Proof of a conjecture of Wiegold for nilpotent Lie algebras

A. A. Skutin

Abstract. Let \( g \) be a nilpotent Lie algebra. By the breadth \( b(x) \) of an element \( x \) of \( g \) we mean the number \( |g : C_g(x)| \). Vaughan-Lee showed that if the breadth of all elements of the Lie algebra \( g \) is bounded by a number \( n \), then the dimension of the commutator subalgebra of the Lie algebra does not exceed \( n(n+1)/2 \). We show that if \( \dim g' > n(n+1)/2 \) for some nonnegative \( n \), then the Lie algebra \( g \) is generated by the elements of breadth \( > n \), and thus we prove a conjecture due to Wiegold (Question 4.69 in the Kourovka Notebook) in the case of nilpotent Lie algebras.

Bibliography: 4 titles.

Keywords: nilpotent Lie algebras, finite \( p \)-groups, breadth of an element, estimate for the size of the commutator subalgebra.

§1. Introduction

In this paper we prove an analogue of a conjecture of Wiegold’s (see [1], 4.69) for nilpotent Lie algebras.

Definition 1. The breadth \( b(x) \) of an element \( x \) of a finite \( p \)-group \( G \) is the number satisfying \( p^{b(x)} = |G : C_G(x)| \), where \( C_G(x) \) is the centralizer of the element \( x \) in \( G \).

The Wiegold conjecture for \( p \)-groups is formulated as follows.

Conjecture 1. Let \( G \) be a finite \( p \)-group and let \( |G'| > p^{n(n-1)/2} \) for some nonnegative integer \( n \). Then \( G \) is generated by the elements of breadth at least \( n \).

A survey of this conjecture can be found in [2]. In [3] Vaughan-Lee proved that for every finite \( p \)-group \( G \) of breadth \( b = \max_{g \in G} b(g) \) we have \( |G'| \leq p^{b(b-1)/2} \).

Conjecture 1 was proved by this author in [4]. Here we prove an analogue of Conjecture 1 suggested by Ant. A. Klyachko for nilpotent Lie algebras.

Definition 2. The breadth \( b(x) \) of an element \( x \) of a nilpotent Lie algebra \( \mathcal{A} \) is the number satisfying \( b(x) = \dim(\mathcal{A}) - \dim(C_{\mathcal{A}}(x)) \), where \( C_{\mathcal{A}}(x) \) denotes the centralizer of the element \( x \) in \( \mathcal{A} \).

Conjecture 2. Let \( g \) be a nilpotent Lie algebra for which it is known that \( \dim g' > n(n-1)/2 \) for some nonnegative \( n \). Then \( g \) is generated by the elements of breadth at least \( n \).

This work was supported by the Russian Foundation for Basic Research (grant no. 19-01-00591-a).

AMS 2020 Mathematics Subject Classification. Primary 17B20; Secondary 17B50.

© 2020 Russian Academy of Sciences (DoM) and London Mathematical Society
In this paper we prove Conjecture 2; moreover, we prove that the following theorems hold.

**Theorem 1.** Let $g$ be a nilpotent Lie algebra over a finite field $F \neq F_2$ and let the dimension of the commutator subalgebra of $g$ be strictly greater than $n(n-1)/2$ for some nonnegative integer $n$. Then the set of elements of $g$ which have breadth at least $n$ cannot be covered by $|F| - 1$ proper subalgebras of $g$.

**Theorem 2.** Let $g$ be a nilpotent Lie algebra over the field $F_2$ and let the dimension of the commutator subalgebra of $g$ be strictly greater than $n(n-1)/2$ for some nonnegative integer $n$. Then the set of elements of $g$ which have breadth at least $n$ cannot be covered by two proper subalgebras of $g$ one of which has codimension at least $2$ in $g$.

**Theorem 3.** Let $g$ be a nilpotent Lie algebra over an infinite field $F$ and let the dimension of the commutator subalgebra of $g$ be strictly greater than $n(n-1)/2$ for some nonnegative integer $n$. Then the set of elements of $g$ which have breadth at least $n$ cannot be covered by finitely many proper subalgebras of $g$.

