Analysis of Remaining Uncertainties and Exponents under Various Conditional Rényi Entropies

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Abstract

In this paper, we analyze the asymptotics of the normalized remaining uncertainty of a source when a compressed or hashed version of it and correlated side-information is observed. For this system, commonly known as Slepian-Wolf source coding, we establish the optimal (minimum) rate of compression of the source to ensure that the remaining uncertainties vanish. We also study the exponential rate of decay of the remaining uncertainty to zero when the rate is above the optimal rate of compression. In our study, we consider various classes of random universal hash functions. Instead of measuring remaining uncertainties using traditional Shannon information measures, we do so using two forms of the conditional Rényi entropy. Among other techniques, we employ new one-shot bounds and the moments of type class enumerator method for these evaluations. We show that these asymptotic results are generalizations of the strong converse exponent and the error exponent of the Slepian-Wolf problem under maximum a posteriori (MAP) decoding.

Index Terms

Remaining uncertainty, Conditional Rényi entropies, Rényi divergence, Error exponent, Strong converse exponent, Slepian-Wolf coding, Universal hash functions, Information-theoretic security, Moments of type class enumerator method

I. INTRODUCTION

In information-theoretic security [1], [2], it is of fundamental importance to study the remaining uncertainty of a random variable $A^n$ given a compressed version of itself $f(A^n)$ and another correlated signal $E^n$. This model, reminiscent of the the Slepian-Wolf source coding problem [3], is illustrated in Fig. 1. A model somewhat similar to the one we study here was studied by Tandon, Ulukus and Ramachandran [4] who analyzed the problem of secure source coding with a helper. In particular, a party would like to reconstruct a source $A^n$ given a “helper” signal (or a compressed version of it) but an eavesdropper, who can tap on $f(A^n)$ is also present in the system. The authors in [4] analyzed the tradeoff between the compression rate and the equivocation of $A^n$ given $f(A^n)$. Villard and Piantanida [5] and Bross [6] considered the setting in which the eavesdropper also has access to memoryless side-information $E^n$ that is correlated with $A^n$. However, there are many ways that one could measure the equivocation or remaining uncertainty. The traditional way, starting from Wyner’s seminal paper on the wiretap channel [7] (and also in [1], [2], [4]–[6]), is to do so using the conditional Shannon entropy $H(A^n|f(A^n), E^n)$, leading to a “standard” equivocation measure. In this paper, we study the asymptotics of remaining uncertainties based on the family of Rényi information measures [8]. The measures we consider include the conditional Rényi entropy $H_{1+s}(A^n|f(A^n), E^n)$ and its so-called Gallager form, which we denote as $H_{1+s}^1(A^n|f(A^n), E^n)$. We note that unlike the conditional Shannon entropy, there is no universally accepted definition for the conditional Rényi entropy, so we define the quantities that we study carefully in Section II-A. Extensive discussions of various plausible notions of the conditional Rényi entropy are provided in the recent works by Teixeira, Matos and Antunes [9] and Fehr and Berens [10].

We motivate our study by first showing that the limits of the (normalized) remaining uncertainty $\frac{1}{n}H_{1+s}(A^n|f(A^n), E^n)$ and the exponent of the remaining uncertainty $-\frac{1}{n}\log H_{1+s}(A^n|f(A^n), E^n)$ (for appropriately chosen Rényi parameters $1+s$) are, respectively, generalizations of the strong converse exponent and the error exponent for decoding $A^n$ given $(f(A^n), E^n)$. Recall that the strong converse exponent [11], [12] is the exponential rate at which the probability of correct decoding tends to zero when one operates at a rate below the first-order coding rate, i.e., the conditional Shannon entropy $H(A|E)$. In contrast, the error exponent [13]–[16] is the exponential rate at which the probability of incorrect decoding tends to zero when one operates at a rate above $H(A|E)$. Thus, studying the asymptotics of the conditional Rényi entropy allows us not only to understanding the remaining uncertainty for various classes of hash functions [17], [18] but also allows us to provide additional information and intuition concerning the strong converse exponent and the error exponent for Slepian-Wolf coding [3]. We also motivate our study by considering a scenario in information-theoretic security where the hash functions we study appear naturally, and coding can be done in a computationally efficient manner. The present work can be regarded a follow-on from the authors’ previous work in [19] on the asymptotics of the equivocations where we studied the behavior of $C_{1+s} := nR - H_{1+s}(f(A^n)|E^n)$.

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1In this paper, we abuse terminology and use the terms Slepian-Wolf coding [3] and lossless source coding with decoder side-information interchangeably.
and \( C_{1+s}^\dagger := nR - H_{1+s}(f(A^n)|E^n) \) (where \( R = \frac{1}{n} \log \| f \| \) is the rate of the cardinality of the range of \( f \)). In [19], we also studied the exponents and second-order asymptotics of the equivocation. However, we note that because we consider the remaining uncertainty instead of the equivocation, several novel techniques, including new one-shot bounds and large-deviation techniques, have to be developed to single-letterize various expressions.

**Paper Organization:** This paper is organized as follows. In Section II, we recap the definitions of standard Shannon information measures and some less common Rényi information measures [9], [10]. We also introduce some new quantities and state relevant properties of the information measures. We state some notation concerning the method of types [16].

In Section III, we further motivate our study by relating the quantities we wish to characterize to the error exponent and strong converse exponent of Slepian-Wolf coding (Proposition 1). In Section IV, we define various important classes of hash functions (e.g., one-shot bounds, large-deviation techniques as well as the moments of type class enumerator method [21]–[25]). Theorems 2, 3 and 4 are proved in Sections V, VI and VII respectively. We conclude our discussion and suggests further avenues for research in Section VIII. Some technical results (e.g., one-shot bounds, concentration inequalities) are relegated to the appendices.

II. INFORMATION MEASURES AND OTHER PRELIMINARIES

A. Basic Shannon and Rényi Information Quantities

We now introduce some information measures that generalize Shannon’s information measures. Fix a normalized distribution \( P_A \in P(A) \) and a sub-distribution (a non-negative vector but not necessarily summing to one) \( Q_A \in P(A) \) supported on a finite set \( A \). Then the relative entropy and the Rényi divergence of order \( 1+s \) are respectively defined as

\[
D(P_A||Q_A) := \sum_{a \in A} P_A(a) \log \frac{P_A(a)}{Q_A(a)}
\]

and

\[
D_{1+s}(P_A||Q_A) := \frac{1}{s} \log \sum_{a \in A} P_A(a)^{1+s}Q_A(a)^{-s},
\]

where throughout, \( \log \) is to the natural base \( e \). It is known that \( \lim_{s \to 0} D_{1+s}(P_A||Q_A) = D(P_A||Q_A) \) so a special (limiting) case of the Rényi divergence is the usual relative entropy. It is also known that the map \( s \mapsto sD_{1+s}(P_A||Q_A) \) is concave in \( s \in \mathbb{R} \) and hence \( D_{1+s}(P_A||Q_A) \) is monotonically increasing for \( s \in \mathbb{R} \). Furthermore, the following data processing or information processing inequalities for Rényi divergences hold for \( s \in [-1,1] \),

\[
D(P_{AW}||Q_{AW}) \leq D(P_A||Q_A) \tag{3}
\]

and

\[
D_{1+s}(P_{AW}||Q_{AW}) \leq D_{1+s}(P_A||Q_A). \tag{4}
\]

Here \( W : A \to B \) is any stochastic matrix (channel) and \( P_{AW}(b) := \sum_a W(b|a)P_A(a) \) is the output distribution induced by \( W \) and \( P_A \).

We also introduce conditional entropies on the product alphabet \( A \times E \) based on the divergences above. Let \( I_A(a) = 1 \) for each \( a \in A \). If \( P_{AE} \) is a distribution on \( A \times E \), the conditional entropy, the conditional Rényi entropy of order \( 1+s \) and the min-entropy relative to another normalized distribution \( Q_E \) on \( E \) as

\[
H(A|E|P_{AE}||Q_E) := -D(P_{AE}||I_A \times Q_E), \tag{5}
\]

\[
H_{1+s}(A|E|P_{AE}||Q_E) := -D_{1+s}(P_{AE}||I_A \times Q_E), \tag{6}
\]

and

\[
H_{\text{min}}(A|E|P_{AE}||Q_E) := -\log \max_{(a,e);Q_E(e) > 0} \frac{P_{AE}(a,e)}{Q_E(e)}. \tag{7}
\]
It is known that \( \lim_{s \to 0} H_{1+s}(A|E|P_{AE}|Q_E) = H(A|E|P_{AE}|Q_E) \) and
\[
\lim_{s \to \infty} H_{1+s}(A|E|P_{AE}|Q_E) = H_{\infty}(A|E|P_{AE}|Q_E) = H_{\min}(A|E|P_{AE}|Q_E).
\] (8)

If \( Q_E = P_E \), we simplify the above notations and denote the conditional entropy, the conditional Rényi entropy of order \( 1+s \) and the min-entropy as
\[
H(A|E|P_{AE}) := H(A|E|P_{AE}|P_E) = -\sum_e P_E(e) \sum_a P_{A|E}(a|e) \log P_{A|E}(a|e),
\] (9)
\[
H_{1+s}(A|E|P_{AE}) := H_{1+s}(A|E|P_{AE}|P_E) = -\frac{1}{s} \log \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1+s},
\] (10)
\[
H_{\min}(A|E|P_{AE}) := H_{\min}(A|E|P_{AE}|P_E) = -\log \max_{(a,e):P_E(e)>0} P_{A|E}(a|e).
\] (11)

The map \( s \mapsto sH_{1+s}(A|E|P_{AE}) \) is concave, and \( H_{1+s}(A|E|P_{AE}|Q_E) \) is monotonically decreasing for \( s \in \mathbb{R} \setminus \{0\} \). The definition of the conditional Rényi entropy in (10) is due to Hayashi [26, Section II.A] and Škorić et al. [27, Definition 7].

We are also interested in the so-called Gallager form of the conditional Rényi entropy and the min-entropy for a joint distribution \( P_{AE} \in \mathcal{P}(A \times E) \):
\[
H_{1+s}^\dagger(A|E|P_{AE}) := -\frac{1+s}{s} \log \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e)^{1+s} \right)^{\frac{s}{1+s}}
\] (12)
\[
H_{\min}^\dagger(A|E|P_{AE}) := H_{\infty}^\dagger(A|E|P_{AE}) = -\log \sum_e P_E(e) \max_a P_{A|E}(a|e)
\] (13)

By defining the familiar Gallager function [13, 14] (parametrized slightly differently)
\[
\phi(s|A|E|P_{AE}) := \log \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e)^{1+s} \right)^{1-s}
\] (14)
we can express (12) as
\[
H_{1+s}^\dagger(A|E|P_{AE}) = -\frac{1+s}{s} \phi \left( \frac{s}{1+s} |A|E|P_{AE} \right),
\] (15)
thus (loosely) justifying the nomenclature “Gallager form” of the conditional Rényi entropy in (12). Note that \( H_{1+s} \) and \( H_{1+s}^\dagger \) are respectively denoted as \( H_{1+s}^\dagger \) and \( H_{1+s} \) in the paper by Fehr and Berens [10]. The Gallager form of the conditional Rényi entropy, also commonly known as Arimoto’s conditional Rényi entropy [28], was shown in [10] to satisfy two natural properties for \( s \geq -1 \), namely, monotonicity under conditioning (or simply monotonicity)
\[
H_{1+s}^\dagger(A|B,E|P_{ABE}) \leq H_{1+s}^\dagger(A|E|P_{AE}),
\] (16)
and the chain rule
\[
H_{1+s}^\dagger(A|B,E|P_{ABE}) \geq H_{1+s}^\dagger(A|E|P_{AE}) - \log |B|.
\] (17)
The monotonicity property of \( H_{1+s}^\dagger \) was also shown operationally by Bunte and Lapidoth in the context of lossless source coding with lists and side-information [29] and encoding tasks with side-information [30]. We exploit these properties in the sequel. The quantities \( H_{1+s} \) and \( H_{1+s}^\dagger \) can be shown to be related as follows [10, Theorem 4]
\[
\max_{Q_E \in \mathcal{P}(E)} H_{1+s}(A|E|P_{AE}|Q_E) = H_{1+s}^\dagger(A|E|P_{AE})
\] (18)
for \( s \in [-1, \infty) \setminus \{0\} \). The maximum on the left-hand-side is attained for the tilted distribution
\[
Q_E(e) = \frac{(\sum_a P_{AE}(a,e)^{1+s})^{\frac{s}{1+s}}}{\sum_e (\sum_a P_{AE}(a,e)^{1+s})^{\frac{s}{1+s}}}. \tag{19}
\]

The map \( s \mapsto sH_{1+s}^\dagger(A|E|P_{AE}) \) is concave and the map \( s \mapsto H_{1+s}(A|E|P_{AE}) \) is monotonically decreasing for \( s \in (-1, \infty) \).

It can be shown by L’Hôpital’s rule that \( \lim_{s \to 0} H_{1+s}(A|E|P_{AE}) = H(A|E|P_{AE}) \). Thus, we regard \( H_{1+s}^\dagger(A|E|P_{AE}) \) as \( H(A|E|P_{AE}) \), i.e., when \( s = 0 \), the conditional Rényi entropy and its Gallager form coincide and are equal to the conditional Shannon entropy.

We also find it useful to consider a two-parameter family of the conditional Rényi entropy\(^2\)
\[
H_{1+s|1+t}(A|E|P_{AE}) := -\frac{1+t}{s} \log \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e)^{1+s} \right) \left( \sum_a P_{A|E}(a|e)^{1+t} \right)^{-\frac{s}{1+s}}.
\] (20)
\(^2\)This new information-theoretic quantity is somewhat related to \( H_{1+s|1+t} \) in the work by Hayashi and Watanabe [31, Eq. (14)-(15)] but is different and not to be confused with \( H_{1+s|1+t} \).
Clearly \( H_{1+s}^1(A|E|P_{AE}) = H_{1+s}^1(A|E|P_{AE}) \), so the two-parameter conditional Rényi entropy is a generalization of the Gallager form of the conditional Rényi entropy in (12).

