Hamiltonicity of the Power Graph of Abelian Groups-I

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Abstract

In this article we discuss the Hamiltonicity of the power graph associated to a finite group.

1 Introduction

Graphs associated to an algebraic structure with a binary operation is in literature for a long time. Most notable of it all would probably be the Cayley graph associated to a group. Association of a graph to a group dates back to Cayley (see [6]), and more recently to Dehn in [4]. Dehn reintroduced the Cayley graph of a group and proposed the word problem of the mapping class group of a surface. In the theory of Cayley graphs it is a long standing problem to solve, that is the conjecture of Lovász, 1969 who conjectured that a vertex transitive graph has a Hamiltonian path and in particular a connected Cayley graph is Hamiltonian. It is an elementary exercise to write down a Hamiltonian cycle when an abelian group acts transitively on a graph. More generally in the framework of Cayley graphs it is easier to get some results on existence of a Hamiltonian cycle when the group is an abelian group.

But the same problem for a non-abelian group still remains open despite repeated attempts by eminent mathematicians. So it is an interesting problem to
ask whether the power graph associated to a group is Hamiltonian. The answer to that question is no, as it is trivial to see from the definition of a power graph that the power graph of $\mathbb{Z}/p\mathbb{Z}$ has a cut vertex when $p$ is a prime. So it is an interesting problem to decide for which groups the associated power graphs are Hamiltonian. The definition of power graph of a finite group occurs in the works of Kelarev et.al in the paper [5], where the directed power graph of a finite group was studied. For a finite group $G$ the graph $G = (V, E)$ is the graph on vertex set $V = G$ where $(g, h)$ is a directed edge from $g$ to $h$ if there is a natural number $k$ such that $h = g^k$. In the survey [1] one can find a very detailed bibliography of the articles related to power graph of a finite group.

In the paper [3] the question of Hamiltonicity of undirected power graph of the group of units in the ring $\mathbb{Z}/\mathbb{Z}_n$ is introduced. It was shown in the same paper that when the number $n$ is a product of more than one Fermat primes [2] the group of units as mentioned above does not give a Hamiltonian graph. But the general question remains open whether the group of units in $\mathbb{Z}/\mathbb{Z}_n = U_n$ gives a Hamiltonian graph. In this article and in the next one we hope to completely answer the question raised in the paper [3].

The question of existence of a Hamiltonian cycle in a graph is a long standing problem in discrete mathematics. Many authors attacked the problem in many ways. In general it remains to be a very active field of research producing a lot of activity in both mathematics and theoretical computer science.

In this article we have developed several techniques to find the answer to the existence of Hamiltonian cycle in a graph, and the concept of a weighted Hamiltonian cycle. A weighted Hamiltonian path is a spanning path where a vertex $v$ is allowed to be visited at most the largest natural number smaller than $w(v)$ times. We describe the Hamiltonian cycles of the power graph in terms of weighted hamiltonian cycles of a related graph. We have broken up the discussion of Hamiltonicity of power graph of a group into two volumes, this volume consist of characterizations of the structure of power graphs and complete characterization of Hamiltonicity of a special types of groups [4,14]. In the next volume we take up the question of general abelian groups.

2 Structure of Power Graphs

In this section we define a few basic concepts and prove a few basic results for general power graphs of groups.
Definition 2.1. Power graph $P_G = (V, E)$ of a finite group $G$ is given by the vertex set $V$ and edge set $E = \{(g, h) : \exists k \in \mathbb{N}, g^k = h\}$.

Note that we have a natural direction on the edges of the power graph of a group. In this article we will consider only the undirected power graph of a group as it is easy to observe that there cannot be any directed Hamiltonian cycle on a power graph (identity vertex is a sink). But many a time we will consider the direction, although we will mention it whenever we assume it, to facilitate our study.

Definition 2.2. A graph $G = (V, E)$ along with a map $f : E \rightarrow \mathcal{P}(\mathbb{N})$, is called a power graph if there is a group $H$ such that $G = P_H$ and $f(g, h) = \{k \in \mathbb{N} : g^k = h\}$.

Lemma 2.3. In a directed power graph $G$ if $(x, y), (y, z)$ are two directed edges, then $(x, z)$ is a directed edge.

Proof. There are $n_x, n_y \in \mathbb{N}$ such that $x^{n_x} = y$ and $y^{n_y} = z$ so putting these two together we have $x^{n_x n_y} = z$. \qed

Edges obtained in a process described in the above lemma will be called transitive edges.

Lemma 2.4. Let $G$ be a group, if $(g, h)$ and $(h, g)$ are two directed edges in the power graph of $G$ then $o(g) = o(h)$.

2.5 Cluster graphs

Throughout this section we will consider the directed power graph of a group.

Definition 2.6. A cluster in a power graph $G$ is a strong component in the associated directed power graph.

Note that a natural graph structure precipitates on the set of clusters $C$ of the power graph $G = (V, E)$ as the quotient graph under the equivalence relation defined by the strong components. Also note that this graph structure acquires a natural direction on the edges and further note that by the lemma 2.3 this direction on the edges of the clusters is not reversible. We will call this to be the cluster graph of the power graph $G$, just as the lemma 2.3. The induced subgraph on the set of vertices of a cluster $c$ is a directed complete subgraph of the graph $G$, as a result the graph $G$ is totally determined if the cluster graph and the size of the clusters.
are known. The cluster graph with the vertex weight defined by the size of the cluster totally characterizes the graph \( G \). Let \( G = (V, E) \) be a vertex weighted graph with vertex weight \( w : V \rightarrow \mathbb{N} \), we define \( \tilde{G}^w \) as a the graph on the set of vertices \( \bigcup_{v \in V} [v] \times [w(v)] \) with ((\( v_1 \), \( v_2 \)), (\( u_1 \), \( u_2 \))) an edge if either \( v_1 = u_1 \) and \( i \neq j \) or \( (v_1, v_2) \in E \), this operation is reversible since the original graph \( G \) is the quotient graph of the graph \( \tilde{G}^w \) where \( (v_1, i) \) is equivalent to \( (u_1, j) \) if \( v_1 = u_1 \). Thus a power graph \( G \) is isomorphic to the graph \( \tilde{C}^w \) where \( C \) is the cluster graph and \( w(c) = |c| \).

We sum up the above discussion in the following three brief lemmas, the proofs of which are more or less automatic.

**Lemma 2.7.** If \( c_1, c_2, c_3 \) such that \( (c_1, c_2) \) and \( (c_2, c_3) \) are directed cluster edges, then there is a directed cluster edge \( (c_1, c_3) \).

