ASYMPTOTIC CURVATURE ESTIMATE FOR STEADY SOLITONS

DAOYUAN HAN

Abstract. In this note, we shall investigate the asymptotic curvature estimate on steady Ricci solitons.

1. Main Theorems

Theorem 1.1. Let \((M, g, f)\) be a complete \(n\)-dimensional \(\kappa\)-noncollapsed steady soliton with nonnegative sectional curvature and positive Ricci curvature. Assuming scalar curvature \(R(x) \to 0\) as \(d(x) \to \infty\) uniformly with respect to distance, then

\[
\liminf_{d(x) \to \infty} R(x)d(x)^\alpha = 0,
\]

for all \(\alpha \in (0, 1)\), where \(d(x)\) is the distance from \(x \in M\) to a fixed point \(x_0 \in M\) and \(C\) is a constant independent of \(x\).

Under the additional assumption

\[
\text{Ric}(\nabla f, \nabla f) \geq \frac{C}{d(x)^2} \quad \text{as} \quad d(x) \to \infty
\]

for some constant \(C > 0\) and

\[
\lim_{d(x) \to \infty} R(x)d(x)^\alpha \quad \text{exists for some} \quad \alpha \in \left[\frac{4}{5}, 1\right),
\]

we can prove the following result.

Theorem 1.2. Let \((M, g, f)\) be a complete \(n\)-dimensional \(\kappa\)-noncollapsed steady soliton with nonnegative sectional curvature and positive Ricci curvature. Assuming scalar curvature \(R(x) \to 0\) as \(d(x) \to \infty\) uniformly with respect to distance, (1.2) and (1.3) then \(R(x)\) must decay at least linearly with respect to \(d(x)\), namely

\[
R(x) \leq \frac{C}{d(x)}.
\]

where \(d(x)\) is the distance from \(x \in M\) to a fixed point \(x_0 \in M\) and \(C\) is a constant independent of \(x\).

Remark 1.3. By Proposition 1.1, the condition (1.3) implies

\[
\lim_{d(x) \to \infty} R(x)d(x)^\alpha = 0 \quad \text{exists for some} \quad \alpha \in \left[\frac{4}{5}, 1\right).
\]

The condition (1.3) can be viewed as an asymptotic control of the curvature.
Remark 1.4. Note that by the result in [17], under the non-collapsing condition, the scalar curvature can’t decay faster than linear, by the uniform decaying condition, we can write

\begin{equation}
R(x) \geq \frac{C'}{d(x)}
\end{equation}

for some constant $C' > 0$, independent of $x$.

2. Preliminary

$(M, g, f)$ is called a gradient Ricci soliton if $\text{Ric}$ of $g$ satisfies

\begin{equation}
\text{Ric} = \nabla^2 f.
\end{equation}

and $f$ is a smooth function on $M$. In this note, we always assume the steady Ricci soliton has positive Ricci curvature. We also assume that $R$ attains a maximum at some point $p$ on $M$, so by the identity (after normalization)

\begin{equation}
|\nabla f|^2 + R = 1.
\end{equation}

we know $p$ is a critical point of $f$ and by the assumption that $\text{Ric} > 0$, $p$ is the unique critical point of $f$.

3. Proof of Theorem 1.1

First, we list a few well-known results for steady solitons valid in higher dimensions.

We also need the following result on the lower bound of volume growth of steady soliton in [20].

Theorem 3.1. [20] If $(M, g)$ is a complete gradient steady Ricci soliton there exists uniform constant $C$ so that for any $r > r_0$

\begin{equation}
\text{Vol}(B_p(r)) \geq Cr
\end{equation}

By the equivalence of $f(x)$ and distance function in [11], we have, under the same conditions as in the Theorem 3.1 above, the following volume lower bound on the sub-level set, $\Sigma_{\leq r} := \{x \in M | f(x) \leq r\}$.

