Policy iteration algorithm for zero-sum multichain stochastic games with mean payoff and perfect information

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Abstract

We consider zero-sum stochastic games with finite state and action spaces, perfect information, mean payoff criteria, without any irreducibility assumption on the Markov chains associated to strategies (multichain games). The value of such a game can be characterized by a system of nonlinear equations, involving the mean payoff vector and an auxiliary vector (relative value or bias). We develop here a policy iteration algorithm for zero-sum stochastic games with mean payoff, following an idea of two of the authors (Cochet-Terrasson and Gaubert, C. R. Math. Acad. Sci. Paris, 2006). The algorithm relies on a notion of nonlinear spectral projection (Akian and Gaubert, Nonlinear Analysis TMA, 2003), which is analogous to the notion of reduction of super-harmonic functions in linear potential theory. To avoid cycling, at each degenerate iteration (in which the mean payoff vector is not improved), the new relative value is obtained by reducing the earlier one. We show that the sequence of values and relative values satisfies a lexicographical monotonicity property, which implies that the algorithm does terminate. We illustrate the algorithm by a mean-payoff version of Richman games (stochastic tug-of-war or discrete infinity Laplacian type equation), in which degenerate iterations are frequent. We report numerical experiments on large scale instances, arising from the latter games, as well as from monotone discretizations of a mean-payoff pursuit-evasion deterministic differential game.

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1 Introduction

The mean-payoff problem for zero-sum two player multichain games We consider a zero-sum stochastic game with finite state space \([n] := \{1, \ldots, n\}\), finite action spaces \(A\) and \(B\) for the first and second player respectively, and perfect information. In the case of the finite horizon problem, in which the payoff of the game induced by a pair of strategies of the two players is defined as the expectation of the sum in finite horizon of the successive rewards (the payments of the first player to the second player), Shapley showed (see [Sha03]) that the value \(v^T\) of the game with horizon \(T\) and initial state \(i \in [n]\) satisfies the dynamic programming equation \(v^{T+1} = f(v^T)\), with a dynamic programming operator \(f : \mathbb{R}^n \to \mathbb{R}^n\) defined as:

\[
[f(v)]_i = \min_{a \in A} \left( \max_{b \in B} \left( \sum_{j \in [n]} P_{ij}^{ab} v_j + r_{ij}^{ab} \right) \right), \quad \forall i \in [n], v \in \mathbb{R}^n.
\]

Here, \(r_{ij}^{ab}\) and \(P_{ij}^{ab}\) represent respectively the reward in state \(i \in [n]\) and the transition probability from state \(i\) to state \(j \in [n]\), when the actions of the first and second players are respectively equal to \(a \in A\) and \(b \in B\).

The above dynamic programming operator \(f\) is order-preserving, meaning that \(v \leq w \implies f(v) \leq f(w)\) where \(\leq\) denotes the partial ordering of \(\mathbb{R}^n\), and additively homogeneous, meaning that it commutes with the addition of a constant vector. These two conditions imply that \(f\) is nonexpansive in the sup-norm (see for instance [CT80], see also [GG04] for more background on this class of nonlinear maps). Moreover, \(f\) is polyhedral, meaning that there is a covering of
\[ \mathbb{R}^n \] by finitely many polyhedra such that the restriction of \( f \) to any of these polyhedra is affine. Kohlberg [Koh80] showed that if \( f \) is a polyhedral self-map of \( \mathbb{R}^n \) that is nonexpansive in some norm, then, there exist two vectors \( \eta \) and \( v \) in \( \mathbb{R}^n \) such that \( f(t \eta + v) = (t+1) \eta + v \), for all \( t \in \mathbb{R} \) large enough. A map of the form \( t \mapsto t \eta + v \) is called a half-line, and \( \eta \) is its slope. It is invariant if it satisfies the latter property. Moreover this property is equivalent to the following system of nonlinear equations for the couple \((\eta, v)\):

\[
\begin{align*}
\eta &= f(\eta), \\
\eta + v &= f_0(v),
\end{align*}
\]

where the maps \( f \) (the recession function) and \( f_0 \) are constructed from \( f \) (see Section 2).

When \( f \) has an invariant half-line with slope \( \eta \), the growth rate of its orbits (also called the cycle time) \( \chi(f) := \lim_{k \to \infty} f^k(v)/k \) exists and is equal to \( \eta \). Here, \( f^k \) denotes the \( k \)-th iterate of \( f \), and \( v \) is an arbitrary vector of \( \mathbb{R}^n \). This shows in particular that the value of the finite horizon game satisfies \( \lim_{T \to \infty} v_T^i / T = \eta_i \) for any final reward. Moreover, \( \eta_i \) gives the value of the game with initial state \( i \), and mean payoff, that is such that the payoff of the game induced by a pair of strategies of the two players is the Cesaro limit of the expectation of the successive rewards. Then a vector \( v \) such that \( t \mapsto t \eta + v \) is an invariant half-line is called a relative value of the game, or bias. It is not unique, even up to an additive constant.

In this paper, we give an algorithm to find an invariant half-line, or equivalently a solution of (1), for general multichain games. This allows us in particular to determine the mean payoff, as well as optimal strategies for both players. By multichain, we mean that there is no irreducibility assumption on the Markov chains associated to the strategies of the two players, which may have in particular several invariant measures.

### Classes of games solvable by earlier policy iteration algorithms

Policy iteration is a general method initially introduced by Howard [How60] in the case of one player problems (Markov decision processes). The idea is to compute a sequence of strategies as well as certain valuations, which serve as optimality certificates, and to use the current valuation to improve the strategy. The algorithm bears some resemblance with the Newton method, as the strategy determines a sub or super-gradient of the dynamic programming operator. The key of the analysis of policy iteration algorithms is generally to show that the sequence of valuations which are computed satisfies a monotonicity property, from which it can be inferred that the same strategy is never selected twice. In the discounted one player case, the valuation which is maintained by the algorithm is nothing but the value vector of the current policy. For one-player games with mean-payoff, in the unichain case (in which every stochastic matrix associated to a strategy has only one final class), the valuation consists of the mean payoff of the current strategy, as well as of a relative value. In both cases, the monotonicity property is natural (it relies on the discrete maximum principle, or properties of monotonicity and contraction of the dynamic programming operator, or on the uniqueness of the invariant measure associated to a strategy). However, even for one player games, the extension to the multichain case is more difficult. It was initially proposed by Howard [How60]. The convergence of his method was established by Denardo and Fox [DF68].

The idea of extending Howard algorithm to the two player case appeared independently in the work of Hoffman and Karp [HK66] for a subclass of mean-payoff games with imperfect information, and in the work of Denardo [Den67] for discounted games. Both algorithms consist of nested iterations; the internal iterations are a simplified version of the one player Howard algorithm. The algorithm for discounted games appeared also, as an adaptation of the Hoffman-Karp algorithm, in the work of Rao, Chandrasekaran, and Nair [RCN73, Algorithm 1] and of Puri [Pur95] (deterministic games with perfect information). More recently, Raghavan and Syed [RS03] developed a related algorithm in which strategy improvements involve only one state at each iteration.

The Hoffman-Karp algorithm requires the game to satisfy a strong irreducibility assumption (each stochastic matrix arising from a choice of strategies of the two players must be irreducible). Without an assumption of this kind, degenerate iterations, at which the mean payoff vector is not improved, may occur, and so the algorithm may cycle (we shall indeed see such an example in Section 5). This pathology appears in particular for the important subclass of deterministic mean payoff games, for which the irreducibility assumption is essentially never satisfied.
A natural idea to solve mean-payoff games, appearing for instance in the work of Puri [Pur95], is to apply the policy iteration algorithm of Denardo [Den67] or Rao, Chandrasekaran, and Nair [RCN73, Algorithm 1] for discounted games, choosing a given discount factor \( \alpha \) sufficiently close to one, which allows one to determine the so-called Blackwell optimal policies. For deterministic games, when the rewards are integers with modulus less or equal to \( W \) and the number of states is equal to \( n \), Zwick and Paterson [ZP96] showed that taking \( 1 - \alpha = 1/(4n^3W) \) is sufficient to determine the mean payoff by a rounding argument. However this requires high precision arithmetics. In the case of stochastic games, the situation is even worse, since examples are known in which the value of \( 1 - \alpha \) to be used for rounding has a denominator exponential in the number of states. In particular, an approach of this kind is impracticable if one works in floating point (bounded precision) arithmetics. Hence, it is desirable to have a policy iteration algorithm for multichain stochastic games relying only on the computation of mean payoffs and relative values as in the algorithm of Howard [How60] and Denardo and Fox [DF68].

The first policy iteration algorithm not relying on vanishing discount, for general (multichain) deterministic mean payoff games, was apparently introduced by Cochet-Terrasson, Gaubert and Gunawardena [CTGG99, GG98]. The former reference concerns the special case in which the mean payoff is the same for all states at each iteration, whereas the second one covers the general case, see also [CT01]. Details of implementation, as well as experimental results were given in [DG06]. The idea of the algorithm of [CTGG99] is to handle degenerate iterations by a tropical (max-plus) spectral projector. The latter is a tropically linear retraction of the whole space onto the fixed point set of the dynamic programming operator associated to a given strategy of the first player.

When the mean payoff or the current strategy is not improved, the new relative value is obtained by applying a spectral projector to the earlier relative value. The proof of termination of the algorithm [CTGG99] relies on a key ingredient from tropical spectral theory, that a fixed point of a tropically linear map is uniquely defined by its restriction to the critical nodes (the nodes appearing infinitely often in a strategy which gives the optimal mean payoff). Then, it was shown in [CTGG99] that at each degenerate iteration, the relative value decreases, and that the set of critical nodes also decreases, from which the termination of the algorithm can be deduced.

A related class of games consists of parity games, which can be encoded as special deterministic games with mean payoff. A policy improvement algorithm for parity games was introduced by Vöge and Jurdziński [VJ00]. This algorithm differs from the one of [CTGG99, GG98] in that instead of the relative value, the algorithm maintains a set of relevant reachable vertices. Other policy algorithm for parity games or deterministic mean payoff games were introduced later on by Bjorklund, Sandberg and Vorobyov [BSV04, BV07], and by Jurdziński, Paterson, and Zwick [JPZ06]. An experimental comparison of algorithms for deterministic games was recently made by Chaloupka [C11], [C19], who also gave an optimized version of the algorithm of [CTGG99, GG98, DG06].

In [BCPS04], Bielecki, Chancelier, Pliska, and Sulem used a policy iteration algorithm to solve a semi-Markov mean-payoff game problem with infinite action spaces obtained from the discretization of a quasi-variational inequality, based on the approach of [CTGG99, GG98]. Their algorithm proceeds in a Hoffman and Karp fashion.

**Policy iteration algorithm for stochastic multichain zero-sum games with mean payoff**

Inspired by the policy iteration algorithm of [CTGG99, GG98] for deterministic games, Cochet-Terrasson and Gaubert proposed in [CTG06] a policy iteration algorithm for general stochastic games (see also [CT01] for a preliminary version). The relative values are now constructed using the nonlinear analogues of tropical spectral projectors. These nonlinear projectors where introduced by Akian and Gaubert in [AG03]. They can be thought of as a nonlinear analogues of the operation of reduction of a super-harmonic function, arising in potential theory. However, no implementation details were given in the short note [CTG06], in which the algorithm was stated abstractly, in terms of invariant half-lines.

We develop here fully the idea of [CTG06], and describe a policy iteration algorithm for multichain stochastic games with mean-payoff (see Section 4.2). We explain how nonlinear systems of the form (1) are solved at each iteration. We show in particular how non-linear spectral projections can be computed, by solving an auxiliary (one player) optimal stopping problem. This relies on the determination of the so called critical graph, the nodes of which (critical nodes) are visited...
An algorithm to compute the critical graph, based on results on \[ AG03 \], is given in Section 5.3. The proof of convergence exploits some results of spectral theory of convex order-preserving additively homogeneous maps, by Akian and Gaubert \[ AG03 \]. Hence, the situation is somehow analogous to the deterministic case \[ CTGG99 \], the technical results of tropical (linear) spectral theory used in \[ CTGG99 \] being now replaced by their non-linear analogues \[ AG03 \]. Note also that the convergence proof of the algorithm of \[ CTGG99, GG98 \] can be recovered as a special case of the present proof.

The convergence proof leads to a coarse exponential bound on the execution time of the algorithm: the number of iterations of the first player is bounded by its number of strategies, and the number of elementary iterations (resolutions of linear systems) is bounded by the product of the number of strategies of both players.

We also show that the specialization of this algorithm to a one-player game gives an algorithm which is similar to the multichain policy algorithm of Howard and Denardo and Fox, see Section 5.2. Then, we discuss an example (see Section 6) involving a variant of Richman games \[ LLP \, 99 \] related with discretizations of the infinity Laplacian \[ ObS05 \], showing that degenerate iterations do occur and that cyclic may occur with naive policy iteration rules. Hence, the handling of degenerate iterations, that we do here by nonlinear spectral projectors, cannot be dispensed with.

The present algorithm has been implemented in the C library \[ PIGAMES \] by Detournay, see \[ Det12 \] for more information. We finally report numerical experiments (see Section 7) carried out using this library, both on random instances of Richman type games with various numbers of states and on a class of discrete games arising from the monotone discretization of a pursuit-evasion differential game. These examples indicate that degenerate iterations are frequent, so that their treatment cannot be dispensed with. They also show that the algorithm scales well, allowing one to solve structured instances with \( 10^6 \) nodes and \( 10^7 \) actions in a few hours of CPU time on a single core processor (the bottleneck being the resolution of linear systems).

We note that our experimental are consistent with earlier experimental tests carried out for simpler algorithms (dealing with one player or deterministic problems) of which the present one is an extension. These tests indicate that policy iteration algorithms are fast on typical instances (although instances with an exponential number of iterations have been recently constructed, as discussed in the next subsection). Indeed, in the case of one-player deterministic games (maximal circuit mean problem), Dasdan, Irani and Gupta \[ DG98 \] concluded that the instrumentation of Howard's policy iteration algorithm by Cochet-Terrasson et al. \[ CTCG+98 \], in which each iteration is carried out in linear time, was the fastest algorithm on their test suite. Dasdan later on developed further optimizations of this method \[ Das04 \]. More recent experiments by Georgiadis, Goldberg, Tarjan, and Werneck \[ GGTW09 \] have indicated that the class of cycle based algorithms (to which \[ CTCG+98, Das04 \] belongs) is among the best performers, close second to the tree based method of Young, Tarjan, and Orlin \[ YTO91 \]. In the deterministic two player case, Chaloupka \[ Cha09 \] compared several algorithms and observed that the one of \[ CTGG99, GG98, DG06 \], with the optimization that he introduced (see also \[ Cha11 \]), is experimentally the best performer.

Alternative algorithms and complexity issues Gurvich, Karzanov and Khachiyan \[ GKK88 \] were the first to develop a combinatorial algorithm (pumping algorithm) to solve zero-sum deterministic games with mean payoff. An alternative approach was developed by Zwick and Paterson \[ ZP96 \], who showed that such a game can be solved by considering the finite horizon game for a sufficiently large horizon, and applying a rounding argument. Both algorithms are pseudo-polynomial. Other algorithms, also pseudo-polynomial, based on max-plus (tropical) cyclic projections, with a value iteration flavor, have been developed by Butkovič and Cuninghame-Green \[ CGB03 \], Gaubert and Sergeev \[ GS07 \], and Akian, Gaubert, Nitiça and Singer \[ AGNS11 \]. Deterministic mean payoff games have been recently proved to be equivalent to decision problems for tropical polyhedra (the tropical analogue of linear programming) \[ ACG12 \]. More generally, the
results there show that stochastic games problems with mean payoff can be cast as tropical convex (non-polyhedral) programming problems.

The pumping algorithm of \cite{GKK88} was recently extended to the case of stochastic games with perfect information by Boros, Elbassioni, Gurvich, and Makino \cite{BEGM10}. They showed that their algorithm is pseudo-polynomial when the number of states of the game at which a random transition occurs remains fixed. No pseudo-polynomial seems currently known without the latter restriction. Their algorithm applies to more general games than the ones covered by the irreducibility assumption of Hoffman and Karp in \cite{HK66}, but it does not apply to all multichain games.

