Quantifier completions, choice principles and applications

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Abstract. We give an expositional account of the quantifier completions by using the language of doctrines. This algebraic presentation allows us to exhaustively analyse the behaviour of the existential and universal quantifiers. In particular there are three points we wish to convey: the first is that they preserve the lattice structure of the fibres under opportune hypotheses which turn out to be preserved as well; the second regards the applications, in particular to the dialectica construction; the third is that these free constructions carry on some relevant choice principles.

Keywords: Doctrines · Completions · Dialectica categories

1 Introduction

In recent years relevant logical completions about quantifiers have been extensively studied in categorical logic. See for instance the work of Hofstra [8] for an application of these completions in the framework of dialectica construction and Frey [5] for applications in homotopy theory. In [3] the author introduces the notion of existential completion of a primary doctrine in order to show how left adjoints can be freely added along projections. This free construction is 2-monadic, i.e. the existential doctrines correspond to the algebras for the associated 2-monad, and this 2-monad is lax-idempotent. In other words the existential structure turns out to be property of a doctrine. Moreover this algebraic characterization allows also to understand which choice principles (such as the Rule of Choice) hold in a given doctrine. As pointed out in [4], the existential doctrines which arise as free-algebras correspond to those logical theories which behave like the regular logic. Hence, for example, they satisfy the Rule of Choice and every formula admits a prenex normal form presentation.

In this work we present the universal completion of a doctrine, i.e. the free addition of right adjoints along projections, and we show it to be in some sense dual to the existential completion. Again this construction rises to a 2-monad, and the algebras for this 2-monad are exactly the universal doctrines. Notice that in this case the 2-monad turns out to be colax-idempotent.
Moreover we show that the universal completion of a doctrine satisfies a property which we call \textit{counterexample property}, which corresponds to saying that if $\forall x \psi(x) \vdash \bot$ then there exists a term $t$ such that $\psi(t) \vdash \bot$. This term $t$ represents the \textit{counterexample}.

Notice that, differently from [3], we work with a notion of doctrine which is more general, since we use $C$-indexed posets $P : C^{\text{op}} \longrightarrow \text{Pos}$. In order to avoid clash of notations, we will call such a functor \textit{slat-doctrine}.

As pointed out by Hofstra [8] in the framework of fibration theory, the interaction of constructions which freely add quantifiers plays a crucial role in the abstract presentation of the dialectica interpretation [7], in particular in the construction of the \textit{dialectica monad}. See [2,9] for more details about the dialectica construction.

Inspired by this work, we study which logical structures are preserved by these two free completions and how they can be combined. First we show under which hypotheses both the existential and universal completions preserve the lattice structure of the fibres of a given doctrine $P : C^{\text{op}} \longrightarrow \text{Lat}$, where \text{Lat} is the category of lattices. Here the crucial assumption is the existence of left-adjoints (right-adjoints respectively) for the family of injections. Moreover, requiring the objects of $C$ to be inhabited, we get the existential completion (universal completion respectively) of $P$ to satisfy this property as well.

Secondly, whenever the base category of a given universal doctrine is also cartesian closed, it is the case that its existential completion preserves the universal structure, i.e. is universal as well. In other words by sequentially applying the universal and the existential completions to a given doctrine $P : C^{\text{op}} \longrightarrow \text{Lat}$, we get a doctrine $(P^{\text{un}})^{\text{ex}} : C^{\text{op}} \longrightarrow \text{Lat}$ which is both existential and universal.

Moreover it validates a form of choice principle called AC by Troelstra in [6]:

$$\forall x \exists y \alpha(x, y) \rightarrow \exists f \forall x \alpha(x, f x).$$

As anticipated, one of the main applications of completions regarding quantifiers arises in the context of dialectica construction. See [29] for more details about the notion of dialectica category associated to a given poset-fibration. For instance, our results yield a proof that the poset reflection of the dialectica category $\text{Dial}(P)$ associated to a doctrine $P : C^{\text{op}} \longrightarrow \text{Lat}$ is a lattice. Such a result gives grounds for hope in further applications of this deep relation between the dialectica construction and quantifier addition, for instance in order to find out the assumptions over a given doctrine making the corresponding dialectica category into a distributive category.

\textit{Synopsis.} In Section 2 we recall definitions about universal and existential doctrines, and we fix the notation.

In Section 3 we recall from [3] the existential completion, we introduce the universal one, and we prove a form a duality between the two notions.

Section 4 is completely devoted to the study of logical structures preserved by quantifier completions.

In Section 5 we present the interaction between the two constructions and applications.
In Section 6 we show which choice principles arise in doctrines which are instances of free quantifier constructions.
Conclusions and future works are treated in Section 7.

2 Brief recap on doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in a series of seminal papers [11, 12], and in the following years this notion has been revisited and generalized in several ways, see [16, 15, 17].

In this work we start by considering a more general notion, which we call slat-doctrine.

Definition 1. A slat-doctrine is a functor \( P: \mathcal{C}^{\text{op}} \rightarrow \text{Pos} \), where \( \mathcal{C} \) is a category with finite products, and \( \text{Pos} \) is the category of posets.

We think of the base category \( \mathcal{C} \) of a slat-doctrine as the category of types or contexts and, for every object \( A \in \mathcal{C} \), we have a poset \( P(A) = (P(A), \vdash) \) of predicates on \( A \).

Example 1. Let \( \mathcal{C} \) be a category with finite limits. The functor:
\[
\text{Sub}_\mathcal{C}: \mathcal{C}^{\text{op}} \rightarrow \text{Pos}
\]
assigning to an object \( A \) in \( \mathcal{C} \) the poset \( \text{Sub}_\mathcal{C}(A) \) of subobjects of \( A \) and such that for an arrow \( B \xrightarrow{f} A \) the morphism \( \text{Sub}_\mathcal{C}(f): \text{Sub}_\mathcal{C}(A) \rightarrow \text{Sub}_\mathcal{C}(B) \) is given by pulling a subobject back along \( f \), is a slat-doctrine.

Example 2. Consider the set-theoretic doctrine \( S: \text{Set}^{\text{op}} \rightarrow \text{Pos} \). In this case \( \text{Set} \) is the category of sets and functions and, for every set \( A \), \( S(A) \) is the poset category of subsets of the set \( A \) together with the inclusions. A functor \( S_f: S(B) \rightarrow S(A) \) acts as the inverse image \( f^{-1}U \) on a subset \( U \) of \( B \).

Example 3. Let \( \mathcal{T} \) be a theory in a first order language \( \mathcal{L} \). We define a slat-doctrine:
\[
LT: \mathcal{C}_\mathcal{T}^{\text{op}} \rightarrow \text{Pos}
\]
where \( \mathcal{C}_\mathcal{T} \) is the category of contexts, i.e. of lists of variables and term substitutions:
- objects of \( \mathcal{C}_\mathcal{T} \) are finite lists of variables \( \vec{x} := (x_1, \ldots, x_n) \), and we include the empty list \( () \);
- a morphisms from \( (x_1, \ldots, x_n) \) into \( (y_1, \ldots, y_m) \) is a substitution:
\[
[t_1/y_1, \ldots, t_m/y_m]
\]
  where the terms \( t_i \) are built in \( \text{Sg} \) on the variable \( x_1, \ldots, x_n \);
- the composition of two morphisms \( \vec{t}/\vec{y}: \vec{x} \rightarrow \vec{y} \) and \( \vec{s}/\vec{z}: \vec{y} \rightarrow \vec{z} \) is given by the substitution.
The functor $LT : C^{op}_T \rightarrow \text{InfSL}$ sends a list $(x_1, \ldots, x_n)$ to the class:

$$LT(x_1, \ldots, x_n)$$

of all well-formed formulas in the context $(x_1, \ldots, x_n)$. We say that $\psi \leq \phi$, where $\phi, \psi \in LT(x_1, \ldots, x_n)$, if $\psi \vdash_T \phi$, and we consider the usual quotient making the class $LT(x_1, \ldots, x_n)$ into a partial order. Given a morphism:

$$[t_1/y_1, \ldots, t_m/y_m] : (x_1, \ldots, x_n) \rightarrow (y_1, \ldots, y_m)$$

of $C_T$, the functor $LT[\bar{t}/\bar{y}]$ acts as the substitution $LT[\bar{t}/\bar{y}](\psi(y_1, \ldots, y_m)) = \psi[\bar{t}/\bar{y}]$.

**Definition 2.** A slat-doctrine $P : C^{op}_T \rightarrow \text{InfSL}$ is existential if, for every $A_1$ and $A_2$ in $C$, for any projection $pr_i : A_1 \times A_2 \rightarrow A_i$, $i = 1, 2$, the functor:

$$P_{pr_i} : P(A_i) \rightarrow P(A_1 \times A_2)$$

has a left adjoint $\exists_{pr_i}$, and these satisfy Beck-Chevalley condition: for any pullback diagram:

$$
\begin{array}{ccc}
X' & \xrightarrow{pr'} & A' \\
\downarrow{f'} & & \downarrow{f} \\
X & \xrightarrow{pr} & A
\end{array}
$$

with $pr$ and $pr'$ projections, for any $\beta$ in $P(X)$ the canonical arrow:

$$\exists_{pr'} P_{f'}(\beta) \leq P_f \exists_{pr}(\beta)$$

is an isomorphism.

**Example 4.** The slat-doctrine $\text{Sub} : C^{op}_T \rightarrow \text{Pos}$ presented in Example 1 is existential if and only if the base category $C$ is regular. See [16].

