Asymptotic Convergence of Deep Multi-Agent Actor-Critic Algorithms

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Abstract

We present sufficient conditions that ensure convergence of the multi-agent Deep Deterministic Policy Gradient (DDPG) algorithm. It is an example of one of the most popular paradigms of Deep Reinforcement Learning (DeepRL) for tackling continuous action spaces: the actor-critic paradigm. In the setting considered herein, each agent observes a part of the global state space in order to take local actions, for which it receives local rewards. For every agent, DDPG trains a local actor (policy) and a local critic (Q-function). The analysis shows that multi-agent DDPG using neural networks to approximate the local policies and critics converge to limits with the following properties: The critic limits minimize the average squared Bellman loss; the actor limits parameterize a policy that maximizes the local critic’s approximation of $Q^*_i$, where $i$ is the agent index. The averaging is with respect to a probability distribution over the global state-action space. It captures the asymptotics of all local training processes. Finally, we extend the analysis to a fully decentralized setting where agents communicate over a wireless network prone to delays and losses; a typical scenario in, e.g., robotic applications.

1 Introduction

Deep reinforcement learning (DeepRL) such as the Multi-Agent Actor-Critic (MAAC) algorithm Lowe et al. (2017) have become popular in optimal multi-agent decision making problems. The effectiveness of these algorithms in many problem domains is due to the efficient use of deep neural networks (thence then name “DeepRL”) to approximate functions like the policy function or the Q-function. These algorithms are versatile in that they are applicable for both cooperative and competitive scenarios involving continuous decision spaces.

In this paper, we analyze the asymptotic properties of the MAAC algorithm. Despite its popularity and usefulness in many practical scenarios, the conditions under which MAAC converges to a required limit are not well studied – this paper addresses this gap and presents sufficient conditions for its convergence, and a thorough asymptotic analysis.
Such convergence guarantees and analyses are in general difficult, even for traditional single-agent DeepRL algorithms. Although RL algorithms with linear function approximations are well studied, ones that use non-linear function approximators like deep neural networks are not well understood. Often, convergence is only characterized under strict assumptions that are difficult to verify in practice, e.g., [Wang et al. (2019)] analyses the case of regular transition kernels. Understanding the behaviour of multi-agent DeepRL algorithms is even more challenging since the various training processes associated with the agents are often intertwined. It is pertinent, both from a theoretical and from a practical standpoint, to analyze, under practical assumptions, the asymptotic properties of multi-agent DeepRL algorithms.

Let us consider the MAAC algorithm more specifically. It may be viewed as a multi-agent version of Deep Deterministic Policy Gradient (DDPG) [Lillicrap et al. (2015)], a popular actor-critic DeepRL algorithm. DDPG uses two neural networks, an actor network and a critic network. The actor network is trained to approximate the optimal policy, and the critic network is trained to approximate an objective function such as the Q-function. The critic network is typically trained using a variant of the squared Bellman error, and the actor network is trained to pick actions that maximize an approximation of the optimal Q-function, as found by the critic. Note that both the critic and the actor networks are trained simultaneously.

In MAAC, every agent trains a local policy to select local actions. The local policies are functions of the local states, which in turn constitute the global (multi-agent) system state. It is assumed that all the agent clocks are discrete and synchronized. At every time step, after all agents have taken actions, they obtain local rewards. Each agent then uses a local copy of DDPG to train a local critic and policy, using samples of the global state and action space collected from experience. As suggested earlier [Mnih et al. (2015)], every local critic network is trained using the Deep Q-Learning algorithm. The local actor network is trained to maximize the best known estimate of the optimal local Q-function \( Q^*_i \), where \( i \) represents the agent index. At any time step, the current estimate is obtained from the local critic network.

Current literature expects that the MAAC algorithm converges to optimal policies. In reality, this is not true and our asymptotic analysis formalizes this. The nature of the limits of MAAC is the following: The limits satisfy that the gradients of all local Bellman errors and all local policy gradients are zero on an average w.r.t. to a limiting measure that is driven by the training process. In that regard, no agent can improve its performance locally w.r.t. to this limiting measure of the training process. As a takeaway for users, our asymptotic analysis formalizes the common intuition that one should not expect generalization of the algorithm well beyond state regions encountered during training! In that regard, the measure may be used to improve the training performance.

Our contributions. To study the convergence of MAAC, we must first understand Deep Q-Learning. Recently, the asymptotic properties of Deep Q-Learning under practical and mild assumptions were studied in [Ramaswamy and Hüllermeier (2021)]. We build on that theory and present an analysis of the asymptotic behaviour of the MAAC algorithm, under similarly mild assumptions. As DDPG is the single-agent special case of MAAC, we also establish convergence of DDPG. In order to understand the limiting properties of MAAC, we construct a limiting probability distribution over the global state-action spaces.
This limiting distribution captures the frequency of the state-action pairs encountered during the local training processes of all agents. Our analysis shows that each local critic network’s weights converge to a minimum of the local expected squared Bellman loss, where the expectation is with respect to that limiting distribution. The limit of each local policy maximizes the expected estimated local $Q^*_i$, where the estimate is obtained from the local critic network. Again, the expectation is with respect to the limiting distribution.

Paper organization. The rest of our work is structured as follows. We start by defining the multi-agent Markov game setting in Section 2. After that, Section 3 describes the main algorithm under consideration and formulates our assumptions. Section 4 then presents our main convergence analysis. Section 5 discusses two extensions: Firstly, we present an alternative implementation of the algorithm by Lowe et al. (2017). Our analysis shows that both algorithms behave asymptotically identical. Nonetheless, we establish that there is a fine difference between the two implementations that might have a significant impact on the training result when the algorithms are run over a finite horizon. Secondly, we extend the algorithm to a fully decentralized setting where agents have to exchange information over a potentially lossy communication network. We show that, under mild conditions on the network, the learning process is asymptotically unaffected by such communication.

