ACC FOR FOLIATED LOG CANONICAL THRESHOLDS

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Abstract. It is known that the set of log canonical thresholds (lcts) on any varieties with fixed dimension satisfies the ascending chain condition. Inspired by the foliated minimal model program, it is intriguing to study the foliated version of lcts and ask whether they have the similar property. We give an affirmative answer in the case of surfaces and threefolds.

Introduction

Singularities play an important role in studying higher dimensional birational geometry. In order to measure the singularities, we would like to find some invariants which behave well under birational maps. For example, the minimal log discrepancies (mlds for short) arise naturally in the study of birational geometry. In [Sho04], Shokurov showed that two following conjectures on minimal log discrepancies implies the termination of flips, which is one of main open problems in the minimal model program.

Conjecture 0.1 ([Sho88, Problem 5]). Fix a positive integer $n$ and a subset $I \subset [0,1]$ satisfying the descending chain condition. Then the set

$$MLD_n(I) := \{ \text{mld}(x,X,B) \mid (X \ni x, B) \text{ is log canonical, dim } X = n, B \in I \}$$

satisfies the ascending chain condition. Here mld$(x,X,B)$ is the minimal log discrepancy of the pair $(X,B)$ with center at $x$, and $B \in I$ means that the coefficients of $B$ belong to $I$.

Conjecture 0.2 ([Amb99, Conjecture 2.4]). Let $X$ be a quasi-projective variety. Then the function $a : |X| \to \mathbb{Q}$, which sends a closed point $x \in X$ to the minimal log discrepancy of $(X,B)$ with center at $x$, is lower semi-continuous.

Conjecture 0.1 is known to hold for surface pairs ([Ale93]), toric varieties ([Bor97]), toric pairs ([Amb96]), exceptional singularities ([HLS19]), and recently terminal threefolds ([HL22]); while conjecture 0.2 holds for $n \leq 3$ ([Ale93] and [Amb99]) and $X$ is toric ([Amb99]).

Another interesting and important invariant is the log canonical threshold, which plays a vital role in the inductive approach for higher dimensional geometry. The set of log canonical thresholds also satisfies the ascending chain condition by the following theorem.

Theorem 0.3 ([HMX14, Theorem 1.1]). Fix $n \in \mathbb{N}$, $I \subset [0,1]$, and a subset $J$ of positive real numbers. If $I$ and $J$ both satisfy the descending chain condition (DCC), then the set

$$\text{LCT}_n(I,J) = \{ \text{lct}(X,\Delta;M) \mid \text{dim } X = n, \Delta \in I, \text{ and } M \in J \}$$

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satisfies the ascending chain condition (ACC).

It was originally conjectured in all dimensions and proved for \( n = 2 \) by Shokurov in [Sho92]. And then Alexeev proved it for \( n = 3 \) in [Ale93]. Later, it was proved for smooth varieties in [dFEM10] and for full generality in [HMX14].

Inspired by the work on the minimal model program on foliated surfaces and threefolds (see, for example, [Bru15], [McQ08], [Spi20], [CS21], [?], [CS20], and the references therein), we are interested in the foliated version of the theorem above. Precisely, we define the foliated log canonical threshold as follows.

**Definition 0.4.** Let \((\mathcal{F}, \Delta)\) be a log canonical pair on a variety \(X\).

1. Given an \(\mathbb{R}\)-Cartier divisor \(M \geq 0\), we define the log canonical threshold of \(M\) with respect to \((\mathcal{F}, \Delta)\) to be

\[
lct(\mathcal{F}, \Delta; M) := \sup \{ t \in \mathbb{R} | (\mathcal{F}, \Delta + tM) \text{ is log canonical} \}.
\]

(See Definition 5.5 for log canonicity.)

2. Let \(I \subset [0, 1]\) and \(J \subset \mathbb{R}_{>0}\). We put \(\text{LCT}_{n,r}(I, J)\) to be the set of all real numbers \(lct(\mathcal{F}, \Delta; M)\) where \((\mathcal{F}, \Delta)\) is a log canonical pair such that \(\mathcal{F}\) is a foliation of rank \(r\) on a variety \(X\) of dimension \(n\) and the coefficients of \(\Delta\) (resp. \(M\)) belong to \(I\) (resp. \(J\)).

In this paper, we show the following theorem.

**Theorem 0.5.** Fix \(I \subset [0, 1]\), a subset \(J\) of positive real numbers, and two positive integers \(r\) and \(n\) with \(r < n\). If \(I\) and \(J\) both satisfy the descending chain condition, then the set \(\text{LCT}_{n,r}(I, J)\) satisfies the ascending chain condition provided that \(n \leq 3\).

With some reformulations as in [HMX14], we also show the following theorem.

**Theorem 0.6.** Fix two positive integers \(r\) and \(n\) with \(r < n \leq 3\) and a set \(I \subset [0, 1]\), which satisfies the descending chain condition. Then there exists a finite subset \(I_0 \subset I\) with the following property: If \((X, \mathcal{F}, D)\) is a triple such that

1. \(X\) is a variety of dimension \(n\),
2. \(\mathcal{F}\) is a foliation of rank \(r\),
3. \((\mathcal{F}, D)\) is log canonical,
4. the coefficients of \(D\) belong to \(I\), and
5. there is a log canonical center \(Z\) which is contained in every component of \(D\),

then the coefficients of \(D\) belong to \(I_0\).

The proof has two steps. We first extract the log canonical center and then apply adjunction to the foliated triple restricting onto some irreducible extracted divisor. The main difficulty is that, in general, there is little control on the singularities when applying adjunction. However, the model that extracts the log canonical center has mild singularities, which gives us more control on singularities. Note that both steps rely on the foliated minimal model program and the resolution of foliation singularities, which are widely open when the dimension is at least 4.

The paper is organized as follows. In section 1, we fix some notations and recall some facts. In section 2, we show Theorem 0.6 in the surface case. In the section 3 and 4, Theorem 0.6 for threefolds is proved in the cases when the rank is 2 and 1, respectively. The last section is to show the set of log canonical thresholds satisfies the ascending chain condition (Theorem 0.5) from our finiteness theorem 0.6.
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1. Preliminaries

We will work over \( \mathbb{C} \).

1.1. Foliations.

Definition 1.1. A foliation \( \mathcal{F} \) on a normal variety \( X \) is a coherent subsheaf \( \mathcal{F} \) of the tangent sheaf \( T_X \) such that

1. it is closed under the Lie bracket and
2. \( \mathcal{F} \) is saturated, that is, the quotient \( T_X/\mathcal{F} \) is torsion free.

The rank of \( \mathcal{F} \) is its rank as a sheaf. The co-rank of \( \mathcal{F} \) is defined as \( \dim X - \operatorname{rank} \mathcal{F} \).

We define the canonical divisor \( K_\mathcal{F} \) for the foliation \( \mathcal{F} \) as \( \mathcal{O}_X(-K_\mathcal{F}) = \operatorname{det}(\mathcal{F}) \).

Let \( \mathcal{F} \) be a foliation of rank \( r \) on a normal variety \( X \). We have a morphism \( \Omega_X^{[r]} \to \mathcal{F}^* \). Taking the double dual of \( r \)-th wedge product, we get a morphism

\[
\varphi : \Omega_X^{[r]} \to \mathcal{O}_X(K_\mathcal{F}),
\]

which yields a map

\[
\phi : \Omega_X^{[r]} \otimes \mathcal{O}_X(-K_\mathcal{F}) \to \mathcal{O}_X
\]

and we define the singular locus of \( \mathcal{F} \) as the co-support of the image of \( \phi \).

Definition 1.2 (Rational transform of foliations). Let \( \mathcal{F} \) be a foliation on a normal variety \( X \).

1. Given \( f : Y \to X \) be a dominant rational map. Let \( U \) be an open subset of \( X \) such that \( f|_V : V \to U \) is an isomorphism where \( V := f^{-1}(U) \). Note that \( \mathcal{F}|_U \subset T_U \cong T_V \). By [Har77, Exercise II.5.15], there is a coherent subsheaf \( \mathcal{G} \) of \( T_Y \) such that \( \mathcal{G}|_V = \mathcal{F}|_U \subset T_V \). Then the pullback foliation \( f^* \mathcal{F} \) is defined to be the saturation of \( \mathcal{G} \). Note that \( \mathcal{G} \) is closed under the Lie bracket and, by [HL21, Lemma 1.8], this definition is well-defined.

2. Given \( g : X \to Z \) be a birational map, then we define the pushforward foliation \( g_* \mathcal{F} \) as \( (g^{-1})^* \mathcal{F} \).

Definition 1.3. Let \( \mathcal{F} \) be a foliation on a normal variety \( X \). Given a subvariety \( W \) of \( X \).

1. \( W \) is tangent to \( \mathcal{F} \) if \( T_W|_U \to T_X|_U \) factors through \( \mathcal{F}|_U \). Otherwise we say \( W \) is transverse to \( \mathcal{F} \).

2. \( W \) is invariant if \( \mathcal{F}|_U \to T_X|_U \) factors through \( T_W|_U \).

Definition 1.4 (Foliated pair). A foliated pair \( (\mathcal{F}, \Delta) \) on a normal variety \( X \) consists of a foliation \( \mathcal{F} \) and an effective \( \mathbb{Q} \)-divisor \( \Delta \) on \( X \) such that \( K_\mathcal{F} + \Delta \) is \( \mathbb{Q} \)-Cartier.

Definition 1.5 ([McQ08, Definition I.1.5]). Let \( (\mathcal{F}, \Delta) \) be a foliated pair on a normal variety \( X \) and \( f : Y \to X \) be a proper birational morphism. For any
divisor $E$ on $Y$, we define the discrepancy of $(\mathcal{F}, \Delta)$ along $E$ to be $a(E, \mathcal{F}, \Delta) = \text{ord}_E(K_\mathcal{F} - f^* (K_\Delta + \Delta))$.

We say $(X, \mathcal{F}, \Delta)$ is terminal (resp. canonical) if $a(E, \mathcal{F}, \Delta) > 0$ (resp. $\geq 0$) for every exceptional divisor $E$ over $X$ and $(X, \mathcal{F}, \Delta)$ is log terminal (resp. log canonical) if $a(E, \mathcal{F}, \Delta) > -\varepsilon(E)$ (resp. $\geq -\varepsilon(E)$) for every divisor $E$ over $X$ where $\varepsilon(E)$ is defined to be $0$ if $E$ is $f^* \mathcal{F}$-invariant, and $1$ otherwise.

**Definition 1.6.** Given a foliated pair $(\mathcal{F}, \Delta)$ on $X$. A subvariety $Z$ of $X$ is called a log canonical center for $(\mathcal{F}, \Delta)$ if $(\mathcal{F}, \Delta)$ is log canonical at the generic point of $Z$ and there is a divisor $E$ over $X$ with discrepancy $a(E, \mathcal{F}, \Delta) = -\varepsilon(E)$ and with center $Z$ on $X$.

