The sn-pole approximation in the Composite Operator Method

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A well-established method to deal with highly correlated systems is based on the expansion of the Green’s function in terms of spectral moments. In the context of the Composite Operator Method one approximation is proposed: a set of \( n \) composite fields is assumed as fundamental basis and the dynamics is considered up to the order \( s \). The resulting Green’s function has a sn-pole structure. The truncation of the hierarchy of the equations of motion is made at the \( s \)-th order and the first \( s - 1 \) equations are treated exactly. A theorem, which rules the conservation of the spectral moments, is presented. The procedure is applied to the Hubbard model and a recurrence relation for the calculation of its electronic spectral moments is derived.

I. INTRODUCTION

The discovery of new materials with unusual properties requires the formulation of new schemes of calculation capable to catch the essential physics of correlations among particles. One powerful tool is the Green’s function formalism. Most of the properties of a many-body system can be fully described in terms of the thermal single-particle Green’s function. For finite systems the spectral function \( \rho(k, \omega) \) can always be written as an infinite weighted sum of delta functions and the retarded Green’s function can be expressed as

\[
g(k, \omega) = \int_{-\infty}^{+\infty} d\omega' \frac{\rho(k, \omega')}{\omega - \omega'} = \sum_{i=1}^{\infty} \frac{\sigma^{(i)}(k)}{\omega - E^{(i)}(k) + i\eta}
\]

where \( E^{(i)}(k) \) and \( \sigma^{(i)}(k) \) are the energy spectrum and the spectral density function, respectively, of the \( i \)-th elementary excitation. In the way of developing approximate methods, one line of thinking is to approximate the Green’s function by a sum on a finite number of excitations (pole approximation). The main question that arises is how the poles must be chosen (i.e., which excitations do describe the physics we are interested to analyze?). Let us consider a Hamiltonian \( H = H[\varphi] \), where \( \varphi \) denotes a field operator. In the spectral density approach (SDA) one truncates the sum in (1.1) by picking up the first \( s \) terms (\( s \)-pole approximation). The unknown functions \( E^{(i)}(k) \) and \( \sigma^{(i)}(k) \) are fixed by the set of \( 2s \) nonlinear equations

\[
m^{(k)}(k) = \sum_{i=1}^{s} E^{(i)}(k) \sigma^{(i)}(k)
\]

where \( m^{(k)}(k) \), the moments of the spectral density, are calculated by means of the equations of motion:

\[
m^{(k)}(k) = F.T. \left[ (i\partial/\partial t_x)^k \langle [\varphi(x), \varphi^\dagger(y)] \rangle \right]_{t_x=t_y}
\]

Eq. (1.2) is derived by making use of the exact relation (conservation of spectral moments)

\[
m^{(k)}(k) = \int_{-\infty}^{+\infty} d\omega \rho(k, \omega) \omega^k
\]

The symbol \([\cdots]_\pm\) denotes the anticommutator (+) or the commutator (−) according to the statistics the field \( \varphi \) obeys; the symbol \( \langle \cdots \rangle \) denotes the thermal average on the grand canonical ensemble; \( F.T. \) indicates the Fourier transform. This procedure, which is an expansion of the Green’s function in terms of the first \( 2s \) spectral moments, gives a good description in the high-energy region but certainly fails in the low-energy region, where one needs \( s \) to be very large.
When we consider systems where the correlation is very high the procedure above illustrated may not be adequate. Due to the strong interactions the original particles are not observable; new excitations appear and it is more convenient to describe the system in terms of new field operators. Collective behaviors in forms of bound states, resonances and diffused modes emerge as the physical fields. Although some of them are not stable excitations, they give considerable contributions to the physical processes and therefore it is sometimes convenient to promote them to the role of well-defined quasi-particle excitations. The choice of new fundamental particles, whose properties have to be self-consistently determined by dynamics, symmetries and boundary conditions, becomes relevant. The necessity of role of well-defined quasi-particle excitations. The choice of new fundamental particles, whose properties have to be considerable contributions to the physical processes and therefore it is sometimes convenient to promote them to the

$$i \frac{\partial}{\partial t} \psi(x) = [\psi(x), H_I] = \epsilon_I (-i \nabla) \psi(x)$$  

(1.5)

The fields \(\psi_n\) are generally constructed as functionals of the original field \(\phi\) and are called composite fields. Then, \(H_{II}\) will be taken into account by some approximate methods that will use \(\psi\) as operatorial basis. Various approximations have been used to estimate the contribution coming from \(H_{II}\). One method, the \(n-\) pole approximation, linearizes the full equation of motion

$$i \frac{\partial}{\partial t} \psi(x) = [\psi(x), H] = \epsilon (-i \nabla) \psi(x) + \delta j(x) \approx \epsilon (-i \nabla) \psi(x)$$  

(1.6)

where the eigenvalue or energy matrix \(\epsilon\) is self-consistently calculated by means of the equation

$$\epsilon(k) M^{(0)}(k) = M^{(1)}(k)$$  

(1.7)

\(M^{(k)}(k)\) is the generalized spectral moment, given by the \(n \times n\) matrix

$$M^{(k)}(k) = F.T. \left[ (i \partial / \partial t_x)^k \left< [\psi(x), \psi^\dagger(y)] \right>_x \right]_{t_x=t_y}$$  

(1.8)

The decomposition in (1.6) is chosen in such a way that the higher-order source \(\delta j(x)\) does not have a projection on the fundamental basis \(\{\psi_i\}\). In this approximation the retarded “single-particle” Green’s function \(G(k,\omega) = F.T. < R[\psi(x)\psi^\dagger(y)] >\) has a \(n\)-pole structure, where the number of poles is equal to the number of composite fields. It is clear that the properties of the Green’s function are controlled by the dynamics and by the choice of the new basis. When this approximation is used the \(COM\) has some similarity with the \(SDA\): both approaches correspond to expanding the Green’s function in terms of spectral moments. However, there is a fundamental difference. In the \(SDA\) there is no freedom in choosing the spectral moments, which are completely fixed by the equations of motion. At the contrary, in the \(COM\) the spectral moments are mainly determined by the choice of the operatorial basis. The idea is that in the framework of approximating the Green’s function by a finite number of excitations, a proper choice of the basis might give a better convergence and a more accurate description of the low-energy region. Let us write

$$M^{(k)}(k) = \tilde{M}^{(k)}(k) + \delta M^{(k)}(k)$$  

(1.9)

where

$$\tilde{M}^{(k)}(k) = \epsilon^k(k) M^{(0)}(k)$$  

(1.10)

$$\delta M^{(k)}(k) = \sum_{m=1}^{k-1} F.T. \epsilon^{k-m-1} (i \partial / \partial t_x)^m < [\delta j(x), \psi^\dagger(y)] >_{t_x=t_y}$$  

(1.11)

\(\tilde{M}^{(k)}(k)\) corresponds to the spectral moment calculated in the \(n\)-pole approximation, where the higher-order field \(\delta j(x)\) in the Heisenberg equation (1.6) is neglected. In Ref. 13 we proved the following theorems.