For future reference, given a joint source \( P_{AE} \), define the critical rates
\[
\hat{R}_s := \frac{d}{dt} t H_{1+s}(A|E|P_{AE}) \bigg|_{t=s}, \quad \text{and,} \quad \hat{R}_s^\dagger := \frac{d}{dt} t H_{1+t}(A|E|P_{AE}) \bigg|_{t=s}.
\]  

(21)  

(22)

B. Notation for Types

The proofs of our results leverage on the method of types [16, Ch. 2], so we summarize some relevant notation here. The set of all distributions (probability mass functions) on a finite set \( \mathcal{A} \) is denoted as \( \mathcal{P}(\mathcal{A}) \). The type or empirical distribution of a sequence \( a \in \mathcal{A}^n \) is the distribution \( Q(a) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{a_i = a\}, a \in \mathcal{A} \). The set of all sequences \( a \in \mathcal{A}^n \) with type \( Q \in \mathcal{P}(\mathcal{A}) \) is the type class and is denoted as \( T_Q \subset \mathcal{A}^n \). The set of all \( n \)-types (types formed from length-\( n \) sequences) on alphabet \( \mathcal{A} \) is denoted as \( \mathcal{P}_n(\mathcal{A}) \). When we write \( a_n \leq b_n \), we mean that inequality on an exponential scale, i.e., \( \lim_{n \to \infty} \frac{\log b_n}{b_n} \leq 0 \). The notations \( \gtrsim \) and \( \lesssim \) are defined analogously. Throughout, we will use the fact that the number of types \( |\mathcal{P}_n(\mathcal{A})| \leq (n+1)^{|\mathcal{A}| - 1} \).

III. Motivation for Studying Remaining Uncertainties

As mentioned in the introduction, in this paper, we study the remaining uncertainty and its rate of exponential decay measured using various Rényi information measures. In this section, we further motivate the relevance of this study by relating the remaining uncertainty to the strong converse exponent for decoding.

A. Relation to the Strong Converse Exponent for Slepian-Wolf Coding

Consider the Slepian-Wolf source coding problem as shown in Fig. 1. For a given function (encoder) \( f_n : \mathcal{A}^n \to \mathcal{M}_n \) and side information vector \( e \in \mathcal{E}^n \), we may define the maximum a-posteriori (MAP) decoder \( g_{f_n} : \mathcal{M}_n \times \mathcal{E}^n \to \mathcal{A}^n \) as follows:
\[
g_{f_n}(m, e) := \arg \max_{a \in A^n : f_n(a) = m} P_{AE}^n(a, e) = \arg \max_{a \in A^n : f_n(a) = m} P_{AE}^n(a|e).
\]  

(23)

Define the probability of correctly decoding \( a \) given the encoder \( f_n \) and the MAP decoder \( g_{f_n} \) as follows:
\[
P_{c}^{(n)}(f_n) := \sum_{a} \sum_{a \neq g_{f_n}(f_n(a), e)} P_{AE}^n(a, e).
\]  

(24)

Then, by the definition of \( H_{1+s}^1 \) in (13), we immediately see that
\[
-\frac{1}{n} \log P_{c}^{(n)}(f_n) = \frac{1}{n} H_{1+s}^1(A^n|f_n(A^n), E^n|P_{AE}^n).
\]  

(25)

When optimized over \( \{f_n\}_{n=1}^\infty \), the quantity on the left of (25) (or its limit) is called the strong converse exponent as its limit characterizes the optimal exponential rate at which the probability of correct decoding the true source \( a \) given \( (f_n(a), e) \) decays to zero. Thus, by studying the asymptotics of \( -\frac{1}{n} \log P_{c}^{(n)}(f_n) \) for all \( s \in [0, \infty) \) and, in particular, the limiting case of \( s \uparrow \infty \) (which we do in (53) in Part (2) of Theorem 2), we obtain a generalization of the strong converse exponent for the Slepian-Wolf problem. In fact, it is known that \( \lim_{n \to \infty} -\frac{1}{n} \log P_{c}^{(n)}(f_n) > 0 \) for any sequence of encoders \( \{f_n\}_{n=1}^\infty \) if and only if the rate \( \lim_{n \to \infty} -\frac{1}{n} \log ||f_n|| < H(A|E|P_{AE}) \) [12, Theorem 2]. This fact will be utilized in the proof of Theorem 3.

B. Relation to the Error Exponent for Slepian-Wolf Coding

Similarly, we may define the probability of incorrectly decoding \( a \) given the encoder \( f_n \) and MAP decoder \( g_{f_n} \) as follows:
\[
P_{c}^{(n)}(f_n) := \sum_{a} \sum_{a \neq g_{f_n}(f_n(a), e)} P_{AE}^n(a, e).
\]  

(26)

Then we have the following proposition concerning the exponent of \( P_{c}^{(n)}(f_n) \).

**Proposition 1.** Assume that \( P_{c}^{(n)}(f_n) \) tends to zero exponentially fast for a given sequence of hash functions \( \{f_n\}_{n=1}^\infty \), i.e., \( \lim_{n \to \infty} -\frac{1}{n} \log P_{c}^{(n)}(f_n) > 0 \) (the existence of the limit is part of the assumption). Then for any \( s \geq 1 \), we have
\[
\lim_{n \to \infty} -\frac{1}{n} \log P_{c}^{(n)}(f_n) = \lim_{n \to \infty} -\frac{1}{n} \log H_{1+s}^1(A^n|f_n(A^n), E^n|P_{AE}^n) = \lim_{n \to \infty} -\frac{1}{n} \log H_{1+s}^1(A^n|f_n(A^n), E^n|P_{AE}^n).
\]  

(27)  

(28)
We recall, by the Slepian-Wolf theorem [3], that there exists a sequence of encoders \( \{f_n\}_{n=1}^\infty \) such that \( P_{c}^{(n)}(f_n) \) tends to zero if and only if \( \lim_{n \to \infty} \frac{1}{n} \log \|f_n\| \geq H(A|E|PAE) \). When optimized over \( \{f_n\}_{n=1}^\infty \), the quantity on the left of (27) is called the optimal error exponent and it characterizes the optimal exponential rate at which the error probability of decoding a given \( (f_n(a),e) \) decays to zero. Thus, Proposition 1 says that the exponents of \( H_1^{s,s} \) and \( H_{1+s}^{s,s} \) for \( s \geq 1 \) are generalizations of the error exponent of decoding \( A^n \) given \( (f_n(A^n),E^n) \). We establish bounds on these limits for certain classes of hash functions in Part (2) of Theorem 4.

**Proof:** We first consider the Gallager form of the conditional Rényi entropy \( H_1^{s,s} \). For brevity, we let \( f = f_n \) (suppressing the dependence on \( n \)) and we also define the probability distributions \( P := P_{f(A^n),E^n} \) and \( Q := P_{A^n|f(A^n),E^n} \). Recall the definition of the MAP decoder \( g_f(m,e) \) in (23). We have

\[
e^{-sH_1^{s,s}(A^n|f(A^n),E^n|P_{A|E})} = \sum_{e,m} P(m,e) \left( \sum_a Q(a|m,e)^{1+s} \right)^{\frac{1}{1+s}} \]

\[
\geq \sum_{e,m} P(m,e) \left( Q(g_f(m,e)|m,e)^{1+s} \right)^{\frac{1}{1+s}} \]

\[
= \sum_{e,m} P(m,e) Q(g_f(m,e)|m,e) \]

\[
= 1 - P_{c}^{(n)}(f) \quad \text{(32)}
\]

In the following chain of inequalities, we will employ Taylor’s theorem with the Lagrange form of the remainder for the function \( t \mapsto (1 + t)^{1+s} \) at \( t = 0^{-} \), i.e.,

\[
(1 + t)^{1+s} = 1 + (1 + s)t + \frac{s(1 + s)}{2}(1 + \xi(t))^{s-1}t^2
\]

for some \( \xi \in [t,0] \). We choose \( t \) to be \( Q(g_f(m,e)|m,e) - 1 \) in our application in (36) to follow. Let \( \xi(m,e) \) be a generic element of \( \{Q(g_f(m,e)|m,e) - 1\} \subset [-1,0] \) taking the role of \( \xi \) in the Taylor series expansion in (33). We bound the conditional Rényi entropy as follows:

\[
e^{-sH_1^{s,s}(A^n|f(A^n),E^n|P_{A|E})} = \sum_{e,m} P(m,e) \sum_a Q(a|m,e)^{1+s} \]

\[
= \sum_{e,m} P(m,e) \left\{ Q(g_f(m,e)|m,e)^{1+s} + \sum_{a:a \neq g_f(m,e)} Q(a|m,e)^{1+s} \right\} \]

\[
= \sum_{e,m} P(m,e) \left\{ 1 + (1 + s)\left[ Q(g_f(m,e)|m,e) - 1 \right] + \frac{s(1 + s)}{2}(1 + \xi(m,e))^{s-1}\left[ Q(g_f(m,e)|m,e) - 1 \right]^2 + \sum_{a:a \neq g_f(m,e)} Q(a|m,e)^{1+s} \right\} \]

\[
\leq \sum_{e,m} P(m,e) \left\{ 1 + (1 + s)\left[ Q(g_f(m,e)|m,e) - 1 \right] + \frac{s(1 + s)}{2}\left[ Q(g_f(m,e)|m,e) - 1 \right]^2 + \sum_{a:a \neq g_f(m,e)} Q(a|m,e) \right\} \]

\[
= 1 - s \sum_{e,m} P(m,e) \sum_{a:a \neq g_f(m,e)} Q(a|m,e) + \frac{s(1 + s)}{2} \left[ \sum_{a:a \neq g_f(m,e)} Q(a|m,e) \right]^2 \]

\[
= 1 - s P_{c}^{(n)}(f) + \frac{s(1 + s)}{2} \sum_{e,m} P(m,e) \left[ \sum_{a:a \neq g_f(m,e)} Q(a|m,e) \right]^2 \quad \text{(40)}
\]

In (37), noting that \( s + 1 \geq 0 \), we uniformly upper bounded \( (1 + \xi(m,e))^{s-1} \) by 1. We also upper bounded \( Q(a|m,e)^{1+s} \) by \( Q(a|m,e) \). In (40), we used the definition of \( P_{c}^{(n)}(f) \) stated in (26). Because \( P_{c}^{(n)}(f) \) is assumed to decay exponentially fast,
we have
\[ P_e^{(n)}(f) \doteq -\log(1 - P_e^{(n)}(f)) \]  
\[ \geq sH_{1+s}(A^n|f(A^n), E^n|P_{AE}^n) \]  
\[ \geq sH_{1+s}(A^n|f(A^n), E^n|P_{AE}^n) \]  
\[ \geq -\log \left\{ 1 - sP_e^{(n)}(f) + \frac{s(1 + s)}{2} \sum_{e,m} P(m,e) \left[ \sum_{a:a \neq g_f(m,e)} Q(a|m,e) \right]^2 \right\} \]  
\[ \doteq sP_e^{(n)}(f) - \frac{s(1 + s)}{2} \sum_{e,m} P(m,e) \left[ \sum_{a:a \neq g_f(m,e)} Q(a|m,e) \right]^2, \]  
where (41) and (45) follow from \(-\log(1 - t) = t + O(t^2)\), (42) uses (32), (43) uses the fact that \(H_{1+s}^{f} \geq H_{1+s}\) (cf. (18)) and (44) uses (40). The second term in (45) is exponentially smaller than \(P_e^{(n)}(f)\) because of the square operation and the fact that \(\sum_{a:a \neq g_f(m,e)} Q(a|m,e) < 1\). Now, since \(s \geq 1\) is constant, the exponents of the quantities on the left and right sides of the above chain are equal. Thus they are equal to the exponents of \(H_{1+s}^{f}(A^n|f(A^n), E^n|P_{AE}^n)\) and \(H_{1+s}(A^n|f(A^n), E^n|P_{AE}^n)\) for every \(s \geq 1\). This completes the proof of Proposition 1.

\[ \square \]

IV. Main Results: Asymptotics of the Remaining Uncertainties

In this section we present our results concerning the asymptotic behavior of the remaining uncertainties and its exponential behavior. As mentioned in Section III, the former is a generalization of the strong converse exponent for the Slepian-Wolf problem [3], while the latter is a generalization of the error exponent for the same problem. Before doing so, we define various classes of random hash functions and further motivate our analysis using an example from information-theoretic security.

A. Definitions of Various Classes of Hash Functions

We now define various classes of hash functions. We start by stating a slight generalization of the canonical definition of a universal 2 hash function by Carter and Wegman [17].

Definition 1. A random\(^3\) hash function \( f_X \) is a stochastic map from \( A \) to \( M := \{1, \ldots, M\} \), where \( X \) denotes a random variable describing its stochastic behavior. The set of all random hash functions mapping from \( A \) to \( M \) is denoted as \( \mathcal{R} = \mathcal{R}(A, M) \). A hash function \( f_X \) is called an \( \epsilon \)-almost universal 2 hash function if it satisfies the following condition: For any distinct \( a_1, a_2 \in A \),

\[ \Pr(f_X(a_1) = f_X(a_2)) \leq \frac{\epsilon}{M}. \]  

(46)

When \( \epsilon = 1 \) in (46), we simply say that \( f_X \) is a universal 2 hash function [17]. We denote the set of universal 2 hash functions mapping from \( A \) to \( M \) by \( U_2 = U_2(A, M) \).

The following definition is due to Wegman and Carter [18].

Definition 2. A random hash function \( f_X : A \to \{1, \ldots, M\} \) is called strongly universal when the random variables \( \{f_X(a) : a \in A\} \) are independent and subject to a uniform distribution, i.e.,

\[ \Pr(f_X(a) = m) = \frac{1}{M} \]  

(47)

for all \( m \in \{1, \ldots, M\} \). If \( f_X \) is a strongly universal hash function, we emphasize this fact by writing \( f_X \).

