MINIMAL SURFACES IN THE 3-SPHERE BY DOUBLING THE CLIFFORD TORUS OVER RECTANGULAR LATTICES

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Abstract. Building on work of Kapouleas and Yang, we construct sequences of minimal surfaces embedded in the round 3-sphere which converge to the Clifford torus counted with multiplicity two and have second fundamental form blowing up at every point of the torus and genus tending to infinity. Each surface in a given sequence resembles a pair of tori close to the limit torus and joined by many catenoidal bridges arranged over a rectangular lattice on the limit. The collection of sequences is indexed by the ratio of the lengths of the lattice edges, which may be any prescribed positive rational. Unlike the surfaces of Kapouleas and Yang, these new embeddings are not symmetric with respect to any isometries of the 3-sphere exchanging the two sides of the limit Clifford torus, except when the corresponding lattice is square.

1. Introduction

The round unit 3-sphere \( S^3 \) accommodates two distinguished families of embedded low-genus minimal surfaces. The first family comprises the totally geodesic, equatorial 2-spheres, each of which is the locus of points equidistant from a pair of antipodal points. The second family comprises the Clifford tori, each of which is the locus of points equidistant from a pair of orthogonally disjoint geodesic circles. These turn out to be the only examples of embedded minimal surfaces of genus 0 and 1 respectively. Indeed it is a theorem of Almgren ([1]) that no genus-0 surface is even immersed minimally except the equatorial spheres, and more recently Brendle ([2]) has affirmed the Lawson conjecture that, while there exist many minimal immersions of genus 1, only the Clifford tori are embedded. Also recently, in their proof of the Willmore conjecture, Marques and Neves ([12]) characterized the Clifford torus (up to isometry) as the least-area minimal immersion of strictly positive genus.

Higher-genus minimal embeddings were first identified by Lawson ([11]), who in fact constructed representatives of each genus, including multiple (anisometric) examples for composite genera. Lawson solves the Plateau problem on suitably selected polygons in \( S^3 \) and extends the solutions by reflection to closed embeddings. Subsequently Karcher, Pinkall, and Sterling ([10]) as well as Choe and Soret ([3]) have produced further minimal embeddings by following a similar approach. We remark that some of the examples in [3] were described in [13] by Pitts and Rubinstein, who proposed minimax (sweep-out) constructions of these and other minimal surfaces.

Here we pursue a third strategy. In [9] Kapouleas and Yang brought gluing techniques to bear on the construction of minimal embeddings in \( S^3 \); in so doing they simultaneously introduced the first example, by gluing, of a doubling construction, whereby two copies of a single minimal surface are glued to form a new minimal surface. Roughly, the data of a doubling construction consist of a minimal embedding (the one to be doubled) and a configuration of points on this surface. For well posed data the procedure should yield a new minimal embedding, which, away from the points in the configuration resembles two close, parallel copies of the original surface with small discs deleted and which, near each point of the configuration, resembles a truncated catenoid, centered on the point in question and having its boundary circles coincide with the boundaries of the discs deleted.
from the copies, one from each copy. See [6] for a more thorough discussion of doubling in general as well as the outline of a doubling of the equator in $S^3$.

Returning to [9], Kapouleas and Yang construct a sequence of minimal embeddings which double a given Clifford torus $T$ over a square lattice parallel to the circles of maximal and minimal curvature on $T$; their surfaces are symmetric with respect to reflections through the great circles on $T$ which intersect the lattice. In the present article we employ the general gluing methodology as developed by Kapouleas (systematically presented along with applications and further references in [4]) to extend their work in another direction by interpreting “doubling” in a generalized sense to encompass tripling, quadrupling, and in fact the incorporation of any number of copies of $T$.)

For convenience we henceforth regard $S^3$ as the unit sphere in $C^2$ with the standard orientation and take $T = \{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\}$ the torus equidistant from the coordinate unit circles $C_1 = \{z_2 = 0\}$ and $C_2 = \{z_1 = 0\}$. Given an oriented great circle in $S^3$ and an angle $\alpha \in \mathbb{R}$, we define $R^\alpha_{C_1} \in SO(4)$ as rotation about $C$ by $\alpha$ according to the usual orientation conventions; thus for example $R^\alpha_{C_2}(z_1, z_2) = (e^{i\alpha}z_1, z_2)$. We also define $X, Y, Z \in O(4)$ as reflection through the great sphere $\{\text{Im } z_1 = 0\}$, reflection through the great sphere $\{\text{Im } z_2 = 0\}$, and reflection through the great circle $\{z_1 = z_2\}$ respectively, so that $X(z_1, z_2) = (\overline{z_1}, z_2)$, $Y(z_1, z_2) = (z_1, \overline{z_2})$, and $Z(z_1, z_2) = (z_2, z_1)$. Here is the result we prove.

**Theorem 1.1.** Fix positive integers $k$ and $\ell$. There exist another positive integer $m_0$ and a sequence $\{\widetilde{M}_m\}_{m = m_0}^\infty$ of closed connected surfaces minimally embedded in $S^3$ such that

(a) the sequence converges in the sense of varifolds to $T$ with multiplicity 2;
(b) $\widetilde{M}_m$ has genus $k\ell m^2 + 1$;
(c) $\widetilde{M}_m$ has stabilizer in $O(4)$

\[ G_m = \begin{cases} 
X, Y, R_{C_1}^{2m}, R_{C_2}^{2m} & \text{for } k \neq \ell \\
X, Y, Z, R_{C_1}^{2m} & \text{for } k = \ell 
\end{cases} \]

(d) $\widetilde{M}_m/\langle R_{C_1}^{2m}, R_{C_2}^{2m} \rangle = \widetilde{T}[{-1}]_m \cup \widetilde{K}_m \cup \widetilde{T}[1]_m$, where $\widetilde{K}_m$ is the intersection of the quotient with the closed tubular neighborhood $C_m$ of radius $\frac{1}{m \max(k, \ell)}$ around the great circle orthogonal to $T$ through $\frac{1}{\sqrt{2}}(1, 1)$, $\widetilde{T}[{-1}]_m$ and $\widetilde{T}[1]_m$ are the closures of the connected components of the complement of $\widetilde{K}_m$ in the quotient, and these regions have the further properties that

(e) there exist a sequence of diffeomorphisms $\{\Phi_m : \mathbb{R}^3 \to S^3\}_{m = m_0}^\infty$ and a bounded sequence $\{\zeta_m\}_{m = m_0}^\infty \subset \mathbb{R}$ such that

\[ \left\{ \Phi_m^{-1}(\widetilde{K}_m) \right\}_{m = m_0}^\infty \text{ is increasing} \]

with limit a standard catenoid, and

\[ \lim_{m \to \infty} m^2[\max(k, \ell)]^2 e^{\frac{k\ell m^2}{2\pi} - 2\zeta_m} \Phi_m^* g_S \to g_E \]

in the sense of uniform smooth convergence on $\Phi_m^{-1}(C_m)$ relative to $g_E$, where $g_S$ is the round metric on $S^3$ and $g_E$ is the Euclidean metric on $\mathbb{R}^3$; and
(f) there exist two sequences of functions

\[ u[±1]_m : \left( \mathbb{T} \left/ \left< \mathbb{R}_{C_2}, \mathbb{R}_{C_1} \right> \right. \right) \rightarrow \mathbb{C}_m \rightarrow \mathbb{R} \]

such that \( u[±1]_m \rightarrow 0 \) smoothly with respect to the rescaling \( m^2 g_T \) of the flat metric \( g_T \) on \( \mathbb{T} \) and \( \mathcal{T}[i]_m \) is the closure of the graph, via the exponential map of \( S^3 \) and a choice of global unit normal on \( \mathbb{T} \), of \( u[i]_m \).

Informal overview. Gluing techniques have been widely applied in differential geometry. The particular methods we use relate most closely to those developed by Schoen in [14] and by Kapouleas in [5], especially as they evolved and were systematized in [7]. In particular the italicized terminology in this introduction is due to Kapouleas; see [1] for a survey of his methodology. The present construction, like many others under the gluing rubric, features three stages. In the first stage the building blocks of the construction are modified and glued together to assemble a manifold which is called the initial surface. For us the ingredients are catenoids and Clifford tori. While the constituents of the initial surface are minimal, the modifications performed during the gluing introduce some mean curvature, with the result that the initial surface may have mean curvature supported everywhere, though it is designed to be approximately minimal.

The idea is to perturb the initial surface to a nearby surface which is exactly minimal. A perturbation is specified by a real-valued function defined on the initial surface; if the initial surface is given by an embedding \( \phi : M \rightarrow S^3 \), then the perturbed surface is given by \( \phi_u : M \rightarrow S^3 \) according to \( \phi_u(p) = \exp_{\phi(p)} u(p) \nu(p) \), where \( \exp : TS^3 \rightarrow S^3 \) is the exponential map on \( S^3 \) and \( \nu : M \rightarrow TS^3 \) is a global unit normal on the initial surface (which is oriented). In this way we obtain the mean curvature functional \( \mathcal{H} \) which assigns to each suitable function \( u : M \rightarrow \mathbb{R} \) the projection \( \mathcal{H}[u] \) of the mean curvature of the perturbed surface \( \phi_u \) along its unit normal (the one obtained as the perturbation of the already chosen \( \nu \)). The goal of course is to solve the quasilinear equation \( \mathcal{H}[u] = 0 \) for \( u \in C^\infty(M) \) small enough to ensure embeddedness. The second stage is therefore devoted to the study of the linearization \( \mathcal{L} \) at \( 0 \) of \( \mathcal{H} \).

In the simplest constructions of this type \( \mathcal{L} \) admits a right inverse \( \mathcal{R} \), with a reasonable bound, between spaces of functions suited to the problem. It remains then only to manage the nonlinear part \( \mathcal{Q} \) of \( \mathcal{H} \) defined by \( \mathcal{Q}[u] = \mathcal{H}[u] - \mathcal{H}[0] - \mathcal{L}u \). Setting \( u = -\mathcal{R}\mathcal{H}[0], \) to solve \( \mathcal{H}[u + v] = 0 \) is to find \( v : M \rightarrow \mathbb{R} \) such that \( \mathcal{L}v = -\mathcal{Q}[u + v]. \) The final stage completes the construction by applying a fixed-point theorem to the map \( v \mapsto -\mathcal{R}\mathcal{Q}[u + v]. \) Some care is required to guarantee the viability of this scheme, and in general (as at present) complications arise which necessitate refinements to all three stages. We now outline the salient features of each as implemented in the construction at hand.

The surfaces we construct are meant to resemble two copies of a single Clifford torus \( \mathbb{T} \) interrupted and connected by catenoidal annuli. We begin with the observation that \( S^3 \setminus (C_1 \cup C_2) \) is foliated by constant-mean-curvature (henceforth abbreviated cmc) tori, each of which is the locus of points a constant distance from \( C_1 \) (and so equivalently a constant distance from \( C_2 \)); the Clifford torus \( \mathbb{T} \) is simply the leaf equidistant from \( C_1 \) and \( C_2 \). This leaf itself admits four foliations by circles. Two of these foliations are by great circles, each leaf of one foliation orthogonally intersecting each leaf of the other. Reflection through any of these great circles exchanges the two sides of \( \mathbb{T} \). The other two foliations are by the principal-curvature circles of \( \mathbb{T} \), the leaves of each intersecting at \( \frac{\pi}{2} \) the leaves of the other and at \( \frac{\pi}{4} \) the leaves of the first two foliations. The circles of principal curvature have radius \( \sqrt{\frac{1}{2}} \) and can be realized as the intersections with \( \mathbb{T} \) of geodesic 2-spheres whose \( z_1 \) or \( z_2 \) coordinate has constant phase. Reflection through any of these spheres preserves the set \( \mathbb{T} \). Equivalently, each circle of principal curvature is parallel, in \( \mathbb{C}^2 \), to either \( C_1 \) or \( C_2 \).
The arrangement of catenoids is specified by three positive integers $k$, $\ell$, and $m$. For simplicity we will assume $k \leq \ell$. To describe the configuration we fix a lattice $L \subset \mathbb{T}$, defined as the orbit of $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ under the group $\left\langle \mathbf{X}, \mathbf{Y}, \mathbf{R}_{\frac{2\pi}{k}}, \mathbf{R}_{\frac{2\pi}{\ell}}, \mathbf{R}_{\frac{2\pi}{m}} \right\rangle < O(4)$. This lattice then has stabilizer $G$ containing $\left\langle \mathbf{X}, \mathbf{Y}, \mathbf{R}_{\frac{2\pi}{k}}, \mathbf{R}_{\frac{2\pi}{\ell}} \right\rangle$, and this containment fails to be proper precisely when $k = \ell$, in which case one must adjoin the generator $\mathbf{Z}$ to complete the symmetry group $G$. Throughout the construction the symmetries of $L$ will be enforced, so that both the initial surfaces and their perturbations shall be invariant under $G$.

The spacing between the tori will be determined momentarily, but it will be assumed small so that the mean curvature of each torus will likewise be small. The cmc leaves of the foliation under consideration are threaded together by great circles orthogonally intersecting each leaf. We designate directions parallel to the leaves as horizontal and directions orthogonal as vertical, interpreting upward to mean toward $C_1$. In this way we obtain a natural coordinate system $(x, y, z)$ adapted to the foliation, whereby the $z$ coordinate indicates the vertical displacement along the geodesics just upward to mean toward $C_1$. Throughout the construction the spacing cannot be prescribed at will, due to obstructions materializing in the second stage. The difficulty can be traced to the existence of small eigenvalues of the linearized operator $\mathcal{L}$. Their presence threatens to render the solutions to the linearized equation too large to ensure embeddedness or even to control the nonlinear terms adequately. Although this problem is analytical in nature and is studied in detail at the second stage, it can be understood geometrically, and the key to its resolution is built into the construction of the initial surface, at the first stage.

The necessary condition for a successful construction is roughly the orthogonality of the initial mean curvature to the eigenspaces, collectively dubbed approximate kernel, with small eigenvalues. Of course the actual equation to be solved is nonlinear, so this condition is insufficient, and additionally it will be necessary to be able to adjust freely, within some small range, the projection of the initial mean curvature onto the approximate kernel. According to Kapouleas' geometric principle such projections are controlled by repositioning the components of the initial surface relative to each other. The orthogonality itself sets the coarse spacing between the tori, uniquely in terms of $k$, $\ell$, and $m$. Somewhat more concretely, the elements of the approximate kernel are themselves approximated by the normal component on the initial surface of the Killing field of the ambient space $S^3$ generating rotations along a vertical great circle. In this way the orthogonality condition
is realized as a balancing condition on portions of the initial surface for forces in the directions of these Killing fields. Enforcing this balancing determines the spacing between the tori as a function of $k$, $\ell$, and $m$. The prescription of the projections onto the approximate kernel will then be accomplished by the introduction of parameters $\zeta$ and $\xi$, which finely tune the waist radius and vertical positioning respectively of the catenoids; equivalently they adjust the vertical positioning of each torus relative to $\mathcal{T}$.

By the end of the first stage we have therefore assembled, for each $k$ and $\ell$, a sequence, indexed by $m$, of two-parameter families of initial surfaces. Every initial surface is covered by regions, called extended standard regions, each of which approximates either a truncated catenoid or a truncated torus. By choosing $m$ large, we shrink each region, reducing the size of the initial mean curvature, as described above. While the total number of extended standard regions of either type grows with $m$, modulating the symmetries there are just three: two, $\mathcal{T}[1]$ and $\mathcal{T}[-1]$, representing respectively the top and bottom tori, and one, $\mathcal{K}[0]$, representing the central catenoids. Although each region contracts to a point in the limit, when rescaled to a fixed size, they tend to the ingredients of the construction—catenoids and tori. This convergence of the standard regions to their minimal prototypes underlies the estimation of $\mathcal{H}_m[0]$, central to the first stage, and also the analysis of $\mathcal{L}_m$, which characterizes the second stage.

