POISSON BOUNDARY OF GROUP EXTENSIONS

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Abstract. Given a finitely generated group, the well-known Stability Problem asks whether the non-triviality of the Poisson-Furstenberg boundary (which is equivalent to the existence of non-constant bounded harmonic functions) depends on the choice of simple random walk on the group. This question was far from being understood even in the class of linear groups. Given an amenable group, e.g. a solvable group, there is no known characterisation, even a conjectural one, of when it admits a simple random walk with non-trivial boundary. We provide a characterisation of groups with non-trivial boundary for finitely generated linear groups of characteristic \( p \). We prove in particular that the Stability Problem has a positive answer in this class of groups. For linear groups of characteristic 0, we prove a sufficient condition for the triviality of the boundary which does not depend on the choice of a simple random walk. We conjecture that our sufficient condition is also necessary. Our arguments are based on a new comparison criterion for group extensions, on new \( \Delta \)-restriction entropy estimates and a criterion for boundary non-triviality, and on a new "cautiousness" criterion for triviality of the boundary.

1. Introduction and statement of main results

Given a group of matrices over a field, Tits alternative [56] implies that the group is non-amenable if and only if it contains a free subgroup. An argument of Milnor [44] and Wolf [63] about the growth of solvable groups, and its refinement due to [50] implies that the growth of such a finitely generated group is exponential if and only if the group contains a free sub-semigroup. A well-known property weaker than subexponential growth and stronger than amenability is the Liouville property of simple random walks on a group, stating that all bounded harmonic functions are constant (for this and other basic definitions related to random walks see Section 2). For fields of positive characteristic we characterise non-Liouville solvable groups as those that contain the three-dimensional lamplighter group as a subgroup of one of its blocks (naturally defined metabelian groups associated to our group in a manner which we describe below). We show that the characteristic zero case is essentially different. For this case we give a sufficient criterion for triviality of the boundary and conjecture that this criterion is also necessary.

Our arguments are based on a new comparison criterion for the Liouville property of groups extensions. A probability measure on a group \( G \) is said to be non-degenerate if its support generates \( G \) as a semi-group.

Theorem A. (= Thm 3.1 Comparison criterion) Let \( 1 \to F \to G \to H \to 1 \) and \( 1 \to F' \to G' \to H' \to 1 \) be short exact sequences where the group \( F \) is Abelian, and suppose that the induced action of \( H \) on \( F \) is the same for both extensions above. Let \( \mu_1 \) and \( \mu_2 \) be finite entropy measures on \( G, G' \) which have the same projections to \( H \). Let

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assume that \( f \) acts non-trivially on the Poisson boundary of \((G, \mu_1)\). Then, if \( \mu_2 \) is non-degenerate, \( f \) acts non-trivially on the Poisson boundary of \((G', \mu_2)\).

The assumption of finite entropy is essential, see Remark 3.3. The assumption that \( \mu_2 \) is non-degenerate cannot be replaced by the weaker condition of irreducibility (that the support of \( \mu_2 \) generates \( G' \) as a group), see Remark 3.3.

We now explain our application of the comparison criterion to linear groups. Since the Poisson boundary is non-trivial for any non-degenerate measure on a non-amenable group, and since any finitely generated amenable linear group is virtually solvable and in particular virtually nilpotent-by-Abelian (Malcev – Kolchin theorem, [41]), the question of which linear groups have the Liouville property is essentially about virtually nilpotent-by-Abelian groups.

The comparison criterion above is well suited to study the Poisson boundary of nilpotent-by-Abelian groups. In particular it allows us to reduce the question of whether or not a given element acts non-trivially on the Poisson boundary to the question about whether corresponding elements act non-trivially on the boundary of associated metabelian groups, we prove this in Thm 4.20. These associated groups are finitely generated \( 2 \times 2 \) metabelian groups of matrices over a field. We recall that there are uncountably many nilpotent-by-Abelian groups (even up to quasi-isometry, in particular they have a continuum of distinct Folner functions, see [19]). In particular, the number of blocks of a nilpotent-by-Abelian groups is not necessarily finite.

For linear groups the number of metabelian blocks is finite, and they can be described explicitly. We consider a partial order \( U \) on \( \{i, j : 1 \leq i, j \leq n\} \), defined as \((i, j) \leq_U (i', j')\) if \( i \leq i' \) and \( j \geq j'\).

![Figure 1. Partial order \( U \) on pairs \((i, j) : 1 \leq i < j \leq n\). \((i, j) \leq_U (i', j')\) if \( i \leq i' \) and \( j \geq j'\); e.g. \((1, 5) \leq_U (2, 4)\).](image)

In the last claim of the theorem below we will also consider complete orders \( T \), extending \( U \). We note that such orders correspond to Young Tableaus (see Remark 5.4). Given a group of \( n \times n \) upper triangular matrices, we consider all matrices

\[
G_{i,j} = \begin{pmatrix}
g_{i,i} & 0 \\
0 & g_{j,j}
\end{pmatrix},
\]

defined when in \( G \) there is a matrix \( g_{i,j} \) with the same diagonal entries (\( g_{i,i} \) and \( g_{j,j} \)). If there is an element in \( G \cap UT(n) \) (Where \( UT(N) \) is the group of uni-upper triangular matrices) with non-zero entry at \((i, j)\) and with zero entries in all positions \((i', j') >_U\)
we consider the group generated by the matrices \( G_{i,j} \) and by the matrix
\[
\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]
and we call this group the \((i, j)\)-basic block of \( G \) and denote it by \( B_{i,j} \). See Section 5 and Definition 5.1 for details.

We show that our random walk \((G, \mu)\) is not Liouville, if and only if for some \((i, j)\) the random walk on the basic metabelian block \( B_{i,j} \) is not Liouville, for some probability measure \( \mu_{i,j} \) on \( B_{i,j} \) with the same projection to \( A \) as \( \mu \). More precisely, we prove Theorem B. (= Thm 5.3)

Let \( G \) be a group of upper-triangular \( n \times n \) matrices over a field.

1. The Poisson boundary of \((G, \mu)\) is non-trivial if and only if there exist \( i, j : 1 \leq i < j \leq n \) such that and for any associated measures \( \mu_{i,j} \) on the \((i, j)\)-block \( B_{i,j} \) (equivalently: for some associated measure on this block) the Poisson boundary of \((B_{i,j}, \mu_{i,j})\) is non-trivial.

2. An element \( g \in G \) acts non-trivially on the boundary \((G, \mu)\) if and only if there exists \( h \in \text{Norm}_G(g) \cap U_n \) and \( i, j : 1 \leq i < j \leq n \) such that the associated measure on the block \( B_{i,j} \) has non-trivial boundary and the matrix corresponding to \( h \) has \( m_{i,j} \neq 1 \) and \( m_{i',j'} = 0 \) for any \((i', j') >_U (i, j)\).

3. Let \( T \) be a partial order on \( \{ (i, j) : 1 \leq i < j \leq n \} \), which extends the partial order \( U \). An element \( g \in G \) acts non-trivially on the boundary \((G, \mu)\) if and only if there exist \( i, j : 1 \leq i < j \leq n \) and \( h \in \text{Norm}_G(g) \cap U_n \) such that the associated measure on the block \( B_{i,j} \) has non-trivial boundary and the matrix corresponding to \( h \) has \( m_{i,j} \neq 1 \) and \( m_{i',j'} = 0 \) for any \((i', j') >_T (i, j)\).

In Theorem 5.20 we explain an analogous reduction statement in a general context of nilpotent-by-Abelian groups.

1.1. Metabelian groups and metabelian blocks. After applying the above theorem, we are left to determine which random walks on metabelian blocks are Liouville. To address this question, we prove a new criterion for triviality of the boundary and a new criterion for non-triviality of the boundary.
Our first new criterion is a cautiousness criterion for the Liouville property, see Theorem 7.9. The argument uses in an essential way the Shannon – McMillan – Breiman type theorem ([13], [32]), and its idea is sketched in the beginning of Section 7. As one of the applications, we obtain Corollary 7.15, a version of which we formulate below.

**Corollary C.** Let $G$ be an (amenable) linear group over $k$ which has a finite index upper-triangular subgroup $G'$. Suppose that none of the basic blocks of $G'$ contains a three dimensional wreath product as a subgroup. If the characteristic of $k$ is zero, we assume additionally that any basic block of $G'$ which contains $\mathbb{Z}^2 \wr \mathbb{Z}$ as a subgroup is isomorphic to this wreath product. Then for any simple random walk (and any any symmetric finite second moment random walk) on $G$ the boundary is trivial.

One of the corollaries of Corollary C is discussed in Corollary 7.19. If $F$ is either a function field over at most 2 variables in positive characteristic or $F$ is a function field over at most 1 variable in characteristic 0 then we prove that any (virtually) solvable group which is linear over $F$ is Liouville: any finite second moment centered measure on this group has trivial boundary. Observe that the case of transcendence degree 0 recovers the previously known case of linear groups over $\mathbb{Q}$ (see [53]; see also [12] for the description of the boundary for random walks on $GL(n, \mathbb{Q})$).

More generally, using a notion of dimension for metabelian groups (see the beginning of Section 6 for a discussion of this notion), we give a complete characterisation of linear amenable groups of characteristics $p$ such that the boundary of simple random walks is non-trivial (see Corollary 8.15 in Section 8, one of its claims is Corollary D below). Passing to a finite index subgroup, these groups correspond to upper-triangular linear groups such that at least one of the valid blocks has dimension $\geq 3$. We mention that characteristic $p$ groups of dimension 0 are virtually Abelian. Taking into account that for characteristic $p$ metabelian groups dimension coincides with the Krull dimension of these groups, triviality of the boundary for simple random walks on such groups follows from combining [29], Thm 1.1 and [52]. Our result holds more generally for any centered finite first moment measure on any characteristic $p$ dimension one group. The main positive characteristic application of our triviality of the boundary criterion is in dimension 2. Here we get new examples of metabelian groups with trivial boundary for simple random walks (such as the Baumslag group for $d = 2$). As before, any linear group with such blocks also provide examples. In dimension 3 we get many new examples of groups with non-trivial boundary.

Now in characteristic 0 the situation was known for simple random walks in dimension 0 (this follows from the already mentioned result of Brofferio – Schapira [12] for linear groups over $\mathbb{Q}$. For linear groups over $\mathbb{Z}$ this result was known earlier due to Kaimanovich [33], [35]). Our result shows that in dimension 1 the boundary of simple (or centered finite second moment) random walks is trivial, and we get some new examples. In dimension 3, we also get many new examples of non-trivial boundary, in particular among metabelian groups containing a three-dimensional wreath product as a subgroup. The only case not covered by our results is therefore dimension 2 and characteristic 0. If the dimension of the group is 2, the situation in characteristic 0 is very different from that of linear groups of characteristic $p$. Namely, we give examples (see Proposition 8.12) of groups of dimension 2 having non-trivial boundary of random walks. Furthermore, we believe (see Question 1 below) that except for the well-known examples of two-dimensional wreath products, and groups that can be reduced to these wreath products as applications of our Theorem B, all simple (or finite entropy non-degenerate) random walks on such groups of dimension 2 have non-trivial boundary. If the answer to Question 1 is positive, one would get a
complete characterisation of metabelian groups with the Liouville property and in view of our theorem, thus for all linear groups of characteristic 0.

More precisely, we conjecture that the assumption of Corollary C above is not only sufficient, but necessary. First we ask this question in characteristic 0. Note that every finitely generated linear group over a field of characteristic 0 is virtually torsion free by Selberg’s lemma so it suffices to analyze torsion-free groups.

**Question 1.** Let $k$ be a field of characteristic 0 and let $G$ be a linear group over $k$. Assume that every finite index upper-triangular subgroup $G'$ of $G$ admits a basic block containing a two-dimensional wreath product but not-isomorphic to such a wreath product. Is it true that every non-degenerate finite entropy random walk on $G$ has non-trivial boundary?

In the case where the field $k$ has positive characteristic case we prove that the sufficient conditions of Corollary C are also necessary.

**Corollary D.** Let $G$ be an amenable linear group over a field of characteristic $p$. Then a simple random walk on $G$ has non-trivial boundary if and only if one of the basic blocks of $G$ contains $\mathbb{Z}^3 \rtimes \mathbb{Z}/p\mathbb{Z}$ as a subgroup.

For more a more general result we refer once more to Corollary 8.15.

We also prove a partial result in this directions in characteristic 0. If some basic block of $G$ (or of $G'$) contains a three-dimensional wreath product, then the boundary of any non-degenerate finite entropy random walk is non-trivial (this is a part of the claim of Corollary 8.10). We also provide further examples in support of our conjecture in Proposition 8.12.

In view of our results, we believe that the Liouville property of finitely generated linear groups is determined by the asymptotics of the Følner function of these groups. For general finitely generated groups, it is known that this is not the case. A group with arbitrarily quick growing Følner function can have the Liouville property. Arbitrarily quick growing Følner functions can even be found among groups of intermediate growth [17]. For a class of elementary amenable groups with prescribed Følner function and trivial boundary see [11]. We conjecture that for linear groups simple random walks are not Liouville if and only if the Følner function is $\geq \exp(Cn^3)$.

We prove our results on the non-triviality of the above mentioned boundaries by developing a $\Delta$-restriction entropy criterion, Theorem 8.1 (and combining this with our comparison criterion). While transience of some associated random walks appears in many examples (including non-degenerate random walks on wreath products) as a sufficient condition for boundary non-triviality, Lemma 8.7 in the proof of Theorem 8.1 uses in essential way a notion of uniform strong transience (see Definition 8.5), and the claim is false under the weaker assumption of transience (see Remark 8.9).

We remind the reader that a well-known open problem asks whether the non-triviality of the Poisson boundary depends on the choice of measure for simple random walks on a finitely generated group $G$. The group structure in this question seems to be important, and it is well-known that a closely related question about quasi-isometric graphs has a negative answer (see T.Lyons [39] for examples of quasi-isometric graphs, one of which is Liouville, and the other is not). Moreover, such examples can even be found among graphs of polynomial growth, see [10]). For the moment, this stability was known only for some classes where all groups have trivial boundary (i.e. subexponential growth, as a well-known corollary of the entropy criterion; polycyclic groups or groups of finite Prufer rank group), groups with (quite) slow decay of return probability ([52], [45]) and in very particular examples (such as wreath products). Our results in Corollary C and its partial converse show stability for any linear groups of positive characteristics (as well for a large
class of groups of zero characteristics). Our conjecture, if shown, would show stability for all linear groups.

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2. Definitions, Notations and Basic Facts

First we recall terminology about group extensions. Given a short exact sequence

\[ 1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1, \]

one says that \( G \) is an extension of \( H \) by a group \( N \). However, if \( N \) and \( H \) have some properties, for example if \( N \) is nilpotent and \( H \) is Abelian, it is said that \( G \) is “nilpotent-by-Abelian” (and not ”Abelian-by-nilpotent”). See [55] for reflections about possible origins of this inconsistent terminology.

Now we recall definitions and background related to random walks. Given a probability measure \( \mu \) on a group \( G \), a (right) random walk \( (G, \mu) \) is a Markov chain with state space \( G \) and with transition probabilities \( p(g|gh) = \mu(h) \), for all \( g, h \in G \). We will assume that the random walks starts at the identity \( e \in G \). We have already mentioned that a measure \( \mu \) on \( G \) is said to be irreducible (also called adapted) if its support generates \( G \) as a group. A measure \( \mu \) on a group \( G \) is non-degenerate if the support of \( \mu \) generates \( G \) as a semi-group. More intuitively nondegeneracy means that the random walk, starting at identity, visits any point \( g \in G \) with positive probability. We now recall the definition of the Poisson boundary.

**Definition 2.1** (Poisson boundary). Consider the space of one-sided infinite trajectories \( G^\mathbb{Z} \). For two trajectories \( X \) and \( Y \) in this space, we say that they are equivalent if they coincide after some instant, possibly up to the time shift. This means there exist \( N, k \geq 0 \) such that \( X_i = Y_{i+k} \) for all \( i > N \). Consider the measurable hull of this equivalence relation in \( G^\mathbb{Z} \). The quotient by the obtained equivalence relation is called the Poisson boundary (also called the Poisson-Furstenberg boundary).

Given an element \( g \in G \), we say that \( g \) acts on the space of trajectories by

\[ g(x_1, x_2, x_3, ...) = (gx_1, gx_2, gx_3, ...). \]

This action preserves the equivalence relation in the definition of the Poisson boundary and thus induces an action on the boundary.

The Poisson boundary can also be defined for any Markov chain. In the case of random walks on groups, there are several equivalent definitions. In particular, if we do not allow the shift by \( k \) in the definition above, and say that \( X \) and \( Y \) are equivalent if \( X_i = Y_i \) for all \( i > N \), then we obtain in this way the definition of the tail boundary. It is not difficult to see that for a general Markov chain this boundary can be larger then the Poisson boundary, and a random walk can have trivial Poisson boundary and non-trivial tail boundary. In the case of random walks on groups, this can not happen and the two definitions are equivalent. For these and other definitions and basic facts about boundaries see [20] [23] [32].

We also mention that in the case when the Markov chain is a random walk on a group, there are structural results where the group structure of the space is essential.

A function \( F : G \rightarrow \mathbb{R} \) is called \( \mu \)-harmonic, if for all \( g \in G \) it holds \( F(g) = \sum_{h \in G} F(gh) \mu(h) \). The Poisson boundary can be equivalently defined in terms of bounded
harmonic functions on the subgroup of $G$, generated by the support of $\mu$ and with values in $\mathbb{R}$. In particular non-triviality of the Poisson boundary is equivalent to the existence of non-constant bounded harmonic functions.

Given a probability measure $\mu$, its entropy $H(\mu)$ is defined as

$$H(\mu) = -\sum_{g \in G} \mu(g) \log(\mu(g)).$$

**Definition 2.2.** We recall that the entropy of a random walk $(G, \mu)$, also called the asymptotic entropy, is defined as the limit, as $n$ tends to infinity, of $H(n)/n$, where $H(n) = H(\mu^{*n})$. $H(n)$ is called entropy function of $(G, \mu)$.

In the definition above $\mu^{*n}$ denotes the $n$-th convolution power of $\mu$, and the above mentioned limit always exists in view of the sub-additivity $H(\mu^{*(n+k)}) \leq H(\mu^{*n}) + H(\mu^{*k})$. The latter is a consequence of the fact that $H(\nu_1 * \nu_2) \leq H(\nu_1) + H(\nu_2)$, for any probability measures $\nu_1$ and $\nu_2$ on a group $G$. Moreover, a Shannon-McMillan-Breiman type theorem holds which states that with probability 1 $\lim_{n \to \infty} -1/n \log(\mu^{*n}(g_n)) = h$ (Thm 2.1 in [32], [13]).

The notion of entropy of random walks was introduced by Avez [1], who showed that for a finitely supported measures $\mu$ if the asymptotic entropy is zero then the Poisson boundary is trivial. [2]. We know now that the statement is if and only if and, moreover, holds true for any finite entropy measure, regardless of the size of the support, because of following Entropy criterion of Derriennic and Kaimanovich – Vershik.

**Theorem** (Entropy criterion. [13][31][32]). Let $\mu$ be a probability measure on a group $G$. If the measure $\mu$ has finite entropy, then the Poisson boundary of the random walk $(G, \mu)$ is trivial if and only if the entropy of the random walk is 0.

The Entropy criterion in particular implies that if the growth of the group $G$ is subexponential, then any finitely supported random walk (and more generally any random walk of finite first moment) has trivial Poisson boundary.

Let $S$ be a generating set of $G$. The word length $l_S(g)$ is the shortest number of terms needed to write $g$ as a product of elements of $S$. The growth function $v_{G,S}(n)$ counts the number of group elements of word length at most $n$.

**Definition 2.3.** Suppose that a probability measure $\mu$ on $G$ has finite first moment, that is, $\sum_g \mu(g) l_S(g) < \infty$. (It is clear that the finiteness of the first moment does not depend on the choice of finite generating set $S$ of $G$). The drift function $L(n)$ is the expected length of the position at time instant $n$

$$L(n) = E[l(x_n)].$$

It is clear again that $L(n + m) \leq L(n) + L(m)$, and thus there exists a limit $l = \lim_{n \to \infty} L(n)/n$. This limit is called the drift (or rate of escape) of the random walk $(G, \mu)$.

It is not difficult to see that $l = 0$ implies $h = 0$ (see [27]). A random walk is said to be symmetric if the defining measure $\mu$ satisfies $\mu(g) = \mu(g^{-1})$, for all $g$. If the random walk is symmetric, the converse statement is true: if $h = 0$ then $l = 0$ (This was shown by Varopoulos [38] under the additional assumption that the support of $\mu$ is finite and by Karlsson – Ledrappier for the general finite first moment case [38]).

Among groups of exponential growth there are many known examples with trivial boundary of simple random walks and there are many known examples with non-trivial
boundary. For many groups it is a challenging problem to understand whether the boundary is non-trivial.

**Definition 2.4.** Given groups $A$ and $B$, a wreath product $A \wr B$ is a semi-direct product of $A$ and $\sum_A B$, where $A$ acts on $\sum_A B$ by shifting the index set.

(Some papers use $B \wr A$ notation for what we denote $A \wr B$). If $B = \mathbb{Z}/2\mathbb{Z}$, wreath products $A \wr B$ are also called lamplighter groups. The elements of lamplighter groups are pairs $(a, f)$, $f : A \to \mathbb{Z}/2\mathbb{Z}$ is a finitely supported function, which we can view as "the lamp configuration". In the sequel we denote by $\delta$ the element $(e_A, \chi_{e_A})$, "the lamp" at the identity of $A$.

One-dimensional and two-dimensional lamplighter groups $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$ provide examples of groups of exponential growth with trivial boundary for simple random walks [32]. On the other hand $\mathbb{Z}^d \wr \mathbb{Z}/2\mathbb{Z}$ where $d > 2$ provides examples of amenable groups where simple random walks with non-trivial boundary [32]. Observe that $\mathbb{Z}^d \wr \mathbb{Z}/2\mathbb{Z}$ are examples of (torsion-abelian)-by-abelian groups. For some groups in this class (e.g. Baumslag groups which will be discussed in the next subsection) the question was open and we give a complete characterisation of groups in this class.

Equivariant quotients of Poisson-Furstenberg boundaries (in the category of measure spaces) are called $\mu$-boundaries of the random walk $(G, \mu)$.

We also recall the definition of conditional entropy. Let $P$ be the measure of the space of one-sided infinite trajectories $G^\infty$. Consider a $\mu$-boundary $B$. Since $B$ is a quotient of this space $P$, the points of $B$ define conditional measures on $P$. Let us say that a probability measure in the space of one-sided infinite trajectories $G^\infty$ has asymptotic entropy $h$ if its one-dimensional distributions $\lambda_n$ (n step distribution of the corresponding conditional process) satisfy

$$-\ln \lambda_n(x_n)/n \to h$$

for almost all $x_n \in G^\infty$.

We recall that the Entropy criterion of Kaimanovich [37], Theorem 4.6 states the following.

**Theorem** (Conditional entropy criterion). Let $\mu$ be a probability measure on a group $G$. If the measure $\mu$ has a finite entropy, then a $\mu$-boundary $B$ is maximal (equal to the Poisson boundary) if and only if the asymptotic entropy is $0$ for almost all $b \in B$.

2.1. **Examples.** In this subsection we introduce notation and list some examples of groups, where our cautiousness criterion can be used to show triviality of the boundary, and other examples to illustrate our non-triviality for the boundary criterion. We also give some illustration of the application of these two criteria, combined with the comparison theorem.

In the examples below and throughout the paper $F_d$ denotes the free group on $d$ generators. In many examples below we consider matrices over $\mathbb{Z}[X_1^{\pm1}, \ldots, X_d^{\pm1}]$. Alternatively by choosing $X_i$ to be mutually transcendental numbers in $\mathbb{C}$, we can consider these matrices to be over $\mathbb{C}$.

A group is said to be metabelian if it is solvable of step 2, that is, if it is an extension of an Abelian group by an Abelian group. We recall that any metabelian group is linear over a commutative ring [48]. If the commutator subgroup of a finitely generated metabelian group is torsion free then it can furthermore be represented by triangular matrices over a field of characteristic zero; if the commutator subgroup is order $p$, for $p$ prime, then the group can be represented by triangular matrices over a field of characteristic $p$ [49]; (for the latter case for a virtually metabelian group see [61]). For free metabelian groups in
the example below representations over a field were studied e.g. in [60], the representation described in the example below is studied in [3].

We now commence with presenting our examples.

