On the Equivalence of SDP Feasibility and a Convex Hull Relaxation for System of Quadratic Equations

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Abstract

We show semidefinite programming (SDP) feasibility problem is equivalent to solving a convex hull relaxation (CHR) for system of quadratic equations. On the one hand this offers a simple description of SDP. On the other hand, this equivalence makes it possible to describe a version of the Triangle Algorithm for SDP feasibility based on solving CHR. Specifically, the Triangle Algorithm either computes an approximation to the least-norm feasible solution of SDP, or using its distance duality, provides a separation when no solution within a prescribed norm exists. The worst-case complexity of each iteration is computing largest eigenvalue of a symmetric matrix arising in that iteration. Alternate complexity bounds on the total number of iterations are derived. Further applications includes solving an SDP optimization problem. The Triangle Algorithm thus provides an alternative to the existing interior-point algorithms for SDP. Finally, gaining from these results and insights, we discuss potential extension to solving general system of polynomial equations.

Keywords: Convex Hull, Quadratic System of Equations, Semidefinite Programming, Approximation Algorithms, Triangle Algorithm

1 Introduction

In this article we prove the feasibility problem in semidefinite programming (SDP) is equivalent to a convex hull relaxation (CHR) for system of quadratic equations. Specifically, consider a system of quadratic equations $q_i(x) = b_i, i = 1, \ldots, m$, where $b_i \in \mathbb{R}$, and $q_i(x) = x^T A_i x$, $A_i$ an $n \times n$ real symmetric matrix. The problem of testing if there is a common solution $x$ is well known to be NP-hard. Rather than testing if the system has a solution, we test if $b = (b_1, \ldots, b_m)^T$ lies in the convex hull of the set of all $m$-tuples $(q_1(x), \ldots, q_m(x))^T$, as $x$ ranges in $\mathbb{R}^n$. We refer to this problem as the convex hull relaxation (CHR). The convex hull of a set is the smallest convex set containing that set. By Carathéodory theorem, if $b$ lies in this convex hull, it can be represented as a convex combination of at most $m+1$ such $m$-tuples. This characterization, in an elementary fashion, gives rise to a positive semidefinite matrix $X$, where for $i = 1, \ldots, m$, $Tr(A_i X) = b_i$. Hence $X$ is a feasible solution of an SDP. Conversely, a feasible solution to the SDP defined by these equations gives rise to a solution in the convex hull of the set of $m$-tuples.

On the one hand this gives a very simple description of SDP. On the other hand, suppose we wish to test if there is a solution $x$ to the system of quadratic equations, where $\|x\|$ is within a given radius $r$. We show how to solve the convex hull relaxation of this problem via a version of the Triangle Algorithm, a fully polynomial-time approximation scheme (FPTAS) designed to test if a given point lies in an arbitrary given compact convex subset of the Euclidean space, [7, 8]. In summary, from the results in this article we gain insights on SDP, including the applicability of the Triangle Algorithm as an alternative algorithm to interior-point algorithms for SDP. The results also offer insights on solving a general system of polynomial equation. In the remaining of this section we give a brief review on SDP and the Triangle Algorithm.

SDP has received much attention in the literature due to its wide range of applications. It is a generalization of LP, where the underlying nonnegativity cone is replaced with the cone of symmetric positive
semidefinite matrices, see e.g. [1], [13], [15]. As a special case of self-concordant optimization problems, SDP can be approximated to within $\varepsilon$ tolerance in polynomial time complexity in terms of the dimensions of the problem and $\ln 1/\varepsilon$, see [16]. The main work in each iteration of interior-point algorithms is solving a Newton system arising in that iteration. The over all complexity in solving an SDP with $n \times n$ matrices can be as large $O(n^{6.5} \ln 1/\varepsilon)$, see [12]. SDP relaxations have found applications in combinatorial optimization, see [4] for approximation of the MAX CUT problem, and [10] for applications in other combinatorial problems. In some SDP relaxations the overall complexity can be reduced, e.g. to $O(n^{3.5} \ln 1/\varepsilon)$, [9].

