OPERATOR IDEALS AND APPROXIMATION PROPERTIES

SILVIA LASSALLE AND PABLO TURCO

ABSTRACT. We use the notion of $\mathcal{A}$-compact sets, which are determined by a Banach operator ideal $\mathcal{A}$, to show that most classic results of certain approximation properties and several Banach operator ideals can be systematically studied under this framework. We say that a Banach space enjoys the $\mathcal{A}$-approximation property if the identity map is uniformly approximable on $\mathcal{A}$-compact sets by finite rank operators. The Grothendieck's classic approximation property is the $\mathcal{K}$-approximation property for $\mathcal{K}$ the ideal of compact operators and the $p$-approximation property is obtained as the $\mathcal{N}^p$-approximation property for $\mathcal{N}^p$ the ideal of right $p$-nuclear operators. We introduce a way to measure the size of $\mathcal{A}$-compact sets and use it to give a norm on $\mathcal{K}_\mathcal{A}$, the ideal of $\mathcal{A}$-compact operators. Most of our results concerning the operator Banach ideal $\mathcal{K}_\mathcal{A}$ are obtained for right-accessible ideals $\mathcal{A}$. For instance, we prove that $\mathcal{K}_\mathcal{A}$ is a dual ideal, it is regular and we characterize its maximal hull. A strong concept of approximation property, which makes use of the norm defined on $\mathcal{K}_\mathcal{A}$, is also addressed. Finally, we obtain a generalization of Schwartz theorem with a revisited $\epsilon$-product.

INTRODUCTION

The Grothendieck's classic approximation property is one of the most important properties in the theory of Banach spaces. A Banach space has the approximation property if the identity map can be uniformly approximated by finite rank operators on compact sets. There are several reformulations of this property, all of them involving at least one of the concepts: compact operators or uniform convergence on compact sets. Ever since Grothendieck's famous Résumé [16] and reinforced by the fact that there are Banach spaces which lack the approximation property (the first example given by Enflo [14]), important variants of this property have emerged and were intensively studied. The main developments on approximation properties can be found in [5, 18] and in the references therein.

The main purpose of this article is to undertake the study of a general method to understand a wide class of approximation properties and different ideals of compact operators which can be equally modeled once the system of compact sets has been chosen. To this end, we use a refined notion of compactness, given by a Banach operator ideal $\mathcal{A}$, introduced by
Carl and Stephani [4]. Then, we say that a Banach space has the \( A \)-approximation property if the identity map is uniformly approximated by finite rank operators on \( A \)-compact sets. Also, the system of \( A \)-compact sets induces in a natural way the class of \( A \)-compact operators, consisting of all the continuous linear operators mapping bounded sets into \( A \)-compact sets. This ideal, which we denote by \( K_A \), was introduced and studied in [4]. However, the authors do not emphasize their study from a geometrical point of view. Here, we introduce a way to measure the size of \( A \)-compact sets and then use our definition to endow \( K_A \) with a norm \( \| \cdot \|_{K_A} \), under which it is a Banach operator ideal.

The paper is divided in three parts. Fixed a Banach operator ideal \( A \), we give the basics of \( A \)-compact sets, then we study the ideal of \( A \)-compact operators \((K_A, \| \cdot \|_{K_A})\) and finally we apply the results obtained to study two natural types of approximation properties induced by \( A \). To exemplify many of our results we appeal to the concept of \( p \)-compact sets, \( p \)-compact operators and the \( p \) and \( \kappa_p \)-approximation properties (see definitions below). More precisely, in Section 1 we examine the class of \( A \)-compact sets in a Banach space \( E \) \((K \subset E\) is relatively \( A \)-compact if there exist a Banach space \( X \), an operator \( T \in A(X;E) \) and a compact set \( M \subset X \) such that \( K \subset T(M) \)). We show that the definition can be reformulated considering only operators in \( A(\ell_1;E) \). Also, we show that the class of \( p \)-compact sets fits in this framework for the ideal of right \( p \)-nuclear operators, \( \mathcal{N}^p \). This fact and the notion of \( A \)-null sequences [4], allow us to solve a question posed in [9] which was also settled independently by Oja in her recent work [20].

Section 2 is devoted to the ideal of \( A \)-compact operators with an appropriate norm. Our main results are obtained for right-accessible ideals \( A \), which include minimal and injective ideals (see definitions below). When \( A \) is right-accessible, we prove that \( K_A \) is a dual operator ideal, it is regular and we characterize its maximal hull. Also we show that coincides with the surjective hull of the minimal kernel of \( A \). Then, different Banach operator ideals may produce the same ideal of compact operators. This is the case of \( \mathcal{N}^{\sigma} \), the dual ideal of the \( p \)-nuclear operators and \( \Pi_p^d \), the dual ideal of the \( p \)-summing operators; both produce the ideal of \( p \)-compact operators, \( K_p \). As a consequence of our results, we give a factorization of \( K_p \) in terms of \( \Pi_p^d \) and \( K \) the ideal of compact operators. The ideal \( K_p \), also coincides with that of \( \mathcal{N}^p \)-compact operators and it is not injective [13 Proposition 3.4]. However, we show that any injective Banach operator ideal \( A \) produces an injective ideal \( K_A \). We finish the section with some results on \( A^d \)-compact operators, with \( A^d \) the dual ideal of \( A \) and apply our results to give some examples.

Finally, the last section deals with two types of approximation properties related with \( A \)-compact sets. The \( A \)-approximation property can be seen as a way to weaken the Grothendieck’s classic approximation property. It is defined by changing the system of compact sets by the system of \( A \)-compact sets. For the particular case of \( \mathcal{N}^p \), we recover the
notion of $p$-approximation property, which was studied for many authors in the last years, see for instance [2, 6, 8, 17, 23]. We prove that a Banach space $E$ enjoys the $A$-approximation property if and only if the set of finite rank operators from $F$ to $E$ is norm-dense in $\mathcal{K}_A(F; E)$, for all Banach spaces $F$. If we take into account the norm $\| \cdot \|_{\mathcal{K}_A}$ instead of the supremum norm, we obtain the $sA$-approximation property, which is stronger than the $A$-approximation property. In this case, when $A$ is $\mathcal{N}^p$, we recover the $\kappa_p$-approximation property, defined in [10] and studied later in [13, 17, 19]. We study in tandem both types of approximation properties and show that in general they differ. Also, we address the $A$-approximation property in terms of a refined notion of the $\epsilon$-product of Schwartz.

Throughout this paper $E$ and $F$ denote Banach spaces, $E'$ and $B_E$ denote the topological dual and the closed unit ball of $E$, respectively. A general Banach operator ideal is denoted by $(A, \| \cdot \|_A)$. When the norm $\| \cdot \|_A$ is understood we simply write $A$. We denote by $\mathcal{L}, \mathcal{F}, \mathcal{F}'$ and $\mathcal{K}$ the operator ideals of linear bounded, finite rank, approximable and compact operators, respectively; all considered with the supremum norm. Often, for $x' \in E'$ and $y \in F$, the $1$-rank operator from $E$ to $F$, $x \mapsto x'(x)y$ is denoted by $x' \otimes y$.

Along the manuscript, we use several classic Banach operator ideals such as the ideal of $p$-nuclear, quasi $p$-nuclear and $p$-summing operators, $1 \leq p < \infty$, denoted by $\mathcal{N}_p, \mathcal{QN}_p$ and $\Pi_p$, respectively. The basics for these ideals may be found in [7, 13, 21] or [23]. Also, to illustrate our results, we appeal to the ideals of right $p$-nuclear operators $\mathcal{N}^p$, and $p$-compact operators $\mathcal{K}_p$. To give a brief description of these ideals, we need some definitions.

Fix $1 \leq p < \infty$, a sequence $(x_n)_n$ in $E$ is said to be $p$-summable if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$ and it is said to be weakly $p$-summable if $\sum_{n=1}^{\infty} \|x'(x_n)\|^p < \infty$, for all $x' \in E'$. As usual, $\ell_p(E)$ and $\ell^u_p(E)$ denote the spaces of all $p$-summable and weakly $p$-summable sequences in $E$, respectively. Both are Banach spaces, the first one considered with the norm $\|(x_n)_n\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$ and the second one with the norm $\|(x_n)_n\|_{p}^u = \sup_{x' \in B_{E'}} \{(\sum_{n=1}^{\infty} \|x'(x_n)\|^p)^{1/p}\}$.

For $p = \infty$, we have the spaces $c_0(E)$ and $c_0^u(E)$ of all null and weakly null sequences of $E$, respectively. Both spaces are endowed with their natural norms.

A mapping $T \in \mathcal{L}(E; F)$ belongs to the ideal of right $p$-nuclear operators $\mathcal{N}^p(E; F)$, if there exist sequences $(x'_n)_n \in \ell_{p'}^u(E')$ and $(y_n)_n \in \ell_p(F)$, \(\frac{1}{p} + \frac{1}{p'} = 1\) (\(\ell_{p'} = c_0\) if \(p = 1\)), such that $T = \sum_{n=1}^{\infty} x'_n \otimes y_n$. The right $p$-nuclear norm of $T$ is defined by $\nu^p(T) = \inf\{\|(x'_n)_n\|_{\ell_{p'}^u(E')}\|(y_n)_n\|_{\ell_p(F)} : T = \sum_{n=1}^{\infty} x'_n \otimes y_n\}$.

