Analysis of Bell Based Euler Polynomials and Their Application

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Abstract
In the present article, we study Bell based Euler polynomials of order \( \alpha \) and investigate some correlation formula, summation formula and derivative formula. Also, we introduce some relations of Stirling numbers of the second kind. Moreover, we derive several important formulae of Bell based Euler polynomials by using umbral calculus.

Keywords
Bell polynomial · Euler polynomial · Stirling polynomial · Stirling number of second kind · Sheffer sequence

Mathematics Subject Classification
11B68 · 33B15 · 33C05 · 33C10 · 33C15 · 33C45 · 33E20

Introduction
The polynomials and numbers play an important role in the multifarious areas of science such as Mathematics, Applied science, Physics and Engineering sciences and some related research areas involving number theory, quantum mechanics, differential equations and mathematical physics (see [1,3,4,18]). Some well known polynomials are Bell polynomials, Genocchi polynomials, Euler polynomials, Bernoulli polynomials and Hermite polynomials.

In particular, the Bell polynomials are considered extremely important due to their various applications in different mathematical frameworks (see [1,3,18]). It is used in Statistics, Combinatorial analysis, representing Lucas and other polynomials. Moreover, they are also used as a tool for representing the \( n \)th derivative of a composite function. Stirling numbers are usually defined as the coefficients in an expansion of positive integral power of a variable in terms of factorial powers. They have been constantly studied and rediscovered due to their interesting properties and applications in solving problems. In the last few decades,
several extensions of Stirling numbers are studied related to combinatorial, probabilistic and statistical applications.

The properties of polynomials arising from umbral calculus are studied by many authors. Kim et al. [11] studied the properties of Bernoulli, Euler, Abel, Bell and other polynomials by using umbral calculus and gave some useful identities. They also investigated partially degenerate Bell numbers and polynomials associated with umbral calculus (see also, [7,12, 14,15]).

Motivated by the above-mentioned work, in the present article we introduce Bell based Euler polynomials of order \( \alpha \) and investigate some useful correlation formula, summation formula and derivative formula of it. We obtain some implicit summation formula and special cases of Bell based Euler polynomials of order \( \alpha \). Also, we derived some relation between the Bell based Euler polynomials of order \( \alpha \) and Stirling numbers of the second kind. Further, we define Bell based Euler polynomials treated as a Sheffer sequence and using properties of Sheffer sequence we derive some application of Bell based Euler polynomials arising from umbral calculus.

**Preliminaries**

In the present paper, we take the symbols \( \mathbb{Z}, \mathbb{N}, \mathbb{N}_0, \mathbb{R}, \) and \( \mathbb{C} \) to be the set of integers, set of natural numbers, set of non negative integers, set of real numbers and set of complex numbers respectively.

The bivariate Bell polynomials are described by the following generating function (see [7]):

\[
\sum_{n \geq 0} B_n(x; y) \frac{t^n}{n!} = e^{xt} e^{y(e^t - 1)}. \quad (1)
\]

When we take \( x = 0 \), \( B_n(0; y) = B_n(y) \) are called classical Bell polynomials (or exponential polynomials), which are described by following generating function (see [2–4,13,17]):

\[
\sum_{n \geq 0} B_n(y) \frac{t^n}{n!} = e^{y(e^t - 1)}. \quad (2)
\]

If we take \( y = 1 \) in (2) i.e. \( B_n(0; 1) = B_n(1) = B_n \) are called Bell numbers which are defined as follows (see [2–4,13,17]):

\[
\sum_{n \geq 0} B_n \frac{t^n}{n!} = e^{(e^t - 1)}. \quad (3)
\]

The generating functions of Euler polynomials of order \( \alpha \) (see [8–11,19,20]) is given as:

\[
\sum_{n \geq 0} E_n^{(\alpha)}(x) \frac{t^n}{n!} = e^{xt} \left( \frac{2}{e^t + 1} \right)^\alpha \quad (|t| < 2\pi). \quad (4)
\]

If we take \( x = 0 \) in (4) i.e. \( E_n^{(\alpha)}(0) = E_n^{(\alpha)} \) are called Euler numbers (see [8–11,19,20]) which is shown as follows:

\[
\sum_{n \geq 0} E_n^{(\alpha)} \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^\alpha. \quad (5)
\]
The generating function of second kind Stirling polynomials $S_2(n, k; x)$ and Stirling numbers $S_2(n, k)$ are defined as (see [2,3]):

$$\sum_{n \geq 0} S_2(n, k; x) \frac{t^n}{n!} = \left( \frac{e^t - 1}{k!} \right)^{k} e^{tx}. \quad (6)$$

When $x = 0$ in (6) i.e. $S_2(n, k; 0) = S_2(n, k)$ are called Stirling numbers and defined by following exponential generating function (see [2,3]):

$$\sum_{n \geq 0} S_2(n, k) \frac{t^n}{n!} = \left( \frac{e^t - 1}{k!} \right)^{k}. \quad (7)$$

The Stirling numbers of the second kind $S_2(n, k)$ denotes the partition of $n$ object into $k$ nonempty block. So, the Stirling numbers play an important role in every field of mathematics such as combinatorial theory, number theory, numerical analysis etc.

