Hard thermal loop resummation techniques
in hot gauge theories

Randy Kobes

Physics Department and Winnipeg Institute for Theoretical Physics
University of Winnipeg
Winnipeg, Manitoba R3B 2E9 Canada

ABSTRACT

A review is given of the hard thermal loop resummation methods initiated by Braaten, Pisarski, and others. We describe some successes of these techniques as well as instances where modifications are necessary. Some possible directions where these modifications may lead are also discussed.

1. Introduction

The infrared behaviour of gauge theories at high temperature is generally more pronounced than at zero temperature and leads to a number of interesting problems. One of the outstanding paradoxes at the time of the first workshop in this series in Cleveland in 1988 was the gauge dependence of the one–loop gluon damping constant. The problem at the time was the following. If one considers the response of a field to a small external perturbation \( J(t) \),

\[
\delta \phi(t) \sim \int dt' D(t-t') J(t') \sim \int dt' \int dk_0 e^{-ik_0(t-t')} D(k_0) J(t'),
\]

then the poles of the (retarded) propagator \( D(k) \),

\[
D(k) \sim \frac{i}{k_0 - \Sigma(k)} \sim \frac{i}{k_0 - \omega(k) + i\gamma(k)},
\]

will determine a characteristic frequency \( \omega \) and damping rate \( \gamma \) of the oscillations. For \( QCD \) in the high temperature, long wavelength limit one can calculate these quantities at the one–loop level by considering the graphs in Fig. 1, along with Faddeev–Popov ghost terms as required;

Fig. 1. One–loop gluon self–energy
the frequency
\[ \omega(k) \sim \text{Re} \Sigma(k_0 = \omega) \] (3)

turns out to be of order $gT$ and is gauge independent, but the damping rate
\[ \gamma \sim \text{Im} \Sigma(k_0 = \omega) \] (4)
is of order $g^2 T$ but gauge dependent. There was even some controversy over the sign of $\gamma$, which from Eq.(1) would determine if the oscillations were damped or anti-damped in time. These results for $\gamma$ were puzzling because, as well as being a supposedly physical quantity, there exist formal arguments that the poles of a propagator are generally gauge independent, even if the propagator itself is not.

The source of this paradox was argued at the time in particular by Pisarski to be the breakdown of the loop expansion at high temperature. Further work has resulted in what is now known as an effective expansion in terms of hard thermal loops which aims to include in a systematic manner all relevant loop effects to a given order. In this expansion two scales of momenta are relevant: hard ($\sim T$) and soft ($\sim gT$), with $g << 1$. If an internal momentum is hard then ordinary bare propagators and vertices are sufficient, but if the momentum is soft then effective propagators and vertices, as illustrated in Fig. 2, must be used.

Fig. 2. Effective propagators and vertices for QED
The dominant contributions in the loop integrals come from hard internal momenta, and hence these terms are called hard thermal loops. These graphs enjoy some remarkable properties such as gauge invariance and so are interesting in their own right. They have been studied from the point of view of their relation to effective actions, Chern–Simons theory, and kinetic equations.

2. Successes of the effective expansion

One of the first applications of this effective expansion was to the calculation of damping rates for particles at rest. For QCD the relevant graphs are illustrated in Fig. 3, again with Faddeev–Popov ghost terms as required.

The calculation of the damping rate from these graphs is involved and requires numerical methods; nevertheless, the result turns out to be positive (indicating damped modes), gauge independent, and reasonable. Similar calculations were also carried out for the fermion damping rate.

A subtlety in these calculations was soon recognized. The gauge dependent piece of the damping rate, which for covariant gauges is proportional to

$$\zeta (k^2 - m^2)^2 \int \frac{dq}{q^4 [(k + q)^2 - m^2]} ,$$

is finite as the mass–shell limit $k^2 \to m^2$ is taken. This would lead back to a gauge dependent damping rate. A resolution to this paradox was later proposed, where it was shown that if one introduces an infrared cut–off and then take the mass–shell limit before this cut–off is removed, the gauge dependent contribution vanishes.