**Example 2.5** (Free metabelian and free \( p \)-metabelian groups). Let \( \text{Met}_d \) denote the free metabelian group on \( d \) generators, that is, \( F_d/[[F_d, F_d], [F_d, F_d]] \). This group is generated by \( d \) matrices of the form

\[
\begin{pmatrix}
X_i^{-1} & 1 \\
0 & X_i
\end{pmatrix}
\]

over \( \mathbb{Z}[X_1^\pm, \ldots, X_d^\pm] \). Let \( \text{Met}_d(p) \) be the free \( p \)-metabelian group, this group is by definition \( \text{Met}_d/\langle[[\text{Met}_d, \text{Met}_d]]\rangle^p \). It is generated by the same matrices if we consider them over \( \mathbb{Z}/p\mathbb{Z}[X_1^\pm, \ldots, X_d^\pm] \).

**Example 2.6** (\( d \)-dimensional lamplighters). Consider the wreath product \( G_d = \mathbb{Z}^d \wr \mathbb{Z} \). This group is isomorphic to the group generated by the matrices over \( \mathbb{Z}[X_1^\pm, \ldots, X_d^\pm] \)

\[
\delta = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad M_{x_i} = \begin{pmatrix}
1 & 0 \\
0 & X_i
\end{pmatrix}
\]

Let \( G_d,p = \mathbb{Z}^d \wr \mathbb{Z}/p\mathbb{Z} \). This group is generated by the above mentioned matrices when we consider them over \( \mathbb{Z}/p\mathbb{Z}[X_1^\pm, \ldots, X_d^\pm] \).

In the example below the dimension is at most two and in the next example the dimension is three, see [6.20].

**Example 2.7** (Liouville quotients of Lamplighter groups). Any proper quotients of the 3-dimensional Lamplighter over \( \mathbb{Z}/p\mathbb{Z} \), (with \( p \) prime), provide examples of groups with trivial Poisson boundary for simple random walks (or any symmetric measure of finite second moment).

**Example 2.8** (Non-Liouville quotients of Lamplighter groups). Quotients of the \( d \)-dimensional lamplighter group, \( d \geq 4 \), over a single relation are examples of groups where any non-degenerate finite entropy measure has a non-trivial boundary.

A result of Baumslag [5] states that any finitely generated metabelian group can be imbedded into a finitely presented metabelian group. A basic example of his construction is the following imbedding of Lamplighter groups \( \mathbb{Z}^d \wr \mathbb{Z}/p\mathbb{Z} \).

**Example 2.9** (Baumslag groups). (\( \Rightarrow \) Example [7.20]) Consider \( B_d(\mathbb{Z}/p\mathbb{Z}) \) to be a subgroup of \( GL_2(R) \), \( R = (\mathbb{Z}/p\mathbb{Z})[X_1^\pm, \ldots, X_D^\pm] \), generated by \( 2d + 1 \) matrices of the form

\[
\delta = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad M_{x_i} = \begin{pmatrix}
1 & 0 \\
0 & X_i
\end{pmatrix}, \quad M_{x_i+1} = \begin{pmatrix}
1 & 0 \\
0 & X_i + 1
\end{pmatrix}.
\]

For \( D = 2 \) all finite second moment centered random walk on these groups have trivial boundary.

We mention below two one-dimensional examples in characteristic 0. By our results these groups have trivial boundary for centered finite second moment random walks.

**Example 2.10.** (\( \Rightarrow \) Example [7.21]) The group \( G_{2,3,x} \) is the group generated by the matrices \( \delta, M_2, M_3, M_x \) defined below

\[
\delta = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
1 & 0 \\
0 & 2
\end{pmatrix}, \quad M_3 = \begin{pmatrix}
1 & 0 \\
0 & 3
\end{pmatrix}, \quad M_x = \begin{pmatrix}
1 & 0 \\
0 & x
\end{pmatrix}.
\]

Another one-dimension example:
Example 2.11. The group $G_{X,X+1,X+2}$ is generated by the following matrices over $\mathbb{Q}(X)$:

$$\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_X = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}, \quad M_{X+1} = \begin{pmatrix} 1 & 0 \\ 0 & x + 1 \end{pmatrix}, \quad M_{X+2} = \begin{pmatrix} 1 & 0 \\ 0 & X + 2 \end{pmatrix}.$$ 

If we consider the matrices in the example above over $\mathbb{Z}/p\mathbb{Z}(X)$, the group we obtain is linear in characteristic $p$. But even in characteristic $0$ case as in the example above it provides an example of triviality of the boundary.

The examples below for $d = 2$ provide (torsion-free) metabelian examples without three dimension wreath products (as a subgroup) and with non-trivial boundary for simple random walks; as well as examples of symmetric finite first moment measures with non-trivial boundary on groups without two dimensional wreath products as subgroups.

Example 2.12 (Lamplighter-Baumslag-Solitar group). We consider the group $\text{LBS}_d$, generated by $M_{X_1}, \ldots M_{X_d}, \delta$ and

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

We call this group "Lamplighter Baumslag Solitar" since $\delta$ and $M_{X_1}, \ldots M_{X_d}$ generate $\mathbb{Z}^d \wr \mathbb{Z}$, while $M_2$ and $\delta$ generate the solvable Baumslag Solitar group since $M_2^{-1} \delta M_2 = \delta^2$.

A more general example of this type can also be considered if instead of $2$ we consider an algebraic number $\alpha$ which is not a root of unity (see Proposition 8.12). These groups have nontrivial boundary for finite entropy non-degenerate random walks whenever $d \geqslant 2$.

In the example above, if we take $d = 1$, we get a group of dimension $1$, but the moment condition for centered random walks to have trivial boundary resembles known two-dimensional examples. Since, in contrast with the one-dimensional Lamplighter, finite first moment symmetric random walks can have non-trivial boundary.

Below we mention an example of metabelian group, containing the $3$-dimensional wreath product as a subgroup. By our results all such groups have non-trivial boundary for simple random walks.

Example 2.13. Consider the group generated by $2 \times 2$ matrices $\delta, M_{X_1}, M_{X_2}, M_{X_3}$ and

$$M_{X_1+X_2+X_3} = \begin{pmatrix} 1 & 0 \\ 0 & X_1 + X_2 + X_3 \end{pmatrix}.$$ 

We can also consider the the following group (and many other examples) which also have nontrivial boundary for the same reason

Example 2.14. The group, generated by $\delta, M_{X_1}, M_{X_2}, M_{X_3}$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{X_1 + X_2 + X_3 + 1} - \sqrt{X_1 + X_2 + X_3} \end{pmatrix}.$$ 

Another general class of $2 \times 2$ matrices containing three dimensional wreath products (and having thus non-trivial Poisson boundary) is described in the following example.

The examples below will follow from Lemma 8.22 and Corollary 8.10.

Example 2.15. Let $G$ be a non-abelian upper triangular linear group over a field $k$. Consider the homomorphism $\phi$ from $G \to k$ which sends a matrix

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$
to \( a/b \). Let \( k' \) be the minimal field which contains the image of \( \phi(G) \). Then if the transcendence degree of \( k' \) is \( \geq 3 \), then boundary is nontrivial for any finite entropy random walk.

We have mentioned in the introduction (Corollary 7.19) that if the transcendence degree of \( k' \) is \( \leq 1 \) the boundary is trivial for centered finite second moment random walks. Furthermore, by the same corollary if the transcendence degree of \( k' \) is 2 and \( k' \) is of positive characteristic the boundary is also trivial for centered finite second moment random walks.

In the example below the group is defined over a field of transcendence degree 3, while its blocks are two dimensional lamplighters. This is among many examples where the criterion about the transcendence degree of the blocks implies triviality of the boundary of the random walk.

**Example 2.16 (X - Y - Z group).** Consider the group generated by matrices

\[
\begin{align*}
\bar{M}_X &= \begin{pmatrix} X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\bar{M}_Y &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & 1 \end{pmatrix},
\bar{M}_Z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Z \end{pmatrix}
\end{align*}
\]

and

\[
\delta_{1,1,1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]

Then this group has trivial Poisson Boundary for any simple random walk (or more generally any centered finite second moment random walk).

### 3. Comparison Criterion for Group Extensions

The main goal of this section is to prove the following theorem

**Theorem 3.1 (Comparison Criterion: Abelian case).** Let \( 1 \to F \to G \to H \to 1 \) and \( 1 \to F \to G' \to H \to 1 \) be short exact sequences where the group \( F \) is Abelian, and the induced action of \( H \) on \( F \) is the same for both extensions. Let \( \mu_1 \) and \( \mu_2 \) be finite entropy measures on \( G, G' \) which have the same projections to \( H \). Let \( f \in F \), assume that \( f \) acts non-trivially on the Poisson boundary of \((G, \mu_1)\). Then under the assumption that \( \mu_2 \) is non-degenerate, \( f \) acts non-trivially on the Poisson boundary of \((G', \mu_2)\).

**Remark 3.2.** The assumption that the entropy of the measures is finite is essential in the theorem, as well as in its Corollary 3.4 below. Indeed, consider \( G = G' = \mathbb{Z}^3 \setminus \mathbb{Z}/2\mathbb{Z} \). We have

\[
1 \to \sum_{\mathbb{Z}^3} \mathbb{Z}/2\mathbb{Z} \to G \to \mathbb{Z}^3 \to 1.
\]

The wreath product \( G \) is solvable, and in particular amenable. As we have mentioned, by a theorem of Rosenblatt [51] and Kaimanovich-Vershik [32] we know that \( G \) admits non-degenerate measures \( \mu \) (which can be chosen to have full support \( \text{supp} \mu = G \)), such that the boundary \((G, \mu)\) is trivial. Consider a measure \( \mu' \), which has the same projection to \( G \) as \( \mu \), with the support of \( \mu' \) being equal to \( \mathbb{Z}^3 \cup \delta \). The projection of \( \mu' \) (and \( \mu \)) to \( \mathbb{Z}^3 \) is a non-degenerate random walk on \( \mathbb{Z}^3 \), hence this random walk is transient. A well-known argument from [32] implies that the value of the lamp at 0 stabilizes to a non-trivial limit along the trajectories of the random walk \((G, \mu')\), that the boundary of \((G, \mu')\) is thus non-trivial and also that \( f = \delta \) acts non-trivially on this boundary.
Remark 3.3. The assumption of Theorem 3.1 that the support of $\mu_2$ generates $G$ as a semi-group (the measure is non-degenerate) can not be weakened to ask that it generates $G$ as a group. Consider two central extensions, one of which is a direct product:

$$1 \to C \to G \to H \to 1,$$

$$1 \to C \to C \times H \to H \to 1.$$  

For any measure $\mu_2$ on $C \times H$ its Poisson boundary is equal to the boundary of its projection to $H$, and any element of $C$ acts trivially on this boundary. However, there exist central extensions and (not non-degenerate) measures on them where elements of the center act non-trivially. Such examples can be chosen among Hall’s group in such a way that the center is infinite [20][Cor 5.14]. For earlier non-discrete examples with finite center see [24], $SL(2, \mathbb{R})$ example discussed after Thm 5.3 and remarks after [20][Cor 5.14].

As a consequence of the above comparison criterion for actions of an element we get the following corollary relating the triviality of the boundary between two different measures:

**Corollary 3.4 (Comparison criterion: triviality of the boundary).** Let $1 \to F \to G \to H \to 1$ and $1 \to F \to G' \to H \to 1$ be short exact sequences with $F$ Abelian, and suppose that the actions of $H$ on $F$ for both sequences are the same. Let $\mu_1$ and $\mu_2$ be finite entropy measures on $G$, $G'$ which have the same projections to $H$, and where $\mu_2$ is non-degenerate. Then if the Poisson boundary of $(G, \mu_1)$ is non-trivial, the boundary of $(G, \mu_2)$ is also non-trivial.

**Proof.** To see that Corollary 3.4 follows from Theorem 3.1 observe that if the projection of $\mu_1$ to $H$ has non-trivial Poisson boundary, then the same is true for the projection of $\mu_2$ (this is the same projection), and then clearly the boundary of $(G, \mu_2)$ is non-trivial. It is sufficient therefore to consider the case when the random walk on $H$, defined by the projection of $\mu_1$ (and of $\mu_2$) has trivial boundary. In this case note that the Poisson boundary of $\mu_i$, $i = 1$ or 2, is non-trivial if and only if there exists an element of $F$ that acts non-trivially on the boundary. (Indeed, if all elements of $F$ act trivially, observe that a non-constant harmonic function on $(G, \mu)$ is $F$-invariant and induces a non-constant harmonic function on $H$). Hence the claim of the corollary follows from the claim of the Theorem.  

It seems natural to ask whether one can replace the assumption that the group is Abelian by the assumption that this group is hyper-FC central.

**Remark 3.5.** Assume that $F$ is not hyper FC central and that $F$ admits finite entropy non-degenerate measures with trivial boundary. Then the statement of Corollary 3.4 is not true for the following extension:

$$1 \to F \to F \to 1 \to 1.$$  

In this case the assumption that $\mu$ and $\mu'$ have the same projection to the trivial group is obviously verified. But the claim of the Corollary is not true. Indeed, by a result of [21] we know that any not hyper FC central group admits a finite entropy measure with non-trivial boundary.

We also mention that any amenable group admits a non-degenerate measure with trivial boundary (Rosenblatt [32, 51]). This measure can not, however, necessarily be chosen to have finite entropy.
Remark 3.6. Let $\mu$ be a measure on a countable group $G$. Consider $\mu' = \alpha \mu + \beta \chi_e$, $\alpha + \beta = 1$, $\alpha > 0$. Then the Poisson boundary of $(G, \mu)$ is trivial if and only if the Poisson boundary of $(G, \mu')$ is trivial. Moreover, $g \in G$ acts trivially on the boundary of $(G, \mu)$ if and only if $g$ acts trivially on the boundary of $(G, \mu')$.

This follows by noting that the space of harmonic functions for the two measures is the same.

Remark 3.7. We consider a lamplighter group $G = \mathbb{Z}^d \rtimes \mathbb{Z}/2\mathbb{Z}$. Consider a finite entropy non-degenerate measure $\mu$ on $G$, $\mu(e) \neq 0$. Then there exists $\mu'$ on $G$, such that $\mu'$ is supported on the union of $\mathbb{Z}^d$ and $\delta$ and which has the same projection to $\mathbb{Z}^d$ as $\mu$ and such that the following holds. For any such $\mu'$ the action of $\delta$ on the boundary of $(G, \mu)$ is non-trivial if and only the action of $\delta$ on the boundary of $(G, \mu')$ is non-trivial. In particular, $\delta$ acts non-trivially if and only if the projection to the base group is transient.

This provides us in particular a new way to see why all finite entropy non-degenerate random walks on $G = \mathbb{Z}^d \rtimes \mathbb{Z}/2\mathbb{Z}, d \geq 3$ have non-trivial boundary ([16]).

Proof. We consider $\mu'$ with the same projection to $\mathbb{Z}^3$ and with the support as we claim. By comparison criterion the action of $\delta$ is either trivial in both cases or non-trivial. Finally, observe that for $\mu'$ the original argument of Kaimanovich and Vershik can be used to show that the action of $\delta$ is non-trivial whenever the projected random walk is transient and another argument from [32] shows triviality of the boundary in the case of recurrent projection.

3.1. Proof of Theorem 3.1. We start with a corollary of the relative entropy criterion of Kaimanovich. Given $f \in G$, $\text{Norm}_G(f)$ denotes the (normal) subgroup of $G$, generated as a normal subgroup by $f$. If $G$ is clear from context, then we omit $G$ in the notation and write $\text{Norm}(f)$.

Lemma 3.8. Let $G$ be a countable group, and $\mu$ be a probability measure on $G$. Then $f \in G$ acts trivially on the Poisson boundary of $(G, \mu)$ if and only if

$$h(G, \mu) = h(G/\text{Norm}_G(f), \mu),$$

where we use the same notation $\mu$ for its image in the quotient group $G/\text{Norm}_G(f)$.

Proof. Observe that the quotient map from the boundary of $(G, \mu)$ to the boundary of $(G/\text{Norm}_G(f), \mu)$ is an isomorphism if and only if the asymptotic entropies are equal. Indeed, the entropy of $(G, \mu)$ is equal to the asymptotic entropy of $(G/\text{Norm}(f), \mu)$ plus the conditional entropy, conditioned on the convergence to a point of the boundary of the quotient group. The claim thus follows from the conditional entropy criterion [37] (cited at the end of Section 2).

Now we start proving the theorem. We assume that $f \in F$ acts non-trivially on the boundary of $(G, \mu)$. We want to prove that it acts non-trivially on the boundary of $(G', \mu')$.

First observe that Lemma 3.8 implies

Corollary 3.9. Suppose $f$ acts non-trivially on the Poisson boundary of $(G, \mu)$, where $\mu$ has finite entropy. Then for large enough $N$ it holds

$$H(\mu^N, G/\text{Norm}(f)) < Nh(G, \mu)$$
Proof. If $N$ is large enough, then $H(\mu^\ast N, G/\text{Norm}(f))/N$ is close to $h(G/\text{Norm}_G(f), \mu)$, and by Lemma 3.8 we know that $h(G/\text{Norm}_G(f), \mu) < h(G, \mu)$. □

To prove the Theorem, we will apply corollary above for $\mu_1$ and we need to show that the same inequality holds for some $N$ for our second measure $\mu_2$. This will be done by studying the $\Delta$-restriction entropy, defined below.

Definition 3.10 ($\Delta$-restriction entropy). Given a group $G$, a probability measure $\mu$ on $G$ and a finite set $\Delta \subseteq \text{supp}(\mu)$, we define the $\Delta$-restriction entropy $H_\Delta(n)$ as follows. We consider an $n$-step trajectory of $(G, \mu)$. Then we take the conditional entropy of $X_n$, conditioned on prescribing all increments except those that are in $\Delta$.

For brevity, we will often say $\Delta$-entropy instead of $\Delta$-restriction entropy. We note that a special case of $\Delta$-entropy was used in [16] to provide lower estimates of the asymptotic entropy for wreath products and 3-dimensional Baumslag groups.

For our applications, we will be interested in $\Delta$ such that its elements belong to the same coset $G/F$.

If $\mu$ is some measure on a countable space, not necessarily a probability measure, we can also speak about the entropy of $\mu$ defined by
\[
H(\mu) = \sum x \cdot \mu(x) \ln \mu(x).
\]

Lemma 3.11. For any group $G$, any probability measure $\mu$, and any finite $\Delta \subseteq \text{supp}(\mu)$ the function $H_\Delta(n)$ is subadditive:
\[
H_\Delta(n + m) \leq H_\Delta(n) + H_\Delta(m).
\]

In particular, there is a limit $H_\Delta(n)/n$ and if this function $\geq Cn$ for some positive $C$ and an infinite subsequence of $n$, then it holds for all $n$.

Proof. Indeed, conditioning for $n$ first increments and for $m$ last increments is the same as our conditioning for $n + m$ increments. Observe also that conditional entropy of the convolution of two measures is not greater than the sum of the conditional entropies (all conditioned on the same event). The last two claims then follow from Fekete’s lemma. □

Claim 3.12. Let $\mu$ be a finite entropy probability measure on a group $G$, and $\Delta$ is a subset of the support of $\mu$. Then the $\Delta$-entropy of the random walk $(G, \mu)$ is at most the entropy of the restriction of $\mu$ to $\Delta$.

Proof. Observe that $H_\Delta(1)$ is equal to $H(\nu)$. By subadditivity of $\Delta$-entropy the claim follows. □

Lemma 3.13. Consider a probability measure $\mu$ on a group $G$, and a finite set $\Delta = \{g_1, \ldots, g_k\} \subseteq \text{supp}(\mu)$. Consider $f \in G$ and assume that the normal subgroup $\text{Norm}(f)$ generated by $f$ is Abelian. We also assume that all the $g_i$ have the same projection to $G/\text{Norm}(f)$. If the $\Delta$-restriction entropy is linear, then $f$ acts nontrivially on the Poisson boundary of $(G, \mu)$.

Proof. To prove this lemma, observe that the $\Delta$-restriction entropy gives a lower bound for the conditional entropy of $(G, \mu)$, conditioned on the quotient in $G/\text{Norm}(f)$. Indeed, the conditional information of our $n$ step position from the definition of $\Delta$-restriction entropy already entirely determines the projections to $G/\text{Norm}(g)$. In other words we are conditioning on more information in the definition of $\Delta$-restriction entropy than on the
full $n$-step trajectory on the quotient group $G/\text{Norm}(f)$ which is in turn more information than the position at step $n$ of the induced random walk on $G/\text{Norm}(f)$. We can conclude that the asymptotic entropy of $(G, \mu)$ is strictly greater than the asymptotic entropy of the induced random walk $G/\text{Norm}(f)$. Hence we can use Lemma 3.8 and claim that $f$ acts non-trivially on the boundary of $(G, \mu)$.

Lemma 3.16 below shows that, under appropriate assumptions, the converse of the statement in Lemma 3.13 is true.

First we make two straightforward remarks about Abelian normal subgroups, which we will use several times in the sequel.

**Claim 3.14.** If $F$ is an Abelian normal subgroup of $G$, $f_i \in F$, then
\[ g_1 f_1 g_2 f_2 \cdots g_k f_k = g_1 g_2 \cdots g_k f'_1 \cdots f'_k \]
where $f'_i$ is the conjugate of $f_i$ by $g_i+1 g_i+2 \cdots g_k$. That is $f'_i = g_k^{-1} g_{k-1}^{-1} \cdots g_{i+1}^{-1} g_i g_{i+1} \cdots g_k$

This also implies the following.

**Claim 3.15.** Consider a partition of $S_1, \ldots, S_t$ of the elements $1, 2, \ldots, k$. Then
\[ g_1 f_1 g_2 f_2 \cdots g_k f_k = g_1 g_2 \cdots g_k f''_1 f''_2 \cdots f''_t \]
where $f''_j$ is a product of conjugates of $f_j$, $j \in S_t$.

We now give a formula evaluating the $\Delta$-restriction entropy as the entropy of a random variable taking values in $F$.

**Lemma 3.16.** ([Main Lemma for comparison criterion])

Consider an extension
\[ 1 \to F \to G \to K \to 1. \]
Let $f \in F$, $F$ be Abelian. Let $\mu$ be a finite entropy probability measure on $G$ satisfying $h(\mu) > H(G/\text{Norm}(f), \mu)$. (In other words, the asymptotic entropy of $(G, \mu)$ is strictly larger than the (usual) entropy of the of the projection of $\mu$ to the normal subgroup generated by $f$). Then there exists a finite set $\Delta = \{g_1, \ldots, g_k\}$, such that $g_i$ have the same projection to $G/\text{Norm}(f)$ and such that $\Delta$-entropy of $(G, \mu) \geq C n$, for some positive constant $C$.

For the proof of Lemma 3.16 above we start with the following observation:

**Lemma 3.17.** Let $\mu$ be a probability measure of $G$ of finite entropy. Let $F$ be an Abelian normal subgroup of $G$. Then
\[ H(X_n) = H(\mu^{*n}) \leq n H_{\mu|G/F} + \sum_{\alpha} H_{\Delta_{\alpha}}(n), \]
where $\Delta_{\alpha}$ are subsets in the partition of elements in the support of $\mu$ to their projection to $G/F$.

**Proof.** For each $\alpha$, fix an element $g_\alpha \in \Delta_i$. Consider the elements $f_{\alpha,j}$, where $j$ takes values from 1 to $|\Delta_i|$, such that the elements of $\Delta_{\alpha}$ are $g_\alpha f_{\alpha,j}$. When we write $X_n$ as a product of increments of the form $g_\alpha f_{\alpha,j}$, we see a product as in the left hand side in the Claim 3.15.

So we use this Claim, and write
\[ X_n = U_n \prod_{j=1}^n W_\alpha^n \]
where $U_n$ is the product of the corresponding $g_\alpha$. Here instead of increments in $\Delta_{\alpha}$ we write $g_\alpha$, and $W_\alpha^n$ is a product of conjugates of $f_{\alpha,j}$'s with the same value of $\alpha$. 
We denote by $Y_i$ increments of the trajectory $X_n$.

For fixed $U_n$, we have that

$$H(X_n | U_1, \ldots, U_n) \leq \sum_{\alpha} H(W_{\alpha}^n | U_1, \ldots, U_n)$$

Note that $H_{W_{\alpha}^n | U_1, \ldots, U_n} = H_{\Delta_\alpha}(X_n)$.

If we fixed the conditioning from the definition of $\Delta_\alpha$-entropy, then we have fixed $U_1, U_2, \ldots, U_n$, and the distribution we get is isomorphic to the distribution of $W_{\alpha,n}$, conditioned on $U_1, U_2, \ldots, U_n$. Indeed, the conjugates of $f_{\alpha,j}$ are defined by $f_{\alpha,j}$ and $U_1, U_2, \ldots, U_n$ (the conjugate $xy^{-1}$ of $x \in F$ only depends on the coset of $y$ in $G/F$ and the value of $x$).

