ABSOLUTELY CONTINUOUS SPECTRUM FOR
QUASI-PERIODIC SCHRÖDINGER OPERATORS

HELGE KRÜGER

Abstract. I prove that quasi-periodic Schrödinger operators in arbitrary di-

dimension have some absolutely continuous spectrum.

1. Introduction

Let $d \geq 1$ and consider the family of Schrödinger operators $H_{\lambda,\alpha,x} = \Delta + \lambda V_{\alpha,x}$ acting on $\ell^2(\mathbb{Z}^d)$ where $\lambda > 0$ is a coupling constant,

\begin{equation}
\Delta \psi(n) = \sum_{|e|=1} \psi(n + e), \quad |x|_1 = |x_1| + \cdots + |x_d|
\end{equation}

is the discrete Laplacian, and the potential $V_{\alpha,x}$ is the multiplication operator with the sequence

\begin{equation}
V_{\alpha,x}(n) = f(x + \alpha \star n), \quad (\alpha \star n)_j = \alpha_j n_j
\end{equation}

$\alpha \in \mathbb{R}^d$, $x \in \mathbb{T}^d$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $f : \mathbb{T}^d \to \mathbb{R}$ is a non constant real-analytic function.

My main goal will be to show

\begin{equation}
\text{Theorem 1.1.} \quad \text{Let $\varepsilon > 0$. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, there exists a set of frequencies $A_{\lambda} \subseteq [0, 1]^d$ of measure $|A_{\lambda}| \geq 1 - \varepsilon$ such that for $\alpha \in A_{\lambda}$, $x \in \mathbb{T}^d$ the Schrödinger operator $H_{\lambda,\alpha,x} = \Delta + \lambda V_{\alpha,x}$ has some absolutely continuous spectrum.}
\end{equation}

Previously, Bourgain [5], [6] has shown that this class of operators exhibits extended states and in particular the spectrum is purely continuous. Denoting by $\sigma_{\text{ac}}(H_{\lambda,\alpha,x})$ the absolutely continuous spectrum of $H_{\lambda,\alpha,x}$, I will in fact show that

\begin{equation}
|\sigma(\Delta) \setminus \sigma_{\text{ac}}(H_{\lambda,\alpha,x})| \leq \varepsilon, \quad \sigma(\Delta) = [-2d, 2d].
\end{equation}

Unfortunately, I am unable to address the structure of $\sigma_{\text{ac}}(H_{\lambda,\alpha,x})$ as a set or to show that the absolutely continuous spectrum is pure.

I want to mention at this point that the results of this paper can be extended in various ways with minimal effort. Maybe, the most important one is that instead of considering $\psi$ to be real valued, one could take $\psi(n) \in \mathbb{C}^k$ and $V(n)$ to be an appropriate Hermitian matrix. This extension would allow one to first consider Schrödinger operators on so called strips, i.e. defining the Laplacian on $\mathbb{Z} \times \{1, \ldots, W\}$ for some $W \geq 2$. Furthermore, this would allow one to consider

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periodic directions in the definition of $V$. Finally, it should be possible to replace the Laplacian $\Delta$ by a more general hopping operator $T$ of the form
\begin{equation}
T\psi(n) = \sum_{k \in \mathbb{Z}^d} t_k \psi(n + k)
\end{equation}
for $|t_k| \leq e^{-\eta |k|}$ for some $\eta > 0$.

The proof to exhibit absolutely continuous spectrum proceeds by first studying the dual operator $\hat{H}_{\lambda,\alpha,x}$ which exhibits Anderson localization. As already mentioned, this implies that the operator $H_{\lambda,\alpha,x}$ has extended states. In order to obtain absolutely continuous spectrum, I will need to obtain further control on the eigenvalues of $\hat{H}_{\lambda,\alpha,x}$. This will be done by using the methods of [19]. For an implementation of this strategy in the simpler context of limit-periodic Schrödinger operators see [18]. For further consequences of the eigenvalue perturbation results for the skew-shift, see [20]. I expect that the methods of [20] can be used to prove results about the integrated density of states and eigenvalue statistics for quasi-periodic Schrödinger operators. However, the statements will be weaker as the set of eliminated frequencies depends on the energy.

Let us now review what else is known about the absolutely continuous spectrum for Schrödinger operators. Proving absolutely continuous spectrum for the free and periodic Schrödinger operators can be done by fairly standard methods. Proofs in dimension one usually rely on the fact that the Schrödinger equation
\begin{equation}
H_x u(n) = u(n+1) + u(n-1) + f(T^n x) u(n)
\end{equation}
is equivalent to the transfer matrix equation
\begin{equation}
\begin{pmatrix}
u(n+1) \\ u(n)
\end{pmatrix} = A(T^n x) 
\begin{pmatrix}
u(n) \\ u(n-1)
\end{pmatrix}, \quad A(x) = \begin{pmatrix} E - f(x) & -1 \\ 1 & 0 \end{pmatrix},
\end{equation}
where $T x = x + \alpha \pmod{1}$. Using methods based on KAM, one can show absolutely continuous spectrum by showing that the cocycle
\begin{equation}
T \times \mathbb{C}^2 \ni (x,v) \mapsto (x + \alpha, A(x)v)
\end{equation}
is conjugated to a cocycle of rotations. This was first done by Dinaburg and Sinai [10] then improved by Eliasson [11] to prove purely absolutely continuous spectrum. The final improvement was by Avila and Jitomirskaya [4] using input based on duality. Furthermore, Kotani theory [9] allows one to describe the absolutely continuous spectrum by the vanishing of the Lyapunov exponent, $L(E) = \lim_{n \to \infty} \frac{1}{n} \int \log \|A(T^{n-1} x) \cdots A(x)\| dx$. This has allowed Avila [3] to give an excellent description of the absolutely continuous spectrum.

For operators in higher dimensions, the main work on understanding almost-periodic potentials has been done in the continuum. In [15], Karpeshina and Lee have shown that the Schrödinger operator $H = -\Delta + V$ acting on $L^2(\mathbb{R}^2)$ with $V$ a small enough limit-periodic potential has some absolutely continuous spectrum at high energies and that the spectrum contains a semi-axis. In the discrete case, I have shown [18] that limit-periodic potentials which are sufficiently well approximated by periodic potentials lead to purely absolutely continuous spectrum.

Finally, in dimension two and for polyharmonic operators, i.e. $(-\Delta)^\ell$ for $\ell \geq 2$, Karpeshina and Shterenberg have exhibited in [16] the existence of an absolutely continuous component in the spectrum of a quasi-periodic operator on $L^2(\mathbb{R}^2)$. 
Our conceptual understanding of the absolutely continuous spectrum in dimensions $d \geq 2$ is much weaker than in one dimension. Due to subordinacy theory [14], [12], we can morally characterize the absolutely continuous spectrum as the set of energies, such that there exists a bounded solution. The best, we can do in higher dimensions is [17], which is too weak to imply the results of this paper or [18].

In the case of ergodic operators so in particular almost-periodic ones, one even has Kotani theory and thus can describe the absolutely continuous spectrum as the essential closure of the set of energies where the Lyapunov exponent vanishes.

Finally, let me mention that proving pure point spectrum respectively proving that the spectrum is purely continuous is a statement about solutions as one shows that all generalized eigenfunctions are either square integrable or not. It would be very interesting to obtain a description of the absolutely continuous spectrum in terms of solutions in arbitrary dimension.

My methods are based on considering the dual operator. This concept has been introduced in [1], [2] for the almost Mathieu operator. This concept was further developed in [13], [8].

