FORMALITY OF DG ALGEBRAS (AFTER KALEDIN)

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Abstract. We provide proper foundations and proofs for the main results of [Ka]. The results include a flat base change for formality and behavior of formality in flat families of $\mathcal{A}(\infty)$ and DG algebras.

1. Introduction

Let $k$ be a field of characteristic zero. Given a DG algebra $\mathcal{A}$ over $k$ Kaledin [Ka] defines a cohomology class $K_{\mathcal{A}}$ which vanishes if and only if $\mathcal{A}$ is formal. (This class $K_{\mathcal{A}}$ is an element of the second Hochschild cohomology group of a DG algebra $\tilde{\mathcal{A}}$ which is closely related to $\mathcal{A}$.) This is a beautiful result which has many important applications. One of the applications is mentioned in [Ka] (Theorem 4.3): if one has a "flat" family $\mathcal{A}_X$ of DG algebras parametrized by a scheme $X$, then formality of the fiber $\mathcal{A}_x$ is a closed condition on $x \in X$.

Unfortunately, the paper [Ka] is hard to read. There are many misprints and inaccuracies. The definition and treatment of the Hochschild cohomology of a family of DG algebras is unsatisfactory: for example, in the proof of main Theorem 4.3 it is implicitly assumed that the Hochschild cohomology behaves well with respect to specialization.

But nonetheless we found the paper [Ka] inspiring and decided to provide the necessary foundations and proofs of its main results.

Unlike [Ka] we found it more convenient to work with $\mathcal{A}(\infty)$ algebras rather than with DG algebras. Namely, for a commutative ring $R$ we consider $\mathcal{A}(\infty)$ $R$-algebras which are minimal ($m_1 = 0$) and flat, i.e. each $R$-module $H^n(A) = A^n$ is projective. That is what we mean by a flat family of $\mathcal{A}(\infty)$ algebras over $SpecR$. We are mostly interested in the case when the $R$-module $A$ is finite.

The behavior of the ($R$-linear) Hochschild cohomology $HH_R(A)$ with respect to base change $R \to Q$ is hard to control. For $\mathcal{A}(\infty)$ algebras $A$ which are finitely defined (i.e. only finitely many $m_i$'s are not zero) one may consider the Hochschild cohomology with compact supports $HH_{R,c}(A)$. It comes with a natural map $HH_{R,c}(A) \to HH_R(A)$ which is injective in cases which are important for us. The groups $HH_{R,c}$ have better behavior with respect to base change and they contain Kaledin’s cohomology classes, which are

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obstructions to formality. Thus in essential places we work with $HH_c(A)$ and not with $HH(A)$. The good functorial behavior of $HH_c(A)$ allows us to prove a faithfully flat base change result for formality (Proposition 6.2). A similar result for commutative DG algebras over a field was proved by Sullivan [Su] (see also [HaSt]).

The paper is organized as follows. In Section 2 we recall $A(\infty)$ algebras over arbitrary commutative rings, their bar constructions, quasi-isomorphisms and Kadeishvili’s theorem. We also relate the DG formality of flat DG algebras to $A(\infty)$ formality of their minimal models. In Section 3 we recall Hochschild cohomology, introduce Hochschild cohomology with compact supports and discuss its properties. In Section 4 we define Kaledin’s cohomology class and discuss its relation to (infinitesimal) formality. In Section 5 we consider the ”deformation to the normal cone” $\tilde{A}$ of an $A(\infty)$ algebra $A$ and prove the Kaledin’s key result. Section 6 contains applications of this result to the behavior of formality in flat families of $A(\infty)$ (or DG) algebras. Finally in Section 7 we define Kaledin cohomology class in the general context of DG Lie algebras.

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2. $A(\infty)$ algebras

A good introduction to $A(\infty)$ algebras is [Ke]. However there seems to be no systematic treatment of $A(\infty)$ algebras over an arbitrary commutative ring.

2.1. $A(\infty)$-algebras. Fix a commutative unital ring $R$. The sign $\otimes$ means $\otimes_R$. We want to study $A(\infty)$ $R$-algebras and quasi-isomorphisms between them. Let us recall the definitions.

Let $A = \oplus_{n \in \mathbb{Z}} A^n$ be a graded $R$-module. A structure of an $A(\infty)$ $R$-algebra (or, simply, $A(\infty)$ algebra) on $A$ is a collection $m = (m_1, m_2, \ldots)$, where $m_i : A^{\otimes i} \to A$ is a homogeneous $R$-linear map of degree $2 - i$. The maps $\{m_i\}$ must satisfy for each $n \geq 1$ the following identity:

$$\sum (-1)^{r+s+t} m_u(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.

We denote the resulting $A(\infty)$-algebra by $(A, m)$, or $(A, (m_1, m_2, \ldots))$ or simply by $A$.

- If $m_i = 0$ for $i \neq 2$, then $A$ is simply a graded associative $R$-algebra.
- If $m_i = 0$ for $i \neq 1, 2$ then $A$ is a DG $R$-algebra.
• If $m_1 = 0$ then $A$ is called minimal. Note that in this case $A$ is in particular a graded associative $R$-algebra with multiplication $m_2$.

• In any case $A$ is a complex of $R$-modules with the differential $m_1$ and the cohomology $H(A)$ is a graded associative $R$-algebra with multiplication defined by $m_2$.

2.2. $A(\infty)$-morphisms. Given $A(\infty)$ algebras $A, B$ an $A(\infty)$ morphism $f : A \to B$ is a collection $f = (f_1, f_2, \ldots)$, where $f_i : A^\otimes i \to B$ is an $R$-linear map of degree $1 - i$ such that for each $n \geq 1$ the following identity holds.

$$\sum (-1)^{r+s} f_u(1^\otimes r \otimes m_s \otimes 1^\otimes t) = \sum (-1)^s m_r (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_r}) ,$$

where the first sum runs over all decompositions $n = r + s + t$, we put $u = r + 1 + t$, and the second sum runs over all $1 \leq r \leq n$ and all decompositions $n = i_1 + \ldots + i_r$; the sign on the right hand side is given by

$$s = (r-1)(i_1-1) + (r-2)(i_2-1) + \ldots + 2(i_{r-2}-1) + (i_{r-1}-1) .$$

• We have $f_1 m_1 = m_1 f_1$, i.e. $f_1$ is a morphism of complexes.

• We have

$$f_1 m_2 = m_2 (f_1 \otimes f_1) + m_1 f_2 + f_2 (m_1 \otimes 1 + 1 \otimes m_1) ,$$

which means that $f_1$ commutes with the multiplication $m_2$ up to a homotopy given by $f_2$. In particular, if $A$ and $B$ are minimal, then $f_1$ is a homomorphism of associative algebras $f_1 : (A, m_2) \to (B, m_2)$.

We call $f$ a quasi-isomorphism if $f_1 : A \to B$ is a quasi-isomorphisms of complexes. $f$ is called the identity morphism, denoted id, if $A = B$ and $f = (f_1 = \text{id}, 0, 0, \ldots)$.

Let $C$ be another $A(\infty)$ algebra and $g = (g_1, g_2, \ldots) : B \to C$ be an $A(\infty)$ morphism. The composition $h = g \cdot f : A \to C$ is an $A(\infty)$-morphism which is defined by

$$h_n = \sum (-1)^s f_i (g_{i_1} \otimes \ldots \otimes g_{i_r}) ,$$

where the sum and the sign are as in the defining identity.

$A(\infty)$ algebras $A$ and $B$ are called quasi-isomorphic if there exists $A(\infty)$ algebras $A_1, A_2, \ldots, A_n$ and quasi-isomorphisms

$$A \leftarrow A_1 \rightarrow \ldots \leftarrow A_n \rightarrow B .$$

An $A(\infty)$ algebra $A$ is called formal if it is quasi-isomorphic to the $A(\infty)$ algebra $(H(A), (0, m_2, 0, \ldots))$.
2.3. Bar construction. The notions of $A(\infty)$ algebra and $A(\infty)$ morphism can be compactly and conveniently described in terms of the bar construction.

Let $A$ be a graded $R$-module, $A[1]$ its shift $A[1]^n = A^{n+1}$. Let

$$\overline{T}A[1] = \bigoplus_{i \geq 1} A[1]^{\otimes i}$$

be the reduced cofree $R$-coalgebra on the $R$-module $A[1]$ with the comultiplication

$$\Delta(a_1, ..., a_n) = \sum_{i=1}^{n-1} (a_1, ..., a_i) \otimes (a_{i+1}, ..., a_n)$$

so that $\Delta(a) = 0$ and $\Delta(a_1, a_2) = a_1 \otimes a_2$. Denote by $\text{Coder}(\overline{T}A[1])$ the graded $R$-module of homogeneous $R$-linear coderivations of the coalgebra $\overline{T}A[1]$. The composition of a coderivation with the projection to $T^1A[1] = A[1]$ defines an isomorphism of graded $R$-modules

$$\text{Coder}(\overline{T}A[1]) \simeq \text{Hom}_R(\overline{T}A[1], A[1]).$$

Thus a coderivation of degree $p$ is determined by a collection $(d_1, d_2, ...)$, where $d_i : A[1]^{\otimes i} \to A[1]$ is an $R$-linear map of degree $p$.

