Consequences of a Relativistic Pseudospin Symmetry for Radial Nodes and Intruder Levels in Nuclei

A. Leviatan\textsuperscript{1,*} and J.N. Ginocchio\textsuperscript{2,†}

\textsuperscript{1} Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
\textsuperscript{2} Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA

(March 30, 2022)

Abstract

The identification of pseudospin symmetry as a relativistic symmetry of the Dirac Hamiltonian is used to explain the structure of radial nodes occurring in pseudospin doublets and to illuminate the special status of nodeless intruder states in nuclei.

\textit{PACS}: 24.10.Jv, 21.60.Cs, 24.80.+y, 21.10.-k

\textit{Keywords}: Relativistic mean field theory; Symmetry; Dirac Hamiltonian; Pseudospin

\textsuperscript{*}E-mail address: ami@vms.huji.ac.il

\textsuperscript{†}E-mail address: gino@t5.lanl.gov
The concept of pseudospin symmetry [1,2] is based on the empirical observation of quasi-degenerate pairs of certain normal-parity shell-model orbitals with non-relativistic quantum numbers

\[ \left( n_r, \ell, j = \ell + \frac{1}{2} \right) \quad \text{and} \quad \left( n_r - 1, \ell + 2, j = \ell + \frac{3}{2} \right) . \]  

Here \( n_r, \ell, \) and \( j \) are the single-nucleon radial, orbital, and total angular momentum quantum numbers, respectively. The doublet structure, is expressed in terms of a “pseudo” orbital angular momentum \( \tilde{\ell} = \ell + 1 \) and “pseudo” spin, \( \tilde{s} = 1/2 \), which are coupled to total angular momentum \( j = \tilde{\ell} \pm \tilde{s} \). For example, \( (n_r,s_1/2, (n_r-1)d_3/2) \) will have \( \tilde{\ell} = 1 \), \( (n_r,p_3/2, (n_r-1)f_5/2) \) will have \( \tilde{\ell} = 2 \), etc. This pseudospin symmetry has been used to explain features of deformed nuclei [3], including superdeformation [4] and identical bands [5,6] and to establish an effective shell-model coupling scheme [7]. In view of its central role in both spherical and deformed nuclei, there has been an intense effort to understand the origin of this symmetry. The observed reduction of pseudo spin-orbit splitting in the non-relativistic single-particle spectra was shown to follow from nuclear relativistic mean-fields [8,9]. The pseudospin symmetry itself has been shown to arise from a relativistic symmetry of a Dirac Hamiltonian in which the sum of the scalar, \( V_S \), and vector, \( V_V \), potentials cancel, \( V_S + V_V = 0 \) [10,11]. An attractive scalar and repulsive vector potentials of nearly equal magnitudes, \( V_S \sim -V_V \), is an inherent feature of realistic relativistic mean fields in nuclei. Calculations employing such fields in a variety of nuclei have confirmed the existence of an approximate pseudospin symmetry in the energy spectra and wave functions [12–15]. The conditions for an approximate relativistic pseudospin symmetry have been further elaborated [16] including the effect of isospin asymmetry [17]. Implications on magnetic properties and Gamow-Teller transitions in nuclei [18] and on nucleon-nucleus scattering have been considered [19].

