A Comparison of Overconvergent Witt de-Rham Cohomology and Rigid Cohomology on Smooth Schemes

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Abstract

We generalize the functorial quasi-isomorphism in [DLZ11] from overconvergent Witt de-Rham cohomology to rigid cohomology on smooth varieties over a finite field $k$, dropping the quasi-projectiveness condition. We do so by constructing an étale hypercover for any smooth scheme $X$, refined at each level to be a disjoint union of open standard smooth subschemes of $X$. We then find, for large $N$, an $N$-truncated closed embedding into a simplicial smooth scheme over $W(k)$, which allows us to use the results of loc. cit at the simplicial level, and use cohomological descent to prove the comparison.

1 Introduction

Let $X$ be a smooth scheme over a perfect field $k$ of characteristic $p > 0$, and consider its overconvergent de Rham-Witt complex of Zariski sheaves $W^\dagger \Omega^\bullet_{X/k}$, which is defined in [DLZ11] (see Definition 1.1 and Theorem 1.8). One of the main results of loc. cit is that if $X$ is also quasi-projective, then there exists a natural quasi-isomorphism

$$R\Gamma_{\text{rig}}(X/K) \xrightarrow{\sim} R\Gamma(X, W^\dagger \Omega^\bullet_{X/k})_\mathbb{Q},$$

where $K = W(k) \otimes \mathbb{Q}$.

The main result of this paper is Theorem 10, where we drop the quasi-projectivity condition in the comparison. We outline the approach in [DLZ11] and the one used in our paper.

If $X = \text{Spec } A$, [DLZ11] consider pairs $(X, F)$ given by closed immersions $X = \text{Spec } A \hookrightarrow F = \text{Spec } \tilde{A}$ into $W(k)$-schemes, called special frames. To this, the authors associate dagger spaces (in the sense of [Gro00]) $[X^\dagger]_\wp$ functorially in $(X, F)$, which calculate $R\Gamma_{\text{rig}}(X/K)$:

$$R\Gamma_{\text{rig}}(X/K) \xrightarrow{\sim} R\Gamma([X^\dagger]_\wp, \Omega^\bullet_{X^\dagger}[X^\dagger]_\wp).$$

(1)

so using the specialization maps

$$sp_* : [X^\dagger]_\wp \rightarrow X$$

we have that $R\Gamma_{\text{rig}}(X/K) \cong R\Gamma(X, sp_* \Omega^\bullet_{[X^\dagger]_\wp}).$

They also form a quasi-isomorphism of Zariski sheaves on $X$,

$$sp_* \Omega^\bullet_{[X^\dagger]_\wp} \rightarrow W^\dagger \Omega^\bullet_{X/k} \otimes \mathbb{Q},$$

(2)

functorial in $(X, F)$.
For the general case, one could try to work locally by introducing an affine covering of $X$, and let $X_0$ be its disjoint union, fitting into a special frame $(X_0, F_0)$, and then through a 0-coskeleton work on some simplicial frame $(X_\bullet, F_\bullet)$. Unfortunately, by (1) and (2) below this requires proving vanishing of the higher cohomologies of $R^{\bullet, m}_{X_0/F_0}$, which is not known in general. In the case where $X$ is smooth and quasi-projective (though possibly not affine), one can take an open covering by a particular type of affine smooth schemes, standard smooth schemes, which may be lifted nicely over $W(k)$, which are all coming from localizations in a projective space. This gives a nice description of the intersections of such opens in coskel(X) (X_0), which allows them to prove the desired vanishing of higher cohomologies, and then complete the proof by means of cohomological descent.

For our case, when $X$ is not quasi-projective, we do not have a common projective space in which all our open affines are open. So instead of working with the 0-coskeleton, we refine it at each level, getting an étale hypercovering $X_\bullet/X$ so that each level $X_n$ is a disjoint union of affine standard smooth open subschemes of $X$, which we call a special hypercovering. This is done in Section 3.

In Section 4, this hypercovering needs to be embedded into a simplicial smooth scheme over $W(k)$ in order to form a simplicial special frame and dagger space on which to apply (2). We use Tsuzuki’s functor $\Gamma_W(-)$ introduced in [CT03] to form an $N$-truncated simplicial special frame $(X_{\leq N}, F_{\leq N})$, with $N$ large enough so that the $X_m$ for $m > N$ don’t contribute to the calculation of $R^\cdot_{rig}(X/K)$.

The comparison is then proven in in various steps:

- Prove the vanishing of $R^{\bullet, m}_{X_0/F_0}$ for $0 \leq n \leq N$ and $i > 0$: this is done using techniques from the proof of [DLZ11, Proposition 4.35], such as being able to replace the $F_n$ by some $F'_n$ étale over $F_n$ or equal to $F_n \times W(k) A^i_{W(k)}$ for some $r$ fitting into a special frame $(X_n, F'_n)$.
- Prove that the complex $R\Gamma(X_{\leq N}, R^{\bullet, m}_{X_0/F_0})$ calculates $R^\cdot_{rig}(X/K)$ for large enough $N$: this is motivated by [Nak12] and relies on the machinery of [CT03], such as vanishing of higher enough rigid cohomology groups of $X$, independence of the choices of rigid frames and cohomological descent methods.
- Prove that the isomorphism
  $$R^\cdot_{rig}(X/K) \cong R\Gamma(X_{\leq N}, R^{\bullet, m}_{X_0/F_0})$$
  in $D_+(K)$ is independent of choices made, and functorial in $X$: this is done by "refining" any two choices made to a common one.

## 2 Background

### 2.1 Rigid Cohomology

We refer to [CT03] for results and notations involving rigid cohomology. We just make the following changes of notation:

- Given a simplicial morphism of triples $w_\cdot = (\hat{w}_\cdot, \bar{w}_\cdot, \hat{w}_\cdot) : (Y_\cdot, \bar{Y}_\cdot, \bar{X}_\cdot) \rightarrow (X, \bar{X}, \bar{X})$, and a complex of sheaves $E_\bullet$ of coherent $\bar{w}_\cdot^{-1}j_*\mathcal{O}_{[\bar{X}_\cdot, \bar{X}_\cdot]}$-modules we set
  $$Rw_\cdot E := R\mathcal{C}^j((X, \bar{X}, \bar{X}), (Y_\cdot, \bar{Y}_\cdot, \bar{X}_\cdot); E_\bullet).$$

which is defined in Section 4.2. We do this change of notation to match with simplicial notation in [Con03], as both complexes are defined as the total complex associated to

\[
\begin{array}{ccccccc}
... & & & & & & \\
0 & \rightarrow & I_0 & \rightarrow & I_1 & \rightarrow & ... \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & I_0 & \rightarrow & I_1 & \rightarrow & ... \\
\uparrow & & \uparrow & & \uparrow & & \\
0 & & 0 & & & & ...
\end{array}
\]

where \(I_p^\bullet\) is an injective resolution of \(E_p^\bullet\), where the vertical maps come from maps in \(I_p^\bullet\), and the horizontal come from the simplicial structure.

We note that for a \(N\)-truncated simplicial map, we will have a similar complex, but with all columns being 0 after \(N\).

- Given an open immersion of \(X\) into a a proper scheme \(\overline{X}\) over \(k\), and a universally de Rham descendable hypercovering \((Y^\bullet, \overline{Y}^\bullet, Y^\bullet)^{\wedge}\) of \((X, \overline{X})\) (as in [CT03, Definition 10.1.3]) we will set

\[
R\Gamma_{\text{rig}}(X/K) = R\Gamma([\overline{Y}^\bullet, j^! \Omega^\bullet_{Y^\bullet/\overline{Y}^\bullet}])
\]

where the right hand side is seen as coming from the map into the triple \((\text{Spec } k, \text{Spec } k, \text{Spf } W)\).

This definition coincides with theirs since we are dealing with trivial coefficients.

We also recall the following definitions (see Definition 7.2.1., 7.2.2. and 11.3.1 in [CT03]):

**Definition.**

1) For a simplicial scheme \(Y^\bullet\), separated of finite type over a scheme \(X\), we say that \(Y^\bullet \rightarrow X\) is an étale (resp. proper) hypercovering if the canonical morphism

\[
Y_{n+1} \rightarrow \cosk_n^X (sk_n^X (Y^\bullet))_{n+1}
\]

is étale surjective (resp. proper surjective) for any \(n\).

2) For a simplicial pair of schemes \((Y^\bullet, \overline{Y}^\bullet)\) of schemes separated of finite type over some pair \((X, \overline{X})\), we say that \((Y^\bullet, \overline{Y}^\bullet) \rightarrow (X, \overline{X})\) is an étale-proper hypercovering if \(Y^\bullet \rightarrow X\) is an étale hypercovering and \(\overline{Y}^\bullet \rightarrow \overline{X}\) is a proper hypercovering.

