Efficiency and influence function of estimators for ARCH models

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Abstract. This paper proposes a closed-form optimal estimator based on the theory of estimating functions for a class of linear ARCH models. The estimating function (EF) estimator has the advantage over the widely used maximum likelihood (ML) and quasi-maximum likelihood (QML) estimators that (i) it can be easily implemented, (ii) it does not depend on a distributional assumption for the innovation, and (iii) it does not require the use of any numerical optimization procedures or the choice of initial values of the conditional variance equation. In the case of normality, the asymptotic distribution of the ML and QML estimators naturally turn out to be identical and, hence, coincides with ours. Moreover, a robustness property of the EF estimator is derived by means of influence function. Simulation results show that the efficiency benefits of our estimator relative to the ML and QML estimators are substantial for some ARCH innovation distributions.

Keywords: ARCH process; least squares estimation, estimating function approach; quasi-maximum likelihood estimation; asymptotic optimality; asymptotic efficiency.

1. Introduction

Since the seminal papers of Engle (1982) and Bollerslev (1986), autoregressive conditional heteroskedasticity (ARCH) and generalised ARCH (GARCH) models have been proposed for modelling time series with non-constant conditional volatility. Since then, these models have become perhaps the most popular and extensively studied financial econometric models (see e.g., Engle (1995); Gouriéroux (1997); Mikosch (2003); Francq and Zakoian (2004)). The literature on the subject is so vast that we will restrict ourselves to directing the reader to fairly comprehensive reviews by Bollerslev et al. (1992) and Shephard (1996). An excellent survey of the GARCH methodology in finance is also available such as Bauwens et al. (2006).

ARCH model estimation can be achieved using a variety of methods such as conditional least squares (LS) estimation (Tjøstheim (1986)), maximum likelihood (ML) estimation under the assumption of conditional normality, quasi-maximum likelihood (QML) estimation (Weiss (1986); Francq and Zakoian (2004)), generalized method of moments (GMM) estimation (e.g., Rich et al. (1991)). As is well-known, LS, QML and GMM estimation methods yield inefficient and possibly biased estimates relative to ML estimators when the true innovation distribution is known (see for example, Li and Turtle (2000)). However, the possibility for misspecification of the likelihood function for ML and QML estimators motivates our investigation of an alternative estimation method for ARCH models.

The purpose of this paper is to propose an estimator based on the estimating function (EF) approach for ARCH models that improves efficiency without any distributional assumptions for the innovation. This EF estimator admits a closed-form expression which is computationally simple and compares favorably with the ML and QML estimators. Moreover, the EF estimator naturally turns out to have the same limiting distribution as the
ML and QML estimators and hence is also fully efficient when the innovation distribution is Gaussian. It is interesting to note that many standard results in the estimation of ARCH models based on conditional normality are recoverable under the EF approach.

The rest of the paper is organized as follows. Section 2 describes the conditional LS and EF estimation procedures of ARCH models. In addition, the asymptotic efficiency of the EF estimator relative to the LS, ML and QML estimators is discussed. In particular, the lower bound of the asymptotic variance of the LS estimator is formulated. In Section 3, a robustness of the EF estimator is studied by means of influence function. In Section 4, we perform an experiment to examine the asymptotic behavior of EF, LS, ML and QML estimators in terms of mean square errors in a small and a large-sample of observations. The study demonstrates that the efficiency benefits of the EF estimator relative to the LS, ML and QML estimators are substantial for some ARCH innovation distributions.

2. Estimating function formulation and efficiency

In this section, we describe the problem of estimation for a class of ARCH($p$) models characterized by the equations

$$X_t = \sigma_t(\theta_0)\varepsilon_t, \quad \sigma_t^2(\theta_0) = \omega_0 + \sum_{j=1}^{p} \alpha_{0j} X_{t-j}^2, \quad t = 1, \ldots, n, \quad (1)$$

where \{\varepsilon_t\} is a sequence of independent, identically distributed random variables such that $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$, $\theta_0 = (\omega_0, \alpha_{01}, \ldots, \alpha_{0p})^T$ is an unknown vector of true parameters satisfying $\omega_0 > 0$, $\alpha_{0j} \geq 0$, $j = 1, \ldots, p - 1$, and $\varepsilon_t$ is independent of $X_s$, $s < t$. Henceforth, it is tacitly assumed $\alpha_{0p} > 0$ so that the model is of order $p$. We also assume that model (1) is stationary and ergodic. When $p = 1$, Nelson (1990) showed that a sufficient condition for the stationarity is $E(\log(\alpha_{01} \varepsilon_t^2)) < 0$. For a general ARCH($p$) model, Bougerol and Picard (1992) showed that it has a unique non-anticipative strictly stationary solution.

We now turn to describe the conditional least squares estimation of model (1). Write $Y_t = (1, X_{t}^2, \ldots, X_{t-p+1}^2)^T$ and $\eta_t = (\varepsilon_t^2 - 1)\sigma_t^2(\theta_0)$. Then the standard linear autoregressive representation is given by

$$X_t^2 = \theta_0^T Y_{t-1} + \eta_t. \quad (2)$$

Suppose that an observed stretch $X_{1}^2, \ldots, X_{n}^2$ is available. The vector of parameters is $\theta = (\omega, \alpha_1, \ldots, \alpha_p)^T$ which belongs to a compact parameter space $\Theta \subset (0, \infty) \times [0, \infty)^p$, and $\theta_0 \in \Theta$. Let

$$Q_n(\theta) = \sum_{t=1}^{n} (X_t^2 - E(X_t^2|\mathcal{F}_{t-1}))^2 = \sum_{t=1}^{n} (X_t^2 - \theta^T Y_{t-1})^2$$

