Multiply generated dynamical systems and the
duality of higher rank graph algebras

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Abstract
We define a semidirect product groupoid of a system of partially defined
local homeomorphisms $T = (T_1, \ldots, T_r)$. We prove that this construction
gives rise to amenable groupoids. The associated algebra is a Cuntz-like algebra. We use this construction for higher rank graph algebras in order to give
a topological interpretation for the duality in $E$-theory between $C^*(\Lambda)$ and $C^*(\Lambda^{op})$.

Introduction
Toeplitz algebras have been used to define extensions of $C^*$-algebras. The beginning
of this paper is in [19] where a Toeplitz algebra was the main tool in constructing a
$K$-homology class for higher rank graph algebras. Let $(\Lambda, \sigma)$ be a higher rank graph
with shape $\sigma$ (see [16]), $\Lambda^*$ the set of morphism of nonzero shape and $\overline{\Lambda} = \{\Omega\} \cup \Lambda^*$
where $\Omega$ is a symbol (the vacuum morphism) which does not belong to $\Lambda^*$. We
define left and right creations on the Fock space $F_{\Lambda} = F = l^2(\overline{\Lambda})$:

$$L_\lambda \delta_\mu = \begin{cases} \delta_{\lambda \mu} & \text{if } s(\lambda) = t(\mu) \\ 0 & \text{otherwise} \end{cases}$$

$$R_\lambda \delta_\mu = \begin{cases} \delta_{\mu \lambda} & \text{if } s(\mu) = t(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

$$L_\lambda \Omega = R_\lambda \Omega = \delta_\lambda$$

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The left sided Toeplitz algebra is $E_L = C^*(L_\lambda; \lambda \in \Lambda^*)$ the right sided Toeplitz algebra is $E_r = C^*(R_\lambda; \lambda \in \Lambda^*)$ and the two sided Toeplitz algebra is $E = C^*(L_\lambda, R_\lambda; \lambda \in \Lambda^*)$. These algebras are well known in the rank one case since they give rise to short exact sequences. For example, one can use $E$ to define

$$0 \rightarrow K \rightarrow E \rightarrow O_E \otimes O_{E^\text{op}} \rightarrow 0$$

where $E$ is an ordinary oriented graph with certain conditions which give the uniqueness of the algebras $O_E$ and $O_{E^\text{op}}$. $E^\text{op}$ is the graph obtained by reversing the arrows. In the higher rank case the algebra $E$ has a more complicated ideal structure. It is this fact which gives a longer exact sequence. It would give a $KK$-class if the sequence was semisplit [19, 23]. This would follow immediately from the nuclearity of $E$. We tried to give a more conceptual construction of this algebra as a groupoid algebra of an amenable groupoid but we have not been able to do it. However, the construction of the semidirect product appearing in [21] can be generalized to give a groupoid description of one-sided Toeplitz algebras of a higher rank graph. We believe that this construction can be generalized to actions of other semigroups. For example actions of cones in $\mathbb{R}^n$ should give algebras generated by Wiener-Hopf operators (see [17, 18]).

The organization of this paper is as follows. In the first section we give the main construction of this paper, the semidirect product of an action of $\mathbb{N}^r$ by partial local homeomorphism which we also call a multiply generated dynamical system (MGDS). Such an action is obtained by choosing $r$ commuting partially defined local homeomorphisms. The main examples come from shifts on finite, semifinite and infinite paths of a higher rank graph. There are two groupoids associated to such an action, the groupoid of germs of the pseudogroup generated by these local homeomorphism and the semidirect product groupoid. We think that these groupoids have been around in the study of Toeplitz algebras of higher rank graphs as well as in the general theory of crossed products by partial actions (see for example [10, 11]). The semidirect product can be defined only if we have a condition on the domains of the partial maps. Otherwise it may not even give rise to an algebraic groupoid. This condition is fulfilled for one sided Toeplitz algebras associated to higher rank graphs. Following the lines in [21], we prove that these two groupoids are isomorphic if and only if the dynamical system is essentially free. In the general situation we can prove that the natural map from the semidirect product to the groupoid of germs induces a map of corresponding reduced $C^*$-algebras. Next we prove the amenability of the semidirect product. This is done by decomposing the action in subaction and then applying a result on the amenability of extensions.

In the second section we remind several facts about duality in a bivariant theory. We give the definition of Spanier-Whitehead duality and a condition which is often taken as definition in literature. We remind then a way to construct exact sequences starting from an algebra and a tuple of ideals. We show briefly how we can improve the results in [19] for any locally finite higher rank graph with finite set of objects, regardless the uniqueness of the generating relation of our graph algebras as for examples the graph $\mathbb{N}^r$. We obtain a duality for the universal $C^*$-algebras $C^*(\Lambda)$ and $C^*(\Lambda^\text{op})$.

In the last section we give a groupoid approach to the duality of higher rank graph algebras. The $K$-theory fundamental class is given by $r$ partial unitaries which can be best described in the groupoid picture as two-sided shifts. Even if we
do not have a conceptual groupoid approach to the two-sided Toeplitz algebra, we
can define it as a groupoid of germs. The main problem in understanding better
this groupoid is the unit space $X$ which is given by the spectrum of the diagonal
algebra $\mathcal{E} \cap l^\infty(\Lambda)$. The isometries $L^*_xR_\mu$ with $\sigma(\lambda) = \sigma(\mu) = e_j$ are not given by
something like shift equivalence. However, the partial isometries can be viewed as
partial homeomorphisms using Gelfand duality. The space $X$ has three properties
inherited from the graph $\Lambda$:

(i) the shape $\sigma$ extends to a map from $X$ to $\mathbb{N}$

(ii) the multiplication $\lambda\mu\nu$ extends to a multiplication $\lambda x \mu$

(iii) the unique factorization $\alpha = \lambda \alpha' \mu$ where $\sigma(\alpha) \geq \sigma(\lambda) + \sigma(\mu)$ extends to a
unique factorization $x = \lambda x' \mu$ where $\sigma(x) \geq \sigma(\lambda) + \sigma(\mu)$.

These properties are enough to define a MGDS given by shifts on a subspace of
$X \times \Lambda^\infty$ together with an equivariant map to the MGDS $\mathbb{N}^r \times \Lambda^\infty$ given again
by shifts. This map induces a map of the semidirect product groupoids which in
turn induces a morphism between the algebras $T^\otimes r \otimes C^*(\Lambda)$ and $\mathcal{E} \otimes C^*(\Lambda)$. This
morphism is the crucial step in the proof of the duality.

Finally, we want to draw the attention to the results in [8, 9]. Our groupoid
approach may give a hint to what a higher rank hyperbolic group would be.

1 Cuntz-like algebras associated with multiply
generated dynamical systems

Two partial maps on a set $X$, $L : dom(L) \to ran(L)$ and $R : dom(R) \to ran(R)$
can be composed if one defines

$$dom(LR) = \{ x \in dom(R); R(x) \in dom(L) \}$$

and $LR(x) = L(R(x))$ for any $x \in dom(LR)$. We say that two partial maps $L$ and $R$
on a set $X$ commute if $dom(LR) = dom(RL)$ and $LR = RL$ on $dom(LR)$. We set
$L^0 = id$ for any partial map on $X$. If $L$ is injective, we denote by $L^{-1}$ the partial
map $L^{-1} : ran(L) \to dom(L)$. A local homeomorphism is a map $\phi : X \to Y$ with
the property that each point $x$ has a neighborhood $U$ such that $\phi|_U : U \to \phi(U)$ is
a homeomorphism.

A partial homeomorphism on $X$ is a local homeomorphism $S$ from an open set
$dom(S)$ of $X$ onto an open set $ran(S)$ of $X$. A set $\mathcal{G}$ of partial homeomorphisms of
$X$ which is closed under composition, inversion and containing the identity is called
a pseudogroup ([2]). For any set $S$ of partial homeomorphisms on $X$, there exists the
smallest pseudogroup $[S]$ generated by $S$. The semi-direct product groupoid $X \rtimes \mathcal{G}$
of a pseudogroup $\mathcal{G}$ is the set of triples $(x, S, y)$ where $S \in \mathcal{G}, y \in dom(S), x = S(y)$
with the obvious operations

$$(x, S, y)(y, T, z) = (x, ST, y), (x, S, y)^{-1} = (y, S^{-1}, x).$$

The topology is given by the product topology of $X$ and $\mathcal{G}$ where $\mathcal{G}$ has the discrete
topology. The groupoid of germs is a quotient of the semidirect product groupoid by
the equivalence relation \((x_1, S_1, y_1) \sim (x_2, S_2, y_2)\) if and only if \(y_1 = y_2\) and \(S_1 = S_2\) on a neighborhood of \(y_1\). The topology is the quotient topology. These two groupoids are \(r\)-discrete, that is the range and source maps are local homeomorphisms.

In the monoid \(\mathbb{N}^r\) we write \(n = (n_1, \ldots, n_r)\) with \(n_j \in \mathbb{N}\) and \(e_k\) the coordinates \((0, \ldots, 1, \ldots, 0)\).

**Definition 1.1.** (Conform \([21]\)) A multiply generated dynamical system (MGDS) is a pair \((X, T)\) where \(X\) is a topological space and \(T = (T_1, \ldots, T_r)\) a system of \(r\) commuting partial homeomorphisms on \(X\).

**Examples 1.2.**

(i) ([11], Definition 5.1) For \(m \in \overline{\mathbb{N}}^r\) let \(\mathbb{N}^r_m\) be the higher rank graph \(\{(n, n') \in \mathbb{N}^r \times \mathbb{N}^r : n \leq n' \leq m\}\) where \(s(n, k) = k, t(n, k) = n, \sigma(n, k) = k - n\) and the composition is given by \((n, k)(k, p) = (n, p)\). Let \(\Lambda\) be a finitely aligned rank \(r\) graph and

\[X_\Lambda = \{x : \mathbb{N}^r_m \to \Lambda ; m \in \overline{\mathbb{N}}^r\}\]

the space of finite, semifinite and infinite paths. The shape \(\sigma\) can be extended to \(X_\Lambda\), \(\sigma(x) = m\) where \(x\) is defined on \(\mathbb{N}^r_m\). For \(r = 2\) an element in \(X_\Lambda\) can be seen graphically as one of the following:

A basis of a topology on \(X_\Lambda\) is given by the sets \(\{x : x(0, k) = \lambda, k \leq \sigma(x) \leq m\}\) where \(k \in \mathbb{N}^r_m, \lambda \in \Lambda\) and \(m \in \overline{\mathbb{N}}^r_m\). Then the partial homeomorphisms \(T_k\) with \(\text{dom}(T_k) = \{x \in X_\Lambda : \sigma(x) \geq k\}\), \(T_k(x) : \mathbb{N}^r_{\sigma(x)-k} \to \Lambda, T_k(n, n') = x(n + k, n' + k)\) give a MGDS.

(ii) In the example above the restriction to boundary paths \(\partial X\) gives a subsystem. If \(\Lambda\) has no sources then \(\partial X = \Lambda^\infty = \{x : \sigma(x) = (\infty \ldots \infty)\}\) (see [11] Definition 5.10 and [16])

(iii) Let \(X\) be the free monoid on \(r\) letters \(a_1, \ldots, a_r\), that is the set of words \(a_{i_1} \ldots a_{i_k}\). Put on \(X\) the discrete topology and define \(T = (L, R)\) the translations on the left and on the right: \(\text{dom}(L) = \text{dom}(R) = \{a_{i_1} \ldots a_{i_k} : k \geq 1\}\) (the set of nonvoid words), \(L(a_{i_1}a_{i_2} \ldots a_{i_k}) = a_{i_2} \ldots a_{i_k}, R(a_{i_1} \ldots a_{i_{k-1}}a_{i_k}) = a_{i_1} \ldots a_{i_{k-1}}\). For instance, if \(r = 1\) then \(X = \mathbb{N}\) and \(\text{dom}(L) = \text{dom}(R) = \mathbb{N} \setminus \{0\}\), \(L(k) = R(k) = k - 1\). It is clear that \(L\) and \(R\) commute.

