GRADIENT ESTIMATES AND COMPARISON PRINCIPLE FOR SOME NONLINEAR ELLIPTIC EQUATIONS

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(Communicated by Bernd Kawohl)

Abstract. We consider a class of Dirichlet boundary problems for nonlinear elliptic equations with a first order term. We show how the summability of the gradient of a solution increases when the summability of the datum increases. We also prove comparison principle which gives in turn uniqueness results by strengthening the assumptions on the operators.

1. Introduction. Let us consider the class of the homogeneous Dirichlet problems

$$
\begin{cases}
-\text{div} (a(x, \nabla u)) = H(x, \nabla u) + f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. We assume that

$$
a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N
$$

and

$$
H : \Omega \times \mathbb{R}^N \to \mathbb{R}
$$

are Carathéodory functions which satisfy the ellipticity condition

$$
a(x, z) \cdot z \geq |z|^p,
$$

the monotonicity condition

$$
(a(x, z) - a(x, z')) \cdot (z - z') > 0, \quad z \neq z',
$$

and the growth conditions

$$
|a(x, z)| \leq a_0 |z|^{p-1} + a_1, \quad a_0, a_1 > 0,
$$

$$
|H(x, z)| \leq h |z|^q, \quad h > 0
$$

with $1 < p < N$, $p - 1 < q \leq p$, for almost every $x \in \mathbb{R}^N$, for every $z, z' \in \mathbb{R}^N$, and $f$ is in a suitable Lorentz space.

2000 Mathematics Subject Classification. Primary: 35J25; Secondary: 35J60.

Key words and phrases. Gradient estimates, comparison principle, uniqueness, nonlinear elliptic operators, first order terms.

The first and third authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INDAM).
Existence of solutions to problem (1) have been extensively studied in literature. Among them we quote only some contributes and refer to the references therein: [1, 4, 22, 23, 29]. Existence results have been proved under suitable assumptions on summability of the datum \( f \) and all these results require a smallness condition on its norm. Usually one has to distinguish three intervals for \( q \), i.e.

\[
p - 1 < q < \frac{N(p - 1)}{N - 1},
\]

\[
\frac{N(p - 1)}{N - 1} \leq q < p - 1 + \frac{p}{N},
\]

\[
p - 1 + \frac{p}{N} \leq q \leq p.
\]

Depending on these intervals, the notion of solution to problem (1) has to be specified (see Section 2). Indeed \( u \) is the standard weak solution when the datum \( f \) is an element of the dual space \( W^{-1,p'}(\Omega) \), as for example when \( q \) satisfies (8), but this notion does not fit the cases when \( q \) satisfies (6) or (7). In these cases we adopt the notion of “solution obtained as limit of approximations” ([17], see also [18]) which is based on a delicate procedure of passage to the limit. Other equivalent notion of solutions are available in literature, such as renormalized solution ([28, 30]) or entropy solution ([10]).

The purpose of this article is twofold: we consider solutions to (1) and we firstly study how the summability of the gradient of a solution increases when the summability of \( f \) increases. Then we prove comparison principles, and therefore uniqueness results, under more restrictive assumptions on the structure of the operator.

The study of the summability of the gradient of a solution to (1) is faced in Section 3 and the main results are stated in Theorems 3.6 - 3.7. We assume that the datum \( f \) belongs to Lorentz spaces, whose properties are recalled in Section 2. The main ingredients in proving such results are a pointwise estimate of the gradient (see Lemma 3.2), a priori estimates proved in [4] and an Hardy-type inequality (see Lemma 2.1).

As far as the uniqueness concerns we consider elliptic operators which satisfy further standard structural conditions. We change the monotonicity condition (3) in the following “strong monotonicity” condition

\[
(a(x, z) - a(x, z')) \cdot (z - z') \geq \alpha(\varepsilon + |z| + |z'|)^{p-2}|z - z'|^2,
\]

for some \( \alpha > 0 \), with \( \varepsilon \) nonnegative and strictly positive if \( p > 2 \). Moreover we assume the following locally Lipschitz condition on \( H \)

\[
|H(x, z) - H(x, z')| \leq \beta (\eta + |z| + |z'|)^{q-1}|z - z'|,
\]

where \( \beta > 0, \eta \) is nonnegative and strictly positive if \( 1 < p \leq 2 \).

In Section 4 we prove comparison principles, and in turn uniqueness results, for weak solutions or “solution obtained as limit of approximations” to problem (1) when \( p - 1 < q \leq \frac{p}{N} + p - 1 \); the continuous dependence on the data is proved in [15]. Actually classical counter-examples show that uniqueness of weak solutions fails when \( q > p - 1 + \frac{p}{N} \) (see e.g. [31]). In order to avoid technicalities we assume that the datum \( f \) belongs to some Lebesgue space (and not more to the Lorentz spaces as in Section 3). Our main results, Theorems 4.1, 4.3, 4.4, concern the case where \( 1 < p \leq 2 \). The first one gives uniqueness for weak solutions to (1) when \( q = p - 1 + \frac{p}{N} \), while Theorems 4.3, 4.4 give uniqueness for “solution obtained as limit of approximations” whose gradient has a suitable summability when
Finally, in Theorem 4.2 a uniqueness result is stated for “solution obtained as limit of approximations” when \( q \) satisfies (6). Actually this result is already proved (for renormalized solution) in [14] (see also [31]). Uniqueness results when \( p > 2 \) are contained in [15].

2. Preliminary results and notion of solutions. In this section we recall a few properties of Lorentz spaces and the notion of solutions for problem (1) which we use in the following.

Let us begin by introducing the definition of Lorentz spaces. If \( u \) is a measurable function on \( \Omega \) we denote by

\[
\mu(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geq 0,
\]

its distribution function and by

\[
u^*(s) = \sup \{t \geq 0 : \mu(t) > s\}, \quad s \in [0, |\Omega|),
\]

the decreasing rearrangement of \( u \) (see, e.g. [26]).

Lorentz spaces are rearrangements invariant spaces. Given \( r,t \in ]0,\infty[ \), the Lorentz space \( L^{r,t}(\Omega) \) is the set of all measurable functions on \( \Omega \) such that

\[
\|u\|^{*}_{r,t} = \left( \int_0^{|\Omega|} \left[ \nu^*(s) s^{\frac{1}{r}} \right]^t \frac{ds}{s} \right)^{\frac{1}{t}},
\]

if \( t \in ]0,\infty[ \),

\[
\|u\|^{*}_{r,\infty} = \sup_{s>0} \nu^*(s) s^{1/r},
\]

(11)

if \( t = \infty \).

These spaces give in some sense a refinement of the usual Lebesgue spaces. Indeed \( L^{r,r}(\Omega) = L^r(\Omega) \) for any \( r \geq 1 \) and \( L^{r,\infty}(\Omega) \) is the Marcinkiewicz space \( L^r \)-weak. Moreover the following embeddings hold (see [25])

\[
L^{r_1,t_1}(\Omega) \subset L^{r_2,t_2}(\Omega), \quad \text{when } r_1 > r_2, \quad 0 < t_1, t_2 \leq \infty,
\]

(12)

\[
L^{r_1,t_1}(\Omega) \subset L^{r_2,t_2}(\Omega), \quad \text{when } 0 < r_1 \leq \infty, \quad 0 < t_1 < t_2 \leq \infty,
\]

(13)

and

\[
\|u\|^{*}_{r_1,t_2} \leq \left( \frac{t_1}{r} \right)^{\frac{1}{r} - \frac{1}{t_2}} \|u\|^{*}_{r,t_1}.
\]

(14)

In general \( \| \cdot \|^{*}_{r,t} \) is not a norm. However it leads to a topology in \( L^{r,t}(\Omega) \) as follows

\[
u_n \rightarrow u \text{ in } L^{r,t}(\Omega) \iff \lim_{n \rightarrow \infty} \|\nu_n - u\|^{*}_{r,t} = 0.
\]

(15)

We can introduce a metric in the spaces \( L^{r,t}(\Omega) \) in the following way (see [25]). If \( \tau \in ]0,1[ \) with \( \tau < r \) and \( \tau \leq t \), we set

\[
\|u\|^{*}_{r,t} = \|\Pi\|^{*}_{r,t},
\]

where

\[
\Pi(s) = \frac{1}{s} \int_0^s \left[ \nu^*(\sigma) \right]^r d\sigma.
\]

Then

\[
u, v \in L^{r,t}(\Omega) \rightarrow d(u,v) = (\|u - v\|^{*}_{r,t})^r
\]

is a distance in \( L^{r,t}(\Omega) \).
By Hardy inequality (see, for example, [11], Chapter 2, Proposition 3.6), it results
\[ \| u \|_{r,t}^{*} \leq \| u \|_{r,t} \leq \left( \frac{r}{r - \tau} \right)^{1/\tau} \| u \|_{r,t}^{*}. \] (16)
So the convergence induced by the distance \( d \) is the same as (15). Finally these metric spaces are complete and when \( \tau = 1 \) they are Banach spaces.

We explicitly remark that our discussion about Lorentz spaces has mainly the goal to give a meaning to the notion of convergence in \( L(r,t) \).

We finally recall an Hardy-type inequality proved in [5].

\[ \text{Lemma 2.1. Let } f \text{ be a nonnegative decreasing function defined in } [0, +\infty[. \]
For \( \theta \geq 0 \) and \( \gamma \neq 1 \), denote
\[
F_{\gamma}(s) = \begin{cases} \int_{0}^{s} t^{\theta} f(t) dt, & \gamma < 1 \\ \int_{s}^{+\infty} t^{\theta} f(t) dt, & \gamma > 1 \end{cases}
\]
If \( \delta > 0 \), then
\[
\int_{0}^{+\infty} \left( \frac{F_{\gamma}(s)}{s} \right)^{\delta} s^{\delta} ds \leq c \int_{0}^{+\infty} f(s)^{\delta(\theta + \gamma)} \frac{ds}{s}
\] (17)
where \( c \) is a positive constant depending only on \( \theta, \gamma \) and \( \delta \).

