Dehn’s Lemma for immersed loops

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Suppose $\delta$ is a generic immersed closed curve in the boundary of a 3-manifold $M$ and $\delta$ is null-homotopic in $M$. Then $\delta$ can be displaced by a height function in a collar of the boundary of $M$ so that the resulting simple closed curve in the collar bounds a disk in $M$.

We know five theorems whose conclusion is the existence of an embedded disk, perhaps with additional structure, in some larger space. Each introduced an influential technique and had broad consequences. They are: (1) - 1913 Boundary continuity of the Riemann mapping to Jordan domains (Carathéodory \[1\], Osgood and Taylor \[2\]); Carathéodory’s proof introduced “external length”; applications to quasifuschian groups. (2) - 1944, the Whitney disk (with appropriate normal frame extension) \[3\]; applications to Whitney embedding theorem, h-cobordism theorem. (3) - 1957 Dehn’s Lemma - loop theorem (Papakyriakopoulos \[4\]); correctly treated triple points; applications: hierarchy in Haken manifolds, Thurston’s geometrization theorem. (4) - 1982 Disk embedding theorem (Freedman \[5\]); used decomposition theory to identify Casson handles; application: topological classification of simply-connected 4-manifolds. (5) - Existence of Pseudo-holomorphic disks (Gromov \[6\]); brought Kahler manifold techniques into symplectic context; applications: the nonsqueezing theorem, Seiberg-Witten invariants, quantum cohomology. This paper is a comment on (3); we prove a simply stated extension of Dehn’s Lemma.

Let $M$ be a 3-manifold with boundary $\partial M$, and $\delta : S^1 \hookrightarrow \partial M$ be a generic immersion, where $S^1 := [0, 2\pi]/0 \sim 2\pi$. By generic, we mean that $\delta$ has only simple crossings. We say that $\delta'$ is $\delta$ “displaced by a height function $f$” if $\delta' (\theta) = (\delta (\theta), f (\theta))$, where $f : S^1 \to (0, \epsilon)$ is a Morse function and $[0, \epsilon]$ is a normal collar coordinate on $\partial M$ into $M$. We call a simple closed curve (scc) $\alpha \subset \text{int}(M)$ unknotted iff $\alpha$ bounds an embedded disk $\tilde{\alpha} : D^2 \hookrightarrow \text{int}(M)$.

**Theorem 1.** Let $\delta : S^1 \hookrightarrow \partial M$ be a generic immersed loop so that the composition into $M$ is null homotopic. There is a height function $f : S^1 \to (0, \epsilon)$ so that $\delta' = (\delta, f) : S^1 \hookrightarrow \text{int}(M)$ is unknotted.
The theorem readily implies two familiar facts:

1) Dehn’s Lemma: For any scc \( \delta \subset \partial M \) which is null homotopic in \( M \) there is a properly embedded disk \( (D, \partial D) \subset (M, \partial M) \) with \( \partial D \) parameterizing \( \delta \).

2) Given any knot diagram there is always a way to rechoose the crossings to produce an unknot. (This is the case when \( (M, \partial M) \cong (B^3, \partial B^3) \) is a 3-ball.)

Regarding 1: A short argument connects the special case of the theorem where \( \delta \) is one-to-one to Dehn’s Lemma. Let \( C \) be the annular collar joining \( \delta \) to \( \delta' \) and \( D \) a disk with boundary \( \delta' \). If \( C \) and \( D \) are in general position initially there may be arcs of intersection, but a perturbation of \( D \) near \( \partial D = \delta' \) starting from an outermost arc ensures that \( C \) and \( D \) intersect only in sccs contained in \( \text{int}(C) \) and \( \text{int}(D) \). Let \( \sigma \subset D \) be an innermost circle of intersection bounding a subdisk \( \sigma \subset D. \sigma \) may be paired with either an essential or inessential scc in \( C \). Perform disk exchanges to modify \( C \) until either \( C \cup D \) is an embedded disk or some innermost \( \sigma \) is paired to an essential scc in \( C \). In this case a final cut and glue operation yields an embedded disk with boundary \( \delta \).

Regarding 2: We should note a subtlety. The height function produced in the theorem may, in general, be more complicated than the familiar height function which solves the knot diagram problem (see Figure 1b). For knot diagrams the unknotting function may be taken to be any function with a unique local maximum and unique local minimum.

Actually Theorem 1 is a corollary of a stronger Theorem 0, better adapted to the required induction.

