Softening of First-Order Phase Transition on Quenched Random Gravity Graphs

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Abstract

We perform extensive Monte Carlo simulations of the 10-state Potts model on quenched two-dimensional $\Phi^3$ gravity graphs to study the effect of quenched coordination number randomness on the nature of the phase transition, which is strongly first order on regular lattices. The numerical data provides strong evidence that, due to the quenched randomness, the discontinuous first-order phase transition of the pure model is softened to a continuous transition, representing presumably a new universality class. This result is in striking contrast to a recent Monte Carlo study of the 8-state Potts model on two-dimensional Poissonian random lattices of Voronoi/Delaunay type, where the phase transition clearly stayed of first order, but is in qualitative agreement with results for quenched bond randomness on regular lattices. A precedent for such softening with connectivity disorder is known: in the 10-state Potts model on annealed $\Phi^3$ gravity graphs a continuous transition is also observed.
1 Introduction

Systems subject to quenched random disorder often show a completely different behavior than in the pure case. If the pure system has a continuous phase transition it is well known that quenched random disorder can drive the critical behavior into a new universality class, or the transition can even be eliminated altogether.\(^1\) In the case of a first-order phase transition in the pure system the effect of quenched random disorder can also be very dramatic. In fact, phenomenological renormalization-group arguments suggest the possibility for a softening to a continuous transition.\(^2\)

The paradigm for testing the latter prediction is the two-dimensional (2D) \(q\)-state Potts model. This model is exactly known\(^3\) to exhibit on regular lattices for \(q \geq 5\) a first-order transition whose strength increases with \(q\). In Ref.\(^4\) the effect of quenched bond disorder was investigated for the 8-state model. By means of extensive Monte Carlo (MC) simulations the predicted softening was confirmed, and a finite-size scaling (FSS) analysis showed that the critical behavior of the quenched model could be well described by the Onsager Ising model universality class.\(^4\)

In Refs.\(^5,6\) the effect of quenched connectivity disorder was studied by putting the 8-state model on 2D Poissonian random lattices with toroidal topology, constructed according to the Voronoi/Delaunay prescription.\(^7\) Here the bonds are all of equal strength, but the distribution of coordination numbers \((= 3, \ldots, \infty)\) varies randomly from lattice to lattice, giving rise to the quenched random disorder. MC simulations combined with FSS analyses provided clear evidence that for this type of quenched disorder the transition stays first order as in the pure case.

A different, stronger\(^1\), sort of coordination number randomness appears in 2D gravity triangulations or their dual \(\Phi^3\) graphs. In such models one is interested in the coupling of matter to 2D gravity, so the disorder is annealed rather than quenched. Motivated by Wexler's mean field results for \(q = \infty\) Potts models coupled to 2D gravity,\(^8\) simulations of the 10-state and 200-state Potts model coupled to 2D gravity (i.e., on an annealed ensemble of \(\Phi^3\) graphs of spherical topology) gave convincing evidence\(^9\) for a continuous transition, with the measured critical exponents for the 10-state Potts model being consistent with the KPZ exponents of the 4-state Potts model coupled

\(^1\)We provide some justification for the use of “stronger” in what follows.
to 2D gravity.

As the only quenched connectivity disorder seriously investigated to date, 2D Poissonian random lattices, showed no sign of softening for first-order transitions it is interesting to enquire whether the salient feature for the softening in the 2D gravity simulations is the annealed nature of the connectivity disorder or whether it is some intrinsic features of the graphs themselves. The simulations also touch on the question of what constitutes the universality class of the KPZ exponents for matter coupled to 2D gravity. Recent work has suggested that the full 2D gravity curvature distribution is not required on dynamical triangulations in order to stay within the KPZ universality class.\(^{10,11}\) The current work can also be viewed as addressing the question of whether quenching the dynamical lattices has any effect on the exponents.

In this note we present results of a MC study on quenched random lattices drawn from the equilibrium distribution of pure 2D gravity triangulations. To be precise we simulated the 10-state Potts model on the dual of these triangulations, the so-called $\Phi^3$ graphs of spherical topology. As our main result we obtain strong evidence that for the gravity type of random lattices the transition is indeed softened and seems to define a new “quenched” universality class.