Corollary 3.2.

\begin{equation}
\text{Vol}(\Sigma_{\leq r}) \geq Cr
\end{equation}

Proof. We use the equivalence of potential function and distance function from [11]. Let $C_1, C_2$ be constants s.t.

\begin{equation}
C_1 d(x) \leq f(x) \leq C_2 d(x)
\end{equation}

and then we have the inclusion $B_p(r) \subset \Sigma_{\leq C_2 r}$, which implies that

\begin{equation}
\text{Vol}(\Sigma_{\leq C_2 r}) \geq \text{Vol}(B_p(r)) \geq Cr
\end{equation}

\[\square\]

Proposition 3.3. Let $(M, g, f)$ be a complete $n$-dimensional $\kappa$-noncollapsed on all scales steady soliton with $\text{Ric} > 0$, and with scalar curvature $R$ decaying like $r^{-\alpha}$, $\alpha \in (0, 1)$ when $r$ is sufficiently large, namely

\begin{equation}
C_1 r^{-\alpha} \leq R \leq C_2 r^{-\alpha},
\end{equation}

\[\square\]
then we have

\[(3.4)\quad \int_{\Sigma \leq r} R \, d\mu \geq C r^{1-\alpha} \text{ for } r \text{ sufficiently large,}\]

where \(C\) denotes a positive constant depending on \(\alpha\).

**Proof.** Using the result in [20] on the lower bound of volume growth and Corollary 3.2, let \(r_1\) be such that Corollary 3.2 and (3.3) hold, then we know, by coarea formula,

\[
\int_{\Sigma \leq r} R \, dV - \int_{\Sigma \leq r_1} R \, dV = \int_{\{x | r_1 \leq f(x) \leq r\}} R \, dV \geq \int_{r_1}^r \int_{\Sigma_s} R \, d\mu \, ds \geq C r^{1-\alpha},
\]

for \(r\) sufficiently large. The claim follows from the above. \(\square\)

The left hand side of the above is by the computation in [20]

\[(3.5)\quad \int_{\Sigma \leq r} R \, dV = \int_{\Sigma_r} \Delta f \, dV \leq \int_{\Sigma_r} \mid \nabla f \mid \, d\mu \leq Vol(\Sigma_r)
\]

From the above estimate, we have the following estimate.

**Corollary 3.4.** Under the same conditions as in Proposition 3.3, we have

\[(3.6)\quad Vol(\Sigma_r) \geq C r^{1-\alpha}, \text{ for } r \text{ sufficiently large.}\]

where \(C > 0\) is a constant depending on \(\alpha\).

By the above corollary, we have the following lower bound of \(Vol(\Sigma_{\leq r})\),

**Proposition 3.5.** Under the same conditions as in Proposition 3.3, we have

\[(3.7)\quad Vol(\Sigma_{\leq r}) \geq C r^{2-\alpha}, \text{ for } r \text{ sufficiently large.}\]

**Proof.** By the coarea formula, we have

\[
Vol(\Sigma_{\leq r}) \geq \int_0^r \int_{\Sigma_s} d\mu = \int_{r_0}^r \int_{\Sigma_s} d\mu \, ds + \int_0^{r_0} \int_{\Sigma_s} d\mu \, ds,
\]

where \(r_0\) is chosen so that Corollary 3.3 holds. By choosing \(r\) sufficiently large, we have

\[
Vol(\Sigma_{\leq r}) \geq C r^{2-\alpha}
\]

where \(C > 0\) and depends on \(\alpha, r_0\). \(\square\)

We will use the above results to prove Proposition 1.1. The proof is by contradiction and iteration of the above process.

**Proof.** Suppose by contradiction that \((M, g)\) has the limit

\[
\liminf_{d(x) \to \infty} R(x) d(x)^\alpha = C > 0.
\]

We apply the volume estimate (3.7) back to the formula (3.3). Choose \(r_1 > 0\) s.t. Proposition 3.5 holds, we have

\[(3.8)\quad \int_{\Sigma \leq r} R \, d\mu \geq C r^{2-2\alpha},\]

for \(r \geq r_1\).