be the penalty function, where $\mathcal{F}_t = \sigma\{X_s^2, t \leq s\}$. Then from the linear regression theory, we can define the conditional least squares (LS) estimator of $\theta$ by

$$\hat{\theta}_n^{(LS)} = \arg\min_{\theta \in \Theta} Q_n(\theta) = (Y^T Y)^{-1} Y X, \quad (3)$$

where $Y$ is the matrix of order $n \times (1 + p)$ with $t$th row $Y_{t-1}$ and $X = (X_1^2, \ldots, X_n^2)^T$. Note that (3) does not take into account the nuisance parameter $\theta_0$ associated with variance and, hence, serves only as an initial estimator. The asymptotic validity of (3) can be easily established using an appropriate central limit theorem, and as expected, its efficiency is
smaller than that of the QML estimator.

To describe the limiting distribution of (3), we impose an additional condition on \( \theta_0 \), and the moment of \( \varepsilon_t \). Recall that we have assumed that the process is stationary and ergodic. Let

\[
A_{0t} = \begin{pmatrix}
\alpha_{01} \varepsilon_t^2 & \cdots & \alpha_{0p} \varepsilon_t^2 \\
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix},
\]

Introduce the notation \( A_{0t}^{\otimes s} = A_{0t} \otimes \cdots \otimes A_{0t} \), \( \Sigma_s = E(A_{0t}^{\otimes s}) \), where \( \otimes \) denotes the tensor product.

**Assumption 1**

\[
E[|\varepsilon_t|^8] < \infty \quad \text{and} \quad \|\Sigma_4\| < 1,
\]

where \( \| \cdot \| \) is the spectral matrix norm. In the case when \( p = 1 \), and \( \{\varepsilon_t\} \) is Gaussian, it is seen that \( \|\Sigma_4\| < 1 \) implies \( \alpha_{01} < 10^{-\frac{3}{4}} \approx 0.3 \). The following theorem establishes the asymptotic distribution of (3).

**Lemma 1.** Suppose that the assumptions of model (1) and Assumption 1 hold. Then

\[
\sqrt{n}(\hat{\theta}_n^{(LS)} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\varepsilon_t^2)\mathcal{U}^{-1}\mathcal{R}(\theta_0)\mathcal{U}^{-1}) \quad \text{as } n \to \infty,
\]

where the matrices \( \mathcal{U} = E(Y_{t-1}Y_{t-1}^T) \) and \( \mathcal{R}(\theta_0) = E(\sigma_t^4(\theta_0)Y_{t-1}Y_{t-1}^T) \) are positive definite with typically bounded elements.

**Remark 1.** Note that Assumption 1 ensures that \( \mathcal{U} \) and \( \mathcal{R}(\theta_0) \) are all finite. When the errors are standard normal, necessary and sufficient condition for the existence of higher moments of \( X_t \) in terms of the parameter \( \theta_0 \) is given by Engle (1982, Theorems 1 and 2).

**Remark 2.** As an illustration, we verify \( \mathcal{R}(\theta_0) \) is positive definite. Indeed, it is nonnegative definite, i.e., \( c^T\mathcal{R}(\theta_0)c = E(\sigma_t^2(\theta_0)c^TY_{t-1})^2 \geq 0 \) for any given vector \( c = (c_0, \ldots, c_p)^T \in R^{p+1} \). Moreover, if we suppose that \( \mathcal{R}(\theta_0) \) is not positive definite, then there exists a vector \( (c_0, \ldots, c_{j_0}) \) with \( c_{j_0} \neq 0 \) (\( j_0 \leq p \)) such that \( c_0 + c_1X_{s-1}^2 + \cdots + c_{j_0}X_{s-j_0}^2 = 0 \) a.e. This implies \( \sigma_t^2(\theta_0) > 0 \) a.e., because of \( \omega_0 > 0 \). In this case, we can write \( X_{s-j_0}^2 = -\gamma_0 - \gamma_1X_{s-1}^2 - \cdots - \gamma_{j_0-1}X_{s-j_0+1}^2 \), where \( \gamma_k = c_k/c_{j_0} \). Hence, substituting this into the last term of \( \sigma_t^2(\theta_0) \) in (1) with \( s - j_0 = t - p \) entails an ARCH\((p - 1)\) representation, leading to a contradiction.

The conditional least squares estimator \( \hat{\theta}_n^{(LS)} \) typically possesses the properties that it admits a closed-form expression, which is computationally easy. However, \( \hat{\theta}_n^{(LS)} \) in general is not asymptotically efficient. Thus we next discuss an asymptotically efficient estimator proposed by Godambe (1985) which has the following desirable properties: (i) it has an explicit form which is computationally easy (ii) it compares favourably with the ML and QML estimators.
Let $X^{(n)} = (X_1, \ldots, X_n)^T$ be a vector of random variables forming a stochastic process. The distribution family $\mathbb{F}$ of $X^{(n)}$ is specified by an unknown parameter vector $\theta = (\theta_1, \ldots, \theta_p)^T$ and $\theta = \theta(F)$, $F \in \mathbb{F}$ be a real parameter vector. An estimating function $g(X^{(n)}, \theta(F))$ satisfying certain regularity conditions, is called a regular unbiased estimating function if

$$E[g(X^{(n)}, \theta(F))] = 0, \quad F \in \mathbb{F}. \quad (4)$$

For a given set of unbiased estimating functions $g$ belonging to the class $\mathcal{G}$, the estimating function $g^* \in \mathcal{G}$ is said to be optimal for $\theta$ if

$$E[g^2(X^{(n)}, \theta(F))] / \{E[\partial g(X^{(n)}, \theta(F))/\partial \theta|_{\theta=\theta(F)}]\}^2$$

is minimized for all $F \in \mathbb{F}$ at $g = g^*$.