The question of the complexity of deterministic mean payoff games was raised in \cite{GKK88}, and it has remained open since that time. Note in this respect that such games are known to have a good characterization in the sense of Edmonds, i.e., to be in $\text{NP} \cap \text{coNP}$. Indeed, the strategies of one player can be used as concise certificates, as observed by Condon \cite{Con92}, Paterson and Zwick \cite{ZP96}. Such games even belong to the class $\text{UP} \cap \text{coUP}$ as shown by Jurdziński \cite{Jur98}. We refer the reader to the discussion in \cite{BSV04,JPZ06} for more information. The arguments of Condon \cite{Con92} also imply that zero-sum stochastic games with perfect information (and finite state and action spaces) belong to $\text{NP} \cap \text{coNP}$. An important subclass of deterministic games with mean payoff consists of parity games. These can be reduced to mean payoff deterministic games \cite{Pur95}, which in turn can be reduced to discounted deterministic games. The latter ones can be reduced to simple stochastic games (Zwick and Paterson \cite{ZP96}). In \cite{AM09}, Andersson and Miltersen generalized this result showing that stochastic mean payoff games with perfect information, stochastic parity games and simple stochastic games are polynomial time equivalent. In particular, the decision problem corresponding to a game of any of these classes lies in the complexity class of $\text{NP} \cap \text{coNP}$. 

Friedmann has recently constructed an example \cite{Fri09} showing that the Vöge-Jurdziński strategy improvement algorithm for parity games \cite{VJ00} may require an exponential number of iterations. This also yields an exponential lower bound \cite{Fri11} for the Hoffman-Karp strategy improvement rule for discounted deterministic games \cite{Pur95}. The result of Friedmann has also been extended to total reward and undiscounted MDP by Fearnley \cite{Fea10a,Fea10b} and to simple stochastic games and weighted discounted stochastic games by Andersson \cite{And09}.

Moreover, for Markov decision process with a fixed discount factor, some upper bound on the number of policy iterations was given in \cite{MH86}. Recently, Ye gave a first strongly polynomial bound \cite{Ye05,Ye11}. The latter bound has been improved and generalized to zero-sum two player stochastic games with perfect information factor by Hansen, Miltersen and Zwick in \cite{HMZ11}, again for a fixed discount factor, giving the first strongly polynomial bound for these games. Note that a polynomial bound for mean payoff games does not follow from these results (to address the mean payoff case, we need to consider the situation in which the discount factor tends to 1).

Complexity results of a different nature have been established with motivations from numerical analysis (discretizations of PDE), exploiting in particular the relation between policy iteration and the Newton method. The policy iteration algorithm for one-player discounted games with an infinite number of actions has been proved to have a superlinear convergence around the solution under suitable assumptions (see in particular the works of Puterman and Brumelle \cite{PB79}, Akian \cite{Ak90}, and Bokanowski, Maroso, and Zidani \cite{BMZ09}). Chancelier, Messaoud, and Sulem \cite{CMS07} also considered, in view of their application to quasi-variational inequalities, partially undiscounted infinite horizon problems for which they proved the contraction of the policy iteration algorithm.

The plan of the paper is the following: Section 2 is recalling some background on stochastic zero-sum two player games, Section 3 explain the construction of the nonlinear projection, Section 4 gives the algorithm, its practical version and its proof, Section 5 gives the ingredients of the algorithm, Section 6 shows an example with possible cycling of iterations when not using the notion of spectral projector, and Section 7 is for the numerical experiments.
2 Two player zero-sum stochastic games with discrete time and mean payoff

The class of two player zero-sum stochastic games was first introduced by Shapley in the early fifties, see [Sha53]. We recall in this section basic definitions on these games in the case of finite state space and discrete time (for more details see [Sha53], [FY97], [Sor02]).

We consider the finite state space \([n] := \{1, \ldots, n\}\). A stochastic process \((\xi_k)_{k\geq 0}\) on \([n]\) gives the state of the game at each point time \(k\), called stage. At each of these stages, two players, called “MIN” and “MAX” (the minimizer and the maximizer) have the possibility to influence the course of the game.

The stochastic game \(\Gamma(i_0)\) starting from \(i_0 \in [n]\) is played in stages as follows. The initial state \(\xi_0 = i_0\) is known by the players. Player MIN plays first, and chooses an action \(\alpha_0\) in a set of possible actions \(A_{i_0}\). Then the second player, MAX, chooses an action \(\beta_0\) in a set of possible actions \(B_{i_0}\). The actions of both players and the current state determine the payment \(r_{i_0}^{\alpha_0\beta_0}\) made by MIN to MAX and the probability distribution \(j \mapsto P_{i_0}^{\alpha_0\beta_0}\) of the new state \(\xi_1\). Then the game continues in the same way with state \(\xi_1\) and so on.

At a stage \(k\), each player chooses an action knowing the history defined by \(\zeta_k = (\xi_0, \alpha_0, \beta_0, \cdots, \xi_{k-1}, \alpha_{k-1}, \beta_{k-1}, \xi_k)\) for MIN and \((\xi_k, \alpha_k)\) for MAX. We call a strategy or policy for a player, a rule which tells him the action to choose in any situation. There are several classes of strategies. Assume \(A_i \subseteq A\) and \(B_i \subseteq B\) for some sets \(A\) and \(B\). A behavior or randomized strategy for MIN (resp. MAX) is a sequence \(\sigma := (\sigma_0, \sigma_1, \cdots)\) (resp. \(\delta := (\delta_0, \delta_1, \cdots)\)) where \(\sigma_k\) (resp. \(\delta_k\)) is a map which to a history \(h_k = (i_0, a_0, b_0, \cdots, i_{k-1}, a_{k-1}, b_{k-1}, i_k)\) with \(i_j \in [n]\), \(a_j \in A_{i_j}\), \(b_j \in B_{i_j}\) for \(0 \leq l \leq k\) (resp. \((h_k, a_k)\)) at stage \(k\) associates a probability distribution on a probability space over \(A\) (resp. \(B\)) which is supported in the possible actions space \(A_{i_k}\) (resp. \(B_{i_k}\)). A Markovian strategy is a strategy which only depends on the information of the current stage \(k\): \(\sigma_k\) (resp. \(\delta_k\)) depends only on \(i_k\) (resp. \((i_k, a_k)\)), then \(\sigma_k(h_k)\) (resp. \(\sigma_k(h_k, a_k)\)) will be denoted \(\sigma_k(i_k)\) (resp. \(\delta_k(i_k, a_k)\)). It is said stationary if it is independent of \(k\), then \(\sigma_k\) is also denoted by \(\sigma\) and \(\delta_k\) by \(\delta\). A strategy of any type is said pure if for any stage \(k\), the values of \(\sigma_k\) (resp. \(\delta_k\)) are Dirac probability measures at certain actions in \(A_{i_k}\) (resp. \(B_{i_k}\)) then we denote also by \(\sigma_k\) (resp. \(\delta_k\)) the map which to the history assigns the only possible action in \(A_{i_k}\) (resp. \(B_{i_k}\)).

In particular, if \(\bar{\sigma}\) is a pure Markovian stationary strategy, also called feedback strategy, then \(\bar{\sigma} = (\sigma_k)_{k\geq 0}\) with \(\sigma_k = \bar{\sigma}\) for all \(k\) and \(\bar{\sigma}\) is a map \([n] \rightarrow A\) such that \(\sigma(i) \in A_i\) for all \(i \in [n]\). In this case, we also speak about pure Markovian stationary or feedback strategy for \(\sigma\) and we denote by \(A_{\bar{\sigma}}\) the set of such maps. We adopt a similar convention for player MAX: \(B_{\bar{\sigma}} := \{\delta : [n] \times A \rightarrow B \mid \delta(i, a) \in B, \forall i \in [n], a \in A_i\}\).

A strategy \(\bar{\sigma} = (\sigma_k)_{k\geq 0}\) (resp. \(\bar{\delta} = (\delta_k)_{k\geq 0}\)) together with an initial state determines stochastic processes \((\alpha_k)_{k\geq 0}\) for the actions of MIN, \((\beta_k)_{k\geq 0}\) for the actions of MAX and \((\xi_k)_{k\geq 0}\) for the states of the game such that

\[
P(\xi_{k+1} = j \mid \xi_k = h_k, \alpha_k = a, \beta_k = b) = P_{ij}^{\alpha_b} \tag{2a}
\]

\[
P(\alpha_k \in A' \mid \xi_k = h_k) = \sigma_k(h_k)(A') \tag{2b}
\]

\[
P(\beta_k \in B' \mid \xi_k = h_k, \alpha_k = a) = \delta_k(h_k, a)(B') \tag{2c}
\]

where \(\xi_k := (\xi_0, \alpha_0, \beta_0, \cdots, \xi_{k-1}, \alpha_{k-1}, \beta_{k-1}, \xi_k)\) is the history process, \(h_k\) is a history vector at time \(k\): \(h_k = (i_0, a_0, b_0, \cdots, i_{k-1}, a_{k-1}, b_{k-1}, i_k)\) and \(A'\) (resp. \(B'\)) are measurable sets in \(A\) (resp. \(B\)). For instance, for each pair of feedback strategies \((\sigma, \delta)\) of the two players, that is such that for \(k \geq 0\) : \(\sigma_k = \sigma\) with \(\sigma \in A_{\bar{\sigma}}\) and \(\delta_k = \delta\) with \(\delta \in B_{\bar{\delta}}\), the state process \((\xi_k)_{k\geq 0}\) is a Markov chain on \([n]\) with transition probability

\[
P(\xi_{k+1} = j \mid \xi_k = i) = P_{ij}^{\sigma(i)\delta(i, \sigma(i))} \text{ for } i, j \in [n] ,
\]

and \(\alpha_k = \sigma(\xi_k)\) and \(\beta_k = \delta(\xi_k, \alpha_k)\).

When the strategies \(\bar{\sigma}\) for MIN and \(\bar{\delta}\) for MAX are fixed, the payoff in finite horizon \(\tau\) of the game \(\Gamma(i, \sigma, \delta)\) starting from \(i\) is

\[
J^{\tau}(i, \sigma, \delta) = \mathbb{E}_{\bar{\sigma}, \bar{\delta}} \left[ \sum_{k=0}^{\tau-1} r_{\xi_k}^{\alpha_k\beta_k} \right],
\]
Then, the dynamic programming equation of the finite horizon game writes:

\[ J(i, \sigma, \delta) = \limsup_{\tau \to \infty} \frac{1}{\tau} J^\tau(i, \sigma, \delta). \]

When the action spaces \( A_i \) and \( B_i \) are finite sets for all \( i \in [n] \), the finite horizon game and the mean payoff game have a value which is given respectively by:

\[ v^\tau_i = \inf_\sigma \sup_\delta J^\tau(i, \sigma, \delta), \quad (3) \]

and

\[ \rho_i = \inf_\sigma \sup_\delta J(i, \sigma, \delta), \quad (4) \]

for all starting state \( i \in [n] \), where the infimum is taken among all strategies \( \sigma \) for \( \text{MIN} \) and the supremum is taken over all strategies \( \delta \) for \( \text{MAX} \) (see [Sha03] for finite horizon games, and [LL69] for mean payoff games).

Indeed, the value \( v^\tau \) of the finite horizon game satisfies the dynamic programming equation [Sha03]:

\[ v_i^{\tau+1} = \min_{a \in A_i} \left( \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^{ab} v_j^\tau + r_{ij}^{ab} \right) \right), \quad \forall i \in [n], \quad (5) \]

with initial condition \( v_i^0 = 0, \quad i \in [n] \). Moreover, optimal strategies are obtained for both players by taking pure Markovian strategies \( \sigma \) for \( \text{MIN} \) and \( \delta \) for \( \text{MAX} \) such that, for all \( k = 0, \ldots, \tau - 1 \), and \( i \) in \([n]\), \( \sigma_k(i) \) attains the minimum in \((5)\) with \( \tau \) replaced by \( \tau - k - 1 \), and that, for all \( k = 0, \ldots, \tau - 1 \), \( i \) in \([n]\) and \( a \) in \( A_i \), \( \delta_k(i,a) \) attains the maximum in the expression of \( F(v^\tau - k - 1; i, a) \) defined as follows:

\[ F(v; i, a) = \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^{ab} v_j + r_{ij}^{ab} \right). \quad (6) \]

We denote by \( f \) the dynamic programming or Shapley operator from \( \mathbb{R}^n \) (that is here equivalent to \( \mathbb{R}^{[n]} \)) to itself given by:

\[ [f(v)]_i := F(v; i) := \min_{a \in A_i} F(v; i, a), \quad \forall i \in [n], \quad v \in \mathbb{R}^n. \quad (7) \]

Then, the dynamic programming equation of the finite horizon game writes:

\[ v^{\tau+1} = f(v^\tau). \quad (8) \]

The operator \( f \) is order-preserving, i.e. \( v \leq w \implies f(v) \leq f(w) \) where \( \leq \) denotes the partial ordering of \( \mathbb{R}^n \) (\( v \leq w \) if \( v_i \leq w_i \) for all \( i \in [n] \)), and additively homogeneous, i.e. it commutes with the addition of a constant vector, which means that \( f(\lambda + v) = \lambda + f(v) \) for all \( \lambda \in \mathbb{R} \) and \( v \in \mathbb{R}^n \), where \( \lambda + v = (\lambda + v_i)_{i \in [n]} \). This implies that \( f \) is nonexpansive in the sup-norm (see for instance [CT80]). Note that it was observed independently by Kolokoltsov [Kol92], by Gunawardena and Sparrow (see [Gum03]) and by Rubinov and Singer [RS01] that, conversely, if \( f : \mathbb{R}^n \to \mathbb{R}^n \) is order-preserving and additively homogeneous, then \( f \) can be put in the form \((6,7)\), with possibly infinite sets \( A_i \) and \( B_i \).

When the action spaces \( A_i \) and \( B_i \) are finite sets for all \( i \in [n] \), the map \( f \) is also polyhedral, meaning that there is a covering of \( \mathbb{R}^n \) by finitely many polyhedra such that the restriction of \( f \) to any of these polyhedra is affine. Kohlberg [Koh80] showed that if \( f \) is a polyhedral self-map of \( \mathbb{R}^n \) that is nonexpansive in some norm, then, there exist two vectors \( u \) and \( v \) in \( \mathbb{R}^n \) such that \( f(t \eta + v) = (t + 1) \eta + v \), for all \( t \in \mathbb{R} \) large enough. A map \( \omega : t \in [t_0, \infty) \mapsto t \eta + v \in \mathbb{R}^n \), with \( t_0 \in \mathbb{R} \), and \( \eta, v \in \mathbb{R}^n \), is called a half-line with slope \( \eta \). A germ of half-line at infinity is an equivalence class for the equivalence relation on half-lines \( \omega \sim \omega' \) if \( \omega(t) = \omega'(t) \) for \( t \in \mathbb{R} \) large enough. A germ can be identified with the couple \( (\eta, v) \) of vectors of \( \mathbb{R}^n \). Hence, in the sequel, we shall use the expression “half-line” either for a map \( \omega : t \in [t_0, \infty) \mapsto t \eta + v \in \mathbb{R}^n \), for its germ, or...
for the couple \((\eta, v)\). We shall say that it is invariant by \(f\) if it satisfies the latter property, that is \(f(t\eta + v) = (t + 1)\eta + v\), for all \(t \in \mathbb{R}\) large enough. The interest of an invariant half-line is that its slope determines the growth rate of the orbits of \(f\), \(\chi(f) := \lim_{k \to \infty} f^k(w)/k\). Here, \(f^k\) denotes the \(k\)-th iterate of \(f\), and \(w\) is an arbitrary vector of \(\mathbb{R}^n\). When it exists, the growth rate \(\chi(f)\) is called the cycle time of \(f\). Indeed, if \(f(t\eta + v) = (t + 1)\eta + v\) for \(t \geq t_0\), then \(f^k(t_0\eta + v) = (t_0 + k)\eta + v\) for \(k \geq 0\), hence \(\lim_{k \to \infty} f^k(t_0\eta + v)/k = \eta\), and by the nonexpansiveness of \(f\), \(\lim_{k \to \infty} f^k(w)/k = \eta\) for all \(w \in \mathbb{R}^n\), that is \(\chi(f)\) does exist and is equal to \(\eta\). For the game problem this shows that the value of the finite horizon game has a linear growth with respect to time:

\[
\lim_{t \to \infty} \frac{1}{t} v^T = [\chi(f)]_i = \eta_i ,
\]

where \(f\) is the Shapley operator defined in [67]. Moreover, the value \(\rho\) of the mean payoff game defined in [4] coincides with the slope of an invariant half-line of \(f\), and thus with the former limit:

\[
\rho_i = \eta_i = [\chi(f)]_i .
\]

