A CONFORMALLY INVARIANT YANG-MILLS TYPE ENERGY AND EQUATION ON 6-MANIFOLDS

A. ROD GOVER, LAWRENCE J. PETERSON, AND CALLUM SLEIGH

Abstract. We define a conformally invariant action $S$ on gauge connections on a closed pseudo-Riemannian manifold $M$ of dimension 6. At leading order this is quadratic in the gauge connection. The Euler-Lagrange equations of $S$, with respect to variation of the gauge connection, provide a higher-order conformally invariant analogue of the (source-free) Yang-Mills equations.

For any gauge connection $A$ on $M$, we define $S(A)$ by first defining a Lagrangian density associated to $A$. This is not conformally invariant but has a conformal transformation analogous to a $Q$-curvature. Integrating this density provides the conformally invariant action.

In the special case that we apply $S$ to the conformal Cartan-tractor connection, the functional gradient recovers the natural conformal curvature invariant called the Fefferman-Graham obstruction tensor. So in this case the Euler-Lagrange equations are exactly the “obstruction-flat” condition for 6-manifolds. This extends known results for 4-dimensional pseudo-Riemannian manifolds where the Bach tensor is recovered in the Yang-Mills equations of the Cartan-tractor connection.

1. Introduction

Central to both the standard model in physics and our understanding of differentiable 4-manifolds are the extremely well-known gauge field equations known as the Yang-Mills equations [D, DK, H, YM]. A feature of these equations, that is crucially important in both spheres, is their conformal invariance in dimension 4 [D, FP-1978, FP-1984, JR, T]. On the background of a Riemannian or pseudo-Riemannian 4-manifold $(M, g)$, consider a gauge connection $A$ with curvature $F_{ab}$ (where the gauge field indices are suppressed and the tensor indices are abstract).

The Yang-Mills Lagrangian density is (up to a constant factor we shall ignore)

$$\mathcal{L}_A := \text{Trace}(g^{ac} g^{bd} F_{ac} \circ F_{bd}).$$

For any $\Upsilon \in C^\infty(M)$, let $\hat{g}_{ab} = e^{2\Upsilon} g_{ab}$. Under the conformal transformation $g_{ab} \mapsto \hat{g}_{ab}$, the Lagrangian transforms to $e^{-4\Upsilon} \mathcal{L}_A$, and this exactly balances the corresponding transformation of the metric measure $d\mu^g \mapsto d\mu^\hat{g} = e^{4\Upsilon} d\mu^g$. The action $S$ given by

$$S(A) := \int_M \mathcal{L}_A d\mu^g,$$

2020 Mathematics Subject Classification. primary: 53C18, 53C07; secondary: 58E15, 81T13, 53A55.

A.R.G. gratefully acknowledges support from the Royal Society of New Zealand via Marsden Grants 16-UOA-051 and 19-UOA-008. C.S. Acknowledges support of an FRDF Postdoctoral Fellowship at the University of Auckland.
for compactly supported \( F \), is thus conformally invariant. It follows at once that
the functional gradient, with respect to variation of connection, is also conformally invariant. The Euler-Lagrange equations that these determine, that is the (source-free) Yang-Mills equations

\[
\delta_A F = 0,
\]

are thus conformally invariant as well. Here \( \delta_A \) is the formal adjoint of the connection-twisted exterior derivative \( d_A \).

This argument for conformal invariance evidently fails in dimensions other than 4, and it is well-known, and easily verified, that the equations (2) are not conformally invariant in other dimensions \[Br-1981\]. It is thus natural to ask whether, in dimensions \( n \) other than 4, there is a replacement action/energy and corresponding natural Euler Lagrange PDE system that is conformally invariant. That is, on a smooth \( n \)-manifold we seek a map from pairs \((g, A)\), where \( g \) is a pseudo-Riemannian metric of any given signature \((p, q)\), and \( A \) is a principal connection, to corresponding Lagrangian densities \( \mathcal{L}_A \) of conformal weight \( -n \) so that the expression (1) is again conformally invariant. (We will discuss densities and weights in Section 2.2 below.) There is an important question here of whether the conformally invariant 4-dimensional Yang-Mills equations are an isolated phenomenon, or whether rather they are the dimension 4 case of more general picture. From elementary conformal geometry one does not expect such a possibility in odd dimensions.

These questions are interesting from the perspective of gauge theory in general, but they also have an interesting and direct motivation in conformal geometry linked to the Bach tensor and its generalisations. On pseudo-Riemannian 4-manifolds, the Bach tensor is an important conformal invariant given by

\[
B_{ab} := \nabla^c A_{acb} + P^{cd} C_{cadb}.
\]

Here and below, \( A_{acb} \), \( P^{cd} \), and \( C_{cadb} \) are the Cotton, Schouten, and Weyl tensors, respectively, and \( \nabla \) is the Levi-Civita connection. See Section 2.1. In other dimensions \( n \geq 3 \) we will also say that \( B_{ab} \), as given in (3), is the Bach tensor. Metrics \( g \) that are conformal to Einstein metrics are Bach-flat (meaning \( B^g_{ab} = 0 \)). So in dimension 4 the equations \( B^g_{ab} = 0 \) provide, in a sense, a conformally invariant weakening of the Einstein equations, an idea that dates back to the original work of Bach for a “conformal gravity” theory \[B-1921\] and that is still exploited \[AO, CLW, GV, H, M\]. Self-dual 4-manifolds are also Bach-flat and so there is considerable interest in determining to what extent there are other examples on closed Riemannian manifolds \[CGGGR, LeB\]. The tensor (3) is not conformally invariant in dimensions other than 4, but in higher even dimensions there is a conformally invariant higher-order analogue of (3) that takes the form

\[
\mathcal{O}_{ab} = \Delta^{n/2-2} \nabla^c \nabla^d C_{acbd} + \text{lower-order terms}.
\]

Fefferman and Graham discovered the tensor \( \mathcal{O}_{ab} \) as an obstruction to their “ambient metric” construction. See \[FG-1985\]. For this reason, \( \mathcal{O}_{ab} \) is often referred to as the Fefferman-Graham obstruction tensor. In dimension 4, the tensors \( \mathcal{O}_{ab} \) and \( B_{ab} \) are equal. We will often write \( \mathcal{O}_{ab}^n \) instead of \( \mathcal{O}_{ab} \) to emphasise the dimension.
The Bach tensor also arises as the functional gradient (now with respect to metric variations and where we are ignoring overall constant factors) of the integral of the square of the conformal Weyl tensor \(|C|^2_g = C^{abcd}C_{abcd}\). (Here we use the metric to raise and lower indices.) As in the case of \(\mathcal{L}_A\), the weights conspire so that \(\int_M |C|^2_g \, d\mu^g\) is conformally invariant in dimension 4. On closed manifolds, an alternative Lagrangian density is provided by Branson’s much studied \(Q\)-curvature \(Q^g\) of [BrO, Br-1993, Br-1995]. This alternative density generalises, in that in all even dimensions there is such a \(Q\)-curvature, and the Fefferman-Graham obstruction tensor is the functional gradient of (4) 

\[
\int_M Q^g \, d\mu^g
\]

with respect to metric variations [GH]. The situation is slightly more subtle now, however, because the \(Q\)-curvature is not itself conformally invariant (whether we treat it as a function or as a conformal density). Under conformal transformation, \(Q\) changes by a term in which a linear operator acts on the logarithm of the conformal factor. This term is of divergence form, and for this reason, (4) is conformally invariant [Br-1995].

On a general Riemannian or pseudo-Riemannian manifold there is no natural conformally invariant connection on the tangent bundle, but in dimensions \(n \geq 3\) there is a natural conformally invariant connection on a related bundle of rank \(n+2\). This is the conformal tractor connection [BEG] due, in its original form, to Cartan and Thomas [C, T]. See Section 6.1 below. A nice feature of this connection is that in dimension 4, its associated Yang-Mills equations recover exactly the Bach-flat condition \(B_{ab} = 0\) [Mer, GSS]: Lemma 4.2 in [GSS] states that in dimension 4, in an informal tractor notation,

(5) 

\[
\delta_{\nabla^T} F_{\nabla^T} = c \cdot (0, 0, 0, B_{ab}),
\]

where \(\nabla^T\) is the conformal tractor connection, and \(c\) is an (explicit non-zero) constant. One might hope that a higher-dimensional conformally invariant Yang-Mills type theory would generalise this picture.

In the current article, we treat these questions in dimension \(n = 6\). It seems to us that the results should assist with extending many of the studies and directions mentioned above from the confines of dimension 4 to higher even dimensions. For convenience, we will always view the gauge connection \(A\) as a linear connection acting on some vector bundle \(V\), and we let \(F_A\) denote the curvature of \(A\). Our work will rely on the operator \(Q^A_2\) defined in Section 3.2. This operator belongs to a family of operators defined in [BG-2005, BG-2007]. In what follows, we will also work with a bilinear mapping \(\langle \cdot, \cdot \rangle\) which we define in Section 3.

For closed (i.e. compact without boundary) 6-manifolds, we provide the sought map from pairs \((g, A)\), consisting of a metric and a connection, to Lagrangian densities \(\mathcal{L}_A\) so that the action \(S\) given in (1) is conformally invariant. The Lagrangian density we provide, namely \(\langle F_A, Q^A_2 F_A \rangle\), is an analogue of the \(Q\)-curvature. Here we mean that this density is not conformally invariant but rather transforms, under conformal rescaling of the metric \(g\) on \(M\), by a linear divergence-type operator acting on the logarithm of the conformal factor. This is an immediate consequence
of Proposition \ref{proposition} below, which was first derived in [BC-2007]. If \( M \) is closed, it follows that

\[
\mathcal{S}(A) = \int_M \langle F_A, Q^A_{\mathcal{F} \mathcal{F}} \rangle \, d\mu^g
\]

is invariant under conformal change of the metric on \( M \). See Proposition \ref{conformal_invariance}.

An important feature of the Lagrangian density \( \langle F_A, Q^A_{\mathcal{F} \mathcal{F}} \rangle \) is that it is quadratic in the jets of the connection \( A \) at leading order. So if we consider the functional gradient of \( \mathcal{S} \), with respect to variation of connection, we find that this functional gradient is linear at leading order. The functional gradient therefore provides a conformally invariant map \( \mathfrak{D} \) from connections \( A \) to 1-forms taking values in \( \text{End}(V) \):

\[
A \mapsto \mathfrak{D}A.
\]

We thus obtain a higher-order conformally invariant analogue of the Yang-Mills equations. See Proposition \ref{higher_order_Yang-Mills} and Definition \ref{definition}. By construction these equations are conformally invariant, but a direct verification is provided in Proposition \ref{conformal_invariance_Yang-Mills}.

As an application and test of this, in Section \ref{applications} we show that the non-linear operator \( \mathfrak{D} \) applied to the conformal tractor connection exactly recovers the Fefferman-Graham obstruction tensor of conformal 6-manifolds. Thus we obtain a perfect parallel of the situation in dimension 4, as discussed above, and at the same time a new perspective on the Fefferman-Graham obstruction tensor. See Theorem \ref{Fefferman-Graham_obstruction} which, again in an informal tractor notation, states that

\[
\mathfrak{D}\nabla^T = k \cdot (0, 0, 0, \mathcal{O}_{ab}),
\]

where \( \nabla^T \) is the conformal tractor connection, and \( k \) is an (explicit non-zero) constant. Compare this to \ref{Fefferman-Graham_obstruction} above.

