ISOSPECTRAL FINITENESS OF HYPERBOLIC ORBISURFACES

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Abstract. We discuss questions of isospectrality for hyperbolic orbisurfaces, examining the relationship between the geometry of an orbisurface and its Laplace spectrum. We show that certain hyperbolic orbisurfaces cannot be isospectral, where the obstructions involve the number of singular points and genera of our orbisurfaces. Using a version of the Selberg Trace Formula for hyperbolic orbisurfaces, we show that the Laplace spectrum determines the length spectrum and the orders of the singular points, up to finitely many possibilities. Conversely, knowledge of the length spectrum and the orders of the singular points determines the Laplace spectrum. This partial generalization of Huber’s theorem is used to prove that isospectral sets of hyperbolic orbisurfaces have finite cardinality, generalizing a result of McKea n [15] for Riemann surfaces.

1. Introduction

Historically, inverse spectral theory has been concerned with the relationship between the geometry and the spectrum of compact Riemannian manifolds, where “spectrum” means the eigenvalue spectrum of the Laplace operator as it acts on smooth functions on a manifold $M$. Since Milnor’s pair of isospectral (same eigenvalue spectrum) non-isometric flat tori in dimension 16, there has been much work to characterize the properties of smooth manifolds which are spectrally determined. Recently, this study has broadened to the category of orbifolds, which are a natural generalization of manifolds (see, e.g., [6], [12], [13], [20]). Instead of being locally modelled on $\mathbb{R}^n$, an orbifold is locally modelled on $\mathbb{R}^n$ modulo the action of a discrete group of isometries.

We will be interested in compact hyperbolic orbisurfaces, which are a natural generalization of compact hyperbolic Riemann surfaces. A hyperbolic orbisurface can be viewed as the quotient of the hyperbolic plane by a finite group of isometries which is permitted to include elliptic elements. These elliptic elements give rise to conical singularities in the quotient surface. We examine the relationship between the geometric properties of orbisurfaces and the eigenvalue spectrum of the Laplace operator as it acts on smooth functions on the orbisurface. For compact Riemann surfaces, Huber’s theorem is a powerful tool in studying this relationship. It says that the Laplace spectrum determines the length spectrum and vice versa, where the length spectrum is the sequence of lengths of all oriented closed geodesics in the surface, arranged in ascending order. The Selberg Trace Formula is used to derive this result. There has been recent interest in extending Huber’s theorem for Riemann surfaces to more general settings. A trace formula for discrete cocompact groups $\Gamma \subset PSL(2, \mathbb{C})$ is developed by Elstrodt et al. in [8]; they use the formula to find a natural generalization of Huber’s theorem. Parnovskii [16] states a trace

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formula in the case of a cocompact discrete group of isometries of hyperbolic space of arbitrary dimension; a generalized version of Huber’s theorem is stated without proof. In our extension of Huber’s theorem, we emphasize the geometric meaning of the various quantities involved in the Selberg Trace Formula in the orbifold setting. Our proof is completely different from that given in [8] for dimension three. We use our extended Huber’s theorem to show that isospectral sets of hyperbolic orbisurfaces have finite cardinality, generalizing a result of McKean [15] for Riemann surfaces.

The paper is organized as follows. We begin with the necessary definitions and background concerning orbifolds and the eigenvalue spectrum of the Laplace operator in the orbifold context. In section 3 we move into the study of the relationship between the geometry of an orbisurface and its Laplace spectrum. Using Weyl’s asymptotic formula for orbifolds as developed by Farsi [9], we are able to give obstructions to isospectrality of Riemann orbisurfaces; these obstructions involve the genera and number of singularities of our orbisurfaces. We show in section 4 that the Laplace spectrum determines the length spectrum and the orders of the singularities, up to finitely many possibilities. Conversely, knowledge of the length spectrum and the orders of the singular points determines the Laplace spectrum. In the final section, we use this theorem to show finiteness of isospectral sets of Riemann orbisurfaces.

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2. Preliminaries

On orbifolds, we can define coordinate charts which encode information about the local group actions at the singularities. We follow Satake [18] and Stanhope [20]. Additional references include Chapter 2 of [5], [19] and [21].

Definition 2.1. Let $X$ be a Hausdorff space, and let $U$ be an open set in $X$. An orbifold coordinate chart over $U$ is a triple $(\tilde{U}, \Gamma \backslash \tilde{U}, \pi)$ such that:

1. $\tilde{U}$ is a connected open subset of $\mathbb{R}^n$,
2. $\Gamma$ is a finite group of diffeomorphisms acting on $\tilde{U}$ with fixed point set of codimension at least two, and
3. $\pi : \tilde{U} \to U$ is a continuous map which induces a homeomorphism between $\Gamma \backslash \tilde{U}$ and $U$. We require $\pi \circ \gamma = \pi$ for all $\gamma \in \Gamma$.

Now suppose that $U$ and $U'$ are two open sets in a Hausdorff space $X$ with $U \subset U'$. Let $(U, \Gamma \backslash \tilde{U}, \pi)$ and $(U', \Gamma' \backslash \tilde{U}', \pi')$ be charts over $U$ and $U'$, respectively.