We denote by \(\mathcal{G}(X, T)\) the full pseudogroup generated by the restrictions of \(T_j|_U\) where \(U\) is an open subset of \(X\) on which \(T_j\) is injective. Because of the commutation conditions on \(T\), we can define \(T^n = T_1^{n_1}T_2^{n_2} \cdots T_r^{n_r}\) for \(n \in \mathbb{N}^k\). One can see a MGDS as an action of the semigroup \(\mathbb{N}^r\) on \(X\) by partial local homeomorphisms. We write sometimes \(T^n\) instead of \(T^n\) when we want to view \(T\) as a semigroup. We need a technical condition on the domains of \(T^n\) (domain condition):

\[(DC) \quad \text{dom}(T^n) \cap \text{dom}(T^m) \subset \text{dom}(T^{n \vee m})\]
where \( n \lor m \) is the componentwise maximum. The following lemma is basically Lemma 2.4 from [21] for MGDS with (DC).

**Lemma 1.3.** Let \((X, T)\) be a MGDS with (DC)

(i) A partial homeomorphism \( S \) belongs to \( G(X, T) \) if and only if it is locally of the form \((T^m_{|U})^{-1}T^n_{|V}\) where \( m, n \in \mathbb{N}^r \), \( U \) is an open set on which \( T^m \) is injective and \( V \) is an open set on which \( T^n \) is injective.

(ii) Let \( a \in X \). Suppose that \((T^m_{|U})^{-1}T^n_{|V}\) and \((T^p_{|W})^{-1}T^q_{|Y}\) are two partial homeomorphisms as in (i) having \( a \) in their domains and \((T^m_{|U})^{-1}T^n_{|V}a = (T^p_{|W})^{-1}T^q_{|Y}a\). If \( m - n = p - q \), then \((T^m_{|U})^{-1}T^n_{|V}\) and \((T^p_{|W})^{-1}T^q_{|Y}\) have the same germ at \( a \).

**Proof.** It is clear that \((T^m_{|U})^{-1}T^n_{|V} \in G(X, T)\) and, since \( G(X, T) \) is full, it is still true for a partial homeomorphism locally of this form. The inverse of \((T^m_{|U})^{-1}T^n_{|V}\) is \((T^n_{|V})^{-1}T^m_{|U}\) which belongs to \( G(X, T) \). It remains to show that the product of \((T^m_{|U})^{-1}T^n_{|V}\) and \((T^p_{|W})^{-1}T^q_{|Y}\) is locally of the same form. When \( r = 1 \) this is the alternative \( n \geq p \) or \( n \leq p \) given in the proof of Lemma 2.4 of [21]. In our setting this alternative does not work since \( \mathbb{N}^r \) is not totally ordered. For this reason we need the condition (DC). Let \( x \in Y \) such that \((T^p_{|W})^{-1}T^q_{|Y}x = y \in V\). We can suppose that this happens in a neighborhood of \( x \) and so we assume that it is true on \( Y \). Then with \( z = (T^m_{|U})^{-1}T^n_{|V}y \) we have \( T^qx = T^py \) and \( T^ny = T^mz \). From condition (DC) we have \( y \in \text{dom}(T^{p
ot\lor q}) \) and \( T^{p + p
ot\lor q - p}x = T^{p
ot\lor q}y = T^{m + p
ot\lor q - n}z \) for any \( x \in Y \) so that \((T^m_{|U})^{-1}T^n_{|V}(T^p_{|W})^{-1}T^q_{|Y}\) is locally of the form \((T^m_{|Z})^{-1}T^n_{|Z}T^{p
ot\lor q - p}\).

(ii) Taking neighborhoods of \( a \) and \((T^m_{|U})^{-1}T^n_{|V}a\) we may assume that \( W = U \) and \( Y = V, T^p(U) = T^q(V) \) and \( T^{m\lor q} \) is injective on \( U \). If \( x \in U, (T^p_{|W})^{-1}T^q_{|Y}x = y' \in U \) then \( T^m=y \) and \( T^py' = T^qy \). As \( U \subset \text{dom}(T^m) \cap \text{dom}(T^p) \subset \text{dom}(T^{m\lor q}) \) we have \( T^{m\lor q}y = T^{m + m\lor q - m}x = T^{q + m\lor q - p}x = T^{m\lor q}y' \). Therefore \( y = y' \) so \((T^m_{|U})^{-1}T^n_{|V} = (T^p_{|W})^{-1}T^q_{|Y}\).

Having defined a pseudogroup, we denote by \( \text{Germ}(X, T) \) the groupoid of germs of \( G(X, T) \).

Following [21] Definition 2.5 we consider another groupoid, the semidirect product groupoid (simply replacing \( \mathbb{Z} \) with \( \mathbb{Z}^r \)):

**Definition 1.4.** Let \((X, T)\) be a MGDS with (DC). Its semidirect groupoid is

\[
G(X, T) = \{(x, m - n, y); m, n \in \mathbb{N}^r, x \in \text{dom}(T^m), y \in \text{dom}(T^n), T^m x = T^n y\}
\]

with the groupoid structure induced by the product structure of the trivial groupoid \( X \times X \) and of the group \( \mathbb{Z}^r \). The topology is defined by the basic open sets

\[
U(U; m, n; V) = \{(x, m - n, y); (x, y) \in U \times V, T^m x = T^n y\}
\]

where \( U \) (respectively \( V \)) is an open subset of the domain of \( T^m \) (respectively \( T^n \)) on which \( T^m \) (respectively \( T^n \)) is injective.

The family of given subsets is indeed a basis for a topology since

\[
U(U; m, n; V) \cap U(U'; m', n'; V') \supset U(U \cap U'; m \lor m', n \lor n'; V \lor V').
\]

Thus, \( \gamma = (x, z, y) \) and \( \eta = (x', z', y') \) in \( G(X, T) \) are composable if and only if \( y = x' \) and then \( \gamma \eta = (x, z + z', y') \). The range and domain are \( r(x, z, y) = x \) and \( d(x, z, y) = y \).
This is a groupoid indeed since \( y \in \text{dom}(T_{n+m}^{\eta}) \) so \( T_{n+n}^{m+n^m} x = T_{n+m}^{m+n^m} y' \). In the absence of the condition (DC), \( G(X,T) \) may not be a groupoid. Consider, for example, \( (X, L, R) \) as in example 1.2(iii). The condition (DC) is not satisfied since \( \text{dom}(L) \cap \text{dom}(R) \) is the set of nonvoid words, while \( \text{dom}(LR) \) is the set of words with length greater or equal to 2. Let \(| \cdot |\) be the word length on \( X \) and \( \Omega \) the empty word. For \( x, y \in X \) we have \( \gamma = (x, (|x| - |y|, y) \in G(X,T) \) and \( \eta = (y, (|y|, 0), \Omega \in G(X,T) \) since \( T^{(|x|,0)} x = L^{[x]} x = \Omega = R^{[y]} y = T^{(0,|y|)} \) and \( T^{(|y|,0)} y = L^{[y]} y = \Omega = T^0 \Omega \) but \( \gamma \eta = (x, (|x| + |y|, -|y|), \Omega) \notin G(X,T) \) since \( x \notin \text{dom}(T^{m}) \) for any \( n \in \mathbb{N}^2 \) with \( n_1 > |x| \).

We assume again that \((X,T)\) have (DC). According to (ii) of the previous lemma, there is a map \( \pi \) from \( G(X,T) \) onto \( \text{Germ}(X,T) \) which sends \((x, m - n, y)\) into the germ \([x, (T_{n}^{m})^{-1}T_{n}^{m}, y]\) where \( U \) is an open neighborhood of \( x \) on which \( T^m \) is injective and \( V \) is an open neighborhood of \( y \) on which \( T^m \) is injective. This map is continuous and is a groupoid homomorphism. It is an isomorphism when \((X,T)\) is essentially free.

**Definition 1.5.** (Conform Definition 2.6 [21]) We shall say that a MGDS \((X,T)\) is essentially free if for every pair of distinct \( m, n \in \mathbb{N}^r \), there is no open set on which \( T^m \) and \( T^n \) agree.

**Lemma 1.6.** (Conform Lemma 2.7 [21]) Let \((X,T)\) be an essentially free MGDS with (DC). Then:

(i) If \((T_{n}^{m})^{-1}T_{n}^{m} \) and \((T_{n}^{p})^{-1}T_{n}^{q} \), where \( m, n, p, q \in \mathbb{N} \) and \( U, V, W, Y \) are open sets such that \( T_{n}^{m}, T_{n}^{n}, T_{n}^{p}, T_{n}^{q} \) are injective and have the same germ at \( a \), then \( m - n = p - q \).

(ii) The map \( c : \text{Germ}(X,T) \rightarrow \mathbb{Z}^r \) such that \( c([T_{n}^{m}^{-1}T_{n}^{m}, x, (T_{n}^{m})^{-1}T_{n}^{m}, x] = m - n \) is a continuous homomorphism.

**Proof.** (i) By assumption, we have \( T^{m} y = T^{n} x \) and \( T^{p} y = T^{q} x \) for \( x \) and \( y \) in neighborhoods of \( a \) respectively \( b = (T_{n}^{m})^{-1}T_{n}^{m} a \). Then on these neighborhoods we have \( T^{n+m+n+m-n} x = T^{m+n+p-p} y \). The essential freeness implies that \( n + m + p - m = n + m + p - n \) so \( n - m = q - p \).

(ii) We have seen in the proof of the previous lemma that the product of \((T_{n}^{m})^{-1}T_{n}^{m} \) and \((T_{n}^{p})^{-1}T_{n}^{q} \) is locally of the form \((T_{n}^{m+n+p-n}T_{n}^{m+n+p-n})^{-1}T_{n}^{m+n+p-n} \) and that the inverse of \((T_{n}^{m})^{-1}T_{n}^{m} \) is \((T_{n}^{m})^{-1}T_{n}^{m} \). This shows that \( c \) is a homomorphism and by construction it is locally constant.

**Proposition 1.7.** Let \((X,T)\) be a MGDS with (DC). Then \((X,T)\) is essentially free if and only if the above surjection \( \pi : G(X,T) \rightarrow \text{Germ}(X,T) \) is an isomorphism.

**Proof.** Word with word as in Proposition 2.8 [21].

This homomorphism induces a morphism of algebras even if \((X,T)\) is not essentially free. We construct it in the lemma.

**Lemma 1.8.** Let \( \pi : G_1 \rightarrow G_2 \) a surjective morphism between two \( r \)-discrete groupoids with the property that if \( s(\pi(x)) = r(\pi(y)) \) then \( s(x) = r(y) \). Then the correspondence \( C_c(G_1) \ni f \rightarrow \tilde{\pi}(f) \in C_c(G_2) \), \( \tilde{\pi}(f)(x) = \sum_{\pi(x') = x} f(x') \), is a \( * \)-homomorphism between the topological algebras \( C_c(G_1) \) and \( C_c(G_2) \). Composing it with the regular representation of \( C_c(G_2) \) we obtain a bounded representation of \( C_c(G_1) \) therefore a morphism \( \tilde{\pi} : C^* G_1 \rightarrow C^* G_2 \).
Proof. The condition in the statement is $(\pi \times \pi)^{-1}(G_2^u) = G_1^u$. This means that composable pairs lifts only to composable pairs. In particular, the restriction $\pi^0$ of $\pi$ to the unit space must be a bijection. Moreover, $\pi$ satisfies this condition if $\pi^0$ is a bijection which identifies the equivalence relations associated to $G_1$ and $G_2$. In particular if $\pi(u) = r(x)$ then $\{x^\prime; \pi(x^\prime) = x\} \subset G_1^u$. Indeed, if $\pi(x^\prime) = \pi(y^\prime) = x$ then $\pi(y^{-1})$ and $\pi(x^\prime)$ are composable, so $y^{-1}$ and $x^\prime$ are composable, hence $r(x^\prime) = r(y^\prime)$. This means that $\pi(G_1^u) = G_2^{\pi^0(u)}$. If $f \in C_c(G_1)$ then $\{x^\prime; \pi(x^\prime) = x\} \subset supp(f) \cap G_1^u$ which is finite. Therefore the sum which defines $\tilde{\pi}(f)$ is finite. We have supp($\tilde{\pi}(f)$) $\subset \pi(supp(f))$ so $\tilde{\pi}(f) \in C_c(G_2)$.

