The Weil-Petersson Kähler form and affine foliations on surfaces

by

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Abstract The space of broken hyperbolic structures generalizes the usual Teichmüller space of a punctured surface, and the space of projectivized broken measured foliations—or equivalently, the space of projectivized affine foliations of a punctured surface—likewise admits a generalization to projectivized broken measured foliations. Just as projectivized measured foliations provide Thurston’s boundary for Teichmüller space, so too do projectivized broken measured foliations provide a boundary for the space of broken hyperbolic structures. In this paper, we naturally extend the Weil-Petersson Kähler two-form to a corresponding two-form on the space of broken hyperbolic structures as well as Thurston’s symplectic form to a corresponding two-form on the space of broken measured foliations, and we show that the former limits in an appropriate sense to the latter. The proof in sketch follows earlier work of the authors for measured foliations and depends upon techniques from decorated Teichmüller theory, which is also applied here to a further study of broken hyperbolic structures.

0.— Introduction

In the paper [PP], we established a relation between the Weil-Petersson Kähler form on the Teichmüller space of a punctured surface and Thurston’s piecewise-linear symplectic form on the space of measured foliations of compact support on that surface. We extend here this work to the context of broken hyperbolic structures and of broken measured foliations. Both spaces of broken structures were defined in [OP2], and we shall recall the definitions in the next section. The space of broken hyperbolic structures contains Teichmüller space as a proper subspace; the space of broken measured foliations can be identified with the space of affine foliations on the surface (as discussed in Appendix A), and it likewise contains the space of measured foliations as a proper subspace.

More precisely, in this paper, we define a two-form on the space of broken hyperbolic structures which extends the Weil-Petersson Kähler two-form defined on Teichmüller space. Likewise, we define a two-form on the space of broken measured foliations, which extends Thurston’s form defined on measured foliations space. In
the paper [OP2], the space of projective classes of broken measured foliations was described as a boundary for the space of broken hyperbolic structures, generalizing Thurston’s realization of the space of projective classes of measured foliations as a boundary to the Teichmüller space of the surface. In the work here, we exhibit a relation between the two-forms on the spaces of broken hyperbolic structures and broken measured foliations, which is analogous to the relation on Teichmüller space and its boundary that we produced in [PP]. In fact, the whole discussion in this paper takes place (as in the paper [PP]) in the decorated spaces of broken hyperbolic structures and of broken measured foliation, whose definitions we give in the next section. Finally in Appendix B, we further apply techniques from decorated Teichmüller theory to study the space of broken hyperbolic structures.

1.—— Decorated broken hyperbolic structures and broken measured foliations

In what follows, $F_g^s$ is an oriented surface of negative Euler characteristic with genus $g$ and $s$ punctures, $s > 0$. Let $\Delta$ be an ideal triangulation of $F_g^s$, that is, a decomposition of $F_g^s$ into triangles whose vertices are at the punctures. (We do not require that two triangles meet in at most one edge, as in a simplicial decomposition.) Let $\tilde{F}_g^s$ be the topological universal cover of $F_g^s$ (with respect to some basepoint which we suppress) and $\tilde{\Delta}$ the lift of $\Delta$ to $\tilde{F}_g^s$.

**Definition 1.1 (Broken hyperbolic structures).**—— A broken hyperbolic structure on $(F_g^s, \Delta)$ is a Riemannian metric of constant curvature $-1$ on $F_g^s - \Delta$ such that the completion of each face of $\Delta$, with its vertices deleted, is isometric to a hyperbolic ideal triangle. (We recall that a hyperbolic ideal triangle is the convex hull, in the hyperbolic plane $\mathbb{H}^2$, of three distinct points in the boundary of that space.) We require furthermore that conditions (1.1.1) to (1.1.3) be satisfied:

(1.1.1) (Homothety) Consider the surface $F_g^s$ as a quotient of the disjoint union of the ideal triangles (the faces of $\Delta$), by gluing their edges pairwise. Then, each gluing map between two edges of triangles which map to the same edge of $\Delta$ is a homothety, with respect to the metrics on the edges of the triangles which are induced by the hyperbolic metrics on the triangles; that is, across each pair of identified edges, the hyperbolic metrics on the faces are related by an overall homothety. Thus, associated with a pair $f_1, f_2$ of faces sharing the edge $e$, there is a “homothetic scaling factor” $\sigma(f_1, f_2) \in \mathbb{R}_+$ of broken metric across $e$; specifically, if an arc $a \subseteq e$ has length in $f_i$ given by $\rho_i$, for $i = 1, 2$, then $\rho_2 = \sigma(f_1, f_2) \rho_1$. Notice that $\sigma(f_1, f_2) \sigma(f_2, f_1) = 1$.

(1.1.2) (No Horocyclic Holonomy) The holonomy of the broken hyperbolic structure is trivial around each cusp. This means that if $f_1, f_2, \ldots, f_n, f_{n+1} = f_1$ are consecutive faces of $\Delta$ traversed in a counterclockwise sense about a puncture, then $\prod_{i=1}^n \sigma(f_i, f_{i+1}) = 1$. (In particular, for $s = 1$, this condition holds automatically.)

(1.1.3) (Completeness) The broken hyperbolic structures that we consider in this paper are complete in the sense which we describe now. To formulate the definition, we first introduce the horocyclic foliation associated to a broken hyperbolic structure. This is a foliation on $F_g^s$ which is defined as follows. On each ideal triangle in $\mathbb{H}^2$,
there is a well-defined partial foliation (that is, a foliation supported on a subsurface of that triangle) whose leaves are connected pieces of horocycles or “horocyclic arcs”, where the horocycle is centered at the vertices of these triangles and where each leaf has its endpoints on the edges of that triangle making right angles with the edges. The non-foliated region is a central triangle which is bounded by three of these horocyclic arcs, which pairwise meet tangentially at their endpoints (Figure 1).

![Figure 1 The horocyclic partial foliation of an ideal triangle.](image1)

We call this foliation the **horocyclic (partial) foliation of the ideal triangle**. (We note that since all ideal triangles are isometric, it suffices to define the horocyclic foliation of a particular ideal triangle, and then carry by an isometry to an arbitrary triangle.) From this partial foliation we can obtain a foliation of full support on the ideal triangle, by collapsing the non-foliated region onto a **tripod** (see Figure 2).

![Figure 2 The horocyclic foliation of an ideal triangle.](image2)

The horocyclic foliations of the various ideal triangles which constitute the faces...
of $\Delta$ extend naturally to a foliation on $F_g^s$ which we call the horocyclic foliation of the broken hyperbolic structure. By construction, this foliation is well-defined up to an isotopy preserving the edges of $\Delta$. The broken hyperbolic structure is then said to be complete if there exists a neighborhood of each puncture of $F_g^s$ which is topologically an annulus and on which the foliation induced by the horocyclic foliation is a foliation by circles which are homotopic to the puncture.

Let us note that in the case where the broken hyperbolic structure on $(F_g^s, \Delta)$ is a hyperbolic structure (in other words, if all the homothety scales in (1.1.1) are equal to unity), then completeness of broken hyperbolic structure is equivalent to completeness of hyperbolic structure in the usual sense. We recall that in this case, the neighborhood of each puncture is a “cusp”, that is, a surface isometric to the quotient of the region in the upper half space model of $\mathbb{H}^2$ which is above the line $y = 1$ by a parabolic transformation fixing the point $\infty$.

**Equivalence relation.**— Two broken hyperbolic structures on $(F_g^s, \Delta)$ are considered to be equivalent if they differ by an isometry which is isotopic to the identity, with this isotopy preserving setwise the vertices and the edges of $\Delta$.

Let $\mathcal{BH}(\Delta)$ denote the set of equivalence classes of broken hyperbolic structures.

The subset of $\mathcal{BH}(\Delta)$ which consists of the broken hyperbolic structures for which the homothety scales in (1.1.1) are all equal to unity can naturally be identified with the Teichmüller space $\mathcal{T} = \mathcal{T}(F_g^s)$ of $F_g^s$, that is, the space of isotopy classes of complete finite-area hyperbolic metrics on this surface. Thus, we have a natural inclusion $\mathcal{T} \subset \mathcal{BH}(\Delta)$.

**Definition 1.2 (Decorated broken hyperbolic structure).**— A decorated broken hyperbolic structure on $(F_g^s, \Delta)$ is a broken hyperbolic structure together with the choice, for each puncture of $F_g^s$, of a closed leaf of the associated horocyclic foliation which is homotopic to that puncture (that is, a closed leaf contained in one of the annuli that are discussed in the construction of horocyclic foliation following Property 1.1.2 of Definition 1.1).

