ON THE EULER-POINCARÉ CHARACTERISTIC AND MIXED MULTIPlicITIES OF MAXIMAL DEGREES

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ABSTRACT: This paper defines the Euler-Poincaré characteristic of joint reductions of ideals which concerns the maximal terms in the Hilbert polynomial; characterizes the positivity of mixed multiplicities in terms of minimal joint reductions; proves the additivity and other elementary properties for mixed multiplicities. The results of the paper together with the results of [17] seem to show a natural and nice picture of mixed multiplicities of maximal degrees.

1 Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and infinite residue field $k = A/\mathfrak{m}$. Let $M$ be a finitely generated $A$-module. Let $J$ be an $\mathfrak{m}$-primary ideal, $I_1, \ldots, I_d$ be ideals of $A$. Put $e_i = (0, \ldots, 1, \ldots, 0); n = (n_1, \ldots, n_d); k = (k_1, \ldots, k_d); \mathbf{0} = (0, \ldots, 0) \in \mathbb{N}^d; \mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d; |k| = k_1 + \cdots + k_d; I = I_1, \ldots, I_d; \mathbb{I} = I_1^{n_1} \cdots I_d^{n_d}$.

Denote by $P(n_0, n, J, I, M)$ the Hilbert polynomial of the function $\ell\left(\frac{J^{n_0+\mathbb{I}^{n}M}}{J^{n_0+1+\mathbb{I}^{n}M}}\right)$, by $\triangle^{(k_0, k)}P(n_0, n, J, I, M)$ the $(k_0, k)$-difference of the polynomial $P(n_0, n, J, I, M)$. And one can write $P(n_0, n, J, I, M) = \sum_{(k_0, k) \in \mathbb{N}^{d+1}} e(J^{k_0+k+\mathbb{I}}; M)(\binom{n_0+k_0}{k_0}) \binom{n+k}{n}$, here $\binom{n+k}{n} = \binom{n_1+k_1}{n_1} \cdots \binom{n_d+k_d}{n_d}$. Recall that the original mixed multiplicity theory studied the mixed multiplicities concerning the terms of highest total degree in the Hilbert polynomial $P(n_0, n, J, I, M)$, i.e., the mixed multiplicities $e(J^{k_0+1}; \mathbb{I}; M)$ with $k_0 + |k| = \deg P(n_0, n, J, I, M)$ (see e.g. [3–10, 14–16, 18–34]). And in
a recent paper [17], one considered a larger class than the class of original mixed multiplicities concerning the terms of maximal degrees in the Hilbert polynomial $P(n_0, n, J, I, M)$. There is a fact that the condition for $\Delta^{(k_0, k)}P(n_0, n, J, I, M)$ to be a constant is equivalent to that the term $e(J^{[k_0+1]}, I^{[k]}; M)^{(n_0+k_0)(n+k)}$ in the polynomial $P(n_0, n, J, I, M)$ satisfies the condition $e(J^{[h_0+1]}, I^{[h]}; M) = 0$ for all $(h_0, h) > (k_0, k)$. And in this case, $\Delta^{(k_0, k)}P(n_0, n, J, I, M) = e(J^{[k_0+1]}, I^{[k]}; M)$ (see e.g. [17, Proposition 2.4 and Proposition 2.9]). These facts show the natural appearance of the objects given in [17] by the following definition.

**Definition 1.1** (Definition 4.2). We call that $e(J^{[k_0+1]}, I^{[k]}; M)$ is the mixed multiplicity of maximal degrees of $M$ with respect to ideals $J, I$ of the type $(k_0 + 1, k)$ if $e(J^{[h_0+1]}, I^{[h]}; M) = 0$ for all $(h_0, h) > (k_0, k)$.

One has seen the presence of all these mixed multiplicities in [17, Example 2.12]. The results of [17] showed that many important properties of usual mixed multiplicities not only are still true but also are stated more natural in the broader class of the mixed multiplicities of maximal degrees. However, the additivity and other properties of these multiplicities are not yet known. Recall that by using the multiplicity formula of Rees modules [7, Theorem 4.4] (which is a generalized version of [21, Theorem 1.4]), although the additivity of usual mixed multiplicities was solved in [32], but it seems that this method can not be applied to mixed multiplicities of maximal degrees. This paper first studies the additivity and characterizes the positivity in terms of joint reductions for these mixed multiplicities.

Remember that Rees in 1984 [12] gave the notion of joint reductions and considered the Euler-Poincaré series for joint reductions of ideals of dimension 0, and from which he showed that each mixed multiplicity is the multiplicity of a joint reduction.

**Definition 1.2** (Definition 2.2). Let $I_i$ be a sequence consisting $k_i$ elements of $I_i$ for all $1 \leq i \leq d$. Put $x = I_1, \ldots, I_d$ and $(\emptyset) = 0_A$. Then $x$ is called a sequence of the type $k$ in $I$. And $x$ is called a joint reduction of $I$ with respect to $M$ of the type $k$ if $\mathbb{P}^n M = \sum_{i=1}^{d} (I_i)^{\mathbb{P}^n-e_i}M$ for all large $n$. A joint reduction of the type $k$ is called a minimal joint reduction if every sequence of the type $h$ in $I$ with $h < k$ is not a joint reduction of $I$ with respect to $M$.

In 1994, by defining the "Euler-Poincaré polynomial" of joint reductions for graded modules, Kirby-Rees [8] proved the additive property and the additivity and reduction formula for mixed multiplicities of graded modules. And basing on the idea of Serre [13] and Auslander-Buchsbaum [1], the authors in [34] defined the Euler-Poincaré characteristic of mixed multiplicity systems for graded modules. However, the results obtained from these works seem to be not close enough to use in proving even the properties for mixed multiplicities of ideals as in [32, 33].
The above facts are a motivation to encourage us to find a more specific invariant that first becomes an effective tool for proving properties of mixed multiplicities of ideals in Noetherian local rings.

Let \( x = J_1, \ldots, J_d, J_0 \) be a joint reduction of \( I, J \) with respect to \( M \) of the type \((k_0 + 1, k)\), here \( J_0 \subset J, J_i \subset I_i \) (\( 1 \leq i \leq d \)) and \( t_0, t_1, \ldots, t_d \) be variables over \( A \). Set \( X = J_1 t_1, \ldots, J_d t_d, J_0 t_0 \) and \( T^n = t_1^{n_1} \cdots t_d^{n_d} \). Rees in [12] built the Euler-Poincaré series basing on the Koszul complex of the module \( \bigoplus_{n_0 \geq 0, n \geq 0} J^{n_0} I^n M t_0^{n_0} T^n \) with respect to \( X \). Now basing on Kirby and Rees’s approach [8] in a new context of mixed multiplicities of maximal degrees, we consider the Koszul complex of the module \( M = \bigoplus_{n_0 \geq 0, n \geq 0} I^n M t_0^{n_0} T^n \) with respect to \( X \), and build an invariant called the Euler-Poincaré characteristic as follows.

For any \( 0 \leq i \leq n = k_0 + |k| + 1 \), denote by \( H_i(X, M) \) the \( i \)th homology module of the Koszul complex of \( M \) with respect to \( X \). Using results of Kirby and Rees in [8], we obtain Lemma 3.3, and get by Proposition 2.7 and Lemma 3.3 that the sum \( \sum_{i=0}^n (-1)^i \ell_i(H_i(X, M)_{(n_0, n)}) \) is a constant for all large enough \( n_0, n \). And this constant is called the Euler-Poincaré characteristic of the joint reduction \( x \) of the module \( M \) with respect to the ideals \( J, I \), and denoted by \( \chi(x, J, I, M) \). And our goal is completed by Proposition 3.7 which shows that this invariant is additive on \( A \)-modules and characterizes mixed multiplicities of maximal degrees.

Next, to state the main result, we need to explain more the relationship between some objects. The mixed multiplicity of maximal degrees of \( M \) with respect to ideals \( J, I \) of the type \((k_0 + 1, k)\) is defined if and only if \( \Delta^{(k_0, k)} P(n_0, n, J, I, M) \) is a constant. This is equivalent to that there exists a joint reduction of \( I, J \) with respect to \( M \) of the type \((k, k_0 + 1)\) by Proposition 2.7 (see Remark 4.3).

The paper not only proves the additivity, but more noticeably characterizes the positivity in terms of minimal joint reductions for mixed multiplicities of maximal degrees. And as one might expect, we obtain the following theorem.

**Theorem 1.3** (Theorem 4.4). Let \( N \) be an \( A \)-submodule of \( M \). Assume that the mixed multiplicity of maximal degrees of \( M \) with respect to \( J, I \) of the type \((k_0 + 1, k)\) is defined and let \( x \) be a joint reduction of the type \((k_0 + 1, k)\) of \( I, J \) with respect to \( M \). Then \( \chi(x, J, I, M) \) is independent of \( x \) and we have

1. \( e(J^{[k_0 + 1]}, I^{[k]}; M) = \Delta^{(k_0, k)} P(n_0, n, J, I, M) = \chi(x, J, I, M) \).
2. \( e(J^{[k_0 + 1]}, I^{[k]}; M) > 0 \) if and only if \( x \) is a minimal joint reduction.
3. \( e(J^{[k_0 + 1]}, I^{[k]}; M) = e(J^{[k_0 + 1]}, I^{[k]}, N) + e(J^{[k_0 + 1]}, I^{[k]}, M/N) \).

