RESEARCH ARTICLE

Local well-posedness to the 2D Cauchy problem of nonhomogeneous heat-conducting Navier–Stokes and magnetohydrodynamic equations with vacuum at infinity

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This paper concerns the Cauchy problem of nonhomogeneous heat-conducting magnetohydrodynamic (MHD) equations in \( \mathbb{R}^2 \) with vacuum as far-field density. By spatial weighted energy method, we derive the local existence and uniqueness of strong solutions provided that the initial density and the initial magnetic decay not too slowly at infinity. As a byproduct, we get the local existence of strong solutions to the 2D Cauchy problem for nonhomogeneous heat-conducting Navier–Stokes equations with vacuum at infinity.

KEYWORDS
2D Cauchy problem, local well-posedness, nonhomogeneous heat-conducting MHD equations, vacuum at infinity

MSC CLASSIFICATION
76D05; 76D03

1 INTRODUCTION AND MAIN RESULTS

Magnetohydrodynamics is the study of the interaction of electromagnetic fields and conducting fluids. The modeling consists of a coupling between the Navier–Stokes equations of continuum fluid mechanics and the Maxwell equations of electromagnetism. For a detailed derivation of the model, we refer to Gerbeau et al.\(^1\). In this paper, we consider the nonhomogeneous heat-conducting magnetohydrodynamic equations in \( \mathbb{R}^2 \times (0, T) \):

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla P &= H \cdot \nabla H, \\
c_\text{v}[\rho \theta_t + \text{div}(\rho \theta u)] - \kappa \Delta \theta &= \frac{\nu}{2}[\nabla u + (\nabla u)^T]^2 + \nu(\text{curl}H)^2, \\
H_t - \nu \Delta H + u \cdot \nabla H - H \cdot \nabla u &= 0, \\
\text{div} u &= \text{div} H = 0,
\end{aligned}
\]

(1.1)

where \( \rho = \rho(x, t), \theta = \theta(x, t), u = (u^1, u^2)(x, t), H = (H^1, H^2)(x, t) \), and \( P = P(x, t) \) denote the density, the absolute temperature, the velocity, the magnetic field, and the pressure, respectively. The positive constant \( \mu \) is the viscosity coefficient of the fluid, \( \nu > 0 \) is the magnetic diffusive coefficient, while \( c_\text{v} \) and \( \kappa \) are the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity, respectively. \( \text{curl}H \triangleq \partial_1 H^2 - \partial_2 H^1 \).
System (1.1) is supplemented with the initial condition
\[
(\rho, \rho u, \rho \theta, H)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0, H_0)(x), \quad x \in \mathbb{R}^2,
\]
and the far-field behavior
\[
(\rho, u, \theta, H)(x, t) \to (0, 0, 0, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0.
\]  
When we don’t take into account (1.1), (1.1) becomes the classical nonhomogeneous magnetohydrodynamic equations
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla P &= H \cdot \nabla H, \\
H_t - \nu \Delta H + u \cdot \nabla H - H \cdot \nabla u &= 0, \\
\text{div} u &= \text{div} H = 0.
\end{align*}
\]

There are many works on the well-posedness theory of solutions to (1.4). In the absence of vacuum, that is, the initial density has a positive lower bound, the local existence of strong solutions in some Besov spaces was established by Abidi and Paicu\(^2\) and later extended by Chen et al\(^3\) to be the global one. Meanwhile, Chen et al\(^4\) showed global well-posedness to the three-dimensional (3D) Cauchy problem for discontinuous initial density. In the presence of vacuum, that is, in the case that the initial density vanishes in some region, there has been a considerable number of researches on the nonhomogeneous fluid equations since the works of Lions\(^5\) and Choe and Kim\(^6\) where the global-in-time weak solutions and local strong solutions to the nonhomogeneous Navier–Stokes equations were obtained, respectively. Under the compatibility condition
\[
-\mu \Delta u + \nabla P_0 - H_0 \cdot \nabla H_0 = \sqrt{\rho_0 g} \quad \text{for some} \quad (P_0, g) \in H^1 \times L^2,
\]
Chen et al\(^7\) proved the local existence and uniqueness of strong solutions to the 3D Cauchy problem of (1.4). Moreover, they derived the global strong solution provided that the initial data satisfy some smallness condition. However, once we consider 2D problem of (1.4), there is no need to require any additional smallness condition on the initial data for the global existence and uniqueness of strong solutions, as showed in Huang and Wang\(^8\) (see also Zhong\(^9\)) and Lü et al\(^10\). Recently, without using (1.5), Song\(^11\) and Lü et al\(^12\) established the local strong solutions to the 3D and 2D Cauchy problem of (1.4) by time-weighted estimates, respectively.

In contrast to (1.4), model (1.1) is more in line with reality, but the problem becomes challenging. By introducing the compatibility condition
\[
\begin{align*}
-\mu \Delta u_0 + \nabla P_0 - H_0 \cdot \nabla H_0 &= \sqrt{\rho_0 g_1}, \\
-\kappa \Delta \theta_0 - \frac{\nu}{2} |\nabla u_0 + (\nabla u_0)^T|^2 - \nu (\text{curl} H_0)^2 &= \sqrt{\rho_0 g_2},
\end{align*}
\]
with \(P_0 \in H^1\) and \(g_1, g_2 \in L^2\), Wu\(^13\) proved the local existence and uniqueness of strong solutions to the 3D initial-boundary value problem of (1.1). Very recently, with the help of time-weighted estimates, Zhong\(^14,15\) showed global strong solutions to the 3D problem of (1.1) without using (1.6) provided that the initial data satisfy some smallness condition. Such smallness condition is not needed to the 2D initial-boundary value problem.\(^16\) Moreover, he also obtained large time decay rates of the solution. Meanwhile, applying a logarithmic interpolation inequality and delicate energy estimates, Zhong\(^17\) established global strong solution to the 2D Cauchy problem of (1.1) with non-vacuum at infinity for large initial data provided that (1.6) holds true. In addition, Zhu and Ou\(^18\) studied the global well-posedness of strong solutions for 3D initial-boundary value problems with viscosity-dependent density and temperature. It is worth mentioning that the compatibility condition (1.6) is also needed in Zhu and Ou\(^18\) in order to ensure the boundedness of temperature. It should be noted that system (1.1) becomes the nonhomogeneous heat-conducting Navier–Stokes equations when there is no electromagnetic field; the mathematical results concerning the global existence of strong solutions to this model can refer, for example, to several studies.\(^19–23\)

Although some research results have been obtained, yet it is still unknown even for the local existence of strong solutions to 2D Cauchy problems (1.1)–(1.3) \textit{with vacuum at infinity}. The aim of the present paper is to investigate the local well-posedness theory of strong solutions to problems (1.1)–(1.3).

Our main result can be stated as follows.

**Theorem 1.1.** Let \(\eta_0\) be a positive constant and
\[
\bar{x} \triangleq (e + |x|^2)^{\frac{1}{2}} \ln^{1+\eta_0}(e + |x|^2).
\]  

\[
(\rho, \rho u, \rho \theta, H)(x, 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0, H_0)(x), \quad x \in \mathbb{R}^2,
\]
For constants $q > 2$ and $a > 1$, assume that the initial data $(\rho_0 \geq 0, u_0, \theta_0 \geq 0, H_0)$ satisfy

\[
\begin{aligned}
\rho_0 \xi^a \in L^1 \cap H^1 \cap W^{1,q},& \quad H_0 \xi^\frac{a}{2} \in H^1, \\
(\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0) \in L^2, & \quad (\nabla u_0, \nabla \theta_0, \nabla H_0) \in H^1, \\
div u_0 = \text{div} H_0 = 0,
\end{aligned}
\]

and the compatibility condition

\[
\begin{aligned}
-\mu \Delta u_0 + \nabla P_0 - H_0 \cdot \nabla H_0 = \sqrt{\rho_0} g_1, \\
\kappa \Delta \theta_0 + \frac{\kappa}{2} \nabla \theta_0 + (|\nabla u_0|^2 + \nu (\text{curl} H_0)^2) = \sqrt{\rho_0} g_2,
\end{aligned}
\]

for some $P_0 \in H^1(\mathbb{R}^2)$ and $g_1, g_2 \in L^2(\mathbb{R}^2)$. Then there exists a small time $T_0 > 0$ such that problems (1.1)–(1.3) have a unique strong solution $(\rho \geq 0, u, \theta \geq 0, H)$ on $\mathbb{R}^2 \times (0, T_0]$ satisfying

\[
\begin{aligned}
\rho \in C([0, T_0]; L^1 \cap H^1 \cap W^{1,q}), & \quad \rho \xi^a \in L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \\
|\sqrt{\rho} u|, |\sqrt{\rho} \theta|, |\nabla u|, |\nabla \theta|, |\nabla u| \in L^\infty(0, T_0; L^2), \\
\nabla u, \nabla \theta, H \xi^\frac{a}{2} \in L^\infty(0, T_0; H^1), \\
H, \nabla H, H, \nabla^2 H \in L^\infty(0, T_0; L^2), \\
\nabla u, \nabla \theta \in L^2(0, T_0; \mathcal{H}^1) \cap L^2(0, T_0; W^{1,q}) \cap \nabla^\frac{a}{2} L^2(0, T_0; W^{1,q}), \\
\nabla u \in L^2(0, T_0; L^2) \cap L^2(0, T_0; L^2), \\
\nabla H, \nabla \theta, \nabla^2 \xi^\frac{a}{2} \in L^2(0, T_0; H^1), \\
\sqrt{\rho} u, \sqrt{\rho} \theta \in L^2(0, T_0; L^2),
\end{aligned}
\]

and

\[
\inf_{0 \leq t \leq T_0} \int_{B_N} \rho(x, t) dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho_0(x) dx,
\]

for some constant $N > 0$ and $B_N \triangleq \{x \in \mathbb{R}^2 | |x| < N\}$.

**Remark 1.1.** Compatibility condition (1.9) is used for obtaining the $L^\infty(0, T; L^2)$-norm of $\sqrt{\rho} u$ and $\sqrt{\rho} \theta$, which is crucial in tackling the $L^\infty(0, T; L^2)$-norm of the gradient of the temperature. This is very different from the nonhomogeneous case,\textsuperscript{12} where the authors didn’t need to require such compatibility condition via time-weighted techniques. It is certainly interesting to investigate the local well-posedness theory of strong solutions to (1.1)–(1.3) without using (1.9). This will be left for future studies.

**Remark 1.2.** Thanks to the presence of compatibility condition (1.9), the regularities of the initial data required in this paper are stronger than those in Lü et al. and Liang\textsuperscript{12,24} to prove the local well-posedness of strong solutions.

We now comment on the key analysis of this paper. It should be pointed out that the crucial techniques in Wu\textsuperscript{13} cannot be adapted because his arguments are only valid for the case of bounded domains. Moreover, similarly to nonhomogeneous model (1.4), the main difficulty here is the presence of vacuum at infinity and the criticality of Sobolev’s inequality in $\mathbb{R}^2$. However, compared with Lü et al.,\textsuperscript{12} some new difficulties arise due to the appearance of energy equation (1.1)\textsubscript{3} as well as the coupling of the velocity with the temperature. Indeed, if we multiply (1.1)\textsubscript{3} by $\theta$, we get after integration by parts that

\[
\frac{c_v}{2} \frac{d}{dt} \int \rho \theta^2 dx + \kappa \int |\nabla \theta|^2 dx = \int \left[ \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \nu (\text{curl} H)^2 \right] \theta dx.
\]

Since the $L^p(\mathbb{R}^2)$-norm of $\theta$ and spatial weighted estimates on the gradients of the velocity and the magnetic field are unavailable, it is hard to control the term on the right-hand side of (1.12). To this end, motivated by Liang and Shuai,\textsuperscript{25} we...
obtain a spatial weight estimate (see (2.10)) on the quadratic nonlinearity \( \frac{\kappa}{2} |\nabla u + (\nabla u)^r|^2 + \nu (\text{curl} H)^2 \), which plays a crucial role in dealing with the \textit{a priori} estimates of the temperature (see Lemma 3.7). Furthermore, it is worth emphasizing that a Hardy-type inequality (see (2.8)) and Gagliardo–Nirenberg inequality (see (2.4) and (2.5)) are mathematically useful for the analysis.

As a direct corollary of Theorem 1.1, we have the following local existence result for 2D nonhomogeneous heat-conducting Navier–Stokes equations with vacuum at infinity.

**Theorem 1.2.** Let \( \eta_0 \) and \( \tilde{x} \) be as in (1.7). For constants \( q > 2 \) and \( a > 1 \), assume that the initial data \((\rho_0, u_0, \theta_0, H_0)\) satisfy

\[
\rho_0 \tilde{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \quad (\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0) \in L^2, \quad (\nabla u_0, \nabla \theta_0) \in H^1, \quad \text{div} u_0 = 0,
\]

and the compatibility condition

\[
\begin{aligned}
-\mu \Delta u_0 + \nabla P_0 &= \sqrt{\rho_0} \tilde{g}_1, \\
\kappa \Delta \theta_0 + \frac{\kappa}{2} |\nabla u_0 + (\nabla u_0)^r|^2 &= \sqrt{\rho_0} \tilde{g}_2,
\end{aligned}
\]

for some \( P_0 \in H^1(\mathbb{R}^2) \) and \( \tilde{g}_1, \tilde{g}_2 \in L^2(\mathbb{R}^2) \). Then there exists a small time \( T_0 > 0 \) such that the 2D Cauchy problem of nonhomogeneous heat-conducting Navier–Stokes equations (i.e., (1.1)–(1.3) with \( H = 0 \)) has a unique strong solution \((\rho, u, \theta)\) on \( \mathbb{R}^2 \times (0, T_0) \) satisfying (1.10) with \( H = 0 \) and (1.11).

The rest of the paper is organized as follows. In Section 2, we collect some elementary facts and inequalities, which will be needed in later analysis. Section 3 is devoted to the \textit{a priori} estimates, which are needed to obtain the local existence and uniqueness of strong solutions. Finally, the main result Theorem 1.1 is proved in Section 4.