The space of decorated broken hyperbolic structures, up to the equivalence relation generated by isometries isotopic to the identity and fixing setwise the vertices and the edges of $\Delta$, will be denoted by $\tilde{\mathcal{BH}}(\Delta)$.

The decoration of the surface (that is, the choice of a horocycle around each puncture) is a tool which has been proved to be useful in hyperbolic geometry (see [Pe1]). In some sense, it serves as a way for measuring (algebraic) distances from the puncture using the Minkowski inner product, cf. Appendix B.

The decorated Teichmüller space $\tilde{\mathcal{T}}$ from [Pe1] is the total space of the trivial $\mathbb{R}_+^s$ bundle over $\mathcal{T}$, where the fibre over a point is the collection of all $s$-tuples of horocycles, not necessarily embedded or disjoint, with one horocycle about each puncture; the coordinate on the fibre is the hyperbolic length of the specified horocycle. In this paper, we shall consider the subspace $\tilde{\mathcal{T}}' \subset \tilde{\mathcal{T}}$ corresponding to disjointly embedded families of horocycles. Thus, whereas $\tilde{\mathcal{T}} \not\subset \tilde{\mathcal{BH}}(\Delta)$, we have the natural inclusion $\tilde{\mathcal{T}}' \subset \tilde{\mathcal{BH}}(\Delta)$.
Definition 1.3 (Measured foliation).— A foliation $F$ on a surface is said to be a measured foliation if each arc $c$ which is transverse to the leaves of $F$ is equipped with a Borel measure which is equivalent to the Lebesgue measure of an interval, with the property that if $c'$ is another arc which is transverse to $F$ and which is obtained from $c$ by an isotopy during which each point of the arc stays on the same leaf, then the resulting natural map between $c$ and $c'$ (obtained by sliding along the leaves) is measure-preserving.

Definition 1.4. (Broken measured foliation).— A broken measured foliation $F$ on $(F^s_g, \Delta)$ is a foliation which is transverse to $\Delta$ and which satisfies the following four properties:

1. Each puncture of $F^s_g$ has a neighborhood which is topologically an annulus and on which $F$ induces a foliation by circles homotopic to the puncture.
2. The foliation induced by $F$ on each face of $\Delta$ has exactly one singular point, and the local model of that singular point is a tripod, as in Figure 2 above.
3. The foliation induced by $F$ on each face of $\Delta$ is equipped with a transverse measure, and the total measure of each transverse arc having one endpoint at a vertex is infinite.
4. The transverse measures induced on each edge of $\Delta$ from its two sides differ by a homothety.

Equivalence relation.— We shall say that two broken measured foliations on $(F^s_g, \Delta)$ are equivalent if they differ by a transverse measure-preserving isotopy which preserves the set of edges of $\Delta$.

We denote by $BM(\Delta)$ the set of equivalence classes of broken measured foliations relative to the ideal triangulation $\Delta$. Let $MF$ denote the space of equivalence classes of measured foliations so that the each leaf has compact closure, or equivalently, there are no leaves running to the punctures. We have a natural inclusion $MF \subset BM(\Delta)$. Measured foliations correspond to broken measured foliations for which the homothety factors in (1.4.4) are all equal to unity. Indeed, every measured foliation which has no leaves running between punctures can be made transverse to $\Delta$ by an isotopy, so that the subspace $MF(\Delta)$ of broken measured foliations with all the homothety factors equal to unity is canonically identified with $MF$.

There is a natural action of the multiplicative group $\mathbb{R}_+ = \{ t \in \mathbb{R} : t > 0 \}$ on $BM(\Delta)$, obtained by multiplying the transverse measure of each broken measured foliation by a constant factor. We denote the quotient space by $PBM(\Delta)$.

Definition 1.5 (Decorated broken measured foliation).— A decorated broken measured foliation on $F^s_g$ is a broken measured foliation together with the choice, for each puncture of $F^s_g$, of a closed leaf homotopic to this puncture, in one of the annuli that are referred to in Property 1.4.1 of Definition 1.4.

Example 1.6 Again, a basic example of a decorated broken measured foliation is the horocyclic foliation associated to a decorated broken hyperbolic structure. Here, the transverse measure for the induced foliation on each component of $F^s_g - \Delta$ is...
defined so as to coincide on each edge of that component with the Lebesgue measure induced from its structure as a hyperbolic ideal triangle.

The space of decorated broken measured foliations, up to measure-preserving isotopy which fixes the edges of $\Delta$, will be denoted by $\hat{B}\hat{M}(\Delta)$, and the associated projective space by $\hat{P}\hat{B}\hat{M}(\Delta)$. Define the space $\hat{M}\hat{F}$ to be the collection of all measured foliations in $M\hat{F}$ together with a choice of embedded horocyclic leaf about each puncture and its corresponding projectivized version $P\hat{M}\hat{F}$. In the same way as for the non-decorated versions, we have natural inclusions $\hat{M}\hat{F} \subset \hat{B}\hat{M}(\Delta)$ and $P\hat{M}\hat{F} \subset P\hat{B}\hat{M}(\Delta)$.

2.— Topology

A triangle-edge pair of $(F^s, \Delta)$ is a pair $(t, e)$ where $t$ is a triangle of $\Delta$ and $e$ an edge of $t$.

We let $P$ be the set of triangle-edge pairs of $(F^s, \Delta)$.

We define now the shift parameters associated to the edge-pairs, which are useful parameters for the spaces $B\hat{H}(\Delta)$ and $B\hat{M}(\Delta)$. The shift parameters for both spaces are defined in similar manners, and we shall use them to define the topology of these spaces.

We start with the space $B\hat{H}(\Delta)$.

Shift parameters for broken hyperbolic structures.

We first recall that if $t$ is a hyperbolic ideal triangle, then each edge of $t$ is equipped with a distinguished point, which is the foot of the perpendicular on that edge issuing from the center of the triangle. Given a broken hyperbolic structure $H$ on $(F^s, \Delta)$ and given an edge $e$ of $\Delta$, there are two distinguished points on $e$, corresponding to the inclusion of $e$ as an edge of two triangles in $\Delta$, each of these triangles being equipped with the structure of a hyperbolic ideal triangle inherited from $H$.

Let $(t, e)$ be a triangle-edge pair of $(F^s, \Delta)$. We measure the distance between the two distinguished points on $e$ using the metric of the triangle $t$. This distance is equipped with a sign, which is defined using the convention given in Figure 3. We note that this convention depends only on the orientation of the surface $F^s$, and not on any choice of an orientation for the edge $e$. We denote this signed distance by $s_H(t, e)$, and we call it the shift parameter induced on the triangle-edge pair $(t, e)$ by the structure $H$. We note that if $(t', e)$ is the other triangle-edge pair containing the edge $e$, then the shift parameter $s_H(t', e)$ has the same sign as $s_H(t, e)$, and that the values of the two parameters differ by a multiplicative factor equal to the homothety factor in (1.1.1).
Shift parameters for hyperbolic structures.

In the case where the broken hyperbolic structure $H$ is a hyperbolic structure, then the shift parameters $s_{H}(t, e)$ and $s_{H}(t', e)$ are equal, and we shall denote this common value by $s_{H}(e)$.

Topology for broken hyperbolic structures.

It is clear that the collection of shift parameters associated to all the triangle-edge pairs of $(F_{g}^{s}, \Delta)$ determines completely the broken hyperbolic structure. Thus, we have an injective map from $\mathcal{BH}(\Delta)$ into the space $\mathbb{R}^{P}$ of real valued functions on the set $P$ of triangle-edge pairs. Using this injection, we define a topology for the space $\mathcal{BH}(\Delta)$ by taking the product topology on $\mathbb{R}^{P}$ and the induced topology on the space $\mathcal{BH}(\Delta)$. Thus, two broken hyperbolic structures are close if and only if the shift parameters that they induce on triangle-edge pairs are close.

We now define similar parameters for the space $\mathcal{BM}(\Delta)$.

Shift parameters for broken measured foliations.

