New Construction of Mutually Unbiased Bases in Square Dimensions

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Abstract

We show that \( k = w + 2 \) mutually unbiased bases can be constructed in any square dimension \( d = s^2 \) provided that there are \( w \) mutually orthogonal Latin squares of order \( s \). The construction combines the design-theoretic objects \((k, s)\)-nets (which can be constructed from \( w \) mutually orthogonal Latin squares of order \( s \) and vice versa) and generalized Hadamard matrices of size \( s \). Using known lower bounds on the asymptotic growth of the number of mutually orthogonal Latin squares (based on number theoretic sieving techniques), we obtain that the number of mutually unbiased bases in dimensions \( d = s^2 \) is greater than \( s^{1/14.8} \) for all \( s \) but finitely many exceptions. Furthermore, our construction gives more mutually orthogonal bases in many non-prime-power dimensions than the construction that reduces the problem to prime power dimensions.

1 Introduction

Two orthonormal bases \( \mathcal{B} \) and \( \mathcal{B}' \) of the Hilbert space \( \mathbb{C}^d \) are called mutually unbiased if and only if

\[
|\langle \phi | \psi \rangle|^2 = \frac{1}{d}
\]

for all \( |\phi \rangle \in \mathcal{B} \) and all \( |\psi \rangle \in \mathcal{B}' \). The concept of mutually unbiased bases (MUBs) plays an important role in quantum information theory. For example, some quantum cryptographic protocols rely on the fact that no information can be obtained when a quantum system which is initialized in a basis state from \( \mathcal{B}' \) is measured with respect to the basis \( \mathcal{B} \). The protocols in [5, 9, 4] exploit this observation in order to distribute secret keys over a public channel in an information-theoretic secure way.

It is known that the cardinality of any collection of mutually unbiased bases of \( \mathbb{C}^d \) cannot exceed \( d + 1 \) (see [3, 12, 17, 20, 24]). Sets attaining this bound are extremely interesting because they allow quantum state tomography with projective measurements consisting of a minimal number of operators [18]. Furthermore, for some of these cases it is possible to define a discrete version of the Wigner function for states on the Hilbert space \( \mathbb{C}^d \) (see e.g.
for an overview on this subject). An enjoyable application of mutually unbiased bases is the Mean King’s problem. Let $N_{\text{MUB}}(d)$ denote the maximum cardinality of any set containing pairwise mutually unbiased bases of $\mathbb{C}^d$. We have $N_{\text{MUB}}(d) \geq 3$ for any dimension $d$ (see [3]). It is known that $N_{\text{MUB}}(d) = d + 1$ holds when $d$ is a prime power [18, 24, 25, 3, 21].

An open problem is to determine $N_{\text{MUB}}(d)$ when the dimension is not a prime power. The case $d = 6$ is studied in [15]. An elementary lower bound on $N_{\text{MUB}}(d)$ for arbitrary $d$ is derived in [25, Proposition 2.20] and [21]. Let $d = p_1^{e_1} \cdots p_r^{e_r}$ be a factorization of $d$ into distinct prime powers $p_i^{e_i}$. Then

$$N_{\text{MUB}}(d) \geq \min\{N_{\text{MUB}}(p_1^{e_1}), N_{\text{MUB}}(p_2^{e_2}), \ldots, N_{\text{MUB}}(p_r^{e_r})\}. \quad (1)$$

This is proved as follows. Let $m := \min_i N_{\text{MUB}}(p_i^{e_i})$. Choose $m$ mutually unbiased bases $B^{(i)}_1, \ldots, B^{(i)}_m$ of $\mathbb{C}^{p_i^{e_i}}$, for all $i$ in the range $1 \leq i \leq r$. Then

$$\{B^{(1)}_k \otimes \cdots \otimes B^{(r)}_k \mid k = 1, \ldots, m\}$$

is a set of $m$ mutually unbiased bases of $\mathbb{C}^d$. This is easily memorable by $N_{\text{MUB}}(mn) \geq \min\{N_{\text{MUB}}(n), N_{\text{MUB}}(m)\}$ for all $m, n \geq 2$. In the following we will refer to this construction as the “reduce to prime powers” construction.

We present a construction that yields more MUBs than the “reduce to prime powers” construction in many non-prime-power dimensions. However, our construction always produces less MUBs if $d^2$ is a prime power. More precisely, if $d = p^{2e}$ for some prime number and $e > 1$ then our construction gives only $p^e + 1$ MUBs compared to the optimal value $N_{\text{MUB}}(p^{2e}) = p^{2e} + 1$.