## 2 | PRELIMINARIES

In this section, we will recall some known facts and elementary inequalities, which will be used frequently later. First of all, if the initial density is strictly away from vacuum, the following local existence theorem on bounded balls can be shown by similar arguments as those in Choe and Kim.\(^6\)

**Lemma 2.1.** For \( R > 0 \) and \( B_R = \{x \in \mathbb{R}^2 | |x| < R\} \), assume that \((\rho_0, u_0, \theta_0, H_0)\) satisfies

\[
(\rho_0, u_0, \theta_0, H_0) \in H^2(B_R), \quad \inf_{x \in B_R} \rho_0(x) > 0, \quad \text{div} u_0 = \text{div} H_0 = 0. \tag{2.1}
\]

Then there exists a small time \( T_R > 0 \) such that Equation (1.1) with the following initial-boundary-value conditions

\[
\begin{aligned}
(\rho, u, \theta, H)(x, t = 0) &= (\rho_0, u_0, \theta_0, H_0), \quad x \in B_R, \\
(u, \theta, H)(x, t) &= (0, 0, 0), \quad x \in \partial B_R, \quad t > 0,
\end{aligned} \tag{2.2}
\]

has a unique classical solution \((\rho > 0, u, \theta, H)\) on \( B_R \times (0, T_R) \) satisfying

\[
\begin{aligned}
\rho &\in C ([0, T_R]; H^2), \\
(u, \theta, H) &\in C ([0, T_R]; H^3) \cap L^2(0, T_R; H^3), \\
P &\in C ([0, T_R]; H^1) \cap L^2(0, T_R; H^2),
\end{aligned} \tag{2.3}
\]

where we denote \( H^k = H^k(B_R) \) for positive integer \( k \).

Next, the following well-known Gagliardo–Nirenberg inequality (see Ladyzenskaja et al.\(^26\), Chapter II) will be used in the next section frequently.

**Lemma 2.2.** For \( f \in H^r(B_R) \) and \( g \in L^r(B_R) \cap W^{1,q}(B_R) \) with \( r \in (1, \infty) \) and \( q \in (2, \infty) \), there exists a positive constant \( C \) independent of \( R \) such that

\[
\|f\|_{L^p} \leq C \|f\|_{L^2} \|f\|_{H^r}^{p-2}, \quad \forall p \in [2, \infty), \tag{2.4}
\]

\[
\|g\|_{L^q} \leq C \|g\|_{L^r} + C \|g\|_{L^r} \frac{r_1}{r_1+q-2} \|\nabla g\|_{L^2} \frac{2q}{2q+2-2r}, \tag{2.5}
\]

where \( 1 < r_1 < q \).
Next, for $\Omega = \mathbb{R}^2$ or $\Omega = B_R$, the following weighted $L^m$-bounds for elements of the Hilbert space $\tilde{D}^{1,2}(\Omega) \equiv \{ v \in H^{1}_{\text{loc}}(\Omega) | \nabla v \in L^2(\Omega) \}$ can be found in Theorem B.1 of Lions.5

**Lemma 2.3.** For $m \in [2, \infty)$ and $\theta \in \left( 1 + \frac{m}{2}, \infty \right)$, there exists a positive constant $C$ such that for either $\Omega = \mathbb{R}^2$ or $\Omega = B_R$ with $R \geq 1$ and for any $v \in \tilde{D}^{1,2}(\Omega)$,

$$\left( \int_{\Omega} \frac{|v|^m}{(e + |x|^2)^{\theta}} \, dx \right)^{\frac{1}{m}} \leq C \|v\|_{L^2(B_R)} + C \|\nabla v\|_{L^2(\Omega)}. \quad (2.6)$$

A useful consequence of Lemma 2.3 is the following crucial weighted bounds (see Liang24, Lemma 2.4) for elements of $\tilde{D}^{1,2}(\Omega)$.

**Lemma 2.4.** Let $\tilde{x}$ and $\eta_0$ be as in (1.7) and $\Omega$ be as in Lemma 2.3. Assume that $\rho \in L^\infty(\Omega)$ is a nonnegative function such that

$$\int_{B_{N_1}} \rho \, dx \geq M_1, \quad \|\rho\|_{L^\infty(\Omega)} \leq M_2. \quad (2.7)$$

for positive constants $M_1, M_2$, and $N_1 \geq 1$ with $B_{N_1} \subset \Omega$. Then, for $\epsilon, \eta > 0$, there is a positive constant $C$ depending only on $\epsilon, \eta, M_1, M_2, N_1$, and $\eta_0$ such that, for $v \in \tilde{D}^{1,2}(\Omega)$ with $\sqrt{\rho}v \in L^2(\Omega)$,

$$\|v\tilde{x}^{-\eta}\|_{L^\frac{2p}{p-2}(\Omega)} \leq C \left( \|\sqrt{\rho}v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)} \right) \quad (2.8)$$

with $\tilde{\eta} = \min\{1, \eta\}$.

The following $L^p$-bound for elliptic systems is a direct result of the combination of the well-known elliptic theory27,28 and a standard scaling procedure.

**Lemma 2.5.** For $p > 1$ and $k \geq 0$, there exists a positive constant $C$ depending only on $p$ and $k$ such that

$$\|\nabla^{k+2}v\|_{L^p(B_R)} \leq C \|\Delta v\|_{W^{k, p}(B_R)}, \quad (2.9)$$

for every $v \in W^{k+2, p}(B_R)$ satisfying

$$v = 0 \quad \text{on} \quad \partial B_R.$$

Finally, by the same arguments as those in Lemma 3.1 of Liang and Shuai,25 we have the following spatial weighted estimate on the solution.

**Lemma 2.6.** Let $(\rho, u, \theta, H)$ be the solution to problems (1.1), (2.1), and (2.2), then it holds that, for $b_1 > 0$,

$$\int_{B_{b_1}} \left[ \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + v(\text{curl}H) \right] |x|^{b_1} \, dx \leq \int_{B_{b_1}} |c_s (\rho \theta + \rho u \cdot \nabla \theta) ||x|^{b_1} \, dx. \quad (2.10)$$

### 3 A PRIORI ESTIMATES

In this section, for $r \in [1, \infty]$ and $k \geq 0$, we write

$$\int \cdot \, dx = \int_{B_R} \cdot \, dx, \quad L^r = L^r(B_R), \quad W^{k,r} = W^{k,r}(B_R), \quad H^k = W^{k,2}.$$

Moreover, for $R > 4N_0 \geq 4$ with $N_0$ fixed, assume that $(\rho_0, u_0, \theta_0, H_0)$ satisfies, in addition to (2.1), that

$$\frac{1}{2} \leq \int_{B_{4N_0}} \rho_0(x) \, dx \leq \int_{B_R} \rho_0(x) \, dx \leq \frac{3}{2} \quad (3.1)$$
Lemma 2.1 thus yields that there exists some \( T_R > 0 \) such that initial-boundary-value problems (1.1), (2.1), and (2.2) have a unique classical solution \((\rho > 0, u, \theta, H)\) on \( B_R \times (0, T_R) \) satisfying (2.3).

Let \( \bar{x}, \eta_0, a, \) and \( q \) be as in Theorem 1.1, the main aim of this section is to derive the following key a priori estimate on \( \psi \) defined by

\[
\psi(t) \triangleq 1 + \|\sqrt{\rho_u}\|_{L^2} + \|\sqrt{\rho_\theta}\|_{L^2} + \|\sqrt{\rho_{uu}}\|_{L^2} + \|\sqrt{\rho_{\theta\theta}}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla^2 H\|_{L^2} + \|\nabla H\|_{L^2} + \|\rho\|_{L^{2^{(2)}}(H_t)} \cap W^{1,4}.
\]

\[
(3.2)
\]

**Proposition 3.1.** Assume that \((\rho_0, u_0, \theta_0, H_0)\) satisfies (2.1) and (3.1). Let \((\rho, u, \theta, H)\) be the solution to initial-boundary-value problems (1.1), (2.1), and (2.2) on \( B_R \times (0, T_R) \) obtained by Lemma 2.1. Then there exist positive constants \( T_0, L_0, M, N_0, \) and \( E_0 \) such that

\[
\sup_{0 \leq t \leq T_0} (H_t) + \|\nabla P\|_{L^2} + \int_0^{T_0} \left( \|\sqrt{\rho}u\|_{L^2}^2 + \|\sqrt{\rho_\theta}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 \theta H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) dt \leq M.
\]

(3.3)

where

\[
E_0 \triangleq \|\sqrt{\rho_0}u_0\|_{L^2} + \|\sqrt{\rho_\theta_0}\|_{L^2} + \|\nabla u_0\|_{H^1} + \|\nabla \theta_0\|_{H^1} + \|H_0\|_{L^2}^2
\]

\[+ \|\rho_0\|_{L^{2^{(2)}}(H_t)} \cap W^{1,4} + \|H_0\|_{H^1} + \|g_1\|_{L^2} + \|g_2\|_{L^2}.
\]

To show Proposition 3.1, whose proof will be postponed to the end of this section, we begin with the following elementary estimate for the solution.

**Lemma 3.1.** Under the conditions of Proposition 3.1, let \((\rho, u, \theta, H)\) be a smooth solution to initial-boundary-value problems (1.1), (2.1), and (2.2). Then, for any \( t > 0 \),

\[
\sup_{0 \leq s \leq t} (\|\rho\|_{L^{2^{(2)}}(H_t)} + \|\sqrt{\rho_u}\|_{L^2} + \|\nabla u\|_{L^2}) + \int_0^t (\mu \|\nabla u\|_{L^2}^2 + v \|\nabla H\|_{L^2}^2) ds \leq \|\rho_0\|_{L^{2^{(2)}}(H_t)} + \|\sqrt{\rho_0}u_0\|_{L^2} + \|H_0\|_{L^2}^2.
\]

(4.4)

**Proof.** We deduce from (1.1)_1 and (1.1)_5 that the density satisfies a transport equation; thus, we have

\[
\sup_{0 \leq s \leq t} \|\rho\|_{L^{2^{(2)}}(H_t)} \leq \|\rho_0\|_{L^{2^{(2)}}(H_t)}.
\]

(5.5)

Multiplying (1.1)_2 by \( u \) and (1.1)_4 by \( H \), respectively, we get after integrating by parts that

\[
\frac{d}{dt} (\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2) + 2 (\mu \|\nabla u\|_{L^2}^2 + v \|\nabla H\|_{L^2}^2) = 0.
\]

(6.6)

Integrating (6.6) over \([0, t]\) leads to

\[
\sup_{0 \leq s \leq t} (\|\sqrt{\rho}u\|_{L^2}^2 + \|H\|_{L^2}^2) + \int_0^t (\mu \|\nabla u\|_{L^2}^2 + v \|\nabla H\|_{L^2}^2) ds \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2 + \|H_0\|_{L^2}^2.
\]

(7.7)

This together with (3.5) yields the desired (4.4).

Next, we will give some spatial weighted estimates on the density and the magnetic field.
Lemma 3.2. Under the conditions of Proposition 3.1, let \((\rho, u, \theta, H)\) be a smooth solution to initial-boundary-value problems (1.1), (2.1), and (2.2). Then there exists a \(T_1 = T_1(N_0, E_0) > 0\) such that, for all \(t \in (0, T_1)\),

\[
\sup_{0 \leq s \leq t} \left( \|\rho \bar{x}^a\|_{L^1} + \|H \bar{x}^a\|_{L^2} \right) + \int_0^t \|\nabla H \bar{x}^a\|_{L^2} ds \leq C,
\]

(3.8)

where (and in what follows) \(C\) denotes a generic positive constant depending on \(\mu, v, \kappa, q, \eta_0, N_0,\) and \(E_0\), but independent of \(R\).

Proof.

1. For \(N > 1\), let \(\varphi_N \in C_0^\infty(B_N)\) satisfy

\[
0 \leq \varphi_N \leq 1, \; \varphi_N(x) = 1, \text{ if } |x| \leq \frac{N}{2}, \text{ and } |\nabla \varphi_N| \leq 3N^{-1}.
\]

(3.9)

It follows from (1.1) and (3.4) that

\[
\frac{d}{dt} \int \rho \varphi_{2N_0} dx = \int \rho u \cdot \nabla \varphi_{2N_0} dx \geq -CN_0^{-1} \|\rho\|_{L^1}^{\frac{1}{2}} \|\sqrt{\rho} u\|_{L^2} \geq -\tilde{C}(E_0).
\]

(3.10)

Integrating (3.10) and using (3.1) give

\[
\inf_{0 \leq s \leq T} \int_{B_{2N_0}} \rho dx \geq \inf_{0 \leq s \leq T} \int \rho \varphi_{2N_0} dx \geq \int \rho_0 \varphi_{2N_0} dx - \tilde{C} T \geq \frac{1}{4},
\]

(3.11)

where \(T_1 \triangleq \min\{1, (4\tilde{C})^{-1}\}\). From now on, we will always assume that \(t \leq T_1\). The combination of (3.11), (3.4), and (2.8) implies that, for \(\epsilon, \eta > 0\) and \(v \in D^{1,2}(B_R)\) with \(\sqrt{\rho} v \in L^2(B_R),\)

\[
\|v\bar{x}^{-\eta}\|_{L^2}^{\frac{1}{\eta}} \leq C(\epsilon, \eta) \left( \|\sqrt{\rho} v\|_{L^2}^{\frac{1}{\eta}} + \|\nabla v\|_{L^2}^{\frac{1}{\eta}} \right),
\]

(3.12)

where \(\bar{\eta} \triangleq \min\{1, \eta\}\).

