FLOWS WITH THE WEAK TWO-SIDED LIMIT SHADOWING PROPERTY

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Abstract. In this paper we study the weak two-sided limit shadowing for flows on a compact metric space which is different with the usual shadowing, two-sided limit shadowing and L-shadowing, and characterize the weak two-sided limit shadowing flows from the pointwise and measurable viewpoints. Moreover, we prove that if a flow \( \phi \) has the weak two-sided limit shadowing property on its chain recurrent \( \text{CR}(\phi) \) then the set \( \text{CR}(\phi) \) is decomposed by a finite number of closed invariant sets on which \( \phi \) is topologically transitive and has the two-sided limit shadowing property.

1. Introduction. The theory of shadowing in dynamical systems is a rapidly developing branch of modern global theory of dynamical systems. There are various notions of shadowing for dynamical systems (homeomorphisms or flows) on compact metric spaces, and a lot of interesting results of dynamical systems with various type of shadowing were investigated in many papers (e.g., see [2, 4, 6, 7, 9, 11, 13]).

Very recently, Artigue et al. [4] introduced the notion of L-shadowing for homeomorphisms on compact metric spaces, and explored the dynamics of homeomorphisms with the L-shadowing property.

In this paper we study the weak two-sided limit shadowing for flows which is different with the usual shadowing, two-sided limit shadowing and L-shadowing, and prove the spectral decomposition theorem for flows with the weak two-sided limit shadowing property.

Let us first recall the notion of shadowing for a homeomorphism \( f \) on a compact metric space \( X \) with a metric \( d \). For given \( \delta \), a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) is called a \( \delta \)-pseudo orbit of \( f \) if \( d(f(x_i), x_{i+1}) \leq \delta \) for all \( i \in \mathbb{Z} \). We say that \( f \) has the shadowing property if for any \( \varepsilon > 0 \), there is \( \delta > 0 \) such that any \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) is \( \varepsilon \)-shadowed by a point \( x \) in \( X \); i.e., \( d(f^i(x), x_i) \leq \varepsilon \) for all \( i \in \mathbb{Z} \).

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In the light of the rich consequence of the dynamical systems with the shadowing property, there are many works that studied the variants of the shadowing property such as two-sided limit shadowing property and L-shadowing property. A sequence \( \{x_i\}_{i \in \mathbb{Z}} \) in \( X \) is called a two-sided limit pseudo orbit of \( f \) if \( \lim_{|i| \to \infty} d(f(x_i), x_{i+1}) = 0 \). We say that \( f \) has the two-sided limit shadowing property if any two-sided limit pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) is two-sided limit shadowed by a point \( x \) in \( X \); i.e., \( \lim_{|i| \to \infty} d(f^i(x), x_i) = 0 \). The dynamics of homeomorphisms with the two-sided limit shadowing property have been studied in many papers (e.g., see [6, 7, 9]), even though the notion of two-sided limit shadowing is a little strict in some sense. Note that the notion of two-sided limit shadowing property is strong in the following sense: if \( f \) has the two-sided limit shadowing property, then it is topologically mixing, shadowing, average shadowing, asymptotic average shadowing, has the specification property and positive topological entropy (e.g., see [9]).

For any \( \delta > 0 \), a sequence \( \{x_i\}_{i \in \mathbb{Z}} \) is called a two-sided limit \( \delta \)-pseudo orbit of \( f \) if
\[
\lim_{|i| \to \infty} d(f(x_i), x_{i+1}) = 0 \text{ and } d(f(x_i), x_{i+1}) < \delta, \forall i \in \mathbb{Z}.
\]
We say that \( f \) has the L-shadowing property if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that any two-sided limit \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) of \( f \) is two-sided limit \( \varepsilon \)-shadowed by a point \( x \) in \( X \), i.e.,
\[
\lim_{|i| \to \infty} d(f^i(x), x_i) = 0 \text{ and } d(f^i(x), x_i) < \varepsilon.
\]
Important dynamics of the homeomorphisms with the L-shadowing property were investigated by Artigue et al. in [4] recently.

On the other hand, Lee [17] introduced the notion of weak two-sided limit shadowing for homeomorphisms as follows. We say that \( f \) has the weak two-sided limit shadowing property if there exists \( \delta > 0 \) such that any two-sided limit \( \delta \)-pseudo orbit \( \{x_i\}_{i \in \mathbb{Z}} \) of \( f \) is two-sided limit \( \varepsilon \)-shadowed by a point \( x \) in \( X \), i.e.,
\[
\lim_{|i| \to \infty} d(f^i(x), x_i) = 0.
\]
The notion of two-sided limit shadowing property was called weak limit shadowing property in [17] and its dynamics have been studied in many papers (e.g., see [17, 20, 21]). We observe that the notion of weak two-sided limit shadowing property is really weaker than those of two-sided limit shadowing property and L-shadowing property (see Example 2.2). Moreover, it has no relation with shadowing property in general (see Examples 2.2 and 2.3).

In this paper, we introduce the notion of weak two-sided limit shadowing for flows and prove that if a flow \( \phi \) on a compact metric space \( X \) has the weak two-sided limit shadowing property on its chain recurrent set \( CR(\phi) \), then the set \( CR(\phi) \) is decomposed by a finite number of closed invariant sets on which \( \phi \) is topologically transitive and has the two-sided limit shadowing property. Moreover we characterize the weak two-sided limit shadowing property from the pointwise and measure theoretical viewpoints.

Throughout the paper, we let \( \phi \) denote a flow on a compact metric space \( X \), i.e., \( \phi : X \times \mathbb{R} \to X \) is a continuous map satisfying
\[
\phi(x, 0) = x \text{ and } \phi(\phi(x, t), s) = \phi(x, t + s), \forall x \in X, \forall t, s \in \mathbb{R}.
\]
For any $t \in \mathbb{R}$ and $x \in X$, we write
\[ \phi_t(x) = \phi(x,t) \quad \text{and} \quad \phi_\mathbb{R}(x) = \{ \phi_t(x) \mid t \in \mathbb{R} \}. \]

A sequence of pairs of points and times $\{(x_i, t_i)\}_{i \in \mathbb{Z}}$ in $X \times \mathbb{R}$ is called a two-sided limit pseudo orbit of $\phi$ if
\[ t_i \geq 1 \quad \forall i \in \mathbb{Z} \quad \text{and} \quad \lim_{|i| \to \infty} d(\phi_{t_i}(x_i), x_{i+1}) = 0. \]

For a sequence $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ in $X \times \mathbb{R}$, we let
\[ T_i = \begin{cases} \sum_{j=0}^{i-1} t_j & \text{if } i > 0, \\ 0 & \text{if } i = 0, \\ -\sum_{j=1}^{i-1} t_j & \text{if } i < 0, \end{cases} \]

and write
\[ x_k \star t = \phi_{t_{i}} \circ \phi_{T_t - T_x}(x_i) \]
whenever $T_i \leq t + T_k < T_{i+1}$.

Here we call the sequence $\{T_i\}_{i \in \mathbb{Z}}$ by the associated sequence of $\xi$. Denote by $\text{Rep}$ the collection of all increasing homeomorphisms $h$ from $\mathbb{R}$ to itself satisfying $h(0) = 0$. We say that a flow $\phi$ has the two-sided limit shadowing property if for any two-sided limit pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ of $\phi$, there are $h \in \text{Rep}$ and $x \in X$ satisfying
\[ \lim_{|t| \to \infty} d(\phi_{h(t)}(x), x_0 \star t) = 0. \]

In this case, we say that $\xi$ is two-sided limit shadowed by a point $x \in X$, (for more details, see [2]).

For any $\delta > 0$ and $T > 0$, a sequence $\xi = \{(x_i, t_i)\}_{i=a}^{b}$ ($-\infty \leq a < b \leq \infty$) in $X \times \mathbb{R}$ is called a $(\delta, T)$-pseudo orbit (or $(\delta, T)$-chain) of $\phi$ if
\[ t_i \geq T \quad \text{and} \quad d(\phi_{t_i}(x_i), x_{i+1}) < \delta, \quad a \leq i < b. \quad (1) \]

Moreover the sequence $\xi$ is said to be a two-sided limit $(\delta, T)$-pseudo orbit of $\phi$ if (1) is satisfied and
\[ \lim_{|i| \to \infty} d(\phi_{t_i}(x_i), x_{i+1}) = 0. \]

**Definition 1.1.** We say that a flow $\phi$ on a compact metric space $X$ has the weak two-sided limit shadowing property if there exists $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit is two-sided limit shadowed by a point $x \in X$, i.e., there is $h \in \text{Rep}$ satisfying
\[ d(\phi_{h(t)}(x), x_0 \star t) \to 0 \quad \text{as } |t| \to \infty. \]

Similarly to Proposition 1.3 in [23] and Proposition 2.2 in [24], we observe that a flow $\phi$ has the weak two-sided limit shadowing property if and only if there are $T > 0$ and $\delta > 0$ such that any two-sided limit $(\delta, T)$-pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ with $T \leq t_i \leq 2T$ for all $i \in \mathbb{Z}$ is two-sided limit shadowed by a point in $X$.

**Remark 1.** Note that if a flow $\phi$ has the two-sided limit shadowing property then it has the weak two-sided limit shadowing property, but the converse does not hold in general as we can see in Example 2.2. Moreover we can check that the weak two-sided limit shadowing property has no relationship with the shadowing property (e.g., see Examples 2.2 and 2.3).
2. Weak two-sided limit shadowing and two-sided limit shadowing. In this section we study the relationship between weak two-sided limit shadowing property and two-sided limit shadowing property for flows, and characterize the weak two-sided limit shadowing flows from the pointwise and measurable viewpoints.

We first show that the weak two-sided limit shadowing property for flows is a dynamical property, i.e., if two flows $\phi$ and $\psi$ on compact metric spaces $X$ and $Y$, respectively, are topologically conjugate and $\phi$ has the weak two-sided limit shadowing property, then $\psi$ also has the weak two-sided limit shadowing property.

**Proposition 1.** Let $\phi$ and $\psi$ be two flows on compact metric spaces $X$ and $Y$, respectively. Suppose that $\phi$ and $\psi$ are topologically conjugate. Then $\phi$ has the weak two-sided limit shadowing property if and only if $\psi$ has the weak two-sided limit shadowing property.

**Proof.** Since $\phi$ and $\psi$ are topologically conjugate, there is a homeomorphism $\lambda : X \to Y$ and a continuous map $\sigma : X \times \mathbb{R} \to \mathbb{R}$ such that for any $x \in X$ the restriction of $\sigma$ on $\{x\} \times \mathbb{R}$, denoted by $\sigma_x$, is in $\text{Rep}$ and $\lambda(\phi_t(x)) = \psi_{\sigma_x(t)}(\lambda(x))$.

