Convergence to weak solutions of a space-time hybridized discontinuous Galerkin method for the incompressible Navier–Stokes equations

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Abstract
We prove that a space-time hybridized discontinuous Galerkin method for the evolutionary Navier–Stokes equations converges to a weak solution as the time step and mesh size tend to zero. Moreover, we show that this weak solution satisfies the energy inequality. To perform our analysis, we make use of discrete functional analysis tools and a discrete version of the Aubin–Lions–Simon theorem.

Keywords: Navier–Stokes, space-time, hybridized, discontinuous Galerkin, minimal regularity

1. Introduction
In this article, we continue our study of the space-time hybridized discontinuous Galerkin (HDG) method for the evolutionary incompressible Navier–Stokes equations analyzed in [28]. Therein, we proved that the method is pressure-robust (see Section 1.1 below) and derived optimal rates of convergence in space and time for the velocity field which are independent of the pressure. However, as a discontinuous method, the HDG method introduces additional stabilization which is a potential source of consistency error if the exact solution is not sufficiently regular. Consequently, the convergence results deduced from the standard a priori analysis of DG and HDG methods often exclude the case of non-smooth solutions which may be present in physically realistic scenarios.

For this reason, our analysis in [28] considered strong solutions of the Navier–Stokes system, and cannot be used to deduce convergence to weak solutions in the absence of additional regularity. This restricts the applicability of our analysis in two important ways. The first is that in three spatial dimensions, the existence of a strong solution is guaranteed only for sufficiently small problem data (see e.g. [5, 11, 47]). The second is that, for a polygonal ($d = 2$) or polyhedral domain ($d = 3$), the exact solution is known to possess sufficient regularity for consistency only under further geometric assumptions on the domain (e.g. convexity); see [12].

The purpose of this paper is to fill this gap. To circumvent the problems posed by the lack of consistency in our numerical scheme, we instead consider the concept of asymptotic consistency introduced in [14, Section 5.2]. That is, we aim to show the discrete weak formulation resulting from our space-time HDG discretization converges to the exact weak formulation of the Navier–Stokes equations in a suitable sense as the mesh size tends to zero. This is made challenging...
by the discontinuous nature of our numerical method, as standard compactness results like the Rellich–Kondrachov theorem [5, Theorem III.2.34] and the Aubin–Lions–Simon theorem [5, Theorem II.5.16] are lost and appropriate discrete analogues must be derived.

Fortunately, discrete compactness for DG schemes is, at this point, well studied. We mention in particular the works of Buffa and Ortner [7], Di Pietro and Ern [37], and Kikuchi [27], wherein discrete versions of the Rellich–Kondrachov theorem are proven for broken Sobolev and broken polynomial spaces. A common theme among these works is the introduction of a discrete analogue of the gradient operator that incorporates information from the jumps of the discrete solution across its discontinuities. As for a discrete analogue of the Aubin–Lions–Simon theorem in the time-dependent setting, we mention the work of Walkington [50] where it is shown that DG time stepping methods enjoy similar compactness properties to the evolutionary equations they are used to approximate. Unfortunately, the results of [50] are valid only for conforming spatial discretizations. This was remedied in [33], wherein a generalization of the work of Walkington valid for broken Sobolev spaces (and thus, for a broad class of non-conforming discretizations) is obtained.

In this article, we adapt some of the available discrete functional analysis tools [37] to the HDG setting (see also [27] for similar efforts). We also prove a variation of the discrete Aubin–Lions–Simon theorem in [50] valid for our non-conforming discretization. Our result differs slightly from that of [33] in that we stay entirely within the framework of broken polynomial spaces. In an effort to unify the available discrete functional analysis tools for spatial DG discretizations and DG time stepping, we introduce a discrete time derivative operator in analogy with the aforementioned discrete gradient operator using the time lifting operator in [43], and we show that some of the assumptions required in [33, 50] for compactness can be interpreted using this discrete time derivative. The main contributions of this work can be summarized as follows:

(i) We prove that the discrete solution of space-time HDG scheme analyzed in [28] for strong solutions converges to a Leray–Hopf weak solution of the evolutionary Navier–Stokes equations. To our knowledge, this is one of the few minimal regularity convergence results available for HDG discretizations in general, and it is the first for a space-time HDG discretization. As a byproduct, we obtain a new proof of the existence of weak solutions of the Navier–Stokes equations.

(ii) We introduce the notion of a discrete time derivative and show that it serves as an appropriate approximation of the distributional time derivative.

(iii) We obtain a minor variation of the discrete Aubin–Lions–Simon compactness theorem proven in [50, Theorem 3.1] valid for broken polynomial spaces, as a special case of [33, Theorem 3.2], using the discrete functional analysis tools of DiPietro and Ern [37]. Moreover, we show that uniform bounds on the discrete time derivative in a discrete dual space suffice for compactness for DG time stepping schemes, by analogy to the corresponding continuous theorem.

The article is organized as follows: in the remainder of Section 1, we give a brief overview of the relevant literature and introduce some of the notation that will be used throughout. In Section 2, we introduce the space-time HDG method under consideration and recall some of the key results obtained in [28]. In Section 3, we introduce discrete analogues of the gradient operator and time derivative, and recast the numerical scheme in terms of these discrete operators. In Section 4, we
prove that these discrete operators are bounded uniformly with respect to the mesh size and time step, and as a consequence we obtain convergence of the sequence of discrete velocity solutions as the mesh size and time step tend to zero. In Section 5, we show that the limit of this sequence of discrete solutions is a weak solution to the Navier–Stokes equations.

1.1. Related results

A shortfall of many popular discretizations for incompressible flows is that they fail to preserve a fundamental invariance property enjoyed at the continuous level: perturbations in the external forcing term by a gradient field should not influence the velocity field [34]. Violation of this invariance property at the discrete level can lead to a coupling of the approximation errors in the velocity and pressure and introduces an unfavourable scaling in the velocity error with respect to inverse powers of the kinematic viscosity. Discretizations that do preserve this invariance property have been coined pressure-robust, and have recently been the subject of intensive study [20, 26, 30, 31, 32, 34, 44].

The two requisites to ensure that a discretization is pressure-robust are exact satisfaction of the divergence constraint and $H(\text{div}; \Omega)$-conformity of the velocity field. While the latter is automatically satisfied for classical conforming discretizations like the Taylor–Hood element [23], the former is not. In fact, conforming discretizations with exact mass conservation are scarce; examples include the Scott–Vogelius element [45] and the more recent Guzmán–Neilan element [21, 22]. Instead, one can use $H(\text{div}; \Omega)$-conforming DG methods such as the Raviart–Thomas (RT) element or the Brezzi–Douglas–Marini (BDM) element [4] to obtain pressure-robust discretizations. To our knowledge, the use of $H(\text{div}; \Omega)$-conforming DG methods for incompressible flows was first explored in [10].

In [41], a simple class of pressure-robust HDG methods is introduced. By leveraging hybridization, an $H(\text{div}; \Omega)$-conforming discrete velocity field can be obtained while working with completely discontinuous finite element spaces. In [24, 25], a space-time HDG method that remains pressure-robust on moving and deforming domains is introduced based on this discretization; however, a rigorous analysis of this method is lacking. Our work in [28] is a first step in this direction. We mention also the papers [1, 9, 35, 38, 39] wherein alternative space-time finite element discretizations of the Stokes and Navier–Stokes problem are considered.

We remark that our analysis of convergence to weak solutions is greatly facilitated by the $H(\text{div}; \Omega)$-conformity of the discrete velocity as well as the fact that it is point-wise solenoidal. This ensures that, though our finite element spaces are completely discontinuous, the discrete velocity solution belongs to the continuous space $H$ defined below. This is particularly important when we seek to extract a convergent subsequence of approximate velocities in Section 4, as no additional effort will be required to show that accumulation points satisfy the incompressibility constraint. Moreover, this guarantees that the discrete time derivative defined in Section 3 actually belongs to the same continuous dual space as the distributional time derivative of the exact solution.

1.2. Notation

We use standard notation for Lebesgue and Sobolev spaces: given a bounded measurable set $D$, we denote by $L^p(D)$ the space of $p$-integrable functions. When $p = 2$, we denote the $L^2(D)$ inner product by $(\cdot, \cdot)_D$. We denote by $W^{k,p}(D)$ the Sobolev space of $p$-integrable functions whose distributional derivatives up to order $k$ are $p$-integrable. When $p = 2$, we write $W^{k,p}(D) = H^k(D)$. We define $H^1_0(D)$ to be the subspace of $H^1(D)$ of functions with vanishing trace on the boundary of $D$. We denote the space of polynomials of degree $k \geq 0$ on $D$ by $P_k(D)$. We use standard
notation for spaces of vector valued functions with $d$ components, e.g. $L^2(D)^d$, $H^k(D)^d$, $P^k(D)^d$, etc. At times we drop the superscript for convenience, e.g. we denote by $\|\cdot\|_{L^2(\Omega)}$ the norm on both $L^2(\Omega)$ and $L^2(\Omega)^d$.

Next, let $U$ be a Banach space, $I = [a, b]$ an interval in $\mathbb{R}$, and $1 \leq p < \infty$. We denote by $L^p(I; U)$ the Bochner space of $p$-integrable functions defined on $I$ taking values in $U$. When $p = \infty$, we denote by $L^\infty(I; U)$ the Bochner space of essentially bounded functions taking values in $U$ and by $C(I; U)$ the space of (time) continuous functions taking values in $U$. Finally, we let $P_k(I; U)$ denote the space of polynomials of degree $k \geq 0$ in time taking values in $U$.

2. Preliminaries

In this section, we discuss the weak formulation for the continuous Navier–Stokes problem eq. (1), introduce the space-time HDG method that we will use to approximate solutions of eq. (1), and collect a number of useful results for our analysis.

2.1. The continuous problem

Given a suitably chosen body force $f$, kinematic viscosity $\nu \in \mathbb{R}^+$, and initial data $u_0$, we consider the transient Navier–Stokes system posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$:

\[\begin{align*}
\partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p &= f, & \text{in } \Omega \times (0, T], \quad (1a) \\
\nabla \cdot u &= 0, & \text{in } \Omega \times (0, T], \quad (1b) \\
u \cdot u &= 0, & \text{on } \partial \Omega \times (0, T], \quad (1c) \\
\quad u(x, 0) &= u_0(x), & \text{in } \Omega. \quad (1d)
\end{align*}\]

To avoid any complications arising from curved boundaries, we will assume further that in two spatial dimensions $\Omega$ is a polygon and in three spatial dimensions $\Omega$ is a polyhedron. As we are interested in weak solutions, we require no assumption that $\Omega$ is convex.

We begin our discussion of the Navier–Stokes system with the theory of weak solutions of Leray–Hopf type [5, 47]. The starting point is $\mathcal{V} = \left\{ u \in C^\infty_c(\Omega)^d \mid \nabla \cdot u = 0 \right\}$, the space of compactly supported solenoidal smooth vector fields. We define two function spaces, $H$ and $V$, as the closures of $\mathcal{V}$ in the norm topologies of $L^2(\Omega)$ and $H^1_0(\Omega)$, respectively. For an open, bounded Lipschitz set $\Omega$, we have the following characterizations of $H$ and $V$ [47, Theorems I.1.4 and I.1.6]:

\[H = \left\{ u \in L^2(\Omega)^d \mid \nabla \cdot u = 0 \text{ and } u \cdot n|_{\partial \Omega} = 0 \right\}, \quad V = \left\{ u \in H^1_0(\Omega)^d \mid \nabla \cdot u = 0 \right\}. \quad (2)\]

Note that $H \subset H(\text{div}; \Omega) : = \left\{ v \in L^2(\Omega)^d \mid \nabla \cdot u \in L^2(\Omega) \right\}$. We equip $H$ and $V$ with the standard norms on $L^2(\Omega)^d$ and $H^1_0(\Omega)^d$, respectively.

Definition 2.1 (Weak solution). Given a body force $f \in L^2(0, T; H^{-1}(\Omega)^d)$ and an initial condition $u_0 \in H$, a function $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ with $\frac{du}{dt} \in L^1(0, T; V')$ is said to be a weak solution of the Navier–Stokes equations eq. (1) provided it satisfies for all $\varphi \in C^\infty_c(0, T; V)$,

\[
\int_0^T \left\langle \frac{du}{dt}, \varphi \right\rangle_{V' \times V} \, dt + \int_0^T \langle u \cdot \nabla \rangle u, \varphi \rangle \, dt + \nu \int_0^T (\nabla u, \nabla \varphi) \, dt = \int_0^T \langle f, \varphi \rangle_{H^{-1} \times H_0^1} \, dt, \quad (3)
\]

and $u(0) = u_0$ in $V'$ (see e.g. [5, Section V.1.2.2]).
It is well known that weak solutions in the sense of Definition 2.1 are weakly continuous from \([0, T]\) into \(H\), and their distributional time derivative possess the further regularity \(\frac{df}{dt} \in L^{d/4}(0, T; V')\) (see e.g. [5]). If \(d = 2\), this solution is unique and furthermore \(u \in C(0, T; H)\). Uniqueness in three dimensions remains an open problem.

Remark 2.1 (On the regularity of the body force). Our main result (Theorem 5.3) should remain valid for \(f \in L^2(0, T; H^{-1}(\Omega)^d)\) provided there is an appropriate smoothing operator \(E_h : V_h \rightarrow H^1_0(\Omega)^d\); see e.g. [2]. In particular, if \(v_h \rightarrow v\) strongly in \(L^2(\Omega)^d\), we require also that \(E_h v_h \rightarrow v\) strongly in \(L^2(\Omega)^d\) as \(h \rightarrow 0\). For simplicity, we focus on body forces \(f \in L^2(0, T; L^2(\Omega)^d)\).

Remark 2.2 (The energy inequality). In two dimensions, the weak solution to the Navier–Stokes equations satisfies the following energy equality: for all \(s \in (0, T)\),

\[
\|u(s)\|^2_{L^2(\Omega)} + 2\nu \int_0^s \|u\|^2_V \, dt = \|u_0\|^2_{L^2(\Omega)} + 2 \int_0^s \langle f, u \rangle_{H^{-1} \times H^1_0} \, dt. \tag{4}
\]

In three dimensions, we say that a weak solution is of Leray–Hopf type if it satisfies the energy inequality: for a.e. \(s \in (0, T)\),

\[
\|u(s)\|^2_{L^2(\Omega)} + 2\nu \int_0^s \|u\|^2_V \, dt \leq \|u_0\|^2_{L^2(\Omega)} + 2 \int_0^s \langle f, u \rangle_{H^{-1} \times H^1_0} \, dt. \tag{5}
\]

2.2. Space-time setting and finite element spaces

In this subsection, we will introduce the space-time slabs, elements, faces, and finite element spaces required for the space-time HDG discretization. We follow some of the definitions introduced in [13]. We define a simplicial mesh of \(\Omega\) to be a couple \((T_h, F_h)\) where the set of mesh elements \(T_h\) is a finite collection of nonempty, disjoint simplices \(K\) with boundary \(\partial K\) and diameter \(h_K\) such that \(\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}\). We define the mesh size \(h\) of \(T_h\) to be \(h = \max_{K \in T_h} h_K\).

