
\frac{dl(\beta + ky^*)}{dk} = \sum_{y_i = 0} \alpha_i(\varphi) \left[ -b'(\theta_i) \right] \theta'_i z_i + \sum_{y_i = \bar{y}} \alpha_i(\varphi) \left[ \bar{y} - b'(\theta_i) \right] \theta'_i z_i > 0,
\]

with the inequality following because \( b' \) and \( \theta' \) are both positive and because \( b' = \mu < \bar{y} \). Notice also that the inequality is strict because we must have at least one observation for which \( z_i \neq 0 \); otherwise, our full rank assumption would be violated, and we would be in the case of perfect collinearity. Because this expression is always positive, \( l(\beta + ky^*) > l(\beta) \) for any \( k > 0 \) and for any \( \beta \in \mathbb{R}^M \). Thus, there is no finite solution \( \beta^{MLE} \in \mathbb{R}^M \) that maximizes \( l(\cdot) \), and estimates are said not to exist. Intuitively, the objective function will always be increasing as either \( x_i \beta \rightarrow -\infty \) for at least one observation where \( y_i = 0 \) or \( x_i \beta \rightarrow \infty \) for at least one observation where \( y_i = \bar{y} \).

Alternatively, suppose that, for any \( y \), we always have that \( x_i y \neq 0 \) for at least one interior observation (\( 0 < y_i < \bar{y} \)). Importantly, this ensures that \( \lim_{k \rightarrow \infty} l_i(\beta + ky) = -\infty \) for at least one

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11In Verbeek (1989), the relevant theorems are Theorem 6, which establishes conditions under which the likelihood function has a local maximum that lies on the boundary of the parameter space, and Theorem 4, which establishes that any local maximum on the boundary is a global maximum if the likelihood function is concave. Note that we have relaxed the concavity assumption since it is straightforward to show the weaker result that if there is a local maximum at the boundary, the global maximum can only occur at the boundary.
observation. Since \( l_i(\cdot) \) is continuous in \( \beta \) and (by assumption) has a finite upper bound, we can therefore always identify a finite scalar \( k \) such that \( k > k \) implies that \( l(\beta + ky) = \sum_i l_i(\beta + ky) < \sum_i l_i(\beta) = l(\beta) \), for any \( \beta, y \in \mathbb{R}^M \). In other words, searching for an ML or PML solution \( \beta^{\text{MLE}} \) in any direction from any starting point in \( \mathbb{R}^M \) space will always eventually yield a decrease in \( l(\cdot) \). Because \( l(\cdot) \) is continuous, this guarantees the existence of a finite \( \beta^{\text{MLE}} \in \mathbb{R}^M \) maximizing \( l(\cdot) \).

Next, note that, for any \( y_i = 0 \) observation such that \( x_iy > 0 \), \( l_i(\beta + ky) \) is monotonic in \( k \) with \( \lim_{k \to \infty} l_i(\beta + ky) = -\infty \). Similarly, note that \( \mu < \bar{y} \) ensures the same is true for any \( y_i = \bar{y} \) observation such that \( x_iy < 0 \).\(^{12}\) Thus, we can again always find a sufficient \( k \) such that \( k > k \) implies \( l(\beta + ky) < l(\beta) \) so long as we always have that either \( x_iy > 0 \) for at least one observation where \( y_i = 0 \) or \( x_iy < 0 \) for at least one observation where \( y_i = \bar{y} \).\(^{13}\) Finally, note that we do not consider the case where there exists a vector \( y \) such that \( x_iy = 0 \) for all \( i \), as this is the case where \( X \) is not of full rank. Therefore, the only possible scenario in which estimates do not exist is the one where we can find a linear combination of regressors \( z_i = x_iy^* \) satisfying (3)-(5).\(^\Box\)

To tie in some standard terminology from the binary choice literature (cf., Albert and Anderson, 1984), we will say that when estimates maximizing (1) do not exist, the linear combination of regressors defined by \( z_i = x_iy^* \) “separates” the observations for which \( z_i \geq 0 \) from the rest of the sample. For the sake of providing a more unified perspective, we will henceforth refer to the nonexistence with the term “separation”. A particular point of interest for us is how to also adapt the related terms “complete separation” and “quasi-complete separation” to this more general context. For binary choice models, separation is usually considered “complete” if either \( z_i < 0 \) for all \( y_i = 0 \) or \( z_i > 0 \) for all \( y_i = \bar{y} = 1 \), since in these cases the value of \( z_i \) perfectly predicts whether \( y_i \) is 0 or 1. Otherwise, we have only “quasi-complete separation”, where only some \( y_i \) outcomes are perfectly predicted. Outside of binary choice models, however, as long as \( y_i \) takes on at least two positive values, it will never be the case that \( z_i = 0 \) perfectly predicts all positive \( y_i \), regardless of whether \( z_i < 0 \) perfectly predicts all \( y_i = 0 \) outcomes or only some of them. Thus, for lack of an analogous vocabulary for discussing separation in the nonbinary choice case, we would suggest that separation occurring in these models should generally be regarded as “quasi-complete”.

In addition, for those readers more familiar with Santos Silva and Tenreyro (2010)’s results for

\(^{12}\)To be clear, if \( \bar{y} = \infty \), we never have that \( y_i = \bar{y} \); only conditions (3) and (4) are salient. On the other end of the spectrum, models for “fractional” data such as Bernoulli PML (cf., Papke and Wooldridge 1996; Santos Silva et al. 2014) allow the dependent variable to vary continuously over \([0, 1]\). For these models, all three conditions stated in Proposition 1 are relevant.

\(^{13}\)Readers should be wary of the weight carried by the word “always” here. It could be the case, for example, that \( x_iy = 0 \) for all \( y_i > 0 \) with \( x_iy \geq 0 \) for all \( y_i = 0 \). This is still a case where estimates do not exist, since \( y^* = -y \) would satisfy the needed conditions.
Poisson models specifically, another term that is useful for us to clarify for the nonbinary choice context is “overlap”. In Santos Silva and Tenreyro (2010), Poisson estimates are shown to exist so long as there are no regressors that are perfectly collinear over the subsample where \( y_i > 0 \). In our way of phrasing the issue, this criterion equates to saying there exists no linear combination of regressors satisfying equation (3). However, as Santos Silva and Tenreyro (2010) are careful to note, this criterion is only sufficient, rather than necessary and sufficient. As the remaining elements of the preceding proof show, even if such a linear combination exists, separation is still avoided if \( z_i \) takes on both positive and negative values when \( y_i = 0 \), such that its maximum and minimum values over \( y_i = 0 \) “overlap” the \( z_i = 0 \) values it takes on when \( y_i > 0 \).14

Interestingly, we do not know of a widely accepted label for what we have called the “linear combination of regressors that separates the data” (i.e., \( z_i = x_i y^* \)). Clearly, \( z \) plays a central role in the analysis of separation, and the literature could use a concise name for it. We propose the term “certificate of separation.” The idea is that we can easily “certify” whether any such \( z \) separates the data by verifying that i) its values satisfy (3)-(5), and ii) that the \( R^2 \) of a regression of \( z \) against the regressors \( x \) is equal to one.15 Note that there can be multiple \( z \)’s certifying separation of different observations, and that adding up two or more \( z \)’s preserves their properties.16 Thus, we refer to an “overall certificate of separation” \( \mathbf{z} \) that can be used to identify all separated observations. For any \( y^* \) associated with a certificate of separation, we will tend to use the term “separating vector” (although another name for it is the “direction of recession”; see Geyer, 2009). We will use \( \mathbf{z} \) to denote the separating vector associated with \( \mathbf{z} \).

**Results for gamma PML and inverse Gaussian PML.** One stipulation that sticks out in Proposition 1 is our requirement that the individual likelihood contribution \( l_i(\cdot) \) have a finite upper bound. To our knowledge, the implications of relaxing this assumption have not been touched upon in the prior literature. Rewinding some of the last few details behind the above proof, the specific role played by this restriction is that it ensures that if \( \lim_{k \to \infty} l_i(\beta + ky) = -\infty \) for any \( i \), the overall objective function \( l(\beta + ky) = \sum_i l_i(\beta + ky) \) also heads toward \(-\infty\) for large \( k \). However, this might not hold if \( l_i(\cdot) \) is not bounded from above. In this case, even if the data exhibit “overlap” (as defined above), this alone will not be sufficient to ensure that \( l(\cdot) \) has a

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14The question of when overlap occurs is precisely the point left ambiguous in Santos Silva and Tenreyro (2010). See page 311 of their paper.

15Our proposed name borrows from optimization, where phrases such as “certificate of feasibility,” “certificate of convexity,” “certificate of nonnegativity,” and so on are used with a similar purpose.

16If \( z^* \) and \( z^{**} \) are valid certificates of separation with associated coefficients \( y^* \) and \( y^{**} \), then \( z^{***} = z^* + z^{**} \) is also a valid certificate of separation, as (3)-(5) hold trivially and \( z^{***} \) is a linear combination of \( x \) with associated coefficient \( y^{***} = y^* + y^{**} \).
maximum. Instead, stronger conditions may be needed.

For illustration, the two models we will consider where \( l_i(\cdot) \) does not necessarily have a finite upper bound are gamma PML and inverse Gaussian PML.\(^{17}\) As shown in Table 1, the form of the pseudo-likelihood function for gamma PML regression is

\[
l(\beta) = \sum_i l_i(\beta) = \sum_i -\alpha y_i \exp(-x_i \beta) - \alpha x_i \beta \tag{6}
\]

and the form for inverse Gaussian PML is

\[
l(\beta) = \sum_i l_i(\beta) = \sum_i -\alpha \frac{y_i}{2} \exp(-2x_i \beta) + \alpha \exp(-x_i \beta). \tag{7}
\]

In either case, notice that the associated \( b_i \) function from equation (1), \( x_i \beta \) for gamma and \(-\exp(-x_i \beta)\) for inverse Gaussian, has a lower bound of \(-\infty\), as \( \lim_{x_i \beta \to -\infty} b_i = -\infty\). Thus, in either case, while \( l_i(\cdot) \) continues to have a finite upper bound for observations where \( y_i > 0 \), if \( y_i = 0 \), we have that \( \lim_{x_i \beta \to -\infty} l_i(\cdot) = \infty \). With this in mind, the following Proposition collects results that apply to either of these estimators:

**Proposition 2** (Gamma PML and inverse Gaussian PML) Suppose the matrix of regressors \( X = x_1, x_2, \ldots, x_M \) is of full column rank. Also let \( \gamma^* = (\gamma_1^*, \gamma_2^*, \ldots, \gamma_M^*) \in \mathbb{R}^M \) be a nonzero vector of the same dimension as \( \beta \).

(a) If \( l(\beta) \) conforms to gamma PML as stated by (6), PML estimation of \( \beta \) will not have a solution if and only if there exists a linear combination of regressors \( z_i = x_i \gamma^* \) such that

\[
z_i \geq 0 \quad \forall \ i \quad \text{s.t.} \quad y_i > 0 \tag{8}
\]

and either of the following two conditions holds:

\[
\sum_i z_i < 0 \quad \text{or} \quad \sum_i z_i = 0 \text{ with } z_i > 0 \text{ for at least one observation with } y_i > 0. \tag{9}
\]

In addition, if only (8) can be satisfied and \( z_i = 0 \) for all \( y_i > 0 \), PML estimates of \( \beta \) exist but are nonunique.

\(^{17}\)Note that ML estimation of either a gamma distribution or an inverse Gaussian distribution will not admit \( y_i = 0 \) values. Thus, we consider PML versions of these estimators only. In general, what gamma PML and inverse Gaussian PML have in common is that their score functions place a relatively larger weight on observations with a smaller conditional mean. Similar results will apply to other estimators with comparable score functions.
(b) If \( l(\beta) \) conforms to inverse Gaussian PML (i.e., (7)), PML estimation of \( \beta \) will have no solution if and only if there exists a linear combination of regressors \( z_i = x_i \gamma^* \) such that \( z_i \) satisfies (8) and at least 1 \( z_i \) is \( < 0 \) when \( y_i = 0 \).

Part (a) of Proposition 2 follows from again considering the function \( l(\beta + k\gamma) \), this time specifically for gamma PML. Using (6), it is straightforward to show that \( \lim_{k \to \infty} l(\beta + k\gamma) = -\infty \) if \( x_i \gamma < 0 \) for at least one observation with \( y_i > 0 \). By a continuity argument similar to the one used above, this implies that \( l(\beta + k\gamma) \) must eventually become decreasing in \( k \) for sufficiently large \( k \).

