Generalized Schrödinger-Poisson type systems

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Abstract

In this paper we study the boundary value problem

\[
\begin{align*}
-\Delta u + \varepsilon \Phi f(u) &= \eta |u|^{p-1}u \quad \text{in } \Omega, \\
-\Delta \Phi &= 2q F(u) \quad \text{in } \Omega, \\
u = \Phi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain, \( 1 < p < 5, \varepsilon, \eta = \pm 1, q > 0, f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( F \) is the primitive of \( f \) such that \( F(0) = 0 \). We provide existence and multiplicity results assuming on \( f \) a subcritical growth condition. The critical case is also considered and existence and nonexistence results are proved.

Keywords: Schrödinger-Poisson equations, variational methods, mountain pass.

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1 Introduction

This paper deals with the following problem

\[
\begin{align*}
-\Delta u + \varepsilon \Phi f(u) &= \eta |u|^{p-1}u \quad \text{in } \Omega, \\
-\Delta \Phi &= 2q F(u) \quad \text{in } \Omega, \\
u = \Phi &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(P)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary \( \partial \Omega \), \( 1 < p < 5, q > 0, \varepsilon, \eta = \pm 1, f : \mathbb{R} \to \mathbb{R} \) is a continuous function and \( F(s) = \int_0^s f(t) \, dt \).

When the function \( f(t) = t \) and \( \varepsilon = \eta = 1 \), this system represents the well known Schrödinger-Poisson (or Schrödinger-Maxwell) equations, briefly SPE, that have been widely studied in the recent past. In the pioneer paper of Benci and Fortunato [5], the linear version of SPE (where \( \eta = 0 \)) has been approached as an eigenvalue problem. In [16] the authors have proved the existence of infinitely many solutions for SPE when \( p > 4 \), whereas in [19] an analogous result has been found for almost any \( q > 0 \) and \( p \in ]2, 5[ \). A multiplicity result has been obtained in [20] for any \( q > 0 \) and \( p \) sufficiently close to the critical exponent 5, by using the abstract Lusternik-Schnirelmann theory. For the sake of completeness we mention also [15] where Neumann condition on \( \Phi \) is assumed on \( \partial \Omega \), [1] and the references within for results on SPE in unbounded domains and [8, 9] for the Klein-Gordon-Maxwell system in a bounded domain.

If \( f(t) = t \) and \( \varepsilon = -1 \), the system is equivalent to a nonlocal nonlinear problem related with the following well known Choquard equation in the whole space \( \mathbb{R}^3 \)

\[
\Delta u + u - \left( \frac{1}{|x|} \ast |u|^2 \right) u = 0.
\]

We refer to [12, 13] for more details on the Choquard equation and to [14] for a recent result on a system in \( \mathbb{R}^3 \) strictly related with ours.

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Generalized SPE

Up to our knowledge, problem (P) has not been investigated when a more general function $f$ appears instead of the identity. Since problem (P) possesses a variational structure, our aim is to find weak assumptions on $f$ in order to apply the usual variational techniques. In particular, the first step in a classical approach to such a type of systems consists in the use of the reduction method. To this end, we need to assume suitable growth conditions on $f$ which allow us to invert the Laplace operator and thus to solve the second equation of the system. Then, after we have reduced the problem to a single equation, we find critical points of a one variable functional, checking geometrical and compactness assumptions of the Mountain Pass Theorem. If on one hand a suitable use of some a priori estimates makes quite immediate to show that geometrical hypotheses are verified (at least for small $q$), on the other some technical difficulties arise in getting boundedness for the Palais-Smale sequences. If $\eta = 1$, we use a suitable truncation argument based on an idea of Berti and Bolle [6] and Jeanjean and Le Coz [10] (see also [3, 11]) and we are able to show the existence of a bounded Palais-Smale sequence of the functional taking $q$ sufficiently small. If we had the sufficient compactness, we would conclude by extracting any strongly convergent subsequence from this bounded Palais-Smale sequence. However the growth hypothesis we assume on $f$ does not permit to deduce compactness on the variable $\Phi$. Indeed the exponent 4 turns out to be critical and in this sense we are justified to refer to (f) as the critical growth condition for the function $f$.

The first result in this paper is the following.

**Theorem 1.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$|f(s)| \leq c_1 + c_2|s|^4$$  \hspace{1cm} (f)

for all $s \in \mathbb{R}$. Then, there exists $\bar{q} > 0$ such that for all $0 < q \leq \bar{q}$ problem

$$\begin{cases}
-\Delta u + \varepsilon q \Phi f(u) = |u|^{p-1}u & \text{in } \Omega, \\
-\Delta \Phi = 2qF(u) & \text{in } \Omega, \\
u = \Phi = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (P_f)

has at least a nontrivial solution.

Inspired by [14], in the second part of this paper we study (P) with $\varepsilon = \eta = -1$ and $f(s) = |s|^{r-2} s$.

The system becomes

$$\begin{cases}
-\Delta u - q |u|^{r-2} u \Phi + |u|^{p-1} u = 0 & \text{in } \Omega, \\
-r \Delta \Phi = 2q |u|^r & \text{in } \Omega, \\
u = \Phi = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (P_r)

where $1 < r$ and $1 < p < 5$.

