Numerical Study on the Mixed Model in the GOCE Polar Gap Problem

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Abstract Gravity gradients acquired by the Gravity field and steady-state Ocean Circulation Explorer (GOCE) do not cover the entire earth because of its sun-synchronous orbit leaving data gaps with a radius of about 6.5° in the polar regions. Previous studies showed that the loss of data in the polar regions deteriorates the accuracy of the low order (or near zonal) coefficients of the earth gravity model, which is the so-called polar gap problem in geodesy. In order to find a stable solution for the earth gravity model from the GOCE gravity gradients, three models, i.e. the Gauss-Markov model, light constraint model and the mixed model, are compared and evaluated numerically with the gravity gradient simulated with the EGM2008. The comparison shows that the Best Linear Uniformly Unbiased Estimation (BLUUE) estimator of the mixed model can solve the polar gap problem as effectively as the light constraint model; furthermore, the mixed model is more rigorous in dealing with the supplementary information and leads to a better accuracy in determining the global geoid.

Keywords Earth gravity model; satellite gravity; GOCE; mixed model; polar gap problem

CLC number P223

Introduction

The new generation dedicated satellite gravity missions CHAllenging Minisatellite Payload⁴ (CHAMP) and Gravity Recovery and Climate Experiment⁵ (GRACE), equipped with high-accuracy onboard accelerometers and GPS receivers, have brought revolutionary advancements to the mapping of the global Earth Gravity Model (EGM). Gravity gradients are the second-order derivatives of the gravitational potential, which are more sensitive than the gravity for the determination of medium and high degree coefficients. The CHAMP and GRACE missions recover the EGM with SST measurements, while the GOCE mission,⁶ which is managed by the European Space Agency (ESA), measures the external gravity gradients directly, and use of them may significantly improve the accuracy of the determination of the EGM. It is anticipated that the GOCE mission can achieve a 1-cm accuracy global geoid at the spatial resolution of 100 km.

The GOCE satellite moves in a sun-synchronous orbit with an inclination of 96.5°, which indicates that the footprints of the GOCE satellite cannot cover the polar regions leaving a gap of about 13°. However, the spherical harmonic expansion of the external gravitational potential is based on the orthogonality of the associated Legendre functions on a sphere. This conflict has been studied analytically, and it is argued that the
data loss in the polar regions will distort the lower order Spherical Harmonic (SH) coefficients.\cite{4} To solve the polar gap problem, the best method is to fill in the gaps with gravimetric observations directly, for example, with SST or gravity data in the polar regions. An alternative method is to combine them in an indirect way, for example, by introducing a pre-determined EGM together with its statistical information, which contains the gravity information in polar regions already.

We will focus on the alternative method based on the studies of Cai et al.\cite{5} using (1) the Gauss-Markov model, i.e. the rigorous least-squares solution from the gravity gradients; (2) the light constraint model, for this case, the SH coefficients of a pre-determined EGM and their statistical information will be used, and it can be taken as a type of regularization; (3) the mixed model: it can solve the polar gap problem as effectively as the light constraint model because these two models can derive the same dispersion matrix, while the solution of the mixed model is unbiased. The main work of the current study is to compare and evaluate these models numerically, so we will not involve in discussions on other techniques, e.g. Xu and Ditmar.\cite{6,7}

1 Remarks on the numerical simulation

In this section, the mathematical formula of the gravity gradients and the demonstration of the robustness of the software used in the simulation will be presented.

1.1 Mathematical relation between the gravity gradients and the earth gravity model

The external gravitational potential $V$ of the earth can be represented as:\cite{8}

$$V(r, \theta, \lambda) = \frac{GM}{R} \sum_{n=0}^{\infty} \left( \frac{R}{r} \right)^{n+1} \left\{ \sum_{m=0}^{n} \left( \bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda \right) \overline{P}_{nm}(\cos \theta) \right\}$$ (1)

where $(r, \theta, \lambda)$ are the radial distance, co-latitude and longitude, $GM$ is the gravitational constant of the earth, $R$ is the mean radius of the earth, $\overline{P}_{nm}(\cos \theta)$ is the fully normalized associated Legendre function, $n$ and $m$ are the degree and order of the spherical harmonic expansion, respectively, and $\bar{C}_{nm}$ and $\bar{S}_{nm}$ are the fully normalized SH coefficients. For the numerical study, the truncation degree of the EGM is limited to a maximum integer $N_{\text{max}}$. According to Eq.(1), the gravity gradients can be derived easily\cite{8} for example

