α-Parameterized Differential Transform Method

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Abstract

In this paper we propose a new version of differential transform method (we shall call this method as α-parameterized differential transform method), which differs from the traditional differential transform method in calculating coefficients of Taylor polynomials. Numerical examples are presented to illustrate the efficiency and reliability of own method. The result reveal that α-Parameterized differential transform method is a simple and effective numerical algorithm.

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1. Introduction

Many problems in mathematical physics, theoretical physics and chemical physics are modelled by the so-called initial value and boundary value problems in the second-order ordinary differential equations. In most cases, these problems may be too complicated to solve analytically. Alternatively, the numerical methods can provide approximate solutions rather than the analytic solutions of problems. There are various approximation methods for solving a system of differential equations, e.g. Adomian decomposition...
method (ADM), Galerkin method, rationalized Haar functions method, homotopy perturbation method (HPM), variational iteration method (VIM) and the differential transform method (DTM).

The DTM is one of the numerical methods which enables to find approximate solution in case of linear and non-linear systems of differential equations. The main advantage of this method is that it can be applied directly to nonlinear ODEs without requiring linearization. The well known advantage of DTM is its simplicity and accuracy in calculations and also wide range of applications. Another important advantage is that this method is capable of greatly reducing the size of computational work while still accurately providing the series solution with fast convergence rate. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. The concept of the differential transform method was first proposed by [19], who solved linear and nonlinear initial value problems in electric circuit analysis. Afterwards, Chiou and Tzeng [18] applied the Taylor transform to solve nonlinear vibration problems, Chen and Ho [2] developed this method to various linear and nonlinear problems such as two point boundary value problems and Ayaz [3] applied it to the system of differential equations. Abbasov et al. [1] used the method of differential transform to obtain approximate solutions of the linear and non-linear equations related to engineering problems and observed that the numerical results are in good agreement with the analytical solutions. In recent years many authors has been used this method for solving various types of equations. For example, this method has been used for differential-algebraic equations [4], partial differential equations [2, 7, 17], fractional differential equations [16] and difference equations [15]. In [9, 10, 11], this method has been utilized for Telegraph, Kuramoto-Sivashinsky and Kawahara equations. Shahmorad et al. developed DTM to fractional-order integro-differential equations with nonlocal boundary conditions [12] and class of two dimensional Volterra integral equations [13]. Abazari et. al. applied this method for Schrödinger equations [14]. Different applications of DTM can be found in [5, 6]. Even if the differential transform method (DTM) is an effective numerical method for solving many initial value problem, there are also some disadvantages, since this method is designed for problems that have analytic solutions (i.e. solutions that can be expanded in Taylor series).

In this paper we suggest a new version of DTM which we shall called $\alpha$-parameterized differential transform method ($\alpha$-p DTM) to solve initial value and boundary value problems, particularly eigenvalue problems. Note
that, in the special cases $\alpha = 1$ and $\alpha = 0$ of the $\alpha$-p DTM reduces to the standard DTM.

2. The classical DTM

In this section, we describe the definition and some basic properties of the classical differential transform method. An arbitrary analytic function $f(x)$ can be expanded in Taylor series about a point $x = x_0$ as

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}$$  \hspace{1cm} (1)

The classical differential transformation of $f(x)$ is defined as

$$F(k) = \frac{1}{k!} \left[ \frac{d^k f(x)}{dx^k} \right]_{x=x_0}$$ \hspace{1cm} (2)

Then the inverse differential transform is

$$f(x) = \sum_{k=0}^{\infty} (x-x_0)^k F(k).$$ \hspace{1cm} (3)

The fundamental mathematical operations performed by differential transform method are listed in following

i) If $f(x) = g(x) \pm h(x)$ then $F(k) = G(k) \pm H(k)$

ii) If $f(x) = \alpha g(x)$, $\alpha \in \mathbb{R}$, then $F(k) = \alpha G(k)$

iii) If $f(x) = \frac{d^m g}{dx^m}$ then $F(k) = (k+1)(k+2)...(k+m)G(k+m)$

iv) If $f(x) = x^m$ then $F(k) = \delta(k-m) = \begin{cases} 1 & \text{for } k = m \\ 0 & \text{for } k \neq m \end{cases}$

v) If $f(x) = g(x)h(x)$ then $F(k) = \sum_{m=0}^{k} H(m)G(k-m)$

3. $\alpha$-Parameterized differential transform method ($\alpha$-p DTM)

In this section we suggest a new version of classical differential transform method by following.

