Quenched chiral logarithms in lattice QCD with overlap Dirac quarks* †

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We examine quenched chiral logarithms in lattice QCD with overlap Dirac quarks. From our data of \( m_{\pi}^2 \), we determine the coefficient of quenched chiral logarithm \( \delta = 0.203(14), 0.176(17), 0.193(17) \) and 0.200(13) for lattices of sizes \( 8^3 \times 24 \), \( 10^3 \times 24 \), \( 12^3 \times 24 \) and \( 16^3 \times 32 \) respectively. Also, for the first three lattice sizes, we measure the index susceptibility of the overlap Dirac operator, and use the exact relation between the index susceptibility and the \( \eta' \) mass in quenched chiral perturbation theory to obtain an independent determination of \( \delta = 0.198(27), 0.173(24), 0.169(22) \), which are in good agreement with those determined from \( m_{\pi}^2 \).

1. Introduction

In quenched chiral perturbation theory \( (q\chi PT) \), the pion mass to one-loop order reads as

\[
m_{\pi}^2 = C m_q \{ 1 - \delta [\ln(C m_q / \Lambda^2) + 1] \} + B m_q^2 \tag{1}
\]

where \( m_q \) denotes the bare \( (u \) and \( d \) quark mass, \( \Lambda \) is the chiral cutoff which can be taken to be \( 2\sqrt{2\pi} f_{\pi} (f_{\pi} \simeq 132 \text{ MeV}) \), \( C \) and \( B \) are parameters, and \( \delta \) is the coefficient of the quenched chiral logarithm.

Theoretically, \( \delta \) can be estimated to be \( \frac{m_{\eta'}}{8\pi^2 f_{\pi}^2 N_f} \) \( \frac{m_{\eta'}}{8\pi^2 f_{\pi}^2 N_f} \) \( \frac{m_{\eta'}}{8\pi^2 f_{\pi}^2 N_f} \) (2)

where \( m_{\eta'} \) denotes the \( \eta' \) mass in \( q\chi PT \), and \( N_f \) is the number of light quark flavors. For \( f_{\pi} = 132 \) MeV, \( N_f = 3 \), and \( m_{\eta'} = \sqrt{m_q^2 + m_q^2 - 2m_q^2} = 853 \text{ MeV} \), (2) gives

\[
\delta \simeq 0.176 \tag{3}
\]

Evidently, if one can extract \( \delta \) from the data of \( m_{\pi}^2 \), then \( m_{\eta'} \) in \( q\chi PT \) can be determined by (2).

Besides from the data of \( m_{\pi}^2 \), one can also obtain \( \delta \) via (2) by extracting \( m_{\eta'} \) from the propagator of the disconnected hairpin diagram. However, to compute the propagator of the hairpin is very tedious.

Fortunately, with the realization of exact chiral symmetry on the lattice, the quark propagator coupling to \( \eta' \) is \( (D_c + m_q)^{-1} \), thus only the zero modes of \( D_c \) can contribute to the hairpin diagram, regardless of the bare quark mass \( m_q \). Therefore one can derive an exact relation between the \( \eta' \) mass in \( q\chi PT \) and the index susceptibility of any Ginsparg-Wilson lattice Dirac operator, without computing the hairpin diagram at all. Explicitly, this exact relation reads as

\[
(m_{\eta'} a)^2 = \frac{4N_f}{(f_{\pi} a)^2} \frac{\langle (n_+ - n_-) \rangle}{N_s} \tag{4}
\]

where \( N_s \) is the total number of sites, and \( \chi \equiv \langle (n_+ - n_-) \rangle / N_s \) is the index susceptibility of any Ginsparg-Wilson lattice Dirac operator in the quenched approximation. Then (3) and (4) together gives

\[
\delta = \frac{1}{2\pi^2 (f_{\pi} a)^4} \frac{\langle (n_+ - n_-) \rangle}{N_s} \tag{5}
\]

A salient feature of (5) is that \( \delta \) can be determined at finite lattice spacing \( a \), by measuring the index (susceptibility) of the overlap Dirac operator, and with \( f_{\pi} a \) extracted from the pion propagator.

Now it is clear that, in order to confirm the presence of quenched chiral logarithm in lattice QCD, one needs to check whether the coefficient \( \delta \) obtained by fitting (3) to the data of \( m_{\pi}^2 \), agrees with that (5) from the index susceptibility. This is a requirement for the consistency of the theory, since the quenched chiral logarithm in \( m_{\pi}^2 \) is

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due to the \( \eta' \) loop coupling to the pion propagator through the mass term (in the chiral lagrangian), thus \( \delta (m_{\eta'}) \) must be the same in both cases. We regard this consistency requirement as a basic criterion for lattice QCD (with any fermion scheme) to realize QCD chiral dynamics in continuum.

Recently, we have determined \( \delta = 0.203(14) \), from the data of \( m_\pi^2 \), as well as \( \delta = 0.197(27) \) from the index susceptibility, for the \( 8^3 \times 24 \) lattice\(^3\). Their excellent agreement suggests that lattice QCD with overlap Dirac quarks indeed realizes the QCD chiral dynamics in the continuum.

In this paper, we extend our studies to other lattices of larger volumes and/or smaller lattice spacings. Our computations are performed with a Linux PC cluster of 30 nodes at the speed \( \sim 30 \) Gflops\(^4\).