§ 2. Statements and proofs of the main lemmas

**Definition 3.** The breadth $b_{h}(x)$ of an element $x$ of a finite-dimensional Lie algebra $g$ with respect to a proper subalgebra $h \subseteq g$ is the number such that $\dim(h/C_{h}(x)) = b_{h}(x)$, where $C_{h}(x) = \{h \in h \mid [x, h] = 0\}$ is the centralizer of $x$ in $h$. It follows from this definition that $b(x) = b_{g}(x)$.

The next two lemmas are well known in the theory of Lie algebras, and therefore we present them without proof.

**Lemma 1.** Let $g$ be a Lie algebra over a finite field $F$. Then $g$ cannot be covered by $|F|$ proper subalgebras. Moreover, if $g$ is covered by $|F| + 1$ proper subalgebras $h_1, h_2, \ldots, h_{|F|+1}$, then every $h_i$ is of codimension $1$ in $g$.

**Lemma 2.** Let $g$ be a nilpotent Lie algebra over a field $F$. Suppose that there is a central subalgebra $f$ of codimension $2$ in $g$. Then $\dim g' \leq 1$.

**Lemma 3.** Consider an ideal $h$ of a finite-dimensional nilpotent Lie algebra $g$ which has codimension $1$ in $g$. Let $f$ denote the ideal of $g$ generated by elements $x \in h$ such that $b_{h}(x) = b(x)$. In this case

1) if $f = h$, then $g' = h'$;

2) if $f$ is of codimension at most $1$ in $h$, then the commutator subalgebra $b'$ also has codimension at most $1$ in $g'$.

**Proof.** Consider the factorization homomorphism $\pi : g \rightarrow g/h'$. For every element $x$ in $h$ such that $b_{h}(x) = b(x)$ we have $\{[x, y] \mid y \in g\} = \{[x, z] \mid z \in h\} \subseteq h'$. Therefore, $\pi(x)$ is contained in the centre of $\pi(g)$, and $\pi(f)$ is a central Lie subalgebra of $\pi(g)$. If $f = h$, then we see that $\pi(f) = \pi(h)$ is a central Lie subalgebra of $\pi(g)$ of codimension $1$, and therefore $\pi(g)$ is Abelian and $g' = h'$. In the case when $f$ has codimension not exceeding $1$ in $h$ we see that $\pi(f)$ is a central Lie subalgebra of $\pi(g)$ of codimension not exceeding $2$. Therefore, by Lemma 2, we have $\dim \pi(g') \leq 1$ and $h'$ has codimension not exceeding $1$ in $g'$.

This completes the proof of the lemma.
Lemma 4. Let $\mathfrak{g}$ be a finite-dimensional nilpotent Lie algebra. Then for every ideal $\mathfrak{h}$ of codimension 1 of $\mathfrak{g}$ and every element $x$ belonging to $\mathfrak{g} \setminus \mathfrak{h}$ we have $\dim \mathfrak{g}' \leq b(x) + \dim \mathfrak{h}'$.

Proof. The set $V = \{ [x, h] \mid h \in \mathfrak{h} \}$ forms a vector space of dimension not exceeding $b(x)$. Therefore, it suffices to prove that $\mathfrak{g}' = V + \mathfrak{h}'$. The commutator subalgebra $\mathfrak{h}'$ of the ideal $\mathfrak{h}$ is also an ideal of the Lie algebra $\mathfrak{g}$. We claim that $V + \mathfrak{h}'$ is an ideal of $\mathfrak{g}$:

$$[[x, h], g] \in \mathfrak{h}' \quad \text{for all} \quad [x, h], g \in \mathfrak{g} \setminus \mathfrak{h}. $$

It is clear that the image of the element $x$ under the homomorphism $\pi: \mathfrak{g} \to \mathfrak{g}/(V + \mathfrak{h}')$ commutes with itself and with the ideal $\pi(\mathfrak{h})$ of codimension 1 in $\pi(\mathfrak{g})$, and thus $\pi(x)$ belongs to the centre of $\pi(\mathfrak{g})$. Moreover, $\pi(\mathfrak{h})$ is an Abelian Lie algebra, since the commutator subalgebra of $\mathfrak{h}$ is contained in $\ker \pi$. This implies that $\pi(\mathfrak{g}) = \mathfrak{g}/(V + \mathfrak{h}')$ is Abelian and $\mathfrak{g}' \subseteq V + \mathfrak{h}' \subseteq \mathfrak{g}'$, $\mathfrak{g}' = V + \mathfrak{h}'$. This completes the proof of the lemma.

