POSITIVE SEMICLASSICAL STATES FOR A FRACTIONAL SCHRÖDINGER-POISSON SYSTEM

EDWIN GONZALO MURCIA AND GAETANO SICILIANO

Abstract. We consider a fractional Schrödinger-Poisson system in the whole space $\mathbb{R}^N$ in presence of a positive potential and depending on a small positive parameter $\varepsilon$. We show that, for suitably small $\varepsilon$ (i.e. in the “semiclassical limit”) the number of positive solutions is estimated below by the Ljusternick-Schnirelmann category of the set of minima of the potential.

1. Introduction

In the last decades a great attention has been given to the following Schrödinger-Poisson type system

$$
\begin{cases}
-\Delta u + V(x)u + \phi u = |u|^{p-2}u \\
-\Delta \phi = u^2,
\end{cases}
$$

which arises in non relativistic Quantum Mechanics. Such a system is obtained by looking for standing waves solutions in the purely electrostatic case to the Schrödinger-Maxwell system. For a deduction of this system, see e.g. [12]. Here the unknowns are $u$, the modulus of the wave function, and $\phi$ which represents the electrostatic potential. $V$ is a given external potential and $p \geq 2$ a suitable given number.

The system has been studied by many authors, both in bounded and unbounded domains, with different assumptions on the data involved: boundary conditions, potentials, nonlinearities; many different type of solutions have been encountered (minimal energy, sign changing, radial, nonradial...), the behaviour of the solutions (e.g. concentration phenomena) has been studied as well as multiplicity results have been obtained. It is really difficult to give a complete list of references: the reader may see [13] and the references therein.

However it seems that results relating the number of positive solutions with topological invariants of the “objects” appearing in the problem are few in the literature. We cite the paper [34] where the system is studied in a (smooth and) bounded domain $\Omega \subset \mathbb{R}^3$ with $u = \phi = 0$ on $\partial \Omega$ and $V$ constant. It is shown, by using variational methods that, whenever $p$ is sufficiently near the critical Sobolev exponent 6, the number of positive solutions is estimated below by the Ljusternick-Schnirelmann category of the domain $\Omega$.

On the other hand it is known that a particular interest has the semiclassical limit of the Schrödinger-Poisson system (that is when the Plank constant $\hbar$ appearing in the system, see e.g. [12], tends to zero) especially due to the fact that this limit describes the transition from Quantum to Classical Mechanics. Such a situation is studied e.g. in [33], among many other papers. We cite also Fang and Zhao [23] which consider the following doubly perturbed system.

\begin{thebibliography}{99}
\bibitem{} 2000 Mathematics Subject Classification. 35J50, 35Q40, 58E05,
\textit{Key words and phrases.} Fractional Laplace equation, multiplicity of solutions, Ljusternick-Schnirelmann category.
\end{thebibliography}
in the whole space $\mathbb{R}^3$:
\[
\begin{cases}
-\varepsilon^2 \Delta w + V(x)w + \psi w = |w|^{p-2}w \\
-\varepsilon \Delta \psi = w^2.
\end{cases}
\]
Here $V$ is a suitable potential, $4 < p < 6$, and $\varepsilon$ is a positive parameter proportional to $\hbar$.
In this case the authors estimate, whenever $\varepsilon$ tends to zero, the number of positive solutions by the Ljusternick-Schnirelmann category of the set of minima of the potential $V$, obtaining a result in the same spirit of [34].

Recently, especially after the formulation of the Fractional Quantum Mechanics, the derivation of the Fractional Schrödinger equation given by N. Laskin in [27–29], and the notion of fractional harmonic extension of a function studied in the pioneering paper [15], equations involving fractional operators are receiving a great attention. Indeed pseudodifferential operators appear in many problems in Physics and Chemistry, see e.g. [30, 31]; but also in obstacle problems [32, 35], optimization and finance [20], conformal geometry and minimal surfaces [14, 16, 17], etc.

Motivated by the previous discussion, we investigate in this paper the existence of positive solutions for the following doubly singularly perturbed fractional Schrödinger-Poisson system in $\mathbb{R}^N$:
\[
(P_\varepsilon)
\begin{cases}
\varepsilon^{2s} (-\Delta)^s w + V(x)w + \psi w = f(w) \\
\varepsilon^\theta (-\Delta)^{\alpha/2} \psi = \gamma_\alpha w^2,
\end{cases}
\]
where $\gamma_\alpha := \frac{\pi N/2^{s+1} \Gamma(\alpha/2)}{\Gamma(N/2 - \alpha/2)}$ is a constant ($\Gamma$ is the Euler function). By a positive solution of ($P_\varepsilon$) we mean a pair $(w, \psi)$ where $w$ is positive. To the best of our knowledge, there are only few recent papers dealing with a system like ($P_\varepsilon$): in [37] the author deals with $\varepsilon = 1$ proving under suitable assumptions on $f$ the existence of infinitely many (but possibly sign changing) solutions by means of the Fountain Theorem. A similar system is studied in [36] and the existence of infinitely many (again, possibly sign changing) solutions is obtained by means of the Symmetric Mountain Pass Theorem.

In this paper we assume that

(H1) $s \in (0, 1)$, $\alpha \in (0, N)$, $\theta \in (0, \alpha)$, $N \in (2s, 2s + \alpha)$,

moreover the potential $V$ and the nonlinearity $f$ satisfy the assumptions listed below:

(V1) $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and
\[0 < \min_{\mathbb{R}^N} V := V_0 < V_\infty := \liminf_{|x| \to +\infty} V \in (V_0, +\infty);\]

(f1) $f : \mathbb{R} \to \mathbb{R}$ is a function of class $C^1$ and $f(t) = 0$ for $t \leq 0$;

(f2) $\lim_{t \to 0} f(t)/t = 0$;

(f3) there is $q_0 \in (2, 2^*_s - 1)$ such that $\lim_{t \to \infty} f(t)/t^{q_0} = 0$, where $2^*_s := 2N/(N - 2s)$;

(f4) there is $K > 4$ such that $0 < KF(t) := K \int_0^t f(\tau)d\tau \leq tf(t)$ for all $t > 0$;

(f5) the function $t \mapsto f(t)/t^3$ is strictly increasing in $(0, +\infty)$.

The assumptions on the nonlinearity $f$ are quite standard in order to work with variational methods, use the Nehari manifold and the Palais-Smale condition. The assumption (V1) will be fundamental in order to estimate the number of positive solutions and also to recover some compactness.
We recall, once for all, that a $C^1$ functional $J$, defined on a smooth manifold $\mathcal{M}$, is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ ($\text{(PS)}_c$ for brevity) if every sequence $\{u_n\} \subset \mathcal{M}$ such that
\begin{equation}
J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0
\end{equation}
has a convergent subsequence. A sequence $\{u_n\}$ satisfying (1.1) is also named a $\text{(PS)}_c$ sequence.

To stay our result let us introduce $M := \{x \in \mathbb{R}^N : V(x) = V_0\}$ the set of minima of $V$. Our result is the following

**Theorem 1.1.** Under the above assumptions (H1), (V1), (f1)-(f5), there exists an $\varepsilon^* > 0$ such that for every $\varepsilon \in (0, \varepsilon^*]$ problem $(P_\varepsilon)$ possesses at least $\text{cat} \ M$ positive solutions.

Moreover if $\text{cat} \ M > 1$ and $M$ is bounded, then (for suitably small $\varepsilon$) there exist at least $\text{cat} \ M + 1$ positive solutions.

Hereafter, given a topological pair $(X,Y)$, $\text{cat}_X(Y)$ is the Ljusternick-Schnirelmann category of $Y$ in $X$, and, if $X = Y$ this is just denoted with $\text{cat} \ X$.

The proof of Theorem 1.1 is carried out by adapting some ideas of Benci, Cerami and Passaseo [10, 11] and using the Ljusternick-Schnirelmann Theory. We mention that these ideas and techniques have been extensively used to attack also other type of problems, and indeed similar results are obtained for other equations and operators, like the Schrödinger operator [18, 19], the $p$–laplacian [3, 4], the biharmonic operator [7], $p&q$–laplacian, fractional laplacian [24, 25], magnetic laplacian [5, 6] or quasilinear operators [2, 8, 9].

The plan of the paper is the following. In Section 2 we recall some basic facts, we present some preliminaries and the variational setting for the problem. Section 3 is devoted to prove some compactness properties; as a byproduct we prove the existence of a ground state solution for our problem, that is a solution having minimal energy. In Section 4 we introduce the barycenter map, we show some of its properties and prove, by means of the Ljusternick-Schnirelmann Theory, Theorem 1.1.

**Notations.** In the paper we will denote with $| \cdot |_p$ the usual $L^p$ norm in $\mathbb{R}^N$; we denote with $B_r(x)$ the closed ball in $\mathbb{R}^N$ centered in $x$ with radius $r > 0$, with $B_r^c(x)$ its complementary; if $x = 0$ we simply write $B_r$; moreover the letters $C, C_1, C_2, \ldots$ will denote generic positive constants (whose value may change from line to line). Other notations will be introduced whenever we need.

2. Preliminaries

2.1. Some well known facts. Before to introduce the variational setting of our problem, we recall some basic facts concerning the fractional Sobolev spaces and their embeddings.

Given $\beta \in (0, 1)$, the fractional Laplacian $(-\Delta)^\beta$ is the pseudodifferential operator which can be defined via the Fourier transform

$$
\mathcal{F}((-\Delta)^\beta u) = | \cdot |^{2\beta} \mathcal{F}u,
$$
or, if $u$ has sufficient regularity, by

$$
(-\Delta)^\beta u(z) = -\frac{C_{N,\beta}}{2} \int_{\mathbb{R}^N} \frac{u(z+y) - u(z-y) - 2u(z)}{|y|^{N+2\beta}} dy, \quad z \in \mathbb{R}^N,
$$
where $C_{N,\beta}$ is a suitable normalization constant.
For $s \in (0, 1)$ let
\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2} u \in L^2(\mathbb{R}^N) \right\} \]
be the Hilbert space with scalar product and (squared) norm given by
\[ (u, v) = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \int_{\mathbb{R}^N} u v, \quad ||u||^2 = |(-\Delta)^{s/2} u|^2_2 + |u|^2_2. \]
It is known that $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), p \in [2, 2^*_s]$ with $2^*_s := 2N/(N - 2s)$. Moreover the embedding of $H^s(\Omega)$ is compact if $\Omega \subset \mathbb{R}^N$ is bounded and $p \neq 2^*_s$.

