Classical and Quantum Fermions Linked by an Algebraic Deformation

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Abstract

We study the regular representation $\rho_\zeta$ of the single-fermion algebra $A_\zeta$, i.e., $c^2 = c^{+2} = 0$, $cc^+ + c^+ c = \zeta \, 1$, for $\zeta \in [0,1]$. We show that $\rho_0$ is a four-dimensional nonunitary representation of $A_0$ which is faithfully irreducible (it does not admit a proper faithful subrepresentation). Moreover, $\rho_0$ is the minimal faithfully irreducible representation of $A_0$ in the sense that every faithful representation of $A_0$ has a subrepresentation that is equivalent to $\rho_0$. We therefore identify a classical fermion with $\rho_0$ and view its quantization as the deformation: $\zeta : 0 \rightarrow 1$ of $\rho_\zeta$. The latter has the effect of mapping $\rho_0$ into the four-dimensional, unitary, (faithfully) reducible representation $\rho_1$ of $A_1$ that is precisely the representation associated with a Dirac fermion.

1 Introduction

The description of fermions in terms of the Clifford algebra relations

$$cc^+ + c^+ c = 1,$$

$$c^2 = c^{+2} = 0,$$

$\zeta \in [0,1]$.
dates back to early days of quantum physics. This algebra may be obtained by quantizing a classical system with fermionic variables, e.g., a free fermion or a fermionic oscillator. The classical fermionic variables satisfy the Grassmann algebra relations

\[ cc^+ + c^+ c = 0, \]
\[ c^2 = c^{+2} = 0. \]

Therefore similarly to the case of bosonic variables, the quantization of a fermionic variable may be viewed as the deformation of the algebraic relations

\[ cc^+ + c^+ c = \zeta 1, \]
\[ c^2 = c^{+2} = 0, \]

where the deformation parameter \( \zeta \) takes values in \([0, 1]\). Motivated by the method used in [3] to study the representation theory of orthofermions, we investigate in this paper the effect of the deformation \( \zeta \to 0 \) on the representations of the associative algebra \( \mathcal{A}_\zeta \) generated by 1, c, and \( c^+ \) and subject to relations (5) and (6).

It is well-known [4, 3] that the representations of the Clifford algebra \( \mathcal{A}_1 \) are, up to equivalence, direct sums of copies of the trivial representation \( \rho_{\text{trivial}} \):

\[ \rho_{\text{trivial}}(1) = \rho_{\text{trivial}}(c^+) = \rho_{\text{trivial}}(c) = 0, \]

and the two-dimensional unitary (or *) representation \( \rho_\ast \):

\[ \rho_\ast(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_\ast(c) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho_\ast(c^+) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \rho_\ast(c)^\dagger. \]

For \( \zeta \neq 0 \), one can simply absorb the deformation parameter \( \zeta \) in the definition of \( c \) and/or \( c^+ \). Therefore the representations of \( \mathcal{A}_\zeta \) for \( \zeta \neq 0 \) are the same as those of \( \mathcal{A}_1 \). As we shall see below, for \( \zeta = 0 \) the situation is completely different.

Before, we begin our analysis, we wish to make note of the following facts about the Grassmann algebra \( \mathcal{A}_0 \).

1. \( \mathcal{A}_0 \) does not admit nontrivial unitary representations. In order to see this we first note that in view of Eqs. (5) and (6) the algebra \( \mathcal{A}_\zeta \) is spanned by the basis elements

\[ 1, c^+, c, c^{+2} \]
1, \( c^+, c \) and \( n \), where \( n := c^+c \). Now, let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be an inner-product space and \( \rho : \mathcal{A}_0 \to \text{End}(\mathcal{H}) \) be a representation of \( \mathcal{A}_0 \) where ‘End’ abbreviates ‘Endomorphism’ (a linear operator mapping \( \mathcal{H} \) into \( \mathcal{H} \)). By definition, if \( \rho \) is a unitary representation, then \( \rho(c^+) = \rho(c)^\dagger \), where a dagger stands for the adjoint of the corresponding operator. According to Eqs. (3) and (4), the unitarity of \( \rho \) implies for all \( |\psi\rangle \in \mathcal{H} \),

\[
||\rho(n)|\psi\rangle||^2 = \langle \psi|\rho(n)^\dagger\rho(n)|\psi\rangle = \langle \psi|\rho(n)^2|\psi\rangle = \langle \psi|\rho(n^2)|\psi\rangle = 0.
\]

Hence \( \rho(n)|\psi\rangle = 0 \). On the other hand,

\[
||\rho(c)|\psi\rangle||^2 = \langle \psi|\rho(c)^\dagger\rho(c)|\psi\rangle = \langle \psi|\rho(n)|\psi\rangle = 0.
\]

Therefore for all \( |\psi\rangle \in \mathcal{H} \), \( \rho(c)|\psi\rangle = 0 \), so that \( \rho(c) = 0 \), \( \rho(c^+) = 0 \), and \( \rho \) is trivial.