**Definition 3.** A slat-doctrine $P : C^{op}_T \rightarrow \text{InfSL}$ is universal if, for every $A_1$ and $A_2$ in $C$, for any projection $pr_i : A_1 \times A_2 \rightarrow A_i$, $i = 1, 2$, the functor

$$P_{pr_i} : P(A_i) \rightarrow P(A_1 \times A_2)$$

has a right adjoint $V_{pr_i}$, and these satisfy Beck-Chevalley condition.

**Example 5.** The slat-doctrine $LT : C^{op}_T \rightarrow \text{Pos}$ as defined in Example 3 for a first order theory $T$, is universal and existential. The right adjoints are computed by quantifying universally the variables that are not involved in the substitution given by the projection, and similarly left adjoints are computed by quantifying existentially the variables which are not involved in the substitution given by the projection.
Example 6. The slat-doctrine \( S: \text{Set}^{\text{op}} \to \text{Pos} \) presented in Example 2 is existential and universal: on a subset \( U \) of \( A \times B \), for a projection \( \pr_A: A \times B \to A \), the right adjoint \( \V_{\pr_A} \) is given by the assignment \( \V_{\pr_A}(U) := \{ a \in A \mid \pr_A^{-1}(a) \subseteq U \} \), and the left adjoint \( \I_{\pr_A} \) is given by \( \I_{\pr_A}(U) := \{ a \in A \mid \exists b \in A \times B (b \in \pr_A^{-1}(a) \cap U) \} \).

The category of slat-doctrines \( \text{SD} \) is a 2-category, where:

- a 1-cell is a pair \((F, b)\)

\[
\begin{array}{c}
\text{C}^{\text{op}} \\
\downarrow F^{\text{op}} \\
\text{D}^{\text{op}} \\
\downarrow b \\
\text{Pos} \\
\end{array}
\]

such that \( F: \text{C} \to \text{D} \) is a functor and \( b: F \to R \circ F^{\text{op}} \) is a natural transformation.

- a 2-cell is a natural transformation \( \theta: F \to G \) such that for every \( A \) in \( \text{C} \) and every \( \alpha \) in \( PA \), we have:

\[
b_A(\alpha) \leq R\theta_A(c_A(\alpha)).
\]

We denote as \( \text{ExD} \) the 2-full subcategory of \( \text{SD} \) whose elements are existential slat-doctrines, and whose 1-cells are those 1-cells of \( \text{SD} \) which preserve the existential structure. Similarly, we denote by \( \text{UnD} \) the 2-full subcategory of \( \text{SD} \) whose elements are universal slat-doctrines, and whose 1-cells are those 1-cells of \( \text{SD} \) which preserve the universal structure.

3 Quantifiers and completions

In this section we recall from [3] the existential completion and we show how this construction can be dualised to obtain a free construction which adds right adjoints along the projections, called universal completion.

Existential completion. Let \( P: \text{C}^{\text{op}} \to \text{Pos} \) be a slat-doctrine. The existential completion \( \text{Pex}: \text{C}^{\text{op}} \to \text{Pos} \) of \( P \) is a slat-doctrine such that, for every object \( A \) of \( \text{C} \), the poset \( P_{\text{ex}}(A) \) is defined as follows:

- objects: triples \((A, B, \alpha)\), where \( A \) and \( B \) are objects of \( \text{C} \) and \( \alpha \in P(A \times B) \).
- order: \((A, B, \alpha) \leq (A, C, \beta)\) if there exists an arrow \( f: A \times B \to C \) of \( \text{C} \) such that:

\[
\alpha \leq P_{(\pr_f,A)}(\beta)
\]

where \( \pr_A: A \times B \to A \) is the projection on \( A \).
The functor $P^\text{ex}_{f} : P^\text{ex}(C) \to P^\text{ex}(A)$ sends an object $(C,D,\gamma)$ of $P^\text{ex}(C)$ to the object $(A,D,\mu_{f}\cdot \text{pr}_{A},\text{pr}_{D})((\gamma))$ of $P^\text{ex}(A)$, where $\text{pr}_{A},\text{pr}_{D} : A \times D \to D$ are projections.

The logical intuition is that an element $(A,B,\alpha)$ of the fibre $P^\text{ex}(A)$ represents a predicate $[a : A, b : B] | \exists b : B \phi(a,b)$.

Recall from [3] that the previous construction provides a free completion, i.e. it extends to a 2-functor which is left adjoint to the forgetful functor. We summarise the main properties of the existential completion in the following theorem and we refer to [3] for all details.

**Theorem 1.** The slat-doctrine $P^\text{ex}$ is existential, and the 2-monad $(T^\text{ex}, \mu^\text{ex}, \eta^\text{ex})$ on SD is lax-idempotent. Moreover we have the isomorphism $\text{Ex}D \cong T^\text{ex}.\text{Alg}$ between the 2-category of existential slat-doctrines, and that of strict algebras for $T^\text{ex}$.

**Remark 1 (Prenex normal form).** Observe that in the existential completion of a slat-doctrine $P$, every object $(A,B,\alpha) \in P^\text{ex}(A)$ equals:

$$(A,B,\alpha) = T^\text{ex}_{\text{pr}_A} \eta^\text{ex}_{A \times B}(\alpha).$$

By dualizing the previous construction, we define the *universal completion* of a slat-doctrine. Similarly to the existential case, the intuition is that an element $(A,B,\alpha)$ of the fibre $P^\text{un}(A)$ of the new doctrine that we are going to define represents a predicate $[a : A, b : B] | \forall b : B \phi(a,b)$.

**Universal completion.** Let $P : C^{\text{op}} \to \text{Pos}$ be a slat-doctrine. The universal completion $P^{\text{un}} : C^{\text{op}} \to \text{Pos}$ of $P$ is a slat-doctrine such that, for every object $A$ of $C$, the poset $P^{\text{un}}(A)$ is defined as follows:

- **objects:** triples $(A,B,\alpha)$, where $A$ and $B$ are objects of $C$ and $\alpha \in P(A \times B)$.
- **order:** $(A,B,\alpha) \leq (A,C,\beta)$ if there exists an arrow $g : A \times C \to B$ of $C$ such that:

$$P_{(\text{pr}_{A}, g)}(\alpha) \leq \beta$$

where $\text{pr}_{A} : A \times C \to A$ is the projection on $A$.

Whenever $f : A \to C$ is an arrow of $C$, the functor $P^{\text{un}}_{f} : P^{\text{un}}(C) \to P^{\text{un}}(A)$ is defined as for the existential completion.

**Theorem 2.** The slat-doctrine $P^{\text{un}}$ is universal.

**Proof.** Part I. Existence of right adjoints. Let $P : C^{\text{op}} \to \text{Pos}$ be a slat-doctrine and let for every $A_{1}$ and $A_{2}$ be objects of $C$. The assignment $(A_{1} \times A_{2}, B, \beta) \mapsto (A_{1}, A_{2} \times B, \beta)$ defines a functor $\eta^{\text{un}}_{\text{pr}_{1}} : P^{\text{un}}(A_{1} \times A_{2}) \to P^{\text{un}}(A_{1})$ since, whenever $g : A_{1} \times A_{2} \times C \to B$ is an arrow witnessing that:

$$(A_{1} \times A_{2}, B, \beta) \leq (A_{1} \times A_{2}, C, \gamma)$$
in \( P(A_1 \times A_2) \), then the arrow \((\text{pr}_{A_2}, g) : A_1 \times A_2 \times C \rightarrow A_2 \times B\) witnesses that \((A_1, A_2 \times B, \beta) \leq (A_1, A_2 \times C, \gamma)\) in \( P(A_1) \). Let us verify that \( \mathcal{V}^{\text{un}}_{\text{pr}_1} \) is right adjoint to \( \mathcal{P}^{\text{un}}_{\text{pr}_1} \). Let \((A_1, B, \beta)\) be an object of \( \mathcal{P}^{\text{un}}(A_1) \) and let \((A_1 \times A_2, C, \gamma)\) be an object of \( \mathcal{P}^{\text{un}}(A_1 \times A_2) \).

If

\[
P^{\text{un}}_{\text{pr}_1}(A_1, B, \beta) = (A_1 \times A_2, B, \text{pr}_{A_1 \times A_2} \times 1_B(\beta)) \leq (A_1 \times A_2, C, \gamma)
\]

then there is an arrow \( g : A_1 \times A_2 \times C \rightarrow B \) such that:

\[
P_{(\text{pr}_{A_1} \times 1_B)(\text{pr}_{A_1 \times A_2} \times g)}(\beta) = P_{(\text{pr}_{A_1 \times A_2} \times g)}(\text{pr}_{A_1} \times 1_B(\beta)) \leq \gamma.
\]

This means that \((A_1, B, \beta) \leq (A_1, A_2 \times C, \gamma) = \mathcal{V}^{\text{un}}_{\text{pr}_1}(A_1 \times A_2, C, \gamma)\).

Viceversa, if the latter holds, that is, there is an arrow \( h : A_1 \times A_2 \times C \rightarrow B \) such that \( P_{(\text{pr}_{A_1} \times 1_B)(h)}(\beta) \leq \gamma \), then:

\[
P_{(\text{pr}_{A_1 \times A_2} \times g)}(\text{pr}_{A_1} \times 1_B(\gamma)) = P_{(\text{pr}_{A_1} \times 1_B)(h)}(\beta) \leq \gamma
\]

which implies that \( \mathcal{P}^{\text{un}}_{\text{pr}_1}(A_1, B, \beta) \leq (A_1 \times A_2, C, \gamma)\).