We will consider also the homogeneous Sobolev spaces $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $|(-\Delta)^{\alpha/4} u|^2_2$. This is a Hilbert space with scalar product and (squared) norm
\[ (u, v)_{\dot{H}^{\alpha/2}} = \int_{\mathbb{R}^N} (-\Delta)^{\alpha/4} u (-\Delta)^{\alpha/4} v, \quad ||u||^2_{\dot{H}^{\alpha/2}} = |(-\Delta)^{\alpha/4} u|^2_2. \]
It is well known that $\dot{H}^{\alpha/2}(\mathbb{R}^N) \hookrightarrow L^{2^*_\alpha/(\alpha)}(\mathbb{R}^N), 2^*_\alpha = 2N/(N - \alpha)$. For more general facts about the fractional Laplacian we refer the reader to the beautiful paper [22].

We recall here another fact that will be frequently used:
\[ (2.1) \quad \forall \varepsilon > 0 \exists M_\varepsilon > 0 : \int_{\mathbb{R}^N} f(u) u \leq \varepsilon \int_{\mathbb{R}^N} u^2 + M_\varepsilon \int_{\mathbb{R}^N} |u|^{q_0 + 1}, \quad \forall u \in H^s(\mathbb{R}^N). \]
This simply follows by (f2) and (f3).

2.2. The variational setting. It is easily seen that, just performing the change of variables
\[ w(x) := u(x/\varepsilon), \psi(x) := \phi(x/\varepsilon), \]
problem $(P_{\varepsilon})$ can be rewritten as
\[ (P_{\varepsilon}) \quad \left\{ \begin{array}{l}
(-\Delta)^s u + V(\varepsilon x) u + \phi(x) u = f(u) \\
(-\Delta)^{\alpha/2} \phi = \varepsilon^{\alpha-\theta} \gamma_\alpha u^2,
\end{array} \right. \]
to which we will refer from now on.

A usual “reduction” argument can be used to deal with a single equation involving just $u$. Indeed for every $u \in H^s(\mathbb{R}^N)$ the second equation in $(P_{\varepsilon})$ is uniquely solved. Actually, for future reference, we will prove a slightly more general fact.

Let us fix two functions $u, w \in H^s(\mathbb{R}^N)$ and consider the problem
\[ (Q_\varepsilon) \quad \left\{ \begin{array}{l}
(-\Delta)^{\alpha/2} \phi = \varepsilon^{\alpha-\theta} \gamma_\alpha u w, \\
\phi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)
\end{array} \right. \]
whose weak solution is a function $\tilde{\phi} \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ such that
\[ \forall v \in H^{\alpha/2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (-\Delta)^{\alpha/4} \tilde{\phi} (-\Delta)^{\alpha/4} v = \varepsilon^{\alpha-\theta} \gamma_\alpha \int_{\mathbb{R}^N} u w v. \]
For every $v \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$, by the Hölder inequality and the continuous embeddings, we have
\[ \left| \int_{\mathbb{R}^N} u w v \right| \leq |u|^{\frac{4N}{N+\alpha}} |w|^{\frac{4N}{N+\alpha}} |v|^{2^*_\alpha/2} \leq C ||u||_2 ||w||_2 ||v||_{\dot{H}^{\alpha/2}} \]
deducing that the map
\[ T_{u, w} : v \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} u w v \in \mathbb{R} \]
is linear and continuous: then there exists a unique solution \( \phi_{\varepsilon,u,w} \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \) to \((Q_\varepsilon)\). Moreover this solution has the representation by means of the Riesz kernel \( \mathcal{K}_\alpha(x) = \gamma_\alpha^{-1}|x|^\alpha-N \), hence

\[
\phi_{\varepsilon,u,w} = \varepsilon^{\alpha-\theta} \frac{1}{| \cdot |^{N-\alpha}} \ast (uw).
\]

Furthermore

\[
(2.2) \quad \|\phi_{\varepsilon,u,w}\|_{\dot{H}^{\alpha/2}} = \varepsilon^{\alpha-\theta} \|T_{u,w}\|_{\mathcal{L}(\dot{H}^{\alpha/2};\mathbb{R})} \leq \varepsilon^{\alpha-\theta} C \|u\|\|w\|
\]

and then, for \( \zeta, \eta \in H^s(\mathbb{R}^N) \)

\[
(2.3) \quad \int_{\mathbb{R}^N} \phi_{\varepsilon,u,w} \zeta \eta \leq \|\phi_{\varepsilon,u,w}\|_{2_{s/2}} \|\zeta\|_{\frac{4N}{N+s}} \|\eta\|_{\frac{4N}{N+s}} \leq \varepsilon^{\alpha-\theta} C_u \|u\|\|w\|\|\zeta\|\|\eta\|
\]

where \( C_u \) is a suitable embedding constant. Although its value is not important, we will refer to this constant later on.

A particular case of the previous situation is when \( u = w \). In this case we simplify the notation and write

- \( T_u(v) := T_{u,u}(v) = \int_{\mathbb{R}^N} u^2 v, \) and
- \( \phi_{\varepsilon,u} \) for the unique solution of the second equation in \((P_\varepsilon^*)\) for fixed \( u \in H^s(\mathbb{R}^N) \). Then

\[
\|\phi_{\varepsilon,u}\|_{\dot{H}^{\alpha/2}} \leq \varepsilon^{\alpha-\theta} C \|u\|^2
\]

and the map

\[
u \in H^s(\mathbb{R}^N) \mapsto \phi_{\varepsilon,u} \in \dot{H}^{\alpha/2}(\mathbb{R}^N)
\]

is bounded.

Observe also that

\[
(2.4) \quad u_n^2 \rightarrow u^2 \; \text{in} \; L^{2N/(N+\alpha)}(\mathbb{R}^N) \quad \Rightarrow \quad T_{u_n} \rightarrow T_u \; \text{as operators} \quad \Rightarrow \quad \phi_{\varepsilon,u_n} \rightarrow \phi_{\varepsilon,u} \; \text{in} \; \dot{H}^{\alpha/2}(\mathbb{R}^N).
\]

For convenience let us define the map (well defined by \((2.3)\))

\[
A : u \in H^s(\mathbb{R}^N) \mapsto \int_{\mathbb{R}^N} \phi_{\varepsilon,u} u^2 \in \mathbb{R}.
\]

Then

\[
(2.5) \quad |A(u)| \leq \varepsilon^{\alpha-\theta} C_u \|u\|^4
\]

(where \( C_u \) is the same constant in \((2.3)\)). Some relevant properties of \( \phi_{\varepsilon,u} \) and \( A \) are listed below. Although these properties are known to be true, we are not able to find them explicitly in the literature; so we prefer to give a proof here.

**Lemma 2.1.** The following propositions hold.

(i) For every \( u \in H^s(\mathbb{R}^N) \): \( \phi_{\varepsilon,u} \geq 0 \);

(ii) for every \( u \in H^s(\mathbb{R}^N) \), \( t \in \mathbb{R} : \phi_{\varepsilon,t,u} = t^2 \phi_{\varepsilon,u} \);

(iii) if \( u_n \rightharpoonup u \) in \( H^s(\mathbb{R}^N) \) then \( \phi_{\varepsilon,u_n} \rightharpoonup \phi_{\varepsilon,u} \) in \( \dot{H}^{\alpha/2}(\mathbb{R}^N) \);

(iv) \( A \) is of class \( C^2 \) and for every \( u, v, w \in H^s(\mathbb{R}^N) \)

\[
A'(u)[v] = 4 \int_{\mathbb{R}^N} \phi_{\varepsilon,u} uv, \quad A''(u)[v,w] = 4 \int_{\mathbb{R}^N} \phi_{\varepsilon,u} vw + 8 \int_{\mathbb{R}^N} \phi_{\varepsilon,u} uvw,
\]

(v) if \( u_n \rightharpoonup u \) in \( L^r(\mathbb{R}^N) \), with \( 2 \leq r < 2^*_s \), then \( A(u_n) \rightarrow A(u) \);

(vi) if \( u_n \rightharpoonup u \) in \( H^s(\mathbb{R}^N) \) then \( A(u_n - u) = A(u_n) - A(u) + o_n(1) \).
Proof. Items (i) and (ii) follow directly by the definition of $\phi_{\varepsilon,u}$.

To prove (iii), let $v \in C_c^\infty(\mathbb{R}^N)$; we have
\[
\int_{\mathbb{R}^N} (-\Delta)^{\alpha/4}(\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u})(-\Delta)^{\alpha/4} v = \int_{\mathbb{R}^N} (u_n^2 - u^2)v \\
\leq |v|_\infty \left( \int_{\text{supp } v} (u_n - u)^2 \right)^{1/2} \left( \int_{\text{supp } v} (u_n + u)^2 \right)^{1/2} \\
\to 0.
\]
The conclusion then follows by density.

The proof of (iv) is straightforward: we refer the reader to [23].

To show (v), recall that $2 < \frac{4N}{N+\alpha} < 2^*$. Since by assumption $|u_n^2|_{\frac{2N}{N+\alpha}} \to |u^2|_{\frac{2N}{N+\alpha}}$ and $u_n^2 \to u^2$ a.e. in $\mathbb{R}^N$, using the Brezis-Lieb Lemma, $u_n^2 \to u^2$ in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$. But then using (2.4) we get $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ in $L^{2^*/2}(\mathbb{R}^N)$. Consequently
\[
|A(u_n) - A(u)| \leq \int_{\mathbb{R}^N} |\phi_{\varepsilon,u_n} u_n^2 - \phi_{\varepsilon,u} u^2| \leq \int_{\mathbb{R}^N} |(\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u}) u_n^2| + \int_{\mathbb{R}^N} |\phi_{\varepsilon,u} (u_n^2 - u^2)| \\
\leq |\phi_{\varepsilon,u_n} - \phi_{\varepsilon,u}|_{2^*/2} |u_n^2|_{\frac{2N}{N+\alpha}} + |\phi_{\varepsilon,u}|_{2^*/2} |u_n^2 - u^2|_{\frac{2N}{N+\alpha}}
\]
from which we conclude.

