Hamiltonian path integral quantization in polar coordinates

A.K. Kapoor
School of Physics
University of Hyderabad
Hyderabad 500046 INDIA

and

Pankaj Sharan
Physics Department
Jamia Milia Islamia, Jamia Nagar
New Delhi 110025, INDIA

ABSTRACT

Using a scheme proposed earlier we set up Hamiltonian path integral quantization for a particle in two dimensions in plane polar coordinates. This scheme uses the classical Hamiltonian, without any $O(h^2)$ terms, in the polar variables. We show that the propagator satisfies the correct Schrödinger equation.
1 Introduction

The Feynman path integral scheme gives an important route to quantization \([1]\). That in non-cartesian coordinates one needs to add \(O(\hbar^2)\) terms to the potential to arrive at correct path integral, was at first demonstrated in polar coordinates by Edwards and Gulyaev \([2]\) who also computed the free particle propagator in \(r,\theta\) variables using the path integrals. However, for systems with finite degrees of freedom and with Lagrangians quadratic in velocities, the scheme of Pauli and DeWitt \([3, 4]\) has the distinguishing feature that the Lagrangian path integral quantization method can be set up consistently in arbitrary coordinates without addition of ad hoc \(O(\hbar^2)\) terms to the potential. However, the same is not true for the Hamiltonian path integral quantization. It has been known that \(O(\hbar^2)\) terms must be added to the classical Hamiltonian in order to arrive at the correct quantization from most of the available the Hamiltonian path integral schemes. \([5, 6, 7, 8]\)

A Hamiltonian path integral quantization scheme was given by one of us in \([5]\). This scheme is a natural generalization of the Pauli-DeWitt’s scheme for the Lagrangian formulation. However, the Hamiltonian path integral suggested in \([5]\) failed to give the correct Schrödinger equation even for the free particle in two dimension if one used the classical Hamiltonian in the plane polar coordinates; this scheme too required addition of ad hoc \(O(\hbar^2)\) terms to the classical Hamiltonian.

It may be appropriate to recall at this stage that the canonical quantization procedure too does not give the expected Hamiltonian operator as \(-\frac{\hbar^2}{2m}\nabla^2\) in the \(r\theta\) coordinates. To see this note that the classical Hamiltonian \(\frac{p^2}{2m} + \frac{k^2}{2mr^2}\) does not contain a product of two non-commuting factors. The canonical quantization gives the momentum operators conjugate to \(r,\theta\) as

\[
\hat{P} = -i\hbar \frac{\partial}{\partial r} + \frac{i\hbar}{2r}, \quad \hat{p} = -i\hbar \frac{\partial}{\partial \theta}
\]  

Replacing the c-number variables by corresponding operators the quantum mechanical Hamiltonian as the operator is easily seen to be

\[
\hat{H} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \right) + \frac{\hbar^2}{8mr^2} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{\hbar^2}{8mr^2}
\]  

It was soon realized that it is possible to modify the formalism of \([5]\) by incorporating the idea of local scaling of time which had been found a useful technique in exact evaluation of path integrals for several potential problems \([6, 7]\). In
a new scheme of Hamiltonian path integration was suggested incorporating the
idea of local scaling of time in the Hamiltonian path integral method of [8]. This
modified scheme of Hamiltonian path integration with scaling was further studied
in [9, 10]. Working within the Hamiltonian path integral framework, an important
feature of the scheme of [8] is that one can use the classical Hamiltonian in arbitrary
coordinates and still arrive at the correct Hamiltonian path integral representation
for the quantum mechanical propagator. This is in contrast to the well known
fact that in all the other existing Hamiltonian path integral schemes where one
is required to add $\hbar^2$ terms to obtain the correct Schrödinger equation in coordi-
nates other than cartesian coordinates; such terms being absent in the cartesian
coordinates only.

In this paper we shall describe the Hamiltonian path integral scheme of [8, 9, 10]
and set up the propagator for a free particle in two dimensions in plane polar
coordinates and derive the Schrödinger equation for the propagator. This was after
all the first example where the need for addition of $\hbar^2$ terms to the hamiltonian
was demonstrated [2]. We also hope that this paper will make the formalism, the
methods and the results of our earlier papers transparent. In Sec. 2 we summaize
the Hamiltonian path integral quantization scheme of [8, 9] and in Sec 3 we set
up the path integral for propagator of a particle in two dimensions in plane polar
coordinates and show that it satisfies the correct Schrödinger equation.