**Theorem 0.** Let \( \delta : S^1 \hookrightarrow \partial M \) be a generic immersed loop with base point \(* \) (\(* \) is assumed disjoint from multiple points of \( \delta \)) whose composition into \( M \) is null homotopic. There is a height function \( f : S^1 \to [0, \epsilon), \ f(*) = 0, \ f(S^1\setminus *) \subset (0, \epsilon), \) so that \( \delta' = \{ (\delta(\theta), f(\theta)) \} \) bounds an embedded disk \( \Delta \subset M \) with \( \Delta \cap \partial M = * \). We say \( \delta' \) collapses to \(* \), and call \( \Delta \) a “lollypop” for \( \delta' \).

**Proof.** No methods post dating Papakyriakopoulos [4] are required. He would have found this proof rather easy to understand and perhaps to generate. First we build a tower.

\( \delta \) is the pointed, immersed loop, \( D \) the general position null homotopy bounding it, and \( N \) the regular neighborhood of \( D \) in \( M \). Subscript will indicate height in the tower.
(a) This example of a loop on the boundary of a genus 2 handlebody shows that the height function \( f \), constructed in the proof of Theorem 0, cannot always have a unique local maximum, the familiar form for unknotting knot diagrams on the 2-sphere. The unknotting function \( f \), relative to the indicated base point \( * \) with \( f(*) = 0 \), must have at least two local maxima.

(b) For this absolute example, the unknotting function \( f \) produced by Theorem 1 must have at least \( m \) local minima (and maxima)

Figure 1

To build the tower we should ask about the \( \mathbb{Z}_2 \) Betti number \( b_1(N) := \text{rank}(H_1(N; \mathbb{Z}_2)) \). If \( b_1(N) = 0 \), there is no tower. For homological reasons \( \partial N \) is a disjoint union of 2-spheres and \( \delta \) is contained in one of these: \( \delta \leftrightarrow S \subset \partial N \).

If \( b_1(N) > 0 \) choose a 2-fold cover \( \tilde{N}_0 \to N_0 := N \) and choose a lift \( l_1 : D \leftrightarrow \tilde{N}_0 \) and let \( N_1 = \text{neib}(l_1(D)) \subset \tilde{N}_0 \). If \( b_1(N_1) = 0 \) then \( N_1 \) is the top of the tower. If \( b_1(N_1) > 0 \) continue and find a 2-fold cover \( \tilde{N}_1 \to N_1 \), and lift \( l_2 : D \leftrightarrow \tilde{N}_1 \) and set \( N_2 = \text{neib}(l_2(D)) \subset \tilde{N}_1 \). Again if \( b_1(N_2) = 0 \) then \( N_2 \) is the top of the tower. If not, proceed to construct \( l_3 \ldots l_n \) and \( N_3 \ldots N_n \) until \( b_1(N_n) = 0 \). A simple complexity argument, where the complexity can
be the number of simplicies identified by $l_k$ for a fixed triangulation of $D$ making all $l_k$ simplicial, shows that the tower is indeed finite.

\[ N_n \subset \tilde{N}_{n-1} \]
\[ N_{n-1} \subset \tilde{N}_{n-2} \]

\[ l_n \]
\[ l_{n-1} \]

\[ l_0 \]
\[ l_1 \]
\[ l_2 \]

\[ N_2 \subset \tilde{N}_1 \]
\[ N_1 \subset \tilde{N}_0 \]
\[ N_0 \subset M \]

Figure 2. The tower

Observation: $\delta_n := l_n(\partial D)$ can be unknotted by a suitable resolution of its crossings (i.e. a normal function $f$ as in Theorem 1). As in Theorem 0, given for any base point $\ast$ (chosen away from crossings) we can resolve crossings $\delta_n \to \delta'_n$ and produce an embedded disk $\Delta$ with $\partial \Delta = \delta'_n$ and $\Delta \cap \partial N_n = \ast$.

**Explanation of $\delta'_n$ and its null-isotopy.** One might expect to choose $\delta'_n = \{\delta_n(\theta), f(\theta)\}, f(\ast) = 0, f(\theta \neq \ast) > 0$, where $f$ has a unique local maximum and a unique local minimum. However, this solution does not, in general, push down the tower. We prefer to give a second solution. There are two cases. If $l_n$ is one-to-one $\delta_n = l_n(\partial D)$ bounds a hemisphere $E_1 \subset S \subset \partial N_n$. Pushing the disk $E_1$ normally toward interior $N_n$ gives a disk $(\Delta, \ast) \subset (N_n, \partial N_n)$ bounding $\delta'_n$.