The Bell polynomials related to Stirling number of the second kind are defined by the following relation [13]:

$$B_n(y) = \sum_{k=m}^{n} S_2(n, k) y^m. \quad (8)$$

**Bell Based Euler Polynomials and Numbers**

In this section, we introduce a unified class of Bell based Euler polynomials of order $\alpha$ and investigate various correlation formulae like implicit summation formulae, derivative formulae.

For any $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, we define Bell based Euler polynomials of order $\alpha$ as:

$$\sum_{n \geq 0} B_E^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^{\alpha} e^{xt+y(e^t-1)}(|t| < 2\pi). \quad (9)$$

If $x=0$ and $y=1$ in (9) then we get a Bell Based Euler numbers of order $\alpha$, which are defined as follows:

$$\sum_{n \geq 0} B_E^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^{\alpha} e^{(e^t-1)}(|t| < 2\pi). \quad (10)$$

**Special Cases**

The class of Bell based Euler polynomials of order $\alpha$ introduced here generalizes many polynomials present in the existing literature and their generating functions can be obtained directly from (9) (see [7–10,19]). In this section, we introduce some special cases of Bell based Euler polynomials of order $\alpha$, which are obtained by specializing the particular values in (9).
If we choose $\alpha = 0$ in (9), the Bell based Euler polynomials of order $\alpha$ reduce to bivariate Bell polynomials [7]:

$$
\sum_{n \geq 0} B^{(\alpha)}_n(x; y) \frac{t^n}{n!} = e^{xt+y(e^t-1)}.
$$

(2) In case $y=0$ in (9), the Bell based Euler polynomials of order $\alpha$ reduce to the familiar Euler polynomials $E_{n}^{(\alpha)}(x)$ of order $\alpha$ [10,18]

$$
\sum_{n \geq 0} B^{(\alpha)}_n(x) \frac{t^n}{n!} = \left( \frac{2}{e^{\alpha} + 1} \right)^\alpha e^{xt}.
$$

(3) In case $y=0$ and $\alpha=1$ in (9), the Bell Based Euler polynomials $B^{(\alpha)}_n(x; y)$ reduce to usual Euler polynomials $E_{n}(x)$ [18,19]

$$
\sum_{n \geq 0} B^{(\alpha)}_n(x) \frac{t^n}{n!} = \left( \frac{2}{e^{\alpha} + 1} \right) e^{xt}.
$$

Theorem 1 The following relation hold true for $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$
B^{(\alpha)}_n(x; y) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) E_{k}^{(\alpha)}(x) B_{n-k}(y).
$$

Proof By using relation (9), we have

$$
\sum_{n \geq 0} B^{(\alpha)}_n(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^{\alpha} + 1} \right)^\alpha e^{xt+y(e^t-1)}
$$

$$
= \left\{ \left( \frac{2}{e^{\alpha} + 1} \right)^\alpha e^{xt} \right\} \left\{ e^{y(e^t-1)} \right\}
$$

$$
= \left\{ \sum_{k \geq 0} E_{k}^{(\alpha)}(x) \frac{t^k}{k!} \right\} \left\{ \sum_{n \geq 0} B_{n}(y) \frac{t^n}{n!} \right\}
$$

Now, using series rearrangement method, we get

$$
\sum_{n \geq 0} B^{(\alpha)}_n(x; y) \frac{t^n}{n!} = \sum_{n \geq 0} \left\{ \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) E_{k}^{(\alpha)}(x) B_{n-k}(y) \right\} \frac{t^n}{n!}.
$$

By equating the same power of $t$ both side, we get the desired result (11). $\square$

Theorem 2 The Bell based Euler polynomials of order $\alpha$ satisfy the following relation for any $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$:

$$
B^{(\alpha)}_n(x; y) = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) E_{k}^{(\alpha)}(y) B_{n-k}(x).
$$
Proof By using generating function (9), we have
\[
\sum_{n \geq 0} B^{E}(\alpha)_{n} (x; y) \frac{t^{n}}{n!} = \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{x t + y(e^{t} - 1)}
\]
\[
= \{ \alpha \} \left( \frac{2}{e^{t} + 1} \right)^{\alpha} \left\{ e^{x t + y(e^{t} - 1)} \right\}
\]
\[
= \left\{ \sum_{k \geq 0} E^{(\alpha)}_{k} \frac{t^{k}}{k!} \right\} \left\{ \sum_{n \geq 0} B_{n} (x; y) \frac{t^{n}}{n!} \right\}.
\]