Other calculations were subsequently done to test the validity of this effective expansion. One such calculation was the next–to–leading order correction to the plasma frequency in the long–wavelength limit for QCD; this involves the real parts of the graphs of Fig. 3, and is found to be

$$\omega^2 \sim \frac{1}{9} g^2 N^2 \left[ 1 - 0.18 g \sqrt{N} + \ldots \right].$$
Another calculation carried out was the photon production rate for real hard photons \( (k^2 = 0, k_{\mu} \sim T) \), which involves calculating the imaginary part of the graph of Fig. 4.

\[ \text{Fig. 4. Calculation of the photon production rate} \]

A general feature of calculations such as these is that internal momenta can be hard or soft. When hard bare propagators and vertices suffice, but when soft the effective ones may be needed. An intermediate scale \( (\sim \sqrt{g} T) \) is then introduced to separate the contributions from each regime, and when added together the dependence on this intermediate scale must cancel. One finds for this calculation the result

\[
E \frac{dR}{d^3p} \sim \frac{\alpha^2 T^2}{E} \ln \left( \frac{0.31 E}{\alpha T} \right).
\]

(7)

Other successes of the effective expansion have also been found.

3. Problems with the effective expansion

Although successful for the problems it addresses, the effective expansion does not cure all the infrared problems of hot perturbative gauge theories. One example of this is the damping rate of a fast fermion \( (E >> m >> T, v \rightarrow 1) \), which involves the calculation indicated in Fig. 5.

\[ \text{Fig. 5. Calculation of the fast fermion damping rate} \]

One finds for this that even with the effective propagator an infrared divergence remains:

\[
\gamma \sim e^2 T \int e^T \frac{dq}{q}.
\]

(8)

For QCD this divergence can be removed by introduction of a magnetic mass \( m_{\text{mag}} \sim g^2 T \), A resolution which also applies for QED, where a magnetic mass
is not expected to arise, was put forth by Lebedev and Smilga. They suggested to use an internal effective fermion propagator of the form
\[ S(k) \sim \frac{i}{k_0 - E + i\gamma}, \tag{9} \]
where \(\gamma\) is the damping rate to be calculated. The equation indicated in Fig. 5 with this effective internal fermion propagator then becomes an implicit equation for \(\gamma\), which can be solved to yield
\[ \gamma \sim e^2 T \int_{\gamma}^{\infty} dq \frac{e^T q}{q} \sim e^2 T \ln \left(\frac{e^T \gamma}{\gamma}\right) \sim e^2 T \ln \left(\frac{1}{e}\right). \tag{10} \]

A subtlety with this calculation arose concerning the proper on–shell condition to use; if the real condition \(k_0 = E\) is used then the infrared divergence disappears, but if the complex condition \(k_0 = E - i\gamma\) is employed the divergence resurfaces. A resolution to this difficulty could involve the use of a more complicated form of the internal fermion propagator with cuts rather than the simple pole of Eq.(9).

Another calculation for which the effective expansion is inadequate is the photon production rate for soft real photons \((k^2 = 0, k_\mu \sim gT)\). This involves again calculating the imaginary part of the graph of Fig. 4. With the appropriate effective propagators and vertices, however, a mass–shell singularity remains.

Further indications that the effective expansion breaks down near the light–cone is provided by calculating corrections to the dispersion relation in the vicinity of the light–cone. A relatively simple illustration of this is scalar QED. There it is found that the first–order corrections, which involve the analogous graph to Fig. 4, diverge logarithmically near the light–cone \((\sim \ln(k^2/k^2))\). As the “tree–level” effective propagator itself also diverges near the light–cone \((\sim 1/\sqrt{k^2})\), this shows that “loop” corrections are no longer small in this regime, and hence the effective expansion is breaking down.