To prove (vi), for the sake of simplicity we drop the factor $\varepsilon^{\alpha-\theta}$ in the expression of $\phi_{\varepsilon,u,v}$. Defining
\[
\sigma := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(y)u^2(x)}{|x-y|^{N-\alpha}} dy dx,
\]
\[
\sigma_1 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n^2(y)u^2(x)}{|x-y|^{N-\alpha}} dy dx, \quad \sigma_2 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n(y)u(y)u_n(x)u(x)}{|x-y|^{N-\alpha}} dy dx,
\]
\[
\sigma_3 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n(y)u_n(x)u(x)}{|x-y|^{N-\alpha}} dy dx, \quad \sigma_4 := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u_n(y)u(y)u^2(x)}{|x-y|^{N-\alpha}} dy dx,
\]
it is easy to check that
\[
A(u_n - u) - A(u) + A(u) = 2\sigma + 2\sigma_1 + 4\sigma_2 - 4\sigma_3 - 4\sigma_4.
\]
Now we claim that, whenever $u_n \to u$ in $H^s(\mathbb{R}^N)$,
\[
\lim_{n \to \infty} \sigma_i^n = \sigma, \quad i = 1, 2, 3, 4
\]
which readily gives the conclusion.

We prove here only the cases $i = 1, 2$ since the proof of the other cases is very similar. Recall that
\[
\phi_{\varepsilon,u}(x) = \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-\alpha}} dy, \quad \phi_{\varepsilon,u_n}(x) = \int_{\mathbb{R}^N} \frac{u_n^2(y)}{|x-y|^{N-\alpha}} dy.
\]
Since $u^2 \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N) = L^{2^*/2}(\mathbb{R}^N)$ and by item (iii) it holds $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ in $L^{2^*/2}(\mathbb{R}^N)$, we conclude that
\[
\sigma_1^n = \int_{\mathbb{R}^N} \phi_{\varepsilon,u_n} u^2 \to \int_{\mathbb{R}^N} \phi_{\varepsilon,u} u^2 = \sigma
\]
and the claim is true for $i = 1$.

For $i = 2$ recall that
\[
\phi_{\varepsilon,u_n,u}(x) = \int_{\mathbb{R}^N} \frac{u_n(y)u(y)}{|x-y|^{N-\alpha}} dy.
\]
First we show that $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ a.e. in $\mathbb{R}^N$. Given $\xi > 0$ and choosing $R > 1/\xi$, $\frac{N}{2q} < p < \frac{N}{N-\alpha}$ and $\frac{N}{N-\alpha} < q$ (so that $2p', 2q' \in (2, s^*)$), we have, for large $n$:

$$
\| \phi_{\varepsilon,u_n} - \phi_{\varepsilon,u} \| \leq \| u_n - u \|_{L^{2p'}(B_R(x))} \| u \|_{L^{2p'}(B_R(x))} \left( \int_{|y-x|<R} \frac{dy}{|x-y|^{p(N-\alpha)}} \right) ^{1/p} + \| u_n - u \|_{L^{2q'}(B_R(x))} \| u \|_{L^{2q'}(B_R(x))} \left( \int_{|y-x|\geq R} \frac{dy}{|x-y|^{q(N-\alpha)}} \right) ^{1/q}
$$

$$
\leq C_1 \xi + C_2 \xi^{N-\alpha},
$$

concluding the pointwise convergence. Moreover by the Sobolev embedding and using (2.2),

$$
|\phi_{\varepsilon,u_n} u_n |^2 \leq |\phi_{\varepsilon,u_n, u}|_{2N/\alpha} \leq C_1 \| u_n \|^2 \| u \| \leq C_2
$$

and therefore, up to subsequence, $\phi_{\varepsilon,u_n} \to \phi_{\varepsilon,u}$ in $L^2(\mathbb{R}^N)$, by [26, Lemma 4.8]. Since $u \in L^2(\mathbb{R}^N)$

$$
\sigma_n^2 = \int_{\mathbb{R}^N} \phi_{\varepsilon,u_n, u_n} u_n \to \int_{\mathbb{R}^N} \phi_{\varepsilon,u} u^2 = \sigma
$$

and the claim is proved for $i = 2$.

We introduce now the variational setting for our problem. Let us define the Hilbert space

$$
W_{\varepsilon} := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x) u^2 < \infty \right\}
$$

endowed with scalar product and (squared) norm given by

$$
(u, v)_{\varepsilon} := \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \int_{\mathbb{R}^N} V(\varepsilon x) uv
$$

and

$$
\| u \|_{\varepsilon}^2 := \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \int_{\mathbb{R}^N} V(\varepsilon x) u^2.
$$

Then it is standard to see that the critical points of the $C^2$ functional (see Lemma 2.1 (iv))

$$
I_{\varepsilon}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \int_{\mathbb{R}^N} V(\varepsilon x) u^2 + \frac{1}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon, u} u^2 - \int_{\mathbb{R}^N} F(u),
$$

on $W_{\varepsilon}$ are weak solutions of problem $(P_{\varepsilon}^*)$.

By defining

$$
N_{\varepsilon} := \left\{ u \in W_{\varepsilon} \setminus \{0\} : J_{\varepsilon}(u) = 0 \right\},
$$

where

$$
J_{\varepsilon}(u) := I'_{\varepsilon}(u)[u] = \| u \|_{\varepsilon}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon, u} u^2 - \int_{\mathbb{R}^N} f(u) u,
$$

we have, by standard arguments:

**Lemma 2.2.** For every $u \in N_{\varepsilon}$, $J_{\varepsilon}'(u)[u] < 0$ and there are positive constants $h_{\varepsilon}, k_{\varepsilon}$ such that $\| u \|_{\varepsilon} \geq h_{\varepsilon}, I_{\varepsilon}(u) \geq k_{\varepsilon}$. Furthermore, $N_{\varepsilon}$ is diffeomorphic to the set

$$
S_{\varepsilon} := \{ u \in W_{\varepsilon} : \| u \|_{\varepsilon} = 1, u > 0 \text{ a.e.} \}.
$$

$N_{\varepsilon}$ is the Nehari manifold associated to $I_{\varepsilon}$. By the assumptions on $f$, the functional $I_{\varepsilon}$ has the Mountain Pass geometry. This is standard but we give the easy proof for completeness.

**MP1** $I_{\varepsilon}(0) = 0$;
(MP2) since, for every $\xi > 0$ there exists $M_{\xi} > 0$ such that $F(u) \leq \xi u^2 + M_{\xi} |u|^{q_0 + 1}$, we have
\[
I_\varepsilon(u) \geq \frac{1}{2} \|u\|_{\varepsilon}^2 - \int_{\mathbb{R}^N} F(u) \\
\geq \frac{1}{2} \|u\|_{\varepsilon}^2 - \xi C_1 \|u\|_{\varepsilon}^2 - M_{\xi} C_2 \|u\|_{\varepsilon}^{q_0 + 1}
\]
and we conclude $I_\varepsilon$ has a strict local minimum at $u = 0$;

(MP3) finally, since (F4) implies $F(t) \geq Ct^K$ for $t > 0$, with $K > 4$ (and less then $q_0 + 1$), fixed $v \in C_c^\infty(\mathbb{R}^N), \nu > 0$ we have
\[
I_\varepsilon(tv) = \frac{t^2}{2} \|v\|_{\varepsilon}^2 + \frac{t^4}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon,v} v^2 - \int_{\mathbb{R}^N} F(tv) \\
\leq \frac{t^2}{2} \|v\|_{\varepsilon}^2 + \frac{t^4}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon,v} v^2 - Ct^K \int_{\mathbb{R}^N} v^K
\]

concluding that the functional is negative for suitable large $t$.

Then denoting with
\[
c_\varepsilon := \inf_{\gamma \in \mathcal{H}_\varepsilon} \sup_{t \in [0,1]} I_\varepsilon(\gamma(t)), \quad \mathcal{H}_\varepsilon = \left\{ \gamma \in C([0,1], W_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \right\}
\]
the Mountain Pass level, and with
\[
m_\varepsilon := \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u)
\]
the ground state level, it holds, in a standard way, that
\[
(2.6) \quad c_\varepsilon = m_\varepsilon = \inf_{u \in W_\varepsilon \setminus \{0\}} \sup_{t \geq 0} I_\varepsilon(tu).
\]

It is known that for “perturbed” problems a major role is played by the problem at infinity that we now introduce.

2.3. The problem at “infinity”. Let us consider the “limit” problem (the autonomous problem) associated to $(P_\varepsilon)$, that is
\[
(A_{\mu}) \quad \left\{ \begin{array}{l}
(-\Delta)^s u + \mu u = f(u) \\
u \in H^s(\mathbb{R}^N)
\end{array} \right.
\]
where $\mu > 0$ is a constant. The solutions are critical points of the functional
\[
E_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + \frac{\mu}{2} \int_{\mathbb{R}^N} u^2 - \int_{\mathbb{R}^N} F(u).
\]
in $H^s(\mathbb{R}^N)$. Denoting with $H^s_\mu(\mathbb{R}^N)$ simply the space $H^s(\mathbb{R}^N)$ endowed with the (equivalent squared) norm
\[
\|u\|_{H^s_\mu}^2 := |(-\Delta)^{s/2} u|^2 + \mu |u|^2,
\]
by the assumptions of the nonlinearity $f$, it is easy to see that the functional $E_\mu$ has the Mountain Pass geometry with Mountain Pass level
\[
c_{\mu} := \inf_{\gamma \in \mathcal{H}_\mu} \sup_{t \in [0,1]} E_\mu(\gamma(t)), \quad \mathcal{H}_\mu := \left\{ \gamma \in C([0,1], H^s_\mu(\mathbb{R}^N)) : \gamma(0) = 0, E_\mu(\gamma(1)) < 0 \right\}.
\]
Introducing the set
\[
\mathcal{M}_\mu := \left\{ u \in H^s(\mathbb{R}^N) \setminus \{0\} : \|u\|_{H^s_\mu}^2 = \int_{\mathbb{R}^N} f(u) u \right\}
\]
it is standard to see that

- $\mathcal{M}_\mu$ has a structure of differentiable manifold (said the Nehari manifold associated to $E_\mu$),
- $\mathcal{M}_\mu$ is bounded away from zero and radially homeomorphic to the unit sphere,
- the mountain pass value $c_\mu^\infty := \inf_{u \in \mathcal{M}_\mu} E_\mu(u) > 0$.

The symbol “$\infty$” in the notations is just to recall we are dealing with the limit problem. In the sequel we will mainly deal with $\mu = V_0$ and $\mu = V_\infty$ (whenever this last one is finite). Of course the inequality

$$m_\varepsilon \geq m_0^-$$

holds.

