Controlling properties of a hybrid Cooper pair box interacting with a nanomechanical resonator in the presence of Kerr nonlinearities and losses.

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We consider the Jaynes-Cummings model describing the interaction of a Cooper pair box (CPB) and a nanoresonator (NR) in the presence of a Kerr medium and losses. The evolution of the entropy of both subsystems and the CPB population inversion were calculated numerically. It is found that these properties increase when the NR frequency is time-dependent, even in the presence of losses; the effect is very sensitive to detuning and disappears in the resonant regime. The roles played by the losses affecting the CPB and the NR are also compared.

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I. INTRODUCTION

The implementation of the quantum computer is considered as a big challenge in the recent years due to the failure of classical computers ability to simulate and compute the large scale data. The interaction of the single photon field with superconducting circuit to generate quantum bits is studied experimentally [1, 2] and theoretically [3–5]. Wallraff et al. have observed a strong coupling between the superconducting two-level system and a single microwave photon which prove the generation of qubits. Recently, the construction of the first quantum computer by Canadian company (D-Wave) using the system of the superconducting qubits was shown. This quantum computer succeeds to solve famous problems in mathematics, the Ramsey problem [3].

The study of the interaction between a two-level atom and a radiation field is very important in quantum mechanics, as its applications in laser physics and quantum optics. This system has similarities with others, e.g., the interaction between a Cooper pair box (CPB) and a nanomechanical resonator (NR). There are many works in the literature dealing with these systems [7, 8], but only few of them treat these systems in the situation where one of the frequencies [9, 10], or the amplitude [11], varies with time. In the CPB-NR system these variations change the subsystems coupling and modify their dynamic properties, e.g., showing amplification of the excitation transition rate in subsystems [12].

An important focus of quantum optics is concerned with the atom-field coupled system. Inspired by various tests applied to this system, and its limitations, the researchers have passed from the light domain to the microwave domain of superconducting cavities coupled to Rydberg atoms, or to the quantum electrodynamics circuit coupled to nanoresonators. After being extended to broader context the system was used to investigate Landau–Zener transitions [13], atomic physics and quantum optics [14]; mechanisms for photon generation from quantum vacuum [15]; quantum simulation, challenges, and promises of fast-growing field [16]. Here we will assume the laboratory substituting the atom (field) by the CPB (NR) [17].

In this report we will employ the Jaynes-Cummings model to treat a CPB-NR system with losses; a nonlinear Kerr medium is also added to include influences of nonlinearities. We investigate the time evolution of the CPB population inversion ($I(t)$), as well as the statistical properties of both subsystems. We consider the NR initially in a coherent state and the CPB in its excited state. The influence caused by the time-dependent NR frequency upon the properties of both subsystems is studied. We consider the combined effects of nonlinearity and losses upon the dynamics of the CPB population inversion and upon the NR entropy, the latter being used as a measure for the degree of entanglement. We compare the results obtained in the resonant case ($\omega_{NR} = \omega_{CPB}$) with those obtained for small detunings between the CPB and NR. The influences of losses from the NR and CPB upon the mentioned properties are also compared.

II. THE HAMILTONIAN SYSTEM

A superconductor CPB charge qubit is adjusted to the input voltage $V_g$ of the system through a capacitor with an input capacitance $C_1$. Observing the configuration shown in Fig. 1 we see three loops: a small loop in the left, another in the right, and a great loop in the center. The control of the external parameters of the system can be implemented via the input voltage $V_g$ and
Cooper pairs in the superconducting island. So we have,

\[ |\text{number in the input with the input voltage } V | \]

and making the approximation \( \pi B \ell x, \) we find the Hamiltonian in the form,

\[ \hat{H} = \omega_0 \hat{a} \hat{a}^\dagger + 4E_c \left( N_g - \frac{1}{2} \right) \hat{\sigma}_z - 4E_0 \cos \left( \frac{\pi \Phi_x}{\Phi_0} \right) \cos \left( \frac{\pi \Phi_x}{\Phi_0} \right) \hat{\sigma}_x, \]

