Optimization of total population in logistic model with nonlocal dispersals and heterogeneous environments

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Abstract
In this paper, we investigate the issue of maximizing the total equilibrium population with respect to resources distribution \( m(x) \) and diffusion rates \( d \) under the prescribed total amount of resources in a logistic model with nonlocal dispersals. Among other things, we show that for \( d \geq 1 \), there exist \( C_0, C_1 > 0 \), depending on \( \|m\|_{L^1} \) only, such that

\[
C_0\sqrt{d} \leq \text{supremum of total population} \leq C_1\sqrt{d}.
\]

However, when replaced by random diffusion, a conjecture, proposed by Wei-Ming Ni and justified in Bai et al. (Proc. Am. Math. Soc. 144:2161–2170, 2016), indicates that in the one-dimensional case,

\[
\text{supremum of total population} = 3\|m\|_{L^1}.
\]

This reflects serious discrepancies between models with local and nonlocal dispersal strategies. Furthermore, we provide an equivalent characterization about the combination of resource distribution and diffusion rate such that the corresponding total population could reach the optimal order \( \sqrt{d} \) as \( d \) goes to infinity.

Keywords Total population · Nonlocal dispersal · Heterogeneity

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1 Introduction

Dispersal is an important feature of life histories of many organisms and often crucial for their persistence. Understanding the effect of dispersal in heterogeneous environment on population dynamics is an important issue in spatial ecology [9]. Total population is an important indicator for persistence of species. If the quantity is at low level, the risk of extinction will increase, while if the quantity is at high level, it will lead to shortage of resources and intense pressure of competition, which may jeopardize the existing stability of the multi-species systems [19]. Therefore, an interesting problem in spatial ecology is how dispersal strategies of the species and the distribution of resources affect the total population.

Our study is motivated by a series of intriguing questions and work related to total equilibrium population in a single logistic equation with random diffusion as follows

\[
\begin{aligned}
    u_t &= d \Delta u + u[m(x) - u] \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial u}{\partial \nu} &= 0 \quad x \in \partial \Omega, \quad t > 0,
\end{aligned}
\]  

(1.1)

where \( u \) represents the population density of a species at location \( x \in \Omega \) and at time \( t > 0 \), \( d \) is the dispersal rate of the species which is assumed to be a positive constant, the habitat \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \nu \) denotes the unit outward normal vector. The function \( m(x) \) is the intrinsic growth rate or carrying capacity, which reflects the environmental influence on the species \( u \). Unless designated otherwise, we assume that \( m(x) \) satisfies the following condition:

\((\mathbf{M}) \quad m(x) \in L^\infty(\Omega), \quad m(x) \geq 0 \text{ and } m \not\equiv \text{const on } \bar{\Omega}.
\)

It is known that if \( m \) satisfies the assumption \((\mathbf{M})\), then for every \( d > 0 \), the problem (1.1) admits a unique positive steady state, denoted by \( \theta_{d,m}(x) \), which is globally asymptotically stable (see e.g. [9]). In addition, a remarkable property concerning \( \theta_{d,m}(x) \) was first observed in [18]

\[
\int_{\Omega} \theta_{d,m}(x) \, dx > \int_{\Omega} m(x) \, dx \quad \text{for all } d > 0.
\]  

(1.2)

Biologically, this indicates that when coupled with diffusion, a heterogeneous environment can support a total population larger than the total carrying capacity of the environment, which is quite different from homogeneous environment. Simply speaking, heterogeneity of resources can benefit survival of species. This theory is further confirmed experimentally [32]. Moreover, it is known that (see e.g. [18])

\[
\lim_{d \to 0^+} \int_{\Omega} \theta_{d,m}(x) \, dx = \lim_{d \to \infty} \int_{\Omega} \theta_{d,m}(x) \, dx = \int_{\Omega} m(x) \, dx.
\]  

(1.3)

This indicates that for given \( m(x) \), the total population as a function of the diffusion rate \( d \) is not monotone and achieves its maximum at some intermediate value. Examples constructed in [17] show that the local maximum might not be unique. These observations naturally lead to a biological question:

**Question 1** Given the total amount of resources, how should we distribute the resource and/or adjust diffusion rate to maximize the total equilibrium population?
For simplicity, denote
\[ M_1 = \{ m \mid m \text{ satisfies condition (M)}, \int_{\Omega} m(x) dx = 1 \}. \]

In the one-dimensional case, W.-M. Ni conjectured that the supremum of the total population over all \( d > 0 \) and \( m \in M_1 \) is 3. This conjecture is confirmed in [3]. However, for higher dimensional case, it is proved in [14] that the supremum is unbounded. Moreover, when the diffusion rate is fixed, optimization of total population with respect to resources distributions is studied and existing results indicate that the optimal configuration is of bang-bang type. See [21–23, 28]. This question is also studied in patchy environment [27].

The main purpose of this paper is to investigate Question 1 for the following single species model with nonlocal dispersal strategy
\[
\begin{aligned}
&ut(x, t) = dL[u] + u[m(x) - u] \quad x \in \Omega, \quad t > 0, \\
u(x, 0) = u_0 \geq 0, \quad x \in \Omega,
\end{aligned}
\]  
(1.4)

where the nonlocal diffusion operator \( L \), which corresponds to nonlocal homogeneous Neumann boundary condition, is defined as:
\[
L[u] := \int_{\Omega} k(x, y)u(y)dy - \int_{\Omega} k(y, x)dyu(x),
\]
and the dispersal kernel function \( k(x, y) \geq 0 \) describes the probability to jump from one location to another. Although the most popular forms of continuum models have been those given by differential equations (local diffusion), in many situations in ecology (e.g. [6–8, 30]), dispersal is better described as a long range process rather than a local one, and integral operators appear as a natural choice. Depending on the background, there are many forms of nonlocal models. For a detailed introduction of modeling, see the book [11] and the survey paper [12]. The nonlocal diffusion operator studied in this paper is a general form which appears commonly in different types of models in ecology. See [1, 13, 15, 16, 20, 24–26, 29] and the references therein.

From now on, assume that the kernel \( k \) satisfies
\[(K) \quad k(x, y) \in C(\mathbb{R}^n \times \mathbb{R}^n) \text{ is nonnegative and } k(x, x) > 0 \text{ in } \mathbb{R}^n, k(x, y) \text{ is symmetric, i.e., } k(x, y) = k(y, x). \text{ Moreover, } \int_{\mathbb{R}^n} k(x, y)dy = 1. \]

and for simplicity, denote
\[
a(x) = \int_{\Omega} k(y, x)dy \leq 1.
\]

First of all, we prepare the existence and uniqueness result for the model (1.4) provided that \( m(x) \in L^\infty(\Omega) \).

**Theorem 1.1** Assume that \( m(x) \in L^\infty(\Omega) \) is nonconstant and the kernel \( k \) satisfies (K). Define
\[
\mu_0 = \mu_0(m) = \sup_{0 \neq \psi \in L^2(\Omega)} \frac{\int_{\Omega} \left( dL[\psi](x)\psi(x) + m(x)\psi^2(x) \right) dx}{\int_{\Omega} \psi^2(x) dx}.
\]

Then the problem (1.4) admits a unique positive steady state, still denoted by \( \theta_{d, m} \), in \( L^\infty(\Omega) \) if and only if \( \mu_0 > 0 \). Moreover,
\[
\| \theta_{d, m} \|_{L^\infty} \leq \| m \|_{L^\infty}.
\]  
(1.5)
In particular, if \( m \in \mathcal{M}_1 \), then the problem (1.4) admits a unique positive steady state in \( L^\infty(\Omega) \) for any \( d > 0 \).

When \( m \in C(\overline{\Omega}) \), the existence and uniqueness of positive steady state for the model (1.4) has been studied thoroughly. See [5] for symmetric operators in the one dimensional case and [4, 10] for nonsymmetric operators. The proofs of these studies rely on the properties of nonlocal eigenvalue problems, thus the condition \( m \in C(\overline{\Omega}) \) is required. However, to study the questions in this paper, the condition \( m(x) \in L^\infty(\Omega) \) is necessary. For this purpose, we develop a different approach in the proof of Theorem 1.1, which depends on the application of energy functional. Moreover, when the positive steady state to the nonlocal problem (1.4) exists for all \( d > 0 \), it is known that the properties (1.2) and (1.3) are also valid (see e.g. [31]).

