PRE-QUANTIZATION OF QUASI-HAMILTONIAN SPACES

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Abstract. This paper develops the pre-quantization of Lie group-valued moment maps, and establishes its equivalence with the pre-quantization of infinite-dimensional Hamiltonian loop group spaces.

1. Introduction

The notion of relative gerbes for smooth maps of manifolds is introduced in [19]. The equivalence classes of relative gerbes are classified by the relative integral cohomology in degree three. The differential geometry of relative gerbes, consisting of relative connection, relative connection curvature, relative Cheeger-Simons differential character, and relative holonomy are also discussed in [19]. Inspired by the pre-quantization of Hamiltonian $G$-manifolds, the main objective of this paper is to construct a method to pre-quantize the quasi-Hamiltonian $G$-spaces with group-valued moment maps. For this purpose, the premise of relative gerbes is used. Recall that a pre-quantization of a symplectic manifold $(M, \omega)$ is a line bundle $L$ over $M$ with curvature 2-form $\omega$. The symplectic manifold $(M, \omega)$ is prequantizable if the 2-form $\omega$ is integral. In this paper, a notion of a pre-quantization of a space with $G$-valued moment map is introduced, and then a similar criterion for being pre-quantizable is given. It is proven that, given two pre-quantizable quasi-Hamiltonian $G$-spaces, their fusion product is again pre-quantizable. Another objective of this paper is to prove that the pre-quantization of quasi-Hamiltonian $G$-spaces is equivalent with the pre-quantization of corresponding infinite-dimensional Hamiltonian loop group spaces.

The organization of this paper is as follows: in Section 2, the relative gerbes are reviewed. In Section 3, an explicit construction of the basic gerbe for $G = SU(n)$ is given. As well, it is shown that the construction of the basic gerbe over $SU(n)$ is equivalent to the construction of the basic gerbe in Gawedzki-Reis [7]. In Section 4, a relative gerbe for the map Hol : $A_G(S^1) \to G$ is constructed, where $A_G(S^1)$ is the affine space of connections on the trivial bundle $S^1 \times G$. In Section 5, quasi-Hamiltonian $G$-spaces are reviewed. The pre-quantization of quasi-Hamiltonian $G$-spaces is introduced in Section 6, and the pre-quantization conditions for the examples of quasi-Hamiltonian $G$-spaces, described previously, are examined. It is shown that a given conjugacy class $C = G \cdot \exp(\xi)$ of $G$ is pre-quantizable when $\xi \in \Lambda^*$, where $\Lambda^*$ is the weight lattice. It is also proven that $G^{2h}$ has a unique pre-quantization, which enables one to construct a finite dimensional pre-quantum line bundle for the moduli space of flat connections of a closed oriented surface of genus $h$.

Further, recall that there is a one-to-one correspondence between quasi-Hamiltonian $G$-spaces and infinite dimensional loop group spaces [2]. By extending this correspondence in Section 7, it is proven that pre-quantization of a quasi-Hamiltonian
$G$-spaces with group-valued moment map coincides with the pre-quantization of the corresponding Hamiltonian loop group space.

2. Review of Relative Gerbes

2.1. Gerbes. The main references for this section are [19], [12], [5] and [11].

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover for a manifold $M$. It will be convenient to introduce the following notations. Suppose that there is a collection of line bundles $L_{i(0)}, \ldots, i(n)$ on $U_{i(0)}, \ldots, i(n)$. Consider the inclusion maps,

$$\delta_k : U_{i(0)}, \ldots, i(n+1) \to U_{i(0)}, \ldots, i(k), \ldots, i(n+1) \quad (k = 0, \ldots, n + 1)$$

and define Hermitian line bundles $(\delta L)_{i(0)}, \ldots, i(n+1)$ over $U_{i(0)}, \ldots, i(n+1)$ by

$$\delta L := \bigotimes_{k=0}^{n+1} (\delta_k^* L)^{(-1)^k}.$$ 

Notice that $\delta(\delta L)$ is canonically trivial. If one has a unitary section $\lambda_{i(0)}, \ldots, i(n)$ of $L_{i(0)}, \ldots, i(n)$ for each $U_{i(0)}, \ldots, i(n) \neq \emptyset$, then one can define $\delta \lambda$ in a similar fashion. Note that $\delta(\delta \lambda) = 1$ as a section of trivial line bundle.

**Definition 2.1.** A gerbe on a manifold $M$ on an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $M$ is defined by Hermitian line bundles $L_{ii'}$ on each $U_{ii'}$ such that $L_{ii'} \cong (L_{ii'}^1)^{-1}$ and a unitary section $\theta_{ii'}$ of $\delta L$ on $U_{ii'}$ such that $\delta \theta = 1$ on $U_{ii'}$. Denote this data as $G = (\mathcal{U}, L, \theta)$.

Denote the set of all gerbes on $M$ as $Ger(M)$. $Ger(M)$ has a natural group structure and is classified with $H^3(M, \mathbb{Z})$.

**Definition 2.2.** A quasi-line bundle for the gerbe $G$ on a manifold $M$ on the open cover $\mathcal{U} = \{U_i\}_{i \in I}$ is defined as:

1. a Hermitian line bundle $E_i$ over each $U_i$;
2. Unitary sections $\psi_{ii'}$ of

$$(\delta E^{-1})_{ii'} \otimes L_{ii'}$$

such that $\delta \psi = \theta$.

Denote this quasi-line bundle as $L = (E, \psi)$.

Any two quasi-line bundles over a given gerbe differ by a line bundle.

**Definition 2.3.** A relative gerbe for $\Phi \in C^\infty(M, N)$ is a pair $(L, G)$ where $G$ is a gerbe over $N$ and $L$ is a quasi-line bundle for $\Phi^* G$. Denote the set of all relative gerbes for the map $\Phi$ as $Ger(\Phi)$.

$Ger(\Phi)$ has a group structure and is classified with the third degree relative integral cohomology of the map $\Phi$, i.e.,

$$Ger(\Phi) \cong H^3(\Phi, \mathbb{Z}).$$
3. GERBES OVER A COMPACT LIE GROUP

It is well-known that for a compact, simple, simply connected Lie group the integral cohomology $H^\bullet(G, \mathbb{Z})$ is trivial in degree less than three, while $H^3(G, \mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$. The gerbe corresponding to the generator of $H^3(G, \mathbb{Z})$ is called the basic gerbe over $G$. In this section, an explicit construction of the basic gerbe for $G = SU(n)$ is given. This gerbe plays an important role in pre-quantization of the quasi-Hamiltonian $G$-spaces.

3.1. SOME NOTATIONS FROM LIE GROUPS. Let $G$ be a compact, simple simply connected Lie group, with a maximal torus $T$. Let $\mathfrak{g}$ and $\mathfrak{t}$ denote the Lie algebras of $G$ and $T$, respectively. Denote by $\Lambda \subset \mathfrak{t}$ the integral lattice, given as the kernel of $\exp : \mathfrak{t} \to T$.

Let $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$ be its dual weight lattice. Recall that any $\mu \in \Lambda^*$ defines a homomorphism $h_\mu : T \to U(1)$, $\exp \xi \mapsto e^{2\pi \sqrt{-1} \langle \mu, \xi \rangle}$.

