EXPOSING BOUNDARY POINTS OF STRONGLY PSEUDOCONVEX SUBVARIETIES IN COMPLEX SPACES

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Abstract. We prove that all locally exposable points in a Stein compact in a complex space can be exposed along a given curve to a given real hypersurface. Moreover, the exposing map for a boundary point can be sufficiently close to the identity map outside any fixed neighborhood of the point. We also prove a parametric version of this result for bounded strongly pseudoconvex domains in \( \mathbb{C}^n \). For a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) and a given boundary point of it, we prove that there is a global coordinate change on the closure of the domain which is arbitrarily close to the identity map with respect to the \( C^1 \)-norm and maps the boundary point to a strongly convex boundary point.

1. Introduction

Let \( X \) be a complex space. We assume throughout this paper that all complex spaces are reduced, irreducible, and paracompact. Let \( X_{\text{sing}} \) be the set of singular points of \( X \) and let \( X_{\text{reg}} = X \setminus X_{\text{sing}} \) be the set of smooth points of \( X \).

**Definition 1.1.** Let \( X \) be a complex space, let \( K \subset X \) be a compact set, and let \( \zeta \in K \cap X_{\text{reg}} \) be a point in \( X_{\text{reg}} \). We will say that \( \zeta \) is **locally exposable** if there exists an open set \( U \subset X \) containing \( \zeta \) and a \( C^2 \)-smooth strictly plurisubharmonic function \( \rho \) on \( U \) such that

(i) \( \rho(\zeta) = 0 \) and \( d\rho(\zeta) \neq 0 \), and

(ii) \( \rho < 0 \) on \( (K \cap U) \setminus \{\zeta\} \).

Our main concern in this paper is to show that locally exposable points are globally exposable (see Definition 1.2).

The first result of the present paper is the following.

**Theorem 1.1.** Let \( X \) be a complex space, let \( K \subset X \) be a Stein compact, and let \( \zeta \in K \cap X_{\text{reg}} \) be locally exposable. Let \( H \subset X \setminus (K \cup X_{\text{sing}}) \) be a locally closed \( C^2 \)-smooth subset of a real hypersurface in \( X \). Let \( \gamma : [0, 1] \to X_{\text{reg}} \) be a smoothly embedded curve in \( X \) with \( \gamma(0) = \zeta \), \( \gamma(1) \in H \), and \( \gamma(t) \in X \setminus (K \cup H) \) for \( t \in (0, 1) \).

Then for any (small) neighborhood \( V \) of \( \gamma \), and any \( \epsilon > 0 \), there exist an open neighborhood \( U \) of \( K \), an arbitrarily small neighborhood \( V' \subset V \) of \( \zeta \), and a biholomorphic map \( f : U \to f(U) \subset X \) such that the following holds:

(a) \( f(V') \subset V \) and \( f(\zeta) = \gamma(1) \),

(b) \( \text{dist}(f(z), z) < \epsilon \) for \( z \in K \setminus V', \) and

(c) \( f(K) \cap H = \{\gamma(1)\} \).

where \( \text{dist}(\cdot, \cdot) \) is a fixed distance on \( X \). In the case \( X = \mathbb{C}^n \) and \( K \subset \mathbb{C}^n \) is polynomially convex, \( f \) can be taken to be a holomorphic automorphism of \( \mathbb{C}^n \).

By saying that \( K \) is a Stein compact we mean that \( K \) has a Stein neighbourhood basis in which \( K \) is holomorphically convex.
Definition 1.2. The map \( f \) will be said to expose the point \( \zeta \) with respect to \( H \), and \( f(\zeta) \) is said to be exposed. If \( K \subset B_R(0) \subset \mathbb{C}^n \) and \( H = \{ z \in \mathbb{C}^n : \| z \| = R \} \), for any \( R > 0 \), we will say that \( f(\zeta) \) is globally exposed.

From the proof, we will see that Theorem 1.1 can be generalized in various directions as follows:

- one can easily expose finitely many points simultaneously,
- the last statement can be generalized as: if \( X \) is a Stein manifold and has density property (for definition see [12]) and \( K \) is \( \mathcal{O}(X) \)-convex, then \( f \) can be taken in \( \text{Aut}_{\text{hol}} X \),
- if \( X \) is a 1-convex space, the same statement still holds if we assume that \( \zeta \) is outside the exceptional set. This can be proved by Remmert’s reduction and interpolation for the exposing maps constructed in Theorem 1.1. (Exposing boundary points in this setting was proposed by Franc Forstnerič in [9].)

The importance of Theorem 1.1 in the case that \( K \) is the closure of a domain with a strictly pseudoconvex boundary point \( \zeta \), is that it tells us that \( K^0 \) resembles a strictly convex domain in a concrete geometric sense. Convex domains are exceptionally well behaved from a complex analysis point of view. As a comparison, the Levi problem was to show that such a domain \( K^0 \) resembles a strictly convex domain in a much weaker function theoretic sense.

Our next result concerns in which ways a locally exposable point can be locally exposed with global maps. It is quite simple using Andersén-Lempert theory to show that a locally defined exposing map in \( \mathbb{C}^n \) can be approximated by holomorphic automorphisms, but as a consequence one looses completely control of the behaviour on most of \( K \).

Theorem 1.2. Let \( D \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. Then for any \( \zeta \in \partial D \) and any \( \epsilon > 0 \), there is an injective holomorphic map \( F : \overline{D} \to \mathbb{C}^n \) such that \( F(\zeta) = \zeta \) is a strictly convex boundary point of \( F(D) \), \( \partial F(D) \) and \( \partial D \) are tangent at \( \zeta \), and \( \| F - \text{id} \|_{C^1(\overline{D})} < \epsilon \).

In [2] we will construct a strongly pseudoconvex domain such that the map \( F \) in Theorem 1.2 cannot be taken in \( \text{Aut}_{\text{hol}}(\mathbb{C}^n) \) for some boundary points.

Furthermore we are interested in parametric version of the previous two theorems.

Theorem 1.3. Let \( D \subset \subset \mathbb{C}^n \) be a strongly pseudoconvex domain with smooth boundary. Let \( H \subset \subset \mathbb{C}^n \) be a smooth closed real hypersurface with \( \overline{D} \cap H = \emptyset \). Let \( \gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n \) be a fiber-preserving continuous map such that for all \( \zeta \in \partial D \):

1) \( \gamma(\zeta) = \gamma(\zeta,:) : I \to \mathbb{C}^n \) is a smooth embedding;
2) \( \gamma(0) = \zeta, \gamma(1) \in H \) and \( \gamma|_{(0,1)} \subset \mathbb{C}^n \setminus (\overline{D} \cup H) \).

(We implicitly make the identification of \( \{ \zeta \} \times \mathbb{C}^n \) with \( \mathbb{C}^n \).) Then for any neighborhood \( V \) of \( \gamma(\partial D \times [0,1]) \) and any \( \epsilon > 0 \), for any sufficiently small neighborhood \( V' \subset V \) of \( \{ (\zeta, \zeta) : \zeta \in \partial D \} \) in \( \partial D \times \mathbb{C}^n \) there exists a smooth fiber-preserving map \( f : \partial D \times \overline{D} \to \partial D \times \mathbb{C}^n \) such that the following holds for each \( \zeta \in \partial D \):

(a) \( f_\zeta := f(\zeta,:) : \overline{D} \to \mathbb{C}^n \) is injective and holomorphic,
(b) \( f_\zeta(V'_\zeta) \subset \gamma(1) \) and \( f_\zeta(\zeta) = \gamma(1) \),
(c) \( || f_\zeta(z) - z || < \epsilon \) for \( z \in \overline{D \setminus V'_\zeta} \),
where $V_\zeta := V \cap (\{\zeta\} \times \mathbb{C}^n)$. If in addition $\overline{D}$ is polynomially convex, we can take $f$ a smooth map from $\partial D \times \mathbb{C}^n$ to itself such that $f|_{\zeta} \in Aut(\mathbb{C}^n)$ for all $\zeta \in \partial D$.

**Theorem 1.4.** Let $D \subset \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^2$-smooth boundary. For any $\epsilon > 0$, there is a continuous map $f : \partial D \times \overline{D} \to \mathbb{C}^n$ such that for all $\zeta \in \partial D$ the following hold:

1) $f_\zeta = f(\zeta, \cdot) : \overline{D} \to \mathbb{C}^n$ is a holomorphic injective map,
2) $f_\zeta(\zeta) = \zeta$ is a strictly convex boundary point of $f_\zeta(D)$,
3) $\partial f_\zeta(D)$ and $\partial D$ are tangential at $\zeta$; and
4) $||f_\zeta - id||_{C^1(D)} < \epsilon$.

