Boundary Stabilization of the linear MGT equation with Feedback Neumann control

Marcelo Bongarti and Irena Lasiecka
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152 USA

December 10, 2020

Abstract

The Jordan–Moore–Gibson–Thompson (JMGT)\cite{12, 20, 44} equation is a benchmark model describing propagation of nonlinear acoustic waves in heterogeneous fluids at rest. This is a third order (in time) dynamics which accounts for a finite speed of propagation of heat signals (see \cite{14, 15, 20, 21, 26}). In this paper, we study a boundary stabilization of linearized version (also known as MGT–equation) in the critical case, configuration in which the smallness of the diffusion effects leads to conservative dynamics \cite{24}. Through a single measurement in feedback form made on a non–nempty, relatively open portion of the boundary under natural geometric conditions, we were able to obtain uniform exponential stability results that are, in addition, uniform with respect to the space–dependent viscoelasticity parameter which no longer needs to be assumed positive and in fact can be degenerate and taken to be zero on the whole domain. This result, of independent interest in the area of boundary stabilization of MGT equations, provides a necessary first step for the study of optimal boundary feedback control on infinite horizon \cite{6}.

keywords: Nonlinear acoustics, second sound, third order in time, MGT equation, heat-conduction, boundary stabilization, degenerate viscoelasticity.

1 Introduction

Let $\Omega \in \mathbb{R}^3$ bounded with $C^2$–boundary denoted by $\Gamma$. With $T > 0$ (could also be $T = \infty$), the third-order (in time) quasilinear JMGT–equation is given by

$$\tau u_{ttt} + (\alpha - 2ku)u_{tt} - c^2 \Delta u - b \Delta u_t = 2ku^2_t \text{ in } (0, T) \times \Omega$$

(1)
where $k > 0$ is a nonlinearity parameter, $c > 0$ denotes the speed of sound, $\tau > 0$ denotes thermal relaxation parameter, $b := \delta + \tau c^2 > 0$ where $\delta$ denotes the sound diffusivity and $\alpha$ refers to viscoelastic friction. The parameter $\tau > 0$ – introduced when the Fourier’s law of heat conduction is replaced my the Maxwell–Cattaneo’s (MC) law – plays an important role: it removes the paradox of infinite speed of propagation in heat waves.

A multitude of applications ranging from acoustics, image processing, thermodynamics, etc. have brought a considerable attention to the dynamics behind the hyperbolic (third–order in time) acoustic wave models. As a consequence, a rich literature on the topic has been developed during the last decade [1, 2, 3, 4, 5, 7, 10, 11, 13, 16, 17, 18, 22, 30, 31, 32, 33, 36, 37, 39, 40, 41]. The interest behind the propagation of waves through viscous fluids and other heterogeneous media, has been already pointed out – almost two centuries ago – by Professor Stokes in his prominent article [43] in 1851. Stokes’ work was rooted on the idea that heat propagation – in particular within acoustic media – was hyperbolic. Later experimental studies indicated the presence of heat waves in such materials thereby dictating the presence of the (nowadays known as) thermal relaxation parameter $\tau > 0$.

There is a strong connection between Stokes–model and the JMGT model, both revealing the basic principles of acoustic waves propagation in the presence of the heat waves. This also led to a creation of other models accounting for a more detailed information about the material and the medium [21, 12, 44]. In these models, standard time derivative may be replaced by material derivative, which depends on the medium. Dependence of the heat flux upon the media, its surroundings and other thermal–material dependent quantities: thermal inertia, specific heat, velocity field, etc. are taken into account in the respective modeling processes. For a broader understanding of hyperbolic heat and modern nonlinear acoustics, we refer to the works [8, 9, 20, 21, 27, 44] and references therein. Recent review of pertinent modeling aspects of acoustic waves can also be found in [20, 22, 34]. In order to focus our work on boundary stabilization and the related technical details, in the present work we consider $\tau > 0$ and fixed. However, other generalizations may be possible and, indeed, welcome.

Typical boundary conditions associated with the model (1) and its linearization are homogenous Dirichlet $u = 0$ imposed on $\Gamma$. Questions such as wellposedness of solutions and their stability were extensively studied [23, 24, 25, 35]. With $k = 0$ equation (1) is linear and reads

$$\tau u_{ttt} + \alpha u_{tt} - c^2 \Delta u - b \Delta u_t = 0 \text{ in } (0, T) \times \Omega$$

(2)

(here we allow $\alpha \in \mathbb{R}$). The structurally damped case ($b > 0$) corresponds to a group generator defined on the phase space $H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$. In the absence of structural damping ($b = 0$), however, semigroup generation fails and the problem is ill-posed, a property dating back to Fattorini [19] in 1969. Nonlinear semigroups corresponding to (1) were shown to exist in the following phase spaces
First with the data assumed to be small at the level of the underlying phase space \([25]\) and later with smallness required only in the lower topology, namely \(H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)\) [2].

The parameter \(\gamma := \alpha - \frac{\tau c^2}{b}\) is critical for stability of solutions: both linear and nonlinear semigroups are exponentially stable provided that \(\gamma > 0\). When \(\gamma = 0\) the linear dynamics is conservative and therefore there is no decay of solutions. This brings us to the goal of the present paper, namely the linear stabilization: how to stabilize the model with critical \(\gamma\)-i.e \(\gamma\) can be degenerate? It is known by now that specifically constructed memory terms may have stabilizing effects on the critical dynamics \([16, 18, 31]\). In this work we are interested how to achieve stabilization in the critical case via a boundary feedback only. It is known, that boundary of the region is accessible to external manipulations, hence a good place for placing actuators and sensors. Within this spirit, we shall show that a suitable boundary feedback implemented on the boundary \(\Gamma\) will lead to exponential stability of the resulting semigroup. This result is important not only on its own rights, but also within the context of the quasilinear equations where questions of stability are strongly linked to decay properties of linearized solutions \([25]\).

### 1.1 The linearized PDE model with space–dependent viscoelasticity

Let the domain \(\Omega \subset \mathbb{R}^n (n = 2, 3)\) with the boundary \(\Gamma = \partial \Omega\) be of class \(C^2\). Let \(\Gamma\) be divided into two disjoint parts, \(\Gamma_0\) and \(\Gamma_1\), both non-empty, with \(\Gamma_1\) relatively open in \(\Gamma\). We consider the linear version of (1) but with a space–dependent coefficient \(\alpha \in C(\overline{\Omega})\)

\[
\tau u_{ttt} + \alpha(x)u_{tt} - \epsilon^2 \Delta u - b \Delta u_t = 0 \quad \text{in} \quad (0, T) \times \Omega
\]

and subject to the Robin-type (on \(\Gamma_0\)) and Dirichlet (on \(\Gamma_1\)) boundary conditions

\[
\frac{\partial u}{\partial v} + \eta u_t = 0 \quad \text{on} \quad (0, T) \times \Gamma_0 \quad \text{and} \quad u = 0 \quad \text{on} \quad (0, T) \times \Gamma_1
\]

where \(\eta > 0\), and the initial conditions are given by

\[
u(0) = u_0, u_t(0) = u_1, u_{tt}(0) = u_2.
\]

In addition to continuity, we assume that the damping coefficient \(\alpha(x) > 0\) is such that stability parameter \(\gamma \in C(\overline{\Omega})\) satisfies

\[
\gamma(x) := \alpha(x) - \frac{\tau c^2}{b} \geq 0.
\]

Remark 1.1. The above assumption will be used only for the stability estimates. The generation of a semigroup is valid without assuming (6).
Since the damping typically depends on local properties of the material, assuming variability of the damping $a(x)$ is physically relevant and, in most cases, necessary. We shall show that under suitable geometric conditions imposed on $\Omega$ the linear system is exponentially stable in the topology of the natural phase space.

