The appearance of geometric flow in conservation law of particle number in classical particle diffusion, and the one in the conservation law of probability in quantum mechanics are discussed in a geometrical environment: two dimensional curved surface with thickness $\epsilon$ embedded in $\mathbb{R}^3$. In such a system with small parameter $\epsilon$, the usual two dimensional conservation law doesn’t hold and we find the anomaly by the equation $\frac{\partial \rho}{\partial t} + \nabla^{(2)}_i J^i \neq 0$, where the density $\rho$, the flow $J^i$, and $\nabla^{(2)}_i$ is the two dimensional covariant derivative. The anomalous term is obtained in $\epsilon$ (thickness of surface) expansion. We find this term has Gaussian and mean curvature dependence and can be written in total divergence of some geometric flow. In total we have

$$\frac{\partial \rho}{\partial t} + \nabla^{(2)}_i (J^i + J^i_0) = 0.$$  

This fact holds both in classical and quantum mechanics when we confine the particle in curved surface with small thickness.

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I. GEOMETRICAL TOOL

The particle motion on a given curved surface $M^2$ is an interesting problem in a wide range of physics. The classical diffusion process of particles on such a manifold is expressed by changing the Laplacian to the Laplace-Beltrami operator in diffusion equation, however, when the surface has a thickness $\epsilon$, i.e., the configuration space is $M^2 \times \mathbb{R},$ the situation is not simple. To make the problem concrete, we first introduce the 2-dimensional curved manifold $\Sigma$ in $\mathbb{R}^3$, and we also introduce the similar two copies of $\Sigma$ called $\Sigma'$ and $\tilde{\Sigma}$ and place them in both sides of $\Sigma$ at a small distance of $\epsilon/2$.

\[ \mathbf{X}(q^0, q^1, q^2) = \mathbf{x}(q^1, q^2) + q^0 \hat{n}(q^1, q^2), \]

(1)

where $-\epsilon/2 \leq q^0 \leq \epsilon/2$.

Then we obtain the curvilinear coordinate system between two surfaces ($\subset \mathbb{R}^3$) by the coordinate $q^\mu = (q^0, q^1, q^2)$, and metric $G_{\mu\nu}$. (Hereafter Greek indices $\mu, \nu, \cdots$ runs from 0 to 2.)

\[ G_{\mu\nu} = \frac{\partial \mathbf{X}}{\partial q^\mu} \cdot \frac{\partial \mathbf{X}}{\partial q^\nu}. \]

(2)

Each part of $G_{\mu\nu}$ is the following.

\[ G_{ij} = g_{ij} + q^0 \left( \frac{\partial \mathbf{x}}{\partial q^i} \cdot \frac{\partial \hat{n}}{\partial q^j} + \frac{\partial \mathbf{x}}{\partial q^j} \cdot \frac{\partial \hat{n}}{\partial q^i} \right) + (q^0)^2 \frac{\partial \hat{n}_i}{\partial q^j} \cdot \frac{\partial \hat{n}_j}{\partial q^i}, \]

(3)

where

\[ g_{ij} = \frac{\partial \mathbf{x}}{\partial q^i} \cdot \frac{\partial \mathbf{x}}{\partial q^j}. \]

(4)

is the metric (first fundamental tensor) on $\Sigma$. Hereafter indices $i, j, k \cdots$ are lowered or rised by $g_{ij}$ and its inverse $g^{ij}$. We also obtain

\[ G_{0i} = G_{00} = 0, \quad G_{00} = 1. \]

(5)
We can proceed the calculation by using the new variables. We first define the tangential vector to $\Sigma$ by
\[
\vec{B}_k = \frac{\partial \vec{x}}{\partial q^k}.
\] (6)

Note that $\vec{n} \cdot \vec{B}_k = 0$. Then we obtain two relations. Gauss equation:
\[
\frac{\partial \vec{B}_k}{\partial q^j} = -\kappa_{ij} \vec{n} + \Gamma^k_{ij} \vec{B}_k,
\] (7)

Weingarten equation:
\[
\frac{\partial \vec{n}}{\partial q^j} = \kappa^m_{ij} \vec{B}_m,
\] (8)

where
\[
\Gamma^k_{ij} = \frac{1}{2} g^{km} (\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}).
\]