For a further discussion of parametric exposing of points, see Section 5.

2. Transforming a Boundary Point to a Strongly Convex One

In this section, we consider transforming a boundary point of a strongly pseudoconvex domain to a strictly convex one, with certain control of the behavior of the involved transformation. The aim is to prove Theorem 1.2 and Theorem 1.4. We also construct a strongly pseudoconvex domain in $\mathbb{C}^2$ for which the transformation furnished by Theorem 1.2 can not be taken in $Aut(\mathbb{C}^2)$.

We need the following lemmas. The first lemma is to prove the existence of peak functions with certain estimates. The key tool is the existence of embedding of strongly pseudoconvex domains into strongly convex domains with certain boundary conditions, established by the second author in [4].

**Lemma 2.1.** Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$-smooth boundary. Then for any $\zeta \in \partial D$, there is a holomorphic function $f$ defined on some neighborhood of $\overline{D}$ which satisfies the following two conditions:

1) $f(\zeta) = 1$, and $|f(z)| < 1$ for $z \in \overline{D}\{\zeta\}$;
2) the estimate

$$|f(z)| \leq e^{-c||z-\zeta||^2}$$

holds on $\overline{D}$ for some constant $c > 0$.

**Proof.** For the case that $D$ is the ball $B := \{z \in \mathbb{C}^n; |z_1+r|^2+|z_2|^2+\cdots+|z_n|^2 < r^2\}$ and $\zeta = 0$, a simple calculation shows that $f(z) := e^{z_1}$ satisfies the conditions. By the result in [4] mentioned above, the general case can be reduced to the case that $D = B$.

For the proof of Theorem 1.4 we need a parametric version of Lemma 2.1.

**Lemma 2.2.** Let $D \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with smooth boundary. Then there is a smooth map $p : \partial D \times \overline{D} \to \mathbb{C}$ such that:

1) For each $\zeta \in \partial D$, $p_\zeta(\cdot) := p(\zeta, \cdot)$ is a holomorphic function on $\overline{D}$;
2) $p_\zeta(\zeta) = 1$ and $|p_\zeta(z)| < 1$ for $z \in \overline{D}\{\zeta\}$ for all $\zeta \in \partial D$;
3) there exists a constant $c$ such that $|p_\zeta(z)| \leq e^{-c||z-\zeta||^2}$ for $(\zeta, z) \in \partial D \times \overline{D}$.

**Proof.** By a result in [4], there is a proper holomorphic embedding $\sigma$ from some neighborhood of $\overline{D}$ into some neighborhood of a bounded strongly convex domain $W \subset \mathbb{C}^N$ for some $N$ such that $\sigma(\partial D) \subset \partial W$ and $\sigma(\overline{D})$ and $\partial W$ intersect transversely. Let $\rho$ be a defining function for $W$, for each $\xi \in \partial W$ set...
\[ \Lambda_\xi(w) := \sum_{j=1}^N \partial \rho / \partial w_j(\xi) \cdot (w_j - \xi_j), \] and set \( g_\xi(w) = e^{\Lambda_\xi(w)}. \) Then \( g \) satisfies
\[ 1-3 \text{ for } D \text{ replaced by } W. \]
So we may set \( p(\zeta, z) = g_{\sigma(\zeta)}(\sigma(z)). \) \( \square \)

The last lemma we need is the following

**Lemma 2.3.** Let \( D \) be a bounded domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. Let \( F \) be a diffeomorphism from some neighborhood of \( \overline{D} \) to its image in \( \mathbb{C}^n. \) Assume that \( \{F_j\} \) is a sequence of smooth maps defined on some neighborhood of \( \overline{D} \) such that \( ||F_j - F|| \) converges uniformly on \( \overline{D} \) to 0 in \( C^1 \)-norm. Then \( F_j \) is injective on \( \overline{D} \) for \( j \) large enough.

**Proof.** We prove by contradiction. If it were not the case, then, without loss of generality, we can assume that for each \( j \) there exist \( a_j \) and \( b_j \) in \( \overline{D} \) with \( a_j \neq b_j \) such that \( F_j(a_j) = F_j(b_j). \) We may assume \( a_j \to a \) and \( b_j \to b \) as \( j \to \infty. \) Then it is necessary that \( a = b \) since \( F \) is injective. We have \( a \in D \) or \( a \in \partial D. \) Since \( \partial D \) is \( C^2, \) taking a \( C^2 \)-smooth coordinate change near \( a \) if necessary, we may assume that there is a neighborhood \( U \) of \( a \) such that \( U \cap \overline{D} \) is convex. Let \( \gamma_j \) be the line segment in \( \mathbb{C}^n \) given by \( t \to (1 - t)a_j + tb_j, \; t \in [0, 1], \) then \( \gamma_j \subset \overline{D} \) for \( j \) large enough. We have
\[
F_j(b_j) - F_j(a_j) = \int_0^1 \frac{dF_j(\gamma_j(t))}{dt} dt
\]
(1)
which can not be 0 for \( j \) large enough since \( \int_0^1 dF_j(\gamma_j(t)) dt \to dF(a) \) and \( dF(a) \) is nonsingular. Contradiction. \( \square \)

**Proof of Theorem 1.2.** We can assume that \( \zeta = 0 \) and the local defining function \( \rho(z) \) of \( D \) near 0 can be expanded as
\[
\rho(z) = \text{Re}(z_n + Q(z)) + \sum_{i,j} a_{ij} z_i \bar{z}_j + \cdots,
\]
where \( Q(z) = \sum_{i,j} q_{ij} z_i \bar{z}_j \) is a symmetric quadratic form. Let \( f \) be a holomorphic function defined in some neighborhood of \( \overline{D} \) that is furnished by Lemma 2.1. We will show that there are positive integers \( M \) and \( N_j, j = 1, \ldots, M \) such that the transformation \( F(z) = (\tilde{z}_1, \ldots, \tilde{z}_n) \) given by
\[
\begin{cases}
\tilde{z}_k = z_k, \text{ for } k = 1, \ldots, n - 1, \\
\tilde{z}_n = z_n + \sum_{j=1}^M Q_f N_j
\end{cases}
\]
is that expected by this theorem.

First note that
\[
||Qf^N|| \leq ||z|| e^{-Nc||z||^2},
\]
and hence goes to zero uniformly on \( \overline{D} \) as \( N \to \infty. \) Note also that
\[
||Qf||^{N-1} \leq N ||z||^2 e^{-Nc||z||^2}.
\]
Let \( \sigma \) be the function on \( (0, +\infty) \) given by \( x \mapsto xe^{-cx}. \) It is clear that \( \sigma(0) = 0 \) and \( \lim_{x \to +\infty} \sigma(x) = 0, \) and \( \sigma \) attains its maximum \( 1/e \) at \( x = 1/c. \) So the right hand side of (3) attains its maximum \( 1/e \) at \( ||z||^2 \in \mathbb{R}. \) For any \( \epsilon > 0 \) and a sufficiently large integer \( M \) with \( 1/Mc < \epsilon/2, \) by the above discussion, we can find \( r_1 > r_2 > \cdots > r_M > 0 \) and positive numbers \( N_1, \ldots, N_M \) such that \( ||(Qf^N)'|| < \)
the proof of Theorem 1.1. The main aim of this section is to prove the following theorem which is a key step in

$$\exists \rho \in \mathbb{R}_+$$ on \( \{ z \in \mathcal{D} : ||z|| \geq r_j \text{ or } ||z|| \leq r_{j+1} \} \). Define \( \varphi_M = \sum_{j=1}^{M} \frac{1}{M} Q f_j^{N_j} \), then \( \varphi_M \) is a holomorphic function defined on some neighborhood of \( \overline{\mathcal{D}} \), and \( ||\varphi'_M|| < \epsilon \) on \( \overline{\mathcal{D}} \). By taking \( N_j \) large enough, we can also require that \( ||\varphi_M(z)|| < \epsilon \) for \( z \in \mathcal{D} \). Let \( F_M \) be the map defined as in \( \mathcal{D} \). By Lemma 2.3 \( F_M \) is injective on \( \mathcal{D} \) for \( M \) and \( N_j \) large enough. It is clear that \( ||F_M - \text{id}||_{C^1} \) converges to 0 uniformly on \( \mathcal{D} \), and \( F_M(\zeta) \) is a strongly convex boundary point of \( F_M(D) \).