**Notation:** Throughout this paper, $L^2(\Omega)$ denotes the space of Lebesgue measurable functions whose squares are integrable and $H^s(\Omega)$ denotes the $L^2(\Omega)$-based Sobolev space of order $s$. Moreover, we use the notation $H^1_1(\Omega)$ to represent the space

$$H^1_1(\Omega) := \{ u \in H^1(\Omega); u|_\Gamma = 0 \} \quad (7)$$

instead of the standard $H^1(\Omega)$ in order to emphasize the portion of the boundary on what the trace is vanishing. We also denote by $H^2_1(\Omega)$ the space $H^2(\Omega) \cap H^1_1(\Omega)$. We denote the inner product in $L^2(\Omega)$ and $L^2(\Gamma)$ respectively by

$$(u, v) = \int_\Omega uv d\Omega \quad \text{and} \quad (u, v)_\Gamma = \int_\Gamma uv d\Gamma$$

and the respective induced norms in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $\| \cdot \|_2$ and $\| \cdot \|_\Gamma$ respectively.

### 1.2 Main results and discussion

We begin with the abstract version of equation (3). To this end, let $A : \mathcal{D}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the operator defined as

$$A \xi = -\Delta \xi, \quad \mathcal{D}(A) = \left\{ \xi \in H^2(\Omega); \frac{\partial \xi}{\partial n}|_{\Gamma_1} = 0 \right\} . \quad (8)$$

It is well known that $A$ is a positive ($\Gamma_1 \neq 0$), self-adjoint operator with compact resolvent and that $\mathcal{D} \left( A^{1/2} \right) = H^1_{\Gamma_1}(\Omega)$ (equivalent norms). In addition, up to a bit of abuse of notation we denote (also) by $A : L^2(\Omega) \rightarrow [\mathcal{D}(A)]'$ the extension (by duality) of the operator $A : \mathcal{D}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ defined in (8).

Next, for $\varphi \in L^2(\Gamma_0)$, let $\psi = N(\varphi)$ be the unique solution of the elliptic problem

$$\left\{ \begin{array}{ll}
\Delta \psi = 0 & \text{in } \Omega \\
\frac{\partial \psi}{\partial n} = \varphi & \text{on } \Gamma_0 \\
\psi = 0 & \text{on } \Gamma_1.
\end{array} \right.$$
It follows from elliptic theory that $N \in \mathcal{L}(H^s(\Gamma_0), H^{s+3/2}(\Omega))$ \footnote{$\mathcal{L}(X, Y)$ denote the space of linear bounded operators from $X$ to $Y$} ($s \in \mathbb{R}_+^*$) and

$$N^* A \xi = \begin{cases} \xi & \text{on } \Gamma_0 \\ 0 & \text{on } \Gamma_1 \end{cases}$$  \hspace{1cm} (9)

for all $\xi \in \mathcal{D}(A)$, where $N^*$ represents the adjoint of $N$ when the latter is considered as an operator from $L_2(\Gamma_0)$ to $L_2(\Omega)$.

The introduction of $A$ and $N$ will allow us to write equation (3) abstractly as

$$\tau u_{ttt} + \alpha(x)u_{tt} + c^2 A(u + \eta NN^* A u_t) + b A(u_t + \eta NN^* A u_{tt}) = f. \hspace{1cm} (10)$$

The abstract version of our model gives rise to the natural phase space we are going to consider. We define $\mathbb{H}$ as

$$\mathbb{H} := H^1_{\Gamma_1}(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \hspace{1cm} (11)$$

The computations leading to (10) show formally that $u$ is a solution of (3) with boundary condition given by (4) if and only if $\Phi = (u, u_t, u_{tt})^\top$ is a solution for the first other system

$$\begin{cases} \Phi_t = \mathcal{A} \Phi + F \\ \Phi(0) = \Phi_0 = (u_0, u_1, u_2)^\top, \end{cases} \hspace{1cm} (12)$$

with $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H} \rightarrow \mathbb{H}$ given by

$$\mathcal{A}(\xi_1, \xi_2, \xi_3)^\top := (\xi_2, \xi_3, -\alpha \tau^{-1} \xi_3 - c^2 \tau^{-1} A(\xi_1 + \eta NN^* A \xi_2) - b \tau^{-1} A(\xi_2 + \eta NN^* A \xi_3)) \hspace{1cm} (13)$$

where

$$\mathcal{D}(\mathcal{A}) = \left\{ (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{H}; \xi_3 \in \mathcal{D} \left( A^{1/2} \right), \xi_1 + \eta NN^* A \xi_2 \in \mathcal{D}(A), \xi_2 + \eta NN^* A \xi_3 \in \mathcal{D}(A) \right\}$$

$$= \left\{ (\xi_1, \xi_2, \xi_3)^\top \in \left[ H^2_{\Gamma_1}(\Omega) \right]^2 \times \mathcal{D}(A^{1/2}); \left[ \frac{\partial \xi_1}{\partial \nu} + \eta \xi_2 \right]_{\Gamma_0} = \left[ \frac{\partial \xi_2}{\partial \nu} + \eta \xi_3 \right]_{\Gamma_0} = 0 \right\} \hspace{1cm} (14)$$

and $F^\top = (0, 0, f)$.

We now recall that, topologically, the space $\mathbb{H}$ is equivalent to

$$\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega) \hspace{1cm} (15)$$

with the topology induced by the inner product

$$\langle (\xi_1, \xi_2, \xi_3)^\top, (\varphi_1, \varphi_2, \varphi_3) \rangle_{\mathbb{H}} = (A^{1/2} \xi_1, A^{1/2} \varphi_1) + b (A^{1/2} \xi_2, A^{1/2} \varphi_2) + (\xi_3, \varphi_3), \hspace{1cm} (16)$$

for all $(\xi_1, \xi_2, \xi_3)^\top, (\varphi_1, \varphi_2, \varphi_3)^\top \in \mathbb{H}$. Because of this equivalence we will be using the same $\mathbb{H}$ to denote both spaces.

