On Some Applications of Group Representation Theory to Algebraic Problems Related to the Congruence Principle for Equivariant Maps

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Abstract

Given a finite group \(G\) and two unitary \(G\)-representations \(V\) and \(W\), possible restrictions on Brouwer degrees of equivariant maps between representation spheres \(S(V)\) and \(S(W)\) are usually expressed in a form of congruences modulo the greatest common divisor of lengths of orbits in \(S(V)\) (denoted \(\alpha(V)\)). Effective applications of these congruences is limited by answers to the following questions: (i) under which conditions, is \(\alpha(V) > 1\)? and (ii) does there exist an equivariant map with the degree easy to calculate? In the present paper, we address both questions. We show that \(\alpha(V) > 1\) for each irreducible non-trivial \(\mathbb{C}[G]\)-module if and only if \(G\) is solvable. For non-solvable groups, we use 2-transitive actions to construct complex representations with non-trivial \(\alpha\)-characteristic. Regarding the second question, we suggest a class of Norton algebras without 2-nilpotents giving rise to equivariant quadratic maps, which admit an explicit formula for the Brouwer degree.

1 Introduction

1.1 Topological motivation

The methods based on the usage of Brouwer degree and its infinite dimensional generalizations are unavoidable in many mathematical areas which, at first glance, have nothing

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in common: (i) qualitative investigation of differential and integral equations arising
in mathematical physics (existence, uniqueness, stability, bifurcation of solutions (see [28, 33, 15])), (ii) combinatorics (equipartition of mass (see [6, 37]), geometry (harmonic
maps between surfaces (see [17, 24])), to mention a few. In short, given a continuous map
$$\Phi : M \to N$$ of manifolds of the same dimension, the Brouwer degree deg($\Phi$) is an integer
which can be considered as an algebraic count of solutions to equation $\Phi(x) = y$ for a
given $y \in N$ (for continuous functions from $\mathbb{R}$ to $\mathbb{R}$, the Brouwer degree theory can be
traced to Bolzano-Cauchy Intermediate Value Theorem).

In general, practical computation of the Brouwer degree is a problem of formidable
complexity. However, if $\Phi$ respects some group symmetries of $M$ and $N$ (expressed in
terms of the so-called equivariance, see Section 2.1), then the computation of deg($\Phi$) lies
in the interplay between topology and group representation theory. Essentially, symmetries
lead to restrictions on possible values of the degree. These restrictions (typically
formulated in a form of congruencies) have been studied by many authors using different
techniques (see, for example, [27, 29, 22, 7, 5, 2, 36, 26, 14, 31] and references therein (see
also the survey [35])). The following statement (which is a particular case of the so-called
congruence principle established in [29], Theorems 2.1 and 3.1) is the starting point for
our discussion.

**Congruence principle:** Let $M$ be a compact, connected, oriented, smooth $n$-dimensional
manifold on which a finite group $G$ acts smoothly, and let $W$ be an orthogonal $(n + 1)$-
dimensional $G$-representation. Denote by $\alpha(M)$ the greatest common divisor of lengths of
$G$-orbits occurring in $M$. Assume that there exists an equivariant map $\Phi : M \to W \setminus \{0\}.
Then, for any equivariant map $\Psi : M \to W \setminus \{0\}$, one has

$$\text{deg}(\Psi) \equiv \text{deg}(\Phi) \pmod {\alpha(M)}.$$

Clearly, the congruence principle contains a non-trivial information only if $\alpha(M) > 1$
(for example, if a non-trivial group $G$ acts freely on $M$, then $\alpha(M) = |G| > 1$). Also,
the congruence principle can be effectively applied only if there exists a “canonical”
equivariant map $\Phi : M \to W \setminus \{0\}$ with deg($\Phi$) easy to calculate (for example, if $M$
coincides (as a $G$-space) with the unit sphere $S(W)$ in $W$, then one can take $\Phi := \text{Id}$, in
which case, for any equivariant map $\Psi : S(W) \to W \setminus \{0\}$, one has: $\text{deg}(\Psi) \equiv \text{deg}(\text{Id}) = 1$
($\pmod {\alpha(S(W))}$); in particular, $\text{deg}(\Psi) \neq 0$ provided $\alpha(S(W)) > 1$). This way, we arrive
at the following two problems:

**Problem A.** Under which conditions on $M$, is $\alpha(M)$ greater than 1?

**Problem B.** Under which conditions on $M$ and $W$, does there exist an equivariant map
$\Phi : M \to W \setminus \{0\}$ with deg($\Phi$) easy to calculate?

Assume, in addition, that $V$ is an orthogonal $G$-representation and $M = S(V)$ (recall
that $S(V)$ is called a $G$-representation sphere). Then: (i) Problem A can be traced to the
classical result of J. Wolf [38] on classification of finite groups acting freely on a finite-
dimensional sphere, (ii) both Problems A and B are intimately related to a classification
of $G$-representations up to a certain (non-linear) equivalence (see [2, 36, 1, 30]).

A study of numerical properties of orbit lengths of finite linear groups has a long
history and can be traced back to H. Zassenhaus [37]. A special attention was paid to
studying regular orbits, orbits of coprime lengths, etc., in the case of the ground field of
positive characteristic (see [18] for a comprehensive account about the current research
in this area). To the best of our knowledge, the case of zero characteristic was not as well studied as the one of positive characteristic. It seems that the invariant introduced in our paper (the \( \alpha \)-characteristic of a linear representation) has not been studied in detail before.

The goal of this paper is to develop some algebraic techniques allowing one to study Problems \( A \) and \( B \) for finite solvable and 2-transitive groups. We are focused on the situation when \( V \) and \( W \) are complex unitary \( G \)-representations of the same dimension and \( M = S(V) \) is a \( G \)-representation sphere (in this case, we set \( \alpha(V) = \alpha(S(V)) \) and call it \( \alpha \)-characteristic of \( V \)). However, some of our results (see Corollary 7.3) are formulated for equivariant maps of \( G \)-manifolds.

1.2 Main results and overview

(A) If \( V \) and \( U \) are (complex unitary) \( G \)-representations, then \( \alpha(U \oplus V) = \gcd \{ \alpha(U), \alpha(V) \} \). This simple observation suggests to study Problem \( A \) first for \( S(V) \), where \( V \) is an irreducible representation. By combining the main result from [25] with several group theoretical arguments, we obtain the following result: \( G \) is solvable if and only if \( \alpha(V) > 1 \) for any non-trivial irreducible \( G \)-representation (see Theorem 3.8). Among many known characterizations of the class of (finite) solvable groups, we would like to refer to Theorem 3.7 from [8] (where a concept of admissible representations is used) as the result close in spirit to ours. Also, if \( G \) is nilpotent, we show that for any non-trivial irreducible \( G \)-representation, there exists an orbit \( G(x) \) in \( S(V) \) such that \( |G/G_x| = \alpha(V) \) (see Proposition 3.19).

On the other hand, we discovered that a sporadic group (the Janko Group \( J_1 \) (see [23])) satisfies the following property: all irreducible \( J_1 \)-representations have the \( \alpha \)-characteristic equals 1 (recall that \( J_1 \) is of order 175560 and admits 15 irreducible representations). With these results in hand, we arrived at the following question: Given a (finite) non-solvable group \( G \) different from \( J_1 \), does there exist an easy way to point out an irreducible \( G \)-representation \( V \) with \( \alpha(V) > 1 \)? In this paper, we focus on the following setting: Given \( H < G \), take the \( G \)-action on \( G/H \) by left translations and denote by \( V \) the augmentation submodule of the associated permutation \( G \)-representation \( C \oplus V \). It turns out that \( \alpha(V) > 1 \) if and only if \( |G/H| = q^k \), where \( q \) is a prime (see Lemma 4.4). Combining this observation with the classification of 2-transitive groups (see [10], for example) allows us to completely describe faithful augmented modules \( V \) associated with 2-transitive group \( G \)-actions on \( G/H \) such that \( \alpha(V) > 1 \) (see Theorem 4.3).

Finally, it is possible to show that if \( H \trianglelefteq G \), \( V \) is an \( H \)-representation and \( W \) is a \( G \)-representation induced from \( V \), then, \( \alpha(V) \) divides \( \alpha(W) \). This observation suggests the following question: under which conditions, does \( \alpha(V) = 1 \) imply \( \alpha(W) = 1 \)? We answer this question affirmatively assuming that \( V \) is irreducible and \( G/H \) is solvable (see Proposition 5.5).

(B) In general, Problem \( B \) is a subject of the equivariant obstruction theory (see [36, 5] and references therein) and is far away from being settled even in relatively simple cases. On the other hand, if \( W \) is a subrepresentation of the \( m \)-th symmetric power of \( V \), then one can look for a required map in the form of a \( G \)-equivariant \( m \)-homogeneous map \( \Phi : S(V) \to W \setminus \{0\} \), in which case \( \deg(\Phi) = m^n \). In particular case when \( m = 2 \),
Problem 13 reduces to the existence of a commutative (in general, non-associative) bilinear multiplication $\ast : V \times V \to V \subset \text{Sym}^2(V)$ satisfying two properties: (i) $\ast$ commutes with the $G$-actions, and (ii) the complex algebra $(V, \ast)$ is free from 2-nilpotents. Combining this idea with the techniques related to the so-called Norton algebra (see [11]), we establish the existence of an equivariant quadratic map between two non-equivalent $(n - 1)$-dimensional $S_n$-representations (having the same symmetric square) taking non-zero vectors to non-zero ones, provided that $n$ is odd (see Theorem 6.9). For $n = 5$, we give an explicit formula of such a map.

After the Introduction, the paper is organized as follows. Section 2 contains preliminaries related to groups and their representations. In Section 3, we first consider functorial properties of $\alpha$-characteristic (see Proposition 3.2). Next, we focus on solvable and nilpotent groups and prove Theorem 3.8 and Proposition 3.19. 2-transitive actions are considered in Section 4, while induced representations with trivial $\alpha$-characteristic are considered in Section 5. Section 6 is devoted to the existence of quadratic maps relevant to Problem 13. In the concluding Section 7, we consider applications of the obtained results to the congruence principle.

2 Preliminaries

2.1 Groups and Their Actions

This subsection collects some basic facts about finite groups and their actions that are used in our paper. Although the material given here is well-known to any group theorist, we decided to include it here, because we expect that the paper could be of interest for mathematicians working outside the group theory.

Throughout the paper, we consider only finite groups if no otherwise is stated, and by $G$, we always mean a finite group.

For any $G$, denote by Aut($G$) (resp. Inn($G$)) the group of automorphisms (resp. inner automorphisms) of $G$, by $e$ the identity of $G$ and by $1$ the trivial group or the trivial subgroup of $G$.

Given $H, K < G$, set $HK := \{hk \in G : h \in H, k \in K\}$. Given a prime $p$, denote by Syl$_p(G)$ the collection of Sylow $p$-subgroups of $G$. Recall an important characterization of solvable groups from [25]:

**Theorem 2.1.** Let $p_1, p_2, \ldots, p_k$ be a sequence of all distinct prime factors of $|G|$. Then, $G$ is solvable if and only if $G = P_1 P_2 \cdots P_k$ for any choice of $P_j \in \text{Syl}_{p_j}(G)$, $j = 1, \ldots, k$.

Recall that $N \leq G$ is called a minimal normal subgroup if $N$ is non-trivial and contains no other non-trivial normal subgroups of $G$. The socle of $G$ is the subgroup generated by all minimal normal subgroups of $G$. The following result is well-known (see [21]).

**Proposition 2.2.** A minimal normal subgroup of a solvable group is elementary abelian.

Let $X$ be a $G$-space. For any $x \in X$, denote by $G_x$ the isotropy (stabilizer) of $x$ and by $G(x)$ the $G$-orbit of $x$ in $X$. We call the conjugacy class of $G_x$ the orbit type of $x$ and denote by $\Phi(G; S)$ the collection of orbit types of points in $S \subset V$. For any $H < G$, denote by $X^H := \{x \in X : hx = x \text{ for all } h \in H\}$ the set of $H$-fixed points in $X$. 4
If \(|X| \geq 2\), we say that \(G\) acts 2-transitively on \(X\) if for any \(a, b, c, d \in X, a \neq b, c \neq d\), there exists \(g \in G\) such that \(ga = c\) and \(gb = d\). Since any transitive (in particular, 2-transitive) action is equivalent to the \(G\)-action on the coset space \(G/H\) by left translation for some \(H < G\), the existence of 2-transitive \(G\)-action is actually an intrinsic property of \(G\). Therefore, we adopt the following definition.