Definition 2.2. An injection $\lambda : (U, \Gamma \backslash \tilde{U}, \pi) \hookrightarrow (U', \Gamma' \backslash \tilde{U}', \pi')$ consists of an open embedding $\lambda : \tilde{U} \hookrightarrow \tilde{U}'$ such that $\pi = \pi' \circ \lambda$. 
For any $\gamma \in \Gamma$, there exists a unique $\gamma' \in \Gamma'$ for which $\lambda \circ \gamma = \gamma' \circ \lambda$. The correspondence $\gamma \mapsto \gamma'$ defines an injective homomorphism of groups from $\Gamma$ into $\Gamma'$.

**Definition 2.3.** A smooth orbifold $(X, A)$ consists of a Hausdorff space $X$ together with an atlas of charts $A$ satisfying the following conditions:

1. For any pair of charts $(U, \Gamma \backslash \tilde{U}, \pi)$ and $(U', \Gamma' \backslash \tilde{U}', \pi')$ in $A$ with $U \subset U'$ there exists an injection $\lambda : (U, \Gamma \backslash \tilde{U}, \pi) \hookrightarrow (U', \Gamma' \backslash \tilde{U}', \pi')$.
2. The open sets $U \subset X$ for which there exists a chart $(U, \Gamma \backslash \tilde{U}, \pi)$ in $A$ form a basis of open sets in $X$.

Given an orbifold $(X, A)$, we call the topological space $X$ the underlying space of the orbifold. Henceforth orbifolds $(X, A)$ will be denoted simply by $O$. We now give some examples of orbifolds.

1. Let $\Gamma$ be a group acting properly discontinuously on a manifold $M$ with fixed point set of codimension at least two. Then the quotient space $O = \Gamma \backslash M$ is an orbifold. Since $O$ can be expressed as a global quotient (that is, as a subset of $\mathbb{R}^n$ modulo the action of a discrete group), it is called a good or global orbifold. If $M$ is a surface, then $O$ is an orbisurface.

2. Consider the orbisurface whose underlying space is the sphere $S^2$, and which has one singular point. A neighborhood of this singular point is modelled on $\mathbb{Z}_n \backslash \mathbb{R}^2$, where $\mathbb{Z}_n$ is the group of rotations of order $n$. Such a singular point is called a cone point of order $n$, and this orbisurface is known as the $\mathbb{Z}_n$-teardrop. Unlike the previous example, the $\mathbb{Z}_n$-teardrop cannot be expressed as a quotient with respect to the action of a discrete group, and thus is an example of a bad orbifold (see [19]).

We will be interested in orbifolds which have a hyperbolic structure. The construction of a Riemannian metric on an orbifold is as in the manifold case, with the metric being defined locally via coordinate charts and patched together using a partition of unity. In addition, the metric must be invariant under the local group actions. A smooth orbifold with a Riemannian metric is a Riemannian orbifold. An orbisurface with a hyperbolic metric of constant curvature -1 will be called a Riemann orbisurface. Every Riemann orbisurface arises as a global quotient of the hyperbolic plane by a discrete group of isometries (see [19]).

An orbifold $O$ is said to be locally orientable if it has an atlas in which every coordinate chart $(U, \Gamma \backslash \tilde{U}, \pi)$ is such that $\Gamma$ is an orientation-preserving group. If all injections as in Definition 2.2 are induced by orientation-preserving maps, then $O$ is orientable.

Our goal is to study the spectrum of the Laplace operator as it acts on smooth functions on a compact Riemannian orbifold $O$. A map $f : O \to \mathbb{R}$ is a smooth function on $O$ if for every coordinate chart $(U, \Gamma \backslash \tilde{U}, \pi)$ on $O$, the lifted function $\tilde{f} = f \circ \pi$ is a smooth function on $\tilde{U}$. If $O$ is a compact Riemannian orbifold and $f$ is a smooth function on $O$, then we define the Laplacian $\Delta f$ of $f$ by lifting $f$ to $\tilde{f}$. We denote the $\Gamma$-invariant metric on $\tilde{U}$ by $g_{ij}$ and set $\rho = \sqrt{\det(g_{ij})}$. Then we can define the Laplacian locally by

$$\Delta \tilde{f} = \frac{1}{\rho} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (g^{ij} \frac{\partial f}{\partial x^i} \rho).$$
We are really interested in the eigenvalues of the Laplace operator as it acts on smooth functions. In analogy with the manifold case, Chiang proved the following theorem:

**Theorem 2.4.** Let $O$ be a compact Riemannian orbifold.

1. The set of eigenvalues $\lambda$ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on $O$, denoted $\text{Spec}(O)$.
2. Each eigenvalue $\lambda_i$ has finite multiplicity.
3. There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta \phi_i = \lambda_i \phi_i$.

The multiplicity of the $i$th eigenvalue $\lambda_i$ is the dimension of the space of eigenfunctions with eigenvalue $\lambda_i$.