Our first result about Question 1 indicates that the supremum of total equilibrium population \( \int_\Omega \theta_{d,m} \, dx \) over \( m \in \mathcal{M}_1 \) and \( d \geq 1 \) is of order \( \sqrt{d} \), where \( \theta_{d,m} \) is designated in Theorem 1.1.

**Theorem 1.2** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 1 \). Then there exist \( C_0, C_1 > 0 \), independent of \( m \in \mathcal{M}_1 \), such that for \( d \geq 1 \)

\[
C_0 \sqrt{d} \leq \sup \left\{ \int_\Omega \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\} \leq C_1 \sqrt{d},
\]

where \( \theta_{d,m} \) denotes the unique positive steady state to the problem (1.4).

Recall that the studies in [3, 14] for the local model (1.1) reveal that the supremum of total equilibrium population over all \( d > 0 \) and \( m \in \mathcal{M}_1 \) is 3 for the one-dimensional case and unbounded for the higher dimensional case, respectively. However, Theorem 1.2 indicates that the supremum over \( m \in \mathcal{M}_1 \) is always unbounded in any dimensional case when the diffusion rate \( d \) goes to infinity. This demonstrates serious discrepancies between models with local and nonlocal dispersal strategies.

It is worth pointing out that, thanks to the property (1.2) and the estimate (3.4) obtained in the proof of Theorem 1.2, one has for any \( d > 0 \),

\[
\int_\Omega m(x) \, dx < \int_\Omega \theta_{d,m}(x) \, dx \leq 2 \int_\Omega m(x) \, dx + 4 \|k\|_{L^\infty} |\Omega|^2 d.
\]

This provides a range of the total equilibrium population when \( d \) is bounded. Thus it suffices to estimate the total population when \( d \geq 1 \) in Theorems 1.2. This observation and the estimate (1.5) in Theorem 1.1 indicate that, under the prescribed total amount of resources, the unboundedness of total equilibrium population is due to the unboundedness of diffusion rate \( d \) and \( \|m\|_{L^\infty} \) simultaneously.

On the basis of Theorems 1.2 and the above discussions, we further explore how to combine resources distribution \( m(x) \) and diffusion rate \( d \) such that the total equilibrium population of the problem (1.4) is of order \( \sqrt{d} \) as \( d \) goes to infinity. The following result provides an equivalent characterization.

**Theorem 1.3** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^n, n \geq 1 \), \( \theta_{d,m} \) denotes the unique positive steady state to the problem (1.4) and \( S \) is a subset of \( \{d > 0\} \times \mathcal{M}_1 \). Then there exists \( C > 0 \) such that \( \int_\Omega \theta_{d,m} \, dx \geq C \sqrt{d} \) for all \( (d, m) \in S \) if and only if the following assumption is valid in \( S \):

\[
\int_\Omega \theta_{d,m} \, dx \geq C \sqrt{d} \quad \text{for all} \quad (d, m) \in S.
\]
There exist $\varepsilon_0$, $D > 0$ such that for any $(d, m) \in S$ with $d > D$, $m$ satisfies

$$\int_{\{\frac{m}{d} > (1+\varepsilon_0)a\}} m(x)dx \geq \varepsilon_0,$$

where

$$a(x) = \int_{\Omega} k(y, x)dy \leq 1.$$

Intuitively, Theorem 1.3 demonstrates that given total amount of resources, the total equilibrium population could reach the order $\sqrt{d}$ as $d \to \infty$ if and only if certain amount of resources concentrates at certain height related to diffusion rate as described in the assumption (A). As mentioned earlier, a series of results for the local model (1.1) in [21–23, 28] indicate that when the diffusion rate is fixed, bang-bang type is the optimal configuration of the resources for maximizing total population size. Therefore, generally speaking, for both local and nonlocal models, concentration of resources is beneficial for survival of species.

We emphasize that Theorem 1.3 provides an equivalent criterion to determine whether the total equilibrium population could reach the order $\sqrt{d}$ as $d \to \infty$ under the prescribed total amount of resources. With the help of this equivalent criterion, some concrete examples are constructed in Sect. 4.2 to elaborate how to choose $m$ for $d$ large such that the corresponding total population could reach the optimal order $\sqrt{d}$ and how properties of the kernel functions and the locations where the resources concentrate affect the total population.

A surprising and important feature in Theorems 1.2 and 1.3 is that the estimates for the total population are independent of $\|m\|_{L^\infty}$. Indeed, we only require that the total resources, i.e., $\|m\|_{L^1}$, is fixed. Biologically, this is the most natural and basic assumption. Mathematically, it is quite challenging to obtain these estimates, since we could only rely on the $L^1$ norm of $m$ and need avoid the appearance of other norms. Throughout the proofs of Theorems 1.2 and 1.3, the key idea is to divide the domain $\Omega$ in a proper way such that we could establish more precise estimates about $\theta_{d, m}$, $m$ in each sub-domain.

At the end, notice that for Theorems 1.1–1.3, only the assumption (K) is imposed on the kernel function $k(x, y)$. However, if the kernel function $k(x, y)$ is of the special form

$$k(x, y) = J(x - y),$$

it is proved that solutions of nonlocal equations with suitably rescaled kernel functions converge to solutions of local equations [2]. In particular, if $J$ also satisfies

$$\int_{\mathbb{R}^n} |x|^2 J(x)dx < +\infty,$$

then the convergence relation between solutions to equations with rescaled nonlocal operator and random diffusion is verified [2]. This reflects the complexity and diversity of nonlocal operators considered in this paper. Therefore, for nonlocal models, it is possible that bang-bang type is not optimal location of resources maximizing total population for fixed diffusion rate. The answer might depend on the specific properties of kernel functions $k(x, y)$. Additionally, Theorems 1.2 and 1.3 are related to the combined effects of diffusion rate and resources distribution on maximizing total equilibrium population. This remains unknown for local models. We will return to these problems in future work.

This paper is organized as follows. Theorem 1.1 is proved in Sect. 2. Section 3 is devoted to the proof of Theorem 1.2. At the end, the proof of Theorem 1.3 is present and some concrete examples are discussed in Sect. 4.
2 Existence and uniqueness of positive steady state

In this section, we establish the existence and uniqueness of positive steady state to the problem of (1.4) when \( m(x) \in L^\infty(\Omega) \).

**Proof of Theorem 1.1** First, if the problem (1.4) admits a positive steady state, denoted by \( \theta \), in \( L^\infty(\Omega) \), then it is easy to see that \( \mu_0 > 0 \) by choosing \( \psi = \theta \).

The rest of the proof is devoted to proving the other direction. Let \( u \) be the solution of

\[
\begin{align*}
    u_t &= dL[u](x, t) + u(x, t)[m(x) - u(x, t)] & \text{in } \Omega, \quad t > 0, \\
    u(x, 0) &= \|m\|_{L^\infty} & \text{in } \Omega.
\end{align*}
\]

(2.1)

Thus, \( u \) is decreasing in \( t \) and there exists \( \theta^* \in L^\infty(\Omega) \) such that \( u(x, t) \to \theta^*(x) \) pointwisely as \( t \to \infty \). Moreover, \( \theta^* \) is a steady state of (2.1).

