A stochastic model for non-relativistic particle acceleration

G. Palocchia, M. Laurenza, and G. Consolini

1 INAF - Istituto di Astrofisica e Planetologia Spaziali, Via del Fosso del Cavaliere 100, 00133 Roma

A stochastic model is proposed for the acceleration of non-relativistic particles yielding to energy spectra with a shape of a Weibull’s function. Such particle distribution is found as the stationary solution of a diffusion-loss equation in the framework of a second order Fermi’s mechanism producing anomalous diffusion for particle velocity. The present model is supported by in situ observations of energetic particle enhancements at interplanetary shocks, as here illustrated by means of an event seen by STEREO B instruments in the heliosphere. Results indicate that the second order Fermi’s mechanism provides a viable explanation for the acceleration of energetic particles at collisionless shock waves.

*giuseppe.pallocchia@iaps.inaf.it*
One of the most intriguing and unsolved problems of Astrophysics is the particle acceleration to high energies in space plasmas. Fermi’s acceleration mechanism \([\text{1}]\) is a theoretical tool extensively used in astrophysical contexts and also in other research fields like Plasma Physics \([\text{2}]\) and in the theory of dynamical systems \([\text{3, 4}]\). The first-order acceleration, a variant of original Fermi’s mechanism, constitutes the basics for the diffusive shock acceleration (DSA) \([\text{e.g., 3, 6}]\) wherein a particle, repeatedly scattered across the shock front, gains energy through head-on collisions against the converging downstream and upstream plasma irregularities. The DSA naturally produces a power law energy spectrum which is accepted to explain the observed cosmic-ray spectrum up to about \(10^{15}\text{eV}\) \([\text{5}]\). Hence, the DSA approach has received the most attention to interpret particle acceleration at shock waves, although it does not fully address several aspects of phenomenon. For instance, the expected relationship between the power-law spectral index and the shock compression ratio at interplanetary shocks is loose when checked through observations \([\text{8}]\). Moreover, observations of solar energetic particle (SEP) events have shown that the predicted power law is valid on a limited energy interval \([\text{e.g., 9}]\) below a characteristic energy where the spectrum has a rollover. An exponential decay was only heuristically introduced to take into account this feature \([\text{10}]\), where the rollover energy is supposed to depend on several parameters related to the interplanetary shock \([\text{11}]\). On the other hand, stochastic acceleration (SA), also called second-order acceleration and based on the original Fermi’s mechanism, is characterized by an average energy gain due to the particle interaction with randomly moving magnetized clouds or turbulent fluctuations. The SA has been proposed to play a dominant role in many other astrophysical environments where particles can be accelerated in a bounded space region such as Radio galaxies \([\text{12}]\), solar flares \([\text{13}]\), the interstellar medium \([\text{14}]\), supernova remnants \([\text{15}]\). Few theoretical works suggested that SA could be important at shock waves as well \([\text{16–18}]\), although this has not been tested against observations. Recent observational studies \([\text{13, 20}]\) have shown that SEP spectra, as well as spectra of particles accelerated at transient and Corotating Interaction Regions (CIRs) shocks, can be successfully fitted by means of a Weibull’s function \([\text{21}]\). Here we propose a theoretical derivation of such a Weibull’s spectrum through a leaky-box model based on a second-order Fermi’s mechanism wherein the broadening of energy distribution is slower than mean energy gain. The good agreement with observations and the overall physical consistency of the model (both illustrated in an event of acceleration at interstellar shock), provide evidence that SA can be effective at collisionless shock. Hence, the present paper offers a scenario alternative to that depicted by DSA which is generally invoked to account for particle acceleration in the shock-related physical contexts.