Let $\mathbb{L}$ be a class of linear combinations of unbiased estimating functions of the form

$$g = \sum_{t=1}^n a_{t-1} h_t,$$

where the weights $a_{t-1}$ are any function of $X_1, \ldots, X_{t-1}$ and $\theta$, and $h_t$ is a function of $X_1, \ldots, X_t$ and $\theta$ satisfying $E_F(h_t|\mathcal{B}_{t-1}) = 0$, where $\mathcal{B}_t = \sigma\{X_s, t \leq s\}$. Moreover, for all $F \in \mathbb{F}$, the $h_t$'s are mutually orthogonal.

An obvious example of $h_t$ is

$$h_t = X_t - E(X_t|\mathcal{B}_{t-1}), \quad (5)$$

which is the residual between $X_t$ and its best predictor $E(X_t|\mathcal{B}_{t-1})$. We assume that $h_t$ and $a_{t-1}$ are differentiable with respect to $\theta$ for $1 \leq t \leq n$. These considerations motivate the following result, which is due to Godambe (1985).

**Lemma 2.** In the class $\mathcal{G}$ of estimating functions $g$, the optimal estimating function is given by

$$g^* = \sum_{t=1}^n a_{t-1}^* h_t,$$

where $a_{t-1}^* = E[(\partial h_t/\partial \theta)|\mathcal{B}_{t-1}]/E[h_t^2|\mathcal{B}_{t-1}]$.

By virtue of Lemmas 1 and 2, we are now in a position to state our main result. For this purpose, we need the following notation. In view of (5), we can set

$$h_t = X_t^2 - E(X_t^2|\mathcal{F}_{t-1}) = X_t^2 - \sigma_t^2(\theta_0). \quad (6)$$

In general, the choice of an estimating function can be viewed in a manner analogous to the selection of moment conditions in the generalized method of moments (see Hansen (1982)). Now based on (6), we have

$$E\left(\frac{\partial h_t}{\partial \theta} \middle| \mathcal{F}_{t-1}\right) = -\frac{\partial \sigma_t^2(\theta_0)}{\partial \theta} \quad \text{and} \quad E(h_t^2|\mathcal{F}_{t-1}) = E(X_t^4|\mathcal{F}_{t-1}) - \sigma_t^4(\theta_0).$$

Then by virtue of (6) and Lemma 2, the optimal estimating function is

$$g^* = -\sum_{t=1}^n \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta} \frac{X_t^2 - \sigma_t^2(\theta_0)}{E(X_t^4|\mathcal{F}_{t-1}) - \sigma_t^4(\theta_0)}. \quad (7)$$
It should be pointed out that (7) is based on the finite sample, and it does not depend on any distributional assumptions for \(X_t^2\) conditional on \(F_{t-1}\). Noting that \(E(h_t^2|F_{t-1}) = \text{Var}(\varepsilon_t^2/\sigma_t^4(\hat{\theta}_n))\) and using (3), it follows that the solution to \(g^* = 0\) in (7) is the estimating function (EF) estimator given by

\[
\hat{\theta}_n^{(EF)} = \left( \sum_{t=1}^{n} Y_{t-1}Y_{t-1}^T \right)^{-1} \left( \sum_{t=1}^{n} Y_{t-1}X_t^2 \right).
\] (8)

Here, it is assumed that \(\sqrt{n}(\hat{\theta}_n^{(LS)} - \theta_0) = O_p(1)\). We now impose the following additional regularity conditions. Recall the matrix \(A_0\) and write it as \(A_0 = \{A_0\}_t\). In the notation of Bougerol and Picard (1992), the top Lyapunov exponent is defined by

\[
\gamma(A_0) \equiv \inf_{t \geq 1} \frac{1}{t} E(\log \|A_0A_0\cdots A_0\|) = \lim_{t \to \infty} \frac{1}{t} \log \|A_0A_0\cdots A_0\|
\]

under the assumption that \(E(\log^+ \|A_0\|) \leq E\|A_0\| < \infty\).

**Assumption 2**

(i) \(\theta_0 \in \tilde{\Theta}\), where \(\tilde{\Theta}\) denotes the interior of the compact parameter space \(\Theta\).

(ii) \(\gamma(A_0) < 0\).

(iii) \(\varepsilon_t^2\) has a non-degenerate distribution with \(E\varepsilon_t^2 = 1\).

(iv) \(E\varepsilon_t^4 < \infty\).

(v) \(\Gamma(\theta_0) = E(Y_{t-1}Y_{t-1}^T/\sigma_t^4(\theta_0))\) is finite.

Hence, we have the following theorem, which is the main result of the paper. The proofs for Lemma 1 and Theorem 1 are given in Section 5.

