QUASI-UNIFORMITY OF MINIMAL WEIGHTED ENERGY POINTS ON COMPACT METRIC SPACES

D. P. HARDIN, E. B. SAFF, AND J. T. WHITEHOUSE

Abstract. For a closed subset $K$ of a compact metric space $A$ possessing an $\alpha$-regular measure $\mu$ with $\mu(K) > 0$, we prove that whenever $s > \alpha$, any sequence of weighted minimal Riesz $s$-energy configurations $\omega_N = \{x_{i,N}^{(s)}\}_{i=1}^N$ on $K$ (for ‘nice’ weights) is quasi-uniform in the sense that the ratios of its mesh norm to separation distance remain bounded as $N$ grows large. Furthermore, if $K$ is an $\alpha$-rectifiable compact subset of Euclidean space ($\alpha$ an integer) with positive and finite $\alpha$-dimensional Hausdorff measure, it is possible to generate such a quasi-uniform sequence of configurations that also has (as $N \to \infty$) a prescribed positive continuous limit distribution with respect to $\alpha$-dimensional Hausdorff measure. As a consequence of our energy related results for the unweighted case, we deduce that if $A$ is a compact $C^1$ manifold without boundary, then there exists a sequence of $N$-point best-packing configurations on $A$ whose mesh-separation ratios have limit superior (as $N \to \infty$) at most 2.

1. Introduction

Let $A$ be a compact infinite metric space with metric $m : A \times A \to [0, \infty)$ and let $\omega_N = \{x_i\}_{i=1}^N \subset A$ denote a configuration of $N \geq 2$ points in $A$. We are chiefly concerned with two ‘quality’ measures of $\omega_N$; namely, the separation distance of $\omega_N$ defined by
\[
\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} m(x_i, x_j),
\]
and the mesh norm of $\omega_N$ with respect to $A$ defined by
\[
\rho(\omega_N, A) := \max_{y \in A} \min_{1 \leq i \leq N} m(y, x_i).
\]
This quantity is also known as the fill radius or covering radius of $\omega_N$ relative to $A$. The optimal values of these quantities are also of interest and we consider, for $N \geq 2$, the $N$-point best-packing distance on $A$ given by
\[
\delta_N(A) := \max\{\delta(\omega_N) : \omega_N \subset A, |\omega_N| = N\},
\]
and the $N$-point mesh norm of $A$ given by
\[
\rho_N(A) := \min\{\rho(\omega_N, A) : \omega_N \subset A, |\omega_N| = N\}.
\]
where $|S|$ denotes the cardinality of set $S$.

In the theory of approximation and interpolation (for example, by splines or radial basis functions (RBFs)), the separation distance is often associated with some measure of ‘stability’ of the approximation, while the mesh norm arises in the error of the approximation. In this context, the mesh-separation ratio (or mesh ratio)

$$\gamma_{\omega, A} := \rho_{\omega, A}/\delta_{\omega, A},$$

can be regarded as a ‘condition number’ for $\omega$ relative to $A$. If $\{\omega_N\}_{N=2}^\infty$ is a sequence of $N$-point configurations such that $\gamma_{\omega_N, A}$ is uniformly bounded in $N$, then the sequence is said to be quasi-uniform on $A$. Quasi-uniform sequences of configurations are important for a number of methods involving RBF approximation and interpolation (see [9, 15, 17, 19]).

We remark that in some cases it is easy to obtain positive lower bounds for the mesh-separation ratio. For example, if $A$ is connected, then $\gamma_{\omega_N, A} \geq 1/2$. Furthermore, letting

$$B(x, r) = \{y \in A : m(y, x) \leq r\}$$

be the closed ball in $A$ with center $x$ and radius $r$, then $\gamma_{\omega_N, A} \geq \beta/2$ for any $N$-point configuration $\omega_N \subset A$ whenever $A$ and $\beta \in (0, 1)$ have the property that for any $r \in (0, \text{diam}(A))$ and any $x \in A$, the annulus $B(x, r) \setminus B(x, \beta r)$ is nonempty. The diameter of $A$ is defined by

$$\text{diam}(A) := \max \{m(x, y) : x \in A, y \in A\}.$$

In this paper we consider the separation distance and mesh norm of finite point configurations in $A$ that minimize certain weighted energy functionals. We call $w : A \times A \rightarrow [0, \infty)$ an SLP weight on $A$ if it is symmetric and lower semi-continuous on $A \times A$ and is positive on the diagonal, $D(A)$, of $A \times A$. For $s > 0$ and a collection of $N \geq 2$ distinct points $\omega_N = \{x_1, \ldots, x_N\} \subset A$, the $(s, w)$-energy of $\omega_N$ (also known as the weighted Riesz $s$-energy) is

$$E_s^w(\omega_N) := \sum_{i \neq j} \frac{w(x_i, x_j)}{m(x_i, x_j)^s} = \sum_{i=1}^N \sum_{j=1 \atop j \neq i}^N \frac{w(x_i, x_j)}{m(x_i, x_j)^s},$$

and we denote the minimal $N$-point $(s, w)$-energy of $A$ by

$$\mathcal{E}_s^w(N, A) := \inf \{E_s^w(\omega_N) : \omega_N \subset A, |\omega_N| = N\}.$$

Since $A$ is compact and the energy $E_s^w(\omega_N)$ is lower semi-continuous, there exists at least one $N$-point configuration $\omega_N^* \subset A$ such that $E_s^w(\omega_N^*) = \mathcal{E}_s^w(N, A)$. We refer to such an $\omega_N^*$ as an $N$-point $(s, w)$-energy minimizing configuration on $A$. The asymptotics as $N \rightarrow \infty$ of $N$-point $(s, w)$-energy minimizing configurations and their energies are investigated in [1, 10] for $d$-rectifiable sets $A \subset \mathbb{R}^d$ and $s > d$ (see further discussion in the next section).

In our results we shall require that $A$ is either $\alpha$-regular or upper $\alpha$-regular as we next describe. For a positive Borel measure $\mu$ supported on $A$ and $\alpha > 0$, we say that $\mu$ is upper $\alpha$-regular if there is some finite constant $C_0$ such that

$$\mu(B(x,r)) \leq C_0 r^\alpha \quad (x \in A, 0 < r \leq \text{diam}(A)),$$

and we say that $\mu$ is lower $\alpha$-regular if there is some positive constant $c_0$ such that

$$c_0^{-1} r^\alpha \leq \mu(B(x,r)) \quad (x \in A, 0 < r \leq \text{diam}(A)).$$
We shall refer to $A$ as an upper $\alpha$-regular metric space if there exists an upper $\alpha$-regular measure $\bar{\mu}$ on $A$ such that $\bar{\mu}(A) > 0$ and shall refer to $A$ as a lower $\alpha$-regular metric space if there exists a lower $\alpha$-regular measure $\underline{\mu}$ on $A$ such that $\underline{\mu}(A) < \infty$. (Obviously, if $A$ is upper $\alpha$-regular then $A$ has infinitely many points.) If $A$ supports a measure that is both upper and lower $\alpha$-regular, then we say that $A$ is an $\alpha$-regular metric space. If $A$ is $\alpha$-regular, then it is not difficult to show that the Hausdorff dimension of $A$, $\dim_H A$, equals $\alpha$ (cf. [12]). Furthermore, the $\alpha$-dimensional Hausdorff measure of $A$, $\mathcal{H}_\alpha(A)$, is positive and finite.

Many of the constants appearing in this paper, either explicitly or implicitly involve the upper and lower regularity constants $C_0$ and $c_0$ appearing in (1.5) and (1.6). However, in certain cases we are interested in ‘local’ regularity estimates (i.e., for $r$ small) which can substantially improve our explicit estimates for particular metric spaces of interest (e.g., $A$ is the sphere $S^d$ with the Euclidean metric).