The Triangle Algorithm (TA), introduced in [7], is a geometrically inspired algorithm originally designed to solve the convex hull membership problem (CHM): testing if a given $p_o \in \mathbb{R}^m$ lies in $\text{conv}(S)$, where $S = \{v_1, \ldots, v_n\} \subset \mathbb{R}^m$. Triangle algorithm is endowed with distant dualities and offers fast alternative complexity bounds to polynomial-time algorithms, allowing trade-off between dependence on the dimension of the problem and the desired tolerance in approximation. In numerical experimentations TA performs quite well. A generalization of the Triangle Algorithm described in [8] tests if a given pair of arbitrary compact convex sets $C, C' \subset \mathbb{R}^m$ intersect, or if they are separable. Specifically, it can solve any of the following four problems when applicable: (1) compute an approximate point of intersection, (2) compute a separating hyperplane, (3) compute an approximation to the optimal pair of supporting hyperplanes, (4) approximate the distance between the sets. In particular, CHM and the hard margin problem (SVM) are very special cases. In this more general version of the Triangle Algorithm the complexity of each iteration depends on the nature and description of the underlying convex sets. In this article we are interested in the version of the algorithm where one of the sets is a singleton point. We refer to General-CHM as the problem of testing if a given point $p_o \in \mathbb{R}^m$ lies in a given compact convex subset $C$ of $\mathbb{R}^m$. We will summarize the algorithm and its complexity in a subsequent section.

The organization of the article is as follows. Section 2 describes the equivalence of SDP feasibility and the convex hull relaxation for system of quadratic equations. Section 3 describes the Triangle Algorithm and its complexity for the General-CHM. Section 4 specializes the Triangle Algorithm and its complexity for solving the convex hull relaxation. We end with concluding remarks.

## 2 Equivalence of SDP Feasibility and Convex Hull Relaxation

Let $S = \{A_1, \ldots, A_m\}$ be a subset of $\mathbb{S}^n$, the set of $n \times n$ real symmetric matrices. Let $b = (b_1, \ldots, b_m)^T \in \mathbb{R}^m$, $b \neq 0$. The Frobenius inner product of $X, Y \in \mathbb{S}^n$ is denoted by any of the following equivalent notations

$$
\langle X, Y \rangle_F = Tr(XY) = X \cdot Y = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}.
$$

(1)

We write $X \succeq 0$ for $X \in \mathbb{S}^n_+$, the cone of positive semidefinite matrices in $\mathbb{S}^n$. The SDP feasibility problem is testing if $b \in \mathbb{P}$, where

$$
\mathbb{P} = \{ A(X) \equiv (A_1 \cdot X, \ldots, A_m \cdot X)^T : X \succeq 0 \}.
$$

(2)

For $i = 1, \ldots, m$, let $q_i(x) = x^T A_i x$. Let $Q(x) = (q_1(x), \ldots, q_m(x))^T$. Consider testing the solvability of the system of quadratic equations $Q(x) = b$. When each $A_i$ is a diagonal matrix solving this system is equivalent to solving a linear programming feasibility problem. However, it is NP-hard in general. Rather than solving this NP-hard problem we consider the convex hull relaxation of the problem defined as testing if $b$ lies in $C$, where

$$
C = \text{conv} \{ (Q(x) = (q_1(x), \ldots, q_m(x))^T : x \in \mathbb{R}^n) \},
$$

(3)

the convex hull of the set of all $Q(x), x \in \mathbb{R}^n$. Given a real number $r$, let

$$
C(r) = \{ x \in C : \|x\| \leq r \}, \quad \mathbb{P}(r) = \{ X \in \mathbb{P} : Tr(X) \leq r^2 \}.
$$

(4)

**Theorem 1.** $b \in C(r)$ if and only if $b \in \mathbb{P}(r)$. In particular, $b \in C$ if and only if $b \in \mathbb{P}$. 

