Algebraic dimension and complex subvarieties of hypercomplex nilmanifolds

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Abstract. A nilmanifold is a (left) quotient of a nilpotent Lie group by a cocompact lattice. A hypercomplex structure on a manifold is a triple of complex structure operators satisfying the quaternionic relations. A hypercomplex nilmanifold is a compact quotient of a nilpotent Lie group equipped with a left-invariant hypercomplex structure. Such a manifold admits a whole 2-dimensional sphere $S^2$ of complex structures induced by quaternions. We prove that for any hypercomplex nilmanifold $M$ and a generic complex structure $L \in S^2$, the complex manifold $(M, L)$ has algebraic dimension 0. A stronger result is proven when the hypercomplex nilmanifold is abelian. Consider the Lie algebra of left-invariant vector fields of Hodge type $(1,0)$ on the corresponding nilpotent Lie group with respect to some complex structure $I \in S^2$. A hypercomplex nilmanifold is called abelian when this Lie algebra is abelian. We prove that all complex subvarieties of $(M, L)$ for generic $L \in S^2$ on a hypercomplex abelian nilmanifold are also hypercomplex nilmanifolds.

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4 Abelian (hyper-)complex varieties and their Albanese varieties

5 Subvarieties in abelian hypercomplex nilmanifolds

1.1 Complex geometry of non-algebraic manifolds

Algebraic varieties are typically given by a system of polynomial equations. Complex non-algebraic varieties are described in different and widely varying fashions, much less uniform.

One of the main sources of non-algebraic complex manifolds comes from the theory of homogeneous or locally homogeneous spaces. Many compact Lie groups, as well as their homogeneous spaces, admit left-invariant complex structures ([Sam], [T], [J]). When the group $G$ is not compact, one can consider a cocompact lattice $\Gamma$ and take (say) a left quotient $G/\Gamma$ equipped with a left-invariant complex structure. Finally, one could take a quotient of a domain $\Omega \subset \mathbb{C}^n$ by a holomorphic action of a discrete group. This way one can obtain some of the non-Kähler complex surfaces (Hopf, Kodaira, Inoue), as well as the multi-dimensional generalization of the Inoue surfaces, called Oeljeklaus-Toma manifolds ([OT]).

It is interesting that in most of these examples, the complex geometry is more restrictive than in the algebraic case. The Inoue surface and some special classes of Oeljeklaus-Toma manifolds have no complex subvarieties at all ([OV2]). More general Oeljeklaus-Toma manifolds can admit subvarieties, but the subvarieties of smallest possible dimension are also obtained as a quotient of a domain $\Omega \subset \mathbb{C}^n$ by an affine holomorphic action ([OVV, Theorem 1.2]). Moreover, the torsion-free flat connection, intrinsic on the Oeljeklaus-Toma manifolds, induces a torsion-free flat connection on any of these subvarieties.

The complex structures on compact Lie groups and associated homogeneous spaces are also very interesting. By Borel-Remmert-Tits theorem, such manifolds are always principal toric fibrations over a rational projective manifold ([T], [V5]). It is possible to classify all subvarieties of a class of principal toric fibrations, called “positive principal toric fibrations”. In [V5] it was shown that any irreducible subvariety of a positive principal toric fibration is either a pullback from the base of the fibration, or belongs to its fiber.

An interesting example of a non-algebraic complex manifold is a Bogomolov–Guan manifold ([Gu], [Bog]). It is a non-Kähler holomorphic symplectic manifold which is constructed from a Kodaira surface in a similar way as generalized Kummer varieties are constructed from complex 2-dimensional tori (see [Beau], [Bog], [KurV]). A Bogomolov–Guan manifold is equipped with a nat-
ural Lagrangian fibration over the projective space. A partial classification of subvarieties of Bogomolov–Guan manifold was obtained in [BKKY], where the algebraic dimension of Bogomolov–Guan manifolds was computed as well.

It is natural to expect that these results can be generalized. For example, consider a flat affine complex manifold, that is, a manifold equipped with a flat torsion-free connection preserving the complex structure. This class of manifolds contains Hopf manifolds, Oeljeklaus-Toma manifolds and compact tori. Subvarieties in a non-algebraic torus and the Oeljeklaus-Toma manifolds are also flat affine ([Ue], [OV2]), however, for the Hopf manifold and the torus, such result won’t work, indeed, there are tori which are algebraic and Hopf manifolds obtained as principal elliptic bundles over a weighted projective space.

However, in both of these cases, it is enough to deform the complex structure to a general point to obtain a classification of subvarieties. As far as we know, the following questions remains open.

**Question 1.1:** Let $M$ be a compact flat affine complex manifold. Is it true that there exists a finite covering $\tilde{M}$ of $M$ and a flat affine complex manifold $M_1$ in the same deformation class as $\tilde{M}$, such that all complex subvarieties of $M_1$ are also flat affine manifolds?

A similar question can be stated in context of complex structures on locally homogeneous manifolds, defined in Subsection 2.1.

**Question 1.2:** Let $M = G/\Gamma$ be a compact locally homogeneous complex manifold. Is it true that there exists a locally homogeneous complex manifold $M_1$ in the same deformation class as $\tilde{M}$, such that all complex subvarieties of $M_1$ are locally homogeneous submanifolds (Definition 5.1)?

One of the easiest way to provide deformations of complex structures is to use the quaternionic rotation. This is one of the motivation of the present work, where we successfully solve Question 1.2 in the presence of an abelian hypercomplex structure.

### 1.2 Trianalytic subvarieties in hyperkähler manifolds

A great source of non-algebraic Kähler manifolds is provided by the hyperkähler geometry. Consider a compact, Kähler, holomorphically symplectic manifold $M$, such as a K3 surface. Calabi-Yau theorem provides $M$ with a hyperkähler structure (Definition 2.27), that is, a triple $I, J, K$ of complex structure operators satisfying quaternionic relations, and a Riemannian metric which is Kähler with respect to $I, J, K$. One of the main tools of hyperkähler geometry is called “the hyperkähler rotation”: any quaternion $L \in \mathbb{H}$, $L^2 = -1$, induces a complex structure on $M$. This gives the whole 2-dimensional sphere of complex stricitures, called the twistor sphere, or the twistor deformation of a hyperkähler manifold.
Starting from an algebraic manifold and applying the rotation, we usually obtain non-algebraic ones. Recall that the algebraic dimension (Subsection 2.2) \( a(M) \) of a complex manifold \( M \) is the transcendence degree of its field of global meromorphic functions. When \( M \) is algebraic, one has \( a(M) = \text{dim} M \). Hyperkähler rotation provides examples of complex manifolds with algebraic dimension 0:

**Proposition 1.3:** ([Fu, Cor. 5.12]) Let \( X \) be a compact hyperkähler manifold. Then for all but a countable number of complex structures \( L \in H \) the algebraic dimension of the complex manifold \( X_L \) is zero.

This result is a part of a more general phenomenon, which can be used to classify complex subvarieties in general (non-algebraic) deformations of hyperkähler manifolds. For the further use, we formulate the following definition in a more general situation of hypercomplex geometry (Subsection 1.3).

**Definition 1.4:** A subvariety \( Z \) of a hypercomplex manifold \( X \) is called trianalytic if it is complex analytic with respect to every complex structure \( L \in H \).

Trianalytic subvarieties enjoy many good properties: they are completely geodesic ([V3]), and can have only very mild singularities, which are resolved by a normalization ([V4]).

**Proposition 1.5:** ([V1], see also [V2]) Let \( X \) be a hyperkähler manifold (not necessarily compact). Then for all but a countable number of complex structures \( L \in H \), all compact complex subvarieties of \( X_L \) are trianalytic.

This result was often applied in symplectic geometry to infer that a general almost complex structure on \( X \) does not support complex curves ([EV], [SV2]).

**Proposition 1.3** follows directly from **Proposition 1.5**. Indeed, let \( X \) be a compact hyperkähler manifold. Let \( L \in H \) be a complex structure such that \( X_L \) does not contain subvarieties which are not trianalytic. Then a fortiori \( X_L \) does not contain any divisors. Complex varieties with no divisors are of algebraic dimension zero.

### 1.3 Hypercomplex manifolds

Hyperkähler manifolds are well understood, but there are not many compact examples: so far, only two series and two sporadic examples exist. It makes sense to weaken the hyperkähler condition and consider **hypercomplex manifolds** (Definition 2.12). A hypercomplex structure is a triple of complex structures satisfying quaternionic identities. Examples of hypercomplex manifolds are abundant: there are homogeneous examples ([J]), nilmanifolds and solvmanifolds ([BD1], [DF1]), fibered bundles ([V6]), locally conformally hyperkähler examples ([Or]) and many others.
Hypercomplex manifolds were first introduced by Ch. Boyer ([Bo]), who also gave the complete list of compact hypercomplex manifold in real dimension 4: K3 surface, compact torus, and Hopf manifold.

Any element \( L \in \mathbb{H} \), \( L^2 = -1 \) induces an integrable complex structure on \( M \), just like in the hyperkähler case. However, Proposition 1.3 and Proposition 1.5 don’t hold for general hypercomplex manifolds. In fact, for most examples listed above these results are false. Indeed, by Borel-Remmert-Tits theorem, any complex manifold with trivial (or abelian) fundamental group and admitting a transitive action of a compact Lie group is always a principal toric fibration over a rational projective manifold ([T], [V5]). Therefore, any homogeneous hypercomplex manifold of this type which is not a torus always has positive algebraic dimension.

Another counterexample is given by the Hopf manifold \( \mathbb{H}^n \{0\}/\mathbb{Z} \), where the \( \mathbb{Z} \)-action is given by the multiplication by \( \lambda \in \mathbb{R} > 1 \). For any complex structure \( L \in \mathbb{H} \) the manifold \( X_L \) admits an elliptic fibration over \( \mathbb{C}P^{2n-1} \). Hence the algebraic dimension of \( X_L \) is always \( 2n - 1 \) and \( X_L \) always contains an elliptic curve.

It is interesting that all counterexamples existing so far are hypercomplex manifolds with non-trivial canonical bundle. In fact, the anticanonical bundle for all these examples is globally generated and semi-ample.

We expect that a better control over subvarieties and algebraic dimension of hypercomplex manifolds is possible when the canonical bundle is trivial.

Note that the canonical bundle of a hypercomplex manifold can be interpreted in terms of its hypercomplex structure. Recall that the Obata connection of a hypercomplex manifold \( (M, I, J, K) \) is a torsion-free connection \( \nabla \) such that \( \nabla(I) = \nabla(J) = \nabla(K) = 0 \). Such a connection exists and is unique on any hypercomplex manifold ([K],[Ob]). Clearly, the Obata connection acts on the bundle \( \Lambda^{2n,0}(M, I) \), which is equal to the canonical bundle of \( (M, I) \) where \( n = \dim_H M \). This action is compatible with the real structure \( \eta \mapsto J(\eta) \), hence its holonomy is real. The corresponding character on the holonomy group can be identified with the determinant \( \det : GL(n, \mathbb{H}) \to \mathbb{R} > 0 \). Whenever this character vanishes (or, equivalently, when the holonomy of the Obata connection belongs to \( \ker \det = SL(n, \mathbb{H}) \)), the canonical class is also trivial.

**Conjecture 1.6:** Let \( (M, I, J, K) \) be a compact hypercomplex manifold with the holonomy of Obata connection in \( SL(n, \mathbb{H}) \). Then for all complex structures \( L \in \mathbb{H} \) outside of a countable subset, the complex manifold \( (M, L) \) has algebraic dimension 0, and all its complex subvarieties are trianalytic.

