Locally finite knowledge structures

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December 8, 2021

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This research was supported by the Center for Rationality and Interactive Decision Theory (Jerusalem), the Department of Mathematics of the Hebrew University (Jerusalem), the Edmund Landau Center for Research in Mathematical Analysis (Jerusalem) sponsored by the Minerva Foundation (Germany), the German Science Foundation (Deutsche Forschungsgemeinschaft), and a London Mathematical Society Grant to support a visit of Benjy Weiss to London.
Abstract:

In a game of incomplete information, an infinite state space can create problems. When the space is uncountably large, the strategy spaces of the players may be unwieldy, resulting in a lack of measurable equilibria. When the knowledge of a player allows for an infinite number of possibilities, without conditions on the behavior of the other players, that player may be unable to evaluate and compare the payoff consequences of her actions. We argue that local finiteness is an important and desirable property, namely that at every point in the state space every player knows that only a finite number of points are possible. Local finiteness implies a kind of common knowledge of a countable number of points. Unfortunately its relationship to other forms of common knowledge is complex. In the context of the multi-agent propositional calculus, if the set of formulas held in common knowledge is generated by a finite set of formulas but a finite structure is not determined then there are uncountably many locally finite structures sharing this same set of formulas in common knowledge and likewise uncountably many with uncountable size. This differs radically from the infinite generation of formulas in common knowledge, and we show some examples of this. One corollary is that if there are infinitely many distinct points but a uniform bound on the number of points any player knows is possible then the set of formulas in common knowledge cannot be finitely generated.

Key words: Bayesian Games, Cantor Sets, Baire Category, Modal Logic, Common Knowledge
1 Introduction

1.1 The problem with infinity

We want to understand infinite games of incomplete information whose lack of finiteness lies not in the infinite repetition of stages nor in the number of actions of the players but rather in the infinite structure of the information. We assume that nature chooses a point in a state space and then the players are informed partially of nature’s choice, either through partitions of the state space or sigma algebras associated to the various players. If the state space is finite, we remain in the context of a finite game tree, for which Nash’s Theorem shows that there are mixed equilibria. The size of the game tree may grow exponentially in the size of the state space, nevertheless it remains finite and subject to the fixed point theorems underlying the Nash Theorem. With an appropriate topology, one can approximate infinite state spaces with finite state spaces. But how do the equilibria of the approximate finite state space games relate to the equilibria of the original game, if indeed there are any?

The space of all functions from the continuum to a set of only two elements is extremely wild, indeed it is equivalent to the set of all subsets of the continuum, a set of higher cardinality than the continuum. Since the integral of an arbitrary function from $[0, 1]$ to $[0, 1]$ is not well defined, even with one player and an arbitrary strategy we could have trouble defining a payoff. The measurability of the strategies is an essential issue. But once we require that the strategies of the players are measurable in some sense, the existence of an equilibrium in measurable strategies is called into question. For fixed point theorems to work for non-zero-sum games, usually we need that the strategy spaces are compact and convex and that there is continuity from the strategies to the resulting payoffs. By defining the topology weakly we can get compactness of the strategy spaces, however continuity to the payoffs may fail. By defining the topology strongly we can accomplish the continuity to the payoffs but the compactness of the strategy spaces may fail.

Roughly this is the background to the non-zero-sum example presented by this author (Simon 2003), for which equilibria do exist but none of which are measurable. This example demonstrates that approximating by finite state spaces teaches us very little about the original game. In this example
a strong locally finite property holds and local finiteness guarantees that some non-measurable equilibria do exist (Proposition 1, Simon 2003). This focuses attention onto the local finiteness property as an important condition sufficient for the existence of equilibria.

It should be noted however that with zero-sum games of incomplete information the situation is quite different. A weak form of continuity, separable continuity, combined with strategic compactness suffices for the existence of a value and optimal measurable strategies, proven by Mertens, Sorin and Zamir (1994) with the help of Sion’s Theorem (Sion 1958).

Next we describe our main results, followed by how they are related to games of incomplete information.

1.2 Logic and semantic models

Strictly speaking, our main results pertain to modal logic. This perspective introduces indirectly a more topological approach. Its basic structure is that of knowledge without any probability attached, however it can be related closely to probability theory and games of incomplete information, as we will see later.

Let \( X \) be a set of primitive propositions, and let \( J \) be a set of agents. Although it is legitimate to consider the case of either \( X \) or \( J \) infinite, for this paper we will assume throughout that both \( X \) and \( J \) are finite. Construct the set \( \mathcal{L}(X, J) \) of formulas using the sets \( X \) and \( J \) in the following way:
1) If \( x \in X \) then \( x \in \mathcal{L}(X, J) \),
2) If \( g \in \mathcal{L}(X, J) \) then \( \neg g \in \mathcal{L}(X, J) \),
3) If \( g, h \in \mathcal{L}(X, J) \) then \( g \land h \in \mathcal{L}(X, J) \),
4) If \( g \in \mathcal{L}(X, J) \) then \( k_j g \in \mathcal{L}(X, J) \) for every \( j \in J \),
5) Only formulas constructed through application of the above four rules are members of \( \mathcal{L}(X, J) \).

We write simply \( \mathcal{L} \) if there is no ambiguity. \( \neg f \) stands for the negation of \( f \), \( f \land g \) stands for both \( f \) and \( g \). \( f \lor g \) stands for either \( f \) or \( g \) (inclusive) and \( f \rightarrow g \) stands for \( \neg f \land g \).

The connection to games of incomplete information is that an \( x \in X \) could represent a fact about the game, for example the validity of a payoff matrix, the hand that a player holds in a game of cards, or the probability with which
a player believes something.

The relationship between multi-agent logic and games of incomplete information is mediated by semantic models called Kripke Structures. There are several ways to define Kripke structures, but for our purposes we present the partition model (otherwise known as that corresponding to the S5 logic, which will be explained later). A Kripke structure \( K = (S, X, J, (\mathcal{P}_j \mid j \in J), \psi) \) is defined by two sets \( S \) and \( X \), a set \( N \) of agents, a collection \( (\mathcal{P}_n \mid n \in N) \) of partitions of \( S \) and a function \( \psi : S \to \{0, 1\}^X \). The agent \( n \) can distinguish between two points in \( S \) if and only if they belong to different members of the partition \( \mathcal{P}_n \). The statement \( \psi(s)^x = 0 \) means that \( x \) is not true at \( s \) and \( \psi(s)^x = 1 \) means that \( x \) is true at \( s \).

We extend the function \( \psi \) by defining a map \( \alpha^K \) from \( L(X, J) \) to \( 2^S \), the subsets of \( S \), inductively on the structure of the formulas, the set \( \alpha^K(f) \) is where the formula \( f \) is valid:

**Case 1** \( f = x \in X \): \( \alpha^K(x) := \{ s \in S \mid \psi^x(s) = 1 \} \).

**Case 2** \( f = \neg g \): \( \alpha^K(f) := S \setminus \alpha^K(g) \),

**Case 3** \( f = g \land h \): \( \alpha^K(f) := \alpha^K(g) \cap \alpha^K(h) \),

**Case 4** \( f = k_j(g) \): \( \alpha^K(f) := \{ s \mid s \in P \in \mathcal{P}_j \Rightarrow P \subseteq \alpha^K(g) \} \).

A cell of a Kripke Structure is a member of the meet partition \( \wedge_{i=1}^n \mathcal{P}_i \), or equivalently a minimal set \( C \) such that for all \( j \) the properties \( P \in \mathcal{P}_j \) and \( P \cap C \neq \emptyset \) imply that \( P \subseteq C \). A member of \( \mathcal{P}_j \) for some \( j \) is called a possibility set. A cell has finite fanout if at every point in the cell and every agent \( j \) the possibility set of \( \mathcal{P}_j \) containing the point has only finitely many elements. Finite fanout is the logical equivalent of local finiteness. A formula \( f \) is held in common knowledge at a point \( x \) of a Kripke structure if \( f \) is true at \( x \) and \( k_{n_1}k_{n_2}\ldots k_{n_l}f \) is true at \( x \) for every choice of a finite sequence \( n_1, n_2, \ldots, n_l \) of agents. A set of formulas in common knowledge is finitely generated if the common knowledge of some finite subset implies the common knowledge of the whole set.

Given a Kripke structure, construct a sequence \( \mathcal{R}_0, \mathcal{R}_1, \ldots \) of partitions of \( S \) by \( \mathcal{R}_0 = \{ \psi^{-1}(a) \mid a \in \{0, 1\}^X \} \) and \( x \) and \( y \) belong to the same member of \( \mathcal{R}_i \) if and only if \( x \) and \( y \) belong to the same member of \( \mathcal{R}_{i-1} \) and for every person \( j \) the members \( P_x \) and \( P_y \) of \( \mathcal{P}_j \) containing \( x \) and \( y \) respectively intersect the same members of \( \mathcal{R}_{i-1} \). Let \( \mathcal{R}_\infty \) be the limit of the \( \mathcal{R}_i \), namely \( x \) and \( y \) belong to the same member of \( \mathcal{R}_\infty \) if and only if \( x \) and \( y \) belong to
the same member of $\mathcal{R}_i$ for every $i$.

For any set $X$ and set $J$ of persons there is a canonical Kripke structure $\Omega = \Omega(X, J)$ defined by the formulas in $\mathcal{L}(X, J)$ such that from any Kripke structure $\mathcal{K}$ (using the same $X$ and $n$) there is a canonical map to $\Omega$ defined by the map $\alpha^K$ with the property that $y$ and $z$ are mapped to the same point of $\Omega$ if and only if $y$ and $z$ share the same member of $\mathcal{R}_\infty$. Any Kripke structure with finite fanout will be mapped surjectively to a cell of $\Omega$ with finite fanout, so that if $\mathcal{R}_\infty$ separates all the points of a Kripke structure with finite fanout then the structure is isomorphic to a cell of $\Omega$.

Our main result is that for every finitely generated set of formulas that can be known in common either this set determines uniquely a finite cell of $\Omega$ or sharing this same set of formulas in common knowledge there are uncountable many cells of $\Omega$ with finite fanout and also uncountably many cells with infinitely many possibility sets of uncountable size. Furthermore, if there is a uniform bound on the size of the possibility sets and the cell is infinite then the formulas in common knowledge cannot be finitely generated. The situation is very different, however, for sets of formulas held in common knowledge that are not finitely generated – if there are uncountably many corresponding cells then either none of these cells or all of them could have finite fanout. And given that the set of formulas held in common knowledge is not finitely generated and finite fanout doesn’t hold there may be a large difference between the structure of the cells of $\Omega$ and the cells of some other Kripke structure holding the same set of formulas in common knowledge.

Central to understanding our results is point-set topology. For every Kripke Structure $\mathcal{K} = (S; X; N; (P^n \mid n \in N); \psi)$ we define a topology on the set $S$, the same as in Samet (1990). Let $\{\alpha^K(f) \mid f \in \mathcal{L}\}$ be the base of open sets of $S$. We call this the topology induced by the formulas. The topology of a subset $A$ of $S$ will be the relative topology for which the open sets of $A$ are $\{A \cap O \mid O \text{ is an open set of } S\}$.

To some extent this paper follows results from two previous papers of this author. In Simon (1999) we showed that for every set of formulas that can be held in common knowledge either there were uncountably many cells of $\Omega$ corresponding to those formulas or there was only one. The proof used Baire Category in the following way: it was shown that a cell shares its formulas in common knowledge with no other cell if and only if there is some point in
the cell and a positive integer \( n \) such that the set of points that are reachable in \( n \) steps from this point (using alternating possibility sets) is an open set relative to the closure of the cell (meaning that it is not meagre in its closure). A cell that shares its formulas in common knowledge with no other cell is defined to be centered. In Simon (2001) we showed that if the set of formulas held in common knowledge is finitely generated then the maximality of this set of formulas is equivalent to the finiteness of the corresponding cell and is equivalent to its being centered.

Given finite generation of the formulas in common knowledge, constructing uncountably many cells with finite fanout is much more difficult than constructing uncountably many cells of uncountable size. Even constructing just one cell of finite fanout is difficult. To construct a cell of finite fanout dense in \( \Omega \) (meaning only the tautologies in common knowledge), we spliced together an infinite sequence of finite Kripke structures (Simon 1999). For each non-tautological formula that could be true was associated one of these models and a point in that model where this formula was true. The added connections to the other Kripke Structure were distant enough from this point so that after the splicing each such formula was still true at its corresponding point. To copy the same approach to construct just one cell of finite fanout with a fixed formula \( f \) in common knowledge one must keep this formula \( f \) true at all (rather than at just some) points after the new connections are made.

There is an additional interest from logic to our main results. Finite fanout characterizes the uniqueness of the extension of \( \Omega(X, N) \) to the canonical Kripke structures using ordinals beyond the first infinite ordinal \( \omega \) (Fagin 1994). Our main results show that this property of unique extension has little to do with the set of formulas held in common knowledge.

1.3 Local and global models

The example of Simon (2003) reveals that there are at least two levels on which a game of incomplete information is played, calling for two models of the game, a global model and a local model. It is with the local model that local finiteness (equivalently finite fanout) is important.

The global model is that of measurability. There is a sigma algebra \( \mathcal{F} \) on
the state space $\Omega$ for which a probability measure $\mu$ is defined. For every player $i$ there is a corresponding set $A^i$ of actions (either finite or infinite) and for every choice of $a \in A := \prod_{i \in N} A^i$ and every player $i \in N$ there is an $\mathcal{F}$ measurable function $f^i(a) : \Omega \to \mathbb{R}$ that represents the payoff to Player $i$ if the players choose the actions $a$ at the point $x \in \Omega$. Corresponding to each player $i$ is a sigma algebra $\mathcal{F}_i$ that is smaller than or equal to $\mathcal{F}$ (as a collection of subsets of $\Omega$) and represents the private information of Player $i$. For each player $i$ there is a probability distribution $\mu_i$ defined on $\mathcal{F}_i$ that is smaller than or equal to $\mu_i$ and represents the private information of Player $i$.

