PROOF OF THE LOVÁSZ CONJECTURE

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Abstract. To any two graphs $G$ and $H$ one can associate a cell complex $\text{Hom}(G, H)$ by taking all graph multihomorphisms from $G$ to $H$ as cells.

In this paper we prove the Lovász Conjecture which states that if $\text{Hom}(C_{2r+1}, G)$ is $k$-connected, then $\chi(G) \geq k + 4$, where $r, k \in \mathbb{Z}$, $r \geq 1$, $k \geq -1$, and $C_{2r+1}$ denotes the cycle with $2r + 1$ vertices.

The proof requires analysis of the complexes $\text{Hom}(C_{2r+1}, K_n)$. For even $n$, the obstructions to graph colorings are provided by the presence of torsion in $H^*(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z})$. For odd $n$, the obstructions are expressed as vanishing of certain powers of Stiefel-Whitney characteristic classes of $\text{Hom}(C_{2r+1}, K_n)$, where the latter are viewed as $\mathbb{Z}_2$-spaces with the involution induced by the reflection of $C_{2r+1}$.

1. Introduction

The main idea of this paper is to look for obstructions to graph colorings in the following indirect way: take a graph, associate to it a topological space, and then look for obstructions to colorings of the graph by studying the algebraic invariants of this space.

The construction of such a space, which is of interest here, has been suggested by L. Lovász. The obtained complex $\text{Hom}(G, H)$ depends on two graph parameters. The algebraic invariants of this space, which we proceed to study, are its cohomology groups, and, when it can be viewed as a $\mathbb{Z}_2$-space, its Stiefel-Whitney characteristic classes.

1.1. The vertex colorings and the category of Graphs.

All graphs in this paper are undirected. The following definition is a key in turning the set of all undirected graphs into a category.

Definition 1.1. For two graphs $G$ and $H$, a graph homomorphism from $G$ to $H$ is a map $\phi : V(G) \to V(H)$, such that if $(x, y) \in E(G)$, then $(\phi(x), \phi(y)) \in E(H)$.

Here, $V(G)$ denotes the set of vertices of $G$, and $E(G)$ denotes the set of its edges.

For a graph $G$ the vertex coloring is an assignment of colors to vertices such that no two vertices which are connected by an edge get the same color. The minimal needed number of colors is denoted by $\chi(G)$, and is called the chromatic number of $G$.

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Deciding whether or not there exists a graph homomorphism between two graphs is in general at least as difficult as bounding the chromatic numbers of graphs because of the following observation: a vertex coloring of $G$ with $n$ colors is the same as a graph homomorphism from $G$ to the complete graph on $n$ vertices $K_n$. Because of this, one can also think of graph homomorphisms from $G$ to $H$ as vertex colorings of $G$ with colors from $V(H)$ subject to the natural condition.

Since an identity map is a graph homomorphism, and a composition of two graph homomorphisms is again a graph homomorphism, we can consider the category $\textbf{Graphs}$ whose objects are all undirected graphs, and morphisms are all the graph homomorphisms.

We denote the set of all graph homomorphisms from $G$ to $H$ by $\text{Hom}_0(G, H)$. Lovász has suggested the following way of turning this set into a topological space.

**Definition 1.2.** We define $\text{Hom}(G, H)$ to be a polyhedral complex whose cells are indexed by all functions $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $(x, y) \in E(G)$, for any $\tilde{x} \in \eta(x)$ and $\tilde{y} \in \eta(y)$ we have $(\tilde{x}, \tilde{y}) \in E(H)$.

The closure of a cell $\eta$ consists of all cells indexed by $\tilde{\eta} : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, which satisfy $\tilde{\eta}(v) \subseteq \eta(v)$, for all $v \in V(G)$.

We think of a cell in $\text{Hom}(G, H)$ as a collection of non-empty lists of vertices of $H$, one for each vertex of $G$, with the condition that any choice of one vertex from each list will yield a graph homomorphism from $G$ to $H$. A geometric realization of $\text{Hom}(G, H)$ can be described as follows: number the vertices of $G$ with $1, \ldots, |V(G)|$, the cell indexed with $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$ is realized as a direct product of simplices $\Delta^1, \ldots, \Delta^{|V(G)|}$, where $\Delta^i$ has $|\eta(i)|$ vertices and is realized as the standard simplex in $\mathbb{R}^{|\eta(i)|}$. In particular, the set of vertices of $\text{Hom}(G, H)$ is precisely $\text{Hom}_0(G, H)$.

The barycentric subdivision of $\text{Hom}(G, H)$ is isomorphic as a simplicial complex to the geometric realization of its face poset. So, alternatively, it could be described by first defining a poset of all $\eta$ satisfying conditions of Definition 1.2 with $\eta \geq \tilde{\eta}$ iff $\eta(v) \supseteq \tilde{\eta}(v)$, for all $v \in V(G)$, and then taking the geometric realization.

The $\text{Hom}$ complexes are functorial in the following sense: $\text{Hom}(H, -)$ is a covariant, while $\text{Hom}(-, H)$ is a contravariant functor from $\textbf{Graphs}$ to $\textbf{Top}$. If $\phi \in \text{Hom}_0(G, G')$, then we shall denote the induced cellular maps as $\phi_H : \text{Hom}(H, G) \to \text{Hom}(H, G')$ and $\phi_H : \text{Hom}(G', H) \to \text{Hom}(G, H)$.

### 1.2. The statement of the Lovász conjecture.

Lovász has stated the following conjecture, which we prove in this paper.

**Theorem 1.3.** (Lovász Conjecture). Let $G$ be a graph, such that $\text{Hom}(C_{2r+1}, G)$ is $k$-connected for some $r, k \in \mathbb{Z}$, $r \geq 1$, $k \geq -1$, then $\chi(G) \geq k + 4$.

Here $C_{2r+1}$ is a cycle with $2r + 1$ vertices: $V(C_{2r+1}) = \mathbb{Z}_{2r+1}$, $E(C_{2r+1}) = \{(x, x + 1), (x + 1, x) \mid x \in \mathbb{Z}_{2r+1}\}$.

The motivation for this conjecture stems from the following theorem which Lovász has proved in 1978.

**Theorem 1.4.** (Lovász, [13]). Let $H$ be a graph, such that $\text{Hom}(K_2, H)$ is $k$-connected for some $k \in \mathbb{Z}$, $k \geq -1$, then $\chi(H) \geq k + 3$.

One corollary of Theorem 1.4 is the Kneser conjecture from 1955, see [5].
Remark 1.5. The actual theorem from [15] is stated using the neighborhood complexes \( \mathcal{N}(H) \). However, it is well known that \( \mathcal{N}(H) \) is homotopy equivalent to \( \text{Hom}(K_2, H) \) for any graph \( H \), see, e.g., [2] for an argument. In fact, these two spaces are known to be simple-homotopy equivalent, see [18].

We note here that Theorem 1.3 is trivially true for \( k = -1 \): \( \text{Hom}(C_{2r+1}, G) \) is \((-1)\)-connected if and only if it is non-empty, and since there are no homomorphisms from odd cycles to bipartite graphs, we conclude that \( \chi(G) \geq 3 \). It is also not difficult to show that Theorem 1.3 holds for \( k = 0 \) by using the winding number. A short argument for a more general statement can be found in subsection 2.2.

1.3. Plan of the paper.

In Section 2, we formulate the main theorems and describe the general framework of finding obstructions to graph colorings via vanishing of powers of Stiefel-Whitney characteristic classes.

In Section 3, we introduce auxiliary simplicial complexes, which we call \( \text{Hom}_+(\cdot, \cdot) \). For any two graphs \( G \) and \( H \), there is a canonical support map \( \text{supp} : \text{Hom}_+(G, H) \to \Delta_{|V(G)|-1} \), and the preimage of the barycenter is precisely \( \text{Hom}(G, H) \). This allows us to set up a useful spectral sequence, filtering by the preimages of the \( i \)-skeleta.

In Section 4, we compute the cohomology groups \( H^*(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \) up to dimension \( n - 2 \), and we find the \( \mathbb{Z}_2 \)-action on these groups. These computations allow us to prove the Lovász conjecture for the case of odd \( k, k \geq 1 \).

In Section 5, we study a different spectral sequence, this one converging to \( H^*(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \). Understanding certain entries and differentials leads to the proof of the Lovász conjecture for the case of even \( k \) as well.

The results of this paper were announced in [1], where no complete proofs were given. The reader is referred to [12] for a survey on \( \text{Hom} \) complexes, which also includes a lot of background material which is omitted in this paper.

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2. The idea of the proof of the Lovász conjecture.

2.1. Group actions on \( \text{Hom} \) complexes and Stiefel-Whitney classes.

Consider an arbitrary CW complex \( X \) on which a finite group \( \Gamma \) acts freely. By the general theory of principal \( \Gamma \)-bundles, there exists a \( \Gamma \)-equivariant map \( \bar{w} : X \to \mathbb{E}\Gamma \), and the induced map \( w : X/\Gamma \to B\Gamma = \mathbb{E}\Gamma/\Gamma \) is unique up to homotopy.

Specifying \( \Gamma = \mathbb{Z}_2 \), we get a map \( \bar{w} : X \to S^\infty = \mathbb{E}\mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) acts on \( S^\infty \) by the antipodal map, and the induced map \( w : X/\mathbb{Z}_2 \to \mathbb{RP}^\infty = B\mathbb{Z}_2 \). We denote the induced \( \mathbb{Z}_2 \)-algebra homomorphism \( H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2) \) by \( w^* \). Let \( z \) denote the nontrivial cohomology class in \( H^1(\mathbb{RP}^\infty; \mathbb{Z}_2) \). Then \( H^*(\mathbb{RP}^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[2] \) as a graded \( \mathbb{Z}_2 \)-algebra, with \( z \) having degree 1. We denote the image \( w^*(z) \in H^1(X/\mathbb{Z}_2; \mathbb{Z}_2) \) by \( \varpi_1(X) \). This is the first Stiefel-Whitney class of the \( \mathbb{Z}_2 \)-space \( X \).

Clearly, \( \varpi_1^k(X) = w^*(z^k) \), since \( w^* \) is a \( \mathbb{Z}_2 \)-algebra homomorphism. We will be mainly interested in the height of the Stiefel-Whitney class, i.e., largest \( k \), such that \( \varpi_1^k(X) \neq 0 \); it was called cohomology co-index in [3].
Turning to graphs, let $G$ be a graph with $\mathbb{Z}_2$-action given by $\phi : G \to G$, $\phi \in \text{Hom}_i(G,G)$, such that $\phi$ flips an edge, that is, there exist $a, b \in V(G)$, $a \neq b$, $(a, b) \in E(G)$, such that $\phi(a) = b$ (which implies $\phi(b) = a$). For any graph $H$ we have the induced $\mathbb{Z}_2$-action $\phi_H : \text{Hom}(G, H) \to \text{Hom}(G, H)$. In case $H$ has no loops, it follows from the fact that $\phi$ flips an edge that this $\mathbb{Z}_2$-action is free.

Indeed, since $\phi_H$ is a cellular map, if it fixes a point from some cell $\eta : V(G) \to 2^V(H) \setminus \{\emptyset\}$, then it maps $\eta$ onto itself. By definition, $\phi$ maps $\eta$ to $\eta \circ \phi$, so this means that $\eta = \eta \circ \phi$. In particular, $\eta(a) = \eta \circ \phi(a) = \eta(b)$. Since $\eta(a) \neq \emptyset$, we can take $v \in V(H)$, such that $v \in \eta(a)$. Now, $(a, b) \in E(G)$, but $(v, v) \notin E(H)$, since $H$ has no loops, which contradicts the fact that $\eta \in \text{Hom}(G, H)$.

Therefore, in this situation, $\text{Hom}(G, -)$ is a covariant functor from the induced subcategory of Graphs, consisting of all loopfree graphs, to $\mathbb{Z}_2$-spaces (the category whose objects are $\mathbb{Z}_2$-spaces and morphisms are $\mathbb{Z}_2$-maps).

We order $V(C_{2r+1})$ by identifying it with $[1, 2r + 1]$ by the map $q : \mathbb{Z} \to \mathbb{Z}_{2r+1}$, taking $x \mapsto [x]_{2r+1}$. With this notation $\mathbb{Z}_2$ acts on $C_{2r+1}$ by mapping $[x]_{2r+1}$ to $[-x]_{2r+1}$, for $x \in V(C_{2r+1})$. Let $\gamma \in \text{Hom}_0(C_{2r+1}, C_{2r+1})$ denote the corresponding graph homomorphism. This action has a fixed point $2r + 1$, and it flips one edge $r, r + 1$.

Furthermore, let $\mathbb{Z}_2$ act on $K_m$ for $m \geq 2$, by swapping the vertices $1$ and $2$ and fixing the vertices $3, \ldots, m$; here, $K_m$ is the graph defined by $V(K_m) = [1, m]$, $E(K_m) = \{(x, y) \mid x, y \in V(K_m), x \neq y\}$. Since in both cases the graph homomorphism flips an edge, they induce free $\mathbb{Z}_2$-actions on $\text{Hom}(C_{2r+1}, G)$ and $\text{Hom}(K_m, G)$, for an arbitrary graph $G$ without loops.

### 2.2. Non-vanishing of powers of Stiefel-Whitney classes as obstructions to graph colorings.

The connection between the non-nullity of the powers of Stiefel-Whitney characteristic classes and the lower bounds for graph colorings is provided by the following general observation.

**Theorem 2.1.** Let $G$ be a graph without loops, and let $T$ be a graph with $\mathbb{Z}_2$-action which flips some edge in $T$. If, for some integers $k \geq 0$, $m \geq 1$, we have $w^k_i(\text{Hom}(T, G)) \neq 0$, and $w^k_i(\text{Hom}(T, K_m)) = 0$, then $\chi(G) \geq m + 1$.

**Proof.** We have already shown that, under the assumptions of the theorem, $\text{Hom}(T, H)$ is a $\mathbb{Z}_2$-space for any loopfree graph $H$. Assume now that the graph $G$ is $m$-colorable, i.e., there exists a homomorphism $\phi : G \to K_m$. It induces a $\mathbb{Z}_2$-map $\phi^T : \text{Hom}(T, G) \to \text{Hom}(T, K_m)$. Since the Stiefel-Whitney classes are functorial and $w^k_i(\text{Hom}(T, K_m)) = 0$, the existence of the $\mathbb{Z}_2$-map $\phi^T$ implies that $w^k_i(\text{Hom}(T, G)) = 0$, which is a contradiction to the assumption of the theorem. \(\square\)

**Remark 2.2.** If a $\mathbb{Z}_2$-space $X$ is $k$-connected, then there exists a $\mathbb{Z}_2$-map $\phi : S^{k+1}_a \to X$, in particular, $w^{k+1}_i(X) \neq 0$.

**Proof.** To construct $\phi$, subdivide $S^{k+1}_a$ simplicially as a join of $k + 2$ copies of $S^0$, and then define $\phi$ on the join of the first $i$ factors, starting with $i = 1$, and increasing $i$ by 1 at the time. To define $\phi$ on the first factor $\{a, b\}$, simply map $a$ to an arbitrary point $x \in X$, and then map $b$ to $\gamma(x)$, where $\gamma$ is the free involution of $X$. Assume $\phi$ is defined on $Y$ - the join of the first $i$ factors. Extend $\phi$ to $Y \ast \{a, b\}$ by extending it first to $Y \ast \{a\}$, which we can do, since $X$ is $k$-connected, and then extending $\phi$ to the second hemisphere $Y \ast \{b\}$, by applying the involution $\gamma$. 
Since the Stiefel-Whitney classes are functorial, we have \( \phi^*(w_1^{k+1}(X)) = w_1^{k+1}(S^{k+1}_a) \), and the latter is clearly nontrivial. \( \square \)

Let \( T \) be any graph and consider the following equation

\[
(2.1) \quad \varpi^{n-\chi(T)+1}(\text{Hom}(T, K_n)) = 0, \text{ for all } n \geq \chi(T) - 1.
\]

**Theorem 2.3.**

(a) The equation (2.1) is true for \( T = K_m \), \( m \geq 2 \).

(b) The equation (2.1) is true for \( T = C_{2r+1} \), \( r \geq 1 \), and odd \( n \).

**Proof.** The case \( T = K_m \) is [2, Theorem 1.6] and has been proved there. The case \( T = C_{2r+1} \) will be proved in the Section 4.

**Remark 2.4.** For a fixed value of \( n \), if the equation (2.1) is true for \( T = C_{2r+1} \), then it is true for any \( T = C_{2r+1} \), if \( r \geq r \).

**Proof.** If \( r \geq r \), there exists a graph homomorphism \( \phi : C_{2r+1} \rightarrow C_{2r+1} \) which respects the \( \mathbb{Z}_2 \)-action. This induces a \( \mathbb{Z}_2 \)-map

\[
\phi_{K_n} : H^*(\text{Hom}(C_{2r+1}, K_n)) \rightarrow H^*(\text{Hom}(C_{2r+1}, K_n)),
\]

yielding

\[
\tilde{\phi}_{K_n} : H^*(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \rightarrow H^*(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2).
\]

Clearly, \( \tilde{\phi}_{K_n}(\varpi_1(\text{Hom}(C_{2r+1}, K_n))) = \varpi_1(\text{Hom}(C_{2r+1}, K_n)) \). In particular, \( \varpi_1(\text{Hom}(C_{2r+1}, K_n)) = 0 \), implies \( \varpi_1(\text{Hom}(C_{2r+1}, K_n)) = 0 \). \( \square \)

Note that for \( T = C_{2r+1} \) and \( n = 2 \), the equation (2.1) is obvious, since \( \text{Hom}(C_{2r+1}, K_2) = \emptyset \). We give a quick argument for the next case \( n = 3 \). One can see by inspection that the connected components of \( \text{Hom}(C_{2r+1}, K_3) \) can be indexed by the winding numbers \( \alpha \). These numbers must be odd, so \( \alpha = \pm 1, \pm 3, \ldots, \pm (2s+1) \), where

\[
s = \begin{cases} 
(r - 1)/3, & \text{if } r \equiv 1 \mod 3, \\
(r - 2)/3, & \text{otherwise},
\end{cases}
\]

in particular \( s \geq 0 \). Let \( \phi : \text{Hom}(C_{2r+1}, K_3) \rightarrow \{\pm 1, \pm 3, \ldots, \pm (2s+1)\} \) map each point \( x \in \text{Hom}(C_{2r+1}, K_3) \) to the point on the real line, indexing the connected component of \( x \). Clearly, \( \phi \) is a \( \mathbb{Z}_2 \)-map. Since \( H^1(\{\pm 1, \pm 3, \ldots, \pm (2s+1)\}/\mathbb{Z}_2; \mathbb{Z}_2) = 0 \), the functoriality of the characteristic classes implies \( \varpi_1(\text{Hom}(C_{2r+1}, K_3)) = 0 \).