2. Noting that for any \(\delta > 0\), it holds that

\[
|\nabla \bar{x}| \leq C(\eta_0) \ln^{1+\eta_0}(e + |x|^2) \leq C(\eta_0) \bar{x}^\delta.
\]

(3.13)

Multiplying (1.1) by \(\bar{x}^a\) and integrating by parts, we then obtain from Hölder’s inequality, (3.12), (3.13), and (3.4) that

\[
\frac{d}{dt} \|\rho \bar{x}^a\|_{L^1} = \int \rho (u \cdot \nabla) \bar{x}^a dx
\]

(3.14)

\[
\leq C \int \rho |u| \bar{x}^{a-1+\frac{\delta}{4\eta_0}} dx
\]

\[
\leq C \|\rho \bar{x}^{a-1+\frac{\delta}{4\eta_0}}\|_{L^{\frac{4\eta_0}{\delta}}} \|u \bar{x}^{\frac{\delta}{4\eta_0}}\|_{L^{4\eta_0}}
\]

(3.15)

\[
\leq C \|\rho\|_{L^{\frac{4\eta_0}{\delta}}} \|\rho \bar{x}^{a-1+\frac{\delta}{4\eta_0}}\|_{L^{4\eta_0}} \left( \|\sqrt{\rho} u\|_{L^2} + \|\nabla u\|_{L^2} \right)
\]

\[
\leq C \left( 1 + \|\rho \bar{x}^a\|_{L^1} \right) \left( 1 + \|\nabla u\|_{L^2} \right).
\]
This combined with Grönwall’s inequality and (3.4) leads to

\[
\sup_{0 \leq s \leq t} \|\rho x^a\|_{L^1}^2 \leq C \exp \left\{ C \int_0^t \left( 1 + \|\nabla u\|_{L^2}^2 \right) \, ds \right\} \leq C. \tag{3.16}
\]

3. Multiplying (1.1)_4 by \( H\vec{x}^a \) and integrating by parts yield

\[
\frac{1}{2} \frac{d}{dt} \|H\vec{x}^a\|_{L^2}^2 + \nu \|\nabla H\vec{x}^a\|_{L^2}^2 = \frac{\nu}{2} \int |H|^2 \Delta \vec{x}^a \, dx + \int (H \cdot \nabla)u \cdot H\vec{x}^a \, dx + \frac{1}{2} \int |H|^2 u \cdot \nabla \vec{x}^a \, dx 
\triangleq I_1 + I_2 + I_3. \tag{3.17}
\]

Direct calculations lead to

\[
|I_1| \leq C \int |H|^2 \vec{x}^a \, dx \leq C \|H\vec{x}^a\|_{L^2}^2, \tag{3.18}
\]

and

\[
|I_2| \leq \int \|\nabla u\| \|H\vec{x}^a\| \, dx 
\leq \|\nabla u\| \|H\vec{x}^a\|_{L^2}^2 
\leq C \|\nabla u\| \|H\vec{x}^a\|_{L^2}^2 \|H\vec{x}^a\|_{H^1} 
\leq C \|\nabla u\| \|H\vec{x}^a\|_{L^2} \left( \|H\vec{x}^a\|_{L^2} + \|\nabla H\vec{x}^a\|_{L^2} + \|H\nabla \vec{x}^a\|_{L^2} \right) 
\leq C \|\nabla u\| \|H\vec{x}^a\|_{L^2} \left( \|H\vec{x}^a\|_{L^2} + \|\nabla H\vec{x}^a\|_{L^2} + \|H\nabla \vec{x}^a\|_{L^2} \|\vec{x}^{-1} \nabla \vec{x}\|_{L^\infty} \right) 
\leq C \left( 1 + \|\nabla u\|_{L^2}^2 \right) \|H\vec{x}^a\|_{L^2}^2 + \frac{\nu}{4} \|\nabla H\vec{x}^a\|_{L^2}^2, \tag{3.19}
\]
due to (2.4) and (3.13). Similarly, it follows from Hölder’s inequality, (3.13), (2.4), (3.12), and (3.7) that, for \( a > 1 \),

\[
|I_3| \leq C \int |H|^2 \vec{x}^a \, dx 
\leq C \|H\vec{x}^a\|_{L^2} \|H\vec{x}^a\|_{L^2} \|\nabla \vec{x}^{-1} \|_{L^\infty} \|\vec{x}^{-\frac{a}{2}}\|_{L^\infty} 
\leq C \left( 1 + \|\nabla u\|_{L^2}^2 \right) \|H\vec{x}^a\|_{L^2}^2 + \frac{\nu}{4} \|\nabla H\vec{x}^a\|_{L^2}^2. \tag{3.20}
\]

Putting (3.18)–(3.20) into (3.17), we thus deduce from Grönwall’s inequality and (3.7) that

\[
\sup_{0 \leq s \leq t} \|H\vec{x}^a\|_{L^2}^2 + \int_0^t \|\nabla H\vec{x}^a\|_{L^2}^2 \, ds \leq C \exp \left\{ C \int_0^t \left( 1 + \|\nabla u\|_{L^2}^2 \right) \, ds \right\} \leq C. \tag{3.21}
\]

This along with (3.16) gives the desired (3.8).

\[\square\]

**Lemma 3.3.** Let \((\rho, u, \theta, H)\) and \(T_1\) be as in Lemma 3.2. Then there exists a positive constant \( a > 1 \) such that, for all \( t \in (0, T_1) \),

\[
\sup_{0 \leq s \leq t} \left( \|\nabla u\|_{L^2}^2 + \|H\|_{L^4}^4 + \|\nabla H\|_{L^2}^2 \right) + \int_0^t \left( \|\sqrt{\rho} u_s\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|H_s\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \, ds 
\leq C + C \int_0^t \psi^a(s) \, ds. \tag{3.22}
\]
Proof.

1. It follows from Hölder's inequality, (3.2), (3.12), and (3.16) that, for any $\varepsilon, \eta > 0$ and $v \in L^{1,2}(B_R)$ with $\sqrt{\rho v} \in L^2(B_R)$,

$$
\left\| \rho^\alpha u \right\|_{L^{2,\infty}} \leq C \left\| \rho^\alpha \frac{\partial x}{\partial t} \right\|_{L^{4,1}} \left\| \nabla v^\alpha \right\|_{L^{4,1}} \leq C \left( \int \rho^{-\frac{4}{3}} \left\| \nabla v^\alpha \right\|_{L^{4,1}} \right)^\frac{3}{8} \left\| \nabla v^\alpha \right\|_{L^{4,1}} \leq C \left( \int \rho^{-\frac{4}{3}} \right)^\frac{3}{8} \left\| \nabla v^\alpha \right\|_{L^{4,1}} \leq C \sqrt{\rho v} + C \left\| \nabla v \right\|_{L^2},
$$

(3.23)

where $\tilde{\eta} = \min \{1, \eta\}$. This together with (3.12) implies that

$$
\left\| \rho^\alpha u \right\|_{L^{2,\infty}} + \left\| u^\alpha \tilde{v} \right\|_{L^{2,\infty}} \leq C \left( \int \rho \left\| \sqrt{\rho u} \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} \right) \leq C \psi^\alpha.
$$

(3.24)

$$
\left\| \rho^\alpha \theta \right\|_{L^{2,\infty}} + \left\| \theta \tilde{v} \right\|_{L^{2,\infty}} \leq C \left( \int \rho \left\| \sqrt{\rho \theta} \right\|_{L^2} + \left\| \nabla \theta \right\|_{L^2} \right) \leq C \psi^\alpha,
$$

(3.25)

where (and in what follows) we use $\alpha > 1$ to denote a generic constant, which may be different from line to line.

2. Multiplying (1.1) by $u_t$ and integrating by parts, one has

$$
\frac{\mu}{2} \frac{d}{dt} \int \left| \nabla u \right|^2 dx + \int \rho |u_t|^2 dx = - \int \rho u \cdot \nabla u \cdot u_t dx + \int H \cdot \nabla H \cdot u_t dx.
$$

(3.26)

By Cauchy–Schwarz inequality, Hölder's inequality, (2.4), and (3.24), we get

$$
\left| \int \rho u \cdot \nabla u \cdot u_t dx \right| \leq \frac{1}{2} \int \rho |u_t|^2 dx + \frac{1}{2} \int |\nabla u|^2 dx
$$

$$
\leq \frac{1}{2} \left\| \rho u \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla u \right\|_{L^2}^2 + C \left( \int \rho \left\| \sqrt{\rho u} \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} \right)
$$

$$
\leq \frac{1}{2} \left\| \rho u \right\|_{L^2}^2 + C \left\| \nabla u \right\|_{L^2}^2 + C \psi^\alpha.
$$

(3.27)

Integration by parts together with (1.1) and (2.4) indicates that

$$
\int H \cdot \nabla H \cdot u_t dx = - \frac{d}{dt} \int H \cdot \nabla u \cdot H dx + \int H_t \cdot \nabla u \cdot H dx + \int H \cdot \nabla u \cdot H_t dx
$$

$$
\leq - \frac{d}{dt} \int H \cdot \nabla u \cdot H dx + \frac{1}{4} \left\| H_t \right\|_{L^2}^2 + C \left\| H \right\|_{L^2}^2 \left\| \nabla u \right\|_{L^2}^2
$$

$$
\leq - \frac{d}{dt} \int H \cdot \nabla u \cdot H dx + \frac{1}{4} \left\| H_t \right\|_{L^2}^2 + C \left\| H \right\|_{L^2}^2 \left\| \nabla H \right\|_{L^2} \left\| \nabla u \right\|_{L^2} \left\| H \right\|_{H^1}
$$

$$
\leq - \frac{d}{dt} \int H \cdot \nabla u \cdot H dx + \frac{1}{4} \left\| H_t \right\|_{L^2}^2 + C \psi^\alpha.
$$

(3.28)

Inserting (3.27) and (3.28) into (3.26) gives rise to

$$
B'(t) + \left\| \rho u_t \right\|_{L^2}^2 \leq \frac{1}{2} \left\| H_t \right\|_{L^2}^2 + C \psi^\alpha.
$$

(3.29)

where
\[
B(t) \triangleq \mu \|\nabla u\|_{L^2}^2 + 2 \int H \cdot \nabla u \cdot H \, dx
\]
satisfies
\[
\frac{\mu}{2} \|\nabla u\|_{L^2}^2 - \frac{2}{\mu} \|H\|_{L^2}^4 \leq B(t) \leq \frac{3\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{2}{\mu} \|H\|_{L^2}^4,
\]  
(3.30)

owing to
\[
\left| 2 \int H \cdot \nabla u \cdot H \, dx \right| \leq 2 \|\nabla u\|_{L^2} \|H\|_{L^2}^2 \leq \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{2}{\mu} \|H\|_{L^2}^4.
\]

3. Multiplying (1.1) by \(|H|^2H\) and integrating by parts, we derive from (2.4) and Cauchy–Schwarz inequality that
\[
\frac{1}{4} \frac{d}{dt} \|H\|_{L^4}^4 + \nu \|\nabla H\|_{L^2}^2 + \frac{\nu}{2} \|\nabla |H|\|_{L^2}^2 \leq C \|\nabla u\|_{L^2} \||H|\|_{L^4}^2
\]
\[
\leq C \|\nabla u\|_{L^2} \||H|\|_{L^2} \||H|\|_{L^2}^2
\]
\[
\leq C \left(1 + \|\nabla u\|_{L^2}^2\right) \|H\|_{L^4}^4 + \frac{\nu}{4} \|\nabla |H|\|_{L^2}^2,
\]
which implies that
\[
\frac{d}{dt} \|H\|_{L^4}^4 + \nu \|\nabla H\|_{L^2}^2 \leq C \left(1 + \|\nabla u\|_{L^2}^2\right) \|H\|_{L^4}^4.
\]
(3.31)

Thus, adding (3.31) multiplied by \(\frac{3}{\mu}\) to (3.29) leads to
\[
\frac{d}{dt} \left(B(t) + \frac{3}{\mu} \|H\|_{L^4}^4\right) + \|\nabla u\|_{L^2}^2 + \frac{3\nu}{\mu} \|\nabla H\|_{L^2}^2 \leq C \left(1 + \|\nabla u\|_{L^2}^2\right) \|H\|_{L^4}^4 + \frac{1}{2} \|H\|_{L^2}^2 + C\psi^\alpha.
\]
(3.32)

4. Noting that \((u, P)\) satisfies the Stokes system
\[
\begin{cases}
-\mu \Delta u + \nabla P = -\rho u_t - \rho u \cdot \nabla u + H \cdot \nabla H, & x \in B_R, \\
\text{div} u = 0, & x \in B_R, \\
u(x) = 0, & x \in \partial B_R.
\end{cases}
\]
(3.33)

Applying the standard \(L^p\)-estimate to (3.33) yields that, for any \(p \in [2, \infty)\),
\[
\|\nabla^2 u\|_{L^p} + \|\nabla P\|_{L^p} \leq C \|\rho u_t\|_{L^p} + C \|\rho u \cdot \nabla u\|_{L^p} + C \||H|\| \|\nabla H\|_{L^p}.
\]
(3.34)

Then we obtain from (3.34) with \(p = 2\), (3.2), and (3.27) that
\[
\|\nabla^2 u\|_{L^2} + \|\nabla P\|_{L^2} \leq C \|\rho u_t\|_{L^2} + C \|\rho u \cdot \nabla u\|_{L^2} + C \||H|\| \|\nabla H\|_{L^2}
\]
\[
\leq C \|\rho u_t\|_{L^2} + C\psi^\alpha + C \||H|\| \|\nabla H\|_{L^2}.
\]
(3.35)
This combined with (1.1)4, (2.4), (3.7), and (3.24) gives that, for \( \delta > 0 \),

\[
\frac{d}{dt} \left[ \nu ||\nabla H||_{L^2}^2 + ||H||_{L^2}^2 + \nu^2 ||\Delta H||_{L^2}^2 \right]
\]

\[
= \int |H_t - \nu \Delta H|^2 dx
\]

\[
= \int |H \cdot \nabla u - u \cdot \nabla H|^2 dx
\]

\[
\leq C ||H|| ||\nabla u||_{L^2}^2 + C ||u|| ||\nabla H||_{L^2}^2
\]

\[
\leq C ||H|| ||\nabla u||_{L^2}^2 + C ||\Delta^2 \nabla H||_{L^2}||\nabla H||_{L^2} + ||H||^2_{L^2} + ||\Delta H||^2_{L^2}
\]

\[
\leq C \left( 1 + ||H||^4_{L^2} + \delta \left( ||\sqrt{\rho} u||^2_{L^2} + ||H|| ||\nabla H||^2_{L^2} \right) + C ||\nabla H \Delta^2 ||_{L^2}^2 + C \psi^\alpha \right.
\]