Lemma 5. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then for every number $n \leq \dim \mathfrak{g}'$ there is a finite-dimensional Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that $\dim \mathfrak{g}' \geq \dim \mathfrak{h}' \geq n$.

Proof. By assumption, $n \leq \dim \mathfrak{g}'$; hence there are linearly independent elements $a_1, \ldots, a_n \in \mathfrak{g}'$. Let $A \subseteq \mathfrak{g}$ be a finite subset such that the linear space $\langle [A, A] \rangle$ contains $a_1, \ldots, a_n$. The Lie algebra $\mathfrak{g}$ is nilpotent; so the Lie algebra $\mathfrak{h}$ generated by the set $A$ is finite-dimensional. By construction, $\{a_1, \ldots, a_n\} \subseteq \mathfrak{h}'$ and $\dim \mathfrak{h}' \geq n$. This completes the proof of the lemma.

§ 3. Proof of Theorem 1

We argue by contradiction. Assume that proper subalgebras $\mathfrak{h}_1, \ldots, \mathfrak{h}_{|\mathfrak{F}|-1}$ cover all elements whose breadth is not less that $n$. We claim that, in this case, $\dim \mathfrak{g}' \leq n(n - 1)/2$. We apply Lemma 5 to the Lie algebra $\mathfrak{g}$ and the number $\min(\dim \mathfrak{g}', n(n - 1)/2 + 1) \leq \dim \mathfrak{g}'$; we see that there is a finite-dimensional subalgebra $\mathfrak{u}$ of $\mathfrak{g}$ such that $\dim \mathfrak{u}' \geq \min(\dim \mathfrak{g}', n(n - 1)/2 + 1)$. We can also assume that $\mathfrak{u}$ is not contained in any $\mathfrak{h}_i$. Replacing $\mathfrak{g}$ by $\mathfrak{u}$ and the subalgebras $\mathfrak{h}_i$ by $\mathfrak{h}_i \cap \mathfrak{u}$, we may assume that the Lie algebra $\mathfrak{g}$ is finite-dimensional. We use induction on the dimension of $\mathfrak{g}$. We may assume that every $\mathfrak{h}_i$ is a maximal ideal of codimension 1 in $\mathfrak{g}$ (since every proper Lie subalgebra of a finite-dimensional nilpotent Lie algebra is contained in some ideal of codimension 1) and that $\mathfrak{h}_1 \neq \mathfrak{h}_2$ (since every non-Abelian finite-dimensional nilpotent Lie algebra $\mathfrak{g}$ has at least two distinct maximal ideals). Consider an arbitrary ideal $I \neq \mathfrak{h}_i$ of codimension 1 in $\mathfrak{g}$ such that $I \cap \mathfrak{h}_1 = I \cap \mathfrak{h}_2 = \mathfrak{h}_1 \cap \mathfrak{h}_2$. We let $J$ denote the ideal of $\mathfrak{g}$ generated by the elements $x$ belonging to $I$ for which $b_I(x) = b(x)$.

Consider the case $J = I$. Applying Lemma 3 to $I < \mathfrak{g}$ we see that $I' = \mathfrak{g}'$. The rest of the proof follows from the induction assumption: the Lie algebra $I$ has lower dimension, and all its elements of breadth at least $n$ are contained in the union of the proper subalgebras $I \cap \mathfrak{h}_i$ (since $b_I(x) \leq b(x)$).

Now we may assume that $I \neq J$; we note that the set $I \setminus (J \cup \mathfrak{h}_i)$ consists of elements of the breadth no greater than $n - 2$ in $I$ (since $J$ is generated by
the elements \( \{ g \in I \mid b_I(g) = b(g) \} \) and \( b_I(x) \leq b(x) \). Applying the induction assumption to the Lie algebra \( I \) and the \( |F| - 1 \) proper subalgebras \( J, I \cap h_1 = I \cap h_2, I \cap h_3, \ldots, I \cap h_{|F| - 1}, \) we conclude that \( \dim I' \leq (n - 1)(n - 2)/2 \). Finally, consider an arbitrary element \( a \) outside \( I \cup h_i \) (such an element exists by Lemma 1). The breadth of \( a \) does not exceed \( n - 1 \), and therefore \( \dim g' \leq b(a) + \dim I' \leq n(n - 1)/2 \) by Lemma 4.

