A GLUING CONSTRUCTION FOR FRACTIONAL ELLIPTIC EQUATIONS.
PART I: A MODEL PROBLEM ON THE CATENOID

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Abstract. We develop a new infinite dimensional gluing method for fractional elliptic equations. In Part I, as a model problem, we construct a solution of the fractional Allen–Cahn equation vanishing on a rotationally symmetric surface which resembles a catenoid and has sub-linear growth at infinity. In Part II, we construct counterexamples to De Giorgi Conjectures to fractional Allen-Cahn equation.

CONTENTS

1. Introduction 1
  1.1. The Allen–Cahn equation 1
  1.2. The fractional case and non-local minimal surfaces 2
  1.3. A brief description 4
  2. Outline of the construction 5
    2.1. Notations and the approximate solution 5
    2.2. The error 6
    2.3. The gluing reduction 6
    2.4. Projection of error and the reduced equation 9
  3. Computation of the error: Fermi coordinates expansion 9
  4. Linear theory 18
    4.1. Non-degeneracy of one-dimensional solution 18
    4.2. A priori estimates 20
    4.3. Existence 32
    4.4. The positive operator 34
  5. Fractional gluing system 34
    5.1. Preliminary estimates 34
    5.2. The outer problem: Proof of Proposition 2.2 37
    5.3. The inner problem: Proof of Proposition 2.3 38
  6. The reduced equation 41
    6.1. Form of the equation: Proof of Proposition 2.4 41
    6.2. Initial approximation 43
    6.3. The linearization 47
    6.4. The perturbation argument: Proof of Proposition 2.5 50
References 50

1. Introduction

1.1. The Allen–Cahn equation. In this paper we are concerned with the fractional Allen–Cahn equation, which takes the form

\[(−Δ)^su + f(u) = 0 \quad \text{in } \mathbb{R}^n\]  \hspace{1cm} (1.1)

where \(f(u) = u^3 - u = W'(u)\) is a typical example that \(W(u) = \left(\frac{u-\frac{1}{2}}{2}\right)^2\) is a bi-stable, balanced double-well potential.
In the classical case when \( s = 1 \), such equation arises in the phase transition phenomenon [4, 27]. Let us consider, in a bounded domain \( \Omega \), a rescaled form of the equation (1.1),

\[-\varepsilon^2 \Delta u_\varepsilon + f(u_\varepsilon) = 0 \quad \text{in } \Omega.\]

This is the Euler–Lagrange equation of the energy functional

\[J_\varepsilon(u) = \int_\Omega \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right) \, dx.\]

The constant solutions \( u = \pm 1 \) corresponds to the stable phases. For any subset \( S \subset \Omega \), we see that the discontinuous function \( u_\varepsilon = \chi_S - \chi_{\Omega \setminus S} \) minimize the potential energy, the second term in \( J_\varepsilon(u) \). The gradient term, or the kinetic energy, is inserted to penalize unnecessary forming of the interface \( \partial S \).

Using \( \Gamma \)-convergence, Modica [75] proved that any family of minimizers \( (u_\varepsilon) \) of \( J_\varepsilon \) with uniformly bounded energy has to converge to some \( u_S \) in certain sense, where \( \partial S \) has minimal perimeter. Caffarelli and Córdoba [21] proved that the level sets \( \{u_\varepsilon = \lambda\} \) in fact converge locally uniformly to the interface.

Observing that the scaling \( v_\varepsilon(x) = u_\varepsilon(\varepsilon x) \) solves

\[-\Delta v_\varepsilon + f(v_\varepsilon) = 0 \quad \text{in } \varepsilon^{-1} \Omega,\]

which formally tends as \( \varepsilon \to 0 \) to (1.1), the intuition is that \( v_\varepsilon(x) \) should resemble the one-dimensional solution \( w(z) = \tanh \frac{z}{\varepsilon} \) where \( z \) is the normal coordinate on the interface, an asymptotically flat minimal surface. Indeed, we have that

\[J_\varepsilon(v_\varepsilon) \approx \text{Area}(M) \int_\mathbb{R} \left( \frac{1}{2} w'(z)^2 + W(w(z)) \right) \, dz.\]

Thus a classification of solutions of (1.1) was conjectured by E. De Giorgi [37].

**Conjecture 1.1.** Let \( s = 1 \). At least for \( n \leq 8 \), all bounded solutions to (1.1) monotone in one direction must be one-dimensional, i.e. \( u(x) = u(x_1) \) up to a translation and a rotation.

It has been proven for \( n = 2 \) by Ghoussoub and Gui [65], \( n = 3 \) by Ambrosio and Cabré [5], and for \( 4 \leq n \leq 8 \) under an extra mild assumption by Savin [80]. In higher dimensions \( n \geq 9 \), a counterexample has been constructed by del Pino, Kowalczyk and the third author [39]. See also [17, 66, 70].

### 1.2. The fractional case and non-local minimal surfaces

While Conjecture 1.1 is almost completely settled, a recent and intense interest arises in the study of the fractional non-local equations. A typical non-local diffusion term is the fractional Laplacian \( (-\Delta)^s \), \( s \in (0,1) \), which is defined as a pseudo-differential operator with symbol \( |\xi|^{2s} \), or equivalently by a singular integral formula

\[(-\Delta)^s u(x_0) = C_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x_0) - u(x)}{|x_0 - x|^{n+2s}} \, dx, \quad C_{n,s} = \frac{2^{2s}s\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)\pi^{n/2}},\]

for locally \( C^2 \) functions with at most mild growth at infinity. Caffarelli and Silvestre [24] formulated a local extension problem where the fractional Laplacian is realized as a Dirichlet-to-Neumann map. This extension theorem was generalized by Chang and González [30] in the setting of conformal geometry. Expositions to the fractional Laplacian can be found in [2, 12, 44, 68].

In a parallel line of thought, \( \Gamma \)-convergence results have been obtained by Ambrosio, De Philippis and Martinazzi [6], González [67], Savin and Valdinoci [82]. The latter authors also proved the uniform convergence of level sets [85]. Owing to the varying strength of the non-locality, the energy

\[J_\varepsilon(u) = \varepsilon^{2s} \|u\|_{H^s(\Omega)} + \int_\Omega W(u) \, dx\]

\( \Gamma \)-converges (under suitable rescaling) to the classical perimeter functional when \( s \in \left[\frac{1}{2}, 1\right) \), and to a non-local perimeter when \( s \in (0, \frac{1}{2}) \).
A singularly perturbed version of (1.1) was studied by Millot and Sire [73] for the critical parameter $s = \frac{1}{2}$, and also by these two authors and Wang [74] in the case $s \in (0, \frac{1}{2})$.

In the highly non-local case $s \in (0, \frac{1}{2})$, the corresponding non-local minimal surface was first studied by Caffarelli, Roquejoffre and Savin [22]. Concerning regularity, Savin and Valdinoci [84] proved that any non-local minimal surface is locally $C^{1,\alpha}$ except for a singular set of Hausdorff dimension $n - 3$. Caffarelli and Valdinoci [26] showed that in the asymptotic case $s \to (1/2)^-$, in accordance to the classical minimal surface theory, any $s$-minimal cone is a hyperplane for $n \leq 7$ and any $s$-minimal surface is locally a $C^{1,\alpha}$ graph except for a singular set of codimension at least 8. Recently Cabré, Cinti and Serra [15] gave a quantitative version in $\mathbb{R}^3$. Barrios, Figalli and Valdinoci [7] improved the regularity of $C^{1,\alpha}$ $s$-minimal surfaces to $C^\infty$. Graphical properties and boundary stickiness behaviors were investigated by Dipierro, Savin and Valdinoci [49,50].

Non-trivial examples of such non-local minimal surface were constructed by Dávila, del Pino and the third author [36] at the limit $s \to (1/2)^-$, as an analog to the catenoid. Note that the non-local catenoid they constructed has eventual linear, as opposed to logarithmic, growth at infinity; a similar effect is seen in the construction in the present article.

Strongly interests are also seen in a fractional version of De Giorgi Conjecture.

Conjecture 1.2. Bounded monotone entire solutions to (1.1) must be one-dimensional, at least for dimensions $n \leq 8$.

In the rest of this paper we will focus on the mildly non-local regime. For $s \in [\frac{1}{2}, 1)$ positive results have been obtained: $n = 2$ by Sire and Valdinoci [86] and by Cabré and Sire [19], $n = 3$ by Cabré and Cinti [14] (see also Cabré and Solá-Morales [20]), $n = 4$ and $s = \frac{1}{3}$ by Figalli and Serra [61], and the remaining cases for $n \leq 8$ by Savin [81] under an additional mild assumption. A natural question is whether or not Savin’s result is optimal. In a forthcoming paper [29], we will construct global minimizers in dimension 8 and give counter-examples to Conjecture 1.2 for $n \geq 9$ and $s \in (\frac{1}{2}, 1)$.

Some work related to Conjecture 1.2 involving more general operators include [16,51,57,83]. For similar results in elliptic systems, the readers are referred to [8,9,45,54–56,58,90,91] for the local case, and [11,47,59,92] under the fractional setting.

The construction of solution by gluing for non-local equations is a relatively new subject. Du, Gui, Sire and the third author [52] proved the existence of multi-layered solutions of (1.1) when $n = 1$. The first and third authors [28] constructed a non-planar traveling wave solution. Other work involves the fractional Schrödinger equation [32,35], the fractional Yamabe problem [38] and non-local Delaunay surfaces [34].

For general existence theorems for non-local equations, the readers may consult, among others, [31,33,62,63,76–79,87,88,93,94] as well as the references therein. Related questions on the fractional Allen–Cahn equations, non-local isoperimetric problems and non-local free boundary problems are also widely studied in [10,23,41–43,46,48,60,71,72]. See also the expository articles [1,64,89].

Despite similar appearance, (1.1) for $s \in (0, 1)$ is different from that for $s = 1$ in a number of striking ways. Firstly, the non-local nature disallows the use of local Fermi coordinates. Secondly, the one-dimensional solution $w(z)$ only has an algebraic decay of order $2s$ at infinity, in contrast to the exponential decay when $s = 1$. Thirdly, the fractional Laplacian is a strongly coupled operator and hence it is impossible to “integrate in parts” in lower dimensions. Finally the inner-outer gluing using cut-off functions no longer work due to the nonlocality of the fractional operator.

The purpose of this article is to establish a new gluing approach for fractional elliptic equations for constructing solutions with a layer over higher-dimensional sub-manifolds. In particular, in the second part we will apply it to partially answer Conjecture 1.2. To overcome the aforementioned difficulties, the main tool is an expansion of the fractional Laplacian in the Fermi coordinates, a refinement of
the computations already seen in [28], supplemented by technical integral calculations. This can be considered fractional Fermi coordinates. When applying an infinite dimensional Lyapunov–Schmidt reduction, the orthogonality condition is to be expressed in the extension. The essential difference from the classical case [40] is that the inner problem is subdivided into many pieces of size \( R = o(\varepsilon^{-1}) \), where \( \varepsilon \) is the scaling parameter, so that the manifold is nearly flat on each piece. In this way, in terms of the Fermi normal coordinates, the equations can be well approximated by a model problem.

1.3. A brief description. We define an approximate solution \( u^*(x) \) using the one-dimensional profile in the tubular neighborhood of \( M_\varepsilon = \{|x_n| = F_\varepsilon(|x'|)| \} \), namely \( u^*(x) = w(z) \) where \( z \) is the normal coordinate and \( F_\varepsilon \) is close to the catenoid \( \varepsilon^{-1}\cosh^{-1}(\varepsilon|x'|) \) near the origin. In contrast to the classical case we take into account the non-local interactions near infinity and define \( u^*(x) = w(z_+) + w(z_-) + 1 \) where \( z_\pm \) are the signed distances to the upper and lower leaves \( M_\varepsilon^\pm = \{x_n = \pm F_\varepsilon(|x'|)\} \). As hinted in Corollary 6.3, \( F_\varepsilon(r) \sim r^\frac{s}{s-1} \) as \( r \to +\infty \). The parts of \( u^* \) are glued to the constant solutions \pm 1 smoothly to the regions where the Fermi coordinates are not well-defined.

We look for a real solution of the form \( u = u^* + \varphi \), where \( \varphi \) is small and satisfies

\[
(-\Delta)^s \varphi + f'(u^*)\varphi = g. \tag{1.2}
\]

Our new idea is to localize the error in the near interface into many pieces of diameter \( R = o(\varepsilon^{-1}) \) for another parameter \( R \) which is to be taken large. At each piece the hypersurface is well-approximated by some tangent hyperplane. Therefore, using Fermi coordinates, it suffices to study the model problem where \( u^*(x) \) is replaced by \( w(z) \) in (1.2).

As opposed to the local case \( s = 1 \), an integration by parts is not available the fractional Laplacian in only the \( z \)-direction, unless \( n = 1 \). So we develop a linear theory using the Caffarelli–Silvestre local extension [24].

Finally we will solve a non-local, non-linear reduced equation which takes the form

\[
\begin{cases}
H[F_\varepsilon] = O(\varepsilon^{2s-1}) & \text{for } 1 < r \leq r_0 \\
H[F_\varepsilon] = \frac{C_\varepsilon 2s-1}{F_\varepsilon^{2s}} (1 + o(1)) & \text{for } r > r_0
\end{cases}
\]

where \( H[F_\varepsilon] \) denotes the mean curvature of the surface described by \( F_\varepsilon \). (Note that the surface is adjusted far away through the nonlocal interactions of the leafs. A similar phenomena has been observed in Agudelo, del Pino and the third author [3] for \( s = 1 \) and dimensions \( \geq 4 \).) A solution of the desired form can be obtained using the contraction mapping principle, justifying the \textit{a priori} assumptions on \( F_\varepsilon \).

In this setting, our main result can be stated as follows.

**Theorem 1.3.** Let \( 1/2 < s < 1 \) and \( n = 3 \). For all sufficiently small \( \varepsilon > 0 \), there exists a rotationally symmetric solution \( u \) to (1.1) with the zero level set \( M_\varepsilon = \{ (x', x_3) \in \mathbb{R}^3 : |x_3| = F_\varepsilon(|x'|) \} \), where

\[
F_\varepsilon(r) \sim \begin{cases} 
\varepsilon^{-1}\cosh^{-1}(\varepsilon r) & \text{for } r \leq r_\varepsilon \\
\frac{1}{r} & \text{for } r \geq \delta_0|\log \varepsilon| r_\varepsilon
\end{cases}
\]

where \( r_\varepsilon = (|\log \varepsilon|)^{\frac{s-1}{s}} \) and \( \delta_0 > 0 \) is a small fixed constant.

In a forthcoming paper [29], together with Juan Dávila and Manuel del Pino, we will construct similarly a global minimizer on the Simons’ cone. Via the Jerison–Monneau program [70], this provides counter-examples to the De Giorgi conjecture for fractional Allen–Cahn equation in dimensions \( n \geq 9 \) for \( s \in (\frac{1}{2}, 1) \).

**Remark 1.4.** Our approach depends crucially on the assumption \( s \in (\frac{1}{2}, 1) \). Firstly, it is only in this regime that only the local mean curvature appears in the error estimate. A related issue is also seen in the choice of those parameters between 0 and (a factor times) \( 2s - 1 \). Secondly, it gives the \( L^2 \)
integrability of an integral involving the kernel \( w \) in the extension. It will be interesting to see whether this gluing method will work in the case \( s = \frac{3}{2} \) under suitable modifications.

On the other hand, we do not know how to deal with other pseudo-differential operators which cannot be realized locally.

This paper is organized as follows. We outline the argument with key results in Section 2. In Section 3 we compute the error using an expansion of the fractional Laplacian in the Fermi coordinates. In Section 4 we develop a linear theory and then the gluing reduction is carried out in Section 5. Finally in Section 6 we solve the reduced equation.

2. OUTLINE OF THE CONSTRUCTION

2.1. Notations and the approximate solution. Let

- \( s \in (\frac{1}{2}, 1), \alpha \in (0, 2s - 1), \tau \in (1, 1 + \frac{s}{2s}) \),
- \( M \) be an approximation to the catenoid defined by the function \( F \),
- \( \varepsilon > 0 \) be the scaling parameter in \( M_\varepsilon = \varepsilon^{-1}M = \{ x_n = F_\varepsilon(|x'|) = \varepsilon^{-1}F(\varepsilon|x'|) \} \),
- \( z \) be the normal coordinate direction in the Fermi coordinates of the rescaled manifold, i.e. signed distance to the \( M_\varepsilon \), with \( z > 0 \) for \( x_n > F(\varepsilon|x'|) > 0 \),
- \( y_1, z_1 \) respectively the projection onto and signed distance (increasing in \( x_n \)) from the upper leaf
  \[
  M^+_\varepsilon = \{ x_n = F_\varepsilon(|x'|) \},
  \]
- \( y_2, z_2 \) respectively the projection onto and signed distance (decreasing in \( x_n \)) to the lower leaf
  \[
  M^-_\varepsilon = \{ x_n = -F_\varepsilon(|x'|) \},
  \]
- \( \bar{\delta} > 0 \) be a small fixed constant so that the Fermi coordinates near \( M_\varepsilon \) is defined for \( |z| \leq \frac{8\bar{\delta}}{\varepsilon} \),
- \( \bar{R} > 0 \) be a large fixed constant,
- \( R_0 \) be the width of the tubular neighborhood of \( M_\varepsilon \) where Fermi coordinates are used, see (2.1),
- \( R_1 \) be the radius of the cylinder from which the main contribution of \( (-\Delta)^s \) is obtained, see Proposition 2.1,
- \( R_2 > \frac{4\bar{R}}{\varepsilon} \) be the radius of the inner gluing region (i.e. threshold of the end, see Section 2.3),
- \( u^*_n(x) = \text{sign} (x_n - F_\varepsilon(|x'|)) \) for \( x_n > 0 \) and is extended continuously (i.e. \( u^*_n(x) = +1 \) for \( |x'| \leq \varepsilon^{-1} \)),
- \( \eta : \mathbb{R} \to [0, 1] \) be a cut-off with \( \eta = 1 \) on \( (-\infty, 1] \) and \( \eta = 0 \) on \( [2, +\infty) \),
- \( \chi : \mathbb{R} \to [0, 1] \) be a cut-off with \( \chi = 0 \) on \( (-\infty, 0] \) and \( \chi = 1 \) on \( [1, +\infty) \),
- \( \| \kappa \|_\alpha \) \( (0 \leq \alpha < 1) \) be the Hölder norm of the curvature, see Lemma 3.6,
- \( \langle x \rangle = \sqrt{1 + |x|^2} \).

Define the approximate solution

\[
  u^*(x) = \eta \left( \frac{\varepsilon |z|}{\delta R_0(|x'|)} \right) \left( w(z) + \chi \left( |x'| - \frac{\bar{R}}{\varepsilon} \right) (w(z_1) + w(z_2) + 1 - w(z)) \right) + \left( 1 - \eta \left( \frac{\varepsilon |z|}{\delta R_0(|x'|)} \right) \right) u^*_n(x).
\]

where

\[
  R_0 = R_0(|x'|) = 1 + \chi \left( |x'| - \bar{R} \right) (F_\varepsilon^{2s}(|x'|) - 1).
\]

Roughly,

- \( u^*(x) = +1 \) for large \( |z| \), small \( |x'| \) and large \( x_n \),
- \( u^*(x) = -1 \) for large \( |z| \), large \( |x'| \) and small \( x_n \),
- \( u^*(x) = w(z) \) for small \( |z| \) and small \( |x'| \),
• $u^*(x) = w(z_1) + w(z_2) + 1$ for small $|z|$ and large $|x'|$.

The main contributions of $(-\Delta)^* u^*$ come from the inner region with certain radius. We choose such radius that joins a small constant times $\varepsilon^{-1}$ to a power of $F_\varepsilon^*$ as $|x'|$ increases. More precisely, let us set

$$R_1 = R_1(|x'|) = \eta \left(\left|\frac{2\bar{R}}{\varepsilon} + 2\right| \frac{\delta}{\varepsilon} + \left(1 - \eta \left|\frac{2\bar{R}}{\varepsilon} + 2\right\right)\right) F_\varepsilon^* (|x'|),$$

where $\tau \in \left(1, 1 + \frac{\alpha}{2}\right)$. We remark that the factor 2 is inserted to make sure that $u^*(x) = w(z_1) + w(z_2) - 1$ in the whole ball of radius $F_\varepsilon^* (|x'|)$ where the main order terms of $(-\Delta)^* u^*$ are obtained.

2.2. The error. Denote the error by $S(u^*) = (-\Delta)^* u^* + (u^*)^3 - u^*$. In a tubular neighborhood where the Fermi coordinates are well-defined, write $x = y + z\nu(y)$ where $y = y(|x'|) = (|x'|, F_\varepsilon(|x'|)) \in M_\varepsilon$ and $\nu(y) = \nu(y(|x'|)) = \sqrt{1 + F_\varepsilon(|x'|)^2}$ be the unit normal pointing up in the upper half space (and down in the lower half).

Proposition 2.1. Let $x = y + z\nu(y) \in \mathbb{R}^n$. If $|z| \leq R_1$, where $R_1$ as in (2.2), then we have

$$S(u^*)(x) = \begin{cases} c_H H_{M^+}(y) + O(\varepsilon^2), & \text{for } \frac{1}{\varepsilon} \leq r \leq \frac{4\bar{R}}{\varepsilon}, \\ c_H (z_+ h) H_{M^+}(y) + c_H (z_- h) H_{M^-}(y) - 3(w(z_+)) (1 + w(z_+)) (1 + w(z_-)) + O(F_\varepsilon^{-2\tau}), & \text{for } r \geq \frac{4\bar{R}}{\varepsilon}. \end{cases}$$

The proof is given in Section 3.

2.3. The gluing reduction. We look for a solution of (1.1) of the form $u = u^* + \varphi$ so that

$$(-\Delta)^* \varphi + f'(u^*) \varphi = S(u^*) + N(\varphi) \quad \text{in } \mathbb{R}^n,$$

where $N(\varphi) = f(u^* + \varphi) - f(u^*) - f'(u^*) \varphi$. Consider the partition of unity

$$1 = \tilde{\eta}_o + \tilde{\eta}_+ + \tilde{\eta}_- + \sum_{i=1}^{\tilde{i}} \tilde{\eta}_i,$$

where the support of each $\tilde{\eta}_i$ is a region of radius $R$ centered at some $y_i \in M_\varepsilon$, and $\tilde{\eta}_\pm$ are supported on a tubular neighborhood of the ends of $M_\varepsilon^\pm$ respectively. It will be convenient to denote $\mathcal{I} = \{1, \ldots, \tilde{i}\}$ and $\mathcal{J} = \mathcal{I} \cup \{+,-\}$. For $j \in \mathcal{J}$, let $\zeta_j$ be cut-off functions such that the sets $\{\zeta_j = 1\}$ include the supp $\tilde{\eta}_j$, with comparable spacing that is to be made precise. We decompose

$$\varphi = \phi_o + \zeta_+ \phi_+ + \zeta_- \phi_- + \sum_{i=1}^{\tilde{i}} \zeta_i \phi_i = \phi_o + \sum_{j \in \mathcal{J}} \zeta_j \phi_j,$$

in which

• $\phi_o$ solves for the contribution of the error away from the interface (support of $\tilde{\eta}_o$),
• $\phi_\pm$ solves for the that in the far interfaces near $M_\varepsilon^\pm$ (support of $\tilde{\eta}_\pm$),
• $\phi_i$ solves for that in a compact region near the manifold (support of $\tilde{\eta}_i$).