## 2 Model and simulation

We used the standard definition of the $q$-state Potts model,

$$Z_{\text{Potts}} = \sum_{\{\sigma_i\}} e^{-\beta E}; E = -\sum_{\langle ij \rangle} \delta_{\sigma_i\sigma_j}; \sigma_i = 1, \ldots, q,$$

(1)

where $\beta = J/k_B T$ is the inverse temperature in natural units, $\delta$ is the Kronecker symbol, and $\langle ij \rangle$ denotes the nearest-neighbour bonds of random $\Phi^3$ graphs (without tadpoles or self-energy bubbles) with $N = 250, 500, 1000, 2000, 3000, 5000, 10000$ sites. For each lattice size we generated 64 independent replica using the Tutte algorithm,\(^{12}\) and performed long simulations of the 10-state model near the transition point at $\hat{\beta} = 2.20, 2.20, 2.20, 2.22, 2.23, 2.24, 2.242$, respectively, using the single-cluster update algorithm.\(^{13}\) After thermalization we recorded 500 000 measurements (750 000 for $N = 10000$), taken after 6, 6, 6, 20, 15, 15, 15 clusters had been flipped, of the energy $E$ and the magnetization $M = (q \max \{n_i\} - N)/(q - 1)$ in 64
time-series files, where \( n_i \leq N \) denotes the number of spins of “orientation” \( i = 1, \ldots, q \) in one lattice configuration. Obviously it is sufficient to store the integers \( N/q \leq \max \{ n_i \} \leq N \). The corresponding quantities per site are denoted in the following by \( e = E/N \) and \( m = M/N \).

Given this raw data we employed standard reweighting techniques\(^{14} \) to compute, e.g., the specific heat, \( C^{(i)}(\beta) = \beta^2 N (\langle e^2 \rangle - \langle e \rangle^2) \), for each replica labeled by the superindex \( (i) \), and then performed the replica average, \( C(\beta) = [C^{(i)}(\beta)] = (1/64) \sum_{i=1}^{64} C^{(i)}(\beta) \), denoted by the square brackets. To perform the replica average at the level of the \( C^{(i)}(\beta) \) (and not at the level of energy moments) is motivated by the general rule that quenched averages should be performed at the level of the free energy and not the partition function.\(^{15} \) Finally, we determined the maximum, \( C_{\text{max}} = C(\beta_{\text{Cmax}}) \), for each lattice size and studied the FSS behavior of \( C_{\text{max}} \) and \( \beta_{\text{Cmax}} \). The error bars on the two quantities entering the FSS analysis are estimated by jack-kniving\(^{16} \) over the 64 replica. This takes into account the statistical errors on the estimates of each \( C^{(i)}(\beta) \) as well as the fluctuations among the different \( C^{(i)}(\beta), i = 1, \ldots, 64 \), caused by the quenched connectivity disorder of the randomly chosen \( \Phi^3 \) graphs.

The analysis of the magnetic susceptibility, \( \chi(\beta) = \beta N (\langle m^2 \rangle - \langle m \rangle^2) \) and the energetic Binder parameter\(^{2} \), \( V_3(\beta) = 1 - [\langle e^4 \rangle] / 3 [\langle e^2 \rangle]^2 \), proceeds exactly along the same lines, yielding \( \chi_{\text{max}} \) and \( \beta_{\chi_{\text{max}}} \) as well as \( V_{3,\text{min}} \) and \( \beta_{V_{3,\text{min}}} \). In order to be prepared for the possibility of a second-order phase transition, the magnetic Binder parameter, \( U_3(\beta) = 1 - [\langle m^4 \rangle] / 3 [\langle m^2 \rangle]^2 \), was also measured as, in this case, its crossing for different lattice sizes provides an alternative determination of the critical coupling \( \beta_c \), and the FSS of either the maximum slopes or the slopes at \( \beta_c \) can be used to extract the correlation length exponent \( \nu \).

