Chiral Vertices, Fusion Rules and Vacua of Fractional Quantum Hall Systems

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Abstract

Vacua of two-dimensional incompressible systems, such as FQH systems, are characterized by rational conformal field theories. We develop a method to express the wavefunctions of FQH-like systems in terms of chiral vertices. We formulate quantum mechanics on those expressions, which reveals the simple structure underlying the conformal field theory description of the quantum hall states. Also we argue the recent conjecture of Nayak and Wilczek on the spinor statistics of 2n quasihole state in paired quantum hall states.

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1 Introduction

In the universe, various kinds of low dimensional structures exist. For more than a decade, physicists have been paying much effort to explore those low dimensional structures. The regions where low dimensional structures emerge widely spread in many branches of physics. High temperature superconductivity and quantum hall effect in condensed matter systems, string theory which unifies space-time and particle, and various solvable statistical models and their critical behavior and so on. In all cases, two dimensional nature of system plays a prominent role. The common feature is that two dimensional structure seems to have a deep relationship with the phases of the system.

Among them, Fractional Quantum Hall Effect (FQHE) is remarkable from experimental verifiability. In the experiments for this system one can control arbitrarily the parameters which govern the vacuum of the system, that is the strength of a background uniform magnetic field and the electron’s density. The vacua are parametrized by the ratio of electron’s density and the strength of the magnetic field, which is called ‘filling fraction’.

Fractional quantum hall effect was first discovered by Tsui, Stormer and Gossard in 1982 in the two-dimensional system of electron in a uniform magnetic field with odd denominator filling fractions. It was first phenomenologically understood by the first-quantized approach initiated by Laughlin. The remarkable ingredient is fractional statistics obeyed by the quasiparticles sustained by the vacua. Thus fractional quantum hall effect gives us the evidence that, in 2+1 dimension, there are very rich structure of vacua. These approach was extended to so called hierarchical construction to include the general odd denominator filling fractions.

Some years later their dynamics are formulated as a 2+1 dimensional system of second-quantized particle coupled to U(1) Chern-Simons gauge field. Chern-Simons gauge field adds a flux to the particle to which the gauge field couples. The most transparent formulation is through Landau-Ginzburg type theory. In LG formulation, the quasiparticle appears as a classical configuration of charged boson and U(1) Chern-Simons field.

As they are formulated with Chern-Simons gauge field, the relation between FQHE and conformal field theories naturally emerges. In fact in , it was shown that various FQH wavefunctions could be interpreted as certain conformal blocks of proper RCFT. They proposed that the effective theory for the bulk is the corresponding Chern-Simons theory. To prove this proposal, it should be verified quasiparticle’s statistics (defined by Berry’s phase) are given by the braiding property obtained from Chern-Simons theory. Although this is proved for abelian cases, many indications show the validity of this proposal.
for the system obtained from $SU(N)_k$ model and, more recently \cite{9} gives a general argument in favor of this proposal. Assuming that this is true, some FQH states are supposed to have a excitation with so-called nonabelian statistics. These states are some paired quantum hall states \cite{10} \cite{11} which might be realized in FQH systems with even-denominator filling fractions \cite{12}. Recently, those paired quantum hall states have attracted much attention \cite{13}–\cite{16}. Other examples of nonabelian statistics were given in \cite{8} based on $SU(N)_k$ Wess-Zumino-Witten models.

There are other kind of formulation of FQHE, developed in \cite{17}–\cite{18}, which is the approach to the formulation of FQHE from area-preserving diffeomorphism. This approach is based on the incompressible liquid nature of FQH system. A hierarchical construction scheme, which fit to composite fermion construction of \cite{19}, arises from the existence of particular class of 1+1 dimensional field theory($W_{1+\infty}$ minimal model (this is not RCFT)) on the edge. Also excitations with nonabelian statistics are found on these models.

FQHE has been investigated in a system of electron. However, recently, the possibility of FQHE also in a bosonic system was pointed out \cite{20}. This is natural because, as the coupling to Chern-Simons gauge field changes the statistics of particle, the original statistics of the particle does not matter.

In this paper we will study the basic aspects of general 2+1 dimensional incompressible systems described by rational conformal field theories. We will use the canonical quantized formulation of RCFT. This is because there is a system which lacks a suitable dynamical 2+1 dimensional field theory description. We introduce a method to express the wavefunctions of FQH-like systems in terms of chiral vertices. We define some useful operator on chiral vertices. We also argue the recently conjectured spinor statistics of Pfaffian states \cite{9} by these formulation.

Our main philosophy is that RCFT (or its operator algebra) characterizes the vacua of 2+1 dimensional incompressible system. Fusion algebra \cite{22} is the most essential ingredient for the system characterized by rational conformal field theory.

The organization of this paper is as follows. In sect.2 we review generalities on rational conformal field theory. Sect.3 describes the general feature of canonically quantized RCFT as the incompressible system. In particular, we formulate the extended U(1) current algebra description of a background uniform magnetic field, which will be useful when one deals with a charged particle system such as FQHE. In sect.4 we discuss the property of some operators on chiral vertices. Sect.5 gives some examples, which involves some known states of FQHE. Also we describe how one can deduce the spinor statistics of multiple quasihole Pfaffian state.

\footnote{The role of fusion algebra in FQH systems were previously discussed in \cite{11} \cite{13}.}
2 Generalities on RCFT

First we recall some general properties of rational conformal field theory (RCFT). Let Σ be a Riemann surface of genus $g$ with $n$ punctures $z^i, i = 1 \cdots n$. We denote their moduli by $m^j, j = 1, \cdots, 3g - 3$. A conformal field theory is said to be rational if its unnormalized $n$-point function $G$ has an expansion

$$G(z^i, m^j, \bar{z}^i, \bar{m}^j) = \sum_{IJ} h_{IJ} F_I(z^i, m^j) F_J(\bar{z}^i, \bar{m}^j)$$

as a sum of products of holomorphic building blocks $F_I$ and antiholomorphic ones $\overline{F}_J$, and $h_{IJ}$ is a hermitian metric. $F_I$ span a finite dimensional vector space. In general, $F_I$ have a nontrivial monodromy and modular behavior. These properties are further specified by some PDE which arise from the appearence of null states. $F_I$ are in fact holomorphic sections of a vector bundle $\mathcal{V}_{g,n}$ over the moduli space $\mathcal{M}_{g,n}$ of punctured Riemann surfaces. As $\mathcal{V}_{g,n}$ is finite dimensional, rational conformal field theories have only a finite number of Virasoro representations i.e. primary fields.

In this paper we don’t treat the properties associated with the hermitian metric $h_{IJ}$ although these properties are very remarkable. We will mainly treat chiral half of RCFT in the remainder of this paper.

Let us consider some RCFT with $N$ primary fields $\phi_i, (i = 0, \cdots, N - 1)$ corresponding to the irreducible representations $[\phi_i]$. $\phi_0$ is always used to denote the representations which contains the identity. In this RCFT any analytic blocks are built out from three point functions $\langle \phi_i \phi_j \phi_k \rangle$ by so-called sewing (or gluing) procedure. The sewing procedure can be formally written by Feynman diagrams of $\phi^3$ theory by relating the $\phi^3$ vertices to $\langle \phi_i \phi_j \phi_k \rangle$. However the correspondence between Feynman diagrams of $\phi^3$ and the correlation functions on punctured Riemann surfaces with operators at punctures is not one to one. $\phi^3$ graphs are redundant. This redundancy leads to duality between different bases of analytic blocks.