We compute

$$\tilde{\pi}(f \ast g)(x) = \sum_{\pi(x^\prime) = x} f \ast g(x^\prime) = \sum_{\pi(y^\prime z^\prime) = x} f(y^\prime)g(z^\prime)$$

$$\tilde{\pi}(f) \ast \tilde{\pi}(g)(x) = \sum_{y^\prime z^\prime = x} \tilde{\pi}(f)(y)\tilde{\pi}(g)(z)$$

Since $\pi$ is surjective we have

$$\tilde{\pi}(f) \ast \tilde{\pi}(g)(x) = \sum_{\pi(y^\prime \pi(z^\prime) = x} f(y^\prime)g(z^\prime)$$

The first sum runs over $\{(y^\prime, z^\prime): \pi(y^\prime)\pi(z^\prime) = x\}$ and the second over sum runs over $\{(y^\prime, z^\prime): \pi(y^\prime z^\prime) = x\}$. These two sets are equal by the assumption on $\pi$ so $\tilde{\pi}$ is an algebraic morphism.

If $f_n \in C_c(G_1)$, supp$(f_n) \subset K$, $f_n \to u$ on $K$, then $\tilde{\pi}(f_n) \to u \tilde{\pi}(f)$ since the cardinal of the set $\{x^\prime; \pi(x^\prime) = x\}$ is bounded when $x$ runs over a compact set.

Finally, one has

$$\|\tilde{\pi}(f)\|_I = \sup_{u \in G_2^0} \sum_{r(x) = u} |\tilde{\pi}(f)(x)|$$

$$= \sup_{u \in G_2^0} \sum_{r(x) = u} |\sum_{\pi(x^\prime) = x} f(x^\prime)|$$

$$\leq \sup_{u \in G_2^0} \sum_{r(x) = u, \pi(x^\prime) = x} |f(x^\prime)|$$

Since $\pi(G_1^u) = G_2^{\pi^0(u)}$ and $\pi^0$ is a bijection, the last sum is

$$\sup_{u \in G_2^0} \sum_{r(x) = u} |f(x)| = \|f\|_I \geq \|\tilde{\pi}(f)\|_I$$

Therefore $\|f\|_I \geq \|\tilde{\pi}(f)\|_{B(L^2(G_2))}$ since the regular representation is bounded. □

**Proposition 1.9.** The Lemma 1.8 holds for $G_1 = G(X,T)$, $G_2 = Germ(X,T)$ and $\pi$ the morphism of Lemma 1.3.

**Proof.** $\pi$ is continuous and surjective. $\pi(x, z, y)$ and $\pi(x^\prime, z^\prime, y^\prime)$ are composable if and only if $y = x^\prime$ which is the condition for $(x, z, y)$ and $(x^\prime, z^\prime, y^\prime)$ to be composable. □
We assume from now on that $X$ is Hausdorff, second countable and locally compact. Then $G(X,T)$ becomes a Hausdorff, locally compact étale groupoid. In the next theorem we prove the amenability of $G(X,T)$. For the proof we need some notations. The motivation comes from the example 1.2. For $1 \leq j \leq r$ let $X_j = \bigcap_{i \in \mathbb{N}} \text{dom}(T_j^n)$ and for a subset $J \subset \{1, \ldots, r\}$ $(J$ may be void) we denote by $X_J = \cap_{j \in J} X_j \bigcap \cap_{j \notin J} (X \setminus X_j)$. Clearly $X_J \cap X_{J'} = \Phi$ for $J \neq J'$ and for $x \in X$ we have $x \in X_{J_x}$ where $J_x = \{j : x \in X_j\}$ so the subsets $X_J$ provide a partition of $X$. When $r = 1$ we get the partition in the end of the proof of Proposition 2.9 (i) [21]. For $x \in X$ let

$$\sigma(x) = \sup\{n \in \mathbb{N}^r : x \in \text{dom}(T^n)\} \in \mathbb{N}^r$$

For the rank one case $\sigma(x)$ may be considered as the exit time of $T$, the first time when $x$ escapes the domain of $T$. In general $\sigma(x)_j$ can be thought of as the exit time of $T_j$ so we call $\sigma(x)$ the exit time of $T$. The crucial assumption (DC) ensures the existence of this supremum. Then $X_J = \{x \in X : \sigma(x)_j = \infty \text{ for } j \notin J\}$. For $n \in \mathbb{N}^r$ and $J = \{j : n_j = \infty\}$ we have $\sigma^{-1}(n) = \cap_{j \in J} X_j \bigcap \cap_{j \notin J} \text{dom}(T^n) \bigcap \text{dom}(T^n_{n+\epsilon}) \subset X_J$ which is Borel and analytic so that a preimage by $\sigma$ is analytic. Note that from the condition (DC) we have $\cap_{j \notin J} \text{dom}(T^n) \bigcap \text{dom}(T^n_{n+\epsilon}) = \cap_{j \notin J} \text{dom}(T^n_{n+1}) \bigcap \text{dom}(T^n_{n+1})$. Denote by $\mathbb{Z}^J$ the subgroup of $\mathbb{Z}^r$ given by the inclusion $\mathbb{Z}^{J} \ni z \mapsto \sum_{j \in J} z_j e_j \in \mathbb{Z}^r$ and, for $z \in \mathbb{Z}^r$, let $z_J = \sum_{j \in J} z_j e_j \in \mathbb{Z}^J$. We use a similar notation for $N^J$ and $n_J$.

**Lemma 1.10.** (i) For $x, y \in X$ and $n \in \mathbb{N}^r$ we have $\sigma(T^n x) = \sigma(x) - n$.
(ii) For $(x, z, y) \in G(X,T)|_{X_J}$ we have $z_J = \sigma(x)_J - \sigma(y)_J$ where $J^c = \{1, \ldots, r\} \setminus J$.

**Proof.** (i) By definition of $\sigma$ we have $\sigma(T^n x) = \sup\{m \in \mathbb{N}^r : x \in \text{dom}(T^{m+n})\} = \sup\{m - n \in \mathbb{N}^r : m \geq n, x \in \text{dom}(T^m)\} = \sup\{m \in \mathbb{N}^r : m \geq n, x \in \text{dom}(T^m)\} - n = \sigma(x) - n$.
(ii) By definition of $G(X,T)$ there exist $n, m \in \mathbb{N}^r$ such that $T^n x = T^m y$ so by (i) we have $\sigma(T^n x) = \sigma(x) - n = \sigma(T^m y) = \sigma(y) - m$. We can substract the finite $J^c$ coordinates to get $z_J = n_J - m_J = \sigma(x)_J - \sigma(y)_J$. \hfill $\square$

The following lemma is very likely folklore:

**Lemma 1.11.** Let $G$ be a Borel groupoid, $(X_j)_{j \in J}$, $J$ a finite set, a partition of $G^0$ by invariant Borel sets. Then $G$ is measurewise amenable if and only if $G|_{X_J}$ is measurewise amenable.

**Proof.** This follows directly form the definition of amenability ([1] Definition. 3.2.8). Precisely, we denote by $\sqcup$ the disjoint union of Borel set with the obvious Borel structure. We know that $G^0 = \sqcup X_j$, $G = \sqcup G|_{X_j}$ so $L^\infty(G) = \oplus L^\infty(G|_{X_j})$ and $L^\infty(G^0) = \oplus L^\infty(X_j)$. Therefore a mean $m : L^\infty(G) \to L^\infty(G^0)$ is invariant if and only if the restrictions $m_J : L^\infty(G|_{X_j}) \to L^\infty(X_j)$ are invariant. Conversely, we can piece $m_J$ together to define $m = \oplus m_J$ an invariant mean $L^\infty(G) \to L^\infty(G^0)$. \hfill $\square$

**Theorem 1.12.** (Conform Prop.2.9 [21]) Let $(X, T)$ be a MGDS with (DC). Then, i) $G(X,T)$ is amenable;

ii) the full and reduced $C^*$-algebras coincide;

iii) the $C^*$-algebra $C^*(X, T) = C^*(G(X, T))$ is nuclear;
Proof. (i) We will check measurewise amenability (according to [1], 3.3.7, it is equivalent to topological amenability for étale groupoids). Each $X_J$ is an invariant Borel set for $G(X,T)$. By Lemma 1.11 it is enough to prove that the reduction of $G(X,T)$ on $X_J$ is amenable. By the definition of $X_J$, the homomorphism $c_J : G(X,T)|_{X_J} \to \mathbb{Z}^J$ defined by $c_J(x,z,y) = z_J$ is strongly surjective in the sense given in [1] Definition 5.3.7. We shall show that $R_J = c_J^{-1}(0)$ is an amenable equivalence relation.

Once this proven, we can now apply a result on the amenability of an extension ([1], 5.3.14) to conclude that $G(X,T)|_{X_J}$ is amenable, hence $G(X,T)$ amenable. We have

$$R_J = \{(x,m,n,y) : x,y \in X_J, \exists n,m \in \mathbb{N}, \text{ with } m_J = n_J, x \in \text{dom}(T^n), y \in \text{dom}(T^m), T^m(x) = T^n(y)\}.$$ 

If $(x,z,y) \in R_J$ we have $z_J = 0$ and $z_J = \sigma(x,y) - \sigma(y)$ (Conform Lemma 1.10(ii)) so $z$ depends only on $x$ and $y$ and $R_J \subset X_J \times X_J$. We leave $z$ out when we have such an element. We show that $R_J$ is an equivalence relation. If $(x,y), (y,z) \in R_J$ then we can find $n, n', m, m' \in \mathbb{N}$ such that $n_J = n'_J$, $T^n x = T^{n'} y$, $m_J = m'_J$, $T^m y = T^{m'} z$.

As we have seen before, we can assume $n_J = \sigma(x)_{Jc}$, $n'_J = \sigma(y)_{Jc}$, $m_J = \sigma(y)_{Jc}$, $m'_J = \sigma(z)_{Jc}$. (DC) gives again $T^{n_J \vee n'_J + \sigma(x)} x = T^{n_J \vee n'_J + \sigma(y)} y = T^{n_J \vee n'_J + \sigma(z)} z$ and then $(x,z) \in R_J$ which proves that $R_J$ is an equivalence relation.

We shall show that $R_J$ is an inductive limit of amenable equivalence relations.

For $N \in \mathbb{N}^J$ let

$$R_J^N = \{(x,y) \in R_J : \exists n,m \in \mathbb{N} \text{ such that } n_J = m_J \leq N, T^n(x) = T^m(y)\}.$$ 

In view of Lemma 1.10(ii), we can choose $n, m$ such that $n_J = \sigma(x)_{Jc}$ and $m_J = \sigma(y)_{Jc}$ so $R_J^N$ can be described as

$$R_J^N = \{(x,y) \in R_J : \exists n \in \mathbb{N}^J, n \leq N, \text{ such that } T^{n+\sigma(x)}(x) = T^{n+\sigma(y)}(y)\}.$$ 

$R_J^N$ is an equivalence relation on $X_J$ since we have seen before that $T^{n_J \vee n'_J + \sigma(z)} x = T^{n_J \vee n'_J + \sigma(z)} c$ for $(x,y,m,n), (y,z,p,q)$ defining two elements in $R_J^N$ as above. $R_J = \bigcup_{N \in \mathbb{N}^J} R_J^N$, $(R_J^N)^0 = X_J = (R_J^0)^0$ and $R_J^N = R_J^{N+1}$. We shall show that $R_J^N$ is a proper equivalence relation. This ensures that $R_J$ is the inductive limit of $R_J^N$ in the sense of [1] Chapter 5.3.f. Then Prop. 5.3.37 of [1] gives the amenability of $R_J$.