As already mentioned the proof in this paper proceeds by using that Anderson localization holds for the dual operator $\hat{H}_{\lambda,\alpha}$ defined in (2.3) respectively its fibers see (2.4). This implies in particular that for the almost every energy $E$ in the absolutely continuous component exhibited in Theorem 1.1 there exists an extended state. By an extended state, I mean in this context an almost-periodic solution $u : \mathbb{Z}^d \to \mathbb{C}$ of the eigenvalue problem

$$H_{\lambda,\alpha,x}u = Eu.$$  

(1.8)

The main problem in order to carry out such a construction is that one needs to relate the eigenfunctions of $\hat{H}_{\lambda,\alpha}^{(0)}$ and $\hat{H}_{\lambda,\alpha}^{\Lambda_r(0)}$ for $r < R$, where $\Lambda_r(0) = [-r,r]^d$. This is only possible if one knows that the corresponding eigenvalues are simple. In order to ensure this, we use the methods of [19], which put as into a perturbative regime. This problem was solved in [18] by a novel estimate.

I believe that obtaining an understanding of how to prove simplicity of the eigenvalues in a problem of this type, would lead to major improvements in all known results.

2. Strategy of the proof

Let us now deal with the specifics of the proof. First, we can write $f$ in Fourier series as

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e(k \cdot x), \quad e(t) = e^{2\pi it}, \quad k \cdot x = \sum_{j=1}^{d} k_jx_j.$$  

(2.1)

We will restrict ourself for simplicity to the potential $V_\alpha(n) = f(n \ast \alpha)$. For $u \in \ell^1(\mathbb{Z}^d)$, we define its Fourier transform by $\hat{u}(x) = \sum_{k \in \mathbb{Z}^d} e(k \cdot x)u(k)$. The Fourier transform of $(\Delta + \lambda V_\alpha)\psi$ is given by

$$\sum_{j=1}^{d} 2\cos(2\pi x_j)\hat{\psi}(x) + \lambda \sum_{k \in \mathbb{Z}^d} \hat{f}(k)\hat{\psi}(x + k \ast \alpha).$$  

(2.2)
We define the operator $\hat{H}_{\lambda,\alpha} : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by

\begin{equation}
\hat{H}_{\lambda,\alpha}\psi(x) = \sum_{j=1}^{d} 2\cos(2\pi x_j)\psi(x) + \lambda \sum_{k \in \mathbb{Z}^d} \hat{f}(k)\psi(x + k \ast \alpha).
\end{equation}

We see that $\hat{H}_{\lambda,\alpha}\psi(x)$ only depends on $\{\psi(x + k \ast \alpha)\}_{k \in \mathbb{Z}^d}$. It thus makes sense to consider the fibered operator

\begin{equation}
\hat{H}_{\lambda,\alpha,\epsilon}\psi(n) = \left(\sum_{j=1}^{d} 2\cos(2\pi(x_j + n_j \alpha_j))\right)\psi(n) + \lambda \cdot \sum_{k \in \mathbb{Z}^d} \hat{f}(k)\psi(n + k).
\end{equation}

As $f$ is real-analytic, we have that $|\hat{f}(k)| \leq Ce^{-\eta|k|}$ for $C, \eta > 0$. For simplicity, we will assume in the following that $|\hat{f}(k)| \leq e^{-\eta|k|}$, which is possible by changing $\lambda > 0$. We can write $\hat{H}_{\lambda,\alpha,\epsilon} = T + W_{\alpha,\epsilon}$, where $T$ is the hopping operator

\begin{equation}
T\psi(n) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k)\psi(n + k)
\end{equation}

and $W_{\alpha,\epsilon}$ is the multiplication operator by the sequence

\begin{equation}
W_{\alpha,\epsilon}(n) = \sum_{j=1}^{d} 2\cos(2\pi(x_j + n_j \alpha_j)) = W(x + n \ast \alpha), \quad W(x) = \sum_{j=1}^{d} 2\cos(2\pi x_j).
\end{equation}

In particular, $\hat{H}_{\lambda,\alpha,\epsilon}$ is again a quasi-periodic operator but the coupling constant is changed from small to large. Finally, we note that we view $\hat{H}_{\lambda,\alpha,\epsilon}$ as an operator acting on $\ell^2(\mathbb{Z}^d)$.

Given $\Lambda \subseteq \mathbb{Z}^d$, we denote by $A^\Lambda$ the restriction to $\ell^2(\Lambda)$ of an operator $A : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$. We introduce the cube

\begin{equation}
\Lambda_r(n) = \{x \in \mathbb{Z}^d : |n - x|_\infty \leq r\}
\end{equation}

where $|x|_\infty = \max(|x_1|, \ldots, |x_d|)$. Finally, we introduce the following definition.

**Definition 2.1.** An eigenvalue $E$ of a self-adjoint operator $A$ is called $\delta$-simple if

\[\text{tr}(P_{E-E+\delta}(A)) = 1.\]

The following theorem provides the perturbative analysis of eigenfunctions. It is note worthy that the perturbation parameter is the frequency $\alpha \in [0,1]^d$, which enters the problem as the fast variable.

**Theorem 2.2.** Let $\varepsilon \in (0,1)$ and $R_1 \geq 1$ be large enough. There exists $\lambda_1 > 0$ and sequences $R_1 < R_2 < R_3 < \ldots$, $\delta_1 > \delta_2 > \delta_3 > \ldots$ such that for $\lambda \in (0,\lambda_1)$ and $y \in \mathbb{T}^d$, we have

(i) $R_j = (R_{j-1})^{10}$, $\delta_1 = \lambda^\frac{1}{10}$ and $\delta_j = \lambda^\frac{1}{10} \exp(-(R_{j-1})\frac{1}{2})$ for $j \geq 2$.

(ii) There exists $G_y \subseteq [0,1]^d$ of measure $|G_y| \geq 1 - \varepsilon$.

(iii) For $j \geq 1$, there exists a function $E_j : G_y \to \mathbb{R}$ such that for $\alpha \in G_y$

\begin{equation}
E_j(\alpha) = \sigma(\hat{H}_{\lambda,\alpha,\epsilon}^{\Lambda_{R_j}(0)})
\end{equation}

is $\delta_j$-simple.

(iv) We have $|E_j(\alpha) - E_{j-1}(\alpha)| \leq (\delta_j)^{10}$ for $\alpha \in G_y$ and $j \geq 0$. For $\psi_j$ the corresponding normalized eigenfunctions, we have

\begin{equation}
||\psi_j - \psi_{j-1}|| \leq (\delta_j)^3.
\end{equation}
In the theorem, when \( j = 0 \) we formally set \( \psi_{-1} = \delta_0 \) and \( E_{-1} = \sum_{j=1}^{d} 2 \cos(2\pi y_j) \).

We will explain the proof of this theorem in Section 4. Large parts of the proof follow ideas from [19].

In order to deduce Theorem 4.1, we will need to reformulate the conclusions of the previous theorem. In particular, instead of fixing \( L \) in (2.14), \( |A| \geq \) we've constructed so far doesn't necessarily lead to a nice function \( E : G \rightarrow \mathbb{R} \).

By the previous theorem, we have that \( |A| \geq 1 - \varepsilon \). Introduce \( A = \{ x, \alpha \in T^d \times [0,1]^d : \alpha \in G_x \} \).

By construction, we clearly have that \( G \) and a map \( \psi : T^d \rightarrow \ell^2(\mathbb{Z}^d) \) such that

\[
(2.11) \quad A = \{ \alpha : |\{ x : (x, \alpha) \in A \}| \geq 1 - \varepsilon \}. 
\]

One can check that \( |A| \geq 1 - \varepsilon \). Let us now fix \( \alpha \in A \) and define

\[
(2.12) \quad G = \{ x : (x, \alpha) \in A \}. 
\]

By construction, we clearly have that \( |G| \geq 1 - \varepsilon \). For \( x \in G \) the conclusions (ii)-(v) of the previous theorem hold. Unfortunately, taking the limit of the eigenvalues, we've constructed so far doesn't necessarily lead to a nice function \( E : G \rightarrow \mathbb{R} \).

The following proposition remedies this situation.