The closures of the connected components of $t^{reg}$ are called Weyl chambers. Fix a Weyl chamber $t^+$. Let $\mathcal{R} \subset \Lambda^*$ be the set of roots, i.e., the non-zero weights for the adjoint representation. Define $t^{reg} := \mathfrak{t} \setminus \bigcup_{\alpha \in \mathcal{R}} \ker \alpha$.

Any root $\alpha \in \mathcal{R}$ can be uniquely written as $\alpha = \sum k_i \alpha_i$, $k_i \in \mathbb{Z}, \alpha_i \in \mathcal{S}$. The height of $\alpha$ is defined by $ht(\alpha) = \sum k_i$. Since $\mathfrak{g}$ is simple, there is a unique root $\alpha_0$ with $ht(\alpha) \geq ht(\alpha_0)$ for all $\alpha \in \mathcal{R}$, which is called the lowest root. The fundamental alcove is defined as $\mathfrak{A} = \{\xi \in t^+ | \langle \alpha_0, \xi \rangle \geq -1\}$.

The basic inner product on $\mathfrak{g}$ is the unique invariant inner product such that $\alpha.\alpha = 2$ for all long roots $\alpha$, which is used here to identify $\mathfrak{g}^* \cong \mathfrak{g}$. The mapping $\xi \mapsto \text{Ad} G(\exp \xi)$ is a homeomorphism from $\mathfrak{A}$ onto $G/\text{Ad} G$, the space of the conjugacy classes in $G$. Therefore, the fundamental alcove parameterizes conjugacy classes in $G$ [6]. Denote the quotient map by $q : G \to \mathfrak{A}$.

Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left and right invariant Maurer Cartan forms. If $L_g$ and $R_g$ denote left and right multiplication by $g \in G$, then the values of $\theta_g^L$ and $\theta_g^R$ at $g$ are given by

$$\theta_g^L = dL_{g^{-1}} : TG_g \to TG_e \cong \mathfrak{g}, \quad \theta_g^R = dR_{g^{-1}} : TG_g \to TG_e \cong \mathfrak{g}.$$ 

For any $g \in G$,

$$\theta_g^L = \text{Ad}_g(\theta_g^R).$$
For any invariant inner product $B$ on $\mathfrak{g}$, the form
\begin{equation}
\eta := \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \in \Omega^3(G)
\end{equation}
is bi-invariant since the inner product is invariant. Any bi-invariant form on a Lie group is closed. Therefore, $\eta$ is a closed 3-form and its cohomology class represents the generator of $H^3(G, \mathbb{R}) \cong \mathbb{R}$ if one assumes that $G$ is compact and simple. If, in addition, $G$ is simply connected, then $H^3(G, \mathbb{Z}) = \mathbb{Z}$, and one can normalize the inner product such that $[\eta]$ represents an integral generator [9]. $B$ is called the inner product at level $\lambda > 0$ if $B(\xi, \xi) = 2\lambda$ for all short lattice vectors $\xi \in \Lambda$. The inner product at level $\lambda = 1$ is called the basic inner product. (It is related to the Killing form by a factor $2c_\mathfrak{g}$, where $c_\mathfrak{g}$ is the dual Coxeter number of $\mathfrak{g}$.) Suppose that $G$ is simply connected and simple. It is known that the 3-form defined by Equation 3.1 is integral if and only if its level $\lambda$ of $B$ is an integer.

### 3.2. Standard Open Cover of $G$

Let $\mu_0, \ldots, \mu_d$ be the vertices of $\mathcal{A}$, with $\mu_0 = 0$. Let $\mathcal{A}_j$ be the complement of the closed face opposite to the vertex $\mu_j$. The standard open cover of $G$ is defined by the pre-images $V_j = q^{-1}(\mathcal{A}_j)$. Denote the centralizer of $\exp(\mathfrak{a}_j)$ by $G_j$. Then, the flow-out $S_j = G_j. \exp(\mathcal{A}_j)$ is an open subset of $G_j$, and it is a slice for the conjugation action of $G$. Therefore,

$$G \times_{G_j} S_j = V_j.$$  

More generally, let $\mathcal{A}_I = \cap_{j \in I} \mathcal{A}_j$ and $V_I = q^{-1}(\mathcal{A}_I)$. Then, $S_I = G_I. \exp(\mathcal{A}_I)$ is a slice for the conjugation action of $G$, and therefore

$$G \times_{G_I} S_I = V_I.$$  

Denote the projection to the base by

$$\pi_I : V_I \to G/G_I.$$  

### Lemma 3.1. $\eta_G$ is exact over each of the open subsets $V_j$.

**Proof.** $S'_j := G_j \cdot (\mathcal{A}_j - \mu_j)$ is a star-shaped open neighborhood of 0 in $\mathfrak{g}_j$ and is $G_j$-equivariantly diffeomorphic with $S_j$. One can extend this retraction from $S_j$ onto $\exp(\mu_j)$ to a $G$-equivariant retraction from $V_j$ onto $C_j = q^{-1}(\mu_j)$. But, since $d_G(\omega_C + \iota^*_C \eta_G) = 0$, then $\eta_G$ is exact over $V_j$.  

Let $\iota_j : C_j \to V_j$ and $\pi_j : V_j \to G/G_j = C_j$ denote the inclusion and the projection respectively. The retraction from $V_j$ onto $C_j$ defines a $G$-equivariant homotopy operator

$$h_j : \Omega^p(V_j) \to \Omega^{p-1}(V_j).$$  

Thus,

$$d_G h_j + h_j d_G = Id - \pi^*_j \iota^*_j.$$  

Define the equivariant 2-form $\varpi_j$ on $V_j$ by $(\varpi_j)_G = h_j \eta_G - \pi^*_j \omega_C$. Write $(\varpi_j)_G = \varpi_j - \theta_j$, where $\varpi_j \in \Omega^2(V_j)$ and $\theta_j \in \Omega^1(V_j, \mathfrak{g})$. For any conjugacy class $C \subset V_j$, $\iota^*(\varpi_j)_G + \omega_C$ is an equivariantly closed 2-form with $\theta_j$ as its moment map. Therefore, $\iota^*(\varpi_j)_G + \omega_C = \theta_j^*(\omega_C)_G$, where $(\omega_C)_G$ is the symplectic form on the (co)-adjoint orbit $O = \theta_j(C)$.  

Proposition 3.2. Over \( V_{ij} = V_i \cap V_j \), \( \theta_i - \theta_j \) takes values in the adjoint orbit \( O_{ij} \) through \( \mu_i - \mu_j \). Furthermore,

\[
(\omega_i)_G - (\omega_j)_G = \theta^*_{ij} (\omega_{O_{ij}})_G
\]

where \( \theta_{ij} := \theta_i - \theta_j : V_{ij} \to O_{ij} \), and \( (\omega_{O_{ij}})_G \) is the equivariant symplectic form on the orbit.