It is not difficult to see that there are other conformally invariant actions on connections in dimension 6. For example, apart from divergences, we can add cubic terms \( \text{Trace}(g^{af}g^{bc}g^{de}F_{ab} \circ F_{cd} \circ F_{ef}) \) to the Lagrangian density. However Theorem \ref{Fefferman-Graham_obstruction} (i.e. \ref{Fefferman-Graham_obstruction}) suggests that \ref{Fefferman-Graham_obstruction} is distinguished.

Finally, in Section \ref{section} we discuss the link between our action \ref{action}, when specialised to the case that \( A \) is the conformal tractor connection, and a conformally invariant action we construct using the Fefferman-Graham invariant of Proposition 3.4 of [FG-1985].

It seems that recently there has been some interest in “higher order Yang-Mills” flows [K, Z]. In these works, the generation of the equations emphasises the use of broadly analogous higher-order actions on Riemannian manifolds, rather than attention to conformal properties, but it is possible that some of the results could be applied to the equations we develop here. In [K] Kelleher asks the question of whether there might be conformally invariant higher-order Yang-Mills type equations, and suggests that, if so, conformal invariance might be a good distinguishing property. In [Z] the author looks at a notion of Yang-Mills for higher dimensions that is not closely related to the programme here.
2. Notation and background

Throughout our work, $M$ will be a smooth manifold of dimension $n$. Our main interest lies in smooth manifolds of dimension 6, but in this section, we work with general manifolds in general dimensions $n$. We let $TM$ and $T^*M$ denote the tangent and cotangent bundles, respectively, of $M$. We will often use Penrose\'s abstract index notation. Lowercase indices $a$, $b$, $c$, and so forth will be associated to $TM$ and $T^*M$, and uppercase indices such as $B$, $C$, and $D$ will be associated to other finite-dimensional vector bundles. Parentheses will indicate symmetrisation of indices. For example, if $T_{abc}$ is a rank 3 tensor, then $T_{a(bc)} = (1/2)(T_{abc} + T_{acb})$.

We will always assume that $g_{ab}$ (or just $g$) denotes a pseudo-Riemannian metric on $M$ and that $R_{ab}^{cd}$ is the Riemannian curvature tensor of the Levi-Civita connection $\nabla$ associated to $g_{ab}$. We use the sign convention for $R_{ab}^{cd}$ such that

$$ (\nabla_a \nabla_b - \nabla_b \nabla_a)\psi^c = R_{ab}^{cd} \psi^d $$

for all vector fields $\psi^c$ on $M$. Let $k \in \mathbb{Z}_{>0}$ and a smooth tensor field $T^{c_1...c_k}$ on $M$ be given. Then

$$ \nabla_a \nabla_b T^{c_1...c_k} = \nabla_b \nabla_a T^{c_1...c_k} + R_{ab}^{c_1} T^{i_1...c_k} + \cdots + R_{ab}^{c_k} T^{c_1...i_k}, $$

by [1]. We may use the metric $g_{ab}$ to raise or lower any of the free indices in (9).

We let $\Delta := \nabla_a \nabla^a$. For any vector bundle $W$ over $M$, let $\Gamma(W)$ denote the space of all smooth sections of $W$. The notation $W^*$ will always denote the dual of $W$. Our scaling convention for the wedge product of differential forms will be such that for all $p, q \in \mathbb{Z}_{\geq 0}$, all $x \in M$, and all $p$-forms $\omega$ and $q$-forms $\psi$ at $x$,

$$ \omega \wedge \psi = \frac{(p+q)!}{p!q!} \text{Alt}(\omega \otimes \psi). $$

2.1. Conformal geometry. Let a manifold $M$ and pseudo-Riemannian metrics $g$ and $\hat{g}$ on $M$ be given, and suppose there is an $\Upsilon \in C^\infty(M)$ such that $\hat{g} = e^{2\Upsilon} g$. Then we say that $\hat{g}$ is conformal to $g$. Let $\mathfrak{c}$ denote the set of all pseudo-Riemannian metrics on $M$ which are conformal to $g$. We say that $\mathfrak{c}$ is a conformal structure on $M$ or a conformal class on $M$, and we say that the pair $(M, \mathfrak{c})$ is a conformal manifold. Throughout our work, $\hat{g}$ will always denote the metric $e^{2\Upsilon} g$ (for some $g$ and $\Upsilon$). For any object, such as a tensor, that depends on the choice of the metric in $\mathfrak{c}$, a hat $\hat{\ }$ will indicate the object as determined by the metric $\hat{g}$.

We will need to introduce several classical tensors that occur naturally in conformal geometry due to their simple transformation laws under conformal rescaling. The first tensor we will need is the symmetric tensor $P_{ab} \in \Gamma(S^2T^*M)$, called the Schouten tensor, defined in general dimensions $n \geq 3$ by

$$ P_{ab} := \frac{1}{n-2} \left( \hat{R} c_{ab} - \frac{1}{2(n-1)} g_{ab} Sc \right). $$

Here $\hat{R} c_{ab} = R^c_{\ abc}$ is the Ricci tensor, and $Sc = \hat{R} c^a_a$ is the scalar curvature. The tensors $P$, $R$, $\hat{R}$, $\hat{R} c$, $\hat{R} c^a$, and so forth depend on the choice of metric $g$, and we may write $P^g$, $R^g$, and so forth to indicate this dependence. We will usually omit $g$ from our notation, however.
By using the metric to raise indices, we may associate $P_{ab}$ to an element of $\Gamma(\text{End}(T^*M))$. We will let the symbol $P^\#$ denote the corresponding derivation on general tensor fields. For example, the action of $P^\#$ on a two-form $\omega_{ab}$ is

$$(P^\#\omega)_{ab} = P_a^c\omega_{cb} + P_b^c\omega_{ac}.$$  

We will use this later. For all $r, s \in \mathbb{Z}_{\geq 0}$, let $T^{(r,s)}$ denote the bundle of tensors of contravariant rank $r$ and covariant rank $s$. For any vector bundle $W$, we may extend the action of $P^\#$ (trivially) to sections of $T^{(r,s)} \otimes W$ as follows. First, let

$$P^\#(\omega \otimes U) = (P^\#\omega) \otimes U$$

for all $\omega \in \Gamma(T^{(r,s)})$ and all $U \in \Gamma(W)$. Then extend the action $P^\#$ linearly to sums of sections of the form $\omega \otimes U$, where $\omega \in \Gamma(T^{(r,s)})$ and $U \in \Gamma(W)$.

Our work will also involve the scalar field $J$ given by $J = P_a^a = P_{ab}g^{ab}$. From the second Bianchi identity, it follows that

$$\nabla_a P^a_b = \nabla_b J.$$  

The Cotton tensor $A_{abc}$ is defined by

$$A_{abc} := \nabla_b P_{ca} - \nabla_c P_{ba}.$$  

The Cotton tensor is trace-free. The Weyl tensor is given by

$$C_{abcd} := R_{abcd} + g_{cb}P_{ad} - g_{ca}P_{bd} + g_{da}P_{bc} - g_{db}P_{ac}.$$  

In all dimensions $n \geq 3$, the Bach tensor $B_{ab}$ is given by (3). We note here that $B_{ab}$ is trace-free. A short computation shows that $B_{ab}$ is symmetric.

2.2. Conformal density bundles. One may simplify many of the calculations and formulae of conformal geometry by using the language of weighted density bundles. By its definition, a conformal structure $c$ determines a ray sub-bundle $Q$ of $S^2T^*M$. This sub-bundle is actually a principal $\mathbb{R}^+$-bundle $\pi : Q \to M$. For every $g \in c$ and every $x \in M$, the fibre of $Q$ over $x$ consists of all metrics at $x$ which are conformal to $g$ at $x$. For each $w \in \mathbb{R}$ we will denote by $\mathcal{E}[w]$ the induced line bundle arising from the representation of $\mathbb{R}^+$ on $\mathbb{R}$ given by $t \mapsto t^{-w/2}$. We note that each bundle $\mathcal{E}[w]$ is canonically oriented. Sections of $\mathcal{E}[w]$ are called conformal densities of weight $w$ and maybe identified (via the associated bundle construction) with homogeneous functions of degree $w$ on $Q$.

There is a tautological function $g$ on $Q$ taking values in $S^2T^*M$ which simply assigns to a point $(g_x, x) \in Q$ the metric $g_x$ at $x$. This determines a section $g$ (or $g_{ab}$) of $S^2T^*M \otimes \mathcal{E}[2]$ called the conformal metric. For any nonvanishing $\sigma \in \Gamma(\mathcal{E}[1])$, it follows $\sigma^{-2}g_{ab}$ is a metric in the conformal class, and we use the term conformal scale for $\sigma$. In a similar fashion, we may define a section $g^{ab}$ of $S^2TM \otimes \mathcal{E}[-2]$ in such a way that $g^{ab} = g^{-1}$. A given pseudo-Riemannian metric $g$ determines a positive section $\xi^g$ of $\mathcal{E}[1]$ via the relation $g_{ab} = (\xi^g)^{ab}g_{ab}$. We say that $\xi^g$ is the scale density associated to $g$. For any metric $g$, the Levi-Civita connection acts on sections of $\mathcal{E}[w]$ as follows. Let $f \in C^\infty(M)$ be given. Then $\nabla((\xi^g)^w f) = (\xi^g)^w df$, where $d$ is the exterior derivative on functions. Note that $\nabla \xi^g = 0$. 
Note that $g_{ab}$ and $g^{ab}$ are conformally invariant, so we may use $g^{ab}$ and $g_{ab}$ to raise and lower tensor indices in a conformally invariant way. This fits with the notion of weighted tensors. From this point forward, all tensors will implicitly be the product of a tensor and a density of some weight $w$. We say that the tensor is \textit{weighted}, or that it \textit{carries} a weight. The Riemannian curvature tensor $R_{abcd}$ will carry a weight of 0. On the other hand, $g_{ab}$ will carry a weight of 2, and $g^{ab}$ will carry a weight of $-2$. If we raise or lower an index, we will always use the conformal metric to do this. Thus $R_{abcd}$ will have weight 2, and $R^{abcd}$ will carry a weight of $-2$. The tensors $P_{ab}$, $J$, $A_{abc}$, $C_{abcd}$, and $B_{ab}$ will carry weights 0, $-2$, 2, and $-2$, respectively. Elements and sections of vector bundles will also carry weights.

To work with weighted tensors and vectors efficiently, we introduce additional notations. For every $k \in \mathbb{Z}_{\geq 0}$, let $\Lambda^k(M)$ denote the bundle of $k$-forms on $M$, and for every $w \in \mathbb{R}$, let $\Lambda^k(M, w)$ denote $\Lambda^k(M) \otimes \mathcal{E}[w]$. Let $\Omega^k(M, w)$ denote $\Gamma(\Lambda^k(M, w))$. Similarly, for any vector bundle $W$ over $M$, let $\Lambda^k(W)$ denote the bundle of $k$-forms on $M$ with values in $W$, and let $\Lambda^k(W, w)$ denote $\Lambda^k(W) \otimes \mathcal{E}[w]$. Let $\Omega^k(W, w)$ denote $\Gamma(\Lambda^k(W, w))$. Note that $\Lambda^k(M) = \Lambda^k(M, 0)$. Similar remarks apply to $\Lambda^k(W)$, $\Omega^k(M)$, and $\Omega^k(W)$.