In this situation the contrasting nonlocal and local nonlinear terms perturb the functional in a way which in some sense recalls the typical concave-convex power-like nonlinearity. As a consequence, the functional’s geometry depends on the ratio of magnitude between $r$ and $p$. We get the following result.

**Theorem 1.2.** If $\frac{p+1}{2} < r < 5$, problem (P_r) has infinitely many solutions for any $q > 0$.

If $r = \frac{p+1}{2}$, then there exists an increasing sequence $(q_n)_n$ such that, if $q \geq q_n$, problem (P_r) has at least $n$ couples of solutions.

If $1 < r < \frac{p+1}{2}$, then there exists an increasing sequence $(q_n)_n$ such that, if $q \geq q_n$, problem (P_r) has at least $2n$ couples of solutions.

Finally, if $p < 5 \leq r$, then the problem has no nontrivial solution.

The paper is organized as follows: in Section 2 we introduce the functional setting where we study the problem (P_f) and the variational tools we use; in Section 3 we provide the proof of Theorem 1.1; in Section 4 we consider the system (P_r) and prove Theorem 1.2.

Throughout the paper we will use the symbols $H^{-1}$ to denote the dual space of $H^1_0(\Omega)$, $\langle \cdot, \cdot \rangle$ to denote the duality between $H^1_0(\Omega)$ and $H^{-1}$ and $\|u\|_{H^1_0} := (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$ for the norm on $H^1_0(\Omega)$. Moreover $\| \cdot \|_p$ will denote the usual $L^p(\Omega)$-norm.

We point out the fact that in the sequel we will use the symbols $C, C_1, C_2, C_3$ and so on, to denote positive constants whose value might change from line to line.


2 Variational tools

Standard arguments can be used to prove that problem \((P_f)\) is variational and the related \(C^1\) functional
\[ J_q : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \]

is given by
\[ J_q(u, \Phi) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \varepsilon \int_\Omega |\nabla \Phi|^2 dx + \varepsilon q \int_\Omega F(u) \Phi dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} dx. \]

Since, by \((f)\), \(F : L^6(\Omega) \to L^{3/2}(\Omega) \hookrightarrow H^{-1}\), then certainly we have that for all \(u \in H_0^1(\Omega)\) there exists a unique \(\Phi_u \in H_0^1(\Omega)\) which solves
\[ -\Delta \Phi = 2q F(u) \]
in \(H^{-1}\). In particular, we are allowed to consider the following map
\[ u \in L^6(\Omega) \mapsto \Phi_u \in H_0^1(\Omega) \]

which is continuously differentiable by the Implicit Function Theorem applied to \(\partial_q H\) where, for any \((u, \Phi) \in L^6(\Omega) \times H_0^1(\Omega)\),
\[ H(u, \Phi) := \frac{1}{4} \int_\Omega |\nabla \Phi|^2 dx - q \int_\Omega F(u) \Phi dx. \]

Since, for every \(u \in H_0^1(\Omega)\),
\[ \partial_q J_q(u, \Phi_u) = -\varepsilon \partial_q H(u, \Phi_u) = 0, \tag{1} \]

then
\[ \int_\Omega |\nabla \Phi_u|^2 dx = 2q \int_\Omega F(u) \Phi_u dx \tag{2} \]

and
\[ \int_\Omega F(u) \Phi_u dx \geq 0. \tag{3} \]

Moreover, we have the following estimates.

**Lemma 2.1.** For every \(u \in H_0^1(\Omega)\)
\[ \left( \int_\Omega |\nabla \Phi_u|^2 dx \right)^{1/2} \leq C q \left( \int_\Omega |F(u)|^{6/5} dx \right)^{5/6} \tag{4} \]

and
\[ \int_\Omega F(u) \Phi_u dx \leq q(C_1 \|u\|_{L^6}^2 + C_2 \|u\|_6^{10}). \tag{5} \]

**Proof.** By Holder inequality we have
\[ \int_\Omega F(u) \Phi_u dx \leq \left( \int_\Omega |F(u)|^{6/5} dx \right)^{5/6} \left( \int_\Omega |\Phi_u|^{6} dx \right)^{1/6}. \tag{6} \]

Then, from (2), by using Sobolev embedding \(H_0^1(\Omega) \hookrightarrow L^6(\Omega)\), we get
\[ \int_\Omega |\nabla \Phi_u|^2 dx = 2q \int_\Omega F(u) \Phi_u dx \leq 2q \left( \int_\Omega |F(u)|^{6/5} dx \right)^{5/6} \left( \int_\Omega |\Phi_u|^{6} dx \right)^{1/6} \leq C q \left( \int_\Omega |F(u)|^{6/5} dx \right)^{5/6} \left( \int_\Omega |\nabla \Phi_u|^2 dx \right)^{1/2} \]
and then we get (4).

By (6) and (4) we have
\[ \int_\Omega F(u) \Phi_u dx \leq C q \left( \int_\Omega |F(u)|^{6/5} dx \right)^{5/3} \]
and then, since for \((f)\) it is
\[ |F(s)|^{6/5} \leq C_1 |s|^{6/5} + C_2 |s|^6, \tag{7} \]
for all \(s \in \mathbb{R}\), we get (5).
Equation (2) allows us to define on $H_0^1(\Omega)$ the $C^1$ one variable functional

$$I_q(u) := J_q(u, \Phi_u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \varepsilon \frac{q}{2} \int_{\Omega} F(u) \Phi_u dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx.$$ 

By using standard variational arguments as those in [5], the following result can be easily proved.