Let $F$ be a broken measured foliation on $(F_{g}^{s}, \Delta)$ and let $e$ be again an edge of $\Delta$. On each side of $e$, there is a triangle of $\Delta$ equipped with a measured foliation induced from $F$, there is a unique singular point in the interior of this triangle, and there is a well-defined singular leaf issuing from that singular point and hitting the edge $e$. The two hitting points associated to the two sides of $e$ give two distinguished points on $e$. If $(t, e)$ is now a triangle-edge pair in $(F_{g}^{s}, \Delta)$, then the segment in $e$ which joins these points has a well-defined measure, with respect to the transverse measure of the foliation induced on the triangle $t$. Again, we assign a sign to this measure, using the convention given in Figure 4, and we note that the definition of this sign depends on the orientation of the surface $F_{g}^{s}$ and not on a choice of an
orientation on $e$. We denote this signed measure by $s_F(t, e)$, and we call it the \textit{shift parameter} induced on the triangle-edge pair $(t, e)$ by the structure $F$. We note that if $(t', e)$ is the other triangle-edge pair containing $e$, then the quantity $s_F(t', e)$ differs from $s_F(t, e)$ by a multiplicative factor which is equal to the homothety factor in (1.4.4).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{In case (a), the sign is positive, and in case (b), the sign is negative.}
\end{figure}

\textbf{Shift parameters for measured foliations.}

In the case where the broken measured foliation $F$ is a measured foliation, then we have $s_F(t, e) = s_H(t', e)$, and we use the notation $s_F(e)$ to denote this common value, which we call the shift parameter associated to the edge $e$ by the measured foliation $F$.

\textbf{Topology for broken measured foliations.}

The collection of shifts associated to the various triangle-edge pairs of $(F^s_g, \Delta)$ determines completely the broken measured foliation $F$. Thus, we have an injective map from $BM(\Delta)$ to the space $\mathbb{R}^P$ of real valued functions on the set $P$ of triangle-edge pairs. This provides the space $BM(\Delta)$ with a topology for which two broken measured foliations are close if and only if the associated set of shift parameters on all the triangle-edge pairs are close.

There is a natural map $BH(\Delta) \to BM(\Delta)$ which is defined by associating to each broken hyperbolic structure its broken measured foliation defined in example 1.6 above and noting that, in that construction, the equivalence class of the broken measured foliation depends only on the equivalence class on the broken hyperbolic structure.
Conversely, given a broken measured foliation on \((F^s_g, \Delta)\), we can assign to it a broken hyperbolic structure by considering, for each face of \(\Delta\) with its induced foliation, a structure of hyperbolic ideal triangle for which this measured foliation is the horocyclic foliation. Properties (1.4.2) and (1.4.3) insure that this hyperbolic structure exists. Now the structure of hyperbolic ideal triangle on each face of \(\Delta\) makes the surface \(F^s_g\) naturally equipped with a broken hyperbolic structures with the homothety factor in (1.1.1) along each side of \(\Delta\) equal to the homothety factor in (1.4.4). It is clear that the equivalence class of this broken hyperbolic structure depends only on the equivalence class of the broken measured foliation with which we started.

Thus, we have a map \(BH(\Delta) \rightarrow BM(\Delta)\) which is one-to-one. We have the following proposition, whose proof is clear from the description of the topologies on the spaces \(BH(\Delta)\) and \(BM(\Delta)\).

**Proposition 2.1.**—The map \(BH(\Delta) \rightarrow BM(\Delta)\) which assigns to the equivalence class of each broken hyperbolic structure the equivalence class of the associated broken measured foliation is a homeomorphism.

**Topologies for the decorated spaces.**

To define the topologies of the decorated spaces, we use the shift parameters that were defined for the broken hyperbolic structures (respectively the broken measured foliations), together with extra parameters which give, for each puncture of \(F^s_g\), the distance between the distinguished closed curve corresponding to the decoration and a fixed point on an edge abutting on that puncture. In fact, we can choose once and for all, for each puncture of \(F^s_g\), one of the edges abutting on that puncture, and we fix a point on that edge. We then associate to each decoration (that is, to each distinguished closed curve around a puncture), the signed distances between the intersection points of this curve with the chosen edge of \(\Delta\) abutting on that puncture and the point that we fixed on that edge. To measure this distance, we choose one side for each edge, and we use the hyperbolic metric on the chosen side. The extra parameters for the decorated broken measured foliations are defined in the same manner.

It is clear now that the space \(\tilde{BH}(\Delta)\) (respectively \(\tilde{BM}(\Delta)\)) is an \(\mathbb{R}^s\)-bundle over the space \(BH(\Delta)\) (respectively \(BM(\Delta)\)), and we furthermore have the following

**Proposition 2.2.**—The map \(f_\Delta : \tilde{BH}(\Delta) \rightarrow \tilde{BM}(\Delta)\) which assigns to each equivalence class of decorated broken hyperbolic structure the equivalence class of the associated decorated broken measured foliation is a homeomorphism.

3.— The two-form on the space of decorated broken hyperbolic structures

**Definition 3.1 (\(\lambda\)-lengths).**—Let \(H\) be an element of \(\tilde{BH}(\Delta)\) and let \(t\) be a triangle of \(\Delta\). Then \(H\) equips \(t\) with the structure of a hyperbolic ideal triangle together with three distinguished horocyclic arcs centered at the vertices and joining pairwise the three edges of this triangle, as in Figure 5. We note that the three
horocycles are two-by-two disjoint since by assumption they are leaves of the horocyclic partial foliation of that triangle; this is in contrast to the situation in the paper [PP], where the horocycles were allowed to intersect. For each edge $e$ of $t$, let $\delta_H(t,e)$ be the distance between the two horocycles which intersect $e$. (Thus, this distance is nonnegative, unlike the situation in [PP].) We define $\lambda_H(t,e)$, the $\lambda$-length of the triangle-edge pair $(t,e)$ with respect to the broken hyperbolic structure $H$, by the formula

$$\lambda_H(t,e) = \sqrt{2\exp(\delta_H(t,e))}.$$  

**Figure 5** The three horocycles induced by the decoration.

**Proposition 3.2** A decorated broken hyperbolic structure is uniquely determined by its collection of $\lambda$-lengths of triangle-edge pairs.

**Proof.** This can be seen at the level of a face of $\Delta$: Let $t$ be a hyperbolic ideal triangle with vertices $v_1, v_2, v_3$, equipped with its three horocyclic arcs joining the edges pairwise as in Figure 5, with relative distances $\delta_1, \delta_2$ and $\delta_3$. The proof of Proposition 3.2 is a consequence of the following

**Lemma 3.3.**— The three numbers $\delta_1, \delta_2$ and $\delta_3$ determine completely the positions of the three horocyclic arcs in the triangle $t$.

**Proof.** We show that it is not possible to keep the same distances between the horocyclic arcs if we move any of these horocycles from its original position. Suppose for that that we move $h_1$ towards $v_1$. If we want to keep the same distance $\delta_2$ between $h_1$ and $h_3$, then we have to move $h_3$ to a new position that has to be farther away from the vertex $v_3$ than the old position. Now, in order to keep the distance $\delta_1$ between $h_3$ and $h_2$ unchanged, we have to move $h_2$ to a new position which is nearer than the old one to the vertex $v_2$. But then, the new distance between $h_2$ and $h_1$ will be greater than $\delta_3$. Thus, we cannot move $h_1$ towards $v_1$. The same kind of
argument shows that we cannot move $h_1$ away from $v_1$ while keeping unchanged the three relative distances $\delta_1$, $\delta_2$, $\delta_3$ between the horocyclic arcs. This proves Lemma 3.3 and Proposition 3.2.

Thus, the map from $\widetilde{\mathcal{BH}}(\Delta)$ to the set of functions on the collection $P$ of triangle-edge pairs of $\Delta$, which assigns to each triangle-edge pair its $\lambda$-length, is injective. It is clear that this map is continuous and proper (the topology on $\widetilde{\mathcal{BH}}(\Delta)$ being given by the shift parameters). Thus, we have the following

**Proposition 3.4.**— The map $\widetilde{\mathcal{BH}}(\Delta) \to \{ t \in \mathbb{R}_+ : t \geq \sqrt{2}\}^P$ induced by $\lambda$-lengths is a homeomorphism onto its image.

**Definition 3.5 (The two-form $\widetilde{\Omega}$ on $\widetilde{\mathcal{BH}}(\Delta)$).**— We define the form $\widetilde{\Omega}$ on $\widetilde{\mathcal{BH}}(\Delta)$ in terms of the $\lambda$–length coordinates by the formula

$$\widetilde{\Omega} = -2 \sum d \log a \wedge d \log b + d \log b \wedge d \log c + d \log c \wedge d \log a,$$

where the sum is taken over the set of triangles of $\Delta$, and where for each such triangle, $a$, $b$ and $c$ are the $\lambda$–lengths of the three associated triangle-edge pairs taken in a counterclockwise cyclic order.

