A smallness regularity criterion for the 3D Navier-Stokes equations in the largest class

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Abstract

In this paper, we consider the three-dimensional Navier-Stokes equations, and show that if the $\dot{B}^{-1}_{\infty,\infty}$-norm of the velocity field is sufficiently small, then the solution is in fact classical.

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1. Introduction

Consider the following three-dimensional (3D) Navier-Stokes equations:

$$\begin{align*}
\begin{cases}
\dot{u} + (u \cdot \nabla)u - \Delta u + \nabla \pi = 0, \\
\nabla \cdot u = 0, \\
u(x,0) = u_0,
\end{cases}
\end{align*}$$

(1)

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the fluid velocity, $\pi = \pi(x,t)$ is a scalar pressure; and $u_0$ is the prescribed initial velocity field satisfying the compatibility condition $\nabla \cdot u_0 = 0$.

The existence of a global weak solution

$$u \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$$

to (1) has long been established by Leray [10], see also Hopf [9]. But the issue of regularity and uniqueness of $u$ remains open. Initialed by Serrin
[15, 16] and Prodi [14], there have been a lot of literatures devoted to finding sufficient conditions to ensure $u$ to be smooth, see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 12, 13, 17, 18, 19, 20, 21, 22, 23] and references cited therein. Noticeably, the following Ladyzhenskaya-Prodi-Serrin condition ([6, 14, 15, 16]):

$$u \in L^p(0, T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \quad 3 \leq q \leq \infty$$

(2)
can ensure the smoothness of the solution.

Note that the limiting case $L^\infty(0, T; L^3(\mathbb{R}^3))$ in (2) does not fall into the framework of standard energy method, which was proved by Escauriaza, Seregin and Šverák [6] using backward uniqueness theorem. Due to the fact that

$$L^3(\mathbb{R}^3) \subset \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3), \quad \text{but} \quad L^3(\mathbb{R}^3) \neq \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3),$$

we shall consider in this paper the regularity of solutions of (1) in $\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)$. However, we could not prove a regularity criterion as $L^\infty(0, T; \dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3))$, since the function in $\dot{B}^{-1}_{\infty, \infty}(\mathbb{R}^3)$ has no decay at infinity, which ensures that the solution is smooth outside an big ball centered at origin so that the backward uniqueness theorem can be applied.

Before we state the precise result, let us recall the weak formulation of (1).

**Definition 1.** Let $u_0 \in L^2(\mathbb{R}^3)$ satisfying $\nabla \cdot u_0 = 0$, $T > 0$. A measurable vector-valued function $u$ defined in $[0, T] \times \mathbb{R}^3$ is said to be a weak solution to (1) if

1. $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3));$
2. $u$ satisfies (1)$_{1, 2}$ in the sense of distributions;
3. $u$ satisfies the energy inequality:

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \|u_0\|_{L^2}, \quad \text{a.e.} \quad t \in [0, T].$$

Now, our main result reads:
**Theorem 2.** Let \( u_0 \in L^2(\mathbb{R}^3) \) satisfying \( \nabla \cdot u_0 = 0, \ T > 0 \). Assume that \( u \) is a weak solution of (1) in \([0, T]\). If there exists an absolute constant \( \varepsilon_0 > 0 \) such that
\[
\|u\|_{\dot{B}_{\infty, \infty}^{-1}} \leq \varepsilon_0,
\]
then \( u \) is smooth in \((0, T)\).

The rest of this paper is organized as follows. In section 2, we recall the definition of Besov spaces and an interpolation inequality. Section 3 is devoted to proving Theorem 2.

**2. Preliminaries**

We first introduce the Littlewood-Paley decomposition. Let \( \mathcal{S}(\mathbb{R}^3) \) be the Schwartz class of rapidly decreasing functions. For \( f \in \mathcal{S}(\mathbb{R}^3) \), its Fourier transform \( \mathcal{F}f = \hat{f} \) is defined as
\[
\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x) e^{-ix \cdot \xi} \, dx.
\]