**Theorem I** Given a set of composite fields \(\{\psi_i, i = 1, \ldots, n\}\), satisfying linearized Heisenberg equations \(i(\partial / \partial t) \psi = \epsilon \psi\), the approximate spectral moments \(\tilde{M}^{(k)}(k)\), are conserved at any order.
**Theorem II** Given a set of composite fields \( \{ \psi_l, l = 1, \ldots, n \} \), if the subset \( \{ \psi_l, l = 1, \ldots, n - 1 \} \) satisfies linear Heisenberg equations
\[
i \frac{\partial}{\partial t} \psi_l(x) = \sum_{p=1}^{l+1} \gamma_{lp}(-i\nabla) \psi_p(x),
\]
then the first \( 2(n-l+1) \) spectral moments for the field \( \psi_l \) [that is \( M_{ll}^{(s)}(k) \)] are conserved.

The Theorem II shows that the SDA corresponds to COM when a particular choice of the basic fields is considered and a pole approximation is used. In COM we have more flexibility, in the sense that there is no restriction on the choice of the basic fields.

A different approximation has been proposed in Refs. 11 and 12, where in the context of the \( t-J \) and Hubbard models the incoherent part of the Green’s function has been taken into account by a two-site approximation in combined use of the resolvent method. Although this approximation gave excellent results, the numerical calculations are very heavy in the small temperature regime. A pole approximation is much more convenient for practical calculations and there are no restrictions on the range of physical parameters. As shown in a variety of works, the pole approximation gives a satisfactory account of the local and spectral features. Therefore, it is interesting to explore the possibility to improve the approximation of the COM by improving the \( n \)-pole approximation. In this article we propose a new approximation, the \( sn \)-pole approximation, where we combine two ingredients: the choice of the fields and the dynamics. The procedure is the following: (i) we fix \( n \) basic fields by some reasonable criterion based on the physical properties of the model; (ii) we take into account the dynamics up to the \( s \)-th order. In this scheme the Green’s function has a \( sn \)-pole structure
\[
G(k, \omega) = \sum_{i=1}^{sn} \frac{\sigma(i)(k)}{\omega - E_i(k) + i\eta}
\]
where \( E_i(k) \) and \( \sigma(i)(k) \) are expressed in terms of the \( 2s \) spectral moment matrices \( M^{(k)}(k) \) \( [0 \leq k \leq 2s - 1] \). Let \( A^{(m)}(k) \) \( [m = 1, \ldots, s - 1] \) be matrices of rank \( n \times n \) determined by the equations
\[
M^{(s+k)}(k) - \sum_{m=0}^{s-1} A^{(m)}(k) M^{(m+k)}(k) = 0 \quad [\text{for } 0 \leq k \leq s - 1]
\]
Then, the main results of this work can be summarized in the following theorems.

**Theorem III** Given a set of composite fields \( \{ \psi_i, i = 1, \ldots, n \} \), satisfying linearized Heisenberg equations
\[
\left( i\frac{\partial}{\partial t} \right)^s \psi(x) - \sum_{m=0}^{s-1} A^{(m)}(-i\nabla) \left( i\frac{\partial}{\partial t} \right)^m \psi(x) = 0
\]
the first \( 2s \) generalized spectral moments are conserved and the approximate spectral moments \( \tilde{M}^{(k)}(k) \) are conserved at any order.

**Theorem IV** Given a set of composite fields \( \{ \psi_l, l = 1, \ldots, n \} \), if the subset \( \{ \psi_l, l = 1, \ldots, n - 1 \} \) satisfies the Heisenberg equations (1.14), the first \( 2s(n-l+1) \) spectral moments for the field \( \psi_l(x) \) [that is \( M_{ll}^{(s)}(k) \)] are conserved.

When a particular choice of the basic fields is considered, we will show that the \( sn \)-pole approximation is equivalent to SDA with \( s \rightarrow sn \).

**II. EQUATIONS OF MOTION AND GREEN’S FUNCTION**

The composite field \( \psi(x) \) satisfies the equation of motion
\[
i \frac{\partial}{\partial t} \psi(x) = [\psi(x), H] = j^{(1)}(x)
\]
The source \( j^{(1)}(x) \) is expressed, in general, in terms of higher-order composite fields with a dynamics determined by the Heisenberg equation
\[
\frac{\partial}{\partial t} \psi(x) = [\psi(x), H] = j^{(1)}(x)
\]
\[ i \frac{\partial}{\partial t} j^{(1)}(x) = [j^{(1)}(x), H] = j^{(2)}(x) \]  

(2.2)

The process can be continued and, in general, we have for the \( s \)-th derivative the equation

\[ \left( i \frac{\partial}{\partial t} \right)^s \psi(x) = j^{(s)}(x) \quad [\text{for} \quad s \geq 1] \]

(2.3)

Let us rewrite Eq. (2.3) under the form

\[ \left( i \frac{\partial}{\partial t} \right)^s \psi(x) - \sum_{m=0}^{s-1} A^{(m)}(-i\nabla) \left( i \frac{\partial}{\partial t} \right)^m \psi(x) = \delta j^{(s)}(x) \]

(2.4)

where \( A^{(m)}(-i\nabla) \) are matrices of rank \( n \) determined by the equations

\[ < \delta j^{(s)}(x,t), \left( -i \frac{\partial}{\partial t} \right)^k \psi^\dagger(y,t) > = 0 \quad [\text{for} \quad 0 \leq k \leq s - 1] \]

(2.5)

By substituting (2.4) into (2.5) we find that the matrices \( A^{(m)}(k) \) \( [0 \leq m \leq s - 1] \) are determined by the following system of equations

\[ M^{(s+k)}(k) - \sum_{m=0}^{s-1} A^{(m)}(k) M^{(m+k)}(k) = 0 \quad [\text{for} \quad 0 \leq k \leq s - 1] \]

(2.6)

The evaluation of the \( s \) matrices \( A^{(m)}(k) \) requires the knowledge of the first \( 2s \) generalized spectral moments \( M^{(k)}(k) \), that can be calculated by means of (1.8). By using the equation of motion (2.4) the retarded Green’s function \( G(x,y) = < R[\psi(x)\psi^\dagger(y)] > \) satisfies the equation

\[ \sum_{k=0}^{s} A^{(k)}(-i\nabla_x) \left( i \frac{\partial}{\partial t_x} \right)^k G(x,y) \]

\[ = \sum_{k=1}^{s} (i)^k \left( \frac{\partial}{\partial t_x} \right)^{k-1} \delta(t_x - t_y) \sum_{m=k}^{s} A^{(m)}(-i\nabla_x) I^{(m-k)}(x,y) \]

\[ - < R[\delta j^{(s)}(x)\psi^\dagger(y)] > \]

