The local lifting problem for dihedral groups

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Abstract

Let $G = D_p$ be the dihedral group of order $2p$, where $p$ is an odd prime. Let $k$ an algebraically closed field of characteristic $p$. We show that any action of $G$ on the ring $k[[y]]$ can be lifted to an action on $R[[y]]$, where $R$ is some complete discrete valuation ring with residue field $k$ and fraction field of characteristic 0.

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1 Introduction

1.1 Let $k$ be an algebraically closed field of characteristic $p$ and $G$ a finite group. A local $G$-action is a faithful $k$-linear action $\phi : G \hookrightarrow \text{Aut}_k(k[[y]])$.

Problem 1.1 (The local lifting problem) Does there exist a lift of $\phi$ to an $R$-linear action $\phi_R : G \hookrightarrow \text{Aut}_R(R[[y]])$, where $R$ is some complete discrete valuation ring with residue field $k$ and fraction field of characteristic zero?

If a lift $\phi_R$ as above exists then we say that $\phi$ lifts to characteristic 0.

The main motivation for studying Problem 1.1 comes from the global lifting problem: given a smooth projective curve $Y$ over $k$ on which the group $G$ acts, can we lift the curve $Y$ to characteristic 0, together with the $G$-action? It is well known that it suffices to construct such a lift locally, i.e. to lift the completion of the curve $Y$ at closed points together with the action of the stabilizer of that point. The global lifting problem is hence reduced to the local lifting problem. See [4] or [2]. In this paper we solve the local lifting problem for the dihedral group of order $2p$.

Theorem 1.2 Suppose that $p$ is odd and that $G$ is the dihedral group of order $2p$. Then every local $G$-action $\phi : G \hookrightarrow \text{Aut}_k(k[[y]])$ lifts to characteristic 0.

Our approach to the local lifting problem is based on a generalization of the methods of Green–Matignon [5] and Henrio [6] (which treat the case $G = \mathbb{Z}/p$). We roughly do the following. Let $G = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ be a semidirect product of a cyclic group of order $m$ by a cyclic group of order $p$ (with $(p,m) = 1$). Suppose that $G$ acts on the open rigid disc over a $p$-adic field $K$. To this action we associate a certain object, called a Hurwitz tree. This object represents,
some sense, the reduction of the \(G\)-action on the disc to characteristic \(p\). Our first main result is that one can reverse this construction. As a consequence, the local lifting problem for the group \(G = \mathbb{Z}/p \times \mathbb{Z}/m\) can be reduced to the construction of certain Hurwitz trees. Our second main result is that this can always be done if \(G = D_p\). This involves the construction of certain differential forms on the projective line with very specific properties.

1.2 Let \(G\) be a finite group. If there exists a local \(G\)-action \(\phi : G \hookrightarrow \text{Aut}_k(k[[y]])\), then \(G\) is an extension of a cyclic group \(C\) of order prime to \(p\) by a \(p\)-group \(P\). Furthermore, if \(P = 1\) then \(\phi\) always lifts to characteristic 0.

In this paper we always assume that the subgroup \(P\) has order \(p\). Then there exists a character \(\chi : C \to \mathbb{F}_p^\times\) such that \(G = P \rtimes \chi C\) (by this we mean that \(\tau \sigma \tau^{-1} = \chi(\tau) \sigma \) for \(\sigma \in P\) and \(\tau \in C\)). Write \(m\) for the order of \(C\).

Let \(\phi : G \hookrightarrow \text{Aut}_k(k[[y]])\) be a local \(G\)-action. Let \(\sigma\) be a generator of the cyclic group \(P\). The conductor of \(\phi\) is the positive integer

\[
h := \text{ord}_y \left( \frac{\sigma(y)}{y} - 1 \right).
\]

There are certain obvious necessary conditions on the character \(\chi\) and the conductor \(h\) for a lift to exist. In the case where \(G = P\) has order \(p\), this was first noted by Oort, [9, §I.1]. In [1], Bertin found a systematic way to produce necessary conditions for the liftability of \(\phi\), for general groups \(G\). This will be further discussed in [3]. In our case (where \(P\) has order \(p\)), we can summarize these conditions as follows (see Corollary 3.5 below).

**Proposition 1.3** Suppose that \(\phi\) lifts. Then the following holds.

(i) The character \(\chi\) is either trivial or injective.

(ii) If \(\chi\) is injective, \(m|\ h + 1\).

Note that for the group \(G = D_p\), \(\chi\) is injective and \(m = 2\). Moreover, the conductor of any local \(D_p\)-action \(\phi\) is odd, by the Hasse–Arf theorem. Hence the necessary conditions in Proposition 1.3 are automatically verified.

Chinburg–Guralnick–Harbater [3] call a group \(G = P \rtimes \chi C\) a local Oort group at \(p\) if every local \(G\)-action \(\phi : G \hookrightarrow \text{Aut}_k(k[[y]])\) lifts to characteristic zero. This terminology is inspired by the conjecture of Oort [10] that every cyclic cover of smooth projective curves is liftable to characteristic zero. Green–Matignon [4] have shown that a cyclic group of order \(pm\), with \((p, m) = 1\), is a local Oort group at \(p\). Proposition 1.3 shows that, for \(P\) cyclic of order \(p\), the only other possible local Oort group at \(p\) is the dihedral group \(D_p\) of order \(2p\). Theorem 1.2 shows that \(D_p\) is indeed a local Oort group. Hence we now have a complete list of all local Oort groups with Sylow \(p\)-subgroup of order \(p\).

**Question 1.4** Let \(G = P \rtimes \chi C\) be as above, with \(P \cong \mathbb{Z}/p\), and let \(\phi : G \hookrightarrow \text{Aut}_k(k[[y]])\) be a local \(G\)-action. Suppose that \(\phi\) satisfies the necessary conditions given by Proposition 1.3. Is it then true that \(\phi\) always lifts?
In this paper we did not attempt to answer Question 1.4. Nevertheless, the methods presented here reduce this question to solving certain explicit equations over \( \mathbb{F}_p \).

1.3 The local lifting problem (Problem 1.1) makes sense also for groups \( G \) which are the extension of a cyclic group of order prime to \( p \) by an arbitrary \( p \)-group. For \( p \) odd and \( G \cong (\mathbb{Z}/p)^2 \), Green and Matignon have constructed local \( G \)-actions \( \varphi \) which do not lift. They also showed that every local \( G \)-action lifts if \( G \) is a cyclic group of order \( p^n m \) with \( n = 1, 2 \) and \( \gcd(p, m) = 1 \), see [4]. Therefore, it seems reasonable to conjecture that for \( p \) odd the only local Oort groups at \( p \) are the cyclic groups of order \( p^n m \) and the dihedral groups of order \( 2p^n \).

For \( p = 2 \) the situation is somewhat different. The groups \( (\mathbb{Z}/2)^2 \) (\cite[Th. 5.2.1]{11}) and \( A_4 \) (I.I. Bouw, unpublished) are local Oort groups at 2. It seems one needs a more careful analysis of the situation before one can make a reasonable guess on what to expect here.

The paper is organized as follows. In §2 we study \( G \)-action on the boundary of a rigid disc. This study is extended in §3 to \( G \)-actions on the whole disc. The main result here is Theorem 3.6 which allows the construction of a \( G \)-action on the disc, starting from a Hurwitz tree. In §4 we prove our main theorem (Theorem 1.2) in the case where the conductor \( h \) is > \( p \). In §5 we deal with the case \( h < p \).

We thank Leonardo Zapponi for interesting comments on the previous version of this paper. Among other things, he told us how the equations in §5.1 could be simplified.

2 Group actions on the boundary of the disc

In this section we study possible actions of the group \( G = \mathbb{Z}/p \times \mathbb{Z}/m \) on the boundary of an open rigid disc. This is a preparation for the next section, where we will study \( G \)-actions on the whole disc. The main result is Proposition 2.3 which shows that the action of \( G \) on the boundary of a disc is determined, up to conjugation, by three simple invariants. This will allow us in §3 to construct \( G \)-actions on the whole disc by patching. In the case \( G = \mathbb{Z}/p \) this is already contained in [6].

2.1 Let \( R \) be a complete discrete valuation ring with residue field \( k \) and fraction field \( K \). We assume that \( k \) is algebraically closed of characteristic \( p > 2 \) and that \( K \) has characteristic 0. We fix a uniformizing element \( \pi \) of \( R \).

Let

\[ Y_K := \{ y \mid |y|_K < 1 \} . \]

be the rigid open unit disc with parameter \( y \). We let \( S = R[[y]][y^{-1}] \) denote the ring of formal Laurent series \( f = \sum_i f_i y^i \) with \( f_i \in R \) for \( i \in \mathbb{Z} \) and
\[ \lim_{i \to -\infty} f_i = 0. \]

The ring \( S \) is a complete discrete valuation ring with residue field \( \bar{S} = k((y)) \), uniformizing element \( \pi \) and fraction field \( S \otimes_R K \). We can think of elements of \( S \) as bounded functions on the boundary of \( Y \).

We denote by \( v : (S \otimes K)^\times \to \mathbb{Z} \) the exponential valuation normalized by \( v(\pi) = 1 \) and by \( \text{ord}_y : \bar{S}^\times \to \mathbb{Z} \) the exponential valuation normalized by \( \text{ord}_y(y) = 1 \). Furthermore, we define the function \( \sharp : (S \otimes K)^\times \to \mathbb{Z} \) by the formula

\[ \sharp f := \text{ord}_y\left( \frac{f}{\pi^{v(f)}} \right) \mod \pi. \]

A parameter on the boundary of the disc is a unit \( z \in S^\times \) with \( \sharp z = 1 \).

2.2 Let \( P \) be a cyclic group of order \( p \), \( C \) a cyclic group of order \( m \), with \( (m, p) = 1 \), and \( \chi : C \to \mathbb{F}_p^\times \) a character. We set \( G := P \rtimes \chi C \).

An automorphism of the boundary of the disc is an automorphism of \( \text{Spec} S \) which induces a continuous automorphism of the residue field \( \bar{S} = k((y)) \) of \( S \). We write \( \text{Aut}_R(\text{Spec} S) \) for the group of such automorphisms. If \( z \) is any parameter, there exists a unique automorphism \( \tau \) with \( \tau^* y = z \).

Fix a \( G \)-action \( \psi : G \to \text{Aut}_R(\text{Spec} S) \) and consider the group \( G \) as a subgroup of \( \text{Aut}_R(\text{Spec} S) \) by means of \( \psi \). Let \( T := S^P \) be the ring of invariants under the action of \( P \subset G \). Clearly, \( T \) is again a complete discrete valuation ring with parameter \( \pi \), and \( S \) is finite and flat over \( T \) of degree \( p \). It follows from classical valuation theory that the extension of residue fields \( \bar{S}/\bar{T} \) has degree \( p \) and that \( \bar{T} = k((z)) \) is a field of Laurent series over \( k \). Let \( D_{S/T} \) denote the different of the extension \( S/T \).

**Definition 2.1** The **different** of the \( G \)-action \( \psi \) on the boundary of the disc is the unique rational number \( \delta \) with

\[ D_{S/T} = p^\delta \cdot S. \]

The **conductor** of \( \psi \) is the integer

\[ h := \sharp (\sigma^* y - 1). \]

The **tame inertia character** of \( \psi \) is the character \( \lambda : C \to k^\times \) such that

\[ \tau^* y \equiv \lambda(\tau) \cdot y \pmod{y^2} \]

in \( \bar{S} \).
The conductor and the different are already defined by Henrio [6]. He shows that $0 \leq \delta \leq 1$. Moreover, we have the following classification result.

**The étale case:** Suppose that $\delta = 0$. Then the extension $S/T$ is unramified and the extension of residue fields $\bar{S}/\bar{T}$ is Galois, with Galois group $P$. The conductor $h$ of $\psi$ is equal to the usual conductor of the Galois extension $\bar{S}/\bar{T}$. In particular, $h$ is prime to $p$ and positive.

