Pseudoanalytic Extension on $F(p, p - 2, s)$ Spaces and Applications

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Abstract. In this paper, we generalized the main results in [9]. As an applications, we give a characterization of the closure of $F(p, p - 2, s)$ spaces in Lipschitz-type spaces $A_\omega$ by pseudoanalytic extension.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. For $0 < p < \infty$, $H^p$ denotes the Hardy space, which consisting of all functions $f \in H(\mathbb{D})$ satisfied (see [13])

$$\|f\|_{H^p} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$ 

As usual, $H^\infty$ is the set of bounded analytic functions in $\mathbb{D}$ and $A$ denotes the disc algebra.

Let $0 < p < \infty$, $-2 < q < \infty$, $s \geq 0$. The $F(p, q, s)$ ([27]) space is the set of all $f \in H(\mathbb{D})$ such that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q (1 - |z|^2) \left(1 - |\phi_a(z)|^2\right)^s dA(z) < \infty,$$

where $\phi_a(z) = \frac{z - a}{1 - a\bar{z}}$ and $dA(z) = \frac{1}{\pi} dxdy$. When $q = p - 2$, $F(p, p - 2, s)$ is Möbius invariant Besov-type spaces. When $0 < s < 1$, $F(2, 0, s) = Q_s$ ([24, 25]); If $s = 1$, $F(2, 0, 1) = BMOA$, the space of analytic functions in the Hardy space $H^1(\mathbb{D})$ whose boundary functions have bounded mean oscillation. When $s > 1$, $F(2, 0, s) = \mathcal{B}$ (the Bloch space).

Let $\omega : [0, \infty) \to \mathbb{R}$ be a right-continuous with $\omega(0) = 0$. If $\omega$ is increasing and $\frac{\omega(t)}{t}$ is nonincreasing for $t > 0$, there exists constant $C(\omega)$ such that

$$\int_0^\infty \frac{\omega(t)}{t} dt + \delta \int_0^\infty \frac{\omega(t)}{t^2} dt \leq C(\omega) \cdot \omega(\delta),$$

then we say that $\omega$ is a regular majorant, where $0 < \delta < 1$.

2010 Mathematics Subject Classification. Primary 30D50; Secondary 30H25, 46E15

Keywords. $F(p, p - 2, s)$ spaces; pseudoanalytic extension; Lipschitz-type spaces.

Received: 18 October 2019; Revised: 13 September 2020; Accepted: 20 September 2020

Communicated by Miodrag Mateljević

Research supported by NNSF of China (No.11801250) and (No.11871257), Overseas Scholarship Program for Elite Young and Middle-aged Teachers of Lingnan Normal University, Yanling Youqing Program of Lingnan Normal University, the Key Subject Program of Lingnan Normal University (No.1171518004) and (No.LZ1905), and Department of Education of Guangdong Province (No. 2018KTSCX133).

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In this paper, we shall be concerned with the space $\mathcal{A}_\omega = A \cap \Lambda_\omega(D)$. When $\omega(t) = t^s, 0 < \alpha < 1$, it give the classical Lipschitz space $\Lambda_\omega$. For more informations on $\mathcal{A}_\omega$, we refer to [4] and the paper referin there.

Pseudoanalytic extension, as explained in [10], an analytic function in $D$, can be extended to $D_e = \{z : |z| > 1\}$ as a $C^1$ function whose Cauchy-Riemann $\partial$-derivative becomes appropriately small. There are many applications for pseudoanalytic extension, for example: K-property ([9]); inner-outer factorization ([10]); Bernstein-type inequality related to kernel of $H^p$ spaces ([6]) and so on.

In this paper, we generalize the main results in [9] to $F(p, p - 2, s)$ spaces. Moreover, we also give an application on our result to studying the closure of $f(p, p - 2, s)$ in Lipschitz-type spaces $\mathcal{A}_\omega$ (denoted by $\mathcal{C}_\omega(\mathcal{A}_\omega \cap F(p, p - 2, s))$) by pseudoanalytic extension.

In this paper, the symbol $f \approx g$ means that $f \leq g \leq f$. We say that $f \leq g$ if there exists a constant $C$ such that $f \leq Cg$.