where \( \hat{a} \) is the creation (annihilation) operator for the input excitations, corresponding to the frequency \( \omega \) and mass \( m; \) \( E_0 \) and \( E_c \) are respectively the energy of each Josephson junction and the charging energy of a single electron; \( C_1 \) and \( C_2 \) stand for the input capacitance and the capacitance of each Josephson tunnel, respectively; \( \Phi_0 = \frac{\pi}{e} \) is the quantum flux and \( N_g = \frac{C_1 V_0}{e} \) is the charge number in the input with the input voltage \( V. \) We have used the Pauli matrices to describe our system operators, where the states \( |0\rangle \) and \( |1\rangle \) represent the number of Cooper pairs in the superconducting island. So we have, \( \hat{\sigma}_z = |1\rangle \langle 1| - |0\rangle \langle 0| \)

and \( E_c = \frac{\pi^2}{(C_1 + 4C_2)}. \)

The magnetic flux can be written as the sum of two terms,

\[ \Phi_e = \Phi_b + B \ell x, \]

where \( \Phi_b \) is the induced flux, corresponding to the equilibrium position of the \( NR \) and the second term describes the contribution due to the \( NR \) vibration; \( B \) represents the magnetic field created in the loop and \( \ell \) is the length of the \( NR. \) We write the displacement \( \hat{x} \) as \( \hat{x} = x_0 (\hat{a}^\dagger + \hat{a}), \)

where \( x_0 \) represents the \( NR \) amplitude oscillation. Substituting the Eq. (2) in Eq. (1) and controlling the flux \( \Phi_b \) we can adjust \( \cos \left( \frac{\pi \Phi_b}{\Phi_0} \right) = 0 \) to obtain

\[ \hat{H} = \omega_0 \hat{a} \hat{a}^\dagger + 4E_c \left( N_g - \frac{1}{2} \right) \hat{\sigma}_z - 4E_0 \cos \left( \frac{\pi \Phi_x}{\Phi_0} \right) \sin \left( \frac{\pi B \ell x}{\Phi_0} \right) \hat{\sigma}_x \]

Next, we consider a more general scenario, substituting \( \omega \to \omega(t) = \omega_0 + f(t), \)

\[ \lambda_0 \to \lambda(t) = \lambda_0 (1 + f(t)/\omega_0)^{1/2} \]

and \( \chi(t) = \chi_0 + \delta f(t) \) \( [23, 24]. \) In addition we consider the presence of the term \( \kappa(t) \) standing for the time-dependent loss affecting the \( CPB, \) the term \( \delta(t) \) being the same for the \( NR, \) and \( \chi(t) \) is the response time of the Kerr medium. This extended and somewhat realistic scenario requires the substitution of the Hamiltonian \( \hat{H}_{eff} \) by the Hamiltonian \( \hat{\mathcal{H}} \) given by

\[ \hat{\mathcal{H}} = \omega(t) \hat{a} \hat{a}^\dagger + \frac{1}{2} \omega_c(t) \hat{\sigma}_z + \lambda(t) \left( \hat{a} \hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_- \right) + \chi(t) \hat{a}^2 \hat{\sigma}_- - i \frac{\kappa(t)}{2} |1\rangle \langle 1| - i \frac{\delta(t)}{2} \hat{a} \hat{a}^\dagger. \]

\( \kappa(t) \) is the \( CPB \) decay coefficient from the excited level \( |1\rangle \) to the fundamental state \( |0\rangle, \) \( \delta(t) \) is the same for the \( NR \) and \( \lambda(t) \) is the time-dependent coupling between the \( CPB \) and \( NR. \) Actually, the inclusion of loss in the system turns its treatment somewhat more realistic since dissipation is ubiquitous in the real world; as consequence
FIG. 1. Model for the CPB-NR coupling.

the Eq. (5) corresponds to the evolution of a system described by a non-Hermitian Hamiltonian [25–29]. On the other hand, as the response time of the Kerr medium is assumed so fast the medium follows the NR adiabatically and the third-order nonlinear susceptibility can be modulated by the NR frequency \( \omega(t) \).