It is sufficient to show that $\phi$ has the weak two-sided limit shadowing property if $\psi$ has the weak two-sided limit shadowing property. Take $\varepsilon > 0$ such that any two-sided limit $(\varepsilon, 1)$-pseudo orbit of $\psi$ is two-sided limit shadowed by a point in $Y$. By the uniform continuity of $\lambda$, take $\delta > 0$ such that $d_X(x, y) < \delta$ for $x, y \in X$ implies $d_Y(\lambda(x), \lambda(y)) < \varepsilon$. We claim that any two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ can be two-sided limit shadowed by a point in $X$. Indeed, let $\xi = \{(x_t, t)\}_{t \in \mathbb{Z}}$ be a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ in $X$. By the choice of $\delta$ and $\lambda$, for any $i \in \mathbb{Z}$ we have

$$d_Y(\psi_{\sigma_x(t_i)}(\lambda(x_i)), \lambda(x_{i+1})) = d_Y(\lambda(\phi_{t_i}(x_i)), \lambda(x_{i+1})) < \varepsilon,$$

and

$$\lim_{|t| \to \infty} d_Y(\psi_{\sigma_x(t_i)}(\lambda(x_i)), \lambda(x_{i+1})) = \lim_{|t| \to \infty} d_Y(\lambda(\phi_{t_i}(x_i)), \lambda(x_{i+1})) = 0.$$  

Then a sequence $\eta = \{(\lambda(x_i), \sigma_x(t_i))\}_{i \in \mathbb{Z}}$ is a two-sided limit $(\varepsilon, 1)$-pseudo orbit of $\psi$ in $Y$. By the weak two-sided limit shadowing property of $\psi$, there are $y \in Y$ and $h \in \text{Rep}$ such that

$$\lim_{|t| \to \infty} d_X(\phi_{h(t)}(y), \lambda(x_0) \ast t) = 0.$$  

We define an increasing homeomorphism $t : \mathbb{R} \to \mathbb{R}$ as follows. For each $s \in \mathbb{R}$ and $n \in \mathbb{Z}$ such that $T_n^\varepsilon \leq s < T_{n+1}^\varepsilon$, we define $t(s) := \sigma_{x_0}^{-1}(s - T_n^\varepsilon) + T_n^\xi$, where $T_n^\varepsilon$ and $T_n^\xi$ are the $n$th sums of associated sequences of $\eta$ and $\xi$, respectively. We can check that $T_n^\varepsilon \leq t(s) < T_{n+1}^\varepsilon$ and $t$ is an increasing homeomorphism on $\mathbb{R}$. Take $x = \lambda^{-1}(y)$, and define a homeomorphism $g$ on $\mathbb{R}$ by $g(s) = \sigma_{x_0}^{-1}(h(t^{-1}(s)))$ for all $s \in \mathbb{R}$. For $s \in \mathbb{R}$, we observe that

$$x_0 \ast t(s) = \lambda^{-1}(\lambda(x_0) \ast s) \text{ and } \phi_{g(t(s))}(x) = \lambda^{-1}(\psi_{h(t(s))}(\lambda(x))).$$

Then by the uniform continuity of $\lambda$, we get

$$\lim_{|s| \to \infty} d_X(\phi_{g(t(s))}(x), x_0 \ast t(s)) = \lim_{|s| \to \infty} d_X(\lambda^{-1}(\psi_{h(t(s))}(y)), \lambda^{-1}(\lambda(x_0) \ast s)) = 0.$$  

Consequently, $\phi$ has the weak two-sided limit shadowing property. \hfill \Box

By definition, it is clear that if a flow $\phi$ has the two-sided limit shadowing property then it has the weak two-sided limit shadowing property. To show that the
converse does not hold, we use the suspension flow of a homeomorphism on a compact metric space.

Let $f$ be a homeomorphism on a compact metric space $X$. Given a continuous map $\tau : X \to \mathbb{R}$, we let

$$X_\tau = \bigcup_{0 \leq t \leq \tau(x)} (x, t)/(x, \tau(x)) \sim (f(x), 0).$$

The suspension flow $\phi$ of $f$ on $X_\tau$ is defined by

$$\phi((x, t), s)) = (x, t + s), \ 0 \leq t + s < \tau(x)$$

for $(x, t) \in X_\tau$ and $s \in \mathbb{R}$. Here the map $\tau$ is called a height function of the suspension flow $\phi$. To study the weak two-sided limit shadowing property for suspension flows, by Proposition 1, it is sufficient to consider the suspension flow $\phi$ with the height function $\tau = 1$, and $X_1$ will be denoted by $X$ for simplicity. For any $0 \leq s, t \leq 1$ and $x, y \in X$, let

$$l((x, t), (y, t)) = (1 - t)d(x, y) + td(f(x), f(y))$$

and $l((x, t), (x, s)) = |t - s|$. For any two points $x, y$ in $X$, take a finite chain $\xi = \{x = w_0, w_1, \ldots, w_{n+1} = y\}$ from $x$ to $y$ in $X$ such that for each $i$, either $w_i$ and $w_{i+1}$ belong to $X \times \{t\}$ for some $t \in \mathbb{R}$ or $w_i$ and $w_{i+1}$ are on same orbit of $\phi$. Define the length of the chain $\xi$ by $l(\xi) = \sum_{i=0}^{n} l(w_i, w_{i+1})$. Finally, we define the distance function $\rho$ on $X$ by

$$\rho(x, y) = \text{ the infimum of the lengths of all chains from } x \text{ to } y$$

(for more details, see [5]).

**Theorem 2.1.** If a homeomorphism $f$ on a compact metric space $X$ has the weak two-sided limit shadowing property then its suspension flow $\phi$ of $f$ has the weak two-sided limit shadowing property.

**Proof.** Suppose that $f$ has the weak two-sided limit shadowing property. Let $\varepsilon > 0$ be such that any two-sided limit $\varepsilon$-pseudo orbit of $f$ can be two-sided limit shadowed by a point in $X$. By Lemma 2.5 in [23], we take $\delta < 1/4$ with respect to $\varepsilon$. It is sufficient to show that any two-sided limit ($\delta, 2$)-pseudo orbit $\xi = \{(x_k, s_k), t_k\}_{k \in \mathbb{Z}}$ with $2 \leq t_k \leq 4$ can be two-sided limit shadowed by a point in $X_\tau$.

For each $k \in \mathbb{Z}$, let $w_k = [s_k + t_k]$ be the greatest integer number which is less than $s_k + t_k$. Then we can write

$$\phi_{t_k}(x_k, s_k) = (f^{w_k}(x_k), s_k + t_k - w_k).$$

Since $\xi$ is a $\delta$-pseudo orbit of $\phi$ and by the choice of $\delta$, we get $|s_k + t_k - w_k - s_{k+1}| < \delta$, or $|1 + s_k + t_k - w_k - s_{k+1}| < \delta$, or $|1 + s_{k+1} - s_k - t_k + w_k| < \delta$. For each $k \in \mathbb{Z}$, we define

$$n_k = \begin{cases} w_k & \text{if } |s_k + t_k - w_k - s_{k+1}| < \delta, \\ w_k - 1 & \text{if } |1 + s_k + t_k - w_k - s_{k+1}| < \delta, \\ w_k + 1 & \text{if } |1 + s_{k+1} - s_k - t_k + w_k| < \delta. \end{cases}$$

By the choice of $\delta$, we have $d(f^{n_k}(x_k), x_{k+1}) < \varepsilon$ for all $k \in \mathbb{Z}$. Then we can define a $\delta$-pseudo orbit of $f$ as follows. For each $i \in \mathbb{Z}$, define $y_i = f^{i-N_k}(x_k)$ if $N_k \leq i < N_{k+1}$ where

$$N_k = \begin{cases} \sum_{j=0}^{k-1} n_j & \text{if } i > 0, \\ 0 & \text{if } i = 0, \\ \sum_{j=k}^{-1} n_j & \text{if } i < 0. \end{cases}$$
We show that \( \{y_i\}_{i \in \mathbb{Z}} \) is a two-sided limit \( \varepsilon \)-pseudo orbit of \( f \). For any \( \eta > 0 \), take \( \eta_0 > 0 \) with respect to \( \eta \) by Lemma 2.5 in [23]. Since \( \xi \) is a two-sided limit \( (\delta, 2) \)-pseudo orbit, there is \( N > 0 \) such that \( \rho((f^{\nu_k}(x_k), s_k + t_k - w_k), (x_{k+1}, s_{k+1})) < \eta \) for all \( |k| \geq N \). It implies that \( d(f^{\nu_k}(x_k), x_{k+1}) = d(f(y_{N_k-1}), y_{N_k}) < \eta \) for all \( |k| \geq N \), and so \( \{y_i\}_{i \in \mathbb{Z}} \) is a two-sided limit \( \delta \)-pseudo orbit of \( f \).

By the weak two-sided limit shadowing property of \( f \), there is \( x \in X \) such that \( x \) two-sided limit shadows \( \{y_i\}_{i \in \mathbb{Z}} \). We claim that \( \xi \) is two-sided limit shadowed by \((x_0, s_0)\). Indeed, we define a function \( \alpha : \mathbb{R} \to \mathbb{R} \) by

\[
\alpha(t) = \frac{s_{k+1} + n_k - s_k}{t_k}(t - T_k) + s_k + N_k - s_0 \quad \text{if} \quad T_k \leq t < T_{k+1}.
\]

It is easy to check that \( \alpha \) is an increasing homeomorphism on \( \mathbb{R} \) with \( \alpha(0) = 0 \). For any \( \gamma > 0 \), we take \( \gamma_0 < \gamma/2 \) such that \( d(z_1, z_2) < \gamma_0 \) implies that \( d(f^i(z_1), f^i(z_2)) < \gamma \) for all \( 0 \leq i \leq 5 \). Since \( \xi \) is a two-sided limit pseudo orbit of \( \phi \), there is \( M > 0 \) such that

\[
\rho((f^{\nu_k}(x_k), s_k + t_k - w_k), (x_{k+1}, s_{k+1})) < \gamma_0, \quad \forall |k| \geq M.
\]

By Lemma 2.4 in [23], we get \( |s_k + t_k - n_k - s_{k+1}| < \gamma_0 \). Then for each \( T_k \leq t < T_{k+1} \) with \( k \geq M \), we have

\[
|s_0 + \alpha(t) - N_k - s_k - t + T_k| = |\alpha(t) - s_k - N_k + s_0 - (t - T_k)|
\]

\[
= |s_{k+1} + n_k - s_k - t_k|\left|\frac{t - T_k}{t_k}\right| < \gamma_0.
\]

For each \( k \in \mathbb{Z} \) with \( |k| \geq M \), we take \( j \in \mathbb{N} \) such that \( 0 < s_k + t - T_k - j < 1 \). Note that \( j \leq 5 \). Then for any \( T_k \leq t < T_{k+1} \) with \( k \geq M \), we have

\[
\rho(\phi_{\alpha(t)}(x, s_0), \phi_{t-T_k}(x_k, s_k))
\]

\[
= \rho((f_{j+N_k}(x_k), s_0 + \alpha(t) - j - N_k), (x_k, s_k + t - T_k))
\]

\[
\leq \rho((f_{j+N_k}(x_k), s_0 + \alpha(t) - j - N_k), (f^j+N_k(x), s_k + t - T_k))
\]

\[
\quad + \rho((f^j+N_k(x), s_k + t - T_k), \phi_{t-T_k}(x_k, s_k))
\]

\[
= |s_0 + \alpha(t) - N_k - s_k - t + T_k|
\]

\[
+ (1 - s_k - t + T_k + j)d(f_{j+N_k}(x), f^j(x_k))
\]

\[
+ (s_k - t + T_k + j)d(f^{j+1+N_k}(x), f^{j+1}(x_k)) < \gamma.
\]

It implies that \( \xi \) is two-sided limit shadowed by \((x_0, s_0)\). Therefore \( \phi \) has the weak two-sided limit shadowing property.