In the relativistic pseudospin scheme [10,11], the non-relativistic wave functions of Eq. (1) are associated with the upper components of Dirac wave functions which are eigenstates of a
Dirac Hamiltonian with $V_S \sim -V_V$. The pseudospin $\tilde{s}$, and pseudo orbital angular momentum $\tilde{\ell}$, are found to be the ordinary spin, and ordinary orbital angular momentum respectively, of the lower component of the Dirac wave functions. The underlying Dirac structure ensures that the wave function of the upper component of the Dirac eigenfunction has a spherical harmonic of rank $\ell = \tilde{\ell} - 1$ for aligned spin: $j = \tilde{\ell} - 1/2 = \ell + 1/2$, and a spherical harmonic of rank $\ell + 2 = \tilde{\ell} + 1$ for unaligned spin: $j = \tilde{\ell} + 1/2 = (\ell + 2) - 1/2$. This explains the particular angular momenta defining the pseudospin doublet of Eq. (1). A prominent and uncommon feature of this doublet is that its members have different radial wave functions. In particular, the state with aligned spin in Eq. (1) has $n_r$ nodes as compared to $n_r - 1$ nodes of its partner with unaligned spin. One objective of the present letter is to reveal the so far unexplained mechanism by which the relativistic origin of pseudospin symmetry ensures this particular relation between radial quantum numbers of pseudospin partners. A related question concerns the shell-model states with aligned spin and no nodes, e.g., $0s_{1/2}$, $0p_{3/2}$, $0d_{5/2}$, etc. For heavy nuclei such states with large $j$, i.e., $0g_{9/2}$, $0h_{11/2}$, $0i_{13/2}$, are the “intruder” abnormal-parity states which, due to the strong spin-orbit term, intrude from the shell above and defect to the shell below. The original formulation of pseudospin was oriented to a symmetry of normal-parity shell-model orbitals only. The intruder states were discarded from the non-relativistic pseudospin scheme. However, that is clearly not a satisfactory procedure if pseudospin is identified as a relativistic symmetry of the Dirac Hamiltonian, since now both the normal-parity states and abnormal-parity states are eigenstates of the same Hamiltonian. The relativistic attributes of pseudospin symmetry will allow us to examine the properties and special status of these intruder levels and to explain why these states do not have a partner which is an eigenstate.

Properties of Dirac wave functions in a central field were considered in [20,21] for vector potentials only. The discussion below generalizes these results to the case where
both scalar, $V_S(r)$, and vector, $V_V(r)$, spherically symmetric potentials are present in the Dirac Hamiltonian. As usual, in this case the eigenstates can be written in the form

$$\Psi_{\kappa,m} = \left( g_{\kappa} [Y_\ell \chi_m]^{(j)} , i f_{\kappa} [Y_\ell \chi_m]^{(j)} \right)$$

where $g_{\kappa}(r)$ and $f_{\kappa}(r)$ are the radial wave functions of the upper and lower components respectively, $Y_\ell$ and $\chi$ are the spherical harmonic and spin function which are coupled to angular momentum $j$ projection $m$. The labels $\kappa = -(j+1/2)$ and $\ell' = \ell + 1$ for aligned spin $j = \ell + 1/2$, while $\kappa = (j + 1/2)$ and $\ell' = \ell - 1$ for unaligned spin $j = \ell - 1/2$. Setting $\hbar = c = 1$, the radial Dirac equations for a single nucleon of mass $M$ and total energy $E$ are

$$\frac{dG}{dr} = -\frac{\kappa}{r} G + A(r) F \quad (2a)$$

$$\frac{dF}{dr} = \frac{\kappa}{r} F - B(r) G \quad (2b)$$

where $G = rg_{\kappa}$, $F = rf_{\kappa}$ and

$$A(r) = [E + M + V_S(r) - V_V(r)] \quad (3a)$$

$$B(r) = [E - M - V_S(r) - V_V(r)] . \quad (3b)$$

We consider potentials satisfying $rV_S, rV_V \to 0$ for $r \to 0$, and $V_S, V_V \to 0$ for $r \to \infty$. For bound state solutions $-M < E < M$ and the radial wave functions satisfy $G(0) = F(0) = 0$, $G(\infty) = F(\infty) = 0$, and $\int_0^\infty (G^2 + F^2) \, dr = 1$. Specifically, we are interested in bound Dirac valence states for which both the binding energy $(M - E) > 0$ and the total energy $E > 0$ are positive. For relativistic mean fields relevant to nuclei, $V_S$ is attractive and $V_V$ is repulsive with typical values $V_S(0) \sim -400$, $V_V(0) \sim 350$, $E \sim 900$, $M \sim 940$ MeV. Accordingly, in practical applications, the quantity $A(r)$ in Eq. (3a) is positive definite, $A(r) > 0$, with $A(0) \sim 1090$ MeV, while the quantity $B(r)$ in Eq. (3b) is monotonic in $r$ and changes sign from a small positive value at the origin, $B(0) \sim 10$ MeV, to a negative value $(E - M) < 0$ at large $r$. It is now straightforward to obtain from Eq. (2) the asymptotic behavior of $G$ and $F$.
for \( r \rightarrow \infty \) \( G \sim e^{-\lambda r} \), \( F \sim e^{-\lambda r} \), \( F/G \rightarrow -\sqrt{\frac{M-E}{M+E}} \), \( \lambda = \sqrt{M^2 - E^2} \)

