GEOMETRY AND $\mathcal{W}$-GRAVITY

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ABSTRACT

The higher-spin geometries of $\mathcal{W}_\infty$-gravity and $\mathcal{W}_N$-gravity are analysed and used to derive the complete non-linear structure of the coupling to matter and its symmetries. The symmetry group is a subgroup of the symplectic diffeomorphisms of the cotangent bundle of the world-sheet, and the $\mathcal{W}_N$ geometry is obtained from a non-linear truncation of the $\mathcal{W}_\infty$ geometry. Quantum $\mathcal{W}$-gravity is briefly discussed.

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### 1. Introduction

Infinite-dimensional symmetry algebras play a central rô le in two-dimensional physics and there is an intimate relationship between such algebras and two-dimensional gauge theories or string theories. Perhaps the most important example is the Virasoro algebra, which is a symmetry algebra of any two-dimensional conformal field theory. This infinite-dimensional rigid symmetry can be promoted to a local symmetry (two-dimensional diffeomorphisms) by coupling the two-dimensional field theory to gravity, resulting in a theory that is Weyl-invariant as well as diffeomorphism invariant. The two-dimensional metric enters the theory as a Lagrange multiplier imposing constraints which satisfy the Virasoro algebra, and the Virasoro algebra also emerges as the residual symmetry that remains after choosing a conformal gauge. The quantisation of such a system of matter coupled to gravity defines a string theory and if the matter system is chosen such that the world-sheet metric $h_{\mu\nu}$ decouples from the quantum theory (i.e. if the matter central charge is $c = 26$), the string theory is said to be critical [1]. Remarkably, the introduction of gravity on the world-sheet leads to a critical string theory which leads to gravity in space-time.

The situation is similar for each of the cases in the following table.

| Algebra                | Spins | 2-D Gauge Theory | Gauge Fields | String Theory |
|------------------------|-------|------------------|--------------|---------------|
| Virasoro Algebra       | 2     | Gravity          | $h_{\mu\nu}$ | Bosonic String |
| Super-Virasoro         | $2, \frac{3}{2}$ | Supergravity     | $h_{\mu\nu}, \psi_{\mu}$ | Superstring |
| $N = 2$ Super-Virasoro | $2, \frac{3}{2}, 1$ | $N = 2$ Supergravity | $h_{\mu\nu}, \psi_{\mu}, \bar{\psi}_{\mu}, A_{\mu}$ | $N = 2$ Superstring |
| Topological Virasoro   | $2, 2, 1, 1$ | Topological Gravity | $h_{\mu\nu}, g_{\mu\nu}, A_{\mu}, \psi_{\mu}$ | Topological String |
| $\mathcal{W}$-Algebra  | $2, 3, \ldots$ | $\mathcal{W}$-Gravity | $h_{\mu\nu}, B_{\mu\nu\rho}, \ldots$ | $\mathcal{W}$-Strings |

In the first column are extended conformal algebras, i.e. infinite dimensional algebras that contain the Virasoro algebra. Each algebra is generated by a set of currents, the spins of which are given in the second column. Each algebra can arise as the symmetry algebra of a particular class of conformal field theories e.g. the super-Virasoro algebra is a symmetry of super-conformal field theories while the
topological Virasoro algebra is a symmetry of topological conformal field theories. For such theories, the infinite-dimensional rigid symmetry of the matter system can be promoted to a local symmetry by coupling to the gauge theory listed in the third column. In this coupling, the currents generating the extended conformal algebra are coupled to the corresponding gauge fields in the fourth column. In each case, the gauge fields enter as Lagrange multipliers and the constraints that they impose satisfy the algebra given in the first column. Finally, integration over the matter and gauge fields defines a generalisation of string theory which is listed in the last column. In general, the gauge fields will become dynamical in the quantum theory, but for special choices of conformal matter system (e.g. \( c = 26 \) systems for the bosonic string, \( c = 0 \) systems for the topological string or \( c = 15 \) systems for the \( N = 1 \) superstring), the string theory will be ‘critical’ and the gauge fields will decouple from the theory. A row can be added to the table corresponding to any two-dimensional extended conformal algebra.

Consider now the set of models corresponding to the last row of the table. A \( \mathcal{W} \)-algebra might be defined as any extended conformal algebra, i.e. a closed algebra that satisfies the Jacobi identities, contains the Virasoro algebra as a subalgebra and is generated by a (possibly infinite) set of chiral currents (for a review, see [2]). Often the definition of \( \mathcal{W} \)-algebra is restricted to those algebras for which at least one of the generating currents has spin greater than 2, but relaxing this condition allows the definition to include all the algebras in the table and almost all of the results to be reviewed here apply with this more general definition. However, many (but not all) interesting \( \mathcal{W} \)-algebras contain a spin-three current, and for this reason 3 is included as a typical higher spin in the \( \mathcal{W} \)-algebra entry in the table.

The infinite-dimensional \( \mathcal{W} \)-algebra symmetry can be promoted to a local symmetry by coupling to suitable gauge fields, to obtain a theory of matter coupled to \( \mathcal{W} \)-gravity [3-11]; for a review, see [12]. Again, the gauge fields will in general become dynamical in the quantum theory [13] but for matter systems for which the \( \mathcal{W} \)-algebra central charge takes a particular critical value, the \( \mathcal{W} \)-gravity gauge fields decouple and it is possible to interpret the theory as a \( \mathcal{W} \)-string [14]; in a
\( W \)-conformal gauge, this can be thought of as a conformal field theory satisfying constraints which generate a \( W \)-algebra instead of the usual Virasoro constraints.

The simplest \( W \)-algebras are those that are Lie algebras, with the generators \( t_a \) (labelled by an index \( a \) which will in general have an infinite range) satisfying commutation relations of the form

\[
[t_a, t_b] = f_{ab}^\ c t_c + c_{ab}
\]  

for some structure constants \( f_{ab}^\ c \) and constants \( c_{ab} \), which define a central extension of the algebra. However, for many \( W \)-algebras, the commutation relations give a result non-linear in the generators

\[
[t_a, t_b] = f_{ab}^\ c t_c + c_{ab} + g_{abcd} t_c t_d + \ldots = F_{ab}(t_c)
\]  

and the algebra can be said to close in the sense that the right-hand-side is a function of the generators. Most of the \( W \)-algebras that are generated by a finite number of currents, with at least one current of spin greater than two, are non-linear algebras of this type. Classical \( W \)-algebras for which the bracket in (1.2) is a Poisson bracket are straightforward to define, as the non-linear terms on the right-hand-side can be taken to be a product of classical charges. For quantum \( W \)-algebras, however, the bracket is realised as a commutator of quantum operators and the definition of the right-hand-side requires some normal-ordering prescription. The complications associated with the normal-ordering mean that there are classical \( W \)-algebras for which there is no corresponding quantum \( W \)-algebra that satisfies the Jacobi identities [7]. At first sight, it appears that there might be a problem in trying to realise a non-linear algebra in a field theory, as symmetry algebras are usually Lie algebras. However, as will be seen, a non-linear algebra can be realised as a symmetry algebra for which the structure ‘constants’ are replaced by field-dependent quantities.
Consider a field theory in flat Minkowski space with metric $\eta_{\mu\nu}$ and coordinates $x^0, x^1$. The stress-energy tensor is a symmetric tensor $T_{\mu\nu}$ which, for a translation-invariant theory, satisfies the conservation law

$$\partial^{\mu}T_{\mu\nu} = 0 \quad (1.3)$$

A spin-$s$ current in flat two-dimensional space is a rank-$s$ symmetric tensor $W_{\mu_1\mu_2...\mu_s}$ and will be conserved if

$$\partial^{\mu_1}W_{\mu_1\mu_2...\mu_s} = 0 \quad (1.4)$$

A theory is conformally invariant if the stress tensor is traceless, $T_{\mu}^{\mu} = 0$. Introducing null coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, the tracelessness condition becomes $T^{+\pm} = 0$ and (1.3) then implies that the remaining components $T_{\pm\pm}$ satisfy

$$\partial^+ T^{-\pm} = 0, \quad \partial^- T^{++} = 0 \quad (1.5)$$

If a spin-$s$ current $W_{\mu_1\mu_2...\mu_s}$ is traceless, it will have only two non-vanishing components, $W_{++...+}$ and $W_{--...-}$. The conservation condition (1.4) then implies that

$$\partial^{-} W_{++...+} = 0, \quad \partial^{+} W_{--...-} = 0 \quad (1.6)$$

so that $W_{++...+} = W_{++...+}(x^+)$ and $W_{--...-} = W_{--...-}(x^-)$ are right- and left-moving chiral currents, respectively. For a given conformal field theory, the set of all right-moving chiral currents generate a closed current algebra, the right-moving chiral algebra, and similarly for left-movers. The right and left chiral algebras are examples of $\mathcal{W}$-algebras but are often too large to be useful. In studying

\* Recall that, in two dimensions, any tensor can be decomposed into a set of symmetric tensors, e.g. $V_{\mu\nu} = V_{(\mu\nu)} + V_{\nu\mu}$ where $V = \frac{1}{2}\epsilon_{\mu\nu}V_{\mu\nu}$. Thus without loss of generality, all the conserved currents of a given theory can be taken to be symmetric tensors. A rank-$s$ symmetric tensor transforms as the spin-$s$ representation of the two-dimensional Lorentz group.
conformal field theories, it is often useful to restrict attention to all theories whose chiral algebras contain a particular $\mathcal{W}$-algebra; the representation theory of that $\mathcal{W}$-algebra then gives a great deal of useful information about the spectrum, modular invariants etc of those theories and may lead to a classification.

