Groups with non-simply connected asymptotic cones

A.Yu. Ol’shanskii, M.V. Sapir

Abstract

We construct a group (an HNN extension of a free group) with polynomial isoperimetric function, linear isodiametric function and non-simply connected asymptotic cones.

1 Introduction

Recall that a function \( f: \mathbb{N} \to \mathbb{N} \) is an isoperimetric (resp. isodiametric) function of a finite presentation \( \langle X \mid R \rangle \) of a group \( G \) if every word \( w \) in \( X \), that is equal to 1 in \( G \), is freely equal to a product of conjugates \( \prod_{i=1}^{m} x_i^{-1} r_i x_i \) where \( r_i \) or \( r_i^{-1} \) is in \( R \), \( x_i \in (X \cup X^{-1})^* \), and \( m \leq f(|w|) \) (resp. \( |x_i| \leq f(|w|) \) for every \( i \)).

Isoperimetric (resp. isodiametric) functions \( f_1, f_2 \) of any two finite presentations of the same group \( G \) are equivalent, that is \( f_2(n) \leq C f_1(Cn) + Cn \), \( f_1(n) \leq C f_2(Cn) + Cn \) for some constant \( C \). As usual, we do not distinguish equivalent functions. The smallest isoperimetric function of a group is called its Dehn function.

In terms of van Kampen diagrams, \( f \) is an isoperimetric (isodiametric) function of the finite presentation if for every word \( w \), that is equal to 1 in \( G \) there exists a van Kampen diagram with boundary label \( w \) and area (resp. diameter) at most \( f(|w|) \) (see Gersten [3] for details).

Let \( (X, \text{dist}) \) be a metric space, \( o = (o_n) \) be a sequence of points in \( X \), \( d = (d_n) \) be an increasing sequence of numbers with \( \lim d_n = \infty \), and let \( \omega: P(\mathbb{N}) \to \{0, 1\} \) be an ultrafilter. An asymptotic cone \( \text{Con}_\omega(X, o, d) \) of \( (X, \text{dist}) \) is the subset of the cartesian power \( X^\mathbb{N} \) consisting of sequences \( (x_i)_{i \in \mathbb{N}} \) with \( \limsup \frac{\text{dist}(o_i, x_i)}{d_i} < \infty \) where we identify two sequences \( (x_i) \) and \( (y_i) \) with \( \lim \omega \frac{\text{dist}(x_i, y_i)}{d_i} = 0 \). The distance between two elements \( (x_i) \) and \( (y_i) \) in the asymptotic cone \( \text{Con}_\omega(X, o, d) \) is defined as \( \lim \omega \frac{\text{dist}(x_i, y_i)}{d_i} \). Here \( \lim \omega \) is the \( \omega \)-limit defined as follows. If \( a_n \) is a bounded sequence of real numbers then \( \lim \omega (a_n) \) is the (unique) number \( a \) such that for every \( \epsilon > 0 \), \( \omega(\{ n \mid |a_n - a| < \epsilon \}) = 1 \).

If \( G \) is a finitely generated group, then asymptotic cones of \( G \) are, by definition, asymptotic cones of the Cayley graph of \( G \) (with respect to some generating set). Asymptotic cones of \( G \) do not depend on the choice of the sequence \( o \), so we can always assume that \( o = (1) \) where 1 is the identity, and use notation \( \text{Con}_\omega(G, d) \).

Gromov proved [4] that if all asymptotic cones of a group \( G \) (for all \( \omega \) and all \( d \)) are simply connected then \( G \) is finitely presented, has polynomial isoperimetric function and linear isodiametric function. Papasoglu [6] proved that if a finitely presented group has quadratic isoperimetric function then all its asymptotic cones are simply connected. But in general the existence of polynomial isoperimetric functions does not imply that the asymptotic cones are

*Both authors were supported in part by the NSF grant DMS 0245600. In addition, the research of the first author was supported in part by the Russian Fund for Basic Research 02-01-00170 and by the Russian Fund for the Support of Leading Research Groups 1910.2003.1
simply connected. Indeed, it is not even true that polynomial isoperimetric inequality implies linear isodiametric inequality. First examples of groups with polynomial Dehn functions and non-linear isodiametric functions were constructed in [1] and [7].

The question of whether we can guarantee that the asymptotic cones are simply connected by requiring that both the Dehn function is polynomial and the isodiametric function is linear, was open. The question was mentioned, in particular, by Drutu in [2]. The following theorem answers this question.