Proof. Let \( \nu : \mathfrak{A}_j \to \mathfrak{t} \) be the inclusion map. Then,

\[
\tilde{h}_j \circ (\exp |_{\mathfrak{A}_j})_i^\frac{1}{2} (\theta^L + \theta^R) = \tilde{h}_j \circ d\nu = \nu - \mu_j
\]

where \( \tilde{h}_j \) is the homotopy operator for the linear retraction of \( \mathfrak{t} \) onto \( \mu_j \). This proves that \( (\exp |_{\mathfrak{A}_j})^* \theta_j = \nu - \mu_j \). Therefore, for \( \xi \in \mathfrak{A}_{ij} \),

\[
\theta_{ij}(\exp \xi) = (\xi - \mu_i) - (\xi - \mu_j) = \mu_j - \mu_i.
\]

Therefore, \( \theta_{ij} \) takes values in the adjoint orbit through \( \mu_j - \mu_i \) by equivariance. The difference \( \omega_i - \omega_j \) vanishes on \( T \), and is, therefore, determined by its contractions with generating vector fields. But, \( \theta_{ij} \) is a moment map for \( \omega_i - \omega_j \), hence \( \omega_i - \omega_j \) equals to the pull-back of the symplectic form on \( O_{ij} \). \( \square \)

3.3. Construction of the Basic Gerbe. Let \( G \) be a compact, simple, simply connected Lie group, and \( B \) be an invariant inner product at integral level \( k > 0 \). Use \( B \) to identify \( \mathfrak{g} \cong \mathfrak{g}^* \) and \( \mathfrak{t} \cong t^* \). Assume that under this identification, all vertices of the alcove are contained in the weight lattice \( \Lambda^* \subset \mathfrak{t} \). This is automatic if \( G \) is the special unitary group \( A_d = SU(d+1) \) or the compact symplectic group \( C_d = Sp(2d) \). In general, the following table lists the smallest integer \( k \) with this property [4]:

| \( G \) | \( A_d \) | \( B_d \) | \( C_d \) | \( D_d \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( k \) | 1 | 2 | 1 | 2 | 3 | 12 | 60 | 6 | 2 |

For constructing the basic gerbe over \( G \), pick the standard open cover of \( G \), \( \mathcal{V} = \{ V_i, i = 0, \cdots, d \} \). For any \( \mu \in \Lambda^* \), with stabilizer \( G_\mu \), define a line bundle

\[
L_\mu = G \times_{G_\mu} \mathbb{C}_\mu
\]

with the unique left invariant connection \( \nabla \), where \( \mathbb{C}_\mu \) is the 1-dimensional \( G_\mu \)-representation with infinitesimal character \( \mu \). \( L_\mu \) is a \( G \)-equivariant pre-quantum line bundle for the orbit \( O = G \cdot \mu \). Therefore,

\[
\frac{i}{2\pi} \text{curvature}_G(\nabla) = (\omega_\mathcal{O})_G : = \omega_\mathcal{O} - \Phi_\mathcal{O}
\]

where \( \omega_\mathcal{O} \) is a symplectic form for the inclusion map \( \Phi_\mathcal{O} : O \to \mathfrak{g}^* \). Define line bundles

\[
L_{ij} := \theta^*_{ij}(L_{\mu_j - \mu_i})
\]

equipped with the pull-back connection. In three fold intersection \( V_{ijk} \), the tensor product \( (\delta L)_{ijk} = L_{jk} L^{-1}_{ik} L_{ij} \) is the pull-back of the line bundle over \( \tilde{G}/G_{ijk} \), which is defined by the zero weight

\[
(\mu_k - \mu_j) - (\mu_k - \mu_i) + (\mu_j - \mu_i) = 0
\]
of $G_{ijk}$. Therefore, it is canonically trivial with trivial connection. The trivial sections $t_{ijk} = 1$ satisfy $\delta t = 1$ and $(\delta \nabla) t = 0$. Define $(F_j)_G = (\omega_j)_G$. Since

$$\delta (F)_G = \theta^*_i j (\omega_{\mathcal{O}})_G = \frac{1}{(2\pi i - 1)} \text{curve}_G (\nabla^s),$$

then $\mathcal{G} = (\mathcal{V}, L, t)$ is a gerbe with connection $(\nabla, \omega)$. The construction of the basic gerbe is discussed in more general cases in [17, 3].

3.4. The Basic Gerbe Over $SU(n)$. This Section, shows that the construction of the basic gerbe over $SU(n)$, as discussed in the previous section, is equivalent to the construction of the basic gerbe in Gawedzki-Reis [7].

The special unitary group is the classical group:

$$SU(n) = \{ A \in U(n) \mid \det A = 1 \},$$

which is a compact connected Lie group of dimension equal to $n^2 - 1$ with Lie algebra equal to the space:

$$su(n) = \{ A \in L_{\mathfrak{g}}(\mathbb{C}^n, \mathbb{C}^n) \mid A^* + A = 0 \text{ and } tr A = 0 \}.$$ 

Any matrix $A \in SU(n)$ is conjugate to a diagonal matrix with entries

$$\text{diag} \left( \exp((2\pi \sqrt{-1}) \lambda_1 (A)), \ldots , \exp((2\pi \sqrt{-1}) \lambda_n (A)) \right)$$

where $\lambda_1 (A), \ldots , \lambda_n (A) \in \mathbb{R}$ are normalized by the identity $\Sigma_{i=1}^n \lambda_i (A) = 0$, and

$$\lambda_1 (A) \geq \lambda_2 (A) \geq \cdots \geq \lambda_n (A) \geq \lambda_1 (A) - 1. \tag{3.2}$$

Consider the following maximal torus of $SU(n)$

$$T = \{ A \in SU(n) \mid A \text{ is diagonal} \}. $$

Let $\mathfrak{t}$ be the Lie algebra of $T$. Thus, $\mathfrak{t} \cong \{ \lambda \in \mathbb{R}^n \mid \Sigma_{i=1}^n \lambda_i = 0 \}$. The roots $\alpha \in \mathcal{R} \subset \mathfrak{t}^*$ are the linear maps:

$$\alpha_{ij} : \mathfrak{t} \to \mathbb{R}, (\lambda_1, \ldots , \lambda_n) \mapsto \lambda_i - \lambda_j, \quad i \neq j,$$

and the set of simple roots is

$$\mathcal{S} = \{ \alpha_{1,2}, \alpha_{2,3}, \ldots , \alpha_{n-1,n} \}. $$

The lowest root is $\alpha_{n,1} (\{ 14 \}, \text{Appendix C}).$ Choose the following Weyl chamber

$$\mathcal{T}_+ = \{ (\lambda_1, \cdots , \lambda_n) \in \mathfrak{t} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \}. $$

In the above case the fundamental alcove is

$$\mathfrak{A} = \{ (\lambda_1, \cdots , \lambda_n) \in \mathfrak{t} \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_1 - 1 \}. $$

The basic inner product on $\mathfrak{t}$ is induced from the standard basic inner product on $\mathbb{R}^n$. One can use this inner product to identify $\mathfrak{t} \cong \mathfrak{t}^*$. Under this identification $\alpha_{i,j} = e_i - e_j$, where $\{ e_i \}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. The fundamental weights are given by

$$\mu_i = \{ \lambda \in \mathfrak{A} \mid \lambda_1 = \lambda_2 = \cdots = \lambda_i \geq \lambda_{i+1} = \cdots = \lambda_n = \lambda_1 - 1 \}. $$

$SU(n)^{reg} = \{ A \in SU(n) \mid \text{all eigenvalues of } A \text{ have multiplicity one} \} \cong G \times T \text{-int } \mathfrak{A} \cong G/T.$ For $i \in \{ 1, \cdots , n \}$, define

$$\mathfrak{A}_i := \{ \lambda \in \mathfrak{A} \mid \lambda_1 \geq \cdots \geq \lambda_i \geq \lambda_{i+1} \geq \cdots \geq \lambda_n \geq \lambda_1 - 1 \}. $$