3. Compactness properties for $I_\varepsilon, E_\mu$ : existence of a ground state solution

We begin by showing the boundedness of the Palais-Smale sequences for $E_\mu$ in $H_0^s(\mathbb{R}^N)$ and $I_\varepsilon$ in $W_\varepsilon$. Let $\{u_n\} \subset H_0^s(\mathbb{R}^N)$ be a Palais-Smale sequence for $E_\mu$, that is, $|E_\mu(u_n)| \leq C$ and $E_\mu'(u_n) \rightarrow 0$. Then, for large $n$,

$$C + \|u_n\|_{H_0^s} > E_\mu(u_n) - \frac{1}{K} E_\mu'(u_n)[u_n] = \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{H_0^s}^2 + \frac{1}{K} \int_{\mathbb{R}^N} (f(u_n)u_n - KF(u_n))$$

$$\geq \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{H_0^s}^2,$$

and thus $\{u_n\}$ is bounded. Similarly we conclude for $I_\varepsilon$, using that

$$I_\varepsilon(u_n) - \frac{1}{K} I_\varepsilon'(u_n)[u_n] = \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{\varepsilon}^2 + \left(\frac{1}{4} - \frac{1}{K}\right) \int_{\mathbb{R}^N} \phi_{\varepsilon,u_n} u_n^2 + \frac{1}{K} \int_{\mathbb{R}^N} (f(u_n)u_n - KF(u_n))$$

$$\geq \left(\frac{1}{2} - \frac{1}{K}\right) \|u_n\|_{\varepsilon}^2.$$

In order to prove compactness, some preliminary work is needed. Let us recall the following Lions type lemma, whose proof can be found in [21, Lemma 2.3].

**Lemma 3.1.** If $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and for some $R > 0$ and $2 \leq r < 2^*_s$ we have

$$\sup_{x \in \mathbb{R}^N} \int_{B_R(x)} |u_n|^r \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,$$

then $u_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*_s$.

Then we can prove the following

**Lemma 3.2.** Let $\{u_n\} \subset W_\varepsilon$ be bounded and such that $I_\varepsilon'(u_n) \rightarrow 0$. Then we have either

a) $u_n \rightarrow 0$ in $W_\varepsilon$, or

b) there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants $R, c > 0$ such that

$$\liminf_{n \rightarrow +\infty} \int_{B_R(y_n)} u_n^2 \geq c > 0.$$
Proof. Suppose that b) does not occur. Using Lemma 3.1 it follows
\[ u_n \to 0 \text{ in } L^p(\mathbb{R}^N) \text{ for } p \in (2, 2^*_e). \]
Using (2.1), the boundedness of \( \{u_n\} \) in \( L^2(\mathbb{R}^N) \) and the fact that \( u_n \to 0 \) in \( L^{q_0+1}(\mathbb{R}^N) \), we conclude that
\[ \int_{\mathbb{R}^N} f(u_n)u_n \to 0. \]
Finally, since
\[ \|u_n\|_2^2 - \int_{\mathbb{R}^N} f(u_n)u_n \leq \|u_n\|_2^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon,u_n} u_n^2 - \int_{\mathbb{R}^N} f(u_n)u_n = I'_\varepsilon(u_n)[u_n] = o_n(1), \]
it follows that \( u_n \to 0 \) in \( W_\varepsilon \).

In the rest of the paper we assume, without loss of generality, that \( 0 \in M \), that is, \( V(0) = V_0 \).

Lemma 3.3. Assume that \( V_\infty < \infty \) and let \( \{v_n\} \subset W_\varepsilon \) be a \((PS)_d\) sequence for \( I_\varepsilon \) such that \( v_n \to 0 \) in \( W_\varepsilon \). Then \( v_n \not\to 0 \) in \( W_\varepsilon \implies d \geq m_{V_\infty}^\infty \).

Proof. Observe, preliminarily, that by condition (V1) it follows that
\[ \forall \xi > 0 \exists \tilde{R} = \tilde{R}_\xi > 0: V(\varepsilon x) > V_\infty - \xi, \forall x \notin B_{\tilde{R}}. \]
Let \( \{t_n\} \subset (0, +\infty) \) be such that \( \{t_n v_n\} \subset \mathcal{M}_{V_\infty} \). We start by showing the following

Claim: The sequence \( \{t_n\} \) satisfies \( \limsup_{n \to \infty} t_n \leq 1. \)

Supposing by contradiction that the claim does not hold, there exists \( \delta > 0 \) and a subsequence still denoted by \( \{t_n\} \), such that
\[ t_n \geq 1 + \delta \quad \text{for all } n \in \mathbb{N}. \]
Since \( \{v_n\} \) is a bounded \((PS)_d\) sequence for \( I_\varepsilon \), \( I'_\varepsilon(v_n)[v_n] = o_n(1) \), that is,
\[ \|v_n\|_2^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 = \int_{\mathbb{R}^N} f(v_n)v_n + o_n(1). \]
Moreover, since \( \{t_n v_n\} \subset \mathcal{M}_{V_\infty} \), we get
\[ \|t_n v_n\|_{H_\varepsilon^\infty}^2 = \int_{\mathbb{R}^N} f(t_n v_n) t_n v_n. \]
These equalities imply that
\[ \int_{\mathbb{R}^N} \left( f(t_n v_n) \frac{t_n}{t_n} - f(v_n) \right) v_n = \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))v_n^2 - \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 + o_n(1), \]
and thus
\[ \int_{\mathbb{R}^N} \left( f(t_n v_n) \frac{t_n}{t_n} - f(v_n) \right) v_n \leq \int_{\mathbb{R}^N} (V_\infty - V(\varepsilon x))v_n^2 + o_n(1). \]
Using (3.1), the fact that \( v_n \to 0 \) in \( L^2(B_{\tilde{R}}) \) and that \( \{v_n\} \) is bounded in \( W_\varepsilon \), let us say by some constant \( C > 0 \), we deduce by (3.3)
\[ \forall \xi > 0: \int_{\mathbb{R}^N} \left( f(t_n v_n) \frac{t_n}{t_n} - f(v_n) \right) v_n \leq \xi C + o_n(1). \]
Since \( v_n \not\to 0 \) in \( W_\varepsilon \), we may invoke Lemma 3.2 to obtain \( \{y_n\} \subset \mathbb{R}^N \) and \( R, c > 0 \) such that
\[ \int_{B_R(y_n)} v_n^2 \geq c. \]
Defining $\tilde{v}_n := v_n(\cdot + y_n)$, we may suppose that, up to a subsequence, 
$$\tilde{v}_n \rightharpoonup \tilde{v} \text{ in } H^s(\mathbb{R}^N)$$
and, in view of (3.5), there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $\tilde{v} > 0$ in $\Omega$. By (3.2) and (3.4) becomes
$$0 < \int_{\Omega} \left( \frac{f((1 + \delta)\tilde{v}_n)}{(1 + \delta)\tilde{v}_n} - \frac{f(\tilde{v}_n)}{\tilde{v}_n} \right) \tilde{v}_n^2 \leq \xi C + o_n(1).$$
Now passing to the limit and applying Fatou’s Lemma, it follows that, for every $\xi > 0$
$$0 < \int_{\Omega} \left[ \frac{f((1 + \delta)\tilde{v})}{(1 + \delta)\tilde{v}} - \frac{f(\tilde{v})}{\tilde{v}} \right] \tilde{v}^2 \leq \xi C,$$
which is absurd and proves the claim.

Now we distinguish two cases.

**Case 1:** $\limsup_{n \to \infty} t_n = 1$.

Up to subsequence we can assume that $t_n \to 1$. We have,
$$d + o_n(1) = I_\varepsilon(v_n) \geq m^\infty_{V_\infty} + I_\varepsilon(v_n) - E_{V_\infty}(t_nv_n).$$
Moreover,
$$I_\varepsilon(v_n) - E_{V_\infty}(t_nv_n) = \frac{1 - t^2_n}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (V(\varepsilon x) - t^2_n V_\infty) v_n^2$$
$$+ \frac{1}{4} \int_{\mathbb{R}^N} \phi_{x,v_n} v_n^2 + \int_{\mathbb{R}^N} (F(t_nv_n) - F(v_n)),$$
and due to the boundedness of $\{v_n\}$ we get, for every $\xi > 0$,
$$I_\varepsilon(v_n) - E_{V_\infty}(t_nv_n) \geq o_n(1) - C\xi + \int_{\mathbb{R}^N} (F(t_nv_n) - F(v_n)),$$
where we have used again (3.1). By the Mean Value Theorem, $\int_{\mathbb{R}^N} (F(t_nv_n) - F(v_n)) = o_n(1)$, therefore (3.6) becomes
$$d + o_n(1) \geq m^\infty_{V_\infty} - C\xi + o_n(1),$$
and taking the limit in $n$, by the arbitrariness of $\xi$, we deduce $d \geq m^\infty_{V_\infty}$.

**Case 2:** $\limsup_{n \to \infty} t_n = t_0 < 1$.

We can assume $t_n \to t_0$ and $t_n < 1$. Since $t \mapsto \frac{1}{4} f(t)t - F(t)$ is increasing in $(0, \infty)$,
$$m^\infty_{V_\infty} \leq E_{V_\infty}(t_nv_n) = \int_{\mathbb{R}^N} \left( \frac{1}{2} f(t_nv_n)t_nv_n - F(t_nv_n) \right)$$
$$= \int_{\mathbb{R}^N} \frac{1}{4} f(t_nv_n)t_nv_n + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(t_nv_n)t_nv_n - F(t_nv_n) \right)$$
$$= \frac{1}{4} \|t_nv_n\|^2_{H^s_{V_\infty}} + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(t_nv_n)t_nv_n - F(t_nv_n) \right).$$
(3.7)
But
$$\|t_nv_n\|^2_{V_\infty} \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 + \int_{\mathbb{R}^N} t^2_n V_\infty v_n^2,$$
(3.8)
Again by (3.1), given \( \xi > 0 \),
\[
\ell_n^2 V_\infty - \xi < V_\infty - \xi < V(\varepsilon x) \quad \text{for} \ x \notin B_R
\]
and hence
\[
\int_{\mathbb{R}^N} \ell_n^2 V_\infty v_n^2 \leq \int_{B_R} V_\infty v_n^2 + \int_{|x| \geq R} V(\varepsilon x) v_n^2 + \int_{|x| \geq R} \xi^2 v_n^2
\]
\[
\leq o_n(1) + \int_{\mathbb{R}^N} V(\varepsilon x) v_n^2 + C\xi.
\]
From this and (3.8) we have
\[
\|t_n v_n\|_{H^s_{\varepsilon \infty}}^2 \leq \|v_n\|_\varepsilon^2 + C\xi + o_n(1).
\]
Therefore, using (3.7)
\[
m_{V_\varepsilon}^\infty \leq \frac{1}{4} \|v_n\|_\varepsilon^2 + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(v_n) v_n - F(v_n) \right) + C\xi + o_n(1)
\]
\[
= I_\varepsilon(v_n) - \frac{1}{4} I'_\varepsilon(v_n)[v_n] + C\xi + o_n(1)
\]
\[
= d + C\xi + o_n(1).
\]
concluding the proof.

\[\square\]

**Proposition 3.4.** The functional \( I_\varepsilon \) in \( W_\varepsilon \) satisfies the \((PS)_c\) condition

1. at any level \( c < m_{V_\varepsilon}^\infty \), if \( V_\infty < \infty \),
2. at any level \( c \in \mathbb{R} \), if \( V_\infty = \infty \).