Next, consider what happens if there exists a linear combination of regressors \( z_i = x_i \gamma^* \), which is always \( \geq 0 \) when \( y_i > 0 \). In this case, because \( \lim_{k \to \infty} \sum_{z_i \neq 0} -\alpha y_i \exp(-x_i \beta - k z_i) = 0 \), we have that
\[
\lim_{k \to \infty} \frac{d}{dk} l(\beta + k\gamma) = \lim_{k \to \infty} \sum_{i} -\alpha (x_i \beta - k z_i) + \sum_{z_i = 0} -\alpha y_i \exp(-x_i \beta).
\]
There are four possibilities for the above limit. If \( \sum_i z_i < 0 \), the gamma pseudo-likelihood function is always increasing in the direction associated with \( \gamma \), such that finite estimates do not exist. Alternatively, if \( \sum_i z_i > 0 \), the limit equals \( -\infty \) and we are again assured that this function must eventually decrease with \( k \), such that estimates will exist.

The remaining two possibilities occur when \( \sum_i z_i = 0 \). In this case, the effect of an increase in \( k \) on the likelihood function is always given by
\[
\frac{dl(\beta + k\gamma)}{dk} = \sum_{z_i > 0} \alpha y_i \exp(-x_i \beta - k z_i) z_i \geq 0.
\]
Inspecting the above expression, \( dl(\beta + k\gamma)/dk > 0 \) with strict inequality if \( z_i > 0 \) for at least one observation with \( y_i > 0 \), ensuring again that finite estimates do not exist. The final possibility is if \( z_i = 0 \) for all \( y_i > 0 \) observations, in which case \( l(\beta + k\gamma) = l(\beta) \) for any \( k > 0 \). In other words, regardless of which initial \( \beta \) we consider, the likelihood will always be weakly higher when we increment \( \beta \) by some positive multiple of \( \gamma \), implying either that a finite solution for \( \beta \) maximizing \( l(\cdot) \) does not exist (if \( \sum_i z_i = 0 \) with \( z_i > 0 \) for at least one \( y_i > 0 \)) or that any finite solution will be nonunique (if \( z_i = 0 \) for all \( y_i > 0 \)). Thus, taking all of these results together, gamma PML estimation of \( \beta \) will not have a finite solution if there exists a linear combination of regressors satisfying (8) and (9) and may not necessarily have a unique solution even if these conditions are not met.

For proving part (b), which pertains instead to inverse Gaussian PML, it is again convenient to work with the derivative of the \( l(\beta + k\gamma) \) function with respect to \( k \). Continuing to let \( z_i = x_i \gamma^* \),
and after dividing up terms appropriately, this derivative can be expressed as

\[
\frac{dl(\beta + ky)}{dk} = -\alpha \sum_{y_i=0} \exp(-x_i \beta - kz_i) z_i + \alpha \sum_{y_i>0} y_i \exp(-2x_i \beta - 2kz_i) z_i - \alpha \sum_{y_i>0} \exp(-x_i \beta - kz_i) z_i.
\]

(10)

Let us start with the conditions highlighted in part (b), where \( z_i \geq 0 \) for all \( y_i > 0 \) and where \( z_i < 0 \) for at least 1 observation where \( y_i = 0 \). We can see that the second and third terms in (10) will go to 0 in the limit where \( k \) becomes infinitely large. The first term, meanwhile, heads to infinity. Thus, the pseudo-likelihood function increases asymptotically for large \( k \), and it is clear there is no finite solution for \( \beta \).

However, we still need to verify what happens if we cannot find a linear combination \( z_i \) satisfying both of the conditions stated in part (b). This part requires slightly more work. If \( z_i \geq 0 \) for all \( i \), for example, all three terms in (10) go to zero for \( k \to \infty \)—a result that is not in itself all that informative. Likewise, if we consider what happens when \( z_i \) may be less than zero for \( y_i > 0 \), the first and third terms could potentially head toward \(+\infty\), while the second term heads toward \(-\infty\). In all of these seemingly ambiguous scenarios, we can use L’Hôpital’s rule to clarify that \( \frac{dl(\beta + ky)}{dk} < 0 \) for sufficiently large \( k \), indicating that the pseudo-likelihood function will always eventually decrease in the direction associated with \( \gamma \).

\[\blacksquare\]

To our knowledge, we are the first to study the general circumstances under which estimates from gamma PML and inverse Gaussian PML exist.\(^{18}\) That these estimators have not been specifically looked at in this context is perhaps not all that surprising, since these models have not traditionally been used with zeroes and since the increase in popularity of PML estimation in applied work has only occurred relatively recently. Indeed, thanks to contributions such as Manning and Mullahy (2001), Santos Silva and Tenreyro (2006), and Head and Mayer (2014), the main context in which researchers will likely be familiar with gamma PML is in settings where zeroes are common, such as data for international trade flows and health care costs. Inverse Gaussian PML is also sometimes considered for these types of applications (see Egger and Staub, 2015) but is significantly less popular, likely because it is more difficult to work with numerically.

In this light, the results contained in Proposition 2 can be read in one of two ways. On the one hand, we confirm that gamma PML and inverse Gaussian PML can, in principle, be used with datasets that include observed zeroes, even though their ML equivalents cannot. Since the ability

\(^{18}\)Even Wedderburn (1976), in his original derivation of a sufficient condition for the existence of GLM estimates, specifically avoids commenting on what conditions would be needed for gamma estimates to exist if the dependent variable is allowed to be zero.
to admit zeroes on the dependent variable is one of the reasons researchers have recently become curious about these estimators, this confirmation seems useful.\textsuperscript{19} On the other hand, we can see from a comparison of Propositions 1 and 2 that the criteria required for gamma PML and inverse Gaussian PML to have finite solutions are considerably more strict than the equivalent criteria required for most other standard GLM estimators. Furthermore, as we will see in the next section, these fundamental differences also imply that gamma PML and inverse Gaussian PML lack some appealing properties that enable us to more easily remedy situations where estimates do not exist for other models. For these reasons, we recommend researchers to exercise extra caution when using either of these two estimators with datasets that include zeroes in the dependent variable.

3 Addressing separation in practice

This section describes recommendations for dealing with separation in practice, including in high-dimensional environments with many fixed effects and other nuisance parameters. Before digging into these details, it is important to make two general points. First, as we have shown, the implications of separation differ depending on the estimator; thus, the appropriate remedy should similarly depend on the estimator being used. Second, the appeal of our own preferred alternative—dropping separated observations beforehand—is likely to depend on one’s comfort level with allowing the linear predictor $x_i\beta$ to attain what would ordinarily be an inadmissible value. One method we caution against is to simply remove one of the regressors implicated in separation from the model, as this affects the identification and estimation of all remaining parameters, with the effect differing depending on which regressor is dropped.\textsuperscript{20}

In subsection 3.1, we will first show that when $x_i\beta$ is allowed to attain $\pm\infty$, separated observations often do not affect the score function for $\beta$ under fairly general circumstances. This insight provides a theoretical justification for the practice of dropping separated observations from the estimation, at which point the separation problem becomes one of perfect collinearity. This is particularly useful for models with many fixed effects, as perfect collinearity amongst the fixed effects is generally not a problem for fixed effects estimators. Once these results are established, subsections 3.2 and 3.3 then focus on detecting and addressing separation, including in high-dimensional environments.

\textsuperscript{19}The other main reason is that the traditional practice of applying a log transformation to the dependent variable and estimating a linear model is now widely known to introduce a bias whenever the error term is heteroskedastic.

\textsuperscript{20}Furthermore, in fixed effects models computed using dimensionality-reducing techniques (e.g., Figueiredo et al., 2015; Larch et al., 2019; Stammann, 2018), even identifying which combinations of the $x_{mi}$’s induce separation may be infeasible.
3.1 Effects of dropping separated observations

We now turn to discussing how identification of at least some of the model parameters can be achieved when separation occurs. We start with the concept established in Verbeek (1989) and Clarkson and Jennrich (1991) of a “compactified” (or “extended”) GLM where the parameter space is extended to admit its boundary values. We can phrase this compactification in one of several equivalent ways. For example, we could express the domain for $\beta$ as $[-\infty, +\infty]^M$, the compact closure of $\mathbb{R}^M$. However, it is also often convenient to work with the linear predictor $x_i \beta$, which in turn also may vary over $[-\infty, +\infty]$ for each $i$. In particular, note how the conditional mean $\mu_i$ behaves as $x_i \beta$ attains either of its two limits: when $x_i \beta \to -\infty$, we have that $\mu_i \to 0$, whereas when $x_i \beta \to \infty$ (a situation that is only relevant for binary choice models and fractional data models), we have that $\mu_i \to y$. It is straightforward to show that estimates always exist when we compactify the model in this way.

With this adjustment to the parameter space in mind, consider what happens to the score function $s(\beta)$ and information matrix $F(\beta) := \mathbb{E}[\partial s(\beta)/\partial \beta]$ in the limit as $k \to \infty$ in the case of separation outlined above. In other words, consider

$$\lim_{k \to \infty} s(\beta + ky^*) := \lim_{k \to \infty} \sum_i s_i(\beta + ky^*) := \lim_{k \to \infty} \sum_i \alpha_i(\varphi) [y_i - \mu_i(x_i \beta + ky^*)] \theta'(x_i \beta + ky^*; \cdot)x_i$$

and

$$\lim_{k \to \infty} F(\beta + ky^*) := \lim_{k \to \infty} \sum_i F_i(\beta + ky^*) := -\lim_{k \to \infty} \sum_i \alpha_i(\varphi) \mu_i'(x_i \beta + ky^*)\theta'(x_i \beta + ky^*; \cdot)x_ix_i^T$$

where we take $y^*$ to be a vector satisfying the applicable conditions for nonexistence. At this point, it will also be useful to state the following lemma:

**Lemma 1** Suppose that $l(\beta)$ conforms to (1). If the likelihood contribution $l_i(\beta)$ has a finite upper bound, then:

(a) The respective limits of the $i$-specific score term, $s_i$, and $i$-specific information term, $F_i$, each go to 0 as the linear predictor $x_i \beta$ goes to $-\infty$ (i.e., $\lim_{x_i \beta \to -\infty} s_i = 0$ and $\lim_{x_i \beta \to -\infty} F_i = 0$).

---

21 As discussed in Verbeek (1989), one way to justify the inclusion of infinitely large values in the admissible parameter space is to observe that we could just as easily perform the maximization over a homeomorphic space where the parameters of interest are instead bounded by a finite interval (e.g., $[-1, 1]^M$ instead of $[-\infty, \infty]^M$). A version of this concept is also described in Haberman (1974). It is also sometimes referred to as the “Barndorf-Nielsen completion” (Barndorff-Nielsen, 1978).
(b) For models where \( \lim_{x_i \beta \rightarrow \infty} \mu_i = \bar{y} < \infty \) (e.g., binary choice models), the limits of \( s_i \) and \( F_i \) go to 0 as \( x_i \beta \) goes to \( \infty \) as well (i.e., \( \lim_{x_i \beta \rightarrow \infty} s_i = 0 \) and \( \lim_{x_i \beta \rightarrow \infty} F_i = 0 \)).

The utility of this lemma (which we prove in our Appendix) is that, together with (11) and (12), it delivers the following proposition:

Proposition 3 (Effects of dropping separated observations) Suppose the assumptions stated in Proposition 1 continue to hold, except we now consider a “compactified” GLM where the domain for \( \beta \) is \([−\infty, +\infty]^M\). Further, assume the joint likelihood of any non-separated observations satisfies the classical assumptions described in Gourieroux et al. (1984). If there exists a separating vector \( \gamma^* \in \mathbb{R}^M \) meeting the conditions described in Proposition 1, then:

(a) A solution for \( \beta \in [−\infty, +\infty]^M \) maximizing \( l(\beta) \) always exists.

(b) ML and PML estimates for the linear predictors \((x_i \beta)\), canonical parameters \((\theta_i)\), and conditional means \((\mu_i)\)’s of any observations not separated by \( \gamma^* \) (i.e., those with \( x_i \gamma^* = 0 \)) are unaffected by dropping any observations that are separated by \( \gamma^* \) (i.e., those with \( x_i \gamma^* \neq 0 \)).

(c) For any \( m \) with \( \gamma^*_m = 0 \), the associated individual parameter estimate \( \beta_m \) is unaffected by dropping any observations with \( x_i \gamma^* \neq 0 \).

(d) If \( l(\beta) \) is in the linear exponential family and if the relationship between the linear predictor \( x_i \beta \) and the conditional mean \( \mu(x_i \beta) \) is correctly specified, then all finite elements of \( \beta \) are consistently estimated and their asymptotic confidence intervals can be inferred using the subsample of non-separated observations.