3) For a simplicial triple of schemes \((Y^\bullet, \overline{Y}^\bullet, Y^\bullet)\) separated of finite type over some triple \((X, \overline{X}, X')\), we say that \((Y^\bullet, \overline{Y}^\bullet, Y^\bullet) \rightarrow (X, \overline{X}, X')\) is an étale-proper hypercovering if \((Y^\bullet, \overline{Y}^\bullet) \rightarrow (X, \overline{X})\) is one, and the natural maps

\[
\cosk_{n+1}^X (sk_{n+1}^X (Y^\bullet))_k \rightarrow \cosk_n^X (sk_n^X (Y^\bullet))_k
\]

are smooth around \(\cosk_{n+1}^X (sk_{n+1}^X (Y^\bullet))_k\) for any \(n\) and \(k\).

4) Given an étale hypercovering \(Y^\bullet \rightarrow X\), we say that \(V^\bullet\) is a refinement of \(Y^\bullet\) if it fits into a diagram

\[
\begin{array}{ccc}
V^\bullet & \rightarrow & Y^\bullet \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]
where $V_\bullet \to X$ is an étale hypercovering, and the induced morphisms
\[
V_{n+1} \to \cosk_X^n (sk_X^n (V_\bullet))_{n+1} \times_{\cosk_X^n (sk_X^n (Y_\bullet))_{n+1}} Y_{n+1}
\]
are étale surjective for each $n$.

5) Given an étale-proper hypercovering $(Y_\bullet, Y_\bullet) \to (X, X)$, we say that $(V_\bullet, V_\bullet)$ is a refinement of $(Y_\bullet, Y_\bullet)$ if it fits into a diagram of pairs
\[
(V_\bullet, V_\bullet) \longrightarrow (Y_\bullet, Y_\bullet) \quad \downarrow \quad (X, X)
\]
where $V_\bullet$ is a refinement of $Y_\bullet$ over $X$.

### 2.2 Special Frames and Dagger Spaces

The following is a summary of Section 4 of [DLZ11].

**Definition.** A special frame is a pair $(X, F)$ with a closed embedding $X \hookrightarrow F$, where $X$ and $F$ are smooth affine schemes over $k$ and $W(k)$ respectively.

Given a special frame $(X, F)$, we can choose an embedding $F \hookrightarrow \mathbb{A}^n_{W(k)}$ for some $n$, and in turn we have an open embedding $E := \mathbb{A}^n_W \hookrightarrow \mathbb{P}^n_{W(k)} =: P$. Let $Q = \mathbb{P}$ and $\overline{X}$ be the closures of $F$ and $X$ respectively in $P$, and let $\mathcal{F}$ and $Q$ be the $p$-adic completions of $F$ and $Q$ respectively. Then,
\[
X \hookrightarrow \overline{X} \hookrightarrow Q
\]
is a frame for rigid cohomology in the sense of Berthelot (i.e. we have an open immersion of $X$ into a proper scheme $\overline{X}$ over $k$, and a closed immersion $\overline{X} \hookrightarrow Q$ where $Q$ is smooth around $X$). So we may define the rigid cohomology of $X$ as
\[
R\Gamma_{\text{rig}}(X/K) = R\Gamma(\overline{X}|Q, j^! \Omega^\bullet_{\overline{X}|Q}),
\]
where $j$ is the inclusion $|X|_{Q} \hookrightarrow |\overline{X}|_{Q}$. Note also that $|X|_{Q} = |X|_{F}$.

The authors then give an explicit description of a fundamental system of strict neighborhoods of $|X|_{F}$ in $|\overline{X}|_{Q}$, which they use to give a dagger structure (in the sense of [Gro00]) on $|X|_{F}$, denoted by $|X|_{F}^\dagger$, along with a morphism
\[
sp_\ast : |X|_{F}^\dagger \to X
\]
which is independent of the choice of embedding of $F$ into affine and projective space. Thus, we have an association
\[
(X, F) \mapsto |X|_{F}^\dagger
\]
of special frames into dagger spaces, functorial in $(X, F)$.

By [Gro00, Theorem 5.1], this gives quasi-isomorphisms
\[
R\Gamma_{\text{rig}}(X/K) = R\Gamma(|\overline{X}|_Q, j^! \Omega^\bullet_{|\overline{X}|_Q}) \sim R\Gamma(|X|_{F}^\dagger, \Omega^\bullet_{|X|_{F}^\dagger}). \tag{3}
\]

To such a frame $(X, F)$, they also form in [DLZ11, (4.32)] a map
\[
sp_\ast \Omega^\bullet_{|X|_{F}^\dagger} \to W^\dagger \Omega^\bullet_{X/k} \otimes Q, \tag{4}
\]
which is a quasi-isomorphism of Zariski sheaves.
2.3 Standard Smooth Schemes

Definition. We call a ring $A$ a standard smooth algebra (over $k$) if $A$ can be represented in the form

$$A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_m),$$

where $m \leq n$ and the determinant

$$\det \left( \frac{\partial f_i}{\partial X_j} \right), \quad 1 \leq i, j \leq m$$

is a unit in $A$. The scheme $\text{Spec } A$ is then called a standard smooth scheme.

Such schemes are convenient to work with, since for a standard smooth algebra represented as $k[T_1, \ldots, T_n]/(f_1, \ldots, f_r)$, we may choose liftings $\tilde{f}_1, \ldots, \tilde{f}_r$ to $W[T_1, \ldots, T_n]$, and let $\tilde{A}$ be the localization of $W[T_1, \ldots, T_n]/(\tilde{f}_1, \ldots, \tilde{f}_r)$ with respect to $\det \left( \frac{\partial \tilde{f}_i}{\partial T_j} \right)$. Then, $\tilde{A}$ is a standard smooth algebra which lifts $A$ over $W$, which gives a special frame $(\text{Spec } A, \text{Spec } \tilde{A})$. We note that this may be done functorially in $A$; that is, given a homomorphism of standard smooth algebras $\varphi : A \rightarrow B$ with presentations

$$A \cong k[T_1, \ldots, T_n]/(f_1, \ldots, f_r), \quad B \cong k[S_1, \ldots, S_m]/(g_1, \ldots, g_s),$$

after choosing liftings $\tilde{f}_i$ to define $\tilde{A}$, we may chose the representation

$$B \cong k[S_1, \ldots, S_m, T_1, \ldots, T_n]/(g_1, \ldots, g_s, f_1, \ldots, f_r, T_1 - \alpha(T_1), \ldots, T_r - \alpha(T_r))$$

and then take liftings $\tilde{g}_j, \tilde{\alpha}_i$ over $g_j$ and $\alpha(T_i)$ respectively to form $\tilde{B}$.

Note also that for any such standard smooth scheme $F = \text{Spec } \tilde{A}$, we have an étale map

$$F \rightarrow k_W^n$$

for some $n$.

3 The hypercovering

We recall the definition of a split simplicial scheme from [Con03, Definition 4.9]

Definition. We say that a simplicial scheme $Y_\bullet$ is split if there exist subobjects $NY_j$ in each $Y_j$ such that the natural map

$$\bigcup_{\phi : [n] \rightarrow [m]} NY_\phi \rightarrow Y_n$$

is an isomorphism for every $n \geq 0$, where $NY_\phi := NY_m$ for a surjection $\phi : [n] \rightarrow [m]$, and the natural maps are given by the composition

$$NY_\phi \subset Y_m Y_\bullet(\phi) \rightarrow Y_n.$$

The truncated case is defined similarly.
We denote by $NY_{m,\phi}$ the image of $NY_{\phi} \subset Y_m$ under this isomorphism. Notice that this agrees with the definition in [CT03, Section 11.2] as for any epimorphism $\phi : [n] \rightarrow [m]$ we have a commutative map

\[
\begin{array}{ccc}
NY_{m,\phi} & \subset & Y_{m} \\
\downarrow & & \downarrow \\
NY_{n,\phi} & \subset & Y_{n}
\end{array}
\]

Next, by [Con03, Theorem 4.12], given any split $n$-truncated simplicial scheme $Y_{\leq n}/X$ with the splitting given by $\{NY_k\}_{0 \leq k \leq n}$, in order to extend it to a split $(n+1)$-truncated scheme $Y_{\leq n+1}/X$ it suffices to give a scheme $N$ and a morphism

\[\beta : N \rightarrow \cosk^X_{n}(Y_{\leq n})_{n+1}.\]

This coincides with the functor $\Omega^X_{n+1}(Y_{\leq n}, NY_0, \ldots, NY_{n+1})$ given in [CT03, Section 11.2].

**Proposition 1.** Given any étale hypercovering $Z_{\bullet} \rightarrow X$, with $Z_n$ being smooth schemes over $k$, there exists an étale hypercovering $Y_{\bullet} \rightarrow X$ refining $Z_{\bullet} \rightarrow X$ such that for any $n$, $Y_n$ is the disjoint union of affine standard smooth schemes giving a finite open covering of $Z_n$.

