Determination of Eigenvalues in Problems on the Buckling of Compressed Rods (Part I)

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Abstract. In many technical academic courses and in the calculations of mechanical design engineers, buckling is a difficult problem. The applied methods of its solution are of a private nature and are unsuitable for modern, more complex structures in such sectors of the economy as mechanical engineering, construction, instrument making, oil and gas facilities, the nuclear industry, power networks, etc. complex problems of the stability of a compressed rod and at the same time achieve greater clarity and versatility with the proposed methods. Using specific examples, the application of the traditional analytical and the proposed analytical-graphic methods is shown. The proposed mathematical models and decision algorithms are verified using computational experiments. Based on the results obtained, practical conclusions are drawn.

1. Introduction

In the courses of structural mechanics (resistance of materials, technical mechanics, structural mechanics, mechanics of solid deformable bodies) considerable time is devoted to the phenomenon of buckling of compressed rods, since in most cases large deformations appear, the transition of the material into a plastic state and, even, the destruction of the structure. This topic is one of the most difficult in these disciplines and in the practice of design engineers.

The topic of the stability of the equilibrium state of compressed rods has an extensive bibliography and a long history, begun by L. Euler back in 1744. Suffice it to point to publications that are popular among specialists in structural mechanics and university students [1-7]. For non-classical structures with such features as multi-span, variable stiffness, the presence of elastic intermediate supports, etc. solving the problem of the stability of compressed rods presents significant difficulties. For them, it is often not possible to establish the spectra of eigenvalues and forms by analytical methods. Traditional methods and numerous special cases of determining critical forces using tables and graphs given in the technical and reference literature [5] turned out to be ineffective. For the particular case of stability of multi-span rods, analytical methods were proposed [8, 9], and later, a high-precision graphic-analytical method, the efficiency and versatility of which were demonstrated at one of the universal conferences [10].

The complication of the design schemes of structures for the above reasons led to the need to formulate non-classical problems of the stability of compressed rods. In this work, analytical and analytical-graphic methods for determining the critical force are used. A simple universal graphic-
analytical method is proposed that allows finding the values of the first elements of the spectrum of eigenvalues sufficient for practical needs in the environment of the Matlab computing complex.

At the same time, traditional methods and numerous special cases of determining critical forces using tables and graphs given in technical and reference literature turned out to be ineffective.

In this paper, it is proposed to use computer technologies to significantly simplify the solution of complex problems of the stability of a compressed rod and at the same time achieve greater clarity. Its results can be applied in the course of regular classes and in independent study of the discipline, for example, in the distance education system.

The problem becomes much more complicated if the cross-section of the bar is variable along the axis, that is, its axial moment is a function \( J = J(x) \); supports are elastic, non-classical; the load is distributed; forces are applied not only at the ends, but also in the span; the rod is multi-span; rests on an elastic base, etc. In these cases, the technical reference literature \([1, 2]\) offers solutions for specific situations in the form of tables with coefficients, graphs, etc. At present, when technology, equipment, and apparatus are changing extremely rapidly towards complication, there is no reason to hope for an accurate determination of the critical forces under such circumstances, since the number of possible variants of design schemes is unlimited.

2. Formulation of the problem

In fig. 1 shows a design diagram of a compressed rod in its initial position. The loss of stability of the rod will be considered under the following assumptions \([1, 4]\):

1. The axis of the unloaded bar is ideally straight and the force \( F \) acts strictly along this axis until stability is lost.
2. External forces are "dead", i.e. when the bar is deformed, it does not change either in magnitude or in direction.
3. Changes in the geometric dimensions of the bar under subcritical deformations are considered negligible.
4. The relationship between the internal bending moment and the transverse bending of the bar during buckling is described by the usual dependence of the linear theory of bending of beams, based on the hypothesis of flat sections.
5. Transverse deflections \( v(x) \) and its derivatives are infinitesimal of the first order.

As a consequence of item 5, when determining the boundary conditions, the angle of rotation of the section of the bar and its trigonometric functions are taken equal to
In the simplest case of a bar of constant cross-section, at a certain, so-called critical value of the longitudinal force $F = F_\mathrm{c}$, the vertical rectilinear form of equilibrium loses stability and a transition to the curvilinear form $v(x)$ occurs. In this case, instead of simple compression, the rod also begins to experience bending, which significantly reduces its bearing capacity and ultimately leads to structural failure.