The wave function that describes our system can be written as,

\[
|\Psi(t)\rangle = \sum_{n} [C_{1,n}(t)|1,n\rangle + C_{0,n}(t)|0,n\rangle],
\]

where \( C_{1,n}(t) \) and \( C_{0,n}(t) \) are respectively the probability amplitudes of the states \( |1,n\rangle \) and \( |0,n\rangle \), namely, the CPB in its excited state \( |1\rangle \) or ground state \( |0\rangle \) with elementary excitations \( n \) in the NR. As mentioned before, at \( t = 0 \) the system is decoupled, the CPB initially in its excited state \( |1\rangle \) and the NR in a coherent state \( |\alpha\rangle \), given by

\[
|\alpha\rangle = \sum_{n=0}^{\infty} F_n |n\rangle.
\]

The total wave function in the initial state can be written as \( |\Psi(0)\rangle = \sum_{n=0}^{\infty} F_n |1,n\rangle \), with the initial amplitudes \( C_{0,n}(0) = 0 \) and \( \sum_{n=0}^{\infty} |C_{1,n}(0)|^2 = 1 \). The Schrödinger equation for the present system, described by non-Hermitian and time-dependent Hamiltonian in the Eq. (5) is,

\[
\frac{d|\Psi(t)\rangle}{dt} = -i\mathcal{H}|\Psi(t)\rangle.
\]

We obtain the following set of coupled equations of motion for the amplitude probabilities \( C_{1,n}(t) \) and \( C_{0,n+1}(t) \):

\[
\frac{\partial C_{1,n}(t)}{\partial t} = \left( -i\omega(t) - i\frac{\omega_c(t)}{2} - i\chi(t)(n^2 - n) - \frac{1}{2}(\kappa(t) + n\delta(t)) \right) C_{1,n}(t) - i\lambda(t)\sqrt{n+1}C_{0,n+1}(t),
\]

\[
\frac{\partial C_{0,n+1}(t)}{\partial t} = \left( -i(n+1)\omega(t) + i\frac{\omega_c(t)}{2} - i\chi(t)(n^2 + n) - \frac{1}{2}(n+1)\delta(t) \right) C_{0,n+1}(t) - i\lambda(t)\sqrt{n+1}C_{1,n}(t),
\]

whose solutions allow us to calculate the entropy of the NR subsystem and the population inversion of the CPB.

III. TIME EVOLUTION OF THE POPULATION INVERSION

The present approach also allows us to investigate the CPB dynamics in a non-perturbative way. A convenient
way to characterize the response to the NR influence is given by the CPB population inversion. This parameter is defined as

$$I(t) = \sum_{n=0}^{\infty} \left[ |C_{1,n}(t)|^2 - |C_{0,n+1}(t)|^2 \right].$$

To calculate this property \(I(t)\) we will assume the NR frequency varying with time as \(\omega(t) = \omega_0 + f(t)\). The third order nonlinear susceptibility is modulated as \(\chi(t) = \chi_0 + \epsilon f(t)\) and we also assume the NR initially in a coherent state with the mean excitation number \(\bar{n} = 25\) and \(\frac{\lambda_0}{\chi_0} = \frac{\epsilon}{\chi_0} = 20k\). We consider the time evolution of the population inversion for different values of the decay coefficients \(\kappa(t)\) and \(\delta(t)\).

A. Resonant case: \(f(t) = 0\)

If the model is restricted to the usual case, with the NR frequency constant, the Eqs. (9) and (10) can be solved numerically. In Figure (2 a) we plotted the inversion \(I(t)\) as a function of time. In absence of losses the average inversion, defined by its value during the collapse, is greater than zero, close to 0.5, when we consider the value \(\frac{\lambda_0}{\chi_0} = 0.2\). However, as well known, neglecting the presence of the Kerr medium \((\frac{\chi_0}{\lambda_0}) = 0\), the value of the average inversion vanishes.