Denote by $s : A \to A[1]$ the shift operator. Given an $R$-linear map $m_i : A^{\otimes i} \to A$ of degree $2-i$ we can define an $R$-linear map $d_i : A[1]^{\otimes i} \to A[1]$ of degree 1 by commutativity of the following diagram

$$\begin{array}{ccc}
A^{\otimes i} & \xrightarrow{m_i} & A \\
\downarrow s^{\otimes i} & & \downarrow s \\
A[1]^{\otimes i} & \xrightarrow{d_i} & A[1]
\end{array}$$

Thus $d_i(sa_1 \otimes ... \otimes sa_i) = (-1)^n sm_i(a_1 \otimes ... \otimes a_i)$, where $n = \frac{i(i+1)}{2} + (i-1) \deg(a_1) + (i-2) \deg(a_2) + ... + \deg(a_{i-1})$. Then a collection of $R$-linear maps $m = (m_1, m_2, ...)$, $m_i : A^{\otimes i} \to A$ of degree $2-i$, defines a structure of an $A(\infty)$ $R$-algebra on $A$ if and only if the corresponding collection $d = (d_1, d_2, ...)$, $d_i : A[1]^{\otimes i} \to A[1]$ of degree 1, defines an $R$-linear coderivation of the coalgebra $\overline{T}A[1]$ such that $d^2 = 0$. Given an $A(\infty)$ algebra $(A, m)$ we will also denote (abusing notation) by the same letter $m$ the corresponding coderivation of the coalgebra $\overline{T}A[1]$. The resulting DG coalgebra $(\overline{T}A[1], m)$ is called the bar construction of $A$ and is denoted $\mathcal{B}A$.

Let $B$ be another $A(\infty)$ algebra. In a similar manner (using appropriate sign changes) there is a bijection between $A(\infty)$ morphisms $A \to B$ and homomorphisms of degree zero of DG coalgebras $\mathcal{B}A \to \mathcal{B}B$. Again we will usually use the same notation for both.
Let \( f = (f_1, f_2, \ldots) : \mathcal{T}A[1] \to \mathcal{T}B[1] \) be a homomorphism of coalgebras. Then for each \( n \)

\[
f(\bigoplus_{i \leq n} T^i A[1]) \subset \bigoplus_{i \leq n} T^i B[1].
\]

The map \( f \) is an isomorphism if and only if \( f_1 \) is an isomorphism. On the other hand if \( f_1 = 0 \) and \( A = B \), then the map \( f \) is locally nilpotent.

Similar considerations apply to coderivations \( g = (g_1, g_2, \ldots) : \mathcal{T}A[1] \to \mathcal{T}A[1] \). Namely, let \( g \) have degree zero and \( g_1 = 0 \), then \( g \) is locally nilpotent and hence the coalgebra automorphism

\[
\exp(g) : \mathcal{T}A[1] \to \mathcal{T}A[1]
\]

is well defined (provided \( \mathbb{Q} \subset R \)).

2.4. Flat \( A(\infty) \) algebras and their minimal models.

**Definition 2.1.** An \( A(\infty) \) \( R \)-algebra \( A \) is called flat if each cohomology \( H^i(A) \) is a projective \( R \)-module.

Thus if \( R \) is a field then any \( A(\infty) \) algebra is flat. We consider a flat \( A(\infty) \) \( R \)-algebra as a flat family of \( A(\infty) \) algebras over \( \text{Spec}R \). Let us recall the following simple important result of Kadeishvili.

**Theorem 2.2** (Kad1). Let \( A \) be a flat \( A(\infty) \) \( R \)-algebra. Choose a quasi-isomorphism of complexes of \( R \)-modules \( g : H(A) \to A \) (the differential in \( H(A) \) is zero). Then there exists a structure of a minimal \( A(\infty) \) algebra on \( H(A) \) with \( m_2 \) being induced by the \( m_2 \) of \( A \) and an \( A(\infty) \) morphism \( f = (g = f_1, f_2, \ldots) \) from \( H(A) \) to \( A \) (which is a quasi-isomorphism).

We call the \( A(\infty) \) algebra \( H(A) \) as in the above theorem a minimal model of \( A \).

Let \( A \) and \( B \) be \( A(\infty) \) \( R \)-algebras and \( f, g \) morphisms from \( A \) to \( B \). Let \( F, G \) denote the corresponding morphisms of DG coalgebras \( BA \to BB \). One defines \( f \) and \( g \) to be homotopic if \( F \) and \( G \) are homotopic, i.e. if there exists a homogeneous \( R \)-linear map \( H : BA \to BB \) of degree \(-1\) such that

\[
\Delta \cdot H = F \otimes H + H \otimes G \quad \text{and} \quad F - G = m_B \cdot H + H \cdot m_A.
\]

**Lemma 2.3.** In the above notation assume that \( A \) and \( B \) are minimal. Let \( f : A \to B \) and \( g : B \to A \) be morphisms such that \( g \cdot f \) and \( f \cdot g \) are homotopic to the identity (i.e. \( A \) and \( B \) are homotopy equivalent). Then the corresponding morphisms \( F : BA \to BB \) \( G : BB \to BA \) are mutually inverse isomorphisms.
Proof. Let $H : BA \to BA[-1]$ be a homotopy between morphisms $G \cdot F$ and $\text{id}_{BA}$. Then $H$ is defined by a collection of $R$-linear maps $h_i : A[1] \otimes i \to A[1]$, $i \geq 1$, which satisfy some properties ([Le-Ha], 1.2.1.7).

Let $F = (f_1, f_2, ...)$, $G = (g_1, g_2, ...)$, $G \cdot F = (t_1, t_2, ...)$. Then

$$(G \cdot F)|_{A[1]} = t_1|_{A[1]} = g_1 \cdot f_1|_{A[1]}.$$  

Also $H|_{A[1]} = h_1|_{A[1]}$. Since $A$ is minimal the equation

$$G \cdot F - \text{id} = m \cdot H + H \cdot m$$

when restricted to $A[1]$ becomes $g_1 \cdot f_1 - \text{id} = 0 \cdot h_1 + h_1 \cdot 0$. So $t_1 = \text{id}$, i.e. $G \cdot F : BA \to BA$ is an automorphism.

Let us recall another result of Kadeishvili.

**Theorem 2.4** (Kad2). a) Homotopy is an equivalence relation on the set of morphisms of $A(\infty) \ R$-algebras $A \to B$.

Denote by $\mathcal{H}$ the category obtained by dividing the category of $A(\infty) \ R$-algebras by the homotopy relation.

b) Assume that $C$ is an $A(\infty) \ R$-algebra such that the $R$-module $C^n$ is projective for all $n \in \mathbb{Z}$. Then a quasi-isomorphism of $A(\infty) \ R$-algebras $s : A \to B$ induces an isomorphism

$$s_* : \text{Hom}_{\mathcal{H}}(C, A) \to \text{Hom}_{\mathcal{H}}(C, B).$$

**Corollary 2.5.** On the full subcategory of $A(\infty) \ R$-algebras which consists of algebras $C$ such that the $R$-module $C^n$ is projective for all $n \in \mathbb{Z}$ the relation of quasi-isomorphism coincides with the relation of homotopy equivalence.

**Corollary 2.6.** Let $A$ and $B$ be two minimal flat $A(\infty) \ R$-algebras. Then they are quasi-isomorphic if and only if their bar constructions $BA$ and $BB$ are isomorphic.

Proof. The "if" direction is obvious.

Assume that $A$ and $B$ are quasi-isomorphic, i.e. there exists a chain of morphisms of $A(\infty) \ R$-algebras which are quasi-isomorphisms:

$$A \xleftarrow{f} A_1 \xrightarrow{g} A_2 \leftarrow ... \to B.$$  

Choose a flat minimal $A(\infty) \ R$-algebra $A'_1$ and a quasi-isomorphism $i : A'_1 \to A_1$. The quasi-isomorphism $f \cdot i : A'_1 \to A$ between two minimal flat $A(\infty) \ R$-algebras induces an isomorphism of their bar constructions $BA'_1 \to BA$. 

Choose a flat minimal $A(\infty)$ algebra $A'_1$ and a quasi-isomorphism $j : A'_2 \to A_2$. It follows from Theorem 2.4 b) that the induced maps $(g \cdot i)_*: \text{Hom}_R(C, A'_1) \to \text{Hom}_R(C, A_2) \leftarrow \text{Hom}_R(C, A'_2) : j_*$

are isomorphisms if $C$ is a minimal flat $A(\infty)$ algebra. In particular $A'_1$ and $A'_2$ are homotopy equivalent, hence $BA'_1 \simeq BA'_2$ by Lemma 2.3 Continuing this way we arrive at an isomorphism $BA \simeq BB$. □

This last corollary implies in particular that for a flat $A(\infty)$ algebra its minimal model (as in Theorem 2.2) is unique up to a quasi-isomorphism. We will identify a quasi-isomorphism $A \simeq B$ between flat minimal $A(\infty)$ algebras with the corresponding isomorphism $BA \simeq BB$.

**Corollary 2.7.** Let $A$ be a flat $A(\infty)$ algebra and $B$ be a minimal flat $A(\infty)$ algebra which is quasi-isomorphic to $A$. Then there exists a morphism $B \to A$ which is a quasi-isomorphism.

**Proof.** Let $H(A)$ be a minimal flat $A(\infty)$ algebra with a quasi-isomorphism $H(A) \to A$ as in Theorem 2.2. Then the minimal flat $A(\infty)$ algebras $B$ and $H(A)$ are quasi-isomorphic. So by Corollary 2.6 $BB \simeq BH(A)$. Hence there also exists a quasi-isomorphism $B \simeq A$. □

2.5. **Flat DG algebras.** A DG ($R$-)algebra is an $A(\infty)$ algebra $(A, m)$ such that $m_i = 0$ for $i > 2$. A morphism of DG algebras is a homomorphism of graded associative algebras which commutes with the differentials. Thus the category of DG algebras is not a full subcategory of $A(\infty)$ algebras. We say that DG algebras $A$ and $B$ are DG quasi-isomorphic if there exists a chain of morphisms of DG algebras

$$A \leftarrow A_1 \leftarrow \ldots \leftarrow A_n \to B$$

where all arrows are quasi-isomorphisms. It is well known that if $R$ is a field then two DG algebras are quasi-isomorphic (as $A(\infty)$ algebras) if and only if they are DG quasi-isomorphic. For a general ring $R$ we have a similar result for flat DG algebras (Definition 2.1).