**Proof.** The proof is nearly identical to [CT03, Proposition 11.3.2], with the only difference being that when we form a finite affine Zariski covering of

\[\cosk^X_{n}(Y_{\leq n})_{n+1} \times \cosk^X_{n}(Z_{\leq n})_{n+1} Z_{n+1},\]

we require the covering to be by affine standard smooth schemes also.

**Definition.** We say $Y_{\bullet} \rightarrow X$ is a special hypercovering if $Y_{\bullet}$ is a split étale hypercovering of $X$, and each $Y_n$ is a disjoint union of affine standard smooth schemes which give an open covering of $X$.

We prove the existence and some functorial property of such hypercoverings, which will be useful to work on the comparison locally.

**Proposition 2.** Given a smooth scheme $X$:

i) There exists a special hypercovering $Y_{\bullet} \rightarrow X$.

ii) Given two special hypercoverings $Y_{\bullet}, Y'_{\bullet}/X$, there a third special hypercovering $Y''_{\bullet}/X$ refining them.

iii) Given a morphism $X \rightarrow X'$ of smooth schemes, there exist special hypercoverings $Y_{\bullet} \rightarrow X$ and $Y'_{\bullet} \rightarrow X'$ fitting in a commutative diagram

\[
\begin{array}{ccc}
Y_{\bullet} & \longrightarrow & Y'_{\bullet} \\
\downarrow & & \downarrow \\
X & \longrightarrow & X'
\end{array}
\]
Proof. Part i) follows immediately from Proposition 1 by taking the constant simplicial scheme $Z_\bullet = \cosk X_1(X)$ (so $Z_n = X$ for all $n$). For part ii), we just apply Proposition 1 with $Z_\bullet := Y_\bullet \times_X Y'_\bullet$, and for part iii) we find some special hypercovering $Y'_\bullet \to X'$, and then again use Proposition 1 with $Z_\bullet := Y'_\bullet \times_{X'} X$.

The following proposition will be used to find an open embedding of these special hypercoverings into proper hypercoverings of some compactification $\overline{X}$ of $X$ in a functorial manner:

**Proposition 3.** Given a split étale hypercovering $X_\bullet \to X$, and an open embedding $X \hookrightarrow \overline{X}$ into a proper $k$-scheme:

i) There exists a split étale-proper hypercovering $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ such that for each $n$, $X_n \to \overline{X}_n$ is an open embedding.

ii) Given a morphism of compactifications $(X, \overline{X}) \to (Y, \overline{Y})$, and a morphism of split étale-hypercoverings $X_\bullet \to Y_\bullet$ (over $X$ and $Y$) given by morphisms $NX_k \to NY_k$, we may form $\overline{X}_\bullet$ and $\overline{Y}_\bullet$ as in i), with a morphism fitting into the commutative diagram

\[
\begin{array}{ccc}
(X_\bullet, \overline{X}_\bullet) & \longrightarrow & (Y_\bullet, \overline{Y}_\bullet) \\
\downarrow & & \downarrow \\
(X, \overline{X}) & \longrightarrow & (Y, \overline{Y})
\end{array}
\]

Proof. Part i) is \cite[Proposition 11.7.3]{CT03}, but we still explain it. First note that given any compification $V \hookrightarrow \overline{V}$ of $k$-schemes, and a morphism $U \to V$, by Nagata’s compactification theorem for $U \to V$ we may choose a compactification $U \hookrightarrow \overline{U}$ fitting into a diagram

\[
\begin{array}{ccc}
U & \longrightarrow & \overline{U} \\
\downarrow & & \downarrow \\
V & \longrightarrow & \overline{V}
\end{array}
\]

where $\overline{U}$ is proper over $\overline{V}$ (and thus over $k$). In this case we will say that $\overline{U}$ is a compactification of $U$ over $(V, \overline{V})$.

Let $X_\bullet$ be giving by a splitting $\{NX_k\}$. We construct $\overline{X}_\bullet$ by a splitting at each level. First, we set $N\overline{X}_0 = \overline{X}_0$ to be a compactification of $NX_0 = X_0$ over $(X, \overline{X})$.

Next, having constructed a split $n$-truncated $\overline{X}_{\leq n}$ with a splitting at each level. First, we set $N\overline{X}_0 = \overline{X}_0$ to be a compactification of $NX_0 = X_0$ over $(X, \overline{X})$.

Next, having constructed a split $n$-truncated $\overline{X}_{\leq n}$ with a splitting at each level. First, we set $N\overline{X}_0 = \overline{X}_0$ to be a compactification of $NX_0 = X_0$ over $(X, \overline{X})$.

The above is an open immersion since all $X_k \hookrightarrow \overline{X}_k$ and $X \hookrightarrow \overline{X}$ are, and similarly the right hand side is proper. Then, letting $\overline{X}_{\leq n+1} = \Omega(\overline{X}_{\leq n+1}, NX_0, ..., N\overline{X}_n)$ we have an $n + 1$-truncated proper hypercovering of $\overline{X}$, with an open immersion coming from $\overline{X}_{\leq n+1}$.
For ii), we construct $\overline{Y}_n$ as in i). Then, we build $\overline{Y}_n$ similarly, except that at each $n$, we take a compactification $N\overline{X}_{n+1}$ of $NX_{n+1}$ over
\[
\cosk_n(X_n)_{n+1} \times \cosk_n(Y_n)_{n+1} \hookrightarrow \cosk_n(\overline{X}_n)_{n+1} \times \cosk_n(\overline{Y}_n)_{n+1}.
\]
This all fits into a commutative diagram
\[
\begin{array}{c}
\xymatrix{
NX_{n+1} \\
\cosk_n(X_n)_{n+1} \\
\cosk_n(X_n)_{n+1} \\
\cosk_n(\overline{X}_n)_{n+1} \\
N\overline{X}_{n+1}
}
\end{array}
\]
where all horizontal morphisms are open immersions, and the vertical morphisms on the right are all proper. This gives us the desired functoriality.

4 The simplicial special frame

We explain the construction of the Tsuzuki functor $\Gamma_N^C(\dash)$, introduced in [CT03, Section 11.2]. Given a category $\mathcal{C}$ with finite inverse limits, a non-negative integer $N$, and an object $Z$, we construct a $N$-truncated simplicial object $\Gamma_N^C(Z)$ in Simp$_{\leq N}(\mathcal{C})$ as follows:

Set
\[
\Gamma_N^C(Z)_m := \prod_{\phi:[N] \to [m]} Z_\phi
\]
where $Z_\phi = Z$. To define the simplicial maps, given $\alpha : [m'] \to [m]$, we define $\Gamma_\alpha : \Gamma_N^C(Z)_m \to \Gamma_N^C(Z)_{m'}$ by
\[
\Gamma_\alpha : (c_\phi)_{\phi:[N] \to [m]} \mapsto (d_\psi)_{\psi:[N] \to [m']}
\]
where $d_\psi := c_{\alpha \phi \psi}$.

Given any $Y_{\leq N}$ in Simp$_{\leq N}(\mathcal{C})$, and a morphism $f : Y_N \to Z$ in $\mathcal{C}$, we construct a morphism
\[
Y_{\leq N} \to \Gamma_N^C(Z)_{\leq N}
\]
by the commutative diagram
\[
\begin{array}{c}
\xymatrix{
Y_m \\
Y_N \\
Y_N \ar[r]^{f} \\
Z = Z_\phi
}
\end{array}
\]
for any $m$ and $\phi : [N] \to [m]$, where $p_\phi$ is just the projection on to the $\phi : [N] \to [m]$ factor.

Letting $\mathcal{C}$ be the category of schemes over $\text{Spec}(W(k))$, and $Y_{\leq N}$ some simplicial scheme over $k$ or $W(k)$, we have the following:

**Lemma 4.** If $f : Y_N \to W$ is a closed immersion over $W(k)$, and $Y_{\leq N}$ and $Z$ are separated, then the induced morphism
\[
Y_{\leq N} \to \Gamma_N^W(Z)_{\leq N}
\]
is a closed immersion of $N$-truncated schemes.
Proof. For any \(0 \leq m \leq N\), consider any face morphism \(d : Y_m \to Y_N\) (with \(d = id_{Y_N}\) if \(m = N\)), and a corresponding degeneracy map \(s : Y_N \to Y_m\) which is a section to \(d\). Then, we have

\[
\begin{array}{ccc}
Y_N & \longrightarrow & Y_m \\
\downarrow & & \downarrow \\
W(k) & & \\
\end{array}
\]

where the vertical and diagonal maps are separated. This shows that \(d\) is also separated. Then, by the commutative diagram

\[
\begin{array}{ccc}
Y_m & \longrightarrow & Y_N \\
\downarrow \scriptstyle{s} & & \downarrow \scriptstyle{d} \\
Y_m & \longrightarrow & \\
\end{array}
\]

we see that \(s\) is a closed immersion. Finally, by the definition of the map \(g_m : Y_m \to \Gamma^{W(k)}_N(Z)_m\), we have a commutative diagram

\[
\begin{array}{ccc}
Y_m & \xrightarrow{g_m} & \Gamma^{W(k)}_N(Z)_m \\
\downarrow \scriptstyle{s} & & \downarrow \scriptstyle{pr_s} \\
Y_N & \xrightarrow{f} & Z \\
\end{array}
\]

which shows that \(g_m\) is in fact a closed immersion.

