CONCRETE REPRESENTATION OF ATOMIC \((F_4)\) FILTRATIONS

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Abstract. We prove that for any sequence of functions adapted to a biparameter atomic filtration satisfying \((F_4)\) condition there is a sequence having the same joint distribution but adapted to the canonical \((F_4)\) filtration. Even in one parameter case our result is an improvement of the theorem due to Montgomery-Smith, since the construction gives a morphism of filtrations and does not depend on underlying sequence.

1. Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, i.e. \(\Omega\) is a sample space with a \(\sigma\)-field \(\mathcal{F}\) and a probability measure \(\mathbb{P}\). A sequence of \(\sigma\)-fields \((\mathcal{F}_i : i \in \mathbb{N}_0)\) is called filtration if
\[
\mathcal{F}_i \subset \mathcal{F}_{i+1}, \quad \text{for all } i \in \mathbb{N}_0.
\]
A model example of a filtration can be obtained by considering a product space
\[
(S, S, \mu) = \bigotimes_{i=0}^{\infty} (S_i, S_i, \mu_i)
\]
where each \((S_i, S_i, \mu_i)\) is a probability space. Then \(\mathcal{F}_i\) we set to be \(\sigma\)-field generated by the projection onto the first \(i\) coordinates. The resulting sequence \((\mathcal{F}_i : i \in \mathbb{N}_0)\) will be called the canonical filtration on \((S, S, \mu)\).

Suppose that \((\mathcal{F}_i : i \in \mathbb{N}_0)\) is a filtration in a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). A theorem due to Montgomery-Smith (see [2, Theorem 3.1]) asserts that: For any sequence of random variables \((f_n : 0 \leq n \leq N)\) on \((\Omega, \mathcal{F}, \mathbb{P})\) adapted to \((\mathcal{F}_n : 0 \leq n \leq N)\), i.e. each \(f_n\) is \(\mathcal{F}_n\)-measurable, there is \((f_n : 0 \leq n \leq N)\) a sequence of functions on \([0, 1]^N\) adapted to the canonical filtration and having the same joint distribution as \((f_n : 0 \leq n \leq N)\). The construction is clever, but tailored to a given sequence \((f_n : 0 \leq n \leq N)\). One of the goals of the present article is to remove this disadvantage, provided that \(\Omega\) is discrete. To achieve this we use the following observation: Suppose that there are two probability spaces \((S, S, \mu)\) and \((T, \mathcal{T}, \nu)\) equipped with family of \(\sigma\)-fields \((\mathcal{F}_i : i \in \mathcal{I})\) and \((\mathcal{G}_i : i \in \mathcal{I})\), respectively. Assume that there is a mapping
\[
\pi : (S, S, \mu) \to (T, \mathcal{T}, \nu)
\]
so that
\[
\pi^{-1}(U) \in \mathcal{G}_j \text{ for all } U \in \mathcal{F}_i \text{ and } i \in \mathcal{I},
\]
and
\[
\mu(\pi^{-1}(U)) = \nu(U), \text{ for all } U \in \mathcal{T}.
\]
Then \(\pi\) induces a mapping \(^1\)
\[
\pi^* : L^0(\Omega, \mathcal{F}, \mathbb{P}) \to L^0(S, S, \mu)
\]
\[
f \mapsto f \circ \pi
\]
that maps \(\mathcal{F}_n\)-measurable function to \(\mathcal{G}_n\)-measurable function preserving distributions, that is
\[
\mu(\pi^*(f) > \lambda) = \nu(f > \lambda)
\]
for all \(\lambda > 0\) and \(f \in L^0(\Omega, \mathcal{F}, \mathbb{P})\). The main theorem for one parameter case is the following.

\(^1\) By \(L^0(\Omega, \mathcal{F}, \mathbb{P})\) we denote the space of equivalence classes of \(\mathcal{F}\)-measurable functions.