Let $I = [a, b] \subset \mathbb{R}$ be an arbitrary real interval, $f : I \to \mathbb{R}$ is an infinitely differentiable function (in real applications it is enough to required that $f(x)$
is sufficiently large order differentiable function), \( \alpha \in [0, 1] \) any real parameter and \( N \) any integer (large enough). Let us introduce the following notations

\[
D_a(f; k) := \frac{f^{(k)}(a)}{k!}, \quad D_b(f; k) := \frac{f^{(k)}(b)}{k!}
\]  

(4)

\[
D(f, \alpha; k) := \alpha D_a(f; k) + (1 - \alpha)D_b(f; k), \quad \alpha \in [0, 1], \quad k \in \mathbb{N}
\]  

(5)

**Definition 1.** The sequence

\[
(D_\alpha(f)) := (D(f, \alpha; 1), D(f, \alpha; 2), \ldots)
\]

is called the \( \alpha \)-P transformation of the original function \( f(x) \). The so-called ,,differential inverse” transform of \( D_\alpha(f) \) we define as

\[
E_\alpha(D_\alpha(f)) := \sum_{k=0}^{\infty} D(f, \alpha; k)(x - x_\alpha)^k
\]  

(6)

if the series is convergent, where \( x_\alpha = \alpha a + (1 - \alpha)b \).

The function \( \tilde{f}_\alpha(x) \) defined by equality

\[
\tilde{f}_\alpha(x) := E_\alpha(D_\alpha(f))
\]

we called the \( \alpha \)-parameterized approximation of the function \( f(x) \).

**Remark 1.** In the cases of \( \alpha = 1 \) and \( \alpha = 0 \) the \( \alpha \)-p differential transform (5) reduces to the classical differential transform (2) at the points \( x = a \) and \( x = b \) respectively. Namely for \( \alpha = 0 \) and \( \alpha = 1 \) the equality \( \tilde{f}_\alpha(x) = f(x) \) is hold.

**Remark 2.** For practical application, instead of \( \tilde{f}_\alpha(x) \) it is convenient to introduced \( N \)-term \( \alpha \)-parameterized approximation of the function \( \tilde{f}_\alpha(x) \) which we shall define as

\[
\tilde{f}_{\alpha,N}(x) := E_{\alpha,N}(D_\alpha(f)) := \sum_{k=0}^{N} D(f, \alpha; k)(x - x_\alpha)^k
\]  

(7)
Theorem 1. If \( f(x) \) is constant function then \( \tilde{f}_\alpha(x) = f(x) \) and \( \tilde{f}_{\alpha,N}(x) = f(x) \) for each \( N \).

Proof. The proof is immediate from Definition 1 and Remark 2.

Theorem 2. If \( f(x) = cg(x), \ c \in \mathbb{R}, \) then \( D_\alpha(f) = cD_\alpha(g) \) and \( \tilde{f}_\alpha(x) = c\tilde{g}_\alpha(x) \).

Proof. By applying the well-known properties of classical DTM we get
\[
D(f, \alpha; k) = \alpha D_a(f; k) + (1 - \alpha) D_b(f; k) \\
= \alpha c D_a(g; k) + (1 - \alpha) c D_b(g; k) \\
= c(\alpha D_a(g; k) + (1 - \alpha) D_b(g; k)) \\
= cD(g, \alpha; k)
\]
(8)
Consequently \( D_\alpha(f) = cD_\alpha(g) \), from which immediately follows that \( \tilde{f}_\alpha(x) = c\tilde{g}_\alpha(x) \).

Theorem 3. If \( f(x) = g(x) \pm h(x) \) then \( D_\alpha(f) = D_\alpha(g) \pm D_\alpha(h) \) and \( \tilde{f}_\alpha(x) = \tilde{g}_\alpha(x) \pm \tilde{h}_\alpha(x) \).