We are then in position to state our wellposedness of the phase space solutions.
Theorem 1.2. Assume \( f \in L^1((0,T), L^2(\Omega)) \), and \( \eta \geq 0, \alpha \in C(\Omega) \). For every initial data \( \Phi_0 := (u_0, u_1, u_2) \) in \( \mathbb{H} \), there exists a unique semigroup solution \( \Phi = (u, u_t, u_{tt}) \) such that \( \Phi \in C([0, T], \mathbb{H}) \) for every \( T > 0 \). Moreover, if the initial datum belongs to \( \mathcal{D}(\mathcal{A}) \) and \( f \in C^1([0, T], L^2(\Omega)) \) the corresponding solution is in \( C((0, T]; D(\mathcal{A})) \cap C^1([0, T], \mathbb{H}) \).

Our main results is exponential stability of the solutions referred to in Theorem 1.2. In order to formulate the result we need to impose geometric condition:

Assumption 1.3. 1. we choose a point \( x_0 \in \mathbb{R}^n \) outside of \( \overline{\Omega} \). and we define the vector field \( h : \mathbb{R}^n \to \mathbb{R}^n \) given by \( h(x) = x - x_0 \). With \( \nu(x) \) denoting the outwards normal unit vector at \( x \) we define \( \Gamma_0, \Gamma_1 \subseteq \Gamma \) by

\[
\Gamma_0 = \{ x \in \Gamma; \nu(x) \cdot h(x) > 0 \}, \quad \Gamma_1 = \{ x \in \Gamma; \nu(x) \cdot h(x) \leq 0 \}.
\]

Remark 1.4. The geometric condition imposed above can be substantially relaxed. For instance, if the feedback control is active on the full \( \Gamma \), there is no need for any geometric conditions. However, this would require some microlocal analysis and becomes rather technical \([28, 29, 45]\). For this reason we opted for a more restrictive geometry, as to make the exposition fully independent.

Theorem 1.5. Let Assumption 1.3 and condition (6) be satisfied. Then, there exist \( \omega > 0 \) and \( M > 0 \) such that

\[
\|\Phi(t)\|_\mathbb{H} \leq M e^{-\omega t} \|\Phi_0\|_\mathbb{H}
\]

for all \( t \geq 0 \).

As mentioned earlier, the wellposedness of the MGT equation with homogenous boundary conditions is well known by now \([24, 25, 35]\). However, in the case of non-homogeneous boundary conditions, the situation is much more complicated due to the fact that "wave" operator with Neumann boundary data does not satisfy the Lopatinski condition \([42, 45]\), unless \( \Omega \) is one dimensional. This leads to a loss of 1/3 derivative when looking at the map from the boundary with \( L^2 \) data into the \( H^1 \times L^2 \) solutions. This has been known for some time in the case of wave equation, but only recently studied for MGT equation \([5, 7, 46]\). In fact, an open loop control problem for MGT equation with \( L^2 \)-Neumann controls leads to only distributional solutions \([6]\). However, for both the wave and MGT equations, a boundary feedback via Neumann boundary conditions does recover this loss leading to the recuperation of the full energy. This is due to the boundary dissipativity with \( \eta \geq 0 \). The main mathematical issue in dealing with the nonhomogenous Neumann boundary data is to deal with unbounded and uncloseable perturbations within the context of the third order hyperbolic dynamics. In fact, this is the first result on wellposedness of feedback generator for MGT dynamics. Also, notice that applying feedback \( \frac{\partial^2}{\partial t^2} + \eta u_t = 0 \), with \( \eta < 0 \) leads to the ill-posed dynamics. This corresponds to
anti–damping which shifts the spectrum to a “wrong” complex half–plane—thus denying wellposedness of a semigroup. While the analog of stability result in Theorem 1.5 is known for the case of wave equation, this result is new for the MGT ($\tau > 0$) with critical stability parameter $\gamma$. The difficulties encountered in the proof of wellposedness in Theorem 1.2 are compounded, when proving Theorem 1.5, by geometric considerations necessary when studying dissipation with restricted geometric support (such as portion of the boundary). Geometric condition assumed in Assumption (1.3) can be, however, substantially relaxed. For instance, when the dissipation is active on the full boundary there is no need for any geometric constraints \[28, 29\]. However, this brings forward microlocal analysis arguments which are known by now, however tedious. In order to ease readability and focus of the analysis, we have opted for a more restrictive version of geometric assumption.

It should be noted that the result of Theorem 1.5 is critical when studying optimal boundary feedback control problem for MGT equation. While the feedback synthesis for this model has been carried out in \[6\] for a finite horizon problem, analysis of infinite horizon problem requires stabilizability condition, which is provided by Theorem 1.5.

2 Wellposedness: proof of Theorem 1.2

The thermal relaxation parameter $\tau > 0$ plays no significant role in the study of the wellposedness of (3), therefore for the sake of readability we are assuming $\tau = 1$ in this section.

The main goal of this section is to prove that the operator $A : D(A) \subset H \rightarrow H$ generates a strongly continuous semigroup. It is convenient to introduce the following change of variables $bz = bu_1 + c^2u$ (see [35]) which reduces the problem to a PDE–abstract ODE coupled system.

Let $M \in L(H)$ defined by

$$M(\xi_1, \xi_2, \xi_3)^T = \left( \xi_1, \xi_2 + \frac{c^2}{b}\xi_1, \xi_3 + \frac{c^2}{b}\xi_2 \right)$$

which has inverse $M^{-1} \in L(H)$ given by

$$M^{-1}(\xi_1, \xi_2, \xi_3)^T = \left( \xi_1, \xi_2 - \frac{c^2}{b}\xi_1, \xi_3 - \frac{c^2}{b}\xi_2 + \frac{c^4}{b^2}\xi_1 \right)$$

and therefore is an isomorphism of $H$. The next lemma makes precise the translation of (3) to the system involving $z$.

Lemma 2.1. Assume that the compatibility condition

$$\frac{\partial}{\partial v}u_0 + \eta u_1 = 0 \text{ on } \Gamma_0$$

(18)
holds. Then \( \Phi \in C^1(0, T; \mathbb{H}) \cap C(0, T; \mathcal{D}(\mathcal{A})) \) is a strong solution for (12) if, and only if, \( \Psi = M\Phi \in C^1(0, T; \mathbb{H}) \cap C(0, T; \mathcal{D}(\mathcal{A})) \) is a strong solution for

\[
\begin{cases}
    \Psi_t = A\Psi + G \\
    \Psi(0) = \Psi_0 = M\Phi_0 = \left( u_0, u_1 + \frac{c^2}{b}u_0, u_2 + \frac{c^2}{b}u_1 \right)^T,
\end{cases}
\]

(19)

where \( G = M\mathcal{F} \) and \( A = M\mathcal{A}M^{-1} \) with

\[
\mathcal{D}(A) = \left\{ (\xi_1, \xi_2, \xi_3)^\top \in \left[ H^2_{Γ_1}(Ω) \right]^2 \times \mathcal{D} \left( A^{1/2} \right); \left[ \frac{\partial \xi_2}{\partial ν} + η\xi_3 \right]_{Γ_0} = 0 \right\}
\]

(20)

Proof. It is simple to check that if \( \Phi \in C^1(0, T; \mathbb{H}) \cap C(0, T; \mathcal{D}(\mathcal{A})) \) is a strong solution for (12) then \( \Psi = M\Phi \) belongs to \( C^1(0, T; \mathbb{H}) \cap C(0, T; \mathcal{D}(\mathcal{A})) \) and satisfy (19).

