On Indicated Coloring of Some Classes of Graphs

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Abstract
Indicated coloring is a type of game coloring in which two players collectively color the vertices of a graph in the following way. In each round the first player (Ann) selects a vertex, and then the second player (Ben) colors it properly, using a fixed set of colors. The goal of Ann is to achieve a proper coloring of the whole graph, while Ben is trying to prevent the realization of this project. The smallest number of colors necessary for Ann to win the game on a graph $G$ (regardless of Ben’s strategy) is called the indicated chromatic number of $G$, denoted by $\chi_i(G)$. In this paper, we obtain structural characterization of $\{P_5, K_4, Kite, Bull\}$-free graphs and connected $\{P_6, C_5, K_{1,3}\}$-free graphs that contain an induced $C_6$. Also, we prove that $\{P_5, K_4, Kite, Bull\}$-free graphs and connected $\{P_6, C_5, \overline{P_5}, K_{1,3}\}$-free graphs which contain an induced $C_6$ are $k$-indicated colorable for all $k \geq \chi(G)$. In addition, we show that, if $m \geq 1$ and $G$ is a bipartite graph, then $\chi_i(\mathbb{K}[G](m, m, \ldots, m)) = \chi(\mathbb{K}[G](m, m, \ldots, m))$. Further, we show that $\mathbb{K}[C_5]$ is $k$-indicated colorable for all $k \geq \chi(G)$ and as a consequence, we exhibit that $\{P_2 \cup P_3, C_4\}$-free graphs, $\{P_5, C_4\}$-free graphs are $k$-indicated colorable for all $k \geq \chi(G)$. This partially answers one of the questions which was raised by Grzesik (Discret Math 312:3467–3472, 2012).

Keywords Game chromatic number · Indicated chromatic number · $P_5$-free graphs

Mathematics Subject Classification 05C15 · 05C75
1 Introduction

All graphs considered in this paper are simple, finite and undirected. For any positive integer $k$, a proper $k$-coloring of a graph $G$ is a mapping $c: V(G) \rightarrow \{1, 2, \ldots, k\}$ such that for any two adjacent vertices $u, v \in V(G)$, $c(u) \neq c(v)$. A graph is said to be $k$-colorable if it admits a proper $k$-coloring. The chromatic number $\chi(G)$ of a graph $G$ is the smallest $k$ such that $G$ is $k$-colorable. In this paper, $P_n$, $C_n$ and $K_n$ respectively denote the path, the cycle and the complete graph on $n$ vertices. For $S, T \subseteq V(G)$, let $(S)$ denote the subgraph induced by $S$ in $G$ and let $[S, T]$ denote the set of all edges with one end in $S$ and the other end in $T$. $[S, T]$ is said to be complete if every vertex in $S$ is adjacent with every vertex in $T$. For any graph $G$, let $\overline{G}$ denote the complement of $G$.

Let us recall some of the definitions which are required for this paper. Let $\mathcal{F}$ be a family of graphs. We say that $G$ is $\mathcal{F}$-free if it contains no induced subgraph which is isomorphic to a graph in $\mathcal{F}$. For two vertex-disjoint graphs $G_1$ and $G_2$, the join of $G_1$ and $G_2$, denoted by $G_1 + G_2$, is the graph whose vertex set $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and the edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. In this paper, we write $H \subseteq G$ if $H$ is an induced subgraph of $G$. Next, the coloring number of a graph $G$, denoted by $\text{col}(G)$, is defined by $\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$. By Szekeres–Wilf’s inequality, $\chi(G) \leq \text{col}(G)$.

A game coloring of a graph is a coloring of the vertices in which two players Ann and Ben are alternatively coloring the vertices of the graph $G$ properly by using a fixed set of colors $C$. The first player Ann is aiming to get a proper coloring of the whole graph, whereas the second player Ben is trying to prevent the realization of this project. If all the vertices are colored then Ann wins the game, otherwise Ben wins (that is, at that stage of the game there appears a block vertex. A block vertex means an uncolored vertex which has all colors from $C$ on its neighbors). The minimum number of colors required for Ann to win the game on a graph $G$ irrespective of Ben’s strategy is called the game chromatic number of the graph $G$ and it is denoted by $\chi(G)$. There has been a lot of papers on game coloring. See for instance, [10,16,19,20]. The idea of indicated coloring was introduced by Grzesik in [9] as a slight variant of the game coloring in the following way: in each round the first player Ann selects a vertex and then the second player Ben colors it properly, using a fixed set of colors. The aim of Ann as in game coloring is to achieve a proper coloring of the whole graph $G$, while Ben tries to “block” some vertex. The smallest number of colors required for Ann to win the game on a graph $G$ is known as the indicated chromatic number of $G$ and is denoted by $\chi_i(G)$. Clearly from the definition we see that $\omega(G) \leq \chi(G) \leq \chi_i(G) \leq \Delta(G) + 1$. For a graph $G$, if Ann has a winning strategy while using $k$ colors, then we say that $G$ is $k$-indicated colorable.

In [20], Zhu has asked the following question for game coloring. Whether increasing the number of colors will favor Ann? That is, if Ann has a winning strategy using $k$ colors, will Ann have a winning strategy using $k + 1$ colors? The same question was asked by Grzesik for indicated coloring. The question can be equivalently stated as “Whether $G$ is $k$-indicated colorable for every $k \geq \chi_i(G)$”. He also showed by an example that the increase in number of colors does not make life simple for Ann rather it makes it much harder. There has been already some
partial answers to this question. For instance in [8,14], Pandiya Raj et.al. have shown that chordal graphs, cographs, complement of bipartite graphs, \( \{P_5, K_3\} \)-free graphs, \( \{P_5, Paw\} \)-free graphs, and \( \{P_5, K_4 - e\} \)-free graphs are \( k \)-indicated colorable for all \( k \geq \chi(G) \). In addition, Lason in [12] has obtained the indicated chromatic number of matroids. In this paper, we obtain structural characterization of \( \{P_5, K_4, Kite, Bull\} \)-free graphs and connected \( \{P_6, C_5, K_{1,3}\} \)-free graphs which contain an induced \( C_6 \). Also, we prove that \( \{P_5, K_4, Kite, Bull\} \)-free graphs and connected \( \{P_6, C_5, \overline{P_5}, K_{1,3}\} \)-free graphs which contain an induced \( C_6 \) are \( k \)-indicated colorable for all \( k \geq \chi(G) \). In addition, we have shown that, if \( m \geq 1 \) and \( G \) is a bipartite graph, then \( \chi_I(\mathbb{K}[G](m, m, \ldots, m)) = \chi(\mathbb{K}[G](m, m, \ldots, m)) \). Further, we show that \( \mathbb{K}[C_5] \), the complete expansion of \( C_5 \), is \( k \)-indicated colorable for all \( k \geq \chi(G) \) and as a consequence, we exhibit that \( \{P_5 \cup P_3, C_4\} \)-free graphs, \( \{P_5, C_4\} \)-free graphs are \( k \)-indicated colorable for all \( k \geq \chi(G) \).

Notations and terminologies not mentioned here are as in [18].

2 Structural Characterization of Some Free Graphs and their Indicated Coloring

In [3], it has been shown that the game chromatic number of a bipartite graph can be arbitrarily large when compared to the chromatic number which is equal to 2. But while considering the indicated chromatic number of a bipartite graph \( G \), Grzesik in [9] has shown that \( \chi_I(G) = 2 \).

**Theorem 2.1** [9] Every bipartite graph is \( k \)-indicated colorable for every \( k \geq 2 \).

Next, let us recall the definition of complete expansion and independent expansion of a graph \( G \). Let \( G \) be a graph on \( n \) vertices \( v_1, v_2, \ldots, v_n \), and let \( H_1, H_2, \ldots, H_n \) be \( n \) vertex-disjoint graphs. An expansion \( G(H_1, H_2, \ldots, H_n) \) of \( G \) is the graph obtained from \( G \) by

(i) replacing each \( v_i \) of \( G \) by \( H_i, i = 1, 2, \ldots, n \), and

(ii) by joining every vertex in \( H_i \) with every vertex in \( H_j \)

whenever \( v_i \) and \( v_j \) are adjacent in \( G \).

For \( i \in \{1, 2, \ldots, n\} \), if \( H_i \cong K_{m_i} \), then \( G(H_1, H_2, \ldots, H_n) \) is said to be a complete expansion of \( G \) and is denoted by \( \mathbb{K}[G](m_1, m_2, \ldots, m_n) \) or \( \mathbb{K}[G] \). For \( i \in \{1, 2, \ldots, n\} \), if \( H_i \cong \overline{K}_{m_i} \), then \( G(H_1, H_2, \ldots, H_n) \) is said to be an independent expansion of \( G \) and is denoted by \( \mathbb{I}[G](m_1, m_2, \ldots, m_n) \) or \( \mathbb{I}[G] \).

In [17], Sumner studied the structural property of \( \{P_5, K_3\} \)-free graphs.

**Theorem 2.2** [17] Let \( G \) be a \( \{P_5, K_3\} \)-free graph. Then each component of \( G \) is either bipartite or \( \mathbb{I}[C_5](m_1, m_2, \ldots, m_5) \), where \( m_i \geq 1 \) for \( i = 1, 2, 3, 4, 5 \).

Let us start this section with a structural characterization of a family of \( P_5 \)-free graphs. The study of \( P_5 \)-free graphs has been of interest for a lot of coloring parameters. For instance, see [2,4,6,7]. In this direction, we would like to consider connected \( \{P_5, K_4, Kite, Bull\} \)-free graphs that contain an induced \( C_5 \). Here, the graphs \( Kite \) and \( Bull \) are shown in Fig. 1.
Theorem 2.3 If $G$ is a connected $\{P_5, K_4, \text{Kite}, \text{Bull}\}$-free graph that contains an induced $C_5$, then $V(G) = V_1 \cup V_2 \cup V_3$ such that (1) $V_2$ is a complete bipartite graph with bipartitions $B$ and $S$, (2) $V_1 \cup V_3$ is disjoint union of $\mathbb{I}[C_5]$’s and bipartite graphs, (3) $[V_1, B]$ is complete, $[V_1, S] = [V_1, V_3] = [V_3, B] = \emptyset$ and (4) there exists $x^* \in S$ such that $[x^*, V_3]$ is complete.

Proof Let $G$ be a connected $\{P_5, K_4, \text{Kite}, \text{Bull}\}$-free graph that contains an induced $C_5 \cong \langle \{v_0, v_1, v_2, v_3, v_4\} \rangle = \langle N_0 \rangle$, and let $N_i = \{x \in V(G) : \text{dist}(x, N_0) = i\}$, $i \geq 1$.

Claim 1 If $x \in N_1$, then $\langle N(x) \cap N_0 \rangle \cong 2K_1$ or $C_5$.

For $x \in N_1$, the possibilities for $\langle N(x) \cap N_0 \rangle$ are $K_1, K_2, P_3, P_4, 2K_1, K_1 \cup K_2$ and $C_5$. Here (a) if $\langle N(x) \cap N_0 \rangle \cong K_1$ or $K_2$, then $P_5 \subseteq G$, (b) if $\langle N(x) \cap N_0 \rangle \cong P_3$ or $P_4$, then $\text{Kite} \subseteq G$, and (c) if $\langle N(x) \cap N_0 \rangle \cong K_1 \cup K_2$, then $\text{Bull} \subseteq G$, a contradiction. Finally, if $\langle N(x) \cap N_0 \rangle \cong 2K_1$ or $C_5$, we cannot get $P_5$ or $K_4$ (or $\text{Kite}$ or $\text{Bull}$) as an induced subgraph in $\langle N_0 \cup N_1 \rangle$. Hence $\langle N(x) \cap N_0 \rangle \cong 2K_1$ or $C_5$.

Throughout this proof, for any integer $i$, $v_i$ means $v_i \pmod{5}$ and $A_i$ means $A_i \pmod{5}$.

