On deformations of generalized Calabi-Yau, hyperKähler, $G_2$ and Spin(7) structures I

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Abstract. In this paper we will introduce a new notion of geometric structures defined by systems of closed differential forms in term of the Clifford algebra of the direct sum of the tangent bundle and the cotangent bundle on a manifold. We develop a unified approach of a deformation problem and establish a criterion of unobstructed deformations of the structures from a cohomological point of view. We construct the moduli spaces of the structures by using the action of $b$-fields and show that the period map of the moduli space is locally injective under a cohomological condition (the local Torelli type theorem). We apply our approach to generalized Calabi-Yau (metrical) structures and obtain an analog of the theorem by Bogomolov-Tian-Todorov. Further we prove that deformations of generalized Spin(7) structures are unobstructed.

Contents

§0 Introduction
§1 Clifford algebras and Spin groups
§2 Clifford-Lie operators
§3 Deformations of generalized geometric structures
§4 Deformations of generalized Calabi-Yau structures
§5 Generalized hyperKähler, $G_2$ and Spin(7) structures
§0 Introduction

In a paper [Go], the author introduced a notion of geometric structures defined by systems of closed differential forms which are based on the action of the gauge group of the tangent bundle on a manifold. This approach provides a systematic construction of smooth moduli spaces of Calabi-Yau, hyperKähler, $G_2$ and Spin(7) structures. In a paper [Hi1] Hitchin presented a generalized geometry, which depend on a suggestive idea of replacing the tangent bundle by the direct sum of the tangent bundle $T$ and the cotangent bundle $T^*$. The generalized geometry is a current issue which is rapidly studied in differential geometry and mathematical physics. However the idea of generalized geometry is perhaps not as widely appreciated as it should be. A possible reason for this is that the generalized geometry is at present restricted to rather special cases. In this paper we will develop the idea of the generalized geometry from a wide view point as in [Go] which is of general nature together with some new applications. Since there is an indefinite metric on the direct sum $T \oplus T^*$ on a manifold $X$, then the bundle of the Clifford algebras $\text{CL}(X)$ of $T \oplus T^*$ naturally appears and we obtain various fibre bundles of Lie groups such as Spin, Pin and conformal Pin group $\text{Cpin}(X)$ which act on the differential forms on $X$ by the spin representation. Then we introduce a notion of geometric structures which are defined by systems of closed differential forms in an orbit of the action of the conformal pin group. We develop a deformation problem of the structures and establish a criterion for unobstructed deformations of the structures and study a problem for when the local Torelli type theorem holds (theorem 3-7, 8 and 9). Then we apply our approach to interesting special cases of generalized structures discussed in [Hi1], [Gu1]. For instance, generalized $\text{SL}_n(\mathbb{C})$ structures are defined as complex pure spinors with a non-degenerate condition which are a generalization of complex structures with trivial canonical line bundle. Note that we call them generalized $\text{SL}_n(\mathbb{C})$ structures because the special linear group $\text{SL}_n(\mathbb{C})$ naturally arises as the isotropy group. (see [Go] for $\text{SL}_n(\mathbb{C})$ structures). A generalized $\text{SL}_n(\mathbb{C})$ structure $\phi$ induces a generalized complex structure $J_\phi$. Then our criterion easily implies

**Theorem 4-1-6.** If we have the $dd^J$-property for the $J_\phi$ corresponding a generalized $\text{SL}_n(\mathbb{C})$ structure $\phi$, then $\phi$ is a topological structure, so that is, we have unobstructed deformations of $\phi$ on which the local Torelli type theorem holds.

The $dd^J$-property is a generalization of ordinary $\overline{\partial}\partial$-lemma in Kähler geometry and Gualtieri shows that the $dd^J$-property holds for generalized Kähler structures [Gu2]. A Calabi-Yau (metrical) structure in [Gu1] is a pair consisting of generalized $\text{SL}_n(\mathbb{C})$ structures $\phi_0$ and $\phi_1$ such that the corresponding pair of generalized complex structures yields a generalized Kähler structure. Deformations of such pairs seems to be complicated, however it is observed that our systematic approach is adapted to obtain unobstructed deformations of Calabi-Yau (metrical) structures and the local Torelli type theorem of them (theorem 4-2-3).
Li also shows a result of deformations of generalized complex structure [Li]. It is worthwhile to mention that there is a relation between deformations of generalized SL\(_n(C)\) structures and ones of generalized complex structures (Proposition 4-1-7). If we have the \(dd^c\)-property, there is a surjective map from deformations of generalized SL\(_n(C)\) structures to ones of generalized complex structures, so that is, both deformations are essentially same, which yields an another proof of the result by Li.

We give a brief outline of this paper. In section 0, we present an exposition of the Clifford algebras of the direct sum of a real vector space \(V\) and the dual space \(V^*\) and various groups such as Spin, Pin and conformal pin groups. It is important that the exponential \(e^b\) (resp. \(e^\beta\)) for a 2-form \(b \in \wedge^2 V^*\) (resp. a 2-vector \(\beta \in \wedge^2 V\)) gives an element of the spin group. The materials in this section are already well explained in [L-M], [Ha] and [Hi1]. In section 2, we introduce a subbundle \(\text{Cl}^k\) over a manifold \(X\) which gives a filtration of the even Clifford bundle and one of the odd Clifford bundle:

\[
\text{Cl}^0 \subset \text{Cl}^2 \subset \text{Cl}^4 \subset \cdots,
\]
\[
\text{Cl}^1 \subset \text{Cl}^3 \subset \text{Cl}^5 \subset \cdots.
\]

Further we discuss differential operators acting on differential forms on \(X\) which arise as commutators between the exterior derivative \(d\) and the action of the Clifford algebra \(\text{Cl}\). The Clifford-Lie operators of order 3 are introduced in definition 2-2 which are invariant under the adjoint action of \(e^a\) for \(a \in \text{Cl}^2\) (lemma 2-4). The exterior derivative \(d\) is a Clifford-Lie operator of order 3 and it follows that \(e^{-a} \circ d \circ e^a\) is also a Clifford-Lie operator of order 3, which play a significant role in studying the deformation problem. In section 3, a notion of geometric structures is introduced. We start with the direct sum of the real vector space \(V\) of \(n\) dim and the dual space \(V^*\). The conformal pin group \(\text{Cpin}(V \oplus V^*)\) of \(V \oplus V^*\) linearly acts on the direct sum of skew-symmetric tensors \(\oplus^l \wedge^* V^*\). Let \(\Phi = (\phi_1, \cdots, \phi_l)\) be an element of the direct sum \(\oplus^l \wedge^* V^*\) and \(\mathcal{B}(V)\) the orbit of \(\text{Cpin}(V \oplus V^*)\) through \(\Phi\). We fix the orbit \(\mathcal{B}(V)\) and goes to a oriented, compact manifold \(X\) of dim \(n\). The orbit \(\mathcal{B}(V)\) yields the orbit in \(\oplus^l \wedge^* T^*_x X\) for each point \(x \in X\) and we have a fibre bundle \(\mathcal{B}(X)\) by

\[
\mathcal{B}(X) := \bigcup_{x \in X} \mathcal{B}(T_x X) \to X.
\]

The set of global sections of \(\mathcal{B}(X)\) is denoted by \(\mathcal{E}_\mathcal{B}(X)\) and then we define a \(\mathcal{B}(V)\)-structure on \(X\) by a \(d\)-closed section of \(\mathcal{E}_\mathcal{B}(X)\). We denote by \(\widetilde{\mathcal{M}}_\mathcal{B}(X)\) the set of \(\mathcal{B}(V)\)-structures on \(X\):

\[
\widetilde{\mathcal{M}}_\mathcal{B}(X) = \{ \Phi \in \mathcal{E}_\mathcal{B}(X) \mid d\Phi = 0 \}.
\]
Then we define a moduli space of $\mathcal{B}(V)$-structures on $X$ by the quotient space:

$$\mathcal{M}_B(X) = \tilde{\mathcal{M}}_B(X)/\tilde{\text{Diff}}_0(X),$$

where $\tilde{\text{Diff}}_0(X)$ is an extension of the diffeomorphisms of $X$ by the action of $d$-exact $b$-fields (see definition 3-2). Since the de Rham cohomology class of $\Phi$ is invariant under the action of $\tilde{\text{Diff}}_0(X)$, we have the Period map:

$$P_B: \mathcal{M}_B(X) \to H^*_dR(X).$$

In order to discuss deformations of a $\mathcal{B}(V)$ structure $\Phi$, we introduce a suitable deformation complex $\#_B$ (proposition 3-3):

$$0 \to E^{-1}(X) \xrightarrow{d^{-1}} E^0(X) \xrightarrow{d_0} E^1(X) \xrightarrow{d_1} E^2(X) \xrightarrow{d_2} \cdots,$$

Each vector bundle $E^{k-1}(X)$ is defined by the action of the Clifford subbundle $\text{CL}^k$ of $\Phi$, so that is, $E^{k-1}(X) = \text{CL}^k \cdot \Phi$ and the differential operator $d_k$ is the restriction of $d$ to the bundle $E^k(X)$. An orbit $\mathcal{B}(V)$ is an elliptic orbit if the deformation complex $\#_B$ is an elliptic complex. It is observed that the complex $\#_B$ is a subcomplex of the direct sum of the de Rham complex and we have the map $p^k_B$ from the cohomology groups $H^k(\#_B)$ of the complex $\#_B$ to the direct sum of the de Rham cohomology groups. We say a $\mathcal{B}(V)$-structure $\Phi$ is a topological structure if the map $p^k_B$ is injective for $k = 1, 2$ (definition 3-5). Our criterion for unobstructed deformations and the local Torelli type theorem is shown in theorem 3-7:

**Theorem 3-7.** Let $\mathcal{B}(V)$ be an elliptic orbit and $\Phi$ a $\mathcal{B}(V)$-structure on a compact and oriented $n$-manifold $X$. If $\Phi$ is a topological structure, then deformations of $\Phi$ are unobstructed and the deformation space of $\Phi$ is locally embedded into the de Rham cohomology group $H^*_dR(X)$. In particular, if an orbit $\mathcal{B}(V)$ is elliptic and topological, the period map $P_B$ of the moduli space $\mathcal{M}_B(X)$ of $\mathcal{B}(V)$ structures on $X$ is locally injective.

In section 4 we apply our approach to generalized $\text{SL}_n(\mathbb{C})$ structures and generalized Calabi-Yau (metrical) structures. In section 5, we introduce generalized hyperKähler, $G_2$ and Spin(7) structures as special $\mathcal{B}(V)$-structures. The generalized exceptional structure ($G_2$ and Spin(7)) are discussed by Witt [W] from other point of view. Our approach is adapted in these interesting cases. For instance, we will show that deformations of generalized Spin(7) structures are unobstructed. We will discuss the deformation problems of other special structures in a forthcoming paper.
§1. Clifford algebras and Spin groups

§1-1. Let $V$ be an $n$ dimensional real vector space and $V^{\ast}$ the dual space of $V$. We denote by $\eta(v)$ by the natural coupling between $v \in V$ and $\eta \in V^{\ast}$. Then there is an indefinite bilinear form $\langle , \rangle$ on the direct sum $V \oplus V^{\ast}$ which is defined by

$$\langle E_1, E_2 \rangle = \frac{1}{2} \eta_1(v_2) + \frac{1}{2} \eta_2(v_1),$$

where $E_i = v_i + \eta_i \in V \oplus V^{\ast}$ for $i = 1, 2$. (In particular the norm $\|E\|^2 = \langle E, E \rangle$.) We consider $V \oplus V^{\ast}$ as the $2n$ dimensional vector space and denote by $\otimes^k(V \oplus V^{\ast})$ the tensor product of $k$-copies of $V \oplus V^{\ast}$. Let

$$\otimes(V \oplus V^{\ast}) := \sum_{i=0}^{\infty} \otimes^k(V \oplus V^{\ast}).$$

be the the tensor algebra of $(V \oplus V^{\ast})$ (Note that $\otimes^0(V \oplus V^{\ast}) = \mathbb{R}$), and define $\mathcal{I}$ to be the two-sided ideal in $\otimes(V \oplus V^{\ast})$ generated by all elements of the form $E \otimes E - \|E\|1$ for $E \in V \oplus V^{\ast}$. Then the Clifford algebra $\text{CL}(V \oplus V^{\ast})$ is defined to be the quotient algebra with the unit 1 :

$$\text{CL}(V \oplus V^{\ast}) = \otimes(V \oplus V^{\ast})/\mathcal{I}.$$

The product of the Clifford algebra is called the Clifford product which is denoted by $\alpha \cdot \beta$ for $\alpha, \beta \in \text{CL}(V \oplus V^{\ast})$ and for all $E, F \in V \oplus V^{\ast}$,

$$E \cdot F + F \cdot E = \langle E, F \rangle 1.$$ 

Since the ideal $\mathcal{I}$ is generated by tensors of degree 2, the Clifford algebra $\text{CL}(V \oplus V^{\ast})$ is decomposed into the even part and the odd part :

$$\text{CL}(V \oplus V^{\ast}) = \text{CL}^{\text{even}} \oplus \text{CL}^{\text{odd}}.$$

There are two involutions of $\text{CL}(V \oplus V^{\ast})$ which play the smart roles. The first one is defined by the decomposition (1-1-5) :

$$\tilde{\alpha} := \begin{cases} +\alpha, & (\alpha \in \text{CL}^{\text{even}}), \\ -\alpha, & (\alpha \in \text{CL}^{\text{odd}}), \end{cases}$$

for $\alpha \in \text{CL}(V \oplus V^{\ast})$. If we reverse the order in a simple product $\alpha = E_1 \cdot E_2 \cdots E_k \in \text{CL}(V \oplus V^{\ast})$ ($E_i \in V \oplus V^{\ast}$), we obtain the second involution $\sigma$ of $\text{CL}(V \oplus V^{\ast})$ :

$$\sigma(\alpha) = E_n \cdots E_2 \cdot E_1.$$
Since there is the natural isomorphism between the skew-symmetric tensors \( \wedge^*(V \oplus V^*) \) and \( \text{CL}(V \oplus V^*) \) as \( \mathbb{R} \)-module, there is the metric \( \langle , \rangle \) on \( \text{CL}(V \oplus V^*) \) which is written as

\[(1-1-8) \quad \langle \alpha, \beta \rangle = \frac{1}{2} \langle 1, \sigma(\alpha)\beta \rangle,\]

for \( \alpha, \beta \in \text{CL}(V \oplus V^*) \). In particular we denote by \( \|\alpha\| \) the Clifford norm of \( \alpha \) :

\[(1-1-9) \quad \|\alpha\|^2 := \langle \alpha, \alpha \rangle = \frac{1}{2} \langle 1, \sigma(\alpha)\alpha \rangle.\]

Let \( \wedge^p V^* \) be the space of skew-symmetric tensor of degree \( p \) and \( S \) the direct sum of the spaces of skew-symmetric tensors :

\[(1-1-10) \quad S := \bigoplus_{p=0}^{\infty} \wedge^p V^*.\]