Now let us explain what we mean for solution to problem (1). If the datum \( f \) belongs to the dual space \( W^{-1,p'}(\Omega) \), a solution to problem (1) is a standard weak solution. A function \( u \in W^{1,p}_0(\Omega) \) is a weak solution to (1) if
\[
\int_{\Omega} a(x, \nabla u) \nabla \phi = \int_{\Omega} H(x, \nabla u) \phi + \int_{\Omega} f \phi, \quad \forall \phi \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega).
\]
The definition of weak solution does not fit the case when \( p - 1 < q < p - 1 + \frac{N}{p} \), since in general the right-hand side of (1) is not more an element of the dual space \( W^{-1,p'}(\Omega) \). A different notion of solution has to be adopted and we refer to solutions obtained as a limit of weak solutions to approximated problems whose data are smooth enough, the so-called “solution obtained as limit of approximations”.

If \( f \) belongs to the Lebesgue space \( L^{m}(\Omega) \), for some \( m \geq 1 \), a measurable function \( u : \Omega \to \mathbb{R} \) is called “solution obtained as limit of approximations” to problem (1) (see [17], [18]) if
(i) for every \( k > 0 \), \( T_{k}(u) \in W^{1,p}_{0}(\Omega) \);
(ii) a sequence of functions \( f_{n} \in C_{0}^{\infty}(\Omega) \) exists such that
\[
f_{n} \rightarrow f \quad \text{strongly in } \ L^{m}(\Omega)
\] (18)
and a sequence of weak solutions \( u_{n} \in W^{1,p}_{0}(\Omega) \cap L^{\infty}(\Omega) \) to the approximated problems
\[
\begin{cases} -\text{div} (a(x, \nabla u_{n})) = T_{n}(H(x, \nabla u_{n})) + f_{n} & \text{in } \Omega \\ u_{n} = 0 & \text{on } \partial \Omega, \end{cases}
\] (19)
satisfies
\[
u_{n} \rightarrow u \quad \text{a.e. in } \Omega.
\] (20)
\[
\nabla u_{n} \rightarrow \nabla u \quad \text{a.e. in } \Omega,
\] (21)
Lemma 3.1. Let us suppose that $H$ order term been studied in [23]. Theorems 3.6 and 3.7 give results proved in [5] when the lower have, a.e. in $(0,1)$, we obtain the corresponding estimate for $|\nabla f|$ on the summability of the datum $f$. One can deduce the apriori estimates for the decreasing rearrangement of $|\nabla f|$ stated in Theorems 3.6 and 3.7 below. Their proofs are obtained in various steps.

A priori estimates for the gradient.

3. A priori estimates for the gradient. The main results of this section is stated in Theorems 3.6 and 3.7 below. Their proofs are obtained in various steps.

The first step consists in proving Lemma 3.2 below. It gives a pointwise estimate for the gradient of a solution to (19). Then we prove Propositions 3.3 - 3.5 below from which we can take for example $f_n = T_n(f)$.

We explicitly remark that the existence of a weak solution to (19) is assured by classical results (see [27]). Moreover the gradient $\nabla u$ is the generalized gradient of $u$ defined according to Lemma 2.1 in [10] which states the existence of a measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$\nabla T_k(u) = v\chi_{\{|u| \leq k\}} \text{ a.e. in } \Omega, \text{ for every } k > 0. \quad (23)$$

We define the gradient $\nabla u$ as this function $v$. The gradient defined in (23) is not the gradient used in the definition of Sobolev space, since it is possible that the function $u$ does not belong to $L^1_{\text{loc}}(\Omega)$ or $v$ does not belong to $(L^1_{\text{loc}}(\Omega))^N$. However, if $v$ belongs to $(L^1_{\text{loc}}(\Omega))^N$, then $u$ belongs to $W^1_{\text{loc}}(\Omega)$ and $v$ is the distributional gradient of $u$ (see [19]).

The same definition has a meaning if the datum $f$ belongs to a Lorentz space $L(m,k)$ with $k < \infty$, when we substitute the convergence in (18) with the convergence in Lorentz space recalled above. If $f$ belongs to Marcinkiewicz space $L(m,\infty)$, the definition of “solution obtained as limit of approximations” is adapted by taking into account approximated source terms $f_n \in L^\infty(\Omega) \cap W^{-1,p}(\Omega)$; one can take for example $f_n = T_n(f)$.

We begin by recalling the following result proved in [21, Lemma 4.1].

Lemma 3.1. Let us suppose that (2)-(5) hold true with $1 < p < N$ and $p - 1 < q \leq p$.

Let $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ be a weak solution to problem (1) with $f \in L^\infty(\Omega)$. We have, a.e. in $(0,|\Omega|)$,

$$\left( N\omega_N^{1/N} \right)^{-\frac{p+1}{p-1}} \left[ (-u^*)'(s) \right] \leq \left[ \int_s^\infty f^*(\sigma) \exp \left( h(N\omega_N^{1/N})^{q-p} \int_\sigma^\infty \frac{(-u^*)'(r)[q-p+1]}{r^{(q-p+1)/N}} \, dr \right) \, d\sigma \right]^{\frac{1}{p-1}}. \quad (24)$$
Now we prove a pointwise estimate for the gradient of weak solutions to problem (1) with bounded data. This pointwise estimate is not known in literature; the novelty is due to the presence of the first order term \( H \). Our result overlaps with the result in [5] when \( H = 0 \); further estimates for gradient when \( H = 0 \) with Neumann or Dirichlet boundary conditions are proved in [3] and [16] respectively (cf. [20] for local inequality in terms of nonlinear potentials).

**Lemma 3.2.** Let us suppose that (2)-(5) hold true with \( 1 < p < N \) and
\[
p - 1 < q \leq p.
\]
Let \( u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) be a weak solution to problem (1) with \( f \in L^\infty(\Omega) \). We have, a.e. in \((0,|\Omega|)\), for every \( \lambda > \frac{p}{N} - 1 \),
\[
\left( |\nabla u|^{s}(s) \right)^p \leq \frac{\lambda + 1}{(N\omega_N\frac{p}{N})^{p'}} \left\{ \frac{1}{s^{\lambda + 1}} \int_0^s r^{\lambda - \frac{q}{p'}} \left[ \int_0^r \psi(r,\sigma) f^*(\sigma) d\sigma \right]^{\frac{p}{p'}} dr + \right. \\
\left. \frac{1}{s^{\lambda + 1}} \int_0^1 \left[ \int_0^\sigma \psi(r,\sigma) f^*(\sigma) d\sigma \right]^{\frac{p}{p'}} dr \right\},
\]
where
\[
\psi(r,\sigma) = \exp \left( \frac{h}{(N\omega_N\frac{p}{N})^{p-q}} \int_\sigma^r \frac{\left| \frac{(-u^*)'(z)}{z^{\frac{p}{p-q}+1}} \right|^{q-p+1} dz \right)
\]
\[
(26)
\]
**Proof.** We just give a sketch of the proof, since it’s obtained by combining classical arguments used in [32] (see also the proof of Lemma 4.1 in [21]) and classical Hardy’s inequality.

Denote by
\[
\Phi(t) = \int_{|u| \leq t} |\nabla u|^p dx, \quad t \in (0, +\infty),
\]
\[
\Psi(s) = \Phi(u^*(s)), \quad s \in (0, |\Omega|),
\]
and consider the following test function in (1)
\[
\varphi(x) = \begin{cases} 
  k \text{ sign } u, & |u| > t + k \\
  (|u| - t) \text{ sign } u, & t < |u| \leq t + k \\
  0, & |u| \leq t
\end{cases}
\]
with \( k > 0 \) and \( 0 < t < \text{esssup} |u| \). Using (2), (5) and letting \( k \) go to \( 0 \), we have
\[
- \frac{d}{dt} \int_{|u| > t} |\nabla u|^p dx \leq h \int_{|u| > t} |\nabla u|^p dx + \int_{|u| > t} f dx.
\]
(27)
By coarea formula and Hölder inequality, since \( p - 1 < q \leq p \), we have
\[
\int_{|u| > t} |\nabla u|^q dx \leq \int_t^\infty \left( - \frac{d}{d\tau} \int_{|u| > \tau} |\nabla u|^p dx \right)^{\frac{q}{p}} \left( \frac{(-u^*)(\tau)}{\mu_u(\tau)^{q-p+1}} \right)^{\frac{p}{p-q}} d\tau.
\]
(28)
On the other hand, by coarea formula, isoperimetric inequality ([32]; see also the proof of Lemma 4.1 in [21]) and Hardy-Littlewood inequality, it follows
\[
- \frac{d}{dt} \int_{|u| > t} |\nabla u|^p dx \leq \frac{h}{(N\omega_N\frac{p}{N})^{p-q}} \int_t^{+\infty} \left( - \frac{d}{d\tau} \int_{|u| > \tau} |\nabla u|^p dx \right) \left( \frac{(-u^*)(\tau)}{\mu_u(\tau)^{q-p+1}} \right)^{\frac{p}{p-q}} d\tau + \int_0^{\mu_u(t)} f^*(s)ds.
\]
According to Gronwall’s Lemma, we deduce
\[
\Phi'(t) = -\frac{d}{dt} \int_{|u| > t} |\nabla u|^p \, dx \leq \int_t^{+\infty} f^*(\mu_u(\tau))(-\mu'_u(\tau)) \times \exp \left( \frac{h}{(N\omega_N^p)^{p-q}} \int_t^\tau \left( \frac{-\mu'_u(r)}{\mu_u(r)^{p-q}} \right) dr \right) \, d\tau.
\]  
(29)