From the way it is defined, it is clear that $\widetilde{\Omega}$ on $\widetilde{\mathcal{BH}}(\Delta)$ pulls back to the (restriction to $\widetilde{T}' \subseteq \widetilde{T}$ of) form $\tilde{\omega}$ which was defined in [Pe2] on the decorated Teichmüller space $\mathcal{T} \subset \widetilde{\mathcal{BH}}(\Delta)$ of the surface and which we used in the paper [PP]. The form $\tilde{\omega}$, in turn, is the pull-back of the Weil-Petersson Kähler two-form on the Teichmüller space $\mathcal{T}$ of the surface; in this sense $\widetilde{\Omega}$ “extends” the Weil-Petersson two-form.

4.— The two-form on the space of decorated broken measured foliations

**Definition 4.1 (Compactly supported broken measured foliation).**— Consider an element of $\mathcal{BM}(\Delta)$ represented by a foliation $F$ on $F_g^s$. We associate to $F$ a partial foliation $F_0$ on $F_g^s$ by deleting in the neighborhood of each puncture of $F_g^s$ the largest annulus foliated by closed leaves of $F$ which is described in Property (1.4.1) of Definition 1.4. We call this foliation $F_0$ the **compactly supported broken measured foliation on** $F_g^s** associated to** $F$.

We regard two compactly supported broken measured foliations as equivalent if they are associated to two broken measured foliations representing the same element of $\mathcal{BM}(\Delta)$, or, equivalently, if one of these compactly supported broken measured foliations can be obtained from the other one by an isotopy of the surface which preserves setwise each edge of $\Delta$ and the transverse measure in each face of $\Delta$.

We denote by $\mathcal{BM}_0(\Delta)$ the set of equivalence classes of compactly supported broken measured foliations on $F_g^s$.

We let $\mathcal{MF}_0$ be the set of equivalence classes of compactly supported measured foliations on $F_g^s$. Again, we note that any element of $\mathcal{MF}_0$ can be represented by a compactly supported broken measured foliation on $(F_g^s, \Delta)$ having all of its homothety factors equal to unity. Thus, there is a natural inclusion $\mathcal{MF}_0 \subset \mathcal{BM}_0(\Delta)$,
for each ideal triangulation $\Delta$. Furthermore, it is clear that the construction of the compactly supported broken measured foliation in Definition 4.1 gives a natural “identification” $BM(\Delta) \cong BM_0(\Delta)$ (i.e., a homotopy inverse to the inclusion $BM_0(\Delta) \subset BM(\Delta)$) which restricts to an identification between the two subspaces $MF \subset BM(\Delta)$ and $MF_0 \subset BM_0(\Delta)$.

**Definition 4.2 (Compactly supported broken measured foliation with collars)** A compactly supported broken measured foliation with collars is a partial broken measured foliation on $F_s^g$ which is obtained from a broken measured foliation by removing a (not necessarily maximal) annulus neighborhood of each puncture of $F_s^g$, which is foliated by circles. As in the case of compactly supported broken measured foliations, there is a natural equivalence relation on the space of compactly supported broken measured foliations with collars, and we let $\tilde{BM}_0(\Delta)$ be the space of equivalence classes.

We let $\tilde{PBM}_0(\Delta)$ be the space of equivalence classes of compactly supported broken measured foliations with collars, up to the natural action of $\mathbb{R}_+$ on the transverse measures.

To each element of $\tilde{BM}(\Delta)$, we associate an element of $\tilde{BM}_0(\Delta)$, by removing, from the support of a broken measured foliation $F$ representing the element of $\tilde{BM}(\Delta)$, the foliated annulus around each puncture of $F_s^g$ which is bounded by the closed leaf corresponding to the decoration. It is clear that this gives a natural identification $\tilde{BM}(\Delta) \cong \tilde{BM}_0(\Delta)$.

We let $\tilde{MF}'_0$ be the set of equivalence classes of compactly supported measured foliations with collars on $F_s^g$. Note that there is a natural one-to-one correspondence between the space $\tilde{MF}'_0$ and the space $\tilde{MF}$ which is defined at the end of §1 above. Again, we note that there is a natural inclusion $\tilde{MF}'_0 \subset \tilde{BM}_0(\Delta)$, for every ideal triangulation $\Delta$. Recall from [PP] the space $\tilde{MF}_0$ of decorated measured foliations of compact support; in that work, we considered formal collar widths, which may be positive or negative, where the space of collars of non-negative widths is identified with $\tilde{MF}'_0$.

The identification $\tilde{BM}(\Delta) \cong \tilde{BM}_0(\Delta)$ restricts to an identification between the subspace $\tilde{MF} \subset \tilde{BM}(\Delta)$ of decorated measured foliations and the subspace $\tilde{MF}'_0 \subset \tilde{BM}_0(\Delta)$ of compactly supported measured foliations with collars on $F_s^g$.

The next proposition follows directly from the definitions.

**Proposition 4.3** There is a natural fibre bundle $\tilde{BM}_0(\Delta) \to BM_0(\Delta)$ whose fibre above a point is $(\mathbb{R}_+ \cup \{0\})^s$ (the set of collar weights), which restricts to a fibre bundle $\tilde{MF}'_0 \to MF_0$ over the subspace $\tilde{MF}_0$ of $BM_0(\Delta)$.

**The dual graph** $G$. The dual graph to the triangulation $\Delta$ is a graph $G$ which is embedded in $F_s^g$ and which is defined as follows. In each face of $\Delta$, there is a vertex of $G$, and two such vertices are joined by a segment in the graph (and this segment will be the union of two edges of the graph) whenever the corresponding faces share a common edge. The segment intersects the corresponding edge of $\Delta$ in a unique
point, and we take this point to be also a vertex of $G$. Thus, the graph $G$ has two kinds of vertices: trivalent vertices (which are in one-to-one correspondence with the faces of the triangulation $\Delta$), and bivalent ones (in one-to-one correspondence with the edges of $\Delta$). An edge of $G$ is then a connected component of the space $G - \{\text{vertices}\}$.

We note by the way that the graph $G$ together with its embedding in $F^s_g$ is an example of what is usually called a fatgraph (that is, a graph equipped with a cyclic ordering at each of its vertices).

We shall define a two-form on $\widetilde{BM}(\Delta)$ which is an extension of the form $\iota$ which we defined in ([PP], §3) on the space $\mathcal{MF}$ of decorated measured foliation in $F^s_g$ and which in turn was an extension of Thurston’s form on the space of measured foliations on $F^s_g$. For the purpose of making this definition, we need to consider, as we did in ([PP], §2), the null-gon track or “freeway” which is dual to $\Delta$.

**The freeway dual to $\Delta$.** The freeway $\tau$ dual to $\Delta$ is a maximal train track in $F^s_g$ in the usual sense except that the connected components of $F^s_g - \tau$ are allowed to be once-punctured monogons. In fact, there is one such component around each puncture of $F^s_g$. The intersection of $\tau$ with each face $t$ of $\Delta$ is as in Figure 6. Thus, there are three trivalent vertices of $\tau$ in the interior of $t$. Furthermore, there is a bivalent vertex of $\tau$ on each edge of $\Delta$, in the same way as for the graph $G$ dual to $\Delta$. In fact, $\tau$ is obtained from $G$ by blowing up each trivalent vertex of $G$ into a triangle as illustrated in Figure 6.

![Figure 6](image_url)

**Figure 6** The intersection of $\tau$ with a face of $\Delta$.

It follows from these definitions that each edge of $\tau$ is contained in a face of $\Delta$, and that there are two types of edges in each such face: the edges which have one vertex on the boundary of that face, which we call the large edges, and those whose two vertices are in the interior of that face, which we call the small edges (see Figure 7).
We define now a system of weights on the edges of a \( \tau \), associated to a compactly supported foliation \( F_0 \). We deal first with the large edges. Let \( e \) be a large edge of \( \tau \). Then, \( e \) is contained in a triangle \( t \) of \( \Delta \), and \( e \) has one of its endpoints on an edge \( \ell \) of \( \Delta \). The weight \( w(e) \) induced by \( F_0 \) on \( e \) is defined as the total measure of the edge \( \ell \) with respect to the transverse measure of the restriction of the foliation \( F_0 \) to the triangle \( t \). Consider now a face \( t \) of \( \Delta \), and let \( w(a) \), \( w(b) \) and \( w(c) \) be the weights which are defined in this way on the three large edges \( a, b \) and \( c \) of \( \tau \) that are contained in \( t \). There are three weights, \( w(\alpha) \), \( w(\beta) \) and \( w(\gamma) \) induced on the three small edges \( \alpha, \beta \) and \( \gamma \) contained in \( t \). They are defined by the three equations

\[
\begin{align*}
    w(\alpha) &= \frac{1}{2} \left[ (w(b) + w(c) - w(a)) \right], \\
    w(\beta) &= \frac{1}{2} \left[ (w(c) + w(a) - w(b)) \right], \\
    w(\gamma) &= \frac{1}{2} \left[ (w(a) + w(b) - w(c)) \right].
\end{align*}
\]

We note that the triple \( w(a), w(b), w(c) \) satisfies the three large triangle inequalities, as a consequence of the condition of conservation of mass at the trivalent vertices of \( \tau \), and therefore the three weights \( w(a), w(b) \) and \( w(c) \) are \( \geq 0 \).