Thus, for a smooth scheme \(X\) over \(k\), having formed a special hypercovering \(Y_\cdot \to X\) as in the previous section, for any \(N \geq 0\), we may write

\[Y_N = \bigsqcup_{\phi : [N] \to [m]} NY_{N,\phi}\]

where for \(\phi : [N] \to [m]\), \(NY_{N,\phi}\) is identified with \(NY_m\) under the map \(Y_m \xrightarrow{Y^\phi} Y_N\), and each \(NY_{N,\phi}\) is a disjoint product of a finite open covering of \(X\) by affine standard smooth schemes. Thus, it is also affine standard smooth, and as explained in the introduction we may lift them to standard smooth schemes \(NE_{N,\phi}\) over \(W(k)\). This gives a cartesian diagram

\[
\begin{array}{ccc}
Y_N & \longrightarrow & E_N := \bigsqcup_{\phi : [N] \to [m]} NE_{N,\phi} \\
\downarrow & & \downarrow \\
Spec k & \longrightarrow & Spec W(k). \\
\end{array}
\]

Then, we may form the \(N\)-truncated simplicial \(W(k)\)-scheme

\[F_{\cdot \leq N} := \Gamma^{W(k)}_N(E_N)_{\cdot \leq N}\]

and by Lemma 4 we get an \(N\)-truncated special frame

\[(Y_{\leq N}, F_{\leq N}). \quad (5)\]
5 The comparison theorem

We outline the formation of the comparison map, we want to work on a special hypercovering \( X_\bullet \to X \), and for some \( N \) to construct the \( N \)-truncated simplicial frame \((X_{\leq N}, F_{\leq N})\) as in (5), which will in turn give us \( N \)-truncated dagger spaces and rigid frames

\[
(X_{\leq N}, Y_{\leq N}, Q_{\leq N}).
\]

We will use the functorial quasi-isomorphisms

\[
\text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]} \to W^\dagger \Omega^\bullet_{X_{\leq N}, k} \otimes \mathbb{Q}
\]

from [DLZ11] to give us a quasi-isomorphism

\[
R\Gamma(X_{\leq N}, \text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]} ) \xrightarrow{\sim} R\Gamma(X_{\leq N}, W^\dagger \Omega^\bullet_{X_{\leq N}, k} \otimes \mathbb{Q}).
\]

We will then prove vanishing of the higher cohomologies of the \( R\text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]} \) for this particular special frames in Proposition 7 and show that

\[
R\Gamma(X_{\leq N}, R\text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]} )
\]

calculates \( R\Gamma_{rig}(X/K) \) for \( N \) large enough. This last part, and showing independence of choices made and functoriality are quite, and are motivated by [Nak12].

In the course of the proof, we will need some tools from [DLZ11] in order to compare special frames. The first result is proven in the proof of Proposition 4.35 and the second result is Proposition 4.37.

**Proposition 5.**

i) Given a map of special frames

\[
\begin{array}{ccc}
X & \longrightarrow & F' \\
\downarrow & & \downarrow \\
X & \longrightarrow & F
\end{array}
\]

with the right vertical map being étale, then we get a natural isomorphism of dagger spaces

\[
|X^\dagger_{\mathfrak{p}}, \cong |X^\dagger_{\mathfrak{p}}.
\]

ii) Given a special frame \((X, F \times \mathbb{A}^n_{W(k)})\) for any \( n \), such that the map \( X \to \mathbb{A}^n_{W(k)} \) factors through the origin, there exists a natural quasi-isomorphism

\[
R\text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]} \sim R\text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}_{\leq N}]}. 
\]

When proving the comparison, we will need to show vanishing of the higher cohomologies of \( R\text{sp}_* \Omega^\bullet_{Y_{\leq N}[\mathfrak{p}_{\leq N}]} \) for \( 0 \leq m \leq N \), where \((Y_m, F_m)\) are the special frames constructed in sections 3 and 4. The above proposition will allow us to reduce it to the following theorem, which follows from the proof of [Ber97b, Theorem 1.10]:

**Proposition 6.** Given a special frame \((X, F)\), where \( F \) is a lifting of \( X \) over \( W(k) \), then

\[
R^i \text{sp}_* \Omega^\bullet_{X_{\leq N}[\mathfrak{p}]} = 0 \text{ for } i > 0.
\]
We now prove a key ingredient of the comparison theorem:

**Proposition 7.** Given an $N$-truncated simplicial frame $(Y_{\bullet \leq N}, F_{\bullet \leq N})$ as in (5), for $0 \leq m \leq N$ and $i > 0$,

$$R^i sp_\bullet \Omega^\bullet Y_{m[\sigma_m]} = 0.$$

**Proof.** Pick any $0 \leq m \leq N$. By splitness of $Y_\bullet$, we may write

$$Y_m = \bigsqcup_{\phi: [N] \to [m]} NY_{m, \phi}.$$

Fix some degeneracy map $\sigma: [N] \to [m]$. Then, by construction of $\Gamma_N W(k)(-)$, we have a commutative diagram

$$\begin{array}{ccc}
Y_m & \longrightarrow & F_m = \prod_{\alpha: [N] \to [m]} E_\alpha \\
Y_\bullet(\sigma) \downarrow & & \downarrow p_\sigma \\
Y_N & \longrightarrow & E_N = E_\sigma
\end{array}$$

where $E_\sigma = E_N$ was defined in section 4, and $p_\sigma$ is the projection, and both horizontal maps and the left vertical map are closed immersions. This gives us a closed immersion

$$Y_m \hookrightarrow E_\sigma.$$

Let

$$F'_m := \prod_{\alpha: [N] \to [m], \alpha \neq \sigma} E_\alpha,$$

so $F_m = F'_m \times E_\sigma$. Then, since each of the $E_\alpha$ are standard smooth schemes over $W(k)$, so is their product, and we may get an étale morphism

$$F'_m \to \mathbb{A}^n_{W(k)}$$

for some $n$. Thus, considering the commutative diagram

$$\begin{array}{ccc}
Y_m & \longrightarrow & F'_m \times E_\sigma \\
& \downarrow & \downarrow \\
Y_m & \longrightarrow & \mathbb{A}^n_{W(k)} \times E_\sigma
\end{array}$$

where the right vertical morphism is étale, using Proposition 5.i) we may reduce to the case of the special frame $(Y_m, \mathbb{A}^n_{W(k)} \times E_\sigma)$. Furthermore, we may assume that the map $Y_m \to \mathbb{A}^n_{W(k)}$ factors through the origin. To see this, write $Y_m = \text{Spec } (A)$ and $E_\sigma = \text{Spec } B$, so $\mathbb{A}^n_{W(k)} \times E_\sigma = \text{Spec } B[T_1, \ldots, T_n]$. Then, since $B \to A$ is surjective (as $Y_m \to E_\sigma$ is a closed immersion), we may pick $b_1, \ldots, b_n \in B$ which map to the images of $T_1, \ldots, T_n$ respectively in $A$, and replace $T_i$ by $T'_i := T_i - b_i$, giving a special frame

$$(Y_m, \text{Spec } B[T'_1, \ldots, T'_n]) = (Y_m, \mathbb{A}^n_{W(k)} \times E_\sigma)$$

factoring through the origin. Thus, by Proposition 5.ii), we reduce the proof to the special frame $(Y_m, E_\sigma)$.

Now, since

$$[Y_m[\sigma_\sigma]] = \bigsqcup NY_{m, \phi}|_{[\sigma_\sigma]} \cong \bigsqcup NY_{m, \phi}|_{[\sigma_\sigma]}$$
we may reduce to studying the special frames \((NY_{m,\phi}, E_\sigma)\) for any \(\phi : [m] \to [k]\) and \(0 \leq k \leq m\). But notice that by the construction of the frame, for any \(\phi : [m] \to [k]\), we have a commutative diagram

\[
\begin{array}{c}
NY_{m,\phi} \subset Y_{m,\phi} \\
\cong \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ Y_{N,\phi \circ \sigma} \\
NY_{N,\phi \circ \sigma} \subset Y_N \xrightarrow{E_N} E_N \xrightarrow{\bigcup} NE_{N,\psi}
\end{array}
\]

where \(\alpha\) vary over all morphisms \(\psi : [N] \to [k']\) with \(0 \leq k' \leq N\), and the composite map \(NY_{m,\phi} \to E_N\) is the map giving the special frame. Thus, \(NY_{m,\phi}\) is isomorphic to \(NE_{N,\phi \circ \sigma} \subset E_N\), and therefore

\[
\text{sp}^{-1}(NY_{m,\phi}) = \text{sp}^{-1}(NY_{m,\phi})([\sigma]) = \text{sp}^{-1}(NY_{m,\phi})([N,\psi]),
\]

which reduces the proof to the case of the special frame \((NY_{m,\phi}, NE_{N,\phi \circ \sigma})\).