We consider the approximate linear operators

$$L_0 = (-\Delta)^* + 2 \quad \text{for } \phi_o,$$
$$L = (-\Delta_{(y,z)})^* + f'(w) \quad \text{for } \phi_j, \quad j \in \mathcal{J},$$

Notice that $w$ is not the approximate solution in the far interface. We rearrange the equation as

$$(-\Delta)^* \left( \sum_{j \in \mathcal{J}} \zeta_j \phi_j \right) + f'(u^*) \left( \sum_{j \in \mathcal{J}} \zeta_j \phi_j \right) = S(u^*) + N(\varphi),$$
\[ L_0 \phi_0 + \zeta_+ L \phi_+ + \zeta_- L \phi_- + \sum_{i=1}^i \zeta_i L \phi_i \]

\[
= \left( \tilde{\eta}_0 + \tilde{\eta}_+ + \tilde{\eta}_- + \sum_{i=1}^i \tilde{\eta}_i \right) \left( S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_0 - \sum_{j \in J} [(-\Delta_{(y,z)})^s, \zeta_j] \phi_j \right) + \sum_{j \in J} \zeta_j f'(w_j) - f'(u^*) \phi_j - \sum_{j \in J} ((-\Delta_x)^s - (-\Delta_{(y,z)})^s) (\zeta_j \phi_j), \]

where \[ [(-\Delta_{(y,z)})^s, \zeta_j] \phi_j = (-\Delta_{(y,z)})^s (\zeta_j \phi_j) - \zeta_j (-\Delta_{(y,z)})^s \phi_j \], and the summands in the last term means

\[ (-\Delta_x)^s (\zeta_j \phi_j) (Y_j(y) + z\nu(Y_j(y))) = (-\Delta_{(y,z)})^s (\tilde{\eta}_j \tilde{\phi}(y,z)) \]

for \( \zeta_j = \tilde{\eta}_j(y)\tilde{\zeta}(z) \) and \( \phi_j(Y_j(y) + z\nu(Y_j(y))) = \tilde{\phi}_j(y,z) \) with a chart \( y = Y_j(y) \) of \( M_\varepsilon \). In fact, for \( j \in I \) one can parameterize \( M_\varepsilon \) locally by a graph over a tangent hyperplane, and for \( j \in \{+, -, \varepsilon\} \) one uses the natural graph \( M_\varepsilon^\pm = \{(y, \pm F_\varepsilon(|y|)) : |y| \geq R_2\} \).

Let us denote the last bracket of the right hand side of \( (2.3) \) by \( G \). Since \( \tilde{\eta}_j = \zeta_j \tilde{\eta}_j \), we will have solved \( (2.3) \) if we get a solution to the system

\[
\begin{align*}
L_0 \phi_0 &= \tilde{\eta}_0 G & \text{for } x \in \mathbb{R}^n, \\
L \phi_+ &= \tilde{\eta}_+ G & \text{for } (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \\
L \tilde{\phi}_- &= \tilde{\eta}_- G & \text{for } (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}, \\
L \tilde{\phi}_i &= \tilde{\eta}_i G & \text{for } (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R},
\end{align*}
\]

for all \( i \in I \). Except the outer problem with \( L_0 = (-\Delta)^s + 2 \), the linear operator \( L \) in all the other equations has a kernel \( w' \) and so we will use an infinite dimensional Lyapunov–Schmidt reduction procedure.

From now on we consider the product cut-off functions, defined in the Fermi coordinates \( (y, z) \) where \( y = Y(y) \) is given by a chart of \( M_\varepsilon \),

\[ \tilde{\eta}_j(x) = \eta_j(y)\zeta(z), \quad \text{for } j \in J. \]

The diameters of \( \zeta(z) \) and \( \eta_i(y) \) are of order \( R \), a parameter which we choose to be large (before fixing \( \varepsilon \)). We may assume, without loss of generality, that for \( i \in I \), \( \eta_i(y) \) is centered at \( y_i \in M_\varepsilon \), \( B_R(y_i) \subset \{ \tilde{\eta}_i = 1 \} \subset \text{supp} \tilde{\eta}_i \subset B_{2R}(y_i), |D\tilde{\eta}_i| = O(R^{-1}), \text{ and } \frac{2R - R_2}{R} \geq c > 0 \text{ for any } i_1, i_2 \in I. \)

We define the projection orthogonal to the kernels \( w'(z) \),

\[ \Pi g(y, z) = g(y, z) - c(y)w'(z), \quad c(y) = \int_{\mathbb{R}} \zeta(z)g(y, z)w'(\tilde{z})d\tilde{z}, \quad \int_{\mathbb{R}} \zeta(z)w'(\tilde{z})d\tilde{z} = 1. \]

Note that in the region of integration \( |z| \leq 2R < \tilde{\varepsilon}^{-1} \) the Fermi coordinates are well-defined, and that the projection is independent of \( j \in J \).

Motivated by Proposition 2.1 and Lemma 4.6, for each \( i \in I \) we expect the decay

\[ \| \tilde{\phi}_i(y, z) \|_{\mu, \sigma} \leq CR^{\mu+\sigma} (y_i)^{-\frac{4\varepsilon}{2s+1}}. \]

so we define

\[ \| \phi_i \|_{\mu, \sigma} = \langle y_i \rangle^\theta \| \tilde{\phi}_i \|_{\mu, \sigma} = \langle y_i \rangle^\theta \sup_{(y, z) \in \mathbb{R}^n} \langle y \rangle^{\mu} \langle z \rangle^\sigma |\tilde{\phi}_i(y, z)|, \]

with \( 1 < \theta < 1 + \frac{4\varepsilon - 1}{2s+1} = \frac{4s}{2s+1} < 2s \). At the ends \( M_\varepsilon^\pm \) where \( r \geq R_2 \) we have, for \( \mu < \frac{4\varepsilon}{2s+1} - \theta, \)

\[ \| \tilde{\phi}_\pm(y, z) \|_{\mu, \sigma} \leq CR^{\frac{4\varepsilon}{2s+1} - \mu}. \]
This suggests
\[
\|\phi_{\pm}\|_{\pm,\mu,\sigma} = R_2^{\theta} \|\tilde{\phi}_{\pm}\|_{\mu,\sigma} = R_2^{\theta} \sup_{(y,z) \in \mathbb{R}^n} \langle y \rangle^\mu \langle z \rangle^\sigma |\tilde{\phi}_{\pm}(y,z)|,
\]
with \(0 < \theta < \frac{2s-1}{2s+1} - \mu\). Therefore for \(j \in \mathcal{J}\), we consider the Banach spaces
\[
X_j = \left\{ \phi_j : |\phi_j|_{j,\mu,\sigma} < C\delta \right\},
\]
where, under the constraint \(R \leq |\log \varepsilon|, \delta = \delta(R, \varepsilon) = R^{\mu+\sigma} \frac{\log \varepsilon}{2s+1} - \theta\) with \(1 < \theta < 1 + \frac{2s-1}{2s+1} = \frac{4s}{2s+1}\).

For the other parameters we take \(0 < \mu < \frac{1}{2s+1} - \theta < \theta\) sufficiently small and \(R_2\) sufficiently large, so that \(R_2^\theta \delta\) is small and \(2 - 2s < \sigma < 2s - \mu\). The decay of order \(\sigma > 2 - 2s\) in the \(z\)-direction will be required in the orthogonality condition (4.7). That \(R_2^\theta \delta\) is small will be used in the inner gluing reduction. The condition \(\sigma + \mu < 2s\) ensures that the contribution of the term \((2 - f'(u^*))\phi_o\) is small compared to \(S(u^*)\).

We will first solve the outer equation for \(\phi_o\). Let us write \(M_{\varepsilon,R} = \{y + z\nu(y) : y \in M_\varepsilon\text{ and }|z| < R\}\) for the tubular neighborhood of \(M_\varepsilon\) with width \(R\).

**Proposition 2.2.** Consider
\[
\|\phi_o\|_\theta = \sup_{(x',x_0) \in \mathbb{R}^n} \langle x' \rangle^\theta (\text{dist}(x, M_{\varepsilon,R}))^{2s} |\phi_o(x)|,
\]
\[
X_o = \left\{ \phi_o : |\phi_o|_\theta \leq C\varepsilon^\theta \right\}.
\]

If \(\phi_j \in X_j\) for all \(j \in \mathcal{J}\) with \(\sup_{j \in \mathcal{J}} |\phi_j|_{j,\mu,\sigma} \leq 1\), then there exists a unique solution \(\phi_o = \Phi_o((\phi_j)_{j \in \mathcal{J}})\) to
\[
L_o\phi_o = \tilde{\eta}_o G = \tilde{\eta}_o \left( S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_o - \sum_{j \in \mathcal{J}} (\Delta_{(y,z)})^\sigma (\zeta_j)\phi_j 
\]
\[
+ \sum_{j \in \mathcal{J}} \zeta_j (f'(w_j) - f'(u^*))\phi_j - \sum_{j \in \mathcal{J}} ((\Delta_x)^s - (\Delta_{(y,z)})^s)(\zeta_j \phi_j) \right) \quad \text{in } \mathbb{R}^n \quad (2.4)
\]
in \(X_o\) such that for any pairs \((\phi_j)_{j \in \mathcal{J}}\) and \((\psi_j)_{j \in \mathcal{J}}\) in the respective \(X_j\) with \(\sup_{j \in \mathcal{J}} |\phi_j|_{j,\mu,\sigma} \leq 1\),
\[
|\Phi_o((\phi_j)_{j \in \mathcal{J}}) - \Phi_o((\psi_j)_{j \in \mathcal{J}})||_{\theta} \leq C\varepsilon^\theta \sup_{j \in \mathcal{J}} |\phi_j - \psi_j|_{j,\mu,\sigma}. \quad (2.5)
\]

The proof is carried out in Section 5.2.

Then the equations
\[
L_o\tilde{\phi}_j(y,z) = \eta_j(y)\zeta_j G(y,z)
\]
are solved in two steps: (1) eliminating the part of error orthogonal to the kernels, i.e.
\[
L_o\tilde{\phi}_j(y,z) = \eta_j(y)\zeta_j \Pi G(y,z); \quad (2.6)
\]
and (2) adjust \(F_\varepsilon(r)\) such that \(c(y) = 0\), i.e. to solve the reduced equation
\[
\int_{\mathbb{R}} \zeta(z) G(y,z)w'(z) \, dz = 0. \quad (2.7)
\]

Using the linear theory in Section 4, step (1) is proved in the following

**Proposition 2.3.** Suppose \(\mu \leq \theta\). Then there exists a unique solution \((\phi_j)_{j \in \mathcal{J}}, \phi_j \in X_j\), to the system
\[
L_o\tilde{\phi}_j = \eta_j \Pi G = \eta_j \Pi \left( S(u^*) + N(\varphi) + (2 - f'(u^*))\phi_o - \sum_{j \in \mathcal{J}} (\Delta_{(y,z)})^\sigma (\zeta_j)\phi_j 
\]
\[
+ \sum_{j \in \mathcal{J}} \zeta_j (f'(w_j) - f'(u^*))\phi_j - \sum_{j \in \mathcal{J}} ((\Delta_x)^s - (\Delta_{(y,z)})^s)(\zeta_j \phi_j) \right) \quad \text{for } (y,z) \in \mathbb{R}^n. \quad (2.8)
\]
The proof is given in Section 5.3.
Step (2) is outlined in the next subsection.

2.4. Projection of error and the reduced equation. As shown above, the error is to be projected onto $w'_j$ weighted with a cut-off function $\zeta$ supported on $[-2R, 2R]$. In fact we have

**Proposition 2.4 (The reduced equation).** In terms of the rescaled function $F(r) = \varepsilon F_\varepsilon(\varepsilon^{-1}r)$ and its inverse $r = G(z)$ where $G : [0, +\infty) \to [1, +\infty)$, (2.7) is equivalent to the system

$$
\begin{aligned}
H_M(G(z), z) &= \left( \frac{G'(z)}{1 + G'(z)^2} \right)' - \frac{1}{G(z)\sqrt{1 + G'(z)^2}} = N_1[F] \quad \text{for } 0 \leq z \leq z_1, \\
H_M(r, F(r)) &= \frac{1}{r} \left( \frac{rF'(r)}{1 + F'(r)^2} \right) = N_1[F] \quad \text{for } r_1 \leq r \leq 4\tilde{R}, \\
F''(r) + \frac{F'(r)}{r} - \frac{C_0^{2s-1}}{F^{2s}(r)} = N_2[F] \quad \text{for } r \geq 4\tilde{R},
\end{aligned}
$$

subject to the boundary conditions

$$
\begin{aligned}
G(0) &= 1 \\
G'(0) &= 0 \\
F(r_1) &= z_1 \\
F'(r_1) &= \frac{1}{G'(z_1)}.
\end{aligned}
$$

where $z_1 = F(r_1) = O(1)$, $N_1[F] = O(\varepsilon^{2s-1})$ and $N_2[F] = o\left(\frac{\varepsilon^{2s-1}}{F_0^{2s}(r)}\right)$, with $F_0$ as in Corollary 6.3. Moreover, $N_1$ and $N_2$ have a Lipschitz dependence on $F$.

This is proved in Section 61.1. The equation (2.9)–(2.10) is to be solved in a space with weighted Hölder norms allowing sub-linear growth. More precisely, for any $\alpha \in (0, 1)$, $\gamma \in \mathbb{R}$ we define the norms

$$
\|\phi\|_* = \sup_{|r_1, +\infty|} r^{\gamma-2}\phi(r) + \sup_{|r_1, +\infty|} r^{\gamma-1}\phi'(r) + \sup_{|r_1, +\infty|} r^{\gamma}\phi''(r)
$$

and

$$
\|h\|_{**} = \sup_{r \in [1, +\infty)} r^{\gamma}\|h(r)\| + \sup_{r \neq \rho \in [1, +\infty)} \min \{r, \rho\}^{\gamma+\alpha} \frac{|h(r) - h(\rho)|}{|r - \rho|^\alpha}.
$$

**Proposition 2.5.** There exists a solution to (2.9) in the space

$$
X_* = \left\{ (G, F) \in C^{2,\alpha}_c([0, z_1]) \times C^{2,\alpha}_c([r_1, +\infty)) : \|G\|_{C^{2,\alpha}([0, z_1])} < +\infty, \|F\|_* < +\infty, (2.10) \text{ holds} \right\}.
$$

The proof is contained in Section 6.

3. Computation of the error: Fermi coordinates expansion

We prove the following

**Proposition 3.1 (Expansion in Fermi coordinates).** Suppose $0 < \alpha < 2s - 1$ and $F_\varepsilon \in C^{2,\alpha}_c([1, +\infty))$. Let $x_0 = y_0 + z_0\nu(y_0)$ where $y_0 = (x', F_\varepsilon([x']))$ is the projection of $x_0$ onto $M_\varepsilon$, and $u_0(x) = w(z)$. Then for any $\tau \in (1, 1 + \frac{\alpha}{2s})$ and $|z_0| \leq R_1$, we have

$$
(-\Delta)^\tau u_0(x_0) = w(z_0) - w(z_0)^3 + c_H(z_0)H_{M_\varepsilon}(y_0) + N_1[F]
$$
Corollary 3.5. Let \( c_H(z_0) = C_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{2s}} \, dz, \)

\[
R_1 = R_1(|x'|) = \eta \left( |x'| - \frac{2R}{\varepsilon} + 2 \right) \delta \frac{\varepsilon}{R} + \left( 1 - \eta \left( |x'| - \frac{2R}{\varepsilon} + 2 \right) \right) F^*(|x'|),
\]

and \( N_1[f] = O \left( R^{-2s} \right) \) is finite in the norm \( \| \cdot \|_{s*}. \)

Remark 3.2. \( c_H(z_0) \) is even in \( z_0. \) Also

\[
c_H(z_0) = \frac{C_{1,s}}{2s - 1} \int_{\mathbb{R}} \frac{w'(z)}{|z_0 - z|^{2s-1}} \, dz \sim (z_0)^{-2s-1}.
\]

This implies Proposition 2.1. A proof is given at the end of this section.

A similar computation gives the decay in \( r = |x'| \) away from the interface.

**Corollary 3.3.** Suppose \( x_0 = y_0 + z_0\nu(y_0), \) \( y_0 = (x_0, F_\varepsilon(0)) \) and \( z_0 \geq c\varepsilon^{-2s}. \)

\[
(-\Delta)^s u^*(x_0) = O \left( r_0^{-\frac{4s}{2s-1}} \right) \text{ as } r_0 \to +\infty.
\]

**Proof.** Take a ball around \( x_0 \) of radius of order \( r_0^{\frac{2s}{2s-1}}. \) In the inner region one uses the closeness to \( +1 \) of the approximate solution \( u^*. \)

For more general functions one has a less precise expansion. On compact sets, we have

**Corollary 3.4.** Let \( u_1(x) = \phi(y, z) \) in a neighborhood of \( x_0 = y_0 + z_0\nu(y_0) \) where \( |y_0|, |z_0| \leq 4R = o(\varepsilon^{-1}), \) and \( u_1 = 0 \) outside a ball of radius \( 8R. \) Then

\[
(-\Delta_x)^s u_1(x_0) = (-\Delta_{(y,z)})^s \phi(y_0, z_0) \cdot \left( 1 + O \left( R \| \kappa \|_{0} \right) \right) + O \left( R_1^{-2s} \left( |\phi(y_0, z_0)| + \sup_{|z_0 - y_0| \geq R_1} |\phi(y_0, z)| \right) \right).
\]

**Proof.** The lower order terms contain either \( \kappa_i|z_0| \) or \( \kappa_i|y_0|, \) where \( i = 1 \) or \( 2. \)

At the ends we need the following

**Corollary 3.5.** Let \( u_1(x) = \phi(y, z) \) in a neighborhood of \( x_0 = y_0 + z_0\nu(y_0) \) where \( |y_0| \geq R_2, |z_0| \leq 4R = o(\varepsilon^{-1}), \) and \( u_1 = 0 \) when \( z \geq 8R. \) Then

\[
(-\Delta_x)^s u_1(x_0) = (-\Delta_{(y,z)})^s \phi(y_0, z_0) \cdot \left( 1 + O \left( F^{-2s}_{-\varepsilon} \right) \right) + O \left( F^{-2s}_{-\varepsilon} \left( |\phi(y_0, z_0)| + \sup_{|z_0 - y_0| \geq R_1} |\phi(y_0, z)| \right) \right).
\]

To prove Proposition 3.1, we consider \( M_\varepsilon \) as a graph in a neighborhood of \( y_0 \) over its tangent hyperplane and use the Fermi coordinates. Suppose \( (y_1, y_2, z) \) is an orthonormal basis of the tangent plane of \( M_\varepsilon \) at \( y_0. \) Write

\[
C_{R_1} = \left\{ (y, z) \in \mathbb{R}^2 \times \mathbb{R} : |y| \leq R_1, |z| \leq R_1 \right\}.
\]

Then there exists a smooth function \( g : B_{R_1}(0) \to \mathbb{R} \) such that, in the \( (y, z) \) coordinates,

\[
M_{\varepsilon} \cap C_{R_1} = \left\{ (y, g(y)) \in \mathbb{R}^3 : |y| \leq R_1 \right\}.
\]

Then \( g(0) = 0, Dg(0) = 0 \) and \( \Delta g(0) = 2H_{M_\varepsilon}(x_0). \) We may also assume that \( \partial_{y_1y_2} g(0) = 0. \) We denote \( \kappa_i(y) = \partial_{y_2} y_i g(y). \)

We state a few lemmata whose non-trivial proofs are postponed to the end of this section.
Lemma 3.6 (Local expansions). Let \( |y| \leq R_1 \). For \( i = 1, 2 \) we have

\[
|\kappa_i(y) - \kappa_i(0)| \lesssim \|\kappa_i\|_{C^2(B_{2R_1}(|x'|))} |y|^{\alpha} \lesssim \|F_{\varepsilon}^{-2s}\|_{C^0(B_1(|x'|))} |y|^\alpha
\]

\[
\lesssim \begin{cases} 
2^{2s+\alpha} |y|^\alpha & \text{for all } |x'| \leq \frac{2R}{\varepsilon}, \\
F_{\varepsilon}^{-2s}(|x'|)|y|^\alpha & \text{for all } |x'| \geq \frac{R}{\varepsilon}.
\end{cases}
\]

The quantity \( \|F_{\varepsilon}\|_{C^2(B_{R_1}(|x'|))} \lesssim \|F_{\varepsilon}^{-2s}\|_{C^0(B_1(|x'|))} \) will be used repeatedly and will be simply denoted by \( \|\kappa\|_\alpha \), as a function of \( |x'| \), for any \( 0 \leq \alpha < 1 \). We have

\[
g(y) = \frac{1}{2} \sum_{i=1}^2 \kappa_i(0) y_i^2 + O \left( \|\kappa\|_\alpha |y|^{2+\alpha} \right)
\]

\[
Dg(y) \cdot y = \sum_{i=1}^2 \kappa_i(0) y_i^2 + O \left( \|\kappa\|_\alpha |y|^{2+\alpha} \right)
\]

\[
|Dg(y)|^2 = O \left( \|\kappa\|_\alpha^2 |y|^2 \right).
\]

In particular,

\[
g(y) - Dg(y) \cdot y = -\frac{1}{2} \sum_{i=1}^2 \kappa_i(0) y_i^2 + O \left( \|\kappa\|_\alpha |y|^{2+\alpha} \right) = O(\|\kappa\|_\alpha |y|^2)
\]

\[
\sqrt{1 + |Dg(y)|^2} - 1 = O \left( \|\kappa\|_\alpha^2 |y|^2 \right)
\]

\[
1 - \sqrt{1 + |Dg(y)|^2} = O \left( \|\kappa\|_\alpha^2 |y|^2 \right)
\]

\[
g(y)^2 = O \left( \|\kappa\|_\alpha^2 |y|^4 \right).
\]

Lemma 3.7 (The change of variable). Let \( |y|, |z|, |z_0| \leq R_1 \). Under the Fermi change of variable \( x = \Phi(y, z) = y + z\nu(y) \), the Jacobian determinant

\[
J(y, z) = \sqrt{1 + |Dg(y)|^2} (1 + \kappa_1(y)z)(1 + \kappa_2(y)z)
\]

satisfies

\[
J(y, z) = 1 + (\kappa_1(0) + \kappa_2(0))z + O \left( \|\kappa\|_\alpha |y|^2 |z| \right) + O \left( \|\kappa\|_\alpha^2 (|y|^2 + |z|^2) \right),
\]

and the kernel \( |x_0 - x|^{-3-2s} \) has an expansion

\[
|x_0 - x|^{-3-2s} = |(y, z_0 - z)|^{-3-2s} \left[ 1 + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^2 \kappa_i(0) \frac{y_i^2}{|(y, z_0 - z)|^2} \right.
\]

\[
+ O \left( \|\kappa\|_\alpha^2 (|y|^2 |z| + |z_0|^2) \right) + O \left( \|\kappa\|_\alpha^2 |y|^2 (|y|^2 + |z|^2 + |z_0|^2) \right) \right].
\]