Remembering that tensors carry weights, we now state the transformation rules for $P_{ab}$ and $J$ under conformal change of metric. In general dimensions $n$, the Schouten tensor transforms according to the rule

$$
\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} g_{ab} \Upsilon_c \Upsilon^c.
$$

Here $\nabla$ is the Levi-Civita connection associated to the original metric $g$, and (for example) $\Upsilon_a = \nabla_a \Upsilon$. Also, $\Upsilon^c = g^{cd} \Upsilon_d$. We will use these notational conventions throughout our work. Note that $\Upsilon^c$ carries a weight of $-2$. In general dimensions $n$, the conformal transformation rule for $J$ is as follows:

$$
\hat{J} = J - \nabla_a \Upsilon^a + \left(1 - \frac{n}{2}\right) \Upsilon_a \Upsilon^a.
$$

2.3. Integration of densities. We now consider integration of densities. Let a conformal manifold $(M, c)$, a section $\phi$ of $\mathcal{E}[-n]$, which is of compact support, and $g \in c$ be given. Then $\phi = (\xi^g) -n f$ for some $f \in C^\infty(M)$. Let $\mu^g$ denote the pseudo-Riemannian measure on $M$ associated to $g$, and define $\int_M \phi \, \mu$ by

$$
\int_M \phi \, d\mu := \int_M f \, d\mu^g.
$$

This integral is independent of the choice of the $g \in c$ that we use in its definition. To see this, begin by recalling our convention that $\hat{g} = e^{2\Upsilon} g$. One can show that $\xi^g = e^\Upsilon \xi^\hat{g}$. Thus $(\xi^g)^{-n} = e^{-n\Upsilon} (\xi^\hat{g})^{-n}$, and hence with $f$ as above,

$$
\phi = (\xi^g)^{-n} f = e^{n\Upsilon} (\xi^\hat{g})^{-n} f = e^{n\Upsilon} (\xi^g)^{-n} e^{-n\Upsilon} f.
$$

So if we use $\hat{g}$ to define $\int_M \phi \, d\mu$, we find that

$$
\int_M \phi \, d\mu = \int_M e^{-n\Upsilon} f \, d\mu^g = \int_M e^{-n\Upsilon} f e^{n\Upsilon} \mu^g = \int_M f \, d\mu^g.
$$
This is consistent with (14).

3. Bundles and Connections

In our discussions of connections and curvature on general vector bundles, our conventions and basic definitions will essentially follow those given in standard references such as [DK]. In this section, we describe our conventions and define some of the vector bundles and operators that our work will use. We also state some conformal transformation rules for connections and other operators. For consistency with work in other contexts, we work in general dimensions $n$ unless we indicate otherwise.

Let a finite-dimensional vector bundle $W$ over $M$, a connection $A$ on $W$, and a nonnegative integer $k$ be given. Let $\nabla$ denote the standard coupling of $A$ and the Levi-Civita connection. When $\nabla$ acts on weighted tensors associated to $TM$ or $T^*M$, it acts as the Levi-Civita connection. If $\nabla$ acts on a section of $W$, then it acts as $A$. By linearity and the Leibniz property, $\nabla$ is extended to weighted sections of tensor products of these bundles. The symbol $\nabla$ will denote a coupled connection throughout much of our work.

Let $w \in \mathbb{R}$ be given. The connection $A$ induces a twisted exterior derivative $d_A : \Omega^k(W, w) \to \Omega^{k+1}(W, w)$.

By (10),

$$ (d_A \omega)_{a_1 \ldots a_{k+1}}^B = \frac{1}{k!} \sum_{\sigma \in S_k+1} (\text{sgn } \sigma) \nabla_{a_{\sigma(1)}} \omega_{a_{\sigma(2)} \ldots a_{\sigma(k+1)}}^B $$

for all $\omega \in \Omega^k(W, w)$. Here $S_{k+1}$ denotes the set of all permutations of $\{1, \ldots, k+1\}$.

To give a formula for the formal adjoint of $d_A$, we begin by letting a metric $h$ (of some signature) on $W$ be given, and we suppose that $h$ is preserved by the connection $A$. Also let $k \in \mathbb{Z} \geq 0$ and $w_1, w_2 \in \mathbb{R}$ be given, and suppose that $w_1 + w_2 = 2k - n$. We will use the conformal metric $g$ on $M$ together with $h$ to define a bilinear mapping

$$ \langle \cdot, \cdot \rangle : \Lambda^k(W, w_1) \times \Lambda^k(W, w_2) \to \mathcal{E}[-n]. $$

Specifically, for all $x \in M$ and all $\omega \in \Lambda^k(W, w_1)$ and all $\eta \in \Lambda^k(W, w_2)$ at $x$, we let

$$ \langle \omega, \eta \rangle = \frac{1}{k!} \omega_{a_1 \ldots a_k} B \eta^{a_1 \ldots a_k} B. $$

Now again let $k \in \mathbb{Z} \geq 0$ and $w_1, w_2 \in \mathbb{R}$ be given, but now suppose that $w_1 + w_2 = 2(k + 1) - n$. The formal adjoint of $d_A : \Omega^k(W, w_1) \to \Omega^{k+1}(W, w_1)$ with respect to $\langle \cdot, \cdot \rangle$ is easily computed to be the operator $\delta_A : \Omega^{k+1}(W, w_2) \to \Omega^k(W, w_2 - 2)$ given by

$$ (\delta_A \eta)_{a_1 \ldots a_k}^B = -\nabla_b \eta_{a_1 \ldots a_k}^b. $$

Once again let $k \in \mathbb{Z} \geq 0$ be given. For any $w \in \mathbb{R}$, an obvious adaptation of the above discussion defines our conventions for the exterior derivative $d : \Omega^k(M, w) \to$
\( \Omega^{k+1}(M, w) \) and its formal adjoint \( \delta \). By removing the subscript \( A \) and the index \( B \) from (15) and (17), we obtain symbolic formulae for \( (d\omega)_{a_1 \ldots a_{k+1}} \) and \( (\delta \eta)_{a_1 \ldots a_k} \).

We will always assume that the connection \( A \) is independent of any choice of metric from the conformal class \( c \) on \( M \). The coupled connection \( \nabla \) will transform under conformal change of the metric \( g \), however, and we will need certain conformal transformation rules involving \( \nabla \). Specifically, let \( w \in \mathbb{R} \), a one-form \( \eta \), and a two-form \( \omega \) be given. Suppose that \( \eta \) and \( \omega \) have weight \( w \) and that both take values in \( \mathbb{R} \) or \( W \). Then

\[
\hat{\nabla}_a \eta_b = \nabla_a \eta_b + (w - 1) \Upsilon_{a b c} \eta_c,
\]

and

\[
(\hat{\delta_A} \omega)_c = (\delta_A \omega)_c + (4 - n - w) \Upsilon^a \omega_{ac}.
\]

Both of these transformation rules follow from elementary computations and the fact that \( A \) is conformally invariant. From (17) and (18), it follows that

\[
(\hat{\delta_A} \omega)_c = (\delta_A \omega)_c + (4 - n - w) \Upsilon^a \omega_{ac}.
\]

Our work will focus on vector bundles \( adV \) which we will define as follows. Let \( \pi : P \to M \) be an arbitrary \( G \)-bundle, for some Lie group \( G \). Any finite-dimensional \( G \)-module \( \mathbb{V} \) gives rise to an associated vector bundle \( V \) over \( M \). Let \( A \) denote a \( G \)-connection on the bundle \( P \). The same notation will be used to denote the corresponding associated linear connection on the associated vector bundle \( V \). The adjoint representation induces a bundle of Lie-algebras over \( M \), and the derivative of the \( G \)-representation acting on \( \mathbb{V} \) then gives a vector bundle \( adV \), which is a sub-bundle of \( \text{End}(V) \cong V \otimes V^* \). Let \( x \in M \) and elements \( \omega \) and \( \eta \) of \( adV \) at \( x \) be given. We will let \( \omega \eta \) denote the composition \( \omega \circ \eta \). We may write \( \omega \eta \) in index notation as \( \omega^B \eta^D C \). We will use this notation when we define a metric on \( adV \) below. Our work will not require the existence of a metric on \( V \).

We will work with operators acting on \( \Omega^k(adV, w) \) for various \( k \in \mathbb{Z}_{\geq 0} \) and \( w \in \mathbb{R} \). To define these operators, let \( k \in \mathbb{Z}_{\geq 0} \) be given. Let \( \nabla \) denote the connection \( A \), and recall that \( \nabla \) determines a dual connection \( \nabla^* \) on \( V^* \). Specifically, define \( \nabla^* \) by the formula \( (\nabla^*_X u)(v) = X(u(v)) - u(\nabla_X v) \). Here \( X \in \Gamma(TM) \), \( u \in \Gamma(V^*) \), and \( v \in \Gamma(V) \).

By coupling \( \nabla \) and \( \nabla^* \), we determine a connection on \( adV \), and we in turn couple this connection with the Levi-Civita connection on \( M \) to obtain a connection on \( \Omega^k(adV, w) \). Then (15) defines an exterior derivative \( d_A : \Omega^k(adV, w) \to \Omega^{k+1}(adV, w) \) associated to \( A \). When we work with this \( d_A \), we will append a lower index \( C \) to each side of (15).

To give a formula for the formal adjoint of \( d_A \), we will need a metric on \( adV \). For any \( x \in M \) and any \( \omega \) and \( \eta \) in \( adV \) at \( x \), let \( h(\omega, \eta) = \omega^B \eta^C \epsilon_{B C} \). Then by construction, \( h \) is preserved by the connection \( A \). For the general theory, and for the purposes of deriving equations, we will assume that \( G \) is such that \( h \) is non-degenerate, so \( h \) is a bundle metric on \( adV \). (For example this is always the case if \( G \) is semisimple. Then \( h \) is a multiple of the Killing form.) Now let \( k \in \mathbb{Z}_{\geq 0} \) be given. Let \( W = adV \), and let \( \langle \cdot, \cdot \rangle \) be the bilinear mapping described in (16) and
in the discussion preceding (15). In this context, we may rewrite (15) as
\[ \langle \omega, \eta \rangle = \frac{1}{k!} \omega_{a_1 \ldots a_k} B^C \eta^{a_1 \ldots a_k} C. \]
and (17) as
\[ (\delta_A \eta)_{a_1 \ldots a_k} B^C = -\nabla_b \eta^b_{a_1 \ldots a_k} C. \]

**Remark 3.1.** Note that since the \( h \) is defined by the canonical self-duality of \( \text{End}(V) \), it is non-degenerate but with mixed signature. Indeed if, for example, we use any choice of positive definite metric to identify \( V \) with \( V^* \), and hence to identify \( \text{End}(V) \) with \( \otimes^2 V \), then it is easily seen that \( h \) is positive definite on the symmetric part \( S^2 V \subset \otimes^2 V \), and negative definite on the skew part \( \Lambda^2 V \subset \otimes^2 V \). Of course the signature of \( h \) is independent of the choice made.

Although for the general theory we assume assume that \( G \) is such that \( h \) is non-degenerate on \( adV \), the equations that we derive can be used more widely.

We will sometimes consider the induced connections or twisted exterior derivatives on \( V \) and on \( adV \) at the same time, and we will use the same notations (such as \( d_A \)) for the versions of these operators on both \( V \) and \( adV \).