Finally, when the action spaces are finite, one can easily see that the Shapley operator \(f\) in [67] satisfies for all \(\eta, v \in \mathbb{R}^n\),

\[
f(t\eta + v) = t\hat{f}(\eta) + \hat{f}_\eta(v) \quad \text{for } t \text{ large},
\]

where \(\hat{f}\) is the recession function of \(f\) (see [GG04]):

\[
[\hat{f}(\eta)]_i := \lim_{t \to \infty} \frac{[f(t\eta)]_i}{t} = \min_{a \in A_i} \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^a \eta_j \right) , \quad i \in [n] ,
\]

and \(\hat{f}_\eta\) is what we shall call the tangent of \(f\) at infinity around the slope \(\eta\):

\[
[\hat{f}_\eta(v)]_i := \lim_{t \to \infty} [f(t\eta + v) - t\hat{f}(\eta)]_i = \min_{a \in A_{i,\eta}} \max_{b \in B_{i, a, \eta}} \left( \sum_{j \in [n]} P_{ij}^a \eta_j + r_{ij}^{ab} \right) ,
\]

with

\[
\hat{A}_{i, \eta} := \text{argmin}_{a \in A_i} \left\{ \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^a \eta_j \right) \right\} ,
\]

\[
\hat{B}_{i, a, \eta} := \text{argmax}_{b \in B_i} \left\{ \sum_{j \in [n]} P_{ij}^a \eta_j \right\} .
\]

Indeed, for an action \(a \in A_i\) and \(i \in [n]\), we have from the finiteness of the sets \(B_i\) :

\[
F(t\eta + v; i, a) = \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^b (t\eta_j + v_j) + r_{ij}^{ab} \right)
\]

\[
= \max_{b \in B_i} \left( t \sum_{j \in [n]} P_{ij}^b \eta_j + P_{ij}^b v_j + r_{ij}^{ab} \right)
\]

\[
= \max_{b \in B_i} \left( t \sum_{j \in [n]} P_{ij}^b \eta_j \right) + \max_{b \in B_{i, a, \eta}} \left( \sum_{j \in [n]} P_{ij}^b v_j + r_{ij}^{ab} \right) \quad \text{for } t \text{ large}
\]

\[
= t \hat{F}(\eta; i, a) + \hat{F}_\eta(v; i, a)
\]

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where one denotes:
\[
\hat{F}(\eta; i, a) := \max_{b \in B_i} \left( \sum_{j \in [n]} P_{ij}^{ab} \eta_j \right),
\]
\[
\hat{F}_b(v; i, a) := \max_{b \in B_{i,a}} \left( \sum_{j \in [n]} P_{ij}^{ab} v_j + v_i^{ab} \right).
\]
Then, using the finiteness of the sets \(A_i\), and
\[
[f(\eta + v)]_i = F(t\eta + v; i) = \min_{a \in A_i} F(t\eta + v; i, a),
\]
one obtains Equation (9).

From (9), we deduce easily that \((\eta, v)\) is an invariant half-line of \(f\) if, and only if, it satisfies:
\[
\begin{cases}
\eta = \hat{f}(\eta) \\
\eta + v = \hat{f}_b(v).
\end{cases}
\]
This couple system of equations is what is solved in practice, when one looks for the value function \(\rho = \eta\) of the mean payoff game.

### 3 Reduced super-harmonic vectors

We next present the non-linear analogue of a result of classical potential theory, on which the policy iteration algorithm for mean payoff games relies. Recall that a self-map \(f\) of \(\mathbb{R}^n\) is order-preserving if \(v \leq w \implies f(v) \leq f(w)\), where \(\leq\) denotes the partial ordering of \(\mathbb{R}^n\), and that it is additively homogeneous if it commutes with the addition of a constant vector. More generally, it is \(\text{additively subhomogeneous}\), if \(f(\lambda + v) \leq \lambda + f(v)\) for all \(\lambda \geq 0\) and \(v \in \mathbb{R}^n\). It is easy to see that an order-preserving self-map \(f\) of \(\mathbb{R}^n\) is additively subhomogeneous if, and only if, it is nonexpansive in the sup-norm. (See for instance [GG04] for more background on order-preserving additively homogeneous maps.)

We shall now recall some definitions and results of [AG03], where the corresponding proofs can be found, up to an extension from additively homogeneous maps to subhomogeneous maps as in [AG03 §1.4]. To show the analogy with potential theory, we shall say that a vector \(u \in \mathbb{R}^n\) is \(\text{harmonic}\) with respect to an order preserving, additively (sub)homogeneous map \(g\) of \(\mathbb{R}^n\) if it is a fixed point of \(g\), i.e. if \(g(u) = u\), and that it is \(\text{super-harmonic}\) if \(g(u) \leq u\). ([AG03] deals more generally with additive eigenvectors and super-eigenvectors). We shall denote by \(\mathcal{H}(g)\) and \(\mathcal{H}^+(g)\) the set of harmonic and super-harmonic vectors respectively.

We say that a self-map \(g\) of \(\mathbb{R}^n\) is \(\text{convex}\) if all its coordinates \(g_i : \mathbb{R}^n \to \mathbb{R}\) are convex functions.

Then, the \(\text{subdifferential}\) of \(g\) at a point \(u \in \mathbb{R}^n\) is defined as
\[
\partial g(u) := \{ M \in \mathbb{R}^{n \times n} | g(v) - g(u) \geq M(v - u), \forall v \in \mathbb{R}^n \}.
\]
Hence,
\[
\partial g(u) = \{ M \in \mathbb{R}^{n \times n} | M_i \in \partial g_i(u) \},
\]
where \(M_i\) denotes the \(i\)-th row of the matrix \(M\). It can be checked that when \(g\) is order-preserving and additively homogeneous (resp. subhomogeneous), \(\partial g(u)\) consists of stochastic (resp. substochastic) matrices, that is matrices with nonnegative entries and row sums equal to 1 (resp. less or equal to 1), see [AG03 Cor. 2.2 and (4)]. Assume \(g\) has a harmonic vector \(u\). We say that a node is critical if it belongs to a recurrence class of some matrix \(M \in \partial g(u)\), where a recurrence class of \(M\) means a (final) communication class \(F\) of \(M\) such that the \(F \times F\) submatrix of \(M\) is stochastic (note that a recurrence class may not exist if \(g\) is not additively homogeneous), see [AG03 §2.3 and 1.4]. One defines also the critical graph \(G^c(g)\) of \(g\) as the union of the graphs of the \(F \times F\) submatrices of the matrices \(M \in \partial g(u)\), such that \(F\) is a recurrence class of \(M\). The set of critical nodes and the critical graph of \(g\) are independent of the choice of the harmonic vector \(u\) [AG03].
Prop. 2.5. Indeed, when \( g \) arises from a stochastic control problem with ergodic reward, a node is critical if it is recurrent for some stationary optimal strategy.

If \( I \) is any subset of \([n]\), we denote by \( r_I \) the restriction from \( \mathbb{R}^n \) to \( \mathbb{R}^I \), such that \((r_Iv)_i := v_i, \) for all \( i \in I \). For all \( u \in \mathbb{R}^n \), we define \( u_I := r_Iu \), and for all self-maps \( g \) of \( \mathbb{R}^n \), we define \( g_I := r_Ig. \) Let \( J := [n] \setminus I \). We denote by \( r_J \) the canonical map identifying \( \mathbb{R}^I \times \mathbb{R}^J \) to \( \mathbb{R}^n \), which sends \((w,z)\) to the vector \( u \) such that \( u_i = w_i \) for all \( i \in I \) and \( u_i = z_i \) for all \( i \in J \). Then, the transpose \( r_I^* \) of \( r_I \) is the map from \( \mathbb{R}^J \) to \( \mathbb{R}^n \) such that \( r_I^*(w) = r_J(w,0) \). Finally, for all \( I,J \subseteq [n] \), and for all \( n \times n \) matrices \( M \), we denote by \( M_{IJ} \) the \( I \times J \) submatrix of \( M \).

**Lemma 1.** Let \( g \) denote a convex, order preserving, and additively homogeneous self-map of \( \mathbb{R}^n \). Assume that \( u \in \mathbb{R}^n \) is harmonic with respect to \( g \). Denote by \( C \) the set of critical nodes of \( g \) and by \( N = [n] \setminus C \) its complement in \([n]\). Then, the map \( h : \mathbb{R}^N \to \mathbb{R}^N \) with \( h(w) := (r_N \circ g \circ r_J)(w,u_C) \) has a unique fixed point.

**Proof.** Since the map \( g \) is order preserving and additively homogeneous, it is nonexpansive in the sup-norm, and so, the map \( h \) is also order preserving and nonexpansive in the sup-norm, hence it is additively subhomogeneous. Since \( u \) is harmonic with respect to \( g \), that is a fixed point of \( g \), \( u_N \) is a fixed point of the map \( h \). A classical result of convex analysis (Theorem 23.9 of [Rock70]) shows in particular that if \( G \) is a finite valued convex function defined on \( \mathbb{R}^d \), if \( A \) is a linear map \( \mathbb{R}^p \to \mathbb{R}^d \), and if \( H(v) := G(Av) \), then \( \partial H(v) = A^\top \partial G(Av) \). Applying this result to every convex map \( G_i \) defined on \( \mathbb{R}^p \) such that \( G_i(w) := g_i(w + r_N(0,u_C)) \), with \( i \in N \), and to the linear map \( A = r_N^* \), we deduce that \( \partial h_i(u_N) \) is the projection on \( \mathbb{R}^N \) of the subdifferential of \( G_i \) at the point \( r_N^*(u_N) \), or equivalently of the subdifferential of \( g_i \) at the point \( r_N^*(u_N) + r_N(0,u_C) = u_N + u_C = u \). Using \( [15] \), this implies that \( \partial h_i(u_N) = \{M_{NN} \mid M \in \partial g_i(u)\} \). Since \( g \) is order preserving and additively homogeneous, the elements of \( \partial g_i(u) \) are stochastic matrices, and by the above equality, or since \( h \) is order preserving and additively subhomogeneous, the elements of \( \partial h_i(u_N) \) are substochastic matrices. Recall that the set of critical nodes of \( h \) is defined as the set of nodes that belong to a final class \( F \) of some matrix \( P \in \partial h_i(u_N) \) satisfying that \( P_{FF} \) is stochastic. Denote by \( F \) such a class. We have \( F \subseteq N \). Moreover, since \( \partial h_i(u_N) = \{M_{NN} \mid M \in \partial g_i(u)\} \), we can find a matrix \( Q \in \partial g_i(u) \) the \( N \times N \) submatrix of which, \( Q_{NN} \), coincides with \( P \). Since \( F \subseteq N \), \( Q_{FF} \) coincides with \( P_{FF} \). Hence \( Q_{FF} \) is a stochastic matrix, which implies that \( F \) is a recurrent class of \( Q \). This shows that the nodes of \( F \) are critical nodes of \( g \), which contradicts the fact that the set of critical nodes is \( C \) since \( F \subseteq N = [n] \setminus C \). Therefore the set of critical nodes of \( h \) is empty. It follows from Corollary 1.3 of [AG03] that \( h \) has a unique fixed point. \( \square \)

We shall need the following result of [AG03].

**Lemma 2.** ([AG03, (7) and Lemma 3.3]). Let \( g \) be a convex order-preserving additively homogeneous self-map of \( \mathbb{R}^n \), with at least one harmonic vector. Denote by \( C \) the set of critical nodes. If \( u \) is super-harmonic with respect to \( g \), then \( g(u) = u \) on \( C \), and \( g^\omega(u) := \lim_{k \to \infty} g^k(u) \) exists, is harmonic with respect to \( g \) and coincides with \( u \) on \( C \). Moreover, the map \( g^\omega : \mathcal{H}^+(g) \to \mathcal{H}(g) \) is order-preserving, additively homogeneous, convex, and is a projector.

The following result gives other characterizations of \( g^\omega(u) \) that allows one to compute it efficiently.

**Theorem 3.** Let \( g \) denote a convex, order preserving, and additively homogeneous self-map of \( \mathbb{R}^n \). Assume that \( g \) admits at least one harmonic vector. Let \( C \) denote the set of critical nodes of \( g \), and let \( N \) denote its complement in \([n]\), \( N = [n] \setminus C \). For a super-harmonic vector \( u \), the following conditions define uniquely the same vector \( v \):

(i) \( v = g^\omega(u) := \lim_{k \to \infty} g^k(u) \);

(ii) \( v \) is harmonic and coincides with \( u \) on \( C \);

(iii) \( v \) coincides with \( u \) on \( C \) and its restriction to \( N \) is a fixed point of the map \( h : w \mapsto (r_N \circ g \circ r_J)(w,u_C) \);

(iv) \( v \) is the smallest super-harmonic vector that dominates \( u \) on \( C \).
Proof. (i)$\Rightarrow$(ii): This follows from Lemma 2.

(ii)$\Rightarrow$(iii): Assume that the vector $v$ is harmonic and coincides with $u$ on $C$ and let $h$ be defined as in Point (iii). Then, $v_N = h(v_N)$, showing that $v_N$ is a fixed point of $h$.

(iii)$\Rightarrow$(i): Let $v$ and $h$ be as in Point (iii), hence $v_C = u_C$ and $v_N$ is a fixed point of $h$. By Lemma 2, $w := \varphi^*(u)$ is harmonic with respect to $g$ and $w_C = u_C$. Applying Lemma 1 to $g$ and $w$ (instead of $u$), and using $w_C = u_C$, we get that the fixed point of $h$ is unique, and thus equal to $w_N$. This shows that $v_N = w_N$, and since $v_C = u_C = w_C$, we get that $v = w = \varphi^*(u)$, that is Point (i).

(ii)$\Rightarrow$(iv): Let $v$ be as in Point (ii). Since $v$ is harmonic and coincides with $u$ on $C$, it is super-harmonic and dominates $u$ on $C$. By ((ii)$\Rightarrow$(iii)), $v_N$ is a fixed point of $h$, with $h$ as in Point (iii). Assume now that $w$ is super-harmonic and dominates $u$ on $C$, that is $w_C \geq u_C$. Then, $w \geq g(w)$, and since $g$ is order preserving, $w_N \geq g_N(w_N, w_C) \geq g_N(w_N, u_C) = h(w_N)$. Since $h$ is order-preserving, we deduce from $w_N \geq h(w_N)$ that $w_N \geq h^1(w_N) \geq h^2(w_N) \geq \cdots$. Since $h$ is nonexpansive and admits a fixed point, every orbit of $h$ is bounded. Hence, $h^k(w_N)$ has a limit as $k$ tends to infinity, and this limit is a fixed point of $h$. Applying Lemma 1 to $g$ and $v$ (instead of $u$), and using $v_C = u_C$, we get that the fixed point of $h$ is unique and equal to $v_N$. It follows that $w_N \geq v_N$. Since $v$ coincides with $u$ on $C$ and $w_C \geq u_C$, we deduce that $w \geq v$. This shows that $v$ is the smallest super-harmonic vector that dominates $u$ on $C$.

(iv)$\Rightarrow$(ii): Let $v$ be a minimal super-harmonic vector that dominates $u$ on $C$ (or the smallest one if it exists). Since $v$ is a super-harmonic vector, that is $g(v) \leq v$, and $g$ is order-preserving, we get that $g(g(v)) \leq g(v)$, which shows that $g(v)$ is also super-harmonic. Moreover, by Lemma 2, $g(v)$ coincides with $v$ on $C$, hence it dominates $u$ on $C$. Since $g(v) \leq v$, the minimality of $v$ implies $g(v) = v$, which shows that $v$ is harmonic. Since $u$ and $v$ are super-harmonic vectors and $g$ is order-preserving, we get that the infimum $v \wedge u$ of $v$ and $u$ is also a super-harmonic vector. Since $v$ dominates $u$ on $C$, we get that $v \wedge u$ equals $u$ on $C$. Hence by the minimality of $v$, and $v \wedge u \leq v$, we obtain that $v = v \wedge u$, hence $v \leq u$. This implies that $v$ equals $u$ on $C$, hence $v$ satisfies (ii). \qed

Let $g^\omega$ be defined as in Theorem 3. When $g(v) = Mv$ is a linear operator, and $M$ is a stochastic matrix, $g^\omega(u)$ coincides with the reduced super-harmonic vector of $u$ with respect to the set $C$. When $g$ is a max-plus linear operator, the operator $g^\omega$ coincides with the spectral projector which has been defined in the max-plus literature, see [CTGC99]. For this reason, we call $g^\omega$ the (nonlinear) spectral projector of $g$.