Since $R_J^N$ is a discrete equivalence relation we have to show that the space $X_J/R_J^N$ is analytic (Conform [1] Example 2.1.4(2)). According to [3] Corollary 4.12, this follows from the countable separability of $X_J/R_J^N$. Since $\sigma$ is a Borel map, we can find a sequence $Y_i$ of Borel subsets of $X_J$ such that $Y_i \subset \{x \in X_J : \sigma(x)_{Jc} = k\}$ for some $k \in \mathbb{N}^J$ and the restriction of $T^n$ to $Y_i$ is injective. We can suppose furthermore that $(Y_i)_i$ is separating for $X_J$ and is closed under finite intersections.

The saturation of a Borel set $A \subset X_J$ is $[A] = \bigcup_{n \in \mathbb{N}^J, n \leq N}(T^{n+\sigma})^{-1} (T^{n+\sigma}(A))$ which is a Borel set. We use here the notation $T^{n+\sigma}(\cdot)$ for the Borel map $x \mapsto T^{n+\sigma(x)}(x)$. Let $(x,y) \notin R_J^N, x \in Y_i$. We prove that we can choose $Y_j$ such that $y \notin [Y_j]$ so $\pi(Y_j)$ separates $\pi(x)$ and $\pi(y)$ where $\pi$ is the projection from $X_J$ to $X_J/R_J^N$. If $y \notin Y_i$ we are done. If $y \in [Y_i]$ there exists $y' \in Y_i$ such that $(y,y') \in R_J^N$.

Since the family $(Y_i)_i$ separates $X_J$ and is closed under finite intersections, we can find another set $Y_j \subset Y_i$ which separates $x$ and $y'$, that is $x \in Y_j$ and $y' \notin Y_j$. If $y \notin [Y_j]$ we are back to the case $y \notin [Y_i]$. If $y \in [Y_j]$ then $(y,y') \in R_J^N$ for some $y'' \in Y_j \subset Y_i$ so $(y'',y'') \in R_J^N$. This means that there exists $n \in \mathbb{N}^J, n \leq N$ such
that \( T^{n+\sigma(y')}_{C}(y') = T^{n+\sigma(y'')}_{C}(y'') \). Since \( y', y'' \in Y_i \) we have \( \sigma(y')_{C} = \sigma(y'')_{C} \). \( T^{n+\sigma(y')}_{C} \) is injective on \( Y_i \) so \( y' = y'' \). This contradicts the choice of \( Y_j \) (\( y'_j \notin Y_j \)). Therefore, \( y \notin [Y_j] \).

Corollary 1.13. There is a morphism \( \tilde{\pi} : C_{r}(G(X,T)) \to C_{r}(\text{Germ}(X,T)) \) induced by the canonical morphism \( \pi : G(X,T) \to \text{Germ}(X,T) \).

Proof. Conform Proposition 1.9 there is a morphism \( C_{r}^{*}(G(X,T)) = C^{*}(G(X,T)) \to C_{r}^{*}(\text{Germ}(X,T)) \).

2 Duality in bivariant theories

K-theory and K-homology are particular cases of the Kasparov groups. The product in KK-theory provides the analogue of the cup and cap products in algebraic topology. Therefore KK-theory and other bivariant theories are the framework for notions of duality. KK-theory is suitable when geometric classes are available Ext-theory was used in [14, 8, 9]. The novelty in [19] is to use Ext' but a technical condition of semisplintness leads to E-theory. We give first a general notion of duality analogous with the Spanier-Whitehead duality in algebraic topology. Then we give a condition which implies this duality. It appeared for the first time in [15] Chapter 4, Th. 6, but was highlighted by Connes in [6] Chapter VI.4.3. After that we recall the construction of long exact sequences starting from an algebra and a tuple of ideals.

Let \( F \) be any of the bivariant theories: KK-theory, E-theory, Ext-theory or KExt-theory. For \( x \in F(A_{1} \otimes \ldots \otimes A_{n}, B_{1} \otimes \ldots \otimes B_{m}) \) we define the flip maps \( \sigma_{ij}(x) \) and \( \sigma^{ij}(x) \) the induced maps on F-groups by the flip homomorphisms of \( C^{*} \)-algebras \( A_{1} \otimes \ldots \otimes A_{i} \otimes \ldots \otimes A_{j} \otimes \ldots \otimes A_{n} \to A_{1} \otimes \ldots \otimes A_{j} \otimes \ldots \otimes A_{i} \otimes \ldots \otimes A_{n} \) respectively.

Definition 2.1. (Spanier-Whitehead duality, [14], Definition. 2.2) Let \( A, B \) be separable \( C^{*} \)-algebras, \( \Delta \in F^{*}(A \otimes B, \mathbb{C}), \delta \in F^{*}(\mathbb{C}, A \otimes B) \). We say that \( A \) and \( B \) are Spanier-Whitehead r-dual in F-theory if the maps \( \Delta_{i} : K_{i}(A) \to K^{i+r}(B), \Delta_{i}(x) = x \otimes_{A} \Delta \) and \( \sigma^{12}(\Delta)_{i} : K_{i}(B) \to K^{i+r}(A), \sigma^{12}(\Delta)_{i}(x) = x \otimes_{B} \Delta \) are isomorphisms with inverses \( \delta_{i} : K^{i+r}(B) \to K_{i}(A), \delta_{i}(y) = \delta \otimes_{B} y \) respectively \( \sigma_{12}(\delta)_{i} : K^{i+r}(A) \to K_{i}(B), \sigma_{12}(\delta)_{i}(y) = \sigma_{12}(\delta) \otimes_{A} y \).

The duality classes, \( \Delta \) and \( \delta \), are the K-homology and the K-theory classes.

Theorem 2.2. ([9] Theorem 11) Let \( A, B, \Delta, \delta \) be \( C^{*} \)-algebras and classes as above such that the following equations hold:

\[
\delta \otimes_{B} \sigma^{12}(\Delta) = [1_{A}]
\]

\[
\sigma_{12}(\delta) \otimes_{A} \Delta = [(-1)^{i}1_{B}]
\]

Then the maps \( \Delta_{i} \) and \( \delta_{i} \) defined in the above definition are isomorphisms.

The proof can be found in [9]. We shall call such algebras simply r-dual. The sign in the second condition comes from the change of sign under flip maps \( \sigma \). The sign is not given in [15] and [6]. We do not know of any example of two dual \( C^{*} \)-algebras such that the assumptions of Th. 2.2 are not fulfilled. If \( B \) is the same as
A we say that $A$ is a Poincaré duality algebra. The dual is not unique since if $A$ has a dual $B$ and $A$ is $KK$-equivalent to $A'$ then $A'$ is also dual with $B$.

Bivariant theories work well when the algebra in the first argument is separable. For $KK$-theory this is forced by the Kasparov technical theorem. Since in the definition of duality both algebras appear in the first argument we are forced to work with separable algebras. For a separable algebra the $K$-theory group is countable. For this reason not every algebra has a dual. For example $M_{2\infty}$, the CAR algebra, does not have a dual. If there was a dual $B$ for $M_{2\infty}$ we would have $K_i(B) = K^i(M_{2\infty})$ where $i$ is 0 or 1. But $K^i(M_{2\infty})$ is $\mathbb{Z}(2)/\mathbb{Z}$ ([12]), where $\mathbb{Z}(2)$ is the group of 2-adic numbers, which is uncountable. Therefore $B$ cannot be separable.

In [19] the $K$-homology class was given as an $E$-theory class associated to a long exact sequences. We recall the construction of long exact sequences starting from an algebra and a tuple of ideals.

Let $\mathcal{E}$ be an arbitrary $C^*$-algebra and $J_1, \ldots, J_r$ be ideals in it. We shall describe next a procedure of defining an $r$-fold exact sequence starting with $B = \bigcap_{i=1}^r J_i$ and ending with $A = \mathcal{E}/J_1 + \ldots + J_r$. The motivation comes from the Zekri's Yoneda product ([23]) $\epsilon_1 \otimes_\mathbb{C} \epsilon_2$ of two 1-fold extensions $\epsilon_1 \in \text{Ext}(A_1, B_1)$ and $\epsilon_2 \in \text{Ext}(A_2, B_2)$. Let

$$0 \to B_1 \to E_1 \to A_1 \to 0$$

$$0 \to B_2 \to E_2 \to A_2 \to 0$$

be the extensions $\epsilon_1$ and $\epsilon_2$. Forgetting for the moment the troubles caused by tensor products, the product $\epsilon_1 \otimes_\mathbb{C} \epsilon_2$ is computed by tensoring $\epsilon_1$ with $B_2$ on the right, $\epsilon_2$ with $A_1$ on the left and then splicing:

$$0 \to B_1 \otimes B_2 \to E_1 \otimes B_2 \to A_1 \otimes E_2 \to A_1 \otimes A_2 \to 0$$

Define $\mathcal{E} = E_1 \otimes E_2$, $J_1 = B_1 \otimes E_2$, $J_2 = E_1 \otimes B_2$. We have

$$B_1 \otimes B_2 = J_1 \cap J_2 = J_1J_2$$

$$E_1 \otimes B_2 = J_2$$

$$A_1 \otimes E_2 = \mathcal{E}/J_1$$

$$A_1 \otimes A_2 = \mathcal{E}/J_1 + J_2$$

so the exact sequence is

$$0 \to J_1 \cap J_2 \to J_2 \to \mathcal{E}/J_1 \to \mathcal{E}/J_1 + J_2 \to 0$$

For a nonempty subset $S \subseteq \{1, \ldots, r\}$ let us define

$$J^S = \bigcap_{j \in S} J_j \quad \text{and} \quad J_S = \sum_{j \in S} J_j.$$  

We shall use the abbreviation $\{k, k+1, \ldots, l\} = \overline{k,l}$. Define

$$\mathcal{E}_0 = J_1 \cap \ldots \cap J_r = J^{\overline{1,r}}$$

$$\mathcal{E}_1 = J_2 \cap \ldots \cap J_r = J^{\overline{2,r}}$$

$$\mathcal{E}_k = J_{k+1} \cap \ldots \cap J_r / (J_1 + \ldots + J_{k-1}) \cap J_{k+1} \cap \ldots \cap J_r = J^{\overline{k+1,r}} / \overline{J_{1,k-1} \cap J^{k+1,r}},$$
for any \( k \in \{2, \ldots, r-1\} \) and
\[
\mathcal{E}_k = \mathcal{E}/J_1 + \ldots + J_{r-1} = \mathcal{E}/J_{1;r-1}
\]
\[
\mathcal{E}_{r+1} = \mathcal{E}/J_1 + \ldots + J_r = \mathcal{E}/J_{1;r}.
\]
\( \mathcal{E}_0 \subseteq \mathcal{E}_1 \) so we can define \( i_0 : \mathcal{E}_0 \hookrightarrow \mathcal{E}_1 \). Since \( \mathcal{E}_1 \subseteq J_3 \cap \ldots \cap J_r \), there is a map \( i_1 : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) given by the inclusion composed with the quotient map. For \( k \in \{2, \ldots, r-2\} \) we have
\[
J_{k+1;r}^{k+1} = J_{k+1} \cap \ldots \cap J_r \subseteq J_{k+2} \cap \ldots \cap J_r = J_{k+2;r}
\]
and
\[
J_{1;k-1} \cap J_{k+1;r}^{k+1} \subseteq J_{1;k} \cap J_{k+2;r}
\]
so that we can again define a map \( i_k : \mathcal{E}_k \rightarrow \mathcal{E}_{k+1} \) as the bottom line of the diagram
\[
\begin{array}{ccc}
J_{k+1;r}^{k+1} & \rightarrow & J_{k+2;r} \\
\pi_k & & \pi_{k+1} \\
\mathcal{E}_k & \rightarrow & \mathcal{E}_{k+1}
\end{array}
\]
where the vertical arrows are projections. Using the isomorphism \( J/J \cap I \cong J + I/I \) we write
\[
\mathcal{E}_{r-1} = J_r/J_{1;r-2} \cap J_r = J_1 + \ldots + J_{r-2} + J_r/J_1 + \ldots + J_{r-2}
\]
Therefore, there is also a natural homomorphism \( i_{r-1} : \mathcal{E}_{r-1} \rightarrow \mathcal{E}_r \) since \( J_1 + \ldots + J_{r-2} + J_r \subseteq \mathcal{E} \) and \( J_1 + \ldots + J_{r-2} \subseteq J_1 + \ldots + J_{r-1} \). Finally \( i_r : \mathcal{E}_r \rightarrow \mathcal{E}_{r+1} \) is defined since \( J_1 + \ldots + J_{r-1} \subseteq J_1 + \ldots + J_r \).