The analysis of $\mathcal{L}$ is facilitated by a conformal change of metric on the initial surface, from metric $g$ induced by the round metric on $S^3$ to the new metric $\chi = \rho^2 g$, where the conformal factor $\rho^{-1}$ measures the size of the initial surface, in a natural sense, at the point where it is evaluated. It transitions smoothly from, on the catenoidal regions, the radius of the catenoid as a function of height to, on the toral regions, the constant $\frac{1}{\ell m}$. This change of metric has the effect of uniformizing the problem in $m$; the equation $\mathcal{L}u = -\mathcal{H}[0]$ is recast as $\mathcal{L}_\chi = \rho^{-2} \mathcal{H}[0]$, where the operator $\mathcal{L}_\chi = -\rho^{-2} \mathcal{L}$ has an easily identified, nondegenerate limit on each limit standard region. Specifically, with the $\chi$ metric, the catenoidal regions tend to long cylinders on which $\mathcal{L}_\chi$ converges to the Jacobi operator for a standard catenoid parametrized by the cylinder, while the toral regions tend to rectangles of a fixed size on which $\mathcal{L}_\chi$ converges to the Jacobi operator for a plane, which is simply the flat Laplacian. The strategy for completing the second stage is to treat $\mathcal{L}_\chi$ on each extended standard region as a perturbation of the corresponding limit operator so as to construct an inverse to $\mathcal{L}_\chi$ on each region and then to piece together these local inverses into a global one.

The operators act between function spaces adapted to the problem. We choose weighted Hölder spaces. Owing to the vast disparity of scale between the toral and catenoidal regions (which in turn originates in the exponential growth of a catenoid’s radius with its height), in order to ensure embeddedness and control the nonlinear terms it is necessary that the size of a perturbation decay from the toral regions toward the center of the catenoidal regions. This decay is also useful for the purposes of an iteration argument by which one passes from the local inverses of the linearized operator to a global inverse on the entire initial surface. Our function spaces are weighted precisely to enforce this decay, and of course they are also required to respect the symmetry group $\mathcal{G}$.

It turns out that on the catenoidal regions we can impose boundary data so that the limit operator admits a bounded right inverse. By the convergence we can then solve the equation $\mathcal{L}_\chi u = f$ for $u$ satisfying good estimates, independent of $m$, assuming adequate estimates on $f$. On each toral region, however, the limit operator has one-dimensional kernel, so that the operators themselves have approximate kernel. To confront this difficulty we are compelled to introduce, in accordance with the gluing methodology, two-dimensional substitute kernel having basis $\{w_i \in C^\infty(M_m)\}_{i=\pm 1}$, where each $w_i$ is smooth on the entire initial surface but supported on just $\mathcal{T}[i]$, with the result that, given $f$ in an appropriately chosen space of functions on $\mathcal{T}[i]$, we can find $\mu \in \mathbb{R}$ and $u$ in another apt space of functions such that $\mathcal{L}_\chi u = f + \mu w_i$, with estimates independent of $m$ for $\mu$ and $u$ in terms of $f$. The decay implicit in these estimates is obtained at no additional cost; that is we
need not introduce extended substitute kernel, as required by some other constructions, including [15].

The reader familiar with [9] and other constructions of Kapouleas will note some departures from his standard methodology. If we were to impose Dirichlet data on the catenoidal regions as in [9], we would be confronted with one-dimensional approximate kernel on these regions as well, because of the absence of symmetries exchanging the two sides of $\mathbb{T}$. Of course this approximate kernel can be managed within the established framework, but we have opted to circumvent it altogether by relaxing the boundary conditions, the result of which alteration is simply passed to the adjoining toral regions during the iteration. In fact we do not explicitly identify the approximate kernel on any region, nor do we consider the $h$ metric of [9].

In any case, by the end of the second stage, after applying a partition of unity and an iteration scheme, we are able to find, given any suitable $f : M \to \mathbb{R}$, a function $u : M \to \mathbb{R}$, with ample decay, and two real numbers $\mu_{\pm}$, all controlled, independently of $m$, in terms of $f$ and solving the equation

$$L_{\chi} u = f + \mu_1 w_1 + \mu_{-1} w_{-1}.$$  

Thus, taking $f = \rho^{-2}H[0, \zeta, \xi]$ at the start of the third stage, we obtain a healthy perturbation $u$ such that

$$\rho^{-2}H[u, \zeta, \xi] = \mu_1 w_1 + \mu_{-1} w_{-1} + \rho^{-2}Q[u].$$

The twin challenges of this stage are first to perturb the perturbation by the addition of another function $v$ selected to eliminate the nonlinear terms and second to adjust the parameters $\zeta$ and $\xi$ so as to eliminate the substitute kernel. Rather than directly monitor the components $\mu_{\pm}$ of the substitute kernel, we instead estimate the more accessible forces already mentioned, which serve as convenient proxies for these components. The determination of the parameters can then be integrated into the fixed-point argument which proves the theorem.

This concludes the informal overview. The actual construction is organized as follow. In Section 2 we start by defining the extended standard regions. We also prove estimates for their geometry needed later; these estimates are stated in terms of norms described in Appendix A and are supported by results contained in Appendix B. We conclude this section with the assembly of the initial surfaces by gluing together the standard regions, establishing their basic geometric and topological properties and proving in particular a global estimate for the initial mean curvature. In Section 3 we turn to the analysis of the linearized operator, first locally on the standard regions, and then globally. The main result there is the existence of a suitably bounded global inverse, modulo substitute kernel. In the study of the linearized operator on catenoidal regions we appeal to some facts about the limit operator which are collected in Appendix C. Section 4 begins with estimates for the nonlinear terms and the relevant forces. We then apply these estimates in the proof of the main theorem.

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2. **The initial surfaces and their extended standard regions**

Each of the initial surfaces decomposes into (overlapping) regions of two types, those of the first type resembling pieces of catenoids and those of the second pieces of tori. Here we study the details of this resemblance. In a later subsection these extended standard regions will be glued together to build the initial surfaces.
Let $g_E$ denote the Euclidean metric on $\mathbb{R}^3$ and $g_S$ the round metric on $S^3$, which for convenience we embed in $\mathbb{C}^2$ as $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. We are interested in the Clifford torus $T = \{(z_1, z_2) : |z_1| = |z_2| = 1/\sqrt{2}\}$ equidistant from the unit coordinate circles $C_1 = \{z_2 = 0\}$ and $C_2 = \{z_1 = 0\}$. Our construction is facilitated by a coordinate system adapted to $T$. Writing $\exp^T$ and $\exp^S$ for the exponential maps on $T$ and $S^3$ respectively and $\nu^T$ for the unit normal on $T$ pointing toward $C$, we observe that $\Phi$ is a covering of $S^3 \setminus (C_1 \cup C_2)$ and

\[
\Phi(x, y, z) = \exp^{S^3}_{(x, y, z)} \nu^T = \left( e^{i\sqrt{2}x} \sin \left( z + \frac{\pi}{4} \right), e^{i\sqrt{2}y} \cos \left( z + \frac{\pi}{4} \right) \right).
\]

We observe that $\Phi$ is a covering of $S^3 \setminus (C_1 \cup C_2)$ and

\[
\Phi^* g_S = (1 + \sin 2x)dx^2 + (1 - \sin 2x)dy^2 + dz^2 = g_E + \sin 2\alpha (dx^2 - dy^2).
\]

For $\tau \in \mathbb{R}$ define the homothety $S_\tau : \mathbb{R}^3 \to \mathbb{R}^3$ by

\[
S_\tau(x, y, z) = \tau(x, y, z),
\]

so that $S_\tau g_E = \tau^2 g_E$. Define also, for $v \in \mathbb{R}^3$, the translation $T_v : \mathbb{R}^3 \to \mathbb{R}^3$ by

\[
T_v(x, y, z) = (x + v^1, y + v^2, z + v^3),
\]

so that $T_v g_E = g_E$. Then

\[
\Phi^* T_v^* g_S |_{(x, y, z)} = \tau^2 g_E |_{(x, y, z)} + \tau^2 \sin 2\tau \alpha (dx^2 - dy^2) \quad \text{and}
\]

\[
\Phi^* T_v^* g_S |_{(x, y, z)} = g_E |_{(x, y, z)} + \sin 2\alpha (dx^2 - dy^2).
\]

In particular $\Phi$ intertwines horizontal translations with rotations about the coordinate unit circles in $S^3$:

\[
\Phi^* T_{(0,0,0)}^* = R_{C_2}^\alpha \Phi \text{ and } \Phi^* T_{(0,0,0)}^* = R_{C_1}^\alpha \Phi.
\]

Similarly $\Phi$ intertwines reflections through planes with reflections through geodesic spheres:

\[
\Phi^* \hat{X} = \hat{X} \Phi \text{ and } \Phi^* \hat{Y} = \hat{Y} \Phi,
\]

where $\hat{X}(x, y, z) = (-x, y, z)$, $\hat{Y}(x, y, z) = (x, -y, z)$, $\hat{X}(z_1, z_2) = (z_1, z_2)$, and $\hat{Y}(z_1, z_2) = (z_1, \bar{z}_2)$.

Since we shall make frequent use throughout the construction of cutoff functions, we fix now a smooth, nondecreasing $\Psi : \mathbb{R} \to [0, 1]$ with $\Psi$ identically 0 on $(-\infty, -1]$, identically 1 on $[1, \infty)$, and such that $\Psi - \frac{1}{2}$ is odd. We then define, for any $a, b \in \mathbb{R}$, the function $\psi[a, b] : \mathbb{R} \to [0, 1]$ by

\[
\psi[a, b] = \Psi \circ L_{a,b},
\]

where $L_{a,b} : \mathbb{R} \to \mathbb{R}$ is the linear function satisfying $L(a) = -3$ and $L(b) = 3$.

**Catenoideal regions.** The catenoids in $\mathbb{R}^3$ constitute a six-parameter family of complete, properly embedded minimal surfaces. Three parameters specify a catenoid’s center, two more its axis, and one last its size. We have no use for the two directional degrees of freedom and parametrize the vertical catenoid of center $v \in \mathbb{R}^3$ and waist radius $\tau$ by the cylinder $K = \mathbb{R} \times S^1$ according to

\[
\kappa_{v,\tau} : K \to \mathbb{R}^3
\]

\[
\kappa_{v,\tau}(t, \theta) \mapsto v + \tau (\cosh \theta \cos \theta, \cosh \theta \sin \theta, t).
\]

Obviously $\kappa_{v,\tau} = T_v S_\tau \kappa$, where $\kappa = \kappa_{0,1}$ parametrizes the standard catenoid.
Lemma 2.10. The map \( \tilde{\kappa}_{v,\tau} : \mathcal{K} \to \mathbb{R}^3 \) is a proper, minimal embedding. The catenoidal metric \( \tilde{\kappa}_{v,\tau}^* g_E \) on \( \mathcal{K} \) is conformal to the flat metric \( \tilde{\chi}_{\kappa} = dt^2 + d\theta^2 \):

\[
\tilde{\chi}_{\kappa} = \rho^2_s \tilde{\kappa}^* g_E,
\]

where the conformal factor \( \rho_s : \mathcal{K} \to \mathbb{R} \) is defined by

\[
\rho_s(t) = \frac{1}{\tau \cosh t}.
\]

The catenoid \( \tilde{\kappa}_{v,\tau} \) has second fundamental form

\[
\tilde{\mathcal{A}} = \begin{bmatrix}
\text{sech} t & \cosh t & 0 \\
\cosh t & \text{sech} t & 0 \\
0 & 0 & 0
\end{bmatrix} \left( dt^2 - d\theta^2 \right).
\]

so that

\[
\left\| \tilde{\mathcal{A}} \right\|^2_{g_E \otimes (\tilde{\kappa}^* g_{\mathcal{K}})^{-1}} = 2 \left( \text{sech}^4 t + \text{sech}^2 t \right) = C(k) \text{sech}^{k+1} t.
\]

Proof. The claims are standard results, straightforward to calculate. Starting with the expression for \( \tilde{\mathcal{A}} \), we may find the estimate for \( D^k \tilde{\mathcal{A}} \) as follows:

\[
\hat{D} - D = \text{tr} \left( \frac{2 \sinh t \cosh t}{\cosh^2 t} dt \otimes g \otimes g^{-1} \right) = 2 \text{tr} \left( \frac{\sinh t \cosh t}{\cosh^2 t} dt \otimes g \otimes g^{-1} \right),
\]

but every derivative of \( \text{tanh} \) is bounded above in absolute value by a constant, as is every derivative of \( \text{sech} \), so each covariant derivative with respect to \( \tilde{\kappa}^* g_E \) of \( \tilde{\mathcal{A}} \) has \( \tilde{\chi}_\kappa \) norm bounded by a constant, which upon the appropriate scaling yields the estimate. 

We also define the immersion \( \kappa_{v,\tau} : \mathcal{K} \to \mathbb{S}^3 \) by

\[
\kappa_{v,\tau} = \hat{\Phi}_{\kappa_{v,\tau}},
\]

sometimes more profitably realized as the composite

\[
\mathcal{K} \xrightarrow{\tilde{\kappa}} \mathbb{R}^3 \xrightarrow{p_{\hat{\Phi}}} \hat{\Phi}^* \mathbb{S}^3.
\]

When the particular \( v \) and \( \tau \) are clear from context we may write simply \( \kappa \) (whereas \( \hat{\kappa} \) always refers to the case of \( v = 0 \) and \( \tau = 1 \)). The next proposition compares the geometry of \( \kappa \) to the geometry of \( \hat{\kappa} \). We remark that the estimates for the second fundamental form and for the mean curvature especially can be significantly strengthened by a more sensitive analysis, but we have no need for these improvements. The notation of the norms is explained in Appendix A.

Proposition 2.18. Fix \( a, \tau > 0 \), and \( v \in \mathbb{R}^3 \). Let

\[
\chi = \rho^2_s \kappa_{v,\tau}^* g_S,
\]

(2.19)

\[
\mathcal{K}_a = [-a, a] \times \mathbb{S}^1 \subset \mathcal{K}, \quad \text{and } D_{a,v} = (v^3 - a, v^3 + a) \times \mathbb{R}^2 \subset \mathbb{R}^3.
\]

Write \( A \) for the projection of the second fundamental form of \( \kappa_{v,\tau} \) along the global unit normal which points inward at the waist, and write \( H \) for the trace of \( A \) with respect to \( \kappa^* g_S \). Then for each nonnegative integer \( j \) there exists a constant \( C(j) \) depending on \( j \) alone such that, provided \((a + 1) \tau + |v^3| < 1/C(0) \) and \( \tau \cosh a < 1 \),

\[
\begin{align*}
(a) \quad & \left\| \tau^2 \mathcal{S}_{v} \mathcal{T}_{v} \mathcal{P}^* g_{S} - g_{E} \right\|_{\Gamma^0(\tau^2 \mathcal{S}_{v} \mathcal{T}_{v} \mathcal{P}^* g_{S} - g_{E})} \leq C(0); \\
(b) \quad & \left\| D^j[g_{E}] \right\|_{\Gamma^0(\tau^2 \mathcal{S}_{v} \mathcal{T}_{v} \mathcal{P}^* g_{S} - g_{E})} \leq C(j) \text{ for } j \geq 1; \\
(c) \quad & \left\| \kappa_{v,\tau} - \tilde{\chi}_{\kappa} \right\|_{\Gamma^0(\tau^2 \mathcal{S}_{v} \mathcal{T}_{v} \mathcal{P}^* g_{S} - g_{E})} \leq C(j); \\
(d) \quad & \left\| \rho^{-2} \left\| A \right\|_{\mathbb{S}^3} \right\| \leq C(0); \quad \text{and }
\end{align*}
\]
\[(e) \| \rho^{-2}H : C^1 \left( K_\alpha, \tilde{\chi}_\alpha, \tau \right) \leq C(0). \]

Proof. Claims (a) and (b) are easily confirmed by calculating
\[(2.21) \quad \tau^{-2} \tilde{S}_{\tau}^* \Phi^* g_S = g_E + \sin(\tau z + v^3) (dx^2 - dy^2).\]
Applying \( \tilde{\kappa}^* \) to this same equation yields
\[(2.22) \quad \tau^{-2} \kappa^* g_S = \tilde{\kappa}^* g_E + \sin(\tau z + v^3) \left[ (\sinh t \cos \theta dt - \cosh t \sin \theta d\theta)^2 - (\sinh t \sin \theta dt + \cosh t \cos \theta d\theta)^2 \right]\]
and, multiplying by \( \cosh^{-2} t \), we get
\[(2.23) \quad \chi = \tilde{\chi}_\tau + \sin(\tau z + v^3) \left[ \tanh^2 t \cos 2\theta dt^2 - 2 \tanh t \sin 2\theta dt d\theta - \cosh 2\theta d\theta^2 \right],\]
which makes claim (c) obvious, in light of the uniform boundedness of the component functions and all their derivatives. (Of course a more refined estimate, analogous to (b), is available but proves unnecessary.) For the last two items we appeal to [B.15] letting \( g_0 = g_E \), \( g_1 = \tau^{-2} \tilde{S}_{\tau}^* \Phi^* g_S \), and \( \phi = \tilde{\kappa} \), and making use of [2.10]. \( \square \)

**Toral regions.** The Clifford tori in \( S^3 \) constitute a four-parameter family of closed, embedded minimal surfaces, but the toral standard regions we consider are perturbations of patches of the fixed torus \( T = \Phi(\{ z = 0 \}) \). This is the torus to be doubled.