Then \( E = v + \eta \in V \oplus V^* \) acts on \( S \) by the interior and the exterior product :

\[(1-1-11) \quad E \cdot \phi = i_v \phi + \eta \wedge \phi\]

Since we have the identity :

\[(1-1-12) \quad i_v \eta \wedge \phi + \eta \wedge i_v \phi = \|E\|^2 \phi,\]

we have the action of \( \text{CL}(V \oplus V^*) \) on \( S \), (which is called the spin representation). Let \( \text{CL}(V \oplus V^*)^\times \) be the group which consists of invertible elements of \( \text{CL}(V \oplus V^*) \). For each \( g \in \text{CL}(V \oplus V^*)^\times \) we define a linear map \( \tilde{\text{Ad}}_g : \text{CL}(V \oplus V^*) \to \text{CL}(V \oplus V^*) \) by

\[(1-1-13) \quad \tilde{\text{Ad}}_g(\alpha) := \tilde{g}^{-1} \alpha g, \quad (\alpha \in \text{CL}(V \oplus V^*)),\]

where \( \tilde{g} \) is the first involution of \( g \). Note that the image \( \tilde{\text{Ad}}_g(V \oplus V^*) \) is not a subspace of \( V \oplus V^* \) for a general \( g \in \text{CL}(V \oplus V^*)^\times \). The conformal pin group \( \text{Cpin}(= \text{Cpin}(V \oplus V^*)) \) is a subgroup of \( \text{CL}(V \oplus V^*)^\times \) which defined by

\[(1-1-14) \quad \text{Cpin} := \{ g \in \text{CL}(V \oplus V^*)^\times \mid \tilde{\text{Ad}}_g(V \oplus V^*) \subset V \oplus V^* \}.\]

Since \( \tilde{\text{Ad}}_g \) is an orthogonal endomorphism of \( V \oplus V^* \), we have the short exact sequence :

\[(1-1-15) \quad 1 \to \mathbb{R}^\times \to \text{Cpin} \xrightarrow{\tilde{\text{Ad}}} \text{O}(V \oplus V^*) \to 1.\]
Since each element of the conformal pin group \( C_{\text{pin}} \) is written as a simple product, it follows that the Clifford norm of \( g \in C_{\text{pin}} \) is given by \( \|g\|^2 = \sigma(g) \cdot g \). We define the Pin group \( \text{Pin}(= \text{Pin}(V \oplus V^*)) \) by

(1-1-16) \quad \text{Pin} = \{ g \in C_{\text{pin}} \mid \|g\| = \pm 1 \},

and the Spin group \( \text{Spin}(= \text{Spin}(V \oplus V^*)) \) is defined by

(1-1-17) \quad \text{Spin} := \text{Pin} \cap \text{CL}^{\text{even}}.

Then we also have the short exact sequence:

(1-1-18) \quad 1 \to \mathbb{Z}_2 \to \text{Spin} \xrightarrow{\text{Ad}} \text{SO}(V \oplus V^*) \to 1.

We denote by \( \text{Spin}_0(= \text{Spin}_0(V \oplus V^*)) \) the identity component of \( \text{Spin} \). Then \( \text{Spin}_0 \) is given by

(1-1-19) \quad \text{Spin}_0 = \{ g \in \text{Spin} \mid \|g\| = 1 \}.

§1-2. The Lie algebra \( \mathfrak{so}(V \oplus V^*) \) of the Lie group \( \text{SO}(V \oplus V^*) \) is decomposed into three parts:

(1-2-1) \quad \mathfrak{so}(V \oplus V^*) = \text{End}(V) \oplus \wedge^2 V \oplus \wedge^2 V^*.

Each \( a \in \mathfrak{so}(V \oplus V^*) \) is written as a form of matrix:

\[
\begin{pmatrix}
A & \beta \\
b & -A^*
\end{pmatrix},
\]

where \( A \in \text{End}(V) \), \( b \in \wedge^2 V^* \), \( \beta \in \wedge^2 V \) and \( A^* \in \text{End}(V) \) is defined by \( A^*(\eta)(v) = \eta(Av) \) for \( v \in V \) and \( \eta \in V^* \). (Note that \( b : V \to V^* \) and \( \beta : V^* \to V \).) Then the Lie group \( \text{GL}(V) \) is embedded into \( \text{SO}(V \oplus V^*) \):

(1-2-2) \quad \begin{pmatrix}
g & 0 \\
0 & (g^*)^{-1}
\end{pmatrix}, \quad g \in \text{GL}(V).

Further for \( b \in \wedge^2 V^* \) and \( \beta \in \wedge^2 V \) we define \( e^b \) and \( e^\beta \) by:

\[
e^b = 1 + b + \frac{1}{2!}b^2 + \cdots,
\]
\[
e^\beta = 1 + \beta + \frac{1}{2!}\beta^2 + \cdots,
\]
then $e^b$ and $e^\beta$ are elements of Spin$_0$ respectively. Let GL$_0$(V) be the identity component of GL(V). We denote by $q$ the embedding (1-2-2) of GL$_0$(V) into the identity component of SO$_0$(V $\oplus$ V$^*$):

(1-2-3) \quad q: GL_0(V) \to SO_0(V $\oplus$ V$^*$).

Let Ad be the covering map Ad: Spin$_0$ $\to$ SO$_0$(V $\oplus$ V$^*$) as before. Then there is a map $p$: GL$_0$(V) $\to$ Spin$_0$ such that Ad$p$ = $q$, so that is, $p$ is the lift of the map $q$:

(1-2-4) \quad GL_0(V) \xrightarrow{p} \text{Spin}_0 \xrightarrow{\text{Ad}} GL_0(V) \xrightarrow{\varphi} SO_0.

The representation $S$ of the Clifford algebra CL(V $\oplus$ V$^*$) restricts to the representation $\rho_{\text{spin}}$ of Spin$_0$.

(1-2-5) \quad \rho_{\text{spin}}: \text{Spin}_0 \to GL(S).

We also denote by $\rho^*_{\text{GL}}$ the linear representation of GL(V) on $\wedge^* V^*$. The composition $\rho_{\text{spin}} \circ p$ gives rise to a representation of GL$_0$(V).

**Lemma 1-2-1.** The representation $\rho_{\text{spin}} \circ p$ is given by

$$\rho_{\text{spin}} = (\text{det } V^*)^{\frac{1}{2}} \otimes (\rho^*_{\text{GL}})^{-1},$$

where $(\text{det } V^*)^{\frac{1}{2}}$ is the half of the determinant representation.

§2. Clifford-Lie operators

We use the same notation as in section 1. Let X be a real manifold of dim $n$. Then we consider the direct sum $T \oplus T^*$ of the tangent bundle $T = TX$ and the cotangent bundle $T^* = T^*X$. Let CL(X) = CL(T $\oplus$ T$^*$) be the Clifford bundle on X:

$$\text{CL}(X) := \bigcup_{x \in X} \text{CL}(T_x X \oplus T^*_x X) \to X.$$

We also define the conformal pin group-bundle Cpin(X) = Cpin(T $\oplus$ T$^*$) by:

$$\text{Cpin}(X) := \bigcup_{x \in X} \text{Cpin}(T_x X \oplus T^*_x X) \to X.$$
Let $\pi$ be the natural projection,

$$\pi: \otimes(T \oplus T^*) \to CL(X) = \otimes(T \oplus T^*)/I.$$ 

We denote by $CL^{2i}$ the image

$$CL^{2i} := \pi\left(\oplus_{l=0}^{i} \otimes 2l (T \oplus T^*)\right).$$

Then we have a filtration of $CL^{\text{even}}$:

$$CL^0 \subset CL^2 \subset CL^4 \subset \cdots.$$ 

We also have a filtration of $CL^{\text{odd}}$ which defined by

$$CL^1 \subset CL^3 \subset CL^5 \subset \cdots,$$

where

$$CL^{2i+1} := \pi\left(\oplus_{l=0}^{i} \otimes 2l+1 (T \oplus T^*)\right).$$

Let $S(X)$ be the bundle of differential forms $\wedge^*T^*X$ over a manifold $X$. By using the spin representation on each fibre as in section 1, the bundle of the Clifford algebra $CL(X)$ acts on $S(X)$. Let $\mathcal{L}_E$ be the anti-commutator $\{d, E\} = dE + Ed$ for a section $E$ of the bundle $T \oplus T^*$. (For simplicity, we denote it by $E \in CL^1 = T \oplus T^*$.) If we denote $E = v + \theta \in T \oplus T^*$ then $\mathcal{L}_E = \mathcal{L}_v + (d\theta)$, where $\mathcal{L}_v$ is the ordinary Lie derivative and $(d\theta)$ acts on $S(X)$ by the wedge product. Next we consider a bracket $[\mathcal{L}_E, F] = \mathcal{L}_E F - F \mathcal{L}_E$ for $E, F \in T \oplus T^*$.

**Lemma 2-1.** The bracket $[\mathcal{L}_E, F]$ is a section of $T \oplus T^*$.

**proof.** When we write $E = v + \theta, F = w + \eta \in T \oplus T^*$, then we have

$$[\mathcal{L}_E, F] = [\mathcal{L}_v + (d\theta), w + \eta]
= [\mathcal{L}_v, w] + [\mathcal{L}_v, \eta] + [(d\theta), w] + [(d\theta), \eta]
= [v, w] + (\mathcal{L}_v \eta) + [(d\theta), w].$$

Since $[(d\theta), w] \in (T \oplus T^*)$, we have the result. □

In this paper Clifford algebra valued Lie derivatives play an significant role.

**Definition 2-2 (Clifford-Lie operators).** A Clifford-Lie operator of order 3 on $X$ is a differential operator acting on $S(X)$ which is locally written as

$$L = \sum_{i,j} a^{ij} E_i \mathcal{L}_{E_j} + K.$$
on every open set $U$ on $X$ for some $E_i \in \text{CL}^1(TU \oplus T^*U)$, $a_{ij} \in C^\infty(U)$ and $K \in \text{CL}^3(TU \oplus T^*U)$.

Let $\{x_1, \ldots, x_n\}$ be a local coordinates of $X$. We denote by $v_i$ the vector field $\frac{\partial}{\partial x_i}$ and $\theta^i = dx^i$. Then the exterior derivative $d$ is locally written as

$$d = \sum_{i=0}^n \theta^i \wedge \mathcal{L}_{v_i}.$$ 

Hence $d$ is the Clifford-Lie operator of order 3.

Let $a$ be a section of $\text{CL}^2(T \oplus T^*)$. Then we have

**Lemma 2-3.** If $L$ is a Clifford-Lie operator of order 3 then the commutator $[L, a]$ is also a Clifford-Lie operator of order 3.

**proof.** Let $f$ be a function on $X$ and $E = v + \theta$ a section of $T \oplus T^*$. Since we have

$$\mathcal{L}_E f a = (\mathcal{L}_E f)a + f\mathcal{L}_E a,$$

where $\mathcal{L}_E f = \mathcal{L}_v f \in C^\infty(X)$. We have the following equality on an open set $U$ on $X$:

$$[L, fa] = L(fa) - faL$$
$$= \sum_{ij} a_{ij} E_i \mathcal{L}_{E_j} (fa) - f aL + K$$
$$= \sum_{ij} a_{ij} E_i (\mathcal{L}_{E_j} f)a + f[L, a].$$

Since $E_i (\mathcal{L}_{E_j} f)a \in \text{CL}^3(T \oplus T^*)$, it is sufficient to show the lemma in the case $a = F_1 F_2$ for $F_i \in T \oplus T^*(i = 1, 2)$. The bracket $[\mathcal{L}_E, F_1 F_2]$ is given by

$$[\mathcal{L}_E, F_1 F_2] = \mathcal{L}_E F_1 F_2 - F_1 F_2 \mathcal{L}_E$$
$$= [\mathcal{L}_E, F_1] F_2 + F_1 \mathcal{L}_E F_2 - F_1 F_2 \mathcal{L}_E$$
$$= \mathcal{L}_E, F_1] F_2 + F_1 [\mathcal{L}_E, F_2].$$

Hence it follows from lemma 2-1 that $[\mathcal{L}_E, F_1 F_2] \in \text{CL}^2$. The bracket $[E_1 \mathcal{L}_{E_2}, F_1 F_2]$ is given by

$$[E_1 \mathcal{L}_{E_2}, F_1 F_2] = E_1 \mathcal{L}_{E_2} F_1 F_2 - F_1 F_2 E_1 \mathcal{L}_{E_2}$$
$$= E_1 [\mathcal{L}_{E_2}, F_1 F_2] + E_1 F_1 F_2 \mathcal{L}_{E_2} - F_1 F_2 E_1 \mathcal{L}_{E_2},$$
$$= [E_1, F_1 F_2] \mathcal{L}_{E_2} + E_1 [\mathcal{L}_{E_2}, F_1 F_2].$$
Since $[E_1, F_1 F_2] = 2(E_1 F_1) F_2 - 2(E_1, F_2) F_1 \in \text{CL}^1 = (T \oplus T^*)$, it follows that the bracket $[E_1 \mathcal{L}_{E_2}, F_1 F_2]$ is a Clifford-Lie operator of order 3. Then the result follows from the equation:

$$[L, F_1 F_2] = \sum_{i,j} a_{ij} E_i \mathcal{L}_{E_j}, F_1 F_2]$$

$$= \sum_{i,j} a_{ij} [E_i \mathcal{L}_{E_j}, F_1 F_2].$$

\[\Box\]

**Lemma 2-4.** The commutator $[d, a]$ is a Clifford-Lie operator of order 3.

**proof.** Since $d$ is a Clifford-Lie operator of order 3, the result follows from lemma 2-3. We think a following direct proof is more readable. $[d, f a] = df a - f d a = (df)a + f [d, a]$ for a function $f$. Hence it is sufficient to show the lemma in the case $a = E_1 E_2$, where $E_i \in T \oplus T^*$ ($i = 1, 2$). Then the bracket $[d, a]$ is written as

$$[d, a] = dE_1 E_2 - E_1 E_2 d$$

$$= \mathcal{L}_{E_1} E_2 - E_1 dE_2 - E_1 E_2 d$$

$$= \mathcal{L}_{E_1} E_2 - E_1 \mathcal{L}_{E_2}$$

$$= E_2 \mathcal{L}_{E_1} - E_1 \mathcal{L}_{E_2} + [\mathcal{L}_{E_1}, E_2].$$

Hence the result follows from $[\mathcal{L}_{E_1}, E_2] \in \text{CL}^1 \subset \text{CL}^3$. \[\Box\]

**Proposition 2-5.** Let $a_1, a_2 \in \text{CL}^2(T \oplus T^*)$. Then $[[d, a_1], a_2]$ is a Clifford-Lie operator of order 3. Further we denote by $\text{Ad}_a L$ the commutator $[L, a]$. Then the composition $\text{Ad}_{a_1} (\text{Ad}_{a_2} \cdots \text{Ad}_{a_n} d \cdots)$ is a Clifford-Lie operator of order 3 for $a_1, \cdots, a_n \in \text{CL}^2$.