2 Hamiltonian path integral quantization:

In this section, at first, we shall briefly recall Lagrangian path integral as given
in [3, 4]. We summarize the steps of construction of the Hamiltonian path integral
representation for the propagator in arbitrary coordinates from [8].

Lagrangian path integral: Let the classical Lagrangian for a particle with $n$
degrees of freedom, with generalized coordinates $q^k, k = 1, \ldots n$, be given by

$$ L = g_{ij} \frac{\partial q^i}{\partial t} \frac{\partial q^j}{\partial t} + V(q) $$  \hspace{1cm} (3)

The first step in the Lagrangian form of path integral quantization is the short time
propagator (STP)

$$ (qt|q_0t_0) = (2\pi\hbar)^{-n/2} (g(q)g(q_0))^{-1/4} \sqrt{D} \exp \left[ iS(qt, q_0t_0)/\hbar \right] $$  \hspace{1cm} (4)
where
\[ D = \det \left( \frac{\partial^2 S}{\partial q^i \partial q^j} \right) \] (5)
and \( S(q_t, q_{0t}) \) is the classical action along the classical trajectory joining the points \( q_1, t_1 \) and \( q_2, t_2 \). It is also the generator of canonical transformation corresponding to the time evolution. The Lagrangian path integral is obtained by iterating the short time propagator \((q_2t_2|q_1t_1)\).

\[ K(q_t; q_{0t}) = \lim_{N \to \infty} \int \prod_{k=1}^{N-1} \rho(q_k) dq_k \prod_{j=0}^{N-1} (q_{j+1} - q_j) \epsilon \] (6)
where \( \epsilon = t/N \).

The steps that are needed to set up the Hamiltonian path integral quantization scheme in arbitrary coordinates and to arrive at a representation for the propagator are summarized below.

**Short time propagator:** The classical Hamiltonian corresponding to Eq. (3) is given by
\[ H = \frac{g^{ij} p_i p_j}{2m} + V(q) . \] (7)

To define the short time propagator we employ the generators of time evolution in terms of canonical variables. This definition goes as follows. Let \( t_1, \tau, t_2 \), with \( t_1 < \tau < t_2 \), be infinitesimally close times, and \( q_1, q_2, \) and \( p \) be any values of coordinates and momenta. Consider a classical trajectory \( \gamma_1 \) starting from \( q_1 \) at time \( t_1 \) such that at time \( \tau \) its momenta are \( p \). Similarly let \( \gamma_2 \) be the trajectory which has momenta \( p \) at time \( \tau \) and coordinates \( q_2 \) at time \( t_2 \). Next we find the generators \( S_{--}(p\tau t_2, q_1 t_1) \) and \( S_{++}(q_2 t_2, p\tau) \) of evolution along the two trajectories \( \gamma_1, \gamma_2 \) appearing inside the respective arguments. These generators are Legendre transforms of the classical action computed along the trajectories \( \gamma_1, \gamma_2 \) and depend on the Hamiltonian which we shall denote by \( h(q,p) \). We then define the ‘mixed short time propagators’ by
\[ (q_2 t_2 | p\tau) = (2\pi\hbar)^{-n/2} \sqrt{D_{++}} \exp\left[i S_{++}(q_2 t_2, p\tau)/\hbar \right] \] (8)
\[ (p\tau | q_1 t_1) = (2\pi\hbar)^{-n/2} \sqrt{D_{--}} \exp\left[i S_{--}(p\tau, q_1 t_1)/\hbar \right] \] (9)
where
\[ D_{++} = \det \left( \frac{\partial^2 S_{++}}{\partial q^i \partial p^j} \right) \] (10)
\[ D_{-} = \det \left( \frac{\partial^2 S_{-}}{\partial q_i \partial p_j} \right) \quad (11) \]

and finally the canonical short time propagator is defined by

\[ (q_2 t_2 \| q_1 t_1) = \frac{1}{\sqrt{\rho(q_1)\rho(q_2)}} \int d^n p \ (q_2 t_2 \| p\tau) \ (p\tau \| q_1 t_1) \quad (12) \]

which propagate the square integrable wave functions \( \psi(q) \) with measure \( \rho(q) d^n q \).