But by construction, \(NE_{N,\phi \circ \sigma}\) is a smooth lift of \(NY_{N,\phi \circ \sigma} \cong NY_{m,\phi}\) over \(W(k)\), and thus we can apply Proposition 6 to complete the proof.

We will need the following to deal with \(N\)-truncations, which basically says that for some large enough \(N\), we only need the \(N\)-skeleton in the calculations of cohomologies on simplicial objects (such as for rigid cohomology and overconvergent Witt de-Rham).

For a complex \(A^\bullet\) of \(K\) vector spaces, and any \(h\), consider the \(h\)-truncated complex

\[
\tau_{\leq h}(A^\bullet) = \begin{cases} 
A^i & \text{if } i < h \\
\ker(A^h \to A^{h+1}) & \text{if } i > h \\
0 & \text{else.}
\end{cases}
\]

For a double complex \(A^{\bullet \bullet}\), let \(\tau_{\leq h}^{(1)}(A^{\bullet \bullet}) := \tau_{\leq h}(A^{\bullet \bullet})\), and let \(s : C(K) \to K\) be the total complex map.

**Lemma 8.** [Nak12, Lemma 2.2] Consider a double complex \(A^{\bullet \bullet}\) of \(K\) vector spaces such that \(A^{p,q} = 0\) for \(p < 0\) or \(q < 0\). Given any

\[
N > \max\{i + (h - i + 1)(h - i + 2)/2 \mid 0 \leq i \leq h\} = (h + 1)(h + 2)/2,
\]

the natural maps \(s(\tau_{\leq h}^{(1)}(A^{\bullet \bullet})) \to s(A^{\bullet \bullet})\) induce a quasi-isomorphism

\[
\tau_{\leq h}(s(\tau_{\leq h}^{(1)}(A^{\bullet \bullet}))) \sim \tau_{\leq h}(s(A^{\bullet \bullet})).
\]

From this, and the formation of the spectral sequence for cohomology on simplicial objects, it follows for example that for some simplicial rigid frame \((Z_\bullet, \overline{Z}_\bullet, Z_\bullet), h\) and \(N\) as in (6), we get natural quasi-isomorphisms

\[
\tau_{\leq h}R\Gamma([\overline{Z}_\bullet \leq N | z_\bullet \leq N, j^! \Omega^\bullet_{\overline{Z}_\bullet \leq N | z_\bullet \leq N}] \sim \tau_{\leq h}R\Gamma([\overline{Z}_\bullet, j^! \Omega^\bullet]_{\leq N | z_\bullet})),
\]

and that for a smooth simplicial scheme \(X_\bullet\),

\[
\tau_{\leq h}R\Gamma(X_\bullet, W^j \Omega^\bullet_{X_\bullet \leq N}) \sim \tau_{\leq h}R\Gamma(X_\bullet \leq N, W^j \Omega^\bullet_{X_\bullet \leq N}).
\]

This is useful by the following theorem of vanishing of rigid cohomology:

**Theorem 9.** [Tsu04, Theorem 6.4.1] Given a scheme \(X\) over \(k\), there exists an integer \(c\) such that for \(i > c\), \(H^{12/21}_{rig}(X/K) = 0\).
We can now prove the main comparison theorem:

**Theorem 10.** Given a smooth scheme $X$ over $k$, there exists a functorial quasi-isomorphism

$$R\Gamma_{\text{rig}}(X/K) \sim R\Gamma(X, W^i\Omega^\bullet_{X/k}) \otimes \mathbb{Q}.$$ 

**Proof.** We form a special hypercovering

$$X_\bullet \to X.$$ 

For any $h$, we form an $N$-truncated special frame

$$(X_{\leq N}, F_{\leq N})$$

as explained in section 4. Then, for $0 \leq m \leq N$ and $i > 0$,

$$R^i\text{sp}_\ast \Omega^\bullet_{X_m/k} = 0$$

by Proposition 7.

Next, by [DLZ11] we have natural isomorphisms of Zariski sheaves

$$\text{sp}_\ast \Omega^\bullet_{X_m/k} \sim W^i\Omega^\bullet_{X_m/k} \otimes \mathbb{Q}$$

giving a $N$-truncated simplicial version

$$\text{sp}_\ast \Omega^\bullet_{X_{\leq N}} \sim W^i\Omega^\bullet_{X_{\leq N}} \otimes \mathbb{Q}.$$ 

Then, by vanishing of the higher $R^i\text{sp}_\ast$, and applying $R\Gamma(X, R\epsilon_\ast (-))$ to both sides we obtain

$$R\Gamma(X_{\leq N}, W^i\Omega^\bullet_{X_{\leq N}}) \otimes \mathbb{Q}. \quad (7)$$

Then, by Lemma 8 and $X_\bullet \to X$ being an étale hypercovering, we get

$$\tau_{\leq h}(R\Gamma(X_{\leq N}, W^i\Omega^\bullet_{X_{\leq N}})) \sim \tau_{\leq h}(R\Gamma(X, W^i\Omega^\bullet_{X/k})) \sim \tau_{\leq h}(R\Gamma(X, W^i\Omega^\bullet_{X/k})). \quad (8)$$

To complete the proof, we will need to show that the left hand side of (7) calculates $R\Gamma_{\text{rig}}(X/K)$ for $h = c$ as in Proposition 9, that the isomorphism in $D_+(K)$ with $R\Gamma_{\text{rig}}(X/K)$ is independent of choices made, and that it can be done functorially.

*The left hand side of (7) calculates $R\Gamma_{\text{rig}}(X/K)$:

We construct an isomorphism in $D_+(K)$. Firstly, from the $N$-truncated simplicial special frame $(Y_{\leq N}, F_{\leq N})$ we construct a simplicial rigid frame

$$(X_{\leq N}, Y_{\leq N}, Q_{\leq N})$$

as outlined in Section 2.2. Then, since the construction of the dagger spaces $|Y_m|_{\hat{F}_m}$ are functorial in $(Y_m, F_m)$, by a simplicial version of [Gro00, Theorem 5.1] we get quasi-isomorphisms

$$R\Gamma(|X_{\leq N}^{1/\hat{F}_m}, \Omega^\bullet_{X_{\leq N}}|_{\hat{F}_m}) \sim R\Gamma(|Y_{\leq N}^{1/\hat{Q}_m}, j^{1/\Omega^\bullet_{Y_{\leq N}}}|_{\hat{Q}_m}). \quad (9)$$

Now, we construct a complex which calculates $R\Gamma_{\text{rig}}(X/K)$ similar to the proof of [CT03, Theorem 11.1.1]. We take a compactification

$$X \hookrightarrow \overline{X}.$$
for some proper $k$-scheme $X$. Next, take a finite open affine covering of $X$, and let $U$ be its disjoint union, and $U := \overline{U} \times X$. Since $U$ is affine, we may take a closed embedding into some smooth formal $\mathcal{W}$-scheme $\mathcal{U}$. Set
\[
(U_\bullet, U_\bullet, \mathcal{U}_\bullet) := (\cosk^X_0(U), \cosk^X_0(\overline{U}), \cosk^\mathcal{W}_0(U)).
\]
This is a universal de Rham descendable hypercovering of $(X, X)$ (in the sense of [CT03]), and thus by independence of choice of compactification and such a hypercovering (see [CT03, Proposition 10.4.3, Corollary 10.5.4]) we may define
\[
R\Gamma_{\text{rig}}(X/K) := R\Gamma(\mathcal{U}_\bullet[\mathcal{U}_\bullet], j^! \Omega^\bullet_{\overline{U}/\mathcal{U}_\bullet}).
\]

Next, by Proposition 3 we may construct a proper hypercovering $X_\bullet$ over $X$ with an open embedding $X_\bullet \hookrightarrow \overline{X}_\bullet$, making $(X_\bullet, \overline{X}_\bullet)$ into an étale-proper hypercovering of $(X, \overline{X})$. By [CT03, Lemma 7.2.3] and both hypercoverings being preserved by base change, we get an étale-proper hypercoverings
\[
(\cosk^X_0(\sk^X_N(X_\bullet))) \times_X U, \cosk^\mathcal{W}_0(\sk^\mathcal{W}_{m,n})(\overline{U}_m) \to (U, \overline{U}). \tag{10}
\]
Then, using [CT03, Proposition 11.5.1] we may construct an étale-proper hypercovering $(V_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet)$ of $(U_\bullet, U_\bullet)$ such that $(V_\bullet_\bullet, \overline{V}_\bullet_\bullet)$ is a refinement of (10).

