ABSTRACT
A weighted global error minimization (WGEM) is proposed in this study. The goal is to improve the accuracy of the global error minimization (GEM) based on a weighting function. In addition to the first-order approximation, the fourth-order approximation for the Duffing oscillator is demonstrated by coupling a proper weighting function with the GEM. In order to exhibit the advantage of this modification, the obtained result is compared with both the exact frequency and the outcome of the GEM. The corollary outstandingly reveals that approximations using the WGEM have a lower relative error than those from the GEM in the first-order approximation. Also, the modified approach can preserve its accuracy in the fourth-order approximation. The WGEM can be promisingly utilized to other resembling nonlinear problems.

1. Introduction
The accurate estimation of the relationship between frequency and amplitude of nonlinear oscillators is of significant importance for many fields of applied science and engineering. Surveys of the literature express there are different approximate analytical techniques for dealing with the nonlinear problems. Among them, one may attract attention to the weighted linearization technique [1], the energy balance method [2–4], the optimal homotopy asymptotic method [5], the linearized harmonic balance method [6], the global residue harmonic balance method [7], the homotopy analysis method [8,9], Max–Min approach [10,11], Hamiltonian approach [12,13], the variational approach [14–16], the variational iteration method [17], the rational harmonic balance method [18], the Chebyshev polynomial approximation [19] and so on [20–22].

The aim of this investigation is to reach a more accurate relationship between frequency and amplitude to nonlinear vibrating systems in conjunction with the global error minimization (GEM) [23–25]. The precision of the GEM has been improved by combining with an appropriate weighting function. The weighting function plays a special role in acquiring an accurate estimation. Therefore, a suitable weighting function is chosen based on related works and author knowledge [26,27]. Whereas it is integrable and non-negative over the interval. It should be noted that an arbitrary weighting function is inappropriate for the procedure.

The rest of the manuscript is organized as follows. The outline of the proposed modification is presented in Section 2. The relationship between the frequency and the initial amplitude of the Duffing oscillator is provided via either first- and fourth-order approximations in Section 3. This study ends with a brief conclusion in the last section.

2. The weighted GEM
This section gives the basic idea of the weighted global error minimization (WGEM). Consider a general nonlinear oscillator as follows:

\[ \ddot{u} + F(u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \]  

By defining a functional as follows:

\[ E(u) = \int_0^T (\dot{u} + F(u))^2(W(t))^3 \, dt, \quad T = 2\pi \omega^{-1}, \]  

where \( W(t) \) is a weighting function and hereon it is selected as follows:

\[ W(t) = \frac{3a_0^2}{4\pi^3} (2\pi t - \omega t^2). \]  

Also by assuming \( F(u) \) is an odd function. One may employ an approximate trial function in the form of

\[ \ddot{u}(t) = \sum_{n=0}^{\infty} (a_{2n+1}) \cos((2n + 1)\omega t)), \]  

so that

\[ \sum_{n=0}^{\infty} a_{2n+1} = A. \]
And the unknown parameters (i.e. $a_{2n+1}$ & $\omega$) have been obtained through the following conditions:

$$\frac{\partial E(\tilde{u})}{\partial \omega} = 0 \quad \text{and} \quad \frac{\partial E(\tilde{u})}{\partial a_{2n+1}} = 0 \quad \text{for} \quad n \geq 1.$$  \hspace{1cm} (6)

To demonstrate the practicality and effectiveness of the aforementioned procedure, a cubic Duffing oscillator with strong nonlinearity is taken into consideration in the present study. The results are illustrated in the next section.

3. The cubic Duffing oscillator

This section investigates the accuracy of the approach by the cubic Duffing oscillator. The governing equation for this type oscillator is:

$$\ddot{u} + \mu u + \varepsilon u^3 = 0, \quad u(0) = A, \quad \dot{u}(0) = 0.$$  \hspace{1cm} (7)

Equation (7) is a mathematical model of a conservative system. The exact frequency of this oscillator is given as:

$$\omega_e = \frac{\pi}{2} \sqrt{\mu + \varepsilon A^2} \left( K \left( \frac{1}{2} \mu + \varepsilon A^2 \right) \right)^{-1}.$$  \hspace{1cm} (8)

Where $K(m)$ is the complete elliptic integral of first kind and defined as follows:

$$K(m) = \int_0^{\pi/2} (1 - m \sin^2(\theta))^{-1/2} d\theta.$$  \hspace{1cm} (9)

The first- and fourth-order approximations for this nonlinear model are given in the following context.