Let us choose an non-negative radial function \( \varphi \in \mathcal{S}(\mathbb{R}^3) \) such that
\[
0 \leq \hat{\varphi}(\xi) \leq 1, \quad \hat{\varphi}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq 1, \\
0, & \text{if } |\xi| \geq 2,
\end{cases}
\]
and let
\[
\psi(x) = \varphi(x) - 2^{-3}\varphi(x/2), \quad \varphi_j(x) = 2^{3j}\varphi(2^j x), \quad \psi_j(x) = 2^{3j}\psi(2^j x), \quad j \in \mathbb{Z}.
\]

For \( j \in \mathbb{Z} \), the Littlewood-Paley projection operators \( S_j \) and \( \triangle_j \) are, respectively, defined by
\[
S_j f = \varphi_j * f, \quad \triangle_j f = \psi_j * f.
\]

Observe that \( \triangle_j = S_j - S_{j-1} \). Also, it is easy to check that if \( f \in L^2(\mathbb{R}^3) \), then
\[
S_j f \to 0, \ \text{as} \ j \to -\infty; \quad S_j f \to f, \ \text{as} \ j \to \infty,
\]
in the $L^2$ sense. By telescoping the series, we have the following Littlewood-
Paley decomposition

$$f = \sum_{j=-\infty}^{\infty} \triangle_j f,$$

for all $f \in L^2(\mathbb{R}^3)$, where the summation is in the $L^2$ sense.

Let $s \in \mathbb{R}; p, q \in [1, \infty]$, the homogeneous Besov space $\dot{B}_{p,q}^{s}(\mathbb{R}^3)$ is defined
by the full dyadic decomposition such as

$$\dot{B}_{p,q}^{s} = \left\{ f \in Z'((\mathbb{R}^3); \| f \|_{\dot{B}_{p,q}^{s}} = \left\{ \| 2^j \| \triangle_j f \|_{L^p} \right\}_{j=-\infty}^{\infty} \right\},$$

where $Z'((\mathbb{R}^3)$ is the dual space of $Z(\mathbb{R}^3) = \left\{ f \in \mathcal{S}(\mathbb{R}^3); D^\alpha \hat{f}(0) = 0, \ \forall \alpha \in \mathbb{N}^3 \right\}.$

The following interpolation inequality will be need in Section 3,

$$\| f \|_{L^q} \leq C \| f \|_{\dot{H}^{\alpha}(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{q}} \| f \|_{\dot{B}_{\infty,\infty}^{\frac{1}{2}}}^{1 - \frac{2}{q}}, \ \forall f \in \dot{H}^{\alpha}(\mathbb{R}^3) \cap \dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^3), \ (4)$$

where $2 < q < \infty$ and $\alpha > 0$. See [11] for the proof.

3. Proof of Theorem 2

In this section, we shall prove Theorem 2.

By the classical “weak=strong” type uniqueness theorem, we need only
to derive the a priori estimate

$$u \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)). \ (5)$$

Multiplying (1) by $-\triangle u$, integrating over $\mathbb{R}^3$, we obtain

$$\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2} + \| \triangle u \|^2_{L^2} = \int_{\mathbb{R}^3} [(u \cdot \nabla) u] \cdot \triangle u \, dx \equiv I. \ (6)$$

By Hölder inequality,

$$I \leq \| u \|_{L^6} \| \nabla u \|_{L^3} \| \triangle u \|_{L^2}.$$
Invoking (4) with $q = 6$, $\alpha = 1$; and $q = 3$, $\alpha = 2$, we may further estimate $I$ as

$$I \leq C \left( \| u \|_{H^2}^\frac{1}{3} \| u \|_{B_{\infty,\infty}^{-1}}^{\frac{2}{3}} \right) \left( \| \nabla u \|_{H^1}^\frac{1}{3} \| \nabla u \|_{B_{\infty,\infty}^2}^{\frac{2}{3}} \right) \| \Delta u \|_{L^2}^2 \tag{7}$$

Substituting (7) into (6), we see

$$\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \left( 1 - C \| u \|_{B_{\infty,\infty}^{-1}}^{-1} \right) \| \Delta u \|_{L^2}^2 \leq 0.$$ 

Thus, if

$$\| u \|_{B_{\infty,\infty}^{-1}} \leq \frac{1}{C} \equiv \varepsilon_0,$$

we deduce that $\| \nabla u \|_{L^2}$ is decreasing, and thus (5), as desired.

The proof of Theorem 2 is completed.