**Lemma 2.24.** The map \( \Phi|_{\mathbb{R}^2 \times \{ 0 \}} \) descends to an embedding of the torus \( \mathbb{R}^2 \times \{ 0 \} / \langle \hat{T}_{(\sqrt{2}\pi, 0, 0)}, \hat{T}_{(0, \sqrt{2}\pi, 0)} \rangle \) in \( S^3 \). The embedding induces the flat metric \( g_E = dx^2 + dy^2 \) on the torus and has second fundamental form
\[(2.25) \quad A_T = (dx^2 - dy^2) \partial_z.\]

More generally it is easy to see that the slice \( \{ z = c \} \) is a constant-mean-curvature torus with mean curvature \( 2 \tan 2c \). Each toral standard region consists of a rectangular portion of such a slice, parallel to \( T \), with a disc excised and a corresponding truncated catenoidal end glued smoothly to the boundary of the disc. The symmetries we will impose identify opposite edges of the region so that intrinsically each such region is itself simply a (rectangular) torus with a disc removed.

The geometry of each toral region is controlled by several parameters. One parameter, \( z_0 \in \mathbb{R} \), will position the region in \( S^3 \) by specifying the height of the center of the attached catenoid. Another parameter \( \tau \) sets the waist radius of this catenoid. The resulting quantity
\[(2.26) \quad a = \text{arcosh} \frac{1}{\ell m \tau}\]
prescribes the extent of the catenoidal regions described above and indicates where they attach to the tori. A final parameter, \( b > 0 \), limits the overlap of the toral regions with the catenoidal regions, restricting the parametrizing patch to
\[(2.27) \quad T_{b, m, \tau} = \left\{ (x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \geq m \tau \cosh b \right\} / \langle \hat{X}, \hat{Y}, \hat{T}_{(\sqrt{2}\pi/k, 0, 0)}, \hat{T}_{(0, \sqrt{2}\pi/\ell, 0)} \rangle, \]
identifying \( \mathbb{R}^2 \) with \( \{ z = 0 \} \subset \mathbb{R}^3 \). Next we define the function \( r : T_{b, m, \tau} \rightarrow \mathbb{R} \) by \( r(x, y) = \sqrt{x^2 + y^2} \), the flat distance on \( T_{b, m, \tau} \) to the origin, and the functions \( \hat{\varphi}_{\pm 1} : T_{b, m, \tau} \rightarrow \mathbb{R} \) by
\[(2.28) \quad \hat{\varphi}_{\pm 1} = z_0 \pm \left( \tau \text{arcosh} \frac{1}{\ell m \tau} \right) \psi \left[ \frac{1}{\ell}, \frac{2}{\ell} \right] r \pm \left( \tau \text{arcosh} \frac{r}{m \tau} \right) \psi \left[ \frac{2}{\ell}, \frac{1}{\ell} \right] r.\]
Now the embeddings \( \varphi_{m, z_0, \tau, \pm 1} : T_{b, m, \tau} \rightarrow S^3 \) which define the toral standard regions take the form
\[(2.29) \quad \varphi_{\pm 1}(x, y) = \Phi \left( \frac{x}{m}, \frac{y}{m}, \hat{\varphi}_{\pm 1}(x, y) \right).\]
The conformal factor \( \rho \) defined above on \( \mathcal{K} \) may be regarded as a function on a neighborhood of the origin in \( \mathcal{T}_{b,m,\tau} \), which we extend to a function, of the same name, \( \rho : \mathcal{T}_{b,m,\tau} \rightarrow \mathbb{R} \), by

\[
\rho = \frac{m}{r} + \left( \ell m - \frac{m}{\ell} \right) \psi \left[ \frac{1}{\ell m}, \frac{2}{\ell m} \right] \circ r.
\]  

Likewise the metric \( \chi \) originally defined on \( \mathcal{K} \) extends to metrics

\[
\chi_{\pm 1} = \rho^2 \varphi_{\pm 1}^* g_S
\]
on \( \mathcal{T}_{b,m,\tau} \), which are to be compared to the metric

\[
\tilde{\chi}_T = m^{-2} \rho^2 g_E,
\]
g\( E \) being the Euclidean metric on \( \mathcal{T}_{b,m,\tau} \). The next proposition makes this and other comparisons.

**Proposition 2.33.** Fix \( b, m, \tau > 0 \) and \( z_0 \in \mathbb{R} \), with \( \tau \cosh b \leq \ell^{-1} m^{-1} \). Write \( A_{\pm 1} \) for the projection of the second fundamental form of \( \varphi_{\pm 1} \) along the unit normal having positive inner product with \( \pm \partial_x \), and write \( H_{\pm 1} \) for the trace of \( A_{\pm 1} \) with respect to \( \varphi_{\pm 1}^* g_S \). There exists a constant \( C \), depending on just \( k, \ell \), such that, provided

\[
|z_0| + a \tau + \frac{1}{\cosh b} + \frac{1}{m} < \frac{1}{C},
\]

\( a \) is everywhere bounded by 1 and each of

\[
\partial_j (T^* \mathcal{T}_{b,m,\tau}, \tilde{\chi}_T, |z_0| + a \tau + \rho \tau) \leq C;
\]

\[
|\rho^{-2} A_{\pm 1} |^2 - 2 \rho^{-2} : C^1 (T^* \mathcal{T}_{b,m,\tau}, \tilde{\chi}_T, |z_0| + a \tau + \rho \tau) \leq C;
\]

\[
|\rho^{-2} H_{\pm 1} : C^1 (T^* \mathcal{T}_{b,m,\tau}, \tilde{\chi}_T, (\rho^{-2} + \tau) (|z_0| + [a + m^2] \tau + \rho \tau)) \leq C.
\]

**Proof.** We use [3.9] with \( M = S^3 \), \( g = g_S \), \( \Sigma = \mathcal{T}_{b,m,\tau} \), \( \phi_0 (x,y) = \Phi \left( \frac{x}{m}, \frac{y}{m}, 0 \right) \), and \( u = \tilde{\varphi} \), so that \( \phi_u = \varphi \), as is readily verified since \( \Phi \) maps vertical lines to geodesics and \( \partial_x \) is everywhere orthogonal to \( \mathcal{T} \) with length 1. We make a few observations in preparation for the application of the lemma. Obviously \( A_0 \) is uniformly bounded;

\[
\operatorname{arcosh} \frac{2}{\ell m \tau} \leq \operatorname{arcosh} 2 + \operatorname{arcosh} \frac{1}{\ell m \tau} \lesssim a,
\]

provided \( a \geq 2 \); the cutoff functions featured in the definition of \( \tilde{\varphi} \) are bounded by 1 and each of their derivatives (of order at least 1) is supported on \( \left[ \frac{1}{\ell \tau}, \frac{2}{\ell \tau} \right] \) and bounded by constants depending on only \( k, \ell \), and the order of the derivative;

\[
\left\| D^k [g_E] r \right\|_{g_E} \leq C(k) r^{1-k} \text{ and } r \lesssim C;
\]

and

\[
\frac{r}{m \tau} \geq \cosh b.
\]

With these comments in mind and assuming \( b \) sufficiently large and \( m \tau \) sufficiently small, we estimate

\[
|\tilde{\varphi}| \lesssim |z_0| + a \tau \text{ and } \left\| D^j \tilde{\varphi} : \Gamma^0 \left( T^* \mathcal{T}_{b,m,\tau}, m^{-2} g_E, \frac{m^2 \tau}{r^j} \right) \right\| \leq C(j) \text{ for } j \geq 1.
\]

Thus, taking

\[
|z_0| + a \tau + \frac{1}{\cosh b}
\]
sufficiently small (in terms of a constant depending on just $k$ and $\ell$), we can ensure the applicability of the lemma, which delivers the first and third items. The second item follows in conjunction with [B.1] while to secure the last we supplement the above estimates with

$$\left|\Delta [m^{-2} g_E] \widetilde{\varphi}\right| \lesssim \frac{m^4 \tau^3}{\ell^4} \text{ and } \left\|d\Delta [m^{-2} g_E] \widetilde{\varphi} : T^0 \left(T^* T_{b,m,\tau}, \hat{X}_{T}, \frac{m^5 \tau^3}{\rho^5}\right)\right\| \leq C. \tag{2.40}$$

\[\square\]

**The initial surfaces.** Continuing to assume $k \leq \ell$, we now construct a sequence, indexed by $m$, of two-parameter families of initial surfaces. The numbers $k$, $\ell$, and $m$ determine, through balancing, the gross scale

$$\zeta = \frac{1}{\ell m} e^{-\frac{k m^2}{4\pi}}, \tag{2.41}$$

of the waist radius, to be justified later, in [4.13]. The parameter $\zeta$ adjusts this to the exact radius

$$\tau = e^{\ell \zeta}, \tag{2.42}$$

while the parameter $\xi$ places the centers of the catenoidal regions at height $\xi \tau$. We set

$$K[0] = \kappa(0,0,\xi \tau), (K_a) \text{ and } T[\pm 1] = \varphi_{m,\xi \tau,\pm 1} (T_{b,m,\tau}), \tag{2.43}$$

where $b$ will be selected in the next section but in fact has no bearing on the definition of the initial surface, in light of the overlap of the catenoidal and toral regions. Actually the precise value of $b$ will depend on $m$ and $\zeta$, but $b + b^{-1}$ will be bounded independently of both. For simplicity we shall sometimes write $\kappa_{\zeta, \xi}$ and $\varphi_{\pm 1, \zeta, \xi}$ for the above parametrizations.

The initial surface $M_{m}[\zeta, \xi]$ itself is at last constructed as the orbit under the imposed symmetry group $\mathcal{G}$, identified in the main theorem, of the union of these three regions:

$$M_{k,\ell,m,\zeta,\xi} = \left\langle X, Y, R^{2 \pi}_{C_1}, R^{2 \pi}_{C_2} \right\rangle (T[-1] \cup K[0] \cup T[1]). \tag{2.44}$$

It should be born in mind that the initial surfaces as well as many analytic and geometric entitites defined on them naturally depend on $k$, $\ell$, $m$, $\zeta$, and $\xi$, but we reserve the right, at will, to express or suppress in the notation this dependence.

Note that the function $\rho$ defined above on regions of both types extends to a well-defined smooth function on the entire initial surface by decreeing invariance under $\mathcal{G}$. Likewise we have the global metric $\chi = \rho^2 g$, conformal to the metric $g$ induced on $M$ by $g_S$. Note also that there exists a global unit normal $\nu$ on $M$ which points inward along each catenoidal region, upward on each upper toral region, and downward on each lower toral region. Defining then $\mathcal{H}$ as in the introduction we sometimes write simply $H$ for $\mathcal{H}[0]$, the projection along $\nu$ of the mean curvature of the initial surface. The next proposition records some basic properties of the initial surface as well as a crucial estimate of the initial mean curvature.

**Proposition 2.45.** Given nonnegative integers $k$ and $\ell$ and real $\zeta > 0$, there exists $m_0 > 0$ such that whenever $\zeta, \xi \in [-\zeta, \xi]$ and $m > m_0$, the initial surface $\widetilde{M}_{k,\ell,m,\zeta,\xi}$ is smooth, closed, and embedded, with genus $km^2 + 1$ and stabilizer $\mathcal{G}$ in $O(4)$. Furthermore there exists a constant $C$, depending on just $k$ and $\ell$, so that for each $\gamma \in (0, \frac{1}{2})$

$$\left\|\rho^{-2} H : C^1 \left(M, \chi, \frac{m^3}{\rho^3}\right)\right\| \leq C\tau. \tag{2.46}$$

**Proof.** The first few claims are clear from the construction of the initial surfaces. The estimate of the mean curvature is easy to check using [2.18] and [2.33]. The only potentially troublesome terms contributing to $\rho^{-2} H$ are $\rho^{-2}$ from [2.18] and $\rho \tau$ from [2.33] but the first one satisfies the estimate
on \( \{ r \leq \tau^{1/(2-\gamma)} \} \) and the second one on \( \{ r \geq \tau^{1/(1+\gamma)} \} \). Since \( \frac{1}{2-\gamma} \leq \frac{1}{1+\gamma} \) whenever \( 0 < \gamma < \frac{1}{2} \), the estimate holds globally.

### 3. The Linearized Equation

The linearization \( \mathcal{L} \) at 0 of \( \mathcal{H} \), defined by \( \mathcal{L}u = \frac{d}{dt}|_{t=0} \mathcal{H}[tu] \), takes the form

\[
\mathcal{L} = \Delta + \| A \|^2 + 2,
\]

where \( \Delta \) denotes the Laplacian on \( M \) induced by the metric it inherits from \( g_S \), \( A \) denotes the second fundamental form of \( M \) relative to \( S^3 \), and the constant term 2 originates from the Ricci curvature of \( S^3 \). We now proceed to study the operator \( \mathcal{L}_\chi = \rho^{-2} \mathcal{L} \) on \( M \) and to that end introduce, for each \( \beta \in (0,1) \) and \( \gamma \in (0, \frac{1}{2}) \), the weighted Hölder norms

\[
\| u \|_{r,\beta,\gamma} = \| \cdot \|_{C^{k,\beta,\gamma}(M)} = \| \cdot \|_{C^{r,\beta}(M,\chi,\frac{L}{r m})}
\]

and corresponding Banach spaces

\[
C^{r,\beta,\gamma}_G(M) = \left\{ u \in C^{r,\beta}(M,\chi) : \| u \|_{r,\beta,\gamma} < \infty \text{ and } Gu = u \right\},
\]

as instances of the weighted norms and spaces defined in Appendix A.

Obviously these function spaces depend on the parameters \( \zeta \) and \( \xi \) (and also on \( k, \ell, m, \beta, \) and \( \gamma \)). Because the perturbing function will be chosen in parallel with these parameters in the proof of the main theorem, it will be necessary to have some means of identifying functions defined on initial surfaces with the same values of \( k, \ell, \) and \( m \) but different values of \( \zeta \) and \( \xi \). Such identification is achieved by pulling back the functions through diffeomorphisms between the initial surfaces. We define these diffeomorphisms as a compromise between the natural identifications suggested on each standard region.