A field theory with action $S_0$ and symmetric tensor conserved currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_{s_A}}$ (where $A = 1, 2, \ldots$ labels the currents, which have spin $s_A$) will be invariant under rigid symmetries with constant parameters $k^\mu, \lambda_A^{\mu_1\mu_2...\mu_{s_A}-1}$ (translations and ‘$\mathcal{W}$-translations’) generated by the Noether charges $P_\mu, Q^A_{\mu_1\mu_2...\mu_{s_A}-1}$ (momentum and ‘$\mathcal{W}$-momentum’) given by $P_\mu = \int dx^0 T_{0\mu}$ and $Q^A_{\mu_1\mu_2...\mu_{s_A}-1} = \int dx^0 W^A_{\mu_1\mu_2...\mu_{s_A}-1}$. This is true of non-conformal theories (e.g. affine Toda theories) as well as conformal ones. However, if the currents are traceless, then the theory is in fact invariant under an infinite dimensional extended conformal symmetry. The parameters $\lambda_A^{\mu_1\mu_2...\mu_{s_A}-1}$ are then traceless and the corresponding transformations will be symmetries if the parameters are not constant but satisfy the conditions that the trace-free parts of $\partial^\nu k^\mu, \partial^\nu \lambda_A^{\mu_1\mu_2...\mu_{s_A}-1}$ are zero. This implies that $\partial^\pm k^\pm = 0$ and $\partial^\pm \lambda_A^{\pm...\pm} = 0$ so that the parameters are ‘semi-local’, $k^\pm = k^\pm (x^\pm)$ and $\lambda_A^{\pm...\pm} = \lambda_A^{\pm...\pm} (x^\pm)$ and these are the parameters of conformal and ‘$\mathcal{W}$-conformal’ transformations.

The rigid symmetries corresponding to the currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_{s_A}}$ can be promoted to local ones by coupling to the $\mathcal{W}$-gravity gauge fields $h^{\mu\nu}, B^A_{\mu_1\mu_2...\mu_{s_A}}$ which are symmetric tensors transforming as

$$
\delta h^{\mu\nu} = \partial^\nu k^\mu + \ldots, \quad \delta B^A_{\mu_1\mu_2...\mu_s} = \partial^\nu \lambda_A^{\mu_1\mu_2...\mu_{s-1}} + \ldots,
$$

(1.7)

to lowest order in the gauge fields. The action is given by the Noether coupling

$$
S = S_0 + \int d^2 x \left( h^{\mu\nu} T_{\mu\nu} + B^A_{\mu_1\mu_2...\mu_{s_A}} W^A_{\mu_1\mu_2...\mu_{s_A}} \right) + \ldots
$$

(1.8)

plus terms non-linear in the gauge fields. If the currents $T_{\mu\nu}, W^A_{\mu_1\mu_2...\mu_s}$ are traceless, i.e. if there is extended conformal symmetry, then the traces of the gauge fields
decouple and the classical theory is invariant under Weyl and ‘\(W\)-Weyl’ transformations given to lowest order in the gauge fields by

\[
\delta h^{\mu\nu} = \Omega \eta^{\mu\nu} + \ldots, \quad \delta B_A^{\mu_1\mu_2\ldots\mu_s} = \Omega_A^{(\mu_1\mu_2\ldots\mu_{s-2}\eta^{\mu_{s-1}\mu_s})} + \ldots
\]

where \(\Omega(x^\nu), \Omega_A^{\mu_1\mu_2\ldots\mu_{s-2}}(x^\nu)\) are the local parameters. This defines the linearised coupling to \(W\)-gravity. The full non-linear theory is in general non-polynomial in the gauge fields of spins 2 and higher.

The key to the non-polynomial structure of the coupling of matter to gravity is the tensor calculus and the construction of the fundamental density \(\sqrt{g}\). These have a natural interpretation in terms of Riemannian geometry, which is based on a line element \(ds = (g_{\mu\nu}dx^\mu dx^\nu)^{1/2}\). A higher spin generalisation of Riemannian geometry is needed in order to describe the geometry of \(W\)-gravity. A number of approaches have been proposed in the literature [18-24]; here we shall review the approach of [15,16,17]. A spin-\(n\) gauge field could be used to define a geometry based on a line element \(ds = (g_{\mu_1\mu_2\ldots\mu_n}dx^{\mu_1}dx^{\mu_2}\ldots dx^{\mu_n})^{1/n}\) (first considered by Riemann [31]). A further generalization is to consider a line element \(ds = N(x, dx)\) where \(N\) is some function which is required to satisfy the homogeneity condition \(N(x, \lambda dx) = \lambda N(x, dx)\). This defines a Finsler geometry [32] and generalises the spin-\(n\) Riemannian line element. To describe \(W\)-gravity, it seems appropriate to generalise still further and drop the homogeneity condition on \(N\). Instead of working with \(N^2(x, dx)\), which is a function on the tangent bundle of the space-time \(M\) generalising the metric \((N^2 = g_{\mu\nu}dx^\mu dx^\nu + \ldots)\), it will prove more convenient to consider functions \(F(x, y)\) on the cotangent bundle of \(M\), generalising the inverse metric \((F(x^\mu, y_{\mu}) = g^{\mu\nu}(x)y_{\mu}y_{\nu} + \ldots)\). Expanding in the fibre coordinate \(y\) gives a set of higher spin gauge fields (see eqn (6.1)). Such a geometric interpretation of the results reviewed here seems appropriate as they have a natural expression in terms of functions on the cotangent bundle; for further discussion, see [16].

Given a generalised line element \(F(x^\mu, y_{\mu})\), one can demand that it be preserved under diffeomorphisms \(x \rightarrow f(x)\), and so derive the tensorial transformation rules
of the coefficients $g^{\mu\nu}(x)$, $g^{\mu\nu\rho}(x), \ldots$ occurring in the Taylor expansion of $F(x,y)$ with respect to $y$. However, one can consider a much larger group of transformations of the form $x \to f(x,y)$, $y \to g(x,y)$ consisting of diffeomorphisms of the cotangent bundle $T^*M$ and it is natural to ask whether such transformations could lead to the higher-spin symmetries of $\mathcal{W}$-gravity. Expanding the infinitesimal transformation $\delta x^\nu = k^\nu(x, y) = \sum k(x)^{\nu \mu_1 \ldots \mu_n} y_{\mu_1} \ldots y_{\mu_n}$ gives a set of higher-spin transformations ($k^{\nu \mu_1 \ldots \mu_n}$ is the parameter for a local spin-$(n+2)$ transformation).

However, this group of transformations turns out to be too large for the present purposes; it is much larger than the symmetry group of $w_\infty$-gravity. However, the subgroup of this consisting of symplectic diffeomorphisms of $T^*M$ turns out to play an important role in $\mathcal{W}$-gravity and contains the symmetry group of $\mathcal{W}$-gravity.

If $M$ is a $D$-dimensional space with coordinates $x^\mu$, the cotangent bundle $T^*M$ is a $2D$-dimensional space whose coordinates can be taken to be $x^\mu, y_\mu$ where $y_\mu$ transform as covariant vectors under diffeomorphisms of $M$. The cotangent bundle is a symplectic manifold with natural two-form $\Omega = dx^\mu_\Lambda dy_\mu$ and the subgroup of the diffeomorphisms of $T^*M$ that preserve the symplectic structure $\Omega$ (i.e. the canonical transformations for the phase space $T^*M$ with coordinates $x^\mu$ and momenta $y_\mu$) is the group of symplectic diffeomorphisms of $T^*M$.