Proof. By using the definition of transform (5)
\[
D(f, \alpha; k) = \alpha D_a(f; k) \pm (1 - \alpha) D_b(f; k) \\
= \alpha D_a(g + h; k) \pm (1 - \alpha) D_b(g + h; k) \\
= D(g, \alpha; k) \pm D(h, \alpha; k)
\]
(9)
Consequently \( D_\alpha(f) = D_\alpha(g) \pm D_\alpha(h) \), from which immediately follows that \( \tilde{f}_\alpha(x) = \tilde{g}_\alpha(x) \pm \tilde{h}_\alpha(x) \).

Theorem 4. Let \( f(x) = \frac{d^m g}{dx^m} \) and \( m \in \mathbb{N} \). Then
\[
D(f^{(m)}, \alpha; k) = \frac{(k + m)!}{k!} D(f, \alpha; k + m)
\]
and
\[
\tilde{f}_\alpha^{(m)}(x) = \sum_{k=0}^{\infty} \frac{(k + m)!}{k!} D(f, \alpha; k + m)(x - x_\alpha)^k
\]
where \( x_\alpha = \alpha a + (1 - \alpha)b \).
Thus we get

\[ D(f^{(m)}, \alpha; k) = \alpha D_a(f^{(m)}; k) + (1 - \alpha)D_b(f^{(m)}; k) \]
\[ = \alpha(k+1)(k+2)...(k+m)D_a(f; k+m) \]
\[+ (1 - \alpha)(k+1)(k+2)...(k+m)D_b(f; k+m) \]
\[= (k+1)(k+2)...(k+m)(\alpha D_a(f; k) + (1 - \alpha)D_b(f; k+m)) \]
\[= \frac{(k+m)!}{k!}D(f, \alpha; k+m) \]

Thus we get \( D(f^{(m)}, \alpha; k) = \frac{(k+m)!}{k!}D(f, \alpha; k+m) \). Using this we find \( f^{(m)}(x) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!}D(f, \alpha; k+m)(x-x_{\alpha})^k \).

**Theorem 5.** Let \( f(x) = x^m, \ m \in \mathbb{N} \). Then

\[
D(f, \alpha; k) = \begin{cases} 
\binom{m}{k} (\alpha a^{m-k} + (1 - \alpha)b^{m-k}) & \text{for } k < m \\
1 & \text{for } k = m \\
0 & \text{for } k > m
\end{cases}
\]

**Proof.** Let \( k < m \). By using the definition of the transform (5) we have

\[
D(f, \alpha; k) = \alpha D_a(f; k) + (1 - \alpha)D_b(f; k) = \alpha D_a(x^{(m)}; k) + (1 - \alpha)D_b(x^{(m)}; k) = \binom{m}{k} (\alpha a^{(m-k)} + (1 - \alpha)b^{(m-k)})
\]

The equalities \( D(x^m, \alpha; m) = 1 \) and \( D(x^m, \alpha; m+s) = 0 \) for \( s \geq 1 \) is obvious.

**Theorem 6.** If \( f(x) = g(x)h(x) \) then \( D(f, \alpha; k) = \sum_{m=0}^{k} [\alpha D_a(g; m)D_a(h; k-m) + (1 - \alpha)D_b(g; m)D_b(h; k-m)] \).

**Proof.** By using the definition of transform given in Eq. (5) we have

\[
D(f, \alpha; k) = \alpha D_a(f; k) + m(1 - \alpha)D_b(f; k)
\]
\[= \alpha D_a(gh; k) + (1 - \alpha)D_b(gh; k)\]
\[= \frac{\alpha}{k!} \sum_{m=0}^{k} \binom{k}{m} g^{(m)}(a)h^{(k-m)}(a) + \frac{(1 - \alpha)}{k!} \sum_{m=0}^{k} \binom{k}{m} g^{(m)}(b)h^{(k-m)}(b)\]
\[= \sum_{m=0}^{k} [\alpha D_a(g; m)D_a(h; k-m) + (1 - \alpha)D_b(g; m)D_b(h; k-m)]. \quad (11)\]
4. Justification of the \(\alpha\)-p DTM

In order to show the effectiveness of \(\alpha\)-p DTM for solving boundary value problems, examples is demonstrated.