Specifically, if $\bar{\mu}$ is an upper $\alpha$-regular measure, $\underline{\mu}$ is a lower $\alpha$-regular measure and $r^* > 0$, we define

$$C_0(r^*) := \sup \{ \frac{\bar{\mu}(B(x,r))}{r^\alpha} : x \in A, 0 < r \leq r^* \},$$

$$c_0(r^*)^{-1} := \inf \{ \frac{\underline{\mu}(B(x,r))}{r^\alpha} : x \in A, 0 < r \leq r^* \}. $$

(1.7)

We note that both $C_0(r^*)$ and $c_0(r^*)$ are increasing in $r^*$, and we make the definitions

$$C_0(0) := \lim_{r^* \to 0^+} C_0(r^*),$$

$$c_0(0) := \lim_{r^* \to 0^+} c_0(r^*).$$

(1.8)

Furthermore, if $A$ is a compact (i.e., without boundary), $C^1$, $d$-dimensional manifold and $\mu = \mathcal{H}_d$, then $C_0(0) \cdot c_0(0) = 1$. For the largest length scale of interest, with a slight abuse of notation, the global constants for $\bar{\mu}$ and $\underline{\mu}$, respectively, are related by $C_0 = C_0(\text{diam}(A))$ and $c_0 = c_0(\text{diam}(A))$.

One may obtain simple upper bounds for $\delta_N(A)$ (respectively, lower bounds for $\rho_N(A)$) in the case that $A$ is lower (respectively, upper) $\alpha$-regular. Specifically, if $A$ is lower $\alpha$-regular then there is a constant $c_A < \infty$ such that

$$\delta_N(A) \leq c_A N^{-1/\alpha}, \quad (N \geq 2),$$

(1.9)

while if $A$ is upper $\alpha$-regular then there is a constant $\tilde{c}_A > 0$ such that

$$\rho_N(A) \geq \tilde{c}_A N^{-1/\alpha}, \quad (N \geq 2).$$

(1.10)

The bound (1.9) is a consequence of the facts that the balls $\{ B(x, \delta(\omega_N)/2) : x \in \omega_N \}$ are pairwise disjoint and that there exists a lower $\alpha$-regular measure $\mu$ with $\mu(A) < \infty$. Similarly, if $A$ is upper $\alpha$-regular, then the bound (1.10) follows from the covering property of the balls $\{ B(x, \rho(\omega_N, A)) : x \in \omega_N \}$ and the existence of an upper $\alpha$-regular measure $\bar{\mu}$ with $\bar{\mu}(A) > 0$.

The main result of this paper, given in Theorem 5, is that a sequence of $N$-point $(s, w)$-energy minimizing configurations on an $\alpha$-regular compact metric space $A$ is quasi-uniform on $A$ whenever $s > \alpha$. As an application, we deduce that, if $A \subset \mathbb{R}^p$ is $d$-rectifiable for some integer $0 < d \leq p$ with $\mathcal{H}_d(A) > 0$, then a quasi-uniform sequence of $N$-point configurations on $A$ can be found that has a prescribed bounded positive density on $A$ (see Corollary 6 and the preceding discussion). Furthermore we show that if $A$ is a smooth closed manifold, then there exists a sequence of $N$-point
best-packing configurations \( \{ \nu_N \}_{N=2}^\infty \) on \( A \) for which \( \limsup_{N \to \infty} \gamma(\nu_N, A) \leq 2 \) (see Corollary 8).

2. **Main Results**

We first consider the separation distance of \((s, w)\)-energy minimizing configurations on an upper \( \alpha \)-regular compact metric space \( A \). For these separation results, we consider symmetric weight functions \( w \) such that \( \|w(\cdot, x)\|_{L_p(\mu)} \) is uniformly bounded on \( A \) for some \( 1 < p \leq \infty \). Here we use the standard notation,

\[
\|f\|_{L^p(\mu)} := \begin{cases} \left( \int_A |f|^p \, d\mu \right)^{1/p}, & 1 \leq p < \infty, \\ \mu\text{-ess sup} |f|, & p = \infty, \end{cases}
\]

where \( \mu \) is a positive Borel measure and \( f \) is a Borel measurable function on \( A \).

The following theorem extends a result [1, Theorem 4] to a more general class of weight functions and to more general compact metric spaces.

**Theorem 1.** Let \( A \) be a compact, upper \( \alpha \)-regular metric space with respect to \( \bar{\mu} \) and let \( w \) be an SLP weight on \( A \) such that \( \|w(\cdot, x)\|_{L_{p_0}(\bar{\mu})} \) is uniformly bounded on \( A \) for some \( 1 < p_0 \leq \infty \). Suppose \( 1 < p \leq p_0 \), \( s > \alpha \left( 1 - \frac{1}{p} \right) \), and \( N \geq 2 \). If \( \omega_N^* \) is an \( N \)-point \((s, w)\)-energy minimizing configuration on \( A \), then

\[
\delta(\omega_N^*) \geq C_1 N^{-\left( \frac{1}{\alpha + \frac{1}{p}} \right)} \quad (N \geq 2),
\]

where \( C_1 \) is a constant independent of \( N \) indicated below in (3.13).

Taking \( w \) bounded and setting \( p = \infty \) in Theorem 1 produces the following result.

**Corollary 2.** Suppose \( A \) is a compact, upper \( \alpha \)-regular metric space and \( w \) is a bounded SLP weight on \( A \), and let \( s > \alpha \). If \( \omega_N^* \) is an \( N \)-point \((s, w)\)-energy minimizing configuration on \( A \), then

\[
\delta(\omega_N^*) \geq C_2 N^{-1/\alpha} \quad (N \geq 2),
\]

where \( C_2 \) is a constant independent of \( N \). Consequently,

\[
\delta_N(A) \geq C_2 N^{-1/\alpha} \quad (N \geq 2).
\]

For the unweighted case \( w \equiv 1 \), the constant \( C_2 \) satisfies

\[
C_2 \geq \left[ \frac{\bar{\mu}(A)}{C_0} \left( 1 - \frac{\alpha}{s} \right) \right]^{1/\alpha} \left( \frac{\alpha}{s} \right)^{1/s},
\]

where \( C_0 = C_0(\text{diam}(A)) \).

We note that if \( A \) in Corollary 2 is \( \alpha \)-regular, then by inequality (1.9) we see that \( N \)-point \((s, w)\)-energy minimizing configurations on \( A \) have the best possible order of separation as \( N \to \infty \).

With respect to the separation constant of (2.4), if \( d \geq 2 \) and \( A = S^d \) with \( \sigma_d \) denoting the uniform probability distribution on \( S^d \), then we can get an explicit lower bound for \( C_2 \) by calculating the regularity constant \( C_0 \). As stated in [13], for \( x \in S^d, 0 \leq r \leq 2 \), and

\[
\gamma_d := \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma(d/2) \Gamma(1/2)},
\]

and
there holds
\[ \sigma_d(r) := \sigma_d(B(x, r)) = \gamma_d \int_{1-r^2/2}^1 (1-t^2)^{d/2-1} dt \]
from which it follows that
\[ \sigma_d(r) \leq \frac{\gamma_d}{d}, \]
and, as \( r \to 0^+ \),
\[ \sigma_d(r) = \frac{\gamma_d}{d} r^d + \mathcal{O}(r^{d+2}). \]

Therefore, for the uniform probability distribution on \( S^d \), the global upper regularity constant is
\[ C_0 = \sup_{0 < r \leq 2} \frac{\sigma_d(r)}{r^d} = \frac{\gamma_d}{d}, \]
and when applied to (2.4) we obtain
\[ C_2 \geq \left( \frac{d}{\gamma_d} \right)^{1/d} \left( 1 - \frac{d}{s} \right)^{1/d} \left( \frac{d}{s} \right)^{1/s}. \]

With this lower bound for \( C_2 \), (2.2) becomes
\[ \delta(\omega_N^s) \geq \left( \frac{d}{\gamma_d} \right)^{1/d} \left( 1 - \frac{d}{s} \right)^{1/d} \left( \frac{d}{s} \right)^{1/s} N^{-1/d} \quad (N \geq 2, s > d), \]
and, on letting \( s \to \infty \), we deduce for the \( N \)-point best-packing distance
\[ \delta_N(S^d) \geq \left( \frac{d}{\gamma_d} \right)^{1/d} N^{-1/d} \quad (N \geq 2, s > d). \]

A less explicit lower bound for the separation constant of minimal energy points for \( s > d \) on \( S^d \) was obtained in [13, Corollary 4].