Let us start our model derivation from the classical Fermi’s scheme in which particles are stochastically accelerated in a spatial region by interactions with randomly moving magnetic irregularities or turbulent fluctuations. Moreover, let us assume that scattering is effective in making the particles distribution isotropic. In our model the spatial region is homogenous and, consequently, the spatial diffusion is not considered. The number of particles per unit volume and per unit solid angle having kinetic energies in the range \(E\) to \(E + \Delta E\), is then expressed as \((4\pi)^{-1}N(E, t)\Delta E\), that is only as a function of the time and energy. All of the particles are injected in the acceleration process with the same energy \(E_{\text{in}}\) (henceforth we refer all energies to \(E_{\text{in}}\) for notation convenience, thus \(E_{\text{in}} = 0\)) at constant rate of \(q_{\text{in}}\) particles per unit volume and time. Particle leakage from the acceleration region is taken into account through a characteristic time of confinement \(\tau\) indipendent from the energy. Hence the appropriate diffusion-loss equation, expressing the conservation of the number of particles in energy space, reads \([\text{e.g., 22, 24}]\):

\[
\frac{\partial N}{\partial t} = \frac{\partial (b(E)N)}{\partial E} + \frac{1}{2} \frac{\partial^2 (d(E)N)}{\partial E^2} - \frac{N}{\tau} + q_{\text{in}} \delta(E)
\]

where \(b(E) = -\frac{d(E)}{dt}\), \(d(E) = \frac{d(E\Delta E^2)}{dt}\) and \((\langle\cdot\rangle)\) stands for average over a particles ensemble. The four terms on right-hand side account respectively for: 1) the mean “drift” of the particles in energy space \((b(E)\text{ represents the average acceleration rate}); 2) the “broadening” of the particle energy distribution (terms 1 and 2) are connected with the stochastic nature of the acceleration process); 3) particles leakage from the acceleration region; 4) supply from sources of monoenergetic beam of fresh particles with energy \(E_{\text{in}}\).

In the framework of second-order Fermi’s mechanism it is known that anomalous (i.e. nonstandard Brownian) diffusion for particles velocity can arise \([\text{23, 24}]\). \(\langle|\vvec{v}(t) - \vvec{v}_0|^2\rangle \sim t^{2\nu}\) with \(\nu \neq 1/2\) (\(\nu_0\) is the initial velocity). For instance, Bouchet et al. \([\text{25}]\) developed two-dimensional minimal stochastic model in which particles absorb kinetic energy (accelerate) through collisions against magnetic irregularities modeled as localized moving scattering centers. They found for both particle velocity and position an anomalous superdiffusive behaviour. Hence, we assume there exists a non-relativistic implementation of Fermi’s stochastic mechanism in which the particles undergo an anomalous diffusion for velocity yielding to:

\[
\langle E(t)^n\rangle \sim (t/\tau)^{\nu(n)}
\]

where \(\nu(n)\) is a concave function of \(n\) (i.e. its slope continually decreases). The nonlinearity of \(\nu(n)\) indicates that the probability distribution function (PDF) of particle velocity at different times is not self-similar, namely a PDF of the form \(P(|\vvec{v}|, t) = t^{-\nu}F(|\vvec{v}|/\nu(t))\) cannot describe the anomalous diffusion at all time scales by means of the same
value of $\nu$. Actually, numerical studies on the motion of tracer particles in sandpile and in plasma turbulence show that system finite size effects can determine a breakdown of PDF self-similarity characterized by a nearly piecewise linear $n \nu(n)$ function with a smaller slope for high $n$ than for low $n$. Therefore, we justify the assumption of concavity for $n \nu(n)$ as a way to account for finite size effects on velocity diffusion in the model (e.g. the finite value of the probability per unit time $\tau^{-1}$ for a particle to exit from the acceleration process).

