Maximum entropy principle and the form of source in non-equilibrium statistical operator method

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Abstract. It is supposed that the exponential multiplier in the method of the non-equilibrium statistical operator (Zubarev’s approach) can be considered as a distribution density of the past lifetime of the system, and can be replaced by an arbitrary distribution function. To specify this distribution the method of maximum entropy principle as in [Schönfeldt J-H, Jiminez N, Plastino A R, Plastino A, Casas M 2007 Physica A 374 573] is used. The obtained distribution is close to exponential one. Another approach to the maximum entropy principle, as in [Van der Straeten E and Beck C 2008 Phys. Rev. E 78 051101], except exponential distributions yields power-like, log-normal distributions, as well as distributions of other kind and transitions between them.

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1. Introduction

Among possible approaches to the description of non-equilibrium systems the Non-equilibrium Statistical Operator Method (NESOM) especially demonstrated its efficiency \[1, 2, 3\]. NESOM provides a very promising technique that implies in a far-reaching generalization the statistical methods developed by Boltzmann and Gibbs. NESOM was initially built on intuitive and heuristic arguments, apparently it can be incorporated within an interesting approach to the rationalization of statistical mechanics, as contained in the maximization of (informational statistical) entropy (MaxEnt for short) and Bayesian methods. NESOM appears as a very powerful, concise, based on sound principles, and elegant formalism of a broad scope to deal with systems arbitrarily far from equilibrium. The non-equilibrium statistical operator (NSO) introduced in \[1, 2, 3\] has a form

\[
\ln \rho(t) = \int_0^\infty p_{qzub}(u) \ln \rho_q(t-u,-u) du, \quad \ln \rho_q(t,0) = -\Phi(t) - \sum_n F_n(t) P_n; \quad (1)
\]

where

\[
p_q(u) = p_{qzub}(u) = \varepsilon e^{-\varepsilon u}, \quad u = t - t_0,
\]

\(H\) is Hamiltonian, \(\ln \rho(t)\) is the logarithm of the NSO, \(\ln \rho_q(t_1, t_2)\) is the logarithm of the quasi-equilibrium (or relevant) distribution; the first time argument indicates the time dependence of the values of the thermodynamic parameters \(F_n\); the second time argument \(t_2\) in \(\ln \rho_q(t_1, t_2)\) denotes the time dependence through the Heisenberg representation for dynamical variables \(P_n\) on which \(\ln \rho_q(t,0)\) can depend \[1, 2, 3\].

In \[1, 2, 3\] \(p_q(u) = p_{qzub}(u) = \varepsilon \exp\{-\varepsilon u\};\) after the thermodynamic limiting transition \(N \to \infty, V \to \infty, N/V = \text{const}, \varepsilon \to 0.\) From the complete group of solutions of Liouville equation (symmetric in time) the subset of retarded "unilateral" in time solutions is selected by means of introducing a source \(K\) in the Liouville equation for \(\ln \rho(t)\) \((L\) is Liouville operator; \(iL = -\{H, \rho\}\) = \(\sum_k \left[ \frac{\partial H}{\partial p_k} \frac{\partial \rho}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial \rho}{\partial p_k} \right] p_k;\) \(p_k\) and \(q_k\) are pulses and coordinates of particles; \(\{\ldots\}\) are Poisson brackets)

\[
\frac{\partial \ln \rho(t)}{\partial t} + iL \ln \rho(t) = -\varepsilon(\ln \rho(t) - \ln \rho_q(t,0)) = K_{zub}. \quad (3)
\]

In \[4, 5\] a convenient redefinition of the source term is proposed. Although infinitesimally small, the source term introduced by Zubarev into the Liouville equation is shown to influence the macroscopic behaviour of the system in the sense that the corresponding evolution equations do not coincide exactly with those obtained from an initial-value problem which corresponds to a definite experimental situation and a physical set of macroobservables.

In \[6, 7\] it was noted that in place of the function \(p_q(u) = p_{qzub}(u) = \varepsilon e^{-\varepsilon u}\) in \(1\) arbitrary (but having certain properties \[8\]) weight functions \(w(t, t_0)\) can be used.
Zubarev’s nonequilibrium statistical operator does satisfy Liouville equation, but it must be borne in mind that the group of its solutions is composed of two subsets, one corresponding to the retarded and second one to the advanced solutions. The presence of the weight function \( w(t,t_0) \) (Abel’s kernel (2) in Zubarev’s approach) in the time-smoothing or quasi-average procedure that has been introduced selects the subset of retarded solutions from the total group of solutions of the Liouville equation. This consideration is related to the question: how to obtain an irreversible behavior in the evolution of the macroscopic state of the system? In the MaxEnt-NESOM approach the irreversibility is incorporated from the outset using an ad hoc non-mechanical hypothesis. MaxEnt-NESOM yields information on the macrostate of the system at time \( t \), when a measurement is performed, including the evolutionary history (in the interval from the initial time of preparation \( t_0 \) up to time \( t \)) by which the system came into that state (which introduced a generalization of Kirkwood’s time-smoothing formalism [9]). Functions \( w(t,t_0) \) are typically kernels [1, 2, 3, 8] that appear in the mathematical theory of convergence of integrals. In [10, 11, 12] other interpretation of the functions \( w(t,t_0) \), denoted as \( p_q(u) \), is given. With the change of function \( w(t,t_0) = p_q(u) \) the form of source (3) in the Liouville equation also changes. For an arbitrary function \( p_q(u) \) it looks like (4).