2
Proof. Assume \( b \in C(r) \). From the definition of \( C(r) \) and Carathéodory theorem, for some \( t \leq m + 1 \), there exists \( x_1, \ldots, x_t \in \mathbb{R}^n \), each \( \|x_i\| \leq r \), such that

\[
\sum_{i=1}^{t} \alpha_i Q(x_i) = b, \quad \sum_{i=1}^{t} \alpha_i = 1, \quad \alpha_i \geq 0, \quad \forall i.
\]

(5)

Set \( X_i = x_i x_i^T \), \( i = 1, \ldots, t \). Then \( X_i \in \mathbb{S}_+^n \). Hence \( X = \sum_{i=1}^{t} \alpha_i X_i \in \mathbb{S}_+^n \). Moreover, for each \( k = 1, \ldots, m \),

\[
A_k \bullet X = A_k \bullet \sum_{i=1}^{t} \alpha_i X_i = \sum_{i=1}^{t} \alpha_i A_k \bullet X_i = \sum_{i=1}^{t} \alpha_i A_k \bullet x_i x_i^T = \sum_{i=1}^{t} \alpha_i q_k(x_i) = \sum_{i=1}^{t} \alpha_i b_k = b_k.
\]

(6)

Additionally,

\[
Tr(X) = \sum_{i=1}^{t} \alpha_i Tr(x_i x_i^T) = \sum_{i=1}^{t} \alpha_i \|x_i\|^2 \leq \sum_{i=1}^{t} \alpha_i r^2 = r^2.
\]

(7)

Hence \( b \in P(r) \). Conversely, suppose \( b \in P(r) \). Then there exists \( X \in \mathbb{S}^n \) such that \( A_k \bullet X = b_k \), \( k = 1, \ldots, m \). Let the spectral decomposition of \( X \) be \( X = U \Lambda U^T \), with \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), \( U = [u_1, \ldots, u_n] \) corresponding to eigenvalue-eigenvectors. We have \( Tr(X) = \sum_{i=1}^{n} \lambda_i \). For \( i = 1, \ldots, n \), let \( \alpha_i = \lambda_i / Tr(X) \), \( x_i = \sqrt{Tr(X)} u_i \). Then \( \|x_i\| = \sqrt{Tr(X)} \). Thus \( X \) is a convex combination of \( X_i = x_i x_i^T \)’s:

\[
X = \sum_{i=1}^{n} \lambda_i u_i u_i^T = \sum_{i=1}^{n} \alpha_i x_i x_i^T.
\]

(8)

Thus for \( k = 1, \ldots, m \),

\[
A_k \bullet X = A_k \bullet \sum_{i=1}^{n} \lambda_i u_i u_i^T = \sum_{i=1}^{n} \alpha_i A_k \bullet x_i x_i^T = \sum_{i=1}^{n} \alpha_i x_i^T A_k x_i = b_k.
\]

(9)

Hence \( b \in C(r) \).

\( \square \)

Remark 1. Given any approximation to \( Q(x) = b \) as a convex combination \( \sum_{i=1}^{t} \alpha_i Q(x_i) \) in \( C \), it induces the set \( \{ x_i : i = 1, \ldots, t \} \) as approximate solutions to \( Q(x) = b \). We can further improve on this set by replacing each \( x_i \) with \( Q(x_i) \neq 0 \), with \( \gamma_i x_i \), where

\[
\gamma_i^* = \text{argmin} \{ \| \gamma Q(x_i) - b \| : \gamma \in \mathbb{R} \} = b^T Q(x_i) / \| Q(x_i) \|^2.
\]

(10)

Remark 2. Suppose the quadratic system \( Q(x) = b \) is inhomogeneous, say for \( i = 1, \ldots, m \), \( q_i(x) = x^T A_i x + c_i^T x + d_i \), where \( c_i \in \mathbb{R}^n \), \( d_i \in \mathbb{R} \). We can convert \( q_i(x) \) to a homogeneous one as follows. First, the constant term \( d_i \) is absorbed in \( b_i \). If \( c_i \neq 0 \) for some \( i \), we introduce a new variable \( z \) and replace \( q_i(x) \) with \( q_i(x, z) = x^T A_i x + z c_i^T x \). Also we add a new equation \( q_{m+1}(x, z) = z^2 = b_{m+1} = 1 \). Now suppose \( (x, z) \) satisfies

\[
Q(x, z) \equiv (q_1(x, z), \ldots, q_{m+1}(x, z))^T = (b_1, \ldots, b_{m+1})^T.
\]