2. Auxiliary results

If $Q$ is a measurable subset of $C$ and $Q$ varies over all discs in $C$, $|Q|$ will denote the measure (area) of $Q$. Let $\omega$ be a positive measurable function on $C$. We say that $\omega$ is an $A_t$-weight ($t > 1$) if (see [22])

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \omega(z) dA(z) \right)^{t-1} < \infty.$$ 

**Remark 1.** Let $t > 1$ and $\omega$ be an $A_t$-weight and $T$ be a Calderon-Zygmund operator. It is well know that (see [22])

$$\int_C |Tf(z)|^t \omega(z) dA(z) \leq \int_C |f(z)|^t \omega(z) dA(z), \text{ for all } f \in L^t(\omega).$$

Here $L^t(\omega)$ denote the space of functions $f \in L^t$ which satisfy

$$\int_C |f(z)|^t \omega(z) dA(z) < \infty.$$ 

The following lemma generalized [9, Proposition 1].

**Lemma 1.** Suppose that $1 < p < \infty, 0 < s < 1, p + s > 2, z \in C$ and $a \in D$. Then $\left| 1 - |z|^p \right|^{p-2} \left| \frac{1}{|\varphi^s(z)|^2} - 1 \right|^s$ is an $A_{p^s}$-weight.

**Proof.** Since

$$\left| 1 - |z|^p \right|^{p-2} \left| \frac{1}{|\varphi^s(z)|^2} - 1 \right|^s = \frac{(1 - |a|^2)^s|z|^2 - 1|^{p-2+s}}{|z - a|^{2s}}.$$ 

Let

$$M_a(z) = \frac{(1 - |a|^2)^s|z|^2 - 1|^{p-2+s}}{|z - a|^{2s}}$$

and

$$N_a(z) = \frac{|z|^2 - 1|^{p-2+s}}{|z - a|^{2s}}.$$
It is easily to see that $M_r(z)$ is an $A_p$-weight if and only if $N_r(z)$ is an $A_p$-weight. Now, we adopt and modify the method in [9, Proposition 1]. Suppose that $N_r(z) = J(z)K_r(z)$, where

$$J(z) = \left| \frac{z^2}{z - a} \right|^{p-2s}, \quad K_r(z) = \frac{1}{|z|^\frac{s}{2}}, \quad K_0(z) = \frac{1}{|z|^s}.$$ 

From [22, page 218], we known that $K_0(z)$ is an $A_r$-weight ($t > 1$). Since $K_r(z)$ are translates of $K_0(z)$, we have $K_r(z)$ is also an $A_t$-weight, that is,

$$\sup_Q \left( \frac{1}{|Q|} \int_Q K_r(z) dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{K_r(z)} dA(z) \right)^{t-1} < \infty. \quad (*)$$

Let $r \in (1, \frac{p-1}{p-2s})$ and $Q$ be any disc. Let $\frac{1}{t} + \frac{1}{r} = 1$. Then, for any $a \in \mathbb{D}$, we have

$$\left( \frac{1}{|Q|} \int_Q N_a(z) dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{N_a(z)} dA(z) \right)^{t-1} = \left( \frac{1}{|Q|} \int_Q J(z) K_a(z) dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{J(z) K_a(z)} dA(z) \right)^{t-1} \leq \left[ \sup_{z \in Q} J(z) \right] \times \left( \frac{1}{|Q|} \int_Q K_a(z) dA(z) \right) \times \left( \frac{1}{|Q|} \int_Q \frac{1}{J(z) K_a(z)} dA(z) \right)^{t-1} \times \left( \frac{1}{|Q|} \int_Q \frac{1}{K_a(z)} dA(z) \right)^{t-1}.$$ 

By direct calculation (or see [9, page 484]), we obtain

$$\left[ \sup_{z \in Q} J(z) \right] \times \left( \frac{1}{|Q|} \int_Q \frac{1}{J(z) K_a(z)} dA(z) \right)^{t-1} < \infty.$$ 