**Proof.** Let \( \{u_n\} \subset W_\varepsilon \) be such that \( I_\varepsilon(u_n) \to c \) and \( I'_\varepsilon(u_n) \to 0 \). We have already seen that \( \{u_n\} \) is bounded in \( W_\varepsilon \). Thus there exists \( u \in W_\varepsilon \) such that, up to a subsequence, \( u_n \rightharpoonup u \) in \( W_\varepsilon \). Note that \( I'_\varepsilon(u) = 0 \), since by Lemma 2.1 (iv), we have for every \( w \in W_\varepsilon \)
\[
(u_n, w)_\varepsilon \to (u, w)_\varepsilon, \quad A'(u_n)[w] \to A'(u)[w] \quad \text{and} \quad \int_{\mathbb{R}^N} f(u_n) w \to \int_{\mathbb{R}^N} f(u) w.
\]
Defining \( v_n := u_n - u \), we have that \( \int_{\mathbb{R}^N} F(v_n) = \int_{\mathbb{R}^N} F(u_n) - \int_{\mathbb{R}^N} F(u) + o_n(1) \) (see [1]) and by Lemma 2.1 (vi), we have \( A(v_n) = A(u_n) - A(u) + o_n(1) \); hence arguing as in [4], we obtain also
\[
(3.9) \quad I'_\varepsilon(v_n) \to 0.
\]
Moreover
\[
(3.10) \quad I_\varepsilon(v_n) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) = c - I_\varepsilon(u) + o_n(1) =: d + o_n(1)
\]
and (3.9) and (3.10) show that \( \{v_n\} \) is a \((PS)_d\) sequence. By (f4),
\[
I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{4} I'_\varepsilon(u)[u] = \frac{1}{4} \|u\|_\varepsilon^2 + \int_{\mathbb{R}^N} \left( \frac{1}{4} f(u) u - F(u) \right)
\]
\[
\geq \frac{1}{4} \int_{\mathbb{R}^N} \left( f(u) u - 4F(u) \right) \geq 0
\]
and then coming back in (3.10) we have
\[
(3.11) \quad d \leq c.
\]
Then,
1. if \( V_\infty < \infty \), and \( c < m_{V_\infty}^\infty \), by (3.11) we obtain
\[
d \leq c < m_{V_\infty}^\infty.
\]
It follows from Lemma 3.3 that \( v_n \to 0 \), that is \( u_n \to u \) in \( W_\varepsilon \).

2. If \( V_\infty = \infty \), by the compact imbedding \( W_\varepsilon \hookrightarrow L^r(\mathbb{R}^N), 2 \leq r < 2^*_\varepsilon \), up to a subsequence, \( v_n \to 0 \) in \( L^r(\mathbb{R}^N) \) and since \( I'_\varepsilon(v_n) \to 0 \), we have
\[
I'_\varepsilon(v_n)[v_n] = \|v_n\|_\varepsilon^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 - \int_{\mathbb{R}^N} f(v_n)v_n = o_n(1).
\]

By Lemma 2.1 (v), \( A(v_n) = \int_{\mathbb{R}^N} \phi_{\varepsilon,v_n} v_n^2 = o_n(1) \), and since by (2.1) it holds again \( \int_{\mathbb{R}^N} f(v_n)v_n = o_n(1) \), we have by (3.12) \( \|v_n\|_\varepsilon^2 = o_n(1) \), that is \( u_n \to u \) in \( W_\varepsilon \).

The proof is thereby complete.

As a consequence it is standard to prove that

**Proposition 3.5.** The functional \( I_\varepsilon \) restricted to \( N_\varepsilon \) satisfies the \((PS)_c\) condition

1. at any level \( c < m_{V_\infty}^\infty \), if \( V_\infty < \infty \),
2. at any level \( c \in \mathbb{R} \), if \( V_\infty = \infty \).

Moreover, the constrained critical points of the functional \( I_\varepsilon \) on \( N_\varepsilon \) are critical points of \( I_\varepsilon \) in \( W_\varepsilon \), hence solution of \((P^*_\varepsilon)\).

Let us recall the following result (see [24, Lemma 6]) concerning problem \((A_\mu)\).

**Lemma 3.6** (Ground state for the autonomous problem). Let \( \{u_n\} \subset \mathcal{M}_\mu \) be a sequence satisfying \( E_\mu(u_n) \to m_\mu^\infty \). Then, up to subsequences the following alternative holds:

a) \( \{u_n\} \) strongly converges in \( H^s(\mathbb{R}^N) \);

b) there exists a sequence \( \{\tilde{y}_n\} \subset \mathbb{R}^N \) such that \( u_n(\cdot + \tilde{y}_n) \) strongly converges in \( H^s(\mathbb{R}^N) \).

In particular, there exists a minimizer \( w_\mu \geq 0 \) for \( m_\mu^\infty \).

Now we can prove the existence of a ground state for our problem. Assumption (H1) is tacitly assumed.

**Theorem 3.7.** Suppose that \( f \) verifies (f1)-(f5) and \( V \) verifies (V1). Then there exists a ground state solution \( u_\varepsilon \in W_\varepsilon \) of \((P^*_\varepsilon)\),

1. for every \( \varepsilon \in (0,\varepsilon_0] \), for some \( \varepsilon_0 > 0 \), if \( V_\infty < \infty \);
2. for every \( \varepsilon > 0 \), if \( V_\infty = \infty \).

**Proof.** Since the functional \( I_\varepsilon \) has the geometry of the Mountain Pass Theorem in \( W_\varepsilon \) there exists \( \{u_n\} \subset W_\varepsilon \) satisfying
\[
I_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad I'_\varepsilon(u_n) \to 0.
\]

1. If \( V_\infty < \infty \), in virtue of Proposition 3.4, we have only to show that \( c_\varepsilon < m_{V_\infty}^\infty \) for every positive \( \varepsilon \) smaller than a certain \( \varepsilon_0 \).

Let \( \mu \in (V_0, V_\infty) \), so that
\[
m_{V_\infty}^\infty < m_\mu^\infty < m_{V_\infty}^\infty.
\]
For \( r > 0 \) let \( \eta_r \) be a smooth cut-off function in \( \mathbb{R}^N \) which equals 1 on \( B_r \) and with support in \( B_{2r} \). Let \( w_r := \eta_r w_\mu \) and \( s_r > 0 \) such that \( s_r w_r \in \mathcal{M}_\mu \). If it were, for every
$r > 0 : E_\mu(s_r w_r) \geq m_{V_\infty}^\infty$, since $w_r \to w_\mu$ in $H^s(\mathbb{R}^N)$ for $r \to +\infty$, we would have $s_r \to 1$ and then
\[ m_{V_\infty}^\infty \leq \liminf_{r \to +\infty} E_\mu(s_r w_r) = E_\mu(w_\mu) = m_{\mu}^\infty \]
which contradicts (3.13). This means that there exists $\tau > 0$ such that $\omega := s_\tau w_\tau \in \mathcal{M}_\mu$ satisfies (3.14)
\[ E_\mu(\omega) < m_{V_\infty}^\infty. \]

Given $\varepsilon > 0$, let $t_\varepsilon > 0$ the number such that $t_\varepsilon \omega \in \mathcal{N}_\varepsilon$. Therefore
\[ t_\varepsilon^2 \|\omega\|_\varepsilon^2 + t_\varepsilon^4 \int_{\mathbb{R}^N} \phi_{t_\varepsilon \omega} \omega^2 = t_\varepsilon \int_{\mathbb{R}^N} f(t_\varepsilon \omega) \omega \]
implicating that
\[ \frac{\|\omega\|_\varepsilon^2}{t_\varepsilon^2} + \int_{\mathbb{R}^N} \phi_{t_\varepsilon \omega} \omega^2 \geq \int_{B_{t_\varepsilon \omega}^\infty} \frac{f(t_\varepsilon \omega)}{t_\varepsilon^2} \omega^2. \]

Now we claim that there exists $T > 0$ such that $\limsup_{t_\varepsilon \to 0^+} t_\varepsilon \leq T$. If by contradiction there exists $\varepsilon_n \to 0^+$ with $t_{\varepsilon_n} \to \infty$, then by (3.15) and (f5) we have
\[ \frac{\|\omega\|_{\varepsilon_n}^2}{t_{\varepsilon_n}^2} + \int_{\mathbb{R}^N} \phi_{\varepsilon_n \omega} \omega^2 \geq \int_{B_{t_{\varepsilon_n} \omega}^\infty} \frac{f(t_{\varepsilon_n} \omega(\tau))}{(t_{\varepsilon_n} \omega(\bar{\tau}))^3} \int_{B_{t_{\varepsilon_n} \omega}^\infty} \omega^2, \]
where $\omega(\tau) := \min_{B_{t_{\varepsilon_n} \omega}^\infty} \omega(x)$. The absurd is achieved by passing to the limit in $n$, since by (f4) the right hand side of (3.16) tends to $\infty$, while the left hand side tends to 0.

