Friction and diffusion of matter-wave bright solitons

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We consider the motion of a matter-wave bright soliton under the influence of a cloud of thermal particles. In the ideal one-dimensional system, the scattering process of the quasiparticles with the soliton is reflectionless, however, the quasiparticles acquire a phase shift. In the realistic system of a Bose-Einstein condensate confined in a tight waveguide trap, the transverse degrees of freedom generate an extra but small nonlinearity in the system which gives rise to finite reflection and leads to dissipative motion of the soliton. We calculate the velocity and temperature-dependent frictional force and diffusion coefficient of a matter wave bright soliton immersed in a thermal cloud.

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Solitons are localized waves that propagate without spreading and attenuation. They appear from classical systems like ocean waves to optics and quantum systems like Bose-Einstein condensates (BEC) of atomic gases. A BEC of a dilute atomic gas with attractive two-body interactions in three dimensions (3D) is unstable and collapses 1. In one dimension (1D), however, a BEC with attractive interaction is stable against collapse and forms a self-bound particle-like object known as a bright soliton. Recently, bright solitons of Bose-condensed 7Li atoms were observed in quasi-1D waveguide traps at Rice University 2 and at ENS in Paris 3.

One of the most important features of solitons is the dissipationless motion over long distances. Because of this property, optical solitons have important applications in transatlantic fiber-optic communication systems 4. In this Letter we discuss how dissipative effects in the motion of a soliton in a thermal cloud can arise due to the 3D nature of the BEC in a tight waveguide. Although the dynamics is strictly one-dimensional, the transverse extent of the mean field generates extra nonlinear terms in the effective 1D equation, as first discussed in Ref. 5. This formalism can also be applied to other systems described by a cubic nonlinear Schrödinger equation per-
\[ n(x,t) = |\phi(x,t)|^2. \] A simple scaling argument shows that \( \tilde{\mu} = \hbar g f(\alpha n) \), where \( f(\cdot) \) is a dimensionless function, which has been computed numerically in Ref. [3]. Physical solutions of Eq. [3] are found only if \( -\alpha n < 0.47 \), otherwise transverse collapse occurs [11]. In the following we will be interested in the quasi-1D regime of small \( \alpha n \) and expand \( \tilde{\mu}(\alpha n) \) in a power series.

In the quasi-1D limit, when \( |\alpha|n \ll 0.47 \), the radial wavefunction \( \chi \) will be close to the ground state of the 2D harmonic oscillator with a Gaussian profile. We can expand \( \chi \) in terms of the radial eigenmodes \( \varphi_\nu(\rho) \), \( \chi(\rho,x) = \varphi_0(\rho) + \sum_\nu C_\nu(x) \varphi_\nu(\rho) \). The coefficients \( C_\nu \) are small and can be calculated perturbatively. The transverse chemical potential \( \tilde{\mu} \) can be obtained by using second order perturbation theory:

\[ \tilde{\mu} = \hbar \omega + gn - g_2 n^2 + \ldots, \]  

where \( g = 2a_0 \hbar \omega \) and \( g_2 = 24 \ln(4/3) a^2 \hbar \omega \). A correction to the 1D coupling constant \( g \) beyond the GP approach presented here has been found in Ref. [12]. The constant \( g_2 \) was calculated first in Ref. [3] and corrections beyond GP can be obtained by the self-consistent Hartree Fock Bogoliubov approach of Ref. [13]. We obtain the following effective equation describing the condensate in the quasi-1D limit:

\[ \left[-\left(\hbar^2/2m\right)\partial_x^2 + g|\phi|^2 - g_2|\phi|^4\right]\phi = \mu\phi. \]  

This is a nonlinear Schrödinger equation with a cubic and a quintic nonlinearity, as used before in Ref. [3]. The possibility of collapse is inherent in this equation as the quintic nonlinear term is attractive. An estimate from the 3D GP equation gives stability of a single soliton solution if \( N|\alpha|/l < 0.627 \) is fulfilled [14].

