Relative localization of point particle interactions

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Abstract

We review the main concepts of the recently introduced principle of relative locality and investigate some aspects of classical interactions between point particles from this new perspective. We start with a physical motivation and basic mathematical description of relative locality and review the treatment of a system of classical point particles in this framework. We then examine one of the unsolved problems of this picture, the apparent ambiguities in the definition of momentum constraints caused by a non-commutative and/or non-associative momentum addition rule. The gamma ray burst experiment is used as an illustration. Finally, we use the formalism of relative locality to reinterpret the well-known multiple point particle system coupled to 2+1 Einstein gravity, analyzing the geometry of its phase space and once again referring to the gamma ray burst problem as an example.

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Contents

1 Introduction 3

2 The principle of relative locality 4
   2.1 Physical concept ................................................. 4
   2.2 The geometry of phase space .................................... 5

3 Dynamics of point particle interactions 7
   3.1 General considerations ........................................... 7
   3.2 The change of vertex problem .................................... 10
   3.3 Illustration: gamma ray bursts .................................. 12
      3.3.1 Specifying momentum constraints ........................ 14
      3.3.2 Specifying momentum constraints - 1st alternative .......... 15
      3.3.3 Specifying momentum constraints - 2nd alternative .......... 16
      3.3.4 Specifying momentum constraints - 3rd alternative .......... 17

4 Multiparticle systems coupled to 2+1 gravity 18
   4.1 Phase space of multiple point particle systems ................ 18
   4.2 The relative locality interpretation ............................ 23
   4.3 Gamma ray bursts in 2+1 dimensions ............................ 25

5 Conclusions 26

6 Appendix - Review of 2+1 Einstein gravity 28
1 Introduction

The notion of relative locality was originally proposed as a response to criticisms to Double Special Relativity (DSR) [3, 4], a modification of Special Relativity proposed as a step towards quantum gravity. DSR proposes the replacement of the Poincaré symmetry group of SR by the $\kappa$-Poincaré group, while preserving relativity of inertial frames. The practical consequence of the modification is the introduction of an universal invariant energy scale (Planck scale) along with the invariant velocity, $c$.

There are compelling arguments to believe that 2+1 gravity is a DSR theory. Indeed, when coupled to point particles, the theory has [5]

- a universal mass limit independent of the number of particles;
- a noncommutative spacetime Poisson algebra;
- a non-linear 3-momentum addition rule;
- a curved 3-momentum space;

all of which are characteristics of a DSR theory.

One of the main criticisms of DSR is its apparent violation of locality. The construction used to support this claim is the study of a particle collision: if two worldlines intersect in some inertial frame, then there exists a class of inertial observers for which the worldlines do not cross [12]. That is, the notion of locality is no longer invariant - an interaction that is local for an observer can be non-local for a different one - a feature that naturally raised concerns about the theory, since it seemed to indicate a fundamental subjectivity of reality, violating the principle of equivalence.

The concept of relative locality was suggested to argue this is not the case, by proposing that the violation of locality present in DSR theories is not real, but a consequence of misinterpreting the geometry of reality, due to taking spacetime to be an absolute, invariant entity - an assumption motivated by our knowledge and intuition, but effectively unwarranted. Compare this situation with the problem of simultaneity in SR - the counterintuitive statement of two events occurring at the same time for one observer being located at different times for a different one stems from abandoning the Newtonian notion of absolute time. In the framework of relative locality, it is the absoluteness of spacetime that is abandoned.

Relative locality is a change of paradigm with the intention of better understanding quantum gravity. In this context, the problem of studying the properties of Feynman diagrams of a field theory according to the new framework becomes of major importance. A natural first step to approach it is to describe how our understanding of particle interactions changes in the simpler context of classical mechanics, which is the main topic of the present essay.
2 The principle of relative locality

2.1 Physical concept

As observers immersed in the universe, notions of space and time are intuitive to us. We immediately interpret the localization of objects around us in terms of distances, and perceive the ordering of events and the concept of cause and effect as result of the passage of time. Perhaps because these concepts are so natural and intuitive, all physical theories to date take them as fundamental rather than “emergent” from deeper structures. But considering more carefully the mechanisms through which we probe spacetime, we see that within them are the seeds for a drastic change of perspective.

Consider a simple example - measuring the length of a bar. An observer “sees” the bar by receiving photons from it - optical mechanisms convert them in information interpretable by the brain as an image. And, indeed, knowing the time each of two photons, one from each end of the bar, takes to reach the eye, together with the direction in which they were sent, one can derive the length of the bar from basic geometric considerations.

![Figure 1: Deriving the length L of the bar using the data Δt1, Δt2, α obtained from photons.](image)

In more abstract terms, the observer performed a distance measurement using only a calorimeter for photon detection and a clock for time measurement. The same line of thought can be applied to other measurements (velocity, acceleration...), in a way that suggests that the fundamental experimental apparatuses are the calorimeter and the clock - and we effectively probe spacetime through exchange of energy-momentum quanta.

Put this way, it seems almost natural to propose that the “arena” where physical processes happen is not spacetime, but energy-momentum space. Spacetime becomes a construction, an auxiliary entity derived by observers from physical interactions. As such, there is no reason to assume this constructed spacetime is independent of energy and momentum (the parameters of a DSR change of frame), so in general it can be observer-dependent - which, as will be illustrated below, leads to locality of an interaction being itself relative to the observer.

The change of paradigm described above leads to the formulation of the Principle of Relative Locality. Quoting [1]:

*Physics takes place in phase space and there is no invariant global projection that gives a description of processes in spacetime. From their measurements local observers can construct descriptions of particles moving and interacting in a spacetime, but different observers construct different spacetimes, which are observer-dependent slices of phase space.*
2.2 The geometry of phase space

Any theory of quantum gravity must have as fundamental quantities Planck’s constant, $\hbar$, and Newton’s constant, $G$. Taking different limits of these, we obtain different experimental regimes of gravity: taking $\hbar \to 0$ leads us to classical General Relativity, while considering $G \to 0$ should recover special-relativistic quantum mechanics. But there is a third limit that can be taken, if we notice that the two fundamental constants can be combined to form the Planck mass

$$m_p = \sqrt{\frac{\hbar}{G}}$$

: taking $G \to 0$ and $\hbar \to 0$ while keeping $m_p$ finite. In this regime, since the effects of classical gravity and special-relativistic quantum mechanics are negligible, the geometry of spacetime is Minkowski and one can work in the classical mechanics formalism; short-distance quantum phenomena should be irrelevant as well since the Planck length is $l_p = \sqrt{\hbar G} \to 0$. However, the presence of the Planck scale indicates that novel quantum gravity effects could occur at energies of order $m_p$, possibly resulting in a nontrivial geometry of momentum space.

In line with taking momentum space $\mathcal{P}$ as fundamental, our phase space will be its cotangent bundle, $T^*\mathcal{P}$. Spacetime at a given point in $\mathcal{P}$ is the cotangent space to $\mathcal{P}$ at that point, which we denote by $\mathcal{X}(p) = T^*_p\mathcal{P}$. While we will not discuss it here, one can speculate that when $G_N$ and $\hbar$ are present (the hypothetical full quantum gravity theory), the geometry of the whole phase space is something much more complex, intertwining spacetime and momentum space in a nontrivial fiber bundle.

Metric of momentum space

Our observer can construct the metric of momentum space through two classes of measurements:

- Determining the rest mass of a particle gives the geodesic distance of its position in $\mathcal{P}$ to the origin,

$$D^2(p, 0) = m_p^2;$$

(1)

- On the other hand, a measurement of kinetic energy $K$ gives the distance between the moving particle’s momentum, $p$ and that of the same particle at rest, $p'$:

$$D^2(p, p') = -2mK.$$ 

(2)

This information is sufficient to recover the metric of momentum spacetime

$$dk^2 = D^2(k, k + dk) \equiv g^{ab}(k) \, dk_a dk_b.$$ 

(3)

Connection of momentum space

Considering interaction processes between point particles, it becomes evident that a combination rule for momenta is necessary, since it is through “addition” that the principle of conservation of momentum is expressed. We will look into this in more detail in Section 3, but in its essence the addition rule is a map

$$\oplus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$$

$$(p, q) \to p \oplus q.$$ 

(4)

\footnote{Note that, to keep in line with the convention of lower indices for momenta, lower indices are contravariant and upper indices are covariant.}
which can be experimentally determined from particle collision processes, where total momentum is conserved. For very low energies, it should reduce to the usual linear addition, \( p \oplus q \approx p + q \), but in the spirit of DSR, we will assume that in general it is not linear. But if \( \oplus \) is not linear, there is no reason to assume it is commutative or even associative - and we will not do so. However, to turn incoming momenta into outgoing and vice-versa while analyzing point particle interactions, we will require an inversion map, i.e. to assume that each momentum \( p \) has a reciprocal \( \ominus p \) satisfying both the left and right inverse conditions:

\[
(\ominus p) \ominus (p \ominus q) = (q \ominus p) \ominus (\ominus p) = q, \quad \forall p, q \in \mathcal{P}
\]  

(5)

\( \oplus \) is directly related to the notion of a connection in \( \mathcal{P} \), since it can be used to define parallel transport of a vector. As shown in Figure 2, the parallel transport of the vector field \( p \) from \( \mathcal{T}_0 \mathcal{P} \) to \( \mathcal{T}_{dq} \mathcal{P} \) results in their combination \( p \oplus dq \), which admits the following expansion for small \( p \) assuming \( dq \) is infinitesimal:

\[
(p \oplus dq)_a = p_a + dq_a - \Gamma_{bc}^a(0) p_b dq_c + (...) 
\]  

(6)

Figure 2: Momentum addition as a parallel transport.

The definition of the connection of momentum space is then natural:

\[
\Gamma^b_{ac}(k) = -\frac{\partial}{\partial p_b} \frac{\partial}{\partial q_c} (p \oplus k q) |_{p=q=k} 
\]  

(7)

where \( \oplus_k \) is simply the addition rule with the neutral element shifted from 0 to \( k \), which has the expression

\[
p \oplus_k q = k \oplus ((\ominus k \oplus p) \oplus (\ominus k \oplus q)) 
\]  

(8)

and satisfies \( k \oplus_k k = k, \forall k \in \mathcal{P} \) (analogous to \( 0 \oplus 0 = 0 \)).