Here the third-order nonlinear susceptibility represents the coupling of the NR and the Kerr medium. The higher the value of \(\chi\) the stronger the coupling of the NR with the Kerr medium, and reversely. Considering the case \(\frac{\chi_0}{\lambda_0} = 0.2\) and the system with no loss \((\kappa(t) = \delta(t) = 0)\) we observe the collapse-revival effect (see Fig. (2 a)). In this figure, the horizontal line crossing the value \(I(t) = 0.5\) of the average inversion is due to the presence of nonlinearity, whose absence turns the average inversion null. We note in this figure the amplitude of oscillation decreasing with time, due to the presence of the Kerr medium. However, when considering only the CPB loss \((\kappa(t) \neq 0, \delta(t) = 0)\) the collapse-revival effect reappears, with subsequent oscillations that smoothly vanish (see Fig. (2 b)). Contrarily, when considering only loss in the NR \((\kappa(t) = 0, \delta(t) \neq 0)\) these oscillations rapidly vanish. So the spoiling effect caused by the NR loss dominates that coming from the CPB loss, see Fig. (2 c) and (2 b).

B. Off-resonant case: \(f(t) = \tau \sin(\omega t)\)

When the system is non resonant, with \(\frac{\omega}{\lambda_0} = 10\) and \(\frac{\omega}{\chi_0} = 1\), and assuming only CPB loss \((\kappa(t) \neq 0, \delta(t) = 0)\), we observe periodic behavior of the population inversion, its amplitude decaying over time - see Fig. (3 a); in this case the collapse-revival effect is not observed. However, this behavior is modified when the loss comes from the NR; in this case the inversion and the oscillations are destroyed. Considering \(\omega\) increasing and only the

\[\text{FIG. 2. Time evolution of the population inversion for different values of the parameters } \kappa(t) \text{ and } \delta(t) \text{ for: } (n) = 25, \omega_0/\lambda_0 = \omega_\omega/\lambda_\omega = 20k, f(t) = 0; \chi_0/\lambda_0 = 0.2; (a) } \kappa/\lambda_\omega = 0.0 \text{ and } \delta/\lambda_\omega = 0.0; (b) } \kappa/\lambda_\omega = 0.01 \text{ and } \delta/\lambda_\omega = 0.0; (c) \kappa/\lambda_\omega = 0.01 \text{ and } \delta/\lambda_\omega = 0.01.\]

CPB loss - see Fig. (2 b), the system continues displaying the population inversion and oscillations, the latter exhibiting small periods, an effect not shown when the loss.
different values of the parameter $\kappa$ see Fig. (3 c).

Nowadays some devices are based on quantum mechanical phenomena, and this holds also for information transmission. For example, in optical communication a polarized photon can carry information. Now, the entropy of entanglement is defined for pure states as the von Neumann entropy of one of the reduced states, e.g., the NR entropy,

$$S_{NR}(t) = - \left\{ \pi_{NR}^+(t) \ln \pi_{NR}^+(t) + \pi_{NR}^-(t) \ln \pi_{NR}^-(t) \right\},$$

where $\pi_{NR}^\pm(t) = \frac{1}{2}[(C|C) + (S|S) \pm \frac{i}{2}((C|C) - (S|S))^2 + 4|C|S|^2]^{1/2}$ with $(C|C) = \sum_{n=0}^{\infty} |C_{0,n+1}(t)|^2$, $(S|S) = \sum_{n=0}^{\infty} |C_{0,n+1}(t)|^2$, and $(C|S) = \langle S|C \rangle^* = \sum_{n=0}^{\infty} C_{1,n+1}^*(t)C_{0,n+1}(t)$.

In our calculations using the Eq. (12) we will assume the NR frequency varying in the form $\omega(t) = \omega_0 + f(t)$. The third order nonlinear susceptibility is modulated in the form $\chi(t) = \chi_0 + \varepsilon f(t)$ and we also assume the initial NR in a coherent state with the mean number of photon $\bar{n} = 25$ and $\bar{n}_\omega = \bar{n}_\varepsilon = 20k$. Next we will consider the time evolution of the entropy for different values of the decay coefficients $\kappa(t)$ and $\delta(t)$.