Lemma 3.8 (Reducing the kernel). There hold

\[
C_{3,s} \int_{\mathbb{R}^2} \frac{1}{|(y, z_0 - z)|^{3+2s}} dy = C_{1,s} \frac{1}{|z_0 - z|^{1+2s}},
\]

\[
C_{3,s} \int_{\mathbb{R}^2} \frac{y_i^2}{|(y, z_0 - z)|^{5+2s}} dy = \frac{1}{3 + 2s} C_{1,s} \frac{1}{|z_0 - z|^{1+2s}} \quad \text{for } i = 1, 2,
\]

\[
\int_{\mathbb{R}^2} \frac{|y|^\alpha}{|(y, z_0 - z)|^{3+2s}} dy = C \frac{1}{|z_0 - z|^{1+2s-\alpha}}.
\]
**Proof of Proposition 3.1.** The main contribution of the fractional Laplacian comes from the local term which we compute in Fermi coordinates $\Phi(y, z) = y + zv(y)$,

\[
(-\Delta)^{s}u_0(x_0) = C_{3,s} \int_{\Phi(C_{R_1})} \frac{u_0(x_0) - u_0(x)}{|x - x_0|^{3+2s}} \, dx + O(R_1^{-2s})
\]

\[
= C_{3,s} \int_{C_{R_1}} \frac{w(z_0) - w(z)}{\Phi(y_0, z_0) - \Phi(y, z)} \, J(y, z) \, dydz + O(R_1^{-2s}).
\]

By Lemma 3.7 we have

\[
J(y, z) = 1 + (\kappa_1(0) + \kappa_2(0))z + O (\|\kappa\|_{\alpha} |y|^{\alpha} |z|) + O \left( \|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2}) \right)
\]

\[
\frac{1}{\Phi(y_0, z_0) - \Phi(y, z)} = \frac{1}{|(y, z_0 - z)|^{3+2s}} \left[ 1 + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{\infty} \frac{\kappa_i(0)}{|(y, z_0 - z)|^{2}} + O \left( \frac{\|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2})}{|(y, z_0 - z)|^{2}} \right) \right].
\]

Hence

\[
\frac{J(y, z)}{|\Phi(y_0, z_0) - \Phi(y, z)|^{3+2s}} = \frac{1}{|(y, z_0 - z)|^{3+2s}} \left[ 1 + (\kappa_1(0) + \kappa_2(0))z + O (\|\kappa\|_{\alpha} |y|^{\alpha} |z|) + O \left( \|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2}) \right) \right]
\]

\[
\left[ 1 + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{\infty} \frac{\kappa_i(0)}{|(y, z_0 - z)|^{2}} + O \left( \frac{\|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2})}{|(y, z_0 - z)|^{2}} \right) \right]
\]

\[
= \frac{1}{|(y, z_0 - z)|^{3+2s}} \left[ 1 + (\kappa_1(0) + \kappa_2(0))z + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{\infty} \frac{\kappa_i(0)}{|(y, z_0 - z)|^{2}} + O \left( \|\kappa\|_{\alpha} |y|^{\alpha} (|z| + |z_0|) \right) + O \left( \|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2} + |z_0|^{2}) \right) \right].
\]

We have

\[
(-\Delta)^{s}u_0(x_0) = C_{3,s} \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3+2s}} \, J(y, z) \, dydz + O(R_1^{-2s})
\]

\[
= C_{3,s} \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3+2s}} \left[ 1 + (\kappa_1(0) + \kappa_2(0))z + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{\infty} \frac{\kappa_i(0)}{|(y, z_0 - z)|^{2}} \right]
\]

\[
+ O \left( \|\kappa\|_{\alpha} |y|^{\alpha} (|z| + |z_0|) \right) + O \left( \|\kappa\|_{0}^{2} (|y|^{2} + |z|^{2} + |z_0|^{2}) \right) \right]
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5.
\]
where

\[ I_1 = C_{3,s} \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} \, dydz \]

\[ I_2 = C_{3,s} \left( \kappa_1(0) + \kappa_2(0) \right) \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} z \, dydz \]

\[ I_3 = C_{3,s} \frac{3 + 2s}{2} \sum_{i=1}^{2} \kappa_i(0) \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} (z_0 + z)^2 \, dydz \]

\[ I_4 = O \left( ||\kappa||_\infty \right) \int_{C_{R_1}} \frac{|w(z_0) - w(z) - \chi B_1(z_0) w'(z_0)(z_0 - z)|}{|y|^{3 + 2s}} |y|^a |z| \, dydz \]

\[ I_5 = O \left( ||\kappa||_0^2 \right) \int_{C_{R_1}} \frac{|w(z_0) - w(z) - \chi B_1(z_0) w'(z_0)(z_0 - z)|}{|y|^{3 + 2s}} (|y|^2 + |z|^2 + |z_0|^2) \, dydz. \]

In the last terms \( I_4 \) and \( I_5 \), the linear odd term near the origin has been added to eliminate the principal value before being estimated by its absolute value. One may obtain more explicit expressions by extending the domain and using Lemma 3.8 as follows. \( I_1 \) resembles the fractional Laplacian of the one-dimensional solution.

\[ I_1 = C_{3,s} \int_{\mathbb{R}^3} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} \, dydz - C_{3,s} \int_{\mathbb{R} \setminus C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} \, dydz \]

\[ = C_{3,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|y|^{3 + 2s}} \frac{1}{|z_0 - z|^{1 + 2s}} \, dz + O \left( \rho^{-3 - 2s} \rho^2 \, d\rho \right) \]

\[ = C_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1 + 2s}} \, dz + O \left( R_{1}^{-2s} \right) \]

\[ = w(z_0) - w(z_0)^3 + O \left( R_{1}^{-2s} \right). \]

Hereafter \( \rho = \sqrt{|y|^2 + |z_0 - z|^2} \). \( I_2 \) and \( I_3 \) are of the next order where we see the mean curvature.

\[ I_2 = -C_{3,s} \sum_{i=1}^{2} \kappa_i(0) \int_{C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} z \, dydz \]

\[ = -C_{3,s} \sum_{i=1}^{2} \kappa_i(0) \int_{\mathbb{R}^3} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} \, dydz \]

\[ - C_{3,s} \sum_{i=1}^{2} \kappa_i(0) \int_{\mathbb{R} \setminus C_{R_1}} \frac{w(z_0) - w(z)}{|(y, z_0 - z)|^{3 + 2s}} (z_0 + z - z_0) \, dydz \]

\[ = -C_{1,s} \sum_{i=1}^{2} \kappa_i(0) \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1 + 2s}} \, dz \]

\[ + O \left( ||\kappa||_0 \right) \int_{R_1} \frac{1}{\rho^{3 + 2s}} \rho^2 \, d\rho \right) + O \left( ||\kappa||_0 \right) \int_{R_1} \frac{\rho}{\rho^{3 + 2s}} \rho^2 \, d\rho \]

\[ = 2 \left( C_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1 + 2s}} \, dz \right) H_{M_0}(y_0) + O \left( ||\kappa||_0 \right) R_{1}^{-2s} ||\kappa||_0 \right) \]
Also,

\[ I_3 = C_{3,s} \frac{3 + 2s}{2} \sum_{i=1}^{2} \kappa_i(0) \int_{\mathbb{R}^3} \frac{w(z_0) - w(z)}{|y(z_0 - z)|^{5+2s}} (z_0 - z) y_1^2 \, dydz \]

\[ + O(\|\kappa\|_0) \int_{\mathbb{R}^3 \setminus C_{R_1}} \frac{w(z_0) - w(z)}{|y(z_0 - z)|^{5+2s}} (2z_0 - (z_0 - z)) y_1^2 \, dydz \]

\[ = C_{1,s} \frac{1}{2} \sum_{i=1}^{2} \kappa_i(0) \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1+2s}} (z_0 + z) \, dz \]

\[ + O \left( \|\kappa\|_0 \int_{R_1} \frac{\rho^2}{\rho^{5+2s}} d\rho \right) + O \left( \|\kappa\|_0 \int_{R_1} \frac{\rho^3}{\rho^{5+2s}} d\rho \right) \]

\[ = \left( C_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1+2s}} (z_0 + z) \, dz \right) H_{M_2}(y_0) + O(\|\kappa\|_0 R_1^{-2s}(|z_0| + R_1)). \]

The remainder terms \( I_4 \) and \( I_5 \) are estimated as follows.

\[ I_4 = O(\|\kappa\|_\alpha) \int_{C_{R_1}} \frac{|w(z_0) - w(z) - \chi_{B_1(z_0)}(z) w'(z_0)(z_0 - z)|}{|y(z_0 - z)|^{3+2s}} |y|^{\alpha}(|z| + |z_0|) \, dydz \]

\[ = O(\|\kappa\|_\alpha) \int_{\mathbb{R}} \frac{|w(z_0) - w(z) + \chi_{B_1(0)}(z) w'(z_0)(z_0 - z)|}{|y(z_0 - z)|^{3+2s}} |y|^{\alpha}(|z_0 - z| + |z_0|) \, dydz \]

\[ + O \left( \|\kappa\|_\alpha \int_{R_1} \frac{\rho^2}{\rho^{5+2s}} d\rho \right) \]

\[ = O(\|\kappa\|_\alpha^2) \left( 1 + \int_{1}^{R_1} \frac{\rho^2 + |z_0|^2}{\rho^{5+2s}} d\rho \right) \]

\[ = O \left( \|\kappa\|_0^2 \left( 1 + R_1^{-2s} + R_1^{-2s}|z_0|^2 \right) \right). \]

In conclusion, we have, since \(|z_0| \leq R_1 \) and \( \alpha < 2s - 1 \),

\[ (-\Delta)^s u_0(x_0) = w(z_0) - w(z_0)^3 + \left( C_{1,s} \int_{\mathbb{R}} \frac{w(z_0) - w(z)}{|z_0 - z|^{1+2s}} (z_0 - z) \, dz \right) H_{M_2}(y_0) \]

\[ + O \left( R_1^{-2s} \left( 1 + \|\kappa\|_0 R_1 + \|\kappa\|_\alpha R_1^{2s} + \|\kappa\|_0^2 R_1^2 \right) \right) \]

\[ = w(z_0) - w(z_0)^3 + c_\mathcal{H}(z_0) H_{M_2}(y_0) + O(R_1^{-2s}), \]
the last line following from the estimate
\[ \| \kappa \|_\alpha R_1^{2s} \lesssim \begin{cases} \varepsilon^0 & \text{for } |x'| \leq \frac{2R}{\varepsilon} \\ \frac{F_{\varepsilon}^{-2s(\tau-1)}}{|x'|^\alpha} & \text{for } |x'| \geq \frac{R}{\varepsilon} \end{cases} \]
\[ \lesssim \begin{cases} \varepsilon^0 & \text{for } |x'| \leq \frac{2R}{\varepsilon} \\ \varepsilon^{0-2s(\tau-1)}(\varepsilon|x'|)^{-2s(\tau-1)(1-\frac{2}{2s+1})} & \text{for } |x'| \geq \frac{R}{\varepsilon} \end{cases}. \]

The finiteness of the remainder in the norm \( \| \cdot \|_{\infty} \) is a tedious but straightforward computation. An example, the difference of the exterior error with two radii \( F_{\varepsilon}^r \) and \( G_{\varepsilon}^r \) is controlled by
\[ \int_{\Phi(C_{r})} \frac{u_0(x_0) - u_0(x)}{|x - x_0|^{r+2s}} \, dx - \int_{\Phi(C_{r})} \frac{u_0(x_0) - u_0(x)}{|x - x_0|^{r+2s}} \, dx \]
\[ = \int_{C_{r} \setminus C_{r'}} \frac{w(z_0) - w(z)}{|\Phi(y_0, z_0) - \Phi(y, z)|^{3+2s}} J(y, z) \, dydz. \]

Following the computations in the above proof, a typical term would be
\[ O \left( G_{\varepsilon}^{-2s\tau} - F_{\varepsilon}^{-2s\tau} \right) = O \left( r^{-\frac{2(2s+1)}{2s+1}} \right) \]
which implies Lipschitz continuity with decay in \( r \).

Similarly we prove the expansion at the end.

**Proof of Corollary 3.5.** We recall that a tubular neighborhood of an end of \( M_{\varepsilon}^+ \) are parameterized by
\[ x = y + z\nu(y) = (y, F_{\varepsilon}(r)) + z \frac{(-F_{\varepsilon}'(r))^2 + 1}{1 + F_{\varepsilon}'(r)^2} \]
for \( |y| > \rho_0, |z| \leq \frac{\delta}{\varepsilon} \), where \( r = |y| \). In place of Lemma 3.7 we have for \( |z| \leq F_{\varepsilon}'(r) \) with \( 1 < \tau < \frac{2s+1}{2} \),
\[ J(y, z) = \left( 1 + O \left( F_{\varepsilon}'(r)^2 \right) \right) \left( 1 + O \left( F_{\varepsilon}''(r) F_{\varepsilon}'(r) \right) \right)^2 \]
\[ = \left( 1 + O \left( F_{\varepsilon}^{-(2s-1)}(r) \right) \right)^2 \]
\[ \geq \left( 1 + O \left( F_{\varepsilon}^{-(2s-\tau)}(r) \right) \right), \]
\[ |x - x_0|^2 = \left( |y_0 - y|^2 + |z_0 - z|^2 \right) \left( 1 + O \left( F_{\varepsilon}''(r) F_{\varepsilon}'(r) \right) \right). \]

The result follows by the same proof as Proposition 3.1.

We now give a proof of the error estimate stated in Section 2.

**Proof of Proposition 2.1.** Using the Fermi coordinates expansion of the fractional Laplacian (Proposition 3.1), we have, in an expanding neighborhood of \( M_{\varepsilon} \), the following estimates on the error:

- For \( \frac{1}{\varepsilon} \leq |x'| \leq \frac{2R}{\varepsilon} \) and \( |z| \leq \frac{\delta}{\varepsilon} \),
  \[ S(u^\varepsilon)(x) = c_H(z) H_{M_{\varepsilon}}(y) + O \left( \varepsilon^{2s} \right). \]
For $|x'| \geq \frac{4R}{\varepsilon}$ and $|z| \leq F^*_\varepsilon(|x'|)$,

$$S(u^*)(x) = (-\Delta)^s(w(z_+) + w(z_-) + 1) + f(w(z_+) + w(z_-) - 1) + O\left(F^*_\varepsilon^{-2s}\right)$$

$$= f(w(z_+) + w(z_-) + 1) - f(w(z_+)) - f(w(z_-))$$

$$+ c_H(z_+)H_{M^+}(y_+) + c_H(z_-)H_{M^-}(y_-) + O\left(F^*_\varepsilon^{-2s}\right)$$

$$= 3(w(z_+) + w(z_-))(1 + w(z_+))(1 + w(z_-))$$

$$+ c_H(z_+)H_{M^+}(y_+) + c_H(z_-)H_{M^-}(y_-) + O\left(F^*_\varepsilon^{-2s}\right).$$

For $\frac{2R}{\varepsilon} \leq |x'| \leq \frac{4R}{\varepsilon}$, $x_0 > 0$ and $|z| \leq R_1(|x'|)$,

$$S(u^*)(x) = (-\Delta)^s w(z_+) + (-\Delta)^s \left( \left(1 - \eta \left( \frac{|x'| - R}{\varepsilon} \right) (w(z_-) + 1) \right) \right)$$

$$+ f \left( w(z_+) + \left(1 - \eta \left( \frac{|x'| - R}{\varepsilon} \right) (w(z_-) + 1) \right) \right)$$

$$= c_H(z_+)H_{M_+}(y_+) + O(\varepsilon^{2s}).$$

Here the second term is small because of the smallness up to two derivatives.

For $\frac{2R}{\varepsilon} \leq |x'| \leq \frac{4R}{\varepsilon}$, $x_0 < 0$ and $|z| \leq R_1(|x'|)$, we have similarly

$$S(u^*)(x) = c_H(z_-)H_{M_-}(y_-) + O(\varepsilon^{2s}).$$

This completes the proof. \hfill \Box

**Proof of Lemma 3.7.** Referring to Lemma 3.6 and keeping in mind that $\|\kappa\|_0 R_1 = o(1)$, for the Jacobian determinant we have

$$J(y, z) = 1 + (\kappa_1(0) + \kappa_2(0))z + ((\kappa_1 + \kappa_2)(y) - (\kappa_1 + \kappa_2)(0))z$$

$$+ \left( \sqrt{1 + |Dg(y)|^2} - 1 \right) \left(1 + (\kappa_1(y) + \kappa_2(y))z + \kappa_1(y)\kappa_2(y)z^2 \right)$$

$$= 1 + (\kappa_1(0) + \kappa_2(0))z + O\left(\|\kappa\|_0 \|y\|_0 |z|\right) + O\left(\|\kappa\|_0^2 |z|^2 \right)$$

$$+ O\left(\|\kappa\|_0^2 |y|^2 \right) \left(1 + O\left(\|\kappa\|_0 |z|\right)^2 \right)$$

$$= 1 + (\kappa_1(0) + \kappa_2(0))z + O\left(\|\kappa\|_0 \|y\|_0 |z|\right) + O\left(\|\kappa\|_0^2 (|y|^2 + |z|^2) \right).$$

To expand the kernel we first consider

$$x_0 - x = (y, g(y)) - (0, z_0) + z \frac{(-Dg(y), 1)}{\sqrt{1 + |Dg(y)|^2}},$$
\[ |x_0 - x|^2 = |y|^2 + g(y)^2 + z^2 + z_0^2 - \frac{2zz_0}{\sqrt{1 + |Dg(y)|^2}} + \frac{2z(g(y) - Dg(y) \cdot y)}{\sqrt{1 + Dg(y)^2}} - 2z_0g(y) \]
\[ = |y|^2 + |z_0 - z|^2 + 2z(g(y) - Dg(y) \cdot y) - 2z_0g(y) + g(y)^2 - (2zz_0 - 2z(g(y) - Dg(y) \cdot y)) \left( 1 - \frac{1}{\sqrt{1 + |Dg(y)|^2}} \right) \]
\[ = |(y, z_0 - z)|^2 - (z_0 + z)^2 \sum_{i=1}^{2} \kappa_i(0)y_i^2 + O \left( \|\kappa\|_\alpha |y|^{2+\alpha} |z| + |z_0| \right) \]
\[ + O \left( \|\kappa\|_0^2 |y|^4 + O \left( \|\kappa\|_0^2 |y|^2 |z| \right) \left( |z_0| + \|\kappa\|_0 |y|^2 \right) \right) \]
\[ = |(y, z_0 - z)|^2 - (z_0 + z)^2 \sum_{i=1}^{2} \kappa_i(0)y_i^2 \]
\[ + O \left( \|\kappa\|_\alpha |y|^{2+\alpha} (|z| + |z_0|) \right) + O \left( \|\kappa\|_0^2 |y|^2 (|y|^2 + |z| |z_0|) \right) \, . \]

By binomial theorem,
\[ |x_0 - x|^{-3-2s} = |(y, z_0 - z)|^{-3-2s} \left[ 1 + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{2} \kappa_i(0) \frac{y_i^2}{|(y, z_0 - z)|^2} \right] \]
\[ + O \left( \frac{\|\kappa\|_\alpha |y|^{2+\alpha} (|z| + |z_0|)}{|(y, z_0 - z)|^2} \right) + O \left( \frac{\|\kappa\|_0^2 |y|^4 (|z|^2 + |z_0|^2)}{|(y, z_0 - z)|^4} \right) \]
\[ = |(y, z_0 - z)|^{-3-2s} \left[ 1 + \frac{3 + 2s}{2} (z_0 + z) \sum_{i=1}^{2} \kappa_i(0) \frac{y_i^2}{|(y, z_0 - z)|^2} \right] \]
\[ + O \left( \frac{\|\kappa\|_\alpha |y|^{2+\alpha} (|z| + |z_0|)}{|(y, z_0 - z)|^2} \right) + O \left( \frac{\|\kappa\|_0^2 |y|^2 (|y|^2 + |z|^2 + |z_0|^2)}{|(y, z_0 - z)|^4} \right) \] .

\[ \square \]

**Proof of Lemma 3.8.** The first equality follows by the change of variable \( y = |z_0 - z|^\gamma \). To prove the second one, we have
\[
\int_{R^2} \frac{y_i^2}{|(y, z_0 - z)|^{5+2s}} \, dy = \frac{1}{2} \int_{R^2} \left( \frac{|y|^2 + |z_0 - z|^2}{|y|^2 + |z_0 - z|^2} \right)^{\frac{5+2s}{2}} \, dy
\]
\[ = \frac{1}{2} \int_{R^2} \frac{dy}{\left( \frac{|y|^2 + |z_0 - z|^2}{|y|^2 + |z_0 - z|^2} \right)^{\frac{5+2s}{2}}} - \frac{1}{2} |z_0 - z|^2 \int_{R^2} \frac{dy}{\left( \frac{|y|^2 + |z_0 - z|^2}{|y|^2 + |z_0 - z|^2} \right)^{\frac{5+2s}{2}}}
\]
\[ = \frac{1}{2} \frac{C_1, s}{2 C_3, s} \frac{1}{|z_0 - z|^{1+2s}} - \frac{1}{2} \frac{C_3, s}{2 C_5, s} \frac{|z_0 - z|^2}{|z_0 - z|^{1+2s}}
\]
\[ = \frac{1}{2} \frac{C_1, s}{2 C_3, s} \left( 1 - \frac{C_3, s}{C_1, s C_5, s} \right) \frac{1}{|z_0 - z|^{1+2s}} .
\]

Recalling that
\[ C_{n,s} = \frac{2^{2s} s}{\Gamma(1 - s)} \frac{\Gamma(\frac{n+2s}{2})}{\pi^{\frac{n+2s}{2}}} ,
\]

we have
\[ 1 - \frac{C_{3,s}^2}{C_{1,s} C_{3,s}} = 1 - \frac{\Gamma \left( \frac{3+2s}{2} \right)^2}{\Gamma \left( \frac{4+2s}{2} \right) \Gamma \left( \frac{2+2s}{2} \right)} = 1 - \frac{1 + 2s}{3 + 2s} = \frac{2}{3 + 2s} \]
and hence
\[ \int_{\mathbb{R}^2} \frac{y_i^2}{|y_i|^{3+2s}} dy = \frac{1}{3 + 2s} \frac{1}{C_{1,s} C_{3,s}} |y_i|^{-3+2s}. \]

\[ \square \]

4. Linear theory

4.1. Non-degeneracy of one-dimensional solution. Consider the linearized equation of \((-\Delta)^s u + f(u) = 0\) at \(w\), the one-dimensional solution, namely
\[ (-\Delta)^s \phi + f'(w)\phi = 0 \quad \text{for} \ (y,z) \in \mathbb{R}^n, \tag{4.1} \]
or the equivalent extension problem
\[
\begin{cases}
\nabla \cdot (t^s \nabla \phi) = 0 & \text{for} \ (y,z,t) \in \mathbb{R}^{n+1}_+ \\
t^s \frac{\partial \phi}{\partial \nu} + f'(w)\phi = 0 & \text{for} \ (y,z) \in \mathbb{R}^n.
\end{cases} \tag{4.2}
\]

Given \(\xi \in \mathbb{R}^{n-1}\), we define on
\[ X = H^1(\mathbb{R}^2_+, t^s) \]
the bilinear form
\[ (u,v)_X = \int_{\mathbb{R}^2_+} t^s \left( \nabla u \cdot \nabla v + |\xi|^2 uv \right) dz + \int_{\mathbb{R}} f'(w)uv dz. \]

**Lemma 4.1** (An inner product). Suppose \(\xi \neq 0\). Then \((\cdot, \cdot)_X\) defines an inner product on \(X\).

**Proof.** Clearly \((u,u)_X < \infty\) for any \(u \in X\). For \(R > 0\), denote \(B^+_R = B_R(0) \cap \mathbb{R}^2_+\) and its boundary in \(\mathbb{R}^2_+\) by \(\partial B^+_R\). It suffices to prove that
\[ \int_{B^+_R} t^s u^2 \, dz + \int_{\partial B^+_R} t^s u^2 \, |\nabla u|^2 \, dz = \int_{B^+_R} t^s w_z^2 \, |\nabla \left( \frac{u}{w_z} \right)|^2 \, dz. \tag{4.3} \]

Since the right hand side is non-negative, the result follows as we take \(R \to +\infty\). To check the above equality, we compute
\[
\int_{B^+_R} t^s w_z^2 \, |\nabla \left( \frac{u}{w_z} \right)|^2 \, dz \, dt = \int_{B^+_R} t^s \left| \nabla u - \frac{u}{w_z} \nabla w_z \right|^2 \, dz \, dt
\]
\[
= \int_{B^+_R} t^s |\nabla u|^2 \, dz \, dt + \int_{B^+_R} t^s \frac{u^2}{w_z^2} |\nabla w_z|^2 \, dz \, dt - \int_{B^+_R} t^s \nabla \left( u^2 \right) \cdot \nabla w_z \, dz \, dt.
\]

Since \(\nabla \cdot (t^s \nabla w_z) = 0\) in \(\mathbb{R}^2_+\), we can integrate the last integral by parts as
\[ - \int_{B^+_R} t^s \nabla \left( u^2 \right) \cdot \nabla w_z \, dz = - \int_{\partial B^+_R} t^s u^2 \frac{\partial w_z}{w_z} \, dz + \int_{B^+_R} u^2 \nabla \cdot \left( \frac{t^s \nabla w_z}{w_z} \right) \, dz \, dt
\]
\[ = \int_{\partial B^+_R} u^2 \frac{f'(w) w_z}{w_z} \, dz + \int_{B^+_R} t^s u^2 \nabla w_z \cdot \nabla \frac{1}{w_z} \, dz \, dt
\]
\[ = \int_{\partial B^+_R} \frac{t^s}{w_z} f'(w) u^2 \, dz - \int_{B^+_R} t^s \frac{u^2}{w_z} |\nabla w_z|^2 \, dz \, dt.
\]

Therefore, (4.3) holds and the proof is complete. \(\square\)
Lemma 4.2 (Solvability of the linear equation). Suppose $\xi \neq 0$. For any $g \in C_c^\infty (\mathbb{R}^+_1)$ and $h \in C_c^\infty (\mathbb{R})$, there exists a unique $u \in X$ of

$$
\begin{cases}
- \nabla \cdot (t^a \nabla u) + t^a |\xi|^2 u = g & \text{in } \mathbb{R}^+_1 \\
t^a \frac{\partial u}{\partial \nu} + f'(w) u = h & \text{on } \partial \mathbb{R}^+_1.
\end{cases}
$$

(4.4)

Proof. This equation has the weak formulation

$$(u, v)_X = \int_{\mathbb{R}^+_1} t^a \left( \nabla u \cdot \nabla v + |\xi|^2 uv \right) dz + \int_\mathbb{R} f'(w) uv dw = \int_{\mathbb{R}^+_1} gv dz + \int_\mathbb{R} hv dz.
$$

By Riesz representation theorem, there is a unique solution $u \in X$. $\square$

Lemma 4.3 (Non-degeneracy in one dimension [52, Lemma 4.2]). Let $w(z)$ be the unique increasing solution of

$$
(- \partial_z)^s w + f(w) = 0 \quad \text{in } \mathbb{R}.
$$

If $\phi(z)$ is a bounded solution of

$$
(- \partial_z)^s \phi + f'(w) \phi = 0 \quad \text{in } \mathbb{R},
$$

then $\phi(z) = C w'(z)$.