Example 1. (Application to boundary-value problem) Let us consider the differential equation

\[ \ell y := y''(x) + \mu^2 y(x) = 0, \quad x \in [0, 1], \quad \mu \in \mathbb{R}. \quad (12) \]

with the boundary conditions

\[ y(0) = 0, \quad y(1) = 1. \quad (13) \]

Exact solution for this problem is

\[ y(x) = \frac{\sin \mu x}{\sin \mu}. \quad (14) \]

Applying the \(N\)-term \(\alpha\)-p differential transform of both sides (12) and (13) we obtain the following \(\alpha\)-parameterized boundary value problem as

\[ (\tilde{\ell} y)_{\alpha,N} = \tilde{0}_{\alpha,N}, \quad \tilde{y}_{\alpha,N}(0) = \tilde{0}_{\alpha,N}, \quad \tilde{y}_{\alpha,N}(1) = \tilde{1}_{\alpha,N}. \quad (15) \]

By using the fundamental operations of \(\alpha\)-p DTM we have

\[ D(y, \alpha; k + 2) = -\frac{\mu^2 D(y, \alpha; k)}{(k + 1)(k + 2)} \quad (16) \]

The boundary conditions given in (13) can be transformed as follows

\[ \tilde{y}_{\alpha,N}(0) = \sum_{k=0}^{N} D(y, \alpha; k)(\alpha - 1)^k = 0 \quad \text{and} \quad \tilde{y}_{\alpha,N}(1) = \sum_{k=0}^{N} D(y, \alpha; k)\alpha^k = 1. \quad (17) \]

Using (16) and (17) and by taking \(N = 5\), the following \(\alpha\)-p approximate solution is obtained

\[ \tilde{y}_\alpha(x) = A + (x - x_\alpha)B - \frac{\mu^2(x - x_\alpha)^2 A}{2} - \frac{\mu^2(x - x_\alpha)^3 B}{6} + \frac{\mu^4(x - x_\alpha)^4 A}{24} + \frac{\mu^4(x - x_\alpha)^5 B}{120} + O(x^6) \quad (18) \]
where $x_\alpha = (1 - \alpha)$, according to (6), $D(y, \alpha; 0) = A$ and $D(y, \alpha; 1) = B$. The constants $A$ and $B$ evaluated from equations in (16) as follow

$$A = (2880(x_0^4\mu^8 - 4x_0^5\mu^8 + 6x_0^6\mu^8 + 24\mu^4 - 4x_0^7\mu^8 - 480\mu^2 + x_0^8\mu^8 + 2880 - 12x_0^2\mu^6 + 40x_0^3\mu^6 - 60x_0^4\mu^6 + 48x_0^5\mu^6 - 16x_0^6\mu^6)^{-1}) \times (x_0 - \frac{\mu^2x_0^3}{6} + \frac{\mu^4x_0^5}{120})$$

and

$$B = (2880(x_0^4\mu^8 - 4x_0^5\mu^8 + 6x_0^6\mu^8 + 24\mu^4 - 4x_0^7\mu^8 - 480\mu^2 + x_0^8\mu^8 + 2880 - 12x_0^2\mu^6 + 40x_0^3\mu^6 - 60x_0^4\mu^6 + 48x_0^5\mu^6 - 16x_0^6\mu^6)^{-1}) \times (1 - \frac{\mu^2x_0^2}{2} + \frac{\mu^4x_0^4}{24})$$

**Example 2.** (Application to eigenvalue problems) We consider the following eigenvalue problem

$$y'' + \lambda y = 0, \quad x \in [0, 1] \quad (21)$$

$$A_{11}y(0) + A_{12}y'(0) = 0 \quad (22)$$

$$A_{21}y(1) + A_{22}y'(1) = 0 \quad (23)$$

Taking the $\alpha$-p differential transform of both sides (21) we find

$$D(y'' + \lambda y, \alpha; k) = (k + 1)(k + 2)D(y, \alpha; k + 2) + \lambda D(y, \alpha; k) = 0. \quad (24)$$

