Conformal Killing vector fields and a virial theorem

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Received 11 June 2014, revised 27 August 2014
Accepted for publication 16 September 2014
Published 4 November 2014

Abstract

The virial theorem is formulated both intrinsically and in local coordinates for a Lagrangian system of a mechanical type on a Riemann manifold. An important case studied in this paper is that of an affine virial function associated with a vector field on the configuration manifold. The special cases of a virial function associated with a Killing, a homothetic, and a conformal Killing vector field are considered and the corresponding virial theorems are established for these types of functions.

Keywords: virial theorem, Hamiltonian systems, symplectic manifolds, canonical transformations
PACS numbers: 02.40.Yy, 45.20.Jj
Mathematics Subject Classification: 37J05, 70H05, 70G45

1. Introduction

Since the establishment of the so-called virial theorem in 1870, its usefulness and range of applicability have been increasing almost continuously. It was stated by Clausius that the mean vis viva of the system is equal to its virial, where vis viva integral is the total kinetic energy of the system. The Latin word virias was used by Clausius to denote the scalar quantity represented in terms of the forces $F_i$ acting on the system as

$$\frac{1}{2} \left( \sum F_i \cdot r_i \right),$$

and it was shown to be minus one-half of the averaged potential energy of the system [1].
The important point is the wide range of applicability of the virial theorem, as it is applicable to dynamical and thermodynamical systems, it can also be formulated to deal with relativistic (in the sense of special relativity) systems. It is also applicable to systems with velocity-dependent forces and viscous systems, and even if it provides less information than the equations of motion, it is simpler to apply and then it can provide information concerning systems whose complete analysis may defy description. For instance, in astronomy, the virial theorem finds applications in the theory of dust and gas of interstellar space as well as cosmological considerations of the universe as a whole and in other discussions concerning the stability of clusters, galaxies, and clusters of galaxies. For an excellent historical account, one can see [2].

In one-particle Newtonian mechanics of a particle of mass $m$ under the action of a force $F$, the virial function introduced by Clausius is $G(x, \dot{x}) = m x \cdot \dot{x}$, and one can show, using Newton second law, that $dG/dt = m x \cdot \dot{x} + x \cdot F$, and when integrating this expression between $t = 0$ and $t = \tau$, dividing by the total time interval $\tau$ and taking the limit of $\tau$ going to infinity, we find that if the possible values of $G$ are bounded, then $\langle (2T(x) + x \cdot F) \rangle = 0$. In the particular case of a conservative force, $F = -\nabla V$, $\langle (2T(x) - x \cdot \nabla V) \rangle = 0$. When the potential $V$ is homogeneous of degree $k$, Euler’s theorem of homogeneous functions implies that $x \cdot \nabla V = k/V$, and therefore, $\langle (2T(x) - k V(x)) \rangle = 0$, i.e., if $E$ is the total energy, $\langle T(x) \rangle = \frac{kE}{k + 2}$, $\langle V(x) \rangle = \frac{2E}{k + 2}$.

Remark that the existence of the time average of a function depends on the evolution curve and therefore on the initial conditions. The assumption that the function remains bounded guarantees that such an average does exist no matter what the evolution curve is, and, moreover, when the motion is periodic, the average coincides with the average in a time period. On the other side, the relation $\langle (A + B) \rangle = \langle A \rangle + \langle B \rangle$ holds when the three averages do exist. In particular, expressions such as $\langle (A - B) \rangle = 0$ imply $\langle A \rangle = \langle B \rangle$ when $\langle A \rangle$ does exist.

Relevant questions about this result are as follows. Where does the virial function $G$ come from? Why is the relation simpler for power law potentials? What is the reason for the values of the coefficients? Is there any generalization? And what about a quantum mechanical interpretation? The answer to all these questions rests on the geometrical interpretation of the virial theorem for dynamical systems, in particular for systems defined by regular Lagrangians (see [3] and references therein). The standard virial theorem is based on the transformation properties of kinetic and potential energy under dilations, and therefore is only valid for systems with $\mathbb{R}^n$ as the configuration space. To generalize the virial theorem for other systems, we should use the tools of geometric mechanics. First, the problem was analyzed in the framework of Hamiltonian dynamical systems, and therefore for systems described by a regular Lagrangian. But the possibility of establishing a virial-like relation in the case in which we have a vector field $X$, which is a complete lift of a vector field on the base manifold such that $XL = aL$, was also studied in [4], as well as the even more general case of vector fields whose flows are non-strictly canonical transformations. The virial-like relations we obtain are more general than the standard ones and some have been used in tensor virial theorems or the so-called hypervirial theorems [5].

The equivalence of Lagrangian and Hamiltonian formalisms in the regular case is well known. Actually, the geometric approach was first developed in the Hamiltonian formalism in the framework of symplectic geometry in phase spaces (i.e., cotangent bundles) and then the Legendre transformation was used to translate the symplectic structure to the Lagrangian formalism. However, it was soon proven that one can develop the formalism in the framework
of tangent bundle geometry by using the geometric tensors characterizing tangent bundle structures, the vertical endomorphism, and the Liouville vector field \[6–9\]. As virial-like relations can be directly established in terms of the Lagrangian function and are not so easily derivable in the Hamiltonian formalism, we will mainly restrict ourselves to the Lagrangian formalism, even if the final expressions can be translated to the Hamiltonian language. The extension of some of such results to the framework of mechanics in Lie algebroids was developed in \[10\].

This paper attempts to develop analogous results in the particular case of mechanical-type Lagrangians, and in this case conformal Killing vector fields will be shown to play a very relevant role. For mechanical systems, \(L = T - V\), finding infinitesimal symmetries of the metric, i.e., Killing fields, is relatively easy. As it is well known, if such a vector field is also a symmetry of the potential, we get a constant of the motion, which simplifies the problem. If the Killing vector field is not a symmetry of the potential, the virial theorem provides relevant information, namely the average value of the derivative of the potential vanishes. With more generality, for a homothetic, or a conformal Killing vector field, the virial theorem allows us to establish relations between the averages of the kinetic energy and those of certain derivatives of the potential.

The paper is organized as follows. In section 2, some geometrical concepts about Riemann structures are recalled and relevant expressions for tensor fields and functions are written in generalized coordinates. In section 3, the virial theorem is presented for Lagrangian systems of the mechanical type, both in intrinsic form and in terms of local coordinates, and a spherical geometry problem is analyzed using this approach. In section 4, we consider an important particular case of an affine on the velocities virial function, associated with a vector field on the configuration manifold, more specifically, when the vector field is either a Killing, a homothetic, or a conformal Killing vector field. Several examples are used to illustrate the theory. In the last section, we offer final comments about the results presented in the paper.

2. Riemann structures and mechanical-type Lagrangians

Let \((M,g)\) be a pseudo-Riemann manifold, i.e., \(g\) is a non-degenerate symmetric two times covariant tensor field on \(M\). Non-degeneracy means that the map \(\tilde{g}: T^\ast M \to T^\ast M\) is a fibered map over the identity on \(M\) and induces the corresponding map between the spaces of sections of the tangent and cotangent bundles, to be denoted by the same letter \(\tilde{g}: \mathfrak{X}(M) \to \Omega^1(M): \langle \tilde{g}(X), Y \rangle = g(X, Y)\).