$$V_{rr}(r, \theta, \lambda) = \Delta \frac{\partial^2 V}{\partial r^2}$$

$$= \frac{GM}{R^3} \sum_{n=0}^{N_{\text{max}}} (n+1)(n+2) \left( \frac{R}{r} \right)^{n+3} \left\{ \sum_{m=0}^{n} \left( \bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda \right) \overline{P}_{nm}(\cos \theta) \right\}$$ (2)

Now we will come to the demonstration of the robustness of the software used in the study. The source codes are transported from the brute-force dynamic software developed by the School of Geodesy and Geomatics, Wuhan University, designed for the processing of satellite-to-satellite tracking data. To validate the codes, a number of verification tests were performed.

1.1.1 Validation of the integrated orbit by the energy conservation principle

The numerical simulation includes the work of the integration of the GOCE-type orbit. For this case, the energy conservation principle is used for the validation, which reads

$$E = V - \frac{1}{2}\mathbf{v}^2 - \mathbf{\omega}(\mathbf{v}_x \mathbf{v}_y - \mathbf{v}_y \mathbf{v}_z)$$ (3)

where $\mathbf{v}$ is the magnitude of the velocity vector $(v_x, v_y, v_z)$ in the inertial frame, and $\mathbf{\omega}$ is the Earth’s rotational speed. See the energy loss in Fig.1. From Eq. (3), it can be derived that by keeping the main terms

$$\frac{dE}{dr} = \frac{\partial V}{\partial r} dr - v d\mathbf{v}$$ (4)

Note that $V$ is a potential function. Eq.(4) can be

![Fig. 1](image)

The energy loss is defined as the quantity $V - \frac{1}{2}\mathbf{v}^2 - \mathbf{\omega}(\mathbf{v}_x \mathbf{v}_y - \mathbf{v}_y \mathbf{v}_z)$ at any epoch relative to the same quantity at the first epoch. It can be seen that during the whole time period, the energy loss is at the level of $10^{-4}$ m$^2$s$^{-2}$.\cite{9}
simplified as the following:
\[ \delta E = f \delta \mathbf{r} - v \delta v \]  
(5)
where \( \delta E \) is the numerical energy loss, \( f \) is the force acting on the satellite, \( \delta \mathbf{r} \) is the position error and \( v \) is the velocity error. If we assume that the energy loss be \( 10^{-4} \text{ m}^2\text{s}^{-2} \), then it can be derived from Eq.(5) that the error of the satellite positions is about \( 10^{-5} \text{ m} \); thus, it will not affect the simulation of the gradients.

1.1.2 Validation of the simulated gravity gradients by Laplace's equation

The Laplace equation in a Cartesian coordinate system reads:

\[ \nabla^2 V = 0 \]  
(6)

The above relation can be transformed into the corresponding spherical coordinate system:

\[ \Delta V = \frac{1}{r^2 \sin \theta} \nabla_\theta \nabla_\lambda V_\theta + \frac{1}{r^2} \nabla_\lambda V_\lambda + V_{rr} + V_{r\theta} \frac{1}{r} \cot \theta V_\theta + V_{r\lambda} \frac{1}{r^2 \sin^2 \theta} V_{\lambda\lambda} \]  
(7)

By computing all the terms in Eq.(7), the simulation of gravity gradients can be validated with the Laplace equation.

1.2 Parallel computing

For efficient processing of the GOCE data, parallel computing is preferred.\(^{[10,11]}\) We performed all the computations on the workstation IBM P575-2 in the data processing center of the School of Geodesy and Geomatics, Wuhan University. The P575-2 is equipped with 16 processors and was designed for the computation-intensive tasks. Two or more processors share the main memory in the P575-2 Symmetric MultiProcessing (SMP) architecture. Thus, OpenMP is suitable for the development of the parallelized software on this platform. In this study, parallel computing is realized successfully. After optimization of the algorithm, the source codes and the data processing flowchart, the final results of this effort show that computation can be completed in about 3 days using 8 processors.