**Theorem 1.** Suppose that the assumptions of model (1) and Assumption 2 hold. Then as \(n \to \infty\),

\[
\sqrt{n}(\hat{\theta}_n^{(EF)} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \text{Var}(\varepsilon_t^2/\sigma_t^4(\theta_0))^{-1}(\theta_0)).
\]

**Remark 3.** Under the assumption of conditional normality, we have \(E(X_t^4|B_{t-1}) = 3\sigma_t^4(\theta_0)\), and in analogy with Engle (1982) the expression (7) reduces to

\[
g^* = -\sum_{t=1}^{n} \frac{1}{2\sigma_t^2(\theta_0)} \frac{\partial \sigma_t^2(\theta_0)}{\partial \theta} \left( \frac{X_t^2}{h_t} - 1 \right).\] (9)

Comparing (9) with the first-order condition of Engle (1982), we observe that they are equivalent up to a sign change. As is well known, under the additional assumption of normality, the ML estimator of \(\theta\) has the following asymptotic distribution,

\[
\sqrt{n}(\hat{\theta}_n^{(ML)} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 2\Gamma^{-1}(\theta_0)), \quad \text{as } n \to \infty.
\]

Hence, we conclude that the theory of estimating functions and ML method for the estimators of the ARCH model yield essentially the same asymptotic distribution.
Remark 4. As shown by Francq and Zakoïan (2004) for the ARCH model, the QML estimator of $\theta$ is obtained by maximising the normal log-likelihood function although the true probability density function is non-normal. Under the conditions of model (1) and Assumption 2, they showed that the QML estimator is asymptotically normal: 

$$\sqrt{n}(\hat{\theta}^{QML} - \theta_0) \xrightarrow{d} N(0, Var(\varepsilon_t^2)\Gamma^{-1}(\theta_0)),$$ 

as $n \to \infty$.

It is interesting to note that, if the true probability density function is the normal distribution, the asymptotic distribution of the ML and QML estimators is identical and coincides with ours.

Remark 5. Consider the GARCH($p$, $q$) model 

$$X_t = \sigma_t(\vartheta_0)\varepsilon_t, \quad \sigma_t^2(\vartheta_0) = \omega_0 + \sum_{i=1}^{p^*} \alpha_{0i}X_{t-i}^2 + \sum_{j=1}^{q} \beta_{0j}\sigma_{t-j}^2(\vartheta_0), \quad t = 1, \ldots, n$$

(10) 

where $\{\varepsilon_t\}$ is a sequence of independent, identically distributed random variables such that $E\varepsilon_t = 0$, $E\varepsilon_t^2 = 1$, $\vartheta_0 = (\omega_0, \alpha_{01}, \ldots, \alpha_{0p}, \beta_{01}, \ldots, \beta_{0q})^T \in \Theta \subset (0, \infty) \times (0, \infty)^{p+q}$ is an unknown vector of true parameters satisfying $\omega_0 > 0$, $\alpha_{0i} \geq 0$, $i = 1, \ldots, p$, $\beta_{0j} \geq 0$, $j = 1, \ldots, q$ and $\varepsilon_t$ is independent of $X_s, s < t$. Necessary and sufficient conditions under which the GARCH($p$, $q$) equations have a unique, strictly stationary, and non-anticipative solution were found by Nelson (1990) for $p = 1$ and $q = 1$, and by Bougerol and Picard (1992a, b) for arbitrary $p \geq 1$ and $q \geq 1$.

Write $\xi_t = (\varepsilon_t^2 - 1)\sigma_t^2(\vartheta_0)$. Then by analogy with (2), it follows that (10) can be represented as an ARMA($p^*$, $q$) model:

$$X_t^2 = \omega_0 + \sum_{i=1}^{p^*} \phi_{0i}X_{t-i}^2 + \xi_t + \sum_{j=1}^{q} \beta_{0j}\xi_{t-j},$$

(11) 

where $p^* = \max\{p, q\}$ and $\phi_{0i} = \alpha_{0i} + \beta_{0i} \geq 0$, $i = 1, \ldots, p^*$. We have further defined $\alpha_{0i} = 0$ for $i > p$ and $\beta_{0j} = 0$ for $j > q$. Henceforth, it is assumed that $X_t^2$ is covariance-stationary provided that $\xi_t$ has finite variance and that the roots of $1 - \phi_{01}z - \cdots - \phi_{0p^*}z^{p^*} = 0$ are outside the unit circle. Given the nonnegativity restriction, this means that $X_t^2$ is covariance-stationary if $\phi_{01} + \cdots + \phi_{0p^*} < 1$.

Suppose that an observed stretch $X_1^2, \ldots, X_n^2$ is available from $\{X_t^2\}$. Let $R(l)$ denote the autocovariance function of lag $l$, 

$$R(l) = E(X_t^2 - \mu)(X_{t+l}^2 - \mu), \quad l = 0, \pm 1, \ldots, $$

where $\mu = E(X_t^2) = \omega_0/(1 - \phi_{01} - \cdots - \phi_{0p^*})$, with the corresponding estimator

$$\hat{R}(l) = \frac{1}{n} \sum_{t=1}^{n-|l|} (X_t^2 - \hat{\mu})(X_{t+l}^2 - \hat{\mu}), \quad |l| < n,$$

where $\hat{\mu} = \sum_{t=1}^{n} X_t^2/n$. Here the initial values of $X_0^2 = \cdots = X_{-p^*}^2 = \xi_0 = \cdots = \xi_{-q} = 0$ have negligible effect on parameter estimates when the sample size is large. The expression

$$I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^{n} (X_t^2 - \hat{\mu}) e^{it\lambda} \right|^2 = \frac{1}{2\pi} \sum_{l=-n+1}^{n-1} \hat{R}(l)e^{-il\lambda}$$

6
is called the periodogram of the partial realization of \( \{X_t^2\} \). The vector of parameters is \( \varphi = (\omega, \phi_1, \ldots, \phi_p, \beta_1, \ldots, \beta_q)^T \in \Theta \). By the stationarity and ergodicity, set \( \sigma^2_\xi = E \xi_t^2 > 0 \) and write \( \rho = (\varphi^T, \sigma^2_\xi)^T \). Then the spectral density of \( \{X_t^2\} \) is

\[
 f_\rho(\lambda) = \frac{\sigma^2_\xi}{2\pi} \left| 1 - \sum_{j=1}^{q} \beta_j e^{ij\lambda} \right|^2 \left| 1 - \sum_{j=1}^{p} \phi_j e^{ij\lambda} \right|^{-2}.
\]