To describe $p$-compact operators, the notion of $p$-compact sets is required. Following [25], we say that a subset $K \subset E$ is relatively $p$-compact, $1 \leq p \leq \infty$, if there exists a sequence $(x_n)_n \subset \ell_p(E)$ so that $K \subset p\text{-co}\{x_n\}$, where $p\text{-co}\{x_n\} = \{\sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p'}}\}$ is the $p$-convex hull of $(x_n)_n$ and \(\frac{1}{p} + \frac{1}{p'} = 1\) (\(\ell_{p'} = c_0\) if \(p = 1\)). With $p = \infty$, we have the relatively compact sets and the balanced convex hull of $(x_n)_n$, $\text{co}\{x_n\}$. A mapping $T \in \mathcal{L}(E; F)$
belongs to the ideal of \( p \)-compact operators \( \mathcal{K}_p(E;F) \), if it maps bounded sets into relatively \( p \)-compact sets and \( \kappa_p(T) = \inf \{ \|(y_n)\|_p : T(B_E) \subset p\text{-co}\{y_n\} \} \) is the \( p \)-compact norm of \( T \).

All the definitions and notation used regarding operator ideals can be found in the monograph by Defant and Floret [7]. For further reading on operator ideals we refer the reader to the books of Pietsch [21], of Diestel, Jarchow and Tonge [13] and of Ryan [23]. For further information on approximation properties, we refer the reader to the survey by Casazza [5] and to the book of Lindenstrauss and Tzafriri [18], see also [12] and [23].

1. On compact sets and operator ideals

Fix a Banach operator ideal \( (\mathcal{A}, \| \cdot \|_\mathcal{A}) \). Following [4], a set \( K \subset E \) is relatively \( \mathcal{A} \)-compact if there exist a Banach space \( X \), an operator \( T \in \mathcal{A}(X;E) \) and a compact set \( M \subset X \) such that \( K \subset T(M) \). A sequence \( (x_n)_n \subset E \) is \( \mathcal{A} \)-convergent to zero if there exist a Banach space \( X \) and \( T \in \mathcal{A}(X;E) \) with the following property: given \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that \( x_n \in \varepsilon T(B_X) \), for all \( n \geq n_\varepsilon \). There is a handy characterization of \( \mathcal{A} \)-null sequences.

**Lemma 1.1.** [4 Lemma 1.2] Let \( E \) be a Banach space and \( \mathcal{A} \) a Banach operator ideal. A sequence \( (x_n)_n \subset E \) is \( \mathcal{A} \)-null if and only if there exist a Banach space \( X \), an operator \( T \in \mathcal{A}(X;E) \) and a null sequence \( (y_n)_n \subset X \) such that \( x_n = T(y_n) \), for all \( n \).

The following characterization of \( \mathcal{A} \)-compactness was extracted from [4 Section 1].

**Theorem 1.2.** Let \( E \) be a Banach space, \( K \subset E \) a subset and \( \mathcal{A} \) a Banach operator ideal. The following are equivalent.

(i) \( K \) is relatively \( \mathcal{A} \)-compact.

(ii) There exist a Banach space \( X \) and an operator \( T \in \mathcal{A}(X;E) \) such that for every \( \varepsilon > 0 \) there are finitely many elements \( z_i^\varepsilon \in E \), \( 1 \leq i \leq k_\varepsilon \) realizing a covering of \( K \):

\[
K \subset \bigcup_{i=1}^{k_\varepsilon} \{ z_i^\varepsilon + \varepsilon T(B_X) \}.
\]

(iii) There exists an \( \mathcal{A} \)-null sequence \( (x_n)_n \subset E \) such that \( K \subset \text{co}\{x_n\} \).

**Example 1.3.** Compact sets are \( \mathcal{K} \)-compact sets and \( p \)-compact sets are \( N^p \)-compact sets.

**Proof.** Let \( K \subset E \) be a \( p \)-compact set, then \( K \subset p\text{-co}\{x_n\} = \{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\ell_{p'}} \} \) with \((x_n)_n \in \ell_p(E) \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \) \((\ell_{p'} = c_0 \text{ if } p = 1\) ). Take \( \beta = (\beta_n)_n \in B_{c_0} \) such that \((\beta_n x_n)_n \in \ell_p(E) \). Let \( y_n = \frac{x_n}{\beta_n} \) and \((e'_n)_n \) the sequence of coordinate functionals on \( \ell_{p'} \). Define \( T : \ell_{p'} \to E \) the linear operator by \( T = \sum_{n=1}^{\infty} e'_n \otimes y_n \). Then, \( K \subset T(M) \) with \( M = \{ (\alpha_n \beta_n)_n : (\alpha_n)_n \in B_{\ell_{p'}} \} \). The result follows by noting that \( T \in N^p(\ell_{p'};E) \) and \( M \subset B_{\ell_{p'}} \) is relatively compact. \( \square \)

Recently, Delgado and Piñeiro [9] define \( p \)-null sequences, \( p \geq 1 \), as follows. A sequence \((x_n)_n \) in a Banach space \( E \) is \( p \)-null if, given \( \varepsilon > 0 \), there exist \( n_0 \in \mathbb{N} \) and \((z_k)_k \in \varepsilon B_{\ell_{p'}}(E) \) such that \( x_n \in p\text{-co}\{z_k\} \) for all \( n \geq n_0 \). In [9 Theorem 2.5 ], \( p \)-compact sets are characterize
as those which are contained in the convex hull of a $p$-null sequence. Then, the authors prove, under certain hypothesis on the Banach space $E$, that a sequence is $p$-null if and only if it is norm convergent to zero and relatively $p$-compact, [9, Proposition 2.6]. Also, they wonder if the result remains true for arbitrary Banach spaces. We give an affirmative answer to this question as an immediate consequence of the above results.

**Corollary 1.4.** Let $E$ be a Banach space and $1 \leq p \leq \infty$. A sequence $(x_n)_n \subset E$ is $p$-null if and only if $(x_n)_n$ is norm convergent to zero and relatively $p$-compact.

**Proof.** By Example 1.3, the definition of $p$-null sequences coincides with that of $\mathcal{A}$-null sequences for $\mathcal{A} = \mathcal{N}^p$. Then, the result follows from [4, Lemma 1.2] and the equivalence (i) and (iii) of Theorem 1.2. □

When this manuscript was complete we learned that E. Oja has also obtained Corollary 1.4. She gives a description of the space of $p$-null sequences as a tensor product via the Chevet-Saphar tensor norm. As an application, the result is obtained [20, Theorem 4.3].

Now, we introduce a way to measure the size of relatively $\mathcal{A}$-compact sets. Let $K \subset E$ be a relatively $\mathcal{A}$-compact set we define

$$m_\mathcal{A}(K; E) = \inf\{\|T\|_\mathcal{A} : K \subset T(M), T \in \mathcal{A}(X; E), M \subset B_X\},$$

where the infimum is taken considering all Banach spaces $X$, all operators $T \in \mathcal{A}(X; E)$ and all compact sets $M \subset B_X$ for which the inclusion $K \subset T(M)$ holds.

There are some properties which derive directly from the definition of $m_\mathcal{A}$. For instance $m_\mathcal{A}(K; E) = m_\mathcal{A}(\overline{\text{co}}\{K\}; E)$. Also, since $\|T\| \leq \|T\|_{\mathcal{A}}$ for any Banach operator ideal $\mathcal{A}$ and $T \in \mathcal{A}(X; E)$ then, $\sup_{x \in K} \|x\| \leq m_\mathcal{A}(K; E)$. Moreover, if $\mathcal{B}$ is a Banach operator ideal such that $\mathcal{A} \subset \mathcal{B}$, a set $K \subset E$ is $\mathcal{B}$-compact whenever it is $\mathcal{A}$-compact and $m_\mathcal{B}(K; E) \leq m_\mathcal{A}(K; E)$.

**Remark 1.5.** Example 1.3 shows that given a sequence $(x_n)_n \in \ell_p(E)$, there exist an operator $T \in \mathcal{N}^p(\ell_{p'}; E)$ and a relatively compact set $M \subset B_{\ell_{p'}}$ such that $p\text{-co}\{x_n\} = T(M)$. Moreover, fixed $\varepsilon > 0$, we can choose $T$ such that $\|(x_n)_n\|_p \leq \|T\|_{\mathcal{N}^p} \leq \|(x_n)_n\|_p + \varepsilon$. Then, if $K \subset E$ is $p$-compact,

$$m_{\mathcal{N}^p}(K; E) = \inf\{\|(x_n)_n\|_p : K \subset p\text{-co}\{x_n\}\}.$$

Note that $m_{\mathcal{N}^p}$ recovers the size of $p$-compact sets defined in [15, Definition 2.1]. Also, note that if $F$ is a Banach space containing $E$, any set $K \subset E$ is $\mathcal{A}$-compact as a set of $F$ whenever it is $\mathcal{A}$-compact as a set of $E$, and $m_\mathcal{A}(K; F) \leq m_\mathcal{A}(K; E)$. However, the definition of $m_\mathcal{A}$ may depend on the space the sets are considered, as it is shown in [15, Corollary 3.5]. We will prove, with additional hypotheses on $\mathcal{A}$, that the $m_\mathcal{A}$-size of an $\mathcal{A}$-compact set remains the same, see Corollary 2.8 for any pair of Banach spaces $E$ and $F$ such that $E \subset F$, and Corollary 2.15 for the special case when $F = E''$. 
The next result shows that the definition of $A$-compact sets (and therefore the size $m_A$) can be reformulated considering only operators in $A(\ell_1; E)$.

**Proposition 1.6.** Let $E$ be a Banach space, $K \subset E$ a subset and $A$ a Banach operator ideal. The following are equivalent.

(i) $K$ is relatively $A$-compact.

(ii) There exist a Banach space $X$, operators $T \in A(X; E)$ and $S \in \overline{F}(\ell_1; X)$ and a relatively compact set $M \subset B_{\ell_1}$ such that $K \subset T \circ S(M)$. Moreover,

$$m_A(K; E) = \inf\{\|T\|_A\|S\| : K \subset T \circ S(M); M \subset B_{\ell_1}\}.$$

where the infimum is taken over all Banach spaces $X$, operators $T$ and $S$ and sets $M$ as above.