Now note that the entropy of the sequence $U_1, U_2, \ldots, U_n$ is equal to $nH(\mu, G/F)$. Thus adding this equality to (1) and substituting the second half with $H_{W_{\alpha}^n | U_1, \ldots, U_n} = H_{\Delta_\alpha}(X_n)$ yields the desired inequality.

Now we prove Lemma 3.16.

**Proof.** We observe that Lemma 3.17 above and the assumption that $h(\mu, G) > H(\mu, G/F)$ implies that $\sum_{\alpha} H_{\Delta_\alpha}(n)$ grows linearly in $n$. Therefore, if the index set is finite (and this the case when projection of $\mu$ to $G/F$ is finitely supported), then there exists $\alpha$ such that, for some subsequence of $n$, $\Sigma_n$ grows linearly. Here we use the notation $\Sigma_n$ for the $\Delta_\alpha$-entropy, for the set $\Delta_\alpha$ described in Lemma 3.17. So we see that $\Delta_\alpha$ entropy is linear on some subsequence. By Lemma 3.11 $\Delta_\alpha$ entropy is linear.

Now, even if the projection is infinitely supported, we argue as follows. We write $\mu = \mu' + \mu''$, where $\mu''$ has small entropy. By Lemma 3.17

$$\sum_{\alpha} H_{\Delta_\alpha}(X_n)$$

is linear.

And, using Lemma 3.12, we claim that given $\varepsilon$, we can choose a co-finite set $A_\varepsilon$, such that the sum

$$\sum_{\alpha \in A_\varepsilon} H_{\Delta_\alpha}(X_n) \leq \varepsilon n.$$  

Therefore, we can observe that $\sum_{\alpha \notin A_\varepsilon} H_{\Delta_\alpha}(X_n)$ is linear. And thus for some $\alpha \notin A_\varepsilon$, $\Delta_\alpha(X_n)$ entropy is linear on some subsequence. Hence, again by Lemma 3.11 $\Delta_\alpha$ entropy is linear.

Now we have found $\Delta \subset \text{supp} \mu$ such that the $\Delta$-entropy is linear. Observe that we can moreover claim the existence of a finite set $\Delta$ with this property. Indeed, we write $\Delta = \Delta' \cup \Delta''$, such that $\Delta'$ is finite, and the entropy of the non-normalised restriction $\mu''$ of $\mu$ on $\Delta''$ is smaller than $\varepsilon$. Using again Lemma 3.12 we observe that

$$H_{\Delta}(X_n) \leq H_{\Delta'}(X_n) + n \cdot H(\mu'').$$

Therefore, if $\varepsilon$ is small enough, we conclude that $\Delta'$ entropy grows linearly. □

Before we continue the proof of the theorem, we discuss the concept of colored increments.

**Definition 3.18** (Coloured $\Delta$ restriction entropy). Given a subset $S$ of $G$ and $p$, $0 < p < 1$ we say that the coloured $\Delta, p$ restriction entropy is the conditional entropy of $X_n$ conditioned on the following event. For each $n \in \mathbb{N}$ we choose independently and identically at random a Bernoulli random variable which is 0 with probability $1 - p$ and 1 with probability $p$. We condition on all increments where the Bernoulli random variable is 0 and on all increments which are not in $\Delta$.

\[ H \]
We are specifically interested in the case when we have several elements \( \Delta = (g_1, \ldots, g_m) \), which have the same projection to \( G/F \), where \( F \) is in an Abelian normal subgroup of \( G \), and we will usually take \( p = 1/k, k \in \mathbb{Z} \). In this case we can informally think about this as conditioning on all increments except those of one of \( k \) possible colors (as well as all increments not in \( \Delta \)). Our main interest is to understand whether the \( \Delta \), \( p \)-entropy is linear.

**Remark 3.19.** The definition of coloured \( \Delta \)-entropy and the question of its linearity can be reduced to the question about linearity of the usual, not colored, \( \Delta \)-entropy for a random walk on the product of \( G \) with \( \mathbb{Z} \), or some other groups with Liouville property.

**Lemma 3.20** (Changing charges for \( \Delta \)-entropy). Let \( F \) be an Abelian normal subgroup of \( G, f \in F \). Let \( \mu \) and \( \mu' \) be probability measures on \( G \), and let a finite subset \( \Delta \) of \( G \) belong to the support of both of them. We also assume that \( \Delta \) belongs to the same coset \( G/F \), and that \( \mu \) and \( \mu' \) have the same projection to \( G/F \).

If the \( \Delta \)-entropy of \( \mu \) is linear, then the \( \Delta \)-entropy of \( \mu' \) is also linear.

**Proof.** Choose a large enough \( k \) such that for each \( g \in \Delta \)

\[
\mu(g)/k < \mu'(g).
\]

Consider the coloring of \( \Delta \) increments for the random walk \((G, \mu)\) in \( k \) colors. Fix some element \( g \in \Delta \) and write elements of \( \Delta \) as \( gf_j \). We know that \( f_j \in F \).

Use Claim 3.14 to rewrite the product of \( \) increments, putting all \( f_j \)'s on the right. We get a product \( \mathbb{Z}_n \) multiplied by conjugations of \( f_j \). More precisely we use Claim 3.15 to group the conjugations of \( f_j \)'s "of a given color" together (for each fixed \( j \)). For a color \( i \) we denote by \( \Sigma^n_{i,j} \) the product of conjugates of \( f_j \) of color \( i \). Then the product of \( n \) increments is equal to

\[
Z_n \Pi_{1 \leq i \leq k} \Sigma^n_{i,j}
\]

It follows from the definition that the \( \Delta \)-entropy of \( \mu \) is the entropy of \( \Pi_{1 \leq i \leq k} \Sigma^n_{i,j} \). This is at most

\[
\Sigma_{1 \leq i \leq k} H(\Sigma^n_{i,j})
\]

Observe that the distribution of \( \Sigma^r_{i,i} \) does not depend on \( i \) (since all colours have the same role), and in particular the entropy of \( \Sigma^r_{i,j} \) is equal to that of \( \Sigma^r_{j,i} \) for all \( i, l : 1 \leq i, l \leq k \). Thus \( \Delta \)-entropy is at most

\[
k \Sigma_{j=1} H(\Sigma^r_{j,1})
\]

Since the \( \Delta \)-entropy of \( \mu \) is linear in \( n \), we conclude that there exists \( j \) such that \( H(\Sigma^r_{j,1}) \geq C/kn \), for \( C \) not depending on \( n \) (for some subsequence of \( n \)'s and hence by subadditivity of coloured \( \Delta \)-entropy, for all \( n \)).

Let us call the first color (among \( k \) colors) "white". So we know that the \( \Delta \)-entropy of the white color is linear.

Now we look at our random walk \((G, \mu')\) and we want to claim that the \( \Delta \)-entropy of the random walk \((G, \mu')\) is bounded from below by the white \( \Delta \)-entropy of \((G, \mu)\).

We color increments of the random walk \((G, \mu')\) in two colors, white and black. Each increment \( g \in \Delta \subset G' \) we colour in white with probability \( \mu(g)/k \) and in black with probability \( -\mu'(g - \mu(g)/k) \).

We consider any coupling between trajectories \((G, \mu)\) and \((G, \mu')\), where corresponding increments have the same projection on \( G/F \) and white increments in \( G, \mu \) are sent to white increments in \((G, \mu')\) with the same value in \( \Delta \).

Observe that the \( \Delta \)-entropy after \( n \) steps of \((G, \mu')\) is greater or equal to the white entropy of this random walk. (Since the white entropy is conditioned on strictly more information).
Finally, observe that the white $\Delta$ entropy of $(G, \mu)$ is equal to the white $\Delta$-entropy of $(G, \mu')$. By our assumption the projections onto $G/F$ are equal for coupled trajectories. Hence we use the fact that for our coupling both increments at some time $i$ (for $(G, \mu)$ and $(G, \mu')$ trajectories), are the same mod $F$, the fact that $F$ is Abelian and we use Claim \ref{3.15}. Indeed, when we bring $\Delta$-increments of white colour on the right using this remark, observe that the product on the right, corresponding to conjugations of these white elements, depends only on the value of these increments and projection to $G/F$ of the corresponding position of the random walk. 

\begin{corollary}
If $G$ and $G'$ are extensions of an Abelian group $F$ where $G/F$, $G'/F'$ are isomorphic and they induce isomorphic conjugation actions on $F$ and $F'$. Then if measures $\mu$ on $G$ and $\mu'$ on $G'$ have the same projection on $K = G/F$, $\Delta \in \text{supp } \mu$ and $\Delta' \in \text{supp } \mu'$, $\Delta = g\Omega$, $\Delta' \subset g'\Omega$ where $g$ has the same projection to $K$ as $g'$, and $\Omega$ is some subset of $F$. Then the $\Delta$-entropy of $(G, \mu)$ is linear if and only if the $\Delta'$ entropy of $(G', \mu')$ is linear.
\end{corollary}

\begin{proof}
Indeed, this follows from Lemma \ref{3.20} in view of the following observation. Under the assumption of our corollary assume additionally that $\mu(gw) = \mu(g'w)$ for any $w \in \Omega$.

Observe that in this case the $\Delta$-entropy of $(G, \mu)$ is equal to the $\Delta'$ entropy of $(G', \mu')$, as follows from \ref{3.14}. Knowing this, we can then change the values of $\mu(h)$, $h \in \Delta$ using Lemma \ref{3.20}.
\end{proof}

Now we are ready to prove Theorem \ref{3.1}. Since $\mu'$ is non-degenerate by the assumption of the Theorem, replacing the measures if necessary by affine combinations of their convolution powers, $\mu$ by $\sum_{i=1}^{\infty} a_i \mu^{*i}$ and analogously $\mu'$ by $\sum_{i=1}^{\infty} a_i \mu'^{*i}$, we can assume that $\text{supp } \mu' = G'$. (Here we used the easy and well-known fact that the Poisson boundary is the same for $\mu$ and for a convex combination of its convolution powers \cite{30}.)

We assume that $f$ acts non-trivially on the boundary of $(G, \mu)$. If necessary, we choose some $m$, replace $\mu$ by $\mu^{*m}$ and $\mu'$ by $\mu'^{*m}$ so that we can assume

\begin{equation}
\begin{aligned}
h(G, \mu) &> H(G/\text{Norm}(f), \mu).
\end{aligned}
\end{equation}

In other words, we can assume that $\mu$ satisfies the technical assumption of Lemma \ref{3.16} (See Corollary \ref{3.9}). By Lemma \ref{3.16} we know therefore that for some $\Delta \subset G$, $\Delta$ belonging to a coset of $G/F$, the $\Delta$-entropy is linear. Write $\Delta = g\Omega$, $\Omega \subset F$. Consider $\Delta' = g'\Omega$, $g$ has the same projection on $K = G'/F'$ as the projection of $g'$ to $K = G/F$.

Since the support of $\mu'$ is $G'$, we know that $\Delta'$ belongs to the support of $\mu'$. We now use Corollary \ref{3.21} to claim that $\Delta'$ entropy of $(G', \mu')$ is linear. Then we can use Lemma \ref{3.13} to claim that $f'$ acts non-trivially on the boundary of $(G', \mu')$. Since $f'$ belongs to the normal subgroup generated by $f$ we can conclude that $f$ acts non-trivially on the boundary of $(G', \mu')$, and this concludes the proof of the comparison criterion.

\section{Nilpotent-by-Abelian Groups}

In this section we study nilpotent-by-Abelian groups. Given a nilpotent-by-Abelian group $G$, we associate certain metabelian groups, which we call metabelian components of $G$. We will first check that triviality of the boundary is equivalent to triviality of the boundaries (for appropriate measures) on all metabelian components of $G$. Then for each metabelian component we define $p$-primary metabelian components, and reduce the problem to that of the corresponding measures on $(p)$-by-Abelian components as well as on the (torsion free)-by-Abelian component. Torsion components can be in turn reduced to the case when the group is an extension of $p$-primary group by Abelian ones). Finally,
both the order \( p \) extensions and the torsion free extension case are reduced to associated blocks (some groups we associate to each such extension).

**Claim 4.1.** Given \( G = A \times B \), and a measure \( \mu \) on \( G \) of finite entropy. The mapping from the boundary of \( G \) to the product of boundaries of \( A \) and \( B \) is injective.

**Proof.** By our assumption \( H(\mu) \) is finite. Let \( P(A) \) be the Poisson boundary of the projected random to \( A \) ((\( A, \mu \)) and \( P(B) \) the Poisson boundary of the projection to \( B \). By the conditional entropy criterion of Kaimanovich, we know that the conditional entropy of \((A, \mu)\) conditioned on its boundary value along infinite trajectory, is zero. And the same about \((B, \mu)\) and \( P(B) \). Now consider conditional entropy of \((A \times B, \mu)\) conditioned on the limiting values in \( P(A) \) and \( P(B) \). We recall that entropy of the measure on a product of two spaces \( X \) and \( Y \) is at most the sum on the entropies of the projection to \( X \) and to \( Y \). Applying this to spaces with probability measures: the space \( X \) of trajectories of \((A, \mu)\) and the space \( Y \) of trajectories of \((B, \mu)\), we conclude that conditional entropy is at most \( 0 + 0 = 0 \).

\[ \square \]

**Corollary 4.2.** Let \( G = A \times B, \mu \) on \( G \) is a probability measure. Let \( g = (g_a, g_b), g \in G, g_a \in A, g_b \in B \). Then \( g \) acts non-trivially on the Poisson boundary of \((G, \mu)\) if and only if at least one of \( g_a \) or \( g_b \) acts non-trivially on the Poisson boundary of \((A, \mu)\) or \((B, \mu)\) respectively.

**Proof.** First observe that if \( g_a \) acts non-trivially on the boundary of \((A, \mu)\), then \( g \) has the same action on this boundary which is a quotient of the Poisson boundary of \( G \).

Now we prove the other direction. We know that the Poisson boundary of \((G, \mu)\) injects into the Poisson boundary of \((A, \mu)\) times the Poisson boundary of \((B, \mu)\). Observe that \( g = (g_a, e)(e, g_b) \) acts non-trivially on the boundary of \((G, \mu)\), then either \((g_a, e)\) or \((g_b, e)\) acts non-trivially on the boundary of \( G, \mu \). Since \( P(G) \) injects into \( P(A) \times P(B) \), if for example \((g_a, e)\) acts non-trivially on the boundary, then it acts non trivially on \( P(A) \times P(B) \), and it is clear that the action of such elements on \( P(B) \) is trivial. Therefore, under this assumption \((g_a, e)\) acts non-trivially on \( P(A) \times \{e\} \), and thus \( g_a \) acts non-trivially on \( P(A) \).

\[ \square \]

In our reduction procedure in this section we will use several times a corollary from the Comparison Criterion (Thm 3.1), formulated in Lemma 4.3 below.

**Lemma 4.3.** [Reduction Lemma] Let \( G \) be a group, \( H \) be a normal subgroup in \( G \) and \( A \) be a normal subgroup of \( G \) in the center of \( H \). Consider a short exact sequence \( 1 \to H \to G \to K \to 1 \). Let \( C \) be the semi-direct product of \( K \) with \( A \), with action being the conjugation action of \( G \) on \( A \). Consider a non-degenerate finite entropy probability measure \( \mu \) on \( G \). Let \( \nu \) be a non-degenerate finite entropy measure on \( C \) with the same projection to \( K \) as \( \mu \). Then \( a \) acts non-trivially on the boundary of \((C, \nu)\) if and only if it acts non-trivially on the boundary of \((G, \mu)\).

**Proof.** Let \( \phi_1 \) be the natural map from \( C \to K \) and \( \phi_2 \) be the natural map from \( G/A \to G/H = K \). Consider the subgroup \( W \) of \( C \times G/A \) of elements \((c, x)\) such that \( \phi_1(c) = \phi_2(x) \). Note that there is a natural injection from \( A \to W \) coming from the natural injection from \( A \to C \) on the first coordinate and trivial on the second coordinate.

\[ 1 \to A \to W \to U \subseteq K \times G/A \to 1, \]
where $U = G/A$ consists of $(k, y)$, $y \in G/A$ such that projection of $y$ to $K$ is $k$. and we have

(3) \quad 1 \to A \to G \to G/A \to 1

and

(4) \quad 1 \to A \to W \to G/A \to 1

Consider a non-degenerate measure $\rho$ of finite entropy on $W$ which has the same projection to $G$ (coming from the projection to the first coordinate of $U$) as $\mu$ (and thus as $\nu$). We apply the comparison criterion (Thm 3.1) to $p_{W,\rho}$ and $p_{G,\mu}$ (with respect to the short exact sequences in Equation 3 and 4). We conclude that an action of $a \in P$ on the boundary of $p_{G,\mu}$ is non-trivial if and only if its action on the boundary of $p_{W,\rho}$ is non-trivial. We know that $W$ is a subgroup of $\hat{G}$, and we can consider measure $\rho$ as a measure on $\hat{G}$. Now we apply Corollary 4.2 to $p_{\hat{G},\rho}$, and claim that $a$ acts non-trivially on the boundary of $p_{\hat{G},\rho}$ if and only either $a$ acts non-trivially on the quotient random walk on $G$ or on the boundary of the projected random walk on $G/A$. Clearly the second option can not happen, so we know that $a$ acts non-trivially on the boundary of projection of $\rho$ to $G$. Now we apply once more comparison criterion, this time for $p_{\hat{G},\rho}$ and $p_{\hat{G},\nu}$. We conclude that of $a$ acts non-trivially on the boundary of $p_{\hat{G},\rho}$ if $a$ acts non-trivially on the boundary of $p_{\hat{G},\nu}$.

For our reduction procedure, we will also use several times the following not difficult property of actions on Poisson boundary:

**Lemma 4.4.** Let $G$ be a group. Let $\mu$ be a finite entropy measure on $G$. Let $g \in G$. Let $M$ be a normal subgroup of $G$. Then $g$ acts non-trivially on the boundary of $(G, \mu)$ if and only if at least one of the following two conditions hold

1. Some element of $M \cap \text{Norm}_{G}(g)$ acts non-trivially on the boundary of $(G, \mu)$.
2. $g$ acts non-trivially on the boundary of $(G/M, \mu)$.

**Proof.** First observe that if (1) or (2) holds, then $g$ acts non-trivially on the boundary of $(G, \mu)$. Indeed, if (2) holds, then if an element acts non-trivially on the boundary of the quotient group, it also acts non-trivially on the boundary of our group. It is also straightforward that if (1) holds, then there exists some element in $\text{Norm}_{G}(g)$ that acts non-trivially, hence $g$ acts non-trivially.

Consider the map from $G \to G/M \times G/\text{Norm}_{G}(g)$. If (1) does not hold, then all elements of $M \cup \text{Norm}_{G}(g)$ act trivially, and since $M \cup \text{Norm}_{G}(g)$ is the kernel of the map above, we can conclude, using Corollary 4.2, that if $g$ acts non-trivially on the boundary of $(G, \mu)$ then $g$ either acts non-trivially on the projected r.w. on $G/M$ or on $G/\text{Norm}_{G}(g)$. The last possibility clearly can not happen, so in this case $g$ acts non-trivially on the boundary of $G/M$, and thus (2) holds.

4.1. **Reduction to Abelian-by-$K$ components.** Consider a nilpotent-by-Abelian group $G$ and a short exact sequence

$$1 \to N \to G \to A \to 1,$$

where $N$ is nilpotent of degree $d$. We will canonically associate to $G$ certain metabelian groups.
Definition 4.5 (Abelian-by-$K$ components).  

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

$N$ is hyper-central (for example nilpotent). We define (upper) Abelian-by-$K$ components of $G$ in the following way. We consider the canonical central series of $N$, $C_{\alpha+1}$ is the center of $N/C_{\alpha}$. Observe that $C_{\alpha}$ is a characteristic subgroup, for all $\alpha$. We consider quotient $C_{\alpha+1}/C_{\alpha}$ and a short exact sequence

$$1 \rightarrow N/C_{\alpha} \rightarrow G/C_{\alpha} \rightarrow K \rightarrow 1$$

which induces an action of $K$ by conjugation $C_{\alpha+1}/C_{\alpha}$. We consider the semi-direct product, and call it $M_{\alpha}$.

When we have a probability measure $\mu$ on $G$, we associate to it (any) measure on $M_{\alpha}$ with the same projection to $K$. If $\mu$ is irreducible (or non-degenerate), then the projection to $K$ is clearly irreducible (respectively non-degenerate), and we can therefore choose an associated measure to be irreducible (or non-degenerate).

Note that if $K$ is Abelian, then Abelian-by-$K$ components are metabelian. Observe also that if $H$ is nilpotent, we have associated in this case finitely many metabelian components.

Lemma 4.6 (Reduction to Abelian-by-$K$ components). Let $G$ be such that

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1,$$

$H$ is hypercentral and let and $M^\alpha$ be components of $G$ corresponding to this short exact sequence. Let $\mu$ be a finite entropy irreducible measure on $G$. Choose some finite entropy irreducible measure $\mu^\alpha$ on $M^\alpha$ which has the same projection to $K$ as $\mu$. Consider $g \in H$. Let $\alpha$ be the smallest ordinal such that $g \in M^\alpha$. $g$ acts trivially on the Poisson boundary of $(G, \mu)$ if and only if the two following conditions hold.

1. Every element $h$ of the normal subgroups in $G$ (generated by $g$) $h \in \text{Norm}(g)$ which is also contained in $N_{\beta}$ where $\beta < \alpha$ acts trivially on the Poisson boundary of $(M^\beta, \mu^\beta)$.
2. $g$ acts trivially on the Poisson boundary of $(M^\alpha, \mu^\alpha)$.

In particular, if the boundary of the projection to $K$ is trivial, then the Poisson boundary of $(G, \mu)$ is trivial if and only if that of $(M^\alpha, \mu^\alpha)$ is trivial for all $\alpha$.

Proof. First apply Lemma 4.4 for $G = G$, $g = g M = C_{\alpha-1}$ where we suppose that $g \in C_{\alpha}$, and $\alpha$ is minimal with this property). Thus we see that $g$ acts non-trivially if and only if condition (1) fails or $g$ acts non-trivially on $G/C_{\alpha-1}$. Now applying Lemma 4.3 to $G' = G/C_{\alpha-1}$, $H' = N/C_{\alpha-1}$ and $A' = C_{\alpha}/C_{\alpha-1}$ implies that $g$ acting non-trivially on $G/C_{\alpha-1}$ is the same as acting nontrivially on $M^\alpha$.

Note that in particular case when $G$ is hypercentral-by-Abelian, we have reduced the question of non-triviality of the action of elements of $N$ on the boundary $(G, \mu)$ to the question about random walks on corresponding metabelian components. And in a general case ($g$ is not necessarily in $N$) the reduction can be done using Lemma 4.4.

Remark 4.7. In the case when $N$ is nilpotent, we can analogously define lower Abelian-by-$K$ components (metabelian if $K$ is Abelian). And analogously the non-triviality of the action of an element on the boundary of $(G, \mu)$ can be reduced to the same question about random walks on lower components.
**Remark 4.8.** Metabelian components $M_i$ (as well as upper metabelian components $M^i$) are not necessarily finitely generated. Consider for example $N$ to be a free two step nilpotent group on a countable set $n_i$, $i \in \mathbb{Z}$ and $A = \mathbb{Z}$ acts by shift of the index on the set $n_i$. Let $G$ be the corresponding extension. Then $G$ is finitely generated (and $N$ is finitely generated as a normal subgroup in $G$). But $[N, N]$ is freely generated by $[n_i, n_j], i > j$. And the action by $A$ preserves the difference $j - i$, so all $[n_i, n_0], i \in \mathbb{Z}$ lie on distinct orbits and are linearly independent in $G$. So that the extension of $[N, N]$ by $A$ is not a finitely generated subgroup.

**Remark 4.9.** As we mentioned if $G$ is finitely generated, $M_i$ (and $M^i$) are not necessarily finitely generated. Thus we start with a finitely supported measure on (nilpotent-by-Abelian) $G$, to use the lemma above we need to consider infinitely generated measures on $M_i$ (or $M^i$). There are other known situation where the understanding of a Liouville property of a group $G$ one makes a reduction to some not finitely generated groups (and not finitely supported measures). As for example the argument of Kaimanovich and Vershik, that uses exit measure to some Abelian group (e.g. $\mathbb{Z} \wr A$ or more generally $B \wr A$, $G$ is virtually $\mathbb{Z}^2$ or $\mathbb{Z}$. We also mention that Furstenberg-Lyons-Sullivan approximations [34][40], that reduces a question of the boundary on the covering, in particular for universal covers of compact manifolds, to a measure on the deck transformation group, have infinite support.

