On the Equivalence of Two Expressions for Statistical Significance in Point Source Detections

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The problem of point source detection in Poisson-limited count maps has been addressed by two recent papers [M. Lampton, ApJ 436, 784 (1994); D. E. Alexandreas, et al., Nucl. Instr. Meth. Phys. Res. A 328, 570 (1993)]. Both papers consider the problem of determining whether there are significantly more counts in a source region than would be expected given the number of counts observed in a background region. The arguments in the two papers are quite different (one takes a Bayesian point of view and the other does not), and the suggested formulas for computing p-values appear to be different as well. It is shown here that the expressions provided by the authors of these two articles are in fact equivalent.

1. Introduction

Space is big. Stars are big too, but space is bigger. As a consequence, stars and in fact most astronomical objects of interest appear as point sources from our perspective here on earth. Thus many astronomical surveys concentrate on the detection and characterization of point sources in the sky. Particularly for instruments that measure high energy radiation (extreme ultraviolet and beyond), individual photons are counted, and for these instruments the statistical treatment of point source detection requires proper consideration of Poisson statistics.

The particular problem of interest is the following: Given \( N \) counts in a source region of area \( A_{src} \), and \( B \) counts in a background region of area \( A_{bak} \), is there a real point source in the source kernel? More specifically, compute a \( p \)-value associated with the probability of observing \( N \) or more counts in the source region under the null hypothesis that the count rate per unit area is the same for both the source and background regions. In two recent papers [1, 2], this specific problem of point source detection is addressed. The papers take quite different approaches in their derivation, and produce expressions which appear on the surface to be quite different. It will be shown, however, that the expressions are equal.

1.1 Binomial formulation

Lampton [1], hereafter referred to as Paper I, provides an elegant formulation. Rather than consider the source and background as separate Poisson processes, the sum \( N + B \) is treated as a fixed number, and the binomial distribution of \( N + B \) total counts into areas \( A_{src} \) and \( A_{bak} \) is considered. Let \( f = A_{src}/(A_{src} + A_{bak}) \) be the fraction of total counts expected in
the source region; then \(1 - f\) is the fraction of counts expected in the background region. We can write down the likelihood that exactly \(n\) counts would be observed in the source region:

\[
P(n) = \binom{N + B}{n} f^n (1 - f)^{N + B - n}
\]

where

\[
\binom{a}{b} = \begin{cases} 
\frac{a!}{b!(a - b)!} & \text{for } a \geq b \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

is the binomial coefficient. The \(p\)-value is given by the probability of observing \(n \geq N\) photons in the source region:

\[
p_{\text{Lamp}} = \sum_{n=N}^{N+B} \binom{N + B}{n} f^n (1 - f)^{N + B - n}
\]

### 1.2 Bayesian formulation

In Alexandreas et al. [2], hereafter referred to as Paper II, the argument begins with the remark that if the expected number \(\mu\) of counts in the source region were known exactly, then likelihood of seeing exactly \(n\) counts in the source region is given by the Poisson formula:

\[
P(n|\mu) = \frac{\mu^n}{n!} e^{-\mu}.
\]

The \(p\)-value is the probability of observing \(n \geq N\) counts:

\[
p = \sum_{n=N}^{\infty} \frac{\mu^n}{n!} e^{-\mu}.
\]

In the problem at hand, the actual background level is not known exactly, but it can be estimated from the \(B\) counts in the background region. Following the notation in Paper II, let \(\alpha = A_{\text{src}} / A_{\text{bak}}\) be the ratio of areas in the source and background region. Then \(\hat{\mu} = \alpha B\) is an estimate for the expected number of counts in the source region. In Paper II, the authors implicitly take a Bayesian approach (though they do not identify it as Bayesian) and produce a probability distribution on the parameter \(\mu\) that is proportional to the likelihood of observing \(B\) background counts, given \(\mu\).

\[
P(\mu|B) \propto P(B|\mu) = \frac{(\mu/\alpha)^B}{B!} e^{-\mu/\alpha}
\]

