Generalized Number Systems and Application to Hyperoctahedral Groups

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Abstract

In this work, we generalize the integer enumeration basis. We also construct bijections between the elements of special sets and the elements of some groups and treat the special case of hyperoctahedral groups. We then find an analogous of the Lehmer code for the hyperoctahedral groups.

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1 Introduction and notation

A number system is a framework for representing numerals. Classical numbering system or enumeration system with base \( q (q \in \mathbb{N}) \) are expressing all natural number \( n \in \mathbb{N} \) in the form:

\[
n = \sum_{i=0}^{k(n)} n_i q^i , \quad \text{where} \quad n_i \in \{0, 1, 2, \cdots , q - 1\} \quad (1)
\]

Such representation are for example used in number theory or combinatorics as usefull tool for finding exotic congruences. In other hand one have also the factorial number system which is already known in 19th century by Cantor [3]. This is expressing each \( n \in \mathbb{N}_0 \) in the form

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\[ n = \sum_{i=0}^{l} f_i.i! \], where \( f_i \in \{0, 1, 2, \cdots, i\} \), (2)

for some \( l \in \mathbb{N}_0 \), and it is well known that these representation are unique. Known as Lehmer code [6]. Laisant [5] build a code by associating an element of this representation an element of a symmetric group. This association is proven to be a bijection between the set \( \{0, 1, \cdots, n! - 1\} \) and the symmetric group \( S_n \).

This article is organised as follow: in section 2, we will begin by generalizing these notion of representations. For that we given the condition for a infinite sequence of numbers to be an enumeration basis. We will also extend the results to rationals numbers and real numbers. In section 3, we apply this methode for some sequence of positive integers and find a Lehmer code analogous result for the hyperoctahedral groups.

Notation

We make the convention of notation:

- \( \mathbb{N} = \{1, 2, 3, \cdots\} \) is the set of non-negative integers.
- \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).
- \( \alpha = (\alpha_0, \alpha_1, \alpha_2, \cdots), \mathcal{U} = (U_0, U_1, U_2, \cdots) \) are sequences of non-negative integers.
- \( \beta = 1 + \alpha \) is a sequence of integers such that \( \beta_i = 1 + \alpha_i \) for all \( i \in \mathbb{N}_0 \).
- Bold symbols 1, 2, i, j, k will denote respectively some standard vectors \( e_1, e_2, e_i, e_j, e_k \) of the canonical basis of the vector space \( \mathbb{R}^n \).
- \( \mathcal{B} = (B_n)_{n \in \mathbb{N}_0} = (2^n n!)_{n \in \mathbb{N}_0} \) is a sequence of integers.
- \( (B_n)_{n \in \mathbb{N}} \) is a sequence of hyperoctahedral groups and should not be confounded with the sequence \( \mathcal{B} \).
- For a numbering system \( (\mathcal{U}, \alpha) \) (see Definition 2.1), \( <U_0, \ldots, U_n> \) will denote the set \( \{\sum_{i=0}^{n} a_i U_i \mid 0 \leq a_i \leq \alpha_i\} \).

2 General basis for positive integers

2.1 General Number systems

Let us start with two sequences of integers \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) and \( \mathcal{U} = (U_n)_{n \in \mathbb{N}} \). We want to express all positive integer \( n \) with an expression:

\[ n = \sum_{i=0}^{l} k_i.U^i \], where \( k_i \in \{0, 1, 2, \cdots, \alpha_i\} \) and for some \( l \in \mathbb{N}_0 \) (3)

Definition 2.1. Let \( \mathcal{U} \) and \( \alpha \) be two sequences of non-negative integers. We say that the system \( (\mathcal{U}, \alpha) \) is a system of enumeration or a number system if and only if all elements of \( \mathbb{N}_0 \) can be represented as in (3) and this representation is unique.
It is important here that \( U_n \) and \( \alpha_n \) are non-negative integers. Other authors considers for example some \( U_n \) negative or even complex numbers. See for example [4, Chapter 4] for this topic. The unicity is also important in this definition. The representation as in (3) are also called radix system but the representation may not be unique. That means that for a suitable fixed sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \), we need to find the appropriate sequence \( U = (U_n)_{n \in \mathbb{N}} \) such that the representation above is unique. And conversely, for a fixed sequence \( U = (U_n)_{n \in \mathbb{N}_0} \), the question is: are there any appropriate sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) such that all positive integers are representable with the system \((U, \alpha)\)?