More precisely we set

\[
a = \text{arcosh} \left( \frac{1}{\ell m \tau} \right),
\]

that is the value of \( a \) when \( \zeta = 0 \), and we observe that the function \( \tilde{R} : [\tau, \frac{1}{l m}] \rightarrow [\text{arcosh} R, \frac{1}{l m}] \) defined by

\[
\tilde{R}(R) = \tau \cosh \left( \frac{\text{arcosh} R}{a} \right)
\]

is strictly monotonic and that near \( \frac{1}{l m} \) it tends to the identity as \( m \rightarrow \infty \), with the result that \( \tilde{R} : (\tau, \frac{1}{l m}) \rightarrow (\text{arcosh} R, \frac{1}{l m}) \) given by

\[
\tilde{R}(R) = R(R) \cdot \psi \left[ \frac{1}{l m}, \frac{1}{2l m} \right] (R) + R \cdot \psi \left[ \frac{1}{2l m}, \frac{1}{l m} \right] (R)
\]

is another monotonic function, agreeing exactly with the identity on a neighborhood of \( R = \frac{1}{l m} \). Then we can define the diffeomorphism

\[
P_{\zeta,\xi} : M_{k,\ell,m,\zeta,\xi} \rightarrow M_{k,\ell,m,0,0}
\]

by the requirements that it be equivariant with respect to the action of the symmetry group \( G \), that

\[
P_{\zeta,\xi} |_{\tau^{[\pm 1]} \setminus \mathcal{K}[0]} = \varphi_{\pm 1,0,0}^{-1} \circ \varphi_{\pm 1,\zeta,\xi} \big|_{\tau^{[\pm 1]} \setminus \mathcal{K}[0]} ,
\]

and that

\[
P_{\zeta,\xi} \mathcal{K}(t,\theta) = \kappa_{0,0} \left( \text{sgn} t \text{arcosh} \left( \frac{1}{\ell m} \right) \right).
\]

Further, given \( u : M_{0,0} \rightarrow \mathbb{R} \) we set \( \mathcal{P}_{\zeta,\xi}(u) = u \circ P_{\zeta,\xi} \), so that \( \mathcal{P}_{\zeta,\xi} \) is pullback by \( P_{\zeta,\xi} \).
Lemma 3.8. There is a constant $C > 0$ so that, given $c > 0$, there exists $m_0$, depending on just $c$, $k$, and $\ell$, such that whenever $\zeta, \xi \in [-c, c]$ and $m > m_0$, the map $P_{\zeta, \xi}$ above defines a linear homeomorphism

$$P_{\zeta, \xi} : C^{k, \beta, \gamma}(M_{k, \ell, m, 0, 0}, \chi) \to C^{k, \beta, \gamma}(M_{k, \ell, m, \zeta, \xi}, \chi)$$

with $\|P_{\zeta, \xi}\| + \|P_{\zeta, \xi}^{-1}\| \leq C$.

Catenoidal regions. We first construct a right inverse to $L_\chi$ on $K[0]$, which we shall tacitly identify with the cylinder $K_a$ parametrizing it, by comparison to the operator

$$\hat{L}_\kappa = \Delta \hat{\kappa} + \frac{2}{\cosh^2 t},$$

which is the usual Jacobi operator for the standard catenoid parametrized by the cylinder $K$. Analogous to the function spaces above we define

$$C_{\text{sym}}^r(K_a, \hat{\kappa}) = C_{\text{sym}}^r(K_a, \hat{\kappa}, \hat{e}^\gamma(|t| - a))$$

and

$$C_{\text{sym}}^r(K_a, \hat{\kappa}) = C_{\text{sym}}^r(K_a, \hat{\kappa}, \hat{e}^\gamma(|t| - a)),$$

and we note that the linear homeomorphism

$$\hat{P}_a : C_{\text{sym}}^r(K_a, \hat{\kappa}) \to C_{\text{sym}}^r(K_a, \hat{\kappa})$$

is independent of $\xi$ and so depends on just $\zeta$ or equivalently $a$ (for fixed $m$), vindicating the notation.

Lemma 3.13. Fix $\beta \in (0, 1)$ and $\gamma \in (0, \frac{1}{4})$. There exist a unique linear map

$$\hat{R}_{K_a} : C_{\text{sym}}^{0, \beta, \gamma}(K_a, \hat{\kappa}) \to C_{\text{sym}}^{2, \beta, \gamma}(K_a, \hat{\kappa})$$

and a constant $C$, depending on just $\beta$ and $\gamma$, such that if $f \in C_{\text{sym}}^{0, \beta, \gamma}(K_a, \hat{\kappa})$ and $u = \hat{R}_{K_a} f$, then

(a) $\hat{L}_\kappa u = f$;
(b) $\|\hat{R}_{K_a}\| \leq C$;
(c) $\int_0^{2\pi} u(0, \theta) d\theta = \int_0^{2\pi} \partial_t u(0, \theta) d\theta = 0$;
(d) $u(t, \theta) - \frac{1}{2\pi} \int_0^{2\pi} u(t, \theta) d\theta$ satisfies Dirichlet boundary conditions;
(e) $\hat{P}_a^{-1} \hat{R}_{K_a} \hat{P}_a : C_{\text{sym}}^{0, \beta, \gamma}(K_a, \hat{\kappa}) \to C_{\text{sym}}^{2, \beta, \gamma}(K_a, \hat{\kappa})$ depends continuously on $a$.

Proof. Given any $v \in C_{\text{sym}}^{0}(K_a)$ we may define its 2n-th Fourier component

$$v_n(t) = \int_0^{2\pi} v(t, \theta) \cos 2n\theta d\theta$$

for each nonnegative integer $n$; all odd-frequency and sine modes vanish due to the symmetries. If $\hat{L}_\kappa$ had some nontrivial kernel in the subspace of $C_{\text{sym}}^{2, \beta, \gamma}(K_a, \hat{\kappa})$ satisfying the boundary and initial conditions, then, projecting, there would be some nonzero Fourier mode lying in the kernel of $\partial_t^2 - 4n^2$ for some integer $n$. The very existence, however, of the Green’s functions $\Box_{K_a}$ and $\Box_{K_a}$ constructed in Appendix C precludes the possibility of such kernel. Thus, using also the classical Schauder estimates, given any $f \in C_{\text{sym}}^{0, \beta, \gamma}(K_a, \hat{\kappa})$, we are guaranteed the existence of a unique solution $u$ satisfying the boundary and initial conditions as well as the estimate

$$\|u\|_{C^{2, \beta}(B^\infty_p)} \leq C \left( \|f\|_{C^{0, \beta}(B^\infty_p)} + \|u\|_{C^{0, \beta}(B^\infty_p)} \right)$$
for each \(p \in \mathcal{K}_a\), where \(C\) is a universal constant and \(B_p^{\hat{\kappa}}(2)\) is the ball on \(\mathcal{K}_a\) of radius 2, with respect to the flat metric, centered at \(p\).

Furthermore, the Fourier modes \(u_n(t)\) and \(f_n(t)\) are well-defined for each nonnegative integer \(n\) and solve the equation \(\partial_t^2 u_n - 4n^2 u_n = f_n\). Using Corollary 7 and Corollary 8, \(u_n\) is seen to satisfy the estimate

\[
\sup_{t \in [-a,a]} e^{\gamma(a-|t|)} |u_n(t)| \leq \frac{C(\gamma)}{n^2 + 1} \sup_{t \in [-a,a]} e^{\gamma(a-|t|)} |f_n(t)| .
\]

According to this last inequality, the series \(\sum_{n=0}^{\infty} u_n(t)\) converges in \(C^{0,0,\gamma}(\mathcal{K}_a, \hat{\kappa}_\kappa)\), but already we know that this series converges in \(L^2\) to \(u\), so in fact it converges in \(C^{0,0,\gamma}(\mathcal{K}_a, \hat{\kappa}_\kappa)\) to \(u\), yielding the desired \(C^0\) decay estimate, which coupled with (3.16) establishes all claims except the continuity with respect to \(a\). This final assertion follows by observing that the coefficients of \(\hat{P}_a^{-1}\hat{\mathcal{L}}_a\), with respect to the natural coordinates on \(\mathcal{K}_a\), are in fact smooth functions of the coordinates and of \(a\). It is then easy to check that the map

\[
\hat{P}_a^{-1}\hat{\mathcal{L}}_a : C^{2,\beta,\gamma}_{sym}(\mathcal{K}_a, \hat{\kappa}_\kappa) \to C^{0,\beta,\gamma}_{sym}(\mathcal{K}_a, \hat{\kappa}_\kappa)
\]

depends continuously on \(a\). Subject to the stated boundary and initial conditions, which are independent of \(a\), this map has unique inverse \(\hat{\mathcal{L}}_a^{-1}\hat{\mathcal{R}}_a\), which as such is likewise continuous in \(a\).

**Proposition 3.19.** Fix \(\beta \in (0,1)\) and \(\gamma \in (0,\frac{1}{2})\). Given \(\xi > 0\), there is a positive integer \(m_0\)—depending on just \(k, \ell\), and \(\xi\)—and there exists a constant \(C > 0\)—depending on just \(k, \ell, \beta\), and \(\gamma\)—such that whenever \(|\xi| + |\xi| \leq \xi\) and \(m > m_0\) there exists a linear map

\[
\mathcal{R}_{\kappa[0]} : C^{0,\beta,\gamma}_{sym}(\mathcal{K}[0], \chi) \to C^{2,\beta,\gamma}_{sym}(\mathcal{K}[0], \chi)
\]

such that for every \(f \in C^{0,\beta,\gamma}_{sym}(\mathcal{K}_a, \chi)\)

(a) \(L_\chi \mathcal{R}_{\kappa[0]} f = f\); and

(b) \(|\mathcal{R}_{\kappa[0]}| \leq C\); and

(c) \(\hat{P}_{\xi,\xi}^{-1}\mathcal{R}_{\kappa[0]} \mathcal{R}_{\xi,\xi} \mathcal{R}_{\kappa[0]} : C^{0,\beta,\gamma}_{sym}(\mathcal{K}[0],0,\chi) \to C^{2,\beta,\gamma}_{sym}(\mathcal{K}[0],0,\chi)\) depends continuously on \((\xi, \xi)\).

**Proof.** Since

\[
L_\chi - \hat{\mathcal{L}}_a = (\Delta_\chi - \Delta_{\hat{\kappa}_\kappa}) + (\rho^{-2} \|A\|^2 - 2 \text{sech}^2 t) + 2\rho^{-2},
\]

by (2.18) together with (2.4) we can make the norm of \(L_\chi - \hat{\mathcal{L}}_a : C^{2,\beta,\gamma}(\mathcal{K}_a, \hat{\kappa}_\kappa) \to C^{0,\beta,\gamma}(\mathcal{K}_a, \hat{\kappa}_\kappa)\) as small as desired by requiring \(a\tau + |\xi| \tau + \tau \cosh a\) to be sufficiently small, so in particular we can arrange for \(I + \hat{\mathcal{R}}_{\kappa_a}(L_\chi - \hat{\mathcal{L}}_a)\), where \(I\) is the identity operator on \(C^{2,\beta,\gamma}(\mathcal{K}_a, \hat{\kappa}_\kappa)\), to have a bounded inverse. Then

\[
L_\chi = \hat{\mathcal{L}}_a + (L_\chi - \hat{\mathcal{L}}_a) = \hat{\mathcal{L}}_a \left(I + \hat{\mathcal{R}}_{\kappa_a}(L_\chi - \hat{\mathcal{L}}_a)\right)
\]

admits a right inverse

\[
\mathcal{R}_{\kappa[0]} = \left(I + \hat{\mathcal{R}}_{\kappa_a}(L_\chi - \hat{\mathcal{L}}_a)\right)^{-1} \hat{\mathcal{R}}_{\kappa_a},
\]

which is a bounded map \(C^{0,\beta,\gamma}_{sym}(\mathcal{K}_a, \hat{\kappa}_\kappa) \to C^{2,\beta,\gamma}_{sym}(\mathcal{K}_a, \hat{\kappa}_\kappa)\). The lemma (2.4) just invoked also asserts comparability of the \(\chi\) and \(\hat{\kappa}\) norms, so in fact \(\mathcal{R}_{\kappa[0]}\) is defined and bounded as a map \(C^{0,\beta,\gamma}_{sym}(\mathcal{K}[0], \chi) \to C^{2,\beta,\gamma}_{sym}(\mathcal{K}[0], \chi)\). Since \(\theta \mapsto -\theta\) and \(\theta \mapsto \pi - \theta\) are symmetries of both \(\hat{\kappa}\) and \(\kappa\), it is clear that \(\hat{\mathcal{L}}_a\) and \(L_\chi\) equally, in addition to \(\hat{\mathcal{R}}_{\kappa_a}\), preserve these symmetries and therefore so does \(\mathcal{R}_{\kappa[0]}\).
Moreover, we can rewrite (3.25) by revealing the tacit identifications and conjugating by $P_{\zeta, \xi}$ to obtain

$$
\mathcal{P}_{\zeta, \xi}^{-1}\mathcal{R}_{\xi}\mathcal{R}_{\zeta}\mathcal{P}_{\zeta, \xi} = \mathcal{P}_{\zeta, \xi}^{-1}(\kappa_{\zeta, \xi}^*)^{-1} \left( \mathcal{I} + \mathcal{R}_{\xi, \eta} \left( \kappa_{\zeta, \xi}^* \mathcal{L}_{\zeta, \xi} \left( \kappa_{\zeta, \xi}^* \mathcal{L}_{\xi, \eta} \right)^{-1} - \mathcal{L}_{\xi, \eta} \right) \right)^{-1} \mathcal{R}_{\xi, \eta} \kappa_{\zeta, \xi}^* \mathcal{P}_{\zeta, \xi}
$$

(3.24)

$$
= \left( \kappa_{0,0}^* - \left( \mathcal{P}_a^{-1} \mathcal{R}_{\xi, \eta} \mathcal{P}_a \right) \kappa_{0,0}^* \left[ \mathcal{P}_{\zeta, \xi}^{-1} \left( \mathcal{L}_{\zeta, \xi} - \left( \kappa_{\zeta, \xi}^* \mathcal{L}_{\xi, \eta} \right)^{-1} \mathcal{L}_{\xi, \eta} \kappa_{\zeta, \xi}^* \right) \mathcal{P}_{\zeta, \xi} \right] \right)^{-1} \circ \left( \mathcal{P}_a^{-1} \mathcal{R}_{\xi, \eta} \mathcal{P}_a \right) \kappa_{0,0}^*.
$$

which expression, using (3.13) and reasoning similar to the continuity element of its proof, exhibits the continuity in $\zeta$ and $\xi$. \hfill \Box

**Toral regions.** Now we construct a right inverse, modulo a one-dimensional subspace, to $\mathcal{L}_{\xi}$ on $\mathcal{T}[1]$ and $\mathcal{T}[-1]$, each identified with $\mathcal{T}_{b,m,\tau}$. As on the catenoidal region, we accomplish this by comparison with another operator, in this case

$$
\mathcal{L}_{\tau} = \Delta_{\hat{\chi}_{\tau}} : C^{2,\beta}_{\text{sym}} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right) \rightarrow C^{0,\beta}_{\text{sym}} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right),
$$

where now the subscript $\text{sym}$ indicates restriction to only those functions invariant under the group

$$
\mathcal{T}_{b,m,\tau} \times \mathcal{T}_{b,m,\tau} \backslash \mathcal{T}_{b,m,\tau}, \mathcal{T}_{b,m,\tau} \backslash \mathcal{T}_{b,m,\tau}
$$

regarding $\mathcal{T}_{b,m,\tau}$ as a subset of the plane $z = 0$ in $\mathbb{R}^3$.

The one-dimensional substitute kernel just mentioned is spanned by the function $w \in C^\infty(\mathcal{T}_{b,m,\tau})$ defined by

$$
w = \psi \left[ \frac{1}{\ell}, \frac{2}{\ell} \right] \circ r,
$$

giving rise through the above identifications to the functions $w_1, w_{-1} \in C^\infty(M)$ spanning the two-dimensional substitute kernel on the entire initial surface.

At this point, in order to ensure that the diffeomorphisms $P_{\zeta, \xi}$ above preserve the toral standard regions—a technical convenience—we set

$$
b = \frac{a}{\ell}
$$

for some constant $b$, depending on just $k$ and $\ell$ and selected at the end of this subsection. To compare solutions corresponding to different parameters we use the linear homeomorphisms

$$
\hat{\mathcal{P}}_{\tau} : C^{k,\beta} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right) \rightarrow C^{k,\beta} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right)
$$

(3.29)

defined by $\hat{\mathcal{P}}_{\tau} = \varphi_{\zeta, \xi}^* \mathcal{P}_{\zeta, \xi} \varphi_{0,0}^*$, which depends on just $\zeta$ or equivalently $\tau$ (for fixed $m$).