After applying series rearrangement technique, we get the desired result (12). \qed

Theorem 3 If \( \alpha \in \mathbb{C} \) and \( n \in \mathbb{N} \), the following relation hold true:
\[
B^{E}(\alpha)_{n} (x; y) = \sum_{k=0}^{n} \binom{n}{k} E^{(\alpha)}_{k} (y) x^{n-k}.
\]
(13)

Proof Using relation (9), we have
\[
\sum_{n \geq 0} B^{E}(\alpha)_{n} (x; y) \frac{t^{n}}{n!} = \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{x t + y(e^{t} - 1)}
\]
\[
= \{ \alpha \} \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{y(e^{t} - 1)} e^{x t} \left\{ e^{x t} \right\}
\]
\[
= \left\{ \sum_{k \geq 0} B^{E}(\alpha)_{k} (y) \frac{t^{k}}{k!} \right\} \left\{ \sum_{n \geq 0} \frac{(x t)^{n}}{n!} \right\}
\]
\[
= \left\{ \sum_{n \geq 0} \sum_{k \geq 0} B^{E}(\alpha)_{k} (y) \frac{x^{n+k}}{n!} \frac{t^{n+k}}{k!} \right\}.
\]

using series rearrangement, we get
\[
\sum_{n \geq 0} B^{E}(\alpha)_{n} (x; y) \frac{t^{n}}{n!} = \sum_{n \geq 0} \left\{ \sum_{k=0}^{n} \binom{n}{k} B^{E}(\alpha)_{k} (y) x^{n-k} \right\} \frac{t^{n}}{n!}.
\]

By equating the same power of \( t \) both side, we get the desired result (13). \qed

Implicit Summation Formulae

In this section, we define some implicit summation formulae for Bell based Euler polynomials of order \( \alpha \) given by the following theorems:

Theorem 4 For \( \alpha_{1}, \alpha_{2} \in \mathbb{C} \) and \( n \in \mathbb{N} \), the following relation hold true:
\[
B^{E}(\alpha_{1}+\alpha_{2})_{n} (x_{1} + x_{2}; y_{1} + y_{2}) = \sum_{k=0}^{n} \binom{n}{k} B^{E}(\alpha_{1})_{k} (x_{1}; y_{1}) B^{E}(\alpha_{2})_{n-k} (x_{2}; y_{2}).
\]
(14)
Proof Using the following identity

\[
\left( \frac{2}{e^t + 1} \right)^{\alpha_1 + \alpha_2} e^{(x_1 + x_2)t + (y_1 + y_2)(e^t - 1)}
\]

\[= \left\{ \left( \frac{2}{e^t + 1} \right)^{\alpha_1} e^{x_1 t + y_1 (e^t - 1)} \right\} \left\{ \left( \frac{2}{e^t + 1} \right)^{\alpha_2} e^{x_2 t + y_2 (e^t - 1)} \right\}.
\]

By using generating function (9), we have

\[
\sum_{n \geq 0} \mathcal{B}^{\alpha_1 + \alpha_2}_n (x_1 + x_2; y_1 + y_2) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^{\alpha_1 + \alpha_2} e^{(x_1 + x_2)t + (y_1 + y_2)(e^t - 1)}
\]

\[= \left\{ \left( \frac{2}{e^t + 1} \right)^{\alpha_1} e^{x_1 t + y_1 (e^t - 1)} \right\} \left\{ \left( \frac{2}{e^t + 1} \right)^{\alpha_2} e^{x_2 t + y_2 (e^t - 1)} \right\}
\]

\[= \left\{ \sum_{k \geq 0} \mathcal{E}_k^{(\alpha_1)} (x_1; y_1) \frac{t^k}{k!} \right\} \left\{ \sum_{n \geq 0} \mathcal{B}_n^{(\alpha_2)} (x_2; y_2) \frac{t^n}{n!} \right\}
\]

\[= \sum_{n \geq 0} \sum_{k \geq 0} \mathcal{B}_k^{(\alpha_1)} (x_1; y_1) \mathcal{B}_n^{(\alpha_2)} (x_2; y_2) \frac{t^{n+k}}{n! k!}.
\]

using series rearrangement technique, we obtain

\[
\sum_{n \geq 0} \mathcal{B}^{\alpha_1 + \alpha_2}_n (x_1 + x_2; y_1 + y_2) \frac{t^n}{n!} = \sum_{n \geq 0} \left\{ \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k^{(\alpha_1)} (x_1; y_1) \mathcal{B}_n^{(\alpha_2)} (x_2; y_2) \right\} \frac{t^n}{n!}.
\]

Now, equating the same power of \( t \) both side, we get the desired result (14). \( \square \)

