Asymptotically exact heuristics for prime divisors of the sequence \( \{a^k + b^k\}_{k=1}^{\infty} \)

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Abstract

Let \( N_{a,b}(x) \) count the number of primes \( p \leq x \) with \( p \) dividing \( a^k + b^k \) for some \( k \geq 1 \). It is known that \( N_{a,b}(x) \sim c(a,b)x/\log x \) for some rational number \( c(a,b) \) that depends in a rather intricate way on \( a \) and \( b \). A simple heuristic formula for \( N_{a,b}(x) \) is proposed and it is proved that it is asymptotically exact, i.e. has the same asymptotic behaviour as \( N_{a,b}(x) \). Connections with Ramanujan sums and character sums are discussed.

1 Introduction

Let \( p \) be a prime (indeed, throughout this note the letter \( p \) will be used to indicate primes). Let \( g \) be a non-zero rational number. By \( \nu_p(g) \) we denote the exponent of \( p \) in the canonical factorisation of \( g \). If \( \nu_p(g) = 0 \), then by \( \text{ord}_p(g) \) we denote the smallest positive integer \( k \) such that \( g^k \equiv 1 \pmod{p} \). If \( k = p - 1 \), then \( g \) is said to be a \textit{primitive root} mod \( p \). If \( g \) is a primitive root mod \( p \), then \( g^j \) is a primitive root mod \( p \) iff \( \gcd(j, p-1) = 1 \). There are thus \( \varphi(p-1) \) primitive roots mod \( p \) in \((\mathbb{Z}/p\mathbb{Z})^*\), where \( \varphi \) denotes Euler’s totient function.

Let \( \pi(x) \) denote the number of primes \( p \leq x \) and \( \pi_g(x) \) the number of primes \( p \leq x \) such that \( g \) is a primitive root mod \( p \). Artin’s celebrated primitive root conjecture (1927) states that if \( g \) is an integer with \( |g| > 1 \) and \( g \) is not a square, then for some positive rational number \( c_g \) we have \( \pi_g(x) \sim c_g A \pi(x) \), as \( x \) tends to infinity. Here \( A \) denotes \textit{Artin’s constant}

\[
A = \prod_p \left( 1 - \frac{1}{p(p-1)} \right) = 0.3739558136\ldots
\]

Hooley \[3\], under assumption of the Generalized Riemann Hypothesis (GRH), established Artin’s conjecture and explicitly evaluated \( c_g \).

It is an old heuristic idea that the behaviour of \( \pi_g(x) \) should be mimicked by \( H_1(x) = \sum_{p \leq x} \varphi(p-1)/(p-1) \), the idea being that the ‘probability’ that \( g \) is a primitive root mod \( p \) equals \( \varphi(p-1)/(p-1) \) (since this is the density of primitive
roots in \((\mathbb{Z}/p\mathbb{Z})^\ast\)). Using the Siegel-Walfisz theorem (see Lemma 1 below), it is not difficult to show, unconditionally, that \(H_1(x) \sim A\pi(x)\). Although true for many \(g\) and also on average, it is however not always true, under GRH, that \(\pi_g(x) \sim H_1(x)\), i.e., the heuristic \(H_1(x)\) is not always asymptotically exact. Nevertheless, Moree [5] found a quadratic modification, \(H_2(x)\), of the above heuristic \(H_1(x)\) involving the Legendre symbol that is always asymptotically exact (assuming GRH).

A prime \(p\) is said to divide a sequence \(S\) of integers, if it divides at least one term of the sequence \(S\) (see [1] for a nice introduction to this topic). Several authors studied the problem of characterising (prime) divisors of the sequence \(\{a^k + b^k\}_{k=1}^\infty\). Hasse [2] seems to have been the first to consider the Dirichlet density of prime divisors of such sequences. Later authors, e.g., Odoni [9] and Wiertelak [11] strengthened the analytic aspects of his work. The best result to date, in the formulation of [4], seems to be as follows (recall that \(\text{Li}(x) = \int_2^x dt/\log t\) is the logarithmic integral):

**Theorem 1** Let \(a\) and \(b\) be non-zero integers. Put \(r = a/b\). Assume that \(r \neq \pm 1\).

Let \(\lambda\) be the largest integer such that \(|r| = u^{2\lambda}\), with \(u\) a rational number. Let \(\varepsilon = \text{sign}(r)\) and \(L = \mathbb{Q}(\sqrt{u})\). We have

\[
N_{a,b}(x) = \delta(r)\text{Li}(x) + O\left(\frac{x(\log \log x)^4}{\log^3 x}\right),
\]

where the implied constant may depend on \(a\) and \(b\) and \(\delta(r)\), a rational number, is given in Table 1.