Adding (3.36) to (3.32), we obtain after choosing \( \delta \) suitably small that

\[
\frac{d}{dt} \left( B(t) + \frac{3}{\mu} ||H||^4_{L^2} + \nu ||\nabla H||_{L^2}^2 \right) + ||\nabla u||_{L^2}^2 + ||H|| ||\nabla H||_{L^2}^2 + ||H||^2_{L^2} + ||\Delta H||^2_{L^2}
\]

\[
\leq C \left( 1 + ||H||^4_{L^2} + C ||\nabla H \Delta^2 ||_{L^2}^2 + C \psi^\alpha \right.
\]

which together with Grönwall's inequality, (3.30), (3.7), and (3.21) yields that

\[
\sup_{0 \leq s \leq t} \left( ||\nabla u||_{L^2}^2 + ||H||^4_{L^2} + ||\nabla H||_{L^2}^2 \right) + \int_0^t \left( ||\nabla u||_{L^2}^2 + ||H|| ||\nabla H||_{L^2}^2 + ||\Delta^2 H||_{L^2}^2 \right) ds \leq C + \int_0^t \psi^\alpha(s) ds.
\]

This along with (3.35) indicates that

\[
\int_0^t ||\nabla^2 u||_{L^2}^2 \leq C + \int_0^t \psi^\alpha(s) ds.
\]

The proof of Lemma 3.3 is complete. \( \square \)

**Lemma 3.4.** Let \((\rho, u, \theta, H)\) and \(T_1\) be as in Lemma 3.2. Then there exists a positive constant \( \alpha > 1 \) such that, for all \( t \in (0, T_1) \),

\[
\sup_{0 \leq s \leq t} \left( ||\sqrt{\rho} u||_{L^2}^2 + ||H||^2_{L^2} \right) + \int_0^t \left( ||\nabla u||_{L^2}^2 + ||H||^2_{L^2} \right) ds \leq C \exp \left\{ C \int_0^t \psi^\alpha ds \right\}.
\]

**Proof.**

1. Differentiating (1.1)2 with respect to \( t \) gives that

\[
\rho u_t + \rho u \cdot \nabla u_t - \mu \Delta u_t = -\rho_t (u_t + u \cdot \nabla u) - \rho u_t \cdot \nabla u - \nabla P_t + (H \cdot \nabla H)_t.
\]

Multiplying (3.38) by \( u_t \) and integrating the resulting equality by parts over \( B_R \), we obtain after using (1.1)1 and (1.1)5 that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx
\]

\[
\leq C \int \rho |u_t||u_t| (|\nabla u_t| + |\nabla u|^2 + |u||\nabla^2 u|) dx + C \int \rho |u|^2 |\nabla u||\nabla u_t| dx
\]

\[
+ C \int \rho |u_t|^2 |\nabla u| dx + \int H_t \cdot \nabla H \cdot u_t dx + \int H \cdot \nabla H_t \cdot u_t dx \triangleq \sum_{i=1}^5 I_i.
\]
It follows from (3.23), (3.24), and (2.4) that

\[
\hat{I}_1 \leq C\|\sqrt{\rho u}\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2}^\frac{1}{2} \|\sqrt{\rho u_t}\|_{L^2}^\frac{1}{2} (\|\nabla u_t\|_{L^2} + \|\nabla u^2\|_{L^2})
\]

\[
+ C\|\rho u_t\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2}^\frac{1}{2} \|\sqrt{\rho u_t}\|_{L^2}^\frac{1}{2} \|\nabla u\|_{L^2}
\]

\[
\leq C(1 + \|\nabla u\|_{L^2}^2) \|\sqrt{\rho u_t}\|_{L^2} \left(\|\sqrt{\rho u_t}\|_{L^2} + \|\nabla u_t\|_{L^2}\right)^\frac{3}{2}
\]

\[
\times \left(\|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} + \|\nabla^2 u\|_{L^2}\right)
\]

\[
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C\psi^a.
\]  

(3.40)

 Hölder’s inequality combined with (3.23) and (3.24) leads to

\[
\hat{I}_2 + \hat{I}_3 \leq C\|\sqrt{\rho u}\|_{L^2}^2 \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2} \leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C\psi^a.
\]  

(3.41)

From (3.31), Grönwall’s inequality, and (3.7), we have

\[
\sup_{0 \leq s \leq t} \|H\|_{L^4}^4 \leq C,
\]  

(3.42)

which combined with integration by parts, Hölder’s inequality, and (2.4) indicates that

\[
\hat{I}_4 + \hat{I}_5 = -\int H_t \cdot \nabla u_t \cdot Hdx - \int H \cdot \nabla u_t \cdot H_t dx
\]

\[
\leq 2\|\nabla u_t\|_{L^2} \|H_t\|_{L^4} \|H\|_{L^4}
\]

\[
\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C\|H_t\|_{L^2} \|H\|_{L^2}
\]  

(3.43)

Thus, substituting (3.40)–(3.43) into (3.39), we obtain that

\[
\frac{d}{dt} [\|\sqrt{\rho u_t}\|_{L^2}^2 + \mu \|\nabla u_t\|_{L^2}^2] \leq C\|H_t\|_{L^2}^2 + \epsilon \|\nabla H_t\|_{L^2}^2 + C\psi^a.
\]  

(3.44)

2. Differentiating (1.1)_4 with respect to \(t\) shows that

\[
H_{tt} - H_t \cdot \nabla u - H \cdot \nabla u_t + u_t \cdot \nabla H + u \cdot \nabla H_t = \nu \Delta H_t.
\]  

(3.45)

Multiplying (3.45) by \(H_t\) and integrating the resulting equality over \(B_R\) yield that

\[
\frac{1}{2} \frac{d}{dt} \int |H_t|^2 dx + \nu \int |\nabla H_t|^2 dx = \int (H \cdot \nabla) u_t \cdot H_t dx - \int (u_t \cdot \nabla H_t) H_t dx + \int (H_t \cdot \nabla) u \cdot H_t dx - \int (u \cdot \nabla) H_t \cdot H_t dx
\]

\[
\triangleq S_1 + S_2 + S_3 + S_4.
\]  

(3.46)
Integration by parts together with (3.42), Hölder’s inequality, (3.12), and (3.21) leads to

\[ S_1 + S_2 = - \int (H \cdot \nabla) H_t \cdot u_t dx + \int (u_t \cdot \nabla) H_t \cdot H dx \]
\[ \leq 2 \| \nabla H_t \|_{L^2} \| u_t \|_{L^3} \| H \|_{L^4} \]
\[ \leq \frac{v}{4} \| \nabla H_t \|_{L^2}^2 + \frac{4}{v} \| u_t \|_{L^2}^2 \| H \|_{L^4}^2 \]
\[ \leq \frac{v}{4} \| \nabla H_t \|_{L^2}^2 + C(v, a) \| \sqrt{\rho} u_t \|_{L^2}^2 + C(v, a) \| \nabla u_t \|_{L^2}^2. \]

By virtue of Hölder’s inequality and (2.4), one has

\[ S_3 \leq \| H_t \|_{L^2}^2 \| \nabla u \|_{L^2} \leq C \| H_t \|_{L^2} \| H \|_{H^1} \| \nabla u \|_{L^2} \leq \frac{v}{4} \| \nabla H_t \|_{L^2}^2 + C \| u \|_{H^1}^2. \]

We derive from integration by parts and (1.1) that

\[ S_4 = \int (u \cdot \nabla) H_t \cdot H_t dx = -S_4, \]

that is,

\[ S_4 = 0. \]

Inserting (3.47)–(3.49) into (3.46), we get

\[ \frac{d}{dt} \| H_t \|_{L^2}^2 + v \| \nabla H_t \|_{L^2}^2 \leq C \| H_t \|_{L^2}^2 + C \| \sqrt{\rho} u_t \|_{L^2}^2 + C_1 \| \nabla u_t \|_{L^2}^2 \]

for some positive constant \( C_1 \) depending on \( v \) and \( a \). Adding (3.50) multiplied by \( \frac{\mu}{2C_1} \) to (3.44) and then choosing \( \varepsilon = \frac{\mu v}{4C_1} \), we arrive at

\[ \frac{d}{dt} \left( \| \sqrt{\rho} u_t \|_{L^2}^2 + \frac{\mu}{2C_1} \| H_t \|_{L^2}^2 \right) + \frac{\mu}{2} \| \nabla u_t \|_{L^2}^2 + \frac{\mu v}{4C_1} \| \nabla H_t \|_{L^2}^2 \leq C \| u \|_{H^1}^2 \| \sqrt{\rho} u_t \|_{L^2}^2 + C \| H_t \|_{L^2}^2 + C \| u \|_{H^1}^2. \]

3. It follows from (2.5), Young’s inequality, (3.13), (3.12), and (2.4) that

\[ \| u \|_{L^\infty} \leq C \left( \| u x^{-\frac{3}{2}} \|_{L^4} + \| \nabla \left( u x^{-\frac{3}{2}} \right) \|_{L^2} \| u x^{-\frac{3}{2}} \|_{H^1} \right) \]
\[ \leq C \left( \| u x^{-\frac{3}{2}} \|_{L^4} + \| u x^{-\frac{3}{2}} \|_{L^2} \right) \]
\[ \leq C \left( \| u x^{-\frac{3}{2}} \|_{L^4} + \| u x^{-\frac{3}{2}} \|_{L^2} + \| \nabla u \|_{L^2} \right) \]
\[ \leq C \left( \| \sqrt{\rho} u \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla u \|_{H^1} \| u \|_{H^1} \right) \]
\[ \leq C \left( \| \sqrt{\rho} u \|_{L^2} + \| \nabla u \|_{H^1} \right). \]

We deduce from (1.1), (3.42), and (2.4) that

\[ \| H_t \|_{L^2} \leq C \left( \| \Delta H \|_{L^2} + \| u \cdot \nabla H \|_{L^2} + \| H \cdot \nabla u \|_{L^2} \right) \]
\[ \leq C \left( \| u \|_{L^4} \| \nabla H \|_{L^2} + \| u x^{-\frac{3}{2}} \|_{L^2} \| \nabla H x^{-\frac{3}{2}} \|_{L^2} + \| H \|_{L^2} \| \nabla u \|_{L^2} \right) \]
\[ \leq C \left( \| \nabla^2 H \|_{L^2} + \| u x^{-\frac{3}{2}} \|_{L^2} \| \nabla H x^{-\frac{3}{2}} \|_{L^2} + \| H \|_{L^2} \| \nabla u \|_{H^1} \right). \]
which together with (3.52) yields that

$$
\|H_t(0)\|_{L^2}^2 \leq CE_0^a \leq C.
$$  \hfill (3.53)

From (2.5) and (3.52), one has

$$
\|\sqrt{\rho}u\|_{L^{\infty}} \leq \|\sqrt{\xi}\|_{L^{\infty}} + \|\nabla(\rho\xi)\|_{L^q}^{\frac{q}{2-q}} \|\nabla u\|_{L^q}^{\frac{q}{2-q}} \left(\|\sqrt{\rho}u\|_{L^2} + \|\nabla u\|_{L^2}\right),
$$

which combined with (1.1)_2 and (1.9) leads to

$$
\int \rho|u|^2(x,0)dx \leq \limsup_{t \to 0} \int \rho^{-1}|\mu \Delta u + H \cdot \nabla u - \rho u \cdot \nabla u|^2 dx \\
\leq 2\|g_1\|_{L^2}^2 + 2\|\sqrt{\rho}u(0)\|_{L^2}^2 \|\nabla u(0)\|_{L^2}^2 \\
\leq 2\|g_1\|_{L^2}^2 + CE_0^a \leq C.
$$

This along with (3.51), Grönwall’s inequality, and (3.53) leads to (3.37).