3. **Obstructions to Isospectrality**

Much information about the relationship between the Laplace spectrum of an orbifold $O$ and the geometric properties of $O$ can be gleaned by studying the heat equation:

$$\Delta F = -\frac{\partial F}{\partial t},$$

where $F(x, t)$ is the heat at a point $x \in O$ at time $t$. With initial data $f : O \to \mathbb{R}$, $F(x, 0) = f(x)$, a solution of the heat equation is given by

$$F(x) = \int_O K(x, y, t)f(y)dy.$$  

Here $K : O \times O \times \mathbb{R}_+^* \to \mathbb{R}$ is a $C^\infty$ function given by the convergent series

$$K(x, y, t) = \sum_i e^{-\lambda_i t} \phi_i(x)\phi_i(y).$$  

The eigenfunctions $\phi_i$ of $\Delta$ are chosen such that they form an orthonormal basis of $L^2(O)$, the square-integrable functions on $O$. We say that $K$ is the fundamental solution of the heat equation on $O$, or the heat kernel on $O$. The appropriate physical interpretation is that $K(x, y, t)$ is the temperature at time $t$ at the point $y$ when a unit of heat (a Dirac delta-function) is placed at the point $x$ at time $t = 0$.

By considering the asymptotic behavior of $K$ as $t \to 0$, we can recover information about the geometry of $O$. In this direction, Farsi showed (see [9]) that Weyl’s asymptotic formula can be extended to orbifolds. In particular, she proved

**Theorem 3.1.** Let $O$ be a closed orientable Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_i \leq \lambda} 1$ we have

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O)\frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$. Here $B_0^n(1)$ denotes the $n$-dimensional unit ball in Euclidean space, and $n$ is the dimension of $O$.

This theorem implies that, in analogy with the manifold case, the Laplace spectrum determines an orbifold’s dimension and volume.

By looking at the terms of the asymptotic expansion of the trace of the heat kernel, Gordon et al. [12] have given the following obstruction to isospectrality:
Theorem 3.2. Let $O$ be a Riemannian orbifold with singularities. If $M$ is a manifold such that $O$ and $M$ have a common Riemannian cover, then $M$ and $O$ cannot be isospectral.

In particular, this implies that a hyperbolic orbifold with singularities is never isospectral to a hyperbolic manifold.

We want to investigate further obstructions to isospectrality; our focus will be the case of orbisurfaces. In analogy with the surface case, we can define the Euler characteristic and state a Gauss-Bonnet theorem for orbisurfaces (see [21]).

Definition 3.3. Let $O$ be an orbisurface with $s$ cone points of orders $m_1, \ldots, m_s$. Then we define the (orbifold) Euler characteristic of $O$ to be

$$
\chi(O) = \chi(X_0) - \sum_{j=1}^{s} \left(1 - \frac{1}{m_j}\right),
$$

where $\chi(X_0)$ is the Euler characteristic of the underlying space of $O$.

The Gauss-Bonnet theorem for orbisurfaces gives the usual relationship between topology and geometry:

Theorem 3.4. Let $O$ be a Riemannian orbisurface. Then

$$
\int_{O} KdA = 2\pi \chi(O),
$$

where $K$ is the curvature and $\chi(O)$ is the orbifold Euler characteristic of $O$.

Note that we define the curvature of an orbifold $O$ at a point $x \in O$ with coordinate chart $(U, \Gamma \backslash \tilde{U}, \pi)$ to be the curvature at a lift $\tilde{x} \in \tilde{U}$ of $x$.

Combining the Gauss-Bonnet theorem with Weyl’s asymptotic formula, we see that for an orbisurface with given curvature, the spectrum determines the orbifold Euler characteristic. However, since the orbifold Euler characteristic involves both the genus of the underlying surface and the orders of the cone points in the orbisurface, it is not immediately clear that the spectrum determines the genus. This is still an open question.

In the case of orientable orbisurfaces, Gordon et al. [12] have shown that the Euler characteristic can be recovered from the asymptotic expansion of the trace of the heat kernel. Together with some computations for cone points, this allows them to define a spectral invariant which determines whether an orbifold is a football (underlying space $S^2$, cone points at the north and south poles) or teardrop and determines the orders of the cone points. In a similar vein, we give the following obstructions to isospectrality.

Proposition 3.5. Fix $g \geq 1$ and $m \geq 2$. Let $O$ be a compact orientable Riemann orbisurface of genus $g$ with exactly one cone point of order $m$. Let $O'$ be in the class of compact orientable Riemann orbisurfaces of genus $g$, and suppose that $O$ is isospectral to $O'$. Then $O'$ must have exactly one cone point, and its order is also $m$.