**Canonical path integral:** As a next step we define a canonical path integral built up from the STP \( (q_2 t_2 \| q_1 t_1) \) given by Eq. (12); the resulting path integral denoted by \( K[h, \rho](q_t, q_{t_0}) \) is defined for the Hamiltonian \( h(q, p) \) as follows.

\[ K[h, \rho](q_t, q_{t_0}) \overset{def}{=} \lim_{N \to \infty} \int N^{-1} \prod_{k=1}^{N-1} \rho(q_k) dq_k \prod_{j=0}^{N-1} (q_{j+1} \| q_j) \quad (13) \]

This definition is a natural generalization of the Lagrangian path integral quantization scheme of DeWitt described above. But is seen to fail for the ‘benchmark’ case of free particle in two dimensions in \( r, \theta \) coordinates. Identifying \( h(q, p) \) with the classical free particle Hamiltonian does not lead to the correct Schrödinger equation in the \( r, \theta \) variables for the propagator[5].

As already mentioned above, it is possible to modify the above scheme by incorporating the idea of local scaling of time[6, 7] in such a way that it is not necessary to introduce ad hoc \( O(\hbar^2) \) terms in the classical Hamiltonian to obtain the correct Schrödinger equation for each set of coordinates. The modified definition is more general than just getting the right Schrödinger equation for the free particle. In fact it gives us a practical method for relating the path integrals for different problems, thereby helping us to tackle many exactly solvable potentials by purely path integral methods[10]. In the next paragraph we introduce this modified scheme resulting in a Hamiltonian path integral with scaling.

**Canonical path integral with local scaling of time:** Let \( \alpha(q) \) be a strictly positive function of \( q \). It will be called scaling function. Given a hamiltonian \( H \), define for any real \( E > 0 \), the pseudo-Hamiltonian by

\[ H^E_{\alpha} \overset{def}{=} \alpha(H - E) \quad (14) \]

The Hamiltonian path integral \( K \) with scaling \( \alpha \) is defined to be

\[ K[H, \rho, \alpha](q_t, q_{t_0}) \quad (15) \]
\[ \equiv \sqrt{\alpha(q)\alpha(q_0)} \int_0^\infty \frac{dE}{2\pi\hbar} \exp[-iEt/\hbar] \int_0^\infty d\sigma K[H^E_\alpha,\rho](q\sigma,q_0\sigma_0 = 0) \quad (16) \]

(17)

The original canonical short time propagator appears in the propagator \( K \) in the right hand side of the above equation calculated for pseudo-time \( \sigma \) by identifying the function \( h \) with the pseudo-Hamiltonian \( H^E_\alpha \). It can be shown that for \( \alpha = 1 \) the path integral with scaling coincides with the Hamiltonian path integral defined above in Eq. (13). If we take \( H \) as in Eq. (7) then \( K \) satisfies the following equation for arbitrary \( \alpha \) [9].

\[ i\hbar \frac{\partial K}{\partial t} = \hat{H}_{\rho,\alpha}K \quad (18) \]

where

\[ \hat{H}_{\rho,\alpha} = -\frac{\hbar^2}{2m} \rho^{\frac{1}{2}} \alpha^{\frac{1}{2}} \left( \frac{\partial}{\partial q^i} g^{ij} \alpha \frac{\partial}{\partial q^j} \right) \rho^{\frac{1}{2}} \alpha^{\frac{1}{2}} + V \quad (19) \]

and has the normalization

\[ \lim_{t \to t_0} K[H,\rho,\alpha](q,t,q_0) = \frac{1}{\rho(q_0)} \delta^n(q - q_0) \quad (20) \]