**Theorem 2.3.** ([19], Prop. 3.1) The \( r \)-fold sequence
\[
0 \rightarrow B \xrightarrow{i_0} \mathcal{E}_1 \xrightarrow{i_1} \cdots \xrightarrow{i_{r-1}} \mathcal{E}_r \xrightarrow{i_r} A \rightarrow 0
\]
is exact.

Such sequences define \( Ext^r \) classes provided they are semisplit. In [19] this technical condition was avoided using \( E \)-theory, that is thinking of such a sequence as an \( E^r \)-theory class. However, as a consequence of the fact that \( \text{Ext}^1(A, B) = \text{Ext}^1(A, B) \) is a group and the Yoneda product is bilinear, we can prove that \( \text{Ext}^r(A, B) \) is a group. Indeed, we can decompose any \( \epsilon \in \text{Ext}^1(A, B) \) as a Yoneda product \( \gamma(\epsilon_1, \epsilon_2) \) with \( \epsilon_1 \in \text{Ext}^1(A, A_1) \). Since \( \text{Ext}^1(A, A_1) \) is a group, we can find an extension, \( \epsilon' \in \text{Ext}^1(A, A_1) \) such that \( \epsilon_1 \oplus \epsilon' \) is a null (i.e. split) extension in \( \text{ext}^1(A, A_1) \). The inverse of \( \epsilon \) is now \( \gamma(\epsilon'_1, \epsilon_2) \). As a consequence, if \( A \) is nuclear then \( \epsilon \) is always semisplit so \( \gamma(\epsilon'_1, \epsilon_2) \) provides an inverse for \( \epsilon \) in \( \text{Ext}^1(A, B) \). Therefore \( \text{Ext}^r(A, B) \) is a group which contains \( \text{Ext}^r(A, B) \). We conjecture that they are equal if \( A \) is nuclear. More generally we can define a group if in the semisplitness condition we only require that \( \epsilon_r \), the rightmost 1-fold exact sequence, is semisplit. We believe
that this is equal to $\text{Ext}^r(A, B)$. In any case it factors through $KK$ as a consequence of abstract characterizations ([7]).

For example if $G$ is a groupoid and $X_1, \ldots, X_r$ are $r$ open invariant subsets of $G^0$ we get a tuple of ideals of $C^*_r(G) \left( C^*_r(G|X_1), C^*_r(G|X_2), \ldots C^*_r(G|X_r) \right)$. The algebras $\mathcal{E}_k$ corresponds to $C^*_r(G|Y_k)$ where

$$Y_0 = X_1 \cap \ldots \cap X_r$$

$$Y_1 = X_2 \cap \ldots \cap X_r$$

$$Y_k = (X_{k+1} \cap \ldots \cap X_r) \setminus (X_1 \cup \ldots \cup X_{k-1})$$

$$Y_r = X \setminus (X_1 \cup \ldots \cup X_{r-1})$$

$$Y_{r+1} = X \setminus (X_1 \cup \ldots \cup X_r)$$

In particular, if $(X, T)$ is a MGDS with (DC) condition and $X_j$ are the set used in the proof of the amenability of $G(X, T)$ we have

$$Y_k = \left\{ x \in X : \sigma(x)_j < \infty \text{ for } j \geq k + 1 \text{ and } \sigma(x)_j = \infty \text{ for } j \leq k - 1 \right\}$$

One can improve the result of [19] for any higher rank graph $\Lambda$ with the following finiteness condition:

$$(F): \text{The set } \{ \lambda : \sigma(\lambda) = n \} \text{ is nonvoid and finite for any } n \in \mathbb{N}^r$$

such that $0 < \# \Lambda_n(v) < \infty \text{ and } 0 < \# \Lambda^n(v) < \infty$ where $n \in \mathbb{N}^r$, $v \in \Lambda^0$. We shall give up the aperiodicity condition and replace the Toeplitz algebra $\mathcal{E}$ with the subalgebra $\tilde{\mathcal{E}}$ of $\mathcal{E} \oplus C^*(\Lambda) \otimes C^*(\Lambda^o)$ generated by $\tilde{L}_\lambda = L_\lambda \oplus (s_\lambda \otimes 1)$ and $\tilde{R}_\lambda = R_\lambda \oplus (1 \otimes t_\lambda)$. We refer to $\mathcal{E}$ as the first summand of $\tilde{\mathcal{E}}$ and to $C^*(\Lambda) \otimes C^*(\Lambda^o)$ as to the second summand. This trick will give rise to a kind of pull-back for long exact sequences. The constructions are the same as in [19] with $O_\Lambda$ replaced by $C^*(\Lambda)$ and $O_{\Lambda^o}$ replaced by $C^*(\Lambda^o)$. In this way we avoid aperiodicity conditions.

The closed linear span of a set $S$ in a normed linear space is denoted by $[S]$ and the projection onto a closed subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$ by $P_\mathcal{L}$. For any $j \in \{1, \ldots, r\}$ and $a \in \Lambda^o$ we define

$$P_a = P_{[\Omega, \delta_\lambda|o(\lambda)=a]} \quad P_a^j = P_{[\Omega, \delta_\lambda|o(\lambda)=a \text{ and } \sigma(\lambda)_j=0]} \quad Q_a = P_{[\Omega, \delta_\lambda|t(\lambda)=a]} \quad Q_a^j = P_{[\Omega, \delta_\lambda|t(\lambda)=a \text{ and } \sigma(\lambda)_j=0]}$$

$$P^j = P_{[\Omega, \delta_\lambda|\sigma(\lambda)_j=0]}$$

Define $L_\Omega = 1 = R_\Omega$ and for $a \in \Lambda^o$ let $L_a$ be the projection $P_a$ and $R_a$ the projection $Q_a$. It is easy to check that for any $\mu \in \Lambda^*$, $j \in \{1, \ldots, r\}$ and $a \in \Lambda^o$ we have

$$L^*_\mu L_\mu = P_{t(\mu)}, \quad P_a = L_a = \sum_{t(\lambda)=a; \sigma(\lambda)=e_j} L_\lambda \quad P_a^j = \sum_{t(\lambda)=a; \sigma(\lambda)=e_j} L_\lambda^* \quad P_a^j,$$

$$R^*_\mu R_\mu = Q_{t(\mu)}, \quad Q_a = R_a = \sum_{t(\lambda)=a; \sigma(\lambda)=e_j} R_\lambda \quad R_a^* = \sum_{t(\lambda)=a; \sigma(\lambda)=e_j} R_\lambda^* \quad Q_a^j$$

Note also that

$$1 - \sum_{\sigma(\lambda)=e_j} L_\lambda \quad P_a^j = 1 - \sum_{\sigma(\lambda)=e_j} R_\lambda^*$$
and for all \( k \in \mathbb{N}^r \) we have
\[
\sum_{\sigma(\lambda) = k} L_\lambda L_\lambda^* = \sum_{\sigma(\lambda) = k} R_\lambda R_\lambda^* = P_{[\delta_{\mu} | \sigma(\mu) \geq k]}.
\]

We have similar relations in the algebra \( \tilde{\mathcal{E}} \) if we replace \( L \) with \( \tilde{L} \) and \( R \) with \( \tilde{R} \):
\[
\tilde{L}_a = \sum_{t(\lambda) = a; \sigma(\lambda) = e_j} \tilde{L}_\lambda \tilde{L}_\lambda^* + P_a^j,
\]
\[
\tilde{R}_a = \sum_{t(\lambda) = a; \sigma(\lambda) = e_j} \tilde{R}_\lambda \tilde{R}_\lambda^* + Q_a^j.
\]
\[
1 - \sum_{\sigma(\lambda) = e_j} \tilde{L}_\lambda \tilde{L}_\lambda^* = P^j = 1 - \sum_{\sigma(\lambda) = e_j} \tilde{R}_\lambda \tilde{R}_\lambda^*
\]

As in [19] the Toeplitz algebra \( \mathcal{E} \) contains a tuple of \( r \) ideals \( (J_1, ..., J_r) \), \( J_j = \langle P^j \rangle \) the closed two-sided ideals generated by \( P^j \) in \( \mathcal{E} \). Note that the ideals \( J_k \) are in the first summand.

As in [19], Lemma 2.1 we have \( \bigcap_{j=1}^r J_j = \mathbb{K}(F) \) and as in [19] Theorem 2.2 there is a morphism \( \text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op}) \simeq \mathcal{E}/J_{1, ..., r} \) given by \( s_\lambda \otimes 1 \mapsto \tilde{L}_\lambda \) and \( 1 \otimes s_\lambda \mapsto \tilde{R}_\lambda \).

The pull back construction, more exactly the second summand, is used to prove the injectivity of the above isomorphism which used the uniqueness of the Cuntz-Krieger relations associated to the graph. Indeed, since \( J_{1, ..., r} \subset \pi_1(\mathcal{E}) \) we have the following diagram:
\[
\text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op}) \xrightarrow{id} \mathcal{E}/J_{1, ..., r} \xrightarrow{id} \text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op})
\]

where the vertical arrow is the projection onto the second summand: \( \mathcal{E}/J_{1, ..., r} \ni x \oplus y \mapsto y \in \text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op}) \). From the construction in section 2 we get a long exact sequence
\[
0 \rightarrow \mathbb{K}(F) \rightarrow \mathcal{E}_1 \rightarrow \ldots \rightarrow \mathcal{E}_r \rightarrow \text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op}) \rightarrow 0
\]

therefore an \( E \)-theory class \( \Delta \in E^*(\text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op}), \mathbb{C}) = K^*(\text{C}^*(\Lambda) \otimes \text{C}^*(\Lambda^{op})) \).

