ON GENERAL FIBERS OF GAUSS MAPS
IN ARBITRARY CHARACTERISTIC

KATSUHISA FURUKAWA

Abstract. We investigate general fibers of the Gauss map $\gamma$ of a projective variety $X$ in $\mathbb{P}^N$, in terms of the degeneracy map $\kappa$ from $X$ to a Grassmann variety. Here, if $\gamma$ is separable, then a general fiber of $\gamma$ coincides with the linear subvariety $\kappa(x)$ of $\mathbb{P}^N$ for general $x$ in the fiber. In this paper, we show the following results: (1) Equivalence of the separability of the two maps $\gamma$ and $\kappa$. (2) $\kappa(x)$ is constant on $x \in F$ for each irreducible component $F$ of a general fiber of $\gamma$; in particular, $F$ is contained in this constant linear subvariety even if $\gamma$ is inseparable. (3) Inseparability of Gauss maps of strange varieties; this is useful to construct some typical examples.

1. Introduction

Let $X \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field of arbitrary characteristic, and let $\gamma : X \rightarrow \mathbb{G}(\dim(X), \mathbb{P}^N)$ be the Gauss map of $X$, which sends a smooth point $x \in X$ to the embedded tangent space $T_xX$ to $X$ at $x$ in $\mathbb{P}^N$. We denote by

$$d_x\gamma : T_xX \rightarrow T_{\gamma(x)}G(\dim(X), \mathbb{P}^N)$$

the linear map between Zariski tangent spaces at $x$ and $\gamma(x)$, and by $\operatorname{rk}(\gamma)$ the rank of the linear map $d_x\gamma$ with general $x \in X$. Then $\gamma$ is separable if and only if $\operatorname{rk}(\gamma) = \dim(\operatorname{im}(\gamma))$.

We take $m_0 := \dim(X) - \operatorname{rk}(\gamma)$ and define the degeneracy map $\kappa$ of $X$,

$$\kappa : X \rightarrow \mathbb{G}(m_0, \mathbb{P}^N),$$

(1)

to be the rational map which sends a general point $x \in X$ to the $m_0$-plane $\mathbb{L}_x(\ker d_x\gamma)$ in $\mathbb{P}^N$, where we denote by $\mathbb{L}_x(A) \subset T_xX$ the $m$-plane corresponding to an $m$-dimensional vector subspace $A \subset T_xX$. (See [1] p. 97, for the characteristic zero case.)

In characteristic zero, it is well known that the closure of a general fiber of $\gamma$ is equal to a linear subvariety of $\mathbb{P}^N$ (Griffiths and Harris [7 (2.10)], Zak [16 I, 2.3. Thm. (c)]; Kleiman and Piene gave another proof in terms of reflexivity [11 pp. 108–109]), where the fiber of $\gamma$ is indeed equal to the $m_0$-plane $\mathbb{L}_x(\ker d_x\gamma)$ for general $x$ in the fiber. The same statement holds in arbitrary characteristic if $\gamma$ is separable ([5 Thm. 1.1]; see also [Remark 2.1(a)]).

In this paper, we first show the equivalence of the separability of $\gamma$ and $\kappa$.

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Theorem 1.1. Let $X \subset \mathbb{P}^N$ be a projective variety over an algebraically closed field of arbitrary characteristic. Then the Gauss map $\gamma$ of $X$ is separable if and only if the degeneracy map $\kappa$ of $X$ is separable.

In Examples 2.3 and 1.5 (the latter one is simpler), we directly verify that if $\gamma$ is inseparable, then so is $\kappa$; the above theorem explains such situations.

We next focus on general fibers of inseparable Gauss maps. Wallace [15, §7] pointed out that the Gauss map $\gamma$ can be inseparable in positive characteristic. In this case, a general fiber of $\gamma$ is not equal to the $m_0$-plane $L_x(\ker d_x \gamma) \subset \mathbb{P}^N$, since their dimensions are different. Moreover, it is possible that a general fiber of $\gamma$ is not equal to a linear subvariety of $\mathbb{P}^N$; the fiber can be a union of points (Kaji [9, Example 4.1] [10], Rathmann [14, Example 2.13], Noma [13]), and can be a non-linear variety (Fukasawa [2, §7] [3]; see also Example 4.6). The toric case was also studied in [6].

We investigate the relationship between the $m_0$-plane $L_x(\ker d_x \gamma)$ and a general fiber of $\gamma$ (possibly non-linear, as above). In consequence, we have the following theorem.

Theorem 1.2. Let $X \subset \mathbb{P}^N$ be a projective variety, and let $F \subset X$ be an irreducible component of the closure of a general fiber of the Gauss map $\gamma$. Then the $m_0$-plane $L_x(\ker d_x \gamma) \subset \mathbb{P}^N$ is constant on general $x \in F$ (in other words, $F$ is contracted to one point by $\kappa$). Therefore $F$ is contained in this constant $m_0$-plane.

Let $L_F \subset \mathbb{P}^N$ be the above constant $m_0$-plane for $F$. It is possible that $L_{F_1}$ and $L_{F_2}$ can be different for two irreducible components $F_1$ and $F_2$ of a general fiber of $\gamma$ (see Example 4.7).

Finally, we examine Gauss maps of strange varieties, where a projective variety $X \subset \mathbb{P}^N$ is said to be strange for a point $v \in \mathbb{P}^N$ if $v \in T_xX$ (equivalently, $T_xX \subset T_x^v \mathbb{P}^N$ in $T_x^v \mathbb{P}^N$) holds for any smooth point $x \in X$. In previous studies of Gauss maps in positive characteristic, the inseparability of $\gamma$ of strange $X$ was often observed with attractive phenomena (e.g., [3, 4]). Motivated by such observations, we show:

Theorem 1.3. Let $X \subset \mathbb{P}^N$ be a projective variety which is strange for a point $v \in \mathbb{P}^N$. Then we have $v \in L_x(\ker d_x \gamma)$ (equivalently, $d_x \gamma(T_x^v \mathbb{P}^N) = 0$) for any smooth point $x \in X$. In addition, the Gauss map $\gamma$ is inseparable if $X$ is not a cone with vertex $v$.

The paper is organized as follows. The “only if” part of Theorem 1.1 immediately follows from [3] (see Remark 2.1). Thus our purpose is to complete the “if” part. In §2.1 we give a parametrization of $\kappa$. In §2.2 we find that $\gamma$ is equal to the composition $\gamma_{X_0} \circ \kappa$ if $\kappa$ is separable, where $\gamma_{X_0}$ is the expanding map from $X_0 = \kappa(X)$ to a Grassmann variety. Then, from calculation of derivations, the separability of $\gamma_{X_0}$ is proved, which yields our assertion of Theorem 1.1.
§3.1 we construct a closed subvariety $X' \subset X$ such that $\gamma|_{X'}$ is separable and that $X'$ satisfies some additional conditions (see Proposition 3.1). In §3.2 for the closure $Y' \subset G(\dim(X), \mathbb{P}^N)$ of the image of $X'$ under $\gamma$, we consider the shrinking map $\sigma'$ from $Y'$ to a Grassmann variety, and show that $\sigma' \circ \gamma(x)$ corresponds to the $m_0$-plane $L_x(\ker d_x\gamma)$ for a general point $x \in X'$. This implies Theorem 1.2. In §4 we investigate the case where $X$ is strange for $v$. This condition implies that the $m_0$-plane $\sigma'(y) \subset \mathbb{P}^N$ contains $v$ for general $y \in Y'$. Then we have Theorem 1.3. By using this theorem, we construct some typical examples which are relevant to Theorems 1.1 and 1.2.

2. Degeneracy maps and expanding maps

Let $X \subset \mathbb{P}^N$ be an $M$-dimensional projective variety over an algebraically closed field $K$ of arbitrary characteristic. We denote by $\kappa : X \rightarrow G(m_0, \mathbb{P}^N)$ the degeneracy map of $X$ as in [1] and by $\mathcal{X}_0 \subset G(m_0, \mathbb{P}^N)$ the closure of the image of $X$ under $\kappa$. The purpose of this section is to show the “if” part of Theorem 1.1. We note that the “only if” part holds as in the following remark.

Remark 2.1. Assume that the Gauss map $\gamma : X \rightarrow G(M, \mathbb{P}^N)$ is separable. Then the following holds.

(a) From [3] Thm 1.1, the closure $\overline{\gamma^{-1}(y)} \subset X$ of the fiber of $\gamma$ at a general point $y \in \text{im}(\gamma)$ is equal to a linear subvariety of $\mathbb{P}^N$. Indeed, we have $\overline{\gamma^{-1}(y)} = L_x(\ker d_x\gamma)$ for general $x \in \gamma^{-1}(x)$. This is because, since $T_x\gamma^{-1}(y) \subset T_xX$ is contained in $\ker(d_x\gamma)$, it follows that $\gamma^{-1}(y) \subset L_x(\ker d_x\gamma)$, where these two linear varieties have the same dimension.

(b) Let $Y' \subset G(M, \mathbb{P}^N)$ be the closure of the image of $X$ under $\gamma$. In [5], $\mathcal{X}_0$ is defined as the closure of the image of $Y'$ under the shrinking map $\sigma$ of $Y$. Since $\gamma$ is separable, (a) and [5] Cor. 3.2 implies that this $\mathcal{X}_0$ coincides with $\kappa(X)$ and that $\kappa = \sigma \circ \gamma$. In addition, $\sigma$ is a birational map. Hence $\kappa$ is separable.

2.1. Parametrization of the degeneracy map $\kappa$. In this subsection, we will give a parametrization of $\kappa$ as in the formula (13). As a preparation, we first consider how the degeneracy map $\kappa$ is defined by sheaf homomorphisms.