**Convention.** For a fixed number system \((U, \alpha)\), and \( a_n \in \mathbb{N}_0 \) with \( a_n \leq \alpha_n \), we use then the convention

\[
a_n a_{n-1} a_{n-2} \cdots a_2 a_1 a_0 \quad \text{or} \quad a_n : a_{n-1} : a_{n-2} : \cdots : a_2 : a_1 : a_0
\]

to denote the number \( \sum_{i=0}^{n} a_i U_i \).

**Lemma 2.2.** Consider a fixed number system \((U, \alpha)\) and let \( a \in \mathbb{N} \) with

\[
a = a_n a_{n-1} a_{n-2} \cdots a_2 a_1 a_0
\]

with \( a_n \neq 0 \), then

\[
a_n U_n \leq a \leq (a_n + 1) U_n
\]

**Proof.** By Euclidean division, one can write \( a = a_n U_n + r \) with \( r = a_{n-1} \cdots a_2 a_1 a_0 \leq U_n \).

**Lemma 2.3.** If \((U, \alpha)\) is a number system and

\[
a = a_n a_{n-1} a_{n-2} \cdots a_2 a_1 a_0 = b_m b_{m-1} m_{m-2} \cdots b_2 b_1 b_0
\]

with \( a_n \neq 0 \) and \( b_m \neq 0 \), then

\[
n = m \quad \text{and} \quad a_i = b_i \quad \text{for all} \quad i \in \{0, 1, 2, \cdots n\}.
\]

**Proof.** By the Lemma 2.2, the relation \( n = m \) is immediate because if \( m > n \) then \( a \) will be smaller than \( U_m \) which is absurd. So \( n = m \) and one have the unicity by induction and by the unicity of the expression \( a = a_n U_n + r \).

**Theorem 2.4.** Let \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) and \( U = (U_n)_{n \in \mathbb{N}} \) be two sequences of positive integers. Then, \((U, \alpha)\) is an number system if and only if \( \forall n \in \mathbb{N}_0, \alpha_n \geq 1, \ U_0 = 1 \) and

\[
U_n = \prod_{i=0}^{n-1} (1 + \alpha_i) = \prod_{i=0}^{n-1} \beta_i \quad \text{(4)}
\]

i.e all natural number can be expressed as in (3) and the representation is unique.

**Proof.** By construction and the Lemma 2.3, we just need that \( 1 + \sum_{i=0}^{n} \alpha_i U^i \) should be equal to \( U_n \). The last condition means that \( \frac{U_n}{U_{n-1}} = (1 + \alpha_{i-1}) \) which lead to the formula \( U_n = \prod_{i=0}^{n-1} (1 + \alpha_i) \).

The theorem above also means that once, we fix a specific sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) with \( \alpha_0 \geq 1 \), then there is only one sequence \( U = (U_n)_{n \in \mathbb{N}} \) such that \((U, \alpha)\) is a number system. And conversely, for a fixed positive sequence of integer \( U = (U_n)_{n \in \mathbb{N}} \), there will we a corresponding \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) such that \((U, \alpha)\) is a number system if \( U_0 = 1 \) and \( \forall n \in \mathbb{N}, \beta_n := \frac{U_{n+1}}{U_n} \in \mathbb{N} \).
A Horner’s-like procedure. Horner’s scheme is based on expressing a polynomial by a particular expression so that the value of the corresponding polynomial function is quickly obtained. For instance:

\[ a + bx + cx^2 + dx^3 + ex^4 = a + x \cdot (b + x \cdot (c + x \cdot (d + x \cdot e))) \]

For a number system \((\mathcal{U}, \alpha)\), recall that \(\mathcal{U}_n = \prod_{i=0}^{n-1} \beta_i\). To express a positive integer \(a\) in the \((\mathcal{U}, \alpha)\)-system, one proceed to the following: Begin by dividing \(a\) by \(\beta_0\) and take \(a_0\) to be the rest \(a_0 := r_0: a = r_0 + \beta_0 q_0\).

