On the Logarithmic Triviality of Scalar Quantum Electrodynamics

by

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Abstract

Using finite size scaling and histogram methods we obtain numerical results from lattice simulations indicating the logarithmic triviality of scalar quantum electrodynamics, even when the bare gauge coupling is chosen large. Simulations of the non-compact formulation of the lattice abelian Higgs model with fixed length scalar fields on $L^4$ lattices with $L$ ranging from 6 through 20 indicate a line of second order critical points. Fluctuation-induced first order transitions are ruled out. Runs of over ten million sweeps for each $L$ produce specific heat peaks which grow logarithmically with $L$ and whose critical couplings shift with $L$ picking out a correlation length exponent of $0.50(5)$ consistent with mean field theory. This behavior is qualitatively similar to that found in pure $\lambda\phi^4$. 
Do field theories which are strongly coupled at short distances exist in four dimensions? This is an important question to answer from both a purely theoretical and a phenomenological perspective. Theoretically, one wants to know if the Landau zero\(^1\) (complete screening of interactions) is universal in non-asymptotically free field theories in four dimensions, as suggested by perturbation theory. In less than four dimensions, theories with non-trivial high energy interactions are commonplace and perturbation theory in bare coupling parameters is known to be misleading. For example, four Fermi theories in dimensions \(d, 2 < d < 4\), have an ultra-violet stable fixed point where chiral symmetry is broken, as indicated by \(1/N\) expansions while perturbative expansions in the theory’s coupling constant are non-renormalizable.\(^2\) Phenomenologically, one wants to understand the Higgs mechanism in the successful Standard Model and build theories where the appropriate form of spontaneous symmetry breaking can occur at short distances.

Issues such as these have rekindled interest in existence questions for various field theories. Considerable work on \(\lambda \phi^4\) theories strongly suggest that this theory becomes free as its cutoff is removed,\(^3\) although a proof of this property remains elusive. Of course, \(\lambda \phi^4\) theories are unrealistically simple so the phenomenological impact of this program is not great. In this paper we shall study scalar electrodynamics at strong gauge couplings with sufficient numerical resources to make quantitative claims about its ultraviolet behavior. We shall see that our numerical results are consistent with the logarithmic triviality of scalar electrodynamics, qualitatively similar to pure \(\lambda \phi^4\).

We begin with a lattice formulation of scalar electrodynamics which is particularly well suited to answer these physics questions. Consider the non-compact formulation of the abelian Higgs model with a fixed length scalar field,\(^4\):

\[
S = \frac{1}{2} \beta \sum_p \theta^2_p - \lambda \sum_{x,\mu} (\phi_x^* U_{x,\mu} \phi_{x+\mu} + c.c.) \tag{1}
\]

where \(p\) denotes plaquettes, \(\theta_p\) is the circulation of the non-compact gauge field \(\theta_{x,\mu}\) around a plaquette, \(\beta = 1/e^2\) and \(\phi_x = exp(i\alpha(x))\) is a phase factor at each site. We choose this action (the electrodynamics of the planar model) because preliminary work has suggested that it has a line of second order transitions,\(^4\) because it does not require fine tuning and because it is believed to lie in the same universality class as the ordinary lattice abelian Higgs model.
with a conventional, variable length scalar field.\textsuperscript{5} In Fig. 1 we show the phase diagram of the model in the bare parameter space $\beta - \lambda$. A preliminary investigation has indicated that the line emanating from the $\beta \to \infty$ limit of Fig. 1 is a line of critical points which potentially could produce a family of interacting, continuum field theories.\textsuperscript{4} Note that in the $\beta \to \infty$ limit the gauge field in Eq. (1) reduces to a pure gauge transformation so the model becomes the four dimensional planar model which is known to have a second order phase transition which is trivial, i.e. described by a free field. The non-compact nature of the gauge field is important in Fig. 1—the compact model has a line of first order transitions and only at the endpoint of such a line in the interior of a phase diagram can one hope to have a critical point where a continuum field theory might exist.\textsuperscript{4} Since one must fine tune bare parameters to find such a point, the compact formulation of the model is much harder to use for quantitative work.\textsuperscript{6} The fact that Eq. (1) uses fixed length scalar fields avoids another fine tuning—the variable length scalar field formulation would possess a quadratically divergent bare mass parameter which would have to be tuned to zero with extraordinary accuracy to search for critical behavior. Conventional wisdom based on the renormalization group states that Eq. (1) should have the same critical behavior as the fine-tuned variable length model,\textsuperscript{5} so it again emerges as preferable. Note also that in the naive classical limit where the field varies smoothly Eq. (1) reduces to a free massive vector boson. In the vicinity of the strong coupling critical point we investigate here, the fields are rapidly varying on the scale of the lattice spacing and the specific heat scaling law is not that of a Gaussian model.