**Proposition 2.2.** Let $(u, \Phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$, then the following propositions are equivalent:

(a) $(u, \Phi)$ is a critical point of functional $J_q$;

(b) $u$ is a critical point of functional $I_q$ and $\Phi = \Phi_u$.

So we are led to look for critical points of $I_q$. To this end, we need to investigate the compactness property of its Palais-Smale sequences.

It is easy to see that the standard arguments used to prove boundedness do not work. Indeed, assuming that $(u_n) \in (H_0^1(\Omega))^\mathbb{N}$ is a Palais-Smale sequence, namely $(I_q(u_n))_n$ is bounded and $I_q'(u_n) \to 0$ in $H^{-1}$, we obtain the following inequality

$$I_q(u_n) = \frac{1}{p+1} \langle I_q'(u_n), u_n \rangle \leq C_1 + C_2 \|u_n\|.$$  \hfill (8)

Since, by (1),

$$\langle I_q'(u_n), u_n \rangle = \langle \partial_u J_q(u_n, \Phi_{u_n}), u_n \rangle,$$

from (8) we get

$$\left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\Omega} |\nabla u_n|^2 dx + \varepsilon \frac{q}{2} \int_{\Omega} F(u_n) \Phi_{u_n} dx - \frac{\varepsilon q}{p+1} \int_{\Omega} f(u_n) u_n \Phi_{u_n} dx \leq C_1 + C_2 \|u_n\|.$$  \hfill (9)

In the classical SPE ($\varepsilon = 1$ and $f(t) = t$) we should deduce the boundedness of the sequence $(u_n)_n$ for $p \geq 3$. In our general situation we need a different approach.

Let $T > 0$ and $\chi : [0, +\infty) \to [0, 1]$ be a smooth function such that $\|\chi''\|_{L^\infty} \leq 2$ and

$$\chi(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{if } s \geq 2. \end{cases}$$

We define a new functional $I^T_q : H_0^1(\Omega) \to \mathbb{R}$ as follows

$$I^T_q(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \varepsilon \frac{q}{2} \left(\frac{\|u\|_{H^1_0}}{T}\right) \int_{\Omega} F(u) \Phi_u dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx$$  \hfill (9)

for all $u \in H_0^1(\Omega)$. We are going to find a critical point $u \in H_0^1(\Omega)$ of this new functional such that $\|u\|_{H^1_0} \leq T$ in order to get solutions of our problem.

3 Proof of Theorem 1.1

We prove that the functional $I^T_q$ satisfies Mountain Pass geometrical assumptions. More precisely, we have the following result.

**Lemma 3.1.** Under hypothesis (f), there exists $\bar{q} \in \mathbb{R}_+ \cup \{+\infty\}$ such that for all $0 < q < \bar{q}$ functional $I^T_q$ satisfies:

(i) $I^T_q(0) = 0$;

(ii) there exist constants $\rho, \alpha > 0$ such that

$$I^T_q(u) \geq \alpha \quad \text{for all } u \in H_0^1(\Omega) \text{ with } \|u\|_{H^1_0} = \rho;$$

(iii) there exists a function $\bar{u} \in H_0^1(\Omega)$ with $\|\bar{u}\|_{H^1_0} > \rho$ such that $I^T_q(\bar{u}) < 0.$
Proof. Property (i) is trivial. To prove (ii) we distinguish two cases. If \( \varepsilon = 1 \), by (3), for every \( q > 0 \) we deduce that
\[
I_q^T(u) \geq \frac{1}{2} \rho^2 - C \rho^{p+1} \geq \alpha > 0
\]
for suitable \( \rho, \alpha > 0 \). If \( \varepsilon = -1 \), by using (5) and the immersion of \( H^1_0(\Omega) \) into \( L^p(\Omega) \) spaces, we get
\[
I_q^T(u) \geq \frac{1}{2} \|u\|_{H^1_0}^2 - \frac{q}{2} \int_\Omega F(u)\Phi_u dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} dx
\]
Thus, if \( q \) is such that \( q^2 C_1 < 1 \) and \( \rho \) is small enough, there exists \( \alpha > 0 \) such that \( I_q^T(u) \geq \alpha \) for all \( u \in H^1_0(\Omega) \) with \( \|u\|_{H^1_0} = \rho \).

To prove (iii), let us consider \( u \in H^1_0(\Omega) \), \( u \neq 0 \) and \( t > \frac{2T}{\|u\|_{H^1_0}} \) such that \( \chi \left( \frac{\|tu\|_{H^1_0}}{\rho} \right) = \chi \left( \frac{\|tu\|_{H^1_0}}{T} \right) = 0 \).

Then, we have
\[
I_q^T(tu) = \frac{t^2}{2} \int_\Omega |\nabla u|^2 dx - \frac{t^p+1}{p+1} \int_\Omega |u|^{p+1} dx
\]
and so, for \( t \) large enough, \( I_q^T(tu) \) is negative.

