VARIANCE OF OPERATORS AND DERIVATIONS

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Abstract. The variance of a bounded linear operator $a$ on a Hilbert space $\mathcal{H}$ at a unit vector $\xi$ is defined by $D_\xi(a) = \|a\xi\|^2 - |\langle a\xi, \xi \rangle|^2$. We show that two operators $a$ and $b$ have the same variance at all vectors $\xi \in \mathcal{H}$ if and only if there exist scalars $\sigma, \lambda \in \mathbb{C}$ with $|\sigma| = 1$ such that $b = \sigma a + \lambda 1$ or $a$ is normal and $b = \sigma a^* + \lambda 1$. Further, if $a$ is normal, then the inequality $D_\xi(b) \leq \kappa D_\xi(a)$ holds for some constant $\kappa$ and all unit vectors $\xi$ if and only if $b = f(a)$ for a Lipschitz function $f$ on the spectrum of $a$. Variants of these results for $C^*$-algebras are also proved, where vectors are replaced by pure states.

We also study the related, but more restrictive inequalities $\|bx - xb\| \leq |az - xa|$ supposed to hold for all $x \in B(\mathcal{H})$ or for all $x \in B(H^\kappa)$ and all $n \in \mathbb{N}$. We consider the connection between such inequalities and the range inclusion $d_b(B(H)) \subseteq d_a(B(H))$, where $d_a$ and $d_b$ are the derivations on $B(H)$ induced by $a$ and $b$. If $a$ is subnormal, we study these conditions in particular in the case when $b$ is of the form $b = f(a)$ for a function $f$.

1. INTRODUCTION AND NOTATION

The expected value of a quantum mechanical quantity represented by a self-adjoint operator $a$ on a complex Hilbert space $\mathcal{H}$ in a state $\omega$ is $\omega(a)$, while the variance of $a$ is defined by $D_\omega(a) = \omega(a^*a) - |\omega(a)|^2$. If $a$ is the multiplication by a bounded measurable function on $L^2(\mu)$ for a probability measure $\mu$ and $\omega$ is the state $x \mapsto \langle x1, 1 \rangle$, where $1 \in L^2(\mu)$ is the constant function, these notions reduce to the classical notions of probability calculus. We may define the variance by the same formula for all (not necessarily selfadjoint) operators $a \in B(\mathcal{H})$. For a general vector state $\omega(a) := \langle a\xi, \xi \rangle$, coming from a unit vector $\xi \in \mathcal{H}$, the variance $D_\omega(a) = \|a\xi\|^2 - |\langle a\xi, \xi \rangle|^2$ means just the square of the distance of $a\xi$ to the set of all scalar multiples of $\xi$. (Thus $D_\omega(a) = \eta_a(\xi)^2$, where $\eta$ is the function considered by Brown and Pearcy in [7].) We will prove that an operator $a$ is almost determined by its variances: if $a, b \in B(\mathcal{H})$ are such that $D_\omega(a) = D_\omega(b)$ for all vector states $\omega$ then $b = \alpha a + \beta 1$ or $a$ is normal and $b = \alpha a^* + \beta 1$ for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ (Theorem 2.3). We will also deduce a variant of this statement for $C^*$-algebras, where vector states are replaced by pure states.

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Then we will study the inequality

\[(1.1) \quad D_\omega(b) \leq \kappa D_\omega(a),\]

where \(\kappa\) is a positive constant (which may be taken to be 1 if we replace \(b\) by \(\kappa^{-1/2}b\)). If (1.1) holds for all vector states \(\omega\), then we will show that there exists a Lipschitz function \(f : \sigma_{ap}(a) \to \sigma_{ap}(b)\), where \(\sigma_{ap}(\cdot)\) denotes the approximate point spectrum, such that if \(a\) is normal then \(b = f(a)\) (Theorem 3.5). For a general \(a\), however, \(f\) is perhaps not nice enough to allow the definition of \(f(a)\). Therefore we will also consider stronger variants of (1.1).

For \(2 \times 2\) matrices (1.1) implies that \(b = \alpha a + \beta 1\) for some scalars \(\alpha, \beta \in \mathbb{C}\) (Lemma 3.2). But for general operators the condition (1.1) is not very restrictive for it does not even imply that \(b\) commutes with \(a\). For example, if \(a\) is hyponormal (1.1) holds with \(b = a^*\). A simple computation (Lemma 4.1) shows, however, that for a vector state \(\omega = \omega_\xi\) the quantity \(D_\omega(a)\) is just the square of the norm of the operator \(d_a(\xi \otimes \xi^*)\), where \(d_a\) is the derivation on \(B(\mathcal{H})\), defined by \(d_a(x) = ax - xa\), and \(\xi \otimes \xi^*\) is the rank one operator on \(\mathcal{H}\), defined by \((\xi \otimes \xi^*)\eta = \langle \eta, \xi_\xi \rangle\). Thus we will also study the condition

\[(1.2) \quad \|d_b(x)\| \leq \kappa \|d_a(x)\| \quad (\forall x \in B(\mathcal{H})),\]

where \(a, b \in B(\mathcal{H})\) and \(\kappa > 0\) are fixed. We will show (Theorem 4.2) that if equality holds in (1.2) and \(\kappa = 1\) then either \(b = \sigma a + \lambda 1\) for some scalars \(\sigma, \lambda \in \mathbb{C}\) with \(|\sigma| = 1\) or there exist a unitary \(u\) and scalars \(\alpha, \beta, \lambda, \mu \in \mathbb{C}\) with \(|\beta| = |\alpha|\) such that \(a = \alpha u^* + \lambda 1\) and \(b = \beta u + \mu 1\). This will also be generalized to \(C^*\)-algebras.

For a normal \(a\) Johnson and Williams [21] proved that the condition (1.2) is equivalent to the range inclusion \(d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))\). Their work was continued by several researchers, including Williams [37], Fong [16], Kissin and Shulman [23], Bresar [8] and in [9] in different contexts, but still restricted to special classes of operators \(a\) (such as normal, isometric or algebraic). It is known that the range inclusion \(d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))\) does not imply (1.2) in general since it does not even imply that \(b\) is in the bicommutant \((a)''\) of \(a\) [20]. However in Theorem 5.8 we will prove that conversely (1.2) implies the range inclusion if \(a\) is a direct sum \(a_1 \oplus a_2\), where the commutant \((a_1)'\) of \(a_1\) contains a bounded net of trace class operators converging strongly to the identity, while \((a_2)'\) does not contain any nonzero trace-class operator. Examples include all normal operators, isometries and cyclic subnormal operators. The author does not know of any operators \(a, b\) satisfying (1.2) for which the range inclusion does not hold. The corresponding purely algebraic problem for operators on an (infinite dimensional) vector space \(V\), where \(B(\mathcal{H})\) is replaced by the algebra \(L(V)\) of all linear operators on \(V\) and the condition (1.2) is replaced by the inclusion of the kernels \(\ker d_a \subseteq \ker d_b\), has a positive answer [26].

By the Hahn-Banach theorem the inclusion \(d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))\) (the norm closure) is equivalent to the requirement that for each \(\rho \in B(\mathcal{H})^2\) (the dual of \(B(\mathcal{H})\)) the condition \(a_\rho - \rho a = 0\) implies \(b_\rho - \rho b = 0\), where \(a_\rho\) and \(b_\rho\) are functionals on \(B(\mathcal{H})\) defined by \((a_\rho)(x) = \rho(xa)\) and \((b_\rho)(x) = \rho(ax)\). The operator spaces \(B(\mathcal{H})\) and \(B(\mathcal{H})^2\) are quite different (if \(\mathcal{H}\) is infinite dimensional), so in general we can not expect a strong connection between (1.2) and a formally similar condition

\[\|b_\rho - \rho b\| \leq \kappa \|a_\rho - \rho a\| \quad (\forall x \in B(\mathcal{H})^2).\]
Question. Does (1.2) imply at least that the centralizer $C_a$ of $a$ in $B(H)^2$ (that is, the set of all $\rho \in B(H)^2$ satisfying $a\rho = \rho a$) is contained in $C_b$?

Using C*-algebraic tools, the above question can be reduced to the corresponding question in the Calkin algebra, and answered affirmatively if $a$ is essentially normal (Corollary 5.11). We recall that an operator $a$ is essentially normal if $a^*a - aa^*$ is compact.

A stronger condition than (1.2), namely that (1.2) holds for all $x \in M_n(B(H))$ and all $n \in \mathbb{N}$ (where $a$ and $b$ are replaced by the multiples $a^{(n)}$ and $b^{(n)}$ acting on $H^n$), implies that (1.2) holds in any representation of the C*-algebra generated by $a, b$ and $1$ (Lemma 6.1) and that $b$ is contained in the C*-algebra generated by $a$ and $1$ (Corollary 6.2). In a special situation (when $H$ is a cogenerator for Hilbert modules over the operator algebra $A_0$ generated by $a$ and $1$) it follows that $b$ must be in $A_0$ (Proposition 6.3). If $a$ is, say, subnormal (a restriction of a normal operator to an invariant subspace), this means that $b = f(a)$ for a function $f$ in the uniform closure of polynomials on $\sigma(a)$. Perhaps for a general subnormal operator $a$ (1.2) does not imply that $b = f(a)$ for a function $f$, but when it does, it forces on $f$ certain degree of regularity. For example, if $a$ is the operator of multiplication on the Hardy space $H^2(G)$ by the identity function on $G$, where $G$ is a domain in $\mathbb{C}$ bounded by finitely many nonintersecting analytic Jordan curves, (1.2) implies that $b$ is an analytic Toeplitz operator with a symbol $f$ which is continuous also on the boundary of $G$ (Proposition 5.9).

Let us call a complex function $f$ on a compact set $K \subseteq \mathbb{C}$ a Schur function if the supremum over all (finite) sequences $\lambda = (\lambda_1, \lambda_2, \ldots) \subseteq K$ of norms of matrices

$$\Lambda(f; \lambda) = \left[ \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \right],$$

regarded as Schur multipliers, is finite. (Here the quotient is interpreted as $0$ if $\lambda_i = \lambda_j$.) If $a$ is normal the work of Johnson and Williams [21] tells us that $b = f(a)$ satisfies (1.2) if and only if $f$ is a Schur function on $\sigma(a)$. In the ‘only if’ direction we extend this to general subnormal operators (Lemma 5.5), in the other direction only to subnormal operators with nice spectra (Theorem 5.7).

In the last section we will investigate the condition (1.2) in the case when $a$ is subnormal and $b = f(a)$ for a function $f$. If $a$ is normal, a known effective method of studying such commutator estimates is based on double operator integrals (see [2] and the references there), which are defined via spectral projection valued measures. But, since invariant subspaces of a normal operator $c$ are not necessarily invariant under the spectral projections of $c$, we will use a different method. In Section 7 we will ‘construct’ for a given subnormal operator $a$ and suitable function $f$ on $\sigma(a)$ a completely bounded map $T_{a,f}$ on $B(H)$ such that $aT_{a,f}(x) - T_{a,f}(x)a = f(a)x - xf(a)$ for all $x \in B(H)$. For $b = f(a)$ this implies (1.2) and also the range inclusion $d_b(B(H)) \subseteq d_a(B(H))$. Thus, by the above mentioned result from [21] even if $a$ is normal the functions $f$ considered here must be Schur. By [21] every Schur function on $\sigma(a)$ is complex differentiable relative to $\sigma(a)$ at each nonisolated point of $\sigma(a)$ (thus holomorphic on the interior of $\sigma(a)$) and $f'$ is bounded. The construction of $T_{a,f}$ applies to a subclass that includes all functions for which $f'$ is Lipschitz of order $\alpha > 0$; only if $\sigma(a)$ is nice enough are we able to find $T_{a,f}$ for all Schur functions.
We will denote by \( \overline{S} \) the norm closure and by \( \overline{S}^* \) the weak* closure of a subset \( S \) in \( B(\mathcal{H}) \).

2. Variance of operators

**Definition 2.1.** For a bounded operator \( a \) on a Hilbert space \( \mathcal{H} \) and a vector \( \xi \in \mathcal{H} \) let

\[
D_\xi(a) = (\|a\xi\|^2\|\xi\|^2 - |\langle a\xi, \xi \rangle|^2)/\|\xi\|^4.
\]

Thus, if \( \xi \) is a unit vector and \( \omega : x \mapsto \langle x\xi, \xi \rangle \) is the corresponding vector state on \( B(\mathcal{H}) \), then

\[
D_\xi(a) = \omega(a^*a) - |\omega(a)|^2,
\]
and this formula can be used to define the variance \( D_\omega(a) \) of \( a \) in any (not just vector) state \( \omega \).

**Remark 2.2.** (i) It is clear from the definition that \( D_\xi(a) \) is just the square of the distance of \( a\xi \) to the set \( \mathbb{C}\xi \) of scalar multiples of \( \xi \). Hence, if \( D_\xi(b) \leq D_\xi(a) \) for all \( \xi \in \mathcal{H} \), then in particular each eigenvector of \( a \) is also an eigenvector for \( b \).

(ii) \( D_\xi(aa + \beta b) = |\alpha|^2D_\xi(a) \) for all \( a, b \in B(\mathcal{H}) \) and \( \alpha, \beta \in \mathbb{C} \).

(iii) \( D_\xi(a^*) = D_\xi(a) \) for all \( \xi \in \mathcal{H} \) if and only if \( a \) is normal.

**Theorem 2.3.** If operators \( a, b \in B(\mathcal{H}) \) satisfy \( D_\xi(b) = D_\xi(a) \) for all \( \xi \in \mathcal{H} \), then there exist \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| = 1 \) such that \( b = \alpha a + \beta a^* + \beta \).

**Proof.** For any two vectors \( \xi, \eta \in \mathcal{H} \) we expand the function

\[
f(z) := D_{\xi + z\eta}(a) = \|a(\xi + z\eta)\|^2\|\xi + z\eta\|^2 - |\langle a(\xi + z\eta), \xi + z\eta \rangle|^2
\]

of the complex variable \( z \) into powers of \( z \) and \( \overline{z} \),

\[
f(z) = D_\xi(a) + 2Re(D_1z) + 2Re(D_2z^2) + D_3|z|^2 + 2Re(D_4|z|^2z) + D_\eta(a)|z|^4.
\]

Among the coefficients \( D_j \) we will need to know only \( D_2 \), which is

\[
D_2 = \langle \eta, a\xi \rangle \langle \eta, \xi \rangle - \langle \eta, a\xi \rangle \langle \eta, a\xi \rangle.
\]

Thus, from the equality \( D_{\xi + z\eta}(a) = D_{\xi + z\eta}(b) \), by considering the coefficients of \( z^2 \) we obtain

\[
\langle b\eta, b\xi \rangle \langle \eta, \xi \rangle - \langle b\eta, \xi \rangle \langle b\xi, \eta \rangle = \langle \eta, a\xi \rangle \langle \eta, \xi \rangle - \langle \eta, a\xi \rangle \langle \eta, a\xi \rangle.
\]

From this we see that if \( \eta \) is orthogonal to \( \xi \) and \( a\xi \) then \( \eta \) must be orthogonal to \( b\xi \) or to \( b^*\xi \). In other words, if for a fixed \( \xi \) we denote

\[
\mathcal{H}_0(\xi) = \{\xi, a\xi\}^\perp, \quad \mathcal{H}_1(\xi) = \{\xi, a\xi, b\xi\}^\perp, \quad \mathcal{H}_2(\xi) = \{\xi, a\xi, b^*\xi\}^\perp,
\]

then \( \mathcal{H}_0(\xi) = \mathcal{H}_1(\xi) \cup \mathcal{H}_2(\xi) \). Since \( \mathcal{H}_j(\xi) \) are vector spaces, this implies that \( \mathcal{H}_j(\xi) = \mathcal{H}_0(\xi) \) or else \( \mathcal{H}_2(\xi) = \mathcal{H}_0(\xi) \). In the first case we have \( b\xi \in \mathbb{C}\xi + \mathbb{C}a\xi \), while in the second case \( b^*\xi \in \mathbb{C}\xi + \mathbb{C}a\xi \). Since this holds for all \( \xi \in \mathcal{H} \), it follows that \( \mathcal{H} \) is the union of the two sets

\[
F_1 = \{\xi \in \mathcal{H} : b\xi \in \mathbb{C}\xi + \mathbb{C}a\xi\} \quad \text{and} \quad F_2 = \{\xi \in \mathcal{H} : b^*\xi \in \mathbb{C}\xi + \mathbb{C}a\xi\}.
\]

Since \( F_1 \) and \( F_2 \) are closed, by Baire’s theorem at least one of them has nonempty interior \( \overset{0}{F}_j \). We will consider only the case when \( \overset{0}{F}_1 \neq \emptyset \) for the other case is similar (except that in the end the observation that \( D_\xi(a^*) = D_\xi(a) \) (\( \forall \xi \in \mathcal{H} \) implies the normality of \( a \) is used). We may assume that \( a \) is not a scalar multiple of the
identity, otherwise the proof is trivial. Then there exists a vector \( \xi \in \hat{F}_1 \) such that \( \xi \) and \( a\xi \) are linearly independent. (Namely, if \( a\xi = \alpha \xi \) for all \( \xi \in \hat{F}_1 \), where \( \alpha \in \mathbb{C} \), then considering this equality for the vectors \( \xi, \zeta \) and \( 1/2(\xi + \zeta) \) in \( \hat{F}_1 \), where \( \xi \) and \( \zeta \) are linearly independent, it follows easily that \( \alpha \xi \) must be independent of \( \xi \) for \( \xi \) in an open subset of \( \mathcal{H} \), hence \( a \) must be a scalar multiple of 1.) Let

\[
U = \{ \xi \in \hat{F}_1 : \xi \text{ and } a\xi \text{ are linearly independent} \}.
\]

Then for any \( \xi, \eta \in U \) such that the ‘segment’ \( \xi(z) = (1 - z)\xi + z\eta \ (|z| \leq 1) \) is contained in \( U \), we have

\[
(2.1) \quad b\xi(z) = \alpha(z) a\xi(z) + \beta(z)\xi(z)
\]

for some scalars \( \alpha(z), \beta(z) \in \mathbb{C} \). To see that the coefficients \( \alpha \) and \( \beta \) are holomorphic functions of \( z \), for any fixed \( z_0 \) with \( |z_0| \leq 1 \) we take the inner product of both sides of (2.1) with the vectors \( \xi(z_0) \) and \( a\xi(z_0) \) to obtain two equations from which we compute \( \alpha(z) \) and \( \beta(z) \) by Cramer’s rule (if \( z \) is near \( z_0 \)). But from the condition \( D_{\xi(z)}(b) = D_{\xi(z)}(a) \) and (2.1) we also conclude that \( |\alpha(z)| = 1 \), hence \( \alpha \) must be constant (for a fixed \( \xi \) and \( \eta \)). Setting in (2.1) first \( z = 0 \) and then \( z = 1 \) we get

\[
(2.2) \quad b\xi = \alpha a\xi + \beta(0)\xi \quad \text{and} \quad b\eta = \alpha a\eta + \beta(1)\eta,
\]

where the constant \( \alpha \) (with \( |\alpha| = 1 \)) is the same for all vectors \( \xi, \eta \) in an open subset of \( \mathcal{H} \). Namely, the coefficients in (2.2) are unique since \( \eta \) and \( a\eta \) are linearly independent for all \( \eta \in U \), hence by fixing \( \eta \) and varying \( z \) in (2.2) we see that \( \alpha \) must be independent of \( \xi \). From (2.2) we have now that \( (b - \alpha a)\xi \in \mathbb{C}\xi \) for all vectors \( \xi \) in an open subset of \( \mathcal{H} \) (with a constant \( \alpha \)), which easily implies that \( b - \alpha a = \beta 1 \) for some constant \( \beta \in \mathbb{C} \).