The space of all connections on \( V \) will be denoted \( \mathcal{A} \). For any \( A_1, A_2 \in \mathcal{A} \), one can easily show that \( A_1 - A_2 \in \Omega^1(adV) \). So once a connection \( A_0 \in \mathcal{A} \) has been chosen, any other connection \( A \in \mathcal{A} \) can be written as \( A = A_0 + a \), for some \( a \in \Omega^1(adV) \). Thus \( \mathcal{A} \) is an affine space modeled on the vector space \( \Omega^1(adV) \). Suppose we choose a local frame for \( V \) and let \( A_0 = d \) be the trivial connection for this frame (the exterior derivative on components). Then for any \( A \in \mathcal{A} \), there is an \( a \in \Omega^1(adV) \) such that \( A = d + a \). This is the usual local expression for a linear connection.

Now let \( A, A_0 \in \mathcal{A} \) be given. Then \( A = A_0 + a \) for some \( a \in \Omega^1(adV) \), as we noted above. Next, let \( k \in \mathbb{Z}_{\geq 0} \) be given, and note that \( A \) and \( A_0 \) induce connections on \( T^{(0,k)} \otimes \text{End}(V) \). Let \( \tilde{A} \) and \( \tilde{A}_0 \), respectively, denote these induced connections. Then for any \( \phi_{j_1 \ldots j_k} B^C \in \Gamma(T^{(0,k)} \otimes \text{End}(V)) \), elementary computations show that
\[ A_i \phi_{j_1 \ldots j_k} B^C = (A_0)_i \phi_{j_1 \ldots j_k} B^C + a_i ^B E \phi_{j_1 \ldots j_k} E^C - \phi_{j_1 \ldots j_k} B^E a_i E^C. \]

When we regard \( A \) and \( A_0 \) as connections on \( T^{(0,k)} \otimes \text{End}(V) \), we will thus say that \( A = A_0 + [a, \cdot] \). In this context, we will also let \( A_0 + a \) denote \( A_0 + [a, \cdot] \). We do this, for example, in (29).

### 3.1. Curvature

There is a (gauge) invariant section \( F_A \in \Omega^2(adV) \) such that \( d_A S^B = F_A S^B \) for all \( S^B \in \Gamma(V) \). Thus
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) S^B = (F_A)_{ab} B^E E^S \]
for all \( S^B \in \Gamma(V) \). Here \( \nabla \) is again the coupled connection. Because the Levi-Civita connection is torsion free, \( F_A \) is the usual curvature of the connection \( A \). Similarly, there is an \( F^*_A \in \Omega^2(adV^*) \) such that \( (\nabla_a \nabla_b - \nabla_b \nabla_a) S_B = (F^*_A)_{ab} C^B S_C \) for all \( S_B \in \Gamma(V^*) \). One can use the duality of \( V \) and \( V^* \) to show that \( (F^*_A)_{ab} C^B = -(F_A)_{ab} B^C \). Thus
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a) S_B = -(F_A)_{ab} B^E E^S \]
for all $S_B \in \Gamma(V^*)$. The curvature $F_A$ satisfies the Bianchi identity $d_A F_A = 0$, as noted in [DK].

3.2. **The coupled $Q^A_2$-operator.** As we noted in Section 1, the $Q$-curvature is defined in [BrO, Br-1993, Br-1995]. In [BG-2005, BG-2007], the first author and Branson define and discuss a family of linear operators $Q_k$, for $n$ even and $k = 0, 1, 2, \ldots, n/2 + 1$, mapping $\Omega^k(M)$ to $\Omega^k(M, 2k - n)$. The $Q_k$-operators generalise Branson’s $Q$-curvature in several ways. Branson’s original $Q$-curvature is equal to $Q^0_1$. We now restrict to manifolds of dimension $n = 6$. The case of interest to us then is the case $k = 2$, where the operator $Q_2$ is given as follows:

$$Q_2 \omega := dA \delta A \omega - 4P \# \omega + 2J \omega.$$  \hspace{1cm} (24)

In [BG-2007], the above-mentioned authors also discuss a generalised family of operators $Q_k$ which act on bundle-valued $k$-forms. We need only the $k = 2$ case here, and we construct such generalised operators by replacing the exterior derivative and its adjoint in (24) with the twisted exterior derivative $d_A$ and its adjoint $\delta_A$, respectively. We do this in the next definition. In this definition and in much of what follows, we suppress bundle indices. We will do so without further comment, since the context should make this clear.

**Definition 3.2.** Let a pseudo-Riemannian manifold $(M, g)$ of dimension $n = 6$, a vector bundle $W$ over $M$, and a bundle metric $h$ on $W$ be given. Also let $A$ be a conformally invariant connection on $W$ that preserves $h$. Let the operator $Q^A_2 : \Omega^2(W) \to \Omega^2(W, -2)$ be defined as follows:

$$Q^A_2 \omega = dA \delta A \omega - 4P \# \omega + 2J \omega.$$  \hspace{1cm} (25)

The following proposition describes the behaviour of $Q^A_2$ under conformal rescaling.

**Proposition 3.3.** Let $n$, $W$, $A$, and $\omega$ be as in Definition 3.2. Suppose also that $\omega$ is $d_A$-closed. Then under a conformal rescaling of the metric $g$ on $M$, the operator $Q^A_2$ obeys the following transformation rule:

$$\hat{Q}^A_2 \omega = Q^A_2 \omega + 2\delta_A d_A (\Upsilon \omega).$$  \hspace{1cm} (26)

**Proposition 3.4.** The operator $Q^A_2$ is formally self-adjoint.

**Proof.** For any $\eta_{ab}$, $\omega_{ab} \in \Omega^2(W)$,

$$P^c_a \eta^{ab} \omega_{ab} + P^c_b \eta^{cb} \omega_{ac} = P^c_a \eta^{cb} \omega_{ab} + P^c_b \eta^{ac} \omega_{ab} = P^c_a \eta_{cb} \omega^{ab} + P^c_b \eta_{ac} \omega^{ab},$$

so the result follows from Definition 3.2. \hfill \Box
4. THE CONFORMALLY INVARIANT ACTION IN SIX DIMENSIONS

Let $M$ be a closed 6-manifold equipped with a conformal class of pseudo-Riemannian metrics and let notation be as in Section 3 above. So $A$ denotes a connection on a vector bundle $V$. Note that

$$\langle F_A, Q_2^A F_A \rangle \in \Gamma(\mathcal{E}[{-6}]),$$

and define the integration of densities as in Section 2.3. This allows us to define an action on $A$ as follows.

**Definition 4.1.** Suppose that $n = 6$ and that $M$ is closed. Define an action $S(A)$ by

$$S(A) := \int_M \langle F_A, Q_2^A F_A \rangle \, d\mu.$$  

**Proposition 4.2.** The action $S(A)$ is invariant under conformal change of the metric $g$ on $M$.

**Proof.** Note that $d_A F_A = 0$, by the Bianchi identity. Thus

$$\int_M \langle F_A, \dot{Q}_2^A F_A \rangle \, d\mu = \int_M \langle F_A, Q_2^A F_A + 2\delta_A d_A(\mathcal{Y} F_A) \rangle \, d\mu$$

by Proposition 3.3. But $M$ has empty boundary and $A$ preserves the pairing. Thus

$$\int_M \langle F_A, 2\delta_A d_A(\mathcal{Y} F_A) \rangle \, d\mu = \int_M \langle d_A F_A, 2d_A(\mathcal{Y} F_A) \rangle \, d\mu = 0,$$

and the proposition follows.

4.1. Euler-Lagrange equations for $S$. We will now derive the Euler-Lagrange equations for the conformally invariant action $S$. Here and below, $[\delta_A F_A, F_A]$ will denote $\delta_A F_A F_A - F_A \delta_A F_A$. This notation includes an implicit contraction of the free lowercase index of $\delta_A F_A$ with the first lowercase index of the other $F_A$ within the brackets.

**Proposition 4.3.** The Euler-Lagrange equations for the action $S$ are

$$\delta_A Q_2^A F_A - [\delta_A F_A, F_A] = 0.$$  

**Proof.** For $a \in \Omega^1(adV)$ and $t \in \mathbb{R}$, let $A + ta$ be a path of connections on $V$ starting at $A$. Then, since $(F_{A+ta})_{ij}^B C = (F_A)_{ij}^B C + t(d_A a)_{ij}^B C + t^2(a \wedge a)_{ij}^B C$, it follows that

$$\left. \frac{d}{dt} S(A + ta) \right|_{t=0} = \int_M \left( \langle d_A a, Q_2^A F_A \rangle + \langle F_A, Q_2^A d_A a \rangle + \langle F_A, \dot{Q}_2^A F_A \rangle \right) \, d\mu.$$

Here $\dot{Q}_2^A$ denotes the linearisation of $Q_2^A$. Thus by Proposition 3.3

$$\left. \frac{d}{dt} S(A + ta) \right|_{t=0} = \int_M \left( 2\langle d_A a, Q_2^A F_A \rangle + \langle F_A, \dot{Q}_2^A F_A \rangle \right) \, d\mu.$$  

Consider the integral of \( \langle F_A, \dot{Q}^A_F A \rangle \). Only the leading term of \( Q^A_2 \) will contribute to this integral. We find that

\[
\frac{d}{dt} \int_M \langle F_A, d_{A+ta} \delta_{A+ta} F_A \rangle \, d\mu \bigg|_{t=0} = \int_M 2\langle \delta_A F_A, \delta_A F_A \rangle \, d\mu.
\]

By using (20) and (21), one can easily show that

\[
\delta_A F_A + ta \delta_A F_A = -[a, F_A].
\]

Thus \( \dot{\delta} A F_A = -[a, F_A] \). From this it follows that

\[
\int_M \langle F_A, \dot{Q}^A_F A \rangle \, d\mu = -\int_M 2\langle [a, F_A], \delta_A F_A \rangle \, d\mu.
\]

Now again let \( \nabla \) denote the coupling of \( A \) and the Levi-Civita connection, and note that

\[
-2\langle [a, F_A], \delta_A F_A \rangle = -2\langle a_c^H F_c^d H F_d^H I - F_d^G H a^c H I \rangle (\nabla e F^{edij}_G) =
\]

\[
2\langle a_c^H F_c^d H F_d^G H \nabla_e F^{edij}_G - F_d^G H a^c H I \nabla_e F^{edij}_G \rangle =
\]

\[
2\langle a_c^H F_d^H I (-1) \nabla_e F^{edij}_G - a^c H I F_d^G H (-1) \nabla_e F^{edij}_G \rangle =
\]

\[
-2\langle a_c^H F_d^H I (\delta_A F)^{dj}_G + a^c H I (\delta_A F)^{dj}_G F_d^G H \rangle = -2\langle a, [\delta_A F, F_A] \rangle.
\]

In most of the expressions in (31), we have omitted the subscript \( A \) from \( F_A \), but we have included the indices of \( F_A \). From (28), (30), and (31), it follows that

\[
\frac{d}{dt} s(A + ta) \bigg|_{t=0} = \int_M \left( 2\langle d_A a, Q^A_2 F_A \rangle - 2\langle a, [\delta_A F, F_A] \rangle \right) d\mu =
\]

\[
\int_M 2\langle a, (\delta_A Q^A_2 F_A - [\delta_A F_A, F_A]) \rangle \, d\mu,
\]

and the proposition follows. \( \square \)

5. A higher-order analogue of the Yang-Mills equations

We continue the setting and notation of the previous section.