**The additive case:** Suppose that $0 < \delta < 1$. Then the group $P$, considered as constant group scheme over $K$, extends to a finite and flat group scheme $P_R$ over $R$, with special fiber isomorphic to $\alpha_p$. Moreover, the action of $P$ on $\text{Spec}(S \otimes_R K)$ extends uniquely to an action of $P_R$ on $\text{Spec} S$ which makes the map $\text{Spec} S \to \text{Spec} T$ a $P_R$-torsor. There exists an exact differential $\omega \in \Omega^\text{cont}_{T/k}$ which classifies the induced $\alpha_p$-torsor $\text{Spec} \bar{S} \to \text{Spec} \bar{T}$ ([8, Proposition III.4.14]). We have

$$h = -\text{ord}_z \omega - 1.$$  
Moreover, $h$ is prime to $p$ and different from 0.

**The multiplicative case:** Suppose that $\delta = 1$. Then the group $P$, considered as constant group scheme over $K$, extends to a finite and flat group scheme $P_R \cong \mu_{p,R}$ over $R$. Moreover, the action of $P$ on $\text{Spec}(S \otimes_R K)$ extends uniquely to an action of $P_R$ on $\text{Spec} S$ which makes the map $\text{Spec} S \to \text{Spec} T$ a $P_R$-torsor. There exists a logarithmic differential $\omega \in \Omega^\text{cont}_{T/k}$ which classifies the induced $\mu_p$-torsor $\text{Spec} \bar{S} \to \text{Spec} \bar{T}$ ([8, Proposition III.4.14]). Again we have

$$h = -\text{ord}_z \omega - 1.$$  
Moreover, $h$ is prime to $p$ (except if $h = 0$) and less then or equal to 0.

See [6, §1] for details. Note that the invariants $h$ and $\delta$ only depend on the action of $P$. However, the fact that the action of $P$ extends to an action of $G = P \rtimes C$ puts some extra conditions on the conductor $h$.

**Proposition 2.2** The tame inertia character $\lambda$ has order $m = |C|$, and we have

$$\chi = \lambda^{-h}.$$  
Therefore, the order of $h$ in $\mathbb{Z}/m$ is equal to the order of the character $\chi$. In particular:

(i) If $\chi$ is injective then $h$ and $m$ are relatively prime. Moreover, $\lambda$ takes values in $\mathbb{F}_p^\times$ and is uniquely determined by $\chi$ and $h$.

(ii) If $\chi$ is trivial then $m|h$.

**Proof:** Let $U := S^G$. As before, we conclude that $S/U$ is a Galois extension of discrete valuation rings, with Galois group $G$ and residue field extension $\bar{S}/\bar{U}$.
of degree $|G| = \text{pm}$. Since $m$ is prime to $p$, $\tilde{T}/\tilde{U}$ is actually a Galois extension with Galois group $C$.

In the étale case $\tilde{S}/\tilde{U}$ is a Galois extension with Galois group $G$. The filtration of higher ramification groups in the upper numbering has a unique jump at $\sigma = h/m$. In this case, the statements of Proposition 2.2 are well known. For instance, (i) and (ii) follow from the Hasse–Arf theorem. We recall the argument, because this allows us to introduce some notation which we need later on. By Artin–Schreier theory, there exists a generator $w$ of the extension $\tilde{S}/\tilde{T}$ such that $\sigma^*w = w + 1$ and

$$w^p - w = z^{-h} + c_1 z^{-h+1} + \ldots \in \bar{T} = k((z)).$$

(1)

Also, it is a standard fact on tame ramification that we may choose the parameter $z$ in such a way that $\tau^*z = \chi'(\tau) \cdot z$ for all $\tau \in C$ and for some character $\chi' : C \to k^\times$ of order $m$. It follows from (1) that $\text{ord}_y(w) = -h$, where $y$ is a parameter of $S$. Therefore, we may assume (by changing the parameter $y$ of $\bar{S}$) that $w = y^{-h}$. A straightforward computation, using $\tau \sigma \tau^{-1} = \sigma \chi(\tau)$ shows that $\chi(\tau) = \chi'(\tau)^{-h}$ and $\tau^*y = \chi'(\tau) \cdot y \pmod{y^2}$. Hence $\lambda = \lambda'$ and $\lambda^{-h} = \chi$. This proves the proposition in the étale case.

In the additive and multiplicative case the extension $\tilde{S}/\tilde{T}$ is inseparable of degree $p$ and an $\alpha_p$- or $\mu_p$-torsor. The torsor structure is encoded by a differential form $\omega \in \Omega_{\tilde{T}/\tilde{K}}$ with $\text{ord}_\omega = -h - 1$. If $w^p = u$ is an equation for this torsor, then $\omega = du$ (in the case of an $\alpha_p$-torsor) or $\omega = du/u$ (in the case of a $\mu_p$-torsor). See [6] for details. It is easy to check that the rule $\tau \sigma \tau^{-1} = \sigma \chi(\tau)$ implies $\tau^*\omega = \chi(\tau) \cdot \omega$. As in the étale case, we choose a parameter $z$ of $\tilde{T}$ with $\tau^*z = \chi'(\tau) \cdot z$ for a character $\chi'$ of order $m$. Then $\omega = (az^{-h+1} + \ldots) dz$, and hence $\chi = (\chi')^{-h}$. If $h \neq 0$ then we can choose the torsor-equation $w^p = u$ such that $u = 1 + z^{-h} + \ldots$. It follows that $y := (w - 1)^{-1/h}$ is a parameter for $\bar{S}$ with $\tau^*y = \chi'(\tau) \cdot y \pmod{y^2}$. Hence $\lambda = \lambda'$ and $\lambda^{-h} = \chi$. On the other hand, if $h = 0$ then we see that $\chi$ must be trivial, and so the claim of the proposition is true as well. This finishes the proof of the proposition. 

\[ \square \]

2.3 Let $\psi_i : G \to \text{Aut}_R(S)$, $i = 1, 2$, be two $G$-actions on the boundary of the disc. We say that $\psi_1$ and $\psi_2$ are conjugate if there exists an automorphism $\eta \in \text{Aut}_R(\text{Spec } S)$ such that $\psi_2(g) \circ \eta = \eta \circ \psi_1(g)$ holds for all $g \in G$.

**Proposition 2.3** Suppose that the two $G$-actions $\psi_1$ and $\psi_2$ have the same conductor $h$, the same different $\delta$ and the same tame inertia character $\lambda$. Suppose, moreover, that $h \neq 0$. Then $\psi_1$ and $\psi_2$ are conjugate.

**Proof:** It follows from [6, Corollaire 1.8] that the restrictions of $\psi_1$ and $\psi_2$ to the subgroup $P \subset G$ are conjugate. Hence we may assume that $\psi_1|_P = \psi_2|_P$.

The $P$-action on $\text{Spec}(S \otimes K)$ extends to a free action of a certain $R$-group scheme $P_R$ on $S$, where $P_R$ is either the constant group scheme $P$ (étale case) or a connected group scheme with special fiber $\alpha_p$ (additive case) or $\mu_p$ (multiplicative case). For $n \geq 0$, set $S_n := S \otimes_R (R/p^n + 1)$ and $P_n := P_R \otimes_R (R/p^n + 1)$. 

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Claim 2.4 There exists a family \( (\eta_n) \), where \( \eta_n \) is an automorphism of \( \text{Spec} \, S_n \) which commutes with the action of \( P_n \) and such that \( \psi_2(\tau) \circ \eta_n \equiv \eta_n \circ \psi_1(\tau) \) and \( \eta_{n+1} \equiv \eta_n \pmod{\pi^{n+1}} \).

The claim implies that \( \eta := \lim_n \eta_n \) is an automorphism of \( S \) which commutes with the action of \( P \) and such that \( \psi_2(\tau) \circ \eta = \eta \circ \psi_1(\tau) \) for all \( \tau \in C \). Hence the proposition is an immediate consequence of Claim 2.4.

We prove Claim 2.4 by induction on \( n \), and we start at \( n = 0 \). We consider the étale case first. Let \( w \) be an Artin–Schreier generator of the extension \( \bar{S}/\bar{T} \), as in the proof of Proposition 2.2. Then \( \sigma^* w = w + 1 \). If \( f := w^p - w = z^{-h} + \ldots \in \bar{T} \). Replacing \( w \) by \( w + g \), for a suitable \( g \in k[[z]] \cdot z^{-h+1} \), we can achieve that \( f \) has no monomials whose exponents are divisible by \( p \).

Furthermore, choosing the parameter \( z \) of \( \bar{T} \) appropriately, we may assume that \( \psi_1(\tau)^* z = \lambda(\tau) \cdot z \) for all \( \tau \in C \). It follows from Artin–Schreier theory that

\[
\psi_1(\tau)^* w = \chi(\tau)(w + g_\tau),
\]

with \( g_\tau \in k[[z]] \cdot z^{-h+1} \), and hence

\[
\psi_1(\tau)(f)^* = \chi(\tau)(f + g_\tau^p - g_\tau).
\]

Using our assumption on \( f \) and \( \psi_1(\tau)^* z = \lambda(\tau) \cdot z \), it is now easy to see that \( g_\tau = 0 \). We conclude that, in terms of the parameter \( y = w^{-h} \) for \( \bar{S} \), the reduction of \( \psi_1 \) to \( \bar{S} \) is given by

\[
\psi_1(\sigma)^* y = y(1 + y^h)^{-1/h}, \quad \psi_1(\tau)^* y = \lambda(\tau) \cdot y.
\]

This depends visibly only on \( h \) and \( \lambda \). Hence we can find another parameter \( y' \) for \( \bar{S} \) such that the reduction of \( \psi_2 \) to \( \bar{S} \) is given by the same formulas (but with \( y \) replaced by \( y' \)). Now \( \eta_0 y := y' \) defines the desired automorphism \( \eta_0 \). This proves Claim 2.4 for \( n = 0 \).

The proof for \( n = 0 \) in the additive and the multiplicative case is quite similar. We sketch it in the multiplicative case. Then \( \bar{S} \to \bar{T} \) is a \( \mu_p \)-torsor, given by a Kummer equation \( w^p = f \), with \( f \in \bar{T}^\times \). We choose \( z \) such that \( \psi_1(\tau)^* z = \lambda(\tau) \cdot z \). Since \( h < 0 \), we can choose \( f = 1 + z^{-h} + \ldots \), and may assume that \( f \) has no monomial whose exponent is divisible by \( p \). Then \( \psi_1(\tau)^* w = w^\nu g_\tau \), with \( \nu \equiv \chi(\tau) \pmod{p} \) and \( g_\tau \in k[[z]]^\times \). Therefore, \( \psi_1(\tau)^* f = f^\nu g_\tau^\nu \). An argument as above shows that \( g_\tau = 1 \). The conclusion is as in the étale case. This completes the proof of Claim 2.4 for \( n = 0 \).

Let us assume that we have constructed \( \eta_n \) for some \( n \geq 0 \). We claim that we can lift \( \eta_n \) to an automorphism \( \eta' \) of \( S \) which commutes with the action of \( P_R \). For instance, in the multiplicative case we have \( \eta_n(y) = yv_n \), where \( v_n \in T_n = S_n^\times \) and \( y^p = f \) is a Kummer equation for the \( \mu_p \)-torsor \( \text{Spec} \, S \to \text{Spec} \, T \). Then we can simply lift \( v_n \) to an element \( v \in T \) and set \( \eta'(y) := yv \). The construction of the lift \( \eta' \) in the étale and the additive case is similar.