Lemma 4.4 (Non-degeneracy in higher dimensions). Let $\phi(y, z, t)$ be a bounded solution of

$$
\begin{cases}
\nabla_{(y,z,t)} \cdot (t^a \nabla_{(y,z,t)} \phi) = t^a \left( \partial_t + \frac{\partial}{\partial t} + \partial_{zz} + \Delta_y \right) \phi = 0 & \text{in } \mathbb{R}^{n+1}_+ \\
t^a \frac{\partial \phi}{\partial \nu} + f'(w) \phi = 0 & \text{on } \partial \mathbb{R}^{n+1}_+.
\end{cases}
$$

(4.5)

where $w(z, t)$ is the one-dimensional solution so that

$$
\begin{cases}
\nabla_{(z,t)} \cdot (t^a \nabla_{(z,t)} w_z) = t^a \left( \partial_t + \frac{\partial}{\partial t} + \partial_{zz} \right) w_z = 0 & \text{in } \mathbb{R}^2_+ \\
t^a \frac{\partial w_z}{\partial \nu} + f'(w) w_z = 0 & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
$$

Then $\phi(y, z, t) = c w_z(z, t)$ for some constant $c$.

Proof. For each $(z, t) \in \mathbb{R}^+_1$, let $\psi(\xi, z, t)$ be a smooth function in $\xi$ rapidly decreasing as $|\xi| \to +\infty$. The Fourier transform $\hat{\phi}(\xi, z, t)$ of $\phi(y, z, t)$ in the $y$-variable, which is the distribution defined by

$$
\langle \hat{\phi}(\cdot, z, t), \mu \rangle_{\mathbb{R}^{n-1}} = \langle \phi(\cdot, z, t), \hat{\mu} \rangle_{\mathbb{R}^{n-1}} = \int_{\mathbb{R}^{n-1}} \phi(\xi, z, t) \hat{\mu}(\xi) d\xi
$$

for any smooth rapidly decreasing function $\mu$, satisfies

$$
\int_{\mathbb{R}^n_{1+1}} \left( - \nabla \cdot (t^a \nabla \psi) + t^a |\xi|^2 \psi \right) \hat{\phi}(\xi, z, t) d\xi dz = \int_{\mathbb{R}^{n-1}} (-f'(w) \psi + t^a \psi|_{t=0}) \hat{\phi}(\xi, z, 0) d\xi dz.
$$

Let $\mu \in C_c^\infty (\mathbb{R}^{n-1})$, $\varphi_+ \in C_c^\infty (\mathbb{R}^2_1)$ and $\varphi_0 \in C_c^\infty (\mathbb{R})$ such that

$$
0 \notin \text{supp } (\mu).
$$

By Lemma 4.2, for any $\xi \neq 0$ we can solve the equation

$$
\begin{cases}
- \nabla \cdot (t^a \nabla \psi) + t^a |\xi|^2 \psi = \mu(\xi) \varphi_+(z, t) & \text{in } \mathbb{R}^2_+ \\
t^a \frac{\partial \psi}{\partial \nu} + f'(w) \psi = \mu(\xi) \varphi_0(\xi) & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
$$

uniquely for $\psi(\xi, z, t) \in X$ such that

$$
\psi(\xi, z, t) = 0 \quad \text{if } \xi \notin \text{supp } (\mu).
$$
In particular, $\psi(\cdot, z, t)$ is rapidly decreasing for any $(z, t) \in \mathbb{R}^2_+$. This implies
\[
\int_{\mathbb{R}^2_+} \langle \hat{\phi}(\cdot, z, t), \mu \rangle_{\mathbb{R}^{n-1}} \varphi_{+}(z, t) \, dz \, dt = \int_{\mathbb{R}} \langle \hat{\phi}(\cdot, z, 0), \mu \rangle_{\mathbb{R}^{n-1}} \varphi_{0}(z) \, dz
\]
for any $\varphi_{+} \in C_{c}^{\infty}(\mathbb{R}^2_+)$ and $\varphi_{0} \in C_{c}^{\infty}(\mathbb{R})$. In other words, whenever $0 \notin \text{supp}(\mu)$, we have
\[
\langle \hat{\phi}(\cdot, z, t), \mu \rangle_{\mathbb{R}^{n-1}} = 0 \quad \text{for all} \quad (z, t) \in \mathbb{R}^2_+.
\]
Such distribution with $\text{supp}(\hat{\phi}(\cdot, z, t)) \subseteq \{0\}$ is characterized as a linear combination of derivatives up to a finite order of Dirac masses at zero, namely
\[
\hat{\phi}(\xi, z, t) = \sum_{j=0}^{N} a_j(z, t) \delta_{0}^{(j)}(\xi),
\]
for some integer $N \geq 0$. Taking inverse Fourier transform, we see that $\hat{\phi}(y, z, t)$ is a polynomial in $y$ with coefficients depending on $(z, t)$. Since we assumed that $\phi$ is bounded, it is a zeroth order polynomial, i.e. $\phi$ is independent of $y$. Now the trace $\phi(z, 0)$ solves
\[
(-\Delta)^s \phi + f'(w)\phi = 0 \quad \text{in} \quad \mathbb{R}.
\]
By Lemma 4.3,
\[
\phi(z, t) = C\psi_{z}(z, t)
\]
for some constant $C \in \mathbb{R}$. This completes the proof. □

4.2. A priori estimates. Consider the equation
\[
(-\Delta)^s \phi(y, z) + f'(w(z))\phi(y, z) = g(y, z) \quad \text{for} \quad (y, z) \in \mathbb{R}^n. \tag{4.6}
\]
Let $\langle y \rangle = \sqrt{1 + |y|^2}$ and define the norm
\[
\|\phi\|_{\mu, \sigma} = \sup_{(y,z) \in \mathbb{R}^n} \langle y \rangle^\mu \langle z \rangle^\sigma |\phi(y, z)|
\]
for $0 \leq \mu < n - 1 + 2s$ and $2 - 2s < \sigma < 1 + 2s$ such that $\mu + \sigma < n + 2s$.

Lemma 4.5 (Decay in $z$). Let $\phi \in L^{\infty}(\mathbb{R}^n)$ and $\|g\|_{0, \sigma} < +\infty$. Then we have
\[
\|\phi\|_{0, \sigma} \leq C.
\]

With the decay established, the following orthogonality condition (4.7) is well-defined.

Lemma 4.6 (A priori estimate in $y, z$). Let $\phi \in L^{\infty}(\mathbb{R}^n)$ and $\|g\|_{\mu, \sigma} < +\infty$. If the $s$-harmonic extension $\phi(t, y, z)$ is orthogonal to $\psi_{z}(s, t)$ in $\mathbb{R}^{n+1}_+$, namely,
\[
\iint_{\mathbb{R}^2_+} t^s \phi \psi_{z} \, dt \, dz = 0, \tag{4.7}
\]
then we have
\[
\|\phi\|_{\mu, \sigma} \leq C \|g\|_{\mu, \sigma}.
\]

Before we give the proof, we estimate some integrals which arises from the product rule
\[
(-\Delta)^s(uv)(x_0) = u(x_0)(-\Delta)^s v(x_0) + C_{n,s} \int_{\mathbb{R}^n} \frac{u(x_0) - u(x)}{|x_0 - x|^{n+2s}} v(x) \, dx
\]
\[
= u(x_0)(-\Delta)^s v(x_0) + v(x_0)(-\Delta)^s u(x_0) - (u, v)_{s}(x_0),
\]
where
\[
(u, v)_{s}(x_0) = C_{n,s} \int_{\mathbb{R}^n} \frac{(u(x_0) - u(x))(v(x_0) - v(x))}{|x_0 - x|^{n+2s}} \, dx.
\]

Lemma 4.7 (Decay estimates). Suppose $\phi(y, z)$ is a bounded function.
(1) As $|y| \to +\infty$,
\[
(-\Delta)^s (y)^{-\mu} = O \left( (y)^{n-2s-\min(\mu, n-1)} \right)
\]
$\phi, (y)^{-\mu})_s = O \left( (y)^{-2s-\min(\mu, n-1)} \right)$.

(2) As $|z| \to +\infty$,
\[
(-\Delta)^s (z)^{-\sigma} = O \left( (z)^{-2s-\min(\sigma, 1)} \right)
\]
$\phi, (z)^{-\sigma})_s = O \left( (z)^{-2s-\min(\sigma, 1)} \right)$.

(3) As $\min\{|y|, |z|\} \to +\infty$,
\[
((y)^{-\mu}, (z)^{-\sigma})_s = O \left( (|y, z|)^{-n-2s}(|y|^{n-1-\mu} + 1)(|z|^{1-\sigma} + 1) \right)
\]
As $\min\{|y|, |z|\} \to +\infty$,
\[
((y)^{-\mu}, (z)^{-\sigma})_s = O \left( (|y, z|)^{-n-2s}(|y|^{n-1-\mu} + 1)(|z|^{1-\sigma} + 1) \right)
\]
\[
+ O \left( |y|^{-n-2s}(|y|^{n-1-\mu} + 1)|z|^{-\sigma-2} \min\{|y|, |z|\}^3 \right)
\]
\[
+ O \left( |y|^{-\mu-2}|z|^{-n-2s}(|z|^{1-\sigma} + 1) \min\{|y|, |z|\}^{n+1} \right)
\]
\[
+ O \left( |z|^{-\sigma}(|y| + |z|)^{-(n+1-2s)}(|y|^{n-1-\mu} + 1) \right)
\]
\[
+ O \left( |y|^{-\mu}(|y| + |z|)^{-1-2s}(|z|^{1-\sigma} + 1) \right)
\]
\[
+ O \left( |y|^{-\mu}|z|^{-\sigma}(|y| + |z|)^{-2s} \right).
\]

In particular,
\[
((y)^{-\mu}, (z)^{-\sigma})_s = o \left( |y|^{-\mu}|z|^{-\sigma} \right) \text{ as } \min\{|y|, |z|\} \to +\infty.
\]

(4) Suppose $\mu < n-1+2s$ and $\sigma < 1+2s$. As $\min\{|y|, |z|\} \to +\infty$,
\[
(-\Delta)^s (y)^{-\mu} (z)^{-\sigma} = o \left( |y|^{-\mu}|z|^{-\sigma} \right)
\]
$\phi, (y)^{-\mu} (z)^{-\sigma})_s = o \left( |y|^{-\mu}|z|^{-\sigma} \right)$.

(5) Suppose $\eta_R(y) = \eta \left( \frac{\|y\|}{R} \right)$ where $\eta$ is a smooth cut-off function as in (4.11), and $\phi(y, z) \leq C (z)^{-\sigma}$. For all sufficiently large $R > 0$, we have
\[
\|(\Delta)^s \eta_R \phi(y, z)| \leq C \left( |z|^{-1} + |z|^{-\sigma} \right) \max\{|y|, R\}^{-2s}. \tag{4.8}
\]

Let us assume the validity of Lemma 4.7 for the moment.

Proof of Lemma 4.5. It follows from Lemma 4.7(2) and a maximum principle [28].

Proof of Lemma 4.6. We will first the a priori estimate assuming that $\|\phi\|_{\mu, \sigma} < +\infty$. We use a blow-up argument. Suppose on the contrary that there exist a sequence $\phi_m(y, z)$ and $h_m(y, z)$ such that
\[
(-\Delta)^s \phi_m + f'(w)\phi_m = g_m \quad \text{for} \quad (y, z) \in \mathbb{R}^n
\]
and
\[
\|\phi_m\|_{\mu, \sigma} = 1 \quad \text{and} \quad \|g_m\|_{\mu, \sigma} \to 0 \quad \text{as} \quad m \to +\infty.
\]

Then there exist a sequence of points $(y_m, z_m) \in \mathbb{R}^n$ such that
\[
\phi_m(y_m, z_m) \langle y_m \rangle^\mu \langle z_m \rangle^\sigma \geq \frac{1}{2}. \tag{4.9}
\]

We consider four cases.
(1) $y_m, z_m$ bounded:
Since $\phi_m$ is bounded and $g_m \to 0$ in $L^\infty(\mathbb{R}^n)$, by elliptic estimates and passing to a subsequence, we may assume that $\phi_m$ converges uniformly in compact subsets of $\mathbb{R}^n$ to a function $\phi_0$ which satisfies

$(-\Delta)^s \phi_0 + f'(w)\phi_0 = 0, \quad \text{in } \mathbb{R}^n$

and, by (4.7),

$$\int_{\mathbb{R}^n_+} t^s \phi_0 w_z \, dt \, dz = 0.$$ 

By the non-degeneracy of $w'$ (Lemma 4.4), we necessarily have $\phi_0(y, z) = Cw'(z)$. However, the orthogonality condition yields $C = 0$, i.e. $\phi_0 \equiv 0$. This contradicts (4.9).

(2) $y_m$ bounded, $|z_m| \to \infty$:
We consider $\tilde{\phi}_m(y, z) = \langle z_m + z \rangle^\sigma \phi_m(y, z_m + z)$, which satisfies in $\mathbb{R}^n$

$$\langle z_m + z \rangle^{-\sigma} \langle -\Delta \rangle^s \tilde{\phi}_m(y, z) + \tilde{\phi}_m(y, z) \langle -\Delta \rangle^s \langle z_m + z \rangle^{-\sigma} - \left( \tilde{\phi}_m(y, z), \langle z_m + z \rangle^\sigma \right) \nu,$$

or

$$\langle -\Delta \rangle^s \tilde{\phi}_m + \left( f'(w(z_m + z)) + \langle -\Delta \rangle^s \langle z_m + z \rangle^{-\sigma} \langle y_m + y \rangle^{-\mu} \right) \tilde{\phi}_m = g_m(y, z), \quad \text{in } \mathbb{R}^n.$$

Using Lemma 4.7(2), the limiting equation is

$$\langle -\Delta \rangle^s \tilde{\phi}_0 + 2\tilde{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n.$$

Thus $\tilde{\phi}_0 = 0$, contradicting (4.9).

(3) $|y_m| \to \infty, z_m$ bounded:
We define $\tilde{\phi}_m(y, z) = \langle y_m + y \rangle^\mu \phi_m(y_m + y, z)$, which satisfies

$$\langle -\Delta \rangle^s \tilde{\phi}_m(y, z) + \left( f'(w(z)) + \langle -\Delta \rangle^s \langle y_m + y \rangle^{-\mu} \right) \tilde{\phi}_m(y, z)$$

$$= g_m(y_m + y, z), \quad \text{in } \mathbb{R}^n.$$

By Lemma 4.7(1), the subsequential limit $\tilde{\phi}_0$ satisfies

$$\langle -\Delta \rangle^s \tilde{\phi}_0 + f'(w)\tilde{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n.$$

This leads to a contradiction as in case (1).

(4) $|y_m|, |z_m| \to \infty$:
This is similar to case (2). In fact for $\tilde{\phi}_m(y, z) = \langle y_m + y \rangle^\mu \langle z_m + z \rangle^\sigma \phi_m(y_m + y, z_m + z)$, we have

$$\langle -\Delta \rangle^s \tilde{\phi}_m(y, z) + \left( f'(w(z_m + z)) + \langle -\Delta \rangle^s \langle y_m + y \rangle^{-\mu} \langle z_m + z \rangle^{-\sigma} \right) \tilde{\phi}_m(y, z)$$

$$= g_m(y_m + y, z_m + z), \quad \text{in } \mathbb{R}^n.$$
In the limiting situation $\hat{\phi}_m \to \hat{\phi}_0$, by Lemma 4.7(4),

$$(-\Delta)^s \hat{\phi}_0 + 2\hat{\phi}_0 = 0 \quad \text{in } \mathbb{R}^n,$$

forcing $\hat{\phi}_0 = 0$ which contradicts (4.9).

We conclude that

$$\|\phi\|_{\mu,\sigma} \leq C \|g\|_{\mu,\sigma} \quad \text{provided} \quad \|\phi\|_{\mu,\sigma} < +\infty. \quad (4.10)$$

Now we will remove the condition $\|\phi\|_{\mu,\sigma} < +\infty$. By Lemma 4.5, we know that $\|\phi\|_{0,\sigma} < +\infty$. Let $\eta : [0, +\infty) \to [0, 1]$ be a smooth cut-off function such that

$$\eta = 1 \text{ on } [0, 1] \quad \text{and} \quad \eta = 0 \text{ on } [2, +\infty). \quad (4.11)$$

Write $\eta_R(y) = \eta \left( \frac{|y|}{R} \right)$. We apply the above derived a priori estimate to $\psi(y, z) = \eta_R(y)\phi(y, z)$, which satisfies

$$(-\Delta)^s \psi + f'(w)\psi = \eta_R g + \phi(-\Delta)^s \eta_R - \langle \eta_R, \phi \rangle. \quad (4.12)$$

It is clear that $\|\eta_R g\|_{\mu,\sigma} \leq \|g\|_{\mu,\sigma}$ and $\|\phi(-\Delta)^s \eta_R\|_{\mu,\sigma} \leq CR^{-2s}$ because of the estimate $(-\Delta)^s \eta(|y|) \leq C \langle y \rangle^{-(n+1-2s)}$. By Lemma 4.7(5),

$$\|(-\Delta)^s \eta_R \phi(y, 0) \phi_{\mu,\sigma} \leq C \left( |z_0|^{-1} + |z_0|^{-\sigma} \right) \max \{ |y_1|, R \}^{-2s}. \quad (4.13)$$

For $\sigma < 1$ and $0 \leq \mu < 2s$, this yields

$$\|(-\Delta)^s \eta_R \phi\|_{\mu,\sigma} \leq CR^{-(2s-\mu)}. \quad (4.14)$$

Therefore, (4.10) and (4.12) give

$$\|\eta_R \phi\|_{\mu,\sigma} \leq C \|g\|_{\mu,\sigma} + CR^{-2s} + CR^{-(2s-\mu)}. \quad (4.15)$$

Letting $R \to +\infty$, we arrive at

$$\|\phi\|_{\mu,\sigma} \leq C \|g\|_{\mu,\sigma}, \quad (4.16)$$

as desired. \hfill $\Box$

**Proof of Lemma 4.7.** We will only prove the statements regarding the fractional Laplacian of the explicit function. The associated assertion concerning the inner product with $\phi$ will follow from the same proof using the its boundedness, since all the terms are estimated in absolute value.