5.1. Definition of \( \mathcal{D} \). The results in the previous section motivate the following definition:

**Definition 5.1.** Define an operator \( \mathcal{D} : A \rightarrow \Omega^1(\text{End}(V)) \) as follows:

\[
\mathcal{D} A := \delta_A Q^A_2 F_A - [\delta_A F_A, F_A].
\]

By comparing Proposition 4.3 to the usual derivation of the Yang-Mills equations as the Euler-Lagrange equations for the functional \( ||F_A||^2 \), one can view the equation \( \mathcal{D} A = 0 \) as a conformally invariant analogue, for dimension 6, of the usual source-free Yang-Mills equations in four dimensions.
5.2. Conformal invariance of $\mathcal{D}$. The fact that the action $\mathcal{S}(A)$ is conformally invariant suggests that the Euler-Lagrange equations $\mathcal{D}A = 0$ are also conformally invariant. This is verified explicitly in the following proposition:

**Proposition 5.2.** On pseudo-Riemannian 6-manifolds, the operator $\mathcal{D}$ is invariant under conformal change of the metric $g$ on $M$.

**Proof.** In this proof, we often omit indices. Note first that $F_A$ and $Q_2^A F_A$ carry weights 0 and $-2$, respectively. Thus

$$\hat{\delta}_A Q_2^A F_A = \hat{\delta}_A (Q_2^A F_A + 2 \delta_A d_A (\Upsilon F_A)) = \delta_A Q_2^A F_A + 2 \delta_A^2 d_A (\Upsilon F_A),$$

by (19) and (25) since $n = 6$. Similarly,

$$- \hat{\delta}_A F_A, F_A] = [- \hat{\delta}_A F_A, F_A] + 2 \iota (d\Upsilon) F_A, F_A],$$

where $\iota (d\Upsilon)$ denotes interior multiplication by $\Upsilon^a$. In the second term on the right-hand side of (32), the bracket notation includes a contraction of the free lower-case index of $\iota (d\Upsilon) F_A$ with the first lower-case index of the second $F_A$ within the brackets. To complete the proof, it suffices to show that

$$\delta_A^2 d_A (\Upsilon F_A) + [\iota (d\Upsilon) F_A, F_A] = 0.$$  

(33)

To do this, we begin by recalling that $d_A F_A = 0$. From this it follows that

$$\delta_A^2 d_A (\Upsilon F_A) = 2 \nabla^c \nabla^d (\Upsilon d F_{ec}) + \nabla^c \nabla^d (\Upsilon_c F_{de}).$$

(34)

Here and for the rest of this proof, we omit the subscript $A$ from $F$. Again, $\nabla$ denotes the coupling of $A$ and the Levi-Civita connection. By (8), (22), and (23), the right-hand side of (33) is equal to

$$R_{ed}^d \bar{\Upsilon} F_{ec} F^G H + R_{ed}^d \bar{\Upsilon} d F_{ec} F^G H - R_{ed}^d \bar{\Upsilon} d F^{ec} F^G H - \frac{1}{2} R_{ed}^d \bar{\Upsilon} d F_{ec} F^G H + \frac{1}{2} R_{ed}^d \bar{\Upsilon} d F^{ec} F^G H + \frac{1}{2} F_{ed}^d \bar{\Upsilon} d F_{ec} F^G H + \frac{1}{2} F_{ed}^d \bar{\Upsilon} d F^{ec} F^G H.$$

(35)

Here we have included the indices associated to $V$ and $V^*$. The sum of the first six terms of (35) is zero, and the sum of the last two terms of (35) is also zero. Thus

$$\delta_A^2 d_A (\Upsilon F_A) = -(\iota (d\Upsilon) F)_e^j F_{ec} F^G H + F_{ec}^G (\iota (d\Upsilon) F)_e^j H = -[\iota (d\Upsilon) F, F]^c G H.$$ 

This establishes (33). □

6. Applications

In this section we specialise to the case that $A$ is the conformal tractor connection of $[\mathcal{BEG}]$. As usual we work on a closed conformal 6-manifold $(M, c)$. In this case, in terms of a metric $g \in c$, our Lagrangian density (26) is

$$\langle F_A, Q_2^A F_A \rangle = 4 A_{abc} \nabla^b P_{ac} - 3 C_{abcd} C_{abc} + 4 C_{abcd} \nabla^b P_{ac} + 4 P_{abc} C_{ade} C_{bed}.$$

(36)

If we work modulo divergences, we may simplify the right-hand side of (36) further. (By divergences, we mean terms of the form $\nabla_i T^i$, where $T^i \in \Gamma(TM \otimes \mathcal{E}[-6]).$)
We find that
\[ S(A) = \int_M \left( 8A_{abc} \nabla^c P^{ab} - JC_{abcd}C^{abcd} + 4P_{ab}C^{ac}C_{bcde} \right) d\mu. \]

For this action we will compute the Euler-Lagrange equations of Proposition 4.3. The main result is Theorem 6.1 below which shows that these equations recover the condition of \( O^6_{ab} = 0 \), where \( O^6_{ab} \) is the Fefferman-Graham obstruction tensor of [FG-2012] [FG-1985] in dimension 6. Thus Proposition 4.3 and the operator of Definition 5.1 so defined, provide a new perspective and way of constructing a symbolic formula for the Fefferman-Graham obstruction tensor \( O^6_{ab} \). This symbolic formula will express \( O^6_{ab} \) in terms of the Levi-Civita connection and tensors associated to the Riemannian curvature tensor.

6.1. The standard tractor bundle. We begin by reviewing some basic facts concerning the standard tractor bundle and the conformal tractor connection. We are interested in pseudo-Riemannian metrics in dimension \( n = 6 \), but our discussion of tractors in this subsection treats pseudo-Riemannian metrics in general dimensions \( n \). We refer the reader to [BEG] and [GP-2003] for further information.

Let a manifold \( M \) of dimension \( n \geq 3 \) and a conformal class \( c \) of pseudo-Riemannian metrics on \( M \) be given. Let \((p,q)\) denote the signature of the metrics in \( c \). The conformal class \( c \) determines a certain \((n+2)\)-dimensional vector bundle over \( M \) commonly known as the (standard) tractor bundle. We let \( T \) (or \( T^B \) in abstract index notation) denote this bundle. The conformal class also determines a connection on \( T \) which we call the tractor connection. Let \( \nabla \) denote this connection. The symbol \( \nabla \) will also denote the coupled Levi-Civita tractor connection. This connection acts on powers of \( T \) in the obvious way.

A choice of metric \( g \in c \) determines a vector bundle isomorphism
\[ I_g : \mathcal{E}[1] \oplus (T^{(0,1)} \otimes \mathcal{E}[1]) \oplus \mathcal{E}[-1] \rightarrow T^B. \]

The chosen metric \( g \) also determines algebraic splitting operators
\[ Y^B \in \Gamma(T^B \otimes \mathcal{E}[-1]), \quad Z^{Bc} \in \Gamma(T^B \otimes T^{(1,0)} \otimes \mathcal{E}[-1]), \quad X^B \in \Gamma(T^B \otimes \mathcal{E}[1]) \]
such that for all \((\sigma, \mu_c, \rho) \in \mathcal{E}[1] \oplus (T^{(0,1)} \otimes \mathcal{E}[1]) \otimes \mathcal{E}[-1], \)
\[ I_g(\sigma, \mu_c, \rho) = \sigma Y^B + \mu_c Z^{Bc} + \rho X^B. \]

The splitting operator \( X^B \) is conformally invariant. When the coupled Levi-Civita tractor connection acts on the three splitting operators, it obeys the following rules:
\[ \nabla_a Y^B = P_{ac} Z^{Bc}, \quad \nabla_a Z^B_{\ c} = -P_{ac} X^B - g_{ac} Y^B, \quad \nabla_a X^B = Z^B_{\ a}. \]

The conformal structure \( c \) also determines a metric \( h \) on \( T^B \) which has signature \((p + 1, q + 1)\). We say that \( h \) is the tractor metric. The tractor connection \( \nabla \) preserves \( h \). We may use \( h \) to raise and lower tractor indices, and this operation commutes with the action of \( \nabla \). One can show that \( Y_B X^B = 1 \) and \( Z_{Bc} Z^{Bc} = g_{ac} \). All other contractions of tractor indices involving pairs of the splitting operators \( Y^B, Z^{Bc}, \) and \( X^B \) are zero.
Let $\Omega_{ab}^{DE}$ denote the curvature of $\nabla^T$. Thus $(\nabla_a \nabla_b - \nabla_b \nabla_a) V^D = \Omega_{ab}^{DE} V^E$ for all $V^D \in \mathcal{T}$. We say that $\Omega_{ab}^{DE}$ is the tractor curvature. One can show that

\begin{equation}
\Omega_{ab}^{DE} = Z^{De} Z^e_e c_{abc} - X^D Z^e_e A_{eab} + X_E Z^{D_e} A_{eab},
\end{equation}

and

\begin{equation}
\nabla^a \Omega_{ac}^{DE} = (n - 4) Z^{D} Z^e_e A_{cde} - X^D Z^e_e B_{ec} + X_E Z^{D_e} B_{ec}.
\end{equation}

Observe that, according to (19), in dimension $n = 4$ the operator $\delta^T_{\nabla}$ is conformally invariant on weight 0 tractor-bundle-valued 2-forms. So $\nabla^a \Omega_{ac}^{DE}$ is conformally invariant in dimension four. On the other hand, in any dimension

\begin{equation}
X_E Z^{Dd} - X^D Z^d_d
\end{equation}

is conformally invariant, by the conformal transformation rules for $Z^{Dd}$ and $Z^d_d$ given in equation (4) of [GP-2003]. Thus (40) provides, up to a constant factor, the standard and well-known conformally invariant injection of 1-forms into the adjoint tractor bundle $\Lambda^2 \mathcal{T}$. Thus in dimension four, the display (39) is recovering the result (5) mentioned in the Introduction.

### 6.2. The obstruction tensor in dimension 6

In this subsection, we will describe symbolic computations which establish the following theorem:

**Theorem 6.1.** Suppose that $n = 6$. Then

\begin{equation}
(\mathcal{D} \nabla^T)_{c}^{DE} = (X_E Z^{Dd} - X^D Z^d_d) 16 \mathcal{O}_c^{6d},
\end{equation}

and hence

\begin{equation}
\mathcal{O}_c^{6d} = \frac{1}{32} (Y^E Z_{De} - Y_D Z^E_e) (\mathcal{D} \nabla^T)_{c}^{DE}.
\end{equation}

Here and throughout our work, our scaling convention for $\mathcal{O}_{ab}^{6}$ is the same as in [GP-2006]. One can expand the right-hand side of (41) and compute a symbolic formula which expresses $\mathcal{O}_e^{6c}$ in terms of the Weyl tensor, the Schouten tensor, and $J$.

**Proof of Theorem 6.1.** We begin by applying Definition 5.1. In this definition, we replace $A$ with $\nabla^T$ and $F$ with the tractor curvature $\Omega$. We work in dimension 6. We conclude that

\begin{equation}
(\mathcal{D} \nabla^T)_{c}^{DE} = (\delta d\delta \Omega)_{c}^{DE} - 4(\delta (P # \Omega))_{c}^{DE} + 2(\delta (J \Omega))_{c}^{DE} - (\delta \Omega)_{bc}^{D} G \Omega^{b}_{c}^{G} E + \Omega^{b}_{c}^{D} G (\delta \Omega)_{bc}^{G} E.
\end{equation}

In (42) and for the rest of this subsection, we omit the subscript $\nabla^T$ from $\delta$ and $d$. Our plan will be to consider the various summands on the right-hand side of (42) separately.