**Definition 2.3.** \(G\) is called a 2-transitive group if it admits a faithful 2-transitive action, or equivalently, \(G\) acts 2-transitively on \(G/H\) (by left translation) for some \(H < G\).

Suppose \(X\) and \(Y\) are (topological) \(G\)-spaces. A continuous map \(f : X \to Y\) is called \(G\)-equivariant if \(f(gx) = gf(x)\) for any \(g \in G\) and \(x \in X\). Note that in this case, \(f\) takes \(H\)-fixed points in \(X\) to \(H\)-fixed points in \(Y\) (i.e., \(f(X^H) \subset Y^H\) for any \(H < G\)). If, in addition, \(X\) and \(Y\) are linear \(G\)-spaces, a \(G\)-equivariant map \(f : X \to Y\) is called admissible if \(f^{-1}(0) = \{0\}\). We refer to [8] and [20] for the equivariant topology background.

### 2.2 Group Representations

Throughout the paper, we consider only finite-dimensional complex unitary representations, and by \(\rho\) (resp. \(V\) and \(\chi\)), we always mean a \(G\)-representation (resp. the associated vector space and the affording character) if no otherwise is stated.

Let \(K\) be an arbitrary field. Denote by \(K[G]\) the group algebra of \(G\) over \(K\). For any \(\rho\), we will simply denote by the same symbol the extension of \(\rho\) to \(K[G]\) (i.e., depending on the context, \(\rho : G \to GL(V)\) or \(\rho : K[G] \to \text{End}(V)\)).

For any \(G\), denote by \(1_G\) the trivial representation or trivial character of \(G\) (depending on the context) and by \(\text{Irr}(G)\) (resp. \(\text{Irr}^*(G)\)) the collection of irreducible (resp. non-trivial irreducible) \(G\)-representations.

For any representation \(\rho : G \to GL(V)\), denote by \(\rho(G)(x)\) or \(G(x)\) the \(G\)-orbit of \(x\) for any \(x \in V\). In addition, set \(\Phi(\rho) := \Phi(G; S(V))\), where \(S(V)\) stands for the unit sphere in \(V\).

If \(\rho\) and \(\sigma\) are \(G\)-representations, then \([\rho, \sigma]\) will stand for the scalar product of their characters.

Let \(H < G\). For any \(G\)-representation \(\rho\) with character \(\chi\), denote by \(\rho_H\) and \(\chi_H\) the restriction of \(\rho\) and \(\chi\) to \(H\), respectively. On the other hand, for any \(H\)-representation \(\psi\) with character \(\omega\), denote by \(\psi^G\) and \(\omega^G\) the induced representation and the induced character of \(\psi\) to \(G\), respectively.

Let \(\sigma\) be an automorphism of \(G\). Denote by \(\rho^\sigma\) (resp. \(\chi^\sigma\)) the composition \(\rho \circ \sigma\) (resp. \(\chi \circ \sigma\)). It is clear that: (i) \(\rho^\sigma\) is a \(G\)-representation affording character \(\chi^\sigma\), and (ii) if \(\rho\) is irreducible, so is \(\rho^\sigma\). If, in particular, \(\sigma : g \mapsto ugu^{-1}\) for some \(u \in H \supseteq G\), denote by \(\rho^{(u)}\) (resp. \(\chi^{(u)}\)) the composition \(\rho \circ \sigma\) (resp. \(\chi \circ \sigma\)) instead of \(\rho^\sigma\) (resp. \(\chi^\sigma\)). In such a case, \(\rho^{(u)}\) (resp. \(\chi^{(u)}\)) is said to be \(U\)-conjugate to \(\rho\) (resp. \(\chi\)).

Recall the following result for permutation representations associated to 2-transitive actions (see [20]).

**Proposition 2.4.** Let \(G\) act transitively on \(X\). Then, the permutation representation associated to this action is equivalent to \(1_G \oplus \rho\), where all irreducible components of \(\rho\) are non-trivial. If, in addition, \(G\) acts 2-transitively on \(X\), then \(\rho\) is irreducible.
The G-representation \( \rho \) in Proposition 2.4 will play an essential role in our consideration. We adopt the following definition.

**Definition 2.5.** Following [19], we call the representation \( \rho \) from Proposition 2.4 the **augmentation representation** associated to the transitive G-action on \( X \) (resp. \( G/H \) by left translation) and denote it by \( \rho^0_{\rho(G,X)} \) (resp. \( \rho^0_{\rho(G,H)} \)). In particular, denote by \( \rho_2(G) \) the collection of all its non-isomorphic augmentation representations arising from 2-transitive actions of \( G \).

We refer to [34], [9] and [20] for the representation theory background and notation frequently used in this paper.

### 3 \( \alpha \)-characteristic of \( G \)-representations

The following definition is crucial for our discussion.

**Definition 3.1.** For a G-representation \( \rho : G \to \text{GL}(V) \), we call
\[
\alpha(\rho) = \alpha(G, S(V)) := \gcd \{|G(x)| : x \in S(V)\} = \gcd \{|G/H| : (H) \in \Phi(\rho)\}
\]
the \( \alpha \)-characteristic of \( \rho \). We will call the \( \alpha \)-characteristic of a representation **trivial** if it takes value 1.

Note that \( \alpha \)-characteristic admits the following functorial properties.

**Proposition 3.2.** Suppose \( \rho \) is a \( G \)-representation.

(a) Let \( H < G \). Then, \( \alpha(\rho_H) \) divides \( \alpha(\rho) \).

(b) Let \( H \trianglelefteq G \) and \( \theta \) be an \( H \)-representation. Then, \( \alpha(\theta) \) divides \( \alpha(\theta^G) \).

(c) Let \( \sigma \) be an automorphism of \( G \). Then, \( \alpha(\sigma^G) = \alpha(\rho) \).

(d) Let \( \mathbb{F} \) be a splitting field of the group algebra \( \mathbb{Q}[G] \) and \( \sigma \) an automorphism of \( \mathbb{F} \). Then, \( \alpha(\rho^\sigma) = \alpha(\rho) \) for \( \rho \in \text{Irr}(G) \).

(e) Let \( \psi \) be another \( G \)-representation. Then, \( \alpha(\rho \oplus \psi) = \gcd \{\alpha(\rho), \alpha(\psi)\} \).

**Proof.** Here we prove part (b) only since other properties are quite straightforward from Definition 3.1. Denote by \( V \) and \( W \) the representation spaces of \( \theta^G \) and \( \theta \), respectively. Take an arbitrary non-zero \( v \in V \). It suffices to show that \( \alpha(\theta) \) divides \( |N(v)| \). Since \( V \) is induced by \( W \), one has \( v = \sum g_iw_i \), where \( \{g_i\} \) is the complete set of representatives of \( N \)-cosets in \( G \) and \( w_i \in W \). Without loss of generality, assume that \( w_1 \neq 0 \). Take \( n \in N_v \). Since \( \rho v = \sum g_i(g_i^{-1}ng_i)w_i \) and \( N \) is normal, we conclude that \( n \in N_v \) if and only if \( n \in g_iN_wg_i^{-1} \) for every \( i \). In particular, \( n \in g_1N_wg_1^{-1} \) implying \( N_v < g_1N_wg_1^{-1} < N \). Therefore, \( |N(w_1)| = |N : N(w_1)| \) divides \( |N(v)| = |N : N_v| \) and the result follows from the fact that \( \alpha(\theta) \) divides \( |N(w_1)| \).

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\( ^1 \)The corresponding module will be called the augmentation module. We do not use a special notation for it.
Remark 3.3. (i) According to Proposition 3.2(e), given a $G$-representation $\rho$ (possibly reducible), one can evaluate $\alpha(\rho)$ by computing $\alpha$-characteristics of all irreducible components of $\rho$.

(ii) The conclusion of Proposition 3.2(b) is not true if $H$ is not normal in $G$. The simplest example is provided by the group $G = S_3$ with $H$ to be an order two subgroup. If $\theta$ is a non-trivial representation of $H$, then $\theta^G = \rho_1 \oplus \rho_2$ where $\dim(\rho_1) = 1, \dim(\rho_2) = 2$. In this case, $\alpha(\theta) = 2$ while $\alpha(\rho_1) = 2$ and $\alpha(\rho_2) = 3$, so that $\alpha(\theta^G) = 1$.

(iii) One could think that there always exists an irreducible constituent $\rho$ of $\theta^G$ with $\alpha(\theta) = \alpha(\rho)$. But this is not true as the following example shows. Take $G = Q_8$, a quaternion group of order eight, and let $H$ be its cyclic subgroup of order 4. If $\theta$ is a faithful irreducible representation of $H$, then $\theta^G$ is an irreducible 2-dimensional $G$-representation. In this case, $\alpha(\theta) = 4$ while $\alpha(\theta^G) = 8$.

Example 3.4. Computation of $\alpha(\rho)$ for $\rho \in \text{Irr}^*(G)$ involves finding maximal orbit types $(H) \preceq (G)$ of $\rho$. For example, the group $A_5$ admits four non-trivial irreducible representations with the lattices of orbit types shown in Figure 2. Then, for each $\rho \in \text{Irr}^*(A_5)$, $\alpha(\rho)$ is the greatest common divisor of indices of proper subgroups which appear in the lattice. The result is shown in Table 2.

Remark 3.5. Note that the character table of a group does not determine the $\alpha$-characteristic of its irreducible representations. For example, $D_8$ and $Q_8$ have the same character table while the $\alpha$-characteristic of their unique 2-dimensional irreducible representations are distinct (4 for $D_8$ and 8 for $Q_8$).

3.1 $\alpha$-characteristic of Solvable Group Representations

Problem A together with Remark 3.3 give rise to the following questions.

Question A. Does there exist a non-trivial group $G$ such that $\alpha(\rho) = 1$ for any $\rho \in \text{Irr}(G)$?

Question B. Does there exist a reasonable class of groups $\mathcal{A}$ such that for any $G \in \mathcal{A}$, one has

\[ \alpha(\rho) > 1 \text{ for any } \rho \in \text{Irr}^*(G)? \]  

Question C. Given a group $G$ which is neither in the case of Question A nor Question B, how can one find a $G$-representation $\rho$ with $\alpha(\rho) > 1$?

An affirmative answer to Question A is given by the following example.

Example 3.6. The Janko Group $J_1$ has 15 irreducible representations—all of them admit trivial $\alpha$-characteristics.

We give a complete answer to Question B in the rest of this subsection and address Question C in Sections 4 and 5. The following example is the starting point for our discussion.

Example 3.7. If $G$ is abelian or a $p$-group, then (i) is true.

We will show that the following statement is true.
**Theorem 3.8.** $G$ is solvable if and only if $\alpha(\rho) > 1$ for any $\rho \in \text{Irr}^*(G)$.

**Remark 3.9.** As it will follow from the proof, the conclusion of the Theorem [3.8] remains true if one replace the complex field by an algebraically closed field of a characteristic coprime to $|G|$.

Let us first present two lemmas required for the proof of necessity in Theorem 3.8.

**Lemma 3.10.** Let $\rho \in \text{Irr}^*(G)$ and $P \in \text{Syl}_p(G)$. Then $\alpha(\rho_P) = \alpha(\rho)_p$, where $\alpha(\rho)_p$ is the highest $p$-power that divides $\alpha(\rho)$. In addition, the following statements are equivalent.

(i) $p$ divides $\alpha(\rho)$.

(ii) $P x \leq P$ for any $x \in S(V)$.

(iii) $[\chi_P, 1_P] = 0$.

**Proof.** By Proposition 3.12, $\alpha(\rho_P)$ divides $\alpha(\rho)$. Since $\alpha(\rho_P)$ is a $p$-power, we conclude that $\alpha(\rho_P) | \alpha(\rho)_p$.