![Figure 2. Internal forces and the coordinate system.](image)

With the loss of stability of the bar in the cross section, in addition to the longitudinal force, a bending moment $M$ and a transverse force $Q$ appear, shown in Fig. 2. The rectilinear longitudinal axis bends and becomes curved. In stability problems, in contrast to bending problems, "the equilibrium equations are compiled for a deformed system deviated from its initial unloaded position" [1]. This, in turn, leads to the need to abandon the "principle of initial dimensions" of the theory of elasticity and resistance of materials.

In the equations, formulas and images of the curved axis of the bar, the internal forces in the sections, displacements, rotations of the cross sections and curvature of the axis shown in Fig. 2, we will assume positive.

### 3. Analytical solution method

It is clear that determining the value of $F_\mathrm{c}$ is a very urgent problem. The Euler formulation of a simple problem with hinged ends is that the curved axis is described by an ordinary second-order differential equation with respect to the curved axis function $v(x)$

$$v''(x) + k^2 v(x) = 0, \quad k = \frac{F}{\sqrt{EJ}}, \quad x \in (0, l).$$

Hereinafter, the dashes in the superscripts correspond to derivatives with respect to the argument $x$. When deriving and using this equation and its accompanying conditions, a number of assumptions are used based on Bernoulli's hypotheses and the smallness of deformations [3, 4]. In this case, Eq. (1) is supplemented by the boundary conditions at the ends of the rod

$$v(0) = 0, \quad v(l) = 0.$$  

They correspond to the hinged support of the ends of the rod, which is fixed in the transverse direction.

Equation (1) and additional conditions (2) form a mathematical model of the problem. Its analytical solution is well known and consists of its own function

$$\sin kl = 0, \quad kl = n\pi, \quad n = 1, 2, \ldots$$
and eigenvalue

\[ F = F_n = \frac{n^2 \pi^2 E J}{l^2}, \quad n = 1, 2, 3, \ldots \]  

where \( n \) are natural numbers.

In addition to calculating the exact Euler formula (4), the critical forces can also be determined from the graph of the \( \sin[k(F/l)] \) function, which can be easily plotted in the Matlab environment. Those arguments \( F \) that make the function zero and will be the critical forces of \( F_k \). Determination of the critical force by the Euler formula is an analytical method, the second proposed method will be called analytic-graphic below.

The need for alternative methods needs the following clarification:

1. The analytical method consists in solving the homogeneous equation \( \sin[k(\pi/l)] = 0 \), which in such spectral problems is called characteristic. In this case, it is very simple and easy to solve. But in other complex variants of the boundary conditions, the analytical method is faced with the need to solve transcendental equations. In such cases, the analytical-graphical method becomes the simplest and most versatile.

2. Another reason is that boundary conditions (2) take into account very local situations, i.e. in a narrow zone of the end sections of the rod. Therefore, as soon as irregularities appear along the length of the rod of the type \( E(x) \), \( J(x) \), the formulation of the Euler problem and the algorithm for its solution have to be modified or, abandoning the analytical method, go to analytical-numerical-graphic.

3. The third circumstance about the importance of verifying the results obtained stimulates the need to use several verification tools. The coincidence of solutions to problems obtained by different methods is a confirmation of the correctness of the problem statement and the selected problem algorithms.

With the help of the following examples, we will confirm the listed theses and problem statements.

**Example 1.** Vertical bar with the design scheme according to Fig. 1 of steel pipe has original data: \( l = 7 \text{ m} \), \( E = 2,1 \times 10^{11} \text{ Pa} \), \( D = 10,8 \text{ cm} \), \( t = 1,4 \text{ mm} \).

It is required to determine the first three eigenvalues and forms of buckling.

To determine the eigenvalues and forms, two small computer programs were developed in the Matlab environment. They displayed on the monitor the graph of the function \( F = \sin(k(l)) \) (fig. 3) and three eigenforms (fig. 4). In the first figure, bold dots mark the points of intersection of the graph with the \( F \) axis. Their abscissas correspond to the three smallest eigenvalues. At the same time, note that Matlab has a "magnifying glass" that allows you to consider the neighborhoods of these points in a several thousand times enlarged form. Therefore, the eigenvalues read with its help should be recognized as highly accurate, despite their graphic origin. With the number of points \( n = 10001 \), it took only 0.683872 seconds. to obtain results by two methods, which turned out to be as follows:

- analytical \( \{28,175; 112,698; 253,571\} \text{ kN} \);
- analytical-graphic \( \{28,290; 112,801; 253,613\} \text{ kN} \).