For $0 \leq i \leq 4$, let $A_i = \{x \in N_1 : N(x) \cap N_0 = \{v_{i-1}, v_{i+1}\} \cup \{v_i\}\}$ and let $B = \{x \in N_1 : N(x) \cap N_0 = \langle C_5 \rangle\}$.

Claim 2 $\langle \bigcup_{i=0}^{4} A_i \rangle \cong \mathbb{I}[C_5]$.

For every $i$, $0 \leq i \leq 4$, we have (a) $\langle A_i \rangle$ is independent (else, if $x, y \in A_i$ are adjacent, then $x, y \neq v_i$ and $\langle \{x, v_{i+1}, y, v_{i-1}, v_{i-2}\} \rangle \cong \text{Kite} \subseteq G$), (b) $[A_i, A_{i+1}]$ is complete (else, if $x \in A_i$ and $y \in A_{i+1}$ are not adjacent, then $\langle \{x, v_{i-1}, v_{i-2}, v_{i-3}, v_i\} \rangle \cong P_5 \subseteq G$), (c) $[A_i, A_{i+2}] = \emptyset$ (else, if $x \in A_i$ and $y \in A_{i+2}$ are adjacent, then $\langle \{v_{i-1}, x, v_{i+1}, v_{i+2}, y\} \rangle \cong \text{Bull} \subseteq G$). Thus from (a), (b) and (c), we conclude that $\langle \bigcup_{i=0}^{4} A_i \rangle \cong \mathbb{I}[C_5]$.

Claim 3 $\langle \bigcup_{i=0}^{4} A_i, B \rangle$ is complete.

On the contrary, if there exist vertices $x \in A_i, x \neq v_i$ and $y \in B$ such that $xy \notin E(G)$, then $\langle \{v_i, v_{i+1}, x, y, v_{i-2}\} \rangle \cong \text{Bull} \subseteq G$, a contradiction.

Claim 4 $\langle B \rangle$ is independent.

Suppose there exist vertices $x$ and $y$ in $B$ such that $xy \in E(G)$, then $\langle \{v_1, v_2, x, y\} \rangle \cong K_4 \subseteq G$, a contradiction.

Claim 5 If $x \in \bigcup_{i=0}^{4} A_i$, then $N(x) \cap N_2 = \emptyset$.

Let $x \in A_i$ for some $i$ such that $0 \leq i \leq 4$. Suppose if there exists a vertex $y \in N(x) \cap N_2$, then $\langle \{y, x, v_{i+1}, v_{i+2}, v_{i+3}\} \rangle \cong P_5 \subseteq G$, a contradiction.
Note that, if $B = \emptyset$, then by using Claims 2 and 5, we can observe that $G \cong \mathbb{I}[C_5]$. Now, let us assume that $B \neq \emptyset$ and $N_2 \neq \emptyset$.

**Claim 6** $\langle B, N_2 \rangle$ is complete.

Here, if there exist vertices $x \in B$ and $y \in N_2$ such that $xy \notin E(G)$, then by using Claim 5, there exists a vertex $z \in B$ such that $yz \in E(G)$. Now from Claim 4, $xz \notin E(G)$. Hence $\langle \{v_1, v_2, x, y, z\} \rangle \cong \text{Kite} \subseteq G$, a contradiction.

**Claim 7** $\langle N_2 \rangle$ is triangle-free.

On the contrary, assume that there exist vertices $\{u_1, u_2, u_3\} \subseteq N_2$ which induce a $K_3$ in $G$. Then by using Claim 6, for every vertex $x \in B$, $\langle \{x, u_1, u_2, u_3\} \rangle \cong K_4 \subseteq G$, a contradiction.

Since $G$ is assumed to be $P_5$-free and $\langle N_2 \rangle$ is triangle-free, by Theorem 2.2 we see that each component of $\langle N_2 \rangle$ is either isomorphic to $\mathbb{I}[C_5]$ or to a bipartite graph.

Suppose $N_3 = \emptyset$, by using the above claims, we see that $G \cong (\cup_{i=0}^{4} A_i \cup N_2) + \langle B \rangle$. Now, let us assume that $N_3 \neq \emptyset$.

**Claim 8** If $xy$ is an edge in $\langle N_2 \rangle$, then $N(x) \cap N_3 = \emptyset$ and $N(y) \cap N_3 = \emptyset$.

Let $xy$ be an edge in $\langle N_2 \rangle$. Suppose there exists a vertex $z \in N_3$ such that $xz \in E(G)$ or $yz \in E(G)$ (or) $xz, yz \in E(G)$, then $\langle \{v_1, b, x, y, z\} \rangle \cong \text{Bull}$ or $\text{Kite} \subseteq G$ (where $b \in B$), a contradiction.

Let $S$ be the collection of the vertices in $N_2$ which have neighbors in $N_3$. From Claim 8, it can be seen that $S$ is an independent subset of $N_2$ such that $\{S, N_2 \setminus S\} = \emptyset$.

**Claim 9** There exists a vertex $x^* \in S$ such that $[x^*, N_3]$ is complete. Also, $\langle N_3 \rangle$ is triangle-free.

On the contrary, let us assume that there exists no $x^* \in S$ such that $[x^*, N_3]$ is complete. Under this assumption, first let us show that there exist vertices $x, x' \in S$ and $y, y' \in N_3$ such that $xy, x'y' \in E(G)$ and $xy', x'y \notin E(G)$. Let $x_1, x_2, \ldots, x_{|S|}$ and $y_1, y_2, \ldots, y_{|N_3|}$ be the vertices of $S$ and $N_3$ respectively. Consider $x_1$. By our assumption, $x_1$ is non-adjacent to at least one of the vertex in $N_3$, say $y_1$ and the vertex $y_1$ should have a neighbor in $S$, say $x_2$. Now the vertex $x_2$ is also non-adjacent to at least one vertex in $N_3$, say $y_2$. Suppose $x_1 y_2 \in E(G)$, then $x = x_1, x' = x_2, y = y_2$ and $y' = y_1$ will possess the required property. If not, $x_1 y_2 \notin E(G)$ and $y_2$ should have a neighbor in $S$, say $x_3$. Suppose $x_3 y_1 \notin E(G)$, $x = x_2, x' = x_3, y = y_1$ and $y' = y_2$ will have the required property. Otherwise, $x_3 y_1 \in E(G)$ and there is a vertex in $N_3$ which is non-adjacent to $x_3$, say $y_3$. Likewise, if $x_1 y_3 \in E(G)$ or $x_2 y_3 \in E(G)$, then as mentioned above we can get vertices with the required condition. If not, $x_1 y_3 \notin E(G)$ and $x_2 y_3 \notin E(G)$, and hence the vertex $y_3$ should have a neighbor in $S$, say $x_4$. Similarly, even when the vertex $x_4$ is non-adjacent to $y_1$ or $y_2$, we can get the vertices with the required condition. Suppose $x_4$ is adjacent to $y_1$ and $y_2$, the process continues. Since the number of vertices is finite, this process stops at a certain stage having vertices $x_i, x_j \in S$ and $y_{i-1}, y_i \in N_3$ such that $x_i y_{i-1}, y_i x_j \in E(G)$ and $x_j y_{i-1}, x_i y_i \notin E(G)$ for some $i, j \in \{1, 2, \ldots, |S|\}$. Thus there exist vertices $x, x' \in S$ and $y, y' \in N_3$ such that $xy, x'y' \in E(G)$ and $xy', x'y \notin E(G)$. Now by using Claim 8, $xx' \notin E(G)$, and hence for some $b \in B$, $\langle \{v_1, b, x', y\} \rangle \cong P_5$ when $yy' \in E(G)$ or $\langle \{y, x, b, x', y'\} \rangle \cong P_5$ when $yy' \notin E(G)$, a contradiction.

Also note that, $\langle N_3 \rangle$ has to be triangle-free. Otherwise, $K_4 \subseteq G$. 

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Claim 10 \( N_i = \emptyset, \) for all \( i \geq 4. \)

This can be easily observed from the fact that if \( N_4 \neq \emptyset, \) then we will get \( P_5 \subseteq G, \) a contradiction.

Since \( G \) is \( P_5 \)-free and \( \langle N_3 \rangle \) is triangle-free, each component of \( \langle N_3 \rangle \) is isomorphic to \( \mathbb{II}[C_5] \) or to a bipartite graph. Let \( V_1 = \bigcup_{i=0}^{4} A_i \cup (N_2 \setminus S), \) \( V_2 = B \cup S \) and \( V_3 = N_3. \) By using the above claims, we see that \( \langle V_2 \rangle \) is a complete bipartite graph, \( \langle V_1 \rangle \) is a disjoint union of \( \mathbb{II}[C_5] \)'s and bipartite graphs such that \( [V_1, B] \) is complete, and \( \langle V_3 \rangle \) is also a disjoint union of \( \mathbb{II}[C_5] \)'s and bipartite graphs such that there exists a vertex \( x^* \in S \) such that \( [x^*, V_3] \) is complete. Also from Claims 5 and 8, it can be observed that \( [V_1, S] = [V_1, V_3] = [V_3, B] = \emptyset. \) □

By Theorem 2.3, one can easily find the chromatic number of this family.

Corollary 2.4 If \( G \) is a connected \( \{P_5, K_4, Kite, Bull\} \)-free graph that contains an induced \( C_5, \) then \( \chi(G) = 3 \) if and only if \( G \cong \mathbb{II}[C_5], \) otherwise \( \chi(G) = 4. \)

Proof By Theorem 2.3, we see that \( V(G) = V_1 \cup V_2 \cup V_3, \) where \( V_1, V_2 \) and \( V_3 \) have the properties stated in the statement of Theorem 2.3. Since \( \langle V_1 \cup V_3 \rangle \) is a disjoint union of \( \mathbb{II}[C_5] \)'s and bipartite graphs, one can color the vertices of \( V_1 \) and \( V_3 \) with colors \( \{1, 2, 3\} \) and \( \{2, 3, 4\} \) respectively which yields a proper coloring for the subgraph \( \langle V_1 \cup V_3 \rangle. \) Since \( \langle V_2 \rangle \) is a complete bipartite graph with bipartition \( B \) and \( S, \) \( [B, V_1] \) is complete, \( [x^*, V_3] \) is complete and \( [V_1, S] = [V_1, V_3] = [V_3, B] = \emptyset, \) coloring the vertices of \( B \) and \( S \) with 4 and 1 respectively will yields a proper coloring for \( G. \) Thus \( \chi(G) \leq 4. \) Suppose \( B = \emptyset, \) then \( G \cong \mathbb{II}[C_5] \) and hence \( \chi(G) = 3. \) If not, \( B \neq \emptyset \) and \( \langle B \rangle + \mathbb{II}[C_5] \subseteq G. \) Thus \( \chi(G) = 4. \) □

Now we shall consider the indicated coloring for the independent expansion of a graph \( G. \)

Theorem 2.5 For \( 1 \leq i \leq n, \) let \( m_i \)'s be positive integers. For any positive integer \( k, \) if \( G \) is a graph which is \( k \)-indicated colorable, then the graph \( \mathbb{II}[G](m_1, m_2, \ldots, m_n) \) is also \( k \)-indicated colorable.