(1) We have

$$(-\Delta_{(y, z)})^s \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} = (-\Delta_y)^s \langle \langle y \rangle^\mu \rangle_{y=y_0}$$

$$= C_{n-1,s} \int_{\mathbb{R}^{n-1}} \frac{\langle \langle y_0 \rangle^{-\mu} - \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0) - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0)}{|y_0-y|^{n-1+2s}} \, dy$$

$$\equiv I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = C_{n-1,s} \int_{B_{\frac{|y_0|}{2}}(y_0)} \frac{\langle \langle y_0 \rangle^{-\mu} - \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0)}{|y_0-y|^{n-1+2s}} \, dy,$$

$$I_2 = C_{n-1,s} \int_{B_1(0)} \frac{\langle \langle y_0 \rangle^{-\mu} - \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0)}{|y_0-y|^{n-1+2s}} \, dy,$$

$$I_3 = C_{n-1,s} \int_{B_1(0) \setminus B_{\frac{|y_0|}{2}}(y_0)} \frac{\langle \langle y_0 \rangle^{-\mu} - \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0)}{|y_0-y|^{n-1+2s}} \, dy,$$

$$I_4 = C_{n-1,s} \int_{\mathbb{R}^{n-1} \setminus \left( B_{\frac{|y_0|}{2}}(y_0) \cup B_{\frac{|y_0|}{2}}(0) \right)} \frac{\langle \langle y_0 \rangle^{-\mu} - \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} - D \langle \langle y \rangle^{-\mu} \rangle_{y=y_0} (y-y_0)}{|y_0-y|^{n-1+2s}} \, dy.$$
If $|y_0| \leq 1$, it is simple to get boundedness since $\langle y \rangle^{-\mu}$ is smooth and bounded. For $|y_0| \geq 1$, we compute

$$|I_1| \lesssim \int_{B_{\frac{|y_0|}{2}}(y_0)} \frac{|D^2 \langle y \rangle^{-\mu}|_{y=y_0}|y-y|^2}{|y_0-y|^{n-1+2s}} \, dy$$

$$\lesssim |y_0|^{-\mu-2} \int_0^{\frac{|y_0|}{2}} \frac{\rho^2}{\rho^{1+2s}} \, d\rho$$

$$\lesssim |y_0|^{-(\mu+2s)},$$

$$|I_2| \lesssim \int_{B_1(0)} \frac{1}{|y_0|^{n-1+2s}} \, dy$$

$$\lesssim |y_0|^{-(n-1+2s)},$$

$$|I_3| \lesssim |y_0|^{-(n-1+2s)} \int_{B_{\frac{|y_0|}{2}}(y_0) \setminus B_1(0)} \left( \langle y_0 \rangle^{-\mu} + |y|^{-\mu} \right) \, dy$$

$$\lesssim |y_0|^{-(n-1+2s)} \int_1^{\frac{|y_0|}{2}} \left( \langle y_0 \rangle^{-\mu} + \rho^{-\mu} \right) \rho^{-n-2} \, d\rho$$

$$\lesssim |y_0|^{-(n-1+2s)} \left( \langle y_0 \rangle^{-\mu} (|y_0|^{n-1} + |y_0|^{-\mu+n-1} - 1) \right)$$

$$\lesssim |y_0|^{-(\mu+2s)} + |y_0|^{-(n-1+2s)},$$

$$|I_4| \lesssim |y_0|^{-\mu} \int_{\mathbb{R}^{n-1} \setminus \left( B_{\frac{|y_0|}{2}}(y_0) \cup B_{\frac{|y_0|}{2}}(0) \right)} \frac{1}{|y_0-y|^{n-1+2s}} \, dy$$

$$\lesssim |y_0|^{-\mu} \int_{\frac{|y_0|}{2}}^{\infty} \frac{1}{\rho^{1+2s}} \, d\rho$$

$$\lesssim |y_0|^{-(\mu+2s)}.$$

(2) This follows from the same proof as (1).

(3) We divide $\mathbb{R}^{n-1} \times \mathbb{R}$ into a 14 regions in terms of the relative size of $|y|, |z|$ with respect to $|y_0|, |z_0|$ which tend to infinity. We will consider such distance “small” if $|y| < 1$ and “intermediate” if $1 < |y| < \frac{|y_0|}{2}$, similarly for $z$. Once the non-decaying part of $\langle y \rangle^{-\mu}, \langle z \rangle^{-\sigma}$ are excluded, the remaining parts can be either treated radially where we consider $(y_0, z_0)$ as the origin, or reduced to the one-dimensional case. More precisely, we write

$$(\langle y \rangle^{-\mu}, \langle z \rangle^{-\sigma})_s(y_0, z_0) = C_{n,s} \int_{\mathbb{R}^n} \frac{\left( \langle y \rangle^{-\mu} - \langle y_0 \rangle^{-\mu} \right) \left( \langle z \rangle^{-\sigma} - \langle z_0 \rangle^{-\sigma} \right)}{|(y-y_0, z-z_0)|^{n+2s}} \, dydz$$

$$= \sum_{1 \leq i,j \leq 4, \min\{i,j\} \leq 2} I_{ij}^{axcs} + I^{sing} + I^{rest},$$
where

\[
I_{11} = C_{n,s} \int_{|y| < 1, |z| < 1} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{12} = C_{n,s} \int_{|y| < 1, 1 < |z| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{13} = C_{n,s} \int_{|y| < 1, |z - z_0| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{14} = C_{n,s} \int_{|y| < 1, \min\{|z|, |z - z_0|\} > \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{21} = C_{n,s} \int_{1 < |y| < \frac{||y|}{|y_0|}, |z| < 1} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{22} = C_{n,s} \int_{1 < |y| < \frac{||y|}{|y_0|}, 1 < |z| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{23} = C_{n,s} \int_{1 < |y| < \frac{||y|}{|y_0|}, |z - z_0| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{24} = C_{n,s} \int_{1 < |y| < \frac{||y|}{|y_0|}, \min\{|z|, |z - z_0|\} > \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{31} = C_{n,s} \int_{|y - y_0| < \frac{||y|}{|y_0|}, |z| < 1} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{32} = C_{n,s} \int_{|y - y_0| < \frac{||y|}{|y_0|}, 1 < |z| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{41} = C_{n,s} \int_{\min\{|y|, |y - y_0|\} > \frac{||y|}{|y_0|}, |z| < 1} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{42} = C_{n,s} \int_{\min\{|y|, |y - y_0|\} > \frac{||y|}{|y_0|}, 1 < |z| < \frac{||y|}{|y_0|}} \frac{(y)^{-\mu} - (y_0)^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{sing} = C_{n,s} \int_{|y| > \frac{||y|}{|y_0|}, |z| > \frac{||y|}{|y_0|}, |(y - y_0, z - z_0)| < \frac{||y|}{|y_0|} + ||y|}{|y - y_0, z - z_0|^{n+2s}} \left( (y)^{-\mu} - (y_0)^{-\mu} \right) \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz,
\]

\[
I_{rest} = C_{n,s} \int_{|y| > \frac{||y|}{|y_0|}, |z| > \frac{||y|}{|y_0|}, |(y - y_0, z - z_0)| > \frac{||y|}{|y_0|} + ||y|}{|y - y_0, z - z_0|^{n+2s}} \left( (y)^{-\mu} - (y_0)^{-\mu} \right) \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right) dydz.
\]

We will estimate these integrals one by one. In the unit cylinder we have

\[
|I_{11}| \lesssim \frac{1}{|(y_0, z_0)|^{n+2s}} \int_{|y| < 1, |z| < 1} dydz \lesssim |(y_0, z_0)|^{-n-2s}.
\]
On a thin strip near the origin,
\[
|I_{12}| \lesssim \frac{1}{|(y_0, z_0)|^{n+2s}} \iint_{|y|<1, 1<|z|<\frac{|z_0|}{2}} (|z|^{-\sigma} + (z_0)^{-\sigma}) \, dydz \\
\lesssim |(y_0, z_0)|^{-n-2s} \left( |z_0|^{1-\sigma} + 1 \right).
\]
Similarly
\[
|I_{21}| \lesssim \frac{1}{|(y_0, z_0)|^{n+2s}} \iint_{1<|y|<\frac{|y_0|}{2}, |z|<1} (|y|^{-\mu} + (y_0)^{-\mu}) \, dydz \\
\lesssim |(y_0, z_0)|^{-n-2s} \left( |y_0|^{n-1-\mu} + 1 \right),
\]
and in the intermediate rectangle,
\[
|I_{22}| \lesssim \iint_{1<|y|<\frac{|y_0|}{2}, 1<|z|<\frac{|z_0|}{2}} (|y|^{-\mu} + (y_0)^{-\mu}) (|z|^{-\sigma} + (z_0)^{-\sigma}) \, dydz \\
\lesssim |(y_0, z_0)|^{-n-2s} \left( |y_0|^{n-1-\mu} + 1 \right) \left( |z_0|^{1-\sigma} + 1 \right).
\]
The integral on a thin strip far is more involved. We first integrate the \(z\) variable by a change of variable \(z = z_0 + |y_0 - y|\zeta\).
\[
I_{13} = C_{n,s} \int_{|y|<1, |z-z_0|<\frac{|z_0|}{2}} \frac{(\langle y \rangle^{-\mu} - (y_0)^{-\mu}) (\langle z \rangle^{-\sigma} - (z_0)^{-\sigma} - D \langle z \rangle^{-\sigma} |z_0| (z - z_0))}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz \\
= C_{n,s} \int_{|y|<1, |z-z_0|<\frac{|z_0|}{2}} \frac{(\langle y \rangle^{-\mu} - (y_0)^{-\mu}) (z - z_0)^2 \left( \int_0^1 (1 - t) D^2 \langle z \rangle^{-\sigma} |z_0 + t(z - z_0)| \, dt \right)}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz \\
= C_{n,s} \int_{|y|<1} \frac{\langle y \rangle^{-\mu} - (y_0)^{-\mu}}{|y - y_0|^{n-3+2s}} \int_{|z-z_0|<\frac{|z_0|}{2}} \left( \int_0^1 (1 - t) D^2 \langle z \rangle^{-\sigma} |z_0 + t(y - y_0)| \, dt \right) z^2 \frac{d\zeta}{(1 + \zeta^2)^{n+2s}} \, dy.
\]
Observing that in this regime \(|y - y_0| \sim |y_0|\) and that
\[
\int_0^T \frac{t^2}{(1 + t^2)^{\frac{n+3}{2}}} \, dt \lesssim \min \{T^3, 1\},
\]
we have
\[
|I_{13}| \lesssim \int_{|y|<1} \frac{1}{|y - y_0|^{n-3+2s} |z_0|^{-\sigma-2} \min \left\{ \left( \frac{|z_0|}{|y - y_0|} \right)^3, 1 \right\}} \, dy \\
\lesssim |y_0|^{-n-2s} |z_0|^{-\sigma-2} \min \{ |y_0|, |z_0| \}^3.
\]
Similarly, changing \(y = y_0 + |z - z_0|\eta\), we have
\[
I_{31} = C_{n,s} \int_{|y-y_0|<\frac{|y_0|}{2}, |z|<1} \frac{(\langle y \rangle^{-\mu} - (y_0)^{-\mu} - D \langle y \rangle^{-\mu} |y_0 \cdot (y - y_0)|) (\langle z \rangle^{-\sigma} - (z_0)^{-\sigma})}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz \\
= C_{n,s} \int_{|y-y_0|<\frac{|y_0|}{2}, |z|<1} \frac{(\langle z \rangle^{-\sigma} - (z_0)^{-\sigma})}{|(y - y_0, z - z_0)|^{n+2s}} \\
\cdot \left( \sum_{i,j=1}^{n-1} \int_0^1 (1 - t) \partial_{ij} \langle y \rangle^{-\mu} |y_0 + t(y - y_0)| \, dt \right) (y - y_0)_i(y - y_0)_j \, dydz \\
= \sum_{i,j=1}^{n-1} \int_{|z|<1} \frac{(\langle z \rangle^{-\sigma} - (z_0)^{-\sigma})}{|z - z_0|^{2s-1}} \int_{|\eta|<\frac{|z_0|}{2s-1}} \left( \int_0^1 (1 - t) \partial_{ij} \langle y \rangle^{-\mu} |y_0 + t(z - z_0)| \, dt \right) \eta_{i,j} \, d\eta \, dz. 
\]
The \( t \)-integral is controlled by \((y_0)^{-\mu} \) since \(|y_0 + t|z - z_0| \eta| < \frac{|y_0|}{2}\). Then using
\[
\int_{|\eta| < \eta_0} \frac{|\eta_i||\eta_j|}{(|\eta|^2 + 1)^{\nu + 2s}} d\eta \lesssim \int_0^{\eta_0} \frac{\rho^2 \rho^{n-2}}{(\rho^2 + 1)^{\frac{n+2s}{2}}} d\rho \\
\lesssim \min \{\eta_0^{n+1}, 1\},
\]
(noting that here we again require \( s > 1/2 \)) we have
\[
|I_{31}| \lesssim \sum_{i,j=1}^{n-1} \int_{|z| < 1} \frac{1}{|z - z_0|^{n-2s}} \langle y_0 \rangle^{-\mu - 2} \min \left\{ \left( \frac{|y_0|}{|z - z_0|} \right)^{n+1}, 1 \right\} dz \\
\lesssim |z_0|^{-n-2s} \langle y_0 \rangle^{-\mu - 2} \min \{ |y_0|, |z_0| \}^{n+1}.
\]

Next we deal with the \( y \)-intermediate, \( z \)-far regions, namely \( I_{23} \). The treatment is similar to that of \( I_{13} \) except that we need to integrate in \( y \). We have, as above,
\[
I_{23} = C_{n,s} \int_{1 < |y| < \frac{|y_0|}{2}} \int_{|z| < 1} \frac{(\langle y \rangle^{-\mu} - \langle y_0 \rangle^{-\mu}) (\langle z \rangle^{-\sigma} - \langle z_0 \rangle^{-\sigma} - D \langle z \rangle^{-\sigma} |z_0(z - z_0)|)}{|(y - y_0, z - z_0)|^{n+2s}} dy dz \\
= C_{n,s} \int_{1 < |y| < \frac{|y_0|}{2}} \int_{1 < |z| < \frac{|z_0|}{2}} \frac{\langle y \rangle^{-\mu} - \langle y_0 \rangle^{-\mu}}{|y - y_0|^{n-3+2s} |z|^{-2s}} \int_{\tau(z,y)} \left( \int_0^1 (1-t)D^2 \langle z \rangle^{-\sigma} |z_0 + t(y - y_0)\zeta| \right) \frac{\zeta d\zeta}{(1 + \zeta^2)\frac{n+2s}{2}} dy.
\]

Hence
\[
|I_{23}| \lesssim \int_{1 < |y| < \frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0 \rangle^{-\mu}}{|y - y_0|^{n-3+2s} |z_0|^{-\sigma-2}} \min \left\{ \left( \frac{|z_0|}{|y - y_0|} \right)^3, 1 \right\} dy \\
\lesssim |y_0|^{-n-2s} |z_0|^{-\sigma-2} \min \{ |y_0|, |z_0| \}^{3} \int_{1 < |y| < \frac{|y_0|}{2}} (|y|^{-\mu} + \langle y_0 \rangle^{-\mu}) dy \\
\lesssim |y_0|^{-n-2s} |z_0|^{-\sigma-2} \min \{ |y_0|, |z_0| \}^{3} \left( |y_0|^{-1-\mu} + 1 \right).
\]

Similarly, we estimate
\[
I_{32} = C_{n,s} \int_{|y_0| < \frac{|y_0|}{2}} \int_{1 < |z| < \frac{|z_0|}{2}} \frac{(\langle y \rangle^{-\mu} - \langle y_0 \rangle^{-\mu} - D \langle y \rangle^{-\mu} |y_0| \cdot (y - y_0)) (\langle z \rangle^{-\sigma} - \langle z_0 \rangle^{-\sigma})}{|(y - y_0, z - z_0)|^{n+2s}} dy dz \\
= \sum_{i,j=1}^{n-1} \int_{1 < |z| < \frac{|z_0|}{2}} \int_{|y| < \frac{|y_0|}{2}} \frac{(\langle z \rangle^{-\sigma} - \langle z_0 \rangle^{-\sigma})}{|z - z_0|^{2s-1}} \int_{\tau(y_0,z)} \left( \int_0^1 (1-t)\partial_{ij} \langle y \rangle^{-\mu} |y_0 + t(y - y_0)| \zeta dt \right) \frac{|\eta_i\eta_j| d\eta}{(|\eta|, 1)^{n+2s}} dz,
\]
which yields
\[
|I_{32}| \lesssim \sum_{i,j=1}^{n-1} \int_{1 < |z| < \frac{|z_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0 \rangle^{-\sigma}}{|z - z_0|^{2s-1}} \langle y_0 \rangle^{-\mu - 2} \min \left\{ \left( \frac{|y_0|}{|z - z_0|} \right)^{n+1}, 1 \right\} dz \\
\lesssim |z_0|^{-n-2s} |y_0|^{-\mu - 2} \min \{ |y_0|, |z_0| \}^{n+1} \int_{1 < |z| < \frac{|z_0|}{2}} (|z|^{-\sigma} + \langle z_0 \rangle^{-\sigma}) dz \\
\lesssim |z_0|^{-n-2s} |y_0|^{-\mu - 2} \min \{ |y_0|, |z_0| \}^{n+1} \left( |z_0|^{1-\sigma} + 1 \right).
\]
We consider the remaining part of the small strip, namely $I_{14}$ and $I_{41}$. Using the change of variable $z = z_0 + |y_0|\xi$, we have

\[
I_{14} = C_{n,s} \int_{|y| < 1, \min\{|z|,|z-z_0|\} > \frac{|y_0|}{2|z_0|}} \frac{\left( (y)^{-\mu} - (y_0)^{-\mu} \right) \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right)}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz,
\]

\[
|I_{14}| \lesssim (z_0)^{-\sigma} \int_{|y| < 1, \min\{|z|,|z-z_0|\} > \frac{|y_0|}{2|z_0|}} \frac{1}{|y_0, z - z_0|^{n+2s}} \, dz
\]

\[
\lesssim (z_0)^{-\sigma} \frac{1}{|y_0|^{n+1+2s}} \int_{|\xi| > \frac{|y_0|}{2|z_0|}, |z-z_0| > \frac{|y_0|}{2|z_0|}} \frac{1}{|(1, \zeta)|^{n+2s}} \, d\zeta
\]

\[
\lesssim (z_0)^{-\sigma} |y_0|^{-(n+1+2s)} \min \left\{ 1, \left( \frac{|z_0|}{|y_0|} \right)^{-(n+1+2s)} \right\}
\]

\[
\lesssim (z_0)^{-\sigma} \min \left\{ |y_0|^{-(n+1+2s)}, |z_0|^{-(n+1+2s)} \right\}
\]

\[
\lesssim (z_0)^{-\sigma} (|y_0| + |z_0|)^{-(n+1+2s)}.
\]

Similarly, with $y = y_0 + |z_0|\eta$, we have

\[
I_{41} = C_{n,s} \int_{\min\{|y|,|y-y_0|\} > \frac{|y_0|}{2|z_0|}, |z| < 1} \frac{\left( (y)^{-\mu} - (y_0)^{-\mu} \right) \left( (z)^{-\sigma} - (z_0)^{-\sigma} \right)}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz,
\]

\[
|I_{41}| \lesssim (y_0)^{-\mu} \int_{\min\{|y|,|y-y_0|\} > \frac{|y_0|}{2|z_0|}, |z| < 1} \frac{1}{|(y - y_0, z_0)|^{n+2s}} \, dydz
\]

\[
\lesssim (y_0)^{-\mu} |z_0|^{-(1+2s)} \int_{|y| > \frac{|y_0|}{2|z_0|}} \frac{dy}{|y|^2 + 1} \frac{1}{(\rho^2 + 1)^{\frac{n+2s}{2}}}
\]

\[
\lesssim (y_0)^{-\mu} |z_0|^{-(1+2s)} \int_{\frac{|y_0|}{2|z_0|}}^\infty \frac{\rho^{-n-2}}{(\rho^2 + 1)^{\frac{n+2s}{2}}} \, d\rho
\]

\[
\lesssim (y_0)^{-\mu} |z_0|^{-(1+2s)} \min \left\{ \left( \frac{|y_0|}{2|z_0|} \right)^{-(1+2s)}, 1 \right\}
\]

\[
\lesssim (y_0)^{-\mu} (|y_0| + |z_0|)^{-(1+2s)}.
\]
In the remaining intermediate region, we first “integrate” in $z$ by the change of variable $z = z_0 + |y - y_0| \zeta$ as follows.

$$I_{24} = C_{n,s} \int_{1 < |y| < \frac{|y_0|}{2}, \min \{|z|, |z-z_0|\} > \frac{|y_0|}{2}} \frac{\langle (y)^{-\mu} - (y_0)^{-\mu} \rangle \langle (z)^{-\sigma} - (z_0)^{-\sigma} \rangle}{|(y-y_0, z-z_0)|^{n+2s}} dydz,$$

$$|I_{24}| \lesssim \langle z_0 \rangle^{-\sigma} \int_{1 < |y| < \frac{|y_0|}{2}, \min \{|z|, |z-z_0|\} > \frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0 \rangle^{-\mu}}{(y-y_0, z-z_0)|^{n+2s}} dydz$$

$$\lesssim \langle z_0 \rangle^{-\sigma} \int_{1 < |y| < \frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0 \rangle^{-\mu}}{y-y_0|^{n-1+2s}} \int_{\frac{|z|}{y-y_0|^{1+2s}}}^{\frac{|z_0|}{y-y_0|^{1+2s}}} \min \left\{ 1, \left( \frac{|z|}{|y-y_0|} \right)^{(n-1+2s)} \right\} dy$$

$$\lesssim \langle z_0 \rangle^{-\sigma} \int_{1 < |y| < \frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0 \rangle^{-\mu}}{y-y_0|^{n-1+2s}} \int_{1 < |y| < \frac{|y_0|}{2}} \frac{|y|^{-\mu} + \langle y_0 \rangle^{-\mu}}{(y-y_0|^{n+2s}} dy$$

$$\lesssim |y|^{n-1-\mu} \langle z_0 \rangle^{-\sigma} \langle |y_0| + |z_0| \rangle^{-(n-1+2s)}.$$

Similarly,

$$I_{42} = C_{n,s} \int_{\min \{|y|, |y-y_0|\} > \frac{|y_0|}{2}, 1 < |z| < \frac{|y_0|}{2}} \frac{\langle (y)^{-\mu} - (y_0)^{-\mu} \rangle \langle (z)^{-\sigma} - (z_0)^{-\sigma} \rangle}{|(y-y_0, z-z_0)|^{n+2s}} dydz,$$

$$|I_{42}| \lesssim \langle y_0 \rangle^{-\mu} \int_{\min \{|y|, |y-y_0|\} > \frac{|y_0|}{2}, 1 < |z| < \frac{|y_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0 \rangle^{-\sigma}}{(y-y_0, z-z_0)|^{n+2s}} dydz$$

$$\lesssim \langle y_0 \rangle^{-\mu} \int_{1 < |z| < \frac{|y_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0 \rangle^{-\sigma}}{z-z_0|^{1+2s}} \int_{|\eta| > \frac{|y_0|}{2}} \frac{d\eta}{(|\eta|^2 + 1)^{n+2s}} dz$$

$$\lesssim \langle y_0 \rangle^{-\mu} \int_{1 < |z| < \frac{|y_0|}{2}} \frac{|z|^{-\sigma} + \langle z_0 \rangle^{-\sigma}}{z-z_0|^{1+2s}} \min \left\{ \left( \frac{|y_0|}{|z-z_0|} \right)^{-1-2s}, 1 \right\} dz$$

$$\lesssim \langle y_0 \rangle^{-\mu} |z_0|^{1-\sigma} \langle |y_0| + |z_0| \rangle^{-(1+2s)}.$$

Now we estimate the singular part $I^{sing}$. The only concern is that if, say, $|y_0| \gg |z_0|$, then the line segment joining $z_0$ and $z$ may intersect the $y$-axis. To fix the idea we suppose that $|y_0| \geq |z_0|$. Having all estimates for the integrals in a neighborhood of the axes, one can factor out the decay $\langle z \rangle^{-\sigma} - \langle z_0 \rangle^{-\sigma}$ and obtain integrability by expanding the bracket with $y$ to second order, as follows. For simplicity let us write

$$\Omega_{sing} = \left\{ (y, z) \in \mathbb{R}^n : |y| > \frac{|y_0|}{2}, |z| > \frac{|z_0|}{2}, |(y-y_0, z-z_0)| < \frac{|y_0| + |z_0|}{2} \right\}.$$
Then
\[
I^{\text{sing}} = C_{n,s} \int_{\Omega_{\text{sing}}} \frac{(y)^{-\mu} - \langle y \rangle^{-\mu}}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz
\]
\[
= C_{n,s} \int_{\Omega_{\text{sing}}} \frac{(z)^{-\sigma} - \langle z \rangle^{-\sigma}}{|(y - y_0, z - z_0)|^{n+2s}}
\]
\[
\quad \cdot \left( \sum_{i,j=1}^{n-1} \int_0^1 (1 - t) \partial_{ij} (y)^{-\mu} |y_0 + t(y - y_0)| \, dt \right) (y - y_0)_i (y - y_0)_j \, dydz.
\]