**Proposition 2.8.** Let $E$ and $F$ be flat DG $R$-algebras and $A$ and $B$ be their minimal $A(\infty)$-models (Theorem 2.2). The following assertions are equivalent.

a) $E$ and $F$ are DG quasi-isomorphic.

b) $E$ and $F$ are quasi-isomorphic.

c) $A$ and $B$ are quasi-isomorphic.

d) $BA$ and $BB$ are isomorphic.
Proof. Clearly a) ⇒ b) and by definition b) ⇔ c). Corollary 2.6 implies that c) ⇔ d). So it remains to prove that d) ⇒ a).

So assume that $BA \simeq BB$. Choose a DG algebra $\tilde{E}$ such that the $R$-module $\tilde{E}^n$ is projective for all $n \in \mathbb{Z}$, and a DG quasi-isomorphism $\tilde{E} \to E$ (for example $\tilde{E}$ may be a cofibrant replacement of $E$).

By Corollary 2.7 there exists an $A(\infty)$ morphism $A \to \tilde{E}$ which is a quasi-isomorphism. By Corollary 2.5 this is a homotopy equivalence, i.e. the induced morphism of the bar constructions $BA \to B\tilde{E}$ is a homotopy equivalence.

Consider the cobar construction $\Omega$ which is a functor from DG coalgebras to DG algebras [Le-Ha],1.2.2. It is the left adjoint to the bar construction $B$. The same proof as of Lemma 1.3.2.3 in [Le-Ha] shows that the adjunction morphism of DG algebras $\Omega B\tilde{E} \to \tilde{E}$ is a quasi-isomorphism.

But the DG coalgebras $BA$ and $B\tilde{E}$ are homotopy equivalent. Hence their cobar constructions are also homotopy equivalent and in particular the DG algebras $\Omega BA$ and $\Omega B\tilde{E}$ are DG quasi-isomorphic. (The notion of homotopy between morphisms of DG algebras is defined for example in [Le-Ha],1.1.2.) Thus the DG algebras $\Omega BA$ and $E$ are DG quasi-isomorphic.

Similarly one shows that the DG algebras $\Omega BB$ and $F$ are DG quasi-isomorphic. But an isomorphism of DG coalgebras $BA \simeq BB$ induces an isomorphism of DG algebras $\Omega BA \simeq \Omega BB$. This proves the proposition. □

A DG algebra in called DG formal if it is DG quasi-isomorphic to a DG algebra with the zero differential.

Corollary 2.9. Let $E$ be a flat DG $R$-algebra with a minimal $A(\infty)$ model $A$. Then $E$ is DG formal if and only in $A$ is formal (Subsection 2.2). So $E$ is DG formal if and only if it is $A(\infty)$ formal.

Proof. This follows from the equivalence of a) and c) in Proposition 2.8. □

In what follows we will be interested only in flat $A(\infty)$ or DG algebras and hence will usually work with their minimal models.

3. Hochschild cohomology

We assume that $A$ is a minimal flat $A(\infty)$ $R$-algebra.

3.1. Consider the graded $R$-module $\text{Coder}(\overline{T}A[1])$ with the self map of degree 1 given by $d \mapsto [m_A, d] = m_A \cdot d - (-1)^{\deg d} \cdot d \cdot m_A$. Since $m_A^2 = 0$ this makes $\text{Coder}(\overline{T}A[1])$ a complex
of $R$-modules which we denote by $C^\bullet_R(A)$. This complex is called the Hochschild complex of $A$. Its (shifted) cohomology

$$HH^{i+1}_R(A) := H^iC^\bullet_R(A)$$

is the Hochschild cohomology of $A$.

Note that quasi-isomorphic flat minimal $A(\infty)$ algebras have isomorphic bar constructions (Corollary 2.6), hence isomorphic Hochschild complexes and Hochschild cohomology.

The Hochschild cohomology $HH^\bullet_R(A)$ is a functor of $R$ which is hard to control because of the presence of infinite products in the Hochschild complex $C^\bullet_R(A)$. It turns out that under certain finiteness assumptions on $A$ there is a natural subcomplex $C^\bullet_{R,c}(A) \subset C^\bullet_R(A)$ whose cohomology behaves better.

**Definition 3.1.** An $A(\infty)$ algebra $A = (A,(m_1,m_2,...))$ is called finitely defined if $m_n = 0$ for $n >> 0$.

Although the above definition can be made for all $A(\infty)$ algebras (in particular any DG algebra would be a finitely defined $A(\infty)$ algebra) we think it only makes sense for minimal ones.

For the rest of this section we assume that all $A(\infty)$ algebras are finitely defined.

3.2. **Definition of $HH^\bullet_{R,c}(A)$**. Recall that the Hochschild complex $C^\bullet_R(A)$ of an $A(\infty)$ $R$-algebra consists of $R$-modules

$$C^n_R(A) = \prod_{n \geq 1} \text{Hom}^n_R(A[1]^{\otimes n},A[1]).$$

Consider the $R$-submodule

$$C^p_{R,c}(A) = \sum_{n \geq 1} \text{Hom}^n_R(A[1]^{\otimes n},A[1]).$$

Notice that $C^\bullet_{R,c}(A)$ is actually a subcomplex of $C^\bullet_R(A)$ since $A$ is finitely defined.

**Definition 3.2.** We call the elements of $C^\bullet_{R,c}(A)$ the Hochschild cochains with compact supports. The corresponding cohomology $R$-modules

$$HH^n_{R,c}(A) := H^n(C^\bullet_{R,c}(A))$$

are called the Hochschild cohomology of $A$ with compact supports.
3.3. Properties of $HH^\bullet_{R,e}(A)$. By definition we have the canonical map

$$\iota: HH^\bullet_{R,e}(A) \to HH^\bullet_{R}(A).$$

**Lemma 3.3.** Assume that $m_n = 0$ for $n \neq 2$, i.e. $A$ is just a graded associative $R$-algebra. Then the map $\iota$ is injective.

**Proof.** Suppose that $d = (d_1, d_2, \ldots) \in C^n_R(A)$ is a coderivation such that $[m_A, d] = e = (e_1, \ldots, e_n, 0, 0, \ldots) \in C_R^n(A)$. Consider the coderivation $d_{\leq n-1} := (d_1, \ldots, d_{n-1}, 0, 0, \ldots) \in C^n_R(A)$. Then $[m_A, d_{\leq n-1}] = e$ (because $m_n = 0$ for $n \neq 2$), i.e. $e$ is also a coboundary in the complex $C_R^n(A)$. \hfill \Box

**Proposition 3.4.** Assume that $A$ is a finite $R$-module. Let $R \to Q$ be a homomorphism of commutative rings and put $A_Q = A \otimes_R Q$. Then

a) $C^n_{Q,e}(A_Q) = C^n_{R,e}(A) \otimes_R Q$;

b) If $Q$ is a flat $R$-module, then $HH^\bullet_{Q,e}(A_Q) = HH^\bullet_{R,e}(A) \otimes_R Q$.

**Proof.** Clearly a) $\Rightarrow$ b). To prove a) notice the isomorphism of $Q$-modules

$$\text{Hom}_Q(A_Q^\otimes n, A_Q) = \text{Hom}_R(A^\otimes n, A_Q) = \text{Hom}_R(A^\otimes n, A) \otimes_R Q$$

(since $A^\otimes n$ is a finite projective $R$-module). \hfill \Box

**Remark 3.5.** In particular, if $A$ is a finite $R$-module then for each $n$ we obtain a quasi-coherent sheaf $\mathcal{H}_n(A)$ on $\text{Spec}R$ which is the localization of the $R$-module $HH^n_{R,e}(A)$.

**Proposition 3.6.** Assume that the ring $R$ is noetherian, $A$ is a finite $R$-module, and $m_n = 0$ for $n \neq 2$ (i.e. $A$ is just a graded associative $R$-algebra). Also assume that each $R$-module $HH^n_{R,e}(A)$ is projective. Let $R \to Q$ be a homomorphism of commutative rings and put $A_Q = A \otimes_R Q$. Then

$$HH^n_{Q,e}(A_Q) = HH^n_{R,e}(A) \otimes_R Q.$$ 

**Proof.** Since $A$ is just a graded associative algebra, the complex $C^\bullet_{R,e}(A)$ is a direct sum of complexes

$$C^\bullet_{R,e}(A) = \bigoplus_{i \in \mathbb{Z}} C^i_{R,e}(A),$$

where $C^i_{R,e}(A) = \text{Hom}_R^{i+j}(A^\otimes j, A)$. Similarly

$$C^\bullet_{Q,e}(A_Q) = \bigoplus_{i \in \mathbb{Z}} C^i(A_Q).$$

By Proposition 3.4 $C^\bullet_{Q,e}(A_Q) = C^\bullet_{R,e}(A) \otimes_R Q$ and this isomorphism preserves the decomposition $C^\bullet = \bigoplus C^i$. So it suffices to prove that for each $i \in \mathbb{Z}$ the complex of $R$-modules $C^i(A)$ is homotopy equivalent to its cohomology $\oplus_n H^n(C^i(A))[-n]$ (with the trivial differential). We need a lemma.
Lemma 3.7. Let $R$ be a commutative noetherian ring and let

$$K^\bullet := \ldots \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \ldots$$

be a bounded below complex of finite projective $R$-modules such that each $R$-module $H^n(K^\bullet)$ is also projective. Then for each $n$ the $R$-module $\operatorname{Im} d^n$ is projective.