Remark 1 In case, if we choose \( \alpha_1 = \alpha, \alpha_2 = 0, x_1 = x, x_2 = 1, y_1 = y \) and \( y_2 = 0 \) in (14), we have

\[
\mathcal{B}^{\alpha}_n (x + 1; y) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k^{(\alpha)} (x; y),
\]

which is an extension of Euler polynomials defined by

\[
\mathcal{E}_n (x + 1) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k (x).
\]

Theorem 5 If \( \alpha \in \mathbb{C} \) and \( n \in \mathbb{N} \), the Bell based Euler polynomials of order \( \alpha \) satisfy the following summation formula:

\[
\mathcal{B}^{(\alpha)}_n (x + 1; y) - \mathcal{B}^{(\alpha)}_{n-1} (x; y) = \sum_{k=0}^{n} \binom{n+1}{k} \mathcal{B}_k^{(\alpha)} (x; y).
\]
Proof Using (9), we get

\[
\sum_{n \geq 0} \mathcal{B} \mathcal{E}_n^{(\alpha)}(x + 1; y) \frac{t^n}{n!} - \sum_{n \geq 0} \mathcal{B} \mathcal{E}_n^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^\alpha e^{(x+1)t+y(e'^{-1})} - \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt+y(e'^{-1})}
\]

\[
= \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt+y(e'^{-1})} (e^t - 1)
\]

\[
= \left\{ \sum_{k \geq 0} \mathcal{B} \mathcal{E}_k^{(\alpha)}(x; y) \frac{t^k}{k!} \right\} \left\{ \sum_{n \geq 0} \frac{t^{n+1}}{(n+1)!} \right\}.
\]

Using series rearrangement technique, we get the desired result (17). \(\square\)

Theorem 6 If \(\alpha = 1\) and \(n \in \mathbb{N}\) then the Bell based Euler polynomials satisfy the following relation:

\[\mathcal{B} \mathcal{E}_n(x; y) = \frac{\mathcal{B} \mathcal{E}_n(x + 1; y) + \mathcal{B} \mathcal{E}_n(x; y)}{2}.\] (18)

Proof Using the generating function (9) for \(\alpha = 1\) and definition of bivariate Bell polynomials, we get

\[
\sum_{n \geq 0} \mathcal{B} \mathcal{E}_n(x; y) \frac{t^n}{n!} = e^{xt+y(e'^{-1})}
\]

\[
= \frac{e^t + 1}{2} \left\{ \sum_{k=0}^{\infty} \mathcal{B} \mathcal{E}_k(x; y) \frac{t^n}{n!} \right\}
\]

\[
= \frac{e^t + 1}{2} \left\{ \left( \frac{2}{e^t + 1} \right) e^{xt+y(e'^{-1})} \right\}
\]

\[
= \frac{1}{2} \left\{ \left( \frac{2}{e^t + 1} \right) e^{(x+1)t+y(e'^{-1})} + \left( \frac{2}{e^t + 1} \right) e^{xt+y(e'^{-1})} \right\}
\]

\[
= \frac{1}{2} \left\{ \sum_{n \geq 0} \mathcal{B} \mathcal{E}_n(x + 1; y) \frac{t^n}{n!} + \sum_{n \geq 0} \mathcal{B} \mathcal{E}_n(x; y) \frac{t^n}{n!} \right\}.
\]

By equating the same power of \(t\) both side, we get the desired result (18). \(\square\)

Remark 2 The formula (18) is the generalized form of the Euler polynomials given as

\[x^n = \frac{\mathcal{E}_n(x + 1) + \mathcal{E}_n(x)}{2}.\] (19)

Theorem 7 If \(n \geq 0\), then

\[\mathcal{B} \mathcal{E}_n^{(\alpha)}(x; y) = \sum_{j=0}^{n} \sum_{k \geq 0} \binom{n}{j} (x)_k \mathcal{S}_2(j, k) \mathcal{B} \mathcal{E}_n^{(\alpha)}(y).\] (20)
Proof By using (9), we have

\[
\sum_{n \geq 0} BE_n^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{xt+y(e^t-1)}
\]

\[
= \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{y(e^t-1)} e^{xt}
\]

\[
= \left( \frac{2}{e^{t} + 1} \right)^{\alpha} e^{y(e^t-1)} (1 + e^{t} - 1)^{\alpha}
\]

\[
= \left\{ \sum_{n \geq 0} BE_n^{(\alpha)}(y) \frac{t^n}{n!} \right\} \left\{ \sum_{k \geq 0} (x)_k \frac{(e^t - 1)^k}{k!} \right\}
\]

\[
= \left\{ \sum_{n \geq 0} BE_n^{(\alpha)}(y) \frac{t^n}{n!} \right\} \left\{ \sum_{k \geq 0} (x)_k \sum_{j \geq 0} S_2(j, k) \frac{t^j}{j!} \right\}.
\]

By using series rearrangement technique and equating same power of \( t \) both side, we get the desired result (20).

\[\square\]

**Derivative Formulas**

**Theorem 8** The differential operator formula for the Bell based Euler polynomials of order \( \alpha \) w.r.t. \( x \) is defined as follows:

\[
\frac{\partial}{\partial x} BE_n^{(\alpha)}(x; y) = n \ BE_{n-1}^{(\alpha)}(x; y),
\]

which holds for all \( n \in \mathbb{N} \).