For the reverse implication, the only non-trivial step is to prove that boundary conditions match. To this end, assume that \( \Psi = (u, z, z_t) \in C^1(0, T; \mathbb{H}) \cap C(0, T; \mathcal{D}(\mathcal{A})) \) is a strong solution for (19). Let

\[
Ω(t) := \left( \frac{\partial u(t)}{\partial ν} + ηu_t(t) \right)_{Γ_0}, \quad t \geq 0
\]

and notice that \( bY_t + c^2Y = 0 \) for all \( t \). This along with the compatibility condition (\( Y(0) = 0 \)) implies that \( Y \equiv 0 \) hence \( (u, u_t, u_{tt}) \in \mathcal{D}(\mathcal{A}) \) for all \( t \). The proof is then complete.

\[\Box\]

For \( (\xi_1, \xi_2, \xi_3)^\top \in \mathcal{D}(A) \) a basic algebraic computation yields the explicit formula for \( A \).

\[
A(\xi_1, \xi_2, \xi_3)^\top = \left( ξ_2 - \frac{c^2}{b}ξ_1, -γ(ξ_3 - \frac{c^2}{b}ξ_2 + \frac{c^4}{b^2}ξ_1) - bAξ_2 - bηANN^*Aξ_3 \right).
\]

(21)

We are ready for our generation result.

**Theorem 2.2.** The operator \( \mathcal{A} \) generates a strongly continuous semigroup on \( \mathbb{H} \).

**Proof.** Equivalently, we show that \( A \) generates a strongly continuous semigroup on \( \mathbb{H} \). If \( \{S(t)\}_{t \geq 0} \) is the said semigroup then \( \{T(t)\}_{t \geq 0} \) \( (T(t) := M^{-1}S(t)M, t \geq 0) \) will be the semigroup generated by \( \mathcal{A} \).

Write \( A = A_d + P \) where

\[
P(\xi_1, \xi_2, \xi_3) = \left( ξ_2, 0, \frac{γc^2}{b}ξ_2 - (1 - γ)ξ_3 \right), \quad (\xi_1, \xi_2, \xi_3)^\top \in \mathbb{H}
\]

is bounded in \( \mathbb{H} \) and

\[
A_d(\xi_1, \xi_2, \xi_3) = \left( -\frac{c^2}{b}ξ_1, -ξ_3 - bAξ_2 - bηANN^*Aξ_3 \right), \quad (\xi_1, \xi_2, \xi_3)^\top \in \mathcal{D}(A_d),
\]

(22)
where \( \mathcal{D}(A_d) := \mathcal{D}(A) \). It then suffices to prove generation of \( A_d \) on \( H \).

We start by showing dissipativity: for \((\xi_1, \xi_2, \xi_3)^T \in \mathcal{D}(A)\) we have

\[
(A_d(\xi_1, \xi_2, \xi_3), (\xi_1, \xi_2, \xi_3))_H = -\frac{c^2}{b} \|A^{1/2} \xi_1\|^2 + b \left( A^{1/2} \xi_3, A^{1/2} \xi_2 \right) - \|\xi_3\|^2 - b(A^{1/2} \xi_2, A^{1/2} \xi_3) - b \eta \|\xi_3\|^2_{r_0}.
\]

hence, \( A_d \) is dissipative in \( \mathbb{H} \).

For maximality in \( \mathbb{H} \), given any \( L = (f, g, h) \in \mathbb{H} \) we need to show that there exists \( \Psi = (\xi_1, \xi_2, \xi_3)^T \in \mathcal{D}(A) \) such that \((\lambda - A_d)\Psi = L\), for some \( \lambda > 0 \). This leads to the system of equations:

\[
\begin{cases}
\lambda \xi_1 + \frac{c^2}{b} \xi_1 = f, \\
\lambda \xi_2 - \xi_3 = g, \\
\lambda \xi_3 + \xi_3 + bA[\xi_2 + \eta NN^*A\xi_3] = h,
\end{cases}
\]

which implies \( \xi_1 = \left( \lambda + \frac{c^2}{b} \right)^{-1} f \in \mathcal{D}(A^{1/2}) \). Moreover, since \( A^{-1} \in \mathcal{L}(L^2(\Omega)) \) the third equation above yields

\[
[(\lambda^2 + \lambda)A^{-1} + b + b \lambda \eta NN^*A] \xi_3 = \lambda A^{-1} h - bg
\]

and then by the strictly positivity (in \( \mathcal{D}(A^{1/2}) \)) of the operator \( K_\lambda \) defined as

\[
K_\lambda := (\lambda^2 + \lambda)A^{-1} + b + b \lambda \eta NN^*A
\]

(in fact recall \((NN^*A\xi, \xi)_{\mathcal{D}(A^{1/2})} = \|N^*A\xi\|^2_{r_0}\) for all \( \xi \in \mathcal{D}(A^{1/2}) \) therefore \( K_\lambda^{-1} \in \mathcal{L}(\mathcal{D}(A^{1/2})) \)) we can write \( \xi_3 = K_\lambda^{-1}(\lambda A^{-1} h - bg) \in \mathcal{D}(A^{1/2}) \). Finally, \( \xi_2 = \lambda^{-1}(\xi_3 + g) = \lambda^{-1}(K_\lambda^{-1}(\lambda A^{-1} h - bg) + g) \in \mathcal{D}(A^{1/2}) \).

The final step for concluding membership of \((\xi_1, \xi_2, \xi_3)\) in \( \mathcal{D}(A) \) follows from

\[
b\lambda(\xi_2 + \eta NN^*A\xi_3) = -(\lambda^2 + \lambda)A^{-1} \xi_3 + \lambda A^{-1} h \in \mathcal{D}(A).
\]

Generation is then achieved. \( \square \)

Applying standard semigroup arguments \([38]\) to the result of Theorem 2.2, we obtain the following corollary, which in turn completed the proof of Theorem 1.2.

**Corollary 2.3** (Wellposedness and Regularity). Assume \( f \in L^1(\mathbb{R}_+, L^2(\Omega)) \) and that condition (18) is at force. Denote by \( \{T(t)\}_{t \geq 0} \) the semigroup given by Theorem 2.2.

(i) If \( \Phi_0 \in \mathbb{H} \), then the function \( \Phi \in C([0, T]; \mathbb{H}) \) defined as

\[
\Phi(t) \equiv T(t)\Phi_0 + \int_0^t T(t - \sigma)F(\sigma)d\sigma, \quad t \in [0, T]
\]
is the unique mild solution for (12) in \( \mathbb{H} \).