Let us show that $\alpha(\rho)_p$ divides the cardinality of every $G$-orbit in $S(V)$. Let $O \subseteq S(V)$ be a $G$-orbit. By Exercise 1.4.17 in [10], the length of every $P$-orbit in $O$ is divisible by $|O|_p$. Therefore, $\alpha(\rho)_p$ divides length of every $P$-orbit in $O$. Hence, $\alpha(\rho)_p$ divides the length of every $P$-orbit in $S(V)$. Hence, $\alpha(\rho)_p | \alpha(\rho_P)$.

(i) $\implies$ (ii). Since $\alpha(\rho_P) = \alpha(\rho)_p \geq p$, each $P$-orbit in $S(V)$ is non-trivial, i.e., $[P : P_x] \geq p$ for each $x \in S(V)$.

(ii) $\implies$ (i). Suppose (ii) is true. Then, $p$ divides $|P/P_x| = |P(x)|$, which divides $|G(x)|$, for any $x \in S(V)$. It follows that $p$ divides $\alpha(\rho)$.

(ii) $\iff$ (iii). Both (ii) and (iii) are equivalent to $\dim V_P = 0$.

**Remark 3.11.** Notice that in Lemma 3.10, for (ii) to imply (i), it is enough to assume that $P < G$ is a $p$-subgroup.

**Remark 3.12.** In what follows, denote $H := \sum_{g \in H} g \in \mathbb{Z}[G]$ and $\hat{H} := \frac{1}{|H|}H \in \mathbb{Q}[G]$ for any $H < G$. Under this notation, Lemma 3.10 (iii) reads $\chi(P) = 0$ or $\chi(P) = 0$. In addition, note that Lemma 3.10 (iii) is equivalent to saying that $\rho$ is not a constituent of $1_P^{\hat{G}}$.

**Proposition 3.13.** Let $V$ be an non-trivial irreducible $G$-representation and $N \leq G$. Then, $N$ acts non-trivially on $V$ if and only if $N_x \leq N$ for any $x \in S(V)$.

**Proof.** Since $N$ is normal in $G$, the subspace $V^N$ is $G$-invariant. Therefore, either $V^N = \{0\}$ or $V^N = V$ from which the claim follows.

The next result immediately follows from Lemma 3.10 and Proposition 3.13 (see also Remark 3.11).

**Corollary 3.14.** Let $N \leq G$ be a $p$-subgroup and let $V$ be a non-trivial irreducible $G$-representation where $N$ acts non-trivially. Then, $p$ divides $\alpha(\rho)$. 

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As for sufficiency in Theorem 3.8 we will need the following lemma.

**Lemma 3.15.** Let $\rho$ be a $G$-representation with character $\chi$ and $H \leq G$. Then,

(i) $\rho(\hat{H})$ is an idempotent.

(ii) If, in addition, $\chi(\hat{H}) = 0$, then both $\rho(\hat{H})$ and $\rho(\hat{H})$ are zero matrices.

**Proof.** Direct computation shows that $\hat{H} \in \mathbb{Q}[G]$ is an idempotent, therefore, so is $\rho(\hat{H})$. If, in addition, $\chi(\hat{H}) = 0$, i.e., $\rho(\hat{H})$ is an idempotent matrix with zero trace, then $\rho(\hat{H})$ is a zero matrix. In this case, $\rho(\hat{H}) = |H| \rho(\hat{H})$ is also a zero matrix.

The next result follows immediately from Lemmas 3.10 and 3.15 (see also Remark 3.12).

**Corollary 3.16.** Let $\rho \in \text{Irr}^*(G)$ with $\alpha(\rho) > 1$. Then, there exists a prime factor $p$ of $|G|$ such that $\rho(P)$ is a zero matrix for any $P \in \text{Syl}_p(G)$.

The following elementary statement is an immediate consequence of the injectivity of a regular representation of a finite group.

**Proposition 3.17.** Let $\rho$ be the regular $G$-representation. Given two elements $x, y$ of the group algebra $\mathbb{Q}[G]$, if $\rho(x) = \rho(y)$, then $x = y$.

We are now in a position to prove Theorem 3.8.

**Proof of Theorem 3.8 Necessity.** We will prove the necessity by induction. Clearly, (i) is true for $|G| = 1$. For the inductive step, assume that (i) is true for solvable groups of order less than $m$. Suppose $|G| = m$. Let $N$ be a minimal normal subgroup of $G$. Then, $N$ is a $p$-subgroup (see Proposition 2.2). If $N = G$, then the result follows (see Example 3.7). Otherwise, if $N \neq G$, consider an arbitrary $\rho \in \text{Irr}^*(G)$. If $N$ is not contained in the kernel of $\rho$, then $\rho$ divides $\alpha(\rho)$ (see Corollary 3.14) and hence, $\alpha(\rho) > 1$. If $N$ is contained in the kernel of $\rho$, then $\rho$ can be viewed as a non-trivial irreducible $(G/N)$-representation. Since $G/N$ is solvable and $|G/N| < m$, by inductive assumption, $\alpha(\rho) > 1$.

**Sufficiency.** Assume (i) is true. Then, for any $\rho \in \text{Irr}^*(G)$, there exists a prime divisor $p$ (depending on $\rho$) such that $\rho(P)$ is a zero matrix for any $P \in \text{Syl}_p(G)$ (see Corollary 3.16). Let $(p_1, \ldots, p_k)$ be a sequence of all distinct prime divisors of $|G|$ (no matter what the order is). Take an arbitrary collection of Sylow subgroups $\{P_i : P_i \in \text{Syl}_{p_i}(G)\}_{i=1}^k$. We claim that $\rho(G) = \rho(P)$, where $P = P_1 \cdots P_k$, for any $\rho \in \text{Irr}(G)$. Indeed,

(a) if $\rho$ is trivial, then $\rho(P) = |G| = \rho(G)$;

(b) if $\rho$ is non-trivial, then since $\rho(P_j)$ is a zero matrix for some $1 \leq j \leq k$ (see Lemma 3.10, Remark 3.12 and Lemma 3.15), so is $\rho(P)$. On the other hand, since $\rho \in \text{Irr}^*(G)$, it follows that $\chi(\hat{G}) = [\chi, 1_G] = 0$ and therefore, $\rho(G)$ is also a zero matrix (see Lemma 3.15).

Then, $\hat{G} = \hat{P}$ (see Proposition 3.17), from which it follows $G = P_1 \cdots P_k$. Since $P_j$ is arbitrarily taken from $\text{Syl}_{p_j}(G)$, it follows from Theorem 2.4 that $G$ is solvable.

**Example 3.18.** Since $A_5$ is not solvable, there exists $\rho \in \text{Irr}^*(A_5)$ such that $\alpha(\rho) = 1$. According to Table 2, this is the only non-trivial irreducible representation of $A_5$ with trivial $\alpha$-characteristic.
3.2 \( \alpha \)-characteristic of Nilpotent Group Representations

If \( G \) is a nilpotent group, then one can strengthen the necessity part of Theorem \( 3.18 \) as follows.

**Proposition 3.19.** If \( G \) is a nilpotent group, then for any \( \rho \in \text{Irr}(G) \),

\[
\text{there exists } v \in S(V) \text{ such that } \alpha(\rho) = |G(v)|. \quad (\dagger)
\]

We say that \( \alpha(\rho) \) is realized by the orbit \( G(v) \) or simply realizable if it satisfies (\dagger). The proof of Proposition 3.19 is based on the following two lemmas.

**Lemma 3.20.** Let \( \rho : A \to \text{GL}(V) \) and \( \psi : B \to \text{GL}(W) \) be two \( G \)-representations. Then, \( \text{lcm}(\alpha(\rho), \alpha(\psi)) \) divides \( \alpha(\rho \otimes \psi) \) and \( \alpha(\rho \otimes \psi) \) divides \( \alpha(\rho) \alpha(\psi) \).

**Proof.** Take \( \tilde{A} := A \times 1 < A \times B \). Then, \((\rho \otimes \psi) \tilde{A}\) is equivalent to the direct sum of \( \text{dim} \, W \) copies of \( \rho \) and hence, \( \alpha((\rho \otimes \psi) \tilde{A}) = \alpha(\rho) \). Now Proposition 3.19(a) implies \( \alpha(\rho) | \alpha(\rho \otimes \psi) \). Similarly, \( \alpha(\psi) | \alpha(\rho \otimes \psi) \). It follows that \( \text{lcm}(\alpha(\rho), \alpha(\psi)) | \alpha(\rho \times \psi) \).

Let \( v \in V, w \in W \) be non-zero vectors. Then, one has \( A_v \times B_w \leq (A \times B)_{v \otimes w} \leq A \times B \), which implies that \(|(A \times B)(v \otimes w)| \) divides \(|A(v)| \cdot |B(w)| \). Therefore, \( \alpha(\rho \otimes \psi) \) divides every product \(|A(v)| \cdot |B(w)| \). This implies that \( \alpha(\rho \otimes \psi) \) divides

\[
\gcd \{|A(v)| \cdot |B(w)| : v \in S(V), w \in S(W)\} = \gcd \{|A(v)| : v \in S(V)\} \cdot \gcd \{|B(w)| : w \in S(W)\} = \gcd \{|A(v)| : v \in S(V)\} \cdot \alpha(\psi) = \alpha(\rho) \alpha(\psi).
\]

\[ \square \]

**Remark 3.21.** If the orders \(|A| \) and \(|B| \) are coprime, then the numbers \( \alpha(\rho), \alpha(\psi) \) are coprime too, and, therefore, \( \text{lcm}(\alpha(\rho), \alpha(\psi)) = \alpha(\rho) \alpha(\psi) \) implying \( \alpha(\rho \otimes \psi) = \alpha(\rho) \alpha(\psi) \). If the group orders are not coprime, then it could happen that \( \alpha(\rho \otimes \psi) \) satisfies \( \text{lcm}(\alpha(\rho), \alpha(\psi)) < \alpha(\rho \otimes \psi) < \alpha(\rho) \alpha(\psi) \). As an example, one could take \( A = B \) to be an extra special group of order \( p^3, p \) is an odd prime. This group has \( p - 1 \) Galois conjugate representations of dimension \( p \). Each of these representations is induced from a one-dimensional representation of \( \mathbb{Z}_p \times \mathbb{Z}_p \), from which it follows that the \( \alpha \)-characteristic of each representation is equal to \( p^2 \). Let \( \rho \) be one of these representations. Then, the irreducible representation \( \rho \otimes \rho \) of \( A \times A \) has \( \alpha \)-characteristic equal to \( p^3 \).

**Lemma 3.22.** Let \( \rho : A \to \text{GL}(V) \) and \( \psi : B \to \text{GL}(W) \) be two \( G \)-representations with \( \gcd \{|A|, |B|\} = 1 \). Then, \(|(A \times B)(v \otimes w)| = |A(v)| \cdot |B(w)| \) for any \( v \in S(V) \) and \( w \in S(W) \).

**Proof.** Suppose \( \gcd \{|A|, |B|\} = 1 \). Note that by definition,

\[
(A \times B)(v \otimes w) = A(v) \otimes B(w) := \{av \otimes bw : a \in A, b \in B\}
\]

for any \( v \in V \) and \( w \in W \). Hence, it suffices to show that the map \( (av, bw) \mapsto av \otimes bw \) is an injection from \( A(v) \times B(w) \) to \( A(v) \otimes B(w) \) for any \( v \in S(V) \) and \( w \in S(W) \).
Suppose that \( a_1 v \otimes b_1 w = a_2 v \otimes b_2 w \) for some \( a_1, a_2 \in A \) and \( b_1, b_2 \in B \). This is true if and only if

\[
a_2 v = \lambda a_1 v \quad \text{and} \quad b_2 w = \lambda^{-1} b_1 w
\]
or, equivalently,

\[
a_1^{-1} a_2 v = \lambda v \quad \text{and} \quad b_2^{-1} b_1 w = \lambda w
\]

for some \( \lambda \in \mathbb{C} \). Hence, \( \text{order}(\lambda) \) divides both \( \text{order}(a_1^{-1} a_2) \) and \( \text{order}(b_2^{-1} b_1) \), which are factors of \( |A| \) and \( |B| \), respectively. It follows that \( \text{order}(\lambda) = 1 \) and hence, \( \lambda = 1 \). Therefore, \( a_1 u = a_2 u \) and \( b_1 u = b_2 u \) and the result follows. \( \square \)

We are now in a position to prove Proposition 3.19.