In the following sections we make further reduction from components to blocks (in particular from metabelian components to metabelian blocks). We will later discuss sufficient condition on $G$ (e.g. finitely generated linear group) which will guarantee that the reduction can be made for finitely many blocks.

### 4.2. Reduction to torsion-by-$K$ and torsion-free-by-$K$ groups.

Having a group $G$ and a short exact sequence

$$1 \to H \to G \to K \to 1,$$

where $H$ is Abelian, we define **torsion-free by $K$ Abelian component** (in case when $K$ is Abelian, these will be torsion-free-by-Abelian component) $L$ of $G$ (with respect to this short exact sequence) to be $G/T$, where $T$ is the torsion subgroup of $H$ (this group is clearly an extension of $H/T$ by $K$).

We also define **torsion-by-$K$ component** $U$ of $G$ (with respect to this exact sequence) as follows: $U$ is a semi-direct product of $T$ by $K$.

**Lemma 4.10** (Reduction to torsion-free-by-$K$ components and torsion-by-$K$ components. I: torsion case). Consider a short exact sequence

$$1 \to H \to G \to K \to 1,$$

where $H$ is Abelian and $T$ is the torsion subgroup of $H$. Let $\mu$ be a non-degenerate probability measure on $G$, and let $\mu'$ be a non-degenerate associated measure on the torsion-by-$K$ component. An element $g \in T$ acts non-trivially on the boundary of $(G, \mu)$ if and only if it acts non-trivially on the boundary of the random walk defined by $\mu'$.

**Proof.** We apply Lemma 4.3 for $G = G$, $H = H$, $A = T$, $K = K$. □

**Lemma 4.11** (Reduction to torsion-free-by-$K$ components and torsion-by-$K$ components. II: torsion-free case).

$$1 \to N \to G \to K \to 1,$$

$T$ is a torsion subgroup of an Abelian subgroup $N$, $g \in N$. 


Let \( \text{Norm}(g) \) be the normal subgroup generated by \( g \) in \( G \), and consider \( C_g = \text{Norm}(g) \cap T \). \( g \) acts non-trivially on the boundary of \( (G, \mu) \) if and only if either \( g \) acts non-trivially on the boundary of a torsion-free-by-\( K \) component, or there is \( c \in C_g \) that acts non-trivially on the boundary of torsion-by-\( K \) component.

**Proof.** We apply Lemma 4.4 for \( G = G, M = T \). □

Consider a torsion-by-\( K \) component \( M \),

\[ 1 \to T \to M \to K \to 1, \]

where \( T \) is an Abelian torsion group. For a prime number \( p \), let \( T_p \) be the \( p \)-part of \( T \), that is

\[ T_p = \{ t \in T : \text{the order of } t \text{ is a power of } p \} \]

It is easy to see that \( T_p \) do not intersect and that \( T \) is a direct sum of \( T_p \);

\[ T = \bigoplus_p T_p \]

(for more on the structure of torsion Abelian groups see e.g. [22] Ch.10.) Let \( M_p \) be an extension of \( T_p \) by \( K \). We say that \( M_p \) is a \( p \)-torsion-by-\( K \) component of \( M \). Given an element \( t \in T \), we write it in the form \( t = \bigoplus_{p \in \Omega_t} (t_p) \), where \( \Omega_t \) is a prime.

**Claim 4.12.** Let \( M \) be a torsion-by-\( K \) group, of the form

\[ 1 \to T \to M \to K \to 1, \]

where \( T \) is an Abelian torsion group. Then \( t \in T \) acts non-trivially on the boundary of a torsion-by-\( K \) component if and only if \( t = \bigoplus_{p \in \Omega_t} (t_p) \) and there exists a prime \( p \) such that \( t_p \) acts non-trivially on the boundary of \( (M^p, \mu^p) \) (where \( \mu^p \) is a measure with the same projection to \( K \) as \( \mu \)).

**Proof.** Indeed, for each \( t \in T \) we know that there exists a finite set of primes \( \Omega_t \), such that

\[ t = \bigoplus_{p \in \Omega_t} (t_p) \]

Since each \( t_p \) is a multiple of \( t \), it is easy to see that \( t \) acts non-trivially if and only if at least one \( t_p \) acts non-trivially. Thus it suffices to check the claim when an element \( t_p \) acts non-trivially.

Let \( A \) be the set of elements of \( T \) of order coprime to \( p \) (that is, \( A = T = \bigoplus_q T_q \) for primes \( q \) not equal to \( p \)). Applying lemma 4.3 for \( G = G, T = H, A = A \), and \( K = K \) yields the desired result. □

So we already made a reduction to torsion-free-by-\( K \) and \( p \)-torsion-by-\( K \) components.

These \( (p \text{-torsion}) \)-by-\( K \) components \( M_p \) can be reduced to a collection of extensions of elementary Abelian \( p \)-groups (Abelian groups with all elements of order \( p \)) by \( K \).

**Definition 4.13** ((Elementary-\( p \)-by-\( K \) components.). Consider a short exact sequence

\[ 1 \to T_p \to M^p \to K \to 1, \]

\( T_p \) are elements of order which is a power of \( p \). Consider (finite or countable) family of groups \( T^{j,p} \), for each \( j \) the group \( T^{j,p} \) is elementary \( p \)-group defined as \( W^{j+1,p}/W^{j,p} \) (here \( W^{j,p} \) are elements of order \( p^j \)). Put \( M^{j,p} \) to be an extension of \( T^{j,p} \) by \( K \). We call these groups elementary-\( p \)-by-\( K \) components of \( M_p \).
Lemma 4.14 (Reduction of extension by $p$-groups to extensions by elementary Abelian $p$-groups). We consider a short exact sequence

$$1 \to T_p \to M^p \to K \to 1,$$

where $T_p$ is $p$-torsion. Let $\mu_p$ be a non-degenerate measure on $M_p$. We claim $g \in T_p$ acts non-trivially if and only if there exists $h \in \text{Norm}(M^p)\{g\}$, $h \in T^{j+1}_p$, $h \notin T^{j+p}$, and $h$ acts non-trivially on an associated random walk on some primary-by-$K$ component $M_j^p$.

Proof. We proceed by induction on $n$ where $g$ has order $p^n$. In the case $n = 0$, $g = e$ the claim is trivial. Suppose we have proven the statement for all elements $g$ of order $p^{n-1}$. Take an element $g$ of order $p^n$. Consider $U = \text{Norm}_{M^p}(g) \cap T^{p^n}$. Applying Lemma 4.4 for $G = M^p$, $M = T^{p^n}$, we conclude that $g$ acts non-trivially on the boundary of $(M^p, \mu_p)$ if and only if either some element of $U$ acts non-trivially on the Poisson boundary of $(M^p, \mu_p)$ or $g$ acts non-trivially on the boundary of the induced random walk on $M^p/T^{p^n}$.

By induction step we now know that $U$ acts non-trivially if and only if $g$ acts non-trivially on the boundary of one of the components.

Now suppose that $g$ acts non-trivially on the boundary of $M^p/T^{p^n}$. This shows that we can assume that $n = 1$ (because $g$ is of order $p$ in this quotient and since “extension of elementary $p^n$ components of the quotient are also components of our group).

We apply Lemma 4.3 for $G' = M^p/T^{p^{n-1}}$, $H = T^p$ and $A = T^{p^n}/T^{p^{n-1}}$ and conclude that $g$ acts non-trivially on $M^p/T^{p^{n-1}}$ if and only if it acts non-trivially on “extension of elementary $p^n$ components of $G'$.

Finally, observe that if $h \in \text{Norm}_{M^p}(g)$ acts non-trivially on the boundary of one of elementary components $M_{j+p}$ (which is an extension of $W^{j+1+p}/W^{j+p}$ by $K$) of $M_p$, then we can apply Lemma 4.3 for $G' = M^p$, $H = T^p$ and $A = T^{p^n}/T^{p^{n-1}}$ and conclude that
if $h$ acts non-trivially on the component $h$ then $h$ (and therefore $g$) acts non-trivially on the group.

**Definition 4.15.** (Single generated component). Consider a short exact sequence

$$1 \to Q \to G \to K \to 1,$$

where $Q$ is Abelian. Given an element $q \in Q$ define the single generated by $K$ component $G_q$ to be the semi-direct product defined by the short exact sequence

$$1 \to \text{Norm}_{G_q}(q) \to G_q \to K \to 1.$$

If $Q$ is either torsion-free Abelian or an elementary $p$-group, then it is clear that $\text{Norm}_{G_q}(q)$ in the definition above is either generated by a single torsion free element, or by a single element of order $p$.

**Lemma 4.16 (Reduction to single generator).** Consider an extension

$$1 \to Q \to G \to K \to 1.$$

Take a finite entropy non-degenerate measure on $G$. An element $q$ in the center of $Q$ acts non-trivially on the boundary of $(G, \mu)$ if and only if it acts non-trivially on the boundary of $(G_q, \mu_q)$, where we chose some/all measures non-degenerate measure $\mu_q$ on $G_q$ to have the same projection to $K$ as $\mu$. We remind that the group $G_q$ is defined in Definition 4.15.

**Proof.** This follows from Lemma 4.15 applied to $G = G$, $H = Q$ and $A = \text{Norm}_{G_q}(q)$. Observe that every conjugate of $q$ in $G$ is also in the center, so $A$ is also in the center of $Q$, and thus $A$ satisfies the assumption of the lemma that $A$ is central in $Q$. □

4.3. Reduction to blocks. Combining the reduction lemmas (reduction to Abelian-by-$K$ components (Lemma 4.6), reduction to primary-by-$K$ and torsion-free metabelian components 4.10 and 4.11, reduction to (elementary $p$)-by-$K$ 4.14 and reduction to single) 4.16, we obtain

**Theorem 4.17.** Let $1 \to N \to G \to K \to 1$ be a short exact sequence where the group $N$ is hypercentral. Let $\mu$ be a non-degenerate finite entropy measure on $G$. Then there exist a countable family of groups $C_\alpha$, $1 \to A \to C_\alpha \to K \to 1$, where $A$ is Abelian, normally generated by a single element, and either torsion-free or an elementary $p$-group such that the following holds. For each $C_\alpha$ fix a finite entropy non-degenerate measure $\mu_\alpha$ with the same projection to $K$ as $\mu$.

1. $(G, \mu)$ has non-trivial boundary if any only if there exists $\alpha$ such that $(C_\alpha, \mu_\alpha)$ has non-trivially boundary.

2. For each $g \in G$ we can associate $g_\alpha \in C_\alpha$ (not depending on $\mu$) such that $g \in G$ acts non-trivially on the boundary of $(G, \mu)$ if and only if either $g$ acts non-trivially on the projected random walk to $K$ or if there exists $\alpha$ such that $g_\alpha$ acts non-trivially on the boundary of $(C_\alpha, \mu_\alpha)$.

**Remark 4.18.** If we follow the reduction procedure explained in this section, we can describe elements $g_\alpha$. In the next section in the case of linear groups explain more explicitly how blocks $C_\alpha$ and $g_\alpha$ can be chosen.

Now we assume that $K$ is finitely generated and Abelian and that $N$ is nilpotent. It is well-known that the exit measure to a finite index subgroup has the same Poisson boundary (see [25], Lemma 4.2) If the group is finitely generated, then, taking the exit measure on an appropriate subgroup, we can assume that $K = \mathbb{Z}^d$. 


In this situation Theorem 4.17 tells us that we get groups $M$ $1 \to N \to M \to K = \mathbb{Z}^d \to 1,$

$N$ is generated by a single element, either of order $p$ or torsion-free, as a normal subgroup of the Abelian subgroup $K$. In this case $M$ is finitely generated. This group $M$ is either a torsion free quotient of $\mathbb{Z}^d \mapsto \mathbb{Z}$ or a quotient of $\mathbb{Z}^d \mapsto \mathbb{Z}/p\mathbb{Z}$.

**Remark 4.19.** As we have already mentioned, the groups above can be represented by upper-triangular matrices over a field of characteristic zero in the (finitely generated) torsion free metabelian case and over a field of characteristic $p$ in $p$-torsion by Abelian case \[49\].

**Corollary 4.20.** Assume that $G$ is a finitely generated nilpotent-by-Abelian group. There exists a collection of groups $B_\alpha$ with a fixed surjection to $A$ such that the following holds. If $\mu$ is a probability measure on $M$ and $\mu_\alpha$ are some probability measures on $B_\alpha$ with the same projection to $A$ as $\mu$, then the boundary of $(G, \mu)$ is non-trivial if and only if there exists $\alpha$ such that the boundary of $(B_\alpha, \mu_\alpha)$ is non-trivial. This claim is for some/equivalently all choices of measures $\mu_\alpha$ on $B_\alpha$ (with the same projection to $A$ as $\mu$).

**Remark 4.21.** This collection in the corollary is not necessarily finite. This can be seen already for (torsion two step nilpotent)-by-$\mathbb{Z}^d$ groups (even for $d = 1$).

## 5. Linear Groups

In the previous section we established a reduction from nilpotent-by-Abelian groups to metabelian ones. In the case of linear upper-triangular groups the collection of blocks can be chosen to be finite, and the reduction can be made in a more explicit way. This is the main result of this section.

As we have mentioned in the introduction, by Malcev’s theorem any solvable linear group contains a finite index subgroup which is isomorphic to a group consisting of upper-triangular matrices. And that a linear group is amenable if and only if this group is virtually solvable (as follows from Tits alternative).

In introduction we gave the definitions of the partial order $U$ and of blocks. We recall below these definitions.

For pairs $(i, j)$ where $1 \leq i < j \leq n$ we consider the following partial order $U$: $(i, j) \leq U (i', j')$ if $i \leq i'$ and $j \geq j'$. See Picture I

**Definition 5.1.** [Basic blocks] Consider a group of $n \times n$ upper triangular matrices. Consider matrices of the form

$$G_{i,j} = \begin{pmatrix} g_{i,i} & 0 \\ 0 & g_{j,j} \end{pmatrix},$$

defined when in $G$ there is a matrix with entries $g_{i,j}$ and with the same diagonal entries ($g_{i,i}$ and $g_{j,j}$). Recall that $UT(n)$ denotes the group of uni-uppertriangular matrices i.e those upper triangular matrices with only 1 on the diagonal. If there is no element in $G \cap UT(n)$ with non-zero entry at $(i, j)$ and with zero entries in all positions $(i', j') < U (i, j)$, then we say that $(i, j)$-block of $G$ is trivial (a group consisting of an identity element). Otherwise, we consider the group generated by $G_{i,j}$ and by a matrix

$$\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and we call this group the $(i, j)$ basic block of $G$. We denote this subgroup by $B_{i,j}$. 
Remark 5.2. A close version of the definition above could also be used: we could also say that a block \( \tilde{B}_{i,j} \) is a subgroup of a linear group generated by
\[
\tilde{G}_{i,j} = \begin{pmatrix} g_{i,i}^{-1} & 0 \\ 0 & 1 \end{pmatrix}
\]
and \( \delta \). Observe that \( \tilde{B}_{i,j} \) is a quotient of \( B_{i,j} \) over a central subgroup (of matrices having a constant on the diagonal), so that the Poisson boundary of a non-degenerate random walk on \( B_{i,j} \) has non-trivial boundary if and only if it happens for its projection to \( \tilde{B}_{i,j} \).

Given a partial (or complete) order \( T \) on pairs \((i, j), i < j \) a group of upper triangular matrices \( G \) has an \((i, j)\)-block that is valid with respect to the partial order \( T \) if there exists an upper uni-triangular element in \( G \), with matrix \( m_{i,j} \) with \( m_{i,j} \neq 0 \) and zero in all \( i', j' \leq_T (i, j) \).

Note that any non-trivial block of \( G \) is valid with respect to \( U \).

Given a probability measure \( \mu \) on \( G \), on any valid block \( B_{i,j} \) we consider any non-degenerate finite entropy measure \( \mu_{i,j} \) with the same projection to the two diagonal entries \( i, i \) and \( j, j \). We call such any such measure an associated measure on this block.

Theorem 5.3. \( \text{(1) The Poisson boundary of } (G, \mu) \text{ is non-trivial if and only if there exist } i, j : 1 \leq i < j \leq n \text{ such that for every associated measures } \mu_{i,j} \text{ on the } (i, j)-\text{block } B_{i,j} \text{ (equivalently: for some associated measure on this block) the Poisson boundary of } (B_{i,j}, \mu_{i,j}) \text{ is non-trivial.} \)

\( \text{(2) Moreover, we can characterise the elements that act trivially on the Poisson boundary. An element } g \in G \text{ acts non-trivially on the boundary } (G, \mu) \text{ if (and only if) there exist } i, j : 1 \leq i < j \leq n \text{ and } h \in \text{Norm}_G(g) \cap UT_n \text{ such that the matrix corresponding to } h \text{ has } m_{i,j} \neq 0 \text{ and } m_{i',j'} = 0 \text{ for any } (i', j') >_U (i, j) \text{ and the associated measure on the block } B_{i,j} \text{ has a nontrivial boundary.} \)

\( \text{(3) Moreover, fix a partial order } T \text{ on } \{i, j : 1 \leq i < j \leq n\}, \text{ which extends the partial order } U. \text{ An element } g \in G \text{ acts non-trivially on the boundary } (G, \mu) \text{ if (and only if) there exist } i, j : 1 \leq i < j \leq n \text{ and } h \in \text{Norm}_G(g) \cap UT_n \text{ such that the matrix corresponding to } h \text{ has } m_{i,j} \neq 1 \text{ and } m_{i',j'} = 0 \text{ for any } (i', j') >_T (i, j) \text{ and the associated measure on the block } B_{i,j} \text{ has a nontrivial boundary.} \)

Remark 5.4. Take a complete order \( T \), which respects the partial order \( U \). Recall that such complete order correspond to Young Tableaus. See picture for an example of a Young tableau corresponding to the horizontal order. Take note that our picture is a flipped version (with respect to vertical axis) of the standard convention for Young Tableau.

\[
\begin{array}{cccc}
4 & 3 & 2 & 1 \\
7 & 6 & 5 \\
9 & 8 \\
10 & \\
\end{array}
\]

Figure 4. Young tableau

Let \( T \) be a partial order on pairs \((i, j), 1 \leq i < j \leq n \). Observe that \( T \) defines a preorder on elements of \( UT_n \), saying that \( u \leq_T w \) if for every non-zero coordinate \((i, j)\) of \( u \) there exists \((i', j')\) such that \((i', j') \geq_T (i, j)\) and the \((i', j')\) coordinate of \( w \) is non-zero.
Denote by $N_{i,j}^T$ the set of $UT_n$ consisting of all upper-unitriangular elements which are 0 in all coordinates $i',j'$ where $(i',j') \triangleright_T (i,j)$ with respect to $T$. By $\delta_{i,j}$ we denote the matrix with 1 at the entry $(i,j)$ and zero elsewhere. For the convenience of the reader we formulate and explain some elementary properties of $N_{i,j}^T$.

**Claim 5.5.** Let $T$ be a partial order order which extends $U$.

1. $N_{i,j}^T$ is a normal subgroup of the upper triangular matrices $T_{n,n}$.
2. Moreover, if we conjugate $I + \delta_{i,j}$ by $u \in UT_n$ we get $I + \delta_{i,j} - k$ where $k \in N_{i,j}^T$.
3. Moreover let $h$ be an element in $U_n$ which is 0 for all elements $(k,l) \triangleright_T (i,j)$.

Then the image of $h$ (under the quotient map) is central in $U_n/N_{i,j}^T$.

**Proof.**
1. Consider upper-triangular matrices $A$ with 0 on the main diagonal (which form a nilpotent group). Consider a map $t$ that maps a nilpotent upper triangular matrix $A$ to the pair $(k,l)$, $1 \leq k, l \leq n$, where $(k,l)$ is the maximal element, with respect to the order $T$, with $a_{k,l} \neq 0$. We also use the convention $t(0) <_T (k,l)$ for any $k, l$, $1 \leq k, l \leq n$. Note that given two non-zero nilpotent upper triangular matrices $A$ and $B$

$$t(AB) <_T t(A), t(B).(*)$$

Also note that

$$t(A + B) \leq_T \max(t(A), t(B)).(+ *)$$

First we prove that $N_{i,j}^T$ is a subgroup. Note that by the definition $N_{i,j}^T$ is the subset consisting of elements of the form $I + A$ where $A$ is a nilpotent upper triangular matrix such that $t(A) <_T (i,j)$.

Also observe that $g^{-1}$ belongs to $N_{i,j}^T$ if $g \in N_{i,j}^T$. Indeed $(1 + A)^{-1} = 1 - A + A^2 - A^3 ... = 1 + B$ since $B$ is a sum of powers of $A$ and by $(*)$ we know that $t(A^n) <_T t(A)$, thus we conclude by $(+ *)$ that $t((1 + A)^{-1}) \leq_T t(A)$. So $N_{i,j}^T$ is a subgroup.

Now we want to prove that the subgroup is normal in the upper-triangular matrices. First note that given an upper triangular matrix $A$ and an invertible diagonal $D$, we have that $DAD^{-1}$ has zero or non-zero entries at the same positions as $A$, and hence $t(DAD^{-1}) = t(A)$. So it suffices to prove that $N_{i,j}^T$ is preserved by conjugation by uni-upper triangular matrices. Let us conjugate $1 + A$ by $1 + B$. Let $(1 + B)^{-1} = 1 + C$. Then $(1 + B)(1 + A)(1 + C) = (1 + B + C + BC) + (A + BA + AC + BAC) = (1 + B)(1 + C) + (A + BA + AC + BAC) = 1 + A + BA + AC + BAC$ where the last equality holds because $1 + B$ and $1 + C$ are inverses. Applying $(+ *)$ to the terms of $1 + A + BA + AC + BAC$, we see that this element belong to the subgroup, and the normality follows.

2. Let $(1 + A)$ and $(1 + B)$ be inverse uni-upper triangular matrices. Then $(1 + A)\delta_{i,j}(1 + B) = \delta_{i,j} + A\delta_{i,j} + \delta_{i,j} B + A\delta_{i,j} B$. By $(*)$ $t(A\delta_{i,j} + \delta_{i,j} B + A\delta_{i,j} B) <_T t(\delta_{i,j}) = (i,j)$ which implies the claim.

3. First we prove the following: Let $A$ and $C$ be two nilpotent upper triangular matrices such that $t(A - C) <_T (i,j)$ then $(1 + A)(1 + C)^{-1} \in N_{i,j}^T$. To see this rewrite it as $(1 + C + (A - C))(1 + C)^{-1}$. This is equal to $1 + (A - C)(1 + C)^{-1}$. Since $t(A - C) <_T (i,j)$ we have (using $(*)$) $t((A - C)(1 + C)^{-1}) <_T (i,j)$ which proves the above mentioned claim.

Now let $h = I + A$. Take $I + B \in U_n$ we want to show that $(I + A)$ and $I + B$ commute modulo $N_{i,j}^T$.

By the above claim it suffices to prove that $t((I + A)(I + B) - (I + B)(I + A)) <_T (i,j)$. However since $(I + A)(I + B) - (I + B)(I + A) = AB - BA$ so $(*)$ and $(+ *)$ immediately imply that $AB - BA <_T (i,j)$.

$\square$

Now we prove the theorem.
Proof. (2) follows from (3) taking $T = U$. And (2) implies (1) since the boundary is trivial if and only if all elements act trivially.

So to prove the theorem, we have to prove Claim (3). First assume that $T$ is complete. Claim 5.5 states that $N_{i,j}^T \cap G$ is a normal subgroup of $G$, for any total order $T$, compatible with $U$.

Suppose that $g$ acts non-trivially on the boundary of $G$. Consider elements $h$ in $\text{Norm}_G(g) \cap UT_n$ that act non-trivially on the boundary of $(G, \mu)$. For any element $h$ in $\text{Norm}_G(g) \cap U_n$ take the minimal $(i, j)$ with respect to $T$ such that the entries of $h$ are zero in all $(i', j') >_T (i, j)$. We use notation $i(h) = i$, $j(h) = j$. Among $i(h)$, $j(h)$ such that $h$ acts non-trivially on the boundary, we take the minimal one which we denote by $(i, j)$. We consider one of such $h$, with $i(h) = i$, $j(h) = j$. Consider the block $B_{i,j}$. We want to show that the boundary of an associated measure on this block is nontrivial.