Now we show that \( \theta^* \not\equiv 0 \). Suppose that it is not true, that is \( u(x, t) \to 0 \) pointwisely as \( t \to \infty \). Since \( \mu_0 > 0 \), by the definition of \( \mu_0 \) we can choose \( 0 \not\equiv \psi_0 \in L^2 \) such that

\[
    \int_\Omega (dL[\psi_0](x)\psi_0(x) + m(x)\psi_0^2(x)) \, dx \geq \frac{\mu_0}{2} \int_\Omega \psi_0^2 \, dx > 0.
\]

(2.2)

Let \( \psi_i := \min\{\psi_0, i\} \), obviously \( \psi_i \to \psi_0 \) in \( L^2(\Omega) \) as \( i \to \infty \). Combined with (2.2), we can fix \( i = i_0 \) large enough, such that

\[
    \int_\Omega (dL[\psi_{i_0}](x)\psi_{i_0}(x) + m(x)\psi_{i_0}^2(x)) \, dx \geq \frac{\mu_0}{4} \int_\Omega \psi_{i_0}^2 \, dx > 0.
\]

Set \( \phi := \varepsilon_{i_0}\psi_{i_0} \), with \( \varepsilon_{i_0} = \frac{1}{i_0} \min\{\|m\|_{L^\infty}, \frac{\mu_0}{8}\} \). It is routine to verify that

\[
    \int_\Omega (dL[\phi](x)\phi(x) + m(x)\phi^2(x)) \, dx - \frac{2}{3} \int_\Omega \phi^3 \, dx \geq \left[ \frac{\mu_0}{4} - \varepsilon_{i_0} i_0 \right] \int_\Omega \phi^2 \, dx > 0.
\]

(2.3)

Suppose that \( v \) is the solution of

\[
\begin{align*}
    v_t &= dL[v](x, t) + (m - v)v & \text{in } \Omega, \quad t > 0, \\
    v(x, 0) &= \phi & \text{in } \Omega,
\end{align*}
\]

and define

\[
    E[v](t) := \frac{1}{2} \int_\Omega (dL[v]v + mv^2) \, dx - \frac{1}{3} \int_\Omega v^3 \, dx.
\]

By comparison principle, \( \phi \leq \|m\|_{L^\infty} \) implies that \( v \leq u \). Thus, \( v \to 0 \) in pointwisely as \( t \to \infty \), and furthermore

\[
    E[v](t) \to 0 \text{ as } t \to \infty.
\]

(2.4)

However, since \( k(x, y) \) is symmetric, straightforward computation yields that

\[
    \frac{d}{dt} E[v](t) = \int_\Omega v_t^2 \, dx \geq 0.
\]

Together with (2.3), one sees that \( E[v](t) \) is an increasing function with positive initial data, which contradicts to (2.4).
Hence \( \theta^*(x) \geq 0 \) is a nontrivial steady state of (1.4), i.e.,

\[
d\left( \int_\Omega k(x, y)\theta^*(y)dy - a(x)\theta^*(x) \right) + \theta^*(x)[m(x) - \theta^*(x)] = 0.
\] (2.5)

Furthermore, denote \( A := \{ x \in \Omega \mid \theta^*(x) = 0 \} \). Due to the assumption (K), a contradiction can be derived easily by integrating both sides of the equation satisfied by \( \theta^* \) in \( A \) if \( A \) has positive measure. This yields that \( \theta^* > 0 \) a.e. in \( \Omega \).

It remains to show the uniqueness of positive steady state to the problem of (1.4) in \( L^\infty(\Omega) \). Suppose that \( \theta \in L^\infty(\Omega) \) is a positive steady state of (1.4), i.e. \( \theta \) satisfies

\[
d\left( \int_\Omega k(x, y)\theta(y)dy - a(x)\theta(x) \right) + \theta(x)[m(x) - \theta(x)] = 0.
\] (2.6)

By multiplying both sides by \( \theta^{p-1} \) and integrating over \( \Omega \), we have

\[
\int_\Omega \theta^{p+1}(x)dx - \int_\Omega m(x)\theta^p(x)dx = d\int_\Omega \theta^{p-1}(x) \left( \int_\Omega k(x, y)\theta(y)dy - a(x)\theta(x) \right)dx \\
\leq d\int_\Omega \theta^{p-1}(x) \left( \int_\Omega k(x, y)dy \right)^{\frac{p-1}{p}} \left( \int_\Omega k(x, y)\theta^p(y)dy \right)^{\frac{1}{p}}dx - d\int_\Omega a(x)\theta^p(x)dx
\]

\[
\leq d\left( \int_\Omega \int_\Omega k(x, y)dy\theta^p(x)dx \right)^{\frac{p-1}{p}} \left( \int_\Omega \int_\Omega k(x, y)\theta^p(y)dydx \right)^{\frac{1}{p}} - d\int_\Omega a(x)\theta^p(x)dx
\]

\[
= d\int_\Omega \int_\Omega k(x, y)dy\theta^p(x)dx - d\int_\Omega a(x)\theta^p(x)dx = 0
\]

since \( k(x, y) \) satisfies the assumption (K) and \( a(x) = \int_\Omega k(y, x)dy \leq 1 \). Thus it is easy to see that

\[
\| \theta \|_{L^{p+1}} \leq \| m \|_{L^p},
\]

which yields that

\[
\| \theta \|_{L^\infty} \leq \| m \|_{L^\infty},
\]

since \( p \) is arbitrary. Then thanks to (2.1) and the comparison principle, it follows that \( \theta(x) \leq \theta^*(x) \). Now due to the equations (2.5) and (2.6) satisfied by \( \theta^* \) and \( \theta \) respectively, straightforward computation gives

\[
0 \leq \int_\Omega (\theta^* - \theta)\theta^*dx = \int_\Omega (m - \theta)\theta^*dx - \int_\Omega (m - \theta^*)\theta^*dx
\]

\[
= -d\int_\Omega \int_\Omega k(x, y)\theta(y)dy\theta^*(x)dx + d\int_\Omega \int_\Omega k(x, y)\theta^*(y)dy\theta(x)dx = 0,
\]

which implies that \( \theta \equiv \theta^* \).

Obviously, if \( m \in M_1 \), then \( \mu_0(m) > 0 \). The proof is complete. \( \square \)

**Remark 2.1** When the nonlocal diffusion operator \( L[u] \) in the problem (1.4) is replaced by

\[
\int_\Omega k(x, y)u(y)dy - u(x),
\]
which corresponds to nonlocal homogeneous Dirichlet boundary condition, the existence and uniqueness of positive steady state in $L^\infty(\Omega)$ is also equivalent to $\mu_0 > 0$. The proof is the same as that of Theorem 1.1.

### 3 Estimates for the supremum of total population

This section is devoted to the proof of Theorems 1.2, where we estimate the supremum of the ratio between total equilibrium population and total carrying capacity over $d > 0$ and $m \in L^\infty(\Omega)$ for the nonlocal model (1.4).

The proof of Theorems 1.2 consists of two parts: the upper bound and lower bound for

$$
\sup \left\{ \int_{\Omega_1} \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\}
$$

and for clarity, we present the proofs of these two parts in two subsections respectively. For the convenience of readers, some explanations are present as follows.

- In the proof of the upper bound, we divide the domain $\Omega$ based on multiples of $d$ as follows

$$
\Omega_i = \{ x \in \Omega \mid \theta_{d,m} > K_id \}, \quad i = 1, 2,
$$

where the constants $K_i$, $i=1,2$, will be chosen carefully. First we estimate

$$
\int_{\Omega_1} \theta_{d,m}(x)dx, \quad \int_{\Omega \setminus \Omega_1} \theta_{d,m}(x)dx
$$

separately to obtain an upper bound of the total population in the order of $d$, and then improve the upper bound to the order $\sqrt{d}$ by estimating

$$
\int_{\Omega_2} \theta_{d,m}(x)dx, \quad \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x)dx
$$

separately.

- In the proof of the lower bound, we construct examples to demonstrate that the order $\sqrt{d}$ could be achieved for $d$ large under the prescribed total carrying capacity.

#### 3.1 Upper bound for $\sup \left\{ \int_{\Omega} \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\}$

In this subsection, we show that there exists $C_1$, independent of $m \in \mathcal{M}_1$, such that

$$
\sup \left\{ \int_{\Omega} \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\} \leq C_1 \sqrt{d}. \quad (3.1)
$$

**Proof (Proof of the upper bound (3.1))** Thanks to Theorem 1.1, when $m(x)$ satisfies the condition (M), the problem (1.4) always admits a unique positive steady state, denoted by $\theta_{d,m}$, i.e., $\theta_{d,m}$ satisfies

$$
d \left( \int_{\Omega} k(x, y)\theta(y)dy - a(x)\theta(x) \right) + \theta(x)[m(x) - \theta(x)] = 0 \quad x \in \Omega, \quad (3.2)
$$

\(\square\) Springer
where

\[ a(x) = \int_{\Omega} k(y, x) dy \leq 1. \]

In the following proof, we keep \( \int_{\Omega} m(x) \, dx \) in the estimates to emphasize the role played by total carrying capacity, though indeed \( \int_{\Omega} m(x) \, dx = 1 \) since \( m \in \mathcal{M}_1 \).