Thus
\[
|I^{\text{sing}}| \lesssim \|z_0\|^{-\sigma} \|y_0\|^{-\mu - 2} \int_{\Omega_{\text{sing}}} \frac{|y - y_0|^2}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz
\]
\[
\lesssim \|z_0\|^{-\sigma} \|y_0\|^{-\mu - 2} \int_0^{\|y_0 + z_0\|} \frac{\rho^2}{\rho^{1+2s}} \, d\rho
\]
\[
\lesssim \|y_0\|^{-\mu - 2s} \|z_0\|^{-\sigma}.
\]

The same argument implies that if $|z_0| \geq |y_0|$ then
\[
|I^{\text{sing}}| \lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma - 2s}.
\]

Therefore, we have in general
\[
|I^{\text{sing}}| \lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma} \max \{\|y_0\|, \|z_0\|\}^{-2s}
\]
\[
\lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma} (\|y_0\| + \|z_0\|)^{-2s}.
\]

Finally, the remaining exterior integral is controlled by
\[
|I^{\text{ext}}| \lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma} \int_{|y| > \frac{|y_0|}{2}, |z| > \frac{|z_0|}{2}, |(y - y_0, z - z_0)| < \frac{|y_0| + |z_0|}{2}} \frac{1}{|(y - y_0, z - z_0)|^{n+2s}} \, dydz
\]
\[
\lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma} \int_{\frac{|y_0|}{2}}^{\infty} \frac{\rho}{\rho^{1+2s}} \, d\rho
\]
\[
\lesssim \|y_0\|^{-\mu} \|z_0\|^{-\sigma} (\|y_0\| + \|z_0\|)^{-2s}.
\]

(4) This follows from the product rule
\[
(-\Delta)^s \left( (y)^{-\mu} \langle z \rangle^{-\sigma} \right) = (y)^{-\mu} (-\Delta)^s \langle z \rangle^{-\sigma} + \langle z \rangle^{-\sigma} (-\Delta)^s (y)^{-\mu} - (\langle y \rangle^{-\mu} \langle z \rangle^{-\sigma})
\]
\[
= (y)^{-\mu} \langle z \rangle^{-\sigma} \left( O((y)^{-2s}) + O((z)^{-2s}) + o(1) \right).
\]
(5) The $s$-inner product is computed as follows. We may assume that $1 \leq |z_0| \leq \frac{8}{s}$. When $|y_0| \geq 3R$,

$$||(-\Delta)^s, \eta_R\phi(y_0, z_0)|| \leq C \int_{\mathbb{R}^n} \frac{|-\eta_R(y)|}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \int_{\mathbb{R}^n} \int_{|y| \leq 2R} \frac{|z|^{-\sigma}}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C R^{-1} \int_{\mathbb{R}} \left( |y_0|^2 + |z_0 - z|^2 \right)^{\frac{n+2s}{2}} \, dz$$

$$\leq C R^{-1} \left( |z_0|^{-\sigma} |y_0|^{-(n+1+2s)} + (1 + |z_0|^{-\sigma})(y_0, z_0)^{-2s} \right)$$

$$\leq C \left( |z_0|^{-1} + |z_0|^{-\sigma} \right) |y_0|^{-2s}.$$

When $|y_0| \leq \frac{8}{s}$,

$$||(-\Delta)^s, \eta_R\phi(y_0, z_0)|| \leq C \int_{\mathbb{R}^n} \frac{(1 - \eta_R(y))}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \int_{\mathbb{R}^n} \int_{|y| > R} \frac{\langle z \rangle^{-\sigma}}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \int_{\mathbb{R}^n} \int_{|y| > R} \frac{\langle z \rangle^{-\sigma}}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \int_{\mathbb{R}^n} \int_{|y| > R} \frac{\langle z \rangle^{-\sigma}}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \int_{\mathbb{R}^n} \int_{|y| > R} \frac{\langle z \rangle^{-\sigma}}{(y_0, z_0) - (y, z)}^{n+2s} \, dy \, dz$$

$$\leq C \left( R^{-1-2s}(1 + R^{1-\sigma}) + R^{-\sigma} R^{-2s} \right)$$

$$\leq C \left( R^{-1-2s} + R^{-\sigma-2s} \right).$$

When $\frac{8}{s} \leq |y_0| \leq 3R$, we have

$$\partial_{y_i y_j} \eta_R = \frac{1}{R^2} \eta'' \left( \frac{y_i y_j}{R^2} \right) + \frac{1}{R^2} \eta' \left( \frac{y_i y_j}{R^2} \right) \left( \delta_{ij} - \frac{y_i y_j}{|y|^2} \right)$$
which implies that \( \| D^2 \eta_k \|_{L^\infty([y_0, y])} \leq C R^{-2} \) for \( |y_0 - y| \leq \frac{\mu}{2} \), where \([y_0, y]\) denotes the line segment joining \(y_0\) and \(y\). Thus
\[
\| (-\Delta)^{s} \eta_k \|_{H^{s}([y_0, y])} \leq C R^{-2} \frac{\|g\|_{\mu, \sigma}}{|y_0 - y|^s}.
\]

4.3. Existence. In order to solve the linearized equation
\[
(-\Delta)^s \phi + f'(w) \phi = g \quad \text{for } (y, z) \in \mathbb{R}^n,
\]
we consider the equivalent problem in the Caffarelli–Silvestre extension [24]
\[
\begin{cases}
-\nabla \cdot (t^s \nabla \phi) = 0 & \text{for } (t, y, z) \in \mathbb{R}_+^{n+1} \\
\frac{\partial \phi}{\partial n} + f'(w) \phi = g & \text{for } (y, z) \in \partial \mathbb{R}_+^{n+1}.
\end{cases}
\]
(4.13)

We will prove the following

\[\text{Proposition 4.8.} \quad \text{Let } \mu, \sigma > 0 \text{ be small. For any } g \text{ with } \|g\|_{\mu, \sigma} < +\infty \text{ satisfying}
\]
\[
\int_{\mathbb{R}} g(y, z) w'(z) \, dz = 0,
\]
(4.14)

there exists a unique solution \( \phi \in H^1(\mathbb{R}_+^{n+1}, t^s) \) of (4.13) satisfying
\[
\iint_{\mathbb{R}^2_n} t^s \phi(t, y, z) w_z(t, z) \, dt \, dz = 0 \quad \text{for all } y \in \mathbb{R}^{n-1},
\]
(4.15)

such that the trace \( \phi(0, y, z) \) satisfies \( \|\phi\|_{\mu, \sigma} < +\infty \). Moreover,
\[
\|\phi\|_{\mu, \sigma} \leq C \|g\|_{\mu, \sigma}.
\]
(4.16)

Let us recall the corresponding known result [52] in one dimension.
Proof. This is Proposition 4.1 in [52]. In their notations, take \( m = 1, \xi_1 = 0 \) and \( \mu = \sigma \). \( \square \)

Proof of Proposition 4.8. (1) We first assume that \( g \in C_c^\infty(\mathbb{R}^n) \). Taking Fourier transform in \( y \), we solve for each \( \xi \in \mathbb{R}^{n-1} \) a solution \( \hat{\phi}(t, \xi, z) \) to

\[
\begin{cases}
- \nabla \cdot (t^a \nabla \hat{\phi}) + |\xi|^2 t^a \hat{\phi} = 0 & \text{for } (t, z) \in \mathbb{R}^2_+,

\quad t^a \frac{\partial \hat{\phi}}{\partial \nu} + f'(w) \hat{\phi} = \hat{g} & \text{for } z \in \partial \mathbb{R}^2_+,
\end{cases}
\]

with orthogonality condition

\[
\int_{\mathbb{R}^2_+} t^a \hat{\phi}(t, \xi, z) w_z(t, z) \, dt \, dz = 0 \quad \text{for all } \xi \in \mathbb{R}^{n-1}
\]

corresponding to (4.15). One can then obtain a solution for \( \xi = 0 \) by Lemma 4.9 and for \( \xi \neq 0 \) by Lemma 4.2. From the embedding \( H^1(\mathbb{R}^2_+, t^a) \hookrightarrow H^1(\mathbb{R}) \) [18], we have the estimate

\[
\| \hat{\phi}(\cdot, \xi, \cdot) \|_{H^1(\mathbb{R}^2_+, t^a)} \leq C(\xi) \| \hat{g}(\xi, \cdot) \|_{L^2(\mathbb{R})}.
\]

We claim that the constant can be taken independent of \( \xi \), i.e.

\[
\| \hat{\phi}(\cdot, \xi, \cdot) \|_{H^1(\mathbb{R}^2_+, t^a)} \leq C \| \hat{g}(\xi, \cdot) \|_{L^2(\mathbb{R})}. \tag{4.17}
\]

If this were not true, there would exist sequences \( \xi_m \to 0 \) (the case \( |\xi_m| \to +\infty \) is similar), \( \hat{\phi}_m \) and \( \hat{g}_m \) such that

\[
\begin{align*}
\| \hat{\phi}_m(\cdot, \xi_m, \cdot) \|_{H^1(\mathbb{R}^2_+, t^a)} & = 1, \quad \| \hat{g}_m(\xi_m, \cdot) \|_{L^2(\mathbb{R})} = 0, \tag{4.18}

\int_{\mathbb{R}^2_+} t^a \hat{\phi}_m(t, \xi_m, z) w_z(t, z) \, dt \, dz & = 0.
\end{align*}
\]

Elliptic regularity implies that a subsequence of \( \hat{\phi}_m(t, \xi_m, z) \) converges locally uniformly in \( \mathbb{R}^2_+ \) to some \( \hat{\phi}_0(t, z) \), which solves weakly

\[
\begin{cases}
- \nabla \cdot (t^a \nabla \hat{\phi}_0) = 0 & \text{for } (t, z) \in \mathbb{R}^2_+,

\quad t^a \frac{\partial \hat{\phi}_0}{\partial \nu} + f'(w) \hat{\phi}_0 = 0 & \text{for } z \in \partial \mathbb{R}^2_+.
\end{cases}
\]

and

\[
\int_{\mathbb{R}^2_+} t^a \hat{\phi}_0(t, z) w_z(t, z) \, dt \, dz = 0 \quad \text{for all } \xi \in \mathbb{R}^{n-1}.
\]

By Lemma 4.4, we conclude that \( \hat{\phi}_0 = 0 \), contradicting (4.18). This proves (4.17).

Integrating over \( \xi \in \mathbb{R}^{n-1} \) and using Plancherel’s theorem, we obtain a solution \( \phi \) satisfying

\[
\| \phi \|_{H^1(\mathbb{R}^{n+1}_+, t^a)} \leq C \| g \|_{L^2(\mathbb{R}^n)}.
\]

Higher regularity yields, in particular, \( \phi \in L^\infty(\mathbb{R}^n) \). Then (4.16) follows from Lemma 4.6.
(2) In the general case, we solve (4.13) with \( g \) replaced by \( g_m \in C_\infty^{R_n} \) which converges uniformly to \( g \). Then the solution \( \phi_m \) is controlled by

\[
\| \phi_m \|_{\mu, \sigma} \leq C \| g_m \|_{\mu, \sigma} \leq C \| g \|_{\mu, \sigma}.
\]

By passing to a subsequence, \( \phi_m \) converges to some \( \phi \) uniformly on compact subsets of \( R^n \), which also satisfies (4.16).

(3) The uniqueness follows from the non-degeneracy of \( w' \) and the orthogonality condition (4.15).

\[ \square \]

4.4. The positive operator. We conclude this section by stating a standard estimate for the operator \((-\Delta)^s + 2\).

**Lemma 4.10.** Consider the equation

\[ (-\Delta)^s u + 2u = g \quad \text{in} \quad R^n. \]

and \( |g(x)| \leq C \langle x' \rangle^{-\theta} \) for all \( x \in R^n \) and \( g(x) = 0 \) for \( x \in M_{\epsilon, R} \), a tubular neighborhood of \( M_\epsilon \) of width \( R \). Then the unique solution \( u = (-\Delta)^s + 2)^{-1} g \) satisfies the decay estimate

\[ |u(x)| \leq C \langle x' \rangle^{-\theta} (\text{dist} (x, M_{\epsilon, R}))^{-2s} \]

**Proof.** The decay in \( x' \) follows from a maximum principle; that in the interface is seen from the Green’s function for \((-\Delta)^s + 2\) which has a decay \(|x|^{-(n+2s)}\) at infinity [35]. \[ \square \]

5. Fractional gluing system

5.1. Preliminary estimates. We have the following

**Lemma 5.1** (Some non-local estimates). For \( \phi_j \in X_j, j \in J \), the following holds true.

1. (commutator at the near interface)

\[
\left| \langle (-\Delta_{(y,z)})^s, \bar{\eta_+} \zeta_i \phi_i(x) \rangle \right| \leq C \| \phi_i \|_{i, \mu, \sigma} \langle y_i \rangle^{-\theta} R^n (R + |(y, z)|)^{-n-2s}.
\]

As a result,

\[
\sum_{i \in I} \left| \langle (-\Delta_{(y,z)})^s, \zeta_i \phi_i(x) \rangle \right| \leq C R^{-\theta} \sup_{i \in I} \| \phi_i \|_{i, \mu, \sigma} \left( R + \text{dist} \left( x, \text{supp} \sum_{i \in I} \zeta_i \right) \right)^{-2s}.
\]

2. (commutator at the end)

\[
\left| \langle (-\Delta_{(y,z)})^s, \bar{\eta_+} \zeta_i \phi_i(y, z) \rangle \right| \leq C \| \phi_i \|_{+, \mu, \sigma} R_2^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-1-2s},
\]

and similarly for \( \phi_- \);

3. (linearization at \( u^{*} \))

\[
\sum_{j \in J} \left| \zeta_j (f'(w_j) - f'(u^{*})) \phi_j \right|
\leq C \sup_{j \in J} \| \phi_j \|_{j, \mu, \sigma} \left( \sum_{i \in I} \zeta_i R^{\mu+\sigma} \langle y_i \rangle^{-\theta - \frac{d+1}{d+2}} + (\zeta_+ + \zeta_-) R_2^{-\theta} \langle y \rangle^{-\mu} \right).
\]

4. (change of coordinates around the near interface)

\[
\sum_{i \in I} \left| \langle (-\Delta_{z})^s - (-\Delta_{(y,z)})^s (\zeta_i \phi_i)(x) \rangle \right|
\leq C R^{\alpha+1+\mu+\sigma} \langle \bar{\phi}_i \rangle_{i, \mu, \sigma} \left( \sum_{i \in I} \zeta_i \langle y_i \rangle^{-\theta} + \varepsilon \left( \text{dist} \left( x, \text{supp} \sum_{i \in I} \zeta_i \right) \right)^{-2s} \right).
\]
Proof of Lemma 5.1.

(1) (a) Since $\phi_i \in X_i$, we have for $|(y_0, z_0)| \geq 3R$,

$$
\left| \left( -\Delta_{(y,z)} \right)^s \tilde{\phi}_i (y_0, z_0) \right| \leq C \left| \tilde{\phi}_i \right|_{i,\mu,\sigma} \int_{|y-z| \leq 2R} \left| \frac{-\tilde{\eta}(y)\zeta(z)}{(y-y_0)^2 + |z-z_0|^2} \right| R^{\mu+\sigma} \langle y \rangle^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-\sigma} \, dydz 
$$

and similarly for $\phi_-$. In particular, all these terms are less than $S(u^*)$.

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and similarly for $\phi_-$. In particular, all these terms are less than $S(u^*)$.
(a) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0) = 0$ with $|y_0| \geq 2R_2$ and $|z_0| \geq 3R$, 
\[
\left|\left(\Delta_{(y,z)}\right)^s, \bar{\eta}_+ \bar{\zeta}\phi_+(y_0, z_0)\right|
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \int_{|y| > R_2} \int_{|z| < 2R} \frac{\langle y \rangle^{-\mu} (z)^{-\sigma}}{|(y_0, z_0) - (y, z)|^{n+2s}} \, dy \, dz
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) \int_{|y| > R_2} \frac{\langle y \rangle^{-\mu}}{|(y_0 - y)^2 + |z_0|^2|^{\frac{n+2s}{2}}} \, dy
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) \left(\int_{R_2 < |y| \leq \frac{|y_0|}{2}} \frac{\langle y \rangle^{-\mu}}{|(y_0 - y)^2 + |z_0|^2|^{\frac{n+2s}{2}}} \, dy + \int_{|y| > \frac{|y_0|}{2}} \frac{\langle y \rangle^{-\mu}}{|(y_0 - y)^2 + |z_0|^2|^{\frac{n+2s}{2}}} \, dy\right)
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) \frac{|y_0|^{n-1-\mu}}{|(y_0, z_0)|^{n+2s}} \frac{\langle y_0 \rangle^{-\mu}}{|z_0|^{1+2s}}
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) \langle y_0 \rangle^{-\mu} \langle z_0 \rangle^{-1-2s}.
\]

(b) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0) = 0$ with $|y_0| \leq 2R_2$ and $|z_0| \geq 3R$, 
\[
\left|\left(\Delta_{(y,z)}\right)^s, \bar{\eta}_+ \bar{\zeta}\phi_+(y_0, z_0)\right|
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) \int_{|y| > R_2} \frac{\langle y \rangle^{-\mu}}{|(y_0 - y)^2 + |z_0|^2|^{\frac{n+2s}{2}}} \, dy
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} (1 + R^{1-\sigma}) |z_0|^{-1-2s}.
\]

(c) When $\bar{\eta}_+(y_0)\bar{\zeta}(z_0) = 0$ with $|y_0| \leq R_2 - 2R$, 
\[
\left|\left(\Delta_{(y,z)}\right)^s, \bar{\eta}_+ \bar{\zeta}\phi_+(y_0, z_0)\right|
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \int_{|y| > R_2} \int_{|z| < 2R} \frac{\langle y \rangle^{-\mu} (z)^{-\sigma}}{|(y_0, z_0) - (y, z)|^{n+2s}} \, dy \, dz
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \int_{|z| < 2R} \langle z \rangle^{-\sigma} \min\left\{\frac{1}{|z_0 - z|^{1+2s}}, \frac{1}{R^{1+2s}}\right\} \, dz
\leq C\left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \langle z_0 \rangle^{-1-2s}.
\]

(d) When $0 \leq \bar{\eta}_+(y_0)\bar{\zeta}(z_0) \leq 1$ with $|y_0| \geq R_2 - 2R$ and $0 \leq |z_0| \leq 3R$, 
\[
\left|\left(\Delta_{(y,z)}\right)^s, \bar{\eta}_+ \bar{\zeta}\phi_+(y_0, z_0)\right|
\leq C \int_{|y| < R_2} \int_{|z| < 2R} \frac{R^{-2} \langle y_0 - y \rangle^2 + |z_0 - z|^2}{|y_0 - y|^{n+2s}} \, \left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \langle y \rangle^{-\mu} (z)^{-\sigma} \, dy \, dz
\leq CR^{-2} \left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} \langle y_0 \rangle^{-\mu} \int_{|y| < R} \frac{(y)^{-\mu}}{|y_0 - y|^{n-1+2s}} \, dy
\leq C \left\|\bar{\phi}_+\right\|_{+, \mu, \sigma} R_2^{-\theta} |y_0|^{-\mu}.
\]
(3) For the localized inner terms,
\[
\sum_{i \in \mathcal{I}} |\zeta_i(f'(w) - f'(u^*))| \phi_i \leq C \|\phi_i\|_{i,\mu,\sigma} \zeta_i F_{\varepsilon}^{2s} R^{\mu+\sigma} \langle y_i \rangle^{-\theta} \\
\leq C \|\phi_i\|_{i,\mu,\sigma} \sum_{i \in \mathcal{I}} \zeta_i R^{\mu+\sigma} \langle y_i \rangle^{-\theta - \frac{\mu}{2s+1}}.
\]

The two terms at the ends are controlled by
\[
|\zeta_\pm(f'(w) - f'(u^*))| \phi_\pm \leq C \|\phi_\pm\|_{\pm,\mu,\sigma} \zeta_\pm R^\sigma R_2^{-(\theta - \mu)} \langle y \rangle^{-\mu}.
\]

By summing up we obtain the desired estimate.

(4) By using Corollary 3.4 and (2.8), we have in the Fermi coordinates,
\[
\left|\left(\langle (\Delta_{\varepsilon})^s - (\Delta_{(y,z)})^s \rangle \zeta_i \phi_i \right)(x)\right|
\leq CR^2 \left|\left(\langle \tilde{\eta}(y)\tilde{\zeta}(z) \langle (\Delta_{\varepsilon})^s \rangle \phi_i \right)(y,z)\right| + C\varepsilon^2 \left|\left(\tilde{\eta}\tilde{\zeta} \phi_i \right)(y,z)\right|
\leq CR^2 \left(\tilde{\eta}(y)\tilde{\zeta}(z) R^{\mu+\sigma} \left\|\phi_i\right\|_{i,\mu,\sigma} \langle y_i \rangle^{-\theta} \langle y \rangle^{-\mu} \langle y \rangle^{-\sigma} + \left\|\phi_i\right\|_{i,\mu,\sigma} \langle y_i \rangle^{-\theta} R^\sigma (R + |(y,z)|)^{-n-2s}\right).
\]
\[
\leq CR^{n+1+\mu+\sigma} \varepsilon \left\|\phi_i\right\|_{i,\mu,\sigma} \langle y_i \rangle^{-\theta} \left(\tilde{\eta}(y)\tilde{\zeta}(z) + (R + |(y,z)|)^{-n-2s}\right).
\]

Going back to the x-coordinates and summing up over \(i \in \mathcal{I}\), we have
\[
\sum_{i \in \mathcal{I}} \left|\left(\langle (\Delta_{\varepsilon})^s - (\Delta_{(y,z)})^s \rangle \zeta_i \phi_i \right)(x)\right|
\leq CR^{n+1+\mu+\sigma} \varepsilon \left\|\phi_i\right\|_{i,\mu,\sigma} \left(\sum_{i \in \mathcal{I}} \zeta_i \langle y_i \rangle^{-\theta} + \varepsilon \theta \left(\text{dist}(x, \text{supp} \sum_{i \in \mathcal{I}} \zeta_i)\right)^{-2s}\right).
\]

(5) Similarly, using Corollary 3.5 and (2.8),
\[
\left|\left(\langle (\Delta_{\varepsilon})^s - (\Delta_{(y,z)})^s \rangle \zeta_+ \phi_+ \right)(x)\right|
\leq CR^{-\frac{n+1}{2s+1}} \left|\left(\langle \tilde{\eta}(y)\tilde{\zeta}(z) \langle (\Delta_{\varepsilon})^s \rangle \phi_+ \right)(y,z)\right| + CR^{-\frac{n+1}{2s+1}} \left|\left(\tilde{\eta}\tilde{\zeta} \phi_+ \right)(y,z)\right|
\leq CR^{-\frac{n+1}{2s+1}} \left(\tilde{\eta}(y)\tilde{\zeta}(z) \left\|\phi_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y \rangle^{-\mu} + \left\|\phi_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-1-2s}\right)
\leq CR^{-\frac{n+1}{2s+1}} \left\|\phi_+\right\|_{+,\mu,\sigma} R_2^{-\theta} \langle y \rangle^{-\mu} \langle z \rangle^{-1-2s}.
\]

\[
\square
\]

5.2. The outer problem: Proof of Proposition 2.2. We give a proof of Proposition 2.2 and solve \(\phi_o\) in terms of \((\phi_j)_{j \in \mathcal{J}}\).