(2.7)

where

\[ I^{(k)}(x,y) = < j^{(k)}(x), \psi^\dagger(y) > >_{t_x=t_y} \]

(2.8)

with the convention that

\[ A^{(s)}(-i\nabla) = -1 \]

(2.9)

Eq. (2.7) is an exact equation. We shall define our approximation, \( sn \)-pole approximation, by neglecting the term \( < R[\delta j^{(s)}(x)\psi^\dagger(y)] > \), with \( \delta j^{(s)}(x) \) fixed by the requirement of vanishing projections on the basis \( j^{(k)}(x); \quad 0 \leq k \leq s - 1 \) with the convection \( j^{(0)}(x) = \psi(x) \). Then, the solution of (2.7) in momentum space is

\[ G(k,\omega) = \frac{1}{\omega^s - \sum_{k=0}^{s-1} A^{(k)}(k)\omega^k} \sum_{k=1}^{s} C^{(s,k)}(k)\omega^{k-1} \]

(2.10)

where we defined

\[ C^{(s,k)}(k) = - \sum_{m=k}^{s} A^{(m)}(k) M^{(m-k)}(k) \quad [\text{for} \quad 1 \leq k \leq s] \]

(2.11)

Eq. (2.10) shows that \( G(k,\omega) \) is expressed in terms of the first \( 2s \) generalized spectral moments. This expansion is dictated by the choice of the composite field and by the dynamics up to the \( s \)-th order. Expression (2.10) can be rewritten in the spectral form
\[ G(k, \omega) = \sum_{i=1}^{s_n} \frac{\sigma^{(i)}(k)}{\omega - E_i(k) + i\eta} \]  

where the energy spectra \( E_i(k) \) are the roots of the equation

\[ \sum_{m=0}^{s_n} \omega^m a_m(k) = 0 \]  

with coefficients \( a_m(k) \) defined by

\[ a_m(k) = \frac{b^{(i)}(k)}{m!} \left[ \frac{\partial}{\partial \omega} \text{Det} \{ \omega^s - \sum_{k=0}^{s-1} A^{(k)}(k) \omega^k \} \right]_{\omega=0} \text{[for } 0 \leq m \leq s_n - 1 \text{]} \]  

The spectral density functions \( \sigma^{(i)}(k) \) are expressed as

\[ \sigma^{(i)}(k) = \frac{1}{b^{(i)}(k)} \sum_{m=0}^{s_n-1} E_i^m(k) \lambda^{(m)}(k) \]  

where we put

\[ b^{(i)}(k) = \prod_{j=1, j \neq i}^{s_n} [E_i(k) - E_j(k)] \]  

and the \( \lambda^{(m)}(k) \) are \( n \times n \) matrices, determined by the recurrence relation

\[ M^{(m)} = \sum_{p=0}^{s-1} A^{(p)}(k) \lambda^{(k+p)}(k) \]

\[ \sum_{p=0}^{s-1} \sum_{m=p+1}^{s} a_{k+p-s}(k) A^{(m)}(k) M^{(m-p)}(k) \]  

with the convention

\[ \lambda^{(m)}(k) = 0 \quad \text{for } m \notin [0, s_n - 1] \]

\[ a_m(k) = 0 \quad \text{for } m \notin [0, s_n] \]  

When \( k \geq s_n - 2s \) the matrices \( \lambda^{(k)}(k) \) can be simply expressed in terms of the spectral moments by means of the relation

\[ \lambda^{(k)}(k) = \sum_{m=0}^{s_n-s-1} a_{s_n-m}(k) M^{(s_n-k-1-m)}(k) \quad [\text{for } s_n - 2s \leq k \leq s_n - 1] \]  

In particular, we see that for \( n \leq 2 \) the relation (2.19) is valid for all values of \( s \) and for \( k \) varying in the all interval \( 0 \leq k \leq s_n - 1 \).

Summarizing, the calculation of the Green’s function in the \( s_n \)-approximation requires the following steps:

- (i) given a Hamiltonian, we choose a set \( \{ \psi(x) \} \) of \( n \) composite field;
- (ii) we calculate the first \( 2s \) generalized spectral moments \( M^{(k)}(k) \) by means of the formula (1.8);
- (iii) we calculate the matrices \( A^{(m)}(k) \quad [0 \leq m \leq s - 1] \) by solving the linear system (2.6);
- (iv) we calculate the “characteristic” coefficients \( a_m(k) \quad [0 \leq m \leq s_n] \) by means of (2.14);
- (v) the energy spectra \( E_i(k) \quad [1 \leq i \leq s_n] \) are calculated as the roots of the equation (2.13);
- (vi) the matrices \( \lambda^{(m)}(k) \quad [0 \leq m \leq s_n - 1] \) are calculated by means of the recurrence relation (2.17);
- (vii) the spectral functions \( \sigma^{(k)}(k) \quad [1 \leq k \leq s_n] \) are calculated by means of the expression (2.15).

In closing this section, it is worth noting that the truncation the hierarchy of the equations of motion at the order \( s \) implies that the first \( s - 1 \) equations are treated exactly.
III. CONSERVATION OF THE SPECTRAL MOMENTS

The spectral moments play an essential role in the formulation developed in the previous Section. It is important to verify to which extent the exact relation (1.4) is conserved in our approximation. In the $sn$-pole approximation the spectral function has the expression

$$\rho(k, \omega) = \sum_{i=1}^{sn} \sigma^{(i)}(k) \delta[\omega - E_i(k)]$$  \hspace{1cm} (3.1)

Let us denote by $B^{(k)}(k)$ the spectral moments calculated in the present approximation

$$B^{(k)}(k) = \int_{-\infty}^{+\infty} d\omega \omega^k \rho(k, \omega) = \sum_{i=1}^{sn} E_i^k(k) \sigma^{(i)}(k)$$  \hspace{1cm} (3.2)

The point to discuss is the relation between $B^{(k)}(k)$ and the exact spectral moments $M^{(k)}(k)$, calculated by means of (1.8). We shall study separately the two quantities. In Appendix A we prove that the matrices $B^{(k)}(k)$ satisfy the following relations

- (i) For $n=1$
  $$B^{(k)}(k) = M^{(k)}(k) \quad \text{for} \quad 0 \leq k \leq 2s - 1$$
  $$B^{(k)}(k) = - \sum_{m=0}^{s-1} a_m(k) B^{(k-s+m)}(k) \quad \text{for} \quad k \geq s$$  \hspace{1cm} (3.3)

- (ii) For $n \geq 2$
  $$B^{(k)}(k) = M^{(k)}(k) \quad \text{for} \quad 0 \leq k \leq 2s - 1$$
  $$B^{(k)}(k) = \sum_{p=0}^{s-1} A^{(p)}(k) B^{(k-s+p)}(k) \quad \text{for} \quad s \leq k \leq sn - 1$$
  $$B^{(k)}(k) = - \sum_{m=0}^{sn-1} a_m(k) B^{(k-sn+m)}(k) \quad \text{for} \quad k \geq sn$$  \hspace{1cm} (3.4)