As an example, if \( f_X \) independently and uniformly assigns each element of \( a \in A \) into one of \( M \) “bins” indexed by \( m \in M \) (i.e., the familiar random binning process introduced by Cover in the context of Slepian-Wolf coding [32]), then (47) holds, yielding a strongly universal hash function. The hierarchy of hash functions is shown in Fig. 2.

A universal 2 hash function \( f \) can be implemented efficiently via circulant (special case of Toeplitz) matrices. The complexity is low—applying \( f \) to an \( m \)-bit string requires \( O(m \log m) \) operations generally. For details, see the discussion in Hayashi and Tsurumaru [20] and the subsection to follow. So, it is natural to assume that the encoding functions \( f \) we analyze in this paper are universal 2 hash functions.

B. Another Motivation for Analyzing Remaining Uncertainties

To ensure a reasonable level of security in practice, we often send our message via multiple paths in networks. Assume that Alice wants to send an \( m \)-bit “message” \( A \) to Bob via \( l \in \mathbb{N} \) paths, and that Eve has access to side-information \( E \)

\(^3\)For brevity, we will sometimes omit the qualifier “random”. It is understood, henceforth, that all so-mentioned hash functions are random hash functions.
correlated to $A$ and intercepts one of the $l$ paths. We also suppose $m = kl$ for some $k \in \mathbb{N}$. Alice applies an invertible function $f$ to $A$ and divides $f(A)$ into $k$ equal-sized parts $(f(A)_1, f(A)_2, \ldots, f(A)_k) \in \mathbb{F}_2^n \cong \mathbb{F}_2^l \oplus \cdots \oplus \mathbb{F}_2^l$ ($k$ times). See Fig. 3. Bob receives all of them, and applies $f^{-1}$ to decode $A$. Hence, Bob can recover the original message $A$ losslessly. However, if Eve somehow manages to tap on the $j$-th part $f(A)_j$ (where $j \in \{1,2,\ldots,k\}$), Eve can possibly estimate the message $A$ from $E$ and $f(A)_j$ (in Fig. 3, we assume Eve taps on the first piece of information $j = 1$). Eve's uncertainty with respect to $A$ is $H(A|f(A)_j, E|P_{AE})$ ($H$ here is a generic entropy function; it will be taken to be various conditional Rényi entropies in the subsequent subsections). In this scenario, it is not easy to estimate the uncertainty $H(A|f(A)_j, E|P_{AE})$ as it depends on the choice of $j$. To avoid such a difficulty, we propose to apply a random invertible function $f_X$ to $A$. To further resolve the aforementioned issue from a computational perspective, we regard $\mathbb{F}_2^n$ as the finite extension field $\mathbb{F}_{2^m}$. When Alice and Bob choose invertible element $X$ in the finite field $\mathbb{F}_{2^m}$ subject to the uniform distribution, and $f_X(A)$ is defined as $f(A) := XA$, the map $A \mapsto f(A)_j$ is a universal$_2$ hash function. Then, Eve's uncertainty with respect to $A$ can be described as $H(A|f(A)_j, E, X|P_{AE} \times P_X)$. When $(A, E)$ is taken to be $(A^n, E^n) = \{(A_i, E_i)\}_{i=1}^n$ where the $(A_i, E_i)$'s are independent and identically distributed, our results in the following subsections are directly applicable in evaluating Eve's uncertainty measured according to various conditional Rényi entropies. We remark that if $m$ is not a multiple of $l$, we can make the final block smaller than $l$ bits without any loss of generality asymptotically.

Indeed, this protocol can be efficiently implemented with (low) complexity of $O(m \log m)$ [20] because multiplication in the finite field $\mathbb{F}_{2^m}$ can be realized by an appropriately-designed circulant matrix, leading to a fast Fourier transform-like algorithm. Therefore, this communication setup, which contains an eavesdropper, is “practical” in the sense that encoding and decoding can be realized efficiently.

### C. Asymptotics of Remaining Uncertainties

Our results in Theorem 2 to follow pertain to the worst-case remaining uncertainties over all universal$_2$ hash functions. We are interested in $\sup_{f_X \in U_2} \frac{1}{n} H_{1\pm s}^{f_X}$ and $\sup_{f_X \in U_2} \frac{1}{n} H_{1\pm s}^{f_X}$, where $H_{1\pm s}$ is a shorthand for $H_{1\pm s}(A^n|f_X, E^n, X_n|P_{AE} \times P_X)$ (similarly for $H_1^{f_X}$) and $P_{AE}$ is the $n$-fold product measure. We emphasize that the evaluations of $\sup_{f_X \in U_2} \frac{1}{n} H_{1\pm s}^{f_X}$ and $\sup_{f_X \in U_2} \frac{1}{n} H_{1\pm s}^{f_X}$ are stronger than those in standard achievability arguments in Shannon theory where one often uses a
random selection argument to assert that an object (e.g., a code) with good properties exist. In our calculations of the asymptotics of \( \sup_{f_{X_n} \in \mathcal{U}_2} \frac{1}{n} H_{1+s}^{\uparrow} \) and \( \inf_{f_{X_n} \in \mathcal{R}} \frac{1}{n} H_{1+s}^{\uparrow} \), we assert that all hash functions in \( \mathcal{U}_2 \) have a certain desirable property; namely, that the remaining uncertainties can be appropriately upper bounded. In addition, in Theorem 3 to follow, we also quantify the minimum rate \( R \) such that the best-case remaining uncertainties over all random hash functions \( \inf_{f_{X_n} \in \mathcal{R}} \frac{1}{n} H_{1+s}^{\uparrow} \) vanish. For many values of \( s \), we show the minimum rates for the two different evaluations (worst-case over all \( f_{X_n} \in \mathcal{U}_2 \) and best-case over all \( f_{X_n} \in \mathcal{R} \)) coincide, establishing tightness for the optimal compression rates.

Let \( |t|^+ := \max\{0, t\} \). The following is our first main result.

**Theorem 2 (Remaining Uncertainties).** For each \( n \in \mathbb{N} \), let the size\(^4\) of the range of \( f_{X_n} \) be \( M_n = e^{nR} \). Fix a joint distribution \( P_{AE} \in \mathcal{P}(\mathcal{A} \times \mathcal{E}) \). Define the worst-case limiting normalized remaining uncertainties over all universal\(_2\) hash functions as

\[
G(R, s) := \lim_{n \to \infty} \frac{1}{n} \sup_{f_{X_n} \in \mathcal{U}_2} H_{1+s}^{\uparrow}(A^n | f_{X_n}(A^n), E^n, X_n | P_{AE} \times P_{X_n}), \quad \text{and} \quad \tag{48}
\]

\[
G^\dagger(R, s) := \lim_{n \to \infty} \frac{1}{n} \sup_{f_{X_n} \in \mathcal{U}_2} H_{1+s}^{\uparrow}(A^n | f_{X_n}(A^n), E^n, X_n | P_{AE} \times P_{X_n}). \quad \tag{49}
\]

Recall the definitions of the critical rates \( \bar{R}_s \) and \( \bar{R}_s^\dagger \) in (21) and (22) respectively. The following achievability statements hold:

1) For any \( s \in [0, 1] \), we have

\[
G(R, -s) \leq |H_{1-s}(A|E|P_{AE})| - R|^{+}, \quad \tag{50}
\]

and for any \( s \in [0, 1/2] \), we have

\[
G^\dagger(R, -s) \leq |H_{1-s}^{\uparrow}(A|E|P_{AE})| - R|^{+}. \quad \tag{51}
\]

2) For \( s \in (0, \infty) \), we have

\[
G(R, s) \leq \begin{cases} H_{1+s}^{\uparrow}(A|E|P_{AE}) - R & R \leq \bar{R}_s \\ \max_{t \in [0, s]} \frac{1}{2} \log(1+t) - R & R > \bar{R}_s \end{cases}, \tag{52}
\]

and

\[
G^\dagger(R, s) \leq \begin{cases} H_{1+s}^{\uparrow}(A|E|P_{AE}) - R & R \leq \bar{R}_s^\dagger \\ \max_{t \in [0, s]} \frac{1}{2} \log(1+t) - R & R > \bar{R}_s^\dagger \end{cases}. \quad \tag{53}
\]

**Theorem 2** is proved in Section V and uses several novel one-shot bounds on the remaining uncertainties (summarized in Appendix A) coupled with appropriate uses of large-deviation results such as Cranmer’s theorem and Sanov’s theorem \([33]\).

In Figs. 4 and 5, we plot the upper bounds in (50)–(53) for a correlated source \( P_{AE} \in \mathcal{P}(\{0, 1\}^2) \) with \( P_{AE}(0, 0) = 0.7 \) and \( P_{AE}(0, 1) = P_{AE}(1, 0) = P_{AE}(1, 1) = 0.1 \). For the upper bounds in (50) and (51), we see from Fig. 4 that the rates at which the curves transition from a positive quantity to zero are clearly the conditional Rényi entropies \( H_{1-s}(A|E|P_{AE}) \) and \( H_{1-s}^{\uparrow}(A|E|P_{AE}) \). In contrast, from Fig. 5, we observe that the rates at which the normalized remaining uncertainties transition from positive quantities to zero are the same and are equal to the conditional Shannon entropy \( H(A|E|P_{AE}) \approx 0.44 \) nats.

**D. Optimal Rates for Vanishing Remaining Uncertainties**

The tightness of the bounds in Theorem 2 is partially addressed in the following theorem where we are concerned with the minimum compression rates \( \bar{R} \) such that the various normalized remaining uncertainties tend to zero.

To state the next result succinctly, we require a few additional definitions. Let \( P_A \in \mathcal{P}(\mathcal{A}) \) be a given distribution. Let \( \gamma(t) := t H_{1+t}(A|P_A) = -\log \sum_a P_A(a)^{1+t} \) and let \( P_A(t)(a) := P_A(a)^{1+t} e^{\gamma(t)} \) be a tilted distribution\(^5\) relative to \( P_A \). Define

\[
s_0(A|P_A) := \max \{ s \in [0, 1] : H_{1-s}(A|P_A) \leq H(A|P_A^{(s-1)}) \}. \quad \tag{54}
\]

We claim that \( s_0(A|P_A) \) is always positive; this is because \( s \mapsto H_{1-s}(A|P_A) \) and \( s \mapsto H(A|P_A^{(s-1)}) \) are continuous and

\[
H_{1-s}(A|P_A) = \begin{cases} H(A|P_A) & s = 0 \\ \log |A| & s = 1 \end{cases}, \quad \text{and} \quad \tag{55}
\]

\[
H(A|P_A^{(s-1)}) = \begin{cases} \log |A| & s = 0 \\ H(A|P_A) & s = 1 \end{cases}. \tag{56}
\]

If \( A \sim P_A \) is not uniform on \( \mathcal{A} \), \( s_0(A|P_A) \in (0, 1) \). In fact since \( s \mapsto H_{1-s}(A|P_A) \) and \( s \mapsto H(A|P_A^{(s-1)}) \) are monotonically increasing and decreasing\(^6\) respectively, \( s_0(A|P_A) \in (0, 1) \) can also be expressed as the unique solution to the equation

\(^4\)When we write \( M_n = e^{nR} \), we mean that \( M_n \) is the integer \( \lfloor e^{nR} \rfloor \).

\(^5\) \( P_A(t) \) is indeed a valid distribution as \( \sum_a P_A(t)(a) = 1 \).

\(^6\)Intuitively, \( H(A|P_A^{(s-1)}) \) is monotonically decreasing because as \( s \) increases, \( P_A^{(s-1)} \) converges to a deterministic distribution, which has the lowest Shannon entropy 0.
Clearly, by the preceding arguments and the fact that $\mathcal{E}$ is a finite set, $s_0$ is positive.

**Theorem 3** (Optimal Rates for Vanishing Normalized Remaining Uncertainties). For each $n \in \mathbb{N}$, let the size of the range of
Define the best-case limiting normalized remaining uncertainties over all random hash functions as
\[ G(R, s) := \lim_{n \to \infty} \frac{1}{n} \inf_{f_{X_n} \in \mathcal{R}} H_{1+s}(A^n | f_{X_n}(A^n), E^n, X_n | P^n_{AE} \times P_{X_n}), \]  
and
\[ \tilde{G}(R, s) := \lim_{n \to \infty} \frac{1}{n} \inf_{f_{X_n} \in \mathcal{R}} H_{1+s}^{\dagger}(A^n | f_{X_n}(A^n), E^n, X_n | P^n_{AE} \times P_{X_n}). \]

Also define the limiting normalized remaining uncertainty for strongly universal hash functions \( \tilde{f}_{X_n} : A^n \to \{1, \ldots, M_n\} \) as
\[ \tilde{G}(R, s) := \lim_{n \to \infty} \frac{1}{n} H_{1+s}(A^n | \tilde{f}_{X_n}(A^n), E^n, X_n | P^n_{AE} \times P_{X_n}). \]

Now define the optimal compression rates
\[ T_s := \inf \{ R \in \mathbb{R} : G(R, s) = 0 \}, \]
\[ \tilde{T}_s := \inf \{ R \in \mathbb{R} : \tilde{G}(R, s) = 0 \}, \]
\[ T_s^\dagger := \inf \{ R \in \mathbb{R} : G^\dagger(R, s) = 0 \}, \]
\[ \tilde{T}_s^\dagger := \inf \{ R \in \mathbb{R} : \tilde{G}^\dagger(R, s) = 0 \}, \]  
and
\[ T_s := \inf \{ R \in \mathbb{R} : \tilde{G}(R, s) = 0 \}. \]

1) For \( s \in [0, 1] \), we have
\[ T_{-s} \leq H_{1-s}(A|E|P_{AE}). \]

and for \( s \in [0, s_0] \), we have
\[ T_{-s} \geq H_{1-s}(A|E|P_{AE}). \]

2) For \( s \in (0, 1) \), we have
\[ T_s = \tilde{T}_s = H(A|E|P_{AE}). \]

3) For \( s \in [0, 1/2] \), we have
\[ T_s^\dagger = \tilde{T}_s^\dagger = H_{1-s}(A|E|P_{AE}), \]

and for \( s \in [0, \infty) \), we have
\[ T_s^\dagger = \tilde{T}_s = H(A|E|P_{AE}). \]

The proof of this result is provided in Section VI.