Let

\[ G = \langle \theta_1, \theta_2, a, k \mid a^{\theta_i} = a, k^{\theta_i} = ka, i = 1, 2 \rangle \]

where \( x^y = y^{-1}xy \). Thus \( G \) has a balanced presentation with 4 generators and 4 relators. It is clear that \( G \) is a split extension of the free group \( \langle a, k \rangle \) by the free group \( \langle \theta_1, \theta_2 \rangle \).

**Theorem 1.1.** The group \( G \) has a cubic isoperimetric function, a linear isodiametric function, and no simply connected asymptotic cones.

This group is an S-machine in terminology of [5] or a hub-free interpretation of an S-machine in terminology of [7]. As an S-machine, \( G \) has one state letter \( k \), one tape letter \( a \), and two rules, both of the form \( [k \rightarrow ka] \). Applying results from [7] to \( G \), we can deduce that \( G \) has a cubic isoperimetric function (in fact its Dehn function is exactly \( n^3 \) by [5]), and linear isodiametric function. For the sake of completeness, we present below a direct proof of these statements. Then we shall prove that \( G \) has no simply connected asymptotic cones.

**Remark 1.2.** Using the construction from [5], we can find an S-machine (a multiple HNN extension of a free group) with Dehn function \( n^2 \log n \), linear isodiametric function and no simply connected asymptotic cones. (One can also assume that the conjugacy problem for that group is undecidable.) The proof is only technically more difficult than the proof of Theorem 1.1, and the group is much more complicated, so we do not include this example here leaving it as an advanced exercise.

It would be interesting to find out what the asymptotic cones of \( G \) are. In particular, how many non-by-Lipschitz-equivalent asymptotic cones does \( G \) have, and what are the fundamental groups of these cones? More generally, it is interesting to find out what are the asymptotic cones of S-machines? Topological properties of asymptotic cones may reflect computational properties of the S-machines (for a general definition of an S-machine see [7] or [5]).

**2 Proof**

For every letter \( x \), an \( x \)-edge in a van Kampen diagram is an edge labeled by \( x^{\pm 1} \); an \( x \)-cell is a cell whose boundary contains an \( x \)-edge. If \( x \) is a free letter in an HNN extension, then an \( x \)-band in a diagram over \( G \) is a sequence of cells containing \( x \)-edges, such that every two consecutive cells share an \( x \)-edge. Our group \( G \) can be considered as an HNN extension with free letters \( \theta_1, \theta_2 \) (obviously), and also as an HNN-extension with free letter \( k \). Thus we can consider \( \theta \)-bands (i.e. \( \theta_i \)-bands, \( i = 1, 2 \)) and \( k \)-bands in van Kampen diagrams. It is also convenient to consider \( a \)-bands, that is sequences of cells corresponding to the relations \( a^{\theta_i} = a \), such that every two consecutive cells share an \( a \)-edge. The boundary of the union of cells from an \( x \)-band \( T \) has the form \( e^{-1}pfq^{-1} \) where \( e, f \) are the only \( x \)-edges on the boundary. The paths \( p \) and \( q \) are called the sides of the band \( T \), \( e \) and \( f \) are called the start and end edges of the band. The median of the band \( T \) is a polygonal simple line that connects the midpoints of \( e, f \) and is

---

1 Paper [7] appeared as a preprint in 1997, two years earlier than [1]. But results of [1] were announced four years earlier, in 1993.
contained in the interior of the cells composing the band. We fix one median for each band. A median of an $x$-annulus is defined similarly; it is a simple closed curve.

If $x, y$ are two different letters in $\{\theta, k, a\}$ then one can consider an $(x, y)$-annulus that is a union of an $x$-band and a $y$-band sharing the first and the last cells only, and such that the region bounded by the medians of the bands does not contain the start and end edges of the band.

We shall say that a van Kampen diagram over $G$ is reduced if it does not contain a pair of cells that share a boundary edge and are mirror images of each other.

The following lemma is a particular case of Lemma 3.3 from [5].

**Lemma 2.1.** A reduced van Kampen diagram $\Delta$ over $G$ has no $k$-annuli, $\theta$-annuli, $(k, \theta)$-annuli, $a$-annuli, $(a, \theta)$-annuli.

**Proof.** We assume that $\Delta$ is a counterexample with minimal area. This means in particular that the boundary of $\Delta$ is the boundary component of an annulus $A$, where $A$ has one of the types from the formulation of the lemma.