Divide \(q_1\) by \(\beta_1\) and take the \(a_1\) to be the rest \(a_1 := r_1: q_1 = r_1 + \beta_1 q_2\).

Continue the procedure until \(q_n = 0\) for some \(n \in \mathbb{N}_0\), by dividing \(q_i\) by \(\beta_i\) and take \(a_i := r_i: q_i = r_i + \beta_i q_{i+1}\).

Clearly, we have: \(a = a_0 + \beta_0 \cdot (a_1 + \beta_1 \cdot (a_2 + \beta_2 \cdot (a_3 + \cdots)))\).

Classical examples. If we choose \(m \in \mathbb{N}\) such that \(\alpha_i = m - 1\) with \(m \geq 2\), then we recover the classical system \((\mathcal{U}, \alpha)\) with \(\mathcal{U} = \mathcal{U}^{(m)} = (1, m, m^2, m^3, \ldots)\) and the addition and multiplication procedures are well known.

If now we choose \(A := (A_n)_n\) with \(A_n = (n + 1)!\) \(\alpha_A = (n + 1)_n\) for all \(n \in \mathbb{N}_0\), then one obtains

Definition 2.5 (Factorial System). One define the factorial system to be the number system:

\[(A, \alpha_A) := (((n + 1)!)_n \in \mathbb{N}_0, (n + 1)_n \in \mathbb{N}_0). \quad (5)\]

2.2 The hyperoctahedral number system

Let us now take as basis \(B = (B_0, B_1, B_2, B_3, \ldots)\) and the corresponding \(\alpha\) such that

\[B_n = 2^n n!\] corresponds to \(\alpha_n = 2n + 1. \quad (6)\]

This means that \(\alpha = (1, 3, 5, 7, \ldots)\) and the first twenty numbers are represented in table [1].

Definition 2.6 (Hyperoctahedral System). One define the hyperoctahedral system to be the system:

\((B, \alpha_B) := ((2^n n!)_n \in \mathbb{N}_0, (2n + 1)_n \in \mathbb{N}_0). \quad (7)\]

This definition is motivated by the fact that we choose \(U_n = B_n\) to be the cardinal of the hyperoctahedral group \(B_n\). We have:

Theorem 2.7. Every positive integer \(a\) has a unique representation in the hyperoctahedral system i.e \((B, \alpha_B)\) is a number system.

Proof. This is obvious since by the Theorem 2.4 one just need to verify that \(\frac{B_{n+1}}{B_n} = 2n + 2. \quad \square\)
| Classical | Factorial | Hyperoctahedral | Classical | Factorial | Hyperoctahedral |
|-----------|-----------|-----------------|-----------|-----------|-----------------|
| 0         | 0         | 0               | 40        | 1220      | 500             |
| 1         | 1         | 1               | 41        | 1221      | 501             |
| 2         | 10        | 10              | 42        | 1300      | 510             |
| 3         | 11        | 11              | 43        | 1301      | 511             |
| 4         | 20        | 20              | 44        | 1310      | 520             |
| 5         | 21        | 21              | 45        | 1311      | 521             |
| 6         | 100       | 30              | 46        | 1320      | 530             |
| 7         | 101       | 31              | 47        | 1321      | 531             |
| 8         | 110       | 100             | 48        | 2000      | 1000            |
| 9         | 111       | 101             | 49        | 2001      | 1001            |
| 10        | 120       | 110             | 50        | 2010      | 1010            |
| 11        | 121       | 111             | 51        | 2011      | 1011            |
| 12        | 200       | 120             | 52        | 2020      | 1020            |
| 13        | 201       | 121             | 53        | 2021      | 1021            |
| 14        | 210       | 130             | 54        | 2100      | 1030            |
| 15        | 211       | 131             | 55        | 2101      | 1031            |
| 16        | 220       | 200             | 56        | 2110      | 1100            |
| 17        | 221       | 201             | 57        | 2111      | 1101            |
| 18        | 300       | 210             | 58        | 2120      | 1110            |
| 19        | 301       | 211             | 59        | 2121      | 1111            |
| 20        | 310       | 220             | 60        | 2300      | 1120            |
| 21        | 311       | 221             | 61        | 2301      | 1121            |
| 22        | 320       | 230             | 62        | 2310      | 1130            |
| 23        | 321       | 231             | 63        | 2311      | 1131            |
| 24        | 1000      | 300             | 64        | 2320      | 1200            |
| 25        | 1001      | 301             | 65        | 2321      | 1201            |
| 26        | 1010      | 310             | 66        | 3000      | 1210            |
| 27        | 1011      | 311             | 67        | 3001      | 1211            |
| 28        | 1020      | 320             | 68        | 3010      | 1220            |
| 29        | 1021      | 321             | 69        | 3011      | 1221            |
| 30        | 1100      | 330             | 70        | 3020      | 1230            |
| 31        | 1101      | 331             | 71        | 3021      | 1231            |
| 32        | 1110      | 400             | 72        | 3100      | 1300            |
| 33        | 1111      | 401             | 73        | 3101      | 1301            |
| 34        | 1120      | 410             | 74        | 3110      | 1310            |
| 35        | 1121      | 411             | 75        | 3111      | 1311            |
| 36        | 1200      | 420             | 76        | 3120      | 1320            |
| 37        | 1201      | 421             | 77        | 3121      | 1321            |
| 38        | 1210      | 430             | 78        | 3200      | 1330            |
| 39        | 1211      | 431             | 79        | 3201      | 1331            |