Part (a) follows from our proof of Proposition 1 (and is also a central result from Verbeek, 1989). After allowing \( \beta \) to take on either \(-\infty \) or \(+\infty \), we rule out the cases where estimates would otherwise be said not to exist. Parts (b) and (c) are analogous to the insights contained in Clarkson and Jennrich (1991)’s Theorem 2. After invoking Lemma 1, the score function in (11) can be rewritten as

\[
\lim_{k \rightarrow \infty} s(\beta + k\gamma^*) = \sum_{x_i \gamma^* = 0} s_i(\beta) + \lim_{k \rightarrow \infty} \sum_{x_i \gamma^* \neq 0} s_i(\beta + k\gamma^*) = \sum_{x_i \gamma^* = 0} s_i(\beta). \tag{13}
\]

The key insight presented in (13) is that the contribution of any observation with \( x_i \gamma^* \neq 0 \) always drops out of the overall score function under these circumstances. As a result, it must be the case
that any $\beta$ that maximizes $l(\beta)$ in the compactified model must also maximize $\sum_{x_iy^* = 0} l_i(\beta)$ (i.e., the likelihood associated with the observations not separated by $x_iy^*$). Otherwise, we would have that $\sum_{x_iy^* = 0} s_i(\beta) \neq 0$ and $\sum_{x_iy^* \neq 0} s_i(\beta) = 0$, implying that the joint likelihood of the non-separated observations can be increased without affecting that of the separated observations.

Parts (b) and (c) then follow because if $\beta = \beta^*$ maximizes the likelihood of the non-separated observations $\sum_{x_iy^* = 0} l_i(\beta)$, then any coefficient vector of the form $\beta^* + ky^*$ maximizes it as well. That is, the estimates of $\beta$ will be different with the separated observations than without them. But the quantities $x_i\beta, \theta_i,$ and $\mu_i$ will not be affected, as stated in part (b), since $x_i(\beta^* + ky^*) = x_i\beta^*$ over the subsample where $x_iy^* = 0$ (and since $\theta_i$ and $\mu_i$ are functions of $x_i\beta$). Consequently, for any $m$ such that $y_m^* = 0$, the individual parameter estimate $\beta_m^* + ky_m^* = \beta_m^*$ is clearly the same in either case, as stated in part (c).

To prove part (d), we consider a suitable reparameterization of the linear predictor $x_i\beta$ that preserves the same information about any $\beta_m$’s associated with regressors that are not involved in separation. Let $S < M$ be the number of regressors for which no separating vector $y^*$ exists with $y_m^* \neq 0$. We need to allow for the possibility that there could be many such separating vectors affecting the data. Without loss of generality, we can make the following assumptions:

- $z_i^{(1)} = \sum_m x_{mi}y_m^{(1)}, z_i^{(2)} = \sum_m x_{mi}y_m^{(2)}, \ldots, z_i^{(j)} = \sum_m x_{mi}y_m^{(j)}$ are $J \leq S$ distinct, linearly independent certificates of separation.

- $\bar{z}_i := \sum_j z_i^{(j)}$ is the “overall” certificate that identifies all separated observations in the data, and $\bar{y} := \sum_j y_m^{(j)}$ is its associated separating vector.

- The $x_i$’s and $z_i$’s are such that $y_j^{(j)} = 1$ and $y_{-j}^{(j)} = 0$ for all $j, -j \in 1 \ldots J$ and $j \neq -j$. Furthermore, $y_m^{(j)} = 0$ for all $j \in 1 \ldots J$ and for all $m > S$.

The reparameterized linear predictor $\bar{x}_i\tilde{\beta}$ can be obtained by adding and subtracting $\sum_{m=1}^J \beta_mz_i^{(m)}$:

$$
\bar{x}_i\tilde{\beta} := x_i\beta + \sum_{m=1}^J \beta_mz_i^{(m)} - \sum_{m=1}^J \beta_mz_i^{(m)} = \sum_{m=1}^J \beta_mz_i^{(m)} + \sum_{m=S+1}^M x_{mi}\bar{\beta}_m + \sum_{m=S+1}^M x_{mi}\beta_m, \tag{14}
$$

where each $\bar{\beta}_m := \beta_m - \sum_{j=1}^J \beta_jy_m^{(j)}$ must now be interpreted as a combination of multiple different parameters. The new set of regressors is $\bar{x}_i := (z_i^{(1)}, \ldots, z_i^{(J)}, x_{j+1}, \ldots, x_{M})$. Under this reparameterization, we know that $\beta_m = \infty$ for $m \in 1 \ldots J$ and $\beta_m = -\infty$ for $m \in J+1 \ldots S$. The combined $\bar{\beta}_m$ parameters will have finite estimates, however. What’s more, the first-order conditions used to identify estimates of $\beta_{S+1}, \ldots, \beta_M$ are unaffected by this transformation.
Similarly, a standard asymptotic variance is of no use for estimators with potentially unbounded (pseudo-)likelihood. However, a similar result can be obtained for an even more general class of GLMs by extending the proofs of Gourieroux et al. (1984)’s proof of consistency for linear exponential families applies directly after restricting the sample to only the non-separated observations. \(^{22}\) Similarly, a standard asymptotic variance expansion gives us

\[
\left( N_{\bar{z}_i=0} \right)^{\frac{1}{2}} \left( \hat{\beta}^F - \beta^F \right) \rightarrow_d \mathcal{N} \left( 0, \bar{F}^{-1} \bar{B}^{-1} \right),
\]

where \( \bar{F} := \mathbb{E} \left[ \sum_{\bar{z}_i=0} \partial \hat{s}_i(\beta^F)/\partial \beta^F \right] \) is a reduced information matrix pertaining only to the finite parameters and \( \bar{B} := \mathbb{E} \left[ \sum_{\bar{z}_i=0} \hat{s}_i(\beta^F) \hat{s}_i(\beta^F)^T \right] \) captures the variance of the modified score.

For researchers encountering separation problems, the key takeaways from Proposition 3 are likely to be parts (c) and (d): even if one or more of the elements of the MLE for \( \beta \) “does not exist” (i.e., is \( \pm \infty \)), it is still often the case that \( \beta \) has some finite elements that are identified by the model’s first-order conditions and that can be consistently estimated. Specifically, as long as separation is “quasi-complete” and there are at least some observations with \( x_i \bar{y} = 0 \), coefficients for regressors that do not play a role in the separation can be consistently estimated by first dropping any separated observations, and then performing the estimation over the subsample where \( x_i \bar{y} = 0 \). Meanwhile, for the parameters that are estimated to be infinite, one can often still estimate finite combinations of the parameters and perform normal inference on them. \(^{23}\)

The practical implications of these insights vary based on the model and estimator. For binary choice models, if the data exhibit complete separation instead of only quasi-complete separation, then meaningful estimation is impossible with or without the separated observations. Furthermore, Proposition 3 is of no use for estimators with potentially unbounded (pseudo-)likelihood

\(^{22}\)Most GLMs typically used in applied economic research are from the linear exponential family (e.g., Poisson, probit, logit, and negative binomial). However, a similar result can be obtained for an even more general class of GLMs by extending the proofs of Fahrmeir and Kaufmann (1985).

\(^{23}\)For example, in a Poisson model where \( \mu = \exp \left[ \beta_0 + \beta_1 y_1 + \beta_2 x_2 + \beta_3 x_3 \right] \) and \( z_1 = x_{11} + x_{12} \) is a linear combination of \( x_{11} \) and \( x_{12} \) that equals 0 for all \( y_1 = 0 \) and is < 0 for some \( y_1 = 0 \), then \( \exp \left[ \beta_0 + \beta_1 z_1 + (\beta_2 - \beta_1) x_2 + \beta_3 x_3 \right] \) is a reparameterization of \( \mu \) that presents the same information about \( \beta_1 \). Here, we know that \( \beta_1 \) and \( \beta_2 \) will both be \( \infty \). The combined parameter \( \beta_2 - \beta_1 \) is finite, however, and the reparameterized model allows us to take into account its covariance with \( \beta_1 \) in drawing inferences.
functions such as gamma PML, as in these cases the compactified model will have infinitely many solutions when there is separation of any kind. However, as we have discussed, the degree of separation for many other commonly used GLMs can only be quasi-complete. A Poisson model, for example, can always be estimated by first identifying and dropping separated observations from the estimation sample. For these situations, Proposition 3 lends significant theoretical justification to this approach, especially when the researcher’s focus is only on a particular subset of regressors (as is often the case with fixed effect models, for example).

To flesh out some additional intuition behind these results, it is helpful to draw a connection between separation and the better understood result of perfect collinearity between regressors. Under perfect collinearity, there is at least one redundant regressor which, given the other regressors, conveys no additional information about the observed outcomes. Therefore, the estimated effect of the redundant regressor could theoretically take any value without affecting the score function or the estimates of variables it is not collinear with. Separation is similar in that, because the regressors implicated in \( x_i \) are only identifiable from the observations where \( x_i \neq 0 \), they therefore provide no information about the remaining parameters that are not involved in separation. The two issues are still fundamentally distinct, since separation involves estimates of the problematic regressors becoming infinite rather than indeterminate. In either case, however, it is important that a researcher note that the choice of which regressor to drop from the estimation is often arbitrary and that the computed coefficients of some of the remaining regressors (i.e., those that are involved in either separation or perfect collinearity) may need to be interpreted as being relative to an omitted regressor or omitted regressors, as shown in (14).

As a final remark, the above proof also suggests that separation has implications for the computation of standard errors. To fix ideas, consider a simple panel data model with \( \mathbb{E}(y_{it}|x_{it}, \delta_i) = \exp(\delta_i + x_{it}\beta) \), with \( \delta_i \) a fixed effect for individual \( i \) and \( t \) an additional index for time. Let \( I \) be the number of individuals, and let the data-generating process for \( y_{it} \in [0, +\infty) \) be such that it takes on zero values with some positive probability (e.g., as in count data). Under these assumptions, it is straightforward to show that separation involving at least one \( \delta_i \) will occur, almost surely, as \( I \to \infty \). Intuitively, if it randomly happens that all \( y_{it} \) values for a given \( i \) are 0, the first-order

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24 More precisely, this occurs when \( x_{i,\bar{y}} \neq 0 \) over the entire sample, for some nonzero vector \( \gamma \).

25 Another similarity between separation and perfect collinearity is that separation is neither strictly a “small sample” issue nor a “large sample” issue. It may be resolved by obtaining a larger sample if the underlying reason is that there is not enough variation in one or more of the regressors in the current sample. However, it may also occur in large samples either because of fundamental co-dependence between \( y_i \) and some of the regressors (e.g., a trade embargo may always predict zero exports) or because the number of regressors increases with the sample size (as is typically the case with panel data models).
condition for the associated $\delta_i$ is only satisfied for $\delta_i \rightarrow -\infty$. Fortunately, most software packages are usually able to recover the correct estimate of $\beta$ in this relatively simple case by forcing the estimates of any such $\delta_i$’s to large negative values. However, as we have noted above, the correct standard error for the estimated $\hat{\beta}$ should be computed using $F_s = 0$ (i.e., the number of fixed effect groups for which at least one $y_{i,t}$ is positive). If separation has gone undetected, the researcher will instead receive standard errors computed using $I$, which are guaranteed to have at least some downward bias. The more general point is that even if a conventional estimation algorithm is able to recover some of the correct estimates, the computed standard errors are not to be trusted.

3.2 Detecting separation with linear programming

The discussion thus far has been strictly theoretical, but the practical aspects of the separation problem are also interesting. To date, most discussion in the existing literature about how to detect separation has focused on binary choice models, where the only relevant conditions governing separation are (4) and (5). However, for nonbinary choice applications, there will usually be many observations with $0 < y_i < \bar{y}$, such that the third condition stated in (3) becomes key. In some cases, this condition can greatly simplify the task of detection. For instance, Santos Silva and Tenreyro (2010) show that if $X$ is of full column rank over $0 < y_i < \bar{y}$, then equation (3) cannot be satisfied and there is no separation. Likewise, if the rank of $X$ over $0 < y_i < \bar{y}$ is $M - 1$, such that there is only one $\gamma^*$ that satisfies equation (3), it is generally easy to compute values for $z_i = x_i \gamma^*$ over the rest of the sample and check whether or not they satisfy the other conditions for separation.

However, detecting separation becomes much more complicated if there are multiple linear combinations of regressors that satisfy equation (3) (i.e., if rank($X$) < $M - 1$ over $0 < y_i < \bar{y}$). Table 2 gives a simple example of a dataset that presents this issue. In this instance, a check for perfectly collinear regressors over $y_i > 0$ would quickly reveal that both $z_{1i} = x_{3i} - x_{4i}$ and $z_{2i} = x_{2i} - x_{4i}$ are always 0 over $y_i > 0$. The second- and third-to-last columns of Table 2 then show that both $z_{1i}$ and $z_{2i}$ exhibit overlap over $y_i = 0$, suggesting that estimates should exist. However, just by virtue of there being two such linear combinations of regressors satisfying (3), then any other linear combination $z_{3i}$ of the form $z_{3i} = ax_{1i} + (1 - a)z_{1i}$ also satisfies (3). Thus, there are actually an infinite number of linear combinations of regressors one would need to check for overlap in this manner in order to determine existence. In this particular example, it is still possible to determine without too much effort that $z_{3i} = 0.5z_{1i} + 0.5z_{2i}$ separates the first observation. But for more general cases, a more rigorous approach is needed to take into account the many different ways
the data could be separated.