(ii) If \( \Phi_0 \in \mathcal{D}(\mathcal{A}) \) and, in addition, \( f \in C^1(\mathbb{R}_+, L^2(\Omega)) \) then the function \( \Phi \in C^1([0, T]; \mathbb{H}) \cap C((0, T); \mathcal{D}(\mathcal{A})) \) defined as

\[
\Phi(t) \equiv T(t)\Phi_0 + \int_0^t T(t-\sigma)F(\sigma)d\sigma \quad t \in [0, T]
\]
is the unique classical solution for (12) in \( \mathbb{H} \).

**Remark 2.4.** We notice here that condition (18) is not essential for well-posedness of weak solutions. However, it is critical for the regularity of solutions which allows to interpret mild (semigroup) solution in a strong form of equation (3).

### 2.1 Stabilization in \( \mathbb{H} \): proof of Theorem 1.5

In order to allow future tracking stability rates with respect to the parameter \( \tau \) we now resume considering \( \tau > 0 \). Moreover, we will use the notation \( a \leq b \) to say that \( a \leq C b \) where \( C \) is a constant possibly depending on the physical parameters of the model \( (\tau, c, b > 0) \) but independent of space, time and \( \gamma \in C(\overline{\Omega}) \).

We will introduce several energy functionals which will be used to describe long time behavior of mild solutions to (3). For a classical solution \((u, u_t, u_{tt})\) of (12) in \( \mathbb{H} \), we define the corresponding energy by \( E(t) = E_0(t) + E_1(t) \) where \( E_i : [0, T] \rightarrow \mathbb{R}_+ \ (i = 0, 1) \) are defined by

\[
E_1(t) := \frac{b}{2} \left\| A^{1/2} \left( u_t + \frac{\tau c^2}{b^2} u \right) \right\|_2^2 + \frac{\tau}{2} \left\| u_{tt} + \frac{c^2}{b} u_t \right\|_2^2 + \frac{c^2}{2b} \left\| \gamma^{1/2} u_t \right\|_2^2
\]

and

\[
E_0(t) := \frac{1}{2} \left\| A^{1/2} u_t \right\|_2^2 + \frac{c^2}{2} \left\| A^{1/2} u \right\|_2^2
\]

where we have ommited the variable \( x \) in \( \gamma(x) \) and

where we have ommited \( x \) in \( a(x) \).

The next lemma guarantees that stability of solutions in \( \mathbb{H} \) is equivalent to uniform exponential decay of the function \( t \mapsto E(t) \).

**Lemma 2.5.** Let \( \Phi = (u, u_t, u_{tt}) \) be a weak solution for (12) in \( \mathbb{H} \) and assume condition (18) holds. Then the following statements are equivalent:

a) \( t \mapsto \|\Phi(t)\|^2_{\mathbb{H}} \) decays exponentially.

b) \( t \mapsto \|M\Phi(t)\|^2_{\mathbb{H}} = \|(u, z, z_t)\|^2_{\mathbb{H}} \) decays exponentially.
c) $t \mapsto E(t)$ decays exponentially.

Proof. Observe that $\Sigma : \mathbb{H} \to \mathbb{R}_+$ defined as

$$
\Sigma((\xi_1, \xi_2, \xi_3)^T) = \frac{c^2}{2} ||A^{1/2}\xi_1||_2^2 + \frac{1}{2} \left\| \alpha^{1/2} \left( \xi_2 - \frac{c^2}{b} \xi_1 \right) \right\|_2^2 + \frac{c^2}{2b} \left\| \gamma^{1/2} \left( \xi_2 - \frac{c^2}{b} \xi_1 \right) \right\|_2^2 + \frac{b}{2} ||A^{1/2}\xi_2||_2^2 + \frac{T}{2} ||\xi_3||_2^2
$$

is such that $\| : \mathbb{H} \sim \Sigma(\cdot)$. The proof follows by noticing that $E(t) = \Sigma(M\Phi(t))$.

Remark 2.6. The equivalence in Lemma 2.5 is uniform with respect to $\gamma \in C(\overline{\Omega})$.

The next proposition provides the set of main identities for the linear stabilization in $\mathbb{H}$.

Proposition 2.7. If $(u, z, z_t)$ is a classical solution of (19) then for all $0 \leq s < t \leq T$ the following identities hold

$$
E_1(t) + b\eta \int_s^t ||z_t||_{I_0}^2 d\sigma + \int_s^t ||\gamma^{1/2}u_{tt}\|_2^2 d\sigma = E_1(s) + \frac{c^2}{b} \int_s^t (f, \gamma u_{tt}) d\sigma
$$

(26)

$$
\int_s^t \left[ b||A^{1/2}z||_2^2 - ||z_t||_2^2 \right] d\sigma = - \int_s^t (\gamma u_{tt}, z) d\sigma - \left[ (z_t, z) + \frac{b\eta}{2} ||z||_{L_0}^2 \right]_s^t + \int_s^t (f, z) d\sigma
$$

(27)

$$
\int_s^t \left[ \frac{n}{2} ||z_t||_2^2 - \frac{b(n-2)}{2} ||A^{1/2}z||_2^2 \right] d\sigma = (b + 1) \int_s^t \int_{\Gamma_0} |z_t|^2 hvd\Gamma_0 d\sigma - \int_s^t (\gamma u_{tt}, h\nabla z) d\sigma
$$

$$
- (z_t, h\nabla z) |_s^t + \int_s^t (f, h\nabla z) d\sigma
$$

(28)

where $h$ is the vector field defined as $h(x) = x - x_0$, see Assumption 1.3.

Proof. 1. Proof of (26): Let, on $\mathbb{H}$, the bilinear form $\langle \cdot, \cdot \rangle$ be given by

$$
\langle (\xi_1, \xi_2, \xi_3)^T, (\varphi_1, \varphi_2, \varphi_3) \rangle = b \left( A^{1/2} \left( \xi_2 + \frac{c^2}{b} \xi_1 \right), A^{1/2} \left( \varphi_2 + \frac{c^2}{b} \varphi_1 \right) \right)
$$

$$
+ \left( \xi_3 + \frac{c^2}{b} \xi_2, \varphi_3 + \frac{c^2}{b} \varphi_2 \right) + \frac{c^2}{b} (\xi_2, \varphi_2),
$$

(29)

which is clearly continuous. Moreover, recalling that $\Phi(t) = (u(t), u_t(t), u_{tt}(t))$ it follows that $2E_1(t) = \langle \Phi(t), \Phi(t) \rangle$ (see (24)) therefore – mostly omitting the parameters $t \in [0, T]$ and $x$ in $\gamma(x)$ we obtain

$$
\frac{dE_1(t)}{dt} = \left( \frac{d\Phi(t)}{dt}, \Phi(t) \right) = \langle \partial\Phi(t) + F, \Phi(t) \rangle
$$

$$
= \langle (u_t, u_{tt}, -\alpha u_{tt} - c^2 A(u - \eta NN^*Au_t) - bA(u_t - \eta NN^*Au_{tt}), (u, u_t, u_{tt})) + \frac{c^2}{b} (f, \gamma u_{tt})
$$

$$
\langle b A^{1/2}z_t, A^{1/2}z \rangle + (-\gamma u_{tt} - c^2 A(u + NN^*Au_t) - bA(u_t + NN^*Au_{tt}), z_t)
$$