\[
\text{Proof of Theorem 1.3} \]

For \( \zeta \in \partial \mathcal{D} \), denote by \( n_{\zeta} \) the unit outward-pointing normal vector of \( \partial \mathcal{D} \) at \( \zeta \). Let \( \mathcal{L}_{\zeta} = \{ x_n_{\zeta} + iyn_{\zeta} : x, y \in \mathbb{R} \} \) and let \( \pi_{\zeta} : \mathbb{C}^n \to \mathcal{L}_{\zeta} \) be the orthogonal projection. For \( z \in \mathbb{C}^n \), let \( t_{\zeta}(z) = z + iy \) if \( \pi_{\zeta}(z - \zeta) = x_n_{\zeta} + iyn_{\zeta} \) and let \( z_{\zeta} = (z - \zeta) - \pi_{\zeta}(z - \zeta) \). Note that \( z \mapsto (z_{\zeta}, t_{\zeta}(z)) \) is a linear isomorphism from \( \mathbb{C}^n \) to \( (T_{\zeta} \partial \mathcal{D}) \times \mathbb{C} \) which maps \( \zeta \) to the origin, where \( T_{\zeta} \partial \mathcal{D} = T_{\zeta} \partial \mathcal{D} \cap iT_{\zeta} \partial \mathcal{D} \) is the complex tangent space of \( \partial \mathcal{D} \) at \( \zeta \). Let \( \rho \) be a defining function of \( \mathcal{D} \) which is strictly plurisubharmonic on some neighborhood of \( \overline{\mathcal{D}} \). After normalization, near each \( \zeta \in \partial \mathcal{D} \), \( \rho \) can be expanded as follows:

\[
(4) \quad \rho(z) = \text{Re}(t_{\zeta}(z)) + Q_{\zeta}(z - \zeta) + L_{\zeta}(z - \zeta) + o(||z - \zeta||^2),
\]

where \( L_{\zeta} \) is the Levi form of \( \rho \) at \( \zeta \) and \( Q \) is the complex Hessian of \( \rho \) at \( \zeta \).

By some basic results from calculus, the right hand side of (4) can be viewed as a smooth function defined on some neighborhood of \( \partial \mathcal{D} \times \mathcal{D} \) in \( \partial \mathcal{D} \times \mathbb{C}^n \). Let \( p : \partial \mathcal{D} \times \mathcal{D} \to \mathbb{C} \) be the smooth map furnished by Lemma 2.2. Then, by the same estimate as in the proof of Theorem 1.2, we can show that there are positive integer \( M \) and \( N_j, j = 1, \cdots , M \) such that the map \( F(\zeta, z) : \partial \mathcal{D} \times \mathcal{D} \to \mathbb{C}^n \) given by

\[
\begin{align*}
F(\zeta, z) &= z' \\
t_{\zeta}(F(\zeta, z)) &= t_{\zeta}(z) + \sum_{j=1}^{M} \frac{1}{M} Q_{\zeta}(z - \zeta)p_{\zeta}^{N_j}(z)
\end{align*}
\]

satisfies the required conditions. \( \square \)

We now construct a strongly pseudoconvex domain \( D \subset \mathbb{C}^2 \) such that the map \( F \) in Theorem 1.2 can not be taken in \( \text{Aut}_{\text{hol}}(\mathbb{C}^n) \) for some \( p \in \partial \mathcal{D} \).

**Example 2.1.** Let \( D := \{ (z, w) \in \mathbb{C}^2 : ||z||^2 + ||z\overline{z}||^2 + ||w||^2 < 3 \} \). Then \( D \) is a bounded strongly pseudoconvex domain with smooth boundary. The intersection \( A \) of \( D \) and the \( z \)-axis is an annulus. Let \( p \) be an inner boundary point of \( A \). It is clear that there exists a compact set \( V \subset D \) such that the polynomial hull of \( V \) contains an open neighborhood of \( p \). So if \( F_j \in \text{Aut}_{\text{hol}}(\mathbb{C}^2) \) is a sequence that converges to \( \text{id} \) uniformly on \( \overline{\mathcal{D}} \), then \( F_j \) converges to \( \text{id} \) uniformly on a neighborhood of \( p \). So \( F_j(p) \) can not be a strictly convex boundary point of \( F_j(D) \) for \( j \) large enough.

3. **The ball model case in \( \mathbb{C}^n \)**

The main technical problem in proving Theorem 1.1 which differs from the results in \( \mathbb{B}^n \) is to prove an exposing result for balls, which can later be used to pass from local to global exposing by approximately gluing. For \( r \in \mathbb{R} \), we denote the point \((0, \cdots , 0, r) \in \mathbb{C}^n \) by \( p_r \). For \( r > 0 \) and \( a \in \mathbb{C}^n \), \( B_r(a) \) denotes the ball in \( \mathbb{C}^n \) centered at \( a \) with radius \( r \), and we denote by \( \mathbb{B}^n \) the unit ball in \( \mathbb{C}^n \) as usual. For \( r, s \in \mathbb{R}, r < s \), we let \( l_{r,s} \) denote the closed line segment between \( p_r \) and \( p_s \). The main aim of this section is to prove the following theorem which is a key step in the proof of Theorem 1.1.
Theorem 3.1. Let $r, s$ be positive numbers with $s > r + 1$. Then for any open neighborhood $V$ of $l_{1, s-r}$, any open neighborhood $U$ of $l_{1, s-r} \cup B_r(p_*)$, and any sufficiently small $\epsilon > 0$, there exits a sequence of maps $\phi_{\nu, t} : \mathbb{B}^n \to \mathbb{C}^n$, $(t \in [0, 1])$, of injective holomorphic maps such that the following hold:

(i) $\phi_{\nu, t}$ are smooth in $t$ and $\phi_{\nu, 0} = \text{Id},$

(ii) $\phi_{\nu, t} \to \text{Id}$ uniformly on $\mathbb{B}^n \setminus B_r(p_1)$ for $t \in [0, 1]$, as $\nu \to \infty$,

(iii) $\phi_{\nu}(p_1) = p_{s+r}$ for all $\nu \in \mathbb{N}$, where $\phi_{\nu} := \phi_{\nu, 1},$

(iv) $\phi_{\nu}(B_r(p_1) \cap \mathbb{B}^n) \subset V \cup B_r(p_0) \cup \{p_{s+r}\},$

(v) $\phi_{\nu, t}(\mathbb{B}^n \cap B_r(p_1)) \subset U$ for all $\nu$ and $t$,

(vi) the images $\phi_{\nu, t}(\mathbb{B}^n)$ are polynomially convex for all $\nu$ and $t$.

Lemma 3.2. Let $a < b$ be points on the real line with $b - a > 2$, and let $l$ be the closed line segment between $a + 1$ and $b - 1$. Let $\delta_{\nu} \to 0$ and let $\Omega_{\nu}$ be the $\delta_{\nu}$ neighborhood of $D_1(a) \cup l \cup D_1(b)$, where $D_s(a)$ is the disk in $\mathbb{C}$ centered at $a$ with radius $s$. There exist injective holomorphic maps $f_{\nu}, g_{\nu} : \mathbb{D} \to \mathbb{C}$ such that the following holds:

(i) $f_{\nu}(\mathbb{D} \cap \mathbb{R}) \subset \mathbb{R}, g_{\nu}(\mathbb{D} \cap \mathbb{R}) \subset \mathbb{R}$, and $f_{\nu}(\mathbb{D}) = g_{\nu}(\mathbb{D}) \subset \Omega_{\nu}$ for all $\nu$,

(ii) $f_{\nu}(-1) = g_{\nu}(-1) = a - 1, f_{\nu}(1) = g_{\nu}(1) = b + 1$ and $f_{\nu}(0) = a, g_{\nu}(0) = b$, for all $\nu$,

(iii) $f_{\nu}(z) \to z + a$ uniformly on some fixed neighborhood of $\mathbb{D} \setminus D_\epsilon(1)$ for any small $\epsilon > 0$;

(iv) $g_{\nu}(z) \to z + b$ uniformly on some fixed neighborhood of $\mathbb{D} \setminus D_\epsilon(-1)$ for any small $\epsilon > 0$;

(v) $f_{\nu}(z) = g_{\nu}(m_{\nu}(z))$ for all $\nu$, where

$$m_{\nu}(z) = \frac{z - r_{\nu}}{1 - r_{\nu}z}$$

with $r_{\nu} \to 1$ as $\nu \to \infty$.