Let $(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet)$ be
\[
V_{m,n} = \cosk^\mathcal{W}_0(N_\bullet)_{m,n}, \overline{V}_{m,n} = \cosk^\mathcal{W}_0(N_\bullet)_{m,n}, \overline{V}_{m,n} = \cosk^\mathcal{W}_0(N_\bullet \times_w \mathcal{U}_\bullet)_{m,n}.
\]

Defining the simplicial maps as in [CT03, Proposition 11.5.4] we have
\[
(*) (V_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \text{ is an étale-proper hypercovering of } (U_\bullet, \overline{U}_m, \mathcal{U}_\bullet) \text{ for any } m;
\]
\[
(**) (V_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \text{ is a universally de Rham descendable hypercovering of } (Y_\bullet, \overline{Y}_n) \text{ for any } n.
\]

Considering the $(\infty, N)$-truncated version $(V_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet)$, we get a morphism
\[
R\Gamma_{\text{rig}}(X/K) = R\Gamma(\overline{U}_\bullet[\overline{U}_\bullet], j^! \Omega^\bullet_{\overline{U}/\overline{U}_\bullet}) \to R\Gamma(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \to R\Gamma(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \to R\Gamma(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet). \tag{11}
\]

We claim this induces becomes a quasi-isomorphism upon applying $\tau_{\leq h}$. To see this, compare the spectral sequences
\[
E_1^{pq} = H^q((\overline{U}_\bullet[\overline{U}_\bullet], j^! \Omega^\bullet_{\overline{U}/\overline{U}_\bullet}) \Rightarrow H_{rig}^{p+q}(X/K),
\]
\[
E_1^{pq} = H^q((\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \Rightarrow H_{rig}^{p+q}(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet),
\]
and notice that for $q \leq h$, by Lemma 8,
\[
H^q((\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \Rightarrow H_{rig}^{p+q}(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet) \Rightarrow H_{rig}^{p+q}(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet)
\]
where the last isomorphism comes from (*). So we get a quasi-isomorphism after applying $\tau_{\leq h}$ to (11).

By Lemma 8, we get
\[
\tau_{\leq h} R\Gamma_{\text{rig}}(X/K) \cong \tau_{\leq h}(R\Gamma(\overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet, \overline{V}_\bullet_\bullet)). \tag{12}
\]
Now, we want to compare the right hand side of (12) to \( \tau \leq h \) applied to the right hand side of (7). Let \( W_{m,n} := V_{m,n} \), let \( W_{m,n} \) be the scheme theoretical closure of

\[
V_{m,n} = V_{m,n} \times_{X_n} X_n \to V_{m,n} \times_k Y_n
\]

and \( R_{m,n} := V_{m,n} \times_{W} \mathcal{Q}_{n} \) with the obvious maps. This gives maps

\[
RG(\big| V_{*, \leq N}[V_{*, \leq N}, j^! \Omega_{* \leq N}[V_{*, \leq N}])
\]

\[
RG(\big| W_{*, \leq N}[R_{*, \leq N}, j^! \Omega_{* \leq N}[R_{*, \leq N}])
\]

\[
RG(\big| Y_{\leq N}[Q_{\leq N}, j^! \Omega_{\leq N}[Q_{\leq N}])
\]

which we claim become quasi-isomorphisms once we apply \( \tau \leq h \). To see this, consider the induced map on spectral sequences

\[
E_1^{pq} = H^q(\big| V_{*, p}[V_{*, p}, j^! \Omega_{*, p}[V_{*, p}]) \Rightarrow H^{p+q}(\big| V_{*, \leq N}[V_{*, \leq N}])
\]

\[
E_1^{pq} = H^q(\big| W_{*, p}[R_{*, p}, j^! \Omega_{*, p}[R_{*, p}]) \Rightarrow H^{p+q}(\big| W_{*, \leq N}[R_{*, \leq N}])
\]

\[
E_1^{pq} = H^q(\big| Y_{*, q}[Q_{*, q}, j^! \Omega_{*, q}[Q_{*, q}]) \Rightarrow H^{p+q}(\big| Y_{*, \leq N}[Q_{*, \leq N}])
\]

for \( p \leq N \).

The top left vertical arrow is an isomorphism by the independence of choice of compactification of the \( V_{m,p} \) in [CT03, Proposition 8.3.5], since for any \( p,m \), we have a morphism of rigid frames

\[
\begin{array}{ccc}
V_{m,p} & \to & W_{m,p} \\
\downarrow & & \downarrow \\
V_{m,p} & \to & V_{m,p}
\end{array}
\]

with middle and right vertical maps proper and smooth respectively.

Next, notice that both \( E_1^{pq} \) and \( \'E_1^{pq} \) calculate \( H_{rig}^q(Y_{p}/k) \) by (**), which is finitely generated. Thus, the bottom left vertical map in (14) must also be an isomorphism.
Summarizing all of the above, we get isomorphisms in $D_+(K)$

$$
\tau_{\leq h}(R\Gamma_{\text{rig}}(X/K)) \sim (11) \tau_{\leq h}(R\Gamma([\overline{V}_{\bullet, \leq N[V_{\bullet, \leq N} \leq N}, j^!\Omega^*] ) \sim (13) \tau_{\leq h}(R\Gamma([\overline{W}_{\bullet, \leq N[X_{\bullet, \leq N} \leq N}, j^!\Omega^*] ) \sim (13)

\tau_{\leq h}(R\Gamma(X, W^!\Omega^*_{X/k})) \sim (8) \tau_{\leq h}(R\Gamma(X, W^!\Omega^*_{X/k})) \sim (8) \tau_{\leq h}(R\Gamma(Y_{\bullet, \leq N}, W^!\Omega^*_{X_{\bullet, \leq N/k}})) \sim (15)

\text{where we have omitted subscripts in the } j^!\Omega^* \text{ and } \Omega^*.

By Theorem 9, there exists a $c$ such that $H^i_{\text{rig}}(X/K) = 0$ for $i > c$. Varying $h$ (and $N$) we see that

$$
\tau_{\leq c}(R\Gamma_{\text{rig}}(X/K)) \sim R\Gamma_{\text{rig}}(X/K), \quad \tau_{\leq c}(R\Gamma(X, W^!\Omega^*_{X/k})) \sim R\Gamma(X, W^!\Omega^*_{X/k})
$$

so we may set $h = c$ in (15) and drop the truncation terms, giving us an isomorphism in $D_+(K)$

$$R\Gamma_{\text{rig}}(X/K) \cong R\Gamma(X, W^!\Omega^*_{X/k}).$$

**Independence of choices:**

We must prove independence of the choices of the special hypercovering $X_{\bullet}$, $c$ as in Theorem 9, $N$ satisfying (6) for $c$, lifting $E_N$ of $X_N$ over $W(k)$ and its immersion into affine space and projective space $A_{W(k)}^r$ and $P = \mathbb{P}^r_{W(k)}$, the compactification $\overline{X}$, the Zariski covering $U$ of $\overline{X}$ and its closed immersion into $\mathcal{U}$, and the refinement $(V_{\bullet}, \overline{V}_{\bullet}, V_{\bullet})$ of $(X_{\bullet} \times_X U, \overline{X}_{\bullet} \times_X \overline{U}, \overline{U})$ over $(U, \overline{U})$.

1. **Independence of $X_{\bullet}$, $E_N \hookrightarrow A_{W}^r \hookrightarrow P = \mathbb{P}^r_{W}$:**

Suppose we have two choices

$$(X^i_{\bullet}, E^i_N \hookrightarrow A^r_{W} \hookrightarrow P^i = \mathbb{P}^r_{W}, i = 1, 2),$$

with all the other choices the same.

By Proposition 2, there exists a special hypercovering $X^2_{\bullet} \rightarrow X$ refining $X^1_{\bullet}$ and $X^2_{\bullet}$. We can choose some lifting $E^2_N$ of $X^2_N$ over $W(k)$ fitting into the diagram of special frames

$$(X^1_N, E^1_N) \rightarrow (X^1_{N}, E^1_{N}) \rightarrow (X^2_N, E^2_N).$$

This will give us morphisms in the $N$-truncated simplicial rigid frames

$$(X^i_{\bullet, \leq N}, Y^i_{\bullet, \leq N}, Q^i_{\bullet, \leq N}) \rightarrow (X^i_{\bullet, \leq N}, Y^i_{\bullet, \leq N}, Q^i_{\bullet, \leq N})$$

for $i = 1, 2$ by functoriality of the $\Gamma^W_{N}(\cdot)$ functor.
Next, by Proposition 3.ii) (with the argument slightly modified to involve a triple fiber product) we may form a proper étale hypercovering \((X_\bullet, X^{12}_\bullet)\) of \((X, \overline{X})\) with maps to \((X_\bullet, \overline{X}_\bullet)\) for \(i = 1, 2\).