3 Results

The maxima of the specific heat and the susceptibility are shown in Fig. 1 on a linear scale. If the transition was of first-order one would expect for large system sizes an asymptotic FSS behavior of the form \( C_{\text{max}} = a_C + b_C N + \ldots \), and \( \chi_{\text{max}} = a_\chi + b_\chi N + \ldots \)\(^{17} \) As is evident from Fig. 1 a linear scaling with

\(^{2}\)The subindex refers to the specific choice of replica average as usually employed in spin glass simulations. For the other options, see Ref.\(^{5} \)
$N$ is clearly not consistent with our data. Even though it is intrinsically never clear what could happen for much larger lattice sizes, we take this as evidence for the softening to a continuous phase transition. Furthermore, at a first-order phase transition one would expect that the energetic Binder-parameter minima approach in the infinite-volume limit a non-trivial value related to the latent heat. Our data shown in Fig. 2, however, approach the trivial limit of $2/3$ which is another firm indication that the transition of the 10-state Potts model on quenched 2D gravity graphs is not of first-order.

Being thus convinced that the transition is continuous, the next goal is to determine the critical exponents and the corresponding universality class. To this end we have tried to describe the scaling of the specific-heat and susceptibility maxima with the standard FSS ansatz

$$C_{\text{max}} = a_C + b_C N^{\alpha/D\nu},$$

and

$$\chi_{\text{max}} = a_{\chi} + b_{\chi} N^{\gamma/D\nu},$$

where $\alpha$, $\gamma$, and $\nu$ are the usual universal critical exponents at a continuous phase transition, $a_{C,\chi}$ and $b_{C,\chi}$ are non-universal amplitudes, and $D$ is the intrinsic Hausdorff dimension of the graphs.

By performing a non-linear three-parameter fit to the specific-heat maxima we obtained $\alpha/D\nu = 0.22(7)$, with a reasonable goodness-of-fit parameter $Q = 0.10$. Since the background term $a_C$ turned out to be consistent with zero, we also tried linear two-parameter fits with $a_C = 0$ kept fixed. By omitting successively the smaller lattice sizes the resulting exponent estimates varied in the range of $\alpha/D\nu = 0.21 \ldots 0.26$. In particular the fit over all data points gave a fully consistent value of $\alpha/D\nu = 0.222(7)$ (with $Q = 0.15$) but, as expected, a much smaller error bar. The susceptibility maxima grow very fast with $N$, such that also here the constant term $a_{\chi}$ can safely be neglected. The linear fit over all data points yielded $\gamma/D\nu = 0.732(10)$ (with $Q = 0.27$), and omitting the $N = 250$ point we obtained $\gamma/D\nu = 0.719(14)$ (with $Q = 0.31$). The fits are shown together with the data on the log-log scale of Fig. 3.

The pseudo-transition points $\beta_{C_{\text{max}}}$, $\beta_{\chi_{\text{max}}}$, and $\beta_{V_{3,\text{min}}}$ are shown in Fig. 4 together with fits to the standard FSS ansatz

$$\beta_{C_{\text{max}}} = \beta_c + c_C N^{-1/D\nu},$$
Table 1: Fit results of $\beta_c(N) = \beta_c + aN^{-1/D\nu}$ to the pseudo-transition points.

| obs. | $\beta_c$ | $1/D\nu$ | $Q$ |
|------|-----------|----------|-----|
| $\beta_{C_{\text{max}}}$ | 2.2448(29) | 0.56(13) | 0.14 |
| $\beta_{X_{\text{max}}}$ | 2.2450(17) | 0.88(54) | 0.14 |
| $\beta_{V_{3,\text{min}}}$ | 2.2435(20) | 0.74(14) | 0.67 |

etc. Here we fitted again all available data down to $N = 250$ and in this way obtained at least for $\beta_c$ stable estimates. Our results are collected in Table 1, where we also give the exponent $1/D\nu$ which, however, has already quite large errors. By taking the average of the three estimates of $\beta_c$ in Table 1, we arrive at our final estimate of the transition point,

$$\beta_c = 2.2445 \pm 0.0020.$$  \hfill (5)

This value is consistent with the estimate obtained from the crossings of the magnetic Binder parameter $U_3$ for different lattice sizes. It is interesting to note that this value (allowing for a factor of two in the normalization of $\beta$) is very close to the $\beta_c$ measured for the 10-state Potts model on annealed 2D gravity graphs in\textsuperscript{18} \textsuperscript{3}, namely $\beta_c = 1.141(1)$. A similar effect is apparent in the simulations of the Ising model on quenched graphs in\textsuperscript{19} where the measured $\beta_c$ is identical to the annealed graph value.