$\kappa_{ij}$ is a symmetric tensor called Euler-Schauten tensor, or second fundamental tensor defined by above two equations. Furthermore, the mean curvature is given by
\[
\kappa = g^{ij} \kappa_{ij},
\] (9)

and Ricci scalar $R$ (Gaussian curvature) is defined by
\[
R/2 \equiv \det(g^{ik} \kappa_{kj}) = \det(\kappa^j_j) = \frac{1}{2} (\kappa^2 - \kappa_{ij} \kappa^{ij}).
\] (10)

Then we have the formula for metric of curvilinear coordinate in a neighborhood of $\Sigma$. 
\[
G_{ij} = g_{ij} + 2 q^0 \kappa_{ij} + (q^0)^2 \kappa_{im} \kappa^m_j.
\] (11)

Now we have the total metric tensor such as,
\[
G_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & G_{ij} \end{pmatrix}.
\] (12)

By using above relations, we can construct the diffusion equation and Schrödinger equation in our environment.

**II. THE CLASSICAL DIFFUSION FIELD**

For the classical diffusion field, the problem is already solved and discussed in [2], however, to compare the results to the ones in quantum mechanics which will be discussed in next section, we briefly sketch the essential part of the results here at the expense of repetition.

We denote 3 dimensional diffusion field as $\phi^{(3)}$ in our space between $\Sigma'$ and $\tilde{\Sigma}$ which satisfies usual 3-dimensional diffusion equation and normalization condition.
\[
\frac{\partial \phi^{(3)}}{\partial t} = D \Delta^{(3)} \phi^{(3)},
\] (13)

\[
1 = \int \phi^{(3)}(q^0, q^1, q^2) \sqrt{G} \, d^3 q,
\] (14)

where $D$ is the diffusion constant, $\Delta^{(3)}$ is the three dimensional Laplace Beltrami operator, and $G = \det(G_{\mu\nu}) = \det(G_{ij})$. When $\epsilon \to 0$, the theory may reduce to the two dimensional one. Our aim is to construct such a two dimensional equation from 3D equation above. The effective two dimensional diffusion field $\phi^{(2)}(q^1, q^2)$ should satisfy the normalization condition such as,
\[
\int \phi^{(2)}(q^1, q^2) \sqrt{g} \, d^2 q = 1.
\] (15)

where $g = \det(g_{ij})$.

From two normalization conditions, we obtain
\[
1 = \int \phi^{(3)}(q^0, q^1, q^2) \sqrt{G} \, d^3 q,
\]

\[
\int \int_{-\epsilon/2}^{\epsilon/2} dq^0 \phi^{(3)}(q^0, q^1, q^2) \sqrt{G} \, d^2 q,
\]

\[
\int \phi^{(2)}(q^1, q^2) \sqrt{g} \, d^2 q.
\]

Therefore we obtain the relation,
\[
\phi^{(2)}(q^1, q^2) = \int_{-\epsilon/2}^{\epsilon/2} \tilde{\phi}^{(3)} dq^0,
\] (16)

where
\[
\tilde{\phi}^{(3)} \equiv \phi^{(3)} \sqrt{G/g}.
\] (17)

We further suppose the local equilibrium condition that there is no diffusion flow in normal direction in layer $\Sigma'$.
\[
0 = \frac{\partial \phi^{(3)}}{\partial q^0} = g^{1/2} \frac{\partial G^{-1/2} \tilde{\phi}^{(3)}}{\partial q^0}.
\] (18)

Then we can write $\phi^{(3)}$ by using $\phi^{(2)}$.
\[
\tilde{\phi}^{(3)} = \frac{1}{N} (G/g)^{1/2} \phi^{(2)}(q^1, q^2),
\] (19)

\[
N = \int_{-\epsilon/2}^{\epsilon/2} (G/g)^{1/2} dq^0.
\] (20)