**Table 1:** The value of \(\delta(r)\)

| \(L\)       | \(\lambda\) | \(\varepsilon = +1\) | \(\varepsilon = -1\) |
|-------------|-------------|----------------------|----------------------|
| \(L \neq \mathbb{Q}(\sqrt{2})\) | \(\lambda \geq 0\) | 2\(^{1-\lambda}/3\) | 1 \(- 2^{1-\lambda}/3\) |
| \(L = \mathbb{Q}(\sqrt{2})\) | \(\lambda = 0\) | 17/24               | 17/24               |
| \(L = \mathbb{Q}(\sqrt{2})\) | \(\lambda = 1\) | 5/12                | 2/3                 |
| \(L = \mathbb{Q}(\sqrt{2})\) | \(\lambda \geq 2\) | 2\(^{-1-\lambda}/3\) | 1 \(- 2^{-1-\lambda}/3\) |

Starting point in the proof of Theorem 1 is the observation that \(p \nmid 2ab\) divides the sequence \(\{a^k + b^k\}_{k=1}^\infty\) iff \(\text{ord}_r(p)\) is even, where \(r = a/b\). The condition that \(\text{ord}_r(p)\) be even is weaker than the condition that \(\text{ord}_r(p) = p - 1\) and now the analytic tools are strong enough to establish an unconditional result.

Note that \(\delta(r)\) does not depend on \(\varepsilon\) in case \(\lambda = 0\). For a ‘generic’ choice of \(a\) and \(b\), \(L\) will be different from \(\mathbb{Q}(\sqrt{2})\) and \(\lambda\) will be zero and hence \(\delta(a/b) = 2/3\). It is not difficult to show [8] that the average density of elements of even order in a finite field of prime cardinality also equals 2/3.

In this note analogs of \(H_{a,b}^{(1)}(x)\) and \(H_{a,b}^{(2)}(x)\) of \(H_1(x)\) and \(H_2(x)\) will be introduced and it will be shown that \(H_{a,b}^{(2)}(x)\) is always asymptotically exact. This leads to the following main result (where \(\pi(x; k, l)\) denotes the number of primes \(p \leq x\) satisfying \(p \equiv l (\text{mod } k)\) and \((*/p)\) denotes the Legendre symbol):
Lemma 1 Let \( G \) be a cyclic group of order \( n \). Let \( G^h = \{ g^h : g \in G \} \) and \( G_w^h = \{ g^h : \nu_2(\text{ord}(g^h)) = w \} \). We have \( \#G^h = n/(n,h) \) and \( \#G_w^h = 2^{-\nu_2(n/(n,h))}n/(n,h) \). Furthermore, for \( w \geq 1 \), we have

\[
\#G_w^h = \begin{cases} 
2^{w-1-\nu_2(n/(n,h))}n/(n,h) & \text{if } \nu_2(n/(n,h)) \geq w; \\
0 & \text{otherwise.}
\end{cases}
\]

(1)

2) If \( \nu_2(h) \geq \nu_2(n) \), then every element in \( G^h \) has odd order. If \( \nu_2(h) < \nu_2(n) \), then \( G_w^h \subseteq G^{2h} \).

3) We have

\[
G_1^h \subseteq \begin{cases} 
G^h \setminus G^{2h} & \text{if } \nu_2(n) = \nu_2(h) + 1; \\
G^{2h} & \text{if } \nu_2(n) > \nu_2(h) + 1.
\end{cases}
\]

If \( \nu_2(n) \leq \nu_2(h) \), then \( G_1^h \) is empty.

2 Preliminaries

The proof of Theorem 2 requires a result from analytic number theory: the Siegel-Walfisz theorem, see e.g., \cite{10, Satz 4.8.3}. For notational convenience we write \((a, b)\) instead of \(\text{gcd}(a, b)\).

Lemma 1 Let \( C > 0 \) be arbitrary. There exists \( c_1 > 0 \) such that

\[
\pi(x; k, l) = \frac{\text{Li}(x)}{\varphi(k)} + O(xe^{-c_1\sqrt{\log x}}),
\]

uniformly for \( 1 \leq k \leq \log C x \), \( (l, k) = 1 \), where the implied constant depends at most on \( C \).

Our two heuristics will be based on the following elementary observation in group theory.