Thus,

$$\left( \frac{1}{|Q|} \int_Q N_a(z) dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{N_a(z)} dA(z) \right)^{t-1} \leq \left( \frac{1}{|Q|} \int_Q K_a(z) dA(z) \right) \times \left( \frac{1}{|Q|} \int_Q \frac{1}{J(z) K_a(z)} dA(z) \right)^{t-1} \times \left( \frac{1}{|Q|} \int_Q \frac{1}{K_a(z)} dA(z) \right)^{t-1}.$$ 

If $2 - s < p \leq 2$, it easily to see that $\frac{p}{p-1} > 1$. If $p > 2$, noted that $r \in (1, \frac{p-1}{p-2s})$ and $\frac{1}{t} + \frac{1}{r} = 1$, we can also deduce that $\frac{p}{p-1} > 1$. Let $t = \frac{p-1+s'}{p'} > 1$. Combined with $(*)$, we have

$$\left( \frac{1}{|Q|} \int_Q K_a(z) dA(z) \right) \times \left( \frac{1}{|Q|} \int_Q \frac{1}{J(z) K_a(z)} dA(z) \right)^{t-1} \times \left( \frac{1}{|Q|} \int_Q \frac{1}{K_a(z)} dA(z) \right)^{t-1} < \infty.$$
Therefore,
\[
\left( \frac{1}{|Q|} \int_Q N_a(z) \, dA(z) \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{N_a(z)} \, dA(z) \right)^{p-1} < \infty,
\]
for any \( a \in \mathbb{D} \). The proof is completed. \( \square \)

3. Pseudoanalytic extension on \( F(p, p-2, s) \)

Now, let us consider the pseudoanalytic extension on \( F(p, p-2, s) \).

**Theorem 1.** Suppose that \( p > 1, 0 < s < 1, p + s > 2 \) and \( f \in \cap_{0<q<\infty} H^p \). Then the following are equivalent:

1. \( f \in F(p, p-2, s) \);
2. \[
\sup_{a \in \mathbb{D}} \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z) < \infty;
\]
3. There exists a function \( F \in C^1(\mathbb{D}_c) \) satisfying
   \[
   F(z) = O(1), \quad \text{as } z \to \infty, \quad (a)
   \]
   \[
   \lim_{r \to 1} F(re^{i\theta}) = f(e^{i\theta}), \quad a.e \text{ and in } L^1([-\pi, \pi]) \text{ for all } q \in [1, \infty), \quad (b)
   \]
   \[
   \sup_{a \in \mathbb{D}} \int_{\mathbb{D}_c} |\tilde{F}(z)|^p (1 - |z|^2)^{p-2} |z|^2 \, dA(z) < \infty. \quad (c)
   \]

**Proof.** (1) \( \Leftrightarrow \) (2). Since \( F(p, p-2, s) \) space is Möbius invariant, we only need to prove that (the case \( a = 0 \))

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z) = \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} \frac{1}{|z|^2} \, dA(z).
\]

On the one hand, it is obvious that

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z) \leq \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z).
\]

On the other hand, let

\[
M_p(r, f')^p = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta.
\]

Bearing in mind that \( M_p(r, f')^p \) is an increasing function of \( r \), we have

\[
\int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z) = \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r^{1-2s} \, dr
\]
\[
\leq M_p \left( \frac{1}{2} f' \right)^p \int_0^1 (1 - r^2)^{p-2s+1-2s} + 2^s \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r \, dr
\]
\[
\leq (C(p, s) + 4^s) \int_0^1 M_p(r, f')^p (1 - r^2)^{p-2s} r \, dr
\]
\[
\leq \int_D |f'(z)|^p (1 - |z|^2)^{p-2s} |z|^2 \, dA(z),
\]
where
\[
C(p, s) = \int_0^1 \frac{(1 - r^2)^{p-2s}r^{1-2s} dr}{\int_1^r (1 - r^2)^{p-2s} r^2 dr} < \infty.
\]

We get the desired result.