for \( r \rightarrow 0 \) \( G \sim r^{-\kappa} \), \( F \sim r^{1-\kappa} \), \( F/G \rightarrow \mathcal{F}(0) \) \( B(0) < 0 \), \( (\kappa < 0) \) \( 1+2\kappa r < 0 \)

for \( r \rightarrow 0 \) \( G \sim r^{1+\kappa} \), \( F \sim r^{\kappa} \), \( G/F \rightarrow \mathcal{A}(0) \) \( 1+2\kappa r > 0 \), \( (\kappa > 0) \).

Both \( G \) and \( F \) behave as a power law for small \( r \) and exhibit an exponential falloff for large \( r \). We illustrate this behavior of \( G \) and \( F \) in Fig. 1a,c,e for the 2p\( \frac{3}{2} \) \( (\kappa = -2) \), 1f\( \frac{5}{2} \) \( (\kappa = 3) \) and 0g\( \frac{9}{2} \) \( (\kappa = -5) \) eigenstates of a Dirac Hamiltonian with scalar and vector potentials of Woods-Saxon form with parameters tuned to the neutron spectra of \(^{208}\text{Pb} \) [17].

To study further properties of the radial wave functions, it is convenient to introduce \( G = r^\kappa G \) and \( F = r^{-\kappa} F \). Then in the open interval \((0, \infty)\), nodes of \( F \) and \( G \) coincide with nodes of \( F \) and \( G \). From Eq. (2) get

\[
\frac{dG}{dr} = r^{2\kappa} A(r) \mathcal{F} \tag{5a}
\]

\[
\frac{dF}{dr} = -r^{-2\kappa} B(r) G \tag{5b}
\]

A number of observations follow from Eq. (5). First, we note that it is impossible for \( \mathcal{F} \) and \( \mathcal{G} \), or \( F \) and \( G \), to vanish simultaneously at the same point because if they did, then all other higher-order derivatives would vanish at that point and hence the functions themselves would vanish everywhere. Second, we see that a node of \( \mathcal{F} \) corresponds to an extremum of \( \mathcal{G} \), and a node of \( \mathcal{G} \) corresponds to an extremum of \( \mathcal{F} \) (since \( B(r) \) in Eq. (5b) changes sign, \( \mathcal{F} \) can have an additional extremum where \( B(r) = 0 \), which does not correspond to a node of \( \mathcal{G} \), but this can occur only at one point since \( B(r) \) is monotonic). It follows that the nodes of \( F \) and \( G \) alternate; that is, between every pair of adjacent nodes of \( F \) (or \( G \)) there is one node of \( G \) (or \( F \)). If we let \( r_1 \) be a node of \( \mathcal{F} \) and \( r_2 \) be a node of \( \mathcal{G} \), then the nature of the extrema at these points is determined from the second derivatives

\[
\left. \frac{d^2 \mathcal{G}}{dr^2} \right|_{r=r_1} = -A(r_1) B(r_1) \mathcal{G}(r_1) \quad \text{where} \quad \mathcal{F}(r_1) = 0 \tag{6a}
\]

\[
\left. \frac{d^2 \mathcal{F}}{dr^2} \right|_{r=r_2} = -A(r_2) B(r_2) \mathcal{F}(r_2) \quad \text{where} \quad \mathcal{G}(r_2) = 0 \tag{6b}
\]
As bound states, both \( G \) and \( F \) vanish at \( r = \infty \) and their extrema are concave towards the \( r \)-axis. Therefore, the extrema at the nodes \( r_1 \) or \( r_2 \) are minima (maxima) if the functions \( G \) or \( F \) are negative (positive) respectively at these points. It follows from Eq. (3) that nodes of \( F \) and \( G \) can occur only where the product \( A(r)B(r) > 0 \) is positive. Since for practical applications \( A(r) > 0 \), this condition reduces to

\[
B(r) > 0 \text{ at nodes of } F \text{ and } G . \tag{7}
\]