One final example of the inadequacy of the effective expansion is the calculation of the next–to–leading order term of the Debye mass. The definition of this mass involves the potential
\[ \Phi(r) \sim \int_{-\infty}^{+\infty} \frac{k dk}{k^2 + \Pi_{00}(0, k)} \frac{\sin kr}{r}, \tag{11} \]
where \(\Pi_{\mu\nu}\) is the gluon polarization tensor. A gauge invariant definition of the Debye mass then follows by considering the pole of Eq.(11):
\[ m^2_D = \Pi_{00}(0, k^2 = -m^2_D). \tag{12} \]
At zeroth order one finds \(m^2_D \sim g^2 T^2\), but using the effective expansion to calculate the next–to–leading order term from the analogous graphs of Fig. 8 leads to a divergent result unless a magnetic mass \(m_{\text{mag}} \sim g^2 T\) is included:
\[ \frac{\delta m^2_D}{m^2_D} \sim g \ln \left(\frac{2m_D}{m_{\text{mag}}}\right) \sim g \ln \left(\frac{1}{g}\right). \tag{13} \]
Questions, though, have been raised over whether or not a simple pole indicated in Eq. (11) is actually present. Nevertheless, as the magnetic mass for QCD is expected to be non-perturbative in nature, this provides another example of the breakdown of the effective expansion.

4. Beyond the hard thermal loop resummation

Some attempts have been made to go beyond the resummation techniques based on hard thermal loops and address problems such as the ones discussed in the last section. One fairly straightforward one which enjoys some success concerns the behaviour of the effective expansion near the light cone. Specifically, one could imagine including corrections to hard internal lines, in the same manner such corrections are included as in Fig. 2 for soft lines. Normally corrections to hard lines are of higher order than the corrections to soft lines and thus can be ignored in lower-order calculations. However, for processes near the light cone, such as that of the dispersion relation of scalar QED mentioned in the last section, corrections to hard lines turn out to be as important as the soft line corrections, and indeed remove the divergence found in the usual expansion using just hard thermal loops. There has also been some work along these lines to see if such corrections can remove the divergence mentioned in the previous section for the soft photon production rate. Although these corrections to hard lines are known to improve the infrared behaviour in many cases, a systematic approach based on their inclusion has not yet been developed to the level of the usual hard thermal loop expansion.

Another approach which attempts to include the most infrared singular terms perturbatively but whose relation to the hard thermal loop expansion is not completely developed is one based on an eikonal and/or ladder approximation. These general methods have been used with some success in calculations of the pressure which as is well-known are plagued with infrared divergences beyond certain orders, and it is hoped they can be applied in other areas. Consider as an illustration the one-loop vertex function in QED of Fig. 6.

Fig. 6. One-loop vertex function
In the limit that the internal photon momentum \( r \) vanishes, one finds that this function can be written in terms of the one–loop self–energy \( \Sigma(p) \) as

\[
\Gamma_\mu(k, p) = \frac{k_\mu + 2p_\mu}{k^2 + 2k \cdot p} \left[ \Sigma(p) - \Sigma(k + p) \right].
\] (14)

This automatically satisfies the Ward identity \( k \cdot \Gamma(k, p) = \Sigma(p) - \Sigma(k + p) \), and so one has in a sense “solved” the Ward identity for \( \Gamma_\mu \) in terms of \( \Sigma \) in this infrared limit. Although this relation holds at the one–loop level, the functional relation of Eq.(14) holds also at higher–loop orders. As well, in a similar manner higher \( n \)–point functions can be expressed in terms of the self–energy in this particular limit. One could then attempt to use these approximations in, for example, the Schwinger–Dyson equation for the self–energy,

\[
\Sigma(p) = ie^2 \int d^4k \gamma_\mu D^{\mu\nu}(p + k) S(k) \Gamma_\nu(p, k),
\] (15)

thereby obtaining a self–consistent equation for \( \Sigma \). Even though approximate, the momentum dependence makes this equation difficult to solve, and one must do further approximations in order to obtain a solution. One approach along these lines is to use a relatively simple form of a self–energy function with some free parameters and then to employ Eq.(15) to determine these parameters. One such approximation, based on the ansatz of Eq.(14) involving a simple pole structure of the propagator, has been tried. However, the assumption of a constant damping rate \( \gamma \) has proven to be too simplistic to lowest non–trivial order. More involved attempts thus must be made, either by modifying the parameterized form of the self–energy or else by using other self–consistent approximations.