**Part II. Beck-Chevalley condition.** Let us consider the pullback square of a projection along a given arrow \( f \) of \( C \), which is of the form:

\[
\begin{array}{ccc}
B \times C & \xrightarrow{\text{pr}_B} & B \\
\downarrow f \times 1_C & & \downarrow f \\
A \times C & \xrightarrow{\text{pr}_A} & A
\end{array}
\]

Then the corresponding:

\[
\begin{array}{ccc}
P^{\text{un}}(A \times C) & \xrightarrow{\mathcal{V}^{\text{un}}_{\text{pr}_A}} & P^{\text{un}}(A) \\
\downarrow P_{f \times 1_C}^{\text{un}} & & \downarrow P_{f}^{\text{un}} \\
P^{\text{un}}(B \times C) & \xrightarrow{\mathcal{V}^{\text{un}}_{\text{pr}_B}} & P^{\text{un}}(B)
\end{array}
\]

commutes, since it is the case that:

\[
P^{\text{un}}_{f \times 1_C^{\text{pr}_A}}(A \times C, D, \delta) = P^{\text{un}}_{f}(A, C \times D, \delta) = (B, C \times D, P_{(f \times 1_C) \times 1_D}(\delta))
\]

and that:

\[
\mathcal{V}^{\text{un}}_{\text{pr}_B}(A \times C, D, \delta) = \mathcal{V}^{\text{un}}_{\text{pr}_B}(B \times C, D, P_{(f \times 1_C) \times 1_D}(\delta)) = (B, C \times D, P_{(f \times 1_C) \times 1_D}(\delta))
\]

whenever \((A \times C, D, \delta)\) is an object of \( P^{\text{un}}(A \times C)\).
Notation. We denote by \((-\text{op}) : \text{Pos} \to \text{Pos}\) the functor which inverts the order of a poset. In other words, if \((A, \leq_A)\) is a poset, then \((A, \leq_A)^{\text{op}} := (A, \leq_{A^{\text{op}}})\) is the poset whose objects are the ones of \(A\) and \(a \leq_{A^{\text{op}}} b\) if and only if \(b \leq_A a\).

**Proposition 1.** Whenever \(P : \mathcal{C}^{\text{op}} \to \text{Pos}\) is a slat-doctrine, it is the case that:

\[
P^{\text{un}} \cong (-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}.
\]

**Proof.** Let \(A\) be an object of \(\mathcal{C}\). The categories:

\[
((-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}})(A) = \left((-)^{\text{op}}P\right)^{\text{ex}}A
\]

share the object class. Hence an object of the former is a triple:

\[
(A, B, \beta \in (-)^{\text{op}}P)(A \times B) = P(A \times B)^{\text{op}},
\]

that is, a triple of the form \((A, B, \beta \in P(A \times B))\), since \(P(A \times B)\) and \(P(A \times B)^{\text{op}}\) share the object class. This is nothing but an object of \(P^{\text{un}}(A)\). As the functors \((-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}\) and \(P^{\text{un}}\) both send such an object to \((C, B, P_{f \times 1_A}(\beta))\) (being \(f : C \to A\) an arrow of \(C\)), we are left to verify that the morphism classes (that is, the ordering relations) of \((-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}A\) and \(P^{\text{un}}(A)\) coincide.

Let \((A, D, \delta)\) be another object of the common object class and let us assume that \((A, B, \beta) \leq (A, D, \delta)\) in \((-)^{\text{op}}((-)^{\text{op}}P)^{\text{ex}}A\), which means that \((A, D, \delta) \leq (A, B, \beta)\) in \((-)^{\text{op}}P^{\text{ex}}A\). This corresponds to the existence of an arrow \(g : A \times D \to B\) such that:

\[
\delta \leq \left((-)^{\text{op}}P\right)_{(\text{pr}_{A \times D})}(\beta) = \left(P_{(\text{pr}_{A \times D})}\right)^{\text{op}}(\beta) = P_{(\text{pr}_{A \times D})}(\beta)
\]

in \(P(A \times D)^{\text{op}}\), that is, \(P_{(\text{pr}_{A \times D})}(\beta) \leq \delta\) in \(P(A \times D)\). This condition is nothing but the existence of the arrow \((A, B, \beta) \leq (A, D, \delta)\) in \(P^{\text{un}}(A)\). \(\square\)

Notice that by Proposition \[\] one can prove the analogous result of Theorem \[\]

**Theorem 3.** The universal completion extends to a 2-adjunction, and the corresponding 2-monad \((T^{\text{un}}, \mu^{\text{un}}, \eta^{\text{un}})\) on \(\text{SD}\) is colax-idempotent. Moreover we have the isomorphism \(\text{UnD} \cong T^{\text{un}}\text{-Alg}\) between the 2-category of universal slat-doctrines, and that of strict algebras for \(T^{\text{un}}\).

**Remark 2 (Prenex normal form).** Observe that as for the existential completion, in the universal completion of a doctrine we have that every formula admits a prenex normal form.

**Remark 3.** Note that, as \[\], the universal and existential completions can be generalized for an arbitrary class \(A\) of morphisms closed under pullbacks, compositions and which contains identities, i.e. they can be used to freely add right (and left) adjoints along the morphisms of the class \(A\).
Remark 4. Recall from [8] that the dialectica construction [2,9] decomposes into two steps from a modern categorical perspective, following the quantifier pattern of the original translation. The dialectica category $\text{Dial}(p)$ associated to a fibration $p$, hence in particular to an indexed category (see [10]), is obtained by first applying the monad which freely adds simple universal quantification and then applying the monad which freely adds simple existential quantification.

Analogously, by applying the corresponding additions of quantifiers to a given slat-doctrine $P: C^{\text{op}} \rightarrow \text{Pos}$ one gets the poset reflection of the dialectica category $\text{Dial}(P)$ to coincide with the category $(P^\text{un})^\text{ex}(1)$.

4 Preservation of logical structures

When a new free completion is introduced, it is natural to study and to understand which structures are preserved by this construction.

This section is completely devoted to this analysis of the two quantifier completions presented in the previous section. Note that, by Proposition 1, we can easily translate a result of preservation for the existential completion in its dual version for the universal completion. This argument will be heavily used along our presentation.

We start by recalling a first result in this direction provided in [3].

**Proposition 2.** Let $P: C^{\text{op}} \rightarrow \text{Pos}$ be a slat-doctrine such that every fibre $P(A)$ has finite meets and such that every re-indexing functor $P_f$ preserves them. Then the existential doctrine $P^{\text{ex}}: C^{\text{op}} \rightarrow \text{Pos}$ has finite meets in every fibre $P^{\text{ex}}(A)$ and the functors $P^{\text{ex}}_f$ preserve them.

By Proposition 1 the dual result holds for the universal completion.

**Proposition 3.** Let $P: C^{\text{op}} \rightarrow \text{Pos}$ be a slat-doctrine such that every fibre $P(A)$ has finite joins and such that every re-indexing functor $P_f$ preserves them. Then the universal doctrine $P^\text{un}: C^{\text{op}} \rightarrow \text{Pos}$ has finite joins in every fibre $P^\text{un}(A)$ and the functors $P^\text{un}_f$ preserve them.

The two previous propositions reflect the logical distributive property of the quantifiers over disjunctions and conjunctions. Now we show that, under the right hypotheses, the existential completion preserves finite joins, and then, dually, that the universal completion preserves finite meets.

Recall from [1] that a category with finite products and finite sums is said to be distributive if the canonical arrow:

$$\theta: (A \times B) + (A \times C) \rightarrow A \times (B + C)$$
is an isomorphism. The canonical arrow is \( \theta := \left( \langle \text{pr}_A \cdot j'_C \rangle \langle \text{pr}_C \rangle \right) \), i.e. it is the unique arrow such that the diagram:

\[
\begin{array}{ccc}
A \times C & \xrightarrow{j_{A \times C}} & (A \times C) + (A \times B) \\
\downarrow \langle \text{pr}_A \cdot j'_C \rangle & & \downarrow \theta \\
A \times (B + C) & \xrightarrow{j_{A \times B}} & A \times B
\end{array}
\]

commutes, where \( j' \) are the injections in \( B + C \).

To simplify the notation and the readability of the statements we introduce the following definition.

**Definition 4.** A lat-doctrine is a functor \( P : \mathcal{C}^{\text{op}} \rightarrow \text{Lat} \), where \( \mathcal{C} \) is a distributive category and \( \text{Lat} \) is the category of lattices, i.e. finitely complete and finitely cocomplete posets, and finite sup\&inf-preserving maps, i.e. finite limit and finite colimit preserving functors.

**Theorem 4.** Let \( P : \mathcal{C}^{\text{op}} \rightarrow \text{Lat} \) be a lat-doctrine such that the images through \( P \) of the injections \( j_A : A \rightarrow A + B \) have left adjoints \( \exists j_A \) which satisfy Beck-Chevalley condition. Then the following properties hold.

1. The slat-doctrine \( P^{\text{ex}} \) is a lat-doctrine.
2. Suppose that there is an arrow \( c : 1 \rightarrow C \) for every non-initial object \( C \) of \( \mathcal{C} \). Then the images through \( P^{\text{ex}} \) of the injections \( j_A : A \rightarrow A + B \) have left adjoints \( \exists j^{\text{ex}}_A \) which satisfy Beck-Chevalley condition.
3. Suppose that there is an arrow \( c : 1 \rightarrow C \) for every non-initial object \( C \) of \( \mathcal{C} \) and the images through \( P \) of the injections \( j_A : A \rightarrow A + B \) have right adjoints \( V j_A \) which satisfy Beck-Chevalley condition. Then the images through \( P^{\text{ex}} \) of the injections \( j_A : A \rightarrow A + B \) have right adjoints \( V^{\text{ex}} j_A \) which satisfy Beck-Chevalley condition.