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Theorem A. Let \((F_n : 1 \leq n \leq N)\) be a filtration in a discrete probability space \((\Omega,F,\mathbb{P})\). Then there is a sequence of probability spaces \(((S_i, S_i, \mu_i) : i \in \mathbb{N}_0)\) such that for any sequence of random variables \((f_n : 1 \leq n \leq N)\) on \((\Omega,F,\mathbb{P})\) adapted to \((F_n : 1 \leq n \leq N)\) there is a sequence of functions \((\tilde{f}_n : 1 \leq n \leq N)\) adapted to the canonical filtration of

\[ (S, S, \mu) = \bigotimes_{i=0}^{\infty} (S_i, S_i, \mu_i) \]

having the same joint distribution as \((f_n : 1 \leq n \leq N)\).

The advantage of our construction is the ability to extend it to biparameter case which is the main result of the present paper. Let us recall that double-indexed sequence of \(\sigma\)-fields \(((F_{i,j} : i, j \in \mathbb{N}_0)\) is a biparameter filtration if for all \(i, j \in \mathbb{N}_0,\)

\[ F_{i,j} \subset F_{i,j+1}, \quad F_{i,j} \subset F_{i+1,j}. \]

The following condition was introduced in [1], \(^2\)

\[ (F_4) \quad \mathbb{E}(\mathbb{E}(f | F_{i,j}) | F_{i',j'}) = \mathbb{E}(f | F_{i \wedge i', j \wedge j'}), \]

or, equivalently, \(F_{i,j+1}\) and \(F_{i+1,j}\) are conditionally independent given \(F_{i,j}\). The condition \((F_4)\) looks quite restrictive, however it allows maintaining a relatively rich structure, see e.g. the monographs [4, 5] and references therein. The simplest example of biparameter filtrations satisfying \((F_4)\) is a tensor of product filtrations. Namely, let \(((S_i, S_i, \mu_i) : i \in \mathbb{N}_0)\) and \(((T_j, T_j, \nu_j) : j \in \mathbb{N}_0)\), be two sequences of probability spaces. In the product space

\[ (S, S, \mu) \otimes (T, T, \nu) \]

where

\[ (S, S, \mu) = \bigotimes_{i=0}^{\infty} (S_i, S_i, \mu_i), \quad \text{and} \quad (T, T, \nu) = \bigotimes_{j=0}^{\infty} (T_j, T_j, \nu_j), \]

we define \(F_{i,j}\) to be the \(\sigma\)-field generated by the projections

\[ (s, t) \mapsto \left( (s' : 0 \leq i' \leq i), (t' : 0 \leq j' \leq j) \right) \]

which is the product of canonical filtrations on \((S, S, \mu)\) and \((T, T, \nu)\). However, this is not a universal model and is characterized by the property that \(F_{i,j} = F_{i,j}^-\), where

\[ F_{i,j}^- = \sigma \left( A \cup B : A \in F_{i-1,j} \text{ and } B \in F_{i,j-1} \right) \]

\[ = F_{i-1,j} \vee F_{i,j-1}. \]

A more general one is constructed from a double-indexed sequence of probability spaces \(((S_{i,j}, S_{i,j}, \mu_{i,j}) : i, j \in \mathbb{N}_0)\). Namely, in the tensor product

\[ \bigotimes_{i=0}^{\infty} \bigotimes_{j=0}^{\infty} (S_{i,j}, S_{i,j}, \mu_{i,j}), \]

we set \(F_{i,j}\) to be the \(\sigma\)-field generated by the projection

\[ s \mapsto (s_{i'j'} : 0 \leq i' \leq i, 0 \leq j' \leq j). \]

Heuristically, the whole space is generated by the surplus of \(F_{i,j}\) over \(F_{i,j}^-\), and this will be the main idea of the proof of the fact that for any probability space equipped with an \((F_4)\) filtration one can find a map \(\pi\) from a product as \((1.3)\) having the desired properties \((1.1a)\) and \((1.1b)\).