Definition 2.2. Let $\Omega_{G(M, \mathbb{P}^N)}$ and $\mathcal{S}_{G(M, \mathbb{P}^N)}$ be the universal quotient bundle and subbundle of rank $M+1$ and $N-M$ on $G(M, \mathbb{P}^N)$ with the exact sequence:

$$0 \rightarrow \mathcal{S}_{G(M, \mathbb{P}^N)} \rightarrow H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{G(M, \mathbb{P}^N)} \rightarrow \mathcal{O}_{G(M, \mathbb{P}^N)} \rightarrow 0.$$  

Let $X \subset \mathbb{P}^N$ be an $M$-dimensional projective variety, and denote by $X^{sm}$ the smooth locus of $X$. We consider $\xi : \gamma^*\mathcal{O}_{G(M, \mathbb{P}^N)} \rightarrow T_{X^{sm}}(-1)$, a natural homomorphism appeared in the following commutative diagram with exact
Let $d\gamma : T_{X^m} \to \gamma^*T_{G(M,\mathbb{P}^N)}$ be the homomorphism of tangent bundles induced by $\gamma$. We set $\delta\gamma$ to be the following composite homomorphism

$$\delta\gamma = d\gamma(−1) \circ \xi : \gamma^*Q^\vee_{G(M,\mathbb{P}^N)} \to T_{X^m}(−1) \to \gamma^*T_{G(M,\mathbb{P}^N)}(−1),$$

where the projectivization of $\ker(\delta\gamma) \otimes k(x) \subset Q^\vee_{G(M,\mathbb{P}^N)} \otimes k(\gamma(x))$ corresponds to $L_x(ker \ d_x \gamma) \subset T_xX$ for a general point $x \in X$. Taking an open subset $X^o \subset X$ such that $\ker(\delta\gamma)|_{X^o}$ is locally free of rank $m_0 + 1$, by universality of the Grassmann variety, we have

$$\kappa : X^o \to G(m_0, \mathbb{P}^N).$$

This is identified with the map given in (1).

Next, we give a local parametrization of $\kappa$ around a general point $x_o \in X$ (cf. [5] §2.2]). Changing the homogeneous coordinates

$$(Z^0 : Z^1 : \cdots : Z^N)$$
on $\mathbb{P}^N$, we can assume that $x_o = (1 : 0 : \cdots : 0) \in \mathbb{P}^N$, and assume that $\kappa(x_o) \in G(m_0, \mathbb{P}^N)$ and $\gamma(x_o) \in G(M, \mathbb{P}^N)$ are given as the following linear subvarieties of $\mathbb{P}^N$:

$$\kappa(x_o) = (Z^{m_0+1} = \cdots = Z^N = 0),$$

$$\gamma(x_o) = (Z^{M+1} = \cdots = Z^N = 0).$$

We take a local parametrization of $X$ around $x_o$ by

$$(1 : z^1 : \cdots : z^M : f^{M+1} : \cdots : f^N),$$

where $z^1, \ldots, z^M$ are a system of regular parameters of the regular local ring $\mathcal{O}_{X,x_o}$, and $f^{M+1}, \ldots, f^N \in \mathcal{O}_{X,x_o}$ are regular functions.
We have a description of \( \gamma(x) \in G(M, \mathbb{P}^N) \) with \( x \in X \) near \( x_0 \), as follows: We consider the affine open subset \( G_M^N \subset G(M, \mathbb{P}^N) \) consisting of \( M \)-planes not intersecting the \((N - M - 1)\)-plane \((Z^0 = Z^1 = \cdots = Z^M = 0)\). We denote by

\[
Z_0, Z_1, \ldots, Z_N \in H^0(\mathbb{P}^N, \mathcal{O}(1))^\vee
\]

the dual basis of \( Z^0, Z^1, \ldots, Z^N \in H^0(\mathbb{P}^N, \mathcal{O}(1)) \), and so on. Then the sheaves \( \mathcal{Q}_{G(M, \mathbb{P}^N)} \) and \( \mathcal{S}'_{G(M, \mathbb{P}^N)} \) are free on \( G_M^N \), and are equal to \( Q \otimes \mathcal{O}_{G_M^N} \) and \( S' \otimes \mathcal{O}_{G_M^N} \), for the vector spaces

\[
Q := \bigoplus_{0 \leq \rho \leq M} K \cdot q^\rho \quad \text{and} \quad S' := \bigoplus_{M+1 \leq \mu \leq N} K \cdot s_\mu,
\]

where \( K \) is the ground field, and where \( q^\rho \), \( s_\mu \) correspond to \( Z^\rho \), \( Z_\mu \). As in [2, §2.2, Formula (12)], under the identification \( \mathbb{G}_M^o \cong \text{Hom}(Q', S') \cong \mathbb{A}^{(M+1)(N-M)}_e \), we can describe \( \gamma(x) \in G_M^N \) by

\[
(6) \quad \gamma(x) = \sum_{0 \leq \rho \leq M, M+1 \leq \mu \leq N} a^\rho_\mu(x) \cdot q^\rho \otimes s_\mu = (a^\rho_\mu(x))_{\rho, \mu}
\]

with

\[
(7) \quad a^\rho_\mu(x) = (f^\rho + \sum_{1 \leq \nu \leq M} -f^\nu_\mu z^\nu)(x),
\]

\[
(8) \quad f^\nu_\mu(x) = f^\nu_\mu(x) \quad (1 \leq \nu \leq M),
\]

where \( f^\nu_\mu := \partial f^\rho / \partial z^\nu \). As in [2, §2.2, Formula (10)], we have

\[
(9) \quad f^\rho_\mu(x_0) = 0 \quad (1 \leq \nu \leq M, M+1 \leq \mu \leq N).
\]

Next, we describe the homomorphism \( \delta \gamma \) given in Definition 2.2. Let \( x \in X \) be a general point near \( x_0 \), as above. Then the linear map \( d_x \gamma : T_x X \to T_{\gamma(x)} G(M, \mathbb{P}^N) = \text{Hom}(Q', S') \) is described by

\[
(\partial / \partial z^e) \mapsto \sum_{0 \leq \rho \leq M, M+1 \leq \mu \leq N} a^\rho_\mu z^e \cdot q^\rho \otimes s_\mu \quad (1 \leq e \leq M).
\]

On the other hand, \( q_0 \mapsto \sum_{1 \leq e \leq M} -z^e \cdot \partial / \partial z^e \), \( q_e \mapsto \partial / \partial z^e \) \((1 \leq e \leq M)\). Hence \( \delta \gamma := (d \gamma(-1) \circ \xi) : Q' \to T_{\gamma(x)} G(M, \mathbb{P}^N) = \text{Hom}(Q', S') \) is described by

\[
(9) \quad \delta \gamma(q_0) = \sum_{0 \leq \rho \leq M, M+1 \leq \mu \leq N} \sum_{1 \leq e \leq M} -a^\rho_\mu z^e \cdot q^\rho \otimes s_\mu
\]

\[
 \delta \gamma(q_e) = \sum_{0 \leq \rho \leq M, M+1 \leq \mu \leq N} a^\rho_\mu z^e \cdot q^\rho \otimes s_\mu \quad (1 \leq e \leq M).
\]
Now we give the parametrization of $\kappa$. From [4], the $M - m_0$ elements $\delta_x\gamma(q_{m_0+1}), \ldots, \delta_x\gamma(q_M)$ give a basis of the vector subspace $\delta_x\gamma(Q')$. Since $\delta_x\gamma(q_0), \ldots, \delta_x\gamma(q_{m_0})$ are linearly dependent on the basis, we have

$$\delta_x\gamma(q_i) = \sum_{m_0+1 \leq j \leq M} h_i^j \cdot \delta_x\gamma(q_j)$$

with some functions $h_i^j = h_i^j(x)$ for $0 \leq i \leq m_0$. It follows from [9] that

$$\sum_{1 \leq i \leq M} -a_{\nu,z}^\mu z^\nu = \sum_{m_0+1 \leq j \leq M} h_0^j a_{\nu,z}^\mu$$

$$a_{\nu,z}^\mu = \sum_{m_0+1 \leq j \leq M} h_i^{j'} a_{\nu,z}^\mu \quad (1 \leq i' \leq m_0)$$

for each $0 \leq \nu \leq M, M + 1 \leq \mu \leq N$. We take a vector $u = \sum_{0 \leq \nu \leq M} u^\nu q_\nu \in Q'$ with $u^\nu \in K$, which corresponds to a point of $T_x X$ and is expressed by

$$\sum_{0 \leq \nu \leq M} u^\nu Z_\nu + \sum_{0 \leq \nu \leq M} u^\nu a_{\nu,z}^\mu Z_\mu \text{ in } \mathbb{P}^N,$$

as in [5 §2.1(B)]. Assume that $u$ is contained in $\ker \delta_x\gamma$. Then it follows from $\sum_{0 \leq \nu \leq M} u^\nu \cdot \delta_x\gamma(q_\nu) = 0$ that

$$\sum_{m_0+1 \leq j \leq M} u^j \cdot \delta_x\gamma(q_j) = \sum_{0 \leq i \leq m_0} -u^i \cdot \delta_x\gamma(q_i) = \sum_{m_0+1 \leq j \leq M} -u^i h_i^j \cdot \delta_x\gamma(q_j).$$

Hence $u^j = \sum_{0 \leq i \leq m_0} -u^i h_i^j$ ($m_0 + 1 \leq j \leq M$). It follows that [11] is equal to

$$\sum_{0 \leq i \leq m_0} u^i Z_i + \sum_{m_0+1 \leq j \leq M} -u^i h_i^j Z_j + \sum_{0 \leq i \leq m_0} u^i a_{\nu,z}^\mu Z_\mu + \sum_{m_0+1 \leq j \leq M} -u^i h_i^j a_{\nu,z}^\mu Z_\mu,$$

which is the description of the point contained in $\kappa(x) = \mathbb{L}_x(\ker d_x\gamma) \subset \mathbb{P}^N$ and corresponding to $u \in \ker \delta_x\gamma$. Taking the affine open subset $G^0_{m_0} \subset G(m_0, \mathbb{P}^N)$, we have that the sheaves $\mathcal{Q}_{G(m_0, \mathbb{P}^N)}$ and $\mathcal{D}^V_{G(m_0, \mathbb{P}^N)}$ are free on $G^0_{m_0}$, and are equal to $Q_0 \otimes \mathcal{O}_{G^0_{m_0}}$ and $S_0^\nu \otimes \mathcal{O}_{G^0_{m_0}}$ for the vector spaces

$$Q_0 := \bigoplus_{0 \leq i \leq m_0} K \cdot u^i \text{ and } S_0^\nu := \bigoplus_{m_0+1 \leq \rho \leq N} K \cdot v_\rho,$$

where $u^i, v_\rho$ correspond to $Z^i, Z_\rho$. Now, from [12] and [5 §2.1(B)], the map $\kappa : X \to G^0_{m_0} = \text{Hom}(Q_0, S_0^\nu)$ is parametrized around $x_\alpha$ by

$$\sum_{0 \leq i \leq m_0} -h_i^j \cdot v_j \otimes u^i + \sum_{m_0+1 \leq j \leq M} (a_{\nu,z}^\mu + \sum_{m_0+1 \leq j \leq M} -h_i^j a_{\nu,z}^\mu) \cdot v_\mu \otimes u^i.$$
Example 2.3. Let $X \subseteq \mathbb{P}^4$ be a surface of degree 8 defined by two homogeneous polynomials

$$(Z^0)^3 Z^3 + (Z^1)^4, \ Z^0 Z^4 - (Z^2)^2.$$ 

Assume that the characteristic is 3. Then the following holds.