By our assumption that $(i, j)$ is minimal with the above mentioned property, we know that the intersection $\text{Norm}_G(g) \cap N_{i,j}^T$, and thus $\text{Norm}_G(h) \cap N_{i,j}^T$ acts trivially on the Poisson boundary. Thus, by Lemma 4.16 applied to $G = G$ and $M = N_{i,j}^T$, we see that $h$ acts non-trivially on the Poisson boundary of $G/N_{i,j}^T$. Using the last part of the Claim 5.5 we see that $h$ is in the center of $Q' = G/N_{i,j}^T \cap U_n/N_{i,j}^T$.

Apply Lemma 4.16 to the short exact sequence $1 \to Q' \to G' = G/N_{i,j}^T \to D \to 1$ and the image $q$ of $h$ in $Q'$. (Here $D$ is the diagonal group). Denote by $G_q'$ the group obtained in Lemma 4.16. Note that the subgroup $D_{i,j}$ of the diagonal group which is 1 on coordinates $i, i$ and $j, j$ acts trivially on $G_q'$. Observe that $D_{i,j}$ is a central subgroup of $G_q'$. We take a quotient of $G_q'$ over this subgroup. Observe that the map from this quotient that sends $q$ to $\delta$ in $B_{i,j}$ and sends $D/D_{i,j}$ to the diagonal elements $G_{i,j}$ in the definition of the block (see Definition 5.1) induces an isomorphism.

Since central extensions have the same boundary for non-degenerate measures, we can conclude that the boundary of (any) associated measure on the block $B_{i,j}$ is non-trivial.

Thus we have proved the last claim of the theorem under the assumption that the order $T$ is complete.

Until now we have assumed that $T$ is a complete order. Now consider $T = U$. Chose a complete order $T'$ which respects $T = U$. Observe that a valid block with respect to a $T'$ is also valid with respect to $U$. By the claim applied to $T'$ we know that if the boundary of our random walk on $G$ is non-trivial, there exists a valid block with respect to $T'$, thus a valid block with respect to $U$, such that any associated measure on this block has non-trivial boundary.

Finally, take any partial order $T$. If we have a valid block with respect to $T$ which has non-trivial boundary, then since it is valid with respect to $U$ also, we know that the boundary of our random walk is non-trivial. Now consider a total order $T'$ which respects $T$. If the boundary of the random walk on $G$ is non-trivial, then there exists a valid block with respect to $T'$ (and thus valid with respect to $T$) which has non-trivial boundary.

\end{proof}

6. General facts about $k[A]$ modules and dimension

We recall that since the 1954 paper of Hall [28], many results about finitely generated metabelian groups $G$ are proven by considering the commutator group $[G, G]$ as a $\mathbb{Z}[H]$ module, for $H = G/[G, G]$. The properties of finitely generated metabelian groups are thus closely related to that of finitely generated modules over polynomial rings in finitely many variables. In this section we formulate basic properties of such modules and related notions.
Given a ring $R$ and a group $G$, $R[G]$ denotes the group ring. The elements of this ring can be considered as finitely supported functions from $G$ to $R$ (with multiplication given by convolution). If $G = \mathbb{Z}^d$, then the elements of the group ring can be identified with Laurent polynomials; elements with support $\mathbb{Z}^+$ can be thus considered as polynomials in $d$ variables, and elements of $\mathbb{Z}^+$ as monomials.

Given a finitely generated metabelian group $G$ and a short exact sequence

$$1 \rightarrow B \rightarrow G \rightarrow A \rightarrow 1,$$

with $A$ and $B$ Abelian we consider $B$ as a module over $\mathbb{Z}[A]$. When furthermore $B$ is a $p$-torsion group we can tensor by $\mathbb{Z}/p\mathbb{Z}$ and consider it as a module over $\mathbb{Z}/p\mathbb{Z}[A]$. If $B$ is torsion free by taking its tensor product with $\mathbb{Q}$ we can consider it as a $\mathbb{Q}[A]$ module. To address these two cases we consider $k[A]$ modules where $k$ is a field. Although our main interest is in applications to metabelian groups, throughout this section we will use the more general language of modules which is helpful for formulating the notion and basic properties of dimension.

**Definition 6.1.** [k- dimension of $k[A]$ modules.] Let $k$ be a field and $A$ be a finitely generated Abelian group. Given a finitely generated $k[A]$ module $M$ and a subgroup $A'$ of $A$, observe that we can consider $M$ as $k[A']$ module. We say that $M$ is finitely $A'$ generated if $M$ is finite dimensional as a $k[A']$ module. Let $d$ be the minimal number such that there exists $A' = \mathbb{Z}^d + C$, where $C$ is a finite Abelian group and where $M$ is finitely generated as an $A'$ module. We say that $d$ is the $k$-dimension of $M$.

Definition 6.1 above can be formulated in terms of Krull dimension (see Remark 6.13), that admits several equivalent definitions. We will use it in the next subsection to define the dimension of metabelian groups. This definition will however be different from the definition of Krull dimension of metabelian groups, considered in [29], see Remark 6.14.

We use throughout the text only Definition 6.1 and for convenience of the reader explain in this section the properties of the dimension which we use for our results about Poisson boundary.

When the field $k$ is clear from context we will sometimes suppress $k$ in the terminology of $k$-dimension.

**Definition 6.2.** We can also speak about the dimension of elements $m \in M$, setting it to be the dimension of the submodule generated by $m$.

**Remark 6.3.** Observe that if the dimension is $d$, then by definition we can take $A' = \mathbb{Z}^d + C$, $C$ is a finite group. In this case we can also take $A' = \mathbb{Z}^d$.

**Remark 6.4.** Let $1 \rightarrow B \rightarrow G \rightarrow A \rightarrow 1$ be a short exact sequence of groups where $A$ and $B$ are Abelian. Assume that $A$ is a finite extension of $\mathbb{Z}^m$. Assume that $B$ is either $p$-torsion or torsion-free. Put $k = \mathbb{Z}/p\mathbb{Z}$ in the first case and $k = \mathbb{Q}$ in the latter case. As we have already mentioned, we can consider $B$ in the first case and $B \otimes \mathbb{Q}$ as a $\mathbb{Z}^m[k]$ module. We will also say that the rank of this module is the rank of this short exact sequence. We also say that the dimension of an element $g \in B$ is the dimension of the corresponding element in the module.

We first prove an elementary lemma which we use throughout the section.

**Lemma 6.5.** Let $W$ be a finite subset of $\mathbb{Z}^d$. There exists a basis $b_1, \ldots, b_d$ of $\mathbb{Z}^d$ such that every element of $A$ has a different $b_1$ coordinate.
Proof. Let $S$ be the maximal absolute value of any coordinate of $W$ in the standard basis $a_1, \ldots, a_d$. Consider the basis given by the row vectors of the matrix $M_d$ whose first column is $(3S)^{d-1}$, whose diagonal entries are equal to one and which is 0 elsewhere. Put

$$M_d = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 3S & 1 & 0 & \ldots & 0 \\ 9S^2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (3S)^{d-1} & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Then any two distinct points in $[-S, S]^d$ have distinct $b_1$ coordinates. To see this observe that the $b_1$ coordinate of $(x_1, \ldots, x_d)$ is

$$x_1 - \sum_{i=2}^{d} (3S)^{i-1} x_i$$

which is clearly distinct for all points in $[-S, S]^d$. \hfill \Box

**Lemma 6.6.** Let $M$ be a finitely generated module over $k$, of dimension $d$. Let $\mathbb{Z}^d$ be the associated subgroup of $\mathbb{Z}^m$. Consider $M$ as a $k[\mathbb{Z}^d]$ module. Then this module has a non-trivial free submodule.

**Proof.** Suppose not. Let $U = U_1, U_2, \ldots, U_n$ be a finite generating set of the $k[\mathbb{Z}^d]$ module. Then each $U_k$ has some non-zero annihilator in $k[\mathbb{Z}^d]$. Let us choose such annihilators for each $U_i; r_1, r_2, \ldots, r_n$. Then $q = r_1 r_2 \ldots r_n$ annihilates $U$ and hence $M$. The element $q$ is non-trivial because the group ring $k[\mathbb{Z}^d]$ is an integral domain.

Let $X$ be the support of $q$ in $\mathbb{Z}^d$. By Lemma 6.5 we can find a basis for $\mathbb{Z}^d$ so that all elements in $X$ have a distinct $b_1$ coordinate. Let $\mathbb{Z}^{d-1}$ be the subgroup of elements which have 0 first coordinate. It is easy to see that finitely many $b_1$ translates of $U$ generate $M$ as a $\mathbb{Z}^d k$-module, and this contradicts the assumption that the dimension is $d$. \hfill \Box

In the proposition below, we will consider a generating set $S$ of $\mathbb{Z}^d$, and $B_{\mathbb{Z}, S}(r)$ is the ball of radius $r$ in this generating set.

**Lemma 6.7** (The dimension of obtainable vector spaces). Let $k$ be a field and $M$ be a finitely generated module over $k[\mathbb{Z}^m]$ of dimension $d$, and let $U$ be a finite generating set of $M$. Let $V_{B_{\mathbb{Z}, S}(r), U}$ be the $\mathbb{Z}/p\mathbb{Z}$ vector space spanned by $zu$ for $z \in B_r, u \in U$.

Then there exist positive numbers $C_1$ and $C_2$ such that for some $C_1, C_2 > 0$ and all $r \geq 1$ it holds

$$C_1 r^d \leq \dim(V_{B_r, U}) \leq C_2 r^d.$$

**Proof.** First observe that it suffices to prove this for any choice of generating set $U$ (other choices will differ by at most a multiplicative constant).

Observe that since the dimension is at most $d$, by definition we know that there exists $\mathbb{Z}^d \subset \mathbb{Z}^m$ such that our module is finitely generated as $\mathbb{Z}^d$ module. Choose $U$ to be the generating set. Note that every generator of $\mathbb{Z}^m$ sends every $u \in U$ to an element of $B_{\mathbb{Z}^d, r_0}(U)$ for some $r_0$. Let $l$ be the maximum among such $r_0$. Then it is easy to see that any element in $B_{\mathbb{Z}^m, r, U}$ is in $B_{\mathbb{Z}^d, l}(U)$.

This implies the upper bound.

The lower bound follows from Lemma 6.7 since the lower bound holds true for a free module and since $\dim(V_{B_r, U}) \leq \dim(W_{B_r, U})$ if $W$ contains $V$ and the finite generating set $T$ for $W$ contains $U$. \hfill \Box

**Remark 6.8.** Consider a free finitely generated $k[\mathbb{Z}^d]$ module. Then its dimension is $d$. 

Indeed, by definition we know that the dimension is at most \( d \). By the previous lemma, the dimension can not be smaller or equal to \( d - 1 \).

**Remark 6.9.** Let \( M \) be \( k[Z^n] \) module. Take \( m < n \), we have \( Z^m \subset Z^n \), and consider \( M \) a \( Z^m \)-module. Assume that \( M \), considered as a \( k[\mathbb{Z}^m] \)-module, has dimension \( d \). Then its dimension as \( k[\mathbb{Z}^n] \)-module is also \( d \).

**Proof.** The upper bound is immediate from the definition of the dimension. The lower bound follows from Lemma 6.7 about the dimension of obtainable vector spaces. \( \square \)

**Remark 6.10.** Let \( M \) be a finitely generated module over \( k[\mathbb{Z}^r] \) and \( M' \) be a submodule. Then the dimension \( d' \) of \( M' \) satisfies \( d' \leq d \) where \( d \) is the dimension of \( M \).

Indeed, let \( A \) be a subgroup of \( \mathbb{Z}^r \) which is isomorphic to \( \mathbb{Z}^d \) and such that \( M \) is a finitely generated module over \( k[A] \). Since \( k[A] \) is Noetherian, \( M' \) is also a finitely generated module over \( k[A] \) which proves the remark.

The lemma below is useful for computing and estimating the dimension. In the next subsection we explain applications to some metabelian examples (and computing their dimension).

**Lemma 6.11.** Let \( k \) be a field. Then any quotient \( M \) of the free module \( k[\mathbb{Z}^d] \) has dimension strictly less than \( d \). If \( I \) is a nontrivial principle ideal then \( M/I \) has dimension precisely \( d - 1 \).

**Proof.** We can assume that at least one non-trivial element \( r \in k[\mathbb{Z}^d] \) that becomes zero in the quotient module. By Lemma 6.5 we can choose a basis \( c_1, c_2, c_d \) of \( \mathbb{Z}^d \) such that the following holds. We take a Laurent polynomial in \( b_1, \ldots, b_d \). We want to rewrite as a Laurent polynomial \( p' \) in \( c_1, \ldots, c_d \). Without loss of generality, it is enough to assume that both \( p \) and \( p' \) are polynomials. So we have polynomials \( p \) and \( p' \) such that

\[
p(b_1, \ldots, b_d) = p'(c_1, \ldots, c_d).
\]

Given a polynomial in \( c_1, \ldots, c_d \), we consider a its monomials \( c_1^{j_1} \ldots c_d^{j_d} \) and \( M = \max j_1 \) and \( m = \min j_1 \), where the maximum and the minimum are taken over the monomials with non-zero coefficients. By using Lemma 6.5 we can assume that \( p' \) has a single monomial with \( j_1 = M \) (and non-zero coefficient) and a single monomial where \( j_1 = m \) (we call this condition for a Laurent polynomial or corresponding polynomial \( ^* \)-condition).

Remove \( c_1 \) from the basis. We claim that our quotient module is a finite dimensional module over \( c_2, \ldots, c_d \), so the dimension of this module as at most \( d - 1 \). To see that it is finite dimensional over \( c_1, \ldots, c_d \), we use a version of polynomial long division. Namely, if \( u \) and \( v \) are Laurent polynomials in \( c_1, \ldots, c_d \), and \( v \) satisfies \( ^* \)-condition with values \( \max \) and \( \min \), then there exist a Laurent polynomial \( w \) with \( c_1 \) coefficients bounded between \( \max \) and \( \min \) in monomials of the corresponding polynomial and a Laurent polynomial \( t \) so that \( u = vt + w \). Since the coefficient of \( w \) is bounded, we conclude that the module is finite dimensional over \( k[\mathbb{Z}^{d-1}] \) (where \( \mathbb{Z}^{d-1} \) corresponds to the group generated by \( c_2, \ldots, c_d \)).

Now we prove the second claim of the lemma. Let \( p \) be a generator of the ideal. Let \( x_1, x_2, \ldots, x_d \) be free generators of \( \mathbb{Z}^d \). Since the ideal \( I \neq k[\mathbb{Z}^d] \), \( p \) has at least two non-trivial monomials. Thus there is a coordinates \( c_i \) such that the monomials of \( p \) have at least two distinct powers of \( c_i \). Renaming the generators, we can assume that it happens for \( c_1 \). Consider the lexicographic ordering on monomials. Observe that a product of \( p \) and a non-zero Laurent polynomial \( q \) has at least two distinct powers of \( c_1 \) (a largest and a smallest). Therefore every Laurent polynomial in \( c_2, \ldots, c_d \) is not a multiple of \( p \) and
The dimension by 

\( \text{dimension} \) the abelianization module. 

\( Z \) as a description of Krull dimension (see e.g. Axiom D3 in Section 8 of [15], one of the axioms for an axiomatic theory). 

The dimension of Metabelian groups; basic examples. We consider \( G \) in the previous subsection, our motivation here is to study finitely generated metabelian groups.

As we have mentioned in the text, metabelian groups is different from this notion in [29], which studied its relation to isoperimetry and probability to return to the origin in metabelian groups. For torsion free groups we consider them as modules over \( \mathbb{Q} \), so that in particular the dimension of \( Z^d \) is equal to \( d \), and not \( d + 1 \) as in [29], that considers them as modules over \( \mathbb{Z}[A] \) that increases the dimension by 1. Given a metabelian group \( G \), [29], Proposition 2.30 considers \([G, G]\) as \( \mathbb{Z}[G/[G, G]] \)-module and the Krull dimension is defined as the Krull dimension of this module.

### 6.1. Dimension of Metabelian groups; basic examples.

As we have mentioned in the previous subsection, our motivation here is to study finitely generated metabelian groups. We consider \( G \) and a short exact sequence where \( B \) is the commutator subgroup and \( A \) is the abelianization.

\[ 1 \to B \to G \to A \to 1, \]
we can consider $B$ as a module over $\mathbb{Z}[A]$. In the previous subsection, we have discussed the dimension of modules; we use this notion in the particular case of metabelian extensions. Namely, if $B$ is a $p$-torsion group we consider $B$ as a module over $\mathbb{Z}/p\mathbb{Z}[A]$. If $B$ is torsion free by taking its tensor product with $\mathbb{Q}$ we consider it as a $\mathbb{Q}[A]$-module.

For $p$-torsion-by-Abelian metabelian group and for torsion-free metabelian group, we define the dimension of the corresponding extension as the dimension of the above mentioned modules. We also speak in this case about the dimension of the groups.

**Remark 6.15.** The dimension does not increase by taking subgroups and quotient groups.

**Remark 6.16.** For simplicity, we assumed in the definition of the dimension of metabelian group that the group is either $p$-torsion-by-Abelian and torsion-free. For a general f.g metabelian group (or even more generally for a nilpotent-by-Abelian group), we can use Lemmas from Section 4 and define the dimension as the maximum of the dimension of metabelian blocks (which are f.g. $2 \times 2$ linear $p$-torsion-by-Abelian and torsion-free groups associated to our group.

**Example 6.17.** Consider a torsion-free (or $p$-torsion) polycyclic metabelian group $G$. Then the dimension of $G$ is zero.

**Proof.** Indeed, if $G$ is torsion-free polycyclic, then the commutator subgroup of $G$ (as well as the abelianization) is finitely generated. \qed

The assumption that the group is metabelian and torsion-free can be removed, and the polycyclicity assumption can be replaced by the assumption of being a finitely generated group of finite Pruefer rank. Indeed, in this general case observe that all associated blocks have commutator groups, which after tensoring by $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$ become finite dimensional vector spaces. Observe also, that there is a common upper bound for the number of generators of the Abelianisation of blocks.

**Example 6.18.** (1) Let $\text{Met}_d$ be a free metabelian group on $d$-generators where $d \geq 2$. Then the dimension of the corresponding Abelian-by-$\mathbb{Z}^d$ extension is $d$. The same is true $\text{Met}_d(p)$.

(2) Let $G$ be a finitely generated metabelian group whose abelianization is virtually $\mathbb{Z}^d$ then $G$ has dimension at most $d$.

**Proof.** Observe that the commutator subgroup of $\text{Met}_d$ (as well as of $\text{Met}_d(p)$) is generated by the conjugates of commutators of the generators, thus the corresponding module if finitely generated and the dimension is at most $d$. Now fix a commutator of generators, call it $c$, note that $c$ generates a free $\mathbb{Q}[\mathbb{Z}^d]$-module (for the wreath product, $\mathbb{Z}/p\mathbb{Z}[\mathbb{Z}^d]$ [$\mathbb{Z}^d$]-module for $\text{Met}_d(p)$), which by Lemma 6.6 and Remark 6.10 implies that the dimension $\geq d$.

To see the second claim observe again that that the commutator subgroup is generated by conjugates of commutators of the generators and thus the corresponding module is finitely generated and of dimension at most $d$. \qed

**Lemma 6.19 (Wreath product subgroup).** Let $G$ a be a semi-direct product of $A$ and $B$, where $A$ and $B$ are Abelian, and be $B$ is either $p$-torsion or torsion-free.

The dimension of $G$ is $\geq d$ if and only if $G$ contains as subgroup $\mathbb{Z}^d \wr \mathbb{Z}$ (if $B$ is torsion-free) and $\mathbb{Z}^d/\mathbb{Z}/p\mathbb{Z}$ $B$ is $p$-torsion.

**Proof.** First suppose that $G$ contains a wreath product of $\mathbb{Z}^d$ with $H = \mathbb{Z}$ or $H = \mathbb{Z}/p\mathbb{Z}$ as a subgroup.

$$1 \to [G, G] \to G \to G/[G, G] \to 1$$
Observe that $H$ belongs to the commutator of the wreath product, and hence to the commutator $[G, G]$. Observe also that $\mathbb{Z}^d$ injects into $G/[G, G]$. Then by monotonicity of dimension for submodules (Remark 6.10) we know that the dimension of $G$ is $\geq d$. Hence, $\mathbb{Z}^d$ injects into $G$.

Now suppose that the dimension of $G$ is $\geq d$. Consider $\mathbb{Z}^d$ that realizes the dimension $d$.

\textbf{Example 6.20.} For any non-trivial quotient of the wreath product $G = \mathbb{Z}^d \wr \mathbb{Z}/p\mathbb{Z}$ or of the free $p$-metabelian group on $d$ generators $G = \text{Met}_d(p)$, the dimension is smaller than $d$. If we add exactly one non-trivial relation in the commutator group, then the dimension is exactly $d - 1$.

\textit{Proof.} The dimension of $G$ is $d$ (for free metabelian group, see Example 6.18). If there is a non-trivial relation not in the commutator subgroup of $G$, then it follows from the second claim of Example 6.18 that the rank $< d$. Otherwise, there is at least one relation in the commutator group. Then both claims follow from Lemma 6.11 (in case of a quotient of wreath product as well as in the case of a quotient of $\text{Met}_d(p)$).

Below is a particular case of Example 6.20. In contrast with some other examples, the Abelian group $\mathbb{Z}$ that realizes that the dimension is 1 cannot be chosen to be generated by one of the standard generators of wreath product.

\textbf{Example 6.21.} Consider $f : \mathbb{Z}^2 \to \mathbb{Z}/p\mathbb{Z}$ that takes value 1 in $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$ and zero otherwise. Take the quotient of $\mathbb{Z}^2 \wr \mathbb{Z}/p\mathbb{Z}$ over the relator $(0, f)$. The dimension of the corresponding extension is 1.

\textbf{Lemma 6.22.} Let $G$ be a group of $2 \times 2$ upper-triangular matrices with at least one non-trivial unipotent element (i.e. an element which is 1 on the diagonal and not zero on the upper right corner). Given a matrix $M$

$$M_{\alpha} = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix},$$

let $\phi(M) = b/a$. If the minimal field including all $\phi(g), g \in G$ has transcendence degree $d$. Then the rank of this metabelian group $G$ is $d$.

\textit{Proof.} Since the transcendence degree is equal to $d$ there are at least $d$ algebraically independent values $\phi(g), g \in G$ can take. We call these matrices $M_1, \ldots, M_d$. Since conjugating by $M$ induces a multiplication by $\phi(M)$ on all unipotent elements (that is, multiplying the upper right coordinate by $\phi(M)$ and acting trivially on the diagonal). These multiplications are all linearly independent for algebraically independent $M$. Therefore, the submodule of $M$, generated by $M_1, \ldots, M_d$, is free. And hence by Remarks 6.8 and 6.10 the dimension of $M$ is at least $d$. By remark 6.12 the dimension can be at most $d$ which proves the example.

\textbf{Example 6.23.} [Baumslag groups] We recall that Baumslag groups

$$B_d(\mathbb{Z}/p\mathbb{Z}) \subseteq GL_2((\mathbb{Z}/p\mathbb{Z})(X_1, \ldots, X_D))$$

and $B_d(\mathbb{Z}) \subseteq GL_2(\mathbb{Z})(X_1, \ldots, X_D)$ are constructed as follows. These are groups generated by the $2 \times 2$ matrices $B_{d,p} \in GL_2((\mathbb{Z}/p\mathbb{Z})(X_1, \ldots, X_D)$ in particular is the group generated...
by the $2d + 1$ matrices of the form

$$
\delta = \left( \begin{array}{cc}
1 & 1 \\
0 & 1 \\
\end{array} \right), \ M_X, \left( \begin{array}{cc}
1 & 0 \\
0 & X_i \\
\end{array} \right), \ M_{X_i+1}, \left( \begin{array}{cc}
1 & 0 \\
0 & X_i + 1 \\
\end{array} \right).
$$

over $(\mathbb{Z}/p\mathbb{Z})(X_1, ..., X_D)$ and respectively over $(\mathbb{Z})(X_1, ..., X_D)$ (1 ≤ i ≤ d).

The dimension (for the field $k = \mathbb{Z}/p\mathbb{Z}$ and correspondingly for the field $k = \mathbb{Q}$), is equal to $d$.

Proof. Observe that the minimal field which includes $\phi(M)$ for all $M$ in the Baumslag group is $k[X_1, ..., X_D]$. The rank of this field is $D$, and thus the example follows from Lemma 6.22.

7. THE CAUTIOUSNESS CRITERION FOR TRIVIALITY OF THE BOUNDARY

In this section we will prove the cautiousness criterion for triviality of the Poisson boundary. It will imply, in particular, that (under sufficient moment conditions) random walk on a linear group in characteristic $p$ of dimension $d ≤ 2$ or on a linear group in characteristic 0 of dimension $d ≤ 1$ have trivial boundary.