\(^2\) \(i \wedge j = \min\{i, j\}\)
Theorem B. Let \((\mathcal{F}_{i,j} : 1 \leq i \leq N, 1 \leq j \leq N)\) be a biparameter \((F_4)\)-filtration in a discrete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Then there is a double-indexed sequence of probability spaces \((S_{i,j}, S_{k,j}, \mu_{i,j})\) so that for any double-indexed sequence of random variables \((\tilde{f}_{i,j} : 1 \leq i \leq N, 1 \leq j \leq M)\) adapted to the filtration \((\mathcal{F}_{i,j} : 1 \leq i \leq N, 1 \leq j \leq M)\) there is a sequence of \((\tilde{f}_{i,j} : 1 \leq i \leq N, 1 \leq j \leq M)\) adapted to the canonical filtration of

\[
(S, S, \mu) = \bigotimes_{i=1}^{N} \bigotimes_{j=1}^{M} (S_{i,j}, S_{k,j}, \mu_{i,j}),
\]

having the same distribution as \((f_{i,j} : 1 \leq i \leq N, 1 \leq j \leq M)\).

Our purpose for developing Theorem B was to gain an understanding of biparameter decoupling analogous to that presented in [2] in the one parameter case, ultimately leading to a proof of one side of the Davis inequality for \((F_4)\) filtrations which is to appear in a forthcoming paper [3].

Notation. Given atomic \(\sigma\)-field \(\mathcal{F}\) by at \(\mathcal{F}\) we denote the set of atoms of \(\mathcal{F}\). Let \(\mathbb{N}\) denote the set of positive integers and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\).

2. One Parameter Case

In this section we want to prove Theorem A. To do so, we construct a sequences of discrete probability spaces \(((S, S_i, \mu_i) : i \in \mathbb{N}_0)\) and a mapping

\[
\pi : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, S, \mu)
\]
satisfying (1.1a) and (1.1b).

Suppose we are dealing with the simplest nontrivial case, that is \((\Omega, \mathcal{B}, \mathbb{P})\), and \(\mathcal{A} \subset \mathcal{B}\). Over each atom \(A\) of \(\mathcal{A}\), there is a different structure of \(\mathcal{B}\), that can be viewed as an individual probability space \((A, \mathcal{B} \cap A, \mathbb{P}_A)\) where

\[
\mathbb{P}_A(U) = \frac{\mathbb{P}(A \cap U)}{\mathbb{P}(A)},
\]

for any \(U \in \mathcal{B} \cap A\). An atom of \(\mathcal{B}\) is in one-to-one correspondence with the choice of an atom of \(\mathcal{A}\) and an element of \((A, \mathcal{B} \cap A, \mathbb{P}_A)\). Moreover, an atom of \(\mathcal{B}\) can be recovered if we redundantly include a choice of an element of \((A', \mathcal{B} \cap A', \mathbb{P}_A')\) for all other atoms \(A'\) of \(\mathcal{A}\). Next, we identify an element of

\[
\bigotimes_{A \in \mathcal{A}} (A, \mathcal{B} \cap A, \mathbb{P}_A)
\]

with a mapping \(\varphi : \text{at } \mathcal{A} \to \text{at } \mathcal{B}\) such that for an atom \(A \in \mathcal{A}\), \(\varphi(A)\) is an atom of \(\mathcal{B}\). That being said, let us define

\[
\pi : (\Omega, \mathcal{A}, \mathbb{P}) \otimes \bigotimes_{A \in \mathcal{A}} (A, \mathcal{B} \cap A, \mathbb{P}_A) \to (\Omega, \mathcal{B}, \mathbb{P})
\]

\[
(A, \varphi) \mapsto \varphi(A).
\]

Now, the condition (1.1a) is obvious. To check (1.1b) we consider each \(A \in \mathcal{A}\) separately. If \(U = \bigcup_k B_k\), where \(B_k \subset A\) are disjoint atoms of \(\mathcal{B}\), then

\[
\pi^{-1}(U) = A \times \{\varphi : \varphi(A) \subset U\}.
\]