In the following example, which is inspired by Example 4 in [17], we see that the notion of weak two-sided limit shadowing property is really weaker than that of two-sided limit shadowing property. We recall that a flow \( \phi \) is topologically transitive on an invariant set \( \Lambda \) if for any nonempty open subsets \( U, V \) of \( \Lambda \), there is \( T > 1 \) such that \( \phi_T(U) \cap V \neq \emptyset \).

**Example 2.2.** Let \( f \) be a homeomorphism on the unit circle \( S^1 = \{(1, \theta) | \theta \in [0, 2\pi)\} \) with \( \text{Fix}(f) = \{(1, \pi), (1, \pi/2), (1, 0)\} \). For any \( x = (1, \theta) \notin \text{Fix}(f) \), we assume that

- if \( \theta \in (0, \pi/2) \) then \( \alpha(x) = \{(1, \pi/2)\} \) and \( \omega(x) = \{(1, 0)\} \),
- if \( \theta \in (\pi/2, \pi) \) then \( \alpha(x) = \{(1, \pi)\} \) and \( \omega(x) = \{(1, \pi/2)\} \),
- if \( \theta \in (\pi, 2\pi) \) then \( \alpha(x) = \{(1, \pi)\} \) and \( \omega(x) = \{(1, 0)\} \).
Then $f$ does not have the two-sided limit shadowing property. In fact, let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a two-sided limit pseudo orbit of $f$ given by
\[
x_i = \begin{cases} 
(1,0) & \text{if } i \geq 0, \\
(1,\pi) & \text{if } i < 0.
\end{cases}
\]
Then it is clear that $\xi$ can not be two-sided limit shadowed by any point in $S^1$.

On the other hand, we can show that $f$ has the weak two-sided limit shadowing property. Let $\{x_i\}_{i \in \mathbb{Z}}$ be a two-sided limit $\delta$-pseudo orbit of $f$ for sufficiently small $\delta > 0$. Then we have
\[
d(x_i, \text{Fix}(f)) \to 0 \text{ as } |i| \to \infty.
\]
Suppose $x_i \to p$ as $i \to \infty$ and $x_i \to q$ as $i \to -\infty$, where $p, q \in \text{Fix}(f)$. Then the two-sided limit pseudo orbit $\{x_i\}_{i \in \mathbb{Z}}$ is two-sided limit shadowed by a point $x \in S^1$ satisfying $\omega(x) = p$ and $\alpha(x) = q$, where
\[
\omega(z) = \{x \in X : f^{n_i}(x) \to z \text{ for some } n_i \to \infty\}
\]
and
\[
\alpha(z) = \{x \in X : f^{n_i}(x) \to z \text{ for some } n_i \to -\infty\}.
\]
Moreover we see that $f$ have neither the shadowing property nor the L-shadowing property. In fact, for sufficiently small $\delta > 0$, take a two-sided limit $\delta$-pseudo orbit $\{x_i = (1, \theta_i)\}_{i \in \mathbb{Z}}$ of $f$ such that $0 \leq \theta_i \leq \pi$ for all $i \in \mathbb{Z}$,
\[
x_i \to (1, \pi) \text{ as } i \to \infty \text{ and } x_i \to (1,0) \text{ as } i \to -\infty.
\]
Then the sequence $\{x_i\}_{i \in \mathbb{Z}}$ can not be two-sided limit shadowed by any point in $S^1$.

Let $\phi$ be the suspension flow of $f$ with the constant height function $\tau = 1$. By Theorem 2.1, we see that the flow $\phi$ has the weak two-sided limit shadowing property. Note that $\phi$ is not topologically transitive on the phase space $\overline{S^1} = S^1 \times [0,1]/\sim$. This shows that $\phi$ does not have the two-sided limit shadowing property by Theorem D in [2].

We recall that a flow $\phi$ on a compact metric space $X$ has the shadowing property if for any $\varepsilon > 0$, there is $\delta > 0$ such that any $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ of $\phi$ can be $\varepsilon$-shadowed by a point $x \in X$; that is, there is $h \in \text{Rep}$ such that $d(\phi_{h(t)}(x), x_0 \ast t) < \varepsilon$ for all $t \in \mathbb{R}$. It is proved in [23] that any suspension flow of a shadowing homeomorphism on a compact metric space has the shadowing property. We observe that there is no relation between shadowing property and weak two-sided limit shadowing property for flows. Indeed, in Example 2.2, the suspension flow $\phi$ does not have the shadowing property since $f$ does not have the shadowing property on $S^1$. Conversely, we present an example for a flow with the shadowing property which does not have the weak two-sided limit shadowing property, which is inspired by Theorem A in [8].

**Example 2.3.** Let $\Sigma_2$ be the sequence space on two symbols 0 and 1 with the metric
\[
d_0(x,y) = \begin{cases} 
\frac{1}{2^k} & \text{if } n = \max\{k \in \mathbb{N} \mid x_i = y_i \text{ for all } |i| < k\}, \\
0 & \text{if } x = y,
\end{cases}
\]
where $x = (x_i)_{i \in \mathbb{Z}}$, $y = (y_i)_{i \in \mathbb{Z}} \in \Sigma_2$. Consider the shift map $\sigma : \Sigma_2 \to \Sigma_2$ given by
\[
(\sigma(x))_i = x_{i+1}, \quad \forall x = (x_i)_{i \in \mathbb{Z}} \in \Sigma_2.
\]
Then $\sigma$ is a homeomorphism on $\Sigma_2$, and has the shadowing property.

For each $n \in \mathbb{N}$, choose a point $p_n \in \Sigma_2$ with the period $n$, and let $E = \bigcup_{n \in \mathbb{N}} O_\alpha(p_n)$. Take a copy $F$ of $E$ such that $\Sigma_2 \cap F = \emptyset$, and let $X = \Sigma_2 \cup F$. Then
$F$ can be enumerated by a bijection $b: E \to F$ which assigns an element $\sigma^k(p_n)$ of $E$ to an element $b(\sigma^k(p_n)) := q_{nk}$ in $F$, where $n \in \mathbb{N}$ and $0 \leq k < n$. Consider a metric $d$ on $X$ defined by

$$d(x, y) = \begin{cases} 
  d_0(x, y) & \text{if } x, y \in \Sigma_2, \\
  \frac{1}{m} + d_0(x, \sigma^k(p_n)) & \text{if } x \in \Sigma_2, \ y = q_{nk} \in F, \\
  \frac{1}{m} + d_0(\sigma^k(p_n), \sigma^l(p_m)) & \text{if } x = q_{nk}, \ y = q_{ml} \in F.
\end{cases}$$

Consider a homeomorphism $f$ of $X$ given by

$$f(x) = \begin{cases} 
  \sigma(x) & \text{if } x \in \Sigma_2, \\
  q_{n(k+1)} & \text{if } x = q_{nk}, \ 0 \leq k < n - 1, \\
  q_{n0} & \text{if } x = q_{n(n-1)}.
\end{cases}$$

Then we see that $f$ has the shadowing property. We claim that $f$ does not have the weak two-sided limit shadowing property. Indeed, we suppose that there is $\delta > 0$ such that any two-sided limit $\delta$-pseudo orbit of $f$ can be two-sided limit shadowed by a point in $X$. Take $n \in \mathbb{N}$ such that $1/n < \delta$ and let $\xi = \{x_i\}_{i \in \mathbb{Z}}$ be a two-sided limit $\delta$-pseudo orbit of $f$ given by

$$x_i = \begin{cases} 
  f^i(p_n) & \text{if } i < 0, \\
  f^i(q_{n0}) & \text{if } i \geq 0.
\end{cases}$$

By assumption, there is $x \in X$ such that

$$\lim_{|i| \to \infty} d(f^i(x), x_i) = 0.$$

Then there is $M > 0$ such that

$$d(f^{-m}(x), f^{-m}(p_n)) < \frac{1}{2n} \quad \text{and} \quad d(f^m(x), f^m(q_{n0})) < \frac{1}{2n}, \ \forall m \geq M.$$ 

It contradicts to the fact that $1/2n$-neighborhood of $\mathcal{O}_f(q_{n0})$ consists of only one orbit of $q_{n0}$. Consequently $f$ does not have weak two-sided limit shadowing property.

Let $\phi$ be the suspension flow $\phi$ on $\overline{X}$ of $f$ under constant map $\tau = 1$. By Theorem 2 in [23], we see that the suspension flow $\phi$ of $f$ has the shadowing property. We claim that $\phi$ does not have the weak two-sided limit shadowing property. Suppose that there is $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit can be two-sided limit shadowed by a point in $\overline{X}$. Take $n \in \mathbb{N}$ such that $1/n < \delta$, and so define a sequence $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ by

$$(x_i, t_i) = \begin{cases} 
  ((p_n, 0), n) & \text{if } i < 0, \\
  ((q_{n0}, 0), n) & \text{if } i \geq 0.
\end{cases}$$

We see that $\xi$ is a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$, and so by the weak two-sided limit shadowing property of $\phi$, there are $\pi \in \overline{X}$ and $h \in \text{Rep}$ such that

$$\lim_{|t| \to \infty} \rho(\phi_{h(t)}(\pi), \pi \ast t) = 0.$$ 

Then there is $T > 0$ such that

$$\rho(\phi_{h(-t)}(\pi), \phi_{-t}(p_n, 0))) < \frac{1}{2n} \quad \text{and} \quad \rho(\phi_{h(t)}(\pi), \phi_{t}(q_{n0}, 0))) < \frac{1}{2n}, \ \forall t \geq T.$$ 

Since $1/2n$-neighborhood of $\phi_{\mathbb{R}}((q_{n0}, 0))$ consists of only one orbit, $\pi$ must belong to $\phi_{\mathbb{R}}((q_{n0}, 0))$. The contradiction shows that $\phi$ does not have the weak two-sided limit shadowing property.
By Example 2.2, the notion of weak two-sided limit shadowing property is really weaker than two-sided limit shadowing property for flows on compact metric spaces. In the following theorem, we show that they are coincided if the flow is chain transitive. We recall that a flow $\phi$ on a compact metric space $X$ is chain transitive if for any $x, y \in X$ and $\varepsilon > 0$, there is an $(\varepsilon, 1)$-chain $\xi = \{(x_i, t_i)\}_{i=0}^n$ from $x$ to $y$, i.e., $x_0 = x$ and $d(\phi_{t_i}(x_n), y) < \varepsilon$.