Now assume $\delta_n$ is not one-to-one. Let $\alpha \subset \delta_n$, not containing $\ast$, be a subarc so that $l_n(\alpha)$ is a scc $\subset \partial N_n$. Let $E_1$ be one of the two disks in $\partial N_n$ bounded by $l_n(\alpha)$. Begin to resolve the crossings of $\delta$ by following this rule: $l_n(\alpha)$ lies above $l_n(\beta)$, where $\beta$ is the complementary arc, $\delta = \alpha \cup \beta$ and $\alpha \cap \beta = \partial \alpha = \partial \beta$. Also the two endpoints of $\alpha$ are at a crossing: resolve this crossing arbitrarily.

Now an isotopy across the disk $E_1$ bounding $\alpha$ effectively erases the loop $\alpha$. If the simplified diagram is a scc we may continue the unknotting using a hemisphere 2-cell $E_2 \subset S, \partial E_2 = l_n(\beta)$ making no further crossing choices.
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If the simplified diagram is still singular choose another arc $\alpha' \subset \beta$ whose image is again a SCC not containing $*$ and proceed as before. Continuing in this way, guided by a sequence of embedded 2-cells, say $\{E_1, \ldots, E_j\}$, all crossings are eventually assigned so that the diagram resolution is unknotted with $*$ remaining in the final 2-cell $E_j$. The cells $E_1, \ldots, E_j$ determine a sequence of isotopies $I_1, \ldots, I_j$ so that the composition $I_j \circ \cdots \circ I_1$ shrinks $\delta'$ toward the base point $*$. Note: the 2-cells $E_i$ will not generally have disjoint interiors. □

This solution will now be pushed down the tower

**Dissection of disks with double arcs:** Let $E$ be a properly embedded disk $(E, \partial E) \subset (M^3, \partial M)$ in a 3-manifold and $\pi : (M, \partial M) \to (P, \partial P)$ a 2-fold cover with covering translation $t$. Assume $E$ and $tE$ are in generic position and meet only in double arcs and double loops. Double loops are easily removed by an innermost circle argument and will not be discussed further. $E$ is assumed to have a base point $* \subset \partial E$. Call $\pi(E) = F$ and $\pi(*) = *$ in an abuse of notation.

We now describe how to form an ordered list of embedded disks $F_1, \ldots, F_k \subset (P, \partial)$ from pieces (some used several times) of $F$. Constructing $\{F_1, \ldots, F_k\}$ will dictate “crossing choices” for $\partial F := \gamma$, which yield $\gamma'$, bounding a lollypop. Each $F_i$ has base point $*_{i}$ with $*_{k} = * \subset \partial F$. Furthermore, once $\{F_1, \ldots, F_k\}$ are constructed we may view them as the instructions for an isotopy $I = I_k \circ \cdots \circ I_1$, as above, collapsing $\gamma'$ toward $*$.

Let $\beta_1$ be an outermost arc cutting off an outermost disk $D_1 \subset F$ not containing $*$. Let $\beta_1'$ be the partner arc of $\beta_1$ and $D_1'$ be the subdisk of $F$
cut off by $\beta'_1$ which also does not contain *. There are two cases, shown in Figure 4.

In both cases $D_1 \cup D'_1$ can be perturbed into a proper map of a disk $F_0$. There is a residual general position proper map with only double arc singularities of a disk formed from $D_1 \cup (F \setminus D'_1)$. In both cases, label this map $F_1$. Because $D_1$ is outermost, both $F_0$ and $F_1$ have fewer double arcs than $F$.

We need to discuss crossing choices (resolutions) and base point choices. The end points on $\partial D'_1$, and its crossings with other segments of $\partial F_1$, are deemed overcrossings (in both cases). $F_0$ is provided a base point $*_0$ in its copy of $D_1$ and $F_1$ retains $*_1 := *$ as its base point.

Now unless $F_0(F_1)$ is embedded find an outermost arc cutting off an outermost disk $D_\bullet$, with $D'_\bullet$ disjoint from $*_0(*_1)$ and sharing a double arc with $D_\bullet$, not containing $*_0(*_1)$ and as above dissect:

\[
\begin{array}{cc}
F_0 & \rightarrow F_{00} & \text{and} & F_1 & \rightarrow F_{10} \\
F_{01} & \rightarrow & F_{00} & \text{and} & F_{11} & \rightarrow & F_{10}
\end{array}
\]

to obtain general position proper pointed maps $F_{00}, F_{01}, F_{10},$ and $F_{11}$ of disks with only double arc singularities. We call such maps good maps. In
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Each case follow the preceding rule for resolving the crossings of $\partial D’_i$ as overcrossings.