The physical reason of equation (18) is the following. The diffusion into the normal direction to the surface may occurs always with satisfying the equilibrium condition to normal direction in this time scale. By performing the $q^0$ integration of diffusion equation (18) with $\sqrt{G/g}$, and using (10) and (11), we obtain the final form of equation up to $O(\epsilon^2)$ as
\[
\frac{\partial \phi^{(2)}}{\partial t} = D \Delta^{(2)} \phi^{(2)} + \tilde{D} g^{-1/2} \frac{\partial}{\partial q^0} g^{1/2} \times \left\{ (3 \kappa^m \kappa^m - 2 \kappa \kappa^0) \frac{\partial}{\partial q^0} + \frac{1}{2} \kappa^0 \frac{\partial R}{\partial q^0} \right\} \phi^{(2)},
\] (21)
where $\tilde{D} = \frac{\hbar^2}{2m} D$.

We can rewrite the diffusion equation into the form

$$-\frac{\partial \phi^{(2)}}{\partial t} = \nabla_i (J^i + J_G^i),$$

$$= g^{-1/2} \frac{\partial}{\partial q^j} g^{1/2} (J^i + J_G^i),$$

where $\nabla_i$ the covariant derivative, the normal diffusion flow

$$J^i = -Dg^{ij} \frac{\partial \phi^{(2)}}{\partial q^j},$$

and the anomalous diffusion flow

$$J_G^i = -\tilde{D} \left[ (3\kappa m \kappa_m - 2\kappa \kappa^i) \frac{\partial \phi^{(2)}}{\partial q^i} - \frac{1}{2} g^{ij} \frac{\partial R}{\partial q^j} \phi^{(2)} \right].$$

(24)

III. QUANTUM MECHANICAL COUNTERPART

In quantum mechanics, the basic equation is the Schrödinger equation written by using curvilinear coordinate in a three dimensional space between two surfaces $\Sigma'$ and $\Sigma$.

$$i\hbar \frac{\partial}{\partial t} \psi = \left[ -\frac{\hbar^2}{2m} \Delta^{(3)} + V(q) \right] \psi,$$

where we should note the form of Laplace Beltrami operator

$$\Delta^{(3)} \equiv G^{-1/2} \partial_\mu G^{1/2} G^{\mu \nu} \partial_\nu,$$

and we suppose that $V$ does not depend on $t$ and $q^0$.

Starting from this wave function $\psi$, we construct the effective two dimensional theory. From the normalization condition we obtain

$$1 = \int |\psi|^2 \sqrt{G} \, dq = \int \left[ \int_{-\epsilon/2}^{+\epsilon/2} |\psi|^2 \sqrt{\frac{G}{g}} \, dq^0 \right] \sqrt{g} \, dq^2 q,$$

(26)

Our effective two dimensional wave function $\phi$ should satisfy

$$|\phi(q^1, q^2)|^2 = \int_{-\epsilon/2}^{+\epsilon/2} |\psi|^2 \sqrt{\frac{G}{g}} \, dq^2 q,$$

(27)

$$1 = \int |\phi|^2 \sqrt{g} \, d^3 q.$$

(28)

Then how can we obtain the equation for $\phi$? To solve the problem, we first define the new variable $\tilde{\psi}$ as

$$\tilde{\psi} \equiv (G/g)^{1/4} \psi $$

(29)

with

$$|\phi(q^1, q^2)|^2 = \int_{-\epsilon/2}^{+\epsilon/2} |\tilde{\psi}|^2 \, dq^0.$$

(30)

Further we suppose the possibility of the separation of variable.

$$\tilde{\psi} = \phi(q^1, q^2, t) \chi(q^0, t),$$

(31)

$$1 = \int_{-\epsilon/2}^{+\epsilon/2} |\chi|^2 \, dq^0.$$

(32)

Then we can construct the equation for $\phi$ in the followings.