**Proof of Proposition 3.19.** We use induction on the number \( k \) of distinct prime divisors of \( |G| \), i.e., \( |G| = \prod_{i=1}^{k} p_i^{n_i} \).

If \( k = 1 \), then \( G \) is a \( p \)-group and, therefore, the length of every \( G \)-orbit is a power of \( p \). Hence, \( \alpha(\rho) = \min_{v \in S(V)} \{|G(v)|\} \).

Assume now that \( k \geq 2 \) and \( p_1, \ldots, p_k \) are the prime divisors of \( |G| \). Then, the group \( G \) is a direct product of its Sylow subgroups \( P_i, i = 1, \ldots, k \), where \( P_i \) is a Sylow \( p_i \)-subgroup of \( G \). Therefore, \( G \cong G_1 \times G_2 \) where \( G_1 = P_1 \) and \( G_2 = P_2 \times \cdots \times P_k \).

Pick an arbitrary \( \rho \in \text{Irr}(G) \). Then, \( \rho \) is equivalent to \( \psi_1 \otimes \psi_2 \) where \( \psi_1 \in \text{Irr}(G_1) \) and \( \psi_2 \in \text{Irr}(G_2) \) (see [34]).

Hence, it suffices to show that \( \alpha(\psi_1 \otimes \psi_2) \) is realizable for any \( \psi_j \in \text{Irr}(G_j) \) \( (\psi_j : G_j \to \text{GL}(V_j)), j = 1, 2 \). By inductive assumption, \( \alpha(\psi_j) = |G_j(\psi_j)| \) for some \( \psi_j \in S(V_j) \), \( j = 1, 2 \). Since \( \gcd\{|G_1|, |G_2|\} = 1 \),

\[
|G_1 \times G_2|(v_1 \otimes v_2) = |G_1(v_1)| \cdot |G_2(v_2)| = \alpha(\psi_1) \cdot \alpha(\psi_2)
\]  

(see Lemma 3.22). On the other hand, one has

\[
\alpha(\psi_1) \cdot \alpha(\psi_2) \mid \alpha(\psi_1 \otimes \psi_2) \mid |G_1 \times G_2|(v_1 \otimes v_2)|
\]  

(see Lemma 3.20). Combining (2) and (3) yields

\[
\alpha(\psi_1 \otimes \psi_2) = |G_1 \times G_2|(v_1 \otimes v_2)|.
\]

The result follows. \( \square \)

**Example 3.23.** In general, for a solvable group \( G \), \( \alpha(\rho) \) may not be realizable for some \( \rho \in \text{Irr}(G) \). Let \( \rho \) be the \( 3 \)-dimensional irreducible \( A_4 \)-representation. Then, \( \Phi(\rho) = \{(Z_1), (Z_2), (Z_3)\} \), and \( \alpha(\rho) = 2 \) is not realizable.

**Example 3.24.** If \( G \) is a \( p \)-group, then \( \alpha(\rho) \) is realizable for any \( G \)-representation \( \rho \), which is not the case if \( G \) is a nilpotent group but not a \( p \)-group. In fact, consider the (reducible) \( \mathbb{Z}_6 \)-representation \( \rho := \psi_1 \otimes 1_{\mathbb{Z}_3} \oplus 1_{\mathbb{Z}_2} \otimes \psi_2 \), where \( \psi_1 \) and \( \psi_2 \) are arbitrary non-trivial irreducible representations of \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \), respectively. Then, \( \Phi(\rho) = \{(Z_1), (Z_2), (Z_3)\} \), and \( \alpha(\rho) = 1 \) is not realizable.

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4 \(\alpha\)-characteristic of an augmentation module related to 2-transitive Group actions

The following example is the starting point of our discussion.

Example 4.1. Note that \(A_5\), the smallest non-solvable group, is a 2-transitive group (see Definitions 2.3 and 2.5). To be more explicit, \(A_5\) admits two non-equivalent irreducible augmentation representations \(\psi_3 = \rho_{[A_5; A_4]}\) and \(\psi_4 = \rho_{[A_5; D_5]}\) (see Table 2).

Since \(\alpha(\psi_3) = 5\) while \(\alpha(\psi_4) = 1\), we arrive at the following question: given an augmentation submodule \(\rho \in \varrho_2(G)\) associated to the 2-transitive \(G\)-action on \(G/H\) by left translation, under which conditions does one have \(\alpha(\rho) > 1\)?

4.1 2-transitive Groups

Let \(G\) be a 2-transitive group acting faithfully on a set \(X\). According to Burnside Theorem (see [10, Theorem 4.1B]), the socle \(S\) of \(G\) is either a non-abelian simple group or an elementary abelian group which acts regularly on \(X\). Thus, 2-transitive groups are naturally divided into two classes

- **Almost simple groups.** \(G\) is called almost simple if \(S \leq G \leq \text{Aut}(S)\) for some non-abelian simple group \(S\).
- **Affine groups.** If \(S\) is elementary abelian, then \(G\) admits the following description.

Let \(V\) be a \(d\)-dimensional vector space over a finite field \(\mathbb{F}\). A group \(G\) is called affine if \(V \leq G \leq \text{AGL}(V)\), where \(V\) is considered as an additive group and \(\text{AGL}(V)\) is the group of all invertible affine transformations of \(V\). A group \(G\) admits a decomposition \(G = VG_o\) where \(G_o = G \cap \text{GL}(V)\) is a zero stabilizer in \(G\). Thus, \(G \cong V \rtimes G_o\) (\(\rtimes\) stands for the semidirect product). The group \(G\) acts 2-transitively on \(V\) if and only if \(G_o\) acts transitively on the set of non-zero vectors of \(V\). In this case, \(V\) is the socle of \(G\).

Remark 4.2. Note that a solvable 2-transitive group is always affine. However, the converse is not true: for example, the full affine group \(\text{AGL}(V)\) of the vector space \(V\) is solvable if and only if \(\text{GL}(V)\) is. The latter happens only when \(d = 1\) or \(d = 2\) and \(|\mathbb{F}| \in \{2, 3\}\).

4.2 Main Result

Our main result provides a complete description of all augmentation modules related to 2-transitive group actions with non-trivial \(\alpha\)-characteristic.

**Theorem 4.3.** Let \((G; X)\) be a 2-transitive group action.

(i) If \(G\) is affine and acts faithfully on \(X\), then \(\alpha(\rho_{(G;X)}) > 1\).

(ii) If \(G\) is almost simple, then all \(\rho \in \varrho_2(G)\) satisfying \(\alpha(\rho) > 1\) are described in Table 1 provided that \(|X|\) is a prime power.

The proof of Theorem 4.3 is based on the classification of finite 2-transitive groups (see [10]) and the following lemma.
If \( \text{Proof of Theorem 4.3.} \)

The constituent of the augmentation submodule is non-trivial.

abelian group which acts faithfully on \( X \) if and only if |\( G : H \)| is a prime power. In the latter case, \( p \) divides \( \alpha(\rho_H) \) where \( \rho \) is the unique prime divisor of \( |G : H| \).

**Proof.** If \( |G : H| \) is not a prime power, then for each prime divisor \( p \) of \( |G| \) a Sylow subgroup \( P \in \text{Syl}_p(G) \) has at least two orbits on \( G/H \). Therefore, \( |\rho_H^p| \geq 2 \) implying \( \alpha(\rho_H^p) > 0 \). By Lemma \( 3.10 \), \( p \) does not divide \( \alpha(\rho_H^p) \). Since this holds for any prime divisor of \( |G| \), we conclude that \( \alpha(\rho_H^p) = 1 \).

Conversely, suppose \( |G : H| = p^k \) for some prime \( p \). Then, a Sylow \( p \)-subgroup \( P \in \text{Syl}(G) \) acts transitively on the coset space \( G/H \) implying that \( |\rho_H^p| = 1 \). Therefore \( \alpha(\rho_H^p) = 0 \) and we are done by Lemma \( 3.10 \). \hfill \[ \blacksquare \]

**Remark 4.5.** Note that although any non-trivial irreducible representation of a solvable group admits a non-trivial \( \alpha \)-characteristic (see Theorem \( 3.3 \)), it may not be true for their direct sum (see Proposition \( 3.2(e) \)). However, by Lemma \( 4.4 \) an augmentation submodule associated to a transitive \( G \)-set of order prime power would admit non-trivial \( \alpha \)-characteristic. In particular, the \( \alpha \)-characteristic of every non-trivial irreducible constituent of the augmentation submodule is non-trivial.

Now we can prove the aforementioned Theorem \( 4.3 \).

**Proof of Theorem 4.3.** If \( G \) is an affine 2-transitive group, then its socle \( S \) is an elementary abelian group which acts faithfully on \( X \). By the Burnside Theorem, \( S \) acts regularly on \( X \). Therefore, \( |X| = |S| \) is a prime power and we are done by Lemma \( 4.4 \).

If \( G \) is an almost simple 2-transitive group, then all 2-transitive \( G \)-sets of order prime power are obtained by the inspection of Table 7.4 from \( 10 \), which yields Table \( 1 \). \hfill \[ \blacksquare \]

**Remark 4.6.** For the complete description of 2-transitive faithful actions of affine groups, we refer to Table 7.3 in \( 10 \).

### 4.3 Examples

In this subsection, we will give some concrete examples of 2-transitive groups illustrating Theorem \( 4.3 \) and Remark \( 4.4 \).

**Example 4.7.** The group \( G := A\text{GL}_3(2) = \mathbb{Z}_2^3 \rtimes \text{GL}(3, 2) \) is a non-solvable 2-transitive affine group (see Remark \( 1.2 \)) with four augmentation representations (see Table \( 4 \)) arising from 2-transitive actions. By Theorem \( 4.3(i) \), \( \alpha(\rho) > 1 \) for all \( \rho \in \rho_2(G) \).

**Example 4.8.** The group \( S_5 \) is an almost simple group with three 2-transitive actions: \( S_5/A_5 \), \( S_5/S_4 \) and \( S_5/A\text{GL}_3(5) \). The corresponding augmentation representations are
denoted by \(\xi_1, \xi_3\) and \(\xi_6\) in Table 3. According to Lemma 4.4 only \(\xi_1 = \rho_{[S_5; A_5]}^a\) and \(\xi_6 = \rho_{[S_5; S_4]}^a\) admit non-trivial \(\alpha\)-characteristics.

**Remark 4.9.** In some cases, Theorem 4.3 can still help one to determine whether \(\alpha(\rho)\) is trivial even when \(\rho\) is not an augmentation representation. For example, \(S_5\) admits two 4-dimensional irreducible representations \(\xi_2\) and \(\xi_6\) (see Table 3). Note that \(\xi_6\) is an augmentation representation related to a 2-transitive action while \(\xi_2\) is not. Let \(V\) and \(V^-\) be \(S_5\)-modules corresponding to \(\xi_6\) and \(\xi_2\), respectively. In Section 6 we will show that there exists an admissible equivariant map from \(V^-\) to \(V\) from which it follows that \(\alpha(\xi_2) \geq \alpha(\xi_6) > 1\) (see Example 4.8)—this agrees with Table 3. One can apply similar argument to \((n - 1)\)-dimensional irreducible \(S_n\)-representations with \(n > 5\) being a prime power.

Our last example illustrates Remark 4.4.

**Example 4.10.** Consider the solvable group \(G := SL_2(\mathbb{Z}_3)\) acting transitively (but not 2-transitively, in particular, the augmentation representation is reducible, see Definition 2.5) on the set \(X\) of eight non-zero vectors of \((\mathbb{Z}_3)^2\). It follows from Lemma 4.4 that \(2 | \alpha(\rho_{[G; X]}^a)\). Therefore, 2 divides \(\alpha\)-characteristic of every non-trivial constituent of \(\rho_{[G; X]}^a\) and there are three of those: two 2-dimensional and one 3-dimensional.

## 5 Irreducible representations with trivial \(\alpha\)-characteristic

As we already know, a finite group \(G\) admitting an irreducible complex representation \(\rho\) with trivial \(\alpha\)-characteristic is non-solvable.

In this section, given \(N \trianglelefteq G\) and an irreducible \(G\)-representation (resp. irreducible \(N\)-representation) with trivial \(\alpha\)-characteristics, we study the \(\alpha\)-characteristics of its restriction to \(N\) (resp. induction to \(G\)).