We next consider the mesh norm of \((s,w)\)-energy minimizing configurations on an \( \alpha \)-regular compact metric space \( A \). In this case we require that the weight function \( w \) be bounded.

**Theorem 3.** Let \( A \) be a compact, \( \alpha \)-regular metric space with respect to the measure \( \mu \) and \( K \subset A \) be a compact set of positive \( \mu \)-measure. Let \( w \) be a bounded SLP weight on \( K \). If \( s > \alpha \) and \( \omega_N^s \) is an \( N \)-point \((s,w)\)-energy minimizing configuration on \( K \), then
\[ \rho(\omega_N^s, K) \leq C_3 N^{-1/\alpha} \quad (N \geq 2), \]
where \( C_3 \) is a constant independent of \( N \) given below in (3.41).

**Theorem 3** substantially extends a result of [6] that holds for unweighted energy minimizing point configurations when \( K \subset \mathbb{R}^p \) is restricted to be the finite union of bi-Lipschitz images of compact sets in \( \mathbb{R}^d \).

We remark that for \( K \) and \( A \) as in Theorem 3, the set \( K \) need not inherit the lower \( \alpha \)-regularity of \( A \). However, since \( \mu(K) > 0 \), we do have that \( K \) is an upper \( \alpha \)-regular metric space and, consequently, there is a constant \( \tilde{c}_K > 0 \) such that (1.10) holds with \( A \) replaced by \( K \). Hence, the inequality (2.9) has the best possible order with respect to \( N \).

Taking \( w \equiv 1 \) in Theorem 3 immediately yields the following.
Corollary 4. Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. Then there exists a constant $C_4$ such that

$$\rho_N(K) \leq C_4 N^{-1/\alpha} \quad (N \geq 2).$$

Combining Corollary 2 and Theorem 3 we obtain our main result.

Theorem 5. Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. Furthermore, let $w$ be a bounded SLP weight on $K$, and for $s > \alpha$ and $N \geq 2$, let $\omega_N^s$ be an $N$-point $(s, w)$-energy minimizing configuration on $K$. Then $\{\omega_N^s\}_{N=2}^\infty$ is quasi-uniform on $K$.

We remark that there are $\alpha$-regular sets $A$ and values of $s < \alpha$ for which (unweighted) $(s, 1)$-energy minimizing configurations on $A$ have a mesh-separation ratio that goes to $\infty$ with $N$. One such example given in [3] is a ‘washer’ $A$ obtained by revolving a certain rectangle about an axis parallel to one of its sides, where it turns out that for $s < 1/3$, the support of the limit distribution of the $(s, 1)$-energy minimizing configurations on $A$ omits an open subset of $A$. Also, for the logarithmic energy which corresponds to $s = 0$, it is shown in [11] that, for $w \equiv 1$, the support of the limit distribution of the log-energy minimizing configurations on a torus in $\mathbb{R}^3$ is only supported on the positive curvature portion of the torus, so that the mesh-separation ratio for such configurations is again unbounded as $N \to \infty$. Examples also abound in one dimension. For the logarithmic energy, it is well-known [21, Sections 6.7 and 6.21] that for $A = [-1, 1]$ and $w \equiv 1$ the minimum energy points are zeros of Jacobi orthogonal polynomials (together with $\pm 1$) that have separation distance of precise order $1/N^2$ and mesh norm of precise order $1/N$, so that the mesh-separation ratio grows like $N$.

One of our main motivations for considering weighted minimum energy configurations is that for a large class of sets $A$ one can design a weight function $w$ so that a sequence of $N$-point $(s, w)$-energy minimizing configurations have a specified limiting density on $A$ as $N \to \infty$. The following result is a consequence of Theorem 5 and [1, Corollary 2]. Recall that a set in $\mathbb{R}^p$ is $d$-rectifiable if it is the Lipschitz image of a bounded set in $\mathbb{R}^d$.

Corollary 6. Let $d \leq p$ and $A \subset \mathbb{R}^p$ be a compact, infinite set that is $d$-rectifiable and lower $d$-regular with respect to $\mathcal{H}_d$ for some integer $d$. Suppose $\sigma$ is a probability density on $A$ that is continuous almost everywhere with respect to $\mathcal{H}_d$ and is bounded above and below by positive constants. Let $s > d$ and $w : A \times A \to [0, \infty)$ be given by

$$w(x, y) := (\sigma(x)\sigma(y))^{-s/2d}. \quad (2.10)$$

For $N \geq 2$, let $\omega_N^s$ be an $N$-point $(s, w)$-energy minimizing configuration on $A$. Then $\{\omega_N^s\}_{N=2}^\infty$ is quasi-uniform on $A$ and the sequence of normalized counting measures associated with the $\omega_N^s$'s converges weak-star (as $N \to \infty$) to $\sigma \mathcal{H}_d$.

For $A$ an infinite, compact, metric space and $s > 0$, let $\omega_N^s$ be an $N$-point $(s, 1)$-energy minimizing configuration on $A$. Furthermore, let $\nu_N$ be a cluster point (in the product topology on $A^N$) of $\omega_N^s$ as $s \to \infty$. As we now show, $\nu_N$ must be an
$N$-point best-packing configuration on $A$, that is, $\delta(\nu_N) = \delta_N(A)$. For this purpose, let $\tilde{\omega}_N$ be an $N$-point best-packing configuration on $A$. Then we have
\[
\delta(\omega^s_N)^{-s} \leq E_1^s(N, A) \leq E_1^s(\tilde{\omega}_N) \leq N(N - 1)\delta_N(A)^{-s},
\]
and so
\[
(N(N - 1))^{-1/s} \delta_N(A) \leq \delta(\omega^s_N) \leq \delta_N(A),
\]
which gives
\[
\lim_{s \to \infty} \delta(\omega^s_N) = \delta_N(A).
\]
Since $\omega^s_N \to \nu_N$ for some subsequence $s_j \to \infty$, it follows from (2.11) and continuity that $\delta(\nu_N) = \delta_N(A)$ and so $\nu_N$ is an $N$-point best-packing configuration on $A$.

In general, it is not true that a sequence of $N$-point best-packing configurations in $A$ is quasi-uniform on $A$ (e.g., if $A$ is the classical $(1/3)$-Cantor set in $[0,1]$ together with any point outside this interval). However, for $A$ as in Theorem 5, it turns out that by using $(s, 1)$-energy minimizing configurations on $A$ and taking $s \to \infty$ we can construct a sequence of $N$-point best-packing configurations in $A$ that is also quasi-uniform on $A$.

**Theorem 7.** Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. For $N \geq 2$, let $\nu_N$ be a cluster point of a family of $(s, 1)$-energy minimizing configurations on $K$ as $s \to \infty$. Then $\{\nu_N\}_{N=2}^\infty$ is a sequence of $N$-point best-packing configurations on $K$ that is also quasi-uniform on $K$.