This suggestion is supported by a result of [SV1], where trianalyticity was proven for codimension 1 and 2 complex subvarieties of a hypercomplex manifold with holonomy in \( SL(n, \mathbb{H}) \) in the presence of an HKT-metric (Definition 2.28).

In [BDV], it was shown that any complex nilmanifold has trivial canonical bundle. Then, it is natural to expect that Conjecture 1.6 would hold for hypercomplex nilmanifolds.
This is the main purpose of the present paper.

1.4 Hypercomplex nilmanifolds as iterated principal bundles

Recall that a nilmanifold (Definition 2.4) is a compact manifold equipped with a transitive action of a connected nilpotent Lie group. Equivalently ([Ma]), a nilmanifold is a quotient \( M = G/\Gamma \), where \( G \) is a connected, simply connected nilpotent Lie group, and \( \Gamma \subset G \) is a cocompact lattice (that is, a discrete subgroup such that \( G/\Gamma \) is compact). We shall usually consider the left quotient; to emphasize this, we sometimes write \( M = \Gamma \backslash G \).

To give a left-invariant geometric structure (such as a complex structure, or a hypercomplex structure) on a Lie group is the same as to give it on its Lie algebra \( \mathfrak{g} = T_eG \) and extend to \( G \) using the left action. In other words, a left invariant almost complex structure on \( G \) is the same as an operator \( I : \mathfrak{g} \rightarrow \mathfrak{g} \) satisfying \( I^2 = -\text{Id} \). Integrability of the corresponding complex structure is equivalent to the relation

\[
[\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0},
\]

where \( \mathfrak{g}^{1,0} = \{ x \in \mathfrak{g}_R \otimes \mathbb{C} \mid I(x) = \sqrt{-1} x \} \). Indeed, by Newlander-Nirenberg theorem, an almost complex structure on a manifold is integrable whenever \( T^{1,0}M, T^{1,0}M \subset T^{1,0}M \). However, it suffices to check that \( T^{1,0}M \) is generated by vector fields \( \xi_1, ..., \xi_n \) such that \( [\xi_i, \xi_j] \in T^{1,0}M \). This is obviously implied by (1.1).

**Example 1.7:** As an example of a complex nilmanifold, we define the Kodaira surface, following [Ha]. Consider the Lie algebra spanned by vectors \( x, y, z, t \), with \( [x, y] = z \) and all other commutators vanishing. Let the complex structure act as \( I(x) = y, I(y) = -x, I(z) = t \) and \( I(t) = -z \). To show that the corresponding almost complex structure is integrable, we notice that the only possibly non-trivial commutator between the \( (1,0) \)-vectors, \( [x + \sqrt{-1} y, z + \sqrt{-1} t] \), vanishes. From the exact sequence of Lie algebras

\[
0 \rightarrow \langle t, z \rangle \rightarrow \langle x, y, t, z \rangle \rightarrow \langle x, y \rangle \rightarrow 0,
\]

it is possible to see that the Kodaira surface admits a principal fibration over an elliptic curve with elliptic fibers. The type of elliptic curves is determined by the choice of the discrete lattice in the corresponding Lie group, with the lattice different for different Kodaira surfaces.

By “hypercomplex nilmanifold” we always mean a quotient \( M = \Gamma \backslash G \), where \( G \) is equipped with a left-invariant hypercomplex structure.

Let \( Z \subset G \) be the center of a nilpotent Lie group \( G \), and \( \Gamma \subset G \) a cocompact lattice. Denote the nilmanifold \( \Gamma \backslash G \) by \( M \). It is not very hard to see that \( Z \cap \Gamma \) is cocompact in \( Z \), and \( M_1 := \frac{G/\Gamma}{Z/\Gamma} \) is a nilmanifold. Moreover, the natural projection \( M \rightarrow M_1 \) is a fibration with the fiber \( \frac{Z}{Z/\Gamma} \), hence it is a principal toric fibration.
Repeating this construction, we prove that any smooth nilmanifold can be obtained as an iterated principal toric bundle, that is, a total space of a principal toric fibration over a principal toric fibration over... over a point.

In complex category, this result is generally false. A survey paper [R2] by S. Rollenske considers this question in some detail, as well as his Ph. D. thesis [R0]. Recall that the classification of 6-dimensional complex nilmanifolds was obtained in [AGS]. In [R0, Example 1.14], Rollenske produces an example of a 6-dimensional complex nilmanifold which does not admit a structure of an iterated holomorphic principal bundle. In [R2, Corollary 3.10] he proves that all other 6-dimensional complex nilmanifolds admit a structure of an iterated holomorphic principal bundle.

One of the most important observations made by Rollenske deals with the abelian complex structures. Abelian complex structures were introduced by M. L. Barberis in her dissertation [Ba], and explored at some length by Barberis, Dotti and their collaborators ([BD1], [ABD], [BD2]). To appreciate their definition, note that the integrability of a left-invariant complex structure operator is equivalent to (1.1), that is, an almost complex structure \( I \in \text{End}(g) \) is integrable if and only if \( g_{1,0} \) is a Lie subalgebra in \( g \otimes \mathbb{C} \). The complex structure \( I \) is called abelian when the algebra \( g_{1,0} \) is abelian.

S. Rollenske ([R2, Cor. 3.11]) has shown that all abelian complex nilmanifolds admit a structure of an iterated holomorphic principal bundle. In fact, their upper central series are complex-invariant subspaces in \( g \) (Lemma 4.1).

For a hypercomplex nilmanifold \((M, I, J, K)\), the complex structure \( I \) is abelian if and only if all the complex structures induced by quaternions are also abelian ([DF1, Lemma 3.1]). We show that the upper central series of an abelian hypercomplex nilmanifold \( M \) are \( \mathbb{H} \)-invariant (Lemma 4.1). This implies that \( M \) is an iterated holomorphic principal bundle, and, moreover, the corresponding projections are holomorphic with respect to \( I, J, K \).

### 1.5 Subvarieties in hypercomplex nilmanifolds: main results

In this subsection we state the main results of the present paper and give a few hints about their proof. It is independent from the main body of this paper. For a definition of a hypercomplex nilmanifold and more details, see Subsection 2.4.

The first of two main results of this paper:

**Theorem 1.8:** Let \((M, I, J, K)\) be a hypercomplex nilmanifold, and
\[
S := \{ L \in \mathbb{H} \mid L^2 = -1 \}
\]
the 2-dimensional sphere of all complex structures on \( M \) induced by the quaternion action. Then for all \( L \in S \) outside of a countable subset \( R \subset S \), the
algebraic dimension of \((M, L)\) vanishes.

**Proof:** Corollary 3.8. ■

The second result is based on Lemma 4.1, which gives a structural result about nilmanifolds with abelian hypercomplex structure.

**Theorem 1.9:** Let \((M, I, J, K)\) be a nilmanifold equipped with an abelian hypercomplex structure, and

\[
S := \{ L \in H \mid L^2 = -1 \}
\]

the 2-dimensional sphere of all complex structures on \(M\) induced by the quaternion action. Then for all \(L \in S\) outside of a countable subset \(R \subset S\), all complex subvarieties \(X \subset (M, L)\) are trianalytic and locally homogeneous.

**Proof:** Theorem 5.2. ■

The proof of Theorem 1.8 is based on a result of [FGV]. In this paper the authors consider a foliation \(\Sigma\) on a complex nilmanifold \(M\) given by intersection of kernels of all closed holomorphic 1-forms:

\[
\Sigma = \bigcap_{\eta \in \Lambda^1(M), dq=0} \ker \eta.
\]

The same foliation can be obtained by left-translates of the subalgebra \([g, g] + I([g, g])\), where \(M = \Gamma \backslash G\), and \(g = \text{Lie}(G)\). By [FGV, Theorem 1.1], any meromorphic function on \(M\) is constant on the leaves of \(\Sigma\). However, the leaves of \(\Sigma\) are not necessarily closed; all meromorphic functions are constant on the leaves of the foliation obtained by taking the closures of the leaves of \(\Sigma\).

Generally speaking, the Lie algebra \(g\) of a nilmanifold \(\Gamma \backslash G\) is equipped with a rational structure, with the rational lattice generated by logarithms of \(\gamma \in \Gamma\). It is not hard to see that a foliation \(\Sigma_V\) given by translates of a subspace \(V \subset g\) has closed leaves if and only if \(V\) is rational. Let \(\hat{V}_Q\) be the smallest rational subspace of \(g\) containing \(V\). From what we stated, it is clear that \(\Sigma_{\hat{V}_Q}\) is a foliation with closed leaves, and for each leaf of \(\Sigma_V\), its closure is a leaf of \(\Sigma_{\hat{V}_Q}\).

Let now \((M, L)\) be a complex nilmanifold, and \(\Sigma_{L,Q}\) the closed foliation obtained by translates of the smallest \(L\)-invariant rational space \([g, g]_{L,Q}\) containing \([g, g] + L([g, g])\). From [FGV, Corollary 1.11], it follows that all meromorphic functions on \((M, L)\) are constant on the leaves of \(\Sigma_{L,Q}\), and, moreover, factorize through the leaf space, which is a complex torus.

Consider now a continuous family of complex structures \(L\) on \(g\) and the corresponding family of subspaces \([g, g]_{L,Q}\). Let \(V\) be a vector space and \(H(V)\) the set of all subspaces contained in \(V\). We call a map \(\varphi\) from a topological space to the set of subspaces of a vector space \(W\) lower semicontinuous if the preimage of \(H(V)\) is closed for any vector space \(V \subset W\).

Since the closure of a limit is contained in the limit of closures, the map \(L \rightarrow [g, g]_{L,Q}\) is lower semicontinuous. However, there are only countably many
possible rational lattices in \( \mathfrak{g} \), and uncountably many points in any continuous family of complex structures. If we apply that argument to the twistor deformation of a complex structure on a hypercomplex nilmanifold, we obtain that the function \( L \rightarrow [\mathfrak{g}, \mathfrak{g}]_{L, \mathcal{Q}} \) is constant on an open subset of the set \( S^2 \subset \mathbb{H} \) of induced complex structure. A more careful analysis of the quaternionic action implies that the function \( L \rightarrow [\mathfrak{g}, \mathfrak{g}]_{L, \mathcal{Q}} \) is constant outside of a countable set (Lemma 3.3). Indeed, for generic \( L \), the space \( [\mathfrak{g}, \mathfrak{g}]_{L, \mathcal{Q}} \) is identified with the minimal rational quaternionic-invariant subspace of \( \mathfrak{g} \) containing \( [\mathfrak{g}, \mathfrak{g}] \).

Denote the corresponding foliation on \( M = \Gamma \backslash G \) by \( \Sigma_{\mathcal{H}, \mathcal{Q}} \). The above argument implies that all meromorphic functions on \( (M, L) \) for any complex structure \( L \in \mathcal{H} \) outside of a countable set are constant on the leaves and factorize through the leaf space of \( \Sigma_{\mathcal{H}, \mathcal{Q}} \). However, this leaf space is actually a hyperkähler torus, which has no global meromorphic functions for general \( L \) by Proposition 1.3. This was a scheme of a proof of Corollary 3.8; detailed argument is given in Subsection 3.2.

To prove Theorem 1.9, we use the iterated fibration constructed in Lemma 4.1. From this lemma it follows that any nilmanifold with an abelian hypercomplex structure is represented as a principal toric bundle \( \pi : M \rightarrow M_1 \) over a nilmanifold \( M_1 \) with an abelian hypercomplex structure, and this fibration is trianalytic, that is, holomorphic with respect to all complex structures induced by quaternions.