We start by explaining the idea of our cautiousness criterion. Consider the basic and well-known example of $\mathbb{Z}^2/\mathbb{Z}/2\mathbb{Z}$. There are several known arguments to explain why simple random walks on this group have trivial boundary. The original argument of Kaimanovich and Vershik [32] uses the fact that the commutator subgroup is recurrent, considers the exit measures to this subgroup and then concludes by observing that this exit measure has trivial boundary, since the subgroup is Abelian.

We note that this argument can not be applied to many of the groups we are interested in. For example, in the two-dimension Baumslag group the quotient over the commutator is $\mathbb{Z}^4$, and this subgroup is transient.

Another way to prove the triviality of the boundary for the wreath products is to observe that the entropy function is closely related to the drift of the random walk on $\mathbb{Z}^2$, and use that this drift is sublinear for recurrent random walks. Such estimates in general require a detailed understanding of the typical elements visited at time $t$, and we do not know in what generality a detailed description of such elements can be obtained for general group extensions.

Now we explain one more way to see that simple random walks on the two-dimensional Lamplighter group has trivial boundary. Observe that for any $\varepsilon > 0$ the random walk on the projection to $\mathbb{Z}^2$ stays in the ball of radius $\varepsilon \sqrt{t}$ for all time instants between 1 and $t$, with positive probability with a lower bound not depending on $t$. See Figure 5 (Here this property holds for all $t$. When it holds for a subsequence of times instants, the property is called cautiousness in [18], see also Definition 7.1 for a more general version of this definition). Observe that with positive probability the position of the random walk at time $t$ belongs to set of cardinality at most $t^2 \exp(C \varepsilon^2 t)$. By the Shannon – McMillan – Breiman type theorem one concludes that the asymptotic entropy of the random walk is zero.

In this section we will show that this latter argument can be generalised to many group extensions.

Definition 7.1 ($f$-cautious random walk). Let $f : \mathbb{N} \to \mathbb{R}$ be an increasing function. A random walk $X_n$ is $f$-cautious along some subsequence if

$$
\limsup_n \mathbb{P}[ \max_{1 ≤ m ≤ n} (|X_m|/f(n)) < \varepsilon ]
$$

is positive for every $\varepsilon > 0$. 

In other words, the definition says that for each $\varepsilon$ there exists a $p_\varepsilon > 0$ and a subsequence $n_i$ such that the following holds: The probability that during the time interval between $1$ and $n_i$ the random walk stays inside the ball of radius $\varepsilon f(n_i)$ is at least $p_\varepsilon$.

**Remark 7.2** (f-cautious random walks with condition for all $n_i$.) Let $f : \mathbb{N} \to \mathbb{R}$ be an increasing function. A random walk is $f$-cautious if for all $\varepsilon > 0$ there exists $p_\varepsilon > 0$ such that for all sufficiently large $n$ trajectories $X_n$ satisfy the following.

$$\mathbb{P}(\forall m : 1 \leq m \leq n, |X_m|/f(n) < \varepsilon) \geq p_\varepsilon$$

**Remark 7.3.** The Central Limit Theorem shows that symmetric finite second moment random walk on $\mathbb{Z}^d$ (and more generally any finitely generated Abelian group) are $f(n)$-cautious for $f(n) = \sqrt{n}$.

**Remark 7.4.** Let $G$ be a group and $\mu$ is a finite first moment measure. In this situation the drift of the random walk is zero if and only if the r.w. is $f(n)$-cautious for $f(n) = n$.

**Remark 7.5.** If $f(n) = \sqrt{n}$, our notion of $f(n)$-cautious is the same as cautiousness defined in [18].

**Remark 7.6.** We mention that there exists $\sqrt{n}$-cautious random walks where it is important that the condition holds along a subsequence. Lacunary hyperbolic examples with this property are studied in Theorem 5.1 of [19].

**Definition 7.7.** [Span function for actions of $G$.] Suppose that a group $G$ acts by automorphisms on a group $H$. Let $S$ be a finite subset of $G$, and $T$ be a finite subset of $H$. Denote by $T_r$ the set of all elements $h$ in $H$ for which there exists some $g \in G$ of word length at most $r$ (with respect to $S$) and some $t \in T$ such that $g(t) = h$. Let $T_{r,n}$ be the set of all elements which are a product of at most $n$ elements in $T_r$. Define $\text{span}_{S,T,G \acts H}^r(n)$ to be the cardinality of $T_{r,n}$. We also call the set $T_{r,n}^{S,T,G \acts H} = T_{r,n}$ obtainable.

**Remark 7.8.** The asymptotics of $\text{span}_{S,T,G \acts H}^r(n)$ do not depend on the choice of $S$ and $T$ when $S$ is a generating set for $G$, and the union of $g(t)$ for all $g \in G$ and $t \in T$ is a generating set for $H$. 

---

**Figure 5.** With positive probability the random walk on $\mathbb{Z}^2$ stays until time $t$ in the ball of radius $r = \varepsilon \sqrt{t}$.
We state below a criterion for the Liouville property for group extensions using cautiousness of the projected random walk.

**Theorem 7.9** (Cautiousness criterion for extensions). Let \( f : \mathbb{N} \to \mathbb{R} \) be an increasing function. Let \( 1 \to A \to G \to K \to 1 \) be a short exact sequence, where \( A \) is Abelian. Let \( \mu \) be a non-degenerate finite entropy probability measure on \( G \) whose projection \( \mu_K \) to \( K \) is \( f \)-cautious and Liouville. Fix some finite generating set \( S \subset K \) and a finite set \( T \subset A \). Assume that for every \( \varepsilon > 0 \) there exist a \( \delta > 0 \), \( C_{\varepsilon} > 0 \) such that

\[
\text{span}_{S,T,G \to H}(\delta f(n), n) \leq C_{\varepsilon}(1 + \varepsilon)^n(*)
\]

for all \( n \).

(1) If the set \( T \) generates \( H \), then the random walk \((M, \mu)\) is also Liouville.

(2) More generally, take \( h \in H \) and consider \( T = T_h = \{h\} \). Then if the assumption \((*)\) holds for \( T_h \), then \( h \) acts trivially on the boundary \((T, \mu)\).

**Remark 7.10.** The second claim of the theorem is indeed more general than the first one. Indeed, in view of Lemma 4.4 if the boundary of \((M, \mu)\) is non-trivial, then there exists \( h \in H \) acting non-trivially on the boundary. The second claim also applies in the case where \( H \) is not normally finitely generated.

**Remark 7.11.** We have mentioned that for \( f(n) = n \) cautiousness implies the Liouville property, so that if \( f(n) \leq n \) we can omit mentioning the Liouville property in the assumption of the criterion. But the criterion makes sense also for \( f(n) > n \) (in particular this applies for some non-symmetric finite first moment random walks with positive drift). We mention also that for a symmetric finite first moment measure \( \nu \) the Liouville property \((G, \nu)\) is equivalent the fact that \((G, \nu)\) has 0 drift, see [38].

The assumption of the theorem above applies in particular for \( G = \mathbb{Z}^d \) and \( H = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \). In this case we can take \( \phi(n) = Cn^d \) and \( f(n) = \sqrt{n} \). We have \( \phi(f(n)) \sim n^{d/2} \). So we control the span, assuming that \( d = 1 \) and the first moment is finite or \( d = 2 \) and the second moment is finite, and then we recover the well-know fact that in these examples the Poisson boundary on the wreath product is trivial.

Before we prove the theorem we first explain in the claim below a straightforward observation about normal form of the walker on a semi-direct product. In the claim and its proof below we use the same notation for subgroup \( H \), quotient group \( K \) and their image for the canonical embedding to the semi-direct product \( K \ltimes H \).

**Claim 7.12.** Let \( G \) be a finitely generated semi-direct product \( K \ltimes H \). Consider a probability measure on \( G \), with the support \( U \). Let \( V \) be the subset of \( H \), consisting of \( h \) such that there exists \( u \in U, u = hk, h \in H, k \in K \). Let \( X_n \) be a trajectory of the random walk on \( G \) and \( Y_n \) be this trajectory’s quotient to \( K \). Let \( l_K(Y_m) < C \) for \( 1 \leq m \leq n \). Then \( X_n \) can be written in the form \( h'k \) where \( h' \in H, k' \in K \) and where \( h' \) is a product of at most \( n \) elements of the form \( kuv^{-1} \) where \( k \in K : l_K(k) < C \), and \( v \in V \).

**Proof.** We prove the statement by induction on \( m \). The base \( m = 1 \) is immediate. Now suppose the statement is true for \( X_{m-1} \). Consider \( X_m \). By induction we know that \( X_{m-1} \) can be written in the form \( h_{m-1}k_{m-1} \). Here \( l_K(k_{m-1}) < C \) and \( h_{m-1} \) can be written as the product of at most \( n - 1 \) elements of the form \( kuv^{-1}, k \in K : |k| < C, u \in U \). We have \( h_{m-1}k_{m-1}hk = h_{m-1}(k_{m-1}hk_{m-1}^{-1})k_{m-1}g \). Since \( l_K(k_{m-1}) < C \) it follows that \( X_m \) can be written in the form \( hk \) where \( h \) is a product of at most \( n \) elements of the form \( kuv^{-1} \) where \( k \in K : |k| < C \), and \( u \in U \).
Lemma 7.13 (Cautiousness criterion for semi-direct products). Assume that $G$ is a semi-direct product corresponding to the short exact sequence,

$$1 \to H \to G \to K \to 1,$$

where $K$ is finitely generated. Let $\mu$ be a finite entropy probability measure on $G$, $\mu_K$ its projection to $K$. We assume that $\mu_K$ is $f$-cautious and Liouville. Let $S$ be a finite generating set for $K$. Assume that the set $T$ consisting of $k \in K$ such that there exist at least one $h \in H$ where $hk$ is in the support, is finite. Assume also that for every $\varepsilon > 0$ there exist a $\delta > 0$, $C_\varepsilon > 0$ such that

$$\text{span}_{S,T,G \to H}(\delta f(n), n) \leq C_\varepsilon(1 + \varepsilon)^n(*)$$

for all $n$. Then the random walk on $M$ is Liouville.

Proof. We want to prove for all $\varepsilon > 0$ that the entropy of $\mu^{\ast n}$ is at most $\varepsilon n + o(n)$ which will imply that the entropy is $o(n)$ and thus the boundary is trivial.

Observe that by the cautiousness of $(K, \mu_K)$ there exists a subsequence $n_i$ such that the following holds. For each $n_i$ with probability bounded below by some $p_\varepsilon > 0$, the projected random walk $Y_{n_i}$ has word length less than $\varepsilon f(n_i)$ for $0 < n_i \leq n_i$.

Thus, by Claim 7.12 with positive probability bounded below by $p_\varepsilon$, at times $n_i$ our random walk on $G$ is contained in $(T_{\varepsilon f(n), n}B_K(\varepsilon f(n)))$. Here $T_{\varepsilon f(n), n}$ is the obtainable set $T_{r,n}$ for $r = f(n)$ from the definition of the Span function for $K \sideset{\to}{\sim} H, S, T$. We recall $B_K(r)$ denotes the ball of radius $r$ in the group $K$ and we omit in this notation the generating set (if we have fixed some generating set).

By assumption of the lemma, this set is of size at most $\exp(\varepsilon n + o(n))$ and thus by convergence of the entropy with probability 1 to its limiting value (Shannon-McMillan-Breiman type theorem, see [32], Thm 2.1 or [13]), we know that the entropy function of our random walk is $\leq \varepsilon n + o(n)$ for all $\varepsilon$ and thus is sublinear which implies the triviality of the boundary.

We now prove the theorem.

Proof. As we have mentioned in the Remark 7.10 it is enough to prove the second claim of the Theorem. We have $h \in H$ and we assume that the assumption (*) of the theorem holds for the set $T_h = \{h\}$. That is, that for a generating set $S$ of $K$ and for every $\varepsilon > 0$ there exist a $\delta, C_\varepsilon$ such that $\text{span}_{S, h, G \to H}(\delta f(n), n) \leq C_\varepsilon(1 + \varepsilon)^n$ for all $n$. We want to prove that $h$ acts trivially on the boundary $(M, \mu)$.

Apply Lemma 4.16 to $G$, our short exact sequence and $h$. Consider the obtained group $G_h$. By the above mentioned Lemma, it is sufficient to prove that $h$ acts trivially on the boundary of $G_h$, for some measure on $G_h$ with the same projection to $K$. Observe that we can choose the measure on $G_h$, with the same projection to $K$ as $\mu$, in such a way that it satisfies the assumption of Lemma 7.13 in particular about finiteness of the set $T$. This completes the proof of the theorem. □

7.1. Applications to the triviality of the boundary for metabelian groups. In the the proof of the corollary below we will need upper bounds for the size of obtainable sets. In the case of extensions of torsion groups, obtainable are subsets of finite dimension vector subspaces, and their dimension was estimated in the previous section, Lemma 6.7 (this we will use in the proof of the first claim of the corollary). In case of torsion-free Abelian groups, the obtainable sets are subset of a vector space over $Q$, and so to estimate the cardinality we need not only the dimension of an ambient vector space, but also a bound.
on coefficients (for elements in this set) in a fixed basis. This is done in the following Lemma.

**Lemma 7.14 (The cardinality of obtainable sets).** Let $K$ be an Abelian group acting on an Abelian group torsion-free group $= A$, such that $A \otimes \mathbb{Q}$, considered as a $\mathbb{Q}[K]$ module, has dimension $d$ (see Definition [6.7]). Let $S$ be a finite subset of $G$ and $T$ be a finite subset of $A$. Consider the obtainable set $T_{r,n}$ defined in Definition [7.7]. There exist a constant $C > 0$ such that its cardinality $\text{span}_{S,T,K \rightarrow A}(r,n)$ satisfies

$$
\text{span}_{S,T,K \rightarrow A}(r,n) \leq ((n + 2)^{C})^{d}.
$$

**Proof.** We first note that by [6.1] there is a subgroup $K'$ isomorphic to $\mathbb{Z}^d$ of $K$ and a finite set $T'$ of elements of $A \otimes \mathbb{Q}$ (we can assume that they belong to $H$) such that the set of elements of the form $k(t')$ where $k \in K, t' \in T'$ spans $A \otimes \mathbb{Q}$. Thus every element in $T$ is the span (over $\mathbb{Q}$) of finitely many conjugates of $k(t)$.

We now will show that there exists $C, D$ and $m$ so that every element in $\tilde{k}(t)$ where $\tilde{k} \in B_r(K)$ and $t \in T$ can be written as a sum of elements of the form $qk(t')$ where $k \in B_m, t \in T'$ and $q$ is a rational number with denominator $D^{r+1}$ and numerator with absolute value at most $C^{r+1}$. Since a sum of $n$ elements of this form can be written in the same way (replacing the maximal numerator absolute value with $nD^{r+1}$) the lemma will follow.

To see the existence of such $C, D$ and $m$ note that there exists a finite subset $P \subset K$ such that for all $s \in S$ and $t' \in T'$ it holds $s(t') = \sum_{p \in P, t' \in T'} q p t(p)$, where $q_{p,t}$ are rational numbers. Let $D$ be a common denominator of all the $p k s t$ for all $s \in S$ and let $m$ be large enough so that $P \subset B_m(K)$. Let $C'$ be the maximal numerator of $q_{k,t}$ and $C$ be $C'[P||T'||D]$. Apply an element $s \in S$ to a rational linear combination of $k(t')$ with common denominator $D^{r+1}$ and maximal numerators $C^{r+1}$, where $k \in B_m, t' \in T'$ and obtain a rational linear combination of $k(t')$ where $k \in B_{r+1}m$ have common denominator $D^{r+2}$ and numerators bounded above by $C^{r+2}$ (since we are summing up at most $|P|T'$ different rationals, multiplying the denominator by $D$ and each numerator by at most $C'$). This yields the desired result.

□

**Corollary 7.15.**

1. (p-torsion Abelian-by-Abelian case, dimension $\leq 2$) Assume that $G$ has as subgroup $\mathbb{Z}/p\mathbb{Z}^X$, the quotient is finitely generated Abelian, and the dimension is $\leq 2$. Then for any centered finite second moment measure $\mu$ on $G$ the Poisson boundary of the random walk $(G, \mu)$ is trivial. Moreover, if the dimension $\leq 1$, then any centered first moment measure $\mu$ on $G$ has trivial Poisson boundary.

2. (torsion-free metabelian case, dimension $\leq 1$) Let $G$ be a torsion-free f.g. metabelian group, $1 \to A \to G \to K \to 1$ and assume that the dimension of $G$ is $\leq 1$. Take a centered measure $\mu$ of finite second moment on $G$. Then the random walk $(G, \mu)$ has trivial Poisson boundary. Moreover, if the dimension is 0, then it is enough to assume finiteness of the first moment.

**Proof.** 1) First observe that the Central Limit Theorem implies that any centered finite second moment random walk on $\mathbb{Z}^d$ and, more generally, on any f.g. Abelian group, is $\sqrt{n}$-cautious (see Remark 7.3). Apply Theorem 7.9 for $f(n) = \sqrt{n}$. Observe that the obtainable set with $\delta = \delta f(n)$ is contained in a vector space, which in view of Lemma 6.7 has dimension at most $C(\delta f(n)^2) = C\delta^2 n$. Thus the cardinality of these obtainable sets is at most $\exp(\ln p\delta^2 n)$. Therefore the assumption of Theorem 7.9 are satisfied, and
we conclude that our random walk has trivial boundary. Therefore, the claim 1) of the corollary for finite centered second moment measures follows.

Now if we have a finite centered first moment random walk, we apply Theorem 7.9 for \( f(n) = n \). By the Law of Large Numbers, any finite first moment centered measure on \( \mathbb{Z}^d \) (and more generally, on any finitely generated Abelian group) has zero drift, and \( \mu(X_n)/n \to 0 \) almost surely. Thus we have \( f(n) \)-cautiousness for \( f(n) = n \). The fact that dimension of our extension \( \leq 1 \) implies by Lemma 6.7 (Lemma about dimension of obtainable vector spaces) that for all \( \delta \), \( \text{span}(\delta n, n) \leq C \delta n \) for some \( c > 0 \) and all \( n \). Which implies that the bounds of Proposition 7.9 are satisfied proving the second part of the first claim.

2) First we deal with the dimension one case. Fix a generating set \( S_K \) for \( K \). Since \( K \) is finitely generated and virtually Abelian we know that the projected random walk on \( K \) is \( f \)-cautious. By Lemma 7.14 we know that \( \text{span} S, T, K \bowtie A \subset Bn, n \subset (n + 2)^C \delta n \) and again conclude that for every \( \epsilon > 0 \) there exist some \( \delta \) such that \( \text{span} S, T, K \bowtie A \subset C (1 + \epsilon)^n \). Thus by applying Theorem 7.9, we see that the boundary is trivial.

As we have already explained in Section 4, the general metabelian case can be reduced to \((p\text{-torsion})\)-by-Abelian case and \((\text{torsion-free})\)-by-Abelian case. Claim 1) of Corollary 7.15 gives a sufficient condition for boundary triviality for \((p\text{-torsion})\)-by-Abelian group. For second moment centered random walks, we will see in Section 8 that this condition is not only sufficient but necessary. Namely, in the \((\text{torsion})\)-by-Abelian case we will show that dimension \( \geq 3 \) implies non-triviality of the boundary for any finite entropy measure.

Remark 7.16. It is clear that in the claim 1) of the corollary the condition of finite second moment is essential. Indeed, we recall again the well-known example of \( \mathbb{Z}^2 \wr \mathbb{Z}/p\mathbb{Z} \). We know that finite first moment random walks can have transient projection to \( \mathbb{Z}^2 \), and that some of them (for example those where support belong to the union the base group and the lamp) have non-trivial boundary.

In Claim 2) of the corollary the assumption of finite second moment is also important, as we explain in the following example.

Remark 7.17. The Lamplighter Baumslag-Solitar groups for \( d = 1 \) are finite second moment measures that have trivial boundary, as we will see in Example 8.11. In these examples finite first moment random walks have non-trivial boundary.

Remark 7.18. (2-dimensional (torus-free) case with non-trivial boundary). .

Lamplighter Baumslag Solitar group for \( d = 2 \) will be shown to have non-trivial boundary for any non-degenerate finite entropy measure, as we will see in Example 8.11. And a similar more general family of examples will be discussed in Proposition 8.12.
The following corollary explains the implication of corollary \[7.15\] to more general linear groups.

**Corollary 7.19.** Let \( F \) be either a function field over at most 2 variables of positive characteristic or \( F \) is a function field over at most 1 variable in characteristic 0. Then any virtually solvable group, linear over \( F \) is Liouville: any finitely supported (and more generally any finite second moment) centered measure on this group has trivial boundary.

**Proof.** By Theorem \[5.3\] it suffices to prove for \( 2 \times 2 \) basic blocks that the corresponding random walks are Liouville. Observe that these blocks have dimension at most 2 if the characteristics is positive, and at most 1 if characteristics 0. Indeed, this follows from Lemma \[6.12\]. By Corollary \[7.15\] the corresponding random walks have trivial Poisson boundary proving the corollary. \( \square \)

**Example 7.20.** [Baumslag groups \( B_{d,p} \) revisited]
We recall that \( B_{d,p} \in GL_2(\mathbb{Z}/p\mathbb{Z})(X_1, ..., X_d) \) is the group of \( 2d + 1 \) matrices of the form
\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & X_i \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & X_i + 1 \\
\end{pmatrix}.
\]

Here \( 1 \leq i \leq d \). If we consider only the generators of the first two forms, then what we have is the wreath product \( \mathbb{Z}^d \wr \mathbb{Z}/p\mathbb{Z} \) If \( d = 1 \) the Baumslag group is a quotient of the 2 dimensional lamplighter. For \( d = 3 \) the non-triviality of the boundary is proven in \[16\]. For \( d = 2 \) this group is a particular case of 2-dimensional extension of a torsion group (see example \[6.23\]) and so the boundary is trivial for any centered random walk with a finite second moment (by Corollary \[7.15\]).

**Example 7.21.** Consider the subgroup \( G_{2,3,x} \) of \( Gl_2(\mathbb{Q}(X)) \) generated by
\[
\begin{pmatrix}
1 & 1 \\
0 & 1 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & 2 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & 3 \\
\end{pmatrix}, \quad
\begin{pmatrix}
1 & 0 \\
0 & X \\
\end{pmatrix}.
\]

The dimension is 1, so any finite second moment centered random walk on this group has trivial boundary.

To see why the dimension is 1 note that the 1 dimensional lamplighter is a subgroup of our group, and the corresponding module over \( \mathbb{Q} \) are the same for our group and for the lamplighter.

### 8. Non-triviality of the Boundary for \( d \geq 3 \).

In this section we consider a group \( K \) acting on an Abelian group \( B \). A particular case of the statement we will prove is that if the dimension is \( \geq 3 \) and \( \mu \) is a finite entropy non-degenerate measure, then the Poisson boundary is non-trivial. Observe that for torsion-Abelian \( (B) \) by a virtually Abelian group \( (K) \) and simple random walks (and for all non-degenerate finite second moment random walks) we therefore have a complete classification (if the dimension is \( \leq 2 \) then the boundary is trivial and if the dimension is \( \geq 3 \) the boundary is non-trivial).

**8.1. A criterion for the non-triviality of the boundary.** When we have an Abelian group \( B \), we say that a set of elements of \( B \) is linearly independent over \( k = \mathbb{Z}/p\mathbb{Z} \) or over \( k = \mathbb{Q} \) if their images in \( B \otimes k \) are linearly independent.
Theorem 8.1. Suppose we have an extension $1 \to B \to G \to K \to 1$. $B$ is Abelian. Let $A = \mathbb{Z}^3$ be a central subgroup of $K$. Assume that for some $b \in B$ the following holds: the elements of $\mathbb{Z}^3 b \in B$ are linearly independent, either over $k = \mathbb{Z}/p\mathbb{Z}$ or over $k = \mathbb{Q}$. Then for any non-degenerate finite entropy measure $\mu$ on $G$ the Poisson boundary is non-trivial, and moreover $b$ acts non-trivially on the boundary.

Before we start the proof of the proposition, we formulate some auxiliary statements. The lemma below we formulate for any field $k$, we will apply it for $k$ being a field of $p$ elements or $k = \mathbb{Q}$.

Let $B$ be a vector space over a field $k$. Assume that $1 \to B \to G \to K \to 1$ is a short exact sequence, and let $A$ be a subgroup of $K$. We consider the action of $A$ by conjugation on $B$. We say that an element $b \in B$ is small with respect to $A$ (and $K$), if the elements of the orbit $Ab$ are not linearly independent over $k$. Observe that $b$ being small is equivalent to the annihilator of $b$ in $k[A]$ being non-trivial.

