MHD Stagnation Point Flow over a Nonlinear Stretching/Shrinking Sheet in Nanofluids

Nor Hathirah Abd Rahman¹,*, Norfifah Bachok¹,², Haliza Rosali¹

¹ Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia
² Institute for Mathematical Research, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia

ARTICLE INFO

ABSTRACT

Article history:
Received 24 May 2020
Received in revised form 18 September 2020
Accepted 22 September 2020
Available online 28 October 2020

In this study, an investigation of the steady 2-D magnetohydrodynamic (MHD) flow of stagnation point past a nonlinear sheet of stretching/shrinking within a non-uniform transverse magnetic intensity in nanofluids had been analysed. Considered material of nanoparticles such as copper (Cu) in water base fluid with Pr = 6.2 to analyze the influence of volume fraction parameter of nanoparticles and the stretching/shrinking sheet parameter. The governing nonlinear partial differential equations (PDEs) are converted into the nonlinear ordinary differential equations (ODEs) and use the boundary value problem solver bvp4c in Matlab program to solve numerically through the use of a similarity transformation. The impact of the parameter of the magnetic field on the coefficient of skin friction, the local number of Nusselt and the profiles of velocity and temperature are portrayed and explained physically. The analysis reveals that the magnetic field and volume fraction of nanoparticles affect the velocity and temperature. The dual solutions are achieved where for the shrinking sheet case and the solutions are non-unique, different from a stretching sheet.

Keywords:
Magnetohydrodynamic; flow of stagnation point; nonlinear sheet of stretching/shrinking; nanofluids; dual solutions

1. Introduction

In the problem of boundary layer, the stagnation point flow phenomenon was picked up by several researchers due to several of uses in the manufacturing sector for system cooling purposes. Stagnation point flow is the continuous motion near a solid surface’s stagnation district that exists in a fluid in both instances of a static or in motion body. Hiemenz [1] became the first scientist to test the steady of 2-D flow of stagnation point to a static semi-infinite surface and obtained an accurate the Navier-Stokes equation solution. Previously, several scientists had selected curiosity in the inquiry into the flow of boundary layer and heat transfer of the sheet of stretching. Crane [2] also researched
the first problem of the flow of boundary layer through a linear sheet of stretching. Chiam [3], who is researching the flow of stagnation point past a sheet of stretching developed the works of Hiemenz [1] and Crane [2]. Thus Miklavčič and Wang [4] investigated the flow equation of similarity solution past a surface of shrinking and noticed that it relies on the outer mass suction. The heat transfer in the flow of stagnation point past a sheet of stretching over a viscoelastic fluid, respectively, was explored by Mahapatra and Gupta [5, 6]. Next, for both 2-D and axisymmetric situations, the flow of stagnation point past a shrinking surface was explored by Wang [7]. For particular values of the shrinking and stretching rate ratio, the dual and unique solutions are identified.

There are many researchers who have been researching exponential and linear stretching/shrinking surfaces. Magyari and Keller [8] were the earliest to acknowledge the continuous boundary layers and heat transfer through a constant of an exponential stretching surface. Then, the study continued by Rohni et al., [9] who work into the exponential shrinking of vertical sheet with suction and the buoyancy force. Numerical findings showed that the existence of buoyancy force would make a contribution to the presence of triple solutions for a specific value of relevant parameters, while the problem has only dual solutions in the absence of the buoyancy force. Mansur et al., [10] have been using the Buongiorno model to explore the flow of stagnation point to a permeable sheet of stretching/shrinking with the effect of suction in nanofluid. They observed that when the sheet is extended, the skin friction reduces, but rises as the suction impact increases. The MHD flow of boundary layer past the flow of stagnation point of a linear surface of stretching was explored by Jat and Chaudhary [11, 12]. Within of the magnetic field, the continuous flow of stagnation point of two-dimensional in a viscous fluid is observed by Aman et al., [13] through a linear sheet of stretching/shrinking. The results show that there are dual solutions for the sheet of shrinking, while the solution for the sheet of stretching is unique.