**Lemma 3.5.** Let \((\rho, u, \theta, H)\) and \(T_1\) be as in Lemma 3.2. Then there exists a positive constant \(\alpha > 1\) such that, for all \(t \in (0, T_1]\),

$$
\sup_{0 \leq s \leq t} \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 H\|_{L^2}^2 + \|\nabla^2 H\xi\|_{L^2}^2\right) \\
+ \int_0^t \left(\|\nabla^2 u\|_{L^2}^{\frac{q+1}{2}} + \|\nabla P\|_{L^2}^{\frac{q+1}{2}} + \|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 + \|\nabla^2 H\xi\|_{L^2}^2\right) ds \\
\leq C\exp \left\{ C \int_0^r \psi^n ds \right\}.
$$

**Proof.**

1. Multiplying (1.1)_4 by \(\Delta H\xi\) and integrating by parts lead to

\[
\frac{1}{2} \frac{d}{dt} \int |\nabla H|^2 \xi dx + \nu \int |\Delta H|^2 \xi dx \\
\leq C \int |\nabla H||H||\nabla u||\nabla \xi| dx + C \int |\nabla H|^2 |u||\nabla \xi| dx + C \int |\nabla H|\Delta H|\nabla \xi| dx \\
+ C \int |H||\nabla u||\Delta H|\xi dx + C \int |\nabla u||\nabla H|^2 \xi dx = \sum_{i=1}^5 I_i.
\]
Using (3.21), (3.12), Hölder’s inequality, and (2.4), we get by some direct calculations that

\[ J_1 \leq C \| H \hat{x}^\xi \|_{L^2} \| \nabla u \|_{L^2} \| \nabla H \hat{x}^\xi \|_{L^2} \]
\[ \leq C \| H \hat{x}^\xi \|_{L^2} \left( \| \nabla H \hat{x}^\xi \|_{L^2} + \| H \hat{x}^\xi \|_{L^2} \right) ^{\frac 12} \| \nabla H \|_{L^2} \| \nabla H \hat{x}^\xi \|_{L^2} \]
\[ \leq C \psi^a + C \psi^a \| \nabla H \hat{x}^\xi \|_{L^2}^2, \]

\[ J_2 \leq C \| \nabla H |^{2 \frac {n-2} {2} \hat{x}^\xi} + \frac {n-2} {n} \|_{L^\infty} \| u \hat{x}^{\frac {n-2} {2} \hat{x}^\xi} + \| \nabla H |^{\frac {n-2} {2} \hat{x}^\xi} \|_{L^\infty} \]
\[ \leq C \psi^a \| \nabla H \hat{x}^\xi \|_{L^2}^{\frac {n-2} {2} \hat{x}^\xi} \| \nabla H |^{\frac {n-2} {2} \hat{x}^\xi} \|_{L^\infty}, \]
\[ \leq C \psi^a \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \| \nabla H \|_{L^2}^2, \]

\[ J_3 + J_4 \leq \frac {v} {4} \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \| H \hat{x}^\xi \|_{L^2}^2 \| u \|_{L^2}^2 \]
\[ \leq \frac {v} {4} \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \| H \hat{x}^\xi \|_{L^2} \| u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1}, \]
\[ \leq \frac {v} {4} \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \psi^a \| H \hat{x}^\xi \|_{L^2}^2 \]
\[ \leq C \| \nabla H \hat{x}^\xi \|_{L^2}^2 \psi^a \]

Substituting the above estimates into (3.56) and noting the following fact

\[ \int |\nabla H|^2 \hat{x}^\xi dx = \int |\nabla H|^2 \hat{x}^\xi dx - \int \delta \partial_k H \cdot \partial_k \hat{x}^\xi dx + \int \delta \partial_k H \cdot \partial_k \hat{x}^\xi dx \]
\[ \leq \int |\nabla H|^2 \hat{x}^\xi dx + \frac 12 \int |\nabla H|^2 \hat{x}^\xi dx + C \int |\nabla H|^2 \hat{x}^\xi dx, \]

we derive that

\[ \frac {d} {dt} \| \nabla H \hat{x}^\xi \|_{L^2}^2 + v \| \nabla H \hat{x}^\xi \|_{L^2}^2 \leq C \left( \psi^a + \| \nabla^2 u \|_{L^2} \right) \| \nabla H \hat{x}^\xi \|_{L^2}^2 + C \psi^a. \]

\[ (3.57) \]

Now we claim that

\[ \int_0^t \left( \| \nabla^2 u \|_{L^2}^{2} + \| \nabla P \|_{L^2}^{2} + \| \nabla^2 u \|_{L^2}^{2} + \| \nabla P \|_{L^2}^{2} \right) d \psi = C \exp \left\{ C \int_0^t \psi^a (s) ds \right\}, \]

\[ (3.58) \]

whose proof will be given at the end of this proof. Thus, we infer from (3.21), (3.22), (3.58), and Grönwall’s inequality that

\[ \sup_{0 \leq t \leq T} \| \nabla H \hat{x}^\xi \|_{L^2}^2 + \int_0^t \| \nabla^2 H \hat{x}^\xi \|_{L^2}^2 d \psi \leq C \exp \left\{ C \int_0^t \psi^a (s) ds \right\}. \]

\[ (3.59) \]

2. It deduces from (1.14), the standard $L^2$-estimate of elliptic equations, (3.24), Hölder’s inequality, (3.4), (3.42), and Gagliardo–Nirenberg inequality that

\[ \| \nabla H \|_{L^2}^2 \leq C \| H \|_{H^2}^2 + C \| u \|_{L^2} \| \nabla H \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \]
\[ \leq C \| H \|_{H^2}^2 + C \| u \|_{L^2}^2 \| H \|_{L^2} + C \| H \|_{L^2}^2 \| u \|_{L^2}^2 \]
\[ \leq C \| H \|_{H^2}^2 + C \| \nabla H \|_{L^2}^2 + C (1 + \| \nabla u \|_{L^2}^2) \| \nabla H \|_{L^2} \| H \|_{L^2} + C \| \nabla u \|_{L^2} \| u \|_{H^1} \]
\[ \leq C \| H \|_{H^2}^2 + C \| \nabla H \|_{L^2}^2 + \frac 14 \| \nabla^2 H \|_{L^2}^2 + \frac 14 \| \nabla^2 u \|_{L^2}^2 \]
\[ + C (1 + \| \nabla u \|_{L^2}^2) (1 + \| \nabla H \|_{L^2}^2). \]

\[ (3.60) \]
It follows from (3.35), (3.4), (3.24), (3.42), and (2.4) that

$$\| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 \leq C \| \rho u \|_{L^2}^2 + C \| \rho u \cdot \nabla u \|_{L^2}^2 + C \| \rho H \|_{L^2}^2 + 2 \| \nabla u \|_{L^2}^2$$

which combined with (3.35) gives that

$$\| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 + \| \nabla^2 H \|_{L^2}^2 \leq C \left( \| \rho u \|_{L^2}^2 + \| H \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 \right) + C \left( 1 + \| \nabla u \|_{L^2}^2 \right) \left( 1 + \| \nabla H \|_{L^2}^2 \right).$$

This along with (3.22), (3.37), and (3.59) yields that

$$\sup_{0 \leq t \leq T} \left( \| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 + \| \nabla^2 H \|_{L^2}^2 \right) \leq C \exp \left( C \int_0^T \psi^a \, ds \right) + C \left( 1 + \int_0^T \psi^a(s) \, ds \right)^{10}$$

(3.61)

3. To finish the proof of Lemma 3.5, it suffices to show (3.58). Choosing $p = q$ in (3.4), we derive from (3.4), Hölder’s inequality, (3.23), and (2.4) that

$$\| \nabla^2 u \|_{L^q}^2 + \| \nabla P \|_{L^q}^2 \leq C \left( \| \rho u \|_{L^q}^2 + \| \rho u \cdot \nabla u \|_{L^q}^2 + \| \rho H \|_{L^q}^2 \right) \left( \| \nabla u \|_{L^q}^2 + \| \nabla H \|_{L^q}^2 \right)$$

(3.62)

which together with Young’s inequality and (3.37) implies that

$$\int_0^t \left( \| \nabla^2 u \|_{L^2}^{q+1} + \| \nabla P \|_{L^2}^{q+1} \right) \, ds \leq C \int_0^t \left( \| \rho u \|_{L^2}^2 \right)^{\frac{q+1}{2-q}} \left( \| \nabla u \|_{L^2}^2 \right)^{\frac{q-2q+1}{2-q}} \, ds + C \int_0^t \psi^a \, ds$$

(3.63)

$$\leq C \left( \| \rho u \|_{L^2}^2 + \| \nabla u \|_{L^2}^2 \right) ds$$

and

$$\int_0^t \left( \| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 \right) \, ds \leq C \int_0^t \left( \| \rho u \|_{L^2}^2 \right)^{\frac{q+1}{2-q}} \left( \| \nabla u \|_{L^2}^2 \right)^{\frac{q-2q+1}{2-q}} \, ds + C \int_0^t \psi^a \, ds$$

(3.64)

$$\leq C \int_0^t \| \nabla u \|_{L^2}^2 ds + C \int_0^t \psi^a \, ds$$

where we have used \( \frac{(q-2)(q+1)}{2(q-2)} \cdot \frac{q^2-2q}{q-2} \in (0, 1) \) due to \( q > 2 \). One thus obtains (3.58) from (3.63) and (3.64).
Lemma 3.6. Let \((\rho, u, \theta, H)\) and \(T_1\) be as in Lemma 3.2. Then there exists a positive constant \(\alpha > 1\) such that, for all \(t \in (0, T_1)\),

\[
\sup_{0 \leq s \leq t} \|\rho \tilde{x}\|_{L^1 \cap H^1 \cap W^{1, q}} \leq C \exp \left\{ C \int_0^t \psi^s \, ds \right\}. \tag{3.65}
\]

Proof. We derive from (1.1) and (1.1) that \(\rho \tilde{x}\) satisfies

\[
\partial_t (\rho \tilde{x}) + \mathbf{u} \cdot \nabla (\rho \tilde{x}) - a \rho \tilde{x} \mathbf{u} \cdot \nabla (\ln \tilde{x}) = 0. \tag{3.66}
\]

Operating \(\nabla\) to (3.66) and then multiplying the resultant equation by \(|\nabla (\rho \tilde{x})|^r \) for \(r \in [2, q]\), we obtain after integration by parts that

\[
\frac{d}{dt} \|\nabla (\rho \tilde{x})\|_{L^r} \leq C (1 + \|\nabla \mathbf{u}\|_{L^\infty} + \|\mathbf{u} \cdot \nabla (\ln \tilde{x})\|_{L^\infty}) \|\nabla (\rho \tilde{x})\|_{L^r} + C^r \rho^s. \tag{3.67}
\]

By (2.5) and Young’s inequality, we see that

\[
\|\nabla \mathbf{u}\|_{L^\infty} \leq C \|\nabla \mathbf{u}\|_{L^2} + C \|\mathbf{u} \cdot \nabla (\ln \tilde{x})\|_{L^\infty} \leq C \|\nabla \mathbf{u}\|_{L^2} + C^r \rho^s.
\]

Similarly to (3.52), we obtain after using (3.13) that

\[
\|\mathbf{u} \cdot \nabla (\ln \tilde{x})\|_{L^\infty} = \|\mathbf{u} \cdot \tilde{x}^{-1} \nabla \tilde{x}\|_{L^\infty} \leq C \psi^s.
\]

From (2.5), we have

\[
\|\rho \tilde{x}\|_{L^\infty} \leq C \|\rho \tilde{x}\|_{L^2} + C \|\rho \tilde{x}\|_{L^{\frac{4+4s}{3+2s}}} \|\nabla (\rho \tilde{x})\|_{L^\infty} \leq C \psi^s.
\]

Applying (3.13) and (2.4), we get

\[
\|\nabla \mathbf{u}\|_{L^\infty} \leq C \|\nabla \mathbf{u}\|_{L^2} \|\tilde{x}^{-\frac{4s}{3+2s}}\|_{L^\infty} \leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{H^s} \leq C \psi^s.
\]

Moreover, it follows from (3.13) and (3.24) that

\[
\|\mathbf{u} \cdot \nabla^2 (\ln \tilde{x})\|_{L^r} \leq C \psi^s.
\]

As a consequence, inserting the above estimates into (3.67), we derive that

\[
\frac{d}{dt} \|\nabla (\rho \tilde{x})\|_{L^r} \leq C \left( \psi^s + \|\nabla \mathbf{u}\|_{L^2}^2 \right) \left( 1 + \|\nabla (\rho \tilde{x})\|_{L^r} \right),
\]

which combined with Grönwall’s inequality and (3.64) indicates that

\[
\sup_{0 \leq s \leq t} \|\nabla (\rho \tilde{x})\|_{L^r} \leq C \exp \left\{ C \int_0^t \psi^s \, ds \right\}. \tag{3.68}
\]
Similarly, multiplying \((ρ\vec{x})^{-1}\) for \(r \in [2, q]\) and then integrating the resultant equation over \(B_r\), we can also deduce that

\[
\sup_{0 \leq s \leq t} \|ρ\vec{x}\|_{L^r} \leq C \exp\left\{C \int_0^t \psi^a ds\right\}.
\]

This along with (3.8) and (3.68) implies (3.65).

**Lemma 3.7.** Let \((ρ, u, θ, H)\) and \(T_1\) be as in Lemma 3.2. Then there exists a positive constant \(α > 1\) such that, for all \(t \in (0, T_1]\),

\[
\sup_{0 \leq s \leq t} \left(\|\sqrt{ρθ}\|_{L^2}^{\frac{1}{2}} + \|∇θ\|_{L^2}^{\frac{1}{2}} + \|ρ\theta\|_{L^2}^{\frac{1}{2}}\right)
+ \int_0^t \left(\|\sqrt{ρθ}\|_{L^2}^{\frac{1}{2}} + \|∇^2θ\|_{L^2}^{\frac{1}{2}} + \|ρ\theta\|_{L^2}^{\frac{1}{2}}\right) ds
\leq C \exp\left\{C \int_0^t \psi^a ds\right\}.
\]

**Proof.**

1. Choosing \(b_1 \leq \frac{a}{2}\) in Lemma 2.6, then for \(0 < b < \min\{b_1, 1\}\), we have

\[
\vec{x}^b \leq C (1 + |x|^{b_1}) < C\vec{x}^\frac{a}{2}.
\]

Thus, it follows from Lemma 2.6 that

\[
\int \left[\frac{μ}{2} |∇u + (∇u)^r|^2 + \nu(\text{curl}\,H)^2\right] \vec{x}^b dx
\leq C \|∇u\|_{L^2}^{\frac{1}{2}} + C \|∇H\|_{L^2}^{\frac{1}{2}} + C \int (ρ|θ| + \rho|u||∇θ|) |x|^b dx
\leq C \|∇u\|_{L^2}^{\frac{1}{2}} + C \|∇H\|_{L^2}^{\frac{1}{2}} + C \|\sqrt{ρ\vec{x}}\|_{L^2} \left(\|\sqrt{ρθ}\|_{L^2} + \|\sqrt{ρu}\|_{L^2}\right) \leq C\psi^a.
\]

Multiplying (1.1) by \(θ\) and integrating by parts, one has

\[
\frac{C}{2} \int ρθ^2 dx + κ \int |∇θ|^2 dx = \int \left[\frac{μ}{2} |∇u + (∇u)^r|^2 + \nu(\text{curl}\,H)^2\right] θ dx.
\]