**Lemma 2.8.** Let \( c_1, c_2 \) be two clusters in a graph \( G \) and let \( v_i \in c_i \) be vertices such that \( (v_1, v_2) \) is an edge in \( G \), then for every \( v \in c_1 \) and \( v' \in c_2 \) there are edges \( (v, v') \) in \( G \).

**Proof.** Follows from the lemma 2.3.

Throughout in this article we will assume that the product of two graphs is given in the following manner.

**Definition 2.9.** Let \( G_i = (V_i, E_i) \) for \( i = 1, 2 \) be two graphs, then the product graph is defined as the graph \( G_1 \times G_2 = (V_1 \times V_2, E) \) where \( ((v_1, w_1), (v_2, w_2)) \) defines an edge if \( v_1 = v_2 \) and \( (w_1, w_2) \in E_2 \) or \( (v_1, v_2) \in E_1 \) and \( w_1 = w_2 \) or \( (v_1, v_2) \in E_1 \) and \( (w_1, w_2) \in E_2 \).

**Lemma 2.10.** Let \( G \) be a directed graph, let \( C(G) \) be its cluster graph with \( w(c) = |c| \) vertex weights, then \( G \simeq \tilde{C}^w \).

**Lemma 2.11.** Let \( G_1, G_2 \) be two directed graphs and \( C(G_i) \) be the corresponding cluster graphs then \( C(G_1 \times G_2) = C(G_1) \times C(G_2) \).

If a cluster edge is not obtained as a transitive edge then it is called an irreducible edge. The directed cluster graph with irreducible edges and the size of each cluster totally determines the original graph as we will see next. As a convention for drawing figures, whenever there is not a chance of ambiguity, our edges in a cluster graph will be oriented from south to north.

Let us also define a product of power graphs in the following manner.
Definition 2.12. Let $G_1, G_2$ be two power graphs and let $f_1, f_2$ be two edge functions respectively, then we define a product $G_1 \boxtimes G_2 = (G_1 \times G_2, E_1 \boxtimes E_2)$ where $E_1 \boxtimes E_2$ is given by the edges $((v_1, x_1), (v_2, x_2))$ is an edge if $f_1(v_1, x_1) \cap f_2(v_2, x_2) \neq \emptyset$.

Theorem 2.13. If $G = G_1 \times G_2$ then $P_{G_1} \boxtimes P_{G_2} = P_G$, where $P_G$ denotes the power graph of the group $G$.

Proof. Only thing that we have to prove is that the edge sets are equal for both sides. In other words we have to prove that $E_1 \boxtimes E_2 = E$, where $E$ is that edge set of the graph $G$. Let $(x, y), (g, h) \in G$ such that there is an edge $((x, y), (g, h)) \in E$. Or equivalently we have a natural number $k$ such that $(x, y)^k = (g, h)$ or $x^k = g$ and $y^k = h \iff (x, g) \in E_1$ and $(y, h) \in E_2$ and $f_1(x, g) \cap f_2(y, h) \neq \emptyset$ (since it contains $k$) $\iff ((x, y), (g, h)) \in E_1 \boxtimes E_2$. \hfill $\square$

Theorem 2.14. Let $G_p, G_q$ be a $p$ and a $q$ group respectively, where $p \neq q$ then $G_p \boxtimes G_q = G_p \times G_q$. 

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Figure 1: cluster graph with irreducible edges of $\mathbb{Z}_3^2 \times \mathbb{Z}_3$

Figure 2: Cluster graph of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$
Proof. We just have to show that $E_p \bowtie E_q = E_p \times E_q$ to show that it is enough to show that for any $(x, y) \in E_p$ and $(g, h) \in E_q$, $f_p(x, y) \cap f_q(g, h) \neq \emptyset$. Equivalently, there exists $k_1$ and $k_2$ such that $x^{k_1} = g$ and $y^{k_2} = h$. So $f_p(x, y) = \{k_1 + m_1o(x) : m_1 \in \mathbb{Z}\}$, $f_q(g, h) = \{k_2 + m_2o(g) : m_2 \in \mathbb{Z}\}$, where $o(x)$ and $o(g)$ are prime to each other. So we have to show that there is a solution of the following equations: $x \equiv k_1 (\mod o(x))$ and $y \equiv k_2 (\mod o(g))$. Now since $o(x)$ and $o(g)$ being co-prime, by Chinese Remainder theorem we get a solution of the above equations say the solution be $k$. Therefore $x^k = g$ and $y^k = h \Rightarrow (x, y)^k = (g, h)$. Hence follows the theorem. $\square$

**Theorem 2.15.** Let $G_1, G_2$ be two Hamiltonian graphs then $G_1 \times G_2$ is Hamiltonian.

**Proof.** Fix a Hamiltonian cycle $v_0, v_1, \ldots, v_n = v_0$ in the graph $G_1$, let us also fix a Hamiltonian cycle $w_0, w_1, \ldots, w_m = w_0$ in the graph $G_2$. We define a Hamiltonian cycle in the graph $G_1 \times G_2$ in the following way. Define paths $\pi_i = (v_0, w_i), (v_1, w_i), \ldots, (v_{m-1}, w_i)$ on the product graph. Note that there is an edge in the product graph between the end vertex of the path $\pi_i$ and the start vertex of the path $\pi_{i+1}$ for each $i \leq m - 1$. And the end vertex of the path $\pi_m$ has an edge to the start vertex of the path $\pi_1$. So the concatenation $\pi_0\pi_1 \ldots \pi_m$ is a Hamiltonian cycle in the product graph. $\square$

**Definition 2.16.** Let $G = (V, E)$ be a graph and let $\sim$ be an equivalence relation on the set of vertices $V$ such that if $v \sim w$ and $(w', w'')$ is an edge then $v \sim w''$, the quotient graph $G/\sim = (V', E')$ where $V' = \{[v] : v \in V\}$ where $[v]$ stands for the equivalence class of the vertex $v$. $([v], [w]) \in E'$ if for any vertex $v_1 \in [v]$ there is a vertex $w_1 \in [w]$ such that $(v_1, w_1) \in E$.

**Theorem 2.17.** Let $G = (V, E)$ be a graph and let $\sim$ be an equivalence relation on the set of vertices $V$. Let $[v]$ stand for the equivalence class of the vertex $v$. If the quotient graph $G/\sim$ and the induced subgraphs on equivalence classes $[v]$ are Hamiltonian for all $v \in V$ then $G$ is Hamiltonian.