First we construct the Schrödinger equation for $\tilde{\psi}$. This is the same form as (25) except changing the Laplace Beltrami operator to the following one.

$$\tilde{\Delta}^{(3)} \equiv (G/g)^{1/4} \Delta^{(3)} (G/g)^{-1/4}.$$

(33)

With the help of appendix 1, this operator can be expanded as

$$\tilde{\Delta}^{(3)} = \Delta^{(2)} + \frac{\partial^2}{\partial (q^0)^2} + V_0 + q^0 V_1 + (q^0)^2 V_2$$

$$+ q^0 \hat{A}_1 + (q^0)^2 \hat{A}_2 + \mathcal{O}((q^0)^3),$$

(34)

where $V_0, V_1, V_2, \hat{A}_1, \hat{A}_2$ are followings.

$$V_0 = \frac{1}{4} (\kappa^2 - 2R),$$

(35)

$$V_1 = \kappa (R - \frac{\kappa^2}{2}) - \frac{1}{2} \Delta^{(2)} \kappa,$$

(36)

$$V_2 = \frac{3}{4} \kappa^4 - \frac{7}{4} \kappa^2 R + \frac{1}{2} R^2 + \frac{1}{2} \kappa \Delta^{(2)} \kappa$$

$$+ \frac{1}{2} g^{ij} (\partial_i \kappa) (\partial_j \kappa) + \nabla_i (\kappa^{ij} \partial_j \kappa)$$

$$- \frac{1}{2} \Delta^{(2)} \kappa R,$$

(37)

$$\hat{A}_1 = -2 \nabla_i \kappa^{ij} \partial_j,$$

(38)

$$\hat{A}_2 = 3 \nabla_i \kappa \kappa^{ik} \partial_k.$$  

(39)

Then our equation for $\tilde{\psi}$ is given as follows.

$$i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\frac{\hbar^2}{2m} |\Delta^{(2)} + \frac{\partial^2}{\partial (q^0)^2} + V_0 + q^0 V_1 + (q^0)^2 V_2$$

$$+ q^0 \hat{A}_1 + (q^0)^2 \hat{A}_2 |\tilde{\psi} + V \tilde{\psi},$$

(40)

where we have omitted $\mathcal{O}((q^0)^3)$ terms for small $q^0$.

We treat this system by perturbation method. The Hamiltonian can be written as
\[ \hat{H} = \hat{H}_0 + \hat{H}_I, \]
\[ \hat{H}_0 = -\frac{\hbar^2}{2m}[\Delta^{(2)} + \frac{\partial^2}{\partial(q^0)^2}] + V_0 + V, \]
\[ \hat{H}_I = -\frac{\hbar^2}{2m}[q_0 (V_1 + \hat{A}_1) + (q_0)^2(V_2 + \hat{A}_2)]. \]

As an eigenfunction of \( \hat{H}_0 \), we introduce \( \chi_N \) and \( \phi_k \) such as
\[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial(q^0)^2} \chi_N = E_N \chi_N, \]
\[ [-\frac{\hbar^2}{2m}(\Delta^{(2)} + V_0) + V] \phi_k = \lambda_k \phi_k. \]

The eigenfunction \( \chi_N \) should also satisfy the Dirichlet boundary condition \( \chi_N(q^0 = \pm \epsilon/2) = 0 \). Then the orthonormal eigenfunction \( \chi_N \) \((N = 1, 2, 3, \cdots)\) is given as
\[ \chi_N = \tilde{\chi}_N e^{-iE_N t/\hbar}, \]
where
\[ \tilde{\chi}_N = \begin{cases} \sqrt{\frac{2}{\epsilon}} \cos(N\pi q^0/\epsilon) & N = \text{odd}, \\ \sqrt{\frac{2}{\epsilon}} \sin(N\pi q^0/\epsilon) & N = \text{even}. \end{cases} \]

and
\[ E_N = \frac{\hbar^2 \pi^2}{2me^2} N^2. \]

We note that
\[ \int_{-\epsilon/2}^{+\epsilon/2} \tilde{\chi}_N^* \chi_N dq^0 \equiv <M|N> = \delta_{MN}. \]

We also assume the existence of the orthonormal eigenfunction \( \phi_k \) and eigenvalue \( \lambda_k \) without showing the explicit form.
\[ \phi_k = \tilde{\phi}_k e^{-i\lambda_k t/\hbar}, \]
and
\[ \int \tilde{\phi}_k^* \tilde{\phi}_j \sqrt{g} dq^0 \equiv (k|j) = \delta_{kj}. \]

Therefore, the eigenfunction of \( \hat{H}_0 \) is given by direct product of \( \chi \) and \( \phi \).
\[ \hat{H}_0 |M\rangle = (E_M + \lambda_k) |M\rangle |k\rangle. \]