**proof.** It follows form Lemma 2-3 and 4. \[\Box\]

**Remark 2-6.** In the case of $a_1, a_2 \in \text{End}(TX)$, the bracket $[[d, a_1], a_2]$ is given in terms of the Nijenhuis tensor of $a_1$ and $a_2$. In the case $a_1, a_2 \in \wedge^2 T$, the bracket $[[d, a_1], a_2]$ is the Schouten bracket. In general the bracket $[[d, a_1], a_2]$ is not a tensor but a differential operator.

Let $a$ be a section of $\text{CL}^2$ and $L$ an operator acting on $S(X)$. We successively define an operator $(\text{Ad}_a^l) L$ acting on $S(X)$ by

$$(\text{Ad}_a)^l L = [(\text{Ad}_a)^{l-1} L, a].$$

We also define a formal power series $(\exp(\text{Ad}_a)) L$ by

$$(\exp(\text{Ad}_a)) L = \sum_{l=0}^\infty \frac{1}{l!} (\text{Ad}_a)^l L$$

$$= d + [L, a] + \frac{1}{2!} [[L, a], a] + \cdots.$$
Lemma 2-7. The power series \((\exp(\text{Ad}_a)) L\) is given by
\[
(\exp(\text{Ad}_a)) L = e^{-a} \circ L \circ e^a.
\]

proof. It follows from definition of \((\text{Ad}_a)^i L\) that
\[
(\text{Ad}_a)^i L = \sum_{m=0}^{l} \frac{(-1)^m m!}{m!(l-m)!} a^m L a^{l-m}.
\]
Then by a combinatorial calculation we have
\[
L a^k = \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} a^{k-l} (\text{Ad}_a)^l L.
\]
Then we have
\[
Le^a = e^a(L + (\text{Ad}_a)L + \frac{1}{2!}(\text{Ad}_a)^2L + \frac{1}{3!}(\text{Ad}_a)^3L + \cdots) = e^a(\exp(\text{Ad}_a)L).
\]
Hence the result follows. \(\square\)

Proposition 2-8. If \(L\) is a Clifford-Lie operator of order 3 and \(a \in \text{CL}^2\), then \(e^{-a} \circ L \circ e^a = (\exp(\text{Ad}_a) L)\) is also a Clifford-Lie operator of order 3. In particular \((\exp(\text{Ad}_a)d)\) is a Clifford-Lie operator of order 3.

proof. The result follows from lemmas 2-5 and 2-7. \(\square\)

§3. Deformations of generalized geometric structures

§3-1. Let \(V\) be an \(n\) dimensional real vector field and \(V^*\) the dual space of \(V\). As in section 1 the space of the skew-symmetric tensors \(S:=\wedge^* V^*\) is regarded as the spin representation of \(\text{CL}(\text{CL}(V \oplus V^*))\), which restricted to give the representation of \(\text{Cpin}(\text{Cpin}(V \oplus V^*))\). We consider the direct sum of the spin representations of \(\text{Cpin}(V \oplus V^*)\):

\[
\bigoplus^t S := (\wedge^* V^* \oplus \cdots \oplus \wedge^* V^*).
\]
Let \(\Phi_V = (\phi_1, \cdots, \phi_t)\) be an element of the direct sum \(\bigoplus^t S\). Then we have the orbit \(\mathcal{B}(V)\) of \(\text{Cpin}(V \oplus V^*)\) through \(\Phi_V\):

\[
\mathcal{B}(V) := \{ g \cdot \Phi_V \mid g \in \text{Cpin}(V \oplus V^*) \}.
\]
From now on we fix the orbit $\mathcal{B}(V)$. In this paper we think that the orbit of $\text{Cpin}(V \oplus V^*)$ gives rise to a generalized geometric structure on the vector space. Let $\text{GL}_0(V)$ be the connected component of $\text{GL}(V)$ with the identity and $\mathcal{A}(V)$ the orbit of $\text{GL}_0(V)$ through $\Phi_V$. As in the section 1, we have the map $p: \text{GL}_0(V) \to \text{Spin}_0 \subset \text{Cpin}$. It follows from lemma 1-2-1 that for $\Phi \in \oplus^l S$ we have,

$$(p(g)) \cdot \Phi = (\det g)^{\frac{1}{2}} \rho_{\text{GL}^*(V)}(g^{-1})\Phi,$$

for $g \in \text{GL}_0(V)$.

Since $(\det g)^{-\frac{1}{2}} p(g) \in \text{Cpin}$, we have

$$(\det g)^{-\frac{1}{2}} p(g) \cdot \Phi = \rho_{\text{GL}^*(V)}(g^{-1})\Phi.$$  

It implies that the $\text{GL}_0(V)$-orbit $\mathcal{A}(V)$ is embedded into the $\text{CPin}(V \oplus V^*)$-orbit $\mathcal{B}(V)$:

$$(3-1-1)\quad \mathcal{A}(V) \hookrightarrow \mathcal{B}(V).$$

The inclusion (3-1-1) shows that the group $\text{Cpin}$ is suitable for our construction. Let $X$ be an oriented and compact real manifold of dim $n$. As in section 2 we have the Clifford bundle $\text{CL}(X)$ and the conformal pin bundle $\text{Cpin}(X)$ over $X$. When we take an identification between $V$ and $T_xX$ for each $x \in X$, we have the orbit $\mathcal{B}(T_xX)$ of $\text{CPin}(T_xX \oplus T_x^*X)$. It follows from (3-1-1) that the orbit $\mathcal{B}(T_xX)$ is independent of a choice of an identification and thus $\mathcal{B}(T_xX)$ is canonically defined as the submanifold of the direct sum of forms $\oplus^l \wedge^* T_x^*X$. Hence we have the fibre bundle $\mathcal{B}(X) \to X$:

$$\mathcal{B}(X) := \bigcup_{x \in X} \mathcal{B}(T_xX) \to X.$$  

Let $H$ be the isotropy group of the action of $\text{CPin}(V \oplus V^*)$ at $\Phi_V$:

$$H := \{ g \in \text{Cpin}(V \oplus V^*) \mid g \cdot \Phi = \Phi \}.$$  

Then the fibre bundle $\mathcal{B}(X)$ is the fibre bundle with fibre $\text{Cpin}(V \oplus V^*)/H$ and $\mathcal{B}(X)$ is embedded into the direct sum of differential forms $\oplus^l \wedge^* T^*X$. We denote by $\mathcal{E}_B(X)$ the set of $C^\infty$-sections of the fibre bundle $\mathcal{B}(X)$:

$$\mathcal{E}_B(X) := C^\infty(X, \mathcal{B}(X)).$$

Each section $\Phi \in \mathcal{E}_B(X)$ consists of differential forms on which the exterior derivative $d$ acts. Let $\widetilde{\mathcal{M}}_B(X)$ be the set of $d$-closed section of $\mathcal{E}_B(X)$:

$$\widetilde{\mathcal{M}}_B(X) := \{ \Phi \in \mathcal{E}_B(X) \mid d\Phi = 0 \}.$$
Definition 3-1. A generalized geometric structure on $X$ associated with the orbit $B(V)$ is a $d$-closed section $\Phi \in \widetilde{M}_B(X)$. For simplicity, we call a $d$-closed section $\Phi$ a $B(V)$-structure on $X$.

The diffeomorphism $Diff(X)$ naturally acts on $\widetilde{M}_B(X)$ by the pull back. We denote by $Diff_0(X)$ the identity component of $Diff(X)$. Since the exponential $e^{d\gamma}$ is a section of the bundle $Spin_0(X)$ for a 1-form $\gamma$, we have the action of $e^{d\gamma}$ on $B(V)$-structures $\widetilde{M}_B(X)$,

$$\Phi \mapsto e^{d\gamma} \wedge \Phi, \quad (\gamma \in T^*X).$$

Let $\widetilde{Diff}_0(X)$ be the group generated by the composition of the action of $Diff_0(X)$ and $d$-exact 2-forms:

$$\widetilde{Diff}_0(X) := \{ e^{d\gamma} \wedge f^* | \gamma \in T^*, f \in Diff_0(X) \}.$$ 

Here the group $\widetilde{Diff}_0(X)$ is regarded as a subgroup of the automorphisms of the bundle $Spin_0(X)$:

$$\begin{array}{ccc}
Spin_0(X) & \longrightarrow & Spin_0(X) \\
\downarrow & & \downarrow \\
X & \longrightarrow & X.
\end{array}$$

Hence the group $\widetilde{Diff}_0(X)$ is an extension of $Diff_0(X)$ by $d$-exact 2-forms $d(\wedge^1 T^*)$:

$$0 \to d(\wedge^1 T^*) \to \widetilde{Diff}_0(X) \to Diff_0(X) \to 0.$$ 

Definition 3-2. A moduli space $M_B(X)$ of $B(V)$-structures on $X$ is the quotient space of $\widetilde{M}_B(X)$ divided by the action of $Diff_0(X)$:

$$M_B(X) := \widetilde{M}_B(X)/Diff_0(X).$$

§3-2. Let $B(V)$ be the fixed orbit of $CPin(V \oplus V^*)$ as in section 3-1 and $\Phi$ a $B(V)$-structure on a manifold $X$. In order to consider deformations of $\Phi$, we introduce a deformation complex of the $B(V)$-structure $\Phi$. As in section 2 there is the natural filtration of the even Clifford bundle $CL^{even}$ and the one of the odd Clifford bundle $CL^{odd}$:

$$CL^0 \subset CL^2 \subset CL^4 \subset \cdots, $$

$$CL^1 \subset CL^3 \subset CL^5 \subset \cdots.$$ 

Then by using the action of $CL^k$ on $\Phi$, we obtain a vector bundle $E^{k-1}(X)$ over $X$:

$$E^{k-1}(X) := CL^k \cdot \Phi,$$
and the corresponding filtrations of vector bundles:

\[ E^{-1}(X) \subset E^1(X) \subset E^3(X) \subset \cdots, \]
\[ E^0(X) \subset E^2(X) \subset E^4(X) \subset \cdots. \]

(Note that we shift the degree of the filtration of vector bundles.) The vector bundle \( E^{-1}(X) \) is the line bundle generated by \( \Phi \). The vector bundle \( E^0(X) \) is generated by \( E \cdot \phi \) for all \( E \in T \oplus T^* \) over \( C^\infty(X) \) and \( E^1(X) \) is generated by \( E_1 \cdot E_2 \cdot \Phi \) for all \( E_1, E_2 \in T \oplus T^* \). Each \( E^k(X) \) is embedded into the direct sum of differential forms on which the exterior derivative \( d \) acts.

**Proposition 3-3.** There is a differential complex \( \#_{\mathcal{B}}(= \#_{\mathcal{B}, \Phi}) \) for each \( \Phi \in \tilde{\mathcal{M}}_B(X) \),

\[ 0 \rightarrow E^{-1}(X) \xrightarrow{d^{-1}} E^0(X) \xrightarrow{d^0} E^1(X) \xrightarrow{d^1} E^2(X) \xrightarrow{d^2} \cdots, \]

where \( d_k \) is given by the restriction \( d|E^k(X) \). The cohomology groups of the complex \( \#_{\mathcal{B}} \) is denoted by \( H^k(\#_{\mathcal{B}}) \),

\[ H^k(\#_{\mathcal{B}}) := \frac{\ker d_k : \Gamma(E^k(X)) \rightarrow \Gamma(E^{k+1}(X))}{\operatorname{im} d_{k-1} : \Gamma(E^{k-1}(X)) \rightarrow \Gamma(E^k(X))}. \]

Then the first cohomology group \( H^1(\#_{\mathcal{B}}) \) is regarded as the infinitesimal tangent space of the deformations of the \( \mathcal{B}(V) \)-structure \( \Phi \).

**proof.** A section of \( E^{-1}(X) \) is written as \( f \Phi \) for a function \( f \). Hence \( d(f \Phi) = df \wedge \Phi \) and we see that the image \( d(E^{-1}(X)) \) is included into \( E^0(X) \). We denote by \( \mathcal{L}_F \) the anti-commutator \( dF + Fd \) acting on forms where \( F \in T \oplus T^* \). When we write \( F = v + \eta \) for \( v \in T \) and \( \eta \in T^* \), then \( \mathcal{L}_F \) is given by

\[ \mathcal{L}_F = \mathcal{L}_v + (d\eta) \wedge, \]

where \( \mathcal{L}_v \) denotes the Lie derivative. Then we have

\[ \mathcal{L}_F(f \Phi) = \mathcal{L}_v(f \Phi) + (d\eta) \wedge (f \Phi) = (\mathcal{L}_v f) \Phi + f \mathcal{L}_v \Phi + f(d\eta) \wedge \Phi, \]

where \( \mathcal{L}_v f \in C^\infty(X) \). Since \( \text{GL}_0(TX) \) is the subbundle of \( \text{Cpin}(X) \), diffeomorphisms of \( X \) acts on \( \mathcal{E}_{\mathcal{B}}(X) \). Hence we have

\[ \mathcal{L}_v \Phi \in T_\Phi \mathcal{E}_{\mathcal{B}}(X). \]

The conformal Spin\(_0\) bundle \( \text{Cpin}_0(X) \) is given by

\[ \text{Cpin}_0(X) = \{ e^a \mid a \in \text{CL}^2 \}. \]
Since the tangent space $T_{\Phi}\mathcal{E}_B(X)$ is generated by the action of $\text{Cpin}_0(T \oplus T^*)$, we have

$$T_{\Phi}\mathcal{E}_B(X) \cong \text{CL}^2 \cdot \Phi = \mathbf{E}^1(X).$$

Hence we have

$$\mathcal{L}_v\Phi \in \mathbf{E}^1(X).$$

Then it follows that $\mathcal{L}_F(\mathbf{E}^{-1}(X)) \subset \mathbf{E}^1(X)$. We also have

$$d(F \cdot \Phi) = \mathcal{L}_F\Phi - Fd\Phi = \mathcal{L}_F\Phi.$$

Hence we have $d(\mathbf{E}^0(X)) \subset \mathbf{E}^1(X)$. For $F_1, F_2 \in T \oplus T^*$ we have

$$\mathcal{L}_{F_1}(F_2 \cdot \Phi) = [\mathcal{L}_{F_1}, F_2]\Phi + F_2 \cdot \mathcal{L}_{F_1}\Phi.$$