For Part (1) of the above result, unfortunately, we do not have a matching lower bound to \( T_{-s} \). However, for \( s \in [0, s_0] \), the bound in (67) says that restricted to the important class of strongly universal hash functions (e.g., the ubiquitous random binning procedure [32]), the result in (66) is tight as there is a matching lower bound. Hence, (67) serves as a “partial converse” to (66). In other words, (66) is tight with respect to the ensemble average [34] when the ensemble is chosen to be a strongly universal hash function.

The equalities in (68)–(70) imply that in the specified ranges of \( s \), the optimal rates for the best-case remaining uncertainty over all hash functions and worst-case remaining uncertainty over all universal2 hash functions are the same. It is interesting to observe that the optimal rate for the \( -s \) case in (69) depends on \( s \in [0, s_0] \) but the optimal rates for the \( +s \) cases in (68) and (70) do not. This is also clearly observed in Figs. 4 and 5.

The proofs of the achievability parts (upper bounds) of these results follow directly from Theorem 2. For the converse parts (lower bounds), we appeal to the method of types [16, Ch. 2], the moments of type class enumerator method [21–25], and the exponential strong converse for Slepian-Wolf coding [12, Theorem 2]. We also exploit a result by Fehr and Berens [10, Theorem 3] concerning the monotonicity (16) and chain rule (17) for Gallager form of the conditional Rényi entropy \( H_{1-s}(A|E|P_{AE}) \).

E. Exponential Rates of Decrease of Remaining Uncertainties

Lastly, we consider the rate of exponential decrease of the various worst-case remaining uncertainties.

**Theorem 4** (Exponents of Remaining Uncertainties). For each \( n \in \mathbb{N} \), let the size of the range of \( f_{X_n} \) be \( M_n = e^{nR} \). Fix a joint distribution \( P_{AE} \in \mathcal{P}(A \times E) \). Define the exponents of (48) and (49) as
\[ E(R, s) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{f_{X_n} \in \mathcal{T}_2} H_{1+s}(A^n | f_{X_n}(A^n), E^n, X_n | P^n_{AE} \times P_{X_n}), \]
and
\[ E^\dagger(R, s) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{f_{X_n} \in \mathcal{T}_2} H_{1+s}^\dagger(A^n | f_{X_n}(A^n), E^n, X_n | P^n_{AE} \times P_{X_n}). \]
These are the exponents of the worst-case remaining uncertainties over all universal₂ hash functions. The following achievability statements hold:

1) For $s \in [0, 1]$, we have

$$E(R, -s) = \sup_{t \in (s, 1)} t \left( R - H_{1-t}(A|E|P_{AE}) \right)^+,$$

and for any $s \in [0, 1/2]$, we have

$$E^\dagger(R, -s) = \sup_{t \in (s, 1/2)} \frac{t}{1-t} \left( R - H_{1-t}(A|E|P_{AE}) \right)^+.$$

2) For $s \in [0, \infty)$, we have

$$E(R, s) = \sup_{t \in (0, 1/2)} \frac{t}{1-t} \left( R - H_{1-t}(A|E|P_{AE}) \right),$$

and

$$E^\dagger(R, s) = \sup_{t \in (0, 1/2)} \frac{t}{1-t} \left( R - H_{1-t}(A|E|P_{AE}) \right).$$

Theorem 4 is proved in Section VII.

We observe from Proposition 1 that the right-hand-sides of the bounds in Part (2), which can be shown to be non-negative for $R \geq H(A|E|P_{AE})$, are lower bounds on the optimal error exponent [12], [14], [15] for the Slepian-Wolf [3] problem, denoted as $E_{SW}^*(R)$. In fact, it can be inferred from Gallager’s work [14] (or [16, Problem 2.15(a)] for the $E = \emptyset$ case) that if we replace the domain of the optimization over $t$ from $(0, 1/2)$ to $(0, 1)$, the lower bounds in (75)–(76) are equal to $E_{SW}^*(R)$ for a certain range of coding rates above $H(A|E|P_{AE})$. The reason why we obtain a potentially smaller exponent is because we consider the worst-case over all universal₂ hash functions $f_{X_n} \in \mathcal{U}_2$ in the definitions of $E(R, s)$ and $E^\dagger(R, s)$ in (71) and (72) respectively. For the Slepian-Wolf problem, we can choose the best sequence of hash functions.

In Fig. 7, we plot the lower bounds in (73) and (74) for the same source $P_{AE}$ as in Figs. 4 and 5 in Section IV-C. We note that the rates at which the lower bounds on the exponents transition from being zero to positive is given by $H_{1-s}(A|E|P_{AE})$ for (73) and $H^\dagger_{1-s}(A|E|P_{AE})$ for (74). The latter observation corroborates (69) of Theorem 3. Furthermore, as discussed in the previous paragraph, the $s = 0$ case in the right plot of Fig. 7 is a lower bound on $E_{SW}^*(R)$. For this source, if we change the domain of optimization of $t$ from $(0, 1/2)$ to $(0, 1)$, the plot does not change (i.e., the optimal $t < 1/2$) so for rates in a small neighborhood above $H(A|E|P_{AE}) \approx 0.44$ nats, the curve indeed traces out the optimal error exponent $E_{SW}^*(R)$.

V. PROOF OF THEOREM 2

We prove statements (50), (51), (52), and (53) in Subsections V-A, V-B, V-C, and V-D respectively.
A. Proof of (50) in Theorem 2

To prove the upper bound in (50), we use the one-shot bound in (176) in Lemma 3 (Appendix A). We first assume that $H_{1-s}(A|E|P_{AE}) - R > 0$. In this case,

$$M^{-s} e^{sH_{1-s}(A^n|E^n|P_{AE}^n)} \geq 1,$$

(77)

for $n$ sufficiently large. Then the one-shot bound in (176) implies that for any $\epsilon$-almost universal hash function $f_{X_n}$,

$$H_{1-s}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n})$$

$$= \frac{1}{s} \log \mathbb{E}_{X_n} \left( e^{sH_{1-s}(A^n|f_{X_n}(A^n), E^n|P_{AE}^n)} \right)$$

$$\leq \frac{1}{s} \log \left( 1 + e^{sH_{1-s}(A^n|E^n|P_{AE}^n)} \right)$$

$$\leq \frac{1}{s} \log \left( 2 \cdot e^{sH_{1-s}(A^n|E^n|P_{AE}^n)} \right)$$

$$= \frac{1}{s} \log (2e^s) + n \left( H_{1-s}(A|E|P_{AE}) - R \right),$$

(81)

where in (80) we used (77) and in (81) we used the fact that the conditional Rényi entropy is additive for independent random variables, i.e., $H_{1-s}(A^n|E^n|P_{AE}^n) = nH_{1-s}(A|E|P_{AE})$. Since this bound holds for all $\epsilon$-universal hash functions $f_{X_n} \in \mathcal{U}_2$ (including $\epsilon = 1$), normalizing by $n$, taking the $\lim_{n \to \infty}$, and appealing to the definition $G(R, -s)$ in (48) establishes that $G(R, -s) \leq H_{1-s}(A|E|P_{AE}) - R$ if $H_{1-s}(A|E|P_{AE}) - R \geq 0$.

Now, when $H_{1-s}(A|E|P_{AE}) - R \leq 0$, we follow the steps leading to (79) but use $\log(1 + t) \leq t$ to establish that

$$H_{1-s}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n}) \leq \frac{1}{s} e^{s(H_{1-s}(A|E|P_{AE}) - R)}.$$

(82)

From (82), we conclude that if $H_{1-s}(A|E|P_{AE}) - R \leq 0$, we have $G(R, -s) = 0$ (because $G(R, -s)$ cannot be negative). Since the two bounds in (81) and (82) hold for all sequences of universal hash functions $f_{X_n} \in \mathcal{U}_2$ (taking $\epsilon = 1$ above), together they establish (50).

B. Proof of (51) in Theorem 2

To prove the upper bound in (51), we use the one-shot bound in (177) in Lemma 3 (Appendix A). Similarly, to the analysis in Section V-A, we may consider two cases $H_{1-s}^+(A|E|P_{AE}) - R > 0$ or $H_{1-s}^+(A|E|P_{AE}) - R \leq 0$. We will only consider the former since the analysis of the latter parallels that in Section V-A. Under the former condition, we may assume that

$$M^{-\frac{1}{s}} e^{\frac{1}{s}R} e^{\frac{1}{s}H_{1-s}^+(A^n|E^n|P_{AE}^n)} \geq 1,$$

(83)

for $n$ sufficiently large. The one-shot bound in (177) implies that for any $\epsilon$-almost universal hash function $f_{X_n}$,

$$H_{1-s}^+(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n})$$

$$= \frac{1}{s} \log \mathbb{E}_{X_n} \left( e^{\frac{1}{s}R} e^{\frac{1}{s}H_{1-s}^+(A^n|f_{X_n}(A^n), E^n|P_{AE}^n)} \right)$$

$$\leq \frac{1}{s} \log \left( 1 + e^{\frac{1}{s}R} e^{\frac{1}{s}H_{1-s}^+(A^n|E^n|P_{AE}^n)} \right)$$

$$\leq \frac{1}{s} \log \left( 2 \cdot e^{\frac{1}{s}R} e^{\frac{1}{s}H_{1-s}^+(A^n|E^n|P_{AE}^n)} \right)$$

$$= \frac{1}{s} \log \left( 2e^{\frac{1}{s}R} \right) + n \left( H_{1-s}^+(A|E|P_{AE}) - R \right),$$

(87)

where in (86) we used (83). Since this bound holds for all sequences of universal hash functions $f_{X_n} \in \mathcal{U}_2$ (taking $\epsilon = 1$ above), normalizing by $n$ and appealing to the definition $G^+(R, -s)$ in (49), we establish the upper bound in (51).

C. Proof of (52) in Theorem 2

To prove the upper bound in (52), we will resort to the one-shot bound in (156) in Lemma 1 (Appendix A). We first observe by Cramér’s theorem [33, Section 2.2] that

$$\lim_{n \to \infty} -\frac{1}{n} \log P_{AE}^n \left\{ (a, e) : P_{A|E}^n(a|e) \geq e^{-nR} \right\} = \sup_{t \geq 0} \left\{ t(\mathbb{H}_{1+t}(A|E|P_{AE}) - R) \right\}.$$
By differentiating the objective function in (95), we see that if
\[ \epsilon \]
Since (88) is not smaller than (90), the latter dominates. Now using the one-shot bound in (156) in Lemma 1 we see that for
\[ t \]
rate in (21)), the optimal solution is attained at
\[ t \]
This is because the cumulant generating function of the random variable \( \log P_{A|E}(A|E) \) where \( (A, E) \) is distributed as \( P_{AE} \) is
\[
\log E \left[ e^{t \log P_{A|E}(A|E)} \right] = \log \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1+t} = -tH_{1+t}(A|E|P_{AE}).
\]
Next, we apply a generalization of Cramér’s theorem concerning arbitrary finite non-negative measures\(^7\) (not necessarily probability measures) to the sequence of random variables \( -\log P_{A|E}(A^n|E^n) = \sum_{i=1}^n -\log P_{A|E}(A_i|E_i) \) under the sequence of non-negative finite joint measures \( B \mapsto \sum_{(a,e)} B P_{A^nE^n}(a,e)P_{A|E}^n(a|e)^s \) to establish that
\[
\lim_{n \to \infty} -\frac{1}{n} \log \sum_{(a,e): P^n_{A|E}(a|e) < \epsilon e^{-nR}} P^n_{A|E}(a,e)P^n_{A|E}(a|e)^s e^{nR}
\]
\[
= \left\{ \begin{array}{ll}
    s(H_{1+s}(A|E|P_{AE}) - R) & R \leq \hat{R}_s \\
    \max_{t \in [0,s]} t(H_{1+t}(A|E|P_{AE}) - R) & R \geq \hat{R}_s
\end{array} \right.
\]
(90)
The statement in (90) holds because the relevant cumulant generating function is
\[
\tau_s(t) = \log \sum_{a,e} P_{A|E}(a,e)P_{A|E}(a|e)^s e^{-t \log P_{A|E}(a|e)}
\]
from the definition of the conditional Rényi entropy in (10). Hence,
\[
\Gamma_s := \lim_{n \to \infty} -\frac{1}{n} \log \sum_{(a,e): P^n_{A|E}(a|e) < \epsilon e^{-nR}} P^n_{A|E}(a,e)P^n_{A|E}(a|e)^s
\]
\[
= \sup_{t \geq 0} \{ tR - \tau_s(t) \}
\]
\[
= \sup_{t \geq 0} \{ tR + (s-t)H_{1+(s-t)}(A|E|P_{AE}) \}.
\]
(93)
(94)
(95)
By differentiating the objective function in (95), we see that if \( R \leq \hat{R}_s = \frac{d}{dt} H_{1+t}|_{t=s} \) (cf. the definition of the critical rate in (21)), the optimal solution is attained at \( t^* = 0 \) (recall that \( t \mapsto tH_{1+t} \) is concave so \( s \mapsto \hat{R}_s \) is decreasing) and so \( \Gamma_s = sH_{1+s}(A|E|P_{AE}) \), leading to the first clause in (90). Conversely, when \( R > \hat{R}_s \), the optimal solution is attained at \( t^* > 0 \). This leads to the second clause on the right-hand-side of (90) because the left-hand-side of (90) is now
\[
\Gamma_s - sR = \sup_{t \geq 0} \{ (t-s)R + (s-t)H_{1+(s-t)}(A|E|P_{AE}) \}
\]
\[
= \max_{t \in [0,s]} \{ tH_{1+t}(A|E|P_{AE}) - tR \}.
\]
(96)
(97)
Since (88) is not smaller than (90), the latter dominates. Now using the one-shot bound in (156) in Lemma 1 we see that for any sequence of \( \epsilon \)-almost universal\(_2 \) hash functions \( f_{X_n} \),
\[
\lim_{n \to \infty} -\frac{1}{n} H_{1+s}(A^n|f_{X_n}(A^n), E^n, X_n|P_{A|E} \times P_{X_n})
\]
\[
= \lim_{n \to \infty} -\frac{1}{n} \log E_{X_n} \left[ e^{-sH_{1+s}(A^n|f_{X_n}(A^n), E^n|P_{AE})} \right]
\]
\[
\leq \lim_{n \to \infty} -\frac{1}{n} \log \left[ 2^{-s} \sum_{(a,e): P^n_{A|E}(a|e) \geq \epsilon e^{-nR}} P^n_{A|E}(a,e) \right.
\]
\[
\left. + 2^{-s} \sum_{(a,e): P^n_{A|E}(a|e) < \epsilon e^{-nR}} P^n_{A|E}(a,e)P^n_{A|E}(a|e)^s e^{nR} \right]
\]
\[
= \left\{ \begin{array}{ll}
    H_{1+s}(A|E|P_{AE}) - R & R \leq \hat{R}_s \\
    \max_{t \in [0,s]} \frac{1}{s} (H_{1+t}(A|E|P_{AE}) - R) & R \geq \hat{R}_s
\end{array} \right.
\]
(98)
(99)
(100)
Since this bound holds for all sequences of universal\(_2 \) hash functions \( f_{X_n} \in \mathcal{U}_2 \) (taking \( \epsilon = 1 \) above), we have established the upper bound in (52).