**Corollary 2.4.** If elements \( a, b \) in a \( C^* \)-algebra \( A \subseteq B(\mathcal{H}) \) satisfy \( D_\omega(b) = D_\omega(a) \) for all pure states \( \omega \) on \( A \), then there is a projection \( p \) in the center \( Z \) of the weak* closure \( R \) of \( A \) and central elements \( u_1, z_1 \in Rp, \ u_2, z_2 \in Rp^\perp \), with \( u_1, u_2 \) unitary, such that \( bp = u_1 a + z_1 \) and \( bp^\perp = u_2 a^* + z_2 \) and \( ap^\perp \) is normal.

**Proof.** Since the condition \( D_\omega(b) = D_\omega(a) \) persists for all weak* limits of pure states on \( A \) and such states are precisely the restrictions of weak* limits of pure states on \( R \) by [17, Theorem 5], the proof immediately reduces to the case \( A = R \). Let \( Z \) be the center of \( R \), \( \Delta \) the maximal ideal space of \( Z \), for each \( t \in \Delta \) let \( Rt \) be the closed ideal of \( R \) generated by \( t \) and set \( R(t) := R/(Rt) \). For any \( a \in R \) let \( a(t) \) denotes the coset of \( a \) in \( R(t) \). Since each pure state on \( R(t) \) can be lifted to a pure state on \( R \), we have \( D_\omega(b(t)) = D_\omega(a(t)) \) for each pure state \( \omega \) on \( R(t) \) and each \( t \in \Delta \). Since \( R(t) \) is a primitive \( C^* \)-algebra by [19], it follows from Theorem 2.3 that there exist scalars \( \alpha(t), \beta(t) \), with \( |\alpha(t)| = 1 \), such that

\[
(2.3) \quad b(t) = \alpha(t)a(t) + \beta(t)1
\]

or

\[
(2.4) \quad b(t) = \alpha(t)a(t)^* + \beta(t)1 \quad \text{and} \quad a \text{ is normal}.
\]

Let \( F_t \) be the set of all \( t \in \Delta \) for which (2.3) holds, \( F_2 \) the set of all those \( t \) for which (2.4) holds and \( U \) the set of all \( t \) such that \( a(t) \) is not a scalar. Since for each \( x \in R \) the function \( t \mapsto \|x(t)\| \) is continuous on \( \Delta \) by [17], it is easy to see that \( U \) is open and \( F_1, F_2 \) are closed.
To show that the coefficients \( \alpha \) and \( \beta \) in (2.3) and (2.4) are continuous functions of \( t \) on \( U \), let \( t \in U \) be fixed, note that the center of \( R(t) \) is \( \mathbb{C}1 \) and that \( R(t) \) is generated by projections, so there is a projection \( p_t \in R(t) \) such that \( (1-p_t)a(t)p_t \neq 0 \). We may lift \( 1-p_t \) and \( p_t \) to positive elements \( x, y \) in \( R \) with \( xy = 0 \) \cite{22, 4.6.20}. Then from (2.3)

\[
(2.5) \quad x(s)b(s)y(s) = \alpha(s)x(s)a(s)y(s) \quad (\forall s \in \Delta),
\]

and \( \|(xay)(s)\| \neq 0 \) for \( s \) in a neighborhood of \( t \) by continuity. Now let \( c \in Z \) be the element whose Gelfand transform is the function \( s \mapsto \|x(s)a(s)y(s)\| \) and let \( \phi : R \to Z \) be a bounded \( Z \)-module map such that \( \phi(xay) = c \). (Such a map may be obtained simply as the completely bounded \( Z \)-module extension to \( R \) of the map \( Z(xay) \to Z \), \( z \mapsto z(xay) \)), since \( Z \) is injective \cite{6}.) Since \( \phi \) is a \( Z \)-module map, \( \phi \) is just a collection of maps \( \phi_x : R(s) \to Z(s) = C \), hence from (2.5) we obtain \( \alpha(s)c(s) = (\phi(xby))(s) \). Since \( c(t) = \|x(t)a(t)y(t)\| \neq 0 \), it follows that \( \alpha \) is continuous in a neighborhood of \( t \), hence continuous on \( U \). Then, denoting by \( q_0 \) the projection corresponding to a clopen neighborhood \( U_0 \subseteq U \) of \( t \), we have from (2.3) that \( \beta(t)q_0(t) = e(t) \), where \( e = (bq_0 - caq_0) \in Rq_0 \), hence \( \beta U_0 \) represents a central element of \( Rq_0 \) and is therefore continuous.

Since \( \Delta \) (hence also \( \overline{U} \)) is a Stonean space and \( \alpha \) (hence also \( \beta \)) are bounded continuous functions, they have continuous extensions to \( \overline{U} \) (see \cite{22, p. 324}). If \( q \in Z \) is the projection that corresponds to \( \overline{U} \), then \( aq \perp 1 \) is a scalar in \( Rq \perp 1 \), and it follows easily that \( bq \perp 1 \) must also be a scalar. So we have only to consider the situation in \( Rq \), which means that we may assume that \( \overline{U} = \Delta \), hence that \( \alpha \) and \( \beta \) are defined and continuous throughout \( \Delta \). The interior \( F := F_1 \) of \( F_1 \) is a clopen subset such that (2.3) holds for \( t \in F \). Since the complement \( F^c = \overline{\mathbb{F}}_1 \) is contained in \( F_2 \), (2.4) holds if \( t \in F^c \). Finally, let \( p \in Z \) be the projection that corresponds to \( F \), and let \( u_1, z_1 \in Zp \), \( u_2, z_2 \in Zp^\perp \) be elements that corresponds to functions \( \alpha|F \), \( \beta|F \), \( \alpha|F^c \) and \( \beta|F^c \) (respectively).

We note that the converse of Corollary 2.4 also holds, the proof follows easily from the well-known fact \cite{22, p. 268} that if \( \omega \) is a pure state on a \( C^* \)-algebra \( R \) then \( \omega(xz) = \omega(x)\omega(a) \) for all \( x \in R \) and all \( z \) in the center of \( R \).

3. The Inequality \( D_\xi(b) \leq D_\xi(a) \)

**Lemma 3.1.** For any two operators \( a, b \in B(\mathcal{H}) \) and any state \( \omega \) on \( B(\mathcal{H}) \) the following estimate holds:

\[
|D_\omega(b) - D_\omega(a)| \leq 2\|b - a\| \|\omega\| \|a\|.
\]

**Proof.** Since \( |\omega(b^*b - a^*a)| \leq \|b^*b - a^*a\| = \|(b^* - a^*)b + a^*(b - a)\| \leq \|b - a\| (\|b\| + \|a\|) \) and \( \|\omega(b)\|^2 - |\omega(a)|^2 = (|\omega(b)| + |\omega(a)|) \|\omega(b)\| - |\omega(a)| \leq (\|a\| + \|b\|) \|b - a\| \), we have

\[
|D_\omega(b) - D_\omega(a)| = |\omega(b^*b - a^*a) - (|\omega(b)|^2 - |\omega(a)|^2)|
\]

\[
\leq |\omega(b^*b - a^*a) + (|\omega(b)|^2 - |\omega(a)|^2)| \leq 2\|b - a\| (\|a\| + \|b\|).
\]

\( \square \)

**Lemma 3.2.** Let \( a, b \in \mathbb{M}_2(\mathbb{C}) \) (\( 2 \times 2 \) complex matrices), \( 0 < \varepsilon < 1/2 \), and let \( \alpha_i \) and \( \beta_i \) \( (i = 1, 2) \) be the eigenvalues of \( a \) and \( b \) (respectively).
(i) If $D_ξ(b) ≤ D_ξ(a)$ for all unit vectors $ξ ∈ C^2$, then $b = θa + τ$ for some scalars $θ, τ ∈ C$ with $|θ| ≤ 1$.

(ii) If $D_ξ(b) ≤ D_ξ(a) + ε^8$ for all unit vectors $ξ ∈ C^2$, then

$$|β_2 - β_1| ≤ |α_2 - α_1| + 2ε(∥a∥ + 2∥b∥ + 1).$$

Proof. (i) Since $D_ξ(a) = D_ξ(a - λ1)$ for all $λ ∈ C$, we may assume that one of the eigenvalues of $a$ is 0, say $aξ_2 = 0$ for a unit vector $ξ_2 ∈ C^2$. Then from $0 ≤ D_ξ(b) ≤ D_ξ(a) = 0$ we see that $ξ_2$ is also an eigenvector for $b$, hence (replacing $b$ by $b - λ1$ for a $λ ∈ C$) we may assume that $bξ_2 = 0$. So, choosing a suitable orthonormal basis $\{ξ_1, ξ_2\}$ of $C^2$, we may assume that $a$ and $b$ are of the form

$$a = \begin{bmatrix} α_1 & γ \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} β_1 & δ \\ 0 & 0 \end{bmatrix}.$$  

Now we compute for any unit vector $ξ = (λ, µ) ∈ C^2$ (using $|λ|^2 + |µ|^2 = 1$) that

$$D_ξ(a) = ∥aξ∥^2 - |⟨αξ, ξ⟩|^2 = |λ|^2 |µ|^2 - |⟨α, µ\rangle|^2$$

$$= (|µ|^2|λ|^2 + |γ|^2|µ|^2 + 2Re(αγλµ)).$$

Using this and a similar expression for $D_ξ(b)$, the condition $D_ξ(b) ≤ D_ξ(a)$ can be written as

$$∥aξ∥^2 - |⟨αξ, ξ⟩|^2 = |α_1|^2|γ|^2 + (|γ|^2|µ|^2 + 2Re(αγλµ)) \geq 0,$$

which means that the matrix

$$M = \begin{bmatrix} |α_1|^2 - |β_1|^2 & α_1γ - β_1δ \\ \overline{α_1γ - β_1δ} & |γ|^2 - |δ|^2 \end{bmatrix}$$

is nonnegative. This is equivalent to the conditions

$$|β_1| ≤ |α_1|, \quad |δ| ≤ |γ| \quad \text{and} \quad \det M ≥ 0.$$  

Since $det M = -|α_1δ - β_1γ|^2$, the condition $det M ≥ 0$ means that $α_1δ = β_1γ$. If $α_1 ≠ 0$, it follows that $b$ is of the form

$$b = \begin{bmatrix} β_1 \\ 0 \end{bmatrix} = \frac{β_1}{α_1}a = θa, \quad \text{where} \quad θ := \frac{β_1}{α_1}, \quad \text{hence} \quad |θ| ≤ 1.$$  

If $α_1 = 0$, then $β_1 = 0$ (since $|β_1| ≤ |α_1|$), hence again $b = θa$, where $θ = δ/γ$ if $γ ≠ 0$.

(ii) As above, replacing $a$ and $b$ by $a - λ1$ and $b - µ1$, where $λ$ and $µ$ are eigenvalues of $a$ and $b$, we may assume that $a$ and $b$ are of the form

$$a = \begin{bmatrix} α_1 & γ \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} β_1 & δ_1 \\ 0 & 0 \end{bmatrix}.$$  

The norms of the new $a$ and $b$ are at most two times greater than the norm of original ones, which will be taken into account in the final estimate. If $ξ = (1, 0)$, then $D_ξ(b) = |δ_2|^2$ and $D_ξ(a) = 0$, hence the condition $D_ξ(b) ≤ D_ξ(a) + ε^8$ shows that $|δ_2| ≤ ε^4$. Thus, denoting by $b_0$ the matrix

$$b_0 = \begin{bmatrix} β_1 & δ_1 \\ 0 & 0 \end{bmatrix},$$

we have that $∥b - b_0∥ ≤ ε^4$, hence by Lemma 3.1

$$|D_ξ(b) - D_ξ(b_0)| ≤ 2ε^4(∥b∥ + ∥b_0∥) ≤ 4∥b∥ε^4$$

for all unit vectors $ξ ∈ C^2$. 

It follows that
\[ D_\varepsilon(b_0) \leq D_\varepsilon(a) + 4\|b\|\varepsilon^4 + \varepsilon^8 \leq D_\varepsilon(a) + \varepsilon^4(4\|b\| + 1). \]
The same calculation that led to (3.1) shows now that
\[ \|\mu\|^2[(|\alpha_1|^2 - |\beta_1|^2)|\lambda|^2 + (|\gamma|^2 - |\delta_1|^2)|\mu|^2 + 2\text{Re}((\alpha_1\overline{\gamma} - \beta_1\overline{\delta_1})\lambda\overline{\mu})] \geq -\varepsilon^4(4\|b\| + 1) \]
for all \( \lambda, \mu \in \mathbb{C} \) with \( |\lambda|^2 + |\mu|^2 = 1 \). We may choose the arguments of \( \lambda \) and \( \mu \) so that \( (\alpha_1\overline{\gamma} - \beta_1\overline{\delta_1})\lambda\overline{\mu} \) is negative, hence the above inequality implies that
\[ t(|\alpha_1|^2 - |\beta_1|^2)(1 - t) + (|\gamma|^2 + |\delta_1|^2)t - 2|\alpha_1\overline{\gamma} - \beta_1\overline{\delta_1}|\sqrt{t(1 - t)} \geq -\varepsilon^4(4\|b\| + 1) \]
for all \( t \in [0, 1] \). Setting \( t = \varepsilon^2 \), it follows that
\[ (|\alpha_1|^2 - |\beta_1|^2)(1 - \varepsilon^2) + (\|a\|^2 + \|b\|^2)^2 \geq -\varepsilon^2(4\|b\| + 1), \]
hence \( |\alpha_1|^2 - |\beta_1|^2 \geq -\varepsilon^2(\|a\|^2 + \|b\|^2 + 4\|b\| + 1) \), so
\[ |\beta_1| \leq |\alpha_1| + \varepsilon(\|a\| + 2\|b\| + 1). \]
Taking into account that \( \alpha_2 \) and \( \beta_2 \) were initially reduced to 0 (by which the norms of \( a \) and \( b \) may have increased at most by a factor 2), this proves (ii).

The approximate point spectrum of an operator \( a \) will be denoted by \( \sigma_{ap}(a) \).

**Definition 3.3.** If \( a, b \in B(H) \) are such that \( D_\varepsilon(b) \leq D_\varepsilon(a) \) for all \( \varepsilon \in H \), then we can define a function \( f : \sigma_{ap}(a) \rightarrow \sigma_{ap}(b) \) as follows. Given \( \alpha \in \sigma_{ap}(a) \), let \( (\xi_n) \) be a sequence of unit vectors in \( H \) such that \( \lim \|a - \alpha\xi_n\| = 0 \). Then from the condition \( D_\varepsilon(b) \leq D_\varepsilon(a) \) we conclude that \( \lim \|b - \lambda_n\xi_n\| = 0 \), where \( \lambda_n = (b\xi_n, \xi_n) \). We will show that the sequence \( (\lambda_n) \) converges, so we define
\[ f(\alpha) = \lim \lambda_n. \]

**Proposition 3.4.** The function \( f \) is well-defined and Lipschitz: \( |f(\beta) - f(\alpha)| \leq |\beta - \alpha| \) for all \( \alpha, \beta \in \sigma_{ap}(a) \).