Theorem 1.3 and its consequences together with the results of [17] seem to show a natural and pleasant picture of mixed multiplicities of maximal degrees.
Using Theorem 1.3 (iii), we prove the additivity and reduction formulas (Corollary 4.6), and formulas concerning the rank of modules (Corollary 4.8) for mixed multiplicities of maximal degrees. Applying these results, we immediately recover results on mixed multiplicities in [32, 33]. Moreover, from Theorem 1.3 (ii), we get a characterization for the positivity of the usual mixed multiplicities of ideals in terms of minimal joint reductions (see Remark 4.9).

This paper is divided into four sections. Section 2 is devoted to the discussion of joint reductions of ideals and differences of Hilbert polynomials and answers questions: when differences of Hilbert polynomials are constants (Prop. 2.7) and are positive constants (Prop. 2.11). Section 3 builds the Euler-Poincaré characteristic of joint reductions which corresponds with mixed multiplicities of ideals; studies the positivity, the additivity (Prop. 3.7) of this invariant. In Section 4, we prove the main theorem and corollaries for mixed multiplicities of ideals.

2 Joint reductions and differences of Hilbert polynomials

In this section, in terms of joint reductions, we answer two questions: when differences of Hilbert polynomials are constants (see Proposition 2.7) and when differences of Hilbert polynomials are positive constants (see Proposition 2.11).

Let \( k, m, n \in \mathbb{N}^d \). Write \( m > n \) if \( m - n \in \mathbb{N}^d \) and there exists \( 1 \leq i \leq d \) such that \( m_i > n_i \). In addition, for any numerical polynomial \( f(n) \) in \( n \in \mathbb{N}^d \), denote by \( \triangle^k f(n) \) the \( k \)-difference of \( f(n) \), which is defined by \( \triangle^0 f(n) = f(n) \); \( \triangle^e f(n) = f(n) - f(n - e_i) \) and \( \triangle^k f(n) = \triangle^{e_i} (\triangle^{k-e_i} f(n)) \) for \( k \geq e_i \).

We assign \( \dim M = -\infty \) for \( M = 0 \) and the degree \( -\infty \) to the zero polynomial. Set \( I = I_1 \cdots I_d; \overline{M} = M/0_M : I^\infty; q = \dim \overline{M} \). Recall that the Hilbert function \( \ell \left( \frac{J^{n_0}M}{J^{n_0+1}n_{n_0}M} \right) \) is a polynomial of total degree \( q - 1 \) for all large \( n_0, n \) by [22, Proposition 3.1] (see [10]). Denote this Hilbert polynomial by \( P(n_0, n, J, I, M) \).

Remark 2.1. We have \( \deg P(n_0, n, J, I, M) = q - 1 \). Put \( n_1 = \cdots = n_d = m \) and fix large enough \( m \) such that \( P(n_0, m, J, I, M) = \ell_A \left( \frac{J^{n_0}M}{J^{n_0+1}n_{n_0}M} \right) \) for large enough \( n_0 \) and \( 0_M : I^\infty = 0_M : I^m \). Then \( \dim I^m M = \dim \overline{M} \) and \( P(n_0, m, J, I, M) \) is a polynomial in \( n_0 \) of degree \( \dim \overline{M} - 1 \). So \( q - 1 = \deg P(n_0, n, J, I, M) = \deg P(n_0, m, J, I, M) \). Hence \( \triangle^{(k_0, 0)} P(n_0, n, J, I, M) = \triangle^{k_0} P(n_0, m, J, I, M) \) if one of the sides is a constant. From this it follows that \( \triangle^{(1, 0)} P(n_0, n, J, I, M) = 0 \) if and only if \( P(n_0, n, J, I, M) \) is a constant, and it also follows that \( \triangle^{(k_0, 0)} P(n_0, n, J, I, M) \) is a constant if and only if \( k_0 \geq \deg P(n_0, n, J, I, M) = \dim M/0_M : I^\infty - 1 \).
The concept of joint reductions of \( m \)-primary ideals was given by Rees [12] in 1984 and was extended to the set of arbitrary ideals by [11, 24, 25, 28, 35]. In this paper we use this concept in another convenient presentation (see [28]).

**Definition 2.2** (see [12, 28]). Let \( \mathcal{I}_i \) be a sequence consisting \( k_i \) elements of \( I_i \) for all \( 1 \leq i \leq d \). Put \( x = \mathcal{I}_1, \ldots, \mathcal{I}_d \) and \((\emptyset) = 0_A\). Then \( x \) is called a joint reduction of \( I \) with respect to \( M \) of the type \( k \) if \( I^n M = \sum_{i=1}^d (\mathcal{I}_i)I^{n-e_i}M \) for all large \( n \). A joint reduction of the type \( k \) is called a minimal joint reduction if every sequence of the type \( h \) in \( I \) with \( h < k \) is not a joint reduction of \( I \) with respect to \( M \).

And a useful tool used in this paper is the concept of weak-(FC)-sequences which was defined in [22] (see e.g. [4, 10, 27, 28]) as the following definition.

**Definition 2.3** ([22]). An element \( x \in A \) is called a weak-(FC)-element of \( I \) with respect to \( M \) if there exists \( 1 \leq i \leq d \) such that \( x \in I_i \) and the following conditions are satisfied:

(FC1): \( xM \cap I^n M = xI^{n-e_i}M \) for all large \( n \).

(FC2): \( x \) is an \( I \)-filter-regular element with respect to \( M \), i.e., \( 0_M : x \subseteq 0_M : I^\infty \).

Let \( x_1, \ldots, x_t \) be elements of \( A \). For any \( 0 \leq i \leq t \), set \( M_i = M/(x_1, \ldots, x_i)M \). Then \( x_1, \ldots, x_t \) is called a weak-(FC)-sequence of \( I \) with respect to \( M \) if \( x_{i+1} \) is a weak-(FC)-element of \( I \) with respect to \( M_i \) for all \( 0 \leq i \leq t - 1 \). If a weak-(FC)-sequence of \( I \) with respect to \( M \) consists of \( k_1 \) elements of \( I_1, \ldots, k_d \) elements of \( I_d \) \((k_1, \ldots, k_d \geq 0)\), then it is called a weak-(FC)-sequence of \( I \) with respect to \( M \) of the type \( k \). A weak-(FC)-sequence \( x_1, \ldots, x_t \) is called a maximal weak-(FC)-sequence if \( I \not\subset \sqrt{Ann(M_{t-1})} \) and \( I \subseteq \sqrt{Ann(M_t)} \).

The following note recalls some important properties of weak-(FC)-sequences.

**Note 2.4.** Let \( \mathcal{I}_i \) be a sequence of elements of \( I_i \) for all \( 1 \leq i \leq d \). Assume that \( \mathcal{I}_1, \ldots, \mathcal{I}_d \) is a weak-(FC)-sequence of \( I \) with respect to \( M \). Then

\[
(\mathcal{I}_1, \ldots, \mathcal{I}_d)M \cap I^n M = \sum_{i=1}^d (\mathcal{I}_i)I^{n-e_i}M \tag{1}
\]

for all large \( n \) by [25, Theorem 3.4 (i)]. And if \( x \in I_i \) \((1 \leq i \leq d)\) is a weak-(FC)-element of \( I, J \) with respect to \( M \), then

\[
P(n_0, n, J, I, M/xM) = P(n_0, n, J, I, M) - P(n_0, n - e_i, J, I, M) \tag{2}
\]

by [6, (3)] (or the proof of [10, Proposition 3.3 (i)]).
The relationship between weak-(FC)-sequences and differences of Hilbert polynomials is showed by the following proposition.

**Proposition 2.5.** Set $J = I_0$. Let $k_i > 0$, $x \in I_i$ ($0 \leq i \leq d$) be a weak-(FC)-element of $I, J$ with respect to $M$. Then $\dim M/xM : I^\infty \leq \dim M/0M : I^\infty - 1$ and

\[
\Delta^{(k_0,k)}P(n_0, n, J, I, M) = \begin{cases} 
\Delta^{(k_0,k-e_i)}P(n_0, n, J, I, M/xM) & \text{if } 1 \leq i \leq d \\
\Delta^{(k_0-1,k)}P(n_0, n, J, I, M/xM) & \text{if } i = 0.
\end{cases}
\]

**Proof.** By (2) in Note 2.4, we obtain

\[
P(n_0, n, J, I, M/xM) = \begin{cases} 
\Delta^{(0,e_i)}P(n_0, n, J, I, M) & \text{if } 1 \leq i \leq d \\
\Delta^{(1,0)}P(n_0, n, J, I, M) & \text{if } i = 0.
\end{cases} \tag{3}
\]

By (3), it follows that

\[
\deg P(n_0, n, J, I, M/xM) \leq \deg P(n_0, n, J, I, M) - 1.
\]

Hence

\[
\dim M/xM : I^\infty \leq \dim M/0M : I^\infty - 1
\]

by Remark 2.1. Also by (3) we get

\[
\Delta^{(k_0,k)}P(n_0, n, J, I, M) = \begin{cases} 
\Delta^{(k_0,k-e_i)}P(n_0, n, J, I, M/xM) & \text{if } 1 \leq i \leq d \\
\Delta^{(k_0-1,k)}P(n_0, n, J, I, M/xM) & \text{if } i = 0.
\end{cases}
\]

By the way, we have some explanations for choosing the element $x$ in (3).

**Remark 2.6.** In studying mixed multiplicities, one always needs the equation (3). To have (3) one used different sequences: Risler and Teissier in 1973 [16] used superficial sequences of $m$-primary ideals; Viet in 2000 [22] used weak-(FC)-sequences; Trung in 2001 [19] used "bi-filter-regular sequences"; Trung and Verma in 2007 [20] used $(\varepsilon_1, \ldots, \varepsilon_m)$-superficial sequences. However, [4, Remark 3.8] showed that the sequences used in [16, 19, 20] are weak-(FC)-sequences. Moreover, [29, Remark 4.1] and [4, Theorem 3.7 and Remark 3.8] seem to account well for the minimum of conditions of weak-(FC)-sequences that is used in the proof of (3) (see [4, Remark 3.9]). This explains why we need to use weak-(FC)-sequences in this paper.