Comparison of the two results indicates that they are almost identical.
Fig. 3. Graph $F - \sin(kl)$.

Fig. 4. Proper forms.

Fig. 4 shows three proper forms normalized to dimensionless unit. The numbers inscribed on the lines correspond to the numbers of the critical forces.

Example 1 deals with the simplest problem of buckling of a bar by analytical and analytical-graphic methods. At the same time, it was sufficient to use a mathematical model as part of an ordinary second-order differential equation and two boundary conditions about the obvious equality to zero of the displacements of the fixed ends. On these grounds, conclusions cannot be drawn that have sufficient argumentation. Therefore, we will continue to consider the topic with more complex calculation schemes and mathematical models.

We choose the main equation in the mathematical model in the form of an ordinary differential equation in the fourth degree at constant values of the elastic modulus and axial moment of inertia $E$ and $J$

$$bv^{IV}(x) + k^2 v''(x) = 0, \quad b = \frac{F}{EJ}, \quad k^2 = \frac{F}{b}, \quad x \in (0, l). \tag{5}$$

The schemes for the boundary conditions supplementing equation (5) can be very diverse. Some of them are shown in Fig. 5.

The general solution to equation (5) is the function

$$v(x) = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4, \tag{6}$$

where $A_i$ are arbitrary constants of integration to be determined. To do this, it is necessary to compose a system of linear algebraic equations with a quadratic matrix of the fourth order. Let's write it in matrix-vector form

$$BA^T = 0. \tag{7}$$

Here $B$ is the matrix of coefficients determined by the boundary conditions, $A$ is the column vector $(A_1, A_2, A_3, A_4)^T$, $T$ is the vector transposition sign.

First, we write out the derivatives of function (6) for an arbitrary point

$$v'(x) = A_1 k \cos kx - A_2 k \sin kx + A_3.$$
\begin{align*}
  v''(x) &= -A_1 k^2 \sin kx - A_2 k^2 \cos kx, \\
  v'''(x) &= -A_1 k^3 \cos kx + A_2 k^3 \sin kx.
\end{align*}

Let's look at a specific example.

**Example 2.** Let's take a design scheme (Fig. 6), which has great versatility [oily]. A vertical rod made of a steel pipe with a design scheme according to Fig. 6 has initial data: \( l = 7 \text{ m}, \ E = 2,1 \cdot 10^{11} \text{ Pa}, \ D = 10,8 \text{ cm}, \ t = 1,4 \text{ mm}, \ c_1 = 2000 \text{ Nm/rad}, \ c_2 = 1000 \text{ Nm/rad}, \ c_3 = 1500 \text{ N/m}. \)

It is required to determine the first three eigenvalues and forms of buckling by analytical-graphic method.

Consider the system of forces acting on the rod according to Fig. 6, which shows two calculation schemes: before loss and after loss of stability. Fragments show the reactions of the supports and internal forces acting in the end sections. Let's start by considering the equilibrium equations for a flat system of active and reactive forces and moments. As you know, there are three such equilibrium equations.

1) \( \sum X = 0. \ -F + X_0 = 0. \ X_0 = F. \)
2) \( \sum Y = 0. \ -Y_D + Y_0 = 0. \ Y_0 = Y_D = c_3 v(l). \)
3) \( \sum M_0 = 0. \ Y_D l - F v(l) + M = 0. \)

\[ M = F v(l) - Y_D l = F v(l) - c_3 v(l) l = v(l)(F - c_3 l). \]

**Figure 6.** Calculation schemes and boundary conditions.

Here, some force factors have already been replaced by the physical, mechanical and kinematic parameters of the deformed system.

Next, it is necessary to go over to the boundary conditions to equation (5) with the ultimate goal of determining the critical forces.

At the top end:

\[ M(l) = bv''(l) = c_2 v'(l), \quad Q(l) = b v''''(l) = c_3 v(l) - F v'(l). \]
At the lower end:

\[ v(0) = 0, \quad M(0) = bv''(0) = c_1v'(0), \quad Q(0) = b v''''(0) = c_1v(0). \]  

(10)

In (9), (10) there are five boundary conditions in which there are only four coefficients \( A_1 \). Consequently, the mathematical model needs to be refined by introducing one more component of the vector \( A \). For this purpose, we will make changes to the main function (6) and derivatives (8)

\[ v(x) = A_1 \sin kx + A_2 \cos kx + A_3 x^2 + A_4 x + A_5, \]  