We now define a spectral projector acting on half-lines. We assume that $g$ is a polyhedral, convex, order preserving, and additively homogeneous self-map of $\mathbb{R}^n$. This implies in particular that for all $i \in [n]$, the domain of the Legendre-Fenchel transform $g^*_i$ of the coordinate $g_i$ of $g$ is included in the set of stochastic vectors, and that $g_i$ is the Legendre-Fenchel transform of $g^*_i$, hence can be put in the same form as in (6):

$$g_i(v) = \max_{b \in B_i} \left( \sum_{j \in [n]} P^b_{ij} v_j + r^b_i \right),$$

where, for all $i \in [n]$, $P^b_i \in \mathbb{R}^n$ is a stochastic vector, $r^b_i \in \mathbb{R}$, and $B_i$ is the domain of $g^*_i$, see [AG03 Prop. 2.1 and Cor. 2.2]. Since the map $g_i$ is polyhedral, the domain of $g^*_i$ is also a polyhedral convex set, see [RGG70] Th. 19.2, and since it is included in the set of stochastic vectors, it is compact, hence it is the convex envelope of the finite set of its extremals. Then, in (16), $B_i$ can be replaced by this finite set.

Since $g$ is polyhedral, order preserving, and additively homogeneous, we get by Kohlberg theorem [Koh80] recalled in Section 2 that $g$ has an invariant half-line $(\eta, v)$, $\eta$ is necessarily equal to $\chi(g)$, and by (14), $v$ and $\eta$ satisfy $\eta = g(\eta)$ and $\eta + v = g_0(v)$, where $\hat{g}$ and $g_0$ are defined in (10) and (11a) respectively. When $g$ is given by (16), these maps can be rewritten as:

$$[\hat{g}(\eta)]_i = \max_{b \in B_i} \left( \sum_{j \in [n]} P^b_{ij} \eta_j \right), \quad i \in [n],$$

$$g_0(v) = \max_{b \in B} \left( \sum_{j \in [n]} P^b_{0j} v_j + r^b_0 \right),$$

$$v_N = h(v_N).$$

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and

\[ [\hat{g}_n(v)]_i = \max_{b \in B_i, a} \left( \sum_{j \in [n]} P_{ij}^b v_j + r_i^b \right), \]

(18a)

\[ \hat{B}_{i, n} := \arg\max_{b \in B_i} \left\{ \sum_{j \in [n]} P_{ij}^b \eta_j \right\}. \]

(18b)

Let us fix an invariant half-line \((\eta, v)\) of \(g\). Denote \(\check{g}(w) := \hat{g}_n(w) - \eta\), then \(v\) is harmonic with respect to \(\check{g}\): \(\check{g}(v) = v\). We define the set of critical nodes of \(g\), \(C(g)\), to be the set of critical nodes of \(\check{g}\). A half-line \(w: t \mapsto t\eta + v\) is super-invariant if \(g \circ w(t) \leq w(t+1)\), for \(t\) large enough. From \(\text{[9]}\), this property is equivalent to the conditions \(\eta \geq \check{g}(\eta)\) with \(v_i \geq \check{g}_i(v)\) when \(\eta_i = \check{g}_i(\eta)\). In particular when the equality \(\eta = \check{g}(\eta)\) holds, it is equivalent to \(v \geq \check{g}(v)\).

**Corollary 1.** Assume that \(g\) is a polyhedral, convex, order preserving, and additively homogeneous self-map of \(\mathbb{R}^n\). Assume that \(w: t \mapsto t\eta + v\) is a super-invariant half-line of \(g\) with \(\eta = \chi(g)\). Then, there exists a unique invariant half-line of \(g\) which coincides with \(w\) on the set of critical nodes of \(g\). It is given by \(t \mapsto t\eta + \check{g}^\omega(v)\), where \(\check{g}: w \mapsto \hat{g}_n(w) - \eta\).

**Proof.** As said above, an invariant half-line of \(g\) must be of the form \(t \mapsto t\eta + z\), where \(\eta = \chi(g)\) and \(z \in \mathbb{R}^n\) is a fixed point of \(\check{g}\). If \(w: t \mapsto t\eta + v\) is a super-invariant half-line of \(g\) with \(\eta = \chi(g)\), then \(\eta = \check{g}(\eta)\), and by \(\text{[9]}\), we get \(v \geq \check{g}(v)\). From this, we deduce that \(t \mapsto t\eta + z\) is an invariant half-line of \(g\) which coincides with \(w\) on \(C\), if and only if \(z\) is harmonic with respect to \(\check{g}\) and coincides with \(v\) on \(C\). By Theorem \(\text{[3]}\), \(\check{g}^\omega(v)\) is such a harmonic vector, and it is the unique one. The corollary follows. \(\square\)

For any super-invariant half-line \(w\) of \(g\) with \(\eta = \chi(g)\), we define \(\check{g}^\omega(w)\) to be the half-line \(t \mapsto t\eta + \check{g}^\omega(v)\).

## 4 Policy iteration algorithm for stochastic mean payoff games

The following policy iteration scheme was introduced by Cochet-Terrasson and Gaubert in \(\text{[CTG06]}\). We first give, in Algorithm \(\text{I}\), an abstract formulation of the algorithm similar to the one given in \(\text{[CTG06]}\), which is convenient to establish its convergence. A detailed practical algorithm will follow and the proof of the convergence of the algorithm will be given in the last subsection.

### 4.1 The theoretical algorithm

In order to present the algorithm, we assume that every coordinate of \(f: \mathbb{R}^n \to \mathbb{R}^n\) is given by:

\[ f_i(v) = \min_{a \in A_i} f_i^a(v), \]

(19)

where \(A_i\) is a finite set, and \(f_i^a\) is a polyhedral order preserving, additively homogeneous, and convex map from \(\mathbb{R}^n\) to \(\mathbb{R}\). These conditions all together are indeed equivalent to the property that \(f\) is of the form \(\text{[17]}\), since as already observed any polyhedral order preserving, additively homogeneous and convex map \(g\) from \(\mathbb{R}^n\) to \(\mathbb{R}\) can be put in the form \(\text{[16]}\), with \(B_i\) a finite set. For all feedback strategies \(\sigma \in A_M = \{ \sigma: [n] \to A_i \mid i \mapsto \sigma(i) \in A_i \}\), we denote by \(f^{(\sigma)}\) the self-map of \(\mathbb{R}^n\) the \(i\)-th coordinate of which is given by \(f_i^{(\sigma)} = f_i^{a_i}\).

**Algorithm 1** (Policy iteration for multichain mean payoff two player games \(\text{[CTG06]}\)).

**Input:** A map \(f\) the coordinates of which are of the form \(\text{[16]}\).

**Output:** An invariant half-line \(w: t \mapsto t\eta + v\) of \(f\) and an optimal policy \(\sigma \in A_M\).

1. **Initialization:** Set \(k = 0\). Select an arbitrary strategy \(\sigma_0 \in A_M\). Compute an invariant half-line of \(f^{(\sigma_0)}\), \(w^{(0)}: t \mapsto t\eta^{(0)} + v^{(0)}\).
2. If \( f \circ w^{(k)}(t) = w^{(k)}(t + 1) \) holds for \( t \) large enough, the algorithm stops and returns \( w^{(k)} \) and \( \sigma_k \).

3. Otherwise, improve the strategy \( \sigma_k \) for \( w^{(k)} \), by selecting a strategy \( \sigma_{k+1} \) such that \( f \circ w^{(k)}(t) = f^{(\sigma_{k+1})} \circ w^{(k)}(t) \), for \( t \) large enough. The choice of \( \sigma_{k+1} \) must be conservative, meaning that, for all \( i \in [n] \), \( \sigma_{k+1}(i) = \sigma_k(i) \) if \( f \circ w^{(k)}(t) = f^{(\sigma_k)} \circ w^{(k)}(t) \), for \( t \) large enough.

4. Compute an arbitrary invariant half-line \( w'(t) : t \mapsto t \eta^{(k+1)} + v' \) of \( f^{(\sigma_{k+1})} \). If \( \eta^{(k+1)} \neq \eta^{(k)} \), then set \( v^{(k+1)} = v' \), i.e. \( w^{(k+1)} = w' \), and go to step \( 5 \). Otherwise (\( \eta^{(k+1)} = \eta^{(k)} \)), we say that the iteration is degenerate.

5. Compute the invariant half-line \( w^{(k+1)} = (f^{(\sigma_{k+1})})^\omega(w^{(k)}) \) of \( f^{(\sigma_{k+1})} \), and define \( v^{(k+1)} \) and \( \eta^{(k+1)} \) by \( w^{(k+1)}(t) = t \eta^{(k+1)} + v^{(k+1)} \).

6. Increment \( k \) by one and go to step \( 2 \).

Let us give some details about the well posedness of this algorithm. First, the existence of the invariant half-lines in Steps 1 and 4 follows from Kohlberg theorem [Koh80] applied to the polyhedral order preserving additively homogeneous maps \( f^{(\sigma_k)} \) with \( k \geq 1 \). Second, due to the finiteness of the action sets \( A_i \) and the fact that the maps \( f^a \) are polyhedral, the maps \( f \) and \( f^a \) can be rewritten in the form \( (f^a) \). Hence, the test of Step 2 and the asymptotic optimization problem of Step 3 can be rewritten as an equality test for (germs of) half-lines and the pointwise minimization of a finite set of half-lines, which are transformed into systems of equations and lexicographical optimization problems, using the representation of half-lines as couples \( (\eta, v) \) instead of maps \( w : t \mapsto \eta + v \), see the following section for details.

Finally, at each iteration \( k \) of Algorithm 1, \( w^{(k)} : t \mapsto t \eta^{(k)} + v^{(k)} \) is a super-invariant half-line of \( f^{(\sigma_{k+1})} \). Indeed, by construction of \( \sigma_{k+1} \), and since \( w^{(k)} \) is an invariant half-line of \( f^{(\sigma_k)} \), we get

\[
(\sigma^{(k+1)}(w^{(k)}(t))) = f(w^{(k)}(t)) \leq f^{(\sigma_k)}(w^{(k)}(t)) = w^{(k)}(t + 1),
\]

for \( t \) large enough. Moreover, since \( w^{(k)} \) is an invariant half-line of \( f^{(\sigma_k)} \), we have \( \chi(f^{(\sigma_k)}) = \eta^{(k)} \). Hence, in Step 3, \( w^{(k)} \) is a super-invariant half-line of \( f^{(\sigma_{k+1})} \) with slope \( \eta^{(k)} \) equal to \( \eta^{(k+1)} = \chi(f^{(\sigma_{k+1})}) \). By Corollary 1 there exists a unique invariant half-line of \( f^{(\sigma_{k+1})} \) which coincides with \( w^{(k)} \) on the set of critical nodes of \( f^{(\sigma_{k+1})} \) and it is given by \( w^{(k+1)} = (f^{(\sigma_{k+1})})^\omega(w^{(k)}) : t \mapsto t \eta^{(k+1)} + v^{(k+1)} \) with \( v^{(k+1)} = (f^{(\sigma_{k+1})})^\omega(\eta^{(k)}) \). Practical computations are detailed in the following sections.

4.2 The practical algorithm

All the steps of Algorithm 1 involve equality tests or pointwise minimizations of half-lines. However, it would not be robust to do these tests on half-lines just by choosing an arbitrary large number \( t \) in the equations and inequations to be solved. We shall rather use the equivalence between the representation of a half-line as a map \( w : t \mapsto \eta + v \) with \( t \) large and that as a couple \( (\eta, v) \). This allows one to transform all the tests into systems of equations or optimizations of finite sets of half-lines for the pointwise lexicographic order (which is linear, for each coordinate). This means that we are solving the system of equations \( (14) \). Then, using the notations of Section 2, the corresponding practical algorithm of the formal Algorithm 1 is given below in Algorithm 2.

Algorithm 2 (Policy iteration for multichain mean payoff two player games).

Input: A map \( f \) the coordinates of which are of the form \( (f^{(a)}) \) and the notations \( (\eta, v) \) and \( (0) \).

Output: An invariant half-line \( (\eta, v) \) of \( f \) and an optimal policy \( \sigma \in A_M \).

1. Initialization: Set \( k = 0 \). Select an arbitrary strategy \( \sigma_0 \in A_M \). Compute the couple \( (\eta^{(0)}, v^{(0)}) \) solution of

\[
\begin{aligned}
\left\{ \begin{array}{l}
\eta_i^{(0)} = F(\eta^{(0)}; i, \sigma_0(i)) \\
v_i^{(0)} + \eta_i^{(0)} = F_{\eta^{(0)}}(v^{(0)}; i, \sigma_0(i))
\end{array} \right. \quad \text{for all } i \in [n].
\end{aligned}
\]
2. If $\eta^{(k)}$ and $v^{(k)}$ satisfy System (14), or equivalently if $\sigma^{k+1} = \sigma_k$ is solution of (22) below, then the algorithm stops and returns $(\eta^{(k)}, v^{(k)})$ and $\sigma_k$.

3. Otherwise, improve the policy $\sigma_k \in A_M$ for $(\eta^{(k)}, v^{(k)})$ in a conservative way, that is choose $\sigma^{k+1} \in A_M$ such that

$$
\begin{align*}
\sigma^{k+1}(i) &\in \arg\min_{a \in A_i(v^{(k)})} \{ F_{\eta^{(k)}}(v^{(k)}; i, a) \} & \text{for all } i \in [n]. \\
\sigma^{k+1}(i) &= \sigma_k(i) \text{ if } \sigma_k(i) \text{ is optimal},
\end{align*}
$$

(22)

4. Compute a couple $(\eta^{(k+1)}, v')$ for policy $\sigma^{k+1}$ solution of

$$
\begin{align*}
\eta^{(k+1)}(i) &= F(\eta^{(k+1)}; i, \sigma^{k+1}(i)) \\
\eta^{(k+1)}(i) + v'(i) &= F_{\eta^{(k+1)}}(v'; i, \sigma^{k+1}(i)) & \text{for all } i \in [n].
\end{align*}
$$

(23)

If $\eta^{(k+1)} \neq \eta^{(k)}$ then set $v^{(k+1)} = v'$ and go to step 6. Otherwise, the iteration is degenerate.

5.i) Let $g := f^{(\sigma^{k+1})}(g_t = F(v; i, \sigma_k(i)))$. Compute $C(g)$ the set of critical nodes of the map $\tilde{g}$ defined by: $\tilde{g} = g_{\eta^{(k+1)}}(\cdot) - \eta^{(k+1)}$, or equivalently:

$$
\tilde{g}_i(v) = F_{\eta^{(k+1)}}(v; i, \sigma_k(i)) - \eta^{(k+1)}
$$

for which $v'$ is a harmonic vector.

5.ii) Compute $v^{(k+1)} = \tilde{g}_c(v^{(k)})$, that is the solution of:

$$
\begin{align*}
\eta^{(k+1)}(i) &= F_c(v^{(k)}; i, \sigma_k(i)) - \eta^{(k+1)} & i \in [n] \setminus C(g) \\
v^{(k+1)}(i) &= v^{(k)} & i \in C(g).
\end{align*}
$$

(24)

6. Increment $k$ by one and go to Step 2.

It remains to precise how the steps are performed. Step 3 is just composed of lexicographic optimization problems in finite sets. The systems (21) and (23) are the dynamic programming equations of a one player multichain mean payoff game, they can be computed by applying the policy iteration algorithm for multichain Markov decision processes with mean payoff introduced by Howard [How60] and Denardo and Fox [DF68]. Note that one can also choose to solve Systems (21) and (23) by applying Algorithm 2 to the maps $h = f^{(\sigma_0)}$ and $h = f^{(\sigma_{k+1})}$ respectively, while replacing minimizations by maximizations, but in that case the algorithm is almost equivalent to that of Howard [How60] and Denardo and Fox [DF68], see Section 5 below. In Step 5 the set of critical nodes of $g$, that is that of $\tilde{g}$, can be computed using a variant of the algorithm proposed in [AC03] § 6.3 described in Section 5.3. Finally, System (24) is the dynamic programming equation of an optimal control problem with infinite horizon stopped when reaching the set $C(g)$ which can be solved using the original policy iteration algorithm of Howard [How60]. We shall recall all these algorithms in Section 5.