**Conjecture 2.5.** The equation (2.1) is true for \( T = C_{2r+1} \), \( r \geq 1 \), and all \( n \).

2.3. Completing the sketch of the proof of the Lovász Conjecture.

Consider one of the two maps \( \iota : K_2 \rightarrow C_{2r+1} \) mapping the edge to the \( \mathbb{Z}_2 \)-invariant edge of \( C_{2r+1} \). Clearly, \( \iota \) is \( \mathbb{Z}_2 \)-equivariant. Since \( \text{Hom}(-, H) \) is a contravariant functor, \( \iota \) induces a map of \( \mathbb{Z}_2 \)-spaces \( \iota K_n : \text{Hom}(C_{2r+1}, K_n) \rightarrow \text{Hom}(K_2, K_n) \), which in turn induces a \( \mathbb{Z} \)-algebra homomorphism \( \iota^* K_n : H^*(\text{Hom}(K_2, K_n); \mathbb{Z}) \rightarrow H^*(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \).

**Theorem 2.6.** Assume \( n \) is even, then \( 2 \cdot \iota^* K_n \) is a 0-map.

Theorem 2.6 is proved in Section 4. The results of this paper were announced in [3], and the preprint of this paper has been available since February 2004. In the summer 2005 an alternative proof of Theorem 2.6 appeared in the preprint [4], and a proof of Conjecture 2.5 was announced by C. Schultz.
Proof of Theorem 1.3 (Lovász Conjecture).
The case $k = -1$ is trivial, so take $k \geq 0$. Assume first that $k$ is even. By the Remark 2.2, we have $\omega_{k+1}(\text{Hom}(C_{2r+1}, G)) \neq 0$. By Theorem 2.3(b), we have $\omega_{2r+1}(\text{Hom}(C_{2r+1}, K_{k+3})) = 0$. Hence, applying Theorem 2.1 for $T = C_{2r+1}$ we get $\chi(G) \geq k + 4$.

Assume now that $k$ is odd, and that $\chi(G) \leq k + 3$. Let $\phi : G \to K_{k+3}$ be a vertex-coloring map. Combining the Remark 2.2 with the fact that $\text{Hom}(C_{2r+1}, -)$ is a covariant functor from loopfree graphs to $\mathcal{C}$ is defined to be the link in $\text{Hom}(\text{Graphs}, \mathcal{C})$ to the base vertex in $G$ and connecting it by edges to all the vertices of $G$. Combining the Remark 2.2, the fact that the coindex of $\chi$ is defined to be the link in $\text{Hom}(\text{Graphs}, \mathcal{C})$ to the base vertex in $G$ and connecting it by edges to all the vertices of $G$.

Let us make a couple of remarks.

Remark 2.7. As is apparent from our argument, we are actually proving a sharper statement than the original Lovász Conjecture. First of all, the condition “$\text{Hom}(C_{2r+1}, G)$ is $k$-connected” can be replaced by a weaker condition “the coindex of $\text{Hom}(C_{2r+1}, G)$ is at least $k + 1$”. Furthermore, for even $k$, that condition can be weakened even further to “$\omega_{k+1}(\text{Hom}(C_{2r+1}, G)) \neq 0$”. Conjecture 2.5 would imply that this weakening can be done for odd $k$ as well.

Remark 2.8. It follows from [2] Proposition 5.1 that Lovász Conjecture is true if $C_{2r+1}$ is replaced by any graph $T$, such that $T$ can be reduced to $C_{2r+1}$, by a sequence of folds.

3. Hom$_+$ and Filtrations

3.1. The + construction.

For a finite graph $H$, let $H_+$ be the graph obtained from $H$ by adding an extra vertex $b$, called the base vertex, and connecting it by edges to all the vertices of $H_+$ including itself, i.e., $V(H_+) = V(H) \cup \{b\}$, and $E(H_+) = E(H) \cup \{(v, b), (b, v) | v \in V(H_+)\}$.

Definition 3.1. Let $G$ and $H$ be two graphs. The simplicial complex $\text{Hom}_+(G, H)$ is defined to be the link in $\text{Hom}(G, H_+)$ of the homomorphism mapping every vertex of $G$ to the base vertex in $H_+$.

So the cells in $\text{Hom}_+(G, H)$ are indexed by all $\eta : V(G) \to 2^{V(H)}$ satisfying the same condition as in the Definition 1.2. The closure of $\eta$ is also defined identical to how it was defined for $\text{Hom}$. Note, that $\text{Hom}_+(G, H)$ is simplicial, and that $\text{Hom}_+(G, -)$ is a covariant functor from $\text{Graphs}$ to $\text{Top}$. One can think of $\text{Hom}_+(G, H)$ as a cell structure imposed on the set of all partial homomorphisms from $G$ to $H$. 
Cochain complexes for graphs

Proposition 3.2. The complex $\text{Hom}_+(G, H)$ is isomorphic to $\text{Ind}(G \times \check{\text{C}} H)$. In particular, $\text{Hom}_+(G, K_n)$ is isomorphic to $\text{Ind}(G)^*n$, where $*$ denotes the simplicial join.

Proof. By the definition, $V(G \times \check{\text{C}} H) = V(G) \times V(H)$. Let $S \subseteq V(G) \times V(H)$, $S = \{(x_i, y_i) | i \in I, x_i \in V(G), y_i \in V(H)\}$. Then $S \in \text{Ind}(G \times \check{\text{C}} H)$ if and only if, for any $i, j \in I$, we have either $(x_i, x_j) \notin E(G)$ or $(y_i, y_j) \in E(H)$, since the forbidden constellation is when $(x_i, x_j) \in E(G)$ and $(y_i, y_j) \notin E(H)$.

Identify $S$ with $\eta_S : V(G) \rightarrow 2^{V(H)}$ defined by: for $v \in V(G)$, set $\eta_S(v) := \{w \in V(H) | (v, w) \in S\}$. The condition for $\eta_S \in \text{Hom}_+(G, H)$ is that, if $(v_1, v_2) \in E(G)$, and $w_1 \in \eta_S(v_1), w_2 \in \eta_S(v_2)$, then $(w_1, w_2) \in E(H)$, which is visibly identical to the condition for $S \in \text{Ind}(G \times \check{\text{C}} H)$. Hence $\text{Hom}_+(G, H) = \text{Ind}(G \times \check{\text{C}} H)$.

To see the second statement note first that $\check{\text{C}} K_n$ is the disjoint union of $n$ looped vertices. Since taking direct products is distributive with respect to disjoint unions, and a direct product of $G$ with a loop is again $G$, we see that $G \times \check{\text{C}} K_n$ is a disjoint union of $n$ copies of $G$. Clearly, its independence complex is precisely the $n$-fold join of $\text{Ind}(G)$.

3.2. Cohomology complexes for $\text{Hom}(G, H)$ and $\text{Hom}_+(G, H)$.

For any CW complex $K$, let $K^{(i)}$ denote the $i$-th skeleton of $K$. Let $R$ be a commutative ring with a unit. In this paper we will have two cases: $R = \mathbb{Z}$.
and $R = \mathbb{Z}_2$. For any $\eta \in K^{(i)}$, we fix an orientation on $\eta$, and let $C_i(K; R) := R[\eta \mid \eta \in K^{(i)}]$, where $R[\alpha \mid \alpha \in I]$ denotes the free $R$-module generated by $\alpha \in I$. Furthermore, let $C^i(K; R)$ be the dual $R$-module to $C_i(K; R)$. For arbitrary $\alpha \in C_i(K; R)$ let $\alpha^*$ denote the element of $C^i(K; R)$ which is dual to $\alpha$. Clearly, $C^i(K; R) = R[\eta^* \mid \eta \in K^{(i)}]$, and the cochain complex of $K$ is

$$
\ldots \xrightarrow{\partial^{i-1}} C^i(K; R) \xrightarrow{\partial^i} C^{i+1}(K; R) \xrightarrow{\partial^{i+1}} \ldots
$$

For $\eta \in K^{(i)}$, $\tilde{\eta} \in K^{(i+1)}$, we have the incidence number $[\eta : \tilde{\eta}]$, which is 0 if $\eta \notin \tilde{\eta}$. In these notations $\partial^i(\eta^*) = \sum_{\tilde{\eta} \in K^{(i+1)}} [\eta : \tilde{\eta}] \tilde{\eta}^*$. For arbitrary $\alpha \in C_i(K; R)$, resp. $\alpha^* \in C^i(K; R)$, we let $[\alpha]$, resp. $[\alpha^*]$, denote the corresponding element of $H_i(K; R)$, resp. $H^i(K; R)$.

When coming after the name of a cochain complex, the brackets $[\cdot]$ will denote the index shifting (to the left), that is for the cochain complex $C^*$, the cochain complex $C^*[s]$ is defined by $C^*[s] := C^{*+s}$, and the differential is the same (we choose not to change the sign of the differential).

We now return to our context. Let $G$ and $H$ be two graphs, and let us choose some orders on $V(G) = \{v_1, \ldots, v_{|V(G)|}\}$ and on $V(H) = \{w_1, \ldots, w_{|V(H)|}\}$. Through the end of this subsection we assume the coefficient ring to be $\mathbb{Z}_2$; the situation over $\mathbb{Z}_2$ is simpler and can be described by tensoring with $\mathbb{Z}_2$.

Vertices of $\text{Hom}_+(G, H)$ are indexed with pairs $(x, y)$, where $x \in V(G)$, $y \in V(H)$, such that if $x$ is looped, then so is $y$. We order these pairs lexicographically: $(v_{i_1}, w_{j_1}) < (v_{i_2}, w_{j_2})$ if either $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. Orient each simplex of $\text{Hom}_+(G, H)$ according to the order on the vertices. We call this orientation standard, and call the oriented simplex $\eta_+$. If $\tilde{\eta}_+ \in \text{Hom}^{(i+1)}_+(G, H)$ is obtained from $\eta_+ \in \text{Hom}_+^{(i)}(G, H)$ by adding a vertex $v$, then $[\eta_+ : \tilde{\eta}_+]$ is $(-1)^{k-1}$, where $k$ is the position of $v$ in the order of the vertices of $\tilde{\eta}_+$.

Let us now turn to $C^*(\text{Hom}(G, H))$. We can fix an orientation, which we also call standard, on each cell $\eta \in \text{Hom}(G, H)$ as follows: orient each simplex $\eta(i)$ according to the chosen order on the vertices of $H$, then, order these simplices in the direct product according to the chosen order on the vertices of $G$. To simplify our notations, we still call this oriented cell $\eta$, even though a choice of orders on the vertex sets of $G$ and $H$ is implicitly present.

We remark for later use, that permuting the vertices of the simplex $\eta(i)$ by some $\sigma \in S_{|\eta(i)|}$ changes the orientation of the cell $\eta$ by $\text{sgn}(\sigma)$, whereas swapping the simplices with vertex sets $\eta(i)$ and $\eta(i+1)$ in the direct product changes the orientation by $(-1)^{|(\eta(i))| \cdot |(\eta(i+1))|} = (-1)^{\dim \eta(i) \cdot \dim \eta(i+1)}$.

If $\tilde{\eta} \in \text{Hom}^{(i+1)}(G, H)$ is obtained from $\eta \in \text{Hom}^{(i)}(G, H)$ by adding a vertex $v$ to the list $\eta(t)$, then $[\eta : \tilde{\eta}]$ is $(-1)^{k+d-1}$, where $k$ is the position of $v$ in $\tilde{\eta}(t)$, and $d$ is the dimension of the product of the simplices with the vertex sets $\eta(1), \ldots, \eta(t-1)$, that is $d = 1 - t + \sum_{j=1}^{t-1} |\eta(j)|$. To see this, note that $[\eta : \tilde{\eta}] = 1$ if the first vertex in the first simplex is inserted. The general case follows from the previously described rules for changing the sign of the orientation under permuting simplices in the product and permuting vertices within simplices.

### 3.3. The support map and the relation between $\text{Hom}(G, H)$ and $\text{Hom}_+(G, H)$.

For each simplex of $\text{Hom}_+(G, H)$, $\eta : V(G) \rightarrow 2^{V(H)}$, define the support of $\eta$ to be $\text{supp} \eta := V(G) \setminus \eta^{-1}(\emptyset)$. A concise way to phrase the definition of supp
differently is to consider the map \( t^G : \text{Hom}_+(G, H) \to \text{Hom}_+(G, \mathbb{C}K_1) \simeq \Delta_{|V(G)|-1} \) induced by the homomorphism \( t : H \to \mathbb{C}K_1 \). Then, for each \( \eta \in \text{Hom}_+(G, H) \) we have supp \( \eta = t^G(\eta) \), where the simplices in \( \Delta_{|V(G)|-1} \) are identified with the subsets of \( V(G) \).

Let \( \tilde{C}^* \) be the subcomplex of \( C^*(\text{Hom}_+(G, H)) \) generated by all \( \eta_+^* \), for \( \eta : V(G) \to 2^{V(H)} \), such that supp \( \eta = V(G) \) (cf. filtration in subsection 3.3). Set

\[
X^*(G, H) := \tilde{C}^*[[V(G)| - 1].
\]

Note that both \( C^i(\text{Hom}_+(G, H)) \) and \( X^i(G, H) \) are free \( \mathbb{Z} \)-modules with the bases \( \{ \eta^*_+ \} \eta \) and \( \{ \eta^+_+ \} \eta \) indexed by \( \eta : V(G) \to 2^{V(H)} \setminus \emptyset \), such that \( \sum_{j=1}^{\|V(G)\|} |\eta(j)| = |V(G)| + i \).

At this point we introduce the following notations: for \( \eta : V(G) \to 2^{V(H)} \setminus \emptyset \), set

\[
c(\eta) := \sum_{1 \leq i \leq |V(G)|} |\eta(i)|.
\]

For any \( \eta : V(G) \to 2^{V(H)} \setminus \emptyset \), set \( \rho(\eta_+) := (-1)^{c(\eta)} \eta \). Obviously, the induced map \( \rho^* : X^i(G, H) \to C^i(\text{Hom}_+(G, H)) \) is a \( \mathbb{Z} \)-module isomorphism for any \( i \).

**Proposition 3.3.** The map \( \rho^* : X^*(G, H) \to C^*(\text{Hom}_+(G, H)) \) is an isomorphism of the cochain complexes.

**Proof.** Indeed, let \( \tilde{\eta} : V(G) \to 2^{V(H)} \setminus \emptyset \) be obtained from \( \eta \) by adding a vertex \( v \) to the list \( \eta(t) \), and let \( k \) be the position of \( v \) in \( \tilde{\eta}(t) \). By our previous computation:

\[
[\eta_+ : \tilde{\eta}_+] = (-1)^{k+d+t}, \text{ whereas } [\eta : \tilde{\eta}] = (-1)^{k+d-1}, \text{ where } d = 1 - t + \sum_{j=1}^{t-1} |\eta(j)|.
\]

This shows that

\[
[\rho(\eta_+) : \rho(\tilde{\eta}_+)] = (-1)^{c(\eta)+c(\tilde{\eta})}[\eta : \tilde{\eta}] = (-1)^{c(\eta)+c(\tilde{\eta})+t+1}[\eta_+ : \tilde{\eta}_+],
\]

but

\[
c(\eta) + c(\tilde{\eta}) + t + 1 = \sum_{i \text{ is even}} |\eta(i)| + \sum_{i \text{ is even}} |\tilde{\eta}(i)| + t + 1 \equiv 0 \pmod{2}
\]

for any \( t \), hence \([\rho(\eta_+) : \rho(\tilde{\eta}_+)] = [\eta_+ : \tilde{\eta}_+] \). \( \Box \)

### 3.4. Relating \( \mathbb{Z}_2 \)-actions on \( \text{Hom}_+(G, H) \) and \( \text{Hom}_+(G, H) \)

Assume that we have \( \gamma \in \text{Hom}_0(G, G) \), and \( 0 \leq r \leq |V(G)|/2 \), such that

\[
\gamma(i) = \begin{cases} 
2r + 1 - i, & \text{if } 1 \leq i \leq 2r; \\
i, & \text{if } 2r + 1 \leq i \leq |V(G)|,
\end{cases}
\]

where we identified \( V(G) \) with the numbering \([1, V(G)]\). In particular, we have \( \gamma^2 = 1 \).

The homomorphism \( \gamma \) induces \( \mathbb{Z}_2 \)-action both on \( \text{Hom}_+(G, H) \), and on \( \text{Hom}_+(G, H) \). We shall see how \( \rho^* \) behaves with respect to this \( \mathbb{Z}_2 \)-action. For any \( \eta : V(G) \to 2^{V(H)} \setminus \emptyset \), \( \gamma \) takes \( \eta \) to \( \eta \circ \gamma \). By a slight abuse of notations we let \( \gamma \) denote the induced actions both on \( C^*(\text{Hom}_+(G, H)) \) and on \( X^*(G, H) \).

Let \( (u_1, \ldots, u_q) \) be the vertices of the simplex \( \eta_+ \) listed in the increasing order. By definition, \( \gamma(\eta_+) = (\gamma(u_1), \ldots, \gamma(u_q)) \), where \( \gamma((v, w)) := (\gamma(v), w) \), for \( v \in V(G), w \in V(H) \). Clearly, \( \gamma(\eta_+) \) has the same set of vertices as \( (\eta \circ \gamma)_+ \), so we just need to see how their orientations relate. To order the vertices of \( \gamma(\eta_+) \) we need to
invert the order of the blocks with cardinalities $|\eta(1)|, \ldots, |\eta(2r)|$ without changing the vertex orders within the blocks. The sign of this permutation is $(-1)^c$, where $c = \sum_{1 \leq i < j \leq 2r} |\eta(i)| \cdot |\eta(j)|$, so we conclude that

$$
(3.2) \quad \gamma(\eta^*_+) = (-1)^d (\eta \circ \gamma)^*_+.
$$

Consider now the oriented cell $\eta$. It is a direct product of simplices $\Delta^1, \ldots, \Delta^{|V(G)|}$ of dimensions $|\eta(1)|-1, \ldots, |\eta(|V(G)|)|-1$, with the standard orientation as defined above. The cell $\gamma(\eta)$ is the direct product of $\gamma(\Delta^1) = \Delta^{2r}$, $\gamma(\Delta^2) = \Delta^{2r-1}, \ldots, \gamma(\Delta^{2r}) = \Delta^1$, $\gamma(\Delta^{2r+1}) = \Delta^{2r+1}, \ldots, \gamma(\Delta^{|V(G)|}) = \Delta^{|V(G)|}$, with the order of the vertices (hence the orientation) within each simplex being the same as in $\eta$.