Table 1: Representing positive integers in the Classical system $U^{(10)}$, in the Factorial system and in the Hyperoctahedral system.
2.3 Extension to rational and real numbers

One can extend the representation to all integers by adding the opposite of all positive integers. But one can also extend it to representation of rational numbers and real numbers.

**Definition 2.8.** Let \((\mathcal{U}, \alpha)\) a number system, one can extend the system with \(U_n = \frac{1}{\alpha n}\) and \(\alpha_n = \alpha_n\). Then, define the set of number \(A_\alpha\) to be the set of real numbers of the form

\[
\sum_{i=-\infty}^{m} a_i U_i
\]

and

\[
A_\alpha = A_\alpha^+ \cup A_\alpha^- \text{ where } A_\alpha^- = -A_\alpha^+
\]

Note that for this definition holds, it is necessary to verify that the serie is convergent. This exercise is left to the reader.

**Theorem 2.9.** Let \((\mathcal{U}, \alpha)\) an extended number system, then all rational integer can be represented in this system i.e:

\[
\mathbb{Q} \subseteq A_\alpha.
\]

**Proof.** This is an easy exercise for the reader. \(\square\)

**Remark 2.10.** Unfortunately, unlike the Theorem 2.4, the representation is not unique for rational numbers. For instance, for a fixed number system \((\mathcal{U}, \alpha)\), then one has \(U_{n+1} = U_n(1 + \alpha_n)\) which lead to the relation:

\[
\frac{\alpha_n}{U_{n+1}} = \frac{1}{U_n} - \frac{1}{U_{n+1}}.
\]

From that, one deduce easily that

\[
\frac{1}{U_n} = \sum_{i=n}^{\infty} \frac{\alpha_n}{U_{n+1}}
\]

For example, the factoradic representation of a rational number \(\frac{a}{b}\) with \(\gcd(a, b) = 1\) in the open unit interval, i.e. \(0 < \frac{a}{b} < 1\), is defined as

\[
a \frac{b}{b} := \sum_{i=1}^{N} \frac{d_i}{(i+1)!}, \quad 0 \leq d_i \leq i,
\]

where \(d_i, 0 \leq d_i \leq i\), is the ”factoradic digit” for place-value \(\frac{1}{(i+1)!}\), and \(N\) is the number of ”factoradic digits” after the ”factoradic point”.