To compute the product \( \delta \otimes \text{C}^*(\Lambda^{op}) \sigma_{12}(\Delta) = \tau_{\text{C}^*(\Lambda)}(\delta) \otimes \tau_{\text{C}^*(\Lambda)}(\sigma_{12}(\Delta)) \), which is a class in \( E^0(\mathcal{S}^{\otimes r} \otimes \text{C}^*(\Lambda), \mathbb{S}^{\otimes r} \otimes \text{C}^*(\Lambda)) \), we find \( \Theta(\text{C}^*(\Lambda)) = \Theta \), \( \Theta(f)(z) = \gamma(z) \), \( \gamma \) the gauge action, to \( \mathcal{S}^{\otimes r} \otimes \text{C}^*(\Lambda) \). This is defined by restricting the morphism \( \Theta : \text{C}(\mathbb{T}^r) \otimes \text{C}^*(\Lambda) \rightarrow \text{C}(\mathbb{T}^r) \otimes \text{C}^*(\Lambda), \Theta(f)(z) = \gamma(z) \) where \( \gamma \) is the gauge action. Our product is a pull-back ([13] Prop 5.8), that is the top row of the following diagram:
\[
0 \rightarrow \text{C}^*(\Lambda) \otimes \mathcal{E}_0 \rightarrow \mathcal{E}_1' \rightarrow \ldots \rightarrow \mathcal{E}_r' \rightarrow 0
\]
\[
0 \rightarrow \text{C}^*(\Lambda) \otimes \mathcal{E}_0 \rightarrow \text{C}^*(\Lambda) \otimes \mathcal{E}_1 \rightarrow \ldots \rightarrow \text{C}^*(\Lambda) \otimes \mathcal{E}_r \rightarrow 0
\]
To show that it gives the same element with \( \tau_{C^*(\Lambda)}(T \otimes r) \in E^r(S^{\otimes r} \otimes C^*(\Lambda), C^*(\Lambda)) \) (which is \( 1_{C^*(\Lambda)} \in E(C^*(\Lambda), C^*(\Lambda)) \) by Bott periodicity) we shall identify a subsequence which gives \( \tau_{C^*(\Lambda)}(T \otimes r) \). This is done by identifying in \( C^*(\Lambda) \otimes \tilde{E} \) an image \( E'_0 \) of the algebra \( T \otimes r \otimes C^*(\Lambda) \) which preserves the canonical tuple of ideals of \( T \otimes r \otimes C^*(\Lambda) \). Therefore it gives a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C^*(\Lambda) \otimes E_0 & \rightarrow & E'_0 & \rightarrow & \cdots & \rightarrow & E'_r & \rightarrow \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & T_0 \otimes C^*(\Lambda) & \rightarrow & T_1 \otimes C^*(\Lambda) & \rightarrow & \cdots & \rightarrow & T_r \otimes C^*(\Lambda) & \rightarrow \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\rightarrow & & C(T^r) \otimes C^*(\Lambda) & \rightarrow & 0 & & & & & \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\rightarrow & & C(T^r) \otimes C^*(\Lambda) & \rightarrow & 0 & & & & & \\
\end{array}
\]

In addition, the leftmost vertical arrow gives an \( E \)-theoretic equivalence since the full corner \( C^*(\Lambda) \otimes P_\Omega \subset C^*(\Lambda) \otimes E_0 \) corresponds to the full corner \( P_0 \otimes C^*(\Lambda) \subset T_0 \otimes C^*(\Lambda) \) are. To construct the algebra \( E'_0 \) we define

\[
W_j = \sum_{\sigma(\lambda) = e_j} s^*_\lambda \otimes \tilde{R}_\lambda
\]

\[
V_\lambda = (W^\sigma(\Lambda))^* (1 \otimes \tilde{L}_\lambda)
\]

where for a tuple of \( r \) commuting operators \( x = (x_1, \ldots, x_r) \) and \( k \in \mathbb{N}^r \) we define \( x^k = x_1^{k_1} \cdots x_r^{k_r} \). One can prove as in [19] Proposition 6.4 that \( C^*(\Lambda) \cong C^*(V_\lambda | \lambda \in \Lambda^*) \), the isomorphism being given by \( s_\lambda \mapsto V_\lambda \). The pull-back construction replaces the uniqueness of \( O_\Lambda \) invoked in [19] Proposition 6.4 and gives the injectivity. The isometries \( (W_k)_k \) commute with each other and with \( V_\lambda \)'s. So we have a morphism \( T^{\otimes r} \otimes C^*(\Lambda) \rightarrow C^*(W_k, V_\lambda) \) which is the identity on the full corner \( P_\Omega \otimes C^*(\Lambda) \). All these statements follow exactly as in [19].

3 A Groupoid Picture of the duality for higher rank graph algebra

In this section we give this duality a groupoid interpretation. Let us start with the \( K \)-theory class. More precisely, from the groupoid picture of the algebra \( C^*(\Lambda) \otimes C^*(\Lambda) \) the partial unitaries \( w_k \) are two-sided shifts on the double infinite paths space. Indeed, conform [19], Proposition 4.1

\[
w_k w_k^* = w_k^* w_k = \sum_{a \in \Lambda^a} s_a \otimes t_a
\]
In the groupoid \( G(\Lambda^\infty, T) \times G((\Lambda^{op})^\infty, T^{op}) \), this projection corresponds to the set \( 1_{Z} \) where
\[
Z = \{(x, y) \in \Lambda^\infty \times (\Lambda^{op})^\infty; s(x) = s(y)\}
\]
The isometry \( w_k \) is given by the bisection
\[
\sum_{\sigma(\lambda) = e_k} 1_{\{(\lambda x, y), (e_k, -e_k), (x, \lambda y)\}}
\]
Identifying the set \( Z \) with the set of two-sided infinite words we can say that \( w_k \) is the two sided shift in the direction \( k \). The transformation group of the \( n \) shifts on \( Z, Z \rtimes \mathbb{Z}^n \) is an open subgroupoid of \( G(\Lambda^\infty, T) \times G((\Lambda^{op})^\infty, T^{op}) \). Note that here one can see clearly the necessity of the condition (F) on \( \Lambda \). Without it the space \( Z \) may not be locally compact or the two-sided shifts on the double infinite path space may not be well defined on \( Z \) (when the graph has sinks or sources). The morphism \( C(\mathbb{T}^r) \to C^*(\Lambda) \otimes C^*(\Lambda^{op}), z_k \mapsto w_k \) is given by the inclusion of \( C^*(\mathbb{Z}^r) \to C(\mathbb{Z}) \rtimes \mathbb{Z}^r \).

The twist morphism \( \Theta \) is given by the topological isomorphism of groupoids
\[
\mathbb{Z}^r \times G(\Lambda^\infty, T) \ni (z, x) \mapsto (z + b(x), x) \in \mathbb{Z}^r \times G(\Lambda^\infty, T)
\]
where \( b \) is the cocycle which gives the gauge action: \( b(x, z, y) = z \).

Identifying \( S^{\otimes r} \) with \( C^*(\mathbb{R}^r) \) the restriction of this morphism to \( S^{\otimes r} \otimes C^*(\Lambda) \) is given by
\[
\mathbb{R}^r \times G(\Lambda^\infty, T) \ni (t, x) \mapsto (t + b(x), x) \in \mathbb{R}^r \times G(\Lambda^\infty, T).
\]
The homotopy between this morphism and the identity is obtained by multiplying \( b(x) \) with a parameter \( s \in [0, 1] \).

To give a groupoid description of the \( K \)-homology class it would be ideal to have a conceptual groupoid description (like semidirect product) of the two-sided Toeplitz algebra \( \mathcal{E} \). We have not been able to do it but we can still use a groupoid of germs. Let \( X_l, X_r, X \) be the Gelfand spectrum of the commutative unital algebras \( \mathcal{E} \cap l^\infty(\mathbf{A}) \), \( \mathcal{E} \cap l^\infty(\mathbf{A}) \) respectively \( \mathcal{E} \cap l^\infty(\mathbf{A}) \) where \( l^\infty(\Lambda) \subset L(\mathbf{A}) \) is the algebra of diagonal operators. There is an isomorphism from \( L_\lambda^* L_\lambda C(X) L_\lambda^* L_\lambda \) to \( L_\lambda L_\lambda^* C(X) L_\lambda L_\lambda^* \) given by \( f \mapsto L_\lambda f L_\lambda^* \). Therefore the isometries \( \{L_\lambda : \lambda \in \Lambda\} \) give rise by Gelfand duality to partial homeomorphisms of \( X_l \), so they generate a pseudogroup \( \mathcal{G}_l \). Similarly, \( \{R_\lambda : \lambda \in \Lambda\} \) give rise to partial homeomorphisms of \( X_r \). The set of partial isometries \( \{L_\lambda, R_\lambda : \lambda \in \Lambda\} \) gives also partial homeomorphisms \( \{l_\lambda, r_\lambda : \lambda \in \Lambda\} \) of \( X \) which generate a pseudogroup \( \mathcal{G} \). We denote by \( G_l, G_r, G \) the corresponding groupoids of germs. It is important to have in mind that the maps \( l_\lambda \) and \( r_\lambda \) extend the maps \( s(\lambda) : \lambda x \mapsto \lambda x \in \lambda \Lambda \) respectively \( \Lambda t(\lambda) \equiv x \mapsto x \lambda \in \Lambda \lambda \) so we shall freely use the notation \( \lambda x = l_\lambda x \) and \( y \lambda = r_\lambda y \).
whenever \( x \in \text{dom}(l_\lambda) \) and \( y \in \text{dom}(r_\lambda) \). Here our convention is that \( \Omega x = x \) for any \( x \in X \).

\( X_t \) is easily described since the algebra \( \mathcal{E}_t \cap l^\infty(\Lambda) \) is generated by the projections \( L_\lambda L_\lambda^* \). It is the space appearing in Example 1.2(i) [11]. \( X_r \) is the same space when we replace \( \Lambda \) with \( \Lambda^{\text{op}} \). However, the space \( X \) seems to be much more complicated.

The difficulty appears from the projections \( P_j L_\lambda L_\lambda^* R_\mu R_\mu^* P_j \) with \( \sigma(\lambda) = \sigma(\mu) = e_j \) which are not in the algebra generated by the projections \( L_\lambda L_\lambda^* \) and \( R_\lambda R_\lambda^* \). This projection is the domain of the partial isometry \( R_\mu^* L_\mu P_j \).

\[
\begin{array}{c}
\mu \\
\mid \\
R_\mu^* L_\lambda \\
\mid \\
\lambda
\end{array}
\]

We do not know something like shift equivalence to describe this isometry. This does not happen in the rank one case since left and right creations commute up to compact operators. In the rank one case the set \( X \) is the disjoint union \( \overline{\Lambda} \bigcup (\Lambda^\infty \times (\Lambda^{\text{op}})^\infty) \).

**Lemma 3.1.**

(i) The set \( \overline{\Lambda} \) is an open dense discrete subset of \( X \) (\( X \) is a compactification of \( \overline{\Lambda} \))

(ii) The algebra \( \mathcal{E} \) is the image of the canonical representation \( \pi \) of \( C^*_r(G) \) on \( l^2(\overline{\Lambda}) = F \)

**Proof.**

(i) \( \mathcal{E} \) contains the ideal of compact operators \( \mathbb{K}(F) \) so \( \mathcal{E} \cap l^\infty(\overline{\Lambda}) \) contains the diagonal algebra \( \mathbb{K}(F) \cap l^\infty(\overline{\Lambda}) = C_0(\overline{\Lambda}) \) as an essential ideal. Therefore \( X \) contains the discrete set \( \overline{\Lambda} \) as a dense open subset. \( X \) is compact since it is the spectrum of an unital algebra.

(ii) The set \( \overline{\Lambda} \) is invariant with respect to the partial homeomorphisms \( l_\lambda, r_\lambda \) so it is invariant for the groupoid \( G \). It is also open so it gives rise to an ideal of \( C^*_r(G) \).

Since the reduction \( G|_\overline{\Lambda} \) is the transitive groupoid, this ideal is the ideal of compact operators. We have a representation \( \pi \) of \( C^*_r(G) \) on \( l^2(\overline{\Lambda}) \) induced by the canonical representation of the ideal \( \mathbb{K}(l^2(\overline{\Lambda})) \): \( \pi(T)\delta_\lambda = \pi(TP_{\delta_\lambda})\delta_\lambda, \; T \in C^*_r(G) \). The image \( \pi(C^*_r(G)) \) in \( \mathbb{L}(l^2(\overline{\Lambda})) \) is exactly the algebra \( \mathcal{E} \). Indeed, a basis for the topology of the groupoid \( G \) is given by equivalence classes (modulo passing to germs) of \((Y, w)\) where \( 1_Y \) is a projection in \( C(X) = \mathcal{E} \cap l^\infty(\overline{\Lambda}) \) and \( w \) is a finite word in \( l_\lambda, l_\lambda^{-1}, r_\lambda, r_\lambda^{-1} \) such that \( v^{-1} \) is a homeomorphism on \( Y \). Therefore \( \pi(1_{(Y, w)}) = 1_Y w \in \mathcal{E} \), where \( w \) is the word in \( L_\lambda, L_\lambda^{-1}, R_\lambda, R_\lambda^{-1} \) obtained by replacing \( l \) and \( r \) with \( L \) and \( R \). Therefore, the image of the representation \( \pi \) is contained in \( \mathcal{E} \) and it is clear that \( L_\lambda, R_\lambda \) are in this image so \( \pi(C^*_r(G)) = \mathcal{E} \). \( \square \)
Proposition 3.2. (i) There are canonical continuous surjective maps \( p_i, p_r \) and \( \sigma \) such that the following diagram is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & \Lambda \\
\downarrow{p_i} & & \downarrow{\sigma} \\
X_i & & \Lambda^r \\
\downarrow{p_r} & & \downarrow{\sigma} \\
X_r & & \\
\end{array}
\]

where \( p_i(\lambda) = p_r(\lambda) = \lambda \) for any \( \lambda \in \Lambda \) and \( \sigma \) extends the shape on \( \Lambda \).