The set of mesh faces \(F_h\) is a finite collection of nonempty, disjoint subsets of \(\overline{\Omega}\) such that, for any \(F \in F_h\), \(F\) is a non-empty, connected subset of a hyperplane in \(\mathbb{R}^d\). We assume further that for each \(F \in F_h\), either there exist distinct mesh elements \(K_1, K_2 \in T_h\) such that \(F = \partial K_1 \cap \partial K_2\), in which case we call \(F\) an interior face, or there exists one mesh element \(K \in T_h\) such that \(F = \partial K \cap \partial \Omega\) and we call \(F\) a boundary face. Moreover, we assume that the set of mesh faces forms a partition of the mesh skeleton; that is, \(\partial T_h = \bigcup_{K \in T_h} \partial K = \bigcup_{F \in F_h} F\). We collect interior faces in the set \(F_h^i\) and boundary faces in the set \(F_h^b\). Note that \(F_h = F_h^i \cup F_h^b\).

Let \(k_s \geq 1\) be a fixed integer. We introduce a pair of discontinuous finite element spaces on \(T_h\):

\[
V_h := \{ v_h \in L^2(\Omega)^d \mid v_h \in P_{k_s}(K)^d \ \forall K \in T_h \}, \tag{6a}
\]

\[
Q_h := \{ q_h \in L^2_0(\Omega) \mid q_h \in P_{k_s-1}(K) \ \forall K \in T_h \}, \tag{6b}
\]

and on \(\partial T_h\), we introduce a pair of discontinuous facet finite element spaces:

\[
\bar{V}_h := \{ \bar{v}_h \in L^2(\partial T_h) \mid \bar{v}_h \in P_{k_s}(F)^d \ \forall F \in F_h, \ \bar{v}_h|_{\partial \Omega} = 0 \},
\]

\[
\bar{Q}_h := \{ \bar{q}_h \in L^2(\partial T_h) \mid \bar{q}_h \in P_{k_s}(F) \ \forall F \in F_h \}.
\]

Next, we partition the time interval \((0, T)\) into a series of \(N + 1\) time-levels \(0 = t_0 < t_1 < \cdots < t_N = T\) of length \(\Delta t_n = t_{n+1} - t_n\), and we define \(\tau = \max_{0 \leq n \leq N-1} \Delta t_n\). We assume this time
partition is quasi-uniform, i.e. there exists a \( C_U > 0 \) such that \( \tau \leq C_U \Delta t_n \) for all \( n = 0, \ldots, N - 1 \). A space-time slab is then defined as \( \mathcal{E}^n = \Omega \times I_n \), with \( I_n = (t_n, t_{n+1}) \). We tessellate the space-time slab \( \mathcal{E}^n \) with space-time prisms \( \mathcal{K} = K \times I_n \). We denote this tessellation by \( \Sigma^h \) and note that \( \mathcal{E}^n = \bigcup_{\mathcal{K} \in \Sigma^h} \mathcal{K} \). Combining each space-time slab \( n = 0, \ldots, N - 1 \), we obtain a tessellation of the space-time domain \( \mathcal{T}^h = \bigcup_{n=0}^{N-1} \mathcal{T}^h_n \).

We next discuss space-time faces. Consider a single space-time slab \( \mathcal{E}^n \). We define a space-time face \( \mathcal{F} = F \times I_n \) to be an interior face if \( F \in \mathcal{F}^i_h \), and a boundary face if \( F \in \mathcal{F}^b_h \). We collect the set of space-time interior faces in the set \( \mathcal{F}^i_h \) and the set of space-time boundary faces in the set \( \mathcal{F}^b_h \).

We collect all space-time faces in the space-time slab \( \mathcal{E}^n \) in the set \( \mathcal{F}^n_h \). Combining each space-time slab \( n = 0, \ldots, N - 1 \), we obtain the set of all space-time interior faces \( \mathcal{F}^i_h = \bigcup_{n=0}^{N-1} \mathcal{F}^i_{h,n} \), the set of all space-time boundary faces \( \mathcal{F}^b_h = \bigcup_{n=0}^{N-1} \mathcal{F}^b_{h,n} \), and the set of all space-time faces \( \mathcal{F}^h_h = \mathcal{F}^i_h \cup \mathcal{F}^b_h \).

We assume that the set of mesh faces forms a partition of the space-time mesh skeleton; that is, \( \partial \mathcal{T}^h = \bigcup_{n=0}^{N-1} \partial K \times I_n = \bigcup_{\mathcal{F} \in \mathcal{F}^n_h} \mathcal{F} \).

We define a space-time prismatic mesh of \( \Omega \times (0, T) \) to be the couple \((\mathcal{T}_h, \mathcal{F}_h)\). We will say that a space-time prismatic mesh \((\mathcal{T}_h, \mathcal{F}_h)\) is shape-regular if the underlying spatial mesh \((T_h, F_h)\) is shape-regular; i.e. there exists constants \( C_R \) such that \( h_K/\rho_K \leq C_R \) for all \( K \in T_h \), where \( \rho_K \) is the radius of the largest ball \((d = 2)\) or sphere \((d = 3)\) inscribed in \( K \). We will say that \((\mathcal{T}_h, \mathcal{F}_h)\) is quasi-uniform if both the underlying spatial mesh \((T_h, F_h)\) is quasi-uniform, i.e. there exists \( C_U \) such that \( h \leq C_U h_K \) for all \( K \in T_h \) and the time partition is quasi-uniform. We will say that \((\mathcal{T}_h, \mathcal{F}_h)\) is conforming if the underlying spatial mesh \((T_h, F_h)\) is conforming: given two elements \( K_1, K_2 \in T_h \), either \( K_1 \cap K_2 = \emptyset \) or \( K_1 \cap K_2 \) is a common vertex \((d = 2)\) or edge \((d = 3)\), or a common face of \( K_1 \) and \( K_2 \). Finally, we will assume that every face \( F \) in the underlying spatial mesh \((T_h, F_h)\) satisfies an equivalence condition: that is, given \( h_F = \text{diam}(F) \), there exist constants \( C_c, C_e \) such that \( C_c h_K \leq h_F \leq C_e h_K \) for all \( K \in T_h \) and for all \( F \in F_h \) where \( F \subset \partial K \).

We now complete our definition of the space-time finite element spaces associated with the space-time prismatic mesh \((\mathcal{T}_h, \mathcal{F}_h)\) of \( \Omega \times (0, T) \). Let \( k_t \geq 0 \) be a fixed integer (not necessarily chosen to be equal to \( k_s \)). We consider the following tensor-product space-time finite element spaces in which we will seek our approximation on each space-time slab \( \mathcal{E}^n \):

\[
\mathcal{V}_h := \{ v_h \in L^2(0, T; L^2(\Omega)^d) \mid v_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \mathcal{V}_h) \},
\]

\[
\mathcal{Q}_h := \{ q_h \in L^2(0, T; L^0(\Omega)) \mid q_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \mathcal{Q}_h) \},
\]

\[
\mathcal{V}_b := \{ \bar{v}_h \in L^2(0, T; L^2(\partial T_h)) \mid \bar{v}_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \mathcal{V}_b) \},
\]

\[
\mathcal{Q}_b := \{ \bar{q}_h \in L^2(0, T; L^2(\partial T_h)) \mid \bar{q}_h|_{\mathcal{E}^n} \in P_{k_t}(I_n; \mathcal{Q}_b) \}.
\]

As the spaces \( \mathcal{V}_h \) and \( \mathcal{Q}_h \) are non-conforming, we make use of broken differential operators. For \( v_h \in \mathcal{V}_h \), we introduce the broken gradient operator \( \nabla_h v_h \) by the restriction \( (\nabla_h v_h)|_K = \nabla(v_h|_K) \) and the broken time derivative \( \partial_t v_h \) by the restriction \( (\partial_t v_h)|_{I_n} = \partial_t(v_h|_{I_n}) \). Moreover, the trace of a function \( v_h \in \mathcal{V}_h \) may be double-valued on space-time interior faces \( \mathcal{F} \in \mathcal{F}^i_h \) as well as across two space-time slabs \( \mathcal{E}^n \) and \( \mathcal{E}^{n+1} \). For fixed \( n \), on an interior face \( \mathcal{F} \in \mathcal{F}^i_{h,n} \) shared by two space-time elements \( \mathcal{K}^L \) and \( \mathcal{K}^R \), we denote the traces of \( v_h \in \mathcal{V}_h \) on \( \mathcal{F} \) by \( v_h^L = \text{trace of } v_h|_{\mathcal{K}^L} \) on \( \mathcal{F} \) and \( v_h^R = \text{trace of } v_h|_{\mathcal{K}^R} \) on \( \mathcal{F} \). We denote by \( u_h^{n\pm} \) the traces at time level \( t_n \) from above and below, i.e. \( u_h^{n\pm} = \lim_{\epsilon \to 0} u_h(t_n \pm \epsilon) \).

We introduce the jump [] and average {} of \( v_h \in \mathcal{V}_h \) across a space-time interior face \( \mathcal{F} \) component-wise: let \( [v_{h,i}] = v_{h,i}^L - v_{h,i}^R \) and \( \{v_{h,i}\} = (v_{h,i}^L + v_{h,i}^R)/2 \) with \( v_{h,i} \) denoting the \( i \)th
Cartesian component of $v_h$. The quantities $\|v_h\|$ and $\{v_h\}$ are then the vectors with $i$th Cartesian component $[v_{h,i}]$ and $\{v_{h,i}\}$, respectively. On space-time boundary faces $\mathcal{F} \in \mathcal{S}_h$, we set $[v_h] = \{v_h\} =$ trace of $v_h|_{\mathcal{X}}$ on $\mathcal{F}$, where $\mathcal{X}$ is the element such that $\mathcal{F} \subset \partial \mathcal{X} \cap [\partial \Omega \times (0,T)]$. Lastly, we define the time jump of $v_h \in V_h$ across the space-time slab $\mathcal{E}_n$ by $[v_h]_n = v_{n+}^+ - v_{n-}^-$. We adopt the following notation for various product spaces of interest in this work: $V_h = V_h \times \bar{V}_h$, $Q_h = Q_h \times \bar{Q}_h$, $V_h = V_h \times \bar{V}_h$, and $Q_h = Q_h \times \bar{Q}_h$. Pairs in these product spaces will be denoted using boldface; for example, $v_h := (v_h, \bar{v}_h) \in V_h$. Lastly, we introduce two mesh-dependent norms on the spaces $V_h$ and $V_h$, both of which are standard in the study of interior penalty methods:

$$
\begin{align*}
\|v_h\|_{1,h}^2 &:= \sum_{K \in T_h} \|\nabla v_h\|_K^2 + \sum_{F \in \mathcal{F}_h} \frac{1}{h_F} \|[v_h]\|_{L^2(F)}^2, & \forall v_h \in V_h, \\
\|v\|^2_{V_h} &:= \sum_{K \in T_h} \|\nabla v_h\|_K^2 + \sum_{K \in T_h} \frac{1}{h_K} \|v_h - \bar{v}_h\|_{\partial K}^2, & \forall v_h \in V_h.
\end{align*}
$$

Throughout we use the notation $a \lesssim b$ to denote $a \leq Cb$ where $C$ is a constant independent of the mesh parameters $h$ and $\tau$, the viscosity $\nu$, but possibly dependent on the polynomial degrees $k_t$ and $k_s$, the spatial dimension $d$, and the domain $\Omega$.

Thanks to the equivalence condition on faces, we have

$$
\|v_h\|_{1,h} \lesssim \|v_h\|_{V_h}, \quad \forall v_h \in V_h,
$$

and hence we can conclude the following discrete Poincaré inequality holds [14, Corollary 5.4]: for all $v_h \in V_h$,

$$
\|v_h\|_{L^2(\Omega)} \lesssim \|v_h\|_{V_h}.
$$

### 2.3. The space-time HDG method

In this subsection, we describe the fully discrete numerical method under consideration in this paper. We discretize the Navier–Stokes problem eq. (1) using the exactly mass conserving space-time HDG method on space-time prismatic meshes studied in [28]. This method combines the point-wise divergence free and $H(\text{div};\Omega)$-conforming HDG method introduced in [41] and analyzed in [29] with a discontinuous Galerkin time stepping scheme; see also [24, 25] for related discretizations on space-time tetrahedral meshes on time-dependent domains.

Due to the use of discontinuous-in-time finite element spaces, the discrete space-time HDG formulation can be localized to a single space-time slab; see e.g. [48, Chapter 12]. We first consider the discrete formulation on a single space-time slab, and in Section 3.3 we will introduce the equivalent discrete formulation obtained by summing over all space-time slabs to aid us in our analysis. For $n = 0, \ldots, N-1$, the space-time HDG method for the Navier–Stokes problem in each space-time slab $\mathcal{E}_n$ reads: find $(u_h, p_h) \in V_h \times Q_h$ such that for all test functions $(v_h, q_h) \in V_h \times Q_h$:

$$
\begin{align*}
- \int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \, dt + (u_{n+1}^+, v_{n+1}^-)_{\mathcal{T}_h} + \int_{I_n} (\nu a_h(u_h, v_h) + o_h(u_h; u_h, v_h)) \, dt \\
+ \int_{I_n} b_h(p_h, v_h) \, dt - \int_{I_n} b_h(q_h, u_h) \, dt = (u_n^-, v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \, dt,
\end{align*}
$$

(10)
where \((u, v)_T = \sum_{K \in T_h} (u, v)_K\). Once we have solved eq. (10) for \(u_h\) in the space-time slab \(E^n\), the trace \(u_{n+1}^-\) serves as an initial condition when solving eq. (10) on the next space-time slab \(E^{n+1}\).

The process is initiated by choosing \(u_0 = \Pi_h^{\text{div}} u_0\) in the first space-time slab \(E^0\), where \(u_0 \in H\) is the prescribed initial condition to the continuous problem eq. (1), and \(\Pi_h^{\text{div}} : L^2(\Omega) \to V_h^{\text{div}}\) is the \(L^2\)-projection onto the discretely divergence free subspace \(V_h^{\text{div}} \subset V_h\); see eq. (15) below and the discussion following.