The combination \( V_S + V_V \) appearing in \( B(r) \), Eq. (3b), is the average potential felt by the nucleon in the non-relativistic limit. Therefore, the condition of Eq. (7) is similar to the statement in the non-relativistic case, that nodes of the radial wave function can occur only in the region of classically allowed motion, \( i.e., \) where the kinetic energy is positive.

We now use the above results to obtain a relation between the radial nodes of \( F \) and \( G \). For that purpose we consider the equation for \( GF = G F \) as derived from Eq. (2),

\[
(GF)' = A(r) F^2 - B(r) G^2 . \tag{8}
\]

For large \( r \), \( (GF)' \sim (E + M)F^2 - (E - M)G^2 > 0 \) is positive, since the binding energy \( (M - E) > 0 \) for bound states. At small \( r \), \( (GF)' = -B(0) G^2 < 0 \) for \( \kappa < 0 \), while \( (GF)' = A(0) F^2 > 0 \) for \( \kappa > 0 \), by employing Eq. (3). Since \( GF \) vanishes both at \( r = 0 \) and \( r = \infty \) we see that it is an increasing negative function at large \( r \), while at small \( r \), \( GF \) is a decreasing negative function for \( \kappa < 0 \) and an increasing positive function for \( \kappa > 0 \),

\[
\begin{align*}
\text{for } r &\to \infty \quad GF < 0 & \quad \text{(9a)} \\
\text{for } r &\to 0 \quad GF < 0 \quad (\kappa < 0) & \quad \text{(9b)} \\
\text{for } r &\to 0 \quad GF > 0 \quad (\kappa > 0) & \quad \text{(9c)}
\end{align*}
\]

consistent with Eq. (4). Furthermore, since \( A(r) > 0 \) and by using Eq. (7) we find that

\[
\begin{align*}
(GF)' \bigg|_{r=r_1} &= -B(r_1) G^2(r_1) < 0 \quad \text{where } F(r_1) = 0 \\
(GF)' \bigg|_{r=r_2} &= A(r_2) F^2(r_2) > 0 \quad \text{where } G(r_2) = 0 . \tag{10}
\end{align*}
\]
Thus $GF$ is a decreasing function at the nodes of $F$, and an increasing function at the nodes of $G$. Exploiting all these derived properties, we observe that for $\kappa > 0$, $GF$ is positive at small $r$ and negative at large $r$, and hence has an odd number of zeroes. By Eq. (10) the first and last zeroes of $GF$ correspond to nodes of $F$, and since the nodes of $F$ and $G$ alternate, then the number of nodes of $F$ exceed by one the number of nodes of $G$. On the other hand, for $\kappa < 0$, $GF$ has the same (negative) sign near both end points, and hence has an even number of zeroes. By similar arguments we find that in this case the first and last zeroes of $GF$ are nodes of $G$ and $F$ respectively, and that $G$ and $F$ have the same number of nodes. These properties of $GF$ are illustrated in Fig. 1b,d for the $2p_{3/2}$ ($\kappa = -2$), $1f_{5/2}$ ($\kappa = 3$) states. Altogether we have,

$$\kappa < 0 : \quad n_F = n_G$$

$$\kappa > 0 : \quad n_F = n_G + 1$$

(11)

where $n_F$ and $n_G$ denote the number of internal nodes of $F$ and $G$ respectively. For $\kappa > 0$, the first and last nodes (considering $F$ and $G$ together) are $F$ nodes. For $\kappa < 0$, the nodes of $G$ precede those of $F$ as $r$ increases. The same results can be obtained by considering the Ricatti equation for the ratio $F/G$ and its asymptotic values, as shown in [20,21] for vector potentials.