**Proof.** Let us fix a Hamiltonian path on the quotient graph $[v_1], [v_2], \ldots, [v_l]$. Now also fix a Hamiltonian cycle for each of the induced subgraphs $[v_1]$ let these be $w_{i1}, w_{i2}, \ldots, w_{ir_i}$, let us call this path $\pi_i$. So we have a cycle $\pi_1\pi_2 \ldots \pi_l$, this is a path since from the end vertex of $\pi_i$ that is $w_{ir_i}$ there is an edge to the vertices $\{w_{i+11}, w_{i+12}, \ldots, w_{i+1r_{i+1}}\}$ without loss of generality let us say to the vertex

\[6\]
So the concatenation works. From the last vertex \( w_{1_1} \), there is an edge to a vertex in the set \( \{ w_{1_2}, w_{1_2}, \ldots, w_{1_{1_1}} \} \) if it is not the vertex \( w_{1} \) we can relabel so that we have a cycle.

Note that although the special case in form of the theorem 2.15 follows from the above theorem 2.17 we have included a separate proof.

**Theorem 2.18.** Strong product of a pair of directed graphs, each of whose underlying undirected graphs are Hamiltonian, is undirected Hamiltonian.

**Proof.** Let \( v_0, v_1, v_2, \ldots v_n \) be an undirected Hamiltonian in the first graph namely \( G_1 \), where we choose \( v_n \) in a way that there are some edges directed towards it. Let similarly choose \( w_1, w_2, \ldots, w_m \) be an undirected Hamiltonian cycle in the second graph namely \( G_2 \), where we choose \( w_0 \).

2.19 \( p \)-primary groups

In this subsection we will consider an Abelian \( p \)-group \( G_p \) for a prime \( p \), which using the structure theorem of Abelian group can be written as \( \bigoplus_{i=1}^{n} \mathbb{Z}/p^{a_i}\mathbb{Z} \) for some natural numbers \( a_i \). For \( a_i \in \mathbb{N}_0 \) such that \( a_i \leq \alpha_i \) we define \( N_{(a_1, a_2, \ldots, a_n)} \) or simply by \( N \) to be the set \( \bigcap N_{a_i} \) where \( N_{a_i} = \{ g \in \mathbb{Z}/p^{a_i}\mathbb{Z} : o(g) = p^{\alpha_i-a_i} \} \). For a tuple \( \alpha = (a_1, a_2, \ldots, a_n) \) we can define \( \alpha + 1 = (\min\{a_1 + 1, \alpha_1\}, \min\{a_2 + 1, \alpha_2\}, \ldots, \min\{a_n + 1, \alpha_n\}) \). Let us define a notation at this juncture \( L_{a_1, a_2, \ldots, a_n} = \{(\alpha) = (a_1, a_2, \ldots, a_n) : \text{ where } 0 \leq a_i \leq \alpha_i \} \), if we declare an edge between \( a \) and \( a + 1 \) then the resulting graph structure on the set \( I_{\alpha} \) is a tree. For the Abelian \( p \)-primary group \( G_p \) the tree \( I_{\alpha} \) is called \( N_{\alpha} \) tree of the group \( G_p \).

**Remark 2.20.**
1. There is a map \( f : N_{\alpha} \to N_{\alpha+1} \) where \( f(g_1, g_2, \ldots, g_n) = (pg_1, pg_2, \ldots, pg_n) \)

2. The group \( H = (\mathbb{Z}/p^a\mathbb{Z})^* \) acts on the set \( N_{\alpha} \) where \( \alpha = \max\{a_1, a_2, \ldots, a_n\} \), by the action: \( x \in H \) then \( x \ast (g_1, g_2, \ldots, g_n) = (\bar{x}g_1, \bar{x}g_2, \ldots, \bar{x}g_n) \). Where \( \bar{x} \) stands for the image of \( x \) under the map \( \mathbb{Z}/p^a\mathbb{Z} \to \mathbb{Z}/p^a \).

**Lemma 2.21.** under the action of the group \( H = (\mathbb{Z}/p^a\mathbb{Z})^* \) on the \( N_{\alpha} \) the orbits are precisely the clusters of the power graph of \( G_p \).
Figure 3: $N_{rs}$ tree of $\mathbb{Z}_{p^3} \times \mathbb{Z}_{p^2}$

**Proof.** Let $g = (g_1, g_2, \ldots, g_n) \in N_a$ and let $O_g$ be the orbit through $g$. Let $x \ast g \in O_g$ for some $x \in H$, now choosing an appropriate representative $x'$ in $\mathbb{Z}$ for the element $x$, we write $x \ast g = (xg_1, xg_2, \ldots, xg_n) = (x'g_1, x'g_2, \ldots, x'g_n) = xg$. Hence there is an edge between $g$ and $x \ast g$. Now note that the edge is reversible since for $y \in H$ such that $xy = 1$ we have $y(x \ast g) = yx \ast g = g$ thus we have an edge from $x \ast g$ to $g$. Now if there is $z \in G$ such that there is a reversible edge between $z$ and $g$, then $z = kg$ for some $k \in \mathbb{N}_0$ hence $z$ is in the orbit of $g$. $\square$

Let us denote the $p$-primary group $\bigoplus_{i=1}^n \mathbb{Z}/p^{\alpha_i}\mathbb{Z}$ by $G_p$, observe that the abelian group gets a ring structure by coordinatewise multiplication, we will call this ring to be $RG_p$. Also let us denote the cluster of an element $\underline{a} = (a_1, a_2, \ldots, a_n)$ by $c(\underline{a})$, recall that we call support($\underline{a}$) = \{i \leq n|a_i \neq 0\}. The following lemma is provided to make the size of the cluster $c(\underline{a})$ clear.

**Lemma 2.22.** Let $\underline{a} \in G_p$ and let $m$ be the maximum among the $\alpha_i$ such that $i$ is in the support of $\underline{a}$ then we have $|c(\underline{a})| = |(\mathbb{Z}/p^m\mathbb{Z})^n|$.