Proof. By Theorem 3.2, $O'$ must contain at least one cone point. We have $\chi(X_0) = \chi(X_{O'})$ by hypothesis, and the observation following Theorem 3.4 implies that $\chi(O) = \chi(O')$. 
Suppose that \( O' \) has one cone point of order \( n_1 \). It follows that
\[
\frac{1}{m} = \frac{1}{n_1},
\]
or \( m = n_1 \). Now suppose that \( O' \) has two cone points of orders \( n_1 \) and \( n_2 \). Then
\[
\frac{1}{m} + 1 = \frac{1}{n_1} + \frac{1}{n_2}.
\]
But \( n_i \geq 2 \) for \( i = 1, 2 \), so \( \frac{1}{n_1} + \frac{1}{n_2} \leq 1 \). This is a contradiction, hence \( O \) and \( O' \) are not isospectral. This argument is easily extended to the case when \( O' \) is assumed to have more than two cone points. \( \square \)

We can extend Proposition 3.5 to the case of two orbisurfaces with different underlying spaces.

**Proposition 3.6.** Let \( O \) be a compact orientable Riemann orbisurface of genus \( g_0 \geq 0 \) with \( k \) cone points of orders \( m_1, \ldots, m_k \), where \( m_i \geq 2 \) for \( i = 1, \ldots, k \). Let \( O' \) be a compact orientable Riemann orbisurface of genus \( g_1 \geq g_0 \) with \( l \) cone points of orders \( n_1, \ldots, n_l \), where \( n_j \geq 2 \) for \( j = 1, \ldots, l \). Let \( h = 2(g_0 - g_1) \). If \( l \geq 2(k + h) \), then \( O \) is not isospectral to \( O' \).

**Proof.** Suppose \( O \) is isospectral to \( O' \). As in the preceding proof, we have that \( O' \) must contain at least one cone point. Also, \( \chi(O) = \chi(O') \), i.e.
\[
2 - 2g_0 - k + \frac{1}{m_1} + \cdots + \frac{1}{m_k} = 2 - 2g_1 - l + \frac{1}{n_1} + \cdots + \frac{1}{n_l}.
\]
So
\[
\frac{1}{m_1} + \cdots + \frac{1}{m_k} = 2(g_0 - g_1) + k - l + \frac{1}{n_1} + \cdots + \frac{1}{n_l}
\]
or equivalently
\[
\frac{1}{m_1} + \cdots + \frac{1}{m_k} \leq h + k - \frac{l}{2}.
\]
But \( k + h \leq \frac{l}{2} \) by hypothesis, which gives the desired contradiction. \( \square \)

If we assume that \( h = 0 \), i.e. that \( g_0 = g_1 \), then we have the following special case of Proposition 3.6.

**Corollary 3.7.** Fix \( g \geq 0 \). Let \( O \) be a compact orientable Riemann orbisurface of genus \( g \) with \( k \) cone points of orders \( m_1, \ldots, m_k \), \( m_i \geq 2 \) for \( i = 1, \ldots, k \). Let \( O' \) be a compact orientable Riemann orbisurface of genus \( g \) with \( l \geq 2k \) cone points of orders \( n_1, \ldots, n_l \), \( n_j \geq 2 \) for \( j = 1, \ldots, l \). Then \( O \) is not isospectral to \( O' \).

Note that in all of the above results, we have the hypothesis that \( O' \) is hyperbolic. In the usual case of surfaces, we know that any surface isospectral to a given one of fixed constant curvature must have the same constant curvature. The proof of this uses the asymptotic expansion of the trace of the heat kernel; in the case of orbisurfaces, this expansion is more complicated and it is no longer clear that fixed constant curvature is a spectral invariant.
4. Huber’s Theorem

Huber’s theorem is a powerful tool in the study of questions of isospectrality of compact Riemann surfaces. It allows us to translate information about eigenvalues into information about the geometry of the surface, and specifically about the lengths of closed geodesics on the surface (see [2]).

Theorem 4.1. (Huber) Two compact Riemann surfaces of genus \( g \geq 2 \) have the same spectrum of the Laplacian if and only if they have the same length spectrum.

The length spectrum is the sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order.

The idea of the proof is as follows. First, a fundamental domain argument leads to a length trace formula. The known eigenfunction expansion of the heat kernel as given in [1] is then plugged into this length trace formula to obtain the Selberg Trace Formula. The Selberg Trace Formula contains information about the eigenvalues on one side and information about the lengths of closed geodesics and the area of the Riemann surface on the other; it is then a matter of showing that the eigenvalues determine the lengths of the closed geodesics and vice versa. Further background and the case of the Selberg Trace Formula for compact hyperbolic manifolds can be found in Randol (in [3]). Other sources for the Riemann surface case include [2], [14] and [17].

We want to extend Huber’s theorem to the class of compact orientable Riemann orbisurfaces. To begin, we need to exhibit a Selberg Trace Formula for such objects. We can state this formula in terms of a function \( h(r) \) and its Fourier transform \( g(u) \), which are not required to have compact support; namely, \( h(r) \) satisfies the following (weaker) conditions:

Assumption 4.2.

- \( h(r) \) is an analytic function on \( |Im(r)| \leq \frac{1}{2} + \delta \);
- \( h(-r) = h(r) \);
- \( |h(r)| \leq M[1 + |Re(r)|]^{-2-\delta} \).