As hinted in Eq. (11), the case when $n_F = 0$ requires a special attention. Indeed, it can be shown that when the radial parts of the wave functions $G$ and $F$ have no nodes, the corresponding bound states can appear only in the $j = \ell + 1/2$ state ($\kappa < 0$), but not in the $j = \ell - 1/2$ state ($\kappa > 0$). This result was obtained previously in [22] for attractive scalar potentials of Coulomb type and separately for vector potentials of similar type. It is possible to generalize this result to the scalar and vector potentials under discussion. To prove the statement we note that according to Eq. (9a), $GF$ is negative for $r$ sufficiently large. If both $G$ and $F$ have no nodes, then the relation $GF < 0$ holds for any $r$. Since by Eqs. (9b,c),
$GF$ is negative near the origin only for states with $\kappa < 0$, it follows from continuity that bound states without nodes ($n_F = n_G = 0$) must have $\kappa < 0$, and hence appear only in the $j = \ell + 1/2$ state, but not in the $j = \ell - 1/2$ state. An example is shown in Fig. 1f for the case of the $0g_{9/2}$ ($\kappa = -5$) intruder state.

We have obtained in Eq. (7) that the quantity $B(r)$ is positive at the nodes of $F$ and $G$. Even if the potentials $V_S$ and $V_V$ support only eigenstates with no nodes, there must still be a region where $B(r) > 0$, in order that bound states exist. This can be inferred from the fact that as shown $GF$ is an increasing negative function at large $r$, and to enable it to vanish at $r = 0$, its derivative $(GF)'$ must change sign from positive to negative in some region. A glance at Eq. (8) shows that since $A(r) > 0$, a necessary (but not sufficient) condition for $(GF)'$ to become negative is that

$$B(r) > 0 \text{ for some } r .$$

The above condition, $- [V_S(r) + V_V(r)] > M - E$, means that in order that bound states exist, there has to be a region where the depth of the average attractive single-nucleon potential is larger than the binding energy.

All of the above results are relevant for understanding properties of states in the relativistic pseudospin scheme [10,11]. The generators for the relativistic pseudospin SU(2) algebra, $\hat{\tilde{S}}_\mu$, which commute with the Dirac Hamiltonian, $[H, \hat{\tilde{S}}_\mu] = 0$, for the case when $V_S = -V_V$ are given by [12]

$$\hat{\tilde{S}}_\mu = \begin{pmatrix} \hat{s}_\mu & 0 \\ 0 & \hat{s}_\mu \end{pmatrix} = \begin{pmatrix} U_p \hat{s}_\mu U_p & 0 \\ 0 & \hat{s}_\mu \end{pmatrix},$$

(13)

where $\hat{s}_\mu = \sigma_\mu/2$ are the usual spin generators, $\sigma_\mu$ the Pauli matrices, and $U_p = \sigma \cdot \hat{p}$ is the momentum-helicity unitary operator introduced in [1]. If in addition the potentials are spherically symmetric, the Dirac Hamiltonian has an additional invariant SU(2) algebra, $[H, \hat{\tilde{L}}_\mu] = 0$, with the relativistic pseudo-orbital angular momentum operators given by
\[ \hat{L}_\mu = \begin{pmatrix} \hat{\ell}_\mu & 0 \\ 0 & \hat{\ell}_\mu \end{pmatrix} = \begin{pmatrix} U_p \hat{\ell}_\mu U_p & 0 \\ 0 & \hat{\ell}_\mu \end{pmatrix}, \]  