Consider the first summand, namely $(\delta d\delta \Omega)_{c}^{DE}$. From (15), (20), and (39), it follows that

\begin{equation}
(\delta \Omega)_{a_1 a_2}^{D} E = -4 \nabla_{a_1} (Z^{Dd} Z^e_e A_{a_2 de}) + 2 \nabla_{a_1} (X^D Z^e_e B_{e a_2}) - 2 \nabla_{a_1} (X_E Z^{Dc} B_{e a_2}).
\end{equation}
From this and from (20), it follows that in Figure 1. In this formula, we antisymmetrise the indices $a_2d$. Here we again antisymmetrise the indices $a_2d$. The index $c$ participates in the antisymmetrisation. The index $a_2d$ is a free index. One may replace this index with the index $c$ to conform with (42).

Now consider the second summand on the right-hand side of (44), namely $-4(\delta(P\#\Omega))_{a_2d}D_E$. By (37),

$$-4(\delta(P\#\Omega))_{a_2d}D_E = -4P_b^kZ^{\alpha\beta}\partial^c{X}^{c}_{\alpha\beta}\kappa_{cde} + 4P_b^kX^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde} - 4P_b^kX^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde} - 4P_b^kX^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde}. $$

From this and from (20), it follows that

$$-4(\delta(P\#\Omega))_{c}D_E =$$

$$4\nabla^{b}(P^{b}_{c}Z_{\alpha\beta}\partial^c{X}^{c}_{\alpha\beta}\kappa_{cde}) - 4\nabla^{b}(P^{b}_{c}X^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde}) + 4\nabla^{b}(P^{b}_{c}X^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde}) + 4\nabla^{b}(P^{b}_{c}X^{\alpha\beta}Z^{c}_{\alpha\beta}\kappa_{cde}).$$
follows that From (37) and from the fact that the Cotton and Weyl tensors are trace-free, it follows that

\[ -4X^D Z_E^e P_b^k P^{bd} C_{kcede} - 4Y^D Z_E^e P^{dk} C_{kced} - 4Z^{Dd} X_E P_b^k P^{be} C_{kced} \\
- 4Z^{Dd} X_E P_b^k P^{be} A_{kce} + 4X^D Y_E P^{ek} A_{kce} - 4X^D Z_E^e P_b^k \nabla^A A_{kce} \\
- 4X^D Z_E^e (\nabla^b P_b^k) A_{kce} - 4Z^{Db} Z_E^e P_b^k A_{kce} - 4X^D X^D P_b^k P^{bd} A_{kdc} \\
- 4X^D Y^D P^{dk} A_{kdc} + 4X^D Z^{Dd} P_b^k \nabla^A A_{kdc} + 4X^D Z^{Dd}(\nabla^b P_b^k) A_{kdc} \\
+ 4Z^b Z^{Dd} P_b^k A_{dck} - 4X^D Z_E^e P_c^k P^{bd} C_{cbdk} - 4Z^{Dd} X_E P_c^k P^{be} C_{bkde} \\
+ 4Z^{Dd} Z_E^e (\nabla^b P_c^k) C_{bkde} + 4Z^{Dd} Z_E^e P_c^k \nabla^b C_{bkde} + 4X^D X_E P_c^k P^{be} A_{ebk} \\
- 4X^D Z_E^e P_c^k \nabla^b A_{ebk} - 4X^D Z_E^e (\nabla^b P_c^k) A_{ebk} - 4Z^{Db} Z_E^e P_c^k A_{ebk} \\
- 4X^D X^D P_c^k P^{bd} A_{dbk} + 4X^D Z^{Dd} P_c^k \nabla^b A_{dbk} + 4X^D Z^{Dd}(\nabla^b P_c^k) A_{dbk} \\
+ 4Z^b Z^{Dd} P_c^k A_{dbk} \]

Figure 2. Symbolic formula for \(-4(\delta(P\#\Omega))_{c}^{D} E\) in dimension 6

Thus from (37) and the fact that the Cotton tensor and the Weyl tensor are trace-free, it follows that \(-4(\delta(P\#\Omega))_{c}^{D} E\) is given by the formula in Figure 2.

We now consider the third summand on the right-hand side of (42), namely \(2(\delta(J\Omega))_{c}^{D} E\). By (20) and (38),

\[
2(\delta(J\Omega))_{c}^{D} E = -2\nabla^a (J\Omega_{ac}^{D} E) = -2\nabla^a (JZ^{Dd} Z_E^e C_{acde} - JX^D Z_E^e A_{ac} + JX_E Z^{De} A_{ac}).
\]

From (37) and from the fact that the Cotton and Weyl tensors are trace-free, it follows that

\[
2(\delta(J\Omega))_{c}^{D} E = \\
-2Z^{Dd} Z_E^e (\nabla^a J) C_{acde} + 2X^D Z_E^e J P^{ad} C_{acde} - 2Z^{Dd} Z_E^e J \nabla^a C_{acde} \\
+ 2X^D Z^{Dd} J P^{ae} C_{acde} + 2X^D Z_E^e (\nabla^a J) A_{ac} - 2X_E Z^{De} (\nabla^a J) A_{ac} \\
+ 2Z^a Z_E^e J A_{ac} - 2Z_E^e Z^{De} J A_{ac} + 2X^D Z_E^e J \nabla^a A_{ac} \]

\[
-2X_E Z^{De} J \nabla^a A_{ac}. 
\]

The fourth summand is \(-\delta(\Omega)^b_{D G}\Omega^b_{c} G\). By (20), (38) and (39),

\[
-\delta(\Omega)^b_{D G}\Omega^b_{c} G = \\
(2Z^{Dd} Z_G^k A_{bdk} - X^D Z_G^k B_{bk} + X_G Z^{Dk} B_{kb}) \times \\
(Z^a Z_E^e C_{cie} - X^G Z_E^e A_{cie} + X_E Z^{Ge} A_{c}^b). 
\]

From this and from the rules for the tractor metric as applied to the splitting operators \(Y^B, Z^Bc,\) and \(X^B\), it follows that

\[
-\delta(\Omega)^b_{D G}\Omega^b_{c} G = \\
2Z^{Dd} Z_E^e A_{bd} i^C C_{bie} + 2Z^{Dd} X_E A_{bd} e^C C_{bie} - X^D Z_E^e B_{bi} C_{cie} \\
- X^D X_E B^b_i A_{c}^b. 
\]
The fifth summand is $\Omega^b_{\cdot \cdot \cdot} G(\delta\Omega)_{b}^{G} E$. By (20), (38) and (39),

$$\Omega^b_{\cdot \cdot \cdot} G(\delta\Omega)_{b}^{G} E = (Z^{Dd}Z_{G}^{\cdot \cdot \cdot}C_{\cdot \cdot \cdot}^{cdk} - X^{D}Z_{G}^{\cdot \cdot \cdot}A_{\cdot \cdot \cdot}^{b} c + X_{G}Z^{Dk}A_{\cdot \cdot \cdot}^{b} c) \times
$$

$$(-2Z^{G}Z_{E}^{\cdot \cdot \cdot}A_{\cdot \cdot \cdot}^{bie} + X^{G}Z_{E}^{\cdot \cdot \cdot}B_{eb} - X^{G}Z_{G}^{\cdot \cdot \cdot}B_{eb}).$$

Thus

$$\Omega^b_{\cdot \cdot \cdot} G(\delta\Omega)_{b}^{G} E =$$

$$-2Z^{Dd}Z_{E}^{\cdot \cdot \cdot}C_{\cdot \cdot \cdot}^{cd}A_{\cdot \cdot \cdot}^{bie} - Z^{Dd}X_{E}^{\cdot \cdot \cdot}C_{\cdot \cdot \cdot}^{bd}eB_{eb} + 2X^{D}Z_{E}^{\cdot \cdot \cdot}A_{\cdot \cdot \cdot}^{bie} A_{\cdot \cdot \cdot}^{bie} + X^{D}X_{E}^{\cdot \cdot \cdot}A_{\cdot \cdot \cdot}^{eb} B_{eb}. \tag{47}$$

Our next step will be to combine (implicitly) several of the symbolic formulæ we computed above. Note first, however, that in the symbolic formula appearing in Figure 1 the index $a_0$ is a free index. After we perform the antisymmetrisation of the lower indices $a_1$ and $a_2$, we may replace the index $a_0$ with the index $c$. We can then combine the symbolic formulæ appearing in figures 1 and 2 together with the symbolic formulæ appearing on the right-hand sides of (45), (46), and (47). The result will be a symbolic formula for $(\mathfrak{D} \nabla^T)^{D E}$, which we will refer to as the “total” symbolic formula for $(\mathfrak{D} \nabla^T)^{D E}$. As we will now show, one can simplify this total symbolic formula and obtain a new total symbolic formula for $(\mathfrak{D} \nabla^T)^{D E}$ in which every term contains an occurrence of either $X_{E}Z^{Dd}$ or $X^{D}Z^{Dd} e$. Note first that none of the terms in the total symbolic formula for $(\mathfrak{D} \nabla^T)^{D E}$ contain an occurrence of $Y_{E}Y^{D}$.

Now consider the terms in the total symbolic formula that contain $X^{D}Y_{E}$ or $X_{E}Y^{D}$. The sum of these terms is

$$(X^{D}Y_{E} - X_{E}Y^{D})(2P_{a_{1}d}A_{cd}^{a_{1}} - 2P_{c}^{d}A_{a_{1}a_{1}}^{a_{1}} + 2P^{a_{1}d}A_{da_{1}}^{a_{1}} - 2P^{a_{1}d}A_{cd}^{a_{1}} + \nabla_{a_{1}}A_{cd}^{a_{1}a_{1}} - \nabla_{e}B_{a_{1}a_{1}a_{1}} + 4P^{e}a_{1}A_{ekc}). \tag{48}$$

This follows by inspection and the fact that $A_{abc}$ is antisymmetric in $b$ and $c$. But $P_{ab}$ is symmetric, and $A_{abc}$ and $B_{ab}$ are trace-free. Thus (48) is equal to

$$(X^{D}Y_{E} - X_{E}Y^{D})(\nabla_{a}B_{ac}^{a} + 2P^{e}a_{1}A_{ekc}).$$

A short symbolic computation using [W] and [L], along with (3), (9), (11), (12), and (13), shows that $\nabla_{a}B_{ac}^{a} + 2P^{e}a_{1}A_{ekc} = 0$ in dimension 6. Thus (48) is zero.

Next, consider the terms in the total symbolic formula for $(\mathfrak{D} \nabla^T)^{D E}$ in which $X^{D}X_{E}$ appears. The sum of these terms is

$$X^{D}X_{E}(2P_{a_{1}d}P_{a_{1}e}A_{cde} - 2P_{c}^{d}P_{a_{1}e}A_{a_{1}de} + 2P^{a_{1}d}P_{a_{1}e}A_{a_{1}de} - 2P^{a_{1}d}P_{a_{1}e}A_{a_{1}de} + 4P_{b}^{k}P^{b}A_{a_{1}b} - 4P_{b}^{k}P^{b}A_{a_{1}b} + 4P_{b}^{k}P^{b}A_{a_{1}b} - 4P_{b}^{k}P^{b}A_{a_{1}b} + A_{eb}^{e}B_{eb}). \tag{49}$$

Note that $P_{a_{1}d}P_{a_{1}e}$ is symmetric in $d$ and $e$ and that $A_{cde}$ is antisymmetric in $d$ and $e$. Thus in the parenthesised expression in (49), terms 1 and 3 are zero. Since $A_{a_{1}de}$ is antisymmetric in $d$ and $e$, it follows that terms 2 and 4 cancel. Terms 5 and 6 cancel, terms 7 and 8 cancel, and terms 9 and 10 cancel. Thus (49) is zero.
This follows from inspection of the total symbolic formula, from the fact that the symmetry of\( Y \) simplification of its left-hand side. To perform this simplification, one may use \( [W] \) and \( [L] \), one can show that the right-hand side of the equation is zero. To do

\[
(Y^D Z_{Ei} - Y_E Z^{D_i})(4B_{ci} - 4(\nabla^a_1 A_{a1i} + P^{kd} c_{kdci}) + 2\nabla^a_1 A_{a1ci}).
\]

It is well-known that \( \nabla_a A^a_{bc} = 0 \). (See formula (2.7) of \[GN\], for example.) Thus (51) is zero, by (3), and hence (50) is zero as well.