The relative weight of the second to the first term on RHS of Eq.(11) can be easily estimated, through dimensional considerations, by the ratio:

$$R(\langle E \rangle) = \frac{d(\langle E \rangle)}{b(\langle E \rangle)(\langle E \rangle)} \sim \frac{\langle E^2 \rangle}{\langle E \rangle^2}$$

(3)

where we consider $\langle (\Delta E)^2 \rangle \sim \langle E^2 \rangle$. Hence, using Eq.(2) in Eq.(3) and dropping the bracket notation (hereafter no longer necessary), we obtain the scaling law:

$$R(\lambda E) = \lambda^{-2\alpha} R(E)$$

(4)

where $\alpha = [1 - \nu(2)/\nu(1)]$ and $\lambda > 0$ is a scale factor.

Since $\alpha > 0$ due to the concavity of $n \nu(n)$, Eq.(4) implies that $R(E) \ll 1$ if $E = \lambda E_\ast \gg E_\ast$, being $E_\ast$ approximately defined through $R(E_\ast) \simeq 1$. Therefore, in energy regime $E \gg E_\ast$, the second term on RHS of Eq.(1) can be neglected and the steady state spectrum $(\partial N/\partial t = 0)$ obtained by solving Eq.(1) reduced to more simple form:

$$N = -E_\tau^\beta \frac{\partial (E^{1-\beta} N)}{\partial E}$$

(5)

where $E_\tau = \langle E(\tau) \rangle$ and $\beta = 1/\nu(1) > 0$. A straightforward integration yields:

$$N(E) = A(E/E_\tau)^{(\beta-1)}e^{-\langle E(E_\tau) \rangle^\beta}$$

(6)

where $A$ is an integration constant. Therefore, the accelerated particles are distributed according to the Weibull’s statistics.

Transients and corotating shocks are systems where particles are assumed to be locally accelerated, as energetic particle enhancements [e.g., 29, 31, and references therein] are usually associated with their passage. Hence, we illustrate the consistency of our model in case the acceleration region is a collisionless shock wave. We remark that the two fundamental assumptions of the model are consistent with physical conditions at interplanetary shock where turbulent fluctuations are observed upstream and/or downstream of the shock front [e.g., 32, 33]. In fact, from the theoretical point of view, turbulence can provide efficient particle scattering (thus supporting the first basic assumption of the model) [25, 34] so that the energy of the turbulent field is transferred to particles through a stochastic Fermi’s mechanism [e.g., 34, 53, 37].

On 3 October 2011 at 22:23 UT, STEREO B spacecraft (located at 1.08 AU, -98.09° and 1.08° heliographic longitude and latitude, respectively) observed a quasi-perpendicular fast shock moving radially outward from Sun with a speed $v_{sh} \simeq 700 \text{ km/s}$. At the same time, a particle enhancement was recorded by the onboard instruments SEPT, LET and HET in the energy range 0.1 – 100 MeV. Figure 1 reports a quicklook of the main plasma and particle parameters along with magnetic field intensity measurements. Data used to study this event are 1 minute averaged proton fluxes measured by the three instruments. This event occurs on a quiet background and the intensities start to rise sharply at the shock passage. Figure 1 shows that the proton peak is found at 22:23 UT, when the shock can be identified by the abrupt changes in the solar wind parameters.