(1) ⇒ (3). Suppose \( f \in F(p, p - 2, s) \), let \( z' = \frac{1}{2} \) and
\[
F(z) = f(z'), \quad z \in \mathbb{D}.
\]
Hence, \( F \in C^1(\mathbb{D},) \) and satisfies (a) and (b). Let \( a \in \mathbb{D} \). Using the fact that \( |\partial_z F(z)| = |f'(z')||z'|^2 \), making change of variables \( z = w' \), and combining with (1) ⇔ (2), we deduce that
\[
\int_{D_1} |\partial_{w'} F(z)| (|z|^2 - 1)^{p-2}(|\rho(z)|^2 - 1)^s dA(z)
\]
\[
= \int_D |f'(w)| (1 - |w|^2)^{p-2}(|\rho(w)|^2 - 1)^s dA(w)
\]
\[
= \int_D |f'(w)| (1 - |w|^2)^{p-2} \left( \frac{1}{|\rho(w)|^2} - 1 \right)^s dA(w) < \infty.
\]

(3) ⇒ (1). Let \( z \in \mathbb{D} \) and \( R > 1 \). Using Cauchy-Green formula we obtain
\[
f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{g(w)}{w - z} d\zeta - \frac{1}{\pi} \int_{1<|w|<R} \frac{\overline{\partial g(w)}}{w - z} dA(w).
\]
Notice the fact that
\[
\int_{|w|=R} \frac{g(w)}{(w - z)^2} dw \to 0, \quad \text{as} \quad R \to \infty.
\]
We deduce
\[
f'(z) = -\frac{1}{\pi} \int_{D_1} \frac{\overline{\partial g(w)}}{(w - z)^2} dA(w).
\]
Let \( G \) be defined by
\[
G(z) = \begin{cases} 
\overline{\partial g(z)}, & z \in \mathbb{D}, \\
0, & z \in \mathbb{D}.
\end{cases}
\]
Let \( T \) denote the Calderón-Zygmund operator defined by
\[
Tg(z) = p.v. \int_C \frac{g(w)}{(w - z)^2} dA(w).
\]
It is not hard to see that
\[
f'(z) = -\frac{1}{\pi} (TG)(z), \quad z \in \mathbb{D}.
\]
Hence, using the boundedness of Calderón-Zygmund operators (see Remark 1) and Lemma 1, we deduce
that
\[ \int_{D} |f'(z)|^p (1 - |z|^2)^{p-2} \left| \frac{1}{|\varphi(z)|^2} - 1 \right|^p dA(z) \]
\[ = \frac{1}{\pi} \int_{D} |(TG(z))|^p (1 - |z|^2)^{p-2} \left| \frac{1}{|\varphi(z)|^2} - 1 \right|^p dA(z) \]
\[ \leq \int_{C} |(TG(z))|^p |1 - |z|^2|^{p-2} \left| \frac{1}{|\varphi(z)|^2} - 1 \right|^p dA(z) \]
\[ \leq \int_{C} |G(z)|^p |1 - |z|^2|^{p-2} \left| \frac{1}{|\varphi(z)|^2} - 1 \right|^p dA(z) \]
\[ \leq \int_{D_2} |\mathcal{B}g(z)|^p (|z|^2 - 1)^{p-2} (|\varphi(z)|^2 - 1) dA(z) < \infty. \]

The proof is completed. □

**Remark 2.** Such function $F$ is said to be a pseudonalytical extension of $f$, clearly it is not uniquely determined by $f$.

**Remark 3.** $T$ is also known as Ahlfors - Beoruling operator, which appears in discussions related to different topics in complex analysis, like Beltrami equation.

Given a function $v \in L^\infty(\partial D)$, the associated Toeplitz operator $T_v$ is defined by

\[ (T_v f)(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{v(\xi)f(\xi)}{\xi - z} d\xi, \quad f \in H^1, z \in D. \]

Recall that a subspace $X$ of $H^1$ is said to have the $K$-property if $T_{\psi}(X) \subset X$ for any $\psi \in H^\infty$.