\text{(s)}
$$
= b \left( A^{1/2}z_t, A^{1/2}z \right) + (-\gamma u_{tt} - c^2 A(u + NN^*Au_t) - bA(u_t + NN^*Au_{tt}), z_t)$$

11
\[
+ \frac{c^2}{b}(\gamma u_{tt}, u_t) + \frac{c^2}{b}(f, \gamma u_{tt}) = -\|\gamma^{1/2}u_{tt}\|_2^2 - b\eta\|z_t\|_{I_0}^2 + \frac{c^2}{b}(f, \gamma u_{tt}),
\]

(30)

where, in (\textasteriskcentered) above we have used the definition of \(\langle \cdot, \cdot \rangle\) along with \(bz(t) = u_t(t) + c^2u(t)\) from what it follows that \(-\alpha u_{tt}(t) + c^2\nu u_{tt}(t) = -\gamma u_{tt}(t)\) and we directly computed

\[
(bA(z + \eta NN^*Az_t), z_t) = b(A^{1/2}(z + \eta NN^*Az_t), A^{1/2}z_t)
\]

\[
= b(A^{1/2}z, A^{1/2}z_t) + b\eta\|N^*Az_t\|_{L^2(\Gamma)}^2
\]

\[
= b(A^{1/2}z, A^{1/2}z_t) + b\eta\|z_t\|_{L^2(\Gamma)}^2
\]

(31)

by using self-adjointness of \(A\) and the characterization (9). The identity (26) then follows by integration on \((s, t)\) for \(0 \leq s < t \leq T\).

2. **Proof of (27):** Observe that taking the \(L^2\)-inner product of

\[
z_{tt} + bA(z + \eta NN^*Az_t) = -\gamma u_{tt} + f
\]

with \(z\) we have, for the left hand side:

\[
(z_{tt} + bA(z + \eta NN^*Az_t), z) = \frac{d}{dt}(z_t, z) - \|z_t\|_2^2 + b\|A^{1/2}z\|_2^2 + b\eta \frac{d}{dt}\|z_t\|_{I_0}^2
\]

while for the right hand side we have

\[
(-\gamma u_{tt} + f, z) = (-\gamma u_{tt}, z) + (f, z).
\]

Then, putting right and left hand sides together and integrating on \((s, t)\) for \(0 \leq s < t \leq T\) yields (27).

3. **Proof of (28):** From divergence theorem, recall that for a vector field \(h\) and a function \(\varphi : \mathbb{R}^n \rightarrow \mathbb{R}\) we have that

\[
\int_{\Omega} h \nabla \varphi d\Omega = \int_{\Gamma} \varphi h \cdot v d\Gamma - \int_{\Omega} \varphi \text{div}(h) d\Omega.
\]

(32)

Considering \(h(x) = x - x_0, x \in \overline{\Omega}\) we go back to the original (non-abstract) \(z\)-equation

\[
z_{tt} - b\Delta z = -\gamma u_{tt} + f
\]

(33)

- with boundary conditions \(\frac{\partial z}{\partial \nu} + \eta z_t = 0\) on \(\Gamma_0\) and \(z = 0\) on \(\Gamma_1\) – and multiply it by \(h\nabla z\). We next analyze each of the involved terms in the resulting expression, omitting the variable \(t\) in most steps. For the first term we obtain

\[
(z_{tt}, h\nabla z) = \frac{d}{dt}(z_t, h\nabla z) - (z_t, h\nabla z_t)
\]
which, by chain rule $\nabla(\theta^2) = 2\theta\nabla\theta$ can be rewritten as

$$ (z_{tt}, h\nabla z) = \frac{d}{dt} (z_t, h\nabla z) - \frac{1}{2} \int_{\Omega} h\nabla(z_t^2(t)) d\Omega $$

and then we can apply the Divergence Theorem (with $\varphi = z_t^2$). For this we notice that since $h = x - x_0 = (x_1 - x_{01}, \cdots, x_n - x_{0n})$, we have

$$ \text{div}(h) = \sum_{k=0}^{n} \frac{\partial(x_i - x_{0i})}{\partial x_i} = n. $$

Hence recalling that $z_t(t) = 0$ on $\Gamma_1$ for all $t$, we can further rewrite (34) as

$$ (z_{tt}, h\nabla z) = \frac{d}{dt} (z_t, h\nabla z) - \frac{1}{2} \int_{\Gamma_0} z_t^2 h\nu d\Gamma_0 + \frac{n}{2} ||z_t||_2^2. $$

(35)

Moving to the next term, we first apply Green’s first Theorem to get

$$ (\Delta z, h\nabla z) = -(\nabla z, \nabla (h\nabla z)) + \left( h\nabla z, \frac{\partial z}{\partial v} \right)_{\Gamma_0} $$

(36)

and then recalling the product rule for gradients along with the fact that the Jacobian Matrix of $h$ is the identity we have

$$ \nabla z\nabla (h\nabla z) = \nabla z(h\nabla (\nabla z) + Jh\nabla z) $$

$$ = h\nabla z\nabla (\nabla z) + Jh||\nabla z||^2 = \frac{h}{2} \nabla(||\nabla z||^2) + ||\nabla z||^2 $$

which then allows us to rewrite (36) as

$$ (\Delta z, h\nabla z) = -\frac{1}{2} \int_{\Omega} h\nabla(||\nabla z||^2) d\Omega - ||\nabla z||_2^2 + \left( h\nabla z, \frac{\partial z}{\partial v} \right)_{\Gamma_0}, $$

(37)

and then again application of divergence theorem (with $\varphi = ||\nabla z||^2$) gives

$$ (\Delta z, h\nabla z) = -\frac{1}{2} \int_{\Gamma_0} ||\nabla z||^2 h\nu d\Gamma_0 + \left( \frac{n}{2} - 1 \right) ||\nabla z||_2^2 + \left( h\nabla z, \frac{\partial z}{\partial v} \right)_{\Gamma_0}, $$

(38)

and finally recalling the definition of normal derivative $\left( \frac{\partial z}{\partial v} = \nabla \cdot v \right)$ we get

$$ (\Delta z, h\nabla z) = \frac{1}{2} \int_{\Gamma_0} ||\nabla z||^2 h\nu d\Gamma_0 + \left( \frac{n}{2} - 1 \right) ||\nabla z||_2^2. $$

(39)

The identity (28) then follows by putting together equations (39) (multiplied by $-b$) and (35) along with the right hand side of (33) multiplied by $h\nabla z$. 

\square
**Remark 2.8.** Take $f = 0$. Observe that if $\gamma \equiv 0$ and one has no other source of dissipation – zero boundary data and no interior damping – then it follows from identity (26) that the $E_1(t) \equiv E_1(0)$, for all $t \geq 0$. With the presence of the boundary dissipation, however, $E_1$ is decreasing even for $\gamma \equiv 0$.

