Mathematical justification of the point vortex dynamics in background fields on surfaces as an Euler–Arnold flow

Yuuki Shimizu

Received: 25 October 2021 / Revised: 30 June 2022 / Accepted: 10 July 2022 / Published online: 4 August 2022
© The JJIAM Publishing Committee and Springer Japan KK, part of Springer Nature 2022

Abstract
The point vortex dynamics in background fields on surfaces is justified as an Euler–Arnold flow in the sense of de Rham currents. We formulate a current-valued solution of the Euler–Arnold equation with a regular-singular decomposition. For the solution, we first prove that, if the singular part of the vorticity is given by a linear combination of delta functions centered at \( q_n(t) \) for \( n = 1, \ldots, N \), \( q_n(t) \) is a solution of the point vortex equation. Conversely, we next prove that, if \( q_n(t) \) is a solution of the point vortex equation for \( n = 1, \ldots, N \), there exists a current-valued solution of the Euler–Arnold equation with a regular-singular decomposition such that the singular part of the vorticity is given by a linear combination of delta functions centered at \( q_n(t) \). As a corollary, we generalize the Bernoulli law to the case where the flow field is a curved surface and where the presence of point vortices is taken into account. From the viewpoint of the application, the mathematical justification is of significance since the point vortex dynamics in the rotational vector field on the unit sphere is regarded as a mathematical model of geophysical flow in order to take effect of the Coriolis force on inviscid flows into consideration.

Keyword Euler equations · Point vortex dynamics · De Rham current

Mathematics subject classification Primary 35Q31 · Secondary 76B47

1 Introduction
The motion of incompressible and inviscid fluids in the Euclidean plane is governed by the Euler equation,
\[ \partial_t v_t + (v_t \cdot \nabla)v_t = -\text{grad}p_t, \quad \text{div} v_t = 0, \]

where \( (v_t \cdot \nabla) = v_t^1 \partial_1 + v_t^2 \partial_2 \). The fluid velocity field \( v_t \in \mathcal{X}(\mathbb{R}^2) \) and the scalar pressure \( p_t \in C(\mathbb{R}^2) \) are the unknowns at time \( t \in (0,T) \). A solution of the Euler equation (1) is called an Euler flow. Applying the curl operator to the Euler equation, we obtain the following equation for the vorticity \( \omega_t = \text{curl} v_t = \partial_1 v_t^2 - \partial_2 v_t^1 \):

\[ \partial_t \omega_t + (v_t \cdot \nabla)\omega_t = 0, \]

which is called the vorticity equation. Owing to \( \text{div} v_t = 0 \), the velocity becomes a Hamiltonian vector field, i.e., \( v_t = -\mathcal{J} \text{grad} \psi_t = (\partial_2 \psi_t, -\partial_1 \psi_t) \) for some \( \psi_t \in C(\mathbb{R}^2) \), where \( \mathcal{J} \) is the symplectic matrix. The unknowns \( \psi_t, v_t \) and \( p_t \) are presented by \( \omega_t \). Since \( \psi_t \) satisfies

\[ \omega_t = -\text{curl} \mathcal{J} \text{grad} \psi_t = -(\partial_1^2 + \partial_2^2) \psi_t = -\Delta \psi_t, \]

it is given by \( \psi_t = \langle G_H, \omega_t \rangle \) and \( v_t = -\mathcal{J} \text{grad}(G_H, \omega_t) \), where \( G_H \) is the Green function for the Laplacian \(-\Delta\). To determine \( p_t \) from \( v_t \), Applying the divergence operator to the Eq. (1), we have

\[ \text{div}(v_t \cdot \nabla) = -\text{div} \text{grad}p_t = -\Delta p_t. \]

Hence, the pressure is determined as a solution of this Poisson equation. We thus obtain

\[ v_t = -\mathcal{J} \text{grad}(G_H, \omega_t), \quad p_t = \langle G_H, \text{div}(v_t \cdot \nabla)v_t \rangle. \]

Note that it is derived under the assumption that \( (v_t, p_t) \) is an Euler flow.

On the other hand, the formulae (2) still make sense in the sense of distributions when we give a time-dependent distribution \( \Omega_t = \sum_{n=1}^N \Gamma_n \delta_{q_n(t)} \) as a linear combination of delta functions centered at \( q_n(t) \) for \( n = 1, \ldots, N \). Replacing \( \omega_t \) by \( \Omega_t \) in the Eq. (2), we formally obtain a velocity field \( V_t \in \mathcal{X}(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N) \) and a pressure \( \Pi_t \in C(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N) \). However, the pair \((V_t, \Pi_t)\) is no longer an Euler flow in a regular sense, since it does not belong to \( L^2_{\text{loc}}(\mathbb{R}^2) \). Hence, we cannot define the dynamics of \( q_n(t) \) from the Euler equation. Instead, to determine the evolution of \( q_n(t) \) by \( V_t \), Helmholtz considered the following regularized equation for \( q_n(t) \) \[27\].

\[ \dot{q}_n = \lim_{q \to q_n} \left[ V_t(q) + \mathcal{J} \text{grad}(G_H, \Gamma_n \delta_{q_n(t)})(q) \right] \]

\[ = -\mathcal{J} \text{grad} \sum_{m=1}^N \Gamma_m G_H(q_n, q_m) \equiv v_n(q_n). \]

It is called the point vortex equation, and the solution of (3) is called the point vortex dynamics. Then, there arises a natural question; How can we interpret \((V_t, \Pi_t)\) as an Euler flow in an appropriate mathematical sense?

The research on connection between the point vortex dynamics and Euler flows was pioneered by Marchiolo and Pulvirenti \[21\]. They proved that the point vortex

\[ \text{Springer} \]
Euler–Arnold flow in the sense of de Rham currents
dynamics can be regarded as a limit of $L^\infty$ solutions of the Euler equation whose initial vorticity is concentrated in sufficiently small region. A good reference for the research in this direction is given in [22]. As another connection, it has been recently demonstrated that the point vortex dynamics can be obtained as a limit of the motion of a rigid body in an Euler flow when the radius and mass of the rigid body with fixed circulation go to zero [15]. A similar result in this direction has been also obtained by a different method when the flow field is a general curved surface [14]. Although there are numerous studies that connect point vortex dynamics with Euler flows, the space of solutions to Euler equations that directly include point vortex dynamics has not been constructed. Delort establishes the global existence of weak solutions to the Euler equation such that the initial vorticity is a bounded Radon measure with compact support and with distinguished sign and in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$ [6]. The Delort solution contains a large class of examples in which the support of the initial vorticity consists of a curve, known as vortex sheets. However, the point vortices, where the support of the initial vorticity is a set of points, can not be applied to the Delort solution due to the fact that the delta function is not in $H^{-1}_{\text{loc}}(\mathbb{R}^2)$, i.e., the velocity field whose vorticity is the delta function is not in $L^2_{\text{loc}}(\mathbb{R}^2)$. Based on the method of constructing the Delort solution, the point vortex dynamics is characterized as a discretization of Euler flows [31]. Thus, due to the singularity of the vorticity and the nonlinearity of the Euler equation, the justification of the point vortex dynamics as an Euler flow is an important but difficult problem in the analysis of the two-dimensional Euler equation.

The point vortex dynamics is sometimes considered in the presence of the velocity field $X_t \in \mathfrak{X}(\mathbb{R}^2)$ of an Euler flow, in which the evolution of $q_n(t)$ is governed by the following equation.

$$\dot{q}_n(t) = \beta X_t(q_n(t)) + \beta \omega v_n(q_n(t)), \quad n = 1, \ldots, N$$

for a given pair of parameters $(\beta X, \beta \omega) \in \mathbb{R}^2$. The velocity field $X_t$ is called the background field. An experimental study confirm the importance of background fields in two-dimensional turbulence [34]. In the same way as in the absence of a background field, a velocity field $V_t \in \mathfrak{X}(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$ and a pressure $\Pi_t \in C(\mathbb{R}^2 \setminus \{q_n(t)\}_{n=1}^N)$ are defined:

$$V_t(q) = X_t(q) - \mathcal{J} \text{grad} \sum_{n=1}^N \Gamma_n G_H(q, q_n(t)), \quad \Pi_t = \langle G_H, \text{div}(V_t \cdot \nabla) \rangle.$$

Even if the regularity of the background field is simply of class $C^r$, it remains open whether the pair is a weak solution of the Euler equation as it is open in the case with no background field.

The purpose of this paper is justifying the point vortex dynamics in background fields as an Euler flow by constructing a space of weak solutions to the Euler equation which contains the point vortex dynamics. To this end, we establish a weak formulation of the Euler equation in the space of currents, which is developed in the theory of geometric analysis and geometric measure theory. The notion of currents
is defined not only for the Euclidean plane but also general curved surface. Hence, the formulation established here can be naturally generalized for surfaces. The Euler equation is generalized for the case of Riemannian manifolds by Arnold [1], called the Euler–Arnold equation. From the viewpoint of the application, it is of significance to justify the point vortex dynamics in a background field on curved surfaces as an Euler–Arnold flow, since the point vortex dynamics in the rotational vector field on the unit sphere is regarded as a mathematical model of a geophysical flow in order to take effect of the Coriolis force on inviscid flows into consideration [24] for instance. As for prerequisites, it is helpful to be familiar with the theory of manifolds and calculus of differential forms, although some basic notions such as vector calculus on curved surfaces related with fluid dynamics, de Rham currents, and the Euler–Arnold equation are reviewed in this paper.

This paper is organized as follows. In Sect. 2, we collect some definitions and conventions from differential geometry that will be used throughout the paper. In Sect. 3, we derive the point vortex dynamics on curved surfaces from the Euler–Arnold equation in the similar manner to the case of the Euclidean plane. In Sect. 4, we review basic concepts of the theory of de Rham currents. Some notions in vector calculus are reformulated in terms of currents for the application to fluid dynamics. In Sect. 5, we examine the Euler–Arnold equation for more details from the viewpoint of currents. We formulate a current-valued solution of the Euler–Arnold equation on surfaces with a regular-singular decomposition, which we call a $C^r$-decomposable weak Euler–Arnold flow. In Sect. 6, our main results are stated and proved. In the first theorem, we prove that, for a given $C^r$-decomposable weak Euler–Arnold flow, if the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$ for $n = 1, \ldots, N$, $q_n(t)$ is a solution of the point vortex equation (31) Conversely, in the second theorem, we next prove that, if $q_n(t)$ is a solution of the point vortex equation (31) for $n = 1, \ldots, N$, there exists a $C^r$-decomposable weak Euler–Arnold flow such that the singular part of the vorticity is given by a linear combination of delta functions centered at $q_n(t)$. As a consequence, the point vortex dynamics in a background field on a surface is justified as a $C^r$-decomposable weak Euler–Arnold flow. In Sect. 7, we apply the main results to some problems in fluid dynamics. As a corollary, we generalize the Bernoulli law to the case where the flow field is a curved surface and where the presence of point vortices is taken into account.

2 Background materials

Let $(M, g)$ be a connected orientable 2-dimensional Riemannian manifold, called a surface, with or without boundary, which can be compact or non-compact. Here, $g$ denotes the Riemannian metric. Let $\mathfrak{X}(M)$ be the space of all vector fields on $M$, that is, measurable sections of the tangent bundle of $M$. Let $\Omega^p(M)$ be the space of all $p$-forms on $M$, that is, measurable section of the $p$-th exterior power of the cotangent bundle. Let $\mathcal{D}^p(M)$ be the space of all smooth $p$-forms with compact support. In this section, every vector field and every differential form are smoothly defined on $M$. 

Springer
By abuse of notation, the same symbol \( p \) is often used for the element of \( M \) and the order of differential forms and currents throughout this paper.

The Riemannian metric defines the musical isomorphism between \( \mathfrak{X}(M) \) and \( \Omega^1(M) \). Every vector field is converted to a 1-form by the flat operator \( b : X \in \mathfrak{X}(M) \to X^b = g(X, \cdot) \in \Omega^1(M) \). The inverse of the flat operator is called the sharp operator \( \sharp : \alpha \in \Omega^1(M) \to \alpha_\sharp \in \mathfrak{X}(M) \). Then, the 1-form \( X^\flat \) and the vector field \( \alpha_\sharp \) is called the velocity form and the dual vector field. Moreover, the Riemannian metric, which is defined on the tangent bundle is extended to the metric \( \tilde{g} \) on the cotangent bundle such that \( \tilde{g}(\alpha, \beta) = g(\alpha_\sharp, \beta_\sharp) \) for all \( \alpha, \beta \in \Omega^1(M) \). For simplicity of notation, we use the same letter \( g \) for \( \tilde{g} \).

Let us examine the musical isomorphism by using a local frame. The local frame \( (e_1, e_2) \) is defined as two vector fields on some open subset \( U \) such that for each \( p \in U, (e_1(p), e_2(p)) \) forms an orthonormal basis for \( T_p M \). Associated with the local frame, a pair of two 1-forms \( (\theta^1, \theta^2) \) on \( U \) is defined as the dual basis of \( T_p^\ast M \) and called the dual coframe. Then, we have

\[
g(e_i, e_j) = \delta_{i,j}, \quad \theta^i(e_j) = \delta^i_j,
\]

where \( \delta_{i,j} \) and \( \delta^i_j \) denote the Kronecker delta. It follows from \( \theta^i = e^\flat_i \) and \( e_j = \theta^j_\flat \) that for each vector field \( X = X^1 e_1 + X^2 e_2 \in \mathfrak{X}(U) \) and each 1-form \( \alpha = \alpha_1 \theta^1 + \alpha_2 \theta^2 \in \Omega^1(M) \),

\[
X^\flat = X^1 \theta^1 + X^2 \theta^2, \quad \alpha_\sharp = \alpha_1 e_1 + \alpha_2 e_2.
\]

A 2-form \( \text{dVol}_g \in \Omega^2(M) \) is called the Riemannian volume form if for each oriented local frame \( (e_1, e_2) \), \( \text{dVol}_g(e_1, e_2) = 1 \). Using the dual coframe \( (\theta^1, \theta^2) \), we can write \( \text{dVol}_g \) as \( \text{dVol}_g = \theta^1 \wedge \theta^2 \) on \( U \). The Riemannian volume form on surfaces is the differential form corresponding to the area element. The area of \( M \) is defined by \( \int_M \text{dVol}_g \). The \( L^2 \) inner product on \( \mathfrak{X}(M) \) is defined by

\[
(X, Y)_{L^2} = \int_M g(X, Y) \text{dVol}_g
\]

for each \( X, Y \in \mathfrak{X}(M) \).

The Hodge-\( * \) operator is defined as \( \alpha \wedge * \beta = g(\alpha, \beta) \text{dVol}_g \) for all \( \alpha, \beta \in \Omega^\rho(M) \). For each dual coframe \( (\theta_1, \theta_2) \), we have

\[
* 1 = \text{dVol}_g, \quad * \text{dVol}_g = 1,
\]

\[
* \theta_1 = \theta_2, \quad * \theta_2 = -\theta_1.
\]

Owing to the orientability of \( M \), the Hodge-\( * \) operator induces an isomorphism \( \mathcal{J} : TM \to TM \) such that \( \mathcal{J} X = (\ast X^\flat)_\sharp \) for each \( X \in \mathfrak{X}(M) \), which is called the complex structure. In particular, \( \mathcal{J} \) rotates vectors by the degree \( +\pi/2 \). We will see \( \mathcal{J} \) is a generalization of the symplectic matrix in the plane to the curved surface. Let \( d \) denote the differential operator \( d : \Omega^\rho(M) \to \Omega^\rho+1(M) \). The codifferential operator is defined by \( \delta = * d * : \Omega^\rho(M) \to \Omega^{\rho-1}(M) \). We denote the Hodge Laplacian by \( \Delta = d\delta + \delta d : \Omega^\rho(M) \to \Omega^\rho(M) \). The \( L^2 \) inner product on \( \Omega^\rho(M) \) is given by
\[(\alpha, \beta)_{L^2} = \int_M \alpha \wedge * \beta \]

for each \(\alpha, \beta \in \Omega^p(M)\). By definition, we have

\[(X, Y)_{L^2} = (X^\flat, Y^\flat)_{L^2}\]  \hspace{1cm} (4)

for each \(X, Y \in \mathfrak{X}(M)\).

**Remark 1** Let us derive a useful presentation of the Hodge-* operator. For each \(p \in M\), there exists a geodesic polar coordinate \((\rho, \theta) = (d(p, q), \theta)\) in the neighborhood \(V\) of \(p\) such that

\[g = (d\rho)^2 + \rho^2(1 + \rho h)^2(d\theta)^2\]

for some \(h \in C^\infty(V)\) (see, for instance, [11]). In this coordinate, the Riemannian volume form is written as

\[d \text{Vol}_g = \rho(1 + \rho h) d\rho \wedge d\theta.\]

Hence, we obtain

\[* d \log \rho = \rho^{-1} d \rho = (1 + \rho h) d\theta.\]

In particular, if \(q \in B_\varepsilon(p)\) for sufficiently small \(\varepsilon > 0\), we deduce

\[* d \log \rho = d \theta + \varepsilon h d\theta = d \theta + O(\varepsilon). \]  \hspace{1cm} (5)

Some notions in vector calculus such as the divergence, the curl, the gradient and the Hamiltonian vector field are rewritten in terms of differential forms. Let us define the divergence operator \(\text{div} : \mathfrak{X}(M) \to \Omega^0(M)\) and the curl operator \(\text{curl} : \mathfrak{X}(M) \to \Omega^0(M)\) by

\[\text{div} X = \delta X^\flat, \quad \text{curl} X = * d X^\flat. \] \hspace{1cm} (6)

Note that \(\Omega^0(M) = C^\infty(M)\). A vector field \(X \in \mathfrak{X}(M)\) is said to be incompressible (or divergence free) if \(\text{div} X = 0\). Given \(X \in \mathfrak{X}(M)\), we call the function \(\omega = \text{curl} X\) the vorticity. A vector field \(X \in \mathfrak{X}(M)\) is said to be irrotational if \(\omega = 0\) on \(M\). Let us define the gradient operator \(\text{grad} : \Omega^0(M) \to \mathfrak{X}(M)\) by

\[\text{grad} \phi = (d\phi)^\sharp.\]

A vector field \(X \in \mathfrak{X}(M)\) is called a gradient vector field if there exists a function \(\phi \in \Omega^0(M)\) such that \(X^\flat = d\phi\), i.e., \(X = \text{grad} \phi\). In contrast, a vector field \(X \in \mathfrak{X}(M)\) is called a Hamiltonian vector field if there exists a function \(\psi \in \Omega^0(M)\) such that \(X^\flat = -* d\psi\), i.e., \(X = - \mathcal{J} \text{grad} \psi\). Here, the symplectic form on the surface is taken as the Riemannian volume form. By definition, we obtain
Euler–Arnold flow in the sense of de Rham currents

Hence, the vorticity $\omega$ of $X = -J \nabla \psi$ satisfies

$$\omega = -\nabla^2 \psi,$$

which yields that $\psi$ is determined by a solution of the Poisson equation (7). Owing to the slip boundary condition for $X$, $\psi$ obeys the Dirichlet boundary condition with a constant boundary value, since

$$d\psi = * X^0 = 0 \quad \text{on } \partial M.$$ 

We define a function $G_H \in C^\infty(M \times M \setminus \Delta)$ as the fundamental solution of the Poisson problem and call it the hydrodynamic Green function, where $\Delta$ is the diagonal set $\Delta = \{(x, x_0) \in M \times M | x = x_0\}$. 