The discrete multilinear forms \(a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}\), \(b_h(\cdot, \cdot) : V_h \times Q_h \to \mathbb{R}\), and \(o_h(\cdot, \cdot, \cdot) : V_h \times V_h \times V_h \to \mathbb{R}\) appearing in eq. (10) serve as approximations to the viscous, pressure-velocity coupling, and convection terms, respectively. We define them as in [29, 41]:

\[
a_h(u, v) := \sum_{K \in T_h} \int_K \nabla u : \nabla v \, dx + \frac{\alpha}{h} \int_{\partial K} (u - \bar{u}) \cdot (v - \bar{v}) \, ds \quad (11a)
\]

\[
- \sum_{K \in T_h} \int_K [(u - \bar{u}) \cdot \partial_n v + \partial_n u \cdot (v - \bar{v})] \, ds,
\]

\[
o_h(w; u, v) := - \sum_{K \in T_h} \int_K w \otimes w : \nabla v \, dx + \frac{1}{2} \int_{\partial K} \nabla \cdot \nabla \cdot (u + \bar{u}) \cdot (v - \bar{v}) \, ds \quad (11b)
\]

\[
+ \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} \nabla \cdot \nabla (u - \bar{u}) \cdot (v - \bar{v}) \, ds,
\]

\[
b_h(p, v) := - \sum_{K \in T_h} \int_K p \nabla \cdot v \, dx + \sum_{K \in T_h} \int_{\partial K} v \cdot n \, dp \, ds. \quad (11c)
\]

The parameter \(\alpha > 0\) appearing in the bilinear form \(a_h(\cdot, \cdot)\) is a penalty parameter typical of interior penalty type discretizations. The bilinear form \(a_h(\cdot, \cdot)\) is continuous and for sufficiently large \(\alpha\) enjoys discrete coercivity [40, Lemmas 4.2 and 4.3], i.e. for all \(u_h, v_h \in V_h\),

\[
\|v_h\|_v^2 \lesssim a_h(v_h, v_h) \quad \text{and} \quad |a_h(u_h, v_h)| \lesssim \|u_h\|_v \|v_h\|_v. \quad (12)
\]

The trilinear form \(o_h(\cdot, \cdot, \cdot)\) satisfies [8, Proposition 3.6]

\[
o_h(w_h; v_h, v_h) = \frac{1}{2} \sum_{K \in T_h} \int_{\partial K} |w_h \cdot n||v_h - \bar{v}_h|^2 \, ds \geq 0, \quad w_h \in V_h^{\text{div}}, \forall v_h \in V_h. \quad (13)
\]

The trilinear form also satisfies [28] for all \(u_h, v_h \in V_h\) and \(d \in \{2, 3\},\)

\[
|o_h(u_h; u_h, v_h)| \lesssim \|u_h\|_{L^2(\Omega)}^{1/(d-1)} \|u_h\|_v^{d/2} \|v_h\|_v, \quad \forall u_h, v_h \in V_h. \quad (14)
\]

Frequent use will also be made of functions in the subspace of discretely divergence free velocity fields:

\[
V_h^{\text{div}} := \{ v_h \in V_h : b_h(v_h, q_h) = 0, \forall q_h \in Q_h \},
\]

\[
V_h^{\text{div}} := \{ v_h \in V_h : \int_0^T b_h(v_h, q_h) \, dt = 0, \forall q_h \in Q_h \}. \quad (15)
\]

We note that \(V_h^{\text{div}} \subset H(\text{div}; \Omega)\), and further \(\nabla \cdot v_h = 0\) and \(v_h \cdot n|_{\partial \Omega} = 0\) for all \(v_h \in V_h^{\text{div}}\) (see e.g. [41, Proposition 1]). As \(\Omega\) is assumed to have a Lipschitz boundary, we therefore have \(V_h^{\text{div}} \subset H\). In fact, it can be shown that \(V_h^{\text{div}} = V_h \cap H\).
2.4. Properties of the space-time HDG scheme

Here, we collect a number of useful results concerning the solution of the space-time HDG scheme eq. \( (10) \). The existence of solutions to the nonlinear algebraic system arising from eq. \( (10) \) was shown in [28]. It was also proven in [28] that the discrete velocity \( u_h \) computed using the space-time HDG scheme is \textit{conforming} in \( L^2(0,T;H) \), i.e., if \( u_h \in V_h \) is the element velocity solution of eq. \( (10) \), then \( \nabla \cdot u_h = 0 \), \( u_h|_{\mathcal{E}^n} \in P_k(I_n; H(\text{div}; \Omega)) \), and the normal trace of \( u_h \) vanishes on the spatial boundary \( \partial \Omega \).

Next, we recall an energy estimate that will allow us to conclude that the discrete velocity pair \( u_h \in V_h \) computed using eq. \( (10) \) is bounded uniformly with respect to the mesh parameters \( h \) and \( \tau \):

**Lemma 2.1.** Let \( d \in \{2,3\} \), \( k_s \geq 1 \) and \( k_t \geq 0 \), and suppose that \( u_h \in V_h \) is solution of space-time HDG scheme eq. \( (10) \) for \( n = 0, \ldots, N - 1 \). For all \( 0 \leq m \leq N - 1 \),

\[
\|u_{m+1}^t\|^2_{L^2(\Omega)} + \sum_{n=0}^{m} \|u^n_h\|^2_{L^2(\Omega)} + v \int_{0}^{t_{m+1}} \|u_h\|^2_{L^2(\Omega)} dt \leq C(f,u_0,\nu).
\]  \( (16) \)

Furthermore, if \( k_t \geq 0 \) when \( d = 2 \) and \( k_t \in \{0,1\} \) when \( d = 3 \), it holds that

\[
\|u_h\|^2_{L^\infty(0,T;L^2(\Omega)^d)} \leq C(f,u_0,\nu).
\]  \( (17) \)

Here, \( C(f,u_0,\nu) \) denotes a constant that depends on the data \( f \), \( u_0 \), and \( \nu \).

The bounds in Lemma 2.1 were proven in [28] under the assumption that \( k_t = k_s \geq 1 \) for simplicity of presentation; we remark that the proofs are equally valid for the general case \( k_s \geq 1 \) and \( k_t \geq 0 \). Note that for the lower order schemes \( k_t \in \{0,1\} \), eq. \( (17) \) follows directly from the eq. \( (16) \). This can be seen immediately when considering constant polynomials in time \( (k_t = 0) \). For linear polynomials in time \( (k_t = 1) \), this follows from the bound (see [49, Section 3]):

\[
\|u_h\|^2_{L^\infty(0,T;L^2(\Omega)^d)} \leq \max_{0 \leq m \leq N-1} \|u^m_h\|^2_{L^2(\Omega)} + \max_{0 \leq m \leq N-1} \||u|^m_h\|^2_{L^2(\Omega)}.
\]

3. Lifting operators and discrete differential operators

In this section, we introduce two discrete differential operators that serve as natural approximations to the distributional gradient and distributional time derivative in the space-time HDG setting. These discrete operators enjoy convergence to their continuous counterparts in the weak topologies of appropriate Bochner spaces.

3.1. Discrete gradient

First, we introduce a discrete gradient operator that will serve as an approximation of the distributional gradient operator following ideas in [7, 37, 27]. The basic building block of the discrete gradient operator is the following observation [7]: as functions \( v_h \in V_h \) are discontinuous, their distributional gradient has a contribution from the jumps of \( v_h \) across element interfaces. Therefore, an appropriate approximation of the distributional gradient in the HDG setting must incorporate the contribution from the jumps between the element solution and the facet solution across element boundaries. We do so by constructing an HDG lifting operator following ideas in [27, 36]. For this, we need to introduce the scalar broken polynomial spaces
\( W_h := \{ w_h \in L^2(\Omega) \mid w_h|_K \in P_{k_h}(K) \} \) and \( \tilde{W}_h := \{ w_h \in L^2(\partial \Omega) \mid w_h|_{\partial K} \in P_{k_h}(\partial K) \} \). We first define a local lifting \( R_h^{K} : L^2(\partial K) \rightarrow P_{k_h}(K)^d \) satisfying
\[
\int_K R_h^{K}(\mu) \cdot w_h \, dx = \int_{\partial K} \mu w_h \cdot n \, ds, \quad \forall w_h \in P_{k_h}(K)^d. \tag{18}
\]
We then define the global lifting \( R_h^{K} : L^2(\partial \Omega) \rightarrow V_h \) by the restriction \( R_h^{K}(\mu)|_K = R_h^{0}(\mu|_{\partial K}) \) for all \( K \in T_h \). Note that \( R_h^{K} \) satisfies for all \( w_h \in V_h \),
\[
\sum_{K \in T_h} \int_K R_h^{K}(\mu) \cdot w_h \, dx = \sum_{K \in T_h} \int_{\partial K} \mu w_h \cdot n \, ds, \tag{19}
\]
and it can be shown using the Cauchy–Schwarz inequality and a standard local discrete trace inequality that
\[
\| R_h^{K}(w_h - \bar{w}_h) \|_{L^2(\Omega)}^2 \lesssim \sum_{K \in T_h} \frac{1}{h_K} \| w_h - \bar{w}_h \|_{L^2(\partial K)}^2, \quad \forall w_h \in W_h \times \tilde{W}_h. \tag{20}
\]
Using the global HDG lifting, we introduce the discrete gradient operator \( G_h^{K} : W_h \times \tilde{W}_h \rightarrow V_h \) in the same spirit as in [37, 27]: given \((v, \bar{v}) \in W_h \times \tilde{W}_h\), we set
\[
G_h^{K}(v, \bar{v}) = \nabla_h v - R_h^{K}(v - \bar{v}), \tag{21}
\]
where \( \nabla_h \) is the broken gradient operator. Crucially, this operator satisfies for all \( v_h \in V_h \) and \( w_h \in V_h \) the identity
\[
\int_{\Omega} G_h^{K}(v_h, \bar{v}_h) \cdot w_h \, dx = \int_{\Omega} \nabla_h v_h \cdot w_h \, dx - \sum_{K \in T_h} \int_{\partial K} (v_h - \bar{v}_h) w_h \cdot n \, ds,
\]
where \( v_h \) and \( \bar{v}_h \) denote the ith Cartesian components of \( v_h \) and \( \bar{v}_h \), respectively.

3.2. Discrete time derivative

To define a discrete time derivative operator that serves as an appropriate approximation of the distributional time derivative, we proceed by analogy with the discrete gradient constructed in the previous section. We follow [43] by introducing a local time lifting operator \( R_{loc,n}^{k_i} : V_h \rightarrow P_{k_i}(I_n; V_h) \) satisfying
\[
\int_{I_n} (R_{loc,n}^{k_i}(u_h), v_h)_{\mathcal{T}_h} \, dt = ([u_h]_n, v^+_h)_{\mathcal{T}_h}, \quad \forall v_h \in P_{k_i}(I_n; V_h), \tag{22}
\]
\[
R_{loc,n}^{k_i}(u_h) = \frac{u^n_i - u^-_i}{2} \sum_{m=0}^{k_i} (-1)^m (2m + 1) L_m^n(t), \tag{23}
\]
where the latter representation formula follows from [43, Lemma 6]. Here \( L_m^n(t) \) are mapped Legendre polynomials; see [43, Section 3]. We then define a global time lifting \( R^{k_i} : V_h \rightarrow V_h \) by the restriction \( R^{k_i}|_{I_n} = R_{loc,n}^{k_i} \). This lifting satisfies:
\[
\int_{0}^{T} (R^{k_i}(u_h), v_h)_{\mathcal{T}_h} \, dt = \sum_{n=0}^{N-1} ([u_h]_n, v^+_n)_{\mathcal{T}_h}, \quad \forall v_h \in V_h. \tag{24}
\]
With the global time lifting in hand, we define the discrete time derivative $D_t^{k_t} : \mathcal{V}_h \to \mathcal{V}_h$ of $v_h \in \mathcal{V}_h$ by setting
\[
D_t^{k_t}(v_h) = \partial_t v_h + R^{k_t}(v_h).
\]

**Lemma 3.1.** Suppose that $u_h \in \mathcal{V}_h^{\text{div}}$. Then, it holds that $D_t^{k_t}(u_h)|_{E^n} \in P_{k_t}(I_n; H)$ for all $0 \leq n \leq N - 1$. 

**Proof.** That $D_t^{k_t}(u_h)$ is divergence free and $H(\text{div}; \Omega)$-conforming follows from the fact that the broken time derivative commutes with the divergence operator and the representation formula eq. (23). It remains to show that $D_t^{k_t}(u_h)|_{\partial \Omega} = 0$. Note that $u_h \cdot n|_{\partial \Omega} = 0$ implies $(\partial_t u_h) \cdot n|_{\partial \Omega} = 0$. This can be seen by considering a single space-time slab $E^n$ and expanding $u_h$ in terms of a basis $\{\psi_i\}_{i=0}^m$ of $P_{k_t}(I_n)$ to find $(\partial_t u_h)|_{I_n} = \sum_{i=0}^{k_t} \partial_t \psi_i u_i$, where $u_i \in \mathcal{V}_h$ is such that $u_i \cdot n|_{\partial \Omega} = 0$. Lastly, since $u_h^+ \cdot n|_{\partial \Omega} = u_h^- \cdot n|_{\partial \Omega} = 0$, the representation formula eq. (23) shows that indeed $R^{k_t}_{\text{loc},n}(u_h) \cdot n|_{\partial \Omega} = 0$. \( \square \)

### 3.3. Rewriting the HDG scheme

We now recast the space-time HDG scheme into a form more amenable to the convergence analysis in Section 5 using the discrete differential operators introduced above. In what follows, $(u_h,i)_{1 \leq i \leq d}$, $(\bar{u}_h,i)_{1 \leq i \leq d}$, $(v_h,i)_{1 \leq i \leq d}$ and $(\bar{v}_h,i)_{1 \leq i \leq d}$ will denote the Cartesian components of $u_h$, $\bar{u}_h$, $v_h$ and $\bar{v}_h$, respectively. We will adopt the convention of summation over repeated indices. Restricting our attention to test functions $v_h \in \mathcal{V}_h^{\text{div}}$ in eq. (10) to remove the contribution from the pressure-velocity coupling term, integrating by parts and summing over all space-time slabs $E^n$, and using the definitions of the lifting operators, we arrive at the problem: find $u_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$ satisfying for all $v_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$,
\[
\int_0^T (D_t^{k_t}(u_h), v_h)_{T_h} \, dt + \int_0^T (\nu u_h, v_h) + o_h(u_h; u_h, v_h)) \, dt = \int_0^T (f, v_h)_{T_h} \, dt, \tag{26}
\]

where
\[
a_h(u_h, v_h) = \int_{\Omega} G_h^{k_s}(u_h,i) \cdot G_h^{k_s}(v_h,i) \, dx - \int_{\Omega} R_h^{k_s}(u_h,i - \bar{u}_h,i) \cdot R_h^{k_s}(v_h,i - \bar{v}_h,i) \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_K} (u_h,i - \bar{u}_h,i)(v_h,i - \bar{v}_h,i) \, ds, \tag{27}
\]
\[
o_h(u_h; u_h, v_h) = \int_{\Omega} u_h \cdot G_h^{2k_s}(u_h,i)v_h,i \, dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{1}{2} (u_h \cdot n + |u_h \cdot n|) (u_h - \bar{u}_h) \cdot (v_h - \bar{v}_h) \, ds. \tag{28}
\]

### 4. Uniform bounds on the discrete differential operators

In this section, we derive uniform bounds on the discrete differential operators of the discrete velocity solution introduced in the previous section. In what follows, we suppose that $u_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h$ is a discrete velocity pair solving the space-time HDG formulation eq. (10) on a given space-time prismatic mesh $(\mathcal{T}_h, \mathcal{S}_h)$ for $n = 0, \ldots, N - 1$. We then show that subsequences of the discrete derivatives converge weakly to their continuous counterparts.
4.1. Bounding the discrete gradient

Before bounding the discrete gradient of \( u_h \), we pause to mention an immediate consequence of the energy bound Lemma 2.1. From the discrete Sobolev embeddings for broken polynomial spaces \([37, \text{Theorem 6.1}]\), we can infer using eq. (8) that

\[
\int_0^T \| u_h \|_{L^q(\Omega)}^2 \, dt \leq C(f, u_0, \nu),
\]

where \( 1 \leq q < \infty \) if \( d = 2 \) and \( 1 \leq q \leq 6 \) if \( d = 3 \). Thus, \( (u_h)_{h \in H} \) is bounded in \( L^2(0, T; L^q(\Omega) \, d) \) for \( 1 \leq q \leq 6 \) and in particular in \( L^2(0, T; H) \).