By the same proof as in (3.5), we get

\[(3.9)\quad Vol(\Sigma_r) \geq C r^{2-2\alpha}.
\]
Now we iterate the process (3.8-3.9), then after \( k \) iterations, we have
\[
V o l(\Sigma_r) \geq C r^{k(1-\alpha)}, \text{ for } r \text{ sufficiently large.}
\]
So when \( k \) is sufficiently large, we get
\[
V o l(\Sigma_r) \geq C r^n, \text{ for } r \text{ sufficiently large.}
\]
Using the equivalence of potential function and distance function from [11]. Let
\[ C'_1, C'_2 \] be constants s.t.
\[
 C'_1 f(x) \leq d(x) \leq C'_2 f(x)
\]
and then we have the inclusion \( \Sigma_{\leq C'_1 r} \subset B_p(r) \), which implies that
\[
V o l(B_p(r)) \geq V o l(\Sigma_{\leq C'_1 r}) \geq C r^n,
\]
which contradicts the result that asymptotic volume ratio equals 0 for \( \kappa \)-noncollapsed steady solitons.

\[ \square \]

4. Proof of Theorem 1.2

The proof follows similar ideas of choosing barrier functions as in the work of Chan [9]. We first show that the function
\[
e^{-1/R^2} \]
is subharmonic with respect to the weighted Laplacian associated to the function \( f \).

\textbf{Lemma 4.1.} Let \((M, g, f)\) be a complete gradient steady soliton satisfying the conditions in Theorem 1.2. Then the function \( e^{-1/R^2} \) satisfies
\[
\Delta f(e^{-1/R^2}) \geq 0 \text{ for } d(x) \text{ sufficiently large.}
\]

\textbf{Proof.} The proof follows from direct computation. Firstly,
\[
\Delta f(e^{-1/R^2}) = e^{-1/R^2} (4R^{-6} - 6R^{-4})|\nabla R|^2 + 2e^{-1/R^2} \cdot R^{-3} \cdot \Delta f R
\]
\[
\geq e^{-1/R^2} (4R^{-6} - 6R^{-4})|\nabla R|^2 - 2e^{-1/R^2} \cdot R^{-1}
\]
\[
= e^{-1/R^2} R^{-6} \left[ 4 - 6R^2 |\nabla f|^2 - 2R^5 \right]
\]
\[
(4.2)
\]
\[
\geq e^{-1/R^2} R^{-6} \left[ 4(4 - 6R^2) |\nabla f|^2 - 2R^5 \right]
\]
\[
\geq e^{-1/R^2} R^{-6} \left[ 4(4 - 6R^2) f(x)^{-4} - cf(x)^{-5\alpha} \right],
\]
where in the last inequality we use Theorem 1.1, the condition 112 and the condition 113. By choosing \( \alpha \in [4/5, 1) \), we know that
\[
\Delta f(e^{-1/R^2}) \geq 0,
\]
when \( f(x) \) is sufficiently large.

\[ \square \]

Next we show that \( e^{-f^2(x)} \) is supersolution of \( \Delta f \).

\textbf{Lemma 4.2.} Let \((M, g, f)\) be a complete gradient steady soliton satisfying the conditions in Theorem 1.2. Then the function \( e^{-f^2} \) satisfies
\[
\Delta f(e^{-f^2}) \leq 0.
\]
Proof. The proof follows from direct computation.
\[
\Delta_f(e^{-f^2}) = (4e^{-f^2} \cdot f^2 - 2e^{-f^2})|\nabla f|^2 - 2e^{-f^2} \cdot f \cdot \Delta f
\]
\[
= 2e^{-f^2} \left[(2f^2 - 1)|\nabla f|^2 - f\right]
\]
\[
\leq 0,
\]
for \(d(x)\) sufficiently large since \(f(x)\) is equivalent to \(d(x)\) when \(d(x)\) is sufficiently large. And in the second line we use the fact that \(\Delta f = 1\) with respect to the normalization in (2.2).

Now we use the above lemmas to prove Theorem 1.2.
Proof. Using Lemma 4.1 and Lemma 4.2 we can choose a large ball \(B(R_0)\) such that, on \(M \setminus B(R_0)\),

\[\Delta_f(e^{-1/R^2}) \geq 0\]

and

\[\Delta_f(e^{-f^2}) \leq 0\]

Pick a large \(b > 0\) such that on \(\partial B(R_0)\),

\[S(x) := e^{-1/R^2} - be^{-f^2} < 0.\]