Our construction combines the design-theoretic objects nets or, more precisely, their description by incidence vectors, and generalized Hadamard matrices. These concepts are briefly introduced in Section 2. The construction is presented in Section 3. In Section 4 we discuss briefly the equivalence of nets to other design-theoretic objects. We relate mutually orthogonal Latin squares and nets in more detail. This shows that there are incidence vectors with good parameters. In Section 5 we examine the cases where our construction yields more mutually orthogonal bases.

2 The ingredients

2.1 Incidence vectors from nets

Our construction of mutually unbiased bases makes use of incidence vectors satisfying certain conditions. The idea to use them is based on the observation that mutually unbiased bases satisfy “similar” conditions. Let

$$B_1 := \{ |\psi_{11}\rangle, |\psi_{12}\rangle, \ldots, |\psi_{1d}\rangle \},$$

$$B_2 := \{ |\psi_{21}\rangle, |\psi_{22}\rangle, \ldots, |\psi_{2d}\rangle \},$$

where
be a collection of \( k \) mutually unbiased bases in \( \mathbb{C}^d \). For such a collection we have
\[
|\langle \psi_{b_i}|\psi_{b_j} \rangle|^2 = \delta_{ij}
\]
for all \( 1 \leq b \leq k \) and all \( 1 \leq i, j \leq d \) and
\[
|\langle \psi_{b_i}|\psi_{c_j} \rangle|^2 = \frac{1}{d}
\]
for all \( 1 \leq b < c \leq k \) and all \( 1 \leq i \leq j \leq d \).

Now we consider a collection of incidence vectors that satisfy “similar” conditions. We say that a (column) vector \( \mathbf{m} := (m[1], \ldots, m[d])^T \) of size \( d \) is an incidence vector if its entries take only the values 0 and 1. The Hamming weight of \( \mathbf{m} \) is the number of 1’s. The support \( \text{supp}(\mathbf{m}) \) of the incidence vector \( \mathbf{m} \) (of Hamming weight \( s \)) is the set of indices \( j_1, \ldots, j_s \) such that the corresponding entries \( m[j_1], \ldots, m[j_s] \) of \( \mathbf{m} \) are all 1. We always sort the indices, i.e., \( j_1 < \ldots < j_s \).

Now we introduce the notion of nets, i.e., collections of incidence vectors satisfying special properties.

**Definition 1 (Net)**

Let \( \{\mathbf{m}_{11}, \ldots, \mathbf{m}_{1s}, \mathbf{m}_{21}, \ldots, \mathbf{m}_{2s}, \ldots, \mathbf{m}_{k1}, \ldots, \mathbf{m}_{ks}\} \) be a collection of \( ks \) incidence vectors of size \( d = s^2 \) that are partitioned into \( k \) blocks where each block contains \( s \) incidence vectors. The incidence vectors are denoted by \( \mathbf{m}_{bi} \), where \( b = 1, \ldots, k \) identifies the block and \( i = 1, \ldots, s \) the vector within a block. If the incidence vectors satisfy the following conditions we say that they form a \((k, s)\)-net.

1. The supports of all vectors within one block are disjoint, i.e.,
\[
\mathbf{m}_{bi}^T \mathbf{m}_{bj} = 0
\]
for all \( 1 \leq b \leq k \) and all \( 1 \leq i \neq j \leq s \).

2. The intersection of any incidence vectors from two different blocks contains exactly one element, i.e.,
\[
\mathbf{m}_{bi}^T \mathbf{m}_{cj} = 1
\]
for all \( 1 \leq b \neq c \leq k \) and all \( 1 \leq i, j \leq s \).

Note that our definition of \((k, s)\)-nets is in accordance with the usual definition of nets in design theory [11, page 172]. The incidence vectors are just the characteristic functions of the subsets used there.
Example 2 (Incidence vectors)

For \( d = s^2 \) with \( s = 2 \) we have \( k = 3 \) blocks. The incidence vectors are defined in the table below for a \((3,2)\)-net:

| \( m_{11} \) | \( m_{12} \) | \( m_{21} \) | \( m_{22} \) | \( m_{31} \) | \( m_{32} \) |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |

In Section [1] we describe briefly that nets are equivalent to other combinatorial objects like transversal designs, mutually orthogonal Latin squares, and orthogonal arrays. Due to the correspondence of equations (2) \( \leftrightarrow \) (4) and (3) \( \leftrightarrow \) (5) one could consider mutually unbiased bases as the quantum generalization of nets. Indeed, this and other quantum generalizations of many design-theoretic objects are considered in [25].