For simplicity, setting \(Z \triangleq \left[\frac{μ}{2} |∇u + (∇u)^r|^2 + \nu(\text{curl}\,H)^2\right]\), then we infer from (3.25) and (3.70) that

\[
\int \left[\frac{μ}{2} |∇u + (∇u)^r|^2 + \nu(\text{curl}\,H)^2\right] θ dx \leq C [\sqrt{Z} \vec{x}^{\frac{1}{b}}]_{L^2} \left(\|\sqrt{ρu}\|_{L^2} + \|∇H\|_{L^2}\right) \leq C\psi^a,
\]

due to \(\sqrt{Z} \leq C (|∇u| + |∇H|)\). Thus, we integrate (3.71) over \([0, t]\) and obtain that

\[
\sup_{0 \leq s \leq t} \|\sqrt{ρθ}\|_{L^2}^{\frac{1}{2}} + \int_0^t \|∇θ\|_{L^2}^{\frac{1}{2}} ds \leq E_0 + C \int_0^t \psi^a ds \leq \exp\left\{C \int_0^t \psi^a ds\right\}.
\]

2. Multiplying (1.1)_3 by \(θ_1\) gives that

\[
\frac{κ}{2} \frac{d}{dt} \|∇θ_1\|_{L^2}^2 + c_v \|\sqrt{ρθ_1}\|_{L^2}^2 = -c_v \int ρu \cdot ∇θ_1 dx + \int Zθ_1 dx.
\]

(3.74)
By virtue of Hölder’s inequality, (3.24), and (3.12), one has

\[
-c_v \int \rho u \cdot \nabla \theta_t dx \leq c_v \| \rho x^a \|_{L^\infty} \| u x^{-\frac{a}{2}} \|_{L^\infty} \| \theta_t x^{-\frac{a}{2}} \|_{L^\infty} \| \nabla \theta \|_{L^2}
\]
\[
\leq C \psi^a (\| \sqrt{\rho} \theta_t \|_{L^2} + \| \nabla \theta_t \|_{L^2})
\]
\[
\leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \psi^a.
\]

We deduce from Hölder’s inequality, (3.12), and (3.70) that

\[
\int Z \theta_t dx \leq C \| \theta_t \|_{L^2} \| \sqrt{Z} \theta_t \|_{L^2} (\| \nabla u \|_{L^2} + \| \nabla H \|_{L^2}) \leq \frac{\kappa}{8} \| \nabla \theta_t \|_{L^2}^2 + C \psi^a.
\]

(3.76)

Substituting (3.75) and (3.76) into (3.74) leads to

\[
\kappa \frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + c_v \| \sqrt{\rho} \theta_t \|_{L^2}^2 \leq \frac{\kappa}{4} \| \nabla \theta_t \|_{L^2}^2 + C \psi^a.
\]

(3.77)

3. Differentiating (1.1)\textsubscript{3} with respect to \(t\) and multiplying the resulting equation by \(\theta_t\) yield that

\[
c_v \frac{d}{dt} \| \sqrt{\rho} \theta_t \|_{L^2}^2 + \kappa \| \nabla \theta_t \|_{L^2}^2 = -c_v \int \rho |\theta_t|^2 dx - c_v \int (\rho u_t \cdot \nabla \theta_t) dx + \int Z \theta_t dx \triangleq \sum_{i=1}^{3} L_i.
\]

(3.78)

It follows from (1.1)\textsubscript{1}, integration by parts, and (3.54) that

\[
L_1 = -c_v \int \rho |\theta_t|^2 dx
\]
\[
= -2c_v \int \rho u \cdot \nabla \theta_t dx
\]
\[
\leq \frac{\kappa}{12} \| \nabla \theta_t \|_{L^2}^2 + C \| \sqrt{\rho} u_t \|_{L^2}^2 \| \sqrt{\rho} \theta_t \|_{L^2}
\]
\[
\leq \frac{\kappa}{12} \| \nabla \theta_t \|_{L^2}^2 + C \psi^a.
\]

In view of (1.1)\textsubscript{1}, (3.12), and (3.24), we obtain from Hölder’s inequality, (2.4), and (2.5) that

\[
L_2 = - \int (\rho u_t \nabla \theta_t) dx
\]
\[
= \int (u \cdot \nabla \rho) \partial_t u \cdot \nabla \theta dx - \int \rho u_t \Theta_t dx
\]
\[
\leq \| \rho x^a \|_{L^\infty} \| u x^{-\frac{a}{2}} \|_{L^\infty} \| \theta_t x^{-\frac{a}{2}} \|_{L^\infty} \| \nabla \theta \|_{L^2} + \| \rho \|_{L^\infty} \| \sqrt{\rho} x^a \|_{L^\infty} \| \nabla \theta \|_{L^2} \| \theta_t x^{-\frac{a}{2}} \|_{L^2} \| \sqrt{\rho} u_t \|_{L^2}
\]
\[
\leq \frac{\kappa}{12} \| \nabla \theta_t \|_{L^2}^2 + C \psi^a.
\]

Direct calculation gives that

\[
Z_i \leq C \sqrt{Z} (|\nabla u_t| + |\nabla H_t|),
\]
which combined with Hölder’s inequality and (3.12) ensures that

\[
L_3 \leq C \int |\theta_t| \sqrt{Z} (|\nabla u_t| + |\nabla H_1|) \, dx
\]

\[
\leq C \|\theta_t \xi^{-\frac{1}{2}}\|_{L^2} \|Z^2 \xi^{-\frac{1}{2}}\|_{L^2} \|Z^2 \|_{L^2} \|\nabla u_t| + |\nabla H_1|\|_{L^2}
\]

\[
\leq C \|\theta_t \xi^{-\frac{1}{2}}\|_{L^2} \|\sqrt{Z} \xi^{-\frac{1}{2}}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\nabla H_1\|_{L^2}
\]

\[
\leq \frac{\kappa}{12} \|\nabla \theta_t\|_{L^2}^2 + C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha dt \right\} \right\} \left( 1 + \|\sqrt{\rho_t}\|_{L^2} + \|\nabla \theta\|_{L^2} \right) \left( \|\nabla u_t\|_{L^2}^2 + \|\nabla H_1\|_{L^2}^2 \right) + C \psi^\alpha.
\]

where in the last inequality we have used

\[
\int Z \xi^2 \, dx \leq C \|\nabla u\|_{L^2}^2 + C \|\nabla H\|_{L^2}^2 + C \|\sqrt{\rho \xi^2}\|_{L^2} \cap L^\infty \left( \|\sqrt{\rho \theta_t}\|_{L^2} + \|\sqrt{\rho u}\|_{L^2} \|\nabla \theta\|_{L^2} \right)
\]

\[
\leq C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha dt \right\} \right\} \left( 1 + \|\sqrt{\rho \theta_t}\|_{L^2} + \|\nabla \theta\|_{L^2} \right) .
\]

de to (3.70), (3.7), and (3.65). Therefore, inserting the above estimates on \(L_1 - L_3\) into (3.78) and combining (3.77), we find that

\[
\frac{\kappa}{t} (c_v \|\sqrt{\rho \theta_t}\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2) + \kappa \|\nabla \theta_t\|_{L^2}^2 + c_v \|\sqrt{\rho \theta_t}\|_{L^2}^2,
\]

\[
\leq C \exp \left\{ C \int_0^t \psi^\alpha dt \right\} \left( 1 + \|\sqrt{\rho \theta_t}\|_{L^2} + \|\nabla \theta\|_{L^2} \right) \left( \|\nabla u_t\|_{L^2}^2 + \|\nabla H_1\|_{L^2}^2 \right) + C \psi^\alpha.
\]

4. It follows from (1.1)_3, (1.9), and (3.54) that

\[
\int \rho \theta_t^2 \, dx \leq \limsup_{t \to 0} \int \rho^{-1} \left[ \frac{\kappa}{c_v} \Delta \theta + \frac{\mu}{2c_v} |\nabla u + (\nabla u)^I|^2 + \frac{\gamma}{c_v} (\text{curl} H)^2 - \rho u \cdot \nabla \theta \right)^2 \, dx
\]

\[
\leq C \|g_2\|_{L^2}^2 + C \|\sqrt{\rho u(0)}\|_{L^2}^2 \|\nabla \theta(0)\|_{L^2}^2
\]

\[
\leq C \|g_2\|_{L^2}^2 + CE_0^\alpha \leq C,
\]

which combined with (3.79) and Grönwall’s inequality gives that, for \(t \in (0, T_1]\),

\[
\sup_{0 \leq s \leq t} (\|\sqrt{\rho \theta_s}\|_{L^2}^2 + \|\nabla \theta_s\|_{L^2}^2) + \int_0^t (\|\sqrt{\rho \theta_s}\|_{L^2}^2 + \|\nabla \theta_s\|_{L^2}^2) \, ds
\]

\[
\leq C + C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \int_0^t \left( 1 + \|\sqrt{\rho \theta_s}\|_{L^2} + \|\nabla \theta_s\|_{L^2} \right) \left( \|\nabla u_s\|_{L^2} + \|\nabla H_1\|_{L^2} \right) \, ds
\]

\[
+ C \int_0^t \psi^\alpha ds.
\]
By Young's inequality and (3.37), it holds that

\[
C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\} \int_0^t \left( 1 + \| \sqrt{\rho \theta} \|_{L^2}^{\frac{1}{2}} + \| \nabla \theta \|_{L^2}^{\frac{1}{2}} \right) \left( \| \nabla u_1 \|_{L^2}^{2} + \| \nabla H_1 \|_{L^2}^{2} \right) ds \\
\leq \frac{1}{2} \sup_{0 \leq s \leq t} \left( \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{2} \right) \\
+ C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\} \left( \int_0^t \left( \| \nabla u_1 \|_{L^2}^{2} + \| \nabla H_1 \|_{L^2}^{2} \right) ds \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} \sup_{0 \leq s \leq t} \left( \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{2} \right) + C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\} .
\]  
(3.81)

Thus, putting (3.81) into (3.82) leads to

\[
\sup_{0 \leq s \leq t} \left( \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{2} \right) + \int_0^t \left( \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{2} \right) ds \leq C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\} .
\]  
(3.82)

5. We deduce from (1.1), the standard \( L^2 \)-estimate of elliptic equations, (3.4), (3.24), and (2.4) that

\[
\| \nabla^2 \theta \|_{L^2}^{2} \leq C \left( \| \rho \theta \|_{L^2}^{2} + \| \rho u \cdot \nabla \theta \|_{L^2}^{2} + \| \nabla u \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} \right) \\
\leq C \left( \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \rho u \|_{L^2}^{2} \| \nabla \theta \|_{L^2}^{2} + \| \nabla u \|_{L^2}^{2} \| \nabla u \|_{L^2}^{2} \right) + \| \nabla \theta \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} \right) \\
\leq C \| \sqrt{\rho \theta} \|_{L^2}^{2} + C \left( 1 + \| \nabla u \|_{L^2}^{4} \right) \| \nabla \theta \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} \right) \\
\leq C \| \sqrt{\rho \theta} \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{2} + C \left( 1 + \| \nabla u \|_{L^2}^{4} \right) \| \nabla \theta \|_{L^2}^{2} + \| \nabla \theta \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} + \| \nabla H \|_{L^2}^{4} \right) .
\]  
(3.83)

which together with (3.37), (3.22), (3.55), and (3.82) leads to

\[
\sup_{0 \leq s \leq t} \| \nabla^2 \theta \|_{L^2}^{2} \leq C \exp \left\{ C \exp \left\{ C \int_0^t \psi^\alpha ds \right\} \right\} .
\]  
(3.84)

Obviously, one gets from (3.83) that

\[
\int_0^t \| \nabla^2 \theta \|_{L^2}^{2} ds \leq C \int_0^t \psi^\alpha ds.
\]  
(3.85)

The standard \( L^4 \)-estimate of elliptic equations together with (1.1), (3.2), Hölder’s inequality, (3.23), and (2.4) yields that

\[
\| \nabla^2 \theta \|_{L^4} \leq C \left( \| \rho \theta \|_{L^4} + \| \rho u \cdot \nabla \theta \|_{L^4} + \| \nabla u \|_{L^2}^{2} + \| \nabla H \|_{L^2}^{2} \right) \\
\leq C \| \sqrt{\rho \theta} \|_{L^2} \| \sqrt{\rho \theta} \|_{L^2} + C \| \rho u \|_{L^2} \| \nabla \theta \|_{L^2} \\
+ C \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla \theta \|_{L^2} \| \nabla \theta \|_{L^2} \\
\leq C \| \sqrt{\rho \theta} \|_{L^2} \| \sqrt{\rho \theta} \|_{L^2} + C \left( \| \rho u \|_{L^2} + \| \nabla u \|_{L^2} \right) \| \nabla \theta \|_{L^2} \| \nabla \theta \|_{L^2} \\
+ C \| \nabla u \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla \theta \|_{L^2} \| \nabla \theta \|_{L^2} \\
\leq C \| \sqrt{\rho \theta} \|_{L^2} \| \sqrt{\rho \theta} \|_{L^2} + C \psi^\alpha.
\]  
(3.86)
Consequently, similarly to (3.63) and (3.64), we infer from (3.86), Young’s inequality, and (3.82) that

\[
\int_0^t \left( \| \nabla^2 \theta \|_{L^2}^{\frac{3}{2}} + \| \nabla^2 \theta \|_{L^2} \right) ds \leq C \int_0^t \left( \psi^n + \| \nabla \psi \|_{L^2} \right) ds \leq C \exp\left\{ C \int_0^t \psi^n ds \right\}.
\]

(3.87)

The proof of Lemma 3.7 is finished.

The proof of Proposition 3.1 is a direct consequence of Lemmas 3.1–3.7.

**Proof of Proposition 3.1.** It follows from (3.4), (3.8), (3.22), (3.37), (3.55), (3.65), and (3.69) that

\[
\psi(t) \leq C \exp\left\{ C \exp\left\{ C \int_0^t \psi^n ds \right\} \right\}.
\]

Standard arguments yield that for

\[
M \triangleq C e^{C\varepsilon} \text{ and } T_0 \triangleq \min\{ T_1, (CM^n)^{-1} \},
\]

we have

\[
\sup_{0 \leq t \leq T_0} \psi(t) \leq M,
\]

which together with (3.8), (3.22), (3.37), (3.55), (3.58), and (3.69) gives (3.3). The proof of Proposition 3.1 is thus complete.