Given that $A$ is finite, for any strategy profile $\sigma = (\sigma^i \mid i \in N) \in \mathcal{S}$ and every player $i \in N$ a corresponding payoff $V^i(\sigma)$ is defined as the integration according to $\mu_i$ of $f^i$ over $\Omega$ and the strategy profile $\sigma$. The strategy profile $\sigma$ is a Harsanyi equilibrium if for every alternative strategy profile $\hat{\sigma}$ and $i \in N$ $V^i(\sigma | \hat{\sigma}^i) \leq V^i(\sigma)$ where $(\sigma | \hat{\sigma}^i)$ is the strategy profile that is determined by $\sigma$ for all players other than $i$ but determined by $\hat{\sigma}^i$ for the player $i$.

The local model starts from different assumptions, though both models may have a synthesis. For every player $i \in N$ there is a set $T_i$ representing the types of the player $i$. Let $T := \prod_{i \in N} T_i$ and for every $t^i \in T_i$ define the cross section $C_{t^i}$ to be $\{t^i\} \times \prod_{j \neq i} T_j$. For every $t^i \in T_i$ there is a sigma algebra $\mathcal{F}_{t^i}$ of subsets of $C_{t^i}$ and a probability distribution $\mu_{t^i}$ on $C_{t^i}$ that is defined on $\mathcal{F}_{t^i}$. As before $A^i$ stands for the actions of player $i$ (perhaps dependent on the value of $t^i$), and for every choice of $a \in A := \prod_{i \in N} A^i$ and every player $i \in N$ there is a payoff function $f^i(a) : T \to \mathbb{R}$ such that its restriction to any $C_{t^i}$ is $\mathcal{F}_{t^i}$ measurable. $f^i(a)$ represents what Player $i$ receives at any $x \in T$ if the players choose the actions $a$. A strategy for a player $i \in N$ is any function $\sigma^i$ from $T_i$ to the probability distributions on its actions, but there may be no measurability requirements. A strategy profile $\sigma$ is a choice of a strategy for each player $i$.

In general, a strategy profile may not define expected payoffs for some fixed $t^i \in T_i$, as there may be problems of measurability. However if the actions are
finite in number and for any fixed \( t^i \in T_i \) the strategies of the other players when restricted to \( C_{t^i} \) are measurable with respect to \( \mathcal{F}_{t^i} \) then one can define for each of her actions an expected payoff for Player \( i \) at the point \( t^i \in T_i \). A strategy profile is a Bayesian equilibrium for a point \( t = (t^i \mid i \in N) \in T \) if for all \( i \in N \) the expected payoffs corresponding to the strategies of player \( i \) can be evaluated in the corresponding cross section \( C_{t^i} \) and Player \( i \) cannot obtain a higher payoff according to that evaluation by choosing a different strategy. Notice that if the state space has the locally finite property and there are finitely many actions then such local evaluations are not problematic even if the strategies used by the other players are wild. As stated above, through approximation by finite games local finiteness does imply the existence of a Bayesian equilibrium (Proposition 1 of Simon 2003).

In some ways the global model is the stronger one. Under natural conditions (such as a Polish state space) a global model induces a local model through regular conditional probability distributions (conditional probabilities for all the Borel sets done in a consistent manner, see Breiman 1992). Likewise under natural conditions an equilibrium of the global model will induce an equilibrium of the local model almost everywhere (see for example Proposition 2 of Simon 2003, whose proof idea was explained to me by J.-F. Mertens). However a lack of a global Harsanyi equilibrium in the presence of local Bayesian equilibria shows that the local model has its advantages, especially when combined with local finiteness.

**Question 1:** Let the player set \( N \) be finite, let there be finitely many stages and on each stage each player has finitely many actions. Assume that the state space \( T = \prod_{i \in N} T_i \) is compact and Polish with a Borel probability distribution \( \mu \). Assume that there is a uniform bound on the payoffs and for every fixed collection of actions the payoffs to each player is a Borel measurable function on \( T \). Is there such a game that lacks a Harsanyi equilibrium and for every choice of regular conditional probability distributions for the players there are no strategies that define a Bayesian equilibrium almost everywhere?

### 1.4 Ergodic Games

It is worthwhile to look briefly at the structure behind the Simon (2003) example. This example belongs to the category of ergodic games, which will
be defined below.

The Bernoulli space can be represented as the set \( \{a, b\}^\mathbb{Z} \), where \( \mathbb{Z} \) stands for the integers (including the negative integers) and \( a \) and \( b \) are distinct symbols. A point of \( \{a, b\}^\mathbb{Z} \) is a doubly infinite sequence \( x = (\ldots, x^{-1}, x^0, x^1, \ldots) \) where for every \( i \in \mathbb{Z} \) \( x^i \) is either \( a \) or \( b \). The probability distribution on the state space is the canonical one that gives every choice of \( a \) or \( b \) at any position equal weight and independently of the choices for \( a \) or \( b \) at the other positions. The function \( T : \{a, b\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z} \), called the shift operator, is defined by \( T(x)^i = x^{i-1} \).

In the Simon (2003) example there are two sets of payoff matrices corresponding to \( a \) and \( b \) and the 0-coordinate of a point in \( \Omega \) determines the payoff matrices, so that if \( y \in \Omega \) and \( y^0 = a \) then payoffs at the point \( y \) are determined by the matrices corresponding to \( a \). There are three players, Players One, Two, and Three. Let \( \sigma : \Omega \to \Omega \) be the measure preserving involution defined by \( (\sigma(y))^i := y^{-i} \), where \( y^i \) is the \( i \)th coordinate of \( y \in \Omega \). \( \sigma \) is the reflection of the sequence about the position zero. Let \( \tau : \Omega \to \Omega \) be the measure preserving involution defined by \( (\tau(y))^i := y^{1-i} \). It follows that \( T := \tau \circ \sigma \) is the usual Bernoulli shift operator \( (T(y))^i = y^{i-1} \). Define the beliefs \( t^1 \), \( t^2 \) and \( t^3 \) of the players according to \( \sigma \) for Player One and \( \tau \) for the other players; this means that at any point \( y \in \Omega \) Player One considers only \( y \) and \( \sigma(y) \) to be possible, and with equal probability; (if \( \sigma(y) = y \) then Player One believes in \( y \) with full probability). At any \( y \), both Player Two and Player Three believe that only \( y \) and \( \tau(y) \) are possible, and with equal probability. Player Two and Three have the same beliefs. Player One always knows the payoff but not always what the other players might know.

The secret to this example is that the shift operator \( T \) is an ergodic operator that acts almost everywhere upon any equilibrium of the game in a way so that this equilibrium cannot be measurable.

The game example in Simon (2003) is also an ergodic game. An ergodic game is a game satisfying the following properties:

1. there is one stage of play with moves chosen simultaneously,
2. there are finitely many players,
3. each player has finitely many moves,
there is a compact Polish space $\Omega$ with an atomless Borel probability distribution $\mu$ representing a choice by nature,

for every combination of moves, one for each player, the payoff to each player is a continuous function from $\Omega$ to $\mathbb{R}$,

at every point $x \in \Omega$ every player $j$ has a discrete probability distribution on $\Omega$, called his belief, with a finite support set $S^j(x)$ containing $x$ such that at all the other points in this finite support set $S^j(x)$ the player $j$ has the same discrete distribution,

discrete beliefs of the players change continuously (with respect to the weak$^*$ topology), and for any player $j$ it forms a regular conditional probability of $\mu$ with respect to the sigma algebra $\mathcal{F}^j := \{B \mid B$ is Borel and $x \in B \iff S^j(x) \subseteq B\}$,

If $B$ is a Borel set with $B \in \mathcal{F}^j$ for all players $j$ then $\mu(B)$ is equal to either zero or one.

By Property 6 all ergodic games have the local finiteness property.

A measure preserving transformation $T$ on a probability space with a Borel probability distribution $\mu$ is called ergodic if the only Borel sets $B$ with $T^{-1}(B) = B$ satisfy $\mu(B) = 0$ or $\mu(B) = 1$. Why is the term ergodic used to describe these games? The ergodic aspect of these games is contained in Property 8. It guarantees that the game cannot not decompose into two different subgames of positive probability.

There is an easy way to create an ergodic game with $n$ players. Let $\Omega$ be a compact Polish probability space with a Borel probability measure $\mu$. A function $f : S \rightarrow S$ is called an involution if $f$ is not the identity function but $f^2$ is the identity function. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be a set of $n$ continuous measure preserving involutions such that some combination of the $\sigma_i$'s is an ergodic transformation. For each player $i$ define $\mathcal{F}_i$ to be $\{\{B \mid B \text{ Borel}, x \in \sigma_i(x) \in B\}$ and $T_i$ to be the equivalence classes defined by $x \sim_i y \text{ if and only if } x = y \text{ or } y = \sigma_i(x)$. For every player $i$ define the belief function $t_i : \Omega \rightarrow \Delta(\Omega)$ by

$t(x)(B) =
0 \text{ if both } x \text{ and } \sigma_i(x) \text{ are not in } B,$

$1 \text{ if both } x \text{ and } \sigma_i(x) \text{ are in } B,$

$1/2 \text{ otherwise.}$
$t_i$ is continuous with respect to the weak$^*$ topology and the function $t : \Omega \to \Delta(\Omega)$ is a regular conditional probability induced by $\mu$ and $\mathcal{F}_i$ (Lemma 0 of Simon 2003). The rest of the properties of an ergodic game can be constructed easily. Indeed the same can be done for any collection of finite groups $(G_i \mid i \in N)$ that act measure preserving on a Polish probability space. In this case at any point $x$ the player $i$ would consider the points of the orbit $G_ix$ to be possible with each point $y = gx$ weighted according to the number of group elements $g \in G_i$ satisfying $gx = y$.

### 1.5 Absolute Continuity

There is a condition on a game of incomplete information which, along with natural assumptions on the payoffs, guarantees the existence of measurable equilibria (as demonstrated by Milgrom and Weber 1985), *absolute continuity with respect to the marginals*. With a weak topology on the strategies, it implies continuity from a compact strategy space to the payoffs. It is worthwhile to see how this condition may fail, especially when local finiteness holds.

If $S = A \times B$ and $\mu$ is a probability distribution on $S$ then the marginal probability distribution on $A$ gives $\mu(C \times B)$ to any measurable subset $C$ of $A$ (and likewise define the marginal probability distribution on $B$). A measure $\mu$ is *absolutely continuous* with respect to another measure $m$ if and only if $m(A) = 0$ implies that $\mu(A) = 0$. One way for a measure $\mu$ to be absolutely continuous with respect to another measure $m$ is for $\mu(B) = \int f(x)1_Bd\mu$ for some non-negative and integrable function $f$. The Radon-Nikodym Theorem states that in many common situations the converse holds, namely that absolute continuity implies the existence of such a function.

A game of incomplete information with a probability measure $\mu$ on a state space $\Omega = \prod_{j \in N}T_j$ (a synthesis of the global and local models) has the *absolutely continuous* property if $\mu$ is absolutely continuous with respect to the product of the marginals induced by $\mu$ on each of the $T_i$.

Notice that no ergodic game can have the absolute continuity property, indeed neither can any game with the local finiteness property where the Radon-Nikodym and Fubini theorems hold and every singleton set has zero measure. It suffices to prove this for two players, as the proof for more play-
ers introduces no new ideas. For each player $i \in \{1, 2\}$ let $\mu_i$ be the marginal distribution of $\mu$ on $T_i$. As any single point of $\Omega$ is given zero probability by $\mu$ and every cross section of $\Omega$ defined by a point of $T_i$ contains only finitely many points it must hold that every single point of $T_i$ is given zero probability by $\mu_i$. By Fubini’s Theorem we can re-write $\mu(\Omega) = \int h(x_1, x_2) \, d(\mu_1 \times \mu_2)$ as $\int(\int h(x_1, x_2) \, d\mu_1(x_1)) \, d\mu_2(x_2)$, where $h$ is the function implied by the Radon-Nikodym Theorem. But for every fixed $x_2 \in T_2$ there are only finitely many points $x_1$ in $T_1$ with $(x_1, x_2) \in \Omega$. Because $\mu_1$ gives zero probability to every single point it must follow that $\int h(x_1, x_2) \, d\mu_1(x_1)$ is zero for every fixed choice of $x_2$, hence $\mu(\Omega) = 0$, a contradiction.

Any game where two distinct players always share identical information and the marginals are atomless will also fail the absolute continuity property, because any set that projects canonically onto the diagonal of the product of these two players’ type spaces will be a set of measure zero in the product topology.

Consider also the following information structure on which a game can be defined for which local finiteness holds but the absolute continuity property fails. Let $\alpha$ be an irrational real number with $0 < \alpha < 1$ and let $C$ be the subset of the square $[0, 1] \times [0, 1]$ defined by $C = \{(x, x)\} \cup \{(x, y) \mid y = x + \alpha \text{ or } y = x + \alpha - 1\}$. Let $\mu$ be the probability measure on $C$ defined by

1. $\mu(\{(x, x) \mid x \in [a, b]\}) = \frac{1}{2}(b - a)$,
2. $\mu(\{(x, y) \mid y = x + \alpha \text{ or } y = x + \alpha - 1, x \in [a, b]\}) = \frac{1}{2}(b - a)$.