By definition of \(\varphi\), the second factor is just a condition on the \(A\)-th coordinate of \(\varphi\) and its measure equals to

\[
\mathbb{P}_A(U) = \frac{\mathbb{P}(U)}{\mathbb{P}(A)}
\]

verifying (1.1b).
In a general case, we have a filtration \( (\mathcal{F}_n : 1 \leq n \leq N) \) in a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We define a map
\[
\pi : (\Omega, \mathcal{F}_0, \mathbb{P}) \otimes \bigotimes_{n=1}^N \bigotimes_{A \in at \mathcal{F}_{n-1}} (A, \mathcal{F}_n \cap A, \mathbb{P}_A) \rightarrow (\Omega, \mathcal{F}_N, \mathbb{P})
\]
where, as previously, an atom of \( \bigotimes_{A \in at \mathcal{F}_{n-1}} (A, \mathcal{F}_n \cap A, \mathbb{P}_A) \) we treat as a function \( \varphi_n : at \mathcal{F}_{n-1} \rightarrow at \mathcal{F}_n \) satisfying \( \varphi_n (A_{n-1}) \subset A_n \). From definition of \( \pi \) it is obvious that for an atom \( B_n \) of \( \mathcal{F}_n \), the condition \( \pi (A, \varphi_1, \ldots, \varphi_N) = B_n \) is equivalent to \( \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1 (A) = B_n \), so it depends only on \( A \) and \( \varphi_i \) for \( i \leq n \) proving (1.1a). The condition (1.1b) can be check on atoms of \( \mathcal{F}_N \). If \( A_N \in at \mathcal{F}_N \), then, denoting its ancestors by \( A_n \in at \mathcal{F}_n \), we have
\[
\pi (A, \varphi_1, \ldots, \varphi_N) = A_N
\]
if and only if \( A = A_0 \), and
\[
\varphi_n (A_{n-1}) = A_n, \quad \text{for all } 1 \leq n \leq N.
\]
The probability of this event equals to
\[
\mathbb{P} (A_0) \prod_{n=1}^N \frac{\mathbb{P} (A_n)}{\mathbb{P} (A_{n-1})} = \mathbb{P} (A_N)
\]
which concludes the proof of Theorem \( A \).

3. Two parameter case

In this section we prove Theorem \( B \). In the two parameter case it is convenient to use the following variant of mathematical induction.

**Lemma 3.1.** Suppose that a set \( X \subset \mathbb{N}^2 \) satisfies

1. \((1, 1) \in X, \)
2. if \((i, 1) \in X, \) then \((i + 1, 1) \in X, \)
3. if \((1, j) \in X, \) then \((1, j + 1) \in X, \)
4. if \((i + 1, j), (i, j + 1), (i, j) \in X, \) then \((i + 1, j + 1) \in X, \)
then \( X = \mathbb{N}^2. \)

Again, our aim is to construct a double-indexed sequence of probability spaces \((S_{i,j}, S_{i,j}, \mu_{i,j})\), and a mapping
\[
\pi : \bigotimes_{i=1}^N \bigotimes_{j=1}^M (S_{i,j}, S_{i,j}, \mu_{i,j}) \rightarrow (\Omega, \mathcal{F}_{N,M}, \mathbb{P})
\]
satisfying (1.1a) and (1.1b). We use similar idea as in the one parameter case. For \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \), we set
\[
(S_{i,j}, S_{i,j}, \mu_{i,j}) = \bigotimes_{A \in at \mathcal{F}_{i,j}} (A, \mathcal{F}_{i,j} \cap A, \mathbb{P}_A)
\]
where
\[
\mathcal{F}_{i,j} = \mathcal{F}_{i-1,j} \lor \mathcal{F}_{i,j-1}
\]
with
\[
\mathcal{F}_{i,0} = \{ \emptyset, \Omega \}, \quad \text{and} \quad \mathcal{F}_{0,j} = \{ \emptyset, \Omega \}.
\]
Hence, the atoms in \( S_{i,j} \) are sequences having a form \( (B_A : A \in at \mathcal{F}_{i,j}) \) where \( B_A \) denotes an atom of \( \mathcal{F}_{i,j} \) contained in \( A \). Observe that for such an atom we have
\[
\mu_{i,j} \left( B_A : A \in at \mathcal{F}_{i,j} \right) = \prod_{A \in at \mathcal{F}_{i,j}} \mathbb{P}_A (B_A).
\]
Consequently, atoms of the domain of $\pi$ are

\[(3.1) \quad \mathbf{B} = \left( B_{A}^{i,j} : A \in \mathcal{F}_{i,j}^{-} \right) : 1 \leq i \leq N, 1 \leq j \leq M \].