**4 PROOF OF THEOREM 1.1**

With the a priori estimates in Section 3 at hand, we are now in a position to prove Theorem 1.1.

**Step 1. Local existence of strong solutions.** Let \((\rho_0, u_0, \theta_0, H_0)\) be as in Theorem 1.1. Without loss of generality, we assume that the initial density \(\rho_0\) satisfies

\[
\int_{\mathbb{R}^2} \rho_0 dx = 1,
\]

which indicates that there exists a positive constant \(N_0\) such that

\[
\int_{B_{N_0}} \rho_0 dx \geq \frac{3}{4} \int_{\mathbb{R}^2} \rho_0 dx = \frac{3}{4}.
\]

(4.1)

We construct \(\bar{\rho}_0^R = \rho_0^R + R^{-1} e^{-|x|^2}\), where \(0 \leq \rho_0^R \in C_0^\infty(\mathbb{R}^2)\) satisfies

\[
\begin{cases}
\int_{B_{N_0}} \rho_0^R dx \geq \frac{1}{2}, \\
\bar{x}^a \rho_0^R \to \bar{x}^a \rho_0 \text{ in } L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \cap W^{1,q}(\mathbb{R}^2) \text{ as } R \to \infty.
\end{cases}
\]

(4.2)

Noting that \(H_0 \bar{x}^a \in H^1(\mathbb{R}^2)\) and \(\nabla H_0 \in H^1(\mathbb{R}^2)\), we choose \(H_0^R \in \{ w \in C_0^\infty(B_R) \mid \text{div } w = 0 \}\) satisfying

\[
H_0^R \bar{x}^a \to H_0 \bar{x}^a, \ \nabla H_0^R \to \nabla H_0 \text{ in } H^1(\mathbb{R}^2) \text{ as } R \to \infty.
\]

(4.3)

Since \(\nabla u_0 \in H^1(\mathbb{R}^2)\), we select \(\bar{v}_i^R \in C_0^\infty(B_R) (i = 1, 2)\) such that for \(i = 1, 2\),

\[
\lim_{R \to \infty} \| \bar{v}_i^R - \partial_i u_0 \|_{H^1(\mathbb{R}^2)} = 0.
\]

(4.4)
Consider the unique smooth solution \( u_0^R \) to the elliptic problem

\[
\begin{cases}
-\mu \Delta u_0^R + \rho_0^R u_0^R + \nabla P_0 = \sqrt{\rho_0^R} h_1^R - \partial_1 v_1^R, & \text{in } B_R, \\
\text{div} u_0^R = 0, & \text{in } B_R, \\
u_0^R = 0, & \text{on } \partial B_R,
\end{cases}
\]

where \( h_1^R = (\sqrt{\rho_0} u_0) * j_\delta \) with \( j_\delta \) being the standard mollifying kernel of width \( \delta \). Extending \( u_0^R \) to \( \mathbb{R}^2 \) by defining \( 0 \) outside \( B_R \) and denoting it by \( \tilde{u}_0^R \), we claim that, up to the extraction of subsequences,

\[
\lim_{R \to \infty} \left( \left\| \nabla \tilde{u}_0^R - \nabla u_0 \right\|_{H^1(\mathbb{R}^2)} + \left\| \sqrt{\rho_0^R} \tilde{u}_0^R - \sqrt{\rho_0} u_0 \right\|_{L^2(\mathbb{R}^2)} \right) = 0.
\]

Indeed, it is not hard to find that \( \tilde{u}_0^R \) is also a solution of (4.5) in \( \mathbb{R}^2 \). Multiplying (4.5) by \( \tilde{u}_0^R \) and integrating the resulting equation over \( \mathbb{R}^2 \) lead to

\[
\left\| \sqrt{\rho_0^R} \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 + \mu \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 \\
\leq \left\| \sqrt{\rho_0^R} \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)} \left\| h_1^R \right\|_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)} \left\| \partial_1 \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)} \\
\leq \frac{1}{2} \left\| \sqrt{\rho_0^R} \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \left\| h_1^R \right\|_{L^2(\mathbb{R}^2)}^2 + \mu \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2\mu} \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2,
\]

which implies that

\[
\left\| \sqrt{\rho_0^R} \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 + \mu \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 \leq \left\| h_1^R \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{ \mu } \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 \leq C.
\]

This together with (4.2) yields that there exist a subsequence \( R_j \to \infty \) and a function \( \tilde{u}_0 \in \{ \tilde{u}_0 \in H_{loc}^1(\mathbb{R}^2) \} \) \( \sqrt{\rho_0} \tilde{u}_0 \in L^2(\mathbb{R}^2), \nabla \tilde{u}_0 \in L^2(\mathbb{R}^2) \) such that

\[
\sqrt{\rho_0^R} \tilde{u}_0^R \to \sqrt{\rho_0} \tilde{u}_0, \nabla \tilde{u}_0^R \to \nabla \tilde{u}_0 \text{ weakly in } L^2(\mathbb{R}^2).
\]

We claim that

\[
\tilde{u}_0 = u_0.
\]

In fact, multiplying (4.5) by a test function \( \pi \in C_0^\infty(\mathbb{R}^2) \) with \( \text{div} \pi = 0 \), it holds that

\[
\int_{\mathbb{R}^2} \partial_i \left( \tilde{u}_0^R - v_1^R \right) \cdot \partial_i \pi dx + \int_{\mathbb{R}^2} \sqrt{\rho_0^R} \left( \sqrt{\rho_0^R} \tilde{u}_0^R - h_1^R \right) \cdot \pi dx = 0.
\]

Then, letting \( R_j \to \infty \), it follows from (4.2), (4.4), and (4.8) that

\[
\int_{\mathbb{R}^2} \partial_i (\tilde{u}_0 - u_0) \cdot \partial_i \pi dx + \int_{\mathbb{R}^2} \rho_0 (\tilde{u}_0 - u_0) \cdot \pi dx = 0,
\]

which implies (4.9). Furthermore, multiplying (4.5) by \( \tilde{u}_0^R \) and integrating the resulting equation over \( \mathbb{R}^2 \), by the same arguments as (4.10), we have

\[
\lim_{R_j \to \infty} \left( \left\| \nabla \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sqrt{\rho_0^R} \tilde{u}_0^R \right\|_{L^2(\mathbb{R}^2)}^2 \right) = \left( \left\| \nabla u_0 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sqrt{\rho_0} u_0 \right\|_{L^2(\mathbb{R}^2)}^2 \right).
\]
which combined with (4.8) leads to

$$\lim_{R_j \to \infty} \|\nabla \hat{u}_0^R - \nabla \bar{u}_0\|_{L^2(\mathbb{R}^2)} = \|\nabla \bar{u}_0\|_{L^2(\mathbb{R}^2)}, \quad \lim_{R_j \to \infty} \left\| \sqrt{\rho_0^R} \hat{u}_0^R \right\|_{L^2(\mathbb{R}^2)} = \left\| \sqrt{\rho_0} \bar{u}_0 \right\|_{L^2(\mathbb{R}^2)}.$$  

This along with (4.9) and (4.8) guarantees that

$$\lim_{R_j \to \infty} \left( \left\| \nabla \hat{u}_0^R - \nabla \bar{u}_0 \right\|_{L^2(\mathbb{R}^2)} + \left\| \sqrt{\rho_0^R} \hat{u}_0^R - \sqrt{\rho_0} \bar{u}_0 \right\|_{L^2(\mathbb{R}^2)} \right) = 0. \quad (4.11)$$  

Moreover, if we differentiate (4.5) and then multiply the resultant equality by \(\nabla \hat{u}_0^R\), it is not hard to infer from (1.9), (4.8), and (4.9) that, up to the extraction of subsequences,

$$\lim_{R_j \to \infty} \left\| \nabla^2 \hat{u}_0^R - \nabla^2 \bar{u}_0 \right\|_{L^2(\mathbb{R}^2)} = 0, \quad (4.12)$$  

which together with (4.11) implies (4.6).

Next, we consider the following elliptic equations in terms of \(\theta_0^R\)

$$\begin{cases} -\kappa \Delta \theta_0^R + \rho_0^R \theta_0^R = \frac{\mu}{2} \nabla u_0^R + (\nabla u_0^R)^T \nabla (\nabla u_0^R)^T + \nu (\text{curl} H_0^R)^2 + \sqrt{\rho_0^R} H_0^R, & \text{in } B_R, \\ \theta_0^R = 0, & \text{on } \partial B_R, \end{cases} \quad (4.13)$$

where \(h_0^R = (\sqrt{\rho_0} \theta_0 + g_2) \ast j_\xi\). Extending \(\theta_0^R\) to \(\mathbb{R}^2\) by defining 0 outside \(B_R\) and denoting it by \(\bar{\theta}_0^R\). Similarly to (4.11) and (4.12), we can also deduce that (see Liang and Shuai)  

$$\lim_{R \to \infty} \left( \left\| \nabla \bar{\theta}_0^R - \nabla \theta_0^R \right\|_{H^1(\mathbb{R}^2)} + \left\| \sqrt{\rho_0^R} \bar{\theta}_0^R - \sqrt{\rho_0} \theta_0 \right\|_{L^2(\mathbb{R}^2)} \right) = 0. \quad (4.14)$$

Hence, by virtue of Lemma 2.1, initial-boundary-value problems (1.1), (2.1), and (2.2) with the initial data \((\rho_0^R, u_0^R, \theta_0^R, H_0^R)\) have a classical solution \((\rho^R, u^R, \theta^R, H^R)\) on \(B_R \times (0, T_R]\). Moreover, Proposition 3.1 shows that there exists a \(T_0\) independent of \(R\) such that (3.3) holds for \((\rho^R, u^R, \theta^R, p^R, H^R)\).

For simplicity, in what follows, we write

$$L^p = L^p(\mathbb{R}^2), \quad W^{k,p} = W^{k,p}(\mathbb{R}^2), \quad H^k = W^{k,2}.$$  

Extending \((\rho^R, u^R, \theta^R, p^R, H^R)\) by zero on \(\mathbb{R}^2 \setminus B_R\) and denoting it by

$$(\hat{\rho}^R, \hat{u}^R, \hat{\theta}^R, \hat{p}^R, \hat{H}^R)$$

with \(\varphi_R\) satisfying (3.9). From (3.3), we have

$$\sup_{0 \leq t \leq T_0} \left( \|\sqrt{\hat{\rho}^R} \hat{u}^R\|_{L^2} + \|\sqrt{\hat{\rho}^R} \hat{\theta}^R\|_{L^2} + \|\nabla \hat{u}^R\|_{H^1} + \|\nabla \hat{\theta}^R\|_{H^1} + \|\hat{H}^R\|_{H^1} + \|\hat{H}^R \hat{x}^2\|_{H^1} \right)$$

$$\leq \sup_{0 \leq t \leq T_0} \left( \|\sqrt{\rho^R} u^R\|_{L^2(B_R)} + \|\sqrt{\rho^R} \theta^R\|_{L^2(B_R)} + \|\nabla u^R\|_{H^1(B_R)} + \|\nabla \theta^R\|_{H^1(B_R)} \right)$$

$$+ \|\hat{H}^R\|_{H^1(B_R)} + \|\hat{H}^R \hat{x}^2\|_{H^1(B_R)} \right) \leq C, \quad (4.15)$$

and

$$\sup_{0 \leq t \leq T_0} \|\hat{p}^R \hat{x}^2\|_{L^1} \leq C.$$
Similarly, it follows from (3.3) that, for \( q \) as in Theorem 1.1,

\[
\sup_{0 \leq t \leq T_0} \left( \left\| \sqrt{\rho R} \bar{\theta}^R \right\|_{L^2} + \left\| \sqrt{\rho R} \bar{\mu}^R \right\|_{L^2} + \left\| \bar{P}^R \right\|_{L^2} \right) \\
+ \int_{0}^{T_0} \left( \left\| \sqrt{\rho R} \bar{\theta}^R \right\|_{L^2}^2 + \left\| \sqrt{\rho R} \bar{\theta}^R \right\|_{L^2}^2 + \left\| \nabla^2 \bar{\mu}^R \right\|_{L^2}^2 + \left\| \nabla^2 (\bar{\theta}^R \bar{\mu}^R) \right\|_{L^2}^2 + \left\| \bar{P}^R \right\|_{L^2} \right) dt \\
+ \int_{0}^{T_0} \left( \left\| \nabla^2 \bar{P}^R \right\|_{L^2}^2 + \left\| \nabla^2 (\bar{\theta}^R \bar{\mu}^R) \right\|_{L^2}^2 + \left\| \nabla^2 \bar{P}^R \right\|_{L^2}^2 + \left\| \nabla (\bar{\theta}^R \bar{\mu}^R) \right\|_{L^2}^2 + \left\| \bar{P}^R \right\|_{L^2} \right) dt \\
+ \int_{0}^{T_0} \left( \left\| \nabla^2 \bar{P}^R \right\|_{L^2}^2 + \left\| \nabla^2 (\bar{\theta}^R \bar{\mu}^R) \right\|_{L^2}^2 + \left\| \nabla^2 \bar{P}^R \right\|_{L^2}^2 + \left\| \nabla (\bar{\theta}^R \bar{\mu}^R) \right\|_{L^2}^2 + \left\| \bar{P}^R \right\|_{L^2} \right) dt \leq C. \tag{4.16}
\]