Proof. Being a projective module is a local property, so we may and will assume that $R$ is a local noetherian ring. We also may assume that $K^n = 0$ for $n < 0$.

Recall the Auslander-Buchsbaum formula: if $M$ is a finite $R$-module of finite projective dimension pd $M$ then

$$\text{pd } M + \text{depth } M = \text{depth } R.$$ 

In particular $\text{pd } M \leq \text{depth } R$.

First we claim that $\text{pd } \operatorname{Im} d^n < \infty$ for any $n$. Indeed, consider the complex

$$0 \to K^0 \xrightarrow{d^0} K^1 \xrightarrow{d^1} \ldots \xrightarrow{d^{n-1}} K^n \to \operatorname{Im} d^n \to 0.$$ 

This may not be a projective resolution of $\operatorname{Im} d^n$ (since the complex $K^\bullet$ may not be exact), but we can easily make it into one:

$$0 \to H^0(K^\bullet) \to K^0 \oplus H^1(K^\bullet) \to K^1 \oplus H^2(K^\bullet) \to \ldots \to K^{n-1} \oplus H^n(K^\bullet) \to K^n \to \operatorname{Im} d^n \to 0$$

where the differential $H^i(K^\bullet) \to K^i$ is any splitting of the projection $\ker d^i \to H^i(K^\bullet)$. Thus we have $\text{pd } \operatorname{Im} d^n \leq n$ hence in particular $\text{pd } \operatorname{Im} d^n \leq \text{depth } R$.

But we claim that in fact $\text{pd } \operatorname{Im} d^n = 0$. The proof is similar. Indeed, put $\delta = \text{depth } R$ and consider the complex

$$0 \to \operatorname{Im} d^n \to K^{n+1} \xrightarrow{d^{n+1}} \ldots \xrightarrow{d^{n+\delta}} K^{n+\delta} \xrightarrow{d^{n+\delta}} \operatorname{Im} d^{n+\delta} \to 0.$$ 

Again we can turn it into an exact complex

$$0 \to \operatorname{Im} d^n \oplus H^{n+1}(K^\bullet) \to K^{n+1} \oplus H^{n+2}(K^\bullet) \to \ldots \to K^{n+\delta} \to \operatorname{Im} d^{n+\delta} \to 0$$

which shows that $\text{pd}(\operatorname{Im} d^n \oplus H^{n+1}(K^\bullet)) = \text{pd } \operatorname{Im} d^n = 0$ (since $\text{pd } \operatorname{Im} d^{n+\delta} \leq \delta$). This proves the lemma.

The lemma implies that for each $n$ we have

$$K^n \simeq \operatorname{Im} d^{n-1} \oplus H^n(K^\bullet) \oplus \operatorname{Im} d^n.$$ 

It follows easily that $K^\bullet$ is homotopy equivalent to its cohomology $\bigoplus_n H^n(K^\bullet)[-n]$. Now apply this to $K^\bullet = C^*_i(A)$.

Remark 3.8. We do not know if Proposition 3.6 remains true without the assumption that $m_n = 0$ for $n \neq 2$. \hfill $\Box$
The following seemingly trivial example is actually an important one.

**Example 3.9.** Let $k$ be a field and $R$ be a $k$-algebra. Let $B$ be a finitely defined $A(\infty)$ $k$-algebra such that $\dim_k B < \infty$. Put $A = B \otimes_k R$. Then for each $n$ we have

$$HH^\bullet_{R,c}(A) = HH^\bullet_{k,c}(B) \otimes_k R$$

and hence in particular the corresponding quasi-coherent $O_{\text{Spec} R}$-module $\mathcal{H}_c^\bullet(A)$ is free. Moreover for any homomorphism of commutative $k$-algebras $R \to Q$ we have

$$HH^\bullet_{Q,c}(A \otimes_R Q) = HH^\bullet_{k,c}(B) \otimes_k Q = HH^\bullet_{R,c}(A) \otimes_R Q.$$ 

In particular, if $x \in \text{Spec} R$ is a $k$-point, then

$$HH^\bullet_{k,c}(A_x) = HH^\bullet_{k,c}(B).$$

3.4. **Invariance of $HH_{R,c}(A)$**. Let $A$ and $B$ be two flat minimal $A(\infty)$ $R$-algebras which are finitely defined. Suppose that $A$ and $B$ are quasi-isomorphic. It is natural to ask whether $HH^\bullet_{R,c}(A) \simeq HH^\bullet_{R,c}(B)$? This is so at least when there exist mutually inverse isomorphisms of the bar constructions $f : BA \to BB$, $g : BB \to BA$, such that $f_n = g_n = 0$ for $n >> 0$. In particular this is true if $A$ and $B$ are usual associative graded $R$-algebras (which are isomorphic).

4. **Kaledin’s cohomology class**

We thank the referee for suggesting that the material of this section be presented in a general context of DG Lie algebras. We do this in Section 7. (The connection being that the Hochshild complex of an $A(\infty)$-algebra is naturally a DG Lie algebra.) However, since we are interested in $A(\infty)$-algebras, we decided to also present this special case explicitly.

4.1. Let $k$ be a field of characteristic zero and $R$ be a commutative $k$-algebra. For an $R$ module $M$ we denote by $M[[h]]$ the $R[[h]]$-module

$$M[[h]] = \lim_{\leftarrow} M[h]/h^n = \lim_{\leftarrow} (M \otimes_R R[h]/h^n)$$

We call an $R[[h]]$-module $P$ $h$-free complete if it is isomorphic to $\bar{P}[[h]]$, where $\bar{P}$ is the $R$-module $P/h$.

Notice that $M[[h]]$ is canonically identified with the set of power series $\Sigma_{i=0}^\infty m_i h^i$, $m_i \in M$. To get the analogous identification for an arbitrary $h$-free complete $R[[h]]$-module one needs to choose a splitting $\bar{P} \to P$ (a map of $R$-modules).

There is a canonical isomorphism of $R[[h]]$-modules

$$\text{Hom}_{R[[h]]}(M[[h]] \otimes_{R[[h]]} \ldots \otimes_{R[[h]]} M[[h]], M[[h]]) = \{\Sigma_{i=0}^\infty f_i h^i | f_i \in \text{Hom}_R(M \otimes_R \ldots \otimes_R M, M)\}$$
4.2. Let $B$ be an $h$-free complete $R[[h]]$-module which has a structure of a minimal $A(\infty)$ $R[[h]]$-algebra $(B, m)$. Assume that the minimal $A(\infty)$ $R$-algebra $(\bar{B}, m^{(0)}) = B/h$ is flat. Choose a splitting $\bar{B} \to B$ of $R$-modules. Then we can write

$$m = m^{(0)} + m^{(1)}h + m^{(2)}h^2 + ...$$

for some coderivations $m^{(i)} \in C^1_R(B)$. Notice that the Hochschild complex $C^\bullet_{R[[h]]}(B)$ is isomorphic to the inverse limit of the sequence $\{C^\bullet_{R/h^n}(B/h^n)\}$ where all maps are surjective. In particular

$$HH^\bullet_{R[[h]]}(B) = \lim_{\leftarrow} HH^\bullet_{R/h^n}(B/h^n).$$

Consider the coderivation

$$\partial_h m = m^{(1)} + 2m^{(2)}h + 3m^{(3)}h^2 + ... \in C^1_{R[[h]]}(B).$$

Then

$$[m, \partial_h m] = m \cdot \partial_h m + \partial_h m \cdot m - \partial_h (m \cdot m) = 0,$$

i.e. $\partial_h m$ is a cocycle and hence defines a cohomology class $[\partial_h m] \in HH^2_R(B)$.

**Lemma 4.1.** Let $f : \overline{TB}[1] \to \overline{TB}[1]$ be a coalgebra automorphism which is the identity modulo $h$. Put $f(c) := f \cdot c \cdot f^{-1}$ for $c \in C^\bullet_{R[[h]]}(B)$. Then the cocycles $\partial_h (f(m))$ and $f(\partial_h m)$ are cohomologous (with respect to the differential $[f(m), -]$).

**Proof.** It suffices to show this modulo $h^n$ for all $n$.

Notice that $f$ has the following canonical decomposition

$$f = \ldots \cdot \exp(g^{(2)}h^2) \cdot \exp(g^{(1)}h)$$

for some coderivations $g^{(1)}, g^{(2)}, \ldots \in C^0_R(B)$. Namely, let $f \equiv \text{id} + f^{(1)}h(\text{mod}h^2)$, where $f^{(1)} = (f^{(1)}_1, f^{(1)}_2, ...), \text{ Let } g^{(1)} = \text{ the coderivation of degree zero defined by the same sequence, i.e. } g^{(1)} = (f^{(1)}_1, f^{(1)}_2, ...). \text{ Then the coalgebra automorphisms } f \text{ and } \exp(g^{(1)}h) \text{ are equal modulo } h^2. \text{ Now replace } f \text{ by } f \cdot \exp(g^{(1)}h)^{-1} \equiv \text{id} + f^{(2)}h^2(\text{mod}h^3). \text{ Let } g^{(2)} = \text{ the coderivation } g^{(2)} = (f^{(2)}_1, f^{(2)}_2, ...), \text{ etc.}

Fix $n \geq 1$. Then

$$f \equiv \exp(g^{(n-1)}h^{n-1}) \ldots \exp(g^{(1)}h)(\text{mod}h^n),$$

and we may and will assume that $f = \exp(g^i)$ for some coderivation $g \in C^0_R(\bar{B})$. We have

$$\partial_h (f(m)) = \partial_h f \cdot m \cdot f^{-1} + f \cdot \partial_h m \cdot f^{-1} - f \cdot m \cdot f^{-1} \cdot \partial_h f \cdot f^{-1}.$$ 

So

$$f \cdot \partial_h m \cdot f^{-1} - \partial_h (f(m)) = [f(m), \partial_h f \cdot f^{-1}].$$
Proposition 4.5
Proof. Recall that we identify a quasi-isomorphism of two minimal flat algebras to an isomorphism of coalgebras. Consider the coalgebra automorphism $f$ with respect to the differential $\partial_h$ so $\partial_h f \cdot f^{-1} \in C^0_R(h^n)(B)$ and hence $\partial_h(f(m))$ and $f(\partial_h m)$ are cohomologous modulo $h^n$ with respect to the differential $[f(m), -]$. \hfill \Box

Corollary 4.2. The class $[\partial_h m] \in HH^2_R[h^n](B)$ is well defined, i.e. is independent of the choice of the splitting $R \to R$.