Furthermore, the mesh-separation ratios satisfy
\[
\limsup_{N \to \infty} \gamma(\nu_N, K) \leq 2 \left( \frac{\mu(A)}{\mu(K)} \right)^{1/\alpha} \frac{[c_0(0) C_0(0)]^{1/\alpha}},
\]
where $c_0(0)$ and $C_0(0)$ are given in (1.8) for the set $A$.

We note that the constant on the right-hand side of (2.12) is at least 2 per (1.7) and (1.8). One can also establish an analogous result concerning the existence of quasi-uniform sequences of weighted best-packing configurations (cf. [2]). We leave this extension to the reader.

When applied to a compact $C^1$ manifold without boundary, Theorem 7 has the following corollary.

**Corollary 8.** If $A$ is a compact $d$-dimensional $C^1$ manifold without boundary and $K \subset A$ is a compact subset of positive $\mathcal{H}_d$ measure, then there exists a sequence of $N$-point best-packing configurations $\{\nu_N\}_{N=2}^\infty$ on $K$ such that
\[
\limsup_{N \to \infty} \gamma(\nu_N, K) \leq 2 \left( \frac{\mathcal{H}_d(A)}{\mathcal{H}_d(K)} \right)^{1/\alpha}.
\]
In particular, if $K = A$ we have
\[
\limsup_{N \to \infty} \gamma(\nu_N, A) \leq 2.
\]

The proof of Corollary 8 simply amounts to noting that for a $C^1$ manifold without boundary $A$, and $\mathcal{H}_d|_A = : \mu$, the constants $c_0(0)$ and $C_0(0)$ of (1.8) are reciprocals.
In comparison with (2.14), we remark that it is easy to see that there is always an increasing sequence \( \{N_k\}_{k=1}^\infty \) of positive integers such that for each \( k \) there is an \( N_k \)-point best-packing configuration \( \hat{\nu}_{N_k} \) on \( A \) such that \( \gamma(\hat{\nu}_{N_k}, A) < 2 \) for all \( k \); indeed, this follows from the observation that if \( \omega_N \) is any \( N \)-point configuration on \( A \) such that \( \gamma(\omega_N, A) \geq 2 \) then there is some \( x \in A \setminus \omega_N \) such that \( \delta(\omega_N \cup \{x\}) = \delta(\omega_N) \).

We conclude this section with further references to related results. Separation theorems for the case \( s \leq d = \text{dim} H(A) \) have been established only for rather special sets and values of \( s \). Dahlberg [5] proved that (unweighted) optimal \( ((p - 2), 1) \)-energy configurations \( \omega_N^* \) on \( A \) are well-separated (i.e., they satisfy \( \delta(\omega_N^*, A) \geq CN^{-1/d} \) for some positive constant \( C \)) if \( A \subset \mathbb{R}^p \) \((p \geq 3)\) is a smooth \( d = p - 1 \)-dimensional closed surface in \( \mathbb{R}^p \) that separates \( \mathbb{R}^p \) into two components. For the critical value \( s = d \) and \( A \) a \( d \)-rectifiable subset of a smooth \( d \)-dimensional manifold in \( \mathbb{R}^p \), it is shown in [1] that the following weaker separation result holds

\[
(2.15) \quad \delta(\omega_N^*) \geq C(N \log N)^{-1/d},
\]

for some positive constant \( C \).

For the case that \( A = S^d \), the \( d \)-dimensional unit sphere in \( \mathbb{R}^{d+1} \), well-separation was proved in [14] for the range of values \( d - 1 < s < d \) and further extended by Dragnev and Saff [8] to the range \( d - 2 < s < d \) with explicit estimates for the separation constant \( C \). Well-separation for \( s = d - 2 \) and \( d \geq 3 \) was established in [6].

Thus, for the important case of \( A = S^2 \) it is known that optimal \( s \)-energy configurations on \( S^2 \) are well-separated for all nonnegative values of \( s \neq 2 \) (well-separatedness for \( s = 0 \) was established in [18]; see also [7]); for the critical value \( s = 2 \), the only known separation results are of the weak form given in (2.15).

Much less is known with regard to covering (mesh norm) theorems in the case that \( s \leq d \) (see [20, Sec. 1.3]).

3. Proofs

In the proofs we shall need that an SLP weight \( w \) is bounded below in a neighborhood of the diagonal \( D(A) \). Indeed, the positivity and lower semi-continuity of \( w \) on \( D(A) \) and the compactness of \( A \) imply that there are positive numbers \( \eta \) and \( \kappa \) such that

\[
(3.1) \quad w(x, y) \geq \eta \quad (x, y \in A, m(x, y) \leq \kappa).
\]

**Proof of Theorem 1.** The initial part of this argument proceeds as in [13]. Let \( N \geq 2 \) be fixed and let \( \omega_N^* = \{x_1, \ldots, x_N\} \subset A \) be a fixed \((s, w)\)-energy minimizing configuration in \( A \). For \( x \in A \) and \( 1 \leq i \leq N \), let

\[
U_i(x) := \sum_{j=1}^{N} \frac{w(x, x_j)}{m(x, x_j)^s},
\]

Since \( \omega_N^* \) is a minimizing configuration we have the lower bound

\[
(3.2) \quad U_i(x_i) \leq U_i(x) \text{ for all } x \in A.
\]
Fix \( r_1 \leq \text{diam}(A) \) such that

\[
\bar{\mu} \left( \bigcup_{j=1}^{N} B(x_j, r_1) \right) \geq \bar{\mu}(A).
\]

The radius \( r_1 \) can clearly be chosen independent of \( N \), for example \( r_1 = \text{diam}(A) \), and we note for future reference that it suffices to take \( r_1 > \rho(\omega_N^*, A) \). For the rest of this proof we fix \( r_1 = \text{diam}(A) \).

Now let \( 0 < \theta < 1 \) and define

\[
r_0 := \left( \frac{\theta \bar{\mu}(A)}{N C_0(r_1)} \right)^{1/\alpha},
\]

where \( C_0(r_1) = C_0 \) is the upper regularity constant of \( \bar{\mu} \) as in (1.7). We note that \( r_0 < r_1 \) as can be seen from the fact that \( \bar{\mu}(A) \leq C_0(r_1) r_1^\alpha \).

For \( B(x, r_0, r_1) := B(x, r_1) \setminus B(x, r_0) \), let

\[ D := \bigcup_{j=1}^{N} B(x_j, r_0, r_1). \]

Using the upper regularity of \( \bar{\mu} \) and (3.3) we see that

\[
\bar{\mu}(D) \geq \bar{\mu}(A) - \sum_{j=1}^{N} \bar{\mu}(B(x_j, r_0)) \geq (1 - \theta) \bar{\mu}(A) > 0,
\]

and thus by inequality (3.2) we have

\[
U_i(x_i) \leq \frac{1}{\bar{\mu}(D)} \int_{D} U_i(x) \, d\bar{\mu}(x) \leq \frac{1}{(1 - \theta)\bar{\mu}(A)} \sum_{j=1}^{N} \int_{B(x_j, r_0, r_1)} \frac{w(x, x_j)}{m(x, x_j)^s} \, d\bar{\mu}(x).
\]

Applying Hölder’s inequality with \( 1/q = 1 - 1/p \) we obtain

\[
U_i(x_i) \leq \frac{1}{(1 - \theta)\bar{\mu}(A)} \sum_{j=1}^{N} \|w(\cdot, x_j)\|_{L_p(\bar{\mu})} \left( \int_{B(x_j, r_0, r_1)} \frac{1}{m(x, x_j)^{sq}} \, d\bar{\mu}(x) \right)^{1/q}.
\]