(ii) The source and terminal maps \( s \) and \( t \) on \( \Lambda \) extends to continuous maps on \( X \).

(iii) \( X \) has the following factorization property: if \( x \in X, \sigma(x) \geq k + p, k, p \in \mathbb{N}^r \) then there exists unique \( \lambda, \mu \in \Lambda \) with \( \sigma(\lambda) = k, \sigma(\mu) = p \) and \( y \in X \) with \( \sigma(y) = \sigma(x) - k - p \) such that \( x = \lambda \mu y \).

(iv) \( J_k = \pi(C^*_r(G|_{X_k})) \) where \( X_k = \{ x \in X : \sigma(x)_k < \infty \} \)

(v) Let \( Z = \{(x, y) \in X \times \Lambda^\infty : s(x) = t(y)\} \). Then the map

\[
\phi = (\sigma, \psi) : (\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto (\sigma(x), xy) \in \mathbb{N}^n \times \Lambda^\infty
\]

extends to a continuous surjective map \( \phi : Z \to \mathbb{N}^n \times \Lambda^\infty \)

Proof. (i) We have the inclusion of algebras \( \mathcal{E}_i \cap l^\infty(\overline{\Lambda}) \subset \mathcal{E} \cap l^\infty(\overline{\Lambda}) \), that is \( C(X_i) \subset C(X) \), with identity on \( C_0(\Lambda) \) hence there is a continuous surjection \( X \to X_i \) which is the identity on the set \( \Lambda \). Similarly, there is a continuous surjection \( X \to X_r \) which is the identity on the set \( \overline{\Lambda} \). The diagram is commutative because \( \sigma p_i = \sigma p_r \) on \( \overline{\Lambda} \) which is a dense set. We denote by \( \sigma \) the shape extended to \( X_i, X_r \) or \( X \).

(ii) \( X \) is a disjoint union of \( dom(l_a) \) so we can define \( s(x) = a \) if \( x \in dom(l_a) \) and \( t(x) = a \) if \( x \in dom(r_a) \). \( s \) and \( t \) are continuous since \( s^{-1}\{\{a\}\} = dom(l_a) \) and \( t^{-1}\{\{a\}\} = dom(r_a) \).

(iii) We have a disjoint union

\[
X = \bigcup_{\sigma(\lambda) = k} dom(l_\lambda) = \bigcup_{\sigma(\lambda) = k} dom(r_\lambda)
\]

Therefore there is one and only one \( \lambda \) with \( \sigma(\lambda) = k \) such that \( x \in dom(l_\lambda) \). We can take \( x' = l_\lambda^{-1}(x) \) and then \( x = \lambda x' \). Moreover \( \sigma(x') = \sigma(x) - k \). This can be checked on the set \( \{\lambda \mu : \mu \in \overline{\Lambda}\} \) which is a dense set in \( dom(l_\lambda^{-1}) = ran(l_\lambda) \).

We repeat now this reason for right factorization of \( x' \).
(iv) The projection \( P_k \) corresponds to the open set \( \{ x \in X : \sigma(x)_k = 0 \} \) which generates the open \( G \)-invariant set \( \{ x \in X : \sigma(x)_k < \infty \} \).

(v) We have to show that the maps \( (\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto \sigma(x) \in \mathbb{N}^r \) and \( (\Lambda \times \Lambda^\infty) \cap Z \ni (x, y) \mapsto xy \in \Lambda^\infty \) extend to continuous maps. The first claim is proven in (i) above. To prove the second let \( (\lambda_n, y_n) \) be a convergent sequence in \( Z \). We have to prove that \( (\lambda_n y_n)(0, m) \) is eventually constant. Since \( y_n \) is convergent in \( \Lambda^\infty \) we know that \( y_n(0, m) \) is eventually constant so we can suppose it is \( \mu \). We know that the sequence \( (\lambda_n \mu)_n \) converges in \( X \) since \( R_\mu \) is continuous so it converges also in \( X_l \). This implies that \( (\lambda_n \mu)(0, m) = (\lambda_n y_n)(0, m) \) is eventually constant. For the surjectivity of \( \phi \) we use the density of the set \( \mathbb{N}^r \times \Lambda^\infty \) in \( \mathbb{N}^r \times \Lambda^\infty \). This dense set is in the image of \( \phi \) which is closed since \( Z \) is compact.

\[ \square \]

From the factorization in (iii) of the proposition above, we can define commuting left and right semigroups of shifts \((L, R)\) on \( X \) with \( \text{dom}(L_k) = \text{dom}(R_k) = \{ x \in X : \sigma(x) \geq k \} \)

\[ L_k x = x', \quad R_k x = x'' \quad \text{where} \quad x = x'x'' \quad \text{with} \quad \sigma(\lambda) = \sigma(\mu) = k \]

These shifts are continuous since their restrictions to \( \text{dom}(l_\lambda) \), respectively \( \text{dom}(r_\lambda) \), are \( l_\lambda^{-1} \) and \( r_\lambda^{-1} \).

**Proposition 3.3.**

(i) The maps

\[ T_m : \{ (x, y) \in Z : x = x'x'' , \sigma(x'') = m \} \ni (x, y) \mapsto (x', x''y'') \in Z \]

\[ V_m : Z \rightarrow Z, \quad V_m(x, y) = (L_m(x(y(0, m))), L_m(y)) \]

define a MGDS \((T, V)\) on \( Z \) which satisfies the condition (DC).

(ii) The exit time map \( \sigma \) defined in section 1 is the same with the map \( \sigma \) in the previews Proposition.
Proof. (i) We note that \( \text{dom}(T_m) = \{(x, y) \in Z : \sigma(x) \geq m\} \) and \( \text{dom}(V_m) = Z \) so \( \text{dom}(T_m V_k) = \text{dom}(V_k T_m) = \text{dom}(T_m) \). \( T_m \) is a local homeomorphism from \( Z \cap (R \times \Lambda^\infty) \) to \( Z \cap (X \times s \times s^\Lambda) \) and \( V_m \) is a homeomorphism from \( \{(x, y) \in Z : xy(0, m) \in L \Lambda^\Lambda\} \) to \( \{(x, y) \in Z : t(x) = s(\lambda)\} \) with \( \sigma(\lambda) = m \). Using the map \( \phi \) from Proposition 3.2, we can see the map \( (T_m, V_k) \) as a homeomorphism from the set \( \phi^{-1}(\{(p, x) : p \geq m, x \in \text{dom}(l_\lambda)\}) \) to \( \phi^{-1}(\{(p, x) : p \geq 0, x \in \text{dom}(l_{s(\lambda)})\}) \). The semigroup condition of \( T \) and \( V \) is easily verified. We prove now that \( T_m V_k = V_k T_m \). Let \( (x, y) \in Z \) with \( \sigma(x) \geq m \). Write \( x = x' x'', y = y' y'' \), \( x'' y' = \alpha \beta \) with \( \sigma(x'') = \sigma(\beta) = m \), \( \sigma(y') = \sigma(\alpha) = k \).

\[
\begin{array}{c|c|c|c}
 x' & x'' & \sigma(x'') = m \\
---&---&---
 y' & y'' & \sigma(y') = k \\
---&---&---
\end{array}
\]

One has
\[
T_m V_k (x, y) = T_m (L_k (x' x'', y''), y'') = T_m (L_k (x' \alpha \beta), y'') = T_m (L_k (x' \alpha), \beta y'') = V_k T_m (x' x'', y' y'') = V_k (x', \alpha y' y'') = (L_k (x' \alpha), \beta y'')
\]

The \( (DC) \) condition is satisfied since \( \text{dom}(T_m V_k) \cap \text{dom}(T_m' V_k') = \text{dom}(T_m) \cap \text{dom}(T_m') = \{(x, y) \in Z : \sigma(x) \geq m \vee m'\} = \text{dom}(T_{m \vee m'}). \)

(ii) Recall that the map \( \sigma \) in section 1 was \( \sigma(x) = \sup\{m : x \in \text{dom}(T_m)\} \). In our case this supremum is \( \sup\{m : m \leq \sigma(x)\} = \sigma(x) \).

\[\square\]

There is a natural equivariant map between MGDS \((Z, (T, V))\) and the MGDS \((\mathbb{N}^n \times \Lambda^\infty, S \times W)\) which gives the algebra \( T^\otimes \otimes C^*(\Lambda) \). Here, an equivariant map between two MGDS \((X, T)\) and \((Y, S)\) means a map \( \phi : X \to Y \) such that \( \phi \circ T_m = S_m \circ \phi \).

**Proposition 3.4.** Let \((\mathbb{N}^n, S), (\Lambda^\infty, W)\) be the MGDS of Example 1.2 (i), (ii). The map \( \phi \) of Proposition 3.2(v) is an equivariant map between \((Z, (T, V))\) and \((\mathbb{N}^n \times \Lambda^\infty, S \times W)\).

**Proof.** Regarding domains, it is enough to check that \( \text{dom}(S_k \phi) = \text{dom}(\phi T_k) \) since \( W_k \) and \( V_k \) are defined everywhere.

\[
\phi(\text{dom}(T_k)) = \phi(\{(x, y) \in Z ; \sigma(x) \geq k\}) = \{(n, x) ; n \geq k\} = \text{dom}(S_k) \phi
\]

Now it remains to check that \( S_k \phi (x, y) = \phi T_k (x, y) \) and \( W_k \phi (x, y) = \phi V_k (x, y) \) for \( (x, y) \in Z \) with \( \sigma(x) \in \mathbb{N}^n \) since \( \Lambda \) is a dense set in \( X \). If \( x = x' x'' \) with \( \sigma(x'') = k \) one has:

\[
\begin{align*}
S_k(\phi(x, y)) &= S_k(\sigma(x), xy) = (\sigma(x) - k, xy) \\
&= (\sigma(x'), x' x'' y) = \phi(x', x'' y) = \phi(T^k(x, y))
\end{align*}
\]

\[\ ]
If \( y = y'y'' \) and \( xy' = \alpha \beta \) with \( \sigma(y') = \sigma(\alpha) = k \) (so \( \sigma(x) = \sigma(\beta) = k \) ) one has
\[
W_k(\phi(x, y)) = W_k(\sigma(x), xy'y'') = W_k(\sigma(x), \alpha \beta y'')
= (\sigma(x), \beta y'') = (\sigma(\beta), \beta y'') = \phi(\beta, y'') = \phi(V_k(x, y))
\]

\[\square\]

An equivariant map \( \phi : (X, T) \to (Y, S) \) induces a natural map of the associated groupoids: \( \phi : G(X, T) \to G(Y, S), \phi(x, z, y) = (\phi(x), z, \phi(y)) \). The next lemma gives a condition for a morphism of r-discrete groupoids to induce a morphism of the corresponding reduced algebras.