To describe duality (and for later convenience), it is convenient to use the concept chiral vertex (see, for example [24]). It is the operator which is defined when we see a representation $[\phi_i]$ as inducing an operator $\Phi_{jk}^i(z) : [\phi_k] \rightarrow [\phi_j]$. This definition is specified by the matrix element between primary fields $\langle j | \Phi_{jk}^i(z) | k \rangle = \delta_{jk} z^{-\Delta_i}, \Delta_i = \Delta_i + \Delta_k - \Delta_j$. Here $\delta_{jk}$ are invariant tensors of the theory. Its actions on descendants are deduced from this formula. This chiral vertex itself represents the sphere with $\phi_i$ at $z$, $\phi_k$ ($\phi_j$) at the origin (the infinity).

There are two fundamental operations on chiral vertices. One of the fundamental
operations is fusing\footnote{For simplicity, we use the notation which is valid for the case where the fusion rule coefficient $N^{k}_{ij}$ (defined in the latter part of this section) are 0 or 1. Typical models ($SU(2)$ WZW models, Ising model etc) satisfy this condition.}:

$$
\Phi_{kp}^i(z_1) \Phi_{pl}^j(z_2) = \sum_q F_{pq} \left[ \begin{array}{cc} i & j \\ k & l \end{array} \right] \sum_q \langle Q | \Phi_{qj}^i(z_1) | j \rangle \Phi_{ql}^q(z_2), \quad (2.2)
$$

where $z_{12} = z_1 - z_2$. This is nothing but the operator product expansion. The other fundamental operation is exchange or braiding\footnote{For simplicity, we use the notation which is valid for the case where the fusion rule coefficient $N^{k}_{ij}$ (defined in the latter part of this section) are 0 or 1. Typical models ($SU(2)$ WZW models, Ising model etc) satisfy this condition.}:

$$
\Phi_{kp}^i(z_1) \Phi_{pl}^j(z_2) = \sum_q R_{pq} \left[ \begin{array}{cc} i & j \\ k & l \end{array} \right] \Phi_{kq}^j(z_1) \Phi_{ql}^i(z_2). \quad (2.3)
$$

$R$ depends on which region one compare the blocks. We will take $R$ in Im$(z_1 - z_2) > 0$. $R$ is related to the monodromy of analytic blocks. In fact

$$
\langle k | \Phi_{kp}^i(e^{2\pi i z}) \Phi_{pl}^j(0) | l \rangle = \sum_{q,s} R_{ps} \left[ \begin{array}{cc} k & j \\ i & l \end{array} \right] R_{sq} \left[ \begin{array}{cc} j & k \\ i & l \end{array} \right] \langle k | \Phi_{kq}^i(z) \Phi_{ql}^j(0) | l \rangle. \quad (2.4)
$$

$F$ and $R$ can be casted as base change transformations in the space of analytic blocks. This interpretation comes from the fact that chiral vertices represent the three-point functions on the sphere and analytic blocks are built up from these functions. More precisely this transformation are formulated as the bundle transformations of the vector bundle over the moduli space $\overline{M}$ as first formulated by Friedan and Shenker\footnote{For simplicity, we use the notation which is valid for the case where the fusion rule coefficient $N^{k}_{ij}$ (defined in the latter part of this section) are 0 or 1. Typical models ($SU(2)$ WZW models, Ising model etc) satisfy this condition.}. In terms of $F$ and $R$, the condition for the equivalence of different descriptions of analytic blocks, i.e. duality is reduced to the equations among $F$ and $R$ and conformal weight. These are polynomial equations such as pentagon and hexagon relations of\footnote{For simplicity, we use the notation which is valid for the case where the fusion rule coefficient $N^{k}_{ij}$ (defined in the latter part of this section) are 0 or 1. Typical models ($SU(2)$ WZW models, Ising model etc) satisfy this condition.}. From those relations $R$ can be written in terms of $F$ as

$$
R_{pq} \left[ \begin{array}{cc} i & l \\ k & j \end{array} \right] = e^{i\pi(\Delta_k + \Delta_j - \Delta_p - \Delta_l)} F_{pq} \left[ \begin{array}{cc} i & j \\ k & l \end{array} \right]. \quad (2.5)
$$

Thus the exchange property is determined from the fusing property and conformal weights of primary fields.

The concept of chiral vertex will be essential for nonabelian excitations discussed in later sections.

Finally let us recall fusion rules. Let $\chi_{ijk}$ be the vector bundle corresponding to the three puctured sphere with $\phi_i, \phi_j, \phi_k$ at the punctures. We put $N_{ijk}$ to be
\[ \dim V_{0,ijk} \] The indices are raised by \( N_{ij0} \). Actually \( N_{ij0} \) is conjugate matrix \( C \) and \( N_{ij}^k \) is the dimension of the space of \( t_{ij}^k \). Then fusion rules are defined as
\[
\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k
\] (2.6)

(2.6) shows how many ways there are to fuse \([\phi_i]\) and \([\phi_j]\) to \([\phi_k]\). From duality we have following equation,
\[
\sum_k N_{ij}^k N_{klm} = \sum_k N_{il}^k N_{jkm}.
\] (2.7)

This relation (2.7) implies the associativity of the fusion rules. Fusion rules therefore forms a representation of chiral ring. A highly nontrivial fact is this representation is determined by the modular property of Virasoro characters \[22\]. Famous Verlinde formula states that fusion rules have a deep relation with the modular behavior of Virasoro characters.

The degeneracy of analytic blocks on a general Riemann surface can be computed from the fusion rules. By \( \phi^3 \)-diagram representation of the given analytic block we can built up any analytic blocks on Riemann surfaces. From the definition of fusion rules and duality, this description gives a consistent number for degeneracy. By this method, we have general formulas for degeneracy as follows:
\[
\begin{align*}
\langle \phi_i \phi_j \phi_k \rangle & \quad N_{ijk} \quad \sum_{lm} N_{lm}^m N_{jk}^m \\
\langle \phi_i \phi_j \phi_k \phi_l \rangle & \quad N_{ijm} N_{kl}^m \\
\cdots & \quad \cdots 
\end{align*}
\] (2.8)

For more details of RCFT, see \[23\].

3 2+1 Dimensional System and RCFT

3.1 Generalities

We will consider how a given RCFT looks like from 2+1 dimensional point of view in this section.

The most famous connection between RCFT and 2+1 dimensional physics is through Chern-Simons theory \[26\]. It is known that chiral algebra of RCFT can be obtained from quantizing Chern-Simons theory on the disk and conformal blocks can be obtained from Chern-Simons theory with Wilson lines \[27\]. Chern-Simons theory is useful to argue qualitative properties of RCFT. However we’d rather give a basic characterization of RCFT as incompressible system in this section.
As in the last section, let $[\phi_i]$ be the Virasoro representations of the chiral half of some RCFT. These fields may represent excitations of $2+1$ dimensional system. First of all a Virasoro representation $[\phi_i]$ should correspond to a $2+1$ dimensional single particle excitation. When we apply some infinitesimal conformal reparametrization on 2 dimensional space, primary field $\phi_i$ transforms to a linear combination of descendant fields. Since this is merely an effect of reparametrization of space, they should be physically the same. This is the first reason for the above statement. Also it is seen in the previous section that $F$ and $R$ is defined for Virasoro representations. Since those operations give us the statistical property in $2+1$ dimension (see sect 3.2), again a Virasoro representation $[\phi_i]$ should correspond to a $2+1$ dimensional single particle excitation. Thus it is natural to see a Virasoro representation $\phi_i$ as quasiparticle in $2+1$ dimension. We will consider these $\phi_i$ as a quasiparticle excitation of some $2+1$ dimensional system.