Since \( \Psi'(s) = \Phi'(u^*(s))(u^*(s))' \) a.e \( s \in (0,|\Omega|) \), by (29) and Lemma 24, we obtain
\[
-\Psi'(s) \leq F(s)
\]  
(30)

where, for any \( s \in (0,|\Omega|) \), we denote
\[
F(s) = \frac{1}{N^q \omega_N^q} \left[ \int_0^s f^*(\sigma) \exp \left( \frac{h}{(N\omega_N^p)^{p-q}} \int_\sigma^s \frac{[(-u^*)'(z)]^{q-p+1}}{z^\frac{pq}{N^q}} \, dz \right) \, d\sigma \right]^{\frac{1}{p-1}}.
\]

Observe that
\[
\int_s^{[\Omega]} |\nabla u|^p (r)^p \, dr \leq \int_{|u| \leq u^*(s)} |\nabla u|^p \, dx = \Psi(s) = \int_s^{[\Omega]} -\Psi'(r) \, dr, \quad 0 < s < |\Omega|.
\]

By Hardy’s inequality (see, for example, [11, Chapter 2, Proposition 3.6]), we get
\[
\int_0^{[\Omega]} |\nabla u|^p (r)^p V(r) \, dr \leq \int_0^{[\Omega]} -\Psi'(r) V(r) \, dr,
\]  
(31)

for every non-decreasing function \( V : (0,|\Omega|) \to [0,+\infty) \). On the other hand (30) ensures that
\[
\int_0^{[\Omega]} -\Psi'(r) V(r) \, dr \leq \int_0^{[\Omega]} F(r) V(r) \, dr.
\]  
(32)

Therefore combining (31) and (32), we get
\[
\int_0^{[\Omega]} |\nabla u|^p (r)^p V(r) \, dr \leq \int_0^{[\Omega]} F(r) V(r) \, dr.
\]

For any \( s \in (0,|\Omega|) \), we deduce
\[
|\nabla u|^p (s) \int_0^s V(r) \, dr \leq \int_0^{[\Omega]} F(r) V(r) \, dr.
\]  
(33)

Inequality (25) follows by choosing
\[
V(r) = \begin{cases} 
    r^\lambda, & 0 \leq r \leq s \\
    s^\lambda, & s \leq r \leq |\Omega|.
\end{cases}
\]

Now we can prove some new regularity results for the gradients of solutions to approximated problems given by Propositions 3.3 - 3.5.

Our first result concerns the case where \( p - 1 < q < \frac{N(p-1)}{N-1} \). In the following we denote by \( t^* \), for \( 1 < t < N \), the Sobolev exponent of \( t \), i.e. \( t^* = \frac{Nt}{N-t} \).

**Proposition 3.3.** Let \( 1 < p < N \). Assume (2) - (5) with
\[
p - 1 < q < \frac{N(p - 1)}{N - 1}.
\]  
(34)
Proof.

Let $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ be a weak solution to (1) with $f \in L^\infty(\Omega)$ such that

$$
\|f\|_{L^1} < K_1,
$$

(35)

where

$$
K_1 = \left( \frac{N\omega_N^{1/N}}{|\Omega|} \right)^{\frac{q}{q-p+r}} \left( q - N(q-p+1) \right)^{\frac{p-1}{q-p+r}}.
$$

(36)

Then, for any $m, k$ such that

$$
1 < m < (p^*)', \ 0 < k \leq \infty,
$$

we have

$$
\|\nabla u\|_{m^*(p-1),k(p-1)} \leq C,
$$

(37)

where $C$ is a positive constant depending on $N$, $p$, $q$, $h$, $|\Omega|$, $m$, $k$ and on $\|f\|_{m,k}$. Moreover $C$ depends on $\|f\|_{m,k}$ in such a way that it is bounded when $f$ varies in sets which are bounded and equi-integrable in $L(m,k)$.

Remark 1. Under the assumptions of Proposition 3.3 we can apply a pointwise estimate proved in [4, Theorem 4.1] which says that, if $q$ satisfies (34) and $f \in L^\infty(\Omega)$ satisfies (35), then the following pointwise estimate for $(-u^*)(s)'$ holds true

$$
(-u^*)(s)' \leq \frac{1}{(N\omega_N^{1/N})^{\frac{1}{q-p+r}}} \frac{s^{-\frac{\rho(N-1)}{N(p-1)}}}{K_1} \left[ 1 - \left( \frac{\|f\|_{L^1}}{K_1} \right)^{\frac{q-p+1}{p-r}} \left( \frac{s^\frac{1}{(N-1)} \|f\|_{L^1}}{K_1} \right)^{\frac{q-p+1}{p-r}} \right],
$$

(38)

for every $s \in (0, |\Omega|]$.

Finally observe that the first index of summability of $|\nabla u|$ in (37) can be also less than 1. Indeed $m^*(p-1) < 1$ when $1 \leq m < \frac{N}{Np-N+1}$ and $1 < p \leq 2 - \frac{1}{N}$.

Proof. In order to prove (37) we apply Lemma 3.2. We begin by evaluating $\psi(r, \sigma)$ and we prove that

$$
\psi(r, \sigma) \leq c,
$$

(39)

where $\psi$ is defined by (26).

Here and in the following $c$ will denote a positive constant which can vary from line to line and depends only on the data of the problem but not on $\|f\|_{m,k}$.

To this aim we use the pointwise estimate (38). Since $q < \frac{N(p-1)}{N-1}$, we get

$$
\int_0^r \left[ (-u^*)(z) \right]^{q-p+1} \frac{dz}{z^{\frac{1}{N}+(p-q)}} \leq \frac{\|f\|_{L^1}^{q-p+1}}{(N\omega_N^{1/N})^{\frac{1}{p-r}}} \left( \frac{q-N(q-p+1)}{N(q-p+1)} \right)^{\frac{p-1}{q-p+r}} \left( \frac{z}{|\Omega|} \right)^{\frac{q-N(q-p+1)}{N(p-1)}}
$$

(40)

which implies (39). By using (25) and (39), we obtain

$$
(|\nabla u|^s(s))^p \leq \frac{c(s+1)}{(N\omega_N)^p} \left\{ \frac{1}{s^{\lambda+1}} \int_0^s r^{\lambda-p'(1-\frac{1}{s})} \left[ \int_0^r f^*(\sigma) d\sigma \right]^{\frac{p}{p-\sigma}} dr + \frac{1}{s} \int_0^{|\Omega|} \frac{1}{r^{p'(1-\frac{1}{s})}} \left[ \int_0^r f^*(\sigma) d\sigma \right]^{\frac{p}{p-\sigma}} dr \right\}.
$$

(41)
Now we proceed by distinguishing the case where \(0 < k < \infty\) and the case where \(k = \infty\).

Assume \(0 < k < \infty\). By (41) and (16), we get

\[
\|\nabla u\|_{m^\ast(p-1),k(p-1)}^{(p-1)} \leq c \int_0^{[\Omega]} (|\nabla u|^s)^{k(p-1)} s^{\frac{p}{s}} \frac{ds}{s} \\
\leq c \left\{ \int_0^{[\Omega]} \left( \frac{1}{s^{\lambda+1}} \int_0^s r^{\lambda+\frac{p}{s}} (\mathcal{F}(r))^{p'} dr \right) s^{\frac{\lambda}{s}} \frac{ds}{s} + \right.
\]
\[
\left. + \int_0^{[\Omega]} \left( \frac{1}{s} \int_s^{[\Omega]} r^{\lambda+\frac{p}{s}} (\mathcal{F}(r))^{p'} dr \right) s^{\frac{\lambda}{s}} \frac{ds}{s} \right\}.
\]

where \(\mathcal{F}(r) = \frac{1}{r} \int_0^r f^\ast(\sigma) d\sigma\). Now we apply Lemma 2.1 we have

\[
\int_0^{[\Omega]} \left( \frac{1}{s^{\lambda+1}} \int_0^s r^{\lambda+\frac{p}{s}} (\mathcal{F}(r))^{p'} dr \right) s^{\frac{\lambda}{s}} \frac{ds}{s} \leq c \int_0^{[\Omega]} (\mathcal{F}(s))^{k\frac{p}{s}} s^{\frac{\lambda}{s}} \frac{ds}{s} = c f_{m,k}^k,
\]
\[
\int_0^{[\Omega]} \left( \frac{1}{s} \int_s^{[\Omega]} r^{\lambda+\frac{p}{s}} (\mathcal{F}(r))^{p'} dr \right) s^{\frac{\lambda}{s}} \frac{ds}{s} \leq c \int_0^{[\Omega]} (\mathcal{F}(s))^{k\frac{p}{s}} s^{\frac{\lambda}{s}} \frac{ds}{s} = c f_{m,k}^k,
\]
which implies (37).

Finally assume \(k = \infty\). If \(f \in L(m, \infty)\), by (41) and (16), we get

\[
\left[ |\nabla u|^s(s) \right]^p \leq c \int_0^{[\Omega]} f_{m,\infty}^{p'} s^{\lambda+1} \int_0^s r^{\lambda-\frac{p'}{s} + (1-\frac{1}{p})p'} dr \\
+ c \int_0^{[\Omega]} f_{m,\infty}^{p'} s^{\lambda+1} \int_0^s r^{\lambda-\frac{p'}{s} + (1-\frac{1}{p})p'} dr \leq c f_{m,\infty}^{p'} s^{-\frac{p'}{s}}.
\]

This completes the proof.

\[\Box\]

**Remark 2.** If \(m = k = 1\), by Lemma 3.2 and (39), it is easy to verify that

\[
\left[ |\nabla u|^s(s) \right]^p \leq C s^{-\frac{p}{s}}
\]

which implies

\[
\|\nabla u\|_{N^{-1},\infty}^{N(p-1)} \leq c \sup_{s > 0} \|\nabla u\|^s(s) s^{\frac{N-1}{s}} \leq C.
\]

This a priori estimate overlaps with the well-known a priori estimate for \(|\nabla u|\) proved in [4] and in [23]. Moreover, observe that, since we assume \(m > 1\), it results \(m^\ast(p-1) > \frac{N(p-1)}{N-1}\), i.e. (37) says that the summability of the gradient increases when the summability of \(f\) increases.