**Proof.** First note that any set $K$ as in (ii) is relatively $A$-compact. Indeed, suppose that there exists a Banach space $X$ such that $K \subset T \circ S(M)$ for $T \in A(X; E)$, $S \in \overline{F}(\ell_1; F)$ and $M \subset B_{\ell_1}$ relatively compact. Then, $K \subset \|S\|T(S(M)/\|S\|)$ with $S(M)/\|S\| \subset B_X$ a relatively compact set. Also, $m_A(K; E) \leq \|S\||T||A\|

For the converse, since $K \subset E$ is relatively $A$-compact, given $\varepsilon > 0$ there exist a Banach space $X$, a compact set $L \subset B_X$ and a operator $T \in A(X; E)$ such that $K \subset T(L)$ and $\|T\|_A \leq m_A(K; E) + \varepsilon$. Since $L \subset B_X$ is compact, there exists a sequence $(y_n)_n \in c_0(X)$ such that $L \subset \{\sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n)_n \in B_{\ell_1}\}$ and $\sup_{n \in \mathbb{N}} \|y_n\| \leq 1 + \varepsilon$. Choose a sequence $(\beta_n)_n \in B_{\ell_1}$ such that $(\frac{\alpha_n}{\beta_n})_n \in c_0(X)$ and $\sup_{n \in \mathbb{N}} \|\frac{\alpha_n}{\beta_n}\| \leq \sup_{n \in \mathbb{N}} \|y_n\| + \varepsilon \leq 1 + 2\varepsilon$. Call $z_n = \frac{\alpha_n}{\beta_n}$ and $M = \{(\gamma_n)_n \in \ell_1 : \gamma_n = \beta_n \alpha_n, \ (\alpha_n)_n \in B_{\ell_1}\}$. Note that $M \subset B_{\ell_1}$ is a relatively compact set. Define the operator $S : \ell_1 \to X$ as $S(\gamma_n) = \sum_{n=1}^{\infty} \gamma_n z_n$. Since $S(B_{\ell_1}) \subset \overline{c_0}(z_n)$ and $\ell_1'$ has the approximation property, $S$ is approximable and $\|S\| \leq \sup_{n \in \mathbb{N}} \|z_n\| \leq 1 + 2\varepsilon$.

Moreover,

$$S(M) = \left\{\sum_{n=1}^{\infty} \gamma_n z_n : (\gamma_n)_n \in M\right\} = \left\{\sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n)_n \in B_{\ell_1}\right\},$$

then $L \subset S(M)$ and $K \subset T(L) \subset T \circ S(M)$.

We also have

$$\|T\|_A \|S\| \leq \|T\|_A(1 + 2\varepsilon) \leq (m_A(K; E) + \varepsilon)(1 + 2\varepsilon),$$

and the result follows by letting $\varepsilon \to 0$. \hfill $\Box$

**Corollary 1.7.** Let $E$ be a Banach space, $K \subset E$ a subset and $A$ a Banach operator ideal. Then, $K$ is relatively $A$-compact if and only if $K$ is relatively $A \circ \overline{F}$-compact and $m_A(K; E) = m_{A \circ \overline{F}}(K; E)$. 
Proof. Since $\mathcal{A} \circ \overline{\mathcal{F}} \subset \mathcal{A}$, every relatively $\mathcal{A} \circ \overline{\mathcal{F}}$-compact set is relatively $\mathcal{A}$-compact and $m_{\mathcal{A}}(K; E) \leq m_{\mathcal{A} \circ \overline{\mathcal{F}}}(K; E)$. The other implication is given by the item (ii) of the above proposition, which also gives that

$$m_{\mathcal{A}}(K; E) = \inf \{\|T\|_A\|S\|: K \subset T \circ S(M); M \subset B_{\ell_1}\}$$

$$\geq \inf \{\|T\|_A\|S\|: K \subset T \circ S(M); M \subset B_F\}$$

$$= m_{\mathcal{A} \circ \overline{\mathcal{F}}}(K; E)$$

and the proof is complete. \hfill \Box

**Corollary 1.8.** Let $E$ be a Banach space, $\mathcal{A}$ a Banach operator ideal and $K \subset E$ a relatively $\mathcal{A}$-compact set. Then,

$$m_{\mathcal{A}}(K; E) = \inf \{\|T\|_A: K \subset T(M), T \in \mathcal{A}(\ell_1; E) \text{ and } M \subset B_{\ell_1}\},$$

where the infimum is taken over all operators $T \in \mathcal{A}(\ell_1; E)$ and all relatively compact sets $M \subset B_{\ell_1}$ such that $K \subset T(M)$.

**Proof.** With standard notation, Proposition 1.6 gives

$$m_{\mathcal{A}}(K; E) = \inf \{\|T\|_A\|S\|: K \subset T \circ S(M); M \subset B_{\ell_1}\}$$

$$\geq \inf \{\|T\|_A: K \subset T(M); M \subset B_{\ell_1}\}$$

$$\geq m_{\mathcal{A}}(K; E),$$

which proofs the result. \hfill \Box

### 2. The ideal of $\mathcal{A}$-compact operators

Associated to the concept of $\mathcal{A}$-compact sets we have the notion of $\mathcal{A}$-compact operators, which generalizes that of compact operators. An operator $T \in \mathcal{L}(E; F)$ is said to be $\mathcal{A}$-compact if $T(B_E)$ is a relatively $\mathcal{A}$-compact set in $F$, \cite[Definition 2]{4}. The space of $\mathcal{A}$-compact operators, denoted by $\mathcal{K}_{\mathcal{A}}$, becomes an Banach operator ideal if endowed with the norm defined, for any $T \in \mathcal{K}_{\mathcal{A}}(E; F)$, by

$$\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(B_E); F).$$

With Example 1.3 and Remark 1.3 we obtain our first example.

**Example 2.1.** The ideal $\mathcal{K}_p$ and the ideal of $\mathcal{N}^p$-compact operators coincide isometrically.

We propose to study some properties enjoyed by $\mathcal{K}_{\mathcal{A}}$ and by the operator ideals obtained by the procedures

$$\mathcal{A} \rightarrow \mathcal{A}^{d}, \quad \mathcal{A} \rightarrow \mathcal{A}^{\min}, \quad \mathcal{A} \rightarrow \mathcal{A}^{\max}, \quad \mathcal{A} \rightarrow \mathcal{A}^{\text{sur}}, \quad \mathcal{A} \rightarrow \mathcal{A}^{\text{inj}}, \quad \mathcal{A} \rightarrow \mathcal{A}^{\text{reg}}.$$

The definitions of these procedures will be given opportune. We start by recalling the dual ideal of $\mathcal{A}$, $\mathcal{A}^{d}$. Given $T \in \mathcal{L}$ denote its adjoint by $T'$, then $\mathcal{A}^{d}(E; F) = \{T \in \mathcal{L}(E; F): T' \in \mathcal{A}^{d}(F; E)\}$.
\( \mathcal{A}(F', E') \) and \( \|T\|_{\mathcal{A}'} = \|T'\|_{\mathcal{A}} \). Also, recall that the minimal kernel of \( \mathcal{A} \), \( \mathcal{A}^{\text{min}} \) is the composition ideal \( \mathcal{A}^{\text{min}} = \mathcal{F} \circ \mathcal{A} \circ \mathcal{F} \) considered with its natural norm. The ideal is said to be minimal if \( \mathcal{A} = \mathcal{A}^{\text{min}} \), isometrically.

Most of our results are obtained for right-accessible operator ideals. By \cite[Proposition 25.2 (2)]{7} we may consider right-accessible Banach operator ideals as those which satisfy \( \mathcal{A}^{\text{min}} = \mathcal{A} \circ \mathcal{F} \), isometrically. By \cite[Corollary 21.3]{7} the left-accessible Banach operator ideals as those satisfying that its dual operator ideal is right-accessible. Also, \( \mathcal{A} \) is totally-accessible if for every finite rank operator \( T \in \mathcal{L}(E; F) \) and \( \varepsilon > 0 \) there exist \( Y \subset F \) a finite-dimensional subspace, \( X \subset E \) a subspace of finite-codimensional and \( S \in \mathcal{L}(E/X; Y) \) such that \( T = I_F S Q_E \) and \( \|S\|_{\mathcal{A}} \leq (1 + \varepsilon)\|T\|_{\mathcal{A}} \), where \( Q_E: E \to E/X \) and \( I_F: Y \to F \) are the canonical quotient mapping and the inclusion, respectively. If \( \mathcal{A} \) is totally-accessible then it is right and left-accessible, see \cite[21.2]{7}.

2.1. On \( \mathcal{A} \)-compact operators related with surjective and injective hulls. Recall that an operator \( T \in \mathcal{L}(E; F) \) belongs to the surjective hull of \( \mathcal{A}(E; F) \), \( \mathcal{A}^{\text{sur}}(E; F) \), if and only if \( T \circ q_E \) belongs to \( \mathcal{A} \) where \( q_E: \ell_1(B_E) \to E \) is the canonical surjection and \( \|T\|_{\mathcal{A}^{\text{sur}}} = \|T \circ q_E\|_{\mathcal{A}} \). On the other hand, an operator \( T \in \mathcal{L}(E; F) \) belongs to the injective hull of \( \mathcal{A}(E; F) \), \( \mathcal{A}^{\text{inj}}(E; F) \), if and only if \( \iota_F \circ T \in \mathcal{A} \), where \( \iota_F: F \to \ell_\infty(B_{F'}) \) is the canonical injection and \( \|T\|_{\mathcal{A}^{\text{inj}}} = \|\iota_F \circ T\|_{\mathcal{A}} \). The ideal \( \mathcal{A} \) is surjective if \( \mathcal{A} = \mathcal{A}^{\text{sur}} \) and it is injective if \( \mathcal{A} = \mathcal{A}^{\text{inj}} \), isometrically. Any injective ideal is right-accessible while any surjective ideal is left-accessible \cite[21.2]{7}.