We next consider what kind of 'hamiltonian' we should take for this system. We consider a second-quantized (in a sense) heuristic hamiltonian which contains the essential physical content the given RCFT has as a $2+1$ dimensional object. Among various CFT operator the most simple choice is:

$$H = L_0. \quad (3.1)$$

Although this seems to be too simple, we will see in sect.3.6 that we must use the projection for conformal weight to form a suitable $2+1$ dimensional quantum mechanical hamiltonian $\mathcal{H}$. In that description (3.1) sees the essence of qualitative nature of the system described by a given RCFT. (3.1) is also the hamiltonian of $1+1$ dimensional physics on the edge and measures the energy of $1+1$ dimensional excitations.

However, as we regard a conformal family corresponding to a single quasiparticle in the bulk, we must project the state to some representative state before we apply $H$ as in the way of gauge fixing. To this end we may simply take the projection operator $P_L$ to the lowest level state of conformal family:

$$M_0 = P_L H P_L. \quad (3.2)$$

By taking this projection, we can reduce descendant fields for this operator. Also we can see conformal blocks as representatives of physically equivalent analytic blocks. So we will treat $M_0$ as a heuristic ' hamiltonian ' or a mass of excitations. This is somewhat similar to the situation in string theory. By taking $M_0$, the vacuum supports a gap since this is RCFT, it means this system is incompressible.

Quasiparticles $\phi_i$ 's short-distance behaviors or interactions are obtained from the operator product expansion. However in the long-wavelength limit one may
ignore the detail of short-distance physics, and an information which neglects such
details is essential. Such a information is provided by the fusion rules \( (2.6) \)
\[
\phi_i \times \phi_j = \sum_k N_{ij}^k \phi_k.
\] (3.3)

Although there is the case when different CFT have a same fusion rules, fusion rules
are at least decisive for whether the system have abelian statistics or nonabelian
statics and what kinds of bound state exist in the system. It is known that fusion
rules lead to a rather hard restriction on the central charge and conformal weight.

Let us see how (3.3) arises. The meaning of (3.3) is from Deligne-Mumford stable
compactification of moduli space \( \mathcal{M}_{g,n} \). The process in which two points
\( z_1 \) and \( z_2 \) come close to each other can be described as the process in which a sphere, that
contains \( z_1 \) and \( z_2 \) at fixed distance, pinches off the surface by forming a neck of
length \( \log |z_1 - z_2| \). The end of this process is the surface with \( z_1 = z_2 \) and a sphere
with two vertex operators \( \phi_i(z_1), \phi_j(z_2) \) plus one extra marked point where another
vertex operator \( \phi_k \) is inserted. This is why \( N_{ij}^k \) must appear on the righthand side
of (3.3). This process also clearly shows how chiral vertices \( \Phi_{ij}^k \) appears.

For a Virasoro representation corresponding to a quasiparticle of 2+1 dimensional system, a conformal block
\( \langle \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) \rangle \) is considered as the amplitude
for quasiparticles to be exist at \( z_1 \cdots z_n \). This means \( \langle \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) \rangle \) may be the
wavefunction for some many-body system. A meaning of chiral vertex \( \Phi_{ij}^k \) in this
context is that it is a expression that represents concisely the situation when \( \phi_k \)
is on some place within a region conformally isomorphic to annulus or tube and its
boundary conditions or asymptotic Virasoro representations for path-integration are
\( \phi_i \) and \( \phi_j \). On each region, we consider the local Hilbert space for the given RCFT
over \( SL(2, \mathbb{C}) \) invariant vacuum. Each region is mapped to the Riemann sphere with
three marked points by a local conformal map (in the limit i.e. boundaries mapped
to points. In this limit we are in \( \overline{\mathcal{M}} \) again.).

From this viewpoint on chiral vertices, let us consider the amplitude for \( n \) excita-
tions \( \phi_{i_1}, \cdots \phi_{i_n} \) on a closed Riemann surface \( \Sigma \) and introduce the canonical expression
of \( \langle \phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n) \rangle \) in terms of chiral vertices. First we make a simply
connected region on \( \Sigma \) which only contains the point at which the excitation \( \phi_{i_n} \) is,
and no other excitations. Second, we make a region isomorphic to annulus which
only contains \( \phi_{i_{n-1}} \). We continue this procedure, to end up with the simply con-
connected region which contains \( \phi_{i_1} \). When \( \Sigma \) has genus \( \geq 1 \), we must insert \( 2g \) region
\( C_1, \cdots, C_g, D_1, \cdots, D_g \) with three boundaries, which don’t contain any excitations.
This procedure defines one direction of flow on \( \Sigma \). This flow must not be stopped
up except at punctures or real boundaries (in case they exist). \( C_j \) arises when the
flow on \( \Sigma \) splits into two, and \( D_j \) arises when two flows joint into one. By this pro-
cedure \( \Sigma \) is divided into \( n \) regions \( A_1, \cdots, A_n \) and \( 2g \) region \( C_1, \cdots, C_g, D_1, \cdots, D_g \)
Now we can express the given amplitude through chiral vertices. The amplitude $\langle \phi_1 \cdots \phi_n \rangle$ corresponds to the expression of chiral vertices

$$\Phi^{i_1}_{l_1} \ldots \Lambda^{d_{3j}-1}_{d_{3j-2}d_{3j}} \cdots \Phi^{i_n}_{a_{2l-2}a_{2l-1}} \cdots \Lambda^{c_{3k-1}}_{c_{3k-2}c_{3k}} \cdots \Phi^{i_n}_{a_{2n-2}1};$$

(3.4)

where $\Lambda^{c_{3k-1}}_{c_{3k-2}c_{3k}}, k = 1, \cdots, g$ are from $C_k$ and $\Lambda^{d_{3j}-1}_{d_{3j-2}d_{3j}}$ are from $D_j$. All $a_i$ and $c_m$ and $d_m$ should be sewed to some $a_i$ or $c_k$ or $d_l$. $a_{2l}$ are sewed to $a_{2l-1}$ or $d_{3j}$ or $d_{3j-1}$, and $a_{2l+1}$ are sewed to $a_{2l+2}$ or $c_{3j}$ or $c_{3j-1}$. $c_{3k-2}$ and $c_{3k-1}$ are sewed to $a_{2l+1}$ or $d_{3j}$ or $d_{3j-1}$, and $c_{3k}$ are sewed to $a_{2l}$ or $d_{3j-2}$. By the above division of $\Sigma$ we know which of $a_i$, $c_k$ and $d_l$ should be the same and sewed. This amplitude has a degeneracy which is computed from the fusion rule coefficients of the given RCFT.

On a sphere they can be simply written as

$$\Phi^{i_1}_{l_1} \Phi^{i_2}_{a_1a_2} \cdots \Phi^{i_{n-1}}_{a_{n-2}a_{n-1}} \Phi^{i_n}_{a_n1};$$

(3.5)

The different ordering of $\phi_i$ gives other expression. However the property of duality reviewed in the previous section ensures all of those expressions lead to the same amplitude.