**Theorem 2.9.** Let $\Psi = (u, z, z_t)$ be a classical solution of (19) in $\mathbb{R}$. Then for all $0 \leq s < t \leq T$, if $f = 0$, the following estimate holds

$$E(t) + \int_s^t E(\sigma) d\sigma \lesssim E(s).$$

**Proof.** **Step 1.** Take $f = 0$ and let $\varepsilon > 0$ to be given. Hölder’s Inequality\(^2\) along with Trace Theorem\(^3\) and identity (26) allow the left hand side of identity (27) to be estimated as

$$\int_s^t \left[ b \| A^{1/2}z \|^2_2 - \| z_t \|^2_2 \right] d\sigma \leq E_1(s) + \varepsilon \overline{\gamma} \int_s^t \| A^{1/2}z \|^2_2 d\sigma,$$

where $\overline{\gamma} = \sup_{x \in \Omega} \gamma(x)$. In fact, we estimate the terms on the right side of (27) as follows:

$$- \int_s^t (y u_{tt}, z) d\sigma - \left[ (z_t, z) + \frac{b}{2} \| z \|^2 \right] |_{s}^{t} \lesssim C_{\varepsilon} \int_s^t \| y^{1/2}u_{tt} \|^2_2 + \varepsilon \overline{\gamma} \int_s^t \| A^{1/2}z \|^2_2$$

$$+ E_1(t) + E_1(s)$$

and then, under (26) with $f = 0$, (41) follows.

**Step 2.** Next, recalling that $\max_{x \in \Omega} |h(x)| < \infty$ and using again identity (26) we also estimate the left hand side of (28) as

$$\int_s^t \left[ \frac{n}{2} \| z_t \|^2_2 - \frac{b(n - 2)}{2} \| A^{1/2}z \|^2_2 \right] d\sigma \leq E_1(s) + \varepsilon \overline{\gamma} \int_s^t \| A^{1/2}z \|^2_2 d\sigma,$$

for which we have used the fact that, for $f = 0$, identity (26) allows us to control the time integrals $\int_s^t \| z_t \|^2_{1,0} d\sigma$ and $\int_s^t \| y^{1/2}u_{tt} \|^2_2 d\sigma$ along with

$$(y u_{tt}, h \nabla z) \lesssim \varepsilon \overline{\gamma} \| \nabla z \|^2_2 + C_{\varepsilon} \| y^{1/2}u_{tt} \|^2_2$$

**Step 3.** Now notice that adding (43) with $(n - 1)/2$ times (41) gives

$$\int_s^t \left[ \frac{b}{2} \| A^{1/2}z \|^2_2 + \frac{1}{2} \| z_t \|^2_2 \right] d\sigma \leq E_1(s) + \varepsilon \overline{\gamma} \int_s^t \| A^{1/2}z \|^2_2 d\sigma,$$

from where it follows, by taking $\varepsilon$ small, that

$$\int_s^t \left[ \| A^{1/2}z \|^2_2 + \| z_t \|^2_2 \right] d\sigma \leq E_1(s).$$

\(^2\)Hölder’s Inequality: $\|fg\|_1 \lesssim \|f\|_p \|g\|_q$ for $p, q \in [1, \infty)$ such that $p + q = pq$. Here we used for $p = q = 2$.

\(^3\)Trace Theorem: $\|f\|^2_1 \lesssim \| A^{1/2}f \|^2_2$. 

14
Remark 2.10. Notice that none of the arguments for Steps 1-3 depend on the requirement that \( \gamma > 0 \). Therefore it is valid for \( \gamma \geq 0 \).

Step 4. Finally, from \( bz = bu_t + c^2u \) we get, from (44)

\[
\|A^{1/2}u\|_2^2 + \int_s^t \|A^{1/2}u\|_2^2 \, d\sigma \leq \int_s^t \|A^{1/2}z\|_2^2 \, d\sigma \leq E_1(s),
\]

and then (40) follows by adding (45), (44) and (26).

This proves the inequality in Theorem 2.9 valid for classical solutions. The extension to mild solutions follows from the density of the domain of the generator in \( H \) and from weak lower-semicontinuity of the energy functions. \( \square \)

The proof of the final result in Theorem 1.5 follows from Datko’s Theorem [38].

References

[1] **Alves**, M. O., **Caixeta**, A. H., **Jorge Silva**, M. A., and **Rodrigues**, J. H. Moore–Gibson–Thompson equation with memory in a history framework: a semigroup approach. Z. Angew. Math. Phys. 69, 4 (Aug. 2018), 106.

[2] **Bongarti**, M., **Charoenphon**, S., and **Lasiecka**, I. Singular Thermal Relaxation Limit for the Moore-Gibson-Thompson Equation Arising in Propagation of Acoustic Waves. In *Semigroups of Operators – Theory and Applications*, vol. 325. Springer, 2020, pp. 147–182.

[3] **Bongarti**, M., **Charoenphon**, S., and **Lasiecka**, I. Vanishing relaxation time dynamics of the Jordan Moore-Gibson-Thompson equation arising in nonlinear acoustics. *arXiv:2011.11141* (Nov. 2020).

[4] **Boulaaras**, S., **Zaraï**, A., and **Draifia**, A. Galerkin method for nonlocal mixed boundary value problem for the Moore-Gibson-Thompson equation with integral condition. *Math Meth Appl Sci* 42, 8 (May 2019), 2664–2679.

[5] **Bucci**, F., and **Eller**, M. The Cauchy-Dirichlet problem for the Moore-Gibson-Thompson equation. *arXiv:2004.11167* (Apr. 2020).

[6] **Bucci**, F., and **Lasiecka**, I. Feedback control of the acoustic pressure in ultrasonic wave propagation. *Optimization* 68, 10 (Oct. 2019), 1811–1854.

[7] **Bucci**, F., and **Pandolfi**, L. On the regularity of solutions to the Moore–Gibson–Thompson equation: a perspective via wave equations with memory. *J. Evol. Equ.* 20, 3 (Sept. 2020), 837–867.
[8] Cattaneo, C. A Form of Heat-Conduction Equations Which Eliminates the Paradox of Instantaneous Propagation. *Comptes Rendus* 247 (1958), 431.

[9] Cattaneo, C. Sulla Conduzione Del Calore. In *Some Aspects of Diffusion Theory*, A. Pignedoli, Ed. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011, pp. 485–485.

[10] Cavalcanti, V. N. D., Lasiecka, I., and Caixeta, A. H. On long time behavior of Moore-Gibson-Thompson equation with molecular relaxation. *EECT* 5, 4 (Oct. 2016), 661–676.

[11] Chen, W., and Palmieri, A. Nonexistence of global solutions for the semilinear Moore-Gibson-Thompson equation in the conservative case. *Discrete & Continuous Dynamical Systems - A* 40, 9 (2020), 5513–5540.

[12] Christov, C. I., and Jordan, P. M. Heat Conduction Paradox Involving Second-Sound Propagation in Moving Media. *Phys. Rev. Lett.* 94, 15 (Apr. 2005), 154301.