Theorem 2 Let \( a \) and \( b \) be non-negative natural numbers. Let \( N_{a,b}(x) \) count the number of primes \( p \leq x \) that divide some term \( a^k + b^k \) in the sequence \( \{a^k + b^k\}_{k=1}^{\infty} \). Put \( r = a/b \) and \( \epsilon = \text{sgn}(a/b) \). Assume that \( r \neq \pm 1 \). Let \( h \) be the largest integer such that \( |r| = r_0^h \) for some \( r_0 \in \mathbb{Q} \) and \( h \geq 1 \). Put \( \epsilon = \nu_2(h) \). If \( \epsilon = 1 \), then

\[
N_{a,b}(x) = \pi(x; 2^{e+1}, 1) - 2^{e+1} \sum_{p \leq x, \ (r_0/p)=1, \ \nu_2(p-1) > \epsilon} 2^{-\nu_2(p-1)} + O\left(\frac{x(\log \log x)^4}{\log^3 x}\right),
\]

and if \( \epsilon = -1 \), then

\[
N_{a,b}(x) = \pi(x) - \sum_{p \leq x, \ (r_0/p)=-1, \ \nu_2(p-1) = e+1} 1 - 2^{e+1} \sum_{p \leq x, \ (r_0/p)=-1, \ \nu_2(p-1) > e+1} 2^{-\nu_2(p-1)} + O\left(\frac{x(\log \log x)^4}{\log^3 x}\right),
\]

where the implied constants depend at most on \( a \) and \( b \).
Proof. 1) Let $g_0$ be a generator of $G$. On noting that $g_0^{m_1} = g_0^{m_2}$ iff $m_1 \equiv m_2 \pmod n$, the proof becomes a simple exercise in solving linear congruences. In this way one infers that $G^h = \{g_0^{hk} : 1 \leq k \leq n/(n, h)\}$ and hence $\#G^h = n/(n, h)$. Note that ord($g_0^{hk}$) is the smallest positive integer $m$ such that $n/(n, h)$ divides $mk$. Thus ord($g_0^{hk}$) will be odd iff $\nu_2(k) \geq \nu_2(n/(n, h))$. Using this observation we obtain that

$$G^h_0 = \{g_0^{hk} : 1 \leq k \leq \frac{n}{(n, h)}, \nu_2(k) \geq \nu_2(\frac{n}{(n, h)})\}$$

(2)

and hence $\#G^h_0 = 2^{-\nu_2(n/(n,h))}n/(n, h)$. Similarly

$$G^h_w = \{g_0^{hk} : 1 \leq k \leq \frac{n}{(n, h)}, \nu_2(k) = \nu_2(\frac{n}{(n, h)}) - w\}$$

and hence we obtain \((\text{I})\).

2) If $\nu_2(h) < \nu_2(h)$, then using \((\text{II})\) we infer that

$$G^h_0 \subseteq \{g_0^{hm} : 1 \leq m \leq \frac{n}{(n, h)}, \nu_2(m) \geq 1\} = \{g_0^{2hk} : 1 \leq k \leq \frac{n}{(n, 2h)}\} = G^{2h},$$

where we have written $m = 2k$ and used that $(n, 2h) = 2(n, h)$.

3) Similar to that of part 2. \hfill \Box

Remark. Note that $G^h$ and $G^h_0$ with the induced group operation from $G$ are actually subgroups of $G$.

3 Two heuristic formulae for $N_{a,b}(x)$

In this section we propose two heuristics for $N_{a,b}(x)$; one more refined than the other. Starting point is the observation that if $p \nmid 2ab$ divides the sequence $\{a^k + b^k\}_{k=1}^{\infty}$ if and only if ord$_r(p)$ is even, where $r = a/b$. Let $h$ be the largest integer such that we can write $|r| = r_0^h$ with $r_0$ a rational number. Let $\epsilon = \text{sgn}(r)$.

We will use Lemma 2 in the case $G = G_p := (\mathbb{Z}/p\mathbb{Z})^* \cong \mathbb{F}_p^*$. The first heuristic approximation we consider is

$$K^{(1)}_{a,b}(x) = \sum_{p \leq x, p \nmid 2ab} \frac{\#G^h_{p,(1-\epsilon)/2}}{\#G^h_p},$$

where $K^{(1)}_{a,b}(x)$ is supposed to be an heuristic for the number of primes $p \leq x$ such that ord$_r(p)$ is odd. From our results below it will follow that $\lim_{x \to \infty} K^{(1)}_{a,b}(x)/\pi(x)$ exists. Note that in case $h = 1$, this limit is the average density of elements of odd order (if $\epsilon = 1$), respectively of order congruent to 2(mod 4) (if $\epsilon = -1$). For a more detailed investigation of the average number of elements having order $\equiv a(\text{mod} d)$ vide [8].