Proof. For simplicity we assume $a = -2, b = 2$. We will construct maps $f_{\nu}$ that satisfy (i)-(iii) and $f_{\nu}(\mathbb{D})$ is invariant under the transformation $z \mapsto -z$. By symmetry, if we put $g_{\nu}(z) = -f_{\nu}(-z)$, then all conditions about $f_{\nu}$ and $g_{\nu}$ in the lemma hold. The convergence of $r_{\nu}$ to 1 in (v) will be guaranteed by the convergence of $f_{\nu}$ as presented in (iv).

We now construct such $f_{\nu}$. By Mergelyan’s Theorem there exists a sequence of embeddings $\phi_{\nu} : D_1(-2) \cup l \cup D_1(2) \to \mathbb{C}$ such that $\phi_{\nu} \to \text{Id}$ on $D_1(-2)$, $\phi_{\nu}(l)$ shrinks to the point $-1$, as $\nu \to \infty$, and such that $\phi_{\nu}$ are are as close as we like to a translation and a scaling on $D_1(2)$, such that the diameter of $\phi_{\nu}(D_1(2))$ shrinks to zero. We may also interpolate to get $\phi_{\nu}(-2) = -2, \phi_{\nu}(-2) = 1$. Replacing $\phi_{\nu}$ by the map given by $z \mapsto \frac{\phi_{\nu}(z) + \phi_{\nu}(-2)}{2}$ if necessarily, we can assume that $\phi_{\nu}(z) = \overline{\phi_{\nu}(\overline{z})}$, and hence $\phi_{\nu}$ maps real numbers to real numbers. Let $W_{\nu} = D_1(-2) \cup l(\delta'_{\nu}) \cup D_1(2)$, where $l(\delta'_{\nu})$ is the $\delta'_\nu$-neighborhood of $l$ and $0 < \delta'_\nu < \delta_{\nu}$. If the $\delta'_\nu$ are chosen small enough, then $\phi_{\nu}(W_{\nu})$ converges to the ball $D_1(-2)$ in the sense of Goluzin.

Let $\psi_{\nu} : \mathbb{D} \to \phi_{\nu}(W_{\nu})$ be the Riemann map with $\psi_{\nu}(0) = -2, \psi'(0) > 0$, we have that $\psi_{\nu}(z) \to z - 2$ uniformly on $\mathbb{D}$ (see [7], Theorem 2, p. 59). Moreover, $\psi_{\nu}(z) = \overline{\psi_{\nu}(\overline{z})}$. Then it is clear that $\tilde{f}_{\nu} := \phi_{\nu}^{-1} \circ \psi_{\nu}$ satisfy (i), (ii), and (iii).

\[\Box\]

Lemma 3.3. Let $0 < r < 1$ and define $\varphi_{r}(z) = \frac{z - r}{1 - rz}$. Then either $|\varphi_{r}(z)| < |z|$ or $\text{Re}(\varphi_{r}(z)) < 0$ for all $z \in \mathbb{D}$.
Proof. Note first that any circle \( \Gamma_a = \{ |z| = a \} \) with \( a > 0 \) is mapped by \( \varphi_r \) to a circle symmetric with respect to the real line, and \( \varphi_r(a) < a \). A straightforward computation shows that \( |\varphi_r(a) - \varphi_r(-a)| = 2a \frac{1 - r^2}{1 - a^2} \) and so the radius of \( \varphi_r(\Gamma_a) \) is less than \( a \). Since also \( \varphi_r(a) < a \) this implies that \( \varphi_r(\Gamma_a) \cap \Gamma_a \) is either empty or is contained in \( \{ \text{Re}(z) < 0 \} \). □

**Lemma 3.4.** Let \( r_j, s_j \) be real numbers for \( j = 1, 2 \) with \( s_j > r_j + 1 \). Then there exists a sequence

\[
(5) \quad \phi_\nu : B^n \cup l_{1,s_1} \cup B_{r_1}(p_{s_1}) \to \mathbb{C}^n
\]

of injective holomorphic maps such that the following hold:

(i) \( \phi_\nu \to \text{Id} \) uniformly on \( B^n \),

(ii) \( \phi_\nu(z) \to \psi(z) = p_{s_2} + \frac{z - p_{s_1}}{|z - p_{s_1}|} \) uniformly on \( B_{r_1}(p_{s_1}) \),

(iii) \( \phi_\nu \) converges uniformly on \( l_{1,s_1} \) to a smooth embedding of \( l_{1,s_1} \) onto \( l_{1,s_2} \).

Moreover, the maps \( \phi_\nu \) are of the form \( \phi_\nu(z) = (h_\nu(z_n) \cdot z', f_\nu(z_n)) \), where \( h_\nu, f_\nu \) are holomorphic functions in one variable. Finally, the maps \( \phi_\nu \) can be chosen to match \( \psi \) to any given order at the point \( p_{s_1} + r_1 \).

Proof. By Mergelyan’s theorem there exists one variable functions \( f_\nu(z_n) \) satisfying (i)-(iii) on the intersection with the \( z_n \)-axis. Again by Mergelyan’s theorem there exist a sequence \( h_\nu(z_n) \) converging to the constant function \( r_2/r_1 \) on the disk of radius \( r_1 \) centred at \( s_1 \), to 1 on the closed unit disk, and some fixed real function increasing/decreasing from \( r_2/r_1 \) to 1 on the line segment connecting the two disks.

We now give the proof of Theorem 3.1.

**Proof.** We prove it first without the parameter \( t \), i.e., we prove the existence of the sequence \( \phi_{\nu,1} \). By Lemma 3.4 it is enough to prove this for some special case of \( r, s \). We take the special case that \( r = 1.5, s = 3 \). Let \( f_\nu \) be maps furnished by Lemma 3.2 with \( a = 0, b = 3.5 \). We define \( \phi_\nu(z) := (z_1, \ldots, z_{n-1}, f_\nu(z_n)) \). Then the properties (ii), (iii), (v) and (vi) hold (we have that (vi) holds because \( f_\nu^{-1} \) is approximable by entire maps).

We now prove (iv). Since \( \epsilon \) can be chosen arbitrarily small, the only place where the inclusion in (iv) can fail is near the point \( p_{4.5} \). Consider first the sequence

\[
\tilde{\phi}_\nu(z) = (z_1, \ldots, z_{n-1}, g_\nu(z_n)).
\]

Then \( \tilde{\phi}_\nu \to z + p_6 \) uniformly on \( B^n \setminus B_r(p_{-1}) \), and since \( g_\nu(1) \) is real (and goes to 1) we have that \( \tilde{\phi}(B^n \setminus B_r(p_{-1})) \) is eventually contained in \( B_{3.5}(p_3) \). Now let \( z = (z_1, \ldots, z_n) \) be a point in \( B^n \setminus B_r(p_1) \). If \( \text{Re}(m_\nu(z_n)) \leq 0 \) then \( \tilde{\phi}_\nu(z) \) is far away from the point \( p_{4.5} \), since \( \phi_\nu(z) = (z_1, \ldots, z_{n-1}, g_\nu(m_\nu(z_n))) \). If \( \text{Re}(m_\nu(z_n)) > 0 \) it follows from Lemma 3.3 that \( (z_1, \ldots, z_{n-1}, m_\nu(z_n)) \in \mathbb{B} \setminus B_r(p_{-1}) \), and so \( \phi_\nu(z) \in B_{1.5}(p_3) \).

To construct isotopies we simply define

\[
\phi_{\nu,t}(z) = \frac{1}{t} \tilde{\phi}_\nu(tz).
\]

Note first that we can choose \( \epsilon > 0 \) arbitrarily small, and that by the construction, \( \mathbb{B} \cap B_r(p_1) \) gets mapped by \( \phi_\nu \) to a relatively compact subset \( \hat{U} \) of \( U \) which is independent of \( \epsilon \). Write

\[
\phi_\nu(z) = A_\nu(z) + G_\nu(z),
\]
where $A_\nu \to \text{Id}$ and $\|G_\nu(z)\| \leq \delta_\nu \|z\|^2$ for $\|z\| < 1 - \epsilon$ where $\delta_\nu \to 0$. Consider a point $z \in B_\nu(p_1) \cap \mathbb{B}^n$. If $t < 1 - \epsilon$ then $\phi_{\nu,t}(z) = A_\nu z + \frac{1}{t}G_\nu(tz)$, with $\|G_\nu(tz)\| \leq \delta_\nu t^2 \|z\|^2$, so if $\nu$ is large we are in $U$. If $t \geq 1 - \epsilon$ we consider two cases. If $tz \notin B_\nu(p_1)$ we may assume that $\phi_\nu$ is as close to the identity as we like, and so we are still in $U$. If $tz \in B_\nu(p_1)$ then $\phi_\nu$ maps $tz$ into $\tilde{U}$, and $\frac{1}{t}U \subset U$ provided $\epsilon$ was chosen small enough.