The next proposition answers the question when the $(k_0,k)$-difference of the Hilbert polynomial is a constant in terms of joint reductions and weak-(FC)-sequences.
Proposition 2.7. The following statements are equivalent:

(i) There exists a weak-(FC)-sequence of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) which is a joint reduction of \( M \) with respect to \( I, J \).

(ii) There exists a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \).

(iii) \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) \) is a constant.

Proof. (i) \( \Rightarrow \) (ii): holds trivially. (ii) \( \Rightarrow \) (i): Let \( x = x_1, \ldots, x_n \) be a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \). Set \( J = I_0 \). Assume that \( x_i \in I_i \) for \( 0 \leq i \leq d \). Now, if \( IJ \not\subseteq \sqrt{\text{Ann}(M)} \), then by [15, Lemma 17.3.2] and [28, Proposition 2.3], there exists a weak-(FC)-element \( y_1 \in I_i \) with respect to \( M \) such that \( y_1, x_2, \ldots, x_n \) is also a joint reduction. If \( IJ \subset \sqrt{\text{Ann}(M)} \), then it can be verified that \( x = x_1, \ldots, x_n \) is a weak-(FC)-sequence by Definition 2.3. Hence there exists a weak-(FC)-sequence \( y_1, y_2, \ldots, y_n \) of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) such that \( y_1, y_2, \ldots, y_n \) is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) by induction. (i) \( \Leftrightarrow \) (iii): By [28, Proposition 2.3] (see [22, Remark 1]), there exists a weak-(FC)-sequence \( x = x_1, \ldots, x_n \) of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) with \( x_n \in J \). Then \( P(n_0, n, J, I, M/(x)M) = 0 \) if and only if \( J^{n_0}[n]M \subset (x)M \) for all large \( n_0, n \). This is equivalent to that \( x \) is a joint reduction by (1) in Note 2.4 (see also [35, Corollary 3.5]). On the other hand, from Proposition 2.5, we obtain

\[
P(n_0, n, J, I, M/(x)M) = \Delta^{(1,0)} P(n_0, n, J, I, M/(x')M),
\]

here \( x' = x_1, \ldots, x_{n-1} \). So \( x \) is a joint reduction if and only if

\[
\Delta^{(1,0)} P(n_0, n, J, I, M/(x')M) = 0.
\]

This is equivalent to \( P(n_0, n, J, I, M/(x')M) \) is a constant by Remark 2.1. Note that

\[
\Delta^{(k_0,k)} P(n_0, n, J, I, M) = P(n_0, n, J, I, M/(x')M)
\]

by Proposition 2.5. Hence \( x \) is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) if and only if \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) \) is a constant. \( \square \)

The existence of joint reductions with respect to modules on an exact sequence is shown by the following result.

Corollary 2.8. Let \( 0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0 \) be an exact sequence of \( A \)-modules. If \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) \) is a constant, then so are \( \Delta^{(k_0,k)} P(n_0, n, J, I, N) \) and \( \Delta^{(k_0,k)} P(n_0, n, J, I, P) \). And if \( x \) is a joint reduction of \( I, J \) with respect to \( M \), then \( x \) is also a joint reduction of \( I, J \) with respect to \( N, P \).
Proof. By Proposition 2.7, \( \Delta^{(k_0,k)} P(n_0, \mathbf{n}, J, I, M) \) is a constant if and only if there exists a joint reduction \( \mathbf{x} \) of \( I, J \) with respect to \( M \) of the type \( (k, k_0 + 1) \). In this case, \( \mathbf{x} \) is also a joint reduction of \( I, J \) with respect to \( N, P \) because \( \text{Ann}(N) \supset \text{Ann}(M) \) and \( \text{Ann}(P) \supset \text{Ann}(M) \) (see e.g [15, Lemma 17.1.4]). Hence \( \Delta^{(k_0,k)} P(n_0, \mathbf{n}, J, I, N) \) and \( \Delta^{(k_0,k)} P(n_0, \mathbf{n}, J, I, P) \) are also constants. \( \square \)

To characterize the positivity of the \( (k_0, k) \)-difference of the Hilbert polynomial, first we need to prove the following lemma.

**Lemma 2.9.** We have the following.

(i) \( \dim \overline{M} \leq 0 \) if and only if \( I \subseteq \sqrt{\text{Ann}(M)} \).

(ii) \( P(n_0, \mathbf{n}, J, I, M) \) is a positive constant if and only if \( \dim \overline{M} = 1 \). And in this case, \( I \not\subseteq \sqrt{\text{Ann}(M)} \).

Proof. The proof of (i): Note that \( \dim A/\text{Ann}(M) : I^\infty = \dim \overline{M} \). On the other hand we have \( \dim A/\text{Ann}(M) : I^\infty \leq 0 \) if and only if \( A/\text{Ann}(M) : I^\infty = 0 \) because \( \text{Ann}(M) : I^\infty : I = \text{Ann}(M) : I^\infty \). So we get (i). \( P(n_0, \mathbf{n}, J, I, M) \) is a positive constant if and only if \( \dim \overline{M} = 1 \) by Remark 2.1. This is equivalent to \( \dim \overline{M} = 1 \). In this case, \( \dim A/\text{Ann}(M) : I^\infty = 1 \). Hence \( I \not\subseteq \sqrt{\text{Ann}(M)} \) by (i). \( \square \)

Let \( (B, \mathbf{n}) \) be an Artinian local ring with maximal ideal \( \mathbf{n} \) and infinite residue field \( B/\mathbf{n} \). Let \( G = \bigoplus_{\mathbf{n} \in \mathbb{N}^d} G_{\mathbf{n}} \) be a finitely generated standard \( \mathbb{N}^d \)-graded algebra over \( B \) (i.e., \( G \) is generated over \( B \) by elements of total degree 1) and let \( E = \bigoplus_{\mathbf{n} \in \mathbb{N}^d} E_{\mathbf{n}} \) be a finitely generated \( \mathbb{N}^d \)-graded \( G \)-module. Set \( G_{++} = \bigoplus_{\mathbf{n} \geq 1} G_{\mathbf{n}} \). Recall that a homogeneous element \( a \in G \) is called a \( G_{++} \)-filter-regular element with respect to \( E \) if \( (0_E : a)_{\mathbf{n}} = 0 \) for all large \( \mathbf{n} \). And a sequence \( x_1, \ldots, x_t \) in \( G \) is called a \( G_{++} \)-filter-regular sequence with respect to \( E \) if \( x_i \) is a \( G_{++} \)-filter-regular element with respect to \( E/(x_1, \ldots, x_{i-1})E \) for all \( 1 \leq i \leq t \) (see e.g. [17, Definition 2.5]).

Denote by \( P_E(\mathbf{n}) \) the Hilbert polynomial of \( \ell_B[E_{\mathbf{n}}] \). Then [17, Proposition 2.4(i)] stated that if \( \Delta^k P_E(\mathbf{n}) \) is a constant, then \( \Delta^k P_E(\mathbf{n}) \geq 0 \). However, the argument that proved this fact in [17] is incorrect. So we need the following note.

**Note 2.10.** For each \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d \), there exists a \( G_{++} \)-filter-regular sequence \( \mathbf{x} \) in \( \bigcup_{i=1}^d G_{e_i} \) with respect to \( E \) consisting of \( k_i \) elements of \( G_{e_i} \) for all \( 1 \leq i \leq d \) by [17, Remark 2.6(i)] (see [29, Proposition 2.2 and Note (ii)]). And then \( \Delta^k P_E(\mathbf{n}) = P_{E/xE}(\mathbf{n}) \) by [17, Remark 2.6(ii)] (see [29, Remark 2.6]). Hence \( \Delta^k P_E(\mathbf{n}) = \ell_B[(E/xE)_{\mathbf{n}}] \) for all large \( \mathbf{n} \). So \( \Delta^k P_E(\mathbf{n}) \geq 0 \) for all large \( \mathbf{n} \). From this it follows that if \( \Delta^k P_E(\mathbf{n}) \) is a constant, then \( \Delta^k P_E(\mathbf{n}) \geq 0 \). Consequently, we get the proof of [17, Proposition 2.4(i)].
Return to $\triangle^{(k_0,k)}P(n_0, n, J, I, M)$. Then by Note 2.10, if $\triangle^{(k_0,k)}P(n_0, n, J, I, M)$ is a constant, then this constant is non-negative. And the following result answers the question when $\triangle^{(k_0,k)}P(n_0, n, J, I, M)$ is a positive constant in terms of minimal joint reductions and maximal weak-(FC)-sequences.

**Proposition 2.11.** Assume that $\triangle^{(k_0,k)}P(n_0, n, J, I, M)$ is a constant. Then the following statements are equivalent:

(i) $\triangle^{(k_0,k)}P(n_0, n, J, I, M)$ is positive.

(ii) Every weak-(FC)-sequence $x_1, \ldots, x_n$ with $x_n \in J$ of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$ is maximal.

(iii) There exists a maximal weak-(FC)-sequence $x_1, \ldots, x_n$ with $x_n \in J$ of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$.

(iv) For any joint reduction $x_1, \ldots, x_n$ of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$ with $x_n \in J$, then $x_1, \ldots, x_{n-1}$ is not a joint reduction of $I, J$ with respect to $M$.