This direct proportionality implies that a uniform Bayesian prior was used (i.e., \(P(\mu) = \text{constant}\)); since \(\mu\) is unbounded, this is a so-called “improper” prior. (Loredo [3] has suggested that the prior \(P(\mu) = 1/\mu\) is more appropriate.) Based on this distribution, an “average” \(p\)-value is computed, by integrating the expression for \(p\) in Eq. (5) against \(P(\mu|B)\). That is,

\[
p = \sum_{n=N}^{\infty} \frac{1}{n!} \int_{0}^{\infty} \mu^n e^{-\mu} (\frac{\mu/\alpha)^B}{B!} e^{-\mu/\alpha} d\mu
\]
After performing the integral, a closed form series solution is obtained, and the form of this expression given in Paper II is

$$p_{\text{Alex}} = 1 - \sum_{n=0}^{N-1} \frac{\alpha^n}{(1 + \alpha)^{(n+B+1)}} \frac{(n+B)!}{n!B!}. \quad (8)$$

To facilitate comparison with the formula in Paper I, we will use $\alpha = f/(1 - f)$, employ the standard notation for binomial coefficients, and use the fact that $\sum_{n=0}^{\infty} = 1$ to write an equivalent form:

$$p_{\text{Alex}} = (1 - f)^{B+1} \sum_{n=N}^{\infty} \binom{n + B}{n} f^n \quad (9)$$

2. Proof of Equivalence

We will begin with three simple lemmas.

**Lemma 1.** The following is an identity:

$$\binom{a}{b} + \binom{a}{b+1} = \binom{a+1}{b+1}. \quad (10)$$

This is in fact a well-known identity, but we will invoke it several times in the proofs below.

**Lemma 2.** The following expression holds for $n \geq 0$ and $0 \leq f < 1$:

$$\frac{f^n}{(1-f)^{n+1}} = \sum_{k=0}^{\infty} \binom{k}{n} f^k. \quad (11)$$

The proof of this is fairly straightforward. The binomial theorem for $(1-f)^{-n-1}$ produces the infinite series, and then both sides are multiplied by $f^n$. Note that the summation starts at $k = 0$, even though the first nonzero term is $k = n$.

**Lemma 3.** The following is an identity:

$$\binom{n+N+B}{n+N} = \sum_{k=0}^{B} \binom{N+B}{N+k} \binom{n}{k}. \quad (12)$$

This is proved by induction on $n$. First note that $n = 0$ produces $\binom{N+B}{N}$ on both the left and right sides, so the statement is true for $n = 0$. Now, suppose it is true for $n = n_o$, and consider the case $n_o + 1$. First expand out the right hand side, using Lemma 1:

$$\sum_{k} \binom{N+B}{N+k} \binom{n_o+1}{k} = \sum_{k} \binom{N+B}{N+k} \left[ \binom{n_o}{k} + \binom{n_o}{k-1} \right]. \quad (13)$$

Rearrange the terms

$$\sum_{k} \binom{N+B}{N+k} \left[ \binom{n_o}{k} + \binom{n_o}{k-1} \right] = \sum_{k} \left[ \binom{N+B}{N+k} + \binom{N+B}{N+k+1} \right] \binom{n_o}{k}. \quad (14)$$
Again, apply Lemma 1:

\[ \sum_k \left[ \binom{N + B}{N + k} + \binom{N + B}{N + k + 1} \right] \binom{n_o}{k} = \sum_k \binom{N + B + 1}{N + k + 1} \binom{n_o}{k}. \]  

(15)

We have inductively assumed that Eq. (12) is valid for \( n = n_o \), so

\[ \sum_k \binom{N + B + 1}{N + k + 1} \binom{n_o + B}{n_o + k} = \binom{n_o + N + B + 1}{n_o + N + 1} \]  

(16)

Combining all of these produces

\[ \sum_k \binom{N + B}{N + k} \binom{n_o + 1}{k} = \binom{n_o + N + B + 1}{n_o + N + 1}, \]  

(17)

which is Eq. (12) for \( n = n_o + 1 \). Thus, our induction was successful, and Lemma 3 is proved.