In \cite[Theorem 2.1]{4}, the operator ideal \( \mathcal{K}_A \) is described in terms of \( \mathcal{A}^{\text{sur}} \) via the identities:
\[
\mathcal{K}_A = (\mathcal{A} \circ \mathcal{K})^{\text{sur}} = \mathcal{A}^{\text{sur}} \circ \mathcal{K}.
\]
Then, we have two direct consequences. First, \( \mathcal{K}_A = \mathcal{K}_{\mathcal{K}_A} \), and the process only may produce a new operator ideal the first time it is applied. Also, \( \mathcal{K}_A \) is surjective. From this second fact we observe that the ideal of nuclear operators \( \mathcal{N} \) does not coincide with \( \mathcal{K}_A \), for any Banach operator ideal \( \mathcal{A} \). With the next proposition we give a slight improvement of \cite[Theorem 2.1]{4} by considering approximable instead of compact operators.

**Proposition 2.2.** Let \( \mathcal{A} \) be a Banach operator ideal. Then
\[
\mathcal{K}_A = \mathcal{K}_{\mathcal{A} \circ \mathcal{F}} = (\mathcal{A} \circ \mathcal{F})^{\text{sur}} = \mathcal{A}^{\text{sur}} \circ \mathcal{K}, \quad \text{isometrically.}
\]

**Proof.** The isometric result is obtained by using the definition of \( \|\cdot\|_{\mathcal{K}_A} \) along the proof of \cite[Theorem 2.1]{4}. An application of Corollary \cite[Lemma 7]{4} completes the proof. \qed

**Corollary 2.3.** Let \( \mathcal{A} \) be a right-accessible Banach operator ideal. Then,
\[
\mathcal{K}_A = (\mathcal{A}^{\text{min}})^{\text{sur}}, \quad \text{isometrically.}
\]
Proposition 2.4. If $A$ is right-accessible, then $K_A$ is totally-accessible. In particular, 
\[ K_A = (K_{A_{\text{min}}}^{\text{sur}}), \text{ isometrically.} \]

Proof. By Proposition 2.2, $K_A = A_{\text{sur}} \circ K$. As $A$ is right-accessible, by [7, Ex 21.1], $A_{\text{sur}}$ is totally-accessible. Also, $K$ is injective and surjective, then $K$ is injective and left-accessible, [7, 21.2]. Hence, by [7, Proposition 21.4], $K_A$ is totally-accessible. Now, since $K_A = K_{K_A}$ and $K_A$ is totally-accessible, a direct application of Corollary 2.3 gives the result. \[ \square \]

Notice that for $A$ and $B$ two Banach operator ideals such that $A_{\text{sur}} = B_{\text{sur}}$, Proposition 2.2 gives $K_A = K_B$. If, in addition, $A$ and $B$ are right-accessible and $A_{\text{min}} = B_{\text{min}}$, then we also have $K_A = K_B$. Combining this two facts, we obtain different descriptions of the ideal of $p$-compact operators and a factorization via the dual ideal of the $p$-summing operators.

This last result was recently also obtained by Ain, Lillemets and Oja [1, Corollary 4.9] independently.

Example 2.5. Let $1 \leq p \leq \infty$. The following isometric identities hold.

(a) $K_p = (K_p^{\text{min}})_{\text{sur}}$.

(b) $K_p = K_{\Pi_p} = K_{\mathcal{N}_p}$.

(c) $K_p = \Pi_p \circ K_{\cdot}$. 

Proof. By [15, Proposition 3.9], $K_p$ is totally-accessible. Then, an application of Corollary 2.3 gives $K_p = (K_p^{\text{min}})_{\text{sur}}$, and (a) is obtained.

To prove that $K_p = K_{\Pi_p}$, note that as a consequence of [15, Proposition 3.9], both ideals $K_p^{\text{min}}$ and $(\Pi_p^{d})^{\text{min}}$ coincide isometrically. Also, $\Pi_p$ is totally-accessible [15, Remark 3.7]. Then, using (a) and Corollary 2.3 we have $K_p = (\Pi_p^{d})^{\text{min}}_{\text{sur}} = K_{\Pi_p}$.

For the other identity, note that $\mathcal{N}_p = (\mathcal{N}_p^{d})^{\text{min}}$. Since $\mathcal{N}_p$ is left-accessible, $\mathcal{N}_p^{d}$ is right-accessible. Another application of Corollary 2.3 gives $K_{\mathcal{N}_p} = (\mathcal{N}_p^{d})^{\text{min}}_{\text{sur}} = \mathcal{N}_p_{\text{sur}} = K_p$, and (b) is proved.

Statement (c) is a direct application of (b) and Proposition 2.2. In fact, $\Pi_p$ is a surjective ideal and therefore $K_p = K_{\Pi_p} = \Pi_p^{d} \circ K$ holds isometrically. \[ \square \]

In general, $K_A$ is not an injective ideal (consider, for instance, the ideal of $p$-compact operators [15, Proposition 3.4]). However, an injective ideal $A$ gives an injective ideal of $A$-compact operators. To show this, we need a preliminary lemma. Although, we believe that it should be a known result, we have not found it in the literature as stated here and we prefer to include a proof.

Lemma 2.6. Let $A$ be a Banach operator ideal. Then, $(A^{\text{inj}}_{\text{min}})_{\text{sur}} = (A_{\text{min}}^{\text{inj}})_{\text{sur}}$, isometrically.

Proof. Let $E$ and $F$ be Banach spaces. An operator $T$ belongs to $(A^{\text{inj}}_{\text{min}})_{\text{sur}}(E; F)$ if and only if $q_E \circ T \in A^{\text{inj}}_{\text{min}}(\ell_1(B_E); F)$, where $q_E : \ell_1(B_E) \to E$ is the canonical surjection. Since
Proposition 2.7. Let $\mathcal{A}$ be an injective Banach operator ideal then, $\mathcal{K}_A$ is also injective. That is,

$$\mathcal{K}_A = \mathcal{K}_A^{\text{inj}},$$

isometrically.

Proof. Since $\mathcal{A}$ is injective, it is right-accessible. Applying Corollary 2.3 and Lemma 2.6 we get

$$\mathcal{K}_A^{\text{inj}} = (\mathcal{A}^{\text{min \ inj}})^{\text{inj}} = (\mathcal{A}^{\text{inj min}})^{\text{sur}} = (\mathcal{A}^{\text{inj min}})^{\text{sur}} = \mathcal{K}_A.$$

All the identities are isometric identifications, thus the proof is complete. \(\Box\)

Notice that there are non injective Banach operator ideals that may induce an injective ideal of compact operators, this is the case of $\mathcal{F}$ and $\mathcal{K}_F = \mathcal{K}$. For operators ideals $\mathcal{A}$ such that $\mathcal{K}_A$ is injective, we show that a set is $\mathcal{A}$-compact regardless it is considered as set of a Banach space $F$ or as set of a closed subspace $E$ of $F$, with equal size.

Corollary 2.8. Let $E$ be a Banach space, $K \subset E$ a subset and $\mathcal{A}$ a Banach operator ideal such that $\mathcal{K}_A$ is injective. Then, $K$ is relatively $\mathcal{A}$-compact in $E$ if and only if $K$ is relatively $\mathcal{A}$-compact in $F$, for every Banach space $F$ containing $E$. Moreover, $m_\mathcal{A}(K; E) = m_\mathcal{A}(K; F)$.

In particular, the result applies to any injective Banach operator ideal $\mathcal{A}$.

Proof. One implication is clear. For the other one, let $\iota: E \hookrightarrow F$ a metric injection such that $\iota(K) \subset F$ is relatively $\mathcal{A}$-compact. We may assume that $K$ is convex, balanced and closed. Since, $K$ is compact, by [23, Lemma 4.11], there are a Banach space $G$ and a operator $T \in \mathcal{L}(G; E)$ such that $T(B_G) = K$. Then, $\iota \circ T \in \mathcal{K}_\mathcal{A}(G; F)$. Since $\mathcal{K}_A = \mathcal{K}_A^{\text{inj}}$, $K$ is relatively $\mathcal{A}$-compact in $E$ and $m_\mathcal{A}(K; E) = m_\mathcal{A}(K; F)$. \(\Box\)

2.2. On the maximal hull of $\mathcal{A}$-compact operators. We examine the maximal hull of $\mathcal{K}_A$ for right-accessible ideals $\mathcal{A}$. The maximal hull of $\mathcal{A}$, $\mathcal{A}^{\text{max}}$ consists of all the operators $T$ such that $R \circ T \circ S \in \mathcal{A}$ for any approximable operators $R$ and $S$ and $\|T\|_{\mathcal{A}^{\text{max}}} = \sup\{|R \circ T \circ S|_{\mathcal{A}}: \|R\|, \|S\| \leq 1\}$. The ideal is maximal if $\mathcal{A} = \mathcal{A}^{\text{max}}$, isometrically.

Proposition 2.9. Let $\mathcal{A}$ be a right-accessible Banach operator ideal, then

$$\mathcal{K}_A^{\text{max}} = (\mathcal{A}^{\text{max}})^{\text{sur}},$$

isometrically.

In particular, if $\mathcal{A}$ is maximal, then $\mathcal{K}_A^{\text{max}} = \mathcal{A}^{\text{sur}}$. 

Proof. Any Banach operator ideal satisfies the isometric identities: $(A^{\max})^{\sur} = (A^{\sur})^{\max}$ and $(A^{\min})^{\max} = A^{\max}$, see \cite[Proposition 8.7.14]{21} and \cite[Proposition 8.7.15]{21}. Now, by Corollary \ref{cor:2.3} we have

$$K^{\max}_A = (A^{\min \sur})^{\max} = (A^{\min})^{\max \sur} = (A^{\max})^{\sur}. \quad \square$$

The following corollary follows from Proposition \ref{prop:2.2} and Proposition \ref{prop:2.9}.

