σ-RELATIONS, σ-FUNCTIONS AND σ-ANTIFUNCTIONS.

IVAN GATICA ARAUS

Abstract. In this article we develop the concepts of σ-relation and σ-function, following the same steps as in Set Theory. First we define the concept of ordered pair and then we build the Cartesian Product of σ-sets so that we can define the concepts of σ-relation and σ-function.

Now, as in σ-Set Theory there exist the concepts of σ-antielement and σ-antiset, we can build the new concepts of σ-antifunction, antidentity and antinverse. Finally, in the case that a σ-function $f : A \to B$ is bijective and there exist $A^*$ and $B^*$ σ-antiset of $A$ and $B$, we get 16 different σ-functions which are related in a diagram of σ-functions.

1. Introduction

The context in which we developed the definitions and results will be the σ-Set Theory (see [1]). In this sense we consider the following axiom system.

Axiom 1.1. (Empty σ-set). There exists a σ-set which has no σ-elements, that is

$$(\exists X)(\forall x)(x \notin X).$$

Axiom 1.2. (Extensionality). For all σ-classes $\hat{X}$ and $\hat{Y}$, if $\hat{X}$ and $\hat{Y}$ have the same σ-elements, then $\hat{X}$ and $\hat{Y}$ are equal, that is

$$(\forall X, Y) [(\forall z)(z \in X \leftrightarrow z \in Y) \rightarrow X = Y].$$

Axiom 1.3. (Creation of σ-Class). We consider an atomic formula $\Phi(x)$ (where $\hat{Y}$ is not free). Then there exists the classes $\hat{Y}$ of all σ-sets that satisfies $\Phi(x)$, that is

$$(\exists \hat{Y})(x \in \hat{Y} \leftrightarrow \Phi(x)), 
\text{with } \Phi(x) \text{ a atomic formula where } \hat{Y} \text{ is not free.}$$

Axiom 1.4. (Scheme of Replacement). The image of a σ-set under a normal functional formula $\Phi$ is a σ-set.

Axiom 1.5. (Pair). For all $X$ and $Y$ σ-sets there exists a σ-set $Z$, called fusion of pairs of $X$ and $Y$, that satisfy one and only one of the following conditions:

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(a): $Z$ contains exactly $X$ and $Y$,  
(b): $Z$ is equal to the empty $\sigma$-set,  
that is  
$$(\forall X,Y)(\exists Z)(\forall a)[(a \in Z \leftrightarrow a = X \lor a = Y) \lor (a \notin Z)].$$

Axiom 1.6. (Weak Regularity). For all $\sigma$-set $X$, for all $\ll x, \ldots, w \gg \in CH(X)$ we have that $X \notin \ll x, \ldots, w \gg$, that is  
$$(\forall X)(\forall \ll x, \ldots, w \gg \in CH(X))(X \notin \ll x, \ldots, w \gg).$$

Axiom 1.7. (non $\epsilon$-Bounded $\sigma$-set). There exists a non $\epsilon$-bounded $\sigma$-set, that is  
$$(\exists X)(\exists y)[(y \in X) \land (\min(X) = \emptyset \lor \max(X) = \emptyset)].$$

Axiom 1.8. (Weak Choice). If $\hat{X}$ be a $\sigma$-class of $\sigma$-sets, then we can choose a singleton $Y$ whose unique $\sigma$-element come from $\hat{X}$, that is  
$$(\forall \hat{X})(\forall x \in \hat{X}) (x \in \hat{X} \rightarrow (\exists Y)(Y = \{x\})).$$

Axiom 1.9. ($\epsilon$-Linear $\sigma$-set). There exist $\sigma$-set $X$ such that $X$ has the linear $\epsilon$-root property, that is  
$$(\exists X)(\exists y)(y \in X \land X \in LR).$$

Axiom 1.10. (One and One $^*$ $\sigma$-set). For all $\epsilon$-linear singleton, there exists a $\epsilon$-linear singleton $Y$ such that $X$ is totally different from $Y$, that is  
$$(\forall X \in SG \cap LR)(\exists Y \in SG \cap LR)(X \neq Y).$$

Axiom 1.11. (Completeness (A)). If $X$ and $Y$ are $\sigma$-sets, then  
$${X} \cup {Y} = \{X,Y\},$$

if and only if $X$ and $Y$ satisfy one of the following conditions:  
(a): $\min(X,Y) \neq |1 \lor 1^*| \land \min(X,Y) \neq |1^* \lor 1|$.  
(b): $\lnot(X \neq Y)$.  
(c): $(\exists w \in X)[w \notin \min(X) \land \lnot\Psi(z,w,a,Y)]$.  
(d): $(\exists w \in Y)[w \notin \min(Y) \land \lnot\Psi(z,w,a,X)]$.

Axiom 1.12. (Completeness (B)). If $X$ and $Y$ are $\sigma$-sets, then  
$${X} \cup {Y} = \emptyset,$$

if and only if $X$ and $Y$ satisfy the following conditions:  
(a): $\min(X,Y) = |1 \land 1^*| \lor \min(X,Y) = |1^* \land 1|$;  
(b): $X \neq Y$;  
(c): $(\forall z)(z \in X \land z \notin \min(X)) \rightarrow \Psi(z,w,a,Y)$;  
(d): $(\forall z)(z \in Y \land z \notin \min(Y)) \rightarrow \Psi(z,w,a,X)$. 