(14)

where \( \hat{\ell}_\mu = r \times p \) are the usual orbital angular momentum operators. The sets \( \{ \hat{S}_\mu, \hat{s}_\mu, \hat{s}_\mu \} \) and \( \{ \hat{L}_\mu, \hat{\ell}_\mu, \hat{\ell}_\mu \} \) form two triads of \( SU(2) \) algebras. The \( \hat{S}_\mu \) and \( \hat{L}_\mu \) operators act on the four-components Dirac wave functions. The \( \hat{s}_\mu \) and \( \hat{\ell}_\mu \) operators form the non-relativistic pseudospin and pseudo-orbital angular momentum algebras respectively, and act on the upper components of the Dirac wave functions. The \( \hat{s}_\mu \) and \( \hat{\ell}_\mu \) act on the “small” lower components of the Dirac wave functions. The total angular momentum operators \( \hat{J}_\mu = \hat{L}_\mu + \hat{S}_\mu \) have block diagonal form with entries \( \hat{j}_\mu = \hat{\ell}_\mu + \hat{s}_\mu = U_p (\hat{\ell}_\mu + \hat{s}_\mu) U_p = \hat{\ell}_\mu + \hat{s}_\mu \).

The eigenfunctions of the Dirac Hamiltonian are also eigenfunctions of \( \hat{S} \cdot \hat{S}, \hat{L} \cdot \hat{L}, \) and \( \hat{J} \cdot \hat{J} \), with eigenvalues \( \tilde{s}(\tilde{s} + 1), \tilde{\ell}(\tilde{\ell} + 1) \) and \( j(j + 1) \) respectively. The pseudospin \( \tilde{s} = 1/2 \) and pseudo-orbital angular momentum \( \tilde{\ell} \) are seen from Eqs. (13)-(14) to be the ordinary spin and ordinary orbital angular momentum of the lower component of the Dirac wave functions. This algebraic structure implies that the two states in the pseudospin doublet will have Dirac wave functions of the form

\[
\Psi_{\kappa<0,m} = \left( \frac{G_{\kappa<0}(r)}{r} [Y_{\tilde{\ell}-1} \chi_m^{(j)}] \right) \left( \frac{F_{\kappa<0}(r)}{r} [Y_{\tilde{\ell}} \chi_m^{(j)}] \right) \quad \kappa = -\tilde{\ell} < 0, \quad j = \tilde{\ell} - \frac{1}{2} \tag{15a}
\]

\[
\Psi_{\kappa'>0,m} = \left( \frac{G_{\kappa'>0}(r)}{r} [Y_{\tilde{\ell}+1} \chi_m^{(j')}] \right) \left( \frac{F_{\kappa'>0}(r)}{r} [Y_{\tilde{\ell}} \chi_m^{(j')}] \right) \quad \kappa' = \tilde{\ell} + 1 > 0, \quad j' = \tilde{\ell} + \frac{1}{2}. \tag{15b}
\]

The members of the doublet share a common pseudo orbital angular momentum \( \tilde{\ell} \), which is coupled to a pseudospin \( \tilde{s} = 1/2 \). The state with negative \( \kappa < 0 \) involves unaligned pseudospin, \( j = \tilde{\ell} - 1/2 \), while its partner with positive \( \kappa' > 0 \) involves aligned pseudospin, \( j' = \tilde{\ell} + 1/2 \).

In the pseudospin symmetry limit the two states in Eq. (15) form a degenerate doublet \( (S = 1/2) \), and are connected by the pseudospin generators \( \hat{S}_\mu \) of Eq. (13). The corresponding upper components are a doublet with respect to the set \( \hat{s}_\mu \) (the non-relativistic
pseudospin algebra). Since the latter, by definition, intertwine space and spin, they
can connect states for which the upper components have different radial wave functions,
\( G_{\kappa<0}(r) \neq G_{\kappa'>0}(r) \). On the other hand, the corresponding lower components are a doublet
with respect to the ordinary spin \( \hat{s}_\mu \), and hence, in the pseudospin limit, their radial wave
functions are equal up to a phase,
\[ F_{\kappa<0}(r) = F_{\kappa'>0}(r) . \tag{16} \]