However, the earliest researcher to analyse the flow in viscous fluid past a nonlinear sheet of stretching is Vajravelu [14]. They found that the heat flow is consistently come from the sheet of stretching to the fluid. Bachok and Ishak [15] had been investigated the similarity solutions past a nonuniform transverse magnetic intensity \( B = B_0 x^{(n-1)/2} \) applied perpendicular to the sheet, where \( B_0 \) is an uniform magnetic intensity. It is concluded that the sheet’s velocity is \( U_w(x) = ax^n \) and the velocity outside the boundary layer is \( U_s(x) = bx^n \) where \( a \) is the stretching/shrinking rate.
\( b > 0 \) is a constant denotes the strength of the flow of stagnation and \( n \) is the stretching index. The simplified 2-D MHD equations of governing are

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0, \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_s \frac{dU_s}{dx} + \frac{\mu_{nf}}{\rho_{nf}} \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B_0^2}{\rho_f} (U_s - u), \tag{2}
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha_{nf} \frac{\partial^2 T}{\partial y^2}, \tag{3}
\]

corresponding to boundary conditions

\[
u = U_w(x), v = 0, T = T_w(x) \text{ at } y = 0, \tag{4}
\]

\[
u \to U_s(x), T \to T_\infty \text{ as } y \to \infty, \tag{4}
\]

which are \( u \) is the \( x - \) axes and \( v \) is \( y - \) axes along the velocity components and \( \sigma \) is the fluid's electrical conductivity. \( T \) is the nanofluid's temperature, \( T_w \) is the variable sheet temperature and \( T_\infty \) is the free flow temperature assuming it is fixed, which are defined as follows (Oztop and Abu Nada [20])

\[
\alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}}, \rho_{nf} = (1 - \varphi) \rho_f + \varphi \rho_s, \mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}}, \tag{5}
\]

\[
(\rho C_p)_{nf} = (1 - \varphi) (\rho C_p)_f + \varphi (\rho C_p)_s, \quad \frac{k_{nf}}{k_f} = \frac{(k_s + 2k_f) - 2\varphi (k_f - k_s)}{(k_s + 2k_f) + \varphi (k_f - k_s)}. \tag{6}
\]

Here, \( \varphi \) is the nanofluid's nanoparticle volume fraction parameter, \( \rho_{nf} \) is the nanofluid's density, \( \alpha_{nf} \) is the nanofluid’s thermal diffusivity, \( k_{nf} \) is the fluid fraction's thermal conductivity, \( k_s \) is the nanoparticle volume fraction's thermal conductivity, \( \rho_f \) is the solid fraction's reference density, \( \mu_{nf} \) is the fluid fraction's viscosity and \( (\rho C_p)_{nf} \) is the nanofluids' heat capacitance, where \( C_p \) at constant pressure is the specific heat. Brinkman [21] computed the nanofluid’s viscosity of \( \mu_{nf} \) as the base fluid viscosity of \( \mu_f \) involving diluted suspension of good spherical particles.

The similarity transformation is used in order to gain a similar solution for the equation of momentum and energy Eq. (1) - (3)

\[
\eta = \eta \sqrt{\frac{b(n+1)}{2u_f} x^{\frac{n-1}{2}}}, \quad \psi = \sqrt{\frac{2u_f b}{n+1} x^{\frac{n+1}{2}}} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{7}
\]

which are \( \eta \) is the variable of similarity and \( \psi \) is the function of stream described as \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \) which equivalently fulfilled with continuity equation, Eq. (1).