A more precise estimate of the exponent $1/D\nu$ can be obtained by analyzing the FSS of the magnetic Binder-parameter slopes in the vicinity of $\beta_c$. Both, the maximum slopes and the slopes at $\beta_c$, are expected to scale as $dU_3/d\beta \propto N^{1/D\nu}$. The fit to the maximum slopes for $N \geq 500$ gave $1/D\nu = 0.616(29)$ (with $Q = 0.29$), and for the slopes at $\beta_c = 2.2445$ we obtained $1/D\nu = 0.614(30)$ (with $Q = 0.78$).

We now take stock of our numerical results. For convenience we show in Tables 2 and 3 the analytically calculated critical exponents for the 2-state Potts (Ising) and 4-state Potts model on flat 2D lattices, coupled to annealed 2D gravity (the standard KPZ result), and on quenched 2D gravity graphs (calculated in\textsuperscript{20} by taking $n$ matter copies with $n \to 0$ in the KPZ formulae).
Table 2: Critical exponents for Ising models

| Type     | $\alpha/D\nu$ | $\gamma/D\nu$ | $1/D\nu$ |
|----------|---------------|---------------|-----------|
| Fixed    | 0             | $7/8$         | $1/2$     |
| Annealed | $-1/3$        | $2/3$         | $1/3$     |
| Quenched | $-0.303...$   | $0.709...$    | $0.349...$|

Table 3: Critical exponents for $q = 4$ Potts models

| Type     | $\alpha/D\nu$ | $\gamma/D\nu$ | $1/D\nu$ |
|----------|---------------|---------------|-----------|
| Fixed    | $1/2$         | $7/8$         | $3/4$     |
| Annealed | 0             | $1/2$         | $1/2$     |
| Quenched | $0.177...$    | $0.709...$    | $0.589...$|

Given the measured value of $\gamma/D\nu \simeq 0.72(2)$ it might seem tempting to conclude, similar to the random bond case, that the critical behavior can be related to an Ising universality class. While our estimate for $\gamma/D\nu$ is very far off the Onsager value of $\gamma/D\nu = 0.875$ for 2D regular lattices, it is remarkably close to the value for an Ising model on annealed gravity graphs, $\gamma/D\nu = 0.666...$, and even fully compatible with the theoretical prediction for the Ising model on quenched $\Phi^3$ graphs $^4$, $\gamma/D\nu = 0.709...$. However, the same value is also predicted for the quenched 4-state Potts model and the estimates for $\alpha/D\nu \simeq 0.22(1)$ and $1/D\nu \simeq 0.61(3)$ itself are closer to the critical exponents of the 4-state Potts model on quenched 2D gravity graphs listed in the bottom line of Table 3. The results are not consistent with the KPZ critical exponents of the 4-state Potts model on annealed graphs because of the relatively large difference (compared with the Ising case) between the quenched and annealed exponents for $q = 4$.

$^4$As $\gamma/D\nu$ depends only on the conformal weight of the spin operator on both fixed and quenched lattices the values for the Ising and 4-state Potts model are “accidentally” equal.
On the evidence of our estimates, it would thus seem that the critical exponents of the 10-state Potts model on quenched 2D gravity graphs are best fitted by those of the 4-state Potts model on quenched 2D gravity graphs as listed in Table 3. The situation appears to be entirely analogous to that for the 10-state Potts model on annealed 2D gravity graphs, where the measured critical exponents are consistent with the annealed (KPZ) exponents for the 4-state Potts model coupled to 2D gravity.