**Corollary 2.10.** If $A$ is right-accessible and maximal then

$$K_A = K^{\max}_A \circ K, \quad \text{isometrically.}$$

When we apply the above results to the ideal of $p$-compact operators, we obtain \cite[Theorem 11]{22} and \cite[Theorem 12]{22}, see also \cite[Corollary 3.6]{15}.

**Example 2.11.** For $1 \leq p \leq \infty$, the following isometric identities hold.

(a) $K^p = \Pi^d_p$.
(b) $(K^d_p)^{\max} = \Pi_p$.

Proof. By \cite[Proposition 8.7.12]{21}, $(A^d)^{\max} = (A^{\max})^d$, for any ideal $A$. Then, statement (b) follows from (a). To prove (a), note that $\Pi^d_p$ is maximal and surjective, thus we have the result from Example \ref{ex:2.5} item (b) and Proposition \ref{prop:2.9}. \hfill $\square$

2.3. **On the regular hull of $K_A$ and dual operator ideals.** The dual ideal of any Banach operator ideal is regular \cite[Ex.22.6]{7}. Now, we show that the regular hull of $K_A$ is a dual operator ideal. Let us recall the definitions. The regular hull of $A$, $A^{reg}$, is the class of all $T \in \mathcal{L}(E; F)$ such that $J_F \circ T \in A(E; F'')$ and $\|T\|_{A^{reg}} = \|J_F \circ T\|_A$, where $J_F : F \to F''$ is the canonical inclusion. The operator ideal $A$ is regular if $A = A^{reg}$, isometrically.

**Proposition 2.12.** Let $A$ be a Banach operator ideal. Then,

$$K^{reg}_A = K^{dd}_A, \quad \text{isometrically.}$$

Proof. We always have $K^{dd}_A \subset K^{reg}_A$, with $\| \cdot \|_{K^{reg}_A} \leq \| \cdot \|_{K^{dd}_A}$. For the other inclusion, let $E$ and $F$ be Banach spaces and take $T \in K^{reg}_A(E; F)$. In particular, $T$ is a compact operator. Thus,

$$T''(B_E) \subset J_F \circ T(B_E)'' = J_F \circ T(B_E).$$

Then, $T'' \in K_A(E''; F'')$. Moreover,

$$\|T\|_{K^{dd}_A} = \|T''\|_{K_A} \leq m_A(J_F \circ T(B_E); F'') = \|J_F \circ T\|_{K_A} = \|T\|_{K^{reg}_A},$$

and the isometric result holds. \hfill $\square$

For right-accessible ideals we have the following.
Proposition 2.13. Let $\mathcal{A}$ be a right-accessible Banach operator ideal, then $\mathcal{K}_\mathcal{A}$ is regular. That is,

$$\mathcal{K}_\mathcal{A} = \mathcal{K}_\mathcal{A}^{\text{reg}}, \quad \text{isometrically.}$$

Proof. We always have $\mathcal{K}_\mathcal{A} \subset \mathcal{K}_\mathcal{A}^{\text{reg}}$ and $\| \cdot \|_{\mathcal{K}_\mathcal{A}^{\text{reg}}} \leq \| \cdot \|_{\mathcal{K}_\mathcal{A}}$. Now, let $E$ and $F$ be Banach spaces and $T \in \mathcal{K}_\mathcal{A}^{\text{reg}}(E; F)$, then $J_F \circ T \in \mathcal{K}_\mathcal{A}(E; F'')$. Since $\mathcal{A}$ is right-accessible, by Corollary 2.3, $J_F \circ T \in (\mathcal{A}^{\text{min}})_{\text{sur}}(E; F'')$. As the dual of $\ell_1(B_E)$ has the approximation property, by [7, Proposition 9.8], if $q_E : \ell_1(B_E) \to E$ is the canonical surjection, then $J_F \circ T \circ q_E \in \mathcal{A}^{\text{min}}(\ell_1(B_E); F'')$. Another application of [7, Proposition 9.8] and Corollary 2.3 gives that $T \in (\mathcal{A}^{\text{min}})_{\text{sur}}(\ell_1(B_E); F'')$ and $\| T \|_{\mathcal{K}_\mathcal{A}^{\text{reg}}} = \| T \|_{\mathcal{K}_\mathcal{A}^{\text{reg}}}$. □

As a consequence of the above, we prove that whenever $\mathcal{A}$ is right-accessible, $\mathcal{K}_\mathcal{A}$ is a dual operator ideal.

Proposition 2.14. Let $\mathcal{A}$ be a Banach operator ideal. Then, $\mathcal{K}_\mathcal{A} \subset \mathcal{K}_\mathcal{A}^{\text{dd}}$ and $\| \cdot \|_{\mathcal{K}_\mathcal{A}^{\text{dd}}} \leq \| \cdot \|_{\mathcal{K}_\mathcal{A}}$. Moreover, if $\mathcal{A}$ is right-accessible, then

$$\mathcal{K}_\mathcal{A} = \mathcal{K}_\mathcal{A}^{\text{dd}}, \quad \text{isometrically.}$$

Proof. Since $\mathcal{K}_\mathcal{A} \subset \mathcal{K}_\mathcal{A}^{\text{reg}}$ and $\| \cdot \|_{\mathcal{K}_\mathcal{A}^{\text{reg}}} \leq \| \cdot \|_{\mathcal{K}_\mathcal{A}}$, the first statement holds by Proposition 2.12. For a right-accessible ideal $\mathcal{A}$, an application of Proposition 2.13 completes the proof. □

For operator ideals $\mathcal{A}$ such that $\mathcal{K}_\mathcal{A}$ is regular we can show that a set $K$ is $\mathcal{A}$-compact regardless it is considered as a subset of a Banach space $E$ or as a subset of its bidual $E''$, with equal size. For $p$-compact sets, this was shown in [15, Theorem 2.4], see also [11, Corollary 3.6].

Corollary 2.15. Let $E$ be a Banach space, $K \subset E$ a subset and $\mathcal{A}$ be a Banach operator ideal such that $\mathcal{K}_\mathcal{A}$ is regular. Then, $K$ is relatively $\mathcal{A}$-compact if and only if $K \subset E''$ is relatively $\mathcal{A}$-compact and $m_\mathcal{A}(K; E) = m_\mathcal{A}(K; E'')$.

In particular, the result applies to any right-accessible Banach operator ideal $\mathcal{A}$.

Proof. The result is obtained with a similar proof to that given in Corollary 2.8. □

2.4. On the ideal of $\mathcal{A}^d$-compact operators. When we consider a left-accessible ideal $\mathcal{A}$ and inspect the ideal of $\mathcal{A}^d$-compact operator, we can push a little bit further. We finish this section with two results that we apply to recover some relations satisfied by the ideals of $p$-compact and quasi $p$-nuclear operators. We need a preliminary lemma.

Lemma 2.16. Let $\mathcal{A}$ be a Banach operator ideal. Then, $(\mathcal{A}^d_{\text{min}})_{\text{sur}} = (\mathcal{A}^{\text{min}}_{\text{d}})_{\text{sur}}$, isometrically.
Proof. The same proof given in Lemma \ref{lemma2.6} works here using \cite[Corollary 22.8.1]{7} instead of \cite[Proposition 25.11.2]{7}.

Proposition 2.17. Let $\mathcal{A}$ be a left-accessible Banach operator ideal. Then

$$\mathcal{K}_{\mathcal{A}^d} = (\mathcal{A}^{\text{min inj}})^d,$$

isometrically.

Proof. Since $\mathcal{A}^d$ is right-accessible, we apply Corollary \ref{corollary2.3} the above lemma and \cite[Theorem 8.5.9]{21} to obtain the isometric identities

$$\mathcal{K}_{\mathcal{A}^d} = (\mathcal{A}^d)^{\text{sur}} = (\mathcal{A}^{\text{min inj}})^d.$$  \hfill \Box

Proposition 2.18. Let $\mathcal{A}$ be a left-accessible Banach operator ideal. Then

$$\mathcal{K}_{\mathcal{A}^d} = \mathcal{A}^{\text{min inj}},$$

isometrically.

Proof. One inclusion is obtained by Proposition \ref{proposition2.17} and the fact that injective ideals are always regular, then

$$\mathcal{K}_{\mathcal{A}^d} = (\mathcal{A}^{\text{min inj}})^d \subset (\mathcal{A}^{\text{min inj}})^{\text{reg}} = \mathcal{A}^{\text{min inj}}.$$  \hfill \Box

For the reverse inclusion, notice that $\mathcal{A}^{\text{min inj}} = (\mathcal{A}^{dd})^{\text{min inj}} \subset (\mathcal{A}^{dd})^d$, for any $\mathcal{A}$. Now, considering the injective hulls and applying \cite[Theorem 8.5.9]{21}, we obtain

$$\mathcal{A}^{\text{min inj}} \subset (\mathcal{A}^{\text{min inj}})^d \subset ((\mathcal{A}^{\text{min inj}})^d)^{\text{sur}}.$$  \hfill \Box

By Lemma \ref{lemma2.16} and Corollary \ref{corollary2.3}

$$((\mathcal{A}^{\text{min inj}})^d)^{\text{sur}} = ((\mathcal{A}^{\text{min inj}})^d)^{\text{sur}} = \mathcal{K}_{\mathcal{A}^d}^d.$$  \hfill \Box

All the inclusions considered are given by contractive maps, which completes the proof. \hfill \Box

We illustrate the above propositions with the following examples. In particular, the third statement recovers a result of \cite[Corollary 3.2]{26}. The other two identifications appear in \cite{11} and \cite{15}.

Example 2.19. Let $1 \leq p < \infty$. The following isometric identities hold.