**Theorem.** The expressions for \( p \)-value in Eq. (9) and in Eq. (3) are equivalent. That is:

\[ p_{\text{Alex}} = p_{\text{Lamp}}. \]  

(18)

To see this, start with the expression for \( p \)-value in Eq. (9), and substitute \( k = n - N \):

\[ p_{\text{Alex}} = (1 - f)^{B+1} \sum_{n=N}^{\infty} \binom{n + B}{n} f^n = (1 - f)^{B+1} f^N \sum_{k=0}^{\infty} \binom{k + N + B}{k + N} f^k \]  

(19)

Use the identity in Lemma 3:

\[ p_{\text{Alex}} = (1 - f)^{B+1} f^N \sum_{k=0}^{\infty} \binom{k + N + B}{k + N} f^k = (1 - f)^{B+1} f^N \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{B} \binom{N + B}{N + j} \binom{k}{j} \right] f^k. \]  

(20)

Switch the order of summation

\[ p_{\text{Alex}} = (1 - f)^{B+1} f^N \sum_{j=0}^{B} \left[ \left( \sum_{k=0}^{\infty} \binom{k + N + B}{k + N} f^k \right) \binom{N + B}{N + j} \right] f^j. \]  

(22)

Substitute the result from Lemma 2 in Eq. (11):

\[ p_{\text{Alex}} = (1 - f)^{B+1} f^N \sum_{j=0}^{B} \left[ \left( \sum_{k=0}^{\infty} \binom{k + N + B}{k + N} \frac{f^j}{(1 - f)^{j+1}} \right) \right] f^j. \]  

(23)

\[ p_{\text{Alex}} = \sum_{j=0}^{B} \binom{N + B}{N + j} f^{N+j} (1 - f)^{B-j}. \]  

(24)

Finally, substitute \( n = j + N \) to obtain

\[ p_{\text{Alex}} = \sum_{n=N}^{N+B} \binom{N + B}{n} f^n (1 - f)^{N+B-n} \]  

(25)

\[ = p_{\text{Lamp}}. \]  

(26)
3. Discussion

Though the same answer is ultimately obtained, this does not imply that the two approaches are equivalent. Hypothesis testing in the presence of a nuisance parameter, in this case the background level, is always problematic.

In Paper I, a trick is employed which enables us to express the null hypothesis in terms that are independent of the nuisance parameter. Crucial use is made of the identity which expresses the joint distribution of two Poisson processes (the counts in the source and background regions) as a product of a single Poisson process and a binomial process:

\[
\mathcal{P}(N, \mu) \mathcal{P}(B, \mu/\alpha) = \frac{\mu^N}{N!} e^{-\mu} \times \frac{(\mu/\alpha)^B}{B!} e^{-\mu/\alpha} = \frac{(\mu + \mu/\alpha)^{N+B}}{(N+B)!} e^{-\mu+\mu/\alpha} \times \frac{(N+B)!}{N!B!} \frac{\mu^N(\mu/\alpha)^B}{(\mu + \mu/\alpha)^{N+B}}
\]

\[
= \mathcal{P}(N + B, \mu + \mu/\alpha) \mathcal{B}(N, B, \frac{\alpha}{1 + \alpha})
\]

(27)

The single Poisson process describes the statistics on the total \(N + B\), while the binomial process describes the partition of this total into the source and background regions. Since the joint distribution is a simple product, the two processes are independent. It is important to note that the binomial process does not depend on \(\mu\); thus, there is no need to worry about estimating this parameter in testing the null hypothesis.

Such decompositions are not always available, particularly as the problems get more complicated, but Paper II provides a methodology that is more adaptable to such situations: one estimates a distribution for the nuisance parameter, and then integrates over it. However, this uses a Bayesian derivation to produce a fundamentally “frequentist” product, namely a \(p\)-value. Given that the choice of prior is arbitrary (the choice in this case was quite natural, though Loredo [3] has argued that other choices might be preferred), one might say that the authors of Paper II were “lucky” to get the same answer that was obtained in Paper I without any free choices.

Loredo [3] argues (quite strenuously!) for a purely Bayesian approach, producing in the end a probability distribution on a source strength parameter. It is not clear how to compare this result with the \(p\)-values produced by the methods [1, 2] discussed here.

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