Without the extra nonlinearity associated with \( g_2 \), Eq. [3] is integrable. For attractive interactions at \( a < 0 \), the bosons form a self-bound particle-like state known as a bright soliton with the wavefunction \( \phi(x) = \sqrt{N/2\tilde{m}} \text{sech}(x/b) \) and the chemical potential \( \mu = -\hbar^2/2mb^2 \), where \( b = l^2/(N|\alpha|) \). We notice that for a weak soliton parameter \( N|\alpha|/l \lesssim 1 \), the system becomes quasi-one-dimensional (\( b \gtrsim 1 \)).

A soliton can be considered as a macroscopically coherent particle of mass \( mN \), moving in the bath of thermal excitations. Dissipative motion of the soliton arises due to the scattering of thermal atoms. Here we consider the interaction of thermally excited particles with the soliton within the Bogoliubov formalism [3],

\[ [H_0 + H_1] \psi = \epsilon(k) \psi \]  

where, \( \psi = (u,v) \) is a two component vector of particle \( (u) \) and hole \( (v) \) amplitudes, and \( \epsilon \) is the quasiparticle energy. The unperturbed Hamiltonian \( H_0 \) and the perturbation \( H_1 \) are given by

\[ \hat{H}_0 = \begin{pmatrix} -\frac{\hbar^2}{2m} \partial_x^2 - \mu & 0 \\ 0 & \frac{\hbar^2}{2m} \partial_x^2 + \mu \end{pmatrix} \]  

\[ \hat{H}_1 = \begin{pmatrix} V_1(x) & V_2(x) \\ -V_2^*(x) & -V_1(x) \end{pmatrix}, \]

where \( V_1 = 2g|\phi|^2 - 3g_2|\phi|^4 \) and \( V_2 = g|\phi|^2 - 2g_2|\phi|^2 \). The scattering states have energy \( \epsilon(k) = \frac{\hbar^2 k^2}{2m} + |\mu| \). In one dimension, neglecting the extra nonlinearity \( (g_2 = 0) \), we obtain the exact solution of the scattering states:

\[ u_k = A(k) \left[ kb + i \tanh(x/b) \right] e^{ikx}, \]

\[ v_k = A(k) \text{sech}^2(x/b) e^{ikx}, \]

where \( A(k) = 1/(k^2b^2 - 1) \) is a normalisation constant. The transmittance is given by

\[ t = (kb + i)^2/(kb - i)^2, \]

and the transmission probability is \( |t|^2 = 1 \). Hence, the quasiparticles scatter without reflection on the soliton but only acquire a phase shift and a time delay in the scattering process. Reflectionless scattering on a soliton in the integrable nonlinear Schrödinger equation (with \( g_2 = 0 \)) is a well-known result of mathematical soliton theory and is also found in an exact solution of the quantum many-body model in the limit of large particle number [12]. In the quasi-1D limit, the soliton thus becomes transparent and exhibits dissipationless motion in a thermal cloud.

Now we consider the scattering problem of quasiparticles in the presence of an extra nonlinearity that breaks the integrability. Assuming that the coupling constant \( g_2 \) of the extra nonlinear term is small, we can solve the scattering problem using Green’s function techniques. In order to solve Eq. [3] for the particle amplitude \( u \) we construct the Green’s function for the \( u \) component of \( H_0 \) satisfying

\[ \left[-(\hbar^2/2m)\partial_x^2 - \mu - \epsilon\right] G_1(x-x') = -\delta(x-x'), \]

which is given by \( G_1(x-x') = (m/\hbar^2k) \sin(k|x-x'|) \). Since the potential is symmetric, the scattering states can be constructed with even or odd symmetry. The Lippmann-Schwinger equation for the particle channel can be written as

\[ u_{e/o} = u_{e/o}^0 + \int G_1(x-x')V_1(x')u_{e/o}(x') \, dx' \]

\[ + \int G_1(x-x')V_2(x')v_{e/o}(x') \, dx', \]

where \( u_{e/o} \) denotes even [odd] wave functions of the particle states and \( u_{e/o}^0 = \cos(kx), [u_{e/o}^0 = \sin(kx)] \). The most general wave function can be constructed from even and odd eigenstates: \( u_k = A_k e^k + B_k e^{-k} \). Asymptotically this
wave function becomes \( \lim_{x \to -\infty} u_k = e^{ikx} + re^{-ikx} \) and \( \lim_{x \to \infty} u_k = te^{ikx} \) where \(|t|^2\) and \( R = |r|^2\) are the transmission and the reflection coefficient, respectively. We obtain \( R(k) \) by solving Eqs. (4) numerically and matching with the asymptotic solutions, see Fig. 1.