4. Exposing a single boundary point

The aim of this section is to prove Theorem 1.1. Recall that a pair $(A,B)$ of compact sets in a complex space $X$ is called a Cartan pair if $A,B,A\cap B,A\cup B$ all admit Stein neighborhood bases in $X$ and $A\setminus B \cap B \setminus A = \emptyset$. The following lemma due to Forstneric will be used to (approximately) glue locally defined exposing maps to define global ones.

**Lemma 4.1 ([5][6]).** Assume that $X$ is a complex space and $X'$ is a closed complex subvariety of $X$ containing the singular locus $X_{\text{sing}}$. Let $(A,B)$ be a Cartan pair in $X$ such that $C := A \cap B \subset X \setminus X'$. For any open set $\tilde{C} \subset X$ containing $C$ there exist open sets $A' \supset A, B' \supset B, C' \supset C$ in $X$, with $C' \subset A' \cap B' \subset \tilde{C}$, satisfying the following property. For every number $\eta > 0$ there exists a number $\epsilon_\eta > 0$ such that for each holomorphic map $\gamma : \tilde{C} \to X$ with $\text{dist}_{\tilde{C}}(\gamma, \text{Id}) < \epsilon_\eta$ there exist biholomorphic maps $\alpha = \alpha_\gamma : A' \to \alpha(A') \subset X$ and $\beta = \beta_\gamma : B' \to \beta(B') \subset X$ satisfying the following properties:

(i) $\gamma \circ \alpha = \beta$ on $C'$,
(ii) $\text{dist}_{A'}(\alpha, \text{Id}) < \eta$ and $\text{dist}_{B'}(\beta, \text{Id}) < \eta$, and
(iii) $\alpha$ and $\beta$ are tangent to the identity map to any given finite order along the subvariety $X'$ intersected with their respective domain.

Moreover, the maps $\alpha_\gamma$ and $\beta_\gamma$ can be chosen to depend continuously on $\gamma$ such that $\alpha_{\text{Id}} = \text{Id}$ and $\beta_{\text{Id}} = \text{Id}$.

We start by embedding a neighbourhood of the curve $\gamma$ suitably into $\mathbb{C}^n$, where we will define local exposing maps.

**Lemma 4.2.** There exists an open Stein neighbourhood $W$ of $\gamma$, an open neighbourhood $U_\zeta \subset W$ of $\zeta$, and a holomorphic embedding $\phi : W \to \mathbb{C}^n$ ($n = \dim(X)$) such that the following holds:

(i) $\phi(\zeta) = 0$,
(ii) $\phi(U_\zeta \cap K) \setminus \{\zeta\} \subset \{z \in \mathbb{C}^n : 2\text{Re}(z_n) + \|z\|^2 < 0\}$

**Proof.** First let $M \subset X$ be a totally real manifold (with boundary) of dimension $n$ that contains the curve $\gamma$, and choose a smooth embedding $g : M \to \mathbb{R}^n \subset \mathbb{C}^n$. Then $M$ admits a Stein neighbourhood $W_0$ such that $g$ may be approximated in $C^1$-norm on $M$ by holomorphic maps $\phi_0 : W_0 \to \mathbb{C}^n$. So $\phi_0$ may be taken to be an embedding of an open Stein neighbourhood $W$ of $\gamma$ into $\mathbb{C}^n$. By assumption there is a (local) strictly pseudoconvex hypersurface $\Sigma \subset \mathbb{C}^n$ and an open set $U_\zeta$ such that $\Sigma$ touches $\phi_0(U_\zeta \cap K)$ only at the point $\phi_0(\zeta)$. Let $F \in \text{Aut}_{\text{hol}}\mathbb{C}^n$ such that $F(\Sigma)$ is strictly convex near $F(\phi_0(\zeta))$. After a translation and scaling, the map $\phi = F \circ \phi_0$ will ensure the conclusions of the lemma.

We may now modify $\gamma$ such that near the origin, the curve $\tilde{\gamma} := \phi(\gamma)$ coincides with the line segment $l_{0,r}$ for some $r > 0$, and such that $\tilde{\gamma}$ is perpendicular to the
(local) hypersurface $\tilde{H} := \phi(W \cap H)$ where they intersect. We keep the notation $\gamma$ for the modified curve. Let $\gamma_{\zeta}$ denote the piece $\phi^{-1}(l_{0, r})$ of $\gamma$.

**Lemma 4.3.** There exists a a compact set $K' \subset X$ with $K \subset \text{int}(K')$ such that for any $\epsilon > 0$ there exists an open neighbourhood $\Omega'$ of $K' \cup \gamma_{\zeta}$ and a holomorphic embedding $\psi : \Omega \to X$ such that the following holds:

(i) $\text{dist}(z, \psi(z)) < \epsilon$ for all $z \in K'$,  
(ii) $\psi(\gamma_{\zeta}) \subset V \cap W$,  
(iii) $\psi(\gamma_{\zeta}) \cap H = \psi(\phi^{-1}(p_{r}))$ and the intersection is perpendicular to $H$ in the coordinates furnished by $\phi$.

**Proof.** We will first define a map $\tilde{h}$ accomplishing (i)-(iii) in the local coordinates furnished by $\phi$, and then we will (approximately) glue the map $\phi^{-1} \circ \tilde{h} \circ \phi$ to the identity map on $K'$.

We start by defining a suitable Cartan pair. Writing $\phi = (\phi_1, ..., \phi_n)$ note that the pluriharmonic function $\rho := \text{Re}(\phi_n)$ extends to a plurisubharmonic function on an open neighbourhood of $K$ which agrees with $\rho$ on an open neighbourhood of $\zeta$, and which is negative and uniformly bounded away from zero away from where they agree. We keep the notation $\rho$ for the extended function. Then for a sufficiently small smoothly bounded strictly pseudoconvex open neighbourhood $\Omega'$ of $K \cup \gamma_{\zeta}$ and sufficiently small $\tau > 0$, the sets

$$A_{\tau} = \overline{\Omega} \cap \{ \rho \geq -2\tau \} \text{ and } B_{\tau} = \overline{\Omega} \cap \{ \rho \leq -\tau \}$$

define a Cartan pair $(A_{\tau}, B_{\tau})$. Let $A_{\tau} = \phi(A_{\tau})$ and $B_{\tau} = \phi(B_{\tau} \cap W)$. If $\Omega'$ and $\tau$ are chosen sufficiently small we have that $A_{\tau} \cap B_{\tau} \subset W$ and $A_{\tau} \cap B_{\tau} \subset B_{\tau/2}(0)$. Set $C_{\tau} = B_{\tau/2}(0)$. Fix $K'$ a Stein compact such that $K' \subset K$ and $K' \subset B_{\tau/2}(0)$.