Axiom 1.13. (Exclusion). For all σ-sets X, Y, Z, if Y and Z are σ-elements of X then the fusion of pairs of Y and Z contains exactly Y and Z, that is

\[(∀X, Y, Z)(Y, Z \in X \rightarrow \{Y\} \cup \{Z\} = \{Y, Z\}).\]

Axiom 1.14. (Power σ-set). For all σ-set X there exists a σ-set Y, called power of X, whose σ-elements are exactly the σ-subsets of X, that is

\[(∀X)(∃Y)(∀z)(z \in Y \iff z \subseteq X).\]

Axiom 1.15. (Fusion). For all sets X and Y, there exists a σ-set Z, called fusion of all σ-elements of X and Y, such that Z contains σ-elements of the σ-elements of X or Y, that is

\[(∀X, Y)(∃Z)(∀b)(b \in Z \rightarrow (∃z)((z \in X \lor z \in Y) \land (b \in z))).\]

Axiom 1.16. (Generated σ-set). For all sets X and Y there exists a σ-set, called the σ-set generated by X and Y, whose σ-elements are exactly the fusion of the σ-subsets of X with the σ-subsets of Y, that is

\[(∀X, Y)(∃Z)(∀a)(a \in Z \leftrightarrow (∃A \in 2^X)(∃B \in 2^Y)(a = A \cup B)).\]

Where:

- \(\langle x, \ldots, z \rangle \in CH(X) := x \in \cdots \in z \in X.\)
- \(X \notin \langle x, \ldots, z \rangle := X \neq x \land \cdots \land X \neq z.\)
- \(LR = \{X : X \text{ has the linear } ε\text{-root property}\}\).
- \(SG = \{X : X \text{ is a singleton}\}\).
- \(Ψ(z, w, a, x) := (∃!w)(\{z\} \cup \{w\} = \emptyset) \land (∀a)(\{z\} \cup \{a\} = \emptyset \rightarrow a \in x).\)
- 1 is the One σ-set and \(1^*\) is the One* σ-set.
- \(min(X) = \{y \in X : y \text{ is an } ε\text{-minimal σ-element of } X\}\).
- \(min(X, Y) \neq [1 \lor 1^*] := min(X) \neq 1 \lor min(Y) \neq 1^*.\)
- \(min(X, Y) = [1 \land 1^*] := min(X) \land min(Y) = 1^*.\)

Also, we remember that the generic object of σ-Set Theory is called σ-class. However, in what follows we consider only the properties of σ-sets. In this sense when we write \(x \in A\), it should be understood that \(x\) and \(A\) are σ-sets, where \(x\) is a σ-element of \(A\).

Now, we present the following definitions and results introduced by I. Gatia in [1] which are necessary for the development of this paper.

**Definition 1.17.** Let A and B be σ-sets. Then we have that

1. If \(\{A\} \cup \{B\} = \emptyset\), then B is called the σ-antielement of A.
2. If \(A \cup B = \emptyset\), then B is called the σ-antiset of A.

To denote the σ-antielements and σ-antisets we use the following notation:

- Let \(x \in A\), then we use \(x^*\) to denote the σ-antielement of \(x\).
- Let \(A\) be a σ-set, then we use \(A^*\) to denote the σ-antiset of \(A\).
Figure 1. Generated Space by \{1_e, 2_e, 1, 2\} and \{1_e, 2_e, 1^*, 2^*\}

We observe that Gatica in [1] uses $A^\star$ in order to denote $\sigma$-antisets. However, in order to unify notation, we use $A^\ast$.

**Definition 1.18.** Let $A$ and $B$ be $\sigma$-sets. We define:

1. $A \cap B = \{ x \in A : x \in B \}$;
2. $A - B = \{ x \in A : x \notin B \}$;
3. $A^\cap B := \{ x \in A : x^* \in B \}$;
4. $A \neq B := A - (A^\cap B)$;
5. $A \cup B = \{ x : (x \in A \neq B) \lor (x \in B \neq A) \}$.

We observe that if $A^\cap B = \emptyset$ and $B^\cap A = \emptyset$ then

$$A \cup B = \{ x : x \in A \lor x \in B \}.$$
Thus, in this case the fusion coincides with the definition of union in a standard set theory.

**Example 1.19.** Let $X$ be a nonempty $\sigma$-set and $2^X$ the power $\sigma$-set of $X$. Then for all $A, B \in 2^X$ we have that

$$A \cup B = \{x : x \in A \lor x \in B\},$$

because $2^X$ is a $\sigma$-antielement free $\sigma$-set (see [1], Definition 3.44 and Theorem 3.48).

**Definition 1.20.** Let $A$ and $B$ be $\sigma$-sets. The **Generated space by $A$ and $B$** is given by

$$\langle 2^A, 2^B \rangle = \{x \cup y : x \in 2^A \land y \in 2^B\}.$$  

We observe that in general $\langle 2^{A \cup B}, 2^{A \cup B^*} \rangle \neq \langle 2^A, 2^B \rangle$. Consider $X = \{1_\Theta, 2^*\}$ and $Y = \{1_\Theta, 2\}$, then $X \cup Y = \{1_\Theta\}$. Therefore $2^{X \cup Y} = \emptyset, \{1_\Theta\}$ and $(2^X, 2^Y) = \{\emptyset, \{1_\Theta\}, \{2\}, \{2^*, \{1_\Theta, 2\}, \{1_\Theta, 2^*\}\}$.