First, we establish a rough estimate for \( \int_{\Omega} \theta_{d,m}(x) \, dx \). Set

\[ \Omega_1 = \{ x \in \Omega \mid \theta_{d,m}(x) > K_1 d \}, \quad \text{where} \quad K_1 = 2 \| k \|_{L^\infty} |\Omega|. \]

For any \( x \in \Omega_1 \),

\[
\theta_{d,m}(x) = m(x) + \frac{d \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) \, dy - a(x) \theta_{d,m}(x) \right)}{\theta_{d,m}(x)} \\
\leq m(x) + \frac{d \| k \|_{L^\infty} \int_{\Omega} \theta_{d,m}(x) \, dx}{K_1 d} \\
= m(x) + \frac{1}{2 |\Omega|} \int_{\Omega} \theta_{d,m}(x) \, dx, \tag{3.3}
\]

which implies that

\[
\int_{\Omega_1} \theta_{d,m}(x) \, dx \leq \int_{\Omega} m(x) \, dx + \frac{|\Omega_1|}{2 |\Omega|} \int_{\Omega} \theta_{d,m}(x) \, dx \leq \int_{\Omega} m(x) \, dx + \frac{1}{2} \int_{\Omega} \theta_{d,m}(x) \, dx.
\]

Then for any \( d > 0 \),

\[
\int_{\Omega} \theta_{d,m}(x) \, dx = \int_{\Omega_1} \theta_{d,m}(x) \, dx + \int_{\Omega \setminus \Omega_1} \theta_{d,m}(x) \, dx \\
\leq \int_{\Omega} m(x) \, dx + \frac{1}{2} \int_{\Omega} \theta_{d,m}(x) \, dx + K_1 d |\Omega \setminus \Omega_1|.
\]

Thus for any \( d > 0 \),

\[
\int_{\Omega} \theta_{d,m}(x) \, dx \leq 2 \int_{\Omega} m(x) \, dx + 2 K_1 d |\Omega \setminus \Omega_1|, \tag{3.4}
\]

Notice that this estimate is valid for all \( d > 0 \). From now on, we focus on \( d \geq 1 \) as required in this theorem. In particular, the estimate (3.4) implies that for all \( d \geq 1 \),

\[
\int_{\Omega} \theta_{d,m}(x) \, dx \leq 2 \left( \int_{\Omega} m(x) \, dx + K_1 |\Omega| \right) d. \tag{3.5}
\]

Next, set

\[ \Omega_2 = \{ x \in \Omega \mid \theta_{d,m}(x) > K_2 d \}, \quad \text{where} \quad K_2 = \frac{4 \left( \int_{\Omega} m(x) \, dx + K_1 |\Omega| \right) \| k \|_{L^\infty}}{\min_{\Omega} a(x)} \\
+ 2 \| k \|_{L^\infty} |\Omega| \]

and we prepare an estimate for \( |\Omega_2| \) in term of \( d \). Denote

\[ \hat{\Omega}_2 = \left\{ x \in \Omega \mid m(x) \geq \frac{d}{2} a(x) \right\}. \]
Obviously,
\[
\int_{\Omega} m(x) dx \geq \int_{\tilde{\Omega}_2} m(x) dx \geq \frac{d}{2} \min a(x)|\tilde{\Omega}_2|,
\]
which implies that
\[
|\tilde{\Omega}_2| \leq \frac{2}{d \min a(x)} \int_{\Omega} m(x) dx.
\]

We claim that \(\Omega_2 \subseteq \tilde{\Omega}_2\). If the claim is true, then one has
\[
|\Omega_2| \leq \frac{2}{d \min a(x)} \int_{\Omega} m(x) dx. \tag{3.6}
\]

To prove this claim, fix any \(x \in \Omega \setminus \tilde{\Omega}_2\), i.e., \(m(x) < \frac{d}{2} a(x)\). Based on the equation (3.2),
\[
\theta_{d,m}(x) = \frac{1}{2} \left[ m(x) - da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy} \right]
\]
\[
= \frac{2d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy}{-m(x) + da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy}}
\]
\[
\leq \frac{2d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy}{da(x)} \leq \frac{2\|k\|_{L^\infty} \int_{\Omega} \theta_{d,m}(x) dx}{\min a(x)}
\]
\[
\leq \frac{4 (\int_{\Omega} m(x) dx + K_1|\Omega|) \|k\|_{L^\infty}}{\min a(x)} d,
\]
where the last inequality is due to (3.5). Hence \(\theta_{d,m}(x) < K_2 d\), i.e., \(x \in \Omega \setminus \Omega_2\). The claim is proved and thus (3.6) is valid.

Now we are ready to improve the estimate for \(\int_{\Omega} \theta_{d,m}(x) dx\). For \(x \in \Omega_2\), since \(\Omega_2 \subseteq \Omega_1\), the estimate (3.3) still holds, i.e.,
\[
\theta_{d,m}(x) \leq m(x) + \frac{1}{2|\Omega|} \int_{\Omega} \theta_{d,m}(x) dx.
\]
Then
\[
\int_{\Omega_2} \theta_{d,m}(x) dx \leq \int_{\Omega} m(x) dx + \frac{|\Omega_2|}{2|\Omega|} \int_{\Omega} \theta_{d,m}(x) dx
\]
\[
\leq \int_{\Omega} m(x) dx + \frac{1}{2} \int_{\Omega_2} \theta_{d,m}(x) dx + \frac{|\Omega_2|}{2|\Omega|} \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) dx,
\]
which yields that
\[
\int_{\Omega_2} \theta_{d,m}(x) dx \leq 2 \int_{\Omega} m(x) dx + \frac{|\Omega_2|}{|\Omega|} \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) dx. \tag{3.7}
\]
Moreover, we analyze the solution $\theta_{d,m}$ in $\Omega \setminus \Omega_2$. According to the equation (3.2), the estimates (3.6) and (3.7) and the fact that $a(x) \leq 1$, one has

$$
\int_{\Omega \setminus \Omega_2} \theta_{d,m}^2(x) \, dx = \int_{\Omega \setminus \Omega_2} m(x) \theta_{d,m}(x) \, dx + d \int_{\Omega \setminus \Omega_2} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) \, dy - a(x) \theta_{d,m}(x) \right) \, dx
$$

$$
\leq K_2 d \int_{\Omega \setminus \Omega_2} m(x) \, dx - d \int_{\Omega_2} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) \, dy - a(x) \theta_{d,m}(x) \right) \, dx
$$

$$
\leq K_2 d \int_{\Omega} m(x) \, dx + d \int_{\Omega_2} \theta_{d,m}(x) \, dx
$$

$$
\leq \left( K_2 + 2d + \frac{2}{\min_{\Omega} a(x)} \right) \frac{1}{|\Omega|} \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) \, dx \int_{\Omega} m(x) \, dx
$$

$$
\leq (K_2 + 2d) \int_{\Omega} m(x) \, dx + \frac{2}{\min_{\Omega} a(x)} \left( \int_{\Omega} m(x) \, dx \right)^2 + \frac{1}{2} \int_{\Omega \setminus \Omega_2} \theta_{d,m}^2(x) \, dx.
$$

This indicates that for $d \geq 1$,

$$
\int_{\Omega \setminus \Omega_2} \theta_{d,m}^2(x) \, dx \leq 2(K_2 + 2d) \int_{\Omega} m(x) \, dx + \frac{4}{|\Omega|} \left( \int_{\Omega} m(x) \, dx \right)^2 \leq K_3 d,
$$

where

$$
K_3 = 2(K_2 + 2) \int_{\Omega} m(x) \, dx + \frac{4}{|\Omega|} \left( \frac{1}{\min_{\Omega} a(x)} \right)^2 \left( \int_{\Omega} m(x) \, dx \right)^2.
$$