Suppose that $p \nmid 2ab$. By assumption $r \in \epsilon G^h_p$. In the case $\epsilon = 1$, the latter set has $\#G^h_{p,0}$ elements having odd order and so, in some sense, $\#G^h_{p,0} / \#G^h_p$ is the probability that ord$_r(p)$ is odd. This motivates the definition of $K^{(1)}_{a,b}(x)$ in case
\(\epsilon = 1\). In case \(\epsilon = -1\) we use the observation that for \(p\) odd, \(-r_0^h\) has odd order iff \(r_0^h\) has order congruent to \(2\pmod{4}\). Thus the elements in \(-G_p^h\) of odd order are precisely the elements having order \(2\pmod{4}\) in \(G_p^h\) and hence have cardinality \#\(G_{p,1}^h\). On using part 1 of Lemma \(2\) we infer that \(K_{a,b}^{(1)}(x) = \sum_{p \leq x, \atop p \nmid 2ab} k_{a,b}^{(1)}(p)\) with

\[
k_{a,b}^{(1)}(p) = \begin{cases} 
\frac{1 + \epsilon}{2} & \text{if } \nu_2(p - 1) \leq \epsilon; \\
2^{\nu_2(p - 1)} & \text{if } \nu_2(p - 1) > \epsilon.
\end{cases}
\] (3)

An heuristic \(H_{a,b}^{(1)}(x)\) for \(N_{a,b}(x)\) is now obtained on merely setting \(H_{a,b}^{(1)}(x) = \pi(x) - K_{a,b}^{(1)}(x)\). Put \(\omega(n) = \sum_{p|n} 1\). On using \(3\) we then infer that

\[
H_{a,b}^{(1)}(x) = \pi(x) - 2^e \sum_{p \leq x} 2^{-\nu_2(p - 1)} + O(\omega(ab)),
\]
if \(\epsilon = 1\) and

\[
H_{a,b}^{(1)}(x) = \pi(x) - 2^e \sum_{p \leq x} 2^{-\nu_2(p - 1)} + O(\omega(ab)).
\]
if \(\epsilon = -1\).

In the context of (near) primitive roots it is known that the ana\(\text{loga}\) of \(H_{a,b}^{(1)}(x)\) do not always, assuming GRH, exhibit the correct asymptotic behaviour, but that an appropriate ‘quadratic’ heuristic, i.e. an heuristic taking into account Legendre symbols, always has the correct asymptotic behaviour \([5, 6, 7]\) (in \(7\) the main result of \(6\) is proved in a different and much shorter way). With this in mind, we propose a second, more refined, heuristic: \(H_{a,b}^{(2)}(x)\).

If \(\nu_p(r) = 0\) we can consider \(|r| = r_0^h\) and \(r_0\) as elements of \(G_p\). We write \(r_0/p = 1\) if \(r_0\) is a square in \(G_p\) and \((r_0/p) = -1\) otherwise.

First consider the case where \(\epsilon := \text{sgn}(r) = 1\). If \(\nu_2(p - 1) \leq \epsilon := \nu_2(h)\), then \(r\) has odd order by part 2 of Lemma \(2\). If \(\nu_2(p - 1) > \nu_2(h)\) and \((r_0/p) = -1\), then \(r \in G_p^h\), but \(r \notin G_p^{2h}\) (by part 2 of Lemma \(2\) again). It then follows that \(r\) has even order. On the other hand, if \((r_0/p) = 1\) then \(r \in G_p^{2h}\). This suggests to take

\[
K_{a,b}^{(2)}(x) = \sum_{p \leq x, \atop \nu_2(p - 1) \leq \epsilon} 1 + \sum_{p \leq x, \atop \nu_2(p - 1) > \epsilon} \frac{\#G_p^h}{\#G_p^{2h}},
\]
where furthermore we require that \(p \nmid 2ab\). A similar argument, now using part 3 instead of part 2 of Lemma \(2\) leads to the choice

\[
K_{a,b}^{(2)}(x) = \sum_{p \leq x, \atop \nu_2(p - 1) = \epsilon + 1} \frac{\#G_p^h}{\#G_p^{2h}} + \sum_{p \leq x, \atop \nu_2(p - 1) > \epsilon + 1} \frac{\#G_p^h}{\#G_p^{2h}},
\]
in case \(\epsilon = -1\), where again we furthermore require that \(p \nmid 2ab\). We obtain \(K_{a,b}^{(2)}(x) = \sum_{p \leq x, \atop p \nmid 2ab} k_{a,b}^{(2)}(p)\), with