In [10, 11, 12] it was noted that the function \( p_q(u) = p_{q_{zub}}(u) = \varepsilon e^{-\varepsilon u} \) in NESOM [1] for the non-equilibrium distribution function can be interpreted as the exponential probability distribution of the lifetime \( \Gamma \) of a system. \( \Gamma \) is a random variable of lifetime (time span) from the moment \( t_0 \) of its birth till the current moment \( t \); \( \varepsilon^{-1} = \langle t-t_0 \rangle = \langle \Gamma \rangle \), where \( \langle \Gamma \rangle = \int u p_q(u) du \) is the average lifetime of the system. This time period can be called the time period of getting information about system from its past. Instead of the exponential distribution \( p_{q_{zub}}(u) \) (2) in (1) used in [1, 2, 3] any other sample distribution \( p_q(u) \) could be taken; integration by parts in time is performed at \( \int p_q(u) du \big|_{u=0} = -1; \int p_q(u) du \big|_{u=\infty} = 0 \). If \( p_q(u) = p_{q_{zub}}(u) = \varepsilon e^{-\varepsilon u} \) by (2), \( \varepsilon = 1/\langle \Gamma \rangle \) the expression for NSO passes in (1) from [1] to [11].

The same interpretation of the distribution \( p_q(u) \) is given in [2], where this value is understood as the distribution of the initial moment of time \( t_0 \). Since the random (past) lifetime is equal to \( \Gamma = u = t-t_0 \), the distribution of the past lifetime \( u \) coincides with the distribution of the initial time values \( t_0 \). The moment \( t_0 \) will be the moment of the first passage in the inverse time, if the moment \( t \) is taken as initial. In [2] the uniform distribution for an initial moment \( t_0 \) is chosen, which after the transition from Abel integration to Cesàro integration passes to the exponential distribution \( p_q(u) = p_{q_{zub}}(u) = \varepsilon e^{-\varepsilon u} \). Such distribution serves as the limiting distribution of the lifetime [13], the first-passage time of a certain level. In the general case it is possible to choose a lot of functions for the obvious type of distribution \( p_q(u) \), which was noted in [10, 11, 12].

In [13] the lifetimes of the system are introduced as random moments of the first-passage time till the moment when a random process describing system reaches a certain limit, for example, a zero value. In [13] approximate exponential expressions for the
probability density function (with a single parameter) and probability distribution of lifetime are obtained, and the accuracy of these expressions is estimated.

In [14] it was noted, that the role of the form of the source term in the Liouville equation in the NSO method has never been investigated. In [15] it is stated that the exponential distribution is the only one which possesses the Markovian property of the absence of afteraction, that is whatever is the actual age of a system, the remaining time does not depend on the past and has the same distribution as the lifetime itself. It is known [1, 2, 3] that the Liouville equation for NSO contains the source \( K_{zub} = -\varepsilon [\ln \rho(t) - \ln \rho_0(t, 0)] \) which becomes vanishingly small after taking the thermodynamic limit and setting \( \varepsilon \to 0 \), which in the spirit of [10] corresponds to the infinitely large lifetime value of an infinitely large system. For a system with finite size this source is not equal to zero. In [8] this term enters the modified Liouville operator and coincides with the form of Liouville equation suggested by Prigogine [16] (the Boltzmann-Prigogine symmetry), when the irreversibility is introduced in the theory at the microscopic level.

In [10] a new interpretation of the method of the NSO is given, in which the operation of taking the invariant part [1, 2] or the use of an auxiliary "weight function" (in the terminology of [6, 7, 8]) in NSO are treated as averaging the quasi-equilibrium statistical operator over the distribution of past lifetime of a system. This approach agrees with the approach of the general theory of random processes, the renewal theory, and also with the conception of Zubarev work [2] where the NSO is conceived as some averaging over the initial moment of time.