(a) $\mathcal{K}_p = \mathcal{QN}_p^d$.

(b) $\mathcal{K}_{p}^d = \mathcal{QN}_p^d$.

(c) $\mathcal{K}_{p}^d = (\Pi_p^{\text{min inj}})^d$.

Proof. By Example \ref{example2.5} (a), write $\mathcal{K}_p = \mathcal{K}_{N_p^d}$. Note that $\mathcal{N}_p$ is minimal, hence it is left-accessible. Now we use that $\mathcal{N}_p = \mathcal{QN}_p^{\text{min inj}}$ with Proposition \ref{proposition2.17} to obtain (a) and with Proposition \ref{proposition2.18} to obtain (b). For the proof of (c), by Example \ref{example2.5} (b), write $\mathcal{K}_p = \mathcal{K}_{\Pi_p^d}$. Now, use that $\Pi_p$ is left-accessible and Proposition \ref{proposition2.18}. \hfill \Box
3. Approximation properties given by operator ideals

In this section we study two different types of approximation properties defined through \(\mathcal{A}\)-compact sets. To this end we consider two different topologies on \(\mathcal{L}(E; F)\).

**Definition 3.1.** Let \(\mathcal{A}\) be an operator ideal. On \(\mathcal{L}(E; F)\), we consider the topology of uniform convergence on \(\mathcal{A}\)-compact sets, \(\tau_\mathcal{A}\), which is given by the seminorms

\[
q_K(T) = \sup_{x \in K} \|T(x)\|,
\]

where \(K\) ranges over all \(\mathcal{A}\)-compact sets. When \(F = \mathbb{C}\), we simply write \(E'_\mathcal{A} = (\mathcal{L}(E; \mathbb{C}); \tau_\mathcal{A})\).

Note that if \(\mathcal{A} = \mathcal{K}\) we obtain \(E'_\mathcal{c}\), the dual space of \(E\) endowed with the topology of uniform convergence on compact sets.

The other topology we consider is induced by the size of the \(\mathcal{A}\)-compact sets \(m_\mathcal{A}\), defined in Section 1.

**Definition 3.2.** Let \(\mathcal{A}\) be an operator ideal. On \(\mathcal{L}(E; F)\), we define the topology of strong uniform convergence on \(\mathcal{A}\)-compact sets, \(\tau_{s\mathcal{A}}\), which is given by the seminorms

\[
q_K(T) = m_\mathcal{A}(T(K); F),
\]

where \(K\) ranges over all \(\mathcal{A}\)-compact sets.

The following statements have straightforward proofs.

**Remark 3.3.** Let \(\mathcal{A}\) be a Banach operator ideal and let \(E\) and \(F\) be Banach spaces.

(a) The topologies \(\tau_{s\mathcal{K}}\) and \(\tau_\mathcal{K}\) coincide on \(\mathcal{L}(E; F)\).

(b) The topologies \(\tau_{s\mathcal{A}}\) and \(\tau_\mathcal{A}\) coincide on \(\mathcal{L}(E; \mathbb{C})\).

(c) \(\text{Id}: (\mathcal{L}(E; F), \tau_{s\mathcal{A}}) \to (\mathcal{L}(E; F), \tau_\mathcal{A})\) is continuous.

(d) \(\text{Id}: (\mathcal{L}(E; F), \tau_\mathcal{B}) \to (\mathcal{L}(E; F), \tau_\mathcal{A})\) is continuous, for any Banach operator ideal \(\mathcal{B}\) such that \(\mathcal{A} \subset \mathcal{B}\).

Based on the Grothendieck’s classic approximation property we have the following definitions.

**Definition 3.4.** Let \(E\) be a Banach space and \(\mathcal{A}\) a Banach operator ideal.

We say that \(E\) has the \(\mathcal{A}\)-approximation property if \(\mathcal{F}(E; E)\) is \(\tau_\mathcal{A}\)-dense in \(\mathcal{L}(E; E)\).

Also, \(E\) has the (strong) \(s\mathcal{A}\)-approximation property if \(\mathcal{F}(E; E)\) is \(\tau_{s\mathcal{A}}\)-dense in \(\mathcal{L}(E; E)\).

It is clear that the \(\mathcal{K}\), the \(s\mathcal{K}\) and the classic approximation properties coincide for any Banach space. Also, the classic approximation property implies the \(\mathcal{A}\)-approximation property, for any \(\mathcal{A}\). However, it may not imply the \(s\mathcal{A}\)-approximation property, see Example 3.5 (a) below or the comments below Proposition 3.10 in [15]. From Remark 3.3 (c), we see that
the $s \mathcal{A}$-approximation property is stronger than the $\mathcal{A}$-approximation property, although the converse might be false as Example 3.5 (a) below shows. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are two Banach operator ideals and $\mathcal{A} \subset \mathcal{B}$, from Remark 3.3 (d), the $\mathcal{B}$-approximation property implies the $\mathcal{A}$-approximation property. Nonetheless, a Banach space may have the $s \mathcal{B}$-approximation property and fail to have the $s \mathcal{A}$-approximation property, see Example 3.5 (b).

Notice that $\mathcal{N}^p$-approximation property is the $p$-approximation property introduced by Sinha and Karn in [25] and then studied in [2, 6, 8, 17]. On the other hand, the $s \mathcal{N}^p$-approximation property coincides with the $\kappa^p$-approximation property defined by Delgado, Piñeiro and Serrano (see [10, Remark 2.2]) and studied later in [15, 17, 19]. In many cases, the $s \mathcal{N}^p$ and the $\mathcal{N}^p$-approximation properties coincide. This happens, for instance, on any closed subspace of $L_p(\mu)$. Moreover, in any closed subspace of $L_p(\mu)$, the $s \mathcal{N}^p$-approximation property coincides with the $\mathcal{K}$-approximation property [19, Theorem 1]. However, these properties may differ as it is summarized below.

**Example 3.5.** Let $1 < p < 2$. There exists a Banach space

(a) with the $s \mathcal{N}^p$-approximation property which fails to have the approximation property.

(b) with the $\mathcal{N}^p$-approximation property which fails to have the $s \mathcal{N}^p$-approximation property.

(c) with the $s \mathcal{N}^2$-approximation property which fails to have the $s \mathcal{N}^p$-approximation property.

**Proof.** Fix $1 < q < 2$ and let $X$ be a subspace of $\ell_q$, without the approximation property. Note that $X$ is reflexive and has cotype 2. Now, combining the comment below [7, Proposition 21.7] and [10, Corollary 2.5] we see that $X'$ the $\kappa^p$-approximation property for any $1 < p < 2$. Then, $E = X'$ is an example satisfying (a).

If $1 \leq p < 2$, every Banach space has the $\mathcal{N}^p$-approximation property [25, Theorem 6.4]. But given $1 < p < 2$ there exists a Banach space which fails to have the $s \mathcal{N}^p$-approximation property [10, Theorem 2.4]. Which proves (b).

Finally, every Banach space has the $s \mathcal{N}^2$-approximation property [10, Corollary 3.6], but given $1 < p < 2$ there exists a Banach space which fails to have the $s \mathcal{N}^p$-approximation property, which proves (c).

**3.1. On finite rank and compact operators and approximation properties.** Now, we inspect the $\mathcal{A}$ and the $s \mathcal{A}$-approximation properties in relation with the ideal of finite rank and compact operators. The first proposition refines a classical result on approximation properties and finite rank operators. Its proof is standard and we omit it.

**Proposition 3.6.** Let $E$ be a Banach space. The following are equivalent.

(i) $E$ has the $\mathcal{A}$-approximation property ($s \mathcal{A}$-approximation property).
(ii) \( \text{Id} \in \mathcal{F}(E; E)^{\tau_A} (\text{Id} \in \mathcal{F}(E; E)^{\tau_A}) \).

(iii) For every Banach space \( F \), \( \mathcal{F}(E; F) \) is \( \tau_A \)-dense \( (\tau_{sA} \text{-dense}) \) in \( \mathcal{L}(E; F) \).

(iv) For every Banach space \( F \), \( \mathcal{F}(F; E) \) is \( \tau_A \)-dense \( (\tau_{sA} \text{-dense}) \) in \( \mathcal{L}(F; E) \).

A Banach space \( E \) has the approximation property if and only if the space of finite rank operators from any Banach space \( F \) into \( E \) is norm dense in the ideal of compact operators. The analogous result remains valid for the \( \mathcal{A} \)-approximation property and the \( sA \)-approximation property. Here, the ideal of \( \mathcal{A} \)-compact operators replaces \( \mathcal{K} \).

**Proposition 3.7.** Let \( E \) be a Banach space and \( \mathcal{A} \) a Banach operator ideal. The following are equivalent.

(i) \( E \) has the \( \mathcal{A} \)-approximation property.

(ii) For all Banach spaces \( F \), \( \mathcal{F} \circ \mathcal{K}_A(F; E) \) is \( \| \cdot \| \)-dense in \( \mathcal{K}_A(F; E) \).

(iii) For all Banach spaces \( F \), \( \mathcal{F}(F; E) \) is \( \| \cdot \| \)-dense in \( \mathcal{K}_A(F; E) \).

**Proof.** To prove that (i) implies (ii) take \( T \in \mathcal{K}_A(F; E) \) and \( \varepsilon > 0 \). Since \( T(B_F) \) is relatively \( \mathcal{A} \)-compact and \( E \) has the \( \mathcal{A} \)-approximation property, by Proposition 3.6 there exists \( S \in \mathcal{F}(E; E) \) such that \( \sup_{x \in T(B_F)} \| Sx - x \| \leq \varepsilon \). Then, \( \| ST - T \| \leq \varepsilon \) and (ii) holds.