**Theorem 4.1.** Let \( (T_h, \mathcal{Q}_h) \) be a conforming and shape-regular space-time prismatic mesh. Let \( d \in \{2, 3\} \) and suppose \( k_t \geq 0 \) if \( d = 2 \) and \( k_t \in \{0, 1\} \) if \( d = 3 \). Let \( u_h \) be the solution of the space-time HDG scheme eq. (10). Then, provided the penalty parameter \( \alpha > 0 \) is chosen sufficiently large, it holds that

\[
\int_0^T \| G_{k_t}^k(u_h,i) \|_{L^2(\Omega)}^2 \, dt \leq C(f, u_0, \nu).
\]

**Proof.** The result follows from eq. (27) and the energy bound in Lemma 2.1, provided \( \alpha > 0 \) is chosen sufficiently large, since for all \( u_h \in V_h \) we have by eq. (20) for \( i = 1, \ldots, d \) that

\[
-\| R_h^{k_t}(u_h,i - \bar{u}_h,i) \|_{L^2(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} \frac{\alpha}{h_K} \| u_h,i - \bar{u}_h,i \|_{L^2(\partial K)}^2 \\
\geq (\alpha - C) \sum_{K \in \mathcal{T}_h} \frac{1}{h_K} \| u_h,i - \bar{u}_h,i \|_{L^2(\partial K)}^2,
\]

and therefore,

\[
a_h(u_h, u_h) \geq \sum_{i=1}^d \| G_{k_t}^k(u_h,i) \|_{L^2(\Omega)}^2.
\]

Consequently, the sequence \( G_{k_t}^k(u_h,i) \) is bounded in \( L^2(0, T; L^2(\Omega)) \). \( \square \)

4.2. Bounding the discrete time derivative

We now turn our focus to bounding the discrete time derivative of \( u_h \in V_h^{\text{div}} \) uniformly, first in the dual space of \( V_h^{\text{div}} \times \tilde{V}_h \), and second in \( L^{4/d}(0, T; V') \). The former is required to obtain a strong compactness result needed for passage to the limit as \( h \to 0 \) in the nonlinear convection term, and the second is essential to ensure the distributional time derivatives of accumulation points of the sequence \( \{u_h\}_{h \in H} \) are sufficiently regular to satisfy Definition 2.1. That \( D_{k_t}^h(u_h) \) can be identified with an element of \( L^{4/d}(0, T; V') \) follows from Lemma 3.1 since \( V \subset H \) with continuous and dense embedding (see e.g. [6, Chapter 5.2]).
4.2.1. Uniform bound in the dual space of $V_h^{\text{div}} \times \bar{V}_h$

To apply the compactness theorem Theorem A.1 later on to prove Theorem 4.3, we will require $F_h : \mathbf{v}_h \mapsto (D_t^{k_i}(u_h), v_h)_{\mathcal{T}_h}$ to be uniformly bounded $L^{4/d}(0, T; (V_h^{\text{div}} \times \bar{V}_h))$, with $(V_h^{\text{div}} \times \bar{V}_h)$ the dual space of $V_h^{\text{div}} \times \bar{V}_h$. We shall see that it suffices to bound $F_h(\mathbf{v}_h)$ in the dual space of the fully discrete space $V_h^{\text{div}} \times \bar{V}_h$, which we equip with the norm

$$\|F_h\|_{(V_h^{\text{div}} \times \bar{V}_h)'} = \sup_{0 \neq \mathbf{v}_h \in V_h^{\text{div}} \times \bar{V}_h} \frac{\left| \int_0^T F_h(\mathbf{v}_h) \, dt \right|}{\int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt}^{(4-d)/4}.$$ 

This motivates the following result (where we choose $F_h : \mathbf{v}_h \mapsto (D_t^{k_i}(u_h), v_h)_{\mathcal{T}_h}$):

**Lemma 4.1.** Let $(\mathcal{T}_h, \partial_h)$ be a conforming and shape-regular space-time prismatic mesh. Let $d \in \{2, 3\}$ and suppose $k_i \geq 0$ if $d = 2$ and $k_i \in \{0, 1\}$ if $d = 3$. Let $\mathbf{u}_h$ be the discrete velocity pair arising from the solution of the space-time HDG scheme eq. (10). It holds for all $\mathbf{v}_h \in V_h^{\text{div}} \times \bar{V}_h$

$$\left| \int_0^T (D_t^{k_i}(u_h), v_h)_{\mathcal{T}_h} \, dt \right| \leq C(u_0, f, \nu, T) \left( \int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt \right)^{(4-d)/4}. \tag{32}$$

**Proof.** Let $\mathbf{v}_h \in V_h^{\text{div}} \times \bar{V}_h$, and use eq. (26) to write

$$\int_0^T (D_t^{k_i}(u_h), v_h)_{\mathcal{T}_h} \, dt = \int_0^T \left( (f, v_h)_{\mathcal{T}_h} - \nu a_h(u_h, \mathbf{v}_h) - o_h(u_h; \mathbf{u}_h, \mathbf{v}_h) \right) \, dt. \tag{32}$$

We now bound each of the three terms on the right hand side of eq. (32), beginning with the first term on the right hand side. The Cauchy-Schwarz inequality, Hölder’s inequality, and the discrete Poincaré inequality eq. (9) yield

$$\int_0^T |(f, v_h)_{\mathcal{T}_h}| \, dt \leq C(f, T) \left( \int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt \right)^{(4-d)/4}. \tag{33}$$

To bound the linear viscous term on the right hand side of eq. (32), we begin by using the boundedness of $a_h(\cdot, \cdot)$ eq. (12) and Hölder’s inequality with $p = 4/d$ and $q = 4/(4-d)$ to find

$$\int_0^T |a_h(u_h, \mathbf{v}_h)| \, dt \leq C \left( \int_0^T \|u_h\|_v^{4/d} \, dt \right)^{d/4} \left( \int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt \right)^{(4-d)/4}. \tag{34}$$

If $d = 2$, directly using the uniform bound in Lemma 2.1, and if $d = 3$, applying Hölder’s inequality to the first integral on the right hand side of eq. (34) with $p = 3$ and $q = 3/2$, followed by the uniform bound in Lemma 2.1, we find

$$\int_0^T |a_h(u_h, \mathbf{v}_h)| \, dt \leq C(f, u_0, \nu, T) \left( \int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt \right)^{(4-d)/4}. \tag{35}$$

Lastly, we must bound the nonlinear convection term on the right hand side of eq. (32). For this, we use the bound eq. (14), apply the generalized Hölder’s inequality with $p = \infty$, $q = 4/d$, and $r = 4/(4-d)$, and use Lemma 2.1, to find

$$\int_0^T |o_h(u_h; \mathbf{u}_h, \mathbf{v}_h)| \, dt \leq C(f, u_0, \nu) \left( \int_0^T \|\mathbf{v}_h\|_v^{4/(4-d)} \, dt \right)^{(4-d)/4}. \tag{36}$$

Collecting eq. (33), eq. (35), and eq. (36) yields the result. \qed
4.2.2. Construction of suitable test functions

To prove a uniform bound on the discrete time derivative in $L^{4/d}(0,T;V')$ (see Theorem 4.2), we will need to construct a suitable set of test functions in the discrete space $V^\text{div}_h \times \bar{V}_h$. This will require two preparatory results. The first is a density result for functions of tensor-product type in $C_c(0,T;V)$ taken from [5, Lemma V.1.2] with minor modification:

**Lemma 4.2.** The set $\mathcal{M}$ of functions $\varphi$ of the form

$$\varphi(t,x) = \sum_{k=1}^{M} \eta_k(t) \psi_k(x), \quad (37)$$

where $M \geq 1$ is an integer, $\eta_k \in C^\infty_c(0,T)$, and $\psi_k \in \mathcal{V}$, is dense in $C_c(0,T;V)$.

Denote by $\Pi_n^i : L^2(I_n) \to P_{k_i}(I_n)$, $\Pi^\text{div}_h : L^2(\Omega) \to V^\text{div}_h$, and $\bar{\Pi}_h : H^1(\Omega)^d \to \bar{V}_h$ the orthogonal $L^2$-projections onto the discrete spaces $P_{k_i}(I_n)$, $V^\text{div}_h$, and $\bar{V}_h$, respectively. We define the *global* $L^2$-projection $\Pi^i$ in time by the restriction $\Pi^i|_{I_n} = \Pi^i_n$. Given a function $\varphi \in \mathcal{M}$, consider for all $n = 0, \ldots, N-1$,

$$\Pi^i \varphi|_{E^n} = \sum_{k=1}^{M} \Pi^i_n \eta_k(t) \Pi^\text{div}_h \psi_k(x) \quad \text{and} \quad \bar{\Pi} \varphi|_{E^n} = \sum_{k=1}^{M} \Pi^i_n \eta_k(t) \bar{\Pi}_h \psi_k(x). \quad (38)$$

By construction, $(\Pi^i v, \bar{\Pi} v) \in V^\text{div}_h \times \bar{V}_h$. We remark that the approximation properties of $\Pi^\text{div}_h$ obtained in [28] and listed in Lemma B.1 require quasi-uniformity of the underlying spatial mesh $\mathcal{T}_h$, which we assume henceforth wherever necessary.

**Proposition 4.1.** Let $(\Sigma_h, \bar{\Sigma}_h)$ be a conforming and quasi-uniform space-time prismatic mesh, and let $d \in \{2,3\}$. Let $\varphi \in \mathcal{M}$ and let $(\Pi^i \varphi, \bar{\Pi} \varphi) \in V^\text{div}_h \times \bar{V}_h$ be the discrete test functions constructed in eq. $(38)$. Then, the following stability property holds:

$$\int_0^T \|\Pi^i \varphi, \bar{\Pi} \varphi\|_{V}^{4/(4-d)} \, dt \leq \int_0^T \|\varphi\|_{V}^{4/(4-d)} \, dt, \quad \forall \varphi \in \mathcal{M}. \quad (39)$$

**Proof.** See Appendix B.2. \hfill $\Box$

4.2.3. Uniform bound in $L^{4/d}(0,T;V')$

With Lemmas 4.1 and 4.2, and Proposition 4.1 in hand, we can now prove the main result of this subsection. Since $V'$ is separable, we can identify $L^{4/d}(0,T;V') \cong (L^{4/(4-d)}(0,T;V'))' \cong (L^{4/(4-d)}(0,T;V'))' \cong (L^{4/(4-d)}(0,T;V'))'(0,T;V')$ (see e.g. [42, Proposition 1.38]), and since $(V, H, V')$ form a Gelfand triple, we have

$$\|D^k_t(u_h)\|_{L^{4/d}(0,T;V')} = \sup_{0 \neq v \in L^{4/(4-d)}(0,T;V)} \frac{\|f_0^T (D^k_t(u_h), v)_{\mathcal{T}_n} \, dt\|_{L^{4/(4-d)}(0,T;V')}}{\|v\|_{L^{4/(4-d)}(0,T;V')}}. \quad (40)$$

**Theorem 4.2** (Uniform bound on the discrete time derivative). Let $(\Sigma_h, \bar{\Sigma}_h)$ be a conforming and quasi-uniform space-time prismatic mesh. Let $d \in \{2,3\}$ and suppose $k_t \geq 0$ if $d = 2$ and $k_t \in \{0,1\}$ if $d = 3$. Let $u_h$ be the discrete velocity pair arising from the solution of the space-time HDG scheme eq. (10). Then $\|D^k_t(u_h)\|_{L^{4/d}(0,T;V')} \leq C(f, u_0, \nu, T).$

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Proof. We follow the strategy used in the proof of [33, Theorem 3.2]. The density of $C_c(0,T;V)$ in $L^p(0,T;V)$ for $1 \leq p < \infty$, gives us also the density of $\mathcal{M}$ in $L^{4/(4-d)}(0,T;V)$. We therefore replace the supremum over $v \in L^{4/(4-d)}(0,T;V)$ in eq. (40) with the supremum over $\varphi \in \mathcal{M}$. Now let $\varphi \in \mathcal{M}$ be arbitrary. Using the expansion of $\varphi$ eq. (37) the definitions of the $L^2$-projections $\Pi^t$ and $\Pi_h$, Proposition 4.1, and Lemma 4.1, we have

$$
\|D^k_{t,h}(u_h)\|_{L^{4/(4-d)}(0,T;V)}^2 = \sup_{0 \neq \varphi \in \mathcal{M}} \left| \int_0^T (D^k_{t,h}(u_h), \varphi)_{\mathcal{T}_h} dt \right| \\
= \sup_{0 \neq \varphi \in \mathcal{M}} \frac{\left| \int_0^T (D^k_{t,h}(u_h), \Pi \varphi)_{\mathcal{T}_h} dt \right|}{\left( \int_0^T \| (\Pi \varphi, \Pi \varphi) \|_{L^{4/(4-d)}}^2 dt \right)^{(4-d)/4}} \\
\lesssim \sup_{0 \neq \varphi \in \mathcal{M}} \frac{\left| \int_0^T (D^k_{t,h}(u_h), \Pi \varphi)_{\mathcal{T}_h} dt \right|}{\left( \int_0^T \| (\Pi \varphi, \Pi \varphi) \|_{L^{4/(4-d)}}^2 dt \right)^{(4-d)/4}} \\
\lesssim \sup_{0 \neq \varphi \in \mathcal{M}} \frac{\left| \int_0^T (D^k_{t,h}(u_h), \varphi)_{\mathcal{T}_h} dt \right|}{\left( \int_0^T \| \varphi \|_{L^{4/(4-d)}}^2 dt \right)^{(4-d)/4}} \leq C(f, u_0, \nu, T).
$$

\[\square\]

4.3. Compactness

We end this section by summarizing the significance of the uniform bounds on the discrete velocity collected in Lemma 2.1, Theorem 4.1, and Theorem 4.2. In particular, we conclude that subsequences of the discrete velocity solution computed by the space-time HDG scheme eq. (10) converge to a limit function $u$ in suitable topologies. The goal of Section 5 will be to show that $u$ is in fact a weak solution to the Navier–Stokes problem in the sense of Definition 2.1.