**Proof.** Given \( \varepsilon > 0 \), choose unit vectors \( \xi, \eta \in H \) such that
\[ \|(a - \alpha1)\xi\| < \varepsilon \quad \text{and} \quad \|(a - \beta1)\eta\| < \varepsilon. \]
Let \( p \) be the projection onto the span of \( \{\xi, \eta\} \) and let \( c \) be the operator on \( pH \) defined by \( c\xi = \alpha\xi \) and \( c\eta = \beta\eta \). Then
\[ \|a|_{pH} - c\|^2 \leq \|(a - c)\xi\|^2 + \|(a - c)\eta\|^2 < 2\varepsilon^2. \]
Let \( \lambda = \langle b\xi, \xi \rangle \), \( \mu = \langle b\eta, \eta \rangle \) and \( d \) the operator on \( pH \) defined by \( d\xi = \lambda\xi \) and \( d\eta = \mu\eta \). Then, using the conditions \( D_\varepsilon(b) \leq D_\varepsilon(a) \) and \( D_\varepsilon(b) \leq D_\varepsilon(a) \), we have
\[ ||b|_{pH} - d||^2 \leq ||(b - \lambda1)\xi||^2 + ||(b - \mu1)\eta||^2 < 2\varepsilon^2. \]
Now by Lemma 3.1 and Remark 2.2(i) and since \( ||d|| \leq ||b|| \), \( ||c|| \leq ||a|| \) we infer from (3.2) and (3.3) that
\[ D_\varepsilon(d) \leq D_\varepsilon(b) + 4||d||_{pH}\|b\| < D_\varepsilon(b) + 4\varepsilon\sqrt{2} \]
and \( D_\varepsilon(c) > D_\varepsilon(a) - 4\varepsilon\sqrt{2} \), hence (since \( D_\varepsilon(b) \leq D_\varepsilon(a) \))
\[ D_\varepsilon(d) \leq D_\varepsilon(c) + 8\varepsilon\sqrt{2} \]
for all \( \xi \in H \) with \( \|\xi\| = 1 \).

By Lemma 3.2 (ii) we now conclude that
\[ |\mu - \lambda| \leq |\beta - \alpha| + \kappa\varepsilon, \]
where \( \kappa \) is a constant. This completes the proof of Proposition 3.4.
where \( \kappa \) is a constant.

If \( (\xi_n) \) and \( (\eta_n) \) are two sequences of unit vectors in \( \mathcal{H} \) such that \( \lim \| (a - a_0)\xi_n \| = 0 \) and \( \lim \| (a - b_0)\eta_n \| = 0 \), we infer from (3.4) (since \( \varepsilon \) can be taken to tend to 0 as \( n \to \infty \)) that

\[
\lim \sup |\mu_n - \lambda_n| \leq |\beta - \alpha|.
\]

Further, if \( \beta = \alpha \) and we put in (3.4) \( \lambda_n = \langle b\xi_n, \xi_n \rangle \) instead of \( \lambda_n = \langle b\xi_n, \xi_n \rangle \) instead of \( \mu_n \), we conclude that \( (\lambda_n) \) is a Cauchy sequence, hence it converges to a point \( \lambda \in \mathbb{C} \). From \( \lim \| (b - \lambda_1)\xi_n \| = 0 \) it follows now that \( \lim \| (b - \lambda_1)\xi_n \| = 0 \), hence \( \lambda \in \sigma_{ap}(b) \). Similarly the sequence \( (\mu_n) = \langle (b\eta_n, \eta_n) \rangle \) converges to some \( \mu \) and (3.5) implies that

\[
|\mu - \lambda| \leq |\beta - \alpha|.
\]

This shows that \( f \) is a well-defined Lipschitz function. \( \square \)

**Theorem 3.5.** Let \( a, b \in B(\mathcal{H}) \). If \( a \) is normal, then there exists a constant \( \kappa \) such that \( D_\xi(b) \leq \kappa D_\xi(a) \) for all \( \xi \in \mathcal{H} \) if and only if \( b = f(a) \) for a Lipschitz function \( f \) on \( \sigma(a) \). In this case \( D_\omega(b) \leq \kappa D_\omega(a) \) for all states \( \omega \).

**Proof.** Assume that \( D_\xi(b) \leq D_\xi(a) \) for all \( \xi \in \mathcal{H} \). We may assume that \( a \) is not a scalar (otherwise the proof is trivial). First consider the case when \( a \) can be represented by a diagonal matrix \( \text{diag}(\alpha_j) \) in some orthonormal basis \( (\xi_j) \) of \( \mathcal{H} \). If \( f : \sigma(a) \to \sigma_{ap}(b) \) is defined as in Definition 3.3, then \( b\xi_j = f(\alpha_j)\xi_j \) for all \( j \), hence \( b = f(a) \).

For a general normal \( a \), first suppose that \( \mathcal{H} \) is separable. Then by Voiculescu’s version of the Weyl-von Neumann-Beigl theorem [36], given \( \varepsilon > 0 \), there exists a diagonal normal operator \( c = \text{diag}(\gamma_j) \) such that \( \| a - c \| < \varepsilon \), where \( \| \cdot \| \) denotes the Hilbert-Schmidt norm. Let \( (\xi_j) \) be an orthonormal basis of \( \mathcal{H} \) consisting of eigenvectors of \( c \), so that \( c\xi_j = \gamma_j\xi_j \). Since \( D_\xi(b) \leq D_\xi(a) \), by Remark 2.2(i) there exist scalars \( \beta_j \in \mathbb{C} \) such that

\[
\| (b - \beta_j\xi_j) \| \leq \| (a - \gamma_j\xi_j) \| = \| (a - c)\xi_j \|.
\]

Hence

\[
\sum_j \| (b - \beta_j)\xi_j \|^2 \leq \sum_j \| (a - c)\xi_j \|^2 < \varepsilon^2.
\]

In particular \( \| b - d \| < \varepsilon \), where \( d \) is the diagonal operator defined by \( d\xi_j = \beta_j\xi_j \). Since \( d \) and \( c \) commute, it follows that

\[
\| bc - cb \| = \| (b - d)c - c(b - d) \| < 2\varepsilon \| c \| \leq 2\varepsilon(\| a \| + \varepsilon) \leq 4\varepsilon \| a \| \quad (\text{if } \varepsilon \leq \| a \|),
\]

hence also

\[
\| ba - ab \| = \| (bc - cb) + b(a - c) - (a - c)b \| \leq 4\varepsilon(\| a \| + \| b \|).
\]

Since this holds for all \( \varepsilon > 0 \), it follows that \( a \) and \( b \) commute. If \( a \) has a cyclic vector this already implies that \( b \) is in \( (a)^\sigma \) hence a measurable function of \( a \), but in general we need an additional argument to prove this. Let \( f : \sigma(a) \to \sigma_{ap}(b) \) be defined as in Definition 3.3. (Note that \( \sigma(a) = \sigma_{ap}(a) \) since \( a \) is normal.) Let \( e(\cdot) \) be the projection valued spectral measure of \( a, \xi \in \mathcal{H} \) any separating vector for the von Neumann algebra \( (a)^\sigma \) generated by \( a \) and \( \varepsilon > 0 \). If \( \alpha \) is any point in \( \sigma(a) \), \( U \) is any Borel subset of \( \sigma(a) \) containing \( a \) and \( \xi_U := e(U)\xi \in (a)^\sigma(\xi) \), then \( \| (a - \alpha\xi_U) \| \to 0 \) as the diameter of \( U \) shrinks to 0. For each \( U \) let \( \beta_U = \langle b\xi_U, \xi_U \rangle \) so that \( \| (b - \beta_U)\xi_U \| \leq \| (a - \alpha\xi_U) \| \); then \( f(\alpha) = \lim_{U \to \alpha} \beta_U \) by the definition of \( f \). Thus, since by Proposition 3.4 \( f \) is a Lipschitz function, for each \( \alpha \in \sigma(a) \) there is an open neighborhood \( U_{\alpha} \) with the diameter at most \( \varepsilon \) such that
\[ |f(\alpha) - \beta_U| < \varepsilon \] for all Borel subsets \( U \subseteq U_\alpha \) and \( |f(\alpha_2) - f(\alpha_1)| < \varepsilon \) if \( \alpha_1, \alpha_2 \in U_\alpha \).

By compactness we can cover \( \sigma(\alpha) \) with finitely many such neighborhoods \( U_\alpha \), and this covering then determines a partition of \( \sigma(\alpha) \) into finitely many disjoint Borel sets \( \Delta_j \) (say \( j = 1, \ldots, n \)) such that each \( \Delta_j \) is contained in some \( U_{\alpha_{ij}(j)} \). Let \( e_j = e(\Delta_j) \). Now we can estimate, denoting \( \beta_j = \beta_{\Delta_j} \),

\[
\|(b - f(a)) e_j \xi\| \leq \|(b - \beta_j) e_j \xi\| + ||\beta_j - f(a)\| e_j \xi\| + \|(f(a_{ij(j)})) - f(a)\| e_j \xi\| \\
\leq ||(a - \alpha_j) e_j \xi\| + ||(\beta_j - f(a)\| e_j \xi\| + \|(f(a_{ij(j)})) - f(a)\| e_j \xi\| \\
\leq 3\varepsilon \|e_j \xi\|.
\]

(Here we have used the spectral theorem to estimate the term \( \|(f(a_{ij(j)})) - f(a)\| e_j \xi\| \) from above by \( \sup_{\alpha \in \Delta_j} |f(a_{ij(j)})) - f(a)\| ||e_j \xi\| \). Since \( b \) commutes with \( a \), hence also with all spectral projections of \( a \), it follows that

\[
\|(b - f(a)) e_j \xi\|^2 = \left\| \sum_{j=1}^{n} e_j (b - f(a)) e_j \xi\right\|^2 = \sum_{j=1}^{n} \|e_j (b - f(a)) e_j \xi\|^2 \\
\leq 9\varepsilon^2 \sum_{j=1}^{n} \|e_j \xi\|^2 = 9\varepsilon^2 \|\xi\|^2.
\]

Thus \( \|(b - f(a)) \xi\| \leq 3\varepsilon \|\xi\| \) and, since this holds for all \( \varepsilon > 0 \) and separating vectors are dense in \( \mathcal{H} \), we conclude that \( b = f(a) \).

If \( \mathcal{H} \) is not necessarily separable, \( \mathcal{H} \) can be decomposed into an orthogonal sum of separable subspaces \( \mathcal{H}_k \) that reduce both \( a \) and \( b \) and are such that \( \sigma(\alpha|\mathcal{H}_k) = \sigma(\alpha) \).

For each \( k \) there exists a Lipschitz function \( f_k \) such that \( b|\mathcal{H}_k = f(a|\mathcal{H}_k) \). Since for any two \( k, j \) the space \( \mathcal{H}_k \oplus \mathcal{H}_j \) is also separable, there also exists a function \( f \) such that \( b|\mathcal{H}_j \oplus \mathcal{H}_k = f(a|\mathcal{H}_j \oplus \mathcal{H}_k) \) and it follows easily that \( f_k = f = f_j \). Thus \( b = f(a) \).

Conversely, if \( b = f(a) \) for a function \( f \) such that

\[ |f(\alpha_2) - f(\alpha_1)| \leq \kappa |\alpha_2 - \alpha_1| \]

for all \( \alpha_1, \alpha_2 \in \sigma(\alpha) \) and some constant \( \kappa \), then for a fixed unit vector \( \xi \in \mathcal{H} \) denote by \( \mu \) the probability measure on Borel subsets of \( \sigma(\alpha) \) defined by \( \mu(\cdot) = \langle e(\cdot) \xi, \xi \rangle \).

Since \( D_\xi(a) \) is just the square of the distance of \( a \xi \) to \( \mathbb{C} \xi \) and similarly for \( D_\xi(b) \), the estimate

\[
\|(f(a) - f(a)) \xi\|^2 = \int_{\sigma(\alpha)} |f(\lambda) - f(\alpha)|^2 d\mu(\lambda) \\
\leq \int_{\sigma(\alpha)} \kappa |\lambda - \alpha|^2 d\mu(\lambda) = \kappa \|a - \alpha\| \xi\|^2
\]

implies that \( D_\xi(b) \leq \kappa D_\xi(a) \).

Finally, since any state \( \omega \) is in the weak*-closure of the set of all convex combinations of vector states and each such combination can be represented as a vector state on \( B(\mathcal{H}^n) \) for some \( n \in \mathbb{N} \), the argument of the previous paragraph (applied to \( a^{(n)} \) and \( b^{(n)} = f(a^{(n)}) \)) implies that \( D_\omega(b) \leq D_\omega(a) \).

A variant of the above Theorem 3.5 was proved in [21] and generalized to \( C^* \)-algebras in [9], but both under the much stronger hypothesis that \( \| [b, x]\| \leq \kappa \|a, x]\| \) for all elements \( x \), where \( [a, x] \) denotes the commutator \( ax - xa \). (See Lemma 4.1 below for the explanation of the connection between the two conditions.)
The following Corollary was proved in [9, 5.2] for prime C*-algebras, but under a much stronger assumption about the connection between \( a \) and \( b \) instead of the inequality \( D_\omega(b) \leq D_\omega(a) \) for pure states \( \omega \).

**Corollary 3.6.** Let \( A \) be a unital C*-algebra, \( a, b \in A \), a normal. If \( D_\omega(b) \leq D_\omega(a) \) for all states \( \omega \) on \( A \), then \( b = f(a) \) for a function \( f \) on \( \sigma(a) \) such that \(|f(\mu) - f(\lambda)| \leq |\mu - \lambda|\) for all \( \lambda, \mu \in \sigma(a) \). If \( A \) is prime, it suffices to assume the condition for pure states only.

**Proof.** The first statement follows immediately from Theorem 3.5 since we may assume that \( A \subseteq B(\mathcal{H}) \) for a Hilbert space \( \mathcal{H} \) and each vector state on \( B(\mathcal{H}) \) restricts to a state on \( A \). For the second statement, we note that the C*-algebra generated by \( a \) and \( b \) is contained in a separable prime C*-subalgebra \( A_0 \) of \( A \) by [15, 3.1] (an elementary proof of this is in [25, 3.2]), and \( A_0 \) is primitive by [29, p. 102], hence we may assume that \( A_0 \) is an irreducible C*-subalgebra of \( B(\mathcal{H}) \). But then each vector state on \( B(\mathcal{H}) \) restricts to a pure state on \( A_0 \), and each pure state on \( A_0 \) extends to a pure state on \( A \). \( \square \)

**Corollary 3.7.** Let \( a, b \in B(\mathcal{H}) \) satisfy \( D_\xi(b) \leq D_\xi(a) \) for all \( \xi \in \mathcal{H} \). If \( a \) is essentially normal, then this implies that \( b = f(\hat{a}) \) for a Lipschitz function \( f \) on the essential spectrum of \( a \), where \( \hat{a} \) denotes the coset of \( a \) in the Calkin algebra.

**Proof.** Any state \( \omega \) on the Calkin algebra can be regarded as a vector state on \( B(\mathcal{H}) \) annihilating the compact operators. By Glimm’s theorem (see [22, 10.5.55] or [17]) such a state \( \omega \) is a weak* limit of vector states, hence \( D_\omega(b) \leq D_\omega(a) \). The conclusion follows now from Corollary 3.6. \( \square \)

**Theorem 3.8.** Let \( A \subseteq B(\mathcal{H}) \) be a C*-algebra \( a, b \in R \) and a normal. Denote by \( \overline{R} \) the weak* closure of \( R \) and by \( Z \) the center of \( \overline{R} \). Then the inequality \( D_\omega(b) \leq D_\omega(a) \) holds for all pure states \( \omega \) on \( A \) if and only if \( b \) is in the norm closure of the set \( S \) of all elements of the form \( \sum_j p_j f_j(a) \) (finite sum), where \( p_j \) are orthogonal projections in \( Z \) with the sum \( \sum_j p_j = 1 \) and \( f_j \) are functions on \( \sigma(a) \) such that \(|f_j(\mu) - f_j(\lambda)| \leq |\mu - \lambda|\) for all \( \lambda, \mu \in \sigma(a) \).

**Proof.** Note that \( g(a(t)) = g(a)(t) \) for each continuous function \( g \) on \( \sigma(a) \). We will use the notation from the proof of Corollary 2.4. Similarly as in that proof, the condition that \( D_\omega(b) \leq D_\omega(a) \) for all pure states \( \omega \) on \( A \) implies the same condition for all pure states on \( R(t) \) for all \( t \in \Delta \) and it follows then from Corollary 3.6 that for each \( t \) there exists a Lipschitz function \( f_t \) on \( \sigma(a(t)) \) with the Lipschitz constant 1 such that \( b(t) = f_t(a(t)) \). By Kirzbraun’s theorem each \( f_t \) can be extended to a Lipschitz function on \( \sigma(a) \), denoted again by \( f_t \), with the same Lipschitz constant 1. Given \( \varepsilon > 0 \), since \( \Delta \) is extremely disconnected and for each \( x \in R \) the function \( t \mapsto \|x(t)\| \) is continuous on \( \Delta \) by [17], each \( t \in \Delta \) has a clopen neighborhood \( U_t \) such that \( \|f_t(a(s)) - b(s)\| \leq \varepsilon \) for all \( s \in U_t \). Let \( (U_j) \) be a finite covering of \( \Delta \) by such neighborhoods \( U_j := U_{t_j} \) and for each \( j \) let \( p_j \) be the central projection in \( R \) that corresponds to the clopen set \( U_j \), and set \( f_j := f_{t_j} \). Then

\[
\|b - \sum_j f_j(a)\| \leq \varepsilon.
\]

Since this can be done for all \( \varepsilon > 0 \), \( b \) is in the closure of the set \( S \) as stated in the theorem.
Conversely, suppose that for each $\varepsilon > 0$ there exists an element $c \in \mathbb{R}$ of the form $c = \sum_j p_j f_j(a)$, where $p_j \in \mathbb{R}$ are projections with the sum 1 and $f_j$ are Lipschitz functions with the Lipschitz constant 1. Then for each pure state $\omega$ on $\mathcal{A}$ and $x \in \mathcal{A}$, the inequality (4.2) simply means that the residual norm $\|\omega(x)\| \leq \|\omega(x)\|$ implies $\|\omega(x)\|$ for all $x \in \mathcal{A}$.