The inclusion \( \mathcal{F} \circ \mathcal{K}_A \subset \mathcal{F} \) proves that (ii) implies (iii). To show that (iii) implies (i) take \( K \subset E \) an \( \mathcal{A} \)-compact set and \( \varepsilon > 0 \). There exist a Banach space \( G \), an absolutely convex compact set \( L \subset G \) and an operator \( T \in \mathcal{A}(G; E) \) such that \( K \subset T(L) \). Now, we may find an absolutely convex compact set \( \tilde{L} \subset G \), a Banach space \( F \) and an injective operator \( i \in \mathcal{L}(F; G) \) such that \( L \subset \tilde{L} \) and \( i(B_F) = \tilde{L} \), see for instance [23, Lemma 4.11]. In particular \( i \) is a compact operator and \( i^{-1}(L) \subset B_F \) is compact.

Consider the following diagram

\[
\begin{array}{ccc}
F & \xrightarrow{i} & G \\
\downarrow q & & \downarrow T \\
F/\ker T \circ i & = & E
\end{array}
\]

where \( q \) is the quotient map and \( T \circ i = \overline{T \circ i} \circ q \). Since \( \overline{T \circ i} \circ (B_F/\ker T \circ i) = T \circ i(B_F) = T(\tilde{L}) \) is an \( \mathcal{A} \)-compact set, then \( \overline{T \circ i} \in \mathcal{K}_A(F/\ker T \circ i; E) \). By hypothesis, there exists a finite rank operator \( S: F/\ker T \circ i \to E \), with \( \| S - \overline{T \circ i} \| \leq \varepsilon/2 \). Write \( S = \sum_{j=1}^{n} y_j' \otimes x_j \) with \( x_1, \ldots, x_n \in E \) and \( y_j', \ldots, y_n' \in (F/\ker T \circ i)' \). To find \( R: E \to E \) a finite rank operator approximating the identity on \( K \), note that \( q(i^{-1}(L)) \subset F/\ker T \circ i \) is compact. Then, for \( \delta = \varepsilon/2 \sum_{j=1}^{n} \| x_j \| \), take \( x_1', \ldots, x_n' \in E' \) such that

\[
\sup_{w \in q(i^{-1}(L))} |y_j'(w) - (\overline{T \circ i})'(x_j')(w)| \leq \delta.
\]
for all \( j = 1, \ldots, n \), and define \( R \) by \( R = \sum_{j=1}^{n} x_j' \otimes x_j \). Then,

\[
\sup_{x \in K} \|(Id - R)(x)\| \leq \sup_{y \in L} \|(Id - R)(Ty)\|
\]

\[
= \sup_{y \in L} \|(T \circ i - R \circ T \circ i)(i^{-1}(y))\|
\]

\[
\leq \sup_{y \in L} \|(T \circ i - S \circ q)(i^{-1}(y))\| + \sup_{y \in L} \|(S \circ q - R \circ T \circ i)(i^{-1}(y))\|.
\]

On one hand we have

\[
\sup_{y \in L} \|(T \circ i - S \circ q)(i^{-1}(y))\| \leq \sup_{z \in B_F} \|(T \circ i - S \circ q)(z)\|
\]

\[
\leq \sup_{w \in B_F \cap ker T \circ i} \|(T \circ i - S)(w)\|
\]

\[
= \|S - T \circ i\|.
\]

On the other hand,

\[
\sup_{y \in L} \|(S \circ q - R \circ T \circ i)(i^{-1}(y))\| = \sup_{y \in L} \|(\sum_{j=1}^{n} y_j' \circ q \otimes x_j - \sum_{j=1}^{n} x_j' \circ T \circ i \otimes x_j)(i^{-1}(y))\|
\]

\[
\leq \sum_{j=1}^{n} \sup_{w \in q(i^{-1}(L))} \|y_j'(w) - T \circ i (x_j')(w)\||x_j||x_j\|
\]

\[
\leq \sum_{j=1}^{n} \delta \|x_j\|.
\]

Therefore, \( \sup_{x \in K} \|(Id - R)(x)\| \leq \varepsilon \), proving (i). \( \square \)

The analogous result concerning the \( s\mathcal{A} \)-approximation property requires the norm \( \|.|\|_{\mathcal{K}_A} \). Its proof is similar to that given above and we omit it.

**Proposition 3.8.** Let \( E \) be a Banach space and \( \mathcal{A} \) a Banach operator ideal. The following are equivalent.

(i) \( E \) has the \( s\mathcal{A} \)-approximation property.

(ii) For all Banach spaces \( F \), \( \mathcal{F} \circ \mathcal{K}_A(F; E) \) is \( \|.|\|_{\mathcal{K}_A} \)-dense in \( \mathcal{K}_A(F; E) \).

(iii) For all Banach spaces \( F \), \( \mathcal{F}(F; E) \) is \( \|.|\|_{\mathcal{K}_A} \)-dense in \( \mathcal{K}_A(F; E) \).

The notion of the minimal kernel of an operator ideal allows us to give another characterization of the \( s\mathcal{A} \)-approximation property.

**Corollary 3.9.** Let \( E \) be a Banach space and \( \mathcal{A} \) a Banach operator ideal. Then, \( E \) has the \( s\mathcal{A} \)-approximation property if and only if \( \mathcal{K}_A(F; E) = \mathcal{K}_{\mathcal{A}}^{\min}(F; E) \), for all Banach spaces \( F \).

**Proof.** Since \( \mathcal{K}_A \) is a surjective ideal, it is left-accessible. Thus, by [7, Proposition 25.2], \( \mathcal{K}_{\mathcal{A}}^{\min} = \mathcal{F} \circ \mathcal{K}_A \). An application of Proposition 3.8 completes the proof. \( \square \)

The above corollary recovers the characterization of the \( \kappa_p \)-approximation property in terms of the ideals \( \mathcal{K}_p \) and \( \mathcal{K}_p^{\min} \), see the comments below [15, Proposition 3.9]. In general, the approximation property does not imply the \( s\mathcal{A} \)-approximation property. However, for right-accessible ideals the claim is true as it happens with the \( \kappa_p \)-approximation property [15, Proposition 3.10].

**Proposition 3.10.** Let \( E \) be a Banach space and \( \mathcal{A} \) be a right-accessible Banach operator ideal. If \( E \) has the approximation property, then \( E \) has the \( s\mathcal{A} \)-approximation property.
Proof. If $E$ has the approximation property, by [7, Proposition 25.11],

$$(k_A^{\text{min}})^{\text{sur}}(F; E) = (k_A^{\text{sur}})^{\text{min}}(F; E) = k_A^{\text{min}}(F; E),$$

for every Banach operator ideal $\mathcal{A}$ and for every Banach space $F$. Since $\mathcal{A}$ is right-accessible, applying Proposition 2.3 we see that $k_A(F; E) = k_A^{\text{min}}(F; E)$ for every Banach space $F$. Therefore, by the above corollary, $E$ has the $s\mathcal{A}$-approximation property. ∎

3.2. On the dual space $E_A'$ and the $\epsilon$-product of Schwartz. The classic approximation property has a well known reformulation in terms of the $\epsilon$-product of Schwartz. More precisely, $E$ has the approximation property if and only if $E \otimes F$ is dense in $\mathcal{L}_\epsilon(E'; F)$ for every locally convex space $F$, [24, Exposé 14]. This result remains valid for the $\mathcal{A}$-approximation property with a general Banach operator ideal $\mathcal{A}$. We denote by $\mathcal{L}_\epsilon(E_A'; F)$ the space of all linear continuous maps from $E_A'$ to a locally convex space $F$, endowed with the topology of uniform convergence on all equicontinuous sets of $E'$. The topology on $\mathcal{L}_\epsilon(E_A'; F)$ is generated by the seminorms $\beta(T) = \sup_{x' \in B_{E'}} \alpha(T x')$, where $\alpha \in \text{cs}(F)$, the set of all continuous seminorms on $F$. As usual, $U^o$ denotes the polar set of a set $U$.

**Theorem 3.11.** Let $E$ be a Banach space and $\mathcal{A}$ a Banach operator ideal. The following statements are equivalent.

(i) $E$ has the $\mathcal{A}$-approximation property.

(ii) $E \otimes F$ is dense in $\mathcal{L}_\epsilon(E_A'; F)$ for all locally convex spaces $F$.

(iii) $E \otimes E'$ is dense in $\mathcal{L}_\epsilon(E_A'; E_A')$.

Proof. Suppose (i) holds. Fix a locally convex space $F$, a continuous seminorm $\beta$ so that $\beta(T) = \sup_{x' \in B_{E'}} \alpha(T x')$, where $\alpha \in \text{cs}(F)$ and $\epsilon > 0$. Take $T \in \mathcal{L}_\epsilon(E_A'; F)$, then we may find an absolutely convex $\mathcal{A}$-compact set $K \subset E$ such that $\sup_{x' \in K^0} \alpha(T(x')) \leq 1$. If $U = \{y \in F : \alpha(y) \leq 1\}$, then $U$ is a neighborhood of $F$ and $T'(U^o) \subset K$. Since $E$ has the $\mathcal{A}$-approximation property, there exists a finite rank operator $S$ such that $\|Sx - x\| < \epsilon$ for all $x \in K$. In particular, $\|Sx - x\| < \epsilon$ for all $x \in T'(U^o)$. Then, for all $x' \in B_{E'}$ and $y' \in U^o$ we have $|x'(S(T'(y'))) - T'(y'))| < \epsilon$. Now, suppose that $S = \sum_{j=1}^n x'_j \otimes x_j$ with $x_1, \ldots, x_n \in E$, $x_1', \ldots, x_n' \in E'$, then

$$|y'\left(\sum_{j=1}^n x_j(x_j)T(x_j') - T(x')\right)| < \epsilon;$$

for any $y' \in U^o$ and $x' \in B_{E'}$. Therefore, taking $R = \sum_{j=1}^n x_j \otimes T(x_j')$ we have $\beta(R - T) < \epsilon$, which proves (ii).