### 2.2 Generalized Hadamard matrices

For our construction we need to introduce generalized Hadamard matrices. We call an \( s \times s \)-matrix \( H \) with entries from \( \mathbb{C} \) a generalized Hadamard matrix if all its entries have modulus one and \( H \) satisfies \( HH^\dagger = s \mathbb{1}_s \). Usual Hadamard matrices have entries \( \pm 1 \). A necessary condition for the existence of (usual) Hadamard matrix is that \( s = 1, 2 \) or \( s \) is a multiple of 4 (see [15]).

Note that generalized Hadamard matrices exist for any dimension \( s \). Just take the (not normalized) Fourier matrix \( DFT_s \), whose entries \((k,l)\) are given by the \( s \)-th roots of unity

\[
DFT_{k,l}^{(s)} := \omega^{kl}
\]

where \( \omega := e^{2\pi i / s} \) and \( k, l = 0, \ldots, s - 1 \). Other examples of generalized Hadamard matrices are character tables of finite abelian groups [19]. This follows from the orthogonality relations of characters.

### 3 Mixing the ingredients: our construction

Now we have all ingredients we need for our construction. We just need the following simple definition. Let \( \mathbf{m} \in \{0,1\}^d \) be an incidence vector of Hamming weight \( s \) and \( \mathbf{h} \in \mathbb{C}^s \) an arbitrary column vector. Then we define the embedding of \( \mathbf{h} \) into \( \mathbb{C}^d \) controlled by \( \mathbf{m} \), denoted by \( \mathbf{h} \uparrow \mathbf{m} \), to be the following vector in \( \mathbb{C}^d \)

\[
\mathbf{h} \uparrow \mathbf{m} := \sum_{r=1}^{s} \mathbf{h}[r] |j_r\rangle,
\]

where \( \mathbf{h}[r] \) is the \( r \)-th entry of the vector \( \mathbf{h} \), \( \{j_1, j_2, \ldots, j_s\} \) the support of \( \mathbf{m} \) with the ordering \( j_1 < j_2 < \ldots < j_s \) and \( |j_r\rangle \) the \( j_r \)-th standard basis vector of \( \mathbb{C}^d \). A less formal way to define
this vector is: the first non-zero entry of \( m \) is replaced by the first entry of \( h \), the second non-zero entry of \( m \) by the second entry of \( h \), etc.

This operation is best illustrated by a simple example:

\[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{bmatrix} \in \{0,1\}^9, \quad h := \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix} \in \mathbb{C}^3, \quad h \uparrow m := \begin{bmatrix} 1 \\ 0 \\ \omega \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \omega^2 \end{bmatrix} \in \mathbb{C}^9
\]

**Theorem 3 (Construction of MUBs)**

Let \( \{m_{11}, \ldots, m_{1s}, m_{21}, \ldots, m_{2s}, \ldots, m_{k1}, \ldots, m_{ks}\} \) be a \((k,s)\)-net and \( H \) an arbitrary generalized Hadamard matrix of size \( s \). Then the \( k \) sets for \( b = 1, \ldots, k \)

\[
B_b := \left\{ \frac{1}{\sqrt{s}} (h_l \uparrow m_{bi}) \mid l = 1, \ldots, s, \ i = 1, \ldots, s \right\}
\]

(7)

are \( k \) mutually orthogonal bases for the Hilbert space \( \mathbb{C}^d \).

**Proof.** All vectors \( h_l \uparrow m_{bi} \) have length \( \sqrt{s} \) because all incidence vectors \( m_{bi} \) have Hamming weight \( s \). Therefore, the vectors in eq. (7) are normalized.

Let us now consider one fixed block \( b \). Then the vectors \( h_l \uparrow m_{bi} \) and \( h_{l'} \uparrow m_{bi'} \) are orthogonal for \((l, i) \neq (l', i')\). This is seen as follows: If \( i \neq i' \) then orthogonality follows from the fact that the incidence vectors \( m_{bi} \) and \( m_{bi'} \) have disjoint supports due to eq. (4). If \( i = i' \) and \( l \neq l' \) then orthogonality follows from the fact that \( h_l \) is orthogonal to \( h_{l'} \).