Therefore, together with (3.7), for $d \geq 1$

$$
\int_{\Omega} \theta_{d,m}(x) \, dx = \int_{\Omega} \theta_{d,m}(x) \, dx + \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) \, dx
$$

$$
\leq 2 \int_{\Omega} m(x) \, dx + \frac{\Omega_2}{|\Omega|} \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) \, dx + \int_{\Omega \setminus \Omega_2} \theta_{d,m}(x) \, dx
$$

$$
\leq 2 \int_{\Omega} m(x) \, dx + 2 \left( |\Omega \setminus \Omega_2| \int_{\Omega \setminus \Omega_2} \theta_{d,m}^2(x) \, dx \right)^{\frac{1}{2}}
$$

$$
\leq 2 \left( \int_{\Omega} m(x) \, dx + \sqrt{K_3 |\Omega|} \right) \sqrt{d}.
$$

Set

$$
C_1 = 2 \left( \int_{\Omega} m(x) \, dx + \sqrt{K_3 |\Omega|} \right).
$$

The desired estimate (3.1) follows. The proof is complete.  

\[\square\]

### 3.2 Lower bound for $\sup \left\{ \int_{\Omega} \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\}$

In this subsection, we show that there exists $C_0 > 0$, independent of $m \in \mathcal{M}_1$, such that

$$
\sup \left\{ \int_{\Omega} \theta_{d,m} \, dx \mid m \in \mathcal{M}_1 \right\} \geq C_0 \sqrt{d}.
$$

(3.8)
For this purpose, we choose \( m_d \) for \( d \) large as follows

\[
m_d(x) = \begin{cases} 
0 & x \in \Omega \setminus \Omega_{0,d}, \\
M_d & x \in \Omega_{0,d},
\end{cases}
\]

where \( \Omega_{0,d} \) denotes a ball with center \( x_0 \in \Omega \), radius \( (M_d \omega_n)^{-\frac{1}{d}} \) with \( M_d \) large enough such that \( \Omega_{0,d} \subset \Omega \) and \( \omega_n \) denoting the volume of the unit ball in \( \mathbb{R}^n \). Moreover, assume that

\[
\lim_{d \to \infty} \frac{1}{a(x_0)} \frac{M_d}{d} = \alpha \in (1, \infty].
\]

(3.9)

Obviously \( |\Omega_{0,d}| = \frac{1}{M_d} \) and thus \( m_d \in \mathcal{M}_1 \). To show the lower bound (3.8), it suffices to show that there exists \( C_0 > 0 \), independent of \( d \), such that

\[
\int_{\Omega} \theta_{d,m_d}(x) \, dx \geq C_0 \sqrt{d},
\]

(3.10)

where \( \theta_{d,m_d} \) denotes the unique positive steady state to the problem (1.4) with \( m \) replaced by \( m_d \).

**Proof** *(Proof of the lower bound (3.10))* First of all, it is routine to show that

\[
\theta_{d,m_d}(x) = \begin{cases} 
\frac{1}{2} \left[ -da(x) + \sqrt{d^2a^2(x) + 4d \int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy} \right] & x \in \Omega \setminus \Omega_{0,d}, \\
\frac{1}{2} \left[ M_d - da(x) + \sqrt{(M_d - da(x))^2 + 4d \int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy} \right] & x \in \Omega_{0,d}.
\end{cases}
\]

Thanks to Theorem 1.2, one sees that

\[
\lim_{d \to \infty} \frac{1}{d} \frac{M_d}{\int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy} = 0
\]

(3.11)

uniformly in \( \Omega \).

For \( x \in \Omega \setminus \Omega_{0,d} \), by Taylor expansion, we can derive

\[
\theta_{d,m_d}(x) = \frac{1}{2} \left[ -da(x) + \sqrt{d^2a^2(x) + 4d \int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy} \right] = \frac{d}{2} a(x) \left[ -1 + \sqrt{1 + \frac{4}{a^2(x)} \frac{\int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy}{d}} \right]
\]

\[
= \frac{\int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy}{a(x)} - (1 + \xi)^{-\frac{1}{2}} a^{-\frac{3}{2}} d \left( \frac{\int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy}{d} \right)^2,
\]

where

\[
0 < \xi(x) \leq \frac{4}{a^2(x)} \frac{\int_{\Omega} k(x, y)\theta_{d,m_d}(y) \, dy}{d}.
\]
This yields that

\[
\int_{\Omega \setminus \Omega_0,d} (1 + \xi)^{-\frac{3}{2}} a^{-2}(x) \frac{1}{d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy \right)^2 dx
\]

\[
= \int_{\Omega \setminus \Omega_0,d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy \right) dx - \int_{\Omega \setminus \Omega_0,d} a(x)\theta_{d,m_d}(x)dx
\]

\[
= \int_{\Omega \setminus \Omega_0,d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(x)dy \right) dx - \int_{\Omega \setminus \Omega_0,d} a(x)\theta_{d,m_d}(x)dx
\]

\[
= \int_{\Omega_0,d} a(x)\theta_{d,m_d}(x)dx - \int_{\Omega_0,d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(x)dy \right) dy
\]

\[
= \int_{\Omega_0,d} \frac{a(x)}{2} \left[ M_d - da(x) + \sqrt{(M_d - da(x))^2 + 4d \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy} \right] dx
\]

\[
- d \int_{\Omega_0,d} \int_{\Omega_0,d} k(x, y)\theta_{d,m_d}(y)dy dx.
\]

Then, thanks to the assumption (3.9), (3.11) and \(|\Omega_0,d| = \frac{1}{M_d}\), one can easily estimate the last term of the above identity as follows

\[
0 \leq \lim_{d \to \infty} \frac{1}{d} \int_{\Omega_0,d} k(x, y)\theta_{d,m_d}(y)dy dx \leq \lim_{d \to \infty} d|\Omega_0,d| \sup_{x \in \Omega} \frac{k(x, y)\theta_{d,m_d}(y)dy}{d} = 0
\]

and thus

\[
\lim_{d \to \infty} \int_{\Omega \setminus \Omega_0,d} (1 + \xi)^{-\frac{3}{2}} a^{-2}(x) \frac{1}{d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy \right)^2 dx
\]

\[
= \lim_{d \to \infty} \frac{1}{d} \left[ M_d - da(x) + \sqrt{\left( \frac{M_d}{d} - a(x) \right)^2 + 4 \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy} \right] dx
\]

\[
= \lim_{d \to \infty} \frac{M_d}{d - a(x)} \left[ \frac{M_d}{d} - a(x) + \left| \frac{M_d}{d} - a(x) \right| \right] dx
\]

\[
= \lim_{d \to \infty} \frac{1}{M_d} \frac{d(a(\xi_d)}{2} \left[ \frac{M_d}{d} - a(\xi_d) + \left| \frac{M_d}{d} - a(\xi_d) \right| \right]
\]

\[
= \left( 1 - \frac{1}{\alpha} \right) a(x_0), \quad (3.12)
\]

where \(\xi_d \in \Omega_0,d\) and \(\alpha \in (1, \infty]\) is needed. Notice that \(a(x) = \int_{\Omega} k(y, x)dy\) is strictly positive and continuous in \(\Omega\), and \(\lim_{d \to +\infty} \xi(x) = 0\) uniformly in \(\Omega\). Hence (3.12) indicates that there exists a constant \(C > 0\) such that for \(d\) large,

\[
\int_{\Omega \setminus \Omega_0,d} \frac{1}{d} \left( \int_{\Omega} k(x, y)\theta_{d,m_d}(y)dy \right)^2 dx \geq C.
\]
Now under the assumption \( \lim_{d \to \infty} \frac{1}{a(x_0)} \frac{M_d}{d} = \alpha \in (1, \infty) \), for \( d \) large, one has
\[
Cd \leq \int_{\Omega} \left( \int_{\Omega} k(x, y) \theta_{d,m_d}(y) \, dy \right)^2 \, dx \leq \|k\|_{L^\infty}^2 |\Omega| \left( \int_{\Omega} \theta_{d,m_d}(y) \, dy \right)^2.
\]
Therefore,
\[
\int_{\Omega} \theta_{d,m_d}(x) \, dx \geq \sqrt{\frac{C}{|\Omega|} \frac{1}{\|k\|_{L^\infty}}} \sqrt{d}.
\]
By setting
\[
C_0 = \sqrt{\frac{C}{|\Omega|} \frac{1}{\|k\|_{L^\infty}}},
\]
the lower bound (3.10) follows immediately. \( \square \)

The estimates (3.1) and (3.10) yield Theorem 1.2.