Proof of Proposition 2.2. We solve it by a fixed point argument. By Corollary 3.3 and Lemma 5.1, the right hand side \(g_o = g_o(\phi_o)\) of (2.4) satisfies \(g_o = 0\) in \(M_{\varepsilon,R}\) and
\[
\|g_o\|_{\theta} \leq C\varepsilon^\theta + \|\tilde{\eta}_o N(\varphi)\|_{\theta} + \|\tilde{\eta}_o (2 - f'(u^*))\phi_o\|_{\theta}
\leq C\varepsilon^\theta + \|\phi_o\|_{L^\infty(\mathbb{R}^n)} \|\phi_o\|_{\theta} + CR^{-2s} \|\phi_o\|_{\theta},
\]
so that by Lemma 4.10,
\[
\|((\Delta)^s + 2)^{-1} g_o\|_{\theta} \leq \left(C + \tilde{C}^2\varepsilon^\theta + \tilde{C}R^{-2s}\right) \varepsilon^\theta \leq \tilde{C}\varepsilon^\theta.
\]
Next we check that for \( \phi_o, \psi_o \in X_o \), \( g_o(\phi_o) - g_o(\psi_o) = 0 \) in \( M_{\varepsilon,R} \) as well as

\[
\|g_o(\phi_o) - g_o(\psi_o)\|_\theta \leq \left\| N\left( \phi_o + \sum_{j \in J} \zeta_j \phi_j \right) - N\left( \psi_o + \sum_{j \in J} \zeta_j \phi_j \right) \right\| + \|\eta_o(2 - f'(u^*)) (\phi_o - \psi_o)\|_\theta \\
\leq C(\varepsilon^6 + R^{-2s}) \|\phi_o - \psi_o\|_\theta.
\]

Hence

\[
\left\| (-(\Delta)^s + 2)^{-1} (g_o(\phi_o) - g_o(\psi_o)) \right\|_\theta \leq C(\varepsilon^6 + R^{-2s}) \|\phi_o - \psi_o\|_\theta.
\]

By contraction mapping principle, there is a unique solution \( \phi_o = \Phi_o((\phi_j)_{j \in J}) \). The Lipschitz continuity of \( \Phi_o \) with respect to \( (\phi_j)_{j \in J} \) can be obtained by taking a difference. \( \square \)

5.3. The inner problem: Proof of Proposition 2.3. Here we solve the inner problem for \( (\phi_j)_{j \in J} \), with the solution of the outer problem \( \phi_o = \Phi_o((\phi_j)_{j \in J}) \) plugged in.

**Proof of Proposition 2.3.** Let us denote the right hand side of (2.8) by \( g_j \). We notice that the norms can be estimated without the projection (up to a constant). Indeed, for any function \( \tilde{h} \) with \( \|\tilde{h}\|_{\mu,\sigma} < +\infty \),

\[
\left\| \left( \int_{-2R}^{2R} \tilde{\zeta}(t) \tilde{h}(y, t) w'(t) \, dt \right) w'(z) \right\|_{\mu,\sigma} \leq C \|\tilde{h}\|_{\mu,\sigma} \sup_{z \in \mathbb{R}} \langle z \rangle^{-1-2s+\sigma} \\
\leq C \|\tilde{h}\|_{\mu,\sigma}.
\]

Then, keeping in mind that a barred function denotes the corresponding one in Fermi coordinates, we have

\[
\|\tilde{h}_i S(u^*)\|_{i,\mu,\sigma} \leq \langle y_i \rangle^\theta \sup_{|y_i|,|z| \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \cdot \langle y_i \rangle^{-\frac{\mu}{\mu+\sigma}} \langle z \rangle^{-(2s-1)} \\
\leq CR \langle y_i \rangle^{-(\frac{\mu}{\mu+\sigma} - \theta)} \\
\leq C \delta_i,
\]

\[
\|\tilde{h}_i (2 - f'(u^*)) \Phi_o((\phi_j)_{j \in J})\|_{i,\mu,\sigma} \leq \|\tilde{h}_i \Phi_o((\phi_j)_{j \in J})\|_{i,\mu,\sigma} \\
\leq \langle y_i \rangle^\theta \sup_{|y_i|,|z| \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \cdot \|\Phi_o((\phi_j)_{j \in J})(y, z)\| \\
\leq \langle y_i \rangle^\theta \sup_{|y_i|,|z| \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \cdot \langle y_i \rangle^{-\theta} \|\Phi_o((\phi_j)_{j \in J})\|_\theta \\
\leq CR^{\mu+\sigma} \varepsilon^\theta \sup_{j \in J} \|\phi_j\|_{j,\mu,\sigma} \\
\leq CR^{\mu+\sigma} \varepsilon^\theta C \delta_i.
\]
and

\[
\left\| \tilde{\eta}_i N \left( \Phi_o((\phi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \phi_j \right) \right\|_{i, \mu, \sigma} \\
\leq C (y_i) \theta \sup_{|y|, |z| \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \left| \Phi_o((\phi_j)_{j \in J})(y, z) + \sum_{j \in J} \tilde{\eta}_j \zeta_j \phi_j(y, z) \right|^2 \\
\leq CR^{\mu+\sigma} (y_i)^\theta \sup_{|y|, |z| \leq 2R} \langle y \rangle^{-2\theta} \left( \sup_{j \in J} \| \phi_j \|_{j, \mu, \sigma} \right)^2 + \sum_{j \in J} \langle y_j \rangle^{-2\theta} \left( \sup_{j \in J} \| \phi_j \|_{j, \mu, \sigma} \right)^2 \\
\leq CR^{\mu+\sigma} (y_i)^{-\theta} C\delta \sup_{j \in J} \| \phi_j \|_{j, \mu, \sigma} \\
\leq CR^{\mu+\sigma} \varepsilon^\theta C\delta^2.
\]

Using Lemma 5.1 and estimating as in the proof of Proposition 2.2, we have for all \(i \in I\),

\[
\left\| g_i \right\|_{i, \mu, \sigma} \leq C\delta(1 + R^{\mu+\sigma} \varepsilon^\theta C + R^{\mu+\sigma} \varepsilon^\theta C\delta + o(1)).
\]

Now we estimate the functions \(\phi_{\pm}\) at the ends. We have similarly

\[
\left\| \tilde{\eta}_+ S(u^*) \right\|_{+ \mu, \sigma} \leq CR^\theta \sup_{y \geq R_2, z \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \langle y \rangle^{\frac{1}{2(1-\mu)}} \langle z \rangle^{-(2s-1)} \\
\leq CR^\theta \left( \frac{1}{\mu - \theta} \right) \\
\leq C\delta \quad \text{for } R_2 \text{ chosen large enough,}
\]

\[
\left\| \tilde{\eta}_+(2 - f'(u^*)) \Phi_o((\phi_j)_{j \in J}) \right\|_{+ \mu, \sigma} \leq CR^\theta \sup_{y \geq R_2, z \leq 2R} \langle y \rangle^\mu \langle z \rangle^\sigma \left| \Phi_o((\phi_j)_{j \in J})(u, z) \right| \\
\leq CR^\theta R^\theta \sup_{y \geq R_2, z \leq 2R} \langle y \rangle^\mu \cdot \langle y \rangle^{-\theta} \varepsilon^\theta \sup_{j \in J} \| \phi_j \|_{j, \mu, \sigma} \\
\leq CR^\theta \varepsilon^\theta C\delta \quad \text{(since } \mu \leq \theta) \\
\leq C\tilde{C} \varepsilon^\frac{1}{2} \delta \quad \text{for } \mu \text{ chosen small enough,}
\]
and

\[
\| \eta N \left( \Phi_o((\phi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \phi_j \right) \|_{+,\mu,\sigma} \leq C R_2^\theta \sup_{y \geq R_2, \zeta \leq 2R} \langle y \rangle^\mu \left( \frac{\langle z \rangle^\sigma}{\Phi_o((\phi_j)_{j \in J})(y, z) + \sum_{j \in J, \sup \eta_j \cap \sup \zeta_j \neq \emptyset} \tilde{\eta}_j \zeta_j \phi_j(y, z)} \right)^2.
\]

\[
\leq C R^\sigma \sup_{y \geq R_2, \zeta \leq 2R} \langle y \rangle^\mu \left( \langle z \rangle^\sigma \Phi_o((\phi_j)_{j \in J})(y, z) + \sum_{j \in J, \sup \eta_j \cap \sup \zeta_j \neq \emptyset} \tilde{\eta}_j \zeta_j \phi_j(y, z) \right)^2
\]

\[
\leq C R^\sigma \left( R_2^{-\theta} + \sum_{j \in J, \sup \eta_j \cap \sup \zeta_j \neq \emptyset} \langle y_j \rangle^{-\theta} \right) \left( \sup_{j \in J} \| \phi_j \|_{j,\mu,\sigma} \right)^2
\]

\[
\leq C R^\sigma \varepsilon \tilde{C} \delta \left( \sup_{j \in J} \| \phi_j \|_{j,\mu,\sigma} \right)
\]

\[
\leq C R^\sigma \varepsilon \tilde{C}^2 \delta^2.
\]

Putting together these estimates together with the non-local terms yields, using the linear theory (Proposition 4.8 and Lemma 4.6),

\[
\sup_{j \in J} \| L^{-1} g_j \|_{j,\mu,\sigma} \leq C \sup_{j \in J} \| g_j \|_{j,\mu,\sigma}
\]

\[
\leq C \delta (1 + o(1))
\]

\[
\leq \tilde{C} \delta.
\]

It suffices to check the Lipschitz continuity with respect to \( \phi_j \in X_j \). Suppose \( \phi_j, \psi_j \in X_j \). Using (2.5), we have for instance

\[
\langle y_i \rangle^\theta \sup_{|y|, |z| \leq 2R} \langle y \rangle^\mu \left( \frac{\langle z \rangle^\sigma}{\Phi_o((\phi_j)_{j \in J})(y, z) - \Phi_o((\psi_j)_{j \in J})(y, z)} \right)
\]

\[
+ N \left( \Phi_o((\phi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \phi_j \right) - N \left( \Phi_o((\psi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \psi_j \right)
\]

\[
\leq C R^{\mu+\sigma} \sup_{|y|, |z| \leq 2R} \left( 1 + \delta \right) \left| \Phi_o((\phi_j)_{j \in J})(y, z) - \Phi_o((\psi_j)_{j \in J})(y, z) \right|_\theta
\]

\[
+ \delta \langle y \rangle^\theta \sum_{j \in J, \sup \eta_j \cap \sup \zeta_j \neq \emptyset} \tilde{\eta}_j \zeta_j \phi_j(y, z) - \psi_j(y, z)
\]

\[
\leq C R^{\mu+\sigma} \delta \sup_{j \in J} \| \phi_j - \psi_j \|_{j,\mu,\sigma}.
\]
and
\[
R'^{R_2}_2 \sup_{|y| \geq R_2, |z| \leq 2R} (y) R [\Phi_o((\phi_j)_{j \in J})(y, z)]
\]
\[
+ N \left( \Phi_o((\psi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \phi_j \right) - N \left( \Phi_o((\psi_j)_{j \in J}) + \sum_{j \in J} \zeta_j \psi_j \right)
\]
\[
\leq CR^\delta R'^{R_2}_2 \sup_{|y| \geq R_2, |z| \leq 2R} \left( (1 + \delta) (y) + \delta (y) \sum_{j \in J} \eta_j \zeta_j \phi_j - \psi_j \right)
\]
\[
\leq CR^\delta R'^{R_2}_2 \sup_{j \in J} \|\phi_j - \psi_j\|_{j, \mu, \sigma}.
\]
Therefore
\[
\sup_{j \in J} \left\| L^{-1} g_j((\phi_j)_{j \in J}) - L^{-1} g_j((\psi_j)_{j \in J}) \right\|_{j, \mu, \sigma} \leq o(1) \sup_{j \in J} \|\phi_j - \psi_j\|_{j, \mu, \sigma}
\]
and \((\phi_k)_{k \in J} \rightarrow L^{-1} g_j((\phi_k)_{k \in J})\) defines a contraction mapping on the product space endowed with the supremum norm for suitably chosen parameters \(R, R_2\) large and \(\varepsilon, \mu\) small. This concludes the proof.

\[\square\]

6. The reduced equation

6.1. Form of the equation: Proof of Proposition 2.4.

**Proof of Proposition 2.4.** Recalling Proposition 2.1, in the near and intermediate regions \(r \in \left[ \frac{1}{\varepsilon}, \frac{4R}{\varepsilon} \right]\),
\[
\Pi S(u^*)(r) = \tilde{C} H_{M_{\varepsilon}}(r) + O(\varepsilon^{2s}),
\]
where
\[
\tilde{C} = \int_{-2R}^{2R} c_{H}(z) \zeta(z) w'(z) \ dz.
\]
For the far region \(r \geq \frac{4R}{\varepsilon}\), let us assume that \(x_n > 0\) to fix the idea. Denote by \(\Pi \pm\) the projections onto the kernels \(w_{\pm}'(z)\) of the upper and lower leaves respectively, where \(w_{\pm}(z) = w(z_{\pm})\). Then \(z_+ = -2F_{\varepsilon}(r)(1 + o(1)) - z_+\) and so from the asymptotic behavior \(w(z) \sim_{z \to +\infty} 1 - \frac{2F_{\varepsilon}}{z}\), we have
\[
\Pi_3(w(z_+) + w(z_-))(1 + w(z_+))(1 + w(z_-))(r)
\]
\[
= \int_{-2R}^{2R} 3(w(z) + w(-2F_{\varepsilon}(r)(1 + o(1)) - z))(1 + w(z))(1 + w(-2F_{\varepsilon}(r)(1 + o(1)) - z))\zeta(z)w'(z) \ dz
\]
\[
= -\frac{\tilde{C}_\pm}{F_{\varepsilon}^2(r)} (1 + o(1)),
\]
where
\[
\tilde{C}_\pm = \int_{-2R}^{2R} 3c_{w}(1 - w(z)^2)\zeta(z)w'(z) \ dz.
\]
Similarly this is also true for the projection onto \( w'_-(z) \) with the same coefficient \( C_±(r) \),

\[
\Pi_- 3(w(z_+) + w(z_-))(1 + w(z_+))(1 + w(z_-))(r) = -\tilde{C}_±(r) F_ε^{2s}(r)(1 + o(1)).
\]

The other projections are estimated as follows.

\[
\Pi_+ c_H(z_+ H_{M_+}(y_+) = \int_{-2R}^{2R} c_H(z)\zeta(z)w'(z) dz \cdot H_{M_+}(y_+) = \bar{C}_{H_{M_+}}(y_+)
\]

\[
\Pi_+ c_H(z_- H_{M_+}(y_-)(r) = \int_{-2R}^{2R} c_H(2F_ε(r)(1 + o(1)) - z)\zeta(z)w'(z) dz \cdot H_{M_+}(y_-)
\]

\[
\Pi_+ c_H(z_- H_{M_+}(y_-)) = O\left(F_ε^{-(2s-1)} , F_ε^{-2s}\right)
\]

\[
\Pi_+ c_H(z_- H_{M_+}(y_-)) = O\left(F_ε^{-(4s-1)}\right)
\]

We conclude that for \( r ≥ \frac{4R}{ε} \),

\[
\Pi_± S(u_+)(r) = \bar{C}_{H_{M_±}}(y) - \tilde{C}_±(r) F_ε^{2s}(r)(1 + o(1)).
\]

By a scaling \( F_ε(r) = ε^{-1}F(εr) \), it suffices to solve

\[
\begin{cases}
    \bar{C}_{H[F_ε]}(r) = o(ε^{2s}) & \text{for } \frac{1}{ε} ≤ r ≤ \frac{4R}{ε},
    \\
    \bar{C}_{H[F_ε]}(r) = \tilde{C}_± F_ε^{2s}(r)(1 + o(1)) & \text{for } r ≥ \frac{4R}{ε}.
\end{cases}
\]

For large enough \( r \) one may approximate the mean curvature by \( \Delta F = \frac{1}{r}(rF')' \). Hence, we arrive at

\[
\begin{cases}
    \frac{1}{r} \left( \frac{rF'(r)}{\sqrt{1 + F'(r)^2}} \right)' = O(ε^{2s-1}) & \text{for } 1 ≤ r ≤ 4R,
    \\
    \frac{1}{r} \left( \frac{rF'(r)}{\sqrt{1 + F'(r)^2}} \right)' = \frac{C_0 ε^{2s-1}}{F_ε^{2s}(r)(1 + o(1))} & \text{for } r ≥ 4R.
\end{cases}
\]

Then the inverse \( G \) of \( F \) is introduced to deal with the singularity at \( r = 1 \) in the usual coordinates. Finally, the Lipschitz dependence of the error follows directly from the previously involved computations.
6.2. Initial approximation. In this section we study an ODE which is similar to the one in \cite{36}. The reduced equation for \( F_\varepsilon : [\varepsilon^{-1}, +\infty) \to [0, +\infty) \) can be approximated by

\[
F''_\varepsilon(r) + \frac{F'_\varepsilon(r)}{r} = \frac{1}{F^2_\varepsilon(r)}, \quad \text{for all } r \text{ large.}
\]

Under the scaling \( F_\varepsilon(r) = \varepsilon^{-1} F(\varepsilon r) \), the equation for \( F : [1, +\infty) \to [0, +\infty) \) is

\[
F''(r) + \frac{F'(r)}{r} = \frac{\varepsilon^{2s-1}}{F^{2s}(r)}, \quad \text{for all } r \text{ large.}
\]

For \( r \) small, we approximate \( F \) by the catenoid. More precisely, let \( f_C(r) = \log(r + \sqrt{r^2 - 1}) \), \( r = |x'| \geq 1 \), \( r_\varepsilon = \left( \frac{\log \varepsilon}{\varepsilon} \right) \frac{e^{\varepsilon^2}}{e^{\varepsilon^2} - 1} \), and consider the Cauchy problem

\[
\begin{cases}
  f''_\varepsilon + \frac{f'_\varepsilon}{r} = \frac{\varepsilon^{2s-1}}{f^{2s}_\varepsilon} & \text{for } r > r_\varepsilon, \\
  f_\varepsilon(r_\varepsilon) = f_C(r_\varepsilon) = \frac{2s-1}{2} (|\log \varepsilon| + \log |\log \varepsilon|) + \log 2 + O \left( \varepsilon^{-2} r^{-2}_\varepsilon \right), \\
  f'_\varepsilon(r_\varepsilon) = f_C'(r_\varepsilon) = r^{-1}_\varepsilon \left( 1 + O \left( \varepsilon^{-2} r^{-2}_\varepsilon \right) \right).
\end{cases}
\]

Then an approximation \( F_0 \) to \( F \) can be defined by

\[ F_0(r) = f_C(r) + \chi (r - r_\varepsilon) (f_\varepsilon(r) - f_C(r)), \quad r \geq 1, \]

where \( \chi : \mathbb{R} \to [0, 1] \) is a smooth cut-off function with

\[ \chi = 0 \text{ on } (-\infty, 0] \quad \text{and} \quad \chi = 1 \text{ on } [1, +\infty). \quad (6.1) \]

Note that \( f'_\varepsilon(r) \geq 0 \) for all \( r \geq r_\varepsilon \).

**Lemma 6.1** (Estimates near initial value). For \( r_\varepsilon \leq r \leq |\log \varepsilon| r_\varepsilon \), we have

\[
\frac{1}{2} |\log \varepsilon| \leq f_\varepsilon(r) \leq C |\log \varepsilon|,
\]

\[
f'_\varepsilon(r) \leq Cr^{-1}_\varepsilon,
\]

\[
|f''_\varepsilon(r)| \leq \frac{1}{r^2} + \frac{C}{|\log \varepsilon| r^{2s}_\varepsilon}.
\]

In fact the last inequality holds for all \( r \geq r_\varepsilon \).

**Proof.** It is more convenient to write

\[ f_\varepsilon(r) = |\log \varepsilon| \tilde{f}_\varepsilon \left( r^{-1}_\varepsilon r \right) \]

so that \( \tilde{f}_\varepsilon \) satisfies

\[
\begin{cases}
  \tilde{f}''_\varepsilon + \frac{\tilde{f}'_\varepsilon}{r} = \frac{1}{|\log \varepsilon| \tilde{f}^{2s}_\varepsilon}, & \text{for } r > 1, \\
  \tilde{f}_\varepsilon(1) = \frac{2s - 1}{2} + \frac{2s - 1}{2} |\log \varepsilon| + \log 2 + O \left( \varepsilon^{2s-1} |\log \varepsilon|^{2s} \right), \\
  \tilde{f}'_\varepsilon(1) = \frac{1}{|\log \varepsilon|} + O \left( \frac{\varepsilon^{2s-1}}{|\log \varepsilon|^{2s}} \right).
\end{cases}
\]

To obtain a bound for the first derivative, we integrate once to obtain

\[
r f'_\varepsilon(r) - \tilde{f}'_\varepsilon(1) = \frac{1}{|\log \varepsilon|} \int_1^r \tilde{f}(\tilde{r}) d\tilde{r} \quad \text{for } r \geq 1.
\]
By the monotonicity of \( f_\varepsilon \), hence \( \tilde{f}_\varepsilon \), we have
\[
\tilde{f}_\varepsilon''(r) \leq \frac{1}{r} \left( \tilde{f}_\varepsilon'(1) + \frac{1}{2\log \varepsilon} \right) \frac{1}{2f_\varepsilon(r)^2 r^2} \\
\leq \frac{1}{r|\log \varepsilon|} + \frac{C r}{|\log \varepsilon|^2}
\]
for \( r \geq 1 \). In particular,
\[
\tilde{f}_\varepsilon''(r) \leq \frac{C}{|\log \varepsilon|} \quad \text{for } 1 \leq r \leq |\log \varepsilon|.
\]
This also implies
\[
\tilde{f}_\varepsilon'(r) \leq C \quad \text{for } 1 \leq r \leq |\log \varepsilon|.
\]
From the equation we obtain an estimate for \( \tilde{f}_\varepsilon'' \) by
\[
\left| \tilde{f}_\varepsilon''(r) \right| \leq \frac{1}{r} \frac{\tilde{f}_\varepsilon'(r)}{2} + \frac{1}{2|\log \varepsilon|^2} \frac{1}{f_\varepsilon^2} \\
\leq \frac{1}{r^2} \frac{1}{|\log \varepsilon|^2} + \frac{C}{|\log \varepsilon|^2},
\]
for all \( r \geq 1 \). □

To study the behavior of \( f_\varepsilon(r) \) near infinity, we write
\[
f_\varepsilon(r) = |\log \varepsilon| g_\varepsilon \left( \frac{r}{|\log \varepsilon|r_\varepsilon} \right).
\]
Then \( g_\varepsilon(r) \) satisfies
\[
\begin{aligned}
g''_\varepsilon + \frac{g'_\varepsilon}{r} &= \frac{1}{g^2_\varepsilon}, & \text{for } r \geq 1 \frac{1}{|\log \varepsilon|} \\
g\left( \frac{1}{|\log \varepsilon|} \right) &= \frac{2s-1}{2} + \frac{2s-1}{|\log \varepsilon|} \log |\log \varepsilon| + \frac{2}{|\log \varepsilon|} + O \left( \frac{\varepsilon^{2s-1}}{|\log \varepsilon|^2s} \right), \\
g'_\varepsilon \left( \frac{1}{|\log \varepsilon|} \right) &= 1 + O \left( \frac{\varepsilon^{2s-1}}{|\log \varepsilon|^{2s}} \right).
\end{aligned}
\]