Converting the integral on the right-hand side of (3.6) to the appropriate integral of the distribution function, and noting that \( sq > \alpha \) by assumption, we have

\[
\int_{B(x_j, r_0, r_1)} \frac{1}{m(x, x_j)^{sq}} \, d\bar{\mu}(x) = \int_{0}^{\infty} \bar{\mu} \left( \{ x \in B(x_j, r_0, r_1) : m(x, x)^{-sq} > t \} \right) dt \leq \int_{r_1^{-sq}}^{r_0^{-sq}} \bar{\mu} \left( B(x_j, t^{-1/sq}) \right) dt \leq \frac{C_0(r_1)^{sq}}{sq - \alpha} \int_{r_1^{-sq}}^{r_0^{-sq}} t^{\alpha - sq} dt \leq \frac{C_0(r_1)^{sq}}{sq - \alpha} \left( \frac{\theta \bar{\mu}(A)}{NC_0(r_1)} \right)^{1-(sq)/\alpha},
\]
which, combined with (3.6), gives
\begin{equation}
U_i(x_i) \leq \frac{\|w\|_{p,\infty}}{(1 - \theta)\bar{\mu}(A)} \left( \frac{C_0(r_1) sq}{sq - \alpha} \right)^{1/q} (N - 1) \left( \frac{\theta\bar{\mu}(A)}{NC_0(r_1)} \right)^{1/q - s/\alpha}
\end{equation}
(3.8)
\[ < \frac{1}{\bar{\mu}(A)} \left( \frac{C_0(r_1)}{s/\alpha} \right)^{s/\alpha} \left( \frac{\|w\|_{p,\infty}}{(1 - \theta)\bar{\mu}(A)} \frac{(sq\bar{\mu}(A))}{sq - \alpha} \right)^{1/q} N^{1/p + s/\alpha}, \]
where \( \|w\|_{p,\infty} := \sup_{x \in A} \|w(\cdot, x)\|_{L_p(\bar{\mu})} < \infty. \)

Choosing
\begin{equation}
\theta_0 := \frac{sq - \alpha}{sq - \alpha + \alpha q} = \left( \frac{s}{\alpha} - \frac{1}{q} \right) \left( \frac{s}{\alpha} + \frac{1}{p} \right)^{-1} < 1,
\end{equation}
(3.9)
which minimizes the right-hand side of (3.8) with respect to \( \theta \), we obtain
\begin{equation}
U_i(x_i) \leq c_1 N^{s/\alpha + 1/p},
\end{equation}
(3.10)
where after a bit of arithmetic we have
\begin{equation}
c_1 := \|w\|_{p,\infty} \left( \frac{C_0(r_1) s/\alpha + 1/p}{\bar{\mu}(A)} \right)^{s/\alpha} \left( \frac{s/\alpha + 1/p}{s/\alpha - 1/q} \right)^{1/p} (s/\alpha)^{1/q}.
\end{equation}
(3.11)

Next, select the indices \( 1 \leq i_s \neq j_s \leq N \) so that \( \delta(\omega_N^*) = m(x_{i_s}, x_{j_s}) \) and let \( \kappa \) and \( \eta \) be as in (3.1). If \( \delta(\omega_N^*) \leq \kappa \), then
\begin{equation}
\frac{\eta}{\delta(\omega_N^*)^s} \leq \frac{w(x_{i_s}, x_{j_s})}{m(x_{i_s}, x_{j_s})} \leq U_i(x_{i_s}) \leq c_1 N^{s/\alpha + 1/p},
\end{equation}
(3.12)
and therefore
\begin{equation}
\delta(\omega_N^*) \geq \left( \frac{\eta}{c_1} \right)^{1/s} N^{-\frac{1}{s} - \frac{1}{sp}}.
\end{equation}

Hence, (2.1) holds with
\begin{equation}
C_1 := \min\{\kappa, (\eta/c_1)^{1/s}\}.
\end{equation}
(3.13)

We remark that for the case when \( w \equiv 1 \) and \( p = \infty \), we can take \( \kappa = \infty \), \( \eta = 1 \), and so from (3.13) we deduce the separation estimate
\begin{equation}
\delta(\omega_N^*) \geq C_2 N^{-1/\alpha} \quad (N \geq 2),
\end{equation}
where
\begin{equation}
C_2 := \left[ \frac{\bar{\mu}(A)}{C_0(r_1)} \right]^{1/\alpha} (1 - \alpha/s)^{1/s}, \quad r_1 = \text{diam}(A).
\end{equation}
(3.14)

For the proof of Theorem 3, we utilize the following.

**Lemma 9.** Let \( A \) be a compact, infinite, lower \( \alpha \)-regular metric space with lower \( \alpha \)-regular measure \( \mu \), \( w : A \times A \to [0, \infty) \) be an SLP weight on \( A \), and \( s > \alpha \). Then there exists a positive integer \( N_0 \) independent of \( s \), such that
\begin{equation}
\mathcal{E}_s^w(N, A) \geq C_5 N^{1+s/\alpha} \quad (N \geq N_0),
\end{equation}
(3.15)
where \( C_5 \) is a constant independent of \( N \) given below in (3.19).
Proof. Let $\kappa$ and $\eta$ be as in (3.1) and let $0 < r_2 \leq \kappa$. Since $A$ is compact, there is some $M$ such that the $M$-point best-packing distance satisfies
\begin{equation}
\delta_M(A) \leq r_2.
\end{equation}

Let $N > M$ and let $\omega_N = \{x_1, \ldots, x_N\} \subset A$ be an arbitrary $N$-point configuration of distinct points. For $1 \leq i \leq N$, let $y_i \in \omega_N$ be a fixed nearest neighbor to $x_i$ in the configuration $\omega_N$, and set
\[
\delta_i := m(x_i, y_i) = \min_{1 \leq j \leq N, j \neq i} m(x_i, x_j) > 0.
\]

We assume an ordering on $\omega_N$ so that $\delta_i \leq \delta_{i+1}$ for $i = 1, \ldots, N - 1$. We note that $\omega_N \setminus \{x_1, \ldots, x_{N-M}\}$ is of cardinality $M$ and thus for all $i \leq N' := N - M$ we have that $\delta_i \leq r_2 \leq \kappa$.

The energy of $\omega_N$ then has the lower bound
\begin{equation}
E_\alpha^w(\omega_N) \geq \sum_{i=1}^{N'} \frac{w(x_i, y_i)}{\delta_i^s} \geq \sum_{i=1}^{N'} \eta \left( \frac{1}{\delta_i^s} \right)^{s/\alpha} \geq \eta \left( \sum_{i=1}^{N'} \frac{1}{\delta_i^s} \right)^{s/\alpha} (N')^{1-s/\alpha}
\end{equation}
\begin{equation}
\geq \eta \left( \sum_{i=1}^{N'} \delta_i^{-s/\alpha} \right)^{-s/\alpha} (N')^{1+s/\alpha} = \eta 2^{-s} \left( \sum_{i=1}^{N'} \left( \frac{\delta_i}{2} \right) ^{-s/\alpha} \right)^{-s/\alpha} (N')^{1+s/\alpha}.
\end{equation}
where the last inequality in the first line follows from Jensen’s inequality and the subsequent inequality follows from the harmonic-arithmetic mean inequality.