**Definition 1** A function $G_H \in C^\infty(M \times M \setminus \Delta)$ is called a hydrodynamic Green function, if the function $G_H$ is a solution of the following boundary value problem. For each $(x, x_0) \in M \times M \setminus \Delta$ and each $\phi \in \mathcal{D}_1(M),$

$$-\nabla^2 G_H(x, x_0)[\phi] = \begin{cases} * \phi(x_0) - \frac{1}{\text{Area}(M)} \int_M \phi, & \text{if } M \text{ is a closed surface}, \\ * \phi(x), & \text{otherwise}, \end{cases}$$

$G_H(x, x_0) = G_H(x_0, x).$

If $M$ has a boundary, for each $x_0 \in \partial M$ and each boundary component $\gamma \subset \partial M$, there exists a constant $c(x_0, \gamma)$ depending on $x_0$ and $\gamma$ such that $G_H(x, x_0) = c(x_0, \gamma)$ for each $x \in \gamma$.

Let us note that a closed surface stands for a compact surface without boundary. A regularized hydrodynamic Green function, called Robin function $R \in C^\infty(M)$ [10, 14], is defined by

$$R(x) = \lim_{x_0 \to x} \left[ G_H(x, x_0) + \frac{1}{2\pi} \log d(x, x_0) \right],$$

where $d \in C^\infty(M \times M)$ is the geodesic distance on $M$. For each Hamiltonian vector field $X = -\nabla \psi$, by solving the Poisson problem (7), we obtain $\psi = \psi^0 + \langle G_H, \omega \rangle$ for some harmonic function $\psi^0$, which gives the Biot–Savart law on the surface.

$$X = -\nabla \psi = -\nabla (\psi^0 + \langle G_H, \omega \rangle).$$

Since the point vortex dynamics is formulated with a hydrodynamic Green function, it is necessary for the Green’s function to exist. One the other hand, there exists a surface with no Green function. Hence, throughout the paper, we assume that for a given surface $(M, g)$, there exists a hydrodynamic Green function. For instance, if
the surface is compact, the existence is guaranteed in [3]. Otherwise, the existence is discussed from the viewpoint of complex analysis [30]. We note that the hydrodynamic Green function \( G_M \) is determined up to harmonic functions which satisfy the Dirichlet constant boundary condition. Hence, according to the choice of the boundary value and asymptotic behavior near ends of surfaces, there are several definitions of the hydrodynamic Green function [10, 14]. In addition, we assume that if \( M \) is a surface with boundary, every vector field \( X \in \mathcal{X}(M) \) satisfies the slip boundary condition on \( \partial M \), that is \( X|_{\partial M} \in \mathcal{X}(\partial M) \). Note that the slip boundary condition is also written as \( \ast X^b = 0 \) in \( \Omega^1(\partial M) \).

**Remark 2** Let us recall that in the plane, every incompressible vector field becomes a Hamiltonian vector field. However, owing to the existence of genus, there exists an incompressible vector field which is not a Hamiltonian vector field. To see this, let us review the notion of (co)closedness, (co)exactness and the de Rham cohomology group. A \( p \)-form \( \alpha \in \Omega^p(M) \) is said to be closed if \( d\alpha = 0 \). A \( p \)-form \( \alpha \in \Omega^p(M) \) is said to be exact if there exists a \( (p-1) \)-form \( \beta \in \Omega^{p-1}(M) \) such that \( \alpha = d\beta \). In contrast, a \( p \)-form \( \alpha \in \Omega^p(M) \) is said to be coexact if there exists a \( (p+1) \)-form \( \beta \in \Omega^{p+1}(M) \) such that \( \alpha = \delta\beta \). Owing to \( \delta = * d * \), the condition \( \delta\alpha = 0 \) is equivalent to the condition \( d * \alpha = 0 \). Thus, a \( p \)-form \( \alpha \in \Omega^p(M) \) is coclosed if and only if \( * \alpha \) is closed. For the same reason, a \( p \)-form \( \alpha \in \Omega^p(M) \) is coexact if and only if \( * \alpha \) is exact.

Using these notions, we see that a vector field \( X \in \mathcal{X}(M) \) is irrotational if and only if the velocity form \( X^b \) is a closed 1-form. A vector field \( X \in \mathcal{X}(M) \) is a gradient vector field if and only if the velocity form \( X^b \) is exact. Similarly, a vector field \( X \in \mathcal{X}(M) \) is incompressible if and only if the velocity form \( X^b \) is coclosed, or equivalently, the 1-form \( * X^b \) is closed. A vector field \( X \in \mathcal{X}(M) \) is a Hamiltonian vector field if and only if the velocity form \( X^b \) is coexact, or equivalently, the 1-form \( * X^b \) is exact. Owing to \( d\delta \omega = \delta d\omega \), if the velocity form \( X^b \) is (co)exact, then \( X^b \) is (co)closed. Hence, every gradient vector field is irrotational as well as every Hamiltonian vector field is incompressible.

On the other hand, the converse does not always hold true. The notion that describes this gap quantitatively is the de Rham cohomology group. The \( p \)-th de Rham cohomology group, say \( H^p(M) \), is defined by the quotient vector space of the space of all closed \( p \)-forms over the space of all exact \( p \)-forms, i.e., \( H^p(M) = \text{Kerd/Imd} \). Clearly, every closed 1-form \( \alpha \) is exact if and only if \( H^1(M) = 0 \). Hence, for each irrotational (resp. incompressible) vector field \( X \in \mathcal{X}(M) \), since \( X^b \) (resp. \( * X^b \)) is closed, the irrotational (resp. incompressible) vector field becomes a gradient (resp. Hamiltonian) vector field if and only if \( H^1(M) = 0 \). For example, when \( M \) is the plane, by Poincaré lemma, we obtain \( H^1(\mathbb{R}^2) = 0 \). On the other hand, when the genus of \( M \) is more than 0, or the number of connected components of the boundary of \( M \) is more than 1, \( H^1(M) \) is not equal to 0 [30]. Note that this problem arises even for multiply connected domains in the plane.
Let us check the above notions are consistent with the notions in the case where the surface is given as the Euclidean plane $(\mathbb{R}^2, (dx^1)^2 + (dx^2)^2)$. Every vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ is conventionally written as a vector-valued function $X = \gamma(X^1, X^2)$. However, in the theory of manifolds, it is rewritten as $X = X^1 \partial_1 + X^2 \partial_2$ by using an orthogonal basis $(\partial_1, \partial_2)$ of the tangent space of $\mathbb{R}^2$. Here, the orthogonal basis is identified with the partial derivative. Writing the dual basis in the cotangent space $X^\flat \in \Theta^1(\mathbb{R}^2)$ by using an orthogonal basis $(\partial_1, \partial_2)$ of the tangent space of $\mathbb{R}^2$, we obtain the velocity form $X^\flat = X^1 \partial_1 + X^2 \partial_2 \in \mathfrak{X}(\mathbb{R}^2)$ and the dual vector field $\alpha_\flat = \alpha_1 \partial_1 + \alpha_2 \partial_2 \in \mathfrak{X}(\mathbb{R}^2)$ for each $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 \in \Omega^1(\mathbb{R}^2)$. Obviously, the Euclidean metric, which is $(X, Y) = X^1 Y^1 + X^2 Y^2$ for each $X, Y \in \mathfrak{X}(\mathbb{R}^2)$, is extended to the metric on the cotangent bundle such that $(\alpha, \beta) = \alpha^1 \beta^1 + \alpha^2 \beta^2$ for each $\alpha, \beta \in \Omega^1(\mathbb{R}^2)$. For each $p$-form $\alpha \in \Omega^p(\mathbb{R}^2)$, $p = 0, 1, 2$, $d\alpha \in \Omega^{p+1}(\mathbb{R}^2)$ and $\ast \alpha \in \Omega^{2-p}(\mathbb{R}^2)$ satisfy

$$d\alpha = \begin{cases} \partial_1 \alpha dx^1 + \partial_2 \alpha dx^2, & \text{if } p = 0, \\ (\partial_1 \alpha_2 - \partial_2 \alpha_1) dx^1 \wedge dx^2, & \text{if } p = 1, \\ 0, & \text{if } p = 2, \end{cases}$$

$$\ast \alpha = \begin{cases} \alpha_0 dx^1 \wedge dx^2, & \text{if } p = 0, \\ -\alpha_2 dx^1 + \alpha_1 dx^2, & \text{if } p = 1, \\ \alpha_{12}, & \text{if } p = 2. \end{cases}$$

Hence, the complex structure $\mathcal{J} : T\mathbb{R}^2 \rightarrow T\mathbb{R}^2$ is exactly the symplectic matrix since

$$\mathcal{J} \partial_1 = \partial_2, \quad \mathcal{J} \partial_2 = -\partial_1.$$ 

Moreover, we obtain

$$\text{div} X = \delta X^\flat = \ast d \ast (X^1 \partial_1 + X^2 \partial_2)$$

$$= \ast (X^1 \partial_1 + X^2 \partial_2) dx^1 \wedge dx^2$$

$$= \partial_1 X^1 + \partial_2 X^2,$$

$$\text{curl} X = \ast dX^\flat = \ast d(X^1 \partial_1 + X^2 \partial_2)$$

$$= \ast (\partial_1 X^2 - \partial_2 X^1) dx^1 \wedge dx^2$$

$$= \partial_1 X^2 - \partial_2 X^1.$$
\[
\text{grad} \phi = (d\phi)_{\sharp} \\
= (\partial_1 \phi dx^1 + \partial_2 \phi dx^2)_{\sharp} \\
= \partial_1 \phi \partial_1 + \partial_2 \phi \partial_2,
\]
\[
- J \text{grad} \psi = -(* d\psi)_{\sharp} \\
= (\partial_2 \psi dx^1 - \partial_1 \psi dx^2)_{\sharp} \\
= \partial_2 \psi \partial_1 - \partial_1 \psi \partial_2,
\]
which gives the gradient vector field and the Hamiltonian vector field in the plane.

3 Point vortex dynamics on surfaces

Also for curved surfaces, the point vortex dynamics is derived from the Euler–Arnold equation in a similar manner to the case of the Euclidean plane. The Euler–Arnold equation on a surface \((M, g)\) is written as

\[
\partial_t v_t + \nabla_v v_t = -\text{grad} \rho_t, \quad \text{div} v_t = 0.
\]

The fluid velocity \(v_t \in \mathfrak{X}(M)\) and the pressure \(\rho_t \in C(M)\) are the unknowns at time \(t \in (0, T)\). The fluid velocity satisfies the slip boundary condition. The solution \((v_t, \rho_t)\) is called an Euler–Arnold flow. The references to the Euler–Arnold equation are found in [1, 2, 9, 32]. The advection term \((v_t \cdot \nabla) v_t\) in the Euler equation (1) is replaced as \(\nabla v_t v_t\) in the Euler–Arnold equation (9) by using the Levi-Civita connection \(\nabla\).

Applying the curl operator \(\text{curl} = \ast d\) to the Eq. (9) and writing the Lie derivative by \(\mathcal{L}\), we deduce from \(\nabla_v v_t = \mathcal{L}_{v_t} v_t - d|v|^2/2\) that

\[
\partial_t \omega_t + \mathcal{L}_{v_t} \omega_t = 0.
\]

The Eq. (10) is the vorticity equation on the surface. As we see in the Euclidean plane, we present \(v_t\) and \(\rho_t\) by \(\omega_t\) since every incompressible vector field becomes a Hamiltonian vector field, which makes it possible to utilize the Biot–Savart law. On the other hand, as we see in Remark 2, on general curved surfaces, there exists an incompressible vector field which is not a Hamiltonian vector field. For such incompressible vector field, we can not recover the vector field from the vorticity since we can not apply the Biot–Savart law to the incompressible vector field. Hence, to use the Biot–Savart law, it is necessary to assume that \(v_t\) is a Hamiltonian vector field, thereby deriving the presentation (2) for the surface. Then \(v_t\) is written as \(v_t = -\mathcal{J} \text{grad} \psi_t\), for some \(\psi_t \in C(M)\). Hence, we obtain

\[
\omega_t = -\text{curl} \mathcal{J} \text{grad} \psi_t = -\Delta \psi_t.
\]

By the Green function \(G_H\) for the Hodge Laplacian \(-\Delta\), \(\psi_t\) is given by \(\psi_t = \langle G_H, \omega_t \rangle\), which yields \(v_t = -\mathcal{J} \text{grad} \langle G_H, \omega_t \rangle\). Similarly, applying the divergence operator to the Eq. (9), we obtain
\[
\text{div}\nabla v_t = -\text{div}\text{grad}p_t = -\Delta p_t.
\]

Hence, under the assumption that \((v_t, p_t)\) is an Euler–Arnold flow and \(v_t\) is a Hamiltonian vector field, we deduce that the velocity field \(v_t\) and the pressure \(p_t\) are given by

\[
v_t = -\mathcal{J}\text{grad}\langle G_H, \omega_t \rangle, \quad p_t = \langle G_H, \text{div}\nabla v_t \rangle. \tag{11}
\]

As the time-dependent distribution, taking a linear combination of delta functions \(\Omega_t = \sum_{n=1}^{N} \Gamma_n \delta_{q_n(t)}\) centered at \(q_n(t) \in M\), we obtain a time-dependent vector field \(V_t \in \mathfrak{X}(M \setminus \{q_n(t)\}_{n=1}^{N})\) and a time-dependent function \(H_t \in C(M \setminus \{q_n(t)\}_{n=1}^{N})\) which is defined by (11) in the sense of distributions. Let us recall Helmholtz proposed that the motion of point vortices in the plane is governed by the evolution equation (3). Note that the velocity field \(v_n\) is written as

\[
v_n = \lim_{q \to q_n} \left[ V_t(q) + \frac{\Gamma_n}{2\pi} \mathcal{J}\text{grad} \log d(q, q_n(t)) \right], \tag{12}
\]

where \(d(q, q_n(t))\) is the Euclidean distance between \(q\) and \(q_n(t)\). By replacing the Euclidean distance in (12) with the geodesic distance \(d\) on the surface, we formulate the evolution equation of point vortices on the surface:

\[
\dot{q}_n = \lim_{q \to q_n} \left[ V_t(q) + \frac{\Gamma_n}{2\pi} \mathcal{J}\text{grad} \log d(q, q_n(t)) \right] = -\mathcal{J}\text{grad} q_n \left[ \sum_{m=1}^{N} \Gamma_n G_H(q_n, q_m) + \Gamma_n R(q_n) \right] \equiv v_n(q_n). \tag{13}
\]

The Eq. (13) is also called the point vortex equation, and the solution is called the point vortex dynamics on the surface.

Point vortex dynamics on surfaces is originally motivated by the applications to geophysical fluids [4]. In addition to the above derivation, which is based on the analogy of Helmholtz’s principle, there are several other derivations of the point vortex dynamics on surfaces. The point vortex dynamics is investigated in the many surfaces: a sphere [18], a hyperbolic disc [17], multiply connected domains [28], a cylinder [23], a flat torus [33], the Bolza surface [13], surfaces of revolution which is diffeomorphic to the plane [16], the sphere [8] and the torus [29]. For general curved surfaces, another derivation of point vortex dynamics from the vorticity equation (10) and the generalized Newton law has been recently developed in [14]. The generalized Newton law is a generalization of the Newton law in the plane to the mechanical system on curved surfaces (see, for instance [26]). Ragazzo and Viglioni compute the external force upon a fluid particle and derive the equation of motion of point vortices with mass by assuming the point vortices is moved by the external force.
As is the case of the Euclidean plane, we ask whether \((V_t, \Pi_t)\) is an Euler–Arnold flow. However, the problem is still open for curved surfaces nor the plane. Let us recall that even when the flow field is the plane, it is hard to justify the point vortex dynamics as an Euler flow owing to the nonlinearity of the vorticity equation and the singularity of the delta function. For general curved surfaces, this justification problem becomes more difficult due to the following reasons. First, the difficulty comes from the generalization of Helmholtz’s principle to the surface. In the analogy of the Helmholtz’s principle, the fluid velocity is regularized by the geodesic distance. There is no particular reason why the geodesic distance should be taken to regularize the fluid velocity. Another choice of the regularizing function has been recently discussed in [15]. Second, as we see in Remark 2, there exists an incompressible vector field which is not a Hamiltonian vector field. For the same reason, there exists an Euler–Arnold flow which is not a Hamiltonian vector field. For example, by using Hodge decomposition, we can construct an irrotational vector field which is not a Hamiltonian vector field. Clearly, the irrotational vector field is a steady solution of the Euler–Arnold equation. On the other hand, in order to derive the point vortex dynamics from an Euler–Arnold flow, we assume the Euler–Arnold flow is a Hamiltonian vector field. Hence, there is a gap between non-Hamiltonian Euler–Arnold flows and the point vortex dynamics.

As for the plane, the point vortex dynamics in the background field is considered. The background field is now taken as a fluid velocity \(X_t\) classical solution of the Euler–Arnold equation. Let us define the point vortex dynamics on surfaces in a background field by a solution of the following equation.

\[
\dot{q}_n(t) = \beta_X X_t(q_n(t)) + \beta_\omega \omega_n(q_n(t)), \quad n = 1, \ldots, N,
\]

for a given \((\beta_X, \beta_\omega) \in \mathbb{R}^2\), where \(X_t(q_n(t))\) is the velocity vector of the fluid velocity \(X_t\) at the point \(q_n(t)\). For example, in the application to geophysical flows, the point vortex dynamics in a background field on the unit sphere is adapted as a mathematical model of incompressible and inviscid fluid flows on the unit sphere with Coriolis force [24]. On the other hand, the problem remains open as to whether the point vortex dynamics in a background field is an Euler–Arnold flow.

### 4 de Rham current

#### 4.1 Basic concepts and properties

We review some basic notions in the theory of de Rham currents, which are alternatives of the Schwartz distributions in the Euclidean space that are required to give a weak formulation of the Euler–Arnold equation on curved surfaces. Roughly speaking, currents are differential forms with distribution coefficients in local charts. A good reference of the theory of de Rham currents is given in [7].

A \(p\)-current is defined as a continuous linear functional over \(\mathbb{R}\) on \(\mathcal{D}^{2-p}(M)\). Let \(T[\phi]\) denote the coupling of the \(p\)-current \(T\) and \(\phi \in \mathcal{D}^{2-p}(M)\). The elements of \(\mathcal{D}(M)\) are often called test forms. The space of all \(p\)-currents on \(M\) equipped with
the weak-* topology is denoted by $\mathcal{D}_p(M)$. The calculus of differential forms such as the differential operator $d$, the Hodge-* operator, the codifferential operator $\delta = * d *$ and the Hodge Laplacian $\Delta = d \delta + \delta d$ can be extended to currents via test forms. The differential operator $d : T \in \mathcal{D}_p(M) \to dT \in \mathcal{D}_{p+1}(M)$ is defined by

$$dT[\phi] = (-1)^{p+1} T[d\phi]$$

(14)

for each $\phi \in \mathcal{D}^{-p}(M)$. The Hodge-* operator $*: T \in \mathcal{D}_p(M) \to * T \in \mathcal{D}_p(M)$ is defined by

$$* T[\varphi] = (-1)^{p(2-p)} T[* \varphi]$$

(15)

for each $\varphi \in \mathcal{D}(M)$. Thus the notions for differential forms such as (co)closeness, (co)exactness and harmonicity can also be defined with respect to currents. A time-dependent current $T$ on $M$ is defined as a current on $(0, T) \times M$. Note that for each time-dependent current $T \in \mathcal{D}_p((0, T) \times M)$ and each $\tau \land \phi \in \mathcal{D}^{-p}((0, T) \times M)$ with $\tau \in \mathcal{D}((0, T))$ and $\phi \in \mathcal{D}^{-p}(M)$, the coupling of $T$ and only $\phi$ defines a 0-current on the time interval $(0, T)$. We also denote this time-dependent 0-current by $T[\phi] \in \mathcal{D}_0((0, T))$ and denote the coupling of $T[\phi]$ and $\tau$ by $T[\phi][\tau] = T[\tau \land \phi]$.