**Theorem 2.4.** A flow $\phi$ on a compact metric space $X$ is chain transitive and has the weak two-sided limit shadowing property if and only if it has the two-sided limit shadowing property.

**Proof.** Suppose that $\phi$ has the two-sided limit shadowing property. Clearly, $\phi$ has the weak two-sided limit shadowing property and it is chain transitive since it is topologically transitive by Theorem D in [2].

Conversely suppose that $\phi$ is chain transitive and has the weak two-sided limit shadowing property, and take $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit can be two-sided limit shadowed by a point in $X$. We claim that $\phi$ has the two-sided limit shadowing property. Let $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ be a two-sided limit pseudo orbit of $\phi$. Take $N \in \mathbb{N}$ such that $\{(x_i, t_i)\}_{i=N}^\infty$ and $\{(x_i, t_i)\}_{i=-\infty}^{-N}$ are $(\delta, 1)$-pseudo orbits of $\phi$. Since $\phi$ is chain transitive, we take $\chi = \{(y_i, c_i)\}_{i=-l+1}^{k-1}$ as a $(\delta, 1)$-pseudo orbit of $\phi$ from $\phi_{t-N}(x_{-N})$ to $x_N$. Adding more periodic $(\delta, 1)$-chains to $\chi$ if necessary, we can assume that $k > N, T_k^x > T_k^\xi$ and $T_{-l}^x < T_{-l}^\xi$, where $T_n^\xi$ and $T_n^x$ are the $n^{th}$ sums of associated sequences of $\xi$ and $\chi$, respectively. We define a two-sided limit $(\delta, 1)$-pseudo orbit $\eta$ for $\phi$ by

$$
(z_i, s_i) = \begin{cases} 
(x_{-N+i+l}, t_{-N+i+l}) & \text{if } i \leq -l, \\
(y_i, c_i) & \text{if } -l+1 \leq i \leq k-1, \\
x_{N+i-k}, t_{N+i-k}) & \text{if } i \geq k.
\end{cases}
$$

By the weak two-sided limit shadowing property of $\phi$, there are $z \in X$ and $h \in \text{Rep}$ such that

$$
\lim_{|t| \to \infty} d(\phi_{h(t)}(z), z \ast t) = 0.
$$

We define a homeomorphism $g \in \text{Rep}$ on $\mathbb{R}$ by

$$
g(t) = \begin{cases} 
h(t - T_{-N}^\xi + T_{-1}^\eta) & \text{if } t \leq T_{-N}^\xi, \\
\frac{t}{T_{-N}^\xi}h(T_{-1}^\eta) & \text{if } T_{-N}^\xi \leq t \leq 0, \\
\frac{t}{T_{N}^\xi}h(T_k^\eta) & \text{if } 0 \leq t \leq T_{N}^\xi, \\
h(t - T_{N}^\xi + T_k^\eta) & \text{if } t \geq T_{N}^\xi,
\end{cases}
$$

where $T_n^\eta$ is the $n^{th}$ sum of associated sequence of $\eta$.

For $t \geq T_{N}^\xi$, we have $x_0 \ast t = x_N \ast (t - T_{N}^\xi) = z_k \ast (t - T_{N}^\xi) = z_0 \ast (t - T_{N}^\xi + T_k^\eta)$, and so

$$
\lim_{t \to \infty} d(\phi_{g(t)}(z), x_0 \ast t) = \lim_{t \to \infty} d(\phi_{h(t-T_{N}^\xi + T_k^\eta)}(z), z_0 \ast (t - T_{N}^\xi + T_k^\eta)) = 0.
$$

For $t \leq T_{-N}^\xi$, we have $x_0 \ast t = x_{-N} \ast (t - T_{-N}^\xi) = z_{-l} \ast (t - T_{-N}^\xi) = z_0 \ast (t - T_{-N}^\xi + T_{-l}^\eta)$, and so

$$
\lim_{t \to -\infty} d(\phi_{g(t)}(z), x_0 \ast t) = \lim_{t \to -\infty} d(\phi_{h(t-T_{-N}^\xi + T_{-l}^\eta)}(z), z_0 \ast (t - T_{-N}^\xi + T_{-l}^\eta)) = 0.
$$

This completes the proof.
Therefore $\xi$ is two-sided limit shadowed by $y$, and so $\phi$ has the two-sided limit shadowing property.

Remarkably, the result in Theorem 2.4 does not hold for homeomorphisms on compact metric spaces. In fact, let $f$ be a homeomorphism on $X = \{a, b\}$ with discrete topology given by $f(a) = b$ and $f(b) = a$. It is clear that $f$ is chain transitive. We observe that any two-sided limit $1/2$-pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is a real orbit of $f$, and so $\xi$ is two-sided limit shadowed by $x_0$. On the other hand, let $\chi = \{x_i\}_{i \in \mathbb{Z}}$ be a two-sided limit pseudo orbit of $f$ defined by $x_i = f^i(b)$ if $i < 0$ and $x_i = f^i(a)$ if $i \geq 0$. It is easy to see that $\chi$ can not be two-sided limit shadowed by any point in $X$, and so $f$ does not have the two-sided limit shadowing property.

In [3], Aponte and Villavicencio introduced the notion of shadowable points for flows and proved that a flow $\phi$ on a compact metric space $X$ has the shadowing property if and only if every point in $X$ is shadowable for $\phi$. Adapting the idea, we introduce the notion of weak two-sided limit shadowable points for flows. A point $x \in X$ is said to be weak two-sided limit shadowable for a flow $\phi$ on a compact metric space $X$ if there is $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ through $x$, i.e., $x_0 = x$, can be two-sided limit shadowed by a point in $X$. The set of all weak two-sided limit shadowable points of $\phi$ will be denoted by $wTSLSh(\phi)$. We observe that a point $x$ is weak two-sided limit shadowable if and only if there is $\delta > 0$ such that any two-sided limit $(\delta, T)$-pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ through $x$ with $T \leq t_i \leq 2T$ for all $i \in \mathbb{Z}$ can be two-sided limit shadowed by a point in $X$, where $T > 0$.

**Lemma 2.5.** Let $\phi$ be a flow on a compact metric space $X$. Then $wTSLSh(\phi)$ is open and invariant.

**Proof.** We first show that $wTSLSh(\phi)$ is an open set. Without loss of generality, we assume that $wTSLSh(\phi)$ is nonempty. Let $y \in wTSLSh(\phi)$ and $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit through $y$ can be two-sided limit shadowed by a point in $X$. Take $0 < \varepsilon < \delta/2$ such that $d(z_1, z_2) < \varepsilon$ implies $d(\phi(z_1), \phi(z_2)) < \delta/2$ for any $0 \leq t \leq 2$. Let $x \in B(y, \varepsilon)$, and $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ be a two-sided limit $(\varepsilon, 1)$-pseudo orbit through $x$ with $1 \leq t_i \leq 2$ for all $i \in \mathbb{Z}$. Define a sequence $\eta = \{(y_i, s_i)\}_{i \in \mathbb{Z}}$ by

$$(\eta, s_i) = \begin{cases} (x_i, t_i) & \text{if } i \neq 0, \\ (y, t_0) & \text{if } i = 0. \end{cases}$$

Since

$$d(\phi_{t-1}(x-1), y) \leq d(\phi_{t-1}(x-1), x_0) + d(x_0, y) \leq \delta,$$

and

$$d(\phi_{0}(y), x_1) \leq d(\phi_{0}(y), \phi_{0}(x_0)) + d(\phi_{0}(x_0), x_1) \leq \delta,$$

we see that $\eta$ is a two-sided limit $(\delta, 1)$-pseudo orbit through $y$. Then there are $z \in X$ and $h \in \text{Rep}$ such that

$$\lim_{|t| \to \infty} d(\phi_{h(t)}(z), y_0 \ast t) = 0.$$

Since $x_0 \ast t = y_0 \ast t$ for any $t > t_0$ and $t < 0$, we have

$$\lim_{|t| \to \infty} d(\phi_{h(t)}(z), x_0 \ast t) = 0.$$
and so $\xi$ is two-sided limit shadowed by $z$. Hence $x \in wTSLS\phi$, and so $B(y, \varepsilon) \subset wTSLS\phi$. It implies that $wTSLS\phi$ is an open set.

Next we prove that the set $wTSLS\phi$ is invariant. Let $x \in wTSLS\phi$ and $s \in \mathbb{R}$. We claim that $\phi_s(x) \in wTSLS\phi$. Take $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit through $x$ can be two-sided limit shadowed by a point in $X$. By uniform continuity of $\phi_s$, let $\varepsilon > 0$ be such that $d(z_1, z_2) < \varepsilon (z_1, z_2 \in X)$ implies $d(\phi_s(z_1), \phi_s(z_2)) < \delta$. Let $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ be a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ through $\phi_s(x)$. Consider the sequence $\eta = \{(\phi_s(x_i), t_i)\}_{i \in \mathbb{Z}}$. Since $d(\phi_{ti}(x_i), x_{i+1}) < \varepsilon$, we have $d(\phi_{-s} (\phi_{ti}(x_i)), \phi_{-s}(x_{i+1})) \leq \delta$ for all $i \in \mathbb{Z}$. Moreover, since $\lim_{|t| \to \infty} d(\phi_{ti}(x_i), x_{i+1}) = 0$, we have $\lim_{|t| \to \infty} d(\phi_{-s} (\phi_{ti}(x_i)), \phi_{-s}(x_{i+1})) = 0$. Then $\eta$ is a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ through $x$. Since $x$ is weak two-sided limit shadowable for $\phi$, there are $y \in X$ and $h \in Rep$ such that
\[
\lim_{|t| \to \infty} d(\phi_{h(t)}(y), \phi_{-s}(x_0) \ast t) = 0,
\]
and so
\[
\lim_{|t| \to \infty} d(\phi_{h(t)}(\phi_s(y)), x_0 \ast t) = 0.
\]
It implies that $\xi$ is two-sided limit shadowed by $\phi_s(y)$. Hence $\phi_s(x) \in wTSLS\phi$, and so $wTSLS\phi$ is invariant.