(v) Every joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$ is minimal.

(vi) There exists a minimal joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$.

In the following proof, joint reductions satisfying Proposition 2.11 (iv) are called temporarily proper joint reductions.

**Proof.** Note that by [28, Proposition 2.3] (see [22, Remark 1]), there always exists a weak-(FC)-sequence $z = z_1, \ldots, z_n$ of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$, here $z_1, \ldots, z_{|k|} \subset I$ and $z_{|k|+1}, \ldots, z_n \subset J$.

(i) $\Rightarrow$ (ii): Let $x = x_1, \ldots, x_n$ with $x_n \in J$ be a weak-(FC)-sequence of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$. Then by Proposition 2.5 we have

$$\triangle^{(k_0,k)}P(n_0, n, J, I, M) = P(n_0, n, J, I, M/(x_1, \ldots, x_{n-1})M).$$

So $P(n_0, n, J, I, M/(x_1, \ldots, x_{n-1})M)$ is a positive constant by (i). Therefore by Lemma 2.9 (ii), we get $I \not\subseteq \sqrt{\text{Ann}[M/(x_1, \ldots, x_{n-1})M]}$ and

$$\dim M/(x_1, \ldots, x_{n-1})M : I^\infty = 1.$$ 

Hence $IJ \not\subseteq \sqrt{\text{Ann}[M/(x_1, \ldots, x_{n-1})M]}$ since $J$ is $m$-primary. Furthermore

$$\dim M/(x_1, \ldots, x_n)M : I^\infty \leq \dim M/(x_1, \ldots, x_{n-1})M : I^\infty - 1.$$
by Proposition 2.5. Hence \( \dim M/(x_1, \ldots, x_n)M : I^\infty \leq 0 \). So by Lemma 2.9 (i), \( I \subset \sqrt{\text{Ann}[M/(x_1, \ldots, x_n)M]} \). Consequently, \( IJ \subset \sqrt{\text{Ann}[M/(x_1, \ldots, x_n)M]} \). Therefore \( x \) is a maximal weak-(FC)-sequence. (ii) \( \Rightarrow \) (iii) is clear. (iii) \( \Rightarrow \) (i): Assume that \( x = x_1, \ldots, x_n \) with \( x_n \in J \) is a maximal weak-(FC)-sequence, then \( I \not\subset \sqrt{\text{Ann}[M/(x_1, \ldots, x_{n-1})M]} \). So we obtain \( \dim M/(x_1, \ldots, x_{n-1})M : I^\infty > 0 \) by Lemma 2.9 (i). Hence by Remark 2.1, \( \deg P(n_0, n, J, I, M/(x_1, \ldots, x_{n-1})M) \geq 0 \). Moreover, by Proposition 2.5, we have

\[
\Delta^{(k_0,k)} P(n_0, n, J, I, M) = P(n_0, n, J, I, M/(x_1, \ldots, x_{n-1})M).
\]

Thus we get \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) \neq 0 \). Therefore (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iii).

(iii) \( \Rightarrow \) (iv): By (i) \( \Leftrightarrow \) (iii), \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) > 0 \). Now let \( x = x_1, \ldots, x_n \) be a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) with \( x_n \in J \). If \( x \) is not proper, then \( x_1, \ldots, x_{n-1} \) is also a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0) \). By [15, Lemma 17.3.2] and [28, Proposition 2.3], there exists a weak-(FC)-sequence \( y_1, \ldots, y_{n-1} \) such that \( y_1, \ldots, y_{n-1} \) is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0) \). In this case, \( j_{n_0}[M/(y_1, \ldots, y_{n-1})M] = 0 \) for all large \( n_0, n \). Hence we have \( P(n_0, n, J, I, M/(y_1, \ldots, y_{n-1})M) = 0 \). Recall that by Proposition 2.5, \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) = P(n_0, n, J, I, M/(y_1, \ldots, y_{n-1})M) \). So \( \Delta^{(k_0,k)} P(n_0, n, J, I, M) = 0 \), we get a contradiction. Hence \( x \) is a proper joint reduction. Thus (iii) \( \Rightarrow \) (iv) is proved. (iv) \( \Rightarrow \) (iii): By Proposition 2.7, there exists a weak-(FC)-sequence \( x_1, \ldots, x_n \) which is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) with \( x_n \in J \). Then in this case, \( IJ \subset \sqrt{\text{Ann}[M/(x_1, \ldots, x_n)M]} \). Now if \( x \) is not maximal, then \( IJ \subset \sqrt{\text{Ann}[M/(x_1, \ldots, x_{n-1})M]} \). Therefore we have \( j_{n_0}M \subset (x_1, \ldots, x_{n-1})M \) for all large \( n_0, n \). Thus in (1) in Note 2.4, \( x_1, \ldots, x_{n-1} \) is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0) \). So \( x \) is not a proper joint reduction. Consequently (iii) \( \Leftrightarrow \) (iv).

(vi) \( \Rightarrow \) (iv) By (vi), it follows that any sequence of the type \( (h, h_0) < (k, k_0+1) \) of \( I, J \) is not a joint reduction of \( I, J \) with respect to \( M \) (see Definition 2.2). Now assume that \( x_1, x_n \) is a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) with \( x_n \in J \). Then \( x_1, \ldots, x_{n-1} \) is a sequence of the type \( (k, k_0) \). Since \( (k, k_0) < (k, k_0+1) \), \( x_1, \ldots, x_{n-1} \) is not a joint reduction of \( I, J \) with respect to \( M \). (iv) \( \Rightarrow \) (v): Let \( x = x_1, \ldots, x_n \) with \( x_n \in J \) be a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \). If \( x \) is not a minimal joint reduction, then there exists a joint reduction \( u \) of \( I, J \) with respect to \( M \) of the type \( (h, h_0) \) with \( (h, h_0) < (k, k_0+1) \). Then there exists a joint reduction \( y = y_1, \ldots, y_n \) of \( I, J \) with respect to \( M \) of the type \( (k, k_0+1) \) with \( u \subsetneq y \). Since \( y \) is a proper joint reduction, it follows that \( y_n \in u \). Hence \( u \) has the type \( (h, t_0+1) < (k, k_0+1) \) since \( y_n \in J \). So \( \Delta^{(u_0,h)} P(n_0, n, J, I, M) \) is a constant by Proposition 2.7. Therefore
\( \Delta^{(k_0,k)}P(n_0, n, J, I, M) = 0 \). But by (i) \( \Leftrightarrow \) (iv), \( \Delta^{(k_0,k)}P(n_0, n, J, I, M) \neq 0 \), which is a contradiction. Hence (iv) \( \Rightarrow \) (v). By Proposition 2.7, there exists a joint reduction of \( I, J \) with respect to \( M \) of the type \( (k, k_0 + 1) \). So (v) \( \Rightarrow \) (vi). The proof is complete. \( \square \)

## 3 Euler-Poincaré characteristic of joint reductions

In this section, we build the Euler-Poincaré characteristic of joint reductions of ideals whose properties are shown in Proposition 3.7.

Although in the context of graded modules, Kirby-Rees [8] built the Euler-Poincaré characteristic of joint reductions, but our goal is to build an object called the Euler-Poincaré characteristic of joint reductions of ideals which is used to prove important properties of mixed multiplicities of maximal degrees of ideals. This reality leads us to choose Rees modules as initial objects. However, to prove properties of this invariant, we need to use properties of more general graded modules. This is the reason why we begin with the case of the following graded modules.

Let \( S = \bigoplus_{n \geq 0} S_n \) be a finitely generated standard \( \mathbb{N}^d \)-graded algebra over \( A \) (i.e., \( S \) is generated over \( A \) by elements of total degree 1); \( V = \bigoplus_{n \geq 0} V_n \) be a finitely generated \( \mathbb{N}^d \)-graded \( S \)-module. Set \( S_i = S_{e_i} \) for \( 1 \leq i \leq d \), \( S = S_1, \ldots, S_d \). Let \( J \) be an \( m \)-primary ideal of \( A \). For the case of these graded modules, first we would like to recall the concept of joint reductions (see [8]).

**Definition 3.1.** Set \( J = S_0 \). Let \( \mathfrak{R}_i \) be a sequence consisting \( k_i \) elements of \( S_i \) for all \( 0 \leq i \leq d \) and \( k_0, \ldots, k_d \geq 0 \). Put \( \mathfrak{R} = \mathfrak{R}_1, \ldots, \mathfrak{R}_d, \mathfrak{R}_0 \) and \( (\emptyset) = 0_S \). Then \( \mathfrak{R} \) is called a **joint reduction** of \( S, J \) with respect to \( V \) of the type \( (k, k_0) \) if

\[
J^{n_0}V_n = (\mathfrak{R}_0)J^{n_0-1}V_n + \sum_{i=1}^{d} (\mathfrak{R}_i)J^{n_0}V_{n-e_i} \quad \text{for all large } n_0, n.
\]

Let \( t \) be a variable over \( A \). Then with the notations \( S \) and \( V \) as the above, we have the \( \mathbb{N}^{d+1} \)-graded algebra \( S = S[t] = \bigoplus_{n_0 \geq 0, n \geq 0} S_{nt^{n_0}} \) and the \( \mathbb{N}^{d+1} \)-graded \( S \)-modules \( \mathcal{V} = \bigoplus_{n_0 \geq 0, n \geq 0} V_{nt^{n_0}} \) and \( \mathcal{V}' = \bigoplus_{n_0 \geq 0, n \geq 0} J^{n_0}V_{nt^{n_0}} \).