**Proposition 2.3.** Let \( \varepsilon_1 > \varepsilon \). There exist \( \delta > 0, G_1 \subseteq T^d \), a function \( \gamma : T^d \rightarrow \mathbb{R} \), and a map \( \psi : T^d \rightarrow \ell^2(\mathbb{Z}^d) \) such that

(i) \( |G_1 \cap G| \geq 1 - \varepsilon_1 \).

(ii) \( |\nabla \gamma(x)| \geq \delta \) for \( x \in G_1 \).

(iii) \( \forall x \in G_1 \cap G \) we have

\[
(2.13) \quad \gamma(x) \in \sigma(\hat{H}_{\lambda,\alpha,x}). 
\]

(iv) \( \|\psi(x)\| = 1 \) for \( x \in G_1 \cap G \) and \( \psi(x) = 0 \) for \( x \in T^d \setminus (G_1 \cap G) \).

(v) \( \hat{H}_{\lambda,\alpha,x} \psi(x) = \gamma(x) \psi(x) \) for \( x \in G_1 \cap G \).

(vi) \( \forall x \in G_1 \cap G \), we have \( \|\psi(x)\|_{\ell^1(\mathbb{Z}^d)} \leq 2 \).

(vii) \( \forall x \in T^d \) and let

\[
(2.14) \quad L = \{ \ell : x + \ell = x_1 \cap G \}, \quad \psi_\ell(x; n) = \psi(x - \ell \cdot \alpha; n + \ell).
\]

Then the \( \{ \psi_\ell \}_{\ell \in L} \) form an orthonormal set in \( \ell^2(\mathbb{Z}^d) \) consisting of eigenfunctions of \( \hat{H}_{\lambda,\alpha,x} \). Finally \( \hat{H}_{\lambda,\alpha,x} \psi_\ell(x) = \gamma(x - \ell \cdot \alpha) \psi_\ell(x) \).

We denote by \( \Gamma : L^2(T^d) \rightarrow L^2(T^d) \) the multiplication operator by \( \chi_{G \cap G_1} \gamma \). By (vi), we can define for \( g \in L^\infty(T^d) \)

\[
(2.15) \quad Qg(x) = \sum_{k \in \mathbb{Z}^d} q_k(x) g(x + k \cdot \alpha), \quad q_k(x) = \chi_G(x + k \cdot \alpha) \cdot \psi(x + k \cdot \alpha; -k).
\]

A formal computation shows that \( \hat{H}_{\lambda,\alpha} Q = Q \Gamma \) is equivalent to

\[
(2.16) \quad \sum_{j=1}^{d} 2 \cos(2\pi x_j) q_\ell(x) + \lambda \sum_{k \in \mathbb{Z}^d} \hat{f}(k) q_{\ell-k}(x + k \cdot \alpha) = \gamma(x + \ell \cdot \alpha) q_\ell(x).
\]

Using that \( q_\ell(x) = \psi_{-\ell}(x; 0) \) and \( q_{-\ell}(x + k \cdot \alpha) = \psi_{-\ell}(x; k) \) in the notation of Proposition 2.3 one easily verifies this. The next lemma establishes that \( Q \) is a bounded operator and thus that we can make this computation.

**Lemma 2.4.** The operator \( Q \) is bounded \( L^2(T^d) \rightarrow L^2(T^d) \).
Lemma 3.1. For $\ell$ in (3.1) $\|g\|_{L^2(\mathbb{T}^d)}^2 = \int_{\mathbb{T}^d} \sum_{n \in \mathbb{Z}^d} g(y)g(y + n \cdot \alpha) \cdot \sum_{k \in \mathbb{Z}^d} \psi(y; -k)\psi(y + n \cdot \alpha; -n - k) dy.$

By Proposition 2.3 (iv), we have that the sum over $k$ is equal 1 if $n = 0$ and equal to 0 otherwise. The claim follows.

One can now formally compute the adjoint of $Q$ to be

$$(2.17) \quad Q^* g(x) = \sum_{k \in \mathbb{Z}^d} \chi_G(x)\psi(x; -k)g(x - k \cdot \alpha).$$

A quick computation shows that $Q^* Q = QQ^* = \chi_G$. In particular, we have that $\|Q\| = 1$. We are now ready for

Proof of Theorem 1.1. The previous lemma shows that $Q$ conjugates the restriction $\tilde{\mathcal{H}}_{\lambda,\alpha}^{G(1)}$ to $L^2(\mathbb{G} \cap \mathbb{G}_1)$ to the multiplication operator by $\gamma$. Hence, it suffices to prove that for $E \in \mathbb{R}$ and $s > 0$, we have that

$$\{|x \in G : \gamma(x) \in [E - s, E + s]\| \lesssim \frac{s}{\delta}.$$  

This follows by the first part of Proposition 2.3.

3. Niceness of the eigenvalue parametrisation: Proof of Proposition 2.3

In order to prove Proposition 2.3 we will need to reformulate the conclusions of Theorem 2.2 for fixed $\alpha \in \mathcal{A}$. There exists a sequence of sets $G_j \subseteq \mathbb{Z}^d$, functions $E_j : G_j \to \mathbb{R}$, and $\psi_j : G_j \to \ell^2(\Lambda_{R_j}(0))$ with the following properties

(i) $G_j \subseteq G_{j-1}$, $|G_j| \geq 1 - \varepsilon^\frac{1}{4}$.
(ii) For $x \in G_j$, we have $E_j(x) \in \sigma(\tilde{\mathcal{H}}_{\lambda,\alpha}^{(0)\otimes_j})$ is $\delta_j$ simple.
(iii) For $x \in G_j$, $\tilde{\mathcal{H}}_{\lambda,\alpha}^{(0)\otimes_j} \psi_j(x) = E_j(x)\psi_j(x)$.
(iv) For $x \in G_j$, $\|\psi_j(x)\| = 1$.
(v) For $x \in G_j$, $\|E_j(x) - E_{j-1}(x)\| \leq (\delta_j)^1$ and $\|\psi_j(x) - \psi_{j-1}(x)\| \leq (\delta_j)^3$.

We begin by understanding the limit of the functions $\psi_j(x)$.

Lemma 3.1. For $x \in G = \bigcap_{j \geq 1} G_j$ with $|G| \geq 1 - \varepsilon^\frac{1}{4}$, there exists $\psi(x) \in \ell^2(\mathbb{Z}^d)$ such that $\psi$ solves $\tilde{\mathcal{H}}_{\lambda,\alpha} \psi(x) = E(x)\psi(x)$ for some $E(x)$. We have that $\psi_j \to \psi$ in $\ell^2(\mathbb{Z}^d)$ and

$$(3.1) \quad \|\psi(x)\|_{\ell^1(\mathbb{Z}^d)} \leq 2.$$  

Proof. By (v) the $\psi_j(x)$ form a Cauchy sequence. Hence $\psi(x)$ exists. By continuity of the norm $\|\psi(x)\| = 1$. Finally, $\psi(x)$ solves the eigenvalue problem for $E(x) = \lim_{j \to \infty} E_j(x)$.

As $\psi_j(x)$ is supported in a set containing less than $(3R_j)^d$ many elements, we have

$$\|\psi_j(x) - \psi_{j-1}(x)\|_{\ell^1(\mathbb{Z}^d)} \leq (3R_j)^d(\delta_j)^3 \leq (\delta_j)^2.$$  

Hence, the $\psi_j(x)$ are also Cauchy in $\ell^1(\mathbb{Z}^d)$ and thus also is $\psi(x)$. Finally, we have

$$\|\psi(x)\|_{\ell^1(\mathbb{Z}^d)} \leq 1 + \sum_{j=0}^{\infty} \|\psi_j(x) - \psi_{j-1}(x)\|_{\ell^1(\mathbb{Z}^d)} \leq 1 + 2\lambda^{\frac{1}{d}} \leq 2.$$
as $\|\psi_{-1}\|_{L^2(\mathbb{Z}^d)} = 1$. □

The next step in our analysis will be to understand the derivative of the functions $E_j(x)$ for $x \in G_j$. The next lemma implies in particular, that $\nabla E_j(x)$ makes sense.