We see that $\gamma(\eta)$ is, up to the orientation, the same cell as $\eta \circ \gamma$. To relate their orientations, we need to permute the simplices $\Delta^2, \ldots, \Delta^1$ back in order, which, by the previous observations, changes the orientation by $(-1)^d$, where

$$
\tilde{d} = \sum_{1 \leq i < j \leq 2r} \dim \Delta^i \cdot \dim \Delta^j = \sum_{1 \leq i < j \leq 2r} (|\eta(i)| - 1)(|\eta(j)| - 1)
= c - (2r - 1) \sum_{i=1}^{2r} |\eta(i)| + \binom{2r}{2}.
$$

Reducing modulo 2, we conclude that

$$
(3.3) \quad \gamma(\eta^*_+) = (-1)^{\tilde{d}} (\eta \circ \gamma)^*_+,
$$

where $d = c + \sum_{i=1}^{2r} |\eta(i)| + r$.

Let us now see how $\rho^*$ interacts with $\gamma$. We have

$$
(3.4) \quad \rho^*(\gamma(\eta^*_+)) = (-1)^c \rho^*(\eta \circ \gamma)^*_+ = (-1)^{c+c(\eta \circ \gamma)} (\eta \circ \gamma)^*_+,
$$

where the first equality is by (3.2) and the second one is by definition of $\rho$, and

$$
(3.5) \quad \gamma(\rho^*(\eta^*_+)) = (-1)^{c(\eta)} (\eta \circ \gamma) = (-1)^{d+c(\eta)} (\eta \circ \gamma)^*_+,
$$

where the first equality is by definition of $\rho$ and second one is by (3.3). Comparing (3.4) with (3.5), and using the computation

$$
c(\eta) + c(\eta \circ \gamma) = \sum_{\substack{i \text{ is even} \\ 1 \leq i \leq |V(G)|}} |\eta(i)| + \sum_{\substack{i \text{ is even} \\ 1 \leq i \leq |V(G)|}} |\eta \circ \gamma(i)|
= \sum_{i=1}^{2r} |\eta(i)| + 2 \sum_{\substack{i \text{ is even} \\ 2r+1 \leq i \leq |V(G)|}} |\eta(i)|,
$$

we see that, for any $\eta$

$$
(3.6) \quad \rho^*(\gamma(\eta^*_+)) = (-1)^r \gamma(\rho^*(\eta^*_+)�).
$$

3.5. The filtration of $C^*(\text{Hom}_+(G, H); \mathbb{Z})$ and the $E_{0}^{0,*}$-tableau.

We shall now filter $C^*(\text{Hom}_+(G, H); \mathbb{Z})$. Define the subcomplexes of $C^*(\text{Hom}_+(G, H); \mathbb{Z})$, $F^p = F^p C^*(\text{Hom}_+(G, H); \mathbb{Z})$, as follows:

$$
F^p : \ldots \xrightarrow{\partial^{p-1}} F^{p,q} \xrightarrow{\partial^p} F^{p,q+1} \xrightarrow{\partial^{p+1}} \ldots.
$$
where
\[ F^{p,q} = F^p \mathcal{E}^q(\operatorname{Hom}_+(G, H); \mathbb{Z}) = \mathbb{Z} \left[ \eta^*_+ \left| \eta_+ \in \operatorname{Hom}_+(G, H), |\text{supp } \eta| \geq p + 1 \right. \right], \]
and \( \partial^* \) is the restriction of the differential in \( C^*(\operatorname{Hom}_+(G, H); \mathbb{Z}) \). Then,
\[ C^q(\operatorname{Hom}_+(G, H); \mathbb{Z}) = F^{0,q} \supseteq F^{1,q} \supseteq \cdots \supseteq F^{|V(G)|-1,q} \supseteq F^{|V(G)|,q} = 0. \]

**Proposition 3.4.** For any \( p \),
\[ (3.7) \quad F^p/F^{p+1} = \bigoplus_{S \subseteq V(G)} C^q(\operatorname{Hom}(G[S], H); \mathbb{Z})[-p]. \]

Hence, the 0th tableau of the spectral sequence associated to the cochain complex filteration \( F^* \) is given by
\[ (3.8) \quad E^0_{p,q} = C^{p+q}(F^p, F^{p+1}) = \bigoplus_{S \subseteq V(G)} C^q(\operatorname{Hom}(G[S], H); \mathbb{Z}). \]

**Proof.** By construction
\[ F^{p,q}/F^{p+1,q} = \mathbb{Z} \left[ \eta^*_+ \left| \eta_+ \in \operatorname{Hom}_+(G, H), |\text{supp } \eta| = p + 1 \right. \right] = \bigoplus_{S \subseteq V(G)} X^{p-q}(G[S], H; \mathbb{Z}) \overset{\rho^*}{\rightarrow} \bigoplus_{S \subseteq V(G)} C^{q-p}(\operatorname{Hom}(G[S], H); \mathbb{Z}), \]
where \( X^* \) is defined in (3.1) and \( \rho^* \) is the map defined in subsection 3.3. \( \square \)

Note, that in particular we have \( F^{|V(G)|-1,q}/F^{|V(G)|,q} = F^{|V(G)|,q} = C^q(\operatorname{Hom}(G, H); \mathbb{Z})[1 - |V(G)|]. \)

4. \( \mathbb{Z}_2 \)-Action on \( H^*(\operatorname{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}) \)

In this section we shall derive some information about the \( \mathbb{Z}_2[\mathbb{Z}_2] \)-modules \( H^*(\operatorname{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \), for \( r \geq 2, n \geq 4 \). For \( r = 2 \) our computation will be complete.

We adopt the following convention: we think of \( C_{2r+1} \) as a unit circle on the plane with \( 2r + 1 \) marked points with numbers \( 1, \ldots, 2r + 1 \) following each other in the clockwise increasing order. The directions left, resp. right on this circle will denote counterclockwise, resp. clockwise.

Furthermore, before we start our computation, we introduce the following terminology. For \( S \subset V(C_{2r+1}) \), we call those connected components of \( C_{2r+1}[S] \) which have at least 2 vertices the arcs. For \( x, y \in \mathbb{Z} \), we let \([x, y]_{2r+1}\) denote the arc starting from \( x \) and going clockwise to \( y \), that is \([x, y]_{2r+1} = \{[x]_{2r+1}, [x + 1]_{2r+1}, \ldots, [y - 1]_{2r+1}, [y]_{2r+1}\}. \)

4.1. The simplicial complex of partial homomorphisms from a cycle to a complete graph.

Here and in the next subsection we summarize some previously published results which are necessary for our present computations. To start with, recall that the homotopy type of the independence complexes of cycles was computed in [3].
Proposition 4.1. ([5] Proposition 5.2]).
For any integer \(m \geq 2\), we have
\[
\text{Ind}(C_m) \simeq \begin{cases} 
S^{k-1} \cup S^{k-1}, & \text{if } m = 3k; \\
S^{k-1}, & \text{if } m = 3k \pm 1.
\end{cases}
\]

Combining Propositions 3.2 and 3.1 we get the following formula.

Corollary 4.2. For any integers \(m \geq 2\), \(n \geq 1\), we have
\[
\text{Hom}(C_m, K_n) \simeq \begin{cases} 
\bigvee_{2^n \text{ copies}} S^{n^{k-1}}, & \text{if } m = 3k; \\
S^{n^{k-1}}, & \text{if } m = 3k \pm 1.
\end{cases}
\]

The following estimates will be needed later for our spectral sequence computations.

Corollary 4.3. We have \(\overline{H}^i(\text{Hom}_+(C_{2r+1}, K_n)) = 0\) for \(r \geq 2\), \(n \geq 4\), and \(i \leq n + 2r - 2\); except for the two cases \((n, r) = (4, 3)\) and \((5, 3)\).

Proof. Note, that if \(2r + 1 = 3k + \epsilon\), with \(\epsilon \in \{-1, 0, 1\}\), then we have \(\overline{H}^i(\text{Hom}_+(C_{2r+1}, K_n)) = 0\), for \(i \leq nk - 2\).

Assume first \(2r + 1 = 3k\). The inequality \(nk - 2 \geq n + 2r - 2\) is equivalent to \(n \geq 3 + 2/(k - 1)\), and the latter is always true since \(k \geq 3\) and \(n \geq 4\).

Assume now \(2r + 1 = 3k + 1\). This time, \(nk - 2 \geq n + 2r - 2\) is equivalent to \(n \geq 3 + 3/(k - 1)\). If \(k \geq 4\), this is always true, since \(n \geq 4\). If \(k = 2\), this reduces to saying that \(n \geq 6\). This yields the two exceptional cases: \(r = 3\) and \(n = 4, 5\).

Finally, assume \(2r + 1 = 3k - 1\). Here, \(nk - 2 \geq n + 2r - 2\) is equivalent to \(n \geq 3 + 1/(k - 1)\), which is always true, since \(k \geq 2\), \(n \geq 4\).

The Corollary 4.2 can be strengthened to include the information on the \(\mathbb{Z}_2\)-action.

Proposition 4.4. For any positive integers \(r\) and \(n\) we have
\[
\text{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2 \simeq \begin{cases} 
\bigvee_{2^{n-1} \text{ copies}} S^{n^{k-1}}, & \text{if } 2r + 1 = 3k; \\
S^{kn}/2 - 1 + \mathbb{R}P^{kn}/2 - 1, & \text{if } 2r + 1 = 3k \pm 1.
\end{cases}
\]

Proof. By Proposition 3.2 we know that \(\text{Hom}_+(C_{2r+1}, K_n)\) is isomorphic to \(\text{Ind}(C_{2r+1})^n\). We analyze \(\mathbb{Z}_2\)-action on \(\text{Ind}(C_{2r+1})\) in more detail.

Assume first \(2r + 1 = 3k - 1\), in particular, \(k\) is even. It was shown in [5] Proposition 5.2 that \(X = \text{Ind}(C_{2r+1}) \setminus \{1, 4, \ldots, 2r-3, 2r\}\) is contractible (here “\(\setminus\)” just means the removal of an open maximal simplex). It follows from the standard fact in the theory of transformation groups, see e.g., [5] Theorem 5.16, p. 222], that \(X/\mathbb{Z}_2\) is contractible as well. Hence \(\text{Ind}(C_{2r+1})\) is \(\mathbb{Z}_2\)-homotopy equivalent to the unit sphere \(S^{k-1} \subset \mathbb{R}^k\) with the \(\mathbb{Z}_2\) acting by fixing \(k/2\) coordinates and multiplying the other \(k/2\) coordinates by \(-1\).

Assume \(2r + 1 = 3k + 1\). The link of the vertex \(2r + 1\) is \(\mathbb{Z}_2\)-homotopy equivalent to a point. Hence, deleting the open star of the vertex \(2r + 1\) produces a complex \(X\), which is \(\mathbb{Z}_2\)-homotopy equivalent to \(\text{Ind}(C_{2r+1})\). It was shown in [5] Proposition 5.2, that \(X \setminus \{2, 5, \ldots, 2r - 4, 2r - 1\}\) is contractible. By an argument, similar to the previous case, we conclude that \(\text{Ind}(C_{2r+1})\) is \(\mathbb{Z}_2\)-homotopy equivalent to the unit sphere \(S^{k-1} \subset \mathbb{R}^k\) with the \(\mathbb{Z}_2\) acting by fixing \(k/2\) coordinates and multiplying the other \(k/2\) coordinates by \(-1\).
In both cases we see that $\text{Hom}_+(C_{2r+1}, K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to $\text{Susp}^{kn/2}S^{kn/2-1}$, with the $\mathbb{Z}_2$-action and the latter space being induced by the antipodal action on $S^{kn/2-1}$. It follows that $\text{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2$ is homotopy equivalent to $\text{Susp}^{kn/2}\mathbb{R}^{kn/2-1}$.

Consider the remaining case $2r + 1 = 3k$. It was shown in [2, Proposition 5.2] that $\text{Ind}(C_{2r+1})$ becomes contractible if one removes the simplices $\{1, 4, \ldots, 2r-1\}$ and $\{2, 5, \ldots, 2r\}$. It follows that $\text{Ind}(C_{2r+1})$ is $\mathbb{Z}_2$-homotopy equivalent to the wedge of two unit spheres $S^{k-1}$ with the $\mathbb{Z}_2$ acting by swapping the spheres. Thus $\text{Hom}_+(C_{2r+1}, K_n)$ is $\mathbb{Z}_2$-homotopy equivalent to a wedge of $2^n$ $(nk - 1)$-dimensional spheres, with the $\mathbb{Z}_2$-action swapping them in pairs, and so $\text{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2$ is homotopy equivalent to a wedge of $2^{n-1}$ $(nk - 1)$-dimensional spheres.

We summarize the estimates which we will need later.

**Corollary 4.5.** We have $\bar{H}^i(\text{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2) = 0$ for $r \geq 2$, $n \geq 5$, and $i \leq n + r - 2$. Except for the case $r = 3$.

**Proof.** If $2r + 1 = 3k$, the inequality $nk - 2 \geq n + r - 2$ is equivalent to $n \geq 3r/(2r - 2)$, which is true for $n \geq 3$, $r \geq 2$. If $2r + 1 = 3k - 1$, then $nk/2 \geq n + r - 2$ is equivalent to $(n - 3)(k - 2) \geq 0$, again true for our parameters.

If $2r + 1 = 3k + 1$, then $nk/2 \geq n + r - 2$ is equivalent to $(n - 3)(k - 2) \geq 2$. This is true for all parameters $n \geq 5$, $k \geq 2$, except for $k = 2$. □

**4.2. The cell complex of homomorphisms from a tree to a complete graph.**

In the next proposition we summarize several results proved in [2, 11].

**Proposition 4.6.** [2, Propositions 4.3, 5.4, and 5.5], [11].

Let $T$ be a tree with at least one edge.

(i) The map $i_{K_n} : \text{Hom}(T, K_n) \to \text{Hom}(K_2, K_n)$ induced by any inclusion $i : K_2 \hookrightarrow T$ is a homotopy equivalence.

(ii) $\text{Hom}(K_2, K_n)$ as a boundary complex of a polytope of dimension $n - 2$, in particular $\text{Hom}(T, K_n) \approx S^{n-2}$.

(iii) Given a $\mathbb{Z}_2$-action determined by an invertible graph homomorphism $\gamma : T \to T$, if $\gamma$ flips an edge in $T$, then $\text{Hom}(T, K_n) \approx \mathbb{Z}_2 S^{n-2}_n$, otherwise $\text{Hom}(T, K_n) \approx \mathbb{Z}_2 S^{n-2}_n.$

Here $S^n_m$ denotes the $m$-sphere equipped with an antipodal $\mathbb{Z}_2$-action, whereas $S^n_l$ is the $m$-sphere equipped with the trivial one.

Let $F$ be any graph, with $F_1, \ldots, F_t$ being the list of all those connected components of $F$ which have at least 2 vertices. For any $\emptyset \neq S \subseteq [1, t]$, and $V = \{v_i\}_{i \in S}$, such that $v_i \in V(F_i)$, for any $i \in S$, set

$$\alpha_+(F, V) := \sum_{\eta} \eta^*_+, \quad \alpha(F, V) := \sum_{\eta} \eta^*,$$

where both sums are taken over all $\eta : V(F) \to 2^{[1, n]} \setminus \{\emptyset\}$, such that

- $\eta(v_i) = [1, n - 1]$, for all $i \in S$;
- $|\eta(w)| = 1$, for all $w \in V(F) \setminus V$.

Note that, for fixed $S$ and $V$, $(-1)^{\ell(\eta)}$ does not depend on the choice of $\eta$ as long as $\eta$ satisfies these two conditions. In our previous notations we have $\alpha_+(F, V) \in X^{(|S|(n-2)}(F, K_n)$, and $\alpha(F, V) \in C^{(|S|(n-2)}(\text{Hom}(F, K_n))$. When $|S| = 1$, $V = \{v\}$, we shall simply write $\alpha_+(F, v)$ and $\alpha(F, v)$. 


Assume now that $F$ is a forest. For $w \in V(F_1)$, such that $(v_1, w) \in E(F)$, set
\[ W := \{ v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_k \} = (V \cup \{w\}) \setminus \{v_i\}. \]
We have a graph homomorphism $K_2 \to (v, w)$, which induces a $\mathbb{Z}_2$-equivariant map $\varphi^*: H^*(\text{Hom}(K_2, K_n)) \to H^*(\text{Hom}(F, K_n))$. We know that $\text{Hom}(K_2, K_n) \cong \mathbb{Z}_2 S_n^{n-2}$, and that the dual of any $(n-2)$-dimensional cell of $\text{Hom}(K_2, K_n)$ is a generator of $H^{n-2}(\text{Hom}(K_2, K_n); \mathbb{Z})$.

Comparing orientations of the cells of $\text{Hom}(K_2, K_n)$ we see that $[\alpha(K_2, 1)] = (-1)^{n-1}[\alpha(K_2, 2)]$, where 1 and 2 denote the vertices of $K_2$. Applying $\varphi^*$ we conclude that
\[ [\alpha(F, V)] = (-1)^{n-1}[\alpha(F, W)]. \]

Since $\rho^*$ is a cochain isomorphism and $\rho^*(\alpha_+(F, V)) = (-1)^{c(n)}[\alpha(F, V)]$, we have
\[ [\alpha_+(F, V)] = \begin{cases} -[\alpha_+(F, W)], & \text{if } v \text{ and } w \text{ have different parity in the order on } V(F); \\ (-1)^{n-1}[\alpha_+(F, W)], & \text{if they have the same parity}. \end{cases} \]

### 4.3. The $E_{0}^{r,q}$-tableau for $E_{1}^{p,q} \Rightarrow H^{p+q}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z})$. \(^1\)

Let us fix integers $r \geq 2$ and $n \geq 4$. Let $(F^p)_{p=0, \ldots, |V(G)|-1}$ be the filtration on $C^\ast(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z})$ defined in subsection 3.6 and consider the corresponding spectral sequence. The entries of the $E_1$-tableau are given by $E_{1}^{p,q} = H^{p+q}(F^p, F^{p+1})$. Since all proper subgraphs of $C_{2r+1}$ are forests, we can now use the formula \((\ref{eq:formula})\) to obtain almost complete information about the $E_1$-tableau. See Figure 4.1, where all the entries outside of the shaded area are equal to 0.

Let $\emptyset \neq S \subset V(C_{2r+1})$, and let $S_1, \ldots, S_{l(S)}$ be the connected components of $C_{2r+1}[S]$, with $|S_1| \geq |S_2| \geq \cdots \geq |S_{l(S)}| > |S_{d(S)}| = \cdots = |S_{l(S)}| = 1$, where $l(S) \geq 1$, but we may have $d(S) = 0$ or $d(S) = l(S)$. By Proposition 2.6 together with property (3) from 2 subsection 2.4] we see that
\[ \text{Hom}(C_{2r+1}[S], K_n) \approx \prod_{i=1}^{d(S)} S_n^{n-2}. \]

Combining with the formula \((\ref{eq:formula})\) we conclude
\[ H^{p+q}(F^p, F^{p+1}) = \bigoplus_{S \subset V(C_{2r+1})} H^q \left( \prod_{i=1}^{d(S)} S_n^{n-2}; \mathbb{Z} \right), \]
for $p \leq 2r-1$.