In light of these complexities, Silvapulle and Burridge (1986) and Clarkson and Jennrich (1991) have suggested using linear programming methods to detect separation. A suitable illustration of these approaches can be expressed using the following constrained maximization problem:

$$\max_{\gamma^S} \sum_{y_i=0} \mathbb{1}_{x_i^s < 0} + \sum_{y_i = \bar{y}} \mathbb{1}_{x_i^s > 0}$$

s.t. $-x_i^s \geq 0$ if $y_i = 0$

$$x_i^s \geq 0 \quad \text{if} \quad y_i = \bar{y}$$

$$x_i^s = 0 \quad \text{if} \quad 0 < y_i < \bar{y}. \quad (15)$$

where $\mathbb{1}_{x_i^s < 0}$ and $\mathbb{1}_{x_i^s > 0}$ respectively denote indicator functions for observations with $x_i^s < 0$ and $x_i^s > 0$. If a nonzero vector, $\gamma^S$, can be found that solves the problem defined by (15), then the linear combination $x_i^s \gamma^S$ clearly satisfies the conditions for separation described in Proposition 1.

Furthermore, since $\gamma^S$ must maximize the number of separated observations, it follows that $\gamma^S = \bar{y}$. A simplex solver or a variety of other similar linear programming methods may be used to solve for $\gamma^S$; see Konis (2007) for a thorough discussion.

A common weakness of linear programming methods in this context is that they suffer from the curse of dimensionality. Notice that the number of constraints associated with (15) is equal to the number of observations, $N$, and the number of $\gamma$-parameters that need to be solved for is equal to the number of regressors, $M$. While there are standard operations that may be used to reduce the size of the problem to one with only $N - M$ constraints (cf., Konis, 2007, p. 64), an obvious problem nonetheless arises if either $M$ or $N - M$ is a large number, as is increasingly the case in applied economics research.26 In these cases, the standard approach just described necessitates solving a high-dimensional linear programming problem, which may be difficult to solve even using the most computationally efficient linear programming solvers currently available.27 The following discussion, therefore, turns to the question of how to equip researchers to deal with the separation problem in models with many fixed effects and other nuisance parameters.

26As noted in the introduction, this popularity is largely driven by the wide adoption of fixed effects Poisson PML (FE-PPML) estimation for estimating gravity models. For example, Figueiredo et al. (2015) estimate a gravity model for patent citations with $N \approx 26$ million and $M \approx 27,000$, and Larch et al. (2019) estimate a similar model for international trade flows with $N \approx 880,000$ and $M \approx 55,000$. However, high-dimensional fixed effects estimation is also likely to become more attractive for other GLM estimators aside from PPML as well; see Stammann et al. (2016), Stammann (2018), and Fernández-Val and Weidner (2016) for some relevant innovations that have appeared in the past few years.

27Computationally efficient linear programming solvers typically involve inverting an $M \times M$ basis matrix (cf., Hall and Huangfu, 2011), a step we would prefer to avoid.
3.3 Addressing separation in high-dimensional environments

To introduce a notion of high dimensionality, we will now suppose the set of regressors can be partitioned into two distinct components: a set of $P$ non-fixed effects regressors $w_i = w_{1i}, \ldots, w_{Pi}$, which we will treat as countable in number, and a set of $Q$ indicator variables $d_i = d_{1i}, \ldots, d_{Qi}$, where $Q$ is allowed to be a large number. The total number of regressors $M = P + Q$ is therefore also large, and the combined matrix of fixed effect and non-fixed effects regressors can be expressed as $X = \{w_i, d_i\}$. Note that this partition does not depend on the indexing of the fixed effects, but they could easily be subdivided into multiple levels (e.g., “two-way” or “three-way” fixed effects specifications) depending on the application. The number of observations, $N$, is assumed to be greater than $M$, with $N - M$ also generally treated as a large number.

Before describing our preferred method for solving this problem, we first briefly discuss the shortcomings of other currently feasible methods that might otherwise seem appealing. One strategy is to reduce the dimensionality of the above linear programming problem using the Frisch-Waugh-Lovell theorem to one we can more easily compute, extending an earlier strategy proposed by Larch et al. (2019). As we discuss further in the Appendix, the weakness of this approach is that it is not able to discover solutions for $\gamma^*$ that involve only the fixed effects. Alternatively, we could simply attempt to compute estimates without any precautions and consider any observation for which the conditional mean appears to be converging numerically to either 0 or $\gamma$ to be separated. As discussed in Clarkson and Jennrich (1991), this latter method is generally not guaranteed to detect separation correctly.

Our algorithm, which is based on an application of weighted least squares, does not suffer from these types of issues. It can be applied to a very general set of estimation settings, is guaranteed to detect separation, and is both simple to understand and fast. Moreover, it can be implemented in any standard statistical package (without the need for a linear programming solver), and our proof of its effectiveness relies only on textbook least-squares algebra.

We now turn to describing how the algorithm works for the estimation of Poisson models and similar models with only a lower bound. We will then explain how it may be readily applied to

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28 In addition, note that the high-dimensional portion of the regressor set need not consist of only indicator variables; the methods we describe can also be applied to models where $d_i$ contains linear time trends, fixed effects interacted with non-fixed effect variables, and so on without loss of generality.

29 We describe in our Appendix an iteratively reweighted least squares (IRLS) algorithm that can accommodate high-dimensional models in a computationally efficient way. The iterative output from this algorithm can in principle be used to detect observations whose $\mu$’s are converging to inadmissible values. However, leaving these observations in the regression sample will nonetheless slow down convergence and is not guaranteed to work. Furthermore, implementing this method is harder when the true distribution of $\mu$ is very skewed, as it becomes very difficult to numerically distinguish true instances of $\mu = 0$ from mere small values of $\mu$. 


binomial and multinomial models without loss of generality. To proceed, let $u_i$ be an artificial regressand such that $u_i \leq 0$ when $y_i = 0$ and $u_i = 0$ when $y_i > 0$. Also, let $\omega_i$ be a set of regression weights, given by

$$\omega_i = \begin{cases} 1 & \text{if } y_i = 0 \\ K & \text{if } y_i > 0, \end{cases}$$

with $K$ an arbitrary positive integer. The purpose behind these definitions is that we can choose a sufficiently large $K$ such that a weighted regression of $u_i$ on $x_i$ can be used to detect if the equality constraint in (3) can be satisfied by the data. This result is an application of what is sometimes called the “weighting method” (Stewart, 1997).\(^{30}\) We clarify how this technique works using the following lemma:

**Lemma 2** For every $\epsilon > 0$, there is an integer $K > 0$ such that $e_i$, the residual from the weighted least-squares regression of $u_i$ on $x_i$ using $\omega_i$ as weights, is within $\epsilon$ of zero ($|e_i| < \epsilon$) for the observations where $y_i > 0$.

To prove this statement, note first that the residual sum of squares (RSS) minimized by this regression will be at most $u' u$. It is then useful to let $K$ equal the smallest integer that is $> u' u/\epsilon^2$. If the weighted least-squares residual $e_i$ is greater than $\epsilon$ in absolute magnitude, then that observation will contribute more than $K \epsilon^2$ to the RSS and the RSS will be at least $K \epsilon^2$. If $K > u' u/\epsilon^2$ then RSS $> u' u$, which is a contradiction.

Because we can force the predicted values of $u_i$ from this regression to zero for observations with $y_i > 0$, the coefficients computed from this regression therefore satisfy (3). The only remaining step is to choose $u_i$ so that all separated observations have predicted values less than zero and all non-separated observations have predicted values equal to zero. We achieve this goal via the following algorithm:

1. Given a certain $\epsilon > 0$, define the working regressor $u_i$ and regression weight $\omega_i$ as

$$u_i = \begin{cases} -1 & \text{if } y_i = 0 \\ 0 & \text{if } y_i > 0; \end{cases} \quad \omega_i = \begin{cases} 1 & \text{if } y_i = 0 \\ K & \text{if } y_i > 0. \end{cases}$$

\(^{30}\)It is also similar to penalty methods and barrier methods, which have been used for decades as alternatives to simplex-based algorithms for solving linear programming problems (Forsgren et al., 2002). The value-added of our approach is that it also takes advantage of recent computational innovations that are specific to the estimation of least-squares regressions and that readily accommodate models with arbitrarily high dimensionality.
Observe that: (i) the regressand is either zero or negative; (ii) $u'u$ is equal to the number of $y_i = 0$ observations (denoted as $N^{(0)}$).

2. Iterate on these two steps until all residuals are smaller in absolute magnitude than $\epsilon$ (i.e., until all $|e_i| < \epsilon$):

(a) Regress $u_i$ against $x_i$ using weights $\omega_i$. Compute the predicted values $\hat{u}_i = x_i\hat{\gamma}$ and residuals $e_i = u_i - \hat{u}_i$.

(b) For observations with $y_i = 0$, update $u_i = \min(\hat{u}_i, 0)$, ensuring that the regressand remains $\leq 0$.

The unweighted $R^2$ of the last regression iteration is always equal to 1.0 when it converges (i.e., $u_i = \hat{u}_i$ for all $i$). The following proposition establishes the convergence properties of this algorithm and its effectiveness at detecting separation:

**Proposition 4** (Convergence to the correct solution) The above algorithm always converges. Furthermore, if all $\hat{u}_i = 0$ upon convergence, there exists no nonzero vector $\gamma^* \in \mathbb{R}^M$ that solves the system defined by (3) and (5) and there is no separation. Otherwise, the observations that are found to have $\hat{u}_i < 0$ are separated and all the observations with $\hat{u}_i = 0$ are not separated.

We provide proof of this proposition in our Appendix. The main observation for our current purposes is that none of the above steps are significantly encumbered by the size of the data and/or the complexity of the model. Thanks to the recent innovations of Correia (2017), weighted linear regressions with many fixed effects can be computed in almost-linear time (as can more general high-dimensional models using time trends or individual-specific continuous regressors).\(^\text{32}\) The above method can therefore be applied to virtually any estimation setting for which (3) and (5) are necessary and sufficient conditions for existence, even when the model features many levels of fixed effects and other high-dimensional parameters. Notably, this includes frequency table models—the original object of interest in Haberman (1974)—which themselves may be thought of as multi-way fixed effects models without non-fixed effect regressors.

\(^\text{31}\)One could also update $K$ and $\epsilon$ with each iteration as well. In theory, this would lead to exact convergence. In practice, we would typically need to insist $\epsilon$ be no smaller than $1e-16$, which is the machine precision of most modern 64 bit CPUs.

\(^\text{32}\)As discussed in Guimarães and Portugal (2010), this is because we can use the Frisch-Waugh-Lovell theorem to first "partial out" the fixed effects, $d_i$, from either side of the problem via a within-transformation operation and then regress the within-transformed residuals of $u_i$ on those of the non-fixed effect regressors, $w_i$, to obtain $e_i$. Correia (2017) then shows how to solve the within-transformation sub-problem in nearly linear time.
The above algorithm still needs a name. Its defining features are that it iteratively uses weighted least squares in combination with a “linear rectifier” function\(^{33}\) to ensure \(u_i\) eventually converges to the overall certificate of separation that identifies all separated observations. Thus, we have settled on the name “iterative rectifier” (or IR for short).\(^{34}\)

Finally, it is important to clarify that our iterative rectifier algorithm can be easily adapted to the binary choice case (or, more generally, to the case of a multinomial dependent variable) using a simple transformation of the model. A logit model can always be rewritten as a Poisson model, for example (Albert and Anderson, 1984).\(^{35}\) For larger problems involving binary choice models, we can use this transformation to write down an equivalent Poisson model that is separated if and only if the original binary choice model is separated. The same is also true for fractional response models where \(y_i\) can vary continuously over \([0, \overline{y}]\). Our Appendix provides further discussion as well as a proof.

### 4 Concluding remarks

In this paper, we have provided an updated treatment of the concept of separation in the estimation of GLMs. While the result that all GLMs with bounded individual likelihoods suffer from separation under similar circumstances has been shown before by several authors, these results arguably have not been given sufficient attention. Now that estimation techniques have progressed to the point where nonlinear models are regularly estimated via (pseudo-)maximum likelihood with many fixed effects, there is considerable ambiguity over whether the estimates produced by these models are likely to exist, what it means when they do not exist, and what can be done to ensure that the model can be successfully estimated.

We have brought more clarity to each of these topics by building on the earlier work of Verbeek (1989) and Clarkson and Jennrich (1991), which we have extended to incorporate estimators that have not been previously examined and that have their own more idiosyncratic criteria governing existence. An important takeaway from this analysis is that some, but not all, GLM estimators can still deliver uniquely identified, consistent estimates of at least some of the model parameters.

\(^{33}\)We borrow this term from the machine learning literature, where \(\min(y, 0)\) and \(\max(y, 0)\) are known as linear rectifiers or ReLUs (Rectified Linear Units). Despite their simplicity, ReLUs have played a significant role in increasing the accuracy and popularity of deep neural networks (Glorot et al., 2011).