Let us choose a lift \( \eta' \) as above. Conjugating \( \psi_2 \) by \( \eta' \), we may assume that \( \psi_1 \equiv \psi_2 \pmod{\pi^{n+1}} \). Then for \( \tau \in C \) we define \( \alpha_\tau := \psi_2(\tau) \circ \psi_1(\tau)^{-1} \)
mod $\pi^{n+2}$. This is an automorphism of $\text{Spec} S_{n+1}$ which is the identity on $\text{Spec} S_n$. Therefore, for all $w \in S_{n+1}$ we have

$$\alpha^*_\tau w = w + \pi^{n+1} \cdot D_\tau(\bar{w}),$$

where $D_\tau$ is a (continuous) $k$-derivation of $\bar{S}$. It is clear that $\alpha_\tau$ commutes with the action of $P_{n+1}$. Therefore the derivation $D_\tau$ is invariant under the action of $P_0$. One also checks that the map

$$C \to \text{Der}_{S/k}^{P_0}, \quad \tau \mapsto D_\tau$$

is a cocycle (note that $\text{Der}_{S/k}^{P_0}$ is a $C$-module in a natural way). Since $\text{Der}_{S/k}^{P_0}$ is a $k$-vector space and the order of $C$ is prime to $p$, this cocycle is a coboundary. This means that there exists a $P_0$-invariant derivation $D$ such that $D_\tau = \tau \circ D \circ \tau^{-1} - D$. Set $\eta_{n+1}^* w := w - \pi^{n+1} \cdot D(\bar{w})$. Then $\psi_2(\tau) \circ \eta_{n+1} = \eta_{n+1} \circ \psi_1(\tau)$ for all $\tau$, as desired. By induction, we get a proof of Claim 2.4, and hence of Proposition 2.3.

\[\square\]

Remark 2.5 The proof of Proposition 2.3 in the étale case for $n = 0$ shows the following. Let $\phi_1, \phi_2 : G \hookrightarrow \text{Aut}_k(k[[y]])$ be two local $G$-actions with the same conductor and the same tame inertia character. Then $\phi_1$ and $\phi_2$ are conjugate (inside $\text{Aut}_k(k[[y]])$).

3 Group actions on the disc

In this section we study actions of the group $G = \mathbb{Z}/p \rtimes \mathbb{Z}/m$ on the open rigid disc. We generalize the results of Green–Matignon [5] and Henrio [6], which treat the case $G = \mathbb{Z}/p$.

Throughout, we suppose that $p$ is an odd prime and that $k$ is an algebraically closed field of characteristic $p$.

3.1 Hurwitz trees We start by giving a formal definition of a Hurwitz tree, following Henrio [6]. Our definition differs slightly from the definition given in [6]; for instance, we do not distinguish between Hurwitz trees and realizations of Hurwitz trees.

Definition 3.1 A decorated tree is given by the following data:

- a semistable curve $Z$ over $k$ of genus 0,
- a family $(z_b)_{b \in B}$ of pairwise distinct smooth $k$-rational points of $Z$, indexed by a finite nonempty set $B$,
- a distinguished smooth $k$-rational point $z_0 \in Z$, distinct from any of the points $z_b$. 

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We require that $Z$ is stably marked by the points $(z_b; z_0)$, see [7].

The *combinatorial tree* underlying a decorated tree $Z$ is the graph $T = (V, E)$, defined as follows. The vertex set $V$ of $T$ is the set of irreducible components of $Z$, together with a distinguished element $v_0$, called the root. The edge set $E$ is the set of singular points of $Z$, together with a distinguished element $e_0$. We write $Z_v$ for the component corresponding to a vertex $v \neq v_0$ and $z_e$ for the singular point corresponding to an edge $e \neq e_0$. An edge $e$ corresponding to a singular point $z_e$ is adjacent to the vertices corresponding to the two components which intersect in $z_e$. The edge $e_0$ is adjacent to the root $v_0$ and the vertex $v$ corresponding to the (unique) component $Z_v$ containing the distinguished point $z_0$. For each edge $e \in E$, the source (resp. the target) of $e$ is the unique vertex $s(e) \in V$ (resp. $t(e) \in V$) adjacent to $e$ which lies in the direction of the root (resp. in the direction away from the root).

Note that, since $(Z, (z_b), z_0)$ is stably marked of genus 0, the components $Z_v$ have genus zero, too, and the graph $T$ is a tree. Moreover, we have $|B| \geq 2$.

An *admissible action* of a group $C$ on a decorated tree $(Z, (z_b), z_0)$ is a $k$-linear action of $C$ on the curve $Z$ which satisfies the following. Firstly, the action permutes the points $z_b$ and fixes the point $z_0$. Secondly, for any singular point $z_e, e \neq e_0$, the stabilizer $C_e \subset C$ of $z_e$ is of order prime-to-$p$, and the two characters describing the action of $C_e$ on the tangent spaces of the two branches of $Z$ at the point $z_e$ are inverse to each other. For every edge $e$ (including $e_0$) we shall write $\lambda_e : C_e \rightarrow k^*$ for the character which describes the action of $C_e$ on the tangent space of the component $Z_{t(e)}$ at $z_e$.

For a vertex $v \in V$, we write $U_v \subset Z_v$ for the complement in $Z_v$ of the set of singular and marked points.

**Definition 3.2** Let $C$ be a cyclic group of order $m$ (with $(m, p) = 1$) and $\chi : C \rightarrow \mathbb{F}_p^*$ a character. A *Hurwitz tree* of type $(C, \chi)$ is given by the following data:

- A decorated tree $Z = (Z, (z_b), z_0)$ with underlying combinatorial tree $T = (V, E)$, together with an admissible action of $C$.
- For each $v \in V - \{v_0\}$, a differential form $\omega_v$ on $U_v \subset Z_v$ without zeroes or poles.
- For every $v \in V$, a rational number $0 \leq \delta_v \leq 1$, called the *different* of $v$.
- For every $e \in E$, a positive rational number $\epsilon_e$, called the *thickness* of $e$.

These objects are required to satisfy the following conditions.

(i) For every $v \in V - \{v_0\}$ and $\tau \in C$, we have $\tau^* \omega_{\tau(v)} = \chi(\tau) \cdot \omega_v$.

(ii) Let $v \in V$. We have $\delta \neq 0$ if and only if $v \neq v_0$. Moreover, if $\delta_v = 1$ (resp. $0 < \delta_v < 1$) the differential $\omega_v$ is logarithmic (resp. exact). (If this holds, then we call the vertex $v$ *multiplicative* (resp. *additive*).
(iii) For every edge $e \in E - \{e_0\}$, we have the equality
\[ \text{ord}_e \omega_{s(e)} = -\text{ord}_e \omega_{t(e)} - 2. \]

(iv) For every edge $e \in E$, we have
\[ \delta_{t(e)} = \delta_{s(t)} + (p - 1) \cdot \epsilon_e \cdot h_e, \]
where
\[ h_e := \text{ord}_e \omega_{t(e)} + 1. \]

(v) For $b \in B$, let $Z_v$ be the component containing the point $z_b$. Then the differential $\omega_v$ has a simple pole in $z_b$.

The integer $h := h_{e_0}$ is called the conductor of the Hurwitz tree. The rational number $\delta := \delta_{e_0}$ is called the different. The tame inertia character is the character $\lambda = \lambda_{e_0} : C \to k^\times$ which describes the action of $C$ on the tangent space of $Z$ at the point $z_0$. The family $(a_b)_{b \in B}$, with
\[ a_b := \text{res}_{z_b} \omega_v, \]
is called the type.

**Lemma 3.3** Let $(Z, \omega_v, \delta_v, \epsilon_e)$ be a Hurwitz tree.

(i) Fix an edge $e \in E$ and let $Z_e \subset Z$ be the union of all components $Z_v$ corresponding to vertices $v$ which are separated from the root $v_0$ by the edge $e$. Then
\[ h_e = |\{ b \in B \mid z_b \in Z_e \}| - 1 > 0. \]
In particular, $h = |B| - 1 > 0$.

(ii) The following conditions on a vertex $v$ are equivalent:
   (a) $v$ is multiplicative,
   (b) $v$ is a leaf of the tree $T$,
   (c) the component $Z_v$ contains one of the marked points $z_b$.

(iii) Recall that $\lambda_e : C_e \to k^\times$ is the character which describes the action of the stabilizer of the point $z_e$ on the cotangent space of $Z_{s(e)}$ at $z_e$. We have that
\[ \lambda_e^{-h_e} = \chi|_{C_e}. \]

(iv) For $b \in B$ and $\tau \in C$ we have $a_{\tau(b)} = \chi(\tau)a_b$.

(v) The character $\chi$ is either trivial or injective. If $\chi$ is injective then $m|h + 1$. 

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Proof: Statements (i) and (ii) are proved in [5] and [6]. Statement (iii) is proved exactly as in the proof of Proposition 2.2. Statement (iv) is an immediate consequence of Condition (i) of Definition 3.2. It remains to prove (v).

Write \( m = m_1 m_2 \), where \( m_1 \) is the order of \( \chi \) and \( m_2 \) is the order of the kernel of \( \chi \). Assume that \( m_2 \neq 1 \). We want to show that this implies \( m_1 = 1 \).

Let \( Z_v \) be the unique component of \( Z \) which contains the point \( z_0 \). By definition, the action of \( C \) fixes the point \( z_0 \). Since \( Z_v \cong \mathbb{P}^1_k \), there exists a unique point \( z_1 \neq z_0 \) on \( Z_v \) with nontrivial stabilizer, and in fact \( z_1 \) is stabilized by the whole group \( C \). It follows from the same computation that we used to prove (iii) that \( m_2 \mid \text{ord}_z \omega_v + 1 \). In particular, \( \text{ord}_z \omega_v \neq 0 \). Therefore, \( z_1 \) is either equal to a marked point \( z_b \) or to a singular point \( z_e \). In the first case, it follows immediately from (iv) that \( m_1 = 1 \), as desired. In the second case, we consider the subtree \( Z_e \subset Z \) as in (i). It is easy to check that \( Z_e \) inherits from \( Z \) the structure of a Hurwitz tree of type \((C, \chi)\), with distinguished point \( z_1 \). Now the claim \( m_1 = 1 \) follows by induction on the number of components of the Hurwitz tree. This proves the first part of (v). Now assume that \( m_2 = 1 \). Then (iv) implies that the action of \( C \) on the set \( B \) is free. Therefore, \( m | h + 1 \) follows from (i). This proves the second part of (v), and completes the proof of the lemma. □

3.2 The Hurwitz tree associated to a \( G \)-action on the disc

Fix a cyclic group \( C \) of order \( m \), with \((p, m) = 1\), and a character \( \chi : C \to \mathbb{F}_p^\times \). We let \( P \) denote a cyclic group of order \( p \), with generator \( \sigma \). We set \( G := P \rtimes \chi(C) \).

Let \( R \) be a complete discrete valuation ring with residue field \( k \). Let \( K \) denote the fraction field of \( R \). We assume that \( K \) has characteristic 0 and contains the \( p \)th roots of unity. Let \( Y_K := \{ y \mid |y|_K < 1 \} \) be the rigid open unit disc over \( K \). Suppose we are given a faithful action of \( G \) on \( Y_K \),

\[
\phi_K : G \hookrightarrow \text{Aut}_K(Y_K).
\]

Obviously, \( \phi_K \) induces a \( G \)-action on the boundary of the disc. We let \( h \) be the conductor, \( \delta \) the different and \( \lambda \) the tame inertia character associated to this action (Definition 2.1). We assume that \( h > 0 \).

Following [6], we will now associate to \( \phi_K \) a Hurwitz tree of type \((C, \chi)\).

Let \( y_{b,K} \in Y_K \) be the fixed points of the automorphism \( \phi_K(\sigma) \), indexed by the finite set \( B \). We assume that the points \( y_{b,K} \) are all \( K \)-rational. It is proved in [5] that \(|B| = h + 1 \). Hence \(|B| \geq 2 \).