Next, for \( p \in [2, q] \), we obtain from (3.3) and (3.65) that

\[
\sup_{0 \leq t \leq T_0} \left\| \nabla (\rho R \bar{P}^R) \right\|_{L^p} \leq C \sup_{0 \leq t \leq T_0} \left( \left\| \nabla (\rho R \bar{P}^R) \right\|_{L^p(B_0)} + \left\| \nabla (\rho R \bar{P}^R) \right\|_{L^p(B_0)} \right) \\
\leq C \sup_{0 \leq t \leq T_0} \left\| \rho R \bar{P}^R \right\|_{L^p(B_0)} \leq C, \tag{4.17}
\]

which together with (3.52) and (3.3) yields

\[
\int_{0}^{T_0} \left\| \nabla \bar{P}^R \right\|_{L^2}^2 dt \leq C \int_{0}^{T_0} \left\| \nabla \bar{u}^R \right\|_{L^2}^2 (B_0) dt \\
\leq C \int_{0}^{T_0} \left\| \nabla \bar{u}^R \right\|_{L^2}^2 (B_0) \left\| \nabla \bar{P}^R \right\|_{L^2}^2 dt \leq C. \tag{4.18}
\]

By virtue of the same arguments as those of (3.55) and (3.58), one gets

\[
\sup_{0 \leq t \leq T_0} \left\| \nabla \bar{P}^R \right\|_{L^2} + \int_{0}^{T_0} \left( \left\| \nabla \bar{P}^R \right\|_{L^2}^2 + \left\| \nabla \bar{P}^R \right\|_{L^2}^2 \right) dt \leq C. \tag{4.19}
\]

With estimates (4.15)–(4.19) at hand, we find that the sequence \((\bar{\rho}^R, \bar{u}^R, \bar{\theta}^R, \bar{P}^R, \bar{H}^R)\) converges, up to the extraction of subsequences, to some limit \((\rho, u, \theta, P, H)\) in some weak sense, that is, as \( R \to \infty \), we have

\[
\bar{\rho}^R \rightarrow \rho x \quad \text{in} \quad C(B_N \times [0, T_0]) \quad \text{for any} \quad N > 0, \tag{4.20}
\]

\[
\bar{\rho}^R \bar{u}^a \rightarrow \rho x^a \quad \text{weakly* in} \quad L^\infty(0, T_0; L^1 \cap H^1 \cap W^{1,q}), \tag{4.21}
\]

\[
\nabla \bar{u} \rightarrow \nabla u, \quad \nabla \bar{\theta} \rightarrow \nabla \theta, \quad H^R \bar{u}^a \rightarrow H \bar{x}^a \quad \text{weakly* in} \quad L^\infty(0, T_0; H^1), \tag{4.22}
\]

\[
\bar{H}^R \rightarrow H \quad \text{weakly* in} \quad L^\infty(0, T_0; H^2), \tag{4.23}
\]

\[
\sqrt{\rho R} \bar{u} \rightarrow \sqrt{\rho u}, \quad \sqrt{\rho R} \bar{\theta} \rightarrow \sqrt{\rho \theta}, \quad \bar{H}^R \rightarrow H_t \quad \text{weakly* in} \quad L^\infty(0, T_0; L^2), \tag{4.24}
\]

\[
\nabla^2 \bar{u} \rightarrow \nabla^2 u, \quad \nabla^2 \bar{\theta} \rightarrow \nabla^2 \theta, \quad \bar{P}^R \rightarrow P \quad \text{weakly in} \quad L^{\frac{n+1}{q}}(0, T_0; L^q) \cap L^2(0, T_0; L^q), \tag{4.25}
\]

\[
\bar{H}^R_t \rightarrow H_t, \quad \nabla H^R \bar{u}^a \rightarrow \nabla H \bar{x}^a, \quad \nabla^2 \bar{H}^R \rightarrow \nabla^2 H \quad \text{weakly in} \quad L^2(0, T_0; L^2), \tag{4.26}
\]

\[
\nabla^2 \bar{u} \rightarrow \nabla^2 u, \quad \nabla^2 \bar{\theta} \rightarrow \nabla^2 \theta \quad \text{weakly in} \quad L^2(0, T_0; L^2). \tag{4.27}
\]
\[
\sqrt{\rho_R \tilde{u}_t^R} \to \sqrt{\rho u_t}, \sqrt{\rho_R \tilde{\theta}_t^R} \to \sqrt{\rho \theta_t}, \nabla P \to \nabla \tilde{P} \text{ weakly* in } L^\infty(0, T_0; L^2),
\]
\[
\sqrt{\rho_R \tilde{u}_t^R} \to \sqrt{\rho u_t}, \sqrt{\rho_R \tilde{\theta}_t^R} \to \sqrt{\rho \theta_t} \text{ weakly in } L^2(0, T_0; L^2),
\]
\[
\nabla \tilde{u}_t^R \to \nabla u_t, \nabla \tilde{\theta}_t^R \to \nabla \theta_t, \nabla \tilde{H}_t^R \to \nabla H_t \text{ weakly in } L^2(0, T_0; L^2),
\]

with
\[
\rho \tilde{\rho}^a \in L^\infty(0, T_0; L^1), \inf_{0 \leq t \leq T_0} \int_{B_{2\tilde{r}_0}} \rho(x, t) dx \geq \frac{1}{4}.
\]

Letting \( R \to \infty \), standard arguments together with (4.20)–(4.31) show that \((\rho, u, \theta, P, H)\) is a strong solution of (1.1)–(1.3) on \( \mathbb{R}^2 \times (0, T_0] \) satisfying (1.10) and (1.11). Indeed, the existence of a pressure \( P \) follows immediately from the (1.1)\_2 and (1.1)\_3 by a classical consideration.

**Step 2. Uniqueness of strong solutions.** Let \((\rho, u, \theta, P, H)\) and \((\tilde{\rho}, \tilde{u}, \tilde{\theta}, \tilde{P}, \tilde{H})\) be two strong solutions satisfying (1.10) and (1.11) with the same initial data, and denote

\[
\Theta \triangleq \rho - \tilde{\rho}, \quad U \triangleq u - \tilde{u}, \quad \Psi \triangleq \theta - \tilde{\theta}, \quad \Phi \triangleq H - \tilde{H}.
\]

First, subtracting mass equation (1.1)\_1 satisfied by \((\rho, u)\) and \((\tilde{\rho}, \tilde{u})\) gives that

\[
\Theta_t + \tilde{u} \cdot \nabla \Theta + U \cdot \nabla \rho = 0.
\]

Multiplying (4.32) by \( 2\Theta \tilde{x}^{2r} \) for \( r \in (1, \tilde{a}) \) with \( \tilde{a} = \min \{ 2, a \} \) and integrating by parts over \( \mathbb{R}^2 \), we deduce from Sobolev’s inequality, (1.11), (3.12), and (3.52) that

\[
\frac{d}{dt} \| \Theta \tilde{x}^r \|_{L^2}^2 \leq C \| \tilde{x}^{\frac{r}{2}} \|_{L^\infty} \| \Theta \tilde{x}^r \|_{L^2}^2 + C \| \Theta \tilde{x}^r \|_{L^2} \| \tilde{U} \tilde{x}^{-(\tilde{a} - r)} \|_{L^2} \| \tilde{x}^{\tilde{a}} \nabla \rho \|_{L^\frac{2}{\tilde{a} - r}} \| \tilde{x}^{\tilde{a}} \nabla \rho \|_{L^\frac{2}{\tilde{a} - r}}
\]

\[
\leq C (1 + \| \nabla \tilde{u} \|_{W^{1, r}}) \| \Theta \tilde{x}^r \|_{L^2}^2 + C \| \Theta \tilde{x}^r \|_{L^2}^2 \left( \| \nabla U \|_{L^2} + \| \sqrt{\tilde{\rho}} U \|_{L^2} \right).
\]

This combined with Grönwall’s inequality shows that, for all \( 0 \leq t \leq T_0 \),

\[
\| \Theta \tilde{x}^r \|_{L^2} \leq C \int_0^t \left( \| \nabla U \|_{L^2} + \| \sqrt{\tilde{\rho}} U \|_{L^2} \right) ds.
\]

Next, subtracting (1.1)\_2 and (1.1)\_4 satisfied by \((\rho, u, H)\) and \((\tilde{\rho}, \tilde{u}, \tilde{H})\) leads to

\[
\rho U_t + \rho u \cdot \nabla U - \mu \Delta U = -\rho U \cdot \nabla \tilde{u} - \Theta(\tilde{u}_t + \tilde{u} \cdot \nabla \tilde{u}) - \nabla (P - \tilde{P}) - \frac{1}{2} \nabla \left( |H|^2 - |	ilde{H}|^2 \right) + H \cdot \nabla \Phi + \Phi \cdot \nabla H,
\]

\[
\Phi_t - \nu \Delta \Phi = H \cdot \nabla U + \Phi \cdot \nabla \tilde{u} - u \cdot \nabla \Phi - U \cdot \nabla \tilde{H}.
\]

Multiplying (4.34) by \( U \) and (4.35) by \( \Phi \), respectively, and adding the resulting equations together, we obtain after integration by parts that

\[
\frac{d}{dt} \int (\rho |U|^2 + |\Phi|^2) \ dx + \int (\mu |\nabla U|^2 + \nu |\nabla \Phi|^2) \ dx
\]

\[
\leq C \| \nabla \tilde{u} \|_{L^\infty} \int (\rho |U|^2 + |\Phi|^2) \ dx + C \int |\Theta||U| (|\tilde{u}_t| + |\tilde{u}||\nabla \tilde{u}|) \ dx - \int \Phi \cdot \nabla U \cdot \tilde{H} \ dx - \int U \cdot \nabla \Phi \cdot \tilde{H} \ dx
\]

\[
\leq C \| \nabla \tilde{u} \|_{L^\infty} \int (\rho |U|^2 + |\Phi|^2) \ dx + K_1 + K_2 + K_3.
\]
By Hölder’s inequality, (1.11), (2.8), (3.3), and (4.33), we get that, for \( r \in (1, \tilde{a}) \),
\[
K_1 \leq C(\Theta \|x\|_{L^2}) \|U \xi^{-\frac{1}{2}}\|_{L^2} \left( \|\tilde{u} x^{-\frac{1}{2}}\|_{L^2} + \|\nabla \tilde{u}\|_{L^2} + \|\bar{u} x^{-\frac{1}{2}}\|_{L^2} \right)
\leq C(\varepsilon) \left( \|\sqrt{\rho} \bar{u} \|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|\bar{u}\|_{L^2}^2 \right) \|\Theta \|_{L^2}^2 + \varepsilon \left( \|\sqrt{\rho} U \|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right)
\leq C(\varepsilon) \left( 1 + \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla^2 \bar{u}\|_{L^2}^2 \right) \int_0^t \left( \|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right) ds + \varepsilon \left( \|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2 \right).
\] (4.37)

For the term \( K_2 \), we derive from Gagliardo–Nirenberg inequality that
\[
K_2 \leq C(\|H\|_{L^2}) \|\Phi\|_{L^2} \|\nabla U\|_{L^2} \leq \varepsilon \|\nabla U\|_{L^2}^2 + \|\nabla \Phi\|_{L^2}^2 + \varepsilon(\|\Theta\|_{L^2}^2).
\] (4.38)

The last term \( K_3 \) can be estimated as follows:
\[
K_3 \leq C(\|U \xi^{-a}\|_{L^2}) \|\nabla H\|_{L^2} \|\xi^{a}\|_{L^2} \|\nabla H\|_{L^2} \|\Phi\|_{L^2}
\leq C(\|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2) \|\nabla H\|_{L^2} \|\Phi\|_{L^2}
\leq \varepsilon(\|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2) + C(\varepsilon) \|\nabla \Phi\|_{L^2}^2
\leq \varepsilon(\|\sqrt{\rho} U\|_{L^2}^2 + \|\nabla U\|_{L^2}^2) + \varepsilon \|\nabla \Phi\|_{L^2}^2 + C(\varepsilon) \|\nabla H\|_{L^2} \|\Phi\|_{L^2}^2,
\] (4.39)

owing to (1.11), (2.8), and (3.3). Denoting
\[
G(t) \triangleq \|\sqrt{\rho} U\|_{L^2}^2 + \|\Phi\|_{L^2}^2 + \int_0^t \left( \|\nabla U\|_{L^2}^2 + \|\nabla \Phi\|_{L^2}^2 + \|\sqrt{\rho} U\|_{L^2}^2 \right) ds.
\]
Then, substituting (4.37)–(4.39) into (4.36) and choosing \( \varepsilon \) suitably small, we have
\[
G'(t) \leq C \left( 1 + \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla H\bar{x}^\frac{3}{2}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|\nabla^2 \bar{u}\|_{L^2}^2 \right) G(t),
\]
which together with Grönwall’s inequality and (1.10) implies that \( G(t) = 0 \). This gives that
\[
U(x, t) = 0, \Phi(x, t) = 0,
\] (4.40)
for almost every \((x, t) \in \mathbb{R}^2 \times (0, T_0]\). Moreover, one infers from (4.33) that, for almost every \((x, t) \in \mathbb{R}^2 \times (0, T_0]\),
\[
\Theta = 0.
\] (4.41)

Finally, subtracting (1.1), satisfied by \((\rho, u, \theta, H)\) and \((\bar{ho}, \bar{u}, \bar{	heta}, \bar{H})\), we deduce from (4.40) and (4.41) that
\[
c_v \rho \Psi_t + c_v \rho u \cdot \nabla \Psi - \kappa \Delta \Psi = 0.
\] (4.42)

Multiplying (4.42) by \( \Psi \) and integrating the resultant over \( \mathbb{R}^2 \), we have
\[
c_v \rho \frac{d}{dt} \|\sqrt{\rho} \Psi \|_{L^2}^2 + \kappa \|\nabla \Psi\|_{L^2}^2 = 0.
\] (4.43)

Integrating (4.43) over \([0, T_0]\), we infer from \( \|\sqrt{\rho} \Psi(0)\|_{L^2} = 0 \) that, for almost every \((x, t) \in \mathbb{R}^2 \times (0, T_0]\),
\[
\Psi = 0.
\]

Thus, we finish the proof of the uniqueness of solutions.
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CONFLICT OF INTERESTS
There are no conflicts of interest to this work.

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