First consider the measurements of the internal energies,

$$E_\gamma = \frac{1}{2} < \sum_p \theta_p^2 >, \quad E_h = < \sum_{x,\mu} \phi_x^* U_{x,\mu} \phi_{x+\mu} + \text{c.c.} >$$

and their associated specific heats $C_\gamma = \partial E_\gamma / \partial \beta$, and $C_h = \partial E_h / \partial \lambda$. Non-analytic behavior in the specific heats at critical couplings can be used to find and classify phase transitions. On a $L^4$ lattice the size dependence of a generic specific heat at a second order critical point should scale as,\textsuperscript{7}

$$C_{\text{max}}(L) \sim L^{\alpha/\nu}$$

where $\alpha$ and $\nu$ are the usual specific heat and correlation length critical
indices, respectively. Here $C_{\text{max}}$ denotes the peak of the specific heat. A measurement of the index $\nu$ can be made from the size dependence of the position of the peak. In a model which depends on just one coupling, call it $g$, then:

$$g_c(L) - g_c \sim L^{-1/\nu}$$

where $g_c(L)$ is the coupling where $C_{\text{max}}(L)$ occurs and $g_c$ is its $L \to \infty$ thermodynamic limit. The scaling laws Eq. (3) and (4) characterize a critical point with powerlaw singularities. This is a possible behavior for scalar electrodynamics, but there is also the possibility suggested by perturbation theory, that the theory is logarithmically trivial. Consider $\lambda\phi^4$ as the simplest, well-studied theory which apparently has this behavior. In this case the theory becomes trivial at a logarithmic rate as the theory’s momentum space cutoff $\Lambda$ is taken to infinity. Then the scaling laws of Eq. (3) and (4) become:

$$C_{\text{max}}(L) \sim (\ln L)^p$$

and

$$g_c(L) - g_c \sim \frac{1}{L^2(\ln L)^q}$$

where $p$ and $q$ are powers predictable in one-loop perturbation theory ($p = \frac{1}{3}$ and $q = \frac{1}{6}$ in $\lambda\phi^4$). Note the differences between these scaling laws and those of the usual Gaussian model, obtained from Eq. (3) and (4) setting $\alpha = 0$ and $\nu = 0.5$: in the Gaussian model the specific heat should saturate as $L$ grows, and the position of the peaks should approach a limiting value at a rate $L^{-2}$.

It is particularly interesting in scalar electrodynamics to consider a large value of the bare (lattice) gauge coupling to see if that can induce non-trivial interactions which survive in the continuum limit. So, we ran extensive simulations on lattices ranging from $6^4$ through $20^4$ at $e^2 = 5.0$ and searched in parameter space $(\beta, \lambda)$ for peaks in $C_{\gamma}$ and $C_{h}$. We used histogram methods to do this as efficiently as possible. For example, on a $6^4$ lattice at $\beta = 0.2000$ and $\lambda = 0.2350$ we found a specific heat peak near $\lambda_c(6) \approx 0.2382$ from the histogram method. The $\lambda$ value in the lattice action was then tuned to 0.2382 and additional simulations and histograms produced
specific heats, found from the variances of $E_\gamma$ and $E_h$ measurements, at a $\lambda_c$ very close to .2382. Using this strategy, measurements of $\lambda_c(L), C_\gamma(L)$ and $C_h(L)$ could be made without relying on any extrapolation methods. We thus avoided systematic errors, although critical slowing down on the larger lattices limited our statistical accuracy. In Table 1 we show a subset of our results that will be analyzed and discussed here. A more complete discussion with additional measurements and analysis will appear elsewhere. The columns labeled $\lambda_c(L), C_\gamma^{\max}(L)$ and $C_h^{\max}(L)$ in Table 1 need no further explanation except to note that the error bars were obtained with standard binning procedures which account for the correlations in the data sets produced by Monte Carlo programs. The Monte Carlo procedure used here was a standard multi-hit Metropolis for the non-compact gauge degrees of freedom and an over-relaxed plus Metropolis algorithm\textsuperscript{12} for the compact matter field. Over-relaxation reduced the correlation times in the algorithm by typically a factor of 2–3. Accuracy and good estimates of error bars are essential in a quantitative study such as this. Unfortunately, cluster and acceleration algorithms have not been developed for gauge theories, so very high statistics of our over-relaxed Metropolis algorithm were essential—tens of millions of sweeps were accumulated for each lattice size as listed in column 7. Specific heats were measured as the fluctuations in internal energy measurements ($C_h = (\langle E_h^2 \rangle - \langle E_h \rangle^2)/4L^4$, etc.), and very high statistics and many $L$ values are needed to distinguish between logarithmic triviality (Eq. 5) and powerlaw behavior (Eq. 3). The other entries in the Table, $K_\gamma(L)$ and $K_h(L)$, are the Binder Cumulants (Kurtosis)\textsuperscript{13} for each internal energy. At a continuous phase transition each Kurtosis should approach $2/3$ with finite size corrections scaling as $1/L^4$. The Kurtosis is a useful probe into the order of a phase transition, although an examination of the internal energy and specific heat histograms are often just as valuable. Since the order of the transitions in lattice and continuum scalar electrodynamics are controversial, we studied these quantities with some care.