The time derivative of the time-dependent current $T$ is defined as the time derivative of $T[\phi] \in \mathcal{D}_0((0, T))$. For each time-dependent current $T \in \mathcal{D}_p((0, T) \times M)$, we define $\partial_1 T \in \mathcal{D}_p((0, T) \times M)$ by $\partial_1 T[\phi][\tau] = * \partial_1 T[\phi][\tau] = -T[\phi][* \partial_1 \tau]$. Note that for each $\phi \in \mathcal{D}^{-p}(M)$ and each $\tau \in \mathcal{D}((0, T))$.

For instance, the space $\mathcal{D}_0(M)$ corresponds to the space of distributions on $M$. For $p \in M$, we define the delta current, say $\delta_p \in \mathcal{D}_0(M)$, by $\delta_p[\phi] = * \phi(p)$. This is the counterpart of the delta function in the theory of distributions. As another example, every smooth $p$-form defines a $p$-current. For each smooth $p$-form $\alpha \in \Omega^p(M)$, it is naturally identified with a $p$-current by an inclusion map $I : \alpha \in \Omega^p(M) \hookrightarrow I(\alpha) \in \mathcal{D}_p(M)$ such that $I(\alpha)[\phi] = \int_M \alpha \land \phi$ for each $\phi \in \mathcal{D}^{-p}(M)$. The $p$-current $I(\alpha)$ is called the integral current of the $p$-form $\alpha$. Then, the inclusion map $I : \Omega^p(M) \hookrightarrow \mathcal{D}_p(M)$ commutes the differential operator $d$ and the Hodge-* operator, that is,

$$dI(\alpha) = I(d\alpha), \quad * I(\alpha) = I(* \alpha).$$

(16)

Indeed, we see that for each $\phi \in \mathcal{D}^{-p}(M)$ and each $\varphi \in \mathcal{D}^{-p}(M)$,

$$dI(\alpha)[\phi] = (-1)^{p+1} I(\alpha)[d\phi] = \int_M (-1)^{p+1} \alpha \land d\phi$$

$$= \int_M d\alpha \land \phi = I(d\alpha)[\phi],$$

$$* I(\alpha)[\varphi] = (-1)^{p(2-p)} I(\alpha)[* \varphi] = \int_M (-1)^{p(2-p)} \alpha \land * \varphi$$

$$= \int_M \star \alpha \land \varphi = I(* \alpha)[\varphi],$$

since

 Springer
Remark 1. We define a Cartesian coordinate pact subset gral current In the coordinate sense of Let us take the geodesic polar coordinate Proof Fix an open subset Proposition 1 vector field around a geodesic circle as follows. In general, if a p-form α is locally 2-integrable, which is defined as follows, the integral current I(α) is well defined. Let us recall that the space \( \Omega^p(M) \) becomes a normed space by the \( L^2 \) norm \( ||\alpha||_{L^2(M)} = (\alpha, \alpha)_{L^2(M)} \). Let \( L^2(M) \) be the quotient space of all measurable forms \( \alpha \) with \( ||\alpha||_{L^2} < \infty \) over the kernel of the \( L^2 \) norm. A p-form \( \alpha \) is said to be locally 2-integrable or of class \( L^2_{\text{loc}}(M) \) if \( ||\alpha||_{L^2(K)} < +\infty \) for every compact subset \( K \subset M \). The topology of the space \( \Omega^p(M) \cap L^2_{\text{loc}}(M) \) is defined as follows. A sequence \( \{\alpha_n\}_{n=1}^\infty \subset \Omega^p(M) \cap L^2_{\text{loc}}(M) \) converges to \( \alpha \in \Omega^p(M) \cap L^2_{\text{loc}}(M) \) in the sense of \( L^2_{\text{loc}}(M) \) if \( ||\alpha_n - \alpha||_{L^2(K)} \to 0 \) for each compact subset \( K \subset M \). A p-form \( \alpha \in \Omega^p(M) \cap L^2_{\text{loc}}(M) \) is said to of class \( H^1_{\text{loc}}(M) \) if there exists \( \beta \in \Omega^{p+1}(M) \cap L^2_{\text{loc}}(M) \) such that

\[
dI(\alpha) = I(\beta).
\]

The p-form \( \beta \) is called the weak derivative of \( \alpha \) and denoted by \( d\alpha \). Hence, even for each \( \alpha \in \Omega^p(M) \cap H^1_{\text{loc}}(M) \), the Eq. (16) is hold true. Note that the above argument of the regularity for measurable forms is applicable the regularity of measurable vector fields owing to the Eq. (4). From this reason, we use the same notation about the regularity of vector field as forms.

We now introduce \( \chi_p : v \in \mathfrak{X}^1(U) \to \mathfrak{X}_p v \in \mathfrak{D}_1(M) \) for a given submanifold \( U \subset M \) and \( p \in U \) by \( \chi_p v[\phi] = \phi_p[v_p] \) for each \( \phi \in \mathfrak{D}_1(M) \). For a given sufficiently small \( \varepsilon > 0 \), we denote by \( B_\varepsilon(p) \) the geodesic \( \varepsilon \)-ball centered at \( p \in M \), i.e.,

\[
B_\varepsilon(p) = \{ q \in M | d(q, p) < \varepsilon \}.
\]

It is characterized as a limit of the mean value for the vector field around a geodesic circle as follows.

**Proposition 1** Fix an open subset \( U \subset M \), \( p \in U \) and \( v \in \mathfrak{X}^1(U) \). Then for any \( \phi \in \mathfrak{D}_1(M) \), we have

\[
\chi_p v[\phi] = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\partial B_\varepsilon(p)} g(v, \mathcal{J} \text{grad log } d(p, q)) \phi.
\]

**Proof** Let us take the geodesic polar coordinate \( (\rho, \theta) = (d(p, q), \theta) \) that is given in Remark 1. We define a Cartesian coordinate \( (x^1, x^2) \) by

\[
x^1 = \rho \cos \theta, \quad x^2 = \rho \sin \theta.
\]

In the coordinate \( (x^1, x^2) \), we write the coordinate representations of \( v \) and \( \phi \) by

\[
v = v^1 \partial_1 + v^2 \partial_2, \quad \phi = \phi_1 dx^1 + \phi_2 dx^2,
\]

where \( \partial_i = \partial / \partial x^i \).

It follows from the Eq. (5) that

\[
g(v, \mathcal{J} \text{grad log } d(p, q)) = \ast d \log d(p, q)[v] = \ast d \log \rho[v]
\]

\[
= (1 + \rho h) d\theta[v].
\]

(17)
Since
\[ \partial_1 = \cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta}, \quad \partial_2 = \sin \theta \frac{\partial}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta}, \]
we obtain
\[ d\theta[v] = d\theta \left[ (v^1 \cos \theta + v^2 \sin \theta) \frac{\partial}{\partial \rho} + \left( -\frac{v^1}{\rho} \sin \theta + \frac{v^2}{\rho} \cos \theta \right) \frac{\partial}{\partial \theta} \right] = \frac{(-v^1 \sin \theta + v^2 \cos \theta)}{\rho}. \] (18)

Owing to
\[ dx^1 = \cos \theta d\rho - \rho \sin \theta d\theta, \quad dx^2 = \sin \theta d\rho + \rho \cos \theta d\theta, \]
we see that on \( \partial B_\varepsilon(p), \)
\[ \phi = \rho(-\phi_1 \sin \theta + \phi_2 \cos \theta) d\theta. \] (19)

Hence, we deduce from the Eqs. (17), (18) and (19) that for each \( q \in \partial B_\varepsilon(p), \)
\[ g(v, \mathcal{F} \text{grad log } d(p,q))\phi = (1 + \varepsilon h) \]
\[ \left\{ v^1 \phi_1 \sin^2 \theta + v^2 \phi_2 \cos^2 \theta - (v^1 \phi_2 + v^2 \phi_1) \sin \theta \cos \theta \right\} d\theta. \]

By continuous differentiability, we see that on \( \partial B_\varepsilon(p), \)
\[ v^j \phi_j = v^j(p)\phi_j(p) + O(\varepsilon) \]
for each \( i, j \in \{1, 2\}. \) Therefore, we conclude that
\[
\int_{\partial B_\varepsilon(p)} \frac{1}{\pi} g(v, \mathcal{F} \text{grad log } d(p,q))\phi = v^1(p)\phi_1(p) \int_{\partial B_\varepsilon(p)} \frac{\sin^2 \theta}{\pi} d\theta \\
+ v^2(p)\phi_2(p) \int_{\partial B_\varepsilon(p)} \frac{\cos^2 \theta}{\pi} d\theta \\
- (v^1(p)\phi_2(p) + v^2(p)\phi_1(p)) \int_{\partial B_\varepsilon(p)} \frac{\cos \theta \sin \theta}{\pi} d\theta + O(\varepsilon) \\
= v^1(p)\phi_1(p) + v^2(p)\phi_2(p) + O(\varepsilon) \\
= \chi_p v[\phi] + O(\varepsilon). \]

which complete the proof. \( \square \)

Let us recall that the velocity field whose vorticity is the delta function is not included in \( L^2_{\text{loc}}(M), \) which yields the Delort solution is not applicable to the point vortex dynamics. We now extend the space \( L^2_{\text{loc}}(M) \) slightly. A \( p \)-form \( \alpha \in \Omega^p(M) \) is said to be \textbf{almost locally 2-integrable} if there exists a subset \( U_a \subset M \) such that the complement \( M \setminus U_a \) is a countable set of points and that \( \alpha \) is of class \( L^2_{\text{loc}}(U_a). \) Let \( \Omega^p_{\text{aloc}}(M) \) denote the set of all almost locally 2-integrable \( p \)-form. Clearly, every \( p \)-form \( \alpha \in \Omega^p(M) \cap L^2_{\text{loc}}(M) \) is of class \( \Omega^p_{\text{aloc}}(M) \) since we can
take $U_\alpha$ as the whole space $M$. The topology of $\Omega^p_{\text{alloc}}(M)$ is defined as follows. A sequence \( \{\alpha_n\}_{n=1}^{\infty} \subset \Omega^p_{\text{alloc}}(M) \) converges to $\alpha_0 \in \Omega^p_{\text{alloc}}(M)$ in the sense of $\Omega^p_{\text{alloc}}(M)$ if \( ||\alpha_n - \alpha_0||_{L^2(K)} \to 0 \) for each compact subset $K \subset M \setminus \bigcup_{n=0}^{\infty} S(\alpha_n)$. For each $\alpha \in \Omega^p_{\text{alloc}}(M)$, there exists a maximal subset $U$ such that $\alpha \in L^2_\text{loc}(U)$. Then, the countable set $S(\alpha) = M \setminus U$ of points is called the singular support of $\alpha$.

We next define a singular integral current as an integral current whose integral kernel is an almost locally $2$-integrable $p$-form. What we would like to do here is to define an integral current for every almost locally $2$-integrable $p$-form $\alpha$ as well as for every locally $2$-integrable $p$-form. Unfortunately, since $\alpha$ is not integrable on a neighborhood of $S(\alpha)$, for each $\phi \in \mathcal{D}^2(M)$, the integral $I(\alpha)[\phi]$ does not make sense. However, if $\phi \in \mathcal{D}^2(M \setminus S(\alpha))$, the integral $I(\alpha)[\phi]$ always makes sense. Therefore, for this test form $\phi$, using its zero extension $\tilde{\phi} \in \mathcal{D}^2(M)$ to the whole space $M$, we can associate the integral $I(\alpha)[\phi]$ with the coupling $T[\phi]$ of a $p$-current $T \in \mathcal{D}^p(M)$. Namely, a $p$-current $T \in \mathcal{D}^p(M)$ is called a pseudointegral $p$-current, if there exists a $p$-form $\alpha \in \Omega^p_{\text{ alloc}}(M)$ such that for every $\phi \in \mathcal{D}^2(M \setminus S(\alpha))$, \( T[\phi] = I(\alpha)[\phi] \), where $\tilde{\phi} \in \mathcal{D}^2(M)$ is the zero extension of $\phi$ over $S(\alpha)$, i.e.,

\[
\tilde{\phi} = \begin{cases} 
\phi, & \text{on } M \setminus S(\alpha), \\
0, & \text{otherwise}.
\end{cases}
\]

By Zorn lemma, a pseudointegral $p$-current $T \in \mathcal{D}^p(M)$, there uniquely exists a $p$-form $\alpha_T \in \Omega^p_{\text{ alloc}}(M)$ such that $T[\tilde{\phi}] = I(\alpha_T)[\phi]$ for every $\phi \in \mathcal{D}^2(M \setminus S(\alpha_T))$ and that if $\alpha \in \Omega^p_{\text{ alloc}}(M)$ satisfies $T[\tilde{\phi}] = I(\alpha)[\phi]$ for any $\phi \in \mathcal{D}^2(M \setminus S(\alpha))$, $S(\alpha_T) \subset S(\alpha)$ and $\alpha_T|_{M \setminus S(\alpha)} = \alpha$. For each pseudointegral $p$-current $T$, the almost locally $2$-integrable $p$-form $\alpha_T$ is called the density of $T$. We define a subset $S(T) \subset M$ by $S(T) = S(\alpha_T)$ and call it the singular support of $T$. Let us note that the space of all pseudointegral $p$-currents is equipped with the relative topology of $\mathcal{D}^p(M)$. Since every pseudointegral $p$-current $T$ is a $p$-current, the differential operator $dT$ and the Hodge-$\ast$ operator $\ast$ $T$ is defined as one for the $p$-current.

Let us define the weak derivative of an almost locally $2$-integrable $p$-form $\alpha$. For a locally $2$-integrable form, the weak derivative is defined using the differential operator on an integral current. Unfortunately, neither $dI(\alpha)$ nor, for that matter, even $I(\alpha)$ can be defined as a $p$-current on the whole space $M$. On the other hand, if $\alpha$ comes from the density of a pseudointegral $p$-current, the weak derivative of $\alpha$ can be defined as follows. We will discuss when $\alpha$ comes from the density of a pseudointegral $p$-current later in Remark 4.

For each pseudointegral $p$-current $T$, we assume the $p+1$-current $dT$ becomes a pseudointegral $p + 1$-current again. Note that since every pseudointegral $p$-current is a $p$-current on $M$, $dT$ is defined as a $p + 1$-current. We then have the densities $\alpha_T \in \Omega^p_{\text{ alloc}}(M)$ and $\alpha_{dt} \in \Omega^{p+1}_{\text{ alloc}}(M)$ of $T$ and $dT$, respectively. We define the weak derivative of $\alpha_T$ by $\alpha_{dt}$ and denote it by $d\alpha_T$. Let us note that $\alpha_T \in \Omega^p(M) \cap L^2_\text{loc}(M \setminus S(\alpha_T))$ and $\alpha_{dt} \in \Omega^{p+1}(M) \cap L^2_\text{loc}(M \setminus S(\alpha_{dt}))$ and that $S(\alpha_{dt})$ is not equal to $S(\alpha_T)$ in general. For instance, $T = I(\log |z|) \in \mathcal{D}_0^p(C)$ gives $\alpha_T = \log |z| \in \Omega^p(C) \cap L^2_\text{loc}(C)$ and $d\alpha_T = (1/2)(z^{-1}dz + \bar{z}^{-1}d\bar{z}) \in \Omega^2(C) \cap L^2_\text{loc}(C \setminus \{0\})$, which yields $S(\alpha_T) = \emptyset$ but $S(d\alpha_T) = \{0\}$. 
Let \( \mathcal{D}_p^i(M) \) be the space of all pseudointegral \( p \)-currents \( T \) such that the \( p+1 \)-current \( dT \) is also a pseudointegral \( p+1 \)-current. The map \( K : T \in \mathcal{D}_p^i(M) \rightarrow K(T) = \alpha_T \in \Omega^p_{\text{aloct}}(M) \) is called the derivative. By definition, we can write the weak derivative by

\[
dK(T) = K(dT) \in \Omega^{p+1}_{\text{aloct}}(M).
\]

Moreover, the derivative \( K \) commutes the Hodge-* operator,

\[
* K(T) = K(\ast T) \in \Omega^p_{\text{aloct}}(M)
\]

for any \( T \in \mathcal{D}_p^i(M) \). To see this, let us substitute \( K(T) \) for \( \alpha \) in the Eq. (16). For each \( \phi \in \mathcal{D}^{2-p}(M \setminus S(T)) \) with its zero extension \( \tilde{\phi} \in \mathcal{D}^{2-p}(M) \), we have

\[
I(\ast K(T))[\phi] = \ast I(K(T))[\phi] = \ast T[\tilde{\phi}] = I(K(\ast T))[\phi],
\]

which gives

\[
I(\ast K(T) - K(\ast T))[\phi] = 0.
\]

Owing to the fundamental lemma of calculus of variation, we deduce the commutation relation. As a result, we obtain

\[
dK(T) = K(dT), \quad \ast K(T) = K(\ast T).
\]  

As a trivial example, for each \( p \)-form \( \alpha \in \Omega^p(M) \cap L^2_{\text{loc}}(M) \), the integral current \( I(\alpha) \) of the \( p \)-form is a pseudointegral \( p \)-current with singular support \( S(I(\alpha)) = \emptyset \), since for each \( \phi \in \mathcal{D}^{2-p}(M \setminus S(I(\alpha))) \) with its zero extension \( \tilde{\phi} \in \mathcal{D}^{2-p}(M) \), \( I(\alpha)[\tilde{\phi}] = I(\alpha)[\phi] \). Hence, we obtain

\[
K(I(\alpha)) = \alpha.
\]  

As another example, for each \( p \in M \), the delta current \( \delta_p \) is a pseudointegral \( 0 \)-current since, for each \( \phi \in \mathcal{D}^2(M \setminus \{p\}) \) with its zero extension \( \tilde{\phi} \in \mathcal{D}^2(M) \), \( \delta_p[\tilde{\phi}] = I(0)[\phi] = 0 \). Roughly speaking, when a \( p \)-current \( T \) is a pseudointegral current, the \( p \)-current is identified with the \( p \)-form \( K(T) \in \Omega^p_{\text{aloct}}(M) \) which has singularities in the singular support \( S(T) \). For example, since for each \( p \in M \), the delta current \( \delta_p \) is a pseudointegral \( 0 \)-current, the delta current \( \delta_p \) is identified with the \( 0 \)-form \( K(\delta_p) = 0 \in \Omega^0_{\text{aloct}}(M) \) which has a singularity at the point \( p \).