\(^7\)The standard Cramér’s theorem [33, Section 2.2] (or Sanov’s theorem [33, Section 2.1]) is a large-deviations result concerning the exponent of \( P^n(B) \) where \( P \) is a probability measure and \( B \) is an event in the sample space \( \Omega \). If \( P \) is not necessarily a probability measure but a finite non-negative measure (as it is in our applications), say \( \mu \), Cramér’s theorem clearly also applies by defining the new probability measure \( B \mapsto \hat{P}(B) := \mu(B)/\mu(\Omega) \).
Thus (106) reduces to

\[\lim_{n \to \infty} -\frac{1}{n} \log \sum_e P^n_E(e) \sum_{\{a: P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \}^{1+s} - \frac{t}{1+s}} \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

\[= \lim_{n \to \infty} -\frac{1}{n} \log \sum_{\{a: P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \}^{1+s} - \frac{t}{1+s}} \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

\[= \max_{t \geq 0} \left( \log \left( P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \right) - \frac{t}{1+s} \right) \]

\[= \max_{t \geq 0} \left( \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \right) \]

Let us justify the claim in (107) carefully. This step follows because the relevant cumulant generating function is

\[\tau_s(t) := \log \sum_e P^n_{A|E}(a|e) \exp \left( t \log \left[ \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \right] \right) \]

\[= \log \sum_e P^n_{A|E}(e) \left( \sum_{a} P^n_{A|E}(a|e) \right) \]

\[= -\frac{t}{1+s} H_{1+s}(A|E|P_{AE}) \]

where the last step results from the definition of the two-parameter conditional Rényi entropy in (20). By an application of Cramér’s theorem, the corresponding exponent is (102).

In addition, we apply the generalized version of Cramér’s theorem (see footnote 7) to compute another large deviations quantity. Consider the sequence of random variables \(- \log P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \) distributed according to the sequence of non-negative finite joint measures \( \sum_{a,e} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \) \( - \frac{t}{1+s} \). We claim that the exponent can be calculated to be

\[\lim_{n \to \infty} -\frac{1}{n} \log \sum_e P^n_{A|E}(e) \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

\[= \left\{ \begin{array}{ll}
\frac{s}{1+s} H_{1+s}(A|E|P_{AE}) - R & R \leq \bar{R}_s \\
\max_{t \in [0,1]} \left( \frac{t}{1+s} H_{1+s}(A|E|P_{AE}) - R \right) & R > \bar{R}_s
\end{array} \right. \]

Let us justify the claim in (107) carefully. This step follows because the relevant cumulant generating function is

\[\tau_s(t) := \log \sum_e P^n_{A|E}(e) \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

\[= \log \sum_e P^n_{A|E}(e) \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

\[= \log \sum_e P^n_{A|E}(e) \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \]

Thus (106) reduces to

\[\max_{t \geq 0} \left\{ \left( -\tau_s(t) - \frac{s - t}{1+s} R \right) \right\} \]

\[= \max_{t \leq s} \left\{ -\log \sum_e P^n_{A|E}(e) \sum_{a} P^n_{A|E}(a|e) \sum_{\hat{a}} P^n_{A|E}(\hat{a}|e) \right\} \]

\[= \max_{t \leq s} \left\{ \frac{t}{1+s} H_{1+s}(A|E|P_{AE}) - \frac{t}{1+s} R \right\} \]
where the last step follows from the definition of the two-parameter conditional Rényi entropy in (20). Now from the definition of the critical rate $\tilde{R}_s$ in (22) and the fact that $H_{1+s}(A^n|X^n) = H^n$ we know that if $R \leq \tilde{R}_s$, the maximization in (111) is attained $t = s$, resulting in the first case in (107). Conversely, the second case results from $R > \tilde{R}_s$ where the domain of $t$ is $[0, s]$ since the eventual exponent cannot be negative. This proves (107).

Since (107) is not greater than (102), the former dominates the exponential behavior of $G^T(R, s)$, and so plugging these evaluations into the one-shot bound in (167) which holds for any $\epsilon$-almost universal $2$ hash function $f_X$, 

$$
\lim_{n \to \infty} \frac{1}{n} H_{1+s}(A^n|X^n)_{X^n}[P_{AE} \times P_{X^n}) \\
= \lim_{n \to \infty} -\frac{1}{n s} \log E_X \left[ e^{-\frac{1}{n s} H_{1+s}(A^n|X^n)_{X^n}[P_{AE} \times P_{X^n})} \right] \\
\leq \lim_{n \to \infty} -\frac{1}{n s} \log \left[ 2^{-\frac{1}{n s} \sum_e P^n_E(e)} \sum_{a \in A} P^n_{AE}(a|e)_{e}^{1+s} \right] \\
\times \left[ \sum_{a \in A} P^n_{AE}(a|e)_{e}^{1+s} \left( \sum_{a \in A} P^n_{AE}(a|e)_{e}^{1+s} \right)^{-1} \right] \\
= \left\{ \begin{array}{ll}
\max_{t \in [0, s]} \frac{1}{t} \left( H_{1+s}(A^n|X^n)_{X^n}[P_{AE} \times P_{X^n}) - R \right) & R \leq \tilde{R}_s \\
\left( H_{1+s}(A^n|X^n)_{X^n}[P_{AE} \times P_{X^n}) - R \right) & R > \tilde{R}_s 
\end{array} \right.
$$

Since this bound holds for all sequences of universal $2$ hash functions $f_X \in \mathcal{U}_2$ (taking $\epsilon = 1$ above), we have established the upper bound in (53).

VI. PROOF OF THEOREM 3

The bounds on the optimal compression rates corresponding to the conditional Rényi entropy and Gallager form of the conditional Rényi entropy are proved in Subsections VI-A and VI-B respectively.

A. Proofs of (66), (67), and (68)

Proof: Recall the definitions of the optimal rates $T_s$, $\bar{T}_s$, and $\tilde{T}_s$ in Theorem 3. Since $\bar{G}(R, s) \leq G(R, s)$, and both functions are monotonically non-increasing in $R$, it holds that $\bar{T}_s \leq T_s$ for all $s$.

First, we prove the upper bounds to $T_{-s}$ and $T_s$ in Section VI-A1; next we prove the lower bound to $\bar{T}_s$ in Section VI-A2; and finally we prove the lower bound to $\tilde{T}_s$ in Section VI-A3. For the $+s$ case, the upper and lower bounds match for all $s \in (0, 1)$ and so this proves (68).

1) Upper Bounds: We refer to the statement in (50). We observe that if $R \geq H_{1-s}(A^n|P_{AE})$, $G(R, -s) = 0$ since $G(R, -s)$ is upper bounded by $|H_{1-s}(A^n|P_{AE}) - R|$ and $G(R, -s)$ is non-negative. Hence, $T_{-s} \leq H_{1-s}(A^n|P_{AE})$ for all $s \in [0, 1]$. This proves (66).

Next we refer to the statement in (52). If $R \geq H(A^n|P_{AE})$, we know from the monotonically decreasing nature of $s \mapsto H_{1+s}(A^n|P_{AE})$ that $H_{1+t}(A^n|P_{AE}) - R$ is non-positive for $t \in [0, s]$. Thus, the optimal $t$ in the optimization in $\max_{t \in [0, s]} \frac{1}{t} \left( H_{1+t}(A^n|P_{AE}) - R \right)$ is $t^* = 0$ and consequently, the optimal objective value is also 0. On the other hand, for $R \in [\bar{R}_s, H(A^n|P_{AE})]$, the optimal $t^* \in [0, s]$ and so the optimal objective value is positive. We conclude for $s \in [0, \infty)$ that the optimal key generation rate is upper bounded by the conditional Shannon entropy $H(A^n|P_{AE})$. In summary, we conclude that $T_s \leq H(A^n|P_{AE})$ for all $s \in [0, \infty)$, proving the upper bound for (68).

2) Lower Bound to $\bar{T}_{-s}$: We now consider strongly universal hash functions [18] (cf. Definition 2) and $s \in [0, s_0]$, where $s_0 = s_0(A|P_{AE})$ is defined in (57). More precisely, we shall show that for the sequence of strongly universal hash functions $(\mathcal{T}_X^n : A^n \rightarrow \{1, \ldots, e^nR\})_{n \in \mathbb{N}}$ and any $s \in [0, s_0]$, we have

$$
\bar{G}(R, -s) = \lim_{n \to \infty} \frac{1}{n} H_{1-s}(A^n|\mathcal{T}_X^n(A^n), E^n, X^n|P_{AE} \times P_{X^n}) \geq |H_{1-s}(A^n|P_{AE}) - R|^+, \tag{115}
$$

immediately implying that $\bar{T}_{-s} \geq H_{1-s}(A^n|P_{AE})$. In fact, for this range of $s$, not only is it true that the minimum value of $R$ such that $\bar{G}(R, -s) = 0$ coincides with that in (66), the bound in (115) serves as a tight lower bound to the achievable (upper) bound for $G(R, -s)$ in (50) (at least for strongly universal hash functions). In the following, we make use heavy use of the method of types; relevant notation is summarized in Section II-B.
For ease of exposition, we first consider the case in which \(|\mathcal{E}| = 1\) or equivalently, \(\mathcal{E} = \emptyset\). Subsequently, we generalize our result to the general case in which \(|\mathcal{E}| > 1\). Starting with the one-shot bound in (180) (in Lemma 3), we have

\[
\mathbb{E}_{X_n} \left[ e^{sH_{1-s}(A^n|\mathcal{E},X_n)} \right] = \mathbb{E}_{X_n} \left[ \sum_{\mathcal{A}} P_\mathcal{A}^n(a)^{1-s} \left( \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right)^s \right] \tag{116}
\]

\[
\geq 2^{s-1} \mathbb{E}_{X_n} \left[ \sum_{\mathcal{A}} P_\mathcal{A}^n(a)^{1-s} \left( P_\mathcal{A}^n(a)^s + \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right)^s \right] \tag{117}
\]

\[
\geq \mathbb{E}_{X_n} \left[ \sum_{\mathcal{A}} P_\mathcal{A}^n(a)^{1-s} \left( P_\mathcal{A}^n(a)^s \right) + \mathbb{E}_{X_n} \left[ \max_{\mathcal{A} \in \mathcal{P}_n(A)} \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \right] \tag{118}
\]

\[
= \sum_{\mathcal{A}} P_\mathcal{A}^n(a)^{1-s} \left( P_\mathcal{A}^n(a)^s + \sum_{\mathcal{Q} \in \mathcal{P}_n(A)} P_\mathcal{A}^n(\mathcal{Q}) \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right) \tag{119}
\]

where in (117) we used the bound \((b+c)^s \geq 2^{s-1}(b^s + c^s)\) for \(b,c \geq 0\) and \(s \in [0,1]\) (a consequence of Jensen’s inequality applied to the concave function \(t \rightarrow t^s\) for \(s \in [0,1]\)), in (118), we split the inner sum into \(\mathcal{A}\)-types on \(\mathcal{A}\), and in (119) and (121) we used the fact that there are polynomially many types so we can interchange sums over types with maximums over types and vice versa. This derivation is similar to [21, Eqn. (20)].