**Proof.** Recall that $\underline{b} \in c(\underline{a}) \iff \exists x, y \in \mathbb{Z}$ such that $x\underline{a} = \underline{b}$ and $y\underline{b} = \underline{a}$. Note that we can take the numbers $x, y$ from the ring $\mathbb{Z}/p^m\mathbb{Z}$, where $m = \max\{\alpha_i|i \in$ support($\underline{a}$)$\}$. Since $x\underline{a} = \underline{b} \iff xy\underline{b} = \underline{b} \iff xy = 1 \in RG_p$, and if $x\underline{a} = y\underline{a}$ then $(x - y)\underline{a} = 0 \iff (x - y) = 0 \in \mathbb{Z}/p^m\mathbb{Z}$ for all $i \in$ support($\underline{a})$. Putting these two together we get the result. $\square$
Corollary 2.23. \(|c(a)| = p^{\alpha_i - 1}(p - 1)|\), where \(\alpha_i\) is maximum of the set \(\{\alpha_i|i \in \text{support}(a)\}\).

Let us prove a small lemma concerning the group \(H = \mathbb{Z}/p^m\mathbb{Z}\), for an element \(x \in H\) let us define \(r(x) = |\{y \in H|py = x\}|\).

Lemma 2.24. With the notation as above \(r(x) = 0\) if \(x \in (\mathbb{Z}/p^m\mathbb{Z})^*\) else \(r(x) = p\). Further if \(y \in \mathbb{Z}/p^n\mathbb{Z}^*\) for some \(n \geq m\) then \(r(yx) = r(x)\).

In the next lemma we calculate the degree of the clusters in the graph \(C(G_p)\). Note that in the cluster graph we only take the so called irreducible edges in consideration and we leave the “implied” edges.

Lemma 2.25. Let \(c(a)\) where \(a = (a_1,a_2,a_3,\ldots,a_n)\) be a cluster in the cluster graph \(C(G_p)\) then the indegree of the cluster \(\text{indeg}(c(a)) = \prod_{i=1}^{n} r(a_i)\).

Proof. The required indegree is given by the number \(|\{c(b) : pb = xa\text{ such that } x \in (\mathbb{Z}/p^\alpha_i\mathbb{Z})^*\}|\), where \(x\) is the maximum of the numbers \(\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n\). Or the number of \(b\) such that \(p(b_1,b_2,b_3,\ldots,b_n) = x(a_1,a_2,a_3,\ldots,a_n)\) or equivalently \(pb_i = xa_i\). So the required number is \(\prod_{i=1}^{n} r(xa_i)\) where \(x \in (\mathbb{Z}/p^\alpha_i\mathbb{Z})^*\) fixed. Now by the lemma 2.24 we are done. \(\Box\)

So as a result of the lemmas 2.20,2.21,2.22 and 2.25 the cluster tree of the group \(G_p\) is now completely clear. In the next section we will pick up the question of the discussion of the Hamiltonicity of these graphs.

3 Investigation of Hamiltonicity

3.1 Weighted Hamiltonicity

Given an weighted graph \(G = (V,E,w)\) where \(w : V \rightarrow \mathbb{N}\) is the weight function we define the graph \(\overline{G}^w = (\overline{V}, \overline{E})\), where \(\overline{V} = \cup_{v \in E}v \times \{1,\ldots,w(v)\}\) and there is an edge between \((v,i)\) and \((v',i')\) if either \(v = v'\) or \((v,v') \in E\). A weighted graph as defined above is called a \(w\)-Hamiltonian graph if there is a path \(\pi = v_1v_2v_3\ldots v_n\) such that \(|v_i|1 \leq i \leq n\) = \(V\) and \(|i|_{v_1} = v| \leq w(v)\). When the weight is clear from the context we will denote \(\overline{G}^w\) by \(G\).

In the lemma below we note a very useful fact, which we will henceforth refer to as the generalized cut lemma.
Lemma 3.2. Let \( G = (V, E, w) \) be a weighted graph, let \( C \subseteq V \), if \( G \) is \( w \)-Hamiltonian then \( w(C) = \sum_{v \in C} w(v) \geq N \) where \( N \) is the number of connected components in \( G \setminus C \). Further if there exist a \( w \)-Hamiltonian path then \( w(C) \geq N - 1 \).

Proof. Clearly since there are \( N \) connected components in the graph \( G \setminus C \) one must visit \( C \) (counting multiplicity) at least \( N - 1 \) times to get a \( w \)-Hamiltonian graph. And that is precisely the condition in the statement. \( \square \)

Lemma 3.3. In a graph \( G = (V, E) \) if there is a special vertex \( v_0 \) such that \( (v_0, v) \in E \) for all \( v \in V \) other than \( v_0 \) then there exist a Hamiltonian cycle in the graph \( G \) if and only if there is a Hamiltonian path in the graph \( G \setminus v_0 \).

Proof. It is clear that if \( \pi = v_1, v_2, \ldots, v_n \) be a Hamiltonian path in \( G \setminus v_0 \), then \( \pi_1 = v_0, v_1, v_2, \ldots, v_n, v_0 \) gives a Hamiltonian cycle in \( G \). \( \square \)

In the lemma below we give a necessary and sufficient condition on a weight function \( w \) for which any graph \( G \) is a \( w \)-Hamiltonian graph.

Lemma 3.4. A weighted tree \( T = (V, E, w) \) is \( w \)-Hamiltonian if and only if \( w(v) \geq \deg(v), \forall v \in V \).

Proof. Let the tree \( T = (V, E, w) \) be \( w \)-Hamiltonian and if possible let there exists \( v \in V \) such that \( \deg(v) > w(v) \), then \( \{v\} \) gives a cut, therefore by cut lemma we have \( w(v) \geq \deg(v) \) the number of connected components of \( G \setminus v \). So we arrive at a contradiction. Hence \( w(v) \geq \deg(v), \forall v \in V \). Conversely, let \( w(v) \geq \deg(v), \forall v \in V \). Now choose any \( v \in V \) and \( T_1, T_2, \ldots, T_k \) be the components of \( T \setminus v \). Then \( T_i \) also satisfies \( w(v_i) \geq \deg(v_i), \forall v_i \in T_i \). If there are Hamiltonian cycles \( \alpha_1, \alpha_2, \ldots, \alpha_k \) in \( T_1, T_2, \ldots, T_k \) then \( v_0 \alpha_1, v_0 \alpha_2, \ldots, v_0 \alpha_k \) is a Hamiltonian path in \( T \) with \( w(v) \geq k \) and \( w(v_i) \geq \deg^T(v_i) = \deg^T(v_i) - 1 \). Hence it follows. \( \square \)

Lemma 3.5. Let \((T, w)\) be a weighted tree with weight \( w \) (here \( T \) may or may not be Hamiltonian). Then \((T, \lambda w)\) is Hamiltonian if and only if \( \lambda \geq \max_{v \in V(T)} \frac{\deg(v)}{w(v)} \).