\[
k_{a,b}^{(2)}(p) = \begin{cases} 
(1 + \epsilon)/2 & \text{if } \nu_2(p - 1) \leq \epsilon; \\
(1 + \epsilon(\frac{r_0}{p}))2^{\nu_2(p - 1)} & \text{if } \nu_2(p - 1) = \epsilon + 1; \\
(1 + (\frac{r_0}{p}))2^{\nu_2(p - 1)} & \text{if } \nu_2(p - 1) > \epsilon + 1.
\end{cases}
\] (4)
Now we put $H_{a,b}^{(2)}(x) = \pi(x) - K_{a,b}^{(2)}(x)$ as before. On invoking Lemma 2, $H_{a,b}^{(2)}(x)$ can then be more explicitly written as

$$H_{a,b}^{(2)}(x) = \pi(x; 2^{e+1}, 1) - 2^{e+1} \sum_{p \leq x, \ (r_0/p) = 1 \atop \nu_2(p-1) > e} 2^{-\nu_2(p-1)} + O(\omega(ab)), \quad (5)$$

if $\epsilon = 1$ and

$$H_{a,b}^{(2)}(x) = \pi(x) - \sum_{p \leq x, \ (r_0/p) = -1 \atop \nu_2(p-1) = e+1} 1 - 2^{e+1} \sum_{p \leq x, \ (r_0/p) = 1 \atop \nu_2(p-1) > e+1} 2^{-\nu_2(p-1)} + O(\omega(ab)), \quad (6)$$

if $\epsilon = -1$.

### 4 Asymptotic analysis of the heuristic formulae

In this section we determine the asymptotic behaviour of $H_{a,b}^{(1)}(x)$ and $H_{a,b}^{(2)}(x)$. We adopt the notation from Theorem 2 and in addition write $D$ for the discriminant of $\mathbb{Q}(\sqrt{r_0})$. Note that $D > 0$.

**Theorem 3** Let $A > 0$ be arbitrary. The implied constants below depend at most on $A$.

1) We have

$$H_{a,b}^{(1)}(x) = \delta_1(r) \text{Li}(x) + O(x \log^{-A} x) + O(\omega(ab)),$$

where

$$\delta_1(r) = \begin{cases} 2^{1-e}/3 & \text{if } \epsilon = +1; \\ 1 - 2^{-e}/3 & \text{if } \epsilon = -1. \end{cases}$$

In particular, if $L \neq \mathbb{Q}(\sqrt{2})$, then $H_{a,b}^{(1)}(x)$ is an asymptotically exact heuristic for $N_{a,b}(x)$.

2) We have

$$H_{a,b}^{(2)}(x) = \delta(r) \text{Li}(x) + O(D^2 x \log^{-A} x) + O(\omega(ab)).$$

In particular, $H_{a,b}^{(2)}(x)$ is an asymptotically exact heuristic for $N_{a,b}(x)$.

The proof of part 2 requires a few facts from algebraic number theory; the proof of part 1 does not even require that and is an easier variant of the proof of part 2 (and is left to the interested reader). The proof of part 2 rests on a few lemmas.

**Lemma 3** Let $n$ be a non-zero integer and $K = \mathbb{Q}(\sqrt{n})$ a quadratic number field of discriminant $\Delta$. Let $A > 1$ and $C > 0$ be positive real numbers. Then

$$\sum_{p \leq x, \ (n/p) = 1 \atop \nu_2(p-1) = k} 1 = \text{Li}(x) \left( \frac{1}{[K(\zeta_2^k) : \mathbb{Q}]} - \frac{1}{[K(\zeta_2^{k+1}) : \mathbb{Q}]} \right) + O \left( \frac{|\Delta| x}{\log^A x} \right),$$

uniformly in $k$ with $k$ satisfying $2^{k+3}|\Delta| \leq \log^C x$, where the implied constant depends at most on $A$ and $C$.
Lemma 5

We have \( \lfloor \varphi / 4 \rfloor \) where we used that if \( \epsilon > 0 \) and \( \varepsilon / (p-1) \geq m \)

set of congruences classes modulo \( 4 |\Delta| \), but do not belong to certain congruence classes of modulus \( 2^{k+3} |\Delta| \). The total number of congruence classes involved is less than \( 8 |\Delta| \). Now apply Lemma 3.

This yields the result but with an, as yet, unknown density.