Lemma 6.2 (Long-term behavior). For any fixed \( \delta_0 > 0 \), there exists \( C > 0 \) such that for all \( r \geq \delta_0 \),
\[
\left| g_\varepsilon(r) - r \frac{2s-1}{2s+1} \right| \leq C r^{-\frac{2s-1}{2s+1}}, \\
\left| g'_\varepsilon(r) - \frac{2}{2s+1} \right| \leq C r^{-\frac{2s-1}{2s+1}}, \\
\left| g''_\varepsilon(r) \right| \leq C r^{-\frac{4s-1}{2s+1}}.
\]

Proof. Consider the change of variable of Emden–Fowler type,
\[
g_\varepsilon(r) = r^{\frac{2s-1}{2s+1}} \tilde{h}_\varepsilon(t), \quad t = \log r \geq -|\log \varepsilon|.
\]
Then \( \tilde{h}_\varepsilon(t) > 0 \) solves
\[
\tilde{h}_\varepsilon'' + \frac{2}{2s+1} \tilde{h}_\varepsilon' + \left( \frac{2}{2s+1} \right)^2 \tilde{h}_\varepsilon = \frac{1}{\tilde{h}_\varepsilon^2} \quad \text{for } t \geq -|\log \varepsilon|.
\]
The function \( h_\varepsilon \) defined by \( h_\varepsilon(t) = \left( \frac{2s+1}{2s+1} \right)^{\frac{2s+1}{2s+1}} \tilde{h}_\varepsilon \left( \frac{2s+1}{2s+1} t \right) \) satisfies
\[
h_\varepsilon'' + 2h_\varepsilon' + h_\varepsilon = \frac{1}{h_\varepsilon^2} \quad \text{for } t \geq -\frac{2s+1}{2} |\log \varepsilon|.
\]
(6.3)
We will first prove a uniform bound for \( h_\varepsilon \) with its derivative using a Hamiltonian
\[
G_\varepsilon(t) = \frac{1}{2}(h_\varepsilon')^2 + \frac{1}{2} \left( h_\varepsilon^2 - 1 \right) + \frac{1}{2s-1} \left( \frac{1}{h_\varepsilon^{2s-1}} - 1 \right),
\]
which satisfies
\[
G'_\varepsilon(t) = -2(h_\varepsilon')^2 \leq 0.
\] (6.4)

By Lemma 6.1, we have
\[
h_\varepsilon(0) = O(\tilde{h}_\varepsilon(0)) = O(g_\varepsilon(1)) = O(1),
\]
\[
h_\varepsilon'(0) = O(\tilde{h}_\varepsilon'(0)) = O\left( \frac{g_\varepsilon'(1)}{2s+1} \right) = O(1).
\]
Therefore, \( G_\varepsilon(0) = O(1) \) as \( \varepsilon \to 0 \) and by (6.4), \( G_\varepsilon(t) \leq C \) for all \( t \geq 0 \) and \( \varepsilon > 0 \) small. This implies that for some uniform constant \( C_1 > 0 \),
\[
0 < C_1^{-1} \leq h_\varepsilon(t) \leq C_1 < +\infty \quad \text{and} \quad |h_\varepsilon'(t)| \leq C_1, \quad \text{for all } t \geq 0.
\] (6.5)

In fact, (6.4) implies
\[
\int_0^t h_\varepsilon''(\tilde{t})^2 \, d\tilde{t} = 2G_\varepsilon(0) - 2G_\varepsilon(t) \leq 2G_\varepsilon(0) \leq C,
\]
with \( C \) independent of \( \varepsilon \) and \( t \), hence
\[
\int_0^\infty h_\varepsilon''(\tilde{t})^2 \, d\tilde{t} \leq C,
\]
uniform in small \( \varepsilon > 0 \). In particular, \( |h_\varepsilon''(t)| \to 0 \) as \( t \to \infty \). We claim that the convergence is uniform and exponential. Indeed, let us define the Hamiltonian
\[
G_{1,\varepsilon} = \frac{1}{2}(h_\varepsilon'')^2 + \frac{1}{2} (h_\varepsilon')^2 \left( 1 + \frac{2s}{h_\varepsilon^{2s+1}} \right)
\]
for the linearized equation
\[
h_\varepsilon''' + 2h_\varepsilon'' + \left( 1 + \frac{2s}{h_\varepsilon^{2s+1}} \right) h_\varepsilon' = 0.
\]
We have
\[
G_{1,\varepsilon}' = -2(h_\varepsilon'')^2 - s(2s+1) \frac{h_\varepsilon^3}{h_\varepsilon^{2s+2}}.
\]
By the uniform bounds in (6.5), if we choose \( 2C_2 = s(2s+1)C_1^{2s+3} + 1 \), then \( \tilde{G}_\varepsilon = C_2G_\varepsilon + G_{1,\varepsilon} \) satisfies
\[
\tilde{G}_\varepsilon' \leq -(h_\varepsilon'')^2 - (h_\varepsilon')^2.
\]
Using (6.5) and the vanishing of the zeroth order term together with its derivative at \( h_\varepsilon = 1 \), we have
\[
\tilde{G}_\varepsilon = C_2 \left( \frac{1}{2}(h_\varepsilon')^2 + \frac{1}{2} (h_\varepsilon^2 - 1) + \frac{1}{2s-1} \left( \frac{1}{h_\varepsilon^{2s-1}} - 1 \right) \right) + \frac{1}{2} (h_\varepsilon'')^2 + \frac{1}{2} (h_\varepsilon')^2 \left( 1 + \frac{2s}{h_\varepsilon^{2s+1}} \right)
\]
\[
\leq C \left( (h_\varepsilon'')^2 + (h_\varepsilon')^2 + (h_\varepsilon - \frac{1}{h_\varepsilon^{2s}})^2 \right)
\]
\[
\leq -C\tilde{G}_\varepsilon'.
\]
It follows that for some constants \( C, \delta_0 > 0 \) independent of \( \varepsilon > 0 \) small,
\[
\tilde{G}_\varepsilon(t) \leq Ce^{-\delta_0 t} \quad \text{for all } t \geq 0
\]
and, in particular,
\[
|h_\varepsilon(t) - 1| + |h_\varepsilon'(t)| \leq Ce^{-\frac{\delta_0 t}{2}}, \quad \text{for all } t \geq 0.
\]
Then (6.1) implies that after a fixed \( t_1 \) independent of \( \varepsilon \), the point \((h_\varepsilon(t_1), h'_\varepsilon(t_1))\) is sufficiently close to \((1, 0)\). Let
\[
\begin{align*}
v_1 &= h_\varepsilon, \\
v_2 &= h'_\varepsilon + h_\varepsilon.
\end{align*}
\]

Then (6.3) is equivalent to
\[
\begin{pmatrix}
v_1 \\ v_2
\end{pmatrix}' = \begin{pmatrix} -v_1 + v_2 \\ v_1 - 2s - v_2 \end{pmatrix}.
\]

For \( t \) large the point \((v_1(t), v_2(t))\) is equivalent to \((1, 1)\), which implies in turn
\[
\epsilon_0(r, t) > 0, \quad \text{there exists } C > 0 \text{ such that for all } r \geq r_0,
\]
\[
\left| g_\varepsilon(r) - r^{\frac{2s+1}{s+1}} \right| \leq Cr^{-\frac{2s+1}{s+1}} \text{ and } \left| g'_\varepsilon(r) - \frac{2}{2s+1} r^{-\frac{2s}{s+1}} \right| \leq Cr^{-\frac{2s}{s+1}}
\]
and, in view of (6.2),
\[
\left| g''_\varepsilon(r) \right| \leq Cr^{-\frac{4s}{s+1}}.
\]

Corollary 6.3 (Properties of the initial approximation). We have the following properties of \( F_0 \).

- For \( 1 \leq r \leq r_\varepsilon \), \( F_0(r) = f_C(r) = \log(r + \sqrt{r^2 - 1}) \) and
\[
\begin{align*}
F_0(r) &= \log(2r) + O(r^{-2}), \\
F'_0(r) &= \frac{1}{\sqrt{r^2 - 1}} + \frac{1}{r} + O(r^{-3}), \\
F''_0(r) &= -\frac{1}{r^2} + O(r^{-4}), \\
F'''_0(r) &= \frac{2}{r^3} + O(r^{-5}).
\end{align*}
\]

- For \( r_\varepsilon \leq r \leq \delta_0 \log |\varepsilon| r_\varepsilon \) where \( \delta_0 > 0 \) is fixed,
\[
\frac{1}{2} \log |\varepsilon| \leq F_0(r) \leq C |\log |\varepsilon||, \\
F'_0(r) \leq Cr^{-1}, \\
\left| F''_0(r) \right| \leq C \left( \frac{1}{r^2} + \frac{1}{|\log |\varepsilon|| r_\varepsilon^2} \right), \\
\left| F'''_0(r) \right| \leq Cr^{-1} \left( \frac{1}{r^2} + \frac{1}{|\log |\varepsilon|| r_\varepsilon^2} \right).
\]
Proof. They follow from Lemmata 6.1 and 6.2. For the third derivative, we differentiate the equation and use the estimates for lower order derivatives. □

6.3. The linearization. Now we build a right inverse for the linearized operator

\[ \mathcal{L}_0(\phi)(r) = (1 - \chi_\varepsilon(r)) \frac{1}{r} \left( \frac{r \phi'}{(1 + F_0(r)^2)\varepsilon^2} \right)' + \chi_\varepsilon(r) \left( \phi'' + \frac{\phi'}{r} + \frac{2s \varepsilon^{2s-1}}{F_0(r)^{2s+1}} \phi \right), \]

where \( \chi_\varepsilon \) is any family of smooth cut-off functions with \( \chi_\varepsilon(r) = 0 \) for \( 1 \leq r \leq r_\varepsilon \) and \( \chi_\varepsilon(r) = 1 \) for \( r \geq \delta_0 \log \varepsilon |r_\varepsilon| \) where \( \delta_0 > 0 \) is a sufficiently small number. The goal is to solve

\[ \mathcal{L}_0(\phi)(r) = h(r) \quad \text{for } r \geq 1. \] (6.7)

in a weighted function space which allows the expected sub-linear growth. Let us recall the norms \( \| \cdot \|_s \) and \( \| \cdot \|_s^* \) defined in (2.11) and (2.12).

**Proposition 6.4.** Let \( \gamma \leq 2 + \frac{2}{2s+1} \). For all sufficiently small \( \delta_0, \varepsilon > 0 \), there exists \( C > 0 \) such that for all \( h \) with \( \|h\|_s^* < +\infty \), there exists a solution \( \phi = T(h) \) of (6.7) with \( \|\phi\|_s < +\infty \) that defines a linear operator \( T \) of \( h \) such that

\[ \|\phi\|_s \leq C \|h\|_s^* \]

and \( \phi(1) = 0 \).

We start with an estimate of the kernels of the linearized equation in the far region, namely

\[ Z'' + \frac{Z'}{r} + \frac{2s \varepsilon^{2s-1}}{g_\varepsilon(r)^{2s+1}} Z = 0, \quad \text{for } r \geq \delta_0 \log \varepsilon |r_\varepsilon|. \] (6.8)

**Lemma 6.5.** There are two linearly independent solutions \( Z_1, Z_2 \) of (6.8) so that for \( i = 1, 2 \), we have

\[ |Z_i(r)| \leq C \left( \frac{r}{r_\varepsilon |\log \varepsilon|} \right)^{-\frac{2s-1}{2s+1}} \quad \text{and} \quad |Z_i'(r)| \leq \frac{C}{r_\varepsilon |\log \varepsilon|} \left( \frac{r}{r_\varepsilon |\log \varepsilon|} \right)^{-\frac{2s-1}{2s+1}} \]

for \( r \geq \delta_0 |\log \varepsilon| r_\varepsilon \) where \( \delta_0 > 0 \) is fixed and \( r_\varepsilon = \left( \frac{|\log \varepsilon|}{\varepsilon} \right)^{\frac{2s-1}{2s+1}} \).

**Proof.** We will show that the elements \( \tilde{Z}_i \) of the kernel of the linearization around \( g_\varepsilon \), which solve

\[ \tilde{Z}'' + \frac{\tilde{Z}'}{r} + \frac{2s}{g_\varepsilon(r)^{2s+1}} \tilde{Z} = 0 \quad \text{for } r \geq \frac{1}{|\log \varepsilon|}, \] (6.9)

satisfies

\[ \left| \tilde{Z}_i(r) \right| \leq C r^{-\frac{2s-1}{2s+1}} \quad \text{and} \quad \left| \tilde{Z}_i'(r) \right| \leq C r^{-\frac{2s-1}{2s+1}} \]

for all \( r \geq \delta_0 \) for \( i = 1, 2 \); the result then follows by setting \( Z_i(r) = \tilde{Z}_i \left( \frac{r}{r_\varepsilon |\log \varepsilon|} \right) \).

A first kernel \( \tilde{Z}_1 \) can be obtained from the scaling invariance \( g_\varepsilon, \lambda(r) = \lambda^{-\frac{2s}{2s+1}} g_\varepsilon(\lambda r) \) of (6.2), giving

\[ \tilde{Z}_1(r) = r g_\varepsilon'(r) - \frac{2}{2s+1} g_\varepsilon(r). \]
Then for $\tilde{Z}_2$ we solve (6.9) with the initial conditions

$$
\tilde{Z}_2(\delta_0) = -\frac{\tilde{Z}_1'(\delta_0)}{\delta_0 (\tilde{Z}_1(\delta_0)^2 + \tilde{Z}_1'(\delta_0)^2)}, \quad \tilde{Z}_2'(\delta_0) = -\frac{\tilde{Z}_1(\delta_0)}{\delta_0 (\tilde{Z}_1(\delta_0)^2 + \tilde{Z}_1'(\delta_0)^2)}
$$

for a fixed $\delta_0 > 0$. In particular the Wronskian $\tilde{W} = \tilde{Z}_1\tilde{Z}_2' - \tilde{Z}_2\tilde{Z}_1'$ computed exactly as

$$
\tilde{W}(r) = \frac{\delta_0 \tilde{W}(\delta_0)}{r} = \frac{1}{r} \quad \text{for all } r > \frac{1}{|\log \varepsilon|}. \quad (6.10)
$$

As in the proof of Lemma 6.2, we write $t = \log r$ and consider the Emden–Fowler change of variable $\tilde{Z}(r) = r^{2s+1} \tilde{v}(t)$ followed by a re-normalization $\tilde{v}(t) = \left(\frac{2}{2s+1}\right)^{-\frac{2s+1}{2}} v\left(\frac{2}{2s+1}t\right)$ which yield respectively

$$
\tilde{v}'' + 2s \frac{2}{2s+1} \tilde{v}' + \left(\frac{2}{2s+1}^2 + \frac{2s}{h\varepsilon^{2s+1}}\right) \tilde{v} = 0, \quad \text{for } t \geq -\log|\log \varepsilon|,
$$

$$
v'' + 2v' + (1 + 2s) v = 2s \left(1 - \frac{1}{h\varepsilon^{2s+1}}\right) v, \quad \text{for } t \geq -\frac{2s + 1}{2} \log|\log \varepsilon|.
$$

From this point we may express $v_2(t)$, and hence $\tilde{Z}_2(r)$, as a perturbation of the linear combination of the kernels

$$
e^{-t} \cos(\sqrt{2s} t) \quad \text{and} \quad e^{-t} \sin(\sqrt{2s} t).
$$

Now we show the existence of the right inverse.

**Proof of Proposition 6.4.** We sketch the argument by obtaining a solution in a weighted $L^\infty$ space. The general case follows similarly.

1. Note that we will need to control $\phi$ up to two derivatives in the intermediate region. For this purpose, for any $\gamma \in \mathbb{R}$ and any interval $I \subseteq [r_1, +\infty)$ we define the norm

$$
\|\phi\|_{\gamma, I} = \sup_I r^{-\gamma-2} |\phi(r)| + \sup_I r^{-\gamma-1} |\phi'(r)| + \sup I r^{\gamma} |\phi''(r)|.
$$

By solving the linearized mean curvature equation in the inner region using the variation of parameters formula, we obtain the estimate

$$
\|\phi\|_{\gamma, [r_1, r_\varepsilon]} \leq C \|r^\gamma h\|_{L^\infty([1, +\infty))},
$$

which in particular gives a bound for $\phi$ together with its derivatives at $r_\varepsilon$.

2. In the intermediate region we write the equation as

$$
\phi'' + \frac{\phi'}{r} = h - \tilde{h}, \quad r_\varepsilon \leq r \leq \tilde{r}_\varepsilon,
$$

where

$$
\tilde{r}_\varepsilon = \delta_0 |\log \varepsilon| r_\varepsilon,
$$

and

$$
\tilde{h}(r) = \chi_\varepsilon(r) \frac{2s \varepsilon^{2s-1}}{F_0'(r)^{2s+1}} \phi(r) + (1 - \chi_\varepsilon(r)) \left(\left(1 - \frac{1}{(1 + F_0'(r)^2)^\gamma}\right) \left(\phi'' + \frac{\phi'}{r}\right) + \frac{3F_0'(r)F_0''(r)}{(1 + F_0'(r)^2)^{\gamma+1}} \phi'\right)
$$

is small. Again we integrate to obtain

$$
\phi(r) = \phi(r_\varepsilon) + r_\varepsilon \phi'(r_\varepsilon) \log \frac{r}{r_\varepsilon} + \int_{r_\varepsilon}^r \frac{1}{\gamma} \int_{r_\varepsilon}^t \tau(h(t) - \tilde{h}(t)) \, d\tau \, dt,
$$

$$
\phi'(r) = \frac{r_\varepsilon \phi'(r_\varepsilon)}{r} + \frac{1}{r} \int_{r_\varepsilon}^r t(h(t) - \tilde{h}(t)) \, dt,
$$

$$
\phi''(r) = -\frac{r_\varepsilon \phi''(r_\varepsilon)}{r^2} + \tilde{h}(r) - \tilde{h}(r) - \frac{1}{r^2} \int_{r_\varepsilon}^r t(h(t) - \tilde{h}(t)) \, dt.
$$
Using Corollary 6.3 we have, for small enough \( \delta_0 \) and \( \varepsilon \),
\[
\| r^\gamma \hat{h} \|_{L^\infty([r_x, \hat{r}_e])} \leq C \varepsilon^{2s-1} \left( \frac{\varepsilon}{\log \varepsilon} \right)^{2s-1} \rho r^2 \| \phi \|_{\gamma, [r_x, \hat{r}_e]} + C \left( \frac{\varepsilon}{\log \varepsilon} \right)^{2s-1} \rho r^2 \| \phi \|_{\gamma, [r_x, \hat{r}_e]}
\]
\[
+ C \left( \frac{\varepsilon}{\log \varepsilon} \right)^{2s-1} \left( \frac{1}{r^2} + \frac{\varepsilon^{2s-1}}{\log \varepsilon} \right) r \| \phi \|_{\gamma, [r_x, \hat{r}_e]}
\]
\[
\leq C \left( \delta_0^2 + \delta_0 \left( \frac{\varepsilon}{\log \varepsilon} \right)^{2s-1} \log \varepsilon \right) \| \phi \|_{\gamma, [r_x, \hat{r}_e]}
\]
\[
\leq \delta_0 \| \phi \|_{\gamma, [r_x, \hat{r}_e]}.
\]
This implies
\[
\| \phi \|_{\gamma, [r_x, \hat{r}_e]} \leq C \| r^\gamma h \|_{L^\infty((1, +\infty))} + \delta_0 \| \phi \|_{\gamma, [r_x, \hat{r}_e]},
\]
or
\[
\| \phi \|_{\gamma, [r_x, \hat{r}_e]} \leq C \| r^\gamma h \|_{L^\infty((1, +\infty))}
\]
(6.11) which is the desired estimate.

(3) In the outer region, we need to solve
\[
\phi'' + \phi' + \frac{2s \varepsilon^{2s-1}}{\varepsilon^2 + 1} \phi = h, \quad r > \hat{r}_e.
\]
In terms of the kernels \( Z_i \) given in Lemma 6.5, the Wronskian \( W = Z_1 Z_2' - Z_1' Z_2 \) is given by
\[
W(r) = \frac{1}{r \varepsilon |\log \varepsilon|} \left( \frac{r}{r \varepsilon |\log \varepsilon|} \right) = \frac{1}{r} \quad (6.12)
\]
using (6.10). Using the variation of parameters formula, we may write
\[
\phi(r) = c_1 Z_1(r) + c_2 Z_2(r) + \phi_0(r),
\]
where
\[
\phi_0(r) = -Z_1(r) \int_{\hat{r}_e}^r \rho Z_2(\rho) h(\rho) \, d\rho + Z_2(r) \int_{\hat{r}_e}^r \rho Z_1(\rho) h(\rho) \, d\rho
\]
and the constants \( c_i \) are determined by
\[
\phi(\hat{r}_e) = c_1 Z_1(\hat{r}_e) + c_2 Z_2(\hat{r}_e)
\]
\[
\phi'(\hat{r}_e) = c_1 Z_1'(\hat{r}_e) + c_2 Z_2'(\hat{r}_e)
\]
By Lemma 6.5, (6.12) and (6.11), we readily check that for \( i = 1, 2 \),
\[
|\phi_0(r)| \leq C \left( \frac{r}{\hat{r}_e} \right)^{-\frac{2s-1}{2s}} \int_{\hat{r}_e}^r \rho \left( \frac{\rho}{\hat{r}_e} \right)^{-\frac{2s-1}{2s}} \rho^{-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))} \, d\rho
\]
\[
\leq Cr_\varepsilon^{2-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))},
\]
\[
|c_i| \leq C r_\varepsilon \left( C r_1^{2-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))} + Cr_1^{1-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))} \right)
\]
\[
\leq Cr_\varepsilon^{2-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))},
\]
\[
|c_i| |Z_i(r)| \leq C \left( \frac{r}{\hat{r}_e} \right)^{-\frac{2s-1}{2s} - (2-\gamma)} r^{2-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))}
\]
\[
\leq Cr_\varepsilon^{2-\gamma} \| r^\gamma h \|_{L^\infty((1, +\infty))} \text{ since } \gamma \leq 2 + \frac{2s-1}{2s+1}
\]
from which we conclude
\[
\| r^\gamma -2 \phi \|_{L^\infty((\hat{r}_e, +\infty))} \leq C \| r^\gamma h \|_{L^\infty((1, +\infty))}.
\]
\( \square \)
6.4. The perturbation argument: Proof of Proposition 2.5. We solve the reduced equation
\[ \mathcal{L}(F) = N_1[F] \quad \text{for } r \geq 1, \]  
using the knowledge of the initial approximation \( F_0 \) and the linearized operator \( \mathcal{L}_0 \) at \( F_0 \) obtained in Sections 6.2 and 6.3 respectively. We look for a solution \( F = F_0 + \phi \). Then \( \phi \) satisfies
\[ \mathcal{L}_0 \phi = A[\phi] = N_1[F_0 + \phi] - \mathcal{L}(F_0) - N_2[\phi], \]
where \( N_2[\phi] = \mathcal{L}(F_0 + \phi) - \mathcal{L}(F_0) - \mathcal{L}'(F_0) \phi \) and \( \phi(0) = 0 \). In terms of the operator \( T \) defined in Proposition 6.4, we can write it in the form
\[ \phi = T(A[\phi]). \]
We apply a standard argument using contraction mapping principle as in [36]. First we note that the approximation \( \mathcal{L}(F_0) \) is small and compactly supported in the intermediate region. The non-linear terms in \( A[\phi] \) are also small in the norm \( \|\cdot\|_{m^*} \). Hence \( T(A[\phi]) \) defines a contraction mapping in the space \( X^* \). The details are left to the interested readers.

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