Thus we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 allows us to define, for \( q < \hat{q} \),
\[
m_q^T = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_q^T(\gamma(t)) > 0
\]
where
\[
\Gamma = \{ \gamma \in C([0,1], H^1_0(\Omega)) \mid \gamma(0) = 0, I_q^T(\gamma(1)) < 0 \}.
\]
Certainly there exists a Palais-Smale sequence at mountain pass level \( m_q^T \), that is a sequence \( (u_n) \) in \( H^1_0(\Omega) \) such that
\[
I_q^T(u_n) \to m_q^T
\]
and
\[
(I_q^T)'(u_n) \to 0.
\]
As a first step we prove that there exists \( T > 0 \) and \( \hat{q} > 0 \) such that for any \( 0 < q \leq \hat{q} \) there exists a Palais-Smale sequence \( (u_n) \) at the level \( m_q^T \) such that, up to a subsequence, \( \|u_n\|_{H^1_0(\Omega)} \leq T \) for any \( n \in \mathbb{N} \).

Let \( T > 0, q > 0 \) and \( (u_n) \) in \( H^1_0(\Omega) \) be a Palais-Smale sequence of \( I_q^T \) at level \( m_q^T \). We make some preliminary computations. The first is an estimate on the mountain pass level \( m_q^T \).

Let \( u \in H^1_0(\Omega), u \neq 0 \) and \( t > 0 \) be such that the path \( \gamma(t) = tu \) belongs to \( \Gamma \). For all \( t \in [0,1] \) it is
\[
I_q^T(\gamma(t)) = \frac{t^2}{2} \int_\Omega |\nabla tu|^2 dx + \frac{q}{2} \chi \left( \frac{\|tu\|_{H^1_0}}{T} \right) \int_\Omega F(tu)\Phi_{tu} dx - \frac{t^p+1}{p+1} \int_\Omega |tu|^{p+1} dx.
\]
From (5) we obtain
\[
\max_{t \in [0,1]} I_q^T(\gamma(t)) \leq \max_{t \in [0,1]} \left( C_1 t^2 \int_\Omega |\nabla u|^2 dx - C_2 t^{p+1} \int_\Omega |u|^{p+1} dx \right)
\]
\[
+ \frac{q^2}{2} \max_{t \in [0,1]} \left[ \chi \left( \frac{\|tu\|_{H^1_0}}{T} \right) \left( C_4 t^2 \|tu\|_{H^1_0}^2 + C_4 t^{10} \|tu\|_{H^1_0}^{10} \right) \right]
\]
\[
\leq C + q^2 (C_5 T^2 + C_6 T^{10}).
\]
Then, from (10) we get
\[ m_T^q \leq C + q^2(C_6T^2 + C_6T^{10}). \]

From (9) we deduce that
\[ ((I_q^T)'(u_n), u_n) = \int_{\Omega} |\nabla u_n|^2 dx + A_n + B_n + C_n - \int_{\Omega} |u_n|^{p+1} dx \]
where
\[ A_n = \varepsilon \frac{q}{2} \chi \left( \frac{\|u_n\|_{H_0^1}}{T} \right) \int_{\Omega} F(u_n) \Phi_{u_n} dx \]
\[ B_n = \varepsilon \frac{q}{2} \chi \left( \frac{\|u_n\|_{H_0^1}}{T} \right) \int_{\Omega} f(u_n) u_n \Phi_{u_n} dx \]
and
\[ C_n = \varepsilon \frac{q}{2} \chi \left( \frac{\|u_n\|_{H_0^1}}{T} \right) \int_{\Omega} F(u_n) \Phi_{u_n}'[u_n] dx. \]

Since the functional
\[ S(u) = \int_{\Omega} |\nabla \Phi_u|^2 dx - 2q \int_{\Omega} F(u) \Phi_u dx, \quad u \in H_0^1(\Omega) \]
is identically equal to 0, we have that
\[ 0 = \frac{1}{2} (S'(u), u) = \int_{\Omega} (\nabla \Phi_u | \nabla \Phi_u'[u]) dx - q \int_{\Omega} f(u) \Phi_u u dx - q \int_{\Omega} F(u) \Phi_u'[u] dx. \]

On the other hand, multiplying the second equation of (P) by \( \Phi'_u[u] \) and integrating, we have that
\[ \int_{\Omega} (\nabla \Phi_u | \nabla \Phi_u'[u]) dx = 2q \int_{\Omega} F(u) \Phi'_u[u] dx \]
and then
\[ \int_{\Omega} F(u) \Phi'_u[u] dx = \int_{\Omega} f(u) \Phi_u u dx. \]

We deduce that \( B_n = C_n \) for any \( n \in \mathbb{N} \). By using (11) and (12) we get
\[ m_T^q + o_n(1)\|u_n\|_{H_0^1} + o_n(1) = I_q^T(u_n) - \frac{1}{p + 1} ((I_q^T)'(u_n), u_n) \]
\[ = \frac{p - 1}{2(p + 1)} \|u_n\|_{H_0^1}^2 + D_n - \frac{1}{p + 1} A_n - \frac{2}{p + 1} B_n, \]
where
\[ D_n = \varepsilon \frac{q}{2} \chi \left( \frac{\|u_n\|_{H_0^1}}{T} \right) \int_{\Omega} F(u_n) \Phi_{u_n} dx. \]

For (f) and (5) we have also the following estimate
\[ \max (|A_n|, |B_n|, |D_n|) \leq q^2(C_1 T^{2} + C_2 T^{10}). \]

We show that, if \( T \) is sufficiently large, then \( \limsup_n \|u_n\|_{H_0^1} \leq T \).