The statistical operator depends on the information-gathering interval \((t_0, t)\), but it must be borne in mind that this is the formal point consisting in that (as Kirkwood pointed out) that the description to be built must contain all the previous history in the development of the macrostate of the system. In [6, 7, 8] several basic steps for the construction of the NESOM formalism are indicated: a third basic step has just been introduced, namely, the inclusion of the past history (other terms used are retro-effects or historicity) of the macrostate of the dissipative system. A fourth basic step needs now to be considered, which is a generalization of Kirkwood’s time-smoothing procedure: the one that accounts for the past history and future dissipative evolution. The time-smoothing procedure introduces a kind of Prigogine’s dynamical condition for dissipativity. The procedure introduces a kind of evanescent history as the system macrostate evolves toward future from the initial condition at time \( t_0 \). The function \( w(t; t_0) \) [6, 7, 8] introduces the time-smoothing procedure. In principle, any kernel provided by the mathematical theory of convergence of trigonometrical series and transform integrals provides is acceptable for these purposes. Kirkwood, Green, Mori [9, 17, 18] and others have chosen what in mathematical terminology is Fejér (or Cesàro-1) kernel. Meanwhile Zubarev introduced the one consisting in Abel’s kernel for \( w \) in Eq. [1] - which apparently appears to be the best choice, either mathematically but mostly physically: that is, taking \( w(t; t_0) = \varepsilon e^{\varepsilon(t_0-t)} \), where \( \varepsilon \) is a positive infinitesimal value which tends to zero after the calculation of averages has been performed, and with
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t₀ going to −∞. Therefore a process with fading memory is introduced. In Zubarev’s approach this fading process occurs in an adiabatic-like form towards the remote past: as time evolves memory decays exponentially with lifetime ε⁻¹ [8]. The approach suggested in [10] [11] [12] and in the present work enables to use a family of functions w(t, t₀) = pₒ(u) and makes clear both their physical sense and those physical situations in which one or another function w(t, t₀) = pₒ(u) can be used.

Besides the Zubarev’s form of NSO [1] [2] [3], the NSO formulation in the Green-Mori form [17] [18] is known, where one assumes the auxiliary weight function [6] [7] [8] to be equal to W(t, t₀) = 1 − (t − t₀)/τ; w(t, t₀) = dW(t, t₀)/dt₀ = 1/τ; τ = t − t₀. After averaging one sets τ → ∞. This choice at pₒ(u = t − t₀) = w(t, t₀) coincides with the uniform lifetime distribution. The source in the Liouville equation takes the form K = ln ρₒ/τ. In [1] this form of NSO is compared to the Zubarev’s form. One could name many examples of explicitly setting the function pₒ(u). Each and every definition implies some specific form of the source term K in the Liouville equation, some specific form of the modified Liouville operator and NSO [10] [11] [12]. Thus the whole family of NSO is defined.

It is possible to make different assumptions about the form of the function of pₒ(u), getting different expressions for the source in the Liouville equation and for non-equilibrium characteristics of the system. It is possible to show [12] that certain choices of the function of pₒ(u) result in the changes in non-equilibrium characteristics in the limit of infinitely large average lifetimes as well. In [10] [11] [12] an analogy is traced to the passage to the thermodynamic limit of systems of infinite size. So explicit form of the function pₒ(u) is important for describing non-equilibrium systems by the NSO method.

Setting the form of the function pₒ(u) reflects not only the internal properties of a system, but also the influence of the environment on an open system, the particular character of its interaction with the environment [8]. In [2] a physical interpretation of the exponential distribution for the function pₒ(u) is given: a system evolves freely like an isolated system governed by the Liouville operator. Besides that the system undergoes random transitions, and the phase point representing the system switches from one trajectory to another one with an exponential probability under the influence of the "thermostat"; the average intervals between successive push events increase infinitely. This takes place if the parameter of the exponential distribution tends to 0 after the transition to the thermodynamic limit. Real physical systems have finite sizes. The exponential distribution describes completely random systems. The influence of the environment on a system can have organized character as well, for example, this is the case of systems in a stationary non-equilibrium state with input and output fluxes. The character of the interaction with the environment can also vary; therefore different forms of the function pₒ(u) can be used.

The adequate choice of the function pₒ(u) is important for correct description of the non-equilibrium properties of statistical systems. To find the type of function pₒ(u), it is necessary to resort to some general principles, such as MaxEnt principle. In this
work two variants of MaxEnt are used for this purpose: the one introduced in [19] for the Liouville equation with a source (Section 2) and that suggested in [26] for the superstatistics (Section 3).

2. Maximum entropy principle for Liouville equations with source

In this paper we apply the maximum entropy principle for the determination of the function \( p_q(u) \). The same approach was applied in [19] for the evolution equations with source terms. In [7, 10, 11, 12] a general form for the source in the Liouville equation for \( \ln \rho(t) \) (3) is obtained. For our case the source term has the following form

\[
K = p_q(0) \ln \rho_q(t, 0) + \int_0^\infty \frac{\partial p_q(u)}{\partial u} \ln \rho_q(t - u, -u) du.
\]

In [19] in the Liouville equation the distribution function \( \rho(\vec{z}, t) \) is written in a form

\[
\rho(\vec{z}, t) = N f_{ME}(\vec{z}, t) = N \frac{Z}{Z} \exp\{-\sum_{i=1}^{M} \lambda_i A_i\},
\]

where \( A_i(\vec{z}) \) are \( M \) appropriate quantities that are functions of the phase space point \( \vec{z} \); the quantities \( A_i(\vec{z}) \) correspond to the values \( P_i \) from (1). The partition function \( Z \) is given by

\[
Z = \int \exp\{-\sum_{i=1}^{M} \lambda_i A_i\} d^N z.
\]