To make this example into a two-player game one can define payoff matrices that change continuously according to the location on $C$. Due to the special locations where the cross sections will be three points instead of the usual two, strictly speaking such a game will not be ergodic. However by identifying the values of 0 and 1 for both players an ergodic game can be constructed. It would be interesting to discover whether a non-zero-sum game can be so constructed where no measurable equilibria exists.

1.6 Finite additivity

One could think that the countably additive axiom of the conventional definition of a probability distribution is to blame for the discrepancy between local and global equilibria, in particular the possibility of Bayesian local equilibria.
where there is no global Harsanyi equilibria. (It should be noticed, however, that the proof that a global Harsanyi equilibrium induces a local Bayesian equilibrium almost everywhere does use countable additivity.) Indeed on the Bernoulli shift space if one required only finite additively there will be many finitely additive shift-invariant measures that are defined on all subsets. This is due to the fact that the shift transformation defines an amenable group action. However with $G_i$ subgroups defining the beliefs of the players (as described above) generating a non-amenable group action there will be state spaces for which there are no finitely additive measures defined on all the subsets that are also invariant with regard to this group. If this groups acts on any measurable equilibria in an appropriate way it may be possible to demonstrate the non-existence of finitely additive global Harsanyi equilibria, though there will be many local Bayesian equilibria.

In this context it may be relevant to review the related Banach-Tarsky Paradox from 1924, which speaks directly to the non-existence of finitely additive measures defined on all subsets that are rotation invariant. The paradox states that there is a way to partition the sphere $S^2$ into finitely many parts $A_1, \ldots, A_k, B_1, \ldots, B_l$ such that after rotating these parts two copies of $S^2$ are created, one sphere from the $A_1, \ldots, A_k$ and another sphere from the $B_1, \ldots, B_l$. The group of rotations of $S^2$ is a non-amenable group. To review the relations between amenability, the paradox, and other issues, see Wagon (1985).

With inspiration from the Banach-Tarsky paradox and its relation to ergodic games through amenability, we pose the following related open questions.

**Question 2:** Does there exist an ergodic game that has no equilibria measurable with respect to any finitely additive probability measure of the state space?

The analogy between Question 2 and the Banach-Tarsky paradox is the following. Assume the information structure of the ergodic game is generated through finite groups representing the beliefs of the players as described above. The subsets $A_1, \ldots, A_k, B_1, \ldots, B_l$, dependent on the strategies used, could cover the state space and represent the locations where certain subsets of strategies are used by the players. One would like to show that in any Bayesian equilibrium there will be group elements acting on these sets in a way similar to the rotations in the Banach-Tarsky paradox, demonstrat-
ing that the equilibrium cannot be measurable with respect to any finitely additive measure. There is a parallel in the main proof of Simon (2003), where it is shown that if some player alternates her behavior throughout the state space between two pure strategies and the measurable subsets $A, B$ represent the locations where these two pure strategies are used then the ergodicity of the square $T^2$ of the shift operator $T$ and the property $T(A) = B$ and $T(B) = A$ almost everywhere implies that either both $A$ and $B$ are of measure zero or both $A$ and $B$ are of measure one, both contradictions.

A question related indirectly to Question 2 is the following:

**Question 3:** Does there exist an ergodic game and some positive $\epsilon > 0$ such that the game does not have a global Harsanyi $\epsilon$-equilibrium in Borel measurable strategies?

The example in Simon (2003) relies heavily on the non-linear aspects of how the payoff can change when one player fixes her action and the other two players vary their mixed strategies. This inspires the following question, which can be amended to refer to finite additivity and measurable $\epsilon$-equilibria:

**Question 4:** Is there a two-person non-zero-sum ergodic game that has no global Borel measurable equilibrium?

One could think that local finiteness is not appropriate for the context of equilibrium existence, rather the locally countable property, namely that every player at every point knows that one of countably many points are possible. Indeed a countably additive measure on a countable set will be determined by the weights given to all the individual points, and indeed the proof of local Bayesian equilibria in Simon (2003) is extended to such a context. But there are limitation to local countability that are not present in local finiteness. For finitely additive measures there is no such unique determination, as there can be many distinct finitely additive measures on the same countable set that assign zero to all singleton sets. Furthermore the infinity of any set introduces topological questions that do not arise with finite sets, namely which sequences of points converge, and if so will this convergence point be a member of that same set. This topological complexity manifests itself further when we consider the logical aspects.
1.7 Meagre and null sets

Although measure theory and Baire Category are distinct approaches, they have many parallels. In most contexts a meagre or no-where dense set will have zero measure, so that a first Baire Category set, the countable union of meagre sets, will also have zero measure.

More parallels can be found when comparing Kripke structures to state spaces of games of incomplete information. If \( C \) is a cell of \( \Omega(X, N) \) then the formulas in common knowledge in \( C \) define the closure of the cell, parallel to the definition of the support of a probability measure on a compact and separable probability space as the smallest closed set of probability one. Property 8 of ergodic games, that a set known in common by the players must be either measure zero or measure one, finds its parallel in the Baire Category argument underlying the centered vs uncentered distinction of the cells of \( \Omega(X, N) \).

It should be noted that it is easy to construct a probability distribution for a Kripke structure, forming the basis of a game of incomplete information. Consistent determinations of probabilities for all the formulas will induce a Borel probability distribution by way of the Kolmogorov extension theorem (where Borel refers to the topology defined by the formulas). Such consistent determinations start with the probabilities for the validity of each of the \( x \in X \) and then the probabilities that the players know or don’t know any of the \( x \in X \) to be valid. Because this distribution will be unique on the Borel sets, given that there is Hausdorff separation there will be no need to consider anything but the Kripke structure’s canonical map into \( \Omega \), unless one wishes to extend this measure beyond the (null set) completion of the Borel sets relative to this probability measure. (Because the base of open sets are defined by the formulas and they define both open and closed sets, any topological separation of points will be Hausdorff.) Given that the topology induced by the formulas is not Hausdorff (meaning that the map into \( \Omega \) is not injective), one could accomplish separation by introducing more elements to the set \( X \). In this context it may noticed that \( X \) can be extended to a countable set without altering the Baire Category structure to the cells (Simon 1999).

It is plausible, with a probability distribution constructed for a Kripke structure and the main results of this paper, that one could answer Question
by demonstrating a game of incomplete information where there are no measurable Harsanyi equilibria and Bayesian equilibria exist for only an unmeasurable subset or a subset of measure strictly less than one.

Going the other way, from either a local or global model for a game of incomplete information to a corresponding Kripke structure, it is easy as long as there are well defined supports for the local beliefs of the players (defined directly or indirectly by the model).

For example, consider the Bernoulli shift space \( \{a, b\}^\mathbb{Z} \) where \( X = \{a, b\} \) and \( a \) is true at \( \omega \in \{a, b\}^\mathbb{Z} \) if and only if \( \omega^0 = a \) (and otherwise \( b \) is true), and let the knowledge of Players One and Two be that as defined above by the involutions in the example of Simon (2003), and for now we drop the third player who has identical information to the second player. Because the shift operator is related to the operations \( k_1 \) and \( k_2 \) on the formulas, one can show easily that the Bernoulli shift space as a Kripke structure will map isomorphically to a compact subset of \( \Omega(X, \{1, 2\}) \) With probability one, any point will be in a dense cell and due to local finiteness there must be uncountably many dense cells. Because of some exceptional points in the shift space, such as \( (\ldots, a, a, a, \ldots) \), there will be some cells that are not dense. However by adding a new third player associated with the involution \( \pi \) defined by \( \pi(z)_i^i = z^i \) if \( i \neq 0 \) and otherwise \( \pi \) switches the \( a \) with the \( b \) and vice-versa on the 0-position, one gets a compact subset of \( \Omega(X, \{1, 2, 3\}) \) that is comprised entirely of uncountably many dense cells.

The rest of the paper is organized as follows. Section 2 provides background information. Sections 3 and 4 contain the proofs of the two main claims, Theorem 1 and Theorem 2. The last section discusses the lack of finite generation in more detail.

2 Background

2.1 Formulas and logic

Recall the above definition of the formulas \( \mathcal{L} \). There is a very elementary logic defined on the formulas in \( \mathcal{L} \) called \( S5 \). For a longer discussion of the \( S5 \) logic, see Cresswell and Hughes (1968); and for the multi-person variation, see Halpern and Moses (1992) and also Bacharach, et al, (1997). Briefly, the
$S5$ logic is defined by two rules of inference, modus ponens and necessitation, and five types of axioms. Modus ponens means that if $f$ is a theorem and $f \rightarrow g$ is a theorem, then $g$ is also a theorem. Necessitation means that if $f$ is a theorem then $k_j f$ is also a theorem for all $j \in J$. The axioms are the following, for every $f, g \in \mathcal{L}(X, J)$ and $j \in J$:

1) all formulas resulting from theorems of the propositional calculus through substitution,
2) $(k_j f \land k_j (f \rightarrow g)) \rightarrow k_j g$,
3) $k_j f \rightarrow f$,
4) $k_j f \rightarrow k_j(k_j f)$,
5) $\neg k_j f \rightarrow k_j(\neg k_j f)$.

A set of formulas $\mathcal{A} \subseteq \mathcal{L}(X, J)$ is called complete if for every formula $f \in \mathcal{L}(X, J)$ either $f \in \mathcal{A}$ or $\neg f \in \mathcal{A}$. A set of formulas is called consistent if no finite subset of this set leads to a logical contradiction, meaning a deduction of $f$ and $\neg f$ for some formula $f$. We define

$$\Omega(X, J) := \{S \subseteq \mathcal{L}(X, J) \mid S \text{ is complete and consistent}\}.$$ 

A tautology of $\mathcal{L}(X, J)$ is a formula that is true at every point of $\Omega(X, J)$, or in other words a formula $f$ is a tautology if and only if for every $z \in \Omega(X, J)$ the formula $f$ is in $z$.

The $\Omega(X, J)$ is itself a Kripke Structure $(\Omega(X, J); X; J; (Q^j(X, J) \mid j \in J); \overline{\psi}(X, J))$ with for every $j \in J$ the partition $Q^j(X, J)$ being that generated by the inverse images of the function $\beta^j : \Omega(X, J) \rightarrow 2^\mathcal{L}(X, J)$ defined by

$$\beta^j(z) := \{f \in \mathcal{L}(X, J) \mid k_j f \in z\},$$

the set of formulas known by person $j$ and $\overline{\psi}$ defined by $\overline{\psi}(z) = 1$ if and only if $x \in z$. Due to the fifth set of axioms $\beta^j(z) \subseteq \beta^j(z')$ implies that $\beta^j(z) = \beta^j(z')$. We will write $\Omega$, $\mathcal{L}$, $\overline{\psi}$ and $Q^j$ if there is no ambiguity.

### 2.2 Common knowledge

The expression $E(f) = E^1(f)$ is defined to be $\land_{j \in J} k_j f$, $E^0(f) := f$, and for $i \geq 1$, $E^i(f) := E(E^{i-1}(f))$. A formula $f \in \mathcal{L}(X, J)$ is common knowledge in a subset of formulas $A \subseteq \mathcal{L}(X, J)$ if $E^n f \in A$ for every $n < \infty$. 

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The set of formulas that are held in common knowledge is constant within any cell and equal to all the formulas true everywhere in that cell (Halpern and Moses 1992). In other words:

**Lemma A:** For any cell \( C \) of \( \Omega(X, J) \) \( \{ f \in \mathcal{L}(X, J) \mid f \) is common knowledge in \( z \) for some \( z \in C \} = \{ f \in \mathcal{L}(X, J) \mid f \) is common knowledge in \( z \) for all \( z \in C \} = \{ f \in \mathcal{L}(X, J) \mid f \in z \) for all \( z \in C \}.

Due to Lemma A, we have a map \( F \) from the set \( Q = \wedge_{j \in J} Q^j \) of cells to subsets of formulas defined by \( F(C) := \{ f \mid f \) is common knowledge in any (equivalently all) members of \( C \} \). For any subset of formulas \( T \subseteq \mathcal{L} \) define \( C_k(T) := \{ f \in \mathcal{L} \mid \) there exists an \( i < \infty \) and a finite set \( T' \subseteq T \) with \( (\wedge_{i \in T'} E^i(t)) \rightarrow f \) a tautology \( \} \). We define \( T(X, J) = \{ C_k(T) \mid T \subseteq \mathcal{L}(X, J)\} \cup \{ \mathcal{L}(X, J) \} \), and we say that \( T \) generates \( C_k(T) \). If there is no ambiguity, we can write simply \( T \). \( C_k(T) \) is the set of formulas whose common knowledge is implied by the common knowledge of the formulas in \( T \).

An \( S \in T \) is finitely generated if there exists a finite subset \( T \subseteq S \) such that \( C_k(T) = S \). For every set of formulas \( T \subseteq \mathcal{L} \) define the set

\[
C_k(T) := \{ z \in \Omega \mid \) every member of \( T \) is common knowledge in \( z \}\}.
\]

For any \( T \subseteq \mathcal{L} \), \( C_k(T) \) is a closed set, since the \( C_k(T) \) is the intersection of the sets \( \alpha(E^i f) \) for all \( l < \infty \) and all formulas \( f \) in \( T \).

A cell \( C \) is defined to be centered if and only if there is no other cell \( C' \) with \( F(C') = F(C) \).

In Simon (1999) we proved that if a cell \( C \) is not centered then there are uncountably many other cells \( C' \) such that \( F(C') = F(C) \). Since \( C_k(T) = F(C) \) means that the cell \( C \) is dense in \( C_k(T) \); the cell \( C \) being not centered is equivalent to the existence of uncountably many other cells \( C' \) that are also dense in \( C_k(F(C)) = C_k(T) \).