Now let $D$ be an open neighbourhood of $\overline{B}_{\tau/2}(0) \cup l_{0, r}$ with $\tilde{g} : D \to \mathbb{C}^n$ a smooth embedding such that $\tilde{g} = \text{id}$ near $\overline{B}_{\tau/2}(0)$, $\tilde{g}$ stretches $l_{0, r}$ to cover $\tilde{\gamma}$, $\tilde{g}^{-1}(\tilde{H})$ is perpendicular to $l_{0, r}$, and such that $\tilde{g}$ is $\overline{\partial}$-flat to order one along $l_{0, r}$. Then $\tilde{g}$ is uniformly approximable on $\overline{B}_{\tau/2}(0) \cup l_{0, r}$ in $C^1$-norm with jet interpolation at the point $p_{r}$. So there exists a holomorphic embedding $\tilde{h} : \overline{B}_{0, r/2}(0) \cup l_{0, r} \to \mathbb{C}^n$, as close to the identity as we like on $\overline{B}_{\tau/2}(0)$, with the image $\tilde{h}(l_{0, r})$ as close as we like to $\gamma$ and with $\tilde{h}^{-1}(\tilde{H})$ perpendicular to $l_{0, r}$. Set $h := \phi^{-1} \circ \tilde{h} \circ \phi$. Now if $h$ is close enough to the identity on $B_{\tau/2}$ we have that Lemma 4.1 furnishes maps $\alpha$ and $\beta$ such that the map $\psi$ defined as $\psi := h \circ \alpha$ on $A_{\tau}$ and $\psi = \beta$ on $B_{\tau}$ will satisfy the claims of the lemma.

\[\square\]

**Proof of Theorem 1.1.** We use the extended function $\rho$ from the proof of the previous lemma to define a Cartan pair:

$$A_{\tau} = K \cap \{ \rho \geq -2\tau \} \text{ and } B_{\tau} = K \cap \{ \rho \leq -\tau \}$$

Then for $\tau$ small we have that $\phi$ maps $A_{\tau}$ into $B_1(p_{-1}) \cup \{0\}$, and $C_{\tau} = A_{\tau} \cap B_{\tau}$ gets mapped into the ball $B_1(p_{-1})$. Choose a small $\delta > 0$ such that the ball $B_3(p_{r-\delta})$ touches $h^{-1}(H)$ only at the point $p_{r}$. Now Theorem 3.1 applied to the dumbbell $B_1(p_{-1}) \cup l_{0, r-2\delta} \cup B_3(p_{r-\delta})$ furnishes local exposing maps $\phi_{\nu} : \overline{B}_1(p_{-1}) \to \mathbb{C}^n$ such that the maps $\phi_{\nu} := \phi^{-1} \circ \tilde{h} \circ \phi$ may be approximately glued to the identity map...
over a neighbourhood of $C_\tau$ to conclude the proof of the first part of Theorem 1.1 by setting $f = \psi \circ \phi_\nu$.

For the last part we now assume that $K \subset \mathbb{C}^n$ is polynomially convex. Let $\rho$ be a non-singular strictly plurisubharmonic function on an open neighbourhood $U_\zeta$ of $\zeta$ such that $\rho(\zeta) = 0$ and $\rho < 0$ on $(K \cap U_\zeta) \setminus \{\zeta\}$. Then for any sufficiently small Runge and Stein neighbourhood $\Omega$ of $K$ the function $\rho$ extends to a plurisubharmonic function on $\Omega$ which is strictly negative on $\Omega \setminus U_\zeta$, and so $\Omega' = \Omega \cap \{\rho < 0\}$ is a Runge and Stein domain with $\zeta \in b\Omega'$. Moreover, choosing $\Omega$ to have a Runge and Stein neighbourhood basis, we have that $\Omega'$ will have a Runge and Stein neighbourhood basis, and so by Theorem 1.3 in [3] there exists $F \in \text{Aut}_{\text{hol}}\mathbb{C}^n$ such that $F(K \setminus \{\zeta\}) \subset \mathbb{B}^n$ and $F(\zeta) = p_1$. Since we can conjugate by $F$ we may as well assume that $K \setminus \{\zeta\} \subset \mathbb{B}^n$ and that $\zeta = p_1$.

My slightly modifying $\gamma$ we may assume that $\gamma$ agrees with $l_{1,1+r}$ near $p_1$ and that it is perpendicular to $H$ where they intersect. First let $D$ be an open neighbourhood of $\mathbb{B}^n \cup l_{1,1+r}$ and $g : [0, 1] \times D \rightarrow \mathbb{C}^n$ be an isotopy of embeddings such that $g_0 = \text{id}$, $g$ stretches $l_{1,1+r}$ along $\gamma$, $g_t = \text{id}$ near $\mathbb{B}^n$ for all $t$, $g^{-1}(H)$ is perpendicular to $l_{1,1+r}$, and $g_t$ is $\mathcal{F}$-flat on $l_{1,1+r}$ to order one. Then the whole parameter family $g_t$ may be approximated by a parameter family $h : [0, 1] \times \mathbb{C}^n$ of holomorphic maps in $C^1$-norm on $\mathbb{B}^n \cup l_{1,1+r}$ with jet interpolation at the end point of $l_{1,1+r}$. So $h_t$ is a family of holomorphic embeddings on some open neighbourhood of $\mathbb{B}^n \cup l_{1,1+r}$, and by the Andersén-Lempert theory $h_1$ may be approximated by a holomorphic automorphism $G$ with interpolation at the end point of $l_{1,1+r}$.

Let $\tilde{H} = G^{-1}(H)$ which is now perpendicular to $l_{1,1+r}$. Choose $s > 0$ such that $B_s(p_{1+r-s})$ touches $\tilde{H}$ only at a single point. Now Theorem 3.1 applied to the dumbbell $\mathbb{B}^n \cup l_{1,1+r} \cup B_s(p_{1+r-s})$ furnishes a one-parameter family of maps $\phi_t$ such that $G \circ \phi_1$ is the exposing map we are after, except that $\phi_1$ is not an automorphism. However, by (vi), $\phi_t(\mathbb{B}^n)$ is polynomially convex for each $t$, and so $\phi_1$ is approximable by holomorphic automorphisms. The proof is complete.

\[\square\]

5. REGULARITY OF EXPOSING MAPS WITH PARAMETERS

In this section we will prove Theorem 1.3. Before we proceed with the proof, we give a further discussion on parametric exposing. The terminology of being exposed will refer to being globally exposed in the sense of Definition 1.2, and $R$ may vary with the parameter.

**Definition 5.1.** Let $D$ be a bounded domain in $\mathbb{C}^n$ with smooth boundary. Then we say that $D$ satisfies:

1) Property (E), if there exists a continuous map $F : \partial D \times \tilde{D} \rightarrow \mathbb{C}^n$ such that for each fixed $p \in \partial D$ the map $F_\zeta := F(\zeta, \cdot) : \tilde{D} \rightarrow \mathbb{C}^n$ is an exposing of $\partial D$ at $\zeta$; or

2) Property (AE), if there exists a continuous map $F : \partial D \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that for each fixed $\zeta \in \partial D$ we have that $F_\zeta \in \text{Aut}_{\text{hol}}\mathbb{C}^n$ is an exposing of $D$ at $\zeta$; or

3) Property (ASE), if there exists a continuous map $F : \partial D \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that for each fixed $\zeta \in \partial D$ we have that $F_\zeta \in \text{Aut}_{\text{diff}}\mathbb{C}^n$ and $F_\zeta(D)$ is exposed at $F_\zeta(\zeta)$.

In the case 2) we say that $F_\zeta$ is ambient exposing, and in case 3) we say that $F_\zeta$ is ambient smoothly exposing. Note that Property (ASE) has nothing to do with the complex structure of $D$ and $\mathbb{C}^n$. In [2], the following question was proposed:
**Question 1.** Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Does $D$ satisfy Property (E)?

The following theorem, which can be viewed as a parametric version of Theorem 1.1, partially answers the above question.

**Theorem 5.1.** Let $D \subset \mathbb{C}^n (n > 1)$ be a bounded strongly pseudoconvex domain with smooth boundary. If $D$ satisfies Property (ASE), then for any positive numbers $\epsilon, \delta$, there exists a continuous map $f: \partial D \times \mathcal{D} \to \mathbb{C}^n$ such that for each $\zeta \in \partial D$:

1) $f_\zeta(\cdot) = f(\zeta, \cdot): \mathcal{D} \to \mathbb{C}^n$ is holomorphic and injective;
2) $f_\zeta(D)$ is exposed at $f_\zeta(\zeta)$; and
3) $||f_\zeta(z) - z|| < \epsilon$ for $z \in \mathcal{D} \setminus B_\delta(\zeta)$, where $B_\delta(\zeta) = \{z \in \mathbb{C}^n; ||z - \zeta|| < \delta\}$.

In particular, $D$ satisfies Property (E). If in addition $\mathcal{D}$ is polynomially convex, then $f$ can be taken to be a smooth map $f: \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that $f_\zeta \in \text{Aut}(\mathbb{C}^n)$ for all $\zeta \in \partial D$. In particular, $D$ satisfies Property (AE).

When trying to apply Theorem 1.3 to Question 1, we meet a topological obstruction. Even we originally just interested in exposing but not ambient exposing, it turns out that Property (ASE) is needed in our argument. It is clear that Property (ASE) is a necessary condition for Property (AE), but we don’t know if it is also necessary for Property (E).