2 Background on Markov Games

In multi-agent reinforcement learning, agents interact with an environment via local actions. To take actions, the agents take local states into account and their behaviour is evaluated via local feedback/reward signals. Typically, the associated global state of the environment is the concatenation of the local states. The global state transitions to a new state after each agent has taken its local action. However, the global state is usually unobservable for every agent. The structure of the local reward functions depend on the nature of interaction between the agents: do they cooperate or compete?

The $D$ agent Markov game model, Littman (1994), is formally defined as the 4-tuple $\langle S, A, p, \{r^i \mid 1 \leq i \leq D} \rangle$, where:

$S = \prod_{i=1}^{D} S_i$ is the global state space, with $S_i$ the local state space of agent $i$. Typically, $S_i \subseteq \mathbb{R}^{k_i}, k_i > 0$.

$A = \prod_{i=1}^{D} A_i$ is the global action space, where an action $a = (a^1, \ldots, a^D) \in A$ denotes the joint action that is the concatenation of local actions $a^i \in A_i \subseteq \mathbb{R}^{d_i}, d_i > 0$.

$p$ is the Markov transition kernel, i.e. $p(\cdot \mid s, a)$ is the distribution of the successor state of state $s$ after action $a$ is executed.

$r^i$ is the local reward function associated with agent $i$. Specifically, $r^i(s, a)$ is the local reward that agent $i$ observes when the system is in state $s$ and the global action $a$ is taken.

In many cooperative Markov games, the local reward functions coincide, i.e., $r^i \equiv r$ for $1 \leq i \leq D$. Such models are called factored decentralized MDPs Bernstein et al. (2002). Moving forward, we assume that the problem at hand can be
modelled using the more general Markov game. As a consequence, our analysis will cover a wide variety of scenarios, including cooperative and competitive ones.

3 The Multi-Agent Actor-Critic Algorithm

$D$ agents interact with the environment at discrete time steps $n \in \mathbb{N}$. At every time step $n$, agent $i$ observes a local state $s^i_n \in \mathcal{S}_i \subseteq \mathbb{R}^{k_i}$, takes a local (continuous) action $a^i_n \in \mathcal{A}_i \subseteq \mathbb{R}^{d_i}$, for which it receives a scalar reward $r^i_n$. The joint (global) action $a_n := (a^1_n, \ldots, a^D_n)$ is an element of the global action space $\mathcal{A} := \prod_{i=1}^D \mathcal{A}_i$.

The local behaviour of agent $i$ is defined by the policy $\pi_i(s^i; \phi^i)$, parameterized by vector $\phi^i$. Define the associated global policy as $\pi := (\pi_1, \ldots, \pi_D)$. For each local reward function $r^i$, the return starting from time-step 1 is given by $R_i = \sum_{n=1}^{\infty} \gamma^{n-1} r^i(s^i_n, a^i_n)$ with discount factor $0 < \gamma < 1$. Given a global policy $\pi$, the associated action-value function $Q_i$ of agent $i$ is given by $Q_i(s, a) := \mathbb{E}_\pi[R_i | s_1 = s, a_1 = a]$.

Algorithm

Lowe et al. [2017]’s MAAC algorithm, an actor-critic algorithm for the Markov game setting, runs the DDPG algorithm at every agent. Each agent $i$ uses a deep neural network approximator $Q_i(s, a; \theta^i)$ for its local critic network; $\theta^i$ represents the vector of network weights. The local critic network is trained using the Deep Q-Learning algorithm to find $\theta^*_i$ such that $Q_i(s, a; \theta^*_i) \approx Q^*_i(s, a)$ for all state-action pairs $(s, a)$, where $Q^*_i$ denotes the optimal action-value function associated with agent $i$. The actor network $\pi_i(s^i; \phi^i)$ is parameterized by $\phi^i$. It is trained to find $\phi^*_i$ such that the global policy $\pi(s; \phi) \approx \arg\max_{a \in \mathcal{A}} Q^*_i(s, a)$ for all states $s$.

We assume that every agent $i$ maintains a local replay memory of up to $N$ transitions, $(s^i_m, a^i_m, r^i_m, s^i_{m+1})$ for transition from time $m$ to $m+1$. At time step $n$, agent $i$ samples a minibatch of $M < N$ transitions from its replay memory. It uses these samples to update its actor and critic:

$$
\begin{align*}
\theta^i_{n+1} &= \theta^i_n + \alpha(n) \frac{1}{M} \sum_m \nabla_{\theta^i} Q_i(s^i_m, a^i_m; \theta^i_n) \left( r^i_m + \gamma Q_i(s_{m+1}, \pi(s_{m+1}; \phi_n); \theta^i_n) - Q_i(s^i_m, a^i_m; \theta^i_n) \right), \\
\phi^i_{n+1} &= \phi^i_n + \beta(n) \frac{1}{M} \sum_m \nabla_{\phi^i} \pi_i(s^i_m; \phi^i_n) \\
&\quad \times \nabla_{a^i} Q_i(s^i_m, a^i_m, \ldots, \pi_i(s^i_m; \phi^i_n), \ldots, a^D_m; \theta^i_n),
\end{align*}
$$

(1)

with step-size sequences $\alpha(n)$ and $\beta(n)$. Note that the policy gradient update (second iteration in (1)) requires the knowledge of actions taken by other agents, $(a^j_m)_{j \neq i}$. Typically, those are obtained from the experience replay of agents $j \neq i$. In our Section 5 we will discuss an alternative to this implementation that uses the knowledge of other agent’s policies, $(\pi^j)_{j \neq i}$, for the update. We shall informally argue that the latter may result into faster training at the cost of increased variance.
In order to update the critic network (first iteration in (1)), knowledge of all local policies, their action sequences, and the global state, are needed – albeit in a delayed manner. This quasi-central perspective is justified since training is often performed in simulated environments. But suppose we wish to train in an online decentralized manner; then, local information needs to be exchanged using, e.g., a wireless network. We discuss this further in Section 5.