Then there exists $\varepsilon_1 > 0$ such that
\[ \forall \varepsilon \in (0, \varepsilon_1] : \ t_\varepsilon \in (0, T]. \]
Condition (V1) implies also that there exists some $\varepsilon_2 > 0$ such that
\[ \forall \varepsilon \in (0, \varepsilon_2] : \ V(\varepsilon x) \leq \frac{V_0 + \mu}{2}, \ \text{for all} \ x \in \text{supp}\omega. \]
Finally let
\[ \varepsilon_3 := \left( \frac{(\mu - V_0) \|\omega\|_2^2}{C_\varepsilon T^2 \|\omega\|^4} \right)^{1/(\alpha - \theta)}, \]
where $C_\varepsilon$ is the same constant appearing in (2.5), hence in particular
\[ \forall \varepsilon \in (0, \varepsilon_3] : \ \int_{\mathbb{R}^N} \phi_{\varepsilon \omega} \omega^2 \leq \varepsilon^{\alpha - \theta} C_\varepsilon \|\omega\|^4 \ \text{and} \ T^2 \varepsilon^{\alpha - \theta} C_\varepsilon \|\omega\|^4 \leq (\mu - V_0) \int_{\mathbb{R}^N} \omega^2. \]
Let $\bar{\varepsilon} := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. By using (3.17)-(3.19) we have, for every $\varepsilon \in (0, \bar{\varepsilon}]$
\[ \int_{\mathbb{R}^N} V(\varepsilon x) \omega^2 + \frac{t_{\varepsilon}^2}{2} \int_{\mathbb{R}^N} \phi_{\varepsilon \omega} \omega^2 \leq \frac{V_0 + \mu}{2} \|\omega\|_2^2 + \frac{1}{2} T^2 \varepsilon^{\alpha - \theta} C_\varepsilon \|\omega\|^4 \leq \mu \int_{\mathbb{R}^N} \omega^2, \]
from which we infer $I_\varepsilon(t_\varepsilon \omega) \leq E_\mu(t_\varepsilon \omega)$. Then by (2.6) and (3.14),
\[ c_\varepsilon \leq I_\varepsilon(t_\varepsilon \omega) \leq E_\mu(t_\varepsilon \omega) \leq E_\mu(\omega) < m_{V_\infty}^\infty, \]
which concludes the proof in this case.

2. If $V_\infty = \infty$, by Proposition 3.4, $\{u_n\}$ strongly converges to some $u_\varepsilon$ in $H^s(\mathbb{R}^N)$, which satisfies
\[ I_\varepsilon(u_\varepsilon) = c_\varepsilon \ \text{and} \ I'_\varepsilon(u_\varepsilon) = 0. \]
and $u_\varepsilon$ is the ground state we were looking for. \qed
4. Proof of Theorem 1.1

In this Section we introduce the barycenter map in order to study the "topological complexity" of suitable sublevels of the functional $I_\varepsilon$ in the Nehari manifold. Let us start with the following

**Proposition 4.1.** Let $\varepsilon_n \to 0^+$ and $u_n \in N_{\varepsilon_n}$ be such that $I_{\varepsilon_n}(u_n) \to m_{V_0}^\infty$. Then there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + \tilde{y}_n)$ has a convergent subsequence in $H^s(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$.

Recall that $M$ is the set where $V$ achieves the minimum $V_0$.

**Proof.** We begin by showing that $\{u_n\}$ is bounded in $H^s_{V_0}(\mathbb{R}^N)$. By assumptions, $I_{\varepsilon_n}'(u_n)[u_n] = 0$ and $I_{\varepsilon_n}(u_n) \to m_{V_0}^\infty$ write as

$$\|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n} u_n^2 = \int_{\mathbb{R}^N} f(u_n)u_n$$

and

$$\frac{1}{2}\|u_n\|_{\varepsilon_n}^2 + \frac{1}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon_n} u_n^2 - \int_{\mathbb{R}^N} F(u_n) = m_{V_0}^\infty + o_n(1)$$

which combined together give

$$\frac{1}{4} \int_{\mathbb{R}^N} f(u_n)u_n - \int_{\mathbb{R}^N} F(u_n) = \frac{1}{4} \left( \|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n} u_n^2 \right) - \int_{\mathbb{R}^N} F(u_n) \leq m_{V_0}^\infty + o_n(1).$$

Using (4) we get

$$0 \leq \left( \frac{1}{4} - \frac{1}{K} \right) \int_{\mathbb{R}^N} f(u_n)u_n \leq m_{V_0}^\infty + o_n(1),$$

and therefore, coming back to (4.1), for some positive constant $C$ (independent on $n$)

$$\|u_n\|_{H^s_{V_0}} \leq \|u_n\|_{\varepsilon_n} \leq C.$$

We prove the following

**Claim:** there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ and $R, c > 0$ such that $\liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0$.

Indeed, if it were not the case then

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} u_n^2 = 0, \text{ for every } R > 0.$$

By Lemma 3.2, $u_n \to 0$ in $L^p(\mathbb{R}^N)$, for $2 < p < 2^*_s$ and then

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0.$$

Therefore $\|u_n\|_{\varepsilon_n}^2 + \int_{\mathbb{R}^N} \phi_{\varepsilon_n} u_n^2 = o_n(1)$, and also from

$$0 \leq \int_{\mathbb{R}^N} F(u_n) \leq \frac{1}{K} \int_{\mathbb{R}^N} f(u_n)u_n$$

we have $\int_{\mathbb{R}^N} F(u_n) = o_n(1)$. But then $\lim_{n \to \infty} I_{\varepsilon_n}(u_n) = m_{V_0}^\infty = 0$ which is a contradiction and proves our claim.

Then the sequence $v_n := u_n(\cdot + \tilde{y}_n)$ is also bounded in $H^s(\mathbb{R}^N)$ and

$$v_n \to v \not\equiv 0 \text{ in } H^s(\mathbb{R}^N)$$
since

\[
\int_{B_R} v^2 = \liminf_{n \to \infty} \int_{B_R} v_n^2 = \liminf_{n \to \infty} \int_{B_R(\tilde{y}_n)} u_n^2 \geq c > 0,
\]

by the claim.

Let now \( t_n > 0 \) be such that \( \tilde{v}_n := t_nv_n \in \mathcal{M}_{V_0} \); the next step is to prove that

(4.4) \[ E_{V_0}(\tilde{v}_n) \to m_{V_0}^\infty. \]

For this, note that

\[
m_{V_0}^\infty \leq E_{V_0}(\tilde{v}_n) = \frac{1}{2}||\tilde{v}_n||_{V_0}^2 - \int_{\mathbb{R}^N} F(u_n)
\]

\[
= t_n^2 \int_{\mathbb{R}^N} \left[ |(-\Delta)^{s/2}u_n(x + \tilde{y}_n)|^2 + V_0u_n^2(x + \tilde{y}_n) \right] dx - \int_{\mathbb{R}^N} F(t_nu_n(x + \tilde{y}_n)) dx
\]

\[
= t_n^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u_n(x)|^2 dx + \frac{t_n^2}{2} \int_{\mathbb{R}^N} V_0u_n^2(x) dx - \int_{\mathbb{R}^N} F(t_nu_n(x)) dx
\]

\[
= \int_{\mathbb{R}^N} (t_nu_n) \] and then

\[
m_{V_0}^\infty \leq E_{V_0}(\tilde{v}_n) \leq I_{\mathbb{R}^N}(t_nu_n) \leq I_{\mathbb{R}^N}(u_n) = m_{V_0}^\infty + o_0(1)
\]

which proves (4.4).

We can prove now that \( v_n \to v \) in \( H^s(\mathbb{R}^N) \). As in the first part of the proof (where we proved the boundedness of \( \{u_n\} \) in \( H^s_{V_0}(\mathbb{R}^N) \)), it is easy to see that

\[
\{\tilde{v}_n\} \subset \mathcal{M}_{V_0} \quad \text{and} \quad E_{V_0}(\tilde{v}_n) \to m_{V_0}^\infty \implies ||\tilde{v}_n||_{H^s_{V_0}} \leq C
\]

and an analogous claim as before holds for the sequence \( \{\tilde{v}_n\} \). Then \( \tilde{v}_n \to \tilde{v} \) in \( H^s_{V_0}(\mathbb{R}^N) \) and (as before) there exists \( \delta > 0 \) such that

(4.5) \[ 0 < \delta \leq ||\tilde{v}_n||_{H^s_{V_0}}. \]

This implies

\[
0 < t_n\delta \leq ||t_nv_n||_{H^s_{V_0}} = ||\tilde{v}_n||_{H^s_{V_0}} \leq C,
\]

showing that, up to subsequence, \( t_n \to t_0 \geq 0 \). If now \( t_0 = 0 \) using (4.2) we derive

\[
0 \leq ||\tilde{v}_n||_{H^s_{V_0}} = t_n||v_n||_{H^s_{V_0}} \leq t_nC \to 0,
\]

so that \( \tilde{v}_n \to 0 \) in \( H^s_{V_0}(\mathbb{R}^N) \). From this and (4.4) it follows \( m_{V_0}^\infty = 0 \) which is absurd. So \( t_0 > 0 \). Then \( t_nv_n \to t_0\tilde{v} =: \tilde{v} \in H^s(\mathbb{R}^N) \) and by (4.5) \( \tilde{v} \neq 0 \). By Lemma 3.6 applied to \( \{\tilde{v}_n\} \) we get \( \tilde{v}_n \to \tilde{v} \) in \( H^s(\mathbb{R}^N) \) and then \( v_n \to \tilde{v} \). By (4.3) we deduce \( v_n \to v \) and the first part of the proposition is proved.

We proceed to prove the second part. We first state that \( \{y_n\} \) is bounded in \( \mathbb{R}^N \) (here \( y_n = \epsilon_n\tilde{y}_n \) with \( \tilde{y}_n \) given in the above claim). Assume the contrary; then
1. if $V_\infty < \infty$, since $\tilde{v}_n \to \tilde{v}$ in $H^s(\mathbb{R}^N)$ and $V_0 < V_\infty$, we have

$$m_{V_0}^\infty = \frac{1}{2} \|\tilde{v}\|^2_{H^s_{V_0}} - \int_{\mathbb{R}^N} F(\tilde{v}) < \frac{1}{2} \|\tilde{v}\|^2_{H^s_{V_\infty}} - \int_{\mathbb{R}^N} F(\tilde{v})$$

$$\leq \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \tilde{v}_n| dx + \int_{\mathbb{R}^N} \frac{1}{2} V(\varepsilon_n x + y_n) \tilde{v}_n^2(x) dx - \int_{\mathbb{R}^N} F(\tilde{v}_n)$$

$$= \liminf_{n \to \infty} \left( \frac{t_n^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u_n|^2 dx + \frac{t_n^2}{2} \int_{\mathbb{R}^N} V(\varepsilon_n x) u_n^2(x) dx - \int_{\mathbb{R}^N} F(t_n u_n) \right)$$

$$\leq \liminf_{n \to \infty} \left( \frac{1}{2} \|t_n u_n\|^2_{\infty} - \int_{\mathbb{R}^N} F(t_n u_n) + \frac{t_n^4}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon_n, u_n} u_n^2(x) \right)$$

from which

$$m_{V_0}^\infty < \liminf_{n \to \infty} I_{\varepsilon_n} (t_n u_n) \leq \liminf_{n \to \infty} I_{\varepsilon_n} (u_n) = m_{V_0}^\infty$$

which is a contradiction.