The function \( f_{ME}(\vec{z}, t) \) is normalized to unity:

\[
\int f_{ME}(\vec{z}, t) d^N z = 1;
\]

\[
\int \rho(\vec{z}, t) d^N z = N(t); \quad \frac{dN}{dt} = \int K d^N z; \quad \frac{\partial \rho}{\partial t} + \vec{w} \nabla \rho = \frac{d\rho}{dt} = K; \quad \nabla \vec{w} = 0.
\]

The probability distribution \( f_{ME}(\vec{z}, t) \) is the one that maximizes the entropy \( S[f] \) under the constraints imposed by normalization and relevant mean values \( \langle A_i \rangle = \int A_i \rho d^N z \) (or \( a_i = \langle A \rangle_i / N \)). The re-scaled mean values \( a_i \) and the associated Lagrange multipliers \( \lambda_i \) are related by the Jayne’s relations [20, 21]

\[
\lambda_i = \frac{\partial S}{\partial a_i}, \quad a_i = \frac{\langle A_i \rangle}{N} = -\frac{\partial}{\partial \lambda_i}(\ln Z),
\]

\[
S = -\int f \ln f d^N z = \ln Z + \sum_i \lambda_i a_i.
\]

If we choose for \( \ln \rho(z, t) \) the function

\[
\ln \rho = \ln \varrho(t) = \int_0^\infty p_q(u) \ln \varrho_q(t-u, -u) du = \int_{-\infty}^t p_q(t-t_0) \ln \varrho_q(t_0, t_0-t) dt_0
\]
from (1), which is included in the Liouville equation (3) (without integrating over time) with $p_{qzub} \rightarrow p_q$, $K_{zub} \rightarrow K$, and, following [1], choose

$$\lambda_i = p_q(t - t_0)F_i(t_0), \quad \text{ (11)}$$

then

$$\ln \rho(\vec{z}, t) = \int_{-\infty}^{t} p_q(t - t_0) \ln \rho(t_0, t_0 - t) dt_0 =$$

$$- \int_{-\infty}^{t} \left( \sum_i \lambda_i A_i + p_q(t - t_0) \ln Z_1 \right) dt_0 = \ln f'_{ME} + \ln N,$$

where

$$\ln f'_{ME} = - \int_{-\infty}^{t} \sum_i \lambda_i A_i dt_0 - \ln Z_\lambda;$$

$$\ln N = \Delta Z = \ln Z_\lambda - \int_{-\infty}^{t} p_q(t - t_0) \ln Z_1 dt_0;$$

$$Z_\lambda = \int \exp[- \int_{-\infty}^{t} \sum_i \lambda_i A_i dt_0] d^N z, \quad Z_1 = \int \exp[- \sum_i F_i A_i] d^N z$$

($F_i$ are taken from (1)). The values $Z_\lambda$ and $Z_1$ in the terminology of [1] are related to the partition functions for a non-equilibrium and relevant statistical operator accordingly.

In [19] for the Liouville equation of the kind (3) with constant sources equation one gets for $d\lambda_i/dt$

$$\frac{d\lambda_i}{dt} = \left( \sum_{j=1}^{M} C_{ji} \lambda_j \right) - \frac{1}{N} \frac{\partial}{\partial a_i} \int K \ln f_{ME} d^N z,$$

where the Zubarev-Peletminskiy selection rule [22, 23, 1, 7, 8]

$$\vec{w} \nabla A_i = \sum_{j=1}^{M} C_{ij} A_j, \quad (i = 1, \ldots, M), \quad \frac{d\vec{z}}{dt} = \vec{w}(\vec{z}); \quad \frac{\partial \rho}{\partial t} + \nabla(\rho \vec{w}) = K \quad \text{ (14)}$$

is used; $i, j = 1, 2; \ldots$; the $C_{ij}$ are c-numbers. In other representations the quantities $A_i$ can depend on the space variable, that is, when considering local densities of dynamical variables, and then the $C_{ij}$ can depend on the space variable as well or be differential operators.

If more complex shape of the source (4) is considered, the equation for $d\lambda_i/dt$ takes on the form

$$\frac{d\lambda_i}{dt} = \left( \sum_{j=1}^{M} C_{ji} \lambda_j \right) - \frac{1}{N} \frac{\partial}{\partial a_i} \left( \int K \ln f_{ME} d^N z \right) -$$