References

[1] H. Beirão da Veiga, A new regularity class for the Navier-Stokes equations in $\mathbb{R}^n$, *Chinese Ann. Math. Ser. B*, 16 (1995), 407–412.

[2] H. Beirão da Veiga, L.C. Berselli, On the regularizing effect of the vorticity direction in incompressible viscous flows, *Differential Integral Equations*, 15 (2002), 345–356.

[3] C.S. Cao, E.S. Titi, Global regularity criterion for the 3D Navier-Stokes equations involving one entry of the velocity gradient tensor, *Arch. Rational Mech. Anal.*, 202 (2011), 919–932.

[4] D. Chae, J. Lee, Regularity criterion in terms of pressure for the Navier-Stokes equations, *Nonlinear Anal.*, 46 (2001), 727–735.

[5] P. Constantin, C. Fefferman, Direction of vorticity and the problem of global regularity for the Navier-Stokes equations, *Indiana Univ. Math. J.*, 42 (1993), 775–789.

[6] L. Escauriaza, G. Seregin, V. Šverák, Backward uniqueness for parabolic equations, *Arch. Ration. Mech. Anal.*, 169 (2003), 147–157.
[7] J.S. Fan, S. Jiang, G.X. Ni, On regularity criteria for the $n$-dimensional Navier-Stokes equations in terms of the pressure, *J. Differential Equations*, **244** (2008), 2963–2979.

[8] X.W. He, S. Gala, Regularity criterion for the weak solutions to the Navier-Stokes equations in terms of the pressure in the class $L^2(0,T;\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3))$, *Nonlinear Analysis: RWA.*, **12** (2011), 3602–3607.

[9] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachr.*, **4** (1951), 213–231.

[10] J. Leray, Sur le mouvement d’un liquide visqueux emplissant l’espace, *Acta Math.*, **63** (1934), 193–248.

[11] Y. Meyer, P. Gerard, F. Oru, Inégalités de Sobolev précisées, Séminaire Équations aux dérivées partielles (Polytechnique) (1996–1997), Exp. No. 4, 8 pp.

[12] J. Neustupa, P. Penel, Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D NavierCStokes equations, in Mathematical Fluid Mechanics (Recent Results and Open Problems), Advances in Mathematical Fluid Mechanics, edited by J. Neustupa, and P. Penel (Birkhäuser, Basel-Boston-Berlin, 2001), 239–267.

[13] P. Penel, M. Pokorný, Some new regularity criteria for the Navier-Stokes equations containing the gradient of velocity, *Appl. Math.*, **49** (2004), 483–493.

[14] G. Prodi, Un teorema di unicitá per le equazioni di Navier-Stokes, *Ann. Mat. Pura Appl.*, **48** (1959), 173–182.

[15] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, **9** (1962), 187–191.

[16] J. Serrin, The initial value problems for the Navier-Stokes equations. in Nonlinear Problems, edited by R. E. Langer (University of Wisconsin Press, Madison, WI, 1963).

[17] Z.F. Zhang and Q.L. Chen, Regularity criterion via two components of vorticity on weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$, *J. Differential Equations*, **216** (2005), 470–481.
[18] Z.J. Zhang, A regularity criterion for the Navier-Stokes equations via two entries of the velocity Hessian tensor, arXiv: 1103.1196.

[19] Z.J. Zhang, A Serrin-type regularity criterion for the Navier-Stokes equations via one velocity component, Commun. Pure Appl. Anal., 12 (2013), 117–124.

[20] Z.J. Zhang, Z.A. Yao, P. Li, C.C. Guo, M. Lu, Two new regularity criteria for the 3D Navier-Stokes equations via two entries of the velocity gradient tensor, Acta Appl. Math., (2012), doi: 10.1007/s10440-012-9712-4.

[21] Y. Zhou, A new regularity criterion for weak solutions to the Navier-Stokes equations, J. Math. Pures Appl., 84 (2005), no. 11, 1496–1514.

[22] Y. Zhou, M. Pokorný, On a regularity criterion for the Navier-Stokes equations involving gradient of one velocity component, J. Math. Phys., 50 (2009), 123514, 11 pp.

[23] Y. Zhou, M. Pokorný, On the regularity of the solutions of the Navier-Stokes equations via one velocity component, Nonlinearity, 23 (2010), 1097–1107.