In this paper we prove that if \( C \) is a cell and \( F(C) \) is finitely generated then either \( C \) is finite and there is no other cell \( C' \) with \( F(C) = F(C') \) or there is a continuum of distinct cells \( C' \) with continuum cardinality such that \( F(C') = C \) and there is a continuum of distinct cells \( C' \) of finite fanout such that \( F(C') = C \).
2.3 Kripke Structures

If $\mathcal{K} = (S; X; J; (P^j \mid j \in J); \psi)$ is a Kripke structure we define a map $\phi^\mathcal{K}: S \rightarrow \Omega(X, J)$ by

$$\phi^\mathcal{K}(s) := \{ f \in \mathcal{L}(X, J) \mid s \in \alpha^\mathcal{K}(f) \},$$

where $\alpha^\mathcal{K}$ is the map defined above. This is the canonical map, also contained in Fagin, Halpern, and Vardi (1991).

Theorem: For every $f \in \mathcal{L}(X, J)$, $f$ is a theorem of the multi-agent S5 logic if and only if $f$ is a tautology. Furthermore, $\phi^\Omega(z) = z$ for every $z \in \Omega$.

For a proof of the first part of this theorem, see Halpern and Moses (1992) and Cresswell and Hughes (1968), and for how the second part follows from the first part see Aumann (1999). We will call this result the “Completeness Theorem.”

For a Kripke Structure $\mathcal{K} = (S; X; J; (P^j \mid j \in J); \psi)$, if $s \in \alpha^\mathcal{K}(f)$, or equivalently $f \in \phi^\mathcal{K}(s)$, we say that $f$ is true at $s$ with respect to $\mathcal{K}$. We say that $f$ is valid with respect to the Kripke Structure $\mathcal{K}$ if $f$ is true at $s$ with respect to $\mathcal{K}$ for every $s \in S$. The Kripke Structure is connected if the meet partition $\wedge j \in J P^j$ is a singleton (equal to $\{S\}$). We define a connected component of a Kripke Structure to be a member of this meet partition. Two points $s, s' \in S$ are adjacent if they share some member of $P^j$ for some $j \in J$. We define the adjacency distance between any two points $s$ and $s'$ in $S$ as $\min \{d \mid \text{there is a sequence } s = s_0, \ldots, s_d = s' \text{, a function } a : \{1, \ldots, d\} \rightarrow J \text{ and sequence of sets } D_1 \in P^{a(1)}, \ldots, D_d \in P^{a(d)} \text{ such that } \text{for all } 1 \leq i \leq d \text{, } s_i \text{ and } s_{i-1} \text{ both belong to } D_i \}$, with zero distance between any point and itself and infinite distance if there is no such sequence from $s$ to $s'$. Such a sequence we call an adjacency path.

Given a Kripke Structure $\mathcal{K} = (S; X; J; (P^j \mid j \in J); \psi)$ and a subset $A \subseteq S$, we define another Kripke Structure $\mathcal{V}^\mathcal{K}(A) := (A; X; J; (P^j|_A \mid j \in J)); (\psi|_A)$ where for all $j \in J P^j|_A := \{ F \cap A \mid F \cap A \neq \emptyset \text{ and } F \in P^j \}$ and for all $x \in X$ and $a \in A \psi^x|_A(a) = \psi^x(a)$. If there is no ambiguity concerning the initial model $\mathcal{K}$, we can replace $\mathcal{V}^\mathcal{K}(A)$ by $\mathcal{V}(A)$.

Now we can see why a Kripke Structure with the finite fanout property is essentially a cell with finite fanout. It is easy to prove that for every set
corresponding to an agent \( j \in J \) in a Kripke Structure \( K \) that \( \phi^K(P) \) is a dense subset of some member of \( Q^j \) (Lemma 5, Simon 1999). Fagin (1994) proved that a cell has a unique extension to all canonical Kripke Structure corresponding to the transfinite ordinal numbers beyond the first infinite ordinal if and only if it is of finite fanout, and that representation in all these canonical Kripke Structure characterizes the interactive knowledge of the agents. Combining these results, restricting ourselves to \( \Omega \) is sufficient for understanding Kripke Structures with finite fanout.

### 2.4 Canonical Finite Models

Behind our main results is a hierarchical construction of \( \Omega \). Every formula has a “depth”, an inductively defined natural number representing the extent to which the knowledge operators \( k_j \) of the agents \( j \in J \) have been used to define the formula. (Formulas of depth zero are those of the conventional propositional calculus, constructed without the knowledge operators of any agents.) For every natural number \( i \) there is a finite Kripke Structure \( \Omega_i \) that represents the knowledge of the agents up to the depth of \( i \). Furthermore \( \Omega \) is the inverse limit of the \( \Omega_i \), meaning that a point in \( \Omega \) is defined by a sequence of extensions from \( \Omega_i \) to \( \Omega_{i+1} \) for all the \( i \). If the set of formulas held in common knowledge is finitely generated, then these formulas have a maximal depth \( d \). By exploiting the choices in how one could extend a point in \( \Omega_i \) to \( \Omega_{i+1} \) for some of the \( i \) that are greater than \( d \), one can construct cells in the limit of the process that do or do not have finite fanout. If the corresponding set of formulas held in common knowledge is not finitely generated, the lack of a maximal depth for a generating subset renders the hierarchical construction meaningless for our purposes.

We define the **depth** of a formula inductively on the structure of the formulas. If \( x \in X \), then \( \text{depth} (x) := 0 \). If \( f = \neg g \) then \( \text{depth} (f) := \text{depth}(g) \); if \( f = g \land h \) then \( \text{depth}(f) := \max (\text{depth}(g), \text{depth}(h)) \); and if \( f = k_j(g) \) then \( \text{depth}(f) := \text{depth}(g) + 1 \).

For every \( 0 \leq i < \infty \) we define \( \mathcal{L}_i := \{ f \in \mathcal{L} \mid \text{depth}(f) \leq i \} \) and define \( \Omega_i \) to be the set of maximally consistent subsets of \( \mathcal{L}_i \). If there may be ambiguity, we will write \( \Omega_i(X,J) \). We must perceive an \( \Omega_i \) in two ways, as a Kripke Structure in its own right and as a canonical projective image of \( \Omega \) inducing a partition of \( \Omega \) through inverse images. We define \( \pi_i : \Omega \rightarrow \Omega_i \) to be the
canonical projection $\pi_i(z) := z \cap L_i$. Due to an application of Lindenbaum’s Lemma, the maps $\pi_i$ are surjective. For all $i \geq k$ define the map $\pi_k$ to be $\pi_k \circ \pi_i^{-1}$. For any Kripke Structure $K = (S; X; J; (\mathcal{P}^j \mid j \in J); \psi)$ and $i \geq 0$ we define $\phi^K_i : S \rightarrow \Omega_i(X, J)$ by $\phi^K_i(s) := \phi^K(s) \cap L_i(X, J) = \pi_i(\phi^K_i(s))$.

For every $0 \leq i < \infty$ we consider the Kripke Structure $\Omega_i = (\Omega_i; X; J; (F^j \mid j \in J); \bar{\psi}_i)$, where $\bar{\psi}_i = \bar{\psi} \circ \pi_i^{-1}$ and for $i > 0$ the function $\beta_i^j : \Omega_i \rightarrow 2^{L_{i-1}(X, J)}$ by $\beta_i(j)(w) := \{ f \in L_{i-1}(X, J) \mid k_j(f) \in w \}$. We define $F_i^j = \{ \Omega_0 \}$ for every $j \in J$.

Now we consider $\Omega_i$ again as a canonical projective image. $G_i$ is defined to be the partition of $\Omega$ induced by the inverse images of $\pi_i$, $G_i := \{ \pi_i^{-1}(w) \mid w \in \Omega_i \}$. By the definition of $\Omega$, the join partition $\vee_{i=1}^\infty G_i$ is the discrete partition of $\Omega$, meaning that it consists of singletons. Let $F_i^j$ be the partition on $\Omega$, coarser than $G_i$, defined by $F_i^j := \{ \pi_i^{-1}(B) \mid B \in F_i^j \}$. From the definitions of the $\Omega_i$ and the $F_i^j$ it follows that $\vee_{i=0}^\infty F_i^j = \Omega^i$. Since $X$ and $J$ are finite, there are several important properties of the Kripke Structure $\Omega_i$, all of which are used in this paper.

(i) $\Omega_i$ is finite for every $0 \leq i < \infty$. (For a more general statement, see Lismont and Mongin 1995.)

(ii) For every $w \in \Omega_i$ we can define a formula $f(w)$ of depth $i$ or less such that $\alpha^{\Omega_i}(f(w)) = \{ w \}$, meaning that the formula $f(w)$ is true with respect to $\Omega_i$ only at $w \in \Omega_i$. This follows from the finiteness of $\Omega_i$. For any subset $A \subseteq \Omega_i$ define $f(A) := \vee_{w \in A} f(w)$, a formula that is true with respect to $\Omega_i$ only in the subset $A$.

(iii) It is easy to extend a member of $\Omega_i$ to a member of $\Omega_{i+1}$. Fix $0 \leq i < \infty$ and $w \in \Omega_i$. For every $j \in J$ define $F_i^j$ by $w \in F_i^j \in F_i^j$. If $(M_i^j \mid j \in J)$ are subsets of $(F_i^j \mid j \in J)$, respectively, such that $F_i^j \cap \pi_i(B) \neq \emptyset$ implies that $M_i^j \cap \pi_i(B) \neq \emptyset$, then there is a unique $v \in \Omega_{i+1}$ such that $\pi_i^{i+1}(v) = w$ and for every $u \in \Omega_i$ $\neg k_j \neg f(u) \in v$ if and only if $u \in M_i^j$. Furthermore, this is the only way to extend a member of $\Omega_i$ to a member of $\Omega_{i+1}$: this is Lemma 4.2 of Fagin, Halpern, and Vardi (1991). For any $i \geq 0$ and $v \in \Omega_k$ with $k > i$ we define
\[ M_i^j(v) := \{ w \in \Omega_i \mid \neg k_j \neg f(w) \in v \}. \]
Notice that if \( w \in F \in \mathcal{F}_i^j \) then \( M_{i-1}^{j-1}(w) \)
is equal to \( \pi_i^{j-1}(F) \), which could be a proper subset of the member of \( \mathcal{F}_{i-1}^{j-1} \)
that contains \( \pi_i^{j-1}(w) \).

(iv) For every formula \( f \in \mathcal{L}_i \) and \( l \geq i \) \( \pi_i^{-1}(\alpha\Omega_i(f)) = \alpha\Omega(f) \). This follows
from (iii) and the Completeness Theorem. (See also Lemma 2.5 in Fagin, Halpern, and Vardi 1991.)

(v) As a Kripke Structure, every \( \Omega_i \) is connected. This was proven first by Fagin, Halpern, and Vardi (1991) and it can be proven in several ways (for example from Proposition 1 of Simon, 1999).

### 2.5 The Common Knowledge of a Formula

Fagin, Halpern and Vardi (1991) investigated what happens when the agents have common knowledge of a finite set of formulas, equivalent the common knowledge of a single formula. Following their definition for “closed” and not wanting to create confusion with topologically closed, for all \( i > 0 \) we define a non-empty subset \( A \subseteq \Omega_i \) to be semantically closed (Simon 2001) if for every \( j < i \), every \( B \in \mathcal{G}_{i-1} \) and every \( w \in A \) if \( \pi_i^{-1}(w) \subseteq F \in \mathcal{F}_i^j \) and \( F \cap B \neq \emptyset \) then \( F \cap B \cap \pi_i^{-1}(A) \neq \emptyset \). Any non-empty subset of \( \Omega_0 \) is allowed to be semantically closed. Let \( f \in \mathcal{L} \) be a formula with \( d = \text{depth}(f) \). Fagin, Halpern, and Vardi (1991) proved that \( \text{Ck} \{ \{ f \} \} \) is not empty if and only if the subset \( \alpha\Omega_d(f) \) is a semantically closed subset of \( \Omega_d \) and that there exists a cell dense in \( \text{Ck} \{ \{ f \} \} \) (equivalently \( \text{Ck} \{ \{ f \} \} = F(C) \) for some cell \( C \)) if and only if the Kripke Structure \( \mathcal{V} (\alpha\Omega_d(f)) \) is connected. For all \( i \geq d = \text{depth}(f) \) we define \( \Omega_i^d := \pi_i(\text{Ck} \{ \{ f \} \}) \); it follows from property (iv) that \( \Omega_i^d \subseteq \alpha\Omega_i(E_{i-d}(f)) \). Define \( \mathcal{F}_i^d(f) \) by \( \mathcal{F}_i^d(f) := \{ F \cap \Omega_i^d \mid F \in \mathcal{F}_i^j \} \). Define the Kripke Structure \( \Omega_i^d = (\Omega_i^d; J; (\mathcal{F}_i^d(f) \mid j \in J); X; \psi_i|_f) \) where \( \psi_i|_f \) refers to the restriction of \( \psi_i \) to \( \Omega_i^d \). Likewise define \( \mathcal{G}_i^d(f) \) by \( \mathcal{G}_i^d(f) := \{ \pi_i^{-1}(F) \mid F \in \mathcal{F}_i^d(f) \} \) and define \( \mathcal{G}_i(f) \) by \( \mathcal{G}_i(f) := \{ G \in \mathcal{G}_d \mid G \subseteq \alpha\Omega(E_{i-d}(f)) \} \) (where \( d \) is the depth of \( f \)).