Theorem 4.3. Let $\mathcal{H} \subset (0, |\Omega|)$ be a countable set of mesh sizes whose unique accumulation point is 0. Assume that $\{(\mathcal{T}_h, \mathcal{S}_h)\}_{h \in \mathcal{H}}$ is a sequence of conforming, shape-regular, and quasi-uniform space-time prismatic meshes, and suppose that $\tau \to 0$ as $h \to 0$. Suppose that $k_s \geq 1$ and $k_{t,i} \geq 0$ if $d = 2$ and $k_{t,i} \in \{0, 1\}$ if $d = 3$, and for each fixed $h \in \mathcal{H}$, let $u_h \in \mathcal{V}_h$ be the discrete velocity pair computed on $(\mathcal{T}_h, \mathcal{S}_h)$ using the space-time HDG scheme eq. (10). Collect each of these solutions in the sequence $\{u_h\}_{h \in \mathcal{H}}$. Then, there exists a function $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ with $\frac{du}{dt} \in L^{4/(4-d)}(0,T;V')$ such that, up to a (not relabeled) subsequence:

(i) $u_h \rightharpoonup u$ in $L^\infty(0,T;H)$ as $h \to 0$,
(ii) $u_h \to u$ in $L^2(0,T;L^2(\Omega)^d)$ as $h \to 0$,
(iii) $G^k_{t,h}(u_{h,i}) \to \nabla u_i$ in $L^2(0,T;L^2(\Omega)^d)$ as $h \to 0$,
(iv) $D^k_{t,h}(u_h) \to \frac{du}{dt}$ in $L^{4/(4-d)}(0,T;V')$ as $h \to 0$.

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Proof. (i) Weak∗ convergence. The existence of a \( u \) satisfying (i) is a direct consequence of the uniform \( L^\infty(0,T;L^2(\Omega)^d) \) bound in Lemma 2.1 and [6, Corollary 3.30].

(ii) Strong convergence. This follows from Theorem A.1 due to the uniform energy bound in Lemma 2.1, the uniform bound on \( D^k_h(u_h) \) in Lemma 4.1 (see also Remark A.1 and [33, Theorem 3.2]), and the uniqueness of distributional limits.

(iii) Weak convergence of the discrete gradient. By Theorem 4.1 there exists \( w \in L^2(0,T;L^2(\Omega)^d) \) such that, upon passage to a subsequence, \( G^k_h(u_{h,i}) \rightharpoonup w \) in \( L^2(0,T;L^2(\Omega)^d) \) as \( h \to 0 \). Let \( \phi \in C^\infty_c(\Omega)^d \) and \( \eta \in C^\infty_c(0,T) \) be arbitrary and let \( \Pi_h \) be the orthogonal \( L^2 \)-projection onto \( V_h \). Extending \( u_{h,i}, G^k_h(u_{h,i}), R^k_h(u_{h,i} - \bar{u}_{h,i}), u, \) and \( w \) by zero outside of \( \Omega \), and integrating by parts element-wise in space, we have for all \( \eta \in C^\infty_c(0,T) \) and \( \phi \in C^\infty_c(\Omega)^d \) that

\[
\int_0^T \left( \int_{\mathbb{R}^d} G^k_h(u_{h,i}) \cdot \phi \, dx \right) \eta \, dt = 0
\]

where we have used eqs. (19) and (21), that \( \phi \) and \( \bar{u}_{h,i} \) are single-valued on element boundaries, and that \( \bar{u}_{h,i}|_{\partial \Omega} = 0 \). Moreover,

\[
\int_0^T \| \eta(\phi - \Pi_h \phi) \cdot n \|_{L^2(\partial K)}^2 \, dt \leq h^{2l+1} \int_0^T \eta^2 \| \phi \|_{H^{l+1}(\Omega)}^2 \, dt.
\]

As a consequence of eqs. (41) and (42) and the strong convergence in \( L^2(0,T;L^2(\Omega)^d) \) of \( u_h \) to \( u \), it holds for all \( \eta \in C^\infty_c(0,T) \) that

\[
\int_0^T \left( \int_{\mathbb{R}^d} w \cdot \phi \, dx \right) \eta \, dt = \lim_{h \to 0} \int_0^T \left( \int_{\mathbb{R}^d} G^k_h(u_{h,i}) \cdot \phi \, dx \right) \eta \, dt = \int_0^T \left( - \int_{\mathbb{R}^d} u_i \nabla \cdot \phi \, dx \right) \eta \, dt.
\]

Hence \( w = \nabla u_i \) as elements of \( L^2(0,T;L^2(\Omega)^d) \), so \( u_i \in L^2(0,T;H^1(\Omega)^d) \). As \( u_i \) vanishes outside of \( \Omega \), the \( H^1(\Omega)^d \)-regularity ensures that \( u_i \) vanishes on the boundary. As \( u \in H \), its distributional divergence vanishes, and thus \( u \in L^2(0,T;V) \).

(iv) Weak convergence of the discrete time derivative. By Theorem 4.2, there exists a \( z \in L^{4/d}(0,T;V') \) such that, upon passage to a subsequence, \( D^k_t(u_h) \rightharpoonup z \) in \( L^{4/d}(0,T;V') \). For arbitrary \( v \in V \) and \( \eta \in C^\infty_c(0,T) \), we use the definition of \( D^k_t(u_h) \) eq. (25) and integrate by parts in time to find

\[
\begin{aligned}
\int_0^T \langle D^k_t(u_h), v \eta \rangle_{V',V} \, dt &= -\int_0^T (u_h,v)_{\mathcal{T}_h} \partial_t \eta \, dt + \sum_{n=0}^{N-1} \left( (u_{n+1}^-,v)_{\mathcal{T}_h} \eta(t_{n+1}) - (u_n^-,v)_{\mathcal{T}_h} \eta(t_n) \right).
\end{aligned}
\]

The telescoping sum on the right hand side of eq. (44) vanishes since \( \eta(0) = \eta(T) = 0 \). Thus, we can take the limit as \( h \to 0 \) to find that for all \( \eta \in C^\infty_c(0,T) \),

\[
\int_0^T \eta(z,v)_{V',V} \, dt = \lim_{h \to 0} \int_0^T \langle D^k_t(u_h), v \eta \rangle_{V',V} \, dt = -\int_0^T \partial_t \eta(u,v)_{\mathcal{T}_h} \, dt,
\]

since \( D^k_t(u_h) \rightharpoonup z \) in \( L^{4/d}(0,T;V') \) and \( u_h \to u \) in \( L^2(0,T;L^2(\Omega)^d) \) as \( h \to 0 \). Therefore, \( z = \frac{du}{dt} \). \qed
5. Convergence to weak solutions

The remainder of this article is dedicated to showing that the limiting function \( u \in L^\infty(0; T, H) \cap L^2(0; T; V) \) guaranteed by Theorem 4.3 is actually a weak solution of the Navier–Stokes problem in the sense of Definition 2.1. The plan is as follows: we first construct a set of test functions in the discrete space that will allow us to conclude upon passage to the limit that \( u \) solves eq. (3). We will then show that the viscous term \( a_h(\cdot, \cdot) \) and the nonlinear convection term \( o_h(\cdot, \cdot, \cdot) \) enjoy asymptotic consistency in the sense of [14, Definition 5.9], and use this to pass to the limit in eq. (26). Finally, we discuss the energy (in)equality and conclude that the constructed weak solution \( u \in L^\infty(0, T; H) \cap L^2(0, T; V) \) is a solution in the sense of Leray–Hopf.

From this point on, we assume that \( \mathcal{H} \subset (0, |\Omega|) \) is a countable set of mesh sizes whose unique accumulation point is 0, \( \{(\mathfrak{H}_h, \mathfrak{F}_h)\}_{h \in \mathcal{H}} \) is a sequence of conforming and quasi-uniform space-time prismatic meshes, and \( \tau \to 0 \) as \( h \to 0 \). For each fixed \( h \in \mathcal{H} \), we let \( \mathbf{u}_h = (u_h, \bar{u}_h) \in V_h^{\text{div}} \times \bar{V}_h \) be a discrete velocity pair computed on \( (\mathfrak{H}_h, \mathfrak{F}_h) \) using the space-time HDG scheme eq. (10). We collect each of these solutions in the sequence \( \{\mathbf{u}_h\}_{h \in \mathcal{H}} \).

5.1. Strong convergence of test functions

Passing to the limit in eq. (26) will require a suitable set of discrete test functions. We will again use the set \( \mathcal{M} \) of functions defined in Lemma 4.2 as our basic building block, as it is sufficiently rich to ensure density in \( C_c(0, T; V) \) while its tensor-product structure allows us to easily combine spatial and temporal projections onto the discrete spaces. In particular, given \( \varphi \in \mathcal{M} \), we will work with the discrete functions \( \Pi \varphi \) and \( \Pi \varphi_u \), constructed in eq. (38). To set notation, we denote by \( \Pi \varphi_i \) and \( \Pi \varphi_i \) the \( i \)th Cartesian component of the vector functions \( \Pi \varphi \) and \( \Pi \varphi_u \), respectively. We first show a strong convergence result for the sequence of discrete test functions \( \{\Pi \varphi, \Pi \varphi_u\}_{h \in \mathcal{H}} \).

Proposition 5.1. Let \( k_s \geq 1, k_t \geq 0 \) if \( d = 2 \), and \( k_t \in \{0, 1\} \) if \( d = 3 \). Suppose that \( \varphi \in \mathcal{M} \) and consider the sequence of discrete test functions \( \{\Pi \varphi, \Pi \varphi_u\}_{h \in \mathcal{H}} \) defined in eq. (38). Then, it holds that \( \Pi \varphi \to \varphi \) strongly in \( L^\infty(0, T; L^\infty(\Omega)^d) \) and \( C^k_h((\Pi \varphi_i, \Pi \varphi_i)) \to \nabla \varphi_i \) strongly in \( L^2(0, T; L^2(\Omega)^d) \) as \( h \to 0 \).

Proof. We first record the following consequences of Lemma B.1:

\[
\|\Pi^I \eta_k \|_{L^\infty(0, T)} \lesssim T^2 \|\eta_k\|_{W^{1, \infty}(0, T)},
\]

\[
\sum_{K \in T_h} \|\nabla (\Pi_h^{\text{div}} \psi_k - \psi_k)\|_{L^2(K)}^2 \lesssim h^2 \|\psi_k\|_{H^2(\Omega)}^2,
\]

\[
\|\psi_k - \Pi_h^{\text{div}} \psi_k\|_{L^\infty(\Omega)} \lesssim h^{1/2} \|\psi_k\|_{H^2(\Omega)}.
\]

That \( \Pi \varphi \to \varphi \) strongly in \( L^\infty(0, T; L^\infty(\Omega)^d) \) follows from eq. (45), since

\[
\|\varphi - \Pi \varphi\|_{L^\infty(0, T; L^\infty(\Omega)^d)} \lesssim \sum_{k=1}^m \left( \|\eta_k\|_{L^\infty(0, T)} \|\psi_k - \Pi_h^{\text{div}} \psi_k\|_{L^\infty(\Omega)} + \|\eta_k\|_{L^\infty(0, T)} \|\psi_k\|_{H^2(\Omega)} \right).
\]
We now prove the strong convergence of $G_k^h((\Pi\varphi_i, \bar{\Pi}\varphi_i))$ to $\nabla \varphi_i$ in $L^2(0, T; L^2(\Omega)^d)$. Using the definition of the discrete gradient eq. (21), the triangle inequality, and eq. (20), we find

\[
\int_0^T \|G_k^h((\Pi\varphi_i, \bar{\Pi}\varphi_i)) - \nabla \varphi_i\|^2_{L^2(\Omega)} \, dt \\
\leq \sum_{K \in T_h} \int_0^T \|\nabla \Pi_h \varphi - \nabla \varphi\|^2_{L^2(K)} \, dt + \sum_{K \in T_h} \int_0^T h_K^{-1} \|\Pi_h \varphi - \bar{\Pi}\varphi\|^2_{L^2(\partial K)} \, dt.
\]

(47)

We start with the first term on the right hand side of eq. (47). By the definition of $\Pi \varphi$, the triangle inequality, and eq. (45), we can write

\[
\sum_{K \in T_h} \int_0^T \|\nabla \Pi_h \varphi - \nabla \varphi\|^2_{L^2(K)} \, dt \\
\lesssim \sum_{k=1}^M \sum_{K \in T_h} \left( \int_0^T (\Pi^t \eta_k)^2 \|\nabla ([\Pi_h^{\text{div}} \psi_k - \psi_k])\|^2_{L^2(K)} \, dt + \int_0^T (\Pi^t \eta_k - \eta_k)^2 \|\nabla \psi_k\|^2_{L^2(K)} \, dt \right) \\
\lesssim \sum_{k=1}^M \|\eta_k\|^2_{W^{1,\infty}(0,T)} \left( h^2 \int_0^T \|\psi_k\|^2_{H^2(\Omega)} \, dt + \tau^2 \int_0^T \|\psi_k\|^2_{L^2(\Omega)} \, dt \right),
\]

which can be seen to vanish as $h \to 0$. Turning now to the second term on the right hand side of eq. (47), we find

\[
\sum_{K \in T_h} \int_0^T h_K^{-1} \|\Pi_h \varphi - \bar{\Pi}\varphi\|^2_{L^2(\partial K)} \, dt \\
\lesssim \sum_{k=1}^M \|\Pi^t \eta_k\|^2_{W^{1,\infty}(0,T)} \sum_{K \in T_h} \int_0^T h_K^{-1} \|\Pi_h^{\text{div}} \psi_k - \bar{\Pi}_h \psi_k\|^2_{L^2(\partial K)} \, dt.
\]

(48)

Using a discrete local trace inequality, the assumed quasi-uniformity of the space-time prismatic mesh, and the approximation properties of $\Pi_h^{\text{div}}$ and $\bar{\Pi}_h$, we find

\[
h_K^{-1} \|\Pi_h^{\text{div}} \psi_k - \bar{\Pi}_h \psi_k\|^2_{L^2(\partial K)} \lesssim h^2 \|\psi_k\|^2_{H^2(\Omega)},
\]

(49)

and thus the right hand side of eq. (48) vanishes as $h \to 0$. The result follows. \hfill \square

5.2. Asymptotic consistency of the linear viscous term

We are now in a position to show that the linear viscous term is asymptotically consistent in the sense of [14, Definition 5.9]:

**Theorem 5.1.** Let $k_s \geq 1$, $k_t \geq 0$ if $d = 2$, and $k_t \in \{0, 1\}$ if $d = 3$. Let $\varphi \in \mathcal{M}$, denote by $(\Pi\varphi, \bar{\Pi}\varphi)$ the discrete test functions constructed as in eq. (38), and let $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ be the limit (up to a subsequence) of $\{u_h\}_{h \in H}$ guaranteed by Theorem 4.3. Then, the following asymptotic consistency result holds for the linear viscous term:

\[
\lim_{h \to 0} \int_0^T a_h(u_h, (\Pi\varphi, \bar{\Pi}\varphi)) \, dt = \int_0^T \int_{\Omega} \nabla u : \nabla \varphi \, dx \, dt.
\]

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Proof. Since $G_h^k_r((\Pi \varphi, \tilde{\Pi} \varphi)) \rightarrow \nabla \varphi_i$ strongly in $L^2(0; T; L^2(\Omega)^d)$ by Proposition 5.1 and by Theorem 4.3 (iii), $G_h^{k_s}(u_h, i) \rightarrow \nabla u_i$ weakly in $L^2(0; T; L^2(\Omega)^d)$ as $h \rightarrow 0$, we can pass to the limit in the first term of eq. (27) to find that
\[
\lim_{h \rightarrow 0} \int_0^T \int_\Omega G_h^k(u_h, i) \cdot G_h^{k_s}((\Pi \varphi, \tilde{\Pi} \varphi)) \, dx \, dt = \int_0^T \int_\Omega \nabla u_i \cdot \nabla \varphi_i \, dx \, dt.
\]

Turning to the second term of eq. (27), we have by the Cauchy-Schwarz inequality, the bound on the global spatial lifting operator eq. (20), the definition of $\Pi \varphi$ and $\tilde{\Pi} \varphi$, and uniform bound Lemma 2.1,
\[
\int_0^T \int_\Omega R_h^{k_s}(u_h, i - \tilde{u}_h, i) \cdot R_h^{k_s}(\Pi \varphi, \tilde{\Pi} \varphi) \, dx \, dt \leq C(f, u_0, \nu) \left( \sum_{k=1}^M \sum_{K \in T_h} \int_0^T h_K^{-1} \Pi^t \eta_k \left\| \Pi_h^{k_s} \psi_k - \tilde{\Pi}_h \psi_k \right\|^2_{L^2(\partial K)} \right)^{1/2},
\]
which can be seen to vanish as $h \rightarrow 0$ by eq. (45a) and eq. (49). In an identical fashion, we find
\[
\lim_{h \rightarrow 0} \sum_{K \in T_h} \int_0^T \int_{\partial K} \frac{\alpha}{h_K} (u_h - \tilde{u}_h) \cdot (\Pi \varphi, \tilde{\Pi} \varphi) \, ds \, dt = 0.
\]
The result follows.