2. The only irreducible representation of \( \mathcal{A}_0 \) is the one-dimensional representation defined by

\[
\rho_\theta^{(1)}(1) = 1, \quad \rho_\theta^{(1)}(c^+) = \rho_\theta^{(1)}(c) = 0. \tag{7}
\]

To see this let \( \rho : \mathcal{A}_0 \to \text{End}(V) \) be an arbitrary representation. Then \( V_\theta = \text{Im}(\rho(n)) := \{\rho(n)v|v \in V\} \) is an invariant (\( \rho \)-stable) subspace \([4]\), because for all \( x \in \mathcal{A}_0 \) and for all \( v \in V_\theta \), \( \rho(x)v \in V_\theta \). This shows that \( \rho \) is reducible. Furthermore, the subrepresentation obtained by restricting \( \rho \) to \( V_\theta \) is clearly equivalent to \( \rho_\theta^{(1)} \).

Next, consider the regular representation \( \rho_\zeta : \mathcal{A}_\zeta \to \text{End}(\mathcal{A}_\zeta) \) of \( \mathcal{A}_\zeta \) that is defined by

\[
\forall x, y \in \mathcal{A}_\zeta, \quad \rho_\zeta(x)y := xy. \tag{8}
\]

In the basis \( \{1, c^+, c, n\} \), where

\[
1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c^+ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},
\]
we have

\[
\rho_\zeta(1) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \rho_\zeta(c^+) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

(9)

\[
\rho_\zeta(c) = \begin{pmatrix}
0 & \zeta & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & \zeta \\
0 & -1 & 0 & 0 \\
\end{pmatrix}, \quad \rho_\zeta(n) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \zeta & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & \zeta \\
\end{pmatrix}.
\]

(10)

Here we have made use of Eqs. (5), (6), and (8).

It is not difficult to show that \( \zeta \neq 0 \) if and only if \( \rho_\zeta \) is a pseudo-unitary representation \([6]\). This is equivalent to the requirement that there is a linear Hermitian invertible operator \( \eta \) such that

\[
\rho_\zeta(c^+) = \rho_\zeta(c)^\sharp := \eta^{-1} \rho_\zeta(c)^\dagger \eta. \tag{11}
\]

This can be easily checked by taking \( \eta \) to be an arbitrary \( 4 \times 4 \) matrix and imposing the condition \( \eta \rho_\zeta(c^+) = \rho_\zeta(c)^\dagger \eta \) to determine the matrix elements of \( \eta \). It follows that the determinant of \( \eta \) is proportional to \( \zeta \). Therefore \( \rho_0 \) is not pseudo-unitary. For \( \zeta \neq 0 \) there are many invertible matrices \( \eta \) satisfying (11), e.g.,

\[
\eta = \begin{pmatrix}
0 & \zeta^{-1} & \zeta^{-1} & 0 \\
\zeta^{-1} & 0 & 0 & 1 \\
\zeta^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}.
\]

(12)

Furthermore, in this case, there are similarity transformations

\[
\rho_\zeta(x) \to \rho'_\zeta(x) := S^{-1} \rho_\zeta(x) S \tag{13}
\]

that reduce \( \rho_\zeta \) into the direct sum of two nontrivial two-dimensional irreducible represen-
tions. A convenient choice is

\[
S = \begin{pmatrix}
\zeta & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}.
\] (14)

Using Eqs. (9), (10), (13), and (14), we have

\[
\rho_\zeta'(1) = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}, \quad \rho_\zeta'(c^+) = \begin{pmatrix}
0 & 0 \\
\zeta & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix},
\] (15)

\[
\rho_\zeta'(c) = \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & \zeta \\
0 & 0
\end{pmatrix}, \quad \rho_\zeta'(n) = \begin{pmatrix}
0 & 0 \\
0 & \zeta \\
0 & 0 \\
0 & \zeta
\end{pmatrix},
\] (16)

where the empty entries are zero. Clearly \(\rho_1'\) is the direct product of two copies of the basic unitary representation \(\rho_\ast\) of the Clifford algebra \(\mathcal{A}_1\). Also note that for \(\zeta = 0\) the matrix \(S\) is not invertible, and the above construction does not apply.