**Proof (Sketch of Proof. See the details in the Appendix).**

1. Let \( A \) be an object of \( \mathcal{C} \). Then \( (A, 0, \bot_{A \times 0}) \) and \( (A, 1, \top_{A \times 1}) \) are initial and terminal in \( P^{\text{ex}}(A) \) whenever 0 and 1 are initial and terminal in \( \mathcal{C} \) (respectively). Moreover, whenever \( (A, B, \beta) \) and \( (A, C, \gamma) \) are objects of \( P^{\text{ex}}(A) \), then the objects:

\[
(A, B + C, P_{\theta^{-1}}(\exists j_{A \times B} \beta \lor \exists j_{A \times C} \gamma)) \quad \text{and} \quad (A, B \times C, P_{\langle \text{pr}_A \cdot \text{pr}_B \rangle} \beta \land P_{\langle \text{pr}_A \cdot \text{pr}_C \rangle} \gamma)
\]

are the coproduct and the product (respectively) of \( (A, B, \beta) \) and \( (A, C, \gamma) \) in \( P^{\text{ex}}(A) \).
2. Whenever \( (A, D, \delta) \) is an object of \( P^{\text{ex}}(A) \), we define:

\[
\exists^{\text{ex}}_{j_A}(A, D, \delta) := (A + B, D, P_{\theta^{-1}} \exists j_{A \times D}(\delta))
\]
being \( \theta \) the isomorphism \((A \times D) + (B \times D) \to (A + B) \times D\) and \(j_{A \times D}\) the injection \(A \times D \to (A \times D) + (B \times D)\). Then the functor \(\mathcal{I}^\text{ex}\) is left adjoint to \(P^\text{ex}\) (one uses the existence of constants in order to verify this) and Beck-Chevalley condition is satisfied.

3. Analogously, whenever \((A, D, \delta)\) is an object of \(P^\text{ex}(A)\), we define:

\[
\forall^\text{ex}_{j_A}(A, D, \delta) := (A + B, D, P_\theta^{-1} \mathcal{I}_{j_A \times D}(\delta))
\]

being \(\theta\) the isomorphism \((A \times D) + (B \times D) \to (A + B) \times D\) and \(j_{A \times D}\) the injection \(A \times D \to (A \times D) + (B \times D)\).

Example 7. Let us consider the subobject lat-doctrine \(\text{Sub}_C : C^{\text{op}} \longrightarrow \text{Lat}\) over a finitely complete category \(C\) with disjoint coproducts which are stable under pullback. Observe that these are the usual assumptions over \(C\) in [2], and they imply that \(C\) is a distributive category. Let \(A, B\) be objects of \(C\) and let \(j_A\) be the monic injection \(A \to A + B\). Observe that \(\text{Sub}_{j_A}\) has the left adjoint, which is given by the functor \(\mathcal{I}_{j_A} : \text{Sub}(A) \longrightarrow \text{Sub}(A + B)\) which acts as the post-composition by \(j_A\).

Again by Proposition [1] the dual result of Theorem 4 holds for the universal completion.

Theorem 5. Let \(C\) be a distributive category and let \(P : C^{\text{op}} \longrightarrow \text{Lat}\) be a lat-doctrine such that the images through \(P\) of the injections \(j_A : A \longrightarrow A + B\) have right adjoints \(V_{j_A}\) which satisfy Beck-Chevalley condition. Then the following properties hold.

1. The slat-doctrine \(P^\text{un}\) is a lat-doctrine.
2. Suppose that there is an arrow \(c : 1 \longrightarrow C\) for every non-initial object \(C\) of \(C\). Then the images through \(P^\text{un}\) of the injections \(j_A : A \longrightarrow A + B\) have right adjoints \(V_{j_A}^\text{un}\) which satisfy Beck-Chevalley condition.
3. Suppose that there is an arrow \(c : 1 \longrightarrow C\) for every non-initial object \(C\) of \(C\) and the images through \(P\) of the injections \(j_A : A \longrightarrow A + B\) have left adjoints \(\mathcal{I}_{j_A}^\text{un}\) which satisfy Beck-Chevalley condition. Then the images through \(P^\text{un}\) of the injections \(j_A : A \longrightarrow A + B\) have left adjoints \(\mathcal{I}_{j_A}^\text{un}\) which satisfy Beck-Chevalley condition.

Proof. Let \(Q := (-)^{\text{op}}P\). Then \(Q\) is a lat-doctrine satisfying the hypotheses of Theorem 4. Hence \(Q^\text{ex}\) satisfies the theses of Theorem 4 which corresponds to saying that \(P^\text{un} = (-)^{\text{op}}Q^\text{ex}\) (see Proposition [1]) satisfies the theses of the current statement.

Example 8. Observe that the subobject lat-doctrine \(\text{Sub} : C^{\text{op}} \longrightarrow \text{Lat}\) presented in Example 7 has right adjoints along the inclusions. In particular the right adjoint \(V_{j_A} : \text{Sub}(A) \longrightarrow \text{Sub}(A + B)\) of \(\text{Sub}_{j_A}\) is the functor sending a subobject represented by a mono \(s : S \longrightarrow A\) to the subobject represented by the arrow:

\[
s + 1_B : S + B \longrightarrow A + B
\]
which is indeed a mono: if \( a \) be an arrow \( X \to S + B \) then \( X \) has a structure of coproduct together with the injections \( a^*i_S: X_S \to X \) and \( a^*i_B: X_B \to X \) obtained by pulling back along \( a \) the injections:

\[
i_S: S \to S + B \quad \text{and} \quad i_B: B \to S + B
\]

respectively. Let us denote as \( a_1 \) and \( a_2 \) the unique arrows \( X_S \to S \) and \( X_B \to B \) respectively such that \( a = a_1 + a_2 \). Observe that the injections \( i_S: S \to S + B \) and \( i_B: B \to S + B \) are the pullbacks along \( s + 1_B \) of the arrows \( j_A \) and \( j_B \), hence \( a^*i_S \) and \( a^*i_B \) are the pullbacks of \( j_A \) and \( j_B \) along \( s + 1_B \). This implies that, whenever \( b \) is another arrow \( X \to S + B \) such that \( (s + 1_B)a = (s + 1_B)b \) then, by applying the same procedure to \( b \), we obtain the same coproduct structure \( (a^*i_S, a^*i_B) \) over \( X \). In particular \( j_A a_1 = (s + 1_B)a(a^*i_S) = (s + 1_B)b(a^*i_S) = j_As b_1 \), which implies that \( a_1 = b_1 \), as \( j_A a \) is a monomorphism. Analogously \( a_2 = b_2 \), hence \( a = b \) and \( s + 1_B \) is proven to be a monomorphism.

Let \( s: S \to A \) be a subobject of \( A \) and let \( t: T \to A + B \) be a subobject of \( A + B \). Then the conditions \( j_A^* t = \text{Sub}_{j_A}(t) \leq s \) and \( t \leq j_A^* s = s + 1_B \) are equivalent: if the latter holds then a mono witnessing the former exists by the universal property of the pullback; if the former holds for a mono \( m: j_A^* T \to S \) then \( m + 1_B: T \to S + B \) witnesses the latter.

5 Combining quantifier completions

In the previous section we showed some preservation properties of the existential and the universal completion. Here we show how to combine these two constructions.

**Theorem 6.** Let \( P: \mathcal{C}^{\text{op}} \to \text{Pos} \) be a universal slat-doctrine and suppose that \( \mathcal{C} \) has exponents. Then \( P^{\text{ex}}: \mathcal{C}^{\text{op}} \to \text{Pos} \) is existential and universal, i.e. the existential completion preserves the universal structure.

**Proof.** Part I. Existence of right adjoints along projections. Let \( A_1, A_2 \) be objects of \( \mathcal{C} \), and let \( \text{pr}_{A_1}: A_1 \times A_2 \to A_1 \) be the first projection. Let:

\[
V_{\text{pr}_{A_1}}^{\text{ex}}: \quad P^{\text{ex}}(A_1 \times A_2) \to P^{\text{ex}}(A_1)
\]

be defined by:

\[
(A_1 \times A_2, B, \alpha) \mapsto (A_1, B^{A_2}, V_{\{\text{pr}_1, \text{pr}_2\}} P_{\{\text{pr}_1, \text{pr}_2, \text{ev}_{\{\text{pr}_2, \text{pr}_3\}}\}}(\alpha))
\]

where \( \text{pr}_i \) are the projections from \( A_1 \times A_2 \times B^{A_2} \) and \( \text{ev}: A_2 \times B^{A_2} \to B \) is the evaluation map. The intuition is that the right adjoints act by mapping a formula \( \exists b: B \alpha(a_1, a_2, b) \mapsto \exists f: B^{A_2} \forall \alpha_2: A_2 \alpha(a_1, a_2, f(a_2)) \). Let us verify that \( V_{\text{pr}_{A_1}}^{\text{ex}} \) is right adjoint to \( P_{\text{pr}_{A_1}}^{\text{ex}} \).