One important property of our constructions of the metric and connection of momentum space is that they are independent of one another - they result from different measurements. This means that, in general, the connection and the metric might not be compatible. We can quantify the incompatibility using the non-metricity tensor, \( N^{abc} = \nabla^a g^{bc} \). It is possible to decompose the full connection in terms of the Levi-Civita connection, the torsion and the non-metricity. To do this, decompose the connection components in their symmetric and antisymmetric parts with respect to the top indices:

\[
\Gamma_{ab}^{\ c} = \Gamma_{ab}^{(\ c)\ +} + \Gamma_{ab}^{[\ c\ )} = \Gamma_{ab}^{c \ +} + \frac{1}{2} T^{ab}_c + N^{ab}_c
\]  

(9)
where $N^{ab}_{c} = \Gamma^{(ab)}_{c} - \nabla^{a}_{c} \nabla^{b}_{c}$ is symmetric in the top indices. Now it can be shown using the definition of covariant derivative that (defining the covariant torsion $T^{abc} = T^{ab}_{d} g^{dc}$)

\[
N^{abc} = \frac{1}{2} (T^{abc} + T^{acb}) - N^{a}_{d} g^{dc} - N^{a}_{c} g^{db}
\]

so that the connection can be written as

\[
\Gamma^{ab}_{c} = \{^{ab}_{c}\} + \frac{1}{2} T^{ab}_{c} - \frac{1}{2} g^{ci} (N^{abi} + N^{bai} - N^{iab} + T^{iab} + T^{iba})
\]

Curvature of momentum space

From the connection introduced above, we can construct the Riemann tensor of momentum space. The definition in terms of the combination rule is

\[
R^{abc}_{d}(r) = 2 \frac{\partial}{\partial p^{[a}} \frac{\partial}{\partial q^{b]} \frac{\partial}{\partial k^{c}} ([p \oplus_r q \oplus_r k] - [p \oplus_r (q \oplus_r k)])_d \bigg|_{p=q=k=r}
\]

While the connection describes the non-linearity of $\oplus$, the curvature is a measure of its non-associativity.

3 Dynamics of point particle interactions

3.1 General considerations

Consider a system of $N$ point particles labeled by indices $I$ interacting locally at vertices $\alpha$, with worldlines $k_{I}^{a}(s) \in \mathcal{P}, s \in \mathcal{I} \subset \mathbb{R}$. This system is described by the action

\[
S = \sum_{I} S_{\text{free}} + \sum_{\alpha} S_{\text{int}} = \sum_{I} \int ds \left( x^{a}_{I} k_{I}^{a} + N^{I} C^{I}(k^{I}) \right) + \sum_{\alpha} \left( -\kappa_{a}^{(\alpha)} z_{(\alpha)}^{a} \right).
\]

In this expression, we introduce spacetime coordinates $x^{a}_{I}$ as elements of $T^{*}_{k} \mathcal{P}$ canonically dual to the momenta, satisfying the Poisson brackets $\{x^{a}, k_{b}\} = \delta^{a}_{b}$. The metric of spacetime plays no role though; only the metric of momentum space appears in the action through the constraints

\[
C^{I}(k) = D^{2}(k) - m^{2}_{I},
\]

which specify the particle masses. The interaction term imposes conservation of momentum in each vertex $\alpha$ through the constraints $\kappa^{(\alpha)}_{a} = k^{(\alpha)}_{a} \left( k^{(\alpha)}_{(\alpha)} \right)$, which depend on the vertex endpoints of the interacting worldlines and are related via equations of motion to the interaction coordinates $z^{a}_{(\alpha)}$ in spacetime.

2Review of content described in [1], [2].
The free equations of motion for this action,

\[
\frac{\delta S}{\delta k^a_I} = 0 \Rightarrow \dot{x}^a_I = N^i_I \frac{\delta C^I}{\delta k^a_k} \tag{16}
\]

\[
\frac{\delta S}{\delta x^a_I} = 0 \Rightarrow \dot{k}^I_a = 0 \tag{17}
\]

\[
\frac{\delta S}{\delta N^i_I} = 0 \Rightarrow C^I(k) = 0, \tag{18}
\]

are analogous to the usual spacetime-centered perspective in classical mechanics. There are two more equations to be derived though - we can still vary the \(z^{(\alpha)}_{a}\) and the \(k^{I}_{(\alpha)}\) at the interaction vertices.

- Varying with respect to \(z^{a}_{(\alpha)}\) results in the momentum conservation constraints in each vertex, given by

\[
\mathcal{K}^{(\alpha)}_a = 0; \tag{19}
\]

- varying with respect to \(k^{I}_{(\alpha)}\) results in the relative locality relations between the location in spacetime of the worldline endpoints and the vertex coordinates:

\[
x^{a}_{I,\alpha} = x^{b}_{(\alpha)} \left( \pm \frac{\delta k^{I}_{b}}{\delta k^{I}_{a}} \right) \equiv z^{b}_{(\alpha)} \left( W^{I}_{x_I} \right) \, ^a_b \tag{20}
\]

where the + sign refers to incoming particles and - to outgoing ones. Note that this variation requires paying attention to whether the corresponding worldline is incoming or outgoing with respect to the vertex, i.e. assuming for simplicity that \(I = [0, 1]\), \(k^{I}_{(\alpha)} = k^{I}(1) \) or \(k^{I}(0)\) if the particle is incoming or outgoing respectively.

From the relative locality relations we see that not only do the worldline endpoints in spacetime not coincide with the vertex location, but, since the operators \(W_{x_I}\) are related to parallel transport (as we will see in more detail later), they do so in a way that directly reflects the fact that the different particles are in “different spacetimes”. For very low energies (small momenta), we recover the usual notion of locality, \( (W_{x_I})^a_b \approx \delta^a_b \Rightarrow x^{a}_{I,\alpha} = z^{a}_{(\alpha)} \).

For the following, we will require some definitions and two useful results.
Translation and parallel transport operators

The aforementioned addition rule for momenta defines left- and right- translation operators:

\[ p \oplus q \equiv L_p(q) \equiv R_q(p), \quad \forall p, q \in \mathcal{P} \quad (21) \]

Based on our interpretation of parallel transport through addition described in section 2.2, it is natural to define parallel transport operators as derivatives of the translation operators:

\[ (U^q_{p \oplus q})^b_a = \frac{\partial (p \oplus q)_a}{\partial q_b} = (d_q L_p)^b_a, \quad (22) \]

\[ (V^p_{p \oplus q})^b_a = \frac{\partial (p \oplus q)_a}{\partial p_b} = (d_p R_q)^b_a. \quad (23) \]

We define an additional operator as the derivative of the inversion map \( \ominus \):

\[ (I^p)_a^b = \frac{\partial (\ominus p)_a}{\partial p_b} = (d_p \ominus)^b_a. \quad (24) \]

Note that \( L, R \) are diffeomorphisms in \( \mathcal{P} \), while \( U, V, I \) are linear operators in \( T \mathcal{P} \), and that the usual chain rule of differentiation applies, \( d_p (A \o B) = d_{B(p)} A \o d_p B, \quad \forall A, B \in \mathcal{E}(\mathcal{P}) \). We have the following results [2]:

1. Inverses of \( U, V, I \):

\[ (U^q_p)^{-1} = U^q_{p \ominus q}; \quad (V^q_p)^{-1} = V^q_{p \ominus q}; \quad (I^p)^{-1} = I^{\ominus p}. \quad (25) \]

To see this, start by writing \( U^q_p = U^q_{(p \ominus q) \oplus p} = d_p L_{q \ominus p} \) and \( U^q_p = U^q_{(p \ominus q) \oplus q} = d_q L_{p \ominus q} \). Then it is clear that \[ 1 = d_p (I \ominus q) = d_p L_{(p \ominus q) \ominus (q \ominus p)} = d_p (L_{p \ominus q} L_{q \ominus p}) = d_q L_{p \ominus q} d_p L_{q \ominus p} = U^q_p U^q_p. \] The calculation for \( V \) is entirely analogous. For \( I \) note that \( 1 = d_p (I \ominus q) = d_p (\ominus p) = d_{p \ominus q} \ominus d_p \ominus = I^{\ominus p} I^p. \)

2. Relation between \( U, V, I \):

\[ -U^0_0 \ominus I^p = V^0_0. \quad (26) \]

The fact that \( (\ominus p) \oplus (p \ominus q) = q \) implies that \( d_p ((\ominus p) \oplus (p \ominus q)) = 0 \). We can express this in terms of the parallel transport operators as

\[ 0 = d_p ((\ominus p) \oplus (p \ominus q)) = [d_p ((\ominus p') \oplus (p \ominus q)) + d_p ((\ominus p) \oplus (p' \ominus q))]|_{p=p'} = [d_p (L_{p' \ominus q} R_q) + d_p (R_{p' \ominus q} \ominus)]|_{p=p'} = d_{R_{p \ominus q} L_{p'} \ominus} \cdot d_{p \ominus q} \cdot d_p \ominus = U^q_{q \ominus q} U^p_{p \ominus q} + V^q_{q \ominus q} I^q_{p \ominus q}. \quad (27) \]

Fixing \( q = 0 \), we then get

\[ U^q_0 V^p + V^q_0 \ominus I^p = 0. \quad (28) \]

In point 1 we showed that \( V^p_0 = 1 \) and \( (I^p)^{-1} = I^{\ominus p} \), so applying \( I^{\ominus p} \) to the equation and renaming \( p \to \ominus p \), we get the final result.

---

3. \( \mathcal{E}(\mathcal{P}) \equiv \text{diffeomorphisms in } \mathcal{P} \)

4. We use the property \( \ominus (p \ominus q) = q \ominus p \). Proof is as follows: \( \ominus (p \ominus q) = q \ominus p \iff q \ominus (p \ominus q) = q \ominus (q \ominus (p \ominus q)) = (q \ominus (p \ominus q)) \ominus (p \ominus q) = \ominus p \ominus (p \ominus q) \iff q = p \).

5. Note that \( \ominus : \mathcal{P} \to \mathcal{P}, \ p \to \ominus p \in \mathcal{E}(\mathcal{P}) \) is its own inverse of since \( \ominus (\ominus p) = p \).
The bivalent vertex

To better understand the relative locality formalism, we will consider the simplest example of an “interaction” vertex: propagation of a free particle. There are two momenta in the corresponding diagram - one incoming and one outgoing, as Figure 4 illustrates:

![Figure 4: The bivalent vertex.](image)

Physically, we do not expect relative locality to introduce any effects in this situation, since any worldline of a free particle can be broken into segments separated by bivalent vertices, introducing momentum constraints in each one - and we do not expect the different worldline segments created by this process to be in different spacetimes. Let us verify this in practice by computing the relative locality relations. Write the conservation rule as

\[ \mathcal{K} = p \oplus q = 0. \]  

The relative locality operators are

\[ W_{xp} = d_p \mathcal{K} = d_p R_{\oplus q} = V^p_{p \oplus q} = V^p_0 = V^q_0, \]  
\[ W_{xq} = -d_q \mathcal{K} = -d_q (L_p \ominus q) = -d_{\ominus q} L_p \ominus d_q \ominus = -U^\ominus q I^q = -U^\ominus q I^q, \]

where we used the momentum constraint to simplify results. Using (26), we get \( W_{xq} = W_{xp} \), so that the worldline endpoint coordinates are

\[ x^a_p = x^a_q = z^b (V^q_0)^a_b, \]

and the “interaction” is local to every observer - the only effect of a coordinate change is to alter its location with respect to the chosen frame, as one would expect in ordinary Special Relativity.