A diffeomorphism \(F: M \to M\) induces a new pseudo-Riemann structure \(F^\ast g\) on \(M\). Such a transformation \(F\) is called a conformal symmetry, when there exists a function \(f \in C^\infty(M)\) such that \(F^\ast g = f \cdot g\). In particular, when \(f\) is a constant (different from one), \(F\) is said to be a (proper) homothety and, finally, when \(F^\ast g = g\), the map \(F\) is called isometry. In the infinitesimal approach, we say that a vector field \(X \in \mathfrak{X}(M)\) is either a conformal, a homothetic, or a Killing vector field, when its flow \(\phi_t\) is made of conformal maps, homotheties, or isometries, respectively:
conformal vector field: \( \mathcal{L}_X g = fg \), \( f \in C^\infty(M) \),

homothetic vector field: \( \mathcal{L}_X g = \lambda g \), \( \lambda \in \mathbb{R} \),

Killing vector field: \( \mathcal{L}_X g = 0 \).

Proper conformal vector fields are those vector fields for which the conformal factor \( f \) is non-constant and similarly a proper homothetic vector field is when \( \lambda \neq 0 \). Using the well-known property \( \mathcal{L}_X \circ \mathcal{L}_Y = \mathcal{L}_Y \circ \mathcal{L}_X = \mathcal{L}_{[X,Y]} \), one sees that the set of conformal vector fields is a Lie algebra and those of homothetic and Killing vector fields are subalgebras. For more details, see [11–14].

Given a symmetric covariant two-tensor field \( K \) in \( M \), we denote by \( T_K \in C^\infty(TM) \) the function

\[
T_K(v) = \frac{1}{2} K(v, v), \quad v \in TM.
\]

This rule identifies symmetric covariant two-tensor fields with quadratic homogeneous functions on the fiber coordinates. In particular, when \( g \) is a Riemann structure in \( M \)

\[
T_g(v) = \frac{1}{2} g(v, v), \quad v \in TM,
\]

is the kinetic energy defined by the metric.

Given a local chart \( (U, q^1, ..., q^n) \) on \( M \), we can consider the coordinate basis of \( \mathfrak{X}(U) \) usually denoted \( \{\partial/\partial q^i\} | i = 1, ..., n \) and the dual basis for \( \mathfrak{Q}(U), \{dq^i\} | i = 1, ..., n \). Then, a vector in a point \( q \in U \) is \( v = v^i (\partial/\partial q^i)_q \) and a covector is \( \xi = p_i (dq^i)_q \), with \( v^i = (dq^i, v) \) and \( p_i = (\xi, \partial/\partial q^i) \) being the usual velocities and momenta, respectively. The local expression for \( g \) is

\[
g = g_{ij}(q)dq^i \otimes dq^j. \tag{1}
\]

We can define Lagrangians of the mechanical type for systems with configuration space \( M, L \in C^\infty(TM) \), by choosing a pseudo-Riemann structure \( g \) on \( M \) and a potential function \( V \in C^\infty(M) \) as follows:

\[
L_{g,V}(q, v) = \frac{1}{2} g_{ij}(v, v) - (\tau^*_M V)(q, v) = \frac{1}{2} g_{ij}(v, v) - V(q), \tag{2}
\]

i.e., the Lagrangian function is of the form \( L_{g,V} = T_g - \tau^*_M V \), where the function \( T_g \in C^\infty(TM) \) represents the kinetic energy given previously, which can be rewritten as

\[
T_g = \frac{1}{2} g(T_{\tau_M} \circ D, T_{\tau_M} \circ D),
\]

with \( D \) being any second-order differential equation vector field, i.e., a vector field on \( TM \) such that \( \tau_M \circ D = id_{TM} \), whereas the potential energy \( \bar{V} = \tau^*_M V \) is a basic function, i.e., the pull-back of a smooth function \( V \) on the base manifold \( M \).

Given a Riemann structure \( g \) on a manifold \( M \) with local expression in a local chart (1), the expression for the corresponding (free, i.e., \( V = 0 \) Lagrangian, i.e., the function \( T_g \), is

\[
T_g(q, v) = \frac{1}{2} g_{ij}(q)v^i v^j. \tag{3}
\]
whereas the coordinate expression of an arbitrary second-order vector field is

\[ D(q, v) = v^j \frac{\partial}{\partial q^j} + f^j(q, v) \frac{\partial}{\partial v^j}. \]  

(4)

Given a vector field on \( M \),

\[ X = X^i(q) \frac{\partial}{\partial q^i} \in \mathfrak{X}(M), \]

(5)

the Lie derivative with respect to the vector field \( X \) of the metric tensor field \( g \) is

\[ \mathcal{L}_X g = X^i \frac{\partial g_{ij}}{\partial q^i} dq^i \otimes dq^j + g_{ij} \left( \frac{\partial X^i}{\partial q^j} dq^j \otimes dq^i + \frac{\partial X^j}{\partial q^i} dq^i \otimes dq^j \right), \]

or using the symmetry property of the metric tensor field,

\[ \mathcal{L}_X g = \left( X^k \frac{\partial g_{ij}}{\partial q^k} + g_{ik} \frac{\partial X^k}{\partial q^i} + g_{jk} \frac{\partial X^k}{\partial q^j} \right) dq^i \otimes dq^j, \]

(6)

and then the condition for \( X \) to be a Killing vector field, i.e., \( \mathcal{L}_X g = 0 \), is written in the previously mentioned local coordinates as

\[ \left( X^k \frac{\partial g_{ij}}{\partial q^k} + g_{ik} \frac{\partial X^k}{\partial q^i} + g_{jk} \frac{\partial X^k}{\partial q^j} \right) dq^i \otimes dq^j = 0. \]

Therefore, the set of conditions for the vector field \( X \in \mathfrak{X}(M) \) given by equation (5) to be a Killing symmetry are

\[ X^k \frac{\partial g_{ij}}{\partial q^k} + g_{ik} \frac{\partial X^k}{\partial q^i} + g_{jk} \frac{\partial X^k}{\partial q^j} = 0, \quad i, j = 1, \ldots, n. \]

(7)

Consider now the complete lift \( X^c \in \mathfrak{X}(TM) \) with flow \( T\phi_t \), where \( \phi_t \) is the flow of the vector field \( X \in \mathfrak{X}(M) \) with local expression (5). Then, the local coordinate expression of \( X^c \) is

\[ X^c(q, v) = X^i(q) \frac{\partial}{\partial q^i} + v^j \frac{\partial X^i}{\partial q^j}(q, v) \frac{\partial}{\partial v^i} = X^i(q) \frac{\partial}{\partial q^i} + (DX^i)(q, v) \frac{\partial}{\partial v^i}, \]

for any second-order differential equation vector field \( D \).