Since the spectral sequence converges to $\tilde{H}^*(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})$, and, since by Corollary 4.3 $\tilde{H}^i(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}) = 0$ for $i \leq n + 2r - 2$, we know that the entries on the diagonals $p+q = n + 2r - 2$, and $p+q = n + 2r - 3$, should eventually all become 0.

\(^1\)The calculations performed in the subsections 3.6, 4.3 have been verified and generalized in 14.
4.4. The cochain complex \((D^*_{0}, d_{1}) = (E^*_{1}, d_{1}), \text{ for } E^*_{1} \Rightarrow H^{p+q}(\text{Hom}_{+}(C_{2r+1}; K_{n}); Z))\).

Let \((D^*_{0}, d_{1})\) denote the cochain complex in the \(i\)-th row of \(E^*_{1}\), for any \(i = 0, \ldots, \left\lfloor \frac{2r + 1}{3} \right\rfloor\). Next we show that \((D^*_{0}, d_{1})\) is isomorphic to the cochain complex of a simplex.

**Lemma 4.7.** We have \(E^*_{1,0} = Z\), and \(E^*_{1,0} = E^*_{2,0} = \cdots = E^*_{2r,0} = 0\).

**Proof.** Let \(\Delta_{2r}\) denote and abstract simplex with \(2r + 1\) vertices indexed by \([1, 2r + 1]\), and identify simplices of \(\Delta_{2r}\) with the subsets of \([1, 2r + 1]\). Let \((C^*(\Delta_{2r}; Z), d^*)\) be the cochain complex of \(\Delta_{2r}\) corresponding to the order on the vertices given by this indexing. By \([2, \text{ Proposition 2.1}]\), each \(S \subseteq V(C_{2r+1})\), \(|S| = p + 1\), contributes one independent generator (over \(Z\)) to \(E^*_{1,0}\). Identifying it with the generator in \(C^*(\Delta_{2r}; Z)\) of the corresponding \(p\)-simplex in \(\Delta_{2r}\), we see that \((D^*_{0}, d_{1})\) and \((C^*(\Delta_{2r}; Z), d^*)\) are isomorphic as cochain complexes.

Indeed, for such \(S\), \(\tau_S := \sum_{\varphi \in \text{Hom}(C_{2r+1}, K_{n})} \varphi^*\) is a representative of the corresponding generator in \(E^*_{1,0}\). This is true even for \(S = V(C_{2r+1})\), since \(\text{Hom}(C_{2r+1}, K_{n})\) is connected for \(n \geq 4\), as was shown in \([2, \text{ Proposition 2.1}]\). Clearly,

\[
d_1(\tau_S) = \sum_{\varphi \in \text{Hom}(C_{2r+1}[S], K_{n})} \sum_{v \notin S} \sum_{\psi \mid S = \varphi} [\psi_{+} : \psi_{-}] \psi^*_+ = \sum_{v \notin S} |S : S \cup \{v\}| \sum_{\psi \in \text{Hom}(C_{2r+1}[S \cup \{v\}], K_{n})} \psi^*_+ = \sum_{v \notin S} |S : S \cup \{v\}| \tau_{S \cup \{v\}},
\]
where the second equality is true since \([\varphi_+ : \psi_+]\) only depends on \(S\) and \(v\), not on the specific choice of \(\varphi\) and \(\psi\). This shows that the following diagram commutes:

\[
\begin{array}{ccc}
C^p(\Delta_{2r}; \mathbb{Z}) & \xrightarrow{d^p} & C^{p+1}(\Delta_{2r}; \mathbb{Z}) \\
\tau \downarrow & & \downarrow \tau \\
E_1^{p,0} & \xrightarrow{d_1} & E_1^{p+1,0},
\end{array}
\]

where \(\tau : C^*(\Delta_{2r}; \mathbb{Z}) \to E_0^{1,0}\) is the linear extension of the map taking \(S\) to \(\tau_S\), for \(S \subseteq V(C_{2r+1})\). It follows that \((D_0^*, d_0)\) is isomorphic to \((C^*(\Delta_{2r}; \mathbb{Z}), d^*)\), therefore \(E_2^{0,0} = \mathbb{Z}\), and \(E_2^{1,0} = E_2^{2,0} = \cdots = E_2^{d,0} = 0\). \(\square\)

4.5. The cochain complexes \((D_1^*, d_1) = (E_1^*, (n-2)t, d_1)\), for \([(2r+1)/3] \geq t \geq 2\), and \(E_1^{p,q} \Rightarrow H^{p,q}(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z})\).

We shall perform only a partial analysis of the cohomology groups of \((D_1^*, d_1)\), which will however be sufficient for our purpose.

For \(S \subseteq V(C_{2r+1})\) and \(v \in S\), let \(a(S, v)\) denote the arc of \(S\) to which \(v\) belongs (assuming this arc exists). Furthermore, for an arbitrary arc \(a\) of \(S\), let \(a = |a_\bullet \cdot a^\bullet|_{2r+1}\). Let \(|a|\) denote the number of vertices on \(a\), and set \(\tilde{a} := |a\bullet - 1, a^\bullet + 1|_{2r+1}\) (so \(|\tilde{a}| = |a| + 2\), if \(|a| \leq 2r - 1\).

For any \(V \subseteq S \subseteq V(C_{2r+1})\), as in subsection 12.2 set \(\sigma_{S, V} := \alpha_+(C_{2r+1}[S], V)\). By our previous observations, \(E_1^{0,t(n-2)} = 0\). Furthermore, for any \(1 \leq i \leq 2r - 1\), \(E_1^{i,t(n-2)}\) is a free \(\mathbb{Z}\)-module with the basis \(\{[\sigma_{S, V}]\}\), where \(S \subseteq V(C_{2r+1})\), \(|S| = i + 1\), \(|V| = t\), and \(v = a_\bullet(S, v)\) (i.e., \([v-1]_{2r+1} \notin S\)) for all \(v \in V\). Since \(\sigma_{S, V}\) is a cocycle in \(X^{n-2}[C_{2r+1}[S], K_n; \mathbb{Z}]\), we have

\[
d_1([\sigma_{S, V}]) = \sum_{w \notin S} (-1)^{z(w)}[\sigma_{S \cup \{w\}, v}],
\]

where

\[
z(w) = \begin{cases} 
|S \cap \{1, w - 1\}|, & \text{if } v \notin \{1, w - 1\}; \\
n + |S \cap \{1, w - 1\}|, & \text{if } v \in \{1, w - 1\}.
\end{cases}
\]

Note, that if \(i \leq 2r - 2\) and \(|w|_{2r-1} = |v - 1|_{2r+1}\), then \(v \neq a_\bullet(S \cup \{w\}, w)\), so \([\sigma_{S \cup \{w\}, v}]\) may differ by a sign from one of the elements in our chosen basis. We shall not need the analog of the equation (4.6) for the case \(|V| \geq 2\).

Let \(A_1^*\) be the subcomplex of \(D_1^*\) defined by:

\[
A_1^* : 0 \to E_1^{2r-2, n-2} \xrightarrow{d_1} E_1^{2r-1, n-2} \xrightarrow{d_1} E_1^{2r, n-2} \to 0,
\]

where the \(\mathbb{Z}\)-modules indexed with \(0, \ldots, 2r - 3\) are equal to 0, and \(E_1^{2r-2, n-2}\) is generated by \(\{[\sigma_{S, v}]\}\), such that \(S\) and \(v\) satisfy all the previously required conditions and, in addition, \(C_{2r+1}[S]\) is connected.

In general, let \(A_j^*\) be the subcomplex of \(D_j^*\) generated by all \(\{[\sigma_{S, V}]\}\), such that (4.7)

\[
\bigcup_{v \in V} \hat{a}(S, v) = V(C_{2r+1}).
\]

In words: the gaps between those arcs of \(S\) which have points in \(V\) are of length at most 2. For future reference, we note, that (4.7) implies that \(|S| + 2|V| \geq 2r + 1\), i.e., \(|S| - 1 \geq 2r - 2t\), hence \(A_j^* = 0\) for \(j < 2r - 2t\).

**Lemma 4.8.** We have \(H^*(D_1^*) = H^*(A_j^*)\).
Lemma 4.9. We have \( \text{homology of the relative complex } (D^n, A^n) \).

Proof. Let us set up another spectral sequence for computing the cohomology of the relative complex \((D^n, A^n)\). We filter by \( \sum_{v \in V} |a(S, v)| \).

More precisely, \( F^p(D^n, A^n) = \mathbb{Z}[[\sigma_{S,v}] | \sum_{v \in V} |a(S, v)| \geq p] \). We see that \( F^p(D^n, A^n)/F^{p+1}(D^n, A^n) = \mathbb{Z}[[\sigma_{S,v}] | \sum_{v \in V} |a(S, v)| = p] \), hence

\[
E_1^{p,q}(D^n, A^n) = H^{p+q}(F^p(D^n, A^n)/F^{p+1}(D^n, A^n)) = \bigoplus_{a_1, \ldots, a_t} H^{p+q}(M_{a_1, \ldots, a_t}),
\]

where the sum is taken over all possible \( t \)-tuples of arcs \( a_1, \ldots, a_t \) such that

1. \( a_i \cap a_j = \emptyset \), for any \( i \neq j, i \in [1, t] \);
2. \( |a_1| + \cdots + |a_t| = p \);
3. \( \bigcup_{v \in V} a(S, v) \neq V(C_{2r+1}) \),

and \( M_{a_1, \ldots, a_t} \) is the cochain subcomplex generated by all \( \{\sigma_{S,v}\} \), such that the arcs with vertices in \( V \) are precisely \( a_1, \ldots, a_t \), i.e., \( \{a(S, v) | v \in V\} = \{a_1, \ldots, a_t\} \).

Restricting the formula (4.6) to \( M_{a_1, \ldots, a_t} \), we see that \( M_{a_1, \ldots, a_t} \) is isomorphic to the cochain complex \( C^* (\Delta_{2r-p-2}; \mathbb{Z}) \). More generally, we see that \( M_{a_1, \ldots, a_t} \) is isomorphic to \( C^* (\Delta_{2r-p}; \mathbb{Z}) \), where \( \tilde{p} = \bigcup_{v \in V} a(S, v) \).

As mentioned, \( \tilde{p} \leq 2r \), hence \( M_{a_1, \ldots, a_t} \) is acyclic for any \( a_1, \ldots, a_t \) satisfying the above conditions. We conclude that \( (D^n, A^n) \) is acyclic. The long exact sequence for the relative cohomology implies that \( H^*(A^n) = H^*(D^n) \). \( \square \)

Now, we can show that \( E_2^{i,t(n-2)} = 0 \) for \( t \geq 2, i < 2r - (t-1)(n-2) \), that is \( E_2^{*} \) is 0 in the region strictly above row \( n-2 \) and strictly below the diagonal \( x + y = 2r + n - 2 \), see Figure 4.2. Indeed, this is immediate when \( 2r - 2t \geq 2r - (t-1)(n-2) \), which after cancellations reduces to \((n-4)(t-1) \geq 2 \). The only cases when this inequality is false are \((t,n) = (2,5)\), and \( n = 4 \).

![Figure 4.2](image-url) The possibly nonzero entries in \( E_2^{*} \)-tableau, for \( E_2^{p,q} \Rightarrow H^{p+q}(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}) \).

4.6. The case \( n = 5 \), for \( E_1^{p,q} \Rightarrow H^{p+q}(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}) \).

Assume now that \( n = 5 \), \( t = 2 \).

Lemma 4.9. We have \( E_2^{2r-4,6} = 0 \).
Proof. By dimensional argument, this is true if $2r + 1 < 8$, so we can assume that $r \geq 4$.

By our previous arguments we need to see that $d_1 : A_1^{2r-4,6} \to A_1^{2r-3,6}$ is an injective map. The generators of $A_1^{2r-4,6}$ can be indexed with unordered pairs $(v, w)$, $v, w \in V(C_{2r+1})$, such that

$$[v - 1, v + 2]_{2r+1} \cap [w - 1, w + 2]_{2r+1} = \emptyset,$$

whereas the generators of $A_1^{2r-3,6}$ can be indexed with ordered pairs $(v, w)$, $v, w \in V(C_{2r+1})$, such that

$$[v - 1, v + 2]_{2r+1} \cap [w - 1, w + 1]_{2r+1} = \emptyset.$$

In these notations we have

$$(4.8) \quad d_1((v, w)) = \epsilon_1(v, w) + \epsilon_2(v, [w + 1]_{2r+1}) + \epsilon_3(w, v) + \epsilon_4(w, [v + 1]_{2r+1}),$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \{-1, 1\}$.

Take $0 \neq \sum_{v, w} \alpha_{v, w} (v, w) \in \ker d_1$. Choose $v, w$ such that $\alpha_{v, w} \neq 0$, and the minimum of the two distances between the arcs $(v, [w + 1]_{2r+1})$ and $(w, [v + 1]_{2r+1})$ is minimized. By symmetry we may assume $[w - v - 1]_{2r+1} \leq [v - w - 1]_{2r+1}$. Then, it follows from (4.3) that $\alpha_{[v+1]_{2r+1}, w} \neq 0$ as well.

Either $(v, [w + 1]_{2r+1}, w)$ is not a well-defined pair or the minimal distance between the two arcs is smaller for this pair, than for $(v, w)$: $[w - v - 1]_{2r+1} \geq [v - w - 2]_{2r+1}$. Both ways we get a contradiction to the assumption that there exists $(v, w)$, such that $\alpha_{v, w} \neq 0$. We conclude that $d_1 : A_1^{2r-4,6} \to A_1^{2r-3,6}$ is injective, hence $E_2^{2r-4,6} = 0$.

This shows, that when $n \geq 5$, there are no higher differentials $d_i$, $i \geq 2$, in our spectral sequence, originating in the region above row $n - 2$ and below diagonal $x + y = 2r + n - 2$. Hence, to figure out what happens to the entries $E_\infty^{2r, n-2}$ and $E_\infty^{2r, n-3}$, it is sufficient to consider rows $n - 2$ and $n - 3$.

4.7. The case $n = 4$, for $E_3^{p,q} = H^{p+q}(\hom_+ (C_{2r+1}, K_n); \mathbb{Z})$.

For $n = 4$ the nonzero rows of $E_3^{*, *}$ are too close to each other, so we are to do the computation by hand in a somewhat detailed way. Since the reduction from $(D_1^*, d_1)$ to $(A_1^*, d_1)$ described in subsection 4.3 was valid when $n = 4$, we may concentrate on the study of the latter complex. Let us first deal with $(A_1^*, d_1)$.

Lemma 4.10. We have $H^{2r-2}(A_2^*) = H^{2r-3}(A_2^*) = \mathbb{Z}$, and $H^i(A_2^*) = 0$, for $i \neq 2r - 2, 2r - 3$.

Proof. We filter $A_2^*$ by

$$F^p A_2^* = \mathbb{Z} \left[ |\sigma_{S, \{v_1, v_2\}}| \min(|\sigma(S, v_1)|, |\sigma(S, v_2)|) \geq p \right].$$

Clearly, $\cdots \subseteq F^p \subseteq F^{p+1} \subseteq \cdots$. Inspecting the case $p \leq r - 2$, we see that in this situation

$$C^*(F^p A_2^*/F^{p+1} A_2^*) = \bigoplus_{i=1}^{2r+1} B_i,$$

where each $B_i$ is isomorphic to $C^*(\Delta_1)$, hence is acyclic.

It follows that $H^*(A_2^*) = H^*(F^{r-1} A_2^*/F^r A_2^*)$. Let $\sigma_i = \sigma_{S_i, V_i}$, where $S_i = \{i + 1, i + r - 1\}_{2r+1} \cup \{i + r + 2, i + 3\}_{2r+1}$, $V_i = \{i + 1, i + r + 2\}$, and let $\tau_i = \sigma_{\bar{S}_i, \bar{V}_i}$, where $\bar{S}_i = S_i \cup \{i + r\}_{2r+1}$. Clearly, $d_1(\sigma_i) = \pm \tau_i \pm \tau_{i+r}$, and to verify the
statement of the lemma we need to show that the number of those $\sigma_i$, for which
$d_1(\sigma_i) = \pm(\tau_i + \tau_{i+r})$ is even. By (4.3) and (4.6) we see that $d_1(\sigma_i) = \pm(\tau_i - \tau_{i+r})$
if $i + r \neq 2r, 2r + 1$, i.e., the only cases we need to consider are $i = r$ and $i = r + 1$.

If $i = 2r + 1$, the different sign comes from (4.4), and the sign contribution is
$2r + 2$. This is an even number, hence again $d_1(\sigma_i) = \pm(\tau_i - \tau_{i+r})$. If $i = 2r$, the
different sign comes from (4.3), but since $n = 4$ is even, the sign remains the
same. □

Next, we consider $(A_t^*, d_1)$, for $t \geq 3$. First, we introduce some additional nota-
tions. Since the sign will not matter in our argument, we write $\sigma_S$ instead of $\sigma_{S;V}$, it is
then defined only up to a sign. For $S \subset V(C_{2r+1})$, $S = V(C_{2r+1}) \setminus \bar{S}$; the
connected components of $C_{2r+1}[S]$ are called gaps. Each gap consists of either one
or two elements, we call the first ones singletons, and the second ones double gaps.
Let $m(S)$ be the leftmost element of the gap which contains $\min(\bar{S} \cap [2, 2r]_{2r+1})$.
For $s \in \bar{S}$, let $s$ be the leftmost element of the first gap to the left of the gap
containing $s$, and let $\bar{s}$ be the leftmost element of the first gap to the right of the
gap containing $s$. For $x, y \in V(C_{2r+1})$, let $d(x, y)$ denote $|[x, y]_{2r+1} - 1$.

Lemma 4.11. We have $E_{2r-2t, 2t}^{2r-2t+1} = E_{2r-2t+1, 2t}^{2r-2t+1} = 0$.