(a) The Gauss map $\gamma$ and degeneracy map $\kappa$ of $X$ are inseparable, as follows. For the local parametrization $(1 : z^1 : z^2 : f^3 : f^4)$ of $X$, it follows that $f^3 = (z^1)^4$ and $f^4 = (z^2)^2$. Renaming $z^1, z^2$ by $s, t$, we can express the parametrization by

$$(14) \quad (1 : s : t : s^4 : t^2) \in \mathbb{P}^4$$

(here, $s^a, t^a$ mean the $a$-th powers of $s, t$). Then the Gauss map $\gamma : X \rightarrow \mathbb{G}(2, \mathbb{P}^4)$ is parametrized on $\mathbb{G}_2^5 \cong \mathbb{A}^6$ by

$$\begin{bmatrix} a_0^3 & a_0^4 \\ a_1^3 & a_1^4 \\ a_2^3 & a_2^4 \end{bmatrix} = \begin{bmatrix} f^3 - (s f^3 + t f^4) & f^4 - (s f^4 + t f^4) \end{bmatrix}. $$

Thus $\gamma$ is inseparable. Next, from (9) we have

$$\begin{bmatrix} \delta_x \gamma(q_0) \\ \delta_x \gamma(q_1) \end{bmatrix} = \begin{bmatrix} -a_{0,s}^3 s - a_{0,t}^3 t & \cdots & -a_{2,s}^4 s - a_{2,t}^4 t \\ a_{0,s}^3 & \cdots & a_{2,s}^4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -t^2 & 0 & t \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & -1 \end{bmatrix}. $$

This implies $h_0^2 = -t, h_1^2 = 0$. Therefore $\kappa : X \rightarrow \mathbb{G}(1, \mathbb{P}^4)$ is parametrized on $\mathbb{G}_1^5 \cong \mathbb{A}^6$ by

$$\begin{bmatrix} -h_0^2 & a_0^3 - h_0^2 a_2^3 \\ -h_1^2 & a_1^3 - h_1^2 a_2^3 \end{bmatrix} = \begin{bmatrix} t & 0 & t^2 \\ 0 & s^3 & 0 \end{bmatrix}. $$

Thus $\kappa$ is inseparable. Here, $X_0 = \kappa(X) \subseteq \mathbb{G}(1, \mathbb{P}^4)$ is equal to the closure of the image of

$$(15) \quad \text{Spec } K[\tilde{s}, t] \subseteq \mathbb{G}_1^5 : (\tilde{s}, t) \mapsto A = \begin{bmatrix} t & 0 & t^2 \\ 0 & \tilde{s} & 0 \end{bmatrix}. $$

(b) Let $U_{X_0} \subseteq \mathbb{G}(1, \mathbb{P}^4) \times \mathbb{P}^4$ be the universal family of $X_0$. Then the projection $\pi_0 : U_{X_0} \rightarrow \mathbb{P}^4$ is locally parametrized by

$$((\tilde{s}, t), u) \mapsto (1 : u : t : u \tilde{s} : t^2) = [1 : u] \cdot [E_2 \ A].$$

where $A$ is the matrix in (15) and $E_2$ is the unit matrix of size 2. Hence $\pi_0(U_{X_0}) \subseteq \mathbb{P}^4$ is the singular quadric 3-fold defined by the polynomial $Z^0 Z^4 - (Z^2)^2$.

(c) Let $V \subseteq \mathbb{P}^4$ be the vertex of the quadric $\pi_0(U_{X_0})$, which is equal to the line $(Z^0 = Z^2 = Z^4 = 0)$. For general $x \in X$, we denote by $v_x \in \mathbb{P}^N$ the intersection of $V$ and the line $\kappa(x)$. Under the description (14) of $x$ with parameters $s$ and $t$, it follows that $v_x = (0 : 1 : 0 : s^3 : 0 : 0)$, which depends only on $s$. Here the line $\kappa(x)$ is given by $\mathbb{P} v_x$. 
2.2. Expanding map. Let $X_0 \subset \mathbb{G}(m_0, \mathbb{P}^N)$ be the closure of the image of $X \subset \mathbb{P}^N$ under the degeneracy map $\kappa$. We consider the expanding map of $X_0$, 

$$\gamma_{X_0} : X_0 \longrightarrow \mathbb{G}(m_0^+, \mathbb{P}^N),$$

with an integer $m_0^+ \geq m_0$ (see \cite{Furukawa} Def. 2.1 for the precise definition). Our first step is to show:

**Proposition 2.4.** Assume that $\kappa : X \longrightarrow X_0$ is separable. Then $m_0^+ = M$ holds, and the composite map $\gamma_{X_0} \circ \kappa$ coincides with the Gauss map $\gamma$ of $X$.

**Remark 2.5.** Let $\mathcal{Q}_{G(M, \mathbb{P}^N)}$ and $\mathcal{S}_{G(M, \mathbb{P}^N)}$ be as in \textbf{Definition 2.2}. Set $\mathcal{Q}_{X_0} = \mathcal{Q}_{G(M, \mathbb{P}^N)|_{X_0}}$ and so on.

(a) As in \cite{Furukawa} Def. 2.1, $\gamma_{X_0}$ is constructed from the homomorphism

$$\varphi_0 : \mathcal{S}_{X_0} \rightarrow \text{Hom}(\text{Hom}(\mathcal{S}_{X_0}, \mathcal{Q}_{X_0}), \mathcal{Q}_{X_0}) \rightarrow \text{Hom}(T_{X_0}, \mathcal{Q}_{X_0}),$$

where $\ker(\varphi_0) \otimes k(a) \subset H^0(\mathbb{P}^N, \mathcal{O}(1))$ gives the defining polynomials of the $m_0^+$-plane $\gamma_{X_0}(a) \subset \mathbb{P}^N$ for general $a \in X_0$. Note that

$$\dim(\varphi_{0,a}(\mathcal{S}_{X_0} \otimes k(a))) = m_0^+ - m_0.$$

(b) Let us consider the pull-back of $\varphi_0$ under $\kappa$. We have a homomorphism $\psi_0 : \kappa^*\mathcal{S}_{X_0} \rightarrow \text{Hom}(T_{X_0}, \kappa^*\mathcal{Q}_{X_0})$ on an open subset $X^0 \subset X$ with the following commutative diagram:

$$
\begin{array}{ccc}
\kappa^*\mathcal{S}_{X_0} & \longrightarrow & \text{Hom}(\text{Hom}(\kappa^*\mathcal{S}_{X_0}, \kappa^*\mathcal{Q}_{X_0}), \kappa^*\mathcal{Q}_{X_0}) \\
\psi_0 |_{X^0} & \downarrow & \text{Hom}(\kappa^*T_{X_0}, \kappa^*\mathcal{Q}_{X_0}) \\
& \longrightarrow & \text{Hom}(T_{X^0}, \kappa^*\mathcal{Q}_{X_0}),
\end{array}
$$

where $d\kappa : T_{X^0} \rightarrow \kappa^*T_{G(m_0, \mathbb{P}^N)|_{X_0}} \cong \text{Hom}(\kappa^*\mathcal{S}_{X_0}, \kappa^*\mathcal{Q}_{X_0})$ is the homomorphism of tangent bundles induced from $\kappa$.

(c) Assume that $\kappa$ is separable. Then the vertical arrow of the above diagram is injective. Then we have

$$\dim(\varphi_{0,x}(\mathcal{S}_{X_0} \otimes k(x'))) = \dim(\psi_{0,x}(\mathcal{S}_{X_0} \otimes k(x'))),$$

$$\ker(\varphi_{0,x}) = \ker(\psi_{0,x}),$$

for a general point $x \in X$ and $x' := \kappa(x) \in X_0$.

**Remark 2.6.** $m_0^+ \geq M$ always holds, as follows. Recall that $M = \dim(X)$. We set $U_{X_0} \subset \mathbb{G}(m_0, \mathbb{P}^N) \times \mathbb{P}^N$ to be the universal family of $X_0$ with the second projection

$$\pi_0 : U_{X_0} \rightarrow \mathbb{P}^N.$$

Let $\tilde{\kappa} : X \longrightarrow X_0 \times \mathbb{P}^N$ be the graph map of $\kappa$, which sends a general point $x$ to $(\kappa(x), x)$. We denote by $U' \subset \mathcal{X}_0$ the image of $X$ under $\tilde{\kappa}$. Since each point $x \in X$ is contained in $L_x(d\kappa \gamma)$, we have $U' \subset U_{X_0}$.

Let $u' = \tilde{\kappa}(x) \in U'$ be a general point such that $\kappa(x) \in X_0$. Here $u'$ is a smooth point of $U_{X_0}$. Since $\pi_0|_U$ is birational onto $X$, we have $\text{rk}d_{u'}\pi_0 \geq M$. 


From [5] Prop 4.3(a), an inequality $m_0^+ \geq \text{rk } d_0 \pi_0$ holds for smooth $u \in U_{\chi_0}$. Applying this to $t^j$, we have $m_0^+ \geq M$.