For a 1-form \( \alpha \) on \( M \), we denote by \( \tilde{\alpha} \) the associated linear function on \( TM \) given by \( \tilde{\alpha}(v) = \langle \alpha(\tau_M(v)), v \rangle \), for \( v \in TM \). In local tangent bundle coordinates, if \( \alpha = \alpha_i(q) dq^i \), the function \( \tilde{\alpha} \) is \( \tilde{\alpha}(q, v) = \alpha_i(q) v^i \). In particular, for an exact 1-form \( \alpha = df \) the associated linear function is \( \tilde{f}^j(q, v) = v^i \frac{df}{dq^i} \), i.e., \( \tilde{f} \) looks like the total derivative of the function \( f \), and we denote \( \tilde{f} = \tilde{df} \), which can also be obtained by \( \tilde{f} = \mathcal{L}_D(\tau_M f) \) for an arbitrary second-order differential equation vector field \( D \). Complete lifts are determined by the action on this kind of function: given a vector field \( X \) on \( M \), its complete lift \( X^c \) is the only vector field on \( TM \), which satisfies

\[ \mathcal{L}_X \tilde{\alpha} = \tilde{\mathcal{L}}_X \alpha \]

(8)

for every 1-form \( \alpha \) on \( M \). It is clear that the preceding condition determines a vector field on \( TM \). Let us show that the complete lift satisfies such a relation. If \( \phi_t \) is the flow of \( X \), then the flow of \( X^c \) is \( T\phi_t \), so that for \( v \in TM \), with \( q = \tau_M(v) \), we have
\[
(\mathcal{L}_{X^*}\tilde{\alpha})(v) = \frac{d}{dt}(T\phi_t(v)) \bigg|_{t=0} = \frac{d}{dt} \left( \alpha_{\tau_M(T\phi_t(v)), T\phi_t(v)} \right) \bigg|_{t=0}
= \frac{d}{dt} \left( \alpha_{\tau_M(q), T\phi_t(v)} \right) \bigg|_{t=0} = \frac{d}{dt} \left( \left( \phi^{\alpha}_q \right)_q, v \right) \bigg|_{t=0}
= \left( (\mathcal{L}_X\alpha)_{\tilde{\alpha}}, v \right) = \hat{\mathcal{L}}_{\tilde{\alpha}}\alpha(v).
\]

In particular, for \( \alpha = df \), we have \( \mathcal{L}_{X^*}\tilde{f} = (\mathcal{L}_Xf) \).

A remarkable property to be used later on is that for a given a vector field \( X \in \mathfrak{X}(M) \), \([X^*, D]\) is a vertical vector field in \( TM \) for any second-order differential equation vector field \( D \), because \( X^*D(q^i) = D(X^i) = DX^i(q^i) = v^k \partial X^i/\partial q^k \). The preceding property (8) can also be used to give an intrinsic proof as follows. Indeed, the action on basic functions is

\[
\tau_{\tilde{M}}^f \tau_{\tilde{M}}^f = \tau_{\mathcal{L}_Xf}^f - \mathcal{L}_X\tau_{\mathcal{L}_Xf} = (\mathcal{L}_Xf) - \mathcal{L}_{X^*}\tilde{f} = 0
\]

from where it follows that \([X^*, D]\) is vertical.

One of the more important properties of complete lifts is the following relationship:

\[
X^*T_g = T_{\mathcal{L}_g}\tilde{f}.
\] (9)

In fact,

\[
(X^*T_g)(q, v) = \frac{1}{2} \left( X^k(q) \frac{\delta g}{\delta q^i}(q) v^i v^j + g_{ij}(q) \frac{\partial X^k}{\partial q^i}(q) v^j + g_{ij}(q) \frac{\partial X^k}{\partial q^j}(q) v^i \right)
= \frac{1}{2} \left( X^k(q) \frac{\delta g}{\delta q^i}(q) + g_{ij}(q) \frac{\partial X^k}{\partial q^j}(q) + g_{ij}(q) \frac{\partial X^k}{\partial q^i}(q) \right) v^j
\]

and therefore, according to equation (6), relation (9) follows. This relation may also be proven intrinsically by using the definitions of Lie derivative and of \( T_g \) mentioned previously in the text: for all \( v \in TM \),

\[
X^*T_g(v) = \frac{d}{dt} T_g \circ T\phi_t(v) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{1}{2} g \left( T\phi_t(v), T\phi_t(v) \right) \right) \bigg|_{t=0}
= \frac{1}{2} \frac{d}{dt} \left( \phi^{\alpha}_q g \right)(v, v) \bigg|_{t=0} = \frac{1}{2} (\mathcal{L}_Xg)(v, v) = T_{\mathcal{L}_g}\tilde{f}(v, v).
\]

Consequently, \( X \in \mathfrak{X}(M) \) is a Killing vector field for the Riemann structure \( g \) if and only if \( X^* \in \mathfrak{X}(TM) \) is a symmetry for the corresponding free Lagrangian, i.e., the conditions for \( X^* \) to be a symmetry of \( T_g \) are given by equation (7).

3. A virial theorem for mechanical-type Lagrangians

A (regular) Lagrangian determines a symplectic structure on the tangent bundle \( TM \), the Cartan two-form \( \omega_T = -d\theta_T = -d(\mathcal{L}_S) \). Here, \( S \) is the vertical endomorphism \([8, 9]\), which is defined using the natural identification of the tangent space \( T_pM \) with the vertical subspace of the tangent space in any point of \( \tau_M^{-1}(q) \). Such a vertical lift allows us to lift a tangent vector field \( X \in \mathfrak{X}(M) \) to a vertical vector field \( X^v \in \mathfrak{X}(TM) \). This vector field is related to the complete lift by \( S(X^v) = X^* \), and if the local coordinate expression of \( X \) is

\[
\sum_i X^i \frac{\partial}{\partial q^i} = \sum_i X^i_{\text{local}} \frac{\partial}{\partial x^i}.
\]
equation (5), that of $X^v$ is

$$X^v(v) = X^v(q) \frac{\partial}{\partial v^i} \in \mathfrak{X}(TM).$$

The energy of a Lagrangian system is defined by $E_L = \Delta L - L$, where $\Delta$ is the Liouville vector field, generator of dilations along the fibers, given by

$$\Delta f(q, v) = \frac{d}{dt}f(q, e^t v)|_{t=0}$$

for all $(q, v) \in TM$ and $f \in C^\infty(M)$. Hence, as $\Delta(T_q) = 2 T_q$ and $\Delta(V) = 0$, the total energy of a Lagrangian system of the mechanical type is $E_L = T_q + V$.

The dynamics are then given by the dynamical vector field $I_L$ defined for a regular Lagrangian $L$ by

$$i(I_L)\omega_L = dE_L. \quad (10)$$

In particular, the coordinate expression of the Cartan 1-form $\theta_L = dL \circ S$ for the Lagrangian (2) is given by

$$\theta_L(q, v) = g_{ij}(q) v^i dq^j$$

and the symplectic form $\omega_L = -d\theta_L$ by

$$\omega_L = g_{ij} dq^i \wedge dv^j + \frac{1}{2} \left( \frac{\partial g_{ij}}{\partial q^k} v^i v^j - \frac{\partial g_{ij}}{\partial q^k} v^j v^i \right) dq^i \wedge dq^k.$$

A regular Lagrangian system on $M$ can be seen as a Hamiltonian system $(TM, \{\cdot, \cdot\}, H)$, where the Hamiltonian $H$ is the energy $E_L$ and the Poisson bracket $\{\cdot, \cdot\}$ is defined by the symplectic two-form $\omega_L$, i.e., $[F_1, F_2] = \omega_L(X_{F_1}, X_{F_2})$, where $i(X_F)\omega_L = dF$. Recall that a Poisson bracket is a skew-symmetric bilinear map on the algebra of smooth functions on the manifold that obeys the Jacobi identity and the Leibniz rule w.r.t. the first argument. In this case, the virial theorem states (see [3] and references therein) that for a smooth bounded function $G$, the time average of the Poisson bracket $\langle \langle G \rangle \rangle = 0$.