The converted ODEs are

\[
\frac{1}{(1-\varphi)^{2.5}(1-\varphi + \varphi \rho_s/\rho_f)} f''' + f' + \frac{2n}{n+1} (1 - f'/2) + \frac{2M}{n+1} (1 - f') = 0, \tag{8}
\]
corresponding to the boundary conditions (4)

\[ f(0) = 0, \quad f'(0) = \varepsilon, \quad \theta(0) = 1, \]
\[ f'(\eta) \to 1, \quad \theta(\eta) \to 0, \quad \text{as} \quad \eta \to \infty \]  

From the equations, the primes indicate differentiation in respect of \( \eta \), where
\[ Pr = \frac{\nu_f}{\alpha_f} \]  

is the number of Prandtl, \( M = \frac{\sigma B_0^2}{b \rho_f (x^{1-n})} \) is the magnetic parameter and \( \varepsilon = a/b \) is the stretching/shrinking parameter which are \( \varepsilon > 0 \) is for stretching sheet case and \( \varepsilon < 0 \) is for shrinking sheet case.

The coefficient of skin friction \( C_f \) and the local Nusselt number \( N_u_x \) in the physical quantities of interest are

\[ C_f = \frac{\tau_w}{\rho_f u_x^2}, \quad N_u_x = \frac{x q_w}{k_f (T_w - T_\infty)} \]  

which are the surface shear stress \( \tau_w \) and the surface heat flux \( q_w \) are defined as

\[ \tau_w = \mu_n f \left( \frac{\partial u}{\partial y} \right)_{y=0}, \quad q_w = -k_n \left( \frac{\partial T}{\partial y} \right)_{y=0} \]  

Applying the similarity variables (7), we obtained

\[ C_f Re_x^{1/2} = \frac{1}{(1-\varphi)^{1.5}} \sqrt{\frac{n+1}{2}} f'''(0), \]  
\[ N_u_x Re_x^{1/2} = -k_n \frac{1}{k_f} \sqrt{\frac{n+1}{2}} \theta'(0). \]  

where \( Re_x = U_x x / \nu_f \).

3. Results

The ODEs equations Eq. (8) - (10) are nonlinear and coupled, and therefore their accurate analytical solutions are impossible. These can be solved numerically using Matlab’s bvp4c solver because of its usefulness in finding a solution for boundary value problems that are far more complicated than initial value problems of various parameter values such as \( M, \varphi \) and \( \varepsilon \). By choosing various initial guesses for the lost values for \( f'''(0) \) and \( \theta'(0) \) the dual solutions were obtained. The guess must meet the boundary conditions (10) asymptotically, while maintaining the solution’s behavior. For copper (Cu) as the working fluid, an analysis of the influence of \( \varphi \), \( Pr \) and \( M \) and water as the fluid base that works are consider. The Pr is considered \( Pr = 6.2 \) and \( \varphi \) is assumed from 0 to 0.2 \((0 \leq \varphi \leq 0.2)\), where \( \varphi = 0 \) refers to the regular fluid. Table 1 lists the thermophysical characteristics of the base fluid and nanoparticles. Table 2 and Table 3 show a comparison of the numerical values of \( Re_x^{1/2} C_f \) and \( Re_x^{-1/2} N_u_x \) for Cu-water between the past work in Rahman et al.,
[22] and the current work showing a favorable agreement. With various values of $\epsilon$, $M$ and $\varphi$, numerical computations were conducted.

### Table 1
Properties of thermophysical of base fluid and nanoparticles [20]

| Properties of physical | Fluid phase (water) | Cu |
|------------------------|---------------------|----|
| $C_p$ (J/kgK)          | 4179                | 385 |
| $\rho$ (kg/m$^3$)      | 997.1               | 8933 |
| $k$ (W/mK)             | 0.613               | 400 |