### 4.3 Convergence of the algorithm

In this subsection, we show in Theorem 7 that Algorithm 1 or equivalently Algorithm 2 terminates after a finite number of steps. This result is proved using Theorem 3. Let first show some intermediate results.

The following lemma is known, see for instance Sorin [Sor04].

**Lemma 4** (See [Sor04]). Let $g$ denote an order preserving self-map of $\mathbb{R}^n$, that is nonexpansive in the sup-norm, and has a cycle time $\chi(g)$. If $w : t \mapsto t\eta + v$ is a super-invariant half-line of $g$, then, $\chi(g) \leq \eta$.

**Proof.** We reproduce the argument, for completeness: if $w : t \mapsto t\eta + v$ is a super-invariant half-line of $g$, that is $g(w(t)) \leq w(t+1)$ for $t \geq t_0$ for some $t_0 \geq 0$, then, $g^k(w(t)) \leq w(t+k)$, for all $k \geq 0$, and $t \geq t_0$, and so $\chi(g) \leq \lim_{k \to \infty} w(t_0 + k)/k = \eta$, which shows Lemma 4. \qed
Since, by (20), \( w^{(k)} \) is a super-invariant half-line of \( f^{(\sigma_k+1)} \), with slope \( \eta^{(k)} = \chi(f^{(\sigma_k)}) \), it follows from Lemma 4 that:

**Lemma 5.** The sequence of strategies defined in Algorithm 7 is such that

\[
\chi(f^{(\sigma_k+1)}) \leq \chi(f^{(\sigma_k)})
\]

We now examine degenerate iterations.

**Lemma 6.** Let \((\sigma_k)_{k \geq 1}\) be the sequence of strategies defined in Algorithm 7 and assume that \(\chi(f^{(\sigma_k+1)}) = \chi(f^{(\sigma_k)})\). Then, the following statements hold.

1. The half-line \( w^{(k+1)} \) agrees with \( w^{(k)} \) on the set of critical nodes of \( f^{(\sigma_k+1)} \).
2. Every critical node of \( f^{(\sigma_k+1)} \) is a critical node of \( f^{(\sigma_k)} \).
3. \( w^{(k+1)} \leq w^{(k)} \).

**Proof.** Let us use the notations: \( g := f^{(\sigma_k+1)} \) (as in Algorithm 2) and \( h = f^{(\sigma_k)} \). By construction and assumption, we have \( \eta^{(k)} = \chi(h) = \chi(g) = \eta^{(k+1)} \), that we shall also denote by \( \eta \).

**Point 1.** Since, by (20), \( w^{(k)} \) is a super-invariant half-line of \( g \), with slope \( \eta^{(k)} = \chi(g) \), and since \( w^{(k+1)} \) is defined as \( g^\circ(w^{(k)}) \), the result follows from Corollary 4.

**Point 2.** Again, since \( w^{(k)} \) is a super-invariant half-line of \( g \), with slope \( \eta^{(k)} = \chi(g) \), we deduce from the definition of \( \tilde{g} \) and (20), that

\[
\tilde{g}(v^{(k)}) \leq v^{(k)}.
\]

(25)

Then by Lemma 2, \( \tilde{g}(v^{(k)}) \) agrees with \( v^{(k)} \) on \( C(\tilde{g}) = C(g) \), the set of critical nodes of \( g \), and so, the equality \( g(w^{(k+1)}(t)) = w^{(k)}(t+1) \) holds on \( C(g) \) for \( t \) large. Since \( w^{(k)} \) is an invariant half line of \( f^{(\sigma_k)} \), we get that \( f_i(w^{(k)}(t)) = f_i^{(\sigma_k+1)}(w^{(k+1)}(t)) = f_i^{(\sigma_k)}(w^{(k+1)}(t)) \) for \( t \) large enough and \( i \in C(g) \). Hence, the conservative selection rule ensures that \( \sigma_{k+1}(i) = \sigma_k(i) \) for all \( i \in C(g) \). This implies that \( g_i = h_i \) for all \( i \in C(g) \), and since \( \chi(g) = \chi(h) \), we get from the definitions of \( \tilde{g} \) and \( \tilde{h} \) that

\[
\tilde{g}_i = \tilde{h}_i \quad \text{for all } i \in C(g).
\]

(26)

Observe that \( v^{(k+1)}(t) \) is a fixed-point of \( \tilde{g} \), and that \( \tilde{g} \) is a polyhedral additively homogeneous order preserving convex selfmap of \( \mathbb{R}^n \). Hence the critical nodes of \( g \) are the indices that belong to a final class of a matrix \( M = \partial g(v^{(k+1)}) \) (since the elements of \( \tilde{g}(v^{(k+1)}) \) are stochastic matrices, all their final classes are recurrent). Let \( F \) be such a final class. From (15), the line \( M_i \in \partial \tilde{g}_i(v^{(k+1)}) \) for \( i \in F \), that is \( \tilde{g}_i(v) = \tilde{g}_i(v^{(k+1)}) \geq M_i(v-v^{(k+1)}) \) for all \( v \in \mathbb{R}^n \). Since \( v^{(k+1)} \) is a fixed point of \( \tilde{g} \), \( v^{(k)} \) a fixed point of \( h \), and \( v^{(k+1)} \) agrees with \( v^{(k)} \) on \( C(g) \) (from Point 1), we get that \( \tilde{g}_i(v^{(k+1)}) = v_i^{(k+1)} = v_i^{(k)} = h_i(v^{(k)}) \) for all \( i \in C(g) \). From (20), we deduce that \( \tilde{g}_i(v) = \tilde{g}_i(v^{(k+1)}) = h_i(v) = h_i(v^{(k)}) \) for all \( i \in C(g) \) and \( v \in \mathbb{R}^n \). Now, since \( F \) is a final class of \( M \), hence \( F \subset C(g) \), and \( M_{ji} = 0 \) for \( i \in F \) and \( j \not\in C(g) \), we get that \( M_{ji}(v^{(k+1)}) = M_{ji}(v^{(k)}) \) for \( i \in F \). This implies that \( h_i(v) = h_i(v^{(k)}) \geq M_i(v-v^{(k)}) \) for all \( v \in \mathbb{R}^n \) and \( i \in F \), which shows that \( M_i \in \partial h_i(v^{(k)}) \) for \( i \in F \). Let \( N := [n] \setminus F \) and define the matrix \( Q \) such that \( Q_i = M_i \) if \( i \in F \) and \( Q_i = 0 \) otherwise. Hence, the \( F \times F \) submatrix of \( M \) is also a \( F \times F \) submatrix of \( Q \), and so \( F \) is a final class of \( Q \). Since \( v^{(k)} \) is a fixed point of \( h \), this implies that \( F \) is included in the set of critical nodes of \( h \), which is also by definition the set of critical nodes of \( h \). This shows that all critical nodes of \( g \) are also critical nodes of \( h \), and shows Point 2.

**Point 3.** From (25), we get that \( \tilde{g}(v^{(k)}) \leq v^{(k)} \), hence the sequence \( \tilde{g}^k(v^{(k)}) \) is nonincreasing and \( \tilde{g}^k(v^{(k)}) \leq v^{(k)} \). Since \( \eta^{(k)} = \eta^{(k+1)} \), we get that \( w^{(k+1)} = g^\circ(w^{(k)}) = \eta^{(k)} + \tilde{g}(v^{(k)}) \leq w^{(k)} \).

Finally, we prove that the algorithm terminates.

**Theorem 7.** A strategy cannot be selected twice in Algorithm 7, and so, the algorithm terminates after a finite number of iterations.
which contradicts the existence of iteration $m$ since a strategy cannot be selected twice, Algorithm 1 stops after a finite number of iterations, in Algorithm 1.

So by Lemma 6, Part 1, we have that agree on $\sigma_m$. Hence, by Lemma 6, Part 2, we have that $C(f(\sigma_m)) = C(f(\sigma_{m-1})) = \cdots = C(f(\sigma_1))$. So by Lemma 6, Part 3, $w(s)$ and $w^{(m)}$ are both invariant half-lines of $f(\sigma_1)$ with slope $\chi(f(\sigma_1))$, that agree on $C(f(\sigma_1))$. Hence, by Corollary 1, $w(s) = w^{(m)}$. Since by Lemma 6, Part 3 we have $w(s) \geq w^{(s+1)} \geq \cdots \geq w^{(m)}$, it follows that $w(s) = \cdots = w^{(m)}$. In particular, $w(s) = w^{(s+1)}$. Hence, $w(s)(t+1) = w^{(s+1)}(t+1) = f^{(s+1)}(t) = f^{(s+1)}(t) = f \circ w^{(s)}(t)$ for $t$ large enough. It follows that $w(s)$ is an invariant half-line of $f$, and so, the algorithm stops at step $s$, which contradicts the existence of iteration $m$, and so the same strategy cannot be selected twice in Algorithm 1.

Since the sets $A_i$ are finite, the number of strategies (the elements of $A_M$) is also finite, and since a strategy cannot be selected twice, Algorithm 1 stops after a finite number of iterations, that is bounded by the number of strategies.

\section{Ingredients of Algorithm 1 or 2: one player games algorithms}

As said in Section 1.2, each basic step of the policy iteration algorithm for multichain mean payoff zero-sum two player games (Algorithm 1 or 2) concerns the solution of one player games, also called stochastic control problems or Markov decision processes, with finite state and action spaces:

A mean payoff problem for Systems (21) and (23), an infinite horizon problem stopped at the boundary for System (24), and the set of critical nodes of the corresponding dynamic programming operator in Step 3. We recall here the policy iteration algorithm for solving stochastic control problems, with either infinite horizon or mean payoff, and the algorithm proposed in [AG03, §6.3] for computing a critical graph, and explain how all these algorithms are applied in Algorithm 1 or 2. By doing so, we shall also see that the classical Howard / Denardo-Fox algorithm can be thought of as a special case of these algorithms, in which the second player has no choices of actions.

In all the section, we consider the following dynamic programming or Shapley operator of a one player game with finite state and action spaces: $g$ is a map from $\mathbb{R}^n$ to itself, given by:

$$[g(v)]_i := \max_{b \in B_i} G(v; i, b) \quad \forall i \in [n], \ v \in \mathbb{R}^n,$$

where

$$G(v; i, b) = \sum_{j \in [n]} P_{ij}^b v_j + r_{i}^b,$$

the vectors $P_{ij}^b$ are substochastic vectors, for all $i \in [n]$ and $b \in B_i$, and $B_i$ are finite sets, for all $i \in [n]$. Equivalently, $g$ is a convex additively subhomogeneous order preserving polyhedral selfmap of $\mathbb{R}^n$.

Since player $\text{min}$ does not exist, the set of feedback strategies for player $\text{max}$, $B_M$, is given by $B_M := \{ \delta : [n] \to B \mid \delta(i) \in B_i, \forall i \in [n] \}$, where $B$ contains all the sets $B_i$. For each $\delta \in B_M$, we denote by $g(\delta)$ the self-map of $\mathbb{R}^n$ given by:

$$g_i(\delta)(v) := G(v; i, \delta(i)) \quad \forall i \in [n], \ v \in \mathbb{R}^n.$$

We also denote by $\nu(\delta)$ the vector of $\mathbb{R}^n$ such that $r_i(\delta) = P_{ij}(\delta(i))$, and $P(\delta)$ the $n \times n$ matrix such that $P_{ij}(\delta) = P_{ij}(\delta(i))$, then $g(\delta) : v \mapsto P(\delta)v + \nu(\delta)$.

\subsection{Policy iterations for one player games with discounted payoff}

System (24) consists in finding the solution $v$ of the equation $v = \bar{g}(v)$ with $v = u$ on $C(\bar{g})$ where $u \in \mathbb{R}^n$ is super-harmonic with respect to $\bar{g}$, $\bar{g}(u) \leq u$, and $g$ is as in (27) with (28). The solution $v$ is thus the value of an one player game with infinite horizon stopped when reaching the set $C(\bar{g})$.
whose transition probabilities are given by the $P_{ij}$, instantaneous reward is given by the $r_i$, and final reward is given by $u_i$, when the game is in state $i \in C(g)$. This value function can be obtained using the classical policy iteration algorithm of Howard \cite{How60} for a one player game. From Theorem \[3], $v$ is solution of the above equation, if and only if $v_C = u_C$ and $v_N$ is a fixed point of the convex polyhedral additively subhomogeneous order preserving selfmap $h$ of $\mathbb{R}^N$, with $C = C(g), N = [n] \setminus C,$ and $h$ defined as in Theorem \[3] Point (iii), with $g$ replaced by $\hat{g}$. One can also consider the equivalent equation $v = h(v)$ with $h_i = \hat{g}_i$ for $i \in N$ and $h_i(v) = u_i$ for $i \in C$ and $v \in \mathbb{R}^n$. In that case, $h$ is a convex polyhedral additively subhomogeneous order preserving selfmap of $\mathbb{R}^n$.

In these two settings, we need to solve an equation of the form $v = g(v)$, where $g$ is of the form \[27], and $g$ has no critical node: $C(g) = \emptyset$. From \cite{AG03} Corollary 1.3, $g$ has a unique fixed point and all the maps $g^{(i)}$ with $i \in B_M$ have a unique fixed point (since their critical nodes are necessarily critical nodes of $g$). The policy iteration algorithm of Howard applied to this equation is then given by Algorithm \[3].

\textbf{Algorithm 3} (Policy iteration of Howard \cite{How60} for stochastic control problems).

\textbf{Input}: A map $g$ of the form \[27] with no critical node.

\textbf{Output}: The fixed point of $g$ and an optimal policy $\delta \in B_M$.

1. \textbf{Initialization}: Set $k = 0$. Select an arbitrary strategy $\delta_0 \in B_M$.

2. Compute the value of the game $v^{(k)}$ with fixed feedback strategy $\delta_k$, that is the solution of the linear system:

$$v^{(k)} = g^{(\delta_k)}(v^{(k)}) .$$

3. If $v^{(k)} = g(v^{(k)})$, or equivalently if $\delta_{k+1} = \delta_k$ is solution of \[29] below, then the algorithm stops and returns $v^{(k)}$ and $\delta_k$.

4. Otherwise, improve the policy $\delta_{k+1} \in B_M$ for the value $v^{(k)}$:

$$\delta_{k+1}(i) \in \arg\max_{b \in B_i} G(v^{(k)}; i, b) \quad \forall i \in [n]. \quad \tag{29}$$

5. Increment $k$ by one and go to Step 2.

It is known \cite{How60} that $v^{(k+1)} \leq v^{(k)}$ and that the algorithm stops after a finite number of steps.

5.2 Policy iteration for multichain one player games

Consider a one player game with dynamic programming operator $g$ given by \[27] and mean payoff. Then, as explained in Section 2 in the more general two player case, the mean payoff of the game is the slope $\eta$ of any invariant half line $(\eta, v)$ of $g$, which is also any solution of the following couple system (see Equation \[14]):

$$\begin{cases}
\eta &= \hat{g}(\eta) \\
\eta + v &= \hat{g}_v(v) .
\end{cases} \tag{30}$$

where $\hat{g}$ and $\hat{g}_v$ are defined in \[10] and \[11] respectively. In the present one player case, they are reduced to:

$$[\hat{g}(\eta)]_i := \max_{b \in B_i} \hat{G}(\eta; i, b) \quad \text{and} \quad [\hat{g}_v(v)]_i := \max_{b \in B_i, \eta} G(v; i, b) , \tag{31}$$

with

$$\hat{G}(\eta; i, b) = \sum_{j \in [n]} P_{ij}^{b} \eta_j \quad \text{and} \quad \hat{B}_i, \eta := \arg\max_{b \in B_i} \left\{ \sum_{j \in [n]} P_{ij}^{b} \eta_j \right\} . \tag{32}$$

for all $\eta, v \in \mathbb{R}^n$, $i \in [n]$, $b \in B$. We refer also to \cite{DF68, Put94} for the existence of solutions to System \[30], and for the proof that $\eta$ solution of this system is the mean payoff of the game in this one player context. The following algorithm for multichain mean payoff Markov decision processes was introduced by Howard \cite{How60} and proved to converge by Denardo and Fox \cite{DF68}:
Algorithm 4 (Policy iteration algorithm for multichain mean payoff one player games).