![Figure 7](image)

**Figure 7** Case (a) is a large edge and case (b) a small edge.

Any system of nonnegative weights on the edges of \( \tau \) gives a well-defined element of the space \( BM_0(\Delta) \) of equivalence classes of compactly supported broken measured foliations with collars on \( F_0^s \); provided these weights satisfy the following condition:

(4.1.2) At each trivalent vertex of \( \tau \), there is a condition of conservation of mass, as in usual train track theory: if at this trivalent vertex, \( \alpha \) and \( \beta \) are the two edges abutting from one side, and \( c \) the edge abutting from the other side, and if \( w(\alpha), w(\beta) \) and \( w(c) \) are the weights on these three edges, then we have \( w(\alpha) + w(\beta) = w(c) \).

**Definition 4.4 (Broken measure on the dual freeway).** — We shall say that a system of nonnegative weights on the edges of \( \tau \) satisfying condition (4.1.2) on the conservation of mass a *broken measure* on \( \tau \).
We note again that a broken measure on $\tau$ can be obtained by taking an arbitrary system of nonnegative weights on the large edges of $\tau$ satisfying the three large triangle inequalities and condition (4.1.2), since such a system of weights induces a unique system of weights (on all the edges of $\tau$).

To each element of $B(\tau)$ which is not identically zero, we associate a partial foliation by the usual construction which replaces each edge of $\tau$ having nonzero weight by a rectangle foliated by leaves parallel to its “horizontal” sides, these sides being nearly parallel to that edge. Each foliated rectangle is equipped with a transverse measure whose total mass is equal to the weight of the corresponding edge. The various rectangles are then glued along their “vertical” sides by measure-preserving maps at the trivalent vertices of $\tau$ and by affine maps at the bivalent ones, and we obtain a partial foliation on $F_g^*$. It is clear that this partial foliation is an element of $\tilde{BM}_0(\Delta)$. Thus, we have a natural map which assigns to each element of $B(\tau)$ which is not non-identically zero an element of $\tilde{BM}_0(\Delta)$. We assign to the zero element of $B(\tau)$ the empty foliation, and we thus obtain a natural map $B(\tau) \to \tilde{BM}_0(\Delta)$ (where we have included the empty foliation as an element of $\tilde{BM}_0(\Delta)$). We recall now that there is a natural identification between the spaces $\tilde{BM}_0(\Delta)$ of compactly supported broken measured foliations with collars and the space $\tilde{BM}(\Delta)$ of decorated broken measured foliations. Composing the two maps, we obtain a map $h_\Delta : B(\tau) \to \tilde{BM}(\Delta)$. We have the following

**Proposition 4.5.**— The natural map $h_\Delta : B(\tau) \to \tilde{BM}(\Delta)$ is a homeomorphism onto.

**Proof.** The only non-trivial fact is the injectivity of this map, and this is can be seen in the same way as in the proof of Theorem 3.1 of [Pa], where the same injectivity property is proved in the case of closed surfaces.

Proposition 4.5 provides a set of global coordinates for the space $\tilde{BM}(\Delta)$.

**Definition 4.6 (The two-form $\tilde{\iota}$ on $\tilde{BM}_0(\Delta)$).**— Using the coordinates provided by proposition 4.5 on $\tilde{BM}(\Delta)$, we define the two form $\tilde{\iota}$ on this space by the formula

$$\tilde{\iota} = -(1/2) \sum dw(\alpha) \wedge dw(\beta) + dw(\beta) \wedge dw(\gamma) + dw(\gamma) \wedge dw(\alpha),$$

where the sum is taken over the set of triangles of $\Delta$, and where for each such triangle, $\alpha, \beta$ and $\gamma$ are the small edges of $\tau$ which are contained in this triangle, in counterclockwise order with respect to the orientation of the surface, and where $w(\alpha), w(\beta)$ and $w(\gamma)$ are the weights on these edges which are induced by the element of $\tilde{BM}_0(\Delta)$. By the natural homeomorphism $\tilde{BM}_0(\Delta) \to \tilde{BM}(\Delta)$, we can consider the form $\tilde{\iota}$ as being defined on the space $\tilde{BM}(\Delta)$.

It is clear from the definitions that this form is an extension of the form $\iota$ defined in ([PP], §3) on the space $\tilde{MF}_0$ of decorated compactly supported measured foliations on $F_g^*$, which in turn is an extension of Thurston’s symplectic form on the space $\tilde{MF}_0$ of compactly supported measured foliations on $F_g^*$.

5.— The relation between the various two-forms
By Proposition 2.3, we have a natural homeomorphism $f_\Delta : \widetilde{BH}(\Delta) \to \widetilde{BM}(\Delta)$.

**Proposition 5.1.**— The homeomorphism $f_\Delta : \widetilde{BH}(\Delta) \to \widetilde{BM}(\Delta)$ preserves the forms $\Omega$ on $\widetilde{BH}(\Delta)$ and $\bar{\iota}$ on $\widetilde{BM}(\Delta)$.

**Proof.** Of course, we use the $\lambda$–length coordinates on $\Delta$ for the space $\widetilde{BH}(\Delta)$ and the broken measures on the dual freeway $\tau$ for the space $\widetilde{BM}(\Delta)$. Since the two-forms are defined in both cases by taking sums over triangles of $\Delta$, it suffices to consider the contributions of the forms to each such triangle.

Let $H \in \widetilde{BH}(\Delta)$. We describe the transformation $f_\Delta$ on the coordinates associated to $t$. Consider the dual graph $G$. Since each edge of $G$ is contained in a unique triangle-edge pair of $(F^s_g, \Delta)$, we can consider the $\lambda$–lengths associated to $H$ as being defined on the edges of the graph $G$, and we denote $\lambda$–length of such an edge $e$ by $\lambda(e)$. To each such edge $e$ is naturally associated a large edge $e'$ of the dual freeway track $\tau$. By examining the way the homeomorphism of Proposition 2.3 is defined, we can see that the weight $w(e')$ on $e'$ which is induced by the transformation $f_\Delta$ is given by

$$w(e') = 2\log \lambda(e) + \log(1/2).$$

Taking differentials, we have $dw(e') = 2d\log \lambda(e)$.

We now use equations (4.1.1) which relate the three small edges of the dual null-track $\tau$ which are in $\Delta$ to the weights on the large edges, and the formula for the two-form $\bar{\iota}$ given in Definition 4.6. This tells us that the contribution of $\bar{\iota}$ to the triangle $t$ of $\Delta$, in which the three edges of the dual graph $G$ in counterclockwise order are denoted by $a$, $b$ and $c$, is equal to $-(1/2)$ times the following quantity:

$$[d\log \lambda(b) + d\log \lambda(c) - d\log \lambda(a)] \land [d\log \lambda(c) + d\log \lambda(a) - d\log \lambda(b)]$$

$$+ [d\log \lambda(c) + d\log \lambda(a) - d\log \lambda(b)] \land [d\log \lambda(a) + d\log \lambda(b) - d\log \lambda(c)]$$

$$+ [d\log \lambda(a) + d\log \lambda(b) - d\log \lambda(c)] \land [d\log \lambda(b) + d\log \lambda(c) - d\log \lambda(a)].$$

Simplifying this formula, we find that the contribution of $\bar{\iota}$ to the triangle $t$ is equal to

$$-2d\log \lambda(b) \land d\log \lambda(c) + 2d\log \lambda(c) \land d\log \lambda(a) - 2\log \lambda(a) \land d\log \lambda(b).$$

Comparing with the formula for the two-form $\bar{\Omega}$ given in Definition 3.5, we see that the last quantity is also equal to the contribution of $\bar{\Omega}$ to the triangle $t$. This proves Proposition 5.1.