Thus the following recurrence relation is obtained

$$D(y, \alpha; k + 2) = -\frac{\lambda D(y, \alpha; k)}{(k + 1)(k + 2)} \quad (25)$$

Using definition of $\alpha$-p differential transform we get

$$\tilde{y}_\alpha(x) = \sum_{k=0}^{\infty} D(y, \alpha; k)(x - x_\alpha)^k \quad (26)$$

$$\tilde{y}'_\alpha(x) = \sum_{k=0}^{\infty} kD(y, \alpha; k)(x - x_\alpha)^{k-1} \quad (27)$$
Consequently
\[
\tilde{y}_\alpha(0) = \sum_{k=0}^{\infty} D(y, \alpha; k)(\alpha - 1)^k = \sum_{k=0}^{\infty} (-1)^k D(y, \alpha; k)(1 - \alpha)^k \tag{28}
\]
\[
\tilde{y}'_\alpha(0) = \sum_{k=0}^{\infty} k D(y, \alpha; k)(\alpha - 1)^{k-1} = \sum_{k=0}^{\infty} (-1)^k k D(y, \alpha; k)(1 - \alpha)^{k-1} \tag{29}
\]

Thus the boundary condition given in (22) can be transformed as follows
\[
A_{11}\tilde{y}_\alpha(0) + A_{12}\tilde{y}'_\alpha(0) = \sum_{k=0}^{\infty} (A_{11}(\alpha - 1)^k + k A_{12}(\alpha - 1)^{k-1}) D(y, \alpha; k) = 0 \tag{30}
\]

Similarly we have
\[
\tilde{y}_\alpha(1) = \sum_{k=0}^{\infty} D(y, \alpha; k) \alpha^k \tag{31}
\]
and
\[
\tilde{y}'_\alpha(1) = \sum_{k=0}^{\infty} k D(y, \alpha; k) \alpha^{k-1} \tag{32}
\]

In this case the boundary condition given in (23) can be written as follows
\[
A_{21}\tilde{y}_\alpha(1) + A_{22}\tilde{y}'_\alpha(1) = \sum_{k=0}^{\infty} (A_{21}\alpha^k + k A_{22}\alpha^{k-1}) D(y, \alpha; k) = 0 \tag{33}
\]

Let \( D(y, \alpha; 0) = A \) and \( D(y, \alpha; 1) = B \). Substituting these values in (25) we have the following recursive procedure
\[
D(y, \alpha; k) = \begin{cases} 
\frac{A(-\lambda)^\ell}{(2\ell)!}, & \text{for } k = 2\ell \\
\frac{B(-\lambda)^\ell}{(2\ell+1)!}, & \text{for } k = 2\ell + 1 
\end{cases} \tag{34}
\]

Substituting (34) in (30) and (33) we find
\[
A \{ \sum_{\ell=0}^{\infty} (A_{11}(\alpha - 1)^{2\ell} + 2\ell A_{12}(\alpha - 1)^{2\ell-1}) \frac{(-\lambda)^\ell}{(2\ell)!} \} + B \{ \sum_{\ell=0}^{\infty} (A_{11}(\alpha - 1)^{2\ell+1} + (2\ell + 1) A_{12}(\alpha - 1)^{2\ell}) \frac{(-\lambda)^\ell}{(2\ell + 1)!} \} = 0 \tag{35}
\]
\[
A \left\{ \sum_{\ell=0}^{\infty} (A_{21} \alpha^{2\ell} + 2\ell A_{22} \alpha^{2\ell-1}) \frac{(-\lambda)^\ell}{(2\ell)!} \right\} \\
+ \ B \left\{ \sum_{\ell=0}^{\infty} (A_{21} \alpha^{2\ell+1} + (2\ell + 1)A_{22} \alpha^{2\ell}) \frac{(-\lambda)^\ell}{(2\ell + 1)!} \right\} = 0. \quad (36)
\]

respectively. In this case we have a linear system of the equations with respect to the variables \(A\) and \(B\) as

\[
AP_{11}(\lambda) + BP_{12}(\lambda) = 0 \quad (37)
\]
\[
AP_{21}(\lambda) + BP_{22}(\lambda) = 0 \quad (38)
\]