Consider a subvariety \( X \subset M \). Using induction, we can assume that for all induced complex structures \( L \in S^2 \subset \mathbb{H} \), except a countable set, the image \( \pi(X) \) is trianalytic and locally homogeneous. Replacing \( M_1 \) by \( \pi(X) \), we may actually assume that the restriction \( \pi|_X \) is surjective. Applying Proposition 1.5, and excising more complex structures from \( S^2 \), we may assume that the fibers \( T \) of the principal toric bundle \( \pi \) are generic enough, and all irreducible complex subvarieties of these fibers are trianalytic subtori (Proposition 5.4).

Then for a general \( x \in M_1 \) there exists a subtorus \( T' \subset T \) such that \( \pi^{-1}(x) \) is a trianalytic subtorus, or a collection of subtori, if \( \pi^{-1}(x) \) has several irreducible components. In the latter case, \( T' \) can be considered as a function of a smooth point \( y \in \pi^{-1}(x) \). Clearly, the correspondence \( y \mapsto T' \) is continuous on the set of smooth points of \( X \). Since \( X \) is irreducible, its smooth locus is connected, hence the torus \( T' \) is actually constant as a function of \( y \). Replacing \( M \) by the quotient \( M/T' \), we can assume that \( X \) is a meromorphic multisection of \( \pi : M \rightarrow M_1 \). The branch locus and the exceptional locus of \( \pi : X \rightarrow M_1 \) produce divisors in \( M_1 \), which is impossible, because by induction hypotheses, all subvarieties of \( M_1 \) are trianalytic. Therefore, the map \( \pi : X \rightarrow M_1 \) is etale. Passing to a finite covering, we may assume that it is a section of the principal bundle \( \pi : M \rightarrow M_1 \). However, any section of a principal bundle trivializes it, giving \( M = M_1 \times T \), and its summand \( X = M_1 \) is trianalytic because this decomposition is compatible with a hypercomplex structure.

This was a scheme of a proof of Theorem 1.9; detailed argument is given in Subsection 5.4.
The hypercomplex rotation makes the argument much more accessible and easy to state. However, a similar result can be proven for any iterated fibration of tori, provided that these tori have algebraic dimension zero and are sufficiently generic. The “sufficiently generic” bit becomes ugly if one is interested in the maximal generality, and we won’t state it here.

2 Preliminaries

2.1 Complex nilmanifolds

Definition 2.1: Let $g$ be a Lie algebra.

- The lower central series of $g$ is the descending filtration on $g$

$$
\ldots \subset g_i \subset \ldots \subset g_1 \subset g_0 = g
$$

(2.1)

where $\forall i > 0: g_i := [g, g_{i-1}]$. In particular, $g_1 = [g, g]$ is the commutator subalgebra of $g$.

- The upper central series of $g$ is the ascending filtration on $g$

$$
0 = z^0 \subset z^1 \subset \ldots \subset z^i \subset \ldots
$$

(2.2)

where $\forall i: z^i := \{ x \in g | [x, g] \subset z^{i-1} \}$. In particular, $z^1 = z(g)$ is the center of $g$.

If $G$ is a group one defines the lower and upper central series of $G$ in a similar way.

Definition 2.2: A Lie algebra $g$ is called nilpotent if the following holds: $\exists k: g_k = 0$. The minimal number $k$ with this property is called the number of steps of a nilpotent Lie algebra $g$. If the number of steps of $g$ equals $k$, then one calls $g$ a $k$-step nilpotent Lie algebra. In a similar fashion one defines a nilpotent group, the number of steps of a nilpotent group and a $k$-step nilpotent group.

The following proposition is well known. For the convenience of the reader, we give its proof here.

Proposition 2.3: Let $g$ be a $k$-step nilpotent Lie algebra, $0 = g_k \subset g_{k-1} \subset \ldots g_1 \subset g_0 = g$ be its lower central series and $0 = z^0 \subset z^1 \subset \ldots$ be its upper central series. Then one has $z^k = g$ and $k$ is the minimal number with this property. Moreover, $\forall i = 0, \ldots, k$ one has

$$
g_{k-1} \subset z^i
$$

(2.3)

Proof: We prove the assertion by induction on the number $k$ of steps of $g$. For $k = 1$ (the case of abelian Lie algebras) the assertion is clear. Let $g$ be a $k$-step
nilpotent Lie algebra. Consider the quotient map \( \pi: g \to g/\mathfrak{z}^1 =: \tilde{g} \). The Lie algebra \( \tilde{g} \) is \((k-1)\)-step nilpotent. Define \( \tilde{z}^i \) and \( \tilde{g}_i \) to be respectively the upper and the lower central series of \( \tilde{g} \). One has

\[
\tilde{z}^i = \pi^{-1}(\tilde{z}^{i-1}), \quad g_j \subset \pi^{-1}(\tilde{g}_j)
\]

By induction hypothesis \( \tilde{z}^{k-1} = \tilde{g} \) while \( \tilde{z}^{k-2} \not\subset \tilde{g} \). Hence \( \tilde{z}^k = \pi^{-1}(\tilde{z}^{k-1}) = g \) while \( \tilde{z}^{k-1} = \pi^{-1}(\tilde{z}^{k-2}) \not\subset g \), and the first assertion follows. By induction hypothesis one has \( \tilde{g}^{k-1} = \tilde{g} \) while \( \tilde{g}^{k-2} \not\subset \tilde{g} \). Hence \( z^k = \pi^{-1}(\tilde{z}^{k-1}) = g \) while \( z^{k-1} = \pi^{-1}(\tilde{z}^{k-2}) \not\subset g \), and the first assertion follows. By induction hypothesis one has \( \tilde{g}^k - i \subset \tilde{z}^i \). We obtain that \( g^k - i \subset \pi^{-1}(\tilde{g}^k - i) \subset \pi^{-1}(\tilde{z}^{i-1}) = z^i \).

**Definition 2.4:** Let \( G \) be a nilpotent Lie group and \( \Gamma \subset G \) be a cocompact lattice\(^1\) in \( G \). Then the quotient \( X := \Gamma \backslash G \) is called a nilmanifold.

It was shown by Mal’cev ([Ma]) that the nilpotent Lie group \( G \) from **Definition 2.4** is uniquely determined by the nilmanifold \( X \).

**Proposition 2.5:** ([Ma]) Let \( \Gamma \) be a nilpotent finitely generated group. Then there exists a simply connected nilpotent Lie group \( G \) such that \( \Gamma \) embeds into \( G \) as a cocompact lattice. The group \( G \) is unique up to isomorphism and depends functorially on \( \Gamma \).

**Definition 2.6:** The group \( G \) from **Proposition 2.7** associated to a nilpotent group \( \Gamma \) is called the Mal’cev completion of \( \Gamma \).

**Proposition 2.7:** ([Ma]) Let \( X = \Gamma \backslash G \) be a nilmanifold, where \( G \) is a simply connected nilpotent Lie group and \( \Gamma \subset G \) a cocompact lattice. Then \( \Gamma \) is naturally isomorphic to \( \pi_1(X) \), and \( G \) to the Mal’cev completion of \( \pi_1(X) \).

Let \( \Gamma \subset G \) be a cocompact lattice. Consider the set \( \Lambda := \{ x \in g \mid \exp(x) \in \Gamma \} \). One can show that \( \Lambda \) is a maximal rank lattice in \( g \) ([Ma]). To be short, we will say that \( \Lambda = \log(\Gamma) \) in the sequel.

Let \( L \) be an almost complex structure\(^2\) on the Lie algebra \( g \) of \( G \). Consider the decomposition \( g \otimes \mathbb{C} = g^{1,0} \oplus g^{0,1} \) into the direct sum of eigenspaces of \( L \). The operator \( L \) multiplies \( g^{1,0} \) by \( \sqrt{-1} \) and multiplies \( g^{0,1} \) by \( -\sqrt{-1} \).

**Definition 2.8:** An almost complex structure \( L \) on \( g \) is called an integrable complex structure or simply a complex structure if \( g^{1,0} \) is a subalgebra of \( g \otimes \mathbb{C} \).

Be aware that the notion of a Lie algebra with an integrable complex structure \( L \) is much broader than the notion of a complex Lie algebra. Indeed, a

---

\(^1\)A cocompact lattice \( \Gamma \) is a discrete subgroup of \( G \) such that the quotient \( \Gamma \backslash G \) is compact.

\(^2\)An almost complex structure \( L \) on a vector space is a linear operator such that \( L^2 = -1 \).
complex structure $L$ defines a structure of a complex Lie algebra on $\mathfrak{g}$ if and only if
\[ \forall x, y \in \mathfrak{g}: L[x, y] = [Lx, y] = [x, Ly], \quad (2.4) \]
or equivalently, $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0$ (then $\mathfrak{g}^{1,0}$ is an ideal). If $L$ is an integrable complex structure on $\mathfrak{g}$ then we denote the corresponding left-invariant complex structure on $G$ again by $L$. The group $G$ with the complex structure $L$ is, generally speaking, not a complex Lie group, because $L$ is not in general right-invariant.

**Definition 2.9:** Let $G$ be a nilpotent group with a left-invariant complex structure. Then the left quotient $X$ of $G$ by a cocompact lattice $\Gamma \subset X$ is called a **complex nilmanifold**.

**Definition 2.10:** ([Ba], see also [BD1]) Let $\mathfrak{g}$ be a Lie algebra with a complex structure $L$ and let $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ be its eigenspace decomposition. If the subspace $\mathfrak{g}^{1,0}$ is an abelian subalgebra of $\mathfrak{g} \otimes \mathbb{C}$, then the complex structure $L$ is called an **abelian complex structure** on $\mathfrak{g}$.

The subspace $\mathfrak{g}^{1,0}$ consists of elements of the form $x - \sqrt{-1}Lx$, $x \in \mathfrak{g}$. The abelianess of $L$ is equivalent to the equality
\[ 0 = [x - \sqrt{-1}Lx, y - \sqrt{-1}Ly] =
\]
\[ = ([x, y] - [Lx, Ly]) - \sqrt{-1}([Lx, y] + [x, Ly]) \]
for every $x, y \in \mathfrak{g}$. The real and the imaginary part of the expression above must both vanish. Therefore, a complex structure $L$ is abelian if and only if
\[ \forall x, y \in \mathfrak{g}: [x, y] = [Lx, Ly] \quad (2.5) \]
or equivalently,
\[ \forall x, y \in \mathfrak{g}: [Lx, y] = -[x, Ly] \quad (2.6) \]

**Definition 2.11:** Assume that the Lie algebra $\mathfrak{g}$ is nilpotent. Equip the corresponding Lie group $G$ with the left-invariant complex structure induced by $L$. If $\Gamma \subset G$ is a cocompact lattice, then the nilmanifold $\Gamma \backslash G$ is called an **abelian complex nilmanifold**.

**Definition 2.12:** A **hypercomplex structure** on a Lie algebra $\mathfrak{g}$ is a triple of integrable complex structures $(I, J, K)$ on $\mathfrak{g}$ satisfying quaternionic relations (see equation $(2.19)$ below). We call a hypercomplex structure on a Lie algebra $\mathfrak{g}$ **abelian** if the complex structures $I, J, K$ are abelian.

By [DF1, Lemma 3.1] for any hypercomplex structure $(I, J, K)$ the complex structures $J$ and $K$ are abelian whenever $I$ is abelian. Therefore, in the definition above it is actually enough to require that only one of the complex structures $I, J, K$ is abelian.
The definition of hypercomplex nilmanifolds and abelian hypercomplex nilmanifolds is analogous to Definition 2.9. The complex (resp. hypercomplex etc) structures on complex (resp. hypercomplex etc) nilmanifolds are examples of so called locally $G$-invariant structures, as explained in the following definition.