§ 4. Proof of Theorem 2

Theorem 4. Let \( g \) be a finite-dimensional nilpotent Lie algebra over a finite field \( F \) such that the following two conditions hold for some nonnegative integers \( n \) and \( k \), \( n \leq k + 1 \):

1) the set of all elements of breadth at least \( n \) is covered by \( |F| \) proper subalgebras \( h_i \) of the Lie algebra \( g \);

2) the Lie algebra \( g \) is generated by the set of elements of breadth not exceeding \( k \).

Then \( \dim g' \leq (n - 1)(n - 2)/2 + k \).

Proof. We use induction on the dimension of \( g \). We may assume that the \( h_i \) are maximal ideals of codimension 1 in \( g \) (since every proper Lie subalgebra of a finite-dimensional nilpotent Lie algebra is contained in some ideal of codimension 1) and \( h_1 \neq h_2 \) (since every non-Abelian finite-dimensional nilpotent Lie algebra \( g \) has at least two distinct maximal ideals). Consider an arbitrary ideal \( I \neq h_i \) of codimension 1 in \( g \) such that \( I \cap h_1 = I \cap h_2 = h_1 \cap h_2 \). We denote the ideal of \( g \) generated by the elements \( x \) of \( I \) such that \( b_I(x) = b(x) \) by \( J \).

Consider the case \( J = I \). Applying Lemma 3 to \( I \triangleleft g \), we see that \( I' = g' \). The rest of the proof follows from the induction assumption: the Lie algebra \( I \) and all its elements of breadth at least \( n \) are contained in the union of the proper subalgebras \( I \cap h_i \) (since \( b_I(x) \leq b(x) \)); moreover, the set \( I \setminus \bigcup_{i \geq 2} h_i \) generates \( I \) (by Lemma 1) and contains the elements whose breadth does not exceed \( n - 1 \leq k \).

Thus, we may assume that \( J \) is a proper ideal of the Lie algebra \( I \). Consider the case when the codimension of \( J \) in \( I \) is 1. Applying Lemma 3 to \( I \triangleleft g \) we see that \( \dim g' \leq \dim I' + 1 \). Every element of the set \( I \setminus (h_1 \cap h_2) \cup h_3 \cup \cdots \cup h_{|F|} \) has breadth not exceeding \( n - 1 \) in \( I \), and this set generates \( I \) (by Lemma 1). Moreover, for every element \( g \) of \( I \setminus (J \cup (h_1 \cap h_2) \cup h_3 \cup \cdots \cup h_{|F|}) \) we have \( b_J(g) \leq n - 2 \). Applying the induction assumption to the Lie algebra \( I \), its proper subalgebras \( J, h_1 \cap h_2, h_3, \ldots, h_{|F|} \), and the parameters \( n - 1 \) and \( n - 1 \) we see that \( \dim I' \leq (n - 3)(n - 2)/2 + n - 1 \). Therefore, in the case when \( k \geq 2 \) we have

\[
\dim g' \leq \dim I' + 1 \leq \frac{(n - 3)(n - 2)}{2} + n = \frac{(n - 2)(n - 1)}{2} + 2 \leq \frac{(n - 2)(n - 1)}{2} + k\]