2. If $V_\infty = \infty$, we have

$$\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) v_n^2(x) dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} v_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) v_n^2(x) dx$$

$$+ \int_{\mathbb{R}^N} \phi_{\varepsilon_n, v_n} v_n^2(x) dx$$

$$= \int_{\mathbb{R}^N} f(v_n(x)) v_n(x) dx,$$

and by the Fatou’s Lemma we obtain the absurd

$$\infty = \liminf_{n \to \infty} \int_{\mathbb{R}^N} f(v_n) v_n = \int_{\mathbb{R}^N} f(v) v.$$

Then \{y_n\} has to be bounded and we can assume $y_n \to y \in \mathbb{R}^N$. If $y \notin M$ then $V_0 < V(y)$, and similarly to the computation made in case 1. above (simply replace $V_\infty$ with $V(y)$) we have a contradiction. Hence $y \in M$ and the proof is thereby complete.

For $\delta > 0$ (later it will be fixed conveniently) let $\eta$ be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that

$$\eta(s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq \delta/2 \\
0 & \text{if } s \geq \delta.
\end{cases}$$

Let $w_{V_0}$ be a ground state solution given in Lemma 3.6 of problem $(A_\mu)$ with $\mu = V_0$ and for any $y \in M$, let us define

$$\Psi_{\varepsilon, y}(x) := \eta(|\varepsilon x - y|) w_{V_0}(\frac{\varepsilon x - y}{\varepsilon}).$$

Let $t_\varepsilon > 0$ verifying $\max_{t \geq 0} I_\varepsilon(t \Psi_{\varepsilon, y}) = I_\varepsilon(t_\varepsilon \Psi_{\varepsilon, y})$, so that $t_\varepsilon \Psi_{\varepsilon, y} \in N_\varepsilon$, and let

$$\Phi_\varepsilon : y \in M \mapsto t_\varepsilon \Psi_{\varepsilon, y} \in N_\varepsilon.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$ and it is easy to see that $\Phi_\varepsilon$ is a continuous map.

The next result will help us to define a map from $M$ to a suitable sublevel in the Nehari manifold.

**Lemma 4.2.** The function $\Phi_\varepsilon$ satisfies

$$\lim_{\varepsilon \to 0^+} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0}^\infty \text{ uniformly in } y \in M.$$
Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ such that
\begin{equation}
|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0}^\infty| \geq \delta_0.
\end{equation}
Using Lebesgue’s Theorem, we have
\begin{equation}
\lim_{n \to \infty} \|\Psi_{\varepsilon_n,y_n}\|_{H_{V_0}^\varepsilon}^2 = \|\Psi_{V_0}\|_{H_{V_0}^\varepsilon}^2,
\end{equation}
\begin{equation}
\lim_{n \to \infty} \int_{\mathbb{R}^N} F(\Psi_{\varepsilon_n,y_n}) = \int_{\mathbb{R}^N} F(\Psi_{V_0}),
\end{equation}
\begin{equation}
\lim_{n \to \infty} \|\Psi_{\varepsilon_n,y_n}\|_{H_{V_0}^\varepsilon}^2 = \|\Psi_{V_0}\|_{H_{V_0}^\varepsilon}^2.
\end{equation}
This last convergence implies that $\{\|\Psi_{\varepsilon_n,y_n}\|\}$ is bounded. From (2.3)
\begin{equation}
\int_{\mathbb{R}^N} \phi_{\varepsilon_n,y_n} \Psi_{\varepsilon_n,y_n}^2 \phi_{\varepsilon_n,y_n} \leq \varepsilon_n^{-\theta} C \varepsilon_n \|\Psi_{\varepsilon_n,y_n}\|^4,
\end{equation}
and then
\begin{equation}
\lim_{n \to \infty} \int_{\mathbb{R}^N} \phi_{\varepsilon_n,y_n} \Psi_{\varepsilon_n,y_n}^2 = 0.
\end{equation}
Remembering that $\{t_{\varepsilon_n,y_n}\} \subset N_{\varepsilon_n}$ (see few lines before the Lemma), the condition $I'_{\varepsilon_n}(t_{\varepsilon_n,y_n})[t_{\varepsilon_n,y_n}] = 0$ means
\begin{equation}
\|\Psi_{\varepsilon_n,y_n}\|_{H_{V_0}^\varepsilon}^2 + t_{\varepsilon_n} \int_{\mathbb{R}^N} \phi_{\varepsilon_n,y_n} \Psi_{\varepsilon_n,y_n}^2 = \int_{\mathbb{R}^N} \frac{f(t_{\varepsilon_n,y_n})}{t_{\varepsilon_n,y_n}} \Psi_{\varepsilon_n,y_n}^2.
\end{equation}
We now prove the following
\textbf{Claim:} $\lim_{n \to \infty} t_{\varepsilon_n} = 1$.
We begin by showing the boundedness of $\{t_{\varepsilon_n}\}$. Since $\varepsilon_n \to 0^+$, we can assume $\delta/2 < \delta/(2\varepsilon_n)$ and then from (4.9), using (f5) and making the change of variable $z := (\varepsilon_n x - y_n)/\varepsilon_n$, we get
\begin{equation}
\frac{\|\Psi_{\varepsilon_n,y_n}\|^2_{H_{V_0}^\varepsilon}}{t_{\varepsilon_n}^2} + \int_{\mathbb{R}^N} \phi_{\varepsilon_n,y_n} \Psi_{\varepsilon_n,y_n}^2 \geq \int_{\mathbb{R}^N} \frac{f(t_{\varepsilon_n,y_n})}{t_{\varepsilon_n,y_n}} \Psi_{\varepsilon_n,y_n}^2,
\end{equation}
where $w_{V_0}(z) := \min_{B_{\delta/2}} w_{V_0}(z)$. If $\{t_{\varepsilon_n}\}$ were unbounded, passing to the limit in $n$ in (4.10), the left hand side would tend to 0 (due to (4.7) and (4.8)), the right hand side to $+\infty$ (due to (f4)). So we can assume that $t_{\varepsilon_n} \to t_0 \geq 0$.
For given $\xi > 0$, by (2.1), there exists $M_\xi > 0$ such that
\begin{equation}
\int_{\mathbb{R}^N} \frac{f(t_{\varepsilon_n},\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}} \Psi_{\varepsilon_n,y_n}^2 \leq \xi \int_{\mathbb{R}^N} \Psi_{\varepsilon_n,y_n}^2 + M_\xi t_{\varepsilon_n}^{-1} \int_{\mathbb{R}^N} \Psi_{\varepsilon_n,y_n}^{q+1}.
\end{equation}
Since $\{\Psi_{\varepsilon_n,y_n}\}$ is bounded in $H^s(\mathbb{R}^N)$, if $t_0 = 0$, from (4.11) we deduce
\begin{equation}
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(t_{\varepsilon_n},\Psi_{\varepsilon_n,y_n})}{t_{\varepsilon_n}} \Psi_{\varepsilon_n,y_n}^2 = 0,
\end{equation}
which joint with (4.8) and (4.9) led to $\lim_{n \to \infty} \|\Psi_{\varepsilon_n,y_n}\|_{H_{V_0}^\varepsilon}^2 = 0$ contradicting (4.7). Then $t_{\varepsilon_n} \to t_0 > 0$. Now taking the limit in $n$ in (4.9) we arrive at
\begin{equation}
\|w_{V_0}\|_{H_{V_0}^\varepsilon}^2 = \int_{\mathbb{R}^N} \frac{f(t_0 w_{V_0})}{t_0} w_{V_0},
\end{equation}
and since $w_{V_0} \in M_{V_0}$, it has to be $t_0 = 1$, which proves the claim.
Finally, note that
\[
I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = \frac{t_{\varepsilon_n}^2}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} \Psi_{\varepsilon_n,y_n}|^2 \, dx + \frac{t_{\varepsilon_n}^2}{4} \int_{\mathbb{R}^N} V(x) \Psi_{\varepsilon_n,y_n}^2 \, dx + \frac{t_{\varepsilon_n}^4}{4} \int_{\mathbb{R}^N} \phi_{\varepsilon_n,\Psi_{\varepsilon_n,y_n}} \Psi_{\varepsilon_n,y_n}^2 \, dx - \int_{\mathbb{R}^N} F(t_{\varepsilon_n} \Psi_{\varepsilon_n,y_n}).
\]
and then (by using the claim) \( \lim_{n \to \infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = E_{V_0}(w_{V_0}) = m_{V_0}^\infty \), which contradicts (4.6). Thus the Lemma holds. □

The remaining part of the paper mainly follows the arguments of [24]. By Lemma 4.2, \( h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty| = o(1) \) for \( \varepsilon \to 0^+ \) uniformly in \( y \), and then \( I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0}^\infty \leq h(\varepsilon) \). In particular the sublevel set in the Nehari manifold
\[
\mathcal{N}_{\varepsilon}^{m_{V_0}^\infty + h(\varepsilon)} := \{ u \in \mathcal{N}_{\varepsilon} : I_\varepsilon(u) \leq m_{V_0}^\infty + h(\varepsilon) \}
\]
is not empty, since for sufficiently small \( \varepsilon \),
\[
(4.12) \quad \forall y \in M : \Phi_\varepsilon(y) \in \mathcal{N}_{\varepsilon}^{m_{V_0}^\infty + h(\varepsilon)}.
\]

From now on we fix a \( \delta > 0 \) in such a way that \( M_{2\delta} \) and \( M_{2\delta} := \{ x \in \mathbb{R}^N : d(x, M) \leq 2\delta \} \) are homotopically equivalent (\( d \) denotes the euclidean distance). Take a \( \rho = \rho(\delta) > 0 \) such that \( M_{2\delta} \subset B_\rho \) and \( \chi : \mathbb{R}^N \to \mathbb{R}^N \) be defined as follows
\[
\chi(x) = \begin{cases} 
  x & \text{if } |x| \leq \rho \\
  \rho \frac{x}{|x|} & \text{if } |x| \geq \rho.
\end{cases}
\]
Define the barycenter map \( \beta_\varepsilon \)
\[
\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^N} \chi(x) u^2(x) \, dx}{\int_{\mathbb{R}^N} u^2(x) \, dx} \in \mathbb{R}^N
\]
for all \( u \in W_\varepsilon \) with compact support.

We will take advantage of the following results (see [24, Lemma 8 and 9]).