We see that the first $2s$ spectral moments are conserved. This is true for any element of the matrix $M^{(k)}(k)$, and therefore for all the fields $\psi_l [1 \leq l \leq n]$. In Appendix B we show that the spectral moments for $k \geq s$ can be written as

$$M^{(k)}(k) = \tilde{M}^{(k)}(k) + \delta M^{(k)}(k)$$  \hspace{1cm} (3.5)

where $\tilde{M}^{(k)}(k)$ and $\delta M^{(k)}(k)$ satisfy the recurrence relations

$$\tilde{M}^{(k)}(k) = \sum_{m=0}^{s-1} A^{(m)}(k) \tilde{M}^{(m+k-s)}(k) \quad \text{for} \quad k \geq s$$  \hspace{1cm} (3.6)

$$\delta M^{(k)}(k) = F^{(k-s)}(k) + \sum_{m=0}^{s-1} A^{(m)}(k) \delta M^{(m+k-s)}(k) \quad \text{for} \quad k \geq 2s$$  \hspace{1cm} (3.7)

with

$$F^{(k)}(k) = F.T. \left< \left[ \frac{\partial}{\partial t} \right]^k \delta j^{(s)}(x), \psi^{\dagger}(y) \right>_{\pm t_x = t_y}$$  \hspace{1cm} (3.8)

Use of time translational invariance and of the constrains (2.5) leads to

$$\delta M^{(k)}(k) = 0 \quad \text{for} \quad 0 \leq k \leq 2s - 1$$  \hspace{1cm} (3.9)
We note that $\tilde{M}^{(k)}(k)$ represents the spectral moment in the approximation considered, when the higher-order field $\delta_j^{(s)}$ is neglected. Furthermore, in Appendix B we show that the matrices $\tilde{M}^{(k)}(k)$ satisfy the recurrence relation

$$\tilde{M}^{(k)}(k) = -\sum_{m=0}^{s-1} a_m(k)\tilde{M}^{(k-m)}(k) \quad [\text{for } k \geq sn]$$  \hspace{1cm} (3.10)

By comparing (3.10) and (3.3)-(3.4) we see that for $k \geq sn$, $\tilde{M}^{(k)}(k)$ and $B^{(k)}(k)$ satisfy the same recurrence relation. But, we have shown that for $k \leq sn - 1$

$$B^{(k)}(k) = \tilde{M}^{(k)}(k)$$  \hspace{1cm} (3.11)

therefore the two matrices coincide at any order. This shows an internal consistency of the approximation, in the sense that the exact relation (1.4) is preserved at any order when calculated on the same ground. The above discussion can be summarized in the Theorem III stated in Section I.

IV. SPECIAL CHOICE OF THE BASIS

Up to now the basic set $\psi(x)$ has not been specified. Let us consider the case where the $n$ fields are chosen as

$$\left(i \frac{\partial}{\partial t}\right)^s \psi_1(x) = \sum_{p=1}^{l+1} \hat{A}^{(0)}_{l,p}(-i\nabla)\psi_1(x) + \sum_{m=1}^{l} \sum_{p=1}^{l} A^{(m)}_{l,p}(-i\nabla) \left(i \frac{\partial}{\partial t}\right)^m \psi_1(x)$$  \hspace{1cm} (4.1)

where $\hat{A}^{(0)}$ is a matrix of $n$ rows and $n + 1$ columns and $A^{(m)} [1 \leq m \leq s - 1]$ are $n \times n$ matrices. That is, we choose the first field $\psi_1(x)$; then, all the other fields $\psi_l [2 \leq l \leq n]$ are determined by means of the equation

$$\hat{A}^{(0)}_{l-1,l}(-i\nabla)\psi_1(x) = \left(i \frac{\partial}{\partial t}\right)^s \psi_1(x) - \sum_{p=1}^{l-1} \hat{A}^{(0)}_{l-1,p}(-i\nabla) \left(i \frac{\partial}{\partial t}\right)^m \psi_1(x)$$  \hspace{1cm} (4.2)

It is clear that our basis satisfies the equation of motion

$$\left(i \frac{\partial}{\partial t}\right)^s \psi(x) = \sum_{m=0}^{s-1} A^{(m)}(-i\nabla) \left(i \frac{\partial}{\partial t}\right)^m \psi(x) + \delta_j^{(s)}(x)$$  \hspace{1cm} (4.3)

where $A^{(0)}$ is the square matrix obtained from $\hat{A}^{(0)}$ by taking the first $n$ rows and $n$ columns. The matrices $\hat{A}^{(0)}$, $A^{(m)} [1 \leq m \leq s - 1]$ and the vector $\delta_j^{(s)}$ have the structure

$$\hat{A}^{(0)}_{l,p} = 0 \quad [\text{for } 1 \leq l \leq n - 1 \text{ and } l + 2 \leq p \leq n + 1]$$

$$A^{(m)}_{l,p} = 0 \quad [\text{for } 1 \leq l \leq n - 1 \text{ and } l + 1 \leq p \leq n]$$  \hspace{1cm} (4.4)

$$\delta_j^{(s)} = 0 \quad [\text{for } l \leq n - 1]$$

We shall take the matrices $A^{(m)} [1 \leq m \leq s - 1]$ in such a way to satisfy the constrain (2.8). We have seen in Section 3 (cfr. Eq. (3.5)) that the spectral moments can be written as $M^{(k)}(k) = \tilde{M}^{(k)}(k) + \delta M^{(k)}(k)$. For the special choice of the basis considered in this Section, it is possible to show that the elements of the matrices $F^{(k)}(k)$, defined by Eq. (3.8), satisfy the following rules

$$F^{(k)}_{l,p}(k) = 0 \quad \text{for } l \leq n - 1 \quad [\text{for any value of } p \text{ and } k]$$

$$F^{(k)}_{n,l}(k) = 0 \quad \text{for } k \leq s(n - p + 1) - 1$$  \hspace{1cm} (4.5)

Then, the elements of the matrix $\delta M^{(k)}(k)$ satisfy the following rule

$$\delta M^{(k)}_{l,l}(k) = 0 \quad \text{for } k \leq 2s(n - l + 1) - 1$$  \hspace{1cm} (4.6)

By recalling the result $B^{(k)}(k) = \tilde{M}^{(k)}(k)$, we have thus proved the Theorem IV, stated in Section I.
V. SOME EXAMPLES

In order to illustrate the formulation let us consider some specific examples.