An average differential flux was calculated on the time interval 22:14 - 22:31 UT around the shock arrival and a best-fit was performed by means of a function derived from Eq.(5) taking into account the conversion from the particle spectrum to the differential flux $(dJ/dE = C \times N(E) \times E^{1/2})$. The obtained values for the best-fit parameters are: $C \sim [2.0 \pm 0.5] \times 10^9 \text{ cm}^{-2} \text{s}^{-1} \text{sr}^{-1} \text{MeV}^{-1}$, $\beta = [0.50 \pm 0.07]$ and $E_\tau = [95 \pm 5] \text{KeV}$. As shown in Figure 2, there is an excellent agreement between our model and the experimental data over the wide energy range $0.3 \div 30 \text{ MeV}$ spanning around two orders of magnitude. As expected from the model, the agreement is worst at lower energies. In turbulent plasmas the theoretical escape time from the acceleration region due to the spatial transport is $\tau(E) \sim E^{-\gamma}$ [e.g., 37]. If we assumed such $\tau(E)$ in the model, the resulting (softer) spectrum would differ from that in Eq.(6) just for the replacement $(E/E_\tau)^\beta \rightarrow \beta/(\beta + \gamma)(E/E_\tau)^{\beta+\gamma}$ in the exponential factor. However, we verified in the present case that such a correction to Weibull’s spectrum ($\gamma = 0$) results to be negligible for energies lower than several tens of $\text{MeV}$. Therefore, in spite of extreme simplicity, our assumption of constant escape time proves to be reasonable by virtue of the good agreement between the present leaky-box model and observations.
In our model, $\tau$, $\beta$ and $E_\tau$ are free parameters which can assume, in principle, any value independently from each other. We show that the obtained estimates are congruent with a physical picture of the event. The value of $\beta = 0.5$ (viz $\nu(1) = 2$) implies superdiffusion for velocity. In general terms, a high degree of persistence of the anomalous diffusion is expected for an efficient particle acceleration. Moreover, as already mentioned, the same superdiffusive behaviour spontaneously arises in a minimal model of second order Fermi’s acceleration proposed by Bouchet et al. Thus, the above $\beta$ value proves to be fairly meaningful from a physical point of view. In case of efficient energization, the mean energy $E_\tau$ gained in a characteristic time $\tau$ has to be much higher than the typical injection energy. As matter of fact, $E_\tau = 95$ KeV considerably exceeds both typical bulk flow $E_{\text{bulk}} = 1/2m_p V_{sw}^2 \sim 5$ KeV and thermal $E_{th} = k_B T_p \sim 0.15$ KeV energies of the upstream solar wind protons (see Fig.1). Hence, it is consistent with the reasonable hypothesis that the energetic particle population is accelerated directly out of the ambient solar wind.

The confinement time $\tau$ cannot be directly obtained through the best-fit procedure. Nevertheless, observations can provide upper and lower limits for its value. In fact, taking into account $\beta = 0.5$ and $E_\tau = 95$ KeV, it is seen from Eq.(2) that a particle energy of $\sim 30$ MeV (viz the highest energy in Fig.2) is reached after a time $T_{\text{high}} \simeq 18\tau$. Obviously, $T_{\text{high}}$ can equal, at most, the shock travelling time from the Sun to the spacecraft position $R_{s/c} = 1.08 AU$, that is $T_{\text{stt}} = R_{s/c}/v_{sh} \simeq 2.7$ days. Hence, the upper limit is $\tau_{up} \simeq 3.6$ hr. On the contrary, in case of nearly

![Graph showing time history of solar wind plasma parameters and energetic particle fluxes as recorded by STEREO B s/c between 21:00 UT and 24:00 UT on October 3rd 2011. From top to bottom: the proton density $n_p$ and temperature $T$, bulk speed $v$, magnetic field magnitude and the proton differential fluxes for a selected number of energy channels ($E \sim 0.53, 1.05, 2.10, 4.74, 6.93$ and $10.95$ MeV from top to bottom).](image)
local acceleration, $T_{\text{high}}$ must be of order of the time width of particle enhancement that, in present case, is around $10 \div 20 \text{ min}$. The lower limit is, therefore, $T_{\text{low}} \approx 1 \text{ min}$. When calculated from Eq.(2) with the above values of $\beta$, $E_T$ and $\tau$, the acceleration time scales of our superdiffusive model result to be comparable with DSA ones or even shorter. For instance, Zhang and Lee [38] estimate that DSA accelerates a proton to an energy of $\sim 10 \text{ MeV}$ in a time of $\sim 12 \text{ hr}$ at 1 AU (see their Fig.1). In our model, the same energy is reached after a time $\sim 10\tau$ which may range from $\sim 10 \text{ min}$ to $\sim 36 \text{ hr}$ depending on the actual $\tau$ value. It is conceivable that a second order Fermi’s acceleration may be more efficient than DSA. For instance, Ostrowski [39] has showed that, under the hypothesis of negligible damping of very low frequency Alfvén waves, statistical acceleration by high-amplitude MHD turbulence can transfer the energy of a weak parallel shock to the particles more efficiently than a first order process. Moreover, Schlickeiser and Achat [16] proposed that due to efficient momentum diffusion of particles in the downstream region of the shock, the acceleration can be dominated by the second-order acceleration mechanism.