$$\sum_j \lambda_j \frac{\partial}{\partial a_i} \left( \frac{1}{N} \int A_j K d^N z \right) - \ln Z_1 \frac{\partial}{\partial a_i} \left( \frac{\dot{N}}{N} \right).$$
We replace the operators $\frac{\partial}{\partial a_i}$ and $\frac{\partial}{\partial \lambda_i}$ taking into account (9)-(13) by the functional differentiation of the kind
\[
\frac{\partial}{\partial a_i} \rightarrow \frac{\delta}{\delta a_i} = N \frac{\delta}{\delta \langle A_i \rangle}, \quad \frac{\partial}{\partial \lambda_i} \rightarrow \frac{\delta}{\delta \lambda_i},
\]
which takes off the integration over time. For example
\[
\frac{\delta}{\delta \langle A_i \rangle} \ln Z^\lambda = - p_q(t - t_0) \sum_k \frac{\partial F_k}{\partial \langle A_i \rangle} \langle A_k \rangle; \quad \frac{\delta \ln Z^\lambda}{\delta \lambda_i} = - \langle A_i \rangle.
\]
The relations (5-8) and (9) are thus hold. If to take into account that $\int . . . \rho d^N z = \int . . . z d^N z = \langle . . . \rangle$ in the NSO method, then
\[
\frac{\partial}{\partial a_i} N = \frac{\partial}{\partial a_i} \left( \frac{\dot{N}}{N} \right) = 0. \quad \tag{16}
\]
Let us consider the integrals in the rhs of (15) of the form $\int KB(z)dz$, $B$ being an arbitrary function of the dynamic variables $z$, and the source term $K$ taken from (4); for Eq. (8), (14) $K = [p_q(0) \ln \rho_q(t,0) + \int_{-\infty}^{t} \frac{\partial p_q(t-t_0)}{\partial t} \ln \rho_q(t_0, t_0 - t) dt] \rho$. Assume that $p_q(u)$ does not depend on $z$. Integrating by parts and assuming $p_q(u)_{u=\infty} \rightarrow 0$, we get:
\[
\int KBd^N z = - \int_{\infty}^{0} p_q(u) \frac{d}{du} (B \ln \rho_q(t - u, -u)) du; \quad u = t - t_0. \quad \tag{17}
\]
Taking into account (16) and the fact that the operation $\frac{\partial}{\partial a_i} \rightarrow \frac{\delta}{\delta a_i}$ eliminates the integration by time, the equation (15) takes on the form
\[
F_i(t_0) \frac{dp_q(t - t_0)}{dt} = -p_q(t - t_0)C_i - p_q(t - t_0)r_1 - p_q^2(t - t_0)r_2, \quad \tag{18}
\]
where $C_i = \sum_j C_{ji} F_j(t_0)$,
\[
r_1 = -\frac{\partial}{\partial \langle A_i \rangle} \frac{d}{dt} (\langle \ln \rho(t) - \ln N \rangle \ln \rho_q(t_0, t_0 - t)); \quad \tag{19}
\]
\[
r_2 = -\sum_j F_j(t_0) r_{2j}; \quad r_{2j} = \frac{\partial}{\partial \langle A_i \rangle} \frac{d}{dt} \langle A_j(t) \ln \rho_q(t_0, t_0 - t) \rangle. \quad \tag{20}
\]
An unknown function $p_q(u)$ enters the expression (19) through the terms $\ln \rho(t)$ and $\ln N$. To get rid of this dependence, we use the averaging theorem. For the expressions for $\ln \rho(t)$ and $\ln N$ in (19) we take all terms besides $p_q(u)$ out of the time integration. For each of these function however a different effective average time value should be used. The remaining integrals over $p_q(u)$ are equal to unity. We get:
\[
\ln N \simeq \ln Z_1(c_3) - \ln Z_1(c_4), \quad \tag{21}
\]
\[
\ln \rho(t) \simeq \ln \rho_q(c_1, c_1 - t) = -\left( \sum_m F_m(c_1)A_m(c_1 - t) + \ln Z(c_1) \right). \tag{22}
\]

Let us make another approximation and change the order of the operations \(\partial/\partial \langle A_i \rangle\) and \(d/dt\) in the expressions (19)-(20). The value \(\int_{t_0}^t r_1 dt = D(t) - D(t_0)\) enters the expression (18), where

\[
D(t) = -\frac{\partial}{\partial \langle A_i \rangle} \langle (\ln \rho(t) - \ln N) \ln \rho_q(t_0, t_0 - t) \rangle \\
= F_i(t_0) [\ln Z(c_3) - \ln Z(c_4) - \ln Z(c_1)] - \sum_{m,n} \left( \langle A_m A_n \rangle - \langle A_m \rangle \langle A_n \rangle \right) \left[ \frac{F_m(t_0)}{\langle A_m A_i \rangle - \langle A_m \rangle \langle A_i \rangle} \right] - \sum_{m,n} F_m(c_1) F_n(t_0) \sum_k \frac{\langle A_k A_m A_n \rangle - \langle A_m A_n \rangle \langle A_k \rangle}{\langle A_i A_k \rangle - \langle A_i \rangle \langle A_k \rangle};
\tag{23}
\]

\[
r_{2j} = \frac{\partial}{\partial t} \left[ \sum_m F_m(t_0) \sum_k \frac{\langle A_k A_j A_m \rangle - \langle A_j A_m \rangle \langle A_k \rangle}{\langle A_i A_k \rangle - \langle A_i \rangle \langle A_k \rangle} + \delta_{ij} \ln Z(t_0) \right. \\
\left. - \sum_m \frac{\langle A_j A_m \rangle - \langle A_j \rangle \langle A_m \rangle}{\langle A_i A_m \rangle - \langle A_i \rangle \langle A_m \rangle} \right]. \tag{24}
\]