In order to estimate \( \rho \), Hosoya and Taniguchi (1982) proposed to minimise

\[
 D(f_\rho(\lambda), I_n) = \int_{-\pi}^{\pi} \left\{ \log f_\rho(\lambda) + \frac{f_\rho(\lambda)}{I_n(\lambda)} \right\} d\lambda
\]

with respect to \( \rho \). Let \( \hat{\rho}^{(QML)} = (\hat{\omega}^{(QML)}, \hat{\phi}^{(QML)}_1, \ldots, \hat{\phi}^{(QML)}_p, \hat{\beta}^{(QML)}_1, \ldots, \hat{\beta}^{(QML)}_q)^T \) be a quasi-Gaussian maximum likelihood estimator of \( \rho \) which minimizes \( D(f_\rho(\lambda), I_n) \). Under certain conditions, they showed that

\[
 \sqrt{n}(\hat{\rho}^{(QML)} - \rho) \xrightarrow{d} \mathcal{N}(0, 4\pi \left( \int_{-\pi}^{\pi} \frac{\partial}{\partial \rho} \log f_\rho(\lambda) \frac{\partial}{\partial \rho^T} \log f_\rho(\lambda) d\lambda \right)^{-1}).
\]

Note that (11) does not take into account the nuisance parameter \( \vartheta_0 \) associated with variance. Hence, \( \hat{\rho}^{(QML)} \) indeed serves only as an initial estimator.

The vector of parameters is \( \vartheta = (\omega, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)^T \in \Theta \). To apply Godambe’s estimating function method to (11), let

\[
 \hat{\sigma}^2_t \equiv \sigma^2_t(\hat{\vartheta}^{(QML)}) = \hat{\omega}^{(QML)} + \sum_{i=1}^{p} \hat{\phi}^{(QML)}_i X_{t-i}^2 + \sum_{j=1}^{q} \hat{\beta}^{(QML)}_j \sigma^2_{t-j}(\hat{\vartheta}^{(QML)}),
\]

with \( \hat{\sigma}^2_0 = \hat{\sigma}^2_1 = \cdots = \hat{\sigma}^2_{q+1} = 0 \). Then we can construct \( \hat{\sigma}^2_t, t = 1, \ldots, n, \) iteratively. Once this is done, we can find an estimator \( \hat{\rho}^{(EF)}_n = \hat{\rho}^{(EF)}(\hat{\rho}^{(QML)}_n) \) of \( \rho \) by means of Godambe’s method.

To gain a further insight into Theorem 1, it is interesting to compare the asymptotic variances of \( \hat{\theta}^{(LS)}_n \) and \( \hat{\theta}^{(EF)}_n \) in terms of their efficiency. This motivates to state the following result, whose proof is given in Section 4.

**Theorem 2.** Under the conditions of Lemmas 1 and 2, the asymptotic variance of \( \hat{\theta}^{(LS)}_n \) typically satisfies the inequality that \( \text{Var}(\varepsilon^2_t) \mathcal{U}^{-1} \mathcal{R}(\theta_0) \mathcal{U}^{-1} \geq \text{Var}(\varepsilon^2_t) \Gamma^{-1}(\theta_0) \).

### 3. Influence function

An influence function is a statistical tool which provides rich qualitative information of how an estimator responds to a small amount of contamination at any point. In the following we introduce a robustness measure for the estimator given by (8). This robust measure by means of influence function will show how the initial estimator \( \hat{\theta}^{(LS)}_n \) affects \( \hat{\theta}^{(EF)}_n \).

Let us first study a robustness property of \( \hat{\theta}^{(LS)}_n \). Write \( S_t = (X^2_t, \ldots, X^2_{t-p+1})^T \) and \( Z_{S,t} = (1, S_t^T)^T \). Then we have \( \hat{\theta}^{(LS)}_n = \hat{U}_S^{-1} \hat{\gamma}_S \), where

\[
 \hat{\gamma}_S = \frac{1}{n} \sum_{t=1}^{n} S^{(1)}_t Z_{S,t-1} \quad \text{and} \quad \hat{U}_S = \frac{1}{n} \sum_{t=1}^{n} Z_{S,t-1} Z_{S,t-1}^T.
\]
Here, $S_{t}^{(1)}$ is the first component of $S_t$. We can now define the corresponding least squares functional as

$$T_S \equiv U_S^{-1} \gamma_S,$$

where

$$\gamma_S = E(S_{t}^{(1)} Z_{S,t-1}) \quad \text{and} \quad U_S = E(Z_{S,t-1} Z_{S,t-1}^T),$$

As a measure of robustness property of $\hat{\theta}_n^{(LS)}$, we consider the following contaminated process

$$S_{\delta,t} = (1 - \delta) S_t + \delta U_t \equiv S_t + \delta V_t,$$

where $\delta \in (0, 1)$. For $S_{\delta} = \{S_{\delta,t}\}$, we can introduce an influence function

$$T'_S = \lim_{\delta \to 0} \frac{T_{S_\delta} - T_S}{\delta}.$$

Noting the formula for differentiation of the inverse of a matrix $dA^{-1} = -A^{-1}(dA)A^{-1}$, we obtain

$$\left. \frac{d}{d\delta} U_{S_\delta}^{-1} \right|_{\delta=0} = -U_S^{-1}(\Delta + \Delta^T)U_S^{-1}, \quad (12)$$

where $\Delta = E(\tilde{V}_t Z_{S,t-1})$ with $\tilde{V}_t = (0, V_t^T)^T$. Also,

$$\left. \frac{d}{d\delta} \gamma_{S_\delta} \right|_{\delta=0} = E(V_{t}^{(1)} Z_{S,t-1}) + E(S_{t}^{(1)} \tilde{V}_{t-1}) \equiv \gamma'_S, \quad (13)$$

where $V_{t}^{(1)}$ is the first component of $V_t$. Hence,

$$T'_S = U_S^{-1}[\gamma'_S - (\Delta + \Delta^T)T_S].$$

The quantity $T'_S$ will reveal how outliers in the dependent and independent variables may combine to affect $\hat{\theta}_n^{(LS)}$.