where \(P_{11}(\lambda) := \sum_{\ell=0}^{\infty} (A_{11}(\alpha - 1)^{2\ell} + 2\ell A_{12}(\alpha - 1)^{2\ell-1}) \frac{(-\lambda)^\ell}{(2\ell)!} \), \(P_{12}(\lambda) := \sum_{\ell=0}^{\infty} (A_{11}(\alpha - 1)^{2\ell+1} + (2\ell + 1)A_{12}(\alpha - 1)^{2\ell}) \frac{(-\lambda)^\ell}{(2\ell + 1)!} \), \(P_{21}(\lambda) := \sum_{\ell=0}^{\infty} (A_{21} \alpha^{2\ell} + 2\ell A_{22} \alpha^{2\ell-1}) \frac{(-\lambda)^\ell}{(2\ell)!} \), and \(P_{22}(\lambda) := \sum_{\ell=0}^{\infty} (A_{21} \alpha^{2\ell+1} + (2\ell + 1)A_{22} \alpha^{2\ell}) \frac{(-\lambda)^\ell}{(2\ell + 1)!} \). Since the system (37)-(38) has a nontrivial solution for \(A\) and \(B\) the characteristic determinant is zero i.e.

\[
P(\lambda) = \begin{vmatrix} P_{11}(\lambda) & P_{12}(\lambda) \\ P_{21}(\lambda) & P_{22}(\lambda) \end{vmatrix} = 0.
\]

Thus we have characteristic equation for eigenvalues by \(\alpha\)-p DTM. Now, let us find the exact eigenvalues and eigenfunctions of the Sturm-Liouville problem (21)-(23). The general solution of equation (23) have the form

\[
y(x) = C \cos \mu x + D \sin \mu x \quad (39)
\]

where \(\lambda = \mu^2\) and \(C, D\) are arbitrary constants. Applying the boundary conditions (21), (22) we get

\[
A_{11}C + \mu A_{12}D = 0 \quad (40)
\]
\[
(A_{21} \cos \mu A_{22} \sin \mu)C + (A_{21} \cos \mu A_{22} \sin \mu)D = 0 \quad (41)
\]

Because we cannot have \(C = D = 0\), this implies

\[
(A_{11} A_{21} + \mu^2 A_{12} A_{22}) \cos \mu - \mu (A_{12} A_{21} - A_{11} A_{22}) \cos \mu = 0. \quad (42)
\]
This is transcendental equation which is solved graphically. Let \( \mu = \mu_n, n \in \mathbb{N} \) are points of intersection of the graphs of the functions

\[
y = (A_{11}A_{21} + \mu^2A_{12}A_{22}) \sin \mu \quad \text{and} \quad y = \mu(A_{12}A_{21} - A_{11}A_{22}) \cos \mu
\]

The eigenvalues and corresponding eigenfunctions are therefore given by

\[
\lambda_n = \mu_n^2 \quad \text{and} \quad y_n(x) = C_n \cos \mu_n x + D_n \sin \mu_n x, \ n \in \mathbb{N}. \tag{45}
\]

Now we consider special case of the Sturm-Liouville problem \((23)-(21)\),

\[
\begin{align*}
y'' + \lambda y &= 0 \quad \text{(46)} \\
y(0) + y'(0) &= 0 \quad \text{(47)} \\
y(1) - y'(1) &= 0. \quad \text{(48)}
\end{align*}
\]

The eigenvalues of this problem are determined by the equation

\[
\tan \mu = \frac{2\mu}{1 - \mu^2}. \tag{49}
\]

This equation can be solved graphically by the points of intersections of the graphs of functions

\[
y = \tan \mu \quad \text{and} \quad y = \frac{2\mu}{1 - \mu^2}
\]

as shown by the sequence \((\mu_n)\) in Figure 1.