Now, we assume that \(R < H_{1-s}(A|P_\mathcal{A})\) and also that \(s \leq s_0(A|P_\mathcal{A})\). The latter assumption means that \(H_{1-s}(A|P_\mathcal{A}) \leq H(A|P_\mathcal{A}^{(s-1)})\) (see Section IV-D). In this case, we may use the bound in (205) in Lemma 4 (in Appendix B) to lower bound the (inner) sum over expectations in (121). We have

\[
\sum_{\mathcal{Q} \in \mathcal{P}_n(A)} \mathbb{E}_{X_n} \left[ \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \geq \sum_{\mathcal{Q} \in \mathcal{P}_n(A)} \mathbb{E}_{X_n} \left[ \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \tag{122}
\]

\[
= \sum_{\mathcal{Q} \in \mathcal{P}_n(A)} \mathbb{E}_{X_n} \left[ \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \tag{123}
\]

\[
= \sum_{\mathcal{Q} \in \mathcal{P}_n(A)} \mathbb{E}_{X_n} \left[ \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \tag{124}
\]

\[
= \sum_{\mathcal{Q} \in \mathcal{P}_n(A)} \mathbb{E}_{X_n} \left[ \sum_{\hat{a} \in \mathcal{T}_{X_n}^s(\mathcal{A})} P_\mathcal{A}^n(\hat{a}) \right]^s \tag{125}
\]

where (122) uses (205) in Lemma 4, and (123) uses the inequality \(\sum_i b_i^s \geq (\sum_i b_i)^s\) which holds for non-negative \(b_i\) and \(s \in [0,1]\) [13, Problem 4.15(f)] and (126) uses the fact that \(\mathcal{T}_{X_n}\) is a strongly universal hash function (cf. Definition 2).
Substituting (126) into (121), we obtain
\[
\mathbb{E}_{X_n} \left[ e^{sH_{1-s}(A^n|T_{X_n}(A^n),X_n|P^n_A)} \right] 
\geq \sum_a P^n_A(a)^{1-s} \left( P^n_A(a)^s + \left\{ e^{-nsR} \sum_{\tilde{a} \neq a} P^n_A(\tilde{a}) \right\}^s \right) 
= 1 + e^{-nsR} \sum_a P^n_A(a)^{1-s} \left\{ 1 - P^n_A(a) \right\}^s 
\geq 1 + \exp \left( n s \left| H_{1-s}(A|P_A) - R \right| \right) 
\geq \exp \left( n s \left( |H_{1-s}(A|P_A)| - R^+ \right) \right),
\]
(127)
where in (129) we used the fact that \( \max_a P^n_A(a) \leq \frac{1}{2} \) (say) for \( n \) large enough so \( 1 \geq \left\{ 1 - P^n_A(a) \right\}^s \geq 1 - \left( \frac{1}{2} \right)^s > 0 \), and in (130), we used the condition \( H_{1-s}(A|P_A) > R \) so the expression in (129) is exponentially large.

In the other case, when \( R \geq H_{1-s}(A|P_A) \), we simply lower bound the sum over types term in (121) by 0 and hence, the entire expression in (121) can be lower bounded by 1. Thus, we conclude that
\[
\mathbb{E}_{X_n} \left[ e^{sH_{1-s}(A^n|T_{X_n}(A^n),X_n|P^n_A)} \right] \geq \exp \left( n s \left| H_{1-s}(A|P_A) - R^+ \right| \right).
\]
(131)

For the case \( |E| = 1 \), this establishes (115) for \( s \in [0, s_0(A|P_A)] \).

Now we extend our analysis to \( |E| > 1 \). Naturally, we operate on a type-by-type basis over \( E^n \). Analogously to the derivation of (131) via Lemma 4, we see that if \( 0 \leq s \leq s_0 = \min_e s_0(A|P_A|E=e) \), we have
\[
\mathbb{E}_{X_n} \left[ e^{sH_{1-s}(A^n|T_{X_n}(A^n),E^n,X_n|P^n_{A,E})} \right] 
\geq \sum_{Q_{CE} \in P^n_{A,E}} P^n_{E}(Q_{CE}) \exp \left( n s \left| \sum_{e \in E} Q_{CE}(e)H_{1-s}(A|P_A|E=e) - R^+ \right| \right).
\]
(132)

See Remark 2 in Appendix B for a detailed description of this step. In fact, his derivation is similar to the corresponding calculations in Merhav’s work in [21, Section IV-C] and [22, Section IV-D]. Because \( P^n_{E}(Q_{CE}) = e^{-nD(Q_{CE}||P_E)} \),
\[
\mathbb{E}_{X_n} \left[ e^{sH_{1-s}(A^n|T_{X_n}(A^n),E^n,X_n|P^n_{A,E})} \right] 
\geq \exp \left( n \max_{Q_{CE} \in P^n_E} \left\{ -D(Q_{CE}||P_E) + s \left| \sum_{e \in E} Q_{CE}(e)H_{1-s}(A|P_A|E=e) - R^+ \right| \right\} \right).
\]
(133)

Denote the optimizer in the maximization in (133) as \( Q_{CE}^* \). If the \( | \cdot |^+ \) is inactive for \( Q_{CE}^* \), by straightforward calculus,
\[
Q_{CE}^*(e) = \frac{P_E(e)e^{sH_{1-s}(A|P_A|E=e)}}{\sum_{e' \in E} P_E(e')e^{sH_{1-s}(A|P_A|E=e')}} \quad \forall e \in E,
\]
(134)
while if the \( | \cdot |^+ \) is active for \( Q_{CE}^* \), obviously \( Q_{CE}^*(e) = P_E(e) \) for all \( e \in E \). By using these forms of the optimizer \( Q_{CE}^* \) and the fact that the conditional Rényi entropy \( H_{1-s}(A|P_A|E=e) \) (defined in (10)) is related to the distribution \( P_E \) and the unconditional Rényi entropies \( \{ H_{1-s}(A|P_A|E=e) : e \in E \} \) as follows
\[
sH_{1-s}(A|E|P_{AE}) = \log \sum_{e \in E} P_E(e)e^{sH_{1-s}(A|P_A|E=e)},
\]
(135)
we see that the maximization in (133) reduces to \( s|H_{1-s}(A|E|P_{AE}) - R|^+ \). Upon taking the log, dividing by \( ns \), and taking the limit, we complete the proof of (115) for the case where \( |E| > 1 \), assuming \( s \in [0, s_0] \). As such, we have completed the proof of the lower bound on the optimal compression rate for strongly universal hash functions in (67).

3) Lower Bound to \( T_s \): For the lower bound to \( T_s \), we note from the work by Hayashi in [35, Lemma 5] that for any \( s \in (-1, 1) \setminus \{0\} \) that the conditional Rényi entropy and its Gallager form satisfy
\[
H_{1+s}(A|E|P_{AE}) \geq H^\dagger_{1+s}(A|E|P_{AE}).
\]
(136)
Furthermore, because \( s \mapsto H^\dagger_{1+s} \) is monotonically non-increasing,
\[
\frac{1}{n}H_{1+s}(A^n|f_n(A^n),E^n|P^n_{AE}) 
\geq \frac{1}{n}H^\dagger_{1+s}(A^n|f_n(A^n),E^n|P^n_{AE}) 
= \frac{1}{n}H^\dagger_{1+s}(A^n|f_n(A^n),E^n|P^n_{AE}).
\]
(137)
We require that the term on the leftmost of this inequality to vanish to since the constraint $\tilde{G}(R, s) = 0$ is present. This implies that the normalized Gallager min-entropy $\frac{1}{n}H_{\infty}^{\uparrow}$ necessarily vanishes. From the relation between the strong converse exponent and $H_{\infty}^{\uparrow}$ in (25), we see that $-\frac{1}{n} \log P^{(n)}_c(f_n) \to 0$, where $P^{(n)}_c(f_n)$ is the probability of correct optimal (MAP) decoding under encoder $f_n$. By the exponential strong converse for Slepian-Wolf coding by Oohama and Han [12, Theorem 2], we know that if $R < H(A|E|P_{AE})$, it is also necessarily true that $\lim_{n \to \infty} -\frac{1}{n} \log P^{(n)}_c(f_n) > 0$. Hence, by contraposition, $R \geq H(A|E|P_{AE})$. Thus, $T_s \geq H(A|E|P_{AE})$. This, together with the corresponding upper bound proved in Section VI-A1, immediately establishes the lower bound to (68) for all $s \in (0, 1)$.

\section*{B. Proofs of (69) and (70)}

\textbf{Proof:} To prove (69) and (70), first recall the definitions of the optimal rates $T^{\uparrow}$ and $\bar{T}^{\uparrow}$ in Theorem 3. Since $\bar{G}^{\uparrow}(R, s) \leq G^{\uparrow}(R, s)$, and both functions are monotonically non-increasing in $R$, it holds that $T^{\uparrow}_s \leq \bar{T}^{\uparrow}_s$ for all $s$.

1) \textbf{Upper Bounds:} The upper bounds for $T^{\uparrow}_s$ and $\bar{T}^{\uparrow}_s$ can be shown in the same way as the arguments to upper bound $T^{-}_s$ and $T_s$ in Section VI-A1 and using the results in (51) and (53). Details are omitted for brevity.

2) \textbf{Lower Bounds:} For the lower bound to $\bar{T}^{\uparrow}_s$, we use an important result by Fehr and Berens [10, Theorem 3], which in our context states that for any hash function $f_n : A \to \{1, \ldots, M\}$, we have

\begin{equation}
H^{\uparrow}_{1-s}(A|f_n(A), E|P_{AE}) \geq H^{\uparrow}_{1-s}(A|E|P_{AE}) - \log M
\end{equation}

(139)

for all $s \in (-\infty, 1)$. Note that $s \in (-\infty, 1)$ includes the range of interest for $\bar{T}^{\uparrow}_s$ which is $s \in [0, 1/2]$. The inequality in (139) is a consequence of monotonicity under conditioning and the chain rule for the Gallager form of the conditional Rényi entropy. See (16) and (17). Since $M = e^{e^{\text{R}}} \text{ and } H^{\uparrow}_{1-s}(A^n|E^n|P_{AE}) = nH^{\uparrow}_{1-s}(A|E|P_{AE})$, this implies that for $s \in [0, 1/2]$, we have $\bar{G}^{\uparrow}(R, s) \geq |H^{\uparrow}_{1-s}(A|E|P_{AE}) - R|^{+}$ which immediately leads to the bound $\bar{T}^{\uparrow}_s \geq H^{\uparrow}_{1-s}(A|E|P_{AE})$, establishing (69).

Now, for the lower bound of $T^{\uparrow}_s$ in (70), note from the monotonically decreasing nature of $s \mapsto H^{\uparrow}_{1+s}$ that

\begin{equation}
\frac{1}{n} H_{1+s}^{\uparrow}(A^n|f_n(A^n), E^n|P_{AE}) \geq \frac{1}{n} H_{1-s}^{\uparrow}(A^n|f_n(A^n), E^n|P_{AE})
\end{equation}

(140)

for all $s \in [0, \infty)$. We require that the term on the left to vanish since the constraint $\bar{G}^{\uparrow}(R, s) = 0$ is present. Hence, $\frac{1}{n} H_{1-s}^{\uparrow}$ also vanishes. Similarly to the argument in Section VI-A3 after (138), by invoking the exponential strong converse to the Slepian-Wolf theorem [12, Theorem 2], we know that $\bar{T}^{\uparrow}_s \geq H(A|E|P_{AE})$. This establishes (70).

\textbf{Remark 1.} We remark that the proof of the lower bound to $\bar{T}^{\uparrow}_s$ above is much simpler than the proof of the lower bound of $T^{-}_s$ for strongly universal hash functions in Section VI-A2 because we can leverage two useful properties of $H^{\uparrow}_{1-s}$ (monotonicity and chain rule) leading to (140). In contrast, $H_{1-s}$ does not possess these properties. Observe that the proof in the first half of Section VI-B2 also allows us to conclude that the upper bound in (51) for the direct part is coincident with the lower bound on $\bar{G}^{\uparrow}(R, -s)$ for all $s \in [0, 1/2]$, i.e.,

\begin{equation}
\bar{G}^{\uparrow}(R, -s) = \tilde{G}^{\uparrow}(R, -s) = |H^{\uparrow}_{1-s}(A|E|P_{AE}) - R|^{+}.
\end{equation}

(141)

\section*{VII. PROOF OF THEOREM 4}

We prove statements (73) and (74) in Subsections VII-A and VII-B respectively. Statements (75) and (76) are jointly proved in Subsection VII-C.

\subsection*{A. Proof of (73) in Theorem 4}

First we note that all the exponents are non-negative since $H_{1-s}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE} | X_n) = O(n)$ and similarly for all the other Rényi information quantities. This gives the $| \cdot |^{+}$ signs in all lower bounds in (73)–(76).

Fix $t \in [s, 1]$. The one-shot bound in (176) in Lemma 3 implies that for any $\epsilon$-almost universal 2 hash function $f_{X_n}$,

\begin{align*}
- \log & \mathop{\mathbb{E}}_{X_n} [H_{1-s}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE} | X_n)] \\
\geq & - \log \mathop{\mathbb{E}}_{X_n} [H_{1-t}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE} | X_n)] \\
= & - \log \left\{ \frac{1}{t} \mathop{\mathbb{E}}_{X_n} \left[ \log \left( e^{tH_{1-t}(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE} | X_n)} \right) \right] \right\} \\
\geq & - \log \left\{ \frac{1}{t} \log \left[ 1 + e^{tH_{1-t}(A^n|E^n|P_{AE} | X_n)} \right] \right\} \\
\geq & - \log \left\{ \frac{1}{t} \cdot \frac{e^{tH_{1-s}(A|E|P_{AE})}}{M^t} \right\} \\
= & - \log \left\{ \frac{1}{t} e^{t} + nt(R - H_{1-t}(A|E|P_{AE})) \right\},
\end{align*}

(142)–(147)
where in (142) we used the fact that \( t \mapsto H_{1-t} \) is monotonically non-decreasing. In (144) we applied Jensen’s inequality to the concave function \( t \mapsto \log t \), and in (146) we employed the bound \( \log(1 + t) \leq t \). The bound in (147) holds for all \( f_{X_n} \in \mathcal{U}_2 \) and all \( t \in [s, 1] \). Now, we normalize by \( n \) and take the \( \lim \) as \( n \to \infty \). Finally, we maximize over all \( t \in [s, 1/2] \). This yields (73), concluding the proof.