Also when the system is on a Riemann surface with some boundary, we can do the same procedure described above with the orientation of the boundaries. This again gives us the canonical expression of the amplitude. Let us consider a Riemann surface with some boundaries, say a disk $D$. Then we can do the procedure such that the region $A_1$ has a real boundary $\partial D$. Now $A_1$ is not simply connected region but becomes an annulus-like region. The amplitude now corresponds to

$$\Phi^{i_1}_{b_{a_1}} \Phi^{i_2}_{a_1 a_2} \cdots \Phi^{i_{n-1}}_{a_{n-2} a_{n-1}} \Phi^{i_n}_{a_n1};$$

(3.6)

where $b$ represents the boundary state. In the case of an annulus, the expression becomes like

$$\Phi^{i_1}_{b_1 a_1} \Phi^{i_2}_{b_2 a_2} \cdots \Lambda^{c_2}_{c_1 c_3} \cdots \Phi^{i_{n-1}}_{a_{2n-5} a_{2n-4}} \Phi^{i_n}_{a_{2n-3}1};$$

(3.7)

where $b_1, b_2$ represent the two boundary states of the annulus. These two states are constrained by each other through the fusion rules. Therefore the fusion rules of the theory gives a correlation of the two edge states. When there are $m$ boundaries, by inserting $\Lambda^{c_{3k-1}}_{c_{3k-2}c_{3k}}, k = 1, \cdots, m-1$, we get the same kind of the expression of the amplitude in terms of chiral vertices $\Phi$. The multiple boundary states are again correlated through the fusion rules.

\[1\] This canonical expression have a similarity with 'temporal gauge' introduced in 2D gravity.
In this expression, the creation of quasiparticle $\phi_i$ at $z$ on some state is achieved by $\Phi_{jk}^i(z)$ where $j$ and $k$ should be suitably chosen (depends on the state on which $\phi_i$ is created).

Now let us define ‘mass’ operators for chiral vertices, which will be shown to have a direct relation to 2+1 dimensional many-body system interpretation in sect.4. By (3.2), we define $M$ and $\overline{M}$ as follows:

\[
M\Phi_{jk}^i = (\Delta_i - \Delta_j + \Delta_k)\Phi_{jk}^i, \quad (3.8)
M\Lambda_{jk}^i = (\Delta_i - \Delta_j + \Delta_k)\Lambda_{jk}^i, \quad (3.9)
\]

\[
\overline{M}\Phi_{jk}^i = (-\Delta_i - \Delta_j + \Delta_k)\Phi_{jk}^i, \quad (3.10)
\overline{M}\Lambda_{jk}^i = (-\Delta_i - \Delta_j - \Delta_k)\Lambda_{jk}^i, \quad (3.11)
\]

\[
\overline{M}\Lambda_{jk}^i = (-\Delta_i + \Delta_j - \Delta_k)\Lambda_{jk}^i, \quad (3.12)
\overline{M}\Lambda_{jk}^i = (-\Delta_i + \Delta_j + \Delta_k)\Lambda_{jk}^i, \quad (3.13)
\]

if $N_{jk}^i \neq 0$, and 0 otherwise, where $\Delta_i$ are the eigenvalue of $M_0$ defined in (3.2),

\[
M_0\phi_i = \Delta_i\phi_i. \quad (3.14)
\]

We also adopt (3.14) to the state at the boundaries. This means that we only treat the state at the boundaries like an asymptotic state of quasiparticle in the bulk when we apply $M$ and $\overline{M}$. We assume $M$ and $\overline{M}$ satisfy Leibniz rule when we apply them to a monomial $\Phi$, $\Lambda$ and $\overline{\Lambda}$. It implies, for example,

\[
M\left[\phi_{b_1a_1}^{i_1} \cdots \Phi_{d_{3j-2}d_{3j}}^{i_{3j-1}} \cdots \phi_{a_{2l-1}a_{2l-2}}^{i_{2l-1}} \cdots \phi_{a_{2n-2}a_{2n-3}}^{i_{2n-2}}\right]
= (\sum_k \Delta_{ik} - \sum_m \Delta_{bm}) \left[\phi_{b_1a_1}^{i_1} \cdots \Phi_{d_{3j-2}d_{3j}}^{i_{3j-1}} \cdots \phi_{a_{2l-1}a_{2l-2}}^{i_{2l-1}} \cdots \phi_{a_{2n-2}a_{2n-3}}^{i_{2n-2}}\right], \quad (3.15)
\]

\[
\overline{M}\left[\phi_{b_1a_1}^{i_1} \cdots \Phi_{d_{3j-2}d_{3j}}^{i_{3j-1}} \cdots \phi_{a_{2l-1}a_{2l-2}}^{i_{2l-1}} \cdots \phi_{a_{2n-2}a_{2n-3}}^{i_{2n-2}}\right]
= (\sum_k \Delta_{ik} + \sum_m \Delta_{bm}) \left[\phi_{b_1a_1}^{i_1} \cdots \Phi_{d_{3j-2}d_{3j}}^{i_{3j-1}} \cdots \phi_{a_{2l-1}a_{2l-2}}^{i_{2l-1}} \cdots \phi_{a_{2n-2}a_{2n-3}}^{i_{2n-2}}\right], \quad (3.16)
\]

where all $a, c, d$ should be sewed to each other plausibly explained as above, and $b_i$ are the states on the boundaries. From these formula, $M$ and $\overline{M}$ depend only on $\phi_{ik}$ and the states on the boundaries $\phi_{bm}$. These operators act invariantly under duality. Expressions like (3.4) are also eigenstates for $M$ and its eigenvalue is the sum of conformal weights of its excitation. We will see this operator has a relation with 2+1 quantum mechanical interpretation in sect.4.

The discussion in this section can also be extended to antichiral half of RCFT.
3.2 Statistics of Quasiparticles

To argue the definition of statistics of quasiparticles, we must first recall the fact that the fundamental object in CFT is not a single operator $\phi_i$. The fundamental object is the sphere with 3 inserted operator, that is represented by chiral vertex $\Phi_{ijk}$. So statistics should be considered on $\Phi_{ijk}$, not on a single Virasoro representation $[\phi]$. For example we will see the statistics in Pfaffian state can be naturally understood as a property of chiral vertex $\Phi_{ijk}$, not on $[\phi_i]$.

Statistics of quasiparticle is defined by Berry’s phase. Whether this phase can be calculated from the braiding matrix of conformal blocks is not proved although some arguments in favor of this are given in [8] [9]. We assume it generally holds.

In section 2.1, we already define the exchange operation on chiral vertices:

$$\Phi_{kp}(z_1)\Phi_{pl}(z_2) = \sum_q R_{pq} \left[ \begin{array}{cc} i & j \\ k & l \end{array} \right] \Phi_{kq}(z_1)\Phi_{ql}(z_2).$$

(3.17)

This is our definition of statistics. This means statistics is considered as a base change of the space of wavefunctions. Among the matrices $R$, the most important one is

$$R_{pq} \left[ \begin{array}{cc} i & i \\ i & i \end{array} \right].$$

(3.18)

This matrices should be considered as the statistics for the excitations $\phi_i$. However other matrices also should be taken as the definition of statistics since the fundamental object is chiral vertex as explained above.

Let us consider the state with $2n$ identical quasiparticles. We can consider the exchange or braiding of $n$ pairs of quasiparticles for this state. Generally the expression in terms of chiral vertices for the creation of $2n$ quasiparticles has the degeneracy $D$ obtained from the fusion rules. As the braiding is half-monodromy around the other particle, this operation can be thought to be $\pi$ rotation, therefore induces a representation of $so(2)$ except the overall phase. Those $so(2)$ can be seen as subalgebra of $so(2n)$ and forms a basis of Cartan subalgebra of $so(2n)$. Thus $n$ $so(2)$ representations specify the highest weight of a representation of Lie algebra $so(2n)$. Also the overall phase determines a representation of $U(1)$. Therefore the $D$ dimensional vector space of degenerate expressions forms thus specified representation of $SO(2n) \times U(1)$. We can see this phenomenon as a extension of statistics to a $D$ dimensional (spinor or ordinary ) representation of $SO(2n) \times U(1)$. Also when we consider the state with $2n + 1$ identical quasiparticles, we end up with a $D$ dimensional (spinor or ordinary ) representation of $SO(2n + 1) \times U(1)$. Whether these representation are spinor or not is determined by the matrix $R$. 