A Borel measure on $X$ is a non-negative $\sigma$-additive map $\mu$ defined on the Borel $\sigma$-algebra $\beta(X)$. For simplicity, we assume that Borel measure implies Borel probability measure, i.e., $\mu(X) = 1$. We recall that a Borel measure $\mu$ is invariant for $\phi$ if $\mu(A) = \mu(\phi_t(A))$ for any Borel set $A$ in $X$ and $t \in \mathbb{R}$. A Borel measure $\mu$ is said to have weak two-sided limit shadowing property for $\phi$ if there are $\delta > 0$ and a Borel set $B$ such that $\mu(B) = 1$ and any two-sided limit $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ through $B$, i.e., $x_0 \in B$, can be two-sided limit shadowed by a point in $X$. We observe that if $\mu$ has the shadowing property for $\phi$, then $\mu(wTSLS\phi) = 1$.

In the following theorem, we present characterizations of weak two-sided limit shadowing property of flow $\phi$ from the measure and pointwise viewpoints.

**Theorem 2.6.** Let $\phi$ be a flow on a compact metric space $X$. Then the followings are pairwise equivalent.

(i) $\phi$ has the weak two-sided limit shadowing property,

(ii) any invariant Borel measure $\mu$ on $X$ has the weak two-sided limit shadowing property for $\phi$,

(iii) $wTSLS\phi = X$.

**Proof.** (i) $\Rightarrow$ (ii). It is clear by definition.

(ii) $\Rightarrow$ (iii). For any $x \in X$, there is an invariant measure $\mu$ such that $\mu(\omega(x)) = 1$. Since $\mu$ has the weak two-sided limit shadowing property for $\phi$, we observe that $\mu(wTSLS\phi) = 1$, and so there is $y \in \omega(x) \cap wTSLS\phi$. By Lemma 2.5, there is $\varepsilon > 0$ such that $B(y, \varepsilon) \subset wTSLS\phi$. Take $T > 0$ such that $\phi_T(x) \in B(y, \varepsilon) \subset wTSLS\phi$. Since $wTSLS\phi$ is invariant, $x \in wTSLS\phi$, and so $wTSLS\phi = X$.

(iii) $\Rightarrow$ (i) By contradiction, suppose that for each $k \in \mathbb{N}$, there is a two-sided limit $(1/k, 1)$-pseudo orbit $\xi^k = \{(x^k_i, t^k_i)\}_{i \in \mathbb{Z}}$ with $1 \leq t^k_i \leq 2$ for all $i \in \mathbb{Z}$ which can not be two-sided limit shadowed by a point in $X$. Since $X$ is compact, we suppose that $x^k_i$ converges to $p \in X$ as $k \to \infty$. Since $p \in wTSLS\phi = X$, we take $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit through $p$ can be
two-sided limit shadowed. Take \( k \in \mathbb{N} \) such that \( d(x_0^k, p) < \delta/2, \ 1/k < \delta/2 \) and 
\( d(\phi_t(p), \phi_t(x_0^k)) < \delta/2 \) for all \( 0 \leq t \leq 2 \). Define a sequence \( \xi = \{(y_i, s_i)\}_{i \in \mathbb{Z}} \) by
\[
(y_i, s_i) = \begin{cases} 
(x_i^k, t_i^k) & \text{if } i \neq 0 \\
(p, t_i^0) & \text{if } i = 0.
\end{cases}
\]
Since
\[
d(\phi_{t_{i-1}}(x_{i-1}^k), p) \leq d(\phi_{t_{i-1}}(x_{i-1}^k), x_0^k) + d(x_0^k, p) \leq \delta,
\]
and
\[
d(\phi_{t_i}(p), x_i^k) \leq d(\phi_{t_i}(p), \phi_{t_i}(x_0^k)) + d(\phi_{t_i}(x_0^k), p) \leq \delta,
\]
\( \xi \) is a two-sided limit \((\delta, 1)\)-pseudo orbit of \( \phi \) through \( p \). Since \( p \) is weak two-sided limit shadowable, there are \( y \in X \) and \( h \in \text{Rep} \) such that
\[
\lim_{|t| \to \infty} d(\phi_{h(t)}(y), x_0 \ast t) = 0,
\]
and so
\[
\lim_{|t| \to \infty} d(\phi_{h(t)}(y), x_0^k \ast t) = 0.
\]
The contradiction completes the proof. 

3. **Spectral decomposition.** The famous spectral decomposition theorem by Smale [22] says that the nonwandering set \( \Omega(f) \) of an Axiom A diffeomorphism \( f \) on a compact smooth manifold admits the spectral decomposition, i.e., \( \Omega(f) \) can be decomposed as a disjoint union of finite basic sets \( B_i \) that is, each \( B_i \) is a compact invariant set such that \( f|_{B_i} \) is topologically transitive. There are many works that generalize the Smale’s spectral decomposition theorem to more general settings (e.g., see [1, 10, 11, 12, 14, 16, 18, 19]). Komuro [14] proved the topological version of the Smale’s spectral decomposition theorem for flows on compact metric spaces. Precisely, if \( \phi \) is expansive on its nonwandering set \( \Omega(\phi) \) and has the shadowing property on \( \Omega(\phi) \), then \( \Omega(\phi) \) can be decomposed as a finite union of disjoint compact invariant sets on each of which \( \phi \) is topologically transitive. Unfortunately, the statement of the spectral decomposition by Komuro is not correct. It can be revised by replacing \( \Omega(f) \) by \( CR(f) \) (for more details, see [18]).

In this section, we present a spectral decomposition theorem for flows with weak two-sided limit shadowing property on compact metric spaces, which is a generalization of the topological version of Smale’s spectral decomposition theorem by Komuro.

**Theorem 3.1.** If a flow \( \phi \) on a compact metric space \( X \) has the weak two-sided limit shadowing property on \( CR(\phi) \), then \( \phi \) has the spectral decomposition, i.e., the nonwandering set \( \Omega(\phi) \) is decomposed by a disjoint union of finitely many invariant and closed subsets
\[
\Omega(\phi) = B_1 \cup \cdots \cup B_l
\]
such that \( \phi \) is topologically transitive on each \( B_i \) for \( 1 \leq i \leq l \). Moreover, \( \phi \) has the two-sided limit shadowing property on each \( B_i \) for \( 1 \leq i \leq l \).

Before proving Theorem 3.1, we first show that if \( \phi \) is expansive and has the shadowing property, then it has the weak two-sided limit shadowing property (see Proposition 2) whose proof is based on Lemma 2.3 and Theorem 2.4 in [4] and Lemma 3.9 in [14]. Note that the converse is not true in general as we can see in Example 2.2. To prove Proposition 2, we first recall some notations.
Komuro [14] introduced a stronger notion of shadowing property for flows, which
is called strong shadowing property. For given \( \varepsilon > 0 \), we denote by \( \text{Rep}(\varepsilon) \) the
collection of \( h \in \text{Rep} \) such that
\[
\left| \frac{h(t) - h(s)}{t - s} - 1 \right| < \varepsilon \text{ if } t \neq s.
\]
We say that \( \phi \) has the strong shadowing property if for any \( \varepsilon > 0 \) there is \( \delta > 0 \)
such that any \((\delta, 1)\)-pseudo orbit \( \xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) of \( \phi \) can be strongly \( \varepsilon \)-shadowed by a
point in \( X \); that is, there are \( y \in X \) and \( h \in \text{Rep}(\varepsilon) \) such that
\[
d(\phi_{h(t)}(y), x_0 \ast t) \leq \varepsilon, \quad \forall t \in \mathbb{R}.
\]
We observe that, by Theorem 4 in [14], a flow \( \phi \) without fixed points has the strong
shadowing property if and only if it has the shadowing property.

For any \( r_1 > 0 \), take \( \tau_0 > 0 \) such that \( d_{C^0}(\phi_s, \text{id}) < r \) for all \( s \in [-\tau_0, \tau_0] \). Let \( \phi \) be
an expansive flow with the shadowing property, and let \( \varepsilon_0 > 0 \) be such that \( 2\varepsilon_0 \)
is an expansive constant of \( \phi \) with respect to \( \tau_0 \). Let \( \xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) be a two-sided
limit pseudo orbit of \( \phi \). Then there are \( q_0 \in X \), \( g_0 \in \text{Rep}(\varepsilon_0) \) and \( M_0 \in \mathbb{N} \) such that
\[
d(\phi_{g_0(t)}(q_0), x_0 \ast t) \leq \varepsilon_0, \quad \forall t \in (-\infty, T_{-M_0}] \cup [T_{M_0}, \infty).
\]
With the notations, we have the following lemma which is crucial for the proof of
Proposition 2.

**Lemma 3.2.** Suppose \( \phi \) is expansive and has the shadowing property. For any
\( \varepsilon_1 < \varepsilon_0 \), there are \( q_1 \in X \), \( g_1 \in \text{Rep}(\varepsilon_0 \varepsilon_1 + \varepsilon_0 + \varepsilon_1) \) and \( M_1 \in \mathbb{N} \) \((M_1 > M_0)\) such that
\[
d(\phi_{g_1(t)}(q_1), x_0 \ast t) \leq \begin{cases} d(\phi_{g_0(t)}(q_0), x_0 \ast t) + \varepsilon_1 + r_1 & \text{if } t \in (T_{-M_1}, T_{M_1}), \\ 2\varepsilon_1 & \text{if } t \in (-\infty, T_{-M_1}] \cup [T_{M_1}, \infty). \end{cases}
\]

**Proof.** Since \( \phi \) is expansive, every fixed point of \( \phi \) is isolated. Hence we may assume
that the \( \phi \) does not have a fixed point.