We are now ready to define the mapping $\pi$, namely for an atom of the form (3.1) we set

\[(3.2) \quad \pi(\mathbf{B}) = \bigcap_{i=1}^{N} \bigcap_{j=1}^{M} B_{A}^{i,j}. \]

Observe that the right hand-side of (3.2) can be written as a disjoint union of sets of the form

\[(3.3) \quad \bigcap_{i=1}^{N} \bigcap_{j=1}^{M} B_{A_{i,j}}^{i,j} \]

while $A_{i,j}$ runs over all atoms of $\mathcal{F}_{i,j}^{-}$. Suppose that there is a sequence $(A_{i,j} : 1 \leq i \leq N, 1 \leq j \leq M)$ so that $A_{i,j} \in \mathcal{F}_{i,j}^{-}$ and

\[ \bigcap_{i=1}^{N} \bigcap_{j=1}^{M} B_{A_{i,j}}^{i,j} \neq \emptyset. \]

We are going to use the induction procedure given by Lemma 3.1. For $2 \leq i \leq N$ and $2 \leq j \leq M$, we have

\[ B_{A_{i-1,j}}^{i-1,j} \cap B_{A_{i,j-1}}^{i,j-1} \cap B_{A_{i,j}}^{i,j} \neq \emptyset. \]

Since

\[ B_{A_{i-1,j}}^{i-1,j} \cap B_{A_{i,j-1}}^{i,j-1} \in \mathcal{F}_{i-1,j}^{-} \subset \mathcal{F}_{i,j}, \]

we conclude that

\[(3.4) \quad B_{A_{i-1,j}}^{i-1,j} \cap B_{A_{i,j-1}}^{i,j-1} \supset B_{A_{i,j}}^{i,j}. \]

Moreover, $B_{A_{i-1,j}}^{i-1,j} \cap B_{A_{i,j-1}}^{i,j-1}$ is an atom of $\mathcal{F}_{i,j}^{-}$, thus

\[(3.5) \quad A_{i,j} = B_{A_{i-1,j}}^{i-1,j} \cap B_{A_{i,j-1}}^{i,j-1}. \]

Similarly, for $2 \leq i \leq N$,

\[ B_{A_{i-1,1}}^{i-1,1} \cap B_{A_{i,1}}^{i,1} \neq \emptyset \]

and since $B_{A_{i-1,1}}^{i-1,1}$ and $B_{A_{i,1}}^{i,1}$ are atoms of $\mathcal{F}_{i-1,1}$ and $\mathcal{F}_{i,1}$, respectively, we obtain that

\[(3.6) \quad B_{A_{i-1,1}}^{i-1,1} \supset B_{A_{i,1}}^{i,1}. \]

Because

\[ \mathcal{F}_{i-1,1} = \mathcal{F}_{i-1,1} \vee \mathcal{F}_{i,0} = \mathcal{F}_{i,1}, \]

$B_{A_{i-1,1}}^{i-1,1}$ is an atom of $\mathcal{F}_{i,1}^{-}$, thus

\[(3.7) \quad A_{i,1} = B_{A_{i-1,1}}^{i-1,1}. \]

Analogously, for $2 \leq j \leq N$, we conclude

\[(3.8) \quad B_{A_{1,j-1}}^{1,j-1} \supset B_{A_{1,j}}^{1,j}, \]

and

\[(3.9) \quad A_{1,j} = B_{A_{1,j-1}}^{1,j-1}. \]

Finally, $B_{A_{1,1}}^{1,1}$ is any atom of $\mathcal{F}_{1,1}$, thus

\[(3.10) \quad A_{1,1} = \Omega. \]
Using (3.4), (3.6) and (3.9), we obtain

\[ \pi(B) = \bigcap_{i=1}^{N} \bigcap_{j=1}^{M} B^{i,j}_{A_{i,j}} = B^{N,M}_{A_{N,M}}, \]

provided that \( \pi(B) \neq \emptyset. \)