This shows that each \( B_b \) is an orthogonal basis of \( \mathbb{C}^d \).

Now let us consider two different blocks \( b \) and \( b' \). Then for arbitrary \( i, i' = 1, \ldots, s \) and \( l, l' = 1, \ldots, s \) there is exactly one pair \((r, r')\) such that \( j_r, j_{r'} \) are both 1, where \( \text{supp}(m_{bi}) = \{j_1, \ldots, j_s\} \) and \( \text{supp}(m_{bi'}) = \{j'_1, \ldots, j'_s\} \). This is ensured by eq. (5). Therefore, we have for the scalar product of the embedded vectors

\[
(h_l \uparrow m_{bi} \mid h_{l'} \uparrow m_{bi'}) = h_l[r] \cdot \overline{h_{l'}[r']}
\]

It is the product of two complex numbers of modulus one. This proves that the square norm of the scalar product of \( \frac{1}{\sqrt{s}} h_l \uparrow m_{bi} \) and \( \frac{1}{\sqrt{s}} h_{l'} \uparrow m_{bi'} \) is \( 1/s^2 \). Consequently, the bases \( B_b \) and \( B_{b'} \) are mutually unbiased for \( b \neq b' \). \( \square \)
Example 4 (MUBs for $d = 4$)

Let us apply our construction to the incidence vectors in Example 2 and the Hadamard matrix of size 2

$$H := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Then we obtain the following three mutually unbiased bases:

$$B_1 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\},$$

$$B_2 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\},$$

$$B_3 := \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Note that, as mentioned before, the construction is not optimal in this case.

4 Construction of the incidence vectors

The problem of finding a large collection of incidence vectors satisfying the conditions in eq. 4 and eq. 5 is a well studied problem in design theory. We have already mentioned that such a collection is equivalent to a net. There are many more objects that are equivalent: transversal designs, mutually orthogonal Latin squares, and orthogonal arrays [6, 7, 11]. We concentrate on the equivalence of mutually orthogonal Latin squares and nets.

Mutually orthogonal Latin squares

We refer the reader to [6, 7, 11, 16] for an extensive introduction to Latin squares. A Latin square of order $s$ is an $s \times s$ array with entries from a set $S$ of cardinality $s$ such that each element of $S$ appears equally often in every row and every column. It is easily seen that a Latin square of order $s$ exists for every positive integer $s$. For example, one may label the rows and columns by $0, 1, \ldots, s - 1$ and define the entry $L_{ij}$ in row $i$ and column $j$ to be $i + j$, where the addition is modulo $s$. The resulting Latin square for $s = 4$ is shown below:

$$\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 0 \\
2 & 3 & 0 & 1 \\
3 & 0 & 1 & 2 \\
\end{array}$$
Two Latin squares $L$ and $L'$ of order $s$ are said to be orthogonal to each other if when one is superimposed on the other the ordered pairs $(L_{ij}, L'_{ij})$ of corresponding entries consist of all possible $s^2$ pairs. A collection of $w$ Latin squares of order $s$, any pair of which is orthogonal, is called a set of mutually orthogonal Latin squares (MOLS). We use $N_{MOLS}(s)$ to denote the maximal value $w$ such that there are $w$ MOLS of order $s$. It is known that $N_{MOLS}(s) \leq s - 1$ for all $s$. If this bound is attained we say that there is a complete set of mutually orthogonal Latin squares of order $s$.

A construction for complete sets of MOLS of order $s$ is known if $s$ is a prime power [15, Theorem 8.3]. No construction for a complete set of mutually orthogonal Latin squares for any other value is known at the present time. In fact, if $s$ is not a prime power, the largest value of $w$ for which it is known that there are $w$ MOLS of order $s$ is usually considerably smaller than $s - 1$, and for several values it is known that $s - 1$ MOLS of order $s$ cannot exist. A table with the largest known values for $w$ is presented in [1] for $s < 10^3$. In [16] these numbers are shown for $s \leq 100$.

Let us now summarize some known results on the number of MOLS of a given order. A result due to Wilson [6] on the existence of six MOLS shows that there are $w \geq 6$ MOLS for orders $s$ with $s \geq 76$. Chowla, Erdős and Strauss [10] proved the important fact that

$$\lim_{s \to \infty} N_{MOLS}(s) = \infty.$$  

They were even able to show that there is a number $s_0$ so that we have for all $s > s_0$

$$N(s) \geq \frac{1}{3}s^{1/91}.$$  

Wilson [23] improved this to $N_{MUB}(s) \geq s^{1/17}$, and Beth [8] obtained the exponent $\frac{1}{14}$. These proofs involve number-theoretic sieve methods.