**Definition 2.13:** Let $X = \Gamma \backslash G$ be a nilmanifold and $\mathfrak{g}$ be the Lie algebra of $G$. Consider a tensor $\eta \in \mathfrak{g}^k \otimes (\mathfrak{g}^*)^l$. Let $\tilde{\eta}$ be the left-invariant tensor on $G$ induced by $\eta$. The tensor $\tilde{\eta}$ clearly descends to a tensor on $X$. In such a situation the induced tensor on a nilmanifold $X$ is called a locally $G$-invariant tensor and the corresponding geometric structure is called a locally $G$-invariant geometric structure.

Be aware that a locally $G$-invariant tensor on a nilmanifold $X = \Gamma \backslash G$ is generally not invariant with respect to the right action of $G$ on $X$. In other words, a complex nilmanifold is generally speaking not a homogeneous complex manifold, though its universal cover is $G$-homogeneous. We will call such structures on $\Gamma \backslash G$ locally homogeneous complex structures.

### 2.2 Algebraic dimension

**Definition 2.14:** Let $X$ be a compact complex manifold. Consider the field $K(X)$ of meromorphic functions on $X$. The transcendence degree of $K(X)$ over $\mathbb{C}$ is called the algebraic dimension of $X$ and denoted by $\dim_{alg}(X)$.

One always has $\dim_{alg}(X) \leq \dim(X)$. Recall that a complex manifold $X$ is called Moishezon if $X$ is bimeromorphic to a complex projective variety. The equality $\dim_{alg}(X) = \dim(X)$ holds if and only if $X$ is Moishezon ([Mo]).

Let $X$ be a compact complex manifold. By [Ca] there exists a meromorphic dominant map $r$ from $X$ to a projective variety $X^{alg}$ which induces an isomorphism $r^* : K(X^{alg}) \to K(X)$ on the fields of meromorphic functions. The variety $X^{alg}$ can be constructed as follows. Let $(t_1, t_2, \ldots, t_N)$ be generators of $K(X)$ over $\mathbb{C}$ as a field. They define a meromorphic map $r : X \to \mathbb{C}P^N$ sending a point $x \in X$ outside of the poles of $t_i$’s to $[1 : t_1(x) : \ldots : t_N(x)] \in \mathbb{C}P^N$. By [Ca] the closure $X^{alg}$ of the image of $r$ in $\mathbb{C}P^N$ is a projective variety such that the map $r^* : K(X^{alg}) \to K(X)$ is an isomorphism.

**Definition 2.15:** Let $X$ be a compact complex manifold. An algebraic reduction of $X$ is a projective variety $X^{alg}$ together with a meromorphic dominant map $r : X \to X^{alg}$ such that the induced map $r^* : K(X^{alg}) \to K(X)$ is an isomorphism.

An algebraic reduction $X^{alg}$ of $X$ is defined uniquely up to a birational isomorphism. Moreover, every meromorphic map from $X$ to an algebraic variety is uniquely factorized through the meromorphic map $r : X \to X^{alg}$ ([Ca]).
Lemma 2.16: Let \( f: X \rightarrow Y \) be a dominant map of complex manifolds. Then \( \dim_{\text{alg}}(X) \geq \dim_{\text{alg}}(Y) \). If, moreover, the map \( f \) is finite, then the equality holds.

Proof: The map \( f \) induces the embedding of the field \( K(Y) \) into \( K(X) \). Hence the transcendence degree of \( K(Y) \) does not exceed that of \( K(X) \). If \( f \) is a finite map, then the extension \( K(Y) \subset K(X) \) is finite. Hence the transcendence degrees of both fields coincide. 

2.3 Albanese variety

Let \( X \) be a compact complex manifold. Denote by \( d\Theta_X \) the sheaf of closed holomorphic differentials\(^3\) on \( X \). Consider a short exact sequence of sheaves on \( X \)

\[
0 \rightarrow \mathbb{C} \rightarrow \Theta_X \rightarrow d\Theta_X \rightarrow 0 \quad (2.7)
\]

It induces an embedding

\[
H^0(X, d\Theta_X) \hookrightarrow H^1(X, \mathbb{C}) \quad (2.8)
\]

of cohomology groups. Let us dualize the map (2.8) to get a surjection

\[
H_1(X, \mathbb{C}) \twoheadrightarrow H^0(X, d\Theta_X)^* \quad (2.9)
\]

Definition 2.17: ([Bl],[Ue]) Let \( X \) be a compact complex manifold. Consider the composition of maps

\[
H_1(X, \mathbb{Z}) \rightarrow H_1(X, \mathbb{C}) \rightarrow H^0(X, d\Theta_X)^* \quad (2.10)
\]

The quotient \( \text{Alb}(X) \) of the space \( H^0(X, d\Theta_X)^* \) by the minimal complex closed subgroup of \( H^0(X, d\Theta_X)^* \) containing the image of \( H_1(X, \mathbb{Z}) \) is called the Albanese variety of \( X \).

Proposition 2.18: Let \( X \) be a compact complex manifold. Then its Albanese variety \( \text{Alb}(X) \) is a compact complex torus.

Proof: By definition, \( \text{Alb}(X) \) is a quotient of \( H^0(X, d\Theta_X)^* \) by a closed subgroup \( R \) which contains the closure of the image of \( H_1(X, \mathbb{Z}) \). To prove that \( \text{Alb}(X) \) is compact, it is enough to show that the \( \mathbb{R} \)-vector space generated by \( R \) in \( H^0(X, d\Theta_X)^* \) coincides with the whole vector space \( H^0(X, d\Theta_X)^* \). Hence it suffices to check that the map (2.9) sends \( H_1(X, \mathbb{R}) \) surjectively onto \( H^0(X, d\Theta_X)^* \), or equivalently, that the map

\[
H^0(X, d\Theta_X) \xrightarrow{\text{Re}} H^1(X, \mathbb{R}) \quad (2.11)
\]

\(^3\)A closed holomorphic differential on a complex manifold \( X \) is a closed \((1,0)\)-form.
is injective. Consider a closed holomorphic differential $\eta \in H^0(X, d\Theta_X)$ such that $\text{Re} \eta = df$ for some smooth function $f$. Then $\eta = 2\partial f$ and $\overline{\partial} f = -\frac{i}{2} \partial \eta = 0$. We obtain that $f$ is a pluri-harmonic function on a compact complex manifold, hence constant by maximum principle. Hence $\eta = 2\partial f = 0$. We have proved that the map (2.11) is injective. ■

Fix a point $x_0 \in X$. For any point $x \in X$ choose a path $\gamma$ connecting the points $x_0$ and $x$. The path $\gamma$ defines a functional $a_\gamma$ on the space $H^0(X, d\Theta_X)$ of closed holomorphic differentials on $X$ in the following way:

$$a_\gamma(\eta) := \int_\gamma \eta$$  \hspace{1cm} (2.12)

If $\gamma'$ is another path connecting the points $x_0$ and $x$ then the functional $a_{\gamma'} - a_\gamma \in H^0(X, d\Theta_X)^*$ lies in the image of $H_1(X, \mathbb{Z})$ in $H^0(X, d\Theta_X)^*$ by the map (2.10). We obtain a correctly defined map

$$A: X \to \text{Alb}(X) \quad x \mapsto a_\gamma$$  \hspace{1cm} (2.13)

where $\gamma$ is any path connecting $x_0$ and $x$.

**Definition 2.19:** Let $X$ be a compact complex manifold. The map $A: X \to \text{Alb}(X)$ defined in (2.13) is called the **Albanese map** of $X$.

One can show that the Albanese map $A: X \to \text{Alb}(X)$ is holomorphic ([Bl]).

Let $X = \Gamma \backslash G$ be a nilmanifold (Definition 2.9). Consider a left-invariant vector field $\xi$ on $G$ corresponding to a vector $\xi \in \mathfrak{g}$. Then $\xi$ descends to a locally $G$-invariant vector field on $X = \Gamma \backslash G$ denoted also by $\xi$. Let $\text{Vol}_X$ be the volume form on $X$ induced by a left-invariant volume form on $G$ such that $\int_X \text{Vol}_X = 1$. Given a 1-form $\eta$ on $X$ we define a covector $\eta_{\text{inv}} \in \mathfrak{g}^*$ as follows

$$\eta_{\text{inv}}(\xi) := \int_X \eta(\xi) \text{Vol}_X$$  \hspace{1cm} (2.14)

One can show that if $\eta$ is a closed 1-form on $X$ then $\eta_{\text{inv}}$ is a closed 1-form on $\mathfrak{g}$. Moreover, if $\eta$ is exact, the form $\eta_{\text{inv}}$ vanishes ([No]). Therefore, the map sending a form $\eta \in \Gamma(\Omega^1 X)$ to $\eta_{\text{inv}} \in \mathfrak{g}^*$ descends to a map $A\nu: H^1(X, \mathbb{R}) \to H^1(\mathfrak{g}, \mathbb{R})$ of cohomology groups.

**Definition 2.20:** ([FGV]) Let $X = \Gamma \backslash G$ be a nilmanifold. The map

$$A\nu: H^1(X, \mathbb{R}) \to H^1(\mathfrak{g}, \mathbb{R}) = \left( \frac{\mathfrak{g}}{[\mathfrak{g}, \mathfrak{g}]} \right)^*$$  \hspace{1cm} (2.15)

sending the cohomology class of a 1-form $\eta$ on $X$ to the cohomology class of the form $\eta_{\text{inv}} \in \mathfrak{g}^*$ is called the **averaging map**.
Proposition 2.21: ([No]) Let \( X = \Gamma \backslash G \) be a nilmanifold. Every closed 1-form \( \eta \) on \( X \) is cohomologous to the locally \( G \)-invariant 1-form on \( X \) associated to \( \eta_{\text{inv}} \in g^* \). Moreover, the averaging map \( \text{Av}: H^1(X, \mathbb{R}) \rightarrow H^1(g, \mathbb{R}) \) is an isomorphism. ■

Let \( X = \Gamma \backslash G \) be a complex nilmanifold. Denote by \( \Lambda = \log(\Gamma) \subset g \) a maximal rank lattice in \( g \) and by \( L \) the complex structure on \( g \). For any \( \mathbb{R} \)-subspace \( W \subset g \) let us denote by \( W_{Q,L} \) the minimal rational \( L \)-invariant subspace of \( g \) containing \( W \). Here we call a subspace \( W \subset V \) rational if \( \Lambda \cap W \) is a maximal rank lattice in \( W \). The subspace \( g[g,g]_{Q,L} \) is a normal subalgebra of \( g \), indeed, one has the following inclusions:

\[
[g,g]_{Q,L} \subset [g,g] \subset [g,g]_{Q,L}
\]

The quotient Lie algebra \( t := g/(g,g)_{Q,L} \) is an abelian Lie algebra. Let \( \Lambda_T \) denote the image of \( \Lambda \) in \( t \). The subgroup \( \Lambda_T \subset t \) is a lattice of maximal rank in \( t \). The quotient map \( p: g \rightarrow t \) induces a map on nilmanifolds:

\[
P: X \rightarrow T := t/\Lambda_T
\] (2.16)

Proposition 2.22: (see also [R1]) Let \( X = \Gamma \backslash G \) be a complex nilmanifold. The complex torus \( T \) defined in equation (2.16) is the Albanese variety of \( X \) (Definition 2.17).