By contradiction, we will assume that there exists a subsequence (relabelled \((u_n)_n\)) such that for all \( n \in \mathbb{N} \) we have \( \|u_n\|_{H_0^1} > T \). By our contradiction hypothesis, (13) and (14), we obtain, for \( n \) large enough,
\[ T^2 - \sigma T \leq \|u_n\|_{H_0^1}^2 - \sigma \|u_n\|_{H_0^1} \leq C + q^2(C_1 T^{2} + C_2 T^{10}), \]
with \( \sigma > 0 \) small. If \( T^2 - \sigma T > C \), we can find \( \bar{q} \) such that for any \( q \leq \bar{q} \) the previous inequality turns out to be a contradiction. The contradiction arises from the assumption that \( \limsup_n \|u_n\|_{H_0^1} \geq T \). So we have that the sequence \((u_n)_n\) possesses a subsequence which is bounded in the \( H_0^1(\Omega) \) norm by \( T \) and such that, for every \( n \in \mathbb{N} \), \( I_q^T(u_n) \) coincides with \( I_q(u_n) \).

The last step is to prove that there exists \( \bar{q} > 0 \) such that for any \( 0 < q \leq \bar{q} \) there exists a Palais-Smale sequence \((u_n)_n\) of \( I_q \) which is, up to a subsequence, weakly convergent to a nontrivial critical point of
Let $T$ and $\tilde{q}$ be given by the first step, and consider any $0 < q \leq \tilde{q}$. We know that there exists a Palais-Smale sequence of the functional $I_q$ at the level $m_q := m_q^T$, such that
\[
\|u_n\|_{H^1_0(\Omega)} \leq T, \text{ for any } n \in \mathbb{N}. \tag{15}
\]
Up to subsequences, there exist $u_0 \in H^1_0(\Omega)$ and $\Phi_0 \in H^1_0(\Omega)$ such that
\[
u_n \rightharpoonup u_0 \text{ in } H^1_0(\Omega), \quad \Phi_{u_n} \rightarrow \Phi_0 \text{ in } H^1_0(\Omega). \tag{16}
\]
By (f) and (16) we also have
\[
F(u_n) \rightarrow F(u_0) \text{ in } L^{\frac{2}{q}_q}(\Omega), \tag{18}
\]
\[
f(u_n)u_n \rightarrow f(u_0)u_0 \text{ in } L^{\frac{2}{q}_q}(\Omega). \tag{19}
\]
Now we show that $\Phi_0 = \Phi_{u_0}$ and that $(u_0, \Phi_0)$ is a weak nontrivial solution of $(P_T)$.
Let us consider a test function $\psi \in C^\infty_0(\Omega)$. From the second equation of our problem we obtain
\[
\int_\Omega (\nabla \Phi_{u_n} \nabla \psi) \, dx = 2q \int_\Omega F(u_n) \psi \, dx.
\]
Passing to the limit and using (17) and (18), we have that
\[
\int_\Omega (\nabla \Phi_0 \nabla \psi) \, dx = 2q \int_\Omega F(u_0) \psi \, dx.
\]
So $\Phi_0$ is a weak solution of $-\Delta \Phi = 2qF(u_0)$, and then, by uniqueness, it is $\Phi_0 = \Phi_{u_0}$.
Since $(u_n)_n$ is a Palais-Smale sequence, for any $\psi \in C^\infty_0(\Omega)$ we obtain that
\[
\int_\Omega (\nabla u_n \nabla \psi) \, dx + \varepsilon q \int_\Omega \Phi_{u_n} f(u_n) \psi \, dx = \int_\Omega |u_n|^{p-1} u_n \psi \, dx + o_n(1).
\]
Passing to the limit, by (16) and (19) we have
\[
\int_\Omega (\nabla u_0 \nabla \psi) \, dx + \varepsilon q \int_\Omega \Phi_{u_0} f(u_0) \psi \, dx = \int_\Omega |u_0|^{p-1} u_0 \psi \, dx
\]
that is $(u_0, \Phi_{u_0})$ is a weak solution of $(P_T)$.
It remains to prove that $u_0 \neq 0$.
Assume by contradiction that $u_0 = 0$. By compactness we obtain that $u_n \rightharpoonup 0$ in $L^{p+1}_0(\Omega)$. On the other hand, since $(I_q'(u_n), u_n) \to 0$, we deduce that, up to subsequences,
\[
\lim_n \int_\Omega |\nabla u_n|^2 \, dx = \lim \varepsilon q \int_\Omega f(u_n)u_n \Phi_{u_n} \, dx =: l_q > 0. \tag{20}
\]
Of course $l_q > 0$. Otherwise from (20) we would deduce that $u_n \to 0$ in $H^1_q(\Omega)$ and then $0 < m_q = \lim_n I_q(u_n) = 0$.
By (15) we also have that $l_q \leq T$. By using Holder inequality, Sobolev inequalities, (f), (4) and (7) we have that
\[
-\varepsilon q \int_\Omega f(u_n)u_n \Phi_{u_n} \, dx \leq C_q^2 \left( \int_\Omega |f(u_n)u_n|^{6/5} \, dx \right)^{5/6} \left( \int |F(u_n)|^{6/5} \, dx \right)^{5/6} \\
\leq q^2 \left( C_1 \|u_n\|_{H^1_q(\Omega)}^2 + C_2 \|u_n\|_{H^1_q(\Omega)}^{10} \right).
\]
Passing to the limit, by (20) we deduce that
\[
l_q \leq q^2 (C_1 l_q^2 + C_2 l_q^{10})
\]
that is
\[
1 \leq q^2 (C_1 l_q + C_2 l_q^9) \leq q^2 (C_1 T + C_2 T^9).
\]
We conclude observing that the previous inequality does not hold if we take $q \leq \tilde{q} < \min \left\{ \tilde{q} : \frac{1}{C_1 T + C_2 T^9} \right\}$. \qed
4 Proof of Theorem 1.2