Especially, for each pseudointegral \( 1 \)-current, we can formulate a vector field with singularities by using the sharp operator \( \# : \alpha \in \Omega^1(M) \rightarrow \alpha_\# \in \mathfrak{X}(M) \). For each \( T \in \mathcal{D}^1_i(M) \), we define \( T_\# \in \mathfrak{X}(M \setminus S(T)) \) by \( T_\# = (K(T))_\# \). As an example, for each \( \psi \in \mathcal{D}^1_0(M) \), \( \mathcal{I}_{\text{grad}}K(\psi) = (K(\ast d\psi))_\# \) is a vector field on \( M \setminus S(\psi) \). The vector field \( \mathcal{I}_{\text{grad}}K(\psi) \) stands for the Hamiltonian vector field induced by the singular Hamiltonian \( \psi \in \mathcal{D}^1_0(M) \) with singularities in \( S(\psi) \). As an application of the calculus of currents, we will derive the Biot–Savart kernel from the Hamiltonian vector field whose vorticity is given by the delta current in Remark 5. This Hamiltonian vector field is regarded as a vector field generated by a point vortex.
Remark 3 In Definition 1, we state the definition of the hydrodynamic Green function in the sense of currents. For each \( x_0 \in M \), we define \( G_{x_0} \in \mathcal{D}_0^1(M) \) by \( K(G_{x_0})(x) = G_H(x, x_0) \). Then, the definition of the hydrodynamic Green function \( G_H \in C^0(M \times M \setminus \Delta) \) is rewritten in terms of the current \( G_{x_0} \) as follows. For each \( x_0 \in M \) and each \( \phi \in \mathcal{D}^2(M) \),

\[
- \Delta G_{x_0} [\phi] = \begin{cases} 
\ast \phi(x_0) - \frac{1}{\text{Area}(M)} \int_M \phi, & \text{if } M \text{ is a closed surface}, \\
\ast \phi(x_0), & \text{otherwise},
\end{cases}
\]

\[
K(G_{x_0})(x) = K(G_x)(x_0),
\]

\[
dK(G_{x_0}) = 0 \quad \text{on } \partial M.
\]

We define the principal value p.v. : \( \alpha \in \Omega_{\text{aloc}}^p(M) \rightarrow \text{p.v. } \alpha \in \mathcal{D}_p^1(M) \) by

\[
\text{p.v. } \alpha[\phi] = \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(S(\alpha))} \alpha \wedge \phi
\]

for each \( \phi \in \mathcal{D}^2(M) \) if the limit exists, where

\[
B_\varepsilon(S(\alpha)) = \bigcup_{x \in S(\alpha)} B_\varepsilon(x).
\]

The domain of p.v., say \( \text{Dom(p.v.)} \), is defined as the space of \( p \)-currents in which the limit exists for every \( \phi \in \mathcal{D}^2(M) \). Let us note that given \( \alpha \in \Omega_{\text{aloc}}^p(M) \) such that the singular support \( S(\alpha) \) has a limit point \( x_0 \in S(\alpha) \), \( \alpha \) does not contained in \( \text{Dom(p.v.)} \) since the integral \( \int_{M \setminus B_\varepsilon(S(\alpha))} \alpha \wedge \phi \) diverges for any \( \varepsilon > 0 \) and any \( \phi \in \mathcal{D}^2(M) \). Hence, every almost locally \( 2 \)-integrable \( p \)-form in the domain \( \text{Dom(p.v.)} \) must have only a discrete singular support. In what follows, we assume every almost locally \( 2 \)-integrable \( p \)-form has only finite singular points.

For example, for each \( p \)-form \( \alpha \in \Omega_p(M) \cap L^2_{\text{loc}}(M) \), we have

\[
I(\alpha) = \text{p.v. } \alpha,
\]

since for each \( \phi \in \mathcal{D}^2(M) \),

\[
\text{p.v. } \alpha[\phi] = \int_M \alpha \wedge \phi = I(\alpha)[\phi].
\]

In particular, it follows from the Eq. (21) that \( I(\alpha) = \text{p.v. } K(I(\alpha)) \). On the other hand, for any \( p \)-current \( T \in \mathcal{D}_p^1(M) \), \( T = \text{p.v. } K(T) \) does not always hold true. As a matter of fact, if \( T = \delta_p \), we have \( \text{p.v. } K(T) = 0 \).

Remark 4 Given a \( p \)-form \( \alpha \in \Omega_{\text{aloc}}^p(M) \), if \( \alpha \in \text{Dom(p.v.)} \), there exists a pseudointegral \( p \)-current \( T_\alpha \in \mathcal{D}_p^1(M) \) such that the density of \( T_\alpha \) is given by \( \alpha \). For each \( p \)-form \( \alpha \in \text{Dom(p.v.)} \), p.v. \( \alpha \) becomes a pseudointegral \( p \)-current since for each \( \phi \in \mathcal{D}^2(M \setminus S(\alpha)) \) with its zero extension \( \tilde{\phi} \in \mathcal{D}^2(M) \),
Euler–Arnold flow in the sense of de Rham currents

We thus obtain

\[ K(p.v. \alpha) = \alpha, \]

which is a generalization of the equality (21) from \( \alpha \in L^2_{\text{loc}}(M) \) to \( \alpha \in \text{Dom}(p.v.) \subset \Omega^p_{\text{aloc}}(M) \).

**Remark 5** Let us apply the calculus of currents to the derivation of the Biot–Savart kernel in the Euclidean plane \((\mathbb{C}, |dz|^2)\). First, let us consider a 0-current \( \psi = (G_H, \delta_0) \in \mathcal{D}'(\mathbb{C}) \), which is the singular Hamiltonian induced by the vorticity \( \delta_0 \). Since for each \( \phi \in \mathcal{S}(\mathbb{C}) \), \( \psi[\phi] = I(G_H(z, 0))[\phi] \), we deduce that \( \psi \) is a pseudointegral 0-current and that \( S(\psi) = \emptyset \). Defining \( u \in \mathcal{D}'(\mathbb{C}) \) by \( u = -*d\psi \), we have

\[-\int \text{grad}(\psi[\tilde{\phi}]) (\psi[\tilde{\phi}]) = (K(u))_{2}.\]

We see that \( u = I(-*dG_H(z, 0)) \) in \( \mathbb{C} \setminus \{0\} \), i.e., \( K(u) = -*dG_H(z, 0) \) and \( S(u) = \{0\} \), since for each \( \phi \in \mathcal{D}(\mathbb{C} \setminus \{0\}) \) with its zero extension \( \tilde{\phi} \in \mathcal{D}(\mathbb{C}) \),

\[ u[\tilde{\phi}] = d\psi[\tilde{\phi}] = -\psi[d*\tilde{\phi}] = \int_{\mathbb{C} \setminus \{0\}} G_H(z, 0) d*\phi \]

\[ = \int_{\mathbb{C} \setminus \{0\}} -d(G_H(z, 0) * \phi) + dG_H(z, 0) \wedge \phi \]

\[ = \int_{\mathbb{C} \setminus \{0\}} *dG_H(z, 0) \wedge \phi = I(-*dG_H(z, 0))[\phi]. \]

By using the Wirtinger derivative,

\[ \partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \]

\[ \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), \]

it follows from \( *d\bar{z} = -idz \) and \( *d\bar{z} = id\bar{z} \) that

\[-* dG_H(z, 0) = i(\partial G_H(z, 0)dz - \bar{\partial}G_H(z, 0)d\bar{z}), \]

which yields

\[-(*dG_H(z, 0))_\bar{z} = i(2\partial G_H(z, 0) \bar{\partial} - 2\bar{\partial} G_H(z, 0)\partial). \]

Therefore we obtain the Biot–Savart kernel as follows.
Remark 6  We compute the vorticity of \(-J\ \text{grad} K(u)\), in which \(p.v.\) plays a key role in the computation by the density of \(\psi\). Indeed, owing to \(K(u) = -\star dG_H(z,0)\), we deduce from (6) that for each \(z \in \mathbb{C} \setminus \{0\}\),

\[
\text{curl}(-J\ \text{grad} K(u)) = \star dK(u) = -\star d * dG_H(z,0) = -\Delta G_H(z,0) = 0.
\]

Hence, \(* dK(u)\) gives the vorticity of the vector field \(-J\ \text{grad} K(u)\) defined on \(\mathbb{C} \setminus \{0\}\) and equals to 0. In contrast, we have

\[
* du = - * d * d\psi = - \Delta \psi = \delta_0 \quad \text{in} \quad \mathcal{D}'_0(\mathbb{C}).
\]

Hence, when we derive the vorticity from the density \(K(u)\) as a pseudointegral 0-current, we need to modify \(* dK(u)\) so that the vorticity of \(u\) derived from \(K(u)\) is equal to the delta function \(\delta_0\) in \(\mathcal{D}'_0(\mathbb{C})\).

Now, we can see that

\[
* \text{p.v. } K(u) = \delta_0 \quad \text{in} \quad \mathcal{D}'_0(\mathbb{C}). \tag{23}
\]

Let us first check \(K(u) \in \text{Dom}(p.v.)\), i.e., the following limit exists:

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} K(u) \wedge \phi < \infty,
\]

which yields that \(p.v.\) \(K(u)\) becomes a pseudointegral 1-current. For each \(\phi \in \mathcal{D}(M)\) and each \(\varepsilon > 0\), it follows from

\[
K(u) \wedge \phi = - * dG_H(z,0) \wedge \phi
= dG_H(z,0) \wedge * \phi
= d(G_H(z,0) * \phi) - G_H(z,0) d * \phi
\]

that

\[
\int_{\mathbb{C} \setminus B_{\varepsilon}(0)} K(u) \wedge \phi = \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} d(G_H(z,0) * \phi) - G_H(z,0) d * \phi
= - \int_{\partial B_{\varepsilon}(0)} G_H(z,0) * \phi - \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} G_H(z,0) d * \phi.
\]

Writing \(\phi\) by \(\phi = \phi_1 dx^1 + \phi_2 dx^2\), owing to
\[\ast \, dx^1 = dx^2 = \varepsilon \cos \theta d\theta,\]
\[\ast \, dx^2 = -dx^1 = \varepsilon \sin \theta d\theta \quad \text{on } \partial B_{\varepsilon}(0),\]
we obtain
\[\int_{\partial B_{\varepsilon}(0)} G_H(z, 0) \ast \phi = \int_{\partial B_{\varepsilon}(0)} \left( -\frac{1}{2\pi} \log \varepsilon \right) \ast (\phi_1 dx^1 + \phi_2 dx^2)\]
\[= -\frac{1}{2\pi} \varepsilon \log \varepsilon \int_{\partial B_{\varepsilon}(0)} (\phi_1 \cos \theta + \phi_2 \sin \theta) d\theta\]
\[\to 0,\]
\[\int_{\mathbb{C} \setminus B_{\varepsilon}(0)} G_H(z, 0) d \ast \phi \to \int_{\mathbb{C}} G_H(z, 0) d \ast \phi < \infty.\]
Hence, we deduce \(K(u) \in \text{Dom}(p.v.)\) and \(p.v. K(u) \in \mathcal{D}^1_1(\mathbb{C}).\) Owing to the equalities (14) and (15), we see that for each \(\phi \in \mathcal{D}^2(\mathbb{C}),\)
\[\ast \text{ d.p.v. } K(u)[\phi] = \text{ p.v. } K(u)[d \ast \phi] = \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} -\ast dG_H(z, 0) \wedge d \ast \phi\]
\[= \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} d(\ast \phi \ast dG_H(z, 0)) - \ast \phi d \ast dG_H(z, 0)\]
\[= \lim_{\varepsilon \to 0} -\int_{\partial B_{\varepsilon}(0)} \ast \phi \ast dG_H(z, 0)\]
\[\quad - \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_{\varepsilon}(0)} \ast \phi \bigtriangleup G_H(z, 0) dz \wedge d\bar{z}\]
\[= \lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(0)} \ast \phi i(\partial G_H(z, 0) dz - \bar{\partial} G_H(z, 0) d\bar{z})\]
\[= \ast \phi(0),\]
which gives the desired equality (23). Therefore, in order to recover the vorticity from the density, we need to consider \(\ast \text{ d.p.v. } K(u)\) instead of \(\ast dK(u)\).

Let us introduce the operator \(\mathcal{Q}\) by \(\mathcal{Q} : \alpha \in \Omega^p_{\text{aloc}}(M) \to \mathcal{Q}\alpha \in \mathcal{D}^{p+1}_p(M),\) which is called the localizing operator. \(\mathcal{Q}\alpha \in \mathcal{D}^{p+1}_p(M)\) is defined by
\[\mathcal{Q}\alpha[\phi] = (-1)^{p+1} \text{ p.v. } \alpha[\text{d}\phi]\]
for each \(\phi \in \mathcal{D}^{1-p}(M).\) The domain of \(\mathcal{Q}, \text{Dom}(\mathcal{Q}),\) is the space of all \(p\)-currents \(\alpha\) in which \(\text{p.v. } \alpha[\text{d}\phi]\) is well-defined for every \(\phi \in \mathcal{D}^{1-p}(M).\) As we see in Remark 4, for a given \(\alpha \in \Omega^p_{\text{aloc}}(M),\) if \(\alpha\) is contained in \(\text{Dom}(\text{p.v.}),\) then \(\text{p.v. } \alpha\) becomes a pseudointegral 0-current. Hence, by the equality (14), for each \(\phi \in \mathcal{D}^{1-p}(M)\) we obtain
\[\text{p.v. } \alpha[\text{d}\phi] = (-1)^{p+1} \text{ d.p.v. } \alpha[\phi],\]
which yields that p.v. $\alpha[d\phi]$ is well-defined and that $\alpha \in \text{Dom}(\mathcal{U})$. Namely, \(\text{Dom}(p.v.) \subset \text{Dom}(\mathcal{U})\) holds true. In particular, if $\alpha \in \text{Dom}(p.v.)$, then
\[
\mathcal{U}\alpha = d\text{p.v. } \alpha.
\]
For instance, for each $p$-form $\alpha \in \mathcal{U}'(M) \cap H^1_{\text{loc}}(M)$, we deduce from the Eq. (16) and (22) that $\alpha \in \text{Dom}(\mathcal{U})$ and
\[
\mathcal{U}\alpha = d\text{p.v. } \alpha = dI(\alpha) = I(d\alpha). \tag{24}
\]
If $\alpha \in \text{Dom}(\mathcal{U})$ satisfies $d\alpha = 0$ in $M \setminus S(\alpha)$, then we obtain
\[
\mathcal{U}\alpha[\phi] = (-1)^{p+1} \text{p.v. } \alpha[d\phi] = (-1)^{p+1} \lim_{\epsilon \to 0} \int_{M \setminus \mathcal{B}_\epsilon(S(\alpha))} \alpha \wedge d\phi
\]
\[
= -\lim_{\epsilon \to 0} \int_{\mathcal{B}_\epsilon(S(\alpha))} \alpha \wedge \phi,
\]
since $(-1)^{p+1} \alpha \wedge d\phi = -d(\alpha \wedge \phi) + d\alpha \wedge \phi$. Hence, $\mathcal{U}\alpha$ is determined by the asymptotic behavior of $\alpha$ near the singular support $S(\alpha)$. The name of \textit{localizing} operator is coming from this property.

### 4.2 Weak formulation of vector fields

Since the space $\mathfrak{X}^{\infty}(M)$ of all smooth vector fields is isomorphic to the space $\Omega^1(M)$ of all smooth 1-forms through the flat operator $\flat : v \in \mathfrak{X}^{\infty}(M) \rightarrow \flat v = g(v, \cdot) \in \Omega^1(M)$, we can obtain a weak extension of the notions associated with vector fields such as the divergence, the vorticity and the slip-boundary condition in the sense of currents by replacing the velocity form with a 1-current. We will use these notions to formulate the Euler–Arnold equations in the sense of currents.

As we see in Sect. 2, the divergence and the vorticity of $v \in \mathfrak{X}^{\infty}(M)$ is defined by $\delta v^\flat \in \Omega^0(M)$ and $\ast d v^\flat \in \Omega^0(M)$. The slip boundary condition is written by
\[
\ast X^\flat = 0, \quad \text{in } \Omega^1(\partial M). \tag{25}
\]
Hence, it is reasonable to define the divergence and the vorticity of a 1-current $\alpha \in \mathcal{A}_L(\partial M)$ by $\delta \alpha \in \mathcal{A}_L^0(\partial M)$ and $\ast d \alpha \in \mathcal{A}_L^0(\partial M)$. On the other hand, for each $\alpha \in \mathcal{A}_L^1(M)$ with $\partial M \cap S(\alpha) = \emptyset$, we define the slip boundary condition as $\ast K(\alpha) = 0$ in $\Omega^1(\partial M)$. Note that the slip boundary condition can not be defined for any 1-current straightforwardly since the restriction of the current on the boundary does not in general make sense.

**Remark 7** As we see in Remark 2, in the plane, every incompressible vector field is a Hamiltonian vector field. Hence, for any incompressible vector field $X \in \mathfrak{X}^{\infty}(\mathbb{R}^2)$, the vector field $X$ can be recovered from its vorticity $\omega$ by use of the Biot–Savart law $X = -\flat \text{grad}(G_H, \omega)$. However, for a general curved surface $M$, even for a multiply connected domain in the plane, not every incompressible vector field becomes a Hamiltonian vector field as long as $H^1(M) \neq 0$. With no restriction to Hamiltonian

\[\text{Springer}\]
Euler–Arnold flow in the sense of de Rham currents

Let $X \in \mathfrak{X}^{\infty}(M)$ be an incompressible vector field. For a given incompressible vector field $Y \in \mathfrak{X}^{\infty}(M)$, let us assume that the relative vector field $Y - X$ is a Hamiltonian vector field: $Y - X = -J \nabla \psi$ for some $\psi \in \Omega^{0}(M)$. Then we note that the vector field $Y$ does not need to be a Hamiltonian vector field, but just the difference between $Y$ and $X$ need to be a Hamiltonian vector field. The relative vorticity $\omega = * (Y - X)$ now satisfies

$$\omega = * d(-J \nabla \psi) = -\bigtriangleup \psi.$$ 

Hence, we obtain $Y = X - J \nabla (G_{H}, \omega)$.

Let us extend the argument in Remark 7 with respect to vector fields to currents by replacing $Y$ with $\alpha \in D_{i}^{1}(M)$. Let us fix incompressible vector field $X \in \mathfrak{X}(M)$ arbitrarily. For a given current $\alpha \in D_{i}^{0}(M)$ with $\partial M \cap S(\alpha) = \emptyset$, let us assume that $\alpha$ satisfies the slip boundary condition and the relative current $\alpha - I(X^{\oplus})$ is coexact, i.e., $\alpha - I(X^{\oplus}) = -*d\psi$ for some $\psi \in D_{i}^{1}(M)$. Defining the relative vorticity $\omega \in D_{i}^{0}(M)$ to $X$ by $\omega = * (\alpha - I(X^{\oplus}))$, we obtain

$$-\bigtriangleup \psi = -*d\psi = * (\alpha - I(X^{\oplus})) = \omega,$$

which gives $\alpha = I(X^{\oplus}) - *d(G_{H}, \omega)$.

In the present paper, we consider a special form of a singular vorticity as follows.