On the other hand, if we consider $A = \{1_\Theta, 2_\Theta\}$ and $B = \{1, 2\}$ we obtain that the generated space by $A \cup B$ and $A \cup B^*$ is the following:

$$\langle 2^{A \cup B}, 2^{A \cup B^*} \rangle = \emptyset, \{1\}, \{1^*\}, \{2\}, \{2^*\}, \{1_\Theta\}, \{2_\Theta\}, \{1^*_\Theta, 2\}, \{1^*_\Theta, 2^*\}, \{1_\Theta, 1\}, \{1_\Theta, 1^*\}, \{1_\Theta, 2\}, \{1_\Theta, 2^*\}, \{2_\Theta, 1\}, \{2_\Theta, 1^*\}, \{2_\Theta, 2\}, \{2_\Theta, 2^*\}, \{1_\Theta, 1^*_\Theta, 2\}, \{1_\Theta, 1^*_\Theta, 2^*\}, \{1_\Theta, 2_\Theta, 1\}, \{1_\Theta, 2_\Theta, 1^*\}, \{1_\Theta, 2_\Theta, 2\}, \{1_\Theta, 2_\Theta, 2^*\}, \{1_\Theta, 2_\Theta, 1^*_\Theta, 2\}, \{1_\Theta, 2_\Theta, 1^*_\Theta, 2^*\}, \{1_\Theta, 2_\Theta, 1_\Theta, 2^*\}, \{1_\Theta, 2_\Theta, 1^*_\Theta, 2^*\}.\]  

See Figure [1] Also if $A = \{1, 2, 3\}$ and $A^* = \{1^*, 2^*, 3^*\}$, then

$$\langle 2^A, 2^{A^*} \rangle = 3^A = \emptyset, \{1\}, \{2\}, \{3\}, \{1^*\}, \{2^*\}, \{3^*\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1^*, 3\}, \{2^*, 3\}, \{1^*, 2^*\}, \{1^*, 3^*\}, \{2^*, 3^*\}, \{1, 2^*\}, \{1, 3^*\}, \{2, 3^*\}, \{1, 2, 3\}, \{1^*, 2, 3\}, \{1^*, 2^*, 3\}, \{1, 2^*, 3\}, \{1^*, 2^*, 3^*\}, \{1, 2^*, 3^*\}, \{1^*, 2^*, 3^*\}.\]  

Now, in Figure [2] we present the patterns of containments of the $3^X$. We observe that the $\sigma$-elements $\{1^*, 2, 3^*\}$ and $\{1^*, 2, 3\}$ can be represented in three dimensions as one of the vertexes of the pyramids

$$\Delta_1 := \{\{1, 3\}, \{2^*, 3\}, \{1, 2^*\}, \{1, 2^*, 3\}\}$$

and

$$\Delta_2 := \{\{1^*, 3^*\}, \{2, 3^*\}, \{1^*, 2\}, \{1^*, 2^*, 3\}\}.$$
Theorem 1.21. (see [1], Theorem 3.29). If $A$ is a $\sigma$-set, then

(a): $\{\emptyset\} \cup \{A\} = \{\emptyset, A\}$.

(b): $\{\alpha\} \cup \{A\} = \{\alpha, A\}$.

(c): $\{\beta\} \cup \{A\} = \{\beta, A\}$.

Theorem 1.22. (see [1], Theorem 3.32). Let $A$ and $B$ be $\sigma$-set. Then $\{A\} \cup \{B\} = \emptyset$ if and only if $\{B\} \cup \{A\} = \emptyset$.

Theorem 1.23. (see [1], Theorem 3.34). Let $A$ be a $\sigma$-set. If there exists the $\sigma$-antielement of $A$, then it is unique.

Theorem 1.24. (see [1], Theorem 3.58). Let $A$ be a $\sigma$-set. If there exists $A^*$, the $\sigma$-antiset of $A$, then $A^*$ is unique.
2. Order Pairs

Following the same steps as in the Set Theory, we define the ordered pairs. Next we define the Cartesian product of the \( \sigma \)-sets. Therefore we can define the concept of \( \sigma \)-relations, \( \sigma \)-functions and \( \sigma \)-antifunctions.

**Definition 2.1.** Let \( X \) and \( Y \) be \( \sigma \)-sets. The ordered pair of \( X \) and \( Y \) is defined by

\[
(X, Y) := \{\{X\}\} \cup \{\{X\} \cup \{Y\}\}.
\]

We observe that for all \( X \) and \( Y \) \( \sigma \)-sets, the ordered pair \((X, Y)\) is a \( \sigma \)-set.

**Lemma 2.2.** Let \( X \) and \( Y \) be \( \sigma \)-set. Then \( (X, Y) = \{\{X\}, \{X\} \cup \{Y\}\} \).

**Proof.** Consider \( A = \{X\} \) and \( B = \{X\} \cup \{Y\} \). By Axiom 4 (Pairs) we have that \( B = \{X, Y\} \) or \( B = \emptyset \). If \( B = \{X, Y\} \) then \( B \) is not totally different from \( A \). Therefore by the condition (b) of Axiom 7 (Completeness (A)) we obtain that \( (X, Y) = \{\{X\}, \{X, Y\}\} \). Now, if \( B = \emptyset \) then by Theorem 1.21 we obtain that \( (X, Y) = \{\{X\}, \emptyset\} \). Therefore \( (X, Y) = \{\{X\}, \{X\} \cup \{Y\}\} \).

**Theorem 2.3.** Let \( X, Y, Z \) and \( W \) be \( \sigma \)-sets. Then \( (X, Y) = (Z, W) \) if and only if \( X = Z \land Y = W \).