\(^{34}\)“Iteratively Rectified Weighted Least Squares” would have introduced acronym ambiguity with “Iteratively Reweighted Weighted Least Squares”.

\(^{35}\)Albert and Anderson (1984) have previously conjectured that this type of equivalence between logit and Poisson models could be used to simplify the problem of detecting separation in frequency table models. The notes we provide in our Appendix include a proof of Albert and Anderson (1984)’s conjecture.
even if other parameter estimates are technically infinite.

We have also introduced a new method to detect separation in models with multiple levels of high-dimensional fixed effects, a task that would otherwise require solving an unpractical or even unfeasible high-dimensional linear programming problem. As GLM estimation with high-dimensional fixed effects increasingly becomes faster and more appealing to researchers, the need for methods that can detect and deal with separation in these models represents an important gap that we aim to fill.

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Log (Pseudo-)likelihood ($\sim - \frac{1}{\text{summation}}$).

First-order condition for $u_1 \cdots u_3 - 1 \log P_{\text{Papke and Wooldridge}}(\exp)$.

Table 1: Mapping different regression models onto GLM

| Model          | Log (Pseudo-)likelihood ($l$) | $\theta(x; \beta; \nu)$ | $b(\theta_l)$ | $\mu_l = \beta^*$ | First-order condition for $\beta_m$ |
|----------------|--------------------------------|-------------------------|---------------|-------------------|-----------------------------------|
| Probit         | $\sum_i \{ y_i \log \mu_i + (1 - y_i) \log (1 - \mu_i) \}$ | $\log \left( \frac{\Phi_i(x; \beta)}{1 - \Phi_i(x; \beta)} \right)$ | $\log (1 + \exp (\theta_l))$ | $\exp(\theta_l)$ | $\frac{\exp(\theta_l)}{1 + \exp(\theta_l)}$ |
| Logit          | $\sum_i \{ y_i \log \mu_i + (1 - y_i) \log (1 - \mu_i) \}$ | $x_i \beta$ | $\log (1 + \exp (\theta_l))$ | $\exp(\theta_l)$ | $\frac{\exp(\theta_l)}{1 + \exp(\theta_l)}$ |
| Poisson        | $\sum_i \{ y_i x_i \beta - \exp (x_i \beta) - \ln y_i \}$ | $x_i \beta$ | $\exp (\theta_l)$ | $\exp (\theta_l)$ | $\sum_i \{ y_i - \mu_i \} x_{mi} = 0$ |
| Negative Binomial | $\sum_i \{ y_i \log \left( \frac{\exp(x_i \beta)}{\nu + \exp(x_i \beta)} \right) - \nu \log (\nu + \exp (x_i \beta)) + c (\nu, y_i) \}$ | $\log \left( \frac{\exp(x_i \beta)}{\nu + \exp(x_i \beta)} \right)$ | $\nu \log \left( \frac{\nu}{1 - \exp(\theta_l)} \right)$ | $\nu \frac{\exp(\theta_l)}{1 - \exp(\theta_l)}$ | $\sum_i \{ y_i - \mu_i \} \left( 1 + \nu^{-1} \mu_i \right)^{-1} x_{mi} = 0$ |
| Gamma (PML)   | $\sum_i -\alpha y_i \exp (-x_i \beta) - \alpha x_i \beta$ | $\exp (-x_i \beta)$ | $\log (-1/\theta_l)$ | $\exp (-x_i \beta)$ | $\sum_i \{ y_i - \mu_i \} \exp (-x_i \beta) x_{mi} = 0$ |
| Gaussian      | $\sum_i \left\{ \frac{1}{2 \pi \sigma^2} \left[ y_i - \exp (x_i \beta) \right]^2 - \frac{1}{2} \log (2 \pi \sigma^2) \right\} + \frac{1}{\sigma^2} \left[ y_i \exp (x_i \beta) - \frac{1}{2} \exp (2x_i \beta) \right]^{1/2}$ | $\exp(x_i \beta)$ | $\frac{\theta_l^2}{2}$ | $\theta_l$ | $\sum_i \{ y_i - \mu_i \} \exp (x_i \beta) x_{mi} = 0$ |
| Inverse Gaussian (PML) | $\sum_i \alpha \left[ -\frac{1}{2} \exp (-2x_i \beta) + \exp (-x_i \beta) \right]$ | $\frac{-\exp(-2x_i \beta)}{2}$ | $\left(-2\theta_l\right)^{1/2}$ | $\left(-2\theta_l\right)^{-1/2}$ | $\sum_i \{ y_i - \mu_i \} \exp (-2x_i \beta) x_{mi} = 0$ |

$\Phi(\cdot)$ is the cdf of a standard normal distribution. $\phi(\cdot)$ is its pdf. $\alpha$ and $\sigma^2$ are dispersion/scaling factors to be estimated, which do not affect identification of $\beta$, $\nu$, which does affect identification of $\beta$, is the dispersion parameter for the negative binomial regression. Note that for gamma and inverse Gaussian, we consider only the pseudo-maximum Likelihood (PML) version of these estimators (the standard likelihood functions for these models do not admit $y_i = 0$ values.) These PML estimators each use a “log link” as opposed to the canonical link. We do the same for the Gaussian GLM shown, since Gaussian (log link) PML is another common PML estimator. The logit and probit likelihood functions can also be applied to fractional data using Bernoulli PML; see Papke and Wooldridge (1996).
In this example, a typical (iterative) check for perfect collinearity over $y_i > 0$ would first reveal that $z_{1i} = x_{3i} - x_{4i}$ is always 0 over $y_i > 0$ and then subsequently also find the same for $z_{2i} = x_{2i} - x_{4i}$. Since both $z_{1i}$ and $z_{2i}$ take on positive as well as negative values over $y_i = 0$, it would appear the model does not suffer from separation. However, the linear combination $z_{3i} = .5z_{1i} + .5z_{2i}$ only takes on values $\leq 0$ over $y_i = 0$, implying separation. $x_1$ is an explicit constant.

### Table 2: Separation may not be detected because of multiple redundant regressors over $y_i > 0$

| $y_i$ | $x_{1i}$ | $x_{2i}$ | $x_{3i}$ | $x_{4i}$ | $z_{1i}$ | $z_{2i}$ | $z_{3i}$ |
|-------|----------|----------|----------|----------|----------|----------|----------|
| 0     | 1        | -1       | 5        | 3        | 2        | -4       | -1       |
| 0     | 1        | 2        | 0        | 1        | -1       | 1        | 0        |
| 0     | 1        | 0        | -6       | -3       | -3       | 3        | 0        |
| 0     | 1        | 0        | 0        | 0        | 0        | 0        | 0        |
| 1     | 1        | 3        | 3        | 3        | 0        | 0        | 0        |
| 2     | 1        | 6        | 6        | 6        | 0        | 0        | 0        |
| 3     | 1        | 5        | 5        | 5        | 0        | 0        | 0        |
| 4     | 1        | 7        | 7        | 7        | 0        | 0        | 0        |
| 5     | 1        | 4        | 4        | 4        | 0        | 0        | 0        |
Online-only Appendix for
“Verifying the existence of maximum likelihood estimates for generalized linear models”

by Sergio Correia, Paulo Guimarães, & Thomas Zylkin

(not for publication)

Additional proofs

Proof of Lemma 1. Recall from our discussion of the likelihood function in (1) that \( b(\theta_i) \) is stipulated to be increasing and convex with respect to \( \theta_i \). Thus, the function \( l_i = y_i \theta_i - b(\theta_i) + c_i \) has a unique, finite maximum as long as \( y_i \) is positive. In turn, we need only concern ourselves with how \( l_i \) behaves either when \( y_i = 0 \) or (for part (b)) when \( y_i = \overline{y} \). In the case where \( y_i = 0 \), note that observation \( i \)'s contribution to the score function—\( s_i(\beta) = \partial l_i / \partial \beta \)—is given by

\[
s_i(\beta) = -\alpha_i(\varphi) b'(\theta_i) \theta'(x_i \beta; \cdot) x_i = -\alpha_i(\varphi) \mu_i \theta'(x_i \beta; \cdot) x_i.
\]

To complete the proof of part (a), first note that \( \mu_i \) and \( \theta'_i \) are both bounded from below by 0; \( \mu_i \times \theta'_i < 0 \) is therefore not possible. Now suppose that \( \lim_{x_i \beta \to -\infty} \mu_i \times \theta'_i > 0 \). In this case, the sign of \( s_i(\beta) \) is equal to the sign of \( -x_i \) at the limit where \( x_i \beta \to -\infty \). As a result, the individual likelihood contribution \( l_i(\cdot) \) can be perpetually increased by increasing \( \beta \) in the direction opposite to \( x_i \) (i.e., by decreasing \( x_i \beta \)). It therefore does not have a finite upper bound. We also need to show that

\[
\lim_{k \to \infty} F_k(\beta) = 0.
\]

However, this follows directly from the fact that \( \lim_{x_i \beta \to -\infty} s_i(\beta) \to 0 \). To see this, let \( F^{(m,n)}_i \) denote the \( m, n \)th element of \( F_i(\beta) \) and let \( s^{(m)}_i \) denote observation \( i \)'s contribution to the \( m \)th element of the score vector. If observation \( i \) is separated at the lower bound by some vector \( \gamma^* \), each element of \( F_i(\beta) \) can be written as

\[
\lim_{k \to \infty} F^{(m,n)}_i \equiv \lim_{k \to \infty} \mathbb{E} \left[ \partial s^{(m)}_i (\beta + k \gamma^*) / \partial \beta_n \right] \equiv \lim_{k \to \infty} \mathbb{E} \left[ \lim_{\tau \to 0} \left( \frac{s^{(m)}_i (\beta + k \gamma^* + \mathbf{1}_n \tau) - s^{(m)}_i (\beta + k \gamma^*)}{\tau} \right) \right] = 0,
\]

where \( \mathbf{1}_n \) is a unit vector with length \( M \) and with its \( n \)th element equal to 1. Finally, for part (b), we also need to consider the case where \( \lim_{x_i \beta \to -\infty} \mu_i \to \overline{y} < \infty \) (where we generally take \( \overline{y} \) to be 1, as is the case in binary choice models). A similar reasoning applies here as well: if \( \lim_{x_i \beta \to -\infty} s_i(\beta) = \lim_{x_i \beta \to -\infty} (\overline{y} - \mu_i) \theta'_i x_i > 0 \), it will always be possible to increase \( l_i(\cdot) \) by increasing \( \beta \) in the same direction as \( x_i \). And we can rule out \( \lim_{x_i \beta \to -\infty} s_i(\beta) < 0 \) in these cases since \( \theta'_i \)}
cannot be negative and since \( \mu_i \) cannot exceed \( \overline{y} \). The reasoning as to why \( \lim_{k \to \infty} F_i(\beta) = 0 \) is the same as in part (a).

**Proof of Proposition 4.** This proof is split into two parts. First, we show the algorithm described in Section 3.3 always converges. Then we show that if the algorithm converges, it always converges to the correct results.

**Proof of convergence.** Let \( u_i^{(k)} \) denote the \( i \)th observation of the working dependent variable at iteration \( k \) and let \( \hat{u}_i^{(k)} := x_i \hat{y}^{(k)} \) be its predicted value, with \( \hat{y}^{(k)} \) denoting the vector of weighted least-squares coefficients estimated at iteration \( k \). Also, let \( u_i^{2(k)} \) and \( \hat{u}_i^{2(k)} \) respectively denote the squares of \( u_i^{(k)} \) and \( \hat{u}_i^{(k)} \). \( e_i := u_i - \hat{u}_i \) will continue to denote a residual, with \( e_i^{(k)} \) denoting the residual from the \( k \)th iteration and \( e_i^{2(k)} \) denoting its square. In addition, it will occasionally be convenient to let \( u^{(k)} \), \( \hat{u}^{(k)} \), and \( e^{(k)} \) respectively denote the vector analogues of \( u_i^{(k)} \), \( \hat{u}_i^{(k)} \), and \( e_i^{(k)} \).

When the algorithm converges, all of the residuals from the weighted least-squares step converge to zero: \( u_i = \hat{u}_i \leq 0 \ \forall i \iff e_i = 0 \ \forall i \). It would be cumbersome to show that all residuals indeed converge, so we instead take a simpler route and work with the sum of squared residuals (SSR). We can do this because \( |e_i| \leq \sqrt{\sum_i e_i^2} \), which implies that if the SSR converges to zero, all residuals must converge to zero as well.