In the above notation, we can similarly derive an influence function of $\hat{\theta}_n^{(EF)} = \hat{U}_{S,w}^{-1} \hat{\gamma}_{S,w}$, where

$$\hat{\gamma}_{S,w} = \frac{1}{n} \sum_{t=1}^{n} S_{t}^{(1)} \hat{w}_{S,t-1} Z_{S,t-1} \quad \text{and} \quad \hat{U}_{S,w} = \frac{1}{n} \sum_{t=1}^{n} \hat{w}_{S,t-1} Z_{S,t-1} Z_{S,t-1}^T$$

with

$$\hat{w}_{S,t} = [w((\hat{\theta}_n^{(LS)})^T Z_{S,t} Z_{S,t}^T \hat{\theta}_n^{(LS)})]^{-1}.$$

Here note that $\hat{w}_{S,t-1} = \sigma_t^{-4}(\hat{\theta}_n^{(LS)})$. Since $\hat{\gamma}_{S,w}$ and $\hat{U}_{S,w}$ are the respective sample versions of

$$\gamma_{S,w} = E(S_{t}^{(1)} w_{S,t-1} Z_{S,t-1}) \quad \text{and} \quad U_{S,w} = E(w_{S,t-1} Z_{S,t-1} Z_{S,t-1}^T),$$

with

$$w_{S,t} = [w((T_S^T Z_{S,t} Z_{S,t}^T T_S^T)]^{-1},$$

the functional analogue of $T_S$ is

$$T_{S,w} = U_{S,w}^{-1} \gamma_{S,w},$$

Write

$$Q_t = \tilde{V}_t Z_{S,t}^T \quad \text{and} \quad L_{w,t} = S_{t+1}^{(1)} w_{S,t} \hat{\theta}_n^{(LS)} T_S^T Q_t T_S.$$
Then by analogy with (12),
\[
\frac{d}{d\delta} U_{S,w}^{-1}\bigr|_{\delta=0} = -U_{S,w}^{-1}\{\tilde{\Delta}_w + \tilde{\Delta}_w^T - (\Phi_w + \Phi_w^T)\} U_{S,w}^{-1},
\]
where
\[
\tilde{\Delta}_w = E(w_{S,t-1}\tilde{V}_{t-1}Z_{S,t-1}^T)
\quad \text{and} \quad
\Phi_w = E(L_{w,t-1}Z_{S,t-1}Z_{S,t-1}^T)
\]
and with (13),
\[
\frac{d}{d\delta} \gamma_{S,w}\bigr|_{\delta=0} = E(V_t^{(1)}w_{S,t-1}Z_{S,t-1}) + E(S_t^{(1)}w_{S,t-1}\tilde{V}_{t-1}) - E(L_{w,t-1}Z_{S,t-1}) - E(L_{w,t-1}Z_{S,t-1})
\equiv \gamma_{S,w}'.
\]
Hence
\[
T_{S,w}' = U_{S,w}^{-1}\{\gamma_{S,w}' - [\tilde{\Delta}_w + \tilde{\Delta}_w^T - (\Phi_w + \Phi_w^T)]T_{S,w}\}.
\]
This expression will facilitate the fundamental description of sensitiveness or insensitiveness of \(\hat{\theta}_{n}^{(EF)}\).

4. Simulations

A finite sample experiment is performed to assess the asymptotic efficiency of the EF estimator given by (8) relative to LS, ML and QML estimators for a small and a large sample of observations.

To facilitate meaningful comparisons, we generate an ARCH(1) of length \(n = 50\) or \(n = 500\) for values of \(\theta_0 = (\omega_0, \alpha_{01}) = (1, 0.1)\) or \(1, 0.3\) based on 5000 replications. Without loss of efficiency, we assume \(\omega_0 = 1\) and estimate \(\alpha_{01}\) using LS, ML, QML and EF methods. For this purpose, we consider two error distributions—a normal distribution and a Student-\(t\) distribution with \(v = 5\) degrees of freedom. In both cases, the unconditional mean and variance are 0 and 1, respectively.

Tables 1 and 2 report the results in terms of bias, variance and mean square error (MSE) for \(\hat{\alpha}_{n}^{(LS)}, \hat{\alpha}_{n}^{(ML)}, \hat{\alpha}_{n}^{(QML)}\) and \(\hat{\alpha}_{n}^{(EF)}\) of \(\alpha_{01}\). A closer examination of the MSE values in the tables reveals some interesting features. We first note that the values are intrinsically stable with respect to the choice of parameters and sample sizes. In every case, we observe that the finite-sample MSE of the EF estimator is desirable relative to other alternatives such as LS, ML and QML methods. The efficiency of \(\hat{\alpha}_{n}^{(EF)}\) increases against its counterpart as the sample size increases or \(\alpha_{01}\) decreases. In the case of normality, the MSE results for the ML, QML and EF estimators are approximately the same. To this end, note that the result of Tables 1 and 2 also holds true for normalized error distributions such as double exponential, logistic and gamma having zero mean and unit variance.