Now we consider the perturbation theory for \( \hat{H}_I \). The precise calculations are given in appendix 2. The important point is that the transition of quantum number \( N, M, \cdots \) does not occur. (adiabatic approximation) Because in the limit of \( \epsilon \rightarrow 0 \), we have \( \Delta E \rightarrow \infty \) as well as \( E_N \rightarrow \infty \). Therefore our system has definite quantum number for \( \chi \) field. In this approximation our effective Hamiltonian can be expressed as
\[ \hat{H}_N = \langle N| \hat{H} |N \rangle \]
\[ = E_N - \frac{\hbar^2}{2m}[\Delta^{(2)} + V_0] + V \\
\[ - \frac{\hbar^2}{2m}[(N) q^0 (N) (V_1 + \hat{A}_1) + N] (q_0)^2 (N) (V_2 + \hat{A}_2)]. \]

The expectation value of \( q^0 \), and \((q^0)^2\) are
\[ \langle N| q^0 |N \rangle = 0, \]
\[ \langle N| (q^0)^2 |N \rangle = \frac{1}{12} \frac{6}{\pi^2 N^2} \epsilon^2 = \epsilon^2. \]

So we have an approximated Schrödinger equation
\[ i\hbar \frac{\partial \phi}{\partial t} = [-\frac{\hbar^2}{2m} \{\Delta^{(2)} + V_0} \\
\[ + (V_2 + \hat{A}_2)^2 \} + V + E_N\phi. \]

From this Schrödinger equation, we obtain
\[ - \frac{\partial}{\partial t} |\phi|^2 = \frac{\hbar}{2mi} \{(\phi^* \Delta^{(2)} \phi - \phi \Delta^{(2)} \phi^*) \\
\[ + \epsilon^2 (\phi^* \hat{A}_2 \phi - \phi \hat{A}_2 \phi^*) \}
\[ = \frac{\hbar}{2mi} \nabla_i \{ g^{ij} (3g^{ij} \kappa i \kappa_j \phi^* \partial_j \phi - \phi \partial_j \phi^*) \}
\[ = \nabla_i (J^i + J_G^i), \]

where
\[ J^i = \frac{\hbar}{2mi} \{ g^{ij} (\phi^* \partial_j \phi - \phi \partial_j \phi^*) \}, \]
\[ J_G^i = \frac{3\hbar \epsilon^2}{2mi} \kappa i \kappa_j \phi^* \partial_j \phi - \phi \partial_j \phi^*. \]

IV. CONCLUSION

We have discussed the conservation law in an effective two dimensional system between two curved surfaces \( \Sigma' \) and \( \Sigma \) with small distance \( \epsilon \). Then we find the anomalous flow depending on the curvature of surface \( \Sigma \). In classical diffusion process, we have
\[ J_G^i = -\frac{\epsilon^2 D}{12} \left[ (3\kappa^{im} \kappa^j_m - 2\kappa \delta^{ij}) \frac{\partial \phi^{(2)}}{\partial q^j} - \frac{1}{2} \frac{\partial}{\partial q^i} \frac{\partial R}{\partial q^j} \phi^{(2)} \right]. \]
In quantum process we obtain instead
\[ J'_Q = \frac{\hbar c^2}{8m} (1 - \frac{6}{\pi^2 N^2}) \kappa^{ik} \kappa^i_k (\phi^* \partial_j \phi - \phi \partial_j \phi^*), \]
where \( N \) is the quantum number defining the energy state of motion in \( q^0 \) direction.

The classical anomalous flow and quantum mechanical anomalous flow are somewhat similar. Both are starting from \( \mathcal{O}(c^2 \kappa^2) \) and proportional to the gradient of fields except the last term in classical flow. The usual two dimensional conservation law can be obtained at \( \epsilon \to 0 \).