**Definition 2** Let $Q_{N}$ denote $(\text{Int } M)^{N} \setminus \{(q_{n})_{n=1}^{N} \in (\text{Int } M)^{N} | \exists i, j, q_{i} = q_{j}\}$. Fix $N \in \mathbb{Z}_{\geq 1}$, $(\Gamma_{n})_{n=1}^{N} \in (\mathbb{R} \setminus \{0\})^{N}$ and $(q_{n})_{n=1}^{N} \in Q_{N}$. A 0-current $\omega \in D_{i}^{0}(M)$ is called a singular vorticity of point vortices placed on $\{q_{n}\}_{n=1}^{N} \subset M$, if for each $\phi \in D^{2}(M)$,

$$\omega[\phi] = \sum_{n=1}^{N} \Gamma_{n} * \phi(q_{n}) + c \int_{M} \phi,$$

where

$$c = \begin{cases} \frac{1}{\text{Area}(M)} \sum_{n=1}^{N} \Gamma_{n}, & \text{if } M \text{ is a closed surface,} \\ 0, & \text{otherwise.} \end{cases}$$

As we see in Remark 3, the hydrodynamic Green function $G_{H}$ is written in terms of the pseudointegral 0-current $G_{x_{0}}$. From this, every solution $\psi \in D_{i}^{0}(M)$ of the Poisson problem $-\bigtriangleup \psi = \omega$ is presented by

$$\psi = \sum_{n=1}^{N} \Gamma_{n}(G_{q_{n}} + I(\psi^{0}(\cdot, q_{n})))$$  \hspace{1cm} (26)

up to a harmonic function $\psi^{0}$, where
for each \((x, x_0) \in M \times M \setminus \Delta\), since for each \(\phi \in \mathcal{D}^2(M)\),

\[
- \Delta \psi^0(x, x_0) = 0, \\
\psi^0(x, x_0) = \psi^0(x_0, x)
\]

Let us recall that the hydrodynamic Green function is determined up to a harmonic function. Namely, we define a function \(G_H \in C^\infty(M \times M \setminus \Delta)\) by

\[
G_H(x, x_0) = G_H(x, x_0) + \psi^0(x, x_0).
\]

It is easy to see that the function \(G_H^\prime\) is also a hydrodynamic Green function. For each \(x_0 \in M\), defining \(G_{x_0} \in \mathcal{D}_0^j(M)\) by \(K(G_{x_0}^\prime)(x) = G_H^\prime(x, x_0)\), we have

\[
G_{x_0} + I(\psi^0(\cdot, x_0)) = G_{x_0}^\prime,
\]

which gives

\[
\psi = \sum_{n=1}^{N} \Gamma_n G_{q_n}.
\]

Hence, without loss of generality, we can assume that \(\psi\) satisfies

\[
\psi = \sum_{n=1}^{N} \Gamma_n G_{q_n}.
\]

Moreover, \(\omega\) and \(\psi\) are pseudointegral 0-currents with \(S(\omega) = \{q_n\}_{n=1}^{N}\) and \(S(\psi) = \emptyset\), since for each \(\phi \in \mathcal{D}^2(M \setminus \{q_n\}_{n=1}^{N})\) with its zero extension \(\tilde{\phi} \in \mathcal{D}^2(M)\) and each \(\varphi \in \mathcal{D}^2(M)\),

\[
\omega \equiv \psi.
\]
Euler–Arnold flow in the sense of de Rham currents

Hence, we obtain

\[ K(\omega) = c, \quad K(\psi)(p) = \sum_{n=1}^{N} \Gamma(p, q_n). \]  \tag{27} \]

Let us define \( u \in \mathcal{D}'(M) \) by \( u = -*d\psi \). Then, it follows from the Eqs. (20) and (27) that

\[ K(u) = -*dK(\psi) = \sum_{n=1}^{N} \Gamma_n(-*d_pG_H(p, q_n)). \]  \tag{28} \]

and that \( S(u) = \{ q_n \}_{n=1}^{N} \). Let us note that the equalities (27) for \( K(\omega) \) and (28) hold true in \( M \setminus \{ q_n \}_{n=1}^{N} \) and \( \Omega^0_{\text{loc}}(M) \).

In the present paper, let us focus on the following vector field, which governs the evolution of point vortices. Let us fix \( N \in \mathbb{Z}_{\geq 1}, \ (\Gamma_n)_{n=1}^{N} \subseteq (\mathbb{R} \setminus \{0\})^N \) and \( (q_n)_{n=1}^{N} \subseteq Q_N \). For each \( n \in \{1, \ldots, N\} \), a vector field \( v_n \in \mathcal{X}(B_n(q_n)) \) of class \( C^1 \) with sufficiently small \( r \in \mathbb{R}_{>0} \) is defined by

\[ v_n(q) = -\mathcal{J} \text{grad}_q \left[ \sum_{m=1}^{N} \Gamma_m G_H(q_m, q) + \frac{\Gamma_n}{2\pi} \log d(q_n, q) \right]. \]  \tag{29} \]

It follows from the regularity theorem for a linear elliptic operator [3] that \( v_n \) is of class \( C^1 \). In particular, by the Eq. (28), we have

\[ K(u) = v_n^b + \frac{\Gamma_n}{2\pi} \ast d \log d(q, q_n). \]  \tag{30} \]

Note that

\[ v_n(q_n) = -\mathcal{J} \text{grad}_{q_n} \left[ \sum_{m\neq n}^{N} \Gamma_m G_H(q_m, q_n) + \Gamma_n R(q_n) \right]. \]
Since \( \sum_{m \neq n}^{N} \Gamma_{m} G_{H}(q_{m}, q_{n}) + \Gamma_{n} R(q_{n}) \) is smooth on \( Q_{N} \), \( v : (q_{n})_{n=1}^{N} \in Q_{N} \rightarrow (v_{n}(q_{n}))_{n=1}^{N} \in TQ_{N} \) is a smooth vector field on \( Q_{N} \). The point vortex dynamics is defined as a solution of the following ordinary differential equation,

\[
\dot{q}_{n}(t) = v_{n}(q_{n}(t)), \quad n = 1, \ldots, N.
\]

\( \mathfrak{X}'(M) \) denotes the space of vector fields of class \( C^{r} \) on \( M \). For a given classical Euler–Arnold flow \( (X_{t}, P_{t}) \in \mathfrak{X}'(M) \times C^{r}(M) \) and \( (\beta_{X}, \beta_{\omega}) \in \mathbb{R}^{2} \), The point vortex dynamics in the background field \( X_{t} \) with the vorticity \( \omega_{X} \) is defined as a solution of the following ordinary differential equation, called the point vortex equation,

\[
\sum_{n=1}^{N} \Gamma_{n} \mathcal{X}_{q_{n}} \{ \dot{q}_{n} - (\beta_{X} X + \beta_{\omega} v_{n}) \} = dI(K(\psi) d\omega_{X}) \quad \text{in} \quad \mathcal{T}_{2}(M). \tag{31}
\]

In particular, if \( \omega_{X} \) is constant, we deduce

\[
\dot{q}_{n}(t) = \beta_{X} X_{t}(q_{n}(t)) + \beta_{\omega} v_{n}(q_{n}(t)), \quad n = 1, \ldots, N. \tag{32}
\]

We will see later that the term \( dI(K(\psi) d\omega_{X}) \) comes from a part \( \ast (\ast d\psi) \ast \psi \) of the advection term of the Euler–Arnold equation.

Let us derive a presentation of the point vortex equation in a coordinate \((x^{i})\). We choose a test form \( \phi_{n} \in \mathcal{T}^{2}(M) \) for each \( n = 1, \ldots, N \) and each \( i = 1, 2 \) such that \( \text{supp}(\phi_{n}) \cap \{ q_{m} \}_{m=1}^{N} = \{ q_{n} \} \) and that \( (d\phi_{n})_{q_{n}} = dx_{q_{n}}^{i} \in T_{q_{n}}^{*} M \). Writing \( \dot{q}_{n}, X \) and \( v_{n} \) by

\[
\dot{q}_{n} = \dot{x}_{1}^{i} \partial_{1} + \dot{x}_{2}^{i} \partial_{2}, \\
X = x_{1}^{i} \partial_{1} + x_{2}^{i} \partial_{2}, \\
v_{n} = v_{1}^{i} \partial_{1} + v_{2}^{i} \partial_{2},
\]

we then have

\[
\sum_{n=1}^{N} \Gamma_{n} \mathcal{X}_{q_{n}} \{ \dot{q}_{n} - (\beta_{X} X + \beta_{\omega} v_{n}) \} \{ \phi_{n} \} = \Gamma_{n} \mathcal{X}_{q_{n}} \{ \dot{q}_{n} - (\beta_{X} X + \beta_{\omega} v_{n}) \} \{ \phi_{n} \} = \Gamma_{n} \mathcal{X}_{q_{n}} \{ \dot{q}_{n} - (\beta_{X} X + \beta_{\omega} v_{n}) \} \{ \phi_{n} \} = \Gamma_{n} \mathcal{X}_{q_{n}} \{ \dot{q}_{n} - (\beta_{X} X + \beta_{\omega} v_{n}) \} \{ \phi_{n} \}.
\]

Hence, in the coordinate, the point vortex equation (31) is presented by

\[
\dot{q}_{n}^{i} = \beta_{X} x_{t}^{i}(q_{n}) + \beta_{\omega} v_{n}^{i}(q_{n}) + \Gamma_{n}^{-1} dI(K(\psi) d\omega_{X})[\phi_{n}^{j}].
\]

\( \Gamma_{n}^{-1} dI(K(\psi) d\omega_{X})[\phi_{n}^{j}] \) becomes a non-local term since

\[
\Gamma_{n}^{-1} dI(K(\psi) d\omega_{X})[\phi_{n}^{j}] = \Gamma_{n}^{-1} \int_{M} (K(\psi) d\omega_{X} \wedge d\phi_{n}^{j}).
\]
Therefore, we see that if the background vorticity $\omega_X$ is not constant, the point vortex equation becomes an integro-differential equation.

## 5 Euler–Arnold flow

Let us derive some equivalent presentations of the Euler–Arnold equation. They are derived from the following dual presentation.

$$\partial_t v^\flat + \nabla_v v^\flat = -dp.$$  \hfill (33)

Based on the fact that

$$\nabla_v v^\flat = \mathcal{L}_v v^\flat - d|v|^2/2,$$
$$\nabla_v v^\flat = i_v dv^\flat + d|v|^2/2,$$

we obtain two equivalent formulations of (33),

$$\partial_t v^\flat + \mathcal{L}_v v^\flat - d|v|^2/2 = -dp,$$  \hfill (34)

$$\partial_t v^\flat + i_v dv^\flat + d|v|^2/2 = -dp,$$  \hfill (35)

where $\mathcal{L}$ and $i$ are the Lie derivative and the interior multiplication respectively. Applying the differential operator $d$ to the Eq. (34), we obtain

$$\partial_t dv^\flat + \mathcal{L}_v dv^\flat = 0.$$  \hfill (36)

Owing to $dv^\flat = \omega d\text{Vol}_g$, the equation is equivalent to the vorticity equation:

$$\partial_t \omega + \mathcal{L}_v \omega = 0.$$  

The Eq. (36) contains both $v$ and $v^\flat$. On the other hand, in order to obtain a weak formulation of the Euler–Arnold equation in terms of currents, we need to present the Euler–Arnold equation only in terms of $v^\flat$. We notice that

$$i_v dv^\flat = \omega i_v d\text{Vol}_g = (* dv^\flat) * v^\flat,$$

owing to $\dim M = 2$. In addition, let us recall the Riemannian metric can be extended to the metric on the cotangent bundle, which yields $|v^\flat|$ makes sense and $|v^\flat| = |v|$ holds. Hence, from the Eq. (35), we deduce that

$$\partial_t v^\flat + (* dv^\flat) * v^\flat + d|v^\flat|^2/2 = -dp.$$  \hfill (37)

Before we replace differential forms in (37) with currents, we need to deal with the nonlinear term carefully in order to avoid multiplication of currents. Let us recall multiplication of almost locally $2$-integrable $p$-forms is still valid and for each $\alpha \in \Omega^p_{\text{loc}}(M)$, if $\alpha \in \text{Dom}(p.v.)$, $\alpha$ is converted to a current by taking the principle
value. Hence, the Euler–Arnold equation is reformulated for a pair of a time-dependent currents \((a_t, p_t) \in \mathcal{D}_t^I((0, T) \times M) \times \mathcal{D}_0^I((0, T) \times M)\) as follows.

\[
\partial_t \text{p.v.} K(a_t) + \text{p.v.} \{K(\ast d a_t) K(\ast a_t) + d|K(a_t)|^2/2\} = -\text{p.v.} K(dp_t)
\]

in \(\mathcal{D}_t^I((0, T) \times M)\), if each of terms is contained in \(\text{Dom}(\text{p.v.})\) and for each \(\tau \wedge \phi \in \mathcal{D}_1((0, T) \times M)\) with \(\tau \in \mathcal{D}_1((0, T))\) and \(\phi \in \mathcal{D}(M)\), \(a_t(\phi), dp_t(\phi) \in \mathcal{D}_0^I((0, T))\) satisfy \(S(a_t(\phi)) = S(dp_t(\phi)) = 0\). When we focus on evolution of vorticity, the vorticity equation is useful rather than the Euler–Arnold equation. Let us remember that the vorticity equation is obtained by applying the curl operator \(\text{curl} = \ast d\) to the Euler–Arnold equation. Based on this, applying the differential operator \(d\) to the Eq. (38), we obtain the vorticity equation corresponding to the Eq. (38) if each terms in the Eq. (38) is contained in \(\text{Dom}(\mathcal{Q})\).

**Definition 3** A pair of a time-dependent currents \((a_t, p_t) \in \mathcal{D}_t^I((0, T) \times M) \times \mathcal{D}_0^I((0, T) \times M)\) is called a weak Euler–Arnold flow, if the following conditions are satisfied:

1. for each \(\tau \wedge \phi \in \mathcal{D}_1((0, T) \times M), S(a_t(\phi)) = S(dp_t(\phi)) = 0\);
2. \(K(a_t) \in \text{Dom}(\mathcal{Q})\) and \(K(\ast d a_t) K(\ast a_t) + d|K(a_t)|^2/2 \in \text{Dom}(\mathcal{Q})\);
3. \(K(dp_t) \in \text{Dom}(\mathcal{Q})\);
4. \(\partial_t \mathcal{Q} K(a_t) + \mathcal{Q} \{K(\ast d a_t) K(\ast a_t) + d|K(a_t)|^2/2\} = -\mathcal{Q} K(dp_t)\) in \(\mathcal{D}_2((0, T) \times M)\);
5. \(\delta a_t = 0\) in \(\mathcal{D}_0((0, T) \times M)\);
6. \(\partial M \cap \text{S}(a_t) = \emptyset\) and \(\ast K(a_t) = 0\) on \(\partial M\).

In particular, we call the fourth condition the weak Euler–Arnold equation and \(a_t\) the velocity current.

**Remark 8** Let us examine the relation between the original Euler–Arnold equation and the weak Euler–Arnold equation (Definition 3–4). Let \((X_t, P_t) \in \mathcal{X}(M) \times \mathcal{C}^r(M)\) be a classical solution of the Euler–Arnold equation, that is,

\[
\partial_t X_t^\phi + (\ast dX_t^\phi) \ast X_t^\phi + d|X_t^\phi|^2/2 = -dP_t, \quad \delta X_t^\phi = 0.
\]

Then, we now show that the pair of a time-dependent currents \((I(X_t^\phi), I(P_t)) \in \mathcal{D}_t^I((0, T) \times M) \times \mathcal{D}_0^I((0, T) \times M)\) is a weak Euler–Arnold flow. Let us recall that for each \(p\)-form \(\alpha \in \Omega^p(M)\) of class \(C^r\), \(\alpha\) also belongs to \(L^2_{\text{loc}}(M)\) so that the integral current \(I(\alpha)\) of the \(p\)-form \(\alpha\) is clearly a pseudointegral \(p\)-current with singular support \(S(I(\alpha)) = \emptyset\). With this in mind, let us check that the pair \((I(X_t^\phi), I(P_t)) \in \mathcal{D}_t^I((0, T) \times M) \times \mathcal{D}_0^I((0, T) \times M)\) satisfies the six conditions in Definition 3. First, since \(X_t\) and \(P_t\) are of class \(C^r\), we have \(S(I(X_t^\phi)) = S(I(P_t)) = \emptyset\). Second, it follows from the Eqs. (16) and (21) that
Euler–Arnold flow in the sense of de Rham currents

Hence, we deduce from the Eq. (24) that

\[ \mathcal{Q} K(I(X_i^p)) = I(dX_i^p), \]
\[ \mathcal{Q}[K(* dI(X_i^p))K(* I(X_i^p))] = I(d\{(dX_i^p) * X_i^p\}), \]
\[ \mathcal{Q}d|K(I(X_i^p))|^2/2 = I(d|X_i^p|^2/2) = I(0), \]

which gives the second condition. Third, by the Eq. (24), we obtain

\[ \mathcal{Q}dP_t = I(dP_t) = I(0), \]

which yields the third condition. Fourth, applying the differential operator \( d \) to the Eq. (39), we obtain

\[ \partial_t dX_i^p + d\{(dX_i^p) * X_i^p\} = 0. \]

Hence, for each \( \tau \wedge \phi \in \mathcal{D}^1((0, T) \times M) \), we have

\[ \int_0^T \int_M \phi \tau \wedge \partial_t dX_i^p + \int_0^T \int_M \phi \tau \wedge d\{(dX_i^p) * X_i^p\} = 0. \]

Since

\[ \int_0^T \int_M \phi \tau \wedge \partial_t dX_i^p = -I(dX_i^p)[\phi][\partial_t \tau] = \partial_t I(dX_i^p)[\phi][\tau], \]

It is rewritten in terms of currents as

\[ \partial_t I(dX_i^p)[\tau \wedge \phi] + I(d\{(dX_i^p) * X_i^p\})[\tau \wedge \phi] = I(0)[\tau \wedge \phi]. \]

It follows from the Eq. (40) and (41) that

\[ \left[ \partial_t \mathcal{Q} K(I(X_i^p)) + \mathcal{Q}[K(* dI(X_i^p))K(* I(X_i^p))] + d|K(I(X_i^p))|^2/2 \right][\tau \wedge \phi] = -\mathcal{Q} K(dP_t)[\tau \wedge \phi], \]

which is exactly the weak Euler–Arnold equation. Fifth, we see from the Eq. (16) that

\[ \delta I(X_i^p) = * d * I(X_i^p) = I(* d * X_i^p) = I(0), \]

which yields the fifth condition. Lastly, the vector field \( X_t \) now satisfies the slip boundary condition (25). Owing to \( \mathcal{S}(I(X_i^p)) = \emptyset \) and the Eq. (21), it follows that
\[ * K(I(X^b_t)) = * X^b_t = 0, \]

which gives the last condition. As a consequence, we conclude that the original Euler–Arnold flow \((X_t, P_t) \in \mathcal{X}'(M) \times C'(M)\) gives a weak Euler–Arnold flow \((I(X_t^b), I(P_t)) \in \mathcal{D}'_1((0, T) \times M) \times \mathcal{D}'_0((0, T) \times M)\).