**Proof** we know that

\[
\frac{\partial}{\partial x} e^{xt+y(e^t-1)} = t \ e^{xt+y(e^t-1)}.
\]

By using definition (9) in (22), we obtain the desired result (21).

\[\square\]

**Theorem 9** The difference operator formula for the Bell based Euler polynomials of order \( \alpha \) w.r.t. \( y \) is defined as:

\[
\frac{\partial}{\partial y} BE_n^{(\alpha)}(x; y) = (-2) \ \left\{ BE_n^{(\alpha)}(x; y) - BE_{n-1}^{(\alpha)}(x; y) \right\},
\]

which holds for all \( n \in \mathbb{N} \).

**Proof** By using well known derivative properties

\[
\frac{\partial}{\partial y} e^{xt+y(e^t-1)} = (e^t - 1) \ e^{xt+y(e^t-1)}.
\]
Now, using definition (9), we get
\[
\begin{align*}
\frac{\partial}{\partial y} \left\{ \sum_{n \geq 0} \mathcal{B}_{n}^{(\alpha)}(x; y) \frac{t^n}{n!} \right\} &= \frac{\partial}{\partial y} \left\{ \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt + y(e^t - 1)} \right\} \\
&= \left\{ \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt + y(e^t - 1)} \right\} (e^t - 1) \\
&= (-2 + e^t + 1) \left\{ \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt + y(e^t - 1)} \right\} \\
&= (-2) \left( 1 - \frac{e^t + 1}{2} \right) \left\{ \left( \frac{2}{e^t + 1} \right)^\alpha e^{xt + y(e^t - 1)} \right\}.
\end{align*}
\]

Now, using (9) in l.h.s of above equation and equating the same power of \( t \) both sides, we get the desired result (23).

\[\Box\]

**Bell Based Euler Polynomials Associated with Umbral Calculus**

In this section, we exploit the concepts of umbral calculus (see [5–7,11,12,14,16]) to derive some useful relations of Bell based Euler polynomials.

Let \( \mathcal{G} \) be set of all formal power series in the variable \( t \) over complex field \( \mathbb{C} \) with
\[
\mathcal{G} = \left\{ g(t) = \sum_{k \geq 0} \frac{c_k}{k!} t^k \text{ s.t. } c_k \in \mathbb{C} \right\}.
\]

Let \( \mathcal{P} \) be set of polynomials in the single variable \( t \) and \( \mathcal{P}^* \) be the set of vector space of all linear functional on \( \mathcal{P} \). In the umbral calculus, we denote \( \langle T | q(x) \rangle \) be linear functional \( T \) on the polynomials \( q(x) \). Now, we define the vector space operations on \( \mathcal{P}^* \) as follows:
\[
\langle T_1 + T_2 | q(x) \rangle = \langle T_1 | q(x) \rangle + \langle T_2 | q(x) \rangle
\]
and
\[
\langle \beta T | q(x) \rangle = \beta \langle T | q(x) \rangle
\]
for any constant \( \beta \) in \( \mathbb{C} \).

The formal power series
\[
g(t) = \sum_{k \geq 0} \frac{c_k}{k!} t^k \in \mathcal{G}
\]
defined a linear functional on \( \mathcal{P} \) as
\[
\langle g(t) | x^n \rangle = c_n
\]
for all \( n \in \mathbb{N} \cup \{0\} \).

If we choose \( g(t) = t^k \) in (26) and (27), we get
\[
\langle t^k | x^n \rangle = n! \delta_{n,k},
\]
for all \( n, k \in \mathbb{N} \cup \{0\} \) and
\[
\delta_{n,k} = \begin{cases} 
0 & \text{if } n \neq k \\
1 & \text{if } n = k
\end{cases}.
\]
Since any linear functional $T$ in $\mathcal{P}^*$ has in the form of (26) i.e.,

$$g_T(t) = \sum_{k \geq 0} \langle T | x^k \rangle \frac{t^k}{k!}$$

and

$$\{ g_T(t) | x^n \} = \{ T | x^n \}.$$  

So, the linear functional $T = g_T(t)$. We know that, the map $T \rightarrow g_T(t)$ is a vector space isomorphism from the set of vector space of all linear functionals on $\mathcal{P}$ onto set of formal series $\mathcal{G}$. Therefore, set of formal series $\mathcal{G}$ have vector space of all linear functionals on $\mathcal{P}$, Also $\mathcal{G}$ have algebra of formal power series and so for $g(t) \in G$ will be treated as both a formal power series and a linear functional. From (27), we get

$$\langle e^{yt} | x^n \rangle = y^n$$

and so, we have

$$\langle e^{yt} | q(x) \rangle = q(y)$$

for all $q(x) \in \mathcal{P}$.