Much more interesting is the case of the trivalent vertex, which can represent physical situations such as the decay of one particle into two or the fusion of two particles. This is the simplest situation where we expect relative locality to introduce novel effects, and it will be the primary kind of vertex studied in the following sections.

3.2 The change of vertex problem

A generic momentum constraint \( \mathcal{K} \) can naively be written as

\[ \mathcal{K}_b = \mathcal{N} \bigoplus k^j_b \]

generalizing the usual linear conservation laws. However, because the addition rule is in general non-commutative and non-associative, \([33]\) is ambiguous: while in the linear case the ordering of additions is naturally irrelevant, in this more general setting it changes the form of the constraint. It is then important to evaluate if these changes affect the physics of the interaction, and in the event they do, determine a consistent, physical way of choosing vertex orderings. A bivalent vertex has two possible ways of writing the momentum constraint, \( \mathcal{K} = p \oplus q \) or \( \mathcal{K} = q \oplus p \), while a trivalent vertex has 12, corresponding to all the different permutations and operation orderings of the 3 momenta (if they are all regarded as incoming for simplicity, some of the possibilities are \( \mathcal{K}^{(1)} = (p \oplus q) \oplus k, \)
\[ \mathcal{K}^{(2)} = p \oplus (q \oplus k), \mathcal{K}^{(3)} = p \oplus (k \oplus q) \ldots \]

Notice that changing the form of writing a single momentum constraint corresponds effectively to applying a
diffeomorphism \( \lambda \) on the original one. For example, \( \mathcal{K}^{(2)} = R_{q\oplus k}R_{\oplus q}R_{\oplus k}\mathcal{K}^{(1)} \) and \( \mathcal{K}^{(3)} = R_{k\oplus q}R_{\oplus k\oplus q}\mathcal{K}^{(2)} \). The diffeomorphism that relates two different constraints is not unique - we could also have written \( \mathcal{K}^{(2)} = L_pL_qL_{\oplus p}\mathcal{K}^{(1)} \).

If we proceed under the assumption there should be at least a class of vertex changes that does not affect the physics of the interaction, we should try to find something analogous to a \textit{gauge transformation} on our system based on the above considerations, i.e. a redundancy in the definition of momentum constraints and vertex coordinates that does not affect the equations of motion. Indeed, there is such a transformation, as described originally in [7]:

\[ \mathcal{K}_{\alpha}^{(a)} \rightarrow \lambda_{\alpha}^{(a)} \left( \mathcal{K}_{\alpha}^{(a)} \right); \quad z_{\alpha}^{a} \rightarrow z_{\alpha}^{b} \frac{\partial K_{\alpha}^{(a)}}{\partial \lambda_{\alpha}^{(a)}} \quad \lambda^{(a)} \in \mathcal{E}(\mathcal{P}), \lambda^{(a)}(0) = 0 \quad (34) \]

This rewriting neither alters the constraints, since \( \mathcal{K}^{(a)} = 0 \Leftrightarrow \lambda^{(a)} \left( \mathcal{K}^{(a)} \right) = 0 \), nor affects the worldline endpoint coordinates given by the relative locality relations:

\[ x_{I,\alpha} = z_{\alpha}^{b} \left( \pm \frac{\delta K_{\beta}^{(a)}}{\delta k_{\alpha}^{a}} \right) \rightarrow z_{\alpha}^{c} \frac{\partial K_{c}^{(a)}}{\partial \lambda_{\alpha}^{(a)}} \cdot \left( \pm \frac{\delta \lambda_{b}^{(a)} \left( \mathcal{K}_{\alpha}^{(a)} \right)}{\delta k_{\alpha}^{a}} \right) \]

\[ = z_{\alpha}^{c} \left( \pm \frac{\delta K_{c}^{(a)}}{\delta k_{\alpha}^{a}} \right) = x_{I,\alpha} \quad (35) \]

It is clear that the physics of an interaction is unaffected by the map above, which means we can regard it as a
gauge symmetry of the theory.

As an application of this idea, consider once again the bivalent vertex. In Section 3.1 we used the momentum
constraint \( \mathcal{K}^{(1)} = p \ominus q \). Let us see what happens if we use \( \mathcal{K}^{(2)} = q \ominus p \). The relative locality operators are then

\[ W^{(2)}_{x_p} = U_0^{\ominus p} p = -V_0^p = -V_0^q \quad \text{and} \quad W^{(2)}_{x_q} = -V_0^q, \quad (36) \]

so that the worldline endpoint coordinates are given by

\[ x_{p}^{a} = x_{q}^{a} = \left( -z^{b} \right) (V_0^q)^a_b \equiv z^{b} (V_0^q)_b^a. \quad (37) \]

This can be described as the result of a gauge transformation if we note that \( \mathcal{K}^{(2)} = R_{\ominus p} \ominus \cdot L_{\ominus p}\mathcal{K}^{(1)} \equiv \lambda \left( \mathcal{K}^{(1)} \right) \) and calculate

\[ z' = z \frac{\partial \mathcal{K}^{(1)}}{\partial \lambda} = z \left( \frac{\partial \lambda}{\partial \mathcal{K}^{(1)}} \right)^{-1} = z \left[ d_0 \left( R_{\ominus p} \ominus \cdot L_{\ominus p} \right) \right]^{-1} \]

\[ = z \left( V_0^p U_0^{\ominus p} \right)^{-1} = -z \left( U_0^{\ominus p} V_0^p \right)^{-1} \]

\[ = -z. \quad (38) \]

We have seen a simple example of how a certain class of vertex changes do not affect the interaction physics, since it
can actually be matched to a redundancy in the definition of the vertex coordinates together with the conservation
laws. However, notice that not all diffeomorphisms corresponding to vertex changes obey the condition \( \lambda(0) = 0 \); an
example was given above, $R_{k\otimes q}R_{k\otimes q}(0) = (\otimes k \otimes q) \oplus (k \otimes q) \neq 0$ unless $\otimes$ is commutative. These cases motivate further study, and some concrete examples will be given in the next section.

### 3.3 Illustration: gamma ray bursts

A typical gamma ray burst event consists of a high-energy gamma ray emission followed by an “afterglow” of lower energy radiation (typically X-rays)\cite{13}. It has been argued in \cite{2} that observational measurements of phenomena of this kind can serve as a detector of non-metricity and/or torsion of momentum space, via the time delay between the detection of two photons of different energies emitted by the same source and the angle deflection relative to the direction of the sources (dual gravitational lensing), respectively. We will redo the calculation of \cite{2} in a slightly different way, to make transparent where a change of vertex could alter the form of the results and/or the physics, and we will consider three possible changes in the form of the momentum constraints and reflect on what the similarities and differences between the calculations in each one are.

The idealized description of the gamma ray burst is summarized in Figure 5. The “experimental set-up” consists of an emitter and a detector, represented by particles with masses $m_1$ and $m_2$ respectively. The emitter sends a photon with energy $E_1$, and, after a proper time $S_1$, it sends another photon with energy $E_2 \neq E_1$. The light rays take proper times $T_1$ and $T_2$ respectively to reach the detector, and the proper time interval between the two detections is $S_2$.

![Figure 5: Notation of positions and momenta in the gamma ray burst setup \cite{2}]

Special Relativity predicts that, assuming the emitter and the detector are at rest with respect to each other, $\frac{k^1}{m_1} = \frac{k^2}{m_2}$, the relation between the intervals of emission and detection is the intuitive one, $S_1 = S_2$. However, working in the relative locality formalism it is possible to have a time delay between the two events, depending on the energies of the photons sent, as we will see.
The derivation of the time delay $\Delta S = S_2 - S_1$ begins with noting that spacetime metric is Minkowski (since we are still working on the $G_N \to 0$, $\hbar \to 0$ limit), so the equation of motion for the spacetime worldlines is simply (picking the $N_I$ appropriately) $\dot{x}_n^n = g^{ab}k^b_i \equiv k^b_i$, and it is then possible to establish kinematic relations between the momenta and the proper times of photon propagation, which are the invariant quantities that we need to calculate $\Delta S$:

$$\begin{align*}
x_2 - u_1 &= \hat{k}_i S_i; \\
y_3 - y_1 &= \hat{p}_i T_i;
\end{align*}$$

where $\hat{k}_i = \frac{k_i}{m_i}$ and $\hat{p}_i = \frac{p_i}{E_i}$, $\forall i$.  

The next step is to use (39) to construct a series of identities relating the proper times $S_i, T_i$ by taking two different routes from $z_1$ to $z_4$, forming a loop in spacetime. We would like to eliminate the interaction coordinates from these identities, so that the expression obtained only includes diffeomorphism-covariant quantities. We use the relative locality relations $x_{I,\alpha} = z_{(\alpha)} W_{x_I}$ to obtain

$$T_1 \hat{p}_1 W_{y_3}^{-1} = (y_3 - y_1) W_{y_3}^{-1} = z_3 - z_1 W_{y_1} W_{y_3}^{-1}$$  (40)

$$S_2 \hat{k}_2 W_{u_3}^{-1} = (x_4 - u_3) W_{u_3}^{-1} = z_4 W_{x_4} W_{u_3}^{-1} - z_3$$  (41)

Eliminate $z_3$ from these to get

$$T_1 \hat{p}_1 W_{y_3}^{-1} + S_2 \hat{k}_2 W_{u_3}^{-1} = z_4 W_{x_4} W_{u_3}^{-1} - z_1 W_{y_1} W_{y_3}^{-1}$$  (42)

Similarly, eliminating $z_2$ from the identities

$$T_2 \hat{p}_2 W_{y_4}^{-1} = (y_4 - y_2) W_{y_4}^{-1} = z_4 - z_2 W_{y_2} W_{y_4}^{-1} \leftrightarrow T_2 \hat{p}_2 W_{y_2}^{-1} = z_4 W_{y_4} W_{y_2}^{-1} - z_2$$  (43)

$$S_1 \hat{k}_1 W_{x_2}^{-1} = z_2 - z_1 W_{x_1} W_{x_2}^{-1}$$  (44)

we get

$$S_1 \hat{k}_1 W_{x_2}^{-1} + T_2 \hat{p}_2 W_{y_2}^{-1} = z_4 W_{y_4} W_{y_2}^{-1} - z_1 W_{x_1} W_{x_2}^{-1}$$  (45)