On the other hand, the primes \( p \) that are counted are precisely the primes \( p \leq x \) that split completely in the normal number field \( K(\zeta_{2k}) \), but do not split completely in the normal number field \( K(\zeta_{2k+1}) \). If \( M \) is any normal extension then it is a consequence of Chebotarev’s density theorem that the set of primes that split completely in \( M \) has density \( 1/[M : Q] \). On using this, the proof is completed.

\[ \sum_{p \leq x, \ (n/p) = 1} \frac{1}{2^k} \left( \frac{1}{[K(\zeta_{2k}) : Q]} - \frac{1}{[K(\zeta_{2k+1}) : Q]} \right) = O \left( \frac{\Delta^2 x}{\log^4 x} \right), \]

where the implied constant depends at most on \( A \).

Proof. We have

\[ \sum_{p \leq x, \ (n/p) = 1} 2^{-\nu_2(p-1)} = \sum_{k=m}^{m_1} 2^{-k} + O \left( \frac{x}{4^{m_1}} \right), \]

where we used the trivial bound \( \sum_{p \leq x, \ (n/p) = 1} 2^{-\nu_2(p-1)} \leq O(x/4^{m_1}) \). Choose \( m_1 \) to be the largest integer such that \( 2^{m_1+3} |\Delta| \leq \log^C x \). Apply Lemma 3 with any \( C > A/2 \). It follows that

\[ \sum_{p \leq x, \ (n/p) = 1} 2^{-\nu_2(p-1)} = \sum_{k=m}^{m_1} \frac{1}{2^k} \left( \frac{1}{[K(\zeta_{2k}) : Q]} - \frac{1}{[K(\zeta_{2k+1}) : Q]} \right) + O \left( \frac{x}{4^{m_1}} \right); \]

\[ = \sum_{k=m}^{m_1} \frac{1}{2^k} \left( \frac{1}{[K(\zeta_{2k}) : Q]} - \frac{1}{[K(\zeta_{2k+1}) : Q]} \right) + O \left( \frac{x}{4^{m_1}} \right), \]

where we used that \( \varphi(2^k) \leq [K(\zeta_{2k}) : Q] \leq 2^k \varphi(2^k) \). On noting that \( O(x/4^{m_1}) = O(\Delta^2 x \log^{-A} x) \), the result follows.

\[ \delta_2(r) = \frac{1}{2^r} - 2^{r+1} \sum_{k=r+1}^{\infty} \frac{1}{2^k} \left( \frac{1}{[L(\zeta_{2k}) : Q]} - \frac{1}{[L(\zeta_{2k+1}) : Q]} \right) \]

if \( \epsilon = 1 \) and

\[ \delta_2(r) = 1 - 2^{r+1} \sum_{k=r+1}^{\infty} \frac{1}{2^k \left[ L(\zeta_{2k+1}) : Q \right]} - \frac{1}{[L(\zeta_{2k+2}) : Q]} \]

\[ - 2^{r+1} \sum_{k=r+2}^{\infty} \frac{1}{2^k \left[ L(\zeta_{2k}) : Q \right]} - \frac{1}{[L(\zeta_{2k+1}) : Q]} \],

if \( \epsilon = -1 \).
Proof. This easily follows on combining the previous lemma with equation (5), respectively (6). □

Remark. From (7) and (8) we infer that
\[
\delta_2(-|r|) - \delta_2(|r|) = 1 - \frac{3}{2^{e+1}} + \frac{2}{L(\zeta_{2e+1}) : \mathbb{Q}} - \frac{2}{L(\zeta_{2e+2}) : \mathbb{Q}}.
\]
The number \(\delta_2(r)\) can be readily evaluated on using the following simple fact from algebraic number theory:

**Lemma 6** Let \(K\) be a real quadratic field. Let \(k \geq 1\). Then
\[
[K(\zeta_{2^k}) : \mathbb{Q}] = \begin{cases} 2^k & \text{if } k \leq 2 \text{ or } K \neq \mathbb{Q}(\sqrt{2}); \\ 2^{k-1} & \text{if } k \geq 3 \text{ and } K = \mathbb{Q}(\sqrt{2}). \end{cases}
\]

**Proof.** If \(K\) is a quadratic field other than \(\mathbb{Q}(\sqrt{2})\) then there is an odd prime that ramifies in it. This prime, however, does not ramify in \(\mathbb{Q}(\zeta_{2^k})\), so in this case \(K\) and \(\mathbb{Q}(\sqrt{2})\) are linearly disjoint. Note that \(\zeta_8 + \zeta_8^{-1} = \sqrt{2}\) and hence \(\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_8)\). Using the well-known result that \([\mathbb{Q}(\zeta_n) : \mathbb{Q}] = \varphi(n)\), the result is then easily completed. □

The result of this evaluation is stated below.