Let \( \mathfrak{R} = \mathfrak{R}_1, \ldots, \mathfrak{R}_d, \mathfrak{R}_0 \) be a joint reduction of \( S, J \) with respect to \( V \) of the type \( (k, k_0) \), where \( \mathfrak{R}_0 \subset J, \mathfrak{R}_i \subset S_i \) for \( 1 \leq i \leq d \). Put \( \mathfrak{R}_0t = \{at \mid a \in \mathfrak{R}_0 \} \); \( X = \mathfrak{R}_1, \ldots, \mathfrak{R}_d, \mathfrak{R}_0t \); \( n = k_0 + |k| \). Considering the Koszul complex of \( \mathcal{V} \) with respect to \( X \):

\[
0 \to K_n(X, \mathcal{V}) \to K_{n-1}(X, \mathcal{V}) \to \cdots \to K_1(X, \mathcal{V}) \to K_0(X, \mathcal{V}) \to 0,
\]
one obtain the sequence of the homology modules $H_0(X, \mathcal{V}), H_1(X, \mathcal{V}), \ldots, H_n(X, \mathcal{V})$.

To be able to apply [8, Theorem 4.2] in building the main object of this section, we need the following facts.

**Note 3.2.** Considering [8, Theorem 4.2] in the context of $\mathcal{R}$ and $V$, we see that:

(i) [8, Theorem 4.2] requires that $\mathcal{R}$ is a joint reduction with respect to $S$. However, the proof of [8, Theorem 4.2] only uses the fact that $\mathcal{R}$ is a joint reduction with respect to $V$. So one can apply [8, Theorem 4.2] with the assumption that $\mathcal{R}$ is a joint reduction of $S, J$ with respect to $V$.

(ii) Although [8, Theorem 4.2] includes an assumption on the lower bound of $k_0 + |k|$, but this assumption is only used for [8, Theorem 4.2 (iv)]. This means that [8, Theorem 4.2 (i), (ii), (iii)] are still true without the assumption on the lower bound of $k_0 + |k|$.

Now, since $\mathcal{R}$ is a joint reduction of $S, J$ with respect to $V$, $(X\mathcal{V}')_{(n_0, n)} = \mathcal{V}'_{(n_0, n)}$ for all large $n_0, n$ (i.e., $X$ is a joint reduction of $\mathcal{V}'$ as in [8]). On the other hand, since $J$ is $m$-primary, $\ell_A((V/\mathcal{V}')_{(n_0, n)}) < \infty$ for all $n_0, n$. So $\ell_A((H_i(X, \mathcal{V})_{(n_0, n)}) < \infty$ for all large enough $n_0, n$ and for all $0 \leq i \leq n$ by [8, Theorem 4.2 (i)] together with Note 3.2. And moreover for all large enough $n_0, n$,

$$\sum_{i=0}^{n} (-1)^i \ell_A((H_i(X, \mathcal{V})_{(n_0, n)})$$

is a polynomial in $n_0, n$ by [8, Theorem 4.2 (ii)] together with Note 3.2. Then we denote by $\chi(n_0, n, \mathcal{R}, J, V)$ this polynomial.

Consider the Hilbert function $\ell_A((V/\mathcal{V}')_{(n_0, n)}) = \ell_A(\frac{V}{J_{n_0}V_n})$. Then by [7, Theorem 4.1], this function is a polynomial for all large $n_0, n$. Denote by $F(n_0, n, J, V)$ this polynomial, and $\triangle(k_0, k)F(n_0, n, J, V)$ the $(k_0, k)$-difference of the polynomial $F(n_0, n, J, V)$.

Let $z_0, z_1, \ldots, z_d$ be variables and put $Z = z_1, \ldots, z_d, Z^n = z_1^{n_1} \cdots z_d^{n_d}$. One writes

$$\sum_{n_0 \geq 0, n \geq 0} f_1(n_0, n)z_0^{n_0}Z^n \sim \sum_{n_0 \geq 0, n \geq 0} f_2(n_0, n)z_0^{n_0}Z^n$$

if $f_1(n_0, n) = f_2(n_0, n)$ for all large $n_0, n$. Set

$$\chi(z_0, Z, \mathcal{R}, J, V) = \sum_{n_0 \geq 0, n \geq 0} \chi(n_0, n, \mathcal{R}, J, V)z_0^{n_0}Z^n,$$

$$F(z_0, Z, J, V) = \sum_{n_0 \geq 0, n \geq 0} F(n_0, n, J, V)z_0^{n_0}Z^n.$$
Recall that $k = (k_1, \ldots, k_d)$ and $\mathfrak{R}$ is a joint reduction of the type $(k, k_0)$. Then using [8, Theorem 4.2 (iii)], together with Note 3.2, we get

$$
\chi(z_0, Z, \mathfrak{R}, J, V) \sim \prod_{i=0}^{d} (1 - z_i)^{k_i} F(z_0, Z, J, V).
$$

(4)

Now by using the expressions as in [12, Lemma 2.2] (or see [8, pages 223-224]), (4) follows that $\chi(n_0, n, \mathfrak{R}, J, V)$ is a constant if and only if $\Delta^{(k_0,k)}F(n_0, n, J, V)$ is a constant and in this case, $\chi(n_0, n, \mathfrak{R}, J, V) = \Delta^{(k_0,k)}F(n_0, n, J, V)$. So we obtain the following result which presents the relationship between $\chi(n_0, n, \mathfrak{R}, J, V)$ and $\Delta^{(k_0,k)}F(n_0, n, J, V)$.

**Lemma 3.3.** Let $\mathfrak{R}$ be a joint reduction of $S$, $J$ with respect to $V$ of the type $(k, k_0)$. Then $\chi(n_0, n, \mathfrak{R}, J, V)$ is a constant if and only if $\Delta^{(k_0,k)}F(n_0, n, J, V)$ is a constant. In this case, we have $\chi(n_0, n, \mathfrak{R}, J, V) = \Delta^{(k_0,k)}F(n_0, n, J, V)$.

The above facts yield the following remarks which will be used in the next part.

**Remark 3.4.** Set $k! = k_1! \cdots k_d!$ and $n^k = n_1^{k_1} \cdots n_d^{k_d}$. Let $\mathfrak{R}$ be a joint reduction of $S$, $J$ with respect to $V$ of the type $(k, k_0)$. Then we have the following.

(i) $\chi(n_0, n, \mathfrak{R}, J, _{-})$ is additive on short exact sequence of $S$-modules because of the additivity of the length (see e.g. [2, Lemma 4.6.5]).

(ii) Suppose that $\mathfrak{R}$ also is a joint reduction of $S$, $J$ with respect to an $S$-graded modules $U$, and $U_n \cong_A V_{n+m}$ for all large $n$ and a fixed $m$, and $\chi(n_0, n, \mathfrak{R}, J, V)$ is a constant. Then by the above construction we have

$$
\chi(n_0, n, \mathfrak{R}, J, V) = \chi(n_0, n, \mathfrak{R}, J, U).
$$

(iii) Assume that $\chi(n_0, n, \mathfrak{R}, J, V)$ is a constant. Then $\Delta^{(k_0,k)}F(n_0, n, J, V)$ is a constant by Lemma 3.3 and hence $\Delta^{(k_0,k)}F(n_0, n, J, V) \geq 0$ by Note 2.10. So $\chi(n_0, n, \mathfrak{R}, J, V) \geq 0$ by Lemma 3.3.

Next, we will define the Euler-Poincaré characteristic of joint reductions of $A$-modules via applying the above facts for the case that $S$ is the Rees algebra $\mathfrak{R}(I; A)$ and $V$ is the Rees module $\mathfrak{R}(I; M)$ as follows.

Recall that $I = I_1 \cdots I_d; I = I_1, \ldots, I_d$ and $\mathbb{F}^n = I_1^{n_1} \cdots I_d^{n_d}$. Let $t_1, \ldots, t_d$ be variables over $A$. Set $T = t_1, \ldots, t_d$ and $T^n = t_1^{n_1} \cdots t_d^{n_d}$. We denote by

$$
\mathfrak{R}(I; A) = \bigoplus_{n>0} \mathbb{F}^n T^n \quad \text{and} \quad \mathfrak{R}(I; M) = \bigoplus_{n>0} \mathbb{F}^n MT^n
$$
the Rees algebra of $I$ and the Rees module of $I$ with respect to $M$, respectively. Then $R(I; A)_{e_i} = I t_i$ for all $1 \leq i \leq d$ and $F(n_0, n, J, R(I; M))$ is the Hilbert polynomial of the Hilbert function $\ell_A \left( \frac{\mathbb{n} M}{\mathbb{m} M} \right)$, i.e.,

$$F(n_0, n, J, R(I; M)) = \ell_A \left( \frac{\mathbb{n} M}{\mathbb{m} M} \right)$$

for all large $n_0, n$. Recall that $P(n_0, n, J, I, M)$ is the Hilbert polynomial of the Hilbert function $\ell_A \left( \frac{n_0 \mathbb{n} M}{n \mathbb{m} M} \right)$. Then it is easily seen that for any $(k_0, k) \in \mathbb{N}^{d+1}$,

$$\triangle^{(k_0+1, k)} F(n_0, n, J, R(I; M)) = \triangle^{(k_0, k)} P(n_0, n, J, I, M).$$

**Note 3.5.** Let $x = J_1, \ldots, J_d, J_0$ be a joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$, where $J_0 \subset J, J_i \subset I_i$ for $1 \leq i \leq d$. Set $x_T = J_1 t_1, \ldots, J_d t_d, J_0$ and $I_T = I_1 t_1, \ldots, I_d t_d$. Then it is easily seen that $x_T$ is a joint reduction of $I_T, J$ with respect to $R(I; M)$ of the type $(k, k_0 + 1)$. Since $x$ is a joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$, $\triangle^{(k_0, k)} P(n_0, n, J, I, M)$ is a constant by Proposition 2.7. Hence $\triangle^{(k_0+1, k)} F(n_0, n, J, R(I; M))$ is a constant. Therefore by Lemma 3.3, $\chi(n_0, n, x_T, J, R(I; M))$ is a constant. Then we denote this constant by $\chi(x, J, I, M)$ and call it the Euler-Poincaré characteristic of $x$ with respect to $J, I$ and $M$ of the type $(k_0 + 1, k)$. These facts yield:

**Remark 3.6.** Let $x$ be a joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$. Then we obtain the following.

(i) By Lemma 3.3, it implies that $\chi(x, J, I, M) = \triangle^{(k_0, k)} P(n_0, n, J, I, M)$. This also follows that $\chi(x, J, I, M)$ only depends on the type $(k, k_0 + 1)$, does not depend on the joint reduction $x$.