(1) Let $A$ be a $k$-annulus. Then it consists of $(k, \theta)$-cells. Hence there is a maximal $\theta$-band $T$ in $\Delta$, whose first cell $\pi_1$ and the last cell $\pi_2$ belong to $A$. Being members of the same $k$-band $A$ and $\theta$-band $T$, the cells $\pi_1$ and $\pi_2$ cannot be neighbors in $T$ (the diagram is reduced). Hence $T$ and a part of $A$ form a $(\theta, q)$-annulus. The area of the subdiagram bounded by this annulus is smaller than that of $\Delta$. This contradicts the choice of $\Delta$.

(2) Let $A$ be a $\theta$-annulus. If it contains $k$-cells, then we come to a contradiction as in (1). Otherwise $\Delta$ has no $k$-cells since there is no counter-example of smaller area. The inner part of $A$ (i.e. the subdiagram bounded by the median of $A$) has no $\theta$-edges for the same reason. Hence $\Gamma$ has no cells corresponding to the relations of the group $G$. So, on the one hand, the inner label of $A$ is a cyclically reduced non-empty word in $Y$ since $\Delta$ is a reduced diagram, and on the other hand, this word is freely equal to 1, a contradiction.

(3) Let $A$ be a $(k, \theta)$-annulus. Then the maximal $k$-band $T$ of $A$ cannot have more than two cells because otherwise $\Delta$ would contain a smaller counterexample as in (2). Hence the length of $T$ is 2, and its cells are mirror copies of each other (since they belong to the same $\theta$-band), a contradiction (we assumed that the diagram is reduced).

(4) Let $A$ be an $a$-annulus. Then its boundary labels are words in $\theta_1, \theta_2$. This leads, as in (1), to a smaller $(a, \theta)$-annulus, a contradiction.

(5) Let $A$ be a $(\theta, a)$-annulus and let $T$ be the maximal $a$-band of it. It cannot have more than two $(\theta, a)$-cells because otherwise there would be a smaller $(\theta, a)$-annulus. Hence the length of $T$ is 2, and its cells are mirror copies of each other, a contradiction. The lemma is proved.

The following lemma is a part of Theorem 1.1.

**Lemma 2.2.** $G$ has a cubic isoperimetric function and a linear isodiametric function.

**Proof.** Let $\Delta$ be a reduced van Kampen diagram over $G$ with perimeter $n$. We need to estimate the area of $\Delta$ and its diameter in terms of $n$.

Suppose that $\Delta$ is reduced. Every $k$-cell in $\Delta$ is an intersection of a $\theta$-band and a $k$-band. Since the bands intersect only once by Lemma 2.1, the number of $k$-cells does not exceed the product of the number of maximal $\theta$-bands and the number of maximal $k$-bands. By Lemma 2.1, every maximal $\theta$-band and every maximal $k$-band connect two edges on the boundary of $\Delta$. Hence the number of maximal $k$- and $\theta$-bands does not exceed $n$, and the number of $k$-cells does not exceed $\frac{n^2}{4}$. Every $a$-cell that is not a $k$-cell is the intersection of a $\theta$-band and an $a$-band. Every $a$-band in $\Delta$ either starts on the boundary of $\Delta$ or on the boundary of a $k$-cell. Since each
k-cell contains exactly one a-edge, the number of maximal a-bands is at most \( \frac{n^2}{8} + \frac{n}{2} \). Since an a-band and a \( \theta \)-band can intersect only once (Lemma 2.1 again) the total number of a-cells that are not \( k \)-cells in \( \Delta \) is a most \( \left( \frac{n^2}{8} + \frac{n}{2} \right) \frac{n}{2} = \frac{n^3}{16} + \frac{n^2}{4} \), and the total number of cells in \( \Delta \) does not exceed \( \frac{n^3}{16} + \frac{n^2}{2} \).

Let \( s \) be the number of \( \theta \)-bands in \( \Delta \). Then \( s \leq \frac{n}{2} \). By Lemma 2.1, there exists a \( \theta \)-band that whose side is a part of the boundary \( \partial \Delta \). Then \( \Delta \) is obtained by gluing \( \mathcal{T} \) and a reduced diagram \( \Delta_1 \) with \( s - 1 \) \( \theta \)-bands. Every vertex on a side of \( \mathcal{T} \) can be connected to the boundary of \( \Delta \) by a path of length at most 2. Therefore by induction on \( s \), we can deduce that every vertex inside \( \Delta \) can be connected to the boundary of \( \Delta \) by a path of length at most \( 2s \leq 2 \frac{n}{2} = n \). Hence the diameter of \( \Delta \) is at most \( 5n/2 \).