Proof. \((T, \lambda w)\) is Hamiltonian \( \iff \lambda w(v) \geq \deg(v), \forall v \in V(T) \iff \lambda \geq \frac{\deg(v)}{w(v)}, \forall v \in V(T) \iff \lambda \geq \max_{v \in V(T)} \frac{\deg(v)}{w(v)}. \) \( \square \)
Theorem 3.6. A weighted graph $G$ with weight $w$ is $w$-Hamiltonian if and only if $\tilde{G}$ is Hamiltonian.

Proof. Let the weighted graph $(G, w)$ be $w$-Hamiltonian. Then let $v_0, v_1, v_2, \ldots, v_n$ be a weighted Hamiltonian cycle in $G$ such that $\bigcup\{v_i\} = V$. Let $w_i$'s be the corresponding weights of $v_i$'s. Now consider the cycle $(v_0, w_0), (v_1, w_0), (v_0, w_1), (v_1, w_1), \ldots, (v_n, w_1), (v_0, w_n), \ldots, (v_n, w_n), (v_0, w_0)$, which gives a Hamiltonian cycle in $\tilde{G}$. For the other direction, let we have Hamiltonian cycle in $\tilde{G}$, say $(v_0, i_0), (v_1, i_1), \ldots, (v_r, i_r), (v_0, i_0)$. Then take the cycle according to the vertices $v_i$'s we get the weighted Hamiltonian cycle. \hfill \Box

Theorem 3.7. Let $(G_1, w_1)$ and $(G_2, w_2)$ be two weighted graphs, if $G_1$ is $w_1$ Hamiltonian and $w_1(G_1)$ minus the spent weights in the $w_1$ cycle is more than the number $|G_2|$ then $G_1 \times G_2$ is $w_1 \times w_2$ Hamiltonian.

Proof. It is clear that with the remaining weights we can go to any section $(G_1, v)$ and using the fact that this is $w_1$ Hamiltonian as $G_1$ we can finish off this section, as the number or remaining weights is more than such sections we can finish off all of the graph. \hfill \Box

3.8 The grid

Let $G_p = \bigoplus_{i=1}^l (\mathbb{Z}_p)^{n_i}$ for some numbers $n_i \in \mathbb{N}$, we define the level sets $L_k = \{ [x] \in G_p | ht(x) = k \}$, where $[x]$ denotes the cluster of the element $x$ and $ht(x)$ is the number $\log_p(o(x))$ or equivalently the number of irreducible edges it takes to reach the cluster of 0 from the cluster of $x$. Now observe that the induced subgraph on $L_0 \cup L_1$ is a star. We will call this the associated star to the group $G_p$. Note that the cluster graph of the group $G_p$ gets a graded structure on the set of vertices through these level sets. Or in other words we have $V = \bigcup_{i=1}^l L_i$.

3.9 Cuts and some necessary conditions

Recall a subset $S$ of the set of vertices of a graph $G$ is called a cut if the number of connected components in $G \setminus S$ is larger than the number of elements in the set $S$. In this section we will investigate several cuts in the power graph of a group $G$. Consider the class of groups given by $G = \mathbb{Z}_p^m \times \mathbb{Z}_q^n$ let $G(m, n, w)$ be the weighted cluster grid of the group.
Lemma 3.10. In the graph $G(m, n, w)$ as in above we have:

1. $S = \{(x, y)|x = 0, y \leq n\}$ is a cut if $n > m(p - 1) + 1$.

2. $S = \{(x, y)|x \leq m, y = 0\}$ is a cut if $m > n(q - 1) + 1$.

3. $S = \{(x, y)|x = 0 \text{ or } y = 0\}$ is a cut if $mn > m(p - 1) + n(q - 1) + 1$.

Theorem 3.11. The power graph of a the group $G = (\mathbb{Z}_p)^m \times (\mathbb{Z}_p)^n$ is not Hamiltonian if $n > m(q - 1) + 1$ or $m > n(q - 1) + 1$ or $mn > m(q - 1) + n(q - 1)$.

Proof. Follows trivially from the lemma 3.10. □

We can generalize the above lemma for the general grid $\text{Grid}^{m_1, m_2, \ldots, m_r}_{u_1, u_2, \ldots, u_r}$. Let us take a subset $I \subset [r] = \{1, 2, \ldots, r\}$ We define the cut $\mathcal{C}$ associated to the subset $I$ is the set of vertices $\cup_{i \in I} C_i$ where $C_i = \{(a_1, a_2, \ldots, a_r) : a_i = 0\}$. In this following lemma we will calculate the total weight of the cut and calculate the number of components in it's complement. For a subset $A \subset \text{Grid}^{m_1, m_2, \ldots, m_r}_{u_1, u_2, \ldots, u_r}$ let us call the dimension of the span of $A$ as the dimension of $A$. So using this notation dimension of $C_i$ is $r - 1$ or the co-dimension is 1.

Lemma 3.12. Let $A$ be a subset of the grid $\text{Grid}^{m_1, m_2, \ldots, m_r}_{u_1, u_2, \ldots, u_r}$ if the co-dimension of the subset $A$ is more than 1 then it does not disconnect the grid.

Proof. We will show that $C_{ij} = \{(a_1, a_2, \ldots, a_r)|a_i = a_j = 0\}$ for $i \neq j$ and unions of these does not disconnect the grid. And a similar argument will prove that for even a higher co-dimension subsets will not disconnect these. Let us show that the points $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$ with $\alpha_i, \alpha_j$ not both zero is connected by a path to $(\beta_1, \beta_2, \beta_3, \ldots, \beta_r)$ with $\beta_i, \beta_j$ not both zero. First let us show the case that $\alpha_i \neq 0$ and $\beta_j \neq 0$, we have the path $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r) \rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_i, 0, \ldots, \alpha_r)$ (where we have 0 in the $j$th position, $\rightarrow (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_i, \beta_j, \ldots, \alpha_r)$. Now we use the fact $\beta_j \neq 0$ to continue the path with $\rightarrow (\alpha_1, \alpha_2, \alpha_3, \ldots, 0, \ldots, \beta_j, \ldots, \alpha_r)$ where the 0 appears in the $i$th position. Continue as $\rightarrow (\alpha_1, \alpha_2, \alpha_3, \ldots, \beta_i, \ldots, \beta_j, \ldots, \alpha_r)$. Now we can change the rest of the points similarly. For the case $\alpha_i = \beta_i = 0$ we will start with $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_j, \ldots, \alpha_r) \rightarrow (\alpha_1, \alpha_2, \alpha_3, \ldots, 1, \ldots, \alpha_j, \ldots, \alpha_r)$ now the final point is connected to $(\beta_1, \beta_2, \beta_3, \ldots, \beta_i, \ldots, \beta_j, \ldots, \beta_r)$ as the first point has 1 in the $i$th place and the second one has a nonzero entry in the $j$th place. □

As a corollary of the above lemma one can say that the only subsets that disconnects the grid are unions of $C_i$ for $i \in I$. 12
Lemma 3.13. Let Grid\(m,u,m,n\), I and \(\mathcal{S}\) be as above then,

1. \(w(I) = w(\mathcal{S}) = \sum_{a \in \mathcal{S}} w(a) = \prod_{j \notin C} u_j \sum_{A \subseteq C, j \notin A} m_j u_j + 1\)

2. number of component in Grid\(m,u,m,n\) \(= \prod_{j \in C} m_j\).