(42) and (45) involve only $z_4$ and $z_1$. Eliminate $z_4$ to obtain

$$z_1 \left( W_{u_1} W_{x_2}^{-1} W_{y_2} W_{y_4}^{-1} - W_{y_1} W_{x_4}^{-1} W_{u_1} W_{x_4}^{-1} \right) = S_2 K_2 - S_1 K_1 - T_2 P_2 + T_1 P_1$$  (46)

where we defined the auxiliary momenta

$$\begin{align*}
K_1 &\equiv \hat{k}_1 W_{y_2}^{-1} W_{y_4}^{-1} \\
K_2 &\equiv \hat{k}_2 W_{x_4}^{-1} \\
P_1 &\equiv \hat{p}_1 W_{y_3}^{-1} W_{u_3} W_{x_4}^{-1} \\
P_2 &\equiv \hat{p}_2 W_{y_4}^{-1}
\end{align*}$$

(46) is the key equation that will allow us to derive the time delay, since if its LHS vanishes, we have successfully derived a relation between the invariant proper times and diffeomorphism-covariant momenta.
We will now specify momentum constraints and calculate the relevant quantities for our study for each of our choices. For simplicity, we will only do computations to first order in the momenta. Definitions (7), (22), (23) and (24) allow us to expand the parallel transport operators:

\[
(U_{p_2}^{p_1})_a^b = \delta_a^b - \Gamma_a^{bc}(p_2 - p_1)_c + O(p_2 - p_1)^2
\]
\[
(V_{p_2}^{p_1})_a^b = \delta_a^b - \Gamma_a^{bc}(p_2 - p_1)_c + O(p_2 - p_1)^2
\]
\[
(P)^b_a = -\delta_a^b - (\Gamma^b_{ac} + \Gamma^b_{a})p_c + O(p^2)
\] (48)

3.3.1 Specifying momentum constraints

The choice of conservation laws in [2] is

\[
\mathcal{K}^1 = (q^1 \otimes k^1) \otimes p^1 = 0 \quad \mathcal{K}^2 = (k^1 \otimes r^1) \otimes p^2 = 0 \quad \mathcal{K}^3 = p^1 \otimes (\otimes k^2 + q^2) = 0 \quad \mathcal{K}^4 = p^2 \otimes (\otimes r^2 + k^2) = 0
\] (49)

from which we can compute the relative locality operators:

\[
W_{x_1} = V_0^{p^1} V_{p^1}
\]
\[
W_{x_2} = V_0^{p^2} V_{p^2}
\]
\[
W_{x_3} = U_0^{p^1} U_{p^2}
\]
\[
W_{x_4} = U_0^{p^2} U_{p^1}
\] (50)

Performing the 1st order expansions of these operators using (48) and computing the quantities in (46), it was found in [2] that

- \((W_{x_1}, W_{x_2}, W_{x_3}, W_{x_4})_a^b \approx \delta_a^b + T_{a}^{bc}p_c \approx W_{y_1} W_{y_2} W_{y_3} W_{y_4}, so that the LHS of (46) conveniently vanishes to 1st order;

- The momenta (47) have the approximate expressions

\[
K_1^b \approx \hat{k}_1^b - m_1 k_1^0 \hat{k}_1^b \approx \hat{k}_1^0 \hat{V}_k^0
\]
\[
K_2^b \approx \hat{k}_2^b - m_2 k_2^0 \hat{k}_2^b \approx \hat{k}_2^0 \hat{V}_k^0
\]
\[
P_1^b \approx \hat{p}_1^b - E_1 \hat{p}_1 \Gamma_1^b \hat{p}_1^b \approx \hat{p}_1^0 \hat{V}_p^0
\]
\[
P_2^b \approx \hat{p}_2^b - E_2 \hat{p}_2 \Gamma_2^b \hat{p}_2^b \approx \hat{p}_2^0 \hat{V}_p^0
\] (51)

where we observe that each uppercase momentum depends only on the corresponding lowercase momentum. The approximate expressions in terms of \(U\) and \(V\) mean that the uppercase momenta can be interpreted as parallel transports of the lowercase ones from \(T_{p_i,k_i}^*,\) back to \(T_0^* P\).

The paper [2] goes on to relate the existence of non-metricity to a nonzero time delay through the first order version of (16), \(K_2 S_2 - K_1 S_1 = P_2 T_2 - P_1 T_1.\) Working in Riemann normal coordinates at the origin of momentum space, so that the metric near 0 is \(g^{ab}(k) \approx g^{ab}, |k| \ll 1,\) and supposing that the torsion vanishes, the connection is entirely given by the non-metricity portion in (12) and the norms of the uppercase momenta can be calculated:

\[
K_1^2 \approx -1 + m_1 N^{abc} \hat{k}_1^a \hat{k}_1^b \hat{k}_1^c; \quad P_1^2 \approx E_1 N^{abc} \hat{p}_1^a \hat{p}_1^b \hat{p}_1^c
\] (52)
We will now rework the problem with the following set of momentum constraints:

3.3.2 Specifying momentum constraints - 1st alternative

time delay. To first order in the momenta, this can only happen if there is non-metricity present, as shown by (52).

ray burst, when usually one of them has a much higher energy than the other, it is possible to obtain a measurable

hand, if parallel to each other - define $\hat{K} = \hat{K}_1 = \hat{K}_2$. (46) then resolves to

\[ \hat{K}\Delta S \equiv \hat{K} (|K_2| S_2 - |K_1| S_1) = T_2 P_2 - T_1 P_1 \]  \hspace{1cm} (53)

We can now decompose the photon momenta in a component parallel to $\hat{K}$ and a component parallel to unit spacelike vectors $R_i \perp \hat{K}$,

\[ P_i = \left( \hat{K} \cdot P_i \right) \hat{K} + \sqrt{\left( \hat{K} \cdot P_i \right)^2 - P_i^2} R_i. \]  \hspace{1cm} (54)

Notice that taking the scalar product of (53) with $\hat{K}$ we get the formula for the time delay,

\[ \Delta S = \left( \hat{K} \cdot P_2 \right) T_2 - \left( \hat{K} \cdot P_1 \right) T_1, \]  \hspace{1cm} (55)

and taking the scalar product of (53) with $\hat{R}_i$ we get $(T_2 P_2 - T_1 P_1) \cdot R_i = 0$, $i \in \{1, 2\}$, so that under the assumptions taken for $\hat{R}_i$ we obtain $\hat{R}_1 = \hat{R}_2$. These two equations imply

\[ T_2 \sqrt{\left( \hat{K} \cdot P_2 \right)^2 - P_2^2} - T_1 \sqrt{\left( \hat{K} \cdot P_1 \right)^2 - P_1^2} = 0. \]  \hspace{1cm} (56)

Computing (53) - (56) we obtain the expression for $\Delta S$ in a form that makes it easier to examine:

\[ \Delta S = T_2 \left[ \left( \hat{K} \cdot P_2 \right) - \sqrt{\left( \hat{K} \cdot P_2 \right)^2 - P_2^2} \right] - T_1 \left[ \left( \hat{K} \cdot P_1 \right) - \sqrt{\left( \hat{K} \cdot P_1 \right)^2 - P_1^2} \right], \]  \hspace{1cm} (57)

since it is now evident that if the photons are null ($P_i^2 = 0$), $\Delta S = 0$, so there is no time delay. On the other hand, if $P_i^2 \neq 0$, $\Delta S$ depends nontrivially in the photons’ energies, and in the experimental situation of a gamma ray burst, when usually one of them has a much higher energy than the other, it is possible to obtain a measurable time delay. To first order in the momenta, this can only happen if there is non-metricity present, as shown by (52).

3.3.2 Specifying momentum constraints - 1st alternative

We will now rework the problem with the following set of momentum constraints:

\[ \mathcal{K}^1 = \ominus p^1 \ominus (q^1 \ominus k^1) = 0 \hspace{1cm} \mathcal{K}^2 = \ominus p^2 \ominus (k^1 \ominus r^1) = 0 \]  \hspace{1cm} (58)

\[ \mathcal{K}^3 = p^1 \ominus (\ominus k^2 \ominus q^2) = 0 \hspace{1cm} \mathcal{K}^4 = p^2 \ominus (\ominus r^2 \ominus k^2) = 0 \]

Only the first two are changed through the diffeomorphisms $\mathcal{K}^i = L_{\ominus p^i} R_{\frac{\lambda}{\ominus p^i}} \mathcal{K}^i_{(\ominus)} \equiv \lambda^i \mathcal{K}^i_{(\ominus)}$. The relative locality operators corresponding to the altered constraints are

\[ W_{x_1} = U_{0}^{2} V_{p^1}^{1} \hspace{1cm} W_{u_1} = -U_{0}^{2} V_{p^1}^{1} T^{k^1} \hspace{1cm} W_{y_1} = U_{0}^{2} \]

\[ W_{x_2} = U_{0}^{2} V_{p^2}^{1} \hspace{1cm} W_{u_2} = -U_{0}^{2} V_{p^2}^{1} T^{r^1} \hspace{1cm} W_{y_2} = U_{0}^{2} \]  \hspace{1cm} (59)

which all obey similar relations to the original ones, $W_{A_1} = U_{p^1}^{2} V_{p^1}^{0} W_{A_{(1)}}$, $A \in \{x, u, y\}$. We can then interpret the change in the relative locality relations for these worldlines using gauge transformations, where the interaction coordinates change as

\[ z_i' = z_i \left( \frac{\partial \lambda^i}{\partial \mathcal{K}^i_{(\ominus)}} \right)^{-1} = z_i d_0 \left( L_{\ominus p^i} R_{p^i} \right) = z_i U_{0}^{p^i} V_{p^i}^{0}, \]  \hspace{1cm} (60)
agreeing with the changes in $W_{A_t}$. When can we be sure that gauge transformations such as this one do not affect the physics? They tell us that after a change of vertex, one can obtain the same relative locality relations given by the original vertex at the cost of redefining the interaction coordinates $z_{(a)}$. So a physical relation can only be affected if it depends on $z_{(a)}$, which is the case in our fundamental equation (46). Luckily, to 1st order in momenta for the original momentum constraints, this dependence vanishes, so the physics of the problem should be unaltered by any modification in the constraints that preserves this cancellation.

For our first attempt at a vertex change, 1st order calculations show that the cancellation is indeed preserved - we see that $(W_{u_1}W_{x_2}^{-1}W_{y_2}W_{y_1}^{-1})^b_a \approx \delta^b_a \approx (W_{y_1}W_{y_3}W_{u_3}W_{x_4}^{-1})^b_a$ so that the LHS of (46) still vanishes to 1st order, and the expressions (51) for the “uppercase momenta” remain valid (an equality can be readily verified to all orders) - so (46) has exactly the same form.

### 3.3.3 Specifying momentum constraints - 2nd alternative

We will now consider this set of conservation laws:

\[
\begin{align*}
\mathcal{K}^1 &= \left( \otimes k^1 \oplus q^1 \right) \otimes p^1 = 0 \\
\mathcal{K}^2 &= \left( \otimes r^1 \oplus k^1 \right) \otimes p^2 = 0 \\
\mathcal{K}^3 &= p^1 \oplus (q^2 \otimes k^2) = 0 \\
\mathcal{K}^4 &= p^2 \oplus (k^2 \otimes r^2) = 0
\end{align*}
\]

All are modified in the same way with respect to the originals - the order of addition within brackets is reversed.