The eigenvalues of the considered problem are given by \( \lambda_n = \mu_n^2 \) and corresponding eigenfunctions are given by

\[
y_n(x) = C_n \cos \mu_n x + D_n \sin \mu_n x, \ n \in \mathbb{N}. \tag{51}
\]

Taking the \( \alpha\)-p differential transform of both sides the equation \((46)\) the following recurrence relation is obtained

\[
D(y, \alpha; k + 2) = -\frac{\lambda D(y, \alpha; k)}{(k + 1)(k + 2)}. \tag{52}
\]
Applying N-term $\alpha$-$p$ differential transform the boundary conditions (47)-(48) are transformed as follows:

\[
\tilde{y}_\alpha(0) + \tilde{y}'_\alpha(0) = \sum_{k=0}^{N} ((\alpha - 1)^k + k(\alpha - 1)^{k-1}) D(y, \alpha; k) = 0 \quad (53)
\]

\[
\tilde{y}_\alpha(1) - \tilde{y}'_\alpha(1) = \sum_{k=0}^{N} (\alpha^k - k\alpha^{k-1}) D(y, \alpha; k) = 0 \quad (54)
\]

By using (52), (53) and (54) we obtain the following equalities (for N=6)

\[
A \left[ 1 + (\alpha - 1)^2 + 2(\alpha - 1) \frac{(-\lambda)}{2!} + ((\alpha - 1)^4 + 4(\alpha - 1)^3) \frac{\lambda^2}{4!} + ((\alpha - 1)^6 + 6(\alpha - 1)^5) \frac{(-\lambda^3)}{6!} \right] + B \left[ \alpha + ((\alpha - 1)^3 + 3(\alpha - 1)^2) \frac{(-\lambda)}{3!} + ((\alpha - 1)^5 + 5(\alpha - 1)^4) \frac{\lambda^2}{5!} \right] = 0 \quad (55)
\]

and

\[
A \left[ 1 + (\alpha^2 - 2\alpha) \frac{(-\lambda)}{2!} + (\alpha^4 - 4\alpha^3) \frac{\lambda^2}{4!} + (\alpha^6 - 6\alpha^5) \frac{(-\lambda^3)}{6!} \right] + B \left[ (\alpha - 1) + (\alpha^3 - 3\alpha^2) \frac{(-\lambda)}{3!} + (\alpha^5 - 5\alpha^4) \frac{\lambda^2}{5!} \right] = 0 \quad (56)
\]
Since the system \((55)-(56)\) has a nontrivial solution for \(A\) and \(B\) the characteristic determinant is zero i.e.

\[
a(\lambda) = \begin{vmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{vmatrix} = 0
\]

where 

\[
a_{11} = 1 + ((\alpha - 1)^2 + 2(\alpha - 1))\frac{(-\lambda)}{2!} + ((\alpha - 1)^4 + 4(\alpha - 1)^3)\frac{\lambda^2}{3!} + ((\alpha - 1)^6 + 6(\alpha - 1)^5)\frac{(-\lambda^3)}{4!},
\]

\[
a_{12} = \alpha + ((\alpha - 1)^3 + 3(\alpha - 1)^2)\frac{(-\lambda)}{3!} + ((\alpha - 1)^5 + 5(\alpha - 1)^4)\frac{\lambda^2}{4!},
\]

\[
a_{21} = 1 + (\alpha^2 - 2\alpha)\frac{(-\lambda)}{2!} + (\alpha^4 - 4\alpha^3)\frac{\lambda^2}{4!} + (\alpha^6 - 6\alpha^5)\frac{(-\lambda^3)}{5!},
\]

\[
a_{22} = (\alpha - 1) + (\alpha^3 - 3\alpha^2)\frac{(-\lambda)}{3!} + (\alpha^5 - 5\alpha^4)\frac{\lambda^2}{5!}.
\]

Taking \(\alpha = \frac{1}{2}\) we have the following algebraic equation for approximate eigenvalues:

\[
-1 - \lambda + \frac{11\lambda^2}{6} - \frac{89\lambda^3}{120} - \frac{299\lambda^4}{15360} - \frac{11\lambda^5}{9830400} = 0 \quad (57)
\]

This equation can be solved by various numerical methods.