Input: A map \( g \) of the form (27) with (28), and the notations (31,32).

Output: An invariant half-line \((\eta,v)\) of \( g \) and an optimal policy \( \delta \in B_M \).

1. Initialization: Set \( k = 0 \). Select an arbitrary strategy \( \delta_0 \in B_M \).

2. For each final class \( F \) of \( P(\delta_0) \), denote by \( i_F \) the minimal index of the elements of \( F \), and define \( S \) as the set of all these indices \( i_F \). Compute the couple \((\eta^{(k)},v^{(k)})\) for policy \( \delta_k \) solution of

\[
\begin{aligned}
\eta_i^{(k)} &= \hat{G}(\eta^{(k)};i,\delta_k(i)) \quad i \in [n] \setminus S \\
\eta_i^{(k)} + v_i^{(k)} &= G(v^{(k)};i,\delta_k(i)) \quad i \in [n] \\
v_i^{(k)} &= 0 \quad i \in S.
\end{aligned}
\]  

(33)

3. If \((\eta^{(k)},v^{(k)})\) is solution of (30), or equivalently if \( \delta_{k+1} = \delta_k \) is solution of (34) below, then the algorithm stops and returns \((\eta^{(k)},v^{(k)})\) and \( \delta_k \).

4. Otherwise, improve the policy \( \delta_{k+1} \in B_M \) for \((\eta^{(k)},v^{(k)})\) in a conservative way, that is choose \( \delta_{k+1} \in B_M \) such that :

\[
\begin{aligned}
\delta_{k+1}(i) &\in \arg\max_{b \in B_{i,\delta_k}} G(v^{(k)};i,b) \\
\delta_{k+1}(i) &= \delta_k(i) \text{ if } \delta_k(i) \text{ is optimal},
\end{aligned}
\]  

(34)

5. Increment \( k \) by one and go to Step 2.

The justifications and details of Algorithm 4 can be found in [DF68, Put94] and are recalled in Appendix. Solving System (33) turns out to be a critical step. This can be optimized by exploiting the structure of the system, we discuss this issue in Appendix. As explained in Section 4.2 another way to solve a multichain mean payoff Markov decision process may be to use Algorithm 4 or 2 in the particular case of a one-player game, with maximizations instead of minimizations. In order to compare it with Algorithm 4 we rewrite below Algorithm 2 in that case, with the above notations. Note that in the one-player case, the map \( g \) of Step 5 of Algorithm 2 is affine, hence its critical graph reduces to the final graph of its tangent matrix.

Algorithm 5 (Specialization of Algorithm 2 to the one player case).

Input: A map \( g \) of the form (27) with (28), and the notations (31,32).

Output: An invariant half-line \((\eta,v)\) of \( g \) and an optimal policy \( \delta \in B_M \).

1. Initialization: Set \( k = 0 \). Select an arbitrary strategy \( \delta_0 \in B_M \). Compute the couple \((\eta^{(0)},v^{(0)})\) solution of

\[
\begin{aligned}
\eta_i^{(0)} &= \hat{G}(\eta^{(0)};i,\delta_0(i)) \quad \text{for all } i \in [n]. \\
\eta_i^{(0)} + v_i^{(0)} &= G(v^{(0)};i,\delta_0(i)) \quad \text{for all } i \in [n].
\end{aligned}
\]  

(35)

2. If \( \eta^{(k)} \) and \( v^{(k)} \) satisfy System (30), or equivalently if \( \delta_{k+1} = \delta_k \) is solution of (36) below, then the algorithm stops and returns \((\eta^{(k)},v^{(k)})\) and \( \delta_k \).

3. Otherwise, improve the policy \( \delta_k \in B_M \) for \((\eta^{(k)},v^{(k)})\) in a conservative way, that is choose \( \delta_{k+1} \in B_M \) such that

\[
\begin{aligned}
\delta_{k+1}(i) &\in \arg\max_{b \in B_{i,\delta_k}} G(v^{(k)};i,b) \\
\delta_{k+1}(i) &= \delta_k(i) \text{ if } \delta_k(i) \text{ is optimal},
\end{aligned}
\]  

(36)

4. Compute a couple \((\eta^{(k+1)},v')\) for policy \( \delta_{k+1} \) solution of

\[
\begin{aligned}
\eta_i^{(k+1)} &= \hat{G}(\eta^{(k+1)};i,\delta_{k+1}(i)) \quad \text{for all } i \in [n]. \\
\eta_i^{(k+1)} + v_i' &= G(v';i,\delta_{k+1}(i)) \quad \text{for all } i \in [n].
\end{aligned}
\]  

(37)

If \( \eta^{(k+1)} \neq \eta^{(k)} \) then set \( v^{(k+1)} = v' \) and go to step 6. Otherwise, the iteration is degenerate.
5.1) Compute $C$, the set of final nodes of the matrix $P^{(\delta_{k+1})}$.

5.2) Compute the solution $v^{(k+1)}$ of:

$$
\begin{align*}
\begin{cases}
v^{(k+1)}_i &= G(v^{(k+1)}; i, \delta_{k+1}(i)) - \eta^{(k+1)}_i & i \in [n] \setminus C \\
v^{(k+1)}_i &= v^{(k)}_i & i \in C.
\end{cases}
\end{align*}
$$

(38)

6. Increment $k$ by one and go to Step 2.

Systems (35) and (37) are of the form:

$$
\begin{align*}
\begin{cases}
\eta &= P\eta \\
\eta + v &= P\eta + r,
\end{cases}
\end{align*}
$$

(39)

where $r = \gamma^{(d)} \in \mathbb{R}^n$ and $P = P^{(d)}$ is a stochastic matrix, with $\delta = \delta_0$ or $\delta_{k+1}$. It can be shown that the solution $\eta$ of such a system is unique, that one can eliminate for each final class $F$ of $P$ one of the equations $\eta_i = (P\eta)_i$ with index $i \in F$, and that $v$ is defined up to an element of the kernel of $I - P$, the dimension of which is equal to the number of final classes of $P$. When this number is strictly greater than one, and $v^{(k+1)}$ is chosen to be any solution $v^i$ of (37) in Algorithm 5, the algorithm may cycle, see Section 6 for an example in the two player case. One way to handle this is either to fix to zero the value of $\mu_F v$ for each invariant measure $\mu_F$ of $P$ with support in a final class $F$ of $P$, or to fix to zero the components of $v$ with indices in some set $S$ containing exactly one node of each final class of $P$. In these two cases, the solution $v$ of (39) become unique. Moreover, if in Algorithm 5 (37) is combined with either the conditions $\mu_F v = 0$ or the conditions $v^i_0 = 0$ with $S$ chosen in a conservative way, that is such that the same index is chosen in $F$ for iterations $k$ and $k + 1$, if $F$ is a final class of $P^{(\delta_{k+1})}$ which is also a final class of $P^{(\delta_k)}$, then $v' = v^{(k)}$ on the set of final nodes of $P^{(\delta_{k+1})}$ when $\eta^{(k+1)} = \eta^{(k)}$, which implies that $\eta^{(k+1)} = \eta^{(k)}$, hence Step 5 of Algorithm 5 becomes useless. This shows that Algorithm 4 is equivalent to Algorithm 5, where (37) is combined with the conditions $v^{i}_0 = 0$, where $S$ is the set of minimal indices of each final class of $P^{(\delta_{k+1})}$. In other words, Algorithm 4 is a particular realization of Algorithm 5, where one chooses one special solution $v' = v^{(k+1)}$ of (37) at each iteration of the algorithm, even when $\eta^{(k+1)} \neq \eta^{(k)}$. Denardo and Fox proved [DF68, Put94] that the sequence of couples $(\eta^{(k)}, v^{(k)})_{k \geq 1}$ of Algorithm 4 is non decreasing in a lexicographical order, meaning that $\eta^{(k+1)} \geq \eta^{(k)}$, with $v^{(k+1)} \geq v^{(k)}$ when $\eta^{(k+1)} = \eta^{(k)}$, and that Algorithm 4 stops after a finite number of iterations (when the sets of actions are finite). Indeed, the convergence of Algorithm 4 proved in Section 4.3 shows that this also holds for the little more general Algorithm 5.

5.3 Critical graph

When a degenerate iteration ($\eta^{(k+1)} = \eta^{(k)}$) occurs in Step 4 of Algorithm 4 one has to compute the critical nodes of $g := f^{(\sigma_{k+1})}$, that is that of $\tilde{g}$. This can be done by applying the techniques of [AG03] § 6.3, leading to Algorithm 6 below. More precisely, one applies first the followings steps to the map $\tilde{g}$ and its harmonic vector $v'$, then apply Algorithm 6.

Consider an additively homogeneous map $g$ whose coordinates are defined as in (27) with (28), and $u$ a harmonic vector of $g$. For any set $\mathcal{P}$ of stochastic matrices, we define $\mathcal{G}^i(\mathcal{P})$ as the union of the graphs of the matrices $M_{\mathcal{P}}$, where $M \in \mathcal{P}$ and $F$ is a final class of $M$. Define

$$
\tilde{B}_i = \left\{ b \in B_i \mid G(u; i, b) = u \right\} \quad \text{and} \quad \mathcal{P}_i = \left\{ P^b_i \mid b \in \tilde{B}_i \right\}.
$$

(40)

Then, the critical graph of $g$ is given by

$$
\mathcal{G}^i(g) = \mathcal{G}^i(\partial g(u)), \quad \text{where} \quad \partial g(u) = \text{co}(\mathcal{P}_1) \times \cdots \times \text{co}(\mathcal{P}_n),
$$

(41)

and co(·) denotes the convex hull of a set. The following algorithm computes the graph in (41) for a general family $\left\{ \mathcal{P}_i \right\}_{i \in [n]}$, where $\mathcal{P}_i \subset \mathbb{R}^n$ is a nonempty finite set of stochastic vectors. Note that any such family $\left\{ \mathcal{P}_i \right\}_{i \in [n]}$ corresponds to the map $g : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$
[g(v)]_i = \max_{P \in \mathcal{P}_i} Pf_i \quad \text{for all} \quad i \in [n],
$$

(42)
which has $u = 0$ as a harmonic vector, and is of the above form. Hence the algorithm below corresponds also to the computation of the critical graph of this map $g$.

Before writing the algorithm, we recall some definitions of graph theory (see for instance [CLRS01]). We define a graph $G := (V, E)$ as a finite set of vertices (or nodes) $V$ and a set of edges (or arcs) $E := \{(i, j) \mid i, j \in V\}$. A path of length $l \geq 0$ is a sequence $(i_0, \ldots, i_l)$ such that $i_k \in V$ for $k \in \{0, \ldots, l\}$ and $(i_k, i_{k+1}) \in E$ for $k < l$. A strongly connected component of $G$ is the restriction $G|_{V'}$ of $G$ to some subset of nodes $V' \subseteq V$, that is the graph $(V', E')$ with $E' := \{(i, j) \in E \mid i, j \in V'\}$, where $V'$ is such that there exists a path from each node $i \in V'$ to every node $j \in V'$. A strongly connected component $G'$ is called trivial if it consists in exactly one node and no arcs. We define a final class of $G = (V, E)$ as a non trivial strongly connected component $G' = (V', E')$ of $G$ such that there exists no arc $(i, j) \in E$ with $i \in V'$ and $j \in V \setminus V'$.

Note that the strongly connected components of a graph can be found using Tarjan algorithm, see [CLRS01].

Algorithm 6 (Algorithm to compute the critical graph, compare with [AG03 § 6.3]).

**Input:** $(\mathcal{P}_1, \cdots, \mathcal{P}_n)$ where $\mathcal{P}_i \subset \mathbb{R}^n$ is a finite set of stochastic vectors for $i \in [n]$.

**Output:** A graph depending on $\mathcal{P}_1, \cdots, \mathcal{P}_n$, equal to $G^i(\text{co}(\mathcal{P}_1) \times \cdots \times \text{co}(\mathcal{P}_n))$ if all the $\mathcal{P}_i$ are nonempty; and its set of nodes.

1. Set $F(0) = \emptyset$, $I(0) = [n]$, $G(0) = \emptyset$, $Q^{(0)}_i = \mathcal{P}_i$ for $i \in [n]$, and $k = 0$.
2. If all the sets $\{Q^{(k)}_i\}_{i \in I(k)}$ are empty, then the algorithm stops and returns $G^{(k)}$ and $F^{(k)}$.
3. Otherwise, build the graph $G = (I^{(k)}, E)$ with set of nodes $I^{(k)}$, and set of arcs $E = \{(i, j) \in I^{(k)} \times I^{(k)} \mid p_j \neq 0 \text{ for some } p \in Q^{(k)}_i\}$. Set $F$ as the union of final classes of $G$.
4. Put $I^{(k+1)} = I^{(k)} \setminus F$ and $E^{(k+1)} = F^{(k)} \cup F$.
5. Set $G^{(k+1)} = G^{(k)} \cup G|_F$ where $G|_F$ denotes the restriction of $G$ to $F$.
6. For all $i \in I^{(k+1)}$, define the sets $Q^{(k+1)}_i \subset \mathbb{R}^{I^{(k+1)}}$ of row vectors obtained by restricting to $I^{(k+1)}$ the vectors $p \in Q^{(k)}_i$ such that $\sum_{j \in I^{(k+1)}} p_j = 1$.
7. Increment $k$ by one, and go to Step 2.

The convergence (after at most $n$ iterations) of this algorithm follows from variants of Lemmas 4.7 and 4.9 of [AG03], applied to the maps $g_k$ constructed by (12) from the families $(Q^{(k)}_i)_{i \in [n]}$. Indeed, if all the $Q^{(k)}_i$ with $i \in I^{(k)}$ are nonempty, the map $g_k$ is a map from $\mathbb{R}^n$ to itself and Lemma 4.7 says that $g_k$ has at least one invariant critical class, which implies that the set $F$ of Step 5 is nonempty. Moreover, Lemma 4.9 says that, if all the $Q^{(k+1)}_i$ with $i \in I^{(k+1)}$ are nonempty, the critical graph of $g_k$ is equal to the union of $G|_F$ with the critical graph of the map $g_{k+1}$.

In order to generalize these arguments, one need to extend the notion of critical graph to the case of a map $g$ from $(\mathbb{R} \cup \{-\infty\})^n$ to itself, of the form (22) with general families $(\mathcal{P}_i)_{i \in [n]}$ of (possibly empty) finite sets of stochastic vectors (or of the form (27) with (28), with a harmonic vector $u \in (\mathbb{R} \cup \{-\infty\})^n$). For instance, define the critical graph of $g$ as the restriction to the set of nodes $i \in [n]$ such that $\mathcal{P}_i$ is nonempty (or $u_i \neq -\infty$) of the critical graph of $g \lor \text{id}$, where $\text{id}$ is the identity map and $\lor$ denotes the supremum operation. Then, the identically $-\infty$ map has no critical class, any map $g$ which is not identically $-\infty$ has an invariant critical class, and the above recurrence formula for critical graphs is true even if $g_{k+1}$ takes $-\infty$ values. This shows that Algorithm 6 computes the critical graph of the map $g$ associated to the family $(\mathcal{P}_i)_{i \in [n]}$, even if some of the sets of the family are empty.

Note that since Tarjan algorithm has a linear complexity in the number of arcs of a graph, the complexity of the above algorithm is at most in the order of $nm$, where $m$ is the sum of the number of arcs of all the elements of $\mathcal{P}_i, i \in [n]$. This is comparable with the complexity of solving the linear systems of the form (33) by LU solvers, hence with the other steps of Algorithm 2.
6 An example with degenerate iterations

In this section, we present an example of zero-sum two player stochastic game for which we encounter a degenerate iteration when using the policy iteration algorithm for the mean payoff problem, and showing that Step 5 of Algorithm 1 is essential to obtain the convergence of the algorithm.