Computing the relative locality operators:

\[
\begin{align*}
W_{x_1} &= V_0^p U_p^q \\
W_{x_2} &= V_0^p U_p^k \\
W_{x_3} &= U_0^p V_p^q \\
W_{x_4} &= U_0^p V_p^k
\end{align*}
\]

\[
\begin{align*}
W_{u_1} &= -V_0^p V_p^k I_{k^1} \\
W_{u_2} &= -V_0^p V_p^r r^1 \\
W_{u_3} &= -U_0^p U_p^k I_{k^2} \\
W_{u_4} &= -U_0^p U_p^r r^2
\end{align*}
\]

\[
\begin{align*}
W_{y_1} &= V_0^p \\
W_{y_2} &= V_0^p \\
W_{y_3} &= V_0^p \\
W_{y_4} &= V_0^p
\end{align*}
\]

The diffeomorphisms relating the new momentum constraints to the original ones are

\[
\begin{align*}
\lambda^1 &= L_{\otimes k^1 \oplus q^1} L_{k^1 \oplus q^1} \\
\lambda^2 &= L_{\otimes r^1 \oplus k^1} L_{r^1 \oplus k^1} \\
\lambda^3 &= R_{q^2 \otimes k^2} R_{q^2 \otimes k^2} \\
\lambda^4 &= R_{k^2 \otimes r^2} R_{k^2 \otimes r^2}
\end{align*}
\]

We cannot, however, apply the same gauge formalism as before, because these diffeomorphisms do not respect $\lambda^i(0) = 0$. They correspond to legitimate changes in the conservation law imposed: $\mathcal{K}^{(1)} = 0 \leftrightarrow p^1 = \otimes k^1 \oplus q^1$ but $\mathcal{K}^{(2)} = 0 \leftrightarrow p^1 = q^1 \oplus k^1$, which is obviously different if $\oplus$ is not commutative. The fact that this is not a mere gauge transformation is further supported by the fact that the $W_{A_t}$’s relations with the $W_{(orig)}^{(orig)}$ are dependent on $A$: for example $W_{x_1} = W_{x_1}^{(orig)} V_{q^1} U_{p^1}$, $W_{u_1} = W_{u_1}^{(orig)} I_{k^1} U_{p^1} V_{k^1}$ and $W_{y_1} = W_{y_1}^{(orig)}$.

Carrying on with the 1st order computation of (46), though, we verify that $(W_{u_1}W_{x_2}^{-1}W_{y_2}W_{y_1}^{-1})^b_a \approx \delta^b_a - T_{a\beta} T_{\beta}^2 \approx (W_{y_1}W_{y_3}W_{u_3}W_{x_4}^{-1})^b_a$ so that the LHS vanishes once again, and the new uppercase momenta are given by
We can multiply (46) by \( p \) where 3.3.4 Specifying momentum constraints - 3rd alternative same result as the paper does. parallel transported momenta: is for the original time delay calculation we can still carry on with the derivation in a covariant way. Computing the however, since the offending term is also proportional to the torsion, in the torsion-free momentum space considered result in terms of diffeomorphism-covariant quantities only; there appears to be a legitimate change in the physics the calculation we followed in 3.3.1 to obtain the delay would not (in the full case with torsion) allow us to obtain a we get a term the most remarkable result of computing the time delay equation: we obtain all the corresponding relative locality operators have been computed in the previous sections, so we will just state changes considered alter the physics. Indeed, this statement can be generalized to any vertex change - to this order, momentum addition is associative, and without torsion it is also commutative, which means all forms of writing \( K_b = \sum_{j=1}^{N} k_j^b = 0 \) become equivalent.

3.3.4 Specifying momentum constraints - 3rd alternative

We will now consider a similar modification of momentum constraints as in 3.3.3, but only commuting the momenta within brackets in the first two:

\[
\begin{align*}
\mathcal{K}^1 &= (q^1 \otimes k^1) \otimes p^1 = 0 \\
\mathcal{K}^3 &= p^1 \otimes (q^2 \otimes k^2) = 0 \\
\mathcal{K}^2 &= (k^1 \otimes r^1) \otimes p^2 = 0 \\
\mathcal{K}^4 &= p^2 \otimes (k^2 \otimes r^2) = 0
\end{align*}
\]

All the corresponding relative locality operators have been computed in the previous sections, so we will just state the most remarkable result of computing the time delay equation: we obtain \( (W_{a_1} W_{a_2}^{-1} W_{a_3} W_{a_4}^{-1})_a \approx \delta^b_a + T_{a}^{bc} p_c \) but \( (W_{a_1} W_{b_1}^{-1} W_{a_2} W_{b_2}^{-1})_a \approx \delta^b_a - T_{a}^{bc} p_c \), so the LHS of (66) does not vanish to 1st order in the momenta. Instead we get a term \( z_1 T_{a}^{bc} (p^1 + p^2) p_c \), which depends on an interaction coordinate. This is problematic, because it means the calculation we followed in 3.3.1 to obtain the delay would not (in the full case with torsion) allow us to obtain a result in terms of diffeomorphism-covariant quantities only; there appears to be a legitimate change in the physics following the change of vertex in this case.

However, since the offending term is also proportional to the torsion, in the torsion-free momentum space considered for the original time delay calculation we can still carry on with the derivation in a covariant way. Computing the parallel transported momenta is:

\[
\begin{align*}
K_1^b &\approx \hat{k}_1^b - m_1 \hat{k}_1^a \Gamma_{a}^{bc} \hat{k}_c^1 \approx \left( \hat{k}_1 U^0_1 \right)^b \\
K_2^b &\approx \hat{k}_2^b - m_2 \hat{k}_2^a \Gamma_{a}^{bc} \hat{k}_c^2 - E_2 \hat{k}_2^a T_{a}^{bc} \hat{p}_c^2 \approx \left( \hat{k}_2 V^0_2 U^p_0 V^0_{p_2} \right)^b \\
P_1^b &\approx \hat{p}_1^b - E_1 \hat{p}_1^a \Gamma_{a}^{bc} \hat{p}_c^1 \approx \left( \hat{p}_1 V^0_1 U^p_0 V^0_{p_2} \right)^b \\
P_2^b &\approx \hat{p}_2^b - E_2 \hat{p}_2^a \Gamma_{a}^{bc} \hat{p}_c^2 \approx \left( \hat{p}_2 V^0_2 U^p_0 V^0_{p_2} \right)^b
\end{align*}
\]

Again, the difference between these and the ones resulting from the original choice of momentum constraints is proportional to the torsion. Thus we find that to first order, if momentum space is torsion-free, none of the vertex changes considered alter the physics. Indeed, this statement can be generalized to any vertex change - to this order, momentum addition is associative, and without torsion it is also commutative, which means all forms of writing \( K_b = \sum_{j=1}^{N} k_j^b = 0 \) become equivalent.
4 Multiparticle systems coupled to 2+1 gravity

In this section we will outline the known picture of the phase space of a model of $N$ point particles coupled to 2+1 Einstein gravity, which is described in detail in [8]. In the following, $\mathcal{M}$ is the spacetime manifold, and index notation is as follows: $a, b, c...$ are tangent space indices; $i, j...$ are Lie algebra indices and $\pi_1, \pi_2...$ are particle labels.

4.1 Phase space of multiple point particle systems

The starting point to understanding the construction of the multiparticle phase space is the result shown in [10] for the spacetime metric induced by a point particle of mass $m$ and spin $s$:

$$ds^2 = -(dt + 4Gs \, d\phi)^2 + dr^2 + r^2 [(1 - 4Gm) d\phi]^2$$

(67)

Physically, the geometry represented by this metric is flat (as can be seen from computing the Riemann tensor), but with a time offset of $8\pi Gs$ and a conical deficit angle of $8\pi Gm$, which means that, even though there is no local gravitational force acting on the particles, they still interact with each other due to the topological effects, a peculiar characteristic of this model. This geometry also imposes a maximum bound on the total mass of the system: the sum of all deficit angles cannot exceed $2\pi$, so we have that $M = \sum_{\pi=1}^{N} m_\pi < \frac{1}{4G}$.

![Figure 6: Geometry of the 2+1 multiparticle system coupled to gravity.](image)

We use the first-order formalism of Einstein gravity (see Appendix for a brief review; [9] for a more detailed one) with the frame fields and the spin connection as dynamical variables, which suggests an infinite dimensional phase space. However, this space can be reduced via application of constraints and quotienting out of gauge symmetries, to an extent where it is actually finite-dimensional, with a set of dynamical variables that are essentially the positions and momenta of the point particles. Throughout, we will work in the ADM formalism [11] for gravity, by splitting $\mathcal{M} \approx \mathbb{R} \times \Sigma$, where $\Sigma$ has the topology of $\mathbb{R}^2$ with $N$ “holes” cut out, corresponding to the particles.

The need to redefine phase space variables

To better understand the formalism at hand we will start by considering the problem without gravity. We then have a system of free particles, which can be parameterized by the variables $x_\pi = x_\pi^i \gamma_i$ and $p_\pi = p_\pi^i \gamma_i$ (using the isomorphism $\mathfrak{M}^{2,1} \approx sl(2)$) satisfying the Poisson brackets $\{p_\pi^i, x_\pi^j\} = \eta^{ij}$. The phase space is $6N$-dimensional, and the physical phase space is the submanifold defined by the mass shell and positive energy conditions:

$$\frac{1}{2} Tr (p_\pi^2) = -m_\pi^2, \quad p_\pi^0 = \frac{1}{2} Tr (p_\pi^2) > 0.$$  

(68)
The evolution equations are similar to those derived in Section 3.1, but from the traditional spacetime perspective,

\[ \dot{p}_\pi = 0, \quad \dot{x}_\pi = N_\pi p_\pi. \]  

(69)

Dots represent derivatives with respect to ADM time, which can of course be freely reparameterized. This is the gauge freedom of the system, which means we can fix \( N_\pi \) to be whatever we want. However, there is a gauge restriction to be imposed: that at a given ADM time \( t \), all particles are located in the same spacelike surface \( \Sigma(t) \), which translates to

\[ x_{\pi_2} - x_{\pi_1} \text{ is spacelike } \forall \pi_1, \pi_2 \in \{1, \ldots, N\}. \]  

(70)

We are still left with the Poincaré symmetry, which can be partially gauged away by moving to the center of mass frame (suppose such a frame exists by excluding the case where all the particles are massless), characterized by the expressions for total momentum and total angular momentum of the system

\[ P = \sum_\pi p_\pi = M_\gamma 0, \quad J = \frac{1}{2} \sum_\pi [p_\pi, x_\pi] = S_\gamma 0, \]  

(71)

leaving only time translations and spatial rotations as ungauged symmetries, and a phase space of dimension \( 6N - 4 \). However, this raises a problem. The constraints (71) are second class, which is undesirable for a Hamiltonian description. One way of solving the problem is to consider a different picture of phase space for which (71) become 1st class constraints. This is done by reparameterizing their solutions. The convenient reparameterization is the triangulation picture, which we will describe first for the model without gravity and then adapt to the gravity model.