In summary, we have introduced a simple stochastic model to obtain a Weibull’s spectrum for accelerated energetic particles. The fundamental assumption was that acceleration is given by an anomalous diffusion in momentum space characterized by a broadening of the energy distribution slower than average energy gain. Afterwards, through the analysis of an event registered in the interplanetary space, we showed that the model can account for the observations at collisionless shock over a wide energy range and that its acceleration time scales are competitive with those of the diffusive shock acceleration (DSA).

In conclusion, the present study is particularly important since it provides evidence that a second order Fermi’s process may efficiently accelerate particles at shock waves, viz, in a physical environment where instead DSA is usually thought to play the dominant role. Moreover, we point out that the parameters of the Weibull’s spectrum acquire a clear physical meaning within our model and, hence, their experimental estimates represent a helpful tool in interpreting the observations of energetic particles connected with several solar and interplanetary phenomena such as SEP, CIR and transient collisionless shocks. Nevertheless, further theoretical and observational efforts are needed to better understand the details of the microphysics of the magnetic field turbulence around the collisionless shock front and how it can affect the trapping and acceleration of energetic particles [e.g., 10].

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[1] E. Fermi, Phys. Rev. 75, 1169 (1949).
| Reference | Details |
|-----------|---------|
| [2]       | G. Michalek, M. Ostrowski, and R. Schlickeiser, Solar Phys. **184**, 339 (1999) |
| [3]       | G. M. Zaslavskii and B. V. Chirikov, Soviet Physics Doklady **9**, 989 (1965). |
| [4]       | M. S. Lichtenberg and M. A. Lieberman, *Springer-Verlag, New York, 1991, —c1991, 2nd ed.* (1991). |
| [5]       | G. F. Krymskii, Akademiia Nauk SSSR Doklady **234**, 1306 (1977). |
| [6]       | R. D. Blandford and J. P. Ostriker, Astrophys. J. Lett. **221**, L29 (1978). |
| [7]       | R. D. Blandford and D. Eichler, *Phys. Rep.* **154**, 1 (1987). |
| [8]       | P. van Nes, R. Reinhard, T. R. Sanderson, K.-P. Wenzel, and R. D. Zwickl, J. Geophys. Res. **89**, 2122 (1984). |
| [9]       | R. A. Mewaldt, C. M. S. Cohen, A. W. Labrador, R. A. Leske, G. M. Mason, M. I. Desai, M. D. Looper, J. E. Mazur, R. S. Selesnick, and D. K. Haggerty, J. Geophys. Res. **110**, A09S18 (2005). |
| [10]      | D. C. Ellison and R. Ramaty, Astrophys. J. **298**, 400 (1985). |
| [11]      | M. A. Lee, R. A. Mewaldt, and J. Giacalone, Space Sci. Rev. **173**, 247 (2012). |
| [12]      | J. A. Eilek, Astrophys. J. **230**, 373 (1979). |
| [13]      | V. Petrov and S. Liu, Astrophys. J. **610**, 550 (2004), astro-ph/040155. |
| [14]      | E. S. Seo and V. S. Ptuskin, Astrophys. J. **431**, 705 (1994). |