**Lemma 3.30.** Fix $\beta \in (0, 1)$. There exist a constant $C$, depending on only $k$, $\ell$, and $\beta$, and a linear map

$$
\hat{\mathcal{R}}_{b,m,\tau} : C^{0,\beta}_{\text{sym}} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right) \rightarrow C^{2,\beta}_{\text{sym}} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right) \times \mathbb{R}
$$

(3.31)

such that if $f \in C^{0,\beta}_{\text{sym}} \left( \mathcal{T}_{b,m,\tau}; \hat{\chi}_{\tau}, \frac{m^2}{\rho^2} \right)$ and $(u, \mu) = \hat{\mathcal{R}}_{b,m,\tau} f$, then

(a) $\mathcal{L}_{\tau} u = f + \mu w$;

(b) $\| \hat{\mathcal{R}}_{b,m,\tau} \| \leq C$; and
Given \( f \in C^{0, \beta}(\mathcal{T}_{b,m,T}, \hat{\chi}_T, \frac{m^2}{\rho^2}) \) and extend it to a function

\[
F \in C^{0, \beta}\left(R = \left[ -\frac{\pi}{\sqrt{2k}}, \frac{\pi}{\sqrt{2k}} \right] \times \left[ -\frac{\pi}{\sqrt{2\ell}}, \frac{\pi}{\sqrt{2\ell}} \right], \hat{\chi}_T \right),
\]

say by higher-order reflection and smooth cutoff, which is identically zero in a neighborhood of the origin and which satisfies \( \|F\|_{C^{0, \beta}(R, \hat{\chi}_T, \frac{m^2}{\rho^2})} \lesssim \|f\|_{C^{0, \beta}(\mathcal{T}_{b,m,T}, \hat{\chi}_T, \frac{m^2}{\rho^2})} \). Extend also \( w \), without relabellng, to a smooth function on all of \( R \) by defining the extension to be 0 outside the original function’s domain.

Setting

\[
\mu = -\frac{\int_R \frac{\rho^2}{m^2} F \ d\text{vol}[g_E]}{\int_R w \ d\text{vol}[g_E]},
\]

we get \( |\mu| \lesssim \|f\|_{C^{0, \beta}(\mathcal{T}_{b,m,T}, \hat{\chi}_T, m^2/\rho^2)} \) and \( \int_R \left( \frac{\rho^2}{m^2} F + \mu w \right) \ d\text{vol}[g_E] = 0 \). Obviously \( \frac{\rho^2}{m^2} F + \mu w \in L^2(R, g_E) \), so there exists a unique \( U \in H^2(R, g_E) \) such that

\[
\Delta_{g_E} U = \frac{\rho^2}{m^2} F + \mu w \quad \text{and} \quad \|U\|_{C^0(R)} \lesssim \|f\|_{C^{0, \beta}(\mathcal{T}_{b,m,T}, \hat{\chi}_T, m^2/\rho^2)}.
\]

Actually, since \( F \in C^{0, \beta}(R, g_E) \), we have in fact \( U \in C^{2, \beta}(R, g_E) \). On the disc \( D \) of \( g_E \) radius \( \frac{1}{2} \) centered at 0, the solution \( U \) differs from the solution to the Poisson problem on \( D \), with the same source \( F \) but trivial boundary data for the nonconstant Fourier components (in the angular variable) and trivial initial data at the center of the disc for the constant component, by the (bounded) harmonic function with the same corresponding data as \( U \). We separate radial and angular variables and decompose \( U|_D \) into its angular Fourier components. Because of the reflectional symmetries through 0, the projections of \( U|_D \) onto the first harmonics \( \cos \theta \) and \( \sin \theta \) both vanish. For the higher harmonics, it is easy to see, using say the Green’s function, that the corresponding solutions to the Dirichlet problem have the desired \( C^0 \) decay, and it is also obvious that the associated harmonic functions (simply multiples of the real and imaginary parts of \( z^n \)) likewise have the desired decay.

In light of the Schauder estimates above it remains only to manage the constant component of \( U|_D \). We arrange for this to vanish by subtracting from \( U \), without relabelling and without altering either assertion of (3.35) the constant \( U(0,0) \). Taking \( u = U|_{\mathcal{T}_{b,m,T}} \) then establishes all requirements except the continuous dependence on the parameters. To get the continuity we first observe that the extension of \( f \) to \( F \) described above can in fact be performed after first pulling back \( f \) to the parameter domain corresponding to \( \zeta = \xi = 0 \). Then \( F \) will depend continuously on the parameters in the obvious sense. The functions \( w \) and \( \rho \) are continuous in the same sense, so the expression (3.33) for \( \mu \) is then manifestly continuous. The operator \( \Delta_{g_E} \) being trivially continuous
in the parameters, the solution $U$ to (3.35) is continuous as well, and consequently its value at the origin is too, concluding the proof.

**Proposition 3.36.** Fix $\beta \in (0, 1)$ and $i \in \{1, -1\}$. Given $\varepsilon > 0$, there is a positive integer $m_0$—depending on just $k$, $\ell$, and $\varepsilon$—and there is a constant $C$—depending on only $k$, $\ell$, and $\beta$—such that, provided $|\zeta| + |\xi| \leq \varepsilon$, $b > C$, and $m > m_0$, there exists a linear map

$$\mathcal{R}_{\mathcal{T}[i]}: C^{0,\beta}_\text{sym} \left( \mathcal{T}[i], \chi, \frac{m^2}{\rho^2} \right) \to C^{2,\beta}_\text{sym} \left( \mathcal{T}[i], \chi, \frac{m^2}{\rho^2} \right) \times \mathbb{R}$$

such that if $f \in C^{0,\beta}(\mathcal{T}[i], \chi, \frac{m^2}{\rho^2})$ and $(u, \mu) = \mathcal{R}_{\mathcal{T}[i]} f$, then

(a) $\mathcal{L}_x u = f + \mu w$;

(b) $\|\mathcal{R}_{\mathcal{T}[i]}\| \leq C$; and

(c) $(\mathcal{P}_{\zeta, \xi}^{-1} \times \mathcal{P}_{\zeta, \xi}) \mathcal{R}_{\mathcal{T}[i]_{|\zeta, \xi}} \mathcal{P}_{\zeta, \xi}|_{\mathcal{T}[i]|_{\zeta, \xi}, 0} : C^{0,\beta}(\mathcal{T}[i], \chi, \frac{m^2}{\rho^2}) \to C^{2,\beta}(\mathcal{T}[i], \chi, \frac{m^2}{\rho^2}) \times \mathbb{R}$ depends continuously on $(\zeta, \xi)$.

**Proof.** First we observe that, by the estimates in 2.33 above together with 3.4, we have the equivalence, through constants independent of the parameters, of the $C^{2,\beta}(\mathcal{T}[i], \chi, \frac{m^2}{\rho^2})$ and $C^{2,\beta}(\mathcal{T}[b, \mu], \chi, 
\mathcal{T}, \frac{m^2}{\rho^2})$ norms (and likewise of the corresponding weighted $C^{0,\beta}$ norms), provided that

$$|z_0| + a\tau + \frac{1}{\cosh b} + \frac{1}{m}$$

is sufficiently small in terms of an absolute constant. Now define

$$L, \tilde{L} : C^{2,\beta}_\text{sym}(\mathcal{T}[b, \mu], \chi, \frac{m^2}{\rho^2}) \times \mathbb{R} \to C^{0,\beta}_\text{sym}(\mathcal{T}[b, \mu], \chi, \frac{m^2}{\rho^2})$$

by

$$L(u, \mu) = \mathcal{L}_x u - \mu w$$

and

$$\tilde{L}(u, \mu) = \tilde{\mathcal{L}} u - \mu w.$$
(a) \( L_{\chi}u = f + \mu_1 w_1 + \mu_{-1} w_{-1} \);
(b) \( \|R\| \leq C \); and
(c) the map \((P^{-1} \times I_{\mathbb{R}^2})RP : C^{0,\beta,\gamma}_G(M_0,\chi) \times [-\ell,\ell]^2 \to C^{2,\beta,\gamma}_G(M_0,\chi) \times \mathbb{R}^2\), given by
\begin{equation}
(f,\zeta,\xi) \mapsto \left( P^{-1}_{\zeta,\xi} \times I_{\mathbb{R}^2} \right) RP_{\zeta,\xi} f,
\end{equation}
is continuous.

Proof. We will need \( \bar{R}_K : C^{0,\beta,\gamma}_G(M,\chi) \to C^{2,\beta,\gamma}_G(M,\chi) \times \mathbb{R}^2 \) given by
\begin{equation}
\bar{R}_K f = (\psi [\ell m, 2\ell m](\rho) R_{K[0]}(\psi [\ell m, 2\ell m](\rho) f), 0, 0)
\end{equation}
and \( \bar{R}_T : C^{0,\beta}_{sym}(T[-1] \cup T[1], \chi, \frac{m^2}{\rho^2}) \to C^{2,\beta,\gamma}_G(M,\chi) \) given by
\begin{equation}
\bar{R}_T f = \left( \psi \left[ \frac{1}{\tau \cosh b}, \frac{1}{2\tau \cosh b} \right] (\rho) \pi_1 \times \pi_2 \times 0 \right) R_{T[1]} f |_{T[1]}
\end{equation}
\begin{equation}
+ \left( \psi \left[ \frac{1}{\tau \cosh b}, \frac{1}{2\tau \cosh b} \right] (\rho) \pi_1 \times 0 \times \pi_2 \right) R_{T[-1]} f |_{T[-1]},
\end{equation}
where \( \pi_i \) denotes the obvious projection. Next we define \( L : C^{2,\beta,\gamma}_G(M,\chi) \times \mathbb{R}^2 \to C^{0,\beta,\gamma}_G(M,\chi) \) by
\begin{equation}
L(u, \mu_1, \mu_{-1}) = L_{\chi} u - \mu_1 w_1 - \mu_{-1} w_{-1}
\end{equation}
and \( \bar{R} : C^{0,\beta,\gamma}_G \to C^{2,\beta,\gamma}_G \times \mathbb{R}^2 \) by
\begin{equation}
\bar{R} = \bar{R}_K + \bar{R}_T \left( I - L \bar{R}_K \right),
\end{equation}
where \( I \) denotes the identity operator on \( C^{0,\beta,\gamma}_G(M,\chi) \).

Then \( \|I - L \bar{R}\| \) can be made arbitrarily small by choosing \( m \) large, so that \( L \bar{R} \) is invertible and we can take
\begin{equation}
\bar{R} = \bar{R} \left( L \bar{R} \right)^{-1}.
\end{equation}
The continuity in \((\zeta,\xi)\) of \( \left( P^{-1}_{\zeta,\xi} \times I_{\mathbb{R}^2} \right) RP_{\zeta,\xi} \) follows from the last two propositions. Its (joint) continuity in \((\zeta,\xi)\) and \( f \) then follows from its uniform (in \((\zeta,\xi)\)) boundedness. \( \square \)

Corollary 3.51. The map
\begin{equation}
(P^{-1} \times I_{\mathbb{R}^2}) R \rho^{-2} H : \mathbb{R}^2 \to C^{2,\beta,\gamma}_G(M_0,\chi) \times \mathbb{R}^2
\end{equation}
is continuous.

Proof. The composite \( \bar{\phi}_{\zeta,\xi} = \phi_{\zeta,\xi} \circ P^{-1}_{\zeta,\xi} : M_{0,0} \to S^3 \) of the defining embedding \( \phi_{\zeta,\xi} \) of the initial surface \( M_{\zeta,\xi} \) in \( S^3 \) with the diffeomorphism \( P^{-1}_{\zeta,\xi} \) defined at the beginning of this section realizes the initial surface \( M_{\zeta,\xi} \) as an embedding of \( M_{0,0} \) in \( S^3 \). The projection \( \bar{H} \left[ \bar{\phi}_{\zeta,\xi} \right] \) along the selected unit normal of the mean curvature of \( \bar{\phi}_{\zeta,\xi} \) is then a smooth function of both the parameters and the points in \( M_{0,0}, \) so \( P^{-1}_{\zeta,\xi} H_{\zeta,\xi} = \bar{H} \left[ \bar{\phi}_{\zeta,\xi} \right] \), as an element of \( C^{2,\beta,\gamma}_G(M_{0,0},\chi) \), is certainly continuous in \((\zeta,\xi)\). Since \( P^{-1}_{\zeta,\xi} \rho^{-2} \) is also continuous in the parameters, we have secured continuity in \((\zeta,\xi)\) of \( P^{-1}_{\zeta,\xi} \rho^{-2} H_{\zeta,\xi} \). The claim then follows from the last assertion of 3.43. \( \square \)
4. The nonlinear terms and the substitute kernel

In this section we prove the main theorem as outlined in the introduction. In order to solve for the nonlinear terms in the proof we will need the following estimate for them. Recall $Q[u] = H[u] - H[0] - Lu.$

**Lemma 4.1.** Given any constants $C, \xi > 0$, there exists a positive integer $m_0$ depending on only $k$, $\ell$, $C$, and $\xi$ such that whenever $m \geq m_0$, $|\xi| + |\xi| \leq \xi$, and $u : M, \zeta, \xi \rightarrow \mathbb{R}$ satisfies $\|u\|_{2, \beta, \gamma} \leq C\tau$, we have

\[
\|\rho^{-2}Q[u]\|_{0, \beta, \gamma} \leq \xi^{1+\frac{1}{2}}.
\]

Furthermore the map

\[
P^{-1}Q[P] : C^{2, \beta, \gamma}(M, 0, \chi) \times [-\xi, \xi] \rightarrow C^{0, \beta, \gamma}(M, 0, \chi)
\]

is continuous.

**Proof.** The estimate relies on just the fundamental theorem of calculus and a simple scaling argument. To express it succinctly we first generalize the definition of $H[u]$; given a metric $h$ on $S^3$ we write $H_h[u]$ for the projection along $\nu_h$ of the mean curvature, relative to $(S^3, h)$, of the immersion $\phi_{u,h} : M \rightarrow S^3$ defined by $\phi_{u,h}(p) = \exp^h(p) u(p) \nu_h(p)$, where $\exp^h : TS^3 \rightarrow S^3$ is the exponential map determined by $h$ and $\nu_h$ is the unit normal to $M$ having everywhere positive inner product with $\nu$. Correspondingly we define

\[
Q_h[u] = H_h[u] - H_h[0] - \frac{d}{dt} \bigg|_{t=0} H_h[tu].
\]

Then it is easy to see, for any constant $\lambda > 0$, that $H_{\lambda^2 h} = \lambda^{-1} H_h[u]$ and so, taking $h = g_s$,

\[
\lambda^{-2} Q[u] = \lambda^{-1} Q_{\lambda g_s}[\lambda u].
\]

Moreover, the value of $H_h[u]$ at a point $p \in M$ depends on only $h, p, u(p), du(p),$ and $D^2 u(p)$, so there exists a smooth (nonlinear) function

\[
H_h : (M \times \mathbb{R}) \oplus T^* M \oplus T^* M \rightarrow \mathbb{R}
\]

such that

\[
H_h : (p, z, v, A) = \mathcal{H}_{[u_{p,z,v},A]}(p),
\]

for some (any) $u_{p,z,v,A} \in C^2(M, g)$ satisfying $u_{p,z,v,A}(p) = z, du_{p,z,v,A}(p) = v,$ and $D^2 u_{p,z,v,A}(p) = A$. Given $u \in C^2(M, g)$ and $p \in M$, we provisionally define $u(p) = (u(p), du(p), D^2 u(p)).$ Then

\[
Q_h[u](p) = H_h(p, u_p) - H_h(p, 0) - \frac{d}{dt} \bigg|_{t=0} H_h(p, tu(p))
\]

\[
= \int_0^1 \frac{d}{dt} H(p, tu(p)) dt - \partial_t H||_{[p,0]} u^i(p)
\]

\[
= \int_0^1 \partial_t H_{u_{p, tu(p)}} u^i(p) dt - \int_0^1 \partial_i H_{u_{p, 0}} u^i(p) dt
\]

\[
= \int_0^1 \int_0^1 \frac{d}{ds} \partial_j H_{u_{p, stu(p)}} u^i(p) ds dt
\]

\[
= \int_0^1 \int_0^1 \partial_i \partial_j H_{u_{p, stu(p)}} u^i(p) u^j(p) ds dt.
\]
Now, for any \( p \in M \) and \( u \in C^2(M, g) \),
\[
\rho^{-2}(p) Q[u](p) = \rho^{-1}(p) Q_{\rho^2(p)g_S}[\rho(p)u](p)
\]
(4.8)
\[
= \rho(p)u'(p)u'(p) \int_0^1 \int_0^1 \partial_i \partial_j \mathcal{H}_{\rho^2(p)g_S}[\rho(p)u](p, ds dt),
\]
where, crucially (see for instance the proof of (15.9) \( \mathcal{H}_{\rho^2(p)} \) and all its derivatives, with respect to the metric \( \rho^2(p)g_S \), are bounded uniformly in \( m, p, \zeta, \xi \), and every \( u \) in question. From here, noting \( \|d\rho^{-2}(p)\|_\chi + \|D^2\rho^{-2}(p)\|_\chi \lesssim \rho^{-2}(p) \) and bearing in mind the multiplicative properties of Hölder norms, the estimate follows immediately.