**Proof.** \((\rightarrow)\) Consider \( (X, Y) = (Z, W) \) then we will prove that \( (X = Z \land Y = W) \).

**(case a):** Suppose that \( X = \emptyset \). By Theorem 1.21 and Lemma 2.2 we obtain \( (X, Y) = \{\emptyset, \emptyset, Y\} \). Since \( (X, Y) = (Z, W) \) then

\[
\{\emptyset, \emptyset, Y\} = \{\{Z\}, \{Z\} \cup \{W\}\}.
\]

It is clear by Axiom 4 (Pairs) that \( \{Z\} \cup \{W\} = \{Z, W\} \) or \( \{Z\} \cup \{W\} = \emptyset \). Since \( (X, Y) = \{\{Z\}, \{Z\} \cup \{W\}\} \) then by Axiom 2 (Extensionality) we obtain that \( \{Z\} \cup \{W\} \neq \emptyset \), in consequence \( \{Z\} \cup \{W\} = \{Z, W\} \). Therefore \( \{\emptyset\} = \{Z\} \) and \( \{\emptyset, Y\} = \{Z, W\} \).

Finally, \( Z = \emptyset \) and so \( Y = W \).

**(case b):** Suppose that \( Y = \emptyset \). By Theorem 1.21 and Lemma 2.2 we obtain \( (X, Y) = \{\{X\}, \{X, \emptyset\}\} \). Now, if we use the same argument as in (case a) we have that \( X = Z \) and \( W = \emptyset \).

**(case c):** Suppose that \( X \neq \emptyset \) and \( Y \neq \emptyset \). By Lemma 2.2 we obtain \( (X, Y) = \{\{X\}, \{X\} \cup \{Y\}\} \) and \( (Z, W) = \{\{Z\}, \{Z\} \cup \{W\}\} \). It is clear by Axiom 4 (Pairs) that \( \{X\} \cup \{Y\} = \{X, Y\} \) or \( \{X\} \cup \{Y\} = \emptyset \).

**(c.1):** Consider \( \{X\} \cup \{Y\} = \{X, Y\} \). Now, if we use the same argument as in (case a) we have that \( X = Z \) and \( Y = W \).

**(c.2):** If \( \{X\} \cup \{Y\} = \emptyset \) then \( Y \) is the \( \sigma \) antielement of \( X \). Since \( (X, Y) = (Z, W) \), by Axiom 2 (Extensionality) we obtain that \( \{X\} = \{Z\} \) and \( \{Z\} \cup \{W\} = \emptyset \). Finally, we obtain that \( X = Z \), in consequence by Theorem 1.23 \( Y = W \).
(←) Consider \((X = Z \land Y = W)\) then by Axiom 2 (Extensional) it is clear that \((X, Y) = (Z, W)\). □

**Lemma 2.4.** Let \(A\) and \(B\) be \(\sigma\)-sets. If \(x \in A\) and \(y \in B\), then \((x, y) \in 2^{(A, B)}\).

**Proof.** Let \(x \in A\) and \(y \in B\). By Definition 1.20 we have that
\[ \langle 2^A, 2^B \rangle = \{ a \cup b : a \in 2^A \land b \in 2^B \} . \]
It is clear that \(\{x\} \in 2^A\) and \(\{y\} \in 2^B\). Therefore \(\{x\}, \{y\}\) and \(\{x\} \cup \{y\}\)
are \(\sigma\)-elements of \(\langle 2^A, 2^B \rangle\). Finally, since \((x, y) = \{\{x\}, \{x\} \cup \{y\}\}\) then
\((x, y) \in 2^{(A, B)}\).

□

**Definition 2.5.** Let \(A\) and \(B\) be \(\sigma\)-sets. The **Cartesian product of \(A\) and \(B\)** is the \(\sigma\)-set of all order pairs \((x, y)\) such that \(x \in A\) and \(y \in B\), that is
\[ A \times B = \{ (x, y) : x \in A \land y \in B \} . \]

The Cartesian product \(A \times B\) is a \(\sigma\)-set because \(A \times B \subseteq 2^{(A, B)}\).

3. **\(\sigma\)-Relations, \(\sigma\)-functions and \(\sigma\)-antifunctions.**

Now, we present the definition of binary \(\sigma\)-relations, as in the Set Theory.

**Definition 3.1.** Let \(A\) and \(B\) be \(\sigma\)-sets.

(a): A **binary \(\sigma\)-relation** on \(A \times B\) is a \(\sigma\)-subset \(R\) of \(A \times B\).

(b): A **binary \(\sigma\)-relation** on \(A\) is a \(\sigma\)-subset \(R\) of \(A \times A\).

If \(R\) is a binary \(\sigma\)-relation on \(A \times B\), then we also use \(R(x) = y\) for \((x, y) \in R\). As in the Set Theory we define the **domain** of \(R\) as
\[ \text{dom}(R) = \{ x \in A : (\exists y \in B)(R(x) = y) \} , \]
and the **range** of \(R\) as
\[ \text{ran}(R) = \{ y \in B : (\exists x \in A)(R(x) = y) \} . \]

**Definition 3.2.** Let \(A\) and \(B\) be \(\sigma\)-sets. A **binary \(\sigma\)-relation** \(f\) on \(A \times B\) is a **\(\sigma\)-function** from \(A\) to \(B\) if for all \(x \in A\) there exists a unique \(y \in B\) such that \(f(x) = y\). The unique \(y\) such that \(f(x) = y\) is the **value** of \(f\) at \(x\).