Proof: The averaging map \( \text{Av}: H^1(X, \mathbb{C}) \rightarrow H^1(g, \mathbb{C}) \) sends the subspace \( H^0(X, d\mathcal{O}_X) \subset H^1(X, \mathbb{C}) \) isomorphically onto the subspace

\[
\{ \alpha \in (g^{1,0})^* | d\alpha = 0 \} = \left( \frac{g}{[g,g] + L[g,g]} \right)^* \subset H^1(X, \mathbb{C})
\]

The image of \( H_1(X, \mathbb{Z}) \) in the space \( H^0(X, d\mathcal{O}_X)^* = g/(g,g) + L[g,g] \) by the map (2.10) is precisely the image of the lattice \( \Lambda \subset g \) in this space ([No]). Consider a complex subgroup \( [g,g]_{Q,L} + \Lambda \) of \( g \). This subgroup is mapped onto a closed complex subgroup of \( g/(g,g) + L[g,g] \) under the quotient map \( g \rightarrow g/(g,g) + L[g,g] \). Moreover, the image of \( [g,g]_{Q,L} + \Lambda \) in \( H^0(X, d\mathcal{O}_X)^* \) is the minimal closed complex subgroup of \( H^0(X, d\mathcal{O}_X)^* \) containing the image of \( H_1(X, \mathbb{Z}) \). Hence,

\[
\text{Alb}(X) = \frac{g}{[g,g]_{Q,L} + \Lambda}
\] (2.17)

as desired. ■

Proposition 2.23: ([FGV]) Every meromorphic map from a complex nilmanifold \( X \) to a Kähler manifold is uniquely factorized through the Albanese map (2.16).
As an immediate corollary of Proposition 2.23 one has:

**Corollary 2.24:** ([FGV]) Let $X$ be a complex nilmanifold and $T$ the Albanese variety of $X$. Then the following holds.

$$ \dim_{\text{alg}}(X) = \dim_{\text{alg}}(T) \quad (2.18) $$

**Proof:** Lemma 2.16 implies that $\dim_{\text{alg}}(X) \geq \dim_{\text{alg}}(T)$. Let $r: X \to X^{\text{alg}}$ be an algebraic reduction of $X$. The map $r$ factorizes uniquely through the Albanese map $P: X \to T$ as follows from Proposition 2.23. We use Lemma 2.16 again to obtain that $\dim_{\text{alg}}(T) \geq \dim_{\text{alg}}(X^{\text{alg}}) = \dim_{\text{alg}}(X)$. ■

### 2.4 Hypercomplex manifolds

Recall that the algebra $\mathbb{H}$ of quaternions is a 4-dimensional algebra generated over $\mathbb{R}$ by $I, J, K$ subject to the relations

$$ I^2 = J^2 = K^2 = -1; \quad IJ = -JI = K \quad (2.19) $$

An element $L = t + aI + bJ + cK \in H$ satisfies $L^2 = -1$ if and only if $t = 0$ and $a^2 + b^2 + c^2 = 1$. Therefore such elements form a 2-dimensional sphere inside of $\mathbb{H}$.

**Definition 2.25:** Let $X$ be a smooth manifold equipped with a linear action of the algebra $\mathbb{H}$ on $TX$ i.e. an $\mathbb{R}$-linear map $\mathbb{H} \to \text{End}(TX)$. Such a manifold is called **almost hypercomplex**. Every element $L \in \mathbb{H}$ such that $L^2 = -1$ induces an almost complex structure on $X$ which we denote by the same symbol $L$. If every complex structure $L$ on $X$ induced from quaternions is integrable, $X$ is called a **hypercomplex manifold**.

In the definition of a hypercomplex manifold it is actually enough to require the integrability of only two linearly independent complex structures ([K],[Ob]). For any complex structure $L \in \mathbb{H}$ we denote by $X_L$ the corresponding complex manifold.

Every (almost) hypercomplex manifold $X$ admits a **hyper-Hermitian metric** $g$ i.e. a Riemannian metric which is Hermitian with respect to $I, J, K$. One can obtain such a metric by averaging an arbitrary metric on $X$ by the group $SU(2)$ of quaternions of unit norm.

**Definition 2.26:** A hypercomplex manifold equipped with a fixed hyper-Hermitian metric is called a **hyper-Hermitian manifold**.

For every complex structure $L \in \mathbb{H}$ on a hyper-Hermitian manifold $X$ one defines a 2-form $\omega_L$ on $X$ as $\omega_L(x, y) := g(Lx, y)$.

**Definition 2.27:** A hyper-Hermitian manifold $X$ is called **hyperkähler** if the 2-forms $\omega_L$ are closed for every complex structure $L \in \mathbb{H}$.
Define $\Omega_I := \omega_J + \sqrt{-1}\omega_K$. One can easily check that $\Omega_I$ is a $(2,0)$-form.

**Definition 2.28:** A hyper-Hermitian manifold $X$ is called an **HKT-manifold** ("hyperkähler with torsion") if $\partial\Omega_I = 0$. Here $\partial: \Lambda^{p,q}_I X \to \Lambda^{p+1,q}_I X$ is a $(1,0)$-part of the de Rham differential.

Be aware that the HKT-condition (Definition 2.28) is weaker than the hyperkähler condition (Definition 2.27).

A complex nilmanifold $X = \Gamma\backslash G$ admits a Kähler structure if and only if $X$ is a complex torus ([BeGo]). Hence the only example of a nilmanifold with a hyperkähler structure is a hypercomplex torus, i.e. a quotient of a quaternionic vector space by a maximal rank lattice. It was shown in [FG] that a hypercomplex nilmanifold $X = \Gamma\backslash G$ admits an HKT-metric if and only if it admits a $G$-invariant HKT-metric.

**Proposition 2.29:** ([BDV, Thm. 4.6], [DF2, Prop. 2.1]) Let $X = \Gamma\backslash G$ be a hypercomplex nilmanifold with an invariant hyper-Hermitian metric $g$. The metric $g$ is HKT if and only if the hypercomplex structure on $X$ is abelian.

### 3 $H$-Albanese variety

#### 3.1 Definition of an $H$-Albanese variety

For a compact hypercomplex manifold $X$ consider the vector space $(\Omega^1 X)_{par}$ of real 1-forms $\eta \in \Omega^1 X$ such that

$$d\eta = dI\eta = dJ\eta = dK\eta = 0$$

**Proposition 3.1:** Let $X$ be a hypercomplex manifold. A 1-form $\eta$ satisfies the condition (3.1) if and only if $\eta$ is parallel with respect to the Obata connection $\nabla$ on $X$. \footnote{For the definition of Obata connection, see Subsection 1.3.}

**Proof:** For any tensor $\alpha \in \Omega^1 \otimes \Omega^1 = S^2(\Omega^1 X) \oplus \Omega^2 X$ consider the decomposition $\alpha = \alpha_{sym} + \alpha_{skew}$ where $\alpha_{sym} \in S^2(\Omega^1 X)$ and $\alpha_{skew} \in \Omega^2 X$. The Obata connection $\nabla$ is torsion-free, hence for any 1-form $\eta$ one has $(\nabla\eta)_{skew} = \frac{1}{2}d\eta$.

Assume that $\eta$ is a $\nabla$-parallel 1-form. Then $d\eta = 2(\nabla\eta)_{skew} = 0$. For any complex structure $L \in H$ one has $\nabla L\eta = L\nabla\eta = 0$. Hence $dL\eta = 0$.

Assume that $\eta$ satisfies the condition (3.1). Let $\eta^{1,0}$ be the $(1,0)$-part of $\eta$ with respect to the complex structure $I$. By [So, Prop. 2.2] we have

$$\nabla\eta^{1,0} = \overline{\partial}\eta^{1,0} - J\partial J\eta^{1,0} = 0$$

\footnote{For the definition of Obata connection, see Subsection 1.3.}
Hence $\nabla \eta = 2 \text{Re}(\nabla \eta^{1,0}) = 0$. The proposition is proved.

Proposition 3.1 justifies the notation $(\Omega^1 X)^{\text{par}}$ introduced above.

The space $(\Omega^1 X)^{\text{par}}$ is preserved by the linear action of $\mathbb{H}$ on 1-forms. Take a 1-form $\eta \in (\Omega^1 X)^{\text{par}}$. For every complex structure $L \in \mathbb{H}$ the form $\eta$ is the real part of a closed $L$-holomorphic 1-form $\eta_{L}^{1,0} := \frac{\eta + \sqrt{-1} L \eta}{2}$. Denote by $\mathcal{O}^L_X$ the sheaf of holomorphic functions on the complex manifold $X_L$. For every complex structure $L \in \mathbb{H}$ we have the following embeddings

$$
(\Omega^1 X)^{\text{par}} \xrightarrow{\eta \mapsto \eta_{L}^{1,0}} H^0(X, d\mathcal{O}^L_X) \xrightarrow{\text{Re}} H^1(X, \mathbb{R})
$$

We introduce the following definition.

Definition 3.2: Let $X$ be a compact hypercomplex manifold. Consider the composition of maps

$$
H_1(X, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{R}) \twoheadrightarrow ((\Omega^1 X)^{\text{par}})^*
$$

The quotient $\text{Alb}_{\mathbb{H}}(X)$ of the space $((\Omega^1 X)^{\text{par}})^*$ by the minimal trianalytic closed subgroup of $((\Omega^1 X)^{\text{par}})^*$ containing the image of $H_1(X, \mathbb{Z})$ is called the $\mathbb{H}$-Albanese variety of $X$.

For any complex structure $L \in \mathbb{H}$ the surjection $H^0(X, d\mathcal{O}^L_X)^* \twoheadrightarrow ((\Omega^1 X)^{\text{par}})^*$ induces a holomorphic map

$$
\varphi_L : \text{Alb}(X_L) \longrightarrow (\text{Alb}_{\mathbb{H}}(X))_L
$$

Fix a point $x_0 \in X$. Similarly to the complex case we can construct a map

$$
A_{\mathbb{H}} : X \rightarrow \text{Alb}_{\mathbb{H}}(X)
$$

The map $A_{\mathbb{H}}$ is defined as follows. For any $x \in X$ choose a path $\gamma$ connecting $x_0$ and $x$. The path $\gamma$ defines a functional $a_{\gamma} \in ((\Omega^1 X)^{\text{par}})^*$ as

$$
a_{\gamma}(\eta) = \int_{\gamma} \eta \quad \forall \eta \in (\Omega^1 X)^{\text{par}}
$$

The map $A_{\mathbb{H}}$ sending $x \in X$ to $a_{\gamma}$ is a well-defined map from $X$ to $\text{Alb}_{\mathbb{H}}(X)$. This map is called the $\mathbb{H}$-Albanese map of $X$.

By construction, for every complex structure $L \in \mathbb{H}$ the $\mathbb{H}$-Albanese map $A_{\mathbb{H}} : X \rightarrow \text{Alb}_{\mathbb{H}}(X)$ is equal to the composition of maps

$$
X \xrightarrow{A_L} \text{Alb}(X_L) \xrightarrow{\varphi_L} \text{Alb}_{\mathbb{H}}(X)
$$

where $A_L$ is the Albanese map of $X_L$. Both maps $A_L : X \rightarrow \text{Alb}(X_L)$ and $\varphi_L : \text{Alb}(X_L) \rightarrow (\text{Alb}_{\mathbb{H}}(X))_L$ are holomorphic. It follows that the $\mathbb{H}$-Albanese map preserves the hypercomplex structure.
3.2 Albanese varieties of a hypercomplex nilmanifold

Let $V$ be a real vector space equipped with an action of $\mathbb{H}$ by linear endomorphisms and $\Lambda \subset V$ be a lattice of maximal rank. Recall that a real subspace $W \subset V$ is called rational if $\Lambda \cap W$ is a maximal rank lattice in $W$. Let $L \in \mathbb{H}$ be a complex structure. For any real vector subspace $W \subset V$ let $W_{Q,L}$ denote the minimal rational $L$-invariant subspace of $V$ containing $W$ and let $W_{Q,H}$ denote the minimal rational $H$-invariant subspace of $V$ containing $W$ (see also Subsection 2.2).