Then, by the proof of [CT03, Proposition 11.5.2] we may choose refinements \((V'_\bullet, V'_\bullet, V'_\bullet)\) and \((V''_\bullet, V''_\bullet, V''_\bullet)\) of \((U \times_X X^{12}, \overline{U} \times_X \overline{X}^{12})\) over \((U, \overline{U})\) fitting into diagrams

\[
(V'_\bullet, V'_\bullet, V'\bullet) \longrightarrow (V^1_\bullet, V^1_\bullet, V^1_\bullet) \\
(U \times_X \cosk^X_N(X^{12}_{\leq N}), \overline{U} \times_X \cosk^X_N(\overline{X}^{12}_{\leq N})) \longrightarrow (U \times_X \cosk^X_N(X^1_{\leq N}), \overline{U} \times_X \cosk^X_N(\overline{X}^1_{\leq N}))
\]

\[
(V''_\bullet, V''_\bullet, V''_\bullet) \longrightarrow (V^2_\bullet, V^2_\bullet, V^2_\bullet) \\
(U \times_X \cosk^X_N(X^{12}_{\leq N}), \overline{U} \times_X \cosk^X_N(\overline{X}^{12}_{\leq N})) \longrightarrow (U \times_X \cosk^X_N(X^2_{\leq N}), \overline{U} \times_X \cosk^X_N(\overline{X}^2_{\leq N}))
\]

where the vertical maps are only seen as morphisms of pairs. Taking \((V^{12}_\bullet, V^{12}_\bullet, V^{12}_\bullet)\) to be the fiber product of these two refinements, we get by [CT03, Proposition 11.5.1.(2)] that this is also a refinements of \((U \times_X X^{12}, \overline{U} \times_X \overline{X}^{12})\) over \((U, \overline{U})\), with maps to \((V^i_\bullet, V^i_\bullet, V^i_\bullet)\) for \(i = 1, 2\) compatible with the other maps.

Next, having taking closures \(\overline{W}_{m,n}^i\) of \(V_{m,n}^i\) in \(V_{m,n}^i \times_k Y_n^i\) with closed immersion into \(R_{m,n}^i := V_{m,n}^i \times_k Q_n^i\) for all \(m, n \leq N\), and \(i = 1, 2, 12\). We have the diagram

\[
V^1_{m,n} \longrightarrow \overline{W}^1_{m,n} \longrightarrow V^1_{m,n} \times_k Y_n^1 \\
V^{12}_{m,n} \longrightarrow \overline{W}^{12}_{m,n} \longrightarrow V^{12}_{m,n} \times_k Y_n^{12} \\
V^2_{m,n} \longrightarrow \overline{W}^2_{m,n} \longrightarrow V^2_{m,n} \times_k Y_n^2
\]

which by universal property of closures of a map, gives a factorization

\[
V^1_{m,n} \rightarrow \overline{W}^{12}_{m,n} \rightarrow (\overline{W}^{12}_{m,n} \times_k Y_n^{12}) \times ((\overline{W}^i_{m,n} \times_k Y_n^i) \overline{W}^1_{m,n}) \times ((\overline{W}^2_{m,n} \times_k Y_n^2) \overline{W}^{12}_{m,n} \rightarrow V^{12}_{m,n} \times_k Y_n^{12})
\]

This gives us a diagram of closed immersions into smooth formal schemes

\[
(\overline{W}^1_{m,n}, R^1_{m,n}) \\
(\overline{W}^{12}_{m,n}, R^{12}_{m,n}) \\
(\overline{W}^2_{m,n}, R^2_{m,n})
\]
Then, we have maps from the \( i = 1, 2 \) versions of (15) to a common one with superscript 12, where all maps are quasi-isomorphisms.

2) Independence of \( X, \mathcal{U}, U' \):

Given choices of \( X, \mathcal{U} \) and \( \mathcal{U}' \) for \( i = 1, 2 \), fix the other choices. We may let \( X^{12} \) be the closure of \( X \) in \( X^1 \times \bar{X}^2 \). Then, take an open affine covering of \( X^{12} \) such that each affine is projected into one of the opens in \( \bar{X}^i \) giving \( \mathcal{U}^i \) for \( i = 1, 2 \), so that their disjoint union \( \bar{U}^{12} \) has compatible morphisms to \( U^i \) for \( i = 1, 2 \). Set \( U^{12} = \bar{U}^{12} \times_{X^{12}} X \).

Now, having refinements \( (V^i, \nabla^i, \psi^i) \) of \( (X^i \times X, X^i \times \bar{X}^i \mathcal{U}^i) \) over \( (U^i, \mathcal{U}^i) \) for \( i = 1, 2 \), we use the same argument used above to construct \( (\nabla^{12}, \psi^{12}, \psi^{12}) \) from \( (\nabla^i, \psi^i, \psi^i) \) and \( (\nabla^{12}, \psi^{12}, \psi^{12}) \), which also gives us \( (V^{12}, \nabla^{12}, \psi^{12}) \). We also use the same argument to obtain \( W^i \nabla^{12}, R^{12} \). As before, this gives independence of these choices.

3) Independence of \( c, N'\):

The independence of \( c \) is clear. Suppose we have some \( N^1 \leq N^2 \) satisfying (6). Then, given choices of embeddings \( X_{N_i} \hookrightarrow E_{N_i} \hookrightarrow \mathcal{P}_{N_i} \hookrightarrow \mathcal{P}_{N_i} \) for \( i = 1, 2 \), we will get \( Y'_{\leq N^1}, Q'_{\leq N^1}, W'_{\leq N^1}, R'_{\leq N^1} \), and \( R'_{\leq N^1} \), with all the other choices being the same. But since we are applying \( \tau_{\leq c} \), we may replace the \( N^2 \) truncations by \( N^1 \) truncations through a natural quasi-isomorphism by Lemma 8. Thus, it is reduced to case 1).

**Functoriality:** Given \( f : X \to X' \) of smooth schemes, we must choose as above in a compatible way. Firstly, we may choose compatible special hypercoverings \( X \to X \) and \( X' \to X' \) by Proposition 2.iii).

By Proposition 3 (and its proof), we can always pick compatible compactifications \( X \to \bar{X} \) and \( X' \to \bar{X}' \) and find simplicial compactifications \( X^* \) and \( X'^* \) over \( \bar{X} \) and \( \bar{X}' \) fitting into a commutative diagram

\[
\begin{CD}
(X, X^*) @>>> (X', X'^*) \\
\downarrow @VVV \\
(X, \bar{X}) @>>> (X', \bar{X}').
\end{CD}
\]

After choosing \( \mathcal{U}^{12} \) and \( \mathcal{U}'^{12} \), we may choose a compatible affine covering of \( \bar{X} \) to give \( \mathcal{U}^{12} \) and a closed embedding into some \( \mathcal{U}'^{12} \) compatible.

Fix \( N \) as in (6) for some \( h = c \) satisfying Theorem 9 for both \( X \) and \( X' \). After choosing any lifting \( E'_{N_i} \) of \( X'_{N_i} \) and an embedding into affine and projective space, we may pick lifts of \( E_N \) and embeddings compatible as explained in the remark after introducing standard smooth algebras. This will give compatible morphisms

\[
[X_{\leq N}[\phi_{\leq N}^i] \to Y'_{\leq N}[\phi'_{\leq N}],
\]

\[
(X_{\leq N}, Y_{\leq N}, Q_{\leq N}) \to (X'_{\leq N}, Y'_{\leq N}, Q'_{\leq N})
\]

of \( N \)-truncated dagger spaces and rigid frames respectively.

By [CT03, Proposition 11.5.2], we may also pick compatible refinements \( V^* \) and \( V'^* \) of

\[
\begin{align*}
\text{(cosk}_N^X \text{sk}_N^X (X^*)) \times U, \cosk_N^X \text{sk}_N^X (X^*) \times \bar{X}^i \mathcal{U}^i \text{ and}
\cosk_N^X (\text{sk}_N^X (X'^*)) \times X' \times U', \cosk_N^X (\text{sk}_N^X (X'^*)) \times \bar{X}^i \mathcal{U}'^i.
\end{align*}
\]

over \( (U, \mathcal{U}) \) and \( (U', \mathcal{U}') \) respectively.
Finally, when choosing $W'_{m,n}$ to be the closure of $V'_{m,n}$ in $V'_{m,n} \times_k Y'_n$, from the diagram

\[
\begin{array}{ccc}
V_{m,n} & \longrightarrow & V'_{m,n} \\
\downarrow & & \downarrow \\
V'_{m,n} & \longrightarrow & V''_{m,n}
\end{array}
\]

by the universal property of the schematic closure, we get a closed immersion

$$W_{m,n} \hookrightarrow W'_{m,n} \times_k (V'_{m,n} \times_k Y_n)$$

where $W_{m,n}$ is the closure of $V_{m,n}$ in $V_{m,n} \times_k Y_n$. This gives compatible morphisms from the $X'$ version of (15) to the $X$ version.

\[\square\]

6 Application

As an application, we consider the following $p$-adic étale motivic cohomology on smooth $k$-varieties (generalized in [FM16, Appendix B] to general $k$-varieties):

$$R\Gamma_c(X_{\text{et}}, \mathbb{Z}_p(n)) := \varprojlim R\Gamma_c(X_{\text{et}}, \mathbb{Z}(n)/p^\bullet)$$

and

$$R\Gamma_c(X_{\text{et}}, \mathbb{Q}_p(n)) := R\Gamma_c(X_{\text{et}}, \mathbb{Z}_p(n))\otimes \mathbb{Q}_p.$$ 

Here $\mathbb{Z}(n)$ is Suslin-Voevodsky’s motivic complex defined in [SV00, Definition 3.1] on the big category of smooth $k$-schemes $\text{Sm}/k$. However, since we will be interested in a $p$-adic completion of this cohomology, we will use the identification

$$\mathbb{Z}_p/n\mathbb{Z}_p \cong W_n \Omega_{\log}^n[-n]$$

on $\text{Sm}/k$ from [GL00, Theorem 8.5], where $W_n \Omega_{\log}^n$ (denoted $\nu^n_n$ there) is the subsheaf of $W_n \Omega^n$ étale locally generated by sections of the forms $\text{dlog}f_1...\text{dlog}f_n$, defined in [Ill79, II.5.7].