The inequality (4.2) follows now by a straightforward computation that $D_\omega(c) = D_\omega(f_j(a))$, which is at most $D_\omega(a)$ by the same computation as in the last part of the proof of Theorem 3.5. Now, since $\|b - c\| < \varepsilon$, it follows from Lemma 3.1 that $D_\omega(b) \leq D_\omega(a)$. \hfill $\square$

4. Is a derivation determined by the norms of its values?

Given an operator $a \in \mathcal{B}(\mathcal{H})$, we will denote by $d_a$ the derivation on $\mathcal{B}(\mathcal{H})$ defined by

$$d_a(x) = ax - xa.\]$$

For any vectors $\xi, \eta \in \mathcal{H}$ we denote by $\xi \otimes \eta^*$ the rank one operator on $\mathcal{H}$ defined by $\langle \xi \otimes \eta^* \rangle = \langle \langle \zeta, \eta \rangle \xi \rangle$. The following lemma enables us to interpret the results of the previous section in terms of derivations.

**Lemma 4.1.** For each $\xi \in \mathcal{H}$ and $a \in \mathcal{B}(\mathcal{H})$ the equality $\|d_a(\xi \otimes \eta^*)\|^2 = D_\xi(a)$ holds.

**Proof.** Denote $x = \xi \otimes \eta^*$. The square of the norm of $d_a(x) = a\xi \otimes \eta^* - \xi \otimes (a^*\eta)^*$ is equal to the spectral radius of the operator $T := d_a(x)^*d_a(x)$, which is the largest eigenvalue of the restriction of $T$ to the span $\mathcal{H}_0$ of $\xi$ and $a^*\eta$. If $\xi$ and $a^*\eta$ are linearly independent, then the matrix of $T|\mathcal{H}_0$ in the basis $\{\xi, a^*\eta\}$ can easily be computed to be

$$\begin{bmatrix}
D_\xi(a) & \langle \xi, a\eta \rangle (\|a\xi\|^2 - \|a^*\eta\|^2) \\
0 & D_\xi(a)
\end{bmatrix}.$$ 

Thus $\|d_a(x)\|^2 = D_\xi(a)$. By continuity (considering perturbations of $a$) we see that this equality holds even if $\xi$ and $a^*\eta$ are linearly dependent. \hfill $\square$

**Theorem 4.2.** If $a, b \in \mathcal{B}(\mathcal{H})$ are such that

$$\|b, x\| = \|a, x\| \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}),$$

then either $b = \sigma a + \lambda 1$ for some scalars $\sigma, \lambda \in \mathbb{C}$ with $|\sigma| = 1$ or there exist a unitary $u$ and scalars $a, \beta, \lambda, \mu$ in $\mathbb{C}$ with $|\beta| = |\alpha|$ such that $a = \alpha u^* + \lambda 1$ and $b = \beta u + \mu 1$.

A variant of this theorem was proved in [9, 5.3, 5.4] in general $C^*$-algebras, but under the additional assumption that $a$ and $b$ are normal. The methods in [9] are different from those we will use below. The author is not able to deduce Theorem 4.2 as a direct consequence of the previous results; for a proof we will need two additional lemmas. We denote by $a^{(n)}$ the direct sum of $n$ copies of an operator $a \in \mathcal{B}(\mathcal{H})$, thus $a^{(n)}$ acts on $\mathcal{H}_n$. We will also use the usual notation $[x, y] := xy - yx$, so that $d_a(x) = [a, x]$.

**Remark 4.3.** We will need the following, perhaps well-known, general fact: for any bounded linear operators $S, T : X \to Y$ between Banach spaces the inequality

$$\|Tx\| \leq \|Sx\| \quad (x \in X)$$

implies $\|T\| \leq \|S\|$ ($v \in X^n$). This follows from [21, 1.1, 1.3], but here it is a slightly more direct proof. The inequality (4.2) simply means that there is a
Lemma 4.4. in the proof of [23, 3.3].
for all \( x \parallel (4.3) \) contraction \( Q \), which clearly implies the desired conclusion.

The content of the following lemma was observed already by Kissin and Shulman in the proof of [23, 3.3].

**Lemma 4.4.** [23] Let \( a, b \in \mathcal{B}(\mathcal{H}) \) and suppose that
\[
(4.3) \quad \| [b, x] \| \leq \| [a, x] \|
\]
for all \( x \in \mathcal{K}(\mathcal{H}) \). If \( a \) is normal, then \( \| [b^{(n)}, x] \| \leq \| [a^{(n)}, x] \| \) for all \( x \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \) (\( n \times n \) matrices with the entries in \( \mathcal{B}(\mathcal{H}) \)) and all \( n \in \mathbb{N} \).

**Proof.** Since \( d_n = (d_n|\mathcal{K}(\mathcal{H}))^{\sharp\sharp} \) (the second adjoint in the Banach space sense), it follows from Remark 4.3 that (4.3) holds for all \( x \in \mathcal{B}(\mathcal{H}) \).

Suppose now that \( a \) is normal and note that \((a)^*\) is a \( C^*\)-algebra by the Fuglede-Putnam theorem. Since (4.3) holds for all \( x \in \mathcal{B}(\mathcal{H}) \), \( b \in (a)^\prime\). Further, by (4.3) the map \([a, x] \mapsto [b, x]\) is a contraction from \( d_n(\mathcal{B}(\mathcal{H})) \) to \( d_b(\mathcal{B}(\mathcal{H})) \). Clearly this map is a homomorphism of \((a)^*\)-bimodules, hence by \([32, 2.1, 2.2, 2.3]\) it is a complete contraction, which is equivalent to the conclusion of the lemma. \( \square \)

**Remark 4.5.** We will use below the following well-known fact. Given \( c_j, e_j \in \mathcal{B}(\mathcal{H}) \), an identity of the form \( \sum_{j=1}^n c_jxe_j = 0 \), if it holds for all \( x \in \mathcal{B}(\mathcal{H}) \), implies that all \( c_j \) must be 0 if the \( e_j \) are linearly independent. (See e. g. [3, Theorem 5.1.7]).

We refer to [6] or [28] for the definition of the injective envelope of an operator space used in the following lemma.

**Lemma 4.6.** Let \( \mathcal{R} = d_a(\mathcal{B}(\mathcal{H})) \) and let \( \mathcal{S} \) be the operator system
\[
\mathcal{S} = \left\{ \begin{bmatrix} \lambda & y \\ z^* & \mu \end{bmatrix} : \lambda, \mu \in \mathbb{C}, y, z \in \mathcal{R} \right\}.
\]

If \( a \) does not satisfy any quadratic equation over \( \mathbb{C} \) then the \( C^*\)-algebra \( C^*(\mathcal{S}) \) generated by \( \mathcal{S} \) is irreducible and the injective envelope \( I(\mathcal{S}) \) of \( \mathcal{S} \) is \( \mathcal{M}_2(\mathcal{B}(\mathcal{H})) \).

**Proof.** Since \( \mathcal{S} \) contains the diagonal \( 2 \times 2 \) matrices with scalar entries, each element of \( \mathcal{S}' \) (the commutant of \( \mathcal{S} \)) is a block diagonal matrix, that is, of the form \( c \oplus e \), where \( c, e \in \mathcal{B}(\mathcal{H}) \). To prove the irreducibility of \( C^*(\mathcal{S}) \) means to prove that each selfadjoint such element \( c \oplus e \) is a scalar multiple of 1. Since \( c \oplus e \) commutes with elements of \( \mathcal{S} \), we have that \( cy = ye \) for all \( y \in \mathcal{R} \). Setting \( y = ax - xa \) in the last identity we obtain
\[
(4.4) \quad cax - cxa - axe +xae = 0 \quad \text{for all} \quad x \in \mathcal{B}(\mathcal{H}).
\]

Since in (4.4) the left coefficients \( ca, -c, -a \) and 1 are not all 0, it follows that 1, \( a, e, ace \) are linearly dependent. Thus, if 1, \( a \) and \( e \) are linearly independent, then \( ae = \alpha 1 + \beta a + \gamma e \) for some scalars \( \alpha, \beta, \gamma \in \mathbb{C} \). Using this, we may rearrange (4.4) into
\[
(4.5) \quad (ca + \alpha 1)x + (\beta 1 - c)x a + (\gamma 1 - a)x e = 0.
\]

If 1, \( a \) and \( e \) were linearly independent, then (4.5) would imply that \( a = \gamma 1 \), but this would be in contradiction with the assumption about \( a \). Hence
\[
(4.6) \quad e = \delta 1 + \eta a \quad \text{for some} \quad \delta, \eta \in \mathbb{C}.
\]
Now (4.5) may be rewritten as
\[(\alpha + \gamma \delta)1 + (c - \delta 1)a)x + [(\beta + \gamma \eta)1 - c - \eta a]xa = 0,\]
which implies (since 1 and a are linearly independent) that
\[(\alpha + \gamma \delta)1 + (c - \delta 1)a = 0 \quad \text{and} \quad (\beta + \gamma \eta)1 - c - \eta a = 0.\]
From these two equalities it follows (by putting into the first equality the expression for c obtained from the second equality) that a satisfies the quadratic equation
\[\eta a^2 + (\delta - \beta - \gamma \eta)a - (\alpha + \gamma \delta)1 = 0.\]
From the assumption about a it follows now in particular that \(\eta = 0\), hence we see from (4.6) that \(e = \delta 1\). But then the identity \(ce = ye\) (y \(\in \mathcal{R}\)) implies that \(c = \delta 1\), hence \(c \oplus e\) is a scalar multiple of the identity.

It remains to consider the case when \(1, a\) and \(e\) are linearly dependent, say \(e = \alpha 1 + \beta a\) \((\alpha, \beta \in \mathbb{C})\). Then (4.4) can be rewritten as
\[(c - \alpha 1)ax + (\alpha 1 - \beta a - c)xa + \beta xa^2 = 0.\]
Since \(1, a\) and \(a^2\) are linearly independent by assumption, we infer from (4.7) that \(\beta = 0\). But then \(e = \alpha 1\) and we conclude as above that \(c \oplus e\) is also a scalar multiple of 1. This proves the irreducibility of \(C^*(S)\).

Since \(S\) contains nonzero compact operators, the identity map on \(S\) has a unique completely positive extension to \(C^*(S)\) by the Arveson boundary theorem [5], which implies that \(C^*(S) \subseteq I(S)\). (Otherwise a projection \(B(\mathcal{H}) \rightarrow I(S)\) restricted to \(C^*(S)\) would be a completely positive extension of \(id_S\), different from \(id_{C^*(S)}\).) But since \(C^*(S)\) is irreducible and contains nonzero compact operators, it follows that \(C^*(S) \supseteq M_2(K(\mathcal{H}))\), hence \(I(S)\) must contain the injective envelope \(I(M_2(K(\mathcal{H})))\), which is known to be \(M_2(B(\mathcal{H}))\) [6]. \(\square\)

**Proof of Theorem 4.2.** If a (or b) is a scalar multiple of 1 the proof is easy, so we assume from now on that this is not the case. If a satisfies a quadratic equation of the form
\[a^2 + \beta a + \gamma 1 = 0 \quad (\beta, \gamma \in \mathbb{C}),\]
then each element of \((a)^{''n}\) is a polynomial in a (this holds for any algebraic operator a by [34]), hence in particular b is a linear polynomial in a, say \(b = \sigma a + \lambda 1\). Then the condition (4.1) obviously implies that \(|\sigma| = 1\). Hence we may assume that a does not satisfy any quadratic equation over \(\mathbb{C}\). Further, if b is not of the form \(\sigma a + \lambda 1\) (which we assume from now on), then by Theorem 2.3 and Lemma 4.1 a is normal and (replacing b by \(\alpha b + \beta\) for suitable \(\alpha, \beta \in \mathbb{C}\)) we may assume without loss of generality that \(b = a^*\).

Denote by \(\mathcal{R}_a\) and \(\mathcal{R}_b\) the ranges of the derivations \(d_a\) and \(d_b\) and by \(\mathcal{S}_a\) and \(\mathcal{S}_b\) the corresponding operator systems (as in Lemma 4.6). Since a is normal, by Lemma 4.4 the map
\[\phi : \mathcal{R}_a \rightarrow \mathcal{R}_b, \quad \phi([a, x]) := [b, x] \quad (x \in B(\mathcal{H}))\]
is completely contractive and the same holds for its inverse, hence \(\phi\) is completely isometric and consequently the map
\[\Phi : \mathcal{S}_a \rightarrow \mathcal{S}_b, \quad \Phi \left( \begin{array}{c} \alpha \\ z^* \\ y \\ \beta \end{array} \right) := \begin{array}{c} \alpha \\ \phi(z)^* \\ \phi(y) \\ \beta \end{array} \]
is completely positive with completely positive inverse, hence also completely isometric (see [28]). But then \(\Phi\) extends to a complete isometry \(\psi\) between the injective
envelopes \( I(S_a) \) and \( I(S_b) \) (since both \( \Phi \) and \( \Phi^{-1} \) extend to complete contractions which must be each other’s inverse by rigidity). Since \( a \) (and \( b = a^* \)) does not satisfy any quadratic equation over \( \mathbb{C} \), these injective envelopes are both \( \text{M}_2(B(H)) \) by Lemma 4.6, hence \( \psi \) is a unital surjective complete isometry of \( \text{M}_2(B(H)) = B(H^2) \), thus by [6, 4.5.13] or [22, Ex. 7.6.18] (and since all automorphisms of \( B(H^2) \) are inner) \( \psi \) is necessarily of the form

\[
\psi(y) = w^*yw \ (y \in B(H^2)),
\]

where \( w \in B(H^2) \) is unitary. Since by definition \( \psi \) fixes the projections of \( H^2 \) on the two summands, \( w \) must commute with these two projections (by the multiplicative domain argument, see [28, p. 38]), consequently \( w \) is of the form \( w = u \oplus v \) for unitaries \( u, v \in B(H) \). It follows now from the definition of \( \psi \) that \( \phi \) is of the form

\[
\phi(y) = uuyv \ (y \in R_a),
\]

that is \( \phi([a, x]) = u[a, x]v \). Hence \( u[a, x]v = [b, x] \) for all \( x \in B(H) \), which can be rewritten as

\[
(ua - \gamma u)xv - (\alpha u + b)x + (1 - \beta u)xb = 0 \quad (x \in B(H)).
\]

Thus by Remark 4.5 we see from (4.8) that \( v, av, 1, \) and \( b \) are linearly dependent. Hence, if \( 1, v \) and \( b \) are linearly independent, then \( av = \alpha 1 + \beta b + \gamma v \), where \( \alpha, \beta, \gamma \in \mathbb{C} \), and (4.8) can be rewritten as

\[
(ua - \gamma u)xv - (\alpha u + b)x + (1 - \beta u)xb = 0.
\]

But by Remark 4.5 this implies in particular that \( ua - \gamma u = 0 \), hence \( a = \gamma 1 \), a possibility which we have excluded in the first paragraph of this proof. So we may assume that \( 1, v \) and \( b \) are linearly dependent. Then, since \( b = a^* \) is not a scalar, \( v \) can not to be a scalar. Hence \( b = \alpha 1 + \beta v \) for suitable \( \alpha, \beta \in \mathbb{C} \). Since \( v \) is unitary and \( a = b^* \), this concludes the proof.

To extend Theorem 4.2 to \( C^* \)-algebras we need a lemma.

**Lemma 4.7.** Let \( A \subseteq B(H) \) be a \( C^* \)-algebra, \( J \) a closed ideal in \( A \), and let \( a, b \in A \) satisfy \( \|[b, x]\| \leq \|[a, x]\| \) for all \( x \in A \). Then the same inequality holds for all \( x \in A \overline{\Delta} \) and also for all cosets \( \hat{x} \in A/J \).

**Proof.** The statement about the quotient was observed already in [9, Proof of 5.4] and follows from the existence of a quasi-central approximate unit \( \{e_k\} \) in \( J \) [4]. Namely, the conditions \( \|[a, e_k]\| \leq \|[b, e_k]\| \to 0 \) (from the definition of the quasi-central approximate unit) and the well-known property that \( \|[\hat{g}]\| = \lim_k \|[g(1-e_k)]\| (y \in A) \) imply that

\[
\|[\hat{b}, \hat{x}]\| = \lim_k \|[b, x](1-e_k)\| = \lim_k \|[b, x(1-e_k)]\| \leq \lim_k \|[a, x(1-e_k)]\| = \|[a, \hat{x}]\|.
\]

Let \( A^{\sharp\sharp} \) be the universal von Neumann envelope of \( A (= \text{bidual of } A) \) and regard \( A \) as a subalgebra in \( A^{\sharp\sharp} \) in the usual way. Since \( d^{\sharp\sharp}_a \) is just the derivation induced by \( a \) on \( A^{\sharp\sharp} \), it follows from Remark 4.3 that the condition \( \|[b, x]\| \leq \|[a, x]\| \) holds for all \( x \in A^{\sharp\sharp} \). Since \( \overline{A} \) is a quotient of \( A^{\sharp\sharp} \), it follows from the previous paragraph (applied to \( A^{\sharp\sharp} \) instead of \( A \)) that the condition holds also in \( \overline{A} \). \( \square 

**Corollary 4.8.** If \( A \) is a \( C^* \)-algebra and \( a, b \in A \) are such that \( \|[b, x]\| = \|[a, x]\| \) for all \( x \in A \), then there exist a projection \( p \) in the center \( Z \) of \( \overline{A} \) and elements \( s, d \in Zp \) with \( s \) unitary, and \( u, v, c, g, h \in Zp^\perp \) with \( u, v \) unitary, such that \( bp = sa + d \) and \( ap^\perp = cu^* + g, bp^\perp = vcu + h \).
Proof. If $A$ is primitive the corollary follows immediately from Theorem 4.2 and Lemma 4.7 since $A = B(H)$ if $A$ is irreducibly represented on $H$. In general, Lemma 4.7 reduces the proof to von Neumann algebras, where the arguments are similar as in the proof of Corollary 2.4, so we will omit the details. \hfill \Box

**Corollary 4.9.** If $\|[b,x]\| \leq \|[a,x]\|$ for all $x \in A$ then $D_\omega(b) \leq D_\omega(a)$ for all pure states $\omega$ on $A$.