Let \( SSR^{(k)} \) denote the SSR from the \( k \)th iteration. We will prove that \( \lim_{k \to \infty} SSR^{(k)} = 0 \) by first proving that the sum given by \( SSR^{(1)} + SSR^{(2)} + \ldots + SSR^{(k)} \) converges to a finite number. To see this, note that we have by the normal equations that \( \sum \hat{u}_i e_i = 0 \). Thus, the SSR can be computed as \( \sum_{i=1}^n e_i^2 = \sum_{i=1}^n u_i^2 - \sum_{i=1}^n \hat{u}_i^2 \) for all iterations, including \( k + 1 \):

\[
SSR^{(k+1)} = \left( \sum_{i=1}^n u_i^2 \right)^{(k+1)} - \left( \sum_{i=1}^n \hat{u}_i^2 \right)^{(k+1)}.
\]

By construction, \( u_i^{(k+1)} = \min(\hat{u}_i^{(k)}, 0) \), and thus

\[
\left( \sum_{i=1}^n u_i^2 \right)^{(k+1)} = \left( \sum_{\hat{u}_i^{(k)} < 0} \hat{u}_i^2 \right)^{(k)}.
\]

We can also split \( \sum \hat{u}_i^2^{(k+1)} \) based on the values of \( \hat{u}^{(k+1)} \):

\[
\left( \sum_{i=1}^n \hat{u}_i^2 \right)^{(k+1)} = \left( \sum_{\hat{u}_i^{(k+1)} < 0} \hat{u}_i^2 \right)^{(k+1)} + \left( \sum_{\hat{u}_i^{(k+1)} \geq 0} \hat{u}_i^2 \right)^{(k+1)}.
\]
Putting the last three equations together,
\[
SSR^{(k+1)} = \left( \sum_{\hat{u}_i^{(k)} < 0} \hat{u}_i^2 \right)^{(k)} - \left( \sum_{\hat{u}_i^{(k+1)} < 0} \hat{u}_i^2 \right)^{(k+1)} - \left( \sum_{\hat{u}_i^{(k+1)} \geq 0} \hat{u}_i^2 \right)^{(k+1)}.
\]

If we move the last equation forward to \((k+2)\) and then add \(SSR^{(k+1)}\), we notice this is a telescoping series where one term cancels:
\[
SSR^{(k+1)} + SSR^{(k+2)} = \left( \sum_{\hat{u}_i^{(k)} < 0} \hat{u}_i^2 \right)^{(k)} - \left( \sum_{\hat{u}_i^{(k+1)} < 0} \hat{u}_i^2 \right)^{(k+1)} - \left( \sum_{\hat{u}_i^{(k+1)} \geq 0} \hat{u}_i^2 \right)^{(k+2)} - \left( \sum_{\hat{u}_i^{(k+2)} \geq 0} \hat{u}_i^2 \right)^{(k+2)}.
\]

More generally, the infinite sum of the sequence starting at \(k = 2\) is equal to
\[
\sum_{k=2}^{\infty} SSR^{(k)} = \left( \sum_{\hat{u}_i^{(1)} < 0} \hat{u}_i^2 \right)^{(1)} - \sum_{k=2}^{\infty} \left( \sum_{\hat{u}_i^{(k)} \geq 0} \hat{u}_i^2 \right)^{(k)} - \lim_{k \to \infty} \left( \sum_{\hat{u}_i^{(k)} < 0} \hat{u}_i^2 \right)^{(k)} \leq \left( \sum_{\hat{u}_i^{(1)} < 0} \hat{u}_i^2 \right)^{(1)}.
\]

After adding \(SSR^{(1)}\) on both sides, and applying \(\sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n} \hat{u}_i^2 + \sum_{i=1}^{n} e_i^2\), we have that
\[
\sum_{k=1}^{\infty} SSR^{(k)} \leq \left( \sum_{i=1}^{n} \hat{u}_i^2 \right)^{(1)} + SSR^{(1)} \leq \left( \sum_{i=1}^{n} \hat{u}_i^2 \right)^{(1)} + SSR^{(1)} = \left( \sum_{i=1}^{n} u_i^2 \right)^{(1)}.
\]

Therefore,
\[
\sum_{k=1}^{\infty} SSR^{(k)} \leq \left( \sum_{i=1}^{n} u_i^2 \right)^{(1)} = \sum_{y_i = 0} \leq 1,
\]
where the last equality follows from how we initialize \(u_i\), with \(u_i^{(1)} = -1\) for all \(y_i = 0\) observations and \(u_i^{(1)} = 0\) otherwise. Thus, the series of SSRs is bounded above by the number of boundary observations where \(y_i = 0\) (a finite number). We can now show that \(\lim_{k \to \infty} SSR^{(k)} = 0\) (i.e., that the sequence of SSRs converges to 0). To see this, note that if
\[
\lim_{k \to \infty} SSR^{(k)} = c > 0, \quad \text{but} \quad \sum_{k=1}^{\infty} (SSR)^{(k)} \leq C
\]
for some finite \(C\), then, by iteration \(k^* := \left[ \frac{C}{c} \right]\), the sum of the sequence will have exceeded \(C\), a contradiction. Therefore, the SSR converges to zero, with the same necessarily being true for all of the individual residuals.
Proof of convergence to the correct solution. The above proof tells us that our iterative “rectifier” algorithm will eventually converge, but of course it doesn’t tell us that it will converge to the correct solution. What we will prove now is exactly that:

\[ \lim_{k \to \infty} \hat{u}_i^{(k)} < 0 \text{ iff the observation } i \text{ is separated.} \]

As in the main text, \( z \) is the name we will give to the “certificates of separation” used to detect separated observations. By Proposition 1, any such \( z \) must be a linear combination of regressors: \( z = X\gamma \), with \( z_i = 0 \) if \( y_i > 0 \) or if \( y_i = 0 \) but is not separated and with \( z_i < 0 \) for the observations that are separated. It is also important to keep in mind that there can be multiple \( z \) vectors. We do not just want to find some of the \( z \)'s that induce separation; rather, we want to identify a \( z \) that is as large as possible, in the sense of having the most nonzero rows.

Our proof that our algorithm accomplishes this task can be outlined in two steps. We first need to show that if \( \lim_{k \to \infty} \hat{u}_i^{(k)} < 0 \), then observation \( i \) is separated. This is very simple to show, since the above proof of convergence implies that \( \lim_{k \to \infty} \hat{u}_i^{(k)} = \lim_{k \to \infty} u_i^{(k)} \) and since, by construction, \( u_i^{(k)} = 0 \) if \( y_i > 0 \) and \( u_i^{(k)} \leq 0 \) if \( y_i = 0 \). Recalling that \( \gamma^{(k)} \) is the vector of coefficients computed from the weighted least-squares regression in each iteration \( k \), it is now obvious that \( \lim_{k \to \infty} \hat{u}_i^{(k)} = \lim_{k \to \infty} X\gamma^{(k)} \) is a linear combination of regressors that meets the criteria for separation described in Proposition 1 if there are any \( i \) such that \( \lim_{k \to \infty} \hat{u}_i^{(k)} < 0 \).

The second step, showing that \( \lim_{k \to \infty} \hat{u}_i^{(k)} \) is necessarily \( < 0 \) for all separated observations, is more complicated. To prove this part, we will rely on the following lemma:

**Lemma 3** For every possible \( z \) satisfying the criteria for separation described in Proposition 1, we must have that \( \lim_{k \to \infty} u_i^{(k)} < 0 \) on at least one row where \( z_i < 0 \).

The underlined portion of Lemma 3 that clarifies that it applies to “every possible” \( z \) is important. As we will soon see, the fact that the algorithm discovers at least one separated observation associated with every possible linear combination of regressors that induces separation will be sufficient to prove that \( \lim_{k \to \infty} \hat{u}_i^{(k)} < 0 \) for all separated observations, completing our proof of Proposition 4. Before reaching this final step, we first need to prove Lemma 3:

**Proof of Lemma 3.** To prove Lemma 3, it will first be useful to document the following preliminaries. First, recall that each iteration \( k \) involves a regression of our working dependent variable \( u^{(k)} \) on our original regressors \( X \) that produces a set of residuals \( e^{(k)} \). Thus, the normal equations for each of these regressions imply that \( X'e^{(k)} = 0 \implies z'e^{(k)} = 0 \) \( \forall k \) and \( \forall z \) (since \( z'e^{(k)} \)
is just a linear combination of $X' e^{(k)}$ that consists of premultiplying $X' e^{(k)}$ by $\gamma$). Second, we can always decompose each vector of predicted values for our working dependent variable into its positive and negative components using the appropriate rectifier functions: $\hat{u} = \hat{u}^{(+)} + \hat{u}^{(-)}$, where $u^{(+)}_i = \max(\hat{u}_i, 0)$ and $u^{(-)}_i = \min(\hat{u}_i, 0)$. Third, using this notation, we also have that $u_i^{(k+1)} = \hat{u}_i^{(k)(-)}$ (i.e., the working dependent variable inherits the rectified predicted values from the prior iteration).

We start with the observation noted above that the normal equations imply $z' e^{(k)} = 0$ for every iteration (i.e., $\sum z_i e_i^{(k)} = 0 \ \forall k$.) Now let’s focus on iterations $k$ and $k + 1$:

$$\sum z_i e_i^{(k)} + \sum z_i e_i^{(k+1)} = 0.$$  

After grouping terms and using the definition of $e_i$, we have

$$\sum z_i \left[ u_i^{(k)} - \hat{u}_i^{(k)} + u_i^{(k+1)} - \hat{u}_i^{(k+1)} \right] = 0.$$  

Using our decomposition, $\hat{u}_i^{(k)} = u_i^{(k),(+)} + u_i^{(k),(-)}$ (and likewise for $k + 1$):

$$\sum z_i \left[ u_i^{(k)} - \hat{u}_i^{(k),(+)} - \hat{u}_i^{(k),(-)} + u_i^{(k+1)} - \hat{u}_i^{(k+1),(+)} + \hat{u}_i^{(k+1),(-)} \right] = 0.$$  

Also recall that $u_i^{(k+1)} = \hat{u}_i^{(k)(-)}$ (and likewise for $k + 2$):

$$\sum z_i \left[ u_i^{(k)} - \hat{u}_i^{(k),(+)} - u_i^{(k+1)} + u_i^{(k+1)} - \hat{u}_i^{(k+1),(+)} - u_i^{(k+2)} \right] = 0.$$  

After canceling out terms and rearranging, we have

$$\sum z_i \left[ u_i^{(k+2)} - u_i^{(k)} \right] = - \sum z_i \left[ \hat{u}_i^{(k),(+)} + \hat{u}_i^{(k+1),(+)} \right].$$  

Notice that we can focus on the observations where $z_i < 0$ without loss of generality, as the elements of the sum with $z_i = 0$ are obviously zero.\footnote{Also, there must be negative elements of $z$, as otherwise $z$ wouldn’t be a valid certificate of separation.} Thus,\footnote{Also, there must be negative elements of $z$, as otherwise $z$ wouldn’t be a valid certificate of separation.}

$$\sum_{z_i < 0} z_i \left[ u_i^{(k+2)} - u_i^{(k)} \right] = - \sum_{z_i < 0} z_i \left[ \hat{u}_i^{(k),(+)} + \hat{u}_i^{(k+1),(+)} \right].$$  

Notice that unless we have reached convergence before iteration $k + 2$, then the righthand term is strictly positive. This is because $z_i < 0$, and $\hat{u}_i^{(k),(+)}$ is nonnegative, with at least one strictly
positive observation (otherwise, \( \hat{u}_i^{(k)} \) would meet the stopping criteria because it would be strictly nonpositive and the next iteration would return \( \hat{u}_i^{(k+1)} = u_i^{(k+1)} = \hat{u}_i^{(k)} \).)

Thus,

\[
\sum_{z_i < 0} z_i u_i^{(k+2)} > \sum_{z_i < 0} z_i u_i^{(k)}.
\]

By itself, this statement is interesting, because it tells us that the weighted sum of \( u \) is increasing as we iterate. But we can get a useful bound if we recall that on the first iteration, \( u_i = -1 \) when \( y_i = 0 \), which implies \( \sum z_i u_i^{(1)} = \sum z_i \). Then, for every odd iteration with \( k \geq 3 \), we know that

\[
\sum z_i u_i^{(k)} > -\sum z_i.
\]

Denote the minimum (i.e., most negative) \( z_i \) as \( z_{\text{min}} \). Then,

\[
z_{\text{min}} \sum u_i^{(k)} \geq \sum z_i u_i^{(k)} > -\sum z_i \text{ for } k = 3, 5, 7, \ldots
\]

\[
\implies \sum u_i^{(k)} < -\frac{\sum z_i}{z_{\text{min}}} \text{ for } k = 3, 5, 7, \ldots
\]

Given that both \( z_{\text{min}} \) and \( \sum z_i \) must be \(< 0 \) for there to be separation, it follows that \( \sum_{z_i < 0} u_i^{(k)} \) must be negative on every odd iteration starting with \( k = 3 \). Obviously, this is not possible unless at least one \( u_i^{(k)} \) is negative for an observation where \( z_i < 0 \) for each of these iterations. The lemma follows by considering \( k \to \infty \), since \( u_i^{(k)} \) must eventually converge to the same result for both odd and even \( k \).