To obtain the correct quantization scheme in arbitrary coordinates one needs to select \( \alpha = \sqrt{g}(g = \det(g_{ij}) = 1/\det(g^{ij})) \). With this choice of the scaling function the Hamiltonian operator in Eq. (18) becomes

\[ \hat{\mathcal{H}}_{\sqrt{g},\sqrt{g}} = -\frac{\hbar^2}{2m} \sqrt{g} \left( \frac{\partial}{\partial q^i} g^{ij} \sqrt{g} \frac{\partial}{\partial q^j} \right) + V \quad (21) \]

which has the invariant Laplace Beltrami operator as the kinetic energy part of the Hamiltonian operator. It should be noted that the Hamiltonian path integral with scaling is not obtained from any short time propagator but from the full finite Hamiltonian path integral \( K \).

### 3 Propagator in plane polar coordinates

We shall illustrate the detailed calculation of the Schrödinger equation for \( K[H,\sqrt{g},\sqrt{g}] \) for the case of free particle in two dimensions in \( r\theta \) coordinates. For this problem the momenta conjugate to \( (r,\theta) \) will be denoted by \( (P,p) \) and

\[ H = \frac{P^2}{2m} + \frac{p^2}{2mr^2} \quad (22) \]
\[ g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^{-2} \end{pmatrix} \]  

(23)

\[ \rho = \sqrt{g} = r \]  

(24)

\[ H^F_{\sqrt{g}} = r(H - E) = \frac{p^2}{2m} + \frac{p^2}{2mr} - E \rho = h \]  

(25)

Let

\[ \sigma_0 = 0, \sigma_1 = \epsilon, \sigma_2 = 2\epsilon, \ldots, \sigma_N = N\epsilon = \sigma \]  

(26)

be the pseudo time grid or slicing for the interval \((0, \sigma)\). The \(p\)-integrations are placed midway between \(\sigma_j\) and \(\sigma_{j+1}\), i.e., at \(\frac{1}{2 \epsilon}, \frac{3}{2 \epsilon}, \frac{1}{2 \epsilon}, \ldots\). To first order in \(\epsilon\), if \(r_j\) and \(\theta_j\) are coordinates chosen at \(\sigma_j\), \(S_{\pm\pm}\) are given by

\[ S_{++}(r_{j+1}\theta_{j+1}, P_j, p_j) \approx P_j r_{j+1} + p_j \theta_{j+1} - \frac{\epsilon}{2} h_{j+1} \]  

(27)

\[ S_{--}(P_j, p_j, r_j \theta_j) \approx -P_j r_j - p_j \theta_j - \frac{\epsilon}{2} h_j \]  

(28)

\[ h_{j+1} = \frac{r_{j+1} P_j^2}{2m} + \frac{p_j^2}{2mr_{j+1}} - E r_{j+1} \]  

(29)

\[ h_j = \frac{r_j P_j^2}{2m} + \frac{p_j^2}{2mr_j} - E r_j \]  

(30)

The factors \(D_{\pm\pm}\) do not trouble us because these are equal to \(1 + O(\epsilon^2)\). The short time propagator is

\[ \left( r_{j+1}\theta_{j+1} \mid r_j\theta_j \right) = \frac{1}{(2\pi \hbar)^2} \int dP_j \int dp_j \frac{1}{\sqrt{r_{j+1} P_j}} \exp[iS_j/\hbar] \]  

(31)

\[ S_j = P_j (r_{j+1} - r_j) + p_j (\theta_{j+1} - \theta_j) - \frac{\epsilon}{2} (h_{j+1} - h_j) \]  

(32)

and

\[ K[\hbar, \sqrt{g}] (r, \theta, \sigma, r_0, \theta_0 \sigma_0 = 0) \]  

(33)

\[ = \lim_{N \to \infty} \int (\prod_{k=1}^{N-1} r_k dr_k d\theta_k) (r\theta \epsilon \mid r_{N-1}\theta_{N-1}0) \cdots (r_1 \theta_1 \epsilon \mid r_0 \theta_0 0) \]  

\[ = \lim_{N \to \infty} \int d(N-1) \cdots d(1) dP_0 dp_0 \exp \left[ i \sum_{j=0}^{N-1} S_j/\hbar \right] \]  

(34)