We next recall some important geometric properties of connections. Recall that a linear connection on a Riemann manifold $(M, g)$ is compatible with the Riemann structure $g$, i.e., the parallel transport along any curve is an isometry, if and only if

$$X(g(Y, Z)) = g(V_X Y, Z) + g(Y, V_X Z), \quad \forall X, Y, Z \in \mathfrak{X}(M). \quad (11)$$

The main result is that there exists a unique torsion-free metric connection on $M$, called the Levi-Civita connection, which is given by the Koszul formula:

$$2 g(V_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [Z, X]) + g(Z, [X, Y]). \quad (12)$$

In particular, when a coordinate chart is considered, the Christoffel symbols of the second kind, defined by

$$\nabla_{\partial q^i} \left( \frac{\partial}{\partial q^j} \right) = \Gamma^k_{ji} \frac{\partial}{\partial q^k},$$
are given by
\[ \Gamma_{jk}^{\ell}(q) = \frac{1}{2} g^{\ell i}(q) \left( \frac{\partial g_{ij}}{\partial q^k}(q) + \frac{\partial g_{ik}}{\partial q^j}(q) - \frac{\partial g_{jk}}{\partial q^i}(q) \right). \]  
(13)
where \( g^{\ell i} \) are the inverse matrix entries of the Riemann structure \( g \).

Then, the linear connection is given by
\[ \nabla_X Y = X^i \left( \frac{\partial Y^k}{\partial q^i} + Y^j \Gamma_{jk}^{\ell}(q) \right) \frac{\partial}{\partial q^\ell}, \]
and correspondingly,
\[ \nabla_X \alpha = X^k \left( \frac{\partial \alpha}{\partial q^k} - \alpha_i \Gamma_{jk}^{\ell}(q) \right) dq^j. \]

Using these covariant derivatives, the Killing condition \( \mathcal{L}_X g = 0 \), i.e., equation (7), can be written in an intrinsic way as the condition for the covariant derivative of the vector field \( X \), to be a skew-symmetric endomorphism with respect to the metric \( g \), that is (see proposition 4.10 of [15]), for every \( Y, Z \in \mathfrak{X}(M) \),
\[ g \mathcal{L}_X (Y, Z) + g (Y, \nabla_Z X) = 0. \]  
(14)

Another remarkable relation is that if \( \alpha \) is the 1-form \( \alpha = \tilde{\varpi} = \tilde{\varpi}(X) \), where \( X \in \mathfrak{X}(M) \), then, using that the relation \( \mathcal{L}_Z (\alpha, Y) = \{ \nabla_Z \alpha, Y \} + \langle \alpha, \nabla_Z Y \rangle \), for any two vector fields \( Y, Z \in \mathfrak{X}(M) \) can be rewritten then as
\[ Z \mathcal{L}_X (\alpha, Y) = \{ \nabla_Z \alpha, Y \} + g (X, \nabla_Z Y), \]
and remembering the property of the compatibility of the connection with the metric, we see that
\[ \{ \nabla_Z \alpha, Y \} = g (\nabla_Z X, Y). \]  
(15)

The dynamical vector field, solution of the dynamical equation (10) turns out to be
\[ I_L(q, v) = v^i \frac{\partial}{\partial q^i} - \left( \Gamma_{jk}^{\ell}(q) v^j v^k + g^{\ell i}(q) \frac{\partial V}{\partial q^i}(q) \right) \frac{\partial}{\partial v^\ell}, \]
where \( \Gamma_{jk}^{\ell} \) are the Christoffel symbols of the second kind, with respect to the Levi-Civita connection, defined by the metric \( g \), as given by equation (13).

The Hamiltonian vector field of a smooth function \( G \) on \( TM \) is determined by the equation \( i(X_G) \omega_L = dG \) and in local coordinates is given by
\[ X_G(q, v) = g^{\ell i}(q) \frac{\partial G}{\partial v^i}(q, v) \frac{\partial}{\partial q^\ell} + g^{\ell i}(q) \left( \frac{\partial g_{in}}{\partial q^k}(q) v^n - \frac{\partial g_{kn}}{\partial q^i}(q) v^n \right) \times g^{\ell j}(q) \frac{\partial G}{\partial v^j}(q, v) - \frac{\partial G}{\partial q^i}(q, v) \frac{\partial}{\partial v^j} \right). \]  
(16)

Because the total energy of the system is \( E_L = T + V \), then,
\[ X_G(E_L) = -I_L(G) = \frac{\partial G}{\partial v^\ell} \left( \Gamma_{jk}^{\ell} v^j v^k + g^{\ell i} \frac{\partial V}{\partial q^i} \right) - \frac{\partial G}{\partial q^i} v^i. \]  
(17)
The virial theorem, \( \langle X_G(E_L) \rangle = 0 \) (see [3] for a geometric approach), establishes the following relation between time averages:

\[
\left\langle \left( \frac{\partial G}{\partial \nu^i} \Gamma^i_{jk} \nu^j \nu^k + g^{ij} \frac{\partial V_i}{\partial q^j} \right) \right\rangle = 0.
\]

We will see that the preceding expression is much simpler when the vector field \( X_G \) is a complete lift.

A relevant result concerning the virial theorem is that if \( X \) is a vector field on \( M \) and \( X^c \) its complete lift, then the function \( G \) defined by \( G = \langle \theta_L, X^c \rangle \) is such that \( \mathcal{L}_{\Gamma_L} G = \mathcal{L}_X L \), that is,

\[
\Gamma_L(G) = X^c(L).
\]  

In fact, as \( L \) is assumed to be regular the vector field \( \Gamma_L \) satisfies \( \mathcal{L}_L \theta_L = dL \), and then

\[
\mathcal{L}_L \theta_L(X^c) = i(X^c)\mathcal{L}_L \theta_L = \mathcal{L}_L i(X^c)\theta_L + i([X^c, \Gamma_L])\theta_L.
\]

But the Cartan 1-form \( \theta_L \) is a semi-basic 1-form and \( \Gamma_L \) is a vertical vector field because \( \Gamma_L \) is a second-order vector field and then \( i([X^c, \Gamma_L]) = 0 \). Therefore,

\[
\mathcal{L}_L \theta_L(X^c) = \mathcal{L}_L i(X^c)\theta_L = \mathcal{L}_L i(X^c) \theta_L.
\]

From the expression \( \Gamma_L(G) = X^c(L) \), evaluating the time evolution and averaging the interval \( \tau \), in the limit when \( \tau \to \infty \), we find as we did in [3] in an analogous case, that if \( G \) remains bounded,

\[
\langle X^c(L) \rangle = 0 \iff \langle X^c(T) \rangle = 0,
\]

whose local coordinate expression is

\[
\left\langle \frac{X^k}{2} \frac{\partial g^j_{\nu^i}}{\partial q^j} \nu^j \nu^i + \frac{\partial X^k}{\partial q^j} g^{ij} \nu^i \nu^j - X^k \frac{\partial V_i}{\partial q^j} \right\rangle = 0.
\]  

Example 1  (Spherical geometry). Consider as an illustrative example, the motion of a unity mass point on a sphere of radius \( R = 1/\sqrt{2} \) centered at the origin and the usual spherical polar coordinates, i.e., a point \( P \) on the sphere is fixed by two coordinates \( \theta, \phi \) such that

\[
\mathbf{x}(\theta, \phi) = (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta),
\]

and then

\[
g_{\theta \theta} = R^2, \quad g_{\phi \phi} = 0, \quad g_{\theta \phi} = R^2 \sin^2 \theta,
\]

i.e., the arc-length is

\[
d s^2 = R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right).
\]  