We know that the order of power series $g(t)$ (i.e. $o(g(t))$) will be the smallest positive integer $k$ such that the coefficient of $t^k$ does not vanish. We know that $g(t)$ is invertible series if the order of formal power series $g(t)$ is zero. Also the formal power series $g(t)$ is a delta series if the order of $g(t)$ is one (i.e. $o(g(t) = 1$). (see [5–7,11,12,14,16]).

For $g_1(t), \ldots, g_m(t) \in G$, then

$$\{ g_1(t), \ldots, g_m(t) | x^n \} = \sum_{i_1+i_2+\ldots+i_m=n} \binom{n}{i_1, \ldots, i_m} \{ g_1(t) | x^{i_1} \} \ldots \{ g_m(t) | x^{i_m} \},$$

where

$$\binom{n}{i_1, \ldots, i_m} = \frac{n!}{i_1! \ldots i_m!}.$$

If $g(t), h(t) \in G$, then

$$\langle g(t)h(t) | q(x) \rangle = \langle g(t) | h(t)q(x) \rangle = \langle h(t) | g(t)q(x) \rangle.$$

Thus, $\forall g(t) \in G$

$$g(t) = \sum_{k \geq 0} \{ g(t) | x^k \} \frac{t^k}{k!},$$

and $\forall$ polynomials $q(x)$

$$q(x) = \sum_{k \geq 0} \{ t^k | q(x) \} \frac{t^k}{k!}.$$

By using (33), we get

$$q^k(x) = D^k q(x) = \sum_{k \geq 0} \frac{\{ t^l | q(x) \} \frac{t^k}{l!}}{l} x^{l-k} \prod_{s=1}^{k} (l-s+1).$$
From (34), we have
\[ q^k(0) = \left\{ t^k \mid q(x) \right\} \quad \text{and} \quad q^k(0) = \left\{ 1 \mid q^k(x) \right\}. \] (35)

By (35), we note that
\[ t^k q(x) = q^k(x). \] (36)

Let \( g(t) \) and \( h(t) \) be an element of formal power series \( G \) such that \( g(t) \) be a delta and \( h(t) \) be an invertible series. Then \( \exists \) a unique sequence \( S_n(x) \) of the polynomials with following properties
\[ \left\{ h(t) g(t)^k \mid S_n(x) \right\} = n! \delta_{n,k}. \] (37)

For all \( n, k \in \mathbb{N} \cup \{0\} \), which is orthogonality condition for the Sheffer sequence (see [5–7,11,12,14,16]).

The sequence \( S_n(x) \) is said to be Sheffer sequence for \((h(t), g(t))\), which is denoted by \( S_n(x) \sim (h(t), g(t)) \).

Let \( S_n(x) \) is the Sheffer sequence for \((h(t), g(t))\). Then for \( f(t) \in G \) and for \( q(x) \), we have the following relation:
\[ f(t) = \sum_{k \geq 0} \frac{\langle f(t) \mid S_k(x) \rangle}{k!} h(t) g(t)^k, \] (38)

and
\[ q(x) = \sum_{k \geq 0} \frac{\langle h(t) g(t)^k \mid q(x) \rangle}{k!} S_k(x), \] (39)

and the sequence \( S_n(x) \) be a Sheffer sequence for \((h(t), g(t))\), iff
\[ \frac{1}{h(\tilde{g}(t))} e^{y \tilde{g}(t)} = \sum_{k \geq 0} \frac{S_k(y)}{k!} t^k, \quad \forall \ y \in \mathbb{C}. \] (40)

Here, \( \tilde{g}(t) \) is composition inverse of \( g(t) \) i.e. \( \tilde{g}(g(t)) = g(\tilde{g}(t)) = t \).

Suppose that \( S_n(x) \) be an Appell sequence for \( h(t) \). From (40), we get
\[ S_n(x) = \frac{1}{h(t)} x^n \Leftrightarrow t S_n(x) = n S_{n-1}(x). \] (41)

Recently, many authors have studied Euler polynomials, Bernoulli polynomials and Bell polynomials under the theory of umbral calculus (see [5–7,11,12,14–16]).