Thus, the standard open cover for $SU(n)$ is $\mathcal{V} = \{ V_i \}_{i=1}^n$, where $V_i = q^{-1}(\mathfrak{A}_i)$. Each $SU(n)_{ij}$ is isomorphic to $U(n-1)$ with the center isomorphic to $U(1)$. Over the set
of regular elements all the inequalities are strict, and one has $n$ equivariant line bundles $E_1, \ldots, E_n$ defined by the eigenlines for the eigenvalues $\exp((2\pi \sqrt{-1})\lambda_i(A))$. For $i < j$, the tensor product $E_i \otimes \cdots \otimes E_j \rightarrow SU(n)^{eg}$ extends to a line bundle $E_{ij} \rightarrow V_{ij}$. For $i < j < k$, one can have a canonical isomorphism $E_{ij} \otimes E_{jk} \cong E_{ik}$ over $V_{ijk}$. These line bundles together with corresponding isomorphisms define a gerbe over $SU(n)$, in Gawedzki-Reis sense, which represents the generator of $H^3(SU(n), \mathbb{Z})$. Each $E_i$ is equal to $G \times_T C_\nu$, for some $\nu \in \Lambda^*$. In fact, by using the standard action of $T \subset SU(n)$ on $\mathbb{C}^n$, one can see that

$$\nu_i = e_i - \frac{1}{n}(1, \ldots, 1).$$

Since $\mu_i = \Sigma^1_{k=1}\nu_k$, $\mu_i = \Sigma^1_{k=1}\nu_k$. Recall from Section 3.3 that $L_{ij} := \theta_j^*(L_{\mu_j-\mu_i})$ on $V_{ij}$. Thus, for $i < j$

$$L_{ij} = \theta_j^*(L_{\mu_j-\mu_i})$$

$$= \theta_j^*(L_{\Sigma^1_{k=1}\nu_k})$$

$$= \theta_j^*(L_{\Sigma^1_{k=1}\nu_k}) = E_{ij}

4. The Relative Gerbe for $\text{Hol} : A_G(S^1) \rightarrow G$

Denote the affine space of connection on the trivial bundle $S^1 \times G$ by $A_G(S^1)$. Thus, $A_G(S^1) = \Omega^1(S^1, g)$. The loop group $LG = \text{Map}(S^1, G)$ acts on $A_G(S^1)$ by gauge transformations:

$$(4.1) \quad g \cdot A = Ad_g(A) - g^* \theta^R.$$

Taking the holonomy of a connection defines a smooth map

$$\text{Hol} : A_G(S^1) \rightarrow G$$

with equivariance property $\text{Hol}(g \cdot A) = Ad_g(0) \text{Hol}(A)$. If $g$ carries an invariant inner product $B$, write $Lg^*$ instead of $A_G(S^1)$ using the natural pairing between $\Omega^1(S^1, g)$ and $Lg = \Omega^0(S^1, g)$. Let’s refer to this action as the coadjoint action. However, notice that the action 4.1 is not the point-wise action. Recall from Section 3.3,

a) An open cover $\mathcal{V} = \{V_0, \ldots, V_d\}$ of $G$ such that $V_j/G = \mathfrak{A}_j$, where $d = \text{rank} G$.

b) For each $V_j$, a unique $G$-equivariant deformation retraction on to a conjugacy class $C_j = G \cdot \exp(\mu_j)$, where $\mu_j$ is the vertex of $\mathfrak{A}_j$. This deformation retraction descends to the linear retraction of $\mathfrak{A}_j$ to $\mu_j$.

c) 2-forms $\omega_j \in \Omega^2(V_j)$, with $d\omega_j = \eta|_{V_j}$, such that the pull-back onto $C_j$ is the invariant 2-form for the conjugacy class $C_j$.

**Lemma 4.1.** There exists a unique $LG$-equivariant retraction from $\widetilde{V}_j := \text{Hol}^{-1}(V_j)$ onto the coadjoint orbit $O_j = LG \cdot \mu_j$, descending to the linear retraction of $\mathfrak{A}_j$ onto $\mu_j$.

**Proof.** The holonomy map sets up a one-to-one correspondence between the sets of $G$-conjugacy classes and coadjoint $LG$-orbits, hence both are parameterized by points in the alcove. The evaluation map $LG \rightarrow G$, $g \mapsto g(1)$ restricts to an isomorphism $(LG)_j \cong G_j$ [16]. Hence,

$$LG \times_{(LG)_j} \widetilde{S}_j = \widetilde{V}_j$$
where $\tilde{S}_j = \text{Hol}^{-1}(S_j)$ and $S_j = G_j \cdot \exp(\mathfrak{A}_j)$. Therefore, the unique equivariant retraction from $V_j$ onto $C_j$, which descends to the linear retraction of $\mathfrak{A}_j$ onto the vertex $\mu_j$, lifts to the desired retraction. □

Consider the following commutative diagram:

$$
\begin{array}{ccc}
\tilde{V}_j & \xrightarrow{\tilde{\pi}_j} & O_j \\
\text{Hol} & & \text{Hol} \\
V_j & \xrightarrow{\pi_j} & C_j
\end{array}
$$

where $\tilde{\pi}_j : \tilde{V}_j \rightarrow O_j$ is the projection that is obtained from retraction. Let $\sigma_j \in \Omega^2(\tilde{V}_j)$ denote the pull-backs under $\tilde{\pi}_j$ of the symplectic forms on $O_j$.

Lemma 4.2. On overlaps $\tilde{V}_j \cap \tilde{V}_j'$, $\sigma_j - \sigma_j' = \text{Hol}^*(\varpi_j - \varpi_{j'})$.

Proof. Both sides are closed $LG$-invariant forms, for which the pull-back to $t \subset Lg^*$ vanishes. Hence, it suffices to check that at any point $\mu \in O_j \cap O_{j'} \subset t \subset Lg^*$, the contraction with $\zeta_{Lg^*}$ is equal for $\zeta \in Lg$. Also,

$$
\iota(\zeta)\sigma_j = \tilde{\pi}_j^* d(B(\Phi_j, \zeta)) = dB(\tilde{\pi}_j^* \Phi_j, \zeta)
$$

where $\Phi_j : O_j \hookrightarrow Lg^*$ is inclusion. But

$$
(\tilde{\pi}_j^* \Phi_j - \tilde{\pi}_j'^* \Phi_{j'})|_{g.\mu} = g \cdot \mu_j - g \cdot \mu_{j'} = (Ad_g(\mu_j) - \theta^R) - (Ad_g(\mu_{j'}) - \theta^R) = Ad_g(\mu_j - \mu_{j'}).
$$

This, however, is another moment map for $\text{Hol}^*(\varpi_j - \varpi_{j'})$. □

Thus, the locally defined forms

$$
\varpi |_{\tilde{V}_j} = \text{Hol}^*(\varpi_j) - \sigma_j
$$

patch together to define a global 2-form $\varpi \in \Omega^2(Lg^*)$. Form the properties of $\varpi_j$ and $\sigma_j$ we read off:

(i) $d\varpi = \text{Hol}^* \eta$,

(ii) $\iota(\zeta_{Lg^*})\varpi = \frac{1}{2}B(\text{Hol}^*(\theta^L + \theta^R), \zeta(0)) - dB(\mu, \zeta)$.