### Table 2
Values of $C_f Re_{\infty}^{1/2}$ for several values of $M$, $\epsilon$, and $\varphi$

| $M$ | $\epsilon$ | $\varphi$ | Rahman et al., [22] | Current results |
|-----|-------------|------------|----------------------|-----------------|
|     | Cu-water    |            |                      |                 |
| -0.5| 0           | 2.1182     | 2.1182               |                 |
|     | 0.1         | 3.2382     | 3.2382               |                 |
|     | 0.2         | 4.5071     | 4.5071               |                 |
| 0   | 0           | 1.6872     | 1.6872               |                 |
|     | 0.1         | 2.5793     | 2.5793               |                 |
|     | 0.2         | 3.5901     | 3.5901               |                 |
| 0.5 | 0           | 0.9604     | 0.9845               |                 |
|     | 0.1         | 1.4682     | 1.5051               |                 |
|     | 0.2         | 2.0436     | 2.0950               |                 |
| -0.5| 0           | 2.1078     | 2.1708               |                 |
|     | 0.1         | 3.3186     | 3.3186               |                 |
|     | 0.2         | 4.6191     | 4.6191               |                 |
| 0   | 0           | 1.7165     | 1.7165               |                 |
|     | 0.1         | 2.6241     | 2.6241               |                 |
|     | 0.2         | 3.6524     | 3.6524               |                 |
| 0.5 | 0           | 0.9733     | 0.9971               |                 |
|     | 0.1         | 1.4879     | 1.5243               |                 |
|     | 0.2         | 2.0710     | 2.1216               |                 |
| -0.5| 0           | 2.2222     | 2.2222               |                 |
|     | 0.1         | 3.3971     | 3.3971               |                 |
|     | 0.2         | 4.7284     | 4.7284               |                 |
| 0   | 0           | 1.7453     | 1.7453               |                 |
|     | 0.1         | 2.6681     | 2.6681               |                 |
|     | 0.2         | 3.7137     | 3.7137               |                 |
| 0.5 | 0           | 0.9860     | 1.0094               |                 |
|     | 0.1         | 1.5073     | 1.5432               |                 |
|     | 0.2         | 2.0980     | 2.1478               |                 |
Table 3

| $M$ | $\epsilon$ | $\varphi$ | Rahman et al., [22] | Currents results |
|-----|-------------|-----------|---------------------|------------------|
|     |             |           | Cu-water            |                  |
| -0.5| 0           | 0.6870    | 0.6870              |                  |
|     | 0.1         | 1.1432    | 1.1432              |                  |
|     | 0.2         | 1.5185    | 1.5185              |                  |
| 0   | 0           | 1.7148    | 1.7148              |                  |
|     | 0.1         | 2.1358    | 2.1358              |                  |
|     | 0.2         | 2.5400    | 2.5400              |                  |
| 0.5 | 0           | 1.4874    | 3.0095              |                  |
|     | 0.1         | 2.9149    | 3.5235              |                  |
|     | 0.2         | 3.3565    | 4.0560              |                  |
| -0.5| 0           | 0.7079    | 0.7079              |                  |
|     | 0.1         | 1.1649    | 1.1649              |                  |
|     | 0.2         | 1.5419    | 1.5419              |                  |
| 0   | 0           | 1.7220    | 1.7220              |                  |
|     | 0.1         | 2.1442    | 2.1442              |                  |
|     | 0.2         | 2.5494    | 2.5494              |                  |
| 0.5 | 0           | 2.4897    | 3.0119              |                  |
|     | 0.1         | 2.9251    | 3.5266              |                  |
|     | 0.2         | 3.3597    | 4.0560              |                  |
| -0.5| 0           | 0.7279    | 0.7279              |                  |
|     | 0.1         | 1.1857    | 1.1857              |                  |
|     | 0.2         | 1.5642    | 1.5642              |                  |
| 0   | 0           | 1.7291    | 1.7291              |                  |
|     | 0.1         | 2.1524    | 2.1524              |                  |
|     | 0.2         | 2.5586    | 2.5586              |                  |
| 0.5 | 0           | 2.4919    | 3.0143              |                  |
|     | 0.1         | 2.9205    | 3.5296              |                  |
|     | 0.2         | 3.3629    | 4.0632              |                  |