An analytical estimate of the reflection coefficient can be obtained from Eq. (13) by approximating \( \phi = \sqrt{N/2b} \text{sech}(x/b) \) and \( u_{e,o} \) and \( v_{e,o} \) with the properly symmetrised solutions \( A \) \( B \) \( C \). This approximation becomes exact for \( g_2 = 0 \) and relies on \( g_2 \) being a small parameter. The reflection coefficient is given by \( R = |r|^2 \) and

\[
r(k) = -i \left( \frac{I_+ + I_-}{(I_+ - i)(I_- + i)} \right),
\]

where the terms \( I_+ \) and \( I_- \) are given by

\[
I_{\pm} = \mp 2A(k) \left[ kb + 6 \ln(4/3) \frac{N^2|a|^2 Q_{\pm}(kb)}{kb} \right]
\]

with \( Q_{\pm}(x) = (1/3 + x^2 \pm (1 + x^2)^2/3 \sinh(\pi x)) \). By using the Lippmann-Schwinger formalism instead of a simple Born approximation we obtain the correct limiting behaviour for small \( k \) where \( R \to 1 \). Total reflection is expected whenever the special resonant conditions leading to reflectionless scattering at \( g_2 = 0 \) are broken, as \( k \to 0 \) implies a vanishing group velocity \( \partial \epsilon / \partial \hbar k \) to the soliton. This case is very different from phonons scattering on a perturbed dark soliton, which becomes transparent for small \( k \) as found in Ref. \( 2 \). The approximation \( 14 \) reproduces the qualitative features but slightly overestimates the exact values of \( R \) as seen in Fig. 1.

The reflection coefficient \( R \) is a function of dimensionless momentum \( kb \) and the soliton parameter \( N|a|/l \). In the dissipative dynamics of a macroscopic object like a soliton, the microscopic parameter \( N|a|/l \) enters through the reflection coefficient of the quasiparticles. Once we know the interaction of particles with a soliton from the microscopic theory, we can describe its motion in the bath of thermal particles at a given temperature. A bright soliton is a mesoscopic object with mass \( mN \), and its dynamics is governed by classical motion. Therefore, we can define a phase space distribution function of soliton’s center of mass coordinate \( f(p,q,t) \). When the soliton follows the classical trajectory, then the distribution function takes a simple form \( f(p,q,t) = \delta(p - p(t))\delta(q - q(t)) \), where \( p(t), q(t) \) are classical phase space trajectories. In the presence of a bath of thermal atoms, the atoms impart a momentum to the soliton in the scattering process. While the soliton is at rest, the force imparted on the soliton cancels on the average but, nevertheless, the stochastic nature of the force introduces a diffusive motion of the soliton. For a moving soliton, the average force imparted by the thermal particles does not vanish and gives rise to a frictional force on the soliton. To include the dissipative effects in the soliton’s motion we write down the kinetic equation for the phase space distribution function of the soliton \( 10 \):

\[
\frac{\partial f}{\partial t} - \frac{\partial}{\partial p} \left( \frac{\partial H}{\partial q} f \right) + \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} f \right) = I_{\text{coll}},
\]

where, for small momentum transfer, the collision integral \( I_{\text{coll}} \) can be written as

\[
I_{\text{coll}} = \frac{\partial}{\partial p} \left[ Af + Bf \right].
\]

The terms \( A \) and \( B \) gives rise to friction and diffusion of the soliton respectively. The frictional force \( A \) can be computed from the following expression,

\[
A = \int \frac{dk}{2\pi} (-2\hbar k) R(k) \frac{\partial \epsilon(k)}{\hbar \partial k} N(E, k_B T)
\]

where \( N(E, k_B T) \) describes the distribution of thermal particles in the frame of the moving soliton with velocity \( v \) and the energy \( E \) takes the value \( E(k) = (\hbar k - mn)^2/2m \). In each collision, the particle with momentum \( k \) has a probability \( R \) to reflect back and transfer the momentum \( -2\hbar k \) to the soliton. This momentum transfer multiplied with the number of particles coming from each direction per unit time gives rise to a frictional force. When the soliton is at rest, the momentum transfer on each direction cancels on the average and as a result the friction vanishes.