**Lemma 7** We have \(\delta_2(r) = \delta(r)\).

After all this preliminary work, it is straightforward to prove the two main results of this note:

**Proof of Theorem 3** 1) Left to the reader. 2) Combine the latter lemma with Lemma 5. Comparison with Theorem 1 shows that \(H^{(2)}_{a,b}(x) \sim N_{a,b}(x)\) as \(x \to \infty\) and thus \(H^{(2)}_{a,b}(x)\) is an asymptotically exact approximation of \(N_{a,b}(x)\). □

**Proof of Theorem 2** Combine part 2 of Theorem 3 (with any \(A > 3\)), Theorem 1 and equations (5) and (6). □

5 Two alternative formulations

5.1 An alternative formulation using Ramanujan sums

Recall that the Ramanujan sum \(c_n(m)\) is defined as \(\sum_{1 \leq k \leq n, \ (k,n)=1} e^{2\pi i km/n}\). It is well-known that \(c_n(m) \in \mathbb{Z}\) and, more in particular, that
\[
c_n(m) = \varphi(n) \frac{\mu(n/(n,m))}{\varphi(n/(n,m))}.
\]
This is known as Hölder’s identity. It implies that \(c_n(m) = c_n((n,m))\). For our purposes the following weak version of Hölder’s identity will suffice:
\[
c_{2 \nu}(t) = \begin{cases} 0 & \text{if } \nu_2(t) < \nu; \\ -\varphi(2^n) & \text{if } \nu_2(t) = v - 1; \\ \varphi(2^n) & \text{if } \nu_2(t) \geq v. \end{cases}
\]
Another elementary property of Ramanujan sums we need is that for arbitrary natural numbers $n$ and $m$

$$\frac{1}{n} \sum_{d|n} c_d(m) = \begin{cases} 1 & \text{if } n|m; \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Suppose that $\nu_p(r) = 0$, then $\text{ord}_e(p)\left[\mathbb{F}_p^* : \langle r \rangle \right] = p - 1$. Note that $\text{ord}_e(p)$ is off iff $2^{\nu_2(p-1)}\left[\mathbb{F}_p^* : \langle r \rangle \right]$. Using identity (10) it then follows that

$$N_{a,b}(x) = \pi(x) - \sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{v \leq \nu_2(p-1)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] + O(\omega(ab)). \quad (11)$$

Corollary 1 below shows that if in the latter double sum the summation is restricted to those $v$ satisfying in addition $v \leq e$, respectively $v \leq e + 1$, then $K_{a,b}^{(1)}(x)$, respectively $K_{a,b}^{(2)}(x)$ is obtained. This in combination with Theorems 1 and 3 leads to the following theorem:

**Theorem 4** We have, in the notation of Theorem 2 $N_{a,b}(x) = \pi(x) - \sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{v \leq \nu_2(p-1)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] + O\left(\frac{x(\log \log x)^4}{\log^3 x}\right),$

and

$$\sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{e+2 \leq v \leq \nu_2(p-1)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] = O\left(\frac{x(\log \log x)^4}{\log^4 x}\right),$$

where the implied constant depends at most on $a$ and $b$.

**Remark.** Note that $v \leq \min(\nu_2(p - 1), e + 1)$ is equivalent with $2^v|(p - 1, 2h)$.

**Lemma 8** Let $a, b, e$ and $e$ be as in Theorem 2 and let $p \nmid 2ab$.

1) We have

$$2^{-\nu_2(p-1)} \sum_{v \leq \min(\nu_2(p-1), e)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] = k_{a,b}^{(1)}(p).$$

2) We have

$$2^{-\nu_2(p-1)} \sum_{v \leq \min(\nu_2(p-1), e+1)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] = k_{a,b}^{(2)}(p).$$

**Corollary 1** For $1 \leq j \leq 2$ we have

$$\sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{v \leq \min(\nu_2(p-1), e+j-1)} c_{2^v}\left[\mathbb{F}_p^* : \langle r \rangle \right] = K_{a,b}^{(j)}(x).$$