Proof Let \( G \) be the graph and let \( k \) be any positive integer such that \( G \) is \( k \)-indicated colorable. Then clearly, \( k \geq \chi_i(G). \) Also let \( \mathbb{II}[G] = \mathbb{II}[G](m_1, m_2, \ldots, m_n) \) be an independent expansion of \( G, \) for \( m_i \geq 1, 1 \leq i \leq n. \) Let \( H \) be an induced subgraph got from \( \mathbb{II}[G] \) by choosing one vertex from each of the independent set of size \( m_i, \) \( 1 \leq i \leq n, \) which has replaced the vertices of \( G. \) Clearly it can be observed that \( H \cong G \) is an induced subgraph of \( \mathbb{II}[G] \) and \( \chi(H) = \chi(\mathbb{II}[G]). \) Let \( \{1, 2, \ldots, k\} \) be the set of colors. Let Ann first present the vertices of \( H \) by applying the winning strategy for \( G \) using \( k \) colors and then present the remaining vertices of \( \mathbb{II}[G] \) in any order. For \( 1 \leq i \leq n, \) since the set of neighbors of any two vertices of \( K_{m_i} \) is the same, the color given to the vertex \( V(H) \cap K_{m_i} \) is available to the other vertices of \( K_{m_i}. \) Hence Ben will always have an available color for all the vertices of \( \mathbb{II}[G]. \) Thus Ann has a winning strategy for \( \mathbb{II}[G] \) with \( k \) colors. □

We know that, for the union of two graphs \( G_1 \) and \( G_2, \) \( \chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}. \) The same holds even for the indicated chromatic number.
Theorem 2.6 [14]
Let $G = G_1 \cup G_2$. If $G_1$ is $k_1$-indicated colorable for every $k_1 \geq \chi_i(G_1)$ and $G_2$ is $k_2$-indicated colorable for every $k_2 \geq \chi_i(G_2)$, then $\chi_i(G) = \max\{\chi_i(G_1), \chi_i(G_2)\}$ and $G$ is $k$-indicated colorable for all $k \geq \chi_i(G)$.

Next, we see that Corollary 2.7 which was proved in [14] is a simple consequence of Theorems 2.1, 2.2, 2.5 and 2.6.

Corollary 2.7 [14] Every $\{P_5, K_3\}$-free graph $G$ is $k$-indicated colorable for all $k \geq \chi(G)$.

Now, let us consider the indicated coloring of $\{P_5, K_4, Kite, Bull\}$-free graphs which contains an induced $C_5$.

Theorem 2.8 Let $G$ be a connected $\{P_5, K_4, Kite, Bull\}$-free graph which contains an induced $C_5$. Then $G$ is $k$-indicated colorable for all $k \geq \chi(G)$.

Proof Let $G$ be a connected $\{P_5, K_4, Kite, Bull\}$-free graph that contains an induced $C_5$. Then by Theorem 2.3, $V(G) = V_1 \cup V_2 \cup V_3$ where (1) $(V_2)$ is a complete bipartite graph with bipartition, say, $B$ and $S$, (2) $(V_1 \cup V_3)$ is a disjoint union of $\{G_5\}$'s and bipartite graphs, (3) $[V_1, B]$ is complete, $[V_1, S] = \emptyset$, $[V_1, V_3] = \emptyset$, $[V_3, B] = \emptyset$ and (4) there exists a vertex, say $x^* \in S$ such that $[x^*, V_3]$ is complete.

Suppose $B = \emptyset$, then $G \cong \{G_5\}$. Thus by Theorem 2.5, $G$ is $k$-indicated colorable for all $k \geq \chi(G)$. If not, $B \neq \emptyset$ and hence by Corollary 2.4, $G \cong \{G_5\}$ and $\chi(G) = 4$. For $k \geq 4$, let $\{1, 2, \ldots, k\}$ be the set of colors. We shall show that $G$ is $k$-indicated colorable. Let Ann start by presenting $x^*$ and a vertex $b \in B$. Without loss of generality, let the color used by Ben for $b$ and $x^*$ be 1 and 2 respectively. Since $[b, V_1]$ is complete and $[x^*, V_3]$ is complete, the set of available colors for $V_1$ and $V_3$ are $\{2, 3, \ldots, k\}$ and $\{1, 3, 4, \ldots, k\}$ respectively. Since $[V_1, V_3] = \emptyset$, $(V_1 \cup V_3)$ is a disjoint union of $\{G_5\}$'s and bipartite graphs, by Theorems 2.1 and 2.5, $(V_1 \cup V_3)$ is $l$-indicated colorable for all $l \geq 3$. That is, Ann has a winning strategy for $(V_1)$ while using the colors $\{2, 3, \ldots, k\}$ and a winning strategy for $(V_3)$ while using the colors $\{1, 3, 4, \ldots, k\}$. After presenting the vertices of $V_1$ and $V_3$ by using these winning strategies, Ann will present the remaining vertices of $B$ and $S$ in any order. Clearly, the color of the vertices $b$ and $x^*$, namely 1 and 2 are available for the uncolored vertices of $B$ and $S$ respectively. Thus Ann wins the game on $G$ with $k$ colors, $k \geq 4$. \[\square\]

In [2], Bacsó and Tuza obtained the complete characterization of $\{P_5, C_5\}$-free graphs in terms of dominating clique.

Theorem 2.9 [2] In a graph $G$, every connected subgraph contains a dominating clique if and only if $G$ is $\{P_5, C_5\}$-free.

Next, we have determined the structure of $\{P_5, C_5, K_4, Kite, Bull\}$-free graphs which is necessary for establishing an indicated coloring and have shown that these graphs are $k$-indicated colorable for all $k \geq \chi(G)$.

Theorem 2.10 If $G$ is a $\{P_5, C_5, K_4, Kite, Bull\}$-free graph, then $G$ is $k$-indicated colorable for all $k \geq \chi(G)$.
Proof By using Theorem 2.6, it is enough to prove the result for a connected \(\{P_5, C_5, K_4, \text{Kite, Bull}\}\)-free graph. Let \(G\) be such a graph. Let \(k\) be a positive integer such that \(k \geq \chi(G)\) and let \(\{1, 2, \ldots, k\}\) be the color set. We shall show that \(G\) is \(k\)-indicated colorable. Let us first make some observations.

Observation 2.10.1 (i) In any indicated coloring, if \(u, v \in V(G)\) such that \(N(v) \subseteq N(u)\), then Ben will always have the color of \(u\) available for the vertex \(v\).

(ii) In the case when a non-trivial subgraph \(H\) of \(G\) contains a dominating \(K_i\), let \(\chi(H) \geq i\). Let \(x \in V(H)\) and \(Y = \{y \in V(H) : y \neq x\}\) such that \(x\) and \(Y\) have an available color for the vertices of \(\langle x \rangle\) and \(\langle Y \rangle\) respectively. Since no \(K_1\) dominates \(G\), \(X_1\) and \(Y_1\) are non-empty.

By using Theorem 2.9, we can observe that \(G\) contains a dominating \(K_i\), for some \(i \in \{1, 2, 3\}\). Let us divide the proof into 3 cases depending upon the value of \(i\).

Case 1 \(G\) contains a dominating \(K_1\).

Let \(x\) be a dominating vertex in \(G\). Here \(G \cong K_1 + \langle N(x) \rangle\). Clearly \(\langle N(x) \rangle\) is \(\{P_3, C_5, K_3\}\)-free and hence by Theorem 2.2, each component of \(N(x)\) is bipartite.

Here by using Theorem 2.1, Ann has a winning strategy for \(\langle N(x) \rangle\) using \(k - 1\) colors, and hence Ann has a winning strategy for \(G\) using \(k\) colors by presenting the vertex \(x\) and then the vertices of \(N(x)\) by following the winning strategy for \(\langle N(x) \rangle\) using \(k - 1\) colors.

Case 2 \(G\) contains a dominating \(K_2\).

Let \(xy\) be a dominating \(K_2\) in \(G\). Similar to Case 1, \(\langle N(x) \rangle\) and \(\langle N(y) \rangle\) are bipartite graphs. Let \(X_1, X_2\) and \(Y_1, Y_2\) be the bipartition of \(\langle N(x) \rangle\) and \(\langle N(y) \rangle\) respectively such that \(S \subseteq X_1\) and \(T \subseteq Y_1\) where \(S\) and \(T\) denote the set of all isolated vertices of \(\langle N(x) \rangle\) and \(\langle N(y) \rangle\) respectively. Since no \(K_1\) dominates \(G\), \(X_1\) and \(Y_1\) are non-empty. Let \(A\) be the set of all common neighbors of \(x\) and \(y\). Since \(G\) is \(K_4\)-free, \(A\) is an independent set. Let \(G' = \langle V(G) \setminus \{x, y\} \rangle\). Let us further do case by case analysis.

Subcase 2.1 \(A = \emptyset\) and \(X_2 = Y_2 = \emptyset\).

Here \(G\) is a bipartite graph with bipartition \((\{x\} \cup Y_1, \{y\} \cup X_1)\) and hence \(G\) is \(k\)-indicated colorable.

Subcase 2.2 \(A = \emptyset\) and either \(X_2 = \emptyset\) or \(Y_2 = \emptyset\).

Without loss of generality, let \(Y_2 = \emptyset\). Clearly \(V(G)\) can be partitioned into three independent sets namely, \((\{x\} \cup Y_1, \{y\} \cup X_1, X_2)\) such that \(N(u) \subseteq N(x)\) for all \(u\) in \(Y_1\) and hence \(\chi(G) = 3\).

Let Ann start by presenting the vertices in \(\{x\} \cup N(x)\) by using the winning strategy as mentioned in Case 1. Finally, let Ann present the vertex \(y\) and then vertices of \(Y_1\). Since \(N(u) \subseteq N(x)\) for all \(u\) in \(Y_1\), by using (i) of Observation 2.10.1, Ben always has an available color for the vertices of \(Y_1\). Hence \(G\) is \(k\)-indicated colorable.

Subcase 2.3 \(A = \emptyset\), \(X_1, X_2, Y_1\) and \(Y_2\) are non-empty and \(G'\) is bipartite.

Here \(G'\) is \(K_3\)-free graph. While considering \(G'\), if there exist an edge \(a_1b_1, a_1 \in V(G_1) \cap X_1\) and \(b_1 \in Y_1\) where \(G_1\) is a non-trivial component of \(\langle N(x) \rangle\), then let us determine \([X_1, Y_1]\). Since \(Y_2\) is non-empty, there exists at least one non-trivial component say \(H_1\) in \(\langle N(y) \rangle\). Let \(a_2 \in V(G_1) \cap X_2\), \(b_2 \in V(H_1) \cap Y_2\) and \(b_3 \in V(H_1) \cap Y_1\) such that \(a_1a_2\) and \(b_2b_3\) are edges in \(G_1\) and \(H_1\) respectively.

Claim 1 \([a_1, Y_1]\) is complete.

Suppose there exists \(v \in Y_1\) such that \(a_1v \notin E(G)\), then \(\langle\langle a_2, a_1, b_1, y, v\rangle\rangle \cong P_5\) or \(C_5 \subseteq G\), a contradiction.
Subcase 2.4

In \(N(y)\), suppose there exists another component having an edge \(pq\), where \(p \in Y_1\) and \(q \in Y_2\), then \(a_1\) is adjacent to \(p\) and non-adjacent to \(q\) (otherwise we get a \(K_3\)). Hence \(\langle a_2, a_1, b_3, y, q \rangle \cong P_5\) or \(C_5 \subseteq G\), a contradiction. Thus if there exists an edge between a vertex in \(N(y)\) and a non-trivial component \(G_1\) in \(N(x)\), then \(\langle N(y) \rangle \cong H_1 \cup T\) and \(\langle N(x) \rangle \cong G_1 \cup S\). Clearly \(V(G)\) can be partitioned into three independent sets, namely, \(\langle X \rangle \cup Y_1, \langle y \rangle \cup X_2, X_1 \cup Y_2\) and hence \(\chi(G) = 3\).

By Theorem 2.9, \(G_1\) and \(H_1\) contains a dominating \(K_1\) or \(K_2\). By using (ii) of Observation 2.10.1, let us assume that \(G_1\) and \(H_1\) contains a dominating \(K_2\), say \(x_1x_2\) and \(y_1y_2\) respectively. In this case, let Ann start by presenting the vertices in the order \(x, y, x_1, x_2\). Without loss of generality, let the colors assigned by Ben be 1, 2, 3 to the vertices \(x, y, x_1, x_2\) respectively. Next, let Ann present \(y_1\). If Ben assigns \(y_1\) with the color 1 or a new color, then let Ann next present the vertices in the order \(y_2, y, x_1, x_2\). If Ben assigns \(y_1\) with the color 3, then let Ann next present the vertices in the order \(y_2, y, x, x_1, x_2\). In any case, Ben has an available color for the vertices \(y_1, y_2, y\). Next, let Ann present the remaining uncolored vertices of \(G\) in any order. By using (i) of Observation 2.10.1, Ben has \(c(x_1) = 2, c(x_2) = 3, c(y_1), c(y_2)\) as available colors for the remaining vertices of \(X_1, X_2, Y_1, Y_2\), respectively. Thus \(G\) is \(k\)-indicated colorable.