For the remainder of the proof of the overall theorem, let \( \hat{u}_i^{(\infty)} := \lim_{k \to \infty} u_i^{(k)} \) denote the solution obtained by our algorithm. Thanks to the insights established by Lemma 3, we can now prove that \( \hat{u}_i^{(\infty)} < 0 \) if and only if there exists a \( z \) that separates observation \( i \). We can do so by considering two cases. First, note that if \( \hat{u}_i^{(\infty)} = 0 \) for all \( i \), then Lemma 3 implies there cannot be any such \( z \) and the data are not separated. The more interesting case is if \( \hat{u}_i^{(\infty)} < 0 \) for at least one \( i \). In that case, recall that \( \hat{u}_i^{(\infty)} \) is an admissible \( z \) (because it will have converged to a vector that is \(< 0 \) for some observations where \( y_i = 0 \) and is \( 0 \) otherwise.) Thus, for the algorithm to fail to identify a separated observation, it would have to be the case that there exists some other certificate of separation \( z^* \) that is \(< 0 \) for at least one observation where \( \hat{u}_i^{(\infty)} = 0 \). To see why this cannot
Then, given $z^*$ and our solution $\hat{u}_i^{(\infty)}$, we can construct a third certificate $z^{**}$ that also separates the data:

$$z_i^{**} := z_i^* - \alpha^* \hat{u}_i^{(\infty)} \leq z_i^* - z_i^* = 0,$$

where the inequality follows from the definition of $\alpha^*$. By construction, $z_i^{**} < 0$ for at least one observation where $\hat{u}_i^{(\infty)} = 0$, and $z_i^{**} = 0$ for at least one observation where $\hat{u}_i^{(\infty)} < 0$. If $z_i^{**} = 0$ for all observations where $\hat{u}_i^{(\infty)} = 0$, we have a contradiction, since Lemma 3(b) tells us at least one observation separated by $z_i^{**}$ must also be separated by our solution $\hat{u}_i^{(\infty)}$. If not, we repeat: let

$$\alpha^{**} := \sup_{\hat{u}_i^{(\infty)} < 0} \frac{z_i^{**}}{\hat{u}_i^{(\infty)}}$$

and

$$z_i^{***} = z_i^{**} - \alpha^{**} \hat{u}_i^{(\infty)},$$

which gives us yet another certificate of separation $z_i^{***}$ that will equal 0 for at least one observation where $\hat{u}_i^{(\infty)} < 0$ and $z_i^{***} < 0$. We can repeat this process as many times as needed until we eventually obtain a $z$ that does not separate any observations for which $\hat{u}_i^{(\infty)} < 0$. Lemma 3 again provides the needed contradiction indicating that this cannot happen.

**Example code.** The number of steps needed in the above proof may suggest the iterative rectifier algorithm is rather complicated. However, in practice, it requires only a few lines of code to implement. Below, we provide some generic “pseudo code” that should be simple to program in virtually any statistical computing language (e.g., R, Stata, Matlab).
Pseudo code:

Set \( u_i = -1 \) if \( y_i = 0 \); 0 otherwise
Set \( \omega_i = K \) if \( y_i > 0 \); 1 otherwise

Begin loop:

Regress \( u \) on \( X \), weighting by \( \omega \) (produces coefficients \( \hat{\gamma} \))
Set \( \hat{u} = X \hat{\gamma} \)
Set \( \hat{u} = 0 \) if \( |\hat{u}| < \epsilon \)
Stop if \( \hat{u}_i \leq 0 \) for all \( i \) (all separated observations have been identified)
Replace \( u_i = \min (\hat{u}_i, 0) \)

End loop.

For readers interested in more details, we have created a website that provides sample Stata code and datasets illustrating how all of the methods for detecting separation described in this paper can be implemented in practice. Also see our companion paper for the \texttt{ppmlhdfe} Stata command (Correia et al., 2020), which provides further useful information related to technical implementation and testing.

An alternative method using within-transformation and linear programming

Larch et al. (2019) have also recently proposed a method for detecting separation in Poisson-like models in the presence of high-dimensional fixed effects. In their paper, this is accomplished by first “within-transforming” all non-fixed effect regressors with respect to the fixed effects, then checking whether the within-transformed versions of these regressors satisfy conditions for separation. As they discuss (and as we will document here as well), any method based on this strategy is only able to detect instances of separation that involve at least one non-fixed effect regressor; it cannot be used to detect separation involving only the fixed effects. Another difference is that Larch et al. (2019) describe how to detect linear combinations of regressors that satisfy (3) only. Detecting linear combinations of regressors that satisfy both of the relevant conditions described in Proposition 1 (i.e., both (3) and (5)) requires an appropriate extension of their methods that incorporates the linear programming problem in (15).

The first step is to regress each non-fixed effect regressor \( w_{pi} \) on every other regressor (includ-
ing the fixed effects) over \(0 < y_i < \bar{y}\). If we find that \(w_{pi}\) is perfectly predicted over \(0 < y_i < \bar{y}\), then we know there is a linear combination of regressors involving \(w_{pi}\) that satisfies (3), as shown by Larch et al. (2019). Larch et al. (2019) do not discuss how this step is applicable to the linear programming problem in (15), but focusing on these “candidate” linear combinations that we already know to satisfy (3) turns out to be an effective way of reducing the dimensionality of the problem (for nonbinary choice models at least). More formally, we can determine candidate solutions for \(\gamma^*\) by first computing the following linear regression for each \(w_{pi}\):

\[
w_{pi} = \tilde{w}_i^{-p} \delta_p + d_i \tilde{\xi}_p + r_{pi} \quad \text{for} \quad 0 < y_i < \bar{y},
\]

(17)

where \(\tilde{w}_i^{-p}\) is the set of other non-fixed effect regressors (i.e., excluding \(w_{pi}\)). \(\delta_p\) and \(\tilde{\xi}_p\) are the coefficient vectors to be estimated. Our focus is on the residual error \(r_{pi}\) obtained from each of these regressions. If \(r_{pi}\) is uniformly zero, then some combination of the fixed effects and the other non-fixed effect regressors perfectly predicts \(w_{pi}\) over \(0 < y_i < \bar{y}\). Or, to cement the connection with Proposition 1, we would have that \(r_{pi} = \tilde{w}_i^{-p} \delta_p - d_i \tilde{\xi}_p\) is a linear combination of regressors that satisfies condition (8).

Because the estimation expressed in (17) is a linear regression, it can generally be computed very quickly using the algorithm of Correia (2017), even for models with very large \(M\). The main advantage of this first step is that it greatly reduces the dimension of the linear programming problem we need to solve. This is for two reasons. First, it allows us to effectively perform a change of variables from \(x_i\) (which is of dimension \(M\)) to the set of \(r_{pi}\) associated with any regressors that are perfectly predicted over \(0 < y_i < \bar{y}\) (which will have a much smaller dimension \(\leq P \ll M\)). Second, since any linear combination of these \(z_{pi}\)’s is assured to satisfy (3), we no longer need the third set of constraints stipulated in (15). Since we very often have that \(0 < y_i < \bar{y}\) for a majority of the observations in nonbinary choice models, changing variables in this way is likely to also greatly reduce the number of constraints.\(^{37}\)

A suitable reparameterization of our original linear programming problem in (15) helps to illustrate the idea behind this change of variables. Let \(r_i^* := \{r_{pi}|r_{pi} = 0 \text{ if } 0 < y_i < \bar{y}\}\), i.e., a vector consisting of the predicted residuals from (17) associated with any \(w_{pi}\) that are perfectly predicted

\(^{37}\)For this reason, this first step of regressing each regressor on every other regressor can be beneficial even in non-high-dimensional environments when the number of observations with \(0 < y_i < \bar{y}\) is large. A similar first step also appears in Santos Silva and Tenreyro (2010) and Larch et al. (2019), but both of these papers stop short of verifying the “overlap” conditions described in (4)-(5). Addressing the latter complication requires one of the methods described in this paper.
over $0 < y_i < \overline{y}$. The modified linear programming problem based on $r_i^*$ instead of $x_i$ is

$$\begin{align*}
\max_\phi & \sum_{y_i=0} 1 (r_i^* \phi < 0) + \sum_{y_i=\overline{y}} 1 (r_i^* \phi > 0) \\
\text{s.t.} & -r_i^* \phi \geq 0 \text{ if } y_i = 0, \\
& r_i^* \phi \geq 0 \text{ if } y_i = \overline{y},
\end{align*}$$

(18)

where, as noted, the number of parameters we need to solve for (i.e., the length of the vector $\phi$ in this case) is only equal to the number of $w_{pi}$ that we found to be perfectly predicted by other regressors in the first step. Furthermore, the number of constraints we need to take into account is only $N_{y_i=0} + N_{y_i=\overline{y}}$ instead of $N$. To appreciate why this approach works, consider what happens when a nonzero vector $\phi^*$ can be found solving (18). In that case, $r_i^* \phi^* = \sum_{p|r_p \in r_i^*} \phi^*_p r_p = \sum_{p|r_p \in r_i^*} \phi^*_p (w_{pi} - w_i^- p \delta_p - d_i \xi_p)$ is a linear combination of regressors that satisfies (3)-(5), indicating separation.

However, while this approach is able to quickly identify separation involving complex combinations of both fixed effect and non-fixed effect regressors, it cannot be easily used to identify separation involving only fixed effects (at least not without estimating (17) $M$ times in the first step, which is likely to be time consuming). For some standard fixed effect configurations, this latter problem is not so severe. For example, the trivial case where a fixed effect dummy is always equal to zero when $0 < y_i < \overline{y}$ is very easy to find. Models with only one level of fixed effects are thus easy to deal with in this regard. Similarly, in models with two levels of fixed effects (e.g., exporter and importer, firm and employee), the graph-theoretical approach of Abowd et al. (2002) can be applied to identify any combinations of fixed effects that are perfectly collinear over $0 < y_i < \overline{y}$, which then can be added as needed to the linear programming step in (18).

For more general cases, such as nonbinary choice models with more than two levels of high-dimensional fixed effects, it has been known since Haberman (1974, Appendix B) that separation only involving categorical dummies (i.e., fixed effects) can be difficult to verify (see also Albert and Anderson, 1984, p. 9.) To our knowledge, this problem has remained unresolved in the literature and Abowd et al. (2002)’s method cannot be used to solve the problem for general cases either.\footnote{Abowd’s method can still be used to detect separation involving only one or two levels of fixed effects, but not separation involving three or more levels of fixed effects, or separation also involving the non-fixed effect regressors. In addition, it is worth clarifying that neither perfect collinearity between fixed effects nor separation involving only the fixed effects (by Proposition 3) poses an issue for identification of the non-fixed effect parameters. However, separation can affect an estimation algorithm’s ability to reach convergence, the speed at which it converges, and even whether the algorithm converges to the correct estimate values.}

Thus, unless we have a nonbinary choice model with either one or two levels of fixed effects, we
require a different method for detecting separation. Noting that a logit model can be transformed into a Poisson model by adding a fixed effect (as we discuss next), the same is also true for binary choice models with more than one fixed effect.

**Verifying separation in binary choice models using the logit-Poisson transformation**

While the discussion in Section 3.3 focuses on the case of a model with only a lower bound at zero, our methods can be applied to binary choice models without loss of generality. The only further complication that is needed is that we must first transform the model by taking advantage of the following property:

**Definition 1** *(The logit-equivalent Poisson model)* Any logit model with \( p(y_i = 1|x_i) = \exp(x_i\beta)/(1+\exp(x_i\beta)) \) can be rewritten as a logit-equivalent Poisson model via the following steps:

1. Let each observation now be given by \( y_{i,a} \) and be indexed by \( i \) and \( a = 1, 2 \). A “\( y_{i,1} \)” will henceforth indicate an “original” observation from the original logit model and a “\( y_{i,2} \)” will indicate an “artificial” observation. The construction of artificial observations is described in the next step.

2. For every original observation with \( y_{i,1} = 0 \), create an artificial observation with \( y_{i,2} = 1 \). For every original observation with \( y_{i,1} = 1 \), similarly create an artificial observation with \( y_{i,2} = 0 \). For all artificial observations, set all corresponding elements of \( x_{i,2} \) equal to 0. The number of observations should now be \( 2N \), where \( N \) is the original sample size.

3. Add a set of \( i \)-specific fixed effects to the model, to be given by \( \delta_i \). These may be thought of as the coefficients of a set of dummy variables \( d_i \), which equal 1 only for the two observations indexed by a particular \( i \) (one original observation and one artificial observation).