From [13] $d_{x,\kappa} : T_x X \to T_x \mathbb{G}(m_0, \mathbb{P}^N) = \text{Hom}(S_0, Q_0)$ is described by

$$d_{x,\kappa}(\partial/\partial \varepsilon^e) = \sum_{0 \leq i \leq m_0} \left( -h_{i,z}^j \cdot v_j \otimes u^i \right) + \sum_{0 \leq i \leq m_0, m_0 + 1 \leq j \leq M} \left( a_{i,z}^\mu + \sum_{m_0 + 1 \leq j \leq M} -(h_{i,z}^j a_j^\mu + h_{i,z}^j a_j^\mu) \right) \cdot v_\mu \otimes u^i.$$  

Here,

$$a_{i,z}^\mu + \sum_{m_0 + 1 \leq j \leq M} -(h_{i,z}^j a_j^\mu + h_{i,z}^j a_j^\mu) = \sum_{m_0 + 1 \leq j \leq M} -h_{i,z}^j a_j^\mu;$$

this follows from [10] in the case of $i \geq 1$, and additionally follows from [7] and $a_{i,z}^\mu = \sum_{1 \leq \nu \leq M} -f_{i,z}^\nu v_\nu \otimes z^\nu = \sum_{1 \leq \nu \leq M} -a_{i,z}^\nu v_\nu$ in the case of $i = 0$. As a result, we have

$$d_{x,\kappa}(\partial/\partial \varepsilon^e) = \sum_{m_0 + 1 \leq j \leq M} -h_{i,z}^j \cdot v_j \otimes u^i + \sum_{0 \leq i \leq m_0, M + 1 \leq j \leq M} \sum_{m_0 + 1 \leq j \leq M} -h_{i,z}^j a_j^\mu \cdot v_\mu \otimes u^i.$$  

Hence the dual map $\text{Hom}(\text{Hom}(S_0, Q_0), K) \to \text{Hom}(T_x X, K)$ is given by

$$v^j \otimes u_i \mapsto \sum_{1 \leq \nu \leq M} -h_{i,z}^j \cdot d_\nu^e \otimes u^i \quad (m_0 + 1 \leq j \leq M, 0 \leq i \leq m_0),$$

$$v^\mu \otimes u_i \mapsto \sum_{1 \leq \nu \leq M} \sum_{m_0 + 1 \leq j \leq M} -h_{i,z}^j a_j^\mu \cdot d_\nu^e \otimes u^i \quad (M + 1 \leq \mu \leq N, 0 \leq i \leq m_0).$$

On the other hand, $S_0 \to \text{Hom}(\text{Hom}(S_0, Q_0), Q_0) \simeq S_0 \otimes Q_0 \otimes Q_0'$ is given by

$$v^\rho \mapsto \sum_{0 \leq i \leq m_0} v^\rho \otimes u_i \otimes u^i \quad (m_0 + 1 \leq \rho \leq N).$$

Hence $\psi_{0,x} : S_0 \to \text{Hom}(\text{Hom}(S_0, Q_0), Q_0) \to \text{Hom}(T_x X, Q_0)$ is described by

$$\psi_{0,x}(v^j) = \sum_{1 \leq \nu \leq M} -h_{i,z}^j \cdot d_\nu^e \otimes u^i \quad (m_0 + 1 \leq j \leq M),$$

$$\psi_{0,x}(v^\mu) = \sum_{1 \leq \nu \leq M} \sum_{m_0 + 1 \leq j \leq M} -h_{i,z}^j a_j^\mu \cdot d_\nu^e \otimes u^i \quad (M + 1 \leq \mu \leq N).$$

Proof of Proposition 2.4. From [16] $\psi_{0,x}(v^{M+1}), \ldots, \psi_{0,x}(v^N) \in \text{Hom}(T_x X, Q_0)$ are dependent on $\psi_{0,x}(v^{m_0+1}), \ldots, \psi_{0,x}(v^M)$. It follows that

$$\dim(\psi_{0,x}(S_0)) \leq M - m_0.$$  

Assume that $\kappa$ is separable. Then, since $\dim(\psi_{0,x}(S_0)) = \dim(\psi_{0,x}(S_0)) = m_0^+ - m_0$ as in [Remark 2.5], we find that $m_0^+ \leq M$ holds. From [Remark 2.6] we have $m_0^+ = M$. In particular, $\psi_{0,x}(v^{m_0+1}), \ldots, \psi_{0,x}(v^M)$ give a basis of the $(M - m_0)$-dimensional vector subspace $\psi_{0,x}(S_0)$. 
Next, we show $\gamma X_0 \circ \kappa = \gamma$, as follows. From Remark 2.5(c), we have $\ker(\varphi_{0,\kappa(x_0)}) = \ker(\psi_{0,x_0})$, where the right hand side is spanned by $N - M$ vectors $v_i^{M+1}, \ldots, v_i^N$ in $S_0$; this follows from (7) [8] (16). This means that $\gamma X_0(\kappa(x_0))$ coincides with the $M$-plane $(Z^{M+1} = \cdots = Z^N = 0) \subset \mathbb{P}^N$, which is equal to $\gamma(x_0)$. Since $x_0 \in X$ was taken as a general point, the assertion holds. 

Assume that $\kappa$ is separable. Then Proposition 2.4 implies that $\gamma = \gamma X_0 \circ \kappa$. We set $\mathcal{Y} \subset G(M, \mathbb{P}^N)$ to be the closure of the image of $X$ under $\gamma$. Then the following sequence of function fields is induced:

$$K(\mathcal{Y}) \subset K(\mathcal{X}_0) \subset K(X),$$

where the field extension $K(X)/K(\mathcal{X}_0)$ is separably generated.

**Proposition 2.7.** If $\kappa$ is separable, then $K(\mathcal{X}_0)/K(\mathcal{Y})$ is a separable algebraic extension.

**Remark 2.8.** Let $x_0 \in X$ be general. Choosing suitable coordinates on $\mathbb{P}^N$, in [5] we can assume that $x_0 = (1 : 0 : \cdots : 0)$ and that $X$ is locally parametrized as

$$X = \sum_{a \in A} a \gamma + \{ h_i^j \}_{0 \leq i \leq M, m_0 + 1 \leq j \leq M},$$

in [5] where we can also assume that the regular parameters $z^1, \ldots, z^M$ give a separating transcendence base of $K(X)/K$, i.e., the algebraic field extension $K(X)/K(z^1, \ldots, z^M)$ is separable. The reason is as follows. We can choose an $(N - M - 1)$-plane $L \subset \mathbb{P}^N$ such that $T_xX \cap L = \emptyset$ for general $x \in X$. Then the linear projection $\pi_L|_X : X \setminus L \to \mathbb{P}^M$ from $L$ is separable and generically finite; this is because, it follows that $T_xX \cap \ker(\pi_L|_X) = 0$ in $T_x\mathbb{P}^N$ and that $\dim(\ker(\pi_L|_X)) = M$. For a general point $x_0 \in X$ such that $x_0 \notin L$, changing coordinates on $\mathbb{P}^N$, we can assume that $x_0 = (1 : 0 : \cdots : 0)$ and $L = (Z^0 = \cdots = Z^M = 0)$. Then the field extension $K(X)/K(\mathbb{P}^M)$ induced from $\pi_L$ is separable, where we have $K(K(\mathbb{P}^M)) = K(z^1, \ldots, z^M)$.

**Proof of Proposition 2.7.** By (6) we have

$$K(\mathcal{Y}) = \bigoplus_{a \in A} \{ a^i \}_{0 \leq i \leq M, 1 \leq a \leq N}. $$

By (13) we have $K(\mathcal{X}_0) = K(\{ h_i^j \}, \{ a^i + \sum_{m_0 + 1 \leq j \leq M} h_i^j a^j \})$. It follows from $K(\mathcal{Y}) \subset K(\mathcal{X}_0)$ that

$$K(\mathcal{X}_0) = K(\mathcal{Y})(\{ h_i^j \}_{0 \leq i \leq m_0, m_0 + 1 \leq j \leq M})$$

For a field extension $B/A$, we denote by $\text{Der}_A(B) = \text{Der}_A(B, B)$ the set of $A$-derivations $D : B \to B$ (see [12] Ch. 9). We show that $\text{Der}_{K(\mathcal{Y})}(K(\mathcal{X}_0)) = 0$, as follows.

Let us take $D \in \text{Der}_{K(\mathcal{Y})}(K(\mathcal{X}_0))$. Since $K(X)/K(\mathcal{X}_0)$ is separably generated, $D$ is extended to a derivation $\bar{D}$ of $K(X)$. Since $z^1, \ldots, z^M$ form separating transcendence base over $K$, we can regard $M$ elements $\partial/\partial z^1, \ldots, \partial/\partial z^M$ as a basis of the $K(X)$-vector space $\text{Der}_{K(\mathcal{X}_0)}(K(X))$. Thus $\bar{D}$ is expressed by

$$\bar{D} = \sum_{1 \leq e \leq M} b^e \cdot \partial / \partial z^e$$
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with \( b^1, \ldots, b^M \in K(X) \). For a function \( g \in K(X) \), it follows that

\[
\tilde{D}(g) = \sum_{1 \leq e \leq M} b^e g_{ze}.
\]

We show

\[
\sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j a^\mu_{\nu,z}) = 0
\]

for any \( 0 \leq i \leq m_0, 0 \leq \nu \leq M, M + 1 \leq \mu \leq N \), as follows. Since \( a^\nu_\nu \in K(Y) \), we have \( \tilde{D}(a^\nu_\nu) = D(a^\nu_\nu) = 0 \). Thus \( \tilde{D}(a^\mu_{\nu,z}) = \tilde{D}(a^\mu_\nu) = 0 \). It follows from the second formula of \((10)\) that

\[
\tilde{D}(\sum_{m_0 + 1 \leq j \leq M} h^i_j a^\mu_{\nu,z}) = 0 \\
(1 \leq i \leq m_0),
\]

where the left hand side is equal to

\[
\sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j a^\mu_{\nu,z}) + \sum_{m_0 + 1 \leq j \leq M} \sum_{m_0 + 1 \leq j \leq M} h^i_j \tilde{D}(a^\mu_{\nu,z}) = \sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j a^\mu_{\nu,z}).
\]

Hence \( \sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j a^\mu_{\nu,z}) = 0 \) for \( 1 \leq i \leq m_0 \). Next, from the first formula of \((10)\) we have

\[
\tilde{D}(\sum_{1 \leq e \leq M} -a^\mu_{\nu,z} z^e) = \tilde{D}(\sum_{m_0 + 1 \leq j \leq M} h^i_j a^\mu_{\nu,z}),
\]

where the right hand side is equal to \( \sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j a^\mu_{\nu,z}) \). On the other hand, the left hand side is equal to \( \sum_{1 \leq e \leq M} -a^\mu_{\nu,z} \tilde{D}(z^e) \); since \( \tilde{D}(z^e) = b^e \), this is equal to

\[
\sum_{1 \leq e \leq M} -b^e a^\mu_{\nu,z} = -\tilde{D}(a^\mu_\nu) = 0
\]

as in \((18)\). Hence \( \sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j) a^\mu_{\nu,z} = 0 \).