Suppose that the motion is under the action described by a potential function \( V(\theta) \) that does not depend on \( \phi \) but only on the distance to the North pole. Then, if \( X \) is the vector field on the base \( X = \tan \theta \, \partial/\partial \theta \), with complete lift
\[ X^e = \tan \theta \frac{\partial}{\partial \theta} + \sec^2 \theta \nu_\theta \frac{\partial}{\partial \nu_\theta}, \]

as the kinetic energy is
\[ T = \frac{1}{2} R^2 (\nu_\theta^2 + \sin^2 \theta \nu_\phi^2) \]
and
\[ X^e(T) = R^2 \left( \sec^2 \theta \nu_\theta^2 + \sin^2 \theta \nu_\phi^2 \right), \quad X(V) = \tan \theta \frac{\partial V}{\partial \theta}, \]

the virial theorem establishes that
\[ \left\langle R^2 \left( \sec^2 \theta \nu_\theta^2 + \sin^2 \theta \nu_\phi^2 \right) \right\rangle = \left\langle \tan \theta \frac{\partial V}{\partial \theta} \right\rangle. \]

The points of the lower half sphere can be described by the points obtained by central projection onto the tangent plane \( x_3 = -R \), i.e., points \((q_1, q_2, -R)\), such that
\[
\begin{cases}
q_1 = x_1 R = -\frac{R^2 \sin \theta \cos \phi}{R \cos \theta} = -R \tan \theta \cos \phi \\
q_2 = x_2 R = -\frac{R^2 \sin \theta \sin \phi}{R \cos \theta} = -R \tan \theta \sin \phi
\end{cases}
\]
or eliminating the south pole and using polar coordinates \((r, \phi)\) centered at \((0, 0, -R)\), i.e., \(r = -R \tan \theta\), remembering that
\[ \frac{d\theta}{dr} = \frac{1}{R} \frac{1}{1 + (r/R)^2} = \frac{1}{R} \frac{1}{1 + \lambda r^2}, \]

the expression of the arc-length becomes
\[ ds^2 = \frac{1}{\left(1 + \lambda r^2\right)^2} dr^2 + \frac{r^2}{\left(1 + \lambda r^2\right)^2} d\phi^2. \]

In terms of the new coordinates, as \( \tan \theta = -r/R \),
\[ \sec^2 \theta = 1 + \lambda r^2, \quad \sin^2 \theta = \frac{r^2}{R^2} \left(1 + \lambda r^2\right)^{-1}, \quad \nu_\theta = R \left(1 + \lambda r^2\right) \nu_\phi, \]
and then we can rewrite the preceding equation as
\[ \left\langle \left(1 + \lambda r^2\right)^{-1} \left(\nu_\theta^2 + r^2 \nu_\phi^2\right) \right\rangle = \left\langle r \left(1 + \lambda r^2\right) \frac{\partial V}{\partial r} \right\rangle. \tag{21} \]

which coincides with expression (14) of [16]. However, in [16], such expression was only proven for two special cases and it was proposed as a guess for the general case.

### 4. Affine virial functions

As mentioned previously, the virial theorem for a given smooth bounded function \( G \) is \( \langle X_G(E_\perp) \rangle = 0 \), which for systems of the mechanical type reduces to \( \langle X_G(T) + X_G(V) \rangle = 0 \). A particularly simple case would be when \( X_G \) is a complete lift and this property constrains the possible form of \( G \).

Note first that the expression (16) for the vector field \( X_G \) shows that the necessary and sufficient condition for \( X_G \) to be \( \tau_M \) projectable is that \( \partial G/\partial \nu \theta \) be a basic function, i.e., \( G \) is an affine in velocities function, or in more geometric language, there must be a 1-form \( \alpha = \alpha_k(q) dq^k \) on \( M \) and a function \( \varphi \) on \( M \) such that

\[ \alpha = \alpha_k(q) dq^k \text{ on } M \text{ and a function } \varphi \text{ on } M \text{ such that} \]
\[ G = \hat{\alpha} + \tau_M \phi, \]

and then the \( \tau_M \)-related vector field is \( \tilde{g}^{-1}(\alpha) \).

### 4.1. Killing vector fields

For the vector field \( X_G \) to be a complete lift, the \( \tau_M \)-related vector field must be \( \tilde{g}^{-1}(\alpha) \), and the \( n \) functions \( \alpha_k \) and the function \( \phi \) on the base manifold must satisfy, for any index \( i \),

\[
\frac{\partial}{\partial q^k} \left( g^{ij}(\alpha_j) \right) v^k = g^k \left[ \left( \frac{\partial g_{lm}}{\partial q^k} \right) v^m - \frac{\partial g_{ln}}{\partial q} v^n \right] g^{ij} \alpha_j - v^i \frac{\partial \alpha_j}{\partial q^k} - \frac{\partial \phi}{\partial q^k}. \]

These conditions can be rewritten for any pair of indices \((i,k)\), as

\[
\alpha_j \frac{\partial g^{ij}}{\partial q^k} + g^{ij} \frac{\partial \alpha_j}{\partial q^k} = g^{ij} \frac{\partial \alpha_k}{\partial q^l} - g^{im} \frac{\partial g_{mk}}{\partial q^l} g^{ij} \alpha_j - \frac{\partial \phi}{\partial q^k} = 0,
\]

and therefore, as follows

\[
g^{ij} \left( \frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^l} \right) = \alpha_n g^{ij} \left( -\frac{\partial g^{im}}{\partial q^k} + g^{il} \frac{\partial g_{lm}}{\partial q^l} - g^{im} \frac{\partial g_{mk}}{\partial q^l} \right), \quad \frac{\partial \phi}{\partial q^k} = 0.
\]

Using now that

\[
\frac{\partial g^{ij}}{\partial q^k} = -g^{ij} \frac{\partial g_{lm}}{\partial q^k},
\]

the preceding equation becomes

\[
g^{ij} \left( \frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^l} \right) = \alpha_n g^{ij} \left( \frac{\partial g_{lk}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right) = 2 \alpha_n g^{ln} \Gamma_{jk}^i,
\]

or equivalently

\[
g^{ij} \left( \frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^l} \right) = \alpha_n g^{ij} \left( \frac{\partial g_{lk}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right) = 2 \alpha_n g^{ln} \Gamma_{jk}^i,
\]

which can be rewritten as

\[
\frac{\partial \alpha_j}{\partial q^k} + \frac{\partial \alpha_k}{\partial q^l} = 2 \alpha_i \Gamma_{jk}^i,
\]

or in other words, for any pair of indices \( i, k \),

\[
\left( \frac{\partial \alpha_j}{\partial q^k} - \alpha_i \Gamma_{jk}^i \right) + \left( \frac{\partial \alpha_k}{\partial q^l} - \alpha_i \Gamma_{ij}^i \right) = 0.
\]

Multiplying both sides by \( Z^i Y^k \) and summing on repeated indices, we see that this equation is the coordinate expression of the intrinsic one

\[
\langle V_Y \alpha, Z \rangle + \langle V_Z \alpha, Y \rangle = 0, \quad \forall \ Y, Z \in \mathfrak{X}(M),
\]

so that the two-covariant tensor field \( V \alpha \) is skew-symmetric. But, as \( \alpha = \tilde{g}(X) \), relation (15) allows us to express this condition as \( g(V_Y X, Z) + g(Z, V_Y X) = 0 \), which means that \( X \) satisfies the Killing condition (14). The preceding result can be summarized in the following proposition, whose intrinsic proof is also given:
Proposition 1. The vector field $X \in \mathfrak{X}(M)$ is a Killing vector w.r.t. the Riemann structure $g$ iff $X_\alpha = X^c$, where $\alpha$ is the linear in the fiber’s function defined by the 1-form $\alpha = \tilde{g}(X)$.