The values of the correlators \(\langle A_i \rangle, \langle A_i A_k \rangle, \langle A_j A_k A_m \rangle\) are averaged with \(\rho(t)\) and are \(t\)-dependent. In deriving (23), (24) we used the relations like

\[
\frac{\partial \ln Z(c)}{\partial \langle A \rangle} = \sum_n \frac{\langle A_n \rangle}{\langle A A_n \rangle - \langle A \rangle \langle A_n \rangle}; \quad A_n(-c) = e^{-icL} A_n;
\]

\[
\frac{\partial F_m}{\partial \langle A_i \rangle} = \frac{1}{\langle A_i A_m \rangle - \langle A_i \rangle \langle A_m \rangle}.
\]

One can proceed with the expression (24) using the relations

\[
\frac{\partial F_i}{\partial t} = \sum_k \frac{\partial F_i}{\partial \langle A_k \rangle} \frac{\partial \langle A_k \rangle}{\partial t} = \sum_k \frac{\partial F_k}{\partial \langle A_i \rangle} \frac{\partial \langle A_k \rangle}{\partial t}; \quad \frac{\partial F_i}{\partial \langle A_k \rangle} = \frac{\partial F_k}{\partial \langle A_i \rangle} = \frac{\partial^2 S}{\partial \langle A_k \rangle \partial \langle A_i \rangle};
\]

\[
\frac{\partial \ln Z}{\partial t} = -\sum_m \frac{\partial F_m}{\partial t} \langle A_m \rangle.
\]

The solution to (18) has the form

\[
p_q(t - t_0) = \frac{p_q(0)M}{1 + p_q(0) \int_{t_0}^t \frac{1}{F_i} r_2 M dt}. \tag{25}
\]
Since $C_i$ and $F_i(t_0)$ does not depend on $t$,

$$M(t) = \exp \left\{ -\frac{C_i}{F_i} (t - t_0) - \frac{1}{F_i} \int_{t_0}^{t} r_1 dt \right\} =$$

$$\exp \left\{ -\frac{C_i}{F_i} (t - t_0) - \frac{1}{F_i} (D(t) - D(t_0)) \right\},$$

(26)

$M(t_0) = 1$, where $\int_{t_0}^{t} r_1 dt$ is written in (23).

Integrating by parts the second term in the denominator of (25) write it in the following form:

$$p_q(t - t_0) = \frac{p_q(0) M(t)}{1 - L};$$

(27)

$$L = p_q(0) \frac{1}{F_i} \left[ \left( \int r_2 dt \right)_{t_0} - \left( \int r_2 dt \right)_{t} M(t) - \int_{t_0}^{t} \left( \int r_2 dt \right) \left( \frac{C_i}{F_i} + \frac{1}{F_i} \frac{dD(t)}{dt} \right) M(t) dt \right],$$

where

$$\int r_2 dt = \sum_j \sum_m F_m(t_0) F_j(t_0) \sum_k \frac{\langle A_k A_j A_m \rangle - \langle A_j A_m \rangle \langle A_k \rangle}{\langle A_i A_k \rangle - \langle A_i \rangle \langle A_k \rangle} +$$

(28)

$$\sum_j \sum_m F_j(t_0) \langle A_j A_m \rangle - \langle A_j \rangle \langle A_m \rangle;$$

$$\frac{dD(t)}{dt} = \sum_{m,n} \frac{\langle A_n A_m \rangle - \langle A_m \rangle \langle A_n \rangle}{\langle A_m A_i \rangle - \langle A_m \rangle \langle A_i \rangle} \left[ \frac{d}{dt} \ln(\langle A_m A_n \rangle - \langle A_m \rangle \langle A_n \rangle) - \frac{d}{dt} \ln(\langle A_m A_i \rangle - \langle A_m \rangle \langle A_i \rangle) \right] -$$

(29)

$$- \frac{d}{dt} \ln(\langle A_m A_i \rangle - \langle A_m \rangle \langle A_i \rangle) + F_m(c_1) \frac{\langle A_m A_n \rangle - \langle A_m \rangle \langle A_n \rangle}{\langle A_i A_n \rangle - \langle A_i \rangle \langle A_n \rangle} \times$$

$$\left[ \frac{d}{dt} \ln(\langle A_m A_n \rangle - \langle A_m \rangle \langle A_n \rangle) - \frac{d}{dt} \ln(\langle A_i A_n \rangle - \langle A_i \rangle \langle A_n \rangle) \right] -$$

$$- F_m(c_1) F_n(t_0) \sum_k \frac{\langle A_k A_m A_n \rangle - \langle A_m A_n \rangle \langle A_k \rangle}{\langle A_i A_k \rangle - \langle A_i \rangle \langle A_k \rangle} \left[ \frac{d}{dt} \ln(\langle A_k A_m A_n \rangle - \langle A_m A_n \rangle \langle A_k \rangle) - \frac{d}{dt} \ln(\langle A_i A_k \rangle - \langle A_i \rangle \langle A_k \rangle) \right].$$

The expression for $M(t)$ is given in (26), and $p_q(0)$ is determined from the conditions for the norm $\int_0^{\infty} p_q(u) du = 1$ and for the average lifetime $\langle \Gamma \rangle = \int_0^{\infty} u p_q(u) du$.