Finally for the cases when any edge between \(N(x)\) and \(N(y)\) is between two isolated vertices or when there are no edges between \(N(x)\) and \(N(y)\), a similar strategy as given above or even a simpler strategy will yield a winning strategy for Ann using \(k\) colors.

Subcase 2.4 \(A = \emptyset, X_1, X_2, Y_1\) and \(Y_2\) are non-empty and \(G'\) is non-bipartite.

In this case, \(G'\) is non-bipartite and \(\{P_5, C_5\}\)-free graph. Hence \(G'\) contains a \(K_3\). Let us first consider the case when \(G'\) contains a \(K_3\) such that none of the vertices in that \(K_3\) is isolated in \(N(x)\) or \(N(y)\). Let \(\langle x_1, x_2, y_1 \rangle\) be such a \(K_3\), where \(x_1 \in X_1, x_2 \in X_2\) and \(y_1 \in Y_1\). Let \(G_1\) be the component of \(N(x)\) containing \(x_1\) and \(x_2\). Also let \(H_1\) be the non-trivial component containing \(y_1\) and \(y_1y_2 \in E(H_1)\). Suppose \(x_1y_2, x_2y_2 \notin E(G)\) or \(x_1y_2, x_2y_2 \in E(G)\), \(\langle x, x_1, x_2, y_1, y_2 \rangle \cong Kite \subseteq G\) or \(\langle x_1, x_2, y_1, y_2 \rangle \cong K_4 \subseteq G\) respectively and hence a contradiction in both the cases. Thus exactly one of these have to be an edge. Without loss of generality, let it be \(x_1y_2\).

Let us determine all possible edges in \(G'\).

Claim 3 \([y_1, X_1 \cup X_2]\) and \([x_1, Y_1 \cup Y_2]\) are complete.

Suppose \([y_1, X_1]\) is not complete, there exists \(u \in X_1\) such that \(uy_1 \notin E(G)\). Then \(\langle u, x, x_1, x_2, y_1 \rangle \cong Kite\) (if \(ux_2 \notin E(G)\)) or \(\langle u, x_1, x_2, y_1, y_2 \rangle \cong Bull\) (if \(ux_2 \in E(G)\)), a contradiction. Similarly \([y_1, X_2]\) is complete. Thus \([y_1, X_1 \cup X_2]\) is complete. By the symmetry of \(x_1, y_1\), we get \([x_1, Y_1 \cup Y_2]\) is also complete.

Claim 4 \([y_2, X_1]\) and \([x_2, Y_1]\) are complete.

Suppose there exists \(u \in X_1\) such that \(uy_2 \notin E(G)\), then \(\langle u, x, x_1, x_2, y_2 \rangle \cong Bull\) (if \(ux_2 \notin E(G)\)) or \(\langle x, x_1, x_2, u, y_2 \rangle \cong Kite\) (if \(ux_2 \in E(G)\)), a contradiction. Thus \([y_2, X_1]\) is complete. Similarly, we get that \([x_2, Y_1]\) is also complete.
Subcase 2.5

A vertices finally, Ann wins the game using edge between a vertex in $A$ and $X$ respectively then $\langle \{x, y, 1, 2, 3\} \rangle \equiv Bull \subseteq G$, a contradiction. Thus if the common neighbor of an edge in $G_1$ or $H_1$ is adjacent to a vertex in $H_1$ or $G_1$ respectively then $\langle N(y) \rangle \subseteq H_1 \cup T$ and $N(x) \subseteq G_1 \cup S$. Clearly in this case $V(G)$ can be partitioned into three independent sets namely, $\langle \{x \cup Y_1, \{y\} \cup X_1, X_2 \cup Y_2\} \rangle$ and hence $\chi(G) = 3$.

By Theorem 2.9 and (ii) of Observation 2.10.1, let $G_1$ and $H_1$ contain dominating $K_2$’s. Without loss of generality, let us assume that $x_1x_2$ and $y_1y_2$ are dominating $K_2$ in $G_1$ and $H_1$ respectively. In this case, let Ann start by presenting the vertices in the order $x, x_1, y_1, y_2, y$. Let the colors assigned by Ben be 1, 2, 3 to the vertices $x, x_1, x_2$ respectively. For $y_1, y_2, y$, Ben has 1, 3, 2 respectively as available colors and hence cannot block a vertex at this stage. Next, let Ann present the remaining uncolored vertices of $G$ in any order. By using (i) of Observation 2.10.1, Ben has 2, 3, $c(y_1), c(y_2)$ as available colors for the remaining vertices of $X_1, X_2, Y_1, Y_2$ respectively. Thus $G$ is $k$-indicated colorable.

Let us next consider the case when any $K_3$ in $G'$ has one vertex which is isolated in $\langle N(x) \rangle$ or $\langle N(y) \rangle$. The graph $G \setminus S \cup T$ is isomorphic to a graph that has already been considered in Subcase 2.3 and hence $k$-indicated colorable using $k$ colors. Also for $s \in S$ and $t \in T$, $N(s) \subseteq N(x)$ and $N(t) \subseteq N(y)$. Thus by presenting these vertices finally, Ann wins the game using $k$ colors.

Subcase 2.5

$A, X_1, Y_1$ are non-empty, $X_2 = \emptyset$ and $Y_2 = \emptyset$.

Here in this case we claim the following.

Claim 6 For any vertex $x_1 \in X_1$ either $[x_1, Y_1]$ is complete or $[x_1, A]$ is complete.

Otherwise there exist vertices $y_1 \in Y_1, z_1 \in A$ such that $x_1y_1 \notin E(G)$ and $x_1z_1 \notin E(G)$, then $\langle \{z_1, x, y, x_1, y_1\} \rangle \equiv Kite$ (if $y_1z_1 \in E(G)$) or $\langle \{z_1, x, y, x_1, y_1\} \rangle \equiv Bull$ (if $y_1z_1 \notin E(G)$). Similarly for any vertex $y_1 \in Y_1$, either $[y_1, X_1]$ is complete or $[y_1, A]$ is complete.

Clearly $\chi(G) = 3$. Let Ann first present the vertices $x, y$. Next let Ann present the vertices of $A, X_1, Y_1$. Since $k \geq 3$, Ben always has an available color for the vertices $x, y$ and $A$. Also the color of $x$ and the color of $y$ are available to the vertices of $Y_1$ and $X_1$ respectively. Hence $G$ is $k$-indicated colorable.

Subcase 2.6

$A, X_1, Y_1$ are non-empty and either $X_2 = \emptyset$ or $Y_2 = \emptyset$.

Without of loss of generality, let $Y_2 = \emptyset$. Then by Subcase 2.5, $X_2 \neq \emptyset$. Let $z_1, x_1, x_2, y_1$ be the vertices in $A, X_1, X_2, Y_1$ respectively such that $x_1x_2$ be an edge. If $z_1$ is adjacent to both $x_1$ and $x_2$ then $\langle \{z_1, x, x_1, x_2\} \rangle \equiv K_4$, a contradiction. Thus any vertex in $A$ cannot be adjacent to both the ends of any edge in $\langle X_1 \cup X_2 \rangle$. Suppose $[A, (X_1 \cup X_2) \setminus S] = \emptyset$, then $V(G)$ can be partitioned into three independent sets namely, $\langle \{x\} \cup Y_1, \{y\} \cup X_1, X_2 \cup A \rangle$ and hence $\chi(G) = 3$. Suppose there exists an edge between a vertex in $A$ and a vertex in $\langle X_1 \cup X_2 \rangle \setminus S$, without loss of generality, let it be $z_1x_1$.

Claim 7 $[x_1, A]$ is complete.
Otherwise there exists a vertex \( z_2 \in A \) such that \( z_2 x_1 \notin E(G) \), then \( \langle \{ z_2, y, z_1, x_1, x_2 \} \rangle \cong C_5 \) or \( P_5 \), a contradiction.

Claim 8 \([X_2, A] = \emptyset \) and \([X_2, Y_1]\) is complete.

For any neighbor \( u \) of \( x_1 \) in \( X_2 \), it is immediate from the Claim 7 that \([u, A] = \emptyset \). Since \( X_2 \) does not contain isolated vertices, for any \( q \in X_2 (q x_1 \notin E(G)) \), there exists a \( p \in X_1 \), such that \( pq \in E(G) \). Suppose \( q z_1 \in E(G) \), then \( \langle \{ p, q, z_1, x_1, x_2 \} \rangle \cong P_5 \) (if \( x_2 p \notin E(G) \)) or \( \langle \{ p, q, z_1, x_1, x_2 \} \rangle \cong C_5 \) (if \( x_2 p \in E(G) \)), a contradiction. Thus \([A, X_2] = \emptyset \) and by Claim 6, \([X_2, Y_1]\) is complete.

By the arguments similar to Claim 8, it is easy to observe that any vertex in \( A \) cannot have neighbors in more than one component in \( \langle (X_1 \cup X_2) \rangle \). Let \( G_1 \) be the component contains \( x_1 \) in \( (X_1 \cup X_2) \). Now we claim that \([V(G_1) \cap X_1, A]\) is complete. Otherwise, there exist vertices \( u \in V(G_1) \cap X_1, v \in V(G_1) \cap X_2 \) and \( z_1 \in A \) such that \( \langle \{ u, v, x_1 \} \rangle \cong P_3 \subseteq G_1 \) and \( u z_1 \notin E(G) \) then \( \langle \{ u, v, x_1, y_1 \} \rangle \cong P_3 \subseteq G \), a contradiction.

By Theorem 2.9 and (ii) of Observation 2.10.1, let us assume that each of the non-trivial component in \( \langle N(x) \rangle \) contains a dominating \( K_2 \). Let Ann start by presenting the vertices in the order \( x, y, z_1 \) and let \( 1, 2, 3 \) be the colors given by Ben. Next, let Ann present the dominating \( K_2 \)'s in each of the non-trivial component. It can be seen that the color of \( y, z_1 \) and \( x \) are always available for the vertices of \( X_1, X_2 \) and \( Y_1 \) respectively. Thus by using Claims 7 and 8, if Ann presents the remaining vertices in any order, Ben cannot force a block vertex. Thus \( G \) is \( k \)-indicated colorable.

Subcase 2.7 \( A, X_1, Y_1, X_2, Y_2 \) are non-empty.

Let \( z_1, x_1, x_2, y_1, y_2 \) be the vertices in \( A, X_1, X_2, Y_1, Y_2 \) respectively such that \( x_1 x_2, y_1 y_2 \) be the edges. As observed already, \( z_1 \) cannot be adjacent to both the ends of the edges \( x_1 x_2 \) and \( y_1 y_2 \) respectively. Also \( z_1 \) must be adjacent to one end of each of the edges \( x_1 x_2 \) and \( y_1 y_2 \), otherwise by Claim 6, \( \langle \{ x_1, x_2, y_1, y_2 \} \rangle \cong K_4 \), a contradiction. This is true for all the edges in \( \langle X_1 \cup X_2 \rangle \) and \( \langle Y_1 \cup Y_2 \rangle \). Without loss of generality, let \( x_1 \) and \( y_1 \) be the vertices adjacent to \( z_1 \). By Claim 8, \([X_2, A] = [Y_2, A] = \emptyset \) and thus \([X_1 \setminus S, A] \) and \([Y_1 \setminus T, A] \) are complete. By Claim 6, \([Y_2, X_1 \cup X_2]\) and \([X_2, Y_1 \cup Y_2]\) are complete. Also \([X_1 \setminus S, Y_1 \setminus T] = \emptyset \), otherwise \( K_4 \subseteq G \). If there exists another component containing an edge \( x_3 x_4 \) in \( \langle X_1 \cup X_2 \rangle \), then \( \langle \{ x_1, x_2, x_3, x_4 \} \rangle \cong P_5 \subseteq G \), a contradiction. Thus \( \langle (X_1 \cup X_2) \setminus S \rangle \) is a single component, say \( G_1 \). Similarly \( \langle (Y_1 \cup Y_2) \setminus T \rangle \) is also a single component, say \( H_2 \).