**Theorem 1.7** ([Miy87], [SB92] Theorem 9.0.2, or [BM16]). Let $(X, \mathcal{F})$ be a normal foliated variety of dimension $n$ and $H_1, \ldots, H_{n-1}$, and $A$ be ample divisors. Let $C$ be a general intersection of element $D_i \in |m_i H_i|$ for some $m_i \gg 0$. If $K_\mathcal{F} \cdot C < 0$, then through a general point of $C$ there is a rational curve $\Sigma$ tangent to $\mathcal{F}$ such that

$$A \cdot \Sigma \leq 2n \frac{A \cdot C}{-K_\mathcal{F} \cdot C}.$$

1.2. **Indices on foliated surfaces.** Most definitions in this subsection follow from [Bru15] with some generalizations.

Let $p \in \text{Sing}(\mathcal{F}) \setminus \text{Sing}(X)$. That is, $p$ is a smooth point on $X$ but a singular point of the foliation. Let $v$ be the vector field around $p$ generating $\mathcal{F}$. Since $p \in \text{Sing}(\mathcal{F})$, we have $v(p) = 0$. Then we can consider the eigenvalues $\lambda_1, \lambda_2$ of $(Dv)_p$, which do not depend on the choice of $v$.

**Definition 1.8.** If one of the eigenvalues is non-zero, say $\lambda_2$, then we say $p$ is semi-reduced and define the eigenvalue of the foliation $\mathcal{F}$ at $p$ to be

$$\lambda := \frac{\lambda_1}{\lambda_2}.$$

For $\lambda \neq 0$, this definition is unique up to reciprocal $\lambda \sim \frac{1}{\lambda}$.

If $\lambda = 0$, then $p$ is called a saddle-node; otherwise, we say $p$ is non-degenerate. If $\lambda \notin \mathbb{Q}^+$, then $p$ is called a reduced singularity of $\mathcal{F}$.

Reduced singularities arise naturally. Indeed, blowing up a smooth foliation point will introduce a reduced singularity with $\lambda = -1$.

1.2.1. **Non-invariant curves.** We first consider the non-invariant curves and define the tangency order for them.

**Definition 1.9.** Let $(X, \mathcal{F})$ be a foliated surface and $C$ be a non-invariant reduced curve. Let $p \in C \setminus \text{Sing}(X)$ and $v$ be the vector field generating $\mathcal{F}$ around $p$. Let $f$ be the local defining function of $C$ at $p$. We define the tangency order of $\mathcal{F}$ along $C$ at $p$ to be

$$\text{tang}(\mathcal{F}, C, p) := \dim \frac{\mathcal{O}_{X,p}}{(f, v(f))}.$$  

Note that $\text{tang}(\mathcal{F}, C, p) \geq 0$ and is independent of the choices of $v$ and $f$. Moreover, if $\mathcal{F}$ is transverse to $C$ at $p$, then $\text{tang}(\mathcal{F}, C, p) = 0$. Therefore, if $C$ is compact, then we can define

$$\text{tang}(\mathcal{F}, C) := \sum_{p \in C} \text{tang}(\mathcal{F}, C, p).$$
Theorem 1.10 ([Adjunction for non-invariant divisors] [Bru97, Bru15, Proposition 2.2]). Let $\mathcal{F}$ be a foliation on a smooth projective surface $X$. Let $C$ be a non-invariant irreducible curve on $X$ and $C'$ be the normalization of $C$. Then there is an effective divisor $\Delta$ on $C'$ such that $(K_{\mathcal{F}} + C)|_{C'} = \Delta$ with $\deg \Delta = \operatorname{tang}(\mathcal{F}, C)$.

Corollary 1.11. Let $C$ be a non-invariant curve on a foliated surface $(X, \mathcal{F})$. If $C$ is contained in the smooth locus of $X$, then we have $(K_{\mathcal{F}} + C) \cdot C \geq 0$.

1.2.2. Invariant curves. Now we study the invariant curves.

Definition 1.12. Let $(X, \mathcal{F})$ be a foliated surface and $C$ be an invariant curve. Let $p \in C \setminus \operatorname{Sing}(X)$ and $\omega$ be a 1-form generating $\mathcal{F}$ around $p$. If $C$ is an invariant curve and $f$ is the local defining function of $C$ at $p$, then we can write
\[ g\omega = hdf + f\eta \]
where $g$, $h$ are holomorphic functions, $\eta$ is a holomorphic 1-form, and $h$, $f$ are relatively prime functions.

We define the index $Z(\mathcal{F}, C, p)$ to be the vanishing order of $\frac{h}{g}|_C$ at $p$. This definition is independent of the choices of $f$, $g$, $h$, $\omega$, and $\eta$. (For a reference, see [Bru15, page 15 in Chapter 2 and page 27 in Chapter 3].)

Note that if $p \notin \operatorname{Sing}(\mathcal{F})$, then $Z(\mathcal{F}, C, p) = 0 = \operatorname{CS}(\mathcal{F}, C, p)$. Therefore, if $C$ is compact, then we can define
\[ Z(\mathcal{F}, C) := \sum_{p \in C} Z(\mathcal{F}, C, p) \]
where the sums are taken over only finitely many points.

Theorem 1.13 ([Adjunction for invariant divisors] [Bru97, Bru15, Proposition 2.3]). Let $\mathcal{F}$ be a foliation on a smooth projective surface $X$. Let $C$ be an invariant irreducible curve on $X$ and $C'$ be its normalization. Then there is an effective divisor $\Delta$ on $C'$ such that $K_{\mathcal{F}}|_{C'} = K_{C'} + \Delta$ with $\deg \Delta = Z(\mathcal{F}, C) + \deg \operatorname{Diff}_{C}(0)$ where $\operatorname{Diff}_{C}(0)$ is the different with $(K_{X} + C)|_{C'} = K_{C'} + \operatorname{Diff}_{C}(0)$. In particular, we have $K_{\mathcal{F}} \cdot C = Z(\mathcal{F}, C) + 2p_a(C) - 2$ where $p_a(C)$ is the arithmetic genus of $C$.

Lemma 1.14. Given a foliated surface $(X, \mathcal{F})$. Suppose there is a morphism $f : X \to C$ where $C$ is a curve. Let $F$ be the general fiber of $f$.

1. If $K_{\mathcal{F}} \cdot F < 0$, then $F$ is a smooth invariant rational curve with $K_{\mathcal{F}} \cdot F = -2$.
2. If $K_{X} \cdot F < 0$, then $F$ is a smooth rational curve with $K_{X} \cdot F = -2$.

Proof. Since $F$ is a general fiber, we may assume that $F$ has no singularity of $X$ and $\mathcal{F}$.

1. If $F$ is not invariant, then by adjunction for non-invariant divisors (Theorem 1.10), we have
\[ 0 > K_{\mathcal{F}} \cdot F = (K_{\mathcal{F}} + F) \cdot F = \operatorname{tang}(\mathcal{F}, F) \geq 0, \]
which is impossible. Thus, $F$ is invariant. Now by adjunction for invariant divisors (Theorem 1.13), we have
\[ 0 > K_{\mathcal{F}} \cdot F = Z(\mathcal{F}, F) + 2p_a(F) - 2 = 2p_a(F) - 2. \]
Therefore, $p_a(F) = 0$ and $K_{\mathcal{F}} \cdot F = -2$. 

(2) By adjunction, we have
\[ 0 > K_X \cdot F = (K_X + F) \cdot F = 2p_a(F) - 2. \]
Therefore, \( p_a(F) = 0 \) and \( K_X \cdot F = -2 \).

\[ \square \]

1.3. **DCC sets.** Suppose \( I \subset [0,1] \). We define
\[ I_+ := \{0\} \cup \left\{ j : j = \sum_{p=1}^{\ell} i_p \text{ for some } i_1, \ldots, i_\ell \in I \right\} \]
and
\[ D(I) := \left\{ a \leq 1 : a = \frac{m - 1 + f}{m}, m \in \mathbb{N}, f \in I_+ \right\}. \]

We say a set \( I \) of real numbers satisfies the **descending chain condition** (DCC) if it does not contain any infinite strictly decreasing sequence. Also, if \( I \subset [0,1] \) satisfies the DCC, then the minimum among \( I \setminus \{0\} \) exists, which is denoted by \( I_{\text{min}} \).

**Proposition 1.15.** We have the following properties:

1. If \( I \subset [0,1] \) satisfies the DCC, then so do \( I_+ \) and \( D(I) \).
2. \( D(D(I)) = D(I) \).

**Proof.** This is straightforward. For a reference, see [MP04, Lemma 4.4]. \( \square \)

**Lemma 1.16.** Fix a DCC set \( I \) and a positive real number \( \alpha \). Then there is a finite subset \( I_0 \subset I \) depending only on \( I \) and \( \alpha \) such that if \( \sum_{j=1}^{\ell} k_j i_j = \alpha \) for some \( k_j \in \mathbb{N} \), \( i_j \in I \setminus \{0\} \), and \( \ell \in \mathbb{N} \), then all \( i_j \in I_0 \).

**Proof.** Note that
\[ \alpha = \sum_{j=1}^{\ell} k_j i_j \geq \sum_{j=1}^{\ell} k_j I_{\text{min}}. \]
Thus the sum \( \sum_{j=1}^{\ell} k_j \) is bounded, and so is the length \( \ell \). Without loss of generality, we may fix \( \ell \) and \( k_1, k_2, \ldots, k_\ell \). Now we write
\[ \alpha - k_1 i_1 = \sum_{j=2}^{\ell} k_j i_j \]
where the left hand side belongs to a ACC set while the right hand side belongs to a DCC set. The only set which satisfies both the ACC and the DCC is the finite set. Thus we have only finitely many possible \( i_1 \). Similarly, there are only finitely many possible \( i_j \)'s. Hence, there is a finite subset \( I_0 \subset I \) such that all \( i_j \)'s belong to \( I_0 \) provided that \( \sum_{j=1}^{\ell} k_j i_j = \alpha \). \( \square \)

2. **Foliations of rank one on surfaces**

In this section, we show Theorem 2.4, which is Theorem 0.6 when \( r = 1 \) and \( n = 2 \). We first recall the following lemma.
Lemma 2.1. Let \( C = \bigcup C_i \) be a set of proper curves on a smooth surface. Assume that the intersection matrix \((C_i \cdot C_j)_{i,j}\) is negative definite. Let \( A = \sum a_i C_i \) be an \( \mathbb{R} \)-linear combination of the curves \( C_i \)'s. If \( A \cdot C_j > 0 \) for all \( j \), then \( a_i \leq 0 \) for all \( i \). If, moreover, \( C \) is connected, then either \( a_i = 0 \) for all \( i \) or \( a_i < 0 \) for all \( i \).

In particular, if \( C \) is connected and \( A \cdot C_j > 0 \) for some \( j \), then \( a_i < 0 \) for all \( i \).

Proof. For a reference, see [KM98, Lemma 3.41]. □

Lemma 2.2. Given a foliated pair \((\mathcal{F}, \Delta)\) of rank 1 of a surface. Suppose \((\mathcal{F}, \Delta)\) is log canonical at a point \( p \). If \( p \in \text{Supp}(\Delta) \), then \( \mathcal{F} \) is terminal at \( p \).