Proof. Clearly $A_t^* = 0$, unless $2r - 2t \leq i \leq 2r - t$. Note that if $\sigma_S$ is a generator
of $A_t^{2r-2t+1}$, then $S$ has exactly one singleton. We decompose $A_t^{2r-2t+1} = B_1 \oplus B_2 \oplus B_3 \oplus B_4$, where each $B_i$ is spanned by the generators $\sigma_S$, for which certain
conditions are satisfied, see Figure 4.3.

\begin{itemize}
  \item[(B1)] $m(S)$ is the singleton and $d(m(S), m(S)) = 3$, or $m(S)$ is the singleton and
  $d(m(S), m(S)) = 4$;
  \item[(B2)] $m(S)$ is the singleton, and $d(m(S), m(S)) \geq 4$;
\end{itemize}

\footnote{It is possible to rephrase this argument in terms of matchings on chain complexes, see [10].}
(B₃) \( m(S) \) and \( m(S) \) are in double gaps;
(B₄) \( m(S) \) is the singleton, and \( d(m(S), m(S)) \geq 5 \).

\[
\begin{align*}
B₁ & \quad \begin{array}{c}
\overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet}
\end{array} \quad \text{or} \quad \begin{array}{c}
\overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet}
\end{array} \\
B₂ & \quad \begin{array}{c}
\overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet}
\end{array} \\
B₃ & \quad \begin{array}{c}
\overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet}
\end{array} \\
B₄ & \quad \begin{array}{c}
\overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet} \quad \overset{m(S)}{\bullet}
\end{array}
\end{align*}
\]

**Figure 4.4.** The 4 cases in the proof of Lemma 4.11

Let \( \overrightarrow{Aₖ'/Aₖ}, d₁ \) be the complex spanned by \( B₁, B₂, B₃, \) and \( Aₖ' \), for \( 2r - 2t + 2 \leq i \leq 2r - t \). The relative complex \( (Aₖ'/Aₖ, d₁) \) has only cochains in dimensions \( 2r - 2t \) and \( 2r - 2t + 1 \). It is easy to see that the projection \( d₁ : Aₖ^{2r-2t} \rightarrow Aₖ^{2r-2t+1} / \overrightarrow{Aₖ^{2r-2t+1}} = B₄ \) is an isomorphism, with the inverse given by \( \sigma_S \mapsto \sigma_S \setminus \{m(S) - 1\} \), where \( \sigma_S \) is a generator from \( B₄ \). Hence \( (Aₖ'/Aₖ, d₁) \) is acyclic, and we are led to study the complex \( (\overrightarrow{Aₖ'}, d₁) \).

Next, we show that \( H^{2r-2t+1}(\overrightarrow{Aₖ'}) = 0 \), which is the same as to say that \( d₁ \) is injective on \( B = B₁ \oplus B₂ \oplus B₃ \). Let \( \sigma \neq 0 \) be in \( \ker d₁(B) \). We think of \( \sigma \) as a linear combination of the generators from the descriptions of \( B₁, B₂, \) and \( B₃ \), and let \( M \) be the set of generators which have a nonzero coefficient in \( \sigma \).

Assume \( M \) contains a generator \( \sigma_S \) from \( B₃ \), and choose \( \sigma_S \) so that \( d(m(S), m(S)) \) is minimized. The coboundary \( d₁(\sigma_S) \) contains a copy of \( \sigma_{S \cup \{m(S)\}} \). At most 3 other generators will contain \( \sigma_{S \cup \{m(S)\}} \) in the coboundary, depending on which element we remove from \( S \cup \{m(S)\} \) instead of \( m(S) \). Since we have chosen \( d(m(S), m(S)) \) to be minimal, we cannot remove \( m(S) + 2 \). Hence, we must remove an element extending the singleton gap which is not \( m(S) + 1 \). This gives a generator of \( B₄ \), yielding a contradiction.

From now on we may presume that \( M \) contains no generators from \( B₄ \) or \( B₃ \). Assume now \( M \) contains a generator \( \sigma_S \) from \( B₂ \). Again, choose \( \sigma_S \) so that \( d(m(S), m(S)) \) is minimized, and note that \( d₁(\sigma_S) \) contains a copy of \( \sigma_{S \cup \{m(S)\}} \). Examining the generators which contain \( \sigma_{S \cup \{m(S)\}} \) in the coboundary, we see again that, since removing \( m(S) + 2 \) from \( S \cup \{m(S)\} \) would contradict the minimality, we must remove an element extending the singleton gap \( m(S) \). This way we will produce a generator of \( B₄ \), except for one case: when \( m(S) = 1 \), and we remove vertex 2. In this case we produce a generator from \( B₃ \), hence again a contradiction.

Finally, assume \( M \) consists only of generators from \( B₁ \). Choose \( \sigma_S \) so that \( m(S) \) is maximized. Assume first that \( m(S) \) is in a double gap, and consider the copy of \( \sigma_{S \cup \{m(S)\}} \) in \( d₁(\sigma_S) \). There are 3 possibilities. Removing \( m(S) + 2 \) gives a generator of \( B₂ \), whereas removing \( m(S) - 4 \) gives a generator of \( B₄ \). Removing \( m(S) - 2 \) gives either a generator of \( B₃ \), if \( m(S) = 4 \), or a generator \( \sigma_T \) of \( B₁ \), such that \( m(T) = m(S) + 1 \). In either case we get a contradiction.
Assume now that \( m(S) \) is a singleton, and examine the copy of \( \sigma_{S \mapsto \overline{m(S)}} \) in \( d_1(s) \). There is only one possibility for deletion: remove \( m(S) + 1 \). This will produce a generator \( \sigma_T \) of \( B_1 \), with \( m(T) = m(S) \), but such that \( m(T) \) is in a double gap, the case which we have already dealt with.

This finishes the proof that \( H^{2r-2t+1}(A^*_t) = 0 \), which, combined with the acyclicity of \( (A^*_t/A^*_1, d_1) \), and the fact that \( H^*(A^*_t) = H^*(D_t) \), yields \( E_2^{2r-2t+1,2t} = E_2^{2r-2t+1,2t} = 0 \).

4.8. **Finishing the computation of** \( H^{n-2}(\text{Hom}(C_{2r+1}, K_n); R) \) **and of** \( H^{n-3}(\text{Hom}(C_{2r+1}, K_n); R) \), **for** \( R = \mathbb{Z}_2 \) **or** \( \mathbb{Z} \).

Let us now turn our attention to the cochain complex \( A^*_1 \). The generators of \( \overline{E}_1^{2r-2,n-2} \) correspond to arcs of length \( 2r - 1 \) and can be indexed with the elements of \( \text{V}(C_{2r+1}) \): we set \( \tau_{v,2} := \sigma_{v'(C_{2r+1}) \setminus \{v-2,v-1\},v} \), for any \( v \in \text{V}(C_{2r+1}) \). The same way, the generators of \( E_1^{2r-1,n-2} \) correspond to arcs of length \( 2r \), we denote them by setting \( \tau_{v,1} := \sigma_{v'(C_{2r+1}) \setminus \{v-1\},v} \), for any \( v \in \text{V}(C_{2r+1}) \). It follows from (4.10) that

\[
\begin{align*}
d_1([\tau_{v,2}]) &= (-1)^{n+1} [\tau_{v,1}] + (-1)^n [\tau_{v-1,1}], \\
d_1([\tau_{v,2}]) &= (-1)^{n+1} [\tau_{v,2}] - [\tau_{1,1}],
\end{align*}
\]

for \( v = 3, \ldots, 2r + 1 \), where the second sign follows from (1.8);

\[
\begin{align*}
d_1([\tau_{v,2}]) &= (-1)^{n+1} [\tau_{2,1}] - [\tau_{1,1}],
\end{align*}
\]

where the first sign is determined by the fact that there are \( n + 2r - 3 \) vertices before the one inserted at position \( 2r + 1 \);

\[
\begin{align*}
d_1([\tau_{1,2}]) &= (-1)^{n+1} [\tau_{1,1}] + (-1)^n [\tau_{2r+1,1}],
\end{align*}
\]

where we use again that there are \( n + 2r - 3 \) vertices before the inserted one, and, for determining the second sign, we use the fact that positions 1 and \( 2r + 1 \) have different parity in \( [1, 2r+1] \setminus \{2r\} \).

Summarizing, we have the following matrix for the first differential in \( A^*_1 \):

\[
M = \begin{bmatrix}
(-1)^{n+1} & 0 & 0 & 0 & \cdots & 0 & (-1)^n \\
-1 & (-1)^{n+1} & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & -1 & 1
\end{bmatrix}
\]

Assume first that \( n \geq 5 \), and \( (n, r) \neq (5, 3) \).

**Case 1:** \( n \) **is odd**. It is easy to see that the kernel of the differential \( d_1 : \overline{E}_1^{2r-2,n-2} \rightarrow \overline{E}_1^{2r-1,n-2} \) is one-dimensional and is spanned by \( [\tau_{1,2}] + [\tau_{2,2}] + [\tau_{3,2}] - [\tau_{4,2}] - [\tau_{5,2}] - [\tau_{6,2}] + \cdots + [\tau_{2r+1,2}] \), while the image is

\[
\left\{ \left[ \sum_{i=1}^{2r+1} c_i \tau_{i,1} \right] \mid \left[ \sum_{i=1}^{2r+1} c_i \right] = 0 \right\}.
\]

It follows that \( E_2^{2r-2,n-2} = \mathbb{Z} \). Recall that, by Corollary 4.3, the cohomology groups of \( \text{Hom}(C_{2r+1}, K_n) \) vanish in dimension \( n + 2r - 2 \) and less. Hence, since \( d_2 : E_2^{2r-2,n-2} \rightarrow E_2^{2r,n-3} \) must be an isomorphism, we have \( E_1^{2r,n-3} = E_2^{2r,n-3} = \mathbb{Z} \). On the other hand, the map \( d_3 : E_2^{2r-1,n-2} \rightarrow E_2^{2r,n-2} \) is surjective, and \( E_2^{2r-1,n-2} = E_2^{2r,n-2} = 0 \), so \( E_1^{2r,n-2} = \mathbb{Z} \).
Case 2: $n$ is even. In this case the map $d_1 : E_1^{2r-2,n-2} \to E_1^{2r-1,n-2}$ is injective. It follows that $E_2^{2r-2,n-2} = 0$, and hence $E_1^{2r,n-3} = E_2^{2r,n-3} = 0$. The image on the other hand is not the whole $E_1^{2r-1,n-2}$, but only

$$\left\{ \sum_{i=1}^{2r+1} c_i \tau_{i,1} \right\} \left\{ \sum_{i=1}^{2r+1} c_i \equiv 0 \pmod{2} \right\}.$$  

The fact that $E_2^{2r-1,n-2} = E_2^{2r,n-2} = 0$ and the surjectivity of the map $d_1 : E_1^{2r-1,n-2} \to E_1^{2r,n-2}$ imply that $E_1^{2r,n-2} = \mathbb{Z}_2$. Again, we used that $H^i(\text{Hom}_+(C_{2r+1}, K_n))$ vanish in dimension $n + 2r - 2$ and less.

If $(n, r) = (5, 3)$, then the argument above essentially holds, with the exception that $d_1 : E_1^{2r-1,n-2} \to E_1^{2r,n-2}$ does not have to be surjective. Instead, $\text{Im} d_1 = \mathbb{Z}$ and $E_1^{2r,n-2}/\text{Im} d_1 = \mathbb{Z}$. Thus $H^{n-2}(\text{Hom}(C_7, K_5); \mathbb{Z}) = E_1^{2r,n-2} = \mathbb{Z}^2$.

| $(n, r)$ | $R$ | $H^{n-2}$ | $H^{n-3}$ |
|----------|-----|---------|---------|
| $2 \nmid n, n \geq 5, (n, r) \neq (5, 3)$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $(n, r) = (5, 3)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| $2 \mid n, n \geq 6$, or $n = 4, r \leq 3$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $0$ |
| $n = 4, r \geq 4$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $0$ |
| $n \geq 5, (n, r) \neq (5, 3)$, or $n = 4, r \leq 3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| $(n, r) = (5, 3)$, or $n = 4, r \geq 4$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2$ |

Table 4.1.

Assume, finally, that $n = 4$. If $2r + 1 = 5$, then the computations above hold. If $2r + 1 \geq 7$, the argument above still shows that the image of the map $d_1 : E_1^{2r-1,2} \to E_1^{2r,2}$ is $\mathbb{Z}[d_1(\tau_{1,r})]$, and $2d_1(\tau_{1,r}) = d_1(2\tau_{1,r}) = 0$. If $2r + 1 \geq 9$, we can compute $E_1^{2r,2}$ and $E_1^{2r,1}$ completely, since $H^i(\text{Hom}_+(C_{2r+1}, K_n))$ vanish in dimension $n + 2r - 2$ and less. In this case, the image of the map $d_1 : E_1^{2r-1,2} \to E_1^{2r,2}$ is $\mathbb{Z}_2$, and the map $d_2 : E_2^{2r-3,4} \to E_2^{2r,2}$ is an isomorphism. Since, as we have shown earlier, $E^{2r-3,4}_2 = \mathbb{Z}$, we conclude that $E_1^{2r,2} = \mathbb{Z} \oplus \mathbb{Z}_2$, and $E_1^{2r,1} = 0$.

It follows that, for all $(n, r)$, $H^i(\text{Hom}(C_{2r+1}, K_n); \mathbb{R}) = 0$, if $i \in [1, n - 4]$, and $R = \mathbb{Z}$ or $\mathbb{Z}_2^3$. We summarize our computations of the next two cohomology groups in Table 1, where $H^i = H^i(\text{Hom}(C_{2r+1}, K_n); \mathbb{R})$, and the case $(n, r) = (4, 3)$ is conjectural.

**Proof of Theorem 2.20**

For $n \geq 6$, and $n = 4, r \leq 3$, this follows from the fact that the target group of the map is $\mathbb{Z}_2$. For $n = 4$, $r \geq 4$, we have shown above, that $2d_1(\tau_{1,r}) = 0$. By the construction, $\tau_{1,r} = \sigma_V(C_{r+1}) \setminus \{r-1\}, r$, so $d_1(\tau_{1,r}) = \pm \sigma_V(C_{r+1}, r, 1)$. Let $V(K_2) = \{1, 2\}$, and pick a nontrivial element $\alpha \in H^{n-2}(\text{Hom}(K_2, K_n); \mathbb{Z})$ by setting $\alpha := \eta^*, \eta(1) := [1, n - 1], \eta(2) := \{n\}$. Clearly, $\iota^*_{K_2}(\alpha) = \pm \sigma_V(C_{r+1}, r, 1)$, where $\iota(1) = r, \iota(2) = r + 1$. Thus, we see that $2 \cdot \iota^*_{K_2}(\alpha) = \pm 2d_1(\tau_{1,r}) = 0$. \[3\] This has been strengthened to yield connectivity in \[4\], later a shorter proof appeared in \[5\].
4.9. The $\mathbb{Z}_2$-action on the cohomology groups of $\text{Hom}(C_{2r+1}, K_n)$ for odd $n$.

Throughout this subsection we assume that $n$ is odd, and that $(n, r) \neq (5, 3).$

We tensor all our groups with $C$ to simplify the representations. We denote by $\chi_i$ the one-dimensional representation of $\mathbb{Z}_2$ given by the multiplication by $(-1)^i$.

Lemma 4.12. We have $E_1^{2r,n-2} = \chi_r$, as a $\mathbb{Z}_2$-module.

Proof. Recall that $\sigma_{V(C_{2r+1})}^{2r+1} := \sum_{\eta} \eta^*$, where the sum is taken over all $\eta$, such that $\eta(2r + 1) = 1$, and $|\eta(i)| = 1$, for all $i = 1, \ldots, 2r$. $\sigma_{V(C_{2r+1})}^{2r+1}$ is a representative of the generator of $E_1^{2r,n-2}$. Clearly, $\{\eta \circ \gamma\} = \{\eta\}$ as a collection of cells. To orient the cells in the standard way we need to reverse $\gamma$ as the permutation of $V(C_{2r+1})$. The sign of this is $(-1)^r$, hence $\gamma([\sigma_{V(C_{2r+1})}^{2r+1}]) = (-1)^r [\sigma_{V(C_{2r+1})}^{2r+1}]$.

Lemma 4.13. We have $E_1^{2r-1,n-2} = r\chi_0 + r\chi_1 + \chi_{n+r+1}$, as a $\mathbb{Z}_2$-module.

Proof. $\tau_1, \ldots, \tau_{2r+1}$ can be taken as the representatives of the generators of $E_1^{2r-1,n-2}$. We see first that

(4.9) $\gamma([\tau_{1,1}]) = (-1)^{n+r+1} [\tau_{1,1}]$.

Indeed, $\gamma([\tau_{1,1}]) = \text{sgn} \pi \cdot [\sigma_{1,2r,2r}]$, where $\pi$ is the permutation induced by $\gamma$ on the vertices of each support simplex of $\tau_{1,1}$, i.e.,

$\pi = (n + 2r - 2, n + 2r - 3, \ldots, n + 1, n, 1, \ldots, n - 1).$

Since $\pi$ consists of inverting the sequence $(1, \ldots, n + 2r - 2)$, and then inverting the subsequence $(1, \ldots, n - 1)$, we see that

$$\text{sgn} \pi = (-1)^{|\frac{n+2r-2}{2}| + \frac{n-1}{2}} = (-1)^{r-1 + \frac{n-1}{2}} = (-1)^{r+n},$$

where we used the fact that the sign of inverting a sequence $[1, \ldots, m]$ is $(-1)^{\frac{m}{2}}$, and that, for any natural number $m$, we have $\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m-1}{2} \right\rfloor = m - 1$. Additionally, $\sigma_{(1,2r,2r)]} = -[\sigma_{1,2r,1}]$ by (4.8), hence (4.9) follows.

Next, we shall see that

(4.10) $\gamma([\tau_{2r+2-i,1}]) = (-1)^{r}[\tau_{1,1}]$, for $i = 1, \ldots, 2r$. Again, $\gamma([\tau_{2r+2-i,1}]) = \text{sgn} \pi \cdot [\sigma_{V(C_{2r+1}) \setminus \{i\}, i-1}]$, where $\pi$ consists of inverting the sequence $(1, \ldots, n + 2r - 3)$, and then inverting some subsequence of length $n - 1$ back. It follows that

$$\text{sgn} \pi = (-1)^{|\frac{n+2r-2}{2}| + \frac{n-1}{2}} = (-1)^{r-1 + \frac{n-1}{2}} + \frac{n-1}{2} = (-1)^{r+1}.$$

On the other hand, by (4.8), $[\sigma_{V(C_{2r+1}) \setminus \{i\}, \frac{i-1}{2}]} = -[\sigma_{V(C_{2r+1}) \setminus \{i\}, \frac{i+1}{2}]}$, hence we get (4.10). The actual sign has no bearing on our final conclusion.

Since the permutation action of $\mathbb{Z}_2$ on a 2-dimensional space decomposes as $\chi_0 + \chi_1$, the formulae (4.9) and (4.10) together yield the claim of the lemma.