Continuing in this way a dyadic tree of good maps is obtained. The leaves of this tree are called great maps as they have the additional property of being embeddings; their imbedded images are called great disks. (No disjointness has been constructed or assumed for these great disks.) The leaves are now linearly ordered by the base 2 numerical value of their subscripts considered as decimals. By the time we reach the leaves all crossings have been resolved. These great disks, monotonically reindexed, become the ordered list \{$F_1, \ldots, F_k$\}, built from pieces of $F$, which we sought. We call this list a great sequence (for $\partial F$) guiding a collapse of ($\partial F’$) to *. Note that the great sequence uniquely defines the crossing resolution ($\partial F’$) of $\partial F$. The simple expedient of successively declaring $\partial D’_i$ to over-cross other segments of the boundary has given us the well-defined crossing choices and thus defines ($\partial F’$). The isotopy $I$ shrinks ($\partial F’$) toward *.

![Figure 5. Sample tree with great maps as leaves](image)

To summarize the Dissection section:

The tree, as constructed above, of good disks terminating in great disks is a dissection of $F$. The same tree, according to our convention, determines crossing choices $\gamma’ = (\partial F’)$ for $\gamma = \partial F$. The great sequence \{$F_1, \ldots, F_k$\} (the leaves of the dissection) are said to guide a sequence of isotopies $I_k \circ \cdots \circ I_1 := I$ with $I_i$ supported near $F_i$, $1 \leq i \leq k$, so that $I$ collapses $\gamma’$ toward its base point.

Example. We illustrate (Figure 6) dissection, the crossing resolutions (for $F_1 \rightarrow F_{01}$ viewed from opposite ends of the double curve, i.e. from the exterior of $M$), and the collapsing isotopy with an example. Our crossing convention implies that middle sheet of the three sheets of the outermost $D$ in $F_{..0} \amalg F_{..1}$ should be compressed slightly into interior ($M$) near $\partial\beta_1$ when interpreting the disk $F_j$, $1 \leq j \leq k$, which contain this sheet of $D$ as guiding the isotopies $I_j$. 


We now finish the proof of Theorem 0.

At the top of the tower the collapse of \(((l_n(\partial D))', \ast)\) is guided by an initial sequence of properly embedded 2-cells \(\{E_1, \ldots, E_j\}\), where the base point of each \(E_1, \ldots, E_j\) is taken to be the self intersection point of its boundary, and with the original \(\ast \subset E_j\) serving as the base point for the final \(E_j\).

Now consider each of these cells \(E_i, 1 \leq i \leq j\), and its singular image \(F_i\) under the 2-fold cover \(\pi_{n-1} : \tilde{N}_{n-1} \to N_{n-1}\). Again we may remove double loops and only need consider double arcs. As above, dissect \(F_i\) into a sequence...
of great disks \( \{F_{i,1}, \ldots, F_{i,k_i}\} \) and replace each \( E_i \) with that ordered sublist to obtain a great sequence:

\[
\{F_{i,1}, \ldots, F_{1,k_1}, \ldots, F_{j,1}, \ldots, F_{j,k_j}\}
\]

which guides the collapse of \( (l_{n-1}(\partial D))' \) to \( * \) (and defines the crossing resolution indicated by the “prime”).

Now proceed all the way down the tower. At height \( m \) by induction assume a great sequence \( \{G_1, \ldots, G_p\} \) for \( (l_m(\partial D))' \). Project each \( G_i \), \( 1 \leq i \leq p \), by \( \pi_{m-1} \) to \( N_{m-1} \) and construct, as above, a great sequence for each \( (\pi_{m-1}(\partial G_i)) \). The great sequence for \( (l_{m-1}(\partial D)) \) is obtained by concatenating these \( p \) great sequences. The great sequence for \( (l_{m-1}(\partial D)) \) is obtained by concatenating these \( p \) great sequences. The great sequence for \( (l_{m-1}(\partial D)) \) determines an isotopy compressing \( (l_{m-1}(\partial D))' \) to its base point. It may be viewed as an ‘elaboration’ of the corresponding null isotopy, one stage higher, for \( (l_m(\partial D))' \), which accounts for the double circles and arcs induced by the projection \( \pi_{m-1} \). Finally, setting \( m = 1 \) we obtain a great sequence guiding the null isotopy which establishes Theorem 0. \( \square \)

**Note:** There is a potentially exponential efficiency in describing a null isotopy as a long composition, as we have done in the proof of Theorem 0, compared with attempting a direct description of the bounding disk. The Fox-Artin-like unknot in Figure 7 makes this clear: any bounding disk must pass \( 2^k \) times over the rightmost protuberance. This motivated our proof strategy.

![Figure 7](image-url)
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