Synthesis of the optimal control law for the reorientation of a nanosatellite using the procedure of analytical construction of optimal regulators

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Abstract. The method is described in the article and an algorithm for the synthesis of the optimal control law for the reorientation of the nanosatellite using the procedure of analytical construction of optimal regulators is proposed. The algorithm provides a search of control for reorienting the nanosatellite along a phase trajectory close to the nominal one, subject to restrictions on the magnitude of the control torque. The nominal trajectory is divided into local sections, on each of which the required control is calculated.

1. Introduction

A large number of publications are devoted to the study of the problem of controlling the angular motion of spacecraft in various formulations and using a wide range of solution methods. For example, solving the reorientation problem based on the principle of maximum L.S. Pontryagin is considered in [1-2]. The maximum principle is especially important in control systems with maximum speed or minimum energy consumption, where relay type controls are used. Another approach to realize a complex spatial rotation of a spacecraft using smooth controls is a quasi-optimal reorientation algorithm. This approach is based on the use of the concept of inverse dynamics problems and allows one to form program trajectories under known boundary conditions on states and controls. In this case, the program trajectory is searched for in the class of polynomials of a given degree, the coefficients of which are determined by the known values of the state and control variables of the spacecraft at the boundary points of the trajectory. The synthesis of control laws based on quasi-optimal algorithms is considered in [3-5]. When solving the problems of analysis and synthesis of control systems for moving objects, one of the most powerful tools is the theory of linear dynamic systems. A qualitatively new level of development of the theory of control of dynamic systems is achieved by solving the problem of analytical matrix synthesis of arbitrary control laws with a given set of properties. The study of this problem, in particular, the synthesis of the laws of control of the spacecraft, ensuring the optimal placement of the poles of the closed-loop control system, is devoted to [6]. In this paper, we propose an algorithm for synthesizing the optimal control law, based on the method of analytical construction of optimal regulators (ACOR) [7-8], using the set of reference points of the nominal trajectory for reorientation. When constructing the algorithm, external aerodynamic and gravitational moments are taken into account, which have a significant effect on the angular motion of nanoclass spacecraft.
2. Problem statement

From a given initial position \( x_0 = (\gamma_0, \dot{\gamma}_0, \psi_0, \dot{\psi}_0, \theta_0, \dot{\theta}_0) \) it is necessary to transfer the nanosatellite, on which gravitational and aerodynamic moments act, to the required final position \( x_k = (\gamma_k, \dot{\gamma}_k, \psi_k, \dot{\psi}_k, \theta_k, \dot{\theta}_k) \), and the trajectory of the phase parameters of the nanosatellite during the reorientation \( x(t) = (\gamma(t), \dot{\gamma}(t), \psi(t), \dot{\psi}(t), \theta(t), \dot{\theta}(t)) \) should have the form corresponding to the desired purpose. In this case, a restriction on the magnitude of the control action should be established. This limitation is due to the technical capabilities of the executive elements of the nanosatellite.

Another important point is the accuracy with which it is required to carry out the movement of the nanosatellite along the phase trajectory dictated by the features of the space mission.

The synthesis of the optimal law for controlling the reorientation of a nanosatellite using ACOR in this paper includes the following steps:

- Selection of the nominal trajectory for the reorientation maneuver.
- Search for reference points of the nominal trajectory.
- Implementation of transitions between reference points.
- Analytical construction of optimal regulators.

Depending on the desired purpose of the nanosatellite, a nominal trajectory is selected. The formation of optimal nominal trajectories – a separate task in control theory – was not considered in this paper. Since the construction of optimal controllers takes place for linearized equations of motion, the nominal trajectory must be divided into local sections. It is assumed that the optimal nominal trajectory is found and all reference points are determined on it with a certain step along one of the phase coordinates. Note that the deviation between the reference points within each section is taken 5 degrees for angles and 5 degrees per second for the rates of change of angles to achieve the required accuracy of approximation of the linearized equations. Depending on the curvature of the trajectory and the number of control points, the step can be constant or variable. Figure 1 schematically shows the reference points on the path, which has the form of a straight line segment for the pitch channel.