Lemma 4.14. We have $E_1^{2r-2,n-2} = r\chi_0 + r\chi_1 + \chi_{r+1}$, as a $\mathbb{Z}_2$-module.

Proof. $\tau_1, \ldots, \tau_{2r+1,2}$ can be taken as the representatives of the generators of $E_1^{2r-2,n-2}$. We see first that

(4.11) $\gamma([\tau_{r+2,2}]) = (-1)^{r+1}[\tau_{r+2,2}]$. 
We have $\gamma([r_{r+2}]) = \text{sgn} \gamma(\sigma_{V(C_{2r+1}) \setminus \{r, r+1\}})$, where $\gamma$ consists of inverting the sequence of length $n + 2r - 4$, and then inverting some subsequence of length $n - 1$ back. It follows that
\[
\text{sgn} \gamma = (-1)^{\lfloor \frac{n+2r-4}{2} \rfloor + \lfloor \frac{n-1}{2} \rfloor} = (-1)^{r-2+\lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor} = (-1)^{r+n+1}.
\]
Furthermore, by (4.3)
\[
\sigma_{V(C_{2r+1}) \setminus \{r, r+1\}, r-1} = (-1)^n \sigma_{V(C_{2r+1}) \setminus \{r, r+1, r+2\}},
\]
where $(-1)^n$ is composed of $2r - 3$ steps changing the sign, and one step changing the sign by $(-1)^{n+1}$, since $1$ and $2r + 1$ have the same parity in $V(C_{2r+1}) \setminus \{r, r+1\}$. Summarizing we get (4.11).

Second we note that
\[
\gamma([r_{r+2-i, 1}]) = \pm [r_{r+2, 1}],
\]
for $i \in V(C_{2r+1}) \setminus \{r + 2\}$. Indeed, as before we see that $\gamma([r_{r+2-i, 1}]) = \pm [\sigma_{V(C_{2r+1}) \setminus \{i, i+1, i+2\}}]$.

Equations (4.11) and (4.12) show that the $\mathbb{Z}_2$-representation splits into $\chi_{r+1}$ and the $r$-fold permutation action, yielding the claim of the lemma. \qed

**Corollary 4.15.** The group $\mathbb{Z}_2$ acts trivially on $H^{n-2}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z}$, and as a multiplication by $-1$ on $H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z}$.

**Proof.** It follows from Lemmata (4.12) and (4.13) and (4.14) that $E_1^{2r, n-3} = \chi_{r+1}$, as a $\mathbb{Z}_2$-module. The result follows now from the equation (4.10). \qed

5. **Cohomology groups of $\mathbb{Z}_2$-quotients of products of spheres.**

From now on, unless explicitly stated otherwise, we shall only work with $\mathbb{Z}_2$-coefficients.

We begin by introducing another piece of terminology: for a positive integer $d$, let $d$-symbols be elements of the set $\{*, \infty\}$, where * will denote a open $d$-cell, and $\infty$ denote a 0-cell. We assume throughout this section that $d \geq 2$. For example, $S^d$ is decomposed into $*$ and $\infty$, whereas a direct product of $t$ $d$-dimensional spheres decomposes into cells, indexed by all possible $t$-tuples of $d$-symbols. We let $\dim * = d$, $\dim \infty = 0$, and we set the dimension of a tuple of $d$-symbols be the sum of the dimensions of the constituting symbols.

5.1. **Cohomology groups of $\mathbb{Z}_2$-quotients of products of odd number of spheres.**

Let $X$ be a direct product of $2t + 1$ $d$-dimensional spheres, and let $\mathbb{Z}_2$ act on $X$ by swapping spheres numbered $2i + 1$ and $2i$, for $i \in [1, t]$, and acting on the first sphere by an antipodal map. We shall decompose $X/\mathbb{Z}_2$ into cells, and describe its cohomology groups.

Clearly, $X/\mathbb{Z}_2$ is a total space of a fiber bundle over $\mathbb{RP}^d$ with fiber homeomorphic to a direct product of $2t$ $d$-dimensional spheres. Consider the standard cell decomposition of $\mathbb{RP}^d$ with one cell in each dimension $i \in [0, d]$.

**Proposition 5.1.** The space $X/\mathbb{Z}_2$ can be decomposed into cells indexed with $(i, x, y)$, where $x$ and $y$ are $t$-tuples of $d$-symbols, $0 \leq i \leq d$. The dimension of this cell is $\dim (i, x, y) = i + \dim x + \dim y$.

The coboundary is given by the equation
\[
d^{i+\dim x+\dim y}((i, x, y)^*) = (i + 1, x, y)^* + (i + 1, y, x)^*,
\]
where the cochains are considered with $\mathbb{Z}_2$ coefficients.
Proof. Divide $X$ into cells, by taking the product cell structure, where spheres 2 to $2t + 1$ have one 0-cell and one $d$-cell, whereas the first sphere is subdivided as a join of $d + 1$ 0-spheres, with $\mathbb{Z}_2$ acting antipodally on each of these 0-spheres. The cells can then be indexed with triples $(i, x, y)_+$ and $(i, x, y)_-$. The coboundary is given by

$$d((i, x, y)_+) = (i + 1, x, y)_+ + (i + 1, x, y)_-.$$  

This cell structure is $\mathbb{Z}_2$-equivariant, and no cells are preserved by the involution. This means that it induces a cell structure on $X/\mathbb{Z}_2$. Let $(i, x, y)$ denote the orbit $\{(i, x, y)_+, (i, y, x)_-\}$. After taking the quotient, (5.2) becomes (5.1). □

It follows from the Proposition 5.1 that the generators of $H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ are indexed with

- $(i, x, x)$, for any $0 \leq i \leq d$, and a $t$-tuple of $d$-symbols $x$, here $(i, x, x)^*$ is the cocycle;
- $(0, x, y)$, for any $t$-tuples of $d$-symbols $x \neq y$, here $(0, x, y)^* + (0, y, x)^*$ is the cocycle; $(0, x, y)$ and $(0, y, x)$ index the same generator;
- $(d, x, y)$, for any $t$-tuples of $d$-symbols $x \neq y$, here $(d, x, y)^*$ is the cocycle; $(d, x, y)$ and $(d, y, x)$ index the same generator.

In other words, the cohomology generators are indexed by pairs $((A), i)$, where $A$ is a $2 \times t$ array of $d$-symbols, and $i \in [0, d]$, if $A$ is fixed by $\mathbb{Z}_2$, while $i \in [0, d]$, if $A$ is not fixed by $\mathbb{Z}_2$. Here $\mathbb{Z}_2$ acts on the set of all $2 \times t$ arrays of $d$-symbols by swapping the two rows, and $\langle - \rangle$ denotes an orbit of this action.

For future reference, we remark the following property: these generators behave functorially, under the maps which insert additional pairs of spheres. More specifically, assume $q \geq t$, and let $f : [1, t] \hookrightarrow [1, q]$ be an injection. Let $\hat{f} : \underbrace{S^d \times \cdots \times S^d}_{2q+1} \to \underbrace{S^d \times \cdots \times S^d}_{2t+1}$ be the following map: $\hat{f}$ is identity on the first sphere, it maps isomorphically the spheres indexed $2i$ and $2i + 1$, for $i \in \text{Im } f$, to the spheres indexed by $2f^{-1}(i)$ and $2f^{-1}(i) + 1$, and it maps the remaining spheres to the base point. Then, the induced map on the cohomology $\hat{f}^*$ maps the generator $((A), i)$ to the generator $((\hat{A}), i)$, where $\hat{A}$ is the $2 \times q$ array obtained from $A$ as follows: the column $f(i)$ in $\hat{A}$ is equal to the column $i$ in $A$, and, for $j \notin \text{Im } f$, the column $j$ in $\hat{A}$ consists of two $\infty$’s.

5.2. Cohomology groups of $\mathbb{Z}_2$-quotients of products of even number of spheres.

Let $X$ be a direct product of $2t$ $d$-dimensional spheres, and let $\mathbb{Z}_2$ act on $X$ be swapping spheres $2i - 1$ and $2i$, for $i \in [1, t]$. A customary notation for $X/\mathbb{Z}_2$ is $SP^d(\underbrace{S^d \times \cdots \times S^d}_t)$. Again, we shall decompose $X/\mathbb{Z}_2$ into cells, and describe its cohomology groups.

Proposition 5.2. The space $X/\mathbb{Z}_2$ can be decomposed into cells indexed with two types of labels:

Type 1. the unordered pairs $\{x, y\}$, where $x$ and $y$ are $t$-tuples of $d$-symbols, $x \neq y$; the dimension is $\dim x + \dim y$;

Type 2. $(x, x, k)$, where $x$ is a $t$-tuple of $d$-symbols, and $0 \leq k \leq \dim x$; the dimension is $\dim x + k$. 
With $\mathbb{Z}_2$ coefficients, the coboundary is equal to 0 for all generators, except for $(x, x, 0)^+$, when $\dim x \geq 1$, in which case $\delta^{\dim x}(x, x, 0)^* = (x, x, 1)^*$. In particular, the generators of $\widetilde{H}^i(X/\mathbb{Z}_2; \mathbb{Z}_2)$ are indexed with the same symbols as the cells in our decomposition, except for $(x, x, 0)$ and $(x, x, 1)$.

**Proof.** Start with a usual subdivision of a direct product of $2t$ $d$-spheres, with the cells indexed by pairs $(x, y)$ of $t$-tuples of $d$-symbols. For $x \neq y$, the set $(x, y) \cup (y, x)/\mathbb{Z}_2$ is a cell in $X/\mathbb{Z}_2$, which we label with $\{x, y\}$.

To do the same for $x = y$, we need to take a finer subdivision of $(x, x)$. Let $(x, x, k)^+$, resp. $(x, x, k)^-$, be the set of all points $\bar{x} \in \mathbb{R}^{2 \dim x}$, $\bar{\alpha} = (\alpha_j)_{j \in [2 \dim x]}$, such that $\alpha_j = \alpha_{j+\dim x}$, for $k + 1 \leq j \leq \dim x$, and $\alpha_k > \alpha_{k+\dim x}$, resp. $\alpha_k < \alpha_{k+\dim x}$. Obviously, $(x, x, k)^+$ and $(x, x, k)^-$ are cells, which are mapped to each other by the $\mathbb{Z}_2$-action. These cells are different for $k \geq 1$, whereas $(x, x, 0)^+ = (x, x, 0)^-$ is fixed pointwise.

Set $(x, x, 0) := (x, x, 0)^+$, and $(x, x, k) := (x, x, k)^+ \cup (x, x, k)^- / \mathbb{Z}_2$, for $k \geq 1$. The statements about the coboundary map and the indexing of the cohomology generators follow immediately from our construction. \hfill \Box

Rephrasing Proposition 5.2 in the language of arrays, the generators of $H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ are indexed with $\mathbb{Z}_2$-orbits $\langle A \rangle$ of $2 \times t$ arrays of $d$-symbols, with an additional index $2 \leq i \leq \dim A/2$, if $A$ is fixed by the $\mathbb{Z}_2$-action. Here $\dim A$ is the sum of the dimensions of all entries of $A$.

Again, we have functoriality in the following sense: if $q \geq t$, and $f : [1, t] \hookrightarrow [1, q]$ is an injection, define $\tilde{f} : S^d \times \cdots \times S^d \to S^d \times \cdots \times S^d$ analogously to the one in subsection 5.1. Then $\tilde{f}$ maps $\langle A \rangle$, resp. $(\langle A \rangle, i)$, to $\langle \tilde{A} \rangle$, resp. $(\langle \tilde{A} \rangle, i)$, where $\tilde{A}$ is a $2 \times q$ array of $d$-symbols obtained from $A$ by inserting the columns consisting entirely of $\infty$’s in the places indexed by $[1, q] \setminus \text{Im } f$.

6. Spectral sequence for $H^*(\text{Hom}_+(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$

Next, we would like to show Theorem 2.3(b). We assume that $\varpi_1^{n-2}(\text{Hom}_+(C_{2r+1}, K_n)) \neq 0$, and arrive to a contradiction by doing computations in a spectral sequence, which we now proceed to set up.

6.1. $\mathbb{Z}_2$-equivariant cell decomposition of $\text{Hom}_+(C_{2r+1}, K_n)$.  

For convenience, we give following names to the vertices of $C_{2r+1}$: $c := [0]_{2r+1}$, $a_i := [r+i]_{2r+1}$, and $b_i := [r+1-i]_{2r+1}$, for $i \in [1, r]$. That is $\gamma : C_{2r+1} \to C_{2r+1}$ fixes $c$, and $\gamma(a_i) = b_i$, for any $i \in [1, r]$. Identify $V(C_{2r+1})$ with the vertices of an abstract simplicial $\Delta_{2r}$ of dimension $2r$. It is also convenient to have multiple notations for $c$, namely $a_{r+1}, b_{r+1} := c$, see Figure 5.1.

We subdivide the simplex $\Delta_{2r}$ by adding $r$ more vertices, which we denote $c_1, c_2, \ldots, c_r$, and defining a new abstract simplicial complex $\Delta_{2r}$ on the set $\{c, a_1, \ldots, a_r, b_1, \ldots, b_r, c_1, \ldots, c_r\} = V(\Delta_{2r})$. The simplices of $\Delta_{2r}$ are all the subsets of $V(\Delta_{2r})$ which do not contain the subset $\{a_i, b_i\}$, for any $i \in [1, r]$. We set $\mathcal{C} = \{c, c_1, \ldots, c_r\}$. The complex $\Delta_{2r}$ comes equipped with a simplicial $\mathbb{Z}_2$-action, which fixes $\mathcal{C}$ and swaps $a_i$ and $b_i$, for all $i \in [1, r]$. For $S \subseteq V(\Delta_{2r})$ we let $\langle S \rangle$ denote the $\mathbb{Z}_2$-orbit of $S$. 


One can think of this new complex $\tilde{\Delta}_{2r}$ as the one obtained from $\Delta_{2r}$ by representing it as a topological join $\{c\} \ast \{a_1, b_1\} \ast \cdots \ast \{a_r, b_r\}$, with the additional simplicial structure defined by inserting an extra vertex $c_i$ into the middle of each $\{a_i, b_i\}$, and then taking the join of $\{c\}$ and the subdivided intervals. For $\tilde{\sigma} \in \tilde{\Delta}_{2r}$ we obtain $\partial(\tilde{\sigma}) \in \Delta_{2r}$ by replacing every $c_i$ in $\tilde{\sigma}$ by $\{a_i, b_i\}$, i.e., $\partial(\tilde{\sigma}) = (\tilde{\sigma} \setminus \{c_1, \ldots, c_r\}) \cup \bigcup_{c_i \in \tilde{\sigma}} \{a_i, b_i\}$.

The simplicial complex $\tilde{\Delta}_{2r}$ has an additional property: if a simplex of $\tilde{\Delta}_{2r}$ is $\gamma$-invariant, then it is fixed pointwise. This allows us to introduce a simplicial structure (strictly speaking - a structure of triangulated space) on $\Delta_{2r} / \mathbb{Z}_2$ by taking the orbits of the simplices of $\tilde{\Delta}_{2r}$ as the simplices of $\Delta_{2r} / \mathbb{Z}_2$.

**6.2. The chain complex of the subdivision of $\text{Hom}_+(C_{2r+1}, K_n)$.**

Since we are working over $\mathbb{Z}_2$, from now on we shall drop the $+$ notation for the simplices of $\text{Hom}_+(C_{2r+1}, K_n)$, e.g., we shall write $\eta^*$ instead of $\eta^*_+$. (Here we refer to notations introduced in subsection 3.2.)

Let us now describe a cochain complex $\bar{C}^*(\text{Hom}_+(C_{2r+1}, K_n); \mathbb{Z}_2)$, which comes from a triangulation of the simplicial complex $\text{Hom}_+(C_{2r+1}, K_n)$. The cochain complex consists of vector spaces over $\mathbb{Z}_2$, whose generators are pairs $(\eta, \sigma)^*$, where $\eta \in \text{Hom}_+(C_{2r+1}, K_n)$, and $\sigma \in \Delta_{2r}$, such that $\partial(\sigma) = \text{supp} \eta$. Such a pair indexes the cochain which is dual to the cell $\eta \cap \text{supp}^{-1}(\sigma)$. The coboundary of $(\eta, \sigma)^*$ is the sum of the following generators:

1. $(\tilde{\eta}, \sigma)^*$, if $\text{supp} \tilde{\eta} = \text{supp} \eta$ and $\text{dim} \tilde{\eta} = \text{dim} \eta + 1$;
2. $(\eta, \sigma \cup \{x\})^*$, if $x \in V(\Delta_{2r}) \setminus \sigma$, and $\partial(\sigma) = \partial(\sigma \cup \{x\})$;
3. $(\tilde{\eta}, \sigma \cup \{x\})^*$, if $x \in V(\Delta_{2r}) \setminus \sigma$, $\tilde{\eta}|_{\partial(\sigma)} = \eta$, and all the values of $\tilde{\eta}$ on $\partial(\sigma \cup \{x\}) \setminus \partial(\sigma)$ have cardinality 1.

The degree of $(\eta, \sigma)^*$ in $\bar{C}^*(\text{Hom}_+(C_{2r+1}, K_n))$ is given by

$$\deg(\eta, \sigma)^* = |\sigma| - 1 + \sum_{v \in \text{supp} \eta} (|\eta(v)| - 1) = \deg \eta + |\sigma| - |\partial(\sigma)|.$$
6.3. The filtration of $\tilde{C}^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2)$.

This time, we consider the natural filtration $(\tilde{F}^0 \supseteq \tilde{F}^1 \supseteq \ldots)$ on the cochain complex $\tilde{C}^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2)$ by the cardinality of $\sigma$. Namely, $\tilde{F}^p = \tilde{F}^p C^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2)$, is a cochain subcomplex of $C^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2)$ defined by:

$$\tilde{F}^p : \ldots \rightarrow \tilde{F}^{p,q} \rightarrow \tilde{F}^{p,q+1} \rightarrow \ldots,$$

where

$$\tilde{F}^{p,q} = Z_2 \langle (\eta, \sigma)^* \mid (\eta, \sigma) \in C^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2), |\sigma| \geq p + 1 \rangle,$$

and $\partial^*$ is the restriction of the differential in $C^*_{\tilde{Z}_2}(\text{Hom}_+(C_{2r+1}, K_n); Z_2)$.

The following formula is the analog of (6.2).

**Proposition 6.1.** For any $p$,

$$\tilde{F}^p / \tilde{F}^{p+1} = \bigoplus_{\sigma} C^*(\text{Hom}(C_{2r+1}[\theta(\sigma)], K_n)/Z_2; Z_2)[-p]$$

$$\bigoplus_{\langle \tau \rangle} C^*(\text{Hom}(C_{2r+1}[\theta(\tau)], K_n); Z_2)[-p],$$

where the first sum is taken over all $\sigma \subseteq C$, $|\sigma| = p + 1$, and the second sum is taken over all orbits $\langle \tau \rangle$, such that $\tau \subseteq V(\tilde{\Delta}_{2r})$, $|\tau| = p + 1$, $\tau \not\subseteq C$.