**Corollary 1.** Let $p > 1$, $0 < s < 1$ and $p + s > 2$. The $F(p, p - 2, s)$ has the $K$-property.

**Proof.** The proof is similar to [9, Theorem 2]. For completeness, we give the proof. Suppose that $f \in F(p, p - 2, s)$, $h \in H^\infty$. We need to show that

\[ g_1 =: T_f f \in F(p, p - 2, s). \]

Since, by definition of Toeplitz operator, $g_1$ is the orthogonal projection of $f_{1h}$ onto $H^2$, then, we have

\[ f_{1h} = g_1 + g_2, \]

where $g_2 \in H^2_0$. Therefore, we obtain that

\[ g_1 = f_{1h} - g_2 \quad a.e. \ on \ \partial D. \]

From Theorem 1, we know that there is a function $F \in C^1(\mathbb{D})$ satisfying (a), (b) and (c). We let

\[ H(z) =: h(z'), \quad G_2 =: g_2(z'), \quad G_1(z) =: F(z)H(z) - G_2(z) \quad z \in \mathbb{D}. \]

Hence, using the fact that

\[ F|_{\partial D} = f, \quad H|_{\partial D} = h, \quad G_2|_{\partial D} = \overline{g_2}, \]

we get

\[ G_1|_{\partial D} = g_1. \]
Since $H$ and $G_2$ are holomorphic in $\mathbb{D}$, we obtain

$$\bar{\partial}H = 0, \quad \bar{\partial}G_2 = 0.$$  

Thus, we have $\bar{\partial}G_1 = H \cdot \bar{\partial}F$ on $\mathbb{D}$. Furthermore,

$$|\bar{\partial}G_1| \leq ||H||_{\infty} |\bar{\partial}F|.$$  

It is clear that $G_1$ is $C^1$-smooth in $\mathbb{D}$ and bounded at $\infty$. Using the fact of above and Theorem 1, we easy to get (a), (b) and (c) hold true with $G_1$ and $g_1$ in place of $F$ and $f$. The proof is completed. $ \Box$

4. Closure of $F(p, p-2, s)$ spaces in $\mathcal{A}_w$

Let us recall the following result.

**Lemma 2.** [4, Lemma 7] Let $\omega$ be a regular majorant. Suppose that $f \in \mathcal{A}$. Then $f \in \Lambda_{\omega}$ if and only if there exists a bounded function $g \in C^1(\mathbb{D})$ satisfying

$$\lim_{r \to 1^+} g(re^{i\theta}) = f(e^{i\theta});$$  

$$\sup_{z \in \mathbb{D}} |g(z)| < \infty.$$  

Moreover,

$$|f|_{\Lambda_{\omega}} \approx \inf_{g \in \mathcal{A}} \sup_{z \in \mathbb{D}} (|z|^2 - 1)|\bar{\partial}g(z)|.$$  

**Lemma 3.** Let $\omega$ be a regular majorant. Then

$$\int_{\mathbb{D}} \frac{\omega(|w|^2 - 1)}{|w - z|^2} d\Lambda(w) \leq \frac{\omega(1 - |z|^2)}{(1 - |z|^2)^2}, \quad z \in \mathbb{D}.$$  

**Proof.** Making change of variable $w = \frac{1}{t}$, $v \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} \frac{\omega(|w|^2 - 1)}{|w - z|^2} d\Lambda(w) = \int_{\mathbb{D}} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{|1 - \frac{1-|v|^2}{|v|^2}v|^2} \frac{1}{|v|^2} d\Lambda(v)$$

$$= \int_{\mathbb{D}} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{(1 - |v|^2)(1 - \frac{1}{|v|^2})} d\Lambda(v)$$

$$\leq \int_{0}^{1} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{(1 - r^2)(1 - \frac{1}{r^2})} r dr.$$  

Let $t = \frac{1-|v|^2}{|v|^2}$. Then $r^2 = \frac{1}{1-t}$ and $rdt = \frac{dt}{2t(1-t)}$. We obtain

$$\int_{0}^{1} \frac{\omega(\frac{1-|v|^2}{|v|^2})}{(1 - r^2)(1 - \frac{1}{r^2})} r dr$$

$$\leq \int_{0}^{\infty} \frac{\omega(t)}{t(t + (1 - |v|^2))} dt$$

$$= \int_{0}^{1-|v|^2} \frac{\omega(t)}{t(t + (1 - |v|^2))} dt + \int_{1-|v|^2}^{\infty} \frac{\omega(t)}{t(t + (1 - |v|^2))} dt.$$
We have
\[ \int_0^\infty \frac{\omega(t)}{t} \, dt + \delta \int_0^\infty \frac{\omega(t)}{t^2} \, dt \leq C(\omega) \cdot \omega(\delta). \]