Let \(f\) be a \(\sigma\)-function on \(A \times B\), then it is clear that \(\text{dom}(f) = A\) and \(\text{ran}(f) \subseteq B\). Also, the \(\sigma\)-function from \(A\) to \(B\) will be denoted by
\[ f : A \to B . \]

A **binary operation** on \(A\) is a \(\sigma\)-function from \(A \times A\) to \(A\)
(i.e. \(f : A \times A \to A\)).
Example 3.3. Let $A$ be a $\sigma$-set. If we define $\oplus : 2^A \times 2^A \to 2^A$ were $\oplus(x, y) = x \cup y$, then $\oplus$ is a binary operation on $2^A$.

In fact, let $(x, y) \in 2^A \times 2^A$ then $x, y \in 2^A$. Since $x \subseteq A$ and $y \subseteq A$ then it is clear that there exist $x \cup y \in 2^A$ and by Axiom 2 (Extensionality) it is unique.

The definition of binary operation, is important for the study of the algebraic properties of the Integer Space. In this sense, we present the following conjecture:

Conjecture 3.4. Let $A$ be a $\sigma$-set such that there exists $A^*$, the $\sigma$-antiset of $A$. If we define $\oplus : 3^A \times 3^A \to 3^A$ where $\oplus(x, y) = x \cup y$, then $\oplus$ is a binary operation on $3^A$.

For convenience, we shall use the following notation: Let $A, B, C$ be $\sigma$-set then

$$\delta_A := \{A\}, \delta_{AB} := \{A, B\} \text{ and } \delta_{ABC} := \{A, B, C\}.$$  

For example, if we consider the integer space $3^{\{1,2,3\}}$ (see, figure 2) we obtain that $\oplus$ is a binary operation on $3^{\{1,2,3\}}$.

| $\oplus$ | $\emptyset$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $\delta_{123}$ |
|---------|------------|-----------|-----------|-----------|-------------|-------------|-------------|-------------|
| $\emptyset$ | $\emptyset$ | $\delta_1$ | $\delta_2$ | $\delta_3$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $\delta_{123}$ |
| $\delta_1$ | $\delta_1$ | $\emptyset$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $\delta_{123}$ | $\emptyset$ | $\delta_{12}$ |
| $\delta_2$ | $\delta_2$ | $\delta_{12}$ | $\emptyset$ | $\delta_{13}$ | $\delta_{23}$ | $\delta_{123}$ | $\delta_{12}$ | $\delta_2$ |
| $\delta_3$ | $\delta_3$ | $\delta_{13}$ | $\delta_{23}$ | $\emptyset$ | $\delta_{123}$ | $\delta_{12}$ | $\delta_2$ | $\delta_{12}$ |
| $\delta_{12}$ | $\delta_{12}$ | $\delta_{13}$ | $\delta_{23}$ | $\emptyset$ | $\delta_{123}$ | $\delta_{12}$ | $\delta_2$ | $\delta_{12}$ |
| $\delta_{13}$ | $\delta_{13}$ | $\delta_{23}$ | $\emptyset$ | $\delta_{123}$ | $\delta_{12}$ | $\delta_2$ | $\delta_{12}$ | $\delta_{12}$ |
| $\delta_{23}$ | $\delta_{23}$ | $\emptyset$ | $\delta_{123}$ | $\delta_{12}$ | $\delta_2$ | $\delta_{12}$ | $\delta_{12}$ | $\emptyset$ |

Conjecture 3.4 will be studied in future works. Now, we present the definition of $\sigma$-antifunctions.

Definition 3.5. Let $A$, $B$ and $C$ be $\sigma$-sets. If $f : A \to B$ and $f^* : A \to C$ are $\sigma$-functions, then we say that $f^*$ is the $\sigma$-antifunction of $f$ if for all $x \in A$ we have that $\{f(x)\} \cup \{f^*(x)\} = \emptyset$ (i.e. $(f^*(x))^* = f(x)$).

Example 3.6. Consider the following $\sigma$-sets $A = \{1_\Theta, 2_\Theta, 3_\Theta\}$, $B = \{1, 2, 3\}$ and $C = \{1^*, 2^*, 4\}$. Now, we define $f : A \to B$ such that $f(1_\Theta) = 1$, $f(2_\Theta) = 2$ and $f(3_\Theta) = 2$ and $f^* : A \to C$ such that $f(1_\Theta) = 1^*$, $f(2_\Theta) = 2^*$ and $f(3_\Theta) = 2^*$. It is clear that $f^*$ is the $\sigma$-antifunction of $f$. Also, we obtain that $\text{ran}(f) = \{1, 2\}$ and $\text{ran}(f^*) = \{1^*, 2^*\}$.

In this sense we obtain the following Theorem.
Theorem 3.7. Let $A$, $B$ and $C$ be $\sigma$-sets. If $f : A \to B$ is a $\sigma$-function and $f^* : A \to C$ is the $\sigma$-antifunction of $f$, then $\text{ran}(f)\cdot$ is the $\sigma$-antisets of $\text{ran}(f^*)$ (i.e. $\text{ran}(f^*) = \text{ran}(f^*),$ and $\text{ran}(f) = \text{ran}(f^*)$).