4 Conclusions

To summarize, we have obtained strong numerical evidence that due to connectivity disorder the phase transition in the 10-state Potts model on quenched random gravity graphs is softened to a continuous phase transition. This result is in contrast to a recent simulation of the 8-state model on Poissonian Delaunay/Voronoi random lattices where the transition stays first order as on regular lattices. It is, however, in qualitative agreement with the quenched random bond case, and in particular it does follow the pattern of where the Ising transition on a quenched ensemble of 2D gravity graphs was of similar order to its counterpart on annealed graphs. Like the simulations in of the 10-state Potts model on annealed 2D gravity graphs we see exponents associated with the 4-state Potts model, in this case the “quenched” exponents. An obvious extension of the current work is to look at the 200-state Potts model on quenched graphs to see if, analogously to, Ising exponents are obtained (presumably the quenched exponents rather than the KPZ exponents in this case), and to study the associated cross-over as a function of $q$. As the current work provides rather stronger evidence for the quenched exponents than the Ising simulation of it would also be a worthwhile exercise to improve the statistics of this work to pin down the (small) difference between the annealed and quenched Ising exponents in the measurements.

As a parenthetical remark, it is worth noting that we have carried out our simulations on an ensemble of pure 2D gravity graphs, as this seemed the most natural choice from the theoretical considerations, where the back reaction of the matter on the geometry is essentially switched off when taking the quenched limit. It is not clear what to expect if one simulates the models on a quenched ensemble with the “wrong” background charge, such as $c = -2$. 
Simply switching off the back reaction in an annealed simulation leads to instability as there is insufficient separation between the graph and spin model time scales in such a case.

Without further theoretical work it is presumably very difficult to isolate the distinguishing parameters that classify the disorder in the coordination numbers as “strong” (gravity graphs) and “weak” (Poissonian lattices). One might think that the very different coordination number distributions for Poissonian Voronoi/Delaunay lattices and random gravity triangulations (the dual of $\Phi^3$ graphs) as shown in Fig. 5 were an important factor, but carried out a fairly brutal truncation of the 2D gravity distribution and still found the KPZ exponents for an Ising transition. It is possible that some other geometrical features are the deciding factor: by Euler’s relation, for both types of random lattices the average coordination number is 6 (up to a $1/N$ correction for spherical topology), but the higher moments are obviously very different. Long-range correlations in the lattice geometry, however, play presumably an even more important role than the local moments. As a first step in this direction it would be interesting to compare nearest-neighbour correlations as described by Aboav’s or Weaire’s law for the two different random lattice geometries.\(^1\)

The current work also throws an interesting light on,\(^1\) where it was observed that fluctuating connectivity in a flat 2D geometry was sufficient to obtain the KPZ exponents for the Ising model which are normally presumed to be produced by the curvature fluctuations of 2D gravity. In our simulations we have found the softening effects of 2D gravity persisting even when there is no fluctuating connectivity during the course of a simulation, even though we appear to have obtained a different, quenched set of exponents. The question of exactly what characterizes the universality class of the modified 2D gravity exponents (fluctuating connectivities, graph geometry,...?), whether annealed or quenched, is still open.

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$a + b N^{\alpha/Dv}$

$\alpha/Dv = 0.22(7)$

$Q = 0.10$
Figure 1: The specific-heat (a) and susceptibility (b) maxima vs the number of sites, $N$, showing that the data are incompatible with a scaling $\propto N$, which would be characteristic for a first-order phase transition.
Figure 2: Finite-size scaling of the energetic Binder-parameter minima. The horizontal dashed line shows the trivial limit $2/3$. 
\( a + b N^{\alpha/Dv} \)
\( \alpha/Dv = 0.22(7) \)
\( Q = 0.10 \)
Figure 3: Finite-size scaling of (a) specific-heat and (b) susceptibility maxima on a log-log scale, together with the fits discussed in the text.
Figure 4: Finite-size scaling of the pseudo-transition points, together with the fits discussed in the text.
Figure 5: Distribution of coordination numbers for gravity triangulations dual to $\Phi^3$ graphs and Poissonian Voronoi/Delaunay random lattices.