Let $\Lambda > 1$ and $N_0 := M\Lambda/(\Lambda - 1)$. Then $N' = N - M \geq \Lambda^{-1}N$ for $N \geq N_0$. Noting that the balls $B(x_i, \delta_i/2)$ are pairwise disjoint, we may apply the lower regularity of $\mu$ (with regularity constant $c_0(r_2)$) to obtain
\begin{equation}
E_\alpha^w(\omega_N) \geq \eta 2^{-s} \left( c_0(r_2) \sum_{i=1}^{N'} \mu(A) \left( \frac{B(x_i, \delta_i/2)}{2} \right) \right)^{-s/\alpha} (N')^{1+s/\alpha}
\end{equation}
\begin{equation}
\geq \eta \frac{\Lambda^{-1-s/\alpha}}{(2^s c_0(r_2) \mu(A))^{s/\alpha}} N^{1+s/\alpha}.
\end{equation}
Since (3.18) holds for arbitrary $N$-point configurations $\omega_N \subset A$ with $N \geq N_0$, we obtain that (3.15) holds with
\begin{equation}
C_5 := \Lambda^{-1-s/\alpha} \eta 2^{-s} (c_0(r_2) \mu(A))^{-s/\alpha}.
\end{equation}
We remark that $N_0$ depends on $\Lambda$ and $r_2$, but is independent of $s$. \hfill \Box

Proof of Theorem 3. Appealing to the generality provided by Theorem 1 and Lemma 9, we can substantially extend and improve upon the arguments used in the proof of Theorem 3.6 in [6].

Let $\omega_N' = \{x_1, \ldots, x_N\}$ be an $N$-point $(s, w)$-energy minimizing configuration for the compact set $K$, and, for $y \in K$, consider the function
\begin{equation}
U(y) := \frac{1}{N} \sum_{i=1}^{N} \frac{w(y, x_i)}{m(y, x_i)^s}.
\end{equation}
For fixed $1 \leq j \leq N$, the function $U(y)$ can be decomposed as

\begin{equation}
U(y) = \frac{1}{N} \sum_{i=1}^{N} \frac{w(y, x_i)}{m(y, x_i)^{s}},
\end{equation}

and, since $\omega_N^*$ is a minimizing configuration on $K$, the point $x_j$ minimizes the sum over $i \neq j$ on the right-hand side of equation (3.21). Thus for each fixed $j$ and $y \in K$

\begin{equation}
U(y) \geq \frac{1}{N} \sum_{i=1}^{N} \frac{w(x_j, x_i)}{m(x_j, x_i)^{s}}.
\end{equation}

Summing over $j$ gives

\begin{equation}
NU(y) \geq \frac{1}{N} \sum_{j=1}^{N} \frac{w(y, x_j)}{m(y, x_j)^{s}} + \frac{1}{N} \sum_{j=1}^{N} \sum_{i \neq j} \frac{w(x_j, x_i)}{m(x_j, x_i)^{s}}
\end{equation}

\begin{equation}
= U(y) + \frac{1}{N} E^w_s(N, K),
\end{equation}

and thus

\begin{equation}
U(y) \geq \frac{1}{N(N-1)} E^w_s(N, K) \geq \frac{E^w_s(N, K)}{N^2} \quad (y \in K).
\end{equation}

Since $K$ is compact, there exists a point $y^* \in K$ such that

\begin{equation}
\min_{1 \leq i \leq N} m(y^*, x_i) = \rho(\omega_N^*, K) =: \rho(\omega_N^*).
\end{equation}

Using the fact that a function is lower semi-continuous if and only if it is the limit of an increasing sequence of continuous functions, it is not difficult to show that since $w$ is a bounded SLP weight on $K$, it may be extended to a bounded SLP weight on $A$. Then, by Lemma 9, there are constants $N_0$ and $C_5 > 0$ such that

\begin{equation}
E^w_s(N, K) \geq E^w_s(N, A) \geq C_5 N^{1+s/\alpha} \quad (N \geq N_0).
\end{equation}

We note that the constant $C_5$ of (3.27) does not depend on $K$, but rather on $A$ (specifically on the lower regularity constant of $A$ and on $\mu(A)$) as well as on the extended weight $w$.

Since (3.25) holds for the point $y^*$ of (3.26), we combine (3.25) with (3.27) to obtain

\begin{equation}
U(y^*) \geq \frac{E^w_s(N, K)}{N^2} \geq C_5 N^{s/\alpha - 1} \quad (N \geq N_0).
\end{equation}

Next we determine an upper bound for $U(y^*)$ using the $\alpha$-regularity of the superset $A$. Since $A$ is upper $\alpha$-regular, we see that $K$ is also because $\mu(K) > 0$. Hence, Corollary 2 applied to $K$ implies that there is some $C_2 > 0$ such that $\delta(\omega_N^*) \geq C_2 N^{-1/\alpha}$ for $N \geq 2$. We note that the constant $C_2$ here depends on $K$, specifically $\mu(K)$.

Let $\mathcal{N}$ consist of those $N \geq N_0$ such that

\begin{equation}
\rho(\omega_N^*) \geq \frac{C_2}{2} N^{-1/\alpha}.
\end{equation}
If $\mathcal{N}$ is empty (or finite) then we are done. Assuming that $\mathcal{N}$ is nonempty, let $N \in \mathcal{N}$ be fixed.

For $0 < \epsilon < 1/2$, let

$$r_0 = r_0(N, \epsilon) := \epsilon C_2 N^{-1/\alpha}.$$  

Note that any two of the balls $B(x_i, r_0) \subset A$, for $1 \leq i \leq N$, do not intersect since $r_0 < \delta(\omega_N^*)/2$.

For any $x \in B(x_i, r_0)$, inequalities (3.26) and (3.29) imply

$$m(x, y^*) \leq m(x, x_i) + m(x_i, y^*) \leq r_0 + m(x_i, y^*) \leq 2 \epsilon \rho(\omega_N^*) + m(x_i, y^*) \leq (1 + 2 \epsilon)m(x_i, y^*).$$

For fixed $1 \leq i \leq N$, using (3.31) and taking an average value on $B(x_i, r_0)$ we obtain

$$\frac{w(x_i, y^*)}{m(x_i, y^*)^s} \leq \frac{\|w\|_\infty (1 + 2 \epsilon)^s}{\mu(B(x_i, r_0))} \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s} \leq \frac{\|w\|_\infty (1 + 2 \epsilon)^s c_0(r_0)}{r_0^\alpha} \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s},$$

where $\|w\|_\infty$ denotes the sup-norm of $w$ on $A \times A$ and $c_0(r_0)$ is the localized constant of (1.7) for the set $A$.

Inequality (3.29) and definition (3.30) imply $2 \epsilon \rho(\omega_N^*) \geq r_0$ and thus, for $x \in B(x_i, r_0)$, we obtain

$$m(x, y^*) \geq m(x_i, y^*) - m(x, x_i) \geq m(x_i, y^*) - r_0 \geq m(x_i, y^*) - 2 \epsilon \rho(\omega_N^*) \geq (1 - 2 \epsilon) \rho(\omega_N^*).$$

Inequality (3.33) implies

$$\bigcup_{i=1}^N B(x_i, r_0) \subset A \setminus B(y^*, (1 - 2 \epsilon) \rho(\omega_N^*)), $$

and since the left-hand side is a disjoint union, averaging the inequalities of (3.32) we have

$$U(y^*) \leq \frac{\|w\|_\infty (1 + 2 \epsilon)^s c_0(r_0)}{N r_0^\alpha} \sum_{i=1}^N \int_{B(x_i, r_0)} \frac{d\mu(x)}{m(x, y^*)^s} \leq \frac{\|w\|_\infty (1 + 2 \epsilon)^s c_0(r_0)}{N r_0^\alpha} \int_{A \setminus B(y^*, (1 - 2 \epsilon) \rho(\omega_N^*))} \frac{d\mu(x)}{m(x, y^*)^s}.$$ 