| [15]      | J. S. Scott and R. A. Chevalier, Astrophys. J. **197**, L5 (1975). |
| [16]      | R. Schlickeiser and U. Achat, J. Plasma Phys. **50**, 85 (1993). |
| [17]      | M. Ostrowski and R. Schlickeiser, Astron. Astrophys. **268**, 812 (1993). |
| [18]      | A. Afanasiev, R. Vainio, and L. Kocharov, Astrophys. J. **790**, 36 (2014). |
| [19]      | M. Laurenza, G. Consolini, M. Storini, and A. Damiani, in *American Institute of Physics Conference Series* American Institute of Physics Conference Series, Vol. 1539, edited by G. P. Zank, J. Borovsky, R. Bruno, J. Curtin, S. Cranmer, H. Elliott, J. Giacalone, W. Gonzalez, G. Li, E. Marsch, E. Moebius, N. Pogorelov, J. Spann, and O. Verkhoglyadova (2013) pp. 219–222. |
| [20]      | M. Laurenza, G. Consolini, M. Storini, and A. Damiani, *Journal of Physics Conference Series* **632**, 012066 (2015). |
| [21]      | W. Weibull, J. Appl. Mech **18**, 293 (1951). |
| [22]      | V. L. Ginzburg and S. I. Syrovatskii, *The Origin of Cosmic Rays*, edited by New York: Macmillan (1964). |
| [23]      | J. A. Miller, N. Gessoun, and R. Ramaty, *Astrophys. J.* **361**, 701 (1990). |
| [24]      | M. S. Longair, *High energy astrophysics*. Vol. 2, edited by Cambridge: Cambridge University Press —c1994, 2nd ed. (1994). |
| [25]      | P. Bouchet, F. Cecconi, and A. Vulpiani, Phys. Rev. Lett. **92**, 040601 (2004). |
| [26]      | S. Perri, F. Lepreti, V. Carbone, and A. Vulpiani, Europhys. Lett. **78**, 40003 (2007). |
| [27]      | B. A. Carreras, V. E. Lynch, D. E. Newman, and G. M. Zaslavsky, Phys. Rev. E **60**, 4770 (1999). |
| [28]      | B. A. Carreras, V. E. Lynch, and G. M. Zaslavsky, Phys. Plasmas **8**, 5096 (2001). |
| [29]      | T. P. Armstrong, S. M. Krimigis, and K. W. Behannon, J. Geophys. Res. **75**, 5980 (1970). |
| [30]      | J. T. Gosling, J. R. Asbridge, S. J. Bame, W. C. Feldman, R. D. Zwickl, G. Paschmann, N. Schopke, and R. J. Hynds, J. Geophys. Res. **86**, 547 (1981). |
| [31]      | D. Lario, G. C. Ho, R. B. Decker, E. C. Roelof, M. I. Desai, and C. W. Smith, in *Solar Wind Ten* American Institute of Physics Conference Series, Vol. 679, edited by M. Velli, R. Bruno, F. Malara, and B. Bucci (2003) pp. 640–643. |
| [32]      | C. F. Kennel, F. V. Coroniti, F. L. Scarf, E. J. Smith, and D. A. Gurnett, J. Geophys. Res. **87**, 17 (1982). |
| [33]      | G. P. Zank, G. Li, V. Florinski, Q. Hu, D. Lario, and C. W. Smith, J. Geophys. Res. **111**, A06108 (2006). |
| [34]      | B. A. Tverskoi, Soviet Phys. JETP **26**, 821 (1968). |
| [35]      | V. Petrov, Space Sci. Rev. **173**, 535 (2012). |
| [36]      | A. M. Bykov, D. C. Ellison, S. M. Osipov, and A. E. Vladimirov, Astrophys. J. **789**, 137 (2014). |
| [37]      | Y. Fedorov, B. Shakhow, and M. Stahlik, J. Phys. **45**, 165702 (2012). |
| [38]      | M. Zhang and M. A. Lee, Space Sci. Rev. **176**, 133 (2013). |
| [39]      | M. Ostrowski, Astron. Astrophys. **283**, 344 (1994). |
| [40]      | O. P. Verkhoglyadova and J. A. Le Roux, J. Geophys. Res. **110**, A10S03 (2005). |