### 5.1 Motivating Examples

Keeping in mind Proposition 3.2 consider the following example.

**Example 5.1.** Consider \(N := A_5 \trianglelefteq S_5 =: G\) \((N\) is simple and \(G \simeq Aut(N))\). The isotypical decomposition of \(\xi_N\) for \(\xi \in \text{Irr}(G)\) are as follows (see also Tables 2 and 3):

\[
\begin{align*}
(\xi_0)_N &= (\xi_1)_N = \psi_0, \\
(\xi_2)_N &= (\xi_6)_N = \psi_3, \\
(\xi_3)_N &= (\xi_5)_N = \psi_4, \\
(\xi_4)_N &= \psi_1 \oplus \psi_2.
\end{align*}
\]

Observe that

(i) \(\alpha(\psi_0) = 1\) divides both \(\alpha(\xi_0) = 1\) and \(\alpha(\xi_1) = 2\);

(ii) \(\alpha(\psi_3) = 5\) divides both \(\alpha(\xi_2) = 10\) and \(\alpha(\xi_6) = 5\);

(iii) \(\alpha(\psi_4) = 1\) divides both \(\alpha(\xi_3) = 1\) and \(\alpha(\xi_5) = 1\);

(iv) both \(\alpha(\psi_1) = 2\) and \(\alpha(\psi_2) = 2\) divide \(\alpha(\xi_4) = 2\).
Remark 5.2. Clearly, Example 5.1 is in the complete agreement with Proposition 3.2 (b). On the other hand, it also gives rise to the following question: under which condition, does $\alpha(\theta) = 1$ imply $\alpha(\theta^G) = 1$ for $\theta \in \text{Irr}(N)$ and $N \leq G$? The answer is given in the next subsection.

5.2 Induction and restriction of representations with trivial $\alpha$-characteristic

Let $N$ be a non-trivial proper subgroup of $G$. Pick an arbitrary $\rho \in \text{Irr}(G)$. By Proposition 3.2 (a), $\alpha(\rho_N)$ divides $\alpha(\rho)$. Therefore, if $\alpha(\rho) = 1$, then $\alpha(\rho_N) = 1$. Decomposing $\rho_N$ into a direct sum of $N$-irreducible representations $\rho_N = \sum_{i=1}^{k} \theta_i$, we obtain $\text{gcd}(\alpha(\theta_1), ..., \alpha(\theta_k)) = 1$ (see Proposition 3.2 (e)). If $N$ is normal in $G$, then all constituents $\theta_i$ are $G$-conjugate by Clifford’s theorem (see [20]), and have the same $\alpha$-characteristic (see Proposition 3.2 (c)). Hence, $\alpha(\theta_i) = 1$ for each $i = 1, ..., k$. In other words, trivial $\alpha$-characteristic of an irreducible $G$-representation $\rho$ is inherited by all constituents of its restriction $\rho_N$. But this does not happen for the induction. More precisely, if $\theta \in \text{Irr}(N)$ has trivial $\alpha$-characteristic, then some of the constituents of $\theta^G$ may have non-trivial $\alpha$-characteristic even in the case of $N$ being normal. For example, take $G = A_5 \times \mathbb{Z}_7$, $N = A_5$ and $\theta = \rho_{A_5 \times D_7}$. Then $\theta^G = \sum_{i=0}^{7} \theta \otimes \lambda^i$ where $\lambda \in \text{Irr}(\mathbb{Z}_7)$. Clearly, $\alpha(\theta \otimes 1_{\mathbb{Z}_7}) = \alpha(\theta) = 1$. By Lemma 3.20, $\alpha(\theta \otimes \lambda^i) = 7$ if $i \neq 0$. Thus, $\theta^G$ contains only one irreducible constituent with trivial $\alpha$-characteristic.

Proposition 5.3. Let $N \leq G$ and $\theta \in \text{Irr}(N)$ with $\alpha(\theta) = 1$. Then, $\alpha(\theta^G) = 1$.

Proof. Let $W$ (resp. $V$) be the $N$-representation (resp. $G$-representation) corresponding to $\theta$ (resp. $\theta^G$). It suffices to show that $\alpha(\theta^G)_p = 1$ for each prime divisor $p$ of $|G|$.

Pick a Sylow $p$-subgroup $P \leq G$. Then, $P \cap N$ is a Sylow $p$-subgroup of $N$. It follows from $\alpha(\theta) = 1$ that the subspace $W_1 := W^{P/N}$ is non-trivial (see Lemma 3.11). Pick an arbitrary non-zero $w \in W_1$. Then, the vector $v := \sum_{g \in P} gw$ is fixed by any element of $P$, that is, $Pv = v$. We claim that $v \neq 0$. Let $T_1$ be a transversal of $P/(P \cap N)$. By isomorphism $P/(P \cap N) \cong PN/N$, the set $T_1$ is a transversal of $PN/N$. Now we complete $T_1$ to a transversal $T$ of $G/N$ and set $V = \oplus_{t \in T} tw$.

Now $P = T_1(P \cap N)$ implies $v = |P \cap N| \sum_{t \in T_1} tw \neq 0$. Thus, $V^P$ is non-trivial, and, consequently, $\alpha(\theta^G)_p = 1$. \hfill $\Box$

In general, it is not clear whether $\alpha(\theta) = 1$ implies that $\theta^G$ contains an irreducible constituent with trivial $\alpha$-characteristic. Proposition 5.3 below provides sufficient conditions for that. Its proof is based on the following lemma (see [20]).

Lemma 5.4. Suppose $N \leq G$. Let $\chi$ and $\omega$ be irreducible characters of $G$ and $N$, respectively, such that $[\chi_N, \omega] > 0$. If $p = |G : N|$ is a prime, then for the decompositions of $\chi_N$ and $\omega^G$, one of the following two statements takes place:

(a) $\chi_N = \sum_{i=0}^{p-1} \omega(g_i)$ and $\omega^G = \chi$.

(b) $\chi_N = \omega$ and $\omega^G = \sum_{i=0}^{p-1} \chi \phi_i$, where $\{\phi_i\}_{i=0}^{p-1}$ is the set of all irreducible characters of $G/N \cong \mathbb{Z}_p$, which can be viewed as irreducible characters of $G$ as well.
Proposition 5.5. Let \( N \leq G \) and \( \omega \in \text{Irr}(N) \) with \( \alpha(\omega) = 1 \). If \( G/N \) is solvable then, \( \omega^G \) admits an irreducible component \( \rho \) satisfying \( \alpha(\rho) = 1 \).

Proof. We use induction over \( |G/N| \). Pick a maximal normal subgroup \( M \) of \( G \) which contains \( N \). Then, \( M/N \) is a maximal normal subgroup of \( G/N \), and, by solvability of \( G/N \), \( M \) is of prime index, say \( p \), in \( G \). If \( M \neq N \), then, by induction hypothesis, the representation \( \omega^M \) has an irreducible component, say \( \sigma \), with \( \alpha(\sigma) = 1 \). Applying induction hypothesis to the pair \( M \leq G \) and \( \sigma \) we conclude that \( \sigma^G \) contains an irreducible component \( \rho \in \text{Irr}(G) \) with \( \alpha(\rho) = 1 \). Now it follows from \( \omega^G = (\omega^M)^G \) that \( \rho \) is a constituent of \( \omega^G \).

Assume now that \( M = N \), that is \( |G : N| = p \) is prime. By Lemma 5.3 either \( \omega^G \) is irreducible or \( \omega^G = \sum_{i=0}^{p-1} \chi_i \) (hereafter, \( \omega, \chi, \zeta \) stand for characters rather than for representations). In the first case we are done by Proposition 5.3. Consider now the second case:

\[
\omega^G = \sum_{i=0}^{p-1} \chi_i. \quad \text{In this case, in suffices to show that there exists } 0 \leq j \leq p-1 \text{ such that } \\
\chi_j(R) > 0 \text{ for any prime factor } r \text{ of } |G| \text{ and } R \in \text{Syl}_r(G). \quad \text{Indeed, note that } \sum_{i=0}^{p-1} \chi_i(g) = p \text{ if } g \in N \text{ and } 0 \text{ otherwise. Therefore,} \\
\omega^G(R) = \sum_{i=0}^{p-1} \chi_i(R) = \sum_{g \in R} \sum_{i=0}^{p-1} \chi_i(g) = \sum_{g \in R} \chi(g) \left( \sum_{i=0}^{p-1} \xi_i(g) \right) = \sum_{g \in R \cap N} \omega(g) = p \omega(R \cap N).
\]

Since \( R \cap N \) is either trivial or a Sylow \( r \)-subgroup of \( N \), one always has

\[
\sum_{i=0}^{p-1} \chi_i(R) = p \omega(R \cap N) > 0.
\]

If \( r = p \), then \( \chi_j(R) > 0 \) for some \( 0 \leq j \leq p-1 \). If \( r \neq p \), then \( R \leq N \) and it follows that \( \chi_j(R) = \chi(R) > 0 \) for the same \( j \) as well. The result follows.

The following example illustrates Proposition 5.5.

Example 5.6. Consider \( N := \text{PSL}(2,8) \leq \text{PGL}(2,8) =: G \) (\( N \) is simple, \( G \cong \text{Aut}(N) \) and \( |G : N| = 3 \) is a prime). The relation between \( \text{Irr}(G) \) and \( \text{Irr}(N) \) with respect to restriction and induction are described as follows (see also Tables 5 and 6):

\[
\begin{align*}
\xi_0 N &= \xi_1 N = \xi_2 N = \psi_0, & \psi_0^G &= \xi_0 \oplus \xi_1 \oplus \xi_2, \\
\xi_3 N &= \xi_4 N = \xi_5 N = \psi_1, & \psi_1^G &= \xi_3 \oplus \xi_4 \oplus \xi_5, \\
\xi_6 N &= \xi_7 N = \xi_8 N = \psi_5, & \psi_5^G &= \xi_6 \oplus \xi_7 \oplus \xi_8, \\
\xi_9 N &= \psi_2 \oplus \psi_3 \oplus \psi_4, & \psi_2^G &= \psi_2 \oplus \psi_4 \oplus \xi_9, \\
\xi_{10} N &= \psi_6 \oplus \psi_7 \oplus \psi_8. & \psi_6^G &= \psi_7 = \psi_8 = \xi_{10}.
\end{align*}
\]

One can observe that \( \alpha(\psi_0) = \alpha(\psi_7) = \alpha(\psi_8) = \alpha(\xi_{10}) = 1 \), which agrees with Proposition 5.5.
5.3 Groups with totally trivial $\alpha$ characteristic

This section is devoted to the groups which have no irreducible representation with non-trivial $\alpha$-characteristic. In what follows we denote this class of groups as $\mathcal{T}$. We know that this class is non-empty, since $J_1 \in \mathcal{T}$. We also know that all groups in $\mathcal{T}$ are non-solvable. Below we collect elementary properties of $\mathcal{T}$.

**Proposition 5.7.** Let $G \in \mathcal{T}$. Then

(a) If $N$ is a normal subgroup of $G$, then $N, G/N \in \mathcal{T}$;

(b) All composition factors of $G$ belong to $\mathcal{T}$;

(c) If $H \in \mathcal{T}$, then $G \times H \in \mathcal{T}$

**Proof.** Part (a). The inclusion $G/N$ follows from the fact that every irreducible representation of $G/N$ may be considered as an irreducible representation of $G$. The inclusion $N \in \mathcal{T}$ follows from Proposition 3.2 (b).

Part (b) is a direct consequence of (a).

Part (c) is a direct consequence of Lemma 3.20, since each irreducible representation of $G \times H$ is a tensor product $\psi \otimes \phi$ where $\psi \in \text{Irr}(G), \phi \in \text{Irr}(H)$.

Our computations in GAP suggest the following conjecture.