Recalling from (9), we have
\[ \sum_{n \geq 0} B E_n^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right) e^{xt + y(e^t - 1)} \quad (|t| < 2\pi). \] (42)

As \( t \) goes to zero in (42) gives \( B E_n(x; y) \) equal to one (i.e. \( B E_n(x; y) = 1 \)) which means that \( o \left( \frac{2}{e^t + 1} \right) e^{xt + y(e^t - 1)} = 0 \), which implies that (9) is an invertible series and treated as a Sheffer sequence.

We define some properties of Bell based Euler polynomials arising from umbral calculus as follows:
From (40) and (42), we have
\[ \mathcal{B}E_n(x; y) \sim \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)}, t \right), \tag{43} \]
and
\[ t \mathcal{B}E_n(x; y) = n \mathcal{B}E_{n-1}(x; y). \tag{44} \]

Then from (43) and (44) we say that \( \mathcal{B}E_n(x; y) \) is an Appell sequence for \( \frac{e^t + 1}{2} e^{-y(e^t - 1)} \).

**Theorem 10** If \( q(x) \in \mathcal{P} \), there exist constants \( b_0, b_1, \ldots, b_n \) such that
\[ q(x) = \sum_{k=0}^{n} b_k \mathcal{B}E_n(x; y), \tag{45} \]
where
\[ b_k = \frac{1}{k!} \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)} t^k \right| q(x) \right). \tag{46} \]

**Proof** From (37), (40) and (43), we noted that
\[ \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)} t^k \right| \mathcal{B}E_n(x; y) = n! \delta_{n,k} \quad n, k \in \mathbb{N} \cup \{0\}. \]

By using (45), we obtain
\[ \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)} t^k \right| q(x) = \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)} t^k \right| \sum_{l=0}^{n} b_l \mathcal{B}E_n(x; y) \right) = \sum_{l=0}^{n} b_l \left( \frac{e^t + 1}{2} e^{-y(e^t - 1)} t^k \right| \mathcal{B}E_n(x; y) \right) \]
\[ = \sum_{l=0}^{n} b_l l! \delta_{l,k} = k! b_k. \]

We get the desired result (46). \( \square \)

**Theorem 11** If \( n \in \mathbb{N} \), we have
\[ \int_{x}^{x+z} \mathcal{B}E_n(v; y) \, dv = \frac{e^{zt} - 1}{t} \mathcal{B}E_n(x; y). \tag{47} \]

**Proof** By using (44), we have
\[ \int_{x}^{x+z} \mathcal{B}E_n(v; y) \, dv = \frac{1}{n+1} \left( \mathcal{B}E_{n+1}(x + z; y) - \mathcal{B}E_{n+1}(x; y) \right) \]
\[ = \frac{1}{n+1} \sum_{k \geq 1} \binom{n+1}{k} \mathcal{B}E_{n+1-k}(x; y) z^k \]
\[ = \frac{1}{n+1} \left( \sum_{k \geq 0} \frac{z^k}{k!} t^k - 1 \right) \mathcal{B}E_n(x; y) \]
\[ = \frac{e^{zt} - 1}{t} \mathcal{B}E_n(x; y), \]
we get the desired result. □

**Corollary 1** If \( n \in \mathbb{N} \cup \{0\} \), we have

\[
\int_0^z B^E_n(v; y) \, dv = \left\{ \frac{e^{zt} - 1}{t} \right\} B^E_n(y). \tag{48}
\]

**Proof** From (44), we get

\[
B^E_n(x; y) = \frac{t}{n+1} B^E_{n+1}(x; y), \tag{49}
\]

and by using (49), we have

\[
\left\{ \frac{e^{zt} - 1}{t} \right\} B^E_n(y) = \left\{ \frac{e^{zt} - 1}{t} \right\} \frac{1}{n+1} B^E_{n+1}(y) \]
\[
= \frac{1}{n+1} \{ B^E_{n+1}(z; y) - B^E_{n+1}(0; y) \}
\]
\[
= \int_0^z B^E_n(v; y) \, dv. \]

□

For any \( \mu \in \mathbb{N} \cup \{0\} \) from (9) the Bell based Euler polynomials of order \( \mu \) are given as:

\[
\sum_{n \geq 0} B^E_n^{(\mu)}(x; y) \frac{t^n}{n!} = \left( \frac{2}{e^t + 1} \right)^\mu e^{xt+y(e^t-1)} \quad (|t| < 2\pi). \tag{50}
\]

When \( t \) goes to zero then \( B^E_n^{(\mu)}(x; y) = 1 \), which means that \( o\left( \left( \frac{2}{e^t + 1} \right)^\mu e^{xt+y(e^t-1)} \right) = 0 \). Hence the generating function (50) of order \( \mu \) is an invertible and it will be treated as Sheffer sequence.