The numbers \( \delta \) and \( M \) are some positive constants.

Hejhal [14] obtains the following version of the Selberg Trace Formula for the case of interest:

Theorem 4.3. Suppose that

- \( \Gamma \subset PSL(2, \mathbb{R}) \) is a Fuchsian group with compact fundamental region;
- \( h(r) \) satisfies Assumption 4.2;
- \( \{\phi_n\}_{n=0}^{\infty} \) is an orthonormal eigenfunction basis for \( L^2(\Gamma\backslash \mathbb{H}) \).

Then

\[
\sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} rh(r) \tanh(\pi r) dr \\
+ \sum_{\{R\}_{\text{elliptic}}} \frac{1}{2m(R) \sin \theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1 + e^{-2\pi r}} h(r) dr \\
+ \sum_{\{P\}_{\text{hyperbolic}}} \frac{\ln N(P_s)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)].
\]
where all the sums and integrals in sight are absolutely convergent.

Note that the left side of (2) encodes information about the eigenvalues of the Laplacian, as $\lambda_n = -\frac{1}{4} - r^2_n$. (For easier reference, we follow Hejhal's convention of nonpositive eigenvalues for this section.) The first and last terms on the right side of (2) match the terms which appear in the Selberg Trace Formula for compact Riemann surfaces, where $\mu(F)$ is the area of a fundamental domain for $\Gamma$ and the last term is the sum over hyperbolic conjugacy classes. Every hyperbolic element $P$ in $PSL(2, \mathbb{R})$ is conjugate to a dilation $z \mapsto m^2 z, m > 1$; we call $m^2$ the norm of $P$ and denote it by $N(P)$. A primitive hyperbolic element is one which cannot be written as a nontrivial power of another hyperbolic element. In (2), $P_c$ is a primitive hyperbolic element with $P = P_c^k$ for some $k \geq 1$. Note that [14, Prop. 2.3]

$$\inf_{z \in \mathbb{H}} d(z, Tz) = \ln N(T),$$

where $d$ is the distance on $\mathbb{H}$. The infimum is realized by all points $z$ which lie on the geodesic in $\mathbb{H}$ which is invariant under the action of $T$. In particular, we allow geodesics on $\mathcal{O} = \Gamma \backslash \mathbb{H}$ to pass through cone points, and include the lengths of such geodesics in the length spectrum of $\mathcal{O}$. In the sum over elliptic conjugacy classes, $m(R)$ denotes the order of the centralizer (in $\Gamma$) of a representative $R$ and $\theta(R)$ is half the angle of rotation. We have $Tr(R) = 2 \cos \theta(R)$, and $0 < \theta < \pi$.

In the proof of our partial extension of Huber’s theorem, we will need the following result of Stanhope [20].

**Theorem 4.4.** Let $S$ be a collection of isospectral orientable compact Riemannian orbifolds that share a uniform lower bound $\kappa(n-1)$, $\kappa$ real, on Ricci curvature. Then there are only finitely many possible isotropy types, up to isomorphism, for points in an orbifold in $S$.

We are now prepared to state our partial extension of Huber’s theorem to the setting of compact orientable Riemann orbisurfaces.

**Theorem 4.5.** If two compact orientable Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra and the orders of their cone points, up to finitely many possibilities. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.

**Proof.** We will consider Theorem 4.3 for a specific function $h(r)$. Fix $t > 0$ and let $h(r) = e^{-r^2t}$. Then $h(r)$ satisfies Assumption 4.2 and we have

$$g(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r)e^{-iru}dr$$

$$= \frac{1}{\sqrt{4\pi t}} e^{-u^2/4t}$$

where the first line is the definition of $g(u)$ as the Fourier transform of $h(r)$ and the second line follows from Fourier analysis using a standard polar coordinates trick.
By Theorem 3.3 we have

\[
\sum_{n=0}^{\infty} e^{-\gamma_n^2 t} = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r e^{-r^2 t} \tanh(\pi r) dr \\
+ \sum_{(\ell)} \frac{1}{2m(\ell) \sin \theta(\ell)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(\ell)r}}{1+e^{-2\pi r}} e^{-r^2} dr \\
+ \sum_{(P)} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} \frac{1}{\sqrt{4\pi t}} e^{-(\ln N(P))^2/4t}. \tag{3}
\]

Let \( O \) and \( O' \) be compact orientable Riemann orbisurfaces with the same Laplace spectrum. By Theorem 3.1 the Laplace spectrum determines an orbifold’s volume. So we must have \( \text{vol}(O) = \text{vol}(O') \), and thus the first term on the right side of (3) must be the same for \( O \) and \( O' \).

Note that both \( O \) and \( O' \) have a metric of constant curvature -1, and hence share a uniform lower bound on their Ricci curvature. By Theorem 4.4 we know that there are only finitely many possible isotropy types, up to isomorphism, for points in \( O \) or \( O' \). It is well-known (e.g. [10, p. 16]) that there are only finitely many cone points in \( O \) or \( O' \), up to finitely many possible lists of lengths.