Definition 4.3. The class $[\partial_h m] \in HH^2_R[h^n](B)$ is called the Kaledin class of $B$ and denoted $K_B$.

Remark 4.4. The definition of Kaledin class and the above lemma remain valid for flat minimal $A(\infty)$ $R[h]/h^{n+1}$-algebras. We consider the class $K_{B/h^n+1}$ of the $A(\infty)$ $R[h]/h^{n+1}$-algebra $B/h^{n+1}$ as an element in $HH^2_R[h^n](B/h^n)$.

Proposition 4.5 (Ka). Fix $n \geq 1$. Then the class $K_{B/h^n+1} \in HH^2_R[h^n/h^n](B/h^n)$ is zero if and only if there exists a quasi-isomorphism of $A(\infty)$ $R[h]/h^{n+1}$-algebras $f : B/h^n+1 \to \bar{B}[h]/h^{n+1}$ such that $f \equiv (id, 0, 0, ..., (mod h)$.

Proof. Recall that we identify a quasi-isomorphism of two minimal flat $A(\infty)$ algebras with an isomorphism of their bar constructions.

One direction is clear: if $f : B/h^n+1 \to \bar{B}[h]/h^{n+1}$ is a quasi-isomorphism which is the identity modulo $h$, then $K_{B/h^n+1} = 0$ (since by Lemma 4.1 and Remark 4.4 it corresponds to $K_{\bar{B}[h]/h^{n+1}} = 0$ under $f$).

Suppose $K_{B/h^n+1} = 0$. By induction on $n$ we know that there exists a quasi-isomorphism $B/h^n \to \bar{B}[h]/h^n$ which is the identity modulo $h$. Lift this quasi-isomorphism arbitrarily to an isomorphism of coalgebras $\mathcal{T}(B/h^{n+1}[1]) \to \mathcal{T}(\bar{B}[h]/h^{n+1}[1])$. Then by Lemma 4.1 and Remark 4.4 we may and will assume that
\[ m = m_{B/h^n+1} = m^{(0)} + m^{(n)}h^n \]
and hence $K_{B/h^n+1} = [nm^{(n)}h^{n-1}]$. Since $K_{B/h^n+1} = 0$ there exists a coderivation $g \in C^0_R(\bar{B})$ such that
\[ [m, gh^{n-1}] = [m^{(0)}, gh^{n-1}] = nm^{(n)}h^{n-1}. \]
Consider the coalgebra automorphism $f = \exp(n^{-1}gh^n) : \mathcal{T}(\bar{B}[h]/h^{n+1}[1]) \to \mathcal{T}(\bar{B}[h]/h^{n+1}[1])$. Then $m^{(0)} \cdot f = f \cdot m$, i.e. $f$ is an isomorphism of the bar constructions $f : B(B/h^{n+1}) \to B(\bar{B}[h]/h^{n+1})$ and hence is a quasi-isomorphism from $B/h^{n+1}$ to $\bar{B}[h]/h^{n+1}$ (which is the identity modulo $h$). \hfill \Box
Corollary 4.6. In the notation of Proposition 4.5 assume that \( m_{B/h^{n+1}} = m^{(0)} + m^{(n)}h^n \). Then there exists a quasi-isomorphism of \( A(\infty) \ R[h]/h^{n+1} \)-algebras \( f : B/h^{n+1} \to \tilde{B}[h]/h^{n+1} \) such that \( f \equiv (\text{id}, 0, 0, \ldots)(\text{mod}h) \) if and only if the class \([m^{(n)}] \in HH^2_R(\tilde{B})\) is zero.

Proof. By Proposition 4.5 there exists such a quasi-isomorphism \( f \) if and only if the class \([nm^{(n)}h^{n-1}] \in HH^2_R(B/h^n)\) is zero. Clearly, this is equivalent to the class \([m^{(n)}] \in \tilde{HH}^2_R(\tilde{B})\) being zero. \(\Box\)

5. Deformation to the normal cone

5.1. Let \( k \) be a field of characteristic zero and \( R \) be a commutative \( k \)-algebra. Let \( A = (A, m) \) be a minimal flat \( A(\infty) \ R \)-algebra. Consider the \( A(\infty) \ R[h] \)-algebra \( \tilde{A} = (A[h], \tilde{m} = (m_2, m_3h, m_4h^2, \ldots)) \).

Lemma 5.1. The map \( \tilde{m} \) indeed defines a structure of an \( A(\infty) \ R[h] \)-algebra on \( A[h] \).

Proof. The defining equation as in Subsection 2.1 above are homogeneous: after the substitution of \( m_i \tilde{h}^{i-2} \) instead of \( m_i \), the equation is multiplied by \( h^{n-3} \). \(\Box\)

Denote by \( A(2) \) the \( A(\infty) \ R \)-algebra \( (A, (m_2, 0, 0, \ldots)) \).

Lemma 5.2. We have the following isomorphisms of \( A(\infty) \ R \)-algebras.

a) \( \tilde{A}/h \simeq A(2) \),

b) \( \tilde{A}/(h-1) \simeq A \).

Proof. This is clear. \(\Box\)

Definition 5.3. The \( A(\infty) \ R[h] \)-algebra \( \tilde{A} \) is called the deformation of \( A \) to the normal cone.

Proposition 5.4. The \( A(\infty) \ R \)-algebras \( A \) and \( A(2) \) are quasi-isomorphic if and only if the \( A(\infty) \ R[h] \)-algebras \( \tilde{A} \) and \( A(2)[h] \) are quasi-isomorphic. That is \( A \) is formal if and only if \( \tilde{A} \) is such.

Proof. Given a quasi-isomorphism \( f : \tilde{A} \to A(2)[h] \) we may reduce it modulo \( (h-1) \) to get a quasi-isomorphism between \( A \) and \( A(2) \). Vice versa, let \( f = (f_1, f_2, \ldots) : A \to A(2) \) be a quasi-isomorphism of \( A(\infty) \ R \)-algebras. Then \( \tilde{f} = (f_1, f_2h, f_3h^2, \ldots) \) is a quasi-isomorphism between \( \tilde{A} \) and \( A(2)[h] \). \(\Box\)

Remark 5.5. If \( A \) and \( A(2) \) are quasi-isomorphic, then there exists a quasi-isomorphism \( \tilde{f} : \tilde{A} \to A(2)[h] \) which is the identity modulo \( h \). Indeed, the last proof produces an \( \tilde{f} \), such that \( \tilde{f} \equiv (f_1, 0, 0, \ldots)(\text{mod}h) \), where \( f_1 \) is an algebra automorphism of \( A(2)[h] \). Thus we may take the composition of \( \tilde{f} \) with \( (f_1^{-1}, 0, 0, \ldots) \).
Definition 5.6. The $A(\infty)$ $R$-algebra $A$ is called $n$-formal if there exists a quasi-isomorphism of $A(\infty)$ $R[h]/h^{n+1}$-algebras $\gamma : \tilde{A}/h^{n+1} \to A(2)[h]/h^{n+1}$, such that $\gamma \equiv (\text{id}, 0, 0, \ldots) \pmod{h}$.

Notice that Proposition 4.5 above provides a cohomological criterion for $n$-formality of $A$:

Corollary 5.7. a) The $A(\infty)$ $R$-algebra $A$ is $n$-formal if and only if the Kaledin class $K_{\tilde{A}/h^{n+1}} \in HH^2_R(\tilde{A}/h^n)$ is zero.

b) Assume that $m_{\tilde{A}/h^{n+1}} = m_2 + m_{n+2}h^n$. Then $A$ is $n$-formal if and only if $[m_{n+2}] \in HH^2_R(A(2))$ is zero (see Corollary 4.6).

The next proposition relates $n$-formality to formality.

Proposition 5.8. The $A(\infty)$ $R$-algebra $A$ is formal if and only if it is $n$-formal for all $n \geq 1$.

Proof. One direction is clear: If $A$ and $A(2)$ are quasi-isomorphic, then by Proposition 5.4 and Remark 5.5 there exists a quasi-isomorphism of $A(\infty)$ $R[[h]]$-algebras $\tilde{A} \to A(2)[h]$ which is the identity modulo $h$. It remains to reduce this quasi-isomorphism modulo $h^{n+1}$.

Assume that $A$ is $n$-formal for all $n \geq 1$. By Proposition 5.4 above it suffices to prove that the $A(\infty)$ $R[h]$-algebras $\tilde{A}$ and $A(2)[h]$ are quasi-isomorphic.