We have the following important equivalence (see [11, Remark 2.10 and Theorem 2.12]):

**Fact 1** The existence of $w$ mutually orthogonal Latin squares is equivalent to the existence of an $(s, k)$-net with $k = w + 2$.

## 5 How good is the construction?

Example [4] shows that our construction yields maximally three mutually orthogonal bases for dimension $d = 2^2$ because there are no $w$ MOLS of order 2 with $w > 1$. The maximal value $N_{MUB}(4)$ is 5 because 4 is a prime power (use the optimal construction in [18, 24, 3] for prime powers).

More generally, it is clear that our construction can never beat the “prime power” construction for dimension of the form $d = p^{2e}$, where $p$ is an arbitrary prime and $e \geq 1$. Our construction gives $p^e + 1$ MUBs because there are $p^e - 1$ MOLS of order $p^e$, whereas the prime power construction yields $p^{2e} + 1$ MUBs.
The advantage of our construction is that it yields more MUBs for many square dimensions than the “reduce to prime power” construction. Let us characterize the cases in which it is better.

Let \( d = s^2 \) be a square dimension, i.e., its prime power factorization has the form

\[
d = p_1^{2e_1} p_2^{2e_2} \cdots p_r^{2e_r},
\]

where the \( p_i \)'s are different primes and \( e_i > 0 \) for \( i = 1, \ldots, r \). Then the “reduce to prime powers” gives \( n + 1 \) MUBs, where

\[
n := \min_{i=1,\ldots,r} \{ p_i^{2e_i} \}. \tag{9}
\]

To see if our construction yields more MUBs for the dimension \( d = s^2 \) we have to check if

\[
N_{MOLS}(s) + 2 > n + 1 \tag{10}
\]

where \( n \) is the minimal prime power contained in \( s^2 \) (as given by eq. (9)). The first \( s \) for which we know that this is true is \( s = 26 \). We have \[11, \text{Theorem 2.44 in Chapter II}\]

**Fact 2** \( N_{MOLS}(26) \geq 4 \).

Therefore, we can construct \( N_{MOLS}(26) + 2 \geq 6 \) MUBs in dimension \( d = 26^2 \), whereas the “reduce to prime power” construction gives only 5 MUBs since 4 is the smallest prime power contained in \( d \).

More generally, let us consider the numbers

\[
\mathcal{M} := \{ s \in \mathbb{N} \mid s \equiv 2 \mod 4 \}.
\]

For each \( s \in \mathcal{M} \) the minimal prime power contained in \( s \) is 2. Therefore, the “reduce to prime powers” construction gives only 5 MUBS in dimension \( d = s^2 \). Due to Wilson’s result we know that \( N_{MOLS}(s) \geq 6 \) for all \( s \geq 76 \). Therefore, our construction gives at least 8 MUBs for dimensions \( d = s^2 \) for \( s \geq 76 \). This shows that there are infinitely many dimension in which our construction yields better results.

It follows from our construction and the asymptotic result on the growth of mutually orthogonal Latin square that:

**Corollary 5** We have \( N_{MUB}(s^2) \geq s^{1/14.8} \) for all \( s \) but finitely many exceptions.

To determine the cases in which our construction is better one should consult Table 2.72 in \[11, \text{Chapter II}\]. This table shows the best known lower bounds on \( N_{MOLS}(s) \) for all \( s \leq 10000 \).

It is important that our construction can also be combined with the idea of the “reduce to prime power” construction. Instead of decomposing the dimension \( d \) into prime powers we may decompose it into squares and prime powers. This could be advantageous e.g. for numbers of the form \( d = s^2p \), where \( s^2 \) contains a small prime power and \( p \) is a prime number.
that is relatively prime to \(s^2\) and is larger than the best known lower bound on \(N_{\text{MUB}}(s)\). Consider e.g. the dimension \(d = 26^2 \cdot 7\). Then the “reduce to prime powers” construction gives only 5 MUBs. We can construct 6 MUBs in dimension \(26^2\) with our construction and 8 MUBs in dimension 7. Therefore, we obtain 6 MUBs in dimension \(d = 26^2 \cdot 7\). This shows that our construction has implications not only for square dimensions.