Our second result concerns the case where \(\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}\).  

**Proposition 3.4.** Let \(1 < p < N\). Assume (2) - (5) with

\[
\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}.
\]

Let \(u \in W^1_0\Omega \cap L^{\infty}(\Omega)\) be a weak solution to (1) with \(f \in L^{\infty}(\Omega)\) such that

\[
\|f\|_{N(p-1)+1,\infty} < K_2,
\]

(43)
with
\[ K_2 = \frac{q - p + 1}{q} \left( \frac{\omega_1^{1/N} N(q - p + 1) - q}{q - p + 1} \right)^{\frac{q}{p+q}} \frac{p-1}{hq} \left( \frac{p - 1}{h^q} \right)^{\frac{p-1}{p+q}}. \] (44)

Then, for any \( m, k \) such that
\[ \frac{N(q - p + 1)}{q} < m < \left( p^* \right)', \quad 0 < k \leq \infty, \] (45)
we have
\[ \| \nabla u \|_{m^*(p-1), k(p-1)} \leq C, \] (46)
where \( C \) is a positive constant depending on \( N, p, q, h, m, k, |\Omega| \) and \( \| f \|_{m,k} \).

Moreover \( C \) depends on \( \| f \|_{m,k} \) in such a way that it is bounded when \( f \) varies in sets which are bounded and equi-integrable in \( L(m, k) \).

**Remark 3.** In order to prove Proposition 3.4 we need to apply a pointwise estimate proved in [4, Theorem 4.2] which says that, if \( q \) satisfies (42) and \( f \in L^\infty(\Omega) \) satisfies the sharp smallness assumption (43), the following pointwise estimate for \((-u^*(s))'\) holds true
\[ (-u^*(s))' \leq \frac{Y_0^{\frac{1}{p-1}}}{(N\omega_1^{1/N})^{\frac{p-1}{p}} - \frac{q}{(N\omega_1^{1/N})^{\frac{p}{p+q}} + 1}} s^{-1 - \frac{p-q}{N(N-1)}} \log \left( \frac{r}{\sigma} \right) = C_0 \log \left( \frac{r}{\sigma} \right) \] (47)
where \( Y_0 \geq 0 \) is the smallest nonnegative solution to the equation
\[ \frac{h}{(N\omega_1^{1/N})^{\frac{p}{p-1}}} Y^{\frac{p}{p-1}} - \left( 1 - \frac{q}{N(q - p + 1)} \right) Y + \| f \|_{\frac{N(q - p + 1)}{q}, \infty} = 0. \] (48)

**Proof.** In order to prove (46) we apply Lemma 3.2. We begin by evaluating \( \psi(r, \sigma) \).

Since \( f \in L^\infty(\Omega) \) and we assume (43), the pointwise estimate (47) holds true (see Remark 3 above). Therefore, we have
\[ \frac{h}{(N\omega_1^{1/N})^{\frac{p}{p-1}}} \int_{\sigma}^{r} \frac{(-u^*)(z)(z)^{q-p+1}}{z^{\frac{p-q}{N(N-1)}}} dz \leq \frac{hY_0^{\frac{1}{p-1}}}{(N\omega_1^{1/N})^{\frac{p}{p-1} + p-q + 1}} \log \left( \frac{r}{\sigma} \right) = C_0 \log \left( \frac{r}{\sigma} \right) \] (49)
where
\[ C_0 = \frac{hY_0^{\frac{1}{p-1}}}{(N\omega_1^{1/N})^{\frac{p}{p-1}}}. \]

Now we proceed by distinguishing the case where \( 0 < k < \infty \) and the case where \( k = \infty \).

**The case** \( 0 < k < \infty \). We prove that under these bounds on \( k \), we have
\[ \psi(r, \sigma) \leq c, \] (50)
where \( \psi \) is defined by (26).
Let us begin by assuming \( k > 1 \). By Hölder inequality, inequalities (16) and estimate (49), we get

\[
\int_0^r f^*(\sigma) \exp \left( \frac{h}{(N\omega_N)^{p-q}} \int_0^r \frac{[(-u^*)'(z)]^{q-p+1}}{z^{\frac{m}{N}} \bar{\sigma}} \, dz \right) \, d\sigma
\]

\[
\leq \|f\|_{m,k} \left( \int_0^r \sigma^{(\frac{1}{q} - \frac{C_0}{m})} \, d\sigma \right)^{1 - \frac{1}{q}}
\]  \hspace{1cm} (51)

\[
\leq \|f\|_{m,k} \left( \int_0^r \left( \frac{r}{\sigma} \right) \sigma^{(\frac{1}{q} - \frac{C_0}{m})} \, d\sigma \right)^{1 - \frac{1}{q}}
\]

Observe that the last integral in (52) is finite if

\[
C_0 < 1 - \frac{1}{m}.
\]  \hspace{1cm} (52)

Now if \( Y_0 = 0 \), then \( C_0 = 0 \) and (50) is obviously satisfied. Assume that \( Y_0 > 0 \). Then, by definition of \( C_0 \) and \( Y_0 \) (see Remark 3 above), (52) is satisfied if

\[
1 - \frac{q}{N(q - p + 1)} - \frac{\|f\|_{\frac{N(q-p+1)}{q},\infty}}{Y_0} < 1 - \frac{1}{m}.
\]

This last condition holds true since we assume \( m > \frac{N(q-p+1)}{q} \) which implies

\[
1 - \frac{q}{N(q - p + 1)} < \frac{\|f\|_{\frac{N(q-p+1)}{q},\infty}}{Y_0}.
\]

Finally by (51), we deduce

\[
\int_0^r f^*(\sigma) \exp \left( \frac{h}{(N\omega_N)^{p-q}} \int_0^r \frac{[(-u^*)'(z)]^{q-p+1}}{z^{\frac{m}{N}} \bar{\sigma}} \, dz \right) \, d\sigma \leq C \|f\|_{m,k} r^{1 - \frac{1}{q}}.
\]  \hspace{1cm} (53)

Therefore, when \( 1 < k < \infty \), estimate (53) and Lemma 24 give

\[
|(-u^*)'(s)| \leq C \|f\|_{m,k} s^{1 - \frac{1}{q} - \frac{1}{p-1} + \frac{1}{p^*}}.
\]  \hspace{1cm} (54)

This inequality holds true also when \( 0 < k < 1 \). Indeed, if \( f \in L(m,k) \) with \( 0 < k < 1 \), then by (13), (14) and (16), \( f \in L(m,k+1) \) and

\[
\|f\|_{m,k+1} \leq \frac{m}{m - 1} \left( \frac{m}{m} \right)^{\frac{1}{p+1}} \|f\|_{m,k}.
\]  \hspace{1cm} (55)

Therefore we can firstly apply (54) with \( k \) replaced by \( k + 1 \), and then we use (55). This yields (54) when \( 0 < k < 1 \). Then, (54) holds true for any \( 0 < k < \infty \).

Now we are able to evaluate \( \psi(r,\sigma) \). By definition of \( \psi(r,\sigma) \), since \( m > \frac{N(q-p+1)}{q} \), we deduce

\[
\psi(r,\sigma) = \exp \left( \frac{h}{(N\omega_N)^{p-q}} \int_0^r \frac{[(-u^*)'(z)]^{q-p+1}}{z^{\frac{m}{N}} \bar{\sigma}} \, dz \right)
\]

\[
\leq \exp \left( C \|f\|_{m,k} r^{1 - \frac{1}{q} - \frac{1}{p-1} + \frac{1}{p^*}} \right)
\]

\[
\leq \exp \left( c \|f\|_{m,k} \int_0^r z^{1 - \frac{1}{m} (\frac{1}{p-1} + \frac{1}{p^*})} \, dz \right)
\]

that is (50) is proved. Now the proof proceeds as the previous one, starting from (41), to arrive to (46).
The case \( k = \infty \). If \( f \in L(m, \infty) \), by (49) and (16), we get
\[
\int_0^r f^*(\sigma) \exp \left( \frac{h}{(N \omega_N^{1/2})^{p-q}} \int_\sigma^r \frac{|(-u^*)'(z)|^{q-1}}{(z^{(p-q)(N-1)-1})^N} \, dz \right) \, d\sigma 
\leq \|f\|_{m, \infty} \int_0^r \left( \frac{r}{\sigma} \right)^{C_0} \sigma^{-\frac{1}{m}} \, d\sigma = \|f\|_{m, \infty} r^{1-\frac{1}{m}}
\]
Therefore by using Lemma 3.2, we deduce
\[
|\nabla u^*|^p(s) \leq C r^{-\frac{1}{m(p-1)}}.
\]
This completes the proof. \( \square \)

**Remark 4.** Let us make some remarks on the case \( m = \frac{N(q-p+1)}{q} \). We explicitly observe that if \( m = \frac{N(q-p+1)}{q} \) and \( k = \infty \), Lemma 3.2 and (49) allows to prove an a priori estimate for \(|\nabla u|\) in \( L(t, p) \), with \( t < N(q - p + 1) \), which overlaps with the result proved in [4, Theorem 5.3]. Analogously if \( m = \frac{N(q-p+1)}{q} \), \( k < \infty \) and \( f \) has norm in \( L(N(q-p+1), k) \) small enough we get also an a priori estimate of \(|\nabla u|\) in \( L(t, k(p-1)) \) with \( t < N(q - p + 1) \).

Our third result concerns the limit case \( q = \frac{N(p-1)}{N-1} \).