Proof. Let $f : A \to B$ a $\sigma$-function and $f^* : A \to C$ the $\sigma$-antifunction of $f$. We will prove that $\text{ran}(f) \cup \text{ran}(f^*) = \emptyset$. It is clear that
\[
\text{ran}(f) = \{y \in B : (\exists x \in A)(f(x) = y)\},
\]
\[
\text{ran}(f^*) = \{y \in C : (\exists x \in A)(f^*(x) = y)\}.
\]
Now, by Definition we obtain that
\[
\text{ran}(f) \cup \text{ran}(f^*) = \{x : (x \in \text{ran}(f) \cup \text{ran}(f^*)) \lor (x \in \text{ran}(f^*) \cup \text{ran}(f))\}.
\]
Then, in order to prove that $\text{ran}(f) \cup \text{ran}(f^*) = \emptyset$ it is enough to prove that
\begin{enumerate}[(a):]
\item $\text{ran}(f) \cap \text{ran}(f^*) = \text{ran}(f),$
\item $\text{ran}(f^*) \cap \text{ran}(f) = \text{ran}(f^*).$
\end{enumerate}
We will only prove (a) because the proof of (b) is similar. It is clear by Definition that $\text{ran}(f^*) \subseteq \text{ran}(f)$. Now, let $y \in \text{ran}(f)$ then there exists $x \in A$ such that $f(x) = y$. Since $f^*$ is the $\sigma$-antifunction of $f$, then $\{f(x)\} \cup \{f^*(x)\} = \emptyset$. Therefore, $y \notin \text{ran}(f^*)$ and so $y \in \text{ran}(f) \cap \text{ran}(f^*)$. Finally, $\text{ran}(f) = \text{ran}(f) \cup \text{ran}(f^*)$. \hfill \qed

Theorem 3.8. Let $A$ and $B$ be $\sigma$-sets and $f : A \to B$ a $\sigma$-function. If there exists the $\sigma$-antisets of $\text{ran}(f)$ (i.e. $\text{ran}(f^*)$), then there exists $f^*$ the $\sigma$-antifunction of $f$.

Proof. Since there exists $\text{ran}(f^*)$, then we can define a $\sigma$-set $C$ such that $\text{ran}(f^*) \subseteq C$. Now, we define $f^* : A \to C$ such that $f^*(x) = (f(x))^*$. It is clear by definition that for all $x \in A$ we have that $\{f(x)\} \cup \{f^*(x)\} = \emptyset$. Therefore $f^*$ is the $\sigma$-antifunction of $f$. \hfill \qed

Let $f : A \to B$ and $g : A \to C$ be $\sigma$-functions. We say that $f$ and $g$ are equal if and only if for all $x \in A, f(x) = g(x)$.

Theorem 3.9. Let $A$, $B$ and $C$ be $\sigma$-sets. If $f : A \to B$ is a $\sigma$-function and $f^* : A \to C$ is the $\sigma$-antifunction of $f$, then $f^*$ is unique.

Proof. This proof is obvious by Theorems 1.24 and 3.7 \hfill \qed

We observe that some consequences of the uniqueness of $\sigma$-antielements, $\sigma$-antisets and $\sigma$-antifunctions are the following:
\begin{itemize}
\item If $x^*$ is the $\sigma$-antielement of $x$, then $(x^*)^* = x$.
\item If $A^*$ is the $\sigma$-antiset of $A$, then $(A^*)^* = A$.
\item If $f^*$ is the $\sigma$-antifunction of $f$, then $(f^*)^* = f$.
\end{itemize}

Now, following the same steps as in Set Theory we define the following: If $f : A \to B$ is a $\sigma$-function then
\begin{itemize}
\item $f$ is a $\sigma$-function \textbf{onto} $B$ if $B = \text{ran}(f)$.
f is a $\sigma$-function one-one iff for all $x, y \in A$, $x \neq y \rightarrow f(x) \neq f(y)$.

• $f$ is a $\sigma$-function bijective iff it is both one-one and onto.

Also, we define

• The image of $A$ by $f$, $f(A) = \{y \in B : (\exists x \in A)(f(x) = y)\} = \text{ran}(f)$.
• The preimage of $A$ under $f$, $f^{-1}(A) = \{x \in A : f(x) \in B\}$.

Now, it is clear that: If $f$ is bijective there is a unique $\sigma$-function $f^{-1} : B \rightarrow A$ (called the inverse of $f$) such that

$$(\forall x \in A)(f^{-1}(f(x)) = x).$$

We have chosen the notation $f^{-1}$ to denote the inverse $\sigma$-functions for convenience.

Now, we consider $f : A \rightarrow B$ and $g : C \rightarrow D$ be $\sigma$-functions. If $\text{ran}(g) \cap \text{dom}(f) \neq \emptyset$, then we can define the composition of $f$ and $g$ is the $\sigma$-function $f \circ g$ with domain $\text{dom}(f \circ g) = \{x \in \text{dom}(g) : g(x) \in \text{ran}(g) \cap \text{dom}(f)\}$ such that $(f \circ g)(x) = f(g(x))$ for all $x \in \text{dom}(g)$, that is

$f \circ g : \text{dom}(f \circ g) \rightarrow B.$

Also, we will use the standard definition of identity function: Let $Id_A : A \rightarrow A$ a $\sigma$-function, then we say that $Id_A$ is the identity $\sigma$-function of $A$ iff for all $x \in A$ we have that $Id_A(x) = x$. Now, if we consider the $\sigma$-antiset we have the following definition.

**Definition 3.10.** Let $A$ be a $\sigma$-set. If there exists $A^*$ the $\sigma$-antiset of $A$, then we say that $Id'_A : A \rightarrow A^*$ is the antidentity $\sigma$-function of $A$ iff for all $x \in A$ we have that $Id'_A(x) = x^*$.

We observe that the antidentity $\sigma$-function of $A$ is well defined. In fact, we consider $x \in A$ then by Theorem 1.23 there exists a unique $x^*$ such that $Id'_A(x) = x^* \in A^*$.