**Triangulations**

Consider the foliation of spacetime given by (70), \( \mathcal{M} = \mathbb{R} \times \Sigma(t) \). A **triangulation** \( \Gamma \) is defined by a set of oriented links between particles, \( \lambda \), and links from particles to infinity, \( \eta \), which split the ADM surface into finite polygons \( \Delta_\lambda \) and polygons with a vertex at infinity, \( \Delta_\eta \). The preferred orientation for the links is chosen *a priori* and all particles are connected by links in both directions, so that the triangulation is split in the sets of positive-oriented, negative-oriented and infinity-bound (external) links, \( \Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_\infty = \Gamma_0 \cup \Gamma_\infty \). Also define the set of links that *end* at a particle \( \pi \), \( \Gamma_\pi \), and the set of links that *begin* at \( \pi \), \( \Gamma_{-\pi} \). The particle in which \( \lambda \) ends is called \( \pi_\lambda \), and the particle in which \( \lambda \) begins is called \( \pi_{-\lambda} \).

![Figure 7: Triangulation of a Minkowski ADM surface.](image)
We define the relative momenta $q_\lambda$, related to the particles’ momenta by the equations

$$ p_\pi = \sum_{\lambda \in \Gamma_\pi} q_\lambda, \quad q_{-\lambda} = -q_\lambda, \quad \lambda \in \Gamma_0, \quad (72) $$

the relative positions $z_\lambda$ given by

$$ z_\lambda = x_{\pi_\lambda} - x_{\pi_{-\lambda}}, \quad (73) $$

and the variables associated with external links $6$

$$ q_\eta = -M_\eta \gamma_0, \quad z_\eta = \gamma (\phi_\eta), \quad T_\eta = x_{\pi_{-\lambda}}^0, \quad \eta \in \Gamma_\infty, \quad (74) $$

where $\phi_\eta$ is the polar angle of $\eta$ and the $T_\eta$ represent the time coordinates of particles at the ends of the external links as measured by a clock at infinity.

With this setup, the CM frame constraints (71) are automatically solved if

$$ \sum_{\eta \in \Gamma_\infty} M_\eta = M, \quad \sum_{\lambda \in \Gamma_+} L_\lambda = \sum_{\lambda \in \Gamma_+} \frac{1}{4} Tr ([q_\lambda, z_\lambda] \gamma_0) = S. \quad (75) $$

(75) are first class - the primary goal of the triangulation procedure. It is important to notice the link variable definitions have in them several redundancies, which can be accounted for as extra kinematical constraints. The end result is that the particles’ momenta in the center of mass frame can be completely specified in terms of link variables, and while absolute positions can only be derived up to an overall translation from the relative ones, the angular momentum constraint fixes them up to a time translation $x_\pi \rightarrow x_\pi + \tau \gamma_0$ and rotations of the external links $\phi_\eta \rightarrow \phi_\eta + \epsilon_\eta$ that can be achieved by smooth deformation of the ADM surface.

Introducing the kinematical and dynamical constraints into the ordinary free-particle Hamiltonian, the resulting nonzero Poisson brackets of the system are

$$ \{T_\eta, M_\eta\} = 1 \quad \{L_\eta, \phi_\eta\} = 1 \quad \{q^i_\lambda, z^j_{-\lambda}\} = -\eta^{ij}, \quad \{q^i_\lambda, z^j_\lambda\} = \eta^{ij} \quad (76) $$

Phase space in the presence of gravity

The biggest geometrical difference when applying the above formalism to gravity is the presence of the conical deficits. This means spacetime is not simply connected, so it is no longer possible to describe the system by global Minkowski coordinates - instead, the ADM surface is covered by an atlas whose charts are Minkowski, since geometry is still flat everywhere outside the particles.

$6\gamma(\phi) = \cos(\phi)\gamma_1 + \sin(\phi)\gamma_2$
The Minkowski charts $\Phi_{\Delta}$ are chosen so that each one covers a polygon $\Delta$, and for the atlas to be well-defined there must be diffeomorphisms relating two different sets of coordinates in intersecting patches, the transfer functions. Since the intersection of two polygons is a link, the transfer functions are labeled by the links, and necessarily correspond to isometries of Minkowski space mapping $-\lambda$ to $\lambda$, of which we will only consider the Lorentz rotations $g_{\lambda} \in SL(2)$. They relate space coordinates in the two orientations of a link:

$$z_{-\lambda} = -g_{\lambda} z_{\lambda} g_{\lambda}^{-1}.$$  \hspace{1cm} (77)

Since the relative positions and the transfer functions contain all information about the shape of the polygons and how to “glue” them together to form the ADM surface, they effectively encode spacetime geometry - up to redundancy transformations that correspond to diffeomorphism invariance.

All link variables given for flat space are still definable for the current case, except for the relative momenta, which are now related to the transfer functions. The relation is given by considering a small-$G$ approximation of the problem and checking consistency of the definition with the flat-space formulas given above: one can write (using the exponential map $\exp : sl(2) \rightarrow SL(2)$):

$$g_{\lambda} = e^{4\pi G q_{\lambda}} \approx 1 + 4\pi G q_{\lambda} + O(q_{\lambda}^2), \quad 4\pi G \cdot |q_{\lambda}| \ll 1.$$  \hspace{1cm} (78)

Additionally, the transfer functions in external links are defined to allow satisfaction of the center of mass constraints, as before:

$$g_{\pm \eta} = e^{\mp 4\pi G M_{\eta} \gamma_0}, \quad \eta \in \Gamma_{\infty}.$$  \hspace{1cm} (79)

More remarkable is the assertion that the absolute momenta of the point particles can be deduced from the transfer functions, through structures called holonomies. The holonomy is defined by considering the parallel transport of a spacetime vector along a closed path around a particle. With the present geometry, parallel transport is trivial within polygons, but picks up a Lorentz rotation from a transfer function at each conical deficit, which adds up to a rotation of $8\pi G m$ along a timelike axis. From this Lorentz group transformation one can derive the 3-momentum of the particle. Put more formally, the holonomy of a particle $\pi$ evaluated at a polygon $\Delta$ is the product of the

\footnote{Factor of $4\pi G$ for dimensional correctness.}
transfer functions,
\[ u_{\pi,\Delta} = \prod_{\lambda \in \Gamma_{\pi,\Delta}} g_{\lambda}. \tag{80} \]

Under Lorentz rotations of the charts parametrized by \( \{ h_{\Delta} \}_{\Delta \in \Gamma} \), where \( h_{\Delta} \in SL(2) \), the transfer functions transform as
\[ g_{\lambda} \rightarrow h_{\Delta_{-\lambda}} g_{\lambda} h_{\Delta_{-\lambda}}^{-1}, \tag{81} \]
and hence, the holonomy transforms correctly under \( SL(2) \), thus holonomies at different polygons only differ by the corresponding Lorentz rotations relating the corresponding charts,
\[ u_{\pi,\Delta} \rightarrow h_{\Delta} u_{\pi,\Delta} h_{\Delta}^{-1}, \quad \forall \Delta \Rightarrow u_{\pi,\Delta_{\lambda}} = g_{\lambda}^{-1} u_{\pi,\Delta_{-\lambda}} g_{\lambda}, \quad \forall \lambda. \tag{82} \]

3-momentum is deduced from the canonical projection \( SL(2) \rightarrow sl(2) \) and expressed as follows:
\[ u_{\pi} = U_{\pi} 1 + 4\pi G p_{\pi}^i \gamma_i, \tag{83} \]
where \( U_{\pi} \) is the holonomy scalar. The factor \( 4\pi G \) is not only necessary for dimensional correctness, it fixes the correct mass shell constraint for the model - which we derive by demanding the holonomy to be a rotation of \( 8\pi G m \) along the timelike axis as above. The dynamical constraints are then
\[ U_{\pi} = \cos (4\pi G m_{\pi}), \quad p_{\pi}^0 = \frac{1}{2} Tr (u_{\pi} u_{\pi}^0) > 0. \tag{84} \]

The fact that \( u_{\pi} \in SL(2) \Rightarrow \det (u_{\pi}) = 1 \) gives an extra relation \( (4\pi G)^2 \frac{1}{2} Tr (p_{\pi}^2)^2 - U_{\pi}^2 = -1 \), which allows us to rewrite the mass shell constraint as
\[ \frac{1}{2} Tr (p_{\pi}^2)^2 = \eta_{ij} p_{\pi}^i p_{\pi}^j = -\frac{\sin^2 (4\pi G m_{\pi})}{(4\pi G)^2}, \tag{85} \]
with the correct pre-factor of \( 4\pi G \) giving the usual flat space mass shell relation when \( G \rightarrow 0 \).

The introduction of deficit angles \( \theta_{\eta}^\pm \) and time offsets \( \tau_{\Delta} (\phi_{\eta}) \) of the new geometry leads to updated definitions for the external link variables,
\[ M_{\eta} = \frac{1}{8\pi G} (\theta_{\eta}^+ - \theta_{\eta}^-), \quad z_{\pm \eta} = \pm \gamma (\phi_{\eta} + \theta_{\eta}^\pm), \quad T_{\eta} = x_0^{\pi,\Delta_{\lambda}} - \tau_{\Delta} (\phi_{\eta}), \tag{86} \]
new CM frame constraints which remain first class,
\[ \sum_{\eta \in \Gamma_{\pi}} M_{\eta} = M, \quad \sum_{\lambda \in \Gamma_{\pi}} L_{\lambda} = \sum_{\lambda \in \Gamma_{\pi}} \frac{1}{16\pi G} Tr ((g_{\lambda} z_{\lambda} g_{\lambda}^{-1} z_{\lambda}) \gamma^0) = S, \tag{87} \]
and new kinematical constraints with terms proportional to the masses and spins of the particles. More importantly, the new nonvanishing Poisson brackets are
\[ \{ T_{\eta}, M_{\eta} \} = 1, \quad \{ L_{\eta}, \phi_{\eta} \} = 1, \quad \{ g_{\lambda}, z_{\lambda}^i \} = 4\pi G g_{\lambda} \gamma^i, \quad \{ g_{-\lambda}, z_{\lambda}^i \} = -4\pi G \gamma^i g_{\lambda} = \{ g_{\lambda}, z_{-\lambda}^i \}, \quad \{ \xi_{\lambda}, \eta_{\lambda}^i \} = 8\pi G \epsilon_{ijk} z_{\lambda}^k \tag{88} \]
with the key difference of having a noncommutative Poisson algebra between relative positions.
In [8], it is described in detail how to derive the above outlined phase space picture from Einstein gravity, which consists essentially in writing the first-order action (113) in ADM form and adding extra particle terms and kinematic constraints to accurately reproduce the algebra (88). We just note that it is straightforward to find the frame field and spin connection one forms that describe the metric (67),

\[
e_a = \left( \partial_a t + 4 G s \partial_a \phi \right) \gamma_0 + \partial_a r \gamma(\phi) + (1 - 4 G m) r \partial_a \phi \gamma'(\phi),
\]

\[
\omega_a = -2 G M \partial_a \phi \gamma_0,
\]

and the holonomy scalar is computed in terms of the spin connection in the boundary of the “particle hole” \( B_\pi \):

\[
U_\pi = \frac{1}{2} \text{Tr} \left( \mathcal{P} \exp \int_{B_\pi} d\phi \wedge \omega \right).
\]

4.2 The relative locality interpretation

We will now draw the picture of phase space in the 2+1 multiparticle system from a different perspective: that of relative locality. In order to do so, we need to characterize the geometry of momentum space. Two pieces of data are needed:

- the metric of momentum space;
- the combination rule between two momenta.