5.3. Asymptotic consistency of the nonlinear convection term

We now prove that the nonlinear convection term is asymptotically consistent in the sense of [14, Definition 5.9]:

Theorem 5.2. Suppose that $k_s \geq 1$ and $k_l \geq 0$ if $d = 2$ and $k_l \in \{0, 1\}$ if $d = 3$. Let $\varphi \in M$, denote by $(\Pi \varphi, \tilde{\Pi} \varphi)$ the discrete test functions constructed as in eq. (38), and let $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ be an accumulation point of $\{u_h\}_{h \in H}$ guaranteed by Theorem 4.3. Then, the following asymptotic consistency result holds for the nonlinear convection term:
\[
\lim_{h \rightarrow 0} \int_0^T o_h(u_h; u_h, (\Pi \varphi, \tilde{\Pi} \varphi)) \, dt = \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, dx \, dt.
\]

Proof. We start with the first term on the right hand side of eq. (28). By Hölder’s inequality, we have
\[
\int_0^T \|u \varphi_i - u_h \Pi \varphi_i\|^2_{L^2(\Omega)} \, dt \lesssim \|\varphi - \Pi \varphi\|^2_{L^\infty(0, T; L^\infty(\Omega)^d)} \int_0^T \|u\|^2_{L^2(\Omega)} \, dt + \|\Pi \varphi\|^2_{L^\infty(0, T; L^\infty(\Omega)^d)} \int_0^T \|u - u_h\|^2_{L^2(\Omega)} \, dt,
\]
which can be seen to vanish as $h \rightarrow 0$ by Proposition 5.1 and Theorem 4.3. Therefore, $u_h \Pi \varphi_i \rightarrow u \varphi_i$ strongly in $L^2(0, T; L^2(\Omega)^d)$ as $h \rightarrow 0$, and this combined with the fact that $G_h^{k_s}(u_h, i) \rightarrow \nabla u$ yields
\[
\lim_{h \rightarrow 0} \int_0^T \int_\Omega u_h \cdot G_h^{k_s}(u_h, i) \Pi \varphi_i \, dx \, dt = \int_0^T \int_\Omega (u \cdot \nabla u) \cdot \varphi \, dx \, dt.
\]
It remains to show that the facet term appearing in eq. (28) converges to 0 as \( h \to 0 \). By the definitions of \( \Pi \varphi \) and \( \tilde{\Pi} \varphi \), proceeding as in the proof of [8, Proposition 3.4], and using the fact that \( \int_0^T \| u_h \|_{V,h}^2 \, dt \) is uniformly bounded by Lemma 2.1,

\[
\left| \sum_{K \in T_h} \int_0^T \int_{\partial K} \frac{1}{2} (u_h \cdot n + |u_h \cdot n|) (u_h - \bar{u}_h) \cdot (\Pi \varphi - \tilde{\Pi} \varphi) \, ds \, dt \right| \leq C(f,u_0,\nu) \sum_{k=1}^M \| \Pi \eta_k \|_{L^\infty(0,T)} \left( \sum_{K \in T_h} h_K^{-1} \| \Pi_h \psi_k \|_{L^2(\partial K)}^2 \right)^{1/2},
\]

which can be seen to vanish as \( h \to 0 \) by using the second bound in eq. (45a) and eq. (49).

\[ \square \]

5.4. Passing to the limit

With the asymptotic consistency of the linear viscous term (Theorem 5.1) and the nonlinear convection term (Theorem 5.2), we are ready to pass to the limit as \( h \to 0 \) in eq. (26). Extract from \( \{ u_h \}_{h \in \mathcal{H}} \) the subsequence satisfying the convergence results listed in Theorem 4.3. Let \( \varphi \in \mathcal{M} \) and choose \( \psi_h = (\Pi \varphi, \tilde{\Pi} \varphi) \in \mathcal{V}_h^{div} \times V_h \) as a test function in eq. (26):

\[
\int_0^T (D_t^{k_t}(u_h), \varphi)_T \, dt + \int_0^T (\nu a_h(u_h, (\Pi \varphi, \tilde{\Pi} \varphi)) + o_h(u_h, (\Pi \varphi, \tilde{\Pi} \varphi)) \, dt = \int_0^T (f, \varphi)_T \, dt.
\]

(50)

By the definition of \( \Pi \varphi \), we have

\[
\int_0^T (D_t^{k_t}(u_h), \varphi)_T \, dt = \int_0^T (D_t^{k_t}(u_h), \varphi)_T \, dt = \int_0^T \langle D_t^{k_t}(u_h), \varphi \rangle_{V',V} \, dt.
\]

(51)

Thus, the weak convergence of \( D_t^{k_t}(u_h) \) to \( \frac{du}{dt} \) in \( L^{4/d}(0,T;V') \) yields

\[
\lim_{h \to 0} \int_0^T (D_t^{k_t}(u_h), \varphi)_T \, dt = \int_0^T \langle \frac{du}{dt}, \varphi \rangle_{V',V} \, dt.
\]

(52)

This, in combination with Theorem 5.1 and Theorem 5.2, shows that upon passage to the limit as \( h \to 0 \) in eq. (50) that the limit \( u \in L^\infty(0,T;H) \cap L^2(0,T;V) \) of the subsequence \( \{ u_h \}_{h \in \mathcal{H}} \) given by Theorem 4.3 satisfies for all \( \varphi \in \mathcal{M} \),

\[
\int_0^T \left\langle \frac{du}{dt}, \varphi \right\rangle_{V',V} \, dt + \int_0^T \left( (u \cdot \nabla)u, \varphi \right)_V \, dt + \nu \int_0^T (\nabla u, \nabla \varphi)_V \, dt = \int_0^T (f, \varphi)_T \, dt.
\]

(53)

By the density of the set \( \mathcal{M} \) in \( C_c(0,T;V) \), eq. (53) holds also for all \( \varphi \in C_c(0,T;V) \).

We now show that \( u \in L^\infty(0,T;H) \cap L^2(0,T;V) \) satisfies the initial condition in the sense that \( u(0) = u_0 \) in \( V' \). Our starting point is the definition of the discrete time derivative eq. (25) in a single space-time slab \( E_n \):

\[
\int_{I_n} (\partial_t u_h, v_h)_{T_h} \, dt + (u_{h+} - u_{h-}, v_{h+})_{I_n} = \int_{I_n} (D_t^{k_t}(u_h), v_h)_{T_h} \, dt, \quad \forall v_h \in V_h.
\]

(54)

Let \( \psi \in V \) and \( \eta \in C^\infty(0,T) \) such that \( \eta(T) = 0 \). Define a function \( w_h \in \mathcal{V}_h^{div} \) by setting \( w_h|_{E_n} = \Pi_h^{n+} \eta \Pi_h^{n+} \psi \) with \( \Pi_h^{n+} : H^1(I_n) \to P_k(I_n) \) the DG time projection defined as in [19, 20]...
Section 69.3.2. Define also the global projection $\Pi^{k_t}|_{t_n} = \Pi^{k_t}_{0}$. We note that by definition, $(\psi, v_h)_{t_n} = (\Pi^{\text{div}}_{k_t} \psi, v_h)_{t_n}$ for all $v_h \in V_{h}^{\text{div}}$, and by the defining properties of the projection $\Pi^{k_t}_{n}$ (see [19, Eq. (69.26)]),

$$
\int_{I_{n}}(\partial_t u_h, \Pi^{\text{div}}_{k_t} \psi)_{t_n} \Pi^{k_t}_{n} \eta \ dt = \int_{I_{n}}(\partial_t u_h, \Pi^{\text{div}}_{k_t} \psi)_{t_n} \eta \ dt \quad \text{and} \quad (\Pi^{k_t}_{n} \eta)(t_n^+) = \eta(t_n),
$$

since $\partial_t u_h \in P_{k_t-1}(I_{n})$ (with the convention that $P_{-1}(I_{n}) = \{0\}$). Choosing $v_h = w_h$ in eq. (54), we find

$$
\int_{I_{n}}(\partial_t u_h, \psi)_{t_n} \eta \ dt + (u_n^+ - u_n^-, \psi)_{t_n} \eta(t_n) = \int_{I_{n}}(\mathcal{D}^{k_t}(u_h), \psi)_{t_n} \Pi^{k_t} \eta \ dt. \quad (55)
$$

Integrating by parts in time on the left hand side of eq. (55), summing over all space-time slabs, and using that $\eta(T) = 0$, we have

$$
-\int_{0}^{T} (u_h, \psi)_{t_n} \partial_t \eta \ dt - (u_0^-, \psi)_{t_n} \eta(0) = \int_{0}^{T} (\mathcal{D}^{k_t}(u_h), \psi)_{t_n} \Pi^{k_t} \eta \ dt. \quad (56)
$$

From Theorem 4.3 (i) and (iii), and since $u_n^- = \Pi^{\text{div}}_{k_t} u_0 \to u_0$ strongly in $H$, and $\Pi^{k_t} \eta \to \eta$ strongly in $L^{4/(4-d)}(0,T)$ by eq. (45a), we can pass to the limit as $h \to 0$ in eq. (56) to find

$$
-\int_{0}^{T} (u, \psi)_{t_n} \partial_t \eta \ dt - (u_0^-, \psi)_{t_n} \eta(0) = \int_{0}^{T} \left\langle \frac{du}{dt}, \psi \right\rangle_{V' \times V} \eta \ dt. \quad (57)
$$

Comparing eq. (57) with [5, Theorem II.5.12], we find that

$$
0 = (u(0) - u_0, \psi)_{t_n} = (u(0) - u_0, \psi)_{V' \times V}, \quad \forall \psi \in V \implies u(0) = u_0 \in V'.
$$

Therefore, we have proven:

**Theorem 5.3.** Let $u_0 \in H$ and $f \in L^2(0,T;L^2(\Omega)^d)$ be given and let $\mathcal{H} \subset (0,|\Omega|)$ be a countable set of mesh sizes whose unique accumulation point is $0$. Assume that $\{ (\mathcal{T}_h, \mathcal{S}_h) \}_{h \in \mathcal{H}}$ is a sequence of conforming and quasi-uniform space-time prismatic meshes, and suppose that $\tau \to 0$ as $h \to 0$. Suppose that $k_\tau \geq 1$, $k_t \geq 0$ if $d = 2$ and $k_t \in \{0,1\}$ if $d = 3$, and for each fixed $h \in \mathcal{H}$, let $u_h \in \mathcal{V}_h$ be the discrete velocity pair computed on $(\mathcal{T}_h, \mathcal{S}_h)$ using the space-time HDG scheme eq. (10) for $n = 0, \ldots, N-1$. Collect these discrete solutions in the sequence $\{ u_h \}_{h \in \mathcal{H}}$. Then, upon passage to a subsequence, $\{ u_h \}_{h \in \mathcal{H}}$ converges as $h \to 0$ (in the sense of Theorem 4.3) to a weak solution of the Navier–Stokes problem eq. (3) $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ with $\frac{du}{dt} \in L^{4/(4-d)}(0,T;V')$.

### 5.5. The energy inequality

In three dimensions, we are not guaranteed uniqueness of weak solutions and cannot conclude a priori that the weak solution obtained from Theorem 5.3 satisfies the energy inequality eq. (5). We show below that the weak solution in fact does satisfy eq. (5).

**Lemma 5.1.** Let $d = 3$, $k_\tau \geq 1$, and $k_t \in \{0,1\}$. The weak solution $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ given by Theorem 5.3 satisfies the energy inequality for a.e. $s \in (0,T)$:

$$
\|u(s)\|_{L^2(\Omega)}^2 + 2\nu \int_{0}^{s} \|u\|_{V}^2 \ dt \leq \|u_0\|_{L^2(\Omega)}^2 + 2 \int_{0}^{s} (f, u)_{L^2(\Omega)} \ dt. \quad (58)
$$
Proof. Let \( s \in (0, T) \) be fixed and choose \( n_s \in \{0, 1, \ldots, N - 1\} \) such that \( t_{n_s} \leq s \leq t_{n_{s+1}} \). Testing eq. (10) with \( \mathbf{v}_h = \mathbf{u}_h \in \mathcal{V}_h \times \mathcal{V}_h \), using eq. (31) and the stability of \( \Pi_h \) in \( L^2(\Omega)^d \), and summing from \( n = 0 \) to \( n = n_s \), we have

\[
\| u_{n_s+1}^- \|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^3 \int_0^s \| G_h^{k_i}(\mathbf{u}_h) \|_{L^2(\Omega)}^2 \, dt \leq \| u_0 \|_{L^2(\Omega)}^2 + 2 \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt. \tag{59}
\]

Let us first suppose that \( k_t = 0 \). Since \( u_{t_{n_{s+1}}} = u_h(s) \) for \( s \in (t_{n_s}, t_{n_{s+1}}) \) in this case, eq. (59) yields

\[
\| u_h(s) \|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^3 \int_0^s \| G_h^{k_i}(\mathbf{u}_h) \|_{L^2(\Omega)}^2 \, dt \leq \| u_0 \|_{L^2(\Omega)}^2 + 2 \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt. \tag{60}
\]

Our goal is to justify passage to the limit in eq. (60).