We consider now the product “Yamabe” space $\bar{Y} = \widetilde{BH}(\Delta) \times ]0, \infty[ \; \text{in which we view each element as the equivalence class of a decorated broken structure obtained by gluing ideal triangles, with the ideal triangles having constant curvature, but not necessarily equal to } -1$. More precisely, if $H$ is the equivalence class of a decorated broken hyperbolic structure on $F^s_g$ and if $x \in ]0, \infty[ \;$, then the element $(H, x)$ of $\bar{Y}$ denotes the equivalence class (for the relation of isotopy preserving the vertices, the edges and the faces of $\Delta$) of a structure on $F^s_g$ obtained by replacing each hyperbolic
ideal triangle by an ideal triangle in the complete simply connected Riemann surface with Gaussian curvature equal to $-x^2$, and gluing the new triangles using the same $\lambda$-lengths as for the structure $H$.

We let $q : \mathcal{Y} \to \mathcal{BH}(\Delta)$ denote the projection map onto the first factor.

There is a natural topology on the union $\mathcal{Y} = \mathcal{Y} \cup \mathcal{BM}_0(\Delta)$ which makes $\mathcal{Y}$ homeomorphic to the space $\mathcal{BM}_0(\Delta) \times [0, \infty]$ and which extends the topology which we considered in §5 of [PP] on the union $\mathcal{T} \cup \mathcal{MF}_0(\Delta)$. This topology on $\mathcal{Y}$ is defined as in [PP], §5, using the homeomorphism $f_\Delta : \mathcal{BH}(\Delta) \to \mathcal{BM}(\Delta)$ of Proposition 2.3, instead of the map $F_\Delta$ that we used in [PP]. The topology on $\mathcal{Y}$ is defined from the usual Thurston techniques for the compactification of Teichmüller space, and it has the property that the quotient of the space $\mathcal{Y}$ by the action of $\mathbb{R}_+$ (by homotheties on the second factor of the space $\mathcal{Y} = \mathcal{BH}(\Delta) \times [0, \infty]$ and by multiplying the transverse measure by a constant factor on the space $\mathcal{BM}_0(\Delta)$) gives a topology on the union $\mathcal{BH}(\Delta) \cup \mathcal{PB}\mathcal{M}_0(\Delta)$ which is a natural extension of the topology on $\mathcal{BH}(\Delta) \cup \mathcal{PB}\mathcal{M}_0(\Delta)$ which was defined in [OP2].

There is a criterion, which is equivalent to a criterion stated as Theorem 5.5 in [OP1], for the convergence of a sequence of covering hyperbolic structures (see appendix B) to the projective class of an affine foliation. The criterion in [OP1] uses the notion of covering hyperbolic structures instead of the broken hyperbolic structures that we use here, and it uses affine foliations instead of the broken measured foliation that we use here. The convergence criterion that we state below is tailored to our setting of broken hyperbolic structures and broken measured foliations, and it is valid because the topology of the union of the two spaces, as it is defined in [OP2], §7, like the topology of the corresponding spaces in [OP1], as an endpoint compactification along each ray. We shall not repeat the details of the proof, which are a straightforward adaptation of those of [OP1].

In the setting of this paper, the convergence criterion relies upon the map $f_\Delta : \mathcal{BH}(\Delta) \to \mathcal{BM}(\Delta)$ of Proposition 2.3, which, by the canonical identifications between the spaces $\mathcal{BM}(\Delta)$ and $\mathcal{BM}_0(\Delta)$, we can consider as a map

$$f_\Delta : \mathcal{BH}(\Delta) \to \mathcal{BM}_0(\Delta).$$

**Convergence criterion:** Let $(H_n, x_n)$ be a sequence of elements in the space $\mathcal{Y} = \mathcal{BH}(\Delta) \times [0, \infty]$ such that $H_n$ eventually leaves every compact set in $\mathcal{BH}(\Delta)$, and such that $x_n \to 0$. Then, $(H_n, x_n)$ converges to an element $F \in \mathcal{BM}_0(\Delta)$ in the topology of $\mathcal{Y}$ if and only if the sequence $x_n f_\Delta(H_n)$ converges to $F$ in the topology of $\mathcal{BM}_0(\Delta)$.

We define now the map

$$\bar{h} : \mathcal{Y} \to \mathcal{BM}_0(\Delta) \times [0, \infty]$$

by the formula

$$(H, x) \mapsto (x f_\Delta(H), x)$$
for every \((H, x) \in \mathcal{B}\mathcal{H}(\Delta) \times ]0, \infty[\).

The image of the map \(\tilde{h}\) is the subspace \(\mathcal{B}\mathcal{M}_0(\Delta) \times ]0, \infty[\).

By the convergence criterion above, this map extends to a homeomorphism

\[
\mathcal{T} : \mathcal{Y} \to \mathcal{B}\mathcal{M}_0(\Delta) \times [0, \infty[\]

in which the image of the set \(\mathcal{Y} - \mathcal{Y}\) of ideal points is the subset \(\mathcal{B}\mathcal{M}_0(\Delta) \times \{0\}\).

We consider the two-form \(\tilde{\Omega}\) on \(\mathcal{B}\mathcal{H}(\Delta)\) and its pull-back \(q^* (\tilde{\Omega})\) on \(\mathcal{Y}\), which we normalize to a form \(\Omega' = x^{-2} q^*(\tilde{\Omega})\) on \(\mathcal{Y}\). Likewise, we let \(\iota'\) be the pull-back of the two-form \(\iota\) on \(\mathcal{B}\mathcal{M}_0(\Delta)\) by the projection of \(\mathcal{B}\mathcal{M}_0(\Delta) \times ]0, \infty[\) onto the first factor. Then, using Proposition 5.1, \(\Omega'\) extends continuously to a two-form \(\tilde{\Omega}'\) on \(\mathcal{Y}\) whose restriction on \(\mathcal{Y} - \mathcal{Y} \approx \mathcal{B}\mathcal{M}_0(\Delta)\) is the form \(\tilde{\iota}\) which extends Thurston’s form on \(\mathcal{B}\mathcal{F}_0(\Delta)\).

We summarize these results as

**Theorem 5.2.**— The Weil-Petersson Kähler form on the Teichmüller space \(\mathcal{T}\) of \(F_{g,s}\) extends to a two-form \(\tilde{\Omega}\) on the space \(\mathcal{B}\mathcal{H}(\Delta)\) of decorated broken hyperbolic structures on \((F_{g,s}, \Delta)\), which induces a two-form \(\Omega'\) on the space \(\mathcal{Y} = \mathcal{B}\mathcal{H}(\Delta) \times ]0, \infty[\), which extends continuously to a two-form \(\tilde{\Omega}'\) on \(\mathcal{Y} \cup \mathcal{B}\mathcal{M}_0(\Delta)\). The restriction of this form \(\tilde{\Omega}'\) to the space \(\mathcal{B}\mathcal{M}_0(\Delta)\) is an extension of Thurston’s symplectic form on the space \(\mathcal{M}\mathcal{F}_0\) of compactly supported measured foliations on the surface.

It seems to us a basic question as to whether \(\tilde{\Omega}\) is non-degenerate (as are both the Weil-Petersson and Thurston forms).

We finally remark that one of the innovations in [PP] which has subsequently proved useful is the PL isomorphism of \(\mathcal{M}\mathcal{F}_0\) with the vector space of measures on the freeway, which requires consideration of both positive and negative collars. In the current paper, we consider only positive collars in \(\mathcal{M}\mathcal{F}_0\) and have the suspicion that there may be a nice synthesis of broken structures with negative collars.

**Appendix A — Affine foliations and broken measured foliations.**

For completeness, we recall the definition of affine foliation from [OP1], which was adapted from [HO] to the case of punctured surfaces, and compare with broken measured foliations.

**Definition A.1.**—(Affine foliation) A foliation \(F\) on \(F^s\) is an affine foliation if the following four properties are satisfied:

(A.1.1) There is a neighborhood of each puncture of \(F^s\) which is topologically an annulus on which \(F\) induces a foliation by closed leaves which are homotopic to the puncture.

(A.1.2) The foliation \(F\) has no Reeb components.

(A.1.3) The lift \(\tilde{F}\) of \(F\) to the universal cover \(\tilde{F}^s\) is equipped with a transverse measure \(\mu\) such that for each covering translation \(\alpha\), there exists a positive real
number $\phi(\alpha)$ with the property that for every arc $c$ in $\bar{F}_g^s$ which is transverse to $\bar{F}$, we have $\mu(\alpha(c)) = \phi(\alpha)\mu(c)$.

(A.1.4) If $c$ is an arc which is transverse to $F$ and which has one of its endpoints at a puncture of $F_g^s$, then the $\mu$-transverse measure of each lift of $c$ to $\bar{F}_g^s$ is infinite.