We will prove by induction on $n$ that there exists a sequence of maps $g_2, g_3, \ldots$, where $g_i \in \text{Hom}_R^0(A[1] \otimes i, A[1])$ so that for each $n \geq 2$ the following assertion is true:

E(n): Consider maps $g_i$ as coderivations $g_i = (0, \ldots, 0, g_i, 0, \ldots)$ of degree zero of the coalgebra $\mathcal{T}\tilde{A}[1]$. Then the coalgebra automorphism

$$\gamma_n := \exp(g_nh^n) \cdot \ldots \cdot \exp(g_2h) : \mathcal{T}\tilde{A}[1] \to \mathcal{T}\tilde{A}[1]$$

when reduced modulo $h^n$ becomes a quasi-isomorphism between $\tilde{A}/h^n$ and $A(2)[h]/h^n$.

Then the infinite composition $\tilde{f} := \ldots \exp(g_2h^2)\exp(g_2h)$ is the required quasi-isomorphism between $\tilde{A}$ and $A(2)[h]$.

In order to prove the existence of the $g_i$'s it is convenient to introduce the following $k^*$-action on the $R$-module $\mathcal{T}\tilde{A}[1]$. For $\lambda \in k^*$ put

$$\lambda \star x := \lambda^i x, \quad \text{if} \quad x \in (A[1])^\otimes i, \quad \text{and} \quad \lambda \star h = \lambda h.$$ 

Notice that both $m_2$ and $\tilde{m}$ are maps of degree $-1$ with respect to this action.

Now assume that we found $g_2, \ldots, g_n$ so that E(n) holds. Then

$$\gamma_n \cdot \tilde{m} \cdot \gamma_n^{-1} \equiv m_2 + m_n'h^n \pmod{h^{n+1}}$$
for some coderivation $m'_n \in C^1_R(A)$. Notice that the map $\gamma_n$ is of degree zero with respect to the $k^*$-action. Hence the coderivation $\gamma_n \cdot \tilde{m} \cdot \gamma_n^{-1}$ is again of degree $-1$. This forces the coderivation $m'_n$ to be defined by a single map in $\text{Hom}_R^1(A[1]^\otimes n+2, A[1])$. Since $A$ is $n$-formal, by Corollary 4.6 the class $[m'_n]$ is zero in $HH^2_R(A(2))$. So there exists a coderivation $g_{n+1} \in C^0_R(A)$ such that $[m_2, g_{n+1}] = m'_n$. It is clear that we can choose $g_{n+1}$ to be defined by a single map $g_{n+1} \in \text{Hom}_R^0(A[1]^\otimes n+1, A[1])$. Then the coalgebra isomorphism

$$\gamma_{n+1} := \exp(g_{n+1}h^n) \cdot \gamma_n : TA[1] \to TA[1]$$

induces a quasi-isomorphism between $\tilde{A}/h^{n+1}$ and $A(2)[h]/h^{n+1}$. This completes the induction step and proves the proposition. □

5.2. Notice that for each $n \geq 1$ the $A(\infty)$ algebra $\tilde{A}/h^n$ is finitely defined. Thus the Hochshild cohomology with compact supports $HH^\bullet_R[h]/h^n,c(\tilde{A}/h^n)$ is defined. Moreover the Kaledin class $K_{\tilde{A}/h^{n+1}}$ obviously belongs to the image of $HH^\bullet_R[h]/h^n,c(\tilde{A}/h^n)$ in $HH^2_R[h]/h^n(\tilde{A}/h^n)$. Therefore it is useful to notice the following fact.

**Lemma 5.9.** For any $n \geq 1$ the canonical map

$$HH^\bullet_R[h]/h^n,c(\tilde{A}/h^n) \to HH^\bullet_R[h]/h^n(\tilde{A}/h^n)$$

is injective.

**Proof.** This is easy to see by considering the weights of the $k^*$-action as in the proof of Proposition 5.8. □

**Remark 5.10.** Thus we may and will consider the obstruction to $n$-formality of $A$ (i.e. the Kaledin class $K_{\tilde{A}/h^{n+1}}$) as an element of $HH^2_R[h]/h^n,c(\tilde{A}/h^n)$. In particular in Corollaries 4.6 and 5.7 we can use the Hochschild cohomology with compact supports.

6. Applications

6.1. **Formality of $A(\infty)$ algebras.** Let $k$ be a field of characteristic zero and $R$ be a commutative $k$-algebra. Let $A = (A, m)$ be a minimal flat $A(\infty)$ $R$-algebra and $\tilde{A}$ be its deformation to the normal cone. If $m = (m_2, m_3, ...)$ denote as before $A(2) := (A, (m_2, 0, 0, ...))$, i.e. $A(2)$ is the underlying associative algebra of $A$. We have $A(2) = \tilde{A}/h$. By definition $A$ is formal if it is quasi-isomorphic to $A(2)$.

**Remark 6.1.** Let $R \to Q$ be a homomorphism of commutative $k$-algebras. If $A$ is formal then clearly the $A(\infty)$ $Q$-algebra $A_Q = A \otimes_R Q$ is also formal.
Proposition 6.2. Assume that $A$ is a finite $R$-module. Let $R \to Q$ be a homomorphism of commutative rings. Put $A_Q = A \otimes_R Q$. Assume that $Q$ is a faithfully flat $R$-module. Then $A$ is formal if and only if the $A(\infty)$ $Q$-algebra $A_Q$ is formal.

Proof. By Proposition 5.8 $A$ (resp. $A_Q$) is formal if and only if it is $n$-formal for all $n \geq 1$.

Fix $n \geq 1$. Notice that $Q[h]/h^n$ is faithfully flat over $R[h]/h^n$. By Proposition 3.4 we have $HH^2_Q[h]/h^n,c(A_Q/h^n) = HH^2_{R[h]/h^n,c}(A/h^n) \otimes_{R[h]/h^n} Q[h]/h^n$. And by faithful flatness the class $K_{A/h^n+1} \in HH^2_{R[h]/h^n,c}(A/h^n)$ is zero if and only if the class $K_{A_Q/h^n+1} = K_{A/h^n+1} \otimes 1 \in HH^2_{Q[h]/h^n,c}(A_Q/h^n)$ is zero. Hence the proposition follows from Corollary 5.7 a) and Remark 5.10.

□

Proposition 6.3. Assume that $R$ is an integral domain with the generic point $\eta \in \text{Spec} R$. Assume that $A$ is a finite $R$-module and that the $R$-module $HH^2_{R,c}(A(2))$ is torsion free. If the $A(\infty)$ $k(\eta)$-algebra $A_\eta$ is formal then $A$ is also formal. In particular the $A(\infty)$ $k(x)$-algebra $A_x$ is formal for all points $x \in \text{Spec} R$.

Proof. By Proposition 5.8 it suffices to prove that $A$ is $n$-formal for all $n \geq 1$. We do it by induction on $n$. Fix $n \geq 1$ and assume that $A$ is $(n-1)$-formal. Then we may and will assume that $m_{A/h^n+1} = m_2 + m_{n+2}h^n$. By Corollary 5.7 b) and Remark 5.10 $A$ is $n$-formal if and only if the class $[m_{n+2}] \in HH^2_{R,c}(A(2))$ is zero. This class vanishes at the generic point $\eta$ (since $HH^2_{R,c}(A(2)) \otimes_R k(\eta) = HH^2_{k(\eta),c}(A_\eta(2))$ by Proposition 3.4) and hence vanishes identically, since the $R$-module $HH^2_{R,c}(A(2))$ is torsion free. This completes the induction step and proves the proposition.

□

Proposition 6.4. Let $R$ be noetherian. Assume that $A$ is a finite $R$-module and that for each $n$ the $R$-module $HH^n_{R,c}(A(2))$ is projective. Then the subset

$$F(A) := \{ x \in \text{Spec} R \mid \text{the } A(\infty) \ k(x)-\text{algebra } A_x \text{ is formal} \}$$

is closed under specialization.

Proof. We may assume that $F(A)$ is not empty. Choose $\eta \in F(A)$ and consider its closure $\overline{\eta} =: \text{Spec} \overline{R} \subset \text{Spec} R$. Then $\overline{R}$ is an integral domain and $A_{\overline{R}} = A \otimes_R \overline{R}$ is an (flat minimal) $A(\infty)$ $\overline{R}$-algebra which is a finite $\overline{R}$-module. By Proposition 3.6 above $HH^n_{R,c}(A(2)_{\overline{R}}) = HH^n_{R,c}(A(2)) \otimes_R \overline{R}$. This is a projective $\overline{R}$-module, in particular, torsion free. Hence the assumptions of the previous proposition hold for $A_{\overline{R}}$ and thus $A_{\overline{R}}$ is formal. So $A_x$ is formal for all $x \in \text{Spec} \overline{R}$.

□

Proposition 6.5. Let $R$ be noetherian and $I \subset R$ be an ideal such that $\cap_n I^n = 0$. Assume that $A$ is a finite $R$-module and for each $n$ the $R$-module $HH^n_{R,c}(A(2))$ is
projective. Assume that the $A(\infty) \ R/I^n$-algebra $A_n := A/(I)^n$ is formal for all $n \geq 1$. Then $A$ is formal.