Thus,
\[ \delta \int_0^\infty \frac{\omega(t)}{t^2} \, dt \leq \omega(\delta). \]

Note that
\[ (i) \quad f \in C_{\mathcal{A}_r}(\mathcal{A}_r \cap F(p, p - 2, s)). \]
\[ (ii) \quad \text{For any } \epsilon > 0, \]
\[ \int_{\Omega_{\epsilon}(F)} \frac{\omega(\epsilon^2 - 1)}{\omega(\epsilon^2 - 1)} \, dA(z) < \infty, \]
where \( \Omega_{\epsilon}(F) = \{ z \in \mathbb{D}_r : \frac{\partial F(z)}{\partial |z|^2} \geq \epsilon \} \) and \( F \) is pseudoanalytic extension of \( f \).

Proof. (i) \( \Rightarrow \) (ii). Suppose that \( f \in C_{\mathcal{A}_r}(\mathcal{A}_r \cap F(p, p - 2, s)) \subseteq \mathcal{A}_w \). Then for any \( \epsilon > 0 \), there exist a function \( g \in \mathcal{A}_w \cap F(p, p - 2, s) \), such that
\[ \| f - g \|_{\mathcal{A}_r} \leq \frac{\epsilon}{2}. \]

From Lemma 2, there exist functions \( F, G \in C^1(\mathbb{D}_r) \), such that
\[ \| F - G \|_{\mathcal{A}_r} \leq \frac{\epsilon}{2}. \]

Here \( F, G \) are its pseudoanalytic extension of \( f \) and \( g \), respectively. Since
\[ \| F - G \|_{\mathcal{A}_r} \leq \frac{\epsilon}{2}, \]
we have \( \Omega_{\epsilon}(F) \subseteq \Omega_{\epsilon}(G) \). By Theorem 1, we can deduce that
\[ \int_{\Omega_{\epsilon}(F)} \frac{\omega(\epsilon^2 - 1)}{\omega(\epsilon^2 - 1)} \, dA(z) \leq \frac{2^p}{\epsilon^2} \int_{\Omega_{\epsilon}(G)} \left| \frac{\partial G(z)}{\partial |z|^2} \right|^{p-2} (\omega(\epsilon^2 - 1)) \, dA(z) < \infty. \]
(ii) ⇒ (i). Let $f \in \mathcal{A}_\omega$. Using Cauchy-Green formula we obtain

$$f(z) = \frac{1}{2\pi i} \int_{|w|=R} \frac{F(w)}{w-z} \, dw - \frac{1}{\pi} \int_{1<|w|<R} \frac{\overline{F}(w)}{w-z} \, dA(w).$$

Noting the fact that $\int_{|w|=R} \frac{F(w)}{(w-z)^2} \, dw \rightarrow 0$, as $R \rightarrow \infty$, we obtain

$$f'(z) = -\frac{1}{\pi} \int_{D_\lambda} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w).$$

Let

$$f'_1(z) = -\frac{1}{\pi} \int_{\Omega_\lambda(F)} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w)$$

and

$$f'_2(z) = -\frac{1}{\pi} \int_{D_\lambda \setminus \Omega_\lambda(F)} \frac{\overline{F}(w)}{(w-z)^2} \, dA(w).$$