Proof. The linear in the fiber’s function $G = (\theta_{T_e}, X^c)$ is nothing but the function $\tilde{\alpha}$, because

$$\left\{ \theta_{T_e}, X^c \right\} = \left\{ dT_g \circ S, X^c \right\} = \left\{ dT_g, S(X^c) \right\} = X^c(T_g),$$

where the vector field $X^c$ is the vertical lift of $X$ [8, 9], and therefore,

$$\left\{ \theta_{T_e}, X^c \right\}(v) = \frac{d}{ds} T_g(v + sX(\tau_M(v))) \bigg|_{s=0} = g(X(\tau_M(v)), v) = \tilde{\alpha}(v),$$

for every $v \in TM$.

If the Hamiltonian vector field $X_G$ is the complete lift $X^c$, then relation (18) shows that $X^c(E_L) = -X^c(L)$, because $X^c(L) = \Gamma G = -X_G(E_L) = -X^c(E_L)$. Therefore, $X^c(T_g) - X^c(V) = -X^c(T_g) - X^c(V)$, i.e., $X^c(T_g) = 0$, and then $X$ is a Killing vector. On the other hand, if $X$ is a Killing vector we have that $T_{\epsilon x y} = 0$. Because $i(x_G - X^c)\omega_T = \theta_{T_{x y}} = 0$, then $X_G = X^c$.

Let $X$ be a Killing vector field, and $\alpha = \tilde{g}(X)$ the associated 1-form. As we have seen, $X_\alpha = X^c$, from where we have

$$\left\{ E_L, \tilde{\alpha} \right\} = X_\alpha E_L = X^cE_L = E_{X^cL} = T_{\epsilon x y} + \tau_M(L_XV) = \tau_M(L_XV).$$

Taking mean values, we find that for every Killing vector field $X$:

$$\langle [L_XV] \rangle = 0.$$

Therefore, if $X$ is not a symmetry of the potential energy, then the mean value of the derivative $L_XV$ vanishes along any trajectory of the Lagrangian dynamical system.

Example 2 (Spherical geometry revisited). Returning to the case of the spherical geometry, we can say that the vector field

$$X = X_\theta \frac{\partial}{\partial \theta} + X_\phi \frac{\partial}{\partial \phi}$$

is a Killing vector field if and only if its complete lift

$$X^c = X_\theta \frac{\partial}{\partial \theta} + X_\phi \frac{\partial}{\partial \phi} + \left( \frac{\partial X_\theta}{\partial \theta} v_\theta + \frac{\partial X_\theta}{\partial \phi} v_\phi \right) \frac{\partial}{\partial v_\theta} + \left( \frac{\partial X_\phi}{\partial \theta} v_\theta + \frac{\partial X_\phi}{\partial \phi} v_\phi \right) \frac{\partial}{\partial v_\phi}$$

is a symmetry of the kinetic energy

$$T(\theta, \phi, v_\theta, v_\phi) = \frac{1}{2} \left( v_\theta^2 + \sin^2 \theta \ v_\phi^2 \right).$$

From the condition

$$\left( \frac{\partial X_\theta}{\partial \theta} v_\theta + \frac{\partial X_\theta}{\partial \phi} v_\phi \right) v_\theta + \sin^2 \theta \left( \frac{\partial X_\phi}{\partial \theta} v_\theta + \frac{\partial X_\phi}{\partial \phi} v_\phi \right) v_\phi + X_\theta \sin \theta \ v_\phi = 0,$$
we obtain the conditions:
\[
\begin{align*}
\frac{\partial X_0}{\partial \theta} &= 0, \\
\frac{\partial X_0}{\partial \phi} + \sin^2 \theta \frac{\partial X_0}{\partial \phi} &= 0, \\
\sin \theta \left( \cos \theta X_0 + \sin \theta \frac{\partial X_0}{\partial \phi} \right) &= 0
\end{align*}
\]

One solution is given by \( X_0 = 0 \) and \( X_\phi = 1 \), i.e., the vector field \( X_3 = \partial/\partial \phi \) is a Killing vector field. Another particular solution is \( X_\theta = \cos \phi \) and \( X_\phi = -\sin \phi \cot \theta \), and then another Killing vector field is

\[
X_1 = \cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi}
\]

The corresponding virial theorem is

\[
\left\langle \left\langle L_X V \right\rangle \right\rangle = 0 \iff \left\langle \left\langle \cos \phi \frac{\partial V}{\partial \theta} \right\rangle \right\rangle = \left\langle \left\langle \sin \phi \cot \theta \frac{\partial V}{\partial \phi} \right\rangle \right\rangle.
\]

**Example 3** (Periodic Toda lattice with \( n \) particles). A periodic Toda lattice system with \( n \) particles without impurities (each particle as the same mass \( m \)), is defined by a mechanical Lagrangian \( L = T - V \) on \( T\mathbb{R}^n \). The kinetic energy is the quadratic function defined by the Euclidian metric on \( \mathbb{R}^n \),

\[
T(q, v) = \frac{1}{2} \sum_{i=1}^{n} m v_i^2,
\]

and the potential is given by

\[
V(q) = \sum_{i=1}^{n} e^{q_i - q_{i+1}},
\]

where \( q_{n+1} = q_1 \). Consider the following vector field, for a fixed \( k = 1, \ldots, n \),

\[
X_k = \frac{\partial}{\partial q_k}.
\]

The vector field is a Killing vector w.r.t. the Euclidean metric.

Then, the virial theorem implies that \( \left\langle \left\langle L_X V \right\rangle \right\rangle = \left\langle \left\langle e^{q_i - q_{i+1}} - e^{q_{n-i} - q_i} \right\rangle \right\rangle = 0 \). Therefore, \( \left\langle \left\langle e^{q_i - q_{i+1}} \right\rangle \right\rangle = \left\langle \left\langle e^{q_{n-i} - q_i} \right\rangle \right\rangle \) for every \( k \) and hence \( \langle V \rangle = n \langle e^{q_i - q_{n-i}} \rangle \).

**Example 4** (Kepler problem in polar coordinates). Consider a particle \( P \) of mass \( m \) moving in a plane under the action of a central force \( F(r) = -\gamma m m'/r^2 \) on the direction of a fixed point \( O \) of mass \( m' \gg m \), where \( \gamma \) is a positive constant and \( r \) represents the distance between \( O \) and the point particle \( P \). Let \( \phi \) be the angle that the line \( OP \) makes with a fixed direction on the plane. In polar coordinates, the arc-length is given by \( ds^2 = dr^2 + r^2 d\phi^2 \).

The kinetic energy of the particle is given by
The potential is the function $V(r) = -\gamma m^2/r$. The vector field

$$X = \cos \phi \frac{\partial}{\partial r} - \frac{1}{r} \sin \phi \frac{\partial}{\partial \phi}$$

is a Killing vector field of the Euclidean metric in polar coordinates. Then, the virial theorem tells us that $\langle \langle \mathcal{L}_X V \rangle \rangle = 0$, that is, $\langle \langle -\cos (\phi) \gamma m^2/r^2 \rangle \rangle = 0$.