The symplectic diffeomorphisms have an interesting field theoretic realisation. Consider the transformation

$$\delta \phi = \sum_{n=2}^{\infty} \lambda^{\mu_1 \mu_2 \ldots \mu_{n-1}}(x^\nu) \partial_{\mu_1} \phi \partial_{\mu_2} \phi \ldots \partial_{\mu_{n-1}} \phi \equiv \Lambda(x^\mu, y_\mu)$$

(1.10)

where $y_\mu = \partial_\mu \phi$ and the $\lambda^{\mu_1 \mu_2 \ldots \mu_{n-1}}(x^\nu)$ ($n = 2, 3, \ldots$) are infinitesimal parameters which are symmetric tensor fields on $M$. These transformations satisfy the algebra

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\{\Lambda, \Lambda'\}}$$

(1.11)
where the Poisson bracket is given by

\[
\{\Lambda, \Lambda'\} = \frac{\partial \Lambda}{\partial x^\mu} \frac{\partial \Lambda'}{\partial y_\mu} - \frac{\partial \Lambda'}{\partial x^\mu} \frac{\partial \Lambda}{\partial y_\mu}
\] (1.12)

(Note that the $\partial_\mu y_\nu$ terms cancel in (1.12).) This is precisely the algebra of symplectic diffeomorphisms of $T^* M$. In the one-dimensional case $M = S^1$, $T^* M$ is the cylinder $S^1 \times \mathbb{R}$ and the algebra of symplectic diffeomorphisms is the algebra $w_\infty$ introduced in [28], while (1.10) is the realisation given in [5].

The $n = 2$ term in (1.10) is $\delta \phi = \lambda^{\mu}_{(2)} \partial_\mu \phi$, which is an infinitesimal coordinate transformation on $M$ generated by the vector field $\lambda^{\mu}_{(2)}$, while for higher $n$, (1.10) gives a set of non-linear higher spin generalisations of this. The transformations (1.10) have been suggested as the basis of higher-spin generalisations of gravity [20,5], but unfortunately actions that are invariant under transformations of the type (1.10) only exist in one space-time dimension, $D = 1$ [7,16]. Nevertheless, in two-dimensions, it is known that a $w_\infty$ gravity theory exists and an implicit construction of the action has been given [5] using the methods of [4]. Just as the coupling of $\phi$ to gravity gives a theory with a spin-two gauge invariance which, on going to conformal gauge, reduces to a residual symmetry consisting of two copies of the Virasoro algebra (the algebra of diffeomorphisms of $S^1$), the coupling of $\phi$ to $w_\infty$-gravity gives a theory with gauge invariances of spins 2, 3, 4, \ldots which, on going to a $\mathcal{W}$-conformal gauge, reduce to a residual symmetry consisting of two copies of the $w_\infty$ algebra (the algebra of symplectic diffeomorphisms of $T^* S^1$). The geometry of two-dimensional $w_\infty$-gravity and the relation of its symmetry to the symplectic diffeomorphisms (1.10) will now be analysed and the truncation to $\mathcal{W}_N$ gravity described.
2. Linearised $w_\infty$ Gravity

Linearised $w_\infty$ gravity can be constructed by perturbing about a flat two-dimensional space $M_0$ with metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 2dzd\bar{z}$, where $z = \frac{1}{\sqrt{2}} (x^1 + ix^2)$, $\bar{z} = \frac{1}{\sqrt{2}} (x^1 - ix^2)$ are complex coordinates if $M$ is Euclidean, while, if $M$ is Lorentzian, $z = \frac{1}{\sqrt{2}} (x^1 + x^2)$, $\bar{z} = \frac{1}{\sqrt{2}} (x^1 - x^2)$ are null real coordinates.

If a rank $s$ symmetric tensor $T_{\mu_1\mu_2...\mu_s}$ is traceless, $\eta^{\mu\nu} T_{\mu\nu\rho\sigma...} = 0$, then it has only two non-vanishing components, $T_{zzz...z}$ and $T_{zzz...\bar{z}}$.

Consider the free scalar field action

$$S_0 = \frac{1}{2} \int d^2x \, \partial_\mu \phi \partial^\mu \phi$$ (2.1)

This has an infinite number of conserved currents, which include [5]

$$W_n = \frac{1}{n} (\partial \phi)^n, \quad n = 2, 3, ....$$ (2.2)

and these satisfy the conservation law $\bar{\partial} W_n = 0$ (where $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$). These currents generate the transformations

$$\delta_L \phi = \sum_{n=2}^\infty \lambda_n(z) (\partial_z \phi)^{n-1} \equiv L(z, \partial_z \phi)$$ (2.3)

which are symmetries of (2.1) provided the parameters are holomorphic, $\bar{\partial} \lambda_n = 0$. For $n = 2$, $W_2 = T_{zz}$ is the stress tensor and the $\lambda_2$ term in (2.3) is a conformal transformation. These transformations satisfy the $w_\infty$ algebra $[\delta_L, \delta_{L'}] = \delta_{\{L, L'\}}$ where, for any $L(z, w)$, $L'(z, w)$ with $w = \partial_z \phi$, the Poisson bracket is $\{L, L'\} = \partial_z L \partial_w L' - \partial_z L' \partial_w L$. Similarly, the anti-holomorphic currents $\bar{W}_n = \frac{1}{n} (\bar{\partial} \phi)^n$ generate a second copy of $w_\infty$ (which commutes with the first) and correspond to transformations $\delta \phi = \sum_{n=2}^\infty \bar{\lambda}_n(\bar{z}) (\bar{\partial} \phi)^{n-1}$. Note that with a Lorentzian signature, $\lambda$ and $\bar{\lambda}$ are independent real parameters, but in Euclidean signature, they are complex and satisfy $\bar{\lambda}_n = (\lambda_n)^*$. 

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The parameters $\lambda_n, \bar{\lambda}_n$ can be regarded as the two components of a rank $n-1$ symmetric tensor $\lambda_{(n)}^{\mu_1\mu_2...\mu_{n-1}}$ satisfying the tracelessness condition

$$\eta_{\mu\nu} \lambda_{(n)}^{\mu\nu...\sigma} = 0 \quad (2.4)$$

with $\lambda_n = \lambda_n^{zz...z}, \bar{\lambda}_n = \lambda_n^{\bar{z}z...\bar{z}}$. The conditions $\bar{\partial}\lambda_n = 0, \partial\bar{\lambda}_n = 0$ become the condition that the trace-free part of $\partial(\tau \lambda_{(n)}^{\mu\nu...\sigma})$ vanishes. Then the $\lambda_n$ and $\bar{\lambda}_n$ transformations can be rewritten as (1.10) with the parameters $\lambda_{(n)}^{\mu...\sigma}$ satisfying these two constraints; the constraint (2.4) can be rewritten as

$$\eta_{\mu\nu} \frac{\partial^2 \Lambda}{\partial y_\mu \partial y_\nu} = 0 \quad (2.5)$$

Consider now the promotion of the 'W-conformal symmetry' with (anti-) holomorphic parameters $\lambda_n(z) (\bar{\lambda}_n(\bar{z}))$ to a full gauge symmetry with local parameters $\lambda_n(z, \bar{z}), \bar{\lambda}_n(z, \bar{z})$ under which $\phi$ transforms as

$$\delta \phi = \sum_{n=2}^{\infty} \left[ \lambda_n(z, \bar{z})(\partial\phi)^{n-1} + \bar{\lambda}_n(z, \bar{z})(\bar{\partial}\phi)^{n-1} \right] \quad (2.6)$$

To do this, it is necessary to introduce gauge fields $h_n, \bar{h}_n$ transforming as

$$\delta h_n = -2\bar{\partial}\lambda_n + O(h), \quad \delta \bar{h}_n = -2\partial\bar{\lambda}_n + O(h) \quad (2.7)$$

Then, the Noether coupling action given by

$$S = \int d^2x \left[ \partial \bar{\phi} \partial \phi + \sum_{n=2}^{\infty} \frac{1}{n} (h_n(\partial\phi)^n + \bar{h}_n(\bar{\partial}\phi)^n) + O(h^2) \right] \quad (2.8)$$

is invariant, to lowest order in the gauge fields, under (2.6),(2.7). The full gauge-invariant action [5,9] is non-polynomial in the gauge fields, and the gauge transformations (2.6),(2.7) are also modified by non-polynomial terms.
The action (2.8) can be rewritten as

\[ S = \int d^2 x \left[ \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sum_{n=2}^\infty \frac{1}{n} \tilde{h}_{(n)}^{\mu_1...\mu_n} \partial_{\mu_1} \phi ... \partial_{\mu_n} \phi + O(h^2) \right] \equiv \int d^2 x \tilde{F}(x, y) \tag{2.9} \]

where the \( \tilde{h}_{(n)}^{\mu_1...\mu_n} \) are symmetric tensor gauge fields satisfying

\[ \eta_{\mu\nu} \tilde{h}_{(n)}^{\mu\nu\rho...\sigma} = 0 + O(h^2) \tag{2.10} \]

at least to lowest order in the gauge fields, so that the only non-vanishing components are \( h_n = \tilde{h}_{(n)}^{zzz...z} \), \( \bar{h}_n = \tilde{h}_{(n)}^{zzz...z} \). The constraint (2.10) can be rewritten, to lowest order, as (using (2.9))

\[ \eta_{\mu\nu} \frac{\partial^2}{\partial y_\mu \partial y_\nu} \left[ \tilde{F} - \frac{1}{2} y_\mu y^\mu \right] = 0 + .... \tag{2.11} \]