**Proposition 3.5.** Let \( 1 < p < N \). Assume (2) - (5) with
\[
q = \frac{N(p-1)}{N-1}.
\]
Let \( u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \) be a weak solution to (1) with \( f \in L^\infty(\Omega) \) such that for a constant \( M > |\Omega| \), satisfies
\[
\int_0^{[\Omega]} f^*(s) \log^{N-1}(M/s) \, ds < K_3,
\]
with
\[
K_3 = \frac{\omega N(N-1)^{N-1} N^N}{h^{N-1}}.
\]
Then for any \( m, k \) such that
\[
1 < m < (p^*)', \quad 0 < k \leq \infty,
\]
we have
\[
\|\nabla u\|_{m^*(p-1), k(p-1)} \leq C,
\]
where \( C \) is a positive constant depending on \( N, p, q, h, m, k, |\Omega| \) and \( \|f\|_{m,k} \).
Moreover \( C \) depends on \( \|f\|_{m,k} \) in such a way that it is bounded when \( f \) varies in sets which are bounded and equi-integrable in \( L(m,k) \).

**Remark 5.** In order to prove Proposition 3.5 we need to apply a pointwise estimate proved in [4, Theorem 4.3] which says that, if \( q \) satisfies (57) and \( f \in L^\infty(\Omega) \) satisfies (58), then the following pointwise estimate for \((-u^*(s))'\) holds true
\[
(-u^*(s))' \leq \frac{X_0^{p-1}}{(N \omega_N^{1/2})^{p-1}} \frac{1}{s^{N(p-1)-1} \log^{N-1}(M/s)} \frac{h}{s^{N-1}} Z^{N-1} - (N-1)Z = 0,
\]
where \( X_0 \) is the smallest positive solution to the equation
\[
\frac{h}{(N \omega_N^{1/2})^{p-1}} Z^{N-1} - (N-1)Z = 0.
\]
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i.e.
\[ X_0 = N^N \omega_N \left( \frac{N - 1}{h} \right)^{N-1}. \]  

**Proof.** As in the previous proofs we get the assert by evaluating \( \psi(r, \sigma) \). Since \( f \in L^\infty(\Omega) \) and it satisfies (58), the pointwise estimate (62) holds true (see Remark 5 above). Therefore we have
\[ f^* (\sigma) \exp \left( \frac{h}{(N \omega_N)^{p-q}} \int_0^r \frac{[(u^*)'(z)]^{q-p+1}}{z^{\frac{p-q}{N}}} \, dz \right) \]
\[ \leq \frac{h X_0^{\frac{1}{N}}} {(N \omega_N^{\frac{1}{N}})^{\frac{N-1}{N}}} \int_0^r \frac{1}{z \log \left( \frac{M}{z} \right)} \, dz = C_1 \log \left( \frac{r \log \frac{M}{r}} {1 + \frac{r \log \frac{M}{r}} {1}} \right), \] where, by (63),
\[ C_1 = \frac{h X_0^{\frac{1}{N}}} {(N \omega_N^{\frac{1}{N}})^{\frac{N-1}{N}}} = N - 1. \]

Now we proceed by distinguishing the case where \( 0 < k < \infty \) and the case where \( k = \infty \).

**The case \( 0 < k < \infty \).** We prove that under these bounds on \( k \), we have \( \psi(r, \sigma) \leq c \), (65) where \( \psi \) is defined by (26).

Let us begin by assuming \( k > 1 \). By Hölder inequality, inequalities (16) and estimate (64), we get
\[ \int_0^r f^* (\sigma) \exp \left( \frac{h}{(N \omega_N)^{p-q}} \int_0^r \frac{[(u^*)'(z)]^{q-p+1}}{z^{\frac{p-q}{N}}} \, dz \right) \, d\sigma \]
\[ \leq \|f\|_{m,k} \left( \int_0^r \sigma^{\frac{k}{r} - \frac{1}{r} + \frac{1}{kr'}} \exp \left( \frac{hk'}{(N \omega_N)^{p-q}} \int_0^r \frac{[(u^*)'(z)]^{q-p+1}}{z^{\frac{p-q}{N}}} \, dz \right) d\sigma \right)^{1 - \frac{k}{r'}} \]
\[ \leq \|f\|_{m,k} \left( \int_0^r \frac{1}{\log \left( \frac{M}{\sigma} \right)} \sigma^{\frac{k}{r} - \frac{1}{r} + \frac{1}{kr'}} \, d\sigma \right)^{1 - \frac{k}{r'}}. \] (66)

Moreover it is easy to verify that the following estimate holds
\[ \int_0^r \sigma^{\frac{k}{r} - \frac{1}{r} + \frac{1}{kr'}} \, d\sigma \leq c \left[ \log \left( \frac{M}{r} \right) \right]^{(N-1)k'} \sigma^{\frac{k}{r} - \frac{1}{r} + \frac{1}{kr'}}. \] (67)

Combining (66) and (67), we get
\[ \int_0^r f^* (\sigma) \exp \left( \frac{h}{(N \omega_N)^{p-q}} \int_0^r \frac{[(u^*)'(z)]^{q-p+1}}{z^{\frac{p-q}{N}}} \, dz \right) \, d\sigma \leq c \|f\|_{m,k} r^{1 - \frac{1}{r}}, \] (68)
which coincides with (53) of the proof of Proposition 3.4. Therefore (54) holds true for \( 0 < k < \infty \) and the conclusion follows as in the previous proof.
The case $k = \infty$. If $f \in L(m, \infty)$, by (64) and (16), we get
\[
\int_0^r f^*(\sigma) \exp \left( \frac{h}{(N \omega^h_N)^{p-q}} \int_\sigma^r \left[ (u^*)'(z) \right]^{q-p+1} \frac{dz}{z^\frac{h}{N}} \right) d\sigma
\leq \frac{\|f\|_{m, \infty}}{\log \left( \frac{M}{\sigma} \right)^{N-1}} \int_0^r \left[ \log \left( \frac{M}{\sigma} \right) \right]^{N-1} \sigma^{-\frac{1}{p-1}} d\sigma \leq \|f\|_{m, \infty} r^{1-\frac{1}{p}}.
\]
Therefore by Lemma 3.2 we deduce
\[
\|\nabla u\|^{s^*}(s)^p \leq C r^{1-\frac{1}{p}}.
\]
This completes the proof.

Remark 6. Let us observe that our pointwise estimate of the gradient and (64) allow to prove in a different way the a priori estimate for the gradient of $u$ proved in [4, Theorem 5.4], i.e.
\[
\int_0^{[\Omega]} (|\nabla u|^*)^q \log^7 (M/s) ds < c,
\]
for any $0 < \tau < N-1$, where $c$ is a constant which depends on $f$ by the integral in (58).

The main results of this section are stated in Theorems 3.6 - 3.7 which say as the summability of the gradient of a “solution obtained as limit of approximations” to problem (1) varies with the summability of the datum.

**Theorem 3.6.** Let $1 < p < N$. Assume (2) - (5) with
\[
p - 1 < q < p - 1 + \frac{p}{N}
\]
and $f \in L(m, k)$ with
\[
\max \left\{ 1, \frac{N(q-p+1)}{q} \right\} < m < (p^*)', \quad 0 \leq k < \infty.
\]
Moreover assume that either
(i) if $p - 1 < q < \frac{N(p-1)}{N-1}$, then $f$ satisfies (35),
or
(ii) if $\frac{N(p-1)}{N-1} < q < p - 1 + \frac{p}{N}$, then $f$ satisfies (43).

Let $u$ be a “solution obtained as limit of approximations” to (1). Then
\[
\|\nabla u\|_{m^*, (p-1), k(p-1)} \leq C,
\]
where $C$ is a positive constant depending on $N, p, q, h, m, k, [\Omega]$ and $\|f\|_{m, k}$.
Moreover $C$ depends on $\|f\|_{m, k}$ in such a way that it is bounded when $f$ varies in sets which are bounded and equi-integrable in $L(m, k)$.

**Theorem 3.7.** Let $1 < p < N$. Assume (2) - (5) with
\[
q = \frac{N(p-1)}{N-1}.
\]
Moreover assume that $f \in L(m, k)$ with
\[
1 < m < (p^*)', \quad 1 \leq k < \infty.
and
\[ \|f\|_{m,k} < |\Omega|^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \frac{1}{s} \log\left(\frac{M}{|\Omega|}\right)^{\frac{k(N-1)}{k-1}}, \]
where \(K_3\) is defined by (59). Let \(u\) be a “solution obtained as limit of approximations” to (1). Then (69) holds true.

**Remark 7.** The sharp smallness assumptions (35), (43) and (58) on the datum \(f\) assure the existence of a “solution obtained as limit of approximations” as proved in Theorems 5.2, 5.5, 5.6 respectively in [4]. Moreover observe that, when \(q = \frac{N(p-1)}{N-1}\), we assume the smallness condition (70) (and not (58)), but (70) implies (58). In fact, by using Hölder inequality and (16), we get
\[ \int_{|\Omega|} f^*(s) \log^{N-1}(M/s) ds \leq \|f\|_{m,k,|\Omega|} \left(\frac{1}{p} - \frac{1}{q}\right) \frac{1}{s} \log\left(\frac{M}{|\Omega|}\right)^{\frac{k(N-1)}{k-1}}, \]
and therefore, by (70), (58) holds.

**Proof of Theorem 3.6.** Assume that (i) holds true, for \(k < \infty\). Since \(u\) is a “solution obtained as limit of approximations”, we can consider a sequence \((u_n)_n\) of weak solutions to the approximated problem (19) with \(f_n \in C_0^\infty(\Omega)\) such that
\[ f_n \to f \quad \text{in } L(m,k), \quad \text{and } |\nabla u_n| \to |\nabla u| \quad \text{a.e. in } \Omega, \]
according to the definition given in Section 2. Since \(L(m,k) \subset L^1(\Omega)\) for \(m > 1\), we get \(f_n \to f\) in \(L^1(\Omega)\), then \(\|f_n\|_1 \to \|f\|_1\). Therefore, since \(f\) satisfies (35), for \(n\) large enough, we have
\[ \|f_n\|_{L^1} < K_1. \]
Therefore by Proposition 3.3, we obtain
\[ \|\nabla u_n\|_{m^*(p-1),k(p-1)} \leq C, \]
where \(C\) depends on \(\|f_n\|_{m,k}\) in such a way that it is bounded when \(f_n\) vary in sets which are bounded and equi-integrable in \(L(m,k)\). Since
\[ |\nabla u_n|^*(s) \to |\nabla u|^*(s) \quad \text{a.e. in } (0,|\Omega|), \]
via Fatou Lemma, we get
\[ \|\nabla u\|_{m^*(p-1),k(p-1)} \leq \liminf_{n \to \infty} \|\nabla u_n\|_{m^*(p-1),k(p-1)} \]
which, together with (72) and the fact that \(\|f_n\|_{m,k} \leq c\|f\|_{m,k}\), gives (69) for \(0 < k < \infty\). Finally we obtain (69) for \(k = \infty\) in analogous way.