Most importantly, Fagin, Halpern, and Vardi (1991) showed how to create extensions of \( \Omega_i^d \) to \( \Omega_{i+1}^d \) for all \( i \geq d = \text{depth}(f) \), with only an additional requirement to the rules of (iii) governing extensions from \( \Omega_i \) to \( \Omega_{i+1} \): We must require that the \( \mathcal{F}_i^d \) are in \( \mathcal{F}_i^j(f) \), which means that the \( (M_i^j \mid j \in J) \) are also subsets of \( \Omega_i^d \). For the existence of such subsets is needed the
semantically closed property. The ability to extend establishes the equality
\[ \Omega_i^f = \alpha^{\Omega_i f}(E^{i-d}(f)) \] for all \( i \geq d = \text{depth}(f) \).

Fix \( w \in \Omega_i^f \) with \( i \geq d = \text{depth}(f) \) and with \( \alpha^{\Omega_i f}(f) \) semantically closed, and
let \( \mathcal{F}_i^j \) be the member of \( \mathcal{F}_i^j(f) \) containing \( w \). The choice of \( M_i^j(p_{i+1}(w)) = \mathcal{F}_i \)
for agent \( j \) was called the “least-information” extension in Fagin, Halpern,
and Vardi (1991). Define \( p_{i+1}(w) \) to be that unique member of \( \Omega_i^{f+1} \) such that
\( \pi_i(p_{i+1}(w)) = w \) and \( M_i^j(p_{i+1}(w)) = \mathcal{F}_i \) for every \( j \in J \). If \( f \) is a tautology,
then it was called the “no-information” extension, and in this case we write
\( p_{i+1} \) instead of \( p_i^j \).

We define a formula \( f \in \mathcal{L}(X,J) \) with \( d = \text{depth}(f) \) and \( |J| \geq 2 \) to be generative
if and only if \( \alpha^{\Omega_i f}(f) \) is semantically closed, \( \mathcal{V}(\alpha^{\Omega_i f}(f)) \) is connected,
and there exists more than one cell dense in \( C^k(\{f\}) \), (meaning that these
cells are uncentered). In Simon (2001), Theorem 1 states that the following
are equivalent:
(a) the formula \( f \) is generative,
(b)there is an uncentered cell \( C \) such that \( F(C) = C_k(\{f\}) \), meaning that
there are uncountably many such cells, (equivalently uncountably many un-
centered cells dense in \( C_k(\{f\}) \)),
(c) \( C_k(f) = F(C) \) for some cell, but \( C_k(f) \) is not a maximal member of \( T \),
(d) there is a cell dense in \( C_k(\{f\}) \) that is not finite.

We will call any member of \( \Omega_i \) an atom, or an atom of \( \Omega_i^f \) if it also belongs
to \( \Omega_i^f \).

3 Uncountably many cells with uncountable cardinality

Our first goal is to prove Theorem 1: If the formula \( f \) is generative then
there is a continuum of distinct cells dense in \( C_k(\{f\}) \) such that there are in-
finitely many possibility sets with continuum cardinality. We prove Theorem
1 with something called the alienated extension. The alienated extension is
an alternation between different ways to extend an element of the finite level
structure \( \Omega_i^f \), with long stretches of least information extensions and long
stretches of confirming the finite models \( \Omega_k^f \) for infinitely many \( k \).
If \( f \) is generative and \( i \geq \) depth \( (f) \) define an \( F \in \mathcal{F}_i^j(f) \) to be proto-generative (for \( f \)) if there exists at least one \( v \in \Omega_i^f \) such that the number of members of \( \Omega_i^f \) in \( F \cap \pi_i \circ \pi_{i-1}^{-1}(v) \) is at least 2; and define such an \( F \in \mathcal{F}_i^j(f) \) to be generative (for \( f \)) if for every \( v \in \Omega_i^f \) such that \( F \cap \pi_i \circ \pi_{i-1}^{-1}(v) \neq \emptyset \) then the cardinality of this intersection is at least 2. Define an atom \( w \in \Omega_i^f \) to be proto-generative (respectively generative) for an agent \( j \) if it is contained in a member of \( \mathcal{F}_i^j(f) \) that is proto-generative (respectively generative).

If \( f \) is generative with depth \( d \) then there must be a proto-generative member of \( \mathcal{F}_d^j(f) \) for some \( j \in J \), since otherwise all extensions from \( \Omega_i^f \) to \( \Omega_i^{f+1} \) would be determined uniquely, and the same would be true for all the following \( \Omega_i^f \) for all \( i > d \), and we would have a contradiction to Theorem 1 of Simon (2001).

**Lemma 1:** Let \( f \) be generative and let \( i \geq d = \) depth \( (f) \).

(a) If \( F \in \mathcal{F}_i^j(f) \) is proto-generative then every \( G \in \mathcal{F}_{i+1}^j(f) \) with \( k \neq j \) and \( \pi_{i+1}(G) \cap \pi_i^{-1}(F) \neq \emptyset \) is also proto-generative.

(b) Let \( F \in \mathcal{F}_i^j(f) \) be given. If every \( G \in \mathcal{F}_i^j(f) \) such that \( k \neq j \) and \( G \cap F \neq \emptyset \) is proto-generative, then every \( F' \in \mathcal{F}_{i+1}^j(f) \) extending \( F \) is generative.

(c) If there are at least three agents then there is a level \( \hat{i} \geq d \) such that for all \( k \geq \hat{i} \) all \( k \)-atoms of \( \Omega_k^f \) are generative for all agents. If there are exactly two agents, then there is a level \( \hat{i} \geq d \) such that for all \( k \geq \hat{i} \) any \( k \)-atom of \( \Omega_k^f \) is generative for either one or the other agent.

(d) There is a level \( \hat{i} \) such that for all \( k \geq \hat{i} \) if \( F' \in \mathcal{F}_{k+2}^j(f) \) was created from the use of the least information extension twice from \( F \in \mathcal{F}_k^j(f) \) and \( B \) is in \( \mathcal{G}_k \) with \( B \subseteq \pi_k^{-1}(F) \) then the intersection \( F' \cap \pi_{k+2}(B) \) has at least two elements.

(e) If there are only two agents and \( i \) is large enough so that all atoms of \( \Omega_i^f \) are generative for one or the other agent but none of them are proto-generative for both agents then there must be one agent \( j \) such that all the atoms of \( \Omega_i^f, \Omega_{i+2}^f, \ldots \) are generative for \( j \), all the atoms of \( \Omega_{i+1}^f, \Omega_{i+3}^f, \ldots \) are generative for the other agent \( j' \neq j \), none of the atoms of \( \Omega_i^f, \Omega_{i+2}^f, \ldots \) are proto-generative for \( j' \), and none of the atoms of \( \Omega_{i+1}^f, \Omega_{i+3}^f, \ldots \) are proto-
generative for $j$.

**Proof:**

(a) Let $F' \in \mathcal{F}_{i+1}^j(f)$ be any extension of $F$ intersecting $G \in \mathcal{F}_{i+1}^k(f)$, and let $B \in \mathcal{G}_i(f)$ be any member such that $\pi_{i+1}^{-1}(F') \cap \pi_{i+1}^{-1}(G) \cap B \neq \emptyset$. Since there is at least two ways for Agent $j$ to extend $F$ that included the possibility of $\pi_i(B) \in \Omega_{i+1}^f$ (and because in extending $\pi_i(B)$ the agents choose their sets $M_i^j$ independently) we conclude that $|G \cap \pi_{i+1}(B)| \geq 2$.

(b) Let $B$ be any member of $\mathcal{G}_i(f)$ such that $\pi_{i+1}^{-1}(F') \cap B \neq \emptyset$, and let $G \in \mathcal{F}_{i}^k(f)$ be such that $F \cap G$ contains $\pi_i(B)$. Because $G$ is proto-generative there must be at least two elements of $\Omega_{i+1}^f$ in $F' \cap \pi_{i+1}(B)$.

(c) If none of the atoms of $\Omega_{i+1}^f$ were proto-generative then there would be only one way to extend all of these atoms to the next level, and so on ad infinitum, and that would contradict the assumption that $f$ is generative. Let $v$ be a atom of $\Omega_{i+1}^f$ that is not proto-generative of adjacency distance one from some proto-generative atom $w$ of $\Omega_{i+1}^f$, with the two atoms sharing membership of $F \in \mathcal{F}_{d+1}(f)$. Since there is only one extension of $F$ to a member of $\mathcal{F}_{d+1}(f)$ it must hold that every extension of $v$ in $\Omega_{i+1}^f$ remains adjacent to every extension of $w$ in $\Omega_{d+1}^f$. Because $w$ is proto-generative, and so for $j' \neq j$ the member of $\mathcal{F}_{d+1}(f)$ containing $v$ is proto-generative, it follows from Part (a) that every extension of $w$ is also proto-generative. By induction on the adjacency distance it follows that if $w \in \Omega_{d+1}^f$ is of adjacency distance $l$ from a proto-generative $v \in \Omega_{d+1}^f$ then every extension of $w$ in $\Omega_{d+1}^f$ is also proto-generative. From the finiteness of the adjacency diameter of $\Omega_{d+1}^f$ there is an $l$ such that all atoms of $\Omega_{d+1}^f$ are proto-generative. Combining Parts (a) and (b), if $v \in \Omega_{d+1}^f$ is proto-generative for agent $j$ then any extension of $v$ in $\Omega_{i+1}^f$ is proto-generative for the other agent, any extension of $v$ in $\Omega_{i+2}^f$ is generative for agent $j$, any extension of $v$ in $\Omega_{i+3}^f$ is generative for the other agent, and so on. The claim concerning more than two agents is now transparent.

(d) It follows directly from Parts (b) and (c).

(e) That for the first level $\hat{i}$ there are generative atoms for only one agent follows from the connectedness of $\Omega_{\hat{i}+1}^f$ and induction on the adjacency distance. The rest follows from Parts (b) and (c).

$\square$
For any generative formula \( f \in \mathcal{L} \) define \( \text{gen} (f) \) to be the first level \( i \geq \text{depth} (f) \) such that if there are two agents then every member of \( \Omega_i^f \) is generative for one or the other agent, and if there are at least three agents then all members of \( \Omega_i^f \) are generative for all agents.

For the rest of this section let a generative \( f \in \mathcal{L} \) be fixed. Let \( 2^{\mathbb{N}_0} \) be the set of subsets of the whole numbers \( \mathbb{N}_0 = \{0, 1, 2, \cdots \} \) with infinite cardinality (\( S \in 2^{\mathbb{N}_0} \) implies \( S \subseteq \mathbb{N}_0 \) and \( |S| = \infty \)). For every pair \( i, k \in S \) with \( k \geq i \geq \text{depth} (f) \) we will define a map \( p^{s,f}_{k,i} : \Omega_i^f \rightarrow \Omega_k^f \). If \( i \in S \in 2^{\mathbb{N}_0} \) define \( n_S(i) := \inf \{ k \in S \mid k > i \} \), the first member of \( S \) strictly larger than \( i \). If \( i \in S \) and \( w \in \Omega_i^f \) then define \( p^{s,f}_{n_S(i)}(w) := \phi^{\Omega_i^f}_{n_S(i)}(w) \) and define \( p^{s,f}_i(w) := w \).

\( p^{s,f}_{n_S(i)}(w) \) is an extension of \( w \), meaning that \( \pi_i^{n_S(i)}(p^{s,f}_{n_S(i)}(w)) = w \) (Lemma 1 of Simon 2001). For every \( k \in S \) and \( w \in \Omega_i^f \) with \( k \geq i \in S \) and \( p^{s,f}_k(w) \in \Omega_k^f \) already defined, define \( p^{s,f}_{n_S(k)}(w) \) to be \( p^{s,f}_{n_S(k)}(p^{s,f}_k(w)) \). Lastly, for all \( i \in S \in 2^{\mathbb{N}_0} \) and \( w \in \Omega_i^f \), define \( p^{s,f} : \Omega_i^f \rightarrow \Omega_i^f \) by

\[
p^{s,f}(w) := \bigcap_{l \in S, l > i} \pi_i^{-1} \circ p^{s,f}_l(w).
\]

For any \( i \in S \in 2^{\mathbb{N}_0} \) and \( w \in \Omega_i^f \) we call \( p^{s,f}(w) \) the alienated extension of \( w \) with respect to \( S \) and \( f \). Define \( p^s \) to be \( p^{s,f} \) for any tautology \( f \in \mathcal{L} \).

An alienated extension involves an infinite number of least-information extensions. For all \( 0 \leq i < \infty \) and \( w \in \Omega_i^f \) it is easy to confirm that \( \phi^{\Omega_i^f}_{i+1}(w) = p^{f}_{i+1}(w) \), meaning also that \( p^{\mathbb{N}_0,f} \) is the infinite repetition of the least-information extension. Define the map \( p^f \) to be \( p^{\mathbb{N}_0,f} \) and \( p \) to be \( p^{\mathbb{N}_0} \).

For any \( S \in 2^{\mathbb{N}_0} \) and positive \( k \) define \( n_S^k(i) \) by \( n_S^1(i) = n_S(i) \) and \( n_S^k(i) = n_S \circ n_S^{k-1}(i) \).

**Lemma 2:** If \( S \in 2^{\mathbb{N}_0} \) and \( f \) is generative, then all alienated extensions with respect to \( S \) and \( f \) share the same dense cell of \( \text{Ck}(\{f\}) \).

**Proof:** If \( i \geq \text{depth} (f) \) and both \( w \) and \( w' \) are members of \( \Omega_i^f \) such that both are contained in the same member of \( \mathcal{F}_i^f (f) \), then from induction and the definition of \( \phi^{\Omega_i^f} \), \( p^{s,f}(w) \) and \( p^{s,f}(w') \) are both contained in the same member of \( \Omega^f_k \), the limit of the \( \mathcal{F}_i^f \).