Note that it is not strictly necessary to impose the trace condition (2.10). If it is not imposed, then the action (2.9) remains invariant to linearised order, provided the extra gauge fields corresponding to the traces of \( \tilde{h}_{(n)}^{\mu\nu\rho...\sigma} \) are inert under the gauge transformations, at least to lowest order in the gauge fields. However, this theory is reducible in the sense that it has more gauge fields than symmetries, and can be consistently truncated to one with gauge fields satisfying (2.10). If (2.10) is imposed, then the constraint can be solved in terms of unconstrained gauge fields \( h_{(n)}^{\mu...\sigma} \) by \( \tilde{h}_{(n)}^{\mu...\sigma} = h_{(n)}^{\mu...\sigma} - traces \), while the constraint (2.4) can be solved in terms of unconstrained parameters \( k_{(n)}^{\mu...\sigma} \) by \( \lambda_{(n)}^{\mu...\sigma} = k_{(n)}^{\mu...\sigma} - traces \). The gauge fields \( h_{(n)}^{\mu...\sigma} \) can be taken to transform as

\[ \delta h_{(n)}^{\mu\nu\rho...\sigma} = \partial^\mu k_{(n)}^{\nu\rho...\sigma} + \eta^{\mu\nu} \sigma_{(n)}^{\rho...\sigma} + O(h) \tag{2.12} \]

where \( \sigma_{(n)}^{\mu_1...\mu_{n-2}}(x) \) is the parameter for local \( \mathcal{W} \)-Weyl transformations which correspond, to linearised order, to shifts of the traces of the gauge fields \( h_{(n)} \). For \( n = 2 \), \( \eta^{\mu\nu} + h_{(2)}^{\mu\nu} \) is the linearised inverse metric tensor and \( \sigma_{(2)} \) is the parameter of Weyl transformations.
3. Linear and Non-Linear Gravity

Before proceeding to non-linear $\mathcal{W}$-gravity, it is useful to compare with gravity.

The linearised coupling of a scalar to gravity is given by setting $h(n) = 0$ for $n > 2$ in (2.9), and the full action is non-polynomial in the gauge field $\tilde{h}^{\mu\nu}$. The field $\phi$ transforms as

$$\delta \phi = \lambda^{\mu} \partial_{\mu} \phi$$  \hspace{1cm} (3.1)

corresponding to the $n = 2$ part of (2.6). The full action must be quadratic in $\partial_{\mu} \phi$ on dimensional grounds and so takes the form

$$S = \frac{1}{2} \int d^{2}x \tilde{g}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$$  \hspace{1cm} (3.2)

for some $\tilde{g}^{\mu\nu}(x)$. Comparing with the linearised theory, one finds that $\tilde{g}^{\mu\nu}$ is a non-polynomial function of $\tilde{h}^{\mu\nu}$. However, it is useful to forget about the original variables $\tilde{h}^{\mu\nu}$ and instead to study the non-linear theory using the variable $\tilde{g}^{\mu\nu}(x)$, which has the virtue that it appears linearly in the action (3.2). The action (3.2) is diffeomorphism invariant if the field $\tilde{g}^{\mu\nu}(x)$ transforms as a tensor density, i.e. (3.2) is invariant under (3.1) and

$$\delta \tilde{g}^{\mu\nu} = \lambda^{\rho} \partial_{\rho} \tilde{g}^{\mu\nu} - 2\tilde{g}^{\rho(\mu} \partial_{\rho} \lambda^{\nu)} + \tilde{g}^{\mu\nu} \partial_{\rho} \lambda^{\rho}$$  \hspace{1cm} (3.3)

This action can be used for any manifold $M$ and makes no reference to any background metric.

This formulation is reducible, as it uses three gauge fields (the components of $\tilde{g}^{\mu\nu}$) for two gauge symmetries. An irreducible theory is obtained by imposing the constraint

$$det (\tilde{g}^{\mu\nu}(x)) = \epsilon$$  \hspace{1cm} (3.4)

(where $\epsilon = 1$ for Euclidean signature or $\epsilon = -1$ for Lorentzian signature). This is preserved under (3.3) and so can be consistently imposed, and reduces to $\eta_{\mu\nu} \tilde{h}^{\mu\nu} = $
0 + ... (the $n = 2$ part of (2.10)) in the linearised theory, where $\tilde{g}^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu} + O(h^2)$. The constraint (3.4) can be solved in terms of an unconstrained tensor $g^{\mu\nu}$ by writing $\tilde{g}^{\mu\nu} = \sqrt{\epsilon} g^{\mu\nu}$ where $g = [\text{det}(g^{\mu\nu})]^{-1}$. This solution is invariant under the Weyl symmetry $g^{\mu\nu} \rightarrow \sigma g^{\mu\nu}$, so that $\tilde{g}^{\mu\nu}$ depends on only two of the three components of $g^{\mu\nu}$, the other component being pure gauge.

4. Non-Linear $\mathcal{W}$-Gravity

The non-linear structure of $w_\infty$ gravity will now be presented, following the approach of the previous section. The proof of these results will be given elsewhere. The action is a non-polynomial function of $\partial_\mu \phi$ and can be written as

$$S = \int_M d^2x \tilde{F}(x, \partial \phi)$$

for some $\tilde{F}$, which has the following expansion in $y_\mu = \partial_\mu \phi$:

$$\tilde{F}(x, y) = \sum_{n=2}^\infty \frac{1}{n} \tilde{g}^{\mu_1\mu_2\ldots\mu_n}_{(n)} (x) y_{\mu_1} y_{\mu_2} \ldots y_{\mu_n}$$

where $\tilde{g}^{\mu_1\mu_2\ldots\mu_n}_{(n)} (x)$ are gauge fields. The action will be invariant under diffeomorphisms of $M$ if these gauge fields are tensor densities on $M$. No constraints will be imposed on the $\tilde{g}_{(n)}$ to start with, so that the linearised form of this action is the reducible theory given by (2.9) without the constraint (2.10). The next step is to seek symmetries of this action whose linearised form agrees with those of section 2. The variation of $\phi$ can be taken to be

$$\delta \phi = \Lambda(x, \partial \phi)$$

where

$$\Lambda(x^\mu, y_\mu) = \sum_{n=2}^\infty \lambda^{\mu_1\mu_2\ldots\mu_{n-1}}_{(n)} (x) y_{\mu_1} y_{\mu_2} \ldots y_{\mu_{n-1}}$$

for some parameters $\lambda^{\mu_1\mu_2\ldots\mu_{n-1}}_{(n)} (x)$. From the linearised analysis, one would expect there to be a symmetry of this kind only if a constraint is imposed on the parameters
which is given by (2.5) plus higher order terms in the gauge fields. This is indeed
the case and the full non-linear constraint is

\[ \epsilon^{\mu\rho} \epsilon^{\nu\sigma} \frac{\partial^2 \Lambda}{\partial y_\mu \partial y_\nu} \frac{\partial^2 \tilde{F}}{\partial y_\rho \partial y_\sigma} = 0 \]  

(4.5)

where \( \epsilon^{\mu\nu} \) is the alternating tensor density on \( M \). This involves no background
metric and expanding in \( y \) gives a sequence of non-linear algebraic constraints on
the parameters which, in the linearised approximation, reduce to (2.4). With this
constraint on the parameters, the action (4.1) is invariant (up to a surface term)
under the transformations given by (4.3),(4.4) and

\[
\delta \tilde{g}^{\mu_1\mu_2...\mu_p}_{(p)} = \sum_{m,n=2}^{\infty} \delta_{m+n,p+2} \left[ (m - 1) \lambda^{(m)}_{(m)} \partial_\nu \tilde{g}^{\nu...\mu_p}_{(n)} - (n - 1) \tilde{g}^{\nu\mu_1...\mu_p}_{(n)} \partial_\nu \lambda_{(m)}^{\lambda...\mu_p} \right] \\
+ \frac{(m - 1)(n - 1)}{p - 1} \partial_\nu \left\{ \lambda^{(m)}_{(m)} \tilde{g}^{\nu...\mu_p}_{(n)} - \tilde{g}^{\nu\mu_1...\mu_p}_{(n)} \lambda^{\lambda...\mu_p}_{(m)} \right\}
\]  

(4.6)

From (4.6), the \( \tilde{g}_{(s)} \) transform as tensor densities under the \( \lambda_{(2)} \) transformations,
as expected.