It follows from lemma 2-1 that $[\mathcal{L}_{F_1}, F_2] \in T \oplus T^*$. Hence from $\mathcal{L}_{F_1}\Phi \in \mathbf{E}^1(X)$ we have $\mathcal{L}_F(\mathbf{E}^0(X)) \subset \mathbf{E}^2(X)$. We will show that $d(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+1}(X)$ by induction on $k$. We assume that $d(\mathbf{E}^{k-2}(X)) \subset \mathbf{E}^{k-1}(X)$ and $\mathcal{L}_F(\mathbf{E}^{k-2}(X)) \subset \mathbf{E}^{k}(X)$ for some $k \geq 1$ and for all $F \in T \oplus T^*$. Then for $F_1, F_2 \in T \oplus T^*$ and $s \in \mathbf{E}^{k-2}(X)$ we have

$$d(F_1 \cdot F_2 \cdot s) = \mathcal{L}_{F_1}(F_2 \cdot s) - F_1 \cdot dF_2 \cdot s = [\mathcal{L}_{F_1}, F_2] \cdot s + F_2 \cdot \mathcal{L}_{F_1}s - F_1 \cdot \mathcal{L}_{F_2}s + F_1 \cdot F_2 \cdot ds.$$

It follows from our assumption ($ds \in \mathbf{E}^{k-1}(X)$ and $\mathcal{L}_s \Phi \in \mathbf{E}^{k}(X)$) that $d(F_1 \cdot F_2 \cdot s) \in \mathbf{E}^{k+1}(X)$ since $[\mathcal{L}_{F_1}, F_2] \cdot s \in \mathbf{E}^{k-1}(X) \subset \mathbf{E}^{k+1}(X)$. Hence $d(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+1}(X)$. For $F_3 \in T \oplus T^*$ we also have

$$\mathcal{L}_{F_3}(F_1 \cdot F_2 \cdot s) = [\mathcal{L}_{F_3}, F_2] \cdot F_1 \cdot s + F_2 \cdot \mathcal{L}_{F_3}(F_1 \cdot s),$$

$$= [\mathcal{L}_{F_3}, F_2] \cdot F_1 \cdot s + F_2 \cdot [\mathcal{L}_{F_3}, F_1] \cdot s + F_2 \cdot F_1 \cdot \mathcal{L}_{F_3}s.$$

Hence it follows from our assumption $\mathcal{L}_s \Phi \in \mathbf{E}^k(X)$ that $\mathcal{L}_{F_3}(F_1 \cdot F_2 \cdot s) \in \mathbf{E}^{k+2}(X)$. Hence $\mathcal{L}_F(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+2}(X)$. We already show that our assumption in cases of $k = 1, 2$. Therefore we have $d(\mathbf{E}^k(X)) \subset \mathbf{E}^{k+1}(X)$ for all $k$ by induction. The tangent space of the orbit of $\text{Diff}_0(X)$ is given by the Lie derivative $\mathcal{L}_v\Phi$ and $d\gamma \wedge \Phi$ for $v \in T$ and $\gamma \in T^*$. Hence it follows that the image $d(\Gamma(\mathbf{E}^0(X)))$ is the tangent space of $\text{Diff}_0(X)$. As we see, the tangent space of $\mathcal{E}_B(X)$ is global sections of $\mathbf{E}^1(X)$. Hence the infinitesimal tangent space of deformations of $\Phi$ is given by the first cohomology group $H^1(\#_B)$.
The direct sum $\oplus^l S(= \oplus^l \Lambda^* T^*)$ is invariant under the action of the exterior derivative $d$ which is the direct sum of the full de Rham complex. For simplicity we call $\oplus^l S$ the de Rham complex. Then the complex $\#_B$ is the subcomplex of the de Rham complex:

\[
0 \rightarrow E^{-1}(X) \xrightarrow{d_{-1}} E^0(X) \xrightarrow{d_0} E^1(X) \xrightarrow{d_1} E^2(X) \rightarrow \cdots ,
\]

\[
\cdots \rightarrow \oplus^l \Lambda^* T^* \xrightarrow{d} \oplus^l \Lambda^* T^* \xrightarrow{d} \oplus^l \Lambda^* T^* \xrightarrow{d} \oplus^l \Lambda^* T^* \rightarrow \cdots
\]

We denote by $H^*_{dR}(X)(= \oplus^l \bigoplus_{p=0}^{\dim X} H^p(X, \mathbb{R}))$ the cohomology group of the de Rham complex. Then we have the map $p_B^k$:

\[
p_B^k : H^k(\#_B) \rightarrow H^*_{dR}(X).
\]

Since the action of $\tilde{\text{Diff}}_0(X)$ on $\tilde{\mathcal{M}}_B(X)$ preserves a de Rham cohomology class $[\Phi]$ of $\mathcal{B}(V)$-structure $\Phi$, we have the map $P_B$:

\[
P_B : \mathcal{M}_B(X) \rightarrow H^*_{dR}(X).
\]

The map $P_B$ is called the period map.

**Definition 3-4.** An orbit $\mathcal{B}(V)$ is completely elliptic if the differential complex $\#_B$ is a elliptic complex. In particular, an orbit $\mathcal{B}(V)$ is elliptic if the complex is elliptic at degrees $k = 1, 2$.

**Definition 3-5.** Let $\mathcal{B}(V)$ be an orbit of $\text{Cpin}(V \oplus V^*)$ as in before. We say a $\mathcal{B}(V)$-structure $\Phi$ on $X$ is topological if the map $p_B^k : H^k(\#_B) \rightarrow H^*_{dR}(X)$ is injective for $k = 1, 2$. An orbit $\mathcal{B}(V)$ is topological if each $\mathcal{B}(V)$-structure $\Phi$ is topological on every compact and oriented $n$-manifold.

The complex $\#_B$ is elliptic if the corresponding symbol complex is exact. Hence the elliptic condition only depends on a choice of an orbit $\mathcal{B}(V)$. However the topological condition is depending on a choice of a $\mathcal{B}(V)$-structure $\Phi$ on $X$.

**Definition 3-6.** A $\mathcal{B}(V)$ structure $\Phi$ on $X$ is unobstructed if for each representative element $\alpha$ of the infinitesimal tangent space $H^1(\#_B)$, there exists one parameter family of deformations $\Phi_t \in \tilde{\mathcal{M}}_B(X)$ with $\Phi_0 = \Phi$ such that

\[
\frac{d}{dt} \Phi_t \big|_{t=0} = a.
\]

If $\Phi$ is unobstructed, each infinitesimal tangent generates an actual deformations and the space of deformations of $\Phi$ is locally given by an open set of $H^1(\#_B)$. From the viewpoint as in [Go], we have the following criterion for unobstructed deformations of $\mathcal{B}(V)$-structures and the Torelli-type theorem:
Theorem 3-7. Let $\mathcal{B}(V)$ be an elliptic orbit and $\Phi$ a $\mathcal{B}(V)$-structure on a compact and oriented $n$-manifold $X$. If $\Phi$ is topological, then deformations of $\Phi$ are unobstructed and the deformations of $\Phi$ is locally embedded into the de Rham cohomology group $H^*_\text{dR}(X)$. In particular, if an orbit $\mathcal{B}(V)$ is elliptic and topological, the period map $P_{\mathcal{B}}$ of the moduli space $\mathcal{M}_{\mathcal{B}}(X)$ of $\mathcal{B}(V)$ structures on $X$ is locally injective.

Our proof of theorem 3-7 is shown by the following theorems 3-8 and 3-9.

Theorem 3-8. Let $\mathcal{B}(V)$ be an elliptic orbit of $\text{Cpin}(V \oplus V^*)$ and $\Phi$ a $\mathcal{B}(V)$-structure on a compact, oriented manifold $X$. If the map $p^1_{\mathcal{B}}$ for $\Phi$ is injective, then there exists a neighborhood $U$ of $\Phi$ in the moduli space $\mathcal{M}_{\mathcal{B}}(X)$ such that the restriction of the period map $P_{\mathcal{B}}|_U : U \to H^*_\text{dR}(X)$ is injective.

(Note that Proposition is regarded as a generalization of the Moser’s stability theorem, for symplectic structures an volume forms.)

Theorem 3-9. Let $\mathcal{B}(V)$ be an elliptic orbit of $\text{Cpin}(V \oplus V^*)$ and $\Phi$ a $\mathcal{B}(V)$-structure on a compact, oriented manifold $X$. If $p^2_{\mathcal{B}}$ for $\Phi$ is injective then deformations of $\Phi$ are unobstructed.

Our proof of theorem 3-9 is shown in next section 3-3. In order to obtain theorem 3-8, we will show the following lemma

Lemma 3-10. Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of $\mathcal{B}(V)$-structures which converges to a $\mathcal{B}(V)$ structure $\Phi$, so that is,

$$\lim_{n \to \infty} \Phi_n = \Phi \in \widetilde{\mathcal{M}}_{\mathcal{B}}(X).$$

We denote by $E^k_n(X)$ the vector bundle which defined by $E^k_n(X) = \text{CL}^{k+1} \cdot \Phi_n$ and $\#_{\mathcal{B},n}$ by the deformation complex $\{E^k_n\}$ with cohomology groups $H^k(\#_{\mathcal{B},n})$. If the map $p^1_{\mathcal{B},n} : H^1(\#_{\mathcal{B},n}) \to H^*_\text{dR}(X)$ is not injective for all $n$, then the map $p^1_{\mathcal{B}}$ with respect to $\Phi$ is also not injective.

Lemma 3-10 shows that the injectivity of the map $p^1_{\mathcal{B}}$ is an open condition, so that is, if $p^1_{\mathcal{B}}$ is injective for $\Phi \in \widetilde{\mathcal{M}}_{\mathcal{B}}(X)$, then there exists an neighborhood $\widetilde{U}$ such that $p^1_{\mathcal{B}}$ is also injective for all $\Psi \in \widetilde{U}$.

proof of lemma 3-10. We take a Riemannian metric on the manifold $X$. Then we have the Laplacian $\Delta_{\mathcal{B},n} = d^*_n d_1 + d_0 d^*_0$ defined by the complex $\{E^*_n\}$ acting on sections of $E^*_n(X)$. We denote by $\mathcal{H}^1(\#_{\mathcal{B},n})$ the kernel of the Laplacian $\Delta_{\mathcal{B},n}$. Since the complex $\#_{\mathcal{B},n}$ is elliptic, the cohomology group $H^1(\#_{\mathcal{B},n})$ is isomorphic to $\mathcal{H}^1(\#_{\mathcal{B},n})$. We also have the ordinary Laplacian $\Delta$ which acts on $\oplus^k S$ and we denote by $\Pi$ the $L^2$-projection to the $\Delta$-Harmonic forms. If $p^1_{\mathcal{B},n}$ is not injective, we have $a_n \in \text{CL}^2$ such that $a_n \cdot \Phi_n$ is a non-zero element of $\mathcal{H}^1(\#_{\mathcal{B},n})$ with $\Pi(a_n \cdot \Phi_n) = 0$. For each
\( \Phi_n \) we can take a section \( g_n \) of the fibre bundle \( \text{CSpin}_0(X) \) with \( g_n \cdot \Phi_n = \Phi \) and \( g_n \to 1 \) as \( n \to \infty \). By the left multiplication \( L_{g_n} \) of \( g_n \), we identify \( E^1_n(X) \) with \( \text{E}^1(X) = \text{CL}^2(X) \cdot \Phi \),

\[
L_{g_n} : E^1_n(X) \to E^1(X), \\
a_n \cdot \Phi_n \mapsto g_n \cdot a_n \cdot \Phi_n = (\text{Ad}_{g_n} a_n) \cdot \Phi.
\]

Then the elliptic operator \( \widetilde{\Delta}_{B,n} \) on \( E^1(X) \) is induced by

\[
\widetilde{\Delta}_{B,n} = L_{g_n} \Delta_{B,n} L_{g_n}^{-1}.
\]

We put \( b_n = \text{Ad}_{g_n} a_n \). Then we have

\[
\widetilde{\Delta}_{B,n} b_n \cdot \Phi = L_{g_n} \Delta_{B,n} (a_n \cdot \Phi_n) = 0.
\]

We take \( a_n \) such that the Sobolev norm of \( b_n \cdot \Phi \) is normalized,

\[
\| b_n \cdot \Phi \|_{L^2_2} = 1.
\]

Then from Rellich lemma there exists a subsequence \( \{ b_m \cdot \Phi \}_m \) which converges to \( b \cdot \Phi \in E^1(X) \) with respect to the norm \( L^2_2 \). Since \( \widetilde{\Delta}_{B,m} b_m \cdot \Phi = 0 \), we have an estimate,

\[
\| b_m \cdot \Phi \|_{L^2_2} \leq C_1 \| b_m \cdot \Phi \|_{L^1} \leq C_2 \| b_m \cdot \Phi \|_{L^2_2},
\]

where \( C_i \neq 0 \) does not depend on \( m \) for \( i = 1, 2 \). Hence we have the bound,

\[
0 \neq C_3 \leq \| b \cdot \Phi \|_{L^2_2}.
\]

The family of elliptic operator \( \{ \widetilde{\Delta}_{B,m} \}_m \) also converges to the operator \( \Delta_B \) as \( m \to \infty \).