Lemma 3.3: Let $V$ be a real vector space with a linear action of $\mathbb{H}$, $\Lambda \subset V$ a maximal rank lattice. Then for all but at most countable number of complex structures $L \in \mathbb{H}$ the following property is satisfied: if $W \subset V$ is a rational $L$-invariant subspace of $V$ then $W$ is $H$-invariant.

Proof: Let $W \subset V$ be a subspace of $V$. If $W$ is invariant with respect to two linearly independent complex structures $L, L' \subset V$ then $W$ is $\mathbb{H}$-invariant. Indeed, every two linearly independent complex structures generate $\mathbb{H}$ as an algebra. Therefore, every subspace $W \subset V$ is either not invariant with respect to all complex structures $L \in \mathbb{H}$, or invariant with respect to exactly two complex structures $L, -L \in \mathbb{H}$, or $\mathbb{H}$-invariant. The set of rational subspaces $W$ which are invariant with respect to exactly two complex structures $L_W, -L_W \in \mathbb{H}$ is countable. Let us exclude the complex structures which arise as such $L_W$. For every other complex structure $L \in \mathbb{H}$ each rational $L$-invariant subspace is $\mathbb{H}$-invariant. □

Lemma 3.4: Let $W \subset V$ be a real subspace, $W_{Q,L}$ and $W_{Q,H}$ be as above. Then for all but at most countable number of complex structures $L \in \mathbb{H}$ one has $W_{Q,L} = W_{Q,H}$.

Proof: By Lemma 3.3 for all but at most countable number of complex structures $L \in \mathbb{H}$ the vector subspace $W_{Q,L}$ is $\mathbb{H}$-invariant. If this is the case then one has $W_{Q,L} = W_{Q,H}$. □

Let $X = \Gamma \backslash G$ be a hypercomplex nilmanifold (Definition 2.12), $\mathfrak{g}$ the Lie algebra of $G$, and $\Lambda := \log(\Gamma)$ the lattice in $\mathfrak{g}$. We consider the abelian Lie algebra $\mathfrak{t} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]_{Q,H}$ with a hypercomplex structure (compare with Subsection 2.2 where $\mathfrak{t}$ was defined as $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}_{Q,L}]$). Let $\Lambda_T$ be the image of the lattice $\Lambda$ in $\mathfrak{t}$. The quotient map $p: \mathfrak{g} \to \mathfrak{t}$ induces a map

$$P_{\mathfrak{t}}: X \to T := \mathfrak{t}/\Lambda_T \quad (3.8)$$

It follows from the construction that the map $P_{\mathfrak{t}}$ is compatible with the hypercomplex structure.

Proposition 3.5: Let $X = \Gamma \backslash G$ be a hypercomplex nilmanifold. The hypercomplex torus $T$ defined in (3.8) is the $H$-Albanese variety of $X$ (Definition 3.2).
Proof: The proof follows the same lines as the proof of Proposition 2.22.

Theorem 3.6: Let $X$ be a hypercomplex nilmanifold and $T$ the $H$-Albanese variety of $X$. Then for all but at most countable number of complex structures $L \in H$ the map $P_H : X \to T$, considered as a morphism of complex manifolds $P_H : X_L \to T_L$, is the Albanese map of $X_L$.

Proof: Lemma 3.4 applied to the subspace $[g, g] \subset g$ implies that for all but at most countable number of complex structures $L \in H$ one has $[g, g]_{Q, L} = [g, g]_{Q, H}$. By the construction in Section 2.2 the Albanese variety of $X_L$ is the quotient of $g/[g, g]_{Q, L}$ by a lattice.

Conjecture 3.7: Theorem 3.6 is true for any hypercomplex manifold $X$.

The following statement is an analogue of Proposition 1.3 for hypercomplex nilmanifolds.

Corollary 3.8: Let $X = \Gamma \backslash G$ be a hypercomplex nilmanifold. Then for all but at most countable complex structures $L \in H$ one has $\dim_{alg}(X_L) = 0$.

Proof: By Corollary 2.24 the algebraic dimension of $X_L$ is equal to the algebraic dimension of the Albanese variety of $X_L$. By Theorem 3.6 for all but at most countable number of complex structures $L \in H$ the Albanese variety of $X_L$ is isomorphic to $T_L$ where $T$ is the $H$-Albanese variety of $X$. Any translation-invariant hyper-Hermitian metric on a hypercomplex torus is hyperkähler. Proposition 1.3 implies $\dim_{alg}(T_L) = 0$ for all but at most countable number of complex structures $L \in H$.

4 Abelian (hyper-)complex varieties and their Albanese varieties

The aim of this section is to deduce several useful observations about the geometry of abelian hyper(complex) varieties from a collection of linear algebra results. As before, we denote by $g$ a nilpotent Lie algebra, by $z^i$ the elements of its upper central series and by $g_i$ the elements of its lower central series.

Lemma 4.1: Let $g$ be a nilpotent Lie algebra with an abelian complex structure $L$ (Definition 2.10). Consider the upper central series $0 = z^0 \subset z^1 \subset \ldots \subset z^k = g$ of $g$. Then for all $i = 0, \ldots, k$, the subalgebras $z^i$ are $L$-invariant. If $g$ is equipped with an abelian hypercomplex structure $(I, J, K)$ then the subalgebras $z^i$ are $H$-invariant.

Proof: The proof goes by induction on $i$. The statement trivially holds for $i = 0$. Assume we know the statement for the subalgebra $z^i$. Take an element $x \in z^{i+1}$. For any $y \in g$ one has $[Lx, y] = -[x, Ly]$ as the complex structure $L$
is abelian (see equation (2.6)). As \(x\) is contained in \(\mathfrak{z}^{i+1}\), one has \([x, L y] \in \mathfrak{z}^i\). Therefore, \(L x\) also lies in \(\mathfrak{z}^{i+1}\). Let us assume that \(\mathfrak{g}\) is equipped with an abelian hypercomplex structure. For every \(i\) the subalgebra \(\mathfrak{z}^i\) is \(L\)-invariant for each complex structure \(L \in \{I, J, K\}\). Hence \(\mathfrak{z}^i\) is \(H\)-invariant. ■

Be aware that the statement of the lemma, generally speaking, does not hold for lower central series. Indeed, an element of the form \(L [x, y]\) does not necessarily lie in the commutator of \(\mathfrak{g}\), even for a Kodaira surface. Indeed, consider the Kodaira surface, determined by the Lie algebra \(\mathfrak{g} \langle x, y, z, t \rangle\) as in Example 1.7. Then \([\mathfrak{g}, \mathfrak{g}] = \langle z \rangle\), and \(I(z) \notin [\mathfrak{g}, \mathfrak{g}]\).

**Lemma 4.2:** Let \(\mathfrak{g}\) be a nilpotent Lie algebra equipped with a maximal rank lattice \(\Lambda \subset \mathfrak{g}\) invariant with respect to the Lie bracket. Then the terms \(\mathfrak{g}_i\) of the lower central series of \(\mathfrak{g}\) and the terms \(\mathfrak{z}_i\) of the upper central series of \(\mathfrak{g}\) are rational subspaces.

**Proof:** Clear, because \(\mathfrak{g}\) is a rational algebra. ■

**Lemma 4.3:** Let \(X = \Gamma \backslash G\) be a complex nilmanifold, \(\mathfrak{g}\) the Lie algebra of \(G\) and \(\mathfrak{z} \subset \mathfrak{g}\) the center of \(\mathfrak{g}\). Consider a rational \(L\)-invariant central subalgebra \(\mathfrak{z}' \subset \mathfrak{z}\) and define \(Z' := \exp(\mathfrak{z}')\). Then the holomorphic map

\[
\pi: X = \Gamma \backslash G \longrightarrow Y := \frac{G/Z'}{\Gamma/(\Gamma \cap Z')} \quad (4.1)
\]

is a holomorphic principal toric fiber bundle. Its fiber is \(T' := Z'/(\Gamma \cap Z')\).

**Proof:** We claim that the right action of \(Z'\) on \(G\) is holomorphic i.e. the map

\[
R: G \times Z' \rightarrow G \quad (g, z) \mapsto gz \quad (4.2)
\]

is holomorphic. Indeed, \(R\) is holomorphic in \(z \in Z'\) because the left action of \(G\) on itself preserves the complex structure. The map \(L\) is holomorphic in \(g \in G\) because the right action of \(Z'\) on \(G\) coincides with the left action. The holomorphic right action of \(Z'\) on \(G\) descends to a holomorphic action of \(Z'/(\Gamma \cap Z')\) on \(Y = \frac{G/Z'}{\Gamma/(\Gamma \cap Z')}\). The group \(Z'/(\Gamma \cap Z')\) acts transitively on the fibers of \(\pi: X \rightarrow Y\), hence \(\pi\) is a principal toric fiber bundle. ■

**Proposition 4.4:** Let \(X = \Gamma \backslash G\) be an abelian complex nilmanifold (Definition 2.10). Then

(i) The Albanese variety of \(X\) is positive dimensional.

(ii) ([R2, Cor. 3.11]) There exists a finite sequence of holomorphic maps

\[
X = X_0 \rightarrow X_1 \rightarrow \ldots \rightarrow X_{N-1} \rightarrow X_N = \{pt\} \quad (4.3)
\]

such that for every \(i\) the manifold \(X_i\) is an abelian complex nilmanifold (Definition 2.11) and the map \(X_i \rightarrow X_{i+1}\) is a principal toric fiber bundle.
Proof of (i): By Proposition 2.3 the commutator subalgebra \([g, g]\) is contained in \(S^{k-1}\) where \(k\) is the number of steps of \(g\). Lemma 4.1 together with Lemma 4.2 imply that \(S^{k-1}\) is a proper \(L\)-invariant rational subspace of \(g\). Hence the minimal rational \(L\)-invariant subspace containing \([g, g]\), which we denote by \([g, g]_{Q, L}\) as before, is contained in \(S^{k-1}\). In particular, \([g, g]_{Q, L}\) is a proper subspace of \(g\). If \(t\) denotes the quotient algebra \(g/\left[g, g\right]_{Q, L}\) and \(\Lambda_T\) the image of the lattice \(\Lambda\) in \(t\), then the Albanese variety of \(X\) is nothing but \(t/\Lambda_T\) (see Subsection 2.2). It follows that the Albanese variety of \(X\) is positive dimensional.

Proof of (ii): Let \(Z \subset G\) be the center of \(G\). Define \(G_1\) to be the quotient group \(G/Z\) and \(\Gamma_1\) to be the image of \(G\) in \(G_1\) by the quotient map. Consider the map
\[
\pi: X = \Gamma \backslash G \to X_1 := \Gamma_1 \backslash G_1
\]
(4.4)
By Lemma 4.3 the map \(\pi\) is a principal toric fiber bundle. We repeat this construction with \(X_1 = \Gamma_1 \backslash G_1\) and so on to obtain the desired sequence of maps.

There exists a similar statement for abelian hypercomplex nilmanifolds. We will omit its proof as it follows the same lines as the proof of Proposition 4.4.