By the fact that \(\lim_{f(x) \to \infty} S(x) \to 0\) and

\[\Delta_f S(x) \geq 0,\]

we use maximum principle to get

\[e^{-1/R^2} - be^{-f^2} \leq 0,\]

on \(M \setminus B(R_0)\), which implies

\[R(x) \leq \frac{C}{f(x)},\]

for \(C > 0\) on \(M \setminus B(R_0)\). Using the equivalence of \(f(x)\) and \(d(x)\) in [11],

\[R(x) \leq \frac{C''}{d(x)}.\]

References

[1] Yuxing Deng, Xiaohua Zhu. Classification of gradient steady Ricci solitons with linear curvature decay. [arXiv:1809.08502]

[2] Yuxing Deng, Xiaohua Zhu. Three-dimensional steady gradient Ricci solitons with linear curvature decay. Int. Math. Res. Not. IMRN 2019, no. 4, 1108-1124.

[3] Yuxing Deng, Xiaohua Zhu. Higher dimensional steady Ricci solitons with linear curvature decay. [arXiv:1710.07815]

[4] Andrew S.Dancer, McKenzie Y.Wang. Some new examples of non-kaehler ricci solitons. Math. Res. Lett. 16 (2009), no. 2, 349-363.

[5] Simon Brendle. Rotational symmetry of self-similar solutions to the Ricci flow. Invent. Math. 194 (2013), no. 3, 731-764.

[6] Simon Brendle. Ancient solutions to the Ricci flow in dimension 3. [arXiv:1811.02559v2]

[7] Pengfei Guan, Peng Lu, Yiyan Xu. A rigidity theorem for codimension one shrinking gradient Ricci solitons in \(\mathbb{R}^{n+1}\). Calc. Var. Partial Differential Equations 54 (2015), no. 4, 4019-4036.

[8] Binglong Chen. Strong uniqueness of the Ricci flow. J. Differential Geom. 82 (2009), no. 2, 363-382.
[9] Pak Yeung Chan. *Curvature estimates for steady Ricci solitons.* TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY Volume 372, Number 12, 15 December 2019, Pages 89859008

[10] Thomas Ivey *New examples of complete Ricci solitons.* Proc. Amer. Math. Soc. 122 (1994), no. 1, 241-245.

[11] Huaidong Cao, Qiang Chen. *On Locally Conformally Flat Gradient Steady Ricci Solitons.* Trans. Amer. Math. Soc. 364 (2012), no. 5, 2377-2391.

[12] Hongxin Guo. *Area growth rate of the level surface of the potential function on the 3-dimensional steady gradient Ricci soliton.* Proc. Amer. Math. Soc. 137 (2009), no. 6, 2093-2097.

[13] Hongxin Guo. *Remarks on noncompact steady gradient Ricci solitons.* Math. Ann. 345 (2009), no. 4, 883894.

[14] Jeff Cheeger, Aaron Naber. *Regularity of Einstein manifolds and the codimension 4 conjecture.* Ann. of Math. (2) 182 (2015), no. 3, 1093-1165.

[15] Alix Deruelle. *Steady gradient Ricci soliton with curvature in L1.* Comm. Anal. Geom. 20 (2012), no. 1, 31-53.

[16] R. E. Greene, H. Wu. *Gap theorems for noncompact Riemannian manifolds.* Duke Math. J. 49 (1982), no. 3, 731-756.

[17] Ovidiu Munteanu, Chiung-Jue Anna Sung, Jiangping Wang. *Poisson equation on complete manifolds.* Adv. Math. 348 (2019), 81145.

[18] Bennett Chow, Peng Lu, Lei Ni. *Hamilton’s Ricci Flow.* Graduate Studies in Mathematics Volume 77, AMS Science Press.

[19] Huai-dong Cao, Xi-Ping Zhu. *A complete proof of the Poincare and geometrization conjecturesapplication of the Hamilton-Perelman theory of the Ricci flow.* Asian J. Math. 10 (2006), no. 2, 165492.

[20] Ovidiu Munteanu, Natasa Sesum. *On Gradient Ricci Solitons.* J Geom Anal 23, 539561 (2013).

DEPARTMENT OF MATHEMATICS, LEHIGH UNIVERSITY, BETHLEHEM, PENNSYLVANIA 18015

Current address: Department of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015

E-mail address: dah517@lehigh.edu