Proof. 1. 
\[w(a) = \sum_{a \in \mathcal{S}} w(a) = \sum_{a \in \mathcal{S}} w(a_1)w(a_2) \ldots w(a_r)\]

Since for \(a_i \neq 0\) \(w(a_j) = u_j\).

\[= \sum_{a \in \mathcal{S}} \prod_{a_i \neq 0} u_j\]

Now we take the product \(\prod_{j \notin C} u_j\) out. Now since for \(A \subseteq C\) for \((a_1, a_2, a_3, \ldots, a_r)\) such that \(a_i = 0\) for \(i \in C \setminus A\), we have \(w(a_1, a_2, a_3, \ldots, a_r) = \prod_{j \notin A} u_j\), and since there are \(\prod_{j \in A} m_j\) such elements we have the next equality.

\[= \prod_{j \notin C} u_j \sum_{A \subseteq C} \prod_{j \notin A} m_j u_j\]

2. For this part note that if \(a, b \in \text{Grid}_{m,u,m,n} \setminus C\) then \(a, b\) are not in the same connected component if \(\{i : a_i \neq b_j\} \cap C \neq \emptyset\). So the number of connected components are equal to the number of \(a_i\) such that \(i \in C\), or \(\prod_{j \in C} m_j\).

\(\Box\)

Lemma 3.14. If for any subset \(C \subset [r]\), \(w(C) > \prod_{j \in C} m_j\) then the grid Grid\(m,u,m,n\) is not Hamiltonian.

3.15 Towards sufficient conditions

4 Grid Algorithms

Let us take the weighted grid graph Grid\(m,u,v\), where \(m, n, u, v \in \mathbb{N}\) is defined as the graph with \(V = \{0, 1, 2, \ldots, m\} \times \{0, 1, 2, \ldots, n\}\) and directed edges are \(((i, j), (l, k))\) if \(i = 0\) and \(j = 0\) or \(i = l\) and \(j = 0\) or \(i = 0\) and \(j = k\). Note that this is just
the product graph of the stars \( S_m = [m] \cup \{0\} \) and edges from \( i \in [m] \) to 0, and \( S_n \). And the weight is given by \( w(i, j) = uv \) if \( j \neq 0 \) and \( j \neq 0 \) and \( w(0, j) = v \) and \( w(i, 0) = u \). In the light of the above section it is noted that the Hamiltonicity of such grids are one of the basic questions that we have to discuss. In this section we will throw light on such graphs. Note that we can generalize the grid to a general grid as follows \( Grid_{m_1, m_2, \ldots, m_n}^{w(u, v)} = Grid_{m+n}^w \) is a graph of the product graph of stars \( S_m \) where the weight of a vertex \( w(x_1, x_2, x_3, \ldots, x_t) = \prod_{i=1}^{t} u_i(x_i) \) with the understanding that \( u_i : \{0, 1, 2, \ldots, m_i\} \to \mathbb{N} \) is a function with \( u_i(0) = 1 \). If we take the functions \( u_i(x) = u_i x \) for \( x \neq 0 \) and 0 otherwise then we call that a simple grid.

**Theorem 4.1.** Let \( Grid_{(m,n)}^{(w,u)} = (V, E, w) \) is \( w \)-Hamiltonian then \( nu + mv \geq mn - 1 \) and \( mv \geq n - 1 \).

**Proof.** Let us take \( C = \{(i, j) \in V : i = 0 \text{ or } j = 0\} \). Note that \( w(C) \) as defined in the lemma \([3,2]\) is \( \sum_{v \in C} w(v) = nu + mv + 1 \). And the number of connected components in the complement of \( C \) is clearly is the set of singleton points \((i, j)\) such that \( i \neq 0 \) and \( j \neq 0 \), so the number of components is \( mn \). Now the result follows from the lemma \([3,2]\). \( \square \)

**Theorem 4.2.** The grid as defined above with all the notations in place with \( n \geq m \) is Hamiltonian if \( nu + mv \geq mn - 1 \) and \( mv \geq n - 1 \).

**Proof.** To prove the above theorem we will make use of the following lemmas. \( \square \)

The complete bipartite graph \( K_{m,n} = (V, E) \) where \( V = [m] \times \{0\} \cup \{0\} \times [n] \) with weights defined as \( w(i, 0) = v \) and \( w(0, j) = u \) will be called the associated complete bipartite graph of the grid \( Grid_{m+n}^{u,v} \).

For a weighted complete bipartite graph \( K_{m,n} \) with vertex set \( V = \{x_1, x_2, x_3, \ldots, x_n\} \cup \{y_1, y_2, y_3, \ldots, y_m\} \) with vertex weight \( w \) we define an acceptable path in the following way. For a path \( \pi = v_1 v_2 v_3 \ldots v_k \) we define \( n^\pi(v) = |\{l : v_l = v\}| \) or in other words the number of times the vertex \( v \) appears in the path \( \pi \).