(ii) Let $m$ be a positive integer. Since $R(I; M)_{n+m1} \cong_A R(I; I^m M)_n$, it follows that $\chi(x, J, I, M) = \chi(x, J, I, I^m M)$ by Remark 3.4 (ii).

The additivity on graded $S$-modules of the Euler-Poincaré characteristic is proven easily. But proving the additivity of the Euler-Poincaré characteristic $\chi(x, J, I, M)$ on $A$-modules is not simple. However, our goal is completed by the following result.

**Proposition 3.7.** Let $N$ be an $A$-submodule of $M$. Assume that $x$ is a joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$. Then

(i) $\chi(x, J, I, M) = \triangle^{(k_0, k)} P(n_0, n, J, I, M)$.

(ii) $\chi(x, J, I, M) > 0$ if and only if $x$ is a minimal joint reduction.

(iii) $\chi(x, J, I, M) = \chi(x, J, I, N) + \chi(x, J, M/N)$. 
Proof. (i) follows from Remark 3.6 (i) and (ii) follows from (i) and Proposition 2.11.

The proof of (iii): Note that \( x \) is also a joint reduction of \( I, J \) with respect to \( N, M/N \) by Corollary 2.8. First, we prove that if \( M' \) is a submodule of \( M \), then

\[
\chi(x, J, I, M') \leq \chi(x, J, I, M).
\]  

Indeed, since \( x \) is a joint reduction of \( I, J \) with respect to \( M \), it follows that \( x_J \) is a joint reduction of \( I_J, J \) with respect to \( \mathcal{R}(I; M) \) as mentioned above. Hence \( x_J \) is also a joint reduction of \( I_J, J \) with respect to \( \mathcal{R}(I; M)/\mathcal{R}(I; M') \). On the other hand, by Corollary 2.8, \( x \) is a joint reduction of \( I, J \) with respect to \( M' \), and so \( x_J \) is a joint reduction of \( I_J, J \) with respect to \( \mathcal{R}(I; M') \). Consider the exact sequence of \( \mathcal{R}(I; A) \)-modules:

\[
0 \rightarrow \mathcal{R}(I; M') \rightarrow \mathcal{R}(I; M) \rightarrow \mathcal{R}(I; M)/\mathcal{R}(I; M') \rightarrow 0.
\]

Since \( \chi(n_0, n, x_T, J, \mathcal{R}(I; M)) \) and \( \chi(n_0, n, x_T, J, \mathcal{R}(I; M')) \) are constants by Note 3.5, it follows by Remark 3.4 (i) that \( \chi(n_0, n, x_T, J, \mathcal{R}(I; M)/\mathcal{R}(I; M')) \) also is a constant and

\[
\chi(n_0, n, x_T, J, \mathcal{R}(I; M)) = \chi(n_0, n, x_T, J, \mathcal{R}(I; M')) + \chi(n_0, n, x_T, J, \mathcal{R}(I; M)/\mathcal{R}(I; M')).
\]

On the other hand, by Remark 3.4 (iii), \( \chi(n_0, n, x_T, J, \mathcal{R}(I; M)/\mathcal{R}(I; M')) \geq 0 \).

Hence \( \chi(n_0, n, x_T, J, \mathcal{R}(I; M)) \geq \chi(n_0, n, x_T, J, \mathcal{R}(I; M')). \) So we get (5), that means

\[
\chi(x, J, I, M) \geq \chi(x, J, I, M').
\]

Since \( N \) is a submodule of \( M \), by Artin-Rees Lemma, there exists an integer \( k > 0 \) such that \( \mathbb{P}^{n+k_1} M \cap N = \mathbb{P}^n(I^k M \cap N) \) for all \( n \geq 0 \). Fix this integer \( k \). Then since \( \mathbb{P}^{n+k_1} M \cap N = \mathbb{P}^n(I^k M \cap N) \) for all \( n \geq 0 \), we have

\[
\mathbb{P}^n(I^k M \cap N) = \frac{\mathbb{P}^{n+k_1} M \cap I^k M \cap N}{\mathbb{P}^{n+k_1} M \cap N} = \frac{\mathbb{P}^{n+k_1} M \cap N}{\mathbb{P}^{n+k_1} M \cap N} = \frac{\mathbb{P}^n(I^k M \cap N)}{\mathbb{P}^n(I^k M \cap N)}
\]

for all \( n \geq 0 \). Hence we get the exact sequence

\[
0 \rightarrow \mathcal{R}(I, I^k M \cap N) \rightarrow \mathcal{R}(I; I^k M) \rightarrow \mathcal{R}
\left(I; \frac{I^k M \cap N}{I^k M \cap N}\right) \rightarrow 0.
\]

Consequently by Remark 3.4 (i),

\[
\chi(x, J, I, I^k M) = \chi(x, J, I, I^k M \cap N) + \chi\left(x, J, I, \frac{I^k M \cap N}{I^k M \cap N}\right).
\]
On the other hand, by Remark 3.6 (ii), \( \chi(x, J, I^k M) = \chi(x, J, I, M) \) and
\[
\chi(x, J, I, \frac{I^k M}{I^k M \cap N}) = \chi(x, J, I, (M/N)) = \chi(x, J, I, M/N).
\]
So \( \chi(x, J, I, I^k M \cap N) = \chi(x, J, I, M/N) \). Next, by (5) and since \( I^k N \subset I^k M \cap N \subset N \), it follows that
\[
\chi(x, J, I, I^k N) \leq \chi(x, J, I, I^k M \cap N) \leq \chi(x, J, I, N).
\]
And by Remark 3.6 (ii), we have \( \chi(x, J, I, I^k N) = \chi(x, J, I, N) \). Hence
\[
\chi(x, J, I, I^k M \cap N) = \chi(x, J, I, N).
\]
Consequently, we obtain \( \chi(x, J, I, M) = \chi(x, J, I, N) + \chi(x, J, I, M/N) \).

\[\square\]

4 Mixed multiplicities of ideals

This section studies mixed multiplicities of maximal degrees. The following facts will show the effectiveness of our approach in this paper.

Set \( I^{[k]} = I^{[k_1]}_1, \ldots, I^{[k_d]}_d \); \( n^k = n_1^{k_1} \cdots n_d^{k_d} \); \( k! = k_1! \cdots k_d! \); \( |k| = k_1 + \cdots + k_d \). Recall that \( I = I_1 \cdots I_d \); \( M = M/0_M : I^\infty \) and \( q = \dim(M) \).

If \( I \notin \sqrt{\text{Ann}(M)} \), then \( M = 0 \). Remember that by [22, Proposition 3.1] (see [10]) the Hilbert polynomial \( P(n_0, n, J, I, M) \) of the Hilbert function \( \ell \left( \frac{J^{n_0} \| n M}{J^{n_0+1} \| n M} \right) \) has total degree \( q - 1 \). And in this section we assume that \( I \notin \sqrt{\text{Ann}(M)} \). If write the terms of total degree \( q - 1 \) of \( P(n_0, n, J, I, M) \) in the form
\[
\sum_{k_0 + |k| = q - 1} e(J^{[k_0+1]}, I^{[k]}; M) \frac{n_0^{k_0} n^k}{k_0! k!},
\]
then \( e(J^{[k_0+1]}, I^{[k]}; M) \) is called the mixed multiplicity (or the original mixed multiplicities) of \( M \) with respect to \( J, I \) of the type \( (k_0 + 1, k) \) (see e.g. [10, 21, 22]).

Now we would like to recall objects mentioned in [17] which concern the maximal terms in the Hilbert polynomial. It is well known that one can write
\[
P(n_0, n, J, I, M) = \sum_{(k_0, k) \in \mathbb{N}^{d+1}} e(J^{[k_0+1]}, I^{[k]}; M) \binom{n_0 + k_0}{k_0} \binom{n + k}{k},
\]
where \( \binom{n+k}{n} = \binom{n+k_1}{n_1} \cdots \binom{n+k_d}{n_d} \). Then \( e(J^{[k_0+1]}, I^{[k]}; M) \) are integers.
Note 4.1. \( \Delta^{(k_0,k)}P(n_0, n, J, I, M) \) is a constant if and only if \( e(J^{[k_0+1]}, I^{[h]}; M) = 0 \) for all \((h_0, h) > (k_0, k)\). In this case, \( \Delta^{(k_0,k)}P(n_0, n, J, I, M) = e(J^{[k_0+1]}, I^{[h]}; M) \) (see [17, Proposition 2.4 (ii)]).