Hence we have

\[
\Delta_B (b \cdot \Phi) = 0.
\]

Since \( g_m \to 1 (m \to \infty) \), the sequence \( \{ a_m \cdot \Phi_m \} = \{ g_m^{-1} \cdot b_m \cdot \Phi \}_m \) converges to \( b \cdot \Phi \) \( (n \to \infty) \). Hence it follows from \( \Pi(a_m \cdot \Phi_m) = 0 \),

\[
\Pi(b \cdot \Phi) = 0.
\]

Hence \( b \cdot \Phi \neq 0 \) is an element of \( \text{ker} p^1_B \) and we have the result. \( \square \)

**proof of theorem 3-8.** Let \( \tilde{U} \) be a neighborhood of \( \Phi \) such that \( p^1_B \) is injective for every \( \Psi \in \tilde{U} \). Let \( \{ \Phi_t \}, 0 \leq t \leq 1 \) be a smooth family of \( \mathcal{B}(V) \)-structures in the neighborhood \( \tilde{U} \). We assume that the \( d \)-closed form \( \Phi_t \) belongs to the same de Rham cohomology class as \( \Phi_0 \) for all \( t \), so that is, there exists \( A_t \) such that

\[
(3-2-1) \quad \Phi_t - \Phi_0 = dA_t.
\]

Since the group \( \widehat{\text{Diff}}_0(X) \) is generated by the action of \( \text{Diff}_0(X) \) and the action of \( d \)-exact \( b \)-fields, theorem is reduced to the followings:

\[
(3-2-1) \quad \Phi_t - \Phi_0 = dA_t.
\]
Proposition 3-11. If the map $p_B^1$ is injective for all $\Phi_t$, then there exist a smooth family of diffeomorphisms $\{f_t\}$ and a smooth family of $d$-exact 2-forms $\{d\gamma_t\}$ such that

$$(3-2-2) \quad e^{d\gamma_t} \wedge f_t^*\Phi_t = \Phi_0, \quad \text{for all } t \in [0,1].$$

proof of proposition 3-11. By differentiating the equation (3-2-2), we have

$$(3-2-3) \quad \frac{d}{dt} (e^{d\gamma_t} \wedge f_t^*\Phi_t) = 0, \quad \forall t \in [0,1],$$

which is equivalent to

$$(3-2-4) \quad e^{d\gamma_t} \wedge d\dot{\gamma}_t \wedge f_t^*\Phi_t + e^{d\gamma_t} \wedge \dot{f}_t^*\Phi_t + e^{d\gamma_t} \wedge \dot{f}^*_t\Phi_t = 0.$$

By the left action of $(f_t^{-1})^*(e^{-d\gamma_t})$, we have

$$(3-2-5) \quad (f_t^{-1})^*(d\dot{\gamma}_t \wedge f_t^*\Phi_t) + (f_t^{-1})^*\dot{f}_t^*\Phi_t + \dot{\Phi}_t = 0.$$

We set $(f_t^{-1})^*\dot{\gamma}_t = \tilde{\gamma}_t$. Since $(f_t^{-1})^*\dot{f}_t^*\Phi_t$ is given as the Lie derivative $\mathcal{L}_{v_t}\Phi_t$ for a vector field $v_t$, it follows from (3-2-1) that

$$(3-2-5) \quad (d\tilde{\gamma}_t) \wedge \Phi_t + \mathcal{L}_{v_t}\Phi_t + d\dot{A}_t = 0.$$

Since $\Phi_t$ is $d$-closed, we have

$$(3-2-6) \quad \mathcal{L}_{v_t}\Phi = d_{v_t}\Phi.$$

We substitute (3-2-6) in (3-2-5) and we have

$$(3-2-7) \quad ((d\tilde{\gamma}_t) \wedge \Phi_t + d_{v_t}\Phi_t) = d (\tilde{\gamma}_t + v_t) \cdot \Phi_t = -d\dot{A}_t,$$

where $(v_t + \tilde{\gamma}) \in T \oplus T^*$ acts on $\Phi$ by the Clifford multiplication. We denote by $E^k_t(X)$ the vector bundle $\text{CL}^{k+1} \cdot \Phi_t$ and $\#_{B,t}$ the complex $\{E^*_t(X)\}$. We denote by $E^0_t(X)$ the vector bundle $\text{CL}^{k+1} \cdot \Phi_t$ and by $\#_{B,t}$ the complex $(E^*_t(X), d)$. Then $(\tilde{\gamma}_t + v_t) \cdot \Phi$ is a section of $E^0_t(X)$ and $-\dot{\Phi}_t = -d\dot{A}_t$ is a section of $E^1_t(X)$. Hence $-d\dot{A}_t$ yields the class $-[d\dot{A}_t] \in H^1(\#_{B,t})$ of the deformation complex $\#_{B,t}$:

$$E^0_t \xrightarrow{d_0} E^1_t \xrightarrow{d_1} \ldots.$$

Then we see that the class $[-d\dot{A}_t] \in H^1(\#_{B,t})$ vanishes since the class $-[d\dot{A}_t]$ is represented by the $d$-exact form and the map $p_B^1$ is injective. If we take a metric on
the manifold $X$, we have the adjoint operator $d^*_t$ and the Green operator $G_t$ of the complex $\mathcal{E}_B^0,\mathcal{E}$. We define a section $B_t$ of $E^0_t(X)$ by

$$(3-2-8) \quad B_t = -d^*_t G_t d\dot{A}_t.$$ 

Then from the Hodge theory of the elliptic complex, we have

$$(3-2-9) \quad dB_t = -d\dot{A}_t.$$ 

Since $B_t$ is written as $E_t \cdot \Phi$ for $E_t \in T \oplus T^*$, we set $v_t + \tilde{\gamma}_t = E_t$ such that a smooth family $\{v_t + \tilde{\gamma}_t\}$ satisfies the equation (3-2-7). By solving the equation $(f_t^{-1})^* \tilde{f}_t^* \Phi_t = \mathcal{L}_{v_t} \Phi$, we have the smooth family $\{f_t\}$ with $f_0 = \text{id}$. Hence we have $\{f_t\}$ and $\{d\gamma_t\}$ which satisfy the equation (3-2-2) \quad \Box$

§3-3 Construction of deformations. This subsection is devoted to proof of theorem 3-9.

proof of theorem 3-9. Let $X$ be an $n$-dimensional, compact and oriented manifold with a $\mathcal{B}(V)$-structure $\Phi$. We take a Riemannian metric on $X$. (Note that this metric is independent to the structure $\Phi$.) The conformal pin bundle $\text{Cpin}(X) = \text{Cpin}(T \oplus T^*)$ acts on the fibre bundle $\mathcal{B}(X)$ transitively. Hence every global section $E^0_B(X)$ is written as $\Phi$ for a section $g \in \text{Cpin}(T \oplus T^*)$. The identity component $\text{CSpin}_0(T \oplus T^*)$ of $\text{Cpin}(T \oplus T^*)$ is given by

$$(3-3-1) \quad \text{CSpin}_0(T \oplus T^*) = \{ e^a | a \in \text{CL}^2(T \oplus T^*) \}.$$ 

Hence every deformation of $\Phi$ in $\mathcal{E}_B(X)$ is given by $e^a \cdot \Phi$ for a section $a$ of $\text{CL}^2(T \oplus T^*)$. In order to obtain a deformation of $\Phi$ in $\mathfrak{M}_B(X)$, we introduce a formal power series in $t$ :

$$(3-3-2) \quad a(t) = a_1 t + \frac{1}{2!} a_2 t^2 + \frac{1}{3!} a_3 t^3 + \cdots ,$$ 

each $a_i$ is a section of $\text{CL}^2(T \oplus T^*)$. We define a formal power series $g(t)$ by

$$(3-3-3) \quad g(t) = \exp(a(t)) \in \text{CSpin}_0(T \oplus T^*)[[t]].$$ 

The group $\text{CSpin}_0(T \oplus T^*)$ acts on differential forms and we have

$$(3-3-4) \quad e^{a(t)} \cdot \Phi = \Phi + a(t) \cdot \Phi + \frac{1}{2!} a(t) \cdot a(t) \cdot \Phi + \cdots ,$$ 

$$= \Phi + (a_1 \cdot \Phi) t + \frac{1}{2!} ((a_2 + a_1 \cdot a_1) \cdot \Phi) t^2 + \cdots .$$
The equation that we want to solve is,

\[
(eq_*) \\ de^a(t) \cdot \Phi = 0.
\]

At first we take \(a_1\) such that \(da_1 \cdot \Phi = 0\) as an initial condition. It follows from lemma 2-7 that we have

\[
(3-3-5) \\ e^{-a(t)} \cdot d \cdot e^{a(t)} = \left( (\exp(\text{Ad}_{a(t)})d) \right);
\]

where \((\exp(\text{Ad}_{a(t)}))\) is the operator acting on differential forms which defined by the power series in \(t\):

\[
\left( (\exp(\text{Ad}_{a(t)})d) \right) = d + \frac{1}{k!} \sum_{k=1}^{\infty} \text{Ad}_{a(t)}^k d, \\
= d + [d, a(t)] + \frac{1}{2!} [[d, a(t)], a(t)] + \cdots, \\
= d + [d, a_1]t + \frac{1}{2!} ([d, a_2] + [[d, a_1], a_1]) t^2 + \cdots,
\]

where \(\text{Ad}_{a(t)}^k d = [\text{Ad}_{a(t)}^{k-1} d, a(t)]\). Hence the \((eq_*)\) is equivalent to the equation

\[
(\tilde{eq}_*) \\ \left( (\exp(\text{Ad}_{a(t)})d) \right) \Phi = 0,
\]

Then it follows from proposition 2-5 that \(( (\exp(\text{Ad}_{a(t)})d) \) is a Clifford-Lie operator of order 3 and we have

\[
(3-3-6) \\ \left( (\exp(\text{Ad}_{a(t)})d) \right) \Phi \in \mathcal{E}^2(X).
\]

From (3-3-5), we have

\[
(3-3-7) \\ de^a(t) \cdot \Phi = e^a(t) \cdot \left( (\exp(\text{Ad}_{a(t)})d) \right) \Phi.
\]

We denote by \((P(t))[i]\) the \(i\)th homogeneous part of a power series \(P(t)\) in \(t\). Then from (3-3-7), we have

\[
(3-3-8) \\ (de^a(t) \cdot \Phi)[k] = \sum_{k=i+j, i,j \geq 0} \left( (\exp(\text{Ad}_{a(t)})d) \right)_i \Phi \cdot \left( (\exp(\text{Ad}_{a(t)})d) \right)_j \Phi
\]

Since \(da_1 \cdot \Phi = 0\), we have

\[
(3-3-9) \\ \left( (\exp(\text{Ad}_{a(t)})d) \right)_0 \cdot \Phi = \left( (\exp(\text{Ad}_{a(t)})d) \right)_1 \cdot \Phi = 0.
\]
Thus it suffices to determine $a_k$ satisfying $(\text{eq}_k)$ by induction $k$. We assume that $a_1, \ldots, a_{k-1}$ have been determined so that

$$ (3-3-10) \quad \left( \exp(\text{Ad}_{a(t)})d \right)_{[l]} \Phi = 0, \quad (l = 0, 1, \ldots, k-1). $$

Then it follows from (3-3-8) that

$$ (3-3-11) \quad (\text{de}a(t) \cdot \Phi)_{[k]} = \left( \exp(\text{Ad}_{a(t)})d \right)_{[k]} \Phi $$

Then form (3-3-6) we see that

$$ (3-3-12) \quad (\text{de}a(t) \cdot \Phi)_{[k]} \in E^2(X). $$

The $k$th part $(\text{de}a(t) \cdot \Phi)_{[k]}$ is written as

$$ (3-3-13) \quad (\text{de}a(t) \cdot \Phi)_{[k]} = \frac{1}{k!} da_k \cdot \Phi + \text{Ob}_k, $$

where $\text{Ob}_k (= \text{Ob}_k(a_1, \ldots, a_{k-1}))$ is the non-linear term depending only on $a_1, \ldots, a_{k-1}$. Since $da_k \cdot \Phi \in dE^1(X) \subset E^2(X)$, it follows from (3-3-12) that

$$ (3-3-14) \quad \text{Ob}_k \in E^2(X). $$

Since $\text{Ob}_k$ is $d$-exact, we have the cohomology class $[\text{Ob}_k] \in H^2(\#B)$. Then we have

**Lemma 3-12.** There exists a section $a_k$ satisfying $(\text{de}a(t) \cdot \Phi)_{[k]} = 0$ if and only if the class $[\text{Ob}_k] \in H^2(\#B)$ vanishes.

**proof.** The equation $(\text{de}a(t) \cdot \Phi)_{[k]} = 0$ is written as

$$ (3-3-15) \quad \frac{1}{k!} a_k \cdot \Phi = -\text{Ob}_k, $$

where $\text{Ob}_k$ only depends on $a_1, \ldots, a_{k-1}$. The L.H.S of (3-3-15) is an element of the image $dE^1(X)$ in the complex $\#B$:

$$ \cdots \xrightarrow{d_{-1}} E^0 \xrightarrow{d_0} E^1 \xrightarrow{d_1} E^2 \xrightarrow{d_2} \cdots. $$

The R.H.S. of (3-3-15) is an $d_2$-closed element of $E^2$ which yields the class $[\text{Ob}_k] \in H^2(\#B)$. If we have $a_k$ satisfying the equation (3-3-15), then the class $[\text{Ob}_k]$ vanishes. The complex $\#B$ is an elliptic complex and we have the Green operator $G_{\#B}$ of the complex $\#B$. If the class $[\text{Ob}_k]$ vanishes, we can obtain $a_k$ by using the Green operator:

$$ \frac{1}{k!} a_k \cdot \Phi = -d^* G_{\#B}(\text{Ob}_k) \in E^1. $$
Then $a_k$ satisfies the equation (3-3-15).

We call $[\text{Ob}_k]$ the $k$-th obstruction class. (Note that $[\text{Ob}_k]$ can be defined if the lower obstruction classes vanish.) Since $\text{Ob}_k$ is $d$-exact, we have that the class $[\text{Ob}_k] \in H^2(\#B)$ is in the kernel of the map $p^2_B: H^2(\#B) \to H^*_d(X)$ is injective then $[\text{Ob}_k]$ vanishes. Hence from 3-3-11, we have $a_k$ satisfying $(\exp(\text{Ad}_a(t))d)_{[k]} = 0$. By induction, we have a formal power series $a(t)$ which is a solution of the equation $\tilde{\epsilon}_a$. The rest is to show the convergence of the power series $a(t)$. The convergence can be shown essentially by the same method as in [Go]. We also have the smoothness of solutions by the standard elliptic regularity method. Hence the result follows. □

§4. Generalized Calabi-Yau (metrical) structures

§4-1 Generalized $\text{SL}_n(\mathbb{C})$ structures. Let $V$ be the real vector space of dim $2n$ and $\mathcal{J}(V)$ the set of complex structures on $V$. We denote by $\wedge_{J}^{n,0}V^*_C$ the space of complex forms of type $(n,0)$ with respect to $J \in \mathcal{J}(V)$. Let $\mathfrak{P}(V)$ be the set of pairs consisting of complex structures $J$ and a non-zero complex form of type $(n,0)$:

$$\mathfrak{P}(V) := \{ (J, \Omega) \mid J \in \mathcal{J}(V), \ 0 \neq \Omega \in \wedge_{J}^{n,0}V^*_C \}.$$ 

Then we have the projection to the second component

$$\pi_2: \mathfrak{P}(V) \to \wedge^nV^*_C.$$ 

**Definition 4-1-1.** A complex $n$-form $\Omega_V$ on $V$ is an $\text{SL}_n(\mathbb{C})$ structure if $\Omega_V$ is in the image of $\pi_2(\mathfrak{P}(V))$. The set of $\text{SL}_n(\mathbb{C})$ structures on $V$ is denoted by $\mathcal{A}_{\text{SL}}(V)$.

Hence each $\text{SL}_n(\mathbb{C})$ structure $\Omega_V$ is a complex form of type $(n,0)$ with respect to a complex structure $J \in \mathcal{J}(V)$. Conversely for each $\text{SL}_n(\mathbb{C})$ structure $\Omega_V$ we define a complex subspace $\ker \Omega_V$ by

$$\ker \Omega_V := \{ v \in V_C \mid i_v \Omega_V = 0 \}.$$ 

Then the complexified vector space $V_C$ is decomposed into $\ker \Omega_V$ and the conjugate space $\overline{\ker \Omega_V}$:

$$V_C = \ker \Omega_V \oplus \overline{\ker \Omega_V}.$$ 

(4-1-1)

Hence we define a complex structure $J$ on $V$ by using decomposition (4-1-1) such that $\Omega_V$ is the complex form of type $(n,0)$ with respect to $J$. Then we have the map from the set of $\text{SL}_n(\mathbb{C})$ structures to the set of complex structures :

$$\mathcal{A}_{\text{SL}}(V) \to \mathcal{J}(V).$$
By taking a suitable basis \( \{ \theta^1, \cdots, \theta^n \} \) of \( \ker \Omega_V \), we can write \( \Omega_V = \theta^1 \wedge \cdots \wedge \theta^n \). Then it follows that the real linear group \( \text{GL}(V) \) acts on \( \mathcal{A}_{\text{SL}}(V) \) transitively with isotropy group \( \text{SL}_n(\mathbb{C}) \) and \( \mathcal{A}_{\text{SL}}(V) \) is the orbit which is described as the homogeneous space:

\[
\mathcal{A}_{\text{SL}}(V) = \text{GL}(V)/\text{SL}_n(\mathbb{C}).
\]

The real conformal pin group \( \text{Cpin}(V \oplus V^*) \) acts on \( \wedge^* V^* \otimes \mathbb{C} \). When we consider complex forms as pairs of real forms, we can apply the construction in section 3.