4.2. Conformal Killing and homothetic vector fields

Conformal Killing vector fields and, in particular, homothetic vector fields, have also been relevant in many problems in physics and more particularly in space-time geometry (see [14, 17, 18]). We now explore the information we can extract from them in the problem of the virial theorem under consideration. With this aim, we first find the difference between the Hamiltonian vector field $\alpha \iota_X$ associated with the 1-form $\alpha = \tilde{g}(X)$, where $X$ is a vector field on $M$ and the complete lift of $X$.

**Proposition 2.** If $X$ is the vector field on $M$ associated to the 1-form $\alpha$, $\alpha = \tilde{g}(X)$ and, as before, $\tilde{\alpha} \in C^\infty(TM)$ is the function $\tilde{\alpha}(v) = g(X(\tau_M(v)), v)$, for $v \in TM$, then the difference of the complete lift $X'$ of $X$ and the Hamiltonian vector field $X_\tilde{\alpha}$ associated to $\tilde{\alpha}$ with respect to the symplectic form $\omega$ is the vertical vector field whose contraction with the symplectic form $\omega_{T} \iota_{X}$ is the semi-basic 1-form $\theta_{T} \iota_{X}$.

**Proof.** Notice first that as both vector fields, $X'$ and $X_\tilde{\alpha}$, are projectable on the vector field $X = \tilde{g}^{-1}(\alpha)$, the difference vector is vertical. Moreover, considering the previously mentioned relation $\langle \theta_{T} \iota_{X'}, X' \rangle = \tilde{\alpha}$, we have

$$i(X_{\tilde{\alpha}} - X')\omega_{T} = i(X_{\tilde{\alpha}})\omega_{T} - i(X')\omega_{T} = d\tilde{\alpha} + i(X')d\theta_{T}, \quad (22)$$

and then

$$i(X_{\tilde{\alpha}} - X')\omega_{T} = d \left( i(X')\theta_{T} \right) + i(X')d\theta_{T} = \mathcal{L}_{X'}\theta_{T} = \theta_{X'\iota_{X}} = \theta_{T \iota_{X}}, \quad (23)$$

where the last equality follows from equation (9).

It is also well known (see [19]) that contraction with the symplectic forms $\omega_{L}$ defined by a regular Lagrangian $L$ establishes a one-to-one correspondence of vertical vector fields with semi-basic 1-forms. More explicitly, in the particular case we are considering of $L = T_{\xi}$, the semi-basic 1-form corresponding to the Liouville vector field $\Delta$, generating dilation along the fibers of $TM$, is $-\theta_{T_{\xi}}$, because, as $\theta_{T_{\xi}}$ is semi-basic,

$$i(\Delta)\omega_{T_{\xi}} = -i(\Delta)d\theta_{T_{\xi}} = -\mathcal{L}_{\Delta}\theta_{T_{\xi}},$$

and as $\theta_{T_{\xi}}$ is homogeneous of degree one in velocities, we find that

$$i(\Delta)\omega_{T_{\xi}} = -\theta_{T_{\xi}}. \quad (24)$$

This allows us to write

$$i(X_{\tilde{\alpha}} - X')\omega_{T} = -i(\Delta)\omega_{T \iota_{X}}.$$
**Theorem 3.** A vector field $X$ on $M$ is a conformal Killing vector field, i.e., there exists a function $f \in C^\infty(M)$ such that $\mathcal{L}_X g = f g$, if and only if $X_\alpha = X^\alpha - f \Delta$, where $\alpha$ is the 1-form $\alpha = \tilde{\xi}(X)$.

**Proof.** Indeed, if $X$ is a conformal Killing vector field, there exists a function $f \in C^\infty(M)$ such that $\mathcal{L}_X g = f g$, and then $\theta_{\mathcal{L}_X} = f \theta_f$. Equation (23) reduces in this case to $i(X_\alpha - X^\alpha) \omega_f = f \theta_f$, and then using equation (24), to $i(X_\alpha - X^\alpha) \omega_f = -i(f \Delta) \omega_f$. As $\omega_f$ is non-degenerate we find $X_\alpha - X^\alpha = -f \Delta$.

Conversely, if there exists a function $f \in C^\infty(M)$ such that $X_\alpha - X^\alpha = -f \Delta$, then

$$i(X_\alpha - X^\alpha) \omega_f = -i(f \Delta) \omega_f = f \theta_f,$$

and, as a consequence of equation (23), we obtain that $\theta_{\mathcal{L}_X} = f \theta_f$, which implies $\mathcal{L}_X g = f g$ and then $X$ is a conformal Killing vector field.

This result is in agreement with the meaning of being a conformal Killing vector field: its flow transforms geodesics in re-parameterized geodesics, the responsible term for reparametrization is $f \Delta$. Of course, for $f = 0$, we recover the result of proposition 1.

The preceding results allow us to introduce a virial theorem for conformal Killing vector fields.

**Theorem 4.** Let us consider a Lagrangian of the mechanical type $L = L_{\varphi, v} = T_\varphi - \tau_M^* V$, a conformal Killing vector field $X$ for $g$, i.e., there exists a function $f \in C^\infty(M)$ such that $\mathcal{L}_X g = f g$, and the associated 1-form $\alpha = \tilde{\xi}(X)$. Then we have that

$$\left\langle \{fT_\varphi - \mathcal{L}_X V\} \right\rangle = 0.$$

**Proof.** If $\alpha = \tilde{\xi}(X)$ is the associated 1-form from the relation $X_\alpha = X^\alpha - f \Delta$, it follows that

$$\left\{ E_L, \tilde{\alpha} \right\} = X_\alpha E_L = X^\alpha E_L - f \Delta E_L = E_X^* E_L = 2fT_\varphi + \tau_M^*(\mathcal{L}_X V),$$

where we have used that $E_X^* = E_{\mathcal{L}_X} + \tau_M^*(\mathcal{L}_X V) = fT_\varphi + \tau_M^*(\mathcal{L}_X V)$. Applying the virial theorem $\left\langle \left\{ E_L, \tilde{\alpha} \right\} \right\rangle = 0$, we obtain the result.

**Example 5.** Consider now the spherical geometry metric (20) for $R = 1$ and look for a conformal vector field of the form $X = X_\varphi(\varphi) \partial / \partial \varphi$. From the relationships

$$\mathcal{L}_X \left( d\varphi^2 \right) = 2X_\varphi d\varphi^2, \quad \mathcal{L}_X \left( \sin^2 \vartheta \, d\varphi^2 \right) = 2 \sin \vartheta \cos \vartheta \, X_\varphi \, d\varphi^2,$$

we see that, in order to be a conformal vector field, one must have:

$$2 X_\varphi = 2 \cot \vartheta \, X_\varphi = f(\varphi),$$

from where we obtain $X_\varphi = \sin \vartheta$ and $f(\varphi) = 2 \cos \vartheta$. Therefore, the corresponding virial relation reads