and the induction step is proved. In the case when \( k \leq 1 \) we have \( n = 2 \) (the case \( n \leq 1 \) is trivial), and the set \( I \setminus (J \cup (h_1 \cap h_2) \cup h_3 \cup \cdots \cup h_{|F|}) \) is contained in the centre of the Lie algebra \( I \). By Lemma 1 the set \( I \setminus (J \cup (h_1 \cap h_2) \cup h_3 \cup \cdots \cup h_{|F|}) \) generates a central Lie subalgebra of codimension 1 in \( I \), and thus \( I \) is Abelian. By the assumptions of Theorem 4 there is an element \( g \notin I \) such that \( b(g) \leq k \leq 1 \). Thus, the Lie subalgebra \( C_2(g) \cap I \) has codimension not exceeding 2 in \( g \) and is central in \( g \). Applying Lemma 2 to the Lie algebra \( g \), we see that \( \dim g' \leq 1 \leq (n - 1)(n - 2)/2 + k \).
Finally, we may assume that $J$ has codimension at least 2 in $I$. Thus, the set $I \setminus (J \cup (h_1 \cap h_2) \cup h_3 \cup \cdots \cup h_\mathbb{F})$ generates $I$ (by Lemma 1), and so $I$ is generated by elements of breadth at most $n - 2$ in $I$. We apply the induction assumption to the Lie algebra $I$, its proper subalgebras $J$, $h_1 \cap h_2$, $h_3$, $\ldots$, $h_\mathbb{F}$, and the parameters $n - 1$ and $n - 2$; then we see that

$$\dim I' \leq \frac{(n - 3)(n - 2)}{2} + n - 2 = \frac{(n - 2)(n - 1)}{2}.$$ 

Finally, by the assumptions of Theorem 4 there is some $a \notin I$ such that $b(a) \leq k$. Applying Lemma 4 to the maximal ideal $I$ and $a$ we see that

$$\dim g' \leq b(a) + \dim I' \leq \frac{(n - 2)(n - 1)}{2} + k.$$ 

This completes the proof of the theorem.

4.1. **Proof of Theorem 2.** Suppose the contrary: two proper subalgebras $h_1$ and $h_2$ cover all elements of breadth at least $n$; let $h_2$ be of codimension at least 2 in $g$. Applying Lemma 5 to the Lie algebra $g$ and the number $n(n - 1)/2 + 1 \leq \dim g'$ we see that there is a finite-dimensional subalgebra $u$ of $g$ such that $\dim u' \geq n(n - 1)/2 + 1$. We may also assume that $u$ is not contained in any Lie algebra $h_1$ and $u \cap h_2$ has codimension at least 2 in $u$. Replacing $g$ by $u$ and the subalgebra $h_2$ by $h_1 \cap u$, we may assume that the Lie algebra $g$ is finite-dimensional. By Lemma 1 the set $g \setminus (h_1 \cup h_2)$ generates $g$. All elements of $g \setminus (h_1 \cup h_2)$ have breadth not exceeding $n - 1$ in $g$, and therefore we can apply Theorem 4 to $g$, $h_1$, $h_2$ and $k = n - 1$, and then we obtain

$$\dim g' \leq \frac{(n - 1)(n - 2)}{2} + n - 1 = \frac{n(n - 1)}{2},$$

giving a contradiction.

**Remark.** In fact, Theorem 1 also follows from Theorem 4.

§ 5. **Proof of Theorem 3**

We argue by contradiction. Assume that proper subalgebras $h_1$, $h_2$, $\ldots$, $h_k$ cover all elements of breadth at least $n$. In this case we can prove that $\dim g' \leq n(n - 1)/2$. The rest of the proof of Theorem 3 repeats the proof of Theorem 1 verbatim, with $|\mathbb{F}| - 1$ replaced by the number $k$.

**Bibliography**

[1] V.D. Mazurov and E.I. Khukhro (eds.), *The Kourovka notebook. Unsolved problems in group theory*, 18th ed., Sobolev Institute of Mathematics, Novosibirsk 2014, 248 pp., arXiv:1401.0296; English transl., 18th ed., Sobolev Institute of Mathematics, Novosibirsk 2014, 227 pp.; 2017, arXiv:1401.0300v10.

[2] J. Wiegold, “Commutator subgroups of finite $p$-groups”, *J. Austral. Math. Soc.* 10:3–4 (1969), 480–484.

[3] M.R. Vaughan-Lee, “Breadth and commutator subgroups of $p$-groups”, *J. Algebra* 32:2 (1974), 278–285.
[4] A. Skutin, “Proof of a conjecture of Wiegold”, *J. Algebra* 526 (2019), 1–5.

**Alexander A. Skutin**  
Faculty of Mechanics and Mathematics,  
Lomonosov Moscow State University,  
Moscow, Russia  
*E-mail: a.skutin@mail.ru*