Proposition 4.5: Let \(X = \Gamma \backslash G\) be an abelian hypercomplex nilmanifold (Definition 2.12). Then

(i) The \(H\)-Albanese variety (Definition 3.2) of \(X\) is positive-dimensional.
(ii) There exists a finite sequence of submersions of hypercomplex manifolds
\[
X = X_0 \to X_1 \to \ldots \to X_{N-1} \to X_N = \{pt\}
\]
(4.5)
This sequence satisfies the following properties. First, for every \(i\) the manifold \(X_i\) is an abelian hypercomplex nilmanifold and the maps \(X_i \to X_{i+1}\) are morphisms of hypercomplex manifolds. Second, the map \((X_i)_L \to (X_{i+1})_L\) is a holomorphic principal toric fiber bundle for any complex structure \(L \in H\).

5 Subvarieties in abelian hypercomplex nilmanifolds

5.1 Subvarieties in hypercomplex tori and nilmanifolds

Let \(X = \Gamma \backslash G\) be a nilmanifold and \(g\) the Lie algebra of \(G\). Consider the smooth trivialization of the tangent bundle to \(G\) by the left action of \(G\) on itself. For every \(g \in G\) we identify \(g\) with \(T_g G\) by means of the operator \(L_g\) of the left multiplication on \(g\):
This trivialization descends to a smooth trivialization of the tangent bundle to $X = \Gamma \backslash G$. Indeed, the left action of $\Gamma$ on $G$ preserves the trivialization by left-invariant vector fields. Hence we may and will identify $T_x X$ and $\mathfrak{g}$ for every point $x \in X$.

**Definition 5.1:** Let $X = \Gamma \backslash G$ be a nilmanifold and $Y \subset X$ be a submanifold of $X$. The submanifold $Y$ is said to be a *locally homogeneous submanifold* of $X$ if there exists a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that for every point $y \in Y$ the tangent space $T_y Y$ is identified with $\mathfrak{h} \subset \mathfrak{g}$ via (5.1).

Let $G$ be a Lie group with a left-invariant complex structure. By construction, the complex structure on $G$ is preserved by the map (5.1). However, the trivialization of $TG$ via (5.1) is not holomorphic unless $G$ is a complex Lie group. Let us explain this phenomenon in more detail. For any vector $\xi \in \mathfrak{g}$ let $\xi$ denote the corresponding left-invariant vector field on $G$. A vector field on a smooth manifold is holomorphic if and only if its flow acts by holomorphic automorphisms. The flow of a *left-invariant* vector field $\xi$ is the multiplication by $\exp(t\xi)$ on the right. Hence the trivialization of $TG$ is holomorphic if and only if the complex structure is bi-invariant.

If $X = \Gamma \backslash G$ is a complex nilmanifold then the identification of $T_x X$ with $\mathfrak{g}$ preserves the complex structure. The constructed trivialization of $TX$ is generally not holomorphic. It is holomorphic if and only if $G$ is a complex Lie group. If $X = \Gamma \backslash G$ is a hypercomplex nilmanifold then the identification of $T_x X$ with $\mathfrak{g}$ preserves the hypercomplex structure on these spaces.

The main goal of this section is to give a proof of the following theorem.

**Theorem 5.2:** Let $X = \Gamma \backslash G$ be an abelian hypercomplex nilmanifold. Then for all but at most countable number of complex structures $L \in H$, the complex nilmanifold $X_L$ satisfies the following property: if $Z \subset X_L$ is an irreducible complex subvariety of $X_L$, then $Z$ is a trianalytic locally homogeneous submanifold of $X$.

The proof of this theorem will be given in the next subsection. Now we state its corollary, which may be seen as an analogue of Proposition 1.5 for nilmanifolds with an HKT structure (Definition 2.28).

**Corollary 5.3:** Let $X$ be a hypercomplex nilmanifold with an HKT structure. Then for all but countable number of complex structures $L \in \mathbb{H}$, the complex nilmanifold $X_L$ satisfies the following property: if $Z \subset X_L$ is an irreducible complex subvariety of $X_L$, then $Z$ is a trianalytic submanifold of $X$ (Definition 1.4).

**Proof:** By Proposition 2.29 a hypercomplex nilmanifold $X$ admits an HKT metric if and only if the hypercomplex structure on $X$ is abelian. The result
now follows from Theorem 5.2.

**Theorem 5.2** is well known in the case when $X$ is a hypercomplex torus.

**Proposition 5.4:** ([KV1, Lemma 6.3]) Let $T = V/\Lambda$ be a hypercomplex torus, i.e. a quotient of a quaternionic vector space $V$ by a maximal rank lattice $\Lambda$. For all but at most countable number of complex structures $L \in \mathcal{H}$ the complex torus $T_L$ satisfies the following property: if $Z \subset T_L$ is an irreducible complex subvariety of $T_L$ then $Z$ is a hypercomplex subtorus.

**Proof:** We endow $T$ with a flat hyperkähler metric. Proposition 1.5 implies that for all but a countable number of complex structures $L \in \mathcal{H}$ every complex subvariety $Z \subset T_L$ is trianalytic. A trianalytic subvariety $Z$ of a hyperkähler manifold is totally geodesic by [V3, Cor. 5.4]. The preimage $Z'$ of $Z$ in the vector space $V$ has to be an affine subspace. Indeed, a totally geodesic subvariety of a vector space is an affine subspace. Hence $Z$ is a subtorus.

**Remark 5.5:** Though some results of [KV1] are wrong ([KV2]), [KV1, Lemma 6.3] is correct.

### 5.2 Subvarieties in principal toric fiber bundles

Let $X = \Gamma \backslash G$ be a complex nilmanifold with a complex structure $L$, $\mathfrak{g}$ the Lie algebra of $G$ and $\mathfrak{z}$ (resp. $Z$) the center of the Lie algebra $\mathfrak{g}$ (resp. the center of the Lie group $G$). Consider a holomorphic map

$$\pi: \Gamma \backslash G = X \longrightarrow Y := \frac{G/Z}{\Gamma/(\Gamma \cap Z)}$$

This map is principal toric bundle by Lemma 4.3.

**Lemma 5.6:** Let $M \subset X$ be an irreducible complex subvariety, for any $y \in Y$ denote $M_y := \pi^{-1}(y) \cup M$. Assume that $M_y \subset \pi^{-1}(y)$ is a finite union of subtori in a compact torus $\pi^{-1}(y)$. Given $x \in M_y$ we denote by $\mathfrak{z}_x$ the Zariski tangent space to $M_y$ at $x$ in the torus $\pi^{-1}(y)$.

(i) The family of subspaces $(\mathfrak{z}_x), x \in M$ of $\mathfrak{g}$ is constant on a dense open subset $M'$ of $M$.

(ii) Define $\mathfrak{z}' := \mathfrak{z}_x$ for $x \in M'$ and $Z' := \exp(\mathfrak{z}')$. Then the subvariety $M$ is preserved by the right action of $Z'$.

**Proof of (i):** Let $Y'$ be the maximal open subset of $Y$ such that $M' := \pi^{-1}(Y') \cap M$ contains only smooth points of $M$ and the map $\pi|_{M'}: M' \to Y'$ is smooth. The dimension of $\mathfrak{z}_x$ is constant for $x \in M'$ and the subspace $\mathfrak{z}_x$. 

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depends continuously on \( x \). Since the subspace \( \mathfrak{z}_x \) is rational for every \( x \in M \), the family \( \mathfrak{z}_x \) is locally constant on \( M' \). The subset \( Y' \) is the complement to a closed analytic subset of \( Y \). Hence \( M' = \pi^{-1}(Y') \cap M \) is the complement to a closed analytic subset of \( M \). Since \( M \) is irreducible, \( M' \) is connected. Therefore, the family \( \mathfrak{z}_x \) is constant on \( M' \).

Proof of (ii): The right action of \( Z' = \exp(y') \) on \( X \) preserves \( M' \). Indeed, for any \( y \in Y' \) the subvariety \( \pi^{-1}(y) \cap M \) is a finite union of orbits of \( Z' \). Since \( M' \) is dense in \( M \), the subvariety \( M \) must be preserved by the action of \( Z' \).

5.3 Preliminary lemmas from algebraic geometry and topology

We recall two well known results of algebraic geometry to be used in the sequel.

**Proposition 5.7:** Let \( f : X \to Y \) be a birational morphism of complex varieties. Assume that \( Y \) is smooth. Define the exceptional set \( E \subset X \) as the union of positive dimensional fibers of \( f \). Then \( E \) is a divisor in \( X \).

**Proof:** The statement is local on \( Y \), hence we may assume that there exists a nowhere vanishing section \( \omega \) of the canonical bundle of \( Y \). The form \( f^*\omega \) is a section of the canonical bundle of \( X \). The zero set of \( f^*\omega \) is precisely the exceptional set \( E \). Thus \( E \) is a (Cartier) divisor. ■

The proof of the following proposition is almost the same.

**Proposition 5.8:** Let \( f : X \to Y \) be a finite morphism of complex varieties. Assume that \( Y \) is smooth. Then the branch locus of \( f \) is a divisor in \( Y \).

**Proof:** The statement is local on \( Y \), hence we may assume that there exists a nowhere vanishing section \( \omega \) of the canonical bundle of \( Y \). The form \( f^*\omega \) is a section of the canonical bundle of \( X \). The zero set of \( f^*\omega \) is precisely the ramification locus \( R \subset X \). Thus \( R \) is a (Cartier) divisor in \( X \). The branch locus is the image of \( R \) in \( Y \), hence a divisor in \( Y \). ■

We finish this subsection with a lemma of a topological nature. It will be used in the last step of the proof of Theorem 5.2.

**Definition 5.9:** Let \( \pi : X \to Y \) be a submersion of complex varieties. Recall that a multisection of \( \pi \) is a subvariety \( Y_1 \subset Y \) such that the restriction \( \pi : Y_1 \to Y \) is a finite map.

**Lemma 5.10:** Let \( X = \Gamma \backslash G \) be a nilmanifold, \( Z \subset G \) be the center of \( G \).
Consider the map
\[ \sigma: X = \Gamma\backslash G \to \frac{G/Z}{\Gamma \cap Z} =: Y \] (5.2)

Then \( \sigma \) admits no multisections.

**Proof:** We use \( \sigma_\ast \) to denote the map \( \pi_1(X) \to \pi_1(Y) \) induced by \( \sigma \). Notice that the center of \( \pi_1(X) \) is precisely \( \ker \sigma_\ast \) because \( Y = \frac{G/Z}{\Gamma \cap Z} \), and \( Z \) is the center of \( G \).

First, notice that \( \sigma \) admits no sections. Indeed, \( \sigma \) is a principal bundle, hence any section of \( \sigma \) induces a smooth decomposition \( X = T^n \times Y \), with \( T^n = Z/(\Gamma \cap Z) \) a torus; here, \( \sigma \) being the projection to the second component. Then \( \pi_1(X) = \pi_1(T^n) \times \pi_1(Y) \), with \( \pi_1(T^n) \) being the center of \( \pi_1(X) \). This is impossible; indeed, since \( Y \) is a nilmanifold, the center of \( \pi_1(Y) \) has positive rank, hence the center of \( \pi_1(X) = \pi_1(T^n) \times \pi_1(Y) \) is strictly bigger than \( \pi_1(T^n) \): a contradiction.