### A. Resonant case: $f(t) = 0$

We will use the values of parameters in Fig. (4). In an ideal system the parameters $\kappa(t)$ and $\delta(t)$ are null, as assumed in Fig. (4 a): in this figure the maximum of the NR entropy is close to $\ln 2$; after the start of the interaction the NR entropy gradually tends to its minimum, then returns to its maximum and remains oscillating regularly.

For small values of decay in the CPB, as $\frac{\lambda}{\lambda_0} = 0.01$, and ideal NR ($\delta = 0$) for the case $\frac{\omega}{\lambda_0} = 0.2$, the maximum value of the entropy shows no significative changes for small times - see Fig. (4 b). For larger values of time the amplitude of the entropy oscillations decreases, maintaining the periodicity. If instead we add a small loss only in the NR, say $\frac{\delta}{\lambda_0} = 0.01$, the entropy oscillations vanishes rapidly (cf. Fig. (4 c)), showing the entropy being more sensitive to the loss in the NR than that in the CPB. For larger value of $\delta(t)$ and $\kappa(t)$ the entropy moves rapidly to zero, as expected, due to the passage of both subsystems to their respective ground states.

### B. Off-resonant case: $f(t) = \tau \sin(\omega t)$

Let us now consider the variation in the detuning parameter, where $\tau$ and $\omega$ are parameters that modulates the NR frequency. Our discussion is limited to the condition $\tau \ll \omega_c$, $\omega_0$ and also assuming that $\omega$ is small to avoid interaction of the CPB with other modes of the NR. We have chosen various values $\tau$ of amplitude modulations to verify the entanglement properties between the
FIG. 4. Time evolution of the entropy for different values of the parameters $\kappa(t)$ and $\delta(t)$ for: $\langle n \rangle = 25$, $\omega_c/\lambda_0 = \omega_0/\lambda_0 = 20k$, $\chi_0/\lambda_0 = 0.2$, $f(t) = 0$. (a) $\kappa/\lambda_0 = 0, 0.01$ and $\delta/\lambda_0 = 0.0$; (b) $\kappa/\lambda_0 = 0.01$ and $\delta/\lambda_0 = 0.0$; (c) $\kappa/\lambda_0 = 0.0$ and $\delta/\lambda_0 = 0.01$.

$CPB$ and NR. We also use various values $\omega'$ of frequency modulation to see its influence upon the $CPB-NR$ entanglement - see Figs. 5. Comparing the Fig. 4a with the Fig. 5a, we note that the entropy loses its periodical oscillations, and when $\omega'$ increases this behavior...
becomes more evident - see Fig. (5 b). Now, comparing the entropy in Fig. (4 c) with that in Fig. (5 c), keeping the same parameters, we see the entropy going to zero in both cases, but the Fig. (5 c) displays an interesting effect: even in presence losses, the maximum value of the entropy grows, reaching the value \( \ln 2 \), after which it oscillates downward. This effect came from the sinusoidal modulation of the \( NR \) frequency.

V. CONCLUSION

In the present work we have considered the interaction of a \( CPB \) and an \( NR \) in the presence of a Kerr medium and losses affecting both subsystems. Concerning the influence of the loss affecting the entropy both subsystems and the population inversion of the \( CPB \) we have observed the dominant role played by the \( NR \) upon that played by the \( CPB \). The dissipation causes deterioration of the \( CPB \) excited level whereas convenient modulations favors the control of certain properties of the system. It was also shown that certain choice of the time-dependent frequency makes higher the maximum value of the entropy, even in the presence of dissipation (see Figs. (4 c) and Fig. (5 c) the same occurring for the \( CPB \) population inversion - see Figs. (2 c) and Fig. (3 c). Concerning the entropy, this result is very important for information transmission since the transmission of maximum information through a quantum channel is exactly the von Neumann entropy \( S_B \). These effects are very sensitive to detuning (\( \approx 0.04\%) \) and disappear in resonant regime. The results suggest that it is possible to perform a dynamic control of certain properties of this system, via convenient manipulation of the parameters. We hope that these results might shed light in this scenario, furnishing new insights for researchers in this area.

VI. ACKNOWLEDGMENTS

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