Conformal anomaly in 2d dilaton–scalar theory

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Abstract

The discrepancy between the anomaly found by Bousso and Hawking and that of other workers is explained by the omission of a zero mode contribution to the effective action.

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1. Introduction

A number of recent, and not so recent, papers have been concerned with the conformal anomaly in the dilaton–scalar system in two-dimensional gravity. This anomaly takes the general, local form

\[ T_\mu^\mu = T(x) = \frac{1}{24\pi}(R - 6\nabla^\mu \phi \nabla_\mu \phi + \alpha \Box \phi) \]  

where \( \phi \) is the dilaton field. The treatments agree on the first two terms but the coefficient, \( \alpha \), of the total divergence is subject to some discussion. Firstly there is the question of whether the correct two-dimensional reduction of the spherical four-dimensional theory has been taken. The reduction adopted by Elizalde, Naftulin and Odintsov, [1], Mukhanov, Wipf and Zelnikov, [2], and by Kummer, Liebl and Vassilevich (KLV), [3,4], produces \( \alpha = 6 \). (This follows from the choice \( \varphi = \psi = \phi \) in [3]. See also Chiba and Siino, [5]). Bousso and Hawking (BH), [6], effectively choose a measure such that, in the notation of KLV, \( \psi = 0 \) and \( \varphi = \phi \). BH obtain the value \( \alpha = -2 \) while KLV’s formula gives \( \alpha = 4 \). This latter value is also obtained by Ichinose, [7]. In this brief note we consider only this discrepancy since it seems to be a clear mathematical contradiction. The existence of a discrepancy, actually between the \( \alpha = -2 \) and \( \alpha = 6 \) values, was early noted by Nojiri and Odintsov, [8], who ascribed it entirely to total divergence ambiguities. For completeness, by repeating some standard material, we will here confirm the value \( \alpha = 4 \) and then indicate where we think the calculation of BH breaks down.

2. The dynamics and the anomaly

In its simplest form the (matter) action adopted is, in 2d,

\[ S_m = -\frac{1}{2} \int_{\mathcal{M}} e^{-2\phi} \nabla^\mu f \nabla_\mu f \sqrt{g} d^2 x \]

where \( f \) is the scalar matter field, with the corresponding field operator

\[ A = e^{-2\phi}(-\Box + 2\nabla^\mu \phi \nabla_\mu). \]  

The most rapid method of finding the anomaly relies on its standard expression, \( \zeta(0, x) \), in terms of the local \( \zeta \)-function associated with \( A \), or, entirely equivalently, of the heat-kernel coefficient, \( C_1^{(2)}(x) \), [9]. To this end the operator \( A \) is rewritten

\[ A = -e^{-2\phi}((\nabla^\mu - \nabla^\mu \phi)(\nabla_\mu - \nabla_\mu \phi) + V) \]
where

\[ V = \Box \phi - \nabla^\mu \phi \nabla_\mu \phi. \]

Introducing the auxiliary metric

\[ g'_{\mu\nu} = e^{2\phi} g_{\mu\nu} \]

\( A \) can also be written as

\[ A = -\left( (\nabla'^\mu - \nabla'^\mu \phi)(\nabla'_\mu - \nabla'_\mu \phi) + V' \right) \]

with

\[ V' = \Box' \phi - \nabla'^\mu \phi \nabla'_\mu \phi = e^{-2\phi} V, \]

where \( \nabla'^\mu \) is \( \nabla'_\mu \), raised by \( g'^{\mu\nu} \).

When computing the eigenvalues of \( A \), the scalar product of the \( f \)'s is defined using the covariant measure of the \( g \) metric. However, the trivial Weyl potential, \( \nabla'_\mu \phi \), can be removed by the gauge transformation, \( f \rightarrow f' = \exp(-\phi) f \). The \( f' \)'s are normalized using the auxiliary metric, \( g' \), and have the field operator \( A' \) where

\[ A' = e^{-\phi} Ae^\phi = -(\Box' + V'). \]

The formal computation of \( \zeta(0) \) can thus proceed as for the standard Laplacian by treating, temporarily, \( g' \) as the metric. We will therefore find the integrated anomaly

\[ T = \zeta(0) = \frac{1}{4\pi} \int C_1^{(2)}(g', x) \sqrt{g'} d^2x \]

and can use the expression for \( C_1 \) derived many years ago,

\[ C_1^{(n)}(g', x) = \frac{R'}{6} + V', \quad \text{(3)} \]

where the coordinate system has been extended artificially to an \( n \)-dimensional one for later use.

The local trace anomaly, expressed as a density in the auxiliary metric is, \([9]\),

\[ T'(x) = \frac{1}{4\pi} C_1^{(2)}(g', x) = \frac{1}{4\pi} \left( \frac{R'}{6} + V' \right). \quad \text{(4)} \]

In order to obtain a density in the original metric, \( g \), one simply rewrites the \( g' \) in (4) in terms of \( g \) and removes the resulting overall factor of \( e^{-2\phi} \) to allow for the change in the \( \sqrt{g} \)'s. As advertised we find,

\[ T(x) = \frac{1}{24\pi} (R - 6 \nabla^\mu \phi \nabla_\mu \phi + 4 \Box \phi). \]
Being based on standard techniques, this discussion adds nothing material to the earlier treatments. However, as an amusing novelty, one can check the total derivative term, $\Box' \phi$, in (3) in the following way.