Assume now that (ii) holds for \(k < \infty\). Since \(u\) is a “solution obtained as limit of approximations”, we can consider a sequence \((u_n)_n\) of weak solutions to the approximated problem (19) with \(f_n \in C_0^\infty(\Omega)\) such that (71) hold true. Since \(L(m,k) \subset L(\frac{N(q-p+1)}{q},\infty)\) with \(m > \frac{N(q-p+1)}{q}\), we get \(f_n \to f\) in \(L(\frac{N(q-p+1)}{q},\infty)\), then \(\|f_n\|_{N(q-p+1),q,\infty} \to \|f\|_{N(q-p+1),q,\infty}\). Therefore, since \(f\) satisfies (43), for \(n\) large enough, we have
\[ \|f_n\|_{N(q-p+1),q,\infty} < K_2. \]
Therefore by Proposition 3.4, we obtain \(\|\nabla u_n\|_{m^*(p-1),k(p-1)} \leq C\), where \(C\) depends on \(\|f_n\|_{m,k}\) in such a way that it is bounded when \(f_n\) vary in sets which are bounded and equi-integrable in \(L(m,k)\). Then we get the conclusion arguing as in the previous case.
Proof of Theorem 3.7. Assume that $k < \infty$. As in the previous proof, we can consider a sequence $(u_n)_n$ of weak solutions to the approximated problem (19) with $f_n \in C_0^\infty(\Omega)$,

$$f_n \to f \quad \text{ in } L(m,k), \quad \text{and} \quad |\nabla u_n| \to |\nabla u| \quad \text{a.e. in } \Omega.$$  

Then $\|f_n\|_{m,k} \to \|f\|_{m,k}$ and, since $f$ satisfies (70), for $n$ large enough, we have

$$\|f_n\|_{m,k} < |\Omega|^{- \left(\frac{1}{q} - \frac{1}{p}\right)} \log \left(\frac{M}{|\Omega|}\right) - M^{- \frac{k(N-1)}{N+1}} K_0.$$

Therefore by Proposition 3.5, we obtain $\|\nabla u_n\|_{m^*(p-1),k(p-1)} \leq C$, where $C$ depends on $\|f_n\|_{m,k}$ in such a way that it is bounded when $f_n$ vary in sets which are bounded and equi-integrable in $L(m,k)$. The conclusion follows arguing as in the previous proof.

Remark 8. Let us make some remarks on the bounds of $q$ and $m$. Observe that in Proposition 3.4 and in Theorem 3.6 we assume $q < p - 1 + \frac{N}{N-1}$. This is due to the fact that when $p - 1 + \frac{N}{N-1} \leq q \leq p$, weak solutions to problem (1) have to be considered (see Section 2) and therefore it is not natural to ask larger summability for example, [11], Chapter 2, Proposition 3.6) and integration on $(s, |\Omega|)$. As far as the bounds on $m$ concerns, if $m = (p^*)'$, i.e. $f \in L((p^*)', k)$, Propositions 3.3 - 3.5, and therefore Theorems 3.6 - 3.7, can not be proved since we can not apply Lemma 2.1. However if we assume that $f \in L((p^*)', p')$, under the assumptions of Propositions 3.3 - 3.5, one can prove that $|\nabla u| \in L^p(\Omega)$ by using (25) and the fact $\psi(r,\sigma) \leq c$. Moreover for $m = (p^*)'$, Propositions 3.3 - 3.5, and therefore Thorems 3.6 - 3.7, are not sharp since we can not obtain gradient estimates depending on $k$. Indeed if we assume $f \in L((p^*)', k)$ with $0 < k \leq p'$ is just possible to prove that $|\nabla u| \in L^p(\Omega)$ (and not in $L(p, k(p-1))$ as in the case $m < (p^*)'$).

Finally when $m = \max \left\{1, \frac{N(p-1)}{q-1} \right\}$, by Remarks 2, 4 e 6 we can prove estimates of $|\nabla u|$ which overlaps with the estimates proved in [4] for the existence results.

Remark 9. A remark on the summability of $u$ is in order. Under the assumptions of Theorem 3.6, we have

$$\|u\|_{L^\infty} \leq C, \quad \text{if } m > \frac{N}{p}$$

$$u^*(s) \leq C \log \left(\frac{|\Omega|}{s}\right), \quad \text{if } m = \frac{N}{p}$$

$$\|u\|_{m\frac{N(p-1)}{N-mp}, k(p-1)} \leq C, \quad \text{if } m < \frac{N}{p}.$$  

When $\frac{N(p-1)}{N-1} \leq q < p$ this is an easy consequence of (54), Hardy inequality (see, for example, [11], Chapter 2, Proposition 3.6) and integration on $(s, |\Omega|)$. When $q < \frac{N(p-1)}{N-1}$, it is easy to prove firstly the analogous of (54) and then to conclude by integration on $(s, |\Omega|)$. Analogous estimates for $u$ have been proved in [23] for data in Lebesgue spaces.

We explicitly observe that Theorems 3.6 - 3.7 and this Remark hold true also for operators $a$ which depends on $u$ and satisfy usual growth conditions on $u$. 

4. Comparison principles. In this section we present comparison principles which in turn imply uniqueness of a solution to problem (1), under the assumptions (9) and (10). In order to avoid technicalities we assume that the datum $f$ belongs to Lebesgue spaces (and not more to Lorentz spaces). In this section we assume $p - 1 < q \leq p - 1 + \frac{p}{N}$. (73)

Indeed, as pointed out in Introduction a well-known example shows that uniqueness of a weak solution to (1) does not hold if $q > p - 1 + \frac{p}{N}$.

Uniqueness results for weak solutions when $q = p - 1$ are well-known (cf. [2, 8, 12]), while the uniqueness of renormalized solution or "solution obtained as limit of approximations" are proved, for example, in [6, 7, 9, 13, 14, 24, 31].

In the following we distinguish the comparison principle for weak solutions from comparison principle for "solution obtained as limit of approximations".

4.1. Comparison principle for weak solutions. As pointed out we have to consider only the case where

$q = p - 1 + \frac{p}{N}$. (74)

Let us denote

$Qu \equiv -\text{div} \left( a(x, \nabla u) \right) - H(x, \nabla u), \quad u \in W^{1,p}_0(\Omega).$ (75)

By comparison principle we mean that if $u, v$ are weak solutions to the following Dirichlet problems respectively

$Qu = f$ in $\Omega, \quad u = 0$ on $\partial \Omega$ (76)

$Qv = g$ in $\Omega, \quad v = 0$ on $\partial \Omega$ (77)

with $f, g \in L^{(p^*)'}(\Omega)$ and

$f \leq g$ in $D'(\Omega)$, (78)

then

$u \leq v$ a.e. in $\Omega$. (79)

Observe that (75), (76), (77) and definition of weak solution recalled in Section 2, imply

$\int_{\Omega} [a(x, \nabla u) - a(x, \nabla v)] \cdot \nabla \varphi \, dx - \int_{\Omega} [H(x, \nabla u) - H(x, \nabla v)] \varphi \, dx \leq 0$ (80)

for all nonnegative $\varphi \in W^{1,p}_0(\Omega)$.

We begin by proving the following result

**Theorem 4.1.** Let $N \geq 2$ and $p$ such that

$$\begin{cases}
2N \leq p < 2, & \text{if } N = 2 \\
2N \leq p \leq 2, & \text{if } N \geq 3.
\end{cases}$$ (81)

Assume (2), (4), (5), (9) and (10) with

$q = p - 1 + \frac{p}{N}$. (82)

Denote $u, v \in W^{1,p}_0(\Omega)$ weak solutions to the problems (75), (76) respectively with $f, g \in L^{(p^*)'}(\Omega)$ which satisfy (77) and

$\|f\|_{(p^*)', \infty} < K_2, \quad \|g\|_{(p^*)', \infty} < K_2$, (83)
with $K_2$ given by (44). Then (78) holds. In particular problem (75) has an unique weak solution.

**Remark 10.** We explicitly observe that $\frac{N(p-2+1)}{q} = (p^*)'$. In order to assure the existence of a weak solution, we assume that $f \in L^{(p^*)'}(\Omega)$ and it satisfies the sharp assumption (43). Observe that, by (14), since $f \in L^{(p^*)'}(\Omega)$, then $f \in L^{(p^*)',\infty}$.

Moreover the sharp smallness assumptions on the data (82) are made just to assure the existence of a weak solution. Our result improves the uniqueness result proved in [31], since we have uniqueness of weak solutions for a larger interval of values of $p$.