**Proof of Lemma 8** 1) Let us consider the case $e = -1$ and $\nu_2(p - 1) > e$ (the remaining cases are similar and left to the reader). Since $(-1)^{(p-1)/2^e} \equiv 1(\text{mod} \ p)$ we see that $-1$ and hence $r$ is a $2^e$th-power mod $p$ and thus $\nu_2(\left[\mathbb{F}_p^* : \langle r \rangle \right]) \geq e$. Hence the sum in the statement of the lemma reduces to $2^{-\nu_2(p-1)} \sum_{v \leq e} \varphi(2^v) = 2^{e-\nu_2(p-1)} = k_{a,b}^{(1)}(p)$, where $\varphi$, $\varphi$ and the identity $\sum_{d|n} \varphi(d) = n$ are used.
2) The case $\nu_2(p - 1) \leq e$. The quantity under consideration agrees with that of part 1 and by (11) we obtain that $k^{(1)}_{a,b}(p) = (1 + \epsilon)/2 = k^{(2)}_{a,b}(p)$.

The case $\nu_2(p - 1) = e + 1$. Now $(-1)^{\frac{p-1}{e+1}} \equiv 1 (\mod p), (r_0^h)^{\frac{p-1}{e+1}} \equiv (\frac{r_0}{p})(\mod p)$ and hence $r^{\frac{p-1}{e+1}} \equiv (\frac{r_0}{p})(\mod p)$.

It follows that $\nu_2([\mathbb{F}_p^* : \langle r \rangle]) \geq e + 1$ if $\epsilon(\frac{r_0}{p}) = 1$ and $\nu_2([\mathbb{F}_p^* : \langle r \rangle]) = e$ if $\epsilon(\frac{r_0}{p}) = -1$.

Using (9) the quantity under consideration is seen to reduce to

$$2^{-\nu_2(p-1)} \left( \sum_{v \leq e} \varphi(2^v) + \epsilon(\frac{r_0}{p})2^e \right) = \frac{1 + \epsilon(\frac{r_0}{p})}{2}.$$ 

By (11) this equals $k^{(2)}_{a,b}(p)$.

The case $\nu_2(p - 1) > e + 1$. Now $r^{(p-1)/2^{e+1}} \equiv (\frac{r_0}{p})(\mod p)$. Proceeding as before the quantity under consideration reduces to

$$2^{-\nu_2(p-1)} \left( \sum_{v \leq e} \varphi(2^v) + (\frac{r_0}{p})2^e \right) = 2^{e-\nu_2(p-1)}(1 + (\frac{r_0}{p})).$$

5.2 An alternative formulation involving character sums

Let $G$ be a cyclic group of order $n$ and $g \in G$. It is not difficult to show that, for any $d|n$, $\sum_{\text{ord}(\chi) = d} \chi(g) = c_d([G : \langle g \rangle])$. Using this and noting that $\chi(r) = \chi(\epsilon)\chi^h(r_0)$, equation (11) can be rewritten as

$$N_{a,b}(x) = \pi(x) - \sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{\text{ord}(\chi)|2^e2^{(p-1)}} \chi(\epsilon)\chi^h(r_0) + O(\omega(ab)), \quad (12)$$

where the sum is over all characters of $\mathbb{F}_p^*$ having order dividing $2^{\nu_2(p-1)}$. Note that if $\chi$ is of order $2^n$, then $\chi^h$ is the trivial character if $v \leq e$ and a quadratic character if $v = e + 1$. If in the main term of (12) only those characters of order dividing $h$ are retained, i.e. those for which $\chi^h$ is the trivial character, then $H_{a,b}^{(1)}(x)$ is obtained (this is a reformulation of part 1 of Lemma 8) and hence, by part 1 of Theorem 8 the na"ive heuristic. If in (12) only those characters of order dividing $2h$ are retained, i.e. those for which $\chi^h$ is the trivial or a quadratic character, then the asymptotically exact heuristic is obtained. The error term assertion in Theorem 4 can be reformulated as:

**Proposition 1** We have

$$\sum_{p \leq x, \ p|2ab} 2^{-\nu_2(p-1)} \sum_{2^{v+2}\text{ord}(\chi)|2^e2^{(p-1)}} \chi(\epsilon)\chi^h(r_0) = O \left( \frac{x(\log \log x)^4}{\log^3 x} \right),$$

where the implied constant depends at most on $a$ and $b$.

In the setting of near primitive roots it is already known that for the main term of the counting function of (near) primitive roots only the contributions coming from characters that are either trivial or quadratic need to be included [5].
6 Conclusion

There is a naïve heuristic for $N_{a,b}(x)$ that in many, but not all, cases is asymptotically exact. There is a quadratic modification of this heuristic involving the Legendre symbol that is always asymptotically exact. The same phenomenon is observed (assuming GRH) in the setting of Artin’s primitive root conjecture.

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