Case $n = 1$

The solution of the system (1.17) is

$$A^{(k)}(k) = \sum_{i=1}^{s} \left( \frac{1}{D(k)} \right)_{i+k+1,l} M^{(s+l-1)}(k) \quad [for \ 0 \leq k \leq s-1]$$ (5.1)

where $D(k)$ is the $s \times s$ matrix

$$D(k) = \begin{pmatrix} M^{(0)} & \cdots & M^{(s-1)} \\ \vdots & \ddots & \vdots \\ M^{(s-1)} & \cdots & M^{(2s-2)} \end{pmatrix}$$ (5.2)

Then, the retarded Green’s function is given by

$$G(k,\omega) = \sum_{i=1}^{s} \frac{\sigma^{(i)}(k)}{\omega - E_{i}(k) + \eta i}$$ (5.3)

The energy spectra $E_{i}(k)$ are determined as the roots of the equation (2.13), with characteristic coefficients $a_{m}(k) = -A^{(m)}(k)$. The spectral functions are given by (2.15) with

$$\lambda^{(k)}(k) = -\sum_{m=0}^{s-k-1} A^{(s-m)}(k) M^{(s-k-1-m)}(k) \quad [for \ 0 \leq k \leq s-1]$$ (5.4)

This case is exactly equivalent to SDA and the previous formulas give a complete solution to the problem of solving the system (1.2) for any value of $s$.

Case $n = 2$

For $n = 2$ we introduce the basic field

$$\psi(x) = \begin{pmatrix} \psi_{1}(x) \\ \psi_{2}(x) \end{pmatrix}$$ (5.5)

and consider the particular case where the second field is taken according to the special choice (4.1). By noting that the equation of motion for the field $\psi_{2}(x)$ is

$$\left( i \frac{\partial}{\partial t} \right)^{k} \psi_{2}(x) = -[A_{1,2}^{(0)}(-i\vec{v})]^{-1} \sum_{m=0}^{s} A_{1,2}^{(m)}(-i\vec{v}) \lambda^{(k+m)}(x)$$ (5.6)

the generalized spectral moments are given by

$$M_{11}^{(k)}(k) = m^{(k)}(k)$$

$$M_{12}^{(k)}(k) = M_{21}^{(k)}(k) = -[A_{1,2}^{(0)}(k)]^{-1} \sum_{m=0}^{s} A_{1,1}^{(m)}(k)m^{(k+m)}(k)$$

$$M_{22}^{(k)}(k) = [A_{1,2}^{(0)}(k)]^{-2} \sum_{m=0}^{s} \sum_{p=0}^{s} A_{1,1}^{(m)}(k)A_{1,1}^{(p)}(k)m^{(k+m+p)}(k)$$ (5.7)

We see that in order to calculate the matrices $A^{(m)}$ [0 $\leq m \leq s - 1$], we need to know the spectral moments $m^{(k)}(k)$, relative to the first field $\psi_{1}(x)$, for $0 \leq k \leq 4s - 1$. Lengthy calculations show that the equation (2.6) can be expressed as

$$\sum_{m=0}^{2s} a_{m}(k)m^{(k+m)}(k) = 0 \quad [for \ 0 \leq k \leq 2s - 1]$$ (5.8)

This result shows that the case ($n = 2, s$) is equivalent to the case ($n = 1, 2s$), when the special choice (4.1) is considered. We can make the conjecture that for this particular choice the formulation depends only the product $sn$. That is, two approximations characterized by the pairs $(n_{1}, s_{1})$ and $(n_{2}, s_{2})$ are equivalent if $n_{1}s_{1} = n_{2}s_{2}$. 
VI. THE HUBBARD MODEL

As an application of the formulation let us consider the Hubbard model, described by the following Hamiltonian

\[ H = \sum_{ij} (t_{ij} - \mu \delta_{ij}) c^\dagger(i) c(j) + U \sum_i n^\dagger(i) n(i) \]  

(6.1)

The first term is the kinetic term which describes the motion of the electrons among the sites of the Bravais lattice described by the vectors \( \mathbf{R}_i \) \( [i = (\mathbf{R}_i, t)] \); \( c(i), c^\dagger(i) \) are annihilation and creation operators of electrons at site \( \mathbf{R}_i \) in the spinor notation

\[ c = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}, \quad c^\dagger = \begin{pmatrix} c_\uparrow^\dagger \\ c_\downarrow^\dagger \end{pmatrix} \]  

(6.2)

and satisfy canonical anti-commutation relations:

\[ \{ c_\sigma(i), c^\dagger_{\sigma'}(j) \} = \delta_{\sigma,\sigma'} \delta_{i,j}, \quad \{ c_\sigma(i), c_{\sigma'}(j) \} = \{ c^\dagger_\sigma(i), c^\dagger_{\sigma'}(j) \} = 0 \]  

(6.3)

\( \mu \) is the chemical potential. \( t_{ij} \) denotes the transfer integral and describes hopping between different sites. We fix the scale of the energy in such a way that \( t_{ii} = 0 \). In momentum space the hopping matrix can be written as

\[ t_{ij} = \frac{1}{N} \sum_k e^{i \mathbf{k}(\mathbf{R}_i - \mathbf{R}_j)} t(\mathbf{k}) = \frac{\Omega}{(2\pi)^d} \int d^d k e^{i \mathbf{k}(\mathbf{R}_i - \mathbf{R}_j)} t(\mathbf{k}) \]  

(6.4)

where \( N \) is the number of the sites, \( d \) is the dimensionality of the system, \( \Omega \) is the volume of the unit cell in the coordinate space and \( \Omega_B \) is the volume of the Brillouin zone. In addition to the band energy the model contains a term approximating the interaction among the electrons. In the simplest form of the Hubbard model the interaction is between electrons of opposite spin on the same lattice site; the strength of the interaction is described by the parameter \( U \). \( n_{\sigma}(i) = c^\dagger_\sigma(i) c_\sigma(i) \) is the charge density of the electrons with spin \( \sigma \). The equation of motion for the electron field is

\[ i \frac{\partial}{\partial t} c(i) = b(-i \nabla) c(i) + U \eta(i) \]  

(6.5)

where

\[ b(-i \nabla) = -\mu + t(-i \nabla) \]

\[ \eta(i) = c(i) c(i) \]  

(6.6)

In the framework of the \( sn \)-pole approximation we shall consider the case \( n = 2, s = 2 \). Let us take as basis the two component field

\[ \psi(i) = \begin{pmatrix} \psi_1(i) \\ \psi_2(i) \end{pmatrix} \]  

(6.7)

and let us consider two possible choices.