(11)

If \( z = 1 \), then \( Q(x) = b \). Otherwise, \( z = -1 \) but then \( (-x, -z) \) also satisfies the above equation so that \( Q(-x) = b \).

3 Triangle Algorithm for General Convex Hull Membership

Let the general convex hull membership (General-CHM) be defined as follows: Given a compact convex subset \( C \) of \( \mathbb{R}^m \) and a point \( p_0 \in \mathbb{R}^m \), determine if \( p_0 \in C \). Specifically, given \( \varepsilon \in (0, 1) \), either compute \( p' \in C \) such that \( \| p' - p_0 \| \leq \varepsilon \), or find a hyperplane that separates \( p_0 \) from \( C \). The Triangle Algorithm for
solving General-CHM works as follows: Given arbitrary \( p' \in C \), while \( \| p' - p_0 \| > \varepsilon \), test if there exists a pivot, i.e. \( v \in C \) that satisfies the following equivalent conditions

\[
\| p' - v \| \geq \| p_0 - v \|, \quad (p_0 - p')^T v \geq \frac{1}{2}(\| p_0 \|^2 - \| p' \|^2).
\]

If no pivot exists, \( p' \) is called a witness since it provides a hyperplane \( H \) that separates \( p_0 \) from \( C \), proving \( p_0 \notin C \). Specifically, the separating hyperplane \( H \) is given as

\[
H = \{ x : (p' - p_0)^T x = \frac{1}{2}(\| p' \|^2 - \| p_0 \|^2) \}.
\]

A strict pivot is a pivot where \( \angle p' p_0 v \geq \pi/2 \). Equivalently, \( (p' - p_0)^T (v - p_0) \leq 0 \). That is, \( (p_0 - p')^T v \geq \| p_0 \|^2 - p^T p_0 \).

When \( p' \) admits a pivot \( v \), we can get closer to \( p_0 \) by defining \( p'' \) to be the closest point to \( p_0 \) on the line segment connecting \( p' \) to \( v \). Then we replace \( p' \) with \( p'' \) and repeat the above iteration. In the worst-case the test for the existence of a pivot amounts to computing

\[
\max \{ c^T x : x \in C \}, \quad c = p_0 - p'.
\]

Formally, the Triangle Algorithm can be described:

**Algorithm 1** Triangle Algorithm (\( C \subset \mathbb{R}^m \), \( b \in \mathbb{R}^m \), \( \varepsilon \in (0, 1) \))

1. Select arbitrary \( p' \in C \).
2. Given an iterate \( p' \in C \), if \( \| p' - p_0 \| \leq \varepsilon \), stop. Otherwise, test if there exists a pivot \( v \in C \) (i.e. \( \| p' - v \| \geq \| p_0 - v \| \)). If no pivot exists, then \( p' \) is a witness, stop.
3. Compute the step-size \( \alpha = (p_0 - p')^T (v - p')/\| v - p' \|^2 \). Let the new iterate \( p'' \) be defined as the nearest point to \( p_0 \) on the line segment \( [p' : v] \): If \( \alpha \in [0, 1] \), \( p'' = (1 - \alpha)p' + \alpha v \); otherwise, \( p'' = v \). Replace \( p' \) with \( p'' \), go to step 2.

The correctness of the Triangle Algorithm is due to the following (see [7, 8]):

**Theorem 2.** (Distance Duality) \( p_0 \in C \) if and only if for each \( p' \in C \) there exists a (strict) pivot \( v \in C \). Equivalently, \( p_0 \notin C \) if and only if there exists a witness \( p' \in C \). Furthermore, \( p_0 \in C \) if and only if for each \( p' \in C \), \( p' \neq p_0 \), there exists a strict pivot \( v \in C \).