**Conjecture.** If $G \in \mathcal{T}$ is a simple group, then it is one of the sporadic groups.

6 Existence of Quadratic Equivariant Maps

6.1 General Construction

In general, the problem of existence of equivariant maps between $G$-manifolds is rather complicated. We will study Problem B in the following setting: $V$ is a faithful irreducible $G$-representation and $W$ is another $G$-representation of the same dimension. It is well-known that if $V$ is faithful, then there exists a positive integer $k$ such that the symmetric tensor power $\text{Sym}^k(V)$ contains $W$ (see, for example, [20]). Thus, assume $W \subset \text{Sym}^k(V)$ and let

\[
\begin{cases}
\Delta : V \rightarrow V^\otimes k, \\
\Delta(v) = v \otimes v \otimes \cdots \otimes v.
\end{cases}
\]

be the corresponding diagonal map. Observe that $\Delta$ is $G$-equivariant and $\Delta(V)$ spans $\text{Sym}(V)$. Let $A : \text{Sym}^k(V) \rightarrow W$ be a $G$-equivariant linear operator (e.g. orthogonal projection). Then, $\phi = A \circ \Delta$ is a $k$-homogeneous $G$-equivariant map from $V$ to $W$, which admits the following criterion of admissibility (see, for example, [29]).

**Proposition 6.1.** $\phi$ is admissible if and only if $\ker A \cap \Delta(V) = \{0\}$.

In practice, given characters of $V$ and $W$, constructing an admissible homogeneous equivariant map $\phi : V \rightarrow W$ involves the following steps:
Finding \( k \) satisfying \( W \subset \text{Sym}^k(V) \).

Computing matrices representing the \( G \)-actions on \( V \) and \( \text{Sym}^k(V) \), and finding isotypical basis of \( W \subset \text{Sym}^k(V) \).

Verification of the admissibility of \( \phi = A \circ \triangle \) by the Weak Nullstellensatz (see [13]):

**Proposition 6.2.** Let \( Q := \{ q_i \}_{i=1}^m \subset R := \mathbb{C}[x_1, \ldots, x_N] \) be a collection of polynomials and \( I := \{ \sum_{i=1}^m r_i q_i : r_i \in R \} \) the ideal generated by \( Q \). Then, the following statements are equivalent:

- \( Q \) admits no common zeros.
- The Gröbner basis of \( I \) contains the constant polynomial 1.

**Remark 6.3.**

(i) Steps (1) and (2) are related to the classical Clebsch-Gordan problem of an isotypical decomposition of tensor product of representations (see, for example, [12]).

(ii) For Step (S3), one can use Mathematica (see [39]) to compute the Gröbner basis.

(iii) If \( \phi \) is admissible, then \( \deg(\phi) \) is well-defined and equal to \( kn \), where \( n = \dim(V) \).

To illustrate Proposition 6.1 and also give a brief idea about Steps (S1) – (S3), consider the following example (see also [3] and [4]); the details about these steps will be provided later (see Section 6.2). In what follows, denote by \((V,G)\) a faithful \( G \)-representation and \((V,G/H)\) a non-faithful \( G \)-representation with kernel \( H \triangleleft G \).

**Example 6.4.** Consider the symmetric tensor square \((\text{Sym}^2 \mathbb{C}(H), \mathbb{Q}_8/\mathbb{Z}_2)\) of the complex representation \((H, \mathbb{Q}_8)\). One can easily check that

\[
e_1 = \frac{1 \otimes 1 + j \otimes j}{2}, \quad e_2 = \frac{1 \otimes 1 - j \otimes j}{2}, \quad e_3 = \frac{j \otimes 1 + j \otimes 1}{2}
\]

form an isotypical basis of \( \text{Sym}^2 \mathbb{C}(H) \) and

\[
\triangle(z_1 + jz_2) = e_1(z_1^2 + z_2^2) + e_2(z_1^2 - z_2^2) + e_3(2z_1z_2).
\]

Let \( P_1, P_2 \) and \( P_3 \) denote the natural \( \mathbb{Q}_8 \)-equivariant projections onto the subspaces of \( \text{Sym}^2 \mathbb{C}(H) \) spanned by \{\( e_1, e_2 \}\} \{\( e_2, e_3 \}\} and \{\( e_1, e_3 \}\}, respectively. A direct computation shows that \( \ker P_i \cap \triangle(H) = \{0\} \) for \( i = 1, 2, 3 \). Consequently, \( f_i = P_i \circ \triangle, i = 1, 2, 3, \) are admissible \( \mathbb{Q}_8 \)-equivariant maps.

**Remark 6.5.** Example 6.4 shows the existence of an admissible 2-homogeneous \( \mathbb{Q}_8 \)-equivariant map \( f_1 : (H, \mathbb{Q}_8) \to (\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2) \), where \( \mathbb{C}^2 \subset \text{Sym}^2 \mathbb{C}(H) \) is the subrepresentation spanned by \{\( e_1, e_2 \)\}. In addition, \( \alpha(\mathbb{Q}_8, H) = 8 \) since \( \mathbb{Q}_8 \) acts freely on \( S(H) \). Therefore, it follows from the congruence principle that for any admissible \( \mathbb{Q}_8 \)-equivariant map \( f : (H, \mathbb{Q}_8) \to (\mathbb{C}^2, \mathbb{H}/\mathbb{Z}_2) \),

\[
\deg(f) \equiv \deg(f_1) = 2^2 = 4 \quad (\text{mod } 8).
\]

In particular, \( \deg(f) \) is different from 0.
In addition to the congruence principle, one can also analyze Example 6.4 by the following result (see [2]).

**Theorem 6.6** (Atiyah-Tall). Let $G$ be a finite $p$-group and $V$ and $W$ two $G$-representations. There exists an admissible equivariant map $f : V \to W$ with $\deg(f) \not\equiv 0 \pmod{p}$ if and only if the irreducible components of $V$ and $W$ are Galois conjugate in pairs.

**Remark 6.7.** Since $(\mathbb{H}, \mathbb{Q}_8)$ is irreducible while $(\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2)$ is not, the irreducible components of $(\mathbb{H}, \mathbb{Q}_8)$ and $(\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2)$ are not Galois conjugate in pairs. It follows from Theorem 6.6 that

$$\deg(f) \equiv 0 \pmod{2}$$

for any admissible $\mathbb{Q}_8$-equivariant map $f : (\mathbb{H}, \mathbb{Q}_8) \to (\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2)$.

A comparison between Remark 6.5 and Remark 6.7 shows that the result for Example 6.4 obtained from the congruence principle is more informative.

Possible extensions of Example 6.4 to arbitrary $p$-groups were suggested by A. Kushkuley (see [4]). On the other hand, notice that $(\mathbb{H}, \mathbb{Q}_8)$ (resp. $(\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2)$) is induced by the one-dimensional representation $(\mathbb{C}, \mathbb{Z}_4)$ (resp. $(\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$). Furthermore, $\psi(z) = z^2$ is an admissible 2-homogeneous $\mathbb{Z}_4$-equivariant map from $(\mathbb{C}, \mathbb{Z}_4)$ to $(\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$ and $f_1$ (see Example 6.4) is in fact the $\mathbb{Q}_8$-equivariant extension of $\psi$ (see Figure 1).

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (H) at (0,0) {$(\mathbb{H}, \mathbb{Q}_8)$ \ni $g \mapsto g^2$};
  \node (V) at (2,0) {$(\mathbb{C}, \mathbb{Z}_4)$ \ni $z \mapsto z^2 \in (\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$};
  \node (U) at (0,-1) {$(\mathbb{H}, \mathbb{Q}_8) \ni g \mapsto g^2 \in (\mathbb{C}^2, \mathbb{Q}_8/\mathbb{Z}_2)$};
  \node (V2) at (2,-1) {$(\mathbb{C}, \mathbb{Z}_4/\mathbb{Z}_2)$ \ni $z \mapsto z^2$};
  \draw[->] (H) -- (V) node[midway, above] {$f_1$};
  \draw[->] (H) -- (U) node[midway, left] {induced representation};
  \draw[->] (V) -- (V2) node[midway, above] {induced representation};
  \draw[->] (U) -- (V2) node[midway, left] {$\psi$};
\end{tikzpicture}
\caption{$f_1$ as an extension of $\psi$}
\end{figure}

The above example shows that if a $G$-representation $V$ is induced from an $H$-representation $U$ ($H < G$), and $f$ is an $H$-equivariant homogeneous admissible map defined on $U$, then $f$ can be canonically extended to a $G$-equivariant homogeneous admissible map defined on $V$. However, the construction of an admissible homogeneous $G$-equivariant map becomes more involved if we are given a representation which is not induced from a subgroup. The example considered in the next subsection suggests a method to deal with this problem in several cases.

### 6.2 Example: $S_5$-representations

In this subsection, we will construct an admissible 2-homogeneous equivariant map from $V^-$ to $V$ (see Remark 4.9) following Steps (S1)–(S3), which will be illustrated in detail. In what follows, for an $S_5$-representation $X$ and $\sigma \in S_5$, denote by $\rho_X(\sigma)$ and $\chi_X(\sigma)$ the corresponding matrix representation and character, respectively.

(S1) Denote $U := \text{Sym}^2(V^-)$. Recall that $\chi_U(\sigma) = \frac{1}{2} (\chi_{V^-}(\sigma)^2 + \chi_{V^-}(\sigma^2))$ (see, for example, [34]) and using Table 8 one has $U = 1_{S_5} \oplus V \oplus V_5$.  

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Proposition 6.8. To summarize, one has the criterion provided by Proposition 6.1, which involves computing the orthogonal projection matrix and verifying the admissibility of the 2-homogeneous equivariant map. To be more explicit, let \( B_{V^-} := \{ e_i \}_{i=1}^4 \) of \( V^- \) and \( B_U := \{ e_i \otimes e_j \}_{1 \leq i \leq j \leq 4} \) of \( U \). To obtain a basis \( \rho_U(\sigma) \) corresponding to \( B_U, \sigma \in S_5 \), it suffices to let \( \rho_{V^-}((12)) := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \rho_{V^-}((12345)) := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \) be the matrices corresponding to \( B_{V^-} \), and use formula
\[
\rho_U(\sigma)(e_i \otimes e_j) = \rho_{V^-}(\sigma)(e_i) \otimes \rho_{V^-}(\sigma)(e_j).
\]
Substitution of \( \rho_U(\sigma) \), \( \sigma \in S_5 \), in (6) yields
\[
A = \begin{pmatrix}
\begin{array}{cccccccc}
\phi & -\phi & -\phi & -\phi & -\phi & -\phi & -\phi & -\phi \\
-\phi & \phi & -\phi & -\phi & -\phi & -\phi & -\phi & -\phi \\
-\phi & -\phi & \phi & -\phi & -\phi & -\phi & -\phi & -\phi \\
-\phi & -\phi & -\phi & \phi & -\phi & -\phi & -\phi & -\phi \\
-\phi & -\phi & -\phi & -\phi & \phi & -\phi & -\phi & -\phi \\
-\phi & -\phi & -\phi & -\phi & -\phi & \phi & -\phi & -\phi \\
-\phi & -\phi & -\phi & -\phi & -\phi & -\phi & \phi & -\phi \\
-\phi & -\phi & -\phi & -\phi & -\phi & -\phi & -\phi & \phi \\
\end{array}
\end{pmatrix}
\]

(S3) The column vectors of the projection matrix \( A \) span \( V \) and one can obtain a basis \( B_V \) of \( V \subset U \) using the Gram-Schmidt process. With \( B_{V^-} \) and \( B_V \), let \( \phi := A \circ \Delta \) be viewed as a map from \( C^4 \) to \( C^4 \). To be more explicit, \( \phi = [\phi_1, \ldots, \phi_4]^T \), where \( \phi_i = \phi_i(x_1, x_2, x_3, x_4) \), \( i = 1, \ldots, 4 \), is a 2-homogeneous polynomial. Denote \( P := \{ \phi_i \}_{i=1}^4 \) and \( P_k := [\phi_i | x_k = 1]_{i=1}^4 \). Then, \( \phi \) admits no non-trivial zeros, i.e., \( P \) admits no non-trivial common zeros, if and only if \( P_k \) admits no common zeros, i.e., the Gröbner basis of the ideal generated by \( P_k \) contains only 1, for \( k = 1, 2, 3, 4 \) (see Proposition 6.2). One can use Mathematica to show that it is indeed the case and thereby \( \phi \) is admissible.