Hence, the $0$th tableau of the spectral sequence associated to the cochain complex filtration $\tilde{F}^*$ is given by

$$E^0_{1,q} = \bigoplus_{\sigma} C^q(\text{Hom}(C_{2r+1}[\theta(\sigma)], K_n)/Z_2; Z_2)$$

$$\bigoplus_{\langle \tau \rangle} C^q(\text{Hom}(C_{2r+1}[\theta(\tau)], K_n); Z_2),$$

with the summations over the same sets as in (6.1).

6.4. The analysis of the spectral sequence converging to $H^*(\text{Hom}_+(C_{2r+1}, K_n)/Z_2; Z_2)$.

The $E^*_1$-tableau of this spectral sequence is given by $E^*_{1,q} = H^{p+q}(\tilde{F}^p, \tilde{F}^{p+1})$.

It follows immediately from the formula (6.2) that each $E^*_{1,q}$ splits as a vector space over $Z_2$ into direct sums of $H^q(\text{Hom}(C_{2r+1}[S], K_n); Z_2)$, and of $H^q(\text{Hom}(C_{2r+1}[S], K_n)/Z_2; Z_2)$. More precisely,

$$E^*_{1,q} = \bigoplus_{\sigma \subseteq C} H^q(\text{Hom}(C_{2r+1}[\theta(\sigma)], K_n)/Z_2; Z_2)$$

$$\bigoplus_{\langle \tau \rangle, \tau \not\subseteq C} H^q(\text{Hom}(C_{2r+1}[\theta(\tau)], K_n); Z_2).$$

The generators of $E^*_{1,q}$ stemming from $\sigma \subseteq C$ will be called *symmetric*, whereas the generators stemming from $\langle \tau \rangle$ for $\tau \not\subseteq C$ will be called *asymmetric*.

For $i \in [1, r]$, we shall denote the arc $\{a_i, a_{i-1}, \ldots, a_1, b_1, b_2, \ldots, b_i\}$ by $\sim_i$. For $2 \leq i \leq r$, we denote the arc $\{a_i, a_{i-1}, \ldots, a_r, c, b_r, b_{r-1}, \ldots, b_i\}$ by $\sim_i$. For $2 \leq i < j \leq r$, let $\langle i, j \rangle$ denote the arc $\{a_i, a_{i+1}, \ldots, a_j\}$, let $\langle i, j \rangle$ denote the arc $\{b_j, b_{j-1}, \ldots, b_i\}$, and let $\langle i, j \rangle$ denote the symmetric pair of arcs $\langle i, j \rangle$ and $\langle i, j \rangle$. 

Proposition 6.2. The map
\[ q^{n-3} : H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) \to H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2), \]
is a 0-map.

**Proof.** First of all, since we are working over the field \( \mathbb{Z}_2 \), the map \( q^{n-3} \) is dual to the map on homology
\[ q_{n-3} : H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) \to H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2), \]
hence it is enough to prove that \( q_{n-3} \) is a 0-map.

We start by proving that \( q_{n-3} = 0 \) over integers. The map \( q_{n-3} \) commutes with the \( \mathbb{Z}_2 \)-action. Recall that we have proven that
\[ H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = H^{n-2}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z}, \]
so it follows that \( H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) = \mathbb{Z} \). Let \( \xi \) be a generator of the group \( H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \). By our previous computations \( \gamma^{K_n}(\xi) = -\xi \), since \( H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{C}) = \chi_1 \) as a \( \mathbb{Z}_2 \)-module (it is a dual \( \mathbb{Z}_2 \)-module to \( H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{C}) \)), and since \( H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \) is torsion-free. On the other hand, the \( \mathbb{Z}_2 \)-action on \( H_{n-3}(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2) \) is trivial, hence
\[ -q_{n-3}(\xi) = q_{n-3}(-\xi) = q_{n-3}(\gamma^{K_n}(\xi)) = \gamma^{K_n}(q_{n-3}(\xi)) = q_{n-3}(\xi). \]
We conclude that \( q_{n-3}(\xi) = 0 \).

Second, by the universal coefficient theorem the map
\[ \tau : H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \otimes \mathbb{Z}_2 \to H_{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) \]
is injective and functorial. In our concrete situation, this map is also surjective, hence the claim follows from the following commutative diagram:
\[
\begin{array}{ccc}
H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}) \otimes \mathbb{Z}_2 & \xrightarrow{0\text{-map}} & H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) \otimes \mathbb{Z}_2 \\
\tau \downarrow \text{iso} & & \downarrow \\
H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) & \xrightarrow{q^{n-3}} & H^{n-3}(\text{Hom}(C_{2r+1}, K_n); \mathbb{Z}_2) \\
\end{array}
\]

**Lemma 6.3.** We have \( E^{r+1,n-3}_2 = \mathbb{Z}_2 \).

**Proof.** To start with, the only contribution to \( E^{r,n-3}_2 \) comes from \( \sigma = C \), so the fact that \( q^{n-3} \) in (6.4) is a 0-map implies that the differential \( d_1 : E^{r,n-3}_1 \to E^{r+1,n-3}_1 \) is a 0-map as well.

Consider the cochain complex
\[ A^* : E^{r+1,n-3}_1 \xrightarrow{d_1} E^{r+2,n-3}_1 \xrightarrow{d_1} \cdots \xrightarrow{d_1} E^{2r,n-3}_1. \]
The generators of \( E^{r+1,n-3}_1 \) come from \( \tau = C \cup I \), for \( I \subseteq \{a_1, \ldots, a_r, b_1, \ldots, b_r\} \), \( |I| = i \). We can identify the generator indexed by \( \langle \tau \rangle \) with the simplex of \( \mathbb{R}^{r-1} \cong \{a_1, b_1\} \cdots \{a_r, b_r\}/\mathbb{Z}_2 \), indexed by \( \langle I \rangle \), where the \( \mathbb{Z}_2 \)-action swaps \( a_i \) and \( b_i \), for \( i \in [1, r] \).

By inspecting the description of the differential \( d_1 \) we see that \( A^* \) is isomorphic to the chain complex \( C^*(\mathbb{R}^{r-1}; \mathbb{Z}_2) \). It follows that \( E^{r+1,n-3}_2 = H^0(\mathbb{R}^{r-1}; \mathbb{Z}_2) = \mathbb{Z}_2 \).

In the proof of the next lemma we shall often use the chain homotopy between 0 and the identity.
Let \((C^*, d)\) be a cochain complex, and assume there exist linear maps \(\phi^n : C^n \to C^n\), \(\forall n\), such that
\[
\phi^{n+1}(d(\alpha)) + d(\phi^n(\alpha)) = \alpha, \text{ for all } \alpha \in C^n.
\]
Then \(C^*\) is acyclic.

The proof is immediate, since modulo the coboundaries, every \(\alpha \in C^n\) is equal to \(\phi^{n+1}(d(\alpha))\), hence \(d(\alpha) = 0\) implies \(\alpha = 0\) modulo the coboundaries.

**Lemma 6.4.** We have \(E_2^{r-1,n-2} = 0\).

**Proof.** Set
\[
A^* : E_1^{0,n-2} \xrightarrow{d_1} E_1^{1,n-2} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{2r,n-2}.
\]
Clearly, to show \(E_2^{r-1,n-2} = 0\) is the same as to show that \(H^{r-1}(A^*) = 0\).

For dimensional reasons, every generator in \(A^*\) is indexed either by \(\sigma \subset V(\Delta_2)\) with an arc selected in \(\vartheta(\sigma)\) (which we call the indexing arc), or the whole set \(V(C_{2r+1})\) (namely, those coming from \(H^{n-2}(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2)\) and from \(H^{n-2}(\text{Hom}(C_{2r+1}, K_n))\)). To simplify the terminology, we shall call the set \(V(C_{2r+1})\) an arc as well. Filter the cochain complex \(A^* = G^{2r+1} \supseteq G^{2r} \supseteq \cdots \supseteq G^1 = 0\), where \(G^l\) is spanned by the generators whose indexing arc has the length at least \(l\). We shall compute \(H^{r-1}(A^*)\) by considering the corresponding spectral sequence \(\tilde{E}^p_0 : = C^{p+q}(G^p/G^{p-1})\).

In the same pattern as we have already encountered, the cochain complex \((G^p/G^{p-1}, d_0)\) splits into a direct sum of subcomplexes which are indexed by different arcs. For an arc \(a\), let \(B^*_a\) denote the corresponding summand. Hence \(\tilde{E}^p_1 = \bigoplus_a H^{p+q}(B^*_a)\), where the sum is taken over all arcs \(a\) of length \(p\).

Next, by considering all possible arcs case-by-case, we compute the entries \(\tilde{E}_1^{p,r-1-p}\), for \(p = 2, \ldots, 2r + 1\). To start with, since \(\vartheta(\sigma) = V(C_{2r+1})\) implies \(|\sigma| \geq r + 1\), \(\tilde{E}_0^{2r+1,-r-2} = C^{r-1}(G^{2r+1}/G^{2r}) = 0\) for dimensional reasons, hence \(\tilde{E}_1^{2r+1,-r-2} = 0\).
We shall only consider the cases where we cannot use dimensional reasons to immediately conclude that $B_a^{-1} = 0$.

Case 1. Let $a = \omega_r$. Then, $B_a^{-r} = 0$ for dimensional reasons, and $B_a^{-1} = \mathbb{Z}_2$ coming from $\sigma = \{c_1, \ldots, c_r\}$. The differential $d : B_a^{-r} \to B_a^r$ is a 0-map since $f^{n-2} : H^{n-2}(\mathbb{RP}^{n-2}; \mathbb{Z}_2) \to H^{n-2}(S^{n-2}; \mathbb{Z}_2)$ is a 0-map, where $f : S^{n-2} \to S^{n-2}/\mathbb{Z}_2 = \mathbb{RP}^{n-2}$ denotes the covering map. Hence, in this case, $H^{r-1}(B_a^r) = \mathbb{Z}_2$.

Case 2. Let $a = \omega_2$. Again, $B_a^{-r} = 0$ for dimensional reasons, and $B_a^{-1} = \mathbb{Z}_2$ coming from $\sigma = \{c_2, \ldots, c_r, c\}$. However, this time $d(B_a^{-1}) \neq 0$, since it is induced by the map $f^{n-2} : H^{n-2}(\text{Hom}(G, K_a)/\mathbb{Z}_2) \to H^{n-2}(\text{Hom}(G, K_n))$, which, as we have seen, is not a 0-map; here $G$ is the tree on 3 vertices and $\mathbb{Z}_2$ action is swapping the leaves. Hence $H^{r-1}(B_a^r) = 0$.

Case 3. Let $a = \omega_k$, for $1 \leq k < r - 1$. Let $a \in B_a^m$ be a generator indexed by $\sigma \subset V(\overline{\Delta}_2)$. If $\sigma \subset C$, and $x \in \{a_1, \ldots, a_r, b_1, \ldots, b_r\} \setminus \{a_{k+1}, b_{k+1}\}$, then the differential maps $\alpha$ to the generator indexed by $\langle \sigma \cup \{x\} \rangle$ (again $\omega_k$ is selected) as a 0-map, for the reason described in Case 1. This means that the complex $B_a^n$ splits into two direct summands, one containing all generators indexed by $\sigma \subset C$, and the other those indexed by $\sigma$, such that $\sigma \setminus C \neq \emptyset$.

In both summands, define $\phi^m(\alpha)$ to be the generator indexed by $\langle \sigma \setminus \{c\} \rangle$, if $c \in \sigma$, and $\phi^m(\alpha) = 0$ otherwise. The equation $\phi$ is satisfied, so both summands are acyclic, hence so is $B_a^r$, in particular $H^{r-1}(B_a^r) = 0$.

Case 4. Let $a = \omega_k$, for $3 \leq k \leq r$. We do the same as in the case 3 with $c$, replaced with $c_k$. However, in this complex, if $\sigma \subset C$, and $x \in \{a_1, \ldots, a_r, b_1, \ldots, b_r\} \setminus \{a_{k-1}, b_{k-1}\}$, then the differential maps $\alpha$ to the generator indexed with $\langle \sigma \cup \{x\} \rangle$ (again $\omega_k$ is selected) as an identity, hence the complex does not split and the equation $\phi$ can be applied to the whole complex, yielding $H^{r-1}(B_a^r) = 0$.

Case 5. Let $a = \omega_r$. For each indexing orbit $\langle \sigma \rangle$ choose the representative $\sigma$ such that $a_1 \notin \sigma$. Define $\phi^r$ as in the Case 3, taking $b_1$ instead of $c$. The equation $\phi^r$ is rather straightforward. We just need to pay attention to what the differential does to the generator indexed by $\sigma = \{c_1, \ldots, c_r\}$.

It follows from the description of the cell decomposition and the cohomology of $SP_2(S^n)$, given as a special case in subsection 2.2 that the map $f^{n-2} : H^{n-2}(SP_2(S^n)^2; \mathbb{Z}_2) \to H^{n-2}(S^{n-2} \times S^{n-2}; \mathbb{Z}_2)$, induced by the quotient map, takes the nonzero element to the sum of the two generators of $H^{n-2}(S^{n-2} \times S^{n-2}; \mathbb{Z}_2)$ corresponding to each of the two spheres. In $B_a^r$ this means that the differential of $\sigma$ will contain the generator of $\langle \sigma \cup \{b_1\} \rangle$, with the indexing arc $\langle 1, r \rangle$, but not the generator of $\langle \sigma \cup \{b_1\} \rangle$, with the indexing arc $\langle 1, r \rangle$. Thus we conclude again that $B_a^r$ is acyclic, and $H^{r-1}(B_a^r) = 0$.

Case 6. Let $a = \omega_1$. $B_a^{-r} = 0$ for dimensional reasons. The space $B_a^{-1}$ is spanned by the $2^{-1}$ generators which we can index with sets $\{a_1, a_2, \ldots, a_r\}$, where $a_i \in \{a_1, c_i\}$, for $2 \leq i \leq r$. Denote by $\tilde{a}$ the generator indexed by the set $\{a_1, a_2, \ldots, a_r\}$. The coboundary of every generator $\alpha \neq \tilde{a}$ contains some generator $\beta$ indexed by $\{b_1, a_1, \xi_2, \ldots, \xi_r\}$. Since $\alpha$ is uniquely reconstructible from $\beta$, and the coboundary of $\tilde{a}$ does not contain such generators as $\beta$, we see that an element in $\text{ker}(d : B_a^{-1} \to B_a^r)$ cannot contain $\tilde{a}$ with a nonzero coefficient. Thus, the only chance for this kernel to be nontrivial would be that $\tilde{a}$ lies in it, but, obviously, $d(\tilde{a}) \neq 0$. Hence, once again, $B_a^r$ is acyclic, and $H^{r-1}(B_a^r) = 0$. 
Case 7. Let $a$ be an asymmetric arc, such that $a \cap \{a_r, b_r, c\} = \emptyset$. The complex $B_n^a$ is isomorphic to a simplicial complex of a cone with an apex in the vertex $c$. Hence $B_n^a$ is acyclic, and $H^{-1}(B_n^a) = 0$.

Case 8. Let $a$ be such that $a \cap \{a_1, a_2, b_1, b_2\} = \emptyset$. The complex $B_n^a$ is isomorphic to a simplicial complex of a cone with an apex in the vertex $c_1$. Hence $B_n^a$ is acyclic, and $H^{-1}(B_n^a) = 0$.

Case 9. Let $a = \{c, a_i, a_{i+1}, \ldots, a_r, b_r, b_{r-1}, \ldots, b_j\}$, for $2 \leq i < j$, where possibly $j = r + 1$, which means $a$ does not contain any $b_i$’s. The complex $B_n^a$ is isomorphic to a simplicial complex of a cone with an apex in the vertex $b_1$. Hence $B_n^a$ is acyclic, and $H^{-1}(B_n^a) = 0$.

Case 10. Let $a = \{a_i, a_{i-1}, \ldots, a_1, b_1, b_2, \ldots, b_j\}$, for $r \geq i > j \geq 1$. The complex $B_n^a$ is isomorphic to a simplicial complex of a cone with an apex in the vertex $a_1$. Hence $B_n^a$ is acyclic, and $H^{-1}(B_n^a) = 0$.

We can now summarize our computations as follows: $E_1^{r,1-p} = 0$, for $p = 2, \ldots, 2r - 1$, whereas $E_1^{2r,-r-1} = \mathbb{Z}_2$. The generator of $E_1^{2r,-r-1}$ comes from $a = \sigma$, which in turn comes from $x_1^{r-2}(\text{Hom}(K_2, K_n))$. The map $d_1 : E_1^{2r,-r-1} \rightarrow E_1^{2r+1,-r-1}$ is the same as

$$(t_{K_n})^{n-2} : H^{n-2}(\text{Hom}(K_2, K_n)/\mathbb{Z}_2; \mathbb{Z}_2) \rightarrow H^{n-2}(\text{Hom}(C_{2r+1}, K_n)/\mathbb{Z}_2; \mathbb{Z}_2),$$

where $t : K_2 \rightarrow C_{2r+1}$ is either of the two $\mathbb{Z}_2$-equivariant inclusion maps which take the vertices of $K_2$ to $\{a_1, b_1\}$.

Since we assumed that $x_1^{r-2}(\text{Hom}(C_{2r+1}, K_n)) \neq 0$, and the Stiefel-Whitney characteristic classes are functorial, we see that $d_1 : E_1^{2r,-r-1} \rightarrow E_1^{2r+1,-r-1}$ has rank 1, hence $E_1^{2r,-r-1} = 0$. Thus $E_1^{2r,-r-1} = 0$, for $p = 2, \ldots, 2r + 1$, and we conclude that $E_1^{r-2,n-2} = 0$. \hfill \square

Lemma 6.5. We have $E_2^{-i,n-3+i} = 0$, for all $i = 2, 3, \ldots, r$.

Proof. First of all, we note, that the generators in the columns indexed by $r-1$ and less come from $H^*(\text{Hom}(C_{2r+1}[S], K_n)/\mathbb{Z}_2; \mathbb{Z}_2)$ and from $H^*(\text{Hom}(C_{2r+1}[S], K_n)/\mathbb{Z}_2)$, with $S \neq V(C_{2r+1})$ in both cases.

For each row $q$, $q > n - 2$, we shall show that the subcomplex $A_q^\ast = (\tilde{E}_1^{\ast,q}, d_1)$ is acyclic in the entry $n + r - q - 3$.