Proof. If $\pi : A \to B(H)$ is the irreducible representation obtained from $\omega$ by the GNS construction, then $\pi(A) = B(H)$, hence the corollary follows from Lemmas 4.7 and 4.1. \hfill \Box

5. An inequality between norms of commutators

In this section we study the inequality

\[(5.1) \quad \|[b,x]\| \leq \kappa \|[a,x]\| \quad (\forall x \in B(H)),\]

where $a, b \in B(H)$ are fixed and $\kappa$ is a constant. For a normal $a$ it is proved in [21] that (5.1) holds (for some $\kappa$) if and only if

\[(5.2) \quad d_\omega(B(H)) \subseteq d_\omega(B(H)).\]

That for normal $a$ (5.1) implies (5.2) can be easily proved as follows. We have seen in the proof of Lemma 4.4 that for normal $a$ the condition (5.1) is equivalent to the fact that the map

\[d_\omega(x) \mapsto d_\omega(x) \quad (x \in B(H))\]

is a completely bounded homomorphism of $(a)'$-bimodules $d_\omega(B(H)) \to d_\omega(B(H))$. Then this map can be extended to a completely bounded $(a)'$-bimodule endomorphism $\phi$ of $B(H)$ by the Wittstock theorem (see [6, 3.6.2]), hence we have

\[d_\omega(x) = \phi(d_\omega(x)) = d_\omega(\phi(x)) \quad (x \in B(H)).\]

This argument is perhaps easier than the one in [21], but the argument from [21] can be adapted to von Neumann algebras.

**Proposition 5.1.** Let $R \subseteq B(H)$ be a von Neumann algebra, $a, b \in R$ and denote $(a)^c := R \cap (a)'$. If $\|[b,x]\| \leq \|[a,x]\|$ for all $x \in R$ and $a$ is normal, then there exists a normal bounded $(a)^c$-bimodule map $\psi$ on $R$ with $\|\psi\| \leq 4\|b\|$ such that $d_\omega(R = \psi(d_\omega(R) = (d_\omega(R)\omega)$. Hence in particular $d_\omega(R) \subseteq d_\omega(R)$.

Proof. Since $(a)^c$ is an abelian von Neumann algebra there exists a projection $E_0$ from $B(H)$ onto $(a)^c$ which is an $(a)^c$-bimodule map [22, 8.3.12 and 8.7.24]. Let $E = E_0|R$ and $E^\perp = 1 - E$. Since $E^\perp(R)$ (the kernel of $E$) is a complementary subspace to $(a)^c$, the restriction $q_0 := q|E^\perp(R)$ of the quotient map $q : R \to R/(a)^c$ is an isomorphism of $(a)^c$-bimodules. Denote by $d_\text{q}_0 : R/(a)^c \to \overline{d_\omega(R)}$ the map induced by $d_\omega$ and let $\phi : \overline{d_\omega(R)} \to \overline{d_\omega(R)}$ be the continuous extension of the map $d_\omega(x) \mapsto d_\omega(x) \quad (x \in R)$, regarded as a map into $R$. Then the composition $\psi := \phi d_\omega q_0 E^\perp$ is easily seen to be an $(a)^c$-bimodule map such that $\psi(d_\omega(R) = d_\omega(R)$, hence $d_\omega(R = \overline{(d_\omega(R)\psi)}$ (since $a \in (a)^c$) and consequently $d_\omega(R) \subseteq d_\omega(R)$. The map $\psi$ is bounded with $\|\psi\| \leq \|[\phi d_\omega q_0 E^\perp] \| \leq 2\|[\phi d_\omega] \| \leq 4\|b\|$, but not automatically normal. However, if $\psi = \psi_n + \psi_s$ is the decomposition into the normal part $\psi_n$ and the singular part $\psi_s$ [22, Section 10.1], then the identity $d_\omega(R) = \psi(d_\omega(R)\psi_n = (d_\omega(R)\psi_s$
implies that $(d_a|R)\psi_n = 0$ (since the left side of the identity is normal, while the right side is singular). Similarly $\psi(d_a|R) = 0$, hence we may replace $\psi$ by $\psi_n$. □

We do not know if the range inclusion part of Proposition 5.1 can be extended to general C*-algebras, but at least a somewhat weaker version holds.

**Corollary 5.2.** In a C*-algebra $A$, if elements $a, b$ satisfy $\|b, x\| \leq \|a, x\|$ for all $x \in A$ and $a$ is normal, then $d_b(a) \subseteq 4\|b\|d_a(B_A)$, where $B_A$ denotes the closed unit ball of $A$. In particular $d_b(A) \subseteq d_a(A)$.

**Proof.** We may assume that $A \subseteq B(H)$. Then by Lemma 4.7 $\|b, x\| \leq \|a, x\|$ for all $x \in A$, hence it follows from Proposition 5.1 that $d_b(B_A) \subseteq 4\|b\|d_a(B_A)$. Since the Kaplansky density theorem implies that $d_a(B_A) = d_a(A)$, it follows that $d_b(B_A) \subseteq 4\|b\|d_a(B_A) \cap A = 4\|b\|d_a(A)$, where for the last equality we have used [22, 10.1.4]. □

By [23, 6.5], if $a$ is normal, (5.2) implies (5.1) in any C*-algebra $A$. The proof uses the special case $A = B(H)$ proved earlier in [21]. We will sketch a somewhat simplified proof of this case, but first we need to recall a fact concerning operators in $B(X, Y)$, the space of all bounded linear operators from $X$ into $Y$, where $X$ and $Y$ are Banach spaces. Denote by $X^2$ the dual of $X$ and by $T^2$ the adjoint of $T \in B(X, Y)$. The following simple fact is well-known (see [21]).

**Lemma 5.3.** Given $S, T \in B(X, Y)$, the inclusion $T^2(Y^2) \subseteq S^2(Y^2)$ holds if and only if there exists a constant $\kappa$ such that

\[(5.3) \quad \|T\xi\| \leq \kappa\|S\xi\|
\]

for all $\xi \in X$.

Since $d_a = -(d_a(T(H))^2$, where $T(H)$ is the ideal in $B(H)$ of trace class operators, the following is just a special case of Lemma 5.3.

**Corollary 5.4.** Let $a, b \in B(H)$.

(i) The inclusion $d_b(B(H)) \subseteq d_a(B(H))$ holds if and only if there exists a constant $\kappa$ such that $\|d_b(t)\| \leq \kappa\|d_a(t)\|$ for all $t \in T(H)$.

(ii) The inclusion $d_b(T(H)) \subseteq d_a(T(H))$ is equivalent to the existence of a constant $\kappa$ such that $\|d_b(x)\| \leq \kappa\|d_a(x)\|$ for all $x \in K(H)$ or (equivalently, by Lemma 4.7) for all $x \in B(H)$.

**Lemma 5.5.** Let $a \in B(H)$ be a subnormal operator and $f$ a Lipschitz function on $\sigma(a)$. If $a$ is not normal, assume that $f$ is in the uniform closure of the set of rational functions with poles outside $\sigma(a)$, so that $b := f(a)$ is defined. If $\|b, x\| \leq \kappa\|a, x\|$ for all $x \in B(H)$, then for each sequence $(\lambda_i) \subseteq \sigma(a)$ the matrix $\Lambda(f; \lambda)$ defined by the right side of (5.5) is a Schur multiplier with the norm at most $2\kappa$. (That is, $f$ is a Schur function on $\sigma(a)$ as defined in the Introduction). Similarly, the condition $\|b, x\|_1 \leq \kappa\|a, x\|_1$ for all $x \in T(H)$ implies that $\Lambda(f; \lambda)$ is a Schur multiplier on $T(H)$ with the norm $\leq 2\kappa$.

**Proof.** First suppose that $(\lambda_i)_{i=1}^n$ is a finite subset of the boundary $\partial\sigma(a)$ of $\sigma(a)$. Then each $\lambda_i$ is an approximate eigenvalue of $a$ [11], hence there exists a sequence of unit vectors $\xi_{i,n} \in H$ such that $\lim_{n \to \infty} \|(a-\lambda_i)\xi_{i,n}\| = 0$. Since $a-\lambda_i$ is hyponormal, $\|(a-\lambda_i)^*\xi_{i,n}\| \leq \|(a-\lambda_i)\xi_{i,n}\|$ as $n \to \infty$.

\[(\lambda_i - \lambda_j)(\xi_{i,n}, \xi_{j,n}) = \langle \lambda_i \xi_{i,n}, \xi_{j,n} \rangle - \langle \xi_{i,n}, \lambda_j \xi_{j,n} \rangle \]

\[= \left( \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_i} \right) \|a\| \|\xi_{i,n}\|^2 - \langle \xi_{i,n}, \Lambda_f \xi_{j,n} \rangle \]

\[= \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_i} \|a\| \|\xi_{i,n}\|^2 - \langle \xi_{i,n}, \Lambda_f \xi_{j,n} \rangle \approx \frac{\lambda_i - \lambda_j}{\lambda_j - \lambda_i} \|a\| \|\xi_{i,n}\|^2 \]

...
tends to \( \lim_n(\langle a\xi_{i,n}, \xi_{j,n} \rangle - \langle \xi_{i,n}, a^*\xi_{j,n} \rangle) = 0 \). Thus, if \( i \neq j \), then \( \lim(\xi_{i,n}, \xi_{j,n}) = 0 \), so the set \( \{\xi_{1,n}, \ldots, \xi_{m,n}\} \) is approximately orthonormal if \( n \) is large. It follows that for each matrix \( \alpha = [\alpha_{i,j}] \in M_m(\mathbb{C}) \) the norm of the operator \( x := \sum_{i,j=1}^m \alpha_{i,j}\xi_{i,n} \otimes \xi_{j,n}^* \) is approximately equal to the usual operator norm of \( \alpha \). Further, for large \( n \) we have approximate equalities

\[
d_a(x) = \sum_{i,j=1}^m \alpha_{i,j}(a\xi_{i,n} \otimes \xi_{j,n}^* - \xi_{i,n} \otimes (a^*\xi_{j,n})^*) \approx \sum_{i,j=1}^m \alpha_{i,j}(\lambda_i - \lambda_j)\xi_{i,n} \otimes \xi_{j,n}^*,
\]

and

\[
d_b(x) \approx \sum_{i,j=1}^m \alpha_{i,j}(f(\lambda_i) - f(\lambda_j))\xi_{i,n} \otimes \xi_{j,n}^*,
\]

hence it follows from the assumption \( \|d_b(x)\| \leq \kappa\|d_a(x)\| \) that

\[
||[f(\lambda_i) - f(\lambda_j)\alpha_{i,j}]|| \leq \kappa\|[(\lambda_i - \lambda_j)\alpha_{i,j}]\|.
\]

This estimate means that for a finite subset \( \lambda = (\lambda_i)_{i=1}^m \) of \( \partial \sigma(a) \) the matrix \( \Lambda(f; \lambda) \) with the entries

\[
\Lambda_{i,j}(f; \lambda) = \begin{cases} 
\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}, & \text{if } i \neq j \\
0, & \text{if } i = j
\end{cases}
\]

acts as a Schur multiplier with the norm at most \( \kappa \) on the subspace \( E_m \subseteq M_m(\mathbb{C}) \) of matrices of the form \([\lambda_i - \lambda_j]\alpha_{i,j}\]. Since \( E_m \) is just the set of all matrices with zero diagonal and the natural projection from \( M_m(\mathbb{C}) \) onto the subspace \( D_m \) of diagonal matrices has the Schur norm 1 (and \( \Lambda(f; \lambda)(D_m = 0) \)), it follows that the norm of \( \Lambda(f; \lambda) \) as a Schur multiplier on \( M_m(\mathbb{C}) \) is at most \( 2\kappa \).

Now let \( \lambda_1, \ldots, \lambda_m \) be fixed elements of \( \partial \sigma(a) \) and consider the function

\[
\lambda_1 \mapsto \Lambda(f; \lambda_1, \lambda_1, \ldots, \lambda_m)
\]

from \( \sigma(a) \) into the Banach algebra \( M_m(\mathbb{C}) \) equipped with the Schur norm. Since this function is holomorphic on the interior of \( \sigma(a) \) and bounded on \( \partial \sigma(a) \) (by \( 2\kappa \)), it follows that the Schur norm of \( \Lambda(f; \lambda_1, \lambda_2, \ldots, \lambda_m) \) is at most \( 2\kappa \) for all \( \lambda_i \in \sigma(a) \). In the same way we show that the Schur norm of \( \Lambda(f; \lambda_1, \ldots, \lambda_m) \) is at most \( 2\kappa \) for all \( \lambda_i \in \sigma(a) \). Since the bound \( 2\kappa \) is the same for all \( m \), the lemma is proved. \( \square \)

For a rank one operator \( x \) the operators \( d_a(x) \) and \( d_b(x) \) have rank at most two and on such operators the trace class norm is equivalent to the usual operator norm. If \( a \) is normal, we deduce now from Corollary 5.4, Lemma 4.1 and Theorem 3.5 that each of the two range inclusions \( d_b(B(H)) \subseteq d_a(B(H)) \) and \( d_b(T(H)) \subseteq d_a(T(H)) \) implies that \( b \) is of the form \( b = f(a) \) for a Lipschitz function \( f \) on \( \sigma(a) \). Then \( f \) is a Schur function by Corollary 5.4 and Lemma 5.5.

If \( a \) is diagonal, the proof of Lemma 5.5 shows that (5.1) and its analogue for the trace norm hold for any Schur function \( f \) on \( \sigma(a) \). Indeed, if \( \lambda_i \) are the eigenvalues of \( a \) and a matrix \([x_{i,j}]\) represents an operator \( x \) relative to the orthonormal basis consisting of eigenvectors of \( a \), the inequality (5.1) (and its analogue in the trace norm) assumes the form

\[
||[(f(\lambda_i) - f(\lambda_j))x_{i,j}]|| \leq \kappa||[(\lambda_i - \lambda_j)x_{i,j}]||.
\]

Since the obvious projection from \( B(H) \) (or \( T(H) \)) onto the subset of all diagonal matrices in \( B(H) \) (or in \( T(H) \)) is contractive (in the Schur norm), we see that (5.6) and its trace analogue are equivalent to the requirements that the matrix \( \Lambda(f; \lambda) \)
defined as in (5.5) is a Schur multiplier on $B(\mathcal{H})$ and $T(\mathcal{H})$ (respectively). But it is well-known (and easy to see) that a matrix is a Schur multiplier on $T(\mathcal{H})$ if and only if its transpose is a Schur multiplier on $B(\mathcal{H})$ and then the two have the same norm, hence the two conditions on $f$ are equivalent.

For a general normal $a$ and a Schur function $f$ on $\sigma(a)$, given $\varepsilon > 0$, by the Weyl-von Neumann-Berg theorem [12, Corollary 39.6] there exists a diagonal $a_0$ such that $\sigma(a_0) \subseteq \sigma(a)$, $\|a - a_0\| < \varepsilon$ and (approximating $f$ by polynomials) $\|f(a) - f(a_0)\| < \varepsilon$. Then for each $x \in B(\mathcal{H})$ with $\|x\| = 1$ we have $\|[f(a), x]\| \leq \|[f(a_0), x]\| + 2\varepsilon \leq \kappa \|[a_0, x]\| + 2\varepsilon \leq \kappa \|a, x\| + 2\kappa\varepsilon + 2\varepsilon$, so $\|[f(a), x]\| \leq \kappa \|a, x\|$. The same estimate holds also for the trace norm. Thus we may summarize the above discussion in the following theorem most of which was proved already by Johnson and Williams in [21] in a somewhat different way.

**Theorem 5.6.** [21] If $a \in B(\mathcal{H})$ is normal, then for any $b \in B(\mathcal{H})$ the inclusion $d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))$ holds if and only if there exists a constant $\kappa$ such that $\|d_b(x)\| \leq \kappa \|d_a(x)\|$ for all $x \in B(\mathcal{H})$ and this is also equivalent to the condition that $b = f(a)$ for a Schur function $f$ on $\sigma(a)$.

If $a$ is not normal, then the range inclusion (5.2) does not necessarily imply that $b \in (a)''$ [20], hence it does not imply (5.1). But we will prove that conversely (5.1) implies (5.2), if $a$ satisfies certain conditions which are much more general than normality.

**Proposition 5.7.** Denote $R_a := d_a(B(\mathcal{H}))$. If $\overline{R_a} + (a)' = B(\mathcal{H})$, then for each $b \in B(\mathcal{H})$ the condition (5.1) implies that $R_b \subseteq R_a$. Moreover, if $\overline{R_a} = B(\mathcal{H})$, then there exists a weak* continuous $(a)'$-bimodule map $\phi$ on $B(\mathcal{H})$ such that $d_b = \phi d_a = d_a \phi$.