Proof. The proof is similar to the proof of Proposition 6.3. Namely we prove by induction on $n$ that $A$ is $n$-formal. Fix $n \geq 1$ and assume that $A$ is $n-1$-formal. Then we may assume that $m_{A/h^n+1} = m_2 + m_{n+2}h^n$. By Corollary 5.7 b) and Remark 5.10 $A$ is $n$-formal if and only if the class $[m_{n+2}] \in HH^2_{R,c}(A(2))$ is zero. By Proposition 3.6 we have

$$HH^2_{R,c}(A(2)) \otimes_R R/I^1 = HH^2_{R/I^1,c}(A(2)/I^1)$$

and by our assumption the class $[m_{n+2}] \otimes 1 \in HH^2_{R/I^1,c}(A(2)/I^1)$ is zero. Therefore the class $[m_{n+2}] = 0$, because $\cap I^1 = 0$ and the $R$-module $HH^2_{R,c}(A(2))$ is projective. This completes the induction step and proves the proposition. □

Proposition 6.6. Assume that $R$ is noetherian and has the trivial radical (i.e. the intersection of maximal ideals of $R$ is zero). Assume that $A$ is a finite $R$-module. Assume that for each $n$ the $R$-module $HH^0_{R,c}(A(2))$ is projective. If $A_x$ is formal for all closed points $x \in \text{Spec}R$ then $A$ is formal (and hence $A_y$ is formal for all points $y \in \text{Spec}R$).

Proof. Again we use Proposition 5.8: it suffices to prove that $A$ is $n$-formal for all $n \geq 1$. Fix $n \geq 1$ and assume that $A$ is $n-1$-formal. Then we may assume that $m_{A/h^n+1} = m_2 + m_{n+2}h^n$. By Corollary 5.7 b) and Remark 5.10 $A$ is $n$-formal if and only if the class $[m_{n+2}] \in HH^2_{R,c}(A(2))$ is zero. Let $J \subset R$ be a maximal ideal. By Proposition 3.6 we have

$$HH^2_{R,c}(A(2)) \otimes_R R/J = HH^2_{R/J,c}(A(2)/J)$$

and by our assumption the class $[m_{n+2}] \otimes 1 \in HH^2_{R/J,c}(A(2)/J)$ is zero. Therefore the class $[m_{n+2}] = 0$, because the radical of $R$ is trivial and $HH^2_{R,c}(A(2))$ is a projective $R$-module. This completes the induction step and proves the proposition. □

Remark 6.7. Assume that there exists an associative graded $k$-algebra $B$ such that the $A(2) = B \otimes_k R$ and $\dim_k B < \infty$. Then we may consider $A$ as an $R$-family of $A(\infty)$-structures which extend the same associative algebra structure on $B$. In this case for each $n$ the $R$-module $HH^0_{R,c}(A(2))$ is free and the conclusions of Proposition 6.4, 6.5, 6.6 hold without the assumption of $R$ being noetherian (Example 3.9).

6.2. Formality of DG algebras. All the results of this section can be formulated in the language of DG algebras rather than $A(\infty)$ algebras. Namely, assume again that $k$ is a field of characteristic zero and $R$ be a commutative $k$-algebra. Let $A$ be flat DG $R$-algebra, i.e. each cohomology $R$-module $H^n(A)$ is projective. Then by Theorem 2.2 it has a minimal $A(\infty)$ model $A$, which is unique up to a quasi-isomorphism (Corollary 2.6). It
comes with an $A(\infty)$ quasi-isomorphism $A \to A$. By Corollary 2.9 $A$ is formal (as a DG algebra) if and only if $A$ is formal (as an $A(\infty)$ algebra).

We would like to study extended DG algebras $A \otimes_R Q$, for various (commutative) algebra homomorphisms $R \to Q$. In particular we would like to study the fibers $A_x$ of $A$ at various points of $x \in \text{Spec}R$. To do that we should first replace the DG algebra $A$ by a quasi-isomorphic one which is cofibrant.

**Lemma 6.8.** Let $C$ be a cofibrant DG $R$-algebra. Then $C$ is cofibrant as a complex of $R$-modules.

**Proof.** This follows from [Sch-Sh], Theorem 4.1(3). Alternatively, it is easy to see directly if $C$ is semi-free ([Dr]).

So from now on we assume that the flat DG algebra $A$ is cofibrant. The the $A(\infty)$ quasi-isomorphism $A \to A$ remains a quasi-isomorphism after any extension of scalars.

**Corollary 6.9.** Let $A$ be DG $R$-algebra such that the total cohomology $R$-module $H^\bullet(A)$ is projective of finite rank. Let $R \to Q$ be a homomorphism of commutative rings. Assume that $Q$ is a faithfully flat $R$-module. Then $A$ is formal if and only if the DG $Q$-algebra $A \otimes_R Q$ is formal.

**Proof.** Let $A$ be a minimal $A(\infty)$ $R$-algebra with a quasi-isomorphism of $A(\infty)$ $R$-algebras $f : A \to A$. Then $f \otimes \text{id} : A \otimes_R Q \to A \otimes_R Q$ is also a quasi-isomorphism. So the corollary follows from Proposition 6.2.

**Corollary 6.10.** Let $A$ be DG $R$-algebra such that total cohomology $R$-module $H^\bullet(A)$ is projective of finite rank and $A$ is cofibrant as a complex of $R$-modules. We consider the cohomology $H^\bullet(A)$ as an $A(\infty)$ algebra with $m_i = 0$ for $i \neq 2$.

a) Assume that $R$ is an integral domain with the generic point $\eta \in \text{Spec}R$. Assume that the $R$-module $\text{HH}_{R,c}^2(H^\bullet(A))$ is torsion free. If the DG $k(\eta)$-algebra $A_\eta$ is formal then the DG $R$-algebra $A$ is also formal. In particular, $A_x$ is formal for all points $x \in \text{Spec}R$.

b) Let $R$ be noetherian. Assume that for each $n$ the $R$-module $\text{HH}_{R,c}^n(H^\bullet(A))$ is projective. Then the subset

$$F(A) := \{ x \in \text{Spec}R \mid \text{the DG } k(x) \text{-algebra } A_x \text{ is formal} \}$$

is closed under specialization.

c) Let $R$ be noetherian and $I \subset R$ be an ideal such that $\cap_n I^n = 0$. Assume that for each $n$ the $R$-module $\text{HH}_{R,c}^n(H^\bullet(A))$ is projective. Assume that the DG $R/I^n$-algebra $A \otimes_R R/I^n = A/(I)^n$ is formal for all $n \geq 1$. Then $A$ is formal.
d) Assume that $R$ is noetherian and has the trivial radical (i.e. the intersection of maximal ideals of $R$ is zero). Assume that for each $n$ the $R$-module $HH^n_{R,c}(H^\bullet(A))$ is projective. If $A_x$ is formal for all closed points $x \in \text{Spec} R$ then $A$ is formal (and hence $A_y$ is formal for all points $y \in \text{Spec} R$).

**Proof.** This follows from Propositions 6.3, 6.4, 6.5, 6.6 above. Indeed, if $A \to A$ is a minimal flat $A(\infty)$ model for $A$, then $H^\bullet(A) = A(2)$ and for any homomorphism $R \to Q$ of commutative algebras the DG $Q$-algebra $A \otimes_R Q$ is DG formal if and only if the $A(\infty)$ $Q$-algebra $A \otimes_R Q$ is formal. □

**Remark 6.11.** Let $A$ be as in the last corollary. Assume that there exists an associative $k$-algebra $B$ such that $H^\bullet(A) = B \otimes_k R$. Then we may consider $A$ as an $R$-family of DG algebras with the "same" cohomology algebra. In this case for each $n$ the $R$-module $HH^n_{R,c}(H^\bullet(A))$ is free and the conclusions in parts b), c), d) of the corollary hold without the assumption of $R$ being noetherian (Remark 6.7).

### 7. Kaledin cohomology class for DG algebras

#### 7.1. DG Lie algebras

Let $k$ be a field of characteristic zero, $R$ be a commutative $k$-algebra and $L = \bigoplus L^i$ be a graded $R$-module. Assume that there is given an $R$-linear map $[\cdot, \cdot] : L \otimes_R L \to L$ which is homogeneous of degree zero and satisfies the following relations
\[
[\alpha, \beta] + (-1)^{\bar{\alpha}\bar{\beta}}[\bar{\beta}, \bar{\alpha}] = 0,
\]
\[
(-1)^{\bar{x}\bar{\alpha}}[\alpha, [\beta, \gamma]] + (-1)^{\bar{\alpha}\bar{\beta}}[\bar{\beta}, [\gamma, \alpha]] + (-1)^{\bar{\alpha}\bar{\gamma}}[\bar{\gamma}, [\alpha, \beta]] = 0,
\]
where $\bar{x}$ denotes the degree of a homogeneous element $x \in L$. Then $L$ is called a graded Lie $R$-algebra.

A homogeneous $R$-linear map $d : L \to L$ of degree $l$ is called a derivation if
\[
d([\beta, \gamma]) = [d\beta, \gamma] + (-1)^{\bar{\beta}\bar{\gamma}}[\bar{\gamma}, d\beta].
\]

Homogeneous $R$-linear derivations of $L$ form a graded Lie algebra
\[
\text{Der}_R(L) = \text{Der}(L) = \bigoplus \text{Der}^i(L).
\]

We have a natural homomorphism of graded algebras
\[
ad : L \to \text{Der}(L), \quad ad_\alpha(\cdot) := [\alpha, \cdot].
\]

**Definition 7.1.** A DG Lie algebra is a pair $(L, d)$, where $L$ is a graded Lie algebra and $d \in \text{Der}^1(L)$ is such that $d^2 = 0$.