**Theorem 3.11.** Let $A$ be a $\sigma$-set such that there exists $A^*$ $\sigma$-antiset of $A$. Then the $\sigma$-antifunction of $Id'_A$ is the identity $\sigma$-function of $A$, that is $(Id'_A)^* = Id_A$.

**Proof.** Consider $Id'_A : A \rightarrow A^*$ the antidentity $\sigma$-function of $A$ and $Id_A : A \rightarrow A$ the identity $\sigma$-function of $A$. Now, let $x \in A$, then $\{Id'_A(x)\} \cup \{Id_A(x)\} = \{x^*\} \cup \{x\} = \emptyset$. Therefore, the $\sigma$-antifunction of $Id'_A$ is the identity $\sigma$-function of $A$.\[\square\]

We observe that the $\sigma$-antifunction of $Id_A$ is the antidentity $\sigma$-function of $A$, that is $Id'_A = (Id_A)^*$.

Now, suppose that there exists the $\sigma$-antifunction $f^*$ of a $\sigma$-function $f$ and $f$ has a property $p$, then we will study in that case whether the $\sigma$-antifunction satisfies this property or a similar one.
Lemma 3.12. Let $A$ be a $\sigma$-set. If there exists $A^*$ the $\sigma$-antiset of $A$, then for all $B \in 2^A$ there exists $D \in 2^{A^*}$ such that $D$ is the $\sigma$-antiset of $B$ (i.e. $D = B^*$).

Proof. Consider $A$ a $\sigma$-set, $A^*$ the $\sigma$-antiset of $A$ and $B \in 2^A$. Now, we define the $\sigma$-set $D = \{x \in A^* : x^* \in B\}$. It is clear that $D \in 2^{A^*}$, now we will prove that $D = B^*$. By Theorem [1.24] we only prove that $B \cup D = \emptyset$. It is clear that in order to prove that $B \cup D = \emptyset$ it is enough to prove that

(a): $B = B \cap D = \{x \in B : x^* \in D\}$.
(b): $D = D \cap B = \{x \in D : x^* \in B\}$.

Also, we have that $B \cap D \subseteq B$ and $D \cap B \subseteq D$.

(a) Let $x \in B$, then $x \in A$ and in consequence $x^* \in A^*$. Now, since $(x^*)^* = x \in B$ then $x^* \in D$. Therefore $x \in B \cap D$ and so $B \cap D = B$.

(b) Let $x \in D$, then $x \in A^*$ and $x^* \in B$. Therefore it is clear that $x \in D \cap B$ and so $D \cap B = D$.

Finally, we obtain that $B \cup D = \emptyset$. □

We observe that by Theorem [3.12] and Lemma [3.12] if $f : A \to B$ is a $\sigma$-function such that there exists $B^*$ the $\sigma$-antiset of $B$, then there exists $f^* : A \to B^*$ the $\sigma$-antifunction of $f$.

Theorem 3.13. Let $A$ and $B$ be $\sigma$-sets such that there exists $B^*$ the $\sigma$-antiset of $B$, $f : A \to B$ a $\sigma$-function and $f^* : A \to B^*$ the $\sigma$-antifunction of $f$. Then the following statements hold:

(a): $f$ is a $\sigma$-function onto $B$ if and only if $f^*$ is a $\sigma$-function onto $B^*$.
(b): $f$ is a one-one $\sigma$-function if and only if $f^*$ is a $\sigma$-function one-one.
(c): $f$ is a bijective $\sigma$-function if and only if $f^*$ is a $\sigma$-function bijective.

Proof. Let $f : A \to B$ a $\sigma$-function such that there exists $B^*$ the $\sigma$-antiset of $B$ and $f^* : A \to B^*$ the $\sigma$-antifunction of $f$.

(a) ($\rightarrow$) Suppose that $\text{ran}(f) = B$. Since $f^*$ is the $\sigma$-antifunction of $f$, then by Theorem [3.7] $\text{ran}(f^*) = \text{ran}(f)^* = B^*$.

($\leftarrow$) Suppose that $\text{ran}(f^*) = B^*$. Since $f$ is the $\sigma$-antifunction of $f^*$, then by Theorem [3.7] $\text{ran}(f) = \text{ran}(f^*)^* = (B^*)^* = B$.

(b) ($\rightarrow$) Suppose that $f$ is a one-one $\sigma$-function. Let $x, y \in A$ such that $f^*(x) = f^*(y)$ and $x \neq y$. Since $f$ is one-one then $f(x) \neq f(y)$. Now, as $f^*$ is the $\sigma$-antifunction of $f$, then $\{f(x)\} \cup \{f^*(x)\} = \emptyset$ and $\{f(y)\} \cup \{f^*(y)\} = \emptyset$. Therefore, if we define $z = f^*(x) = f^*(y)$ then we obtain two different $\sigma$-antelements of $z$, which is a contradiction by Theorem [1.23].

($\leftarrow$) This proof is similar to the previous one.

(c) This proof is a direct consequence of (a) and (b). □
Corollary 3.14. Let $A$ and $B$ be $\sigma$-sets such that there exists $B^*$ the $\sigma$-antiset of $B$, $f : A \to B$ a $\sigma$-function and $f^* : A \to B^*$ the $\sigma$-antifunction of $f$. Then there exists $f^{-1}$ the inverse $\sigma$-function of $f$ if and only if there exists $f_*^{-1}$ the inverse $\sigma$-function of $f^*$.