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This extension of statistics is first introduced in [9] for Pfaffian states by explicit calculation and proved for four quasihole state. We will give a general RCFT proof for $2n$ quasihole state in sect.5.2 based on the argument above.

We now discuss the relation between nonabelian statistics and the partition function of the edge state of the disk. Let us consider the system on the disk. At the finite temperature $1/\beta$, the system on the edge is the same as the bulk system on the torus. The partition function of the edge state at finite temperature $i\beta = \tau$ is equal to the sum of Virasoro characters (except constant):

$$\chi_i = \text{tr}_{\phi_i}(q^{L_0+\varepsilon})$$  \hspace{1cm} (3.19)

where $q = e^{2\pi i \tau}$ and $\varepsilon = -\frac{1}{24}c$ and $\tau$ is the purely imaginary modular parameter. The modular transformation $S$ is

$$S : \tau \rightarrow -\frac{1}{\tau}.$$ \hspace{1cm} (3.20)

The behavior of $\chi_i$ under $S : \tau \rightarrow -\frac{1}{\tau}$ transforms as a unitary representation:

$$\chi_i \rightarrow \sum_j S^j_i \chi_j.$$ \hspace{1cm} (3.21)

There is the remarkable relation between the matrix $S^j_i$ and the fusion rule coefficients $N^k_{ij}$ of RCFT.

$$N^k_{ij} = \sum_n S^n_j \chi_i^{(n)} S^n_k$$ \hspace{1cm} (3.22)

$$\chi_i^{(n)} = S^n_i / S^n_0$$ \hspace{1cm} (3.23)

This remarkable fact is conjectured in [22] and proved in [23] (In [22] Verlinde states this relation as the modular transformation $S$ diagonalizes the fusion rules.). In our context (3.22) means the $F$, which is the source of nonabelian statistics is determined by the modular behavior of partition function of the edge excitation. This is to say, nonabelian statistics manifest itself in the modular property of partition functions of the edge states.

---

1^The modular properties of the partition function of the edge excitations for the annular geometry was studied in the second paper of [18] for composite fermion construction.
3.3 Operator Algebra and Vacua

Now let us consider what is essential to determine the phase of the given system. That is *vacuum* (or the ground state). What information do we need to characterize the vacuum of the phase? That is the quantum numbers of its excitations and their interaction. As we are considering the systems which are characterized by some RCFT, the statistics of excitations is determined from fusing $F$ and conformal weights as in (2.5). Also we will see in sect.4 that the interaction between the quasi-particles are obtained from these information in many-body system interpretations. Thus it is clear that it is fusion rules and Virasoro primaries (i.e. operator algebra) that characterize the vacuum of the phase.

3.4 Low Energy Physics

Let us consider the low energy physics of the given system. The excitations which cover the long wavelength physics are the excitations which have lowest energy from vacuum. In our description, it means the eigenstates of $M$ (3.2) whose eigenvalue is lowest. Now let $\phi_i^L$ to be lightest ones among $\phi_i$. In general, when the system has a symmetry of a group $G$, $\phi_i^L$ forms a representation of $G$. For example when the system is spin-singlet, the system should have $SU(2)$ symmetry. In this case $\phi_i^L$ can be labeled as $\phi_i^a$, $a = \uparrow, \downarrow$.

3.5 Rational Torus and Uniform Magnetic Field

We consider in particular how to deal with a system in a uniform magnetic field in this section. The problem is how one can reproduce the effect of the uniform magnetic field by RCFT point of view. In this section, we show the factor called the neutralizing background field used in [6][8][9] are deduced from the symmetry consideration and the fusion rules of rational torus.

When a charged system is coupled to a background uniform magnetic field, there should be U(1) gauge symmetry. When one fixes the gauge, global U(1) symmetry remains. In the spirit of current algebra, U(1) current algebra naturally arises. The theory should be described by some effective theory with U(1) current algebra in the low-energy region. As we consider a incompressible system which are characterized qualitively by $M_0$, extended U(1) algebra should appear.

So first let us review extended U(1) current algebra. It is described by the chiral boson field $\varphi$ which is compactified on a circle with a rational value of the square of radius. They have U(1) current algebra as symmetry with some extended operator. It has $N$ primary fields $[\phi_p]$ with U(1) charge $p/\sqrt{N}$, $p \in Z(\text{mod}N)$. 

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The extended operator has conformal spin $N/2$ and the unit U(1) charge. This operator corresponds to 't hooft operator in Chern-Simons theory \[27\]. This means the extended corresponds to a singular gauge transformation. As we discuss with the gauge fixed, the spectrum $p$ which appears is not modulo $N$. This redundancy is reduced when we take account of the singular gauge transformation. In the case of odd $N$ (this is the case for abelian FQHE) the theory is not well defined on arbitrary Riemann surfaces without additional structures. So when $N$ is odd, we implicitly assume the space is simply connected or additional structure exists. The charge current is

$$J_z = i\sqrt{f} \partial_z \varphi,$$

where $f = \frac{1}{N}$. So $\phi_p$ has the charge $p/N$. Thus $\phi_p$ are the operators for fractionally charged excitations.

The fusion rules follow from U(1) charge conservation:

$$\phi_p \times \phi_q = \phi_{p+q}$$

(3.25)

This fusion rules have the same form of multiplication as U(1). From the explicit expression of vertex operators by chiral boson $\varphi$, we see that $\sqrt{f} \varphi$ plays the role of the generator of U(1) transformation. Let us recall U(1) Chern-Simons theory. The constraint for the gauge fixing $A_0 = 0$ was the field strength $F_{z \bar{z}}$ and $F_{z \bar{z}}$ is the generator of gauge symmetry. Magnetic field is nothing but a background field of Chern-Simons gauge field strength. Therefore, if we compare these two theory, it is clear that we should see $\sqrt{f} \varphi$ as $F_{z \bar{z}}$, which is the field strength in Chern-Simons theory. So the appropriate operator to describe the background uniform magnetic field by U(1) current algebra should be

$$\exp \int \frac{d^2z}{2\pi i} \sqrt{f} \varphi(z).$$

(3.26)

When $N$ is odd, (3.20) in fact reproduce the correct lowest-Landau-level factor of Laughlin wavefunction \[6\]. (Note that we need to do a singular gauge transformation to go back to the symmetric gauge. This is also neccesary in Landau-Ginzburg type theory deviation \[4\] of Laughlin wavefunction).

The magnetic factor (3.26) also can be interpreted as the neutralizing background field. This interpretation is understood when one see th operator in (3.24) as a charged excitation. Quasiparticles become quasiholes in the presence of the factor (3.26).