The iteration complexity for Triangle Algorithm is given in the following theorem from [8].

**Theorem 3.** (Complexity Bounds) Let \( R = \max \{ \| x - p_0 \| : x \in C \} \). Let \( \varepsilon \in (0, 1) \).

(i): Triangle in \( O(1/\varepsilon^2) \) iterations either computes \( p_0 \in C \) with \( \| p_0 - p_\varepsilon \| \leq R\varepsilon \), or a witness. In particular, if \( p_0 \notin C \) and \( \delta_* = \min \{ \| x - p_0 \| : x \in C \} \), the number of iterations to compute a witness is \( O(R^2/\delta_*^2) \). Furthermore, given any witness \( p' \in C \), we have

\[
\delta_* \leq \| p' - p_0 \| \leq 2\delta_*,
\]

i.e. a witness estimates the distance to \( C \) to within a factor of two.

(ii) Suppose a ball of radius \( \rho > 0 \) centered at \( p_0 \) is contained in the relative interior of \( C \). If Triangle Algorithm uses a strict pivot in each iteration, it compute \( p_\varepsilon \in C \) satisfying \( \| p_\varepsilon - p_0 \| \leq \varepsilon \) in \( O((R/\rho)^2 \ln 1/\varepsilon) \) iterations.

**Remark 3.** When case (ii) is applicable, we can think of the ratio \( R/\rho \) as a condition number for the problem. If this condition number is not large the complexity is only logarithmic in \( 1/\varepsilon \), hence few iterations will suffice.

In a more elaborate fashion it can be shown that Triangle Algorithm can approximate the distance from \( p_0 \) to \( C \) to within any prescribed accuracy \( \varepsilon \), see [8].
4 Triangle Algorithm for Convex Hull Relaxation

Consider testing if \( b = (b_1, \ldots, b_m)^T \) lies on \( C(r) = \{Q(x) = (q_1(x), \ldots, q_m(x))^T : \|x\| \leq r \} \), where \( q_i(x) = x^T A_i x, A_i \in \mathbb{S}^n \), \( i = 1, \ldots, m \), and \( r \) a given number. We test this via the Triangle Algorithm for General-CHM, setting \( p_0 = b \) and \( C = C(r) \). Given arbitrary \( x_0 \in \mathbb{R}^n \) with \( \|x_0\| \leq r \), let \( p' = Q(x_0) \). To test if there is a pivot for \( p' \), in the worst-case, amounts to computing

\[
\max\{ (p_p - p')^T Q(x) : \|x\| \leq r \}. \tag{17}
\]

Let \( c = (p_p - p') = (c_1, \ldots, c_m)^T, A = \sum_{i=1}^m c_i A_i \). Then the optimization problem above, its optimal objective value and solution are give as

\[
\max\{x^T Ax : \|x\| \leq r\} = r^2 \lambda_s = r^2 u_s^T A u_s, \tag{18}
\]

where \( \lambda_s \) is the largest eigenvalue of \( A \) and \( u_s \) the corresponding eigenvector. The computation of the optimal solution can thus be achieved via the Power Method. In summary, in the worst case, the complexity of each iteration of the Triangle Algorithm is computing the largest eigenvalue of an \( n \times n \) symmetric matrix arising in that iteration. If initially we have no estimate of the norm of a possible solution in \( C \), we start with an estimate \( r_0 \). If \( b \notin C(r_0) \), Triangle Algorithm will eventually compute a witness. In that case we replace \( r_0 \) with \( 2r_0 \) and repeat the above. The following shows how we can compute an initial lower bound.

**Proposition 1.** Let \( r_0 = \min \{ \sqrt{b_k/\|A_k\|} : k = 1, \ldots, m, b_k \neq 0 \} \). Then for any \( r < r_0, b \notin C(r) \).