Proof. This proof is a direct consequence of Theorem 3.13. □

Theorem 3.15. Let $A$ and $B$ be $\sigma$-sets such that there exist $A^*$ and $B^*$ the $\sigma$-antisets of $A$ and $B$ and $f : A \to B$ a bijective $\sigma$-function. Then there exists a unique $\sigma$-function $f_{-1}^* : B^* \to A^*$ (called antinverse of $f$) such that for all $x \in A$ we have that $f_{-1}^*(f^*(x)) = x^*$, where $f^*$ is the $\sigma$-antifunction of $f$.

Proof. Existence: Consider $A$ and $B$, $\sigma$-sets such that there exist $A^*$ and $B^*$ the $\sigma$-antisets of $A$ and $B$ and $f : A \to B$ a bijective $\sigma$-function. By Theorems 3.8 and 3.13 there exists $f^* : A \to B^*$ the $\sigma$-antifunction of $f$ and it is bijective. Therefore, there exists $f_{-1}^*: B^* \to A$ the inverse of $f^*$. Since there exists $A^*$ the $\sigma$-antiset of $A$ by Theorem 3.8 there exists $(f_{-1}^*)^* : B^* \to A^*$ the $\sigma$-antifunction of $f_{-1}^*$. Therefore, we define $f_{-1}^* = (f_{-1}^*)^*$. Since $\text{ran}(f^*) = \text{dom}(f_{-1}^*) = B^*$ then we can define $f_{-1}^* \circ f^* : A \to A^*$. Now, let $x \in A$ then it is clear that $f^*(x) \in B^*$ and $f_{-1}^*(f^*(x)) = x$ because $f_{-1}^*$...
is the inverse \( \sigma \)-function of \( f^* \). Since \( (f_*^{-1})' \) is the \( \sigma \)-antifunction of \( f_*^{-1} \) then 
\[ \{f_*^{-1}(f^*(x))\} \cup \{(f_*^{-1})'(f^*(x))\} = \emptyset. \]
Therefore \( \{x\} \cup \{(f_*^{-1})(f^*(x))\} = \emptyset \) and so \( f_*^{-1}(f^*(x)) = x^* \).

**Uniqueness:** Suppose that there exists \( \hat{f} : B^* \to A^* \) a \( \sigma \)-function such that for all \( x \in A \) we have that \( \hat{f}(f^*(x)) = x^* \) and \( f_*^{-1} \neq \hat{f} \). Since \( f_*^{-1} \neq \hat{f} \) then there exists \( y \in B^* \) such that \( f_*^{-1}(y) \neq \hat{f}(y) \). As \( f \) is bijective, by Theorem 3.13 we obtain that \( f^* \) is bijective. Therefore, since \( y \in B^* \) there exists \( x \in A \) such that \( f^*(x) = y \). Finally, we obtain two different \( \sigma \)-antielements of \( x \), \( f_*^{-1}(y) \) and \( \hat{f}(y) \), which is a contradiction. \( \Box \)

We observe that if we consider that \( A \) and \( B \) are \( \sigma \)-sets such that there exist \( A^* \) and \( B^* \) the \( \sigma \)-antisets of \( A \) and \( B \) and \( f : A \to B \) a bijective \( \sigma \)-function, then we can obtain 16 different \( \sigma \)-functions, that is

1. \( f : A \to B \) a bijective \( \sigma \)-function.
2. \( f_*^{-1} : B \to A \) the inverse \( \sigma \)-function of \( f \).
3. \( (f_*^{-1})' : B \to A \) the \( \sigma \)-antifunction of \( f_*^{-1} \).
4. \( (f_*^{-1})'' : A \to B \) the inverse \( \sigma \)-function of \( (f_*^{-1})' \).
5. \( f^* : A \to B^* \) the \( \sigma \)-antifunction of \( f \).
6. \( f_*^* : B^* \to A^* \) the inverse \( \sigma \)-function of \( f^* \).
7. \( f_*^{-1} : B^* \to A^* \) the antinverse \( \sigma \)-function of \( f \).
8. \( (f_*^{-1})^{-1} : A^* \to B^* \) the inverse \( \sigma \)-function of \( f_*^{-1} \).
9. \( Id_A : A \to A \) the identity \( \sigma \)-function of \( A \).
10. \( Id_B : B \to B \) the identity \( \sigma \)-function of \( B \).
11. \( Id_{A^*} : A^* \to A^* \) the identity \( \sigma \)-function of \( A^* \).
12. \( Id_{B^*} : B^* \to B^* \) the identity \( \sigma \)-function of \( B^* \).
13. \( Id_A^* : A \to A^* \) the antidentity \( \sigma \)-function of \( A \).
14. \( Id_{B^*}^* : A^* \to A^* \) the antidentity \( \sigma \)-function of \( A^* \).
15. \( Id_{B^*} : B \to B^* \) the antidentity \( \sigma \)-function of \( B \).
16. \( Id_A^* : B \to B^* \) the antidentity \( \sigma \)-function of \( B^* \).

See Figure [3] The properties of this \( \sigma \)-function will be studied in future works.

**References**

[1] I. Gatica, \( \sigma \)-Set Theory: Introduction to the concepts of \( \sigma \)-antielement, \( \sigma \)-antiset and Integer Space, (Preprint) (2010).

ECARIO MAE-AECI

**Department of Mathematical Analysis, University of Sevilla, St. Tarfia s/n, Sevilla, Spain**

**Department of Mathematics, University Andrés Bello, Los Fresnos 52, Viña del mar, Chile**

E-mail address: igatica@us.es