**Remark 9** Let us consider a condition for the weak Euler–Arnold equation to become the conventional weak formulation of the Euler equation in the plane [20]. We fix a weak Euler–Arnold flow \((a_t, p_t) \in \mathcal{D}'_1((0, T) \times M) \times \mathcal{D}'_0((0, T) \times M)\). We denote \(K(a_t)\) by \(\nu_t\). We assume \(S(a_t) = S(p_t) = \emptyset\), which is satisfied when \(\nu_t, K(p_t) \in L^2_{\text{loc}}(M)\). Then, owing to the Eq. (20), we have

\[ K(* d\alpha_t) = * d\nu_t, \quad K(* \alpha) = * \nu_t. \]

Each of terms in the weak Euler–Arnold equation is now written for each \(\tau \wedge \phi \in \mathcal{D}'_1((0, T) \times M)\) as follows:

\[
\begin{align*}
\partial_t \mathcal{Q} K(a_t)[\tau \wedge \phi] &= - \int_0^T \int_M \nu_t \wedge d\phi \wedge \partial_t \tau, \\
\mathcal{Q} \{K(* d\alpha_t)K(* \alpha_t)\}[\tau \wedge \phi] &= \int_0^T \int_M (\ast d\nu_t) \ast \nu_t \wedge d\phi \wedge \tau, \\
\mathcal{Q} \{d|K(a_t)|^2/2\}[\tau \wedge \phi] &= \int_0^T \int_M d|\nu_t|^2/2 \wedge d\phi \wedge \tau, \\
\mathcal{Q} K(d\nu_t)[\tau \wedge \phi] &= \int_0^T \int_M dK(p_t) \wedge d\phi \wedge \tau.
\end{align*}
\]

Therefore, we deduce that when the velocity \(\nu_t\), vorticity \(\omega_t = * d\nu_t\) and the gradient of the pressure \(d\nu_t\) comes from \(L^2_{\text{loc}}(M)\), the weak Euler–Arnold equation becomes the conventional weak formulation of the Euler equation:

\[
0 = [\partial_t \mathcal{Q} K(a_t) + \mathcal{Q} \{K(* d\alpha_t)K(* \alpha_t)\}][\tau \wedge \phi] \\
+ [\mathcal{Q} \{d|K(a_t)|^2/2\} + \mathcal{Q} K(d\nu_t)][\tau \wedge \phi] \\
= - \int_0^T \int_M \nu_t \wedge d\phi \wedge \partial_t \tau + \int_0^T \int_M \omega_t \ast \nu_t \wedge d\phi \wedge \tau
\]

Note that neither is the Delort solution fully contained in our weak solution, nor is our weak solution fully contained in the Delort solution since the weak formulation of the Euler equation differs. More specifically, for the Delort solution, the Eq. (42) are equal to zero owing to Stokes theorem. In contrast, as we will see later, if the vorticity of a weak Euler–Arnold flow is given by a singular vorticity of point vortices, then the Eq. (42) does not equal to zero. Moreover, we will show that the term \(\mathcal{Q} \{d|K(a_t)|^2/2\}\) gives the vector field of the point vortex dynamics. Therefore, the behavior of the term \(\mathcal{Q} \{d|K(a_t)|^2/2\}\) changes drastically depending on whether the solution is smooth or has a strong singularity such as a point vortex.
Let us decompose a weak Euler–Arnold flow into a regular part and a singular part, thereby discussing the decomposition of each term in the weak Euler–Arnold equation (Definition 3-4). Let us fix a weak Euler–Arnold flow \((\alpha_r, p_r) \in \Omega^1_{\text{loc}}((0, T) \times M) \times \Omega^1_{\text{loc}}((0, T) \times M)\) and a classical Euler–Arnold flow \((X_t, P_t) \in \mathfrak{X}(M) \times \mathcal{C}(M)\). As we see in Remark 8, denoting \(K(\alpha_r) \in \Omega^1_{\text{loc}}(M)\) by \(\nu_r\), we have

\[
\partial_t \mathcal{L} \nu_r + \mathcal{L} \{(\ast d \nu_r) \ast \nu_r + d|\nu_r|^2/2\} = -\mathcal{L} K(d\rho),
\]

\[
\partial_t \mathcal{L} X_r^\rho + \mathcal{L} \{(\ast d X_r^\rho) \ast X_r^\rho + d|X_r^\rho|^2/2\} = -\mathcal{L} d P_r.
\]

Writing \(u_r = \alpha_r - I(X_r^\rho) \in \mathcal{D}^i(M), \omega_r = \ast du_r \in \mathcal{D}^i_0(M)\) and \(\omega_X = \ast dX_r^\rho\), we obtain

\[
\partial_t \mathcal{L} \nu - \partial_t \mathcal{L} X^\rho = \partial_t K(u),
\]

\[
\mathcal{L} \{(\ast d \nu) \ast \nu\} - \mathcal{L} \{(\ast d X^\rho) \ast X^\rho\} = \mathcal{L} \{(K(\omega) + \omega_X) \ast (K(u) + X^\rho) - \omega_X \ast X^\rho\}
\]

\[
= \mathcal{L} \{(\omega_X + K(\omega)) \ast K(u) + K(\omega) \ast X^\rho\},
\]

\[
\mathcal{L} d|\nu|^2 - \mathcal{L} d|X^\rho|^2 = \mathcal{L} d g(X^\rho + K(u), X^\rho + K(u)) - \mathcal{L} d g(X^\rho, X^\rho)
\]

\[
= \mathcal{L} d g(2X^\rho + K(u), K(u)),
\]

where we omit the subscript \(t\). Hence, the weak Euler–Arnold equation (Definition 3-4) is reduced to

\[
\partial_t \mathcal{L} K(u) + \mathcal{L} \{(\omega_X + K(\omega)) \ast K(u) + K(\omega) \ast X^\rho + d g(2X^\rho + K(u), K(u))/2\}
\]

\[
= -\mathcal{L}(K(dp) - dP).
\]

(43)

In addition, as we see in Sect. 4.2, if \(u_r\) is a coexact 1-current for each time \(t\), we obtain the Biot–Savart law for currents: \(u_r = -\ast d(G_H, \omega_r)\). Then, given a classical flow \((X_t, P_t)\), the Eq. (43) is regarded as an evolution equation for the relative vorticity \(\omega\) and the relative pressure \(p - P\).

**Remark 10** As an example, let us compute these terms in the Eq. (43) in the case where the flow field is the Euclidean plane \((\mathbb{C}, \langle \cdot, \cdot \rangle)\) and \(\omega\) comes from a singular vorticity of point vortices placed on \(\{z_n\}_{n=1}^N\). We fix an incompressible vector field \(X \in \mathfrak{X}(\mathbb{C})\) with the vorticity \(\omega_X\). We take \(\psi \in \mathcal{D}^i_0(\mathbb{C})\) which is given by the Eq. (26) with \(\psi^0 = 0\) and define \(u \in \mathcal{D}^i_0(\mathbb{C})\) by \(u = -\ast d\psi\). Then, we claim that

\[
\mathcal{L} \{(\omega_X + K(\omega)) \ast K(u) + K(\omega) \ast X^\rho\} = -d I(K(\psi) d\omega_X),
\]

\[
\mathcal{L} d(2X^\rho, K(u)) = -\sum_{n=1}^N \Gamma_n dX_{z_n} X,
\]

\[
\mathcal{L} d(K(u), K(u)) = -\sum_{n=1}^N \Gamma_n dX_{z_n} v_n.
\]

First, owing to the Eq. (26), we now have \(K(\omega) = 0\) in \(\Omega^0_{\text{loc}}(\mathbb{C})\) and \(S(K(\omega) \ast K(u)) = S(K(\omega)) = \{z_n\}_{n=1}^N\). Hence, for each \(\phi \in \mathcal{D}(\mathbb{C})\), we obtain
\[\mathcal{L}K(\omega) * K(u)[\phi] = \lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(K(\omega)+K(u)))} K(\omega) * K(u) \wedge d\phi = 0,\]

\[\mathcal{L}K(\omega) * X^\flat[\phi] = \lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(K(\omega)))} K(\omega) * X^\flat \wedge d\phi = 0.\]

We next see \(\mathcal{L}(\omega_X * K(u)) = -dI(K(\psi)d\omega_X)\). Owing to \(* K(u) = dK(\psi)\), for each \(\phi \in \mathcal{D}(\mathbb{C})\), we have

\[\mathcal{L}(\omega_X * K(u))[\phi] = \lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(\omega))} \omega_X dK(\psi) \wedge d\phi.\]

Since

\[\omega_X dK(\psi) \wedge d\phi = d(\omega_X K(\psi) d\phi) - K(\psi) d\omega_X \wedge d\phi,\]

we deduce from Stokes theorem that

\[\lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(\omega))} \omega_X dK(\psi) \wedge d\phi = \lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(\omega))} d(\omega_X K(\psi) d\phi) - K(\psi) d\omega_X \wedge d\phi\]

\[= - \sum_{n=1}^N \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z_n)} \omega_X K(\psi) d\phi - \lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(\omega))} K(\psi) d\omega_X \wedge d\phi.\]

By continuous differentiability of \(\omega_X\) and the Eq. (27), we obtain

\[\int_{\partial B_\epsilon(z_n)} \omega_X K(\psi) d\phi = \omega_X(z_n) \int_{\partial B_\epsilon(z_n)} K(\psi) d\phi + O(\epsilon)\]

\[= \omega_X(z_n) \sum_{m=1}^N \int_{\partial B_\epsilon(z_n)} \Gamma_m G_H(z, z_m) d\phi_z\]

\[= \omega_X(z_n) \left( \sum_{n=1}^N \Gamma_n G_H(z_n, z_m) - \frac{1}{2\pi} \log \epsilon \right) \int_{\partial B_\epsilon(z_n)} d\phi + O(\epsilon)\]

\[= O(\epsilon).\]

In contrast, since \(K(\psi)\) is of class \(L^2_{\text{loc}}(\mathbb{C}), S(K(\psi)) = \emptyset\) holds true. Hence, it follows that

\[\lim_{\epsilon \to 0} \int_{\mathcal{C} \setminus B_\epsilon(S(\omega))} K(\psi) d\omega_X \wedge d\phi = \int_{\mathcal{C}} K(\psi) d\omega_X \wedge d\phi\]

\[= I(K(\psi)d\omega_X)[d\phi]\]

\[= dI(K(\psi)d\omega_X)[\phi],\]

which gives \(\mathcal{L}(\omega_X * K(u))[\phi] = -dI(K(\psi)d\omega_X)[\phi]\). We next see that

\[\mathcal{L}d(2X^\flat, K(u)) = -\sum_{n=1}^N \Gamma_n dX_{z_n} X.\]
It follows from the Stokes theorem that

$$\mathcal{L}d(2X^b, K(u)) [\phi] = \lim_{\epsilon \to 0} \int_{C \setminus B_\epsilon(S(u))} d(2X^b, K(u)) \wedge d\phi$$

$$= - \sum_{n=1}^N \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z_n)} (2X^b, K(u)) d\phi.$$

We deduce from the Eq. (30) that

$$\int_{\partial B_\epsilon(z_n)} (2X^b, K(u)) d\phi = \int_{\partial B_\epsilon(z_n)} (2X^b, v_n^b) d\phi$$

$$+ \frac{\Gamma_n}{\pi} \int_{\partial B_\epsilon(z_n)} (X, \mathfrak{J} \text{grad log } |z - z_n|) d\phi.$$

By continuous differentiability, for each $z \in \partial B_\epsilon(z_n)$, we have

$$(2X^b(z), v_n^b(z)) = (2X^b(z_n), v_n^b(z_n)) + O(\epsilon),$$

which gives

$$\int_{\partial B_\epsilon(z_n)} (2X^b, v_n^b) d\phi = (2X^b(z_n), v_n^b(z_n)) \int_{\partial B_\epsilon(z_n)} d\phi + O(\epsilon) = O(\epsilon).$$

Applying Proposition 1, we have

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z_n)} \frac{1}{\pi} (X, \mathfrak{J} \text{grad log } |z - z_n|) d\phi = \chi_{z_n} X[d\phi] = d\chi_{z_n} X[\phi].$$

Hence, we see that

$$\mathcal{L}d(2X^b, K(u)) [\phi] = - \sum_{n=1}^N \Gamma_n d\chi_{z_n} X[\phi].$$

We next show that

$$\mathcal{L}d(K(u), K(u)) = - \sum_{n=1}^N \Gamma_n d\chi_{z_n} v_n.$$

Let us recall that $v_n$ is defined by the Eq. (29). By Stokes theorem, we obtain

$$\mathcal{L}d(K(u), K(u)) = \lim_{\epsilon \to 0} \int_{C \setminus B_\epsilon(S(u))} d(K(u), K(u)) \wedge d\phi$$

$$= - \sum_{n=1}^N \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z_n)} (K(u), K(u)) d\phi.$$

The Eq. (30) gives
Since
\[|v_n^\phi|^2 = |v_n^\phi|^2(z_n) + O(\varepsilon), \quad |* \log |z - z_n|^2 = \varepsilon^{-2} \quad \text{on } \partial B_{\varepsilon}(z_n),\]
we see that
\[
\int_{\partial B_{\varepsilon}(z_n)} |v_n^\phi|^2 d\phi = |v_n^\phi|^2(z_n) \int_{\partial B_{\varepsilon}(z_n)} d\phi + O(\varepsilon) = O(\varepsilon),
\]
\[
\int_{\partial B_{\varepsilon}(z_n)} |* \log |z - z_n|^2 d\phi = \varepsilon^{-2} \int_{\partial B_{\varepsilon}(z_n)} d\phi = 0.
\]
Owing to
\[
(v_n^\phi, * \log |z - z_n|) = (v_n, J \nabla \log d(p, q)),
\]
we deduce from Proposition 1 that
\[
\lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(z_n)} d(p, q) = - \sum_{n=1}^{N} \frac{1}{\varepsilon} (v_n, J \nabla \log d(p, q)) d\phi = \chi_{z_n} v_n [d\phi] = d\chi_{z_n} v_n [\phi].
\]
Hence, we conclude that
\[
\mathcal{Q}d(K(u), K(u)) = - \sum_{n=1}^{N} \frac{\Gamma_n}{2\pi} d\chi_{z_n} v_n [\phi].
\]

**Remark 11** Let us examine the singular behavior of \(g(X, K(u))\) and \(|K(u)|^2\) around their singular support on a surface \((M, g)\). We assume \(\omega\) comes from a singular vorticity of point vortices placed on \(\{q_n\}_{n=1}^{N}\). As we see in the proof of Proposition 1, by the Eqs. (17) and (18), we now have
\[
g(v, J \nabla d(p, q)) = (-v^1 \sin \theta + v^2 \cos \theta)\rho^{-1}(1 + \rho h).
\]
By the Eq. (30), we obtain
\[
g(X^\phi, K(u)) = g(X^\phi, v_n^\phi) + \frac{\Gamma_n}{2\pi} g(X^\phi, * \log d(p, q_n))
\]
\[
= g(X, v_n) + \frac{\Gamma_n}{2\pi} g(X, J \nabla \log d(p, q_n)),
\]
which yields there exists a constant \(C\) such that
\[
|g(X^\phi, K(u))| \leq C \sum_{n=1}^{N} d(p, q_n)^{-1}.
\]
Hence, we deduce $g(X^p, K(u)) \in \text{Dom}(p.v.)$. On the other hand, $|K(u)|^2$ is not contained in $\text{Dom}(p.v.)$. In fact, since

$$|K(u)|^2 = \left| \sum_{n=1}^{N} - \ast d \Gamma_n G_H(p, q_n) \right|^2$$

$$= \sum_{m=1}^{N} \sum_{n \neq m} \Gamma_n \Gamma_m s(\ast dG_H(p, q_n), \ast dG_H(p, q_m))$$

$$+ \sum_{n=1}^{N} \Gamma_n^2 |\ast dG_H(p, q_n)|^2,$$

and

$$|g(\ast dG_H(p, q_n), \ast dG_H(p, q_m))| \leq Cd(p, q_n)^{-1}d(p, q_m)^{-1}$$

for some constant $C$, the first term of in (44) is contained in $\text{Dom}(p.v.)$ but the second term is not. In particular, the singular part of the second term in (44) derived from $\log d(p, q_n)$ since

$$|\ast dG_H|^2 = |\ast d(G_R - G_S)|^2$$

$$= |\ast dG_R|^2 - 2g(\ast dG_R, \ast dG_S) + |\ast dG_S|^2,$$

where

$$G_S = -\frac{1}{2\pi} \log d, \quad G_R = G_H + G_S.$$

Let us define a function $\sigma$ by

$$\sigma(u) = \sum_{n=1}^{N} \Gamma_n^2 |\ast dG_S(p, q_n)\rho(p, q_n)|^2,$$

where $\rho(p, q_n)$ is a cut-off function in order $\ast dG_S(p, q_n)\rho(p, q_n)$ to be smoothly defined on $M \setminus \{q_n\}$. In particular, $\sigma \in \Omega^0_{\text{aloc}}(M)$ with $S(\sigma) = \{q_n\}_{n=1}^{N}$. As a result, we deduce that instead of $|K(u)|^2$, $|K(u)|^2 - \sigma(u)$ is contained in $\text{Dom}(p.v.)$. Let us note that $\mathcal{L} d\sigma = 0$ in $\mathcal{D}'(M)$ since for each $\phi \in \mathcal{D}(M)$, by Stokes theorem,

$$\int_{M \setminus B_\varepsilon(\sigma)} d\sigma \wedge d\phi = -\sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} \sigma d\phi$$

$$= -\sum_{n=1}^{N} \left( \sum_{m \neq n} \Gamma_m^2 |dG_S(q_n, q_n)\rho(q_n, q_m)|^2 + \Gamma_n^2 \varepsilon^{-2} \right) \int_{\partial B_\varepsilon(q_n)} d\phi$$

$$+ O(\varepsilon)$$

$$= O(\varepsilon).$$
Remark 10 illustrates that the leading terms in the advection term consists of \( \mathcal{Q}(\omega_X \ast K(u)) \), \( \mathcal{Q}g(X^\varphi, K(u)) \) and \( \mathcal{Q}d|K(u)|^2/2 \). It will be confirmed in Lemma 1 that this property holds true for general curved surfaces. Based on this property and Remark 11, we introduce a model for the pressure that \( \mathcal{Q}K(dp) \) is killed out with a linear combination of singular terms \( \mathcal{Q}dg(X^\varphi, K(u)) \) and \( \mathcal{Q}d|K(u)|^2/2 \), that is,

\[
K(p) = P + (2\beta_X - 1)g(X^\varphi, K(u)) + (2\beta_\omega - 1)(|K(u)|^2 - \sigma(u))/2
\]

for some \( (\beta_X, \beta_\omega) \in \mathbb{R}^2 \). This mathematical model can be interpreted that the singular behavior of the pressure is balanced with the interaction energy density \( g(X^\varphi, K(u)) \) and the regularized kinetic energy density \( (|K(u)|^2 - \sigma(u))/2 \) with a parameter \( (\beta_X, \beta_\omega) \).

For example, when \( \omega \) comes from a singular vorticity of point vortices placed on \( \{q_n\}_{n=1}^N \), owing to the Eq. (30), the pressure \( p \) now satisfies

\[
K(p) = P + (2\beta_X - 1)g(X^\varphi, v^\varphi_n) + (2\beta_\omega - 1)|v^\varphi_n|^2/2 + \frac{\Gamma_n}{2\pi} \{ g(X^\varphi, * \ d \log d^{2\beta_X-1}) + g(v^\varphi_n, * \ d \log d^{2\beta_\omega-1}) \}.
\] (45)

The singular part of \( K(p) \) is given by the second line of the Eq. (45). This implies that the singular behavior of \( p \) in a neighborhood of \( q_n \) is governed by \( d^{2\beta_X-1} \) and \( d^{2\beta_\omega-1} \). Hence, the pressure \( p \) blows up at \( q_n \) and its growth rate in the spatial direction is given by the parameter \( (\beta_X, \beta_\omega) \). Summarizing the above, we propose the following regular-singular decomposition of a weak Euler–Arnold flow.

**Definition 4** A weak Euler–Arnold flow \( (\alpha_t, p_t) \in \mathcal{H}^j((0, T) \times M) \times \mathcal{H}^j((0, T) \times M) \) is said to be \( C^r \) decomposable \( (r \geq 1) \), if there exists a classical Euler–Arnold flow \( (X_t, P_t) \in \mathcal{X}'(M) \times C^r(M) \) and \( (\beta_X, \beta_\omega) \in \mathbb{R}^2 \) such that the following conditions are satisfied for each time \( t \).