**Proof.** Since \( b \neq 0 \), \( r_0 \) is well defined. Suppose \( b \in C \). Then there exists \( t \) such that for each \( i = 1, \ldots, t \),

\[
b_k = \sum_{i=1}^t \alpha_i x_i^T A_k x_i, \quad \sum_{i=1}^t \alpha_i = 1, \quad \alpha_i \geq 0.
\]

This implies the following from which the proof follows:

\[
|b_k| = \sum_{i=1}^t \alpha_i \|x_i^T A_k x_i\| \leq \sum_{i=1}^t \alpha_i \|A_k\| \|x_i\|^2 \leq \|A_k\| \max\{\|x_i\|^2 : i = 1, \ldots, t\}. \tag{19}
\]

\[\square\]

**Theorem 4.** (Complexity of Solving Convex Hull Relaxation) Given \( Q(x) = (x^T A_1 x, \ldots, x^T A_m x) \), where \( A_k \in \mathbb{S}^n, k = 1, \ldots, m, b \in \mathbb{R}^m \), and \( r_0 \) be as in Proposition 1. Given \( r \) let

\[
R_r = \max\{\|Q(x) - b\| : r_0 \leq \|x\| \leq 2r\}. \tag{20}
\]

(i) Suppose \( b \in C \). Let \( r_* \) be the smallest \( r \) for which \( b \in C(r) \). With \( r_0 \) as the initial value for \( r \), in \( O(1/\epsilon^2) \) iterations the Triangle Algorithm computes \( x_i \in \mathbb{R}^n, i = 1, \ldots, t, t \leq m + 1 \), where \( \sum_{i=1}^t \alpha_i Q(x_i) \in C(r), \|x_i\| \leq 2r_*, i = 1, \ldots, t, \) and if \( X = \sum_{i=1}^t \alpha_i x_i x_i^T \) and \( A(X) = (A_1 \bullet X, \ldots, A_m \bullet X)^T \), then \( Tr(X) \leq 4r_*^2 \) and we have

\[
\|\sum_{i=1}^t \alpha_i Q(x_i) - b\| = \|A(X) - b\| \leq \varepsilon R_r.. \tag{21}
\]

(ii) Given \( r > 0 \), suppose the relative interior of \( C(r) \) contains the ball of radius \( \rho > 0 \) centered at \( b \), i.e.

\[
C \cap B_\rho(b) \subset C(r), \tag{22}
\]

where \( B_\rho(b) = \{y \in \mathbb{R}^m : \|y - b\| \leq \rho\} \). Then the Triangle Algorithm in \( O((R_r/\rho)^2 \ln 1/\varepsilon) \) iterations computes \( x_i \in \mathbb{R}^n, i = 1, \ldots, t, t \leq m + 1 \), where \( \sum_{i=1}^t \alpha_i Q(x_i) \in C(r), \|x_i\| \leq r, \) and if \( X = \sum_{i=1}^t \alpha_i x_i x_i^T \), then \( Tr(X) \leq r^2 \) and we have

\[
\|\sum_{i=1}^t \alpha_i Q(x_i) - b\| = \|A(X) - b\| \leq \varepsilon. \tag{23}
\]

(iii) Suppose \( b \notin C(r) \). Given \( r > 0 \), let \( \delta_r = \min\{\|Q(x) - b\| : r_0 \leq \|x\| \leq r\} \). Then in \( O(r^2/\delta_r^2) \) iterations the Triangle Algorithm computes a witness \( p' \in C(r) \), i.e. the orthogonal bisecting hyperplane of the line segment \( p'b \) separates \( b \) from \( C(r) \).
Proof. The proof of complexity theorem mainly follows from Theorem 3. To justify (i), note that for any point \( p' = \sum_{i=1}^{t} \alpha_i Q(x_i) \in C(r) \), \( \|p' - b\| \leq \sum_{i=1}^{t} \alpha_i \|Q(x_i) - b\| \leq \max \{\|Q(x_i) - b\| \} \). Hence the quantity \( R_r \) corresponds to \( R \) defined in Theorem 3. Each time a witness is calculated with respect to a current \( r \) we are not able to reduce the gap between \( b \) and the current iterate \( p' \). However, on the one hand by starting at \( r_0 \) as the initial value, we will end up doubling the current value of \( r \) at most \( O(\ln(r/r_0)) \) times. On the other hand, each time a witness is computed we can increase the value of \( r \) until the current iterate \( p' \) admits a pivot in which case the gap between \( b \) and \( p' \) will decrease. The above argument justifies (i). Proof of (ii) and (iii) also follow from Theorem 3 keeping Theorem 1 in view.

