Comparison of calculation results of flexible plates on the basis of difference equations of successive approximation method and generalized equations of finite difference method

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Abstract. In structural engineering, the analysis of plate elements is one of the important problems that based on theory of plates. The analytical solutions using traditional methods and manual calculations are determined in the cases with simple calculation schemes. It will be difficult to solve problems with the complicatedness in loading and boundary conditions. Thereof, the application of numerical methods is necessary to deal with that. The finite difference method and successive approximation method are popular numerical methods to solve relatively thoroughly the plate problems. In addition, with the development of construction technology, the dimensions of the current building projects become greater and lead to increase of building displacements that is required to take into account in analysis. The analysis of plates with non-linear geometrical is relatively complicated and iterative algorithms can help to solve this problem. In this paper, the authors used the difference equations of successive approximation method (MSA) and generalized equations of finite difference method (MFD) to solve the problems of non-linear plates with different boundary conditions and loading. From the comparison of obtained results with Volmir’s analytical results the conclusions and recommendations are proposed.

1. Introduction
In structural analysis, in many cases plate elements should be calculated with consideration of nonlinearity in geometry or material properties. When calculating nonlinear geometrical plate elements, it is based on many researches of Volmir [1], Timoshenko [2], Kornhisin [3], etc… Besides, there are many studies about these mentioned problems, such as [4-7].

In the analysis of non-linear geometrical plates with different boundary conditions, in Volmir’s works, it is used different assumptive functions of displacement for solving differential equations with relationship internal forces, displacements and loads. In this paper, the authors used the difference equations of successive approximation method (MSA) and generalized equations of finite difference method (MFD) to discrete difference equations with same approximation of different boundary conditions.

The effectiveness of this method is shown in using for solving many linear mechanical problems such as: analysis of shallow shells [8], analysis of shells of rotation [9], [10]; analysis of plates on an elastic base and on a non-continuous elastic base [11], [12]; analysis of plates subjected to the static
and dynamic effects in linear formulation [13], [14]; analysis of composite plates [15], [16]. The convergence of obtained results from two methods and the difference between results of mentioned method and Volmir’s analytical result show the high accuracy and applicability of both methods in practice.

2. Theory background
For solving the problems of flexible plates subjected to transverse load, differential equations with relationship deflections \( w \), stress function \( \Phi \) and load \( q \) according to [1] are written in the following form:

\[
\frac{D}{H} \nabla^2 \nabla^2 w = L(\Phi, w) + \frac{q}{H};
\]

\[
\frac{1}{E} \nabla^2 \nabla^2 \Phi = -\frac{1}{2} L(\Phi, w, w) ;
\]

In which, \( L(\Phi, w) = \frac{\partial^2 \Phi}{\partial y^2} \cdot \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \cdot \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \cdot \frac{\partial^2 w}{\partial x \partial y} \)

\( H \) – thickness of plate;
\( D \) – Cylindrical stiffness;
\( q \) – Intensity of transverse load, distributed according to given law;
\( E \) – Modulus of elasticity;
\( x, y \) – Coordinates.

The solution of differential equations (1), (2) is shown in [17] is simplified into together solving the following differential equations of second order:

\[
\frac{\partial^3 m}{\partial \xi^3} + \frac{\partial^3 m}{\partial \eta^3} = -g; \quad (3)
\]

\[
\frac{\partial^3 w}{\partial \xi^3} + \frac{\partial^3 w}{\partial \eta^3} = -m; \quad (4)
\]

\[
\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} = \alpha; \quad (5)
\]

\[
\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = -f, \quad (6)
\]

in which:

\[
g = \lambda + \bar{q}; \quad (7)
\]

\[
\xi = \frac{x}{a}; \eta = \frac{y}{a}; \xi = a \nabla; \psi = \frac{q a^3}{D}; \psi = \frac{\Phi H}{D}; \bar{w} = \frac{w}{a}; \quad (8)
\]

\[
\lambda = \frac{\partial^2 \psi}{\partial \eta^2} \cdot \frac{\partial^3 w}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \xi^2} \cdot \frac{\partial^3 w}{\partial \eta^2} - 2 \frac{\partial^2 \psi}{\partial \xi \partial \eta} \cdot \frac{\partial^3 w}{\partial \xi \partial \eta}; \quad (9)
\]
\[
\alpha = -k \left( \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2} - \frac{\partial^2 w}{\partial \xi \partial \eta} \frac{\partial^2 w}{\partial \xi \partial \eta} \right), \quad \text{where} \quad k = \frac{EHa^2}{D}.
\]  

(10)

3. Calculation examples

Example 1: the flexible plate fixed hinged support, subjected to distributed load is analyzed as calculation example (figure 1).