Let \((A_1, C, \gamma) \in P^{\text{un}}(A_1)\) and \((A_1 \times A_2, B, \alpha) \in P^{\text{un}}(A_1 \times A_2)\). We are left to verify that the conditions:

\[
(A_1 \times A_2, C, P_{\text{pr}_{A_1} \times \text{pr}_{A_2}}(\gamma)) = P_{\text{pr}_{A_1}}^{\text{ex}}(A_1, C, \gamma) \leq (A_1 \times A_2, B, \alpha)
\]
and:

\[(A_1, C, \gamma) \leq V^{\text{ex}}_{\text{pr} A_1}(A_1 \times A_2, B, \alpha) = (A_1, B^{A_2}, V_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha))\]

are equivalent. The former disequality is equivalent to the condition: there is an arrow 

\[g: A_1 \times A_2 \times C \rightarrow B\]

such that \(P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\gamma) \leq P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha),\) that is:

\[\text{there is } g: A_1 \times A_2 \times C \rightarrow B\]

\[\text{such that } \gamma \leq V_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha)\] (1)

while the latter holds precisely when:

\[\text{there is } h: A_1 \times C \rightarrow B^{A_2}\]

\[\text{such that } \gamma \leq P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha)\] (2)

so that we are left to prove (1) and (2) to be equivalent. Let \(\text{pr}_1, \text{pr}_2, \text{pr}_3\) be the three projections from \(A_1 \times A_2 \times C\), \(\text{pr}_1\) the projections from \(A_1 \times A_2 \times C\) and \(\text{pr}_2\) the projections from \(A_2 \times A_1 \times C\). Moreover let us assume that (1) holds for a given arrow \(g\). We define the arrow \(h: A_1 \times C \rightarrow B^{A_2}\) to be the exponential transpose of \(g(\text{pr}_2, \text{pr}_1, \text{pr}_3)\), hence it holds that \(\text{ev}(A_2 \times h) = g(\text{pr}_2, \text{pr}_1, \text{pr}_3)\). We are going to prove that this choice of \(h\) is such that:

\[V_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha) = P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha)\] (3)

allowing to conclude that (2) holds. Moreover observe that, from a given arrow \(h\) satisfying (2), one can always recover the corresponding arrow \(g\) by anti-transposing \(h\) and precomposing by \((\text{pr}_2, \text{pr}_1, \text{pr}_3)\) (as \((\text{pr}_2, \text{pr}_1, \text{pr}_3)\) is an isomorphism whose inverse is indeed \((\text{pr}_2, \text{pr}_1, \text{pr}_3)\)). Therefore (3) would also imply that (1) follows from (2), concluding our adjointness proof. Hence we are left to prove that (3) holds.

Let \(\varphi\) be the arrow \((A_2 \times h)(\text{pr}_2, \text{pr}_1, \text{pr}_3)\): \(A_1 \times A_2 \times C \rightarrow A_2 \times B^{A_2}\), and observe that the equality:

\[(\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)) \circ (\text{pr}_1, \varphi) = (\text{pr}_2, \text{pr}_1, \text{pr}_3) \circ (\text{pr}_1, A_2, \varphi)\]

holds. Hence it is the case that \(P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha) = P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha)\) and therefore (3) follows if we prove that \(V_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha) = P_{\text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3)}(\alpha)\) holds. By Beck-Chevalley condition for \(V\) it is enough to prove that the right-hand square of the commutative diagram:

\[
\begin{array}{ccc}
A_2 \times A_1 \times C & \xrightarrow{\text{pr}_1, \text{pr}_2, h(\text{pr}_1, \text{pr}_3)} & A_1 \times A_2 \times C \\
\downarrow \text{pr}_1, \text{pr}_2, \text{pr}_3 \quad \text{pr}_1, \varphi & & \text{pr}_1, \text{pr}_2, \text{pr}_3 \\
A_2 \times A_1 \times B^{A_2} & \xrightarrow{\text{pr}_2, \text{pr}_1, \text{pr}_3} & A_1 \times A_2 \times B^{A_2} \\
\end{array}
\]

is a pullback. This is the case: the outer square is a pullback, as its horizontal arrows are the projections \(A_2 \times A_1 \times C \rightarrow A_1 \times C\) and \(A_2 \times A_1 \times B^{A_2} \rightarrow A_1 \times B^{A_2}\) and as
\(\langle \text{pr}_1, \text{pr}_2, h(\text{pr}_1, \text{pr}_3) \rangle = 1_{A^2 \times \langle \text{pr}_{A_1}, h \rangle},\) and moreover the horizontal arrows of the left-hand square are isos, therefore the right-hand square is indeed a pullback as well.

**Part II. Beck-Chevalley condition.** Let us consider a pullback of a projection along a given arrow \(f\), which is of the form:

\[
\begin{array}{ccc}
D \times C & \xrightarrow{\text{pr}_D} & D \\
\downarrow f \times 1_C & & \downarrow f \\
A \times C & \xrightarrow{\text{pr}_A} & A.
\end{array}
\]

and let us verify that the corresponding equality \(P_{\text{ex}} f_{\times 1_C} = P_{\text{ex}} f_{\times 1_C} \times 1_{B^C}\) holds. Whenever \((A \times C, B, \beta) \in P(A \times C)\) we get (by applying the left and right member of the wannabe equality respectively) the elements:

\[
(D, B^C, P_{\text{ex}} f_{\times 1_C} V_{\langle \text{pr}_1, \text{pr}_3 \rangle} P_{\langle \text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3) \rangle}(\beta))
\]

and

\[
(D, B^C, V_{\langle \text{pr}_1, \text{pr}_3 \rangle} P_{\langle \text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3) \rangle} P_{f \times 1_{C \times B}}(\beta))
\]

of \(P(D)\), being \(\text{pr}_i\) the projections from \(A \times C \times B^C\) and \(\overline{\text{pr}}_i\) the projections from \(D \times C \times B^C\). We are left to prove them to be equal. By Beck-Chevally condition for \(V\) it is the case that \(P_{f \times 1_{C \times B}} V_{\langle \text{pr}_1, \text{pr}_3 \rangle} = V_{\langle \text{pr}_1, \text{pr}_3 \rangle} P_{f \times 1_{C \times B}},\) hence we are left to observe that:

\[
P_{f \times 1_{C \times B}} P_{\langle \text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3) \rangle} = P_{\langle \text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3) \rangle} P_{f \times 1_{C \times B}}
\]

which holds because the class of arrows

\[
\langle \text{pr}_1, \text{pr}_2, \text{ev}(\text{pr}_2, \text{pr}_3) \rangle : X \times C \times B^C \longrightarrow X \times C \times B
\]

for \(X\) in \(C\) is a natural transformation \((-) \times C \times B^C \rightarrow (-) \times (C \times B)\).

Let \(\text{SD}_{\text{exp}}\) be the 2-full subcategory of \(\text{SD}\) whose objects are slat-doctrines whose base category has exponents, and whose 1-cells are the 1-cells of \(\text{SD}\) which preserve the exponents. Similarly, we denote by \(\text{UnD}_{\text{exp}}\) and \(\text{ExD}_{\text{exp}}\) the two subcategories of \(\text{SD}_{\text{exp}}\) of the universal and existential slat-doctrines whose base category has exponents.

**Theorem 7.** The 2-monad \(\text{T}_{\text{ex}} : \text{SD}_{\text{exp}} \longrightarrow \text{SD}_{\text{exp}}\) induces a 2-monad:

\[
\overline{\text{T}_{\text{ex}}} : \text{UnD}_{\text{exp}} \longrightarrow \text{UnD}_{\text{exp}}.
\]

**Proof (Sketch of Proof. See the details in the Appendix).** By Theorem 6 the 2-monad \(\overline{\text{T}_{\text{ex}}}\) preserves the universal structure. A direct verification provides the right preservation of 1-cells as well as the fact that the unit and the counit preserve the universal structure.
By Theorem 3 it is the case that $\text{UnD}_{\text{exp}} \cong T^{\text{un}} \cdot \text{Alg}_{\text{exp}}$ and then we can conclude that the existential completion induces a 2-monad

$$T^{\text{ex}} : T^{\text{un}} \cdot \text{Alg}_{\text{exp}} \to T^{\text{un}} \cdot \text{Alg}_{\text{exp}}.$$ 

Observe that the 2-monad $T^{\text{ex}}$ is a lifting of $T^{\text{ex}}$ on $T^{\text{un}} \cdot \text{Alg}_{\text{exp}}$, hence by the well-known characterization of the distributive laws of 2-monads, see for example [18,19], we have the following result.

**Theorem 8.** There exists a distributive law $\lambda : T^{\text{un}} \cdot T^{\text{ex}} \to T^{\text{ex}} \cdot T^{\text{un}}$ and the 2-functor $T^{\text{ex}}T^{\text{un}} : \text{SD}_{\text{exp}} \to \text{SD}_{\text{exp}}$ is a 2-monad.

By combining all previous statements, we finally infer the following result about existential and universal completion of lat-doctrines.

**Theorem 9.** Let $P : C^{\text{op}} \to \text{Lat}$ be a lat-doctrine such that:

- for every object $A$ of $C$ there exists an arrow $a : 1 \to A$;
- the category $C$ has exponents;
- the images $P_{jA}$ of the injections $j_A : A \to A + B$ have left and right adjoints $\exists_{jA} \dashv P_{jA} \dashv V_{jA}$, which satisfy BC.

Then $(P^{\text{un}})_{\text{ex}} : C^{\text{op}} \to \text{Lat}$ is an existential and universal lat-doctrine and the images $(P^{\text{un}})_{jA}^{\text{ex}}$ of the injection $j_A : A \to A + B$ have left and right adjoints $(\exists_{jA}^{\text{un}} \cdot (P^{\text{un}})_{jA}^{\text{ex}} \cdot (V^{\text{un}})_{jA}^{\text{ex}})$, which satisfy BC.

**Proof.** The statement is a consequence of Theorem 3, Theorem 5, and Theorem 6. □

**Remark 5.** Let $P : C^{\text{op}} \to \text{Lat}$ be a lat-doctrine which satisfies the hypotheses of Theorem 9. As exposed in Remark 4, remind that $\text{Dial}(P)$ is nothing but the fiber of $(P^{\text{un}})_{\text{ex}}$ over the terminal object of $C$. Hence, by Theorem 5, we can conclude that the poset reflection of the dialectica category $\text{Dial}(P)$ is a lattice.