From this fact, one can consider a larger class than the class of original mixed multiplicities. This is the reason why [17] gave the following concept.

Definition 4.2. We call that \( e(J^{[k_0+1]}, I^{[h]}; M) \) is the mixed multiplicity of maximal degrees of \( M \) with respect to ideals \( J, I \) of the type \((k_0 + 1, k)\) if \( e(J^{[k_0+1]}, I^{[h]}; M) = 0 \) for all \((h_0, h) > (k_0, k)\).

An example in [17] calculated all mixed multiplicities of maximal degrees in a specific case. For any \((k_0, k) \in \mathbb{N}^{d+1}\), it is not true, in general, that the mixed multiplicity of maximal degrees of the type \((k_0 + 1, k)\) is defined. However, the definiteness of mixed multiplicities of maximal degrees is shown by the following.

Remark 4.3. Let \((k_0, k) \in \mathbb{N}^{d+1}\). Then the mixed multiplicity of maximal degrees of \( M \) with respect to ideals \( J, I \) of the type \((k_0 + 1, k)\) is defined if and only if \( \Delta^{(k_0,k)}P(n_0, n, J, I, M) \) is a constant. This is equivalent to that there exists a joint reduction \( x \) of \( I, J \) with respect to \( M \) of the type \((k, k_0 + 1)\) by Proposition 2.7.

Then the main result of the paper is the following theorem.

Theorem 4.4. Let \( N \) be an \( A \)-submodule of \( M \). Assume that the mixed multiplicity of maximal degrees of \( M \) with respect to \( J, I \) of the type \((k_0 + 1, k)\) is defined and let \( x \) be a joint reduction of the type \((k, k_0 + 1)\) of \( I, J \) with respect to \( M \). Then \( \chi(x, J, I, M) \) is independent of \( x \) and we have

(i) \( e(J^{[k_0+1]}, I^{[k]}; M) = \Delta^{(k_0,k)}P(n_0, n, J, I, M) = \chi(x, J, I, M). \)

(ii) \( e(J^{[k_0+1]}, I^{[k]}; M) > 0 \) if and only if \( x \) is a minimal joint reduction.

(iii) \( e(J^{[k_0+1]}, I^{[k]}; M) = e(J^{[k_0+1]}, I^{[k]}; N) + e(J^{[k_0+1]}, I^{[k]}; M/N). \)

Proof. By Remark 3.6(i), \( \chi(x, J, I, M) \) is independent of \( x \). By Proposition 3.7(i) and Note 4.1, we get (i). (ii) follows from (i) and Proposition 2.11. By (i) and Proposition 3.7(iii), we obtain (iii). }
to $J, I$ of the type $(k_0 + 1, k)$ is defined and let $x$ be a joint reduction of the type $(k, k_0 + 1)$ of $I, J$ with respect to $M$. Then

(i) $e(J^{[k_0+1]}, I^k; M) = e(J^{[k_0+1]}, I^k; N) + e(J^{[k_0+1]}, I^k; P)$.

(ii) $\chi(x, J, I, M) = \chi(x, J, I, N) + \chi(x, J, I, P)$.

Now from Theorem 4.4 we prove the additivity and reduction formula for mixed multiplicities of maximal degrees in the following.

**Corollary 4.6.** Assume that the mixed multiplicity of maximal degrees of $M$ with respect to $J, I$ of the type $(k_0 + 1, k)$ is defined and let $x$ be a joint reduction of the type $(k, k_0 + 1)$ of $I, J$ with respect to $M$. Set $\Pi = \text{Min}(A/\text{Ann}(M))$. Then

(i) $e(J^{[k_0+1]}, I^k; M) = \sum_{p \in \Pi} \ell(M_p)e(J^{[k_0+1]}, I^k; A/p)$.

(ii) $\chi(x, J, I, M) = \sum_{p \in \Pi} \ell(M_p)\chi(x, J, I, A/p)$.

**Proof.** (ii) follows from (i) and Theorem 4.4 (i). Now we prove (i). Let

$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_u = M$

be a prime filtration of $M$, i.e., $M_{i+1}/M_i \cong A/P_i$ where $P_i$ is a prime ideal for all $0 \leq i \leq u - 1$. Then $\Pi \subseteq \text{Ass}(M) \subseteq \{P_0, \ldots, P_{u-1}\} \subseteq \text{Supp}(M)$. By Theorem 4.4(iii), we have

$$e(J^{[k_0+1]}, I^k; M) = \sum_{i=0}^{u-1} e(J^{[k_0+1]}, I^k; A/P_i).$$ (6)

Now we prove by induction on $|k|$ that $e(J^{[k_0+1]}, I^k; A/p) = 0$ for any prime ideal $p \in \text{Supp}(M) \setminus \text{Min}(A/\text{Ann}(M))$. Indeed, consider the case that $|k| = 0$. If $I \subset p$, then $I(A/p) = 0$, and so $e(J^{[k_0+1]}, I^k; A/p) = 0$. Hence we consider the case that $p \nsubseteq I$. In this case, $p \in \text{Supp}(M) \setminus \text{Min}(A/\text{Ann}(M))$, here $M = M/0_M : I^\infty$ by [32, Remark 3.3]. So $\dim A/p < \dim M$. Since $\Delta^{(k_0,0)}P(n_0, n, J, I, M)$ is a constant, it follows that $k_0 + 1 \geq \dim M$ by Remark 2.1. Hence $\dim A/p < k_0 + 1$. Thus $\deg P(n_0, n, J, I, A/p) < k_0$ by Remark 2.1. Therefore $\Delta^{(k_0,0)}P(n_0, n, J, I, A/p) = 0$. Hence $e(J^{[k_0+1]}, I^0; A/p) = 0$ by Theorem 4.4 (i).

Consider the case that $|k| > 0$. Recall that if $I \subset p$, then $e(J^{[k_0+1]}, I^k; A/p) = 0$. Suppose that $p \nsubseteq I$. And without loss of generality, we can assume that $k_1 > 0$. Then by [28, Proposition 2.3] (see [22, Remark 1]), there exists $x \in I_1 \setminus p$ such that $x$ is a weak-(FC)-element of $I, J$ with respect to $M$ and $A/p$. So by Proposition 2.5,

$$\Delta^{(k_0,k)}P(n_0, n, J, I, M) = \Delta^{(k_0,k-e_1)}P(n_0, n, J, I, M/xM),$$
\[ \Delta^{(k_0,k)} P(n_0, n, J, I, A/p) = \Delta^{(k_0,k-e_1)} P(n_0, n, J, I, A/(x) + p). \]

Hence \( \Delta^{(k_0,k-e_1)} P(n_0, n, J, I, M/xM) \) and \( \Delta^{(k_0,k-e_1)} P(n_0, n, J, I, A/(x) + p) \) are constants. Therefore the mixed multiplicities of maximal degrees of \( M/xM \) and \( A/(x, p) \) of the type \((k_0 + 1, k - e_1)\) are defined by Remark 4.3, and moreover

\[ e(J^{[k_0+1], I[k]}; A/p) = e(J^{[k_0+1], I[k-e_1]}; A/(x) + p) \]

by Note 4.1. Let \( q \) be an arbitrary ideal in \( \text{Supp}(A/(x) + p) \). Set \( \bar{A} = A/\text{Ann}(M) \). Denote by \( x, p, \bar{x}, \bar{p}, \bar{q} \) the images of \( x, p, (x) + p, q \) in \( \bar{A} \), respectively. Then since \( x \notin p \) and \( p \in \text{Supp}(M) \setminus \text{Min}(A/\text{Ann}(M)) \), it follows that \( \text{ht}_{\bar{A}}(x + p) > 1 \). Consequence, \( \text{ht}_{\bar{A}/(x)[\bar{x}]/(\bar{x})] > 0 \). So \( \text{ht}_{\bar{A}/(x)}[\bar{q}/(\bar{x})] > 0 \). Hence \( q \not\in \text{Min}(A/(x) + \text{Ann}(M)) \).

Note that \( \text{Min}(A/(x) + \text{Ann}(M)) = \text{Min}(A/\text{Ann}(M/xM)) \) and \( q \in \text{Supp}(M/xM) \). Thus \( q \in \text{Supp}(M/xM) \setminus \text{Min}(A/\text{Ann}(M/xM)) \). Since \( |k - e_1| = |k| - 1 \), applying the inductive hypothesis for \( M/xM \), we have \( e(J^{[k_0+1], I[k-e_1]}; A/q) = 0 \). So we obtain

\[ e(J^{[k_0+1], I[k-e_1]}; A/q) = 0 \quad (7) \]

for every \( q \in \text{Supp}(A/(x) + p) \). Now, considering a prime filtration of \( A/(x) + p \), by (6) and (7), we get \( e(J^{[k_0+1], I[k-e_1]}; A/(x) + p) = 0 \). So \( e(J^{[k_0+1], I[k]}; A/p) = 0 \) for every prime ideal \( p \in \text{Supp}(M) \setminus \text{Min}(A/\text{Ann}(M)) \). The induction is complete. Therefore if \( p_i \notin \Pi \), then \( e(J^{[k_0+1], I[k]}; A/P_i) = 0 \). Note that for any \( p \in \Pi \), the number of times \( p \) appears in the sequence \( P_0, \ldots, P_{u-1} \) is \( \ell(M_p) \) since \( p \in \text{Min}(A/\text{Ann}(M)) \).