**Lemma 4.3.** The function \( \beta_\varepsilon \) satisfies
\[
\lim_{\varepsilon \to 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \text{ uniformly in } y \in M.
\]

**Lemma 4.4.** We have
\[
\lim_{\varepsilon \to 0^+} \sup_{u \in \mathcal{N}_{\varepsilon}^{m_{V_0}^\infty + h(\varepsilon)}} \inf_{y \in M_\delta} |\beta_\varepsilon(u) - y| = 0.
\]

In virtue of Lemma 4.4, there exists \( \varepsilon^* > 0 \) such that
\[
\forall \varepsilon \in (0, \varepsilon^*) : \sup_{u \in \mathcal{N}_{\varepsilon}^{m_{V_0}^\infty + h(\varepsilon)}} d(\beta_\varepsilon(u), M_\delta) < \delta/2.
\]

Define now
\[
M^+ := \big\{ x \in \mathbb{R}^N : d(x, M) \leq 3\delta/2 \big\}
\]
so that \( M \) and \( M^+ \) are homotopically equivalent.
Now, reducing $\varepsilon^* > 0$ if necessary, we can assume that Lemma 4.3, Lemma 4.4 and (4.12) hold. Then by standard arguments the composed map

$$M \xrightarrow{\Phi_{\varepsilon}} \mathcal{N}_{\varepsilon}^{\infty_{0} + h(\varepsilon)} \xrightarrow{\beta_{\varepsilon}} M^+$$

is homotopic to the inclusion map.

In case $V_{\infty} < \infty$, we eventually reduce $\varepsilon^*$ in such a way that also the Palais-Smale condition is satisfied in the interval $(m_{\infty_{0}}, m_{\infty_{0}}^\infty + h(\varepsilon))$, see Proposition 3.5. By well known properties of the category, it is

$$\text{cat}(\mathcal{N}_{\varepsilon}^{m_{\infty_{0}} + h(\varepsilon)}) \geq \text{cat}_{M^+}(M)$$

and the Ljusternik-Schnirelman theory ensures the existence of at least $\text{cat}_{M^+}(M) = \text{cat}(M)$ constraint critical points of $I_{\varepsilon}$ on $\mathcal{N}_{\varepsilon}$. The proof of the main Theorem 1.1 then follows by Proposition 3.5.

If $M$ is bounded and not contractible in itself, then the existence of another critical point of $I_{\varepsilon}$ on $\mathcal{N}_{\varepsilon}$ follows from some ideas in [11]. We recall here the main steps for completeness.

The goal is to exhibit a subset $A \subset \mathcal{N}_{\varepsilon}$ such that

i) $A$ is not contractible in $\mathcal{N}_{\varepsilon}^{m_{\infty_{0}} + h(\varepsilon)}$

ii) $A$ is contractible in $\mathcal{N}_{\varepsilon}^\infty = \{ u \in \mathcal{N}_{\varepsilon} : I_{\varepsilon}(u) \leq \bar{c} \}$, for some $\bar{c} > m_{\infty_{0}}^\infty + h(\varepsilon)$.

This would imply, since the Palais-Smale holds, that there is a critical level between $m_{\infty_{0}}^\infty + h(\varepsilon)$ and $\bar{c}$.

First note that when $M$ is not contractible and bounded the compact set $A := \Phi_{\varepsilon}(M)$ can not be contractible in $\mathcal{N}_{\varepsilon}^{m_{\infty_{0}} + h(\varepsilon)}$, proving i).

Let us denote, for $u \in W_{\varepsilon} \setminus \{0\}$, with $t_{\varepsilon}(u) > 0$ the unique positive number such that $t_{\varepsilon}(u)u \in \mathcal{N}_{\varepsilon}$. Choose a function $u^* \in W_{\varepsilon}$ be such that $u^* \geq 0$, $I_{\varepsilon}(t_{\varepsilon}(u^*)u^*) > m_{\infty_{0}}^\infty + h(\varepsilon)$ and consider the compact and contractible cone

$$\mathcal{C} := \left\{ tu^* + (1 - t)u : t \in [0, 1], u \in A \right\}.$$

Observe that, since the functions in $\mathcal{C}$ have to be positive on a set of nonzero measure, it is $0 \notin \mathcal{C}$. Now we project this cone on $\mathcal{N}_{\varepsilon}$: let

$$t_{\varepsilon}(\mathcal{C}) := \left\{ t_{\varepsilon}(w) : w \in \mathcal{C} \right\} \subset \mathcal{N}_{\varepsilon}$$

and set

$$\tau := \max_{t_{\varepsilon}(\mathcal{C})} I_{\varepsilon} > m_{\infty_{0}}^\infty + h(\varepsilon)$$

(indeed the maximum is achieved being $t_{\varepsilon}(\mathcal{C})$ compact). Of course $A \subset t_{\varepsilon}(\mathcal{C}) \subset \mathcal{N}_{\varepsilon}$ and $t_{\varepsilon}(\mathcal{C})$ is contractible in $\mathcal{N}_{\varepsilon}^\infty$: we deduce ii).

Then there is a critical level for $I_{\varepsilon}$ greater than $m_{\infty_{0}}^\infty + h(\varepsilon)$, hence different from the previous ones we have found. The proof of Theorem 1.1 is complete.

References

[1] C.O. Alves, P.C. Carrião and E.S. Medeiros, *Multiplicity of solutions for a class of quasilinear problem in exterior domains with Neumann conditions*, Abstract and Applied Analysis 03 (2004), 251-268. 12

[2] C.O. Alves and G.M. Figueiredo, *Multiple solutions for a quasilinear Schrödinger equation on \( \mathbb{R}^N \)*, Acta Appl. Math. 136 (2015), 91–117. 3

[3] C.O. Alves and S.H.M. Soares, *Multiplicity of positive solutions for a class of nonlinear Schrödinger equations*, Adv. Differential Equations 15 (2010), no. 11-12, 1083–1102. 3

[4] C.O. Alves and G.M. Figueiredo, *Existence and multiplicity of positive solutions to a p-Laplacian equation in \( \mathbb{R}^N \)*, Differential and Integral Equations 19 (2006)143–162. 3, 12
[5] C.O. Alves and G.M. Figueiredo, *Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field*, Milan J. Math. **82** (2014), no. 2, 389–405. 3

[6] C.O. Alves and G.M. Figueiredo, *Multiplicity of solutions for a NLS equations with magnetic fields in $\mathbb{R}^N$*, Monatsh. Math. **175** (2014), no. 1, 1–23. 3

[7] C.O. Alves and G.M. Figueiredo, *Multiplicity of nontrivial solutions to a biharmonic equation via Lusternik-Schnirelman theory*, Math. Methods Appl. Sci. **36** (2013), no. 6, 683–694. 3

[8] C.O. Alves, G.M. Figueiredo and U.B. Severo, *A result of multiplicity of solutions for a class of quasilinear equations*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 2, 291–309. 3

[9] C.O. Alves, G.M. Figueiredo and U.B. Severo, *Multiplicity of positive solutions for a class of quasilinear problems*, Adv. Differential Equations **14** (2009), 911–942. 3

[10] V. Benci and G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rat. Mech. Anal. **114** (1991), 79–83. 3

[11] V. Benci, G. Cerami and D. Passaseo, *On the number of positive solutions of some nonlinear elliptic problems*, Nonlinear Anal., Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, (1991), 93–107. 3, 20

[12] V. Benci and D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Top. Methods Nonlinear Anal. **11** (1998), 283–293. 1

[13] V. Benci and D. Fortunato, *Solitons in Schrödinger-Maxwell equations*, J. Fixed Point Theory Appl. **15** no. 1, (2014), 101–132. 1

[14] L. Caffarelli, J.M. Roquejoffre and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Apple. Math., **63** (2012), 1111–1144. 2

[15] L. Caffarelli and L.E. Silvestre, *An extension problem related to the fractional laplacian*, Comm. in Partial Differential Equations **32** (2007), 1245–1260. 2

[16] L. Caffarelli and E. Valdinoci, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Diff. Equations **32** (2007), 1245–1260. 2

[17] S.-Y. A. Chang and M. del Mar González, *Fractional Laplacian in conformal geometry*, Adv. Math. **226** (2011), 1410–1432. 2

[18] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Top. Methods Nonlinear Anal. **10** (1997), 1–13. 3

[19] S. Cingolani and M. Lazzo, *Multiple Positive Solutions to Nonlinear Schrödinger Equations with Competing Potential Functions*, J. Diff. Equations **160** (2000), 118–138. 3

[20] R. Cont and P. Tankov, *Financial modeling with jump processes*, Chapman&Hall/CRC Financial Mathematics Series, Boca Raton, FL, 2004. 2

[21] P. d’Avenia, G. Siciliano and M. Squassina, *On fractional Choquard equations*, Math. Models Methods Appl. Sci. **25** (2015), 1447–1476. 9

[22] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker’s guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 512–573. 4

[23] Y. Fang and J. Zhang, *Multiplicity of solutions for the nonlinear Schrödinger-Maxwell system*, Communications on Pure and Applied Analysis **10** (2011), 1267–1279. 1, 6

[24] G.M. Figueiredo and G. Siciliano *A multiplicity result via Ljusternick-Schnirelmann category and Morse theory for a fractional Schrödinger equation in $\mathbb{R}^N$*, NoDEA, to appear (arXiv:1502.01243v1). 3, 13, 19

[25] G.M. Figueiredo, Marcos T. Pimenta and G. Siciliano, *Multiplicity results for the fractional laplacian in expanded domains*, arXiv:1511.09406. 3

[26] O. Kavian, Introduction à la Théorie des Points Critiques, Springer-Verlag 1993. 7

[27] N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Rev. E* **66** (2002), 056108. 2

[28] R. Metzler and J. Klafter, *The random walks guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), 1–77. 2

[29] R. Metzler, J. Klafter, *The restaurant at the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A **37** (2004), 161–208. 2

[30] E. Milakis and L. Silvestre, *Regularity for the nonlinear Signorini problem*, Adv. Math., **217** (2008), 1301–1312. 2

[31] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere*, Math. Models Methods Appl. Sci. **15**, no. 1 (2005) 141–164. 1

[32] G. Siciliano, *Multiple positive solutions for a Schrödinger-Poisson-Slater system*, J. Math. Anal. Appl. **365**, (2010), 288–299. 1, 2
[35] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. 13 (1960), 457–468.

[36] Z. Wei, *Existence of infinitely many solutions for the fractional Schrödinger-Maxwell equations*, arXiv:1508.03088v1.

[37] J. Zhang, *Existence and multiplicity results for the fractional Schrödinger-Poisson systems*, arXiv:1507.01205v1.

(E. G. Murcia, G. Siciliano)

Departamento de Matemática
Instituto de Matemática e Estatística
Universidade de São Paulo
Rua do Matão 1010, 05508-090 São Paulo, SP, Brazil

E-mail address: edwingmr@ime.usp.br
E-mail address: sicilian@ime.usp.br