For fixed $\tau \geq 1$ we define the radius $R(N) := \tau(1 - 2 \epsilon) \rho(\omega_N^*)$, and the constant

$$\tilde{C}_0(\tau) := C_0(R(N))(1 - \tau^{\alpha-s}) + C_0 \tau^{\alpha-s}.$$ 

Note that if $\tau = 1$, then $\tilde{C}_0(1) = C_0$. (We retain $\tau$ as a parameter in our estimates as an option for the reader to optimize $C_3$ for a fixed $s$.) Now we break the integral
on the right-hand side of (3.34) into two terms and proceed as in (3.7) to obtain
\begin{equation}
\int_{A\setminus B(y^\star, (1-2\epsilon)\rho(\omega_N^\star))} \frac{d\mu(x)}{m(x, y^\star)^s} = \int_{B(y^\star, (1-2\epsilon)\rho(\omega_N^\star)R(N))} \frac{d\mu(x)}{m(x, y^\star)^s} + \int_{A\setminus B(y^\star, R(N))} \frac{d\mu(x)}{m(x, y^\star)^s} \leq C_0(R(N)) \int_{R(N)^{-s}} t^{-\alpha/s} dt + C_0 \int_0^{R(N)^{-s}} t^{-\alpha/s} dt = \frac{\tilde{C}_0(\tau)}{(1-\alpha/s)(1-2\epsilon)s-\alpha}\rho(\omega_N^\star)_{\alpha-s}.
\end{equation}

It is convenient to define the quantity
\begin{equation}
\beta(\epsilon) := \frac{\|w\|_{\infty}(1+2\epsilon)^s}{(1-\alpha/s)(1-2\epsilon)s-\alpha(\epsilon C_2)^{1/s}},
\end{equation}
and we note that for fixed \(s > \alpha\) it is minimized as a function of \(\epsilon\) for
\begin{equation}
\epsilon_0 := \frac{1}{2(2(s/\alpha) - 1)} < \frac{1}{2},
\end{equation}
with minimal value
\begin{equation}
\beta_0 := \beta(\epsilon_0) = \frac{\|w\|_{\infty}}{(1-\alpha/s)s-\alpha+1} \left(\frac{4s}{\alpha C_2}\right)^{1/s}.\end{equation}

Using \(\epsilon_0\) and combining inequality (3.34) with inequality (3.36) we obtain
\begin{equation}
U(y^\star) \leq c_0(r_0)\beta_0\tilde{C}_0(\tau)\rho(\omega_N^\star)_{\alpha-s}.
\end{equation}
If \(N \in \mathcal{N}\), then (4.10) and (3.28) imply
\begin{equation}
\rho(\omega_N^\star) \leq \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)} N^{-1/\alpha}.
\end{equation}
If \(N \notin \mathcal{N}\), then either \(N \leq N_0\) or \(\rho(\omega_N^\star) < C_2^{-1/2} N^{-1/\alpha}\). Hence (2.9) holds with
\begin{equation}
C_3 := \max\left\{\text{diam}(A)N_0^{1/\alpha}, \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)} \left(\frac{C_2}{2}\right)\right\}.
\end{equation}
We note that if \(N > N_0\), then it suffices to take
\begin{equation}
C_3 = \max\left\{\left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)} \left(\frac{C_2}{2}\right)\right\}.
\end{equation}

\textbf{Proof of Theorem 7}. Starting with Theorem 3 we shall employ a bootstrapping argument whereby the constants \(C_2, C_5\), and subsequently \(C_3\) are redefined so as to depend on \(N\).

We begin by noting that if \(s \geq 2\alpha\), then the constant \(C_3\) of (3.41) has a uniform upper bound in \(s\); indeed, with \(\kappa = \infty, C_2\) as defined in (3.14) and \(C_5\) as defined in (3.19) (with \(\eta = 1\)), each of the three terms appearing in braces in (3.41) is
uniformly bounded above. Thus there exists a constant $C^*$ independent of $N \geq 2$ and of $s \geq 2\alpha$ such that $\rho(\omega_N^{(s)}, K) < C^* N^{-1/\alpha}$, where $\omega_N^{(s)}$ is any $N$-point $(s, 1)$-energy minimizing configuration on $K$.

We next note that $C_0(0)$ of (1.8) is finite and positive, and utilizing the constant $c_A$ of (1.9) we fix

$$C^* := \max \left\{ C^*, \ c_A, \ \left( \frac{\mu(K)}{C_0(0)} \right)^{1/\alpha} \right\},$$

and we now redefine the radius $r_1$ to be a function of $N$,

$$r_1(N) := C^* N^{-1/\alpha} \ (N \geq 2).$$

Returning to the proof of Theorem 1, we note that $r_1(N) > \rho(\omega_N^{(s)}, K)$, and so inequality (3.3) holds. Furthermore, by the choice of $C^*$ we have that for $0 < \theta_0 < 1$ as in (3.9)

$$r_0(N) := \left( \frac{\theta_0 \mu(K)}{N C_0(0)} \right)^{1/\alpha} < r_1(N).$$

Taking $r_0 = r_0(N)$ in the proof and remembering that $q = 1$ in the current context, we see that with $A$ replaced by $K$ the penultimate term on right-hand side of (3.7) becomes

$$s C_0(r_1(N)) \left( \frac{\theta_0 \mu(K)}{N C_0(0)} \right)^{1-s/\alpha},$$

and thus

$$\int_{B(x_j, r_0(N), r_1(N))} \frac{d\mu(x)}{m(x, x_j)^s} \leq \frac{s C_0(r_1(N))}{s - \alpha} \left( \frac{\theta_0 \mu(K)}{N C_0(0)} \right)^{1-s/\alpha} \leq \frac{s}{s - \alpha} \left( \frac{\theta_0 \mu(K)}{N} \right)^{1-s/\alpha} C_0(r_1(N))^{s/\alpha},$$

where the last inequality follows from the fact that $C_0(0) \leq C_0(r_1(N))$ and $s > \alpha$.

For $w \equiv 1$, the constant $C_2$ of (3.14) with $r_1 = r_1(N)$ becomes

$$C_2(N) := \left( \frac{\alpha}{s} \right)^{1/s} \left( \frac{1 - \alpha/s}{C_0(r_1(N))} \right)^{1/\alpha} \mu(K)^{1/\alpha},$$

where $C_0(r_1(N))$ is the local upper regularity constant of (1.7), and we have

$$\delta(\omega_N^{(s)}) \geq C_2(N) N^{-1/\alpha} \ (N \geq 2, \ s \geq 2\alpha).$$

Furthermore, allowing the radius $r_2$ appearing in (3.16) to depend on $N \geq 2$ by taking $r_2 := r_1(N)$, we see via (1.9) and (3.43) that

$$r_1(N) \geq \delta_N(A) \ (N \geq 2),$$

and there is no need to designate the integer $M$ in the proof of Lemma 9. Thus we can take $\Lambda = 1$ in (3.19), and it follows (with $\eta = 1$) that

$$E_1^s(\omega_N^{(s)}) \geq C_5(N) N^{1+s/\alpha} \ (N \geq 2, \ s \geq 2\alpha),$$

where
Theorem 7. The argument leading to equation (2.11) shows that
\[ \nu \]

Therefore, the sequence of configurations
\[ \{3.53\} \]

best-packing configuration on
\[ K \]

limits of the constants
\[ N \]

for fixed
\[ N \]

(3.47)
\[ C_5(N) := \frac{1}{2^s [c_0(r_1(N))\mu(A)]^{s/\alpha}}. \]

We remark that \( C_2(N) \) clearly depends on the subset \( K \), whereas \( C_5(N) \) depends on the superset \( A \).