4 Optimal characterization of total population

In this section, we first present the proof of Theorem 1.3 and then as an application of Theorem 1.3, some concrete examples are constructed to demonstrate for \( d \) large, how to choose \( m \) to support more populations.

4.1 Proof of Theorem 1.3

First, we recall that due to the property (1.2), it is enough to deal with \( d \to \infty \) in the proof of Theorem 1.3. In order to prove the sufficiency of the assumption \((A)\), the main idea is to analyze the solution \( \theta_{d,m} \) in the following two types of sub-domains
\[
\Omega_{\varepsilon_0} = \left\{ x \in \Omega \mid m(x) > d(1 + \varepsilon_0)a(x) \right\}, \quad \Omega_d = \left\{ x \in \Omega \mid m(x) \leq d^3 a(x) \right\},
\]
where we make use of the structure of the equation satisfied by \( \theta_{d,m} \), the relation between \( \Omega_{\varepsilon_0} \) and \( \Omega_d \), and Taylor expansion. However, to prove the necessity of the assumption \((A)\), we argue by contradiction. Suppose that there exist a sequence \( d_\ell \to \infty \) as \( \ell \to \infty \) and \( m_\ell \in M_1 \) such that \( (d_\ell, m_\ell) \in S \), and for any given \( \varepsilon > 0 \), \( \int_{\Omega^\ell_\varepsilon} m_\ell(x) \, dx < \varepsilon \) for \( \ell \) large enough, where
\[
\Omega^\ell_\varepsilon = \left\{ x \in \Omega \mid m_\ell(x) > d_\ell(1 + \varepsilon)a(x) \right\}.
\]
To derive a contradiction, the key technique is to provide a more precise estimate for \( \theta_{d_\ell,m_\ell} \) in \( \Omega \setminus \Omega^\ell_\varepsilon \), i.e., Claim 1 in the proof.

Proof of Theorem 1.3 First, we prove that the assumption \((A)\) is sufficient for the total equilibrium population reaching order \( \sqrt{d} \) as \( d \to \infty \).
Based on the equation (3.2),

\[
\theta_{d,m}(x) = \frac{1}{2} \left[ m(x) - da(x) + \sqrt{(m(x) - da(x))^2 + 4d \int_\Omega k(x, y)\theta_{d,m}(y)dy} \right]
= \frac{da}{2} \left[ \frac{m}{da} - 1 + \sqrt{\left(1 - \frac{m}{da}\right)^2 + 4d \int_\Omega k(x, y)\theta_{d,m}(y)dy} \right].
\]

Obviously there exists \( D_1 > D \) such that for \( d > D_1 \), we have

\[
\frac{m(x)}{da(x)} < \frac{1}{2} \text{ for } x \in \Omega_d.
\]

Then by Taylor expansion \( \sqrt{\beta^2 + z} = \beta + \frac{1}{2} \beta^{-1} z - \frac{1}{8} (\beta^2 + \xi)^{-\frac{3}{2}} z^2 \), \( \beta > 0 \), it is standard to derive that for \( x \in \Omega_d, d > D_1 \),

\[
\theta_{d,m}(x) = \left(a - \frac{m}{d}\right)^{-1} \int_\Omega k(x, y)\theta_{d,m}(y)dy
- \left((1 - \frac{m}{da})^2 + \xi\right)^{-\frac{1}{2}} \frac{1}{a^2 d} \left(\int_\Omega k(x, y)\theta_{d,m}(y)dy\right)^2,
\]

where

\[
0 < \xi < \frac{4}{a^2 d} \int_\Omega k(x, y)\theta_{d,m}(y)dy.
\]

This implies that

\[
\int_{\Omega_d} \left(a - \frac{m}{d}\right) \left((1 - \frac{m}{da})^2 + \xi\right)^{-\frac{1}{2}} \frac{1}{a^2 d} \left(\int_\Omega k(x, y)\theta_{d,m}(y)dy\right)^2 dx
= \int_{\Omega_d} \left(\frac{m(x)}{d}\right) \theta_{d,m}(x) + \int_\Omega k(x, y)\theta_{d,m}(y)dy - a(x)\theta_{d,m}(x) dx.
\] (4.2)

Let us estimate the right hand side of (4.2) first. Notice that for \( d \geq 1, \Omega_{\epsilon_0} \subseteq \Omega \setminus \Omega_d \), and

\[
|\Omega \setminus \Omega_d| \leq \int_{\{m \geq d^\frac{3}{2} a(x)\}} \frac{m(x)}{d^\frac{3}{2} \min_{\Omega_d} a} dx \leq \frac{1}{\min_{\Omega_d} a} d^{-\frac{3}{2}},
\]
since \( m \in \mathcal{M}_1 \). Then it follows from Theorem 1.2 and the assumption \((\text{A})\) that for \( d > D_1 \)
\[
\int_{\Omega_d} \left( \frac{m(x)}{d} \theta_{d,m}(x) + \int_{\Omega} k(x, y) \theta_{d,m}(y) dy - a(x) \theta_{d,m}(x) \right) dx \\
\geq \int_{\Omega_d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy - a(x) \theta_{d,m}(x) \right) dx \\
= \int_{\Omega \setminus \Omega_d} \left( a(x) \theta_{d,m}(x) - \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right) dx \\
\geq \int_{\Omega \setminus \Omega_d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right) dx - \int_{\Omega \setminus \Omega_d} k(x, y) \theta_{d,m}(y) dy dx \\
= \int_{\Omega \setminus \Omega_d} \frac{a}{2} \left[ m - da + \sqrt{(m - da)^2 + 4d \int_{\Omega} k(x, y) \theta_{d,m}(y) dy} \right] dx \\
- \int_{\Omega \setminus \Omega_d} k(x, y) \theta_{d,m}(y) dy dx \\
\geq \int_{\Omega \setminus \Omega_d} a(m - da) dx - C_1 \|k\|_{L^\infty} \sqrt{d} |\Omega \setminus \Omega_d| \\
\geq \int_{\Omega \setminus \Omega_d} a \left( m - \frac{m}{1 + \varepsilon_0} \right) dx - C_1 \|k\|_{L^\infty} \frac{1}{\min_{\Omega} a} d^{-\frac{1}{2}} \\
\geq \frac{\varepsilon_0^2}{1 + \varepsilon_0} \min_{\Omega} a - C_1 \|k\|_{L^\infty} \frac{1}{\min_{\Omega} a} d^{-\frac{1}{2}}.
\]

This, together with (4.2), indicates that there exists \( D_2 > D_1 \) such that for \( d > D_2 \),
\[
\frac{1}{2} \frac{\varepsilon_0^2}{1 + \varepsilon_0} \min_{\Omega} a \leq \int_{\Omega_d} \left( \frac{m(x)}{d} \theta_{d,m}(x) + \int_{\Omega} k(x, y) \theta_{d,m}(y) dy - a(x) \theta_{d,m}(x) \right) dx \\
= \int_{\Omega_d} \left( a - \frac{m}{d} \right) \left( \left( 1 - \frac{m}{da} \right)^2 + \xi \right)^{-\frac{1}{2}} \frac{1}{a^2 d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx \\
\leq \int_{\Omega_d} \left( \frac{1}{4} \right)^{-\frac{1}{2}} \frac{1}{a^2 d} \left( \int_{\Omega} k(x, y) \theta_{d,m}(y) dy \right)^2 dx \\
\leq 8 \left( \min_{\Omega} a \right)^{-2} \|k\|_{L^\infty}^2 |\Omega| \frac{1}{d} \left( \int_{\Omega} \theta_{d,m} dx \right)^2.
\]

It is routine to check that the lower bound estimate \( \int_{\Omega} \theta_{d,m} dx \geq C \sqrt{d} \) is valid with
\[
C = \frac{\varepsilon_0 \left( \min_{\Omega} a \right)^{\frac{3}{2}}}{4 \|k\|_{L^\infty} \sqrt{1 + \varepsilon_0} |\Omega|}.
\]