I Choice

\[ \psi_1(i) = c(i) \]

\[ \psi_2(i) = -[A_{1,2}^{(0)}(-i \nabla)]^{-1} \sum_{m=0}^2 A_{1,1}^{(m)}(-i \nabla) (i \frac{\partial}{\partial t})^m c(i) \]  

(6.8)

This field satisfies the equation of motion

\[ \left( i \frac{\partial}{\partial t} \right)^2 \psi(i) = \sum_{m=0}^1 A^{(m)}(-i \nabla) \left( i \frac{\partial}{\partial t} \right)^m \psi(i) + \delta j(i) \]  

(6.9)

with the constrain

\[ M^{(2+k)}(k) - \sum_{m=0}^1 A^{(m)}(k) M^{(m+k)}(k) = 0 \quad [0 \leq k \leq 1] \]  

(6.10)
As shown in Section 5, by taking this choice the formulation \((n=2, s=2)\) is equivalent to the case \((n=1, s=4)\). Then, the retarded Green’s function has the expression

\[
G(k, \omega) = \sum_{i=1}^{4} \frac{\sigma^{(i)}(k)}{\omega - E^{(i)}(k) + i\eta}
\]  

(6.11)

The energy spectra are the roots of the equation

\[
\sum_{m=0}^{4} a_m(k)\omega^m = 0
\]  

(6.12)

with coefficients determined by the equations

\[
\sum_{m=0}^{4} a_m(k) m^{(k+m)}(k) = 0 \quad [0 \leq k \leq 3]
\]  

(6.13)

The spectral functions are

\[
\sigma^{(i)}(k) = \frac{1}{b^{(i)}(k)} \sum_{m=0}^{3} m^{(m)}(k) \sum_{p=0}^{m} E^{p-m-1}(k)a_p(k)
\]  

(6.14)

We need to calculate the first 8 electronic spectral moments

\[
m^{(k)}(k) = \text{F.T.} <\left(\frac{i}{\partial t_i}\right)^k c(i), c^\dagger(j)\text{E.T.}> \quad (6.15)
\]

The calculation of the electronic spectral moments can be done by means of a recurrence relation, as shown in Appendix D.

II Choice

\[
\psi(i) = \begin{pmatrix} \xi(i) \\ \eta(i) \end{pmatrix} \quad \xi(i) = [1 - c^\dagger(i)c(i)]c(i) \\
\eta(i) = c^\dagger(i)c(i)c(i)
\]  

(6.16)

This field satisfies the equation of motion

\[
\left(\frac{i}{\partial t}\right)^2 \psi(x) - \sum_{m=0}^{1} A^{(m)}(-i\nabla) \left(\frac{i}{\partial t}\right)^m \psi(x) = \delta j(x)
\]  

(6.17)

with the constrain

\[
M^{(2+k)}(k) - \sum_{m=0}^{1} A^{(m)}(k)M^{(m+k)}(k) = 0 \quad [0 \leq k \leq 1]
\]  

(6.18)

The spectral moments \(M^{(k)}(k) = \text{F.T.} <\{i\partial\psi(i)/\partial t_i\}^k, \psi(j)\}_{t_i=t_j}\) can be expressed in terms of the electronic moments (6.15) by means of the following relations

\[
M_{11}^{(k)}(k) = m^{(k)}(k) - 2d^{(k)}(k) + g^{(k)}(k) \\
M_{12}^{(k)}(k) = M_{21}^{(k)}(k) = d^{(k)}(k) - g^{(k)}(k) \\
M_{22}^{(k)}(k) = g^{(k)}(k)
\]  

(6.19)

where

\[
d^{(k)}(k) = U^{-1}[m^{(k+1)}(k) - b(k)m^{(k)}(k)] \\
g^{(k)}(k) = U^{-2}[2b(k)m^{(k+1)}(k) + b^2(k)m^{(k)}(k)]
\]  

(6.20)

We need to calculate the first 6 electronic spectral moments. The retarded Green’s function has a four-pole structure with energy spectra and spectral functions calculated as shown in Section 5. As stated in Section II, we note that
the approximation has been made on the equation of motion for second-order time derivative. No approximation has been done on the equations for the first-order time derivatives. In particular, the following equations

\[ i \frac{\partial}{\partial t} \xi(i) = b(-i\nabla)\xi(i) + t(-i\nabla)\eta(i) - \pi(i) \]
\[ i \frac{\partial}{\partial t} \eta(i) = (U - \mu)\eta(i) + \pi(i) \]

(6.21)

where

\[ \pi(i) = \frac{1}{2} \sigma^\mu n_\mu(i)[t(-i\nabla)c(i)] + c(i)[t(-i\nabla)c^\dagger(i)]c(i) \]

(6.22)

are treated exactly in this approximation.

VII. CONCLUSIONS

Among various analytical methods developed in the last decades to deal with highly correlated electronic systems, the technique of expanding the Green’s function in terms of a finite number of spectral moments has been rather successful in describing the properties of many correlated models. Along this line of thinking, the crucial point is the choice of the basis in terms of which the expansion is realized. A standard way is to choose the spectral moments of the original fields. However, in presence of strong correlation this choice may not provide the better basis for a perturbative approach and other basis may be considered. In particular, a set of composite field operators may be taken as the fundamental set in terms of which the Green’s function formulation is realized. In this process there are two ingredients that should be considered: the choice of the basis and the dynamics. In this work, we have presented a scheme of calculations where both aspects are taken into account. Firstly, a set of \( n \) composite basic fields is taken; the choice of this set is made on the basis of physical criteria: the role played by the interaction and by the value of the external parameters, the phase we are interested to describe, and so on. Secondly, the dynamics is considered up to the \( s \)-th order: we consider time derivatives of the fundamental fields of increasing order and the truncation procedure is applied to the equation of motion for the time derivative of order \( s \), with the advantage that all equations of motion up to order \( s - 1 \) are treated exactly. Two theorems have been proved about the conservation of the spectral moments in the approximation considered. Finally, we have applied the formulation to the case of the Hubbard models where some specific realizations have been considered. In Appendix D we have presented a recurrence relation for the calculation of the spectral moments relative to the Hubbard model.

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APPENDIX A: CONSERVATION OF THE FIRST 2\( s \) SPECTRAL MOMENTS

The spectral moments calculated in the \( sn \)-pole approximation have the expression

\[ B^{(k)}(k) = \sum_{i=1}^{sn} E^{(k)}_i(k)\sigma^{(i)}(k) \]

(A1)

By using the expression (2.15) for the spectral functions, we have

\[ B^{(k)}(k) = \sum_{m=0}^{sn-1} \lambda^{(m)}(k)P^{(k+m)}(k) \]

(A2)

where we have defined

\[ P^{(k)}(k) = \sum_{i=1}^{sn} \frac{E^{(k)}_i(k)}{b^{(i)}(k)} \]