Now suppose we know the length spectrum and the orders of the cone points for \( O \) or \( O' \). The argument that we then know the Laplace spectrum of \( O \) or \( O' \) is as for Riemann surfaces (see [17, p. 45]). We include it for completeness.

Consider the function \( f(t) e^{t^2/4t} \). Take the limit of this function as \( t \downarrow 0 \); we see that there is a unique \( \omega > 0 \) for which this limit is finite and nonzero. Let \( \gamma_1 \) be the shortest primitive closed geodesic in \( O \). Then \( \omega = \ell(\gamma_1) \). We remove the contribution of \( \gamma_1 \) and all its powers from \( f(t) \), and proceed as above to find the length of the next-shortest primitive closed geodesic. In this way, we can determine the lengths of the hyperbolic elements in \( O \) or \( O' \), up to finitely many possible lists of lengths.

Finally, we multiply both sides of (3) by \( e^{-t/4} \) and recall that \( \lambda_n = -\frac{1}{4} - r_n^2 \) to obtain

\[
\sum_{n=0}^{\infty} e^{\lambda_n t} = \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} r e^{-r^2 t} \tanh(\pi r) dr \\
+ \sum_{(\ell)} \frac{e^{-t/4}}{2m(\ell) \sin \theta(\ell)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(\ell)r}}{1+e^{-2\pi r}} e^{-r^2} dr \\
+ \sum_{(P)} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} \frac{e^{-t/4}}{\sqrt{4\pi t}} e^{-(\ln N(P))^2/4t}. \tag{4}
\]
Knowledge of the length spectrum and the orders of the cone points in \( \mathbb{H} \) translates to knowledge of the function
\[
c(t) = \sum_{n=0}^{\infty} e^{\lambda_n t} - \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} re^{-r^2 t} \tanh(\pi r) dr
\]
\[
= \sum_{-\frac{1}{4} \leq \lambda_n < 0} e^{\lambda_n t} - \frac{\mu(F)}{4\pi} e^{-t/4} \int_{-\infty}^{\infty} re^{-r^2 t} \tanh(\pi r) dr + \sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t}
\]
where
\[
\sigma(t) = \frac{1}{2\pi} \int_{0}^{\infty} re^{-r^2 t} \tanh(\pi r) dr.
\]
It is a straightforward but tedious calculation to show that as \( t \to \infty \), \( \sigma(t) \to 0 \) (see [7]).

If \( \lambda_1 \geq -\frac{1}{4} \), then \(-\lambda_1\) is the unique \( \omega > 0 \) such that
\[
0 < \lim_{t \to \infty} e^{\omega t} c(t) < \infty.
\]
In fact, this limit is the multiplicity \( m_1 \) of \( \lambda_1 \). We can therefore subtract \( m_1 e^{\lambda_1 t} \) from \( c(t) \) and continue in this way to find all the small eigenvalues. Once all the small eigenvalues have been found, the function
\[
\tilde{c}(t) = -\sigma(t) e^{-t/4} \mu(F) + \sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t}
\]
has the property that for \( \omega > 0 \),
\[
\lim_{t \to \infty} e^{\omega t} \tilde{c}(t)
\]
is 0 or \( \infty \). So we can now multiply \( \tilde{c}(t) \) by \( -\frac{e^{t/4}}{\sigma(t)} \) and take the limit as \( t \to \infty \) to get \( \mu(F) \). We then know the function
\[
\sum_{\lambda_n < -\frac{1}{4}} e^{\lambda_n t},
\]
and we can determine the remaining eigenvalues in the same way as we found the small eigenvalues. Hence the spectrum of the Laplacian is determined by the length spectrum and the orders of the cone points. \( \square \)

5. Finiteness of Isospectral Sets

McKean [15] showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of compact orientable Riemann orbisurfaces. Specifically, we show

**Theorem 5.1.** Let \( O \) be a compact orientable Riemann orbisurface of genus \( g \geq 1 \). In the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to \( O \).

**Remark 5.2.** Note that there is no need for a dimension restriction on the orbifolds that can be isospectral to \( O \), by Theorem 3.1. In addition, by Theorem 3.2, there can be no Riemann surfaces isospectral to \( O \).
McKean [15] states the following proposition, which he attributes to Fricke and Klein [11]. For the sake of accuracy, we give a complete proof below.

**Proposition 5.3.** Let $M = G \setminus \mathbb{H}$ be a Riemann surface of genus $g \geq 2$, where $G \leq SL(2, \mathbb{R})$. Let the set $\{h_1, \ldots, h_n\}$, $n \leq 2g$, be a generating set for $G$. Then the single, double, and triple traces
\[
tr(h_i),
\]
\[
tr(h_i h_j), \quad i < j
\]
\[
tr(h_i h_j h_k), \quad i < j < k
\]
determine $G$ up to a conjugation in $PSL(2, \mathbb{R})$ or a reflection.