One can observe that, \( \langle z_1, x, y, x_1, x_2, y_1, y_2 \rangle \cong \overline{C_7} \subseteq G \) and \( k \geq \chi(G) = \chi(\overline{C_7}) = 4 \). By Theorem 2.9 and (ii) of Observation 2.10.1, let \( G_1 \) and \( H_1 \) contain a dominating \( K_2 \). Without loss of generality, let the colors assigned by Ben be \( 1, 2, 3 \). Now Ben has \( \{2, 4\}, \{2, 3, 4\}, \{1, 4\}, \{1, 3, 4\}\) as available colors for the vertices \( x_1, x_2, y_1, y_2 \) respectively. Next let Ann present the vertex \( x_2 \). If Ben assigns \( x_2 \) with the color \( 3 \) or \( 4 \) (or a new color), then let Ann present the vertices in the order \( y_1, y_2, x_1 \). If Ben assigns \( x_2 \) with the color \( 2 \), then let Ann next present the vertices in the order \( x_1, y_2, y_1 \). In any case, Ben has available colors for the vertices \( y_1, y_2, x_1 \). Next, let Ann present the remaining uncolored vertices in the order
Let $x, y$ and $z$ be the mutually adjacent vertices which dominates the vertices of $G$. Let $X, Y, Z$ be the individual neighbors of $x, y, z$ respectively and $A, B, C$ be the common neighbors of $xy, yz, zx$ respectively. Since no $K_2$ dominates $G$, we have that $X, Y$ and $Z$ are non-empty. While considering $x, y, z, X, Y$, we have that $[z, X] = \emptyset$ and $[z, Y] = \emptyset$ and by Claim 6, $[X, Y]$ is complete. Similarly $[Y, Z]$ and $[Z, X]$ are complete. Now we claim that $X, Y$ and $Z$ are independent sets. Suppose $X$ is not independent, there exists an edge $x_1x_2$ in $(X), y_1 \in Y$ and $z_1 \in Z$ such that $\langle \{x_1, x_2, y_1, z_1\}\rangle \cong K_4$, a contradiction. Similarly $Y$ and $Z$ are independent sets.

Now consider the sets $A, B$ and $C$. Clearly $A, B$ and $C$ are independent sets. Also $[A, Z], [B, X]$ and $[C, Y]$ are complete, otherwise $Kite \subseteq G$. Next we claim that $[A, X \cup Y] = [B, Y \cup Z] = [C, Z \cup X] = \emptyset$. Suppose $[A, X \cup Y] \neq \emptyset$, there exist vertices $a_1 \in A, x_1 \in X$ such that $a_1x_1 \in E(G)$. Then $\langle \{a_1, x_1, y_1, z_1\}\rangle \cong Bull$ and in order to prevent this $Bull$, $a_1y_1 \in E(G)$. Then $\langle \{a_1, x_1, y_1, z_1\}\rangle \cong Kite$, a contradiction. Thus $[A, X \cup Y] = \emptyset$. Similarly $[B, Y \cup Z] = [C, Z \cup X] = \emptyset$. Further we claim that $[A, B], [B, C]$ and $[C, A]$ are complete. Suppose $[A, B]$ is not complete, there exist vertices $a_1 \in A, b_1 \in B$ such $a_1b_1 \notin E(G)$. Then $\langle \{x_1, y_1, z_1, a_1, b_1\}\rangle \cong Bull$, a contradiction. Thus $[A, B]$ is complete and similarly $[B, C]$ and $[C, A]$ are complete.

Let $x_1, y_1, z_1$ be the vertices in $X, Y, Z$ respectively. Let Ann start by presenting the vertices in the order $x, y, z, x_1$. Since $k \geq 3$, without loss of generality, let the colors assigned by Ben be 1, 2, 3 to the vertices $x, y, z$ respectively. If Ben assigns the color 2 or a new color to the vertex $x_1$, then Ann will present the vertices in the order $z_1, y_1$. If Ben assigns the color 3 to the vertex $x_1$, then Ann will present the vertices in the order $y_1, z_1$. Next, let Ann present the vertices in the order $X \setminus \{x_1\}, Y \setminus \{y_1\}, Z \setminus \{z_1\}, A, B, C$. By using (i) of Observation 2.10.1, Ben has an available colors $c(x_1), c(y_1), c(z_1), c(z), c(x), c(y)$ for all the vertices of $X \setminus \{x_1\}, Y \setminus \{y_1\}, Z \setminus \{z_1\}, A, B, C$ respectively. Thus $G$ is $k$-indicated colorable. □

Corollary 2.11 is an immediate consequence of Theorems 2.6, 2.8 and 2.10.

Corollary 2.11 Let $G$ be a $\{P_5, K_4, Kite, Bull\}$-free graph. Then $G$ is $k$-indicated colorable for all $k \geq \chi(G)$.

Next, let us consider a structural characterization of a family of $P_6$-free graphs. The study of $P_6$-free graphs has also been of interest for a lot of coloring parameters. See for instance, [11,13,15]. Here, we would like to consider connected $\{P_6, C_5, K_{1,3}\}$-free graphs that contain an induced $C_6$.

Theorem 2.12 If $G$ is a connected $\{P_6, C_5, K_{1,3}\}$-free graph which contain an induced $C_6$ then $G$ is isomorphic to the graph given in Fig. 2. Here $V(G) = (\cup_{i=0}^5 A_i) \cup (\cup_{j=0}^5 B_j)$ and the circle denote the complete subgraph induced by the sets $A_i$ and $B_j$ and the double line between any two sets denote the join of the two sets.

Proof Let $G$ be a connected $\{P_6, C_5, K_{1,3}\}$-free graph that contain an induced $C_6 \cong \langle \{v_0, v_1, v_2, v_3, v_4, v_5\}\rangle = \langle N_0 \rangle$, and let $N_i = \{x \in V(G) : \text{dist}(x, N_0) = i\}, i \geq 1$. ☐
Claim 1 If \( x \in N_1 \), then \( \langle N(x) \cap N_0 \rangle \cong P_3 \) or \( 2K_2 \).

For \( x \in N_1 \), the possibilities for \( \langle N(x) \cap N_0 \rangle \) are \( K_1, K_2, P_3, P_4, P_5, 2K_1, 3K_1, 2K_2, K_1 \cup K_2, K_1 \cup P_3 \) and \( C_6 \). Here (a) if \( \langle N(x) \cap N_0 \rangle \cong K_1 \) or \( K_2 \), then \( P_6 \subseteq G \), (b) if \( \langle N(x) \cap N_0 \rangle \cong P_4 \) or \( K_1 \cup K_2 \), then \( C_5 \subseteq G \), (c) if \( \langle N(x) \cap N_0 \rangle \cong P_5 \) or \( C_6 \) (or) \( K_1 \cup P_3 \) (or) \( 3K_1 \) (or) \( 2K_1 \), then \( K_{1,3} \subseteq G \), a contradiction. Finally, if \( \langle N(x) \cap N_0 \rangle \cong P_3 \) or \( 2K_2 \), we see that neither \( P_6 \) nor \( C_5 \) (nor) \( K_{1,3} \) is an induced subgraph of \( \langle N_0 \cup N_1 \rangle \). Thus \( \langle N(x) \cap N_0 \rangle \cong P_3 \) or \( 2K_2 \).

Throughout this proof, for any integer \( i \), \( v_i \) means \( v_i \) (mod 6) and \( A_i \) means \( A_i \) (mod 6).

For \( 0 \leq i \leq 5 \), let \( A_i = \{ x \in N_1 : N(x) \cap N_0 = \{ v_{i-1}, v_i, v_{i+1} \} \} \cup \{ v_i \} \) and \( B_i = \{ x \in N_1 : N(x) \cap N_0 = \{ v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2} \} \} \).

Claim 2 \( \cup_{i=0}^5 A_i \cong K[C_6] \).

For every \( i \), \( 0 \leq i \leq 5 \), we have (a) \( \langle A_i \rangle \) is complete (Suppose there exist vertices \( x, y \in A_i \) such that \( x y \notin E(G) \), then \( \langle \{ v_{i+1}, v_{i+2}, x, y \} \rangle \cong K_{1,3} \subseteq G \)), (b) \([A_i, A_{i+1}] \) is complete, (if not, there exist vertices \( x \in A_i \) and \( y \in A_{i+1} \) such that \( x y \notin E(G) \), and hence \( \langle \{ x, v_i, y, v_{i+2}, v_{i+3}, v_{i+4} \} \rangle \cong P_6 \subseteq G \)), (c) \([A_i, A_{i+2}] \) = \( \emptyset \), (Suppose there exist vertices \( x \in A_i \) and \( y \in A_{i+2} \) such that \( x y \in E(G) \), then \( \langle \{ x, y, v_{i+3}, v_{i+4}, v_{i+5} \} \rangle \cong C_5 \subseteq G \)), (d) \([A_i, A_{i+3}] \) = \( \emptyset \) (otherwise as shown previously, we can find \( x \in A_i \) and \( y \in A_{i+3} \) such that \( x y \in E(G) \), and \( \langle \{ x, v_{i-1}, v_{i+1}, y \} \rangle \cong K_{1,3} \subseteq G \)). Thus from (a), (b), (c) and (d), it can be seen that \( \cup_{i=0}^5 A_i \cong K[C_6] \).

Claim 3 \( B_i \) is complete, for \( i = 0, 1, 2, 3, 4, 5 \).

Here, if there exist vertices \( x, y \in B_i \) such that \( x y \notin E(G) \), then \( \langle \{ v_{i-1}, v_i, x, y \} \rangle \cong K_{1,3} \subseteq G \), a contradiction.

Claim 4 \([B_i, B_{i+1}] \) = \( \emptyset \), for \( i = 0, 1, 2, 3, 4, 5 \).

Suppose there exist vertices \( x \in B_i \) and \( y \in B_{i+1} \) such that \( x y \in E(G) \), then \( \langle \{ x, y, v_{i+1}, v_{i-2} \} \rangle \cong K_{1,3} \subseteq G \), a contradiction.

Claim 5 \([A_i, B_i] \) = \( \emptyset \), \( i = 0, 1, 2, 3, 4, 5 \).

On the contrary, if there exist vertices \( x \in A_i \) and \( y \in B_i \) such that \( x y \in E(G) \), then \( \langle \{ y, x, v_{i-2}, v_{i+2} \} \rangle \cong K_{1,3} \subseteq G \), a contradiction.
Proof
Let us consider the graph \( V \) of \( G \). Without loss of generality, let it be the maximum clique. It is easy to see that \( \chi(\{v_i-1, v_i, v_i+1, v_i+2, v_i+3\}) \cong C_5 \subseteq G \), a contradiction.

Claim 7 [A\(_i\), B\(_{i+2}\)] is complete, for \( i = 0, 1, 2, 3, 4, 5 \).

It is easy to observe that if there exist vertices \( x \in A_i \) and \( y \in B_{i+2} \) such that \( xy \notin E(G) \), then \( \{x, v_i-1, v_i, v_i+1\} \cong C_5 \subseteq G \), a contradiction.

Claim 8 \( N_i = \emptyset \), for all \( i \), \( i \geq 2 \).

It is enough to show that \( N_2 = \emptyset \). Suppose \( N_2 \neq \emptyset \), then there exists a vertex \( x \in N_2 \). Since \( G \) is connected, there exists a vertex \( y \in A_j \) or \( y \in B_j \) for some \( j \in \{0, 1, \ldots, 5\} \) such that \( xy \in E(G) \). Then \( \{y, v_j-1, v_j+1, x\} \cong K_{1,3} \subseteq G \), a contradiction. Thus \( V(G) = N_0 \cup N_1 \).