Let \( Y_R \) be the minimal model of \( Y_K \) which separates the points \( y_{b,K} \). More precisely, \( Y_R \) is an admissible blow-up of the formal unit disc \( \text{Spf} R[[y]] \) such that

- the exceptional divisor \( Y \) of the blow-up \( Y_R \to \text{Spf} R[[y]] \) is a semistable curve over \( k \),
- the fixed points \( y_{b,K} \) specialize to pairwise distinct smooth points \( y_b \) on \( Y \), and
- if \( y_0 \) denotes the unique point on \( Y \) which lies in the closure of \( Y_R \otimes k - Y \), then \((Y, (y_b), y_0)\) is stably marked.
We call \((Y, (y_b), y_0)\) the special fiber of the model \(Y_R\). Note that it is a decorated tree, in the sense of Definition 3.1. Note also that there is a natural action of the group \(G\) on \((Y, (y_b), y_0)\). The element \(\sigma\) fixes the points \((y_b)\) and \(y_0\). Therefore, \(\sigma\) acts trivially on \(Y\), and we obtain a natural action of \(C = G/P\) on \(Y\).

Let \(Z_K := Y_K/P\) be the quotient of the disc under the cyclic group of order \(p\); this is again a rigid open disc. Let \(z_0, K \in Z_K\) denote the image of the fixed point \(y_b, K\). Similarly to what we did above, we define \(Z_R\) as the minimal model of \(Z_K\) which separates the points \(z_0, K\), and we let \((Z, (z_0), z_0)\) denote the special fiber of \(Z_R\). The canonical map \(Y_K \to Z_K\) extends uniquely to a map \(Y_R \to Z_R\).

One shows that the induced map \(Y \to Z\) is a \(C\)-equivariant homeomorphism, purely inseparable of degree \(p\).

Let \((V, E)\) be the combinatorial tree underlying \((Z, (z_b), z_0)\). (We will use freely the notation introduced in §3.1.) For \(v \in V\), let \(U_v \subset Z_v\) be the complement of the singular and marked points and let \(U_{v, K} \subset Z_K\) be the affinoid subdomain with reduction \(U_v\). Let \(V_v \subset Y\) (resp. \(V_{v, K} \subset Y_K\)) denote the inverse image of \(U_v\) (resp. of \(U_{v, K}\)). By construction, \(V_v, K \to U_{v, K}\) is a torsor under the constant \(K\)-group scheme \(P\). One shows that this torsor extends to a torsor \(V_{v, R} \to U_{v, R}\) under a certain finite flat \(R\)-group scheme \(P_v\) of degree \(p\). Therefore, the finite inseparable map \(V_v \to U_v\) is endowed with a structure of a torsor under the finite, flat and local \(k\)-group scheme \(P_v \otimes k\) of degree \(p\). See [6] for details. According to the classification of such group schemes, we have to distinguish two cases (compare to the additive and the multiplicative case in §2.2).

(a) \(P_v \otimes k \cong \mu_{p, R}\): the \(\mu_p\)-torsor \(V_v \to U_v\) can be classified by a logarithmic differential form \(\omega_v\) on \(U_v\). We set \(\delta_v := 1\); this is the exponential valuation of the different of the \(R\)-group scheme \(P_v \cong \mu_{p, R}\).

(b) \(P_v \otimes k \cong \alpha_p\): the \(\alpha_p\)-torsor \(V_v \to U_v\) can be classified by an exact differential form \(\omega_v\) on \(U_v\). We define \(0 < \delta_v < 1\) as the exponential valuation of the different of the \(R\)-group scheme \(P_v\).

Finally, for \(e \in E\) we let \(A_e \subset Y_K\) denote the subset of all points which specialize to the singular point \(y_e \in Y\) corresponding to \(e\). This is an open annulus. We define \(\epsilon_e\) as the thickness of \(A_e\), i.e. the positive rational number such that

\[ A_e \equiv \{ y \mid |p|^\epsilon_e < |y|_K < 1 \}. \]

**Proposition 3.4** The datum \((Z, \omega_v, \delta_v, \epsilon_e)\) defines a Hurwitz tree of type \((C, \chi)\). Moreover, \(h\) is the conductor, \(\delta\) the different and \(\lambda\) the tame inertia character of this Hurwitz tree.

**Proof:** Except for Condition (i) of Definition 3.2 and the statement on the tame inertia character which involve the \(C\)-action, this is already proved in [6]. To check Condition (i), one easily verifies the following two facts: (a) the construction of the datum \((Z, \omega_v, \delta_v, \epsilon_e)\) depends functorially on the pair \((Y_K, \sigma)\), and (b) for \(a \in \mathbb{F}_p^\times\), the pair \((Y_K, \sigma^a)\) gives rise to the datum \((Z, a \cdot \omega_v, \delta_v, \epsilon_e)\).
The statement on the tame inertia character follows from the fact that the action of $C$ on the semistable model $Z_R$ is admissible.

**Corollary 3.5** Let $\phi : G \hookrightarrow \text{Aut}_k(k[[y]])$ be a local $G$-action. If $\phi$ lifts to characteristic zero, then the character $\chi$ is either trivial or injective. Moreover, if $\chi$ is injective then $m|h + 1$.

**Proof:** A lift of $\phi$ induces a $G$-action on the open rigid disc (with discriminant 0). Hence the corollary follows from Proposition 3.4 and Lemma 3.3 (v).

### 3.3 Construction of $G$-actions on the disc

In §3.2 we have associated to a $G$-action $\phi_K : G \hookrightarrow \text{Aut}_K(Y_K)$ on the open unit disc a Hurwitz tree of type $(C, \chi)$. The main result of this section is that this construction can be reversed.

**Theorem 3.6** Every Hurwitz tree $(Z, \omega, \delta, \epsilon)$ of type $(C, \chi)$ is associated to a faithful $G$-action $\phi_K : G \hookrightarrow \text{Aut}_K(Y_K)$ on the open rigid unit disc over $K$, for some finite extension $K$ of $\text{Frac}(W(k))$.

The proof of Theorem 3.6 will be given in §3.6. The relevance for the local lifting problem is summarized in the next corollary.

**Corollary 3.7** Let $\phi : G \hookrightarrow \text{Aut}_k(k[[y]])$ be a local $G$-action with conductor $h$. Then $\phi$ lifts to characteristic 0 if and only if there exists a Hurwitz tree of type $(C, \chi)$, conductor $h$ and discriminant $\delta = 0$.

**Proof:** The ‘only if’ part of the corollary follows already from Proposition 3.4. Let $Z$ be a Hurwitz tree of type $(C, \chi)$, conductor $h$ and discriminant $\delta = 0$. By Theorem 3.6, there exists a $G$-action on the open rigid disc which gives rise to the Hurwitz tree $Z$. Let $\phi_R : G \hookrightarrow \text{Aut}_R(R[[y]])$ be the induced action on the formal model $\text{Spf } R[[y]]$ of $Y_K$. The reduction of $\phi_R$ to $k$ corresponds to a local $G$-action $\phi' : G \hookrightarrow \text{Aut}_k(k[[y]])$ with conductor $h$. Let $\lambda : C \rightarrow k^\times$ be the tame inertia character associated to $\phi_K$. Up to changing $\phi$ by an automorphism of $G$, we may assume that $\phi(\tau)^*y \equiv \lambda(\tau) \cdot y \pmod{y^2}$. Remark 2.5 implies that $\phi$ and $\phi'$ are conjugate in $\text{Aut}_k(k[[y]])$. Hence we may regard $\phi_R$ as a lift of $\phi$.

### 3.4 Realization of the vertices

Let $(C, \chi)$ be as before, and set $G := P \rtimes \chi$. Set $Z := \mathbb{P}^1_k$. We let $C$ act on $Z$ in such a way that $\tau^*z = \lambda(\tau) \cdot z$ for all $\tau$, where $z$ is the standard parameter on $\mathbb{P}^1$ and $\lambda : C \rightarrow k^\times$ is a character of order $m = |C|$. Suppose we are given a differential form $\omega$ on $Z$ and a rational number $0 < \delta \leq 1$ such that the following holds.

- The differential $\omega$ has poles in $r \geq 2$ points $z_1, \ldots, z_r \neq \infty$, a zero at $\infty$ and no other poles or zeroes.
- For all $\tau \in C$ we have $\tau^*\omega = \chi(\tau) \cdot \omega$.
\begin{itemize}
  \item If $\delta = 1$ (resp. $\delta < 1$) then $\omega$ is logarithmic (resp. exact).
\end{itemize}

Note that the datum $(Z, \omega, \delta)$, together with the $C$-action, essentially corresponds to a vertex $v \neq v_0$ of a Hurwitz tree of type $(C, \chi)$. The points $z_i$ (resp. the point $\infty$) correspond to the singular points $z_v$ such that $s(e) = v$ (resp. to the unique singular point $z_\infty$ with $t(e) = v$).

Suppose first that $\delta = 1$ (the multiplicative case). Then we set $R := W(k)[\zeta_p]$ and $K := \text{Frac}(R)$. We fix an isomorphism $P \cong \mu_p(K)$; this choice allows us to view the abstract group $G$ as a group scheme over $K$. Then $G_R := \mu_{p,R} \times_C C$ is the unique finite flat group scheme over $R$ with generic fiber $G$ and with connected subgroup scheme $P_R := \mu_{p,R}$. We write $G_k := G_R \otimes_R k$. We set $U := Z - \{\infty\} = \mathbb{H}_1$. Then the logarithmic differential $\omega$ gives rise to a finite flat and radicial morphism of smooth curves $V \to U$, together with an action of $P_k = \mu_{p,k}$ on $V$ such that $U = V/P_k$. The formula $\tau^*\omega = \chi(\tau) \cdot \omega$ shows that this action extends uniquely to an action of $G_k$ on $V$ which induces the canonical action of $C = G_k/P_k$ on $U$. See [12, §4.1] for details.

Now suppose that $0 < \delta < 1$ (the additive case). Write $\delta = 1 - n/d$, for some $n, d \in \mathbb{N}$. Let $K$ be an extension of $\text{Frac}(W(k)[\zeta_p])$ of degree $d$ and let $\pi$ denote a uniformizer of $K$. Let $\mathcal{H}_n$ be the finite flat group scheme over $R$ defined in [6, §1.1]. We have $\mathcal{H}_n \otimes R \cong \mu_{p,R}$ and $\mathcal{H}_n \otimes k \cong \alpha_p$. Similar to what we did in the multiplicative case, we identify $\mathcal{H}_n(K)$ with $P$ and let $G_R := \mathcal{H}_n \times_C C$ denote the unique finite flat group scheme over $R$ with generic fiber $G$ and with connected subgroup scheme $P_R := \mathcal{H}_n$. We write $G_k := G_R \otimes_R k$ and set $U := Z - \{z_1, \ldots, z_r, \infty\}$. Then the exact differential $\omega$ gives rise to a $P_k = \alpha_{p^r}$-torsor $V \to U$. Moreover, the $P_k$-action extends uniquely to an action of $G_k$ which induces the given action of $C = G_k/P_k$ on $U$.

\textbf{Lemma 3.8} The curve $V$ lifts to a smooth formal scheme $V_R$ over $R$, together with an action of $G_R$ which lifts the action of $G_k$ on $V$.

\textbf{Proof:} We give an abstract proof of this fact which treats the multiplicative case and the additive case simultaneously. It is not hard to give a more down-to-earth proof, using the explicit equations for $\mu_{p,R}$- and $\mathcal{H}_n$-torsors.

Let $\text{Def}(V, G_R)$ be the functor which classifies $G_R$-equivariant deformations of $V$ over local artinian $R$-algebras with residue field $k$. To prove the lemma, it suffices to show that $\text{Def}(V, G_R)$ is unobstructed. By [2] and [12], the obstructions for $\text{Def}(V, G_R)$ are represented by elements of the equivariant cohomology group $H^2(G_k, V, T_{V/k})$ (here $T_{V/k}$ denotes the tangent sheaf on $V$; it affords a natural action of $G_k$). The spectral sequence

$$E_2^{i,j} = H^i(C, H^j(P_k, V, T_{V/k})) \Rightarrow H^{i+j}(G_k, V, T_{V/k}),$$

together with the fact that the order of $C$ is prime to $p$, shows that

$$H^2(G_k, V, T_{V/k}) = H^2(P_k, V, T_{V/k})^C.$$  

Moreover, if an element $c$ of $H^2(G_k, V, T_{V/k})$ represents an obstruction for the functor $\text{Def}(V, G_R)$, then its image in $H^2(P_k, V, T_{V/k})$ represents the induced ob-
struction for the functor $\text{Def}(V, P_R)$. Hence it suffices to show that $\text{Def}(V, P_R)$ is unobstructed.