**Definition 4.3.** A \( w \)-Hamiltonian path \( \pi = v_1, v_2, \ldots, v_k \) in \( K_{m,n} \) is acceptable if there is a partition of edges not in the path \( E \setminus \{(v_i, v_{i+1}) : 1 \leq i \leq k - 1\} = A \sqcup B \), where \( A = \bigcup A_i, B = \bigcup B_j, A_i = \{l : (x_i, y_i) \in A\}, B_j = \{l : (x_j, y_j) \in B\} \), such that the following two statements are true.
1. $w(x_i) \geq n^r(x_i) + |B_i| - 1$ if $v_1 = x_i$, else, $w(x_i) \geq n^r(x_i) + |B_i|$. \\
2. $w(y_j) \geq n^r(y_j) + |A_j| - 1$ if $v_1 = y_j$ else, $w(y_j) \geq n^r(y_j) + |A_j|$. \\

**Proposition 4.4.** \(\text{Grid}^{(v,u)}_{(m,n)}\) is \(w\)-Hamiltonian if and only if there is an acceptable path in the associated complete bipartite graph. \\

**Proof.** Let us prove that if there is a \(w\)-Hamiltonian cycle in the grid then there is an acceptable path in the associated complete bipartite graph. Without loss of generality we assume that the \(w\)-Hamiltonian cycle \(\pi = v_0v_1 \ldots v_rv_{r+1}\) starts at the point \((0,0)\), i.e. \(v_0 = v_{r+1} = (0,0)\). For the rest of the vertices \(v_i = (x_i, y_i)\), we call such a vertex interior vertex if \(x_iy_i \neq 0\). We define a path \(\tilde{\pi}\) on the associated complete bipartite graph by ignoring the interior vertices of the path \(\pi\) and realizing the points \((x_i, 0)\) as \((i, 0)\) and \((0,y_i)\) as \((0, i)\). Clearly this is a \(w\)-Hamiltonian path. For the acceptable part define \(A_i = \{i: (x_i, 0)(y_i)(x_i, 0)\text{ is an expression in } \pi\}\), similarly we define \(B_j = \{j: (0,y_j)(y_j)(0, y_j)\text{ is an expression in } \pi\}\). It is easy to check that this is an acceptable path. For the reverse direction we define the \(w\)-Hamiltonian path in the grid in the following way, we take an acceptable path \(\pi = v_1v_2v_3 \ldots v_r\) on \(K_{m,n}\) and at the first point when \(v_i = (x_i, 0)\) we insert the expression \(\prod_{j \in B_j} (x_j, y_j)(x_j, 0)\) and whenever first we have \(v_k = (0, y_i)\) we insert the expression \(\prod_{s \in A_i} (x_s, y_s)(0, y_i)\). It is easy to prove that this is a \(w\)-Hamiltonian cycle in the grid. \\

Let us consider the path as described below in the complete bipartite graph \(K_{m,n}\). Let us assume that \(V_1 = \{x_1, x_2, x_3, \ldots, x_m\}, V_2 = \{y_1, y_2, y_3, \ldots, y_n\}\) and also without a loss of generality let us assume that \(m \geq n\). Let us also consider the weight \(w\) which is defined as \(w(x_i) = v\) and \(w(y_j) = u\). Also assume that \(m = sn + r\), where \(0 \leq r < n\). Let us look at the following path \(\pi = v - 1v_2v_3 \ldots v_{2n-1}\):

\[x_1y_1x_2y_2, \ldots, x_ny_n\]
\[x_{n+1}y_1x_{n+2}y_2, \ldots, x_{2n}y_n\]
\[
\ldots
\]
\[x_m(y_{s-1} + 1)x_{(s-1) + 2} \cdots x_{ns}y_n\]
\[x_{ns+1}y_1x_{ns+2}y_2 \cdots x_{ns+r}\]
So we can also say that for \( j < m \):

\[
v_j = \begin{cases} 
  x_l & \text{if } j \text{ is odd and } l = (j + 1)/2 \\
  y_k & \text{if } j \text{ is even } k = j \mod (n)
\end{cases}
\]

Let us define for a vertex \( a \in V_1 \cup V_2 \) the number of times the vertex occurs in the path \( \pi \), \( d(a) = |v_j : v_j = a| \), in the following lemma we calculate these numbers precisely for the path.

**Lemma 4.5.** For the path \( \pi \) as above we have the following equalities:

1. 
   \[
   d(y_i) = \begin{cases} 
   s & \text{if } i \geq r \\
   s + 1 & \text{else}
   \end{cases}
   \]

2. \( d(x_i) = 1 \) for all \( i \).

*Proof.* It follows clearly from the definition of the path. □

**Lemma 4.6.** For the path \( \pi \) above we have \( d(y_i) \leq u \).

*Proof.*

1. **Case \( r = 0 \).** In this case \( d(y_i) = m/n \) and we know that \( nu \geq m - 1 \) so \( u \geq (m - 1)/n = m/n - 1/n \) and both \( u \) and \( m/n \) are integers we have the result of the lemma.

2. **Case \( i \geq r \).** We have \( d(y_i) = s \) so \( sd(y_i) = ns = m - r \leq nu + 1 - r \leq nu \) hence \( d(y_i) \leq u \).

3. **Case \( 1 < i < r \).** \( d(y_i) = s + 1 \) or \( nd(y_i) = ns + n = m + n - r \leq nu + 1 + n - r \). Hence \( d(y_i) \leq u + 1 + n - r/n \leq u \) as \( 1 + n - r < n \).

□

Now let us consider the weights we have after the path \( \pi \) for the first \( x_1 \) and the last \( x_m \) visits there will be no weights lost as we are starting from this points or still to move from this points. So the weights used will be for each of the \( x_i \) where \( i \) is other than \( 1, m \) we have used up only one weight. And for the \( y_i \) we have used as many weights as we have visited these points, that is \( d(y_i) \). Let us denote the available weights by \( \theta(a) \) for a vertex \( a \in V_1 \cup V_2 \). And note that we have the following weights left:

\[
\theta(x_i) = \begin{cases} 
  v & \text{if } i = 1, m \\
  v - 1 & \text{else}
\end{cases}
\]
and

$$\theta(y_i) = \begin{cases} 
    u - s & \text{if } i \geq r \\
    u - s - 1 & \text{else}
\end{cases}$$

Consider the subset $A_\pi = \{(i, j) : (x_i, y_j) \text{ does not occur in } \pi\}$ of the grid $[m] \times [n]$, with weights $u_i = \theta(x_i)$ and $v_j = \theta(y_j)$. In the following proposition we will prove that this subset is colorable; a concept that we are defining in the next subsection see 4.10.

**Proposition 4.7.** The subset $A_\pi$ as defined above is colorable.

**Proof.** As a result of 4.10 we have to prove that for any subset $B \subset A_\pi$ we have the inequality:

$$\sum_{i \in \pi_1(B)} u_i + \sum_{j \in \pi_2(B)} v_j \geq |B|$$

or

$$\sum_{i \in \pi_1(B)} \theta(x_i) + \sum_{j \in \pi_2(B)} \theta(y_j) \geq |B|$$

To deal with the case when $1, m \in \pi_1(B)$ we see that the maximum cardinality of such a $B$ is the size of the grid minus the number of edges appearing in the path $\pi$. Clearly there are $2m - 1$ edges in the path $\pi$ so the maximum size of $B$ is $mn - 2m + 1$ in such case the left hand side becomes: $(v - 1)(m - 2) + 2v + (u - s)(n - r) + (u - s - 1)r = mv + nu - 2m + 2$ and since we have $mv + nu \geq mn - 1$ we have the desired colorability in this case. The general case is an easy reiteration of this argument.

\[ \square \]

Let us summarize the above in this following theorem.