Our simulation results highlight the benefits of using the EF formulation for modelling data drawn from non-normal conditional distributions. This approach naturally takes advantage of departures from normality to improve the efficiency of estimators given a finite sample of data. In comparison with asymptotically based methods, the focus on the finite sample in the EF formulation is important. We observe that efficiency gains from the EF approach are substantial. Hence, the EF formulation is potentially useful in cases with serious departures from normality in which efficiency is important.
Table 1: MSE of the LS, ML, QML and EF estimators for $\alpha_{01} = 0.1, 0.3$ with standard normal errors

| Parameter $\alpha_{01}$ | MSE (LS) $\hat{\alpha}_n$ | MSE (ML) $\hat{\alpha}_n$ | MSE (QML) $\hat{\alpha}_n$ | MSE (EF) $\hat{\alpha}_n$ |
|-------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.1                     | 0.1195 0.0098             | 0.0120 0.0013             | 0.0120 0.0013             | 0.0062 0.0012             |
| 0.3                     | 0.7150 0.0953             | 0.5668 0.0524             | 0.5668 0.0524             | 0.5670 0.0524             |
| 0.1                     | 0.8345 0.1051             | 0.5788 0.0537             | 0.5788 0.0537             | 0.5732 0.0536             |
| 0.3                     | 0.1037 0.0978             | 0.0867 0.0420             | 0.0867 0.0420             | 0.0850 0.0405             |
| 0.1                     | 0.7554 0.1899             | 0.6156 0.1324             | 0.6156 0.1324             | 0.5702 0.1320             |
| 0.3                     | 0.8591 0.2877             | 0.7023 0.1744             | 0.7023 0.1744             | 0.6552 0.1725             |

Note: The three values in each cell are from top to bottom: squared bias, variance, and MSE.

Table 2: MSE of the LS, ML, QML and EF estimators for $\alpha_{01} = 0.1, 0.3$ with $t_5$ distributed errors

| Parameter $\alpha_{01}$ | MSE (LS) $\hat{\alpha}_n$ | MSE (ML) $\hat{\alpha}_n$ | MSE (QML) $\hat{\alpha}_n$ | MSE (EF) $\hat{\alpha}_n$ |
|-------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| 0.1                     | 0.2314 0.0124             | 0.0203 0.0088             | 0.0111 0.0050             | 0.0078 0.0013             |
| 0.3                     | 0.7808 0.1090             | 0.6010 0.0552             | 0.5797 0.0537             | 0.5633 0.0556             |
| 0.1                     | 1.0123 0.1214             | 0.6213 0.0640             | 0.5908 0.0587             | 0.5711 0.0569             |
| 0.3                     | 0.2132 0.1096             | 0.1710 0.1013             | 0.0987 0.0268             | 0.0836 0.0277             |
| 0.1                     | 0.9875 0.1711             | 0.6575 0.1579             | 0.6432 0.1410             | 0.6482 0.1377             |
| 0.3                     | 1.2007 0.2807             | 0.8285 0.2592             | 0.7419 0.1678             | 0.7318 0.1654             |

Note: The three values in each cell are from top to bottom: squared bias, variance, and MSE.
5. Proofs

In this section we provide the proofs of Lemma 1, and Theorems 1 and 2.

**Proof of Lemma 1.** Note from (3) that

\[
\sqrt{n}(\hat{\theta}_n^{(LS)} - \theta_0) = \left( \frac{1}{n} \sum_{t=1}^{n} Y_{t-1} Y_{t-1}^T \right)^{-1} \left( n^{-1/2} \sum_{t=1}^{n} Y_{t-1} \eta_t \right). \tag{14}
\]

Since \( \{X_t\} \) is stationary and ergodic, so is \((Y_{t-1} Y_{t-1}^T)\) which is finite by Assumption 1. Thus by the ergodic theorem

\[
\frac{1}{n} \sum_{t=1}^{n} Y_{t-1} Y_{t-1}^T \overset{a.s.}{\rightarrow} U.
\]

Next consider the second factor on the right side of (14). Let \( c = (c_0, \ldots, c_p)^T \) be any vector with \( c \neq 0 \). Recall that \( \eta_t = (\varepsilon_t^2 - 1)\sigma_t^2(\theta_0) \). Then using the martingale central limit theorem and Cramer-Wold device, it follows that

\[
n^{-1/2} \sum_{t=1}^{n} c^T Y_{t-1} \eta_t \overset{d}{\rightarrow} \mathcal{N}(0, \text{Var}(\varepsilon_t^2) c^T \mathcal{R}(\theta_0) c).
\]

To prove this, note that \( Y_{t-1} \eta_t \) is a martingale difference sequence since \( E(Y_{t-1} \eta_t | \mathcal{F}_{t-1}) = 0 \). We now verify the conditional Linderberg condition only since the other conditions can be verified easily.