**Proof.** By (5.1) the correspondence $d_a(x) \mapsto d_b(x)$ extends to a bounded map $\phi_0$ from $\overline{R_a}$ into $\overline{R_b}$ such that $\phi_0 d_a = d_b$. Note that $\phi_0(d_a(K(\mathcal{H}))) \subseteq d_b(K(\mathcal{H}))$. Identifying $d_b(K(\mathcal{H}))^{\sharp\sharp}$ with $d_a(K(\mathcal{H}))^{\sharp\sharp}$ inside $K(\mathcal{H})^{\sharp\sharp} = B(\mathcal{H})$ in the usual way, it follows that $\phi := \phi_0^{\sharp\sharp}$ is the weak* continuous extension of $\phi_0$ to $\overline{d_a(B(\mathcal{H}))} = \overline{d_a(K(\mathcal{H}))}$ satisfying $\phi d_a = d_b$. Since $\phi_0$ is an $(a)'$-bimodule maps, so must be $\phi$ by continuity, hence in particular

$$
 d_b(x) = \phi(d_a(x)) = d_a \phi(x) \quad \text{for all } x \in \overline{R_a} = \overline{d_a(B(\mathcal{H}))}
$$

and consequently $d_b(\overline{R_a}) \subseteq \overline{R_a}$. Finally, to conclude the proof, note that the assumption $\overline{R_a} + (a)' = B(\mathcal{H})$ implies that $R_b = d_b(\overline{R_a})$ since from (5.1) $(a)' \subseteq (b)' = \ker d_b$. \hfill $\Box$

By duality the condition $\overline{d_a(B(\mathcal{H}))} = B(\mathcal{H})$ means that the kernel of $d_a|T(\mathcal{H})$ is 0, that is, $(a)' \cap T(\mathcal{H}) = 0$. There are many Hilbert space operators $a$ which do not even commute with any nonzero compact operator. This is so for example, if $a$ is normal and has no eigenvalues. (Namely, $(a)'$ is a $C^*$-algebra and contains the spectral projection $p$ corresponding to any nonzero eigenvalue of each $h = h^* \in (a)'$. Since $p$ is of finite rank, $ap$, and therefore also $a$, has eigenvalues if $h \neq 0$.) For a general normal $a \in B(\mathcal{H})$ we can decompose $\mathcal{H}$ into the orthogonal sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the closed linear span of all eigenvectors of $a$ and $\mathcal{H}_2 = \mathcal{H}_1^\perp$. Then $a$ also decomposes as $a_1 \oplus a_2$, where $(a_2)'$ contains non nonzero compact operators, while $(a_1)'$ contains a net of finite rank projections converging strongly to the identity.
Another class of operators which admit the decomposition as in the previous paragraph are (rationally) cyclic subnormal operators. Such an operator can be decomposed into the direct sum of the normal and the pure part. An operator \( c \) in the commutant of the pure part \( a_0 \) is subnormal (by Yoshino’s theorem [13, 5.4]), hence normal if compact [18]. But then the (finite dimensional) eigenspace \( \mathcal{H}_0 \) of \( c \) corresponding to a nonzero eigenvalue is invariant under \( a_0 \) and \( a_0|\mathcal{H}_0 \) is normal, contradicting the purity of \( a_0 \). (A general pure subnormal operator, however, can commute with a nonzero trace class operator; an example is in [38, 2.1].) By the Wold decomposition an isometry is a direct sum of a unitary and of copies of the unilateral shift, hence, the following theorem applies also to isometries.

**Theorem 5.8.** Let \( a \in B(\mathcal{H}) \) and suppose that \( \mathcal{H} \) decomposes into the orthogonal sum \( \mathcal{H}_1 \oplus \mathcal{H}_2 \) of two subspaces which are invariant under \( a \), so that \( a = a_1 \oplus a_2 \), where \( a_i \in B(\mathcal{H}_i) \). If \( (a_1') \cap T(\mathcal{H}_1) \) contains a bounded net \((\epsilon_k)\) converging to 1 in the strong operator topology, while \((a_2') \cap T(\mathcal{H}_2) = 0 \), then the condition

\[
\|d_b(x)\| \leq \|d_a(x)\| \quad (\forall x \in B(\mathcal{H}))
\]

implies that \( d_a(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H})) \).

**Proof.** Since \( b \in (a)' \), \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are invariant subspaces for \( b \), so \( b \) also decomposes as \( b = b_1 \oplus b_2 \), where \( b_j \in B(\mathcal{H}_j) \). Relative to the same decomposition of \( \mathcal{H} \) each \( x \in B(\mathcal{H}) \) can be represented by a \( 2 \times 2 \) operator matrix \( x = [x_{i,j}] \) and

\[
d_b(x) = \begin{pmatrix}
b_1x_{1,1} - x_{1,1}b_1 & b_1x_{1,2} - x_{1,2}b_2 \\
b_2x_{2,1} - x_{2,1}b_1 & b_2x_{2,2} - x_{2,2}b_2
\end{pmatrix}.
\]

Thus it suffices to show that for each pair \((i,j)\) of indexes and for each \( x_{i,j} \in B(\mathcal{H}_j, \mathcal{H}_i) \) the element \( b_i x_{i,j} - x_{i,j} b_j \) is in the range of the map \( d_{a_i,a_j} \) defined on \( B(\mathcal{H}_j, \mathcal{H}_i) \) by \( d_{a_i,a_j}(y) = a_i y - y a_j \). In the case \( i = 2 = j \) this follows from Proposition 5.7. We will now consider the case \( i = 1 \) and \( j = 2 \), the remaining two cases are treated similarly.

From the norm inequality in the theorem we have in particular that

\[
\|d_{b_2,b_1}(x)\| \leq \|d_{a_2,a_1}(x)\| \quad (\forall x \in B(\mathcal{H}_1, \mathcal{H}_2)).
\]

This implies that there exists a bounded \((a_2')', (a_1)'\)-bimodule map

\[
\phi_0 : d_{a_2,a_1}(K(\mathcal{H}_1, \mathcal{H}_2)) \to K(\mathcal{H}_1, \mathcal{H}_2)
\]

such that \( \phi_0 d_{a_2,a_1} = d_{b_2,b_1} \). (To prove the bimodule property use that \((a_1)' \subseteq (b_1)'\), which follows from \((a)' \subseteq (b)'\).) Now observe that the closure \( X \) of \( d_{a_2,a_1}(T(\mathcal{H}_1, \mathcal{H}_2)) \) in the trace norm is contained in \( d_{a_2,a_1}(K(\mathcal{H}_1, \mathcal{H}_2)) \) (the closure of \( d_{a_2,a_1}(K(\mathcal{H}_1, \mathcal{H}_2)) \) in the usual operator norm), and that \( X \subseteq T(\mathcal{H}_1, \mathcal{H}_2) \) is a nondegenerate right Banach module over the Banach algebra \( A := (a_1)' \cap T(\mathcal{H}_1) \) since for each \( t \in T(\mathcal{H}_1, \mathcal{H}_2) \) the operators \( t e_k \) converge to \( t \) in the trace norm. Moreover, \((\epsilon_k)\) is a bounded approximate identity for \( A \), hence by the Cohen-Hewitt theorem [10, p. 108] each \( t \in X \) can be factored as \( t = s c \), where \( s \in X \) and \( c \in A \). Since \( \phi_0 \) is a homomorphism of right \((a_1)'\)-modules (hence also of right \( A \)-modules), it follows that \( \phi_0(t) = \phi_0(s)c \), hence \( \phi_0(t) \in T(\mathcal{H}_1, \mathcal{H}_2) \) (since \( c \in T(\mathcal{H}_1) \)). Thus \( \phi_0 \) maps \( X \) into \( T(\mathcal{H}_1, \mathcal{H}_2) \). Moreover, it is easy to verify that the graph of the restriction \( \psi := \phi_0|X \) is closed, hence \( \psi \) is bounded by the closed graph theorem. Now from \( d_{b_1,b_2}(T(\mathcal{H}_1, \mathcal{H}_2)) = \psi(d_{a_2,a_1}(T(\mathcal{H}_1, \mathcal{H}_2)) \) we infer (by taking the dual maps) that \( d_{b_1,b_2} = d_{a_2,a_2} \psi^{\#} \), where \( d_{a_2,a_2} : B(\mathcal{H}_2, \mathcal{H}_1)/\ker d_{a_2,a_2} \to B(\mathcal{H}_2, \mathcal{H}_1) \) is the map induced by \( d_{a_2,a_2} \), hence the range of \( d_{b_1,b_2} \) is indeed contained in the range of \( d_{a_1,a_2} \). \( \square \)
As customary, Rat($K$) denotes the algebra of all rational functions with poles outside a compact subset $K \subseteq \mathbb{C}$ and, if $\mu$ is a positive Borel Measure on $K$, $R^2(K, \mu)$ is the closure in $L^2(\mu)$ of Rat($K$). As before we denote by $\dot{a}$ the coset in the Calkin algebra $C(\mathcal{H})$ of an operator $a \in \mathcal{B}(\mathcal{H})$.

**Proposition 5.9.** Let $K$ be a compact subset of $\mathbb{C}$, $a$ a subnormal operator with $\sigma(a) \subseteq K$ such that $a$ is cyclic for the algebra Rat($K$) and let $c$ be the minimal normal extension of $a$. Assume that $\sigma(c) \subseteq \sigma(\dot{a})$, let $\mu$ be a scalar spectral measure for $c$ such that $c$ is the multiplication on $\mathcal{H} := R^2(K, \mu)$ by the identity function $z$. Denote by $p$ the orthogonal projection from $\mathcal{K} := L^2(\mu)$ onto $\mathcal{H}$ and assume that the only function $h \in C(\sigma(c)) + (L^\infty(\mu) \cap R^2(K, \mu))$ for which the operator $T_h$ defined by $T_h(\xi) := p(f\xi)$ ($\xi \in \mathcal{H}$) is compact is $h = 0$. Then for each $b \in \mathcal{B}(\mathcal{H})$ satisfying $\|b, x\| \leq \|a, x\|$ ($x \in \mathcal{B}(\mathcal{H})$) there exists a function $f \in C(\sigma(c)) \cap R^2(K, \mu)$ such that $b = f(c)|\mathcal{H}$.

Moreover, if $K$ is the closure of a domain $G$ bounded by finitely many nonintersecting analytic Jordan curves and $a$ is the multiplication operator by $z$ on the Hardy space $H^2(G)$, $f$ can be extended to a Schur function on $K$.

**Proof.** It is well-known that a rationally cyclic subnormal operator $a$ can be represented as the multiplication on $R^2(K, \mu)$ by the independent variable $z$ [13, p. 51] and that $(a')' = R^2(K, \mu) \cap L^\infty(\mu)$ by Yoshino’s theorem [13, p. 52]. Since $b \in (a)'$, it follows that $b$ is the multiplication on $R^2(K, \mu)$ by a function $f \in R^2(K, \mu) \cap L^\infty(\mu)$.

On the other hand $a$ is essentially normal by the Berger-Show theorem [13, p. 152], hence by Corollary 3.7 and Lemma 4.1 $\dot{b} = g(\dot{a})$ for a continuous function $g$ on $\sigma(\dot{a})$. Further, since $c$ is normal and $a$ is subnormal and essentially normal, an easy computation with $2 \times 2$ operator matrices (relative to the decomposition $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$) shows that the operator $p^+ c^* p$ is compact, hence (since also $p^+ c p = 0$) the map $h \mapsto \dot{T}_h$ from $C(\sigma(c))$ into the Calkin algebra is a $*$-homomorphism. Thus it must coincide with the $*$-homomorphism $h \mapsto h(\dot{c})$ since they coincide on the generator $c$. It follows in particular that $\dot{b} = g(\dot{a}) = \dot{T}_g$, hence the operator $T_g - f = T_g - b$ is compact. But by the hypothesis this is possible only if $g - f = 0$, hence $f$ is continuous.

In the case $a$ is the unilateral shift, $f$ is a continuous function on the circle and contained in the closure $P^2(\mu)$ of polynomials in $L^2(\mu)$, where $\mu$ is the normalized Lebesgue measure on the circle. It is well known that such a function can be holomorphically extended to disc $D$ such that the extension (denoted again by $f$) is continuous on $\overline{D}$. By Lemma 5.5 $f$ is a Schur function on $\overline{D}$. Similar arguments apply to multiply connected domains by [1, 2.11, 1.1], [27, 4.3, 9.4]).

Perhaps, in general, (5.1) does not even imply that $d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))$, but a possible counterexample is not known to the author. Note, however, that by duality between $T(\mathcal{H})$ and $B(\mathcal{H})$ (5.1) implies that $d_b(B(\mathcal{H})) \subseteq d_a(B(\mathcal{H}))$, hence in particular $d_b(K(\mathcal{H})) \subseteq d_a(K(\mathcal{H}))$ since the weak topology agrees on $K(\mathcal{H})$ with the weak* topology inherited from $B(\mathcal{H})$. More generally, we will see that the problem depends entirely on what happens in the Calkin algebra.

For a $C^*$-algebra $A$ and $a \in A$ note that a functional $\rho \in A^2$ annihilates $d_a(A)$ if and only if $[a, \rho] = 0$, where $[a, \rho] \in A^2$ is defined by $([a, \rho])(x) = \rho(xa - ax)$. In other words, the annihilator in $A^2$ of $d_a(A)$ is just the centralizer $C_a$ of $a$ in $A^2$. 

Proposition 5.10. If \(a, b \in B(H)\) satisfy \(\|b, x\| \leq \|[a, x]\|\) for all \(x \in B(H)\), then \(\|[\hat{b}, \hat{x}]\| \leq \|[\hat{a}, \hat{x}]\|\) in the Calkin algebra \(C(H)\). If this latter inequality implies that \(C_{\hat{a}} \subseteq C_{\hat{b}}\), then \(C_a \subseteq C_b\) also holds, hence \(d_b(B(H)) \subseteq d_a(B(H))\).

Proof. The first statement follows from Lemma 4.7. To prove the rest of the proposition, first note that for any \(x \in B(H)\) and a functional \(\rho \in C_a\) the normal part \(\rho_n\) and the singular part \(\rho_s\) are both in \(C_a\). (Indeed, from \([a, \rho] = 0\) we have \([a, \rho_n] = -[a, \rho_s]\), where the left side is normal and the right side is singular, hence both are 0.) Further, since \(\rho_n\) is given by a trace class operator \(t\), \([a, t] = 0\), hence the hypothesis of the proposition implies that \([b, t] = 0\), so \([b, \rho_n] = 0\). Since singular functionals annihilate \(K(H)\), they can be regarded as functionals on \(C(H)\). Thus, if the condition \(\|[\hat{b}, \hat{x}]\| \leq \|[\hat{a}, \hat{x}]\|\) \((\hat{x} \in C(H))\) implies that \(C_{\hat{a}} \subseteq C_{\hat{b}}\), then we have \(\rho_s \in C_{\hat{b}}\) and consequently also \(\rho \in C_b\). This proves that \(C_a \subseteq C_b\) and the Hahn-Banach theorem then implies that \(d_b(B(H)) \subseteq d_a(B(H))\). \(\square\)

Corollary 5.11. Suppose \(a, b \in B(H)\) satisfy (5.1). If \(a\) is essentially normal, then \(d_b(B(H)) \subseteq d_a(B(H))\).

Proof. By Lemma 4.7 and Corollary 5.2 \(d_b(C(H)) \subseteq d_a(C(H))\). Now Proposition 5.10 completes the proof. \(\square\)

6. Commutators and the completely bounded norm

In this section we will study stronger variants of the condition \(\|[b, x]\| \leq \|[a, x]\|\) \((x \in B(H)\)) in the context of completely bounded maps.

Lemma 6.1. If \(a, b \in B(H)\) satisfy

\[
\|[b^n, x]\| \leq \|[a^n, x]\| \quad \text{for all} \quad x \in M_n(B(H)) \quad \text{and all} \quad n \in \mathbb{N},
\]

then

\[
\|[\pi(b), x]\| \leq \|[\pi(a), x]\| \quad \text{for all} \quad x \in B(H_\pi)
\]

for every unital \(*\)-representation \(\pi : A \to B(H_\pi)\) of the \(C^*\)-algebra \(A\) generated by 1, \(a\) and \(b\).

Proof. First assume that \(H_\pi\) is separable. Let \(J = K(H) \cap A\), \(H_n = [\pi(J)H_\pi]\), and let \(\pi_n\) and \(\pi_s\) be the representations of \(A\) defined by \(\pi_n(a) = \pi(a)|H_n\) and \(\pi_s(a) = \pi(a)|H_s\) \((a \in A)\), so that \(\pi = \pi_n \oplus \pi_s\). By basic theory of representations of \(C^*\)-algebras of compact operators \(\pi_n\) is a subrepresentation of a multiple \(id^{(m)}\) of the identity representation. By Voiculescu’s theorem ([35], [4]) the representation \(\pi \oplus id\) is approximately unitarily equivalent to \(\pi_n \oplus id\), hence \(\pi \oplus id\) is approximately unitarily equivalent to a subrepresentation of \(id^{(m+1)}\). It follows easily from (6.1) that (6.2) holds for any multiple of the identity representation in place of \(\pi\), hence it must also hold for any subrepresentation \(\rho\) of \(id^{(m+1)}\) (to see this, just take in (6.2) for \(x\) elements that live on the Hilbert space \(\rho\)). But then it follows from the approximate equivalence that the condition (6.2) holds for \(\pi \oplus id\) in place of \(\pi\), hence also for \(\pi\) itself.

In general, when \(H_\pi\) is not necessarily separable, \(H_\pi\) decomposes into an orthogonal sum \(\oplus_{j \in J}H_j\) of separable invariant subspaces for \(\pi(A)\). For a fixed \(x \in B(H_\pi)\) there exists a countable subset \(\mathcal{J}\) of \(J\) such that the norm of the operator \([\pi(b), x]\) is the same as the norm of its compression to \(\mathcal{L} := \oplus_{j \in \mathcal{J}}H_j\). Since \(\mathcal{L}\) is separable, it follows from what we have already proved that \(\|[\pi(b), x]\| \leq \|[\pi(a), x]\|\). \(\square\)
Corollary 6.2. If \( a, b \in B(\mathcal{H}) \) satisfy (6.1) then \( b \) is contained in the \( C^* \)-algebra \( B \) generated by \( a \) and 1.

\[ \text{Proof.} \] Let \( \pi \) be the universal representation of \( A = C^*(a, b, 1) \) and \( \mathcal{H}_\pi \) its Hilbert space. It follows from Lemma 6.1 that \( \pi(b) \in (\pi(A))^\prime \), hence also \( \pi(b) \in (\pi(B))^\prime \). But \( (\pi(B))^\prime = \pi(B) \), thus \( \pi(b) \in \pi(B) \cap \pi(A) = \pi(B) \) by [22, 10.1.4]. \[ \square \]

Let us say that a completely contractive Hilbert module over an operator algebra \( A \) (that is, a Hilbert space on which \( A \) has a completely contractive representation) is a cogenerator if every completely contractive Hilbert \( A \)-module is contained (completely isomorphically) in a multiple of \( \mathcal{H} \) (that is, in a direct sum of copies of \( \mathcal{H} \)). Here by an operator algebra we will always mean a norm complete algebra of operators on a Hilbert space.