We are now in the position to verify (1.1a). If \( U \) is an atom of \( \mathcal{F}_{n,m} \), then \( B \in \pi^{-1}(U) \) is equivalent to \( B^{n,m}_{A_{n,m}} = U \). In view of (3.5), (3.7), (3.9) and (3.10), \( B^{n,m}_{A_{n,m}} \) depends only on \( A_{i,j} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), thus \( \pi^{-1}(U) \) belongs to \( S_{n,m}. \) To show (1.1b), it is sufficient to consider \( U \) being an atom of \( \mathcal{F}_{N,M}. \) Suppose that \( B \in \pi^{-1}(U) \). By (3.11), for each \( 1 \leq i \leq N \) and \( 1 \leq j \leq M, \) \( B^{i,j}_{A_{i,j}} \) is the unique atom of \( \mathcal{F}_{i,j} \) containing \( U. \) Therefore,

\[ \mu(\pi^{-1}(U)) = \mu(B) = \prod_{i=1}^{N} \prod_{j=1}^{M} \mathbb{P}(B^{i,j}_{A_{i,j}}). \]

Now it is enough to show that for each \( 1 \leq n \leq N \) and \( 1 \leq m \leq M, \)

\[ \prod_{j=1}^{m} \prod_{j=1}^{n} \mathbb{P}(B^{i,j}_{A_{i,j}}) = \mathbb{P}(B^{n,m}_{A_{n,m}}). \]

For the proof we use the induction procedure given by Lemma 3.1. For \( n = m = 1, \) there is nothing to be proved since \( A_{1,1} = \Omega. \) For \( n > 1 \) and \( m = 1, \) by (3.7), we have

\[ \prod_{i=1}^{n} \prod_{j=1}^{m} \mathbb{P}(B^{i,j}_{A_{i,j}}) = \prod_{i=1}^{n} \mathbb{P}(B^{i,1}_{A_{i,1}}) \]

\[ = \mathbb{P}(B^{1,1}_{A_{1,1}}) \prod_{i=2}^{n} \mathbb{P}(B^{i,1}_{A_{i,1}}) = \mathbb{P}(B^{1,1}_{A_{1,1}}). \]

For \( n = 1 \) and \( m > 1 \) the reasoning is analogous. Now, let us suppose that (3.12) holds true for \((n-1, m-1),\) \((n-1, m),\) and \((n, m-1)\) for some \( 2 \leq n \leq N \) and \( 2 \leq m \leq M. \) Then

\[ \prod_{i=1}^{n} \prod_{j=1}^{m} \mathbb{P}(B^{i,j}_{A_{i,j}}) \]

\[ = \prod_{i=1}^{n-1} \prod_{j=1}^{m} \mathbb{P}(B^{i,j}_{A_{i,j}}) \prod_{i=1}^{n-1} \prod_{j=1}^{m-1} \mathbb{P}(B^{i,j}_{A_{i,j}}) \prod_{i=1}^{n} \prod_{j=1}^{m-1} \mathbb{P}(B^{i,j}_{A_{i,j}}) \]

\[ = \prod_{i=1}^{n-1} \prod_{j=1}^{m} \mathbb{P}(B^{i,j}_{A_{i,j}}) \prod_{i=1}^{n-1} \prod_{j=1}^{m-1} \mathbb{P}(B^{i,j}_{A_{i,j}}) \prod_{i=1}^{n} \prod_{j=1}^{m} \mathbb{P}(B^{i,j}_{A_{i,j}}) \]

(3.13)

Observe that by the conditional independence

\[ \frac{\mathbb{P}(B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(B^{n-1,m}_{A_{n-1,m}})}{\mathbb{P}(B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(A_{n,m})} = \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(A_{n,m}) \]

\[ = \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} \cap B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(A_{n,m}) \]

\[ = \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} \cap B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(B^{n-1,m}_{A_{n-1,m}} | B^{n-1,m}_{A_{n-1,m}}) \mathbb{P}(A_{n,m}) \]

\[ = 1, \]

where the last equality is a consequence of (3.5). Therefore, by (3.13), we conclude that (3.12) holds true proving Theorem B.
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