**Definition 4-1-2.** Let \( \mathcal{B}_{\text{SL}}(V) \) be the orbit of \( \text{Cpin} \) including \( \text{SL}_n(\mathbb{C}) \) structures \( \mathcal{A}_{\text{SL}}(V) \). An element \( \phi_V \) of \( \mathcal{B}_{\text{SL}}(V) \) is a generalized \( \text{SL}_n(\mathbb{C}) \) structure on \( V \) and we call \( \mathcal{B}_{\text{SL}}(V) \) the orbit of generalized \( \text{SL}_n(\mathbb{C}) \) structures.

Let \( X \) be a compact and oriented real manifold of dim \( 2n \). Then by applying the construction as in section 3, we define \( \mathcal{B}_{\text{SL}}(V) \)-structures on \( X \) which are generalized geometric structures corresponding to the orbit \( \mathcal{B}_{\text{SL}}(V) \). Assume that there exists a \( \mathcal{B}_{\text{SL}}(V) \)-structure \( \phi \) on \( X \). Then we have the sequence of vector bundles \( \{ \mathcal{E}^k_{\text{SL}} \} \) over \( X \) and the complex \#\( \mathcal{B}_{\text{SL}} \):

\[
(\#_{\mathcal{B}_{\text{SL}}}) \quad 0 \to \mathcal{E}^{-1}_{\text{SL}} \to \mathcal{E}^0_{\text{SL}} \to \mathcal{E}^1_{\text{SL}} \to \mathcal{E}^2_{\text{SL}} \to \cdots.
\]

Let \( L_{\phi} \) be the vector bundle over \( X \) which is defined by

\[
L_{\phi} = \{ E \in T \oplus T^* \mid E \cdot \phi = 0 \}.
\]

Then we have a decomposition:

\[
(4-1-2) \quad (T \oplus T^*) \otimes \mathbb{C} = L_{\phi} \oplus \overline{L_{\phi}},
\]

where \( \overline{L_{\phi}} \) is the conjugate bundle of \( L_{\phi} \). We denote by \( \wedge^i \overline{L_{\phi}} \) the \( i \)-th wedge product of \( \overline{L_{\phi}} \) which acts on \( \phi \) by the Clifford multiplication. Then we define a vector bundle \( U_{\phi}^i \) by

\[
U_{\phi}^{-n+i} := \wedge^i \overline{L_{\phi}} \cdot \phi,
\]

for \( i = 0, \cdots, 2n \). The bundle \( U_{\phi}^{-n} \) is the line bundle generated by \( \phi \). The vector bundle \( \mathcal{E}^k_{\text{SL}} \) is described in terms of \( U_{\phi}^i \).

**Lemma 4-1-3.** We have the following identification as real vector bundle:

\[
\mathcal{E}^0_{\text{SL}} \cong U_{\phi}^{-n+1},
\]
\[
\mathcal{E}^1_{\text{SL}} \cong U_{\phi}^{-n} \oplus U_{\phi}^{-n+2},
\]
\[
\mathcal{E}^2_{\text{SL}} \cong U_{\phi}^{-n+1} \oplus U_{\phi}^{-n+3}.
\]
In general we have
\[
E^{2k-1}_{SL} \cong \bigoplus_{i=0}^{k} U^{-n+2i}, \\
E^{2k}_{SL} \cong \bigoplus_{i=0}^{k} U^{-n+2i+1}.
\]

proof. We consider the complex form \( \phi = \phi^R + \sqrt{-1} \phi^3 \) as the pair of real forms \((\phi^R, \phi^3)\). Then applying the construction in section 3, we have the vector bundles \( E^k_{SL} \) which generated by
\[
E^k_{SL} = \text{span}\{(a \cdot \phi^R, a \cdot \phi^3) \mid a \in \text{CL}^k\}.
\]
Then we have the complex form \( a \cdot \phi^R + \sqrt{-1} a \cdot \phi^3 = a \cdot \phi \). From the decomposition (4-1-2), we have the identification:
\[
\text{CL}^{2k} \otimes \mathbb{C} \cong \text{CL}^{2k} (L_\phi \oplus \overline{L_\phi}) \cong \bigoplus_{i=0}^{k} \wedge^{2l} (L_\phi \oplus \overline{L_\phi}).
\]
Since \( L_\phi \cdot \phi = \{0\} \), We have an identification:
\[
E^{2k-1}_{SL} = \text{CL}^{2k} \cdot \phi \cong \bigoplus_{i=0}^{k} \wedge^{2l} \overline{L_\phi} \cdot \phi \\
= \bigoplus_{i=0}^{k} U^{-n+2l}.
\]
Similarly we have \( E^{2k}_{SL} \cong \bigoplus_{i=0}^{k} U^{-n+2i+1} \). \( \square \)

Proposition 4-1-4. The complex \( \#_{B_{SL}} \) is elliptic, so that is, the orbit \( B_{SL} \) is an elliptic orbit.

proof. Since there is the inclusion \( \text{CL}^{k-2} \subset \text{CL}^k \), we have the inclusion \( E^{k-2}_{SL} \subset E^k_{SL} \) with the quotient
\[
E^k_{SL}/E^{k-2}_{SL} \cong U^{-n+k+1},
\]
for \( k \geq 0 \). Replacing \( E^{-1} \) by \( E^{-1} \otimes \mathbb{C} \), we have a complex \( \widetilde{\#}_{B_{SL}} \) Hence there is a map of the complex \( \widetilde{\#}_{B_{SL}} \) by shifting its degree from \(* \) to \(* + 2 \) :
\[
\widetilde{\#}_{B_{SL}} \xrightarrow{[2]} \widetilde{\#}_{B_{SL}}.
\]
Hence we have the following commutative diagram :
\[
\begin{array}{cccccccc}
0 & \rightarrow & 0 & \rightarrow & E^{-1}_{SL} \otimes \mathbb{C} & \rightarrow & E^0_{SL} & \rightarrow & E^1_{SL} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & E^{-1}_{SL} \otimes \mathbb{C} & \rightarrow & E^0_{SL} & \rightarrow & E^1_{SL} & \rightarrow & E^2_{SL} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U^{-n} & \xrightarrow{\partial} & U^{-n+1} & \xrightarrow{\partial} & U^{-n+2} & \xrightarrow{\partial} & U^{-n+3} & \xrightarrow{\partial} & U^{-n+4} & \rightarrow & \cdots \\
\end{array}
\]
It follows from $U_{\phi}^{-n+i} = \wedge^i L_\phi \cdot \phi$ that the quotient complex $(U_{\phi}^{-n+i}, \overline{\partial})$ is an elliptic complex. Hence the from the commutative diagram we see that the complex $\#_{B_{SL}}$ is elliptic by induction on degree $k$. □

The complex $(U_{\phi}, \overline{\partial})$ is the deformation complex of generalized complex structures which is introduced by [Gu1]. Then the exterior derivative $d$ acting on $U_{\phi}^p$ is decomposed into two projections $\partial$ and $\overline{\partial}$, so that is,

$$d = \partial + \overline{\partial},$$

$$U_{\phi}^{p-1} \xleftarrow{\partial} U_{\phi}^p \xrightarrow{\overline{\partial}} U_{\phi}^{p+1}.$$ 

Let $\Phi$ be a $B_{SL}(V)$-structure on $X$. Then there is the generalized complex structure $J_{\phi}$ corresponding to $\phi$. We define an operator $d^J$ by

$$d^J := \sqrt{-1}(\overline{\partial} - \partial).$$

The $dd^J$-property is introduced by [Hi], [Gu2] and [Cav]:

**Definition 4-1-5.** A generalized complex manifold $(X, J)$ satisfies the $dd^J$-property iff the following are equivalent:

1. $a \in \wedge^* T^*$ is $d$-closed and $d^J$-exact,
2. $a \in \wedge^* T^*$ is $d$-exact and $d^J$-closed,
3. $a = dd^J b$ for some $b \in \wedge^* T^*$.

**Theorem 4-1-6.** If the $dd^J$-property holds for the $J_{\phi}$ corresponding a $B_{SL}$-structure $\phi$, then $\phi$ is a topological structure, so that is, we have unobstructed deformations of $\phi$ on which the local Torelli type theorem holds.

**proof.** Since $U_{\phi}^p$ is the eigenspace of the action of $J_{\phi}$ with eigenvalue $\sqrt{-1}p$. Hence we have the decomposition $\wedge^* T^* = \bigoplus_{p=-n}^n U_{\phi}^p$. If an exact form $da^{(m)}$ is an element of $U_{\phi}^{m-1}$ for $a^{(m)} \in U_{\phi}^m$, we have $\partial da^{(m)} = \overline{\partial} da^{(m)} = 0$. Hence applying the $dd^J$-property we have

$$da^{(m)} = dd^J b = 2\sqrt{-1}\overline{\partial} \partial b = 2\sqrt{-1}d\overline{\partial} b,$$

for $b \in U_{\phi}^{m-1}$. Then we have $da^{(m)} = d\gamma$ for $\gamma = 2\sqrt{-1} \overline{\partial} b \in U_{\phi}^{m-2}$. From our decomposition, a form $a$ is written as

$$a = \sum_{p=-n}^m a^{(p)}.$$
where \( a^{(p)} \in U^p_\phi \) for some \( m \). If \( da \) is an element of \( \sum_{p=-n}^{k} U^p_\phi \), then applying the \( dd^J \)-property successively, we have \( da = db \) for \( b \in \sum_{p=-n}^{k-1} U^p_\phi \). Similarly if \( da \in \wedge^{even} T^* \) ( resp. \( da \in \wedge^{odd} T^* \) ) then applying the property we see that \( da = db \) for \( b \in \wedge^{even} T^* \) (resp. \( b \in \wedge^{odd} T^* \) ). Hence it follows from lemma 4-1-3 that if \( da \in E^k \rightarrow \) then the map \( p_k^0 : H^k(\#_{B_{SL}}) \rightarrow H^*_{dR}(X) \) is injective for \( k \geq 1 \). □

Gualtieri also shows that the \( dd^c \)-property holds for generalized Kähler structures [Gu2]. By applying his theorem, we have

**Theorem 4-1-7.** Let \( \phi \) be a \( B_{SL}(V) \)-structure on \( X \) with the corresponding generalized complex structure \( J_\phi \). if there exists another generalized complex structure \( I \) such that the pair \((I, J_\phi)\) defines a generalized Kähler structure on \( X \), then \( B_{SL}(V) \)-structure \( \phi \) is a topological structure.

As in the proof of proposition 4-1-4, we have the short exact sequence :

\[
0 \rightarrow \#_{B_{SL}} \rightarrow H^1(\#_{B_{SL}}) \rightarrow (U^*_\phi, \partial) \rightarrow 0.
\]

It follows from \( E^0_{SL} \cong U^{-n+1}_\phi \) that we have the long exact sequence :

\[
0 \rightarrow H^{-1}(\#_{B_{SL}}) \rightarrow H^1(\#_{B_{SL}}) \rightarrow H^2_\partial(U^*_\phi) \rightarrow H^0(\#_{B_{SL}}) \rightarrow H^2(\#_{B_{SL}}),
\]

where \( H^2_\partial(U^*_\phi) \) denotes the cohomology group,

\[
H^2_\partial(U^*_\phi) = \left( \ker \partial : U^{-n+2}_\phi \rightarrow U^{-n+3}_\phi \right) / \partial(U^{-n+1}_\phi).
\]

which is the infinitesimal tangent space of deformations of generalized complex structures ( see Chapter 5 in [Gu1] ). Since the complex \( \#_{B_{SL}} \) is a subcomplex of the complexified de Rham complex we have the map \( \bar{p}_B^0 : H^k(\#_{B_{SL}}) \rightarrow H^*_{dR}(X, \mathbb{C}) \) as in section 3. Then we have a diagram :

\[
\begin{array}{cccc}
H^0(\#_{B_{SL}}) & \rightarrow & H^2(\#_{B_{SL}}) & \\
\bar{p}_B^0 \downarrow & & \downarrow \bar{p}_B^2 & \\
H^*_d(X) & \cong & H^*_{dR}(X) & \\
\end{array}
\]

Hence it implies that if \( \bar{p}_B^0 \) is injective then the map \( H^0(\#_{B_{SL}}) \rightarrow H^2(\#_{B_{SL}}) \) is injective. As in proof of theorem 4-1-6, the \( dd^J \)-property implies the injectivity of the map \( \bar{p}_B^0 \). Hence we have
Proposition 4-1-8. Let $\phi$ be a generalized $\text{SL}_n(\mathbb{C})$ structure on $X$. If the generalized complex structure $J_\phi$ satisfies the $dd^c$-property then we have the exact sequence:

$$0 \to H^{-1}(\#_{B_{\text{SL}}}) \to H^1(\#_{B_{\text{SL}}}) \to H^2_{\partial}(U_\phi^*) \to 0.$$ 

Hence the infinitesimal tangent $H^2_{\partial}(U_\phi^*)$ gives rise to small deformations of generalized complex structure $J_\phi$ which correspond to deformations of generalized $\text{SL}_n(\mathbb{C})$ structures.

§4-2. Generalized Calabi-Yau (metrical) structures. Let $\Omega_V$ be an $\text{SL}_n(\mathbb{C})$-structure and $\omega_V$ a real 2-form on the real vector space of $2n$ dim. As in section 4-1, the $\text{SL}_n(\mathbb{C})$-structure $\Omega_V$ gives rise to the complex structure $J$ on $V$ and we define a bilinear form $g$ by

$$g(u, v) = \omega(Ju, v), \quad (u, v \in V)$$

Definition 4-2-1. A pair $(\Omega_V, \omega_V)$ is a Calabi-Yau structure on $V$ if the following hold:

1. $\Omega_V \wedge \omega_V = 0$,

2. $\Omega \wedge \overline{\Omega} = c_n \omega^n$,

3. The corresponding bi-linear form $g$ is positive-definite.

The condition (1) implies that $\omega_V$ is a form of type $(1, 1)$ with respect to $J$ and then it follows from (3) that $\omega_V$ is a Hermitian form. The equation (2) is called the Monge-Ampère condition. Let $\mathcal{A}_{\text{CY}}(V)$ be the set of Calabi-Yau structures on $V$ which consist of complex $n$-forms and real 2-forms. Then the real linear group $\text{GL}(V)$ acts on $\mathcal{A}_{\text{CY}}(V)$ transitively with the isotropy group $\text{SU}(n)$. Hence $\mathcal{A}_{\text{CY}}(V)$ is the orbit of $\text{GL}(V)$ which is described as a homogeneous space:

$$\mathcal{A}_{\text{CY}}(V) = \text{GL}(V)/\text{SU}(n).$$

Let $(\Omega_V, \omega_V)$ be a Calabi-Yau structure on $V$. Then we consider a pair $(\Omega_V, e^{\sqrt{-1} \omega_V})$ consisting two generalized $\text{SL}_n(\mathbb{C})$ structures $\Omega_V$ and $e^{\sqrt{-1} \omega_V}$.