In fact, it is not difficult to show that the Grassmann algebra \(\mathcal{A}_0\) does not admit one, two, or three-dimensional representations that are faithful. In order to see this, consider an arbitrary representation \(\rho : \mathcal{A}_0 \rightarrow \text{End}(V)\) where \(V\) is a complex (or real) vector space, and suppose that \(\rho\) is faithful (one-to-one). Then there is \(v_1 \in V\) such that \(v_4 := \rho(n)v_1 \neq 0\). This together with the fact that \(\rho(n) = \rho(c^+)\rho(c)\) imply \(v_2 := \rho(c^+)v_1 \neq 0\) and \(v_3 := \rho(c)v_1 \neq 0\). Next let \(\lambda_i \in \mathbb{C}\), with \(i \in \{1, 2, 3, 4\}\), satisfy

\[
\sum_{i=1}^{4} \lambda_i v_i = 0.
\] (17)

Applying \(\rho(n)\) to both sides of this equation yields \(\lambda_1 = 0\). Substituting this equation in (17) and acting by \(\rho(c)\) and \(\rho(c^+)\) on both sides of the resulting equation lead to \(\lambda_2 = 0\) and \(\lambda_3 = 0\), respectively. Therefore \(\lambda_i = 0\) for all \(i \in \{1, 2, 3, 4\}\); \(v_i\) are linearly independent,
and \(\dim(V) \geq 4\). This in particular shows that the regular representation \(\rho_0\) is the ‘lowest’
dimensional faithful representation. In the following we shall use the term ‘faithfully
irreducible representation’ by which we mean a faithful representation that does not admit
a proper faithful subrepresentation. Note that a faithfully irreducible representation may
very well be reducible. The typical example is the regular representation \(\rho_0\).

Next, consider the span of \(v_i\):

\[
V_{v_1} := \text{Span}(v_1, v_2, v_3, v_4) = \left\{ \sum_{i=1}^{4} \lambda_i v_i \right\}, \lambda_i \in \mathbb{C}
\]

It is not difficult to see that for all \(v \in V_{v_1}\) and \(x \in A_0\), \(\rho(x)v \in V_{v_1}\). Hence the restriction
\(\rho_{v_1} : A_0 \to \text{End}(V_{v_1})\) of \(\rho\) to \(V_{v_1}\), which is defined by

\[
\forall x \in A_0 \quad \forall v \in V_{v_1}, \quad \rho_{v_1}(x)v := \rho(x)v,
\]

provides a representation of \(A_0\). Clearly, \(\rho_{v_1}\) is equivalent to the regular representation \(\rho_0\). This proves the following.

**Theorem:** Every faithful representation of the Grassmann algebra \(A_0\) has a sub-
representation that is equivalent to the regular representation \(\rho_0\). In particular, \(\rho_0\)
is (up to equivalence) the unique 4-dimensional faithfully irreducible representation
of \(A_0\).

This is analogous to the well-known fact about the Clifford algebra \(A_1\), namely that every
faithful representation of \(A_1\) has a subrepresentation that is equivalent to the canonical
representation \(\rho_*\). In particular, \(\rho_*\) is (up to equivalence) the unique 2-dimensional faith-
ful irreducible representation of \(A_1\). However there is a stronger result \[\text{[3]}\] indicating that
every representation of \(A_1\) is a direct product of copies of the trivial representation \(\rho_{\text{trivial}}\)
and the canonical representation \(\rho_*\). A similar result does not hold for \(A_0\). This is mainly
because there are, besides the trivial representation, one, two and three-dimensional non-
faitful representations, namely \(\rho_{0}^{(1)} : A_0 \to \text{End}(\mathbb{C}) = \mathbb{C}\) of \([7]\) and \(\rho_{0}^{(2)} : A_0 \to \text{End}(\mathbb{C}^2)\)
and \(\rho_{0}^{(3)} : A_0 \to \text{End}(\mathbb{C}^3)\) defined by

\[
\begin{align*}
\rho_{0}^{(2)}(1) & = 1, & \rho_{0}^{(2)}(c) & = 0, & \rho_{0}^{(2)}(c^+) & = \mu, \\
\rho_{0}^{(3)}(1) & = 1, & \rho_{0}^{(3)}(c) & = \nu, & \rho_{0}^{(3)}(c^+) & = \nu^+,
\end{align*}
\]

\[\text{(18)}\]

\[\text{(19)}\]
where
\[
\mu := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \nu := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nu^+ := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (20)

In view of the above-stated uniqueness property of the regular representation \(\rho_0\) of the Grassmann algebra \(A_0\), we propose to identify a ‘classical fermion’ with \(\rho_0\). Then the quantization of \(\rho_0\) may be viewed as the deformation \(\zeta: 0 \rightarrow 1\) of the regular representation \(\rho_\zeta\) of the one-fermion algebra \(A_\zeta\) that maps the classical fermion \(\rho_0\) to the ‘quantum fermion’ \(\rho_1\). The latter is a four-dimensional unitary reducible representation of the Clifford algebra \(A_1\) that is associated with a Dirac fermion. In this sense Dirac fermions are naturally linked with the quantization of the classical fermions.

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