![Figure 1. The reference points on the trajectory of the channel pitch.](image)

In each new local area, the initial deviation is defined as the difference between the current reference point and the current position obtained in the previous local area with the addition of some random variable that includes unaccounted factors. Upon reaching the required accuracy of hitting a new reference point, the algorithm proceeds to process the next local section, and its position becomes the new initial conditions.

3. Linearization of the equations of motion relative to the center of mass

The motion of the spacecraft relative to the center of mass in a coupled coordinate system, provided that the axes of the coupled coordinate system coincide with the main central axes of inertia of the spacecraft, is described by the dynamic Euler equations:
\[
I_x \dot{\omega}_x - (I_y - I_z) \omega_y \omega_z = M_{sx} + M_{ax} + u_x,
\]
\[
I_y \dot{\omega}_y - (I_z - I_x) \omega_z \omega_x = M_{sy} + M_{ay} + u_y,
\]
\[
I_z \dot{\omega}_z - (I_x - I_y) \omega_x \omega_y = M_{sz} + M_{az} + u_z,
\]
\[\text{(1)}\]

\(I_x, I_y, I_z\) are the moments of inertia of the spacecraft relative to the axes \(Ox, Oy,\) and \(Oz,\) respectively, \(kg \times m^2;\) \(\omega = (\omega_x, \omega_y, \omega_z)^T\) – spacecraft angular velocity vector, \(rad/s;\)

\(\mathbf{M}_g = (M_{gx}, M_{gy}, M_{gz})^T\) – vector of gravitational moment of forces, \(N \times m;\)

\(\mathbf{M}_a = (M_{ax}, M_{ay}, M_{az})^T\) – aerodynamic moment vector, \(N \times m;\)

\(u = (u_x, u_y, u_z)^T\) – control vector, \(N \times m.\)

To achieve a certain description of the motion relative to the center of mass, the system of dynamic equations (1) is supplemented by kinematic equations that establish the relationship between the instantaneous velocity vector \(\mathbf{\dot{v}}\), angles \(\gamma, \psi, \theta\) and orbital velocity \(\omega_o\) [9]:

\[
\begin{align*}
\dot{\omega}_x &= \gamma + \psi \sin \theta + \omega_o \cos \theta \sin \psi, \\
\dot{\omega}_y &= \psi \cos \theta \cos \gamma + \dot{\gamma} \sin \gamma - \omega_o (\cos \psi \sin \gamma + \sin \psi \sin \theta \cos \gamma), \\
\dot{\omega}_z &= \dot{\theta} \cos \gamma - \psi \cos \theta \sin \gamma - \omega_o (\cos \psi \cos \gamma - \sin \psi \sin \theta \sin \gamma).
\end{align*}
\]
\[\text{(2)}\]

Kinematic equations (2), solved concerning the quantities \(\gamma, \psi, \theta\), have the form:

\[
\begin{align*}
\gamma &= \omega_x - \omega_y \cos \theta \sin \gamma - \tan \theta \left[ \left[ \omega_x + \omega_o \left( \cos \psi \sin \gamma + \sin \psi \sin \theta \cos \gamma \right) \right] \cos \gamma - \\
&\quad - \left[ \omega_x + \omega_o \left( \cos \psi \cos \gamma - \sin \psi \sin \theta \sin \gamma \right) \right] \sin \gamma \right],
\end{align*}
\]
\[
\psi &= \left[ \left[ \omega_x + \omega_o \left( \cos \psi \sin \gamma + \sin \psi \sin \theta \cos \gamma \right) \right] \cos \gamma - \\
&\quad - \left[ \omega_x + \omega_o \left( \cos \psi \cos \gamma - \sin \psi \sin \theta \sin \gamma \right) \right] \sin \gamma \right] \cos^{-1} \theta,
\]
\[
\dot{\theta} &= \left[ \omega_x + \omega_o \left( \cos \psi \sin \gamma + \sin \psi \sin \theta \cos \gamma \right) \right] \sin \gamma - \\
&\quad - \left[ \omega_x + \omega_o \left( \cos \psi \cos \gamma - \sin \psi \sin \theta \sin \gamma \right) \right] \cos \gamma.
\]
\[\text{(3)}\]