Definition 4-2-2. The orbit $\mathcal{B}_{\text{CY}}(V)$ of $\text{Cpin}(V \oplus V^*)$ through the pair $(\Omega_V, e^{\sqrt{-1} \omega_V})$ is called the generalized Calabi-Yau orbit. An element $(\phi_{V, 0}, \psi_{V, 0})$ of the orbit $\mathcal{B}_{\text{CY}}(V)$ is a generalized Calabi-Yau structure on $V$. Note that the orbit $\mathcal{B}_{\text{CY}}(V)$ is embedded into pairs of complex forms $\wedge^* V^* \oplus \wedge^* V^*$. Let $X$ be a compact real manifolds of dim $2n$. Then as in section 3, we define generalize Calabi-Yau (metrical) structures on $X$ as $\mathcal{B}_{\text{CY}}(V)$-structures on $X$. 
Let \((\phi_0, \phi_1)\) be a generalized Calabi-Yau structure on \(X\). Since it consists on generalized \(SL_n(\mathbb{C})\) structures, we obtain the pair \((\mathcal{J}_0, \mathcal{J}_1)\) of the corresponding generalized complex structures on \(X\). Then we see that the pair \((\mathcal{J}_0, \mathcal{J}_1)\) is a generalized Kähler structure. By applying Gualtieri’s theorem, we obtain the following theorem of deformations of generalized Calabi-Yau structures (which are deformations of pairs consisting two generalized \(SL_n(\mathbb{C})\) structures with the conditions):

**Theorem 4-2-3.** The generalized Calabi-Yau orbit is an elliptic and topological orbit.

**proof.** Let \((\phi_0, \phi_1)\) be a generalized Calabi-Yau (metrical) structure with the generalized Kähler structure \((\mathcal{J}_0, \mathcal{J}_1)\) on \(X\). We denote by \#_{\text{BCY}} = \{E_{\text{CY}}^*, d\} the deformation complex of generalized Calabi-Yau structure \((\phi_0, \phi_1)\). Then it suffices to show that each map

\[ p_{B_{\text{CY}}}^k : H^k(\#_{\text{BCY}}) \to \bigoplus^2 H^*_{dR}(X, \mathbb{C}) \]

is injective for \(k = 1, 2\). We have the eigenspace decomposition of \(\wedge^* T^* = \bigoplus_{p=0}^n U_{\phi_i}\) for each \(i = 0, 1\). Since \([\mathcal{J}_0, \mathcal{J}_1] = 0\), we have a further decomposition:

\[ \wedge^* T^* = \bigoplus_{|p+q|\leq n, p+q\equiv n \pmod{2}} U^{p,q}, \]

where \(U^{p,q} = U^p_{\phi_0} \cap U^q_{\phi_1}\).

Each \(E_{\text{CY}}^k\) consists of pairs of complex forms. Then the projection \(\pi_1\) to the first component induces a map from the complex \#_{\text{BCY}} to \#_{\text{BSL}}. We denote by \((\ker^*, d)\) the complex defined by the kernel of \(\pi_1\). Then we have a short exact sequence:

\[(4-2-1)\quad 0 \to (\ker^*, d) \to \#_{\text{BCY}} \to \#_{\text{BSL}} \to 0,\]

so that is,

\[
\begin{align*}
\ker^0 & \to \ker^1 \to \ker^2 \to \cdots \\
E_{\text{CY}}^{-1} & \to E_{\text{CY}}^0 \to E_{\text{CY}}^1 \to E_{\text{CY}}^2 \to \cdots \\
E_{\text{SL}}^{-1} & \to E_{\text{SL}}^0 \to E_{\text{SL}}^1 \to E_{\text{SL}}^2 \to \cdots.
\end{align*}
\]

If \(E \cdot \phi_1 = 0\) for real \(E \in T \oplus T^*\) then we see that \(E = 0\). It implies that \(\ker^0 \cong\{0\}\). Similarly \(\ker^1\) and \(\ker^2\) are respectively given by

\[
\begin{align*}
\ker^1 & \cong U^{0, -n+2}, \\
\ker^2 & \cong U^{1, -n+1} \oplus U^{-1, -n+1} \oplus U^{1, -n+3} \oplus U^{-1, -n+3}.
\end{align*}
\]
The complex \((\ker^*, d)\) is a subcomplex and we have the map \(p_k^\ker : H^k(\ker^*) \to H^*_dR(X)\). By applying the Hodge decomposition of generalized Kähler manifold in [Gu2], we see that \(p_k^\ker\) is injective for each \(k\). The short exact sequence (4-2-1) is a subsequence of the following short exact sequence define by the de Rham complex (dR) which yields a splitting long exact sequence,

\[ 0 \to (dR) \to (dR) \oplus (dR) \to (dR) \to 0. \]

Hence we have the diagram of long exact sequences:

\[ \cdots \to H^k(\ker^*) \to H^k(\#B_{CY}) \to H^k(\#B_{SL}) \to \cdots \]

\[
\begin{array}{ccc}
p_k^\ker \downarrow & & p_{B_{CY}}^k \downarrow \\
0 & \rightarrow & H^*_dR(X) \to \oplus^2 H^*_dR(X) \to H^*_dR(X) \to 0,
\end{array}
\]

where the sequence at top is the long exact sequence of (4-2-1). Since \(p_{B_{SL}}^k\) is injective, we see that \(p_{B_{CY}}^k\) is also injective for \(k\). Hence the results follows. \(\square\)

§5. Generalized hyperKähler, \(G_2\) and \(\text{Spin}(7)\) structures

§5-1. Generalized hyperKähler structures. In this section we will introduce two types of generalized hyperKähler structures, so that is, type 1 and type 2. A genralized hyperKähler structure of type 1 is defined by closed differential forms as in section 3. The one of type 2 is based on generalized complex structures satisfying a relations. Let \(V\) be a \(4m\) dimensional real vector space. A hyperKähler structure on \(V\) with a hyperKähler structure \((g, I, J, K)\), where \(I, J\) and \(K\) are three complex structure satisfying the quaternion relations, \((I^2 = J^2 = K^2 = IJK = -1)\) and \(g\) is a Hermitian metric with respect to \(I, J\) and \(K\). Then we have three Hermitian form \(\omega_I, \omega_J\) and \(\omega_K\) with respect to \(I, J\) and \(K\). Conversely such three forms \((\omega_I, \omega_J, \omega_K)\) yields the hyperKähler structure \((g, I, J, K)\). Hence hyperKähler structures on \(V\) can be regarded as geometric structures defined by three special 2-forms (see [Go]). As in section 4, three generalized \(\text{SL}_n(\mathbb{C})\) structures are defined by

\[
\phi_I := e^{\sqrt{-1}\omega_I}, \quad \phi_J := e^{\sqrt{-1}\omega_J}, \quad \phi_K := e^{\sqrt{-1}\omega_K}.
\]

Definition 5-1-1. Let \(B_{HK}(V)\) be the orbit of \(\text{Cpin}\) through the triple \((\phi_I, \phi_J, \phi_K)\). We call \(B_{HK}(V)\) the orbit of generalized hyperKähler structures on \(V\). A generalized hypKähler structure of type 1 on a \(4m\)-fold \(X\) is a \(B_{HK}(V)\)-structure on \(X\).

It is worthwhile to mention that there exist six generalized complex structures defined by generalized hypKähler structures. As in section 4, generalized \(\text{SL}_n(\mathbb{C})\) structures yield generalized complex structures. Hence we have three generalized
complex structures $\mathcal{I}_1$, $\mathcal{J}_1$ and $\mathcal{K}_1$ corresponding to $\omega_I, \omega_J$ and $\omega_K$ respectively. It immediately follows that three compositions $\mathcal{K}_0 := \mathcal{I}_1 \mathcal{J}_1$, $\mathcal{I}_0 := \mathcal{J}_1 \mathcal{K}_1$ and $\mathcal{J}_0 := \mathcal{K}_1 \mathcal{I}_1$ give generalized complex structures respectively. Hence we have six generalized complex structures $\mathcal{I}_0, \mathcal{J}_0, \mathcal{K}_0, \mathcal{I}_1, \mathcal{J}_1, \mathcal{K}_1$ which satisfying relations:

\begin{align*}
(5-1-1) & \quad I_0^2 = J_0^2 = K_0^2 = I_0 J_0 K_0 = -1, \\
(5-1-2) & \quad I_0 I_1 = I_1 I_0 = J_0 J_1 = J_1 J_0 = K_0 K_1 = K_1 K_0 = -G,
\end{align*}

where $G$ is a section of the bundle $\text{SO}(T \oplus T^*)$ with $G^2 = 1$, which is called a generalized metric. The relation (5-1-1) implies that the triple $(\mathcal{I}_0, \mathcal{J}_0, \mathcal{K}_0)$ satisfies the quaternion relations and (5-1-2) shows that we have three generalized Kähler structures with a same $G$.

**Definition 5-1-2.** A generalized hyperKähler structure of type 2 is a system consisting of six generalized complex structures with the relation (5-1-1,1).

The algebra generated by $\langle G, 1 \rangle$ with $G^2 = 1$ is isomorphic to the direct sum $\mathbb{R} \oplus \mathbb{R}$ by taking a basis,

$$
\frac{1}{\sqrt{2}}(1 + G), \quad \frac{1}{\sqrt{2}}(1 - G).
$$

When we set

$$
\mathcal{I}_1 = -G \otimes i, \quad \mathcal{J}_1 = -G \otimes j, \quad \mathcal{K}_1 = -G \otimes k,
$$

then we see that the algebra generated by $\mathcal{I}_\alpha, \mathcal{J}_\alpha, \mathcal{K}_\alpha, G, 1, \alpha = 0, 1$ with relations (5-1-1,2) is isomorphic to $(\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}$. In the case of generalized Kähler structure, we have the algebra $(\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. Hence there are following correspondence:

| generalized metric $\mathbb{R} \oplus \mathbb{R}$ | generalized Kähler structure $\mathbb{C} \oplus \mathbb{C}$ | generalized hyperKähler structure $\mathbb{H} \oplus \mathbb{H}$ |
|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $\uparrow$ | $\uparrow$ | $\uparrow$ |

§5-2. **Generalized $G_2$ structures.** Let $\mathbb{O}$ be the octernions which is regarded as the 8 dimensional real vector space with the metric $\langle , \rangle$ and the non associative multiplication. Consider the 7 dimensional vector space $W := \text{Im}\mathbb{O}$ with a volume form $\text{vol}_7$ and define a 3 form $\phi_{G_2}$ on $W$ by

$$
\phi_{G_2}(x, y, z) := \langle xy, z \rangle.
$$

A 4 form $\psi$ on $W$ is defined by the Hodge star operator $\ast$,

$$
\psi_{G_2} := \ast \phi_{G_2}.
$$

Let $\mathcal{B}_{G_2}$ be the orbit of Cpin through the pair $(\text{vol}_7 - \phi_{G_2}, 1 - \psi_{G_2})$. Then we have
Definition 5-2. A generalized $G_2$ structure is a $B_{G_2}$-structure on a 7-fold $X$.

§5-3. Generalized Spin(7) structures. As in section 5-2, we consider $\mathbb{O}$ as the 8 dimensional vector space $V$ which is decomposed into $V = \mathbb{R} \oplus \text{Im}\mathbb{O}$, where $\mathbb{R}$ denotes the real part of the octernions with a non-zero one form $e^1$. Then the Spin(7) form $\phi_{\text{Spin}}$ (the Cayley form) is a 4-form on $V$ defined by

$$\phi_{\text{Spin}} = e^1 \wedge \phi_{G_2} + \psi_{G_2}.$$  

Let $B_{\text{Spin}}$ be the orbit of the group $C\text{pin}(V \oplus V^*)$ through

$$\Phi_{\text{Spin}} := 1 - \phi_{\text{Spin}} + \text{vol}_8,$$

where $\text{vol}_8$ denotes the volume form of $V \cong \mathbb{O}$. Then we have

Definition 5-3. A generalized Spin(7) structure is a $B_{\text{Spin}}$-structure on a 8-manifold $X$.

Let $(\Omega_I, e^{\sqrt{-1}\omega_I})$ be a Calabi-Yau structure on a 8-manifold $X$. Then the real part $\Phi = \Omega_I^R + (e^{\sqrt{-1}\omega_I})^R$ yields a generalized Spin(7) structure on $X$. Since the group $C\text{pin}$ is real, for a generalized Calabi-Yau structure $(\phi_0, \phi_1)$ on a 8-manifold $X$, the real part $\Phi = \phi_0^R + \phi_1^R$

is a generalized Spin(7) structure.

Proposition 5-4. Let $X$ be a compact, oriented 8-manifold with a generalized Spin(7) structure $\Phi$. Deformations of generalized Spin(7) structures on $X$ are unobstructed.