If one either considers the stationary case or assumes a weak time dependence in the correlators in (23)-(24), $D(t) \simeq D(t_0), r_2 \simeq 0$, and (25)-(27) take on the form

$$p_q(t - t_0) = p_q(0) \exp\left\{ -\frac{C_i}{F_i} (t - t_0) \right\}$$

(30)

with $C_i/F_i = p_q(0) = 1/\langle \Gamma \rangle$. In (30) an exponential distribution for $p_q(u)$, used in [1][2][3] is obtained. However generally the correlators in (23)-(24) are time dependent. Applying the full form of (27) for concrete systems it is possible to state the conditions
where the denominator in (27) is considerably different from unity, hence the lifetime distribution essentially deviates from the exponential one (30).

The term \( L \) in the denominator of (27) is small, since \( p_q(0) \approx 1/\langle \Gamma \rangle \ll 1 \). In the denominator of (27) it stands in the combination with \( 1 - L \approx 1 \). Therefore one can write \( p_q = p_q(0)M[1 + L + L^2 + \ldots] \), and the distribution (27) is close to the exponential distribution (30) used in [1, 2, 3, 7, 8]. This results agrees with the results of [13], where the exponential distribution for the lifetime is shown to be a limiting distribution. But the expression for \( \langle \Gamma \rangle \) is explicitly given in (30), and in (23)-(29). It was already pointed above that the situations where \( L \) is comparable to 1 can arise as well.

For the maximum entropy principle with the Shannon measure of the information entropy the exponential distribution used in [1, 2, 3], is basic. Choosing another measures for the information entropy (e.g. Tsallis and Renyi measures [24, 25]) changes the function \( f_{ME} \), which yields another forms of the lifetime distributions. For the NSO method the functions \( f_{ME} \) are in the form (5), and the information entropy is given by the Shannon measure, hence basic distribution is the exponential one.

The expressions (25), (27) for \( p_q(u) \) depend on \( F_i \), the functions \( C_i = \sum_j C_{ji}F_j \), \( r_k, \ k = 1, 2 \), in (18)-(19) depend on the index \( i \), the function \( p_q(u, i) \) depends on \( i \) as well. One can get an \( i \)-independent function \( p_q(u) \) by symmetrizing the distribution, for example, using the operation

\[
p_q(u) = \left[ \prod_{i=1}^{M} p_q(u, i) \right]^{\frac{1}{M}}.
\]

Such formulation of maxent principle, as in [19], gives the distribution for the lifetime, related to the exponential distribution which serves in this case as the base one. Distributions of other type can be obtained using some other form of maxent principle.

3. Another approach to the method of maximum entropy

Yet another approach to the determination of the type of function of distribution of lifetime is related to the method of maximal entropy inference ("maxent"), developed in [26] for the determination of the distribution of superstatistics. We note a formal similarity between the superstatistics method where the averaging is performed over the parameter \( \beta \), (for example, the inverse temperature)

\[
p(E) \propto \int_{0}^{\infty} f(\beta)e^{-\beta E}d\beta,
\]

and the NSO method where the averaging is performed over the past life spans \( u = t - t_0 \),

\[
\ln \rho(t) = \int_{0}^{\infty} p_q(u) \ln p_q(t - u, -u)du; \ t = t - t_0,
\]
which was already used in [27]. Therefore the maxent method of [26] can be applied for determining the function $p_q(u)$. The analogy here is not merely formal. The principal assumption of [26], that is the separation of the time scales is essential for the NSO method as well [1, 2, 3]. In the approach of superstatistics [28, 29, 30] the system is split into cells and local fluctuations of the value $\beta$ are considered; the fluctuations of the value $u = t - t_0$ affect the complete system.

Closely related is also the research of Crooks [31]. He studies general non-equilibrium systems, without assuming that the system can be divided into different cells that are at local equilibrium. Crooks claims that instead of trying to obtain the probability distribution of the entire non-equilibrium system, one has to try to estimate the “metaprobability,” the probability of the microstate distribution. Crooks also uses the maximum entropy principle but sets in (31) $\lambda_3 = 0$. The main difference is that Crooks does not assume a local equilibrium in the cells, hence his approach, though being an interesting theoretical construction, does not give a straightforward physical interpretation to the fluctuating parameter $\beta$. However such an approach can be applied to the fluctuations of the value $u = t - t_0$. In the approach of [26] one obtains a local fluctuating temperature that coincides with the thermodynamic temperature and which can in principle be measured. The work of Crooks is used by Naudts [32] to describe equilibrium systems. The author shows that some well-known results of the equilibrium statistical mechanics can be reformulated in a very general context with the use of the concepts introduced in [26, 29, 31].