7
where

\begin{align}
  r_N &= r; \quad \theta_N = \theta; \quad (35) \\
  d(j) &= r_j dr_j d\theta_j dP_j dp_j/(2\pi h^2) \quad (36)
\end{align}

Since we are interested only in deriving the Schrödinger equation, for our purpose it is sufficient to compute the propagator \(K[H, \sqrt{g}, \sqrt{g}]\) for small \(t\) only. The expression for \(K\) for the short times becomes

\begin{align}
  K[H, \sqrt{g}, \sqrt{g}](r\theta t, r0\theta0) &= \lim_{N \to \infty} \sqrt{rr_0} \int_0^\infty \frac{dE}{2\pi \hbar} \exp(-iEt/\hbar) \int_0^\infty d\sigma \int d(N-1)...d(1) \int dP0dp0 \exp\left\{i \sum_{j=0}^{N-1} S_j/\hbar\right\} \quad (37)
\end{align}

Note that each \(h_j\) in \(S_j\) carries a term \(-Er_j\) Thus \(E\) integration can be done immediately to give the delta function

\begin{align}
  \delta\left[\epsilon\left(\frac{1}{2}(r + r_0) + r_1 + ... + r_{N-1}\right) - t\right] \quad (38)
\end{align}

which allows us to do the \(\sigma\) integration (\(\epsilon = \sigma/N\)). The net result is that we get an overall factor

\begin{align}
  \frac{2N}{F} = \frac{2N}{r_0 + 2(r_1 + ... + r_{N-1}) + r} \quad (39)
\end{align}

and \(\epsilon \sigma/N\) is replaced by \(2t/F\). Thus

\begin{align}
  K_i = \int dP0dp0 \int d(N-1)...d(1) \left(\frac{2N}{F}\right) \exp\left[i \sum S_j/\hbar\right] \quad (40)
\end{align}

with

\begin{align}
  S_j &= P_j(r_{j+1} - r_j) + p_j(\theta_{j+1} - \theta_j) - \frac{t}{F} \left(\frac{P_j^2}{2m}(r_{j+1} + r_j) + \frac{p_j^2}{2m}(\frac{1}{r_{j+1}} + \frac{1}{r_j})\right) \quad (41)
\end{align}

Here on we write \(K_i\) for \(K[H, \sqrt{g}, \sqrt{g}]\), the arguments being suppressed. We shall also omit the explicit mention of limit \(N \to \infty\), as this limit will be taken in the end only.
Propagator $\mathcal{K}$ for short times: We are interested in the Schrödinger equation. For this purpose it is sufficient to take $t$ infinitesimally small. For $t$ actually equal to zero the propagator $\mathcal{K}_t$ becomes

$$\mathcal{K}_{t=0} = \int dP_0 dp_0 \int d(N-1) \int d(1) \frac{2N}{F} \exp \left( i \sum_j S_j / \hbar \right)$$ (42)

The $p-$ integrations can be carried out and one gets

$$\mathcal{K}_{t=0} = \int \left[ \prod_{j=1}^{N-1} dr_j d\theta_j \right] \frac{2N}{F} \delta(r-r_{N-1})\delta(\theta-\theta_{N-1})...\delta(r_1-r_0)\delta(\theta_1-\theta_0)$$

$$= \frac{1}{r} \delta(r-r_0)\delta(\theta-\theta_0)$$

as $F \to 2Nr$ when all $r_1 = r_2 = ... = r_{N-1} = r$. For finite but small $t$, the exponential is expanded to first order in $t$;

$$\mathcal{K}_t - \mathcal{K}_0 = \left( \frac{-it}{\hbar} \right) \sum_k \int dP_0 dp_0 \int d(N-1) \int d(1) \frac{2N}{F^2} \times$$

$$\left( \frac{P_k^2}{2m}(r_{k+1}+r_k) + \frac{p_k^2}{2m}(\frac{1}{r_{k+1}}+\frac{1}{r_k}) \right) \times$$

$$\exp \left[ \frac{i}{\hbar} \sum_{j=0}^{N-1} \{ P_j(r_{j+1}-r_j) + p_j(\theta_{j+1}-\theta_j) \} \right]$$