To pass from multisections to sections, consider the following operation on principal toric bundles. Let \( \pi: P \to V \) be a principal toric bundle with fiber \( T^n \), and \( P^k := P \times_{\sigma} P \times_{\sigma} \ldots \times_{\sigma} P \) be its \( k \)-th power (as a principal bundle). Taking a quotient of \( P^k \) by a relation \((p_1 + t_1, p_2 + t_2, \ldots, p_i + t_i, \ldots, p_k + t_k) \sim (p_1, p_2, \ldots, p_i, \ldots, p_k)\), for all \( t_1, t_2, \ldots, t_k \in T^n \) such that \( \sum t_i = 0 \), we obtain a principal bundle \( P_k \) with a fiber \( T^n \) again. However, this bundle is not, generally speaking isomorphic to \( P \); the natural map \( P \to P_k \) mapping \( p \) to \((p, p, \ldots, p)\) is a \( k \)-torsion quotient of \( P_k \). We call the bundle \( P_k \) the quotient of \( P \) by \( k \)-torsion. Any order \( k \) multisection of \( P \) gives a section of \( P_k \); hence trivializes \( P_k \).

Return now to the proof of Lemma 5.10. A multisection of \( \sigma \) gives a section of the principal bundle \( \sigma_k: X_k \to Y \), where \( X_k \) is the quotient of the principal bundle \( \sigma: X \to Y \) by \( k \)-th torsion. Indeed, let \( \rho \) be an order \( k \) multisection mapping \( y \in Y \) to \((s_1, \ldots, s_k)\). Then \((s_1, \ldots, s_k)\) defines a point in \( X_k \) which is independent from the ordering of \((s_1, \ldots, s_k)\).

However, \( X_k \) is also a nilmanifold, with the same rational Lie algebra (indeed, a nilmanifold can be defined as a compact manifold which is homogeneous under an action of a nilpotent Lie group). Therefore, the projection \( X_k \to Y \) corresponds to taking the quotient of the corresponding Lie group by its center, and \( \ker ((\sigma_k)_\ast: \pi_1(X_k) \to \pi_1(Y)) \) is the center of \( \pi_1(X_k) \). Since the center of a nilpotent Lie algebra cannot be its direct summand, the projection \( \sigma_k \) also cannot have sections. We proved Lemma 5.10. \( \blacksquare \)

### 5.4 Subvarieties of hypercomplex nilmanifolds: the proof

This subsection is completely dedicated to the proof of Theorem 5.2.

**Theorem 5.2:** Let \( X = \Gamma\backslash G \) be an abelian hypercomplex nilmanifold. Then for all but at most countable number of complex structures \( L \in \mathbb{H} \), the
complex nilmanifold $X_L$ satisfies the following property: if $Z \subset X_L$ is an irreducible complex subvariety of $X_L$, then $Z$ is a trianalytic locally homogeneous submanifold of $X$.

The proof goes by induction on $n = \dim_H X$. It is divided into steps for the convenience of the reader. In **Step 0** we prove the base of induction. In **Step 1** we consider the principal toric fiber bundle $\pi : X \to Y$ as in the second part of Proposition 4.5. We remind the construction of this map and prove the assertion of the theorem in the case then $Y$ is a point. In **Step 2** we define the set of “nice” complex structures on $X$ and prove that this set is a complement to at most countable number of complex structures in $\mathbb{H}$. In the next steps we will prove that for every “nice” complex structure $L$ each irreducible complex subvariety of $X_L$ is a hypercomplex locally homogeneous submanifold of $X$. In **Step 3** we show that it is enough to prove the theorem for complex subvarieties of $X_L$ which are mapped surjectively onto $Y$. In **Step 4** we reduce the theorem further: it is enough to check the statement of the theorem for complex subvarieties $M \subset X_L$ such that the map $\pi|_M : M \to Y$ is surjective and generically finite. In **Step 5** we show that if the map $\pi|_M : X \to Y$ is generically finite then it is étale. It is impossible for $\pi|_M$ to be étale by Lemma 5.10. This finishes the proof.

**Proof of Theorem 5.2.** **Step 0:** The proof goes by induction on $n = \dim_H X$. The only abelian hypercomplex nilmanifold $X = \Gamma \backslash G$ of real dimension 4 is a torus (see also [Bo],[Ha]). Indeed, the center of the Lie algebra $\mathfrak{g}$ must be $\mathbb{H}$-invariant by Lemma 4.1, hence coincide with $\mathfrak{g}$. The statement of the theorem for a hypercomplex torus of real dimension 4 follows from Proposition 5.4 or directly from Proposition 1.5.

**Step 1:** Assume now that $\dim_H X = n > 1$ and the statement of the theorem holds for all abelian hypercomplex nilmanifolds of quaternionic dimension less than $n$. Consider the morphism $\pi : X \to Y$ of hypercomplex manifolds as in the second part of Proposition 4.5. We briefly recall its construction here. Let $\mathfrak{h}$ (resp. $Z$) denote the center of the Lie algebra $\mathfrak{g}$ (resp. the center of the Lie group $G$). Then the quotient map $G \to G/Z$ induces the map $\pi : \Gamma \backslash G = X \to Y := \frac{G/Z}{\Gamma \cap (G/Z)}$. For every complex structure $L \in \mathbb{H}$ the map $\pi$ is a principal holomorphic toric fiber bundle whose fibers are isomorphic to the hypercomplex torus $T := Z/(\Gamma \cap Z)$ (Proposition 4.5).

In the case when the manifold $Y$ is a point, the manifold $X$ is a torus. The assertion in this case follows from Proposition 5.4.

**Step 2:** By Step 1 we may assume that the manifold $Y = \frac{G/Z}{\Gamma \backslash (G/Z)}$ has positive dimension. We introduce the following property of a hypercomplex nilmanifold $N$ and a complex structure $L \in \mathbb{H}$.

\textit{Every irreducible complex subvariety of $N_L$ is a trianalytic locally homogeneous submanifold.}  \hfill (5.3)
Let us call a complex structure $L \in H$ “nice” if it satisfies the following two properties.

1. Take a proper $H$-invariant rational subalgebra $g'$ of the Lie algebra $g$. Let $G'$ denote the exponent of $g'$ in $G$. Clearly, the quotient space $X' := (\Gamma \cap G')\backslash G'$ is a hypercomplex nilmanifold embedded into $X$ as a trianalytically locally homogeneous submanifold. Then for every choice of $g'$ the nilmanifold $X'$ satisfies (5.3).

2. Take an $H$-invariant rational subspace $z'$ of the center $z$ of $g$. Let $Z'$ denote the exponent of $z'$ in $G$. Clearly, the quotient space $Y' := G/Z' \Gamma / (\Gamma \cap Z')$ is an abelian hypercomplex nilmanifold. Then for every choice of $z'$ the nilmanifold $X'$ satisfies (5.3).

We claim that the set of “nice” complex structures $L \in H$ is the complement to at most countable number of complex structures. Indeed, the set of rational subspaces of $g$ is countable. It follows from the induction hypothesis that the first and the second condition above each exclude at most countable number of complex structures.

In the next steps we will prove that every complex subvariety of $X_L$ is trianalytic as soon as the complex structure $L$ is “nice”.

**Step 3:** Take a nice complex structure $L \in H$ and let $M \subset X_L$ be an irreducible complex subvariety of complex dimension $k$. Consider the image $\pi(M)$ of $M$ under the map $\pi: X \to Y = G/Z \Gamma / (\Gamma \cap Z')$ (see Step 1). Suppose that $\pi(M)$ is properly contained in $Y$. The second condition on a “nice” complex structure applied to $z' = z$ implies that $\pi(M)$ is a trianalytically locally homogeneous submanifold of $Y$. By construction, the preimage $X' := \pi^{-1}(\pi(M))$ of $\pi(M)$ in $X$ is a trianalytically locally homogeneous submanifold of $X$. It is associated with some rational $H$-invariant subalgebra $g' \subset g$. By the first condition on a “nice” complex structure every complex subvariety of $X_L$ is a trianalytically locally homogeneous submanifold. Since $M$ is contained in $X'$, it is trianalytically locally homogeneous. We have reduced the theorem to the case when $M$ is mapped surjectively onto $Y$.

**Step 4:** By Step 3 we may assume that $M$ is mapped surjectively onto $Y$. Pick a point $y \in Y$. Consider the fiber $\pi^{-1}(y) \cap M$ of the map $\pi_{|M}: M \to Y$. The first condition on a “nice” complex structure applied to $g' = z$ implies that $\pi^{-1}(y) \cap M$ is a trianalytically locally homogeneous submanifold of $\pi^{-1}(y)$ and hence of $X$. For every $x \in \pi^{-1}(y)$ the tangent space $T_x(\pi^{-1}(y) \cap M)$ is identified with a rational $H$-invariant subspace $\mathfrak{z}_x \subset \mathfrak{z}$ via the trivialization (5.1). By Lemma 5.6 there exists an $H$-invariant rational subspace $\mathfrak{z}' \subset \mathfrak{z}$ such that $\mathfrak{z}' \subset \mathfrak{z}_x$ for every $x \in M$ and $\mathfrak{z}' = \mathfrak{z}_x$ for $x$ in a dense open subset of $X$. Moreover, the right action of the group $Z' := \exp(\mathfrak{z}')$ preserves $M$. 
Suppose that \( \text{dim} \, \mathfrak{z}' > 0 \). Consider the map

\[
\pi': X \rightarrow Y':= \frac{G/Z'}{\Gamma/(\Gamma \cap Z')}
\]  

(5.4)

The map \( \pi' \) is a holomorphic principal toric fiber bundle by Lemma 4.3. By the second condition on a “nice” complex structure, \( \pi'(M) \) is a trianalytic locally homogeneous submanifold of \( Y' \). Since \( M = (\pi')^{-1}(\pi'(M)) \), it is a trianalytic locally homogeneous submanifold of \( X \). We reduced the theorem to the case when \( \text{dim} \, \mathfrak{z}' = 0 \), or in other words, the map \( \pi'|_M : M \rightarrow Y \) is surjective and generically finite.

**Step 5:** Consider a subvariety \( M \subset X \) such that the map \( \pi|_M : M \rightarrow Y \) is generically finite. Consider the Stein factorization of the map \( \pi|_M : 

\[
M \xrightarrow{\pi_0} Y' \xrightarrow{\pi_1} Y
\]

Here \( \pi_0 : M \rightarrow Y' \) has connected fibers and \( \pi_1 : Y' \rightarrow Y \) is a finite morphism. By Proposition 5.8 the branch locus of \( \pi_1 \) is a divisor in \( Y \). Every complex subvariety of \( Y_L \) is trianalytic, in particular, there are no divisors in \( Y_L \). Hence the morphism \( \pi_1 : Y' \rightarrow Y \) is étale. In particular, \( Y' \) is smooth. The map \( \pi_0 : M \rightarrow Y' \) is a birational map to a smooth base. By Proposition 5.7 the exceptional set \( E \subset M \) is a divisor in \( M \). If \( E \) is non-empty then \( E \) has odd dimension. Consider the image \( \pi(E) \) of \( E \) in \( Y \). It is a complex subvariety of \( Y_L \), hence trianalytic. In particular, \( \pi(E) \) is of even dimension. Therefore, for a generic point \( y \in \pi(E) \) the variety \( \pi^{-1}(y) \cap E \) is odd-dimensional. All subvarieties of \( \pi^{-1}(y) \) are trianalytic and hence of even dimension. Hence \( E \) must be empty. We have shown that \( \pi_0 : M \rightarrow Y' \) is an isomorphism and \( \pi_1 : Y' \rightarrow Y \) is étale. Hence the morphism \( \pi|_M : M \rightarrow Y \) is étale. By Lemma 5.10 there are no subvarieties \( M \subset X \) with this property. The theorem is proved. ■

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