3.1. First-order approximation

Based on Section 2, the WGEM is implemented properly. The minimization problem of Equation (7) can be rewritten as:

$$E(u) = \int_0^T (\ddot{u} + \mu u + \varepsilon u^3)^2 \left( \frac{3\omega^2}{4\pi^2} (2\pi t - \omega t^2) \right)^3 dt,$$

$$T = 2\pi \omega^{-1},$$  \hspace{1cm} (10)

for the first-order approximation, the trial function is:

$$\tilde{u}_1(t) = a_1 \cos(\omega t).$$  \hspace{1cm} (11)

Where $a_1 = A$, substituting Equation (11) into Equation (10) yields:

$$E(\tilde{u}_1) = \frac{A^2 \omega^2}{13762560\pi^8} (5A^4(-12324865 + 48\pi^2(69797 + 6912\pi^4)))(\varepsilon^2 + 972A^2(-134925 + 36400\pi^2 + 4096\pi^6)(\varepsilon - \omega^2)^2 + 41472(-1575 + 420\pi^2 + 64\pi^6)(\varepsilon - \omega^2)^2),$$  \hspace{1cm} (12)

by applying $\frac{\partial E(\tilde{u}_1)}{\partial \omega} = 0$, the approximate frequency is obtained as:

$$\omega = \frac{1}{24\sqrt{6}(-1575 + 420\pi^2 + 64\pi^6)}(27A^2$$

$$\times (-134925 + 36400\pi^2 + 4096\pi^6)\varepsilon$$

$$+ 2304(-1575 + 420\pi^2 + 64\pi^6)\mu$$

$$+ (3A^4(1317889636875 + 32\pi^2)(-22324430625$$

$$+ 8\pi^2(378234675 + 8\pi^2(-28698775$$

$$+ 7743120\pi^2 + 331776\pi^6))\varepsilon^2$$

$$+ 31104A^2(-1575 + 420\pi^2 + 64\pi^6)(-134925$$

$$+ 36400\pi^2 + 4096\pi^6)\mu + 1327104(-1575$$

$$+ 420\pi^2 + 64\pi^6\mu^2(1/2)^1/2)).$$  \hspace{1cm} (13)

For more convenience, in current work the value of $\mu$ is selected equal to one ($\mu = 1$). Maximum relative error of Equation (13) is as low as 0.22% in whole of the domain. Furthermore, for large values of the non-linearity parameter ($\mu = \varepsilon A^2$), the relative error of the first-order approximation is as low as 0.01%, viz.,

$$\lim_{\lambda \to \infty} \frac{\omega_{WGEM} - \omega_{exact}}{\omega_{exact}} = 1.0001.$$  \hspace{1cm} (14)

In comparison to the GEM [23], the accuracy of the WGEM is excellent.

3.2. Fourth-order approximation

To illustrate the accuracy of the approach in higher-order approximations, the fourth-order approximation is applied to the Duffing oscillator. In this section, the following trial function is considered:

$$\tilde{u}_4(t) = a_1 \cos(\omega t) + a_3 \cos(3\omega t)$$

$$+ a_5 \cos(5\omega t) + a_7 \cos(7\omega t).$$  \hspace{1cm} (15)

Based on the WGEM algorithm and using Equation (15) as the trial function, the minimization problem of Equation (7) is determined. Thereupon by applying the conditions of Equation (6), the relative error of the approximate frequency and the coefficients of Equation (15) for different values of the nonlinearity parameter are achieved. The results are presented in Table 1. As can be seen, the fourth-order approximation of the algorithm gives an excellent accuracy for whole of the domain. Moreover, in this case the relative error of the WGEM is lower than the GEM slightly.
4. Conclusions

Excellent agreement of the approximate frequency with the exact one has been displayed. This study scrutinizes the accuracy of the modified global error minimization by the Duffing oscillator. This applicable technique successfully provides an accurate approximate frequency for the first-order approximation. Also the higher-order estimation using this approach is more accurate for both small and large amplitudes. The general perspective of this straightforward approach indicates the technique is reliable, simple, powerful and accurate for conservative nonlinear oscillators.

Disclosure statement

No potential conflict of interest was reported by the author.

ORCID

M. K. Yazdi https://orcid.org/0000-0002-5075-4009

References

[1] Agrwal VP, Denman HH. Weighted linearization technique for period approximation in large amplitude nonlinear oscillations. J Sound Vibration 1985;99:463–473.

[2] Durmaz S, Kaya MO. High order energy balance method to nonlinear oscillators. J Appl Math. 2012;2012:1–7. Article ID: 518684.

[3] Yazdi MK, Tehrani PH. The energy balance to nonlinear oscillations via Jacobi collocation method. Alex Eng J. 2015;54:99–103.