For any \( \varepsilon_1 < \varepsilon_0 \), take \( \delta > 0 \) corresponding to \( \varepsilon_1 > 0 \) by the strong shadowing
property of \( \phi \). Take \( m \in \mathbb{N} \) such that \( d_{C^0}(\phi_s, \text{id}) < \delta \) for \( s \in [-\tau_0/m, \tau_0/m] \).
Since \( \{(x_i, t_i)\}_{i \in \mathbb{Z}} \) is a two-sided limit pseudo orbit of \( \phi \), we can take \( N > M_0 \)
such that \( \{(x_i, t_i)\}_{i \geq N} \) and \( \{(x_i, t_i)\}_{i \leq -N} \) are \((\delta, 1)\)-pseudo orbits of \( \phi \).
By the strong shadowing property of \( \phi \), there are \( y \in X \) and \( f_1 \in \text{Rep}(\varepsilon) \) such that
\[
d(\phi_{f_1(t)}(y), x_0 \ast t) < \varepsilon_1, \quad \forall t \geq T_N.
\]
Then we get
\[
d(\phi_{g_0(t)}(q_0), \phi_{f_1(t)}(y)) \leq d(\phi_{g_0(t)}(q_0), x_0 \ast t) + d(x_0 \ast t, \phi_{f_1(t)}(y)) \leq \varepsilon_0 + \varepsilon_1 < 2\varepsilon_0, \quad \forall t \geq T_N,
\]
and so
\[
d(\phi_{g_0(f_1^{-1}(t))}(p), \phi_{t}(y)) < 2\varepsilon_0, \quad \forall t \geq f_1(T_N).
\]
Similarly, there are \( z \in X \) and \( f_2 \in \text{Rep}(\varepsilon) \) such that
\[
d(\phi_{f_2(t)}(p), x_0 \ast t) < \varepsilon_1, \quad \forall t \leq T_N
\]
and
\[
d(\phi_{g_0(f_2^{-1}(t))}(q_0), \phi_{t}(z)) < 2\varepsilon_0, \quad \forall t \leq f_2(T_N).
\]
By Lemma 3.9 in [14], we can take \( M_1 > N \) such that
\[
g_0(T_{M_1}) - m > 1, \quad -m - g_0(T_{-M_1}) > 1,
\]
\[ d(\phi_{g_0}(T_{M_1})+s_1(q_0), \phi_{f_1}(T_{M_1})(q_0)) < \delta \quad \text{and} \quad d(\phi_{g_0}(T_{-M_1})+s_2(q_0), \phi_{f_2}(T_{-M_1})(z)) < \delta \]

for some \( s_1, s_2 \in [-\tau_0, \tau_0] \).

We define a \((\delta, 1)\)-pseudo orbit \( \{(w_i, s_i)\}_{i \in \mathbb{Z}} \) of \( \phi \) by

\[
(w_i, s_i) = \begin{cases}
(k(g_0(t))) & \text{if } t \in [T_{-M_1}, T_{M_1}], \\
k(f_1(t) - f_1(T_{M_1}) + g_0(T_{M_1})) & \text{if } t \in [T_{M_1}, \infty), \\
k(f_2(t) - f_2(T_{-M_1}) + g_0(T_{M_1})) & \text{if } t \in (-\infty, T_{-M_1}].
\end{cases}
\]

By the strong shadowing property of \( \phi \), there are \( q_1 \in X \) and \( k \in \text{Rep}(\varepsilon) \) such that

\[ d(\phi_{k(t)}(q_1), w_0 \ast t) < \varepsilon_1, \quad \forall t \in \mathbb{R}. \]

Define a homeomorphism \( g_1 : \mathbb{R} \to \mathbb{R} \) by

\[
g_1(t) = \begin{cases}
k(g_0(t)) & \text{if } t \in [T_{-M_1}, T_{M_1}], \\
k(f_1(t) - f_1(T_{M_1}) + g_0(T_{M_1})) & \text{if } t \in [T_{M_1}, \infty), \\
k(f_2(t) - f_2(T_{-M_1}) + g_0(T_{M_1})) & \text{if } t \in (-\infty, T_{-M_1}].
\end{cases}
\]

We see that \( g_1 \in \text{Rep}(\varepsilon_1 + \varepsilon_0 + \varepsilon) \).

For \( t \in [0, T_{M_1}] \), we have \( w_0 \ast g_0(t) = \phi_{g_0(t)}(q_0) + \frac{\varepsilon}{m} \) for some \( 1 \leq i \leq m \), and so

\[
d(\phi_{k(g_0(t))}(q_1), \phi_{g_0(t)}(q_0)) < \varepsilon_1.
\]

Hence

\[
d(\phi_{g_1(t)}(q_1), x_0 \ast t) \leq d(\phi_{k(g_0(t))}(q_1), \phi_{g_0(t)}(q_0)) + d(f_1(t), x_0 \ast t) + d(\phi_{g_0(t)}(q_0), x_0 \ast t).
\]

Similarly, for \( t \in (g_0^{-1}(-m), 0] \), we have \( w_0 \ast g_0(t) = \phi_{g_0(t)}(q_0) - \frac{\varepsilon}{m} \) for some \( m \leq i \leq -1 \), and so

\[
d(\phi_{g_1(t)}(q_1), x_0 \ast t) \leq d(\phi_{k(g_0(t))}(q_1), \phi_{g_0(t)}(q_0)) + d(\phi_{g_0(t)}(q_0), x_0 \ast t).
\]

For \( t \in (T_{-M_1}, g_0^{-1}(-m)] \), we have \( w_0 \ast g_0(t) = \phi_{g_0(t)}(q_0) + s_2(q_0) \), and so

\[
d(\phi_{g_1(t)}(q_1), x_0 \ast t) \leq d(\phi_{k(g_0(t))}(q_1), \phi_{g_0(t)}(q_0)) + d(\phi_{g_0(t)}(q_0), x_0 \ast t).
\]

For \( t \in [T_{M_1}, \infty) \), we have \( w_0 \ast (f_1(t) - f_1(T_{M_1}) + g_0(T_{M_1})) = \phi_{f_1(t)}(y) \), and so

\[
d(\phi_{k(f_1(t)-f_1(T_{M_1})+g_0(T_{M_1}))(q_1), \phi_{f_1(t)}(y)) < \varepsilon_1.
\]

Hence

\[
d(\phi_{g_1(t)}(q_1), x_0 \ast t) \leq d(\phi_{k(f_1(t)-f_1(T_{M_1})+g_0(T_{M_1}))(q_1), \phi_{f_1(t)}(y)) + d(\phi_{f_1(t)}(y), x_0 \ast t) \leq 2\varepsilon_1.
\]
Similarly, for $t \in (-\infty, T_{-M_1})$, we have $w_0 \ast (f_2(t) - f_2(T_{-M_1}) + g_0(T_{-M_1})) = \phi_{f_2(t)}(z)$, and so
\[
d(\phi_{g_1(t)}(q_1), x_0 \star t) = \leq d(\phi_k(f_2(t) - f_2(T_{-M_1}) + g_0(T_{-M_1}))(q_1), \phi_{f_2(t)}(z)) + d(\phi_{f_2(t)}(z), x_0 \star t) \leq 2\varepsilon_1.
\]
It completes the proof.

**Proposition 2.** Let $\phi$ be a flow on a compact metric space $X$. If $\phi$ is expansive and has the shadowing property, then $\phi$ has the weak two-sided limit shadowing property.

**Proof.** Let $\{r_i\}_{i=1}^\infty$ be a decreasing sequence in $(0, 1)$ such that $\sum_{i=1}^\infty r_i < 1$. Take a decreasing sequence $\{\tau_i\}_{i=0}^\infty$ in $(0, 1)$ such that $d_{c^0}(\phi_s, id) < r_{i+1}$ for all $s \in [-\tau_i, \tau_i]$ and $i \geq 0$. Choose $\{\varepsilon_i\}_{i=0}^\infty$ as a decreasing sequence in $(0, 1)$ such that $2\varepsilon_i$ is an expansive constant of $\phi$ with respect to $\tau_i > 0$ and $\varepsilon_i < 2^{-2^i}$ for each $i \geq 0$. Take $\delta > 0$ corresponding to $\varepsilon_0$ by the strong shadowing property of $\phi$.

Let $\xi = \{(x_i, t_i)\}_{i \in \mathbb{Z}}$ be a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$. By the strong shadowing property of $\phi$, there are $g_0 \in X$ and $g_0 \in \operatorname{Rep}(\varepsilon_0) \subset \operatorname{Rep}(\frac{1}{2})$ such that
\[
d(\phi_{g_0(t)}(q_0), x_0 \star t) < \varepsilon_0, \forall t \in \mathbb{R}.
\]
For $\varepsilon_1 < \varepsilon_0$, by Lemma 3.2, there are $q_1 \in X$ and $g_1 \in \operatorname{Rep}(\varepsilon_0 \varepsilon_1 + \varepsilon_0 + \varepsilon_1) \subset \operatorname{Rep}(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^2})$ and $M_1 \in \mathbb{N}$ such that
\[
d(\phi_{g_1(t)}(q_1), x_0 \star t) \leq \frac{\varepsilon_0 + \varepsilon_1 + r_1}{2\varepsilon_1} \text{ if } t \in (T_{-M_1}, T_{M_1}),
\]
\[
d(\phi_{g_1(t)}(q_1), x_0 \star t) \leq \frac{2\varepsilon_1 + \varepsilon_2 + r_2}{2\varepsilon_2} \text{ if } t \in (-\infty, T_{-M_1}] \cup [T_{M_1}, \infty).
\]
By applying Lemma 3.2 again, we get $q_2 \in X$, $g_2 \in \operatorname{Rep}(\sum_{j=1}^7 \frac{1}{2^7})$ and $M_2 \in \mathbb{N}$ such that
\[
d(\phi_{g_2(t)}(q_2), x_0 \star t) \leq \frac{\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + r_1 + r_2}{2\varepsilon_1} \text{ if } t \in (T_{-M_2}, T_{M_2}),
\]
\[
d(\phi_{g_2(t)}(q_2), x_0 \star t) \leq \frac{2\varepsilon_1 + \varepsilon_2 + r_2}{2\varepsilon_2} \text{ if } t \in (-\infty, T_{-M_2}] \cup [T_{M_2}, \infty).
\]
Continuing the process, for each $i \in \mathbb{N}$, there are $q_i \in X$, $g_i \in \operatorname{Rep}(\sum_{j=1}^{2^{i+1}-1} \frac{1}{2^7})$ and $M_i > M_{i-1}$ such that for any $k < i$ and $t \in (T_{-M_{k+1}}, T_{M_k}) \cup [T_{M_k}, T_{2M_k+k+1})$, we have
\[
d(\phi_{g_i(t)}(q_i), x_0 \star t) \leq 2 \varepsilon_j \sum_{j=k}^i \sum_{j=k+1}^i r_j.
\]
Note that $g_i \in \operatorname{Rep}(1)$ for all $i \in \mathbb{N}$. We see that $\{g_i\}_{i \in \mathbb{N}}$ is pointwise bounded and equicontinuous. Since $\{g_i\}_{i \in \mathbb{N}}$ has a compactly convergent subsequence and $X$ is compact, there is a subsequence $\{i_n\}_{n \in \mathbb{N}}$ such that
\[
g_{i_n} \text{ compactly converges to } g \text{ and } q_{i_n} \to q \text{ as } n \to \infty,
\]
where $q \in X$ and $g \in \operatorname{Rep}(1)$. For any $t \in \mathbb{R}$, take $k \in \mathbb{N}$ such that $t \in (T_{-M_{k+1}}, T_{-M_k}) \cup [T_{M_k}, T_{2M_k+k+1})$. Then we have
\[
d(\phi_{g(t)}(q), x_0 \star t) = \lim_{n \to \infty} d(\phi_{g_{i_n}(t)}(q_{i_n}), x_0 \star t) \leq 2 \varepsilon_j \sum_{j=k}^\infty \sum_{j=k+1}^\infty r_j,
\]
and so $d(\phi_{g(t)}(q), x_0 \star t) \to 0$ as $|t| \to \infty$. This shows that $\phi$ has the weak two-sided limit shadowing property. \qed
A point \( x \in X \) is said to be nonwandering if for any neighborhood \( U \) of \( x \), there is \( t \geq 1 \) such that \( \phi_t(U) \cap U \neq \emptyset \). The set of all nonwandering points of \( \phi \) is called the nonwandering set of \( \phi \), denoted by \( \Omega(\phi) \). A point \( x \in X \) is called a chain recurrent point of \( \phi \) if for any \( \delta > 0 \), there is a \((\delta,1)\)-chain of \( \phi \) from \( x \) to itself. The set of all chain recurrent points of \( \phi \) is called the chain recurrent set of \( \phi \), and denoted by \( CR(\phi) \). For any \( x, y \in X \), we say that \( x \sim y \) if for any \( \delta > 0 \) there are \((\delta,1)\)-chains from \( x \) to \( y \) and from \( y \) to \( x \). We see that \( \sim \) gives an equivalence relation on the set \( CR(\phi) \). An equivalence class of \( \sim \) is called a chain component of \( \phi \). It is clear that the chain recurrent set and chain components of \( \phi \) are invariant and closed.