Assumptions

Assumptions required for analysis are as follows.

(A1) The critic step size sequence \( \alpha(n) \), is positive, monotonically decreasing and satisfies \( \sum_{n \geq 0} \alpha(n) = \infty \), \( \sum_{n \geq 0} \alpha^2(n) < \infty \). The actor step size sequence \( \{\beta(n)\}_{n \geq 0} \) is chosen such that \( \lim_{n \to \infty} \frac{\beta(n)}{\alpha(n)} = 1 \).

(A2) (a) \( \sup_{n \geq 0} \|\theta_n\| < \infty \) a.s. and \( \sup_{n \geq 0} \|\phi_n\| < \infty \) a.s.
   (b) The state space \( S \) and the action space \( A \) are both compact.

(A3) The state transition kernel \( p(\cdot | s, a) \) is continuous.

(A4) The policies \( \pi_i \) and the critics \( Q_i \) are fully connected feedforward neural networks with twice continuously differentiable activation functions such as Gaussian Error Linear Unit (GELU), sigmoid, etc.

(A5) For \( 1 \leq i \leq D \), the reward function \( r_i : S \times A \to \mathbb{R} \) is continuous.

Most assumptions are standard in literature. (A2)(a) is the strongest assumption as it requires almost sure stability of the algorithm. Assuming stability is a typical first step towards understanding the convergence behaviour of optimization algorithms. Conventionally, this may be enforced by projecting the iterates onto a large compact set after every iteration. Additionally, (A2)(b) assumes compactness of the state space. This is merely to simplify the following presentation and we refer to techniques presented by Ramaswamy and Hüllermeier (2021) for a generalization to \( d \)-dimensional real spaces.

In (A1), we require that the critic and actor step size sequences are chosen such the \( \frac{\beta(n)}{\alpha(n)} \to 1 \). This is not a conventional assumption for AC algorithms Borkar and Konda (1997). However, we will not present a proof based on a two-timescale formulation but instead analyse the whole AC algorithm (1) with respect to the timescale of the critic iterations. Then, the above condition will naturally appear. In practice, we want the critic to converge faster so we would initially choose \( \alpha(n) \) larger than \( \beta(n) \). Our analysis requires that afterwards the iterations asymptotically take steps of the same size. An example step size sequence is illustrated in Figure 1. In (A4), we require twice continuously differentiability of the activations used by policy and actor networks. GELUs are well-known example that satisfy this property Hendrycks and Gimpel (2016). Additionally, GELUs are one of the well-known neural network activation functions with similar high performance across different tasks compared to other well-known activations like ELUs or LeakyReLUs Ramachandran et al. (2017).
4 Analysis

In this section, we analyse the actor-critic iteration given by (1). We will frequently refer to Ramaswamy and Hülkermeier (2021) for details, where a convergence proof of Deep Q-Learning was presented. In essence, we will present a convergence proof for the DDPG algorithm.

Preliminaries

As the first step, we rewrite the AC iteration, (1), such that the minibatch sample size \( L \) equals 1, we get:

\[
\begin{align*}
\theta_{n+1}^i &= \theta_n^i + \alpha(n) \nabla_{\theta} l(\theta_n^i, \phi_n, s_n, a_n), \\
\phi_{n+1}^i &= \phi_n^i + \beta(n) \nabla_{\phi} g(\theta_n^i, \phi_n, s_n).
\end{align*}
\]

The critic loss gradient is given by

\[
\nabla_{\theta} l(\theta_n^i, \phi_n, s_n, a_n) := \nabla_{\theta} Q(s_n, a_n; \theta_n^i) (r_i(s_n, a_n) + \gamma \int Q_i(y, \pi(y; \phi_n); \theta_n^i) p(dy | s_n, a_n) - Q(s_n, a_n; \phi_n)).
\]

and the policy gradient is given by

\[
\nabla_{\phi} g(\theta_n^i, \phi_n, s_n) := \nabla_{\phi} \pi_i(s_n; \phi_n) \times \nabla_{a_i} Q_i(s_n, \pi(s_n; \phi_n); \theta_n^i).
\]

We analyze (2) that does not consider experience replays and uses immediately observed samples instead. An extension to account for experience replays will be discussed in the following section. Additionally, the new critic iteration now uses the gradient of the expected squared Bellman error for each state-action pair \((s_n, a_n)\), instead of the sample squared Bellman error. Ramaswamy and Hülkermeier (2021) showed that the sum of the discounted errors between the version that uses the state transition kernel \( p \) and the one that only uses samples is almost surely convergent. In other words, the error associated with using samples instead of the expected loss vanishes asymptotically. Hence, the two versions have identical asymptotic behaviours.

We now present properties of the gradients \( \nabla_{\phi} g \) and \( \nabla_{\theta} l \).
Lemma 1. (i) $\nabla_{\theta} l(\theta^i_n, \phi_n, s_n, a_n)$ is continuous and locally Lipschitz continuous in the $\theta^i$ and $\phi$-coordinate.

(ii) $\nabla_{\phi} g(\theta^i_n, \phi_n, s_n, a_n)$ is locally Lipschitz continuous in every coordinate.