To summarize, one has

**Proposition 6.8.** There exists an admissible 2-homogeneous \( S_5 \)-equivariant map \( \phi : V^- \to V \).

The construction of an admissible 2-homogeneous equivariant map \( \phi \) in Proposition 6.8 which involves computing the orthogonal projection matrix and verifying the criterion provided by Proposition 6.1 is an ad hoc approach; it is difficult to obtain a global result for arbitrary \( S_n \) by this kind of constructions. In fact, for applications similar to Corollary 7.1, it suffices to know that an admissible homogeneous equivariant map exists while its explicit formula is not necessary. In the next two subsections, we will employ a more universal technique to extend Proposition 6.8 to \( S_n \) for arbitrary odd \( n \).
6.3 Extension of Proposition 6.8

Let us describe explicitly the setting to which we want to extend Proposition 6.8.

Theorem 6.9. Let \((S_n; [n])\) be the natural action of the symmetric group \(S_n\) on the set \([n] = \{1, \ldots, n\}\) and \(V, V^-\) be the modules corresponding to the irreducible representations \(\rho_{(S_n; [n])}^a, \rho_{(S_n; [n])}^b \otimes 1_{S_n}\), respectively (\(1_{S_n}\) is the sign representation). Assume that \(n\) is odd. Then, there exists an admissible 2-homogeneous equivariant map from \(V^-\) to \(V\).

The following statement is a starting point for proving Theorem 6.9.

Proposition 6.10. Let \(V, V^-\) be as in Theorem 6.9 and \(W\) an arbitrary \(S_n\)-representation. Then, there exists an admissible 2-homogeneous equivariant map from \(V^-\) to \(W\) if and only if there exists an admissible 2-homogeneous equivariant map from \(V\) to \(W\).

Proof. By taking the standard basis in \(V\) (resp. \(V^-\)), any map defined on \(V\) (resp. \(V^-\)) can be identified with a map on \(\mathbb{C}^{n-1}\). Let \(\rho_V(\sigma) (\sigma \in S_n)\) be matrices representing the \(S_n\)-action on \(V\). Since \(V^- = V \otimes 1_{S_n}\), one can use the simple character argument to show that the formula

\[
\rho_{V^-}(\sigma) := \begin{cases} 
\rho_V(\sigma), & \text{if } \sigma \in A_n, \\
-\rho_V(\sigma), & \text{if } \sigma \in S_n \setminus A_n,
\end{cases}
\]

(7)

defines matrices representing the \(S_n\)-action on \(V^-\).

Assume that \(\phi : V \rightarrow W\) is an admissible 2-homogeneous equivariant map. Then,

\[
\phi(\rho_{V^-}(\sigma)v) = \begin{cases} 
\phi(\rho_V(\sigma)v), & \sigma \in A_n \\
\phi(-\rho_V(\sigma)v), & \sigma \in S_n \setminus A_n
\end{cases} = \phi(\rho_V(\sigma)v) = \rho_W(\sigma)\phi(v).
\]

Therefore, \(\phi\) can be viewed as an admissible 2-homogeneous equivariant map from \(V^-\) to \(W\) as well. Similarly, one can show that if \(\psi : V^- \rightarrow W\) is an admissible 2-homogeneous equivariant map, then \(\psi\) can be viewed as an admissible 2-homogeneous equivariant map from \(V\) to \(W\) as well. The result follows.

Proposition 6.10 reduces the proof of Theorem 6.9 to providing an admissible 2-homogeneous equivariant map from \(V\) to \(V\). Clearly, the later problem is equivalent to the existence of a bi-linear commutative (not necessarily associative) multiplication \(\star : V \times V \rightarrow V\) commuting with the \(G\)-action on \(V\) such that the algebra \((V, \star)\) does not have 2-nilpotents. In the next subsection, the existence of such multiplication will be studied using the Norton algebra techniques.

6.4 Norton Algebras without 2-nilpotents

In this subsection, we will recall the construction of the Norton Algebra (see also [11]) and apply related techniques to prove Theorem 6.9.

Let \(\Omega\) be a finite \(G\)-set (\(|\Omega| = n\)) and \(U\) the associated permutation representation. With the standard basis \(\{e_g\}_{g \in G}\): \(u \in U\) can be viewed as a vector in \(\mathbb{C}^n\) and, hence, \(U\) is endowed with the natural componentwise multiplication \(u \cdot v := [u_1v_1, \ldots, u_nv_n]^T\) and the scalar product \(\langle u, v \rangle := \sum_{i=1}^n u_i \overline{v_i}\) for \(u = [u_1, \ldots, u_n]^T\) and \(v = [v_1, \ldots, v_n]^T\). Then,
(U, ·) is a commutative and associative complex algebra with the $G$-action commuting with the multiplication "·".

Let $W \subset U$ be a non-trivial $G$-invariant subspace. Denote by $P : U \rightarrow W$ the orthogonal projection with respect to $\langle \cdot, \cdot \rangle$ and define the Norton algebra $(W, \ast)$ as follows: $w_1 \ast w_2 := P(w_1 \cdot w_2)$ for any $w_1, w_2 \in W$. It is clear that the Norton algebra is commutative but not necessarily associative complex algebra with the $G$-action commuting with the multiplication $\ast$. In particular, the quadratic map $w \mapsto w \ast w$ is $G$-equivariant on $W$.

In connection to Theorem 6.9 consider $G = S_n$. Recall that $S_n$ acts naturally on $N_n := \{1, \ldots, n\}$ by permutation. Let $\Omega$ be the set of all two-element subsets of $N_n$, i.e., $\Omega := \{(i, j) : 1 \leq i < j \leq n\}$, on which $S_n$ acts by $\sigma(i, j) = (\sigma(i), \sigma(j))$ for any $\sigma \in S_n$ and $(i, j) \in \Omega$. In this case, the permutation representation associated to $\Omega$ is $U = 1_{S_n} \oplus W \oplus W'$, where $1_{S_n}$, $W$ and $W'$ are irreducible $S_n$-representations with $\dim(1_{S_n}) = 1$, $\dim(W) = n - 1$ and $\dim(W') = n(n - 3)/2$.

Denote by $B := \{e_{ij} : 1 \leq i < j \leq n\}$ the standard basis of $U$ and let $f_i := \sum_{j \neq i} e_{ij}$. Since $\{f_i : i \in N_n\}$ is linearly independent and $\sigma(f_i) = f_{\sigma(i)}$ for any $\sigma \in S_n$ and $i \in N_n$, $\{f_i : i \in N_n\}$ forms a basis of an $S_n$-subrepresentation $F$. To be more explicit, $F = 1_{S_n} \oplus W$, where $1_{S_n} = \{z \sum_{i=1}^n f_i : z \in \mathbb{C}\}$ and $W = \{\sum_{i=1}^n z_i f_i : z_i \in \mathbb{C}, \sum_i z_i = 0\}$. Note that $\{f_n - f_i : j \in N_{n-1}\}$ forms a basis of $W$. Then, $(W, \ast)$ is a Norton algebra satisfying the following property.

**Proposition 6.11.** The Norton algebra $(W, \ast)$ admits 2-nilpotents if and only if $n$ is even. In such a case, $w \ast w = 0$ if and only if $w = \alpha(\sum_{i \in I} f_i - \sum_{i \notin I} f_i)$ for some $\alpha \in \mathbb{C}$ and $I \subset N_n$ with $|I| = n/2$.

**Proof.** Let $P : U \rightarrow W$ be the orthogonal projection. We have to solve the equation $P(w^2) = 0$ for $w \in W$. Note that $P(w^2) = 0$ if and only if $\langle w^2, f_n - f_i \rangle = 0$ for any $i \in N_{n-1}$ or, equivalently,

$$\langle w^2, f_i \rangle = c \quad (8)$$

for any $i \in N_n$ and some $c \in \mathbb{C}$. Set $w = \sum_{i=1}^n z_i f_i$ with $\sum_{i=1}^n z_i = 0$. Then,

$$w = \sum_{1 \leq i < j \leq n} (z_i + z_j) e_{ij},$$

$$w^2 = \sum_{1 \leq i < j \leq n} (z_i + z_j)^2 e_{ij}.$$ 

On the other hand, since

$$\langle e_{ij}, f_k \rangle = \begin{cases} 1, & k \in \{i, j\}, \\ 0, & k \notin \{i, j\}, \end{cases}$$

$$\langle w^2, f_k \rangle = \sum_{i \neq k} (z_i + z_k)^2$$

and it follows from (8) that

$$c = \sum_{i \neq k} (z_i + z_k)^2 = \sum_{i=1}^n (z_i + z_k)^2 = 4z_k^2 = \sum_{i=1}^n z_i^2 + 2z_k \sum_{i=1}^n z_i + (n - 4) z_k^2 = \sigma_2 + (n - 4) z_k^2,$$ 

where

$$\sigma_2 = \sum_{i=1}^n z_i^2.$$ 

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where \( k \in \mathbb{N} \) and \( \sigma_2 := \sum_{i=1}^{n} z_i^2 \). Hence, \( z_k^2 \) is independent of \( k \) and \( z_k^2 = \sigma_2/n \). Let \( \alpha \) be an arbitrary (complex) root of the equation \( z^2 = \sigma_2/n \). Then, \( z_k = \pm \alpha \). Denote by \( I \subset \mathbb{N} \) the set of indices \( k \) such that \( z_k = \alpha \). Since \( \sum_{i=1}^{n} z_i = 0 \), one has either

- \( \alpha = 0 \), or
- \( \alpha \neq 0 \), \( n \) is even and \( |I| = n/2 \).

The result follows.

**Proof of Theorem 6.9** The result simply follows from Proposition 6.10 and Proposition 6.11.

---

### 7 Applications to Congruence Principle

#### 7.1 The Brouwer Degree

Recall the construction of the Brouwer degree. Let \( M \) and \( N \) be compact, connected, oriented \( n \)-dimensional manifolds (without boundary), and let \( f : M \to N \) be a smooth map. Let \( y \in N \) be a regular value of \( f \). Then, \( f^{-1}(y) \) is either empty (in which case, define the Brouwer degree of \( f \) to be zero), or consists of finitely many points, say \( x_1, \ldots, x_k \). In the latter case, for each \( i = 1, \ldots, k \), take the tangent spaces \( T_{x_i}M \) and \( T_yN \) with the corresponding orientations. Then, the derivative \( D_{x_i}f : T_{x_i}M \to T_yN \) is an isomorphism. Define the Brouwer degree by the formula

\[
\deg(f) = \deg(f, y) := \sum_{i=1}^{k} \text{sign}(\det(D_{x_i}f)).
\]

(9)

It is possible to show that \( \deg(f, y) \) is independent of the choice of a regular value \( y \in N \) (see, for example, [33, 15, 32]). If \( f : M \to N \) is continuous, then one can approximate \( f \) by a smooth map \( g : M \to N \) and take \( \deg(g) \) to be the Brouwer degree of \( f \) (denoted \( \deg(f) \)). Again, \( \deg(f) \) is independent of a close approximation.

Finally, let \( M \) be as above and let \( W \) be the oriented Euclidean space such that \( \dim M = \dim W - 1 \). Given a continuos map \( f : M \to W \setminus \{0\} \), define the map \( \tilde{f} : M \to S(W) \) by \( \tilde{f}(x) = \frac{f(x)}{\|f(x)\|} \) (\( x \in M \)). Then, \( \deg(\tilde{f}) \) is correctly defined. Set \( \deg(f) := \deg(\tilde{f}) \) and call it the Brouwer degree of \( f \). In particular, if \( V \) and \( W \) are oriented Euclidean spaces of the same dimension and \( f : V \to W \) is admissible, then \( f \) takes \( S(V) \) to \( W \setminus \{0\} \). Define the Brouwer degree of \( f \) by \( \deg(f) := \deg(f|_{S(V)}) \).

#### 7.2 Congruence Principle for Solvable Groups

Combining Theorem 3.8 and the congruence principle, one immediately obtains the following result.

**Corollary 7.1.** Let \( G \) be a solvable group and let \( V \) and \( W \) be two \( n \)-dimensional representations. Assume, in addition, that \( V \) is non-trivial and irreducible, and suppose that there exists an equivariant map \( \Phi : S(V) \to W \setminus \{0\} \). Then, \( \alpha(V) > 1 \) and for any equivariant map \( \Psi : S(V) \to W \setminus \{0\} \), one has

\[
\deg(\Psi) \equiv \deg(\Phi) \pmod{\alpha(V)}
\]

(10)
In addition, one has the following

**Corollary 7.2.** Let \( G \) be a solvable group and \( V, W \in \text{Irr}^*(G) \). If \( V \) and \( W \) are Galois-equivalent, then \( \alpha(V) > 1 \) and \( \deg(f) \not\equiv 0 \pmod{\alpha(V)} \) for any \( G \)-equivariant map \( f : S(V) \to S(W) \).