Proposition 6.3. If \( a, b \in B(\mathcal{H}) \) satisfy (6.1), where \( \mathcal{H} \) is a cogenerator for the operator algebra \( A_0 \) generated by \( a \) and 1, then \( b \in A_0 \).

\[ \text{Proof.} \] Let \( \pi \) be the universal representation of the \( C^* \)-algebra \( A \) generated by \( 1, a, b \). Then \( \mathcal{H}_\pi \) (the Hilbert space of \( \pi \)) is a cogenerator for \( A_0 \), hence by the Blecher-Solel bicommutation theorem (see [6, 3.2.14]) \( \pi(A_0) = \pi(A_0)^\prime \). From Lemma 6.1 \( \pi(b) \in \pi(A_0)^\prime \), hence \( \pi(b) \in \pi(A_0) \cap \pi(A) = \pi(A) \) by [22, 10.1.4]. \[ \square \]

Remark 6.4. Let \( a, b \in B(\mathcal{H}) \), \( A_0 \) the norm closed operator algebra generated by \( a \) and 1, and \( \mathcal{H}_0 \) the direct sum of infinitely many copies of the Hilbert space of the universal representation of \( C^*_{\max}(A_0) \) (the maximal \( C^* \)-cover of \( A_0 \), see [6]). Then there exists a natural \( * \)-homomorphism \( q \) from \( C^*_{\max}(A_0) \) onto \( C^*(A_0) \) (the \( C^* \) subalgebra of \( B(\mathcal{H}) \) generated by \( A_0 \)). If (6.1) holds, then \( b \in C^*(A_0) \) by Corollary 6.2, hence \( b = q(b_0) \) for some \( b_0 \in C^*_{\max}(A_0) \). If \( b_0 \) can be chosen so that

\[ ||b_0, x|| \leq ||a, x|| \] for all \( x \in B(\mathcal{H}_0) \),

then it follows by Proposition 6.3 that \( b_0 \) can be approximated by polynomials in \( a \) and 1, hence so can be \( b \). If in addition \( a \) is subnormal, then we conclude that \( b = f(a) \) for a function \( f \) in the uniform closure of polynomials on \( \sigma(a) \).

For a general subnormal operator \( a \) the condition (6.1) (in contrast to its more involved version (6.3)) does not imply that \( b \) is in the uniform closure of polynomials in \( a \), however the author does not know if it implies that \( b \) is of the form \( b = f(a) \) for some more general function \( f \).

Problem. If in (6.1) \( a \) is subnormal, is then \( b \) necessarily of the form \( b = f(a) \) for some function \( f \)? Is \( b \) necessarily subnormal?

7. Commutators of Functions of Subnormal Operators

In this section we will show for a subnormal operator \( a \) and a sufficiently regular function \( f \) an estimate of the form

\[ ||[f(a), x]|| \leq \kappa ||a, x|| \] \( \forall x \in B(\mathcal{H}) \).

By Lemma 5.5 such an estimate can hold only for Schur functions, but we are able to prove (7.1) for all Schur functions only if \( \sigma(a) \) is nice (Theorem 7.7). It follows from the proof in [21, Theorem 4.1] that a Schur function \( f \) is complex differentiable in the sense that the limit \( f'(\zeta_0) = \lim_{\zeta \to \zeta_0, \zeta \in \sigma(a)} (f(\zeta) - f(\zeta_0))/(\zeta - \zeta_0) \) exists at each non-isolated point of \( \sigma(a) \). Moreover, from the Lipschitz condition on \( f \) we see that \( f' \) is bounded. However, the boundedness of \( f' \) is not sufficient for \( f \) to
be a Schur function. When $a$ is selfadjoint it is proved in [21, 5.1] that (7.1) holds if $f^{(3)}$ is continuous. We will prove (7.1) for subnormal $a$ under a much milder condition on $f$ (for example, $f'$ Lipschitz suffices), but perhaps when $a$ is normal our condition on $f$ is more restrictive than Peller’s condition that $f$ is a restriction of a function from the appropriate Besov space (see [30] and [2]).

We will start from the special case of the Cauchy-Green formula

\[(7.2)\quad g(\lambda) = -\frac{1}{\pi} \int_{C} \overline{\partial g(\zeta)} (\zeta - \lambda) \, dm(\zeta),\]

which holds for a compactly supported differentiable function $g$ such that $\overline{\partial g}$ is bounded. Here $m$ denotes the planar Lebesgue measure and $\overline{\partial g} = (1/2)(\partial g + i \partial g)$. (The proof in [31, 20.3] is valid for functions with the properties just stated.) We note that an operator calculus based on the Cauchy-Green formula was developed by Dynkin [14], however we will need rather different results, specific to subnormal operators.

**Lemma 7.1.** If $a \in B(H)$ is a subnormal operator and $g : \mathbb{C} \to \mathbb{C}$ is a differentiable function with compact support such that $\overline{\partial g}$ is bounded and $\overline{\partial g}|\sigma(a) = 0$, then

\[(7.3)\quad \langle g(a)\eta, \xi \rangle = -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g(\zeta)} ((\zeta - a)^{-1}\eta, \xi) \, dm(\zeta), \quad (\xi, \eta \in H).\]

**Proof.** Let $c \in B(K)$ be the minimal normal extension of $a$, $e$ the projection valued spectral measure of $c$, $K = \sigma(a)$ and $H = \{\zeta : \overline{\partial g(\zeta)} \neq 0\}$. For $\eta \in H$ and $\xi \in K$ denote by $\mu$ the measure $\langle e(\zeta)\eta, \xi \rangle$. Then by the spectral theorem $g(c) = \int_{K} g(\lambda) \, d\mu(\lambda)$ and $(\zeta - a)^{-1} = \int_{K} (\zeta - \lambda)^{-1} \, d\mu(\lambda)$ for each $\zeta \in \mathbb{C} \setminus K$ (in particular for $\zeta \in H$ since $H \cap K = 0$ because of $\overline{\partial K} = 0$), hence by (7.2)

\[
\langle g(c)\eta, \xi \rangle = \int_{K} g(\lambda) \, d\mu(\lambda) = -\frac{1}{\pi} \int_{H} \int_{\overline{K}} \overline{\partial g(\zeta)} (\zeta - \lambda)^{-1} \, dm(\zeta) \, d\mu(\lambda) = -\frac{1}{\pi} \int_{H} \overline{\partial g(\zeta)} \int_{\overline{K}} (\zeta - \lambda)^{-1} \, d\mu(\lambda) \, dm(\zeta) = -\frac{1}{\pi} \int_{\overline{K}} \overline{\partial g(\zeta)} \int_{H} ((\zeta - a)^{-1}\eta, \xi) \, dm(\zeta) = -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g(\zeta)} ((\zeta - a)^{-1}\eta, \xi) \, dm(\zeta).
\]

For all $\xi \in H^\perp$ the last integrand is 0 since $(\zeta - a)^{-1}\eta \in H$, hence $g(c)\eta \in H$. Thus $H$ is an invariant subspace for $g(c)$ and the usual definition of $g(a)$, namely $g(a) := g(c)|H$ (see [13, p. 85]), is compatible with (7.3). To justify the interchange of order of integration in the above computation, let $M = \sup_{\zeta \in \overline{K}} |\overline{\partial g(\zeta)}|$ and let $R$ be a constant larger than the diameter of the set $H - K$, so that for each $\lambda \in K$ the disc $D(\lambda, R)$ with the center $\lambda$ and radius $R$ contains $H$. Introduce the polar coordinates by $\zeta = \lambda + re^{i\phi}$. Then by Fubini’s theorem

\[
\int_{H} \int_{K} |\overline{\partial g(\zeta)}| \, \mu|(\lambda) \, dm(\zeta) \leq M \int_{H} \int_{K} |\zeta - \lambda|^{-1} \, d\mu|(\lambda) \, dm(\zeta) = M \int_{K} \int_{H} |\zeta - \lambda|^{-1} \, dm(\zeta) \, d\mu|(\lambda) = M \int_{K} \int_{0}^{2\pi} \int_{0}^{R} dr \, d\phi \, d\mu| = 2\pi M R |\mu|(K) < \infty.
\]
Now, if $a$ and $g$ are as in Lemma 7.1 and if $b = g(a)$, we may compute formally for each $x \in B(H)$

$$[b, x] = [g(a), x] = -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g(\zeta)} ((\zeta - a)^{-1} [a, x] (\zeta - a)^{-1} \, dm(\zeta) = [a, T_{a,f}(x)],$$

where

$$T_{a,f}(x) := -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g(\zeta)} ((\zeta - a)^{-1} x (\zeta - a)^{-1} \, dm(\zeta).$$

The problem here is, of course, the existence of the integral in (7.4). We have to show that the map

$$T_{a,f}(x) := -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g(\zeta)} ((\zeta - a)^{-1} x (\zeta - a)^{-1} \, dm(\zeta)$$

is a bounded sesquilinear form on $H$. The following lemma will be helpful.

**Lemma 7.2.** Let $a$, $g$ and $K := \sigma(a)$ be as in Lemma 7.1. If

$$\kappa := \sup_{\lambda \in K} \int_{C \setminus K} |\partial g(\zeta) - \zeta - \lambda|^{-2} \, dm(\zeta) < \infty,$$

then the sesquilinear form defined by (7.5) is bounded by $\frac{2}{\pi} ||x|| \kappa$.

**Proof.** For any $t > 0$, using first the Schwarz inequality and then the inequality $\alpha \beta \leq \frac{1}{8} (t^2 \alpha^2 + t^{-2} \beta^2)$ ($\alpha, \beta \geq 0$) to estimate the inner product in the integral in (7.5), we see that the integral in (7.5) is dominated by

$$\|x\| \int_{K'} |\partial g(\zeta)||((\zeta - a)^{-1} \eta)||((\zeta - a^*)^{-1} \xi) \, dm(\zeta) \leq \|x\|^2 \frac{1}{2} t^2 \int_{K'} |\partial g(\zeta)||((\zeta - a)^{-1} \eta)||((\zeta - a^*)^{-1} \xi) \, dm(\zeta) + t^2 \int_{K'} |\partial g(\zeta)||((\zeta - a)^{-1} \xi) \, dm(\zeta).$$

Using the notation from the proof of Lemma 7.1 (with $\mu(\cdot) := \langle e(\cdot), \xi \rangle$) and (7.6), we have

$$\int_{C \setminus K} |\partial g(\zeta)||((\zeta - a)^{-1} \xi) \, dm(\zeta) = \int_{H} |\partial g(\zeta)||((\zeta - \lambda)^{-2} \, d\mu(\lambda) \, dm(\zeta)$$

$$= \int_{K} \int_{H} |\partial g(\zeta)||\xi - \lambda|^{-2} \, d\mu(\lambda) \, dm(\zeta) \leq \kappa \mu(K) = \kappa \|\xi\|^2.$$

Since a similar estimate holds with $\eta$ in place of $\xi$, it follows that

$$\int_{H} |\partial g(\zeta)||((\zeta - a)^{-1} \eta)||((\zeta - a^*)^{-1} \xi) \, dm(\zeta) \leq \kappa (t^2 \|\eta\|^2 + t^{-2} \|\xi\|^2).$$

Taking the infimum over all $t > 0$ we get

$$\frac{1}{\pi} \int_{H} |\partial g(\zeta)||t((\zeta - a)^{-1} \eta)||t^{-1}((\zeta - a^*)^{-1} \xi) \, dm(\zeta) \leq \frac{2}{\pi} \kappa \|\eta\| \|\xi\|.$$

$\Box$
Remark 7.3. Lemma 7.2 applies, for example, if \( \overline{\partial}g \) is a Lipschitz function of order \( \alpha \), that is \( |\partial g(\zeta) - \partial g(\zeta_0)| \leq \beta |\zeta - \zeta_0|^\alpha \) (\( \zeta, \zeta_0 \in \mathbb{C} \)) for some positive constants \( \alpha \) and \( \beta \), with \( \overline{\partial}g|K = 0 \). In this case the integral (7.6) may be estimated by noting that the Lipschitz condition (together with \( \overline{\partial}g|K = 0 \)) implies that \( |\partial g(\zeta)| \leq \beta \delta(\zeta, K)^\alpha \), where \( \delta(\zeta, K) \) is the distance from \( \zeta \) to \( K \). Let \( R > 0 \) be so large that for each \( \lambda \in K \) the closed discs \( D(\lambda, R) \) with the center \( \lambda \) and radius \( R \) contains \( H \). Introducing the polar coordinates by \( \zeta = \lambda + re^{i\theta} \), for each \( \lambda \in K \) we have

\[
\int_{C \setminus K} |\partial g(\zeta)||\zeta - \lambda|^{-2} dm(\zeta) \leq \beta \int_{H} \frac{\delta(\zeta, K)^\alpha}{|\zeta - \lambda|^2} dm(\zeta) \leq \beta \int_{D(\lambda, R)} |\zeta - \lambda|^\alpha - 2 dm(\zeta) = 2\pi \beta \alpha^{-1} R^\alpha.
\]

Definition 7.4. A function \( f \) on a compact subset \( K \subseteq \mathbb{C} \) is in the class \( L(1 + \alpha, K) \) (where \( \alpha \in (0, 1) \)) if the limit

\[
(f'((\zeta_0)) = \lim_{\zeta \to \zeta_0, \zeta \in \sigma(a)} \frac{f(\zeta) - f(\zeta_0)}{\zeta - \zeta_0}
\]

exists for each (nonisolated) \( \zeta_0 \in K \) and if there exists a constant \( \kappa > 0 \) such that

\[
|f(\zeta) - f(\zeta_0) - f'(\zeta_0)(\zeta - \zeta_0)| \leq \kappa|\zeta - \zeta_0|^{1 + \alpha}
\]

and

\[
|f'(\zeta) - f'(\zeta_0)| \leq \kappa|\zeta - \zeta_0|^{\alpha}
\]

for all \( \zeta, \zeta_0 \in K \).

We need the following consequence of the Whitney extension theorem.

Lemma 7.5. Each \( f \in L(1 + \alpha, K) \) can be extended to a continuously differentiable function \( g \) with compact support such that \( \overline{\partial}g \) is a Lipschitz function of order \( \alpha \) and \( \partial g(\zeta) = 0 \) if \( \zeta \in K \) (even though \( K \) may have empty interior).

Proof. It suffices to extend \( f \) to a differentiable function \( g \) with Lipschitz \( \overline{\partial}g \) and \( \partial g \), for then we simply multiply \( g \) by a smooth function with a compact support which is equal to 1 on \( K \). Let \( \zeta = x + iy \), \( f = f_1 + if_2 \) and \( f'(\zeta) = h_1(\zeta) + ih_2(\zeta) \), where \( f_1, f_2 \) and \( h_1, h_2 \) are real valued functions on \( K \). It follows from (7.8) and (7.9) that for any \( \zeta, \zeta_0 \in K \)

\[
f_1(\zeta) = f_1(\zeta_0) + h_1(\zeta_0)(x - x_0) - h_2(\zeta_0)(y - y_0) + R(\zeta, \zeta_1)
\]

and

\[
h_j(\zeta) = h_j(\zeta_0) + R_j(\zeta, \zeta_0) \quad (j = 1, 2),
\]

where \( R \) and \( R_j \) are function satisfying \( |R(\zeta, \zeta_1)| \leq \kappa|\zeta - \zeta_0|^{1 + \alpha} \) and \( |R_j(\zeta, \zeta_0)| \leq \kappa|\zeta - \zeta_0|^{\alpha} \). By the Whitney extension theorem [33, p. 177] \( f_1 \) can be extended to a differentiable function \( g_1 \) on \( \mathbb{C} \) such that the partial derivatives of \( g_1 \) are Lipschitz of order \( \alpha \) and

\[
\frac{\partial g_1}{\partial x} = h_1, \quad \frac{\partial g_1}{\partial y} = -h_2 \text{ on } K.
\]

Similarly \( f_2 \) can be extended to an appropriate function \( g_2 \) such that

\[
\frac{\partial g_2}{\partial x} = h_2, \quad \frac{\partial g_2}{\partial y} = h_1 \text{ on } K.
\]

Then \( g := g_1 + ig_2 \) is a required extension of \( f \) since (7.10) and (7.11) imply that \( \overline{\partial}g = 0 \) on \( K \). \qed
In all of the above discussion in this section we may replace \( a \) by \( a^{(\infty)} \) acting on \( \mathcal{H}^\infty \), which implies that the map \( T_{a,f} \) defined by (7.4) is completely bounded and, taking in (7.5) \( \xi \) and \( \eta \) to be in \( \mathcal{H}^\infty \), we see that

\[
\langle T_{a,f}(x), \rho \rangle = -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g}(\zeta) \langle x, (\zeta - a)^{-1} \rho(\zeta - a)^{-1} \rangle \, dm(\zeta) = \langle x, (T_{a,f})_\sharp(\rho) \rangle
\]

for each \( \rho = \eta \otimes \xi^* \) in the predual of \( B(\mathcal{H}) \), where

\[
\langle T_{a,f}, \rho \rangle = -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g}(\zeta)(\zeta - a)^{-1} \rho(\zeta - a)^{-1} \, dm(\zeta)
\]

\[
= -\frac{1}{\pi} \int_{C \setminus \sigma(a)} \overline{\partial g}(\zeta)(\zeta - a)^{-1} \eta \otimes ((\zeta - a)^{-1})^{*} \xi \, dm(\zeta).
\]

A similar computation as in the proof of Lemma 7.2 shows that the last integral exists and that \( \|T_{a,f}\| \leq \text{const} \|\rho\| \). Therefore we conclude that \( T_{a,f} \) is \( \text{weak}^* \) continuous. Further, if \( S \) is any \( \text{weak}^* \) continuous \( (a)' \)-bimodule endomorphism of \( B(\mathcal{H}) \), then \( S \) commutes in particular with multiplications by \( (\zeta - a)^{-1} \) and using (7.12) it follows that \( S \) commutes with \( T_{a,f} \). Collecting all the above results, we have proved the following theorem.