5. Analysis of the method

In this study we introduce a new version of classical DTM that will extend the application of the method to spectral analysis of boundary-value problems involving eigenvalue parameter, which arise from problems of mathematical physics. Numerical results reveal that the \(\alpha\)-p DTM is a powerful tool for solving many initial value and boundary value problems. It is concluded that comparing with the standard DTM, the \(\alpha\)-p DTM reduces computational cost in obtaining approximated solutions. This method unlike most numerical techniques provides a closed-form solution. It may be concluded that \(\alpha\)-p DTM is very powerful and efficient in finding approximate for wide classes of boundary value problems. The main advantage of the method is the fact that it provides its user with an analytical approximation, in many cases an exact solution, in a rapidly convergent sequence with elegantly computed terms.

References

[1] A. Abbasov, A. R. Bahadir, The investigation of the transient regimes in the nonlinear systems by the generalized classical method, Math. Prob. Eng. 5 (2005) 503-519.
[2] C. K. Chen, S. H. Ho *Solving partial differential equations by two dimensional differential transform method*, Appl. Math. Comput., 106 (1999), 171-179.

[3] F. Ayaz, *Solutions of the systems of differential equations by differential transform method*, Appl. Math. Comput., 147 (2004), 547-567.

[4] J. D. Cole, *On a quasilinear parabolic equation occurring in aerodynamics*, Quart. Appl. Math., 9 (1951), 225-236.

[5] N. Do˘gan, V. S. Ertürk and Ö. Akın *Numerical Treatment of Singularly Perturbed Two-Point Boundary Value Problems by Using Differential Transformation Method*, Discrete Dynamics in Nature and Society, (2012), doi:10.1155/2012/579431.

[6] V. S. Ertürk and S. Momani *Comparing numerical methods for solving fourth-order boundary value problems*, Appl. Math. Computut., 188(2)(2007), 1963-1968.

[7] M. J. Jang, C. K. Chen, *Two-dimensional differential transformation method for partial differential equations*, Appl. Math. Computut., 121 (2001), 261-270.

[8] F. Kangalgil, F. Ayaz, *Solitary wave solutions for the KDV and mKDV equations by differential transformation method*, Chaos Solitons Fractals, 41 (2009), 464-472.

[9] B. Soltanalizadeh, *Differential transformation method for solving onesspace- dimensional telegraph equation*, Comp. Appl. Math., 30, No. 3 (2011), 639-653.

[10] B. Soltanalizadeh, M. Zarebina, *Numerical analysis of the linear and nonlinear Kuramoto-Sivashinsky equation by using differential transformation method*, Inter. J. Appl. Math. Mechanics, 7, No. 12 (2011), 63-72.

[11] B. Soltanalizadeh, *Application of differential transformation method for numerical analysis of Kawahara equation*, Australian Journal of Basic and Applied Sciences, 5, No. 12 (2011), 490-495.

[12] D. Nazari, S. Shahmorad, *Application of the fractional differential transform method to fractional-order integro-differential equations with non-local boundary conditions*, J. Comput. Appl. Math., 234 (2010), 883-891.
[13] A. Tari, M. Y. Rahimib, S. Shahmoradb, F. Talati, Solving a class of twodimensional linear and nonlinear Volterra integral equations by the differential transform method, J. Comput. Appl. Math., 228 (2009), 70-76.

[14] A. Borhanifar, R. Abazari, Numerical study of nonlinear Schrodinger and coupled Schrodinger equations by differential transformation method, Optics Communications, 283 (2010), 2026-2031.

[15] A. Arikoglu, I. Ozkol, Solution of difference equations by using differential transformation method, Appl. Math. Comput., 174 (2006), 1216-1228.

[16] S. Momani, Z. Odibat, I. Hashim, Algorithms for nonlinear fractional partial differential equations: A selection of numerical methods, Topol. Method Nonlinear Anal., 31 (2008), 211-226.

[17] S. Momani and V. S. Eröfik A numerical scheme for the solution of viscous Cahn- Hilliard equation, Numerical Methods for Partial Differential Equation, 24(2)(2008), 663-669.

[18] J. S. Chiou, J. R. Tzeng, Application of the Taylor transform to nonlinear vibration problems, Journal of Vibration and Acoustics 118(1996),83-87.

[19] J. K. Zhou, Differential transformation and its application for electrical circuits, Huarjung University Press, Wuuhan, China, 1986 (in Chinese).