If the short exact sequence is clear from the context, we will say that $G$ admits small elements with respect to $A$ if there exists an element of $B$, $\neq e_B$, which is small with respect to $A$.

Lemma 8.2 (Reduction to $B$ without small elements). Let $A = \mathbb{Z}^d$. $B$ is a vector space over a field $k$. Assume that $1 \to B \to G \to K \to 1$ is a short exact sequence of groups, let $A$ be a normal subgroup in $K$.

(1) The small elements of $B$ with respect to $A$ form a normal subgroup in $G$, which we denote by $B_{\text{small}}$.

(2) The quotient of $G/B_{\text{small}}$ does not admit small elements

Proof. 1) Observe that $B$ is a $k[A]$-module. Below we use additive notation for the group multiplication in our Abelian group $B$. We first prove that the small elements form a group. Consider $b, c \in B$ with $b$ and $c$ both small with respect to $A$.

Observe that the annihilator in $k[A]$ of $b \in B$ is equal to the annihilator of $-b$. Thus small elements are closed under taking inverses.

Now we prove that the sum of small elements is small. Observe that the annihilator of $b + c$ contains the intersection of the annihilator of $b$ with the annihilator of $c$. By the smallness of $b$ and $c$ we know that annihilators of $b$ and $c$ are non-zero. Since $k[A]$ is an integral domain, we see that this intersection is also a non-zero ideal, and thus $b + c$ is small. Suppose $r$ is an element of the annihilator of $b$. Then $frf^{-1}$, for $f \in K$, (which belongs to $k[A]$ since $A$ is normal in $K$) is in the annihilator of $fbf^{-1}$. Hence a conjugate of a small element is small and thus the subgroup is normal.

2) We now prove that if an element $b \in B$ admits a non-zero element $r \in k[A]$ such that $rb$ is small in $G$ then $b$ itself must be small in $G$. (And this will imply the second claim of the lemma). If $rb$ is small, then it is annihilated by some non-trivial element $s \in k[A]$. Thus $srb = 0$ and so $sr$ is an annihilator of $b$. Since $s, r$ are non-trivial and $k[A]$ is an integral domain, $sr$ is non-trivial and so $b$ is small. $\square$

Given a total preorder $w$, we say that elements $b_1$ and $b_2$ are equivalent if $b_1 \leq_w b_2$ and $b_2 \leq_w b_1$.

Lemma 8.3 (If no small elements, then there is an order with three properties). Consider a short exact sequence of groups

$$1 \to B \to G \to K \to 1.$$ 

$A = \mathbb{Z}^d$ is as normal subgroup of $K$. 

We assume that $B$ does not have non-zero small elements with respect to $A$. There exists a total preorder $w$ on $B$ satisfying

1. Any $A$ orbit of $B$ does not contain equivalent elements with respect to this preorder.
2. The total preorder $w$ is $k[A]$-invariant.
3. If for some subset of $B$ its elements are pairwise not equivalent with respect to $w$, then they are linearly independent over $k$.

**Proof.** Fix a left invariant order $w$ on $A$. This induces a left invariant preorder on $k[A]$ where we say $r \geq r'$ if the non-zero monomial of the largest degree with respect to $A$ of $r$ is larger than the corresponding monomial for $r'$. We slightly abuse the notation and also call this order $w$. We also will use the same notation $w$ for the orders on $B$ and $B_S$ which will be defined below.

Note again that $B$ is a $k[A]$ module. Let $S$ be a maximal $k[A]$ linearly independent subset of $B$. Choose an arbitrary total ordering on $S$. Let $B_S$ be the $k[A]$ module generated by $S$. Consider two elements $x = \sum_i r_i^x s_i^x$, $y = \sum_i r_i^y s_i^y$, where $r_i^x, r_i^y$ are non-zero elements in $k[A]$, and where $s_i^x, s_i^y \in S$. In $B_S$ we say $x \leq_w y$ if either the $w$-largest element of $s_i^x$ is strictly $w$-larger than any element of $s_i^y$, or if the $w$-largest elements of $s_i^x$ and $s_i^y$ are equal and we denote their index by $j$, we require that $r_j^x \geq r_j^y$ with respect to the order on $k[A]$ (which we fixed at the beginning of the proof). So we defined $w$ on $B_S$, now we want to extend $w$ to $B$. To do this, we first check that $w$ defined on $B_S$ has 1), 2), 3) of the lemma (for this we only need, as in definition of $w$ on $B_S$, that $A$ is orderable).

To prove 1) for $w$ on $B_S$, observe that if $x = \sum_i r_i^x s_i^x$, then $hx = h(\sum_i r_i^x s_i^x) = \sum_i hr_i^x s_i^x$, hence these two elements $x$ and $hx$ have the same maximal element $s_j$ with at least one non-zero coefficient in this decomposition, and the maximal coefficients of this $s_j$ are not equal.

To see 2) observe the following. Let $x = \sum_i r_i^x s_i^x$ and $y = \sum_i r_i^y s_i^y$. Assume that $x \geq_w y$. Then either the largest element with non-zero coefficient of $x$ is strictly larger than that of $y$. Then, as we have just mentioned, these largest elements do not change by the action of $h$, and so $hx \geq_w hy$. Otherwise they have the same largest elements $s_i = s_j$, and we have $r_i^x \geq_w r_j^y$, then $hr_i^x \geq_w hr_j^y$ (since $w$ is a left invariant order on $k[A]$).

Finally, 3) holds for $w$ on $B_S$: let $T$ be a set of elements in $B_S$ which have a $k$-linear relation, with non-zero coefficients for each $t \in T$. It suffices to show that there must exist $t_1, t_2 \in T$ which are $w$-equivalent. Indeed, since there is a linear relation for elements of $T$, note that at least two elements of $t_1, t_2 \in T$ have the same maximal elements $s = s_i^{t_1} = s_i^{t_2}$. Also note that among those with this maximal element equal to $s$, there must exist two $(t_1$ and $t_2$) with equivalent $r_i$ in $k[A]$, $t = t_1$ and $t_2$.

We now extend the preorder from $B_S$ to $B$. Given two elements $b_1, b_2 \in B$ we say that $b_1 \leq_w b_2$ if there exist some non-zero $r \in k[A]$ such that $rb_1, rb_2 \in B_S$ and $rb_1 <_w rb_2$. First note that by the maximality of $S$, for every $b \in B$ there exists some $r$ such that $rb \in B_S$.

We now argue that this induced relation is defined for every pair $b_1, b_2$. Let $r_1, r_2$ be such that $r_1 b_1, r_2 b_2 \in B_S$ then $r_1 r_2 b_1, r_1 r_2 b_2$ are also both in $B_S$ and, since $k[A]$ is an integral domain, $r_1 r_2$ is non-zero. Thus, since $r_1 r_2 b_1, r_1 r_2 b_2$ are comparable (by the totality of the preorder on $B_S$) the relation is total on $B$.

We now explain that this relation is indeed a preorder. Note that if $b_1 <_w b_2 <_w b_3$, then there exist $r_1, r_2$ so that $r_1 b_1, r_1 b_2 \in B_S$ and $r_2 b_2, r_2 b_3$ are in $B_S$ then $r_1 r_2 b_1, r_1 r_2 b_2, r_1 r_2 b_3$ are all in $B_S$ and by the multiplication invariance of the order on $B_S$ it holds $r_1 r_2 b_1 <_w r_1 r_2 b_2 <_w r_1 r_2 b_3$ and thus $b_1 <_w b_3$. Therefore the induced relation is a preorder.
We now note that the restriction of this new preorder to $B_S$ induces the original order $w$ on $B_S$. This follows from property 2) of the order on $B_S$. Observe that if there exist $r$ such that $rb_1, rb_2 \in B_S$ and $rb_1 <_w rb_2$ then for all $r'$ such that $r'b_1, r'b_2 \in B_S$ we have $r'b_1 <_w r'b_2$.

Now note that any two elements in $B$ are comparable. These two facts hold because of the following. If $r_1 b_1 <_w r_2 b_1$ and $r_2 b_1, r_2 b_2 \in B_S$, then there exists some common multiple of $r_3 \in k\mathbb{Z}^d$ such that $r_3 = ar_1 = br_2$ (since $k[A]$ is an integral domain, so that the field of fractions is well-defined).

Now let us show that since the order on $B_S$ has 1), 2), 3), then the order on $B$ also has these properties. First we observe that 2) holds: 2) If $b_1, b_2 \in B$ and $b_1 \leq_w b_2$, then $rb_1 \leq_w rb_2$. Indeed, by definition $b_1 \leq_w b_2$ means that there exists $r_1$ such that $r_1 b_1, r_1 b_2 \in B_S$ and $r_1 b_1 <_w r_1 b_2$. Then by the property 2) on the order on $B_S$ we have that $r_1 rb_1 <_w r_1 rb_2$.

Now we check properties 1) and 3). 1): if we have two equivalent elements of an $A = \mathbb{Z}^d$ orbit, then multiplying them from the left we get (by property 2) two equivalent elements in $B_S$ which are also in an $A$ orbit.

3) If a subset of elements of $B$ are pairwise not equivalent with respect to $w$, then they are linearly independent over $k$. Otherwise, we multiply them on the left, and get a linear dependence in $B_S$. Observe that by the construction of the extended order if $rb_1, rb_2$ are not equivalent, for a non-zero element $r$ in $k[A]$, then $b_1, b_2$ are not equivalent. Hence such linear dependence in $B_S$ would be a contradiction.

\[\Box\]

**Definition 8.4.** [Recurrent/transient sequences]
Given a group $\Gamma$, a measure $\mu$ on $\Gamma$. Fix a sequence $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$ and consider the following event. Let $X_1, X_2, \ldots, X_n, \ldots$ be a trajectory of the random walk $(\Gamma, \mu)$. We consider the probability that there exist infinitely many $i$ such that $\gamma_i = X_i$. If this happens with probability 1, we say that the sequence $(\gamma_i, i)$ is a recurrent sequence for $(\Gamma, \mu)$. If all sequences are transient for the random walk $(\Gamma, \mu)$ we say the random walk $(\Gamma, \mu)$ is strongly transient.

**Definition 8.5.** [Uniform strong transience] We say that a random walk on $\Gamma$ is uniformly strongly transient, if for any sequence $\gamma_i \in \Gamma$, $i \in \mathbb{N}$, expected number of bad hits $(X_i = \gamma_i)$ is finite.

**Remark 8.6.** If a random walk is uniformly strongly transient, then there exists a positive constant $C$ (not depending on the sequence $\gamma_i$) such that this expectation is bounded by $C$.

Indeed, this expectation is the sum of expectations $E_i$ to have a bad hit at time $i$. For each $i$ consider the supremum of $\mu^\ast i(\gamma)$. (This is the expectation to have a bad hit at time $i$ if $\gamma_i = \gamma$). Observe that this supremum is indeed a maximum, and choose $\gamma_i$ to be an element that realises this maximum. If the sum is not summable, than for this sequence $\gamma_i$ the expectation would not be finite.

**Lemma 8.7** (Main lemma). Let $G$ be a semi-direct product, corresponding to a short exact sequence $1 \to B \to G \to K \to 1$. Let $A$ be a torsion-free Abelian group which is central in $K$. Assume that $B$ is a vector space over a field $k$. Let $b \in B$ be a non $A$-small element. Suppose $\mu$ is a finite entropy measure on $G$. Consider the normalised restriction (of its projection to $K$) to $A : \mu_A = \frac{\mu|A}{\mu_K(A)}$. We assume that the random walks $(A, \mu_A)$ is uniformly strongly transient. Then $b$ acts non-trivially on the Poisson boundary.
Proof. First observe that we can assume that \( b \in S^+S^- \), where \( S \) is the sub-semigroup generated by the support of \( \mu \), and \( S^- = \{ s^{-1}, s \in S \} \). Indeed, for any measure violating this assumption, on any group, the element \( b \) acts non-trivially on the boundary (see e.g. \([20]\), Remark 5.4; for more general statements see \([36]\), Lemma 2.9). We can therefore assume that there exists \( g \in G \), such that \( g \in \text{supp} \mu^{\ast k} \) and \( gb \in \text{supp} \mu^{\ast l} \) for some \( k, l \geq 1 \).

Observe also, that taking, if necessary, a convex combination with the atomic measure supported on \( e \) (this change does not change the Poisson boundary), we can assume that \( e \in \text{supp} \mu \). We conclude in this case that there exists \( g \in G \) and \( k \geq 1 \) such that \( g, gb \in \text{supp} \mu^{\ast k} \). Replace, if necessary, \( \mu \) by \( \mu^{\ast k} \) and assume that \( g, gb \in \text{supp} \mu \).

Now observe that we can assume that \( B \) does not have \( A \)-small elements (if it is not the case, we know by Lemma 8.3 that small elements form a normal subgroup, and that if we quotient \( G \) over this subgroup the image of not \( A \)-small element \( b \) remains not \( A \)-small. And this quotient does not admit \( A \)-small element. Observe that then if \( b \) acts non-trivially on the Poisson on the quotient group, then it acts non-trivially on the Poisson boundary of \((G, \mu)\).

So now we assume that \( B \) does not have \( A \)-small elements. Then on \( B \) there is an order with 1), 2), 3), which we denote \( w \) (by Lemma 8.3).

Put \( \Delta = \{g, gb\} \). We want to prove that \( \Delta \)-restriction entropy (see Definition 3.10) is positive.

Consider a random walk \( X_1, \ldots, X_n \) on \( G \) and its projection \( A_1, \ldots, A_n \) to \( K \). We fix all increments that are not in \( A \) (this leaves a positive proportion of elements, since \( g, gb \in A \) and these elements belong to the support of \( \mu \)). We fix then also all increments \( \neq g, gb \). Consider corresponding times instants \( t_1, \ldots, t_k \) where multiplication by elements of \( \Delta \) occur. Let \( Z_i \) be the product of all increments not in \( A \) until time \( t_i \) and \( W_i \) be the product of all increments in \( H \) until time \( t_i \). Consider an equivalence relation on \( A \times \mathbb{N} \), saying that \((h, k) \) is equivalent to \((f, l) \) if \( Z_khgbg^{-1}h^{-1}Z_{k-1}f \) is \( w \)-equivalent to \( Z_ifgbg^{-1}f^{-1}Z_{i-1} \).

Observe that property 1) of the order \( w \) implies that the equivalence relation satisfies the condition \((x, k) \) is not equivalent to \((y, k) \) whenever \( x \neq y \), \( x, y \in A \). (this condition will be called ”admissible” in the Appendix). By the Law of Large Numbers, we know that the number of increments \( T_n \) by elements of \( A \) until time instant \( n \) is approximately \( \text{Constant} \).

By Lemma A.9 (proven in the Appendix), we know that the range function with respect to the equivalence relation \( E \) is linear in \( T_n \) (since this relation is admissible), and hence in \( n \).

We conclude that with positive probability there at least \( pn \) linearly independent elements of \( B \), corresponding to possible increments \( gb \) (rather than \( g \)) in the places where multiplication by these two elements occur. This shows that \( \Delta \)-restriction entropy is linear. By Lemma 8.13 we conclude that the action of \( b \) on the boundary is non-trivial. \( \square \)

Corollary 8.8. Take an extension as in Lemma 8.7 and assume that the central subgroup of \( A \) is \( \mathbb{Z}^3 \). Then the claim of this lemma holds for any non-degenerate finite entropy measure on \( G \).

Proof. By Claim 1 of Theorem A.10 (proven in the Appendix) we know that any non-degenerate random walk on \( \mathbb{Z}^3 \) is uniformly strongly transient. Hence, replacing if necessarily the measure by it convolution power (so that its restriction to \( A \) generates \( A \)), we can apply Lemma 8.7. \( \square \)

Now we are ready to prove the theorem of this section.
Proof. By the assumptions of the theorem we have a short exact sequence \( 1 \to B \to G \to K \to 1 \), where \( B \) is Abelian. \( A = \mathbb{Z}^3 \) is a central subgroup of \( K \). By assumption of the Theorem we also know that \( b \in B \) is such that the elements of \( \mathbb{Z}^3 b \) are linearly independent, either over \( k = \mathbb{Z}/p\mathbb{Z} \) or over \( k = \mathbb{Q} \). By definitions of smallness this says that that \( b \) is not small. We need to prove that \( b \) acts non-trivially on the Poisson boundary of \((G,\mu)\) for any finite entropy non-degenerate measure \( \mu \).

Replacing if necessary \( G \) by a semi-direct product \( G' \) with the same action of \( K \) on \( B \), since \( \mu \) is non-degenerate, we can use Theorem [3.1] and see that it is sufficient to prove our theorem for semi-direct products. Observe that if \( b \) is small for \( G \), then it is small for \( G' \) (since the definition depends only on the action of \( K \) on \( B \)). We use Lemma [4.16] for \( Q = B, G = G, K = K, q = b \) and conclude that it is sufficient to assume that \( B \) is generated as a normal subgroup by \( b \) in our group \( G = G_q \). First assume that \( b \) is \( p \)-torsion, \( p \) is prime. Then \( B \) is a vector space over \( \mathbb{Z}/p\mathbb{Z} \). Then \( G, b \) and \( \mu \) satisfy the assumption of Lemma [8.7] and Corollary [8.8]. Thus by the latter corollary we conclude that \( b \) acts non-trivially on the boundary.

Now assume as in the previous case that \( \mathbb{Z}^3 b \) are linearly independent over \( \mathbb{Z}/p\mathbb{Z} \). Consider \( B' = B/pB \) and a short exact sequence

\[ 1 \to B' \to G' \to K \to 1, \]

where \( G' = G/pB \). If the image of \( b \) acts non-trivially on the boundary of the projected r.w. to \( G' \), then \( b \) acts non-trivially on the boundary of \( G \). The image of \( b \) is of order \( p \), so by the previous case we known that the action of \( b' \) (and hence of \( b \)) on the boundary is non-trivially.

Now we assume that \( \mathbb{Z}^3 b \) are linearly independent over \( \mathbb{Q} \). Let \( T_B \) be the torsion subgroup of \( B \). Consider

\[ 1 \to B/T_B \to G/T_B \to K \to 1 \]

Let \( b' \) be the image of \( b \) in \( T_B \). Observe that \( \mathbb{Z}^3 b' \) are linearly independent over \( \mathbb{Q} \), so without loss of the generality we can assume that \( T_B = \{e\} \). Consider \( B' = B \otimes \mathbb{Q}, B \) is a subgroup of \( B', b \in B \in B' \). There exist a semi-direct product \( G' \),

\[ 1 \to B' \to G' \to K \to 1 \]

where the action of \( K \) on \( B' \), restricted to \( B \) is the action we had for our previous short exact sequence. Consider a non-degenerate measure \( \mu' \) on \( G' \), with the same projection to \( K \) as \( \mu \). Applying Lemma [8.7] for \( G', \mu' \) and \( b \), we conclude that \( b \) acts non-trivially on the boundary of \((G',\mu')\). Applying the reduction to single lemma (Lemma [4.16]) twice: once to \( G \) and also to \( G' \), we observe that the reduction groups \( G_q = G_q' (q = b) \). Thus we conclude that \( b \) acts non-trivially on the boundary of \((G,\mu)\).

Remark 8.9. The assumption of (uniform) strong transience (and not simply transience) is essential in Lemma [8.7]. Under assumption of transience there are examples where the action on the boundary is trivial.

Proof. Indeed, take \( G = \mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \). We denote by \( r \) the generator of the cyclic factor above. We denote by \( z \) the generator of the infinite cyclic subgroup of the lamplighter. The subgroup \( B \) is \( \mathbb{Z}/2\mathbb{Z}^2 \), and \( K = \mathbb{Z}^2 \). Let \( K \) be the subgroup generated by \( z, r \) Let \( A \) be the subgroup generated by \( z \). Consider the measure on \( \mathbb{Z}^2 \) with a charge \( 1/3 \) on \( z \), \( (1/6) \) on \( z^{-1} \), \( 1/3 \) on \( zr \) and \( 1/6 \) on \( z^{-1}r^{-1} \) (or any other finitely supported measure on \( \mathbb{Z}^2 \), which has a symmetric projection on \( \mathbb{Z} = \mathbb{Z}^2 / < r > \) and such that its restriction on \( A =< z > \) has a non-zero mean). It is clear that the Poisson boundary of (the central
extension) \((\mathbb{Z} \wr \mathbb{Z}/\mathbb{Z} \times \mathbb{Z}, \mu)\) is equal to the boundary of the projection of the random walk to \(\mathbb{Z} \wr \mathbb{Z}/\mathbb{Z}\), and any element of the configuration acts trivially on the boundary. But \((A, \mu_A)\) is transient.

\[\square\]

**Corollary 8.10.**  
(1) Consider a metabelian basic block \(M\) of a nilpotent-by-Abelian linear group \(a\) with short exact sequence \(1 \to B \to M \to K \to 1\). By our convention such basic blocks are either torsion-free or \((p\text{-torsion})\)-by-Abelian. If \(b \in B\) has dimension \(\geq 3\) (see Remark 6.4 for the notion of the dimension of elements), then \(b\) acts non-trivially on the Poisson boundary. In particular, if \(M\) has dimension \(\geq 3\), then the Poisson boundary of any non-degenerate finite entropy measure is non-trivial.

(2) Moreover, let \(G\) be an upper triangular linear group with a valid block of dimension at least 3. Then the Poisson boundary of any non-degenerate finite entropy measure is non-trivial.

**Proof.** By Lemma 6.6 we know that if \(M\) or \(b\) has dimension \(\geq 3\), then there exists \(b'\) and a subgroup \(A = \mathbb{Z}^3\) of \(K\) such that the module generated by \(b'\) is freely generated as a \(k[A]\) module. The first statement follows therefore from Theorem 8.1. The second statement then follows from 5.3.

\[\square\]

### 8.2. Examples with non-trivial boundary and further applications to linear and metabelian groups.

First we recall that in Example 6.22 we gave a sufficient condition for two by two upper triangular matrices to have dimension at least 3, and thus it follows from Corollary 8.10 that the non-degenerate finite entropy random walks on such groups have non-trivial Poisson boundary.

Below we discuss examples of dimension 2 metabelian groups and measures of non-trivial boundary, which on some of such groups can be chosen finitely supported.

**Example 8.11** (Lamplighter BaumSlag Solitar groups \(d = 1\) and \(d = 2\)). Recall our notation for matrices

\[
\delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},
\]

\[
M_x = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}, \quad M_y = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix}
\]

The group \(G_{x,2}\) (\(d=1\)) generated by \(M_x, M_2\) and \(\delta\) has non-trivial boundary for some finite first moment symmetric random walks.

The group \(G_{x,y,2}\) (\(d=2\)), generated by \(M_x, M_y, M_2\) and \(\delta\) has nontrivial boundary for any finite entropy non-degenerate measure.

**Proof.** Possibly by taking a convolutional power we may assume that \(\mu\) contains \(\delta\) and the identity \(e\) in its support. Note that we have a canonical homomorphism from \(G_{x,2}\) to \(\mathbb{Z}^2\) and from \(G_{x,y,2}\) to \(\mathbb{Z}^3\) given by mapping \(M_x, M_y, M_2\) to distinct generators and sending \(\delta\) to the identity. Put \(\Delta = \{e, \delta\}\). Observe that \(\Delta\)-restriction entropy 3.10 grows linearly.

Indeed, to see this for \(d = 1\) observe

\[
\sum_{j,k} \varepsilon_{j,k} 2^k x^j \neq 0,
\]
and that for $d = 2$ observe that
$$\sum_{i,j,k} \varepsilon_{i,j,k} 2^k x_i y_j \neq 0,$$
if $\varepsilon_{j,k}, \varepsilon_{i,j,k}$ take value $0, 1$ and $-1$ and at least one of the coefficient is $\neq 0$. Here the sum is taken over integers $i, j$ and $k$.

For our application for group with $d = 1$ we use that the existence of finite first moment symmetric random walks with transient projection to $\mathbb{Z}^2$. Indeed, it is well-known that such measures exist and can be chosen to have finite $2 - \varepsilon$ moment (for example, we can choose them of the form $\nu \times \nu$, where $\nu$ is a symmetric measure in the domain of the attraction of a Stable Law, and apply the Local Limit theorem for $\nu$, see e.g. [6].

For our application for $d = 2$ we remind that the projected random walk on $\mathbb{Z}^3$ is transient for any non-degenerate random walk (see e.g. [54]). Hence the number of visited points in $\mathbb{Z}^2$ (correspondingly in $\mathbb{Z}^3$) grows linearly (in the number of steps, in other words the range function grows linearly, see e.g. Theorem 1.4.1 Spitzer [54]). If we condition on all $\Delta$ increments except for the final multiplication at $X_i$ with a projection to given points in $\mathbb{Z}^2$ or correspondingly $\mathbb{Z}^3$, we see that the $\Delta$-restriction entropy is linear and thus the boundary is nontrivial by 3.13.