**Lemma 3.2.** Let $x \in G_j$ and $U = B_{(\delta_j)^2}(x)$. Then there exists an analytic function $f : U \to \mathbb{R}$ such that

1. $f(x) = E_j(x)$.
2. For $y \in U$, $f(y)$ is a $\frac{1}{2}\delta_j$ simple eigenvalue of $\hat{H}_{\lambda,\alpha,y}^{A_{\nu_j}(0)}$.
3. For $y \in U$, $|f(y) - E_j(x)| \leq C|y - x|$ for some $j$ independent $C > 0$.

**Proof.** For $y \in U$, we have

$$\|\hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)} - \hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)}\| \leq \|\nabla W\|_{L^\infty(\mathbb{T}^d)}\delta_j^2 \leq \frac{1}{4}\delta_j,$$

where $W(x) = \sum_{j=1}^d 2\cos(2\pi x_j)$. Define $f$ to be the analytic continuation of the necessarily simple eigenvalue $E_j(x)$. It is clear that (i) and (ii) hold. Furthermore, (iii) follows with $C = \|\nabla W\|_{L^\infty(\mathbb{T}^d)}$. □

**Lemma 3.3.** There exists $\kappa > 0$ and a set $G^\kappa \subseteq \mathbb{T}^d$ of measure $|G^\kappa| \geq 1 - \varepsilon^\frac{1}{2}$ such that for $x \in G^\kappa = G^\kappa \cap G_j$, we have

$$\|\nabla E_j(x)\| \geq \kappa.$$

**Proof.** Recall that $\gamma_{-1}(x) = \sum_{j=1}^d 2\cos(2\pi x_j)$ and define

$$G^\kappa = \{x \in \mathbb{T}^d : |\nabla \gamma_{-1}(x)| \geq 2\kappa\}$$

we have $|G^\kappa| \to 0$ as $\kappa \to 0$. So we may choose $\kappa$ such that $|G^\kappa| = 1 - \varepsilon^\frac{1}{2}$.

By the previous lemma, we can extend $E_j$ to an analytic function in a small neighborhood of $x$. Standard perturbation theory then implies

$$\nabla E_j(x) = \left\langle \psi_j(x), \nabla \hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)} \psi_j(x) \right\rangle.$$ 

For $x \in G^\kappa$, we clearly have that

$$|\left\langle \delta_0, \nabla \hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)} \delta_0 \right\rangle| \geq 2\kappa.$$

We have that

$$\|\psi_j(x) - \delta_0\| \leq \sum_{j=0}^j \|\psi_j(x) - \psi_{j-1}(x)\| \leq \lambda^{\frac{1}{10}}.$$

Then as

$$|\left\langle \psi_j(x), \nabla \hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)} \psi_j(x) \right\rangle - \left\langle \delta_0, \nabla \hat{H}_{\lambda,\alpha,x}^{A_{\nu_j}(0)} \delta_0 \right\rangle| \leq 2\|\nabla W\|_{L^\infty(\mathbb{T}^d)}\|\psi_j(x) - \delta_0\|$$

and $\lambda \leq \kappa^{10}$, the claim follows. □

We will now start to construct the function $\gamma$ described in Proposition 2.3. To do so, we will construct an extension $\gamma_j : G^\kappa \to \mathbb{R}$ of $E_j : G^\kappa_j \to \mathbb{R}$. This extension should have the properties

1. $\gamma_j(x) = E_j(x)$ for $x \in G^\kappa_j$.
2. $|\gamma_j(x) - \gamma_j-1(x)| \leq (\delta_j)^{10}$ for $x \in G^\kappa$.
3. $|\nabla \gamma_j(x)| \geq \kappa - \delta_1 - \cdots - \delta_j$. 
These properties guarantee that (i) through (v) of Proposition 2.3 hold as we have already observed that the eigenfunctions converge. Also (vi) holds, we will prove (vii) at the end of this section. Let us now explain how to construct $g_j$. First, it is clear that the claim holds for $g_{j-1}$, so we only have to construct $g_j$ given $g_{j-1}$.

Define a function $\varphi \colon G_j^\kappa \to \mathbb{R}$ by

$$\varphi(x) = g_{j-1}(x) - E_j(x).$$

We clearly have that $|\varphi(x)| \leq (\delta_j)^{10}$ In order to prove the claim, it suffices to prove that there exists an extension of $\varphi$ to $G^\kappa$ satisfying $|\nabla \varphi| \leq \delta_j$. Let us now recall the conclusions of Lemma 3.2. For $x \in G_j^\kappa$, we can find a function $g_x : U \to \mathbb{R}$ where $U = B_{\delta_j}^2$ such that $\varphi(x) = g_x(x)$ and $|\nabla g_x| \leq C$.

**Lemma 3.4.** For $y \in G_j^\kappa \cap U$, we have $g_x(y) = \varphi(y)$.

**Proof.** We clearly have that $|g_x(y)| \leq (\delta_j)^{10} + C(\delta_j)^2 \leq \delta_j^2$, as $E_j(x)$ is $\delta_j$ simple the claim follows. □

The following lemma now guarantees the existence of $\varphi$.

**Lemma 3.5.** Let $A \subseteq \mathbb{R}^d$ be a set and $f : A \to \mathbb{R}$ a function such that

(i) For $x \in A$, $|f(x)| \leq \varepsilon$.

(ii) For $x \in A$, there exists $g_x : B_{\delta}(x) \to \mathbb{R}$ such that $g_x(y) = f(y)$ for $y \in A \cap B_{\delta}(x)$ and $|g_x(y) - f(x)| \leq C|x - y|$.

Then there exists $F : \mathbb{R}^d \to \mathbb{R}$ such that $f(x) = F(x)$ for $x \in A$ and $|\nabla F(x)| \leq C\frac{x}{\varepsilon}$.

**Proof.** By condition (ii), we can extend $f$ to the $\delta$ neighborhood of $A$ by just setting it equal to the functions $g_x$. Let $f_1$ the extension by 0 of this function to $\mathbb{R}^d$. Let $\eta : \mathbb{R}^d \to \mathbb{R}$ be a mollifier that is $\int \eta(x)dx = 1$, $\eta \geq 0$, and $\text{supp}(\eta) \subseteq B_1(0)$. We set $\eta_{\delta}(x) = \frac{1}{\delta^n}\eta(x/\delta)$.

Finally, we define

$$s(x) = \begin{cases} \delta/6, & \text{dist}(x, A) \geq \delta/6; \\ \text{dist}(x, A), & \text{otherwise}. \end{cases}$$

We are now ready to define

$$F(x) = \begin{cases} f(x), & x \in A; \\ \int f_{\delta}(x, y) \eta_{\delta}(x)(x - y)f_1(y)dy, & x \notin A. \end{cases}$$

First it is clear that $F$ defines a continuous function for $x \notin A$. As $f_1$ is continuous and $\eta_\delta \to \delta$ as $s \to 0$, it follows that $F$ is continuous on $\mathbb{R}^d$. To see the estimates on the gradient, we observe that for $\text{dist}(x, A) \geq \delta$, we have

$$|\nabla F(x)| \leq \|f_1\|_{L^\infty(\mathbb{R}^d)} \cdot \|\nabla \eta_{\delta}(x)\|_{L^1(\mathbb{R}^d)} \leq \varepsilon \cdot \frac{\delta}{6} \cdot \|\nabla \eta\|_{L^1(\mathbb{R}^d)}.$$ 

For $\text{dist}(x, A) < \delta/6$, we have that

$$|\nabla F(x)| \leq \|\nabla f_1\|_{L^\infty(B_{\delta}(x)(x))} \cdot \|\eta_{\delta}(x)\|_{L^1(\mathbb{R}^d)}.$$ 

From this the claim follows. □

In order to prove Proposition 2.3 it remains to prove (vii) that is understand the function $\psi(x)$. First, it is clear that $\psi_j(x) \to \psi(x)$ in $\ell^2(\mathbb{Z}^d)$ for $x \in G = \bigcap_{j=1}^\infty G_j$. 