The metric is easily obtainable from the description of momentum as derived from a particle’s holonomy: the constitutive relation (83) can be rewritten in matricial form (going back to the notation of lower indices for momenta and dropping the \( \pi, \Delta \) subscripts)

\[
u = 4 \pi G \begin{bmatrix} \frac{U_{\pi}}{4 \pi G} - p_2 & p_1 + p_0 \\ p_1 - p_0 & \frac{U_{\pi}}{4 \pi G} + p_2 \end{bmatrix}.
\]

The condition that \( u \in SL(2) \) can be rewritten in the new notation as

\[(4 \pi G)^2 \eta^{ab} p_a p_b - U^2 = -1.
\]

Defining the coordinate \( p_3 = \frac{U}{4 \pi G} \), and \( \eta^{AB} = \text{diag}(-1, +1, +1, -1) \), we see that this equation is simply the embedding equation of 3-dimensional anti-de Sitter space in Minkowski space \( \mathbb{M}^{2,2} \), \( \eta^{AB} p_A p_B = -\rho^2 \), where the curvature radius is

\[\rho = \frac{1}{4 \pi G},\]

and the metric of momentum space is readily written in the quasi-cartesian coordinate system we are working in:

\[g^{ab}(p) = \eta^{ab} - \frac{\eta^{ar} \eta^{bs} p_r p_s}{\eta^{rr} p_r p_s + \rho^2}.
\]

To derive a momentum addition rule we just have to note that putting together two holonomies \( u_{\pi_1, \Delta_1}, u_{\pi_2, \Delta_2} \) (which corresponds physically to the absorption of a particle by the other), the resultant holonomy is simply the group product of the two: \( u_{\pi_1 \circ \pi_2, \Delta_1 \circ \Delta_2} = u_{\pi_1, \Delta_1} \cdot u_{\pi_2, \Delta_2} \). If the corresponding momenta are \( p, q \) respectively, we
obtain (denoting $p^2 = \eta^{ab} p_a p_b$ and $p \cdot q = \eta^{ab} p_a q_b$)

$$u_{\pi_1 \oplus \pi_2, \Delta_1 \oplus \Delta_2} = 4\pi G \left[ \frac{U_{\pi_1 \oplus \pi_2}}{4\pi G} \left( p \oplus q \right)_{2} - \left( p \oplus q \right)_{1} \right] \left( p \oplus q \right)_{0},$$

(94)

where

$$U_{\pi_1 \oplus \pi_2} = \sqrt{1 + (4\pi G)^2 (p \oplus q)^2} = \sqrt{1 + (4\pi G)^2 p^2} \sqrt{1 + (4\pi G)^2 q^2} - (4\pi G)^2 p \cdot q$$

(95)

$$\left( p \oplus q \right)_a = \sqrt{1 + (4\pi G)^2 p^2} q_a + \sqrt{1 + (4\pi G)^2 q^2} p_a + 4\pi G \eta_{ad} \epsilon^{dc} p_b q_c.$$ (96)

An important observation is that, since the combination rule is effectively group multiplication in $SL(2)$, it is associative - the Riemann tensor (13) vanishes. The connection is now obtained from the definition (7), with the result

$$\Gamma^b_{ac} (p) = -(4\pi G)^2 p_c g^{ab} - 4\pi G \epsilon^{abd} \frac{(4\pi G)^2 p_d p_c + \eta_{dc}}{\sqrt{1 + (4\pi G)^2 \eta^{rs} p_r p_s}}$$

(97)

where

$$\{^{ab}_{c}\} = -(4\pi G)^2 p_c g^{ab}$$ (98)

are the Christoffel symbols of the AdS$_3$ metric and

$$T^a_{bc} = -8\pi G \epsilon^{aabd} \frac{(4\pi G)^2 p_d p_c + \eta_{dc}}{\sqrt{1 + (4\pi G)^2 \eta^{rs} p_r p_s}}$$ (99)

is the torsion. We find that the non-metricity tensor vanishes, hence the connection is compatible with the metric.

Lastly, we can re-obtain the mass shell constraint (85) using its definition in the relative locality context, (1).

The geodesic distance $D(p,0)$ is computed in the center of mass frame, by attempting a solution of the form $p_a(\tau) = (p_0(\tau), 0, 0)$, $p_0(0) = 0$, $p_0(1) = p$ to the geodesic equations,

$$\dot{p}_a(\tau) + \Gamma^b_{ac}(p(\tau)) \dot{p}_b(p(\tau)) \dot{p}_c(p(\tau)) = 0.$$ (100)

The ansatz reduces them to one single ODE:

$$\ddot{p}_0 + \frac{p_0 \dot{p}_0^2}{\rho^2 - p_0^2} = 0.$$ (101)

The solution of (101) respecting the boundary conditions which tends to the flat space linear form when $\frac{\rho}{\rho} \rightarrow 0$ is

$$p_0(\tau) = \rho \sin \left( \tau \left( \frac{\pi}{2} - \arctan \sqrt{\left( \frac{\rho}{p} \right)^2 - 1} \right) \right)$$ (102)

\footnote{Also notice that, fortuitously, $T^{cab} + T^{ba} = 0$, so the corresponding term in the decomposition (12) vanishes.}
and the geodesic length is obtained by computing the integral

\[ m = L = \int_0^1 d\tau \sqrt{-g^{ab}(p(\tau)) \dot{p}_a(\tau) \dot{p}_b(\tau)} = \int_0^1 d\tau \sqrt{\frac{p_0^2}{1 - \frac{p_0^2}{\rho^2}}} = \frac{\rho}{2} \left( \pi - 2 \arctan \sqrt{\left( \frac{\rho}{p_0} \right)^2 - 1} \right), \tag{103} \]

which does satisfy \( \rho^2 \sin^2 \left( \frac{m}{\rho} \right) = p_0^2 = -\eta^{ab} p_a p_b \).

The phase space of the 2+1 multiparticle system coupled to gravity is then described as follows: the dynamical variables are the particle’s holonomy momenta \( p_\pi \in \mathcal{P} \approx \text{AdS}_3 \), where the curvature radius of \( \mathcal{P} \) is \( \rho = \frac{1}{4\pi G} \), and their dual spacetime coordinates \( x_\pi \in T^*_p \mathcal{P} = \mathcal{X}(p_\pi) \) with the geometry of spacetime given by \( \text{SL}(2) \) group rule, and the fact that it is curved explains the noncommutative structure of spacetime made evident in the algebra \( \eta^{ab} \).

The constraints on phase space are the mass shell constraints \( \text{(85)} \) and the positive energy constraint \( p_0 > 0 \), along with the 2nd class kinematical constraints to the CM frame, which can be made 1st class by the triangulation procedure.

### 4.3 Gamma ray bursts in 2+1 dimensions

We will now carry out the derivation of the time delay between photon emission and reception for the 2+1 multiparticle system. The calculation follows the same steps performed in Section 3.3, and there are two observations that make our life easier:

- Even though we are now working in a different physical regime than in Section 3.3 (\( \hbar \) is still neglected but \( G \) is present), the fact that spacetime geometry is everywhere flat apart from the conical deficits at the particles’ locations means the kinematic relations \( \text{[67]} \) are still valid, as long as we redefine \( \hat{k}_i = \frac{k_i}{|k_i|} \) according to the mass shell constraints \( \text{[88]} \).

- The fact that \( \oplus \) is associative results in useful formulas for the parallel transport operators:

\[
U^p_q = U^0_q U^p_0; \quad V^p_q = V^0_q V^p_0. \tag{104}
\]

Proof is as follows: \( U^0_q U^p_0 = U^0_{q \oplus 0} U^p_{0 \oplus 0} = d_0 L_q \cdot d_p L_{q \oplus 0} = d_p ( L_q L_{q \oplus 0} ) = d_p L_{q \oplus q} = U^p_q \), where the associativity of \( \oplus \) is used to assert that \( L_q L_{q \oplus 0} = L_{q \oplus q} \). Calculation for \( V \) is entirely analogous.