B. Proof of (74) in Theorem 4

Fix \( t \in [s, 1/2] \). The one-shot bound in (177) in Lemma 3 implies that for any \( \epsilon \)-almost universal \( \epsilon \) hash function \( f_{X_n} \),

\[
- \log \mathbb{E}_{X_n} \left[ H_{1-t}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n) \right] \\
\geq - \log \mathbb{E}_{X_n} \left[ H_{1-t}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n) \right] \\
= - \log \left\{ \frac{1 - t}{t} \mathbb{E}_{X_n} \left[ \log \left( e^{t \frac{1}{t} H_{1-t}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n)} \right) \right] \right\} \\
\geq - \log \left\{ \frac{1 - t}{t} \log \left( 1 + e^{t \frac{1}{t} H_{1-t}^i(A^n|E^n|P_{AE}^n)} \right) \right\} \\
\geq - \log \left\{ \frac{1 - t}{t} e^{t \frac{1}{t} H_{1-t}^i(A|E|P_{AE})} \right\} \\
= - \log \left\{ \frac{1 - t}{t} \epsilon \frac{t}{1 - t} \right\} + n \frac{t}{1 - t} \left( R - H_{1-t}^i(A|E|P_{AE}) \right) \\
\]  

(148)

In (148), we used the fact that for \( t \mapsto H_{1-t}^i \) is monotonically non-decreasing. This bound holds for all \( f_{X_n} \in \mathcal{U}_2 \) and all \( t \in [s, 1/2] \). Now, we normalize by \( n \) and take the \( \lim \) as \( n \to \infty \). Finally, we maximize over all \( t \in [s, 1/2] \). This yields (74), concluding the proof.

C. Proofs of (75) and (76) in Theorem 4

We prove (76) before proving (75). We note that

\[
\lim_{n \to \infty} - \frac{1}{n} \log H_{1+s}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n}) \\
\geq \lim_{n \to \infty} - \frac{1}{n} \log H_{1-s'}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n}) \\
\]  

(154)

for all \( s \in [0, \infty) \) and \( s' \in [0, 1] \). This is because \( \alpha \mapsto H_{1-\alpha} \) is monotonically non-increasing. This bound implies that \( E^\gamma(R, s) \geq E^\gamma(R, -s') \). Combining this with the lower bound on the exponent of \( H_{1-s'}^i \) in (74) and noting that the lower bound is maximized at \( s' = 0 \) immediately establishes (76).

Finally, we note from (18) that \( H_{1+s} \leq H_{1+s}^i \). So

\[
\lim_{n \to \infty} - \frac{1}{n} \log H_{1+s}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n}) \\
\geq \lim_{n \to \infty} - \frac{1}{n} \log H_{1+s}^i(A^n|f_{X_n}(A^n), E^n, X_n|P_{AE}^n \times P_{X_n}). \\
\]  

(155)

Combining this with the lower bound on \( E^\gamma(R, s) \) in (76) completes the proof of (75).

VIII. CONCLUSION AND FUTURE WORK

In this paper, we have developed novel techniques to bound the asymptotic behaviors of remaining uncertainties measured according to various conditional Rényi entropies. This is in contrast to other works [1], [2], [4]–[7] that quantify uncertainty using Shannon information measures. We motivated our study by showing that the quantities we characterize are generalizations of the error exponent and the strong converse exponent for the Slepian-Wolf problem. We studied various important classes of hash functions, including universal \( 2 \) and strongly universal hash functions. Finally, we also showed that in many cases, the optimal compression rates to ensure that the normalized remaining uncertainties vanish can be characterized exactly, and that they exhibit behaviors that are somewhat different to when Shannon information measures are used.

In the future, we hope to derive lower bounds to the normalized remaining uncertainties and upper bounds on their exponents that match or approximately match their achievability counterparts in Theorems 2 and 4. In addition, just as in the authors’ earlier work in [19], we may also study the second-order or \( \sqrt{n} \) behavior [36] of the remaining uncertainties. These challenging endeavors require the development of new one-shot bounds as well as the application of new large-deviation and central-limit-type bounds on various probabilities.
APPENDIX A
ONE-SHOT DIRECT PART BOUNDS

In this appendix, we state and prove several one-shot bounds on the various conditional Rényi entropies. Lemmas 1 and 2 are used in the proofs for the remaining uncertainties (Theorem 2). Lemma 3 is used in the proofs for the exponents (Theorem 4).

**Lemma 1.** For $\epsilon$-almost universal$_2$ hash functions $f_X : A \rightarrow M = \{1, \ldots, M\}$, we have
\[
E_X e^{-sH_{1+\epsilon}(A|f_X(A), E, X|P_{AE})} \geq 2^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e) \geq \epsilon/M} P_{A|E}(a|e) + \left(\frac{\epsilon}{M}\right)^{-s} 2^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e) < \epsilon/M} P_{A|E}(a|e)^{1+s}
\]
for any $s \in [0, 1]$.

**Proof:** We first establish some basic inequalities: For any $(a, e) \in A \times E$, and any $\epsilon$-almost universal$_2$ hash function $f_{X_n} : A \rightarrow M = \{1, \ldots, M\}$,
\[
E_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \leq P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a' \neq a} P_{A|E}(a'|e)
\]
(157)
\[
\leq P_{A|E}(a|e) + \frac{\epsilon}{M}
\]
(158)
\[
\leq 2 \max \{P_{A|E}(a|e), \frac{\epsilon}{M}\}.
\]
(159)

Using (159), we have
\[
E_X e^{-sH_{1+\epsilon}(A|f_X(A), E, X|P_{AE})}
\]
\[
= E_X \sum_e P_E(e) \sum_{a} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right) \left( \sum_{a' \in f_X^{-1}(f_X(a))} \left( \frac{P_{A|E}(a'|e)}{\sum_{a \in f_X^{-1}(f_X(a))} P_{A|E}(a|e)} \right)^{1+s} \right) \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e)^{1+s} \right)^{-s}
\]
(160)
\[
= E_X \sum_e P_E(e) \sum_{a} P_{A|E}(a|e)^{1+s} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s}
\]
(161)
\[
\geq \sum_e P_E(e) \sum_{a} P_{A|E}(a|e)^{1+s} \left( E_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^{-s}
\]
(162)
\[
\geq \sum_e P_E(e) \sum_{a} P_{A|E}(a|e)^{1+s} \left( 2 \max \left\{ P_{A|E}(a|e), \frac{\epsilon}{M} \right\} \right)^{-s}
\]
(163)
\[
= 2^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e) \geq \epsilon/M} P_{A|E}(a|e) + \left( \frac{\epsilon}{M} \right)^{-s} 2^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e) < \epsilon/M} P_{A|E}(a|e)^{1+s}
\]
(164)
Thus, we obtain (156).

**Lemma 2.** For $\epsilon$-almost universal$_2$ hash functions $f_X : A \rightarrow M = \{1, \ldots, M\}$, we have
\[
E_X e^{-sH_{1+\epsilon}(A|f_X(A), E, X|P_{AE})} \geq 2^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} \geq \epsilon/M} P_{A|E}(a|e) + 2^{-s} \left( \frac{\epsilon}{M} \right)^{-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} < \epsilon/M} P_{A|E}(a|e)^{1+s} \left( \sum_{a'} P_{A|E}(a'|e)^{1+s} \right)^{-\frac{s}{1+s}}
\]
(165)
for any $s \in [0, \infty)$. 
Thus, we obtain (167).

\[
\mathbb{E}_X e^{-sH_{1-s}(A|f_X(A),E,X|P_{AE})} = \mathbb{E}_X \sum_{e} P_E(e) \sum_i \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right) \left( \sum_{a' \in f_X^{-1}(i)} \left( \frac{P_{A|E}(a'|e)}{\sum_{a'' \in f_X^{-1}(i)} P_{A|E}(a''|e)} \right)^{1+s} \right)^{\frac{1}{1+s}} \\
= \mathbb{E}_X \sum_{e} P_E(e) \sum_i \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right) \left( \sum_{a' \in f_X^{-1}(i)} P_{A|E}(a'|e)^{1+s} \right) \left( \sum_{a'' \in f_X^{-1}(i)} P_{A|E}(a''|e)^{1+s} \right)^{-\frac{1}{1+s}} \\
= \mathbb{E}_X \sum_{e} P_E(e) \sum_a P_{A|E}(a|e)^{1+s} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e)^{1+s} \right)^{-\frac{1}{1+s}} \\
\geq \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1+s} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e)^{1+s} \right)^{-\frac{1}{1+s}} \\
> \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1+s} \left( 2 \max \left\{ P_{A|E}(a|e)^{1+s}, \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s} \right\} \right)^{-\frac{1}{1+s}} \\
\geq \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} \geq \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s}} P_{A|E}(a|e)^{1+s} \left( 2 \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s} \right)^{-\frac{1}{1+s}} \\
+ \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} < \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s}} P_{A|E}(a|e)^{1+s} \left( 2 \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s} \right)^{-\frac{1}{1+s}} \\
= 2^{-\frac{1}{1+s}} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} \geq \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s}} P_{A|E}(a|e) \\
+ 2^{-\frac{1}{1+s}} \left( \frac{\epsilon}{M} \right)^{1-s} \sum_e P_E(e) \sum_{a : P_{A|E}(a|e)^{1+s} < \frac{\epsilon}{M} \sum_a P_{A|E}(a'|e)^{1+s}} \sum_a P_{A|E}(a'|e)^{1+s} \left( \sum_a P_{A|E}(a'|e)^{1+s} \right)^{-\frac{1}{1+s}} .
\]

Thus, we obtain (167).

\[\text{Lemma 3.} \text{ For } \epsilon\text{-almost universal}_2 \text{ hash functions } f_X : A \rightarrow \mathcal{M} = \{1, \ldots, M\}, \text{ we have} \]

\[
\mathbb{E}_X e^{sH_{1-s}(A|f_X(A),E,X|P_{AE})} \leq 1 + \frac{e^{\epsilon sH_{1-s}(A|E|P_{AE})}}{M^s}, \tag{176}
\]

for all \(s \in [0, 1]\). In addition, we also have

\[
\mathbb{E}_X e^{\frac{s}{1-s}H_{1-s}(A|f_X(A),E,X|P_{AE})} \leq 1 + \frac{e^{\epsilon sH_{1-s}(A|E|P_{AE})}}{M^{1-s}}, \tag{177}
\]

for all \(s \in [0, 1/2]\).

\[\text{Proof:} \text{ We have} \]

\[
\mathbb{E}_X e^{sH_{1-s}(A|f_X(A),E,X|P_{AE})} \\
= \mathbb{E}_X \sum_e P_E(e) \sum_i \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right) \left( \sum_{a' \in f_X^{-1}(i)} \left( \frac{P_{A|E}(a'|e)}{\sum_{a'' \in f_X^{-1}(i)} P_{A|E}(a''|e)} \right)^{1-s} \right) \\
= \mathbb{E}_X \sum_e P_E(e) \sum_i \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right) \left( \sum_{a' \in f_X^{-1}(i)} P_{A|E}(a'|e)^{1-s} \right) \\
= \mathbb{E}_X \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1-s} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^s \\
\leq \sum_e P_E(e) \sum_a P_{A|E}(a|e)^{1-s} \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'|e) \right)^s .
\]


(182) \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left(P_{A|E}(a|e) + \frac{c}{M}\right)^s \\
(183) \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left(P_{A|E}(a|e) + \frac{\epsilon}{M}\right)^s \\
(184) = \sum_e P_E(e) \sum_a \left(P_{A|E}(a|e) + P_{A|E}(a|e) 1^{-s} \left(\frac{\epsilon}{M}\right)^s \right) \\
(185) = 1 + \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left(\frac{\epsilon}{M}\right)^s \\
(186) = 1 + \frac{\epsilon^s e^{H_{1-s}(A|E|P_A|E)}}{M^s}.

In (181), we applied Jensen’s inequality with the concave function $t \mapsto t^s$. Here is where the condition $s \in [0, 1]$ is used. In (182), we used (158) and in (183), we used the fact that $(b+c)^s \leq b^s + c^s$ for $b, c \geq 0$ and $s \in [0, 1]$ [13, Problem 4.15(f)]. Thus, we obtain (176).

For (177), consider,

\[ \mathbb{E}_X e^{\frac{1}{1-s} H_{1-s}(A|f_X(A), E|X|P_{A|E})} \]

\[ = \mathbb{E}_X \sum_e P_E(e) \sum_i \left( \sum_{a \in f_X^{-1}(i)} P_{A|E}(a|e) \right) \left( \sum_{a' \in f_X^{-1}(i)} \left( \frac{P_{A|E}(a'\|e)}{\sum_{a'' \in f_X^{-1}(i)} P_{A|E}(a''\|e)} \right)^{1-s} \right) \left( \frac{1}{\mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'\|e)} \right)^{1-s} \]

\[ = \mathbb{E}_X \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'\|e) \right)^{1-s} \]

\[ \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( \mathbb{E}_X \sum_{a' \in f_X^{-1}(f_X(a))} P_{A|E}(a'\|e) \right)^{1-s} \]

\[ \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a'' \neq a} P_{A|E}(a''\|e) \right) \right)^{1-s} \]

\[ \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a'' \neq a} P_{A|E}(a''\|e) \right) \right)^{1-s} \]

\[ \leq \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( P_{A|E}(a|e) \left( P_{A|E}(a|e) + \frac{\epsilon}{M} \sum_{a'' \neq a} P_{A|E}(a''\|e) \right) \right)^{1-s} \]

\[ = 1 + \sum_e P_E(e) \sum_a P_{A|E}(a|e) 1^{-s} \left( \frac{\epsilon}{M} \sum_{a'' \neq a} P_{A|E}(a''\|e) \right)^{1-s} \]

\[ = 1 + \left( \frac{\epsilon}{M} \right)^{1-s} \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) 1^{-s} \right)^{1+s} \]

\[ = 1 + \left( \frac{\epsilon}{M} \right)^{1-s} \sum_e P_E(e) \left( \sum_a P_{A|E}(a|e) 1^{-s} \right)^{1+s} \]

\[ = 1 + \frac{\epsilon^{1-s} e^{H_{1-s}(A|E|P_A|E)}}{M^{1-s}}. \]
In (191), we applied Jensen’s inequality with the concave function \( t \mapsto t^{r\gamma(t)} \). Here is where the condition \( s \in [0, 1/2] \) is used. The explanations for the other bounds parallel those for the proof of (176) and are omitted for the sake of brevity. This completes the proof of (177).

\[ \Lambda(s, R) := \begin{cases} 
\exp(-n\Lambda(s, R)) & \text{if } s \leq 1 + t_R \\
\exp(-n[s(R + D(P^{(t_R)}_A \| P_A)]) & \text{if } s > 1 + t_R
\end{cases} \] (203)

Furthermore, if \( R < H_1 - s(A|P_A) \), then for all \( s \in [0, s_0(A|P_A)] \) (cf. definition in (54)),

\[ \Psi_n \geq \exp(-n[s(R + D(P^{(t_R)}_A \| P_A))]) \] (204)

The inequality in (204) implies that under the stated conditions on \( A \) and \( s \), the first clause in (203) is active.