**Proof.** Let us denote

$$w = (u - v)_+$$

and

$$D = \{x \in \Omega : w(x) > 0\}.$$

Assume that $D$ has positive measure. Let us fix $t \in [0, \sup w].$ We denote

$$w_t = \begin{cases} w - t & \text{if } w > t \\ 0 & \text{otherwise} \end{cases}$$

and

$$E_t = \{x \in D : w(x) > t\}.$$

Using $w_t$ as test function in (79) we obtain

$$\int_{E_t} [a(x, \nabla u) - a(x, \nabla v)] \nabla w \, dx \leq \int_{E_t} [H(x, \nabla u) - H(x, \nabla v)] w_t \, dx. \quad (83)$$

By assumptions (9) and (10), we have

$$\alpha \int_{E_t} \frac{|
abla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \leq \beta \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{p-2+\frac{2}{N}} |\nabla w_t| w_t \, dx. \quad (84)$$

Let us estimate the integral on the right-hand side of (84) by using Hölder inequality and Sobolev inequality. Since $p \geq \frac{2N}{N+2}$, we obtain

$$\int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{p-2+\frac{2}{N}} |\nabla w_t| w_t \, dx \leq \left( \int_{E_t} \frac{\nabla w_t^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \times \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{1/2} \times \left( \int_{E_t} w_t^{p^*} \, dx \right)^{1/2} \leq C_p \left( \int_{E_t} \frac{\nabla w_t^2}{(|\nabla u| + |\nabla v|)^{2-p}} \, dx \right)^{1/2} \times \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \, dx \right)^{1/2} \times \left( \int_{E_t} w_t^p \, dx \right)^{1/2}, \quad (85)$$
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where \( C_p \) is the best constant in Sobolev embedding \( W^{1,p}_0(\Omega) \subset L^{p^*}(\Omega) \). From (84) and (85) we get

\[
\left( \int_{E_t} \left( \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right) \right)^{\frac{1}{2}} \leq C_p^{\frac{\beta}{\alpha}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \right)^{\frac{p(N+2)-2N}{2Np}} \left( \int_{E_t} |\nabla w_t|^{p} dx \right)^{\frac{1}{p}}.
\]

On the other hand, since \( p \leq 2 \), if \( N \geq 3 \) or \( p < 2 \) if \( N = 2 \), Hölder inequality gives

\[
\int_{E_t} |\nabla w_t|^p dx \leq \int_{E_t} \frac{|\nabla w_t|^p}{(|\nabla u| + |\nabla v|)^{(2-p)/2}} (\eta + |\nabla u| + |\nabla v|)^{(2-p)/2} dx
\]

\[
\leq \left( \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^{\frac{p}{2}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \right)^{\frac{2-p}{2}}.
\]

Then, by using (86) we obtain

\[
1 \leq \frac{\beta C_p}{\alpha} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^p \right)^{\frac{p(N+2)-2N}{2Np} + \frac{2-p}{p}}.
\]

Letting \( t \to \sup w \) the left-hand side goes to zero; this gives a contradiction. Therefore we conclude that \( |D| = 0 \) and we get the assert. \( \Box \)

Remark 11. We explicitly observe that we can not prove a comparison result when \( p > 2 \) and \( q \) satisfies (81) since our approach would require a larger summability of \( |\nabla u| \) which is not natural for a weak solution \( u \). The same occurs also in [31].

4.2. Comparison principle for “solution obtained as limit of approximations”. A comparison principle, and uniqueness result for (1), holds true also for the range of \( q \)

\[ p - 1 < q < p - 1 + \frac{p}{N}. \]

In this case by comparison principle we mean that if \( u, v \) are “solution obtained as limit of approximations” to the following Dirichlet problems respectively

\[
Qu = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (87)
\]

\[
Qv = g \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \quad (88)
\]

with \( Q \) defined in (74), \( f, g \in L^r(\Omega), r \geq 1 \) and

\[
f \leq g \quad \text{a.e. in } \Omega, \quad (89)
\]

then

\[
u \leq v \quad \text{a.e. in } \Omega. \quad (90)
\]

When \( q \) belongs to the interval

\[ p - 1 < q < \frac{N(p-1)}{N-1}, \]

a comparison principle is proved in [14] (see also [31]) for renormalized solution (which are equivalent to “solution obtained as limit of approximations”). Here we just recall its statement by expliciting the sharp assumptions which give existence.
Theorem 4.2. Let $N \geq 2$ and $p$ such that
\[
\begin{cases}
2 - \frac{1}{N} < p < 2, & \text{if } N = 2 \\
2 - \frac{1}{N} < p \leq 2, & \text{if } N \geq 3.
\end{cases}
\] (91)
Assume (2), (4), (5), (9) and (10) with
\[p - 1 < q < \frac{N(p - 1)}{N - 1}.\] (92)
Denote $u, v$ “solution obtained as limit of approximations” to problem (87) and (88) respectively with $f, g \in L^1(\Omega)$ which satisfy (89) and
\[
\|f\|_{L^1} < K_1, \quad \|g\|_{L^1} < K_1
\]
where $K_1$ is given by (36). Then (90) holds. In particular problem (75) has an unique “solution obtained as limit of approximations”.

Assume now
\[\frac{N(p - 1)}{N - 1} \leq q < p - 1 + \frac{p}{N}.\]
We make assumptions on $f$ which assure both existence of a “solution obtained as limit of approximations” and a larger summability of its gradient, (cfr. Theorems 3.6-3.7).

We begin with the case $q > \frac{N(p - 1)}{N - 1}$ and we prove the following comparison principle.

Theorem 4.3. Let $N \geq 2$ and $p$ such that
\[
\begin{cases}
1 < p < 2, & \text{if } N = 2 \\
\frac{2N}{N + 2} \leq p \leq 2, & \text{if } N \geq 3.
\end{cases}
\] (93)
Assume (2), (4), (5), (9) and (10) with
\[\frac{N(p - 1)}{N - 1} < q < p - 1 + \frac{p}{N}.\] (94)
Moreover assume $f, g \in L^m(\Omega)$ with
\[
\max \left\{ \frac{N(q - p + 1)}{q}, \frac{N(2 - p)}{p} \right\} < m < (p^*)' \] (95)
such that (89) holds true and
\[
\|f\|_{\frac{N(q-p+1)}{q}, \infty} < K_2, \quad \|g\|_{\frac{N(q-p+1)}{q}, \infty} < K_2,
\] (96)
where $K_2$ is defined by (44). Denote $u, v$ “solution obtained as limit of approximations” to problem (87) and (88) respectively. Then (90) holds. In particular problem (87) has an unique “solution obtained as limit of approximations”.

Remark 12. We explicitly observe that our result improves the uniqueness result proved in [31] where a comparison principle for renormalized solutions which satisfy a further regularity condition is proved for the smaller interval of values of $p$, that is $\frac{2N}{N+1} < p \leq 2$. 


Proof. Let us denote
\[ w = (u - v)_+ \]
and
\[ D = \{ x \in \Omega : w(x) > 0 \}. \]
Assume that \( D \) has positive measure. Let us fix \( t \in [0, \sup w] \). Denote
\[ w_t = \begin{cases} w - t & \text{if } w > t, \\ 0 & \text{otherwise}, \end{cases} \]
and
\[ E_t = \{ x \in D : w(x) > t \}. \]
Since \( u, v \) are “solution obtained as limit of approximations” to problems (87), and (88) respectively, and since assumptions of Theorem 3.6 are satisfied, we have \( |\nabla u|, |\nabla v| \in L^{m'(p-1)}(\Omega) \). Moreover by definition recalled in Section 2, two sequences of functions \( f_n, g_n \in C_0^\infty(\Omega) \) exist such that
\[ f_n \to f \text{ strongly in } L^m(\Omega) \quad \text{and} \quad g_n \to g \text{ strongly in } L^m(\Omega) \]
and sequences of weak solutions \( u_n, v_n \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) to the approximated problems
\[
\begin{cases} Q_n u_n \equiv -\div (a(x, \nabla u_n)) - T_n(H(x, \nabla u_n)) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial \Omega, \end{cases} \tag{97}
\]
\[
\begin{cases} Q_n v_n \equiv -\div (a(x, \nabla v_n)) - T_n(H(x, \nabla v_n)) = g_n & \text{in } \Omega \\ v_n = 0 & \text{on } \partial \Omega, \end{cases} \tag{98}
\]
respectively, exist such that
\[
\begin{align*}
 u_n & \to u \quad \text{a.e. in } \Omega \quad \text{and} \quad v_n \to v \quad \text{a.e. in } \Omega, \tag{99}
\end{align*}
\]
and
\[
\begin{align*}
 \nabla u_n & \to \nabla u \quad \text{a.e. in } \Omega \quad \text{and} \quad \nabla v_n \to \nabla v \quad \text{a.e. in } \Omega. \tag{100}
\end{align*}
\]
Now we consider
\[ w_n = (u_n - v_n)_+ \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \]
and the set
\[ D_n = \{ x \in \Omega : w_n(x) > 0 \}. \]
For \( n \) large enough \( D_n \) has positive measure. Consider the function
\[ w_{n,t} = \begin{cases} w_n - t & \text{if } w_n > t \\ 0 & \text{otherwise} \end{cases} \]
and denote
\[ E_{n,t} = \{ x \in D_n : w_n(x) > t \}. \]
In the same order of idea of [31], we consider the function
\[ \Psi(x) = [w_{n,t}(x) + \vartheta]^{\lambda} - \vartheta^\lambda, \]
with \( \vartheta > 0 \) and \( \lambda = \frac{[m^*(p - 1)]^s}{m^*} \). Let us explicitly observe that \( \lambda < 1 \), since \( m < (p^*)^s \). This allows to verify that \( \Psi \) belongs to \( W_0^{1,p}(\Omega) \) and we can use it as test function in (97) and in (98). By subtracting the equations we obtain

\[
\lambda \int_{E_{n,t}} [a(x, \nabla u_n) - a(x, \nabla v_n)] (w_{n,t} + \vartheta)^{\lambda - 1} \nabla w_{n,t} \, dx
= \int_{E_{n,t}} [T_n(H(x, \nabla u_n)) - T_n(H(x, \nabla v_n))] [(w_{n,t} + \vartheta)^{\lambda} - \vartheta^\lambda] \, dx
+ \int_{E_{n,t}} (f_n - g_n)(w_{n,t} + \vartheta)^{\lambda} \, dx.
\] (101)

By strong monotonicity of \( a \) (9) and local Lipschitz condition on \( H \) (10), we get

\[
\alpha \lambda \int_{E_{n,t}} \frac{|\nabla w_{n,t}|^2}{(|\nabla u_n| + |\nabla v_n|)^{2-p}} (w_{n,t} + \vartheta)^{\lambda - 1} \, dx
\leq \beta \int_{E_{n,t}} (\eta + |\nabla u_n| + |\nabla v_n|)^{q-1} |\nabla w_{n,t}| (w_{n,t} + \vartheta)^{\lambda} \, dx
+ \int_{E_{n,t} \cap \{f_n - g_n > 0\}} (f_n - g_n)(w_{n,t} + \vartheta)^{\lambda} \, dx.
\]

Observe that the right-hand side is finite since \( q < p - 1 - \frac{p}{2} \), \(|\nabla u_n|, |\nabla v_n| \in L^p(\Omega)\) and \( w_{n,t} \in L^{(m^* (p - 1))^s}(\Omega) \).