Also observe that the dimension our groups is equal to $d$ (for $d = 1$ and $d = 2$). Indeed, tensor by $\mathbb{Q}$ and observe that the group becomes an extension of an Abelian group by $\mathbb{Z}$, correspondingly by $\mathbb{Z}^2$ (generated by the images of $M_x$; and by the images of $M_y$). In view of Remark 6.15, the lower bound follows from the existence 1-dimensional and 2-dimensional lamplighters in the groups.

More generally, we consider
$$M_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$
and the group $G_\alpha$, generated by $M_1, M_\alpha, M_x, M_y$.

Claim 2 and 3 of Proposition 8.12 below classify $G_\alpha$ in terms of triviality/non-triviality of the boundary, in particular providing further examples of 2-dimensional groups with non-trivial boundary. This happens whenever $\alpha$ is algebraic and not a root of unity.

**Proposition 8.12.** (1) If $\alpha$ is algebraic, the group $G_\alpha$ has dimension 2. If not, the group has dimension 3.

(2) If $\alpha$ is a root of unity, then $G_\alpha$ has a finite index subgroup which is a quotient of $\mathbb{Z}^2 \ltimes \mathbb{Z}^\ell$. (In this case by [32] we know the boundary of any centered second moment random walk is trivial).

(3) If $\alpha$ is not the root of unity, then any finite entropy irreducible random walk on $M_\alpha$ has non-trivial boundary.

**Proof.** 1) Follows from Lemma 6.22

2) Now assume that $\alpha$ is a root of unity of degree $d$, $\alpha^d = 1$. Consider the projection to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} / d\mathbb{Z}$. Consider the finite index subgroup of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} / d\mathbb{Z}$ with the third coordinate equal to 0. Its preimage is a finite index subgroup in $G_\alpha$. Observe that the numbers $\alpha^s$, where $1 \leq s < d$ are integers relatively prime with $d$, are linearly independent (and other powers $\alpha^s$ are expressed as their linear combination). We deduce from this that this finite index subgroup is isomorphic to $\mathbb{Z}^2 \ltimes \mathbb{Z}^{\phi(d)}$, where $\phi(d)$ is Euler’s totient function.
3) Observe that we can assume that \( \alpha \) is algebraic since otherwise \( G_\alpha \) has dimension 3. Since \( \alpha \) is not a root of unity, there is some Galois conjugation \( \alpha' \) of \( \alpha \) which has absolute value not equal to 1. Observe that \( G_\alpha \) is isomorphic to \( G_{\alpha'} \). Note also that \( G_{\alpha^{-1}} \) is isomorphic to \( G_\alpha \). So we can assume that the absolute value of \( \alpha > 1 \). Take \( m: |\alpha^m| > 2 \).

We recall that for any group if \((G, \mu)\) does not have \( S^+S^- \) property the Poisson boundary of \((G, \mu)\) is non-trivial, see e.g. [20], Remark 5.4. Hence, we can assume that \((G, \mu)\) satisfy \( S^+S^- \) property. First observe that if \( M \) is non-degenerate , there is \( k \) such that \( M_{\alpha^m}, \epsilon \) are in the support of \( \mu^{\epsilon k} \). We put \( \Delta = \{ \epsilon, M_1 \} \) and similarly to Example 8.11 observe that

\[
\sum_{i,j,k} \epsilon_{i,j,k}(\alpha^m)^k x^i \epsilon^j \neq 0,
\]

if \( \epsilon_{i,j,k} \) take value 0, 1 and \(-1\) and at least one of the coefficient is \(\neq 0\). Observe that points whose projection to \( \mathbb{Z}^3 \) of the from \( \langle mk, \cdot, \cdot \rangle \) (for \( k \in \mathbb{Z} \)) form a finite index subgroup (a lattice) in \( \mathbb{Z}^3 \). Observe that then the number of points after \( n \) steps visited in this lattice grow linearly. (Indeed, with positive probability this number of points is at least \( \text{Const} R_n \), where \( R_n \) is the number of distinct points in \( \mathbb{Z}^3 \), visited up to time instant \( n \). And \( R_n \) is linear in \( n \) since the random walk on \( \mathbb{Z}^3 \) is transient (for this basic fact see e.g. Theorem 1.4.1 in [54]). Then we condition on all \( \Delta \) increments except for the final multiplication at \( X_t \) with a projection to given points of our lattice, we can conclude that \( \Delta \)-restriction entropy is linear.

This implies that \( \Delta \)-restriction entropy is linear, and therefore entropy function grows linearly, and the boundary is non-trivial.

In general, without loss of generality we can assume that \( e \in \text{supp} \mu \). Since \( \mu \) satisfies \( S^+S^- \) property, there is \( r \) such that \( g_1, g_2 \in \text{supp} \mu^{\epsilon r} \), and \( g_1 = M_{\alpha^m} g_2 \). We can put \( \Delta = \{ g_1, g_2 \} \) and show that \( \Delta \)-restriction entropy grows linearly.

\[ \square \]

**Remark 8.13.** In Corollary 7.19 we have seen that for a virtually solvable group, linear over a function field over at most 1 variable, any finite second moment centered measure defines a random walk with trivial boundary. The Example 8.11 shows that in the case of characteristic 0 the assumption of the corollary that the transcendence degree \( \leq 1 \) can not be replace by \( \leq 2 \).

**Remark 8.14.** However, observe that some groups, for example \( \mathbb{Z}^2 \wr \mathbb{Z} \), have trivial boundary for any centered second moment random walk [32]. But this group can not be embedded into linear group of transcendence degree 1 (see Lemma 6.19).

As we have mentioned in the introduction, we conjecture that this is essentially the only 2-dimensional examples of a linear basic block with trivial boundary.

### 8.3. More on groups of characteristic \( p \)

Now we are ready to prove the Corollary \( D \) formulated in the introduction:

**Corollary 8.15.** Let \( G \) be an amenable subgroup of \( GL(n, k) \), \( k \) is a field of characteristic \( p \). The following properties are equivalent

1. \( G \) is commensurable to a group with a basic block \( B_{i,j} \) which contains \( \mathbb{Z}^3 \wr \mathbb{Z}/p\mathbb{Z} \) as a subgroup.
2. The dimension of some block of the group is \( \geq 3 \).
3. There exist a simple random walk on \( G \) with non-trivial Poisson boundary.
4. All non-degenerate random walks of finite entropy on \( G \) have non-trivial Poisson boundary.
6.10 about monotonicity the dimension of our metabelian block has rank 3.

**Remark 8.16.** Consider a free metabelian group $\Gamma$ then it has if for any sequence $p$ sequence of the sequence corresponds to the fact that the set $\gamma$ is.

By Corollary 8.10 we know in this case that any non-degenerate finite entropy measure has non-trivial boundary. Thus (iv) holds.

It remains to show that (1) (wreath products in the blocks condition) is equivalent to (2) (there exists a block of dimension $3$ has non-trivial boundary. By the first part of Lemma 7.15 we know that if the dimension is smaller than 3, then any symmetric finite second moment measure has trivial Poisson boundary. Thus we conclude that dimension is $\geq 3$, in other words (2) holds. Finally, assume (2). Using again Theorem 5.3 and the definition of the dimension, we know that it is enough to consider the case when the group is a metabelian block (of dimension $\geq 3$).

Remark A.1. Consider a group $\Gamma$, a measure $\mu$ on $\Gamma$ with trajectories of the random walk $(\Gamma, \mu)$ denoted by $X_1, X_2, \ldots, X_n, \ldots$.

Given a sequence $\gamma_i$ of elements of $\Gamma$, we say that it is a recurrent sequence for $(\Gamma, \mu)$ if with probability one there exist infinitely many $i$ such that $X_i = \gamma_i$. If $\gamma_i$ is not recurrent we call it transient. If all sequences are transient for the random walk $(\Gamma, \mu)$ we say the random walk $(\Gamma, \mu)$ is strongly transient (as we defined in Definition 8.4).

We give first a note on the terminology: one should not confuse the fact that the sequence $(\gamma_i, i)$ is recurrent with the fact that the set $\{\gamma_i, i \in \mathbb{N}\}$ is recurrent. The recurrence of the sequence corresponds to the fact that the set $\{\gamma_i, i\}$ is recurrent for the time extended random walk $(X_i, i)$.

We recall that a random walk $\Gamma, \mu$ is uniformly strongly transient (see Definition 8.5), if for any sequence $\gamma_i \in \Gamma, i \in \mathbb{N}$, the expected number of $i$ such that $(X_i = \gamma_i)$ is finite.

**Remark A.1.** Consider $\delta_i \geq 0$ such that $\sum_{i=1}^{\infty} \delta_i < \infty$. Assume that $\mu^*(g) \leq \delta_i$ for all $g$. Then the random walk $(G, \mu)$ is uniformly strongly transient.
Proof. Fix a sequence $\gamma_i \in G$. We need to prove that the expected number of bad hits $X_i = \gamma_i$ is finite. This expectation is the sum of the expectation that a bad hit happens at the time instant $i$. This sum is equal to

$$\sum_i P[X_i = \gamma_i] = \sum_i \mu^i(\gamma_i) \leq \sum_i \delta_i < \infty$$

We recall a well-known corollary of the Cauchy Schwarz inequality

**Remark A.2.** For a symmetric measure $\mu$ it holds $\mu^{2n}(g) \leq \mu^{2n} (e)$

**Lemma A.3.** If $\nu$ is a symmetric non-degenerate measure such that the random walk $(\Gamma, \nu)$ is transient then the random walk $(\Gamma, \nu)$ is uniformly strongly transient.

**Proof.** Consider $\mu = \nu^2$. The random walk $(\Gamma, \mu)$ is recurrent if and only if for some/all sequences $\gamma$, the sequence $\gamma_i$ is recurrent. Indeed, we recalled in the previous remark that for any symmetric $\nu$ and any $y \nu^{2n}(y) \leq \nu^{2n}(e)$. Thus we also have that

$$\nu^{2n+1}(y) = \sum_z (\nu^{2n}(z) \mu(z^{-1}y)) \leq \nu^{2n}(e).$$

In view of Remark A.1 we conclude that $\nu$ is uniformly strongly transient. \qed

**Example A.4.** [A transient walk can admit recurrent sequences] Take a random walk on $\mathbb{Z}$, supported on $-1$ and $1$, $\mu(1) = \alpha > \mu(-1) = \beta$. The random walk in transient. Consider $\gamma_n = [(\alpha - \beta)n]$. It is clear that infinitely many times $X_n \geq \gamma_n$, infinitely many times $X_n \leq \gamma_n$, and hence since each jump of our random walk has absolute value one, we can not have for some $n X_n < \gamma_n$ and then for $n+1 X_{n+1} > \gamma_{n+1}$. Hence for infinitely many times $X_n = \gamma_n$

**Remark A.5.** It is clear that $(\Gamma, \mu)$ is uniformly strongly transient if and only if $(\Gamma, \mu^{-1})$ is uniformly strongly transient. Indeed, the probability that $X_i \neq \gamma_i$ for the random walk $(G, \mu)$ is equal to the probability that $X_i \neq \gamma_i^* = \gamma_i^{-1}$ for the random walk $(G, \mu^{-1})$.

Uniformly strongly transience and the lemma below were used in an essential way for our criterion of boundary non-triviality (in particular the application about blocks of dimension $\geq 3$).

**Definition A.6.** [Admissible equivalence relations]. Given an equivalence relations on $X \times \mathbb{N}$, we say that this relation is admissible, if for any $x \neq y$ and any $i \in \mathbb{N}$ it holds $(x, i)$ is not equivalent to $(y, i)$.

**Definition A.7.** [Generalized range with respect to an equivalence relation $E$] Let $X$ be a space equipped with a Markov kernel, and let $E$ be an equivalence relation on $X$. Given an $n$ step trajectory of the random walk $X_1, \ldots, X_n$, we say that the generalized range of $R_E(X_1, \ldots, X_n)$ with respect of $E$ is the number of equivalence classes that $(X_1, 1), \ldots, (X_n, n)$ intersect. We define the range function to be the expectation of the range:

$$R_E(n) = E[R_E(X_1, \ldots, X_n)]$$

**Remark A.8.** Let the equivalence relation $E$ is defined by $(x, i) \sim (y, j)$ if and only if $x = y$. Then the range with respect to $E$ is the same as the range of the random walk, that is, the number of distinct points (in $X$) visited up to the time instant $n$. 

For a random walk on a group, the transience of a random walk is equivalent to linearity of the range function (see Theorem 1.4.1 Spitzer [54] for the case of \( \mathbb{Z}^d \), and a similar argument works for an arbitrary not necessarily commutative group, see e.g. Lemma 1 in [7], for a more general case when instead of random walks on groups we consider the walks on Schreier graphs, there is an analogous statement where we have to consider inverted orbits of random walks, see Lemma 3.1 [8]). In a similar way, linearity of the generalized range function with respect to an equivalence relation is related to uniform strong transience.

**Lemma A.9.** [Generalised range for equivalence relations] Let \( G \) be a group and \( \mu \) be a probability measure on \( G \). Consider an admissible equivalence relation \( E \) on \( G \). Assume that \( \mu \) is uniformly strongly transient. Then the range function satisfies
\[
R_E(p,n) \geq qn, \quad \text{for } q > 0 \text{ and all } n.
\]

**Proof.** We know that \( \mu \), and hence also \( \mu^{-1} \) are uniformly strongly transient. Consider the probability that \( X_n \) is equivalent to none of \( X_i \), for \( 0 \leq i \leq n-1 \). Consider the \( n \) step trajectory of the random walk, started from \( X_n \) and inverting the time (putting \( X'_0 = X_n \), \( X'_1 = X_{n-1} \), \( X'_{n-1} = X_1 \), \( X'_n = X_0 = e \)). Observe that the probability we consider is an expectation that this inverted r.w. trajectory never returns to an element equivalent to its origin element \( X'_0 = X_n \) up to the time instant \( n \). Here we consider an “inverted” equivalence relation \( E' \) on \( X_n \) if \( (x,k) \sim_{E'} (y,m) \) if \( (x,n-k) \sim (y,n-m) \). This auxiliary equivalence relation depends on \( n \), and if we move the base-point of the inverted random walk to \( e \), then it also depends on \( g_n = X_n \).

However, the uniform strong transience allows us to bound from below all considered probabilities (and hence their expectation), implying that the probability to visit at time \( n \) an element, not equivalent to the previous ones, is \( \geq p > 0 \). Finally, observe that the range function with respect to \( E \) is equal to the sum of random variables \( \chi_i \), \( \chi_i \) takes value 1 if \( i \)-th element is not equivalent to the previous ones, and 0 otherwise. Hence we obtain a linear lower bound for the range functions, and this concludes the proof of the lemma. \( \square \)

By a result of Varopoulos [59] any non-degenerate random walk on an infinite group which is not a finite extension of \( \mathbb{Z} \) or \( \mathbb{Z}^2 \) is transient. See e.g. Chapter I, Section 3B, Lemma 3.12 in [62] for a proof based on Nash-Williams criterion and Chapter III, Section 14 [62] for a more general argument based on Coulhon Saloff-Coste isoperimetric inequality. A similar argument is used in 1) of the theorem below to affirm the uniform strong transience.

A result of Dudley [14] states that a countable Abelian group admits a non-degenerate transient random walk if and only if this group does not contain \( \mathbb{Z}^2 \) as a subgroup. (Spitzer refers to this result as the most interesting general results concerning random walks on groups (!), see final remark of section 8 in [54]). The theorem below generalises his result to not necessarily Abelian groups.

**Theorem A.10.** Let \( G \) be a countable group.

1. If \( G \) admits an infinite finitely generated subgroup which is not virtually \( \mathbb{Z} \) or \( \mathbb{Z}^2 \), then any adapted random walk on \( G \) is uniformly strongly transient. In particular, given an admissible equivalence relation \( E \), the range of \( (G, \mu) \) with respect to \( E \) is \( \geq pn \), for some \( p > 0 \) and all \( n \).

2. If all finitely generated subgroups of \( G \) have at most quadratic growth, then \( G \) admits a non-degenerate recurrent measure. This measure can be chosen to be symmetric and with \( \operatorname{supp} \mu = G \).
Remark A.11. The assumption in the first claim of the theorem in that the measure is adapted can not be replaced by the non-degeneracy condition (nor by the $S^+ S^-$ property).

Let $e_1, e_2, \ldots, e_d$ be standard generators of $\mathbb{Z}^d$, consider the measure $\mu$ equidistributed on this set. Random walks $(\mathbb{Z}^d, \mu)$ are essentially $d-1$ dimensional, the $n$ step positions of the random walk belong to $d-1$ subspace of $\mathbb{Z}^d$ (depending on $n$). Put $\gamma_n = (n/d, n/d, n/d)$.

Observe that the probability to hit $\gamma_n$ is equal to 1 for $d = 1$ and also equivalent to $\sim 1/n^{(d-1)/2}$ for all $d \geq 1$. Thus for $d = 3$ the expected number of hitting $\gamma_n$ at time instant $n$ is $\sim \sum_{i=1}^{\infty} 1/i = \infty$, and the sequence $(i, \gamma_i)$ is recurrent.

Proof. 1) To prove the first claim of the theorem, observe that Gromov’s theorem on polynomial growth \cite{20} implies that any group satisfying the assumption of this claim has at least cubic growth: its growth function $\nu_{G,S}(n) \geq A n^3$, for some positive constant $A$, depending on the generating set $S$ and all $n$ (since any nilpotent group has growth $\sim n^d$, and the only groups of (sub)-quadratic growth are finite extensions of $\mathbb{Z}, \mathbb{Z}^2$ or finite ones, see e.g. Corollary 3.18 in \cite{62}; for an exposition of basic facts about growth of groups see also \cite{42} ). Consider a simple random walk on $G$, we denote its symmetric finitely supported measure by $\nu$, and we assume that $\nu = \nu_0^{a^2}$. Recall that by Coulhon Saloff-Coste inequality the Folner function of $G$ satisfies $\text{Fol}_{G,S}(n) \geq \nu_{G,S}(Cn) \geq A' n^3$, where $\nu_{G,S}(n)$ is the growth function of $G$ with respect to $S$. Therefore, the return probability of the random walk $(G, \nu)$ satisfies $\nu^{\ast_n}(e) \leq C' n^{-3/2}$. Indeed, this (well-known fact) follows from the inequality between isoperimetry and return probability, see e.g. Proposition 14.1 and Theorem 14.3 and for this particular Corollary 14.5 in Chapter III \cite{62}.

Now recall that (by an argument that goes back to Baldi et al \cite{4}, see (2.1) in Section 2 of \cite{7} (Varopoulos) the following holds. Take a symmetric random walk $\mu_1$ and a not-necessarily symmetric random walk $\mu_2$. Then there exists $\alpha > 0$ such that for any function $f \in L^2(G)$

$$\langle P_{\mu_2} f, f \rangle \leq \alpha \langle P_{\mu_1} f, f \rangle.$$ 

Indeed, since $\nu_2$ is adapted, we replacing if necessarily $\nu_2$ by a convolution power, we can assume that the Markov kernels satisfy $P_{\mu_2} \geq K P_{\mu_1}$. Then the inequality above is a particular case of the corresponding inequality for Markov kernels. Applying this inequality to $f$ defined by $f(e) = f(g) = 1$ and $f(x) = 0$ otherwise, we conclude that $P_\mu^{\ast_n}(e, e) + P_\mu^{\ast_n}(e, g) \leq C (P_\nu^{\ast_n}(e, e) + P_\nu^{\ast_n}(e, g))$, for a positive constant $C$ not depending on $g$. Since $\nu = \nu_0^{a^2}$ is symmetric, we know that $P_\nu^{\ast_n}(e, g) \leq P_\nu^{\ast_n}(e, e)$ (see Remark A.2), and hence

$$P_\mu^{\ast_n}(g) = P_\mu^{\ast_n}(e, g) \leq 2C P_\nu^{\ast_n}(e, e) \leq \frac{C^n}{n^{3/2}}$$

for a positive $C^n$ not depending on $g$. Thus in view of Remark A.1 the random walk is strongly transient.

Now since we have proved the uniform strong transience, the claim about the range function follows from Lemma A.9.

2) Choose finite generated subgroups $G_i \subset G$ such that $G_i \subset G_{i+1}$ and $\cup_{i \in \mathbb{N}} G_i = G$. Choose a measure $\mu_i$ on $G_i$. We will construct the measure $\mu$ as a convex combination of $\mu_i$. More precisely, we will choose in a recursive way a sequence $a_1 = 1$, $a_i > 0$. On each step we construct $a_1, \ldots, a_n$ and a number $b_n$ such that the numbers $a_i, i > n$, constructed in the sequel will satisfy $a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq b_n$ (and hence
\[
\frac{a_n+1+a_{n+2}+a_{n+3}+\ldots}{a_1+a_2+a_3+\ldots} \leq b_n.
\]
We will check the claim for the measure
\[
\mu = \sum_{i=1}^{\infty} \frac{a_i}{a_1+a_2+a_3+\ldots} \mu_i.
\]

First choose all measures \(\mu_n\) to be finitely supported. Indeed, if the group of growth \(\sim n^d\), then return probability of a simple random walk satisfies \(\sim \frac{C_i}{n^d}\), for all even \(n\), (see e.g. Theorem 15.8 in [12]); Here (in the proof of 2) we use only the case for groups of at most quadratic growth, which, taking in account the polynomial growth theorem, can also be deduced from the local limit theorem for the random walks on \(\mathbb{Z}^2\) and \(\mathbb{Z}\). Indeed \(G_i\) is either finite, or virtually cyclic, or contains \(\mathbb{Z}^2\) as a finite index subgroup.

Observe that the probability that one of the terms corresponding to \(\mu_i\), \(i > n\), occurs at least once among first \(N_n\) increments of the random walk \((G, \mu)\), is at most \(d_n = \frac{N_n b_n}{\sum_{i=1}^{\infty} a_i} \leq N_n b_n\). We choose a sequence \(N_n\) which increases quickly enough and sequence \(b_n\) decreasing quickly enough (in terms of \(N_n\)) so that \(N_n b_n \leq 1/2\). (If we assume moreover that \(\sum N_n b_n < \infty\), then by Borel-Cantelli lemma we know the following. The event: for some \(n\), there is at least one increment corresponding to \(\mu_i\), \(i > n\), among first \(n\) increments of the random walk \((G, \mu)\) occurs only for finitely many \(n\). We show in this case that return probability for \(\mu\) is estimated by these probabilities for its truncations).

Having chosen \(a_1, a_2 \ldots a_n\), consider the probability measure \(\mu_n\) which is the normalised sum \(a_1 \mu_1 + a_2 \mu_2 + \ldots + a_n \mu_n\). Consider \(C_n\) such that the return probability of the random walk \(\tilde{\mu}_n\) satisfies \(\tilde{\mu}_n \geq \frac{C_n}{n}\), for all even \(n\). We choose \(C_n\) in such a way that this sequence is non-increasing \(C_1 \geq C_2 \geq C_3 \ldots\).

Observe that
\[
\mu^{*t}(e) \geq (1 - d_n) \mu_n^{*t}(e) \geq \frac{1}{2} \mu_n^{*t}(e) \geq \frac{C_n}{2n},
\]
for any \(t \leq N_n\), with probability \(\geq 1 - d_n\) no increments of the \(\mu_i\), \(i > n\) occur during first \(N_n\) steps of the r.w. Having fixed \(a_1, a_2 \ldots a_n\) and \(C_n\), we choose \(N_n\) large enough so that
\[
\sum_{t=1}^{N_n} \frac{C_n}{2n} \geq n,
\]
here \(t\) in the sum above is assumed to be even. We conclude therefore that for all \(n \geq 1\)
\[
\sum_{i=1}^{\infty} \mu^{*t}(e) \geq n.
\]
Therefore, this sum is infinite, and thus the random walk \((G, \mu)\) is recurrent. This proves the main claim of 2).

Now if we want to assure that the measure on \(G\) has full support, we choose \(\mu_i\) such that \(\operatorname{supp} \mu_i = G_i\) (rather than having finite support). We assume that \(\mu_i\) is symmetric and has finite second moment. In this case the truncated measure \(\tilde{\mu}_i\) are also symmetric measure of finite second moment. Then we still have \(\mu^{*n}_i(i) \geq C_i/n\), for some \(C_i > 0\), and for all even \(n\) (see Corollary 1.5 in [47]). Thus we can proceed as in the argument explained above.

\[\square\]

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