For the special case when $\mathcal{D}$ is diffeomorphic to the closed unit ball, we can prove the following

**Theorem 5.2.** Let $D$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Assume that either $D$ is diffeomorphic to the unit ball $B^n$ and $n \geq 3$, or the closure $\overline{D}$ of $D$ is diffeomorphic to the closed unit ball and $n = 2$. Then $D$ satisfies Property (E). If in addition $\overline{D}$ is polynomially convex, then $D$ satisfies Property (AE).

**Remark 5.1.** It is known that any smoothly embedded $S^{n-1}$ in $\mathbb{R}^n$ bounds a domain that is diffeomorphic to the unit ball except $n \neq 4$ (see [10]). The case for $n = 4$ is still open. So for $n \neq 2$, the condition in Theorem 5.2 that $D$ is diffeomorphic to $B^n$ can be replaced by that $\partial D$ is diffeomorphic to $S^{2n-1}$.

### 5.1. A parametric version of Forstnerič’s splitting lemma

We first introduce some notations. Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$-smooth boundary and $\delta$ be a sufficiently small positive number. For $\zeta \in \partial D$ we define:

$$A_\zeta = \{z \in \overline{D} : ||z - \zeta|| \leq \delta\},$$

$$B_\zeta = \{z \in \overline{D} : ||z - \zeta|| \geq \delta/2\},$$

$$C_\zeta = A_\zeta \cap B_\zeta.$$

For a closed subset $X$ in $\mathbb{C}^n$ and $a > 0$, we set $X(a) := \{z \in \mathbb{C}^n : ||z - w|| \leq a \text{ for some } w \in X\}$ the $a$-neighborhood of $X$.

The following result is the main result in the thesis of Lars Simon ([11]):

**Lemma 5.3.** Let $D$ and $\delta$ as above. If $\tau > 0$ is small enough (this depends on $\delta$), then for any $\eta > 0$ there exists $\epsilon > 0$ such that the following holds. If $\mu > 5\tau$ and if $\gamma_\zeta$ is a family of injective holomorphic maps $\gamma_\zeta : C_\zeta(\mu) \to \mathbb{C}^n$ satisfying:

- $\gamma : \{(z, \zeta) : z \in C_\zeta(\mu), \zeta \in \partial D\} \to \mathbb{C}^n$, $(z, \zeta) \mapsto \gamma_\zeta(z)$ is continuous,
then there exist families \( \{\alpha_\zeta\}_{\zeta \in \partial D} \) and \( \{\beta_\zeta\}_{\zeta \in \partial D} \) of injective holomorphic maps \( \alpha_\zeta : \partial D \to \mathbb{C}^n \) and \( \beta_\zeta : \partial D \to \mathbb{C}^n \) with the following properties:

1. for all \( \zeta \in \partial D \) we have \( \gamma_\zeta = \beta_\zeta \circ \alpha_\zeta^{-1} \) on \( C_\zeta(\tau) \),
2. \( \text{dist}_{A_\zeta(2\tau)}(\alpha_\zeta, Id) < \eta \) and \( \text{dist}_{B_\zeta(2\tau)}(\beta_\zeta, Id) < \eta \),
3. the maps \( \alpha \) and \( \beta \) are continuous, where

\[
\alpha : \{(z, \zeta) \in \mathbb{C}^n \times \partial D : z \in A_\zeta(2\tau)\} \to \mathbb{C}^n, \quad (z, \zeta) \to \alpha_\zeta(z),
\]

\[
\beta : \{(z, \zeta) \in \mathbb{C}^n \times \partial D : z \in B_\zeta(2\tau)\} \to \mathbb{C}^n, \quad (z, \zeta) \to \beta_\zeta(z).
\]

Moreover, the maps \( \alpha_\zeta \) may be constructed to match the identity to any given order at \( \zeta \), and if we are given a parameter family \( \gamma_{\zeta,t} \) with \( t \in [0, 1] \) we may obtain \( \alpha_{\zeta,t} \) and \( \beta_{\zeta,t} \) also jointly continuous in \((\zeta, t)\).

We note that the two last points were not a part of the statement in \[11\], but they follow from the proof.

5.2. Exposing along normal directions. Let \( \Omega \subset \mathbb{C}^n \) be a domain with a strictly plurisubharmonic defining function \( \rho \), and fix \( p \in \partial \Omega \). By choosing a smooth family of orthogonal frames in \( T_p \partial \Omega \) for \( \zeta \) near \( p \) we may construct a smooth family of locally injective holomorphic maps \( g_\zeta(z) \) such that \( g_\zeta(\Omega) \) has a defining function

\[
\rho_\zeta(z) = 2\text{Re}(z_1 + Q_\zeta(z)) + L_\zeta(z) + \text{h.o.t}
\]

near the origin. Define \( G_\zeta(z) = (z_1 - Q_\zeta(z), z_2, \ldots, z_n) \), so that \( H_\zeta = G_\zeta \circ g_\zeta \) maps \( \partial \Omega \) to a strictly convex surface near the origin. We may cover \( \partial \Omega \) by finitely many open sets \( U_j \) such that we have such families of maps \( H_\zeta \) defined on each \( U_j \), and they all coincide modulo a unitary change of coordinates in \( \{z_1 = 0\} \), sending one frame to another. More precisely, for maps \( g_\zeta^1 \) and \( g_\zeta^2 \) we have that \( g_\zeta^1(z) = V g_\zeta^2(z) \), where \( Vz = (z_1, Uz') \). For the corresponding defining functions we get that \( \rho_\zeta^1(z_1, z') = \rho_\zeta^2(z_1, U^{-1}z') \). Furthermore we have

\[
G_\zeta^1(z) = (z_1 - Q_\zeta(z), z_2, \ldots, z_n),
\]

and

\[
G_\zeta^2(z) = (z_1 - Q_\zeta((z_1, U^{-1}(z'))), z_2, \ldots, z_n).
\]

For a point \( w \) near \( \zeta \), writing \( z = g_\zeta^2(w) \), we see that

\[
G_\zeta^1(g_\zeta^2(w)) = G_\zeta^2(z_1, Uz') = (z_1 - Q_\zeta(z), Uz'),
\]

and in particular we see that the map \( g_\zeta^{-1} \circ g_\zeta \) is independent of the choice of orthonormal frame on the tangent space.

Choose a small \( a > 0 \), let \( l_a \) denote the line segment between 0 and \( a \) in the \( z_1 \)-axis, and let \( \eta_\zeta = (H_\zeta^2)^{-1}(l_a) \). For small \( b, c > 0 \) we let \( B(b, \zeta) \) denote the ball \( (H_\zeta^2)^{-1}(B(a)) \), and \( V_\zeta = (H_\zeta^2)^{-1}(l_a(c)) \), the inverse image of the open \( c \)-tube around \( l_a \).

**Theorem 5.4.** Let \( \Omega \subset \mathbb{C}^n \) be a domain with a strictly plurisubharmonic defining function \( \rho \) and with objects defined above. Then for \( \epsilon_1, \delta_1 > 0 \) and \( a, b, c > 0 \) small enough there exists a continuous map \( F : \partial \Omega \times \Omega \subset \mathbb{C}^n \) such that the following holds.
(i) For each \( \zeta \in \partial \Omega \) the map \( F_\zeta = F(\zeta, \cdot) : \overline{\Omega} \to \mathbb{C}^n \) is a holomorphic embedding.
(ii) \( F_\zeta(\zeta) = (H_\zeta^1)^{-1}(a, 0, 0, 0) =: q_\zeta \).
(iii) \( F_\zeta(\Omega \cap B_3(\zeta)) \subset V_\zeta \cup B(b, \zeta) \cup \{q_\zeta\} \), and
(iv) \( \|F(\zeta, \cdot) - \text{id}\|_{\Omega \setminus B_1(\zeta)} < \varepsilon_1 \).

Moreover, if \( \overline{\Omega} \) is polynomially convex we may achieve that \( F_\zeta \in \text{Aut}_\text{hol}\mathbb{C}^n \) for each \( \zeta \).

**Proof.** We first define local maps achieving (i)-(iii), and then these will be approximately glued to the identity map using Lemma 5.3 above.