This section deals with the study of the system \((P_r)\). We divide the proof of Theorem 1.2 in two parts, concerning respectively the existence and the non existence of a solution according to the value of \(r\).

4.1 The existence result

Here, we suppose that \(1 < r < 5\). In analogy to problem \((P_f)\), we use a variational approach finding the solutions as critical points of the \(C^1\) functional

\[
I_{q,r}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{q}{2r} \int_\Omega |u|^r \Phi_u dx + \frac{1}{p+1} \int_\Omega |u|^{p+1} dx,
\]

where, for any \(u \in H^1_0(\Omega)\), \(\Phi_u \in H^1_0(\Omega)\) is the unique positive solution of

\[
\left\{
\begin{array}{ll}
-r \Delta \Phi = 2q|u|^r & \text{in } \Omega, \\
\Phi = 0 & \text{on } \partial \Omega.
\end{array}
\right.
\]

(21)

We have the following estimates.

**Lemma 4.1.** For every \(u \in H^1_0(\Omega)\)

\[
\|\Phi_u\|_{H^1_0(\Omega)} \leq C\|u\|_r^r.
\]

(22)

and for any \(k > 0\) it is

\[
\frac{kq}{r} \int_\Omega |u|^r \Phi_u dx \geq \frac{2q}{r} \int_\Omega |u|^{r+1} dx - \frac{1}{2k} \int_\Omega |\nabla u|^2 dx.
\]

(23)

**Proof.** The first part of the lemma is a consequence of the fact that, since \(1 < r < 5\), then for any \(u \in H^1_0(\Omega)\) the function \(|u|^r \in L^\frac{2}{r}(\Omega)\) and we can argue as in Section 2 to define the map \(\Phi\) and to deduce (22).

In order to prove (23), we proceed as in [18]: multiplying (21) by \(|u|\) and integrating we get

\[
\frac{2q}{r} \int_\Omega |u|^{r+1} dx = \int_\Omega (\nabla \Phi_u |\nabla |u|) dx = \int_\Omega (\sqrt{k} \nabla \Phi_u |\nabla |u|) dx \\
\leq \frac{k}{2} \int_\Omega |\nabla \Phi_u|^2 dx + \frac{1}{2k} \int_\Omega |\nabla |u|^2 dx.
\]

Inequality (23) follows since it is

\[
\int_\Omega |\nabla \Phi_u|^2 dx = \frac{2q}{r} \int_\Omega |u|^r \Phi_u dx.
\]

\(\Box\)

In the following lemma we establish the compactness of the Palais-Smale sequences of the functional \(I_{q,r}\).

**Lemma 4.2.** If \(1 < r < 5\) the functional \(I_{q,r}\) satisfies the Palais-Smale condition.

**Proof.** Let \((u_n)_n\) be a a Palais-Smale sequence for the functional \(I_{q,r}\), namely \((I_{q,r}(u_n))_n\) is bounded and \(I_{q,r}'(u_n)\) converges to zero in \(H^{-1}\).

We distinguish two cases.

If \(r \geq (p+1)/2\), then

\[
I_{q,r}(u_n) - \frac{I_{q,r}'(u_n)[u_n]}{p+1} = \frac{p-1}{2(p+1)} \int_\Omega |\nabla u_n|^2 dx + q \frac{2r-p-1}{2r(p+1)} \int_\Omega |u_n|^r \Phi_u dx \\
\leq C + o_n(1)\|u_n\|_{H^1_0(\Omega)}
\]

and then \((u_n)_n\) is bounded.

If \(r < (p+1)/2\), then the boundedness of \((u_n)_n\) comes from the boundedness of \((I_q(u_n))_n\) since the
functional is coercive. Indeed, if we suppose that \((u_n)_n\) diverges in the \(H^1_0(\Omega)\)–norm, then, by (22) and the Lebesgue embedding \(L^{p+1}(\Omega) \hookrightarrow L^{2r}(\Omega)\),

\[
I_{q,r}(u_n) \geq C_1\|u_n\|_{H^1_0(\Omega)}^2 - C_2\|u_n\|_{p+1}^{2r} + C_3\|u_n\|_{p+1}^{p+1} \to +\infty.
\]

Thus we can complete the proof of the Theorem 1.2.

**Proof of Theorem 1.2.** We first suppose that \(r\) is subcritical and we deal with each case separately.

**case 1:** \(\frac{p+1}{2} < r < 5\)

We show that for any \(q > 0\) and for any finite dimensional subspace of \(H^1_0(\Omega)\), the functional \(I_{q,r}\) satisfies the assumptions of [4, Theorem 2.23].