Hence, $f'(z) = f'_1(z) + f'_2(z)$. By Lemma 3,

$$\frac{(1-|z|^2)}{\omega(1-|z|^2)} |f'(z) - f'_1(z)| \leq \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{D_\lambda \setminus \Omega_\lambda(F)} \frac{\overline{F}(w)}{|w-z|^2} \, dA(w)$$

$$= \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{D_\lambda \setminus \Omega_\lambda(F)} \frac{|\partial F(w)|}{|w-z|^2} \, dA(w)$$

$$\leq \epsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{D_\lambda \setminus \Omega_\lambda(F)} \frac{\omega(|w|^2-1)}{|w|^2-1} |w-z|^2 \, dA(w)$$

$$\leq \epsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \int_{D_\lambda \setminus \Omega_\lambda(F)} \frac{\omega(|w|^2-1)}{|w|^2-1} |w-z|^2 \, dA(w)$$

$$\leq \epsilon \frac{(1-|z|^2)}{\omega(1-|z|^2)} \frac{\omega(1-|z|^2)}{\omega(1-|z|^2)} = \epsilon,$$

which implies that $f_1 \in \mathcal{A}_\omega$. Now, we are going to prove that $f_1 \in F(p, p-2, s)$.

Let

$$G(z) = \begin{cases} \overline{F}(z), & z \in \Omega_\lambda(F), \\ 0, & z \in \mathbb{C} \setminus \Omega_\lambda(F), \end{cases}$$

and

$$Tg(z) = \text{p.v.} \int_{\mathbb{C}} \frac{g(w)}{(w-z)^2} \, dA(w).$$

It is easy to see that $f'_1(z) = -\frac{1}{\pi} (TG)(z), \ z \in \mathbb{D}$. Hence, using the boundedness of the operator $T$ and Lemma
1, we obtain
\[
\int_{D} |f'(z)|^p(1 - |z|^2)^{p-2} (1 - |\varphi_\omega(z)|^2) dA(z) \\
\leq \frac{1}{\pi} \int_{D} |(TG)(z)|^p(1 - |z|^2)^{p-2} (1 - |\varphi_\omega(z)|^2) dA(z) \\
\leq \frac{1}{\pi} \int_{D} |(TG)(z)|^p(1 - |z|^2)^{p-2} \left( \frac{1}{|\varphi_\omega(z)|^2} - 1 \right)^s dA(z) \\
\leq \int_{C} |G(z)|^p(1 - |z|^2)^{p-2} \left( \frac{1}{|\varphi_\omega(z)|^2} - 1 \right)^s dA(z) \\
\leq \int_{C} \omega^p(1 - |z|^2) \left( |\varphi_\omega(z)|^2 - 1 \right)^s dA(z) \\
\leq \int_{\Omega_0(F)} \omega^p(1 - |z|^2) \left( |\varphi_\omega(z)|^2 - 1 \right)^s dA(z) < \infty.
\]

The proof is completed. \(\square\)

**Corollary 2.** Let \(0 < s < 1, p + s > 2\) and \(\omega\) be a regular majorant. The \(C_{\mathcal{A}_\omega}(A_p \cap F(p, p - 2, s))\) has the K-property.

**Proof.** Let \(f \in C_{\mathcal{A}_\omega}(A_p \cap F(p, p - 2, s)), \varphi \in H^\infty\) and \(g\) be the orthogonal projection of \(f\overline{\varphi}\) onto \(H^2\). Then \(f\overline{\varphi} = g + j\), where \(j \in H^2_{\partial D}\). By pseudoanalytic extension, similar to Corollary 1, there are functions \(G, F, \Phi, J\) on \(\partial D\) with
\[
G = g, \quad F = f, \quad \Phi = \overline{\varphi}, \quad J = j, \quad \text{on} \quad \partial D,
\]
such that \(F\Phi = G + J\). Thus,
\[
|\partial G(z)| \leq \|\varphi\|_{\infty} |\partial F(z)|, \quad z \in \partial D.
\]

Combined with Theorem 2, we have
\[
\int_{z \in \partial D} \frac{\omega^p(1 - |z|^2 - 1)}{(1 - |z|^2)^2} (|\varphi_\omega(z)|^2 - 1)^s dA(z) \\
\leq \int_{z \in \partial D} \frac{\omega^p(1 - |z|^2 - 1)}{(1 - |z|^2)^2} (|\varphi_\omega(z)|^2 - 1)^s dA(z) < \infty.
\]

That is \(g \in C_{\mathcal{A}_\omega}(A_p \cap F(p, p - 2, s))\). The proof is completed. \(\square\)

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