Suppose we are in the multiplicative case, and let $A$ be a local artinian $R$-algebra with residue field $k$. Let $V_A$ be a $P_R$-equivariant lift of $V$. Since $V$ is affine, $V_A$ is affine as well. Therefore, the quotient map $V_A \to U_A := V_A/P_R$ is given by a global Kummer equation $y^p = f$. If $A \to A'$ is a small extension, we can construct a $G_R$-equivariant lift of $V_A$ to $A'$ by first lifting $U_A$ and then lifting the Kummer equation. This shows that $\text{Def}(V, P_R)$ is unobstructed. The proof in the additive case is essentially the same, using the explicit equations for $\mathcal{H}_\infty$-torsors given in [6, §1.1]. This completes the proof of the lemma. □

Let $V_R$ be a lift of $V$ over $R$, together with an action of $G_R$ lifting the action of $G_k$ on $V$. Set $V_K := V_R \otimes_R K$. This is an affinoid with reduction $V$ and carries an action of the abstract group $G = G_K$.

In the multiplicative case we have $U = \mathbb{A}^1_k$ and hence $V \cong \mathbb{A}^1_k$. Therefore, $V_K \cong \{ y \mid |y| \leq 1 \}$ is isomorphic to a closed unit disc.

In the additive case we have $U = \mathbb{A}^1_k - \{ z_1, \ldots, z_r \}$ and hence $V \cong \mathbb{A}^1_k - \{ y_1, \ldots, y_r \}$. Therefore,

$$V_K \cong \{ y \mid |y| \leq 1, |y - y_i| \geq 1 \}$$

is isomorphic to the complement of $r$ open discs inside one closed disc. Let $\text{Spec} S_i$ (resp. $\text{Spec} S_\infty$) be the boundary of the missing open disc containing the point $y_i$ (resp. $\infty$). In other words, $S_i = R[[y - y_i,K]]\{(y - y_i)^{-1}\}$ and $S_\infty = R[[y^{-1}]]\{y\}$. Let $G_i := P \times C_i$ be the stabilizer of the point $y_i$. By construction, we have the following result.

**Proposition 3.9** The action of $G$ on $V_K$ induces an action of $G_i$ on $\text{Spec} S_i$ (resp. of $G$ on $\text{Spec} S_\infty$). This action has conductor $h_i := \text{ord}_{z_i,\omega} + 1$ (resp. conductor $h := -\text{ord}_{\infty,\omega} - 1$) and discriminant $\delta$.

### 3.5 Realization of the edges

We let $G = P \times \chi C$ be as before. Suppose we are given the following data.

- An integer $h \geq 1$ with $\text{ord}(\chi)h + 1$ and $(h, p) = 1$.
- An injective character $\lambda : C \hookrightarrow k^\times$ with $\lambda^{-h} = \chi$.
- Rational numbers $\delta_0, \delta_1$ with $0 \leq \delta_0 < \delta_1 \leq 1$.

These data essentially correspond to an edge of a Hurwitz tree of type $(C, \chi)$. We set

$$\epsilon := \frac{\delta_1 - \delta_0}{(p-1)h}, \quad \kappa_i := \frac{1 - \delta_i}{(p-1)h}, \quad i = 0, 1.$$ 

Let $K$ be a sufficiently large finite and tamely ramified extension of the fraction field of $W(k)$. Let $R$ denote the ring of integers of $K$. Choose a $p$th root of unity $\zeta_p \in R$. Let $\rho_0, \rho_1 \in R$ be an element with $|\rho_0| = |p|^{\epsilon \cdot \kappa_0}$, and set $\pi := \rho_0/\rho_1$. Then

$$B := \{ z_0 \mid |\pi^p| < |z_0| < 1 \}$$
is an open annulus of thickness $\epsilon$, with formal model $\text{Spf} \ R[[z_0, z_1 \mid z_0 z_1 = \pi^n]]$. We let the group $C$ act on $B$ such that $\tau^* z_0 = \lambda(\tau) \cdot z_0$.

Choose a generator $\tau$ of $C$ and a generator $\sigma$ of $P$. Set $\zeta_m := \lambda(\tau)$. Write $m = m_1 m_2$, where $m_1 | (p - 1)$ is the order of the character $\chi$. Choose an integer $\nu$ with $\nu \equiv \zeta^{-h} \pmod{p}$. Then $1 - \nu^m = np$, for some integer $n$. We define

$$f := \prod_{i=0}^{m_1 - 1} (1 + \zeta_m^i \rho_1^{ph} z_1^i)^{\nu^i}. \quad (2)$$

Clearly, $f$ is a bounded invertible function on $B$. A straightforward computation shows that

$$\tau^* f = f^{\nu} \cdot (1 + \rho_1^{ph} z_1^n)^{np}. \quad (3)$$

Let $A \to B$ be the $\mu_p$-torsor defined by the Kummer equation $w^p = f$. One checks that the $C$-action on $B$ extends to a $G$-action on $A$ such that

$$\sigma^* w = \zeta_p \cdot w, \quad \tau^* w = w^{\nu} (1 + \rho_1^{ph} z_1^n). \quad (4)$$

**Proposition 3.10**  
(i) The rigid space $A$ is an open annulus of thickness $\epsilon$, i.e.

$$A \cong \{ y \mid |\pi| < |y| < 1 \}.$$

(ii) Let $\text{Spec} S_0 = \text{Spec} R[[y]][y^{-1}]$ be the ‘outer boundary’ of $A$ (which is mapped to $\text{Spec} R[[z_0]][z_0^{-1}]$). The induced action of $G$ on $\text{Spec} S_0$ has conductor $h$, discriminant $\delta_0$ and tame inertia character $\lambda$.

(iii) Let $\text{Spec} S_1 = \text{Spec} R[[\pi y^{-1}]][\pi y]$ be the ‘inner boundary’ of $A$ (which is mapped to $\text{Spec} R[[z_1]][z_1^{-1}]$). The induced action of $G$ on $\text{Spec} S_1$ has conductor $-h$, discriminant $\delta_1$ and tame inertia character $\lambda^{-1}$.

**Proof:** Equation (2) shows that

$$f = 1 + \left( \sum_{i=0}^{m_1 - 1} (\zeta_m^i \nu)^i \rho_1^{ph} z_0^{-h} + \ldots + 1 + m_1 \rho_0^{ph} z_0^{-h} + \ldots \right)$$

(we have used that $\zeta_m^i \nu \equiv 1 \pmod{p}$). The proposition follows now from the explicit computations in [6, §1].

### 3.6 The proof of Theorem 3.6

Let $Z = (Z, \omega_0, \delta, \epsilon)$ be a Hurwitz tree of type $(C, \chi)$, conductor $h$, discriminant $\delta$ and tame inertia character $\lambda$. We call an action of the group $G = P \rtimes C$ on the open disc a realization of $Z$ if $Z$ is associated to this action by the construction of §3.2. Theorem 3.6 claims that we can realize $Z$. We will prove this claim by induction on the depth of the tree $Z$. Therefore, we may assume that we can realize all Hurwitz trees (of some type $(C', \chi')$) with lower depth than $Z$.

Let $e_0$ be the edge whose source is the root $v_0$, and set $v_1 = t(e_0)$. Then $Z_0 := Z_{v_1}$ is the unique component of $Z$ which contains the distinguished point
z_0 \in Z. Let \( \delta_1 := \delta_{e_1} \). Let \( e_1, \ldots, e_r \) be the edges with source \( v \) and \( z_i := z_{e_i} \), the corresponding singular points. Set \( h_i := h_{e_i} \). Let \( C_i \subset C \) denote the stabilizer of the point \( z_i \), and set \( G_i := P \cdot C_i \subset G \). Note that either \( C_i = C \) or \( C_i = 1 \). Let \( Z_i \subset Z \) be the subtree of \( Z \) which contains the point \( z_i \) but not the component \( Z_0 \). It is clear that \( Z_i \) inherits from \( Z \) the structure of a Hurwitz tree of type \((C_i, \chi_{|C_i})\), with conductor \( h_i \) and different \( \delta_1 \). By our induction hypothesis, there exists an open unit disc \( Y_{i,K} \) over some finite extension \( K \) of \( \text{Frac}(W(k)) \), together with an action of \( G_i \), whose associated Hurwitz tree is \( Z_i \). The extension \( K \) may be taken to be independent of \( i \). We may also assume that the discs \( Y_{i,K} \) corresponding to one \( G \)-orbit of the trees \( Z_i \), together with the group actions, are all isomorphic. Then we can define an action of \( G \) on the disjoint union of the discs \( Y_{i,K} \) which induces the \( G_i \)-action on \( Y_{i,K} \). Let \( \text{Spec} S_i \) be the boundary of \( Y_{i,K} \).

By Lemma 3.3, the differential form \( \omega := \omega_{e_1} \) on \( Z_0 \) has a zero of order \( h - 1 \) at \( z_0 \) and poles of order \( h_i + 1 \) at \( z_i \). Let \( V_K \) be the affinoid with \( G \)-action constructed in §3.4, starting from the datum \((Z_0, \omega, \delta_1)\). By Proposition 2.3 and Proposition 3.9, we can identify the boundary of the missing open disc corresponding to the point \( z_i \) with \( \text{Spec} S_i \), in a way which is compatible with the action of \( G_i \). Clearly, we can choose this identification such that the \( G \)-action on the disjoint union of the boundaries \( \text{Spec} S_i \) induced from the \( G \)-action on the disjoint union of the discs \( Y_{i,K} \) agrees with the \( G \)-action on \( V_K \). We can now use [6, Lemme 3.7] to patch together the affinoid \( V_K \) and the discs \( Y_{i,K} \), in a \( G \)-equivariant way. The result is a closed disc \( W_K \cong \{ w \mid |w| \leq 1 \} \), together with an action of \( G \). By Proposition 3.9 and the construction, the induced action on the boundary of \( \text{Spec} R[[w^{-1}]]\{w\} \) of \( W_K \) has conductor \( h^{-1} \), discriminant \( \delta_1 \) and tame inertia character \( \lambda^{-1} \).

Let \( A_K \) be the open annulus with \( G \)-action constructed in §3.5, starting from the datum \((h, \lambda, \delta_0 := \delta, \delta_1)\). By Proposition 2.3 and Proposition 3.10, we can identify the boundary of \( W_K \) with the ‘inner boundary’ of \( A_K \), in a \( G \)-equivariant way. By [6, Lemme 3.8] we can patch together \( W_K \) and \( A_K \), together with the \( G \)-action, along these boundaries. The result is an open disc \( Y_K \) with \( G \)-action, of conductor \( h \), discriminant \( \delta \) and inertia character \( \lambda \). By construction, \( Y_K \) is a realization of the Hurwitz tree \( Z \). This completes the proof of Theorem 3.6.

\(\square\)

**Remark 3.11**

(i) It would be more satisfactory (and this paper would be shorter) if one could prove Theorem 3.6 using the results of [6] (i.e. the case \( G = \mathbb{Z}/p \)) as a ‘black box’. However, since the automorphism of the disc of order \( p \) constructed in [6] does not depend in a functorial way on the Hurwitz tree, this looks difficult.