To this end, we multiply both sides of eq. (60) by an arbitrary \( \phi \in C_c^\infty(\mathbb{R}) \) satisfying \( \phi \geq 0 \) and integrate from \( s = 0 \) to \( s = T \):

\[
\int_0^T \left( \| u_h(s) \|_{L^2(\Omega)}^2 + 2\nu \sum_{i=1}^3 \int_0^s \| G_h^{k_i}(\mathbf{u}_h) \|_{L^2(\Omega)}^2 \, dt \right) \phi(s) \, ds 
\leq \int_0^T \left( \| u_0 \|_{L^2(\Omega)}^2 + 2 \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt \right) \phi(s) \, ds. \tag{61}
\]

We first consider the integral involving the body force \( f \). By the triangle inequality, the Cauchy-Schwarz inequality, discrete Poincaré inequality eq. (9), and the uniform energy bound in Lemma 2.1, we obtain

\[
\left| \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt - \int_0^s (f, u) \mathcal{T}_h \, dt \right| \leq \left( \int_s^{s+\tau} \| f \|_{L^2(\Omega)} \, dt \right)^{1/2} + \int_0^s \| (f, u_h - u) \mathcal{T}_h \| \, dt. \tag{62}
\]

Since \( f \in L^2(0, T; L^2(\Omega)^d) \), the primitive \( F(\tau) = \int_s^{s+\tau} \| f \|_{L^2(\Omega)} \, dt \) is absolutely continuous. This, combined with the fact \( u_h \to u \) strongly in \( L^2(0, T; L^2(\Omega)^d) \), shows that the right hand side of eq. (62) vanishes as \( h \to 0 \) (since also \( \tau \to 0 \)), and so

\[
\lim_{h \to 0} \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt = \int_0^s (f, u) \mathcal{T}_h \, dt.
\]

Thus, we can apply Lebesgue’s dominated convergence theorem to find

\[
\lim_{h \to 0} \int_0^T \left( \int_0^{t_{n_{s+1}}} (f, u_h) \mathcal{T}_h \, dt \right) \phi(s) \, ds = \int_0^T \left( \int_0^s (f, u) \mathcal{T}_h \, dt \right) \phi(s) \, ds. \tag{63}
\]

With eq. (63) in hand, we pass to the lower limit as \( h \to 0 \) in eq. (61) and use Fatou’s lemma, the weak lower semicontinuity of norms, and Theorem 4.3:

\[
\int_0^T \left( \| u(s) \|_{L^2(\Omega)}^2 + 2\nu \int_0^s \| u \|_{V} \, dt \right) \phi(s) \, ds \leq \int_0^T \left( \| u_0 \|_{L^2(\Omega)}^2 + 2 \int_0^s (f, u) \mathcal{T}_h \, dt \right) \phi(s) \, ds.
\]
As this holds for all \( \phi \in C_c^\infty(\mathbb{R}) \) satisfying \( \phi \geq 0 \), we have for a.e. \( s \in [0,T] \) \cite{47}, pp. 291,
\[
\|u(s)\|^2_{L^2(\Omega)} + 2\nu \int_0^s \|u\|^2_{L^2(\Omega)} \, dt \leq \|u_0\|^2_{L^2(\Omega)} + 2 \int_0^s (f,u)_{T_h} \, dt.
\]

Next, suppose that \( k_t = 1 \). In this case, eq. (59) does not offer direct control over \( \|u_h(s)\|_{L^2(\Omega)} \) for \( s \in (t_{n_s}, t_{n_s+1}) \). Instead, we define \( \tilde{u}_h \) to be piecewise constant (in time) function satisfying \( \tilde{u}_h|_{\mathcal{E}^m} = u_{m+1} = u_h(t_{m+1}) \), so that eq. (59) yields
\[
\|\tilde{u}_h(s)\|^2_{L^2(\Omega)} + 2\nu \sum_{i=1}^{3} \int_0^s \|G_h^k(u_h)\|^2_{L^2(\Omega)} \, dt \leq \|u_0\|^2_{L^2(\Omega)} + 2 \int_{t_{n_s+1}}^{t_{n_s+1}} (f,u_h)_{T_h} \, dt.
\]

Note that if \( u_h \rightharpoonup u \) in \( L^2(0,T;L^2(\Omega)^d) \) as \( h \to 0 \), then also \( \tilde{u}_h \rightharpoonup u \) in \( L^2(0,T;L^2(\Omega)^d) \) as \( h \to 0 \) \cite[Corollary 3.2]{49}. The weak lower semi-continuity of the norm in \( L^2(0,T;L^2(\Omega)^d) \) yields
\[
\int_0^T \|u(s)\|^2_{L^2(\Omega)} \phi(s) \, ds \leq \liminf_{h \to 0} \int_0^T \|\tilde{u}_h(s)\|^2_{L^2(\Omega)} \phi(s) \, ds.
\]
The remainder of the proof is identical to the case \( k_t = 0 \). \( \square \)

A. Discrete compactness of the velocity

In this section, we recall the discrete compactness theory for DG time stepping developed by Walkington in \cite{50} with a minor modification to fit the current non-conforming spatial discretization.

Remark A.1. To our knowledge, the compactness theorem in \cite[Theorem 3.1]{50} for DG-in-time discretizations was first extended to non-conforming spatial approximations in \cite{33}. Note that we can apply \cite[Theorem 3.2]{33} in our setting to conclude that the sequence \( \{u_h\}_{h \in \mathcal{H}} \) is relatively compact in \( L^2(0,T;L^2(\Omega)^d) \) by selecting (using the notation of \cite{33} with \( Y \) and \( X \) replacing \( V \) and \( H \) therein to avoid confusion):
\[
W = H_0^1(\Omega)^d, \quad W(T_h) = H^1(T_h)^d, \quad Y = [BV(\Omega)^d \cap L^4(\Omega)^d ; L^4(\Omega)^d]_{1/2},
\]
\[
X = L^2(\Omega)^d, \quad W' = H^{-1}(\Omega)^d, \quad W_h = V_h,
\]
where \( H^1(T_h)^d \) is the broken \( H^1 \) space equipped with the \( \|\cdot\|_{1,h} \)-norm \cite{7,14,15}, \( BV(\Omega)^d \) is the space of functions of bounded variation \cite{7}, and \( [Y_0,Y_1]_{\theta} \) denotes the complex interpolation between Banach spaces \( Y_0,Y_1 \) with exponent \( \theta \in (0,1) \) \cite{3}.

We present below a simple proof of a special case of \cite[Theorem 3.2]{33} that stays directly within the framework of broken polynomial spaces and their discrete functional analysis tools. This avoids the need to construct a non-conforming space that embeds compactly into \( L^2(\Omega)^d \) and is made possible by the following discrete Rellich–Kondrachov theorem valid for broken polynomial spaces \cite[Theorem 5.6]{14}:

Lemma A.1 (Discrete Rellich–Kondrachov). Let \( \mathcal{H} \subset (0,|\Omega|) \) be a countable set of mesh sizes whose unique accumulation point is 0. We assume \( \{(T_h,F_h)\}_{h \in \mathcal{H}} \) is a sequence of conforming and shape-regular simplicial meshes. Let \( \{v_h\}_{h>0} \) be a sequence in \( \{V_h\}_{h>0} \) bounded in the \( \|\cdot\|_{1,h} \)-norm. Then, for all \( 1 \leq q < \infty \) if \( d = 2 \) and \( 1 \leq q \leq 6 \) if \( d = 3 \), the sequence \( \{v_h\}_{h>0} \) is relatively compact in \( L^q(\Omega)^d \).
Theorem A.1 (Compactness). Let $\mathcal{H} \subset (0, |\Omega|)$ be a countable set of mesh sizes whose unique accumulation point is 0. Assume that $\{(\Sigma_h, \Sigma_h^0)\}_{h \in \mathcal{H}}$ is a sequence of conforming and quasi-uniform space-time prismatic meshes. Let $q = 4/d$. Let the sequence $\{u_h\}_{h \in \mathcal{H}}$ be such that for each $h \in \mathcal{H}$, $u_h \in \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h$. Then, $\{u_h\}_{h \in \mathcal{H}}$ is relatively compact in $L^2(0, T; L^2(\Omega)^d)$ if:

(i) $\{u_h\}_{h \in \mathcal{H}}$ is uniformly bounded in the sense that $\int_0^T \|u_h\|_v^2 \, dt \leq M$ for some $M > 0$ independent of the mesh parameters $h$ and $T$.

(ii) For each $h \in \mathcal{H}$, the following bound on the discrete time derivative of $u_h$ holds uniformly for $q' = 4/(4 - d)$:

$$\left| \int_0^T (\mathcal{D}_t^k(u_h), v_h)_{\mathcal{H}_h} \, dt \right| \lesssim \left( \int_0^T \|v_h\|_{v'}^q \, dt \right)^{1/q'}, \quad \forall v_h \in \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h.$$ 

Proof. The proof, which proceeds in three steps, follows closely the proof of [50, Theorem 3.1] with minor modifications.

**Step one (equicontinuity):** Step one follows exactly the proof of [50, Lemma 3.3]; here we show that the assumptions therein can be interpreted as a uniform bound on the discrete time derivative eq. (25). By definition, it holds that

$$\int_{I_n} (\partial_t u_h, v_h)_{\mathcal{H}_h} + ([u_h]_n, v_h^+)_{\mathcal{H}_h} = \int_{I_n} (\mathcal{D}_t^k(u_h), v_h)_{\mathcal{H}_h} \, dt, \quad \forall v_h \in \mathcal{Y}_h.$$ 

(A.1)

Comparing with [50, Lemma 3.3], we require $F_h : v_h \mapsto (\mathcal{D}_t^k(u_h), v_h)_{\mathcal{H}_h}$ to be uniformly bounded $L^q(0, T; (\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)^\prime)$, where $(\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)^\prime$ is the dual space of $\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h$. We show that assumption (ii) suffices. As the space $\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h$ and its dual are finite-dimensional (hence separable), we make the identification $L^q(0, T; (\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)^\prime) \cong L^{q'}(0, T; \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)$, and we have

$$\|F_h\|_{L^q(0, T; (\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)^\prime)} = \sup_{0 \neq v \in L^{q'}(0, T; \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)} \frac{\left| \int_0^T F_h(v) \, dt \right|}{\left( \int_0^T \|v\|_{v'}^q \, dt \right)^{1/q'}}.$$ 

(A.2)

Choose $F_h$ in eq. (A.2) to be the functional that maps for each $t \in [0, T]$,

$$L^{q'}(0, T; \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h) \ni (v, \bar{v}) \mapsto (\mathcal{D}_t^k(u_h), v)_{\mathcal{H}_h} \in \mathbb{R}.$$ 

Since $v \in L^{q'}(0, T; \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)$, we have $\Pi^t v \in \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h$, and the stability of the $L^2$-projection $\Pi^t$ in $L^q(I_n)$ [16] yields

$$\|F_h\|_{L^q(0, T; (\mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)^\prime)} \lesssim \sup_{v \in L^{q'}(0, T; \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h)} \frac{\left| \int_0^T (\mathcal{D}_t^k(u_h), \Pi^t v)_{\mathcal{H}_h} \, dt \right|}{\left( \int_0^T \|\Pi^t v\|_{v'}^q \, dt \right)^{1/q'}} \lesssim \sup_{v_h \in \mathcal{Y}_h^{\text{div}} \times \tilde{V}_h} \frac{\left| \int_0^T (\mathcal{D}_t^k(u_h), v_h)_{\mathcal{H}_h} \, dt \right|}{\left( \int_0^T \|v_h\|_{v'}^q \, dt \right)^{1/q'}} ,$$

which is uniformly bounded by assumption (ii) of the theorem. Proceeding in an identical fashion to [50, Lemma 3.3], we find that

$$\int_\delta^T \|u_h(t) - u_h(t - \delta)\|_{L^2(\Omega)}^2 \, dt \leq C \max(\tau, \delta)^{1/q'} \delta^{1/2}.$$ 

(A.3)
Step two (relative compactness in $L^2(\theta, T - \theta; L^2(\Omega)^d)$): We aim to show that for all $0 < \theta < T$, the set $\{ u_h |_{\theta,T-\theta} \mid h \in \mathcal{H} \}$ is relatively compact in $L^2(\theta, T - \theta; L^2(\Omega)^d)$. The proof is a minor modification of [50, Theorem 3.2]. To this end, we construct a sequence of regularized functions so that we may leverage the classical Arzelà–Ascoli theorem (see e.g. [46, Lemma 1]). Let $\phi \in C_c^\infty(-1, 1)$ be nonnegative with unit integral. For $\delta > 0$, set $\phi_\delta(s) = (1/\delta)\phi(s/\delta)$. Extend $u_h$ by zero outside of $[0, T]$ and consider the sequence of mollified functions $\{ u_h^\delta \}_{h \in \mathcal{H}}$, where $u_h^\delta(t) = \phi_\delta \ast u_h(t)$.

Since \( \int_0^T \| u_h^\delta(t) \|^2_{L^2} \, dt \) is uniformly bounded by assumption, we have

\[
\| u_h^\delta(t) \|^2_{L^2} \leq \delta \sup_{|s| < \delta} |\phi_\delta(t - s)|^2 \int_0^\delta \| u_h(s) \|^2_{L^2} \, ds \leq M^*,
\]

with $M^*$ a constant independent of $h$ and $\tau$. Thus, by Lemma A.1, the sequence $\{ u_h^\delta \}_{h \in \mathcal{H}}$ is relatively compact in $L^2(\Omega)^d$ for each $t \in [0, T]$. Furthermore, the uniform Lipschitz continuity of the mollifiers $\phi_\delta(s)$ ensures the sequence $\{ u_h^\delta \}_{h \in \mathcal{H}}$ is equicontinuous on $[0, T]$. By the Arzelà–Ascoli theorem, the sequence $\{ u_h^\delta \}_{h \in \mathcal{H}}$ is relatively compact in $C(0, T; L^2(\Omega)^d)$ and thus also $L^2(0, T; L^2(\Omega)^d)$ as the former embeds continuously into the latter. As relatively compact sets are totally bounded,

\[ \forall \epsilon > 0, \exists h_1, \ldots, h_M \in \mathcal{H} \text{ s.t. } \{ u_h^\delta \}_{h \in \mathcal{H}} \subset \bigcup_{i=1}^M B_\epsilon(u_{h_i}), \]

where $B_\epsilon$ is an $\epsilon$-ball in the metric induced by the $L^2(0, T; L^2(\Omega)^d)$-norm. The remainder of the proof that the set $\{ u_h |_{\theta,T-\theta} \mid h \in \mathcal{H} \}$ is relatively compact in $L^2(\theta, T - \theta; L^2(\Omega)^d)$ for all $0 < \theta < T/2$ is identical to that of [50, Theorem 3.2].

Step three (finishing up): The equicontinuity eq. (A.3) and [50, Lemma 3.4] ensure that the sequence $\{ u_h \}_{h \in \mathcal{H}}$ is bounded uniformly in $L^r(0, T; L^2(\Omega)^d)$ for $1 \leq r < 4$. Consequently, for all $\epsilon > 0$ we can find $\theta > 0$ such that

\[ \int_0^\theta \| u_h(t) \|^2_{L^2} \, dt + \int_{T-\theta}^T \| u_h(t) \|^2_{L^2} \, dt \leq \epsilon. \]

It follows that $\{ u_h \mid h \in \mathcal{H} \}$ is the uniform limit of relatively compact sets in $L^2(0, T; L^2(\Omega)^d)$ [46, Section 2]. Thus, $\{ u_h \mid h \in \mathcal{H} \}$ is relatively compact in $L^2(0, T; L^2(\Omega)^d)$. \[\square\]

B. Properties of the projections $\Pi$ and $\bar{\Pi}$

B.1. Approximation properties of $\Pi^i$ and $\Pi_{\alpha}^{\text{div}}$

Lemma B.1 (Approximation properties of $\Pi_{\alpha}^{\text{div}}$ and $\Pi^i$). Let $\ell \geq 0$ and suppose that $\eta \in W^{\ell+1,\infty}(0, T)$ and $\psi \in H^{\ell+1}(\Omega)^d$. Then, for all $n = 0, \ldots, N - 1$,

\[ \| u - \Pi^i u \|_{L^\infty(I_n)} \lesssim \tau^{\ell+1} |\eta|_{W^{\ell+1,\infty}(I_n)}. \quad (B.1) \]

25
and if \( T_h \) is conforming and quasi-uniform, we have for \( 0 \leq m \leq 2 \) and \( K \in T_h \),

\[
\sum_{K \in T_h} \| \psi - \Pi_h^{\text{div}} \psi \|_{H^m(K)}^2 \lesssim h^{2(\ell - m + 1)} | \psi |_{H^{\ell+1}(\Omega)}^2, \tag{B.2} \]

\[
\| \psi - \Pi_h^{\text{div}} \psi \|_{L^\infty(\Omega)} \lesssim h^{1/2} | \psi |_{H^2(\Omega)}, \tag{B.3} \]

the latter requiring \( \ell \geq 1 \).