In deviations from the nominal trajectory, we write equations (1) and (3). Here, deviations are meant the difference between the current value of the angle or rate of change of the angle and some fixed value of the angle or rate of change of the angle, which corresponds to the desired orientation at a given point of flight. Since the deviations are small, the kinematic relations (3) are simplified. Replacing the sines of the angles with the values of their arguments, the cosines with units and preserving the values of the first order of smallness, we obtain:

\[
\begin{align*}
\dot{\gamma} &= \omega_x - \omega_y \cos \theta \sin \gamma, \\
\dot{\psi} &= \omega_x + \omega_o \dot{\gamma}, \\
\dot{\theta} &= \omega_x + \omega_o \dot{\gamma}.
\end{align*}
\]
\[\text{(4)}\]

Substituting the linearized kinematic equations (4) in the dynamic Euler equations (1), we obtain:

\[
\begin{align*}
I_x \ddot{\gamma} + \omega_o \left( I_x + I_y - I_z \right) \dot{\psi} &= M_{sx} + M_{ax} + u_x, \\
I_y \ddot{\psi} - \omega_o \dot{\gamma} \left( I_z - I_x \right) &= M_{sy} + M_{ay} + u_y, \\
I_z \ddot{\theta} &= M_{sz} + M_{az} + u_z.
\end{align*}
\]
\[\text{(5)}\]

Thus, the system of differential equations of motion has the form:
\[ \begin{align*}
\dot{x}_1 &= \dot{\gamma}, \\
\dot{x}_2 &= \dot{\alpha}, \\
\dot{x}_3 &= \dot{\psi}, \\
\dot{x}_4 &= \dot{\phi}, \\
\dot{x}_5 &= \dot{\theta}.
\end{align*} \]

\[ \dot{x}_i = \hat{\gamma}, \dot{x}_i = \hat{\alpha}, \dot{x}_i = \hat{\psi}, \dot{x}_i = \hat{\phi}, \dot{x}_i = \hat{\theta} \quad \text{state vector elements} \quad \hat{x}. \]

For a nanosatellite with different moments of inertia moving in a circular orbit in the central gravitational field of the planet, the following expressions of the projections of the moment of gravitational forces can be written:

\[ \begin{align*}
M_{g_{x}} &= 3\omega_{0}^{2}(I_{z} - I_{x})M_{31}M_{33}, \\
M_{g_{y}} &= 3\omega_{0}^{2}(I_{z} - I_{y})M_{32}M_{33}, \\
M_{g_{z}} &= 3\omega_{0}^{2}(I_{y} - I_{z})M_{33}M_{32},
\end{align*} \]

\[ (7) \]

\[ I_{x}, I_{y}, I_{z} \] are the moments of inertia of the spacecraft relative to the axes \( O_{x}, O_{y}, \) and \( O_{z}, \) respectively, kg \( \times \) m\(^2\); \( \omega_{0} \) – orbital velocity, rad/s; \( M_{31}, M_{32}, M_{33} \) – elements of the matrix of guide cosines:

\[ \begin{align*}
M_{31} &= \sin \theta \sin \psi \cos \gamma + \sin \gamma \cos \psi, \\
M_{32} &= -\cos \gamma \sin \theta, \\
M_{33} &= -\sin \psi \sin \theta \sin \gamma + \cos \gamma \cos \psi.
\end{align*} \]

The expressions for the projections of the moment of aerodynamic forces have the form [10]:

\[ \begin{align*}
M_{ax} &= \frac{1}{2}C_{s}S_{m} \rho V^{2} \left( z_{cp} \sin \theta + y_{cp} \sin \psi \cos \theta \right), \\
M_{ay} &= \frac{1}{2}C_{s}S_{m} \rho V^{2} \left( x_{cp} \sin \psi \cos \theta + z_{cp} \cos \psi \cos \theta \right), \\
M_{az} &= \frac{1}{2}C_{s}S_{m} \rho V^{2} \left( y_{cp} \cos \theta \cos \psi - x_{cp} \sin \theta \right).
\end{align*} \]

\[ (8) \]

\[ C_{s} \] – drag coefficient of 2.2; \( S_{m} \) – midship area, m\(^2\); \( \rho \) – atmospheric density at the point of motion of the nanosatellite, kg/m\(^3\); \( V \) – free-stream velocity equal to nanosatellite speed, m/s; \( x_{cp}, y_{cp}, z_{cp} \) – coordinates of the center of pressure relative to the center of mass in the coordinate system \( Oxyz, m; \gamma, \psi, \theta \) – angles of roll, yaw and pitch, respectively, rad.