Lemma 3.6. Let $x \in \mathbb{T}^d$, the set of functions
\begin{equation}
\psi_\ell(x; n) = \psi(x - \ell \cdot \alpha; n + \ell)
\end{equation}
where $x - \ell \cdot \alpha \in G$ form an orthonormal set in $\ell^2(\mathbb{Z}^d)$ consisting of different eigenfunctions of $\hat{H}_{\lambda, \alpha, x}$.

Proof. One can check that $\hat{H}_{\lambda, \alpha, x} \psi_\ell(x) = \gamma(x - \ell \cdot \alpha) \psi_\ell(x)$. Hence, the $\psi_\ell(x)$ are all eigenfunctions. Furthermore, they are all different as
\[ |\langle \psi_\ell(x), \psi_k(x) \rangle - \langle \delta_\ell, \delta_k \rangle | \leq 12 \lambda^{10}. \]
If the $\gamma(x - \ell \cdot \alpha)$ are all different, we are done as the eigenfunctions of a self-adjoint operator to different eigenvalues are automatically orthonormal.

Let us now show that $E = \gamma(x - \ell \cdot \alpha) = \gamma(x - k \cdot \alpha)$ cannot happen for $k \neq \ell$ and $x - \ell \cdot \alpha, x - k \cdot \alpha \in G$. We can assume that $k = 0$ and choose $j$ so large that $R_j \geq 10 |\ell|$. Let $\varphi_1(n) = \psi_{j-1}(x; n)$, $\varphi_2(n) = \psi_{j-1}(x - \ell \cdot \alpha; n + \ell)$.

We have that
\[ \| (\hat{H}_{\lambda, \alpha, x} - E) \varphi_t \| \leq \sum_{s \geq j+1} (\delta_s)^{10} + e^{-\sqrt{R_j}} \leq \delta_j^5 \]
for $t = 1, 2$ and $|\langle \varphi_1, \varphi_2 \rangle | \leq 12 \lambda^{10}$. This implies that
\[ \text{tr}(P_{|E_j(x) - (\delta_j)^2 E_j(x) + (\delta_j)^2} (\hat{H}_{\lambda, \alpha, x}^{\Lambda R_j(0)})) \geq 2. \]
This is a contradiction finishing the proof. 

4. CONTROL ON THE EIGENVALUES: PROOF OF THEOREM 2.2

The first step of the proof will be to prove the following initial condition.

Proposition 4.1. Let $\varepsilon > 0$ then there exists $\delta > 0$ such that the following holds. Let $R \geq 1, x \in \mathbb{T}^d$. Then there exists $\lambda_2 > 0$ such that there exists a set $G_1 \subseteq [0,1]^d$ of measure $|G_1| \geq 1 - \frac{\varepsilon}{2}$ such that for $\alpha \in G_1$, we have
\begin{equation}
E \text{ is a } \delta\text{-simple eigenvalue of } \hat{H}_{\lambda, \alpha, x}^{\Lambda n(0)}
\end{equation}
for $\lambda \in (0, \lambda_2)$ and some $E$ satisfying
\begin{equation}
|E - \sum_{j=1}^d 2 \cos(2\pi x_j)| \leq \lambda \|f\|_{L^\infty(\mathbb{T}^d)}.
\end{equation}
Furthermore, the corresponding normalized eigenfunction $\psi$ can be chosen to satisfy
\begin{equation}
\|\psi - \delta_0\| \leq \frac{2\|f\|_{L^\infty(\mathbb{T}^d)}}{\delta} \cdot \lambda.
\end{equation}

We will provide the proof of this proposition in Section 5. It is essentially a simple test function construction. We see that $\varepsilon > 0$ dictates that $\delta_1$ must be smaller than $\delta$ thus imposes an additional smallness condition on $\lambda$, i.e. $\lambda \leq \delta^{10}$.

In order to finish the proof of Theorem 2.2 we now need to show that the conclusions for $j - 1$ imply the conclusions of $j$. In order to accomplish this, we
will need to introduce a bit of notation and review some results from \cite{6}. Given
\( \Lambda \subseteq \mathbb{Z}^d \), \( n, m \in \Lambda \), and \( E \in \mathbb{R} \), we introduce the Green’s function by
\[
G_{\lambda,\alpha,x}(E; n, m) = \langle \delta_n, (\hat{H}_{\lambda,\alpha,x} - E)^{-1} \delta_m \rangle.
\]
In order to quantify the behavior of the Green’s function, we introduce

\begin{definition}
Let \( \gamma > 0 \), \( \tau \in (0, 1) \). \( \Lambda_R(0) \) is called \((\gamma, \tau)\)-suitable for \( \hat{H}_{\lambda,\alpha,x} - E \) if
\begin{enumerate}
\item \( \| (\hat{H}_{\lambda,\alpha,x}^R - E)^{-1} \| \leq e^{R^\tau} \).
\item For \( n, m \in \Lambda_R(0) \), \( |n - m| \geq R/2 \), we have
\[
|G_{\lambda,\alpha,x}^R(E; n, m)| \leq e^{-\gamma |n - m|}.
\]
\end{enumerate}
\end{definition}

An adaptation of the argument of \cite{6} shows that

\begin{theorem}
Let \( \varepsilon > 0 \). There exist \( \lambda_3 > 0 \), \( \gamma > 0 \), \( \sigma, \tau \in (0, 1) \) such that for \( \lambda \in (0, \lambda_3) \), there exists \( A_{1,\lambda} \) such that for \( \alpha \in A_{1,\lambda} \) and \( R \geq 1 \), we have for \( E \in \mathbb{R} \) and \( 1 \leq j \leq d \)
\[
\left| \{ x_j \in T : \Lambda_R(0) \text{ is } (\gamma, \tau)\text{-suitable for } \hat{H}_{\lambda,\alpha,x} - E \} \right| \leq e^{-R^\tau}
\]
for each fixed choice of \( x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d \).
\end{theorem}

\begin{proof}
This is basically Proposition 2.2. in \cite{6}. However, Bourgain proves this
result for \( H = \Delta + V \) with \( \Delta \) the usual Laplacian. The modification to treat
operators of the form \( \hat{H} = T + W \), where \( W \) is a quasi-periodic potential and
\( T \) a long range hopping operator are straightforward. The computations at the
beginning of Section 6 would be helpful to write down a proof for the long range
case. \hfill \Box
\end{proof}

Using semi-algebraic set methods and frequency elimination one can show

\begin{proposition}
Let \( x \in \mathbb{T}^d \), \( \gamma > 0 \), \( \tau \in (0, 1) \). There exists \( C_1 = C_1(d) \geq 1 \)
such that for arbitrary \( C_2 > C_1 \) and \( N \geq 1 \) large enough, we have that there exists
\( A \subseteq [0, 1]^d \) satisfying \( |A| \geq 1 - \frac{1}{N^{C_1}} \) such that for \( \alpha \in \mathbb{R} \),
\[
N^{C_1} \leq |n|_\infty \leq N^{C_2}
\]
and \( E \in \mathbb{R} \) satisfying
\[
\text{dist}(E, \sigma(\hat{H}_{\lambda,\alpha,x}^{30}(0))) \leq e^{-cN}
\]
we have
\[
\Lambda_N(n) \text{ is } (\gamma, \tau)\text{-suitable for } \hat{H}_{\lambda,\alpha,x} - E.
\]
\end{proposition}

\begin{proof}
This can be again achieved as in \cite{6} see formulas (3.5) to (3.26). The only
modification necessary is that Bourgain works with \( N \) instead of \( N^{30} \) in \cite{18}. The
changes required for this are minor. \hfill \Box
\end{proof}

With these preparations done, we are now ready to start the proof of Theorem 2.2. We recall that \( r = R_{j-1} \) and \( R = R_j \). We apply Proposition 4.4 with
\( N^{C_1} = \frac{2}{\pi} \). So we let \( N = \lfloor (r/2)^{\frac{1}{C_1}} \rfloor \). In order for this elimination of \( \alpha \) only
contributing \( \frac{\varepsilon}{2} \), we need to choose \( R_1 \) so large that
\[
\sum_{j=2}^{\infty} \frac{1}{(R_j/2)^{C_1}} \leq \frac{\varepsilon}{2}.
\]
This is clearly possible. Furthermore, as $C_2$ is arbitrary, we can choose it such that 
\((R_j/2)^{\frac{1}{10}} \geq R_{j+1} = (R_j)^{10}\).