Now, given any \( i, k \in S \) and \( b \in \Omega_i^f \) and \( d \in \Omega_k^f \), the adjacency distance between \( p^{s,f}(b) \) and \( p^{s,f}(d) \) in \( \Omega_i^f \) is no more than the adjacency distance
Given any generative formula \( f \) define the formula \( g^f_i := \phi^{\Omega^f_i}(\phi^{\Omega^f_{i+1}}) \) of depth \( i+1 \), the formula true in \( \Omega^f_{i+1} \) only in the image \( \phi^{\Omega^f_i}(\phi^{\Omega^f_{i+1}}) \). As we will see, the common knowledge of \( g^f_i \) is closely linked to the Kripke Structure \( \Omega^f_i \). (see also Theorem 4.23 of Fagin, Halpern, and Vardi 1991).

**Lemma 3:** The formula \( g^f_i \) is common knowledge in the Kripke Structure \( \Omega^f_i \). If \( i \in S \leq 2^{\omega_0} \), \( i \geq \text{depth}(f) \), and \( i + 1, i + 2, \ldots, i + l \notin S \) for some \( l \geq 1 \), then \( p^s \circ_i (\Omega^f_i) \leq \alpha^i (E^f(g^f_i)) \), and the same holds for \( \phi^{\Omega^f_i} \) applied to any element of \( \Omega^f_i \).

**Proof:** Because \( \Omega^f_i \) is finite and connected, \( \phi^{\Omega^f_i}(\Omega^f_i) \) is a cell. Because \( \phi^{\Omega^f_i}(\Omega^f_i) \subseteq \alpha^i (g^f_i) \), Property (iv) and Lemma A imply that \( g^f_i \) is common knowledge in the cell \( \phi^{\Omega^f_i}(\Omega^f_i) \). If \( E^f(g^f_i) \), a formula of depth \( i + l + 1 \), were not true at any point of \( \phi^{\Omega^f_{i+1}}(\Omega^f_i) \) then also by Property (iv) we must have that \( g^f_i \) is not common knowledge at some point of \( \phi^{\Omega^f_i}(\Omega^f_i) \), a contradiction.

By Simon (2001) \( \Omega^f_i \) and \( \phi^{\Omega^f_i}(\Omega^f_i) \) are equivalent as Kripke Structures, so that \( g^f_i \) is also common knowledge in the Kripke structure \( \Omega^f_i \).

**Lemma 4:** If \( f \) is generative and \( i \geq \text{gen}(f) \) then \( E^f(g^f_i) \) is not true at any extension of \( p^f_{i+2}(\Omega^f_{i+1}) \).

**Proof:** Because \( \phi^{\Omega^f_{i+1}}(w) = p^f_{i+2}(w) \) for any \( w \in \Omega^f_{i+1} \), given \( w \in \Omega^f_{i+1} \) it suffices to find some \( j \in J \) such that \( k_j(g^f_i) \) is not true at \( p^f_{i+2}(w) \). Let \( j \in J \) be such that \( w \in \Omega^f_{i+1} \) is generative for Agent \( j \) and let \( w \in F \in F_{i+1}(f) \).

For every \( v \in \Omega^f_i \) there is only one member of \( \Omega^f_{i+1} \) in the subset \( \pi_{i+1}(\pi_i^{-1}(v)) \) where \( g^f_i \) is true. But for \( v := \pi_i^{-1}(v) \in \Omega^f_i \) there are at least two \( u \in \Omega^f_{i+1} \) with \( u \in F \cap \pi_{i+1}^{-1}(v) \) (including at least the possibility of \( u = w \)). The \( F' \in F_{i+2}(f) \) containing \( p^f_{i+2}(w) \) must have a non-empty intersection with \( \pi_j^{-1} \circ \pi_i^{-1}(u) \) for all \( u \in F \cap \Omega^f_{i+1} \), and therefore \( F' \) is not contained in \( \phi^{\Omega^f_{i+2}}(\Omega^f_i) \).
Define an equivalence relation on $\mathbb{N}_0$ by $S \sim T$ if and only if there exists an $m \in \mathbb{N}_0$ such that $S \backslash \{0, 1, 2, \ldots, m\} = T \backslash \{0, 1, 2, \ldots, m\}$. The co-sets of this equivalence relation have the cardinality of the continuum.

Let $k$ be such that $2^k \geq \text{gen}(f)$ and let $d = 2^k$. Due to Lemma 2, it suffices to show for some $w \in \Omega^f_d$ that if $S$ and $T$ are both subsets of $\mathbb{N}_0$ with $S \not\sim T$ then $\beta(S) f(w)$ does not share the same cell as $\beta(T) f(w)$. For the sake of contradiction, let us suppose that the adjacency distance in $\text{Ck}\{\{f\}\}$ between $\beta(S) f(w)$ and $\beta(T) f(w)$ equals a finite number $l < \infty$. Because $S \not\sim T$ there exists an $i > \max(\log_2((l + 2)), k)$ such that $i \in S$ and $i \not\in T$, or vice versa. By symmetry, let us assume that $i \in S$ and $i \not\in T$. Lemma 3 applied to $\beta(T) f(w)$ implies that $\beta(T) f(w) \in \alpha f(E^l g_i^f)$. But because the adjacency-distance between $\beta(S) f(w)$ and $\beta(T) f(w)$ is $l$ we have that $\beta(S) f(w) \in \alpha f(E(g_i^f))$, a contradiction to Lemma 4.

If $S$ is infinite (all but one of the uncountably many equivalent relations) then any possibility sets containing an alienated extension is homeomorphic to a Cantor set. This follows from Lemma 1d and the fact that for every such $S$ there are arbitrarily long strings of the least information extension applied. □

Because there are only countably many alienated extensions $\beta(S) f(w)$ for any fixed $S$ (because there are only countably many atoms $w$ on any level) we haven’t shown that there are uncountably many cells where all the possibility sets are equivalent to Cantor sets. That would be a more difficult claim to prove and would require a better understanding of what happens at any finite adjacency distance from a point created by an alienated extension. We leave it as an open question:

**Question 5**: For any generative formula $f$ does there exist uncountably many cells dense in $\text{Ck}\{\{f\}\}$ where all the possibility sets are equivalent to Cantor sets?

**Corollary 1**: If $f$ is generative and $C$ is a cell of finite fanout where $f$ generates all the formulas held in common knowledge then there cannot be a uniform bound on the size of the possibility sets in $C$.

Suppose for the sake of contradiction that $n$ is a uniform bound on the size of the possibility sets in $C$. Let $F$ be any possibility set topologically equivalent to a Cantor set in a cell $C'$ that is also dense in $\text{Ck}\{\{f\}\}$ and let $z$ be any
point of $F$. Let $j$ be the agent such that $F$ is a possibility set for $j$. Because $F$ is infinite, there will be $n + 1$ mutually exclusive formulas $g_0, \ldots, g_n$ such that $\neg k_j \neg g_i$ is true at $z$ for every choice of $i = 0, \ldots, n$ ($\neg k_j \neg g_i$ meaning that $j$ considers the validity of $g_i$ to be possible). Because $C$ is also dense in $\text{Ck}({f})$ there will be a sequence $z_1, z_2, \ldots$ of points in $C$ converging to $z$. At some level $L$ all the $\neg k_j \neg g_i$ will be valid at all $z_l$ for all $l \geq L$ and $0 \leq i \leq n$. Because the $g_i$ are mutually exclusive the possibility set for agent $j$ containing any such point $z_l$ will have cardinality at least $n + 1$, a contradiction.

4 Finite Fanout

Now we construct uncountably many cells of finite fanout dense in $\text{Ck}({f})$ for any generative $f \in \mathcal{L}$. Let such an $f$ be fixed for the rest of this section.

The proof of Theorem 2 is quite different from that of Theorem 1. Again we alternate how extensions are performed, but much more toward confirming finite models and also selectively within any stage, so that some points of $\Omega^f_i$ can be extended by way of the least information extension and other points of $\Omega^f_i$ according to a previous finite model.

For any $w \in \Omega^f_i$ define $F^j_i(w)$ to be that member of $\mathcal{F}^j_i(f)$ containing $w$.

From Proposition 2 of Simon (1999), with finitely many agents a cell is compact if and only if it has finite diameter. Therefore by Theorem 1 of Simon (2001) all cells dense in $\text{Ck}({f})$ do not have finite diameter, and this implies also that there is no bound on the diameters of the $\Omega^f_i$.

Let $S \in 2^\omega$ satisfy

1) $\inf S > \text{gen } (f) + 8,$
2) for every $i \in S$ the differences $n_S(i) - i$ (the next member of $S$ after $i$ minus $i$) start at at least 5 and are strictly increasing,
3) for every $i \in S$ the adjacency diameter of $\Omega^f_{n_S(i)}$ is strictly greater than twice the size of the set $\{k \in S \mid k \leq i\}$ plus 3, and
4) $2^{(n_S(i) - i - 1)/2}$ is strictly greater than the cardinality of $\Omega^f_i$.

Let $T$ be any infinite subset of $S$ with $\inf T = \inf S$. For every such $T$ and $i \geq \inf T$ we define inductively two subsets $A_i$ and $B_i$ of $\Omega^f_i$. If there are only two agents and the levels beyond $\text{gen } (f) + 2$ alternate between being
all generative for one agent and not proto-generative for the other agent (Lemma 1e) then define $w_0$ to be any element of $p^I_{\text{inf}T}(\Omega^I_{\text{gen}(f)+2})$. If there is an atom $v$ of $\Omega^I_{\text{gen}(f)+2}$ that is generative for both agents then let $w_0$ be $p^I_{\text{inf}T}(v)$, in either case $w_0$ is an application of the least information extension from level $\text{gen}(f)+2$ to the level $\text{inf}T$. Define $B_{\text{inf}T}$ to be the singleton \{w_0\} and $A_{\text{inf}T} = \emptyset$. We assume that $A_k$ and $B_k$ have been defined for all $\text{inf}T \leq k < i$, and show how to define $A_i$ and $B_i$. First, we define a extension function $\gamma_i : A_{i-1} \cup B_{i-1} \to A_i \cup B_i$ for all $i > \text{inf}T$; it suffices to determine the sets $M^i_{l-1}(\gamma_i(w))$. If $w \in A_{i-1}$ then $M^i_{l-1}(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^i_{l-1}(w)$. If $w \in B_{i-1}$ and $F^i_{j-1}(w)$ contains some member of $A_{i-1}$, then $M^i_{l-1}(\gamma_i(w)) := (A_{i-1} \cup B_{i-1}) \cap F^i_{l-1}(w)$; otherwise if $A_{i-1} \cap F^i_{l-1}(w) = \emptyset$, then $M^i_{l-1}(\gamma_i(w)) := F^i_{l-1}(w)$. If $i \in T$ we define $B_i$ to be the set $\{p^I_{j}(w) \mid w \in \Omega^I_{l-1}\setminus(A_{i-1} \cup B_{i-1})$, $w \in F^i_{l-1}(b)\}$ for some $b \in B_{i-1}$ and $j \in J$ with $F^i_{l-1}(b) \cap A_{i-1} = \emptyset$ and we define $A_i$ to be the set $\gamma_i(A_{i-1} \cup B_{i-1})$. If $i / \in T$ we define $B_i$ to be the set $\gamma_i(B_{i-1})$. and we define $A_i$ to be the set $\gamma_i(A_{i-1})$. For any $i > \text{inf}T$, $l \geq 0$, and $w \in A_{i-1} \cup B_{i-1}$ we define $\gamma_i+l(w) = \gamma_i+l \circ \ldots \circ \gamma_i(w)$ and we define $\gamma(w) := \cap_{k=1}^{\infty} \pi_{-1}^{-1}(\gamma_k(w))$. We define $C$ to be $\{\gamma(w) \mid i \geq \text{inf}T, w \in A_i\}$.

If there are only two agents, notice from Lemma 1 that any atom of $\Omega^I_i$ with $i \geq \text{inf}S$ and within an adjacency distance of 3 from $B_i$ is either generative for both agents or it is generative for one agent and not proto-generative for the other agent – and the same holds for all extensions of this atom to higher levels. It would be nice, if possible, to prove that if any atom of $\Omega^I_{\text{gen}(f)+2}$ is generative for both agents then there is a level for which all atoms are generative for both agents. Such an argument was easy for non proto-generation for both agents, since the extensions of such atoms were determined and not “running away”. However proto-generation with one but not the other agent still allows for distinct extensions of the same atom to gain distance from each other. The question is whether they can do so fast enough to avoid mutual generation.

Lemma 5 : The extension function $\gamma_i$ is well defined for every $i \geq \text{inf}T$ and if $b \in B_i$ is adjacent in $\Omega^I_i$ to $a \in A_i$, sharing the same member of $\mathcal{T}^f_i$, and $k$ is the largest member of $T$ less than or equal to $i$, then $a = \gamma_i(b')$ for some $b' \in B_{k-1}$ with $F^i_{k-1}(b') \cap A_{k-1} = \emptyset$.

Proof: We prove both claims together by induction on $i$. $\gamma_{\text{inf}T+1}(w_0) =$
Then for every $k \leq k < i$. Let $w \in A_{i-1} \cup B_{i-1}$, and for any given $j \in J$ let us assume that $v \in \Omega_{i-2}^j$ satisfies $\pi_{i-2}(v) \cap \pi_{i-1}(F_{i-1}^j(w)) \neq \emptyset$. We need to show that $\pi_{i-1}(\pi_{i-2}^{-1}(v)) \cap M_{i-1}^j(\gamma_i(w)) \neq \emptyset$. If $i - 1 \notin T$ and $i > \inf T$ then the well definition of $\gamma_i$ shows the same for $\gamma_i$, so for the following cases, we assume that $i - 1 \in T$.