This gives an invariant action which is, however, reducible. To obtain the
irreducible theory, it is necessary to find constraints on the gauge fields that are
preserved by the transformations (4.6) and which reduce to (2.11) in the linearised
limit. The unique such constraint is given by

\[
\text{det} \left( \frac{\partial^2 \tilde{F}(x,y)}{\partial y_\mu \partial y_\nu} \right) = \epsilon
\]

(4.7)

where \( \epsilon = 1 \) for Euclidean signature and \( \epsilon = -1 \) for Lorentzian signature. This
constraint takes a strikingly simple form which does not depend on any background
geometry. Expanding (4.7) in \( y_\mu \) gives an infinite number of constraints on the
density gauge fields \( \tilde{g}^{\mu\nu...}_{(n)} \) (including (3.4) for \( n = 2 \)) which are invariant under the
transformations (4.6). If \( \tilde{F} \) satisfies the constraint (4.7), the constraint (4.5) on the infinitesimal parameters \( \Lambda \) can be rewritten, to lowest order in \( \Lambda \), as

\[
\det \left( \frac{\partial^2}{\partial y_\mu \partial y_\nu} [\tilde{F} + \Lambda](x, y) \right) = \epsilon \tag{4.8}
\]

5. The Non-Linear Constraints and Hyperkahler Geometry

The constraint (4.7) is an equation of Monge-Ampere type [33] and this leads to the following geometrical interpretation. Let \( \zeta_\mu, \bar{\zeta}_\bar{\mu} \) (\( \mu = 1, 2 \)) be complex coordinates on \( \mathbb{R}^4 \). Then, for each \( x^\mu \), a solution \( \tilde{F}(x, y) \) of (4.7) can be used to define a function \( K_x(\zeta, \bar{\zeta}) \) on \( \mathbb{R}^4 \) by

\[
K_x(\zeta, \bar{\zeta}) = \tilde{F}(x^\mu, \zeta_\mu + \bar{\zeta}_{\bar{\mu}}) \tag{5.1}
\]

For each \( x \), \( K_x \) can be viewed as the Kahler potential for a Kahler metric on \( \mathbb{R}^4 \). As a result of (4.7), each \( K_x \) satisfies the complex Monge-Ampere equation, sometimes referred to as the Plebanski equation [34], \( \det(\partial_\mu \partial_{\bar{\mu}} K_x) = \epsilon \) and so the corresponding metric is Kahler and Ricci-flat, which implies that the curvature tensor is either self-dual or anti-self-dual. In the Euclidean case, the metric has signature (4,0) and is hyperkahler, while in the other case the metric has signature (2,2) and holonomy \( SU(1,1) \). As the Kahler potential is independent of the imaginary part of \( \zeta_\mu \), the metric has two commuting (triholomorphic) Killing vectors, given by \( i(\partial / \partial \zeta_\mu - \partial / \partial \bar{\zeta}_{\bar{\mu}}) \). Thus the lagrangian \( \tilde{F}(x, y) \) corresponds to a two-parameter family of Kahler potentials \( K_{x^\mu} \) for (anti) self dual geometries on \( \mathbb{R}^4 \) with two Killing vectors. The parameter constraint (4.8) implies that \( \tilde{F} + \Lambda \) is also a Kahler potential for a hyperkahler metric with two killing vectors, so that for each \( x \), \( \Lambda \) represents an infinitesimal deformation of the hyperkahler geometry.

Techniques for solving the Monge-Ampere equation can be used to solve (4.7). The general solution of the Monge-Ampere equation can be given implicitly by
Penrose’s twistor transform construction [35]. For solutions with one (triholomorphic) Killing vector, the Penrose transform reduces to a Legendre transform solution [37] which was found first in the context of supersymmetric non-linear sigma-models [36]. Writing $y_1 = \zeta, y_2 = \xi$, any function $\tilde{F}(x^\mu, \zeta, \xi)$ can be written as the Legendre transform with respect to $\zeta$ of some $H$, so that

$$\tilde{F}(x, \zeta, \xi) = \pi \zeta - H(x, \pi, \xi) \quad (5.2)$$

where the equation

$$\frac{\partial H}{\partial \pi} = \zeta \quad (5.3)$$

gives $\pi$ implicitly as a function of $x, \zeta, \xi$. The Monge-Ampere equation (4.7) will be satisfied if and only if $H$ satisfies the Laplace equation [37]

$$\frac{\partial^2 H}{\partial \pi^2} + \epsilon \frac{\partial^2 H}{\partial \xi^2} = 0 \quad (5.4)$$

and the general solution of this is

$$H = f(x, \pi + \sqrt{-\epsilon} \xi) + \bar{f}(x, \pi - \sqrt{-\epsilon} \xi) \quad (5.5)$$

where $f, \bar{f}$ are arbitrary independent real functions if $\epsilon = -1$ and are complex conjugate functions if $\epsilon = 1$. Then the general solution of (4.7) is the Legendre transform (5.2),(5.3),(5.5) and the action can be given in the first order form

$$S = \int d^2x \, \tilde{F}(x, y) = \int d^2x \, \left( \pi \partial_\tau \phi - f(x^\mu, \pi + \partial_\sigma \phi) + \bar{f}(x^\mu, \pi - \partial_\sigma \phi) \right) \quad (5.6)$$

where $\tau = x^1$ and $\sigma = -\sqrt{-\epsilon} x^2$. The field equation for the auxiliary field $\pi$ is (5.3) and this can be used in principle to eliminate $\pi$ from the action, but it will not be possible to solve the equation (5.3) explicitly in general. The constraints
(4.8) can be solved similarly. Expanding the functions \( f, \bar{f} \) gives the Hamiltonian form of the \( \omega_\infty \) action [10]

\[
S = \int d^2 x \left( \pi \partial_\tau \phi - \sum_{n=2}^{\infty} \frac{1}{n} \left[ h_n(\pi + \partial_\sigma \phi)^n + \bar{h}_n(\pi - \partial_\sigma \phi)^n \right] \right) \tag{5.7}
\]

consisting of a free term plus a set of Lagrange multiplier gauge fields, \( h_n, \bar{h}_n \), times constraints. The symmetries of the action are simply those generated by these constraints [10] (e.g. \( \delta \phi = L(x^\mu, \pi + \partial_\sigma \phi) + \bar{L}(x^\mu, \pi - \partial_\sigma \phi) \) etc).

A generalisation of the Legendre transform solution that involves transforming with respect to both components of \( y_\mu \) and maintains Lorentz covariance is suggested by the results of [4,5]. Any \( \tilde{F}(x, y) \) can be written as a transform of a function \( H \) as follows:

\[
\tilde{F}(x^\mu, y^\nu) = 2\pi^\mu y_\mu - \frac{1}{2} \eta^{\mu\nu} y_\mu y_\nu - 2H(x, \pi) \tag{5.8}
\]

where the equation

\[
y_\mu = \frac{\partial H}{\partial \pi^\mu} \tag{5.9}
\]

implicitly determines \( \pi_\mu = \pi_\mu(x^\nu, y_\rho) \). Then \( \tilde{F} \) will satisfy (4.7) if and only if its transform \( H \) satisfies

\[
\frac{1}{2} \eta^{\mu\nu} \frac{\partial^2 H}{\partial \pi_\mu \partial \pi_\nu} = \frac{\partial^2 H}{\partial \pi_+ \partial \pi_-} = 1 \tag{5.10}
\]

The general solution of this is (with \( \pi_\pm = \pi_1 \pm \sqrt{-\epsilon} \pi_2 \))

\[
H = \pi_+ \pi_- + f(x, \pi_+) + \bar{f}(x, \pi_-) \tag{5.11}
\]

This solution can be used to write the action

\[
S = \int d^2 x \left( 2\pi^\mu y_\mu - \eta_{\mu\nu} \pi^\mu \pi^\nu - \frac{1}{2} \eta^{\mu\nu} y_\mu y_\nu - 2f(x, \pi_+) - 2\bar{f}(x, \pi_-) \right) \tag{5.12}
\]

The field equation for \( \pi^\mu \) is (5.9), and using this to substitute for \( \pi \) gives the action (4.1) subject to the constraint (4.7). Alternatively, expanding the functions \( f, \bar{f} \) as
\[ f = \sum s^{-1} h_s(x)(\pi_+)^s, \quad \bar{f} = \sum s^{-1} \bar{h}_s(x)(\pi_-)^s \] gives precisely the form of the action given in [5]. The parameter constraint (4.8) is solved similarly, and the solutions can be used to write the symmetries of (5.12) in the form given in [5].

6. Covariant Formulation and \( \mathcal{W} \)-Weyl Invariance

In this section, the covariant solution of the constraints given in [15] will be reviewed. In [16], an alternative covariant solution is given which appears to have a deeper geometrical significance, but it would take too long to describe that here.