Remark 4. When \( Q(x) = b \) is solvable, the Triangle Algorithm computes a relaxed solution, i.e., a set of points \( x_i \in \mathbb{R}^n \), \( i = 1, \ldots, t \), \( t \leq m + 1 \), such that \( \sum_{i=1}^{t} \alpha_i Q(x_i) = b \), \( \sum_{i=1}^{t} \alpha_i = 1 \), \( \alpha_i \geq 0 \). Of course there may be cases where \( Q(x) = b \) is unsolvable but the relaxation is solvable. However, when \( C(r) \) is empty Triangle Algorithm computes a witness, implying \( Q(x) = b \) is not solvable. This is an important feature of the Triangle Algorithm, the ability to produce in some cases a certificate to lack of solvability of a quadratic system. We also remark that under the assumption that input matrices have rational entries, it will be possible to compute an upper bound on the norm of a solution of \( Q(x) = b \), if it exists, and a lower bound to the distance between \( b \) and \( C \), when \( b \not\in C \), so that Triangle Algorithm can solve CHR. We avoid details here.

Remark 5. From the implementation point of view, there are many fine-tuning steps that can be taken to improve the efficiency of the Triangle Algorithm. For instance, we may not need to compute in each iteration the largest eigenvalue in \( \mathbb{R}^{n \times n} \). If we store the pivot as they are generated, there is a good chance that a pivot can be reused one or more times in subsequent iterations. Such discussions are described in [9], where we have proposed a Triangle Algorithm for an SDP version of the convex hull membership (CHM), called spectrahull CHM. That algorithm directly works on SDP. Given substantial computational experiences with the Triangle Algorithm for CHM, and even for the generation of all vertices of the convex hull of a finite set of point, see [2], we would expect the iteration complexity to be quite reasonable for solving SDP feasibility problem, as well as SDP optimization problem, considered in next remark.

Remark 6. Consider the SDP optimization problem, \( \max \{A_0 \cdot X : A(X) = b, Tr(X) \leq r^2, X \succeq 0\} \), for a given \( r \). We can first test the feasibility of \( C(r) \) via the Triangle Algorithm. This produces a set, \( x_i \in \mathbb{R}^n \), \( i = 1, \ldots, t \leq m + 1 \) such that \( \sum_{i=1}^{t} \alpha_i Q(x_i) = b \), \( \sum_{i=1}^{t} \alpha_i = 1 \), \( \alpha_i \geq 0 \). Let \( b_0 = \sum_{i=1}^{t} \alpha_i q_0(x_i) \). We wish to increase the value of \( b_0 \) to a new value, \( b' \), and test via the Triangle Algorithm if \( b = (b_0', b_1, \ldots, b_m)^T \) lies in the augmented convex hull relaxation \( \mathbb{C}(r) = conv(\{Q(x)(q_0(x), q_1(x), \ldots, q_m(x))^T : x \in \mathbb{R}^n\}) \). If \( b_0 \) is not the optimal value, then there must exist a pivot with respect to the augmented convex hull relaxation. When \( b_0 \) has increased sufficiently, we will be able to find a pivot and continue the iteration and increase \( b' \) to a new value if not optimal. In other words it is possible to find the optimal objective value, essentially by solving several interrelated feasibility problems. Many strategies can be described. We will report on these in future work.