### 6 Choice principles and quantifier completions

The constructive features of choice principles play a fundamental role in several areas of mathematics and theoretical computer science. For example in a dependent type theory satisfying the propositions as types correspondence together with the proofs-as-programs paradigm, the validity of the unique choice rule or even more of the choice rule says that the extraction of a computable witness from an existential statement under hypothesis can be performed within the same theory.

However in [13] Maietti shows that the unique choice rule, and hence the choice rule, are not valid in some important constructive theories such as Coquand’s Calculus of Constructions with indexed sum types, list types and binary disjoint sums and in its predicative version implemented in the intensional level of the Minimalist Foundation [13,14].
In this section we show that quantifier completions carry on some strong choice principles, and the preservation of the logical structures suggests that this free-operation could be applied to extend a given theory to another one which satisfies the following principles.

Recall from [4] that the existential completion of a doctrine satisfies the rule of choice.

**Theorem 10 (Rule of Choice).** If the fibres of a slat-doctrine \( P : C^{op} \longrightarrow \text{Pos} \) have finite meets, then the existential completion of a primary doctrine satisfies the Rule of Choice (see [4]), i.e. if:

\[
a : A \mid \top \vdash \exists b : B \alpha(a, b)
\]

then there exists a term \( a : A \mid f(a) : B \) called witness such that:

\[
a : A \mid \top \vdash \alpha(a, f(a)).
\]

Again we can prove the dual property for the universal completion, and we call it counterexample property. In this case, the logical intuition is that the witness plays the role of a counterexample, i.e. if we have \( \forall x : A \alpha(x) \vdash \bot \), then there exists a term \( t \) such that \( \alpha(t) \vdash \bot \). In other words, \( t \) is the counterexample.

**Theorem 11 (Counterexample Property).** If \( P : C^{op} \longrightarrow \text{Pos} \) is a slat-doctrine whose fibres have finite joins, then the universal completion satisfies Counterexample Property, i.e. if:

\[
a : A \mid \forall b : B \alpha(a, b) \vdash \bot
\]

then there exists a term \( a : A \mid g(a) : B \), which represents the counterexample of the previous statement, such that:

\[
a : A \mid \alpha(a, g(a)) \vdash \bot.
\]

We conclude the section by proving that, under the assumption of Theorem 6, the existential completion of a universal doctrine satisfies the so called Skolemization principle.

**Theorem 12 (Skolemization).** Let \( P : C^{op} \longrightarrow \text{Pos} \) be a universal slat-doctrine with exponents, and consider its existential completion \( P^{\text{ex}} \). For every \( A_1, A_2 \) objects of \( C \) and every \( \overline{\alpha} = (A_1 \times A_2, B, \alpha) \in P^{\text{ex}}(A_1 \times A_2) \) it holds that:

\[
\psi^{\text{ex}_{\overline{p}_1}, \overline{p}_2} = \psi^{\text{ex}_{\overline{p}_1}, \overline{p}_3} \eta_{A_1 \times A_2 \times B} = \psi^{\text{ex}_{\overline{p}_1}, \overline{p}_3} \eta_{A_1 \times A_2 \times B A_2} P_{(p_{1, \overline{p}_2, \overline{p}_3})}(\alpha).
\]

**Proof.** Let \( A_1, A_2 \) be objects of \( C \), and let \( \overline{p}_1 : A_1 \times A_2 \longrightarrow A_1 \) be the first projection. Recall that \( \psi^{\text{ex}_{\overline{p}_1}} : P^{\text{ex}}(A_1 \times A_2) \longrightarrow P^{\text{ex}}(A_1) \) is defined by:

\[
(A_1 \times A_2, B, \alpha) \mapsto (A_1, B A_2, \psi_{(p_{1, \overline{p}_2, \overline{p}_3})} P_{(p_{1, \overline{p}_2, \overline{p}_3}, ev(p_{2, \overline{p}_3}))}(\alpha)).
\]

By Remark 5, it is the case that:

\[
\psi^{\text{ex}_{\overline{p}_1}}(\overline{\alpha}) = \psi^{\text{ex}_{\overline{p}_1}, \overline{p}_2} \eta_{A_1 \times B A_2} \psi_{(p_{1, \overline{p}_2, \overline{p}_3})} P_{(p_{1, \overline{p}_2, \overline{p}_3}, ev(p_{2, \overline{p}_3}))}(\alpha).
\]
By Theorem 7, the unit of the 2-monad preserves the universal structure, hence we have that
\[
\eta_{A_1 \times B} V_{(pr_1, pr_3)} = V_{(pr_1, pr_3)}^{ex} \eta_{A_1 \times A_2 \times B} \quad \text{and then:}
\]
\[
V_{pr_1}^{ex}(\tau) = \exists_{pr_1}^{ex} V_{(pr_1, pr_3)}^{ex} \eta_{A_1 \times A_2 \times B} P_{(pr_1, pr_2, ev_{pr_2, pr_3})}(\alpha).
\]
On the other hand, by Remark 1, it is the case that:
\[
V_{pr_1}^{ex}(A_1 \times A_2, B, \alpha) = V_{pr_1}^{ex} \exists_{(pr_1, pr_2)}^{ex} \eta_{A_1 \times A_2 \times B}(\alpha)
\]
that is, the stated equality holds. 

The property proved in Theorem 12 reflects the principle of Skolemization introduced by Gödel in his work on the dialectica interpretation [7]. It is called (AC) by Troelstra in [6], since this principle is a form of choice, and it has the following presentation:
\[
\forall x \exists y \alpha(x, y) \rightarrow \exists f \forall x \alpha(x, f x).
\]

7 Conclusions and further works

We introduced and characterised the universal and existential completions, by showing those logical properties that are preserved by these constructions. We remark that the properties we needed in order to get them preserved (e.g. the existence of left and right adjoints over the injections) are all natural: they are true in many concrete instances enjoying a satisfying "power-set" algebra, like subobject doctrines over lextensive categories (see [2] and Example 2).

One of the major benefits of this property-preservation analysis appears during its application to dialectica construction. During the current essay we only focused on the notion of poset-reflection of the dialectica category associated to a lattice-doctrine. In future work we intend to generalise our approach in order to examine properties of the dialectica category itself and not just of its poset-reflection.

Moreover we are going to find the right hypotheses such that the universal and existential completion preserves the implication of intuitionistic logic, and then possible applications.

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A Proof of Theorem 4

The appendix is devoted to a complete proof of Theorem 4.

Proof.

1. Let $A$ be an object of $C$ and let us verify that $P^{ex}(A)$ has finite coproducts (Part I). Moreover, let us verify that finite coproducts are preserved by the images through $P^{ex}$ of any arrow of $C$ of target $A$ (Part II).

Part I. Finite coproducts in $P^{ex}(A)$. It is the case that $(A, 0, \bot_{A \times O})$ is initial in $P^{ex}(A)$ whenever 0 is initial in $C$. Moreover, whenever $(A, B, \alpha)$ and $(A, C, \beta)$ are two elements of $P^{ex}(A)$, it is the case that:

$$(A, B, \alpha) \lor (A, C, \beta) := (A, B + C, P_{\theta^{-1}}(\exists_{j_{A \times B}}(\alpha) \lor \exists_{j_{A \times C}}(\beta)))$$
is their coproduct. In order to verify this, at first we prove that \((A, B, \alpha) \leq ((A, B + C, \exists_{j_{A \times B}}(\alpha) \lor \exists_{j_{A \times C}}(\beta))\). Observe that the diagram:

\[
\begin{array}{c}
(A \times B) + (A \times C) \\
\downarrow \theta \\
A \\
\end{array}
\]

commutes by definition of \(\theta\). Moreover it is the case that:

\[
P_{\bar{\theta}_{j_{A \times B}}}(P_{\theta^{-1}}(\exists_{j_{A \times B}}(\alpha) \lor \exists_{j_{A \times C}}(\beta))) = P_{j_{A \times B}}(\exists_{j_{A \times B}}(\alpha)) \lor P_{j_{A \times B}}(\exists_{j_{A \times C}}(\beta)) \geq \alpha
\]

since \(\alpha \leq P_{j_{A \times B}}(\exists_{j_{A \times B}}(\alpha))\). Similarly one can prove that \((A, C, \beta) \leq (A, B, \alpha) \lor (A, C, \beta)\).