**Proof.** Let $G$ and $G'$ be two subgroups of $SL(2, \mathbb{R})$ with the same single, double, and triple traces of their generators. Fix $h_1 \in G$. Since the single traces of the generators of $G$ and $G'$ are equal, we can pair $h_1$ with an element in $G'$ that translates the same amount; that is, we can suppose that $h_1 = h'_1$ and that $h_1(z) = m^2 z$ with $m > 1$. Note that any other diagonal element in $G$ fixes the same geodesic in $\mathbb{H}$ as $h_1$ and is thus a multiple of $h_1$. So we can assume that $h_1$ is the only diagonal element in $\{h_1, \ldots, h_n\}$. For $i > 1$, we have
\[
tr(h_i) = a_i + d_i = tr(h'_i) = a'_i + d'_i.
\]

Also,
\[
tr(h_1 h_i) = tr \left( \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right) = tr \left( \begin{pmatrix} ma_i & mb_i \\ m^{-1} c_i & m^{-1} d_i \end{pmatrix} \right) = ma_i + m^{-1} d_i
\]
and similarly for $tr(h'_1 h'_i)$, so that
\[
ma_i + m^{-1} d_i = tr(h_1 h_i) = tr(h'_1 h'_i) = ma'_i + m^{-1} d'_i,
\]
or equivalently
\[
m(a_i - a'_i) = m^{-1}(d'_i - d_i).
\]
From equation (6) we see that $a_i - a'_i = d'_i - d_i$. But we assumed $m > 1$, so we must have
\[
a_i = a'_i \quad \text{and} \quad d_i = d'_i.
\]
We also know that $\det(h_i) = \det(h'_i) = 1$ for all $i$, so
\[
b_i c_i = b'_i c'_i
\]
for all $i$. Straightforward calculations show that
\[
tr(h_i h_j) - tr(h'_i h'_j) = b_i c_j - b'_i c'_j + c_i b_j - c'_i b'_j
\]
and
\[
tr(h_1 h_i h_j) - tr(h'_1 h'_i h'_j) = m(b_i c_j - b'_i c'_j) + m^{-1}(c_i b_j - c'_i b'_j)
\]
for $1 < i < j$. Combining equations (9) and (10) as we combined (5) and (6), we see that $b_i c_j = b'_i c'_j$ for $1 < i < j$. We want to see that none of these numbers are zero. Suppose $c_2 = 0$. Then
\[
h_i^{-n} h_2 h_i^n(\sqrt{-1}) = \frac{a_2 \sqrt{-1} + m^{-2n} b_2}{d_2},
\]
where this is the Möbius action of $SL(2, \mathbb{R})$ on $\mathbb{H}$. So we get infinitely many images of $\sqrt{-1}$ accumulating at $\frac{a_2}{d_2} \sqrt{-1} \in \mathbb{H}$ (unless $b_2 = 0$, which implies that $h_2$ is diagonal, contradicting our assumption that $h_1$ is the only diagonal element in
This contradicts the assumption that $G$ acts properly discontinuously on $\mathbb{H}$. A similar argument with $b_1 = 0$ shows that the off-diagonal entries in the matrix representing the element $h_2$ are nonzero. Our choice of $h_2$ was arbitrary, thus none of the off-diagonal entries in the matrices representing the elements $h_2, \ldots, h_n$ and $h_2', \ldots, h_n'$ are zero. We have

$$\frac{c_j'}{c_j} = \frac{b_i}{b_i'} = \frac{c_i'}{c_i},$$

where the second equality is equation (8), and this common ratio is independent of $i > 1$. Since the traces do not tell us whether the ratio is positive, we must allow the reflection

$$G \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} G \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Suppose that the common ratio is equal to $t^2$, i.e.

$$\frac{b_i}{b_i'} = t^2 = \frac{c_i'}{c_i}$$

for all $i > 1$. Then $b_i = t^2 b_i'$ and $c_i = t^{-2} c_i'$ for $i > 1$. Thus

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} a_i & t^2 b_i' \\ t^{-2} c_i' & d_i \end{pmatrix}$$

for all $i > 1$. We saw that $a_i = a_i'$ and $d_i = d_i'$ for all $i > 1$, so

$$h_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} = \begin{pmatrix} a_i' & t^2 b_i' \\ t^{-2} c_i' & d_i' \end{pmatrix} = s h_i' s^{-1}$$

for $s \in SL(2, \mathbb{R})$ given by $s : z \mapsto t^2 z$. Thus there exists $s \in SL(2, \mathbb{R})$ which, for all $i$, conjugates $h_i$ to $h_i'$. Hence $G$ and $G'$ are the same group up to conjugation in $PSL(2, \mathbb{R})$ or a reflection.

Note that we can easily extend this result to the case of a group $G$ which is the fundamental group of a compact orientable Riemann orbisurface of genus $g \geq 1$. We know that any such group contains a hyperbolic element; without loss of generality, label this element $h_1$. Then the calculations which show that equation (7) holds are still valid, as are the calculations which show that $b_i c_j = b_i' c_j'$ for $i \neq j$. An elliptic element $R$ in $PSL(2, \mathbb{R})$ is conjugate to an element of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

for $0 < \theta < \pi$; a straightforward but messy calculation shows that the off-diagonal entries of $R$ are zero only if $\theta = 0, \pi$. So the remainder of the argument holds in the desired setting.