Next, we prove the necessity of the assumption \((\text{A})\) for the total equilibrium population reaching order \( \sqrt{d} \) as \( d \to \infty \) in the subset \( S \subseteq \{d > 0\} \times \mathcal{M}_1 \). Suppose that the assumption \((\text{A})\) does not hold in \( S \), i.e., there exists a sequence \( d_\ell \to \infty \) as \( \ell \to \infty \) and \( m_\ell \in \mathcal{M}_1 \) such that \( (d_\ell, m_\ell) \in S \) and for any given \( \varepsilon > 0 \), \( \int_{\Omega_{\ell}^\varepsilon} m_\ell dx < \varepsilon \) for \( \ell \) large enough, where \( \Omega_{\ell}^\varepsilon \) is designated in (4.1).
Moreover, according to the definition of $\Omega^\ell$, one has for $\ell > \ell > L_1$

\[
\int_{\Omega \setminus \Omega^\ell} \theta_{d_{\ell}, m_{\ell}}^2 dx = \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy - a(x) \theta_{d_{m_{\ell}}} (x) \right) dx
\]

\[
= \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx - d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy - a(x) \theta_{d_{m_{\ell}}} (x) \right) dx
\]

\[
\leq \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx + d_{\ell} \int_{\Omega \setminus \Omega^\ell} a(x) \theta_{d_{m_{\ell}}} (x) dx
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \frac{1}{2} \left( m_{\ell} (x) - a \right)
\]

\[
+ \sqrt{\left( m_{\ell} - d_{\ell} a \right)^2 + 4d_{\ell} \int \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy dx}
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( m_{\ell} (x) + \sqrt{d_{\ell} \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy} \right) dx
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + \left( \max_\Omega a \right) d_{\ell} \epsilon + \left( C_1 k \right) \left( \max_\Omega a \right) d_{\ell} \epsilon \left( \max_\Omega a \right)
\]

\[
\leq 3 \left( \max_\Omega a \right) d_{\ell} \epsilon + \left( C_1 k \right) \left( \max_\Omega a \right) d_{\ell} \epsilon \frac{\epsilon}{\min_\Omega a} := C(a, k) d_{\ell} \epsilon.
\]

Moreover, according to the definition of $\Omega^\ell$, $m \in M_1$ and Theorem 1.2, it is easy to check that

\[
\int_{\Omega^\ell} \theta_{d_{m_{\ell}}} dx = \int_{\Omega^\ell} \frac{1}{2} \left( m_{\ell} (x) - a \right) + \sqrt{\left( m_{\ell} - a \right)^2 + 4d_{\ell} \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy} dx
\]

\[
\leq \int_{\Omega^\ell} \left( m_{\ell} (x) + \sqrt{d_{\ell} \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy} \right) dx
\]

\[
\leq \epsilon + \left( C_1 k \right) \epsilon \frac{1}{\min_\Omega a} = \epsilon.
\]

Claim 1 There exists $L_1 > 0$, such that for $\ell > L_1$, $\theta_{d_{m_{\ell}}} (x) < 2 \left( \max_\Omega a \right) d_{\ell} \epsilon$ in $\Omega \setminus \Omega^\ell$. □

Assume that the claim is true. Based on the equation satisfied by $\theta_{d_{m_{\ell}}}$ and Theorem 1.2, one has for $\ell > L_1$

\[
\int_{\Omega \setminus \Omega^\ell} \theta_{d_{m_{\ell}}}^2 dx = \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy - a(x) \theta_{d_{m_{\ell}}} (x) \right) dx
\]

\[
= \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx - d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy - a(x) \theta_{d_{m_{\ell}}} (x) \right) dx
\]

\[
\leq \int_{\Omega \setminus \Omega^\ell} m_{\ell} \theta_{d_{m_{\ell}}} dx + d_{\ell} \int_{\Omega \setminus \Omega^\ell} a(x) \theta_{d_{m_{\ell}}} (x) dx
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \frac{1}{2} \left( m_{\ell} (x) - d_{\ell} a \right)
\]

\[
+ \sqrt{\left( m_{\ell} - d_{\ell} a \right)^2 + 4d_{\ell} \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy} dx
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + d_{\ell} \int_{\Omega \setminus \Omega^\ell} \left( m_{\ell} (x) + \sqrt{d_{\ell} \int_{\Omega} k(x, y) \theta_{d_{m_{\ell}}} (y) dy} \right) dx
\]

\[
\leq 2 \left( \max_\Omega a \right) d_{\ell} \epsilon + \left( \max_\Omega a \right) d_{\ell} \epsilon + \left( C_1 k \right) \left( \max_\Omega a \right) d_{\ell} \epsilon \frac{\epsilon}{\min_\Omega a} := C(a, k) d_{\ell} \epsilon.
\]
This, together with (4.4), implies that there exists \( L_2 > L_1 \) such that for \( \ell > L_2 \)

\[
\int_{\Omega} \theta_{d_\ell, m_\ell}(x) dx = \int_{\Omega_\ell} \theta_{d_\ell, m_\ell}(x) dx + \int_{\Omega \setminus \Omega_\ell} \theta_{d_\ell, m_\ell}(x) dx
\]

\[
\leq \varepsilon + (C_1 \|k\|_{L^{\infty}})^{1/2} \left( \min_{\Omega} a \right)^{-1} d_\ell^{-1/2} + |\Omega|^{1/2} \left( \int_{\Omega \setminus \Omega_\ell} \theta_{d_\ell, m_\ell}^2 dx \right)^{1/2}
\]

\[
\leq 1 + |\Omega|^{1/2} (C(a, k)d_\ell \varepsilon)^{1/2}.
\]

Since \( \varepsilon \) is arbitrary, we have

\[
\lim_{d_\ell \to \infty} \frac{\int_{\Omega} \theta_{d_\ell, m_\ell} dx}{\sqrt{d_\ell}} = 0.
\]

This verifies (4.3) and the necessity of the assumption (A).

Now it remains to verify Claim 1. It is equivalent to show that there exists \( L_1 > 0 \), for

\( \ell > L_1 \), \( \theta_{d_\ell, m_\ell}(x) \geq 2 \left( \max_{\Omega} a \right) d_\ell \varepsilon \), \( x \in \Omega \) implies that \( m_\ell(x) > d_\ell a(x) (1 + \varepsilon) \).

Set \( \tau := 2 \left( \max_{\Omega} a \right) \varepsilon \) for clarity. Notice that if \( \theta_{d_\ell, m_\ell}(x) \geq d_\ell \tau \) and \( m_\ell(x) < d_\ell a(x) \)
happen together, then due to the equation satisfied by \( \theta_{d_\ell, m_\ell} \) and Theorem 1.2, one has

\[
d_\ell \tau \leq \theta_{d_\ell, m_\ell}(x) = \frac{1}{2} \left[ m_\ell(x) - d_\ell a(x) + \sqrt{(m_\ell(x) - d_\ell a(x))^2 + 4d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy} \right]
\]

\[
= \frac{2d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy}{-m_\ell(x) + d_\ell a(x) + \sqrt{(m_\ell(x) - d_\ell a(x))^2 + 4d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy}}
\]

\[
\leq \sqrt{d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy}
\]

\[
\leq (C_1 \|k\|_{L^{\infty}})^{1/2} d_\ell^{3/2}.
\]

This yields that

\[
d_\ell < \frac{(C_1 \|k\|_{L^{\infty}})^2}{\tau^4}.
\]

Hence, if \( d_\ell \geq \frac{(C_1 \|k\|_{L^{\infty}})^2}{\tau^4} \), then \( \theta_{d_\ell, m_\ell}(x) \geq d_\ell \tau \) guarantees that

\[
m_\ell(x) \geq d_\ell a(x).
\]

(4.5)

Moreover, if \( \theta_{d_\ell, m_\ell}(x) \geq d_\ell \tau \), direct calculation gives

\[
d_\ell \tau \leq \theta_{d_\ell, m_\ell}(x) = \frac{1}{2} \left[ m_\ell(x) - d_\ell a(x) + \sqrt{(m_\ell(x) - d_\ell a(x))^2 + 4d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy} \right]
\]

\[
\leq \sqrt{(m_\ell(x) - d_\ell a(x))^2 + 4d_\ell \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy},
\]

which, together with Theorem 1.2, yields that

\[
\left( \frac{m_\ell(x)}{d_\ell} - a(x) \right)^2 \geq \tau^2 - \frac{4 \int_{\Omega} k(x, y) \theta_{d_\ell, m_\ell}(y) dy}{d_\ell} \geq \tau^2 - \frac{4C_1 \|k\|_{L^{\infty}}}{\sqrt{d_\ell}} > \frac{\tau^2}{4},
\]

(4.6)
provided that \( d_\ell \) is sufficiently large.