Proof. Since \((\mathcal{F}, \Delta)\) is log canonical at \( p \), we also have \( \mathcal{F} \) is log canonical at \( p \). Let \( E = \bigcup E_i \) be the exceptional divisor for the minimal resolution of \( \mathcal{F} \) at \( p \). Note that

\[(K_\mathcal{F} + \Delta) \cdot E_i \geq K_\mathcal{F} \cdot E_i\]

for all \( i \) and it is strict for some \( i \) because \( p \in \text{Supp}(\Delta) \). By Lemma 2.1, the (foliated) discrepancy \( a(E_i, \mathcal{F}, \Delta) \) is strictly less than \( a(E_i, \mathcal{F}) \) for all \( i \).

Suppose \( \mathcal{F} \) is not terminal at \( p \), then, by [Che21, Lemma 4.2], there is an irreducible exceptional divisor \( E_i \) such that \( a(E_i, \mathcal{F}) = -\varepsilon(E_i) \). But this implies that

\[a(E_i, \mathcal{F}, \Delta) < a(E_i, \mathcal{F}) = -\varepsilon(E_i)\]

which contradicts that \((\mathcal{F}, \Delta)\) is log canonical at \( p \). Therefore, \( \mathcal{F} \) is terminal at \( p \). □

In addition to the existence of a foliated dlt modification on surfaces given in [CS21, Theorem 8.1], the following theorem also gives a foliated dlt modification with more explicit properties. Note that for the case when \( p \notin \text{Supp}(\Delta) \), the foliated dlt modification around \( p \) follows from the classification in [Che21, Theorem 0.1].

Theorem 2.3. Let \((\mathcal{F}, \Delta, p)\) be a germ of foliated pair with \( p \in \text{Supp}(\Delta) \). Suppose \( p \) is a log canonical center of \((\mathcal{F}, \Delta)\). Then we have a foliated dlt modification \( \pi : (Y, \mathcal{F}, \Delta_Y) \to (X, \mathcal{F}, \Delta) \) around \( p \) such that the following hold:

1. The graph of the \( \pi \)-exceptional divisors \( E = \bigcup E_i \) is a chain with \( a(E_i, \mathcal{F}, \Delta) = 0 \).
2. \( K_{\mathcal{F}} + \Delta_Y = \pi^*(K_\mathcal{F} + \Delta) \) where \( \Delta_Y = \pi_*^{-1}\Delta \) is the proper transform of \( \Delta \).
3. There is only one non-reduced foliation singularity \( q \) on \( E \). Precisely, \( q \) is terminal for \( \mathcal{F} \). Moreover, \( q \) is only on the irreducible component \( E_1 \) which is one end of the chain \( E \).
4. \( \Delta_Y \) only meets \( E_1 \).

Proof. Let \( \pi : (Z, \mathcal{H}) \to (X, \mathcal{F}) \) be the minimal resolution of \((X, \mathcal{F})\) at \( p \). By Lemma 2.2 and [Che21, Theorem 0.1], \( p \) is a cyclic quotient singularity and the dual graph of \( \pi \)-exceptional divisors \( F = \bigcup F_i \) is an \( \mathcal{H} \)-chain with \( Z(\mathcal{H}, F_1) = 1 \). Note that

\[K_\mathcal{H} + \Delta_Z = \pi^*(K_\mathcal{F} + \Delta) + \sum a(F_i, \mathcal{F}, \Delta) F_i\]

where \( \Delta_Z = \pi_*^{-1}\Delta \) and \( a(F_i, \mathcal{F}, \Delta) \geq 0 \) for all \( i \) since \( F_i \)'s are all invariant.

Claim. If \( a(F_i, \mathcal{F}, \Delta) = 0 \) for some \( i \), then \( a(F_j, \mathcal{F}, \Delta) = 0 \) and \( \Delta_Z \cdot F_j = 0 \) for all \( j > i \).
Proof (Claim). If \( a(F_i, \mathcal{F}, \Delta) = 0 \) for some \( i_0 \), then we have \((K_{\mathcal{F}} + \Delta_Z) \cdot F_j \geq 0\) for all \( j > i_0 \). Applying Lemma \ref{lem:1.11} on the set of exceptional divisors \( F_j \)'s where \( j > i_0 \), we have \( a(F_j, \mathcal{F}, \Delta) \leq 0 \) and hence \( a(F_j, \mathcal{F}, \Delta) = 0 \) for all \( j > i_0 \). Since \( K_{\mathcal{F}} \cdot F_j = 0 \) for \( j > i_0 \), we have

\[
\Delta_Z \cdot F_j = (K_{\mathcal{F}} + \Delta_Z) \cdot F_j = \left( \sum_{i < i_0} a(F_i, \mathcal{F}, \Delta) F_i \right) \cdot F_j = 0
\]

for \( j > i \). This completes the proof of the Claim.

Suppose \( a(F_i, \mathcal{F}, \Delta) = 0 \) for some \( i \). Then let \( t \) be a positive integer such that \( a(F_i, \mathcal{F}, \Delta) > 0 \) for all \( i < t \) and \( a(F_i, \mathcal{F}, \Delta) = 0 \) for \( i \geq t \). Thus, contracting \( F_1, \ldots, F_{t-1} \) gives a desired foliated dlt modification.

Now suppose \( a(F_i, \mathcal{F}, \Delta) \neq 0 \) for all \( i \). Since \( p \) is a log canonical singularity, there is a higher model \( \phi : (W, L') \rightarrow (Z, \mathcal{H}) \) consisting of a sequence of blowups with \( \phi \)-exceptional divisor \( G = \cup_{j=1}^r G_j \). We may assume that \( G_r \) is the exceptional divisor for the last blowup and \( a(G_r, \mathcal{F}, \Delta) = 0 \) and \( a(G_j, \mathcal{F}, \Delta) > 0 \) for all \( j < r \). Note that \( G_r^2 = -1 \) and \( G_j^2 < -2 \) for \( j < r \). By [Che21, Lemma 4.1(d)], the dual graph of \( (\phi \circ \pi)-\)exceptional divisors is a chain. We relabel all exceptional divisors \( G_j \)'s and the proper transform of \( F_j \)'s as \( \tilde{E}_i \)'s so that \( \tilde{E}_i \cdot \tilde{E}_{i+1} = 1 \) for all \( i \). Let \( \tilde{E}_0 \) be the one such that \( \tilde{E}_0^2 = -1 \). Recall \( a(\tilde{E}_0, \mathcal{F}, \Delta) = 0 \). Using [Che21, Lemma 4.1(b)], we see that the function that maps \( i \) to \( a(\tilde{E}_i, \mathcal{F}, \Delta) \) is convex. Similar as above, contracting all \( \tilde{E}_i \)'s with \( a(\tilde{E}_i, \mathcal{F}, \Delta) > 0 \) gives a desired foliated dlt modification. \( \square \)

**Theorem 2.4.** Fix a set \( I \subset [0,1] \), which satisfies the descending chain condition. Then there exists a finite subset \( I_0 \subset I \) with the following property: If \((X, \mathcal{F}, D)\) is a triple such that

1. \( X \) is a variety of dimension 2,
2. \( \mathcal{F} \) is a foliation of rank 1,
3. \((\mathcal{F}, D)\) is log canonical,
4. the coefficients of \( D \) belong to \( I \), and
5. there is a log canonical center \( Z \) which is contained in every component of \( D \),

then the coefficients of \( D \) belong to \( I_0 \).

**Proof.** If \( Z \) is a divisor, then \( D = Z \) with coefficient one. Then we require \( 1 \in I_0 \).

If \( Z \) is a point, by Theorem 2.3 we have a foliated dlt modification

\[
\pi : (Y, \mathcal{G}, D_Y) \rightarrow (X, \mathcal{F}, D)
\]

with \( K_{\mathcal{G}} + D_Y = \pi^*(K_{\mathcal{F}} + D) \) where \( \mathcal{G} \) is the pullback foliation and \( D_Y \) is the proper transform of \( D \). Let \( E_1 \) be an irreducible component of the exceptional divisor which meets \( D_Y \). Then we have

\[
0 = \pi^*(K_{\mathcal{F}} + D) \cdot E_1 = (K_{\mathcal{G}} + D_Y) \cdot E_1 = \frac{-1}{m} + \sum_j \frac{k_j d_j}{m}
\]

where \( m \) is the index of the only singularity on \( E_1 \) and \( k_j \in \mathbb{N} \) for all \( j \). Here \( k_j \) is the multiple of \( m \) when the corresponding irreducible component does not pass through the only singularity on \( E_1 \). Thus we have \( \sum_j k_j d_j = 1 \). Since all \( d_j \)'s belong to the DCC set \( I \), by Lemma 1.16 we have a finite subset \( I_0 \subset I \) such that \( d_j \in I_0 \) for all \( j \). \( \square \)
Using some ideas in [Ale93, Theorem 5.3], we also show the similar result for the numerically trivial pairs.

**Theorem 2.5** (ACC for numerically trivial pairs). Fix a DCC set $I$. Then there is a finite subset $I_0 \subset I$ with the following property: If $(X, \mathcal{F}, B)$ is a triple such that

1. $X$ is a surface,
2. $\mathcal{F}$ is a foliation of rank one on $X$,
3. $(\mathcal{F}, B)$ is log canonical,
4. $K_F + B$ is numerically trivial, and
5. the coefficients of $B$ belong to $I$,

then all coefficients of $B$ belong to $I_0$.

**Proof.** Given a triple $(X, \mathcal{F}, B)$ satisfying (1)-(5). We write $B = \sum b_j B_j$ where $b_j \in I$ and $B_j$’s are irreducible components.

We may assume that $\mathcal{F}$ has either reduced singularities or terminal singularities by taking a foliated dlt modification. (See Theorem 2.3.) Note that $K_F \equiv -B$ is not pseudo-effective. We would like to run a $K_F$-MMP. Then, either $\rho(X) = 1$ or there is a $K_F$-negative extremal ray that does not contract $B_1$ since $B_1$ is not invariant.

Assume $X$ has Picard number 1. Since the general curves are not numerically trivial, we have $-K_F \cdot C = B \cdot C > 0$ for a general curve $C$. Thus $-K_F$ is ample because $\rho(X) = 1$. By Theorem 1.7 $X$ is uniruled. Hence $K_X$ is not pseudo-effective and therefore $-K_X$ is ample. Moreover, the $(-K_X)$-slope of $\mathcal{F}$ is

$$\mu(-K_X)(\mathcal{F}) = (-K_F) \cdot (-K_X) > 0.$$ 

By [CP19, Theorem 1.1] or [Ou17, Proposition 2.2], $\mathcal{F}$ is algebraically integrable, that is, general leaves are algebraic. Note $\mathcal{F}$ has only reduced or terminal singularities through which there are at most two invariant curves passing. So the algebraic integrability of $\mathcal{F}$ gives a fibration $\pi : X \to C$ onto a curve $C$ and the general fibers are the leaves of $\mathcal{F}$. This shows that $\rho(X) \geq 2$, which gives a contradiction.

Now we suppose $\rho(X) \geq 2$. Then we run a $K_F$-MMP to contract a $K_F$-negative extremal ray. If the contraction is birational, then we notice that the conditions (1)-(5) are preserved. Moreover, the property that the foliation has only finitely many invariant curves passing through each point is also preserved. Besides, each irreducible component $B_i$ of $B$ is not contracted since they are non-invariant. Thus we can repeat the process mentioned above. Since we have shown that Picard number cannot be 1, after finitely many times, the contraction gives a fibration onto a curve.