To prove Theorem 5.1 we will need the following result of Stanhope [20] which gives an upper bound on the diameter of an orbifold:

**Proposition 5.4.** Let $O$ be a compact Riemannian orbifold with Ricci curvature bounded below by $\kappa(n-1)$, $\kappa$ real. Fix an arbitrary constant $r$ greater than zero. Then the number of disjoint balls of radius $r$ that can be placed in $O$ is bounded above by a number that depends only on $\kappa$ and the number of eigenvalues of $O$ less than or equal to $\lambda_1^n(r)$. In particular the diameter of $O$ is bounded above by a number that depends only on $\text{Spec}(O)$ and $\kappa$. 

So isospectral families of orbifolds with a common uniform lower Ricci curvature bound also have a common upper diameter bound.

In Beardon [1], a careful study of the geometry of discrete groups is undertaken. In particular, he looks at Fuchsian groups, which may be considered as discrete groups of isometries of the hyperbolic plane. To every Fuchsian group \( G \), we can associate its Dirichlet polygon; a Dirichlet polygon centered at a point \( w \in \mathbb{H} \) is the set of all points which are, among all their images under the action of \( G \), closest to \( w \). Beardon proves the following theorem about such a polygon \( P \) [1 Thm. 9.3.3]:

**Theorem 5.5.** The set of side-pairing elements \( G^* \) of \( P \) generate \( G \).

Thus every Fuchsian group \( G \) is generated by the side-pairing elements of a Dirichlet polygon associated to it. Note that we can regard the elliptic fixed points of \( G \) as vertices of \( P \).

We are now ready to prove Theorem 5.1.

**Proof of Theorem 5.1.** Let \( C \) be the class of compact orientable hyperbolic orbifolds, and let \( S \) denote the subclass of \( C \) containing those orbifolds which are isospectral to \( O \). Note that any member of \( S \) is a compact orientable Riemann orbisurface, and that it is determined by its fundamental group \( \Gamma \). By Proposition 5.3, specifying \( \Gamma \) (up to a reflection or conjugation) is the same as specifying the single, double and triple traces of a set of generators. We want to show that the spectrum determines finitely many possibilities for these traces, thus only finitely many choices for \( \Gamma \), and hence that \( S \) is a finite set.

Theorem 4.5 tells us that the Laplace spectrum determines (up to finitely many possibilities) the length \( \ell(Q) \) of a shortest closed path in the free homotopy class associated to a given hyperbolic conjugacy class \( Q \) in \( \Gamma \). The relation between the trace of \( Q \) and \( \ell(Q) \) is given by:

\[
\text{tr}(Q) = \pm 2 \cosh \frac{1}{2} \ell(Q).
\]

Note that \( \frac{1}{2} \ell(Q) \) is bounded by \( D \), the diameter of \( \Gamma \backslash \mathbb{H} \). Thus the single traces of the hyperbolic conjugacy classes are bounded by \( 2 \cosh D \). By choosing our generating set to be the set of side-pairing elements of a Dirichlet polygon \( P \) for \( \Gamma \), we can easily bound the traces of hyperbolic conjugacy classes which arise as a product of two or three generators. We fix a point \( p \in P \) and determine an upper bound for \( \text{dist}(p, g_2 \circ g_1(p)) \), where \( g_1 \) and \( g_2 \) are side-pairing elements of \( P \) and \( g_2 \circ g_1 \) is hyperbolic. We have

\[
\text{dist}(p, g_2 \circ g_1(p)) \leq \text{dist}(p, g_1(p)) + \text{dist}(g_1(p), g_2 \circ g_1(p))
\]

and each term on the right side is bounded by \( 2D \). Thus

\[
\text{tr}(g_2 \circ g_1) \leq 2 \cosh 2D.
\]

A similar argument shows that the trace of the product of three side-pairing elements which is hyperbolic is bounded by \( 2 \cosh 3D \).

By Proposition 5.3, there is a common upper bound on the diameter of any orbisurface in \( S \). So there are only finitely many possibilities for the trace of a hyperbolic element which arises as a product of one, two or three side-pairing elements of \( P \).

Finally, we need to consider the case in which a side-pairing element of \( P \) (or a product of such elements) is elliptic. Beardon [1 p.225] notes that \( P \) contains...
representatives of all conjugacy classes of elliptic elements in $\Gamma$. But we know that $\Gamma \setminus \mathbb{H}$ contains only finitely many cone points, and Theorem 4.4 tells us that there can be only finitely many possible isotropy types for the points in $\Gamma \setminus \mathbb{H}$. So there are only finitely many choices for the trace of any elliptic element in $\Gamma$; this implies that if a product of generators of $\Gamma$ is elliptic, there are only finitely many choices for the trace of such a product.

Thus we have shown that there are only finitely many ways to choose the generators of $\Gamma$, up to a reflection or conjugation. □

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