We recall that a flow \( \phi \) has the finite shadowing property on invariant set \( \Lambda \) if for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that any \((\delta,1)\)-pseudo orbit \( \xi = \{(x_i, t_i)\}_{i=0}^{b-1} \) in \( \Lambda \) with \( b \in \mathbb{N} \) can be \( \varepsilon \)-shadowed by a point \( x \) in \( X \), i.e., there is \( h \in \text{Rep} \) such that
\[
d(\phi_{h(i)}(x), x_0 \ast t) < \varepsilon, \quad \forall 0 \leq t \leq T_b.
\]

**Proposition 3.** If a flow \( \phi \) on a compact metric space \( X \) has the weak two-sided limit shadowing property on its chain recurrent set \( CR(\phi) \), then \( \phi \) has the finite shadowing property on \( CR(\phi) \). Moreover, if \( \phi \) has no fixed points, then \( \phi \) has the shadowing property on \( CR(\phi) \).

**Proof.** By contradiction, we suppose that \( \phi \) does not have the finite shadowing property. Then there is \( \varepsilon > 0 \) such that for each \( n \in \mathbb{N} \), there exists a finite \((1/n,1)\)-pseudo orbit \( \alpha_n = \{(x_i^n, t_i^n)\}_{i=0}^{k_n} \) of \( \phi \) in \( CR(\phi) \) which can not be \( \varepsilon \)-shadowed by a point \( x \) in \( X \). Since \( \phi \) has the weak two-sided limit shadowing property on \( CR(\phi) \), by Theorem 3.1, the nonwandering set \( \Omega(\phi) \) can be decomposed by finitely many disjoint, invariant, compact sets; that is,
\[
\Omega(\phi) = \bigcup_{j=1}^{l} B_j,
\]
such that \( \phi \) is topologically transitive on each \( B_j \) for \( 1 \leq j \leq l \). Take \( \delta > 0 \) by the weak two-sided limit shadowing property of \( \phi \) on \( CR(\phi) \). By the proof of Theorem 3.1, we observe that \( d_H(B_i, B_j) > \delta \) for all \( i \neq j \), where \( d_H \) is the Hausdorff distance on \( X \). Let \( N > 0 \) be such that \( 1/N < \delta \). We define a \((\delta,1)\)-pseudo orbit \( \alpha = \{(x_i, t_i)\}_{i=0}^{k} \) of \( \phi \) by \( (x_i, t_i) = \left( \phi_{h(i)}(x_0), i \right) \) for all \( i \leq 0 \).

Assume that \( x_0^N \in B_j \) for some \( 1 \leq j \leq l \), and so \( \alpha, \alpha_n \subset B_j \) for all \( n \geq N \). Since \( \phi \) is topologically transitive on \( B_j \), for any \( \phi_{h_{k_n}}(x_{k_n}^n), x_{k+1}^n \in B_j \) \( (n \geq N) \), there is a \( 1/n \)-chain \( \beta_n \subset B_j \) from \( \phi_{h_{k_n}}(x_{k_n}^n) \) to \( x_{k+1}^n \). It is clear that, by adding the chains, \( \xi = \alpha \alpha_N \beta_N \alpha_{N+1} \beta_{N+1} \ldots \) is a two-sided limit \((\delta,1)\)-pseudo orbit of \( \phi \). By the weak two-sided limit shadowing property of \( \phi \) on \( CR(\phi) \), there are \( x \in X \) and \( h \in \text{Rep} \) such that
\[
\lim_{|t| \to \infty} d(\phi_{h(t)}(x), x_0^N \ast t) = 0.
\]
Then there is \( M > N \) such that
\[
d(\phi_{h(t)+T-h(T)}(\phi_{h(T)}(x)), x_0^M \ast t) < \varepsilon, \quad \forall T \leq t \leq T + \sum_{i=0}^{k_n} t_i^M,
\]
where \( T = \sum_{j=n}^{M-1} \sum_{i=0}^{k_j} t_i^j \). It implies that \( \alpha_M \) can be \( \varepsilon \)-shadowed by \( \phi_{h(T)}(x) \), which is a contradiction.

Moreover, in the case \( \phi \) does not have fixed points and has the finite shadowing property, by Theorem 4.3 in [23], \( \phi \) has the shadowing property. \( \square \)
Let $\mathcal{A}$ be the Lorenz attractor and $\phi$ be the flow on $\mathcal{A}$. Then we get the following corollary.

**Corollary 1.** The geometric Lorenz attractor does not have weak two-sided limit shadowable points.

**Proof.** It is well known that the flow $\phi$ on the geometric Lorenz attractor $\mathcal{A}$ in [15] does not have the finite shadowing property. Since $\phi$ is topologically transitive, by Proposition 3, we obtain that $\phi$ does not have the weak two-sided limit shadowing property. Moreover, we observe that the geometric Lorenz attractor does not have weak two-sided limit shadowable points. Indeed, suppose that $wTSLSh(\phi) \neq \emptyset$. Since $\phi$ is topologically transitive, there is $x \in \mathcal{A}$ which has the dense orbit in $\mathcal{A}$. By Lemma 2.5, $wTSLSh(\phi)$ is an invariant open set, and so $x \in wTSLSh(\phi)$. Then $wTSLSh(\phi)$ is dense in $\mathcal{A}$, and so $wTSLSh(\phi) = \mathcal{A}$. By Theorem 2.6, we know that $\phi$ has the weak two-sided limit shadowing property on $\mathcal{A}$, which is a contradiction.

We observe that the converse of Proposition 3 does not hold in general. To give an example for a flow with finite shadowing property on its chain recurrent set $CR(\phi)$ which does not have the weak two-sided limit shadowing property on $CR(\phi)$, we recall the notion of singular suspension flow of a homeomorphism on a compact metric space.

Let $X$ be a compact metric space and $f : X \to X$ be a homeomorphism on $X$. Let $X$ the suspension space of $f$. Fix $a \in X$ and $e = (a, 1/2) \in X$. Let

$$U = \left\{ \varpi \in X : \rho(e, \varpi) < \frac{1}{4} \right\},$$

and $c : X \to [0, 1]$ a continuous function such that $c(\varpi) = 0$ if $\varpi = e$, $c(\varpi) = 1$ if $\varpi \not\in U$, and $0 < c(\varpi) < 1$ if $\varpi \in U \setminus \{e\}$. Define a flow $\phi$ on $X$ as follows.

$$\phi(x, t) = (x, s + c(x)t), \forall (x, s) \in X, t \in \mathbb{R}.$$  

The flow $\phi$ is called the singular suspension of $f$ with respect to $e$.

**Example 3.3.** Let $\sigma$ be the shift map on the sequence space $\Sigma_2$ as in Example 2.3. We observe that $\sigma$ is topologically transitive on $\Sigma_2$ and has the shadowing property.

Let $\phi$ be the singular suspension flow on $X$ of $\sigma$ with respect to $e = (0, 1/2)$, where $0 \in \Sigma_2$ is such that every coordinate is $0$. We note that $\phi$ is topologically transitive, and so $CR(\phi) = X$. Since $\sigma$ has the shadowing property, by Theorem 6 in [14], we get $\phi$ has the finite shadowing property. We claim that $\phi$ does not have the weak two-sided limit shadowing property. Suppose that there is $\delta > 0$ such that any two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ can be two-sided limit shadowed by a point in $X$. Take a periodic point $\overline{p}$ of $\phi$ such that its period $S > 1$. Since $\phi$ is topologically transitive, there are $\varpi$ and $T > 0$ such that $\rho(e, \varpi) < \delta$ and $\rho(\phi_T(\varpi), \overline{p}) < \delta$. Then we can define a two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ by

$$(x_i, t_i) = \begin{cases} (e, 1) & \text{if } i \leq 0, \\ (\varpi, T) & \text{if } i = 1, \\ (\overline{p}, S) & \text{if } i \geq 2. \end{cases}$$

By the weak two-sided limit shadowing property of $\phi$, there are $\overline{\varpi}$ and $h \in Rep$ such that

$$\lim_{|t| \to \infty} \rho(\phi_{h(t)}(\overline{\varpi}), x_0 \ast t) = 0.$$
Then, we see that $\rho(\phi_t(\overline{y}), e) \to 0$ as $t \to \infty$ and $\rho(\phi_t(\overline{y}), \phi_R(\overline{y})) \to 0$ as $t \to -\infty$. On the other hand, by the construction of singular suspension flow, $\phi_t(\overline{y}) \to e$ as $t \to \infty$ implies that $\overline{y} \in \{e\} \cup \phi_R((0, 0))$. Then $\phi_t(\overline{y}) \to e$ as $t \to -\infty$, which is a contradiction.

Lemma 3.4. If a flow $\phi$ on a compact metric space $X$ has the weak two-sided limit shadowing property on its chain recurrent $CR(\phi)$, then $\Omega(\phi) = CR(\phi)$.