**Remark 2.3.** 1. As we have mentioned before, the Dehn function of \( G \) is \( n^3 \). To obtain the lower bound, it is enough to consider the diagrams with boundary label \( [k^n, \theta_1^n \theta_2^{-n}] \) where \( [x, y] = x^{-1}y^{-1}xy \) (see below). One can also use the general statement from [5].

2. The second part of the proof of Lemma 2.2 works without any significant change for all multiple HNN extensions of free groups. Thus the isodiametric function of any such group is linear.

It remains to show that every asymptotic cone of \( G \) is not simply connected. Suppose that an asymptotic cone \( \text{Con}_\omega(G, d) \) is simply connected. It was noticed by Gromov [4] (see also [6] or [2] for more details) that then for every \( M > 1 \) there exists a number \( k \) such that for every constant \( C \geq 1 \), every loop \( l \) in the Cayley complex of \( G \), such that \( \frac{1}{C}d_m \leq |l| \leq Cd_m \) for any sufficiently large \( m \), bounds a (singular) disc that can be subdivided into \( k \) (singular) subdiscs with perimeters at most \( \frac{|l|}{M} \). (In fact one can assume that the statement is true for some \( M > 1 \); then one can deduce the statement for all \( M \) by further subdividing the subdiscs.)

Fix a natural number \( n = d_m, m >> 1 \). Let \( u_n \) be the commutator \( [k^n, \theta_1^n \theta_2^{-n}] \). Clearly, \( u_n = 1 \) in \( G \) since \( k^{\theta_1^n} = k^{\theta_2^n} = k \). The corresponding van Kampen diagram has the form of a trapezium [7], [5] with the top and the bottom sides \( p, p' \) labeled by \( k^n \) and the left and right sides \( q, q' \), labeled by \( \theta_1^n \theta_2^{-n} \). The perimeter of that diagram is \( 6n \). Let us show that the loop in the Cayley graph of \( G \) corresponding to \( u_n \) cannot bound a disc decomposed into at most \( l \leq \sqrt{n} \) subdiscs of perimeter \( n \). That would contradict the statement from the previous paragraph for \( M = C = 6 \). Suppose that such a decomposition exists. Then we have a (not necessarily reduced) van Kampen diagram \( \Delta \) with boundary label \( u_n \) composed of \( l \) subdiscs \( \Delta_1, \ldots, \Delta_l \) such that the perimeter of each \( \Delta_i \) is at most \( n \). Consider any \( \theta_1 \)-edge \( e \) of the path \( q \). The \( \theta_1 \)-band in \( \Delta \) that starts at \( e \) cannot end on \( p, p' \) (since these paths do not contain \( \theta \)-edges), or on \( q \) (since every edge of \( q \) points from the initial vertex \( q_- \) to the terminal vertex \( q_+ \) of \( q \)). Hence it ends on \( q' \). Since \( \theta \)-bands do not intersect, they connect corresponding \( \theta \)-edges on \( q \) and \( q' \), that is the \( \theta \)-edge number \( i \) on \( q \) is connected with the \( \theta \)-edge number \( i \) on \( q' \).

Let \( e \) be the \( \theta_1 \)-edge number \( n \) on \( q \) and \( \mathcal{T} \) be the maximal \( \theta \)-band starting at \( e \). Let \( r \) be the top side of \( \mathcal{T} \). Then the label \( \phi(r) \) belongs to the free group \( \langle k, a \rangle \) and is equal to \( \theta_1^{-n} k^n \theta_2^n \) in \( G \). Hence this word is freely equal to \( (ka^n)^n \). Since the number of subdiscs \( \Delta_i \) is \( l \leq \sqrt{n} \), there is a subpath \( w \) of \( r \) such that the initial and terminal vertices of \( w \) belong to the boundary of one of the \( \Delta_i \) and the freely reduced form \( W \) of \( \phi(w) \) contains \( ka^n k \) as a subword. Hence, in the group \( G \), we should have an equality \( W = U \) where \( |U| \leq \frac{n}{2} \) since the perimeter of \( \Delta_i \) does not exceed \( n \).