(ii) Let \( \sigma : Y_K \xrightarrow{\sim} Y_K \) be an automorphism of order \( p \) of the disc with conductor \( h < p \). Then [5, Theorem III.3.1] shows that the Hurwitz tree \( Z \) associated to \( \sigma \) is irreducible, i.e. consists of a single component \( Z \cong \mathbb{P}^1 \). Furthermore, [5, Theorem V.6.3.1] shows that the conjugacy class of \( \sigma \) in \( \text{Aut}(Y_K) \) is uniquely determined by the Hurwitz tree \( Z \). If this would be
true for $h > p$ and for arbitrary Hurwitz trees then the proof of Theorem 3.6 could be simplified considerably, as suggested in (i).

4 Local lifts for large conductor

4.1 Existence of differential forms In constructing a Hurwitz tree as in §3.1, the main difficulty is to find the differential forms $\omega_v$. In this section, we illustrate the problems in the easiest case, namely the case that $G$ is cyclic. We will use these results afterwards to prove the local lifting result in case $h > p$.

Lemma 4.1 Let $h > 0$ be an integer which is prime to $p$. There exists a logarithmic differential on $\mathbb{P}^1_k$ which has $h + 1$ simple poles and a single zero of order $h - 1$.

Proof: Choose a primitive $h$th root of unity $\zeta_h \in k$ and define $z_i = \zeta_h^i \in \mathbb{P}^1_k$ for $i = 1, \ldots, h$. Put $z_0 = 0$. Let $u = z^{-h} \prod_{i=1}^{h} (z - z_i)$. Then

$$\omega := \frac{du}{u} = \left( \frac{-h}{z} + \sum_{i=1}^{h} \frac{1}{z - z_i} \right) \, dz = \frac{h}{z^{h+1} - z} \, dz.$$  

Note that $\omega$ has a single zero at $z = \infty$. $\square$

A result in the same direction is proved in [6, §3.5].

We now discuss several problems one encounters while explicitly constructing Hurwitz trees. If $h + 1 \leq p$ then the Hurwitz tree $Z$ is necessarily irreducible ([5]). As soon as $h + 1 > p$ one has greater freedom in choosing the tree $Z$. This makes the problem easier.

Suppose that the Hurwitz tree $Z$ is irreducible, and let $a = (a_b)$ be the type of $Z$ (see Definition 3.2). In general it is not easy to decide which possibilities for $a$ occur. Lemma 4.1 may be rephrased as follows. There exists a logarithmic differential $\omega$ on $\mathbb{P}^1_k$ with residues $a = (1, 1, \ldots, 1, p-h)$ and a single zero. The following lemma shows that not all sets of integers $a$ whose sum is zero in $\mathbb{F}_p$ occur as set of residues of a logarithmic differential.

Lemma 4.2 Let $p = 5$ and $a = (1, 1, 4, 4)$. There exists no logarithmic differential on $\mathbb{P}^1_k$ with a single zero whose set of residues is $a$.

Proof: Suppose that $\omega$ is a logarithmic differential on $\mathbb{P}^1_k$ with a single zero at $z = \infty$ and set of residues $a = (1, 1, 4, 4)$. We may suppose that $\omega$ has poles in the four pairwise distinct points $z = 0, 1, \lambda, \mu$. It follows that $\omega$ is the logarithmic differential of $f = uz(z-1)(z-\lambda)^4(z-\mu)^4$, for some $u \in k^*$. We find that

$$\omega = \frac{1}{z} + \frac{1}{z-1} + \frac{-1}{z-\lambda} + \frac{-1}{z-\mu} = \frac{(1-\lambda-\mu)z^2 + 2\lambda\mu z - \lambda\mu}{z(z-1)(z-\lambda)(z-\mu)}.$$ 

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Since \( \omega \) has a single zero at \( z = \infty \), we should choose \( \lambda, \mu \) such that \( \lambda + \mu = 1 \) and \( \lambda \mu = 0 \). But this contradicts the hypothesis that \( 0, 1, \lambda, \mu \) are pairwise distinct.

The reason the differentials of Lemma 4.1 are so easy to write down is that there is an extra automorphism (of order \( h \)) which fixes the differential. If \( G = D_p \), this trick no longer works, and it will be much harder to find a set of residues \( a \) for which we can find a differential.

**Remark 4.3** Leonardo Zapponi has proved a general criterion for the existence of logarithmic differentials with a single zero which includes Lemma 4.1 and Lemma 4.2 as special cases.

### 4.2 Let \( p \) be an odd prime and \( G \) the dihedral group of order \( 2p \). We write \( C \) for the quotient of \( G \) of order two and let \( \tau \) be its generator. Let \( \chi : C \to \{ \pm 1 \} \) be the character of order 2.

**Theorem 4.4** Let \( \phi : G \to \text{Aut}_k(k[[y]]) \) be a local \( G \)-action. Let \( h \) be the conductor and suppose that \( h > p \). Then \( \phi \) lifts to characteristic zero.

**Proof:** By Corollary 3.7, it suffices to construct a Hurwitz tree \( Z = Z_h \) of type \( (C, \chi) \), conductor \( h \) and discriminant 0. The Hasse–Arf theorem implies that the conductor \( h \) is odd, and we may write \( h + 1 = 2(\alpha + p\beta) \), with \( (p + 3)/2 \leq \alpha \leq (3p - 1)/2 \). Since \( h \) is prime to \( p \), we exclude \( \alpha = (p + 1)/2 \mod p \).

**Case 1:** \( (p + 3)/2 \leq \alpha \leq p \). We construct a tree \( Z \) as follows. The vertex set consists of \( v_0, v_1, w_j, u_j \), for \( j = 0, \ldots, \beta \). There is one edge connecting \( v_0 \) with \( v_1 \), and for \( j = 0, \ldots, \beta \) there is an edge \( e_j \) (resp. \( f_j \)) connecting \( v_1 \) with \( w_j \) (resp. \( u_j \)).

For each \( v \in V \), we choose a curve \( Z_v \) of genus zero together with a coordinate which we denote by \( z \). We suppose that the edge \( e \) with source \( v_1 \) and target \( v_0 \) corresponds to \( z_e = \infty \in Z_{v_1} \). We let \( \tau \) act on \( Z_{v_1} \) by \( \tau^* z = -z \). Choose points \( z_{e_j} \in Z_{v_1} - \{ 0, \infty \} \) and define \( z_{f_j} = \tau(z_{e_j}) = -z_{e_j} \). Choose the points so that all \( 2(\beta + 1) \) points are pairwise distinct. It is no restriction to suppose that \( z_{e_0} = 1 \).

We construct an exact differential \( \omega_{v_1} \) on \( Z_{v_1} \) with a pole of order \( \alpha \) in \( z = \pm z_{e_0} = \pm 1 \) and a pole of order \( p \) in \( z = \pm z_{e_j} \) for \( j \neq 0 \) by

\[
\omega = \frac{dz}{(z^2 - 1)^\alpha \prod_{j=1}^\beta (z^2 - z_{e_j}^2)^p}.
\]

It is easy to see that this differential is exact. Here we use the assumption that \((p + 3)/2 \leq \alpha \leq p\).

On the component \( Z_{w_j} \) we define a logarithmic differential with a single zero in some point \( z_{e_j} \) and \( \alpha \) (resp. \( p \)) simple poles if \( j = 0 \) (resp. \( j \neq 0 \)). The existence of such differentials follows from Lemma 4.1. We define the pair \((Z_{u_j}, \omega_{u_j})\) as a copy of the pair \((Z_{w_j}, \omega_{w_j})\). It is clear that the action of \( C \) on \( Z_{v_1} \)
extends to an admissible action on the whole tree $Z$ which verifies Condition (i) of Definition 3.2.

It remains to choose the discriminants $\delta_v$ and the thicknesses $\epsilon_e$. We set $\delta_v := 0$, $\delta_{wj}, \delta_{uj} := 1$, and for $\delta_{u1}$ we may choose any rational number with $0 < \delta_{u1} < 1$. Then Condition (iv) of Definition 3.2 imposes a unique value for all $\epsilon_e$. This completes the construction of the Hurwitz tree $Z$ and the proof of the theorem in Case 1.

**Case 2:** $p + 1 \leq \alpha \leq (3p - 1)/2$. Write $\alpha = \alpha_1 + \alpha_2$, with $1 < \alpha_2 \leq \alpha_1 < p$, note that this is always possible. We construct a tree $Z$ together with an automorphism $\tau$ of order two as follows. The vertex set of $Z$ consists of vertices $v_0, v_1, w_{01}, w_{02}, u_1, u_2, \ldots, u_\beta$. We choose components $Z_{v_0}, Z_{v_1}, Z_{wu01}, \ldots Z_{wu2}$ together with an action and the automorphism $\tau$ of order two as in Case 1, i.e. $\tau$ sends $w_j$ to $u_j$. We denote again by $e_j$ (resp. $f_j$) the edge with source $v_1$ and target $w_j$ (resp. $u_j$) and choose corresponding points $z_{e_j}$ (resp. $z_{f_j}$) on $Z_{v_1}$, as follows. We take $z_{e_0} = 1$, $z_{f_0} = -1$, $z_{e_2} = \lambda$ and $z_{f_2} = -\lambda$, where $\lambda$ will be specified later on in the proof. For $j = 1, \ldots, \beta$, we choose points $z_{e_j} \in Z_{v_1} - \{0, \infty\}$ and define $z_{f_j} = -z_{e_j}$. We choose these points in such a way that all points $\pm 1, \pm \lambda, \pm z_{e_j}$ are pairwise distinct.

It follows from Lemma 4.1 and the fact that $\alpha_j < p$ that we may choose for $j = 1, 2$ a logarithmic differential $\omega_{wuj}$ which has simple poles in $\alpha_j$ points on $Z_{wu_j} - \{\infty\}$ and a single zero at $\infty$. We define $\omega_{wu_j} = \tau^*\omega_{wu_j}$. Similarly, for $j = 1, \ldots, \beta$, there exist logarithmic differentials $\omega_{w_j}$ on $Z_{w_j}$ with $p$ poles and a single zero.

On $Z_{v_1}$, we want to find a differential $\omega_{v_1}$ with a pole of order $\alpha_j$ in $\pm z_{e_{0j}}$, for $j = 1, 2$, and poles of order $p$ in $\pm z_{e_j}$, for $j = 1, \ldots, \beta$. To simplify the notation, we write $z_j = z_{e_j}$. After multiplying $\omega_{v_1}$ with an element of $k^\times$, the differential is given by

$$\omega_{v_1} = \frac{dz}{(z^2 - 1)^{\alpha_1}(z^2 - \lambda^2)^{\alpha_2} \prod_{j=1}^{2}(z^2 - z_j^2)^p} = \eta\frac{dz}{Q^p},$$

where $\eta := (z^2 - 1)^{p-\alpha_1}(z^2 - \lambda^2)^{-\alpha_2}$ and $Q := (z^2 - 1)(z^2 - \lambda)\prod j (z^2 - z_j^2)$. We claim that $\omega_{v_1}$ is exact, for suitable choice of $\lambda$. Write $\eta = \sum_{i=0}^{2p-\alpha} \eta_i z^{2i}$.

**Lemma 4.5** If $\eta_{(p-1)/2} = 0$, then the differential $\omega_{v_1}$ is exact.

**Proof:** Note that the degree of $\eta$ in $x$ is $4p - 2\alpha \leq 2p - 2$. It follows that the only $0 \leq i \leq p - 1$ for which $2i$ is congruent to $-1$ mod $p$ is $i = (p - 1)/2$. Suppose that $\eta_{(p-1)/2} = 0$ and define

$$G = \frac{1}{Q^p} \sum_{i=0}^{2p-\alpha} \eta_i x^{2i+1}.$$

Then $\omega_{v_1} = dG$. \qed
The lemma implies that we have to choose $\lambda$ in such a way that $\eta_{(p-1)/2} = 0$. One easily computes that

$$\eta_{(p-1)/2} = \pm \sum_{i+j = (3p+1)/2-\alpha} \left( \frac{p - \alpha_1}{i} \right) \left( \frac{p - \alpha_2}{j} \right) \lambda^j.$$

The degree of $\eta_{(p-1)/2}$ in $\lambda$ is $\min(p - \alpha_2, (3p + 1)/2 - \alpha)$. The polynomial has a zero of order $\max(0, (p + 1)/2 - \alpha_2)$ in $\lambda = 0$ and a zero of order $\max(0, (p + 1)/2 - \alpha_2)$ at $\lambda = 1$. (The first two statements are obvious and the second one follows by symmetry.)