Note that \( B_j = B_{j+3} \) for every \( j \in \{0, 1, 2\} \). From all these claims, we see that \( G \) will be isomorphic to the graph shown in Fig. 2.

An immediate consequence of Theorem 2.12 is given in Corollary 2.13.

Corollary 2.13 If \( G \) is a connected \( \{P_6, C_5, \overline{P}_5, K_{1,3}\} \)-free graph that contains an induced \( C_6 \) then \( G \cong \mathbb{K}[C_6] \).

Proof Let \( G \) be a connected \( \{P_6, C_5, \overline{P}_5, K_{1,3}\} \)-free graph that contains an induced \( C_6 \). In this case, it can be seen from the proof of Theorem 2.12 that if \( x \in N_1 \), then \( N(x) \cap N_0 \cong P_3 \) and thereby by using Claims 2 and 8 of Theorem 2.12, we see that \( G \cong \mathbb{K}[C_6] \).

Even though the graph \( G \) shown in Fig. 2 looks simple, it looks challenging to obtain the indicated chromatic number of \( G \). So, we have considered the indicated coloring of \( \mathbb{K}[C_6] \).

Proposition 2.14 For \( 1 \leq i \leq 6 \), let \( m_i \)’s be positive integers. Then the graph \( G \cong \mathbb{K}[C_6](m_1, m_2, m_3, m_4, m_5, m_6) \) is \( k \)-indicated colorable for all \( k \geq \chi(G) \).

Proof Let us consider the graph \( G \cong \mathbb{K}[C_6](m_1, m_2, m_3, m_4, m_5, m_6) \), where \( m_i \geq 1 \) and \( V_i = V(K_{m_i}) \), for \( 1 \leq i \leq 6 \). Let \( k \) be a positive integer such that \( k \geq \chi(G) \) and let \( \{1, 2, \ldots, k\} \) be the set of colors. We shall show that \( G \) is \( k \)-indicated colorable. It is easy to see that \( \chi(G) = \omega(G) \). Let Ann start by presenting the vertices of a maximum clique. Without loss of generality, let it be \( V_1 \cup V_2 \). Since \( k \geq \omega(G) \), Ben has an available color for all the vertices in \( V_1 \cup V_2 \) and let the colors given to \( V_1 \) and \( V_2 \) be \( \{1, 2, \ldots, m_1\} \) and \( \{m_1+1, m_1+2, \ldots, m_1+m_2 = \omega(G)\} \) respectively. Now, let Ann present the vertices of \( V_3 \) and \( V_6 \) (in any order). Since \( V_1 \cup V_2 \) is maximum clique, \( |V_3| \leq |V_1| \) and \( |V_6| \leq |V_2| \), and hence Ben has an available color for all the vertices of \( V_3 \) and \( V_6 \). Since \( [V_i, V_{i+1}] \) is complete, \( \{1, 2, \ldots, m_1\} \) and \( \{m_1+1, m_1+2, \ldots, \omega(G)\} \) are colors available for \( V_5 \) and \( V_4 \) respectively. Without loss of generality, let the number of colors available for \( V_4 \) minus \( |V_4| \) be less than or equal to the number of colors available for \( V_5 \) minus \( |V_5| \). Now, let Ann present the vertices of \( V_4 \) until either the number of available colors for \( V_5 \) is equal to \( |V_5| \) or every vertex in \( V_4 \) is colored. In either case, let Ann proceed by presenting all the vertices of \( V_5 \). If there are some uncolored vertices in \( V_4 \), let Ann present those vertices finally.
Here it can be seen that, the number of available color for the vertices of \( V_4 \cup V_5 \) is at least \( \omega(G) \) and \( |V_4| + |V_5| \leq \omega(G) \). Thus in this ordering, Ben will always have an available color for the remaining vertices. \( \square \)

An immediate consequence of Corollary 2.13 and Proposition 2.14 is given in Corollary 2.15.

**Corollary 2.15** If \( G \) is a connected \( \{P_6, C_5, \overline{P_5}, K_{1,3}\}\)-free graph that contains an induced \( C_6 \), then \( G \) is \( k \)-indicated colorable for all \( k \geq \chi(G) \).

For \( l \geq 4 \), the technique mentioned in Proposition 2.14 will not work for the graph \( \mathbb{K}[C_{2l}](m, m, \ldots, m) \) while using \( k \) colors, \( 2m + 1 \leq k \leq 3m - 1 \). This has been discussed in Sect. 3. Also the indicated coloring of \( \mathbb{K}[C_{2l+1}] \), \( l \geq 2 \) is discussed in Sect. 3.

Next, it can be shown that there is an indicated coloring using \( 2m \) colors for the complete expansion of a bipartite graph by replacing each vertex with exactly \( m \) vertices.

**Theorem 2.16** Let \( G \) be a bipartite graph and \( H \cong \mathbb{K}[G](m, m, \ldots, m) \) be the complete expansion of \( G \), for some \( m \geq 1 \). Then \( \chi_i(H) = 2m \).

**Proof** Without loss of generality, let \( G \) be a connected bipartite graph with bipartition \( X \) and \( Y \). Let \( H \cong \mathbb{K}[G](m, m, \ldots, m) \) be the complete expansion of \( G \), for some \( m \geq 1 \). Let \( H_X \) and \( H_Y \) be the expansion of the vertices of \( X \) and \( Y \) respectively with the subgraph \( K_m \). Let the color set be \( \{1, 2, \ldots, 2m\} \). Let Ann start by presenting the vertices of any \( K_m \) in \( H_X \). Without loss of generality, let the colors used by Ben for these vertices be \( \{1, 2, \ldots, m\} \). Let Ann proceed by presenting all the neighbors of the colored vertices in \( \mathbb{K}[G] \). These neighbors will be in \( H_Y \) and for each of the presented vertex in \( H_Y \), there is a color available for Ben from \( \{m + 1, m + 2, \ldots, 2m\} \). Next, let Ann presents all the neighbors of the colored vertices in \( H_Y \) which are in \( H_X \). Here Ben will have an available color from \( \{1, 2, \ldots, m\} \) for each of the presented vertex. Let Ann continue this process by presenting all the neighbors of the colored vertices until all the vertices in \( H \) are colored. For every vertex in \( H_X \) and \( H_Y \), there is an available color from \( \{1, 2, \ldots, m\} \) and \( \{m + 1, m + 2, \ldots, 2m\} \) respectively at any stage of this process and hence Ben will not be able to create any block vertex. Thus \( H \) is \( 2m \)-indicated colorable and hence \( \chi_i(H) = 2m \). \( \square \)

## 3 Indicated Coloring of \( \mathbb{K}[C_5] \) and Some of its Consequences

Let us start this section by recalling two of the results which were proved in [14].

**Theorem 3.1** [14] Any graph \( G \) is \( k \)-indicated colorable for all \( k \geq \text{col}(G) \).

We know that, for the join of two graphs \( G_1 \) and \( G_2 \), \( \chi(G_1 + G_2) = \chi(G_1) + \chi(G_2) \). The same holds even for the indicated chromatic number.

**Theorem 3.2** [14] Let \( G = G_1 + G_2 \). If \( G_1 \) is \( k_1 \)-indicated colorable for every \( k_1 \geq \chi_i(G_1) \) and \( G_2 \) is \( k_2 \)-indicated colorable for every \( k_2 \geq \chi_i(G_2) \), then \( \chi_i(G) = \chi_i(G_1) + \chi_i(G_2) \) and \( G \) is \( k \)-indicated colorable for all \( k \geq \chi_i(G) \).
Let us recall the structural characterization of \( \{P_2 \cup P_3, C_4\}\)-free graphs, \( \{P_5, C_4\}\)-free graphs and \( \{P_5, (P_2 \cup P_3), \overline{P_5}, \text{Dart}\}\)-free graphs which contains an induced \( C_5 \).

The graphs \((P_2 \cup P_3)\) and Dart are shown in Fig. 1.

**Theorem 3.3** \([5]\) If \( G \) is a connected \( \{P_2 \cup P_3, C_4\}\)-free graph, then \( G \) is chordal or there exists a partition \((V_1, V_2, V_3)\) of \( V(G) \) such that

1. \( \langle V_1 \rangle \cong K_m \), for some \( m \geq 0 \),
2. \( \langle V_2 \rangle \cong K_t \), for some \( t \geq 0 \),
3. \( \langle V_3 \rangle \) is isomorphic to a graph obtained from one of the basic graphs \( G_t \) \((1 \leq t \leq 17)\) shown in Fig. 3 by expanding each vertex indicated in circle by a complete graph (of order \( \geq 1 \)),
4. \([V_1, V_3]\) = \( \emptyset \) and
5. \([V_2, V_3 \setminus S]\) is complete.

**Theorem 3.4** \([7]\) Let \( G \) be a connected \( \{P_5, C_4\}\)-free graph. Then \( V(G) = V_1 \cup V_2 \) such that

1. \( \langle V_1 \rangle \) is a \( P_5 \)-free graph which is also chordal.
2. If \( V_2 \neq \emptyset \), then \( \langle V_2 \rangle = A_1 \cup A_2 \cup \cdots \cup A_l \) where each \( A_i \) is a \( \mathbb{K}[C_5] \), for every \( i \in \{1, 2, \ldots, l\} \) for some \( l \geq 1 \). Also, \( \langle N(A_i) \rangle \) is a complete subgraph of \( V_1 \) and \([A_i, N(A_i)]\) is complete.

Recall that a graph \( G \) is said to be a split graph, if \( V(G) \) can be partitioned into two subsets such that the subgraph induced by one set is a clique and the other is totally disconnected.

**Theorem 3.5** \([1]\) If \( G \) is a connected \( \{P_5, (P_2 \cup P_3), \overline{P_5}, \text{Dart}\}\)-free graph that contains an induced \( C_5 \), then \( G \) is either isomorphic to \( C_5(S_1, S_2, S_3, S_4, S_5) \) or \( C_5(S_1, S_2, S_3, S_4, S_5) + H \), where each \( S_i \) is an induced split subgraph of \( G \), \( H \) is nonempty and \( H \subseteq G \).
We further claim that the subgraph $H$ mentioned in Theorem 3.5 is complete. Let $G$ be a connected $\{P_5, (P_2 \cup P_3), P_5, \text{Dart}\}$-free graph that contains an induced $C_5$. Let $C = \{\{v_0, v_1, v_2, v_3, v_4\}\} \cong C_5 \subseteq G$. Suppose there exist two non-adjacent vertices $x$ and $y$ in $H$. Then $\langle\{v_1, v_2, x, y, v_4\}\rangle \cong (P_2 \cup P_3) \subseteq G$, a contradiction.

It can be noted that $K[C_5]$ is one of the graphs mentioned in Fig. 3 of Theorem 3.3. Also $K[C_5]$ is an induced subgraph of the graphs mentioned in Theorems 3.4 and 3.5. So, we shall first consider the indicated coloring for the complete expansion of $C_5$. Even though $K[C_5]$ looks simple, a technique as done for $K[C_6]$ (see Proposition 2.14) doesn’t look possible for $K[C_5]$. Hence we are forced to adopt a laborious process.

**Theorem 3.6** For $1 \leq i \leq 5$, let $m_i$’s be positive integers. Then the graph $G \cong K[C_5](m_1, m_2, m_3, m_4, m_5)$ is $k$-indicated colorable for all $k \geq \chi(G)$.