The constraint (4.7) can be solved in terms of an unconstrained function

\[ F(x, y) = \sum_{n=2}^{\infty} \frac{1}{n} g_{(n)}^{\mu_1 \mu_2 \ldots \mu_n} (x) y_{\mu_1} y_{\mu_2} \ldots y_{\mu_n} \] (6.1)

by writing

\[ \tilde{F}(x, y) = \Omega(x, y) F(x, y) \] (6.2)

where \( \Omega \) is to be found in terms of \( F \) and has an expansion of the form

\[ \Omega(x, y) = \sum_{n=0}^{\infty} \Omega_{(n+2)}^{\mu_1 \mu_2 \ldots \mu_n} (x) y_{\mu_1} y_{\mu_2} \ldots y_{\mu_n} \] (6.3)

Substituting (6.2) in (4.7) gives a set of equations which can be solved to give the tensors \( \Omega_{(n)} \) in terms of the unconstrained tensors \( g_{(n)} \) in (6.1), giving

\[ \Omega_{(2)} = \sqrt{\epsilon g}, \quad \Omega_{(3)} = -\sqrt{\epsilon g} g_{(2)}^{\mu \nu} g_{\nu \rho}, \] etc where \( g_{\mu \nu} \) is the inverse of \( g_{(2)}^{\mu \nu} \) and \( g = \det[g_{\mu \nu}] \).

Substituting in (6.2) gives

\[ \tilde{g}_{(2)}^{\mu \nu} = \sqrt{\epsilon g} g_{(2)}^{\mu \nu}, \quad \tilde{g}_{(3)}^{\mu \nu \rho} = \sqrt{\epsilon g} \left[ g_{(3)}^{\mu \nu \rho} - \frac{3}{2} \epsilon_{(2)}^{(3)} g_{(2)}^{\nu \rho} g_{\nu \beta} \right], \ldots \] (6.4)

Writing \( \tilde{F} \) in terms of \( F \) gives an action which is invariant under the \( \mathcal{W} \)-Weyl
transformations

$$\delta F(x, y) = \sigma(x, y) F(x, y)$$  \hfill (6.5)

Expanding

$$\sigma(x, y) = \sigma_{(2)}(x) + \sigma_{(3)}^\mu(x) y_\mu + \sigma_{(4)}^{\mu\nu}(x) y_\mu y_\nu + \cdots$$  \hfill (6.6)

these can be written as

$$\delta g_{\mu\nu}^{(2)} = \sigma_{(2)} g_{\mu\nu}^{(2)}, \quad \delta g_{\mu\nu\rho}^{(3)} = \sigma_{(2)} g_{\mu\nu\rho}^{(2)} + \frac{3}{2} \sigma_{(3)}^{\mu} g_{\nu\rho}^{(2)}, \cdots$$  \hfill (6.7)

These transformations can be used to remove all traces from the gauge fields, leaving only traceless gauge fields, as in [5,9].

The constraint (4.8) on the parameters $\lambda_{(n)}$ can be solved in a similar fashion in terms of unconstrained parameters $k_{(n)}^\mu_{1 \cdots \mu_{n-1}}$ and the transformations of the unconstrained gauge fields can be defined to take the form $\delta g_{(n)}^{\mu_1 \mu_2 \cdots \mu_{n}} = \partial_{(\mu_1} k_{(n)}^\mu_{2 \cdots \mu_{n})} + \cdots$ (cf (2.12)). The $g_{(n)}$ might be thought of as gauge fields for the whole of the symplectic diffeomorphisms of $T^*M$ (with parameters $k_{(n)}$), and appear in the action only through the combinations $\tilde{g}_{(n)}$. The transformations of $\tilde{g}_{(n)}$ and $\phi$ then only depend on the parameters $k_{(n)}$ in the form $\lambda_{(n)}$.

7. $\mathcal{W}_N$-Gravity

A free scalar field in two dimensions has the set of conserved currents (2.2), given by $W_n = \frac{1}{n} (\partial \phi)^n$, $n = 2, 3, \ldots$, and these generate a $w_\infty$ algebra. In fact, the finite subset of these given by $W_n$, $n = 2, 3, \ldots, N$ generate a closed non-linear algebra which is a classical limit of the $\mathcal{W}_N$ algebra, and in the limit $N \to \infty$, the classical current algebra becomes the $w_\infty$ algebra. Similarly, the currents $W_n = \frac{1}{n} (\bar{\partial} \phi)^n$ generate a second copy of the $\mathcal{W}_N$ or $w_\infty$ algebra.
The linearised action for $\mathcal{W}_N$ gravity is given by simply truncating the action (2.8) by setting the gauge fields $h_n, \bar{h}_n$ with $n > N$ to zero, giving

$$S = \int d^2 x \left[ \partial\phi \bar{\partial}\phi + \sum_{n=2}^{N} \frac{1}{n} [h_n (\partial\phi)^n + \bar{h}_n (\bar{\partial}\phi)^n] + O(h^2) \right]$$

(7.1)

which is invariant, to lowest order in the gauge fields, under the transformations

$$\delta\phi = \sum_{n=2}^{N} \left[ \lambda_n (z, \bar{z}) (\partial\phi)^{n-1} + \bar{\lambda}_n (z, \bar{z}) (\bar{\partial}\phi)^{n-1} \right]$$

$$\delta h_n = -2\bar{\partial}\lambda_n + O(h), \quad \delta \bar{h}_n = -2\partial\bar{\lambda}_n + O(h)$$

(7.2)

This gives the linearised action and transformations of $\mathcal{W}_N$ or (in the $N \to \infty$ limit) $w_\infty$ gravity. The full gauge-invariant action and gauge transformations are non-polynomial in the gauge fields.

The linearised action for $\mathcal{W}_N$ gravity is then an $N$'th order polynomial in $\partial_{\mu}\phi$. However, the full non-linear action is non-polynomial in $\partial_{\mu}\phi$ and the gauge fields $h_n$, but the coefficient of $(\partial\phi)^n$ for $n > N$ can be written as a non-linear function of the finite number of fundamental gauge fields $h_2, h_3, \ldots, h_N$ that occur in the linearised action. The simplest way in which this might come about would be if the action were given by (4.1),(4.2) and $\tilde{F}$ satisfies a constraint of the form

$$\frac{\partial^{N+1}\tilde{F}}{\partial y_{\mu_1} \partial y_{\mu_2} \ldots \partial y_{\mu_{N+1}}} = 0 + O(\tilde{F}^2)$$

(7.3)

where the right hand side is non-linear in $\tilde{F}$ and its derivatives, and depends only on derivatives of $\tilde{F}$ of order $N$ or less. This is indeed the case; the action for $\mathcal{W}_N$ gravity is given by (4.1) where $\tilde{F}$ satisfies (4.7) and (7.3), and the right hand side of (7.3) can be given explicitly. Just as the non-linear constraint (4.7) had an interesting geometric interpretation, it might be expected that the non-linear form of (7.3) should also be of geometric interest. Here, the results will be summarised; full details will be given in [17].
It will be useful to define

$$F_{\mu_1\mu_2\cdots\mu_n}(x, y) = \frac{\partial^n F}{\partial y_{\mu_1} \partial y_{\mu_2} \cdots \partial y_{\mu_n}}$$  \hspace{1cm} (7.4)$$

and

$$H_{\mu\nu}(x, y) = 2(\tilde{g}^{\mu\nu} + F^{\mu\nu})^{-1}$$  \hspace{1cm} (7.5)$$

where $\tilde{g}^{\mu\nu} = \tilde{g}^{(2)}_{\mu\nu}(x)$.

The action for $W_N$ gravity is then given by the action for $w_\infty$ gravity, but with the function $\tilde{F}$ satisfying one extra constraint of the form (7.3). For $W_3$, this extra constraint is

$$F_{\mu\nu\rho\sigma} = \frac{3}{2} H_{\alpha\beta} F^{\alpha(\mu\nu} F^{\rho\sigma)}_{\beta}$$  \hspace{1cm} (7.6)$$
or, using (7.5),

$$F_{\mu\nu\rho\sigma} = 3(\tilde{g}^{\alpha\beta} + F^{\alpha\beta})^{-1} F^{\alpha(\mu\nu} F^{\rho\sigma)}_{\beta}$$  \hspace{1cm} (7.7)$$

This is the required extra constraint for $W_3$ gravity. Thus the action for $W_3$ gravity is given by (4.1),(4.2), where $\tilde{F}$ is a function satisfying the two constraints (4.7) and (7.7).