In particular, \( F_{\kappa<0}(r) \) and \( F_{\kappa'>0}(r) \) have the same number of nodes, which we denote by
\( n_r \). If we now use the result of Eq. (11), we find for \( n_r \neq 0 \) that \( G_{\kappa<0}(r) \) in Eq. (15)
has also \( n_r \) radial nodes, while \( G_{\kappa'>0}(r) \) has \( n_r - 1 \) nodes. This explains the structure of
nodes in the pseudospin doublets of Eq. (1). The simple relation in Eq. (16) between the
radial wave functions of the lower components of the two states in the doublet, dictates this
particular relation between the radial nodes of the corresponding upper components. This
result cannot be obtained if one considers just the non-relativistic pseudospin algebra, \( \hat{s}_\mu \),
and is a direct outcome of the behavior of nodes of Dirac bound states and the identification
of pseudospin as a relativistic symmetry of the Dirac Hamiltonian.

As we have shown, bound Dirac states, for which both the upper and lower components
have no nodes \( (n_r = 0) \) can occur only for \( \kappa < 0 \) and not for \( \kappa' > 0 \). From Eq. (15a) we
find that such states have pseudo-orbital angular momentum \( \tilde{\ell} \) and total angular momentum
\( j = \tilde{\ell} - 1/2 \). As mentioned, these intruder states are ignored in the non-relativistic pseudospin
scheme, and it is only the relativistic interpretation of pseudospin symmetry, combined with
known properties of Dirac bound states, which enable a classification for these states, as
well as provide a natural explanation why these states do not have a pseudospin partner
which is an eigenstate of the Hamiltonian.

The exact pseudospin limit requires that \( V_S(r) = -V_V(r) \), which implies that \( B(r) = E - M \) in Eq. (3b). It is clear that under such circumstances the condition of Eq. (12) cannot
be fulfilled for bound states with positive binding energy $M - E > 0$. This explains why in the exact pseudospin limit, there are no bound Dirac states and, therefore, by necessity the pseudospin symmetry must be broken in nuclei. Nevertheless, a variety of realistic mean field calculations show that the required breaking of pseudospin symmetry in nuclei is small \[12, 13\]. Quasi-degenerate doublets of normal-parity states and abnormal-parity levels without a partner eigenstate persist in the spectra, and the relation of Eq. (16) is obeyed to a good approximation, especially for doublets near the Fermi surface. As discussed, these features are sufficient to ensure the observed structure of nodes occurring in pseudospin doublets and the special status of intruder levels in nuclei.

In summary, we have shown that identification of pseudospin as a relativistic symmetry of the Dirac Hamiltonian provides a natural explanation for the structure of radial nodes occurring in pseudospin doublets of normal-parity states. The key point in this explanation is that pseudospin symmetry implies a simple relation between the radial wave functions of the lower components of the two states in the doublet, and those in turn govern the radial nodes of the corresponding upper components. The intruder abnormal-parity states which have so far been discarded in non-relativistic treatments, can be accommodated in the relativistic pseudospin scheme and are assigned a pseudo-orbital quantum number $\tilde{\ell}$ and $j = \tilde{\ell} - 1/2$. General properties of Dirac bound states provide a natural explanation for why these nodeless states do not have a partner which is an eigenstate of the Hamiltonian. It is gratifying to note that characteristic features (e.g. radial and angular momentum quantum numbers) of states in the non-relativistic pseudospin scheme, which seem at first ad-hoc without an apparent reason, receive a proper justification once the relativistic origin of pseudospin symmetry in nuclei is taken into consideration.

We thank Dr. Jiri Mares for assistance with his numerical code. This research was supported in part by the U.S.-Israel Binational Science Foundation and in part by the
United States Department of Energy under contract W-7405-ENG-36.
REFERENCES

[1] A. Arima, M. Harvey and K. Shimizu, Phys. Lett. B 30 (1969) 517.

[2] K.T. Hecht and A. Adler, Nucl. Phys. A 137 (1969) 129.

[3] A. Bohr, I. Hamamoto and B.R. Mottelson, Phys. Scr. 26 (1982) 267.