---

\(^{39}\)One possible method is the one discussed in Clarkson and Jennrich (1991) on p. 424, which allows estimation to proceed without precautions and iteratively drops any observations that appear to be converging to a boundary value. The algorithm we describe later in this Appendix could be used in conjunction with this approach. However, as Clarkson and Jennrich (1991) note, this method is not guaranteed to detect separation accurately. Furthermore, in our own implementations, we have noted that removing separated observations mid-estimation generally leads to slower convergence. Yet another problem arises if “cluster-robust” standard errors are used. In that case, the algorithm of Correia (2015) must also be repeatedly applied in order to determine that the correct number of non-singletons clusters that are left as additional separated observations are removed. Otherwise, statistical significance will tend to be overstated.
The resulting logit-equivalent Poisson model is given by 
\[ E[y_{i,t} | x_i, \delta_t] = \exp[\delta_t + x_{i,t} \beta] \]
and is estimated using Poisson regression.

After obtaining a Poisson model in this way, we have the following equivalence:

**Proposition 5** (Logit-Poisson Equivalence) The logit-equivalent Poisson model is equivalent to the original logit model. In particular:

- The first-order conditions (FOCs) for \( \beta \) are the same.
- The parameter estimates for \( \beta \) and their associated asymptotic variances are the same.
- The conditional mean from the Poisson model equals the conditional probability that \( y_i = 1 \) from the logit model.

The properties described in Proposition 5 can be established using the Poisson FOCs for \( \delta_t \) and \( \beta \):

\[
\sum_{i=1}^{N} x_{i,1} \left( y_{i1} - e^{\delta_t + x_{i,1} \beta} \right) = 0, \quad \forall i : \left( 1 - e^{\delta_t} \left( 1 + e^{x_{i,1} \beta} \right) \right) = 0,
\]

where we have used the fact that \( y_{i,1} + y_{i,2} = 1 \) and the fact that all elements of \( x_{i,2} = 0 \). It should be apparent that \( e^{\delta_t} = 1/(1 + e^{x_{i,1} \beta}) \). After plugging in the solution for \( \delta_t \) into the FOC for \( \beta \), we obtain

\[
\sum_{i=1}^{N} x_{i,1} \left( y_{i,1} - \frac{e^{x_{i,1} \beta}}{1 + e^{x_{i,1} \beta}} \right) = 0,
\]

which is the same as the FOC for \( \beta \) from the original logit model. The estimates for \( \beta \) therefore are the same across both models, as are the associated asymptotic variances. Furthermore, the Poisson conditional mean \( e^{\delta_t + x_{i,1} \beta} = \exp(x_i \beta) / [1 + \exp(x_i \beta)] \) is the same as \( p(y_i = 1|x_i) \) from the logit model.

The most important implication of these results for our current purposes is the following:

**Proposition 6** (Equivalence under separation) Suppose that \( l(\beta) \) conforms to (1), the matrix of regressors \( X = x_1, x_2, \ldots, x_M \) is of full column rank, and the individual log-likelihood \( l_i(\beta) \) always has a finite upper bound. Any binary choice model that satisfies these conditions is separated if and only if the logit-equivalent Poisson model is separated.
Suppose we have a binary choice model and there exists a nonzero separating vector \( \gamma^* \in \mathbb{R}^M \) that satisfies (4) and (5). Then, for any separated observation with \( y_i = 1 \), the FOC for \( \delta_i \) in the logit-equivalent Poisson model must satisfy \( e^{\delta_i} = \lim_{x_{i,1}\beta \to \infty} 1/(1 + e^{x_{i,1}\beta}) = 0 \) in the compactified model where such solutions are admissible. Thus, the artificially created observation associated with \( i \) \((y_{i,2})\) has a conditional mean of \( \mu_{i,2} = 0 \) and must be separated. Similarly, for any separated observation with \( y_i = 0 \), the conditional mean for \( y_{i,1}, \mu_{i,1} \), must be 0. This can only be true if \( y_{i,1} \) is separated.

If we instead consider separation in the Poisson model, we can simply focus on cases where either the original observation has a conditional mean of 0 or the artificially created observation has a conditional mean of 0. In the former case, it is obvious there is separation in either model. In the latter case, we must have that \( \delta_i = -\infty \), which can only be true if \( l(\beta) \) is increasing as \( x_{i,1}\beta \to \infty \), implying \( y_{i,1} \) is separated in the original logit model.

Finally, the conditions for a binary choice model to be separated depend only on the configuration of the data and do not depend on the specific choice of model (e.g., logit vs. probit). Therefore, the Poisson model described above can be used to check for separation in any conceivable GLM binary choice model for which the individual likelihood function is bounded from above, not just the logit model.

An IRLS algorithm for GLM estimation with high-dimensional fixed effects

While it is now well known that linear models with seemingly any number of nuisance parameters can be computed very rapidly (cf., Carneiro et al., 2012; Correia, 2017), comparable methods for nonlinear models have been slower to materialize and enter into wide usage. To date, most work in this area has focused on Poisson PML (PPML) estimation with high-dimensional fixed effects (HDFEs) (see Guimarães and Portugal, 2010; Figueiredo et al., 2015; Larch et al., 2019.) This is likely because of the popularity of PPML as an estimator for use with gravity models as well as the nice properties of the Poisson score function that make it both easier to work with in high-dimensional environments and more robust to incidental parameter problems than other estimators in nonlinear settings.\(^{40}\)

\(^{40}\)Weidner and Zylkin (2021) discuss the ”IPP-robustness” properties of the Poisson score functions and describe several common applications where PPML is a consistent estimator despite the presence of multiple levels of fixed effects.
However, the past few years have seen a number of interesting developments that have made fixed effects estimation more appealing in nonlinear environments. In particular, Guimarães (2014) and Stammann et al. (2016) have each independently showed how the classic iteratively reweighted least squares (or “IRLS”) approach to GLM estimation could be combined with the Frisch-Waugh-Lovell theorem to construct an HDFE-IRLS algorithm that avoids the inversion of large matrices ordinarily needed for estimation and that also can, in theory, be used with any GLM. Guimarães (2014) and Correia et al. (2020) have demonstrated versions of this idea for Poisson estimation in particular, whereas Stammann et al. (2016) illustrate an HDFE-IRLS approach for estimating a logit model with two-way fixed effects as well as how to obtain a bias correction to account for the incidental parameter bias. More recently, Stammann (2018) and Bergé (2018) have each described HDFE-GLM algorithms that can be used to estimate any of the GLMs covered in this paper.

Thus, to help fix ideas for readers and raise awareness of these advances, we provide here some brief notes on HDFE-IRLS estimation. A version of this algorithm appears in our own Stata command for HDFE-PPML estimation specifically (Correia et al., 2020), but here we will present a generalized HDFE-GLM algorithm based on IRLS as proof of concept. For notational simplicity, we will assume a doubly indexed panel with two levels of fixed effects. The two panel dimensions will be $i$ and $j$, such that $y_{ij} \geq 0$ will denote the dependent variable and $x_{ij}$ will denote the full matrix of all covariates, including two sets of fixed effects $d_i$ and $d_j$ as well as a set of “main” (non-fixed effect) covariates $w_{ij}$. $\beta$ will continue to denote the full coefficient vector, but we can similarly decompose $\beta$ into the coefficients for the fixed effects, which will be given by $\xi_i$ and $\xi_j$, as well as the main coefficient vector $\delta$. The GLM FOC for $\beta$ from (2) can then be rewritten (now in vector form) as

$$s(\beta) = \sum_{i,j} s_{ij}(\beta) = \sum_{i,j} \alpha_{ij} \left[ y_{ij} - \mu_{ij} \right] \theta'(x_{ij}\beta;\cdot)x_{ij} = 0,$$

(20)

where the functions $\theta_{ij}$, $b(\theta_{ij})$, and $\mu_{ij} = b'(\theta_{ij})$ are all defined analogously. Since IRLS is an iterative procedure, we will use a “0” superscript to denote the current value of an object and a “+” superscript for updated values to be used in subsequent iterations.

The first step is to obtain the classic IRLS linearization of the score function. As is standard, this is achieved by deriving a first-order Taylor approximation for the conditional mean $\mu_{ij}$ around
an initial guess for the parameter vector $\beta^0$. This approximation reads as

$$\mu^*_i \approx \mu_i^0 + b''(\theta_i^0)\theta'(x_i\beta^0; \cdot)x_i(\beta^* - \beta^0). \quad (21)$$

Next, we define the IRLS working dependent variable as

$$q_i^0 := \frac{y_i - \mu_i^0}{b''(\theta_i^0)\theta'(x_i\beta^0; \cdot)} + x_i\beta^0. \quad (22)$$

Substituting (21) and (22) into (20) then delivers the IRLS approximation of the score vector for $\beta$:

$$s(\beta) \approx \sum_{i,j} \psi_{ij}^0 \left(q_{ij}^0 - x_i\beta\right)x_{ij}, \quad (23)$$

where $\psi_{ij}^0 := \alpha_{ij}b''(\theta_{ij}^0)\theta'(x_{ij}\beta^0; \cdot)^2$ will henceforth denote a combined weighting term. Setting the approximated score equal to zero then delivers the following closed-form solution for the updated estimate $\beta^+$:

$$\beta^+ = \left[X'\psi^0 X\right]^{-1} \left[X'\psi^0 q^0\right], \quad (24)$$

where $\psi^0$ is a suitably sized diagonal matrix with $\psi_{ij}^0$ on the diagonal, $X$ is the matrix of regressors, and $q^0$ is a vector containing $q_{ij}^0$. In other words, the IRLS approach allows us to estimate an approximate solution for $\beta$ using weighted least squares, weighting by $\psi_{ij}^0$ and using $q_{ij}^0$ as the dependent variable. The full algorithm uses each new solution for $\beta^+$ to progressively update the values for $\psi_{ij}^0$ and $q_{ij}^0$ used in (24) as well as the underlying values for $\mu_i^0$ and $\theta_i^0$. Eventually, as $\mu_i^0$ and $\theta_i^0$ converge to their true estimated values, the approximation for $\beta$ in (24) then becomes exact and $\beta^+$ converges to the true parameter estimate that solves the original GLM FOC in (20).

Notice that the method just described usually requires inverting the matrix $X'\psi^0 X$ each time we re-estimate $\beta^+$. Thus, IRLS would ordinarily be computationally intensive or even infeasible if the model includes a large number of fixed effects. However, because IRLS gives us a linear regression expression for $\beta$, we can use the Frisch-Waugh-Lovell theorem to derive a more easily computable version of (24) that allows us to obtain an equivalent approximation for our main coefficient vector $\delta$ specifically. The form of this approximation is

$$\delta^+ = \left[w'\psi^0 w\right]^{-1} \left[w'\psi^0 q^0\right], \quad (25)$$
where the bars over both $\bar{w}$ and $\bar{q}^0$ are meant to indicate these are weighted, within-transformed versions of the matrix of non-fixed effect covariates $w$ and of the working dependent variable vector $q^0$. More precisely, $\bar{w}$ and $\bar{q}^0$ can respectively be thought of as the residuals obtained by regressing the untransformed variables $w$ and $q^0$ on the two fixed effects $d_i$ and $d_j$ and weighting by $\psi^0$. By the Frisch-Waugh-Lovell theorem, the solution we obtain for $\delta^+$ will be exactly consistent with the larger vector $\beta^+$ we would have obtained from (24), because we will have already purged $\bar{w}$ and $\bar{q}^0$ of any partial correlation with the fixed effects. Thanks to the methods of Correia (2017), the within-transformation step needed for this type of procedure is generally always feasible and can be computed much faster than a direct estimation of the full model when the number of fixed effects is large.\footnote{The observation that the Frisch-Waugh-Lovell theorem can be used to speed up fixed effects estimation in this way is originally thanks to Guimarães and Portugal (2010) and Gaure (2013). Correia (2017) then demonstrates several computational innovations that can be used to speed up the alternating projections-based methods used to perform the within-transformation step in these earlier papers so that it converges in nearly linear time. Correia et al. (2020) discuss further speed-up tricks that can be used to accelerate HDFE-GLM estimation specifically.}

To fill in the remaining details, the Frisch-Waugh-Lovell theorem also very usefully implies that $\bar{q}^0_{ij} - \bar{w}_{ij} \delta^+ = q^0_{ij} - x_i \beta^+$ (i.e., that the residuals from the two versions of the same regression are equal to one another). This last insight enables us to update the linear predictor term $x_{ij} \beta^+$ using $x_{ij} \beta^+ = q^0_{ij} - (\bar{q}^0_{ij} - \bar{w}_{ij} \delta^+)$, which then in turn allows us to update $\theta^+_{ij} = \theta(x_{ij} \beta^+)$, $\mu^+_{ij} = b'(\theta^+_{ij})$, $\psi^+_{ij} = \alpha_{ij} b''(\theta^+_{ij}) \theta'(x_{ij} \beta^+)$, and $q^+_{ij} = (y_{ij} - \mu^+_{ij})/\psi^+_{ij} + x_{ij} \beta^+$. These steps are exactly equivalent to the updating steps used in the classic IRLS estimation loop. Thus, eventually $\theta^+_{ij}$, $\mu^+_{ij}$, $\psi^+_{ij}$, and $q^+_{ij}$ will converge to their correct values, and $\delta^+$ will converge to the correct estimate as well.