Now assume that \((A, B, \alpha) \leq (A, D, \gamma)\) and \((A, C, \beta) \leq (A, D, \gamma)\), i.e. that there exist \(f_1: A \times B \to D\) and \(f_2: A \times C \to D\) such that \(\alpha \leq P_{(pr_A, f_1)}(\gamma)\) and \(\beta \leq P_{(pr_A, f_2)}\). By the universal property of coproducts, the diagram:

\[
\begin{array}{c}
(A \times C) + (A \times B) \\
\downarrow \langle pr_A, f_2 \rangle \downarrow \langle pr_A, f_1 \rangle \\
A \times D \\
\end{array}
\]

commutes. In order to simplify the notation, let \(c := \langle (pr_A, f_2), (pr_A, f_1) \rangle\). First notice that the following equality holds again by the universal property of coproducts:

\[
pr_A' c = pr_A \theta
\]

where \(pr_A': A \times D \to A\) and \(pr_A: A \times (B + C) \to A\). Hence the diagram:

\[
\begin{array}{c}
(A \times B) + (A \times C) \\
\downarrow \end{array}
\]

\[
\begin{array}{c}
A \times (B + C) \\
\downarrow \theta^{-1} \\
A \times D \\
\downarrow \end{array}
\]

\[
\begin{array}{c}
A \\
\end{array}
\]

\[
\begin{array}{c}
pr_A \\
\end{array}
\]

\[
\begin{array}{c}
pr_A' \\
\end{array}
\]
Part I. Existence of left adjoints. Let \( \alpha \leq P_{j_{A \times B}}(P_c(\gamma)) \). Since \( \exists_{j_{A \times B}} P_{j_{A \times B}} \), it holds that \( \exists_{j_{A \times B}} (\alpha) \leq P_c(\gamma) \) and hence:

\[
P_{\theta^{-1}}(\exists_{j_{A \times B}}(\alpha)) \leq P_{\theta^{-1}}P_c(\gamma).
\]

Similarly one gets that \( P_{\theta^{-1}}(\exists_{j_{A \times C}}(\beta)) \leq P_{\theta^{-1}}P_c(\gamma) \) and therefore we can conclude that:

\[
P_{\theta^{-1}}(\exists_{j_{A \times B}}(\alpha) \lor \exists_{j_{A \times C}}(\beta)) \leq P_{\theta^{-1}}P_c(\gamma)
\]

that is, \((A, B, \alpha) \lor (A, C, \beta) \leq (A, D, \gamma)\).

Part II. Preservation of finite coproducts. Let \( f : D \to A \) be an arrow of \( C \). Then it is the case that \( P_f^{ex}(A, 0, \perp_{A \times 0}) = (D, 0, \perp_{D \times 0}) \). Moreover, one might observe that:

\[
P_{j_{X \times B+C}} P_{\theta^{-1}}(\exists_{j_{A \times B}}(\alpha) \land \exists_{j_{A \times C}}(\beta)) = P_{\theta^{-1}}(\exists_{j_{D \times B}}(P_{j_{X \times B}}(\alpha) \land \exists_{j_{D \times C}}(P_{j_{X \times C}}(\beta)))
\]

by naturality of the class of isomorphisms:

\[
\theta : X \times B + X \times C \to X \times (B + C),
\]

where \( X \) is any object of \( C \), and by Beck-Chevalley property of the class of left adjoints to the injections of coproducts. Observe indeed that the left-hand square of the commutative diagram:

\[
\begin{array}{ccc}
D \times B & \xrightarrow{j_{D \times B}} & (D \times B) + (D \times C) \\
\downarrow{f \times 1_B} & & \downarrow{(f \times 1_B) + (f \times 1_C)} \\
A \times B & \xrightarrow{j_{A \times B}} & (A \times B) + (A \times C)
\end{array}
\quad \quad
\begin{array}{ccc}
(D \times B + (D \times C)) \xrightarrow{\theta} D \times (B + C) \xrightarrow{\pi_D} D \\
\downarrow{f \times 1_{B+C}} & & \downarrow{f} \\
A \times (B + C) \xrightarrow{\theta} A \times (B + C) \xrightarrow{\pi_A} A
\end{array}
\]

is a pullback, since the right-hand square and the outer one (whose horizontal arrows are projections from \( D \times B \) and from \( A \times B \)) are pullbacks (and \( \theta \) are isos). The same holds with with \( j_{D \times C} \) and \( j_{A \times C} \) instead of \( j_{D \times B} \) and \( j_{A \times B} \) respectively. This implies that \( P_f^{ex} \) preserves binary joins.

2. Part I. Existence of left adjoints. Let \((A + B, C, \alpha) \in P^{ex}(A + B)\) and let \((A, D, \delta) \in P^{ex}(A)\). We define \( \exists_{j_{A \times D}}(A, D, \delta) \) to be the object \((A + B, D, P_{\theta^{-1}}\exists_{j_{A \times D}}(\delta))\) being \( \theta \) the isomorphism \((A \times D) + (B \times D) \to (A + B) \times D \) and \( j_{A \times D} \) the injection \( A \times D \to (A \times D) + (B \times D) \). Then the conditions \( \exists_{j_{A \times D}}(A, D, \delta) \leq (A + B, C, \alpha) \) and \((A, D, \delta) \leq P_{j_{A \times B}}(A + B, C, \alpha) \) are equivalent to the conditions:

there is \( g : (A + B) \times D \to C \)

such that \( P_{\theta^{-1}}\exists_{j_{A \times D}}(\delta) \leq P_{(pr_{A + B}, g)}(\alpha) \) \hspace{1cm} (4)

and

there is \( h : A \times D \to C \)

such that \( \delta \leq P_{(pr_A, h)}P_{j_{A \times C}}(\alpha) \) \hspace{1cm} (5)

and

there is \( g : (A + B) \times D \to C \)

such that \( P_{\theta^{-1}}(\exists_{j_{A \times D}}(\delta)) \leq P_{(pr_{A + B}, g)}(\alpha) \) \hspace{1cm} (4)

and

there is \( h : A \times D \to C \)

such that \( \delta \leq P_{(pr_A, h)}P_{j_{A \times C}}(\alpha) \) \hspace{1cm} (5)
respectively. Being $\theta$ an isomorphism and by left adjointness of $\mathfrak{A}_{j_A \times D}$, (4) is equivalent to the existence of an arrow $g : (A + B) \times D \to C$ such that $\delta \leq P_{j_A \times D} \circ P_{(pr_A + g \theta \theta)}(\alpha)$. If (4) holds for some $g$, then let $h := g \theta j_A \times D$. Vice-versa, if (5) holds for some $h$, then let $g := [h, c!]\theta^{-1}$, where $!$ is the unique arrow $B \times D \to 1$ (intuitively, we do not take care of the "$C$-values" that $g$ assumes over "$B \times D$-part of its domain", hence we might define its "restriction" to $B \times D$ as the arrow $c!$, which is the $c$-constant map $B \times D \to C$). In both cases one might verify that:

$$\langle pr_{A + B} \theta, g \theta \rangle j_A \times D = (j_A \times 1_C) \langle pr_A, h \rangle$$

which implies that (4) and (5) are equivalent. Hence $\mathfrak{A}_{j_A}^{ex}$ is left adjoint to $P_{j_A}^{ex}$ and one might as usual verify it to satisfy Beck-Chevalley condition.

**Part II. Beck Chevalley condition.** Let us assume that the square:

$$\begin{array}{ccc}
C & \xrightarrow{j} & C + D \\
\downarrow f & & \downarrow g \\
A & \xrightarrow{j_A} & A + B
\end{array}$$

is a pullback and let $(A, E, \epsilon)$ be an object of $P_{j_A}^{ex}(A)$. We are left to prove that $\mathfrak{A}_{j_A}^{ex} P_{j_A}^{ex}(A, E, \epsilon) = (C + D, E, P_{\theta^{-1} \mathfrak{A}_{j_A \times E}} P_{j_A}^{ex} \epsilon)$ and $P_{\varphi \mathfrak{A}_{j_A \times E}} P_{\varphi^{-1} \mathfrak{A}_{j_A \times E}} (A, E, \epsilon) = (C + D, E, P_{\varphi \times 1_E} P_{\varphi^{-1} \mathfrak{A}_{j_A \times E}} \epsilon)$ are equal, where $\theta$ and $\varphi$ are the isomorphisms $C \times E + D \times E \to (C + D) \times E$ and $A \times E + B \times E \to (A + B) \times E$ respectively. Let us consider the following commutative diagram:

$$\begin{array}{ccc}
C \times E & \xrightarrow{j_{C \times E}} & (C + D) \times E \\
\downarrow f \times 1_E & & \downarrow g \times 1_E \\
A \times E & \xrightarrow{\varphi^{-1} (g \times 1_E) \theta} & (A + B) \times E \\
\downarrow j_{A \times E} & & \downarrow \varphi^{-1} (pr_{A + B}) \epsilon \\
A \times E + B \times E & \xrightarrow{\epsilon} & A + B
\end{array}$$

which is a pullback, since its horizontal arrow $A \times E \to A + B$ equals the arrow $j_A \epsilon$, the arrow $f \times 1_E$ is the pullback of $f$ along $pr_A$ and $f$ is the pullback of $g$ along $j_A$ (hence $f \times 1_E$ is indeed the pullback of $g$ along $j_A \epsilon$). As the right-hand square is pullback, we deduce that the left-hand square is a pullback as well. Therefore, by Beck Chevalley condition on $\mathfrak{A}$, it is the case that:

$$P_{\theta^{-1} \mathfrak{A}_{j_A \times E}} P_{j_{C \times E}} \epsilon = P_{\theta^{-1} \mathfrak{A}_{j_A \times E}} P_{\varphi^{-1} (g \times 1_E) \theta} \mathfrak{A}_{j_A \times E} \epsilon = P_{\varphi \times 1_E} P_{\varphi^{-1} \mathfrak{A}_{j_A \times E}} \epsilon$$

and we are done.