**Case 1:** $w \in A_{i-1}$ and $v \in A_{i-2} \cup B_{i-2}$: $\gamma_{i-1}(v)$ is in $F_{i-1}^j(w)$ because $v$ and $\pi_{i-1}^{-1}(w)$ share the same member of $\overline{F'}_{i-2}(f)$.

**Case 2:** $w \in A_{i-1}$ and $v \notin A_{i-2} \cup B_{i-2}$: This is possible only if $\pi_{i-2}^{-1}(w) \in B_{i-2}$ and $F_{i}^j(\pi_{i-2}^{-1}(w)) \cap A_{i-2} = \emptyset$. Since $v \in F_{i-2}^j(\pi_{i-2}^{-1}(w))$ we have $F_{i-1}^j(v) \in B_{i-2} \cap F_{i-2}^j(w)$.

**Case 3:** $w \in B_{i-1}$ and $B_{i-1}^j(w) \cap A_{i-1} \neq \emptyset$: Let $a \in F_{i-1}^j(w) \cap A_{i-1}$. By the second part of the induction hypothesis $\pi_{i-2}^{-1}(a) \in B_{i-2}$ with $F_{i-2}(\pi_{i-2}^{-1}(a)) \cap A_{i-2} = \emptyset$. It follows that $v$ is in $F_{i-2}^j(\pi_{i-2}^{-1}(a))$ and whether or not $v$ was in $B_{i}$ that there is an extension of $v$ in $A_{i-1} \cup B_{i-1}$.

**Case 4:** $w \in B_{i-1}$ and $F_{i-1}^j(w) \cap A_{i-1} = \emptyset$: Since $w \in p_{i-1}(\Omega_{i-2}^j)$ we have that $p_{i-1}(v) \in F_{i-1}^j(w)$.

For the second part of the claim, suppose for the sake of contradiction that $b' := \pi_k^{-1}(a) \in A_{k-1}$. $b'$ shares the same member of $\overline{F'}_{k-1}(f)$ with $c := \pi_{k-1}^{-1}(b) \in \Omega_{k-1}^j \setminus (A_{k-1} \cup B_{k-1})$. For every $j \in J$ and $D \in G_{k-2}(f)$ if $\pi_k^{-1}(F_k^j(\gamma_k(b'))) \cap D$ then it intersects $D$ in only one member of $G_{k-2}(f)$. If $b'$ is generative for $j$ then by Lemma 1 it is different from $\pi_k^{-1}(F_k^j(\pi_k(b'))) = \pi_k^{-1}(F_k^j(\pi_k(i(b'))))$, a contradiction. If there are only two agents and $b'$ is not generative for $j$ then by the well definition of $\gamma_k - 1 \in c$ must have been in $A_{k-1} \cup B_{k-1}$, also a contradiction. So we conclude that $b'$ was in $B_{k-1}$. Furthermore, if $F_{k-1}^j(b') \cap A_{k-1} \neq \emptyset$ then either $\pi_k(a)$ and $\pi_k(b)$ would not share the same member of $\overline{F'}_{k}(f)$ (the case of $F_{k-1}^j(b')$ generative) or from the well definition of $\gamma_k \pi_{k-1}(b)$ would be in $A_{k-1} \cup B_{k-1}$ (the case of $F_{k-1}^j(b')$ not generative), both contradictions. 

The second part of Lemma 5 shows that if $i \in T$, $b \in B_{i-1}$ and $\gamma_i(b) = a \in A_i$ then for every $k \geq 1$ the only members of $A_{n_k(i)} \cup B_{n_k(i)}$ adjacent to $\gamma_{n_k(i)}(a)$ are already in the set $\gamma_{n_k(i)}(A_i \cup B_i) \subseteq A_{n_k(i)}$. Therefore $C = \{ \gamma(w) \mid i \geq \inf T, w \in A_i \}$ is a cell with finite fanout.
Lemma 6: If \( w \) and \( w' \) are in \( \Omega_i^f \) for some \( i \geq \inf T \) but neither are in \( B_i \) then there is an adjacency path \( w = w_0, w_1, \ldots, w_q = w' \) in \( \Omega_i^f \) between \( w \) and \( w' \) such that \( w_m \notin B_i \) for all \( 1 \leq m \leq q - 1 \).

Proof: We proceed by induction on \( i \). Consider an adjacency path \( v_1, v_2, \ldots, v_l \) in \( \Omega_{i-1}^f \) with \( v_1 = \pi_{i-1}^1(w) \) and \( v_l = \pi_{i-1}^1(w') \). We assume that \( v_k \) and \( v_{k+1} \) share the same member \( F^j_{i-k} \) of \( \mathcal{T}^j_{i-1}(f) \) for all \( 1 \leq k \leq l - 1 \), and that \( j_k \neq j_{k+1} \) for every consecutive pair \( k, k+1 \). We will define an extension \( w_k \in \Omega_i^f \) of \( v_k \) for every \( k \) such that \( M^j_{i-k}(w_k) = M^j_{i-k}(w_{k+1}) \) is a subset \( F^j_k \in \mathcal{T}^j_{i-1}(f) \) containing both \( v_k \) and \( v_{k+1} \) with a non-empty intersection with \( \pi_{i-1}(D) \) for every \( D \in \mathcal{G}_{i-2}(f) \) with \( \pi_{i-1}^{-1}(F^j_k) \cap D \neq \emptyset \) and \( M^j_{i-1}(w_1) = M^j_{i-1}(w) \) for at least one \( j \neq j_1 \) and \( M^j_{i-1}(w_l) = M^j_{i-1}(w') \) for at least one \( j \neq j_{l-1} \). The path \( w, w_1, \ldots, w_l, w' \) will be an adjacency path connecting \( w \) and \( w' \), allowing possibly for the identity of \( w \) and \( w_1 \) or of \( w' \) and \( w_l \). We must show that these extensions can be done so that for all \( 1 < k < l \) no extension \( w_k \) is in \( B_i \).

Case 1: there are at least three agents: Looking at any \( v_k \), let \( j \in J \) be any agent other than \( j_k \) or \( j_{k-1} \). Since all levels involved are generative for all agents and the choice of \( M^j_{i-1}(w_k) \) doesn’t affect the connectivity, the selection of \( M^j_{i-1}(w_k) \) can be made so that \( w_k \) is not a least information extension and therefore not in \( B_i \).

Case 2; there are only two agents \((J = \{1, 2\}) and \( i \notin T \): By \( i \notin T \) the only members of \( B_i \) are extensions of members of \( B_{i-1} \). Since all members of \( B_{i-1} \) are adjacent to members of \( A_{i-1} \), by the induction assumption we can assume that at most \( v_1 \) and \( v_l \) are in \( B_{i-1} \). So if \( v_1 \) and \( v_l \) are not in \( B_{i-1} \), we are done by induction on the stages. Without loss of generality assume for now that \( v_1 = \pi_{i-1}^1(w) \) is in \( B_{i-1} \). Let \( m \) be the largest member of \( T \) that is less than \( i \). If \( m \) is greater than \( \inf T = \inf S \) we know from Lemma 5 that there is a \( j \in \{1, 2\} \) such that \( v_1 \) is a member of \( B_{i-1} \) and \( \pi_{i-1}^1(v_1) \) shared the same member of \( \mathcal{T}^j_{m-1}(f) \) with a \( b \in B_{m-1} \) but with no member of \( A_{m-1} \). In this case define \( b' = \gamma_{i-2}(b) \) (with \( b' = b \) if \( m = i - 1 \)). Otherwise let \( b' \) be any other member of \( \Omega_{i-2}^j \) that shares a member of \( \mathcal{T}^j_{i-2}(f) \) with \( \pi_{i-2}(v) \) (for which there must be several since \( w_0 \) was created from several applications of the non-information extension). Because \( \pi_{i-1}^{-1}(F^j_{i-1}(v_1)) \cap \pi_{i-2}^{-1}(b') \neq \emptyset \) it follows that \( F^j_i(w) \) contains some extension \( u \in \Omega_i^f \) of this \( b' \).
and furthermore no member of either $B_i$ or $B_{i-1}$ is an extension of this $b'$. Therefore, since $w$ and $u$ are adjacent, we can replace $w$ by $u \not\in B_i$, do the same for $u'$ if necessary, and repeat with the induction assumption with the added assumption that none of the $v_k$ are in $B_{i-1}$.

**Case 3:** there are only two agents ($J = \{1, 2\}$) and $i \in T$: All members of $B_i$ are created as extensions of points not in $B_{i-1} \cup A_{i-1}$ and they are in $p_i^j(\Omega_{i-1}^j)$. If possible, for every $k$ let $M_{i-1}^{jk}(w_k) = M_{i-1}^{jk}(u_{k+1})$ be any proper subset of $F_k^j(v_k)$ containing both $v_k$ and $v_{k+1}$, meaning that if this is possible then we keep both $w_k$ and $w_{k+1}$ out of $B_i$. On the other hand, if $M_{i-1}^{jk}(w_k) = F_k^j(v_k)$ is forced for both agents $j$ then we will show that the so defined $w_k$ is also not in $B_i$.

**Case 3a; and $1 < k < l$:** Either $v_k$ is generative for Agent 1 or Agent 2. Without loss of generality assume that $v_k$ is generative for Agent 1, with $F = F_{i-1}^1(v_k) \in \mathcal{F}_{i-1}^1(f)$ connecting $v_k$ to $v^*$, equal to either $v_{k-1}$ or $v_{k+1}$. Because $F$ is generative, if there were only way to define $M_{i-1}^1(v_k)$ so as to include both $v_k$ and $v^*$ it must follow that $F_{i-1}^1(v_k) = \{v_k, v^*\}$ with both $v_k$ and $v^*$ extensions of the same $u \in \Omega_{i-2}^1$ (with $v^* = v_{k+1}$ if $j_k = 1$ and $v^* = v_{k-1}$ if $j_k = 2$). Looking at what is necessary for $w_k$ to be in $B_i$, since $F$ projected to the $i-2$ level contains only one element and either the previous number in $T$ is below the $i-4$ level or $i = \inf T$ and $w_0$ was created through several applications of the least information extension, it is also necessary for there to be some $u' \in F_{i-2}^2(u)$ other than $u$ (with the possible choice of $u' \in B_{i-2}$ if $i > \inf T$; there will be many more than one other member of $F_{i-2}^2(u)$, but one other suffices for the argument). If $v_k$ were generative for Agent 2 we would be able to avoid $w_k$ in $B_i$ from the choice of $M_{i-1}^2(v_k)$. So we have to assume that $v_k$ is not generative for Agent 2 and therefore from Lemma 1 that $F_{i-2}^2(u)$ is generative. Let $H$ be $\pi_{i-3}^{-1} \circ \pi_{i-3}^{i-2}(u')$. There must be at least three members of $F$ corresponding to the three non-empty subsets of $\pi_{i-2}(H) \cap \Omega_{i-2}^j$ combined with $u$, a contradiction to our assumption that $F = F_{i-1}^1(v_k)$ contained only two elements. Therefore there was more than one way to connect our extensions of $v_k$ and $v^*$, one of those ways avoiding membership of $w_k$ in $B_i$.

**Case 3b; $l > 1$ and $k = 1$ or $k = l$):** Without loss of generality we assume that $F \in \mathcal{F}_{i-1}^1(f)$ contains both $v_1$ and $v_2$. As with Case 3a, if $v_1$ were generative for Agent 1 there would be no alternative to $M_{i-1}^1(v_1) =$
only if there were only two elements of $\pi_{i-2}(F)$. As with Case 3a, the generative property of $\pi_{i-2}(v_1)$ for Agent 2 and either the adjacency to $B_{i-2}$ through agent 2 or the definition of $w_0$ results in a contradiction. On the other hand, if $v_1$ is not generative for Agent 1, by the fact that it is also not proto-generative for Agent 1 (by proximity to $\Omega$ see, it does not matter that perhaps $\Omega$ of a member of $\Omega_{i-1}$ is an $\leq 1$ will connect $\pi_i$ and the generative property of $\pi_i$ $(v_1)$ or $F_{i-1}(v)$ contained a member of $B_{i-1}$ and no member of $A_{i-1}$. If either $F_{i-1}(v)$ or $F_{i-1}(v)$ contained a member of $B_{i-1}$ and no member of $A_{i-1}$, then by $w$ and $w'$ both not in $B_i$ it must hold that $M_{i-1}(w) \neq F_{i-1}(v)$ for some $j$ and $M_{i-1}(w') \neq F_{i-1}(v)$ for some $j'$. If $M_{i-1}(w) = F_{i-1}(v)$ and $M_{i-1}(w') = F_{i-1}(v)$ for the same $j$ then $w$ and $w'$ were already adjacent. Otherwise if $j$ can be different from $j'$ define $w* \in \Omega_{i-1}$ by $M_{i-1}(w*) = M_{i-1}(w)$ and $M_{i-1}(w*) = M_{i-1}(w')$ and it follows that $w*$ will connect $w$ and $w'$ without being in $B_i$. □

Lemma 6 implies that the removal of $B_i$ does not disconnect $\Omega_{i-1}^I$. As we will see, it does not matter that perhaps $\Omega_{i-1}^I \setminus B_i$ may be connected through $A_i$. Due to Lemma 1d and Property 4 defining the set $S$, at every level $i$ the set $B_i$ will vastly outnumber the set $A_i$. More importantly, the extensions in $A_{i+1}, A_{i+2}, \ldots$ of a member of $A_i$ do not involve the least information extensions. Notice that every member of $B_i$ (except for $i = \inf T$) is by definition connected to some member of $A_i$.