(A3)
By observing that the energy spectra satisfy the equation
\[
\sum_{m=0}^{s^n} a_m(k) E_i^{m}(k) = 0 \quad \text{[for} \quad i = 1, \ldots, s^n] \quad \text{(A4)}
\]
the quantities \(P^{(k)}(k)\) satisfy the recurrence relation
\[
\sum_{m=0}^{s^n} a_m(k) P^{(k+m)}(k) = 0 \quad \text{[for} \quad k \geq 0] \quad \text{(A5)}
\]
By simple algebraic relations we can show that
\[
P^{(k)}(k) = \begin{cases} 
0 & \text{for} \quad 0 \leq k \leq s^n - 2 \\
1 & \text{for} \quad k = s^n - 1 
\end{cases} \quad \text{(A6)}
\]
Then, Eq. (A2) can be written as
\[
B^{(k)}(k) = \sum_{m=s^n-1-k}^{s^n-1} \lambda^{(m)}(k) P^{(k+m)}(k) \quad \text{[for} \quad 0 \leq k \leq s^n - 1] \quad \text{(A7)}
\]
When \(0 \leq k \leq 2s - 1\) we have always \(s^n - 1 - k \geq s^n - 2s\); then, we can use Eq. (2.19) to obtain
\[
B^{(k)}(k) = \sum_{m=s^n-1-k}^{s^n-1} \sum_{p=0}^{s^n-1-m} M^{(s^n-m-1-p)}(k) a_{s^n-p}(k) P^{(k+m)}(k) \\
= \sum_{m=0}^{s^n} M^{(m)}(k) \sum_{p=s^n-k+m}^{s^n} a_p(k) P^{(p+k-1-m)}(k) \quad \text{(A8)}
\]
where Eq. (A6) has been used. Then, we have the result that in the present approximation the first \(2s\) spectral moments are conserved. In closing this appendix we note that from the above discussion the following relations follow
(i) For \(n = 1\)
\[
B^{(k)}(k) = M^{(k)}(k) \quad \text{for} \quad 0 \leq k \leq 2s - 1 \\
B^{(k)}(k) = - \sum_{m=0}^{s^n-1} a_m(k) B^{(k-s+m)}(k) \quad \text{for} \quad k \geq s \quad \text{(A9)}
\]
(ii) For \(n \geq 2\)
\[
B^{(k)}(k) = M^{(k)}(k) \quad \text{for} \quad 0 \leq k \leq 2s - 1 \\
B^{(k)}(k) = \sum_{p=0}^{s^n-1} A^{(p)}(k) B^{(k-s+p)}(k) \quad \text{for} \quad s \leq k \leq s^n - 1 \quad \text{(A10)}
\]
\[
B^{(k)}(k) = - \sum_{m=0}^{s^n-1} a_m(k) B^{(k-s+n+m)}(k) \quad \text{for} \quad k \geq s^n
\]
\[\text{APPENDIX B:}\]

From the field equations (2.4) we have for \(k \geq s\)
\[
\left( i \frac{\partial}{\partial t} \right)^k \psi(x) = \sum_{m=0}^{s^n-1} A^{(m)}(-i \nabla) \left( i \frac{\partial}{\partial t} \right)^{m+k-s} \psi(x) + \left( i \frac{\partial}{\partial t} \right)^{k-s} \delta^{(s)}(x) \quad \text{(B1)}
\]
Then, for \(k \geq s\) the exact spectral moments satisfy the equation
\[12\]
\[ M^{(k)}(k) = \sum_{m=0}^{s-1} A^{(m)}(k) M^{(m+k-s)}(k) + F^{(k-s)}(k) \]  

(B2)

where we defined

\[ F^{(k)}(k) = F.T. < \left( \frac{\partial}{\partial t} \right)^k \delta^{(s)} \left( \frac{x}{t} \right), \psi^\dagger(y) > \]  

(B3)

Use of time translational invariance and of the constrains (2.5) lead to

\[ F^{(k-s)}(k) = 0 \quad [for \quad s \leq k \leq 2s-1] \]  

(B4)

Then, the spectral moments for \( k \geq s \) can be written as

\[ M^{(k)}(k) = \tilde{M}^{(k)}(k) + \delta M^{(k)}(k) \]  

(B5)

where \( \tilde{M}^{(k)}(k) \) and \( \delta M^{(k)}(k) \) satisfy the recurrence relations

\[ \tilde{M}^{(k)}(k) = \sum_{m=0}^{s-1} A^{(m)}(k) \tilde{M}^{(m+k-s)}(k) \quad [for \quad k \geq s] \]  

(B6)

\[ \delta M^{(k)}(k) = F^{(k-s)}(k) + \sum_{m=0}^{s-1} A^{(m)}(k) \delta M^{(m+k-s)}(k) \quad [for \quad k \geq 2s] \]  

(B7)

with

\[ \delta M^{(k)}(k) = 0 \quad [for \quad 0 \leq k \leq 2s-1] \]  

(B8)

We note that \( \tilde{M}^{(k)}(k) \) represents the spectral moment in the approximation considered, when the higher-order field \( \delta^{(s)} \) is neglected. In Appendix C we prove the following:

**Theorem V** Given \( s \) matrices \( A^{(m)} \quad [0 \leq m \leq s-1] \) of rank \( n \), let us define the characteristic polynomial as

\[ D(\omega) = \text{Det}[-\omega^s + \sum_{m=0}^{s-1} A^{(m)} \omega^m] = \sum_{m=0}^{sn} \omega^m a_m \]  

(B9)

and let us consider the \( n \times n \) matrices \( X^{(m)} \quad [m \geq 0] \) determined by the following recurrence relation

\[ X^{(k)} = \sum_{m=0}^{s-1} A^{(m)} X^{(m+k-s)} \quad [for \quad k \geq s] \]  

(B10)

Then, the following relation holds

\[ \sum_{m=0}^{sn} X^{(k+m)} a_m = 0 \quad [for \quad k \geq 0] \]  

(B11)

for any choice of the matrices \( X^{(m)} \quad [0 \leq m \leq s-1] \).

By means of this theorem the matrices \( \tilde{M}^{(k)}(k) \) satisfy the recurrence relation

\[ \tilde{M}^{(k)}(k) = -\sum_{m=0}^{sn-1} a_m(k) \tilde{M}^{(k-sn+m)}(k) \quad [for \quad k \geq sn] \]  

(B12)
APPENDIX C: PROOF OF THEOREM V

Given $s$ matrices $A^{(m)}$ [$0 \leq m \leq s - 1$] of rank $n$, we consider the matrix

$$P(\omega) = \omega^s I - \sum_{m=0}^{s-1} A^{(m)} \omega^m \quad (C1)$$

where $\omega$ is a parameter, which can generally be complex. $I$ is the unit matrix in $n$ dimensions. Let us suppose that there exists the inverse matrix

$$Q(\omega) = [\omega^s I - \sum_{m=0}^{s-1} A^{(m)} \omega^m]^{-1} \quad (C2)$$

This matrix can be always written as a rational function with respect to the parameter $\omega$

$$Q(\omega) = \frac{B(\omega)}{\varphi(\omega)} \quad (C3)$$

The numerator $B(\omega)$ is a matrix whose elements are polynomials in $\omega$ of degree $s(n - 1)$ . The denominator $\varphi(\omega)$ is the characteristic polynomial of degree $sn$

$$\varphi(\omega) = \text{Det}[-\omega^s + \sum_{m=0}^{s-1} A^{(m)} \omega^m] = \sum_{m=0}^{sn} \omega^m a_m \quad (C4)$$