As an application of Theorem 10, we have the following result, which proves [FM16, Conjecture 7.16.b] for smooth schemes (using [Gei06, Theorem 4.3]):

**Theorem 11.** For a separated, finite type smooth $k$-scheme $X$, and $n \in \mathbb{Z}$, there exists a quasi-isomorphism

$$R\Gamma(X_{\text{et}}, \mathbb{Q}_p(n)) \cong R\lim R\Gamma_c(X_{\text{et}}, W_n \Omega_{X,\log}^n)[-n].$$

Here, $\phi$ is the Frobenius on rigid cohomology, $R\Gamma_rig_c(X/K)^* := \text{RHom}(R\Gamma_rig_c(X/K_0), K)$ and $[A \rightarrow B] := \text{Cone}(A \rightarrow B)[-1]$.

**Proof.** Using (16) we have that

$$R\Gamma_rig(X, \mathbb{Q}_p(n)) \cong R\lim R\Gamma(X_{\text{et}}, W_n \Omega_{X,\text{log}}^n)\otimes \mathbb{Q}[-n].$$

By [Ill79, I.Theorem 5.7.2.] we have a short exact sequence

$$0 \rightarrow W_n \Omega_{X,\text{log}}^n \rightarrow W_n \Omega_X^n \xrightarrow{\text{F}} W_n \Omega_X^n \rightarrow 0$$

in $X_{\text{et}}$, and by the proof of [Ill79, II.Proposition 2.1.],

$$W \Omega_X^n \cong R\lim W_n \Omega_X^n.$$
which gives us

\[ R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left( R\Gamma(X_{et}, W\Omega^\bullet_X) \downarrow \mathbb{F}\right) R\Gamma(X_{et}, W\Omega^\bullet_X) \bigg] \mathbb{Q}[-n]. \quad (17) \]

Next, by [Ert13, Corollary 2.4.12], we have that all logarithmic Witt de-Rham sections are overconvergent, and that \( 1 - F \) is still surjective when restricted to the overconvergent part; so we have a commutative diagram \( X_{et} \) given by

\[
\begin{array}{cccccc}
0 & W\Omega^0_{X,\log} & W^1\Omega^0_X & 1-F & W^1\Omega^0_X & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & W\Omega^0_{X,\log} & W^1\Omega^0_X & 1-F & W^1\Omega^0_X & 0
\end{array}
\]

where the vertical arrows are given by inclusion, and both rows are short exact sequences. Thus, we get a natural quasi-isomorphism

\[
\left[ R\Gamma(X_{et}, W\Omega^\bullet_X) \downarrow \mathbb{F}\right] \mathbb{Q}[-n] \cong \left[ R\Gamma(X_{et}, W^1\Omega^\bullet_X) \downarrow \mathbb{F}\right] \mathbb{Q}[-n].
\]

(18)

We consider the Frobenius on \( W^1\Omega^\bullet_X \) by restricting that on \( W\Omega^\bullet_X \). By the slope decomposition of F-crystals in [Ill79, Corollaire 5.3]

\[ R\Gamma(X, W\Omega^\bullet_{X/k}) \otimes_{W(k)} K \cong \bigoplus \mathbb{Q}(-i) \otimes_{W(k)} K \]

we see the part with slope \( p^n \) must be coming from \( R\Gamma(X, W^1\Omega^n)[-n] \), thus giving

\[
\left[ R\Gamma(X_{et}, W^1\Omega^\bullet_X) \downarrow \mathbb{F}\right] \mathbb{Q}[-n] \cong \left[ R\Gamma(X_{et}, W^1\Omega^\bullet_X) \mathbb{F}\right] \mathbb{Q}[-n].
\]

(19)

Then, by Theorem 10 we get that

\[ R\Gamma_{\text{rig}}(X/K) \Rightarrow R\Gamma(X, W^1\Omega^\bullet_{X/k}) \mathbb{Q} \]

so by (17), (18) and (19) we have

\[ R\Gamma(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (20) \]

Finally, from [Ber97a, Théorème 2.4] we can use Poincaré duality for rigid cohomology to get non-degenerate pairings

\[ H^i_{\text{rig}}(X/K) \times H^{2d-i}_{\text{rig},c}(X/K) \to H^{2d}_{\text{rig},c}(X/K) \Rightarrow K(-d) \]

compatible as F-crystals, where \( K(-d) \) is viewed as \( K \) with a Frobenius action given by multiplication by \( p^d \). Thus, we have a natural quasi-isomorphism

\[ R\Gamma_{\text{rig}}(X/K) \Rightarrow R\Gamma_{\text{rig},c}(X/K)^*[2d] := R\text{Hom}(R\Gamma_{\text{rig},c}(X/K), K)[-2d] \]

and therefore,

\[
R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (21)
\]

and

\[
R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (22)
\]

\[ R\Gamma_{\text{rig}}(X/K) \Rightarrow R\Gamma_{\text{rig},c}(X/K)^*[2d] := R\text{Hom}(R\Gamma_{\text{rig},c}(X/K), K)[-2d] \]

and therefore,

\[
R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (23)
\]

\[ R\Gamma_{\text{rig}}(X/K) \Rightarrow R\Gamma_{\text{rig},c}(X/K)^*[2d] := R\text{Hom}(R\Gamma_{\text{rig},c}(X/K), K)[-2d] \]

and therefore,

\[
R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (24)
\]

\[ R\Gamma_{\text{rig}}(X/K) \Rightarrow R\Gamma_{\text{rig},c}(X/K)^*[2d] := R\text{Hom}(R\Gamma_{\text{rig},c}(X/K), K)[-2d] \]

and therefore,

\[
R\Gamma_{\text{rig}}(X, \mathbb{Q}_p(n)) \cong \left[R\Gamma_{\text{rig}}(X/K) \mathbb{F}\right] \mathbb{Q}[-n]. \quad (25)
\]
References

Ber97a. Pierre Berthelot. Dualité de Poincaré et formule de Künneth en cohomologie rigide. *C. R. Acad. Sci. Paris*, 325(5):493–498, 1997.

Ber97b. Pierre Berthelot. Finitude et pureté cohomologique en cohomologie rigide. *Invent. math.*, 128(2):329–377, 1997.

Con03. Brian Conrad. Cohomological descent. *Online Notes*, 2003.

CT03. Bruno Chiarellotto and Nobuo Tsuzuki. Cohomological descent of rigid cohomology for etale coverings. *Rend. Sem. Mat. Univ. Padova*, 109:63–215, 2003.

DLZ11. Christopher Davis, Andreas Langer, and Thomas Zink. Overconvergent de Rham-Witt Cohomology. *Ann. Scient. Ec. Norm. Sup.*, 44(2):197–262, 2011.

Ert13. Veronika Ertl. Overconvergent Chern Classes and Higher Cycle Classes. *arXiv:1310.3229*, October 2013.

FM16. Matthias Flach and Baptiste Morin. Weil-etale cohomology and Zeta-values of proper regular arithmetic schemes. *arXiv:1605.01277*, 2016.

Gei06. Thomas Geisser. Arithmetic cohomology over finite fields and special values of zeta-functions. *Duke Math. J.*, 133(1):27–57, 2006.

GL00. Thomas Geisser and Marc Levine. The K-theory of fields in characteristic p. *Invent. math.*, 139(3):459–493, March 2000.

Gro00. Elmar Grosse-Klönne. Rigid analytic spaces with overconvergent structure sheaf. *J. reine angew. Math.*, 519:73–95, 2000.

Ill79. Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Scient. Ec. Norm. Sup.*, 12(4):501–661, 1979.

Nak12. Yukiyoshi Nakkajima. Weight Filtration and Slope Filtration on the Rigid Cohomology of a Variety in Characteristic P>0. Number N.S., 130/131 in Mémoires de la Société Mathématique de France. Soc. Math. de France, Paris, 2012.

SV00. Andrei Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. *Arith. Geom. Algebr. Cycles Banff AB 1998*, 548:117–189, 2000.

Tsu04. Nobuo Tsuzuki. Cohomological descent in rigid cohomology. *Geom. Asp. Dwork Theory*, 2:931–982, 2004.