Here, $\mu : Lg^* \rightarrow Lg^*$ is the identity map. Such a 2-form was constructed in [2] using a different method.

Consider the case $G = SU(n)$ or $G = Sp(n)$, which are the two cases where the vertices of the alcove lie in the weight lattice $\Lambda^*$, where we identify $g^* \cong g$ using the basic inner product. Let

$$
U(1) \rightarrow \widehat{LG} \rightarrow LG
$$

denote the $k$-th power of the basic central extension of the loop group [18]. That is, on the Lie algebra level the central extension

$$
\mathbb{R} \rightarrow \widehat{Lg} \rightarrow Lg
$$

is defined by the cocycle

$$
(\xi_1, \xi_2) \mapsto \int_{S^1} B(\xi_1, d\xi_2) \quad \xi \in Lg = \Omega^0(S^1, g))
$$
where $B$ is the inner product at level $k$. The coadjoint action of $\hat{L}G$ on $\hat{L}g^* = Lg^* \times \mathbb{R}$ preserves the level sets $Lg^* \times \{t\}$, and the action for $t = 1$ is exactly the gauge action of $Lg$ considered above. Since $\mu_j \in \Lambda^*$, the orbits $Lg \cdot \mu_j = O_j$ carry $\hat{L}G$-equivariant pre-quantum line bundles $L_{O_j} \to O_j$, given explicitly by

$$L_{O_j} = \hat{L}G \times (\hat{L}G)_j, \mathbb{C}_{(\mu_j, 1)}.$$ 

Here $(\hat{L}G)_j$ is the restriction of $\hat{L}G$ to the stabilizer $(LG)_j$ of $\mu_j \in t \subset Lg^*$, and $\mathbb{C}_{(\mu_j, 1)}$ denotes the 1-dimensional representation of $(\hat{L}G)_j$ with weight $(\mu_j, 1) \in \Lambda^* \times \mathbb{Z}$. Let $E_j \to \hat{V}_j$ be the pull-back $\pi_j^* L_{O_j}$. On overlaps, $\hat{V}_j \cap \hat{V}_j'$. $E_j \otimes E_j'^{-1}$ is an associated bundle for the weight $(\mu_j, 1) - (\mu_j', 1) = (\mu_j - \mu_j', 0)$. Therefore, $E_j \otimes E_j'^{-1}$ is an $LG$-equivariant bundle. It is clear by construction that $E_j \otimes E_j'^{-1}$ is the pull-back of the pre-quantum line bundle over $O_{j'} \subset g^*$. Taking all this information together, we have constructed an explicit quasi-line bundle for the pull-back of the $k$-th power of the basic gerbe under holonomy map $\text{Hol} : Lg^* \to G$ with error 2-form equal to $\varpi \in \Omega^2 (Lg^*)$.

5. Review of Quasi-Hamiltonian $G$-Spaces

Suppose $(M, \omega)$ is a symplectic manifold together with a symplectic action of a Lie group $G$. This action called Hamiltonian if there exists a smooth equivariant map

$$\Phi : M \to g^*$$

such that

$$\iota (\xi_M) \omega + d\Phi, \xi = 0$$

for all $\xi \in g$, where $\xi_M$ is the vector field on $M$ generated by $\xi \in g$, i.e.,

$$\xi_M (m) = \frac{d}{dt} |_{t=0} \exp (t \xi) \cdot m.$$ 

The map $\Phi$ and the triple $(M, \omega, \Phi)$ are known as moment map and Hamiltonian G-manifold respectively [8]. Let $G$ be a compact Lie group. Fix an invariant inner product $B$ on $g$, which we use to identify $g \cong g^*$. Since the exponential map $exp : g \to G$ is a diffeomorphism in a neighborhood of the origin, the composition map

$$\Psi := exp \circ \Phi : M \to G$$

inherits the properties of the moment map $\Phi$ and vice versa.

**Definition 5.1.** A quasi-Hamiltonian $G$-space with group-valued moment map is a triple $(M, \omega, \Psi)$ consisting of a $G$-manifold $M$, an invariant 2-form $\omega \in \Omega^2 (M)$, and an equivariant smooth map $\Psi : M \to G$ such that

1. $d\omega = \Psi^* \eta$ where $\eta \in \Omega^3 (G)$ is the 3-form defined by $B$. This condition is called the relative cocycle condition.
2. $\iota (\xi_M) \omega = \frac{1}{2} B (\Psi^* (\theta^L + \theta^R), \xi)$. This condition is called the moment map condition.
3. The $\ker (\omega_m) \in T_m (M)$ for $m \in M$ consists of all $\xi_M (m)$ such that

$$(\text{Ad}_{\Psi (m)} + 1) \xi = 0.$$ 

This is called the minimal degeneracy condition.
5.1. Examples.

Example 5.1. Consider a Hamiltonian $G$-manifold $(M,\omega,\Phi)$ such that the image of $\Phi$ is a subset of the set of regular values for the exponential map. Then $(M,\Upsilon,\Psi)$ is a quasi-Hamiltonian $G$-space with group-valued moment map, where

$$\Psi = \exp \circ \Phi$$

and

$$\Upsilon := \omega + \Phi^* \varpi$$

where $\varpi \in \Omega^2(g)$ is the primitive for $\exp^* \eta$ given by the de Rham homotopy operator for the vector space $g$. The converse is also true, provided that $\Psi(M)$ lies in a neighborhood of the origin on which the exponential map is a diffeomorphism.

Example 5.2. Let $C \subset G$ be a conjugacy class of $G$. The triple $(C,\omega,\Phi)$ is a quasi-Hamiltonian $G$-space with group-valued moment map where $\Phi : C \rightarrow G$ is inclusion and $\omega_\xi(\xi_C(g),\zeta_C(g)) = \frac{1}{2}B((Ad_g - Ad_{g^{-1}})\xi,\zeta)$ [10].

Example 5.3. Given an involutive Lie group automorphism $\rho \in Aut(G)$, i.e., $\rho^2 = 1$, one defines twisted conjugacy classes to be the orbits of the action $h \cdot g = \rho(h)g\rho^{-1}$. $G$ is a symmetric space

$$G = G \times G / (G \times G)^\rho$$

where $\rho(g_1,g_2) = (g_2,g_1)$. The map $G \times G \rightarrow \mathbb{Z}_2 \ltimes G \times G, (g_1,g_2) \mapsto (\rho^{-1},g_1,g_2)$ takes the twisted conjugacy classes of $G \times G$ to conjugacy classes of the disconnected group $\mathbb{Z}_2 \ltimes G \times G$. Thus by using example 5.2 the group $G$ itself becomes a group-valued Hamiltonian $\mathbb{Z}_2 \ltimes G \times G$, with 2-form $\omega = 0$, moment map $g \mapsto (\rho,g,g^{-1})$ and action $(g_1,g_2) \cdot g = g_2gg_1^{-1}, \rho \cdot g = g^{-1}$.