2 Parameter estimation models

In the frame of the current study, we focus on the evaluation of three parameter estimation models. They will be introduced briefly in this section.

2.1 Linear Gauss-Markov model

The linear Gauss-Markov model reads

\[ y = A \xi + e, \quad E[e] = 0, \quad D[e] = \Sigma \]  
(8)

where \( \xi \) is the unknown parameter vector, containing the SH coefficients. \( A \), the partial derivatives of the observations to the SH coefficients, is the design matrix with full column rank. \( y \) is the observation vector. \( e \) is the random vector of inconsistencies with zero expectation; the dispersion matrix \( \Sigma \) is positive definite. The Best Linear Uniformly Unbiased Estimation (BLUUE) estimator of the Gauss-Markov model is

\[ \hat{\xi}_0 = (A' \Sigma^{-1} A)^{-1} A' \Sigma^{-1} y \]  
(9)

with the dispersion matrix

\[ \Sigma_{\hat{\xi}_0} = (A' \Sigma^{-1} A)^{-1} \]  
(10)

2.2 Mixed model, also called combination with prior information model or mixed model

Theil & Goldberger\(^{[12]}\) and Theil\(^{[13]}\) introduced the mixed estimation technique by unifying the sample and the additional information in a common model; Toutenburg\(^{[14]}\) and Rao & Toutenburg\(^{[15]}\) referred to them as a mixed estimator with additional information as stochastic linear restrictions:

\[ \xi_p, \quad E[\xi_p] = E[\hat{\xi}] = E[\xi] \]  
(11)

together with the stochastic information expressed by a linear stochastic restriction of the type

\[ \xi_p = I \xi + e_p, \quad e_p \sim (0, \Sigma_p) \]  
(12)

Combining Eqs.(8) and (12), we get the mixed model

\[ \begin{bmatrix} y \\ \xi_p \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix} \xi + \begin{bmatrix} e \\ e_p \end{bmatrix}, \quad E \begin{bmatrix} e \\ e_p \end{bmatrix} = 0, \quad D \begin{bmatrix} e \\ e_p \end{bmatrix} = \Sigma \]  
(13)

The BLUUE estimator of the mixed model (13)

\[ \hat{\xi}_p = (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} (A' \Sigma^{-1} y + \Sigma_p^{-1} \xi_p) \]  
(14)

\[ \hat{\xi}_p = \hat{\xi}_0 + (A' \Sigma^{-1} A)^{-1} \left[ \Sigma_p + (A' \Sigma^{-1} A)^{-1} \right]^{-1} \left( \xi_p - \hat{\xi}_0 \right) \]  
(15)
together with the dispersion matrix

$$\Sigma_{\xi_p} = (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} \quad (16)$$

The mixed estimator is unbiased and has a smaller dispersion matrix than BLUUE $\hat{\xi}_p$ in the sense that

$$\Sigma_{\xi} - \Sigma_{\xi_p} = (A' \Sigma^{-1} A)' - (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1}$$

$$= (A' \Sigma^{-1} A)' \Sigma_p^{-1} (A' \Sigma^{-1} A)'^{-1} \geq 0$$

Therefore, the use of stochastic restrictions leads to a gain in accuracy. And this gain is apparently independent of whether $E(\xi_p) = \xi$ holds. It is of interest to introduce the supplementary information available, for example, the pre-determined SH coefficients of an EGM as $\xi_p$ into Eq.(11) and establish the above mixed model, which can be applied in our case study in this paper.

2.3 The light constraint solutions

When the prior information is considered as residuals

$$\xi - \xi_p = 0 \quad (18)$$

together with their stochastic information

$$0 = -I\xi + \xi_p, \xi_p - (0, \Sigma_p) \quad (19)$$

we can combine the models (8) and (19) to build the light constraint model: \[^{[16]}\]

$$\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} A & I \end{pmatrix} \begin{pmatrix} \xi \\ \xi_p \end{pmatrix} + \begin{pmatrix} e \\ \xi_p \end{pmatrix},$$

$$E\left[\begin{pmatrix} e \\ \xi_p \end{pmatrix}\right] = 0, D\left[\begin{pmatrix} e \\ \xi_p \end{pmatrix}\right] = \begin{pmatrix} \Sigma & \Sigma_p \\ \Sigma_p' & \Sigma_p \end{pmatrix}$$