**Proof.** Estimate eq. (B.1) is standard, see e.g. [18, Lemma 11.18]. The proof of eq. (B.2) can be found in [28]; we note that therein it is assumed that \( \ell \geq 1 \) but the proof easily extends to the case \( \ell = 0 \). We now show eq. (B.3). Let \( \hat{K} \) be the reference simplex in \( \mathbb{R}^d \) and suppose that \( F_K : \hat{K} \to K \) is an affine mapping; denote its Jacobian matrix by \( J_K \). As \( H^2(\hat{K}) \subset L^\infty(\hat{K}) \) with continuous embedding for \( d \leq 3 \), we have by repeated use of [18, Lemma 11.7]:

\[
\| \psi - \Pi_h^{\text{div}} \psi \|_{L^\infty(K)} \lesssim \| J_K \|_{L^2}^{-1/2} \| \psi - \Pi_h^{\text{div}} \psi \|_{H^2(K)}. \]

Since \( T_h \) is assumed quasi-uniform and hence shape-regular, we have \( \| J_K \|_{L^2} \lesssim h_K^2 \) and \( \| \psi \|_{H^2} \lesssim h_K^{-d/2} \) (see e.g. [18, Lemma 11.1], [17, Chapter 1.2]). Thus, for \( d \leq 3 \),

\[
\| \psi - \Pi_h^{\text{div}} \psi \|_{L^\infty(K)} \lesssim h^{1/2} | \psi |_{H^2(\Omega)}. \]

The result follows by noting that this bound holds uniformly for all \( K \in T_h \).

B.2. Proof of Proposition 4.1

It suffices to show the inequality in eq. (39) on a single space-time slab \( \mathcal{E}^n \); the result then follows by summing over all space-time slabs. Let \( \varphi \in \mathcal{M} \). By the definitions of the norm \( \| \cdot \|_v \) and the projections \( \Pi \varphi \) and \( \bar{\Pi} \varphi \) given in eq. (38), we have

\[
\| (\Pi \varphi, \bar{\Pi} \varphi) \|_{v}^{4/(4-d)} = \left( \sum_{K \in T_h} \int_K \left| \nabla \Pi_h^{\text{div}} \Pi \varphi \right|^2 \, dx + \sum_{K \in T_h} \int_{\partial K} \left| \Pi_h^{\text{div}} \varphi - \bar{\Pi}_h \varphi \right|^2 \, dS \right)^{2/(4-d)}.
\]

Available approximation results for the projection \( \Pi_h \) and eq. (B.2) yield for \( \Psi \in H^1(\Omega)^d \),

\[
\sum_{K \in T_h} \left( \| \nabla \Pi_h^{\text{div}} \Psi \|_{L^2(K)}^2 + h_K^{-1} \| (\Pi_h^{\text{div}} - \bar{\Pi}_h) \Psi \|_{L^2(\partial K)}^2 \right) \lesssim \| \nabla \Psi \|_{L^2(\Omega)}^2.
\]

Therefore, we have

\[
\| (\Pi \varphi, \bar{\Pi} \varphi) \|_{v}^{4/(4-d)} \lesssim \left( \int_{\Omega} \left| \Pi \sum_{k=1}^M \eta_k \nabla \psi_k \right|^2 \, dx \right)^{2/(4-d)}, \quad \forall \varphi \in \mathcal{M}. \tag{B.4}
\]

If \( d = 2 \), we can integrate eq. (B.4) over \( I_n \) and use Fubini’s theorem and the stability of the projection \( \Pi \) in \( L^2(I_n) \) to find

\[
\int_{I_n} \| (\Pi \varphi, \bar{\Pi} \varphi) \|_v^2 \, dt \lesssim \int_{I_n} \| \varphi \|_v^2 \, dt, \quad \forall \varphi \in \mathcal{M},
\]
as required. On the other hand, if $d = 3$, we integrate eq. (B.4) over $I_n$, and apply a finite-dimensional scaling argument between norms in $L^2(I_n)$ and $L^1(I_n)$ (see e.g. [14, Lemma 1.50]) to find:

$$
\int_{I_n} \| (\Pi \varphi, \bar{\Pi} \varphi) \|_V^4 \, dt \lesssim \tau^{-1} \left( \int_{I_n} \int_{\Omega} |\Pi' \sum_{k=1}^M \eta_k \nabla \psi_k|^2 \, dx \, dt \right)^2.
$$

(B.5)

Using Fubini’s theorem to interchange the temporal and spatial integrals in eq. (B.5) as necessary, we can apply the stability of the projection $\Pi'$ in the $L^2(I_n)$ norm followed by the Cauchy–Schwarz inequality applied to the temporal integral to find

$$
\int_{I_n} \| (\Pi \varphi, \bar{\Pi} \varphi) \|_V^4 \, dt \leq \tau^{-1} \left( \int_{I_n} \| \varphi \|_V^4 \, dt \right)^2 \lesssim \int_{I_n} \| \varphi \|_V^4 \, dt, \quad \forall \varphi \in \mathcal{M}.
$$

References

[1] N. Ahmed and G. Matthies, Higher-order discontinuous Galerkin time discretizations for the evolutionary Navier–Stokes equations,IMA J. Numer. Anal. (2020), draa053.

[2] S. Badia, R. Codina, T. Gudi, and J. Guzmán, Error analysis of discontinuous Galerkin methods for the Stokes problem under minimal regularity,IMA J. Numer. Anal. 34 (2014), no. 2, 800 – 819.

[3] J. Bergh and J. Löfström, Interpolation Spaces: an Introduction, Grundlehren der mathematischen Wissenschaften, vol. 223, Springer-Verlag Berlin Heidelberg, 2012.

[4] D. Boffi, F. Brezzi, and M. Fortin, Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol. 44, Springer, 2013.

[5] F. Boyer and P. Fabrie, Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models, Applied Mathematical Sciences, vol. 183, Springer, 2012.

[6] H. Brezis, Functional Analysis, Sobolev Spaces, and Partial Differential Equations, Springer–Verlag New York, 2011.

[7] A. Buffa and C. Ortner, Compact embeddings of broken Sobolev spaces and applications,IMA Journal of Numerical Analysis 29 (2009), 827–855.

[8] A. Cesmelioglu, B. Cockburn, and W. Qiu, Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier–Stokes equations, Math. Comp. 86 (2017), 1643–1670.

[9] K. Chrysafinos and N. J. Walkington, Discontinuous Galerkin approximations of the Stokes and Navier–Stokes equations, Math. Comp. 79 (2010), 2135–2167.

[10] B. Cockburn, G. Kanschat, and D. Schötzau, A note on discontinuous Galerkin divergence-free solutions of the Navier–Stokes equations, J. Sci. Comput. 31 (2007), no. 1/2, 61–73.

[11] P. Constantin and C. Foias, Navier–Stokes equations, Chicago Lectures in Mathematics, The University of Chicago Press, 1988.

[12] M. Dauge, Stationary Stokes and Navier–Stokes systems on two or three-dimensional domains with corners. Part I. Linearized equations,SIAM J. Math. Anal. 20 (1989), 74–97.

[13] D. A. Di Pietro and J. Droniou, The Hybrid High-Order method for polytopal meshes, Modeling, Simulation and Application, no. 19, Springer–Verlag International Publishing, 2020.

[14] D. A. Di Pietro and A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods, Mathématiques et Applications, vol. 69, Springer–Verlag Berlin Heidelberg, 2012.

[15] V. Dolejší and M. Feistauer, Discontinuous Galerkin Method, Springer Series in Computational Mathematics, no. 48, Springer International Publishing, 2015.

[16] J. Douglas Jr., T. Dupont, and L. Wahlbin, The stability in $L^q$ of the $L^2$–projection into finite element function spaces, Numer. Math. 23 (1975), 193–197.

[17] S. Du and J. F. Sayas, An Invitation to the Theory of the Hybridizable Discontinuous Galerkin Method: Projections, Estimates, Tools, Springer Briefs in Mathematics, Springer International Publishing, 2019.

[18] A. Ern and J. L. Guermond, Finite Elements I, Texts in Applied Mathematics, no. 72, Springer International Publishing, 2021.

[19] A. Ern, Finite Elements III, Texts in Applied Mathematics, no. 74, Springer International Publishing, 2021.

[20] G. Fu, An explicit divergence-free DG method for incompressible flow, Comput. Methods Appl. Mech. Engrg. 345 (2019), 502–517.
[21] J. Guzmán and M. Neilan, *Conforming and divergence-free Stokes elements in three dimensions*, IMA J. Numer. Anal. **34** (2014), 1489–1508.

[22] J. Guzmán and M. Neilan, *Conforming and divergence-free Stokes elements on general triangular meshes*, Math. Comp. **83** (2014), 15–36.

[23] P. Hood and C. Taylor, *Navier–Stokes equations using mixed interpolation*, Finite Element Methods in Flow Problems, University of Alabama in Huntsville Press, 1974, pp. 121–132.

[24] T. L. Horváth and S. Rhebergen, *A locally conservative and energy-stable finite-element method for the Navier–Stokes problem on time-dependent domains*, Int. J. Numer. Meth. Fluids **89** (2019), no. 12, 519–532.

[25] An exactly mass conserving space-time embedded-hybridized discontinuous Galerkin method for the Navier–Stokes equations on moving domains, J. Comput. Phys. **417** (2020).

[26] V. John, A. Linke, C. Merdon, M. Neilan, and L. G. Rebholz, *On the divergence constraint in mixed finite element methods for incompressible flows*, SIAM Rev. **59** (2017), no. 3, 492–544.

[27] F. Kikuchi, *Rellich-type discrete compactness for some discontinuous Galerkin FEM*, Japan J. Indust. Appl. Math. **29** (2012), 269–288.

[28] K. L. Kirk, T. L. Horváth, and S. Rhebergen, *Analysis of an exactly mass conserving space-time hybridized discontinuous Galerkin method for the time-dependent Navier–Stokes equations*, 2021.

[29] K. L. A. Kirk and S. Rhebergen, *Analysis of a pressure-robust hybridized discontinuous Galerkin method for the stationary Navier–Stokes equations*, J. Sci. Comput. **81** (2019), 881–897.

[30] P. L. Lederer, C. Lehrenfeld, and J. Schöberl, *Hybrid discontinuous Galerkin methods with relaxed $H(div)$-conformity for incompressible flows. Part I*, SIAM J. Numer. Anal. **56** (2018), no. 4, 2070–2094.

[31] P. L. Lederer, A. Linke, C. Merdon, and J. Schöberl, *Divergence-free reconstruction operators for pressure-robust Stokes discretizations with continuous pressure finite elements*, SIAM J. Numer. Anal. **55** (2017), no. 3, 1291–1314.

[32] C. Lehrenfeld and J. Schöberl, *High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows*, Comput. Methods Appl. Mech. Engrg. **307** (2016), 339–361.

[33] J. Li, B. Riviere, and N. J. Walkington, *Convergence of a high order method in time and space for the miscible displacement equations*, ESAIM: Mathematical Modelling and Numerical Analysis **49** (2015), no. 4, 953–976.

[34] A. Linke, *On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime*, Comput. Methods Appl. Mech. Engrg. **268** (2014), 782–800.

[35] A. Masud and T. Hughes, *A space-time Galerkin/least–squares finite element formulation of the Navier–Stokes equations for moving domain problems*, Comput. Methods Appl. Mech. Engrg. **146** (1997), 91–126.

[36] I. Oikawa, *Hybridized discontinuous Galerkin method with lifting operator*, JSIAM Letters **2** (2010), 99–102.

[37] D. A. Di Pietro and A. Ern, *Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations*, Math. Comp. **79** (2010), 1303–1330.

[38] S. Rhebergen and B. Cockburn, *A space–time hybridizable discontinuous Galerkin method for incompressible flows on deforming domains*, J. Comput. Phys. **251** (2012), no. 11, 4185–4204.

[39] S. Rhebergen, B. Cockburn, and J. J. W. van der Vegt, *A space–time discontinuous Galerkin method for the incompressible Navier–Stokes equations*, J. Comput. Phys. **233** (2013), 339–358.

[40] S. Rhebergen and G. N. Wells, *Analysis of a hybridized/interface stabilized finite element method for the Stokes equations*, SIAM J. Numer. Anal. **55** (2017), no. 4, 1982–2003.

[41] S. Rhebergen and G. N. Wells, *A hybridizable discontinuous Galerkin method for the Navier–Stokes equations with pointwise divergence-free velocity field*, J. Sci. Comput. **76** (2018), no. 3, 1484–1501.

[42] T. Roubíček, *Nonlinear Partial Differential Equations with Applications*, 2 ed., International Series of Numerical Mathematics, no. 153, Birkhäuser Basel, 2013.

[43] D. Schötzau and T. P. Wihler, *A posteriori error estimates for the hp-version time-stepping methods for parabolic partial differential equations*, Numer. Math. **115** (2010), 475–509.

[44] P. W. Schroeder, C. Lehrenfeld, A. Linke, and G. Lube, *Towards computable flows and robust estimates for inf-sup stable FEM applied to the time-dependent incompressible Navier–Stokes equations*, SeMA J. **75** (2018), 629–653.

[45] L. R. Scott and M. Vogelius, *Conforming finite element methods for incompressible and nearly incompressible continua*, Large-Scale Computations in Fluid Mechanics, Part 2 (La Jolla, CA, 1983), Lectures in Appl. Math. 22., AMS, Providence, RI, 1985, pp. 221–244.

[46] Jacques Simon, *Compact sets in the space $L^p(0; T; B)$*, Annali di Matematica pura ed applicata **146** (1986), no. 1, 65–96.

[47] R. Temam, *Navier–Stokes Equations*, Third (revised) ed., Elsevier Science Publishers B.V., Amsterdam, 1984.

[48] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer Series in Computational Math-
mathematics, vol. 25, Springer–Verlag Berlin Heidelberg, 2006.

[49] N. J. Walkington, *Convergence of the discontinuous Galerkin method for discontinuous solutions*, SIAM J. Numer. Anal. **42** (2005), 1801–1817.

[50] ———, *Compactness Properties of the DG and CG Time Stepping Schemes for Parabolic Equations*, SIAM J. Numer. Anal. **47** (2010), no. 6, 4680–4710.