Example 5.4. Let $D(G)$ be a product of two copies of $G$. On $D(G)$, we can define a $G \times G$ action by

$$(g_1,g_2) \cdot (a,b) = (g_1ag_2^{-1},g_2ag_1^{-1}).$$

Define a map

$$\Psi : D(G) \rightarrow G \times G, \quad \Psi(a,b) = (ab,a^{-1}b^{-1})$$

and let the 2-form $\omega$ be defined by

$$\omega = \frac{1}{2}(B(Pr_1^* \theta^L,Pr_2^* \theta^R) + B(Pr_1^* \theta^R,Pr_2^* \theta^L))$$

where $Pr_1$ and $Pr_2$ are projections to the first and second factor. Then the triple $(D(G),\omega,\Psi)$ is a Hamiltonian $G \times G$-manifold with group-valued moment map.

Example 5.5. Let $G = SU(2)$ and $M = S^4$ the unit sphere in $\mathbb{R}^5 \cong \mathbb{C}^2 \times \mathbb{R}$, with $SU(2)$-action induced from the action on $\mathbb{C}^2$. $M$ carries the structure of a group-valued Hamiltonian $SU(2)$-manifold, with the moment map $\Psi : M \rightarrow SU(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \rightarrow S^2$. For details, see [1]. This example is generalized by Hurtubise-Jeffrey-Sjamaar in [13] to $G = SU(n)$ acting on $M = S^{2n}$ (viewed as unit sphere in $\mathbb{C}^n \times \mathbb{R}$).

The equivariant de Rham complex is defined as

$$\Omega^k_G(M) = \bigoplus_{2i+j=k} (\Omega^j(M) \otimes S^i(g^*))^G$$
where \( S(\mathfrak{g}^*) \) is the symmetric algebra over the dual of the Lie algebra of \( G \). Elements in this complex can be viewed as equivariant polynomial maps from \( \mathfrak{g} \) into the space of differential forms. \( \Omega_G(M) \) carries an equivariant differential \( d_G \) of degree 1,

\[
(d_G \alpha)(\xi) := d\alpha(\xi) + \iota(\xi_M)\alpha(\xi).
\]

Since \((d + \iota(\xi_M))^2 = L(\xi_M)\) and we are restricting on the equivariant maps, \( d_G^2 = 0 \). The equivariant cohomology is the cohomology of this co-chain complex [9]. The canonical 3-form \( \eta \) has a closed equivariant extension \( \eta_G \in \Omega^3_G(G) \) given by

\[
\eta_G(\xi) := \eta + \frac{1}{2} B(\theta_L + \theta_R, \xi).
\]

We can combine the first two conditions of the definition of a group-valued moment map and get the condition

\[
d_G \omega = \Psi^* \eta_G.
\]

5.2. Products. Suppose \((M, \omega, (\Psi_1, \Psi_2))\) is a group-valued Hamiltonian \( G \times G \)-manifold. Then \( \tilde{M} = M \) with diagonal action, moment map \( \tilde{\Psi} = \Psi_1 \Psi_2 \) and 2-form

\[
\tilde{\omega} = \omega - \frac{1}{2} B(\Psi_1^* \theta_L, \Psi_2^* \theta_R)
\]

is a group-valued quasi-Hamiltonian \( G \)-space. If \( \tilde{M} = M_1 \times M_2 \) is a direct product of two group-valued quasi-Hamiltonian \( G \)-spaces, we call \( \tilde{M} \) the fusion product of \( M_1 \) and \( M_2 \). This product is denoted by \( M_1 \oplus M_2 \). If we apply fusion to the double \( D(G) \), we obtain a group-valued quasi-Hamiltonian \( G \)-space with \( G \)-action

\[
g \cdot (a, b) = (Ad_g a, Ad_g b),
\]

moment map

\[
\Psi(a, b) = aba^{-1}b^{-1} \equiv [a, b],
\]

and 2-form

\[
\omega = \frac{1}{2} (B(\Pr^*_1 \theta_L, \Pr^*_2 \theta_R) + B(\Pr^*_1 \theta_R, \Pr^*_2 \theta_L) - B((ab)^* \theta_L, (a^{-1}b^{-1})^* \theta_R)).
\]

Fusion of \( h \) copies of \( D(G) \) and conjugacy classes \( C_1, \cdots, C_r \) gives a new quasi-Hamiltonian space with the moment map

\[
\Psi(a_1, b_1, \cdots, a_h, b_h, d_1, \cdots, d_r) = \prod_{j=1}^h [a_j, b_j] \prod_{k=1}^r d_k.
\]

5.3. Reduction. The symplectic reduction works as usual:

If \((M, \omega, \Psi)\) be a Hamiltonian \( G \)-space with group-valued moment map and the identity element \( e \in G \) be a regular value of \( \Psi \), then \( G \) acts locally freely on \( \Psi^{-1}(e) \) and therefore \( \Psi^{-1}(e)/G \) is smooth. Furthermore, the pull-back of \( \omega \) to identity level set descends to a symplectic form on \( M/G := \Psi^{-1}(e)/G \). For instance, the moduli space of flat \( G \)-bundles on a closed oriented surface of genus \( h \) with \( r \) boundary components, can be written

\[
\mathcal{M}(\Sigma; C_1, \cdots, C_r) = \mathbb{C}^{2h} \oplus C_1 \oplus \cdots \oplus C_r / G = \Psi^{-1}(e)/G
\]

where the \( j \)-th boundary component is the bundle corresponding to the conjugacy class \( C_j \). More details can be found in [2], [1].
6. PRE-QUANTIZATION OF QUASI-HAMILTONIAN G-SPACES

We know that a symplectic manifold \((M, \omega)\) is pre-quantizable (admits a line bundle \(L\) over \(M\) with curvature 2-form \(\omega\)) if the 2-form \(\omega\) is integral. In this Section, we will first introduce a notion of a pre-quantization of a space with \(G\)-valued moment map and then give a similar criterion for being pre-quantizable.

**Definition 6.1.** Let \(G\) be a compact connected Lie group with canonical 3-form \(\eta\). Fix a gerbe \(\mathcal{G}\) on \(G\) with connection \((\nabla, \omega)\) such that \(\text{curv}(\mathcal{G}) = \eta\). A pre-quantization of \((M, \omega, \Psi)\) is a relative gerbe with connection \((\mathcal{L}, \mathcal{G})\) corresponding to the map \(\Psi\) with relative curvature \((\omega, \eta)\).

Since \(\eta\) is closed 3-form and \(\Psi^* \eta = d\omega\), \((\omega, \eta)\) defines a relative cocycle. Recall from Chapter 1 that a class \([[(\omega, \eta)]]\) \(\in H^3(\Psi, \mathbb{R})\) is integral if and only if \(\int_{\beta} \eta - \int_{\gamma} \omega \in \mathbb{Z}\) for all relative cycles \((\beta, \Sigma) \in C_3(\Psi, \mathbb{R})\).

**Remark 6.1.** \((M, \omega, \Psi)\) is pre-quantizable if and only if \([[(\omega, \eta)]]\) is integral by Theorem ??.

**Theorem 6.1.** Suppose \(M_i, i = 1, 2\) are two quasi-Hamiltonian \(G\)-spaces. The fusion product \(M_1 \circledast M_2\) is pre-quantizable if both \(M_1\) and \(M_2\) are pre-quantizable.