To verify the continuity we identify the smooth function
\[
Q : \mathbb{R}^2 \times (M_{0,0} \times \mathbb{R}) \oplus T^*M_{0,0} \oplus T^*M_{0,0}^{\otimes 2}
\]
such that for \( v \in C^2(M_{0,0}) \)
\[
\mathcal{P}^{-1}_{\zeta,\xi} Q_{\zeta,\xi}[\mathcal{P}_{\zeta,\xi}v](p) = Q[\zeta, \xi, v(p), dv(p), D^2v(p)].
\]
(4.10)
The continuity claim then follows from the mean value theorem and the smoothness of \( Q \).

The substitute kernel will be eliminated in the proof of the main theorem by adjusting the parameters \( \zeta \) and \( \xi \). In order to monitor the substitute-kernel content and to prescribe the appropriate parameters we measure vertical forces on the top and the bottom halves of the perturbed surfaces. Specifically we identify the Killing field \( K \) generating rotations about the circle \( \{x = y = \frac{\pi}{2\sqrt{2}}\} \subset S^3 \) of the real plane \( \{\text{Im} z_1 = \text{Im} z_2 = 0\} \subset \mathbb{C}^2 \), so that
\[
K = -\frac{1}{\sqrt{2}} \cot \left(z + \frac{\pi}{4}\right) \sin \sqrt{2}x \cos \sqrt{2}y \partial_x + \frac{1}{\sqrt{2}} \tan \left(z + \frac{\pi}{4}\right) \cos \sqrt{2}x \sin \sqrt{2}y \partial_y
\]
(4.11)
\[
+ \cos \sqrt{2}x \cos \sqrt{2}y \partial_z,
\]
and for \( u \in C^2(M, g) \) we define the forces
\[
\mathcal{F}_{\pm 1}[u] = \int_{\{t \geq 0\} \cap K_{\pm 0} \cup T_{\pm 1}} \mathcal{H}[u](\nu_u, K \circ \phi_u) \, dvol[\phi_u g_S],
\]
(4.12)
where we recall \( \phi_u : M \to S^3 \) is the perturbation by \( u \) of the embedding \( \phi : M \to S^3 \) of the initial surface in \( S^3 \) and where \( \nu_u \) is the unit normal for the perturbed immersion \( \phi_u \) having positive inner product with the parallel translate of the already chosen normal \( \nu \) for \( \phi \) along the geodesics it generates. The dependence of these forces on the parameters is bounded in the next lemma, which also fulfills our promise of justifying the selection of \( \tau \).

**Lemma 4.13.** Given \( c, C > 0 \), there exist a positive integer \( m_0 \), depending on \( k, \ell, C, \) and \( \mathcal{C} \), and a real constant \( c > 0 \), depending on \( k, \ell, \) and \( C \) but not on \( \mathcal{C} \), such that for any \( \zeta, \xi \in [-\mathcal{C}, \mathcal{C}] \) and \( u : M \to \mathbb{R} \) with \( \|u\|_{2,\beta,\gamma} \leq C \tau \), we have for each \( m > m_0 \),
\[
\left| \zeta + \frac{k\ell m^2}{16\pi^2} (\mathcal{F}_1[u] - \mathcal{F}_{-1}[u]) \right| + \left| \xi - \frac{k\ell m^2}{16\pi^2} (\mathcal{F}_1[u] + \mathcal{F}_{-1}[u]) \right| \leq c.
\]
(4.14)
Furthermore each of the maps
\[
\mathcal{F}_{\pm 1}[\mathcal{P} :] : C^2_{\mathcal{G}}(M_{0,0}, \chi) \times [-\mathcal{C}, \mathcal{C}] \to \mathbb{R}
\]
(4.15)
\[
\mathcal{F}_{\pm 1}[\mathcal{P} :] : (v, \zeta, \xi) \mapsto \mathcal{F}_{\pm 1}[\mathcal{P}_{\zeta,\xi}v]
\]
is continuous.
Proof. From the first-variation formula for area (with variation generated by \( K \))

\[
\mathcal{F}_j[u] = \int_{\partial(\{z \geq 0\} \cap \Sigma \cup \Sigma[j])} \langle \eta_j, K \circ \phi_u \rangle \, d\text{vol}[\phi_u^* g_S],
\]

where \( \eta_j \) denotes the outward conormal along the indicated boundary, which has two components: intrinsically (meaning as parametrized by curves on the unperturbed initial surface \( M \)) the circle \( S = \{ t = 0 \} \) and the rectangle \( R_{\pm 1} \)

\[
R_{\pm 1} = \left\{ (x, y, \tau(\xi \pm a)) \mid \frac{\pi}{2\ell m} \leq |x| \leq \frac{\pi}{2\ell m} \text{ and } |y| \leq \frac{\pi}{2\ell m} \right\} \text{ or } \left\{ |x| \leq \frac{\pi}{2\ell m} \text{ and } |y| = \frac{\pi}{2\ell m} \right\}.
\]

We estimate on \( S \)

\[
\| g_S \circ \phi_u - (dx^2 + dy^2 + dz^2) \circ \phi_u \|_{\Gamma_0(\phi_u^* T \otimes \Sigma^2 |_{S}, g_S \circ \phi_u)} \lesssim |\xi| \tau + \| u \|_{C^0(S)}
\]

\[
\| dz(K \circ \phi_u) - 1 \|_{C^0(S)} \lesssim \tau^2, \quad \| dx(K \circ \phi_u) \|_{C^0(S)} + \| dy(K \circ \phi_u) \|_{C^0(S)} \lesssim \tau,
\]

\[
\| \eta_{\pm 1} \pm \partial_\xi \|_{\Gamma_0(\phi_u^* T S^3 |_{S}, g_0 \circ \phi_u)} \lesssim \| u \|_{C^{1}(S, d\theta^2)}, \quad \text{and} \quad \| \partial_\theta \|_{\phi_u^* g_S - \tau} \lesssim (1 + |\xi|) \| u \|_{C^1(S, d\theta^2)} + |\xi| \tau^2,
\]

whence

\[
2\pi \tau \pm \int_S \langle \eta_{\pm 1}, K \circ \phi_u \rangle \, d\text{vol}[\phi_u^* g_S] \lesssim (1 + |\xi|) \left( \| u \|_{C^1(S, d\theta^2)} + \tau^2 \right).
\]

A similar analysis estimates the contribution along \( R_j \), but we prefer the following computation, an elaboration of the replacement argument described in \([9]\). By virtue again of the first-variation formula for area, the integral \( \int_{R_j} \langle \eta_j, K \circ \phi_u \rangle \, d\text{vol}[\phi_u^* g_S] \) measures the force in the direction \( K \) through any surface with boundary \( \phi_u(R_j) \) and conormal \( \eta_j \). In particular we may pick the surface \( \Sigma_j \) which is the graph, over the truncated cmc torus \( T_j \) having boundary \( R_j \), of some function \( v_j \in C^2(T_j) \) agreeing with \( u \) on \( R_j \) and having all derivatives of order at most two bounded (relative to the metric \( g_{T_j} \) induced by \( g_S \)) by \( m^2\tau \). Such a function is easy to construct by identifying \( T_j \) with \( M \) through vertical projection and smoothly cutting off \( u \) near the waist of \( M \). Then the mean curvature of \( \Sigma_j \) is

\[
2 \tan 2\tau(\xi + ja) + \left( \Delta_{T_j} + \| T_j \|^2 + 2 \right) v_j + Q_{T_j}[v_j],
\]

where the first term is just the mean curvature of \( T_j \), the middle terms realize the usual first-order correction, and the quantity \( Q_{T_j}[v_j] \) subsumes all higher-order corrections. Now, writing \( \eta_{T_j} \) for the outward conormal of \( T_j \) and using also Green’s identity and again the fact that \( K \) is a Killing field along with the construction of \( v_j \),

\[
\int_{R_j} \langle \eta_j, K \circ \phi_u \rangle \, d\text{vol}[\phi_u^* g_S] = \int_{T_j} \left( 2 \tan 2\tau(\xi + ja) + Q_{T_j}[v_j] \right) \, dz(K) \, d\text{vol}[g_{T_j}]
\]

\[
- \int_{R_j} u \eta_{T_j} \, dz(K) \, d\text{vol}[g_{R_j}].
\]

From the definitions of \( a \) and \( \tau \) we have

\[
\left| a - \left( \frac{k\ell m^2}{4\pi} - \zeta + \ln 2 \right) \right| \lesssim m^2\tau^2,
\]

and so, with an estimate of the nonlinear terms along the lines of the preceding lemma,

\[
\left| \pm 2\pi \tau + \frac{8\pi^2\tau}{k\ell m^2} (\xi + \zeta) - \int_{R_{\pm 1}} \langle \eta_{\pm 1}, K \circ \phi_u \rangle \, d\text{vol}[\phi_u^* g_S] \right| \lesssim m^{-2}\tau.
\]
Thus

\[
\frac{8\pi^2 \tau}{k\ell m^2} (\xi + \zeta) - \mathcal{F}_{\pm 1}[u] \lesssim m^{-2} \tau + (1 + |\xi|)(\|u\|_{C^1(S, d\pi^2)} + \tau^2),
\]

proving the estimate. The continuity claim is clear from (4.16).

The main result will now be a straightforward consequence of the definition of the initial surfaces coupled with the following theorem.

**Theorem 4.25.** Fix \(\beta \in (0, 1)\), \(\gamma \in \left(0, \frac{1}{2}\right)\), and positive integers \(k\) and \(\ell\). There are absolute constants \(\varepsilon, C > 0\) such that for \(m > C\) we can choose parameters \(\xi, \zeta \in [-c, c]\) so that there exists \(u \in C^\infty_0(M_k, \ell, m, \xi, \zeta)\) with \(\|u\|_{2, \beta, \gamma} \leq C\tau\) such that \(M_u\) is an embedded minimal surface.

**Proof.** Let

\[
B = \left\{ v \in C^2_{\mathcal{F}}(M_{0,0}) : \|v\|_{2, \beta, \gamma} \leq \varepsilon^{1+\frac{k}{2}} \right\} \times [-c, c]^2
\]

and define \(\mathcal{J} : B \to C^2_{\mathcal{F}}(M_{0,0}) \times \mathbb{R}^2\) by

\[
\mathcal{J}(v, \xi, \zeta) = \left( -\mathcal{P}^{-1}_\xi \pi_1 \mathcal{R} \rho^{-2} Q_{\zeta, \xi} [\mathcal{P}_{\zeta, \xi} v - \pi_1 \mathcal{R} \rho^{-2} H], \right. \\
\left. \xi + \frac{k\ell m^2}{16\pi^2 \tau} (\mathcal{F}_1 - \mathcal{F}_{-1}) [\mathcal{P}_{\zeta, \xi} v - \pi_1 \mathcal{R} \rho^{-2} H], \right.
\]

\[
\zeta - \frac{k\ell m^2}{16\pi^2 \tau} (\mathcal{F}_1 + \mathcal{F}_{-1}) [\mathcal{P}_{\zeta, \xi} v - \pi_1 \mathcal{R} \rho^{-2} H].
\]

By 4.11 and 4.13 we can pick \(c\) large enough that \(\mathcal{J}(B) \subseteq B\), and according to 3.43, 3.51, 4.1, and 4.13 \(\mathcal{J} : B \to B\) is continuous with respect to the \(C^2_\Phi\) topology on the first factor of \(B\) and the usual topology on \([-c, c]^2\). Moreover \(B\) is convex and compact under this topology. Therefore the Schauder fixed-point theorem applies to ensure the existence of a fixed point \((v, \xi, \zeta)\). Then, setting \(u = \mathcal{P}_{\zeta, \xi} v - \pi_1 \mathcal{R} \rho^{-2} H_{\xi, \xi}\), by construction of \(\mathcal{J}\) we have \(\mathcal{H}[u, \xi, \zeta] = \mu_1 w_1 + \mu_{-1} w_{-1}\) for some \(\mu_1, \mu_{-1} \in \mathbb{R}\) and simultaneously \(\mathcal{F}_1[u] = \mathcal{F}_{-1}[u] = 0\). It is easy to see, given the sign of \(\langle K, \nu_u \rangle\) on the support of each \(w_i\) along with the positivity of the latter, that both coefficients \(\mu_1\) and \(\mu_{-1}\) must vanish in order for the two forces to vanish. Thus in fact \(\mathcal{H}[u, \xi, \zeta] = 0\). Regularity is an immediate consequence of this last equation, and embeddedness follows from the estimate of \(u\) in light of the geometry of the initial surface.

\[
\square
\]

**Appendix A. Spaces of sections**

Let \(M\) be a smooth manifold, possibly with boundary or corners. For any real vector bundle \(E\) over \(M\) and each nonnegative integer \(r\) we write \(\Gamma^r_{\text{loc}}(E)\) for the vector space of those sections of \(E\) each of whose components, with respect to any local trivialization, is a function \(M \to \mathbb{R}\) all of whose order-\(r\) partial derivatives exist and are continuous. When \(E\) is the trivial bundle \(M \times \mathbb{R}\), we write this space as \(C^r_{\text{loc}}(M)\). We also define \(\Gamma^\infty(E) = \bigcap_{r=0}^\infty \Gamma^r_{\text{loc}}(E)\) and \(C^\infty(M) = \bigcap_{r=0}^\infty C^r_{\text{loc}}(M)\).

Fix a metric \(g\) on \(M\). Many bundles of interest are derived from the tangent bundle \(TM\), by some combination of duality, tensor product, pullback, and orthogonal projection. On such bundles the Levi-Civita connection on \(TM\), induced by \(g\), determines a canonical connection (by imposition of a Leibniz rule or by pullback or projection), which we shall often indiscriminately label \(D\), irrespective of the particular bundle at hand. In particular we get a connection on \(E \otimes T^*M^{\otimes r}\) for
Proof. Work in a parallel frame along Lemma A.5. Given a smooth map (A.4) (A.6) $dP$ (A.7) for each $j \leq r$.