$$\left\langle \cos \vartheta \, T_\varphi \right\rangle = \left\langle \cos \vartheta \left( \nu_\varphi + \sin^2 \vartheta \, \nu_\varphi \right) \right\rangle = \left\langle \sin \vartheta \, \frac{\partial V}{\partial \varphi} \right\rangle.$$
Example 6. Another example with three degrees of freedom is the metric
\[
ds^2 = h(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \quad h(r) > 0.
\]
If we look for a conformal vector field of the form \( X = X_r(r) \partial / \partial r \), we arrive at the relationship
\[
\mathcal{L}_X g = \left( h X_r + 2 h X_r \right)dr^2 + 2 r X_r \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) = f \, g,
\]
and we see that, in order to be a conformal vector field, one must have:
\[
\frac{\dot{h}}{h} X_r + 2 \dot{X_r} = \frac{2 X_r}{r} = f,
\]
from where we can conclude that \( X_r \) is a solution of the differential equation
\[
X_r + \left( \frac{1}{2} \frac{h}{\dot{h}} - \frac{1}{r} \right) X_r = 0 \implies X_r = C \frac{r}{h^{1/2}},
\]
and \( f = 2C / h^{1/2} \). In particular, for \( h(r) = 1 \), the Euclidean metric, we have the homothetic dilation vector field \( X = r \partial / \partial r \), with \( f = 2 \), whereas for \( h(r) = r^2 \), we find the conformal vector field \( X = \partial / \partial r \) with a conformal factor \( f = 2/r \). Therefore, the corresponding virial relations read
\[
\left\langle \left\langle 2T_g \right\rangle \right\rangle = \left\langle \left\langle \mathcal{L}_X V \right\rangle \right\rangle \implies \left\langle \left\{ v^2 + r^2 \left( \nu^2 + \sin^2 \theta \, v^2 \right) \right\} \right\rangle = \left\langle \left\{ \frac{\partial V}{\partial r} \right\} \right\rangle.
\]
and
\[
\left\langle \left\langle 2 \frac{r}{r} T_g \right\rangle \right\rangle = \left\langle \left\langle \mathcal{L}_X V \right\rangle \right\rangle \implies \left\langle \left\{ r \left( v^2 + v^2 + \sin^2 \theta \, v^2 \right) \right\} \right\rangle = \left\langle \left\{ \frac{\partial V}{\partial r} \right\} \right\rangle.
\]

We can prove a similar result when we have two Riemann metrics \( g \) and \( g' \) on \( M \) and the vector field \( X \in \mathfrak{X}(M) \) relates them in the following way \( \mathcal{L}_X g = f \, g' \).

Theorem 5. Consider a Lagrangian of the mechanical type \( L = L_{x,v} = T_g - \tau_M V \). If there exists a function \( f \in C^\infty(M) \) such that \( \mathcal{L}_X g = f \, g' \), and \( \alpha = \tilde{g} (X) \), then,
\[
\left\langle \left\{ f T_g - \mathcal{L}_X V \right\} \right\rangle = 0.
\]

Proof. Because \( \tilde{\alpha} = (\theta, X') \), then \( \Gamma_i (\tilde{\alpha}) = X' (L) \). Hence,
\[
\{ E_L, \tilde{\alpha} \} = X \alpha E_L = -X' (L) = -X' \left( T_g \right) + \tau_M \left( \mathcal{L}_X V \right) = -T_{\mathcal{L}_X g} + \tau_M \left( \mathcal{L}_X V \right).
\]
The virial theorem implies that \( \left\langle \left\{ -f T_g + \mathcal{L}_X V \right\} \right\rangle = 0 \), and the result follows.

Example 7 (Spherical geometry). In example 1, the vector field \( X = \tan(\theta) \partial / \partial \theta \) defines the virial function. In polar coordinates, this vector is given by
The vector field $X$ is not a conformal vector field of the Euclidian metric $g$ given by $ds^2 = dr^2 + r^2 d\theta^2$. In this case, we have $\langle \langle 2(1 + \lambda r^2)^{-1}T_g \rangle \rangle = \langle \langle \mathcal{L}_X V \rangle \rangle$ and this formula is equivalent to equation (21).

A particularly interesting case is when the vector field $X$ is homothetic, i.e., $f = \mu$ is a real constant, $\mathcal{L}_X g = \mu g$, because then $\mathcal{L}_X T_g = \mu T_g$, where $T_g$ is the kinetic energy $T$.

In example 7, when $\lambda \to 0$, the limit vector field is the infinitesimal generator of dilations on $\mathbb{R}^2$ written in polar coordinates, and it is a two-homothetic vector field of the Euclidian metric, so in the limit, the virial theorem implies that $2\langle \langle T_g \rangle \rangle = \langle \langle r\partial_r V \rangle \rangle$.

If $V$ is a $X$-homogeneous function of degree $\nu$, i.e., $\mathcal{L}_X V = \nu V$, then $\langle \langle \mu T_g - \nu V \rangle \rangle = 0$ because $\langle \langle X^i (T_g) - \mathcal{L}_X V \rangle \rangle = 0$. Because the energy is a constant $E$ along a trajectory, we also obtain $\langle \langle T_g + V \rangle \rangle = E$, from where

$$\langle \langle T_g \rangle \rangle = \frac{\nu}{\nu + \mu} E \quad \text{and} \quad \langle \langle V \rangle \rangle = \frac{\mu}{\nu + \mu} E.$$ 

As a particular case, if both degrees of homogeneity are equal $\nu = \mu \equiv a$, then we find that

$$\langle \langle T_g \rangle \rangle = \langle \langle V \rangle \rangle = \frac{1}{2} E.$$ 

On the other hand, this condition is equivalent to $\mathcal{L}_X L = a L$, and hence, we can apply directly a result in [3], obtaining $\langle \langle L \rangle \rangle = 0$, from which we also obtain $E = 2\langle \langle T_g \rangle \rangle = 2\langle \langle V \rangle \rangle$.

5. Summary and outlook

This paper aims to study a recently developed geometric approach to the virial theorem and our attention has been focused on the particularly interesting case of Lagrangian systems of the mechanical type. Geometric properties of the Riemann structure defining the kinetic energy term allow us to identify different types of virial functions and associated vector fields for which a virial-like theorem can be stated. Recall that a first generalization of the virial theorem was established in the framework of the theory of Hamiltonian systems on symplectic manifolds and then for systems defined by regular Lagrangians. Here, this is studied in the particular case of systems of mechanical-type Lagrangians. The general case is $\langle \langle X_G (E_L) \rangle \rangle = 0$. However, there are other types of virial-like theorems of a specifically Lagrangian nature. For instance, we have proven in section 3 that for any complete lift vector field $X'$, the following relation is true: $\langle \langle X^i (T_g) - \mathcal{L}_X V \rangle \rangle = 0$; this is a generalization of the case studied in [3] in which $X^i (L) = a L$. We have displayed in section 4 the most general form of a function $G$ whose associated Hamiltonian vector field is the complete lift of a vector field $X$ on the base manifold, which is a Killing vector field, the function $G$ being determined by the 1-form on $M$ corresponding to $X$ by contraction with the Riemann metric. In this way, we obtain the virial relation $\langle \langle \mathcal{L}_X V \rangle \rangle = 0$. Finally, we have identified the conformal Killing vectors and obtained a virial relation for such vector fields, the case of homothetic vector fields appearing as a particular case: $\langle \langle f T_g - \mathcal{L}_X V \rangle \rangle = 0$. Several examples have been used to illustrate the theory.

The usefulness of this geometric approach suggests the convenience of analyzing the virial theorem in the framework of a non-holonomic system. This approach is receiving increasing interest from the geometric viewpoint [20] and the use of a modern approach to the
concept of quasi-velocity [21] using the geometric tools of Lie algebroids suggests considering virial-like relations in Lagrangian and Hamiltonian systems on Lie algebroids [22] when non-holonomic constraints are present [23]. We believe this kind of application deserves more detailed study.

Acknowledgments

This work was supported by the research projects MTM–2012—33575 (MINECO, Madrid) and DGA-E24/1 (DGA, Zaragoza). We also thank the anonymous reviewers for their valuable comments and suggestions.

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