Before doing this, let us note that some degenerate cases may be not so problematic. Indeed, as observed before, the map \( \tilde{g} \) of Step 5 of Algorithm 2 is a polyhedral order preserving additively homogeneous convex map. By [AG03, Theorem 1.1], the set of fixed points of \( \tilde{g} \) is isomorphic to a convex set which dimension is the number of strongly connected components of the critical graph of \( \tilde{g} \) and which is invariant by the translations by a constant function. In particular, if the number of strongly connected components of the critical graph is equal to one, then the set of fixed points of \( \tilde{g} \) is exactly equal to the translations of \( v' \) by a constant, hence \( \nu(k+1) - v' \) is a constant function. Since all the maps considered in Algorithm 2 are additively homogeneous, this implies that taking \( v' \) instead of \( \nu(k+1) \), that is applying the same steps as in the nondegenerate case, does not change the sequence of policies \((\sigma_k)\), and the invariant half lines are just translated by a constant after this degenerate iteration. Hence, the second part of Step 5 may be avoided in Algorithm 2 when one encounters only such degenerate iterations. However, to know that \( \tilde{g} \) has only one strongly connected component in its critical graph, one need to apply the the first part of Step 5.

We show now an example for which degenerate iterations occur with two strongly connected components of the critical graph of \( \tilde{g} \). We shall call these iterations strongly degenerate.

We consider a directed graph, with a set of nodes (or edges) \([n]\) and a set of arcs \(E \subset [n] \times [n]\), in which each arc \((i,j)\) is equipped with a weight \(r_{ij} \in \mathbb{R}\), and consider the map \( f \) from \(\mathbb{R}^n\) to itself, defined by:

\[
f_i(v) = \frac{1}{2} \left( \max_{j: (i,j) \in E} (r_{ij} + v_j) + \min_{j: (i,j) \in E} (r_{ij} + v_j) \right)
\]

(43)

When the value of \( v \) is fixed at some “boundary” points, and the weights \(r_{ij}\) are independent of \(j\), the map \( f \) arises as the dynamic programming operator of the “tug of war” game [PSSW09], which can viewed also as a discretization of the infinity Laplacian operator. Moreover the case where all the weights \(r_{ij}\) are equal to zero corresponds to a class of auction games, called Richman game [LLP+99]. Therefore, the above map \( f \) appears as the dynamic programming operator of a variant of these games with additive reward and mean payoff.

We apply the policy iteration algorithm to such a game, with a graph of 5 nodes and complete set of arcs \(E = [5] \times [5]\). Hence, the action spaces \(A_i\) and \(B_i\), in every state \(i \in [n]\) can be identified with the set \([5]\). The weight of each arc \((i,j) \in E\) is defined as the entry \(r_{ij}\) of the following matrix:

\[
r = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & -1 & 0 & -1 & 1
\end{pmatrix},
\]

the adjacency graph of which is represented in Figure 4.

Let us fix the initial strategy \(\sigma_0\) for the first player, such that \(\sigma_0(1) = 2\), \(\sigma_0(2) = 2\), \(\sigma_0(3) = 4\), \(\sigma_0(4) = 4\), \(\sigma_0(5) = 2\). Then, the corresponding dynamic programming operator \(f(\sigma_0)\) is given by:

\[
\begin{align*}
f_1^{(\sigma_0)}(v) &= f_2^{(\sigma_0)}(v) = \frac{1}{2} \left( -1 + v_2 + \max(1 + v_1, -1 + v_2, v_3, v_4, v_5) \right) \\
f_3^{(\sigma_0)}(v) &= f_4^{(\sigma_0)}(v) = \frac{1}{2} \left( -1 + v_4 + \max(v_1, v_2, 1 + v_3, -1 + v_4, v_5) \right) \\
f_5^{(\sigma_0)}(v) &= \frac{1}{2} \left( -1 + v_2 + \max(v_1, -1 + v_2, v_3, -1 + v_4, 1 + v_5) \right).
\end{align*}
\]

In Step 3 of Algorithm 2, we compute an invariant half-line of \(f(\sigma_0)\) and obtain for instance \(w^{(0)}(t) = (\eta^{(0)}, v^{(0)}), \) with \(v^{(0)} = (0, 0, -0.5, -0.5, 0)^T\) and \(\eta^{(0)} = (0, 0, 0, 0)^T\). Since \(f(w^{(0)}(t)) < f(\sigma_0) (g^{(0)}(t))\), we need to improve the policy (Step 3) and get the unique solution (even without
the conservative policy): $\sigma_1(1) = 2$, $\sigma_1(2) = 2$, $\sigma_1(3) = 4$, $\sigma_1(4) = 4$, $\sigma_1(5) = 4$. The corresponding operator is then given by:

$$
\begin{align*}
\sigma_i(f^{(\sigma)}) &= f_i^{(\sigma_i)} \\
\sigma_5(f^{(\sigma)}) &= \frac{1}{2}(-1 + v_4 + \max(v_1, -1 + v_2, v_3, -1 + v_4, 1 + v_5))
\end{align*}
$$

We compute then (in Step 4) an invariant half-line $(v^{(1)}, v')$ of $f^{(\sigma)}$, and obtain $\eta^{(1)} = (0, 0, 0, 0)^T$ and for instance $v' = (0, 0, 0.5, 0.5, 0.5)^T$. Since $\eta^{(1)} = \eta^{(0)}$, the iteration is degenerate.

Hence the algorithm enters in Step 5. Set $g := f^{(\sigma)}$. We have to compute the critical graph of $g$, which is here equal to $g$, for instance by applying Algorithm 6 to the sets $P_i$ defined in (40) with $u = v'$. They are given by $P_1 = P_2 = \{0.5, 0.5, 0, 0, 0\}$, $P_3 = P_4 = \{0, 0, 0, 5, 0.5, 0\}$, $P_5 = \{0, 0, 0, 5, 0.5, 0\}$, then the critical graph of $g$ is equal to the final graph of $P_1 \times \cdots \times P_5$, which is composed of two strongly connected components with nodes $\{1, 2\}$ and $\{3, 4\}$. Then, $v^{(1)}$ is the unique solution of:

$$
\begin{align*}
\left\{ \begin{array}{ll}
v_k^{(1)} &= f_k^{(\sigma)}(v) = \frac{1}{2}(-1.5 + \max(0, -1, -0.5, -1.5, v_5^{(1)} + 1)) \\
v_i^{(0)} &= v_i^{(0)}
\end{array} \right. \\
&\quad i \in \{1, 2, 3, 4\}
\end{align*}
$$

We obtain $v^{(1)} = (0, 0, -0.5, -0.5, -0.5)$ and since $f(w^{(1)}(t)) = f^{(\sigma)}(w^{(1)}(t))$, the algorithm stops.

However, if we do not treat the degenerate case by using Step 5 and take for instance $v^{(1)} = v'$, we obtain $f(w^{(1)}(t)) < f^{(\sigma)}(w^{(1)}(t))$, hence we need to improve the strategy, and obtain the unique solution $\sigma_2 = \sigma_0$. This means that the algorithm cycle, showing the necessity of Step 5 in the policy iterations.

7 Implementation and numerical results

The numerical results presented in this section were obtained with a slight modification of the policy iteration Algorithm 2 and of its ingredients of Section 5, all implemented in the C library PIGAMES, see [Det12] for more information. All the tests of this section were performed on a single processor: Intel(R) Xeon(R) W3540 - 2.93GHz with 8Go of RAM.

These slight modifications take into account the fact that (linear or nonlinear) equations may not be solved exactly (in exact arithmetics) because of the errors generated by floating-point computations, and also of the possible use of iterative methods instead of exact methods. Let us explain them briefly. For instance, the stopping criterion in Step 2 of Algorithm 2 can be replaced by a condition on the residual of the mean payoff, $f(\eta^{(k)}) - \eta^{(k)}$ and the residual of the relative value, $\tilde{f}_\epsilon(\eta^{(k)}) - \eta^{(k)} - v^{(k)}$. Here, we consider the infinity norm of the residual of the game that we define as $0.5 \times (\|f(\eta^{(k)}) - \eta^{(k)}\|_\infty + \|\tilde{f}_\epsilon(\eta^{(k)}) - \eta^{(k)} - v^{(k)}\|_\infty)$, where $\| \cdot \|_\infty$ denotes the sup-norm. Then, we stop the policy iterations when the infinity norm of the residual of the game is smaller than a given value $\epsilon_g > 0$ or when the strategies cannot be improved. For the tests of
this section, we took $\epsilon_g = 10^{-12}$. We use the same condition for the stopping criterion of the intern policy iterations, that is for Step 3 of Algorithm 3 and Step 3 of Algorithm 4. Moreover, the optimization problems in Step 3 of Algorithm 2 and Step 3 of Algorithms 3 and 4, are solved up to some precision. This means for instance that in Algorithm 2, one choose $\sigma_{k+1} \in A_M$ such that, for all $i \in [n]$,

$$
\begin{cases}
\hat{F}_{i,\eta}(v^{(k)}; i, \sigma_{k+1}(i)) \leq \epsilon_v + \min_{a \in \hat{A}_{i,\eta,\epsilon}} \left\{ \hat{F}_{i,\eta}(v^{(k)}; i, a) \right\}
\end{cases}
$$

with

$$
\hat{A}_{i,\eta,\epsilon} := \{ a \in A_i \mid \hat{F}(\eta; i, a) \leq \epsilon + \hat{f}(\eta) \}
$$

for some given $\epsilon_\eta$ and $\epsilon_v > 0$. Finally, the linear systems in Step 2 of Algorithms 3 and 4 are solved up to some precision, which may be lower bounded when the matrices of the systems are ill-conditioned. See the appendix for details about the solution of these linear systems.

7.1 Variations on tug of war and Richman games

We now present some numerical experiments on the variant of Richman games defined in Section 6, constructed on random graphs. As in the previous section, we consider directed graphs, with a set of nodes equal to $[n]$ and a set of arcs $E \subset [n]^2$. The dynamic programming operator is the map $f$ defined in (43), where the value $r_{ij}$ is the reward of the arc $(i,j) \in E$. In the tests of Figure 2 to Figure 4, we chose random sparse graphs with a number of nodes $n$ between 1000 and 50000, and a number of outgoing arcs fixed to ten for each node. The reward of each arc in $E$ has value one or zero, that is $r_{ij} = 1$ or 0. The arcs $(i,j) \in E$ and the associated rewards $r_{ij}$ are chosen randomly (uniformly and independently). We start the experiments with a sizer of graph (number of nodes) equal to $n = 1000$, then we increase the size by 1000 until reaching $n = 10000$, after we increase the size by 10000 and end with a number of 50000 nodes. For each size that we consider, we made a sample of 500 tests. The results of the application of the policy iteration (Algorithm 2 with the above modifications) on those games are presented in Figures 2 to 4 and are commented below.

Figure 2 gives for each size $n$, and among the sample of 500 tests, the number of tests that encountered at least one strongly degenerate policy iteration for the first player. Hence, these games require the degenerate case issue presented in this paper, that is Step 5 of Algorithm 1 or 2. Moreover, from the data of Figure 2, we observe that approximately between 10 and 15 percent of the tests have at least one strongly degenerate policy iteration for the first player.

| Number of strongly degenerate iterations | 0   | 1   | 2   | 3   | 6   |
|----------------------------------------|-----|-----|-----|-----|-----|
| Number of tests                        | 6051| 919 | 28  | 1   | 1   |

We observe that in general there is no more than one or two strongly degenerate policy iterations for our sample of tests. Note that in this section, a strongly degenerate policy iteration is to be understood as a strongly degenerate iteration for the first player only, that is for Algorithm 2.
In Figure 3, we draw on the left curves that represent the number of policy iterations for the first player, that is the number of iterations of Algorithm 2, as a function of the size $n$ of the graph. The dashed lines on top and bottom are respectively the maximum and minimum value, over the sample of 500 tests, and the plain line is the average value, all as a function of the size. We observe that the average number of first player’s policy iterations is almost constant as the size increases. Using the same model of representation, we show on the right of Figure 3 respectively the maximum, average and minimum values for the total number of policy iterations for the second player, that is the sum of the numbers of iterations of Algorithm 4 when applied by Algorithm 2 as a function of the size. We also observe that these values do not vary a lot with the size.

In Figure 4, we present on the left the total cpu time (in seconds) needed by the policy iteration to find the solution of the game. As for the two previous figures, the curves from top to bottom show respectively the maximum, average and minimum values, over the sample of 500 tests, as a function of the size of the graphs. Finally, on the right of Figure 4, we give also the average of the total cpu time (in seconds) needed to solve the game but we separated the tests with strongly degenerate policy iteration(s), represented by the dashed line, from the non strongly degenerate ones, represented by the plain curve. We observe that the average cpu time is somewhat greater for the tests with strongly degenerate iteration(s). This is due to the additional steps needed for degenerate iterations. Indeed, the cpu time of a degenerate iteration should be approximately the double of that of a nondegenerate iteration, and since the number of policy iterations is around 10 in the sample of tests, the average of the total cpu time of tests with (strongly) degenerate iterations should be approximately 10 percent greater than that of the other tests.

In addition, in Table 1, we give numerical results for ten tests of the variant of Richman game,
constructed on random large graphs with a number of nodes between $10^5$ and $10^6$. We observe that the number of iterations are of the same order as for the previous sample of tests presented in Figure 3.

Table 1: Numerical results on a variant of Richman game constructed on random large graphs.

| Number of nodes | Iterations | Total number of iterations | Strongly degenerate | Infinity norm of residual | CPU time (s) |
|-----------------|------------|----------------------------|---------------------|--------------------------|--------------|
| 100000          | 12         | 78                         | 1                   | $1.44e - 14$             | 3.24e + 02   |
| 200000          | 12         | 74                         | 0                   | $7.44e - 15$             | 7.90e + 02   |
| 300000          | 11         | 82                         | 0                   | $1.33e - 15$             | 9.38e + 02   |
| 400000          | 12         | 82                         | 1                   | $8.55e - 15$             | 1.42e + 03   |
| 500000          | 12         | 77                         | 1                   | $2.00e - 14$             | 2.16e + 03   |
| 600000          | 12         | 77                         | 0                   | $8.66e - 15$             | 2.61e + 03   |
| 700000          | 11         | 85                         | 0                   | $3.02e - 14$             | 2.61e + 03   |
| 800000          | 12         | 81                         | 1                   | $4.82e - 14$             | 6.79e + 03   |
| 900000          | 12         | 79                         | 1                   | $1.27e - 14$             | 4.17e + 03   |
| 1000000         | 12         | 90                         | 1                   | $3.33e - 15$             | 1.96e + 04   |

7.2 Pursuit games

We consider now a pursuit evasion game with two players: a pursuer and an evader. The evader wants to maximize the distance between him and the pursuer and the pursuer has the opposite objective. See for instance [BFS94, BFS99, LCS08] for a complete description of general pursuit games. To simplify the model, we consider as state of the game, the distance between the two objective. We also restrict the state in the domain $X = \mathbb{R}^2$ centered in the 0-position, that is $x \in X := [-0.5,0.5] \times [-0.5,0.5]$. At each time of the game, the reward for the evader is the euclidean square norm of the distance between the two players, i.e. $\|x\|_2^2$. Such a game is a special class of differential game, the dynamic programming equation of which is an Isaacs partial differential equation. Under our simplifications and assumptions, the Hamiltonian of this equation is given by:

$$H(x,p) = \max_{a \in A(x)} (a \cdot p) + \min_{b \in B(x)} (b \cdot p) + \|x\|_2^2 \quad \forall x \in X, p \in \mathbb{R}^2,$$

meaning that in the case of a finite horizon problem, the Isaacs equation would be given, at least formally (but also in the viscosity sense) by:

$$-\frac{\partial v}{\partial t} + H(x, \nabla v(x)) = 0 \quad x \in X.$$

Here $A(x)$ and $B(x)$ are the sets of possible directions for the evader and the pursuer respectively, when the state is equal to $x \in X$. On the boundary, we consider that only actions keeping the state of the game in the domain $X$ are allowed, hence the above equation has to be satisfied until the boundary.