Proof. By (A4) the neural network activations are twice-continuous differentiability ($C^2$), hence $\pi_i(s; \phi^i)$ and $Q_i(s, a; \theta^i)$ are $C^2$ in their input coordinates. Additionally, it follows from [Ramaswamy and H"ullermeier 2021, Lemma 9] that $\pi_i(s; \phi^i)$ and $Q_i(s, a; \theta^i)$ are $C^2$ in their parameter coordinates $\phi^i$ and $\theta^i$, respectively, for every fixed $s \in S$ and $a \in A$. Note that composition, product and sums of $C^2$ functions are $C^2$. Moreover $C^2$ functions have local Lipschitz gradients. This is because the gradient is $C^1$, and $C^1$ functions are locally Lipschitz [Conway 2019]. Then

For the first part, fix parameter vectors $\phi$, $\theta^i$, and $s \in S$ and $a \in A$. Since, $\pi_i(s; \phi)$ and $Q_i(s, a; \theta^i)$ are $C^2$ in every coordinate, there is some $R > 0$ and continuous functions $L_{Q_i}(y, \theta^i, \phi)$ and $L_{\pi}(y, \phi)$, such that $\forall \phi_1, \phi_2 \in B_R(\phi)$, we have

\[
\int Q_i(y, \pi(y; \phi_1); \theta^i) p(dy \mid s, a) - \int Q_i(y, \pi(y; \phi_2); \theta^i) p(dy \mid s, a) \leq \int L_{Q_i}(y, \theta^i) \|\pi(y; \phi_1) - \pi(y; \phi_2)\|_2 p(dy \mid s, a)
\]

\[
\leq \|\phi_1 - \phi_2\|_2 \int L_{Q_i}(y, \theta^i, \phi) L_{\pi}(y, \phi) p(dy \mid s, a) \leq L(\phi) \|\pi(y; \phi_1) - \pi(y; \phi_2)\|_2
\]

(5)

for some $L(\phi) > 0$. The last inequality follows by (A2). It now follows that $\nabla_{\theta} l(\theta^i, \phi, s, a)$ is locally Lipschitz as a product and sum of locally Lipschitz functions. For part (i), it is left to show that $\nabla_{\theta} l(\theta^i, \phi, s, a)$ is continuous in the $s$ and $a$ coordinate. This directly follows from the continuity of $Q_i(y, \pi(y; \phi); \theta^i)$ and convergence in distribution by continuity of $p(dy \mid s, a)$.

For the second part, we can similarly conclude that $g(\theta^i_n, \phi_n, s_n)$ is locally Lipschitz in its parameter coordinates, as it is the product of $C^2$ functions.

It follows from (A2)(a) that $\theta_n^i$ and $\phi_n^i$ can be restricted to sample path-dependent compact sets. In combination with (A2)(b), we can therefore conclude that the local Lipschitz properties in Lemma 1 are global when we restrict the algorithm iterates to the aforementioned sets.

Convergence Proof

To analyze the asymptotic behaviour of (2), we first construct continuous-time trajectories with identical limiting behaviours. Then, we relegate our analysis to understanding the asymptotic behaviour of these trajectories. For this, we first divide the time axis using $\alpha(n)$ as follows:

\[
t_0 := 0 \text{ and } t_n := \sum_{m=0}^{n-1} \alpha(m) \text{ for } n \geq 1.
\]

(6)

Now define

\[
\overline{\theta}_n(t_n) := \theta_n^i, n \geq 0 \text{ and } \overline{\phi}_n(t_n) := \phi_n^i, n \geq 0.
\]

(7)
Let \( \mathbb{R}^q \) and \( \mathbb{R}^p \) be the parameter spaces of the \( \theta^n \)'s and \( \phi^n \)'s, respectively. Then define \( \overrightarrow{\theta} \in C([0, \infty), \mathbb{R}^q) \) and \( \overrightarrow{\phi} \in C([0, \infty), \mathbb{R}^p) \) by linear interpolation for all \( \overrightarrow{\theta}(t_n) \) and \( \overrightarrow{\phi}(t_n) \), respectively.

To analyze the training process, we formulate a measure process that captures the encountered state-action pairs when using the global policy \( \pi(s_n; \phi_n) \). Therefore, define

\[
\mu(t) = \delta(s_n, a_n), t \in [t_n, t_{n+1}]
\]

where \( \delta(s, a) \) denotes the Dirac measure. This defines a process of probability measures on \( S \times A \). As the policy gradient update (4) is only state-dependent, we also define the associated process on \( S \) by

\[
\mu^s(t) := \mu(t)(\cdot, A) = \delta_{s_n}.
\]

For every probability measure \( \nu \) on \( S \times A \) and associated marginal measures \( \nu^s \) on \( S \), as above, define

\[
\tilde{\nabla} l_i(\theta^i, \phi, \nu) := \int \nabla_{\theta^i} l_i(\theta^i, \phi, s, a) \nu(ds, da),
\]

\[
\tilde{\nabla} g_i(\theta^i, \phi, \nu^s) := \int \nabla_{\phi} g_i(\theta^i, \phi, s) \nu^s(ds).
\]