4. Synthesis of the optimal control law

To find the control of the nanosatellite whose perturbed motion is described by the equation [11]:

\[ \dot{x} = Ax + Bu, \]

we find the matrix \( C^{T} \) of the equation of the regulators:

\[ u = C^{T}\dot{x}, \]

such that, on the asymptotically stable motions of the system (9), (10) excited by arbitrary initial deviations, the functional is minimized
\[ J = \int (\dot{x}^T Q \dot{x} + u^T u) \, dt, \]  

(11)

\( A^{\times \times}, B^{\times \times} \) – given matrices; \( Q^{\times \times} \) – given positive definite matrix.

To find \( C = -P^T B \) the Riccati algebraic equation is solved:

\[ PA + A^T P - PB B^T P + Q = 0, \]  

(12)

\( P \) – symmetric matrix of size \( n \times n \).

The matrix coefficients \( A \) and \( B \) of the Riccati algebraic equation (12) can be easily expressed from the system (6).

To evaluate the operation of the algorithm, mathematical modeling of the reorientation process is carried out with control moments found using the principles of the analytical construction of optimal regulators. The structure of the algorithm can conditionally be divided into several blocks. In the first block, the variables are initialized – setting the parameters of the satellite and the orbit along which it moves, setting the required accuracy of reorientation, setting the integration step. In the second block, reference points of a predetermined nominal trajectory fall into the algorithm. These points represent the sequence of all phase coordinates in which the reorientation should occur. In the third block, the regulators’ matrix is calculated, with the help of which the control vector will be found. The fourth block contains operations that allow moving sequentially along control points and forming data arrays. In the fifth block, a search for control and simulation of the motion of the nanosatellite takes place. In the sixth block, data is output.

5. Simulation results

The simulation results in the form of a control graph are presented in figure 2. For clarity, the graph shows a small portion of time. Modeling parameters:

\[ x_0 = (90 \, \text{deg}, 0.5 \, \text{deg/s}, -90 \, \text{deg}, -0.5 \, \text{deg/s}, -90 \, \text{deg}, 0.5 \, \text{deg/s}); \quad x_s = (0, 0, 0, 0, 0); \quad \text{orbit height} \ h = 400 \, \text{km}; \quad I_x = 0.003 \, \text{kg}\times\text{m}^2; \quad I_y = 0.008 \, \text{kg}\times\text{m}^2; \quad I_z = 0.008 \, \text{kg}\times\text{m}^2; \quad u_{\max} = 1 \times 10^{-3} \, \text{N}\times\text{m}. \]

![Figure 2. Roll channel control.](image)

The simulation results of the algorithm in the form of phase trajectories for the nominal trajectory, which has the form of a straight line segment, are presented in figure 3.

In figure 3, the angles of roll, yaw, and pitch are located along the horizontal axes, and the rate of change of these angles along the corresponding vertical axes. The endpoint of the nominal trajectory is at the origin (0, 0) for each of the graphs.

6. Conclusions

A method is proposed in the work, based on which an algorithm for the synthesis of the optimal control law for a reorientation of a nanosatellite is developed, based on the method of analytical construction of optimal regulators using a variety of reference points of the nominal trajectory for reorientation. When constructing the control search algorithm, disturbing aerodynamic and gravitational moments are taken into account, which have a significant effect on the motion of nanoclass spacecraft.
An analysis of the simulation results showed that the number of intervals for partitioning the nominal trajectory should not be too large, since the maneuver time increases significantly, and at the same time it should be no less than a certain critical value determined by the linearity of the dynamical system. The choice of the optimal number of control points requires additional research.

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