The remark above also show for instance that rational numbers may have multiple factoradic representations:

\[
\frac{1}{m!} = \sum_{i=m}^{\infty} \frac{i}{(i+1)!}, \quad m \geq 1,
\]

where on the left side we have the terminating form, while on the right we have the nonterminating form. This is analogous to

\[
\frac{1}{b^m} = \sum_{i=m}^{\infty} \frac{b-1}{b^{i+1}}, \quad m \geq 0, \ b \geq 2,
\]

for the number system \((\mathcal{U}^{(b)}, (b-1))_{n \in \mathbb{N}}\).
Representation of rational in an extended number system. One can consider without loss of generality that the rational number \( \frac{p}{q} \in ]0, 1[ \). And write:

\[
\frac{p}{q} = b_0 + \frac{b_1}{\beta_0 \beta_1} + \frac{b_2}{\beta_0 \beta_1 \beta_2} + \frac{b_3}{\beta_0 \beta_1 \beta_2 \beta_3} + \cdots .
\] (15)

Then instead of dividing as in the Horner’s like procedure, one multiply by the \( \beta_i \) and take the integer parts:

\[
b_0 = \lfloor \frac{p}{q} \beta_0 \rfloor, b_1 = \lfloor (\frac{p}{q} \beta_0 - b_0) \beta_1 \rfloor, b_2 = \lfloor ((\frac{p}{q} \beta_0 - b_0) \beta_1 - b_1) \beta_2 \rfloor, \cdots .
\]

Note that this procedure also lead to the development of a real number in a certain basis.

Examples. Some rationals and real numbers with a factoradic representation:

\( e = 1 : 0.1 : 1 : 1 : 1 : 1 : 1 : 1 : 1 : \ldots \)
\( \frac{23}{24} = 0 : 1.2 : 3 \)
\( \frac{30}{34} = 1 : 1.0 : 1 : 3 \)

Examples. Some rationals and real numbers with a hyperoctahedral representation:

\( \sqrt{e} = 1.1 : 1 : 1 : 1 : 1 : 1 : \ldots \)
\( \frac{13}{35} = 0.1 : 2 : 3 \)
\( \frac{69}{69} = 2.0 : 1 : 4 \)

3 Applications to some Coxeter groups

3.1 The Lehmer code

Cantor seems to be the first to introduce the factorial number system in [3]. Then, Laisant (cf [5]) introduce a code by associating each element of the symmetric group \( S_n \) to a positive integer in \( 0, 1, 2, \ldots, n! - 1 \). He prove that this association is in fact a bijection. The key ingredient is using a statistic on \( S_n \), namely the inversion statistic.

If a permutation \( \sigma \) is specified by the sequence \( (\sigma_1, \ldots, \sigma_n) \) of its images of \( 1, \ldots, n \), then it is encoded by a sequence of \( n \) numbers, but not all such sequences are valid since every number must be used only once.

The Lehmer code is the sequence

\[ L(\sigma) = (L(\sigma)_1, \ldots, L(\sigma)_n) \quad \text{where} \quad L(\sigma)_i = \# \{ j > i : \sigma_j < \sigma_i \}, \]

in other words the term \( L(\sigma)_i \) counts the number of terms in \( (\sigma_1, \ldots, \sigma_n) \) to the right of \( \sigma_i \) that are smaller than it, a number between 0 and \( n - i \), allowing for \( n + 1 - i \) different values.

A pair of indices \( (i, j) \) with \( i < j \) and \( \sigma_i > \sigma_j \) is called an inversion of \( \sigma \), and \( L(\sigma)_i \) counts the number of inversions \( (i, j) \) with \( i \) fixed and varying \( j \). It follows that

\[ L(\sigma)_1 + L(\sigma)_2 + \cdots + L(\sigma)_n \]

is the total number of inversions of \( \sigma \), which is also the number of adjacent transpositions that are needed to transform the permutation into the identity permutation.

In this section, we will be investigating on construction of an analogue of this code for the Hyperoctahedral groups \( B_n \). The key ingredients of our construction are based on the existence of inversion on this family of groups.
3.2 A code related to hyperoctahedral groups

Recall that the hyperoctahedral group $B_n$ is the group of signed permutations of the coordinates in $\mathbb{R}^n$. We write $i$ for the $i$th standard basis vector. This interaction between simple font for an integer and bold font for the corresponding standard vector is kept in all this section. For instance, for a signed permutation $\pi$, $\pi(i)$ refers to the $|\pi(i)|$th standard basis vector. We use the notation

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{pmatrix}$$

for an element $\pi$ of $B_n$ with $\pi(i) \in \{\pm 1, \ldots, \pm n\}$. The element $\pi$ is an invertible linear map of $\mathbb{R}^n$.