If $\Gamma$ denotes the group of covering translations of the cover $\bar{F}_g^s \to F_g^s$, then it is easy to see that the map $\alpha \mapsto \phi(\alpha)$ from $\Gamma$ to $\mathbb{R}_+^*$ is a homomorphism. It is called the holonomy homomorphism of the affine foliation. Using the canonical isomorphism $\Gamma \simeq \pi_1(S)$, we can thus associate to each affine foliation on $F_g^s$ a homomorphism $\pi_1(S) \to \mathbb{R}_+$, which is also called the holonomy homomorphism.

**Equivalence relation.**— We say that two affine foliations $F$ and $F'$ on $F_g^s$ are equivalent if $F$ can be obtained from $F'$ by an isotopy whose lift to $\bar{F}_g^s$ preserves the transverse measure of the lifted foliations $\bar{F}$ and $\bar{F}'$.

The set of equivalence classes of affine foliations is denoted by $\mathcal{AF}$ and was first studied by Hatcher and Oertel in [HO].

Proposition 7.1 of [OP2] establishes a natural homeomorphism between $\mathcal{BM}(\Delta)$ and $\mathcal{AF}$, for any ideal triangulation $\Delta$ of $F_g^s$. To define this homeomorphism, choose a face $F_1$ of $\bar{\Delta}$, the lift of the triangulation $\Delta$ to the universal cover $\bar{F}_g^s$ of $F_g^s$. The homeomorphism assigns then to each foliation $F$ representing an element of $\mathcal{BM}(\Delta)$ an affine foliation $F$ whose lift $\bar{F}$ to $\bar{F}_g^s$ is equipped with the transverse measure obtained by lifting to the face $F_1$ the transverse measure of $F$ on the image by the covering map, and then scaling the transverse measures on the various faces of $\bar{\Delta}$ so that they agree with each other along the boundaries of these faces.

Appendix B— $\lambda$-lengths and broken hyperbolic structures.

As illustrated in Proposition 3.4, $\lambda$-lengths give global coordinates on $\bar{BH}(\Delta)$. In this appendix, we recall and apply some of the machinery of $\lambda$-lengths to broken hyperbolic structures.

There are several coordinatizations of spaces of broken and unbroken hyperbolic structures on a punctured surface which depend upon the Minkowski inner product $<\cdot, \cdot>$ on $\mathbb{R}^3$ whose quadratic form is given by $x^2 + y^2 - z^2$ in the usual coordinates. As is well-known, the upper sheet

$$H = \{u = (x, y, z) \in \mathbb{R}^3 : < u, u > = -1 \text{ and } z > 0\}$$

of the two-sheeted hyperboloid is isometric to the hyperbolic plane. Furthermore, the open positive light cone

$$L^+ = \{u = (x, y, z) \in \mathbb{R}^3 : < u, u > = 0 \text{ and } z > 0\}$$

is identified with the collection of all horocycles in $H$ via the correspondence $u \mapsto h(u) = \{w \in H : < w, u > = -1\}$. Recall that the lambda length of a pair of
horocycles \( \{h_0, h_1\} \) is defined to be the transform \( \lambda(h_0, h_1) = \sqrt{2} \exp \delta \). Taking this particular transform renders the identification \( h \) geometrically natural in the sense that \( \lambda(h(u_0), h(u_1)) = \sqrt{-< u_0, u_1 >} \), for \( u_0, u_1 \in L^+ \) as one can check.

Three useful lemmas (with computational proofs, which we do not here reproduce) are as follows:

**Lemma B.1** [Pe1;Lemma 2.4] Given three linearly independent rays \( r_0, r_1, r_2 \subseteq L^+ \) from the origin and given three numbers \( \lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}_+ \), there are unique \( u_i \in r_i \), for \( i = 0, 1, 2 \) so that \( \lambda(h(u_i), h(u_j)) = \lambda_k \), where \( \{i, j, k\} = \{0, 1, 2\} \).

**Lemma B.2** [Pe1;Lemma 2.3] Given two points \( u_0, u_1 \in L^+ \), which do not lie on a common ray through the origin, and given two numbers \( \lambda_0, \lambda_1 \in \mathbb{R}_+ \), there is a unique point \( v \) on either side of the plane through the origin containing \( u_0, u_1 \) satisfying \( \lambda(h(v), h(u_i)) = \lambda_i \), for \( i = 0, 1 \).

**Lemma B.3** [Pe1;Proposition 2.8] Suppose that \( \{u_i\}^2_0 \subseteq L^+ \) are linearly independent, let \( \gamma(u_i, u_j) \) denote the geodesic in \( H \) with ideal vertices given by the centers of \( h(u_i) \) and \( h(u_j) \), for \( i \neq j \), and define

\[
-\lambda_i^2 = < u_j, u_k >, \quad \alpha_i = \frac{\lambda_i}{\lambda_j \lambda_k}, \quad \text{for } \{i, j, k\} = \{0, 1, 2\}.
\]

Then \( 2\alpha_i \) is the hyperbolic length along the horocycle \( h(u_i) \) between \( \gamma(u_i, u_j) \) and \( \gamma(u_i, u_k) \), for \( \{i, j, k\} = \{0, 1, 2\} \).

Armed with these lemmas, it is not difficult to give the basic coordinatization of decorated Teichmüller space by lambda lengths:

**Theorem B.4** [Pe1;Theorem 3.1] Fix an ideal triangulation \( \Delta \) of \( F^s_g \). Then the assignment of \( \lambda \)-lengths \( \hat{T}(F) \rightarrow R^3 \) is a homeomorphism onto.

**Proof.** We must describe an inverse to the mapping and thus give the construction of a decorated hyperbolic structure from an assignment of putative \( \lambda \)-lengths. To this end, consider the topological universal cover \( \tilde{F}^s_g \) of \( F^s_g \) and the lift \( \Delta \) of \( \Delta \) to \( \tilde{F}^s_g \); to each component arc of \( \tilde{\Delta} \) is associated the lambda length of its projection.

The proof proceeds by induction, and for the basis step, choose any face \( T_0 \) of \( \tilde{\Delta} \) and any ideal triangle \( t_0 \) in \( H \). The ideal vertices of \( t_0 \) determine three rays in \( L^+ \), so by Lemma A.1, there are three well-defined points in \( L^+ \) realizing the putative \( \lambda \)-lengths on the edges of \( T_0 \). (In effect, the basis step of normalizing a triangle “kills” the conjugacy by the Möbius group in the definition of Teichmüller space.) Of course, the triple of points in \( L^+ \) corresponds by affine duality to a triple of horocycles, one centered at each ideal vertex of \( t_0 \), i.e., a “decoration” on \( t_0 \).

To begin the induction step, consider a face \( T_1 \) adjacent to \( T_0 \) across an arc in \( \tilde{\Delta} \). The two ideal points \( u, v \) which \( T_0 \) and \( T_1 \) share have been lifted to \( L^+ \) in the basis step, and we let \( w \) denote the third ideal point of \( T_0 \) and let \( \Pi \) denote the plane through the origin determined by \( u, v \). According to Lemma A.2, there is a unique lift \( z \) of the third ideal point of \( T_1 \) to \( L^+ \) on the side of \( \Pi \) not containing \( w \) which realizes the putative \( \lambda \)-lengths. Again, \( u, v, z \) gives rise via affine duality to another decorated triangle \( t_1 \) in \( H \) sharing one edge and two horocycles with \( t_0 \).
One continues in this manner serially applying Lemma A.2 to produce a collection of decorated triangles in $H$, where any two triangles have disjoint interiors (because of our choice of the side of the plane in Lemma A.2). Thus, our construction gives an injection $\tilde{F}_g^* \to H$, and we next show that in fact this mapping is also a surjection. To this end, note first that the inductive construction has an image which is open in $H$ by construction. According to Lemma A.3, there is some $\varepsilon > 0$ so that each horocyclic arc inside of each triangle has length at least $\varepsilon$; indeed, there are only finitely many values for such lengths because the surface is comprised of finitely many triangles. Thus, each application of the inductive step moves a definite amount along each horocycle, and it follows easily that the induction has an image which is closed as well. It follows from connectivity that $\tilde{F}_g^* \to H$ is a homeomorphism, so $\tilde{\Delta}$ maps to a tesselation of $H$, i.e., a locally finite collection of geodesics decomposing $H$ into ideal triangles.

Following Poincaré, the hyperbolic symmetry group of this tessellation is the required (normalized) Fuchsian group $\Gamma$ giving a point of Teichmüller space, and the construction likewise provides a decoration on the quotient $H/\Gamma$ as required.