(20)

The BLUUE estimator of the light constraint model reads

$$\hat{\xi}_l = (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} (A' \Sigma^{-1} y + \Sigma_p^{-1} 0)$$

$$= (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} A' \Sigma^{-1} y$$

(21)

together with the dispersion matrix

$$\Sigma_{\xi_l} = (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} \quad (22)$$

The difference between BLUUE estimators of the mixed model and light constraint model:

$$\Delta \xi = \hat{\xi}_p - \hat{\xi}_l$$

$$= (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} (A' \Sigma^{-1} y + \Sigma_p^{-1} \xi_p)$$

$$- (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} A' \Sigma^{-1} y$$

$$= (A' \Sigma^{-1} A + \Sigma_p^{-1})^{-1} \Sigma_p^{-1} \xi_p \quad (23)$$

Although Reigber\[^{[16]}\] mentioned that the Eq.(21) is identical to the signal equations of the least squares collocation method when no systematic parameter (non-stochastic parameters), i.e., least squares filtering is estimated. In fact, the objective function of the light constraint solution with Eq.(20) is given as

$$\begin{vmatrix} e' \hat{\xi}_p \\ \Sigma \\ 0 \end{vmatrix} \Sigma_p^{-1} \begin{vmatrix} e' \hat{\xi}_p \\ \Sigma_p \end{vmatrix}^{-1} = e' \Sigma^{-1} e + \hat{\xi}_p' \Sigma_p^{-1} \hat{\xi}_p \quad (24)$$

which is just the same as the objective function of the estimation with weighting parameters.

Reigber\[^{[16]}\] introduced constraints in the form of a priori weights, such as the Kaula’s rule as a priori information (degree variances) for the geopotential unknowns, to permit stable solutions for high degree and order satellite fields, where the additional parameters are as direct observed quantities with a zero mean and a standard deviation of $\pm 7 \times 10^{-6} / l^2$, or the so-called light constraint solutions.

3 Results and analysis

It is anticipated that the BLUUE estimator of the Gauss-Markov model may reveal the polar gap problem in the numerical sense. From the solutions of the mixed model and the light constraint model, it can be seen that these two methods can both make the solutions more stable because the normal equations have been regularized (precisely speaking, the mixed model is not a regularization method because its solution is strict). We are more interested in the difference between the solution of the SH coefficients and its effects on the geoid.

Taking into account the computation efficiency, only the gravity gradient component $V_{zz}$ is used for the numerical study. \[^{[9]}\] In our numerical simulation, first, the satellite orbit is integrated with the truncated global gravity model EGM2008\[^{[17]}\] up to degree 180 as the true EGM. We then simulate the $V_{zz}$ observables along the satellite orbit with a duration of 29 d and a sampling interval of 5 s, again with EGM2008 up till degree 180 as the true EGM. Before estimating the SH coefficients from the simulated observations, white noise with a STandard Deviation (STD) of $10^{-2}$ E are mixed into the simulated $V_{zz}$.

The minimum, maximum, mean and STD of the noises are $-0.04$ E, $0.04$ E, $-0.13 \times 10^{-4}$ E and $0.01$ E, respectively; see Fig. 2 for their distribution. For the observations $V_{zz}$ and the footprints, see Fig. 3.
The accuracy of the SH coefficients estimated by the Gauss-Markov model is displayed in Fig. 4. For the mixed model and the light constraint model, they have the same dispersion matrix; thus, only one is presented for the improvement of the accuracy after the use of the supplementary information, i.e. the SH coefficients of EGM2008 and their statistical information (see Fig. 5). Comparing Fig. 4 and Fig. 5, it can be seen that the distortion of the low order coefficients has been restrained. The use of the SH coefficients of the EGM2008 up to degree and order 180 as the observations improves the accuracy of the recovered SH coefficients about 3 orders of magnitude higher, especially for the low order and low degree ($\leq 90$) SH coefficients. But for real data processing, we cannot use the true model (EGM 2008) as supplementary information.