Figure 1 - 6 shown the variation of $f''(0)$ and $\theta'(0)$ for certain magnetic field parameter value $M$, volume fraction of nanoparticles $\varphi$ and the stretching index $n$ towards $\epsilon$ for Cu in water base fluid. These figures observed that the solution is unique at the region $\epsilon \geq 1$, dual at the region $\epsilon_c \leq \epsilon < -1$ and at the region $\epsilon < \epsilon_c < 0$ there are no solutions, which is $\epsilon_c$ is the critical value of $\epsilon$. We can show from Figure 1 and 2, that the $M$ increase will increase the $\epsilon_c$ range. Thus, for $M = 0$, the range of $\epsilon$ for which there is a similarity solution is smaller, i.e. $-1.349802 \leq \epsilon < \infty$, while for $M = 0.1$ and $M = 0.2$, the range are $-1.397694 \leq \epsilon < \infty$ and $-1.445707 \leq \epsilon < \infty$, respectively. From these figures, we also can conclude that the rate of boundary layer and the rate of heat transfer increase because of an increment of the values of $M$.

Then, Figure 7 and 8 depict the variations of $C_f R_x^{1/2}$ and $Nu_x R_x^{-1/2}$ for several values of $M$ for Cu in water base fluid with $\epsilon = 1.5$ and $n = 2$. These figures stated that as the $M$ is increasing, the value of $C_f R_x^{1/2}$ and $Nu_x R_x^{-1/2}$ is decreasing. In contrast, the profiles of velocity and temperature of various $M$ and $n$ are shown visually in Figure 9 - 12. These profiles prove the existing of the dual solution in the Figure 1 - 6 that satisfy the far zone boundary conditions (10) asymptotically.
Fig. 1. \( f''(0) \) with \( \epsilon \) for some values of \( M \) for Cu-water, \( Pr = 6.2, \varphi = 0.1 \) and \( n = 2 \)

Fig. 2. \(-\theta'(0)\) with \( \epsilon \) for some values of \( M \) for Cu-water, \( Pr = 6.2, \varphi = 0.1 \) and \( n = 2 \)
Fig. 3. $f''(0)$ with $\epsilon$ for some values of $\varphi$ ($0 \leq \varphi \leq 0.2$) for Cu-water, Pr = 6.2, $M = 0.2$ and $n = 2$

Fig. 4. $-\theta'(0)$ with $\epsilon$ for some values of $\varphi$ ($0 \leq \varphi \leq 0.2$) for Cu-water, Pr = 6.2, $M = 0.2$ and $n = 2$
Fig. 5. $f''(0)$ with $\epsilon$ for some values of $n$ for Cu-water, $Pr = 6.2$, $M = 0.1$ and $\varphi = 0.2$

Fig. 6. $-\theta'(0)$ with $\epsilon$ for some values of $n$ for Cu-water, $Pr = 6.2$, $M = 0.1$ and $\varphi = 0.2$
Fig. 7. $C_f R_e^{1/2}$ with $\varepsilon$ for different magnetic field $M$ for Cu-water, $Pr = 6.2$, $\varepsilon = 1.5$ and $n = 2$

Fig. 8. $Nu_x R_e^{-1/2}$ with $\varepsilon$ for different magnetic field $M$ for Cu-water, $Pr = 6.2$, $\varepsilon = 1.5$ and $n = 2$
Fig. 9. Velocity profiles for different values of $M$ for Cu-water, $\phi = 0.1$, $\epsilon = -1.25$, $n = 2$ and $Pr = 6.2$

Fig. 10. Temperature profiles for different values of $M$ for Cu-water, $\phi = 0.1$, $\epsilon = -1.25$, $n = 2$ and $Pr = 6.2$
4. Conclusions

We have numerically analyzed in a nanofluid how the parameter $M$ impacts the flow of stagnation point past a nonlinear sheet of stretching/shrinking. The analysis of the influence of the $\varphi$ and the heat transfer features of Cu-water was numerically resolved with $Pr = 6.2$. In this study, it reveals that with the increase of the MHD, the solutions range is widespread. And with the increase in
magnetohydrodynamics, the range of solutions is expanded widely. As the factor of $M$ increases, the skin friction and heat transfer also do increase.

Acknowledgement
The authors gratefully appreciate the financial support given in the form of a fundamental research grant scheme (FRGS/1/2018/STG06/UPM/02/4/5540155). Lastly, for all the reviewers, a big thank you for honest feedback and suggestions.