That (ii) implies (iii) is clear. To finish the proof suppose that (iii) holds. Take an absolutely convex $\mathcal{A}$-compact set $K \subset E$ and let $\beta$ be the continuous seminorm given by $\beta(T) = \sup_{x' \in B_{E'}} \alpha(T)$, where $\alpha$ is the Minkowski functional of $K^0$. Since $Id \in \mathcal{L}_\epsilon(E_A'; E_A')$, we have
given $\varepsilon > 0$, there exist $S \in E \otimes E'$ such that $\beta(S - Id) < \varepsilon$. Then, as above, $|x'(S'(x) - x)| < \varepsilon$, for all $x' \in B_{E'}$ and $x \in K$. Therefore, $\|S'(x) - x\| < \varepsilon$ for all $x \in K$, and the proof is complete.

A Banach space $E$ has the classic approximation property if and only if $E'_c$ has the approximation property, [24] Exposé 14]. Aron, Maestre and Rueda show the analogous result for the $p$-approximation property [2] Theorem 4.6]. Here, we present a generalization of these results.

**Proposition 3.12.** Let $E$ be a Banach space and $A$ a Banach operator ideal. Then, $E$ has the $A$-approximation property if and only if $E'_A$ has the approximation property.

**Proof.** The locally convex space $E'_A$ has the approximation property if and only if for any $\varepsilon > 0$, $K \subset E$ an $A$-compact set and $M \subset E'_A$ relatively compact, there exists $S \in \mathcal{F}(E; E)$ such that

\[
| (S' - Id)(x')(x)| \leq \varepsilon, \quad \text{for all } x' \in M, \ x \in K.
\]

The continuity of the identity map $E'_c \rightarrow E'_A \rightarrow (E', w^*)$ says that relatively compact sets in $E'_A$ coincide with $\|\cdot\|$-bounded sets. Then, $E'_A$ has the approximation property if and only if in (1), $M$ is replaced by $B_{E'}$, which is equivalent to say that $E$ has the $A$-approximation property. □

Last but not least, we give a characterization of an $A$-compact operator in terms of the continuity and compactness of its adjoint. This result is well known for compact operators and was studied in the polynomial and holomorphic setting in [3]. Recall that if $E$ and $F$ are locally convex spaces and $U \subset F$ is absolutely convex and closed, for a linear mapping $T: E \rightarrow F$ we have $T(x) \in U$ if and only if $|y'(Tx)| \leq 1$ for all $y' \in U^\circ$, with $x \in E$.

**Proposition 3.13.** Let $E$ and $F$ be Banach spaces, $T \in \mathcal{L}(E; F)$ and $A$ a Banach operator ideal. The following statements are equivalent.

(i) $T \in \mathcal{K}_A(E; F)$.

(ii) $T': F'_A \rightarrow E'$ is continuous.

(iii) $T': F'_A \rightarrow E'_c$ is compact.

(iv) $T': F'_A \rightarrow E_B$ is compact for any Banach operator ideal $B$.

(v) There exists a Banach operator ideal $B$ such that $T': F'_A \rightarrow E'_B$ is compact.

(vi) $T': F'_A \rightarrow E'_w$ is compact.

**Proof.** Suppose (i) holds, then $\overline{T(B_E)} = K$ is $A$-compact and $K^\circ$ is a neighborhood in $F'_A$. Thus, for $y' \in K^\circ$ we have that $\|T'(y')\| = \sup_{x \in B_E} |T'(y')(x)| \leq 1$, proving (ii).

Now, suppose (ii) holds. Then, there exists a relative $A$-compact set $K \subset F$ such that $T'(K^\circ)$ is equicontinuous in $E'$ which, by the Ascoli theorem, is relatively compact in $E'_c$.  

obtaining (iii). Since $Id: E'_c \to E'_B$ is continuous for any Banach operator ideal $\mathcal{B}$, (iii) implies (iv). That (iv) implies (v) and (v) implies (vi) is clear. It remains to show that (vi) implies (i). Let $L \subset E'$ be a $w^*$-compact set (hence $\| \cdot \|$-bounded), and $K \subset F$ an absolutely convex and $\mathcal{A}$-compact set such that $T'(K^o) \subset L$. As $T'(K^o)$ is $\| \cdot \|$-bounded, there exists $c > 0$ such that $|T'(y')(x)| \leq c$ for every $y' \in K^o$ and $x \in B_E$. Therefore, $T(B_E) \subset cK$ which ends the proof.

3.3. On the dual $E'$ and approximation properties given by $\mathcal{A}$. We present a short discussion on approximation properties for dual spaces and the dual ideal of $\mathcal{K}_A$. Recall that, even though $E$ has the approximation property, the space of finite rank operators $\mathcal{F}(E; F)$ may not be dense in the space of compact operators $\mathcal{K}(E; F)$. For this to happen it is required $E'$ with the approximation property. The relations between density of finite rank operators and approximation properties related operator ideals is not without precedent. See [8, Theorem 2.8] for the $\mathcal{N}^p$-approximation property and [10, Theorem 2.3] for the $s\mathcal{N}^p$-approximation property. To give our result, involving a general operator ideal $\mathcal{A}$, we need the following lemma.

**Lemma 3.14.** Let $E$ and $F$ be Banach spaces and $\mathcal{A}$ a Banach operator ideal.

(a) The set $E \otimes F$ is $\tau_{s\mathcal{A}}$-dense in $\mathcal{F}(E'; F)$.

(b) The set $E \otimes F$ is $\tau_{\mathcal{A}}$-dense in $\mathcal{F}(E'; F)$.

**Proof.** Statement (b) follows from (a). To prove the first claim, let $T \in \mathcal{F}(E'; F)$ and suppose $T = \sum_{j=1}^n x''_j \otimes y_j$ with $x''_j \in E''$, $y_j \in F$, $j = 1, \ldots, n$ and $\sum_{j=1}^n \|y_j\| \leq 1$. Fix $\varepsilon > 0$ and $K \subset E'$ an $\mathcal{A}$-compact set (and hence compact). By the Alaoglu theorem, there exist $x_1, \ldots, x_n \in E$ such that $\sup_{x' \in K} |x''_j(x') - x'(x_j)| < \varepsilon$, for $j = 1, \ldots, n$. Take $S$ the finite rank operator, $S = \sum_{j=1}^n x_j \otimes y_j$, then

$$m_{\mathcal{A}}((T - S)(K); F) \leq \sum_{j=1}^n m_{\mathcal{A}}((x''_j - x_j) \otimes y_j)(K); F) = \sum_{j=1}^n \sup_{x' \in K} |x''_j(x') - x'(x_j)||y_j| \leq \varepsilon,$$

and the proof is complete. $\Box$

The $\mathcal{A}$ and the $s\mathcal{A}$-approximation properties are determined on dual spaces by denseness of finite rank operators in the dual ideal of $\mathcal{K}_A$, which is not a surprise at the light of the examples mentioned above, [8, Theorem 2.8] and [10, Theorem 2.3]. We only give a proof for the result concerning the $s\mathcal{A}$-approximation property, so we state this result first.

**Proposition 3.15.** Let $E$ be a Banach space and $\mathcal{A}$ a Banach operator ideal. The following are equivalent.

(i) $E'$ has the $s\mathcal{A}$-approximation property.
(ii) For every Banach space $F$, $\mathcal{F}(E;F)$ is $\| \cdot \|_{\mathcal{K}_A^d}$-dense in $\mathcal{K}_A^d(E;F)$.

Proof. If (i) holds, fix $\varepsilon > 0$ and take $T \in \mathcal{K}_A^d(F;F')$. Since $T' \in \mathcal{K}_A^d(F';F'')$ and $E'$ has the $s\mathcal{A}$-approximation property, by Lemma 3.14 there exists $S \in \mathcal{F}(E;F)$ such that $\|T - S\|_{\mathcal{K}_A^d} = \|S' - T'\|_{\mathcal{K}_A} \leq \varepsilon$ which gives (ii).

Now suppose (ii) holds and take $T \in \mathcal{K}_A^d(F;F')$. By Proposition 2.14, $T' \in \mathcal{K}_A^d(F'';F')$ and $T' \circ J_E \in \mathcal{K}_A^d(E'';F')$. Fix $\varepsilon > 0$, by hypothesis, there exists $S \in \mathcal{F}(E;F')$ such that $\|S - T' \circ J_E\|_{\mathcal{K}_A^d} \leq \varepsilon$. Since $T = (J_E)' \circ T'' \circ J_F$ and $S' \circ J_F \in \mathcal{F}(F;F')$, we see that

$$\|S' \circ J_F - T\|_{\mathcal{K}_A} \leq \|S' - (T' \circ J_E)'\|_{\mathcal{K}_A} = \|S - T' \circ J_E\|_{\mathcal{K}_A^d} \leq \varepsilon.$$  

An application of Proposition 3.8 gives the result. \hfill $\square$

Finally, we have the analogous result for $E'$ and the $\mathcal{A}$-approximation property.

**Proposition 3.16.** Let $E$ be a Banach space and $\mathcal{A}$ a Banach operator ideal. The following are equivalent.

(i) $E'$ has the $\mathcal{A}$-approximation property.

(ii) For every Banach space $F$, $\mathcal{F}(E;F)$ is $\| \cdot \|$-dense in $\mathcal{K}_A^d(E;F)$.

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Department of Mathematics, Universidad de San Andrés, Vito Dumas 284, (B1644BID) Victoria, Buenos Aires, Argentina, FCEN - UBA and IMAS - CONICET.

E-mail address: slassall@dm.uba.ar

Department of Matemática - Pab I, Facultad de Cs. Exactas y Naturales, Universidad de Buenos Aires, (1428) Buenos Aires, Argentina and IMAS - CONICET

E-mail address: paturco@dm.uba.ar