Given a piecewise $C^1$ curve $\gamma : [a, b] \to M$ and any two points $c, d \in [a, b]$, we denote by $P^d_c[\gamma] : E_\gamma(c) \to E_\gamma(d)$ the parallel transport map along $\gamma$ according to $D$, from the fiber over $\gamma(c)$ to the fiber over $\gamma(d)$. (We continue to overload the notation, writing $P^d_c[\gamma]$ regardless of $E$ and delegating to context the identification of the particular bundle.) For future convenience we record a trivial generalization of the fundamental theorem of calculus. First we extend the notion of integration along $\gamma$ so that it becomes a linear map from the space of tensor-valued one-forms on $M$ to the fiber over the terminal point $\gamma(b)$ (or equivalently the space of parallel sections along $\gamma$). Specifically, given $F \in \Gamma^0_{loc}(E \otimes T^*M)$, we define its integral over $\gamma$ to be the tensor

$$\int_{\gamma} F = \int_{a}^{b} P^b_a[\gamma] \left( F_{\gamma(t)} \dot{\gamma}(t) \right) dt.$$

Lemma A.2. Let $\gamma : [a, b] \to M$ be a piecewise $C^1$ curve.

(a) If $F \in \Gamma^1_{loc}(E)$, then

$$\int_{\gamma} DF = F_{\gamma(b)} - P^b_a[\gamma] F_{\gamma(a)}.$$

(b) If $F \in \Gamma^0_{loc}(E \otimes T^*M)$, then the tensor field $t \mapsto \int_{\gamma|[a,t]} F$ on $\gamma$ belongs to $\Gamma^1_{loc}((\gamma^*E)|_{(a,b)})$ and

$$D \int_{\gamma|[a,b]} F = \gamma^* F.$$

Proof. Work in a parallel frame along $\gamma$ and apply the standard fundamental theorem of calculus.

The next lemma, expressing a popular interpretation of curvature, will be useful later as well.

Lemma A.5. Given a smooth map $\gamma : [0, 1]^2 \to M$ such that both $\gamma(0, \cdot)$ and $\gamma(1, \cdot)$ are constant,

$$\frac{dP^1_0[\gamma(\cdot, t)]}{dt} = \int_{\gamma(\cdot, t)} R^E(\cdot, \gamma_t \partial_t) P^0_0[\gamma(\cdot, t)],$$

where $R^E$ is the curvature of the connection on $E$.

Proof. For any $f \in E_{\gamma(0,0)}$ we have, making use of the canonical connection on $\gamma^* E$ induced by the connection on $E$,

$$\frac{dP^1_0[\gamma(\cdot, t)]}{dt} f = \frac{dP^1_0[\gamma(\cdot, t)]}{dt} - P^1_0[\gamma(\cdot, t)] \frac{dP^0_0[\gamma(\cdot, t)]}{dt} f = (D_{\partial_t} P^1_0[\gamma(\cdot, t)] f)_{s=1} - P^1_0[\gamma(\cdot, t)] (D_{\partial_t} P^0_0[\gamma(\cdot, t)] f)_{s=0}$$

$$= \int_{\gamma(\cdot, t)} DD_{\partial_t} P^0_0[\gamma(\cdot, t)] f$$

$$= \int_{0}^{1} P^1_s[\gamma(\cdot, t)] D_{\partial_s} D_{\partial_t} P^0_s[\gamma(\cdot, t)] f ds$$

$$= \int_{0}^{1} P^1_s[\gamma(\cdot, t)] (D_{\partial_s} D_{\partial_t} - D_{\partial_s} D_{\partial_s} - D_{\partial_s, \partial_t}) P^0_s[\gamma(\cdot, t)] f ds$$

$$= \int_{\gamma(\cdot, t)} R(\cdot, \partial_t) P^0_0[\gamma(\cdot, t)] f.$$
Here of course $R$ denotes the curvature of the connection on $\gamma^*E$, and it is easy to see that $R(\partial_s, \partial_t)|_{(s,t)} = R^E(\gamma_*\partial_s, \gamma_*\partial_t)|_{\gamma(s,t)}$. 

Each of our bundles (with rare and unobtrusive exception) inherits from the metric $g$ on $TM$ its own canonical metric, which, with customary indifference to the bundle at hand, we also label $g$. The connection $D$ is compatible with $g$ in the sense that parallel transport is an isometry. To indicate the dependence of a connection on the metric which determines it we will sometimes employ the notation $D[g]$. The metric $g$ establishes a norm $\|\cdot\|_g$ on every fiber of the bundle, which in turn can be used to define various norms of sections. Specifically, given a nonnegative integer $r$ and a section $F \in \Gamma^r_{loc}(E)$, we set

$$\|F \colon \Gamma^r(E,g)\| = \|F\|_{\Gamma^r(E,g)} = \sum_{j=0}^{r} \sup_{p \in M} \|D^j F_p\|_g . \quad (A.8)$$

Given $\beta \in [0,1)$, we measure the corresponding Hölder continuity of $F \in \Gamma^0_{loc}(E)$ at $p \in M$ by

$$[F]_{\beta}(p) = \sup_{v \in \mathcal{N}(p)} \frac{\|F_{\exp_p v} - P_0^1 [t \mapsto \exp_p t v] F_p\|_g}{\|v\|_g^\beta}, \quad (A.9)$$

where

$$\mathcal{N}(p) = \{v \in T_p M : 0 < \|v\|_g \leq 1 \text{ and the curve } t \mapsto \exp_p t v \text{ on } [0,1] \text{ is defined and has length strictly less than any other curve joining } p \text{ and } \exp_p v\}. \quad (A.10)$$

Now we can define Hölder norms by

$$\|F \colon \Gamma^{r,\beta}(E,g)\| = \|F\|_{\Gamma^{r,\beta}(E,g)} = \|F\|_{\Gamma^r(E,g)} + \sup_{p \in M} [D^r F]_{\beta}(p). \quad (A.11)$$

Finally, given a continuous weight function $f : M \to (0,\infty)$, we introduce the weighted Hölder norm

$$\|F \colon \Gamma^{r,\beta}(E,g,f)\| = \|F\|_{\Gamma^{r,\beta}(E,g,f)} = \sup_{p \in M} \frac{\sum_{j=0}^{r} \|D^j F_p\|_g + [D^r F]_{\beta}(p)}{f(p)}. \quad (A.12)$$

From these definitions we get the obvious normed vector spaces

$$\Gamma^r(E,g) = \left\{ F \in \Gamma^r_{loc}(E) : \|F\|_{\Gamma^r(E,g)} < \infty \right\}, \quad (A.13)$$

$$\Gamma^{r,\beta}(E,g) = \left\{ F \in \Gamma^r_{loc}(E) : \|F\|_{\Gamma^{r,\beta}(E,g)} < \infty \right\}, \text{ and}$$

$$\Gamma^{r,\beta}(E,g,f) = \left\{ F \in \Gamma^r_{loc}(E) : \|F\|_{\Gamma^{r,\beta}(E,g,f)} < \infty \right\}. \quad (A.13)$$

Especially useful are the spaces of functions $C^r(M,g)$, $C^{r,\beta}(M,g)$, and $C^{r,\beta}(M,g,f)$, which are simply the spaces just defined with $E$ the trivial bundle $M \times \mathbb{R}$, and whose norms we denote by $\|\cdot\|_{C^r(M,g)}$, $\|\cdot\|_{C^{r,\beta}(M,g)}$, and $\|\cdot\|_{C^{r,\beta}(M,g,f)}$ respectively. Given additionally an action of a group $\mathcal{G}$ on $M$, we indicate the subspaces of symmetric functions with a subscript $\mathcal{G}$, as for example

$$C^{r,\beta}_{\mathcal{G}}(M,g) = \left\{ u \in C^{r,\beta}(M,g) \mid \mathcal{G} u = u \right\}. \quad (A.14)$$

All of these are Banach spaces, enjoying the familiar embedding and compactness results.

**Lemma A.15.** Fix a vector bundle $E$ over $(M,g)$, a nonnegative integer $r$, exponents $\alpha < \beta \in [0,1)$, and a strictly positive weight function $f \in C^0_{loc}(M)$. Then

(a) $\Gamma^{r,\beta}(E,g,f)$ is complete;
(b) the inclusion maps $\Gamma^{r+1,\beta}(E, g) \to \Gamma^{r,\beta}(E, g)$ and $\Gamma^{r,\beta}(E, g) \to \Gamma^{r,\alpha}(E, g)$ are continuous, and, if $M$ is compact, then these inclusions are compact.

We record also some multiplicative properties of these norms.

**Lemma A.16.** Let $E_1$ and $E_2$ be vector bundles (each equipped with a metric $g$ and a metric-compatible connection $D$) over a Riemannian manifold $M$ with weight function $f : M \to \mathbb{R}$, and take sections $F_1 \in \Gamma^{r,\beta}(E_1, g)$, $F_2 \in \Gamma^{r,\beta}(E_2, g, f)$, and $G \in \Gamma^{r,\beta}(E_1^* \otimes E_1 \otimes E_2, g, f)$. Then there exists a constant $C(E_1, E_2, r)$, depending on only $r$ and the dimensions of $E_1$ and $E_2$, such that

(a) $\|F_1 \otimes F_2\|_{\Gamma^{r,\beta}(E_1 \otimes E_2, g, f)} \leq C(E_1, E_2, r) \|F_1\|_{\Gamma^{r,\beta}(E_1, g)} \|F_2\|_{\Gamma^{r,\beta}(E_2, g, f)}$;

(b) $\|\text{tr} G\|_{\Gamma^{r,\beta}(E_1 \otimes E_2, g, f)} \leq C(E_1, E_2, r) \|G\|_{\Gamma^{r,\beta}(E_1^* \otimes E_1 \otimes E_2, g, f)}$, where $\text{tr}$ indicates the obvious contraction.

**Appendix B. Geometric Perturbation**

We now study the response of certain geometric entities defined on a manifold or an immersion to perturbation of the (ambient) metric or of the immersion itself.

**B.1. Perturbation of metrics.** First we make a simple observation in the case of conformal perturbation.

**Lemma B.1.** Let $g$ be a Riemannian metric on a smooth manifold $M$, let $w : M \to \mathbb{R}$ be a strictly positive smooth function, and let $\tilde{g} = w g$. Fix nonnegative integers $k$ and $\ell$, and write $D$ and $\tilde{D}$ for the Levi-Civita connections, induced by $g$ and $\tilde{g}$ respectively, on $TM^\otimes k \otimes T^*M^\otimes \ell$. Then these connections differ by a tensor

\[
\tilde{D} - D = \text{tr} (d \ln w) \otimes g \otimes g^{-1},
\]

where $\text{tr}$ indicates certain contractions.

**Proof.** When $k = 1$ and $\ell = 0$, working in any coordinate system,

\[
\tilde{D} - D = \left( \Gamma^k_{ij} - \Gamma^k_{ij} \right) \partial_k \otimes dx^i \otimes dx^j
\]

\[
= w^{-1} \left( \delta^i_j w, i + \delta^j_i w, j - g_{ij} g^{k\ell} w, k \right) \partial_k \otimes dx^i \otimes dx^j
\]

\[
= \text{tr} (d \ln w) \otimes g \otimes g^{-1},
\]

from which the general case follows immediately through the Leibniz rule. $\Box$

The next, elementary result compares norms and Laplacians on a single manifold defined by different metrics.

**Lemma B.4.** Let $M$ be a smooth manifold, $f : M \to \mathbb{R}$ a (strictly positive) weight function, $r$ a nonnegative integer, and $\beta \in (0, 1)$. There exist a constant $C(M, r)$, depending on only $r$ and the dimension of $M$, and a continuous strictly positive function $W_{M, 0} : M \to \mathbb{R}$, with $W_{M, 0}(p)$ depending on only, with respect to $g_0$, the supremum of the norm of the Riemannian curvature on a ball of radius 1 about $p$, such that if $g_0$ and $g_1$ are Riemannian metrics on $M$ with respective Laplacians $\Delta_{g_0}$ and $\Delta_{g_1}$, then

(a) $C(M, r)^{-1} \|u\|_{C^r(M, g_1, f)} \leq \|u\|_{C^r(M, g_0, f)} \leq C(M, r) \|u\|_{C^r(M, g_0, f)}$

whenever $\|g_1 - g_0\|_{\Gamma^r(T^*M^\otimes 2, g_0)} < 1/C(M, r)$ and $u \in C^r(M, g_0, f)$, so $C^r(M, g_0, f)$ and $C^r(M, g_1, f)$ can be identified as topological vector spaces for $g_0$ and $g_1$ sufficiently close in $\Gamma^r(T^*M^\otimes 2, g_0, f)$;
\[ C(M, r)^{-1} \|u\|_{C^{r, \beta}(M, g_1, f)} \leq \|u\|_{C^{r, \beta}(M, g_1, f)} \leq C(M, r) \|u\|_{C^{r, \beta}(M, g_0, f)} \]

whenever \( \|g_1 - g_0\|_{\Gamma_{\text{max}}(r, 1)(T^* M^{\otimes 2}, g_0, W, M, g_0)} < 1 \) and \( u \in C^{r, \beta}(M, g_0, f) \), so \( C^{r, \beta}(M, g_0, f) \) can be identified as topological vector spaces for \( g_0 \) and \( g_1 \) sufficiently close in \( \Gamma_{\text{max}}(r, 1)(T^* M^{\otimes 2}, g_0, W, M, g_0) \); and

\[ \|\Delta_{g_1} - \Delta_{g_0}\|_{C^{r, \beta}(M, g_0, f)} \leq C(M, r) \|g_1 - g_0\|_{\Gamma_{\text{max}}(r, 1)(T^* M^{\otimes 2}, g_0)} \|u\|_{C^{r+2, \beta}(M, g_0, f)}. \]

**Proof.** With normal coordinates relative to \( g_0 \) it is easy to establish the first item. Continuing to work in these coordinates we see the second by treating the exponential map and parallel transport relative to \( g_1 \) as perturbations of the same relative to \( g_0 \) and by using \[ \text{A.5} \]. Another simple calculation in the same coordinates and an application of \[ \text{A.16} \] complete the proof. \( \square \)

Now we examine the effect of perturbation of ambient metric on the geometry of an isometric immersion. For our purposes it suffices to assume the immersion has two sides and codimension one.

**Lemma B.5.** Let \( \phi: \Sigma \to M \) be a two-sided, codimension-one immersion and \( g_0 \) and \( g_1 \) metrics on \( M \). Fix respective global unit normals \( \nu_0, \nu_1: \Sigma \to TM \) on a single side of \( \Sigma \), and write \( A_i \) for the projection along \( \nu_i \) of the second fundamental form of \( \phi \) relative to \( g_i \) and \( H_i \) for the trace induced by \( g_i \) of \( A_i \), and for each \( p \in \Sigma \) and nonnegative integer \( i \), write \( \Theta^i[A_0, g_1 - g_0](p) \) for the polynomial in \( \left\{ \|D^i[\phi^* g_0]A_0\|_{\phi^* g_0} \right\}_{i=0}^{\infty} \cup \left\{ \|D^i(g_0)(g_1 - g_0)\|_{g_0} \right\}_{i=0}^{\infty} \) each of whose coefficients is 1 and which contains all possible terms with at most one factor of \( \|g_1 - g_0\|_{g_0} \) and homogeneous in \( (g_0, g_1) \) of degree \( \frac{1}{2} \).