Recall that $2 \leq \alpha_2 \leq \alpha_1 \leq p - 1$ and $p + 1 \leq \alpha = \alpha_1 + \alpha_2 \leq (3p - 1)/2$. Therefore $\alpha_1 \geq (p+1)/2$. This implies that the degree of $\eta_{(p-1)/2}$ is $(3p+1)/2 - \alpha$ and that $\eta_{(p-1)/2}$ has no zero at $\lambda = 1$. It follows that the number of zeroes of $\eta_{(p-1)/2}$ different from 0, 1 is

$$\begin{cases} (3p + 1)/2 - \alpha - (p + 1)/2 + \alpha_2 = p - \alpha_1 \geq 1 & \text{if } \alpha \leq (p + 1)/2, \\ (3p + 1)/2 - \alpha \geq 1 & \text{if } \alpha \geq (p + 1)/2. \end{cases}$$

This implies that it is possible to choose $\lambda$ such that $\eta_{(p-1)/2} = 0$ and therefore such that $\omega_{\alpha_1}$ is exact. The proof of the proposition in this case follows now as in Case 1.

\[ \square \]

## 5 Local lifts for small conductor

In this section we prove the local lifting problem for conductor $h < p$. Section 5.1 reformulates the problem in terms of concrete equations, and describes the so-called trivial solutions of these equations. (These are the solutions that do not correspond to a solution of our problem.) In §5.3 we use this to solve the lifting problem for small conductor. Section 5.2 contains a weaker version of this result, namely we suppose that $p$ is large with respect to $h$.

### 5.1 The trivial solutions

Let $p$ be an odd prime and $G$ the dihedral group of order $2p$. Let $C$ be its quotient of order two, $\tau$ the generator of $C$ and $\chi : C \to \{ \pm 1 \}$ the unique character of order 2. Let $0 < h < p$ be an odd integer; write $\alpha = (h + 1)/2$. To prove that all local $G$-actions with conductor $h$ lift to characteristic zero, we need to construct a Hurwitz tree $Z = Z_h$ of type $(C, \chi)$, conductor $h$ and discriminant 0, as in §4. As explained in §4.1, this Hurwitz tree is irreducible, hence we may suppose $Z = \mathbb{P}^1_k$. We may also suppose that the distinguished point $z_0 = \infty$ and that $\tau$ acts on the standard parameter $z$ of $Z$ as $z \mapsto -z$. Our problem is thus reduced to finding a logarithmic differential form $\omega$ on $\mathbb{P}^1_k$ with $h + 1 = 2\alpha$ poles and a single zero at $\infty$ such that $\tau^*\omega = -\omega$.

Let $z = (z_1, \ldots, z_\alpha)$ be points of $Z - \{ 0, \infty \}$ such that $z_i^2 \neq z_j^2$ if $i \neq j$. Suppose $a = (a_1, \ldots, a_\alpha) \in (\mathbb{F}_p^\times)^\alpha$ is a set of residues which we consider to be fixed in this section. Put

$$f = f_{z,a} = \prod_{i=1}^\alpha \left( \frac{z - z_i}{z + z_i} \right)^{a_i}, \quad \text{and} \quad \omega = \omega_{z,a} = \frac{df}{f} = \sum_{i=1}^\alpha \frac{2a_iz_i^2}{z^2 - z_i^2}dz.$$
Lemma 5.1 Suppose that there exists a vector \( z \) as above such that \( \omega \cdot z \) has a single zero at \( \infty \). Then every local \( G \)-action with conductor \( h \) lifts to characteristic zero.

Proof: The hypothesis implies that \((Z, \omega)\) is a Hurwitz tree of type \((C, \chi)\) and conductor \( h \). Hence the lemma follows from Corollary 3.7.

We set \( z := 1/w \) and write

\[
\omega = -\sum_{i=1}^{\alpha} \frac{2a_i z_i}{1 - z_i^2 w^2} dw = -2 \sum_{j \geq 0} d_j w^{2j} dw,
\]

with

\[
d_j = \sum_{i=1}^{\alpha} a_i z_i^{2j+1}.
\]

The condition that \( \omega \) has a single zero at \( z = \infty \) is equivalent to the \( \alpha - 1 \) homogenous equations \( d_0 = d_1 = \ldots = d_{\alpha-2} = 0 \). We denote by

\[ S := \text{Proj}(k[z_1, \ldots, z_{\alpha}]/(d_0, \ldots, d_{\alpha-2})) \]

the subscheme of \( \mathbb{P}^{\alpha-1} \) defined by these equations. A point \([z_1 : \cdots : z_{\alpha}] \in S(k)\) which lies on the hypersurface defined by the equation

\[
\prod_i z_i \cdot \prod_{i < j} (z_i^2 - z_j^2) = 0
\]

is called a trivial solution. If \([z_1 : \cdots : z_{\alpha}] \in S(k)\) is not a trivial solution then we call it a good solution. By construction, there is a bijection between the set of good solutions and the set of Hurwitz trees of type \((C, \chi)\), conductor \( h \), discriminant 0 and type \( a \).

Let \([z_1 : \cdots : z_{\alpha}]\) be a trivial solution. Put \( J = \{1, 2, \ldots, \alpha\} \) and \( J_0 = \{i \in J \mid z_i = 0\} \). Let \( I \subset J - J_0 \) be a maximal subset with the property that \( z_i^2 \neq z_j^2 \) for all \( i \neq j \) in \( I \). For \( l \in I \), we define \( J_l = \{i \in J \mid z_i^2 = z_l^2\} \). Then \( J \) is the disjoint union of the subsets \( J_l \).

Proposition 5.2 (a) Let \([z_1 : \cdots : z_{\alpha}]\) be a trivial solution, and let \((J_l)_{l \in I}\) be the corresponding decomposition of the indices. Then for \( l \in I \) we have

\[
\sum_{i \in J_l} \nu_i a_i \equiv 0 \mod p,
\]

where \( \nu_i \in \{\pm 1\} \) is chosen such that \( z_i = \nu_i z_l \).

(b) Suppose that we can write \( J = J_0 \bigsqcup J_1 \bigsqcup \cdots \bigsqcup J_n \) such that (6) holds for \( l = 1, \ldots, n \) and some choice of \( \nu_i \in \{\pm 1\} \). Then we may define a linear subspace \( S(J_l; \nu_l) \subset S \) of dimension \( n-1 \), contained in the locus of trivial solutions, by putting \( z_i := 0 \) for \( i \in J_0 \) and \( z_i := \nu_i t_i \) for \( i \in J_l \) and \( l = 1, \ldots, n \).
(c) The subspace of all trivial solutions is the union of the subspaces $$\mathcal{S}(J_l; \nu_l).$$

**Proof:** Let $$[z_1 : \cdots : z_\alpha]$$ be a trivial solution, and let $$J_0, \ldots, J_n$$ be the corresponding decomposition of the set of indices. It is no restriction to assume that $$l \in J_l$$ for $$l = 1, \ldots, n.$$ By definition, we have $$d_0 = \ldots = d_{\alpha-2} = 0,$$ where $$d_j$$ is defined by (5). Set $$d_j^{(0)} := d_j$$ and define inductively

$$d_j^{(m)} := d_j^{(m-1)} - z_m^{2j} \cdot d_j^{(m-1)},$$

for $$m = 1, \ldots, n$$ and $$j = m, \ldots, \alpha - 1.$$ By induction and a straightforward computation one shows that

$$d_j^{(m)} = \sum_i a_i \left( \prod_{l=1}^{m} (z_i^{2} - z_l^{2}) \right) z_i^{2j+m+1}.$$ 

Note that the ith term in the above sum is zero for $$i \in J_0 \cup \ldots \cup J_m.$$ In particular, for $$m = n - 1$$ we get

$$d_j^{(n-1)} = \left( \prod_{i \in J_n} \nu_i a_i \right) \left( \prod_{l=1}^{n-1} (z_n^{2} - z_l^{2}) \right) z_n^{2j+n-3}.$$ 

Since $$z_n \neq 0$$ and $$z_n \neq z_l$$ for $$l < n$$ we conclude that

$$\sum_{i \in J_n} \nu_i a_i \equiv 0 \pmod{p}.$$ 

For symmetry reasons, the same holds for $$J_0, \ldots, J_{n-1}.$$ This proves (a).

Part (b) follows by direct verification. Part (c) follows from (a) and (b). \end{proof}

**Proposition 5.3** Let $$z = [z_1 : \cdots : z_\alpha]$$ be a good solution. Then $$z$$ is an isolated point of multiplicity one in the total space $$\mathcal{S}$$ of solutions.

**Proof:** Let $$[z_1 + \epsilon t_1 : \cdots : z_\alpha + \epsilon t_\alpha]$$ be a point of $$\mathcal{S}$$ over the ring $$k[\epsilon],$$ with $$\epsilon^2 = 0.$$ Then

$$\sum_i a_i (z_i + \epsilon t_i)^{2j+1} = \sum_i a_i z_i^{2j+1} + \epsilon \cdot (2j+1)(\sum_i a_i t_i z_i^{2j}) = 0,$$

for $$j = 0, \ldots, \alpha - 2.$$ Since $$2j+1 < p$$ and $$z$$ is a good solution, we conclude that

$$\sum_{i=1}^{\alpha} a_i t_i z_i^{2j} = 0,$$ 

(7)

for $$j = 0, \ldots, \alpha - 2.$$ By assumption we have $$z_i^2 \neq z_j^2$$ for $$i \neq j.$$ Hence the Vandermonde determinant $$\det(z_i^{2j})$$ is nonzero. We conclude that (7), regarded as a linear equation in the $$t_i,$$ has a solution space of dimension 1. Since $$t_i = z_i$$ is a solution to (7), this implies $$[z_1 + \epsilon t_1 : \cdots : z_\alpha + \epsilon t_\alpha] = z.$$ The proposition follows. \end{proof}
5.2 To solve the local lifting problem for $h < p$ one is free to choose the type $a = (a_j)$ (§4.1). In §5.3, we solve the problem by showing that one particular type works. However, it turns out that for a given $p$ most types work. We illustrate this in this section, by using a different type to show that the local lifting problem holds for $p$ large. As this proof is much shorter, it may also serve as an introduction to the type of problems one has to solve. Let $G$ be the dihedral group of order $2p$.

**Theorem 5.4** Let $\phi$ be a local $G$-action. Let $h = 2\alpha + 1$ be its conductor and suppose that $p \geq 2^\alpha$. Then $\phi$ lifts to characteristic zero.

**Proof:** Define $a = (a_1, \ldots, a_\alpha)$ inductively by

$$a_1 = 1, \quad a_{j+1} = 1 + \sum_{i=1}^{j} a_i.$$ 

Then $\sum_{i=1}^{\alpha} a_i = 2^\alpha - 1$. Therefore Proposition 5.2 together with the assumption $p \geq 2^\alpha$ implies that we do not have any trivial solutions in this case. By Bezout’s theorem, the set of good solutions is therefore nonempty. $\square$

**Remark 5.5** (i) In fact, since the homogenous degree of $d_j$ is $2j + 1$, Proposition 5.3 together with Bezout’s theorem shows more precisely that the number of good solutions in Theorem 5.4 is $1 \cdot 3 \cdots (2^\alpha - 3)$.

(ii) One can show something more general then (i). Suppose that, for some type $a$, the locus of trivial solutions has dimension zero. (This can easily be read off from the type.) We can prove a formula for the multiplicities of the trivial solutions. This gives a formula for the number of good solutions, which is recursive in $h$. But it is difficult to prove that this number is positive, although computer experiments suggest that this is almost always the case.