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**References**

[1] R. J. R. Abel, A. E. Brouwer, C. J. Colbourn, and J. H. Dinitz. chapter “Mutually orthogonal Latin squares (MOLS)”, pages 111–142, in *The CRC Handbook of Combinatorial Designs*, Colbourn, C. J. and Dinitz, J. H. (eds.), CRC Press, 1996.

[2] P. K. Aravind. Solution to the king’s problem in prime dimensions. *Z. Naturforschung*, 58:2212, 2003.

[3] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and Vatan F. A new proof of the existence of mutually orthogonal bases. *Algorithmica*, 34:512–528, 2002.

[4] H. Bechmann-Pasquinucci and W. Tittel. Quantum cryptography using larger alphabets. *Phys. Rev. A*, 61(6):062308, 2000.

[5] C. H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of the IEEE Intl. Conf. Computers, Systems, and Signal Processing*, pages 175–179, 1984.

[6] Th. Beth, D. Jungnickel, and H. Lenz. *Design Theory*, volume I of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 2nd edition, 1999.

[7] Th. Beth, D. Jungnickel, and H. Lenz. *Design Theory*, volume II of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, 2nd edition, 1999.

[8] Th. Beth. Eine Bemerkung zur Abschätzung der Anzahl orthogonaler lateinischer Quadrate mittels Sieverfahren. *Abh. Math. Sem. Hamburg*, 53:284–288, 1983.

[9] D. Bruss. Optimal Eavesdropping in Quantum Cryptography with Six State. *Phys. Rev. Lett.*, 81:3018–3021, 1998.

[10] S. Chowla, P. Erdös, and E. G. Straus. On the maximal number of pairwise orthogonal Latin squares of a given order. *Canadian J. Math.*, 12:204–208, 1960.
[11] Ch. J. Colbourn and J. H. Dinitz, editors. *The CRC Handbook of Combinatorial Designs*. CRC Press, Boca Raton FL, 1996.

[12] P. Delsarte, J. M. Goethals, and J. J. Seidel. Bounds for systems of lines, and Jacobi polynomials. *Philips. Res. Repts.*, pages 91–105, 1975.

[13] B.-G. Englert and Y. Aharonov. The mean king’s problem: Prime degrees of freedom. *Phys. Letters*, 284:1–5, 2001.

[14] K. S. Gibbons, M. J. Hoffman, and W. K. Wootters. Discrete phase space based on finite fields. *quant-ph/0401155* 2004.

[15] M. Grassl. On SIC-POVMs and MUBs in dimension 6. *quant-ph/0406175* 2004.

[16] A. S. Hedayat, N. J. A Sloane, and J. Stufken. *Orthogonal Arrays*. Springer Series in Statistics. Springer, 1999.

[17] S. G. Hoggar. $t$-designs in projective spaces. *Europ. J. Combin.*, 3:233–254, 1982.

[18] I. D. Ivanovic. Geometrical description of quantal state determination. *J. Phys. A*, 14:3241–3245, 1981.

[19] G. James and M. Liebeck. *Representations and Characters of Groups*. Cambridge, 2001.

[20] G. A. Kabatiansky and V. I. Levenshtein. Bounds for packings on a sphere and in space. *Problems of Information Transmission*, 14(1):1–17, 1978.

[21] A. Klappenecker and M. Rötteler. Constructions of mutually unbiased bases, in *Finite Fields and Applications: 7th International Conference, Fq7*, Mullen, Gary L. and Poli, A. and Stichtenoth, H. (eds.), Lecture Notes in Computer Science 2984, pages 137–144, 2003. see also *quant-ph/0309120*.

[22] L. Vaidman, Y. Aharonov, and D. Z. Albert. How to ascertain the values of $\sigma_x$, $\sigma_y$, and $\sigma_z$. *Phys. Rev. Lett.*, 58:1385–1387, 1987.

[23] R. M. Wilson. Concerning the number of mutually orthogonal Latin squares. *Discr. Math.*, 9:181–198, 1974.

[24] W. K. Wootters and B. D. Fields. Optimal state-determination by mutually unbiased measurements. *Ann. Physics*, 191:363–381, 1989.

[25] G. Zauner. *Quantendesigns – Grundzüge einer nichtkommutativen Designtheorie*. Dissertation, Universität Wien, 1999.