Another attempt to improve the usual loop expansion of perturbation theory is the use of the renormalization group equations. These equations, which of course are very successful at zero temperature, can be used to give a running coupling constant \( g(\tau, \kappa) \) which hopefully will give information on the scaling of quantities with temperature \( \tau \) and momentum \( \kappa \). This involves calculating the finite temperature \( \beta \)–function from, for example in the background field approach, the transverse piece of the self–energy:

\[
\tau \frac{d g(\tau, \kappa)}{d\tau} = \beta_\tau = -\frac{g}{2\kappa^2} T \left. \frac{d \Pi_T(T, \kappa)}{dT} \right|_{T=T}. \] (16)

However, in high temperature QCD this calculation is plagued with difficulties at the one–loop level of Fig. 1, among which are problems with gauge and process dependence. There are not even definite conclusions on the sign of the \( \beta \)–function, which would indicate asymptotic freedom and whether or not the effective coupling constant decreases in strength as the temperature rises. Attempts were made to improve on this one–loop calculation by including the effects of the hard thermal loops, as in Fig. 3, as well as inclusion by hand of a magnetic mass term for the transverse gluons of order \( g^2 T \). These corrections give results in the direction...
leaning towards asymptotic freedom but they are inconclusive, as in particular the problem of gauge dependence remain. Although one might invoke arguments from the pinch technique or the Vilkovisky–DeWitt effective action that certain gauges might be “preferred”, like the one–loop gluon damping constant this gauge dependence probably indicates that the expansion used is failing. As with the calculation of the next–to–leading order term for the Debye mass described in the previous section, definite results in this regard will probably have to wait until a better understanding of scales approaching the magnetic mass scale of order $g^2T$ is found, if indeed this is possible in perturbation theory.

5. Outlook

Although successful for the problems it was designed for, there are instances where the effective expansion based on hard thermal loops breaks down due to the extreme infrared behaviour. Examples found so far involve processes near the light–cone and/or which begin to probe at the scale $g^2T$ of the magnetic mass. Some of these problems may be solved by inclusion of corrections to hard internal lines, but this particular approach still needs to be developed systematically to the same level as the hard thermal loop expansion. Other problems, however, may require expansions outside of the hard thermal loop expansion; such work is still at a relatively exploratory level. One might then begin to wonder whether or not we have reached the calculational limit of perturbation theory for hot gauge theories. The view in favour of such a limit is supported by the expected non–perturbative nature of the magnetic mass of the gluon, and indeed work has been developed recently in providing an interface between perturbative expansions and non–perturbative lattice gauge theory results. Although certainly of much importance to high temperature QCD, problems in QED would still remain, where a magnetic mass is not expected.

One calculation that might help answer the question of whether or not the perturbative limit has been reached is based on the following argument. The ordinary tree–level propagator becomes of the same order as the bare one–loop self–energy of Fig. 1 when the external momentum is of order $gT$. This necessitates the use of the effective expansion as in Fig. 3, which in particular involves the effective propagator of Fig. 2. One might then ask when this effective propagator becomes of the same order as the effective one–loop terms of Fig. 3. Preliminary results indicate that this might happen at a scale of order $g^{4/3}T$, which is sooner than the probably non–perturbative $g^2T$ scale. Although more work is needed to complete this calculation, if these results are correct it could mean there is still room for perturbation theory applied to hot gauge fields.
6. Acknowledgments

We were all saddened recently to lose two very active members of the thermal field theory community – Tanguy Altherr and Hiroomi Umezawa. As well as being a personal loss to family and friends, their contributions to physics and to workshops such as these will be greatly missed.

I would like to thank the organizers of this workshop, especially Profs. Gui and Khanna, for the opportunity to attend, both in terms of physics and having the chance to visit such an interesting country. Discussions at this workshop and elsewhere on this topic were very beneficial. This work was supported by the Natural Sciences and Engineering Research Council of Canada and by a travel grant from the University of Winnipeg.