**Proof.** Let \(\text{Mult} : G \times G \rightarrow G\) be group multiplication and \(\text{Pr}_i : G \times G \rightarrow G, i = 1, 2\) projections to the first and second factors. Since

\[
\text{Mult}^* \eta = \text{Pr}_1^* \eta + \text{Pr}_2^* \eta + \frac{1}{2} B(\text{Pr}_1^* \theta^L, \text{Pr}_1^* \theta^R),
\]

we get a quasi-line bundle with connection for the gerbe \(\text{Mult}^* \mathcal{G} \otimes (\text{Pr}_1^* \mathcal{G})^{-1} \otimes (\text{Pr}_2^* \mathcal{G})^{-1}\) such that the error 2-form is equal to \(\frac{1}{2} B(\text{Pr}_1^* \theta^L, \text{Pr}_1^* \theta^R)\). Any two such quasi-line bundles differ by a flat line bundle with connection. Let \(\Psi_i, i = 1, 2\) be moment maps for \(M_i, i = 1, 2\) respectively and \(\Psi = \Psi_1 \Psi_2\) be the moment map for their fusion product \(M_1 \circledast M_2\). Thus,

\[
\Psi^* \mathcal{G} = (\Psi_1 \times \Psi_2)^* \text{Mult}^* \mathcal{G} = (\Psi_1^* \times \Psi_2^*)((\text{Pr}_1^* \mathcal{G}) \otimes (\text{Pr}_2^* \mathcal{G})) = \Psi_1^* \mathcal{G} \otimes \Psi_2^* \mathcal{G}.
\]

Therefore \(M_1 \circledast M_2\) is pre-quantizable if and only if both \(M_1\) and \(M_2\) are pre-quantizable.

**Proposition 6.2.** Suppose \(G\) is simple and simply connected. Let \(k \in \mathbb{Z}\) be the level of \((M, \omega, \Psi)\). Suppose \(H^2(M, \mathbb{Z}) = 0\). Then there exists a pre-quantization of \((M, \omega, \Psi)\) if and only if the image of

\[
\Psi^* : H^3(G, \mathbb{Z}) \rightarrow H^3(M, \mathbb{Z})
\]

is \(k\)-torsion.

**Proof.** By assumption, \([[\eta]]\) represents \(k\) times the generator of \(H^3(G, \mathbb{Z})\). If \(H^2(M, \mathbb{Z}) = 0\), the long exact sequence:

\[
\cdots \rightarrow H^2(G, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}) \rightarrow H^3(\Psi, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z}) \xrightarrow{\Psi^*} H^3(M, \mathbb{Z}) \rightarrow \cdots
\]

shows that the map \(H^3(\Psi, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})\) is injective. In particular, \(H^3(\Psi, \mathbb{Z})\) has no torsion, and \((M, \omega, \Psi)\) is pre-quantizable if and only if \([[[\eta]]\) is in the image of the map \(H^3(\Psi, \mathbb{Z}) \rightarrow H^3(G, \mathbb{Z})\), i.e., in the kernel of \(H^3(G, \mathbb{Z}) \rightarrow H^3(M, \mathbb{Z})\). This exactly means that the image of this map is \(k\)-torsion.

\[\Box\]
**Proposition 6.3.** If $H_2(M, \mathbb{Z}) = 0$ a pre-quantization exists. More generally, if $H_2(M, \mathbb{Z})$ is $r$-torsion, a level $k$ pre-quantization exists, where $k$ is a multiple of $r$.

**Proof.** If $rH_2(M, \mathbb{Z}) = 0$, for any cycle $S \in C_2(M)$, there is a 3-chain $T \in C_3(M)$ with $\partial T = r \cdot S$. If $\Psi(S) = \partial B$, $\Psi(T) = rB$ is a cycle and

$$
\int_S k\omega - \int_B k\eta = \frac{k}{r} \left( \int_T d\omega - \int_{rB} \eta \right)
= \frac{k}{r} \left( \int_T \Psi^*\eta - \int_{rB} \eta \right)
= \frac{k}{r} \left( \int_{\Psi(T)} \eta - \int_{rB} \eta \right)
= \frac{k}{r} \left( \int_{\Psi(T)-rB} \eta \right) \in \mathbb{Z}.
$$

(6.1)

By Remark 6.1 $(M, \omega, \Psi)$ is pre-quantizable. □

**Example 6.1.** $M = S^4$ carries the structure of a group-valued Hamiltonian $SU(2)$-manifold, with the moment map $\Psi : M \to SU(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \to S^2$, Example 5.5. By Proposition 6.3 this $SU(2)$-valued moment map is pre-quantizable.

6.1. **Reduction.** Let $G$ be a simply connected Lie group. Fix a pre-quantization $L$ for a space with $G$-valued moment map $(M, \omega, \Psi)$.

\[
\begin{array}{ccc}
L & \xrightarrow{\Psi} & G \\
\downarrow & & \downarrow \\
M & \xrightarrow{\Psi^{-1}(e)} & \{e\}
\end{array}
\]

Since $G|_{\Psi^{-1}(e)}$ is equal to trivial gerbe, $L|_{\Psi^{-1}(e)}$ is a line bundle with connection with curvature $(\iota_{\Psi^{-1}(e)})^*\omega$. Since $G$ is simply connected and the 2-form $(\iota_{\Psi^{-1}(e)})^*\omega$ is $G$-basic, there exists a unique lift of the $G$-action to $L|_{\Psi^{-1}(e)}$ in such a way that the generating vector fields on $L|_{\Psi^{-1}(e)}$ are horizontal. This is a special case of Kostant’s construction [15]. In conclusion, we get a pre-quantum line bundle over $\Psi^{-1}(e)/G$.

6.2. **A Finite Dimensional Pre-quantum Line Bundle for $M(\Sigma)$.** Let $M = G^{2h}$ where $G$ is a simply connected Lie group and consider the map

$$
\Psi : M \to G
$$

with the rule

$$
\Psi(a_1, \ldots, a_h) = \prod_{i=1}^{h} [a_i, b_i].
$$

Let $\mathcal{G}$ be the basic gerbe with the connection on $G$ and $\text{curv}(\mathcal{G}) = \eta$. The moduli space of flat $G$-bundles on a closed oriented surface $\Sigma$ of genus $h$ is equal to

$$
M(\Sigma) = G^{2h}/G = \Psi^{-1}(e)/G.
$$
Since $G$ is simply connected, $H^2(G, \mathbb{Z}) = H^2(G^{2h}, \mathbb{Z}) = 0$. $H^3(G^{2h}, \mathbb{Z}) \cong \mathbb{Z}$ is torsion free therefore by Proposition 4.3.3 there exists a unique quasi-line bundle $\mathcal{L}$ for the gerbe $\Psi^*G$.

Pick a connection for this quasi-line bundle and call the error 2-form $\nu$. Therefore $d(\nu - \omega) = 0$. This together with the fact that $H^2(M, \mathbb{Z}) = 0$ allow us to modify quasi-line bundle with connection such that triple $(M, \omega, \Psi)$ is pre-quantizable. By reduction we get a pre-quantum line bundle over $\Psi^{-1}(e)/G = M(\Sigma)$.

6.3. Pre-quantization of Conjugacy Classes of a Lie Group. Let $G$ be a simple, simply connected compact Lie group. Fix an inner product $B$ at level $k$.

The map

$$\exp : \mathfrak{g} \to G$$

takes (co)adjoint orbits $\mathcal{O}_\xi$ to conjugacy classes $C = G \cdot \exp(\xi)$.