Let $F$ be a general fiber of the fibration. Since $F$ is invariant, by adjunction for invariant divisors (Theorem 1.13), we have $K_F \cdot F = -2$ and thus

$$2 = -K_F \cdot F = B \cdot F = \sum b_j (B_j \cdot F).$$

Note that all $B_j \cdot F > 0$ for all $j$ since $B_j$’s are non-invariant and $F$ is invariant. Therefore, by Lemma 1.16 there is a finite subset $I_0 \subset I$ such that $b_j \in I_0$ for all $j$. \hfill $\square$
3. Foliations of rank two on threefolds

In this section, we show Theorem 3.11, which is Theorem 0.6 when \( r = 2 \) and \( n = 3 \).

3.1. Foliated dlt modification. Most definitions in this subsection follow from [CS21].

**Definition 3.1.** We say \( z_1, \ldots, z_\ell \in \mathbb{C}^* \) satisfy the non-resonant condition if, for any non-negative integers \( a_1, \ldots, a_\ell \) with \( \sum_i a_i z_i = 0 \), we have \( a_i = 0 \) for all \( i = 1, \ldots, \ell \).

**Definition 3.2.** Let \( F \) be a co-rank one foliation on a smooth variety \( X \) of dimension \( n \). We say a point \( p \in X \) is a simple singularity for \( F \) if, in formal coordinates \( x_1, \ldots, x_n \) around \( p \), conormal sheaf \( N^*_F := (T_X/F)^* \) is generated by a 1-form which is in one of the following two forms, for some \( 1 \leq r \leq n \):

1. There are \( \lambda_1, \ldots, \lambda_r \in \mathbb{C}^* \), which satisfy the non-resonant condition, such that
   \[ \omega = x_1 \cdots x_r \cdot \sum_{i=1}^r \lambda_i \frac{dx_i}{x_i}. \]

2. There is an integer \( k \leq r \) such that
   \[ \omega = x_1 \cdots x_r \cdot \left( \sum_{i=1}^k p_i \frac{dx_i}{x_i} + \varphi(x_1^{p_1} \cdots x_k^{p_k}) \sum_{i=2}^r \lambda_i \frac{dx_i}{x_i} \right) \]
   where \( p_1, \ldots, p_k \) are positive integers without a common factor, \( \varphi(s) \) is a formal power series which is not a unit, and \( \lambda_2, \ldots, \lambda_r \in \mathbb{C}^* \) satisfy the non-resonant condition.

**Definition 3.3.** Given a foliated pair \((\mathcal{F}, \Delta)\) of co-rank one on a normal variety \( X \). We say \((\mathcal{F}, \Delta)\) is foliated log smooth if the following hold:

1. \((X, \Delta)\) is log smooth,
2. \( \mathcal{F} \) has simple singularities, and
3. if \( S \) is the support of the non-\( \mathcal{F} \)-invariant components of \( \Delta \), \( p \) is a closed point, and \( \Sigma_1, \ldots, \Sigma_k \) are (possibly formal) \( \mathcal{F} \)-invariant divisors passing through \( p \), then \( S \cup \bigcup_{i=1}^k \Sigma_i \) is a normal crossing divisor at \( p \).

**Proposition 3.4.** Let \( \mathcal{F} \) be a (co-)rank one foliation on a surface \( X \) and \( C \) be a non-invariant curve on \( X \). Suppose \((\mathcal{F}, C)\) is foliated log smooth, then \( (K_{\mathcal{F}} + C) \cdot C = 0 \).

**Proof.** Note that \( X \) is smooth since \((\mathcal{F}, C)\) is foliated log smooth. By adjunction for non-invariant divisors (Theorem 1.10), we have \( (K_{\mathcal{F}} + C) \cdot C = \text{tang}(\mathcal{F}, C) \geq 0 \). Thus it suffices to show that tangency order of \( \text{tang}(\mathcal{F}, C, p) = 0 \) for any \( p \in C \).

If \( p \) is a singularity for \( \mathcal{F} \), then there are two formal invariant divisors \( \Sigma_1 \) and \( \Sigma_2 \) passing through \( p \). Hence \( C \cup \Sigma_1 \cup \Sigma_2 \) is not a normal crossing divisor at \( p \), which contradicts the last requirement that \((\mathcal{F}, S)\) is foliated log smooth.

Therefore, \( p \) is a smooth point for \( \mathcal{F} \). Let \( \Sigma \) be the formal invariant divisor passing through \( p \). Since \( C \cup \Sigma \) is a normal crossing divisor at \( p \), we have \( \text{tang}(\mathcal{F}, C, p) = 0 \). □
Definition 3.5. Given a foliated pair \((\mathcal{F}, \Delta)\) of co-rank one on a normal variety \(X\). A foliated log resolution of \((\mathcal{F}, \Delta)\) is a birational morphism \(\pi: Y \rightarrow X\) such that

1. \(\text{Exc}(\pi)\) is a divisor and
2. \((\mathcal{G}, \pi_*^{-1}\Delta + E)\) is foliated log smooth where \(\mathcal{G}\) is the pullback foliation on \(Y\) and \(E\) is the sum over all \(\pi\)-exceptional divisors.

Remark 3.6. When \(X\) is a surface, the existence of foliated log resolution follows from Seidenberg’s result [Sei68]. For the threefolds \(X\), such a resolution exists by [Can04].

Definition 3.7. We say a foliated pair \((\mathcal{F}, \Delta)\) of co-rank one on a normal variety \(X\) is foliated dlt if

1. every irreducible component of \(\Delta\) is generically transverse to \(\mathcal{F}\) and has coefficient at most one, and
2. there is a foliated log resolution \(\pi: Y \rightarrow X\) of \((\mathcal{F}, \Delta)\) which only extracts divisor \(E\) of discrepancy \(-\varepsilon(E)\).

Definition 3.8. Let \((\mathcal{F}, \Delta)\) be a foliated pair of co-rank one on a normal projective variety \(X\). We call a birational projective morphism \(\pi: Y \rightarrow X\) is a foliated dlt modification if \((\mathcal{G}, \pi_*^{-1}\Delta + \sum \varepsilon(E_i)E_i)\) is foliated dlt and

\[K_{\mathcal{G}} + \pi_*^{-1}\Delta + \sum \varepsilon(E_i)E_i + F = \pi^*(K_{\mathcal{F}} + \Delta)\]

for some effective \(\pi\)-exceptional \(\mathbb{Q}\)-divisors \(F\) on \(Y\) where \(\mathcal{G}\) is the pullback foliation on \(Y\) and the sum is over all \(\pi\)-exceptional divisors.

Theorem 3.9 ([CS21, Theorem 8.1]). Let \((\mathcal{F}, \Delta)\) be a foliated pair of rank two on a normal threefold. Then \((\mathcal{F}, \Delta)\) admits a foliated dlt modification \(\pi: Y \rightarrow X\) such that if \(\mathcal{G}\) is the pullback foliation on \(Y\), then

1. \(Y\) is \(\mathbb{Q}\)-factorial,
2. \(Y\) has at worst klt singularities, and
3. \((\mathcal{G}, \Gamma := \pi_*^{-1}D + E_{\text{inv}})\) is foliated dlt with \(K_{\mathcal{G}} + \Gamma = \pi^*(K_{\mathcal{F}} + D)\) where \(E_{\text{inv}}\) is the sum over all \(\pi\)-exceptional non-invariant divisors.

Moreover, if \((\mathcal{F}, \Delta)\) is log canonical, then we may choose \(\pi: Y \rightarrow X\) such that any log canonical center for \((Y, \Gamma)\) is contained in its codimension one log canonical center.

3.2. Adjunction. The following Proposition slightly generalizes [CS21] Lemma 3.18 by providing more information on the foliated differents.

Proposition 3.10. Let \(\mathcal{F}\) be a co-rank 1 foliation on a normal threefold \(X\), \(S\) be a prime divisor, and \(I\) be a subset of \([0, 1]\). Suppose \((\mathcal{F}, \varepsilon(S)S + \Delta)\) is foliated dlt where \(\Delta\) is an effective \(\mathbb{Q}\)-divisor with coefficients belonging to the set \(I\). Assume that \(K_X, K_X + \Delta,\) and \(S\) are \(\mathbb{Q}\)-Cartier and \(\mathcal{F}\) is non-dicritical (see [CS21, Definition 2.10]).

Then there exists an effective \(\mathbb{Q}\)-divisor \(\Theta\) such that \((K_S + \varepsilon(S)S + \Delta)|_{S^\nu} = K_{\mathcal{G}} + \Theta\) where \(S^\nu\) is the normalization of \(S\) and \(\mathcal{G}\) is the restricted foliation to \(S^\nu\).

Then the following hold:

1. If \(\varepsilon(S) = 1\), then \((\mathcal{G}, \Theta)\) is log canonical, \(\Theta = \Delta|_{S^\nu}\), and the coefficients of \(\Theta\) belong to \(D(I)\).
(2) If \( \varepsilon(S) = 0 \), then \( (S', \Theta') = [\Theta]_\text{red} + \{\Theta\} \) is log canonical and the coefficients of \( \Theta \) belong to \( D(I) \cup \mathbb{N} \). Here \( \{\Theta\} = \Theta - [\Theta] \).

Moreover, any component \( C \) of \( \Theta - \Theta' \) is a log canonical center for \( (\mathcal{F}, \Delta) \) with \( (K_\mathcal{F} + \Delta)|_{C'} = K_{C'} + \Xi \) where the coefficients of \( \Xi \) belong to \( D(I) \cup \mathbb{N} \).

**Proof.** The existence of \( \Theta \) follows from [CS21, Lemma 3.18].

(1) When \( \varepsilon(S) = 1 \), we have \( (\mathcal{G}, \Theta) \) is log canonical by [CS21, Lemma 3.18]. To show other statements in this case, we may reduce to the case when \( X \) is a surface by cutting out by a general hyperplane. Since \( (\mathcal{F}, S) \) is log canonical, \( \mathcal{F} \) has at worst terminal singularities along \( S \) by [Che21, Lemma 5.2].

**Claim.** \( (K_\mathcal{F} + S) \cdot S = 0 \).

**Proof.** We consider \( \pi : Z \rightarrow X \) a foliated log resolution of \( X \) along \( S \) with the pullback foliation \( \mathcal{H} \) on \( Z \) and \( T = \pi^{-1}_*S \) the proper transform of \( S \).

We may write
\[
K_{\mathcal{H}} + T = \pi^*(K_\mathcal{F} + S) + E
\]
where \( E \) is \( \pi \)-exceptional. Note that each component of \( E \) is invariant and \( T \) is non-invariant. So we have \( E \cdot T \geq 0 \). By [Spi20, Proposition 3.4], we have \( (K_\mathcal{F} + S) \cdot S \geq 0 \). Besides, since \( (\mathcal{H}, T) \) is foliated log smooth, we have, by Proposition [CS21, Proposition 3.4], \( (K_{\mathcal{H}} + T) \cdot T = 0 \). Thus, we have
\[
0 = (K_\mathcal{F} + T) \cdot T = (K_\mathcal{F} + S) \cdot S + E \cdot T \geq 0
\]
and therefore \( (K_\mathcal{F} + S) \cdot S = 0 \).