[13] Conejero, J. A., Lizama, C., and Rodenas, F. Chaotic Behaviour of the Solutions of the Moore-Gibson-Thompson Equation. *Applied Mathematics & Information Sciences* 9, 5 (2015), 2233–2238.

[14] Coulouvrat, F. On the equations of non linear acoustics. *Journal d’acoustique (Les Ulis)* 5, 4 (1992), 321–359.

[15] Crighton, D. G. Model Equations of Nonlinear Acoustics. *Annu. Rev. Fluid Mech.* 11, 1 (Jan. 1979), 11–33.

[16] Dell’Oro, F., Lasiecka, I., and Pata, V. The Moore–Gibson–Thompson equation with memory in the critical case. *Journal of Differential Equations* 261, 7 (Oct. 2016), 4188–4222.

[17] Dell’Oro, F., Lasiecka, I., and Pata, V. A note on the Moore–Gibson–Thompson equation with memory of type II. *J. Evol. Equ.* (Dec. 2019).

[18] Dell’Oro, F., and Pata, V. On the Moore–Gibson–Thompson Equation and Its Relation to Linear Viscoelasticity. *Appl Math Optim* 76, 3 (Dec. 2017), 641–655.

[19] Fattorini, H. Ordinary differential equations in linear topological spaces, I. *Journal of Differential Equations* 5, 1 (Jan. 1969), 72–105.

[20] Jordan, P. Nonlinear acoustic phenomena in viscous thermally relaxing fluids: Shock bifurcation and the emergence of diffusive solitons. *The Journal of the Acoustical Society of America* 124, 4 (Oct. 2008), 2491–2491.
[21] Jordan, P. M. Second-sound phenomena in inviscid, thermally relaxing gases. *Discrete & Continuous Dynamical Systems-B* 19, 7 (2014), 2189.

[22] Kaltenbacher, B., and Alpen-Adria-Universität Klagenfurt, Universitätsstrasse 65-67, 9020 Klagenfurt. Mathematics of nonlinear acoustics. *Evolution Equations & Control Theory* 4, 4 (2015), 447–491.

[23] Kaltenbacher, B., and Lasiecka, I. Exponential decay for low and higher energies in the third order linear Moore-Gibson-Thompson equation with variable viscosity. *Palestine Journal of Mathematics* 1, 1 (2012), 1–10.

[24] Kaltenbacher, B., Lasiecka, I., and Marchand, R. Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound. *Control and Cybernetics* 40 (2011), 971–988.

[25] Kaltenbacher, B., Lasiecka, I., and Pospieszalska, M. K. Wellposedness and exponential decay of the energy of the energy in the nonlinear Jordan-Moore-Gibson-Thompson equation arising in high intensity ultrasound. *Math. Models Methods Appl. Sci.* 22, 11 (Nov. 2012), 1250035.

[26] Kaltenbacher, B., and Nikolić, V. On the Jordan-Moore-Gibson-Thompson equation: well-posedness with quadratic gradient nonlinearity and singular limit for vanishing relaxation time. *arXiv:1901.02795* (Oct. 2019).

[27] Lai, W. M., Rubin, D., and Kreml, E. *Introduction to continuum mechanics*, 4th ed ed. Butterworth-Heinemann/Elsevier, Amsterdam ; Boston, 2010.

[28] Lasiecka, I., and Tataru, D. Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. *Differential Integral Equations* 6, 3 (1993), 507–533.

[29] Lasiecka, I., and Triggiani, R. Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions. *Appl Math Optim* 25, 2 (Mar. 1992), 189–224.

[30] Lasiecka, I., and Wang, X. Moore–Gibson–Thompson equation with memory, part II: General decay of energy. *Journal of Differential Equations* 259, 12 (Dec. 2015), 7610–7635.

[31] Lasiecka, I., and Wang, X. Moore–Gibson–Thompson equation with memory, part I: exponential decay of energy. *Z. Angew. Math. Phys.* 67, 2 (Apr. 2016), 17.
[32] Liu, S., and Triggiani, R. Inverse Problem for a Linearized Jordan–Moore–Gibson–Thompson Equation. In New Prospects in Direct, Inverse and Control Problems for Evolution Equations, vol. 10. Springer, 2014, pp. 305–351.

[33] Liu, W., and Chen, Z. General decay rate for a Moore–Gibson–Thompson equation with infinite history. Z. Angew. Math. Phys. 71, 2 (Apr. 2020), 43.

[34] M. Jordan, P., Kaltenbacher, B., U.S. Naval Research Laboratory, USA, and Alpen-Adria-Universität Klagenfurt, Austria. Introduction to the special issue "Nonlinear wave phenomena in continuum physics: Some recent findings". Evolution Equations & Control Theory 8, 1 (2019), 1–3.

[35] Marchand, R., McDevitt, T., and Triggiani, R. An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. Math. Meth. Appl. Sci. 35, 15 (Oct. 2012), 1896–1929.

[36] Nikolić, V., and Said-Houari, B. Mathematical analysis of memory effects and thermal relaxation in nonlinear sound waves on unbounded domains. arXiv:2003.11840 (Oct. 2020).

[37] Nikolić, V., and Said-Houari, B. On the Jordan-Moore-Gibson-Thompson wave equation in hereditary fluids with quadratic gradient nonlinearity. arXiv:2005.07245 (May 2020).

[38] Pazy, A. Semigroups of linear operators and applications to partial differential equations, corr. 2. print ed. No. 44 in Applied mathematical sciences. Springer, New York, NY, 1992.

[39] Pellicer, M., and Said-Houari, B. On the Cauchy problem for the standard linear solid model with heat conduction: Fourier versus Cattaneo. arXiv:1903.10181 (Mar. 2019).

[40] Pellicer, M., and Solà-Morales, J. Optimal scalar products in the Moore-Gibson-Thompson equation. Evolution Equations & Control Theory 8, 1 (2019), 203–220.

[41] Racke, R., and Said-Houari, B. Global well-posedness of the Cauchy problem for the 3D Jordan–Moore–Gibson–Thompson equation. Communications in Contemporary Mathematics (2020), 2050069.

[42] Sakamoto, R. Hyperbolic boundary value problems, 1st english ed ed. Cambridge University Press, Cambridge [Cambridgeshire]; New York, 1982.
[43] **Stokes.** An examination of the possible effect of the radiation of heat on the propagation of sound. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1, 4 (Apr. 1851), 305–317.

[44] **Straughan, B.** *Heat waves.* Springer-Verlag New York, Place of publication not identified, 2014.

[45] **Tataru, D.** On the regularity of boundary traces for the wave equation. *Annali della Scuola Normale Superiore di Pisa- Classe di Scienze* 26, 1 (1998), 185–206.

[46] **Triggiani, R.** Sharp Interior and Boundary Regularity of the SMGTJ-Equation with Dirichlet or Neumann Boundary Control. In *Semigroups of Operators – Theory and Applications*, vol. 325. Springer, 2020, pp. 379–426.