7. References

1. M. Le Bellac and P. Reynaud, in Banff/CAP Workshop on Thermal Field Theory, edited by F. C. Khanna, R. Kobes, G. Kunstatter, and H. Umezawa (World Scientific, Singapore, 1994).
2. Thermal Field Theories and their Applications, edited by K. Kowalski, N. P. Landsman, and Ch. G. van Weert, Physica A158 (1989).
3. K. Kajantie and J. Kapusta, Ann. Phys. (N.Y.) 160, 477 (1985).
4. U. Heinz, K. Kajantie and T. Toimela, Phys. Lett. B183, 96 (1987); Ann. Phys. (N.Y.) 176, 218 (1987).
5. H.-Th. Elze, K. Kajantie and T. Toimela, Z. Phys. C37, 601 (1988).
6. T. H. Hansson and I. Zahed, Phys. Rev. Lett. 58, 2397 (1987); Nucl. Phys. B292, 725 (1987).
7. S. Nadkarni, Phys. Rev. Lett. 61, 396 (1988).
8. M. E. Carrington, T. H. Hansson, H. Yamagishi and I. Zahed, Ann. Phys. (N.Y.) 190, 373 (1989).
9. G. ’t Hooft and M. Veltman, Nucl. Phys. B50, 318 (1972).
10. D. Gross, in Methods in Field Theory, edited by R. Balian and Jean Zinn-Justin, (North Holland, New York, 1976).
11. R. Kobes, G. Kunstatter, and A. Rebhan, Phys. Rev. Lett. 64, 2992 (1990); Nucl. Phys. B355, 1 (1991).
12. R. Pisarski, Physica A158, 146, 246 (1989); Phys. Rev. Lett. 63, 1129 (1989).
13. V. V. Klimov, Sov. Phys. JETP 55, 199 (1982).
14. H. A. Weldon, Phys. Rev. D26, 1384, 2789 (1982).
15. E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569 (1990); B339, 310 (1990).
16. J. C. Taylor and S. M. H. Wong, Nucl. Phys. B346, 115 (1990).
17. J. Frenkel and J. C. Taylor, Nucl. Phys. B334, 199 (1990); Z. Phys. C49, 515 (1991).
18. R. Efraty and V. P. Nair, Phys. Rev. Lett. 68, 2891 (1992); Phys. Rev. D47, 5601 (1993).
19. J. P. Blaizot and E. Iancu, Phys. Rev. Lett. 70, 3376 (1993); Nucl. Phys. B390, 589 (1993).
20. P. Kelly, Q. Liu, C. Lucchesi, and C. Manuel, Phys. Rev. Lett. 72, 3461 (1994); Phys. Rev. D50, 4209 (1994).
21. E. Braaten and R. D. Pisarski, Phys. Rev. Lett. 64, 1338 (1990); Phys. Rev. D42, 2156 (1990).
22. R. Kobes, G. Kunstatter and K. Mak, Phys. Rev. D45, 4632 (1992).
23. E. Braaten and R. Pisarski, Phys. Rev. D46, 1829 (1992).
24. R. Baier, G. Kunstatter, and D. Schiff, Phys. Rev. D45, R4381 (1992); Nucl. Phys. B388, 287 (1992).
25. A. Rebhan, Phys. Rev. D46, 4779 (1992).
26. H. Nakagawa, A. Niégawa, and B. Pire, Phys. Lett. B294, 396 (1992).
27. T. Altherr, E. Petitgirard, and T. del Rio Gaztelurrutia, Phys. Rev. D47, 703 (1993).
28. H. Schulz, Nucl. Phys. B413, 353 (1994).
29. J. Kapusta, P. Lichard and D. Seibert, Phys. Rev. D44, 2774 (1991).
30. R. Baier, H. Nakagawa, A. Niégawa and K. Redlich, Z. Phys. C53, 433 (1992).
31. R. Baier, in *Banff/CAP Workshop on Thermal Field Theory*, edited by F. C. Khanna, R. Kobes, G. Kunstatter, and H. Umezawa (World Scientific, Singapore, 1994).
32. C. P. Burgess and A. L. Marini, Phys. Rev. D45, R17 (1992).
33. A. Rebhan, Phys. Rev. D46, 482 (1992).
34. H. Heiselberg and C. J. Pethick, Phys. Rev. D47, R769 (1993); D48, 2916 (1993).
35. R. D. Pisarski, Phys. Rev. D47, 5589 (1993).