If \( \mathcal{F} \) is smooth along \( S \), then we have \( \Theta = \Delta|_{S'} \) and the coefficients of \( \Theta \) belong to \( I_+ \). Since \( (\mathcal{G}, \Theta) \) is log canonical, the coefficients of \( \Theta \) are at most one, which shows that they belong to \( I_+ \cap [0, 1] \subset D(I) \).

In general, we consider \( \varphi : W \rightarrow X \) the minimal resolution at each terminal singularity of \( \mathcal{F} \) along \( S \) with the exceptional divisor \( F = \sum_i F_i \). We write
\[
K_\mathcal{F}_W + S_W = \varphi^*(K_\mathcal{F} + S) + \sum a_i F_i
\]
where \( \mathcal{F}_W \) is the pullback foliation on \( W \), \( S_W := \varphi^{-1}_*S \) is the proper transform of \( S \), and \( a_i \geq 0 \) since \( (\mathcal{F}, S) \) is log canonical and \( F_i \)'s are invariant.

**Claim.** \( (\mathcal{F}_W, S_W) \) is foliated log smooth along \( S_W \).

**Proof.** Suppose \( (\mathcal{F}_W, S_W) \) is not foliated log smooth along \( S_W \), that is, \( S_W \cup F \) is not a normal crossing divisor. Then, by a direct computation using Mumford’s intersection pairing, the classification of terminal foliation surface singularities (see [Che21, Theorem 0.1]), and the adjunction for invariant divisors (Theorem [1.13]), we have
\[
0 = (K_\mathcal{F} + S) \cdot S \geq (K_\mathcal{F}_W + S_W) \cdot S_W + \frac{1}{m} \geq 0 + \frac{1}{m}
\]
where \( m \geq 2 \) is the index of one of terminal singularities of \( \mathcal{F} \) and the last inequality follows from the adjunction for non-invariant divisors (Theorem [1.10]). This is impossible and thus \( (\mathcal{F}_W, S_W) \) is foliated log smooth along \( S_W \).
Hence, by Proposition 3.4, \[0 = (K_{\mathcal{F} W} + S_W) \cdot S_W = \sum a_i F_i \cdot S_W.\] Since \(F_i \cdot S_W > 0\) for some \(i\), we have \(a_i = 0\) for such \(i\). Therefore, \((\mathcal{F}, S)\) is canonical but not terminal along \(S\). So \(\Delta|_{S^\nu} = 0\); otherwise \((\mathcal{F}, S + \Delta)\) is not log canonical along \(S\), a contradiction.

(2) When \(\varepsilon(S) = 0\), we consider \(\hat{X}\) the formal completion of \(X\) along \(S\) and \(T\) the sum of all formal invariant divisors meeting \(S\). Note that \((\hat{X}, S + T + \Delta)\) is dlt by [CS21, Lemma 3.18] and \(\Theta' = [\Theta]_{\text{red}} + \{\Theta\}\) is the different of \((\hat{X}, S + T + \Delta)\) with respect to \(S\) by the proof of [CS21, Lemma 3.18]. Thus, by adjunction, \((S^\nu, \Theta')\) is log canonical.

Moreover, [CS21, Corollary 3.20] shows that any irreducible component \(C\) of \(\Theta - \Theta'\) is not contained in the singular locus of \(X\) and thus

\[\Theta - \Theta' = (K_{\mathcal{F}|S^\nu} - K_{S^\nu}) - (K_{\hat{X}} + S + T - K_{S^\nu})\]

is a \(\mathbb{Z}\)-divisor since it is the difference of two \(\mathbb{Z}\)-divisors. Therefore, the coefficients of \(\Theta\) belong to \(D(I) \cup \mathbb{N}\) because the coefficients of \(\Theta'\) belong to \(D(I)\).

Besides, each irreducible component \(C\) of \(\Theta - \Theta'\) is a log canonical center for \((\mathcal{F}, \Delta)\). By [CS21, Lemma 3.22], we have

\[\left(K_{\mathcal{F}} + \Delta\right)|_{C^\nu} = K_{C^\nu} + \Xi\]

for some effective divisor \(\Xi\) where \(\nu : C^\nu \to C\) is the normalization of \(C\). We recall the derivation of the equation (1) in [CS21, Lemma 3.22]. Let \(S^\nu\) be the strong separatrix at a general point of \(C\). Then we take the adjunction

\[\left(K_{\mathcal{F}} + \Delta\right)|_{S^\nu} = K_{S^\nu} + C' + \Delta_{S^\nu}\]

where \(\Delta_{S^\nu} \geq 0\) is a \(\mathbb{Q}\)-divisor whose coefficients belong to \(D(I) \cup \mathbb{N}\). Note that \((S^\nu, C' + [\Delta_{S^\nu}]_{\text{red}} + \{\Delta_{S^\nu}\})\) is log canonical and

\[\left(K_{S^\nu} + C' + \Delta_{S^\nu}\right)|_{C^\nu} = K_{C^\nu} + \Xi.\]

From [CS21, Lemma 3.22], \(\nu(P)\) is a log canonical center of \((\mathcal{F}, \Delta)\) for any point \(P\) contained in the support of \(\Xi\). Then by [CS21, Lemma 3.8], \((\mathcal{F}, \Delta)\) is foliated log smooth at \(\nu(P)\). Therefore, the coefficients of \(\Xi\) belong to \(D(D(I)) \cup \mathbb{N} = D(I) \cup \mathbb{N}\).

□

3.3. ACC for LCT when \(n = 3\) and \(r = 2\).

**Theorem 3.11.** Fix a set \(I \subset [0, 1]\), which satisfies the descending chain condition. Then there exists a finite subset \(I_0 \subset I\), depending only on \(I\), with the following property: Suppose that

1. \(X\) is a variety of dimension 3,
2. \(\mathcal{F}\) is a foliation of rank 2 on \(X\),
3. \((\mathcal{F}, D)\) is log canonical,
4. the coefficients of \(D\) belong to \(I\), and
5. there is a log canonical center \(Z\) which is contained in every component of \(D\).

Then the coefficients of \(D\) belong to \(I_0\).
Proof. We may assume that $Z$ is maximal with respect to the inclusion. If $Z$ is a divisor, then $Z = D$ with coefficient one. Thus we require $1 \in I_0$.

Now suppose $Z$ is not a divisor, we have, by Theorem 3.29 a foliated dlt modification $\pi : Y \to X$ such that, if $\mathcal{G}$ is the pullback foliation on $Y$, then

1. $Y$ is $Q$-factorial,
2. $Y$ has at worst klt singularities, and
3. $(\mathcal{G}, \Gamma := \pi_*^{-1}D + \sum_j \epsilon(E_i)E_i)$ is foliated dlt with $K_{\mathcal{G}} + \Gamma = \pi^*(K_Y + D)$ where the sum is over all $\pi$-exceptional divisors.

Let $D_1$ be one of irreducible components of $D$ and $d_1$ be the coefficient of $D_1$ in $D$. We will show that $d_1$ belongs to a finite subset $I_0 \subset I$, depending only on $I$.

Note that the proper transform $\pi_*^{-1}D_1$ has coefficient $d_1$ in $\Gamma$. Without loss of generality, we may assume that $E_1$ intersects $\pi_*^{-1}D_1$. By adjunction for foliated threefolds (Proposition 3.10), we have

$$(K_{\mathcal{G}} + \Gamma)|_{E_1} = K_{\mathcal{G}_1} + \Theta$$

where $\mathcal{G}_1$ is the restricted foliation on $E_1$. Since $\pi_*^{-1}D_1$ meets $E_1$, there is an irreducible component $\Theta_1$ of $\Theta$ whose coefficient has the form

$$b_1 := \frac{m - 1 + f + kd_1}{m}$$

where $m, k \in \mathbb{N}$ and $f \in D(I)$. Note also that $\Theta_1$ dominates $Z$ by the assumption that $Z$ is contained in $D_1$.

We have four cases based on the invariance of $E_1$ and the dimension of $Z$:

3.3.1. $E_1$ is non-invariant. By adjunction (Proposition 3.10), we have $(\mathcal{G}, \Theta)$ is log canonical and the coefficients of $\Theta$ belong to $D(I)$.

1. If $Z$, which is the center of $E_1$, is a point, then $K_{\mathcal{G}_1} + \Theta$ is numerically trivial. By Theorem 2.24 for the DCC set $D(I)$, there is a finite subset $J_1 \subset D(I)$, depending only on $I$, such that all coefficients of $\Theta$, in particular $b_1$, belong to $J_1$. By [HMX14, Lemma 5.2], there is a finite subset $I_1 \subset I$, depending only on $I$, such that $d_1 \in I_1$.

2. If $Z$ is a curve, then $\psi := \pi|_{E_1'} : E_1' \to Z$ is a fibration. Since $\Theta_1$ dominates $Z$, we have $\Theta_1 \cdot F > 0$ for a general fiber $F$ of $\psi$. Therefore, $K_{\mathcal{G}_1} \cdot F = -\Theta \cdot F < 0$ since $(K_{\mathcal{G}_1} + \Theta) \cdot F = 0$. By Lemma 1.14, $F$ is invariant with $K_{\mathcal{G}_1} \cdot F = -2$. Now we write $\Theta = \sum_j b_j \Theta_j$ where $\Theta_j$’s are distinct irreducible components, then

$$2 = -K_{\mathcal{G}_1} \cdot F = \Theta \cdot F = \sum_j (\Theta_j \cdot F)b_j.$$ 

Recall that $b_j \in D(I)$ and $\Theta_1 \cdot F > 0$. Since $(\mathcal{G}, \Theta)$ is log canonical, each $\Theta_j$ is non-invariant. Then $\Theta_j \cdot F$ is a positive integer for all $j$ because $F$ is general and invariant. By Lemma 1.14 for the DCC set $D(I)$ and $\alpha = 2$, there is a finite subset $J_2 \subset D(I)$, depending only on $I$, such that $b_j \in J_2$ for all $j$. By [HMX14, Lemma 5.2], there is a finite subset $I_2 \subset I$, depending only on $I$, such that $d_1 \in I_2$.

3.3.2. $E_1$ is invariant. By adjunction (Proposition 3.10), we have $(E_1', \Theta' = \{\Theta\}_{\text{red}} + \{\Theta\})$ is log canonical and the coefficients of $\Theta$ belong to $D(I) \cup \mathbb{N}$.
(1) If $Z$ is a curve, then $\psi := \pi|_{E'} : E' \rightarrow Z$ is a fibration. Since $\Theta_1$ dominates $Z$, we have $\Theta_1 \cdot F > 0$ for a general fiber $F$ of $\psi$. So $K_{E'} \cdot F = -(\Theta_1 \cdot F) = 0$ because $(K_{E'} + \Theta) \cdot F = 0$. By Lemma 1.14, we have $K_{E'} \cdot F = -2$. If we write $\Theta = \sum b_i \Theta_i$, then
\[
2 = \Theta \cdot F = \sum b_i (\Theta_i \cdot F).
\]
If there is a $j_0$ such that $b_{j_0} \geq 2$ and $\Theta_{j_0} \cdot F \neq 0$, then
\[
2 = \sum b_j (\Theta_j \cdot F) \geq b_{j_0} + b_{j_0} > 0 + 2 = 2,
\]
which is impossible. Hence, we may assume that all $b_j \in D(I)$. Therefore, $b_1 \in J_2$ by the choice of $J_2$ and thus $d_1 \in I_2$ by the choice of $I_2$.