We will now use the following result about general operators.

**Theorem 4.5.** Let $\gamma > 0$, $\tau \in (0, 1)$, $1000\rho \leq r \leq \frac{1000}{900} R$, and $\delta \geq e^{-\frac{1000}{900} \tau}$.

Let $E$ be a $\delta$-simple eigenvalue of $H^{\Lambda_{(0)}(\gamma \tau)}$ and denote by $\psi$ the corresponding normalized eigenfunction. Assume for \(\frac{r}{2} \leq |n|_\infty \leq R\) that

\[
(5.1) \quad \Lambda_{\rho}(n) \text{ is } (\gamma, \tau)\text{-suitable for } H - E.
\]

Then there exists $\hat{E}$ a $e^{-\frac{1000}{900} \tau}$-simple eigenvalue of $H^{\Lambda_{(0)}(\gamma \tau)}$ satisfying $|E - \hat{E}| \leq e^{-\frac{1000}{900} \tau}$. Furthermore, there exists a corresponding normalized eigenfunction $\varphi$ such that

\[
(4.12) \quad \|\psi - \varphi\| \leq e^{-\frac{1000}{900} \tau}.
\]

We are now ready for

**Proof of Theorem 2.2.** We choose $R_1$ by (4.10) and apply Proposition 4.1 with $R = R_1$. We now see that Theorem 2.2 holds with possibly imposing an additional smallness condition on $\lambda$. We finish the proof by the use of induction.

We eliminate frequencies $\alpha$ using Proposition 4.4 obtaining a set $G_j \subseteq G_{j-1}$. Let now $\alpha \in G_j$. Choose $\ell < j$ such that

\[
N^3 \leq R_\ell \leq N^{30}, \quad N = \lfloor (r/2)^{\frac{1}{10}} \rfloor, \quad r = R_{j-1}
\]

where $N$ is as in Proposition 4.4. We then have that

\[
\text{dist}(E_{j-1}(\alpha), \sigma(H^{\Lambda_{R_j} \alpha}(\gamma \tau))) \leq 2\delta_\ell \leq e^{-N^{\frac{1}{10}}}.
\]

As the corresponding eigenfunction is well localized, the same statement holds for $\sigma(H^{\Lambda_{R_j} \alpha}(\gamma \tau))$. Hence, we may conclude from Proposition 4.4 that the assumptions of Theorem 4.5 hold. We set $E_j(\alpha) = \hat{E}$; $\psi_j(\alpha) = \varphi$ so that

\[
|E_j(\alpha) - E_{j-1}(\alpha)| \leq e^{-\frac{1000}{900} \tau}R_{j-1}, \quad |\psi_j(\alpha) - \psi_{j-1}(\alpha)| \leq e^{-\frac{1000}{900} \tau}R_{j-1}
\]

and

\[
E_j(\alpha) \text{ is a } e^{-\frac{1000}{900} \tau}(R_{j-1})^{\frac{1}{10}} \text{-simple eigenvalue of } H^{\Lambda_{R_j} \alpha}(\gamma \tau).
\]

Theorem 2.2 now follows by simple arithmetic. \(\square\)

### 5. Proof of the initial condition

Let $x \in \mathbb{T}^d$, $\alpha \in [0, 1]^d$, we consider

\[
(5.1) \quad W_{x, \alpha}(n) = \sum_{j=1}^{d} 2 \cos(2\pi (x_j + n_j \alpha_j)).
\]

We have that $W_{x, \alpha}(0)$ only depends on $x \in \mathbb{T}^d$, whereas all the $W_{x, \alpha}(n)$ for $n \in \mathbb{Z}^d \setminus \{0\}$ are nontrivial functions of $\alpha \in [0, 1]^d$. This immediately implies

**Proposition 5.1.** Let $\varepsilon > 0$, $R \geq 1$, $x \in \mathbb{T}^d$, and $E = W_{x, \alpha}(0)$. There exists $\kappa > 0$ and $G \subseteq [0, 1]^d$ such that $|G| \geq 1 - \varepsilon$ and for $n \in \Lambda_R(0) \setminus \{0\}$, $\alpha \in G$, we have

\[
(5.2) \quad |W_{x, \alpha}(n) - E| \geq \kappa.
\]
Proof. As \( f(y) = \sum_{j=1}^{d} 2 \cos(2\pi y_j) \) is a real-analytic function. There exists \( \eta_0 > 0 \) and \( \beta > 0 \) such that for each \( j = 1, \ldots, d \)

\[
G_\eta = \{ y_j : |f(y) - E| \leq \eta \}
\]

has measure \( |G_\eta| \leq \eta^\beta \) for \( \eta \in (0, \eta_0) \) and any choice of \( y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_d \).

For \( n \neq 0 \), there is \( j = 1, \ldots, d \) such that \( n_j \neq 0 \). It follows that

\[
|W_{x,\alpha}(n) - E| \geq \eta, \quad n \in \Lambda_R(0) \setminus \{0\}
\]

for \( \alpha \) outside a set of measure \( \#\Lambda_R(0) \cdot \eta^\beta \). Hence, the claim follows for an appropriate choice of \( \eta \). \( \square \)

We thus obtain for \( E \) is a \( \kappa \)-simple eigenvalue of \( W_{x,\alpha}^{\Lambda_R(0)} \) for \( \alpha \in G \). Consider now the operator \( \hat{H}_{\alpha,x}^{\Lambda_R(0)} = \lambda T + W_{x,\alpha} \). We clearly have that

\[
(\hat{H}_{\alpha,x}^{\Lambda_R(0)} - E)\delta_0 \leq \lambda \|T\|
\]

and that for \( \lambda < \frac{\kappa}{2\|T\|} \)

\[
\tr(P_{[E-\frac{\kappa}{2},E+\frac{\kappa}{2}]\hat{H}_{\alpha,x}^{\Lambda_R(0)}}) = 1.
\]

Hence, there exists \( E_1 \) such that \( |E - E_1| \leq \lambda \|T\| < \frac{\kappa}{2} \) and a normalized \( \psi \in \ell^2(\Lambda_R(0)) \) such that

\[
\hat{H}_{\alpha,x}^{\Lambda_R(0)} \psi = E_1 \psi.
\]

**Lemma 5.2.** We have

\[
\|\psi - \delta_0\| \leq \frac{2\lambda \|T\|}{\kappa}.
\]

**Proof.** We have \( 0 = (\hat{H}_{\alpha,x}^{\Lambda_R(0)} - E_1)\psi(n) = (W_{x,\alpha}(n) - E_1)\psi(n) + \lambda(T\psi)(n) \). Thus, we obtain for \( n \neq 0 \) that

\[
\kappa \sum_{n \in \Lambda_R(0) \setminus \{0\}} |\psi(n)|^2 \leq \lambda \|T\|.
\]

This implies \( |\psi(0)|^2 = 1 - \sum_{n \in \Lambda_R(0) \setminus \{0\}} |\psi(n)|^2 \geq 1 - \frac{\lambda \|T\|}{\kappa} \). \( \square \)

This is all we need to prove Proposition 4.1.