1. \( u_t = \alpha_t - I(X^\varphi_t) \) is coexact and the vorticity \( \omega_t = * \ du_t \in \mathcal{H}^0(M) \) is a singular vorticity of point vortices.
2. \( K(p_t) = P_t + (2\beta_X - 1)g(X^\varphi_t, K(u_t)) + (2\beta_\omega - 1)(|K(u_t)|^2 - \sigma(u_t))/2 \).

Then we call \( X_t \) a background field of \( \alpha_t \), \( u_t \) a relative velocity current and \( (\beta_X, \beta_\omega) \) a growth rate of \( p_t \).

Let us note that the \( C^r \) decomposability of the weak Euler–Arnold flow guarantees the existence of the decomposition but there is no mention of the uniqueness of the decomposition. Hence, when we study a \( C^r \) decomposable weak Euler–Arnold flow \( (\alpha_t, p_t) \), we need to fix a classical Euler–Arnold flow \( (X_t, P_t) \in \mathcal{X}'(M) \times C^r(M) \) and a parameter \( (\beta_X, \beta_\omega) \in \mathbb{R}^2 \) such that the velocity field \( X_t \) is a background field of \( \alpha_t \) and the parameter \( (\beta_X, \beta_\omega) \) is a growth rate of \( p_t \). If a weak Euler–Arnold flow \( (\alpha_t, p_t) \) is \( C^r \) decomposable \( (r \geq 1) \), the Eq. (43) is written without the pressure term as follows.
\[
\begin{align*}
\partial_t \mathcal{L} K(u) \\
+ \mathcal{L} \{ (\omega_X + K(\omega)) * K(u) + K(\omega) * X^\psi \} \\
+ \mathcal{L} d g (2\beta_X X^\psi + \beta_\omega K(u), K(u)) \\
= 0.
\end{align*}
\] (46)

6 Main results

Let us fix \( N, r \in \mathbb{Z}_{\geq 1}, \) \((\Gamma_n^N)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N \) and a \( C^r \) one-parameter family \( \Phi : t \in [0,T] \rightarrow \Phi_t \in \text{Diff}^r(Q_N) \) in what follows, where \( \text{Diff}^r(Q_N) \) is the \( C^r \) diffeomorphism group of \( Q_N \). Let us denote by \((q_n(t))_{n=1}^N = \Phi_t((q_n(0))_{n=1}^N)\) an orbit of \( \Phi \).

We first prove that for a given \( C^r \)-decomposable weak Euler–Arnold flow, if the relative vorticity is given by a singular vorticity of point vortices placed on \( \{q_n(t)\}_{n=1}^N \), \( q_n(t) \) is a solution of the point vortex equation (31), which defines the point vortex dynamics in a background field.

**Theorem 1** Let \((\alpha_t, p_t) \in \mathcal{D}_1^1(0,T) \times \mathcal{D}_0^1(0,T) \) be a \( C^r \)-decomposable weak Euler–Arnold flow. Fix a background field \( X_t \) of \( \alpha_t \), a growth rate \((\beta_X, \beta_\omega)\) of \( p_t \). Suppose the relative vorticity \( \omega_t \) is a singular vorticity of point vortices placed on \( \{q_n(t)\}_{n=1}^N \). Then, \( q_n(t)(n = 1, \ldots, N) \) is a solution of the point vortex equation (31).

Conversely, we next prove that if \( q_n(t) \) is a solution of the point vortex equation, there exists a \( C^r \)-decomposable weak Euler–Arnold flow such that the relative vorticity is given by a singular vorticity of point vortices placed on \( \{q_n(t)\}_{n=1}^N \).

**Theorem 2** Fix a classical Euler–Arnold flow \((X_t, P_t) \in \mathfrak{X}(M) \times C^r(M)\) and \((\beta_X, \beta_\omega) \in \mathbb{R}^2 \). Let \( \alpha_t \in \mathcal{D}_0^1(M) \) be a singular vorticity of point vortices placed on \( \{q_n(t)\}_{n=1}^N \). Define a time-dependent current \( u_t \in \mathcal{D}_1^1(0,T) \times \text{Diff}^r(Q_N) \) by \( u_t = - \ast d (G_{\alpha_t}, \omega_t) \). Suppose \( q_n(t)(n = 1, \ldots, N) \) is a solution of the point vortex equation (31). Then, the following pair of time-dependent currents \( \alpha_t \) and \( p_t \) defines a \( C^r \)-decomposable weak Euler–Arnold flow.

\[
\begin{align*}
\alpha_t &= I(X^\psi) + u_t \in \mathcal{D}_1^1((0,T) \times M), \\
p_t &= \text{p.v.} \left\{ P_t + (2\beta_X - 1) g(X^\psi, K(u_t)) \right\} \\
&\quad + \text{p.v.} \left\{ (2\beta_\omega - 1)(|K(u_t)|^2 - \sigma(u_t))/2 \right\} \\
&\in \mathcal{D}_0^1((0,T) \times M).
\end{align*}
\]

The following lemma plays a key role in the proofs of Theorems 1 and 2.

**Lemma 1** Fix a time-dependent vector field \( X_t \in \mathfrak{X}(0,T) \times M \) with \( \text{div} X_t = 0 \) and \((\beta_X, \beta_\omega) \in \mathbb{R}^2 \). Let \( \omega_t \) be a singular vorticity of point vortices placed on \( \{q_n(t)\}_{n=1}^N \).
Define a time-dependent current \( u_t \in \mathcal{D}_1^b((0,T) \times M) \) by \( u_t = -\star dG_{H, \omega_t} \). Then, we have

\[
\partial_t \mathcal{K}(u) = \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} \dot{q}_n,
\]

(47)

\[
\mathcal{K}\{ (\omega_X + K(\omega)) \ast K(u) + K(\omega) \ast X^\flat \} = -dI(K(\psi) d\omega_X),
\]

(48)

\[
\mathcal{K}\{ d\text{g}(2\beta_X X^\flat + \beta_\omega K(u), K(u)) \} = -\sum_{n=1}^{N} \Gamma_n d\chi_{q_n} (\beta_X X + \beta_\omega v_n).
\]

(49)

**Proof** We take the geodesic polar coordinate \((\rho, \theta) = (d(q, q_n), \theta)\) that is given in Remark 1. We fix \( \tau \wedge \phi \in \mathcal{D}_1^b((0,T) \times M) \), where \( \tau \in \mathcal{D}_1^b((0,T)) \) and \( \phi \in \mathcal{D}_0^b(M) \) and sufficiently small \( \varepsilon > 0 \) arbitrarily. We remember the relation

\[
\alpha \wedge d\phi = -d(\phi \alpha) + \phi d\alpha
\]

(50)

for each \( \alpha \in \Omega^1(M) \). To show (47), we first show that \( \mathcal{K}(u) = \ast \omega \). Using the Eq. (50), we have

\[
\mathcal{K}(u)[\phi] = \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(S(u))} K(u) \wedge d\phi
\]

(51)

Applying the Stokes theorem to the first term in the Eq. (51), owing to \( S(u) = \{ q_n \}_{n=1}^{N} \), we obtain

\[
\int_{M \setminus B_\varepsilon(S(u))} -d(\phi K(u)) = \sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} \phi K(u).
\]

By continuous differentiability of \( \phi \), we see that for each \( q \in \partial B_\varepsilon(q_n) \),

\[
\phi(q) = \phi(q_n) + O(\varepsilon).
\]

Owing to the Eqs. (30) and (5), for each \( q \in \partial B_\varepsilon(q_n) \), we have

\[
K(u) = v^\flat_n(q) + \frac{\Gamma_n}{2\pi} \ast d \log \rho(q)
\]

\[
= v^\flat_n(q) + \frac{\Gamma_n}{2\pi} d\theta + O(\varepsilon),
\]

which yields

\[
\phi K(u) = \phi(q_n) v^\flat_n(q) + \phi(q_n) \frac{\Gamma_n}{2\pi} d\theta + O(\varepsilon)
\]
Euler–Arnold flow in the sense of de Rham currents

Since in a local coordinate \((x^1, x^2)\) with \(x^1 = \varepsilon \cos \theta, x^2 = \varepsilon \sin \theta\),

\[
\nu_n^\flat(q) = \nu_n^1(q) dx^1 + \nu_n^2(q) dx^2
= \varepsilon (-\nu_n^1(q_n) \sin \theta + \nu_n^2(q_n) \cos \theta) d\theta + O(\varepsilon),
\]

we obtain

\[
\int_{\partial B_r(q_n)} \nu_n^\flat = O(\varepsilon).
\]

Hence, we deduce that

\[
\int_{\partial B_r(q_n)} \phi K(u) = \phi(q_n) \frac{\Gamma_n}{2\pi} \int_{\partial B_r(q_n)} d\theta + O(\varepsilon)
= \Gamma_n \phi(q_n) + O(\varepsilon)
\]

and the first term in the Eq. (51) satisfies

\[
\lim_{\varepsilon \to 0} \int_{M \setminus B_r(S(u))} -d(\phi K(u)) = \sum_{n=1}^{N} \Gamma_n \phi(q_n).
\]

Regarding the second term in the Eq. (51), the Eqs. (20) and (27) implies

\[
\text{d}K(u) = \text{d}K(- \ast \text{d}\psi)
= K(- \ast \text{d}\psi)
= K(- \ast \Delta \psi)
= \ast K(\omega)
= c \text{dVol}_g.
\]

Hence, the second term in the Eq. (51) becomes

\[
\lim_{\varepsilon \to 0} \int_{M \setminus B_r(S(u))} \phi \text{d}K(u) = \int_M \phi c \text{dVol}_g.
\]

From the above, it follows that

\[
\mathfrak{A} K(u)[\phi] = \sum_{n=1}^{N} \Gamma_n \phi(q_n) + c \int_M \phi \text{dVol}_g = \omega[\ast \phi] = \ast \omega[\phi].
\]

Taking a local coordinate \((x^1, x^2)\) and using the chain rule, we obtain
\[
\frac{d}{dt} \phi(q_n(t)) = \partial_1 \phi(q_n) \frac{dq_1^n}{dt} + \partial_2 \phi(q_n) \frac{dq_2^n}{dt} \\
= d\phi(q_n) \\
= \chi_{q_n} \dot{q}_n \[d\phi] \\
= d\chi_{q_n} \dot{q}_n \[\phi].
\]

Therefore, we deduce that

\[
\partial_t \mathcal{K}(u) \[\phi] = \sum_{n=1}^{N} \Gamma_n \frac{d}{dt} \phi(q_n(t)) = \sum_{n=1}^{N} \Gamma_n \chi_{q_n} \dot{q}_n \[\phi].
\]

which yields the Eq. (47).

In what follows, we fix \( t \in [0, T] \) and omit the subscript \( t \) unless otherwise stated. We next show the Eq. (48). We begin by proving \( \mathcal{K}(u) \equiv 0 \). From the Eq. (28), it follows that

\[
\mathcal{K}(u) \equiv 0 = \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(S(u))} \mathcal{K}(u) \wedge d\phi \\
= \lim_{\varepsilon \to 0} \int_{M \setminus B_\varepsilon(S(u))} d\mathcal{K}(\psi) \wedge d\phi.
\]

By Stokes theorem, we obtain

\[
\int_{M \setminus B_\varepsilon(S(u))} d\mathcal{K}(\psi) \wedge d\phi = - \sum_{n=1}^{N} \int_{\partial B_\varepsilon(q_n)} \mathcal{K}(\psi) d\phi.
\]

Using the Eq. (27), we see that for each \( q \in \partial B_\varepsilon(q_n) \),

\[
\mathcal{K}(\psi)(q) = \sum_{m=1}^{N} \Gamma_m G_H(q, q_m) \\
= \sum_{m \neq n}^{N} \Gamma_m G_H(q, q_m) + \Gamma_n \left( G_H(q, q_n) + \frac{1}{2\pi} \log d(q, q_n) \right) \\
- \frac{\Gamma_n}{2\pi} \log d(q, q_n) \\
= \sum_{m \neq n}^{N} \Gamma_m G_H(q_n, q_m) + \Gamma_n R(q_n) - \frac{\Gamma_n}{2\pi} \log \varepsilon + O(\varepsilon).
\]

We thus obtain

\[
\int_{\partial B_\varepsilon(q_n)} \mathcal{K}(\psi) d\phi = \left( \sum_{m \neq n}^{N} \Gamma_m G_H(q_n, q_m) + \Gamma_n R(q_n) - \frac{\Gamma_n}{2\pi} \log \varepsilon \right) \int_{\partial B_\varepsilon(q_n)} d\phi + O(\varepsilon) \\
= O(\varepsilon),
\] (52)
which yields
\[ \mathcal{Q} \ast K(u)[\phi] = 0. \]

As a result, we can see that
\[ \mathcal{Q} K(\omega) \ast K(u)[\phi] = \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(K(\omega) + K(u)))} K(\omega) \ast K(u) \wedge d\phi \]
\[ = \lim_{\epsilon \to 0} c \int_{M \setminus B_{\epsilon}(S(u))} K(u) \wedge d\phi \]
\[ = c \mathcal{Q} \ast K[\phi] = 0. \]

We next check \( \mathcal{Q}(\omega_X \ast K(u)) = -dI(K(\psi)d\omega_X). \)
Since
\[ \omega_X \ast K(u) \wedge d\phi = \omega_X dK(\psi) \wedge d\phi \]
\[ = d(\omega_X K(\psi)d\phi) - K(\psi)d\omega_X \wedge d\phi, \]
we obtain
\[ \mathcal{Q}(\omega_X \ast K(u))[\phi] = \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(u))} \omega_X dK(\psi) \wedge d\phi \]
\[ = \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(u))} d(\omega_X K(\psi)d\phi) - K(\psi)d\omega_X \wedge d\phi \]
\[ = -\sum_{n=1}^{N} \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(q_n)} \omega_X K(\psi)d\phi - \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(u))} K(\psi)d\omega_X \wedge d\phi. \]

It follows from continuous differentiability of \( \omega_X \) and the Eq. (52) that
\[ \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(q_n)} \omega_X K(\psi)d\phi = \lim_{\epsilon \to 0} \omega_X(q_n) \int_{\partial B_{\epsilon}(q_n)} K(\psi)d\phi = 0. \]

Since \( K(\psi) \) is of class \( L^2_{\text{loc}}(M) \), we have \( S(K(\psi)) = \emptyset \), which yields
\[ \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(u))} K(\psi)d\omega_X \wedge d\phi = \int_{M} K(\psi)d\omega_X \wedge d\phi \]
\[ = I(K(\psi)d\omega_X)[d\phi] \]
\[ = dI(K(\psi)d\omega_X)[\phi]. \]

Concerning \( \mathcal{Q}K(\omega) \ast X^\beta \), it follows from the relation (50) that
Hence, we obtain

\[ \mathcal{Q} \{(\omega_X + K(\omega)) \ast K(u) + K(\omega) \ast X^\flat \} \phi = -dI(K(\psi) d\omega_X)[\phi], \]

which yields the Eq. (48).

We next see the Eq. (49). We deduce from Stokes theorem that

\[
\mathcal{Q} dg(2X^\flat, K(u)) \phi = \lim_{\epsilon \to 0} \int_{M \setminus B_\epsilon(S(u))} dg(2X^\flat, K(u)) \wedge d\phi
\]

The Eq. (30) yields that for each \( q \in \partial B_\epsilon(q_n) \),

\[
g_q(2X^\flat, K(u)) = g_q(2X^\flat, v_n^\flat) + \frac{\Gamma_n}{2\pi} g_q(2X^\flat, * d \log \rho(q))
\]

\[= g_{q_n}(2X^\flat, v_n^\flat) + \frac{\Gamma_n}{2\pi} g_q(2X^\flat, * d \log \rho(q)) + O(\epsilon)\]

\[= g_{q_n}(2X, v_n) + \frac{\Gamma_n}{2\pi} g_q(2X, \mathcal{J} \text{grad} \log \rho(q)) + O(\epsilon).\]

We thus obtain

\[
\int_{\partial B_\epsilon(q_n)} g(2X^\flat, K(u)) d\phi = g_{q_n}(2X, v_n) \int_{\partial B_\epsilon(q_n)} d\phi
\]

\[+ \frac{\Gamma_n}{2\pi} \int_{\partial B_\epsilon(q_n)} g(2X, \mathcal{J} \text{grad} \log \rho) d\phi + O(\epsilon)\]

\[= \frac{\Gamma_n}{2\pi} \int_{\partial B_\epsilon(q_n)} g(2X, \mathcal{J} \text{grad} \log \rho) d\phi + O(\epsilon).\]

It follows from Proposition 1 that

\[
\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(q_n)} g(2X, \mathcal{J} \text{grad} \log \rho) d\phi = 2\pi \chi_{q_n} X[\phi] = 2\pi d\chi_{q_n} X[\phi].
\]
Euler–Arnold flow in the sense of de Rham currents

Hence, we deduce that
\[
\mathcal{L} dg(2X^\phi, K(u))[\phi] = - \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} X[\phi].
\]

What is left is to prove that
\[
\mathcal{L} dg(K(u), K(u))[\phi] = - \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} v_n[\phi].
\] (53)

By a similar argument, we obtain
\[
\mathcal{L} dg(K(u), K(u))[\phi] = \lim_{\epsilon \to 0} \int_{M \setminus B_{\epsilon}(S(u))} dg(K(u), K(u)) \wedge d\phi
\]
\[
= - \sum_{n=1}^{N} \lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(q_n)} g(K(u), K(u)) d\phi.
\]

It follows from the Eq. (30) that for each \( q \in \partial B_{\epsilon}(q_n) \),
\[
g_q(K(u), K(u)) = g_q(v_n^b, v_n^b) + \frac{\Gamma_n}{\pi} g_q(v_n^b, * d \log \rho) + \left( \frac{\Gamma_n}{2\pi} \right)^2 g_q(* d \log \rho, * d \log \rho).
\]

By continuous differentiability, we have
\[
g_q(v_n^b, v_n^b) = g_{q_n}(v_n^b, v_n^b) + O(\epsilon).
\]

From the Eq. (5), we see
\[
g_q(* d \log \rho, * d \log \rho) = g_q(d\theta, d\theta) + O(\epsilon) = \epsilon^{-2} + O(\epsilon).
\]

Hence, we deduce
\[
\int_{\partial B_{\epsilon}(q_n)} g(K(u), K(u)) d\phi = \left( g_{q_n}(v_n^b, v_n^b) + \left( \frac{\Gamma_n}{2\pi} \epsilon^{-2} \right) \right) \int_{\partial B_{\epsilon}(q_n)} d\phi
\]
\[
+ \frac{\Gamma_n}{\pi} \int_{\partial B_{\epsilon}(q_n)} g(v_n^b, * d \log \rho) d\phi + O(\epsilon)
\]
\[
= \frac{\Gamma_n}{\pi} \int_{\partial B_{\epsilon}(q_n)} g(v_n, \mathcal{J} \text{grad} \log \rho) d\phi + O(\epsilon).
\]

Proposition 1 leads to
\[
\lim_{\epsilon \to 0} \int_{\partial B_{\epsilon}(q_n)} g(v_n, \mathcal{J} \text{grad} \log \rho) d\phi = \pi \chi_{q_n} v_n [d\phi] = \pi d\chi_{q_n} v_n[\phi],
\]
which establishes the Eq. (53) and completes the proof. \( \square \)
We now show the two main theorems by using Lemma 1.