(2) If $Z$ is a point, then $K_{E'} + \Theta$ is numerically trivial. When $\Theta = \Theta'$, we have a log canonical pair $(E_i, \Theta)$ such that $K_{E'} + \Theta$ is numerically trivial and the coefficients of $\Theta$ belong to $D(I)$. By [HMX14, Theorem 1.5] for the DCC set $D(I)$, there is a finite subset $J_3 \subset D(I)$ such that all coefficients of $\Theta$ belong to $J_3$. And thus, by [HMX14, Lemma 5.2], $d_1$ belongs to a subset $I_3 \subset I$, which depends only on $I$.

Now suppose $\Theta \neq \Theta'$. The proof of this case will be divided into several claims. Write $[\Theta] = \sum n_i C_i$ and $[\Theta] = \sum b_i \Theta_i$ where $n_i \in \mathbb{N}$, $n_1 \geq 2$, and $b_i \in D(I)$.

Claim. We may assume that no $C_i$ with $n_i \geq 2$ meets $\Theta_1$.

Proof. If there is a curve $C_i$ with $n_i \geq 2$ intersecting $\Theta_1$, then by adjunction (Proposition 3.10), we have
\[
0 = \pi^*(K_{\mathcal{F}} + D) \cdot C_i = (K_{\mathcal{F}} + \Gamma) \cdot C_i = \deg(K_{C_i} + \Xi)
\]
where the coefficients of $\Xi$ belong to $D(I) \cup \mathbb{N}$. Since $\Theta \cdot C_i \neq 0$, we have $\deg(K_{C_i}) = -\deg(\Xi) < 0$. Hence $\deg(\Xi) = \deg(K_{C_i}) = 2$. Therefore, if we write $\Xi = \sum u_s p_s$ where $p_s$'s are points on $C_i$, then we have $\sum u_s = 2$ where $u_s \in D(I) \cup \mathbb{N}$. Recall $\Theta_1 \cdot C_i \neq 0$. Then one of $u_s$'s, say $u_1$, has the form
\[
u_1 = \frac{n - 1 + \ell b_1 + g}{n} \in (0, 1]
\]
where $n_i, \ell \in \mathbb{N}$ and $g \in D(I)$. This shows that all $u_s$'s are less than 2. So we have $u_s \in D(I)$ for all $s$. Replacing $b_1$ by the equation (2), we have
\[
u_1 = \frac{r - 1 + \ell kd_1 + h}{r}
\]
for some $r \in \mathbb{N}$ and $h \in D(I)$. By the choice of $J_2$ and $\sum s u_s = 2$, we have $u_1 \in J_2$, and thus $d_1 \in I_2$ by the choice of $I_2$. This completes the proof of the Claim.

Claim. The Cartier index of $E_i$ at any point on $C_i$ with $n_i \geq 2$ is bounded above by a constant $M$ depending only on $I$.

Proof. To simplify the notation in the proof of this Claim, we use $C_i$ to denote any curve $C_i$ with $n_i \geq 2$. By Proposition 3.10, $C_i$ is a log canonical center for the foliated dlt pair $(\mathcal{F}, \Gamma)$. Then $(\mathcal{F}, \Gamma)$ is foliated log smooth at the generic point of $C_i$ by [CS21, Lemma 3.8]. Thus, there are two
separatrices passing through $C_1$: one is $E_1$, which is the weak separatrix by [CS21 Corollary 3.20], and the other is denoted as $S$, which is the strong separatrix.

Now, by adjunction, we put $(K_{Y^*} + \Gamma)|_{S'} = K_{S'} + \Delta$. By [CS21 Corollary 3.20], we have

$$\left( K_{Y^*} + S + E_1 + \sum S_i + \Gamma \right)|_{S'} = K_{S'} + \Delta'$$

where $\hat{Y}$ is the formal completion of $Y$ along $S$, $\Delta' = \lfloor \Delta \rfloor_{red} + \{ \Delta \}$, and $S, E_1, S_i$'s are all invariant divisors on $\hat{Y}$. Note that if $(\Delta - \Delta') \cdot C_1 \neq 0$, then any point $p$ in the support of $(\Delta - \Delta') \cap C_1$ is a log canonical center for $(S', \Delta')$ and thus a log canonical center for $(\hat{Y}, S + E_1 + \sum S_i + \Gamma)$ by inversion of adjunction. Thus, by [Spi20 Lemma 8.14], $p$ is a log canonical center for $(\mathcal{S}, \Gamma)$ and hence $(\mathcal{S}, \Gamma)$ is foliated log smooth at $p$ by [CS21 Lemma 3.8]. Therefore, $(\Delta - \Delta') \cdot C_1 \in \mathbb{N}$. Also, we have

$$0 = (K_{S'} + \Delta) \cdot C_1$$
$$\geq (K_{S'} + C_1) \cdot C_1 + (\Delta - C_1 - \{ \Delta \}) \cdot C_1$$
$$\geq -2 + (\lfloor \Delta \rfloor - C_1) \cdot C_1$$

where the equality holds since $C_1$ is contracted to a point $Z$ by assumption. Then there is at most one irreducible component of the support of $\Delta - \Delta'$ intersecting $C_1$. Besides, if such irreducible component exists, then it is unique and its coefficient in $\Delta$ is 2. So $(\Delta - \Delta') \cdot C_1 \in \{0, 1\}$.

Next we also notice that

$$-(\Delta - \Delta') \cdot C_1 = (K_{S'} + \Delta') \cdot C_1 - (K_{S'} + \Delta) \cdot C_1$$
$$= \left( K_{Y^*} + S + E_1 + \sum S_i + \Gamma \right) \cdot C_1 - 0$$
$$= (K_{E_1^*} + \Theta') \cdot C_1$$
$$= (K_{E_1^*} + \Theta) \cdot C_1 - (\Theta - \Theta') \cdot C_1$$
$$= -(\Theta - \Theta') \cdot C_1$$

where the second equality comes from the equation (3) and that $C_1$ is contracted to a point $Z$ and the last equality comes from $K_{E_1^*} + \Theta \equiv 0$. Therefore, we have

$$(\Theta - \Theta') \cdot C_1 = (\Delta - \Delta') \cdot C_1 = 0 \text{ or } 1$$

and thus $(K_{E_1^*} + \Theta') \cdot C_1 = 0 \text{ or } -1$. By adjunction, we have

$$(K_{E_1^*} + \Theta')|_{C_1^*} = K_{C_1^*} + \Xi$$

where $\Xi = \sum u_s p_s$ with $u_s \in D(I)$. Note that if $u_s = 1$, then $p_s$ is a log canonical center for $(C_1^*, \Xi)$. So by inversion of adjunction and [Spi20 Lemma 8.14], $p_s$ is a log canonical center for $(\mathcal{S}, \Gamma)$. Since $(\mathcal{S}, \Gamma)$ is foliated dlt, it is foliated log smooth at $p_s$ by [CS21 Lemma 3.8]. Thus the Cartier index at $p_s$ is 1 if $u_s = 1$.

Now since $\deg(K_{C_1^*}) + \sum u_s = 0$ or $-1$, we have $\sum u_s = 2$ or 1. By Lemma [1.16] there is a finite subset $J_4 \subset D(I)$ such that each $u_s$ belongs to $J_4$. We have seen above that the Cartier index at $p_s$ is 1 if $u_s = 1$. For each $u_s < 1$, the index is bounded above. Therefore, because $J_4$ is finite,
the Cartier index of \( E_1^{\nu} \) at any point on \( C_1 \) is bounded above by a constant depending only on \( I \). This completes the proof of the Claim.

Note that \( (E_1^{\nu}, \Theta' - b_1 \Theta_1) \) is log canonical. So we can run the \((K_{E_1^{\nu}} + \Theta' - b_1 \Theta_1)\)-minimal model program (MMP). Let \( R = \mathbb{R}_{\geq 0}[G] \) be a \((K_{E_1^{\nu}} + \Theta' - b_1 \Theta_1)\)-negative extremal ray where \( G \) is an irreducible curve and \( \varphi_R : E_1^{\nu} \to S \) be the contraction associated to \( R \). Note that

\[
0 > (K_{E_1^{\nu}} + \Theta' - b_1 \Theta_1) \cdot G = \left( -b_1 \Theta_1 - \sum (n_i - 1)C_i \right) \cdot G.
\]

So \( G \) must intersect either \( \Theta_1 \) or some \( C_i \) with \( n_i \geq 2 \). We have the following observations:

(a) If \( \varphi_R : E_1^{\nu} \to S \) is a Mori fiber space and \( S \) is a point, then the Picard number of \( E_1^{\nu} \) is 1. So \( \Theta_1 \) is an ample divisor and thus \( \Theta_1 \cdot C_i > 0 \) for any \( C_i \) with \( n_i \geq 2 \), which is a contradiction since \( \Theta_1 \) is disjoint from those \( C_i \)’s.

(b) If \( \varphi_R : E_1^{\nu} \to S \) is a Mori fiber space and \( S \) is a curve, then we may assume \( G \) is a general fiber of \( \varphi_R \). Suppose \( G \cdot \Theta_1 > 0 \), then we have

\[
-K_{E_1^{\nu}} \cdot G = \Theta \cdot G = \sum (\Theta_i \cdot G)b_i + \sum n_iC_i \cdot G \\
\geq b_1 \Theta_1 \cdot G > 0.
\]

So by Lemma 1.14 we have \( K_{E_1^{\nu}} \cdot G = -2 \) and thus \( C_i \cdot G = 0 \) for all \( C_i \) with \( n_i \geq 2 \). Hence, \( b_1 \in J_2 \) by the choice of \( J_2 \) and thus \( d_1 \in I_2 \) by the choice of \( I_2 \).

(c) If \( \varphi_R \) is a divisorial contraction, then \( G^2 < 0 \). Thus \( G \neq \Theta_1 \), otherwise, we have

\[
0 > (K_{E_1^{\nu}} + \Theta' - b_1 \Theta_1) \cdot G = \left( -b_1 G - \sum (n_i - 1)C_i \right) \cdot G = -b_1 G^2
\]

and therefore \( G^2 > 0 \), a contradiction.