Now we want to pass to the limit as \( n \) goes to \(+\infty\) in such inequality. By using (99) and (100), we can apply Fatou lemma and, since \( f_n - g_n \) pointwise converge to \( f - g \leq 0 \) in a.e. \( \Omega \), we obtain

\[
\int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} (w_t + \vartheta)^{\lambda - 1} \, dx
\leq \frac{\beta}{\alpha \lambda} \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{q-1} |\nabla w_t| (w_t + \vartheta)^{\lambda} \, dx.
\] (102)

Observe that the right-hand side of (102) is finite since, by definition of \( \lambda \) and the assumption \( m > \frac{N(q-p+1)}{q} \) we have

\[
\frac{q}{m^* (p - 1)} + \frac{\lambda}{[m^*(p - 1)]^s} < 1.
\]

We begin by estimating the integral on the left-hand side of (102). To this aim let us observe that, by using Sobolev embedding theorem for the function \( [(w_t + \vartheta)^{\frac{q-1}{2}} - \vartheta^{\frac{q-1}{2}}]^s \), we have, for \( s > 1 \)

\[
\left( \int_{E_t} [(w_t + \vartheta)^{\frac{q-1}{2}} - \vartheta^{\frac{q-1}{2}}]^s \, dx \right)^{\frac{1}{s}} \leq C_s \frac{\lambda + 1}{2} \left( \int_{E_t} |\nabla w_t|^s (w_t + \vartheta)^{\frac{q-1}{2}} \, dx \right)^{\frac{1}{s}},
\] (103)

where \( C_s \) is the best constant in Sobolev embedding \( W_0^{1,s}(\Omega) \subset L^s(\Omega) \).
On the other hand, by Hölder inequality, if $1 < s < 2$ we have
\[
\left( \int_{E_t} |\nabla w_t|^s (w_t + \vartheta) \frac{\lambda - 1}{2} dx \right)^{\frac{1}{s}} 
\leq \left( \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} (w_t + \vartheta)^{\lambda - 1} dx \right)^{\frac{1}{2}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|) \frac{(2-p)s}{2-p} dx \right)^{\frac{1}{2} - \frac{1}{s}}.
\]
Then by (103), choosing $s$ such that $\frac{(2-p)s}{2-p} = m^*(p-1)$, i.e. $s = \frac{2m^*(p-1)}{m^*(p-1) + 2-p}$, we get
\[
\left( \int_{E_t} [(w_t + \vartheta)^{\frac{\lambda + 1}{2}} - \vartheta^{\frac{\lambda + 1}{2}}]^{s^*} dx \right)^{\frac{1}{s^*}} 
\leq C_s \frac{\lambda + 1}{2} \times \left( \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} (w_t + \vartheta)^{\lambda - 1} dx \right)^{\frac{1}{2}} \times \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^m (p-1) dx \right)^{\frac{2-p}{2m^*(p-1)}}. 
\]
To estimate the integral on the right-hand side of (102), we have to distinguish the case $q \leq \frac{p}{2}$ and $q > \frac{p}{2}$. Observe that the bounds on $p$ assure that $\frac{p}{2} < p - 1 + \frac{2}{p}$. Let $q \leq \frac{p}{2}$. We can use Hölder inequality, since by the assumption $m \geq \frac{N(2-p)}{2}$, we get $\frac{1}{2} + \frac{1}{s^*} \leq 1$. Therefore we get
\[
\int_{E_t} (\eta + |\nabla u| + |\nabla v|)^{p-1} |\nabla w_t| (w_t + \vartheta)^{\lambda} dx 
\leq \frac{1}{\eta^\frac{p}{2-q}} \left( \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} (w_t + \vartheta)^{\lambda - 1} dx \right)^{\frac{1}{2}} \times \left( \int_{E_t} (w_t + \vartheta)^{s^*} \frac{\lambda + 1}{2} dx \right)^{\frac{1}{s^*}} |E_t|^{\frac{1}{2} - \frac{1}{s^*}},
\]
where, by definition of $s$, $s^* \frac{(\lambda + 1)}{2} = [m^*(p-1)]^*$. By using (102), we deduce
\[
\left( \int_{E_t} \frac{|\nabla w_t|^2}{(|\nabla u| + |\nabla v|)^{2-p}} (w_t + \vartheta)^{\lambda - 1} dx \right)^{\frac{1}{2}} 
\leq \frac{\beta}{\alpha \lambda \eta^\frac{2}{2-q}} \left( \int_{E_t} (w_t + \vartheta)^{m^*(p-1)c} dx \right)^{\frac{1}{2}} |E_t|^{\frac{1}{2} - \frac{1}{s^*}}.
\]
Hence by (104) and (105), we get
\[
\left( \int_{E_t} [(w_t + \vartheta)^{\frac{\lambda + 1}{2}} - \vartheta^{\frac{\lambda + 1}{2}}]^{s^*} dx \right)^{\frac{1}{s^*}} \leq C_s \frac{(\lambda + 1)^{\frac{\lambda + 1}{2}}} {2\alpha \lambda \eta^\frac{2}{2-q}} \left( \int_{E_t} (w_t + \vartheta)^{m^*(p-1)c} dx \right)^{\frac{1}{2}} \times \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^m (p-1) dx \right)^{\frac{2-p}{2m^*(p-1)}} |E_t|^{\frac{1}{2} - \frac{1}{s^*}}.
\]
By letting $\vartheta$ go to zero, we get
\[
1 \leq C_s \frac{(\lambda + 1)^{\frac{\lambda + 1}{2}}} {2\alpha \lambda \eta^\frac{2}{2-q}} \left( \int_{E_t} (\eta + |\nabla u| + |\nabla v|)^m (p-1) dx \right)^{\frac{2-p}{2m^*(p-1)}} |E_t|^{\frac{1}{2} - \frac{1}{s^*}}.
\]
This is a contradiction since the right-hand side goes to zero when $t$ tends to $\sup w$. We conclude that $|D| = 0$, i.e., $u \leq v$ a.e. in $\Omega$ in the case $q \leq \frac{p}{2}$.

Now we evaluate the integral in the right-hand side of (102) in the case where $\frac{p}{2} < q$. We can use Hölder inequality, since by the assumption $m > \frac{N(p-1)}{q}$, we get $\frac{1}{2} + \frac{2q-p}{2m(p-1)} + \frac{1}{s} < 1$. Therefore we get

$$\int_{E_\lambda} \left( \frac{1}{2} \left( \frac{1}{2} \left( (\eta + |u| + |v|)|w_t| (w_t + \vartheta) \right)^{\lambda-1} \right)^{\frac{1}{\lambda}} dx$$

Observe that, as before, $s^* \lambda = [m^*(p-1)]^*$. Therefore by (102), we deduce

$$\left( \int_{E_\lambda} \left( \frac{1}{2} \left( \frac{1}{2} \left( (\eta + |u| + |v|)|w_t| (w_t + \vartheta) \right)^{\lambda-1} \right)^{\frac{1}{\lambda}} dx \right)^{\frac{1}{2}} \leq \left( \int_{E_\lambda} \left( \frac{1}{2} \left( \frac{1}{2} \left( (\eta + |u| + |v|)|w_t| (w_t + \vartheta) \right)^{\lambda-1} \right)^{\frac{1}{\lambda}} dx \right)^{\frac{1}{2}} \right)^{\frac{1}{\lambda}} |E_\lambda|^{\frac{1}{2}} - \frac{2q-p}{2m(p-1)} \lambda.$$

Starting from (104) and arguing as in the previous case, we get the assertion also in the case where $q > \frac{p}{2}$.

An analogous proof gives the comparison principle in the limit case $q = \frac{N(p-1)}{N-1}$.

**Theorem 4.4.** Let $N \geq 2$ and $p$ such that

$$1 < p < 2, \quad \text{if } N = 2$$

$$\frac{2N}{N+2} \leq p \leq 2, \quad \text{if } N \geq 3.$$  \hspace{1cm} (110)

Assume (2), (4), (5), (9) and (10) with

$$q = \frac{N(p-1)}{N-1}. \hspace{1cm} (111)$$

Moreover assume (89) and $f, g \in L^m(\Omega)$ with

$$\max \left\{ 1, \frac{N(2-p)}{p} \right\} < m < (p^*)' \hspace{1cm} (112)$$

and such that

$$\|f\|_{L^m} < \left( \log \left( \frac{\mathcal{M}}{|\Omega|} \right) \right)^{-\frac{m(N-1)}{m}} K_3,$$

$$\|g\|_{L^m} < \left( \log \left( \frac{\mathcal{M}}{|\Omega|} \right) \right)^{-\frac{m(N-1)}{m}} K_3,$$
where $K_3$ is defined by (59). Denote $u, v$ “solution obtained as limit of approximations” to problem “solution obtained as limit of approximations” to problem (87) and (88) respectively. Then (90) holds. In particular problem (87) has an unique “solution obtained as limit of approximations”.

Proof. The proof proceeds exactly as the proof of Theorem 4.3 once we observe that assumptions of Theorem 3.7 hold true and therefore $|∇u|, |∇v| \in L^m(p-1)(Ω)$. □

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Received July 2014; revised January 2015.

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