Now, the formulae \((9)\) and \((19)\) implies

\[
\sum_{m_0 + 1 \leq j \leq M} \tilde{D}(h^i_j) (x) \cdot \delta_x \gamma(q_j) = 0 \\
(0 \leq i \leq m_0)
\]

for general \( x \in X \). Since \( \delta_x \gamma(q_{m_0 + 1}), \cdots, \delta_x \gamma(q_M) \) are linearly independent, we have \( \tilde{D}(h^i_j)(x) = 0 \) and have \( \tilde{D}(h^i_j) = 0 \) for any \( 0 \leq i \leq m_0 \) and \( m_0 + 1 \leq j \leq M \). From \( [17] \) \( D = \tilde{D}|_{K(X_\nu)} = 0 \). Thus the assertion follows.

Since \( \text{Hom}_{K(X_\nu)}(\Omega^1_{K(X_\nu)/K(Y)}, K(X_\nu)) = \text{Der}_{K(Y)}(K(X_\nu)) = 0 \), it follows from the Cartier equality \([12]\ Thm. 26.10\) that

\[
\text{tr.deg}_{K(Y)}(K(X_\nu)) = \text{rk } \Gamma_{K(X_\nu)/K(Y)}/K(0) = 0,
\]

where \( \text{tr.deg}_{A}(B) \) is the transcendence degree of \( B/A \), and \( \Gamma_{A/B/K} \) is the kernel of \( \Omega_{A/K} \otimes_A B \rightarrow \Omega_{B/K} \). Thus the field extension \( K(X_\nu)/K(Y_\nu) \) is algebraic; moreover, the extension is separable since \( \Omega_{K(Y)/K} \otimes_{K(Y)} K(X_\nu) \rightarrow \Omega_{K(Y)/K} \) is injective. \( \square \)
Proof of Theorem 1.1. The “only if” part holds as in Remark 2.1. Assume that $\kappa$ is separable. From Proposition 2.4, we have $\gamma = \gamma_x \circ \kappa$. From Proposition 2.7, the map $\gamma_x$ is separable, and hence so is $\gamma$. \hfill $\square$

3. Fibers of inseparable Gauss maps

Let $X \subset \mathbb{P}^N$ be an $M$-dimensional projective variety, and let $\gamma : X \dashrightarrow \mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ be the Gauss map of $X$, where $\mathcal{Y}$ is the closure of the image of $X$ under $\gamma$. We denote by $r := \text{rk}(\gamma)$, which satisfies $0 \leq r \leq \dim(\mathcal{Y})$. In this section, we give the proof of Theorem 1.2 by showing the following propositions.

Proposition 3.1. A general point of $X$ is contained in an $(M - r + \dim(\mathcal{Y}))$-dimensional closed subvariety $X' \subset X$ which satisfies the following conditions:

(a) The restricted map $\gamma|_{X'}$ is separable and its image is of dimension $r$.

(b) Let $x' \in X'$ be a general point, and let $F \subset X$ be an irreducible component of $\gamma^{-1}(\gamma(x'))$ such that $x' \in F$. Then $F$ is contained in $X'$.

Proposition 3.2. Let $X', F$ be as above. Then the $(M - r)$-plane $\mathbb{L}_x(\ker d_x \gamma)$ is constant on general $x \in F$.

3.1. Restriction of Gauss maps. Let $\alpha \in X$ be a general point such that $\text{rk } d_\alpha \gamma = \text{rk}(\gamma) = r$. We set $L_\alpha := \mathbb{L}_\alpha(\ker d_\alpha \gamma)$ in $\mathbb{T}_\alpha X$, and take an $(N - M + r)$-plane $A \subset \mathbb{P}^N$ such that $A \cap L_\alpha = \{\alpha\}$.

Lemma 3.3. $\mathbb{T}_\alpha (X) \cap A$ is of dimension $r$. Hence $X \cap A$ is smooth at $\alpha$, and $\mathbb{T}_\alpha (X \cap A) = \mathbb{T}_\alpha (X) \cap A$.

Proof. We have $\dim(\mathbb{T}_\alpha X \cap A) \geq r$. Assume that $\dim(\mathbb{T}_\alpha X \cap A) > r$. Then $\mathbb{T}_\alpha X \cap A$ is of codimension $< M - r$ in $\mathbb{T}_\alpha X$, which implies that $\mathbb{T}_\alpha X \cap A \cap L_\alpha$ is of dimension $> 0$, a contradiction. \hfill $\square$

Let $X''$ be the irreducible component of $X \cap A$ containing $\alpha$, where $\dim(X'') = r$ due to Lemma 3.3. Setting $\mathcal{Y}' \subset \mathbb{G}(M, \mathbb{P}^N)$ to be the closure of the image of $X''$ under $\gamma$, we have:

Lemma 3.4. The map $\gamma|_{X''} : X'' \dashrightarrow \mathcal{Y}'$ is separable and generically finite; in particular, $\dim \mathcal{Y}' = r$.

Proof. We have $T_\alpha X'' \cap \ker(d_\alpha \gamma) = 0$, because of $T_\alpha X'' = T_\alpha X \cap T_\alpha A$. It follows that $d_\alpha(\gamma|_{X''}) : T_\alpha X'' \to T_{\gamma(\alpha)} \mathcal{Y}$ is of rank $r$; since $\dim(X'') = r$, we have the assertion. \hfill $\square$

Let $X' \subset X$ be an irreducible component of $\gamma^{-1}\mathcal{Y}'$ containing $X''$.

Lemma 3.5. The map $\gamma|_{X'} : X' \dashrightarrow \mathcal{Y}'$ is separable.

Proof. Let $x$ be a general point of $X''$. Since $\gamma|_{X''}$ is separable, the composite homomorphism $\Omega^1_{\mathcal{Y}', x} \to \Omega^1_{X', x}$ is injective; hence so is the homomorphism $\Omega^1_{\mathcal{Y}',} \gamma(x) \to \Omega^1_{X',}$. Thus $\gamma^* \Omega^1_{\mathcal{Y}',} \to \Omega^1_{X',}$ is injective on an open subset of $X'$, i.e., $\gamma|_{X'}$ is separable. \hfill $\square$
Remark 3.6. Since $\alpha \in X$ is general, we can assume that $\alpha$ is a smooth point of $X$ and that $\gamma(\alpha)$ is a smooth point of $\mathcal{Y}$. Then $X' \cap X^{sm} \cap \gamma^{-1}(\mathcal{Y}^{sm}) \neq \emptyset$.

Now, let us consider general fibers of $\gamma$.

Lemma 3.7. Let $X'^{io} = X^{sm} \cap X'^{sm} \setminus (\bigcup_{i=1}^{s} V_i)$, where $V_1, \ldots, V_s$ are the irreducible components of $\gamma^{-1}(\mathcal{Y}^i)$ not equal to $X'$. Let $x \in X'^{io}$ and let $F$ be an irreducible component of $\gamma^{-1}(\gamma(x)) \subset X$ containing $x$. Then $F$ must be contained in $X'$.

Proof. Since $F$ is irreducible and contained in $\gamma^{-1}(\mathcal{Y}^i)$, $F$ is contained in $X'$ or $V_i$. Since $x$ is not contained in $\bigcup V_i$, and since $x \in F$, we have $F \subset X'$.

Lemma 3.8. $\dim X' = M - \dim(\mathcal{Y}) + r$.

Proof. Since $\alpha$ is general, we can take an open subset $U \subset \mathcal{Y}$ containing $\gamma(\alpha)$ such that each irreducible component $F$ of $\gamma^{-1}(y) \subset X$ is of dimension $M - \dim(\mathcal{Y})$ for any $y \in U$. We take $y \in \mathcal{Y}' \cap U$. Then Lemma 3.7 implies that some $F$ is contained in $X'$. Hence, it follows from $\dim(\mathcal{Y}') = r$ that $\dim(X') = (M - \dim(\mathcal{Y})) + r$.

Proof of Proposition 3.1. For a general point $\alpha \in X$, we take a closed subvariety $X' \subset X$ containing $\alpha$, as above. From Lemma 3.8 we have $\dim X' = M - \dim(\mathcal{Y}) + r$. The statement of (a) follows from Lemmas 3.4, 3.5 The statement of (b) follows from Lemma 3.7.

3.2. Shrinking maps. Landsberg and Piontkowski independently investigated the shrinking map in order to characterize Gauss images (see [12, 2.4.7] and [8, Theorem 3.4.8]). We consider the map to show Proposition 3.2.

Let $\gamma : X \longrightarrow \mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ be the Gauss map of an $M$-dimensional projective variety $X \subset \mathbb{P}^N$. We take a closed subvariety $X' \subset X$ as in Proposition 3.1 and set $\mathcal{Y}' \subset \mathcal{Y}$ to be the closure of the image of $X'$ under $\gamma$. Let $\sigma : \mathcal{Y} \longrightarrow \mathbb{G}(M_{\sigma}, \mathbb{P}^N), \sigma' : \mathcal{Y}' \longrightarrow \mathbb{G}(M'_{\sigma}, \mathbb{P}^N)$

be the shrinking maps of $\mathcal{Y} \subset \mathbb{G}(M, \mathbb{P}^N)$ and $\mathcal{Y}' \subset \mathbb{G}(M', \mathbb{P}^N)$ with integers $M_{\sigma}, M'_{\sigma} \leq M$ (see [5, Def. 2.3] for definition in terms of sheaves).