By a “sector” of $\Delta$ we mean a triangle-vertex pair, and we let $\Sigma = \Sigma(\Delta)$ denote the collection of sectors. To each sector is assigned half the hyperbolic length, or the “h-length”, of the corresponding horocyclic segment as given by Lemma A.3. Suppose that two decorated ideal triangles $t_0, t_1$ meet along a common edge $e$, and suppose that the h-lengths of the sectors incident on the endpoints of $e$ in $t_0$ are $\alpha, \beta$ and in $t_1$ are $\gamma, \delta$; it follows directly from Lemma A.3 that $\alpha \beta = \gamma \delta$, the so-called “coupling equation” on h-lengths, and we have:

**Corollary B.5** [Pe1; Proposition 3.5]. Decorated Teichmüller space is identified with the quadric variety in $\mathbb{R}^{\Sigma^2}_{\geq 0}$ determined by the coupling equations on h-lengths.

The reader will recognize the logs of the h-lengths as the transform in (4.1.1).

In order to apply the foregoing to broken hyperbolic structures, we next recall several definitions from [OP1], whose ideas originate in [Th].

**Definition B.6 (Stretch map).** — For $i = 1, 2$, let $(S_i, \Delta_i)$ be a pair consisting of a hyperbolic surface $S_i$ with punctures, equipped with a geodesic triangulation $\Delta_i$ whose vertices are at the punctures, and such that each face of $\Delta_i$ is isometric to an ideal triangle. Let $K > 1$. A map $f : (S_1, \Delta_1) \to (S_2, \Delta_2)$ is said to be a stretch map with factor $K$ if the restriction of $f$ to each face of $S_1$ is a $K$–Lipschitz stretch map of the following form:

**Definition B.7 (K–Lipschitz stretch map).** — Let $K > 1$ and consider two hyperbolic ideal triangles $T_1$ and $T_2$. Then, a $K$–Lipschitz stretch map from $T_1$ to $T_2$ is is a $K$–Lipschitz homeomorphism such that on each side of $T_1$, the map acts as a homothety, multiplying arclength by $K$, and such that on every closed disk contained in the interior of the triangle, the global Lipschitz constant of the restricted map is strictly less then $K$. For $K = 1$, a $K$–Lipschitz stretch map is an isometry. For $K < 1$, a $K$–Lipschitz stretch map is defined as the inverse map of a $1/K$–Lipschitz stretch map of factor $1/K$. 

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Suppose that \( \phi : \pi_1(F^s_g) \to \mathbb{R}_+ \) is a homomorphism to the multiplicative group of positive reals. (The following definition is adapted from [OP1], where a similar structure is defined on the universal abelian cover of \( F^s_g \); instead of on the universal cover as we do here.)

**Definition B.8 (Covering hyperbolic structure).**— A covering hyperbolic structure for \((F^s_g, \Delta)\) with stretch homomorphism \( \phi \) is a hyperbolic structure \( h \) on \( \tilde{F}^s_g \) such that the lift of the triangulation \( \Delta \) to \( \tilde{F}^s_g \) is a geodesic triangulation \( \tilde{\Delta} \), and such that each covering translation \( \alpha \in \pi_1(F^s_g) \) of the cover \( \tilde{F}^s_g \to F^s_g \) restricts to a stretch map on each triangle complementary to \( \tilde{\Delta} \) whose factor is \( \phi(\alpha) \).

**Construction B.9 (Developing image of broken hyperbolic structure).**— Let \( H \) be a broken hyperbolic structure on \((F^s_g, \Delta)\), and take the corresponding \( \lambda \)-lengths in \( \mathbb{R}_+^P \), where \( P \) is the collection of triangle-edge pairs of \( \Delta \). Choose a face \( T_0 \) of \( \Delta \) to normalize and use the \( \lambda \)-lengths to define a lift to \( L^+ \) as in the basis step of the proof of Theorem B.4. Suppose that \( T_1 \) is another face of \( \Delta \) meeting \( T_0 \) along the edge \( e \), so \( \lambda(T_1, e) = \sigma(T_0, T_1) \lambda(T_0, e) \), where \( \sigma \) denotes the homothety scaling factor in (1.1.1). Let \((T_1, a), (T_1, b)\) denote the triangle-edge pairs of \( T_1 \) other than \((T_1, e)\), and apply Lemma A.2 as before using the \( \lambda \)-lengths \( \sigma(T_0, T_1) \lambda(T_1, a) \) and \( \sigma(T_0, T_1) \lambda(T_1, b) \) to produce the lift \( z \in L^+ \) of the vertex of \( T_1 \) other than the ideal points of \( e \). Notice that the center of the horocycle \( h(z) \) is actually independent of \( \sigma(T_0, T_1) \) by linearity, though the horocycle \( h(z) \) itself does depend upon the \( \sigma(T_0, T_1) \). Continue in this manner as in Theorem B.4, to produce a mapping \( \tilde{F}^s_g \to H \), whose image is called the developing image of the broken hyperbolic structure \( H \).

**Proposition B.10**— The hyperbolic structure on \( \tilde{F}^s_g \) which is produced by Construction B.9 is a covering hyperbolic structure.

**Theorem B.11**— A broken hyperbolic structure on a punctured surface of finite topological type gives rise to a complete hyperbolic structure on the plane; in other words, the developing image of this structure is the entire hyperbolic plane. The universal cover is thus isometric to the hyperbolic plane with the covering transformations acting by quasi-isometries.

**Proof.** It remains only to prove that the developing image of a fixed broken hyperbolic structure is the entire hyperbolic plane. Consider a triangle \( t_1 \subset H \) and its translate \( t_{n+1} \) under a parabolic fixing a vertex \( u \) of \( t_1 \), and let \( t_1, t_2, \ldots, t_{n+1} \) be the consecutive faces of \( \Delta \) in the developing image from \( t_1 \) to \( t_{n+1} \). Since the holonomy homomorphism of a broken hyperbolic structures is trivial around the punctures, we have \( \prod_{i=1}^n \sigma(t_i, t_{i+1}) = 1 \), so in the developing image the horocyclic segment in \( t_1 \) centered at \( u \) has the same hyperbolic length as the horocyclic segment in \( t_{n+1} \) centered at \( u \) by Lemma A.3. Now, let \( N \) majorize the number of intersections of a horocyclic leaf of a horocyclic foliation with \( \Delta \) (so \( N \) can be estimated merely in terms of the topology of the surface), and let \( \varepsilon \) be the smallest \( h \)-length of the broken hyperbolic structure in any sector of \( \Delta \). Thus, traversing \( N \) consecutive faces sharing an ideal point in the developing image moves at least a distance \( \varepsilon \) along the horocycle. It follows that the developing image is the entire hyperbolic plane.
This is an interesting class of geometric structures generalizing hyperbolic structures on surfaces: the universal cover admits the metric structure of the hyperbolic plane with the covering transformations acting by quasi-isometry for this metric. In our investigations here, we have restricted attention to such structures relative to a specified ideal triangulation; we wonder if there is an application of the generalized convex hull construction in Section 7 of [Pe3] (which applies to discrete and radially dense subsets of $L^+$) giving a corresponding cell decomposition of an appropriate space of these generalized hyperbolic structures in the spirit of Section 5 of [Pe1].

Finally, given a tesselation $\tau$ of $H$, define a broken hyperbolic structure of $(H, \tau)$ to be a Riemannian metric of constant curvature $-1$ on $H - \tau$, such that each face is isometric to an ideal triangle, where we furthermore require that metrics on adjacent faces $t_0, t_1$ are related by a homothety scaling factor $\sigma(t_0, t_1)$, where there is no completeness condition and for the no horocyclic holonomy condition, we require that there is some $K > 1$ so that for each edge $e$ of $\tau$ there is some natural number $n = n_e$ so that if $e = e_1, e_2, \ldots, e_{n+1}$ are consecutive edges of $\tau$ sharing an ideal point with consecutive faces $f_1, f_2, \ldots, f_n$, where $e_i, e_{i+1}$ lie in the frontier of $f_i$, for each $i = 1, \ldots, n$, then $K^{-1} < \prod_{i=1}^{n} \sigma(f_i, f_{i+1}) < K$. As in Theorem B.4 or Proposition 3.4, $\lambda$-lengths on triangle-edge pairs of $\tau$ give global coordinates on the space of broken hyperbolic structures on $H$. Furthermore as in Theorem B.11, the developing image of $H$ is the entire hyperbolic plane. This gives a new class of homeomorphisms of the circle as in Section 6 of [Pe3].
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