[4] Yonesian D, Askari H, Saadatnia Z, et al. Frequency analysis of strongly nonlinear generalized Duffing oscillators using He’s frequency-amplitude formulation and He’s energy balance method. Comput Math Appl. 2010;59:3222–3228.

[5] Herisanu N, Marinca V. Accurate analytical solutions to oscillators with discontinuities and fractional-power restoring force by means of the optimal homotopy asymptotic method. Comput Math Appl. 2010;60:1607–1615.

[6] Lai SK, Lim CW. Higher-order approximate solutions to a strongly nonlinear Duffing oscillator. Int J Comput Methods Eng Sci Mech. 2006;7:201–208.

[7] Mohammad M, Akbarzade M. Higher-order approximate analytical solutions to nonlinear oscillatory systems arising in engineering problems. Arch Appl Mech. 2017;87:1317–1332.

[8] Sedighi HM, Shirazi KH, Zare J. An analytic solution of transversal oscillation of quintic non-linear beam with homotopy analysis method. Int J Non-Linear Mech. 2012;47:777–784.

[9] Reza A, Sedighi HM. Nonlinear vertical vibration of tension leg platforms with homotopy analysis method. ADV Appl Math Mech. 2015;7:375–368.

[10] Ganji SS, Ganji DD, Davodi AG, et al. Analytical solution to nonlinear oscillation system of the motion of a rigid rod rocking back using max–min approach. Appl Math Model. 2010;34:2676–2684.

[11] Yazdi MK, Ahmadian H, Mirzabeigy A, et al. Dynamic analysis of vibrating systems with nonlinearities. Commun Theor Phys. 2012;57(2):183–187.

[12] Bayat M, Pakar I, Cvetcianin L. Nonlinear vibration of stringer shell by means of extended Hamiltonian approach. Arch Appl Mech. 2014;84:43–50.

[13] Askari H, Nia ZS, Yildirim A, et al. Application of higher order Hamiltonian approach to nonlinear vibrating systems. J Theor Appl Mech. 2013;51:287–296.

[14] He JH. Variational approach for nonlinear oscillators. Chaos Solitons Fractals. 2007;34:1430–1439.

[15] Yazdi MK. Approximate solutions to nonlinear oscillations via an improved He’s variational approach. Karbala Int J Mod Sci. 2016;2:289–297.

[16] Yonesian D, Askari H, Saadatnia Z, et al. Free vibration analysis of strongly nonlinear generalized Duffing oscillators using He’s variational approach & homotopy perturbation method. Nonlin Sci Lett A. 2011;2:11–16.

[17] Mohammad M. Application of the variational iteration method to nonlinear vibrations of nanobeams induced by the van der Waals force under different boundary conditions. Eur Phys J Plus. 2017;132:19.

[18] Belendez A, Gimeno E, Belendez T, et al. Rational harmonic balance based method for conservative nonlinear oscillators: application to the Duffing equation. Mech Res Commun. 2009;36:728–734.

[19] Jonckheere RE. Determination of the period of nonlinear oscillations by means of Chebyshev polynomials. ZAMM Z Angew Math Phys. 1971;51:389–393.

[20] He JH. Some asymptotic methods for strongly nonlinear oscillators. Chaos Solitons Fractals. 2007;31:395–406.

[21] Mickens RE. Truly nonlinear oscillators: harmonic balance, parameter expansions, iteration, and averaging methods. Singapore: World Scientific; 2010.

[22] Chang DC, Feng SY. Periodic solutions for Hamiltonian equation associated with Gaussian potential. Anal Math Phys. 2017;7:459–477.

[23] Farzaneh Y, Tootoonchi AA. Global error minimization method for solving strongly nonlinear oscillator differential equations. Comput Math Appl. 2010;59:2887–2895.
[24] Yazdi MK, Mirzabeigy A, Abdollahi H. Nonlinear oscillators with non-polynomial and discontinuous elastic restoring forces. Nonlin Sci Lett A. 2012;3:48–53.

[25] Yazdi MK, Tehrani PH. Frequency analysis of nonlinear oscillations via the global error minimization. Nonlinear Eng. 2016;5:87–92.

[26] Yazdi MK, Khan Y, Madani M, et al. Analytical solutions for autonomous conservative nonlinear oscillator. Int J Nonlinear Sci Numer Simul. 2010;11:979–984.

[27] Cveticanin L, Yazdi MK, Askari H. Analytical solutions for a generalized oscillator with strong nonlinear terms. J Eng Math. 2012;77:211–223.