36. V. V. Lebedev and A. V. Smilga, Ann. Phys. (N.Y.) 202, 229 (1990).
37. R. Baier, H. Nakagawa and A. Niégawa, Can. J. Phys. 71, 205 (1993).
38. R. Baier and R. Kobes, Phys. Rev. D50, 5944 (1994).
39. R. Baier, S. Peigné, and D. Schiff, Z. Phys. C62, 337 (1994).
40. P. Aurenche, T. Beccherrawy, and E. Petitgirard, [hep-ph/9403320](http://arxiv.org/abs/hep-ph/9403320) preprint (1994).
41. U. Kraemmer, A. K. Rebhan, and H. Schulz, Ann. Phys. (N.Y.) 238, 286 (1995); [hep-ph/9502324](http://arxiv.org/abs/hep-ph/9502324) preprint (1995); [hep-ph/9505307](http://arxiv.org/abs/hep-ph/9505307) preprint (1995).
42. J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).
43. A. K. Rebhan, Phys. Rev. D48, 3967 (1993); Nucl. Phys. B430, 319 (1994).
44. E. Braaten and A. Nieto, Phys. Rev. Lett. 73, 2402 (1994).
45. S. Peigné and S. M. H. Wong, Phys. Lett. B346, 322 (1995).
46. R. Baier and O. K. Kalashnikov, Phys. Lett. B328, 450 (1994).
47. F. Flechsig and A. K. Rebhan, [hep-ph/9509313](http://arxiv.org/abs/hep-ph/9509313) preprint (1995).
48. A. Niégawa, [hep-th/9408117](http://arxiv.org/abs/hep-th/9408117) preprint (1994).
49. J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D15, 2937 (1977).
50. J. M. Cornwall and W.-S. Hou, Phys. Lett. B153, 173 (1985).
51. A. P. de Almeida and J. Frenkel, Phys. Rev. D47, 640 (1993).
52. M. Carrington, Phys. Rev. D48, 3836 (1993).
53. P. Henning, hep–ph/9508201 preprint (1995).
54. M. B. Kislinger and P. D. Morley, Phys. Rev. D13, 2765 (1976).
55. H. Matsumoto, Y. Nakano and H. Umezawa, Phys. Rev D29, 1116 (1984).
56. L. F. Abbott, Nucl. Phys. B185, 189 (1981).
57. Y. Fujimoto and H. Yamada, Phys. Lett. B195, 231 (1987); B200, 167 (1988).
58. H. Nakagawa, A. Niégawa and H. Yokota, Phys. Rev. D38, 2566 (1988);
   Phys. Lett. B244, 63 (1990); also in Thermal Field Theories, edited by
   H. Ezawa, T. Arimitsu, and Y. Hashimoto (North–Holland, Amsterdam,
   1991).
59. N. P. Landsman, Phys. Lett. B232, 240 (1989).
60. R. Baier, B. Pire and D. Schiff, Phys. Lett. B238, 367 (1990); Z. Phys. C51,
    581 (1991).
61. M. A. van Eijck, C. R. Stephens and Ch. G. van Weert, Mod. Phys. Lett. A9,
    309 (1994); also in Banff/CAP Workshop on Thermal Field Theory, edited by
    F. C. Khanna, R. Kobes, G. Kunstatter, and H. Umezawa (World Scientific,
    Singapore, 1994).
62. P. Elmfors and R. Kobes, Phys. Rev. D51, 774 (1995).
63. J. M. Cornwall, Phys. Rev D26, 1453 (1982).
64. J. Papavassiliou, hep–ph/9504384 preprint (1995).
65. K. Sasaki, hep–ph/9510311 preprint (1995).
66. G. A. Vilkovisky, in Quantum Theory of Gravity, edited by S. M. Christensen,
    (Adam Hilger, Bristol, 1984).
67. B. S. DeWitt, in Quantum Field Theory and Quantum Statistics, edited by
    C. J. Isham and G. A. Vilkovisky, (Adam Hilger, Bristol, 1987).
68. A. Rebhan, Nucl. Phys. B288, 832 (1987).
69. E. Braaten, Phys. Rev. Lett. 74, 2164 (1995).
70. E. Braaten and A. Nieto, Phys. Rev. D51, 6990 (1995).
71. P. Reynaud, INLN preprint 94.23 (1994).