**Theorem 4.8.** The associated complete bipartite to the grid $\text{Grid}^{u,v}_{m,n}$ has an acceptable path if $mv \geq n - 1, nu \geq m - 1$ and $mv + nu \geq mn - 1$.

### 4.9 Color game

In this section we will build up the “unvisits” (the subsets $A, B$) for the above path using a game we will call color game. Let us consider a subset $A \subset [m] \times [n]$ where $[m] = \{1, 2, 3 \ldots, m\}$, and there are natural numbers $u_1, u_2, u_3, \ldots, u_n$ and $v_1, v_2, v_3, \ldots, v_m$. The points $(i, j) \in A$ could be colored by either color red if $u_i \geq 1$ and if it is colored red then $u_i$ is reset by $u_i - 1$ and the point could be
colored blue if \( v_j \geq 1 \) and the number \( v_j \) is reset by \( v_j - 1 \). The goal is to color all the points on \( A \) with some colors. If there is a legal coloring of the subset with the given weights then the subset is called colorable with the given weights.

**Lemma 4.10.** A subset \( A \subset [m] \times [n] \) is colorable if and only if for every subset \( B \subset A \) we have \( \sum_{i \in \pi_1(B)} u_i + \sum_{j \in \pi_2(B)} v_j \geq |B| \).

**Proof.** The necessary condition is obvious. For the sufficient condition let us give an algorithm:

1. Pick a column \( i \) such that \( \pi^{-1}(i) \) has the maximum cardinality.
2. If the color available on the \( X \) axis is not zero color the point with red from \( (i, j_1), (i, j_2), (i, j_3), \ldots, (i, j_r) \) where weights \( v_{j_1} \leq v_{j_2} \leq \ldots v_{j_r} \).
3. Color the rest of the points on \( \pi^{-1}(i) \) blue. This can be done since the condition for this subset guarantees that.
4. Now observe that \( A' = A \setminus \pi^{-1}(i) \) is a subset of a smaller grid and by the lemma \[4.11\] \( A' \) could be colored by the remaining colors.

\[ \square \]

**Lemma 4.11.** Let \( A' \) be as above then for every subset \( B \subset A' \) we have

\[
\sum_{i \in \pi_1(B)} x_i + \sum_{j \in \pi_2(B)} y_j \geq |B|.
\]

**Proof.** If we realize the subset \( B \subset A' \) as a subset of \( A \) we get the desired inequality.

Continuing with the notations from the definition \[4.3\] we will denote the associated complete bipartite weighted graph to the grid \( Grid_{m,n}^w \) as \( K_{m,n} \) with weight \( w \). Let \( \pi = v_1 v_2 v_3 \ldots v_l \) be a \( w \)-Hamiltonian path in \( K_{m,n} \). Let \( E' \) be all the edges encountered in this path, let \( A_\pi = \{(i, j) | (i, j) \notin E'\} \), and let us take \( u_i = w(x_i) - |\{l : v_l = x_i\}| \) if \( x_i \neq v_1 \) and \( w(x_i) - |\{l : v_l = x_i\}| + 1 \) else, similarly we define \( v_j \). In the lemma below we characterize the acceptable paths.

**Lemma 4.12.** The subset \( A_\pi \) is colorable if and only if the path \( \pi \) is acceptable.
Proof. If $A_{x}$ is colorable then define $A_{l} = \{ y_{i} | (l, i) \text{ is colored red} \}$ and similarly we define $B_{l} = \{ x_{i} | (i, l) \text{ is colored blue} \}$. Note that since $A_{x}$ is colorable we have

1. $w(x_{i}) - n^{x}(x_{i}) \geq |A_{l}|$ if $x_{i} \neq v_{1}$ else $w(x_{i}) - n^{x}(x_{i}) + 1 \geq |A_{l}|$.

2. $w(y_{i}) - n^{y}(y_{i}) \geq |B_{l}|$ if $y_{i} \neq v_{1}$ else $w(y_{i}) - n^{y}(y_{i}) + 1 \geq |B_{l}|$.

This finishes the proof of the acceptability and for the reverse part we color the vertices in $A$ red and those in $B$ blue, the condition for acceptability permits us to have a legal coloring with these colors.

□

Corollary 4.13. The grid $\text{Grid}_{(m,n)}^{(v,u)} = (V, E, w)$ is $w$-Hamiltonian if and only if $nu + mv \geq mn - 1$ and $nu \geq m - 1, mu \geq n - 1$.

Proof. Necessary condition was shown in 3.11 and the sufficient condition follows from 4.8 and 4.4.

□

Theorem 4.14. The powergraph of the Abelian groups $(\mathbb{Z}_{p})^{n} \times (\mathbb{Z}_{q})^{m}$, where $p, q$ are distinct primes, is Hamiltonian if and only if $m(q - 1) \geq n - 1$ and $n(p - 1) \geq m - 1$ and $m(q - 1) + n(p - 1) \geq mn - 1$.

Proof. Follows from the corollary 4.13.

□

References

[1] J. Abawajy, A. Kelarev, and M. Chowdhury. Power graphs: a survey. Electron. J. Graph Theory Appl. (EJGTA), 1(2):125–147, 2013.

[2] D. M. Burton. Elementary number theory. W. C. Brown Publishers, Dubuque, IA, second edition, 1989.

[3] I. Chakrabarty, S. Ghosh, and M. K. Sen. Undirected power graphs of semigroups. Semigroup Forum, 78(3):410–426, 2009.

[4] M. Dehn. Papers on group theory and topology. Springer-Verlag, New York, 1987. Translated from the German and with introductions and an appendix by John Stillwell, With an appendix by Otto Schreier.

[5] A. V. Kelarev and S. J. Quinn. Directed graphs and combinatorial properties of semigroups. J. Algebra, 251(1):16–26, 2002.
[6] D. Witte and J. A. Gallian. A survey: Hamiltonian cycles in Cayley graphs. *Discrete Math.*, 51(3):293–304, 1984.