(11)

\[ v'(x) = A_1 k \cos kx - A_2 k \sin kx + 2A_3 x + A_4, \]  

\[ v''(x) = -A_1 k^2 \sin kx - A_2 k^2 \cos kx + 2A_3, \]  

\[ v'''(x) = -A_1 k^3 \cos kx + A_2 k^3 \sin kx. \]  

(12)

We will use (11), (12) and rewrite (9), (10) in the form of five equations containing the vector \( A \) and the matrix \( B \) of the fifth order

\[ \alpha A_1 + \beta A_2 + \lambda A_3 - c_2 A_4 = 0, \quad \gamma A_1 + \delta A_2 + \mu A_3 + FA_4 = 0, \quad A_2 + A_5 = 0, \]  

\[ \epsilon A_1 + \eta A_2 + 2A_3 - c_1 A_4 = 0, \quad \kappa A_1 + \sigma A_2 + \theta A_3 + \xi A_4 + c_3 A_5 = 0. \]  

(13)

Here

\[ \alpha = -b k^2 \sin k l - c_2 k \cos k l, \quad \beta = -b k^2 \cos k l + c_2 k \sin k l, \quad \lambda = -2c_2 l, \]  

\[ \gamma = -b k^3 \cos k l - c_3 \sin k l + F k \cos k l, \quad \delta = b k^3 \sin k l - c_3 \cos k l, \quad \mu = -c_3 l^2 + 2F l, \]  

\[ \epsilon = -c_1 k, \quad \eta = -b k^2, \quad \kappa = -b k^3 - c_3 \sin k l, \quad \sigma = -c_3 \cos k l, \quad \xi = -c_3 l. \]

At this stage, you can start implementing equations (7), (13). Let's write them in the form

\[ BA^T = \begin{bmatrix} \alpha & \beta & \lambda & \mu & F \\ \gamma & \delta & \mu & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \epsilon & \eta & 2 & -c_1 & 0 \\ \kappa & \sigma & \theta & \xi & c_3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]  

(14)

Nonzero solutions of the system of equations (14) can exist only if the determinant of the matrix \( B \) is equal to zero, i.e.

\[ \det B(F) = 0. \]  

(15)

\[ \text{Figure 7. Graph } F - \text{det}. \]

Equation (15) is called characteristic and contains the unknown critical forces \( F_k, \ k = 1, 2, 3 \ldots \). Equation (15) is transcendental and it is not possible to find its analytical solution. A simple way out is to plot \( F - \det (B) \) on a computer and read visually on the monitor the \( F \) values at which the curve crosses the horizontal axis. They are the critical forces of the \( F_k \).
According to this algorithm in the environment of the computing complex, the graph is built, shown in Fig. 7. The result of calculations using the Matlab computing complex by the analytical-graphic method gave the following critical forces:

\[ F = \{10,443; 25,097; 101,514\} \text{ kN}. \]

From these and the following eigenvalues, which are not presented here, we can conclude that the density of eigenvalues along the F axis is greatly reduced.

With the initial data of Example 2, the eigenvalues were calculated for the classical Euler problem with hinged ends at the ends

\[ F = \{28,174; 112,698; 253,571\} \text{ kN}. \]

The strong decrease in the eigenvalues is explained by the fact that the upper hinged-fixed rigid Euler support is replaced by a flexible spring. Remove the coil springs \((c_1=c_2=0)\), and make the coil spring stiff enough \((c_3 = 15000 \text{ kN/m})\) and repeat the calculations. The computer program gave the result

\[ F = \{28,175; 112,698; 253,571\} \text{ kN}. \]

All three eigenvalues coincide with the Euler ones. Such a computational experiment is a verification of the mathematical model of the problem, the developed algorithm for its solution and a computer program.

The second and the following eigenvalues and forms are not of serious interest for applied static problems. But they can be useful in the analysis of high-speed shock and longitudinal variable loads.

4. Conclusions
1. It is impossible to solve the more complicated problems of determining eigenvalues using traditional classical methods. Even simple transcendental equations arising from the boundary conditions are solved with great difficulty and often with insufficient accuracy.

2. Designed for solving technical problems by mathematical methods, the specialized computational complex Matlab is able, with the help of its graphical tools, to significantly simplify the problem of determining eigenvalues.

3. The tasks considered here allow us to move on to even more complex cases of bars with piecewise constant sections, with intermediate supports, with variable sections, etc.

5. References
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