We can also compute the parallel transport operators \( U, V, I \) according to the present combination rule:

\[
(U^q_{p \oplus q})^b_a = \sqrt{1 + (4\pi G)^2 q^2} \delta^b_a - 4\pi G \eta^{ad} \epsilon^{dcb} p_c + (4\pi G)^2 \frac{p_a \eta^{bc} q_c}{\sqrt{1 + (4\pi G)^2 q^2}} \]
\[
(V^p_{p \oplus q})^b_a = \sqrt{1 + (4\pi G)^2 q^2} \delta^b_a + 4\pi G \eta^{ad} \epsilon^{dcb} q_c + (4\pi G)^2 \frac{q_a \eta^{bc} p_c}{\sqrt{1 + (4\pi G)^2 q^2}} \]
\[
(I^p)^b_a = -\delta^b_a. \tag{105}
\]
We will now compute the time delay $\Delta S$ using the momentum constraints as they are given in \cite{[2]}. The formulas \cite{[104]} greatly simplify the relative locality operators:

\[
\begin{align*}
W_{x_1} &= V_0^1 \\
W_{x_2} &= V_0^1 \\
W_{x_3} &= U_0^{i_2} \\
W_{x_4} &= U_0^{i_2} \\
W_{u_1} &= V_0^j U_p^0 V_0^{i_1} \\
W_{u_2} &= V_0^j U_p^0 V_0^{i_1} \\
W_{u_3} &= U_0^{i_2} V_0^{i_2} U_0^{i_2} \\
W_{u_4} &= U_0^{i_2} V_0^{i_2} U_0^{i_2} \\
W_{y_1} &= V_0^j \\
W_{y_2} &= V_0^j \\
W_{y_3} &= V_0^j \\
W_{y_4} &= V_0^j.
\end{align*}
\]

(106)

Computing the terms of \cite{[46]}, we obtain that $W_{u_1} W_{x_2}^{-1} W_{y_2} W_{y_4}^{-1} = V_0^j U_p^0 = W_{y_1} W_{x_4}^{-1} W_{u_3} W_{x_4}^{-1}$, an exact cancellation of the LHS, and the following results for the parallel transported momenta:

\[
\begin{align*}
K_1^b &= \hat{k}_1^a (V_0^a)_{_b} = \sqrt{1 + (4\pi G)^2 k_1^2 k_1^b} = \cos (4\pi Gm_1) \hat{k}_1^b \\
K_2^b &= \hat{k}_2^a (U_0^a)_{_b} = \sqrt{1 + (4\pi G)^2 k_2^2 k_2^b} = \cos (4\pi Gm_2) \hat{k}_2^b \\
P_1^b &= \hat{p}_1^a (U_0^a)_{_b} = \hat{p}_1^b \\
P_2^b &= \hat{p}_2 (U_0^a)_{_b} = \hat{p}_2^b.
\end{align*}
\]

(107)

Each uppercase momenta only depends on the corresponding lowercase one, and the torsion terms turn out to have no effect. The assumption that $K_1$ and $K_2$ are parallel means simply that $\hat{k}_1 = \hat{k}_2$. The calculation of the time delay carries on exactly as done in \cite{[53-57]} and the conclusion is immediate: since from \cite{[85]} we know that photons are null, $P_1^2 = P_2^2 = \hat{p}_1^2 = \hat{p}_2^2 = 0$, and it results that there is no time delay between the intervals of emission and reception in the 2+1 multiparticle model - as we expected from \cite{[2]}, since the connection of momentum space is metric.

More surprising is the fact that the dual gravitational lensing effect derived in \cite{[2]} (angular deviation between the two photons’ momenta) is absent as well: since the decomposition \cite{[54]} implies that $P_i \parallel P_j$ for null $P_i$ and $P_i = \hat{p}_i$, we have $\hat{p}_1 \parallel \hat{p}_2$. This is, however, compatible with the result in \cite{[2]} for the deflection angle,

\[
\Delta \theta = \frac{E_1 + E_2}{2} \sqrt{\eta_{ab} (T^-)_{^a} (T^-)_{^b}},
\]

(108)

where $(T^\pm)_{^a} = e^-_{^b} e^+_{^b} T^{ba}$, $e^a_\pm$ being the null vector giving the direction of $P_i$ and $e^a_\mp = \eta^{ab} e^b_\pm$ the corresponding covector. Indeed, it can be seen from \cite{[99]} that

\[
(T^\pm)_{^a} (P_i) \propto e^-_{^b} e^+_{^b} \epsilon^{bad} (C_{e^c_+ e^d_+} + \eta_{cd}) = 0,
\]

(109)

so the predicted angular deviation vanishes in the 2+1 multiparticle case.

5 Conclusions

The main goal of the work described in this essay was three-fold: to review the main physical concepts and mathematical formalism behind the principle of relative locality, with particular focus on its application to classical interactions between point particles (as a first step to eventually study consequences of the new ideas in field theories and ultimately quantum gravity); to examine an apparent problem of the formalism, the ambiguities in defining the momentum constraints of the classical action, and try to understand whether different definitions produce different physics; and to give an example of how a well-studied system with nontrivial phase space geometry and dynamics, the 2+1 multiparticle model in Einstein gravity, can be understood in terms of the new framework.
Towards the first goal, I tried to give a clear motivation and explanation, based on intuitive ideas, of the material in \[1\] and \[2\]. I believe that the main principle is not hard to grasp for anyone with a basic understanding of differential geometry and special relativity, as long as one gets used to thinking in momentum space - not an easy task at all: in the words of Leonard Susskind, “only perverts think in momentum space”, but in this case, it could be an useful perversion.

The second goal saw us making some considerations on the structure of a vertex change, in particular identifying that identity-preserving diffeomorphisms on a momentum constraint are the mark of a gauge transformation involving the $\mathcal{K}^{(\alpha)}$ and interaction coordinates $z^{(\alpha)}$, therefore just a redundancy in the physical description, and doing several brute-force calculations with different conservation laws based on the gamma delay experiment, which revealed one particular instance where the structure of the derivation is altered - unless torsion effects are not considered.

Finally, regarding the third goal, we constructed a picture of phase space that, despite being that of a classical gravity model, is remarkably similar to the description of relative locality’s phase space in the limit $G_N \to 0$, $\hbar \to 0$ with a finite Planck mass, to the extent where it could be studied using the same language and methods. The 2+1 multiparticle model provided us with an interesting example of a momentum space with a nontrivial but well understood metric (AdS$_3$) and torsion, but no nonmetricity, and an interpretation of the noncommutative nature of spacetime encoded in the relative locality relations.

As for possible further developments on this work, we did not, by any means, exhaust the treatment of the change of vertex problem. The first order results presented in Section 3.3 illustrated how the analysis of the gamma delay problem seemed to change when the 2nd and 3rd alternatives for momentum constraints were introduced, but it was not completely clear whether that change produced an effective modification in the physics - especially because the original calculation presented in [2] was done without torsion, and in this limit all discrepancies disappear. A full calculation of the gamma delay with torsion would probably have shed light on this issue. Alternatively, going to second order and examining the curvature effects could prove enlightening.

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6 Appendix - Review of 2+1 Einstein gravity

For the purposes of this essay, we will introduce the first order formalism for Einstein gravity in 2+1 dimensions, which describes it as a Yang-Mills-like theory with the Lorentz symmetry group $SO(2,1,\mathbb{R}) \approx SL(2,\mathbb{R})$, which Lie algebra is that of traceless $2 \times 2$ matrices, denoted as $sl(2)$. A vector in this algebra can be written as $k = k^i \gamma_i$, where $\gamma_i$ are the following matrices
\[
\gamma_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
\]

The dynamical variables describing geometry are then the frame fields ($dreibein$), which are derived from the natural projection of the tangent bundle $e : TM \rightarrow M$,
\[
e^i_a(p) : T_p M \times sl(2) \rightarrow \mathbb{R} \\
(v, k) \rightarrow e^i_a v^a k_i,
\]
and the spin connection, which is the gauge group that induces sections in $TM$,
\[
\omega^i_a(p) : T_p M \times sl(2) \rightarrow \mathbb{R} \\
(v, k) \rightarrow \omega^i_a v^a k_i.
\]

From these we can construct one-forms in $T_p M$, $e^i = e^i_a dx^a$, $\omega^i = \omega^i_a dx^a$, as well as Lie-algebra valued one-forms
\[
\begin{bmatrix} e \\ \omega \end{bmatrix} = \begin{bmatrix} e^i_a \gamma_i dx^a \\ \omega^i_a \gamma_i dx^a \end{bmatrix} = \begin{bmatrix} e_a dx^a \\ \omega_a dx^a \end{bmatrix} : T_p M \rightarrow sl(2).
\]

Spacetime metric is given by $g_{ab} = \eta_{ij} e^i_a e^j_b = \frac{1}{2} \text{Tr} (e_a e_b)$, while the Levi-Civita connection is related to the spin connection by the following formula,
\[
\omega^{ij} = e^i_b e^j_a + e^k_c \Gamma^{(ab)}_{ca},
\]
where the double-index notation for $\omega$ indicates the usage of a different set of generators for $sl(2)$, $\omega_a = \omega^{ij}_a J_{ij} = \omega^{ij}_a \gamma^i \gamma^j$.

The bulk Einstein-Hilbert-Palatini action for gravity can be rewritten in terms of the new dynamical variables as
\[
S = \frac{1}{16\pi G} \int_M \text{tr} (e \wedge F(\omega)),
\]
where $F(\omega) = d\omega + \omega \wedge \omega$ is the curvature tensor written in terms of the spin connection. The bulk equations of motion are
\[
\frac{\delta S}{\delta e} = 0 \Rightarrow F(\omega) = 0 \\
\frac{\delta S}{\delta \omega} = 0 \Rightarrow de + [\omega, e] = 0
\]
and they state that geometry of spacetime in 2+1-dimensional vacuum is flat. Hence, it becomes clear that three-dimensional gravity has no local degrees of freedom - although it is possible to show that boundary terms in the...
The action of Poincaré symmetry transformations on the dynamical variables takes the usual form for a gauge theory:

\[
\begin{align*}
\text{Lorentz transformations:} & \quad \omega \rightarrow g^{-1} (d + \omega) g, \quad g \in SL(2), \\
\text{Translations:} & \quad \omega \rightarrow \omega, \quad e \rightarrow e + d\phi + [\omega, \phi], \quad \phi \in sl(2).
\end{align*}
\]

References

[1] G. Amelino-Camelia, L. Freidel, J. Kowalski-Glikman, L. Smolin, “The principle of relative locality” [arXiv:1101.0931v2 [hep-th]]

[2] L. Freidel, L. Smolin, “Gamma ray burst delay times probe the geometry of momentum space” [arXiv:1103.5626 [hep-th]]

[3] G. Amelino-Camelia, “Doubly-Special Relativity: First Results and Key Open Problems” [arXiv:gr-qc/0210063v1]

[4] J. Magueijo, L. Smolin, “Generalized Lorentz invariance with an invariant energy scale” [arXiv:gr-qc/0207085v1]

[5] L. Freidel, J. Kowalski-Glikman, L. Smolin, “2+1 gravity and Doubly Special Relativity” [arXiv:hep-th/0307085v2]

[6] L. Freidel, “The geometry of momentum space”, preprint in preparation

[7] L. Freidel, “Synchronization of relative localization”, unpublished notes

[8] H-J. Matschull, “The phase space structure of multi-particle models in 2+1 gravity” [2001 Class. Quantum Grav. 18, 3497 - http://iopscience.iop.org/0264-9381/18/17/309]

[9] A. Ashtekar, Lectures on nonperturbative canonical gravity, World Scientific Publishing, 1991

[10] S. Deser, R. Jackiw, G. ’t Hooft, “Three-Dimensional Gravity: Dynamics of Flat Space” [1984 Annals of Physics 152, 220-235]

[11] R. Arnowitt, S. Deser, C. W. Misner, “The Dynamics of General Relativity” [arXiv:gr-qc/0405109v1]

[12] S. Hossenfelder, “Bounds on an energy-dependent and observer-independent speed of light from violations of locality” [arXiv:1004.0418 [hep-ph]]

[13] G. Ghirlanda, G. Ghisellini, “The present and the future of cosmology with Gamma Ray Bursts” [arXiv:astro-ph/0602498v1]