Proposition 3.9. The $M'_{\sigma}$-plane $\sigma' (\gamma(x)) \subset \mathbb{P}^N$ is equal to $\mathbb{L}_x (\ker d_x \gamma)$ for a general point $x \in X'$. In particular, $M'_{\sigma} = M - r$ for $r := \text{rk}(\gamma)$.

In order to show Proposition 3.9 we study sheaf homomorphisms on $X'$ and $\mathcal{Y}'$ related to $\sigma'$. In this subsection, we set $\mathcal{Q} := \mathcal{Q}_{\mathbb{G}(M, \mathbb{P}^N)}$ and $\mathcal{S} := \mathcal{S}_{\mathbb{G}(M, \mathbb{P}^N)}$ to be the universal quotient bundle and subbundle of rank $M + 1$ and $N - M$ on $\mathbb{G}(M, \mathbb{P}^N)$, with the exact sequence [2]. We denote by $\mathcal{Q}_{\mathcal{Y}} := \mathcal{Q}_{\mathcal{Y}}$ and so on.

As in [5, Def. 2.3], the map $\sigma$ is induced from the following composite homomorphism $\Phi$ on $\mathcal{Y}^{sm}$.

$$\Phi : \mathcal{Q}_{\mathcal{Y}}^{\mathcal{Y}^{sm}} \rightarrow \text{Hom}(\text{Hom}(\mathcal{Q}_{\mathcal{Y}}^{\mathcal{Y}^{sm}}, \mathcal{S}_{\mathcal{Y}^{sm}}), \mathcal{S}_{\mathcal{Y}^{sm}}) \rightarrow \text{Hom}(\mathcal{Q}_{\mathcal{Y}}^{\mathcal{Y}^{sm}}, \mathcal{S}_{\mathcal{Y}^{sm}}).$$
In the same way, $\sigma'$ is induced from the following composite homomorphism $\Phi'$ on $\mathcal{Y}'^{sm}$:

$$
\Phi': \mathcal{Q}'^{\mathcal{Y}^{sm}} \to \mathcal{H}om(\mathcal{H}om(\mathcal{Q}'_{\mathcal{Y}^{sm}}, S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}}), S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}}) \to \mathcal{H}om(T_{\mathcal{Y}^{sm}}, S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}}).
$$

Then, $\Phi'$ is equal to the composition of $\Phi'|_{\mathcal{Y}^{sm}}$ and

$$
\mathcal{H}om(T_{\mathcal{Y}^{sm}}, S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}})|_{\mathcal{Y}^{sm}} = \mathcal{H}om(T_{\mathcal{Y}^{sm}}|_{\mathcal{Y}^{sm}}, S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}}) \to \mathcal{H}om(T_{\mathcal{Y}^{sm}}, S_{\mathcal{Y}^{sm}}^{\mathcal{Y}^{sm}}).
$$

**Remark 3.10.** We note that, by definition, $\sigma'(y) \subset \mathbb{P}^N$ coincides with the $M_{\sigma^{-}}$-plane which is the projectivization of

$$
\ker(\Phi') \otimes k(y) \subset H^0(\mathbb{P}^N, O(1))^\vee.
$$

The $M_{\sigma^{-}}$-plane $\sigma(y) \subset \mathbb{P}^N$ is also described in the same way.

Next, we consider the pull-back of $\Phi, \Phi'$ on $X'$. Denoting by $g := \gamma|_{X'^\circ}$ for a certain open subset $X'^\circ \subset X'$, we have the following commutative diagram:

$$
\begin{array}{cccccc}
g^* \mathcal{Q}' & \to & \mathcal{H}om(g^*T_{\mathcal{Y}'}, g^*S_{\mathcal{Y}'}) & \to & \mathcal{H}om(g^*T_{\mathcal{Y}'}, g^*S_{\mathcal{Y}'}) \\
\downarrow \Phi' & & \downarrow \Psi' & & \downarrow \Psi' \\
\mathcal{H}om(T_{X'^{\text{sm}}}|_{X'^\circ}, g^*S_{\mathcal{Y}'}) & \to & \mathcal{H}om(T_{X'^{\text{sm}}}, g^*S_{\mathcal{Y}'}) & \to & \mathcal{H}om(T_{X'^{\text{sm}}}, g^*S_{\mathcal{Y}'})
\end{array}
$$

where $\Psi$ is given as the composite homomorphism

$$
\gamma^* \mathcal{Q}' \xrightarrow{\gamma^* \Phi'} \mathcal{H}om(\gamma^*T_{\mathcal{Y}'}, \gamma^*S_{\mathcal{Y}'}) \to \mathcal{H}om(T_{\mathcal{Y}'^{\text{sm}}}, \gamma^*S_{\mathcal{Y}'}),
$$

and $\Psi': g^* \mathcal{Q}' \to \mathcal{H}om(T_{X'^{\text{sm}}}, g^*S_{\mathcal{Y}'})$ is given as the composition of the second horizontal arrow and $\Psi'|_{X'^\circ}$.

**Remark 3.11.** Since $g = \gamma|_{X'^\circ}$ is separable, the second vertical arrow of the diagram is injective (by replacing $X'^\circ$ with a smaller open subset, if necessary). Hence we have

$$
\ker(\Psi') = \ker(g^*\Phi').
$$

**Lemma 3.12.** The following equality holds:

$$
\ker(\Psi'|_{X'^\circ}) = \ker(\Psi').
$$

**Proof.** By condition of $X'$, it follows that

$$
d_{x} \gamma(T_xX) = d_{x} \gamma(T_xX') \text{ in } T_y \mathcal{Y}
$$

for a general point $x \in X'$ and $y := \gamma(x) \in \mathcal{Y}$; we denote by $W$ the above vector subspace of $T_y \mathcal{Y}$. In addition, we set $S := S \otimes k(y)$. Then the following diagram is commutative with exact rows:

$$
\begin{array}{cccc}
0 & \to & \mathcal{H}om(T_y \mathcal{Y}/W, S') & \to & \mathcal{H}om(T_y \mathcal{Y}, S') & \to & \mathcal{H}om(T_xX, S') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathcal{H}om(T_y \mathcal{Y}/W, S') & \to & \mathcal{H}om(T_y \mathcal{Y}, S') & \to & \mathcal{H}om(T_xX', S') & \to & 0
\end{array}
$$
This implies that the kernels of the following homomorphisms coincide:

\[ \mathcal{H}om(g^*T_Y, g^*S^V) \rightarrow \mathcal{H}om(T_X|_{X^o}, g^*S^V), \]
\[ \mathcal{H}om(g^*T_Y, g^*S^V) \rightarrow \mathcal{H}om(T_X^o, g^*S^V). \]

It follows from the diagram \[ \text{(20)} \] that \( \ker(\Psi|_{X^o}) = \ker(\Psi') \) holds. \( \square \)

Let us consider a homomorphism \( \xi : \gamma^*Q^V \rightarrow T_{X^o}, \) where the kernel of \( \xi \) is isomorphic to \( \mathcal{O}_X(-1), \) and the projectivization of \( \ker(\xi) \otimes k(x) \subset Q^V \otimes k(\gamma(x)) \) is identified with \( x \in \mathbb{T}_xX \) for a smooth point \( x \in X \) (see the diagram \[ \text{(3)} \]). Then we have

\[ \text{(21)} \quad \ker \Psi|_{X^o} = \ker d\gamma(-1) \circ \xi|_{X^o} \]

for a certain open subset \( X^o \subset X, \) as in \[5\] Prop. 3.5 (we note that the separability of \( \gamma \) is not assumed here).

**Proof of Proposition 3.9.** From Remark 3.11 Lemma 3.12 and the formula \[ \text{(21)} \] we have

\[ \ker \Phi' \otimes k(\gamma(x)) = \ker(d\gamma(-1) \circ \xi) \otimes k(x) \quad \text{in} \quad Q^V \otimes k(\gamma(x)), \]

for a general point \( x \in X^o, \) where the projectivization of the left hand side is the \( M_\sigma \)-plane \( \sigma'(\gamma(x)) \subset \mathbb{P}^N, \) and the projectivization of the right hand side is the \( (M - r) \)-plane \( \mathbb{L}_x(\ker d_\gamma) \subset \mathbb{P}^N. \) \( \square \)

**Proof of Proposition 3.2.** Let \( X^o \subset X \) be the set of points satisfying the statement of Proposition 3.9. Let \( x' \in X \) be a general point with \( x' \in X^o, \) and let \( y = \gamma(x') \in Y. \) As in Proposition 3.1(b) we take an irreducible component \( F \) of the closure of \( \gamma^{-1}(y) \) such that \( x' \in F. \) Here, \( F \subset X'. \) For each point \( x \in F \cap X^o, \) we have \( \mathbb{L}_x(\ker d_\gamma) = \sigma'(\gamma(x)) = \sigma'(y). \) Thus the assertion follows. \( \square \)

Now Theorem 1.2 is proved as follows.

**Remark 3.13.** (a) Let \( \alpha \in X \) be a general point, and let \( X_\alpha^{o} \subset X \) be a closed subvariety containing \( \alpha \) and satisfying the statement of Proposition 3.1. We denote by \( X_\alpha^{o} \subset X_\alpha \) the set of points satisfying the condition of Proposition 3.1(b) and Proposition 3.2. Here,

\[ X = \bigcup_{\alpha \in X; \text{general}} X_\alpha^{o}. \]

(b) Let \( x' \in X \) be a general point. Then, for an irreducible component \( F \subset X \) of the closure of the fiber \( \gamma^{-1}(\gamma(x')) \) such that \( x' \in F, \) it follows that \( \mathbb{L}_x(\ker d_\gamma) \) is constant on general \( x \in F. \) This is because, \( x' \in X_\alpha^{o} \) for some \( \alpha. \)

**Remark 3.14.** Let \( Z \subset X \) be a closed subset with \( Z \neq X, \) i.e., \( \dim(Z) < M. \) Then each irreducible component \( F \) of a general fiber of \( \gamma : X \rightarrow Y \) is not contained in \( Z. \) Otherwise, the dimension of a general fiber of \( Z \rightarrow Y \) is equal to \( M - \dim(Y), \) which implies \( \dim Z = M, \) a contradiction.
Proof of [Theorem 1.2]. Let $X^o$ be the set of points satisfying the statement of [Remark 3.13(b)] and let $Z = X \setminus X^o$. Then, as in [Remark 3.14] each irreducible component $F$ of a general fiber of $\gamma$ is not contained in $Z$. Hence the assertion follows. □

Remark 3.15. Let $X \subset \mathbb{P}^N$ be a projective variety. We consider the degeneracy map $\kappa : X \to G(m_0, \mathbb{P}^N)$.