Hence by (6), we obtain

\[ e(J^{[k_0+1], I[k]}; M) = \sum_{p \in \Pi} \ell(M_p) e(J^{[k_0+1], I[k]}; A/p). \]  

**Remark 4.7.** Set \( \Lambda = \{ p \in \text{Min}(A/\text{Ann}(M)) \mid \text{dim } A/p \geq k_0 + 1 + |k| \} \). Then \( \Lambda \subset \Pi \). From the proof of Corollary 4.6,

\[ e(J^{[k_0+1], I[k]}; A/p) = 0 \]

if \( p \in \Pi \setminus \text{Min}(A/\text{Ann}(M)) \). In the case that \( \text{dim } A/p < k_0 + 1 + |k| \), then

\[ \Delta^{(k_0,k)} P(n_0, n, J, I, A/p) = 0 \]

since \( \text{deg } P(n_0, n, J, I, A/p) = \text{dim } A/p - 1 \) by Remark 2.1. Hence

\[ e(J^{[k_0+1], I[k]}; A/p) = 0 \]

by Note 4.1. Therefore in Corollary 4.6 we can replace the set \( \Pi \) by the set \( \Lambda \), i.e.,

\[ e(J^{[k_0+1], I[k]}; M) = \sum_{p \in \Lambda} \ell(M_p) e(J^{[k_0+1], I[k]}; A/p). \]
Finally, consider the case that $M$ has rank $r > 0$. Recall that $M$ has rank $r$ if $U^{-1}M$ (the localization of $M$ with respect to $U$) is a free $U^{-1}A$-module of rank $r$, where $U$ is the set of all non-zero divisors of $A$. Then since $\text{Ann}(U^{-1}M) = 0$, it follows that $\Pi = \text{Min}(A/\text{Ann}(M)) = \text{Min}A$. Then $M_p \cong (A_p)^r$ for all $p \in \Pi$. So $\ell(M_p) = r \cdot \ell(A_p)$. Therefore by Corollary 4.6 (i),

\[
e(J^{[k_0+1]}, I^{[k]}; M) = \text{rank}(M) \sum_{p \in \Pi} \ell(A_p)e(J^{[k_0+1]}, I^{[k]}; A/p)
\]

and $e(J^{[k_0+1]}, I^{[k]}; A) = \sum_{p \in \Pi} \ell(A_p)e(J^{[k_0+1]}, I^{[k]}; A/p)$. Thus we get

\[
e(J^{[k_0+1]}, I^{[k]}; M) = e(J^{[k_0+1]}, I^{[k]}; A)\text{rank}(M).
\]

Consequently, we obtain the following corollary.

\textbf{Corollary 4.8.} Let $M$ be an $A$-module of positive rank. Assume that the mixed multiplicity of maximal degrees of $M$ with respect to $J, I$ of the type $(k_0 + 1, k)$ is defined and $x$ is a joint reduction of $I, J$ with respect to $M$ of the type $(k, k_0 + 1)$. Then we have

\begin{enumerate}[(i)]
\item $e(J^{[k_0+1]}, I^{[k]}; M) = e(J^{[k_0+1]}, I^{[k]}; A)\text{rank}(M)$.
\item $\chi(x, J, I, M) = \chi(x, J, I, A)\text{rank}(M)$.
\end{enumerate}

Returning to the original mixed multiplicities, we have the following notes.

\textbf{Remark 4.9.} In the case that $k_0 + |k| = q - 1$, from the above results we receive respective results on the original mixed multiplicities as follows.

\begin{enumerate}[(i)]
\item Theorem 4.4 (ii) also characterizes the positivity of the original mixed multiplicities of ideals in terms of minimal joint reductions. Note that this result has not been known in early works.
\item From Theorem 4.4 (iii) we obtain the additivity of original mixed multiplicities [32, Corollary 3.9]. Note that in [32] the additivity was showed via the additivity and reduction formula which had to be proved by another approach. Moreover, it seems that statements of the additivity for multiplicities of maximal degrees are shorter and more natural than for original mixed multiplicities.
\item From Corollary 4.6 (i) and Remark 4.7 we get the additivity and reduction formula of original mixed multiplicities [32, Theorem 3.2].
\item Finally, from Corollary 4.8 (i) we receive [33, Theorem 3.4].
\end{enumerate}
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References

[1] M. Auslander, D. A. Buchsbaum, Codimension and multiplicity, Ann. Math. 68(1958), 625-657.
[2] W. Bruns, J. Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, 39, Cambridge, Cambridge University Press, 1993.
[3] R. Callejas-Bedregal, V. H. Jorge Perez, Mixed multiplicities and the minimal number of generator of modules, J. Pure Appl. Algebra 214 (2010), 1642-1653.
[4] L. V. Dinh, N. T. Manh, T. T. H. Thanh, On some superficial sequences, Southeast Asian Bull. Math. 38 (2014), 803-811.
[5] L. V. Dinh, D. Q. Viet, On two results of mixed multiplicities, Int. J. Algebra 4(1) 2010, 19-23.
[6] L. V. Dinh, D. Q. Viet, On mixed multiplicities of good filtrations, Algebra Colloq. 22, 421 (2015) 421-436.
[7] M. Herrmann, E. Hyry, J. Ribbe, Z. Tang, Reduction numbers and multiplicities of multigraded structures, J. Algebra 197 (1997), 311-341.
[8] D. Kirby, D. Rees, Multiplicities in graded rings I: the general theory, Contemporary Mathematics 159 (1994), 209-267.
[9] D. Katz, J. K. Verma, Extended Rees algebras and mixed multiplicities, Math. Z. 202 (1989), 111-128.
[10] N. T. Manh, D. Q. Viet, Mixed multiplicities of modules over Noetherian local rings, Tokyo J. Math. 29 (2006), 325-345.
[11] L. O’Carroll, On two theorems concerning reductions in local rings, J. Math. Kyoto Univ. 27-1(1987), 61-67.
[12] D. Rees, Generalizations of reductions and mixed multiplicities, J. London. Math. Soc. 29 (1984), 397-414.
[13] J. P. Serre, Algèbre locale. Multiplicités. LNM 11, Springer, 1965.
[14] I. Swanson, Mixed multiplicities, joint reductions and quasi-unmixed local rings , J. London Math. Soc. 48 (1993), no. 1, 1-14.
[15] C. Huneke, I. Swanson, Integral Closure of Ideals, Rings, and Modules, London Mathematical Lecture Note Series 336, Cambridge University Press (2006).
[16] B. Teissier, Cycles évanescents, sections planes, et conditions de Whitney, Singularities à Cargèse, 1972. Astérisque, 7-8 (1973), 285-362.
[17] T. T. H. Thanh, D. Q. Viet, Mixed multiplicities of maximal degrees, J. Korean Math. Soc. 55 (2018), No. 3, 605-622.
[18] T. T. H. Thanh, D. Q. Viet, *Mixed multiplicities and the multiplicity of Rees modules of reductions*, J. Algebra Appl. Vol. 18, No. 9 (2019) 1950176 (13 pages).

[19] N. V. Trung, *Positivity of mixed multiplicities*, J. Math. Ann. 319(2001), 33 - 63.

[20] N. V. Trung, J. Verma, *Mixed multiplicities of ideals versus mixed volumes of polytopes*, Trans. Amer. Math. Soc. 359 (2007), 4711-4727.

[21] J. K. Verma, *Multigraded Rees algebras and mixed multiplicities*, J. Pure and Appl. Algebra 77 (1992), 219-228.

[22] D. Q. Viet, *Mixed multiplicities of arbitrary ideals in local rings*, Comm. Algebra. 28(8) (2000), 3803-3821.

[23] D. Q. Viet, *On some properties of (FC)-sequences of ideals in local rings*, Proc. Amer. Math. Soc. 131 (2003), 45-53.

[24] D. Q. Viet, *Sequences determining mixed multiplicities and reductions of ideals*, Comm. Algebra. 31 (2003), 5047-5069.

[25] D. Q. Viet, *Reductions and mixed multiplicities of ideals*, Comm. Algebra. 32 (2004), 4159-4178.

[26] D. Q. Viet, *The multiplicity and the Cohen-Macaulayness of extended Rees algebras of equimultiple ideals*, J. Pure and Appl. Algebra 205 (2006), 498-509.

[27] D. Q. Viet, *On the Cohen-Macaulayness of fiber cones*, Proc. Amer. Math. Soc. 136 (2008), 4185-4195.

[28] D. Q. Viet, L. V. Dinh, T. T. H. Thanh, *A note on joint reductions and mixed multiplicities*, Proc. Amer. Math. Soc. 142 (2014), 1861-1873.

[29] D. Q. Viet, N. T. Manh, *Mixed multiplicities of multigraded modules*, Forum Math. 25 (2013), 337-361.

[30] D. Q. Viet, T. T. H. Thanh, *Multiplicity and Cohen-Macaulayness of fiber cones of good filtrations*, Kyushu J. Math. 65(2011), 1-13.

[31] D. Q. Viet, T. T. H. Thanh, *On (FC)-sequences and mixed multiplicities of multigraded algebras*, Tokyo J. Math. 34 (2011), 185-202.

[32] D. Q. Viet, T. T. H. Thanh, *On some multiplicity and mixed multiplicity formulas*, Forum Math. 26 (2014), 413-442.

[33] D. Q. Viet, T. T. H. Thanh, *A note on formulas transmuting mixed multiplicities*, Forum Math. 26 (2014), 1837-1851.

[34] D. Q. Viet, T. T. H. Thanh, *The Euler-Poincaré characteristic and mixed multiplicities*, Kyushu J. Math. 69 (2015), 393-411.

[35] D. Q. Viet, T. T. H. Thanh, *On the filter-regular sequences of multi-graded modules*, Tokyo J. Math. 38 (2015), 439-457.