Now consider the terms of the total symbolic formula for \( (\mathcal{D} \nabla^T)^c D_E \) in which \( Z^{Di} Z_E^k \), \( Z^D_c Z_E^e \), or \( Z^{De} Z_E^c \) appears.

We now consider the terms in the total symbolic formula for \( (\mathcal{D} \nabla^T)^c D_E \) in which \( Y^D Z_{Ei} \), \( Y_E Z^{D_i} \), or \( Y_E Z^{D_i} \) appears. Here \( i \) is a dummy index. The total symbolic formula may use some index other than \( i \) here. We will also consider the terms of the total symbolic formula in which \( Y^D Z_{Ec} \) or \( Y_E Z^D \) appears. The sum of the terms under consideration is as follows:

\[
\begin{align*}
&= Z^{Di} Z_E^k \left( -4P_i^d A_{cdk} + 2P_c^d A_{cdk} - 2P^{a1}_i A_{ca1k} + 2P^{a1}_i A_{a1ck} - 2P_c^e A_{cie} + 2P_c^e A_{kie} \\
&\quad - 2P^{a1}_i A_{cia1} + 2P^{a1}_i A_{a1ic} + 2\nabla^a_1 A_{cik} - 2\nabla^a_1 A_{a1ik} - \nabla_i B_{kc} - \nabla_i B_{kc} \\
&\quad + \nabla_c B_{ki} + \nabla_k B_{ic} + \nabla_k B_{ic} - \nabla_c B_{ik} + 4(\nabla^b P^a) C_{acik} + 4P^a \nabla^b C_{acik} \\
&\quad - 4P^a A_{kac} + 4P^a A_{iac} + 4(\nabla^b P^a) C_{bak} + 4P_c^a \nabla^b C_{bak} - 4P^a A_{kia} \\
&\quad + 4P_i^a A_{ika} - 2(\nabla^a J) C_{acik} - 2J \nabla^a C_{acik} + 2J A_{kic} - 2J A_{kic} + 2A_b^c C^b_{cak} \\
&\quad - 2C^b_{ci} A_{bak} \\
&\quad + Z^D_c Z_E^e \nabla^b_{ea1} - \nabla^D_{Ec} \nabla^a_{be1} \\
&\quad = Z^{Di} Z_E^k \left( -4P_i^d A_{cdk} + 2P_c^d A_{cdk} + 4\nabla^a_1 A_{cik} - 2\nabla^a_1 A_{a1ik} - 2\nabla_i B_{kc} + 2\nabla_k B_{ic} \\
&\quad + 2(\nabla^a J) C_{acik} + 4P^a \nabla^b C_{acik} - 4P_i^d A_{kac} + 4P^a A_{iac} + 4(\nabla^b P^a) C_{bak} \\
&\quad + 4P_i^a \nabla^b C_{bak} - 2J \nabla^a C_{acik} + 2J A_{kic} - 2J A_{kic} + 2A_b^c C^b_{cak} - 2C^b_{ci} A_{bak} \\
&\quad + Z^D_c Z_E^e \nabla^b_{ea1} - \nabla^D_{Ec} \nabla^a_{be1} \\
&= (Y^D Z_{Ei} - Y_E Z^{D_i})(-2\nabla^a_1 A_{a1i} + 2\nabla^a_1 A_{a1ci} - 2\nabla_a A_{a1i} + 2\nabla_a A_{a1i} \\
&\quad + \nabla^a_1 B_{ic} - \nabla^a_1 B_{ia1} - B_{ic} - 4P^d k c_{kdci}) \\
&\quad + (Y^D Z_{Ec} - Y_E Z^{D_c})(B^a_{ci}).
\end{align*}
\]

This follows from inspection of the total symbolic formula, from the fact that \( A_{abc} \) is antisymmetric in \( b \) and \( c \), and from the fact that \( C_{abcd} \) is antisymmetric in \( c \) and \( d \). Now recall that the Cotton tensor is trace-free and the Bach tensor is symmetric and trace free. Since \( n = 6 \), it follows that (51) is equal to (52)

\[
(Y^{Di} Z^E_i - Y^E Z^{Di})(4B_{ci} - 4(\nabla^a_1 A_{a1i} + P^{kd} c_{kdci}) + 2\nabla^a_1 A_{a1i}).
\]

This follows from the fact that \( A_{abc} \) is antisymmetric in \( b \) and \( c \), and from the fact that \( C_{abcd} \) is antisymmetric in \( c \) and \( d \). Now recall that the Cotton tensor is trace-free and the Bach tensor is symmetric and trace free. Since \( n = 6 \), it follows that (51) is equal to (52)

\[
(Y^{Di} Z^E_i - Y^E Z^{Di})(4B_{ci} - 4(\nabla^a_1 A_{a1i} + P^{kd} c_{kdci}) + 2\nabla^a_1 A_{a1i}).
\]

This follows from the fact that \( A_{abc} \) is antisymmetric in \( b \) and \( c \), and from the fact that \( C_{abcd} \) is antisymmetric in \( c \) and \( d \). Now recall that the Cotton tensor is trace-free and the Bach tensor is symmetric and trace free. Since \( n = 6 \), it follows that (51) is equal to (52).

Now consider the terms of the total symbolic formula for \( (\mathcal{D} \nabla^T)^c D_E \) in which \( Z^{Di} Z_E^k \), \( Z^D_c Z_E^e \), or \( Z^{De} Z_E^c \) appears. Here \( i, k, \) and \( e \) are dummy indices; in the total symbolic formula, any of these indices may appear as a lower index. The sum of the terms under consideration is given by the symbolic expression on the left-hand side of the equation in Figure 3. This follows from direct inspection of the total symbolic formula for \( (\mathcal{D} \nabla^T)^c D_E \). The equation in the figure follows from simplification of its left-hand side. To perform this simplification, one may use the symmetry of \( B_{ab} \), the antisymmetry of \( A_{abc} \) in \( b \) and \( c \), and (11). By using (11) and (11), one can show that the right-hand side of the equation is zero. To do
\[ X_E Z^{Di} (-2 P_{a_1} e^{\nabla a_1} A_{cie} + 2 P_{e} e^{\nabla a_1} A_{a_1ie} - 2(\nabla^{a_1} P_{a_1} e^{a}) A_{cie} + 2(\nabla^{a_1} P_{e} e^{a}) A_{a_1ie} \\
- 2 P_{a_1} e^{\nabla a_1} A_{cie} + 2 P_{a_1} e^{\nabla a_1} A_{a_1ie} + P_{i}^{e} B_{ee} - P_{a_1}^{e} B_{ic} + P_{a_1}^{e} B_{ia} + \nabla^{a_1}\nabla_{a_1} B_{ic} \\
- \nabla^{a_1}\nabla_{B_{ia}} - 4 P_{k}^{e} B_{bkc} C_{ckie} + 4 P_{k}^{e} B_{bk} A_{ikc} + 4(\nabla_{b} P_{k}^{e}) A_{ikc} - 4 P_{k}^{e} B_{bkc} C_{bkie} \\
+ 4 P_{k}^{e} B_{bk} A_{ikc} + 4(\nabla_{b} P_{k}^{e}) A_{ikc} + 2 J P^{ae} C_{aicie} - 2(\nabla^{a_1} J) A_{iac} - 2 J \nabla^{a} A_{iac} \\
+ 2 A_{bi} e^{a} A_{b}^{i} - C^{b}_{a_1} e^{a} B_{eb}) \\
- Z_{D_{c}}^{D_{c}} X_{E} P^{a_1e} B_{ea_1} \\
+ X_{D}^{D_{E}} (-2 P_{a_1} e^{\nabla a_1} A_{cdi} + 2 P_{e} e^{\nabla a_1} A_{a_1di} - 2(\nabla^{a_1} P_{a_1} d) A_{cdi} + 2(\nabla^{a_1} P_{e} d) A_{a_1di} \\
- 2 P_{a_1} e^{\nabla a_1} A_{cdi} + 2 P_{a_1} e^{\nabla a_1} A_{a_1di} + P_{i}^{e} B_{ce} - P_{a_1}^{e} B_{ic} + P_{a_1}^{e} B_{ia} - \nabla^{a_1}\nabla_{a_1} B_{ic} \\
+ \nabla^{a_1}\nabla_{B_{ic}} - P_{i}^{e} B_{ec} - 4 P_{k}^{e} B_{bkc} C_{ckie} - 4 P_{k}^{e} B_{bk} A_{ikc} - 4(\nabla_{b} P_{k}^{e}) A_{ikc} \\
- 4 P_{k}^{e} B_{bkc} C_{bkie} - 4 P_{k}^{e} B_{bk} A_{ikc} - 4(\nabla_{b} P_{k}^{e}) A_{ikc} + 2 J P^{ae} C_{a dici} + 2(\nabla^{a} J) A_{iac} \\
+ 2 J \nabla^{a} A_{iac} - B_{a}^{b} C_{cai} + 2 A_{bi} e^{a} A_{b}^{i}) \\
+ Z_{E_{c}}^{D_{E}} X_{D}^{D_{E}} P^{a_1e} B_{ea_1} \\
= \\
(X_{E} Z^{Di} - X_{D}^{D_{E}} (-2 P_{a_1} e^{\nabla a_1} A_{cie} + 2 P_{e} e^{\nabla a_1} A_{a_1ie} - 2(\nabla^{a_1} P_{a_1} e^{a}) A_{cie} \\
+ 2(\nabla^{a_1} P_{e} e^{a}) A_{a_1ie} - 2 P_{a_1} e^{\nabla a_1} A_{cie} + 2 P_{a_1} e^{\nabla a_1} A_{a_1ie} + P_{i}^{e} B_{ee} - P_{a_1}^{e} B_{ic} \\
+ P_{a_1}^{e} B_{ia} + \nabla^{a_1}\nabla_{a_1} B_{ic} - \nabla^{a_1}\nabla_{B_{ia}} - 4 P_{k}^{e} B_{bkc} C_{ckie} + 4 P_{k}^{e} B_{bk} A_{ikc} \\
+ 4(\nabla_{b} P_{k}^{e}) A_{ikc} - 4 P_{k}^{e} B_{bkc} C_{bkie} + 4 P_{k}^{e} B_{bk} A_{ikc} + 4(\nabla_{b} P_{k}^{e}) A_{ikc} + 2 J P^{ae} C_{aicie} \\
- 2(\nabla^{a} J) A_{iac} - 2 J \nabla^{a} A_{iac} + 2 A_{bi} e^{a} A_{b}^{i} - C^{b}_{a_1} e^{a} B_{eb} - g^{ie}_{a_1} P^{a_1e} B_{ea_1}) \\
= \\
(X_{E} Z^{Di} - X_{D}^{D_{E}} (-8 P_{ab} \nabla^{a} A_{(ic)b} + 2 P_{e} e^{\nabla a_1} A_{a_1ie} - 4 A_{(ic)a} \nabla^{a} J \\
+ 2(\nabla^{a_1} P_{e} e^{a}) A_{a_1ie} + 2 P^{a_1e} \nabla_{a} A_{a_1ie} + P_{i}^{e} B_{ee} - 3 J B_{ic} + 5 P_{k}^{e} B_{ik} + \Delta B_{ic} \\
- \nabla^{a_1}\nabla_{B_{ia}} + 4 P_{k}^{e} B_{bkc} C_{ckie} + 4(\nabla_{b} P_{k}^{e}) A_{ikc} - 2 A_{bie} A_{c}^{b} + B_{eb} C_{cie}^{b} \\
- g^{ie}_{a_1} P^{a_1e} B_{ea_1}) \\
\]

Figure 4. Sum of the terms in the total symbolic formula for \((D^{\nabla T}c)_{DE}\) in which \(X_{E} Z^{Di}, X_{D}^{D_{E} i}, X_{E} Z^{D_{c}}, \) or \(X_{D}^{D_{E} c}\) occurs.