**Theorem 7.6.** For a subnormal operator \( a \in B(\mathcal{H}) \) and a function \( f \in L(1 + \alpha, \sigma(a)) \) \((\alpha \in (0, 1])\) the map \( T_{a,f} \) defined by (7.4) is a central element in the algebra of all normal completely bounded \( (a)' \)-bimodule endomorphisms of \( B(\mathcal{H}) \) such that \( [f(a), x] = [a, T_{a,f}(x)] = T_{a,f}([a, x]) \) for all \( x \in B(\mathcal{H}) \). (In particular the range of \( d_{f(a)} \) is contained in the range of \( d_a \) and (7.1) holds.)

Now we are going to show that if \( \sigma(a) \) is nice enough, then the Lipschitz type condition on \( f \) in Theorem 7.6 can be relaxed: \( f \) only needs to be a Schur function. First suppose that \( \sigma(a) \) is the closed unit disc \( \overline{D} \). For each \( r \in (0, 1) \) let \( f_r(\zeta) = f(\zeta) \). Thus each \( f_r \) is a holomorphic function on a neighborhood \( \Omega_r \) of \( \overline{D} \) and \( f_r(a) \) can be expressed as \( f(a) = \frac{1}{2\pi i} \int_{\Gamma_r} f_r(\zeta)(\zeta - a)^{-1} \, d\zeta \), where \( \Gamma_r \) is a contour in \( \Omega_r \) surrounding \( \sigma(\alpha) \) once in a positive direction. Then for each \( x \in B(\mathcal{H}) \) we have

\[
[f_r(a), x] = \frac{1}{2\pi i} \int_{\Gamma_r} f_r(\zeta)(\zeta - a)^{-1}, x d\zeta = \frac{1}{2\pi i} \int_{\Gamma_r} f_r(\zeta)(\zeta - a)^{-1}[a, x](\zeta - a)^{-1} d\zeta
\]

and also

\[
[f_r(a), x] = [a, T_r(x)], \quad \text{where } T_r(x) = \frac{1}{2\pi i} \int_{\Gamma_r} f_r(\zeta)(\zeta - a)^{-1} x(\zeta - a)^{-1} d\zeta.
\]

If the set of completely bounded maps \( T_r \) on \( B(\mathcal{H}) \) \((0 < r < 1) \) is bounded, then it has a limit point, say \( T \), in the \( \text{weak}^* \) topology (which the space of all completely bounded maps on \( B(\mathcal{H}) \) carries as a dual space, see e.g. [6, 1.5.14 (4)]). \( T \) commutes with left and right multiplications by elements of \( (a)' \) (since all \( T_r \) do). From (7.13) and (7.14) we see that

\[
[f(a), x] = T([a, x]) = [a, T x] \quad (x \in B(\mathcal{H})).
\]

The same holds for \( a^{(n)} \) in place of \( a \) and \( x \in M_n(B(\mathcal{H})) \), hence in particular (7.1) holds. To estimate the norms of the maps \( T_r \), let \( c \) on \( K \supseteq \mathcal{H} \) be the unitary power dilation of \( a \) (so that \( a^n = pc^n|\mathcal{H} \) for all \( n \in \mathbb{N} \), where \( p \) is the orthogonal projection
from \( K \) onto \( \mathcal{H} \), see e.g. [18] or [28]). Let \( S_r \) be the map on \( B(K) \) defined in the same way as \( T_r \), that is, in the second formula in (7.14) we replace \( a \) by \( c \). Then (since \( f_r \) can be approximated uniformly by polynomials) \( T_r(x) = pS_r(x)\mathcal{H} \) for each \( x \in B(\mathcal{H}) \), where \( x \) is regarded as an operator on \( K \) by setting \( x|\mathcal{H}^\perp = 0 \). Hence \( \|T_r\| \leq \|S_r\| \). In the special case when \( c \) is diagonal (relative to some orthonormal basis of \( K \)) with eigenvalues \( \lambda_i \) and \( x = [x_{i,j}] \), a simple computation shows that \( S_r(x) \) is represented by the matrix \( \left[ \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} \right] \) \( x_{i,j} \) (where the quotient is taken to be 0 if \( \lambda_i = \lambda_j \)). Hence in this case \( \|S_r\| \leq \kappa \) since \( f \) is a Schur function. Since any normal operator \( c \) can be approximated uniformly by diagonal operators, the same estimate must hold for all such \( c \) with \( \sigma(c) \subseteq \mathbb{D} \). A similar reasoning applies also to the completely bounded norm, hence it follows that \( \sup_{0 < r < 1} \|T_r\|_{cb} < \infty \).

Let us now consider the case when \( \sigma(a) \) is the closure of its interior \( U \) and \( U \) is simply connected. Let \( h \) be a conformal map from \( \mathbb{D} \) onto \( U \). If the boundary \( \partial \sigma(a) \) of \( \sigma(a) \) is sufficiently nice, say a Jordan curve of class \( C^3 \), then \( h \) can be extended to a bijection, denoted again by \( h \), from \( \overline{\mathbb{D}} \) onto \( \overline{U} = \sigma(a) \), such that \( h \) and \( h^{-1} \) are in the class \( C^2 \) [24, 5.2.4]. Then by Theorem 7.6 and Lemma 5.5 \( h \) and \( h^{-1} \) are Schur functions. Let \( a_0 = h^{-1}(a) \). Note that \( \{a_0, 1\} \) generates the same Banach subalgebra as \( \{a, 1\} \) since \( h \) and \( h^{-1} \) can both be uniformly approximated by polynomials (by Mergelyan’s theorem). For any Schur function \( f \) on \( \sigma(a) \) the composition \( f_0 := f \circ h \) is a Schur function on \( \mathbb{D} \). (To see this, note that for any \( \lambda \neq \mu \) in \( \mathbb{D} \) we may write \( \frac{f(h(\lambda)) - f(h(\mu))}{h(\lambda) - h(\mu)} = \frac{f(h(\lambda)) - f(h(\mu))}{h(\lambda) - h(\mu)} \) if \( h(\lambda) \neq h(\mu) \) and that the inequality \( ||x_{i,j}y_{i,j}||S \leq ||x_{i,j}||S \|y_{i,j}\|S \) holds for the Schur norm of the Schur product of two matrices.) Note that \( f(a) = f_0(a_0) \) and \( (a_0)' = (a)' \). By the previous paragraph there exists a completely bounded \( (a_0)' \)-bimodule map \( T \) on \( B(\mathcal{H}) \) such that

(7.15) \[ [f_0(a_0), x] = [a_0, Tx] = T([a_0, x]) \] for all \( x \in B(\mathcal{H}) \),

hence \( [f(a), x] = [h^{-1}(a), Tx] = T([h^{-1}(a), x]) \). Now the map \( T \) is not a priori normal, but it can be replaced by its normal part \( T_n \) in (7.15), hence we may achieve that \( T \) is normal. By Theorem 7.6 there exists a completely bounded \( (a)' \)-bimodule map \( S \) on \( B(\mathcal{H}) \) such that \( [h^{-1}(a), y] = [a, Sy] = S([a, y]) \) for all \( y \) in \( B(\mathcal{H}) \) and \( S \) commutes with all normal \( (a)' \)-bimodule maps on \( B(\mathcal{H}) \) (in particular with \( T \)). Hence we have now \( [f(a), x] = [h^{-1}(a), Tx] = [a, STx] \) and \( [f(a), x] = T([h^{-1}(a), x]) = TS([a, x]) \) for all \( x \in B(\mathcal{H}) \). Denoting \( T_{a,f} = TS = ST \), we have deduced the following theorem.

**Theorem 7.7.** For a subnormal \( a \in B(\mathcal{H}) \) suppose that \( \sigma(a) \) is the closure of a simply connected domain bounded by a Jordan curve of class \( C^3 \). Then for each Schur function \( f \) on \( \sigma(a) \) there exists a (normal) completely bounded \( (a)' \)-bimodule map \( T_{a,f} \) on \( B(\mathcal{H}) \) such that \( [f(a), x] = [a, T_{a,f}(x)] = T_{a,f}([a, x]) \) for all \( x \in B(\mathcal{H}) \).

In general, the Lipschitz type condition in Theorem 7.6 can be replaced by a similar, but less restrictive condition, which involves a regular modulus of continuity \( \omega \) in the sense of [33, p. 175] (instead of just \( \omega(t) = t \alpha \)) such that \( \int_0^1 \omega(r)/r \, dr < \infty \). (There exists an appropriate version of Whitney’s extension theorem [33, p. 194].) But probably even this is too restrictive, for we do not need any requirements about \( \partial g \) of the extension \( g \) (only requirements about \( \overline{\partial g} \)). Thus, at least in cases when \( K \) has a nice boundary, it is worthwhile to try a more direct way of extending
functions. We will only consider the case $K \subseteq \mathbb{R}$, from which the result can be generalized to sets bounded by sufficiently regular Jordan curves.

**Proposition 7.8.** If $f$ is a continuously differentiable function on $\mathbb{R}$ with compact support such that

\[
\sup_{\lambda \in \mathbb{R}} \int_{\mathbb{R}} \int_{0}^{1} |f'(x + \lambda) - f'(x + \lambda - h)| \left| \frac{\ln(1 + \frac{y^2}{x^2})}{x^2} \right| dh \, dx < \infty,
\]

then $f$ can be extended to a differentiable function $g$ on $\mathbb{C}$ with a compact support such that $\partial g$ is bounded, $\partial g|\mathbb{R} = 0$ and

\[
\sup_{\lambda \in \mathbb{R}} \int_{\mathbb{C}} |\partial g(\zeta)| |\zeta - \lambda|^{-2} dm(\zeta) < \infty.
\]

The condition (7.16) holds in particular if

\[
\int_{0}^{1} \|\Delta h f'\|_\infty dh < \infty,
\]

where \(\|\Delta h f'\|_\infty = \sup_{x \in \mathbb{R}} |\Delta h f'|\) and $(\Delta h f')(x) = f'(x - h) - f'(x)$.

**Proof.** An extension $g$ of $f$ satisfying $\partial g|\mathbb{R} = 0$ is given by $g(\zeta) = f(x) + iyf'(x)$, where $\zeta = x + iy$. However, for this $g$ to be differentiable we must assume that $f$ is twice differentiable. To avoid this additional assumption on $f$, let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \phi(x) \, dx = 1$, $\phi_y(x) := y^{-1}\phi(y^{-1}x)$, and let

\[
g_0(\zeta) = \begin{cases} 
 f(x) + iy(\phi_y|_y \ast f')(x), & \text{if } y \neq 0 \\
 f(x), & \text{if } y = 0.
\end{cases}
\]

Since $f'$ is continuous, $\phi_y|_y \ast f'$ converges to $f'$ pointwise and it follows by a simple computation that $\partial \phi_0|\mathbb{R} = 0$. If $y > 0$,

\[
g_0(\zeta) = f(x) + i \int_{\mathbb{R}} \phi\left(\frac{x - t}{y}\right) f'(t) \, dt
\]

and

\[
2\partial g_0(\zeta) = \frac{\partial g_0}{\partial x}(x, y) + i \frac{\partial g_0}{\partial y}(x, y)
\]

\[
= f'(x) + \frac{i}{y} \int_{\mathbb{R}} \phi\left(\frac{x - t}{y}\right) f'(t) \, dt + \frac{1}{y} \int_{\mathbb{R}} \phi\left(\frac{x - t}{y}\right) x - t \, f'(t) \, dt
\]

\[
= f'(x) + i \int_{\mathbb{R}} \phi'(s) f'(x - sy) \, ds + \int_{\mathbb{R}} \phi'(s) s f'(x - sy) \, ds.
\]

Denoting $\psi(s) = s\phi(s)$ and noting that $\int_{\mathbb{R}} \phi'(s) \, ds = 0 = \int_{\mathbb{R}} \psi'(s) \, ds$ (since $\phi$ and $\psi$ have compact support) and $\int_{\mathbb{R}} \phi(s) \, ds = 1$, $2\partial g_0(\zeta)$ can be rewritten as

\[
2\partial g_0(\zeta) = i \int_{\mathbb{R}} \phi'(s) f'(x - sy) \, ds + f'(x) + \int_{\mathbb{R}} (\psi'(s) - \phi(s)) f'(x - sy) \, ds
\]

\[
= i \int_{\mathbb{R}} \phi'(s) (f'(x - sy) - f'(x)) \, ds - \int_{\mathbb{R}} \phi(s) (f'(x - sy) - f'(x)) \, ds
\]

\[
+ \int_{\mathbb{R}} \psi'(s) (f'(x - sy) - f'(x)) \, ds.
\]
A similar computation is possible also for \( y < 0 \) and we thus obtain
\[
(7.19) \quad 2\mathcal{D}_g g_0(\zeta) = \int_{-1}^{1} \theta(s)(f'(x - s|y|) - f'(x)) \, ds \ (y \neq 0),
\]
where \( \theta(s) := (i+s)\phi(s) \). Let \( \delta > 0 \) and \( \chi : \mathbb{C} \to [0,1] \) a smooth function depending only on the variable \( y \) which is equal to 1 on the strip \( -\delta \leq y \leq \delta \), and equal to 0 if \( |y| \geq 1 \). Set \( g = \chi g_0 \). Then \( g \) has a compact support, say \( H \), and
\[
(7.20) \quad \int_H |\mathcal{D}_g g_0(\zeta)| |\zeta - \lambda|^{-2} \, dm(\zeta) = \int_H |\mathcal{D}_\chi g_0(\zeta)| |\zeta - \lambda|^{-2} \, dm(\zeta) + \int_H |\chi(\zeta)| |\mathcal{D}_g g_0(\zeta)| |\zeta - \lambda|^{-2} \, dm(\zeta).
\]
Since \( \mathcal{D}_\chi = 0 \) on a neighborhood of the real line, the first integral on the right side in (7.20) is bounded uniformly for \( \lambda \in \mathbb{R} \). Since the support of \( \chi \) is contained in \( \mathbb{R} \times [-1,1] \), the second integral on the right side of (7.20) is dominated by
\[
\int_{\mathbb{R} \times [-1,1]} |\mathcal{D}_g g_0(\zeta)| |\zeta - \lambda|^{-2} \, dm(\zeta).
\]
Using (7.19) (and the boundedness of \( \theta \)) we see that the last integral is dominated by a constant multiple of
\[
\int_{\mathbb{R}} \int_{-1}^{1} \int_{-1}^{1} \frac{|f'(x + \lambda) - f'(x + \lambda - s|y|)|}{x^2 + y^2} \, ds \, dy \, dx.
\]
Interchanging the order of integration over \( y \) and \( s \) and introducing new variable \( h = sy \) instead of \( y \), the last expression transforms into
\[
(7.21) \quad \int_{\mathbb{R}} \int_{-1}^{1} \int_{0}^{1} \frac{|f'(x + \lambda) - f'(x + \lambda - h)|}{x^2 s^2 + h^2} \, |s| \, dh \, ds \, dx.
\]
Interchanging the order of integration over \( s \) and \( h \) and noting that
\[
\int_{0}^{1} \frac{s}{x^2 s^2 + h^2} \, ds = \frac{\ln(1 + \frac{x^2}{h^2}) - \ln(x^2 + 1)}{2x^2},
\]
we transform (7.21) into
\[
(7.22) \quad 2\int_{\mathbb{R}} \int_{0}^{1} \frac{|f'(x + \lambda) - f'(x + \lambda - h)|}{x^2 s^2 + h^2} \, \ln(1 + \frac{x^2}{h^2}) \, dh \, dx - 2\int_{\mathbb{R}} \int_{0}^{1} \frac{|f'(x + \lambda) - f'(x + \lambda - h)|}{x^2 s^2 + h^2} \, \ln(1 + x^2) \, dh \, dx.
\]
Since the function \( x^{-2}\ln(1 + x^2) \) is bounded (by 1) and \( f \) has compact support, the second double integral in (7.22) is bounded uniformly in \( \lambda \). The first double integral in (7.22) is dominated by
\[
\int_{0}^{1} \|\Delta_h f'\|_{\infty} \int_{\mathbb{R}} x^{-2}\ln(1 + \frac{x^2}{h^2}) \, dx = \pi \int_{0}^{1} \frac{\|\Delta_h f'\|_{\infty}}{|h|} \, dh = 2\pi \int_{0}^{1} \frac{\|\Delta_h f'\|_{\infty}}{|h|} \, dh.
\]
We note that the continuity of \( f' \) in Proposition 7.8 is not essential since the area of \( \mathbb{R} \) is 0 and in the proof of Lemma 7.1 it suffices to require that \( \mathcal{D}_g|\sigma(a) = 0 \) m-almost everywhere. (Also (7.2) still holds if \( g \) fails to be differentiable on one line only.) Further, the condition (7.17) for a compactly supported function \( f \) implies that \( \int_{0}^{1} \frac{\|\Delta_h f'\|_{\infty}}{|h|} \, dh < \infty \) (to see this, apply Lagrange’s theorem to the function \( \Delta_h f \)), which means that \( f \) is in the Besov space \( B^1_{\infty,1}(\mathbb{R}) \). Since for selfadjoint \( a \) the condition \( f \in B^1_{\infty,1}(\mathbb{R}) \) suffices for \( \|\langle f(a), x \rangle\| \leq \kappa\|a, x\| \) \( (x \in B(\mathcal{H})_a) \) by [30],
it would be interesting to know if the condition (7.16) allows compactly supported functions which are not in $B_{\infty,1}^1(\mathbb{R})$.

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