References
[1] Hiemenz, K. "Dingler's Poly." J 326 (1911): 321.
[2] Crane, L. J. "Z. angew Flow Past a Stretching Plate." Journal of Applied Mathematics and Physics 21, no. 4 (1970): 645-647. 
https://doi.org/10.1007/BF01587695
[3] TC, Chiam. "Stagnation-point flow towards a stretching plate." Journal of the physical society of Japan 63, no. 6 (1994): 2443-2444. 
https://doi.org/10.1143/JPSJ.63.2443
[4] Miklavčič, M., and C. Wang. "Viscous flow due to a shrinking sheet." Quarterly of Applied Mathematics 64, no. 2 (2006): 283-290. 
https://doi.org/10.1090/S0033-569X-06-01002-5
[5] Mahapatra, Tapas R., and Anadi S. Gupta. "Stagnation‐point flow towards a stretching surface." The Canadian Journal of Chemical Engineering 81, no. 2 (2003): 258-263. 
https://doi.org/10.1002/cjce.5450810210
[6] Wang, C. Y. "Stagnation flow towards a shrinking sheet." International Journal of Non-Linear Mechanics 43, no. 5 (2008): 377-382. 
https://doi.org/10.1016/j.ijnonlinmech.2007.12.021
[7] Magyari, E., and B. Keller. "Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface." Journal of Physics D: Applied Physics 32, no. 5 (1999): 577. 
https://doi.org/10.1088/0022-3773/32/5/012
[8] Mansur, Syahira, Anuar Ishak, and Ioan Pop. "Stagnation-point flow towards a stagnation point over an exponentially shrinking vertical sheet with suction." International journal of thermal sciences 75 (2014): 164-170. 
https://doi.org/10.1016/j.ijthermalsci.2013.08.005
[9] Jat, R. N., Abhishek Neemawat, and Dinesh Rajotia. "MHD boundary layer flow and heat transfer over a continuously moving flat plate." International Journal of Statistiika and Mathematika 3 (2012): 102-108
[17] Rana, P., and R. Bhargava. "Flow and heat transfer of a nanofluid over a nonlinear stretching sheet: A numerical study." *Communication in Nonlinear Science and Numerical Simulation* 17: 212-226. https://doi.org/10.1016/j.cnsns.2011.05.009

[18] Matin, Meisam Habibi, Mohammad Reza Heirani Nobari, and Pouyan Jahanjiri. "Entropy analysis in mixed convection MHD flow of nanofluid over a non-linear stretching sheet." *Journal of Thermal Science and Technology* 7, no. 1 (2012): 104-119. https://doi.org/10.1299/jtst.7.104

[19] Tiwari, Raj Kamal, and Manab Kumar Das. "Heat transfer augmentation in a two-sided lid-driven differentially heated square cavity utilizing nanofluids." *International Journal of heat and Mass transfer* 50, no. 9-10 (2007): 2002-2018. https://doi.org/10.1016/j.ijheatmasstransfer.2006.09.034

[20] Oztop, Hakan F., and Eiyad Abu-Nada. "Numerical study of natural convection in partially heated rectangular enclosures filled with nanofluids." *International Journal of heat and fluid flow* 29, no. 5 (2008): 1326-1336. https://doi.org/10.1016/j.ijheatfluidflow.2008.04.009

[21] Milovanov, Valery Ivanovich, Dmitry Alexandrovich Balashov, Valery Ivanovich Milovanov, and Dmitry Alexandrovich Balashov. "Experimental study of a liquid-steam ejector with a conical mixing chamber." *Chemical Physics* 20, no. 4 (1952): 571-581. https://doi.org/10.1063/1.1700493

[22] Rahman, A. N. H., N. Bachok, and H. Rosali. "Numerical solutions of MHD stagnation-point flow over an exponentially stretching/shrinking sheet in a nanofluid." In *Journal of Physics: Conference Series*, vol. 1366, no. 1, p. 012012. IOP Publishing, 2019.