Thickness of plate, sides \(a = 0.1\, \text{cm}\), load \(q=0.5\, \text{kg/cm}^2\), \(E = 0.75 \cdot 10^6\, \text{kg/cm}^2\) and \(\mu = 0.316\). All data are taken from [1].

![Figure 1. Square plate fixed hinged support.](image)

Determine derivatives (9) and (10) using difference equations of successive approximation method and take into account boundary condition, it will be obtained:

\[
\lambda_{11} = \frac{1}{2} m_{11},
\]

(11)

\[
\alpha_{11} = k \cdot \frac{1}{4} m_{11},
\]

(12)

Writing the difference approximation (3)- (6) according to the equations of successive approximation method [18] and taking into account the boundary conditions, it will be obtained:

\[
-20m_{11} = -6h^2 \cdot g_{11};
\]

(13)

\[
-20w_{11} = -\frac{13h^2}{3} \cdot m_{11};
\]

(14)

\[
-20f_{11} = \frac{13h^2}{3} \cdot \alpha_{11};
\]

(15)
\[ -20 \psi_{11} = -\frac{13h^2}{3} \cdot f_{11}. \]  
\[ (16) \]

From (13) and (14) it will be found:

\[ g_{11} = \frac{200}{13h^4} \cdot w_{11}. \]
\[ (17) \]

Taking into account (11), (12), (14), (15) and (16):

\[ \psi_{11} = -\frac{2,6627k}{h^3} \cdot w_{11}. \]
\[ (18) \]

Using (17) and (18) in (7), it will be obtained:

\[ -\frac{200}{13h^4} \cdot w_{11} = q - \frac{2,6627k}{h^3} \cdot w_{11}. \]
\[ (19) \]

In accounting \( q = \frac{q_{a^3}}{D} = 0.5 \cdot 10^3 \), \( h = l/2 \)

and \( k = \frac{EHa^3}{D} = \frac{0.75 \cdot 10^6 \cdot 10^2}{69,433} = 108817.8 \) it will be obtained:

\[ 4636024,62 \bar{w}_{11} + 246,08 \bar{w}_{11} - 7.2 = 0. \leftrightarrow \bar{w}_{11} = 0,0101 \]  

Then \( w_{11} = 0.101 \text{ cm}. \)

Using the block-scheme in [17] with application of the mathematical modeling software package “Matlab R2017b” it will be obtained the following results (table 1).

**Table 1.** Comparison of calculation results of flexible plate by MSA, MFD and an analytical solution according to example 1.

| Step of grid h | Solution according to MSA | Solution according to MFD | Analytical solution [1] |
|----------------|---------------------------|---------------------------|-------------------------|
|                | \( w_{\text{max}} \) [cm] | \( M_{11}(x) \) [\( \text{kg cm} / \text{cm} \)] | \( w_{\text{max}} \) [cm] | \( M_{11}(x) \) [\( \text{kg cm} / \text{cm} \)] | \( w_{\text{max}} \) [cm] |
| 1/4            | 0.133                     | 0.936                     | 0.134                   | 0.918                   |                  |
| 1/8            | 0.141                     | 0.982                     | 0.141                   | 0.976                   |                  |
| 1/16           | 0.143                     | 0.999                     | 0.142                   | 0.998                   |                  |
| 1/32           | 0.143                     | 1.003                     | 0.143                   | 1.003                   | 0.1519           |

\( M_{11}(x) \)-bending moment for point 1 according to axis \( x \) (\( \xi \)).

Example 2: Flexible square plate fixed along free edges subjected to distributed load (figure 2).

Thickness of plate, sides \( a = 0.1 \text{ cm}, \) load \( q = 0.5 \text{ kg/cm}^2, \) \( E = 0.75 \cdot 10^6 \text{ kg/cm}^2 \) and \( \mu = 0.316. \)
Figure 2. Square plate fixed along nearly free edges.

**Table 2.** Comparison of calculation results of flexible plate by MSA, MFD and an analytical solution according to example 2.

| Step of grid h | Solution according to MSA | Solution according to MFD | Analytical solution [1] |
|----------------|---------------------------|---------------------------|-------------------------|
|                | $w_{\text{max}}$ [cm]     | $M_{11}(x)$ [kg cm]       | $M_{10}(x)$ [kg cm]     | $M_{10}(y)$ [kg cm]     | $w_{\text{max}}$ [cm]     |
| 1/4            | 0.075                     | 0.855                     | -0.680                  | -2.153                  | 0.099                     | 0.853                     | -0.491                  | -1.552                  | 0.08075                  |
| 1/8            | 0.0764                    | 0.904                     | -0.728                  | -2.304                  | 0.085                     | 0.908                     | -0.665                  | -2.106                  |
| 1/16           | 0.0778                    | 0.923                     | -0.738                  | -2.334                  | 0.0802                    | 0.92                      | -0.721                  | -2.283                  |
| 1/32           | 0.0782                    | 0.928                     | -0.740                  | -2.342                  | 0.0788                    | 0.929                     | -0.736                  | -2.329                  |

Example 3: flexible square plate with two sides fixed hinged support and two sides fixed edges subjected to distributed load (figure 3). Thickness of plate, sides $a = 0.1 \text{ cm}$, load $q = 0.5 \text{ kg/cm}^2$, $E = 0.75 \times 10^6 \text{ kg/cm}^2$ and $\mu = 0.316$.