This, one may begin with the right-hand side of the equation and use (3) and (12) and to express \(A_{abc}\) and \(B_{ab}\) in terms of \(P_{ab}\) and \(C_{abcd}\). If one then applies (2) in a suitable way, all derivatives of order greater than 1 cancel. One may then use the Bianchi identities and (13) to show that the remaining terms cancel.

Finally, consider the terms of the total symbolic formula for \((D^{\nabla T}c)_{DE}\) in which \(X_{E} Z^{Di}, X_{D}^{D_{E} i}, X_{E} Z^{D_{c}}, \) or \(X_{D}^{D_{E} c}\) occurs. Here \(i\) is a dummy index, as before. In the total symbolic formula, this \(i\) may appear as a lower index. The sum of the terms under consideration is given by the leftmost member of the equation in Figure 4. This follows from direct inspection of the total symbolic formula for \((D^{\nabla T}c)_{DE}\). The equation in the figure follows from the symmetries of \(A_{abc}\) and \(C_{abcd}\) and from (3) and (11). In dimension \(n = 6\), the rightmost member of this equation is equal to \((X_{E} Z^{Dd} - X_{D}^{D_{E} d}) 16 \mathcal{C}_{cd}\). This follows from a direct
computation using [W] and [L], together with (3), (9), (11), (12), and (13). The proof of Theorem 6.1 is thus complete. □

6.3. A comment on the Fefferman-Graham invariant in dimension 6. Fefferman and Graham introduced an interesting natural scalar conformal invariant $FGI$ in Proposition 3.4 of [FG-1985] (see also [CG]). This invariant has conformal weight $-6$ and so on closed conformal 6-manifolds,

$$\int_M FGI \, d\mu$$

is a well-defined global conformal invariant. Moreover at leading order $FGI$ is quadratic in the jets of the metric. Thus (52) is an important action to consider for metric variations. In fact, it is closely linked to the specialisation to the tractor connection of our action (27) as follows.

Let $A$ denote the tractor connection, and let $F_A$ denote the tractor curvature as given in (38). Then in dimension $n = 6$,

$$FGI = 2 \langle F_A, Q^A F_A \rangle + 8C_{abcd} C^a e^c C^{bedi} - 4C_{abcd} C^a e^c C^{bdie} + (\text{divergences}).$$

Here $\langle \cdot, \cdot \rangle$ and $Q^A$ are as in Definition 4.1. The equality in (53) follows from a short symbolic computation using the computer algebra system Mathematica, [W], together with Lee’s tensor calculus software package Ricci, [L].

In even dimensions, elementary representation theory shows that the Weyl tensor (and its irreducible parts in dimension 4) and the Fefferman-Graham obstruction tensor are the only conformal invariants that at leading order are linear in the jets of the metric [GH]. Using this and Theorem 6.1 it is not difficult to conclude that with respect to metric variations both (27) (with $A$ the conformal tractor connection) and (52) must yield $O^6_{ee}$ as their functional gradient, up to conformally invariant lower-order terms.

References

[AO] G. Anastasiou, R. Olea: From conformal to Einstein gravity. Phys. Rev. D 94 (2016), no. 8, 086008, 4 pp.

[B-1921] R. Bach: Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs. Math. Z. 9 (1921), no. 1-2, 110-135.

[Bz] J.C. Baez: Higher Yang-Mills Theory. arXiv:hep-th/0206130

[BEG] T.N. Bailey, M.G. Eastwood, A.R. Gover: Thomas’s structure bundle for conformal, projective and related structures. Rocky Mountain J. Math. 24 (1994), no. 4, 1191-1217.

[BSS] E. Bergshoeff, A. Salam, E. Sezgin: Supersymmetric $R^2$ actions, conformal invariance and the Lorentz Chern-Simons term in 6 and 10 dimensions. Nuclear Phys. B 279 (1987), no. 3-4, 659-683.

[Br-1993] T.P. Branson: The functional determinant. Global Analysis Research Center Lecture Notes Series, Number 4 (Seoul National University, 1993).

[Br-1981] T.P. Branson: Quasi-invariance of the Yang-Mills equations under conformal transformations and conformal vector fields. J. Differential Geometry 16 (1981), no. 2, 195-203.

[Br-1995] T.P. Branson: Sharp inequalities, the functional determinant, and the complementary series. Trans. Amer. Math. Soc. 347 (1995), no. 10, 3671-3742.
A Conformally Invariant Yang-Mills Type Energy

[BG-2005] T.P. BRANSON, A.R. GOVER: Conformally invariant operators, differential forms, cohomology and a generalisation of Q-curvature. Comm. Partial Differential Equations 30 (2005), no. 10-12, 1611-1669.

[BG-2007] T.P. BRANSON, A.R. GOVER: Pontrjagin forms and invariant objects related to the Q-curvature. Commun. Contemp. Math. 9 (2007), no. 3, 335-358.

[BrO] T.P. BRANSON, B. ØRSTED: Explicit functional determinants in four dimensions. Proc. Amer. Math. Soc. 113 (1991), no. 3, 669-682.

[CGGGR] E. CALVI˜NO-LOUZAO, X. GARCÍA-MARTÍNEZ, E. GARCÍA-RÍO, I. GUTIÉRREZ-RODRÍGUEZ, R. VÁZQUEZ-LORENZO: Conformally Einstein and Bach-flat four-dimensional homogeneous manifolds. J. Math. Pures Appl. (9) 130 (2019), 347-374.

[CG] A. ĆAP, A.R. GOVER: Standard tractors and the conformal ambient metric construction. Ann. Global Anal. Geom. 24 (2003), no. 3, 231-259.

[C] É. CARTAN: Les groupes d’holonomie des espaces généralisés. Acta Math. 48 (1926), no. 1-2, 1-42.

[CLW] X. CHEN, C. LEBRUN, B. WEBER: On conformally Kähler, Einstein manifolds. J. Amer. Math. Soc. 21 (2008), no. 4, 1137-1168.

[D] S.K. DONALDSON: An application of gauge theory to four-dimensional topology. J. Differential Geom. 18 (1983), no. 2, 279-315.

[DK] S.K. DONALDSON, P.B. KRONHEIMER: The geometry of four-manifolds. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1990.

[FG-2012] C. FEFFERMAN, C.R. GRAHAM: The ambient metric. Princeton University Press, Princeton, NJ, 2012.

[FG-1985] C. FEFFERMAN, C.R. GRAHAM: Conformal invariants. In: Elie Cartan et les mathématiques d’aujourd’hui, 95-116. Astérisque, Numéro Hors Série, Société Mathématique de France, Paris, 1985.

[FP-1984] E.S. FRADKIN, M.YA. PALCHIK: Conformal invariance in quantum Yang-Mills theory. Phys. Lett. B 147 (1984), no. 1-3, 86-90.

[FP-1978] E.S. FRADKIN, M.YA. PALCHIK: Recent developments in conformal invariant quantum field theory. Phys. Rep. 44 (1978), no. 5, 249-349.

[GN] A.R. GOVER, P. NUROWSKI: Obstructions to conformally Einstein metrics in n dimensions. J. Geom. Phys. 56 (2006), no. 3, 450-484.

[GP-2006] A.R. GOVER, L.J. PETERSON: The ambient obstruction tensor and the conformal deformation complex. Pacific J. Math. 226 (2006), no. 2, 309-351.

[GP-2003] A.R. GOVER, L.J. PETERSON: Conformally invariant powers of the Laplacian, Q-curvature, and tractor calculus. Comm. Math. Phys. 235 (2003), no. 2, 339-378.

[GSS] A.R. GOVER, P. SOMERBERG, V. SOUČEK: Yang-Mills detour complexes and conformal geometry. Comm. Math. Phys. 278 (2008), no. 2, 307-327.

[GH] C.R. GRAHAM, K. HIRACHI: The ambient obstruction tensor and Q-curvature. In: AdS/CFT correspondence: Einstein metrics and their conformal boundaries, 59-71. IRMA Lect. Math. Theor. Phys., 8, Eur. Math. Soc., Zürich, 2005.

[GV] M.J. GURSKY, J.A. VIACLOVSKY: Rigidity and stability of Einstein metrics for quadratic curvature functionals. J. Reine Angew. Math. 700 (2015), 37-91.

[H] G. HUANG: Rigidity of Riemannian manifolds with positive scalar curvature. Ann. Global Anal. Geom. 54 (2018), no. 2, 257-272.

[JR] R. JACWIW, C. REBBI: Conformal properties of a Yang-Mills pseudoparticle. Phys. Rev. D (3) 14 (1976), no. 2, 517-523.

[K] C. KELLEHER: Higher order Yang-Mills Flow. Calc. Var. Partial Differential Equations 58 (2019), no. 3, Paper No. 100, 45 pp.

[LeB] C. LEBRUN: Bach-flat Kähler surfaces. J. Geom. Anal. 30 (2020), no. 3, 2491-2514.

[L] J.M. LEE: “Ricci” software package. sites.math.washington.edu/~lee/Ricci, 2016, Version 1.61.
[M] J. Maldacena: *Einstein gravity from conformal gravity*. arXiv:1105.5632.

[Mer] S.A. Merkulov: *A conformally invariant theory of gravitation and electromagnetism*. Classical Quantum Gravity 1 (1984), no. 4, 349-354.

[Tb] C. Taubes: *Self-dual Yang-Mills connections on non-self-dual 4-manifolds*. J. Differential Geometry 17 (1982), no. 1, 139-170.

[T] T.Y. Thomas: *On conformal geometry*. Proc. Natl. Acad. Sci. USA 12 (1926), no. 5, 352-359.

[tH] G. ’t Hooft, ed.: *50 years of Yang-Mills theory*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[W] Wolfram Research, Inc.: *“Mathematica” software*. Champaign, Illinois, 2016, Version 11.0.

[YM] C.N. Yang, R.L. Mills: *Conservation of isotopic spin and isotopic gauge invariance*. Phys. Rev. (2) 96 (1954), 191-195.

[Z] P. Zhang: *Gradient flows of higher order Yang-Mills-Higgs Functionals*. J. Aust. Math. Soc. (2021) 1-31. doi:10.1017/S1446788721000057.

A.R.G.: Department of Mathematics, The University of Auckland, Private Bag 92019, Auckland 1142, New Zealand

Email address: r.gover@auckland.ac.nz

L.J.P.: Department of Mathematics, The University of North Dakota, 101 Cornell Street Stop 8376, Grand Forks, ND 58202-8376, USA

Email address: lawrence.peterson@und.edu

Email address: callumsleigh@gmail.com