For all \( \epsilon > 0 \), we show that

\[
L_n = \frac{1}{n} \sum_{t=1}^{n} (\sigma_t^2(\theta_0)c^T Y_{t-1})^2 \times E\{(|\varepsilon_t^2 - 1|^2 I(|\varepsilon_t^2 - 1| > \sqrt{n}\epsilon^2)|\mathcal{F}_{t-1}) \}
\]

where \( I(\Omega) \) is the indicator function of the event \( \Omega \). Observe that

\[
L_n \leq \frac{1}{n} \sum_{t=1}^{n} (\sigma_t^2(\theta_0)c^T Y_{t-1})^2 \times E\{(|\varepsilon_t^2 - 1|^2 I(|\varepsilon_t^2 - 1| > n^{1/4}\epsilon) + I(|\sigma_t^2(\theta_0)c^T Y_{t-1}| > n^{1/4}\epsilon)|\mathcal{F}_{t-1}) \}
\]

\[
= \frac{1}{n} \sum_{t=1}^{n} (\sigma_t^2(\theta_0)c^T Y_{t-1})^2 E\{(|\varepsilon_t^2 - 1|^2 I(|\varepsilon_t^2 - 1| > n^{1/4}\epsilon) \}
\]

\[
+ \text{Var}(\varepsilon_t^2) \frac{1}{n} \sum_{t=1}^{n} (\sigma_t^2(\theta_0)c^T Y_{t-1})^2 I(|\sigma_t^2(\theta_0)c^T Y_{t-1}| > n^{1/4}\epsilon) \}
\]

\[
\equiv T_1 + \text{Var}(\varepsilon_t^2) T_2.
\]

Noting that \( \text{Var}(\varepsilon_t^2) < \infty \) and that

\[
\frac{1}{n} \sum_{t=1}^{n} (\sigma_t^2(\theta_0)c^T Y_{t-1})^2 = E[(\sigma_t^2(\theta_0)c^T Y_{t-1})^2] + o_p(1),
\]

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we obtain $T_1 = o_p(1)$. Moreover, note from Assumption 1 that

$$E(T_2) = E[(\sigma_1^2(\theta_0) c^T Y_{t-1})^2 I(|\sigma_1^2(\theta_0) c^T Y_{t-1}| > n^{1/4}/\epsilon)] = o(1),$$

which implies $T_2 \geq 0$. Hence (15) is satisfied and by Slutsky’s theorem the assertion of Lemma 1 follows.

**Proof of Theorem 2.** From (8), observe that

$$\frac{1}{n} \sum_{t=1}^{n} Y_{t-1}Y_{t-1}^T \{((\hat{\theta}_n^{(LS)})^T Y_{t-1} Y_{t-1}^T \hat{\theta}_n^{(LS)})^{-1} - (\theta_0^T Y_{t-1} Y_{t-1}^T \theta_0)^{-1}\} = o_p(1) \quad (16)$$

and

$$n^{-1/2} \sum_{t=1}^{n} Y_{t-1} \eta_t \{((\hat{\theta}_n^{(LS)})^T Y_{t-1} Y_{t-1}^T \hat{\theta}_n^{(LS)})^{-1} - (\theta_0^T Y_{t-1} Y_{t-1}^T \theta_0)^{-1}\} = o_p(1). \quad (17)$$

In view of Lemma 1, we note that $\hat{\theta}_n^{(LS)} \overset{a.s.}{\to} \theta_0$ for sufficiently large $n$, and thus, $\sigma_1^2(\hat{\theta}_n^{(LS)})$ behaves like $\sigma_1^2(\hat{\theta}_n)$ for each $t = 1, \ldots, n$. This statement holds true, if for $\epsilon > 0$, there exists $N_\epsilon$ such that $||\hat{\theta}_n^{(LS)} - \theta_0|| \leq \epsilon$ for all $n > N_\epsilon$, with probability one. Consequently, we have

$$\sqrt{n}(\hat{\theta}_n^{(EF)} - \theta_0) = \left( \frac{1}{n} \sum_{t=1}^{n} \frac{Y_{t-1}Y_{t-1}^T}{(\theta_0^T Y_{t-1} Y_{t-1}^T \theta_0)^{-1}} \right)^{-1} \left( n^{-1/2} \sum_{t=1}^{n} \frac{Y_{t-1} \eta_t}{(\theta_0^T Y_{t-1} Y_{t-1}^T \theta_0)^{-1}} \right) + o_p(1),$$

for which, the proof is reduced to that of Lemma 1. Since the proof of (16) and (17) is similar, we prove (16) only as follows.

By a Taylor expansion around $\hat{\theta}_n^{(LS)}$ at $\theta_0$, it follows that (16) is dominated by

$$O_p(\sqrt{n}(\hat{\theta}_n^{(LS)} - \theta_0)) \times \left| \frac{1}{n} \sum_{t=1}^{n} Y_{t-1} \eta_t \hat{\theta}_t^T (Y_{t-1} Y_{t-1}^T)(\hat{\theta}_t^T Y_{t-1} Y_{t-1}^T \hat{\theta}_t)^{-2}\right|,$$

where $\hat{\theta}_t$ lies between $\theta_0$ and $\hat{\theta}_n^{(LS)}$. Moreover, from the ergodic theorem, we readily see that

$$\frac{1}{n} \sum_{t=1}^{n} Y_{t-1} \eta_t \hat{\theta}_t^T (Y_{t-1} Y_{t-1}^T)(\hat{\theta}_t^T Y_{t-1} Y_{t-1}^T \hat{\theta}_t)^{-2} \overset{a.s.}{\to} 0.$$
The equality holds if and only if there exists a constant $r \times t$ matrices $C$ such that $\psi A + CB = 0$ almost everywhere.

Using the notation of Lemma 3, write $A = B = Y_{t-1}$ and $\psi = \sigma^4_t(\theta_0)$. Then it follows that $R(\theta_0) \geq U^{T^{-1}}(\theta_0)U$. It is obvious from Lemma 3 that the equality $R(\theta_0) = U^{T^{-1}}(\theta_0)U$ holds if and only if $\sigma^2_t(\theta_0) = k$, a constant almost everywhere. Hence we get the desired result.

6. References

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