**Lemma 3.5.** Let \( \phi : G_1 \to G_2 \) be a morphism of two r-discrete groupoids such that \( \phi : G_1^r \to G_2^{\phi(x)} \) is a bijection for any \( x \in G_1^0 \) and \( \phi_0 : G_1^0 \to G_2^0 \) is proper. Then \( \phi \) is proper and the map \( C_c(G_1) \ni f \mapsto f = f \circ \phi \in C_c(G_2) \) extends to a morphism \( \hat{\phi} : C^*(G_2) \to C^*_r(G_1) \)

**Proof.** It is known [20] that in an r-discrete groupoid the range map is a local homeomorphism. Let \( r_1, r_2 \) be the range map in \( G_1, G_2, K \subset G_2 \) be a compact set such that \( r_2 : K \to r_2(K) \) is a homeomorphism. Since \( \phi_0 \) is proper we know that \( (\phi_0)^{-1}(r_2(K)) \) is compact. Our hypothesis gives that \( r_1 \) is a homeomorphism from \( K \) to \( r_1(K) \). It is enough to show that \( \hat{\phi} \) is a morphism between the topological algebras \( C_c(G_2) \) and \( C_c(G_1) \). Indeed \( \hat{\phi} \) is continuous since \( \phi \) is proper.

\[
\tilde{f} \star g(t) = (f \star g)(\phi(t)) = \sum_{s \in G_2^s(\phi(t))} f(\phi(t)s)g(s^{-1})
\]

We use now the assumption that \( \phi : G_1^r \to G_1^{\phi(x)} \) is a bijection and we get
\[
\tilde{f} \star g(t) = \sum_{v \in G_1^r(t)} f(\phi(t)v)g(\phi(v)^{-1}) = \tilde{f} \star \tilde{g}(t)
\]

hence \( \tilde{f} \star g = \tilde{f} \star \tilde{g} \)

\[\square\]

A simple examples of such maps is given by transformation groups \( \phi : X \rtimes G \to G, \phi(x, g) = g \) where \( X \) is compact.

**Proposition 3.6.** The previous lemma holds if \( G_1 = G(Z, (T, V)) \) and \( G_2 = G(N^\lambda, S \rtimes \Lambda \infty, S \times W) \)

**Proof.** First let us prove that the lemma holds for the map \( \sigma \) and the pairs of groupoids \( G_1 = G(Z, T) \) and \( G_2 = G(N^\lambda, S) \). To prove the surjectivity let \( (\lambda, x) \in Z, (\sigma(\lambda), n - m, k) \in G(N^\lambda, S) \) such that \( n \leq \sigma(\lambda), m \leq k \) and \( \sigma(\lambda) - n = k - m \). We construct \( (\lambda', x') \in Z \) such that \( \sigma(\lambda') = k \) and \( T_n(\lambda, x) = T_m(\lambda', x') \). Since \( n \leq \sigma(\lambda) \) one can decompose \( \lambda = \alpha \beta \) with \( \sigma(\beta) = n \). Then \( \lambda' = \alpha((\beta x)(0, m)) \) and \( x' = (\beta x)(m, \infty) \) have the properties required. To prove the injectivity let \( (\lambda, x) \in Z, (\lambda', x') \in Z, (\lambda'', x'') \in Z, n, m, p, q \in N^\lambda \) such that \( n - m = p - q, \sigma(\lambda') = \sigma(\lambda'') \) \( T_n(\lambda, x) = T_m(\lambda', x') \) and \( T_p(\lambda, x) = T_q(\lambda'', x'') \). Because of the condition (DC) of \( T \) we have \( T_{n+p}(\lambda, x) = T_{m+n+p-n}(\lambda', x') \) and \( T_{n+p}(\lambda, x) = T_{q+n+p-p}(\lambda'', x'') \) so \( T_k(\lambda', x') = T_k(\lambda'', x'') \) where \( k = m + n + p - n = q + n + p - p \), hence \( \lambda' = \alpha \beta', \lambda'' = \alpha \beta'' \)
$\alpha\beta''$ with $\sigma(\beta') = \sigma(\beta) = k$ and $\beta' x' = \beta'' x''$. The uniqueness of the factorization of a word in $\Lambda^\infty$ implies that $\lambda' = \lambda''$ and $x' = x''$.

We prove now that the above lemma is true for $\psi$, $G_1 = G(Z, V)$ and $G_2 = G(\Lambda^\infty, W)$. To prove the surjectivity, let $(\lambda x, n-m, y) \in G(\Lambda^\infty, W)$ with $\sigma(\alpha) = n$, $\sigma(\beta) = m$, $\lambda x = \alpha z, y = \beta z$. One has $\lambda x(0, n) = \alpha y$ so $T_n(\lambda, x) = (\gamma, x(n, \infty))$. Let $\lambda'$ be given by the factorization $\beta' = \lambda' \beta''$ with $\sigma(\beta') = m$ and $x' = \beta' x(n, \infty)$. Then $T_n(\lambda', x') = (\gamma, x(n, \infty)) = T_n(\lambda, x)$ so $((\lambda, x), n-m, (\lambda', x')) \in G(Z, V)$ and $\lambda' x' = \lambda' \beta' x(n, \infty) = \beta' x(n, \infty) = \beta x = y$. To prove the injectivity let $((\lambda, x), m-n, (\lambda', x'))((\lambda, x), p-q, (\lambda'', x'')) \in G(Z, V)$ such that $n-m = p-q, \lambda' x' = \lambda'' x''$. As for $G(Z, T)$ we have $V_k(\lambda', x') = V_k(\lambda'', x'')$ where $k = m+n+p-n = q+n+p-p$. From the definition of $V$ one has the factorizations $\lambda' \alpha = \lambda'' \beta = \beta'' \gamma$, $x' = \alpha y$, $x'' = \beta y$ where $\sigma(\alpha) = \sigma(\alpha') = \sigma(\beta) = \sigma(\beta') = k$. Then we have $\alpha' \gamma y = \beta' \gamma y$, hence $\alpha' = \beta'$ and $'x' = x''$. We also have $\lambda' \alpha = \lambda'' \beta = \alpha' \gamma$ so $\lambda' = \lambda''$.

We show now that lemma holds for $\phi$, $G_1 = G(Z, (T, V))$ and $G_2 = G(\n^n \times \Lambda^\infty, S \times W)$. First we show the surjectivity. Let $(\lambda, x), (\lambda', x'), (\lambda'', x'') \in Z, n, m, n', m', p, q, p', q' \in \n$ such that $V_n(T_m(\lambda_1, x)) = V_{n'}(T_{m'}(\lambda', x'))$, $V_p(T_q(\lambda, x)) = V_{p'}(T_{q'}(\lambda'', x''))$, $n-m = p-p', m-m' = q-q'$, $\sigma(\lambda') = \sigma(\lambda'')$, $\lambda' x' = \lambda'' x''$. As before we have $V_n(T_{m+vq}(\lambda, x)) = V_{n'}(T_{m'+vq-m}(\lambda', x'))$ and $V_p(T_{m+vq}(\lambda, x)) = V_{p'}(T_{q'+m-vq}(\lambda', x'))$ so $V_n(T_{m+vq}(\lambda, x)) = V_{n'}(T_{k}(\lambda', x'))$ and $V_p(T_{m+vq}(\lambda, x)) = V_{p'}(T_{k}(\lambda'', x''))$ with $k = m+m' + q - m = q' + m' + q - q$. As $\psi(T_k(\lambda, x)) = \psi(\lambda, x)$ the injectivity result proven above for $\psi$, $G(Z, V)$ and $G(\Lambda^\infty, W)$ shows that $T_k(\lambda', x') = T_k(\lambda'', x'')$. Now we apply the injectivity result for $\sigma$, $G(Z, T)$ and $G(\n^n, S)$ to get $(\lambda', x') = (\lambda'', x'')$.

To prove the surjectivity let $(\lambda, x) \in Z$ and $((\sigma(\lambda), \lambda x), (n-n', m-m'), (k, y)) \in G(\n^n \times \Lambda^\infty, S \times W)$. We apply the surjectivity proven above for $\psi$, $G(Z, V)$ and $G(\Lambda^\infty, W)$ in the point $T_m(\lambda, x)$. There is $(\alpha, \beta) \in Z$ such that $\alpha \beta = y$ and $V_n(T_m(\lambda, x)) = V_{n'}(\alpha, \beta)$. It follows that $\sigma(\alpha) = \sigma(\lambda) = m - k - m'$. We define now $\lambda' = \alpha \beta(0, m')$ and $x' = \beta(m', \infty)$. One has $\sigma(\lambda') = k$, $\lambda' x' = y$ and $T_{m'}(\lambda', x') = (\alpha, \beta)$ so $V_n(T_m(\lambda, x)) = V_{n'} T_{m'}(\lambda', x')$. 

From Lemma 3.5 we have an induced homomorphism $\tilde{\phi} : T^{\otimes n} \otimes C^*(\Lambda) \rightarrow C^*(Z, (T, V))$. Proposition 1.9 gives a homomorphism $\tilde{\pi} : C^*(Z, (T, V)) \rightarrow C^*(\text{Germ}(Z, (T, V)))$. But $\text{Germ}(Z, (T, V))$ is an open subgroupoid of $G \times \text{Germ}(\Lambda^\infty, W)$. The MGDS $(\Lambda^\infty, W)$ is essentially free so by Proposition 1.7 we have a map from $C^*(\text{Germ}(Z, (T, V)))$ to $E \otimes C^*(\Lambda)$. Composing these homomorphisms we get the map from $T^{\otimes n} \otimes C^*(\Lambda)$ to $E \otimes C^*(\Lambda)$. Schematically we view this in terms of groupoids by the following diagram:

$$
\begin{array}{ccc}
G(Z, (T, V)) & \overset{\phi}{\longrightarrow} & G(\n^n \times \Lambda^\infty, S \times W) \\
\text{Germ}(Z, (T, V)) \text{ open} & \overset{\pi}{\longrightarrow} & G \times \text{Germ}(\Lambda^\infty, W)
\end{array}
$$

$\phi$ reverses and $\pi$ preserves the arrow when passing to the corresponding algebras. The bisection that defines $S_i$ is the set $A_i = \{((n-e_i, x), (-e_i, 0), (n, x)) : n \in \n, n \geq e_i, x \in \Lambda^\infty\}$ and

$$
\hat{\phi}^{-1}(A_i) = \bigcup_{\sigma(\lambda) = e_i} \{((x\lambda, y), (-e_i, 0), (x, \lambda y)) : (x\lambda, y) \in Z\}
$$

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We have $W_i = \tilde{\pi}(1_{\hat{\phi}^{-1}(A_i)}) \in C^*_r(G(Z, (T,V)))$ so $\tilde{\phi}(S_i \otimes 1) = W_i$. Similarly the bisection that defines $s_\lambda$ is the set $B_\lambda = \{(n, \lambda x), (0, -\sigma(\lambda)), (n, x) : n \in \mathbb{N}, x \in \Lambda^\infty, t(x) = s(\lambda)\}.$

$$\hat{\phi}^{-1}(B_\lambda) = \{((x', y'), (0, -\sigma(\lambda)), (x, y)) : (x, y) \in Z,$$

$$\lambda x = x' \mu, y' = \mu y, \sigma(\mu) = \sigma(\lambda)\}$$

Passing to germs with $\tilde{\pi}$, we get $\tilde{\pi}(1_{\hat{\phi}^{-1}(B_\lambda)}) = V_\lambda$. $\hat{\phi}$ restricted to the set $\{\Omega\} \times \Lambda^\infty$ gives an identification between $G(Z, (T,V))|_{\{\Omega\} \times \Lambda^\infty}$ and $G(\mathbb{N} \setminus \Lambda^\infty, S \times W)|_{\{0\} \times \Lambda^\infty}$ so an isomorphism between full corners.

4 Conclusions

As a conclusion we want to draw the attention to the results in [8, 9]. Our groupoid approach to the duality between higher rank graph algebras can lead to a notion of higher rank hyperbolic groups. The higher rank version of a tree is a Euclidean building so the $\tilde{A}_n$-groups in [5] have to fall into this class. One can then think of the one-sided and two-sided Toeplitz algebras as subalgebras of $L(l^2(\Gamma))$, where $\Gamma$ is such a group. For example the left-sided Toeplitz algebra associated to the free group $F_n$ is $C(\overline{F_n}) \rtimes F_n$ where $\overline{F_n}$ is a compactification of $F_n$ using the left-sided distance $d_l(\alpha, \beta) = l(\alpha^{-1} \beta)$, $l$ the word length. Then the right-sided Toeplitz algebras is given by the same algebra $C(\overline{F_n}) \rtimes F_n$ but the compactification is given by the right-sided distance $d_r(\alpha, \beta) = l(\alpha \beta^{-1})$. The two-sided Toeplitz algebra is the algebra generated by these two in their canonical representation on $L(l^2(\mathbb{F}_n))$.

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