Since $Q(\omega)$ is the inverse matrix of $P(\omega)$ we have

$$[\omega^s I - \sum_{m=0}^{s-1} A^{(m)} \omega^m] Q(\omega) = I \quad (C5)$$

Use of Eq. (C3) leads to

$$[\omega^s I - \sum_{m=0}^{s-1} A^{(m)} \omega^m] B(\omega) = \varphi(\omega) I \quad (C6)$$

The two members of this identity are polynomials of degree $sn$ with matricial coefficients. In order that this identity be satisfied, the coefficients for each power $\omega^m$ must be equal. In this identity we can replace the variable $\omega^m$ with a matrix $X^{(m+k)}$ [$k \geq 0$]

$$\omega^m \rightarrow X^{(m+k)} \quad (C7)$$

If the matrices $X^{(m)}$ are chosen in such to satisfy the recurrence relation

$$X^{(s+k)} = \sum_{m=0}^{s-1} A^{(m)} X^{(m+k)} \quad [\text{for } k \geq 0] \quad (C8)$$

then the left hand in (C6) vanishes, and so the second:

$$\varphi(X) = \sum_{m=0}^{sn} a_m X^{k+m} = 0 \quad [\text{for } k \geq 0] \quad (C9)$$

for any choice of the matrices $X^{(m)}$ [$0 \leq m \leq s - 1$].
APPENDIX D: SPECTRAL MOMENTS FOR THE HUBBARD MODEL

We recall the definition of the electronic spectral moments

\[ m^{(k)}(k) = F.T. \langle j^{(k)}(i), c(j) \rangle > E.T. \]  \hspace{1cm} (D1)

with

\[ j^{(k)}(i) = \left( i \frac{\partial}{\partial t} \right)^k c(i) \]  \hspace{1cm} (D2)

In order to calculate the anticommutators, we shall proceed in the following way. From the Heisenberg equation (6.5) we obtain for \( k \geq 1 \)

\[ j^{(k)}(i) = b(-i\vec{\nabla})j^{(k-1)}(i) + U \left( i \frac{\partial}{\partial t} \right)^{k-1} \eta(i) \]  \hspace{1cm} (D3)

with the convention that \( j^{(0)}(i) = c(i) \). By recalling the definition of the Hubbard operator \( \eta(i) \) [cfr. Eq. (6.6)] we have

\[ \left( i \frac{\partial}{\partial t} \right)^k \eta(i) = \sum_{m=0}^{k} \left( \begin{array}{c} k \\ m \end{array} \right) \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) (-1)^lj_{\rho\beta}^{(l)}(i)j_{\rho\beta}^{(m-l)}(i)j_{\alpha\beta}^{(k-m)}(i) \]  \hspace{1cm} (D4)

The field equation (D3) can be written for \( k \geq 1 \) as

\[ j^{(k)}(i) = b(-i\vec{\nabla})j^{(k-1)}(i) + U \sum_{m=0}^{k-1} \left( \begin{array}{c} k-1 \\ m \end{array} \right) \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) (-1)^lj_{\rho\beta}^{(l)}(i)j_{\rho\beta}^{(m-l)}(i)j_{\alpha\beta}^{(k-1-m)}(i) \]  \hspace{1cm} (D5)

We have a recurrence relation which allows us to calculate the source \( j^{(k)}(i) \). Let us define the quantities

\[ f_{\alpha\beta}^{k}(i, j) = \{ j_{\alpha\beta}^{(k)}(i), c_{\beta}^{(j)}(j) \} E.T. \]
\[ g_{\alpha\beta}^{k}(i, j) = \{ j_{\alpha\beta}^{(k)}(i), c_{\beta}^{(j)}(j) \} E.T. \]  \hspace{1cm} (D6)

By means of the previous results, these quantities satisfy the following recurrence relations for \( k \geq 1 \)

\[ f_{\alpha\beta}^{k}(i, j) = b(-i\vec{\nabla})f_{\alpha\beta}^{k-1}(i, j) + U \sum_{m=0}^{k-1} \left( \begin{array}{c} k-1 \\ m \end{array} \right) \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) (-1)^l \left[ j_{\rho\beta}^{(l)}(i)j_{\rho\beta}^{(m-l)}(i)f_{\alpha\beta}^{k-1-m}(i, j) \right. \]
\[ \left. -j_{\rho\beta}^{(l)}(i)f_{\rho\beta}^{m-l}(i, j)j_{\alpha\beta}^{(k-1-m)}(i) + g_{\rho\beta}^{l}(i, j)f_{\rho\beta}^{m-l}(i, j)j_{\alpha\beta}^{(k-1-m)}(i) \right] \]  \hspace{1cm} (D7)

\[ g_{\alpha\beta}^{k}(i, j) = b(-i\vec{\nabla})g_{\alpha\beta}^{k-1}(i, j) + U \sum_{m=0}^{k-1} \left( \begin{array}{c} k-1 \\ m \end{array} \right) \sum_{l=0}^{m} \left( \begin{array}{c} m \\ l \end{array} \right) (-1)^l \left[ j_{\alpha\beta}^{(k-1-m)}(i)j_{\rho\beta}^{(l)}(i)f_{\rho\beta}^{m-l}(i, j) \right. \]
\[ \left. -j_{\alpha\beta}^{(k-1-m)}(i)g_{\rho\beta}^{m-l}(i, j)f_{\rho\beta}^{(l)}(i) + g_{\alpha\beta}^{k-1-m}(i, j)f_{\rho\beta}^{m-l}(i, j)j_{\rho\beta}^{(l)}(i) \right] \]  \hspace{1cm} (D8)

with

\[ f_{\alpha\beta}^{0}(i, j) = \delta_{ij} \delta_{\alpha\beta} \]
\[ g_{\alpha\beta}^{0}(i, j) = 0 \]  \hspace{1cm} (D9)

Then, the spectral moments are given by

\[ m^{(k)}(k) = b(k)m^{(k-1)}(k) + UL^{(k-1)}(k) \]  \hspace{1cm} [for \( k \geq 1 \)]  \hspace{1cm} (D10)
where

\[
L^{(k-1)}_{\alpha\beta}(k) = \sum_{m=0}^{k-1} \binom{k-1}{m} \sum_{l=0}^{m} \binom{m}{l} (-1)^l F.T. \left[ < j^{(l)}_\rho(i) j^{(m-l)}_\rho(i) f^{k-1-m}_{\alpha\beta}(i,j) > E.T. \right]
\]

with

\[
m^{(0)}(k) = 1
\]

\[\text{(D11)}\]

\[\text{(D12)}\]