3. Let $(A + B, C, \alpha) \in P_{j_A}^{ex}(A + B)$ and let $(A, D, \delta) \in P_{j_A}^{ex}(A)$. Then we define $V_{j_A}^{ex}(A, D, \delta)$ to be the object $(A + B, D, P_{\theta^{-1} V_{j_A \times D}^{ex}(\delta)})$ being $\theta$ the isomorphism
\[(A \times D) + (B \times D) \rightarrow (A + B) \times D\] and \(j_{A \times D}\) the injection \(A \times D \rightarrow (A \times D) + (B \times D)\). Then the conditions \((A + B, C, \alpha) \leq V_{j_{A}}(A, D, \delta)\) and \(P_{j_{A}}^{-1}(A + B, C, \alpha) \leq (A, D, \delta)\) are equivalent to the conditions:

there is \(g: (A + B) \times C \rightarrow D\)
such that \(\alpha \leq P_{(pr_{A} + B, g)}P_{(pr_{A}, h)}j_{A \times D}(\delta)\) \hspace{1cm} (6)

and

there is \(h: A \times C \rightarrow D\)
such that \(P_{j_{A} \times 1_{C}}(\alpha) \leq P_{(pr_{A}, h)}(\delta)\) \hspace{1cm} (7)

respectively. Observe that if (6) holds, then:

\[P_{j_{A} \times 1_{C}}(\alpha) \leq P_{(pr_{A} + B, g)}P_{(pr_{A}, h)}j_{A \times D}(\delta)\]

and it is the case that:

\[\theta^{-1}(pr_{A} + B)j_{A} \times 1_{C} = \theta^{-1}(pr_{A}, g(j_{A} \times 1_{C})) = j_{A \times D}(pr_{A}, g(j_{A} \times 1_{C}))\]

because in our case \(\theta^{-1}: (A + B) \times D \rightarrow (A \times D) + (B \times D)\) and

\[\theta_{j_{A \times D}} = (j_{A} pr_{A}, pr_{D}).\]

Moreover \(P_{j_{A} \times D}V_{j_{A} \times D} \leq id\) since \(P_{j_{A} \times D} \dashv V_{j_{A} \times D}\), and then:

\[P_{j_{A} \times 1_{C}}(\alpha) \leq P_{(pr_{A} + B, g)}P_{(pr_{A}, h)}j_{A \times D}(\delta)\]
\[= P_{(pr_{A}, g(j_{A} \times 1_{C}))}P_{j_{A} \times D}V_{j_{A} \times D}(\delta)\]
\[\leq P_{(pr_{A}, g(j_{A} \times 1_{C}))}(\delta).\]

Then we can set:

\[h := g(j_{A} \times 1_{C}): A \times C \rightarrow D\]

so that \(P_{j_{A} \times 1_{C}}(\alpha) \leq P_{(pr_{A}, h)}(\delta)\) as required in (7).

Suppose that (7) holds. First notice that \(j_{A} \times 1_{C} = \theta_{j_{A \times C}}\), hence:

\[\alpha \leq P_{(pr_{A}, h)}j_{A \times D}(\delta).\]

Now we define \(g: (A + B) \times C \rightarrow D\) as the following composition of arrows:

\[(A + B) \times C \xrightarrow{\theta^{-1}} (A \times C) + (C \times B) \xrightarrow{id_{A \times C} + 1} (A \times C) + 1 \xrightarrow{(h, d)} D\]
where ! is the terminal arrow, and \((\frac{h}{d})\) is the coproduct of \(h: A \times C \to D\) and the constant \(d: 1 \to D\). Let us verify that the commutative diagram:

\[
\begin{array}{ccc}
A \times C & \xrightarrow{j_{A \times C}} & (A \times C) + (B \times C) \\
\downarrow_{(pr_A, h)} & & \downarrow_{\theta^{-1}(pr_{A+B}; g)\theta} \\
A \times D & \xrightarrow{j_{A \times D}} & (A \times D) + (B \times D).
\end{array}
\]

is a pullback, in order to use Beck-Chevalley condition. Consider the following diagram:

\[
\begin{array}{c}
E \\
\downarrow_{(a_1, a_2)} \\
A \times C & \xrightarrow{j_{A \times C}} & (A \times C) + (B \times C) & \xrightarrow{\theta} & (A + B) \times C \\
\downarrow_{(pr_A, h)} & & \downarrow_{(pr_{A+B}; g)} \\
A \times D & \xrightarrow{j_{A \times D}} & (A \times D) + (B \times D) & \xrightarrow{\theta} & (A + B) \times D.
\end{array}
\]

and assume that \((pr_{A+B}; g)(a_1, a_2) = j_{A \times D}(a_3, a_4)\). By the equality \(\theta j_{A \times D} = j_{A \times D} (pr_A, pr_D)\), the following ones hold:

- \(a_1 = j_A a_3\);
- \(g(a_1, a_2) = a_4\).

First we prove that the choice of \((a_3, a_2)\) makes the left triangle commutes:

\[\langle a_3, a_4 \rangle = \langle a_3, h(a_3, a_2) \rangle.\]

Then we need to show that \(h(a_3, a_2) = a_4\). By \(g(a_1, a_2) = g(j_A a_3, a_2)\). Observe that \(\langle j_A a_3, a_2 \rangle = (j_A \times 1_C)(a_3, a_2) = \theta j_{A \times C}(a_3, a_2)\), and then, by definition of \(g\), it is the case that:

\[g(a_1, a_2) = \left(\frac{h}{d}\right)(1_{A \times C} + !)\theta^{-1}(\theta j_{A \times C}(a_3, a_2)) = h(a_3, a_2)\]

where the last equality holds as the composition:

\[
\begin{array}{ccc}
A \times C & \xrightarrow{j_{A \times C}} & (A \times C) + (B \times C) \\
& & \xrightarrow{id_{A \times C} + !} (A \times C) + 1
\end{array}
\]

equals the arrow \(A \times C \xrightarrow{j_{A \times C}} (A \times C) + 1\) and, by the universal property of the coproduct, it is the case that \(\frac{h}{d} j_{A \times C} = h\). Hence the left triangle commutes.
Now we consider the top triangle. This commutes because:

\[ \theta j_{A \times C} \langle a_3, a_2 \rangle = \langle j_A \text{pr}_A, \text{pr}_C \rangle \langle a_3, a_2 \rangle = \langle j_A a_3, a_2 \rangle \]

which is equal to \( \langle a_1, a_2 \rangle \) by the hypothesis \( a_1 = j_A a_3 \).

This concludes that the previous commutative square is indeed a pullback, and then, since \( \theta \) is invertible, the first square is a pullback. Therefore, by Beck-Chevalley condition, we get that:

\[ \alpha \leq P_{(\text{pr}_A + B, g)}^\perp P_{\theta^{-1}} \mathcal{V}_{j_A \times B} (\delta). \]

Hence \( P^\perp_{j_A} \vdash \mathcal{V}^\perp_{j_A} \), and one can directly check that these satisfy Beck-Chevalley condition. The proof is analogous to Part II of 2. \( \square \)

\section{Proof of Theorem \textbf{7}}

We divide the proof of Theorem 7 into the following Lemmas.

\textbf{Lemma 1.} The unit of the 2-monad \( T^\text{ex} \) preserves the universal structure.

\textbf{Proof.} Recall from [3] that the unit of the 2-monad \( T^\text{ex} \) is provided by the 1-cell \( \eta^\text{ex}_P : P \rightarrow P^\text{ex} \) of SD, where \( \eta^\text{ex}_P = (\text{id}_C, \iota) \), with \( \iota_A : P(A) \rightarrow P^\text{ex}(A) \) acting as \( \alpha \mapsto (A, 1, \alpha) \). Then one might check that the diagram:

\[
\begin{array}{ccc}
P(A \times B) & \xrightarrow{V^\text{ex}_{\text{pr}_A}} & P(A) \\
\downarrow{\iota_{A \times B}} & & \downarrow{\iota_A} \\
P^\text{ex}(A \times B) & \xrightarrow{V^\text{ex}_A} & P^\text{ex}(A) \\
\end{array}
\]

commutes, since \( 1^B \cong 1 \). Therefore the unit of the 2-monad preserves the universal structure. \( \square \)

\textbf{Lemma 2.} The multiplication of the 2-monad \( T^\text{ex} \) along universal slat-doctrines with exponents preserves the universal structure.

\textbf{Proof.} Recall that the multiplication of the 2-monad \( \mu^\text{ex} : (T^\text{ex})^2 \rightarrow T^\text{ex} \) is given by the assignment \( \mu^\text{ex}_P = \varepsilon^\text{ex}_P \), where \( \varepsilon \) is the counit of the existential completion \( \varepsilon_P : P^\text{ex} \rightarrow P \) and it is given by the pair \( (\text{id}, \zeta) \), with \( \zeta_A : P^\text{ex}(A) \rightarrow P(A) \) acting as \( (A, B, \alpha) \mapsto \exists_{\text{pr}_1} \alpha \), where \( \text{pr}_1 : A \times B \rightarrow A \) is a projection. Then we conclude that the multiplication preserves the universal structure since, for every universal slat-doctrine \( P \), the map \( \mu^\text{ex}_P : (T^\text{ex})^2(P) \rightarrow T^\text{ex}(P) \) has as domain and codomain two slat-doctrines satisfying the (AC) principle by Theorem [12]. \( \square \)

\textbf{Lemma 3.} Let \( (F, f) : P \rightarrow R \) be a 1-cell of universal slat-doctrines with exponents. Then \( T^\text{ex}(F, f) : P^\text{ex} \rightarrow R^\text{ex} \) preserves the universal structure.
Proof. Recall that the 1-cell $T^{\text{ex}}(F, f) = (F, \bar{f}) : P^{\text{ex}} \rightarrow R^{\text{ex}}$ is given by $F$ itself and the functor $\bar{f} : P^{\text{ex}}(A) \rightarrow R^{\text{ex}}F(A)$ acting on the poset $P^{\text{ex}}(A)$ as:

$$(A, B, \alpha) \mapsto (FA, FB, R_{(\text{pr}_A, \text{pr}_B)}f_{A \times B}(\alpha)).$$

Since, by hypothesis, $f$ is natural with respect to $V^P$ and by definition of $V^{\text{ex}}$, one might check, by using the naturality of $f$, that the 1-cell of $(F, \bar{f})$ preserves the universal structure. \qed