6. Understanding eigenfunctions

In order to begin this section, let me review the assumptions about the operator \( H = \lambda T + W : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \), where \( \lambda > 0 \) is a small parameter. We will assume that \( T \) is of the form

\[
T\psi(n) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} \psi(n + k)
\]

where we have that \( |t_{n,k}| \leq e^{-\eta|k|_\infty} \). This is slightly more general than what we need as we have \( t_{n,k} = f(k) \). However, this greater generality doesn’t add any new difficulty. \( W \) is the multiplication operator by a sequence \( W \in \ell^\infty(\mathbb{Z}^d) \). For \( \Lambda \subseteq \mathbb{Z}^d \), we have that

\[
T^\Lambda \psi(n) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \sum_{n + k \in \Lambda} t_{n,k} \psi(n + k)
\]
Finally, we recall that for $n, m \in \Lambda$ and $E \in \mathbb{R}$, the Green’s function is defined by

$$(6.3) \quad G^\Lambda(E; n, m) = \langle \delta_n, (H^\Lambda - E)^{-1}\delta_m \rangle.$$  

Here $H^\Lambda$ denotes the restriction of $H$ to $\ell^2(\Lambda)$. We will now need to prove the following lemma, which allows us to estimate solutions in terms of the Green’s function.

**Lemma 6.1.** Let $\Lambda \subseteq \Xi \subseteq \mathbb{Z}^d$ and $\psi$ solve $H^\Xi \psi = E\psi$. Then for $n \in \Lambda$, we have

$$(6.4) \quad \psi(n) = -\lambda \sum_{m \in \Lambda} G^\Lambda(E; n, m) \sum_{\ell \in \Xi \setminus \Lambda} t_{m, \ell - m}\psi(\ell).$$

**Proof.** We have that $\psi(n) = \langle \delta_n, \psi \rangle = \langle (H^\Lambda - E)^{-1}\delta_n, (H^\Lambda - E)\psi \rangle$. As $(H^\Lambda - E)\psi = \chi_\Lambda(H^\Lambda - E)\psi$ and $(H^\Lambda - E)\psi = (H^\Lambda - E - (H^\Xi - E))\psi = \lambda(T^\Lambda - T^\Xi)\psi$, we obtain

$$\psi(n) = \lambda \langle (H^\Lambda - E)^{-1}\delta_n, \chi_\Lambda(T^\Lambda - T^\Xi)\psi \rangle.$$  

For $m \in \Lambda$, we have that

$$\chi_\Lambda(T^\Lambda - T^\Xi)\psi(m) = -\sum_{\ell \in \Xi \setminus \Lambda} t_{m, \ell - m}\psi(\ell).$$

The claim follows. \qed

This lemma together with our decay assumption on $t_{n, m}$ implies

$$(6.5) \quad |\psi(n)| \leq \lambda \sum_{m \in \Lambda} |G^\Lambda(E; n, m)| \sum_{\ell \in \Xi \setminus \Lambda} e^{-\eta|\ell - m|_\infty} \cdot |\psi(\ell)|.$$  

In particular, we obtain

**Lemma 6.2.** Let $\gamma \in (0, \frac{d}{2})$ and $R \geq 1$ large enough (depending on $\gamma, \eta, d, \tau$).

Let $\Lambda_R(0)$ be $(\gamma, \tau)$-suitable for $H - E$ and $\psi$ solve $H^\Xi \psi = E\psi$ for $\Lambda_R(0) \subseteq \Xi$, $\|\psi\| = 1$. Then for $|n| \in \Lambda_R(0)$,

$$(6.6) \quad |\psi(n)| \leq \lambda e^{-\frac{d}{2} |\text{dist}(n, \partial \Lambda_R(0))|} \max_{m \in \Xi} \left( e^{-\frac{d}{2} |\text{dist}(m, \Lambda_R(0))|} |\psi(m)| \right).$$

**Proof.** Let $\Psi = \max_{m \in \Xi} \left( e^{-\frac{d}{2} |\text{dist}(m, \Lambda_R(0))|} |\psi(m)| \right)$. We begin by observing that (6.5) implies

$$|\psi(n)| \leq \left( \sum_{m \in \Lambda} |G^\Lambda(E; n, m)| \sum_{\ell \in \Xi \setminus \Lambda} e^{-\frac{d}{2}|\ell - m|_\infty} \right) \cdot \Psi.$$  

Thus we have $|\psi(n)| \leq \lambda (I_1 + I_2)\Psi$, where

$$I_1 = \sum_{m \in \Lambda \setminus \Lambda_R(0)} |G^\Lambda(E; n, m)| \sum_{\ell \in \Xi \setminus \Lambda_R(0)} e^{-\frac{d}{2}|\ell - m|_\infty}$$

and

$$I_2 = \sum_{m \in \Lambda_R(0) \setminus \Lambda_{R/2}(0)} |G^\Lambda(E; n, m)| \sum_{\ell \in \Xi \setminus \Lambda_R(0)} e^{-\frac{d}{2}|\ell - m|_\infty}.$$
To estimate $I_1$, we have that $|G^\Lambda(E; n, m)| \leq e^{-\gamma|n-m|}$. Furthermore, we have that $|\ell - m|_\infty \geq (|\ell|_\infty - R) + (R - |m|_\infty)$. From this, we conclude

$$I_1 \leq C_1 \sum_{m \in \Lambda_R(0) \setminus \Lambda_2R(0)} e^{-\gamma|n-m|_\infty - \frac{\gamma}{2}(R - |m|_\infty)} \leq C_2 R^d e^{-\min(\gamma, \frac{\gamma}{2}) \text{dist}(n, \partial \Lambda_R(0))}.$$ 

To estimate $I_2$, we use that $|G^\Lambda(E; n, m)| \leq e^\Gamma$ and $|\ell - m| \geq \frac{R}{2} + (|\ell|_\infty - R)$. From this we conclude

$$I_2 \leq C e^{-\frac{R}{2} + \Gamma}.$$ 

The claim follows.

We will furthermore need the following lemma which allows us to construct test functions by restricting them.

**Lemma 6.3.** Let $\|\psi\| = 1$ solve $H^\Xi \psi = E\psi$ and assume

$$|\psi(n)| \leq \delta, \ n \in \Lambda_{2R}(0) \setminus \Lambda_R(0) \tag{6.7}$$

for $\delta \geq e^{-\frac{\eta}{R}} R$. Then $\varphi = \chi_{\Lambda_{\frac{3}{2}R}(0)} \psi$ satisfies

$$\| (H^\Xi - E) \varphi \| \leq (10R)^{2d} \delta. \tag{6.8}$$

**Proof.** Let $n \in \Xi$, then we have that

$$(H^\Xi - E) \varphi(n) = \lambda \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} \varphi(n + k) + (W(n) - E) \varphi(n).$$

For $n \in \Lambda_{\frac{3}{2}R}(0)$, this is equal to

$$(H^\Xi - E) \varphi(n) = \lambda \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} \psi(n + k).$$

Thus, we may estimate

$$\| (H^\Xi - E) \varphi \| \leq \lambda \left( \delta \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} + \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} \right) \leq \lambda \delta (4R)^d + \lambda \sum_{|k|_\infty \geq \frac{1}{2} R} e^{-\eta|k|_\infty},$$

which clearly satisfies what we need. For $n \in \Xi \setminus \Lambda_{\frac{3}{2}R}(0)$, one has

$$(H^\Xi - E) \varphi(n) = \lambda \sum_{k \in \mathbb{Z}^d \setminus \{0\}} t_{n,k} \varphi(n + k).$$

This sum can be estimated as the previous one and the claim follows.