At the end, choose \( L_1 > 0 \) large enough such that for \( \ell > L_1 \),
\[
d_\ell \geq \frac{(C_1 \| k \|_1)}{\tau^4}
\]
and \( (4.6) \) holds. Then by \((4.5)\) and \((4.6)\), we can further obtain
\[
m_\ell (x) > a(x) + \frac{\tau}{2} \geq a(x) \left( 1 + \frac{\tau}{2 \max_{\Omega} a} \right) = a(x) \left( 1 + \varepsilon \right).
\]
The proof is complete. \( \square \)

4.2 Applications

Based on the equivalent criterion established in Theorem 1.3, a series of examples are constructed to show how the concentration of resources, including the height and locations, and properties of nonlocal kernel functions affect the total population for large diffusion rate.

**Example 1** Define
\[
m_{\alpha, \beta} (x) = \begin{cases} 0 & x \in \Omega \setminus \Omega_{d, \beta}^\alpha, \\ \alpha d^\beta & x \in \Omega_{d, \beta}^\alpha, \end{cases}
\]
where \( \Omega_{d, \beta}^\alpha \subseteq \Omega \) with \( |\Omega_{d, \beta}^\alpha| = (\alpha d^\beta)^{-1} \). \( \alpha, \beta > 0 \). Obviously \( m_{\alpha, \beta} \in \mathcal{M}_1 \). Let \( \theta_{d, \alpha} \) denote the unique positive steady state to the problem \((1.4)\) with \( m(x) \) replaced by \( m_{\alpha, \beta} (x) \). Thanks to Theorem 1.3, we have the following statements.

(i) If \( 0 < \beta < 1 \), then for any \( \alpha > 0 \), we have \( \lim\limits_{d \to \infty} \frac{\int_{\Omega} \theta_{d, \alpha} \, dx}{\sqrt{d}} = 0 \).

(ii) If \( \beta > 1 \), then for any \( \alpha > 0 \), we have \( \int_{\Omega} \theta_{d, \alpha} \, dx \) is of order \( \sqrt{d} \) as \( d \to \infty \).

(iii) For the critical case \( \beta = 1 \), recall that
\[
0 < \min_{\Omega} a(x) \leq \max_{\Omega} a(x) \leq 1, \text{ where } a(x) = \int_{\Omega} k(y, x) \, dy.
\]
Then
- \( \lim\limits_{d \to \infty} \frac{\int_{\Omega} \theta_{d, \alpha} \, dx}{\sqrt{d}} = 0 \) when \( 0 < \alpha \leq \min_{\Omega} a(x) \),
- \( \int_{\Omega} \theta_{d, \alpha} \, dx \) is of order \( \sqrt{d} \) as \( d \to \infty \) when \( \alpha > \max_{\Omega} a(x) \).

Notice that for the cases discussed in Example 1, we only require that \( \Omega_{d, \beta}^\alpha \) is a measurable subset in \( \Omega \) with \( |\Omega_{d, \beta}^\alpha| = (\alpha d^\beta)^{-1} \).

If in addition, assume that \( \min_{\Omega} a(x) < \max_{\Omega} a(x) \), then the case that
\[
\min_{\Omega} a(x) < \alpha \leq \max_{\Omega} a(x) \leq 1, \ \beta = 1
\]
is not mentioned in Example 1. Indeed, in this case, the locations where the resources concentrate, i.e., \( \Omega_{d, \alpha}^1 \), and where \( \alpha > a(x) \) will affect the order of total population as \( d \to \infty \). To better elaborate this point, we construct an example under some extra assumptions.
Example 2 Assume that $\Omega$ is open and convex in $\mathbb{R}^n$, the function $J$ satisfies
\begin{equation}
J(z) \in C(\mathbb{R}^n), \text{ is nonnegative, radially symmetric, } J(0) > 0 \text{ and } \int_{\mathbb{R}^n} J(z)dz = 1.
\end{equation}
Also, assume that $J$ is compactly supported and $diam \{J > 0\} \ll 1$.
Define
\begin{equation}
m_{d,x_0}(x) = \begin{cases}
0 & x \in \Omega \setminus \Omega_{d,x_0}, \\
\frac{1}{d} & x \in \Omega_{d,x_0},
\end{cases}
\end{equation}
where $x_0 \in \Omega$, $\Omega_{d,x_0}$ denotes a ball centered at $x_0$ with $|\Omega_{d,x_0}| = d^{-1}$. Let $\theta_{d,x_0}$ denote the unique positive steady state to the problem (1.4) with $m(x)$ replaced by $m_{d,x_0}(x)$ and the kernel function $k(x, y)$ replaced by $J(x - y)$. Then
\begin{itemize}
  \item $\int_{\Omega} \theta_{d,x_0} dx$ is of order $\sqrt{d}$ as $d \to \infty$ when $x_0$ is close to the boundary of $\Omega$;
  \item $\lim_{d \to \infty} \frac{\int_{\Omega} \theta_{d,x_0} dx}{\sqrt{d}} = 0$ when $x_0$ is away from the boundary of $\Omega$.
\end{itemize}
This follows immediately from Theorem 1.3 and the observation that $a(x) < 1$ if $\text{dist}\{x, \partial \Omega\} < \text{diam}\{J > 0\}/4$ and $a(x) = 1$ if $\text{dist}\{x, \partial \Omega\} > \text{diam}\{J > 0\}/2$.

Contrary to Example 2, to support more population for $d$ large, under certain assumptions, the resources need concentrate away from the boundary. An example is constructed as follows.

Example 3 Assume that $\Omega := B_1(0)$, the function $J$ satisfies the assumption (J) and for some small $\delta > 0$, we assume in addition that $J(z) \equiv \delta$, $z \in B_1(0)$, $J(z)$ is strictly increasing in $|z|$ for $1 < |z| \leq 2$. It easy to check that $a(x)$ is radially symmetric and strictly increasing in $|x|$ for $|x| \leq 1$.

Fix $a(0) < \hat{\alpha} < a(x)$ with $|x| = 1$ and define
\begin{equation}
m_{d,x_0}^{\hat{\alpha}}(x) = \begin{cases}
0 & x \in \Omega \setminus \Omega_{d,x_0}^{\hat{\alpha}}, \\
\hat{\alpha} d & x \in \Omega_{d,x_0}^{\hat{\alpha}},
\end{cases}
\end{equation}
where $x_0 \in \Omega$, $\Omega_{d,x_0}^{\hat{\alpha}}$ denotes a ball centered at $x_0$ with $|\Omega_{d,x_0}^{\hat{\alpha}}| = (\hat{\alpha}d)^{-1}$. Let $\theta_{d,x_0}^{\hat{\alpha}}$ denote the unique positive steady state to the problem (1.4) with $m(x)$ replaced by $m_{d,x_0}^{\hat{\alpha}}(x)$ and the kernel function $k(x, y)$ replaced by $J(x - y)$. Then thanks to Theorem 1.3, it is easy to show that
\begin{itemize}
  \item $\int_{\Omega} \theta_{d,x_0}^{\hat{\alpha}} dx$ is of order $\sqrt{d}$ as $d \to \infty$ when $x_0$ is close to the origin 0;
  \item $\lim_{d \to \infty} \frac{\int_{\Omega} \theta_{d,x_0}^{\hat{\alpha}} dx}{\sqrt{d}} = 0$ when $x_0$ is close to the boundary of $\Omega$.
\end{itemize}

Data Availability Statement The authors confirm that this manuscript has no associated data.

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