Any conjugacy class $C \subseteq G$ is uniquely a $G$-valued quasi-Hamiltonian $G$-space $(C, \omega, \Psi)$, where $\Psi : C \hookrightarrow G$ is inclusion map, as it explained in example 5.2.

Suppose $(\beta, \Sigma) \in \text{Cone}_n(\Psi, \mathbb{Z})$ is a cycle. We want to see under which conditions $C$ is pre-quantizable at level $k$. Equivalently, we are looking for conditions which implies

$$k(\int_\beta \eta - \int_\Sigma \omega) \in \mathbb{Z}$$

where $\eta = \frac{1}{2\pi}B(\theta \mathcal{L}, [\theta \mathcal{L}, \theta \mathcal{L}])$ is canonical 3-form. Consider the basic gerbe $G = (\mathcal{V}, L, \theta)$ with connection on $G$ with curvature $\eta$. For all $C \subseteq G$ there exists a unique $\xi \in \mathfrak{A}$ such that $\exp(\xi) \in C$. Let

$$\iota_C : C \to G$$

be inclusion map assume that $\varpi_0$ is the primitive of $\eta$ on $V_0$, i.e.,

$$\eta|_{V_0} = d\varpi_0$$

where $V_0$ contains $C$. Recall from Section 3.3 that

$$\omega_C = \theta^*_0(\omega_{\mathcal{O}_\xi})_G - \iota^*_C(\varpi_0)_G$$

and pull-back of the $\theta$ to $C$ is zero. Thus,

$$k(\int_\Sigma \omega_C - \int_\beta \iota^*_C \eta) = k(\int_\Sigma \theta^*_0(\omega_{\mathcal{O}_\xi})_G - \iota^*_C(\varpi_0)_G - \int_\beta \iota^*_C(\eta)) = k(\int_\Sigma \theta^*_0(\omega_{\mathcal{O}_\xi})_G).$$

$(\omega_C, \iota^*_C \eta)$ is integral if and only if the symplectic 2-form $k\omega_{\mathcal{O}_\xi}$ is integral. It is a well-known fact from symplectic geometry that $k\omega_{\mathcal{O}_\xi}$ is integral if and only if $B(\xi) \in \Lambda^k : = \Lambda^* \cap k\mathfrak{A}$, by viewing $B$ as a linear map $t \to t^*$. 

7. Hamiltonian Loop Group Spaces

Fix an invariant inner product $B$ on $\mathfrak{g}$. Assume that $G$ is simple and simply connected. Recall that a Hamiltonian loop group manifold is a triple $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$ where $\tilde{M}$ is an (infinite-dimensional) $LG$-manifold, $\tilde{\omega}$ is an invariant symplectic form on $\tilde{M}$, and $\tilde{\Psi} : \tilde{M} \to L\mathfrak{g}^*$ an equivariant map satisfying the usual moment map condition,

$$\iota(\xi_{\tilde{M}})\tilde{\omega} + dB(\tilde{\Psi}, \xi) = 0 \quad \xi \in \Omega^0(S^1, \mathfrak{g}).$$
Example 7.1. Let $O \subset Lg^*$ be an orbit of the loop group action. Then $O$ carries a unique structure for a Hamiltonian $LG$-manifold when the moment map is inclusion and the 2-form is

$$\tilde{\omega}_\mu(\xi_O(\mu), \eta_O(\mu)) = \langle d\mu \xi, \eta \rangle = \int_{S^1} B((d\mu \xi), \eta).$$

The based loop group $\Omega G \subset LG$ consisting of loops that are trivial at the origin of $S^1$, acts freely on $Lg^*$ and the quotient is just the holonomy map. There is a one-to-one correspondence between quasi-Hamiltonian $G$-spaces $(M, \omega, \Psi)$ and Hamiltonian $LG$-spaces with proper moment maps $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$, where

$$M = \tilde{M}/\Omega G,$$

$$\text{Hol} \circ \tilde{\Psi} = \Psi \circ \text{Hol},$$

$$\tilde{\omega} = \text{Hol}^* \omega - \tilde{\Psi} \pi.$$ 

This is called Equivalence Theorem in [2]. We thus, have a commutative diagram:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{\Psi}} & Lg^* \\
\text{Hol} \downarrow & & \text{Hol} \downarrow \\
M & \xrightarrow{\Psi} & G
\end{array}$$

Theorem 7.1. (Equivalence Theorem for Pre-quantization) There is a one-to-one correspondence between pre-quantizations of quasi-Hamiltonian $G$-spaces with group valued moment maps and pre-quantizations of the corresponding Hamiltonian $LG$-spaces with proper moment maps.

Proof. Assume that we have constructed a pre-quantization of a quasi-Hamiltonian $G$-space with group-valued moment map $(M, \omega, \Psi)$ with the corresponding Hamiltonian $LG$-space $(\tilde{M}, \tilde{\omega}, \tilde{\Psi})$. Thus, we have a relative gerbe mapping to the basic gerbe over $G$. Pull-back of this quasi-line bundle under $\text{Hol} : \tilde{M} \to M$, gives a quasi-line bundle of the gerbe $\text{Hol}^* \Psi^* G = \tilde{\Psi}^* \text{Hol}^* G$ over $\tilde{M}$. But recall that, we have a quasi-line bundle for $\text{Hol}^* G$ as it explained in Section 4. Therefore the difference between these two quasi-line bundles with connection is a line bundle with connection $L \to \tilde{M}$ with the curvature 2-form $\tilde{\omega} = \text{Hol}^* \omega - \tilde{\Psi} \pi$ by Remark ??.

Note also that if the quasi-line bundle for $\Psi : M \to G$ is $G$-equivariant, then since the quasi-line bundle for $Lg^* \to G$ is $LG$-equivariant, the line bundle $L$ will be $LG$ equivariant. Conversely, suppose that we are given a $LG$-equivariant line bundle over $\tilde{M}$, where $U(1) \subset \tilde{LG}$ acts with weight 1. The difference of this $LG$-equivariant line bundle and the $LG$-equivariant quasi-line bundle for $\text{Hol}^* \Psi^* G$ (constructed in Section 4), is a quasi-line bundle with error 2-form $\text{Hol}^* \omega$. By descending of this quasi-line bundle to $M$, we can get the desired quasi-line bundle for $\Psi^* G$. \qed

The argument, given here applies in greater generality:

For any $LG$-equivariant line bundle $L \to M$, where the central extension $U(1) \subset \tilde{LG}$ acts with weight $k \in \mathbb{Z}$, there is a corresponding relative gerbe at level $k$ with respect to the map $\Psi : \tilde{M} \to G$. Indeed, the given quasi-line bundle for $\tilde{\Psi}^* \text{Hol}^* G^k$ is given by $LG$ equivariant line bundles over $\text{Hol}^{-1} \Psi^{-1}(V_j)$ at level $k$. Twisting by $L$, we
get new quasi-line bundle where $U(1) \subset \hat{LG}$ acts trivially. The quotient therefore descends to a quasi-line bundle over $M$. For instance, Meinrenken and Woodward construct for any Hamiltonian loop group space a so-called “canonical line bundle” in [16], which is $\hat{LG}$-equivariant at level $2c$, where $c$ is the dual Coxeter number. Therefore this line bundle gives rise to a distinguished element of $H^3(\Phi, \mathbb{Z})$ at level $2c$. Notice that $M$ and $\tilde{M}$ are not pre-quantizable necessarily.

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