$$\equiv \left( \frac{-it}{\hbar} \right) \sum_k X_k$$ (43)

where

$$X_k = \int dP_0 dp_0 \int d(N-1) \int d(1) \frac{2N}{F^2} \left( \frac{P_k^2}{2m}(r_{k+1}+r_k) + \frac{p_k^2}{2m}(\frac{1}{r_{k+1}}+\frac{1}{r_k}) \right)$$

$$\exp \left[ \frac{i}{\hbar} \sum_{j=0}^{N-1} \{ P_j(r_{j+1}-r_j) + p_j(\theta_{j+1}-\theta_j) \} \right]$$ (44)

**Computation of $X_k$**: All the momenta integrations in $X_k$, except over the $k$-th
momenta, can be done giving \(2(N - 1) \delta\) functions.

\[
\delta(r - r_{N-1})...\delta(r_{k+2} - r_{k+1})\delta(r_k - r_{k-1})...\delta(r_1 - r_0) \tag{45}
\]

\[
\delta(\theta - \theta_{N-1})...\delta(\theta_{k+2} - \theta_{k+1})\delta(\theta_k - \theta_{k-1})...\delta(\theta_1 - \theta_0) \tag{46}
\]

This permits all \(r, \theta\) integrations to be done resulting in the replacements

\[
r = r_{N-1} = ... = r_{k+1}; \quad r_0 = r_1 = ... = r_k \tag{47}
\]

So

\[
X_k = \int dPdp \frac{2N}{F_k} \left[ \frac{P^2}{2m} (r + r_0) + \frac{p^2}{2m} \left( \frac{1}{r} + \frac{1}{r_0} \right) \right] \exp \left[ \frac{i}{\hbar} (P(r - r_0) + p(\theta - \theta_0)) \right] \tag{48}
\]

where we have renamed \(P_k, p_k\) as \(P, p\) respectively. The \(k\)-dependence of \(X_k\) resides only in \(F_k\) which is obtained from \(F\) after replacements as in Eq. \(45\)

\[
F_k = \frac{r}{2} + 2((N - k - 1)r + kr_0) + r_0 = (2N - 1 - 2k)r + (2k + 1)r_0 \tag{49}
\]

Writing

\[
F_k^{-2} = \int_0^\infty d\beta \beta \exp(-\beta F_k) \tag{50}
\]

\[
X_k = 2N \int dPdp \left[ \frac{P^2}{2m} (r + r_0) + \frac{p^2}{2m} \left( \frac{1}{r} + \frac{1}{r_0} \right) \right] \exp \left[ \frac{i}{\hbar} (P(r - r_0) + p(\theta - \theta_0)) \right] \times \int_0^\infty d\beta \beta \exp(-\beta r(2N - 2k - 1) - \beta r_0(2k + 1)) \times \\
- \left( \frac{\hbar^2}{2m} \right) \left[ (r + r_0) \delta''(r - r_0) \delta(\theta - \theta_0) + \left( \frac{1}{r} + \frac{1}{r_0} \right) \delta(r - r_0) \delta''(\theta - \theta_0) \right] \tag{51}
\]

where in the last step \(P\) and \(p\) integrations have been computed.

**Schrödinger equation** : The functions propagated in time by \(K\)

\[
\psi(r, \theta, t) = \int K[H, \sqrt{g}, \sqrt{g}(r\theta t, r\theta 0)] \psi(r_0, \theta_0) r_0 dr_0 d\theta_0 \tag{52}
\]
for small $t$, satisfy the equation

\[
\textit{i} \hbar \frac{\partial \psi}{\partial t} = \textit{i} \hbar \int r_0 dr_0 d\theta_0 \left( \frac{K_t - K_0}{t} \right) \psi(r_0 \theta_0)
\]

\[
= \sum_k \int r_0 dr_0 d\theta_0 X_k(r\theta, r_0 \theta_0) \psi(r_0 \theta_0)
\]

(53)

The integrations over $r_0, \theta_0$ are trivial in view of the $\delta-$ functions. For any $k$,

\[
\int r_0 dr_0 d\theta_0 X_k(r\theta, r_0 \theta_0) \psi(r_0 \theta_0)
\]