Instead of introducing the gauge potential we treat the operator $A$ as it stands in (2) and, further, work in $n$ dimensions. The idea here is to use our previous technique, [10], of deriving the total derivative term in the local coefficient from the integrated coefficient (from which of course this term is absent).

To save writing a lot of primes we replace $g'$ in $A$ by $g$ which should be thought of simply as a generic metric. (This is only a notational convenience for the purposes of this check.)

Because $A$ is not conformally covariant in $n$ dimensions the calculation is not quite straightforward, but the necessary formalism is available in [10]. The behaviour of $A$ under scale changes $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$ is easily determined to be

$$(\tilde{A} + U)\tilde{f} = \lambda^{-(n+2)/2} Af, \quad \tilde{f} = \lambda^{(2-n)/2} f,$$

where $U$ measures the loss of conformal covariance and equals, (cf [10]),

$$U = (n-2)\nabla^\mu \omega \nabla_\mu \phi + \xi(n)(n-1)(2\lambda^{-1} \Box \lambda - (n-2)\nabla^\mu \omega \nabla_\mu \omega)$$

with $\xi(n) = (n-2)/4(n-1)$ and $\omega = -\ln \lambda$. The second part of $U$ is connected with the noncovariance of the Laplacian.

Working around $\omega = 0$ (when $U$ vanishes) and applying perturbation theory in $U$ allows one to relate the relevant $\zeta$–functions and thence the heat-kernel coefficients to obtain, [10],

$$\frac{1}{\sqrt{g}} \left. \frac{\delta C_k^{(n)}[e^{-2\omega g}]}{\delta \omega(x)} \right|_{\omega=0} = - (n-2k) C_k^{(n)}(g,x)$$

$$+ \left( (n-2)(\Box \phi + \nabla_\mu \phi \nabla^\mu) + 2(n-1)\xi(n) \Box \right) C_{k-1}^{(n)}(g,x).$$

We use this equation to find the local coefficient on the right from the variation of the integrated one on the left.

As our application we set $k = 1$. Using the fact that $C_0$ is the Weyl volume term, $C_0^{(n)}(g,x) = 1$, we quickly find

$$C_1^{(2)}(g,x) = \Box \phi + \lim_{n \to 2} \frac{1}{n-2} \frac{1}{\sqrt{g}} \left. \frac{\delta C_1^{(n)}[e^{-2\omega g}]}{\delta \omega(x)} \right|_{\omega=0}.$$

(6)
The point of this little exercise is simply to say that, assuming we know only the integrated $C_1^{(n)}$,

$$C_1^{(n)}[g] = \int \left( \frac{R}{6} - (\nabla \phi)^2 \right) \sqrt{g} \, d^n x,$$

then the variation and limit in (6) easily yield

$$C_1^{(2)}(g, x) = \Box \phi + \frac{R}{6} - (\nabla \phi)^2$$

showing the resurrection of the total derivative contribution.

3. Discussion

We now turn to the question raised earlier concerning the origin of the discrepancy with the result of BH, [6]. The problem arises when BH assume, after their eqn. (3.4), that the manifold has the topology of the two-sphere, for then there is a zero mode of the Laplacian and one cannot use the quoted Polyakov form for the effective action (eqn. (3.3)). It is better to use the antisymmetrical cocycle function, $W[\mathcal{G}, g] \sim W[\mathcal{G}] - W[g]$, for the conformal change $g \to \mathcal{G}$.

As we have shown, [11], because of the zero mode, apart from the standard contribution, there is an additional term of the form

$$\Delta W[\mathcal{G}, g] = \frac{1}{48 \pi |\mathcal{M}|} \int \ln \left( \frac{g}{\mathcal{G}} \right) \sqrt{g} \, d^2 x \int R \sqrt{\mathcal{G}} \, d^2 x$$

where $|\mathcal{M}| = |\mathcal{M}(g)|$ is the two-surface area.

Computing this for the uniform rescaling, $\mathcal{G}_{\mu \nu} = \exp(2\phi_c) g_{\mu \nu}$, yields an extra contribution which cancels the change in the effective action used by BH – the last term in eqn. (3.5). (This must be so for consistency and is the whole point of [11].) If we carried on with the analysis as in BH, then we would conclude that $q_3, = \alpha/24\pi$, were zero.

One way of partially retrieving the situation is to use the cocycle function, $W[\mathcal{G}, g]$, (as we should). Then the last term in eqn. (3.3) is replaced by

$$\frac{1}{2} q_3 \int \left( R \sqrt{\mathcal{G}} - R \sqrt{g} \right) \phi \, d^2 x$$

which vanishes when $\phi$ is uniform by topological invariance, but which still has the required variation and everything is consistent. However it is not then possible to
deduce the value of $q_3$ in a simple way, at least not by conformal transformations in two dimensions alone.

If we wish to avoid a zero mode, then it is necessary to include boundary terms in the effective action, *when this last is being evaluated*. In any case, zero mode or boundary, the additional contributions remove any discrepancies and also, incidentally, render nugatory the specific criticisms by Nojiri and Odintsov, [8], of Bousso and Hawking’s choice of term in the effective action. The problem is not so much the ambiguity in this term, rather it is its incorrect behaviour for uniform $\phi$.

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