With these preparations done, we now begin the actual proof of Theorem 4.5.
Lemma 6.4. Let $1000\rho < r < R \leq e^{\frac{1}{2}r^2}$ and $\rho$ be large enough. Assume that for $\frac{1}{2} \leq |n|_{\infty} \leq R$, we have
\begin{equation}
\Lambda_{\rho}(n) \text{ is } (\gamma, \tau)\text{-suitable for } H - E.
\end{equation}

Then we have that
\begin{equation}
\|(H^{\Lambda_0(0)\setminus \Lambda_{\frac{1}{2}}(0)} - E)^{-1}\| \leq e^{5\rho r}, \quad \|(H^{\Lambda_{\rho}(0)\setminus \Lambda_{\frac{1}{2}}(0)} - E)^{-1}\| \leq e^{5\rho r}.
\end{equation}

Proof. As $R \geq r \geq 1000\rho$, it suffices to prove the first claim. Assume by contradiction that the claim fails. Then there exists $|E - E| \leq e^{-5\rho r}$ and $\|\psi\| = 1$ solving
\[H^{\Lambda_0(0)\setminus \Lambda_{\frac{1}{2}}(0)} \psi = \tilde{E} \psi.
\]
Thus there exists $n$ such that $|\psi(n)| \geq \frac{1}{1000\rho r}$. By assumption, we have for $\frac{1}{2} \leq |m - n|_{\infty} \leq \frac{3}{2}$ that $\Lambda_{\rho}(m)$ is $(\gamma, \tau)$-suitable for $H - E$. As $(H - \tilde{E})^{-1} = (\text{Id} + (E - \tilde{E})(H - E)^{-1})^{-1}(H - E)^{-1}$, we have that
\[\|(H^{\Lambda_{\rho}(m)} - \tilde{E})^{-1}\| \leq 2e^{\rho r}.
\]

Now as
\[|G^{\Lambda_{\rho}(m)}(\tilde{E}; k, \ell)| \leq |G^{\Lambda_{\rho}(m)}(E; k, \ell)| + |E - \tilde{E}| \cdot \|(H^{\Lambda_{\rho}(m)} - \tilde{E})^{-1}\| \cdot \|(H^{\Lambda_{\rho}(m)} - E)^{-1}\|
\]
we have that $|G^{\Lambda_{\rho}(m)}(\tilde{E}; k, \ell)| \leq 2e^{-3\rho r}$ for $k \in \Lambda_{\frac{1}{2}}(0)$, $\ell \in \Lambda_{\rho}(0) \setminus \Lambda_{\frac{1}{2}}(0)$. Thus the previous lemma implies that
\[|\psi(m)| \leq e^{-2\rho r}.
\]

\[\varphi = \psi \chi_{\Lambda_{\rho}(n)}\text{ satisfies}
\]
\[\|(H^{\Lambda_{\rho}(n)} - \tilde{E})\varphi\| \leq (10R)^{2d}e^{-2\rho r}, \quad \|\varphi\| \geq \frac{1}{(3R)^d}.
\]
This implies $\|(H^{\Lambda_{\rho}(n)} - \tilde{E})^{-1}\| > \frac{1}{(3R)^d}e^{2\rho r}$, which is a contradiction. \hfill \Box

Let now $\psi$ be the function from the statement of Theorem 4.2. Define the function
\begin{equation}
\psi_1(n) = \begin{cases} 
\psi(n), & n \in \Lambda_\frac{1}{2}(0) \setminus \Xi; \\
0, & \text{otherwise}.
\end{cases}
\end{equation}

Lemma 6.5. We have that $\|\psi_1\| \geq \frac{1}{2}$. Furthermore for $\Lambda_{\frac{1}{2}}(0) \subseteq \Xi$, we have
\begin{equation}
\|(H^\Xi - E)\psi_1\| \leq e^{-\frac{5\pi}{2}}.
\end{equation}

Proof. In order to estimate $\psi(n)$ for $|n|_{\infty} - \frac{1}{2}r \leq \frac{1}{2}r$, we can iterate Lemma 6.2 at least $\frac{1}{2\rho}$ many times to obtain that for these $n$
\[|\psi(n)| \leq e^{-\frac{5\pi}{2}}.
\]
Clearly either $\psi_1$ or $\psi_2 = \psi - \psi_1$ satisfy that $\|\psi_2\| \geq \frac{1}{2}$. By Lemma 6.3 we obtain that
\[\|(H^\Xi - E)\psi_2\| \leq e^{-\frac{5\pi}{2}}.
\]
Thus $\|\psi_2\| \geq \frac{1}{2}$ would contradict the previous lemma. \hfill \Box
From this lemma, it is clear that
\begin{equation}
\sigma(H^{A_\nu(0)}) \cap [E - e^{-\frac{\gamma_0}{300}}, E + e^{-\frac{\gamma_0}{300}}] = \{\tilde{E}\}
\end{equation}
for some $\tilde{E}$. We will now show

**Lemma 6.6.** Let $\varphi$ satisfy $H^{A_\nu(0)} \varphi = \lambda \varphi$ for $\lambda \in [E - e^{-300\gamma_0}, E + e^{-300\gamma_0}]$. Then there exists $|c| = 1$ such that
\begin{equation}
\|\varphi - c\psi\| \leq e^{-\frac{\gamma_0}{300}}.
\end{equation}

**Proof.** For any $n$, $(H^{A_\nu(n)} - E)^{-1} = (E - \lambda)(H^{A_\nu(n)} - \lambda)^{-1}(H^{A_\nu(n)} - E)^{-1}$. Thus, we have that
\begin{equation}
\|(H^{A_\nu(n)} - E)^{-1}\| \leq 2e^{\gamma_0}
\end{equation}
as $|E - \lambda| \cdot \|(H^{A_\nu(n)} - E)^{-1}\| \leq \frac{1}{2}$. This implies in particular, that the estimates on the Green’s function also hold for $\lambda$ up to an negligible factor of 2. Using this, we can show, as in the previous lemma, that $\varphi_1 = \varphi \chi_{A_{\nu}(0)}$ satisfies $\|\varphi_1\| \geq 1 - e^{-\frac{\gamma_0}{300}}$ and
\begin{equation}
\|(H^{A_\nu(0)} - E)\varphi_1\| \leq \frac{1}{2} e^{-\frac{\gamma_0}{300}}.
\end{equation}
Letting $\varphi_1 = \langle \varphi_1, \psi \rangle \psi + \varphi_1^\perp$, we obtain as $\|\varphi_1^\perp\| \geq \frac{\gamma_0}{\delta}\|\varphi_1^\perp\|$ that
\begin{equation}
|\langle \varphi_1, \psi \rangle| \geq \|\varphi_1\| - \frac{10}{\delta} e^{-\frac{\gamma_0}{300}}.
\end{equation}
The claim now follows by simple arithmetic. \[\square\]

**Proof of Theorem 4.5.** The only thing remaining to prove is that
\begin{equation}
\text{tr}(P_{[E - e^{-300\gamma_0}, E + e^{-300\gamma_0}]}(H^{A_\nu(0)})) = 1.
\end{equation}
Assume otherwise, then there would be two orthogonal vectors $\varphi_1$ and $\varphi_2$ that satisfy the conclusions of the previous lemma. So
\begin{equation}
0 = \langle \varphi_1, \varphi_2 \rangle = c_1 c_2 - \langle c_1 \psi - \varphi_1, c_2 \psi \rangle - \langle \varphi_1, c_2 \psi - \varphi \rangle.
\end{equation}
As the last 2 terms are $\leq 2e^{-\frac{\gamma_0}{300}}$, the claim follows. \[\square\]

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*Mathematics 253-37, Caltech, Pasadena, CA 91125*

E-mail address: helge@caltech.edu

URL: *http://www.its.caltech.edu/~helge/*