We begin by dealing with the case $q = n - 1$ separately, that is we analyze the entry $E_2^{r-2,n-1}$. It follows from Propositions 34.1 and 34.2, and for dimensional reasons, that the entries $E_1^{0,n-1}, E_1^{1,n-1}, \ldots, E_1^{r-1,n-1}$ are generated by the contributions whose indexing collections of arcs are $\{\sigma_i, \sigma_j\}$, for $1 \leq i < j - 1 \leq r - 1$.

The contributing spaces are homotopy equivalent to $S^{n-2} \times X/\mathbb{Z}_2$, where $X$ is a direct product of $2t + 1$ $(n - 2)$-dimensional spheres, and $\mathbb{Z}_2$-action is as in Section 4. The generators appearing in the first $r$ entries of the $(n-1)$th row are coming from the $(n-2)$-cocycle of $S^{n-2}$ and the 1-cocycle of $\mathbb{R}P^{n-2}$. The analysis of the differentials shows that the complex $E_1^{0,n-1} \xrightarrow{d_1} E_1^{1,n-1} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E_1^{r-1,n-1}$ computes the nonreduced homology of a simplex with $r - 2$ vertices (which could be identified with the set $\{c_2, \ldots, c_{r-1}\}$). It follows that the entry $E_2^{r-2,n-1}$, which computes the first homology group is equal to 0.

We assume from now on that $q \geq n$. Similar to subsection 4.5, we filter the complexes $A_q^\ast$. To describe the filtration, we sort all generators into 5 groups. The first group (Gr1) contains all asymmetric generators, i.e., those coming from $\langle \sigma \rangle$, for
The symmetric generators, coming from \( \sigma \subseteq \mathcal{C} \), are divided into 4 groups, depending on whether the indexing collection of arcs

- (Gr2) contains both an \( \sim \)-arc, and an \( \sim \)-arc,
- (Gr3) contains an \( \sim \)-arc, but not an \( \sim \)-arc,
- (Gr4) contains an \( \sim \)-arc, but not an \( \sim \)-arc,
- (Gr5) contains no \( \sim \)-arc, and no \( \sim \)-arc.

The groups are ordered as above. We filter the complex \( A^*_Q \) by first sorting the generators by the groups, and then, within each group we filter additionally by the total length of the indexing arcs.

Let \( E_2^{*,*} \) denote the tableaux of the spectral sequence computing the cohomology of \( A^*_Q \). In complete analogy to the situation in subsection 4.5, \( E_0^{*,*} \) splits into pieces indexed by various collections of arcs, which we shall call layers.

We start by analyzing the contributions of the asymmetric generators. Consider the subcomplex \( B^* \) in the splitting indexed by a collection \( A \) of \( t \) arcs of total length \( l \). Since the asymmetric generators come from the direct products of \((n-2)\)-spheres, the only nontrivial cases are \( q = t(n-2) \), for \( t \geq 2 \).

Assume first there is a gap between some pair of arcs of length at least 3, and let \( x \in V(C_{2r+1}) \) be one of the internal points of a gap. If \( x = c \), then \( B^* \) is isomorphic to the chain complex of a cone with apex in \( c \). Without loss of generality, we can assume that \( x = b_i \), for some \( i \). By the previous assumption, \( b_{i-1}, b_{i+1} \notin A \). If \( a_i \notin A \), but either \( a_{i-1} \) or \( a_{i+1} \) (or both) is in \( A \), then \( B^* \) is isomorphic to a chain complex of a cone with apex \( b_i \). If \( a_{i-1}, a_i, a_{i+1} \notin A \), then \( B^* \) is isomorphic to a chain complex of a cone with apex \( c_i \). Finally, assume \( a_i \in A \). Define \( \phi^k : B^k \to B^{k-1} \) as follows: for a generator \( \sigma \in B^k \),

\[
\phi^k(\sigma) = \begin{cases} 
\sigma \setminus \{b_i\}, & \text{if } b_i \in \sigma, \text{ i.e., if } \sigma \cap \{a_i, b_i, c_i\} = \{b_i\}, \text{ or } \{b_i, c_i\}; \\
\sigma \setminus \{c_i\}, & \text{if } \sigma \cap \{a_i, b_i, c_i\} = \{a_i, c_i\}; \\
0, & \text{if } \sigma \cap \{a_i, b_i, c_i\} = \emptyset, \text{ or } \{c_i\}, \text{ or } \{a_i\}.
\end{cases}
\]

Let \( B^* \) be the subcomplex of \( B^* \) generated by all \( \sigma \), such that \( a_i \in \sigma \). Clearly, the L6.3 is fulfilled both for \( B^* \) and for \( B^*/B^* \). It implies that they are both acyclic, hence so is \( B^* \).

If all gaps are of length at most 2, then \( l + 2t \geq 2r + 1 \). On the other hand, \( B^p = 0 \) for \( p \leq l/2 - 1 \), since \( |\partial(\sigma)| \leq 2|\sigma| - 1 \), for \( \sigma \not\subseteq \mathcal{C} \). Recall that \( q = t(n-2) \), it follows that the entry \( n + r - q - 3 \) is 0, since

\[
l/2 - 1 - (n + r - t(n - 2) - 3) > r - t - 1 - n - r + tn - 2t + 3 = tn - 3t - n + 2 = (t - 1)(n - 3) - 1 \geq 1.
\]

Hence \( B^* \) is acyclic in the required entry, for all \( B^* \) in the group (Gr1).

Next, we move on to the symmetric generators. For \( \sigma \subseteq \mathcal{C} \) we call \( |\mathcal{C} \setminus \sigma| \) the total length of gaps. Let \( B^* \) be a subcomplex in the splitting corresponding to a layer from the group (Gr2). The contributing space here is \( S^{n-2} \times X/\mathbb{Z}_2 \), where \( X \) is a direct product of \( 2t + 1 \) \((n-2)\)-spheres and \( \mathbb{Z}_2 \)-action is as in Section 5.

If \( t = 0 \), since the column number is at most \( r - 3 \), the gap between the \( \sim \)-arc and the \( \sim \)-arc is at least 3. This means that \( B^* \) is isomorphic to a cochain complex of the simplex, hence is acyclic.

Assume now \( t \geq 1 \). By examining the cohomology groups of \( S^{n-2} \times X/\mathbb{Z}_2 \), and taking into account that each of the \( t \) pairs of spheres must contribute nontrivially,
we see that the dimension of the contributing cocycle of $S^{n-2} \times X/\mathbb{Z}_2$ is at least $n-2+t(n-2) = (t+1)(n-2)$, hence the total length of gaps is at least $t(n-2) + 1$. Assume the total length of gaps is at most $2(t+1)$, as otherwise $B^*$ is isomorphic to a cochain complex of the simplex. By assumptions, $t \geq 1$ and $n \geq 5$, so unless $(t, n) = (1, 5)$, we have

$$t(n-2) + 1 - (2t + 2) = t(n-4) - 1 > 0,$$

yielding a contradiction.

Consider the remaining case $(t, n) = (1, 5)$. This is the first situation in which we need to analyze the particular entries of $E_1^{*,*}$. Since we must have a precise equality, the total length of gaps is 4, and the only nontrivial case is provided by generators indexed with $\sim_i$, $\sim_j$, and $(i+3, j-3)$. The contributing cohomology generator must be indexed by $(0, *, \infty)$, so just the set of arcs determines everything.

Let $\alpha_{i,j}$ denote such a generator, and let $\beta_{i,j}$ denote the generator whose indexing set of arcs is $\sim_i$, $\sim_j$, and $(i+2, j-3)$, and which is also indexed by $(0, *, \infty)$. Both $\alpha_{i,j}$’s, and $\beta_{i,j}$’s are generators in $E_1^{*,*}$. Consider a linear combination $\sum_{i,j} p_{i,j} \alpha_{i,j}$ lying in the kernel of $d_1$. Since $d_1(\alpha_{i,j})$ contains $\beta_{i,j}$, $\beta_{i+1,j}$, and no other $\beta_{i',j'}$’s we see that $p_{i,j} \neq 0$ implies $p_{i-1,j} \neq 0$. This leads obviously to $p_{i,j} = 0$ for all $i, j$, hence $B^*$ is acyclic in the required entry.

Now consider $B^*$ corresponding to a layer from group $(\text{Gr}3)$. The contributing space here is $X/\mathbb{Z}_2$, where $X$ is a direct product of $2t + 1$ $(n-2)$-spheres and the $\mathbb{Z}_2$-action is as above. Since we are in the row $n$ or higher, we must have $t \geq 1$. The total length of gaps cannot be larger than $2t + 1$, since otherwise $B^*$ is isomorphic to a cochain complex of the simplex. On the other hand, since the dimension of the contributing cohomology generator is at least $t(n-2)$, the total length of gaps must be at least $(t-1)(n-2) + 1$. Comparing these two we see that

$$(t-1)(n-2) + 1 - (2t + 1) = (t-1)(n-4) - 2 > 0,$$

with exceptions: $t = 1$, $n$ is any, $t = 2$, $n = 5, 6$, and $(t, n) = (3, 5)$.

Consider first $t = 1$. Since we can have at most 3 gaps, we must have precisely 3 gaps, so the contributing cohomology generators of $S^{n-2} \times S^{n-2} \times S^{n-2} / \mathbb{Z}_2$ must have dimension $n$. Inspecting the cohomology description of this space from Section 5.5 we see that there are no generators in dimensions between $n-2$ and $2n-4$. Since $2n-4 > n$ we verify this case.

Assume now $(t, n) = (2, 5)$. The only nontrivial case is when the total length of gaps is 4 or 5, and $c$ is in the gaps. Let $\alpha_{i,j}$ denote the generator where the gaps are $\{c, i, i+1, j\}$, $r-2 \geq j \geq i+4$, $i \geq 2$, and $\beta_{i,j}$ denote the generator where the gaps are $\{c, i, j+1, j\}$, $r-3 \geq j \geq i+3$, $i \geq 2$. Let $\gamma_{i,j}$ denote the generator where the gaps are $\{c, i, j\}$, $r-2 \geq j \geq i+3$, $i \geq 2$. Clearly $d_1(\alpha_{i,j}) = \gamma_{i,j} + \gamma_{i+1,j}$, and $d_1(\beta_{i,j}) = \gamma_{i,j} + \gamma_{i+1,j}$. We see that, restricted to the generators $\alpha_{i,j}$, $\beta_{i,j}$, and $\gamma_{i,j}$, we have a chain complex of the graph on Figure 6.3.

The kernel is generated by the elementary squares, so it is enough so see that each square is a coboundary. Indeed, the elementary square with the lower left corner $(i, j)$ is a coboundary of the generator with gaps $\{c, i, i+1, j, j+1\}$.

Finally, assume $(t, n) = (2, 6)$ or $(3, 5)$. These are the tight cases, in the sense that the lengths of all gaps are predetermined: the top gap consists of just $c$, and the other 2, resp. 3, gaps are of length 2. Assume that the kernel of $d_1$ is not zero, and let $\alpha$ be an element in $\ker d_1$. Let $g$ be a generator, which is contained in $\alpha$
with a nonzero coefficient, such that this $g$ maximizes the height of the top gap over all generators appearing with a nonzero coefficient in $\alpha$. Removing the lower element of the top gap of $g$ gives a generator which cannot be cancelled out by the coboundaries of other elements in $\alpha$, due to the assumed maximality property. This yields a contradiction, and hence $\ker d_1 = 0$.

We move on to group (Gr4), and let $B^*$ correspond to a generator indexed by $\sim_j$, and $t$ side arcs. We can have at most $2t + 2$ gaps. The dimension of the contributing cohomology generator is at least $t(n - 2) + n - 2$, thus total length of the gaps is at least $t(n - 2) + 1$. Comparing these inequalities we get

\[ t(n - 2) + 1 - (2t + 2) = t(n - 4) - 1 > 0, \]

with the only exception $t = 1, n = 5$.

Let $(t, n) = (1, 5)$. The interesting dimension here is 6, thus the total length of gaps must be exactly 4. The generators $\alpha_i$ indexed with the collection of arcs $\{(3,i; \sim_{i+3}\},$ for $4 \leq i \leq r - 3$. Since $d_1(\alpha_i)$ contains the generator indexed with $\{(2,i; \sim_{i+3}\},$ and this generator is different for different $\alpha_i$, we see that the only linear combination of $\alpha_i$'s in the kernel of $d_1$ is the trivial one. Hence, we conclude that the contribution to $\widetilde{E}^*_{2,*}$ is 0.

Finally, we consider the case of generators indexed with collections of arcs avoiding all $\sim$- and $\dashv$-arcs. Let us assume there are $t$ such arcs. To avoid a cochain complex of a simplex, the total length of the gaps must be at most $2t + 1$. On the other hand, since the dimension of the generator is at least $t(n - 2)$, the total length of the gaps must be at least $(t - 1)(n - 2) + 1$. Comparing we see that

\[ (t - 1)(n - 2) + 1 - (2t + 1) = (t - 1)(n - 4) - 2 > 0, \]

with the exceptions $t = 1$, any $n$, $n = 5$, $t \leq 3$, and $n = 6$, $t = 2$.

If $t = 1$, the only nontrivial case occurs in the row $n$. Then, in the entry of interest we have only one generator: the one indexed by the arcs $(3,r)$. Its coboundary will contain the generator $(2,r)$, hence it is different from 0.

Let $(t, n) = (2, 5)$. Since we are in the row 6, for dimensional reasons, the total length of gaps in the contributing generator is 4. Thus, we have two types of generators: $\alpha_1^1$ indexed with arc collections $\{(3,i; t+3,r)\}$, $3 \leq i \leq r - 4$, and $\alpha_2^2$ indexed with arc collections $\{(3,i; t+2,r)\}$, $4 \leq i \leq r - 3$. Considering the value of $d_1$ on the generator indexed with $\{(3,i; t+3,r)\}$, we see that for $i > 3$, we have $d_1$ maps to $\{(3,r)\}$, and hence $d_1$ maps to $\{0\}$.

![Figure 6.3.](image_url)
modulo coboundaries, any generator $\alpha_1^i$ is a linear combination of the generators $\alpha_2^j$. The coboundary of $\alpha_1^i$ contains the generator indexed with $\{\jmath, \nu\}$, hence no element in the kernel of $d_1$ can contain $\alpha_1^i$ with a nonzero coefficient. Finally, a nonzero linear combination of $\alpha_2^j$’s cannot lie in the kernel of $d_1$, since $d_1(\alpha_2^j)$ contains the generator indexed with $\{\jmath, \nu\}$, which is different for different $\jmath$.

Again, we conclude that the contribution to $E_*^{\alpha, \nu}$ is 0.

Let $(t, n) = (3, 5)$. For dimensional reasons, the total length of the gaps is precisely 7, thus we have the generators $\alpha_{i,j}$ indexed with arc collections $\{\jmath, \nu\}$, for $\jmath \leq i$, $i + 4 \leq j \leq r$. Since $d_1(\alpha_{i,j})$ contains the generator indexed with $\{\jmath, \nu\}$, and these generators are different for different $\alpha_{i,j}$’s, we see that $d_1$ is injective on the space spanned by $\alpha_{i,j}$’s. Therefore, in this case the contribution to $E_*^{\alpha, \nu}$ is 0. The case $(t, n) = (2, 6)$ is completely analogous.

**Lemma 6.6.** We have $E_*^{i,n,i-1} = 0$, for all $i = 3, \ldots, n - 1$.

**Proof.** Since $|C| = r + 1$, the entries $E_*^{i,n,i-1}$, for $i = 3, \ldots, n - 1$, come from $H_*^{n-1}(\mathcal{C}^{(2r+1)}(\mathcal{K}_n))$, for $\tau \subseteq C$. We have shown before that these cohomology groups vanish in dimension $n - 4$ and less, which implies $E_*^{i,n,i-1} = 0$, hence $E_*^{i,n,i-1} = 0$.

We conclude that $E_*^{i,n-1} = \mathbb{Z}_2$, contradicting the fact that $H_*^{n-2}(\mathcal{C}^{(2r+1)}(\mathcal{K}_n)) = 0$. Therefore, our original assumption that $\nu_*^{n-2}(\mathcal{C}^{(2r+1)}(\mathcal{K}_n)) = 0$ is wrong, and Theorem 2.8(b) is proved.

**REFERENCES**

[1] E. Babson, D.N. Kozlov, *Topological obstructions to graph colorings*, Electron. Res. Announc. Amer. Math. Soc. 9 (2003), 61–68.

[2] E. Babson, D.N. Kozlov, *Complexes of graph homomorphisms*, Israel J. Math., in press.

[3] P.E. Conner, E.E. Floyd, *Fixed point free involutions and equivariant maps*, Bull. Amer. Math. Soc. 66 (1960), 416–441.

[4] S.Lj. Ćukić, D.N. Kozlov, *Higher connectivity of graph coloring complexes*, Int. Math. Res. Not. 25 (2005), 1543–1562.

[5] T. tom Dieck, *Transformation groups*, Studies in Math. 8, de Gruyter, Berlin, New York, 1987.

[6] A. Engström, *A short proof of a conjecture on the higher connectivity of graph coloring complexes*, preprint, 3 pages, 2005.

[7] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[8] M. Kneser, Aufgabe 360, *Jber. Deutsch. Math.-Verein.* 58 (1955).

[9] D.N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A 88 (1999), no. 1, 112–122.

[10] D.N. Kozlov, *Discrete Morse theory for chain complexes*, C. R. Acad. Sci. Paris, Ser. I 340 (2005), pp. 867–872.

[11] D.N. Kozlov, *A simple proof for folds on both sides in complexes of graph homomorphisms*, Proc. Amer. Math. Soc., in press.

[12] D.N. Kozlov, *Chromatic numbers, morphism complexes, and Stiefel-Whitney characteristic classes*, in: *Geometric Combinatorics* (eds. E. Miller, V. Reiner, B. Sturmfels), IAS/Park City Mathematics Series 14, American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, in press.
[13] D.N. Kozlov, *Simple homotopy type of some combinatorially defined complexes*, preprint, 12 pages, 2005.
  arXiv:math.AT/0503613

[14] D.N. Kozlov, *Cohomology of colorings of cycles*, preprint, 24 pages, 2005.
  arXiv:math.AT/0507117

[15] L. Lovász, *Kneser’s conjecture, chromatic number, and homotopy*, J. Combin. Theory Ser. A 25, (1978), no. 3, pp. 319–324.

[16] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Mathematics 341 (1973), pp. 77–139, Springer-Verlag.

[17] G.M. Ziegler, *Generalized Kneser coloring theorems with combinatorial proofs*, Invent. Math. 147 (2002), pp. 671-691.

[18] R.T. Živaljević, *Parallel transport of Hom-complexes and the Lovász conjecture*, preprint, 17 pages, 2005.
  arXiv:math.CO/0506075

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