The calculations here are somewhat similar to those in Merhav’s work in [21, Section IV-C] and [22, Section IV-D] but the whole proof for the case in which \( E = \emptyset \) is included for completeness. See Remark 2 for a sketch of how to extend the analysis to the memoryless but non-stationary case in which \( E \neq \emptyset \).

We remark that when \( s - 1 = t_R \),

\[ \Lambda(s, R) \big|_{s=1+t_R} = (1 + t_R)\gamma'(t_R). \] (206)

By using \( H(P^{(t_R)}_A) = R \), (199), and (200), we immediately see that this “boundary” case coincides with the two cases of (203) so \( s \mapsto \Lambda(s, R) \) is continuous at \( 1 + t_R \).

Proof: Let \( \mathcal{G}_R := \{ Q \in \mathcal{P}(A) : H(Q) > R \} \), \( \mathcal{G}_{R,n} := \mathcal{G}_R \cap \mathcal{P}_n(A) \) and \( \operatorname{cl}(\mathcal{G}_R) \) be the closure of \( \mathcal{G}_R \). We split \( \Psi_n \) into the following two sums

\[ \Psi_n = \sum_{Q \in \mathcal{G}_{R,n}} \mathbb{E}_{X_n} \left\{ \left[ \sum_{\tilde{a} \in \mathcal{T}_Q \setminus \{a\} : \tilde{f}_{X_n}(\tilde{a}) = \tilde{f}_{X_n}(\tilde{a})} P^n_{A}(\tilde{a}) \right]^s \right\}_{\alpha_n} + \sum_{Q \in \mathcal{G}_{R,n}} \mathbb{E}_{X_n} \left\{ \left[ \sum_{\tilde{a} \in \mathcal{T}_Q \setminus \{a\} : \tilde{f}_{X_n}(\tilde{a}) = \tilde{f}_{X_n}(\tilde{a})} P^n_{A}(\tilde{a}) \right]^s \right\}_{\beta_n}. \] (207)
Define $N_Q := \sum_{\tilde{a} \in T_Q \setminus \{a\}} 1\{ \tilde{T}_{X_n}(a) = \tilde{T}_{X_n}(\tilde{a}) \}$. This is a sum of $L_Q := |T_Q \setminus \{a\}|$ independent and identically distributed $\{0, 1\}$-random variables $\{Y_i\}_{i=1}^L$ with $\Pr(Y_i = 1) = e^{-nR} =: p$ (property of strong universal 2 hash functions). Let $a_Q \in T_Q$ be any generic vector of type $Q$. Let us now lower bound $\alpha_n$ and $\beta_n$.

- For the expectation within $\alpha_n$, we have

$$
\mathbb{E}_{X_n} \left\{ \left[ \sum_{\tilde{a} \in T_Q \setminus \{a\}} P^n_{\tilde{A}}(\tilde{a}) \right]^s \right\} = P^n_{\tilde{A}}(a_Q)^s \mathbb{E}_{X_n} \left\{ \left[ \sum_{\tilde{a} \in T_Q \setminus \{a\}} \mathbb{1}_{\tilde{T}_{X_n}(a) = \tilde{T}_{X_n}(\tilde{a})} \right]^s \right\} = P^n_{\tilde{A}}(a_Q)^s \mathbb{E} \left[ N_Q^s \right] \geq P^n_{\tilde{A}}(a_Q)^s \mathbb{E} \left[ N_Q \right]^s \geq \mathbb{E}_{X_n} \left\{ \sum_{\tilde{a} \in T_Q \setminus \{a\}} \mathbb{1}_{\tilde{T}_{X_n}(a) = \tilde{T}_{X_n}(\tilde{a})} P^n_{\tilde{A}}(\tilde{a}) \right\}^s \tag{208} \right.
$$

where (208) follows because all $\tilde{a}$ in the sum have the same type $Q$, (209) from the definition of $N_Q$, (210) follows from Lemma 5 (in Appendix C) under the condition that $Q \in G_{R,n}$ (so $L_Q \cdot p = \mathbb{E}[N_Q] \geq (n+1)^{-|A|} \exp(n[H(Q) - R]) \to \infty$ and (228) applies). Thus, we conclude that

$$
\alpha_n \geq \sum_{Q \in \mathcal{U}_{R,n}} (\mathbb{E}[N_Q])^s P^n_{\tilde{A}}(a_Q)^s \tag{212} \right.
$$

By further using the fact that $P^n_{\tilde{A}}(a_Q) = \exp(-n[D(Q\|P_A) + H(Q)])$ [16, Lemma 2.6], we have

$$
\alpha_n \geq \exp(-n s R) \exp \left( -n s \min_{Q \in \mathcal{U}_{R,n}} D(Q\|P_A) \right) =: \tilde{\alpha}_n. \tag{214} \right.
$$

- Next, we lower bound the expectation in $\beta_n$. The step from (208) to (209) remains the same but we bound $\mathbb{E}[N_Q]$ differently. We have

$$
\mathbb{E} \left[ N_Q^s \right] \geq 1^s \Pr(N_Q = 1) = 1^s \left( \frac{L_Q}{1} \right) p^1 (1-p)^{L_Q-1} \geq L_Q \cdot p \cdot \exp \left( (L_Q - 1) \log(1-p) \right) \geq L_Q \cdot p \cdot \exp \left( - (L_Q - 1) \frac{p}{1-p} \right) \geq L_Q \cdot p = \mathbb{E}[N_Q], \tag{219} \right.
$$

where (218) follows from the basic inequality $\log(1-t) \geq - \frac{t}{1-t}$ and (219) follows from the fact that $\frac{(L_Q - 1)p}{L_Q} \leq \exp(n[H(Q) - R]) \leq 1$ when $Q \in G_{R,n} = \{Q \in \mathcal{P}_n(A) : H(Q) \leq R \}$. Thus, we have

$$
\beta_n \geq \sum_{Q \in \mathcal{V}_{R,n}} \mathbb{E}[N_Q] P^n_{\tilde{A}}(a_Q)^s \geq \max_{Q \in \mathcal{V}_{R}} \mathbb{E}[N_Q] P^n_{\tilde{A}}(a_Q)^s \geq \exp(-n R) \exp \left( -n \min_{Q \in \mathcal{V}_{R}} \left[sD(Q\|P_A) - (1-s)H(Q)\right] \right) =: \tilde{\beta}_n. \tag{222} \right.
$$

We remark that the evaluations in (210) and (219) are, in fact, exponentially tight$^8$ [21, Eqn. (34)]. This implies that $\alpha_n \doteq \tilde{\alpha}_n$ and $\beta_n \doteq \tilde{\beta}_n$. However, we only require the lower bounds.

$^8$The intuition here is that if $H(Q) > R$, the random variable $N_Q$ concentrates doubly-exponentially fast to its expectation, which itself is exponentially large. On the other hand, if $H(Q) < R$, $N_Q$ is typically exponentially small, so $\mathbb{E}[N_Q]$ is dominated by the term $1^s \Pr(N_Q = 1)$. 

It is easy to see that the optimal distribution $Q^*$ in the optimization in the exponent of $\tilde{\alpha}_n$ satisfies $H(Q^*) = R$ (i.e., $Q^*$ lies on the boundary of $\mathcal{G}_R$). In fact, the exponent (which is the lossless source coding error exponent [16, Theorem 2.15]) can be expressed as $D(P_A^{(t_R)}\|P_A)$ where $t_R \geq -1$ is chosen such that $H(P_A^{(t_R)}) = (1 + t_R)\gamma(t_R) - \gamma(t_R) = R$. Thus,

$$\tilde{\alpha}_n = \exp(-nsR)\exp\left(-nsD(P_A^{(t_R)}\|P_A)\right)$$ \hspace{1cm} (223)

$$= \exp(-ns[R + \gamma(t_R) - t_R\gamma(t_R)]).$$ \hspace{1cm} (224)

Now, it is easy to verify (see Shayevitz [37, Section IV-A.8] for example) by differentiating the convex function $g(Q) := sD(Q\|P_A) - (1 - s)H(Q)$ that the unconstrained minimum in the exponent in $\tilde{\alpha}_n$ takes the form of a tilting of $P_A$, i.e.,

$$Q^*(a) := \frac{P_A(a)\gamma(a)}{\sum_a P_A(a)\gamma(a)}, \quad \forall a \in \mathcal{A}.$$ \hspace{1cm} (225)

Now depending on the value of $s$, we have two different scenarios. First, if $s > 1 > t_R$ or equivalently, $Q^* \in \mathcal{G}_R$, then $\tilde{\alpha}_n \equiv \exp(-n[R + g(Q^*)]) = \exp(-n[R + \gamma(s - 1)])$ (using (199) and (200)). On the other hand, if $s - 1 \leq t_R$, the optimal solution in the optimization in $\tilde{\alpha}_n$ is again attained at the boundary of $\mathcal{G}_R$ and $\mathcal{G}_R$ (i.e., the constraint $Q \in \mathcal{G}_R$ is active). Thus, $\tilde{\alpha}_n \equiv \bar{\alpha}_n$ where $\bar{\alpha}_n$ is in (224). In summary, $\bar{\alpha}_n \equiv \exp(-n\Lambda(s, R))$ where $\Lambda(s, R)$ is defined in (203). Now, clearly $\beta_n$ always dominates $\bar{\alpha}_n$ (i.e., $\bar{\alpha}_n$ is exponentially at least as large as $\bar{\alpha}_n$). This is because when $s - 1 \leq t_R$, they are the same, and when $s - 1 > t_R$, we are taking an unconstrained minimum of $g(Q)$ in the exponent, making the overall expression larger. We thus obtain the conclusion in (202).

For the statements in (204)–(205), we assume that $R < H_1-s(A\|P_A)$ and $s \in [0, s_0(A\|P_A)]$. We claim that these imply that $s - 1 \leq t_R$, i.e., the first clause in (203) is active. Note from the definition of $s_0(A\|P_A)$ in (54) that $s \leq s_0(A\|P_A)$ means that

$$H_1-s(A\|P_A) \leq H(A\|P_A^{(s-1)}).$$ \hspace{1cm} (226)

Since $R < H_1-s(A\|P_A)$, it holds that $R < H(A\|P_A^{(s-1)})$, but this in turn implies that $s - 1 \leq t_R$ because $t \mapsto H(A\|P_A^t)$ is monotonically non-increasing. Thus (204) holds.

That (204) is exponentially equal to (205) follows from the fact that when $R < H_1-s(A\|P_A)$ and $s \in [0, s_0(A\|P_A)]$, $\bar{\beta}_n$ is of the same exponential order $\bar{\alpha}_n$ and the latter is lower bounded (on the exponential scale) by (211).

**Remark 2.** To derive a conditional version of Lemma 4 to obtain (132), we assume that the type of $e \in \mathcal{E}^n$ is $Q_E \in \mathcal{P}_\mathcal{E}(\mathcal{E})$. The above derivations go through essentially unchanged by averaging with respect to $Q_E$ everywhere. Specifically, we consider $\mathbf{a}$ and $\mathbf{a}^*$ to belong to various “$\mathcal{V}_A$-shells” $\mathcal{V}_A(e) := \{\mathbf{a} \in \mathcal{A}^n : (\mathbf{a}, e) \in \mathcal{V}_A \times \mathcal{V}_A(e)\}$ [16, Ch. 2]. The entropies $H(Q)$ are replaced by conditional entropies $H(V_A|E|Q_E)$, the relative entropies $D(Q\|P_A)$ by conditional relative entropies $D(V_A|E|P_A|E|Q_E)$ and the tilted conditional distribution is defined as $P_A^t(e) \propto P_A|E|(e)^{1+t}$ and so on. For the analogues of (204)–(205) to hold, we firstly require $R < \sum_e Q_E(e)H_1-s(A\|P_A|E|e)$. If we further assume that $0 \leq s < s_0 = \min_e s_0(A\|P_A|E|e)$, $H_1-s(A\|P_A|E|e) \leq H(A\|P_A^{(s-1)}(\cdot|e))$ for all $e \in \mathcal{E}$, and so $R < \sum_e Q_E(e)H(A\|P_A^{(s-1)}(\cdot|e))$ giving the first clause in the conditional analogue of (203), i.e., that $\Lambda(s, R) = s(R + D(P_A(t_R)\|P_A|Q_E))$ where $t_R \geq -1$ satisfies $H(P_A^{(t_R)}|Q_E) = R$. These observations yield (132) upon averaging over all types on $\mathcal{E}$.

**Appendix C**

**A Useful Concentration Bound**

The following lemma is essentially a restatement of Lemma 10 in [38].

**Lemma 5.** Let $Y_1, \ldots, Y_L$ be independent random variables, each taking values in $\{0, 1\}$ such that $\Pr(Y_i = 1) = p$. Let $N := \sum_{i=1}^L Y_i$. For every $s \in [0, 1]$ and any $0 < \epsilon < 1$,

$$\mathbb{E}[N^s] \geq \lfloor Lp(1 - \epsilon)\rfloor^s \left[1 - \exp\left(-L\frac{p}{2}\epsilon^2\right)\right].$$ \hspace{1cm} (227)

In particular, if $Lp$ is a sequence in $n$ that tends to infinity (as $n$ tends to infinity) exponentially fast, then by taking $\epsilon = 1/2$ (say) in (227),

$$\mathbb{E}[N^s] \geq (Lp)^s = \{\mathbb{E}[N]\}^s.$$ \hspace{1cm} (228)

In fact, we have $\mathbb{E}[N^s] \leq \{\mathbb{E}[N]\}^s$ because $\mathbb{E}[N^s] \leq \{\mathbb{E}[N]\}^s$ by Jensen’s inequality.

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