Then, assuming \( \|g_1 - g_0\|_{g_0} < 1 \), for each nonnegative integer \( r \) there exists a constant \( C(M, r) \), depending on just \( r \) and the dimension of \( M \), such that

\[ \begin{align*}
(\text{a}) & \quad \|D^i[\phi^* g_0](\phi^* g_1 - \phi^* g_0)(\phi^* g_0)\|_{\phi^* g_0} \leq C(M, r)\Theta^i[A_0, g_1 - g_0](p); \\
(\text{b}) & \quad \|D^i[\phi^* g_0](A_1^2 g_1 - A_0^2 g_0)(\phi^* g_0)\|_{\phi^* g_0} \leq C(M, 0)\Theta^{i+2}[A_0, g_1 - g_0](p) \quad \text{for } i = 0, 1; \text{ and} \\
(\text{c}) & \quad \|D^i[\phi^* g_0](H_1 - H_0)(\phi^* g_0)\|_{\phi^* g_0} \leq C(M, 0)\Theta^{i+1}[A_0, g_1 - g_0](p) \quad \text{for } i = 0, 1.
\end{align*} \]

**Proof.** From \( \phi^* g_i = \text{tr} \ d\phi \otimes (g_i \circ \phi) \otimes d\phi \) and the recognition of \( D_i d\phi \) as simply the (vector-valued) second fundamental form \( A_i \), relative to \( g_i \), we obtain the first assertion. Further exploiting the same expression for \( A_i \) we find

\[ (A_1 - A_0)(V, W) = [(D_1 - D_0)(V)](d\phi W) - d\phi ([D_1 - D_0](V)](W), \]

whence, using (a) and setting \( \epsilon^{(1)}(p) = D^i[g_0](g_1 - g_0)(\phi(p)), \)

\[ \|A_1 - A_0\|_{g_0} \leq \|A_0(p)\|_{g_0} \epsilon^{(0)}(p) + \epsilon^{(1)}(p) \]

and similarly

\[ \|D[g_0](A_1 - A_0)(p)\|_{g_0} \leq \left( \|A_0(p)\|_{g_0}^2 + \|DA_0(p)\|_{g_0} \right) \epsilon^{(0)}(p) + \|A_0(p)\|_{g_0} \epsilon^{(1)}(p) + \epsilon^{(2)}(p). \]

From here, using \( \|\nu_1 - \nu_0\|_{g_0} \leq \epsilon^{(0)}(p) \) and \( D[\phi^* g_1]\nu_1 = 0 \), it is simple to obtain (b) and (c). \( \square \)

**B.2. Perturbation of immersions.** Last we restrict attention to codimension-one, oriented immersions into a complete oriented manifold with fixed metric of constant sectional curvature. These simplifications slightly ease the statements that follow and will suffice for our purposes, but it is straightforward to verify analogous estimates under less stringent assumptions. In this next lemma, rather than perturbing the ambient metric we consider perturbations of the immersion itself, which we realize as graphs of sections of the normal bundle, via the ambient exponential map. That is, given an immersion as above \( \phi: \Sigma \to M \), a choice of unit normal \( \nu_0 \) for \( \phi \), and a function \( u: \Sigma \to M \),
we define the perturbed map \( \phi_u : \Sigma \to M \) by \( \phi_u(p) = \exp_{\phi(p)} u(p)\nu_0(p) \). For \( u \) sufficiently small in terms of the sectional curvature, \( \phi_u \) will be another immersion and will admit a unique smooth unit normal \( \nu_u \) having everywhere positive inner product with the parallel translates of \( \nu_0 \) along the geodesics it generates.

**Lemma B.9.** Let \( M \) be a Riemannian manifold and \( \phi : \Sigma \to M \) an immersion as above, and write \( K \) for the absolute value of the sectional curvature of \( M \). There exists a constant \( C \) depending on only the dimension of \( M \) such that if

\[
\tag{B.10} \left\| A_0\|_{\phi_0^* g} |u| + \| du\|_{\phi_0^* g} + K |u|^2 \right\|_{C^0} \leq 1/C,
\]

then \( \phi_u \) is a well-defined immersion satisfying

\[
\begin{align*}
(a) \quad & \left\| \phi_u^* g - \phi_0^* g : \Gamma^0 \left( T^* \Sigma^\otimes 2, \phi_0^* g, \| A_0 \| |u| + \| du\|^2 + K |u|^2 \right) \right\| \leq C; \\
(b) \quad & \left\| D[\phi_0^* g] \phi_u^* g : \Gamma^0 \left( T^* \Sigma^\otimes 3, \phi_0^* g, \| A_0 \| + \| D^2 u\| \right) \right\| \leq C; \\
(c) \quad & \left\| D^2[\phi_0^* g] \phi_u^* g : \Gamma^0 \left( T^* \Sigma^\otimes 4, \phi_0^* g, \| D^2 A_0\| \right) \right\| \leq C;
\end{align*}
\]

\[
\begin{align*}
\tag{d} \quad & \left\| D^3[\phi_0^* g] \phi_u^* g : \Gamma^0 \left( T^* \Sigma^\otimes 5, \phi_0^* g, \| D^3 A_0\| \right) \right\| \leq C;
\end{align*}
\]

\[
\begin{align*}
\tag{e} \quad & \left\| A_u - A_0 : \Gamma^0 \left( T^* \Sigma^\otimes 2, \phi_0^* g, \right) \| D^2 u\| + \left( \| A_0 \| \| du\| + \| D\phi_u^* g\| \right) \right\| \leq C;
\end{align*}
\]

\[
\begin{align*}
\tag{f} \quad & \left\| D[\phi_0^* g](A_u - A_0) : \Gamma^0 \left( T^* \Sigma^\otimes 3, \phi_0^* g, \| D^3 u\| \right) \right\| \leq C;
\end{align*}
\]

\[
\begin{align*}
\tag{g} \quad & \left\| H_u - H_0 - \left( \Delta \phi_u^* g + \| A_0 \|^2 + \text{Ric}(\nu_0, \nu_0) \right) u : C^0(\Sigma, \phi_0^* g, \right) \left( K |u| + \| A_0 \|^2 |u| + \| D^2 u\| \right) \right\| \leq C; \text{ and}
\end{align*}
\]

\[
\begin{align*}
\tag{h} \quad & \left\| d(H_u - H_0) - \left( \Delta \phi_u^* g u + 2(\| A_0 \| D\phi_u^* g) + 2 \text{Ric}(\nu_0, \nu_0) u + \| A_0 \|^2 du + \text{Ric}(\nu_0, \nu_0) du \right) \right\| \leq C;
\end{align*}
\]

**Proof.** We parametrize a neighborhood in \( M \) of \( \phi_0(\Sigma) \) by a subset of the cylinder \( \Sigma \times \mathbb{R} \) using the map \( \Phi : (p, t) \mapsto \exp_{\phi_0(p)} t\nu_0(p) \) and realize an isometric copy of \( \phi_u(\Sigma) \) as the graph of \( u \) in \( \Sigma \times \mathbb{R} \). We write \( T \) for the vector field \( \Phi_* \partial_t \) on \( M \) of unit tangents to the family of geodesics generated by \( \nu_0 \). We parametrize this family by the points \( p \in \Sigma \) and write \( \gamma_t \) for the geodesic \( t \mapsto \exp_{\phi_0(p)} t\nu_0 \), so that \( T(p, t) = \gamma_t^\prime(p) \). A vector field \( V \) on \( \Sigma \) extends in the usual way to a vector field on \( \Sigma \times \mathbb{R} \) and we use the same label \( V \) for its image \( d\Phi V \) in \( M \). Then

\[
\tag{B.11} (D^2 V)(T, T) = D_T D_T V = D_T D_V T = D_V D_T T + \text{R}(T, V) T = \text{R}(T, V) T,
\]

which is simply the Jacobi equation along the geodesics generated by \( \nu_0 \), whereby

\[
\tag{B.12} V_t(p) = P_t^\nu \gamma_t^\nu V_0 + t P_t^\nu \gamma_t^\nu D_T V_0 + \int_{\gamma_t^\nu|_{[0, t]}} \int_{\gamma_t^\nu|_{[0, s]}} R(\cdot, V).
\]
and so
\[
Φ^*g(V, W)_{(p,t)} = φ_0^*g(V, W)_p - 2tA_0(V, W)_p + t^2A_0(S_0V, W)_p + \left\langle V, \int_{γ_p|u_p|} \int_{γ_p|u_s|} R(\cdot, W) \right\rangle_{(p,t)} + \left\langle W, \int_{γ_p|u_p|} \int_{γ_p|u_s|} R(\cdot, V) \right\rangle_{(p,t)}.
\]

(S_0: TΣ → TΣ being the shape operator of φ_0, while of course
\[
Φ^*g(∂_t, ∂_t) = 1 \quad \text{and} \quad Φ^*g(∂_t, V) = 0 \quad \text{for all } V \text{ on } Σ.
\]

Thus
\[
φ_0^*g = φ_0^*g - 2uA_0 + u^2A_0(S_0, ∙) + du ⊗ du + B_u,
\]

where \( B_u(V, W) = C_u(V, W) + C_u(W, V) \) and
\[
C_u(V, W)_p = \int_0^u \int_0^s \left\langle V(p,u(p)), D^u(p)[γ_p]R(T,W)T(p,τ) \right\rangle \, dτ \, ds.
\]

Then
\[
\|B_u(p)\|_{φ_0^*g} \lesssim Ku^2(p) \sup_{t\in[0,1]} \|φ_{tu}^*g\|_{φ_0^*g}
\]

and so
\[
(1 - Ku^2(p)) \sup_{t\in[0,1]} \|φ_{tu}^*g(p)\|_{φ_0^*g} \lesssim 1 + |u(p)| |A_0(p)| + u^2(p)A_0^2(p) + \|du(p)\|^2_{φ_0^*g},
\]

which with \([B.15]\) implies (a) provided
\[
K |u(p)|^2 + \|A_0(p)\| |u(p)| + \|du(p)\|^2 \leq \frac{1}{2}.
\]

Further, whenever \( u = t \) is constant on \( Σ \), using
\[
D_T D_VP^*_s[γ_p]W = R(T, V)P^*_s[γ_p]W,
\]

we obtain the bound
\[
\|D[φ_0^*g]B_t\|_{φ_0^*g} \lesssim Kt^2 \sup_{s\in[0,t]} \left( \|A_s\|_{φ_0^*g} + \|D[φ_0^*g]φ_{0s}^*g\|_{φ_0^*g} \right) + K^2 |t|^3.
\]

Interchanging derivatives at the cost of a curvature term yet again we find also
\[
A_t = A_0 - tA_0(S_0, ∙) + t(R(T, ∙), T) + 2 \int_0^t \int_0^s (⟨(R(S, T)T, ∙), R(\cdot, T)S⟩) \, dτ \, ds,
\]

so
\[
\|A_t - A_0\|_{φ_0^*g} \lesssim \|A_0\|^2 |t| + K |t| + K \|A_0\| |t|^2 \quad \text{and in turn from} \ [B.15] \ \text{and} \ [B.21] \ \text{recalling the assumptions} \ [B.19].
\]

\[
\|D[φ_0^*g](φ_{0t}^*g - φ_0^*g)\|_{φ_0^*g} \lesssim (\|DA_0\| + K) |t|.
\]

On the other hand we note that \( B_u \) and its covariant derivative, with respect to \( φ_0^*g \), vanish at any point \( p \) where \( u(p) = 0 \). In particular, referring to \([B.15]\) at such a point,
\[
D[φ_{0s}^*g](φ_{0s}^*g - φ_0^*g)(p) = -2du(p) \otimes A_0(p) + D^2[φ_t^*g]u(p) \otimes du(p) + du(p) \otimes D^2[φ_t^*g]u(p).
\]

Then for general \( u \), observing \( φ_u = φ_{u(p)u - u(p)} \),
\[
\|D[φ_0^*g](φ_{0t}^*g - φ_0^*g)(p)\|_{φ_0^*g} \leq \|D[φ_0^*g](φ_{u(p)u - u(p)}^*g - φ_{u(p)}^*g)(p)\|_{φ_0^*g} + \|D[φ_0^*g](φ_{u(p)}^*g - φ_0^*g)(p)\|_{φ_0^*g},
\]

which yields (b) in light of the foregoing.
Higher-order derivatives of $B_u$ need not vanish, but at a point $p \in \Sigma$ where $u(p)$ vanishes, we do have the bounds
\begin{equation}
\|D^2[\phi_u^*g]B_u(p)\|_{\phi_u^g} \lesssim K \|du\|^2 \quad \text{and} \quad \|D^3[\phi_u^*g]B_u(p)\|_{\phi_u^g} \lesssim K \|du\|^2 + K \|du\| \|D^2u\|,
\end{equation}

enabling us to establish the next two items in a fashion similar to the above, estimating higher derivatives of $B_t$ and $A_t$.

Now a simple calculation reveals
\begin{equation}
A_u(p) = \left( A_{u(p)} + D^2[\phi_{u(p)}^*g]u\right)_p (T, \nu_u) - (du \otimes (S_{u(p)})^\top, \nu_u) + (S_{u(p)})^\top , \nu_u) \otimes du\right)_p
\end{equation}

\begin{equation}
= \frac{A_{u(p)}(p) + D^2[\phi_{u(p)}^*g]u(p) - du|_p \otimes duS_{u(p)} - duS_{u(p)} \otimes du|_p}{\sqrt{1 + \|du\|^2_{\phi_u^g(p)}},}
\end{equation}

which in conjunction with \[B.22\] \[B.15\] and the already verified items yields the remaining items. 

\(\square\)

APPENDIX C. THE JACOBI OPERATOR ON THE STANDARD CATENOID

Here we state a few elementary results concerning the operator $L_\chi = \partial_t^2 + \partial_\theta^2 + 2 \sech^2 t$ on the cylinder $[-a, a] \times S^1$, used in the proof of \[B.13\] Setting
\begin{equation}
L = \partial_t^2 + 2 \sech^2 t \quad \text{and} \quad A_{\pm} = \partial_t \pm \tanh t,
\end{equation}

we find
\begin{equation}
L = A_- A_+ + 1,
\end{equation}
\begin{equation}
\partial_t^2 = A_+ A_- + 1,
\end{equation}
\begin{equation}
A_+ L = \partial_t^2 A_+ , \quad \text{and} \quad A_- \partial_t^2 = L A_-.
\end{equation}

From these relations we can readily identify, for each integer $n$, the kernel of $L - n^2$ on $[-a, a]$ with Dirichlet data for $n \neq 0$ and with trivial initial data at $t = 0$ for $n = 0$. In particular we can compute the Green’s function $G_{4n}^{D}(s,t)$ for $L - 4n^2$ on $[-a, a]$ with Dirichlet conditions and $n$ a nonzero integer:
\begin{equation}
G_{4n}^{D}(s,t) = \frac{1}{d_n} \int g_-(s)g_+(t) \text{ for } s \leq t \quad \text{and} \quad \frac{1}{d_n} \int g_+(s)g_-(t) \text{ for } s \geq t,
\end{equation}

where
\begin{equation}
g_\pm(t) = (2n \pm \tanh a)(2n - \tanh t)e^{2n(t \mp a)} - (2n \mp \tanh a)(2n + \tanh t)e^{-2n(t \pm a)} \quad \text{and} \quad d_n = (32n^3 - 8n)(4n^2 - 4n \tanh a \coth 4na \mp \tanh^2 a) \sinh 4na.
\end{equation}

We can also compute the Green’s function $G_0^{I}(s,t)$ for $L$ on $[0, a]$ with vanishing initial data at $t = 0$:
\begin{equation}
G_0^{I}(s,t) = \begin{cases} 
[(t - s) \tanh s + 1] \tanh t - \tanh s \text{ for } s \leq t \\
0 \text{ for } s \geq t.
\end{cases}
\end{equation}

Now it is easy to establish, for each $\gamma \in (0, 1)$, the existence of a constant $C(\gamma)$ such that
\begin{equation}
\sup_{t \in [0,a]} e^{\gamma(a-t)} \left| \int_0^a G_0^{I}(s,t)f(s) \, ds \right| \leq C(\gamma) \sup_{s \in [0,a]} e^{\gamma(a-s)} |f(s)|
\end{equation}
for any $f \in C^0([0,a])$. From the reflectional symmetry of $L$ we conclude that if $f \in C^0([-a,a])$ and if $u \in C^2([-a,a])$ solves $Lu = f$ on $[-a,a]$ with $u(0) = u'(0) = 0$, then

$$\sup_{t \in [-a,a]} e^{\gamma(a-|t|)} |u(t)| \leq C(\gamma) \sup_{s \in [-a,a]} e^{\gamma(a-|s|)} |f(s)|.$$  

Moreover, for each nonzero integer $n$ and for $a$ sufficiently large, another estimate yields

$$\sup_{t \in [-a,a]} e^{\gamma(a-|t|)} \left| \int_a^s G_n^D(s,t) f(s) \, ds \right| \leq \frac{C(\gamma)}{n^2} \sup_{s \in [-a,a]} e^{\gamma(a-|s|)} |f(s)|.$$  

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