**Proof** Let the graph $G \cong K[C_5](m_1, m_2, m_3, m_4, m_5)$, where $m_i \geq 1$ and $V_i = V(K_{m_i})$ for $1 \leq i \leq 5$. Also, let $k$ be a positive integer such that $k \geq \chi(G)$ and let $\{1, 2, \ldots, k\}$ be the set of colors. We shall show that $G$ is $k$-indicated colorable. Without much difficulty, it can be seen that $\alpha(G) = 2$. Suppose $\omega(G) \geq \frac{|V(G)|}{\alpha(G)} = \frac{|V(G)|}{2}$, let Ann start by presenting the vertices of a maximum clique. Without loss of generality, let it be $V_1 \cup V_2$. By our choice of $k$, $k \geq \chi(G)$, and hence $|V(G)| \leq 2\omega(G) \leq 2k$. Next, let Ann present the vertices of $V_3$ and $V_5$ in any order. Since $V_1 \cup V_3$ and $V_2 \cup V_3$ induce a clique and $V_1 \cup V_2$ is a maximum clique, we see that $|V_3| \leq |V_2|$ and $|V_3| \leq |V_1|$. Thus Ben will have an available color for each of the vertex in $V_3$ and $V_5$. Finally, let Ann present the vertices of $V_4$. As we have already observed, $|V(G)| \leq 2\omega(G)$ and hence $|V_1| + \cdots + |V_5| \leq 2(|V_1| + |V_2|)$ which in turn implies that $|V_3| + |V_4| + |V_5| \leq k$. Thus here also Ben has an available color for the vertices of $V_4$. Hence, Ann wins the game on $k$ colors.

Now let us consider the case when $\omega(G) < \frac{|V(G)|}{2}$. We know that, $\frac{|V(G)|}{2} \leq \chi(G) \leq k$. In this case, let Ann first present the vertices of $V_1$. For $2 \leq i \leq 5$, let $\mathcal{N}_i$ denote the set of all uncolored vertices in $V_i$ and $\mathcal{C}_i$ denote the set of all available colors for the vertices in $\mathcal{N}_i$. Let $c(v)$ denote the color given by Ben to the vertex $v$.

The following are some of the observation regarding $\mathcal{N}_i$ and $\mathcal{C}_i$.

**Observation 3.6.1** (i) Once when the vertices of $V_1$ are colored by Ben, we have the following values.

$|\mathcal{C}_2| - |\mathcal{N}_i| = k - |V_1| - |V_i| > 0$, for $i \in \{2, 5\}$

$|\mathcal{C}_3| - |\mathcal{N}_i| = k - |V_i| > 0$, for $i \in \{3, 4\}$

$|\mathcal{C}_i \cup \mathcal{C}_{i+1} - |\mathcal{N}_i| - |\mathcal{N}_{i+1}| = k - (|V_i| + |V_{i+1}|) > 0$, for $i \in \{2, 3, 4\}$

(Note that, since $k > \omega(G)$, all the values given in the above equations are positive. Also note that, we shall use $\{\}$ to indicate these values. For instance, $|\mathcal{C}_2| - |\mathcal{N}_2| = k - |V_1| - |V_2|$).

(ii) For $2 \leq i \leq 5$, the sets $\mathcal{N}_i$ and $\mathcal{C}_i$ constantly change during the coloring process.

(iii) For $2 \leq i \leq 5$, if $|\mathcal{C}_i| - |\mathcal{N}_i| \geq 0$, then Ben has an available color at that stage for each of the vertex in $V_i$.

(iv) On coloring the vertices of $V_i$, the value of $|\mathcal{C}_i| - |\mathcal{N}_i|$ remains unchanged (since both $|\mathcal{C}_i|$ and $|\mathcal{N}_i|$ are reduced by 1).
Case 2

Let us now proceed with the other uncolored vertices, namely, $V$.

Observation 3.6.1, Ben has an available color for each of the vertex in $V$ vertices in guarantees that when Ann follows the above strategy for presenting the uncolored vertices in $V$. Hence Ann will have an available color for each of the vertex presented. A similar argument shows that every time a vertex is colored in $V$, Ben always has an available color for the vertices of $V$. Now for presenting the vertices of $V$, let Ann follow the same strategy as given in Case 1 for presenting the vertices in $V$.

Case 1 (i) holds.

In this case, we can easily observe that $|C_4 \cup C_5| - |N_4| - |N_5| \geq 0$ and $|C_i| - |N_i| \geq 0$, $i = 2, 4, 5$. Next, let Ann present the vertices of $V$ in any order. Since $|C_2| - |N_2| \geq 0$, Ben always has an available color for the vertices of $V$. Now for presenting the vertices of $V$, let Ann follow the same strategy as given in Case 1 for presenting the vertices in $V$.

Compare $|C_4| - |N_4|$ and $|C_5| - |N_5|$. Whichever is smaller, Ann presents an uncolored vertex from that vertex set, namely, $V$ or $V_5$. Do this again and again until $N_4 \cup N_5 = \emptyset$.

Note that even after presenting the vertices of $V$, $|C_i| - |N_i| \geq 0$ for $i = 4$ and 5, and $|C_4 \cup C_5| - |N_4| - |N_5| \geq 0$. This together with (iv) and (v) of Observation 3.6.1 guarantees that when Ann follows the above strategy for presenting the uncolored vertices in $V$, Ben will have an available color for each of these vertices.

Case 2 (ii) holds.

For $|C_2| - |N_2| = 0$, Ben should have colored the vertices of $V$ with exactly $|C_2| - |N_2| = k - |V_2|$ colors which are not given to the vertices of $V$. Next, let Ann present the vertices of $V_2$. Since $|C_2| - |N_2| = 0$, by using (iii) of Observation 3.6.1, Ben has an available color for each of the vertex in $V$. One can easily observe that every time a vertex is colored in $V$, the value of $|C_3| - |N_3|$ is reduced by 1. Also $|V_3| + |V_3| \leq \omega(G) < k$ and none of the vertex in $V_4$ is colored. Hence $|C_3| - |N_3| \geq 0$. Now, let Ann present all the uncolored vertices of $V_3$. Again by (iii) of Observation 3.6.1, Ben has an available color for each of the vertex presented. A similar argument shows that $|C_4| - |N_4| \geq 0$ and $|C_5| - |N_5| \geq 0$. As observed earlier, Ben must have colored the vertices of $V_3$ with exactly $|C_2| - |N_2| = k - |V_2|$ colors which are not given to the vertices of $V_1$. Hence, we see that $|c(v) : v \in V_3 \cap \{c(v) : v \in V_1\}| = |V_3| - |c(v) : v \in V_2|$. Also observe that, the value of $|C_4 \cup C_5| - |N_4| - |N_5|$ reduces by one every time Ben colors a vertex of $V_3$ with a color given to a vertex in $V$. Now,

$$|C_4 \cup C_5| - |N_4| - |N_5| = |C_4 \cup C_5| - |N_4|$$

$$- |N_5| = |c(v) : v \in V_3 \cap \{c(v) : v \in V_1\}|$$

$$= (k - |V_4| - |V_3|) - (|V_3| - (k - |V_1| - |V_2|))$$

$$= 2k - |V(G)| \geq 0.$$
Case 3 (iii) holds. For \(|C_4 \cup C_5| - |N_4| - |N_5| = 0\), Ben should have colored the vertices of \(V_3\) with exactly \(|C_4 \cup C_5| - |N_4| - |N_5|\) colors which are given to the vertices of \(V_1\). As observed in Case 2, \(|C_4| - |N_4| \geq 0\) and \(|C_5| - |N_5| \geq 0\). Thus Ann follow the same strategy as given in Case 1 for presenting the vertices in \(V_4 \cup V_5\) to yield a winning strategy. Next for \(i = 2, 3, |C_i| - |N_i| \geq 0\). Since \(\{c(v) : v \in V_3\} \cap \{c(v) : v \in V_4\} = \emptyset\) and \(|\{c(v) : v \in V_1\} \cap \{c(v) : v \in V_3\}| = k - |V_4| - |V_5|\), we have \(|\{c(v) : v \in V_4\} \cap \{c(v) : v \in V_1\}| = |V_1| - \{|C_4 \cup C_5| - |N_4| - |N_5|\}^* = |V_1| - (k - |V_4| - |V_5|)\). As observed in Case 2, the value of \(|C_2 \cup C_3| - |N_2| - |N_3|\) reduces by 1 every time Ben colors a vertex in \(V_4\) with a color given to a vertex in \(V_1\). Now
\[
|C_2 \cup C_3| - |N_2| - |N_3| = |C_2 \cup C_3| - |N_2| \quad - \quad |N_3| \quad - \quad \{|c(v) : v \in V_4\} \cap \{c(v) : v \in V_1\}|
= k - |V_2| - |V_3| - (|V_1| - (k - |V_4| - |V_5|))
= 2k - |V(G)| \geq 0.
\]
Thus again Ann can follow the same strategy as given in Case 1 for presenting the uncolored vertices in \(V_2 \cup V_3\) to yield a winning strategy. Hence \(G\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\). □

Natural question one can ask is the following: Can the techniques used in Theorem 3.6 and Proposition 2.14 be generalised to complete expansion of any odd cycles and even cycles respectively. The answer is no. For instance, let us consider \(G \in \mathbb{K}[C_5]\) with \(V(G) = \bigcup_{i=1}^7 V_i\), where \(|V_i| = 3\), for \(1 \leq i \leq 7\). One can easily observe that the \(\chi(G) = 7\). Since \(|V(G)| = 21\) and \(\alpha(G) = 3\), the number of vertices in each of the color class is exactly 3 for any chromatic coloring. So, if Ben can forbid at least one color class to have three vertices, then Ben will win the game. Let us show that, if Ann follows the technique used in Theorem 3.6, then Ben can forbid at least one color class to have three vertices.

Let \(\{0, 1, \ldots, 6\}\) be the set of colors. If Ann has to imitate the proof of \(\mathbb{K}[C_5]\), then Ann has to first present the vertices of \(V_1 \cup V_2\) or present the vertices of \(V_1\).

Let us first consider the case when Ann starts by presenting the vertices of \(V_1 \cup V_2\) and without loss of generality, let the colors assigned by Ben be \(1, 2, 3\) and \(4, 5, 6\) respectively. Since each of the color class has exactly three vertices, the only possibility for the color 0 are vertices from \(V_3, V_5\) and \(V_7\). Thus if Ben follows the strategy not to use 0 when the vertices in \(V_3, V_5\) and \(V_7\) are presented and to use 0 if any of the vertex in \(V_4\) or \(V_6\) is presented, then Ben will win the game.

Next, when Ann starts by presenting the vertices of \(V_1\), let the colors used by Ben be \(1, 2, 3\). Next, if Ann presents the vertices of \(V_3\) or \(V_4\), Ben will assign the colors \(1, 2, 3\) or \(4, 5, 6\) respectively. In either way, the color 0 cannot be forced by Ann to the vertices of \(V_1, V_2, V_3\) or \(V_4\). Thus in this case also, the color class of the color 0 has at most two vertices and hence Ben wins. Getting a common strategy for all the complete expansion of any odd cycles seems to be a challenging problem. Our main concern was to show that a few families of \(P_5\)-free graphs are \(k\)-indicated colorable for every \(k \geq \chi(G)\) and those families included only complete expansion of \(C_5\). Thus we have stopped with \(\mathbb{K}[C_5]\).
Next, we shall show that for \( l \geq 4 \), the technique mentioned in Proposition 2.14 will not work for the graph \( \mathbb{K}[C_{2l}](m, m, \ldots, m) \) while using \( k \) colors, \( 2m + 1 \leq k \leq 3m - 1 \). The advantage what Ann had in the complete expansion of \( C_6 \) is that the colors given to \( V_1 \) and the colors given to \( V_2 \) where available for the vertices of \( V_3 \) and \( V_4 \) respectively. This guaranteed a proper coloring for the vertices of \( V_4 \) and \( V_5 \). But for a complete expansion of \( C_{2l} \), \( l \geq 4 \), Ann will not have this control. For instance, if Ann follows the connected strategy given in Proposition 2.14 while using \( 2m + 1 \) colors, Ben’s strategy will be to color \( V_n \) colors for \( V_n \) vertices of \( G \). This would have forced Ben to give the vertices of \( V_n \) colors different from those given to \( V_x \). Thus Ann has to stop after presenting one vertex in \( V_n \). This is because the number of available colors for \( V_n \cup V_n \) plus \( |V_n \cup V_n| \) becomes zero. Next, Ann has to present the vertices of \( V_n \cup V_n \) or \( V_n \cup V_n \) depending upon the values of the number of available colors for \( V_n \cup V_n \) plus \( |V_n \cup V_n| \) and the number of available colors for \( V_n \cup V_n \) plus \( |V_n \cup V_n| \) (whichever is smaller). Thus Ann has to present the vertices of \( V_n \cup V_n \). Now Ben will color this vertex with a color given to a vertex in \( V_n \cup V_n \) and thereby the number of available colors for \( V_n \cup V_n \) plus \( |V_n \cup V_n| \) becomes zero. Hence Ann will be forced to present a vertex of \( V_n \cup V_n \) (otherwise Ben will immediately win by presenting another color of \( V_n \cup V_n \) to the vertices of \( V_n \cup V_n \)). In this case Ben will give the color not used in \( V_n \cup V_n \). This will guarantee that at least one of the vertex in \( V_n \cup V_n \cup V_n \cup V_n \) will become a block vertex. Thus the strategy mentioned in Proposition 2.14 will not work in general for larger even cycles.