It follows from Lemma 1 that all \( \tilde{\nabla} l_i \) and \( \tilde{\nabla} g_i \) are continuous in all coordinates and locally Lipschitz in both the \( \theta^i \)- and \( \phi \)-coordinate. We can now define the associated continuous time trajectories in \( C([0, \infty), \mathbb{R}^q) \) and \( C([0, \infty), \mathbb{R}^p) \) that capture the training process starting from time \( t_n \) for \( n \geq 0 \):

\[
\theta^i_n(t) := \overrightarrow{\theta}(t_n) + \int_0^t \tilde{\nabla} l_i(\theta^i_n(s), \phi_n(s), \mu_n(s)) ds,
\]

\[
\phi^i_n(t) := \overrightarrow{\phi}(t_n) + \int_0^t \tilde{\nabla} g_i(\theta^i_n(s), \phi_n(s), \mu^s_n(s)) ds.
\]

where \( \mu_n(t) := \mu(t_n + t) \) and \( \mu^s_n(t) := \mu^s(t_n + t) \). Per definition, the trajectories define solutions to the following families of non-autonomous ordinary differential equations (ODEs):

\[
\begin{align*}
\{ \dot{\theta}^i_n(t) &= \tilde{\nabla} l_i(\theta^i_n(t), \phi_n(t), \mu_n(t)) \}_{n \geq 0}, \\
\{ \dot{\phi}^i_n(t) &= \tilde{\nabla} g_i(\theta^i_n(t), \phi_n(t), \mu^s_n(t)) \}_{n \geq 0}.
\end{align*}
\]

In summary, we obtain that the limiting behaviour of (2) is captured by the limits of the sequences \( \{ \overrightarrow{\theta}([t_n, \infty)) \}_{n \geq 0} \) and \( \{ \overrightarrow{\phi}([t_n, \infty)) \}_{n \geq 0} \) defined by (7). Further, the sequences defined in (11) can then be analyzed as solutions to the ODEs in (12). The following technical Lemma 2 is central for the analysis: it shows that the left-shifted interpolated sequences and the solutions to the above ODE’s are asymptotically identical. To prove Lemma 2 we use that the two step sizes are related by \( \frac{\alpha(n)}{\alpha(n)} \rightarrow 1 \). This is essential since we just constructed the continuous trajectories with respect to the time scale induced by \( \alpha(n) \). The assumption in essence requires that the critic and actor updates asymptotically run on the same time scale.

Lemma 2. For every \( T > 0 \), we have
Lemma 3. 

a) \( \lim_{n \to \infty} \sup_{t \in [0, T]} \| \tilde{\varphi}^i(t_n + t) - \varphi^i_n(t) \| = 0 \)

b) \( \lim_{n \to \infty} \sup_{t \in [0, T]} \| \overline{\varphi}^i(t_n + t) - \phi^i_n(t) \| = 0 \)

Proof. See Appendix A.

The convergence of the trajectories \( \theta^i_n(t) \) and \( \phi^i_n(t) \) now follows from the Arzela-Ascoli theorem, which implies that for each \( i \) and for every interval \([0, T]\) with \( T > 0 \) the family of trajectories is sequentially compact in their respective continuous function spaces. Further, it can be shown that the space of measurable functions from \([0, \infty)\) to the space of probability measures on \( S \times A \) is compact metrizable. It now follows that the product space of each family of trajectories associated with \( \theta^i_n(t) \) and \( \phi^i_n(t) \) together with the above considerations is sequentially compact. Hence, there is a common subsequence such that all considered sequences converge simultaneously. Additionally, we are left with a slight abuse of notation) that \( \theta^i_n \to \theta^i_{\infty}, \phi^i_n \to \phi^i_{\infty} \) and \( \mu_n \to \mu_{\infty} \). Finally, it can be shown that \( \mu_n \) converges in distribution to \( \mu_{\infty} \) (Ramkisun and Hüllermeier 2021, Lemma 4).

The following Lemma 3 shows that the limits just found are solutions to the limits of the families of non-autonomous ordinary differential equations.

Lemma 3. 

a) \( \theta^i_{\infty} \) is a solution to

\[
\tilde{\eta}^i(t) = \nabla l_i(\theta^i(t), \phi^1_{\infty}(t), \ldots, \phi^D_{\infty}(t), \mu_{\infty}(t))
\]

b) \( \phi^i_{\infty} \) is a solution to

\[
\tilde{\phi}^i(t) = \nabla g_i(\theta^i_{\infty}(t), \phi^1_{\infty}(t), \ldots, \phi^D_{\infty}(t), \mu_{\infty}(t))
\]

Proof. Consider the sequence \( \theta^i_n \). The proof for the other parameter sequences is identical. Fix \( T > 0 \). We need to show that

\[
\sup_{t \in [0, T]} \| \theta^i_n(t) - \theta^i_{\infty}(0) \| - \int_0^t \nabla l_i(\theta^i_n(x), \phi^i_{\infty}(x), \mu_{\infty}(x)) dx \xrightarrow{n \to \infty} 0. \tag{13}
\]

The norm in (13) is bounded by

\[
\| \theta^i_n(0) - \theta^i_{\infty}(0) \| + \left\| \int_0^t \nabla l_i(\theta^i_n(x), \phi^i_{\infty}(x), \mu_n(x)) - \nabla l_i(\theta^i_{\infty}(x), \phi^i_{\infty}(x), \mu_{\infty}(x)) dx \right\|. \tag{14}
\]

We can now expand the second term, by successively adding zeros. We can then use Lemma 3 to bound every resulting component (except one) by a term of the form

\[
L \int_0^t \| \theta^i_n(x) - \theta^i_{\infty}(x) \| dx \tag{15}
\]

for a sample path dependent Lipschitz constant \( L \). Additionally, we are left with one term of the form

\[
\left\| \int_0^t \nabla l_i(\theta^i_n(x), \phi^i_{\infty}(x), \mu_n(x)) - \nabla l_i(\theta^i_{\infty}(x), \phi^i_{\infty}(x), \mu_{\infty}(x)) dx \right\|. \tag{16}
\]
Due to the compact convergence of every parameter sequences (Arzela-Ascoli theorem) every parameter sequence will converge uniformly over $[0, T]$. This shows that the above bounds of the form (15) decay to zero. The remaining term in (16) has to decay to zero as $\mu_n \rightarrow \mu_\infty$. This follows as a consequence of the convergence in distribution of $\mu_n$ and the stability of the algorithm.