(d) Furthermore, we suppose \( G \) intersects both \( \Theta_1 \) and one of \( C_i \)’s with \( n_i \geq 2 \). For simplicity, let us assume \( G \) intersects \( C_i \) with \( n_1 \geq 2 \). By adjunction, we have

\[
0 > G^2 = (K_{E_1^{\nu}} + G + \Theta) \cdot G \\
\geq -2 + \frac{m - 1 + n_1 + g}{m} + \frac{n - 1 + b_1 + h}{n} \\
\geq -2 + \frac{m - 1 + 2}{m} + \frac{n - 1}{n} \\
= \frac{1}{m} - \frac{1}{n} > -1
\]

where \( m \) is the Cartier index of \( E_1^{\nu} \) at some point supported on \( C_i \cap G \) and \( n \) is the one at some point supported on \( \Theta_1 \cap G \). Therefore, \( n < m \leq M \). Moreover, the inequality above also shows that the Cartier index of \( E_1^{\nu} \) at any point on \( G \) is bounded above by \( M \).
Since \(0 > G^2 > -1\) and \((M!)^2 G^2\) is an integer, \(G^2\) belongs to a finite set depending only on \(M\), and therefore depending only on \(I\). Thus, by Lemma 14.14 there is a finite subset \(J_5 \subset D(I)\) depending only on \(I\) such that \(b_1\) belongs to \(J_5\). Hence, by [HMX14] Lemma 5.2, \(d_i\) belongs to a subset \(I_5 \subset I\), which depends only on \(I\).

So suppose we have a \((K_{E_i'} + \Theta' - b_1 \Theta_1)\)-negative extremal ray \(R = \mathbb{R}_{>0}[G]\) with \(G \cdot \Theta_1 > 0\). Then, by the observations above, either we complete the proof of the Theorem or \(\varphi_R\) is a divisorial contraction and \(G \cdot C_i = 0\) for any \(C_i\) with \(n_i \geq 2\). Hence, successively contracting down all \((K_{E_i'} + \Theta' - b_1 \Theta_1)\)-negative extremal rays \(R = \mathbb{R}_{>0}[G]\) with \(G \cdot \Theta_1 > 0\), we may assume that \(G \cdot \Theta_1 = 0\) for all \((K_{E_i'} + \Theta' - b_1 \Theta_1)\)-negative extremal rays \(R = \mathbb{R}_{>0}[G]\). Moreover, since all exceptional divisors of divisorial contractions above do not meet \(C_i\)'s with \(n_i \geq 2\), the Cartier index of \(E_i'\) at any point on those \(C_i\)'s is still bounded above by \(M\).

Next we are given any \((K_{E_i'} + \Theta' - b_1 \Theta_1)\)-negative extremal ray \(R = \mathbb{R}_{>0}[G]\) (with \(G \cdot \Theta_1 = 0\)).

**Claim.** \(\varphi_R\) is a divisorial contraction.

**Proof.** Suppose \(\varphi_R : E_i' \to T\) is a Mori fiber space. Note that \(T\) cannot be a point, otherwise \(\rho(E_i') = 1\) and \(G \cdot \Theta_1 > 0\), which contradicts our assumption. So \(T\) must be a curve. We may assume \(G\) is a general fiber of \(\varphi_R\). Since \(G \cdot \Theta_1 = 0\), \(\Theta_1\) is in a fiber of \(\varphi_R\). We notice that

\[
0 > (K_{E_i'} + \Theta' - b_1 \Theta_1) \cdot G
= \left(-b_1 \Theta_1 - \sum (n_i - 1) C_i\right) \cdot G
= -\sum (n_i - 1) C_i \cdot G.
\]

Then one of \(C_i\) with \(n_i \geq 2\) intersects the general fiber of \(\varphi_R\). Let us assume that \(C_i\) with \(n_i \geq 2\) intersects the general fiber of \(\varphi_R\). Thus, \(\Theta_1\) is a component of a singular fiber \(F\) of \(\varphi_R\) with at least two components, otherwise \(\Theta_1 \equiv aG\) for some \(a > 0\) and \(\Theta_1 \cdot C_1 = aG \cdot C_1 > 0\), a contradiction. However, there is another component \(C\) of this singular fiber \(F\) that intersects \(\Theta_1\) because \(F\) is connected. Note that

(a) \(C \neq C_i\) with \(n_i \geq 2\) since \(C \cdot \Theta_1 > 0\) but \(C_i \cdot \Theta_1 = 0\) from our assumption,

(b) \(C^2 < 0\) because \(C\) is an irreducible component of a singular fiber of \(\varphi_R\),

(c) \((K_{E_i'} + \Theta' - b_1 \Theta_1) \cdot C = (-b_1 \Theta_1 - \sum (n_i - 1) C_i) \cdot C < 0\).

So we see that \(\mathbb{R}_{>0}[C]\) is a \((K_{E_i'} + \Theta' - b_1 \Theta_1)\)-negative extremal ray with \(C \cdot \Theta_1 > 0\), which contradicts our assumption. Thus, we complete the proof of the Claim.

Nevertheless, we note that

\[
K_{E_i'} + \Theta' - b_1 \Theta_1 \equiv -b_1 \Theta_1 - \sum (n_i - 1) C_i
\]

is not pseudo-effective. So the outcome of \((K_{E_i'} + \Theta' - b_1)\)-MMP must be a Mori fiber space, which is impossible. This complete the proof of the Theorem. \(\Box\)
4. Foliations of rank one on threefolds

This section is devoted to show Theorem 4.7, which is Theorem 0.6 when \( r = 1 \) and \( n = 3 \).

4.1. Resolution of singularities. Most definitions in this subsection follow from [CS20].

Example 4.1 ([MP13, Example III.iii.3]). Let \( X \) be the quotient of \( \mathbb{C}^3 \) by the \( \mathbb{Z}/2\mathbb{Z} \)-action given by \((x, y, z) \mapsto (y, x, -z)\). We consider a vector field on \( \mathbb{C}^3 \) given by

\[
\partial = \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) + \left( a(xy, z)x \frac{\partial}{\partial x} - a(xy, -z)y \frac{\partial}{\partial y} + c(xy, z) \frac{\partial}{\partial z} \right)
\]

where \( a \) and \( c \) are formal functions in two variables such that \( c \) is not a unit and satisfies \( c(xy, z) = c(xy, -z) \). Note that \( \partial \mapsto -\partial \) under the \( \mathbb{Z}/2\mathbb{Z} \)-action and thus it induces a foliation \( \mathcal{F} \) on \( X \). Moreover, \( \mathcal{F} \) has at worst canonical singularities.

Definition 4.2. Let \( X \) be a threefold and \( \mathcal{F} \) be rank one foliation with at worst canonical singularities. We say \( p \in X \) is a simple singularity of \( \mathcal{F} \) if either

1. \( \mathcal{F} \) is terminal and no component of \( \text{Sing}(X) \) through \( p \) is invariant,
2. \( X \) and \( \mathcal{F} \) are formally isomorphic to the variety and foliation defined in Example 4.1 at \( p \), or
3. \( X \) is smooth at \( p \).

Remark 4.3. The definition above is for the foliation of rank one; while in definition 3.2, we have simple singularity for co-rank one foliation. We keep the same terminologies as in the literature. It should be clear which definition we used from the context since one is for co-rank one foliation and another one is for rank one foliation.

Theorem 4.4 ([MP13, III.iii.4]). Let \( \mathcal{F} \) be a rank one foliation on a normal threefold \( X \). Then there exists a sequence of weighted blowups in foliation invariant centers \( \pi : \tilde{X} \to X \) such that \( \pi^* \mathcal{F} \) has only simple singularities.

Theorem 4.5 ([CS20, Proposition 8.1]). Let \( (\mathcal{F}, \Delta) \) be a log canonical foliated pair of rank one on a normal threefold. Assume \( \mathcal{F} \) has only simple singularities and \( K_{\mathcal{F}} + \Delta \) is pseudo-effective. Then \( (\mathcal{F}, \Delta) \) admits a minimal model \( \varphi : X \to Y \). Moreover, if \( \mathcal{G} := \varphi_* \mathcal{F} \) and \( \Gamma := \varphi_* \Delta \), then we have

1. \( \mathcal{G} \) has only simple singularities,
2. \( (\mathcal{G}, \Gamma) \) is log canonical, and
3. if \( \Theta \geq 0 \) is a \( \mathbb{Q} \)-divisor with \( \mathcal{F} \)-invariant support such that \( (X, \Theta) \) is log canonical, then \( (Y, \varphi_* \Theta) \) is log canonical.

4.2. Adjunction.

Theorem 4.6. Let \( \mathcal{F} \) be a foliation of rank one with at worst simple singularities on a normal threefold \( X \), \( S \) be an invariant prime divisor on \( X \) such that \( (X, S) \) is log canonical, and \( I \) be a subset of \([0, 1]\) containing 1. Suppose \( (\mathcal{F}, \Delta) \) is log canonical where \( \Delta \) is an effective (non-invariant) divisor whose coefficients belong to \( I \).

Let \( \nu : S^\nu \to S \) be its normalization. We write \( (K_{\mathcal{F}} + \Delta)|_{S^\nu} = K_{\mathcal{G}} + \Theta \) where \( \mathcal{G} \) is the pullback foliation on \( S^\nu \) and \( \Theta \) is an effective \( \mathbb{Q} \)-divisor. Then the coefficients of \( \Theta \) belong to \((I_+ \cap [0, 1]) \cup \frac{1}{2}\mathbb{N})\).
4.3. ACC for LCT when \( n = 3 \) and \( r = 1 \).

**Theorem 4.7.** Fix a set \( I \subset [0, 1] \), which satisfies the descending chain condition. Then there exists a finite subset \( I_0 \subset I \) with the following property: If \((X, \mathcal{F}, D)\) is a triple such that

1. \( X \) is a variety of dimension 3,
2. \( \mathcal{F} \) is a foliation of rank 1,
3. \((\mathcal{F}, D)\) is log canonical,
4. the coefficients of \( D \) belong to \( I_0 \), and
5. there is a log canonical center \( Z \) which is contained in every component of \( D \),

then the coefficients of \( D \) belong to \( I_0 \).

**Proof.** Without loss of generality, we may assume that \( Z \) is maximal with respect to the inclusion. If \( Z \) is a divisor, then \( Z = D \) and thus we require \( 1 \in I_0 \).

Otherwise, by the resolution of singularities (Theorem 4.4), we have a birational morphism \( p : X' \to X \) such that the pullback foliation \( \mathcal{F}' := p^*\mathcal{F} \) has only simple singularities. Let \( D' := p_+^{-1}D \). Then we may write

\[
K_{\mathcal{F}'} + D' + F' = p^*(K_\mathcal{F} + D) + F''
\]

where \( F' \) and \( F'' \) are effective \( p \)-exceptional divisors without common components. After possibly taking a higher resolution, we may assume that \((\mathcal{F}', D' + F')\) is log canonical and that \((X', D' + F')\) is log canonical where \( F' \) is the sum of all \( p \)-exceptional divisors. By [CS20, Proposition 8.1], we may run a \((K_{\mathcal{F}}, D'+ F')\)-MMP over \( X \). Let \( \varphi : X' \to Y \) be the output of this MMP with \( \pi : Y \to X \). Let \( \mathcal{G} := \varphi_*\mathcal{F}' \), \( D_Y := \varphi_*D' \), and \( E' := \varphi_*F' \). Then we have, by [CS20, Proposition 8.1 and Lemma 8.2], that

1. \((\mathcal{G}, \Gamma) := D_Y + \sum \varepsilon(E_j)E_j\) is log canonical;
2. we have \( K_{\mathcal{G}} + \Gamma = \pi^*(K_{\mathcal{F}} + D)\);
3. \( \mathcal{G} \) has only simple singularities; and
4. \((Y, E_{\text{inv}})\) is log canonical where \( E_{\text{inv}} \) is the sum of all \( \mathcal{G} \)-invariant exceptional divisors.