V. APPENDIX

A. Geometrical Tools

From the definition of total metric \([11]\) and the definition of the Riemann Scalar \([10]\) we can construct other geometrical quantities.

\[ G = \det G_{ij} = g + 2g\kappa q^0 + g(\kappa^2 + R)(q^0)^2 + \mathcal{O}((q^0)^3). \]

The inverse metric for \( G_{ij} \) is given as

\[ G^{ij} = g^{ij} - 2\kappa^{ij} q^0 + \frac{3}{2}(2\kappa \kappa^{ij} - R g^{ij})(q^0)^2 + \mathcal{O}((q^0)^3). \]

Furthermore,
\[ (\kappa^{-1})^{ij} = \frac{2}{R} (\kappa g^{ij} - \kappa^{ij}), \]
and from this relation we have another relation

\[ \frac{1}{2} R g^{ij} = \kappa \kappa^{ij} - \kappa^i_m \kappa^m_j, \]

\[ R = \kappa^2 - \kappa^i_j \kappa^i_j = 2 \det(\kappa^i_j). \]

These are used to decompose the Laplace-Beltrami operator \([31]\).

B. Perturbation Theory

As is well known in the texts of Quantum Mechanics, By utilizing energy eigen value of \( \hat{H}_0 \): \( E_n \), and state vector \( |n\rangle \), corrected energy eigen value and state vector in first order perturbation are generally given as

\[ E'_n = E_n + \langle n | \hat{H}_I | n \rangle, \]
\[ |n\rangle' = |n\rangle + \sum_{k \neq n} \frac{\langle k | \hat{H}_I | n \rangle | k \rangle}{E_n - E_k}. \]

In our case we have two degrees of freedom \( \chi \) and \( \phi \). Then natural extension is given as

\[ E'_{N,k} = E_N + \lambda_k + \langle N | (k | \hat{H}_I | k \rangle | N \rangle, \]
\[ |N, k\rangle' = |N\rangle | k \rangle + \sum_{M \neq N,k} \frac{\langle M | (j | \hat{H}_I | k \rangle | N \rangle}{(E_N + \lambda_k) - (E_M + \lambda_j)} | M \rangle | j \rangle. \]

When \( M \neq N \), denominator of perturbation energy takes so large and such term can be ignored since in the case of \( \epsilon \to 0 \),

\[ E_{N+1} - E_N > \frac{\hbar^2 \pi^2}{m c^2} \to \infty. \]

That means we can ignore the change of quantum number \( N \) under the perturbation. Therefore we can consider the case only when \( M = N \).

\[ |N, k\rangle' = |N\rangle | k \rangle + \sum_{j \neq k} \frac{\langle N | (j | \hat{H}_I | k \rangle | N \rangle}{\lambda_k - \lambda_j} | N \rangle | j \rangle. \]

Our perturbation theory is given by \(67\) and \(70\). The Hamiltonian that leads to \(67\) and \(70\) directly is

\[ \hat{H}_N \equiv \langle N | \hat{H} | N \rangle = E_N - \frac{\hbar^2}{2m} [\Delta(2) + V_0] + V \]
\[ - \frac{\hbar^2}{2m} |(N | q^0 | N \rangle (V_1 + \hat{A}_1) \]
\[ + (N | (q^0)^2 | N \rangle (V_2 + \hat{A}_2)). \]
S. E. Alvarez, J. Chem. Phys. 140, (2014) 214115; T. Balois, C. Chatelain, M. B. Amar, J. Roy. Soc. Interface 11, (2014) 97.

[2] N. Ogawa, Phys. Rev. E81, 1(2010) 61113; N. Ogawa Phys. Lett. A377 (2013) 2465-2471.

[3] R. C. T. da Costa, Phys. Rev. 23 (1981) 1982; J. Tolar, 1988 Lecture Notes in Physics 313, ed. H. D. Doever, J. D. Henning and T. D. Raev, (Springer-Verlag, Berlin, Heidelberg) 268.

[4] N. Ogawa, K. Fujii, and K. P. Kobushkin, Prog. Theor. Phys. 83 (1990) 894; N. Ogawa, K. Fujii, N. M. Chepilko, and K. P. Kobushkin, Prog. Theor. Phys. 85 (1991) 1189; N. Ogawa, Prog. Theor. Phys. 87 (1992) 513.

[5] K. Fujii and N. Ogawa, Prog. Theor. Phys. 89 (1993) 575.