We shall consider this differential game with a mean-payoff criterion and the above reward. This means that the analogous to System (14) is the following system of Isaacs equations:

$$\begin{cases}
\max_{a \in A(x)} (a \cdot \nabla \eta(x)) + \min_{b \in B(x)} (b \cdot \nabla \eta(x)) = 0, \quad x \in X, \\
-\eta(x) + \max_{a \in \hat{A}_\eta(x)} (a \cdot \nabla v(x)) + \min_{b \in \hat{B}_\eta(x)} (b \cdot \nabla v(x)) + \|x\|_2^2 = 0, \quad x \in X,
\end{cases}$$

where

$$\hat{A}_\eta(x) := \arg\max_{a \in A(x)} (a \cdot \nabla \eta(x)),$$

$$\hat{B}_\eta(x) := \arg\min_{b \in B(x)} (b \cdot \nabla \eta(x)).$$
In classical pursuit-evasion games, such as in [BFS99], the reward is constant and the value function is defined as the time (or the exponential of the opposite of the time) for the pursuer to capture the evader, then the value function is solution of the stationary Isaacs equation that is \( \eta \equiv 0 \), corresponding to the above Hamiltonian with 1 instead of \( \|x\|^2 \). In that case, the value is infinite when the pursuer’s speed is smaller than the evader’s speed, and it is would be difficult to compute an optimal strategy using Isaacs equation. Here by considering a mean-payoff problem, we may solve the problem even when pursuer’s speed is smaller than the evader’s speed, as we shall see below. Note that one may have kept the reward equal to 1, but then the optimal value \( \eta \) would have given less information.

A monotone discretization, for instance a finite difference discretization scheme (see [KD92]), of System (14) yields to System (46) for the dynamic programming operator \( f \) of a discrete time and finite state space game, which then may be solved using our policy iteration Algorithm 2.

In our tests, the domain \( X \) is discretized in each directions with a constant step size \( h \). Then the two players of the discrete game are moving on the discretized nodes of the domain, similarly to the moves in a chess game. We assume also that the evader cannot move when the euclidean norm of the relative distance between him and the pursuer is less than 0.1, i.e when \( x \in B((0, 0); 0.1) \).

We shall call the evader, the mouse and his set of possible actions at each state of the game will be given by:

\[
A(x) := \begin{cases}
(a_1, a_2) | a_l \in \{0, 1, -1\}, & l = 1, 2, \\
(0, 0) & x \in B((0, 0); 0.1)
\end{cases}
\]

where \( \bar{X} \) denotes the interior of \( X \). The pursuer, that we shall call the cat, has the following set of possible actions:

\[
B(x) := \{(b_1, b_2) | b_l \in \{0, b, -b\}, & l = 1, 2, \\
& x \in \bar{X}
\}
\]

where \( b \) is a positive real constant and represents the speed of the cat. Moreover, on the boundary of \( X \), the sets \( A(x) \) and \( B(x) \) are restricted to avoid actions that bring the state out of \( X \).

Numerical results for this game are presented in Table 2 when \( b = 0.999 \), \( b = 1 \) and \( b = 1.001 \) respectively. Note that the solution of the discretization of Equation (46) may differ from the solution of the continuous equation. We observe that for \( b = 0.999 \) and \( b = 1.001 \), we have a strongly degenerate iteration for the first player on the last iteration.

The optimal actions for the discretized problem with \( b = 0.999 \) are represented in Figure 5 at each node of the grid: the actions of the mouse are on the left, and that of the cat are on the right. The optimal actions are approximately the same for the two other values of \( b \). When \( b = 0.999 \), the speed of the cat is smaller than the speed of the mouse (= 1). The numerical results for the discretized game give an optimal mean-payoff \( \eta \) such that \( \eta(x) = 0.492 \) for \( x \in X \setminus B((0, 0); 0.1) \) and \( \eta(x) = 0 \) for \( x \in B((0, 0); 0.1) \). This means that the cat cannot catch the mouse when their starting positions are not too close and the mouse can keep almost the maximum distance between them. The relative value is represented on the left of Figure 5. When \( b = 1 \), the speeds of the cat and the mouse are equal. The numerical results for the discretized game give a relative value \( \nu \) approximately equal to zero for every starting point and an optimal mean-payoff \( \eta(x) \approx \|x\|^2 \), meaning that the cat and mouse keep the same initial distance all along the game. In the last example, the speed of the cat \( \bar{b} = 1.001 \) is greater than that of the mouse (= 1). The numerical results for the discretized game give an optimal mean-payoff \( \eta \) close to zero. The relative value \( \nu \) is given on the right of Figure 5. In this case, the cat catches the mouse.

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Table 2: Numerical results for the mouse and cat example where $\bar{b}$ is the speed of the cat. The second column is the index of the iteration on the cat’s policies and the third column is the corresponding total number of iterations on the mouse’s policies. The last column indicates if the cat’s policy iteration is strongly degenerate. Number of discretization nodes: $257 \times 257$.

| $\bar{b}$ | Cat policy iteration index | Number of mouse policy iterations | Infinite norm of residual | CPU time (s) | Strongly degenerate iteration |
|------------|-----------------------------|----------------------------------|---------------------------|-------------|-------------------------------|
| 0.999      | 1                           | 2                                | $1.25e$ - 06              | $2.59e + 01$ | 0                             |
|            | 2                           | 1                                | $9.93e$ - 12              | $3.95e + 01$ | 0                             |
|            | 3                           | 1                                | $5.68e$ - 14              | $7.35e + 02$ | 1                             |
| 1          | 1                           | 2                                | $1.25e$ - 06              | $2.60e + 01$ | 0                             |
|            | 2                           | 1                                | $3.39e$ - 21              | $3.84e + 01$ | 0                             |
| 1.001      | 1                           | 2                                | $1.25e$ - 06              | $2.59e + 01$ | 0                             |
|            | 2                           | 1                                | $1.96e$ - 14              | $6.51e + 02$ | 1                             |

Figure 5: Optimal actions for the mouse on the left and for the cat on the right.

Figure 6: Relative value $v$ for the mouse and cat game when the speed of the mouse is one, the speed of the cat equals 0.999 on the left and 1.001 on the right.
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A Details of implementation of Policy Iteration for multi-chain one player games

We explain in more details here why System (33) has a unique solution and is selecting one special solution of System (47) with $k$ instead of $k + 1$, and how it is solved practically (see also [DF68, Pnt94]).

Recall that System (37) is of the form (39) rewritten here:

$$
\begin{cases}
\eta &= P\eta \\
\eta + v &= P\eta + r ,
\end{cases}
$$

where $r = r^{(\delta)} \in \mathbb{R}^n$ and $P = P^{(\delta)}$ is a stochastic matrix, with $\delta = \delta_{k+1}$. Moreover, System (39) corresponds to

$$
\begin{cases}
\eta_i &= (P\eta)_i, & i \in [n] \setminus S , \\
\eta_i + v_i &= (Pv)_i + r_i, & i \in [n] , \\
v_i &= 0, & i \in S ,
\end{cases}
$$

(47)

where $r$ and $P$ are as before but with $\delta = \delta_k$, and where $S$ is composed of minimal indices $i_F$ of each final classes $F$ of $P$. Then one need to show that System (37) has a unique solution and is selecting one special solution of System (39).

First, for all final classes $F$ of $P$, $P_{FF}$ is an irreducible Markov matrix, hence the equation $\eta_F = (P\eta)_F$, which is equivalent to $\eta_F = P_{FF}\eta_F$, is also equivalent to the condition that $\eta_i = \eta_j$ for all $i, j \in F$. Moreover, $P_{FF}$ has a unique stationary (or invariant) probability measure $\pi_F$, that is a row probability vector solution of $\pi_F = \pi_F P_{FF}$, and this vector has strictly positive coordinates. This implies that one can eliminate, for each final class $F$ of $P$, one equation with index $i \in F$ in the equation $\eta = P\eta$, without changing the set of solutions. Hence, any solution of System (47) is also solution of (39).

Second, denote by $S$ the union of final classes, and by $T$ the union of transient classes, that is the complement in $[n]$ of $F$. Then, $w$ is in the kernel of $I - P$ if, and only if, it satisfies $w_F = P_{FF}w_F$ for all final classes $F$ of $P$ and $w_T = P_{TT}w_T + P_{TF}w_F$. As said before the first equations are equivalent to the conditions $w_i = w_j$ for all $i, j \in F$. Since $P_{TT}$ has transient classes only, it has a spectral radius strictly less than one, which implies that given the vectors $w_F$ for all final classes $F$, the equation $w_T = P_{TT}w_T + P_{TF}w_F$ has a unique solution $w_T$. Hence, the dimension of the kernel of $I - P$ is equal to the number of final classes of $P$, and any element of this kernel which has one coordinate $i \in F$ equal to zero for each final classes $F$ of $P$, has all its coordinates equal to zero. This implies that given $\eta \in \mathbb{R}^n$, the solution $v$ of System (47) is unique if it exists (the difference between two such solutions satisfies the above conditions). This also implies that the codimension of the image of $I - P$ is equal to the number of final classes. Hence, the image of $I - P$ is exactly equal to the set of vectors $\eta \in \mathbb{R}^n$ such that $P_{FF}\eta_F = 0$, for all final classes $F$ of $P$. A vector $\eta \in \mathbb{R}^n$ is such that System (47) has a solution $v \in \mathbb{R}^n$ if and only if $\eta = P\eta$ and $\eta - r$ is in the image of $I - P$. These conditions are equivalent to the three conditions $\eta_i = \eta_j$ for all $i, j \in F$, for all final classes $F$, $\eta_T = P_{TT}\eta_T + P_{TT}\eta_F$, and $\pi_F\eta_F = \pi_F P_{TF}$, for all final classes $F$. The first and third conditions together are equivalent to $\eta_T = \pi_F P_{TF}$, for all $i \in F$, and all final classes $F$ of $P$, which gives a unique solution $\eta_T$. Since the second one has a unique solution $\eta_T$, given $\eta_F$, we get that there is a unique vector $\eta \in \mathbb{R}^n$ such that System (47) has a solution $v \in \mathbb{R}^n$. In conclusion, System (47) has a unique solution $(\eta, v)$, which finishes the proof what we wanted to show.

One may try to solve System (47) by using usual LU methods, however when $P$ is not irreducible, such a method is not robust. We rather use the decomposition of $P$ in classes, and the previous properties. In particular, since $P_{TT}$ has a spectral radius strictly less than 1, one can compute $(\eta, v)$ solution of System (47) by first computing $\eta_T$ and $v_T$, for all final classes $F$ of $P$, then computing successively $\eta_F$ and $v_F$ which are respectively fixed points of contracting affine systems with tangent linear operator $P_{TT}$:

$$
\begin{cases}
\eta_T &= P_{TT}\eta_T + P_{TF}\eta_F \\
v_T &= P_{TT}\eta_T + P_{TF}v_F + v_T - \eta_T
\end{cases}
$$

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There exists two ways to compute $\eta_F$ and $v_F$. One is to compute the stationary probability $\pi_F$ to determine $\eta_F$ by $\eta_i = \pi_F r_i$, for all $i \in F$, and then solve the following system with unknown $v_F \in \mathbb{R}^F$:

$$v_F = P_F v_F + r_F - \eta_F.$$  

by eliminating one equation (since one equation is redundant) with index $j \in F$, and adding the condition $v_i = 0$ for one element $i \in F$. Another method is to consider $\eta_F$ as constant, say $\eta_i = \bar{\eta}$ for $i \in F$, and solve the system with unknowns $\bar{\eta} \in \mathbb{R}$ and $v \in \mathbb{R}^F$:

$$\bar{\eta} + v_i = \sum_{j \in F} P_{ij} v_j + r_i, \quad i \in F,$$

by adding the condition $v_i = 0$ for one element $i \in F$. In our algorithm, we choose the index $i = j \in F$ to be the minimal index of $F$ (for a fixed total ordering of nodes). This method gives the following algorithm to solve System (47).

**Algorithm 7 (Solution of System (47)).** Decompose the matrix $P$ into irreducible classes and permute nodes without changing the order in each class, such that $P$ takes the following form:

$$P = \begin{pmatrix}
P_{11} & P_{12} & \cdots & \cdots & P_{1m} \\
0 & P_{22} & \cdots & \cdots & P_{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & P_{m-1,m-1} & P_{m-1,m} \\
0 & \cdots & \cdots & 0 & P_{mm}
\end{pmatrix}$$

where $m$ denotes the number of irreducible classes and $P_{II}$ are square irreducible submatrices of $P$, for $I = 1, \ldots, m$. Note that, the class corresponding to a submatrix $P_{II}$ is final if and only if the submatrices $P_{IJ}$ are all null for all $J \neq I$.

For each class $I$ from $m$ to 1, do the following:

**Step 1.** If $I$ corresponds to a final class, that is $P_{II}$ is a stochastic matrix, do one of the two following sequences of operations:

A. (a) Find the stationary probability $\pi_I$ of $P_{II}$: $\pi_I P_{II} = \pi_I$.
   (b) Set $\bar{\eta} = \pi_I r_I$ and $\bar{\eta} = \bar{\eta} \quad i \in I$.
   (c) Solve the system with unknowns $v_I \in \mathbb{R}^I$:

$$\begin{cases}
v_i = \sum_{j \in I} P_{ij} v_j + r_i - \bar{\eta} \quad i \in I\setminus S, \\
v_i = 0 \quad i \in S \cap I,
\end{cases} \quad (48)$$

B. Solve the system with unknowns $v_I \in \mathbb{R}^I$ and $\bar{\eta} \in \mathbb{R}$:

$$\begin{cases}
\bar{\eta} + v_i = \sum_{j \in I} P_{ij} v_j + r_i \quad i \in I, \\
v_i = 0 \quad i \in S \cap I,
\end{cases}$$

and set $\eta_i = \bar{\eta} \quad i \in I$.

**Step 2.** if $I$ corresponds to a transient class, that is if $P_{II}$ is a strictly submarkovian matrix. do the following steps:

(a) compute $\eta_I$ solution of the following system:

$$\eta_I = P_{II} \eta_I + \sum_{J > I} P_{IJ} \eta_J.$$

(b) compute $v_I$ solution of the following system:

$$v_I = P_{II} v_I + \sum_{J > I} P_{IJ} v_J + r_I - \eta_I.$$
In our numerical experiments, the linear system (33) at each intern policy iteration is solved by using Algorithm 7. For the numerical experiments of Section 7.1, on each final class, we used a SOR iterative solver to find the stationary probability \( \pi_I \) and also to compute the corresponding \( v_I \) in method A in Step 1 of Algorithm 7. For the transient class, we used the LU solver of the package [DEG+99].

The Successive Over-Relaxation (SOR) method is an iterative scheme that belongs to the class of splitting methods or relation methods, see for instance [BP94]. It is derived from the Gauss-Seidel relaxation scheme. Consider a matrix \( A \in \mathbb{R}^{n \times n} \) such that \( A = D - L - U \) where \( D, -L, -U \) are respectively the diagonal, lower and upper triangular part of \( A \). The SOR smoothing operator is defined by \( S_w = M^{-1}N \) where \( M = D - wL \) and \( N = [(1 - w)D + wU] \) for \( 0 < w < 2 \).

Consider the irreducible stochastic matrix \( P_{II} \in \mathbb{R}^{I \times I} \) and decompose \( I - P_{II}^T = D - L - U \) where \( I \) is the identity matrix of \( \mathbb{R}^{I \times I} \). Starting from an initial positive approximation \( \pi^{(0)} \in \mathbb{R}^I \), a SOR smoothing step to find the stationary probability of \( P_{II} \) is given by:

\[
\pi^{(k)} = S_w \pi^{(k-1)}
\]

\[
\pi^{(k+1)} = \frac{\pi^{(k)}}{\sum_{i \in [n]} \pi^{(k)}_i}.
\]

The sequence \( (\pi^{(2k)})_{k\geq0} \) converges to the transpose of the stationary probability of \( P_{II} \) when the limit \( \lim_{k \to \infty} S_w^{(2k)} \) exists, see [BP94] for more details. To solve Equation (48), decompose \( I - P_{II} = D - L - U \). Then, starting from an initial approximation \( v^{(0)} \in \mathbb{R}^I \), a SOR smoothing step consists in:

\[
v^{(k)} = (I - 1\mu) (S_w v^{(k-1)} + M^{-1}(r_I - \eta_I))
\]

where \( 1 = (1 \ldots 1)^T \in \mathbb{R}^I \), and \( \mu \in \mathbb{R}^I \) is a row vector such that \( \mu_i = 1 \) for \( i \in S \cap I \), and \( \mu_i = 0 \) otherwise. The sequence \( (v^{(k)})_{k\geq0} \) converges to the solution of Equation (48) when the limit \( \lim_{k \to \infty} S_w^{(k)} \) exists, see [BP94] for more details.

For the numerical tests of Section 7.2 we used the LU solver of the package [DEG+99] in both cases.