It is convenient to think $B_n$ as the Coxeter group of the same type with root system

$$\Phi_n = \{ \pm i, \pm i \pm j \mid 1 \leq i \neq j \leq n \},$$

and positive root system

$$\Phi_n^+ = \{ k, i + j, i - j \mid k \in [n], 1 \leq i < j \leq n \}.$$

We use the number of inversions defined by Reiner [7, 2. Preliminaries] that is: The number of inversions of the element $\pi$ of $B_n$ is

$$\text{inv} \pi = \# \{ v \in \Phi_n^+ \mid \pi^{-1}(v) \in -\Phi_n^+ \}.$$

Let us consider the following subset of $\Phi_n^+$ defined by

$$\Phi_{n,i}^+ = \{ i, i + j, i - j \mid i < j \leq n \},$$

and define the number of $i$-inversions of $\pi$ by

$$\text{inv}_i \pi = \# \{ v \in \Phi_{n,i}^+ \mid \pi^{-1}(v) \in -\Phi_{n,i}^+ \}.$$  \hfill (17)

**Example 3.1.** In the following table, we see the corresponding 1-inversions and 2-inversions of the eight elements of $B_2$:

| \sigma | \text{inv}_1 \sigma | \text{inv}_2 \sigma |
|--------|---------------------|---------------------|
| \sigma_1 | 1                    | 2                   |
| \sigma_2 | 0                    | 2                   |
| \sigma_3 | 1                    | 1                   |
| \sigma_4 | 0                    | 1                   |
| \sigma_5 | 1                    | 0                   |
| \sigma_6 | 0                    | 0                   |
| \sigma_7 | 1                    | 1                   |
| \sigma_8 | 0                    | 2                   |

**Lemma 3.2.** Let $i \in [n]$ and $\pi \in B_n$. If

(i) $\pi(i) = j$, then

$$\text{inv}_i(\pi) = \# \{ k \in \{ i+1, \ldots, n \} \mid j > |\pi(k)| \}.$$

(ii) $\pi(i) = -j$, then

$$\text{inv}_i(\pi) = 1 + \# \{ k \in \{ i+1, \ldots, n \} \mid j > |\pi(k)| \} + 2 \# \{ k \in \{ i+1, \ldots, n \} \mid j < |\pi(k)| \}.$$

**Proof.** Use the definition of the number of $i$-inversions (Equation 16 and Equation 17).

**Lemma 3.3.** Let $\sigma, \tau \in B_n$. If $\sigma(1) \neq \tau(1)$ then $\text{inv}_1 \sigma \neq \text{inv}_1 \tau$.

**Proof.** We use the equations of Lemma 3.2. Let $i, j \in [n]$. If

- $\sigma(1) = i$ and $\tau(1) = j$ with $i < j$, then $\text{inv}_1 \tau - \text{inv}_1 \sigma = j - i$,
\[\pi_0 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \text{inv}_1(\pi_0) \text{ inv}_2(\pi_0) = 0 0 \]

\[\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, \quad \text{inv}_1(\pi_1) \text{ inv}_2(\pi_1) = 0 1 \]

\[\pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \text{inv}_1(\pi_2) \text{ inv}_2(\pi_2) = 1 0 \]

\[\pi_3 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad \text{inv}_1(\pi_3) \text{ inv}_2(\pi_3) = 1 1 \]

\[\pi_4 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \text{inv}_1(\pi_4) \text{ inv}_2(\pi_4) = 2 0 \]

\[\pi_5 = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \quad \text{inv}_1(\pi_5) \text{ inv}_2(\pi_5) = 2 1 \]

\[\pi_6 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \quad \text{inv}_1(\pi_6) \text{ inv}_2(\pi_6) = 3 0 \]

\[\pi_7 = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}, \quad \text{inv}_1(\pi_7) \text{ inv}_2(\pi_7) = 3 1 \]

Table 2: The 1-inversions and 2-inversions of the elements of \(B_2\).

\[\bullet \sigma(1) = -i \text{ and } \tau(1) = -j \text{ with } i < j, \text{ then } \text{inv}_1 \sigma - \text{inv}_1 \tau = j - i,\]

\[\bullet \sigma(1) = i \text{ and } \tau(1) = -j, \text{ then } \text{inv}_1 \sigma < n \leq \text{inv}_1 \tau.\]

\[\Box\]