**Proof.** Take a \( G \)-equivariant map \( f : S(V) \to S(W) \). Since \( V \) and \( W \) are Galois-equivalent, one has \( \dim V^H = \dim W^H \) for any \( H < G \). So, there exists a \( G \)-equivariant map \( g : S(W) \to S(V) \) (see, for example, [5,36,29]). In this case, \( g \circ f : S(V) \to S(V) \) is a \( G \)-equivariant map and, by the congruence principle, \( \deg(g \circ f) \equiv 1 \pmod{\alpha(V)} \). Since \( \alpha(V) > 1 \) (see Theorem 3.8), the result follows.

**Corollary 7.3.** Let \( G \) be a solvable group, \( W \) an \( n \)-dimensional irreducible (complex) \( G \)-representation and \( M \) a (real) compact, connected, oriented smooth \( 2n-1 \)-dimensional \( G \)-manifold. Assume, in addition, that \( \dim_{\mathbb{R}} M^H \leq \dim_{\mathbb{R}} W^H - 1 \) for any \((H) \in \Phi(G, M)\).

Then:

(i) there exists an equivariant map \( f : M \to W \setminus \{0\} \);

(ii) \( \alpha(M) := \gcd \{|G(x)| : x \in M \} > 1 \);

(iii) for any equivariant map \( g : M \to W \setminus \{0\} \), one has

\[
\deg(g) \equiv \deg(f) \pmod{\alpha(M)}.
\]

**Proof.** By condition (11), there exists an equivariant map \( f : M \to W \setminus \{0\} \) (see, for example, [36,29]). Therefore, \( G_{f(x)} \geq G_x \) so that \( |G(f(x))| \) divides \( |G(x)| \) for any \( x \in M \). Since \( \alpha(W) > 1 \) (see Theorem 3.8), it follows that \( \alpha(M) > 1 \). Finally, the congruence principle shows that (12) is true.

**Remark 7.4.** By combining Theorem 3.8 with other versions of the congruence principle given in [29], one can easily obtain many other results on degrees of equivariant maps for solvable groups. We leave this task to a reader.

### 7.3 Congruence Principle for \( S_n \)-representations

In this subsection, we will study the Brower degree of equivariant maps from \( S(V^-) \) to \( S(V) \), where \( V \) and \( V^- \) are as in Theorem 6.9. To this end, we need the following proposition.

**Proposition 7.5.** Let \( V \) and \( V^- \) be \( S_n \)-modules as in Theorem 6.9 with \( n = p^k > 3 \) being an odd prime power. Then:

(i) \( \alpha(V) = p \);

(ii) \( \alpha(V^-) = 2p \).
Proof. (i) According to Lemma 4.4, one has that \( p \) divides \( \alpha(V) \). Hence, it suffices to show that \( V \setminus \{0\} \) admits two \( S_n \)-orbits, say \( O_1 \) and \( O_2 \), such that \( \gcd(|O_1|, |O_2|) = p \).

Indeed, let \( O_1 = S_n(x) \) and \( O_2 = S_n(y) \) be two \( S_n \)-orbits in \( V \setminus \{0\} \) with

\[
\begin{align*}
x &= (n - 1, -1, \ldots, -1), \\
y &= (p - 1, \ldots, p - 1, -1, \ldots, -1).
\end{align*}
\]

Then, \( |O_1| = p^k \), \( |O_2| = \left( \frac{p^k}{p^{k-1}} \right) \), from which it follows that \( \gcd(|O_1|, |O_2|) = p \).

(ii) Since there exists an admissible equivariant map from \( V^- \) to \( V \) (see Theorem 6.9), one has that \( p \) divides \( \alpha(V^-) \). Hence, it suffices to show that

(a) any \( S_n \)-orbit in \( V^- \setminus \{0\} \) is of even length;

(b) \( V^- \setminus \{0\} \) admits two \( S_n \)-orbits, say \( O_1 \) and \( O_2 \), such that \( \gcd(|O_1|, |O_2|) = 2p \).

For (a), take an \( S_n \)-orbit \( O \subset V^- \setminus \{0\} \). If the transposition (12) acts on \( O \) without fixed points, then \( |O| \) is even. Otherwise, let \( x \) be a vector fixed by (12). Then, \( x = (a, -a, 0, \ldots, 0) \) for some \( a \neq 0 \) (see (7)). Since \( n > 3 \), one has \( -x \in O \), from which it follows that \( -O = O \). Thus, the involution \( x \mapsto -x \) acting on \( O \) is without fixed points, which implies that \( |O| \) is even (note that this argument does not work when \( n = 3 \), in which case \( |O| = 3 \)).

For (b), let \( O_1 = S_n(x) \) and \( O_2 = S_n(y) \) be two \( S_n \)-orbits in \( V^- \setminus \{0\} \) with \( x \) and \( y \) given in (13) and (14), respectively. Observe that \( -x \in O_1 \) and \( -y \in O_2 \) (by transposing two \(-1\) components), from which it follows that \( |O_1| = 2p \) and \( |O_2| = 2 \left( \frac{p^k}{p^{k-1}} \right) \). Therefore, \( \gcd(|O_1|, |O_2|) = 2p \).

Combining the congruence principle and Proposition 7.5 yields:

**Corollary 7.6.** Suppose that \( n = p^k > 3 \) is an odd prime power. For any \( S_n \)-equivariant map \( \Psi : S(V^-) \to V \setminus \{0\} \),

\[
\deg(\Psi) \equiv 2^{n-1} \pmod{2p}
\]

(in particular, \( \deg(\Psi) \neq 0 \)).

Proof. Let \( \Phi : S(V^-) \to V \setminus \{0\} \) be the 2-homogeneous equivariant map provided by Theorem 6.9. Since \( \alpha(S_n, V^-) = 2p \) (see Proposition 7.5) and \( \deg(\Phi) = 2^{n-1} \), it follows from the congruence principle that

\[
\deg(\Psi) \equiv \deg(\Phi) \equiv 2^{n-1} \pmod{2p}.
\]

\[\square\]
### A Tables

| $\rho$ | $\alpha(\rho)$ | (2-transitivity) $|G : H|$ | character |
|--------|-----------------|--------------------------|-----------|
| $\psi_0$ | 1 | 1 1 1 1 1 1 | \(1\) |
| $\psi_1$ | 2 | 3 –1 . \(\frac{-\sqrt{5}}{2}\) \(\frac{+\sqrt{5}}{2}\) |
| $\psi_2$ | 2 | 3 –1 . \(\frac{-\sqrt{5}}{2}\) \(\frac{+\sqrt{5}}{2}\) |
| $\psi_3$ | 5 | \(A_4\) 5 | 4 . 1 –1 –1 –1 |
| $\psi_4$ | 1 | \(D_5\) 6 | 5 1 –1 . . |

Table 2: character table of \(G = A_5\)

| $\rho$ | $\alpha(\rho)$ | (2-transitivity) $|G : H|$ | character |
|--------|-----------------|--------------------------|-----------|
| $\xi_1$ | 2 | \(A_5\) 2 | 1 –1 1 1 –1 –1 1 |
| $\xi_2$ | 10 | 4 –2 . 1 1 . –1 |
| $\xi_3$ | 1 | \(AGL(Z_5)\) 6 | 5 –1 1 –1 –1 1 . |
| $\xi_4$ | 2 | 6 . –2 . . . 1 |
| $\xi_5$ | 1 | 5 1 1 –1 1 1 . |
| $\xi_6$ | 5 | \(S_4\) 5 | 4 2 . 1 –1 . –1 |
| $\xi_0$ | 1 | 1 1 1 1 1 1 . |

Table 3: character table of \(G = S_5\)

| $\rho_i = \rho_{(G;X_i)}^\alpha$ | $\alpha(\rho_i)$ | $|X_i|$ | character |
|-----------------|-----------------|-------|-----------|
| $\rho_1$ | 7 7 6 6 2 2 2 | . . . . –1 –1 |
| $\rho_2$ | 2 \(2^3\) 7 –1 3 –1 –1 1 –1 1 –1 . |
| $\rho_3$ | 2 \(2^3\) 7 –1 –1 –1 –1 –1 1 1 1 . |
| $\rho_4$ | 2 \(2^3\) 7 –1 –1 3 –1 –1 1 1 –1 . |

Table 4: irreducible \((V \rtimes \text{GL}(3,2))\)-representations associated to 2-transitive actions
| $\rho$ | $\alpha(\rho)$ | character |
|-------|-----------------|-----------|
| $\psi_0$ | 1               | 1 1 1 1 1 1 1 1 1 1 |
| $\psi_1$ | 2               | 7 -1 -2 1 1 1 . . . |
| $\psi_2$ | 2               | 7 -1 1 A C B . . . |
| $\psi_3$ | 2               | 7 -1 1 B A C . . . |
| $\psi_4$ | 2               | 7 -1 1 C B A . . . |
| $\psi_5$ | 3               | 8 -1 -1 -1 -1 -1 1 1 1 |
| $\psi_6$ | 1               | 9 1 . . . . D F E |
| $\psi_7$ | 1               | 9 1 . . . . E D F |
| $\psi_8$ | 1               | 9 1 . . . . F E D |

Table 5: character table of $\text{PSL}(2, 8)$

| $\rho$ | $\alpha(\rho)$ | character |
|-------|-----------------|-----------|
| $\xi_0$ | 1               | 1 1 1 1 1 1 1 1 1 1 |
| $\xi_1$ | 3               | 1 1 1 1 A A* A* A 1 A A* |
| $\xi_2$ | 3               | 1 1 1 1 A* A A A* 1 A* A |
| $\xi_3$ | 2               | 7 -2 1 -1 1 1 -1 -1 -1 1 1 |
| $\xi_4$ | 6               | 7 -2 1 -1 A* A -A -A* . A* A |
| $\xi_5$ | 6               | 7 -2 1 -1 A A* -A* -A . A A* |
| $\xi_6$ | 3               | 8 -1 1 -1 . 2 2 . 1 -1 -1 |
| $\xi_7$ | 3               | 8 -1 1 -1 . B B* . 1 -A -A* |
| $\xi_8$ | 3               | 8 -1 1 -1 . B* B . 1 -A* -A |
| $\xi_9$ | 2               | 21 3 . -3 . . . . . . . . |
| $\xi_{10}$ | 1            | 27 . 3 . . . . -1 . . . |

Table 6: character table of $\text{PGL}(2, 8)$
B Figures

\begin{figure}
\centering
\begin{tikzpicture}
  \node (A5) at (0,0) {$(A_5)$};
  \node (Z3) at (0,-1) {$(Z_3)$};
  \node (Z2) at (-1,-2) {$(Z_2)$};
  \node (Z1) at (-2,-3) {$(Z_1)$};
  \draw (A5) -- (Z3);
  \draw (A5) -- (Z2);
  \draw (A5) -- (Z1);

  \node (A4) at (1,0) {$(A_4)$};
  \node (D3) at (1,-1) {$(D_3)$};
  \node (Z3) at (2,-2) {$(Z_3)$};
  \node (Z1) at (3,-3) {$(Z_1)$};
  \draw (A4) -- (D3);
  \draw (A4) -- (Z3);
  \draw (A4) -- (Z1);

  \node (V4) at (4,0) {$(V_4)$};
  \node (D5) at (4,-1) {$(D_5)$};
  \node (Z2) at (5,-2) {$(Z_2)$};
  \node (Z1) at (6,-3) {$(Z_1)$};
  \draw (V4) -- (D5);
  \draw (V4) -- (Z2);
  \draw (V4) -- (Z1);

  \node (a) at (-2.5,-1.5) {(a) $\rho_1$ and $\rho_2$};
  \node (b) at (0.5,-1.5) {(b) $\rho_3$};
  \node (c) at (3.5,-1.5) {(c) $\rho_4$};
\end{tikzpicture}
\caption{lattices of orbit types of $\rho \in \text{Irr}^*(A_5)$}
\end{figure}

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