Our proof is a generalization of the proof of deformations of Spin(7) structures in [Go]. Let $C\text{Spin}_0(V \oplus V^*)$ be the identity component of the conformal spin group which acts on $\wedge^*T^*$. There is the metric $g_o$ on the octernions $\mathbb{O}$ which yields a generalized metric $G_o$ by

$$G_o = \begin{pmatrix} o & g_0^* \\ g_0 & 0 \end{pmatrix},$$

where $g^*$ is the dual metric on the dual space $\mathbb{O}^*$. Then we have

Lemma 5-5. Let $H$ be the isotropy group which is defined by

$$H := \{ g \in C\text{Spin}_0(V \oplus V^*) \mid g \cdot \Phi_{\text{Spin}} = \Phi_{\text{Spin}} \}.$$  

Then the identity component $H_0$ of $H$ preserves the generalized metric $G_0$.

proof. Every $g \in C\text{Spin}_0(V \oplus V^*)$ is written as $g = e^a$ for $a \in CL^2$. Then $CL^2$ is decomposed into

$$CL^2 = \mathbb{R} \oplus \text{End}(V) \oplus \wedge^2V^* \oplus \wedge^2V.$$
Then $a$ is written as $a = \lambda + A + b + \beta$ for $A \in \text{End}(V)$, $\beta \in \Lambda^2 V$, $\beta \in \Lambda^2 V$. If $a \cdot \Phi_{\text{Spin}} = 0$ for $a = \lambda + A + \beta + b$, then we see that

\begin{align}
(5-3-1) & \quad (\lambda + A) \cdot \Phi_{\text{Spin}} = 0, \\
(5-3-2) & \quad (b + \beta) \cdot \Phi_{\text{Spin}} = 0
\end{align}

From (5-3-1), we have

\begin{align}
(3-5-3) & \quad \lambda = 0 \quad \text{and} \quad A \in \text{spin}(7).
\end{align}

The space of 2-forms and the space of 2-vectors are respectively decomposed into representation spaces of Spin(7),

\begin{align}
(5-3-4) & \quad \Lambda^2 V^* = \Lambda^2_7 V^* \oplus \Lambda^2_{21} V^*, \\
(5-3-5) & \quad \Lambda^2 V = \Lambda^2_7 V \oplus \Lambda^2_{21} V,
\end{align}

where $\Lambda^2_{21} V^* \cong \Lambda^2_{21} V$ is isomorphic to the Lie algebra spin(7) and $\Lambda^2_7$ denotes the irreducible 7 dimensional representation space. Let $p^*$ be the dual 2-vector of $p \in \Lambda^2_7 V^*$ and $q^*$ the dual 2-vector of $q \in \Lambda^2_{21} V^*$. Then we have

\begin{align}
(5-3-6) & \quad \begin{cases}
    p \wedge \Phi_{\text{Spin}} = p - 3 \ast p, \\
    p^* \cdot \Phi_{\text{Spin}} = 3p - \ast p, \\
    q \wedge \Phi_{\text{Spin}} = q + \ast q, \\
    q^* \cdot \Phi_{\text{Spin}} = -(q + \ast q),
\end{cases}
\end{align}

where $\ast$ denotes the Hodge star operator with respect to the metric $g_0$. From (5-3-6) if $(b + \beta) \cdot \Phi_{\text{Spin}} = 0$, then

\begin{align}
(5-3-7) & \quad b + \beta = q + q^*, \quad \text{(for } q \in \Lambda^2_{21} \cong \text{spin}(7)).
\end{align}

From (5-3-3,7) the Lie algebra of the isotropy group $H$ is given by the direct sum $\text{spin}(7) \oplus \text{spin}(7)$. Hence the identity component $H_0$ is the product Spin(7)×Spin(7) which preserves the generalized metric $G_0$. □

Let $\text{CSpin}_0(X)$ be the fibre bundle over a compact, oriented manifold $X$ with fibre $\text{CSpin}_0$. It suffices to consider that a generalized Spin(7) structure $\Phi$ is a closed section of the fibre bundle $\text{CSpin}_0(X)$. For each $x \in X$, we take an identification $h : \mathbb{O} \cong T_x X$ which gives the identification $\tilde{h} : T_x X \oplus T^*_x X \cong \mathbb{O} \oplus \mathbb{O}^*$. Then there is a $e^a \in \text{CSpin}_0(V \oplus V^*)$ such that

$$
\Phi_{\text{Spin}} = e^a \cdot h^* \Phi \in \Lambda^* \mathbb{O}^*.
$$
We denote by $\text{Ad}_{e^a}$ the adjoint of $e^a$. Then the composition $\hat{h} \circ \text{Ad}_{e^a}$ gives the identification $O \oplus O^* \cong T_x X \oplus T^*_x X$ which defines a generalized metric $G_{\Phi_x}$ by

$$G_{\Phi_x} := \hat{h}_* \circ (\text{Ad}_{e^a})_* G_O.$$ 

When we take another identification $h' : O \cong T_x X$ preserving the orientation and an element $e^{a'} \in \text{CSpin}_0(V \oplus V^*)$ with

$$\Phi_{\text{Spin}} = e^a \cdot h^* \Phi = e^{a'} \cdot (h')^* \Phi,$$

we have $h^{-1}e^a \Phi_{\text{Spin}} = (h')^{-1}e^{-a'} \Phi_{\text{Spin}}$. We can take $e^{a'} h'h^{-1}e^{-a}$ is an element of identity component. It follows from lemma 5-5 that

$$e^{a'} h'h^{-1}e^{-a} \in H_0 \cong \text{Spin}(7) \times \text{Spin}(7).$$

Hence

$$G_{\Phi_x} := \hat{h}_* \circ (\text{Ad}_{e^a})_* G_O = \hat{h'}_* \circ (\text{Ad}_{e^{a'}})_* G_O.$$ 

Thus we have

**Lemma 5-6.** Let $\Phi$ be a generalized Spin(7) structure on $X$. Then there is a generalized metric $G_{\Phi}$ on $X$ which is canonically defined by $\Phi$.

A generalized metric $G_{\Phi}$ yields an operator $\ast$ acting on differential forms with $\ast^2 = 1$ and a generalized metric $G_{\Phi}$ also yields a metric $\langle , \rangle$ on forms and we have the adjoint operator $d^\ast$ which is given by $d^\ast = \ast d\ast$. By using adjoint operator we have the Laplacian $\triangle_{\Phi}$ by $\triangle_{\Phi} = dd^\ast + d^\ast d$. (see [Gu2] for detail).

**Lemma 5-7.** Let $\wedge_{\text{even}}$ be anti-self dual forms of even type with respect to the operator $\ast$, so that is,

$$\wedge_{\text{even}} = \{ s \in \wedge^{\text{even}} | \ast s = -s \}.$$ 

Then the bundle $\wedge_{\text{even}}$ is the subbundle of the vector bundle $E^1_{\text{Spin}}(X) = \text{CL}^2 \cdot \Phi$

**proof.** At first we will show the lemma for the form $\Phi_{\text{Spin}} = 1 - \phi_{\text{Spin}} + \text{vol}_8$ on the octernions $O \cong V$. The action of $g \in \text{GL}(V)$ on forms is given by

$$(\det g)^2 \rho_g^{-1},$$

where $\rho$ denotes the linear representation of GL($V$). Hence for $\lambda \in \mathbb{R}$, $\lambda I \in \text{GL}(V)$ acts on $\Phi_{\text{Spin}}$ by

$$\lambda^4 + \Phi_{\text{Spin}} + \lambda^{-4} \text{vol}_8.$$ 

By differentiating to $\lambda$ we have that

$$(5-3-8) \quad -1 + \text{vol}_8 \in \text{CL}^2 \cdot \Phi_{\text{Spin}}.$$
Note that the the Hodge star * is given by the * operator,

\[(5-3-9) \quad *a = \begin{cases} +a, & a \in \Lambda^0 + \Lambda^1 + \Lambda^4 + \Lambda^7 + \Lambda^8, \\ -a, & a \in \Lambda^2 + \Lambda^3 + \Lambda^5 + \Lambda^6. \end{cases}\]

From (5-3-6), we see that

\[(5-3-10) \quad \Lambda^2 + \Lambda^6 + (\Lambda^2_{21} + \Lambda^6_{21})_+ \subset \text{CL}^2 \cdot \Phi_{\text{Spin}},\]

where \((\Lambda^2_{21} + \Lambda^6_{21})_- = \{ q + *q \mid q \in \Lambda^2_{21} \} \). From (5-3-9) we have \(*q = -*q\) for \(q \in \Lambda^2 V^*\). Hence \(q + *q = q - *q\) is the anti-self dual form with respect to *. We also have

\[gl(8)/\text{spin}(7) = so(8)/\text{spin}(7) \oplus \{ \lambda | \lambda \in \mathbb{R} \} \oplus \text{Sym}_0(8),\]

where \(\text{Sym}_0(8)\) denoted the trace-free symmetric matrix which yields anti-self-dual 4-forms, so that is,

\[(5-3-11) \quad \Lambda^4_\pm = \{ \hat{\rho}_a \Phi \mid a \in \text{Sym}_0(8) \},\]

where \(\hat{\rho}\) denotes the differential representation of \(\text{End}(V)\). Hence \(\Lambda^4_\pm \in \text{CL}^2 \cdot \Phi_{\text{Spin}}\). The anti-self-dual form of even type \(\Lambda^\text{even}_\pm\) is decomposed into there parts :

\[\Lambda^\text{even}_\pm = (\Lambda^0,8)_- \oplus (\Lambda^2,6)_- \oplus (\Lambda^4_\pm),\]

where \(\Lambda^0,8\) and \(\Lambda^2,6\) respectively denotes the anti-self dual 0,8-forms and anti-self dual 2,6-forms. Hence from (5-3-8,10,11) it follow that

\[(5-3-12) \quad \Lambda^\text{even}_\pm \subset \text{CL}^2 \cdot \Phi_{\text{Spin}}.\]

Every generalized \(\text{Spin}(7)\) form \(\Phi_V\) on \(V\) is given by \(k \cdot \Phi_{\text{Spin}}\) for an element \(k \in \text{CSpin}(V \oplus V^*)\). Then the action of \(k\) on forms yields the identification,

\[E^1_{\Phi_{\text{Spin}}} \cong E^1_{\Phi_V}.\]

Then the operator \(*_{\Phi_V}\) with respect to \(\Phi_V\) is given by the adjoint action of \(k\),

\[*_{\Phi_V} = k \cdot (*_{\Phi_{\text{Spin}}}) \cdot k^{-1}.\]

We denote by \(\Lambda^\text{even}_{\Phi_V}\) the anti-self-dual forms of even type with respect to \(\Phi_V\). Then the action of \(k\) also induces the identification between anti-self-dual forms \(\Lambda^\text{even}_{-,*_{\Phi_{\text{Spin}}}} \cong \Lambda^\text{even}_{-,*_{\Phi_V}}\). Then from \(\Lambda^\text{even}_{-,*_{\Phi_{\text{Spin}}}} \subset E^1_{*_{\Phi_{\text{Spin}}}}\) we have \(\Lambda^\text{even}_{-,*_{\Phi_V}} \subset E^1_{*_{\Phi_V}}\). Hence we have the result. □
proof of proposition 5.4. Let \( \Phi \) be a generalized Spin(7) structure on an oriented and compact 8-manifold \( X \) with the differential complex \( \#_{B_{\text{Spin}}} = \{ E^k_{\text{Spin}}(X), d_k \} \) over \( X \). It suffices to show that the map \( p^2_{B_{\text{Spin}}} \) is injective. Note that \( E^1_{\text{Spin}}(X) \subset \wedge^{\text{even}} \) and \( E^2(X) \subset \wedge^{\text{odd}} \). We have the Laplacian \( \Delta_{\Phi} : \wedge^{\text{even/odd}} \rightarrow \wedge^{\text{even/odd}} \) with respect to the generalized metric \( G_{\Phi} \) as in before. We denote by \( G_{\Phi} \) the Green operator of the Laplacian \( \Delta_{\Phi} \) on the manifold \( X \). Let \( d\alpha \) be a \( d \)-exact section of \( E^2(X) \). Then by using the Hodge decomposition with respect to \( \Delta_{\Phi} \), we have

\[
d\alpha = d(dd^*G_{\Phi}\alpha + d^*dG_{\Phi}\alpha).
\]

We define \( \alpha_- \in \wedge^{\text{even}} \) by

\[
\alpha_- = d^*dG_{\Phi} - *d^*dG_{\Phi}.
\]

Then we have \( d\alpha = d\alpha_- \). It follows from lemma 5-7 that \( \alpha_- \in E^1_{\text{Spin}}(X) \) and it implies that the map \( p^2_{B_{\text{Spin}}} \) is injective. \( \square \)

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References

[AGG] V. Apostolov, P. Gauduchon and G. Grancharov, Bihermitian structures on complex surfaces (1999), Proc.London Math. Soc. 79, 414-428.

[BK] Sergey Barannikov and Maxim Kontsevich, Frobenius manifolds and formality of Lie algebras of polyvector fields (1998), International Math. Res. Notices, (4), 201-215.

[Ca] Gil R. Cavalcanti, New aspect of the dd'-lemma, math.DG/0501406 vol.

[F-S] A. Fujiki and G. Schumacher, The moduli space of Extremal compact Kähler manifolds and Generalized Weil-Perterson Metrics, Publ. RIMS, Kyoto Univ vol26. No.1 (1990), 101-183.

[Go] R. Goto, Moduli spaces of topological calibrations, Calabi-Yau, hyperKähler, \( G_2 \), spin(7) structures, International Journal of Mathematics vol15, No. 3 (2004), 211-257.

[Gu 1] Marco Gualtieri, Generalized complex geometry, Oxford D.Phil thesis, math.DG/0401221 (2004).

[Gu2] Marco Gualtiere, Hodge decomposition for generalized Kähler manifolds, math.DG/0409093.

[Ha] F.R. Harvey, Spinors and Calibrations, Perspectives in Mathematics vol9, Academic Press, Inc, 1990.

[Hi1] N. J. Hitchin, Generalized Calabi-Yau manifolds, Q. J. Math. 54 (2003), 281-308.

[Hi2] Nijel, Instantons, Poisson structures and generalized Kähler geometry (2005), preprint.

[Hu] Daniel Huybrechts, Generalized Calabi-Yau structures, K3 surfaces and B-fields, math.AG/0306162.

[Jo1] D.D. Joyce, Compact Riemannian 7-manifolds with holonomy \( G_2 \), I, II, J.Differential Geometry vol43 (1996), 291-328, 329-375.

[Jo2] D.D. Joyce, Compact \( 8 \)-manifolds with holonomy \( \text{Spin}(7) \), Inventiones mathematicae vol128 (1996), 507-552.

[Jo3] D.D. Joyce, Compact Manifolds with Special Holonomy, Oxford mathematical Monographs, Oxford Science Publication, 2000.
[Ka] Y. Kawamata, *Unobstructed deformations- a remark on a paper of Z.Ran*, J. Algebraic Geometry vol1 (1992), 183-190, no. 2.

[Kod] K. Kodaira, *Complex manifolds and deformation of complex structures*, Grundlehren der Mathematischen Wissenschaften, vol283, Springer-Verlag, New York-Berlin, 1986.

[L-M] H.B. Lawson, Jr and M. Michelsohn, *Spin Geometry*, Princeton University press, 1989.

[Li] Yi Li, *On deformations of Generalized Complex structures: the Generalized Calabi-Yau Case*, hep/th0508030.

[Ra] Z. Ran, *Essays on Mirror manifolds*, International Press, Hong Kong, 1992, p. 451-457.

[T1] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Peterson-Weil metric*, Mathematical aspects of string theory (ed. S.-T. Yau), (1987), World Scientific Publishing Co., Singapore, 629–646.

[T2] G. Tian, *Smoothing 3-folds with trivial canonical bundle and ordinary double points*, Essays on Mirror Manifolds (1992), International Press, Hong Kong, 458-479.

[To] A.N. Todorov, *The Weil-Peterson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau) manifolds. I*, Comm. Math. Phys. vol126 (1989), 325–346.

[W] F. Witt, *Special metric structures and closed forms*, math.DG/0502443.

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