Notice that the cohomology of a DG Lie algebra is naturally a graded Lie algebra.
7.2. Gauge group. Let $\mathfrak{g}$ be an graded Lie $R$-algebra. Consider the graded Lie $R[[h]]$-algebra

$$\mathfrak{g}[[h]] := \bigoplus_i \mathfrak{g}^i[[h]],$$

where $\mathfrak{g}^i[[h]]$ consists of power series $\alpha_0 + \alpha_1 h + \alpha_2 h^2 + \ldots$, $\alpha_n \in \mathfrak{g}^i$ with the bracket induces by $[\alpha h^n, \beta h^m] = [\alpha, \beta] h^{n+m}$. Clearly $\mathfrak{g}^i[[h]] = \lim_{\leftarrow} \mathfrak{g}^i[[h]] / h^n$ for each $i$. In particular, the Lie subalgebra $h \mathfrak{g}^0[[h]] \subset \mathfrak{g}[[h]]$ is the inverse limit of nilpotent Lie algebras $\mathfrak{g}^0_n := h \mathfrak{g}^0[[h]] / h^{n+1}$.

Let $G_n$ be the group of $R[h]$-linear automorphisms of the graded Lie algebra $\mathfrak{g}[[h]] / h^{n+1}$ generated by operators $\exp^{\text{ad}_\alpha}$, $\alpha \in h \mathfrak{g}^0[[h]] / h^{n+1}$ which act by the formula

$$\exp(\text{ad}_\alpha) (\beta) = \beta + [\alpha, \beta] + \frac{1}{2!} [\alpha, [\alpha, \beta]] + \ldots$$

Notice that by the Campbell-Hausdorff formula every element of $G_n$ is equal to $\exp^{\text{ad}_\alpha}$, for some $\alpha \in h \mathfrak{g}^0[[h]] / h^{n+1}$.

There are natural surjective group homomorphisms $G_{n+1} \to G_n$ and we denote

$$G = G(\mathfrak{g}) := \lim_{\leftarrow} G_n.$$  

The group $G$ is called the gauge group of $\mathfrak{g}$. It acts naturally by $R[[h]]$-linear automorphisms of the graded Lie algebra $\mathfrak{g}[[h]]$ by the adjoint action. This action is by definition faithful. This induces the action of $G$ on the graded Lie algebra $\text{Der}(\mathfrak{g}[[h]])$. In particular, if $(\mathfrak{g}[[h]], d)$ is a DG Lie algebra and $g \in G$, then $(\mathfrak{g}[[h]], g(d))$ is also such.

7.3. Kaledin class. Let $(\mathfrak{g}, d)$ be a DG Lie $R$-algebra. Consider the DG Lie $R[[h]]$-algebra $(\mathfrak{g}[[h]], d)$. Let $\pi = \pi_1 h + \pi_2 h^2 + \ldots \in h \mathfrak{g}^1[[h]]$ be a solution of the Maurer-Cartan equation

$$d\pi + \frac{1}{2} [\pi, \pi] = 0.$$  

In other words the derivation $d_\pi := d + [\pi, -]$ satisfies $d_\pi^2 = 0$. Consider the element

$$\partial_h(d_\pi) = \partial_h(\pi) = \pi_1 + 2\pi_2 h + 3\pi_3 h^2 + \ldots \in \mathfrak{g}^1[[h]].$$

We have

$$0 = \partial_h(d_\pi^2) = \partial_h(d_\pi) \cdot d_\pi + d_\pi \cdot \partial_h(d_\pi) = [d_\pi, \partial_h(d_\pi)].$$

Thus $\partial_h(d_\pi)$ is a 1-cocycle in the DG Lie algebra $(\mathfrak{g}[[h]], d_\pi)$.

**Definition 7.2.** We call the corresponding cohomology class $[\partial_h(d_\pi)] \in H^1(\mathfrak{g}[[h]], d_\pi)$ the Kaledin class (of $\pi$).
Proposition 7.3. a) The Kaledin class \( \partial_h(d_\pi) \in H^1(\mathfrak{g}[[h]], d_\pi) \) is gauge invariant. That is for \( g \in G \) the classes \( [g(\partial_h(d_\pi))], [\partial_h(g(d_\pi))] \in H^1(\mathfrak{g}[[h]], g(d_\pi)) \) are equal.

b) Moreover, the class \( [\partial_h(\pi)] = 0 \) if and only if \( \pi \) is gauge equivalent to zero, i.e. there exists \( g \in G \) such that \( g(d_\pi) = d \).

Proof. a). Since
\[
H^\bullet(\mathfrak{g}) = \lim_{\to} H^\bullet(\mathfrak{g}[[h]]/h^n)
\]
it suffices to prove that the two classes are congruent modulo \( h^{n+1} \) for all \( n \geq 0 \). So fix \( n \geq 0 \) and \( g \in G \). Since we work modulo \( h^{n+1} \) we may and will assume that \( g \in G_n \).

Lemma 7.4. There exist \( \xi_1, \ldots, \xi_n \in \mathfrak{g}^0 \) such that
\[ g = \exp(\xi_n h^n) \exp(\xi_{n-1} h^{n-1}) \cdots \exp(\xi_1 h). \]

Proof. By induction on \( n \) we assume that the statement of the lemma holds for the image of \( g \) in \( G_{n-1} \). Thus there exist \( \xi_1, \ldots, \xi_{n-1} \in \mathfrak{g}^0 \) so that
\[ \bar{g} := \exp(-\xi_1 h) \cdots \exp(-\xi_{n-1} h^{n-1}) g \]
lies in the kernel of the projection \( G_n \rightarrow G_{n-1} \). Let \( \eta = \eta_1 h + \ldots + \eta_n h^n \in h^0 h^n \) be such that \( \bar{g} = \exp(\eta) \). Since the image of \( \bar{g} \) under the projection \( G_n \rightarrow G_1 \) is trivial we conclude that \( \eta_1 \) is in the center of the graded Lie algebra \( \mathfrak{g} \). Hence we may and will assume that \( \eta_1 = 0 \). Similarly, considering the trivial image of \( \bar{g} \) under the projection \( G_n \rightarrow G_2 \) we may and will assume that \( \eta_2 = 0 \), etc. So \( \bar{g} = \exp(\eta_n h^n) \) and we can take \( \xi_n = \eta_n \). This proves the lemma.

Using the lemma we may and will assume that \( g = \exp(\xi h^1) \) for some \( \xi \in \mathfrak{g}^0 \), \( i > 0 \).

By definition \( g(d_\pi) = g \cdot d_\pi \cdot g^{-1} \), hence
\[
\partial_h(g(d_\pi)) = \partial_h g \cdot d_\pi \cdot g^{-1} + g \cdot \partial_h(d_\pi) \cdot g^{-1} - g \cdot d_\pi \cdot g^{-1} \cdot \partial_h g \cdot g^{-1}.
\]
So
\[
g(\partial_h(d_\pi)) - \partial_h(g(d_\pi)) = [g(d_\pi), \partial g \cdot g^{-1}].
\]

But
\[
\partial_h g \cdot g^{-1} = \partial_h(\exp(\xi h^i)) \cdot \exp(-\xi h^i) = i \xi h^{i-1}.
\]

This proves a).

b). If \( g(d_\pi) = d \) for some \( g \in G \), then \( \partial_h(g(d_\pi)) = 0 \) and hence by part a) also \( [\partial_h(\pi)] = 0 \).

Vice versa, suppose that \( [\partial_h(\pi)] = 0 \). Let \( \pi = \pi_1 h + \pi_2 h^2 + \ldots \). Then in particular \( 0 = [\pi_1] \in H^1(\mathfrak{g}, d) \). So there exists \( \xi_1 \in \mathfrak{g}^0 \) such that \( d(\xi_1) = \pi_1 \). Put \( g_1 := \exp(\xi_1 h) \in G \). Then
\[
g_1(d_\pi) \cong d(\text{mod} h^2).
\]
By induction we may assume that we found $\xi_1, \ldots, \xi_{n-1} \in g^0$ so that

$$g_{n-1} \ldots g_1(d_\pi) \cong d(\text{mod } h^n),$$

where $g_i = \exp(\xi_i h^i)$. Then by part a) we may assume that $\pi_1 = \ldots = \pi_{n-1} = 0$. So by our assumption we have in particular $0 = [n\pi_n h^{n-1}] \in H^1(g[[h]]/h^n, d_\pi)$. This is equivalent to saying that $0 = [n\pi_n] \in H^1(g, d)$. Let $\xi_n \in g^0$ be such that $d(\xi_n) = [\pi_n]$ (recall that $\mathbb{Q} \subset R$) and put $g_n := \exp(\xi_n)$. Then

$$g_n(d_\pi) \cong d(\text{mod } h^{n+1}).$$

This completes our induction step. Put $g := \ldots g_3 g_2 g_1 \in G$. Then

$$g(d_\pi) = d.$$

If we consider the DG Lie $R[[h]]$-algebra $(g[[h]], d_\pi)$ as a deformation of the DG Lie $R$-algebra $(g, d)$, then Proposition 7.3 above asserts that this deformation is trivial if and only if the Kaledin class $[\partial_h(d_\pi)] \in H^1(g[[h]], d_\pi)$ is zero.

All the above can be repeated for DG Lie $R[[h]]/h^n$-algebras $(g[[h]]/h^n, d_\pi)$. In particular we obtain the following corollary.

**Corollary 7.5.** a) The Kaledin class $[\partial_h(d_\pi)] \in H^1(g[[h]]/h^{n+1}, d_\pi)$ is gauge invariant, i.e. for $g \in G_n$ the classes $[g(\partial_h(d_\pi))]$, $[\partial_h(g(d_\pi))] \in H^1(g[[h]]/h^{n+1}, g(d_\pi))$ are equal.

b) Moreover, the class $[\partial_h \pi] = 0$ if and only if $\pi$ is gauge equivalent to zero, i.e. there exists $g \in G_n$ such that $g(d_\pi) = d$.

**Proof.** Same as that of Proposition 7.3. □

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