By (22) and the Sobolev embedding \(H^1_0(\Omega) \hookrightarrow L^{2r}(\Omega)\) it is

\[
I_{q,r}(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - C\|u\|_{H^1_0(\Omega)}^{2r} \geq \frac{1}{2} \|u\|_{H^1_0(\Omega)}^2 - C\|u\|_{H^1_0(\Omega)}^{2r},
\]

then assumption \((I_1)\) holds since \(I_{q,r}(u) \geq \alpha > 0\) if \(\|u\|_{H^1_0(\Omega)}\) is sufficiently small. By Lemma 4.2, also assumption \((I_3)\) holds. \((I_4)\) can be checked by an easy computation. In order to prove \((I_7)\), we show that for any finite dimensional subspace \(E\) of \(H^1_0(\Omega)\) there exists a ball \(B_p\) such that \(I_{q,r}|_{E \cap \partial B_p} < 0\).

Let \(E\) be a finite dimensional subspace of \(H^1_0(\Omega)\). It is easy to see that for \(\rho > 0\) and \(u \in H^1_0(\Omega)\)

\[
\Phi_{\rho u} = \rho^2 \Phi_u.
\]

Since in \(E\) all the norms are equivalent, if \(u \in E \cap \partial B_1\), by (23) we have

\[
I_{q,r}(\rho u) = \frac{\rho^2}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{q}{2r} \rho^2 \int_{\Omega} |u|^{2r} \Phi_u dx + \frac{\rho^{p+1}}{p+1} \int_{\Omega} |u|^{p+1} dx
\]

\[
\leq \frac{\rho^2}{2} \left( - \frac{q}{2r} \rho^2 \int_{\Omega} |u|^{2r} dx + \frac{1}{4k^2} |u|^{2r} + \frac{(c_1(k) - c_2(k))\rho^{2r} + c_3\rho^{p+1}}{2r} \right)
\]

Since \(c_1(k) = O(1/k)\) and \(c_2(k) = O(1/k^2)\) for \(k \to +\infty\), we have that for a \(k\) sufficiently large \(c_1(k) - c_2(k) > 0\). So we can take \(\rho > 0\) as large as needed to have that (24) is negative.

**case 2:** \(\frac{p+1}{2} = r\)

The proof is the same as in the previous case, except for \((I_7)\). In particular we can only prove that for any \(E \subset H^1_0(\Omega)\) finite dimensional subspace there exists \(q > 0\) such that for every \(q > \bar{q}\) and \(\bar{\rho}\) large enough, \(I_{q,r}|_{E \cap \partial B_{\bar{\rho}}} < 0\). Indeed, as in (24), we have that for any \(u \in E \cap \partial B_1\), it is

\[
I_{q,r}(\rho u) \leq \frac{\rho^2}{2} \left( - \frac{q}{r} \rho^2 \|u\|_{r+1} + \frac{\rho^{p+1}}{p+1} \frac{\rho^{2r}}{2r} \right)
\]

so that if \(q\) and \(\rho\) are sufficiently large, \(I_{q,r}(\rho u) < 0\).

**case 3:** \(1 < r < \frac{p+1}{2}\)

In this case it is easy to check that for any \(E \subset H^1_0(\Omega)\) finite dimensional subspace there exists \(q > 0\) such that \(I_{q,r}|_{E \cap \partial B_1} < 0\). In fact, for any \(u \in E \cap \partial B_1\), we have that

\[
I_{q,r}(\rho u) \leq \frac{\rho^2}{2} \left( - \frac{q}{kr} \rho^2 \int_{\Omega} |u|^{r+1} dx + \frac{1}{4k^2} \rho^{2r} + \frac{\rho^{p+1}}{p+1} \int_{\Omega} |u|^{p+1} dx \right)
\]

\[
\leq \frac{\rho^2}{2} \left( - \frac{qk}{2r} \|u\|_{r+1} + \frac{\rho^{p+1}}{p+1} \frac{\rho^{2r}}{2r} \right).
\]
To complete the proof, by [4, Corollary 2.24] we have just to show that for any $q$ the functional $I_{q,r}$ is bounded from below. Indeed, by (22),

$$I_{q,r}(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - C \|u\|_p^{2r} + \frac{1}{p+1} \|u\|_{p+1}^{p+1}$$

$$\geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - C \|u\|_p^{2r} + \frac{1}{p+1} \|u\|_{p+1}^{p+1}$$

$$= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \|u\|_{p+1}^{2r} \left( \frac{1}{p+1} \|u\|_{p+1}^{p+1} - 2r - C \right)$$

where the last quantity cannot diverge negatively.

It is quite natural to wonder if some nonexistence result can be proved in the case $1 < r \leq \frac{p+1}{2}$ and $q$ small. Actually, it can easily be observed that the existence of at least a solution is guaranteed also for small $q$ when a ball with a sufficiently large radius $R$ is contained in $\Omega$. Indeed, consider $u \in C_0^\infty(\Omega)$ such that $\|u\|_\infty \leq \sigma$ where $\sigma > 0$ and $\frac{\sigma}{2r} |x|^{q+1} - \frac{1}{p+1} |x|^{p+1} > 0$ for any $s \in [0, \sigma]$. We set $u_\tau = u(\frac{x}{\tau})$, and we suppose that $\text{Supp}(u_\tau) = t \text{Supp}(u) \subset \Omega$. By a straight computation, using (23), we have that

$$I_{q,r}(u_\tau) \leq t \left( \frac{1}{2} + \frac{1}{4k^2} \right) \int_{\Omega} |\nabla u|^2 dx + t^2 \left( \frac{q}{kr} \int_{\Omega} |u|^{r+1} dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx \right)$$

where this last sum is negative for $t$ sufficiently large. As a consequence, we should have a mountain pass solution for $2r = p + 1$ and also a minimum solution for $2r < p + 1$.