### 2. Results

With the Wilson \( SU(3) \) gauge action and Creutz-Cabbibo-Marinari heat bath algorithm, we generate ensembles of gauge configurations as follows:\(^4\):

| \( L^3 \times T \) | \( a(\text{fm}) \) | \( m_\pi (\text{MeV}) \) | \( a/\lambda \) | \( C_a \) | \( B \) | \( \delta \) | \( m_{\eta'} (\text{MeV}) \) |
|----------------|-------------|----------------|----------|-------|------|------|-------------------|
| \( 8^3 \times 24 \) | 0.147(1) | 418 | 1047 | 2.5 | 0.78 | 1.193(18) | 1.152(56) | 0.203(14) | 916(32) |
| \( 10^3 \times 24 \) | 0.152(1) | 375 | 834 | 2.9 | 0.64 | 1.017(25) | 1.222(71) | 0.176(17) | 853(42) |
| \( 12^3 \times 24 \) | 0.147(1) | 395 | 870 | 3.5 | 0.65 | 1.051(24) | 1.254(73) | 0.193(17) | 893(40) |
| \( 16^3 \times 32 \) | 0.095(1) | 439 | 1101 | 3.4 | 0.53 | 1.117(15) | 2.146(102) | 0.200(13) | 909(31) |

The data of \( f_{\pi a} \) can be fitted by straight line. Thus taking \( f_{\pi a} \) at \( m_q a = 0 \) equal to 132 MeV times the lattice spacing \( a \), then \( a \) can be determined. Fixing \( \lambda \) as \( 2\sqrt{2}\pi f_{\pi a} \), we fit \( G(t) \) to the data of \( (m_\pi a)^2 \), and obtain \( C_a, \delta \) and \( B \). Then \( m_{\eta'} \) is computed from \( Z \). Our results are summarized in Table 1. Evidently, all \( \delta \) values are in good agreement with the theoretical estimate \( \delta \approx 0.176 \) from qQPT. Furthermore, the \( \eta' \) masses are also in good agreement with the theoretical estimate \( m_{\eta'} \approx 853 \text{ MeV} \).

Even though the quenched chiral logarithm may not be easily detected in the graph of \( (m_\pi a)^2 \) vs. \( m_q a \), it can be unveiled by plotting \( (m_\pi a)^2 / (m_q a) \) vs. \( m_q a \), as shown in Fig. 1. Further, plotting \( (m_\pi a)^2 / (m_q a) - B(m_q a) \) vs. \( \log(m_q a) \), the presence of quenched chiral logarithm is evident, as shown in Fig. 2.

Note that, in Fig. 2, for the \( 16^3 \times 32 \) lattice at \( \beta = 6.0 \), the minimum of \( (m_\pi a)^2 / (m_q a) \) occurs at \( m_q a = C a \delta / B \approx 0.1 \), which corresponds to \( m_q \approx 200 \text{ MeV} \) for \( a \approx 0.095 \text{ fm} \) (Table 1). Similarly, for \( 8^3 \times 24, 10^3 \times 24 \), and \( 12^3 \times 24 \) lattices at \( \beta = 5.8 \), the minima of \( (m_\pi a)^2 / (m_q a) \) occur at \( m_q a \approx 0.21, 0.15 \) and 0.16 (i.e., \( m_q \approx 280, 190, \) and 217 MeV) respectively. Now, except for the \( 8^3 \times 24 \) lattice which has the smallest volume, the minima of \( (m_\pi a)^2 / (m_q a) \) for the other three lattice sizes almost occur at the same \( m_q \approx 200 \pm 20 \text{ MeV} \).

\(^3\)The 56 configurations on the \( 16^3 \times 32 \) lattice are retrieved from the gauge connection [http://qcd.nersc.gov/].

\(^4\)The usual formula

\[ G_{\pi}(t) = \frac{Z}{2m_\pi a} [e^{-m_\pi at} + e^{-m_\pi a(T-t)}] \]

to extract the pion mass \( m_\pi a \) and the pion decay constant

\[ f_{\pi a} = 2m_\pi a \frac{\sqrt{Z}}{m_\pi^2 a^2} \]
Table 2
Determination of $\delta$ (from index susceptibility).
The $\eta'$ mass in q\chi PT is computed from (4).

| $L^3 \times T$ | $\chi/10^{-4}$ | $\delta$ | $m_{\eta'}$(MeV) |
|----------------|---------------|----------|-----------------|
| $8^3 \times 24$ | 3.67(50)      | 0.198(27)| 904(61)         |
| $10^3 \times 24$ | 3.65(50)     | 0.173(24)| 817(58)         |
| $12^3 \times 24$ | 3.13(41)     | 0.169(22)| 835(55)         |

This provides rather substantial evidence that the quenched chiral logarithm parameters extracted from our data of $m_\pi^2$ is indeed genuine, not due to any finite size effects or discretization errors.

Next we measure the index susceptibility of the overlap Dirac operator by the spectral flow method \[3\], and then determine $\delta$ via (5). The $\eta'$ mass in q\chi PT is computed from (4). Our results are summarized in Table 2, including our earlier results for the $8^3 \times 24$ lattice. Evidently, each $\delta$ is in good agreement with the corresponding $\delta$ extracted from $m_\pi^2$ (Table 1), as well as with the theoretic estimate 0.176.

Figure 1. $(m_\pi a)^2/(m_q a)$ versus the bare quark mass $m_q a$.

Figure 2. The extraction of the quenched chiral logarithm by plotting $(m_\pi a)^2/(m_q a) - B(m_q a)$ versus log($m_q a$).

3. Conclusion
Our results in this paper provide strong evidences that quenched QCD with overlap Dirac quarks realizes quenched QCD chiral dynamics, as depicted by quenched chiral perturbation theory.

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