As a warmup to the full theory, we checked that our techniques are able to reproduce known results. For example, when $\beta \to \infty$ Eq. (1) reduces to the four dimensional planar spin model which should have an order-disorder transition as a function of $\lambda$ that is described by mean field theory ($\alpha = 0, \nu = .5, etc.$) and produce a free massless field in the continuum limit. We measured the specific heat at the transition for $L = 6, 8, 10, 12$ and 14, and found peak values $20.47(3), 22.80(5), 24.35(9), 25.38(9)$ and $26.24(9)$,
respectively. One million sweeps of our code, tailored for $\beta = \infty$, were run in each case. We note that the specific heat peaks do grow with $L$, but a three parameter fit of the form $aL^\rho + b$ is excellent (confidence level = 92%) producing $\rho = -0.67(11)$, $a = -44(3)$ and $b = 33.7(1.7)$. So, the specific heat is predicted to saturate producing a critical index $\alpha = 0$, and the growth of the peak heights seen in the simulation is a subdominant effect. Other fits to the data such as $a\ln^\rho L + b$ were not stable and over a wide range of fitting parameters produced confidence levels less than a few percent. Certainly much more exacting studies of this model could be made (cluster algorithms), but we are testing here just the simulation and analysis technology available to the gauge model.

Consider the Kurtosis $K_\gamma(L)$, the specific heat $C_{\gamma}^{\max}(L)$ and the critical coupling $\lambda_c(L)$ of scalar electrodynamics. As stated above, we set the lattice (bare) gauge coupling to $e^2 = 5.0$ and then used simulations, enhanced by histogram methods, to locate the transition line in Fig. 1. The Kurtosis $K_\gamma(L)$ is plotted against $10^6/L^4$ in Fig. 2. The size of the symbols include the error bars, but clearly the curve favors a second order transition. A three parameter fit to the $L = 12, 14, 16, 18$ and 20 data using the form $K_\gamma(L) = aL^\rho + b$ is excellent (confidence level = 98%) predicting $\rho = -4.1(4)$ and $K_p(\infty) = 0.666665(2)$. The hypothesis of a line of second order transitions in Fig. 1 appears to be very firm, with no evidence for a fluctuation-induced first order transition. An analysis of $K_h(L)$ gives the same conclusion with somewhat larger error bars. In Fig. 3 we plot our $C_{\gamma}^{\max}(L)$ data vs. $L$. We attempted powerlaw as well as logarithmic finite size scaling hypotheses. The powerlaw hypothesis did not produce a stable fit for any reasonable range of parameters. However, logarithmic fits were quite good. The hypothesis $C_{\gamma}^{\max}(L) = a\ln^\rho L + b$ for $L = 8,10,12,14,16,18$ and 20 fit with a confidence level = 90% producing the estimate $\rho = 1.4(2)$. If we considered the range $L = 8 \rightarrow 18$, the same fitting form predicted $\rho = 1.5(3)$ with confidence level = 84%, and if the range $L = 10 \rightarrow 20$ were taken we found $\rho = 1.4(5)$ with confidence level = 78%. The solid line in Fig. 3 is the $L = 8 \rightarrow 20$ fit. An analysis of $C_{h}^{\max}(L)$ gave consistent results—the same logarithmic dependence should be found in either specific heat—and powerlaw fits to $C_{h}^{\max}(L)$ were also ruled out. In particular, a fit of the form $C_{h}^{\max}(L) = a\ln^\rho L + b$ for $L = 8 \rightarrow 18$ gave $\rho = 0.9(3)$ with confidence level = 82% and for $L = 8 \rightarrow 20$ gave $\rho = 1.0(2)$ with confidence level = 85%. Finally, in Fig. 4 we show $\lambda_c(L)$ vs. $10^4/L^2$. The error bars again fall within the symbols in
the figure. The data is clearly compatible with the correlation length index $\nu = 0.5$ expected of a theory which is free in the continuum limit. In the case of $\lambda \phi^4$ it has proven possible to find the logarithm of Eq. (6) under the dominant $L^{-2}$ behavior by using special techniques. We do not quite have the accuracy to do that here: a powerlaw fit to $\lambda_c(L) = \lambda_c + a/L^{1/\nu}$ using $L = 12 - 20$ predicts $1/\nu = 2.0(1)$, $\lambda_c = .22825(8)$ with confidence level = 92% and using $L = 14 - 20$ predicts $1/\nu = 1.9(3)$, $\lambda_c = .2282(2)$ with confidence level = 97%.