Next, let us consider some of the consequences of Theorem 3.6.

**Corollary 3.7** For \( 1 \leq i \leq 5 \), let \( m_i \)'s be positive integers. Then for the graph \( G \cong \mathbb{K}[C_5](m_1, m_2, m_3, m_4, m_5) \), \( \chi(G) = \max \left\{ \omega(G), \left\lceil \frac{|V(G)|}{2} \right\rceil \right\} \).

**Proof** We know that, \( \chi(G) \geq \max \left\{ \left\lceil \frac{|V(G)|}{2} \right\rceil, \omega(G) \right\} \). If one closely observes Theorem 3.6, it can be seen that \( G \) is \( k \)-indicated colorable for \( k = \max \left\{ \omega(G), \left\lceil \frac{|V(G)|}{2} \right\rceil \right\} \). Thus \( \chi(G) \leq \chi_i(G) \leq \max \left\{ \omega(G), \left\lceil \frac{|V(G)|}{2} \right\rceil \right\} \). \( \square \)

An immediate consequence of Corollary 3.7 is one of the results proved by Fouquet et.al. in [7]. Namely, for \( G \cong \mathbb{K}[C_5](m, m, m, m, m), m \geq 1 \), \( \chi(G) = \left\lceil \frac{5m}{2} \right\rceil = \left\lceil \frac{|V(G)|}{2} \right\rceil \).

**Corollary 3.8** If \( G \) is a \( \{P_5, C_4\}\)-free graph, then \( G \) is \( k \)-indicated colorable for all \( k \geq \chi(G) \).

**Proof** By Theorem 2.6, it is enough to prove the result for a connected \( \{P_5, C_4\}\)-free graph. Let \( G \) be such a graph. Then by Theorem 3.4, \( V(G) = V_1 \cup V_2 \) such that

(i) \( \langle V_1 \rangle \) is a \( P_5 \)-free graph which is also chordal.
(ii) If \( V_2 \neq \emptyset \), then \( \langle V_2 \rangle = A_1 \cup A_2 \cup \cdots \cup A_l \) where each \( A_i \) is a \( \mathbb{K}[C_5] \), for every \( i \in \{1, 2, \ldots, l\} \) for some \( l \geq 1 \). Also, \( \langle N(A_i) \rangle \) is a complete subgraph of \( V_1 \) and \( [A_i, N(A_i)] \) is complete.

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Let the color set be \{1, 2, \ldots, k\}, where \(k \geq \chi(G)\). Since \(\langle V_1 \rangle\) is chordal, col(\langle V_1 \rangle) = \omega(\langle V_1 \rangle) \leq k. By Theorem 3.1, Ann has a winning strategy for the vertices of \(V_1\) using \(k\) colors. Let Ann follow this winning strategy for presenting the vertices of \(V_1\). Let \(Q_i = \langle N(A_i) \rangle \subseteq V_1\), for every \(i \in \{1, 2, \ldots, l\}\). Clearly \(\chi(A_i + Q_i) = \chi(A_i) + \chi(Q_i) \leq \chi(G) \leq k.\) Thus \(k - \chi(Q_i) \geq \chi(A_i)\), for \(1 \leq i \leq l\). By Theorem 3.6, each \(A_i\) is \(k\)-indicated colorable for every \(k \geq \chi(A_i)\). Also for \(1 \leq i \leq l\), since \(Q_i\) is a clique, Ann has a winning strategy for each \(A_i\) while using \(k - \chi(Q_i) \geq \chi(A_i)\) colors, for \(1 \leq i \leq l\). Since \(A_i\)'s are disjoint, if Ann presents the vertices of \(A_i\)'s for \(1 \leq i \leq l\) by using these winning strategies, Ben cannot create a blocked vertex. Thus \(G\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\).

**Corollary 3.9** Let \(S_1, S_2, S_3, S_4, S_5\) be any split graphs. Then the graph \(G \cong C_5(S_1, S_2, S_3, S_4, S_5)\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\).

**Proof** Let \(G\) be the graph \(C_5(S_1, S_2, S_3, S_4, S_5)\). For \(1 \leq i \leq 5\), \(V(S_i) = V_i \cup U_i\), where \(\langle V_i \rangle\) is a maximum clique and \(\langle U_i \rangle\) is an independent set respectively. Let the color set be \(\{1, 2, \ldots, k\}\), where \(k \geq \chi(G)\). By Theorem 3.6, there is a winning strategy for the subgraph \(C_5(V_1, V_2, V_3, V_4, V_5)\) of \(G\) using \(k\) colors. Let Ann follow this winning strategy to present the vertices of \(C_5(V_1, V_2, V_3, V_4, V_5)\). Next, let Ann present the remaining vertices of \(G\), namely the vertices in \(\cup_{i=1}^{5} U_i\), in any order. Since each vertex \(x \in U_i, 1 \leq i \leq 5\), has a non neighbor in \(V_i\), the color of that non neighbor in \(V_i\) will be available for \(x\). Thus Ben cannot create any blocked vertex and hence Ann wins the game on \(G\) with \(k\) colors. □

An immediate consequence of Theorems 2.6, 3.2, 3.5 and Corollary 3.9 is Corollary 3.10.

**Corollary 3.10** If \(G\) is a connected \({P_5, P_2 \cup P_3, P_2, D a r t}\)-free graph that contains an induced \(C_5\), then \(G\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\).

Now let us consider the indicated coloring of connected \({P_2 \cup P_3, C_4}\)-free graphs.

**Theorem 3.11** If \(G\) is a \({P_2 \cup P_3, C_4}\)-free graph, then \(G\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\).

**Proof** By using Theorem 2.6, let us assume that \(G\) is a connected \({P_2 \cup P_3, C_4}\)-free graph. If \(G\) is chordal, then col(G) = ω(G) = χ(G). By Theorem 3.1, \(G\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\). Suppose \(G\) is not chordal, then there exists a partition \(\langle V_1, V_2, V_3 \rangle\) of \(V(G)\) such that \(\langle V_1 \rangle \cong K_m\) for some \(m \geq 0\), \(\langle V_2 \rangle \cong K_t\) for some \(t \geq 0\) and \(\langle V_3 \rangle \cong G_i\) for some \(i, 1 \leq i \leq 17\) (see Fig. 3). Let us divide the proof into two cases as follows.

**Case 1** \(V_2 = \emptyset\)

Since \(\langle V_1, V_3 \rangle = \emptyset\) and \(G\) is connected, \(G \cong G_j\) for some \(j, 1 \leq j \leq 17\). Hence it is enough to show that for \(1 \leq j \leq 17, G_j\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\). Let us first consider the \(G_j\)'s when \(j \in \{1, 2, \ldots, 17\}\) \(\backslash\{6, 7, 8, 9\}\). It is not difficult to observe that for these \(j \in \{1, 2, \ldots, 17\}\) \(\backslash\{6, 7, 8, 9\}\), col(\(G_j\)) = \(\chi(G)\). Hence by Theorem 3.1, \(G_j\) is \(k\)-indicated colorable for all \(k \geq \chi(G)\).
For \( j \in \{6, 7, 8, 9\} \), it can be seen that \( \text{col}(G_j) \neq \chi(G_j) \), so we consider these graphs separately. The graph \( G_6 \cong C_6 \) and hence \( k \)-indicated colorable for all \( k \geq \chi(G_6) \). The graph \( G_7 \cong \mathbb{Z}[C_5]|(m_1, m_2, m_3, m_4, m_5) \), where each \( m_i \geq 1, 1 \leq i \leq 5 \) and hence by Theorem 3.6, \( G_7 \) is \( k \)-indicated colorable for all \( k \geq \chi(G_7) \). Next the graph \( G_8 \cong P \), the Petersen graph. In [14], it has been showed that the Petersen graph \( P \) is \( k \)-indicated colorable for all \( k \geq \chi(P) \). Finally, let us consider the graph \( G_9 \). It is easy to check that \( \chi(G_9) = 3 \) and \( \text{col}(G_9) = 5 \). By Theorem 3.1, it is enough to show that \( G_9 \) is 3 and 4-indicated colorable. Let us first consider \( G_9 \) with 3 colors, namely \( \{1, 2, 3\} \). If Ann presents the vertices of \( G_9 \) in the order \( p, q, r, s, t, u, v, w, x \), then Ben always has an available color for each of the vertices. Now let us consider \( G_9 \) with 4 colors, namely \( \{1, 2, 3, 4\} \). Let Ann start by presenting the vertices \( p, q, r \). Without loss of generality, let Ben color these vertices with 1, 2 and 3 respectively. Now let Ann present the vertex \( u \). Suppose Ben colors \( u \) with 1 or 4, then Ann will present the remaining vertices in the order \( s, t, v, x, w \). Suppose Ben colors \( u \) with 2 or 3, then Ann will present the remaining vertices in the order \( w, x, v, t, s \). This guarantees the fact that Ben cannot block any of the vertex. Thus Ann wins the game with 4 colors.

**Case 2** \( V_2 \neq \emptyset \)

Recall that \( \langle V_2 \rangle \cong K_t \) and \( \langle V_3 \rangle \cong G_j, 1 \leq j \leq 17 \). Since \( V_1 \) is independent and \( [V_1, V_3] = \emptyset \), we can color the vertices of \( V_1 \) with one of the colors of \( V_3 \). Thus \( \chi(G) = \chi(K_t + G_j\setminus S) = t + \chi(G_j\setminus S) \), for \( j \in \{1, 2, \ldots, 17\} \). Let us first consider the graphs \( G \) for which \( \langle V_3 \rangle \cong G_j, j \in \{1, 2, 3, 4, 5\} \), the graph with \( S \neq \emptyset \). Without much difficulty, one can show that \( \text{col}(G) = \chi(G) \). Thus by Theorem 3.1, \( G \) is \( k \)-indicated colorable for all \( k \geq \chi(G) \).

Next, let us consider the graphs \( G \) for which \( \langle V_3 \rangle \cong G_j, j \in \{6, 7, \ldots, 17\} \). We know that, \( K_t \) is \( k_1 \)-indicated colorable for all \( k_1 \geq t \) and by Case 1, \( G_j \) is \( k_2 \)-indicated colorable for all \( k_2 \geq \chi(G_j) \), \( j \in \{6, 7, \ldots, 17\} \). Hence by Theorem 3.2, we see that \( \langle V_2 \cup V_3 \rangle \cong K_t + G_j \) is \( k \)-indicated colorable for all \( k = k_1 + k_2 \geq t + \chi(G_j) = \chi(G) \) and hence Ann has a winning strategy for \( \langle V_2 \cup V_3 \rangle \) while using \( k \) colors for any \( k \geq \chi(G) \). Next, let Ann present the vertices of \( V_1 \) in any order. Since \( V_1 \) is independent and \( [V_1, V_3] = \emptyset \), the colors used in \( V_3 \) are available to Ben for each vertex in \( V_1 \). Thus \( G \) is \( k \)-indicated colorable for all \( k \geq \chi(G) \).

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