In [26] following expression is obtained for the distribution function of the superstatistics $f(\beta; \lambda_i)$

$$f(\beta; \lambda_i) = \frac{Z(\beta)^{-\lambda_1/V}}{Z(\lambda_i)} \exp\left(-\beta \lambda_2 \frac{E(\beta)}{V} - \lambda_3 g(\beta)\right),$$

where $\lambda_i$ are Lagrange multipliers, $V$ being an arbitrary constant (taking out a common factor out of the definition of $\lambda_1$ and $\lambda_2$ will turn out to be useful in the following). Using the well-known formula $S_\beta = \ln Z(\beta) + \beta E(\beta)$ with $Z(\lambda_i)$ being a normalization constant that is fixed by the condition $\langle 1 \rangle_\beta = 1$.

The same approach with said limitations used for the function $p_q(u)$, yields

$$p_q(u; \lambda_i) = \frac{Z(t - u)^{-\lambda_1/V}}{Z(\lambda_i)} \exp\left(-\beta \lambda_2 \sum_m F_m(t - u)\langle A_m \rangle - \lambda_3 g(u)\right),$$  \hspace{1cm} (31)

where $g(u)$ is an arbitrary function of $u$, which is determined by the physical peculiarities of the behaviour of the system in one or another period of its history. Expression (31) is obtained by the optimization of the entropy $S(\lambda_i) = \int p_q(u) \ln p_q(u) du$ with the constraints for entropy

$$\int p_q(u)S(u)du = \int p_q(u) \int p_q(t - u, -u) \ln p_q(t - u, -u)dzdu =$$

$$\int p_q(u)\left[-\sum_m F_m(t - u)\langle A_m \rangle - \ln Z(t - u)\right]du$$
and parameters $\int p_q(u) \sum_m F_m(t-u)\langle A_m\rangle du$. Similarly to [26], one can set the functions $g(u)$ in a different fashion. For example, for $g(u) = u, \lambda_1 = \lambda_2 = 0, \lambda_3 = 1/\langle \Gamma \rangle$, where $\langle \Gamma \rangle$ is the average span of past life of a system (till the present time moment), we get the exponential distribution used in [1, 2, 3]. Setting $g(u) = \ln u$ with appropriate corresponding values of $\lambda$ one gets the power-like distribution for $p_q(u)$; setting $g(u) = (\ln u)^2$ with corresponding $\lambda$ results in the log-normal distribution. Thus setting the function $g(u)$ and $\lambda$ accordingly it is possible to obtain various distributions for the lifetime considered, for example, in [33].

It is possible to examine more difficult cases when the behaviour of the system changes at different stages of its evolution, when, for example the function $g(u) = \ln u$ yields the power-like function $p_q(u)$ at $u < c$, and $g(u)$ gives an exponential shape of $p_q(u)$ at $u > c$.

4. Conclusion

For the determination of the lifetime distribution in the NSO method the method of maximum entropy principle as in [19] is used. The obtained distribution is close to exponential $p_{qzub}(u)$ [2], but does not coincide with it. It is possible to find conditions at which this difference is essential. Using other variants of the maximum entropy principle, as in [26], it is possible to obtain other distributions except exponential one, in particular, power-like and log-normal distributions, transitory ones between them, as well as distributions of other classes.

In the interpretation of [2] it is the random value $t_0$ in $u = t - t_0$ that fluctuates. In [2] the limiting transition is performed for the parameter $\varepsilon, \varepsilon \to 0$ in the exponential distribution $p_q(u) = \varepsilon \exp\{-\varepsilon u\}$ after the thermodynamic limiting transition. In the interpretation of [10] it corresponds to the average lifetime of a system tending to infinity: $\langle \Gamma \rangle = \langle t - t_0 \rangle = 1/\varepsilon \to \infty$. But the average intervals between successive random jumps grow infinitely, getting larger than the lifetime of a system. Therefore the source term in the Liouville equation turns to 0. If however the distribution $p_q(u)$ changes over the interval of the lifetime, the influence of the environment which caused this change, remains within the life span even if the lifetime tends to infinity [12].

There are numerous experimental confirmations for such change of the lifetime distribution $p_q(u)$ over the interval of the system lifetime. The examples thereto are the transition to chaos and the transition from laminar to turbulent flow which are accompanied by the change of the distribution of $p_q(u)$. In [34, 35] the passage from Gaussian to non-Gaussian behaviour of the distribution of the first-passage time for some time moment is demonstrated. Besides the real systems possess finite sizes and finite lifetime. Therefore influence of surroundings on them is always present.

Slow change of the function $g(u) = u = t - t_0$ corresponds in the interpretation of [2] to the slow change of the random value $t_0$ on a temporal scale. Accordingly slow change of the function $g(u) = \ln u$ corresponds to the slow change of $\ln(t - t_0)$. Setting other functions $g(u)$, for example, $g(u) = (\ln u)^2$ and so on is explained on the same
footing.

In the present work by means of two variants of the method of maximum entropy we obtained the expressions for the distribution of the lifetime value. It is noted that the choice of the form of the distribution function for the lifetime value can affect the non-equilibrium behavior of a system even after performing the thermodynamic limiting transition.

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