Concluding Remarks

By employing very basic results from convexity, we have shown the equivalence of solving a convex hull relaxation to a system of quadratic equation, CHR, and the general SDP feasibility problem. On the one hand this demonstrates in an elementary fashion how an SDP feasibility problem may arise, also offering insights on SDP itself. Despite the tremendous literature on SDP, in particular SDP relaxation of quadratic programming optimization problems (e.g. [3]), the equivalence shown in this article appears not to have been given previously. On the other hand, the significance of this equivalence becomes evident by having described a version of the Triangle Algorithm that solves the convex hull relaxation of a system of quadratic equations and at the same time offers a new algorithm for SDP feasibility and optimization. Each iteration of the Triangle Algorithm requires computing a pivot which in the worst-case amounts to computing the largest eigenvalue of a symmetric matrix arising in that iteration and can thus be achieved via the Power Method.
Also, pivots can be stored and reused so that the largest eigenvalue computation is not necessary in each iteration. The Triangle Algorithm thus offers an alternative to the interior-point algorithms for SDP whose iterations need to solve a Newton system which could be expensive for large size problems. The Triangle Algorithm is also equipped with a duality of its own (distance duality) that provides new insights on SDP. On the other hand, the insights gained from SDP can help in solving a system of quadratic equations. For instance, in ways in which we may convert the solution of the convex hull relaxation to a positive semidefinite matrix and then converting it back to an approximate solution to the quadratic system itself. Goemans-Williamson procedure for converting the optimal solution to the SDP relaxation of Max-Cut problem is such approach. The problem of testing if there is a cut of a particular value can be formulated as solving a quadratic system $Q(x) = b$. We solve instead the convex hull relaxation (CHR) and after every so many iterations of the Triangle Algorithm reduce the solution to a positive semidefinite matrix and then represent it as a sum of rank-one matrices (e.g. as in the proof of Theorem 1), then convert these to approximate solutions to the quadratic system. Other schemes are possible.

Viewing the connection between a relaxation to a system of quadratic equations (CHR), SDP feasibility problem, and the Triangle Algorithm suggests that extension of these may be possible when considering a system of homogeneous polynomial equations in several variables. Consider solving a system of equations $Q(x) = b$, where $Q(x) = (p_1(x), \ldots, p_m(x))^T$, each $p_i(x)$ a homogeneous polynomials of degree $d \geq 1$. Consider the relaxation that tests if $b \in \text{conv} \{Q(x) : x \in \mathbb{R}^n\}$. Firstly, it is possible an analogous Theorem 1 can be stated where the $n \times n$ matrices $A_i$ are replaced with multidimensional symmetric matrices. Furthermore, we can then attempt to solve the corresponding relaxation via the Triangle Algorithm. However, the success of such method relies on the computation of a pivot. For instance, if each $p_i(x)$ is a cubic polynomial, the computation of a pivot, in the worst case, amounts to computing the maximum of a trilinear form over the unit ball. It goes without saying that the problem of solving a general system of polynomial equations is a profound problem with deep underlying mathematics, see e.g. [3, 4]. It is no easy task to solve a system of polynomial equations. In particular, the sophisticated Gröbner basis method is very limited in the size of the problems it can solve. The simplicity of the Triangle Algorithm and the results described in this article give rise to the this question: can we solve the convex hull relaxation of the general system of homogeneous polynomial equations via the Triangle Algorithm? Such algorithm would need to compute the maximum of a symmetric multidimensional matrix over the unit ball. The problem of computing eigenvalues of tensors via high-order Power Method has been addressed, see e.g. [5]. It may thus not be too far fetched to consider solving the convex hull relaxation via the Triangle Algorithm, given that the homogeneous degree is small. Further research is of course necessary, complemented with computational experimentation. We plan to first experiment with quadratic systems and will report on the findings in future work. We mention in passing that when the homogeneous degree $d = 1$, the corresponding Triangle Algorithm gives rise to a new iterative algorithm for solving a linear system. Our computational results with this, to be addressed in forthcoming work, establishes the Triangle Algorithm as a new iterative method that is very competitive with the existing iterative algorithms for linear systems.

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