Suppose that

\[
g^\mu(t; y) = \frac{(e^t + 1)^\mu}{2^\mu} e^{-y(e^t-1)}. \]

We know that \( g^\mu(t; y) \) is an invertible series. From (50) we say that \( B^E_n^{(\mu)}(x; y) \) is an Appell sequence for \( g^\mu(t; y) \). Hence from (41), we have

\[
B^E_n^{(\mu)}(x; y) = \frac{1}{g^\mu(t; y)} x^n,
\]

and

\[
t \cdot B^E_n^{(\mu)}(x; y) = n \cdot B^E_{n-1}^{(\mu)}(x; y).
\]

Thus, we have

\[
B^E_n^{(\mu)}(x; y) \sim \left( \frac{(e^t + 1)^\mu}{2^\mu} e^{-y(e^t-1)}, t \right).
\]

Now, using the above result we derive some interesting theorem.
Theorem 12  If \( n \geq 0 \), then
\[
B E_n^{(\mu)}(y) = \sum_{i_1 + \ldots + i_\mu = n} \binom{n}{i_1, \ldots, i_\mu} B E_{i_\mu}(y) \prod_{j=1}^{\mu-1} E_{i_j}.
\]  (51)

Proof  By using (27) and (50), we have
\[
\left\langle \frac{2^\mu}{(e^t + 1)\mu} e^{zt + ty(e' - 1)} \mid x^n \right\rangle = B E_n^{(\mu)}(z; y) = \sum_{l=0}^{n} \binom{n}{l} B E_{n-l}^{(\mu)}(y) z^l,
\]  (52)
and
\[
\left\langle \frac{2^\mu}{(e^t + 1)\mu} e^{zt + ty(e' - 1)} \mid x^n \right\rangle = \left\langle \frac{2}{e^t + 1} \times \ldots \times \frac{2}{e^t + 1} e^{zt + ty(e' - 1)} \mid x^n \right\rangle = \sum_{i_1 + \ldots + i_\mu = n} \binom{n}{i_1, \ldots, i_\mu} B E_{i_\mu}(y) \times E_{i_1} \times \ldots \times E_{i_{\mu-1}}.
\]  (53)

From (52) and (53), we get the result (51). \( \square \)

Theorem 13  If \( q(x) \in P_n \), then
\[
q(x) = \sum_{k=0}^{n} b_k^{\mu} B E_n^{(\mu)}(x; y) \in P_n,
\]
where
\[
b_k^{\mu} = \frac{1}{2^\mu k!} \left( (e^t + 1)\mu e^{-y(e' - 1)} t^k \right) q(x).
\]  (54)

Proof  Let us assume that
\[
q(x) = \sum_{k=0}^{n} b_k^{\mu} B E_n^{(\mu)}(x; y) \in P_n,
\]  (55)
with the help of (55), we can write
\[
\left\langle \left( \frac{e^t + 1}{2} \right)^\mu e^{-y(e' - 1)} t^k \mid q(x) \right\rangle = \left\langle \left( \frac{e^t + 1}{2} \right)^\mu e^{-y(e' - 1)} t^k \mid \sum_{l=0}^{n} b_l^{\mu} B E_n^{(\mu)}(x; y) \right\rangle = \sum_{l=0}^{n} b_l^{\mu} \left\langle \left( \frac{e^t + 1}{2} \right)^\mu e^{-y(e' - 1)} t^k \mid B E_n^{(\mu)}(x; y) \right\rangle
\]  (56)
and we know that
\[
\left\langle \left( \frac{e^t + 1}{2} \right)^\mu e^{-y(e' - 1)} t^k \mid q(x) \right\rangle = \frac{1}{2^\mu} \left( (e^t + 1)\mu e^{-y(e' - 1)} t^k \right) q(x).
\]  (57)
Hence from (56) and (57), we obtain the result (54). \( \square \)
Conclusions

Motivated by the potential of applications in number theory, combinatorics and other fields, we have introduced a unified class of Bell based Euler polynomials of order $\alpha$ and presented various correlation, implicit summation and derivative formulas. Moreover, we have studied some properties of Bell based Euler polynomials using umbral calculus. The results obtained generalizes several well-known results and can be specialized to yield a large number of new and known identities involving basic and unified polynomials given by other authors. The results involving umbral calculus, Bell and Euler polynomials are interesting as they open several new problems of umbral calculus related to other unified polynomials as well as their special cases.

For future direction, we define here a new class of unified polynomials based on Bell and Genocchi polynomials. They are defined by means of the generating function:

$$\sum_{n \geq 0} B_{G}^{(\alpha)}(x; y) \frac{t^n}{n!} = \left( \frac{2t}{e^t + 1} \right)^\alpha e^{xt + y(e^t - 1)} \quad (|t| < 2\pi).$$

This shows that the polynomials that we have introduced open ways to study several new polynomials and similar properties that we have derived using rearrangement techniques and umbral calculus.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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