At finite temperatures the thermal atoms are distributed according to the rules of quantum statistics. Although thermal equilibrium may be reached in an external trap \( 17 \), the subtle conditions of equilibrium are not necessarily fulfilled in a dynamical experimental situation. Here, we consider the motion of a soliton relative to a significantly warmer thermal cloud of atoms. We thus can assume a classical Boltzmann distribution of thermal atoms, \( N(E, k_B T) \sim \exp(-E/k_B T) \). Dissipative effects of the soliton can be enhanced by increasing the density or the temperature of the thermal cloud. We consider the situation where \( 10^4 \) thermal particles are

![FIG. 1: Reflection coefficient of a soliton as a function of momentum, for \( N|a|/l \) = 0.1, 0.2, 0.3, 0.4 (from smaller to larger values of reflection coefficient). The dotted line shows the analytical estimate \( 13 \) for \( N|a|/l = 0.2 \).](attachment:figure1.png)
confined within a length $L = 50b$ ($\approx 70\mu m$ for ENS soliton with $b = 1.4\mu m$), with a density of the thermal gas of $nb = 200$ ($\approx 10^{12}/cm^{3}$) and the velocity distribution of the thermal particles being controlled by changing the temperature. Within a certain range of the soliton velocity the frictional force increases linearly with velocity as seen in Fig. 2. When the velocity is increased further, nonlinear effects take over and the force decreases.

Now we can calculate the diffusion parameter of the transport equation:

$$B = \int \frac{dk}{2\pi} 2(hk)^2 R(k) \left| \frac{\partial \epsilon(k)}{\hbar \partial k} \right| N(E, k_B T).$$

(18)

This term describes the velocity fluctuations of the soliton and gives rise to a diffusion in the momentum space. A graph is shown in Fig. 3.

So far we considered only the low-energy elastic scattering of thermal quasiparticles on the soliton. To restrict our discussion to the quasi-1D case, we neglected the higher energy radial excitations $\sim h\omega$. As an additional effect, the soliton can radiate particles if the colliding quasiparticle has higher energy than the binding energy $|\mu|$. Also nonlinear collective motion of the thermal cloud and the soliton is possible [13]. However, the elastic scattering process discussed in this work will dominate if the condition $k_B T < |\mu| < h\omega$ is fulfilled. A tight radial trapping potential is suitable to avoid inelastic scattering processes. In the ENS experiment [2], the oscillator length of radial confinement was $l = 1.4\mu m$. For a soliton parameter $N|a|/l = 0.4$, the sound velocity at the center of the soliton becomes $c_s \approx 2.5\text{mm/s}$. If a soliton with $N \sim 10^3$ particles moves with a velocity $0.1c_s$, then it decelerates $5.0\text{mm/s}^2$ due to the frictional force. Finally it stops after 0.05s, travelling a distance of $6\mu m$.

The slowing down of a bright soliton can be observed experimentally by suitably manipulating the density and temperature of the thermal cloud.

Due to the friction force, the momentum $\vec{p}$ of the soliton changes as $d\vec{p}/dt = -A \vec{v}$. For small velocities, $A = \gamma v$ and the moving soliton stops after a time scale $\tau = mN/\gamma$. Due to the diffusion process the energy of a resting soliton changes as $E = (B/2\gamma)[1 - e^{-2\gamma/mN}]$. For $t \ll \tau$, the energy of the soliton increases and finally it reaches a steady state with energy $E = B/2\gamma$.

In conclusion, we have investigated the effects of a thermal environment on the dynamics of bright matter-wave solitons and have calculated the frictional force and diffusion coefficient in a microscopic approach. Friction and diffusion effects occur due to the deviation from the quasi-one-dimensional limit. Both of them are generally small and can be controlled by the parameters of the system if unattenuated propagation of solitons is desired. However, the parameters can be chosen such that the dissipative effects become accessible to experimental observation with currently available techniques.

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