**Proof of Theorem 1** Since the background field \(X\) and the relative velocity current \(u\) satisfy the assumptions of Lemma 1, the equalities (47)–(49) hold true. Since \(X\) and \(u\) come from a \(C^r\)-decomposable weak Euler–Arnold flow, they satisfy the Eq. (46). Substituting (47)–(49) into (46), we obtain

\[
\sum_{n=1}^{N} \Gamma_n d\chi_{q_n} \left\{ \dot{q}_n - (\beta_X X + \beta_\omega v_n) \right\} = dI(K(\psi) d\omega_X),
\]

which is the conclusion as desired.

**Proof of Theorem 2** We first prove that the pair \((\alpha_t, p_t)\) is a weak Euler–Arnold flow. As we see in Remark 11, we have \(g(X^b, K(u)), (K(u)^2 - \sigma(u)) \in \text{Dom}(p.v.)\), which yields \(p_t\) is well-defined as a pseudointegral 0-current. Owing to Lemma 1, it is easy to check that \(\alpha_t\) and \(p_t\) satisfy the conditions in Definition 3 except for the weak Euler–Arnold equation. In addition, we see that

\[
\partial_t \mathcal{Q}K(\alpha) = \partial_t \mathcal{Q}(X^b + K(u))
\]

\[
= \partial_t dI(X^b) + \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} \dot{q}_n,
\]

\[
\mathcal{Q}(K(\ast d\alpha)K(\ast \alpha)) = \mathcal{Q}(\ast dX^b) * X^b \mathcal{Q}(\omega_X + K(\omega)) * K(u) + K(\omega) * X^b \mathcal{Q}(\ast dX^b) * X^b - dI(K(\psi) d\omega_X)
\]

and

\[
\mathcal{Q}(K(\ast d\alpha)K(\ast \alpha) + d \{ d\gamma(2\beta_X X^b + \beta_\omega K(u), K(u)) \})
\]

\[
= - \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} (\beta_X X + \beta_\omega v_n).
\]

As we see in Remark 8, since \((X, P)\) is an Euler–Arnold flow, we have

\[
\partial_t dI(X^b) + dI(\ast dX^b) * X^b = 0.
\]

Since \(q_n\) solves the point vortex equation (31), we deduce

\[
\partial_t \mathcal{Q}K(\alpha_t) + \mathcal{Q}(K(\ast d\alpha_t)K(\ast \alpha_t) + d \{ |K(\alpha_t)|^2 / 2 + K(p_t) \})
\]

\[
= \partial_t dI(X^b) + dI(\ast dX^b) * X^b
\]

\[
+ \sum_{n=1}^{N} \Gamma_n d\chi_{q_n} \left\{ \dot{q}_n - (\beta_X X + \beta_\omega v_n) \right\}
\]

\[
- dI(K(\psi) d\omega_X)
\]

\[
= 0,
\]
which yields \((\alpha, p, t)\) is a weak Euler–Arnold flow. By definition, it is obvious that the weak Euler–Arnold flow \((\alpha, p, t)\) is \(C^r\)-decomposable. \(\square\)

7 Applications

As applications of these theorems, we now discuss two examples of point vortex dynamics in a background field: two identical point vortices in a linear shear in the Euclidean plane \((\mathbb{C}, |dz|^2)\) and \(N\)-point vortices on a surface in an irrotational flow.

Let us first check that the point vortex equation without any background field in the plane is obtained from our results as a special case. \(N \in \mathbb{Z}_{\geq 1}, \left(\Gamma_n\right)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N\) and \((z_n)_{n=1}^N \in \mathcal{O}_N\) are arbitrarily fixed. Let us set \(\beta_X = 0\) and \(\beta_{io} = 1\). Then, the point vortex equation (32) is deduced as follows.

\[
\dot{z}_n(t) = v_n(z_n) = -\int_{\partial_a} \sum_{m \neq n} \Gamma_m G_H(z_n, z_m) + \Gamma_n R(z_n).
\]

(54)

Now, the Green function \(G_H\) clearly satisfies that for each \(z \in \mathbb{C}\),

\[
G_H(z, z_n) = -\frac{1}{2\pi} \log |z - z_n| = -\frac{1}{2\pi} \log d(z, z_n),
\]

which yields the Robin function \(R\) is constantly 0. Using the Eq. (8), we obtain

\[
-\int_{\partial} \nabla \nabla K(\psi) = \partial_2 K(\psi) \partial_1 - \partial_1 K(\psi) \partial_2 = -2i\overline{\partial} K(\psi) \partial + 2i \overline{\partial} K(\psi) \overline{\partial}.
\]

Hence, we deduce from the point vortex equation (54) that

\[
\dot{z}_n = -\int_{\partial_a} \sum_{m \neq n} \Gamma_m G_H(z_n, z_m)
\]

\[
= -\sum_{m \neq n} \Gamma_m 2i\overline{\partial} \left( -\frac{1}{2\pi} \log |z_n - z_m| \right)
\]

\[
= \frac{i}{2\pi} \sum_{m \neq n} \frac{\Gamma_m}{z_n - z_m}.
\]

Two identical point vortices in a linear shear Two identical point vortices in a linear shear is used as a model of the vortex merger in [34]. The vortex merger is characterized as a fundamental process of the inverse cascade in 2D turbulence. In the process, vortices with similarly small scales are affected by the shear flow induced from the surrounding vortices. As a result, small vortices combine to form a vortex with large scale. As a simple model for the vortex merger, two identical point vortices in a linear shear is utilized. To use the notations of this paper, let us set \(N = 2\),

\(\square\) Springer
\[ \Gamma_1 = \Gamma_2 = \gamma \] and the linear shear \( X = (cy, 0) \) where \( y = (z - \bar{z})/2i \). Based on [34],
the evolution equation of two identical point vortices in a linear shear placed on
\( \{ q_n(t) \}_{n=1}^2 \) is given by
\[
\dot{q}_n(t) = X(q_n) + v_n(q_n).
\] (55)

Since the Eq. (55) is a Hamiltonian system, plotting the Hamiltonian contours, we
see that the topology of the contours changes around the threshold \( \mu = c\xi_0^2/\gamma \) where
\( \xi_0 \in \mathbb{R} \) is the initial distance between two point vortices. Obviously, for the case
where \( \mu = 0 \), there is no background field and two point vortices moves in a cir-
cle without changing the distance. When \( \mu < 0 \), two point vortices are in periodic
motion and the distance between them becomes shorter, which implies the vortex
merger occurs. Otherwise, the distance is longer. This qualitative understanding is
confirmed to be consistent with experimental studies in [34]. In other words, it is
pointed out that the sign of \( \mu \), especially the orientation of the background field,
determines the occurrence of vortex merger.

Let us apply the main theorems to this fact. First, Theorem 2 shows that there
exists a \( C^\infty \)-decomposable weak Euler flow \( (a_i, p_i) \) such that the relative vorti-
city is a singular vorticity of point vortices placed on \( \{ q_n(t) \}_{n=1}^2 \). This guaran-
tees the dynamics in [34] comes from a weak Euler flow even though the evolu-
tion equation is formally derived with no relation with Euler flows. Second, as
an application of Theorem 1, let us take a \( C^\infty \)-decomposable weak Euler flow
\( (a_i, p_i) \) such that the background field is the linear shear \( X \) and the growth rate is
\( (\beta_X, \beta_\omega) \in \mathbb{R}^2 \) and the relative vorticity is a singular vorticity of point vortices
placed on \( \{ q_n(t) \}_{n=1}^2 \) with \( \Gamma_1 = \Gamma_2 = \gamma \). From Theorem 1, we can deduce that \( q_n(t) \) satisfies
the point vortex equation (32). Then, we can easily see that \( q_n(t) \) comes
from a Hamiltonian system and the Hamiltonian contours are the same as those
given in [34] by changing parameters from \( c \) to \( c\beta_X \) and from \( \gamma \) to \( \gamma\beta_\omega \). In the
similar manner, instead of \( \mu \), we obtain the threshold \( \mu' = c\beta_X\xi_0^2/\gamma\beta_\omega \). Moreover,
the same criterion is valid, that is, the vortex merger occurs if \( \mu' < 0 \). Note that
Theorem 1 tells us that the parameter \( (\beta_X, \beta_\omega) \) is not just a dynamical parameter
contained in the point vortex equation, but also a physical parameter derived from
the pressure of the Euler flow. From this point of view, we conclude that besides
the orientation of the background field, the sign of the growth rate of the pressure
determines the occurrence of vortex merger.

**N-point vortices on a surface in an irrotational flow** In the second case, let us
take a \( C^\infty \)-decomposable weak Euler–Arnold flow \( (a_i, p_i) \) such that the background
field is an irrotational field \( X \in \mathfrak{X}^\infty(M) \) and the relative vorticity is a singular vorti-
city of point vortices placed on \( \{ q_n(t) \}_{n=1}^N \). As we see in Sect. 5, the irrotational
field is the fluid velocity of a steady Euler–Arnold flow \( (X, P) \in \mathfrak{X}^\infty(M) \times C^\infty(M) \)
and the pressure \( P \in C^\infty(M) \) satisfies the Bernoulli law: \( P = -|X|^2/2 \). Let us fix
the growth rate \( (\beta_X, \beta_\omega) \in \mathbb{R}^2 \) and the parameter \( (\Gamma_n)_{n=1}^N \in (\mathbb{R} \setminus \{0\})^N \). For simplicity,
we ignore the interaction between the background field and point vortices, that is,
we assume \( \beta_X = 0 \). Then owing to Theorem 1, \( q_n(t) \) and \( p_i \) satisfy
\[
\begin{aligned}
\{ \dot{q}_n &= \beta_\omega v_n(q_n), \\
K(p_t) &= -(|K(\alpha_t)|^2 - \sigma(u_t))/2 + \beta_\omega(|K(u_t)|^2 - \sigma(u_t)).
\end{aligned}
\]

Let us focus on two cases $\beta_\omega = 0$ and $1$. In the case $\beta_\omega = 0$, it follows from $\dot{q}_n(t) = 0$ that each of point vortices does not move. Hence, we deduce that $(\alpha_t, p_t)$ is a steady solution. Moreover, the pressure satisfies a generalization of the Bernoulli law $K(p) = -(|K(\alpha_t)|^2 - \sigma(u_t))/2$ to the case with non-moving point vortices on curved surfaces. On the other hand, for $\beta_\omega = 1$ we obtain the conventional point vortex equation and the pressure satisfies a modified Bernoulli law $K(p) = -(|K(\alpha_t)|^2 - \sigma(u_t))/2 + |K(u_t)|^2 - \sigma(u_t)$. As a corollary, the Bernoulli law is generalized to the case where the flow field is a curved surface and where the presence of moving or non-moving point vortices is taken into account as follows.

**Corollary 2** (Generalized Bernoulli law with non-moving point vortices on surfaces)

If a $C^\infty$-decomposable weak Euler–Arnold flow $(\alpha_t, p_t)$ on a surface satisfies that the background field of $\alpha_t$ is an irrotational field $X \in \mathfrak{X}^\infty(M)$ and that the pressure is given by $K(p_t) = -(|K(\alpha_t)|^2 - \sigma(u_t))/2$ and that the relative vorticity is a singular vorticity of point vortices, then $(\alpha_t, p_t)$ is a steady solution of the weak Euler–Arnold equations.

**Corollary 3** (Modified Bernoulli law with moving point vortices on surfaces)

If a $C^\infty$-decomposable weak Euler–Arnold flow $(\alpha_t, p_t)$ on a surface satisfies that the background field of $\alpha_t$ is an irrotational field and that $K(p_t) = -(|K(\alpha_t)|^2 - \sigma(u_t))/2 + |K(u_t)|^2 - \sigma(u_t)$ and that the relative vorticity is a singular vorticity of point vortices placed on $(q_n(t))_{n=1}^N$, then for every $n \in \{1, \ldots, N\}$, $q_n(t)$ is a solution of the point vortex equation:

\[
\dot{q}_n(t) = v_n(q_n(t)).
\]

Let us finally discuss the role of the growth rate $\beta_\omega$ in the motion of point vortices and the pressure as an application of Theorem 2. Given a singular vorticity $\omega_t$ of point vortices on $(q_n(t))_{n=1}^N$ and $u_t = - \ast d(G_H, \omega_t)$, we now assume

\[
\dot{q}_n = \beta_\omega v_n(q_n).
\]

Then, the following pair $(\alpha_t, p_t) \in \mathcal{D}^r_1((0, T) \times M) \times \mathcal{D}^r_0((0, T) \times M)$ becomes a $C^r$-decomposable weak Euler–Arnold flow:

\[
\alpha_t = I(X) + u_t \in \mathcal{D}^r_1((0, T) \times M)
\]

\[
p_t = \p_v\{-(|K(\alpha_t)|^2 - \sigma(u_t))/2 + |K(u_t)|^2 - \sigma(u_t)\} \in \mathcal{D}^r_0((0, T) \times M).
\]

Denoting $Q_n(t)$ by the solution of $\dot{Q}_n = v_n(Q_n)$, the solution $q_n(t)$ in (56) can be written as $\dot{q}_n(t) = Q_n(\beta_\omega t)$. Letting $\beta_\omega \to 0$, we see that $q_n$ moves quite slowly on the orbit of $Q_n$. In this sense, $\beta_\omega$ stands for the flexibility of the motion of point vortices besides the growth rate of the pressure relative to the kinetic energy density. We notice that as $\beta_\omega \to 0$ the pressure $p_t$ converges to $\p_v\{-(|K(\alpha_t)|^2 - \sigma(u_t))/2\}$ in $\mathcal{D}^r_0((0, T) \times M)$, which is consistent with the generalized steady Bernoulli law. From
this we can observe that point vortices are frozen if the pressure $p_t$ is sufficiently close to the generalized Bernoulli law in weak-$^*$ topology. As a consequence, we conclude that $\beta_\omega$ describes the flexibility of the motion of point vortices and the growth rate of the pressure and that point vortices are slower to move as the pressure is sufficiently close to the generalized Bernoulli law in weak-$^*$ topology.

Acknowledgements
The author thanks Professor Yoshihiko Mitsumatsu for giving opportunity to reconsider the necessity why point vortex dynamics should be formulated as a Hamiltonian system. He acknowledges the helpful suggestions of Professor Takashi Sakajo. He wishes to express gratitude to Professor Masayuki Asaoka for several helpful comments concerning multiplication of currents. He thanks the anonymous referee for constructive comments that improve the paper significantly. This work was supported by JSPS KAKENHI (no. 18J20037).

References

1. Arnold, V.: Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits. Annales de l’Institut Fourier 16, 319–361 (1966)
2. Arnold, V.I., Khesin, B.A.: Topological Methods in Hydrodynamics. Applied Mathematical Sciences, vol. 125. Springer, New York (1998)
3. Aubin, T.: Some Nonlinear Problems in Riemannian Geometry, Springer Monographs in Mathematics. Springer, Berlin (1998)
4. Bogomolov, V.A.: Two-dimensional fluid dynamics on a sphere. Akademiia Nauk SSSR Fizika Atmosfery i Okeana 15, 29–36 (1979)
5. Chorin, A.J., Marsden, J.E.: A Mathematical Introduction to Fluid Mechanics. Texts in Applied Mathematics, vol. 4, 3rd edn. Springer, New York (1993)
6. Delort, J.M.: Existence de nappes de tourbillon en dimension deux. J. Am. Math. Soc. 4, 553–586 (1991)
7. de Rham, G.: Differentiable manifolds, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 266. Springer, Berlin (1984). Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern
8. Dritschel, D. G., Boatto, S.: The motion of point vortices on closed surfaces. Proc. A 471, 20140890, 25 (2015)
9. Ebin, D. G., Marsden, J.: Groups of dieromorphisms and the motion of an incompressible fluid. Ann. Math. (2) 92, 102–163 (1970)
10. Flucher, M.: Variational Problems with Concentration. Progress in Nonlinear Differential Equations and their Applications, vol. 36. Birkhäuser Verlag, Basel (1999)
11. Gallot, S., Hulin, D., Lafontaine, J.: Riemannian Geometry. Universitext, 3rd edn. Springer, Berlin (2004)
12. Glass, O., Munnier, A., Sueur, F.: Point vortex dynamics as zero-radius limit of the motion of a rigid body in an irrotational fluid. Invent. Math. 27, 171–287 (2018)
13. Grotta Ragazzo, C.: The motion of a vortex on a closed surface of constant negative curvature. Proc. A 473, 20170447, 17 (2017)
14. Grotta Ragazzo, C., Viglioni, H. H. d B.: Hydrodynamic vortex on surfaces. J. Nonlinear Sci. 27, 1609–1640 (2017)
15. Gustafsson, B.: Vortex motion and geometric function theory: the role of connections. Philos. Trans. R. Soc. A 377, 20180341, 27 (2019)
16. Hally, D.: Stability of streets of vortices on surfaces of revolution with a reflection symmetry. J. Math. Phys. 21, 211–217 (1980)
17. Kimura, Y.: Vortex motion on surfaces with constant curvature. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 455, 245–259 (1999)
18. Kimura, Y., Okamoto, H.: Vortex motion on a sphere. J. Phys. Soc. Jpn. 56, 4203–4206 (1987)
19. Kunnen, R., Trieling, R., van Heijst, G.: Vortices in time-periodic shear. Theor. Comput. Fluid Dyn. 24, 315–322 (2010)
Euler–Arnold flow in the sense of de Rham currents

20. Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, vol. 27. Cambridge University Press, Cambridge (2002)
21. Marchioro, C., Pulvirenti, M.: Euler evolution for singular initial data and vortex theory. Commun. Math. Phys. 91, 563–572 (1983)
22. Marchioro, C., Pulvirenti, M.: Mathematical Theory of Incompressible Nonviscous Fluids. Applied Mathematical Sciences, vol. 96. Springer, New York (1994)
23. Montaldi, J., Soulière, A., Tokieda, T.: Vortex dynamics on a cylinder. SIAM J. Appl. Dyn. Syst. 2, 417–430 (2003)
24. Newton, P.K., Shokraneh, H.: The $\n$-vortex problem on a rotating sphere. I. Multi-frequency configurations. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462, 149–169 (2006)
25. Ohkitani, K.: Dynamical equations for the vector potential and the velocity potential in incompressible irrotational Euler flows: a refined bernoulli theorem. Phys. Rev. E 92, 033010 (2015)
26. Oliva, W.M.: Geometric Mechanics, vol. 1798. Springer, Berlin (2002)
27. Saffman, P.G.: Vortex Dynamics. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York (1992)
28. Sakajo, T.: Equation of motion for point vortices in multiply connected circular domains. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 465, 2589–2611 (2009)
29. Sakajo, T., Shimizu, Y.: Point vortex interactions on a toroidal surface. Proc. A. 472, 20160271, 24 (2016)
30. Sario, L., Nakai, M.: Classification Theory of Riemann Surfaces, Die Grundlehren der mathematischen Wissenschaften, vol. 164. Springer, New York (1970)
31. Schochet, S.: The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. Commun. Partial Differ. Equ. 20, 1077–1104 (1995)
32. Taylor, M.E.: Partial differential equations III. Nonlinear equations, Applied Mathematical Sciences, vol. 117, 2nd edn. Springer, New York (2011)
33. Tkachenko, V.K.: Stability of vortex lattices. Sov. J. Exp. Theor. Phys. 23, 1049 (1966)
34. Trieling, R.R., Dam, C.E.C., van Heijst, G.J.F.: Dynamics of two identical vortices in linear shear. Phys. Fluids 22, 117104 (2010)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.