The accuracy of the SH coefficients can only be taken as an internal check on the solution. To study it further, the difference expressed in Eq.(23) is also calculated and plotted in Fig. 6. From the difference between the BLUUE estimator of the Gauss-Markov model and the light constraint model (see Fig. 7), it is clear that the introduction of the EGM as the a priori information has an obvious effect on the coefficients around order zero. Now the comparison between the
geoid heights computed by the recovered EGM and the geoid computed by the EGM2008 will be given. Because the harmonic degree is truncated to the maximum degree 180 in the simulation, the resolution 1° is selected to compute the global geoid grids; see Fig. 7-Fig. 10. The maximum difference is about 68 mm, and the STD of the difference is about 16 mm. In the view of 1-cm global geoid, the difference between these two models is negligible. It is strange that the accuracy of the geoid heights in the polar regions is relatively higher than that in the other areas after the addition of the supplementary information (Fig. 8-Fig. 10), a point which needs further investigation.

Fig. 8  Histogram of geoid differences between the EGMs recovered by the mixed model and the light constraint model. The maximum difference is about 68 mm, and the STD of the differences is about 16 mm

Fig. 9  Latitude dependency of the geoid differences (computed by the EGM recovered by the mixed model vs. that by the light constraint model). The STD of the geoid differences is 16 mm

Fig. 10  Latitude dependency of the geoid differences (computed by the EGM recovered by the light constraint model vs. that by the EGM2008). The STD of the geoid differences is 46 mm

4 Discussion and conclusion

The polar gap problem was investigated numerically using the simulated gravity gradient component $V_{zz}$. A comparison of the Gauss-Markov model, the light constraint model and the mixed model leads to the following conclusions:

(1) The data loss in the polar regions adversely affects the low-order SH coefficients noticeably under the noise level 0.01 E. Additional information must be introduced for a stable solution.

(2) The addition of the pre-determined EGM as the observations can improve the extent of the ill-posedness, which can be seen from the condition number of the normal matrix and the accuracy improvement of the SH coefficients.

(3) The light constraint model is not rigorous in dealing with the supplementary SH coefficients of an EGM as the zero observations, and its effects on the solution are not negligible, which has been revealed by the comparison of the global geoid. The maximum difference can amount to 68 mm, see Fig. 7.

(4) From the comparisons of the geoid with respect to that computed by the EGM2008, the accuracy can neither achieve the objective 1cm at the spatial resolution 100 km of the GOCE mission (see Fig. 9 and Fig. 10). The phenomenon that the accuracy of the geoid in the polar regions appears to be better than in the surrounding areas needs to be studied further. It seems that the normal equation has been over-regularized and the relative weight of the supplementary SH coefficients and the $V_{zz}$ needs to be tuned.

To understand the effects of the polar gaps of the
GOCE mission more comprehensively, many factors need to be studied in a more realistic configuration at least including:

(1) The noise level of the gradients can be adjusted to a more realistic one and colored noise should be used;

(2) The GOCE satellite orbit is tracked by GPS; thus, high-low SST data can be combined for the gravity recovery. Because the high-low SST data are less sensitive to the polar gaps and can compensate for the gravity gradients’ insensitivity to the long wavelength part of the gravity, the addition of the SST data may improve the solution of the low order SH coefficients;

(3) For the real data processing, we cannot use the true model as the supplementary information, as we have done in the text, so the accuracy may get worse if we use the SH coefficients of another slightly different EGM as the observations.

In this sense, we can only conclude that the BLUUE estimator of the mixed model is better than that of the light constraint model. To solve the polar gap problem, we have still much work to do, and we suggest that a more realistic simulation should be performed, for example, simulating the GPS tracking data of the GOCE-type satellite and the gravity gradients along the orbit with a 1-s sampling interval, together with their characteristic errors, then recovering the EGM with the combination of the SST data, the gravity gradients and the supplementary information. Then, the polar gap problem in determining the gravity field based on the GOCE mission might be completely solved.

Acknowledgements

This work was undertaken during the first author’s research visit at the Institute of Geodesy at University of Stuttgart. The financial support from the Ministry of Science, Research and the Arts Baden-Württemberg, Germany, is greatly appreciated.

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