We now return to the proof of Theorem 3 utilizing the constants \( C_2(N) \) and \( C_5(N) \). For \( \beta_0 \) as in (3.39), we see that
\[ \rho(\omega_N^{(s)}, K) \leq C_3(N)N^{-1/\alpha} \quad (N \geq N_0, \ s \geq 2\alpha), \]

where \( N_0 \) is as in Lemma 9, and by (3.42) (choosing \( \tau = 1 \), so that \( \hat{C}_0(\tau) = C_0 \))
\[ C_3(N) := \max \left\{ \frac{c_0(r_0)\beta_0 C_0}{C_5(N)} 1^{(s-\alpha)} \frac{C_2(N)}{2} \right\}. \]

(3.48)

With equations (3.46)-(3.48) in mind, we are ready to complete the proof of Theorem 7. The argument leading to equation (2.11) shows that \( \nu_N \) is an \( N \)-point best-packing configuration on \( K \) for each \( N \geq 2 \). We now need to determine the limits of the constants \( C_2(N) \) of (3.46) and \( C_3(N) \) of (3.48) as \( s \to \infty \). Fixing \( N \) in (3.46) yields
\[ \lim_{s \to \infty} C_2(N) = \left( \frac{\mu(K)}{C_0(r_1(N))} \right)^{1/\alpha} =: \hat{C}_2(N). \]

(3.49)

Since \( c_0(r_0) \) and \( C_0 \) are independent of \( s \) and \( \lim_{s \to \infty} \beta_0^{1/(s-\alpha)} = 1 \), it follows, that for fixed \( N \)
\[ \lim_{s \to \infty} C_3(N) = \max \left\{ \frac{\hat{C}_2(N)}{2}, \lim_{s \to \infty} C_5(N)^{1/(a-s)} \right\} \]
\[ = \max \left\{ \frac{1}{2} \left( \frac{\mu(K)}{C_0(r_1(N))} \right)^{1/\alpha} [c_0(r_1(N))\mu(A)]^{1/\alpha} \right\}. \]

(3.50)

\[ := \hat{C}_3(N) \]

From the continuity of \( \delta(\cdot) \) and \( \rho(\cdot, K) \) on \( K^N \) we deduce that
\[ \delta(\nu_N) \geq \hat{C}_2(N)N^{-1/\alpha} \quad \text{and} \quad \rho(\nu_N, K) \leq \hat{C}_3(N)N^{-1/\alpha} \quad (N \geq N_0). \]

Taking the ratio of these two quantities we have that
\[ \frac{\rho(\nu_N, K)}{\delta(\nu_N)} \leq \frac{\hat{C}_3(N)}{\hat{C}_2(N)} = \max \left\{ \frac{1}{2}, 2 \left( \frac{\mu(A)}{\mu(K)} \right)^{1/\alpha} [c_0(r_1(N))C_0(r_1(N))]^{1/\alpha} \right\}, \]

(3.51)

and hence for \( N \geq N_0 \)
\[ \limsup_{N \to \infty} \frac{\rho(\nu_N, K)}{\delta(\nu_N)} \leq \max \left\{ \frac{1}{2}, 2 \left( \frac{\mu(A)}{\mu(K)} \right)^{1/\alpha} [c_0(0)C_0(0)]^{1/\alpha} \right\} \]
\[ = 2 \left( \frac{\mu(A)}{\mu(K)} \right)^{1/\alpha} [c_0(0)C_0(0)]^{1/\alpha} < \infty. \]

(3.52)

(3.53)

Therefore, the sequence of configurations \( \{\nu_N\}_{N=2}^\infty \) is quasi-uniform on \( K \). \( \square \)
Acknowledgments. We thank the referees for their careful reading and detailed comments.

References

[1] S.V. Borodachov, D.P. Hardin, E.B. Saff, Asymptotics for discrete weighted minimal Riesz energy problems on rectifiable sets, Trans. Amer. Math. Soc. 360 (2008), 1559-1580.
[2] S.V. Borodachov, E.B. Saff, D.P. Hardin, Asymptotics of densest weighted packing on rectifiable sets (Russian) Mat. Sb. 199 (2008), 3-20; translation in Sb. Math. 199 (2008), 1579-1595.
[3] J.S. Brauchart, D.P. Hardin, E.B. Saff, The support of the limit distribution of optimal Riesz energy points on sets of revolution in $\mathbb{R}^3$, J. Math. Phys. 48 (2007), 122901, 24 pp.
[4] J.S. Brauchart, D.P. Hardin, E.B. Saff, Discrete energy asymptotics on a Riemannian circle, preprint (2010)
[5] B.E.J. Dahlberg, On the distribution of Fekete points, Duke Math. J. 45 (1978), 537-542.
[6] S.B. Damelin, V. Maymeskul; On point energies, separation radius, and mesh norm for $s$-extremal configurations on compact sets in $\mathbb{R}^n$, J. Complexity 21 (2005), 845-863.
[7] P.D. Dragnev, On the separation of logarithmic points on the sphere, Approximation Theory X: Abstract and Classical Analysis (C. K. Chui, L. L. Schumaker, and J. Stöckler, eds.) (2002) 137-144.
[8] P.D. Dragnev, E.B. Saff, Riesz spherical potentials with external fields and minimal energy points separation, Potential Anal. 26 (2007), 139-162.
[9] E.J. Fuselier, G.B. Wright, Stability and error estimates for vector field interpolation and decomposition on the sphere with RBFs, SIAM J. Numer. Anal. 47 (2009), 3213-3239.
[10] D.P. Hardin, E.B. Saff, Minimal Riesz energy point configurations for rectifiable $d$-dimensional manifolds, Adv. Math. 193 (2005), 174-204.
[11] D.P. Hardin, E.B. Saff, H. Stahl, Support of the logarithmic equilibrium measure on sets of revolution in $\mathbb{R}^3$, J. Math. Phys., 48 (2007), 022901, 14 pp.
[12] J. Heinonen, Lectures on Analysis on Metric Spaces, Springer-Verlag New York, Universitext (2001)
[13] A.B.J. Kuijlaars, E.B. Saff, Asymptotics for minimal discrete energy on the sphere, Trans. Amer. Math. Soc. 350 (1998), 523-538.
[14] A.B.J. Kuijlaars, E.B. Saff, X. Sun, On a separation of minimal Riesz energy points on spheres in Euclidean spaces, J. Comp. Appl. Math. 199: (2007) 172-180.
[15] Q.T. Le Gia, I.H. Sloan, H. Wendland, Multiscale analysis in Sobolev spaces on the sphere, SIAM J. Numer. Anal. 48 (2010), 20652090, 65J10 (46E35 46N40)
[16] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces; fractals and rectifiability, Cambridge Studies in Advanced Math. vol. 44, Cambridge University Press (1995).
[17] I. Pesenson, A sampling theorem on homogeneous manifolds, Trans. Amer. Math. Soc. 352 (2000), 4257-4269.
[18] E.A. Rakhmanov, E.B. Saff, Y.M. Zhou, Electrons on the sphere. Comput. Methods Funct. Theory, 293–309, Ser. Approx. Decompos., 5, World Sci. Publ., River Edge, NJ, 1995. 78A30 .
[19] R. Schaback, Error estimates and condition numbers for radial basis function interpolation, Adv. Comput. Math. 3 (1995), 251–264.
[20] A. Schürmann, Computational Geometry of Positive Definite Quadratic Forms, University Lecture Series vol. 48, Amer. Math. Soc., Providence, RI (2009).
[21] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, RI, 4th edition (1975).

D. P. Hardin, E. B. Saff, and J. T. Whitehouse: Center for Constructive Approximation, Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: Doug.Hardin@Vanderbilt.Edu
E-mail address: Edward.B.Saff@Vanderbilt.Edu
E-mail address: Tyler.Whitehouse@gmail.com