Proof. Suppose that $\phi$ has the weak two-sided limit shadowing property on $CR(\phi)$. Let $\delta > 0$ be such that any two-sided limit $(\delta, 1)$-pseudo orbit of $\phi$ in $CR(\phi)$ can be two-sided limit shadowed by a point in $X$. It is sufficient to show that $CR(\phi) \subset \Omega(\phi)$. Let $x \in CR(\phi)$ and $N \in \mathbb{N}$ be such that $1/N < \delta$. For each $n \geq N$, since $x \in CR(\phi)$, take a $(1/n, 1)$-pseudo orbit $\alpha_n = \{(x_i^n, t_i^n)\}^k_{i=0}$ from $x$ to itself. We define a two-sided limit $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i)\} \in CR(\phi)$ by adding finite chains

$$\xi = \ldots \alpha_{N+2} \alpha_{N+1} \alpha_N \alpha_{N+1} \alpha_{N+2} \ldots$$

By the weak two-sided limit shadowing property of $\phi$ on $CR(\phi)$, there are $y \in X$ and $h \in Rep$ such that

$$\lim_{|t| \to \infty} d(\phi_{h(t)}(y), x_0 \ast t) = 0.$$ 

For any $\varepsilon > 0$, let $S < 0 < T$ such that $d(\phi_{h(S)}(y), x) < \varepsilon$ and $d(\phi_{h(T)}(y), x) < \varepsilon$. Then $\phi_{h(T) - h(S)}(\phi_{h(S)}(y)) \in \phi_{h(T)-h(S)}(B(x, \varepsilon)) \cap B(x, \varepsilon) \neq \emptyset$. Since $\varepsilon$ is arbitrary, we get $x \in \Omega(\phi)$, and so $CR(\phi) \subset \Omega(\phi)$.

End of Proof of Theorem 3.1. Suppose that $\phi$ has the weak two-sided limit shadowing property on $CR(\phi)$. Let $\delta > 0$ be such that any two-sided limit $(\delta, 1)$-pseudo orbit $\xi \subset CR(\phi)$ of $\phi$ can be two-sided limit shadowed by a point in $X$. By Lemma 3.4, we have $\Omega(\phi) = CR(\phi)$. Then the nonwandering set $\Omega(\phi)$ can be decomposed by the union of chain components

$$\Omega(\phi) = \bigcup_{\lambda \in \Lambda} B_{\lambda},$$

where $B_{\lambda}$'s are chain components of $\phi$. Note that chain component $B_{\lambda}$ is invariant and compact.

We first claim that each chain component of $\phi$ is open in $\Omega(\phi)$. Indeed, for given $\lambda \in \Lambda$, we let $U_{\lambda}$ be a $\delta$-neighborhood of $B_{\lambda}$ in $\Omega(\phi)$; i.e.,

$$U_{\lambda} = \{x \in \Omega(\phi) \mid d(x, B_{\lambda}) < \delta\}.$$ 

For any $x \in U_{\lambda}$, we claim that $x \in B_{\lambda}$. In fact, take $y \in B_{\lambda}$ such that $d(x, y) < \delta/2$. Let $N \in \mathbb{N}$ be such that $1/N < \delta$. Since $x, y \in CR(\phi)$, for each $n \geq N$, we let $\alpha_n$ be a $(1/n, 1)$-chain from $x$ to itself and $\beta_n$ be a $(1/n, 1)$-chain from $y$ to itself. We can take $\alpha_n, \beta_n \subset CR(\phi)$ for all $n \geq N$. Then we define a two-sided limit $(\delta, 1)$-pseudo orbit $\xi = \{(x_i, t_i)\} \in CR(\phi)$ by

$$\xi = \ldots \alpha_{N+1} \alpha_N \beta_{N+1} \ldots$$

By the weak two-sided limit shadowing property of $\phi$ on $CR(\phi)$, there are $z \in X$ and $h \in Rep$ such that

$$\lim_{|t| \to \infty} d(\phi_{h(t)}(z), x_0 \ast t) = 0.$$ 

For $\varepsilon > 0$, let $\varepsilon_0 < \varepsilon$ be such that $d(z_1, z_2) < \varepsilon_0$ implies $d(\phi_{t}(z_1), \phi_{t}(z_2)) < \varepsilon$ for all $0 \leq t \leq 1$. Take $S_1 < 0 < S_2$ such that

$$h(S_2) - h(S_1) > 2, \quad d(\phi_{h(S_1)}(z), x) < \varepsilon_0 \quad \text{and} \quad d(\phi_{h(S_2)}(z), y) < \varepsilon_0.$$
Then we have \( d(\phi_{1+h(S_1)}(z), \phi_1(x)) < \varepsilon \). It is easy to check that 
\[
\{(x, 1), (\phi_{1+h(S_1)}, h(S_2) - h(S_1) - 1)\}
\]
is an \((\varepsilon, 1)\)-chain from \( x \) to \( y \). Similarly we can construct an \((\varepsilon, 1)\)-chain from \( y \) to \( x \). Since \( \varepsilon \) is arbitrary, we get \( x \sim y \), and so \( x \in B_\lambda \). Therefore, \( B_\lambda = U_\lambda \) is open in \( \Omega(\phi) \). Since \( \Omega(\phi) \) is compact, \( \Omega(\phi) \) can be decomposed by finite disjoint union 
\[
\Omega(\phi) = \bigcup_{i=1}^n B_i.
\]
Next we show that \( \phi \) is topologically transitive on \( B_i \) for each \( 1 \leq i \leq n \). Let \( U, V \) be nonempty open sets in \( B_i \), and choose \( 0 < r < \delta \) such that \( B(x, r) \cap B_i \subset U \) and \( B(y, r) \cap B_i \subset V \) for some \( x \in U \) and \( y \in V \). Let \( N \in \mathbb{N} \) be such that \( 1/N < \delta \). For each \( n \geq N \), let \( \gamma_n \) be a \((1/n, 1)\)-chain from \( x \) to \( y \) and \( \tau_n \) be a \((1/n, 1)\)-chain from \( y \) to \( x \). We can take \( \gamma_n, \tau_n \subset B_i \) for all \( n \geq N \). We consider the two-sided limit \((\delta, 1)\)-pseudo orbit \( \xi' = \{(y_i, t_i)\}_{i \in \mathbb{Z}} \) of \( \phi \) defined by adding finite chains 
\[
\xi' = \ldots \gamma_{N+1} \tau_{N+1}, \gamma_{N} \tau_{N} \gamma_{N-1} \tau_{N-1} \ldots
\]
By the weak two-sided limit shadowing property of \( \phi \) on \( CR(\phi) \), there are \( z \in X \) and \( g \in \text{Rep} \) such that 
\[
\lim_{|t| \to \infty} d(\phi_{g(t)}(z), y_0 * t) = 0.
\]
For any \( \varepsilon_1 > 0 \), and \( n \in \mathbb{N} \) with \( 1/n < \varepsilon_1 \), we write \( \tau_n = \{(x^n_i, t^n_i)\}_{i=0}^k \). Take \( S_3 < 0 < S_4 \) such that \( g(S_3), g(S_4) > 1 \). It is easy to see that 
\[
\{(z, g(S_4)), (x^n_0, t^n_0), \ldots, (x^n_k, t^n_k), (\phi_{h(S_3)}(z), -h(S_3))\}
\]
is an \((\varepsilon_1, 1)\)-chain from \( z \) to itself. Since \( \varepsilon_1 \) is arbitrary, we get \( z \in CR(f) \). Take \( 0 < S_5 < S_6 \) such that 
\[
g(S_5) - g(S_6) > 1, d(\phi_{g(S_5)}(z), x) < r \quad \text{and} \quad d(\phi_{g(S_6)}(z), y) < r.
\]
Since \( r < \delta \), we have 
\[
\phi_{g(S_5)}(z) \in B(x, r) \cap B_i \subset U \quad \text{and} \quad \phi_{g(S_6)}(z) \in B(y, r) \cap B_i \subset V.
\]
Consequently, we get \( \phi_{g(S_5)}(z), \phi_{g(S_6)}(z) \in \phi_{g(S_5)} - g(S_5)(U) \cap V \), and so \( \phi \) is topologically transitive on \( B_i \).

Finally we prove that \( \phi|_{B_i} \) has the two-sided limit shadowing property for each \( 1 \leq i \leq n \). By Theorem 2.4, it is sufficient to show that \( \phi|_{B_i} \) has the weak two-sided limit shadowing property for each \( 1 \leq i \leq n \). Let \( \chi = \{(y_i, s_i)\}_{i \in \mathbb{Z}} \) be a two-sided limit \((\delta, 1)\)-pseudo orbit of \( \phi \) in \( B_i \). By the weak two-sided limit shadowing property of \( \phi \), there are \( y \in X \) and \( f \in \text{Rep} \) such that 
\[
\lim_{|t| \to \infty} d(\phi_{f(t)}(y), y_0 * t) = 0.
\]
For any \( \varepsilon > 0 \), take \( 0 < \gamma < \varepsilon \) such that \( d(z_1, z_2) < \gamma \) implies \( d(\phi_t(z_1), \phi_t(z_2)) < \varepsilon \) for all \( 0 \leq t \leq 1 \). For \( \gamma > 0 \), there is \( N > 0 \) such that 
\[
|\langle T_n \rangle - f(T_{-N})| > \gamma, d(\phi_{f(T_n)}(y), y_N) < \gamma \quad \text{and} \quad d(\phi_{f(T_{-n})}(y), y_{-N}) < \gamma,
\]
where \( T_n \) is the \( n \)-th sum of the associated sequence of \( \chi \). It is easy to see that 
\[
\alpha_1 = \{(y, f(T_N))\}
\]
is an \((\varepsilon, 1)\)-chain from \( y \) to \( y_N \), and 
\[
\alpha_2 = \{(y_{-N}, 1), (\phi_1(\phi_{h(T_{-N})}(y)), -h(T_{-N}) - 1)\}
\]
is an \((\varepsilon, 1)\)-chain from \( y_{-N} \) to \( y \). Since \( \phi \) is topologically transitive on \( B_j \), we take \( \alpha \) is an \((\varepsilon, 1)\)-chain from \( y_{-N} \) to \( y_N \). Then the sequence \( \alpha_1 \alpha_2 \) is an \((\varepsilon, 1)\)-chain from \( y \) to itself. Since \( \varepsilon \) is arbitrary, we obtain that \( y \in CR(\phi) \). Moreover, for any \( \varepsilon > 0 \), there are \( \varepsilon \)-chains \( \alpha_1 \) and \( \alpha_2 \) from \( y \) to \( y_N \in B_i \) and \( y_{-N} \) to \( y \), respectively.
Then \( y \in B_i \). Consequently, the two-sided limit \((\delta,1)\)-pseudo orbit \( \chi \) can be two-sided limit shadowed by a point \( y \) in \( B_i \), and so \( \phi|_{B_i} \) has the weak two-sided limit shadowing property. This completes the proof of Theorem 3.1.

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