Now let us consider in a geometrical viewpoint. We specify U(1) current algebra to be the chiral boson $\varphi$ compactified on the circle of (radius)$^2 = 1/N$. Then the
vertex operators $\phi_p$ are

$$
\phi_p(z) = e^{ik_p \varphi(z)},
$$

(3.27)

where $k_p = p/\sqrt{N}, p = 0, 1 \cdots N$ and $\phi_N$ is the extended operator. These vertex operators can be interpreted to represent the rotation of charged particles (or the flux of magnetically charged quasiparticle) in a uniform magnetic field. In other word by realizing the U(1) symmetry by U(1) current algebra, the rotation of charged particles are realized as an internal structure i.e. as a fiber (in this case, circle). In this way, quasiparticles have a internal space which is suppressed to be a flux in the long-wave length limit. And the area enclosed by the circle is $f = 1/N$ times the area of the unit circle. The boson compactified on the unit circle is equivalent two Majorana fermion i.e. Dirac fermion. It might be possible that one can take it as a usual electron (or hole) in the integer quantum hall effect of lowest Landau level. If it is so, one can take $f$ as the filling fraction itself, since the ratio of the area of circles equals to the ratio of magnetic flux. In fact the extended operator

$$
\phi_N = \exp i\sqrt{N} \varphi = \exp \frac{i\varphi}{\sqrt{f}}
$$

(3.28)

is equivalent to a vertex operator of the boson compactified on the unit circle. If $N$ is odd, the statistics for $\phi_N$ is fermion. Also this operator has the unit charge. It is natural that the electron (charge one fermion) is represented by this extended operator in U(1) current algebra at least when $N$ is odd. Then the ratio of enclosed circles $f$ equals to the filling fraction $\nu$. The duality of c=1 CFT corresponds to the duality of the coupling constant (or filling fraction $\nu$) around $\nu = \frac{1}{2}$ of abelian Chern-Simons theory description of FQHE. Among the quasiparticles, the lightest one is

$$
\phi_1 = \exp \frac{i}{\sqrt{N}} \varphi = \exp i\sqrt{f} \varphi.
$$

(3.29)

This is the quasihole operator of $[6]$. This quasiparticle governs the long-wave length physics.

This construction is readily generalized to the multiple rational torus compactified on n-dimensional lattice and reproduces the results of the hierarchical construction $[3] [4]$. The physical condition must be imposed to ensure the net symmetry of the system is U(1) in those cases.

We now generalize our argument to the case in which RCFT has other kind of degree of freedom in addition to the extended U(1) current algebra. We don’t assume this additional degree of freedom is local. For example nonlocal pairing
actually occurs in the case of Ising model. We assume these additional degree of freedom to be some representation $\mathcal{R}$ of some group $G$. $G$ may be $SU(N)$ when the additional degree of freedom is isospin. As before U(1) part can be realized by the chiral boson $\varphi$. The U(1) current becomes

$$J_z = i(\text{dim}\mathcal{R})\sqrt{f} \partial_z \varphi. \quad (3.30)$$

The field which has the unit charge have the part from the rational torus as

$$\exp \left(\frac{i\sqrt{N} \varphi}{\text{dim}\mathcal{R}}\right). \quad (3.31)$$

The effect of magnetic field now amount to be

$$\exp \int \frac{d^2 z}{2\pi i} \text{dim}\mathcal{R} \sqrt{f} \varphi(z). \quad (3.32)$$

This operator create the $\text{dim}\mathcal{R}$ times larger magnetic field than (3.26). So in this case the filling fraction $\nu$ is not $f = 1/N$. It now becomes

$$\nu = (\text{dim}\mathcal{R})^2 f. \quad (3.33)$$

This construction is useful when one sees RCFT as a degree of freedom of a system in a background uniform magnetic field. In that case we can couple U(1) extended algebra to the given RCFT to achieve some physical conditions. This formulation not only describes the long-wave length physics, but also microscopic dynamics i.e. the rotation of charged particle. The reason for this feature is that the symmetry restricts the internal structure. This is a common feature of the worlds described by CFT.

4 Quantum Mechanics of Chiral Vertices and Conformal Blocks as 2+1 Dimensional Wavefunctions

First let us consider when a n-point function $\langle \phi^{i_1}(z_1)\phi^{i_2}(z_2)\phi^{i_2}(z_2)\cdots\phi^{i_n}(z_n) \rangle$ on a disk can be seen as the ground-state wavefunction of some 2+1 dimensional system. Without knowing quantum-mechanical hamiltonian, it is an amplitude for that quasiparticles $\phi_i$ to be at $z_1\cdots z_n$. However some conditions must be imposed to be able to see this amplitude as the ground state wavefunction of a possibly real
physical systems. To apply to realistic system, the primary field $\phi$ which appears in the ground state should be boson or fermion with an integer charge. This gives constraints to $\phi$ which appears in a ground-state wavefunction. Also this ensures us the single-valuedness of $\langle \phi(z_1)\phi(z_2)\phi(z_3)\cdots\phi(z_n) \rangle$. Generally a primary field in a given RCFT can be made to have these properties by combining them with a vertex operator of a suitable rational torus if the primary field is a so-called 'simple current' \[8\].

Simple current is a primary field which has a unique fusion rule with any other primary fields and consequently has an abelian braiding matrix. When the system is under a background uniform magnetic field, introducing the factor (3.32) in the previous section reproduces the effect of magnetic field.

We’d like to give a simple hamiltonian on the space of chiral vertices which have thus obtained wavefunction as the ground state. Let $\Omega$ be the space formed by all the canonical expressions of chiral vertices

$$\Phi^{i_1}(z_1)\Phi^{i_2}(z_2)\cdots\Phi^{i_n}(z_n)$$

where we have omitted the internal indices and boundary states. The omitted internal indices are sewed to some other internal indices as in (3.4). Generally, on $\Omega$, we can define $Quantum Mechanics for (chiral) Vertices (QM\nu)$ by a hamiltonian for some function $V(M, \overline{M})$

$$H = \frac{1}{2m} \partial_z \partial_{\overline{z}} + V(M, \overline{M}), \quad (4.1)$$

where $M$ and $\overline{M}$ were defined in (3.13). This hamiltonian have some useful properties. First of all it does not depend on the space. Rather it acts on the vector bundle $\mathcal{V}$ over the moduli space $\overline{M}$ and its action is compatible with the factorization of $\mathcal{V}$. Second useful property of $QM\nu$ is that nonabelian statistics is manifest. It is not explicit in the ordinary expression of conformal blocks. $V(M, \overline{M})$ is separated if the operator algebra can be divided into several disjoint subalgebra. For example, when we couple rational torus part, $V = V_I + V_{torus}$ where $V_I$ act on internal degree of freedom and $V_{torus}$ act on rational torus part. It is also useful to define the $\Omega_g$ which is the space formed by all $\Phi^{i_1}(z_1)\Phi^{i_2}(z_2)\cdots\Phi^{i_n}(z_n)$ which express analytic blocks on genus $g$ Riemann surfaces. $H$ can be restricted on $\Omega_g$. When the background magnetic field, the elements in $\Omega(= \bigoplus_g \Omega_g)$ or $\Omega_g$ should be multiplied by a magnetic factor (3.32) and $\partial_z$ in $H$ should be replaced by $D_z$. Only the chiral half of operator algebra remains in the symmetric gauge.