(a) [Theorem 1.2] implies the following formula:
\[
\dim(\kappa(X)) \leq \dim(\gamma(X)).
\]
The reason is as follows: Set $X_0 := \kappa(X)$. For an irreducible component $F$ of a general fiber of $\gamma$, we find that $\kappa(F)$ is equal to a set of one point of $X_0$. Hence a general fiber of $\kappa$ contains such a component $F$. It follows that $\dim(X) - \dim(X_0) \geq \dim(F) = \dim(X) - \dim(\gamma(X))$. Hence the assertion holds.

(b) The equality holds in (22) if $\gamma$ is separable. In general, it is possible that $\dim(X_0)$ can be strictly less than $\dim(\gamma(X))$ (see [Example 4.8]).

(c) We can regard $X$ as a subvariety of the union of $m_0$-planes
\[
\bigcup_{x \in X; \text{general}} \mathbb{P}(\ker d_x \gamma),
\]
where the union is equal to the image of the universal family $U_{X_0} \subset X_0 \times \mathbb{P}^N$ of $X_0$ under the second projection, and hence the union is of dimension $\leq \dim(X) + \dim(\gamma(X)) - \text{rk}(\gamma)$. (An example is given in [Example 2.3].)

4. Gauss maps of strange varieties

Assume that $X \subset \mathbb{P}^N$ is an $M$-dimensional projective variety which is strange for a point $v \in \mathbb{P}^N$. Let $\gamma : X \to \mathcal{Y} := \gamma(X) \subset G(M, \mathbb{P}^N)$ be the Gauss map of $X$, and let $X' \subset X$ and $\mathcal{Y}' \subset \mathcal{Y}$ be closed subvarieties given in [Proposition 3.1]. Here we have:

Proposition 4.1. Assume that $X$ is strange for $v$. Let $\sigma' : \mathcal{Y}' \to G(M_{\sigma'}, \mathbb{P}^N)$ be the shrinking map of $\mathcal{Y}' \subset G(M, \mathbb{P}^N)$. Then the $M_{\sigma'}$-plane $\sigma'(y) \subset \mathbb{P}^N$ contains the point $v$ for general $y \in \mathcal{Y}'$.

To show the result, we consider the projection $\pi_v : \mathbb{P}^N \setminus \{v\} \to \mathbb{P}^{N-1}$ from $v$, yielding an inclusion
\[
\mathbb{G}_v := G(M-1, \mathbb{P}^{N-1}) \hookrightarrow G(M, \mathbb{P}^N)
\]
which sends an $(M-1)$-plane $L \subset \mathbb{P}^{M-1}$ to the $M$-plane $\pi_v^{-1}(L) \cup \{v\} \subset \mathbb{P}^N$. Regarding $\mathbb{G}_v \subset G(M, \mathbb{P}^N)$ as the space of $M$-planes in $\mathbb{P}^N$ containing $v$, since $X$ is strange for $v$, we have
\[
\mathcal{Y}' \subset \mathcal{Y} \subset \mathbb{G}_v.
\]

In this section, we denote by $\mathcal{Q} := \mathcal{Q}_{G(M, \mathbb{P}^N)}$ and $\mathcal{S} := \mathcal{S}_{G(M, \mathbb{P}^N)}$ the universal quotient bundle and subbundle of ranks $M+1$ and $N-M$ on $G(M, \mathbb{P}^N)$, and...
by \( Q_v := Q_{G_v} \) and \( S_v := S_{G_v} \) the universal quotient bundle and subbundle of ranks \( M \) and \( N - M \) on \( G_v \).

**Remark 4.2.** (a) We have \( S|_{G_v} = S_v \) and have a natural injection \( Q_v \hookrightarrow Q|_{G_v} \) with the following commutative diagram:

\[
\begin{array}{c}
0 \\ \downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
S_v \\
H^0(P^{N-1}, O(1)) \otimes O_{G_v} \\
Q_v \\
\downarrow \\
0
\end{array}
\quad
\begin{array}{c}
H^0(P^N, O(1)) \otimes O_{G_v} \\
Q|_{G_v} \\
\downarrow \\
0.
\end{array}
\]

(b) Let \( E \) be the kernel of the surjection \( Q|_{G_v}^Y \to Q_v^Y \) induced from the dual of the above injection. Let \( y \) be a member of \( G(M, P^N) \) belonging to \( G_v \). Then the projectivization of \( E \otimes k(y) \subset Q|_{G_v}^Y \otimes k(y) \) corresponds to the point \( v \) contained in the \( m \)-plane \( y \) in \( P^N \).

**Proof of Proposition 4.1.** We set \( \Phi^Y : Q_v^Y \mid y_{\text{sm}} \to \text{Hom}(T_{y_{\text{sm}}}, S_v^Y \mid y_{\text{sm}}) \) as in §3.2 where the projectivization of \( \text{ker}(\Phi^Y) \otimes k(y) \) corresponds to the \( M^- \)-plane \( \sigma'(y) \subset P^N \). In a similar way, we set \( \Phi'_v \) by the composite map

\[
\Phi'_v : Q_v^Y \mid y_{\text{sm}} \to \text{Hom}(\text{Hom}(Q_v^Y \mid y_{\text{sm}}, S_v^Y \mid y_{\text{sm}}), S_v^Y \mid y_{\text{sm}}) \to \text{Hom}(T_{y_{\text{sm}}}, S_v^Y \mid y_{\text{sm}}).
\]

Here the following commutative diagram holds:

\[
\begin{array}{c}
Q_v^Y \mid y_{\text{sm}} \\
\downarrow \\
Q_v^Y \\
\downarrow \\
\end{array}
\quad
\begin{array}{c}
\Phi^Y \\
\text{Hom}(T_{y_{\text{sm}}}, S_v^Y \mid y_{\text{sm}}) \\
\downarrow \\
\Phi'_v \\
\text{Hom}(T_{y_{\text{sm}}}, S_v^Y \mid y_{\text{sm}})
\end{array}
\]

Therefore \( \text{ker}(\Phi^Y) \otimes k(y) \) contains \( E \otimes k(y) \) for general \( y \in Y^\nu \). This means that \( v \in \sigma'(y) \), as in [Remark 4.2](#).

**Proof of Theorem 1.3.** Assume that \( X \) is strange for \( v \). As in [Remark 3.13](#) a general point \( x \in X \) is regarded as a general point of \( X' \) with some closed subvariety \( X' \subset X \). As in [Proposition 3.9](#) the plane \( \sigma'(\gamma(x)) \subset P^N \) coincides with \( L_x(\ker d_x\gamma) \). Thus, it follows from [Proposition 4.1](#) that \( L_x(\ker d_x\gamma) \) contains the point \( v \).

If \( \gamma \) is separable, then \( X \) must be a cone with vertex \( v \). This is because, for general \( x \in X \), \( L_x(\ker d_x\gamma) \) coincides with the closure of the fiber \( \gamma^{-1}(\gamma(x)) \subset X \). As in [Remark 2.1](#), thus \( \pi_{v, X} \subset X \) holds.

**Remark 4.3.** We say that \( X \) is strange for a linear variety \( V \subset P^N \) if \( V \subset T_x X \) for any smooth point \( x \in X \), equivalently, \( X \) is strange for any point \( v \in V \). If \( X \) is strange for \( V \), then \( L_x(\ker d_x\gamma) \) contains \( V \) for general point \( x \in X \). This immediately follows from [Theorem 1.3](#) since we have \( v \in L_x(\ker d_x\gamma) \) for any \( v \in V \).

**Remark 4.4.** Let \( X \subset P^N \) be strange for a point \( v \in P^N \). If \( X \subset P^N \) is not a cone with vertex \( v \), then the linear projection \( \pi_v \mid X : X \dasharrow P^{N-1} \) is
inseparable. The reason is as follows. Since \( X \) is not a cone, \( \pi_v \) is generically finite onto its image. Since \( X \) is strange, \( T_x \mathcal{X} \subset T_x X \) for general \( x \in X \). Since \( d_v \pi_v(T_x \mathcal{X}) = 0 \), we have \( \dim d_v \pi_v(T_x X) < \dim(X) \).

**Example 4.5.** Let \( X \subset \mathbb{P}^N \) be an \( M \)-dimensional projective variety which is strange for \( v \in \mathbb{P}^N \) and satisfies \( \text{rk}(\gamma) = M - 1 \). Assume that \( X \) is not a cone with vertex \( v \). (We can give such a variety \( X \), e.g., Example 4.7 below.) Then Theorem 1.3 implies that \( \gamma \) is inseparable and that \( \mathbb{L}_x(\ker d_x \gamma) = \mathcal{X} \). Hence, in this case, the degeneracy map \( \kappa : X \rightarrow \mathbb{G}(1, \mathbb{P}^N) \) factors into the linear projection \( \pi_1 \vert_X : X \rightarrow \mathbb{P}^{N-1} \) followed by the natural inclusion \( \mathbb{P}^{N-1} \hookrightarrow \mathbb{G}(1, \mathbb{P}^N) \) induced from \( \pi_1 \) as in [23]. In particular, \( \kappa \) is inseparable, since so is \( \pi_1 \) as in Remark 4.4.

According to Fukasawa’s results, general fibers of an inseparable Gauss map can be non-linear. We check Theorem 1.3 in such a case:

**Example 4.6.** Let the characteristic of the ground field \( K \) be 3, and let \( X \subset \mathbb{P}^4 \) be a 3-fold defined by the homogeneous polynomial,\[
g = (Z^1)^6 + (Z^2)^6 + Z^3 Z^4 (Z^0)^4,\]
where \((Z^0 : Z^1 : \cdots : Z^4)\) are the homogeneous coordinates on \( \mathbb{P}^4 \). Then we have:

(a) The Gauss map \( \gamma \) of \( X \) is inseparable, and its general fiber is set-theoretically equal to a smooth conic \( C \subset X \).