[4] J. Dudek, W. Nazarewicz, Z. Szymanski and G.A. Leander, Phys. Rev. Lett. 59 (1987) 1405.

[5] W. Nazarewicz, P.J. Twin, P. Fallon and J.D. Garrett, Phys. Rev. Lett. 64 (1990) 1654.

[6] F.S. Stephens et al., Phys. Rev. Lett. 65 (1990) 301; F.S. Stephens et al., Phys. Rev. C 57 (1998) R1565.

[7] D. Troltenier, C. Bahri, J.P. Draayer, Nucl. Phys. A 586 (1995) 53.

[8] C. Bahri, J.P. Draayer and S.A. Moszkowski, Phys. Rev. Lett. 68 (1992) 2133.

[9] A. L. Blokhin, C. Bahri, and J.P. Draayer, Phys. Rev. Lett. 74 (1995) 4149.

[10] J.N. Ginocchio, Phys. Rev. Lett. 78 (1997) 436.

[11] J.N. Ginocchio and A. Leviatan, Phys. Lett. B 425 (1998) 1.

[12] J.N. Ginocchio and D. G. Madland, Phys. Rev. C 57 (1998) 1167.

[13] G.A. Lalazissis et al., Phys. Rev. C 58 (1998) R45.

[14] J. Meng et al., Phys. Rev. C 58 (1998) R628; J. Meng, K. Sugawara-Tanabe, S. Yamaji and A. Arima, Phys. Rev. C 59 (1999) 154.

[15] J.N. Ginocchio and A. Leviatan, Phys. Rev. Lett. (2001) in press, [nucl-th/0105071].

[16] K. Sugawara-Tanabe and A. Arima, Phys. Rev. C 58 (1998) R3065; K. Sugawara-
Tanabe, S. Yamaji and A. Arima, Phys. Rev. C 62 (2000) 054307; S. Marcos, L. Savushkin, M. López-Quelle and P. Ring, Phys. Rev. C 62 (2000) 054309.

[17] P. Alberto et al. Phys. Rev. Lett. 86 (2001) 5015.

[18] J.N. Ginocchio, Phys. Rev. C 59 (1999) 2487;
    P. von Neumann-Cosel and J.N. Ginocchio, Phys. Rev. C 62 (2000) 014308.

[19] J.N. Ginocchio, Phys. Rev. Lett. 82 (1999) 4599;
    H. Leeb and S. Wilmsen, Phys. Rev. C 62 (2000) 024602.

[20] M.E. Rose and R.R. Newton, Phys. Rev. 82 (1951) 470.

[21] M.E. Rose, Relativistic Electron Theory, (John Wiley, NY, 1961).

[22] M. Hirooka, G. Konisi and T. Satto, Prog. Theor. Phys. 41 (1969) 161.

[23] J.S. Bell and H. Ruegg, Nucl. Phys. B 98 (1975) 151.
FIGURES

Figure 1. (a) The radial upper component \((G)\) and lower component \((F)\) in \((\text{Fermi})^{-1/2}\) and (b) the corresponding product \(GF\) in \((\text{Fermi})^{-1}\) of the \(2p_{3/2}\) \((\kappa = -2)\) state. (c) and (d) The same for the \(1f_{5/2}\) \((\kappa = 3)\) state. (e) and (f) The same for the \(0g_{9/2}\) \((\kappa = -5)\) state.

All states are eigenstates of a Dirac Hamiltonian with scalar \((S)\) and vector \((V)\) potentials:

\[
V_{S,V}(r) = \alpha_{S,V} \left[ 1 + \exp(\frac{r-R}{a}) \right]^{-1}
\]

with \(\alpha_S = -358\), \(\alpha_V = 292\) MeV, \(R = 7\) fm, \(a = 0.6\) fm.
a) $2p_{3/2}$ ($\kappa = -2$)

b) $GF$

c) $1f_{5/2}$ ($\kappa = 3$)

d) $GF$

e) $0g_{9/2}$ ($\kappa = -5$)

f) $GF$