To see how (4.1) works, let us consider the system on a disk described by rational torus with $N = q$ ($N$ is defined as in sect.3.5). Its primary fields are labeled by $p \in \mathbb{Z}_q$ and we label the extended operator as $q$. The extended operator has the unit charge and the statistics of fermion or boson as $q$ is odd or even. Then we can form
the wavefunction which can be seen as a wavefunction of charged boson or electron system:

\[ \langle e^{i\sqrt{q}\varphi(z_1)} e^{i\sqrt{q}\varphi(z_2)} \cdots e^{i\sqrt{q}\varphi(z_n)} \rangle. \]

(4.2)

This is known to become Laughlin wavefunction for \( \nu = \frac{1}{q} \) when \( q \) is odd with the magnetic factor (3.26) \[6\]. (4.2) can be written in the canonical expression of chiral vertices as

\[ \Psi = \Phi^q_{-nq,-(n-1)q}(z_1) \cdots \Phi^q_{-2q,-q}(z_{n-1}) \Phi^q_{-q,0}(z_n). \]

(4.3)

The eigenvalue of \( M \) for this wavefunction is

\[ E_\Psi = n(1-n)q^2. \]

(4.4)

Then, if we take \( V \) to be

\[ V(M) = E_\Psi - M, \]

(4.5) is the exact zero-energy state of \( H \). (4.5) counts the increase of the total multiplicity of zeros of wavefunctions. The gaps of the excitations above this zero-energy state are

\[ V(M)\Phi^p_{-p-nq,-nq}(z_j)\Psi = np\Phi^p_{-p-nq,-nq}\Psi \]

(4.6)

This value does not depend on where we put \( \Phi^p \) in the conformal block. For \( \Phi^p \) which can be multiplied at some boundary state, the excited energy can be obtained by acting \( V(M) \) on \( \Phi^p \) alone. Now it is clear that \( \phi_1 \) is the lightest quasiparticle in this system. Also, one can take similar potentials such as following one:

\[ V(M) = (E_\Psi - M)^2. \]

(4.7)

Generally, the operator \((-M)\) gives the total multiplicities of zero or total angular momentum when it acts on a canonical expression on disk.

Next, let us consider \( QMV \) in more generalities. Let \( \Sigma \) be a genus \( g \) Riemann surface with \( m \) boundaries. General amplitude on \( \Sigma \) can be written in terms of the element in \( \Omega_g \):

\[ \Psi = \Phi^{i_1}_{b_1 a_1} \cdots \Lambda^{d_{j_1} - 1}_{d_{j_2}} \cdots \Lambda^{c_{k-1}}_{c_{k-2} c_{k}} \cdots \Phi^{i_{j_1}}_{a_{2j_1} - 2 a_{2j_1} - 1} \cdots \Phi^{j_{n}}_{a_{2n} - 1}. \]

(4.8)
Let us consider the following potentials:

\[ M_a = \frac{1}{2}(M + \overline{M}), \]
\[ M_b = \frac{1}{2}(\overline{M} - M). \] (4.9) (4.10)

From (3.15)(3.16),

\[ M_a \Psi = \left( \sum_k \Delta_{i_k} \right) \Psi, \] (4.11)
\[ M_b \Psi = \left( \sum_m \Delta_{b_m} \right) \Psi. \] (4.12)

So \( M_a \) only sees the excitations at the bulk. This operator plays the role of mass operator in general QMV. It reproduce the description of ‘mass’ in sect.3.1 at quantum mechanical level (in the case of closed Riemann surface, \( M \) or \( \overline{M} \) is enough). On the other hand, \( M_b \) only sees the states at the boundaries. The boundary states are determined by the fusion rules in the canonical expression of the amplitude. Every potentials \( V(M, \overline{M}) \) can be rewritten in terms of \( M_a, M_b \). This means one can always separate these two contributions. Also we can see how the bulk states and the boundary states correlates each other in the expansion of \( V(M, \overline{M}) \) in terms of \( M_a \) and \( M_b \).

In QMV on a general Riemann surface \( \Sigma \), fusion rules are considered to be a kind of conservation laws. This analogy arises since, in the canonical expression of wavefunctions through chiral vertices, the expression depends on ‘flow’ or Morse function on \( \Sigma \) and the fusion rules controls the joint and split of the ‘flow’ on \( \Sigma \). In ordinary mechanics, the conservation laws are determined by the symmetry group it has. The conserved currents form a representation of its Lie algebra. As explained in [25], RCFT can be seen as a generalization of group theory. QMV has the generalization of group theory as underlying conservation law. Also, as in sect.3.2, fusion rules and mass (conformal weight), i.e. operator algebra determines the statistics of quasiparticle of the system.

Degeneracy of a given wavefunction is computed from the expression through chiral vertices by the fusion rules. For example, the degeneracy of the wavefunction for Laughlin wavefunction for filling factor \( \nu = 1/q \) is shown to be \( q^g \) on genus \( g \) Riemann surface.
5  Some examples

We would like to take some specific examples in this section. For application to real physics, we only consider the system in a background uniform magnetic field. We will consider $SU(N)_k$ WZW model, and Ising model. $SU(2)_1$ model gives the Halperin state of FQHE [23] and Ising model gives so-called Pfaffian state [3].

5.1 $SU(N)_k$

$SU(N)_k$ WZW models with rational torus were already discussed as 2+1 dimensional many-body system in [3]. These model have excitations with nonabelian statistics for $k > 1$. However we mainly consider $SU(N)_1$ WZW model in this section. This model gives us a generalization of Halperin state. The relation between composite fermion and $SU(N)_1$ model is discussed in [18]. Our discussion is simple applications of discussions in sect.3.5 and sect.4.

First let us recall $SU(2)_k$ WZW models. $SU(2)_k$ has $k+1$ representations $[\phi_l]$, namely the ones with $SU(2)$ isospin $\frac{1}{2}l \leq \frac{1}{2}k$. The conformal weights for $\phi_l$ are

$$h_l = \frac{l(l + 2)}{4(k + 2)} .$$

The fusion rules are

$$\phi_l \times \phi_{\ell} = \sum_{j=|l-\ell|}^{\min(l+\ell,2k-l-\ell)} \phi_j ,$$

where $j - |l - \ell|$ is an even integer. From this fusion rules, when $k > 1$, we have excitations with nonabelian statistics when this RCFT becomes the internal degree of freedom of 2+1 dimensional system by coupling a suitable rational torus. The modular behavior of the characters $\chi_l$ are

$$S_{ln} = \left( \frac{2}{k+2} \right)^{1/2} \sin \left( \frac{(l+1)(n+1)}{k+2} \pi \right) .$$

The matrices $F$ and $R$ are known for these models. See, for example [24] for the formulas of these matrices.

Let us now consider $SU(2)_1$. $SU(2)_1$ WZW model has two primary fields ($k = 1$)

$$V^\downarrow(z_\downarrow), V^\uparrow(z_\uparrow) = J_0^+ V^\downarrow(z_\uparrow) ,$$

(5.4)
where $J^+$ are the creation operator in the standard basis of $su(2)$. The conformal weights for these vertex operators are respectively

$$h_l = \frac{l(l + 2)}{4(k + 2)}, \quad (5.5)$$

The fact that $SU(2)$ freedom is nothing but spin degree of freedom suggests its relevance to electronic system. When the system is in a background uniform magnetic field, rational torus with $N = q$ ($N$ defined in sect.3.5) appeared as explained in sect.3.5. From (3.31), the primary fields with the unit charge are

$$V^{\uparrow}(z) e^{i\frac{\sqrt{q}}{2}\varphi(z)}. \quad (5.6)$$

They are in fact fermion when $q = 4m + 2$ when $m$ is even and boson when $m$ is odd.

Thus this field have the same quantum numbers with electron when $q = 4m + 2$ with even $m$. We can construct the $SU(2)$ singlet ground state from these fields;

$$\langle V^{\uparrow}(z_1) e^{i\frac{\sqrt{q}}{2}\varphi(z_1)} \cdots V^{\uparrow}(z_n) e^{i\frac{\sqrt{q}}{2}\varphi(z_n)} V^{\downarrow}(z_1) e^{i\frac{\sqrt{q}}{2}\varphi(z_1)} \cdots \rangle \quad \cdots \quad (5.7)$$

where the factor 2 in the magnetic factor comes from the internal degree of freedom as in (3.32). This state is manifestly $SU(2)$ invariant. Actually this is the spin-singlet state, so-called Halperin state [23] [6].

$$\prod_{i<j}(z_i^\downarrow - z_j^\uparrow)^\frac{q-2}{4} \prod_{i<j}(z_i^\uparrow - z_j^\downarrow)^\frac{q}{4} \times \prod_{i<j}(z_i^\downarrow - z_j^\uparrow)^\frac{q-2}{4} \exp \left[ -\frac{1}{4} \sum_i (|z_i^\uparrow|^2 + |z_i^\downarrow|^2) \right] \quad (5.8)$$

Its filling fraction is $\nu = \frac{4}{q}$ from (3.33). The lightest excitation of above the state (5.7) is given by

$$V^{\downarrow\uparrow}(z) e^{i\frac{\sqrt{q}}{\sqrt{q}} \varphi(z)}. \quad (5.9)$$

These excitations governs the low-energy physics of the system.