(b) \( \mathbb{L}_x(\ker d_x \gamma) \) is constant for general \( x \) in such a conic \( C \), and moreover, is equal to the 2-plane spanned by \( C \).

The reason is as follows. We can identify \( \gamma : X \rightarrow (\mathbb{P}^4)^\vee \) with the rational map sending \((Z^0 : \cdots : Z^4)\) to
\[
(\partial g/\partial Z^i)_{0 \leq i \leq 4} = (Z^3 Z^4 (Z^0)^3 : 0 : Z^4 (Z^0)^4 : Z^3 (Z^0)^4).
\]
Let \( \ell = (Z^0 = Z^3 = Z^4 = 0) \), a line in \( \mathbb{P}^4 \). Then \( X \) is strange for \( \ell \). The image of \( \gamma \) is equal to \( \ell^* \subset (\mathbb{P}^4)^\vee \), the set of hyperplanes containing \( \ell \). Here \( \ell^* \) is a 2-plane of \((\mathbb{P}^4)^\vee \). On the other hand, the rank of \( \gamma \) is equal to 1. For a general point \( x \in X \), it follows from Theorem 1.3 that
\[
\mathbb{L}_x(\ker d_x \gamma) = \langle x, \ell \rangle,
\]
where the right hand side is the 2-plane in \( \mathbb{P}^4 \) spanned by \( x \) and \( \ell \).

Now, we fix a general point \( \alpha = (1 : \alpha_1 : \cdots : \alpha_4) \in X \) such that \( \alpha_3, \alpha_4 \) are nonzero. Then we have \( \gamma(\alpha) = (1 : 0 : 0 : 1/\alpha_3 : 1/\alpha_4) \) in \((\mathbb{P}^4)^\vee \). On the other hand, the 2-plane \( \mathbb{L}_x(\ker d_x \alpha) = \langle \alpha, \ell \rangle \) is defined by two homogeneous polynomials \( \alpha_3 Z^0 - Z^3, \alpha_4 Z^0 - Z^4 \). Let us consider \( C_\alpha = \langle \alpha, \ell \rangle \cap X \), whose defining polynomials are
\[
(Z^1)^6 + (Z^2)^6 + \alpha_3 \alpha_4 (Z^0)^6, \alpha_3 Z^0 - Z^3, \alpha_4 Z^0 - Z^4.
\]
Then \( C_\alpha \subset \mathbb{P}^4 \) is set-theoretically equal to the smooth conic in \( \langle \alpha, \ell \rangle \),
\[
((Z^1)^2 + (Z^2)^2 + \sqrt{\alpha_3 \alpha_4} (Z^0)^2)^2 = \alpha_3 Z^0 - Z^3 = \alpha_4 Z^0 - Z^4 = 0,
\]
Theorem 1.2

We have \( \gamma(C_\alpha) = \{ \gamma(\alpha) \} \), since the coordinates of a general point of \( C_\alpha \) is written by \((1 : * : * : \alpha_3 : \alpha_4)\). If a point \( \beta = (1 : \beta_1 : \cdots : \beta_4) \in X \) satisfies \( \gamma(\alpha) = \gamma(\beta) \), then we have \( \beta_3 = \alpha_3, \beta_4 = \alpha_4 \), and then \( C_\alpha = C_\beta \) holds because of their defining polynomials. Hence \( \gamma^{-1}(\gamma(\alpha)) \) coincides with the conic \( C_\alpha \), which means the statement of (a). For general \( x \in C_\alpha \), the 2-plane \( \langle x, \ell \rangle \) is spanned by \( C_\alpha \), which implies the statement of (b).

In Theorem 1.2 in the case where a general fiber of the Gauss map is not irreducible, the constant \( m_0 \)-planes of irreducible components can be different:

Example 4.7. Let the characteristic be 3, and let \( X \subset \mathbb{P}^3 \) be the surface defined by the homogeneous polynomial,

\[
(24) \quad g = (Z^0)^5 + (Z^1)^5 - (Z^2)^3(Z^3)^2.
\]

Then we have:

(a) The Gauss map \( \gamma \) of \( X \) is inseparable, and its general fiber is set-theoretically equal to a set of 4 points.

(b) \( \mathbb{L}_x(\ker d_x \gamma) \neq \mathbb{L}_{x'}(\ker d_{x'} \gamma) \) for distinct \( x, x' \) of the above 4 points.

The reason is as follows. The map \( \gamma : X \to (\mathbb{P}^3)^\flat \) is given by

\[
(25) \quad (\partial g/\partial Z^i)_{0 \leq i \leq 3} = \left( -(Z^0)^4 : -(Z^1)^4 : 0 : (Z^2)^3(Z^3) \right).
\]

Thus \( X \) is strange for \( v := (0 : 0 : 1 : 0) \), and the rank of \( \gamma \) is equal to 1. In addition, \( \gamma \) is generically finite; thus, each irreducible component of a general fiber of \( \gamma \) is a set of one point.

Let \( x = (1 : a : b : c) \in X \) be a general point such that \( a \neq 0 \). Let \( x' = (w : a' : b' : c') \in X \) be a point satisfying \( \gamma(x') = \gamma(x) \). By (25), we have

\[
(-a^4 : -a^4 : 0 : b^3c) = (-1 : -a^4 : 0 : b^3c).
\]

Since \( w \) must be nonzero, we can set \( w = 1 \) and express \( x' = (1 : a' : b' : c') \). Then the above equality implies \( a^4 = a^4 \) and \( b^3c = b^3c \). Thus \( a' \) is a 4-th root of \( a^4 \), which is equal to a 4-th root of 1 multiplied by \( a \). Let \( \zeta \in K \) be the square root of -1. Then the set of 4-th roots of 1 consists of 4 elements 1, \(-1, \zeta, -\zeta \). Let \( \zeta \in K \) be one of the 4-th roots of 1, and set \( a' = a\zeta \). Here, \( b', c' \in K \) are uniquely determined, as follows. From (24), we have \( 1 + a^5 - b^5c^2 = 0 \), and also have \( 1 + (a\zeta)^5 - b^3c^2 = 0 \). Thus,

\[
a^5\zeta = \zeta(b^3c^2 - 1) = b^3c^2 - 1.
\]

It follows from \( (b')^5c' = b^3c \) that \( \zeta(b^3c^2 - 1) = b^3c^2 - 1 = b^3cc' - 1 \), which implies

\[
c' = \frac{\zeta(b^3c^2 - 1) + 1}{b^3c}.
\]

Hence we also have

\[
b^3 = -\frac{b^3c^2}{\zeta(b^3c^2 - 1) + 1}.
\]
Thus $b'$ is obtained as the unique third root of the right hand side of the above formula.

We set $x_ξ := (1 : aξ : b' : c') ∈ X$, where $b', c' ∈ K$ are uniquely determined from a 4-th root $ξ$ of 1, as above. Then,
\[
γ⁻¹(γ(x)) = \{ x, x₋₁, x_ξ, x₋ξ \}.
\]

For the strange point $v := (0 : 0 : 1 : 0)$, we have $x_v = x_v$ for each $x' ∈ γ⁻¹(γ(x))$ with $x' ≠ x$. On the other hand, \textbf{Theorem 1.3} implies that $L_x(\ker d_γγ) = x_v$ and $L_x'(\ker d'_γγ) = x_v′$. Here the two line are distinct; hence the assertion of \textbf{Remark 3.15} follows.

Finally, we see that “$\dim(κ(X)) ≠ \dim(γ(X))$” can occur in the formula \textbf{(22)} in \textbf{Remark 3.15} if the Gauss map $γ$ is inseparable:

\textbf{Example 4.8.} Let the characteristic be 3, and let $X ⊂ P^6$ be a 4-fold by the following 3 homogeneous polynomials,
\[
(Z^0)^4Z^5 + (Z^1)^2(Z^2)^3 + (Z^0)^2(Z^3)^3, (Z^0)^4Z^6 + (Z^1)^2(Z^4)^3,
\]
\[
(Z^0)^2(Z^2)^3(Z^6) − (Z^0)^2(Z^4)^3(Z^5) − (Z^3)^3(Z^3)^3.
\]

Then we have:

(a) The Gauss map $γ$ of $X$ is inseparable with $\text{rk}(γ) = 1$, and is generically bijective. In particular, we have $\text{dim}(\text{im}(γ)) = 4$.

(b) $X$ is strange for the 2-plane $V = (Z^0 = Z^1 = Z^5 = Z^6 = 0) ⊂ P^6$, and $L_x(\ker d_γγ)$ coincides with 3-plane $\langle x, V \rangle$ for general $x ∈ X$.

(c) $κ(x)$ is constant for general $x ∈ \langle x, V \rangle ∩ X$; this implies $\text{dim}(κ(X)) = 3$.

The reason is as follows: (a) We parametrize $x ∈ X$ around $x_o = (1 : 0 : ⋯ : 0)$ by $(1 : s : t : u : v : g : h)$, where $s,t,u,v$ are the regular local parameters, and where $g,h$ satisfies $g + s^2t^3 + u^3 = h + s^2v^3 = 0$. Then, by using the description \textbf{(6)} we can check that the Gauss map $γ : X → GL_3^0$ is of rank 1 and is generically bijective.

(b) Since $\text{rk}(γ) = 1$ and since $X$ is strange for the 2-plane $V$, it follows from \textbf{Theorem 1.3} and \textbf{Remark 4.3} that $L_x(\ker d_γγ) = \langle x, V \rangle$ for general $x$.

(c) For general 3-plane $W ⊂ P^6$ containing $V$, we have $\text{dim}(W ∩ X) = 1$. For general $x ∈ W ∩ X$, we have $L_x(\ker d_γγ) = \langle x, V \rangle = W$. Hence the fiber of $κ$ at $W ∈ G(3,P^6)$ is of dimension 1.

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E-mail address: katu@toki.waseda.jp

DEPARTMENT OF MATHEMATICS, SCHOOLS OF FUNDAMENTAL SCIENCE AND ENGINEERING, WASEDA UNIVERSITY, OHKUBO 3-4-1, SHINJUKU, TOKYO, 169-8555, JAPAN
URL: http://www.aoni.waseda.jp/katu/index.html