5.3 The proof for general $p$ In this section, we prove the lifting result for small conductor and arbitrary $p$.

**Theorem 5.6** Let $\phi$ be a local $G$-action with conductor $h$. Suppose that $h = 2\alpha + 1 < p$. Then $\phi$ lifts to characteristic zero.

In combination with Theorem 4.4 we obtain:

**Corollary 5.7** The local lifting problem holds for $G = D_p$.

The rest of this section concerns the proof of Theorem 5.6. Put $r = \alpha - 1$ and define $a = (a_1 = 1, \ldots, a_r = 1, a_{\alpha} = \alpha)$. The advantage of considering this type is that there is a large symmetry group acting: our equations are invariant under the symmetric group acting on $z_1, \ldots, z_r$. Therefore we may simplify the
equations for the $z_i$ by considering the elementary symmetric functions in the $z_i$. We may also suppose that $z_\alpha = 1$. The following notation replaces the previous one.

Write $h := c_0 z^r + c_1 z^{r-1} + \cdots + c_r$. We consider the $c_i$ as variables. Set $g(z) := h(-z)$ and

$$f := \frac{g}{h} \cdot \left( \frac{z - 1}{z + 1} \right)^\alpha, \quad \omega := \frac{df}{f} = \frac{(d_0 z^{2r} + d_1 z^{2(r-1)} + \cdots + d_r) \, dz}{gh (z^2 - 1)}.$$

(Note that the definition of $g$ implies that $\omega$ is odd, and hence that the numerator of $\omega$ contains only even powers of $z$.) A straightforward computation shows that

$$d_\ell = \sum_{i=0}^{\ell} (-1)^i c_i c_{2\ell + 1 - i} (2\ell + 1 - 2i) - \sum_{i=0}^{\ell-1} (-1)^i c_i c_{2\ell - i} (2\ell - 1 - 2i) + 2\alpha \sum_{i=0}^{\ell-1} (-1)^i c_i c_{2\ell - i} + \alpha (-1)^\ell c_\ell^2.$$ 

In particular, the $d_\ell$ are homogeneous of degree 2 in $c_0, \ldots, c_r$. We set

$$S := \text{Proj}(k[c_0, \ldots, c_r]/(d_0, \ldots, d_{r-1})).$$

A point $[c_0 : \cdots : c_{\alpha-1}] \in S(k)$ which lies on the hypersurface

$$\text{discr}(z^2 - 1) h \eta = 0$$

is a trivial solution. Otherwise, it is a good solution. As in §5.1, we get a bijection between the set of good solutions and the set of Hurwitz trees of type $(C, \chi)$, conductor $h$, discriminant 0 and type $a = (1, \ldots, 1, \alpha)$. (In contrast to §3, we consider the marked points to be unordered in this section.)

To prove Theorem 5.6, it suffices to find a good solution. In the rest of this paper, we will show that there exists in fact a unique good solution $[c_0 : \cdots : c_r]$. This means that the number of good solutions $[z_1 : \cdots : z_\alpha]$ is $r! = \# S_r$. We start by describing the trivial solutions.

**Lemma 5.8** The trivial solutions are contained in the hyperplane $T \subset \mathbb{P}^{\alpha-1}$ defined by the equation $c_0 = 0$. The hyperplane $T$ does not contain any good solutions.

**Proof:** If $c_0 = 0$ then $\deg(h) < r$. This implies immediately that $T$ does not contain any good solutions.

Suppose that $[c_0 : \cdots : c_{\alpha-1}]$ is a trivial solution with $c_0 \neq 0$. Then we may suppose that $c_0 = 1$. Write $h = \prod_{i=1}^r (z - z_i)$. Then $z = [z_1 : \cdots : z_r : 1]$ is a trivial solution of the equations considered in §5.1. It follows from Proposition 5.2 that there exists a subset $J_\alpha \subseteq \{1, \ldots, \alpha\}$ with $\alpha \in J_\alpha$ such that

$$\sum_{i \in J_\alpha} \nu_i a_i = \nu_\alpha \alpha + \sum_{i \in J_\alpha - \{\alpha\}} \nu_i \equiv 0 \pmod{p}.$$
But this is impossible, because $2\alpha < p$. By contradiction, the lemma is proved. 

For $\ell = 0, \ldots, r - 1$, let 

$$S_\ell = \{d_0 = \ldots = d_\ell = 0\} \subset P^r = \{[c_0 : c_1 : \cdots : c_r]\}.$$ 

Alternatively, $S_\ell$ corresponds to logarithmic differentials of the above sort with a zero of order at least $2(\ell + 1)$ at $z = \infty$. The following claim implies Theorem 5.6.

**Claim:** For $\ell = 0, \ldots, r - 1$, let $S'_\ell$ be the closure of $S_\ell - \mathcal{T}$. Then $S'_\ell \subset P^r$ is a linear subspace of dimension $r - \ell - 1$. In particular, $S'_{r-1}$ contains a unique good solution.

We prove the claim by induction on $\ell$. By (8) we have $d_0 = c_0(c_1 + \alpha c_0)$. Define $S'_0$ by $c_0 - \alpha c_1 = 0$. Then the claim holds for $\ell = 0$.

Suppose that the claim holds for $\ell$. We may choose linear parameters $t_1, \ldots, t_{r-\ell}$ for $S'_\ell$ (i.e. we write $c_j$ as a linear form in the $t_i$). Since $S'_\ell$ is not contained in the hyperplane $\mathcal{T}$, it is no restriction to suppose that $c_0 = t_1 = t$. To prove the claim for $\ell + 1$, it suffices to show that 

$$d_{\ell+1} = t(at + b), \quad \text{for some } a, b \in \mathbb{k}[t_2, \ldots, t_{r-\ell}] \quad \text{with } b \neq 0. \quad (8)$$

Namely, if this holds, $S'_{\ell+1}$ is obtained by intersecting $S'_\ell$ with $at + b = 0$. Since $b \neq 0$, this intersection is not completely contained in $\mathcal{T}$. Since $d_{\ell+1}$ is homogeneous of degree 2, it follows that $at + b$ is linear in $t_1, \ldots, t_{r-\ell}$. Therefore $S'_{\ell+1}$ is a linear subspace of dimension $r - \ell - 2$.

Write $\text{ord}_t$ for the standard valuation in $t$. We have to show that $\text{ord}_t(d_{\ell+1}) = 1$. Suppose that $\text{ord}_t(d_{\ell+1}) \neq 1$. To deduce a contradiction, we distinguish the following cases:

(a) $\text{ord}_t(d_{\ell+1}) = 0$,

(b) $\text{ord}_t(d_{\ell+1}) \geq 2$.

**Case (a):** Suppose that $\text{ord}_t(d_{\ell+1}) = 0$. Write $d_j = d_j^0 + t d_j^1 + t^2 d_j^2$ (resp. $c_j = c_j^0 + t c_j^1$) for the Taylor expansion of $d_j$ (resp. $c_j$) with respect to $t$. It follows from the proof of the claim for $\ell = 0$ that $c_1 = -\alpha c_0 = -\alpha t$, therefore $c_0^0 = c_1^0 = 0$. Let $j$ be the minimal $j \geq 2$ such that $c_j^0 \neq 0$. The Newton polygon of $h$ shows that there are exactly $j$ zeroes of $h$ with negative $t$-valuation $-1/j$. Set $z = yt^{-1/j}$ and write $\omega$ as a series in $t$. Let $\omega_1$ be the first nonzero coefficient. A short computation gives

$$\omega_1 = \frac{d_{\ell+1}^0 dy}{y^{2(\ell-j+1)}(y^j + c_j^0(y^j + (-1)^j c_j^0))}.$$

Using the fact that $\omega$ is logarithmic, it is easy to see that $\omega_1$ is either logarithmic or exact. Suppose that $j$ is even. (The case $j$ odd is similar and left to the
Therefore we claim that \( \omega_1 \) is not exact either. This gives a contradiction.

After multiplying \( \omega_1 \) with a constant, and changing the parameter \( y \), we may write

\[
\omega_1 = \frac{dy}{y^{2(\ell-j+1)}(y^j-1)^2} = \frac{y^{p-2(\ell-j+1)}(y^j-1)^{p-2} dy}{yp(y^j-1)^p}.
\]

Put \((y^j - 1)^{p-2} = \sum_{s=0}^{p-2} \delta_s y^s \). Since \( \omega \) is logarithmic, we claim that there is a unique \( (s, u) \) such that

\[
p - 2\ell + 2j - 2 + js = up - 1.
\]

Then \( u \equiv (p - 2\ell - 1)/p \) mod \( j \). Since \( 2j \leq 2\ell \leq 2r < p \), it follows that \( 1 \leq u \leq j \).

Therefore \( u \) is uniquely determined by (9). Now define

\[
s = \frac{up + 1 - p + 2\ell - 2j}{j} \leq \frac{jp + 1 - p + 2\ell - 2j}{j} \leq p - 2.
\]

This shows that there is a unique \( s \) with the required properties. Note that for \( s \) we have \( \delta_s \neq 0 \).

It follows now from standard properties of the Cartier operator that

\[
C\omega_1 = \delta_{s/p} y^{-1+(p - 2\ell + 2j - 2 + js)/p} \frac{dy}{y^j - 1} \neq 0.
\]

This shows that \( \omega_1 \) is not exact. We conclude that Case (a) does not occur.

**Case (b):** we suppose that \( \text{ord}_d(d_{\ell}) = 2 \). (The case \( d_\ell = 0 \) is similar, and left to the reader.) Then \( d_{\ell+1} = at^2 \), with \( a \in k^\times \).

Let \( j \) be as in Case (a). By assumption, the numerator of \( \omega \) is \( Fdz \) with \( F := d_{\ell+1}z^{2(r-\ell-1)} + \cdots + d_r \). The proof showing that Case (a) does not occur implies that the Newton polygon of \( F \) contains a piece with slope \( 1/j \) which ends in the point \((2(r - \ell - 1), 2) \). Here we use that we are in Case (b). It follows that

\[d_{\ell+1} = \cdots = d_{\ell+j} = 0.\]

Let \( \omega_0 \) be the differential obtained from \( \omega \) by specializing \( t \) to 0. Note that \( \omega_0 \) corresponds to a generic point \([0 : \cdots : c_j^0 : \cdots : c_r^0]\) on \( S_\ell^r \cap T \) and that \( \dim S_\ell^r \cap T = r - \ell - 2 \). On the other hand, we have

\[
\omega_0 = \frac{d_{\ell+j+1}z^{2(r-\ell-j-1)} + \cdots + d_r^0}{h_0 g_0(z^2 - 1)} \, dz,
\]

where \( h_0(z) = c_j^0 z^{r-j} + \cdots + c_r^0 \) and \( g_0(z) = h_0(-z) \). Since \( c_j^0 \neq 0 \), \( \omega_0 \) has a zero of order at least \( 2(\ell + 1) \) at \( z = \infty \). Therefore, the point \([c_j^0 : \cdots : c_r^0]\) is \( \mathbb{P}^{r-1} \) (with \( \hat{r} := r - j \)) lies on the subspace \( S_\ell^r \subset \mathbb{P}^{r-1} \) analogous to \( S_\ell^r \subset \mathbb{P}^{r-1} \). Moreover, this point does not lie on the hyperplane \( T = \{c_0 = 0\} \). By induction.
on \( r \), we may assume that we have already proved our claim for \( \tilde{r} \). In particular, 
\[
\dim \tilde{S}' = r - \ell - 1 - j < r - \ell - 2.
\]
This contradicts the assertion made above that \( \omega_0 \) corresponds to a generic point of dimension \( r - \ell - 2 \). It follows that Case (b) does not occur either.

We conclude that \( \text{ord}_t(\omega) = 1 \). Theorem 5.6 follows. \( \square \)

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