It can now be shown (Ramaswamy and Hüllemeier, 2021, Lemma 6) that the limiting measure $\mu_\infty$ of the probability measures on $\mathcal{S}$, i.e. the limiting marginal distribution of $\mu_\infty$, is stationary with respect to the state Markov process $s$. By construction, $\mu_\infty$ captures the long-term behaviour of the training processes of all agents and therefore directly depends on the local states encountered during training by all agents.

The following Theorem 1 is now an immediate consequence of the discussion so far.

**Theorem 1.** Under (A1)-(A5) the limits $(\theta_i^\infty, \phi_i^\infty)$ of the actor-critic algorithm (2) satisfy $\tilde{\nabla} l_i(\theta_i^\infty, \phi_\infty, \mu_\infty) = 0$ and $\tilde{\nabla} g_i(\theta_i^\infty, \phi_\infty, \mu_\infty) = 0$.

**Proof.** We merely give a short sketch. First, we can append the ODEs defined in (12) to form a new big ODE in the appended parameter space. We can then show that a solution to this ODE is an equilibrium point of the new big ODE. By construction, the appended trajectories (11) define a solution to the new ODE and hence their limit is an equilibrium point. The theorem now follows by applying lemma 2. For additional details we refer to Ramaswamy and Hüllemeier (2021).

5 Extensions

This section discusses extensions to algorithm (1) and the analysis of the previous section. We start by discussing the inclusion of experience replay in the analysis.

Extension to Experience Replays

Stabilizing DeepRL algorithms is one of the most important aspects for practical implementations. One of the most important breakthroughs has been the use of experience replays Mnih et al. (2015). Instead of only using the tuple $(s_n, a_n, r_n, s_{n+1})$ for training at every iteration $n$, the encountered tuples are stored and, at every iteration, a minibatch of i.i.d. samples is used to evaluate an average gradient as in (1). This reduces the bias of the training algorithm towards the current interaction of the agents with the environment and it has been empirically shown to stabilize the training process.

To include the use of experience replays in section 4 the probability measure $\mu(t)$ needs to be redefined. In (1), each agent samples $L < H$ tuples independently from the experience replay $R_n$ at every iteration $n$. We now need to define a measure process for each agent since the sampling processes of the agents will in general be different and we can not expect them to coordinate their sampling. For $t \in [t_n, t_{n+1})$, define $\mu^i(t)$ to be the probability measure on $\mathcal{S} \times \mathcal{A}$ that places a mass of $1/L$ on each pair $(s_{m(n,j,i)}, a_{m(n,j,i)})$ for $1 \leq j \leq L$, where each $m(n,j,i)$ denotes one of the time indices sampled by agent $i$ at time $n$. 10
Now we define $\mu(t)$ as the product measure induced by the $\mu_i(t)$. We need to redefine (10) accordingly. For example, for agent 1, the averaged critic update will take the form

$$\int \nabla_{\theta^1} l_1(\theta^1, \phi, s, a) \nu(ds, da, S_2, A_2, \ldots, S_D, A_D).$$

(17)

Hence, for every $t = t_n$ we find that $\nabla l_1(\Phi^1(t), \phi(t), \mu(t))$ yields the sampling-based critic update in (1) as desired. The analysis then follows the same lines using the redefined measure process and the new versions of (10).

**Alternative implementation to Lowe et al. (2017)**

The policy gradient update of Lowe et al. (2017) in (1) evaluates the following gradient of the approximated $Q$-function:

$$\nabla_{a^i} Q_i(s_m^i, a_m^i, \ldots, \pi_i(s^i_m; \phi^i_n), \ldots, a_D^i; \theta_n^i)$$

(18)

As highlighted earlier, the update conditions on the actions of the other agents sampled from the experience replay. As an alternative, we propose to also condition the policy gradient update on the current policy of the other agents, i.e. to use

$$\nabla_{a^i} Q_i(s_m^i, \pi_1(s^1_m; \phi^1_n), \ldots, \pi_D(s_D^i, \phi^D_n; \theta_n^i)$$

(19)

in place of (18) in (1). The policies of the other agents are already used in the critic update and therefore have to be available. As a consequence of our reduction in Section 4, both versions result in the same simplified iteration (2).

Lowe et al. (2017) discuss that conditioning on the historical actions removes the non-stationarity of the environment from the perspective of each agent since the other agents vary their policy as well. This may only be partially true. We believe that their formulated update will in general lead to a more stable algorithm, though it does not remove the non-stationarity of the environment from the perspective of each agent. However, it does smooth out the non-stationarity. Intuitively, it takes longer for the behaviour of an agent to manifest in the replay memory (in the form of samples) compared to directly conditioning on the behaviour of an agent using its current policy.