Next let us consider the a system with internal $SU(N)$ symmetry. The excitations are in the $N$-dimensional representation of $SU(N)$. The system are realized as $SU(N)_1$ WZW model. The conformal weight for the primary fields is

$$\frac{N^2 - 1}{2N(N + k)}. \quad (5.10)$$
where \( k = 1 \). In this case, by coupling the rational torus with

\[
q = N^2 m + \frac{N(N^2 - 1)}{(N + k)},
\]

the system has a boson or fermion with the unit charge as \( m \) is even or odd respectively. By using antisymmetric tensor, we can again form the \( SU(N) \) invariant ground state wavefunction. From \( (3.32) \), the magnetic factor is

\[
\exp \int \frac{d^2 z}{2\pi i} N \frac{\varphi(z)}{\sqrt{q}}.
\]

The state has filling fraction

\[
\nu = \frac{N}{Nm + \frac{N^2 - 1}{(N+k)}}
\]

from \( (3.33) \).

### 5.2 Ising Model

We’d like to cast Ising model in the present framework. Ising model has three primary fields \( 1, \psi, \sigma \). Here \( \psi \) is a Majorana fermion and \( \sigma \) is the spin field. Their conformal weights are respectively \( \Delta_\psi = \frac{1}{2}, \Delta_\sigma = \frac{1}{16} \). The fusion rules of Ising model are

\[
\psi \times \psi = 1, \psi \times \sigma = \sigma, \sigma \times \sigma = 1 + \psi
\]

Under a suitable normalization, we get the fusion matrices \( F \) as follows:

\[
F \left[ \begin{array}{cc}
\psi & \psi \\
\psi & \psi
\end{array} \right] = 1, F \left[ \begin{array}{cc}
\sigma & \psi \\
\psi & \sigma
\end{array} \right] = -1,
\]

\[
F \left[ \begin{array}{cc}
\sigma & \sigma \\
\sigma & \sigma
\end{array} \right] = \frac{1}{\sqrt{2}} \left( \begin{array}{cc}
1 & 1 \\
1 & -1
\end{array} \right)
\]

From these formulae we get the exchange matrix \( R \),

\[
R \left[ \begin{array}{cc}
\sigma & \sigma \\
\sigma & \sigma
\end{array} \right] = \frac{e^{i\pi/8}}{\sqrt{2}} \left( \begin{array}{cc}
1 & -i \\
-i & 1
\end{array} \right)
\]
Let us make a boson or fermion with the unit charge from $\psi$ by coupling a rational torus with suitable $N = q$. From the fusion rules, $\psi$ can appear only when they are paired. Therefore internal degree of freedom is $Z_2$, and a pair of $\psi$ forms a representation with $\dim \mathcal{R} = 2$. The filling fraction is $\nu = \frac{1}{q}$. From (3.31) we can form a boson or fermion with the unit charge as

$$\psi e^{\i \sqrt{q} \phi}$$

It is a charge one fermion when $q = 8k$ and a charge one boson when $q = 8k + 4$. From this field we can form a ground-state wavefunction for a certain 2+1 dimensional quantum mechanics as explained in sect.4:

$$\langle \psi(z_1) e^{\i \sqrt{q} \phi(z_1)} \cdots \psi(z_m) e^{\i \sqrt{q} \phi(z_m)} \exp \int \frac{d^2 z}{2\pi i} \frac{2\phi(z)}{\sqrt{q}} \rangle$$

This wavefunction is Pfaffian state first derived in [6] when $q = 8k$. The name 'Pfaffian' comes from the fact that this wavefunction equals to

$$\text{Pfaff}(\frac{1}{z_i - z_j}) \prod_{i<j} (z_i - z_j)^q \exp \left[ -\frac{1}{4} \sum_i |z_i|^2 \right].$$

The lightest excitation in this system is the quasi-hole

$$\sigma e^{\i \sqrt{q} \phi}.$$ 

However this field cannot appear alone on the Pfaffian state. This is clear from the fusion rules. We had better use the chiral vertices expression for further consideration. Pfaffian state can be written in terms of chiral vertices as

$$\Phi^\psi \psi \Phi^\psi \psi \cdots \Phi^\psi \psi \psi$$

where the rational torus part are omitted. The potential in (4.1) which have (5.21) as its exact ground state is, for example,

$$V(M, \overline{M}) = E_{\text{Pfaff}} - M,$$

where $E_{\text{Pfaff}}$ is the eigenvalue of $M$ for Pfaffian state. This potential counts the increase of the total multiplicities of zeros of the wavefunctions. There are two possible expressions composed from two chiral vertices to insert into (5.21) which include the spin field $\sigma$. They create 2 quasihole state. They are

$$\Psi = \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma,$$

$$\Psi' = \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma \Phi^\sigma.$$
From (4.10), these are the lightest excitation of the system. Actually these two expressions create the same state up to phase. This situation is same for general 2n case, so we may only consider the expressions of type $\Phi_{\sigma_1}^\sigma \cdots \Phi_{\sigma_1}^\sigma$. For 4 quasihole state, the expressions are:

$$
\Xi_1 = \Phi_{1\sigma}^\sigma \Phi_{\sigma 1}^\sigma \Phi_{1\sigma}^\sigma \Phi_{\sigma 1}^\sigma
$$

(5.25)

$$
\Xi_2 = \Phi_{1\sigma}^\sigma \Phi_{\sigma \psi}^\sigma \Phi_{\psi \sigma}^\sigma \Phi_{\sigma 1}^\sigma
$$

(5.26)

This degeneracy 2 can easily expected from the fusion rules (5.14). Obviously when we exchange two chiral vertices in the middle, non-abelian statistics appears. $\Xi_1$ and $\Xi_2$ are transformed irreducibly in the exchange. Thus this system has excitations with nonabelian statistics. From the general discussion in sect.3.2 for multiple quasiparticle state, this statistics can be extended to a representation of $SO(4)$. From (5.16) we see that the representation formed by $\Xi_1$ and $\Xi_2$ is 2 dimensional spinor representation of $SO(4) \times U(1)$.

Next let us consider 2n quasihole state. As in $n = 2$ case, the degeneracy of the canonical expression for 2n quasihole state is $2^{n-1}$ from the fusion rules (5.14). Again from the general discussion of sect.3.2, we see that the vector space of these degenerated states forms $2^{n-1}$ dimensional representation of $SO(2n) \times U(1)$. and it is spinor from (5.16).

This result is proved in [9] for the four quasihole state by an explicit calculation of conformal block, with the indication for 2n quasihole state. Our discussion doesn’t depend on the explicit form of conformal blocks. This is natural because statistics is determined only from the operator algebra of RCFT.

Among 2n quasihole states, 8 quasihole state ($n = 4$) forms 8 dimensional spinor representation of $SO(8)$. It is interesting that this representation coincides with the space-time interpretation for the ground state of Ramond sector of superstring. Also the triality relation of $SO(8)$ on this state is interesting. These issues will be explored elsewhere.

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