For $W_4$ gravity, the extra constraint is

$$F_{\mu\nu\rho\sigma\tau} = 5H_{\alpha\beta} F^{\alpha(\mu\nu} F^{\rho\sigma\tau)}_{\beta} - \frac{15}{4} H_{\alpha\beta} H_{\gamma\delta} F^{\alpha(\mu\nu} F^{\rho\sigma\tau)}_{\gamma\delta}$$  \hspace{1cm} (7.8)$$

so that the $W_4$ action is (4.1) where $\tilde{F}$ satisfies (4.7) and (7.8), and $H^{\mu\nu}$ is given in terms of $\tilde{F}$ by (7.5). Similar results hold for all $N$. In each case, one obtains an equation of the form (7.3), where the right hand side is constructed from the $n$’th order derivatives $F^{\mu_1\cdots\mu_n}$ for $2 < n \leq N$ and from $H_{\mu\nu}$.

Expanding $\tilde{F}$ in $\partial_\mu \phi$ (4.2) gives the coefficient of the $n$-th order $\partial_\mu_1 \phi \cdots \partial_\mu_n \phi$ interaction, which is proportional to $\tilde{g}^{\mu_1\cdots\mu_n}(n)$. The constraint (7.3) implies that for $n > N$, the coefficient $\tilde{g}(n)$ of the $n$-th order interaction can be written in terms of
the coefficients $\tilde{g}_{(m)}$ of the $m$-th order interactions for $2 \leq m \leq N$. For $\mathcal{W}_3$, the
$n$-point vertex can be written in terms of 3-point vertices for $n > 3$, so that (with
$\tilde{g}_{\alpha\beta} = \left(\tilde{g}_{(2)}^{\alpha\beta}\right)^{-1}$)
\[ \tilde{g}_{(4)}^{\mu\nu\rho\sigma} = \tilde{g}_{\alpha\beta} - \alpha^{(\mu\nu} \tilde{g}_{^{\rho\sigma)}\beta} \]  \hspace{1cm} (7.9)
\[ \tilde{g}_{(5)}^{\mu\nu\rho\sigma\tau} = \frac{5}{4} \tilde{g}_{\alpha\beta} \tilde{g}_{\gamma\delta} \alpha^{(\mu\nu} \tilde{g}_{^{\rho\sigma\gamma|\tilde{\tau}}\delta)} \]  \hspace{1cm} (7.10)

etc, while for $\mathcal{W}_4$, all vertices can be written in terms of 3- and 4-point vertices, e.g.
\[ \tilde{g}_{(5)}^{\mu\nu\rho\sigma\tau} = \frac{5}{2} \tilde{g}_{\alpha\beta} \tilde{g}_{\gamma\delta} \alpha^{(\mu\nu} \tilde{g}_{^{\rho\sigma\gamma|\tilde{\tau}}\delta)} \]  \hspace{1cm} (7.11)

For the derivation of these results, and the form of the transformation rules, see [17].

To attempt a geometric formulation of these results, note that while the second
derivative of $\tilde{F}$ defines a metric, the fourth derivative is related to a curvature, and
the $n$’th derivative is related to the $(n - 4)$’th covariant derivative of the curvature.
The $\mathcal{W}_3$ constraint (7.7) can then be written as a constraint on the curvature,
while the $\mathcal{W}_N$ constraint (7.3) becomes a constraint on the $(N - 3)$’th covariant
derivative of the curvature. One approach is to introduce a second Kahler metric
$\hat{K}_x$ on $\mathbb{R}^4$ given in terms of the potential $K_x$ introduced in (5.1) by
\[ \hat{K}_x = K_x + \tilde{g}_{\alpha\beta} \tilde{\zeta}_\alpha \tilde{\zeta}_\beta \]  \hspace{1cm} (7.12)

The corresponding metric is given by
\[ \hat{G}^{\mu\nu} = \tilde{g}^{\mu\nu} + G^{\mu\nu} \]  \hspace{1cm} (7.13)

Then if $\tilde{F}$ satisfies the $\mathcal{W}_3$ constraint (7.7), the curvature tensor for the metric
\[ \hat{\mathcal{R}}^{\mu\nu\rho\bar{\sigma}} = \frac{1}{2} \hat{G}_{\alpha\bar{\beta}} \left[ T^{\alpha\mu\bar{\nu}} T^{\bar{\beta}\bar{\sigma}\rho} + T^{\alpha\mu\bar{\nu}} T^{\bar{\beta}\bar{\sigma}\rho} + T^{\bar{\beta}\bar{\sigma}\mu} T^{\alpha\rho\bar{\nu}} + T^{\bar{\beta}\bar{\sigma}\mu} T^{\alpha\rho\bar{\nu}} \right] \] (7.14)

where
\[ T^{\mu\nu\bar{\rho}} = \frac{\partial^3 \hat{K}}{\partial \zeta_\mu \partial \zeta_\nu \partial \bar{\zeta}_\bar{\rho}}, \quad T^{\bar{\mu}\bar{\nu}\rho} = \frac{\partial^3 \hat{K}}{\partial \bar{\zeta}_{\bar{\mu}} \partial \bar{\zeta}_{\bar{\nu}} \partial \zeta_\rho} \] (7.15)

This is similar to, but distinct from, the constraint of special geometry [38]. Note that (7.14) is not a covariant equation as the definitions (7.15) are only valid in the special coordinate system that arises in the study of \( \mathcal{W} \)-gravity. However, tensor fields \( T^{\mu\nu\bar{\rho}}, T^{\bar{\mu}\bar{\nu}\rho} \) can be defined by requiring them to be given by (7.15) in the special coordinate system and to transform covariantly, in which case the equation (7.14) becomes covariant, as in the case of special geometry [38]. For \( \mathcal{W}_N \), this generalises to give a constraint on the \( (N - 3) \)th covariant derivative of the curvature, which is given in terms of tensors that can each be written in terms of some higher order derivatives of the Kahler potential in the special coordinate system.

For each \( x^\mu \), the solutions to the constraints for \( \mathcal{W}_N \) gravity are parameterised by the \( 2(N - 1) \) variables \( h_n, \bar{h}_n \) for \( 2 \leq n \leq N \) which are then the coordinates for the \( 2(N - 1) \) dimensional moduli space for the self-dual geometry satisfying the \( \mathcal{W}_N \) constraint. For the \( x \)-dependent family of solutions, the moduli become the fields \( h_n(x), \bar{h}_n(x) \) on the world-sheet.
8. Conclusion

We have seen that symplectic diffeomorphisms of the cotangent bundle of the two-dimensional space-time (or world-sheet) $\mathcal{N}$ play a fundamental role in $\mathcal{W}$-gravity, generalising the role played by the diffeomorphisms of $\mathcal{N}$ in ordinary gravity theories. Further, in the case in which the matter system consists of a single boson, we have completely determined the non-linear structure of the coupling to $\mathcal{W}$-gravity and found that it involves the solution to an interesting non-linear differential equation which can be linearised by a twistor transform. We found an infinite set of symmetric tensor density gauge fields $\tilde{g}^{\mu_1...\mu_n}_{(n)}, n = 2, 3, \ldots$, transforming under the action of a gauge group isomorphic to the symplectic diffeomorphisms of $T^*\mathcal{N}$. The results could be simply stated in terms of the generating function

$$\tilde{F}(x, y) = \sum_{n=2}^{\infty} \frac{1}{n} \tilde{g}^{\mu_1...\mu_n}_{(n)}(x) y_{\mu_1} y_{\mu_2}...y_{\mu_n}$$

and in [16] it is argued that the quantity $\tilde{F}$ has a natural geometrical interpretation as a ‘$\mathcal{W}$-density’. In [16], the concept of a ‘$\mathcal{W}$-scalar’ $F(x, y)$ was also introduced which generated a set of tensor gauge fields $g^{\mu_1...\mu_n}_{(n)}, n = 2, 3, \ldots$, which had natural geometric transformation rules. The first of these gauge fields, $g^{\mu\nu}_{(2)}$, is the inverse of the usual world-sheet metric. Just as in Riemannian geometry one can construct the tensor density $\tilde{g}^{\mu\nu}_{(2)}$ in terms of the tensor $g^{\mu\nu}_{(2)}$ by $\tilde{g}^{\mu\nu}_{(2)} = \sqrt{\gamma} g^{\mu\nu}_{(2)}$, it seems that in $\mathcal{W}$-geometry, a $\mathcal{W}$-density can be constructed from a $\mathcal{W}$-scalar, and the $\mathcal{W}$-scalar can be thought of as giving some kind of generalisation of the Riemannian line element [16]. Further, $\mathcal{W}$-scalars can be constructed in any dimension, but $\mathcal{W}$-densities and hence invariant actions can only be constructed in one and two dimensions [16].

So far, only the case in which the matter system is a single boson has been discussed. It is clearly important to generalise the results given here to less trivial matter systems. It turns out that it is straightforward although non-trivial to generalise to the case of multi-boson realisations. For $n$ bosons $\phi^i, i = 1, \ldots, n$,
one obtains a construction on a bundle with local coordinates \( x^\mu, y^i_\mu \) and fibres \((T^*\mathcal{N})^n\) but the gauge group remains essentially the same as in the one boson case; further details will be given elsewhere.