**Lemma 3.4.** We have \(\text{inv}_1 B_n = \{0, \ldots, 2n - 1\}\).

**Proof.** On one hand, we deduce from Lemma 3.2 that the minimal value of \(\text{inv}_1\) is 0, corresponding to the elements \(\pi \in B_n\) such that \(\pi(1) = 1\), and the maximal is \(2n - 1\), corresponding to the elements \(\pi \in B_n\) such that \(\pi(1) = -1\). One the other hand, we deduce from Lemma 3.3 that \(\text{inv}_1\) has \(2n\) possible values. \[\Box\]

Let us consider the enumeration basis \(B = (B_0, B_1, B_2, B_3, \ldots)\) relative to the hyperoctahedral groups. Recall that \(B_n = 2^n n!\) with corresponding maximal positive integer \(\alpha_n = 2^n + 1\). The aim of this section is to prove the following theorem.

**Theorem 3.5.** There is a one-to-one correspondence between the elements of \(B_n\) and those of \(<B_0, B_1, \ldots, B_{n-1}>\) with respect to the \(i\)-inversions. This bijection is given by

\[b_n : B_n \to <B_0, \ldots, B_{n-1}> \quad \pi \mapsto \sum_{i=1}^{n} \text{inv}_{n-i+1}(\pi)B_{i-1}.\]

**Proof.** For \(n = 1\), we have trivially \(b_1\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = 0\) and \(b_1\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = B_0\). From table 2 of Example 3.1, we can deduce \(b_2\). For instance \(b_2\left(\begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix}\right) = 2B_1 + B_0\).

Now, we assume that the bijection exists for any positive integer smaller than \(n\) and prove by induction that \(b_{n+1}\) also exists.

For \(\pi \in B_{n+1}\) and \(i \in [n+1]\), we write \(\text{sg} \pi(i)\) for the sign of \(\pi(i)\).
Let \( \pi = \left( \begin{array}{cccc} 1 & 2 & \cdots & n+1 \\ \pi(1) & \pi(2) & \cdots & \pi(n+1) \end{array} \right) \in B_{n+1} \). We consider the \( n \) rightmost columns of \( \pi \), i.e. \( \hat{\pi} = \left( \begin{array}{cccc} 2 & \cdots & n+1 \\ \pi(2) & \cdots & \pi(n+1) \end{array} \right) \), and define \( \overline{\pi} = \left( \begin{array}{cccc} 2 & \cdots & n+1 \\ \pi(2) & \cdots & \pi(n+1) \end{array} \right) \) with \( \overline{\pi}(i) = \{ \pi(i) + \text{sg}(\pi(i)) \cdot 1 \text{ if } |\pi(i)| < |\pi(1)|, \pi(i) \text{ if } |\pi(i)| > |\pi(1)|. \}

Then, we get an element \( \overline{\pi} \) of \( B_n \) and we have
\[
\sum_{i=1}^{n} \text{inv}_{n-i+2}(\overline{\pi}) B_{i-1} = b_n(\overline{\pi}).
\]

We can now proceed of the proof of the existence of the bijection \( b_{n+1} \).

Let \( \sigma, \tau \in B_n \) such that \( \sigma \neq \tau \). If
\begin{itemize}
  \item \( \sigma(1) \neq \tau(1) \), then, from Lemma 3.3, we have \( \text{inv}_1(\sigma) B_n \neq \text{inv}_1(\tau) B_n \),
  \item \( \sigma(1) = \tau(1) \), then, from Equation 18, we get
\end{itemize}
\[
\sum_{i=1}^{n} \text{inv}_{n-i+2}(\sigma) B_{i-1} \neq \sum_{i=1}^{n} \text{inv}_{n-i+2}(\tau) B_{i-1}.
\]

So, on one hand, the map \( \pi \mapsto \sum_{i=1}^{n+1} \text{inv}_{n-i+2}(\pi) B_{i-1} \) is injective.

On the other hand, we deduce from Lemma 3.4 that \( \text{inv}_1 B_{n+1} = \{0, \ldots, \alpha_n\} \).

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