4.2 The nonexistence result

Here we assume that $1 < p < 5 \leq r$. Following [21], we adapt the Pohožaev arguments in [17] to our situation (for a similar result see also [7]). Actually the proof is the same as in [2], but we report it here for completeness.

Let $\Omega \subset \mathbb{R}^3$ be a star shaped domain and $u, \Phi \in C^2(\Omega) \cap C^1(\Omega)$ be a nontrivial solution of $(P_r)$. If we multiply the first equation of $(P_r)$ by $x \cdot \nabla u$ and the second one by $x \cdot \nabla \Phi$ we have that

$$0 = (\Delta u + q\Phi |u|^{r-2}u - |u|^{p-1}) (x \cdot \nabla u)$$

$$= \text{div} [(\nabla u)(x \cdot \nabla u)] - |\nabla u|^2 - x \cdot \nabla \left( \frac{|\nabla u|^2}{2} \right) + q \frac{x}{r} \cdot \nabla \Phi |u|^r - \frac{q}{r} (x \cdot \nabla \Phi) |u|^r - \frac{1}{p+1} x \cdot \nabla |u|^{p+1}$$

$$= \text{div} \left[ (\nabla u)(x \cdot \nabla u) - x |\nabla u|^2 \frac{1}{2} + q \frac{x}{r} \Phi |u|^r - \frac{1}{p+1} x |u|^{p+1} \right]$$

$$+ \frac{1}{2} |\nabla u|^2 - \frac{3}{r} q\Phi |u|^r - \frac{q}{r} (x \cdot \nabla \Phi) |u|^r + \frac{3}{p+1} |u|^{p+1}$$

and

$$0 = (r \Delta \Phi + 2q|u|^r) (x \cdot \nabla \Phi)$$

$$= r \text{div} [(\nabla \Phi)(x \cdot \nabla \Phi)] - r |\nabla \Phi|^2 - \frac{r}{2} x \cdot \nabla (|\nabla \Phi|^2) + 2q (x \cdot \nabla \Phi) |u|^r$$

$$= r \text{div} \left[ (\nabla \Phi)(x \cdot \nabla \Phi) - x |\nabla \Phi|^2 \frac{1}{2} \right] + \frac{r}{2} |\nabla \Phi|^2 + 2q (x \cdot \nabla \Phi) |u|^r.$$

Let $n$ be the unit exterior normal to $\partial \Omega$. Integrating on $\Omega$, since by boundary conditions $\nabla u = \frac{\partial u}{\partial n} n$ and $\nabla \Phi = \frac{\partial \Phi}{\partial n} n$ on $\partial \Omega$, we obtain

$$- \frac{1}{2} |\nabla u|^2 \Delta n + \frac{1}{2} \int_{\partial \Omega} \frac{\partial u}{\partial n} n \cdot \nabla u = - \frac{3}{r} q \int_{\Omega} \Phi |u|^r - \frac{q}{r} \int_{\Omega} (x \cdot \nabla \Phi) |u|^r + \frac{3}{r} |u|^{p+1}$$

and

$$- \frac{r}{2} |\nabla \Phi|^2 \Delta n - \frac{r}{2} \int_{\partial \Omega} \frac{\partial \Phi}{\partial n} n \cdot \nabla \Phi = 2q \int_{\Omega} (x \cdot \nabla \Phi) |u|^r.$$
Substituting (26) into (25) we have
\[
-\frac{1}{2} \| \nabla u \|^2_2 - \frac{1}{2} \int_{\partial \Omega} | \frac{\partial u}{\partial n} |^2 \ x \cdot n = -\frac{3}{4} q \int_{\Omega} \Phi |u|^r - \frac{1}{4} \| \nabla \Phi \|^2_2 + \frac{1}{4} \int_{\partial \Omega} | \frac{\partial \Phi}{\partial n} |^2 \ x \cdot n + \frac{3}{p+1} \| u \|_{p+1}^{p+1}. \tag{27}
\]
Moreover, multiplying the first equation of (Pr) by \( u \) and the second one by \( \Phi \) we get
\[
\| \nabla u \|^2_2 = q \int_{\Omega} \Phi |u|^r - \| u \|_{p+1}^{p+1} \tag{28}
\]
and
\[
\frac{r}{4} \| \nabla \Phi \|^2_2 = 2q \int_{\Omega} \Phi |u|^r. \tag{29}
\]
Hence, combining (27), (28) and (29), we have
\[
\frac{r}{4} - \frac{5}{4} \| \nabla \Phi \|^2_2 + \frac{5 - p}{2(p+1)} \| u \|_{p+1}^{p+1} + \frac{1}{2} \int_{\partial \Omega} | \frac{\partial u}{\partial n} |^2 \ x \cdot n + \frac{1}{4} \int_{\partial \Omega} | \frac{\partial \Phi}{\partial n} |^2 \ x \cdot n = 0
\]
and we get a contradiction.

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