One motivation for the study of \( \mathcal{W} \)-geometry is to try to understand finite \( \mathcal{W} \)-transformations (as opposed to those with infinitesimal parameters) and the moduli space for \( \mathcal{W} \)-gravity. The infinitesimal transformations for the scalar field \( \phi \) were derived from studying infinitesimal symplectic diffeomorphisms and it follows that the large \( \mathcal{W} \)-transformations of \( \phi \) are given by the action of large symplectic diffeomorphism transformations on \( y_\mu = \partial_\mu \phi \). It seems natural to conjecture that the transformations of the gauge fields can be defined to give invariance under the full group of symplectic diffeomorphisms, as opposed to invariance under the subgroup generated by exponentiating infinitesimal ones, but this remains to be proved.

Another important issue is that of quantum \( \mathcal{W} \)-gravity. Here we shall briefly review the linearised conformal gauge results of [13]; see [40] for a discussion of light-cone gauge \( \mathcal{W} \)-gravity, including non-linear corrections. We have seen that for each spin \( s, 2 \leq s \leq N \), there is a symmetric tensor gauge field \( g_{\mu_1\mu_2...\mu_s}^{(s)} \) transforming under linearised \( \mathcal{W} \)-gravity and \( \mathcal{W} \)-Weyl transformations as

\[
\delta g_{\mu_1\mu_2...\mu_s}^{(s)} = \partial_{(\mu_1} \lambda_{\mu_2...\mu_s)}^{(s)} + \eta_{(\mu_1\mu_2} \sigma_{\mu_3...\mu_s)} + O(h) \tag{8.2}
\]

The \( \mathcal{W} \)-gravity parameter \( \lambda_{\mu_1...\mu_{s-1}}^{(s)} \) is a rank-\((s - 1)\) symmetric tensor and the \( \mathcal{W} \)-Weyl parameter \( \sigma_{\mu_1...\mu_{s-2}} \) is a rank-\((s - 2)\) symmetric tensor. Note that the part of the transformation (8.2) involving the trace of \( \lambda_{\mu_1...\mu_{s-1}}^{(s)} \) can be absorbed into a redefinition of the \( \mathcal{W} \)-Weyl parameter. The linearised spin-\( s \) curvature

\[
R^{(s)}_{\mu_1\nu_1 \mu_2\nu_2 ... \mu_s\nu_s} = \left( \{ \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_s} g_{\nu_1\nu_2...\nu_s}^{(s)} - (\mu_1 \leftrightarrow \nu_1) \} - (\mu_2 \leftrightarrow \nu_2) \right) \cdots - (\mu_s \leftrightarrow \nu_s) + O(g^2) \tag{8.3}
\]

is invariant under (8.2) to lowest order in the gauge fields. It has \( s \) anti-symmetric pairs of indices and is symmetric under the interchange of any two pairs. The
corresponding linearised curvature scalar is given by

\[ R^{(s)} = \frac{1}{2^s} \epsilon^{\mu_1 \nu_1} \cdots \epsilon^{\mu_s \nu_s} R^{(s)}_{\mu_1 \nu_1 \cdots \mu_s \nu_s} = \epsilon^{\mu_1 \nu_1} \cdots \epsilon^{\mu_s \nu_s} \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_s} g^{(s)}_{\nu_1 \nu_2 \cdots \nu_s} \quad (8.4) \]

The effective action obtained by integrating out the matter takes the form

\[ \Gamma = \sum_{s=2}^{N} a_s \int d^2x \, R^{(s)} \frac{1}{\Box} R^{(s)} \quad (8.5) \]

for some constants \(a_s\). This is invariant, at least to lowest order in the gauge fields, under the \(W\)-gravity transformations but not under the \(W\)-Weyl transformations:

\[ \delta \Gamma = -2 \sum_{s=1}^{N} \int d^2x \left[ a_s R^{(s)} \epsilon^{\mu_1 \nu_1} \cdots \epsilon^{\mu_{s-2} \nu_{s-2}} \partial_{\mu_1} \cdots \partial_{\mu_{s-2}} \sigma_{\nu_1 \cdots \nu_{s-2}} \right] \quad (8.6) \]

Thus, at the linearised level, there are no \(W\)-gravity anomalies but there are \(W\)-Weyl anomalies.

In general, the gauge symmetries can be used to gauge away all but the total traces of the gauge fields (modulo some subtleties discussed in [13]). For even spins \(s = 2r\), this leaves a scalar field \(\rho^{(2r)} \propto g^{(2r)}_{\nu_1 \cdots \nu_r} \), while for odd spins \(s = 2r+1\) this leaves a vector gauge field \(A^{(2r+1)}_{\mu} \propto g^{(2r+1)}_{\mu \nu_1 \cdots \nu_r} \) with field strength \(F^{(2r+1)} = \epsilon^{\mu \nu} \partial_{\mu} A^{(2r+1)}_{\nu} \). The residual spin-one gauge invariances can then be fixed using the Lorentz gauge condition \(\partial_{\mu} A^{(2r+1)}_{\mu} = 0\), which can be solved to give \(A^{(2r+1)}_{\mu} = \epsilon_{\mu \nu} \partial^\nu \rho^{(2r+1)} \) (plus a zero-mode piece) for some scalar \(\rho^{(2r+1)}\). Thus all of the \(W\)-symmetries are fixed by the gauge choices

\[ g^{(2r)}_{\mu_1 \cdots \mu_{2r}} = (-1)^r \eta_{(\mu_1 \mu_2 \cdots \eta_{\mu_{2r-1} \mu_{2r}})\rho^{(2r)}} \]
\[ g^{(2r+1)}_{\mu_1 \cdots \mu_{2r+1}} = (-1)^r \eta_{(\mu_1 \mu_2 \cdots \eta_{\mu_{2r-1} \mu_{2r}} \epsilon_{\nu}) \lambda \partial^\lambda} \rho^{(2r+1)} \quad (8.7) \]

and the linearised curvature scalars (8.3),(8.4) become

\[ R^{(2r)} = \Box^r \rho^{(2r)} \, , \quad R^{(2r+1)} = \Box^{r+1} F^{(2r+1)} = \Box^{r+1} \rho^{(2r+1)} \quad (8.8) \]
Substituting this in (8.5) gives the effective action

\[ \Gamma = \sum_{r=1}^{[N/2]} \left[ a_{2r} \rho^{(2r)} \Box^{2r-1} \rho^{(2r)} + a_{2r+1} F^{(2r+1)} \Box^{2r-1} F^{(2r+1)} \right] = \sum_{s=2}^{N} a_s \rho^{(s)} \Box^{s-1} \rho^{(s)} \]

consisting of the Liouville action for \( s = 2 \) and higher derivative counterparts for the higher spin cases. It is perhaps surprising that the \( \mathcal{W} \)-gravity generalisation of Liouville theory that emerges in this approach is not the Toda theory advocated in [39] and elsewhere, but a higher derivative version of this. (Note, however, that an alternative, but less natural, geometric framework that does lead to Toda-theory version of \( \mathcal{W} \)-gravity was given in [13]; it is related to the one described here by a non-local change of variables.)

Consider now the moduli space \( M_n \) for gauge fields \( \tilde{g}_{(n)} \) subject to the constraints generated by (4.7) (and the \( \mathcal{W}_N \) constraint (7.3), if appropriate) [13]. Linearising about a Euclidean background \( \tilde{F} = \frac{1}{2} \tilde{g}_{(2)\mu\nu} y_\mu y_\nu \) and choosing complex coordinates \( z, \bar{z} \) on the Riemann surface \( \mathcal{N} \) such that the background is \( \tilde{F} = y_\mu z^\mu \), and using the linearised transformations \( \delta \tilde{g}_{(n)} = \partial_{\bar{z}} \lambda_{(n)}^{zz\ldots z} \), it follows by standard arguments that the tangent space to the moduli space \( M_n \) at a point corresponding to the background configuration is the space of holomorphic \( n \)-differentials, \( i.e. \) the \( n \)-th rank symmetric tensors \( \mu_{zz\ldots z} \) with \( n \) lower \( z \) indices satisfying \( \partial_{\bar{z}} \mu_{zz\ldots z} = 0 \) [13]. It follows from the Riemann-Roch theorem that the dimension of this space on a genus-\( g \) Riemann surface (the number of anti-ghost zero-modes) is \( \dim(M_n) = (2n-1)(g-1) + k(n, g) \) where \( k(n, g) \) is the number of solutions \( \kappa_{zz\ldots z} \) (with \( n-1 \) ‘\( z \)’ indices) to \( \partial_{\bar{z}} \kappa_{zz\ldots z} = 0 \) (the number of ghost zero-modes). It would be of great interest to use information about the global structure of the symplectic diffeomorphism group to learn more about the structure of these moduli spaces.
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