(54)

\[
= -\frac{\hbar^2}{2m} (2N) \int d\beta e^{-2\beta rN} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\hbar^2}{2m} (2N) \int d\beta e^{-\beta r(2N-2k-1)} \frac{\partial^2}{\partial r_0^2} \left( (r + r_0)e^{-\beta r_0(2k+1)r_0 \psi} \right) \bigg|_{r = r_0}
\]

(57)

The first term on summing over $k$ becomes

\[
\sum_{k=0}^{N-1} 2N \int d\beta \exp(-2\beta rN) 2 \frac{\partial^2 \psi}{\partial \theta^2} = \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}
\]

(58)

For the second term, we note that

\[
\frac{\partial^2}{\partial r_0^2} \left( (r + r_0)e^{-\beta r_0(2k+1)r_0 \psi} \right) = e^{-\beta r_0(2k+1)} \{ -6\beta r_0(2k+1) \psi + 2 \beta^2 r^2(2k+1) + 2 \psi + (-4 \beta r^2(2k+1) + 6r) \frac{\partial \psi}{\partial r} + 2r^2 \frac{\partial^2 \psi}{\partial r^2} \}
\]

(59)

Therefore, on using

\[
\sum_{k=0}^{N-1} (2k + 1) = N^2
\]

(60)

\[
\sum_{k=0}^{N-1} (2k + 1)^2 = \frac{1}{3} N(2N - 1)(2N + 1) \approx \frac{4}{3} N^3
\]

(61)

\[
\int_0^\infty d\beta \beta^n \exp(-2\beta rN) = \frac{n!}{(2rN)^{n+1}}
\]

(62)
the expression
\[
\sum_k 2N \int_0^\infty d\beta \beta e^{-2\beta r N} \left( (-6\beta r(2k+1) + 2\beta^2 r^2(2k+1)^2 + 2) \psi \\
+ (-4\beta r^2(2k+1) + 6r) \frac{\partial \psi}{\partial r} + 2r^2 \frac{\partial^2 \psi}{\partial r^2} \right)
\]
becomes
\[
\frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}
\]
(64)
Thus we get the desired Schrödinger equation
\[
\frac{i\hbar}{\partial t} = -\frac{k^2}{2m} \left( \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} \right)
\]
(65)
which shows that use of \( K[H, \sqrt{g}, \sqrt{g}] \) leads to the correct propagator for these coordinates. The expression for \( K \) is not yet the final propagator, one has to take into account of the boundary conditions at \( r = 0 \) and for \( \theta = 0, 2\pi \). This can be easily done and the final answer for the propagator is
\[
K(r, \theta; r_0, \theta_0, 0, ) = \\
\sum_{m = -\infty}^{\infty} [K(r, \theta + 2\pi m, t, r_0, \theta_0, 0,) + K(-r, \theta + 2\pi (2m + 1), t, r_0, \theta_0, 0,)]
\]
(66)
References

[1] Marinov M S 1980 Phys. Reports 60 1
[2] Edwards S F and Gulyev Y V 1964 Proc. Roy. Soc. (London) A279 2229
[3] Dewitt B S 1957 Revs. Mod. Phys. 29 377
[4] Pauli W 1973 Selected Topics in Field Quantization (MIT Press Cambridge)
[5] Paek D and Inomata A 1969 J. Math. Phys. 10 1422
[6] Arthurs A M 1970 Proc Roy Soc London A313 445
Arthurs A M 1970 Proc Roy Soc London A318 523
[7] Gervias J L and Sakita B 1976 Nucl. Phys. B110 53
[8] Langguth W and Inomata A 1979 J. Math. Phys. 20 499
[5] Kapoor A K 1984 Phys. Rev. D29 2339
[6] Ho R and Inomata A 1982 Phys. Rev. Lett. 48 231
[7] Inomata A 1982 Phys. Lett. 87A 387
[8] Kapoor A K 1984 Phys. Rev. D30 1750
[9] Kapoor A K and Pankaj Sharan, hep-th/9501013 (unpublished)
[10] Kapoor A K and Pankaj Sharan, hep-th/9501014 (unpublished)