Asymptotically, this is not a problem as our convergence analysis shows. In practice, however, learning will stop after a finite horizon. From the perspective of some agent $i$ during training, the other agents might have initially behaved in a certain way, which was then well represented in the replay memory. During later stages of learning, the other agents might “quickly” converge to completely different behaviour, e.g. due to the use of constant learning rates as it is typically done in practice. At first, agent $i$ then still uses mostly outdated samples that result into a sampled gradient that is significantly biased towards old behaviour. This will not reflect the current policies of the other agents. If training is then stopped too early, the resulting policy might perform very badly. This would be especially undesirable in cooperative environments. Based on this line of argument, we recommend to decay the learning rates if such behavior is observed since then, after some time, the policies of the agents cannot change as much any more and the effect of outdated samples gets reduced over time.
Extension to a fully decentralized setting

The MAAC iteration (1) assumes the paradigm of centralized training with decentralized execution, i.e. the agents train with global information but can be deployed solely based on local information. However, this assumption is often not satisfied. For example in robotics scenarios, states and actions are inherently local information. In those scenarios, the local data \((s^i_n, a^i_n, \phi^i_n)\) is therefore not available globally. To run the MAAC algorithm in such online, fully decentralized settings, we propose an extension that relies on data communicated via a (in this case, wireless) network.

For simplicity, let's suppose that the agents use separate protocols to exchange their state and action pairs \((s^i_n, a^i_n)\) via the wireless network. Usually, the dimension of the state and action spaces will be significantly smaller compared to the dimension of the neural network parameters \(\phi^i_n\), since the number of neurons and layers will be increased as the complexity of the problem grows. It is therefore fair to assume that all pairs \((s^j_n, a^j_n)\) will at some point reach agent \(i\), though with potentially large and varying information delay. As long as all samples reach the other agents at some point, the limiting distribution that describes the limit of the training process will be based on the same encountered state action pairs. This is because all encountered state action pairs have the same probability of being sampled from the experience replay, though potentially in a different order.

Let us now suppose that the agents use a separate protocol to exchange the local policies \(\phi^i_n\). Therefore, we consider that agent \(i\) has only access to \(\phi^j_{n - \tau_{ij}(n)}\) for every agent \(j \neq i\) at every time step \(n\). Here, \(\phi^j_{n - \tau_{ij}(n)}\) denotes the latest available policy parametrization from agent \(j\) at agent \(i\) at time \(n\) and we refer to \(\tau_{ij}(n)\) as the associated information delay. For every agent \(i\), define the global policy associated with the delayed information at time \(n\) by

\[
\pi_{\tau_{i}(n)}(s) := (\pi_1(s^1; \phi^1_{n - \tau_{1i}(n)}), \ldots, \pi_D(s^D; \phi^D_{n - \tau_{Di}(n)}))
\] (20)

We can now rewrite iteration (1) using the redefined delayed global policies:

\[
\theta_{n+1}^i = \theta_n^i + \alpha(n) \frac{1}{M} \sum_m \nabla_{\theta^i} Q_i(s_m, a_m; \theta_n^i) \left( r_i(s_m, a_m) + \gamma Q_i(s_{m+1}, \pi_{\tau_{i}(n)}(s_{m+1}); \theta_n^i) - Q_i(s_m, a_m; \phi_n^i) \right)
\]

\[
\phi_{n+1}^i = \phi_n^i + \beta(n) \frac{1}{M} \sum_m \nabla_{\phi^i} \pi_i(s^i_m; \phi_n^i) \times \nabla_{a^i} Q_i(s_m, \pi_{\tau_{i}(n)}(s_m); \theta_n^i),
\] (21)

Let us now make the following assumptions for the wireless network and delay variables \(\tau_{ij}(n)\).

(A6) (a) The wireless network satisfies a block-independence property. That is, there is \(T_1 \in \mathbb{N}\) such that the success or failure events of transmissions that are separated by at least \(T_1\) steps are independent.

(b) There exists a number \(T_2 \in \mathbb{N}\) and a non-negative, integer-valued random variable \(\tau\) that stochastically dominates all \(\tau_{ij}(n)\) with \(\mathbb{P}(\tau > t) < 1\) for all \(t \geq T_2\).
We refer the reader to [Redder et al. 2021] for a discussion of (A6) and alternative network assumptions. In general, practical communication networks that do not satisfy (A6)(b) are artificial. The strength of (A6) is that it allows us to construct a random variable with finite moments that stochastically dominates all delay variables $\tau_{ij}(n)$. For the subsequent theorem, we merely use that second moment is bounded to prove that the error between iteration (1) and (21) vanishes asymptotically. However, the following lemma may also be used to conclude that this error decays at a required rate after a specific point in time.

**Lemma 4.** Under (A6), there exists a non-negative integer-valued random variable $\tau$ that stochastically dominates all $\tau_{ij}(n)$ with $\mathbb{E}[\tau^2] < \infty$ for all $d > 1$.

**Proof.** (A6)(b) implies that the success probabilities of the communication events associated with the processes $\tau_{ij}(n)$ are uniformly lower-bounded across time. Specifically, the probability of information exchange between any pair $i, j$ during any interval $[n, n + T_2]$ is positive with a uniform lower bound $p > 0$. Using the block-independence property (A6)(a), we can now show that the delay variables are stochastically dominated by a random variable $\tau$ whose complementary CDF is the complementary CDF of a geometrically distributed random variable up to constant factor. As all moments exist for geometrically distributed random variables, the lemma follows.

We can now formulate the following theorem for the decentralized MAAC iteration that locally uses the outdated policy information of other agents. We have therefore extended the MAAC algorithm of Lowe et al. (2017) to fully decentralized training and gave sufficient conditions for its convergence.

**Theorem 2.** Under (A1)-(A6), the decentralized MAAC iteration (21) prone to information delays has the same convergence properties as the central iteration (1).

**Proof.** Using Lemma 4 for $d = 2$ it follows from the Borel-Cantelli Lemma that there exists a sample path-dependent constant $L \in \mathbb{N}$ such that $\tau_{ij}(n) \leq \sqrt{n}$ for all $n \geq L$ and for all $i, j$. We can now use similar arguments as in [Redder et al. 2021], where the effect of age of information has been considered in an offline optimization setting, to show that the gradient errors when comparing the versions of the algorithm with and without information delays vanishes asymptotically. For this, we use the locally Lipschitzness of $\nabla l$ and $\nabla g$ due to Lemma 1 as well as (A1) and (A2). The theorem therefore follows as a corollary to Theorem 1.