**Lemma 7:** If $i \in T$ and the shortest adjacency paths within $\Omega_{i-1}^I \setminus B_i$ between $w \in \Omega_{i-1}^I \setminus \pi_{i-1}(B_{nT(i)})$ and $\pi_{i-1}(B_{nT(i)} \subseteq \Omega_{i-1}^I$ are of length $k \geq 1$, then there is an $1 \leq l \leq k$ with $p_{i-1}^l(w) \in B_{nT(i)}$.

**Proof:** We proceed by induction on $k$. If $k = 1$, let $c \in \pi_{i-1}(B_{nT(i)})$ be adjacent to $w \notin \pi_{i-1}(B_{nT(i)})$ and let $j$ be the agent such that $w$ and $c$ share the same member of $F_j(f)$. The atom $p_{nT(i)}^j(c) \in B_{nT(i)}$ can not share the same member of $F_j(f)$ with a member of $A_{nT(i)}$. Since by Lemma 5 the element $c$ would have shared the same member of $F_j(f)$ with a member of
Theorem 2: If the formula $f$ is generative then there is a continuum of cells with finite fanout that are dense in $\text{Ck}(\{ f \})$. 

Proof: Density of the cell will follow because by choosing any level $i \in T$ and any $w \in \Omega_i^f$ we will show that there is a level $\hat{i} \geq i$ with a member of $A_{\hat{i}}$ extending $w$. By Lemma 7 and the fact that $B_i$ does not disconnect $\Omega_i^f \backslash B_i$ (Lemma 6), we need to show that $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ is not empty for every $i \in T$. We establish this by induction on $i \in T$. The set $B_{\inf T} = \{ w_0 \}$ is not empty. Assume the claim is true for any particular $i \in T$ and all those members of $T$ before $i$. By Lemma 7 all elements of $B_i \cup (\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ are within an adjacency distance of $m := |T \cap \{ 1, 2, \ldots, i \}|$ from $\gamma_i(w_0)$, yet the diameter of $\Omega_i^f$ is at least $2m+1$ by the definition of $S$; this means that both $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ and $\Omega_i^f \backslash \pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ are not empty. Also by Lemma 7, the non-emptiness of $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ and the existence of some $w \in \Omega_i^f \backslash \pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ that is of positive but finite adjacency distance from $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ implies the non-emptiness of $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)}) \subseteq \Omega_i^f$. Density now follows by Lemma 7 and the connectivity of all the $\Omega_i^f$. 

Lemma 8: The cell $C = \{ \gamma(a) \mid i \in T, a \in A_i \}$ is dense in $\text{Ck}(\{ f \})$. 

Proof: Density of the cell will follow because by choosing any level $i \in T$ and any $w \in \Omega_i^f$ we will show that there is some level $\hat{i} \geq i$ with a member of $A_{\hat{i}}$ extending $w$. By Lemma 7 and the fact that $B_i$ does not disconnect $\Omega_i^f \backslash B_i$ (Lemma 6), we need to show that $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ is not empty for every $i \in T$. We establish this by induction on $i \in T$. The set $B_{\inf T} = \{ w_0 \}$ is not empty. Assume the claim is true for any particular $i \in T$ and all those members of $T$ before $i$. By Lemma 7 all elements of $B_i \cup (\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ are within an adjacency distance of $m := |T \cap \{ 1, 2, \ldots, i \}|$ from $\gamma_i(w_0)$, yet the diameter of $\Omega_i^f$ is at least $2m+1$ by the definition of $S$; this means that both $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ and $\Omega_i^f \backslash \pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ are not empty. Also by Lemma 7, the non-emptiness of $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ and the existence of some $w \in \Omega_i^f \backslash \pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ that is of positive but finite adjacency distance from $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)})$ implies the non-emptiness of $\pi^{\text{nr}(i)}_i(B_{\text{nr}(i)}) \subseteq \Omega_i^f$. Density now follows by Lemma 7 and the connectivity of all the $\Omega_i^f$. 

Theorem 2: If the formula $f$ is generative then there is a continuum of cells with finite fanout that are dense in $\text{Ck}(\{ f \})$. 

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Proof: Let $C$ be a cell created from a subset $T \subseteq S$ with $\inf T = \inf S$. Every possibility set of $C$ is created from a least information extension from some level $i \in T$ applied repetitively to a member of $\pi_{n_T(i)}(B_{n_T(i)})$ (sharing its possibility set in $\Omega_i^f$ with a member of $B_i$) up to the stage $n_T(i)$, followed by a limiting of its size according to membership in some possibility set of $\Omega_{n_T(i)}^f$. Due to Property 4 defining $S$ and Lemma 1d, we can read off the subset $T$ from the sizes of the possibility sets in $C$. As there are uncountably many infinite subsets $T$ of $S$ with $\inf T = \inf S$, the theorem is proven. \qed

5 Infinite generation and infinite fanout

Let us review the possibilities for cells of finite fanout. All finite cells are defined by the common knowledge of a single formula (Fagin, Halpern, and Vardi 1991). Combined with results from Simon (1999, 2001) if a cell $C$ has finite fanout it can come in only one of four forms:

1) $C$ is finite, $F(C)$ is finitely generated and maximal in $T$,
2) $C$ is infinite, $F(C)$ is finitely generated and not maximal in $T$, and $C$ is uncentered,
3) $C$ is infinite, $F(C)$ is infinitely generated and not maximal in $T$, and $C$ is centered.
4) $C$ is infinite, $F(C)$ is infinitely generated, and $C$ is uncentered.

Within Case 4, $F(C)$ may or may not be maximal, as with the Bernoulli shift space examples presented above. How can one distinguish Case 3 from the others? A countable cell is centered if and only if it contains at least one isolated point, (Simon 1999), a straightforward application of Baire Category. Difficult to distinguish is Case 2 from Case 4, the distinction being that of finite vs infinite generation.

How wild can things be if finite fanout does not hold? Until now, this paper has been concerned with the existence and properties of uncountably many cells that share the same set of formulas in common knowledge. This is qualitatively different from that of uncountably many distinct Kripke Structure that map injectively to mutually distinct subsets of the same cell of $\Omega$. These issues are different because every possibility set of a cell is a compact set, a property not assumed of the image of a semantic model that maps injectively to $\Omega$. This distinction comes into sharp contrast when considering the finite
fanout property. A cell of finite fanout has the surjective property, meaning that all Kripke Structure that map to it must map to it surjectively. A non-surjective cell may offer many possibilities for disconnected Kripke Structure to map to some cell, but neither this cell nor these Kripke Structure can have finite fanout. This is because the image of a possibility set of a multi-partition with evaluation in $\Omega$ must be a dense subset of a possibility set of $\Omega$ (Lemma 5, Simon 1999).

We present a cell that is centered and compact, meaning also by Proposition 2 of Simon (1999) that it has finite adjacency diameter, and yet there is a Kripke Structure with uncountably many connected components that maps injectively to this cell. Furthermore, the corresponding set of formulas cannot be finitely generated, since the compactness of the cell $C$ implies the maximality of these formulas in $\mathcal{T}$ and by Theorem 1 of Simon (2001) this would imply that this cell must be finite.

To explain our claim, we must first describe Example 3 presented in Simon (2001). This was an example of a compact cell homeomorphic to a Cantor set with an adjacency radius of 2. To construct this example we let $\Omega = \Omega(X, \{1, 2\})$ and defined a sequence of partitions in the following way:

for every $0 < i < \infty$ define $A_i = \{p_i(w) \mid w \in \Omega_{i-1}\}$. Define $P_0 = \{\Omega\}$ and $P_i = P_{i-1} \lor \{\pi^{-1}_i(A_i), \Omega \setminus \pi^{-1}_i(A_i)\}$. We labelled the partitions by $B = (P_i \mid 0 \leq i < \infty)$ and we defined a Kripke Structure

$$K(B) = (\Omega; (Q_j \mid j \in \{1, 2\}), P_\infty; X; \overline{\psi})$$

where the partition $P_\infty$ for the third agent is the limit of the partitions $P_i$, (meaning that $z$ and $z'$ share the same member of $P_\infty$ if and only if they share the same member of $P_i$ for every $i < \infty$), with $\overline{\psi}$ and the $Q_j$ for $j = 1, 2$ the same used to define the Kripke Structure $\Omega$. The third agent can distinguish two points if and only if the no-information extension was applied on different stages. We showed (Simon 2001) that the set $\phi^{K(B)}(\Omega)$ is a cell of $\Omega(X, \{1, 2, 3\})$ equivalent as a semantic model to $K(B)$, and furthermore that the map $\phi^{K(B)}$ is a homeomorphism between $\Omega = \Omega(X, \{1, 2\})$ and the cell that is its image.

Define $A := \{p^S(w) \mid S \in 2^{\infty_0}, i \in S, w \in \Omega_i \subseteq \Omega = \Omega(X, \{1, 2\})$, the set of all alienated extensions with respect to the tautologies. Define $B := \phi^{K(B)}(A)$, the image of the set $A$ in $\Omega(X, \{1, 2, 3\})$. We will show that $B$
defines a Kripke Structure with uncountably many connected components.
To show this, we need some additional results from Simon (1999).

We define a subset $A \subseteq \Omega$ to be good if for every $j \in J$ and every $F \in Q^j$ satisfying $F \cap A \neq \emptyset$ it follows that $F \cap A$ is dense in $F$. By Lemma 5 and Lemma 6 of Simon (1999) $A$ is good if and only if for every $z \in A \phi^V(A)(z) = z$ (where the Kripke structure $V(A)$ is defined above).

First we show that $B$ is a good subset. Let $z = p^S(w) \in \Omega$ for some $i \in S \in 2^{\mathbb{N}_0}$ and $w \in \Omega_i$. Let $j \in \{1, 2\}$, $z \in F \in Q^j = Q^j(X, \{1, 2\})$, and $F \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k$ with $k \in S$ and $k \geq i$. Since $v$ shares the same member of $\mathcal{F}_k$ with $\pi_k(z)$ we have that $p^S(v) \in F$. Otherwise let $z \in P \in \mathcal{P}_\infty$ and let $P \cap \pi_k^{-1}(v) \neq \emptyset$ for some $v \in \Omega_k$ with $k \in S$ and $k \geq i$. Likewise $p^S(v)$ is in $P$, since $\pi_k^{-1}(v)$ shares the same member of $\mathcal{P}_k$ with $z = p^S(\pi_k(z))$. By the above mentioned homeomorphism $B$ is a good subset.

Fix $w_0 \in \Omega_0$. Next we assume that the adjacency distance between $p^S(w_0)$ and $p^T(w_0)$ within the Kripke Structure $V^{K(B)}(A)$ is $l < \infty$ for some pair $S, T \in 2^{\mathbb{N}_0}$ with $S$ and $T$ both containing $\{0\}$. Let $p^S(w_0) = z_0, z_1, \ldots, z_l = p^T(w_0)$ be a path of members of $A$ such that for every $0 \leq k \leq l - 1$ $z_k$ and $z_{k+1}$ share the same member of $Q^1$, $Q^2$, or $\mathcal{P}_\infty$, and for every $0 \leq k \leq l$ $z_k = p^S(v_k)$ for $S_k \in 2^{\mathbb{N}_0}$ for all $k$, $v_k \in \Omega_{n_k}$ and $n_k \in S_k$ (with $S_0 = S$, $S_l = T$, $n_0 = n_l = 0$, and $v_0 = v_l = w_0$.) Let $N = \max_{0 \leq k \leq l}(n_k)$. If $z_k$ and $z_{k+1}$ share the same member of $\mathcal{P}_\infty$ then by the definition of $\mathcal{P}$, we have that $S_k \setminus \{0, 1, \ldots, N - 1\} = S_{k+1} \setminus \{0, 1, \ldots, N - 1\}$. Now assume that $z_k$ and $z_{k+1}$ share the same member of $Q^1$ (respectively $Q^2$.) If $i \geq \max(n_k, n_{k+1})$ it is not possible for $i$ to be in $\mathbb{N}_0 \setminus S_k$ without $i$ being in $\mathbb{N}_0 \setminus S_{k+1}$ (and vice versa). If such an $i \in \mathbb{N}_0 \setminus S_k$ were in $S_{k+1}$ then $\pi_{i+1} \circ p^{S_{k+1}}(v_{k+1})$ would be a no-information extension and therefore $\pi_{i+1} \circ p^{S_k}(v_k)$ could not share the same member of $\mathcal{F}_{i+1}$ with it, given that $i \not\in S_k$. (We use that all members of $\Omega_i$ for all $i \geq 0$ are generative for both agents.) That suffices for $S_k \setminus \{0, 1, \ldots, N - 1\} = S_{k+1} \setminus \{0, 1, \ldots, N - 1\}$ for all $k$ and therefore that $S \setminus \{0, 1, \ldots, N - 1\} = T \setminus \{0, 1, \ldots, N - 1\}$. With $\sim$ defined on $2^{\mathbb{N}_0}$ as before, we see that $S \not\sim T$ implies that $p^S(w)$ and $p^T(w)$ cannot have a finite adjacency distance in the Kripke Structure $V^{K(B)}(A)$.

The above argument that $S \setminus \{0, 1, \ldots, N - 1\} = T \setminus \{0, 1, \ldots, N - 1\}$ works only because all the points concerned are alienated extensions. With respect
to the whole space $\Omega$ the Kripke Structure $\mathcal{K}(\mathcal{B})$ is connected and has an adjacency radius of 2 (see Simon 2001), meaning that there is a point such that all other points can be reached from this point by adjacency paths of length 2 or less!

6 References

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