By scaling we may assume that all (local) images \( H_\zeta^1(\Omega) \) are contained in the ball \( B = \{z : 2\Re z_1 + \|z\|^2 < 0\} \) of radius one centered at the point \((-1, 0, \ldots, 0)\). Let \( \phi_{\nu,t} \) be the maps from Theorem 3.1 defined for the ball \( B \), the line segment \( l_a \), the ball of radius \( b \) centred at the point \( a \), and the set \( V = l_a(c) \). Similarly to the above discussion, since \( \phi_{\nu,t} \) are one variable maps, the maps defined by \( \tilde{F}_{\zeta}(z) := (H_{\zeta}^1)^{-1} \circ \phi_{\nu,t} \circ H_{\zeta}^1 \) do not depend on the choice of orthonormal frame on the tangent space, and so are well defined independently of \( j \), and \( \tilde{F}_{\zeta} \) is a locally defined map (near each \( \zeta \)) achieving (i)-(iii) for \( \nu \) large enough.

Now choose \( \delta > 0 \) small enough, set \( \gamma_{\zeta} = \tilde{F}_{\zeta}|_{C_{\zeta}(\mu)} \), and let \( \alpha_{\zeta}, \beta_{\zeta} \) be the maps from Lemma 5.3. Then the maps \( F_\zeta \) defined by \( \tilde{F}_{\zeta} \circ \alpha_{\zeta} \) on \( A_{\zeta}(2\tau) \) and \( \tilde{F}_{\zeta} \circ \beta_{\zeta} \) on \( B_{\zeta}(2\tau) \) are globally defined on \( \Omega \) and will satisfy (i)-(iv) as long as all constants involved are small enough.

For the last part, assuming polynomial convexity, note that \( F_\zeta \) is isotopic to the identity map through holomorphic embeddings by setting \( \tilde{F}_{\zeta,t} := (H_{\zeta}^1)^{-1} \circ \phi_{\nu,t} \circ H_{\zeta}^1 \), including the \( t \)-parameter from Theorem 3.1 and invoking the last statement in Lemma 5.3. Following the arguments in Section 5 in [3] each image \( F_{\zeta,t}(\Omega) \) will be polynomially convex, and so by the Andersén-Lempert theorem with parameters and jet-interpolation, the map \( F_{\zeta,1} \) may be approximated by a family of holomorphic automorphisms. Note that in the usual statements of the Andersén-Lempert theorems, isotopies are required to by of class \( C^2 \), but it is quite simple to work with \( C^0 \)-isotopies instead, see [4] for details. \( \square \)

### 5.3. Proof of Theorem 1.3

Let \( \eta_{\zeta} \) denote the arcs from the previous section with \( a << 1 \). We leave it to the reader to convince himself/herself that the arcs \( \gamma_{\zeta} \) may be modified so that each of them parametrises \( \eta_{\zeta} \) over an interval \([0, s]\), and so that \( \gamma_{\zeta} \) is perpendicular to \( H \). Let \( \psi_t : \partial \Omega \times \overline{\Omega} \cup (\bigcup_{\zeta} \{\zeta\} \times \eta_{\zeta}) \to \partial \Omega \times \mathbb{C}^n \) be a fiberpreserving smooth map, \( t \in [0, 1] \), that is the identity near \( \partial \Omega \times \overline{\Omega} \), and stretches each \( \eta_{\zeta} \) to cover \( \gamma_{\zeta} \). According to [9] Section 5 we have that \( \psi_t \) may be approximated by an isotopy of injective holomorphic maps such that \( \psi_1(\eta_{\zeta}) \) is still perpendicular to \( H \), keeping the notation \( \psi_t \) for the approximation. Let \( W_{\zeta} \) denote \( \psi_{\zeta,1}(H) \). Then each \( W_{\zeta} \) is transverse to \( \eta_{\zeta} \), and there is a \( b > 0 \) such that the ball of radius \( b \) centred at \( a - b \) is tangent to \( (H_{\zeta}^1)^{-1}(W_{\zeta}) \) for all \( \zeta \), where \( H_{\zeta}^1 \) is as in the previous section. Letting \( F_\zeta \) be maps as in Theorem 5.4, the maps \( \psi_{\zeta,1} \circ F_\zeta \) will satisfy the claims of the theorem. Finally, as before if \( \Omega \) is polynomially convex, the maps \( \psi_{\zeta,1} \) may be taken to be holomorphic automorphisms.

### 5.4. Proof of Theorem 5.1 and Theorem 5.2

In this subsection, we give the proofs of Theorem 5.1 and Theorem 5.2 which are based on Theorem 1.3.
Proof. (of Theorem 5.1) For $R > 0$, let $B_R$ be the ball in $\mathbb{C}^n$ centered at the origin with radius $R$, and let $S_R = \partial B_R$ be the boundary of $B_R$. We want to prove that there is a smooth family of curves $\gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n$ which satisfies the conditions 1)-4) in Theorem 1.3 with $H$ replaced by $S_R$ for $R >> 1$.

By assumption, there is a smooth map $F : \partial D \times \mathbb{C}^n \to \mathbb{C}^n$ such that for each $\zeta \in \partial D$, $F_{\zeta}(\cdot) := F(\zeta, \cdot) : \mathbb{C}^n \to \mathbb{C}^n$ is a diffeomorphism and $F_{\zeta}(0)$ is exposed at $F_{\zeta}(\zeta)$.

We assume that all $F_{\zeta}$ are orientation preserving. For any $\zeta \in \partial D$ there is a natural isotopy $H_{\zeta} : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ from $F_{\zeta}$ to the identity, given by $F_{\zeta,t} = \frac{1}{t}F_{\zeta}(tz)$ for $t \neq 0$, and smoothly extended to 0 by setting $F_{\zeta,0}(z) = z$.

This isotopy gives us a smooth family of vector fields $X_{\zeta}(t, z)$ on $\mathbb{C}^n$. In other words, $F_{\zeta,t}$ is generated by $X_{\zeta}$. It is clear that $X_{\zeta}(t, z)$ is even smooth jointly with respect to $\zeta, t$, and $z$.

Let $r > 0$ such that $F_{\zeta,t}(\mathbb{D}) \subset B_r(0) := \{ z \in \mathbb{C}^n : ||z|| < r \}$ for all $\zeta \in \partial D$ and $t \in [0,1]$. Let $R > r$ be a positive number. Let $\psi$ be a positive smooth function on $\mathbb{C}^n$ such that $\psi \equiv 1$ on $B_r$ and $\psi \equiv 0$ on $\mathbb{C}^n \setminus B_R$. Let $X'_{\zeta}(t, z) = \psi X_{\zeta}(t, z)$, then $X'$ has compact support.

Let $G_{\zeta} : \mathbb{C}^n \to \mathbb{C}^n$ be the diffeomorphism of $\mathbb{C}^n$ generated by $X'_{\zeta}(t, z), t \in [0,1]$. Then $G_{\zeta} = F_{\zeta}$ on $\mathbb{D}$ and $G_{\zeta} = Id$ on $\mathbb{C}^n \setminus B_R$.

For $\zeta \in \partial D$, let $n_{\zeta}$ be the unit out-pointing normal vector of $\partial G_{\zeta}(D)$ at $G_{\zeta}(\zeta)$. Then there is a unique $r_{\zeta} \in (0, +\infty)$ such that $G_{\zeta}(\zeta) + r_{\zeta}n_{\zeta} \in S_R$. We define a smooth family of curves $\gamma : \partial D \times [0,1] \to \partial D \times \mathbb{C}^n$ by setting $\gamma(\zeta, t) = (\zeta, G_{\zeta}^{-1}(G_{\zeta}(\zeta) + r_{\zeta}tn_{\zeta}))$.

Then the theorem follows by applying Theorem 1.3 with $\gamma$ and $H = S_R$. \qed

Proof. (of Theorem 5.2) For $n = 2$, we already assume that $\overline{D}$ is diffeomorphic to the closed unit ball. For $n > 2$, we want to show that $\overline{D}$ is diffeomorphic to the closed unit ball too. By the Collar Theorem (see Theorem 6.1 in [8]), $\partial D$ is simply connected. Let $z_0 \in D$ and let $B$ be an open ball centered at $z_0$ with boundary $S \subset D$. Let $W = \overline{D} \setminus B$, then the triad $(W; S, \partial D)$ is an $h$-cobordism. Note that the relative homology $H_*(W; S) = 0$. By the h-Cobordism Theorem (see Theorem 9.1 in [10]), $W$ is diffeomorphic to $S \times [0,1]$ and hence $\overline{D}$ is contractible. So $\overline{D}$ is diffeomorphic the the unit ball (See Proposition A in §9 in [10]). By a result in differential topology (see Theorem 3.1 in [8]), there is a diffeomorphism $\sigma$ of $\mathbb{C}^n$ such that $\sigma(\overline{D})$ is the closed unit ball. So the Theorem follows from Theorem 5.1. \qed

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