### 6 Conclusions

In this paper, we presented an asymptotic analysis of the Multi-Agent DDPG algorithm by Lowe et al. (2017). We build on the DQN analysis paper [Ramaswamy and Hüllermeier 2021], which only considers a single learning iteration. A key contribution is to extend the procedure to a setting with multiple coupled iterations with different step sizes. We achieve this using the observation that one can append the ODEs associated with each local DDPG iteration and analyse them using our novel step-size choice.
As an important by product we present for the first time a set of practical sufficient conditions for the convergence of the single agent DDPG algorithm for continuous action spaces. Furthermore, we extended the MADDPG algorithm to a fully decentralized setting under mild assumptions on a wireless communication network. This extends the offline analysis done in [Redder et al. (2021)] to the online setting of MA-RL. The impact of communication losses on the data in experience replays has so far not been considered. This is on our agenda for future work.

A Proof of Lemma 2

We define $[t]$ for $t \geq 0$ as $[t] := t_{\sup\{n|t_n \leq t\}}$. As the first step we can show that:

$$\sup_{t \in [0,T]} \| \overline{\phi}(t_{n+l}) - \overline{\phi}([t_n + t]) \| \in \Theta(\beta(n))$$ (22)

$$\sup_{t \in [0,T]} \| \phi_n(t) - \phi_n([t_n + t] - t_n) \| \in \Theta(\beta(n))$$ (23)

To prove the lemma, we need to show that

$$\sup_{t \in [0,T]} \| \overline{\phi}([t_n + t]) - \phi_n([t_n + t] - t_n) \| \to 0.$$ (24)

We have $[t_n + t] = t'_{n+l}$ for some $l \geq 0$. Using this, we have

$$\| \overline{\phi}(t_{n+l}) - \phi_n(t_{n+l} - t_n) \| \leq \| \overline{\phi}(t_{n+l}) - \overline{\phi}(t_n) \| - \int_{t_n}^{t_{n+l} - t_n} \nabla g_i(\theta_m, \phi_m, s_m) dx$$ (25)

The first term can be rewritten using a telescoping series:

$$\overline{\phi}(t_{n+l}) - \overline{\phi}(t_n) = \sum_{m=n}^{n+l-1} \phi_m - \phi_{m+1}$$

$$= \sum_{m=n}^{k_n+l-1} \beta(m) \nabla g_i(\theta_m, \phi_m, s_m)$$ (26)

$$= \sum_{m=n}^{n+l-1} \int_{t_m}^{t_{m+1}} \frac{\beta(m)}{\alpha(m)} \nabla g_i(\theta_m, \phi_m, s_m, x) dx$$

The last step follows from $\alpha(m) = t_{m+1} - t_m$ and using that $\phi_m - \overline{\phi}(t_m) = \overline{\phi}([t])$ for all $t \in [t_m, t_{m+1})$. Now we rewrite the second term in (25) as follows:

$$\int_{0}^{t_{n+l} - t_n} \nabla g_i(\theta_n(x), \phi_n(x), \mu^*_n(x)) dx =$$

$$\sum_{m=n}^{n+l-1} \int_{t_m}^{t_{m+1}} \nabla g_i(\theta_n(x - t_n), \phi_n(x - t_n), \mu^*_n(x - t_n)) dx$$ (27)
We now evaluate the difference of the terms under the integrals in (26) and (27). First, we start by adding and subtracting \( \phi_i(n) ([x] - t_n) \)

\[
\left\| \frac{\beta(m)}{\alpha(m)} \nabla g_i(\bar{\theta}([x]), \bar{\phi}([x]), \mu_n(x-t_n)) \right\|
\]

\[
- \nabla g_i(\theta_i^n(x-t_n), \phi_n(x-t_n), \mu_n(x-t_n)) \right\| \leq \frac{\beta(m)}{\alpha(m)} L \times (\left\| \bar{\theta}([x]) - \bar{\theta}_n([x] - t_n) \right\| + \left\| \bar{\phi}([x]) - \bar{\phi}_n([x] - t_n) \right\|)
\]

\[
+ \frac{\beta(m)}{\alpha(m)} \nabla g_i(\theta_i^n([x] - t_n), \phi_i^n([x] - t_n), \mu_n(x-t_n)) \right\|
\]

\[
- \nabla g_i(\theta_i^n(x-t_n), \phi_n(x-t_n), \mu_n(x-t_n)) dx \right\|
\]

Finally, the last term can be upper bounded by

\[
C \left| \frac{\beta(m)}{\alpha(m)} - 1 \right| + L \left( \left\| \theta_i^n(x-t_n) - \theta_i^n([x] - t_n) \right\| + \left\| \phi_n(x-t_n) - \phi_n([x] - t_n) \right\| \right) \leq \sum_{m=n}^{n+1} \beta(m) L \times (\left\| \theta_i^n(t_m) - \theta_i^n(t_m - t_n) \right\| + \left\| \phi_n(t_m) - \phi_n(t_m - t_n) \right\|)
\]

\[
+ C \sum_{m=n}^{n+1} \alpha(m) \left| \frac{\beta(m)}{\alpha(m)} - 1 \right| + L \Theta(\alpha(m) \beta(m)),
\]

The last term in the above expression goes to zero as \( n \to \infty \) under (A1). Finally, we can sum up all L.H.S and R.H.S. in (30) and apply the discrete version of Gronwall’s Lemma.

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