Let \( D_1 \) be one of irreducible components of \( D \) and \( d_1 \) be the coefficient of \( D_1 \) in \( D \). Then the proper transform \( \pi_{\mathrm{inv}}^{-1}D_1 \) has coefficient \( d_1 \) in \( \Gamma \). Without loss of generality, we may assume that \( E_1 \) intersects \( \pi_{\mathrm{inv}}^{-1}D_1 \). Let \( \Theta_1 \) be an irreducible component of \( \pi_{\mathrm{inv}}^{-1}D_1 \cap E_1 \). By the assumption that \( Z \) is contained in \( D_1 \), we have \( \Theta_1 \) dominates \( Z \).

We have four cases based on the invariance of \( E_1 \) and the dimension of \( Z \):

4.3.1. \( E_1 \) is non-invariant. By adjunction [CS20, Proposition 2.19], we have

\[
(K_{\mathcal{G}} + E_1)|_{E_1^c} \sim_{\mathbb{Q}} \Xi
\]
4.3.2. Where the coefficients belong to \( \Xi \) is an effective \( Q \)-divisor.

(1) If \( Z \), which is the center of \( E_1 \), is a point, then \( K_{\mathcal{H}} + \Gamma \) is numerically trivial and thus we have an effective \( Q \)-divisor \( \Xi + (\Gamma - E_1)|_{E_1^\nu} \sim_0 0 \). Therefore,
\[
\Xi = 0 = (\Gamma - E_1)|_{E_1^\nu},
\]
which shows that all irreducible components of \( \Gamma - E_1 \) do not intersect \( E_1 \). This is impossible since \( \pi_1^{-1}D_1 \) is in the support of \( \Gamma - E_1 \).

(2) If \( Z \) is a curve, then \( \psi := \pi|_{E_1^\nu} : E_1^\nu \to Z \) is a fibration. Let \( \Theta_1 \) be an irreducible component of \( \pi_1^{-1}D_1 \cap E_1 \). Since \( \Theta_1 \) dominates \( Z \), we have \( \Theta_1 \cdot F > 0 \) where \( F \) is a general fiber of \( \psi \). Hence,
\[
0 = \pi^*(K_{\mathcal{H}} + D) \cdot F = (\Xi + (\Gamma - E_1)|_{E_1^\nu}) \cdot F \geq \Theta_1 \cdot F > 0,
\]
which is impossible.

4.3.2. \( E_1 \) is invariant. By Theorem [10] we have
\[
(K_{\mathcal{H}} + \Gamma)|_{E_1^\nu} = K_{\mathcal{H}^\nu} + \Theta
\]
where \( \mathcal{H}^\nu \) is the pullback foliation on \( E_1^\nu \) and \( \Theta \) is an effective divisor whose coefficients belong to \( \frac{1}{2} I_1 \). Let \( \Theta = \sum b_i \Theta_i \) where \( \Theta_i \) be an irreducible component of \( \pi_1^{-1}D_1 \cap E_1 \) which dominates \( Z \). Note that \( 2b_i = kd_1 + f \) where \( k \in \mathbb{N} \) and \( f \in I_1 \).

(1) If \( Z \) is a curve, then \( \psi := \pi|_{E_1^\nu} : E_1^\nu \to Z \) is a fibration. By the assumption that \( Z \) is contained in \( D_1 \), we have \( \Theta_1 \cdot F > 0 \) where \( F \) is a general fiber of \( \psi \). Note that
\[
0 = \pi^*(K_{\mathcal{H}} + D) \cdot F = (K_{\mathcal{H}^\nu} + \Theta) \cdot F
\]
and thus \( -K_{\mathcal{H}^\nu} \cdot F = \Theta \cdot F \geq \Theta_1 \cdot F > 0 \). By Lemma [10] we have \( F \) is invariant and \( K_{\mathcal{H}^\nu} \cdot F = -2 \). Therefore, we have
\[
4 = -2K_{\mathcal{H}^\nu} \cdot F = 2\Theta \cdot F = \sum 2b_i(\Theta_i \cdot F)
\]
where \( \Theta_1 \cdot F \in \mathbb{Z}_{\geq 0} \) and \( \Theta_1 \cdot F > 0 \). Replacing \( 2b_i \) by \( kd_1 + f \), there is a finite subset \( I_1 \subset I \), by Lemma [10] such that \( d_i \in I_1 \) since \( \Theta_1 \cdot F > 0 \).

(2) Now suppose \( Z \) is a point. Then \( K_{\mathcal{H}^\nu} + \Theta \equiv 0 \).

**Claim.** \( \mathcal{H} \) is algebraically integrable with rationally connected leaves.

**Proof.** By [10] Theorem 8.1], \( (\mathcal{H}, \Theta) \) admits a foliated dlt modification \( \pi : S \to E_1^\nu \) such that

(a) \( S \) is \( Q \)-factorial,
(b) \( S \) has at worst klt singularities,
(c) the pullback foliation \( \mathcal{H}_S \) on \( S \) is non-dicritical, that is the exceptional curves over \( S \) are all invariant,
(d) \( (\mathcal{H}_S, \pi_1^{-1}\Theta + \sum \varepsilon(E_i)E_i) \) is foliated dlt, and
(e) \( K_{\mathcal{H}_S} + B := K_{\mathcal{H}_S} + \pi_1^{-1}\Theta + \sum \varepsilon(E_i)E_i + F = \pi^*(K_{\mathcal{H}^\nu} + \Theta) \) where \( F \) is a \( \pi \)-exceptional \( Q \)-divisor on \( S \).

If \( S \) has Picard number 1, then \( -K_{\mathcal{H}_S} \cdot C = B \cdot C > 0 \) for a general curve \( C \). Thus, \( -K_{\mathcal{H}} \) is ample since \( \rho(S) = 1 \). By Theorem [10] \( S \) is uniruled. Hence \( K_S \) is not pseudo-effective and therefore \( -K_S \) is ample. Moreover, we note that the \( (-K_S) \)-slope of \( \mathcal{H}_S \) is
\[
\mu(-K_S)(\mathcal{H}_S) = (-K_{\mathcal{H}_S}) \cdot (-K_S) > 0.
\]
By [CP19 Theorem 1.1] or [On17] Proposition 2.2, $\mathcal{H}_S$ is algebraically integrable with rationally connected leaves, and thus so is $\mathcal{H}$.

Now suppose $\rho(S) \geq 2$. We will run $\mathcal{H}_S$-MMP on $S$. Note that $K_{\mathcal{H}_S} \equiv -B$ is not pseudo-effective. Then we will end up with a Mori fiber space, either a fibration onto a curve or a surface with Picard number 1. In both cases, we see that $\mathcal{H}_S$ is algebraically integrable with rationally connected leaves. Therefore, $\mathcal{H}$ is also algebraically integrable with rationally connected leaves.

**Claim.** Any dicritical singularity for $\mathcal{H}$ is a log canonical center for $\mathcal{G}$.

**Proof.** Let $p$ be a dicritical singularity for $\mathcal{H}$. Assume $p$ is not a log canonical center for $\mathcal{G}$. Since $\mathcal{G}$ has simple singularities which are log canonical, $\mathcal{G}$ is terminal at $p$. Note that $K_\mathcal{G} \cdot L = -\Gamma \cdot L < 0$ for a general leaf $L$ of $\mathcal{H}$. By [CS20] Proposition 3.3 and Theorem 3.2, $\mathcal{G}$ is a fibration locally around $p$, which contradicts that $p$ is a dicritical singularity for $\mathcal{H}$. Therefore, $p$ is a log canonical center for $\mathcal{G}$. This completes the proof of the Claim.

Let $L$ be a general leaf of $\mathcal{H}$. Then by [CS20] Proposition 2.16, we have

$$0 = \pi^*(K_\mathcal{F} + D) \cdot L = (K_\mathcal{G} + \Gamma) \cdot L = \deg(K_{L^\nu} + \Delta)$$

and $\deg \Delta \geq 0$. If there is a dicritical singularity for $\mathcal{H}$, then $\deg \Delta \geq 1$. Thus, $\deg K_{L^\nu} \leq 1$ and therefore $\deg K_{L^\nu} = -2$ and $\deg \Delta = 2$. On the other hand, if there is no dicritical singularity for $\mathcal{H}$, then $\mathcal{G}$ is induced by a fibration $\psi : E_1' \to C$ for some curve $C$. Hence, $\deg \Delta = -\deg K_{L^\nu} = 2$ as $L$ is general and $\psi$ has rationally connected fibers.

Let $\nu : E_1' \to E_1' \hookrightarrow Y$ be the composition of the normalization and the inclusion. If $\Theta_1$ is contained in a fiber of $\psi$, then $\mathcal{F}$ is terminal along $\nu(\Theta_1) \subset \pi^{-1}_1 D_1$. By [CS20] Lemma 2.12, $\nu(\Theta_1)$ is not invariant, a contradiction.

Note that the coefficients of $\Delta$ belong to $D(D(I)) \cup \frac{1}{2} \mathbb{N}$, which satisfies the descending chain condition. By Lemma 1.16 there is a finite subset $I_2 \subset D(D(I)) \cup \frac{1}{2} \mathbb{N}$ depending only on $I$ such that the coefficients of $\Delta$ belong to $I_2$. By [HMX14] Lemma 5.2, there is a finite subset $I_3 \subset D(I)$, depending only on $I$, such that $b_1 \in I_3$. Applying [HMX14] Lemma 5.2 again, there is a finite subset $I_4 \subset I$, depending only on $I$, such that $d_1 \in I_4$.

5. **Proof of Theorem on ACC for LCT**

**Proof (Theorem 1.5).** Given a non-decreasing sequence $\{c^{(I)}\}_{I=1}^\infty$ in $\text{LCT}_{n,r}(I, J)$. We will drop the superscript if no confusion arises. We may assume that both 0 and 1 belong to $I$ and $J$.

For each $c$ in the sequence, we may find a foliation $\mathcal{F}$ of rank $r$ on a variety $X$ of dimension $n$, a divisor $\Delta$ whose coefficients belong to $I$, and an $R$-Cartier divisor $M$ whose coefficients belong to $J$ such that $c = \sup\{t \in \mathbb{R}|(\mathcal{F}, \Delta + tM)\text{ is log canonical}\}$.

Let $D = \Delta + cM$ and $K = \{a + bc \leq 1|a \in I \text{ and } b \in J\}$. Note that $K$ satisfies the descending chain condition and $(\mathcal{F}, D)$ is log canonical with coefficients of $D$ in $K$. Since $c$ is the log canonical threshold, there is a log canonical center $Z$
contained in the support of $M$. Possibly throwing away components of $D$ which do not contain $Z$, then we may assume every component of $D$ contains $Z$. By Theorem [Theorem 0.6], there is a finite subset $K_0 \subset K$ such that the coefficients of $D$ belong to $K_0$. This shows that the sequence $c^{(l)}$ eventually terminates.

□

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