Figure 3. Square plate with two sides fixed hinged support and two sides fixed edges.
Example 4: the square flexible plate fixed along nearly free edges, subjected to distributed load on a half of plate (figure 4).

Thickness of plate, sides $a = 0,1 \text{ cm}$, load $q = 0,5 \text{ kg/cm}^2$, $E = 0,75 \cdot 10^6 \text{ kg/cm}^2$ and $\mu = 0,316$.

Table 3. Comparison of calculation results of flexible plate by MSA, MFD according to example 3.

| Step of grid $h$ | Solution according to MSA | Solution according to MFD |
|-----------------|--------------------------|--------------------------|
|                 | $M_{11}(x)$ | $M_{11}(y)$ | $M_{10}(x)$ | $M_{10}(y)$ | $w_{max}$ | $M_{11}(x)$ | $M_{11}(y)$ | $M_{10}(x)$ | $M_{10}(y)$ | $w_{max}$ |
| 1/4             | 0.097        | 0.771        | 1.045        | -0.832       | -2.634     | 0.115       | 0.837        | 0.946        | -0.569       | -1.8 |
| 1/8             | 0.102        | 0.806        | 1.122        | -0.895       | -2.832     | 0.109       | 0.834        | 0.109        | -0.816       | -2.581 |
| 1/16            | 0.1037       | 0.821        | 1.145        | -0.906       | -2.867     | 0.106       | 0.829        | 1.135        | -0.887       | -2.807 |
| 1/32            | 0.1043       | 0.825        | 1.151        | -0.909       | -2.876     | 0.104/8     | 0.827        | 1.149        | -0.904       | -2.862 |

Table 4. Comparison of calculation results of flexible plate by MSA, MFD according to example 4.

| Step of grid $h$ | Solution according to MSA | Solution according to MFD |
|-----------------|--------------------------|--------------------------|
|                 | $M_{11}(x)$ | $M_{11}(y)$ | $M_{10}(x)$ | $M_{10}(y)$ | $w_{min}$ | $M_{11}(x)$ | $M_{11}(y)$ | $M_{10}(x)$ | $M_{10}(y)$ | $w_{min}$ |
| 1/4             | 0.0387       | 0.5918       | 0.5826       | -1.844       | -0.583     | 0.058       | 0.5644       | 0.5531       | -1.222       | -0.386 |
| 1/8             | 0.0422       | 0.5572       | 0.5512       | -1.925       | -0.608     | 0.048       | 0.5493       | 0.5436       | -1.718       | -0.543 |
| 1/16            | 0.0429       | 0.5465       | 0.5425       | -1.931       | -0.610     | 0.0444      | 0.5443       | 0.5404       | -1.878       | -0.593 |
| 1/32            | 0.0431       | 0.5436       | 0.5402       | -1.933       | -0.611     | 0.0434      | 0.5430       | 0.5397       | -1.919       | -0.606 |
Figure 5. Diagram of bending moment $M(x)$ for $h=1/32$ in example 4.

Figure 6. Diagram of bending moment $M(y)$ for $h=1/32$ in example 4.

Figure 7. Diagram of deflections for $h=1/32$ in example 4.

Figure 8. Diagram of bending moment $M(x)$ for $h=1/32$ in example 4 according to line 01-11-21.

Figure 9. Diagram of bending moment $M(y)$ for $h=1/32$ in example 4 according to line 01-11-21.
Figure 10. Diagram of bending moment \( M(x) \) for \( h=1/32 \) in example 4 according to line 10-11-12.

Figure 11. Diagram of bending moment \( M(y) \) for \( h=1/32 \) in example 4 according to line 10-11-12.

4. Summary

Due to the problems of analysis are geometrically nonlinear, the principle of independence of the action of forces is not valid for them. This paper presents numerical simulation with its results.

Comparing the results of examples 2 and 4, it can be seen that \( 2w_{11} = 2.0.0431cm \) in example 4 is not equal to \( w_{11} = 0.0782cm \) in example 2.

The obtained results imply the efficiency of applying the MSA and MFD equations to the plate analysis in a nonlinear formulation using mathematical software. They confirm each other, practically approach to the same result.

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