Let \( \Gamma \) be a reduced diagram for this equality with boundary \( r_1 r_2^{-1} \) where \( \phi(r_1) = W, \phi(r_2) = U \). Let \( r_1' \) be a subpath of \( r_1 \) with label \( ka^n k \). Consider the two maximal \( k \)-bands \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) starting on the \( k \)-edges \( e \) and \( f \) of \( r_1' \).
Note that since \( r_1 \) does not contain \( \theta \)-edges, every \( \theta \)-band crossing \( T_1 \) or \( T_2 \) must end on \( r_2 \). Hence if \( T_i \) or \( T_2 \) end on \( r_1 \) or the length of that \( k \)-band must be zero. But this would mean that a non-trivial subword of a freely reduced word \( W \) is equal to 1 in the free group \( \langle k, a \rangle \) which is impossible. Hence \( T_i \) and \( T_2 \) end on \( r_2 \).

Let \( \Gamma' \) be the subdiagram of \( \Gamma \) bounded by \( r_1'' = r_1' \setminus \{ e, f \} \), sides of \( T_1 \) and \( T_2 \), and a part \( \bar{r}_2 \) of \( r_2 \). Let \( \partial \Gamma' = \bar{r}_1 \bar{r}_2^{-1} = r_1'' (r_2')^{-1} \). The length of a side of \( T_1 \) or \( T_2 \) is at most twice as large as the number of \( \theta \)-edges in it, and the number of \( \theta \)-edges in these sides is at most \( |r_2| - |\bar{r}_2| - 2 \) since there are two \( k \)-edges on \( r_2 \). Hence we have \( |r_2''| < 2|r_2| - 2|\bar{r}_2| + |\bar{r}_2| = 2|r_2| - |\bar{r}_2| \).

Every \( k \)-band \( T \) in \( \Gamma' \) connects two edges \( e(T), f(T) \) on \( \bar{r}_2 \). We say that a \( k \)-band \( T \) is farther from the boundary than a \( k \)-band \( T' \) if the subpath of \( \bar{r}_2 \) between \( e(T), f(T) \) contains \( e(T') \) and \( f(T') \). Note that the length of a side of \( T \) is at most twice as large as the length of the subpath of \( \bar{r}_2 \) between \( e(T) \) and \( f(T) \).

Let us remove from \( \Gamma' \) all the \( k \)-bands that are the farthest from \( \bar{r}_2 \) together with parts of \( \Gamma' \) between their sides and \( \bar{r}_2 \). Let us denote the resulting subdiagram of \( \Gamma' \) by \( \Gamma'' \). Then \( \partial \Gamma'' = r_1'' r_2'' \). Since \( k \)-bands do not intersect, \( |r_2''| \leq |r_2'| + |\bar{r}_2| \). Hence \( |r_2''| < 2|r_2| - |\bar{r}_2| + |\bar{r}_2| = 2|r_2| \leq n \).

Note that \( \Gamma'' \) does not contain \( k \)-cells. Therefore every \( a \)-band that starts on \( r_1'' \) must end on \( r_2'' \). But that is impossible since \( |r_2''| < n \), and \( \phi(r_1'') = a^n \), a contradiction. This completes the proof of Theorem 1.1.

References

[1] Martin Bridson. Asymptotic cones and polynomial isoperimetric inequalities. Topology 38 (1999), no. 3, 543–554.

[2] Cornelia Drutu. Quasi-isometry invariants and asymptotic cones. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000). Internat. J. Algebra Comput. 12 (2002), no. 1-2, 99–135.

[3] Steve M. Gersten. Isoperimetric and isodiametric functions of finite presentations. Geometric group theory, Vol. 1 (Sussex, 1991), 79–96, London Math. Soc. Lecture Note Ser., 181, Cambridge Univ. Press, Cambridge, 1993.
[4] M. Gromov. Asymptotic invariants of infinite groups. Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser., 182, Cambridge Univ. Press, Cambridge, 1993.

[5] A.Yu. Olshanskii and M. V. Sapir. Groups with small Dehn functions and bipartite chord diagrams. preprint, arXiv math.GR/0411174.

[6] P. Papasoglou. Asymptotic cones and the quadratic isoperimetric inequality. Journal of Differential Geometry, 1996, 44, 789–806.

[7] M. V. Sapir, J. C. Birget, E. Rips. Isoperimetric and isodiametric functions of groups, Annals of Mathematics, 157, 2 (2002), 345–466.

Alexander Yu. Ol’shanskii
Department of Mathematics
Vanderbilt University
alexander.olshanskiy@vanderbilt.edu
http://www.math.vanderbilt.edu/~olsh

and

Department of Higher Algebra, MEHMAT
Moscow State University
olshan@shabol.math.msu.su

Mark V. Sapir
Department of Mathematics
Vanderbilt University
m.sapir@vanderbilt.edu
http://www.math.vanderbilt.edu/~msapir