One of the motivations for this study was the recent finding that the chiral symmetry breaking transition in non-compact lattice electrodynamics with dynamical fermions is not described by a logarithmically trivial model. Powerlaw critical behavior has been found with non-trivial critical indices satisfying hyperscaling. The present negative result for scalar electrodynamics suggests that the chiral nature of the transition for fermionic electrodynamics is an essential ingredient for its non-triviality. It remains to be seen, however, if the chiral transition found on the lattice produces an interesting continuum field theory.

In conclusion, our numerical results support the notion that scalar electrodynamics is a logarithmically trivial theory. We suspect that this result could be made even firmer by additional simulation studies which use more sophisticated techniques such as renormalization groups transformations or partition function methods. Since we did not wish to bias our study toward logarithmic triviality, we did not pursue special methods which require additional theoretical input in order to be quantitative. Certainly our concentration on a line of fixed electric charge in the entire phase diagram should be relaxed. Hopefully, accelerated Monte Carlo algorithms could be developed for scalar electrodynamics so that larger systems could be simulated with better control. A word of warning for the ambitious—the standard CRAY random number generator RANF which uses the linear congruent algorithm with modulus $2^{48}$ proved inadequate for lattices whose linear dimension was a power of 2. Presumably this occurred because for strides of length $2^N$ the period of RANF is reduced from $2^{46}$ to $2^{46-N}$ and the well-known correlations in such generators are expected to have maximum effect if the distribution is sampled with a period of $2^N$. The simplicity of lattice scalar electrodynamics also makes it more susceptible to the correlations in random number generators than other models. We discovered this problem when our $16^4$ simulations were unstable and after considerable investigative
work we isolated the problem in the random number generator. We cured the problem by adding extra calls to RANF to avoid strides of length $2^N$. Problems with generally accepted random number generators have been studied systematically in ref. (16).

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**Figure Captions**

1. The phase diagram of non-compact scalar electrodynamics.

2. The Kurtosis $K_\gamma(L)$ vs. $10^6/L^4$.

3. The specific heat peaks $C^{\text{max}}_\gamma(L)$ vs. $L$. The solid line is the logarithmic fit discussed in the text.

4. The critical coupling $\lambda_c(L)$ vs. $L^{-2}$. 
Table 1: Measurements on non-compact lattice scalar electrodynamics.

| L  | $\lambda_c(L)$ | $C_h^{max}(L)$ | $K_h(L)$ | $C_\gamma^{max}(L)$ | $K_\gamma(L)$ | Sweeps(millions) |
|----|----------------|----------------|----------|----------------------|----------------|------------------|
| 6  | .23815(1)      | 13.81(2)       | .657668(9) | 7.965(9)             | .665784(2)    | 40               |
| 8  | .23375(3)      | 15.83(2)       | .662954(5) | 8.083(3)             | .666374(1)    | 60               |
| 10 | .23173(1)      | 17.23(4)       | .664892(4) | 8.285(6)             | .666544(1)    | 60               |
| 12 | .23070(1)      | 18.43(7)       | .665713(4) | 8.457(9)             | .666606(1)    | 30               |
| 14 | .23004(1)      | 19.38(9)       | .666110(3) | 8.594(15)            | .666633(1)    | 20               |
| 16 | .22962(1)      | 20.25(13)      | .666319(2) | 8.747(17)            | .666647(1)    | 12               |
| 18 | .22933(1)      | 20.85(15)      | .666441(2) | 8.863(26)            | .666654(1)    | 12               |
| 20 | .22912(1)      | 21.76(20)      | .666510(2) | 8.956(20)            | .666658(1)    | 10               |