FIXED POINT PROPERTIES AND REFLEXIVITY IN VARIABLE LEBESGUE SPACES

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ABSTRACT. In this paper the weak fixed point property (w-FPP) and the fixed point property (FPP) in Variable Lebesgue Spaces are studied. Given $(\Omega, \Sigma, \mu)$ a σ-finite measure and $p(\cdot)$ a variable exponent function, the w-FPP is completely characterized for the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ in terms of the exponent function $p(\cdot)$ and the absence of an isometric copy of $L^1[0,1]$. In particular, every reflexive $L^{p(\cdot)}(\Omega)$ has the FPP and our results bring to light the existence of some nonreflexive variable Lebesgue spaces satisfying the w-FPP, in sharp contrast with the classic Lebesgue $L^p$-spaces. In connection with the FPP, we prove that Maurey’s result for $L^1$-spaces can be extended to the larger class of variable $L^{p(\cdot)}(\Omega)$ spaces with order continuous norm, that is, every reflexive subspace of $L^{p(\cdot)}(\Omega)$ has the FPP. Nevertheless, Maurey’s converse does not longer hold in the variable setting, since some nonreflexive subspaces of $L^{p(\cdot)}(\Omega)$ satisfying the FPP can be found. As a consequence, we discover that several nonreflexive Nakano sequence spaces $\ell^{p_m}$ do have the FPP endowed with the Luxemburg norm. As far as the authors are concerned, this family of sequence spaces gives rise to the first known nonreflexive classic Banach spaces enjoying the FPP without requiring of any renorming procedure. The failure of asymptotically isometric copies of $\ell_1$ in $L^{p(\cdot)}(\Omega)$ is also analyzed.

1. Introduction

The class of Variable Lebesgue Spaces (VLSs) arises as a generalization of classic Lebesgue spaces $L^p(\Omega)$, when the constant exponent $p$ is replaced with a variable exponent function $p(\cdot)$. The resulting function space $L^{p(\cdot)}(\Omega)$ shares many of the geometrical properties of the $L^p$-spaces but also differ from them in some sort of interesting and unexpected forms (as an example, VLSs are not translation invariant unless the exponent function $p(\cdot)$ is constant). Variable Lebesgue spaces can be traced back in the literature to 1931 [37] and they lie within the scope of the more general class of modular function spaces, initially defined by H. Nakano [34, 35, 36] and studied by Orlicz and Musielak [38]. However, the last two decades have
witnessed an explosive development in the analysis of the intrinsic structure of VLSs by their own right, in particular, since M. Růžička discovered that they constitute a natural functional setting for the mathematical model of electrorheological fluids \cite{38}. As Banach function spaces, the structure and geometrical properties of VLSs connected to Harmonic Analysis and some other areas within the scope of Functional Analysis have been studied in \cite{9, 11, 26, 30} and the references therein. However, a precise Fixed Point Theory for nonexpansive operators on this family of Banach spaces seems to be in a very incipient state.

We recall that a Banach space \((X, \| \cdot \|)\) is said to have the fixed point property (FPP) if every nonexpansive operator \(T : C \to C\), with \(C\) a closed convex bounded subset of \(X\), has a fixed point. Besides, \(X\) is said to have the weakly fixed point property (\(w\)-FPP) when the above holds for all domains which are convex and weakly compact. Here nonexpansiveness means \(\|Tx - Ty\| \leq \|x - y\|\) for every \(x, y \in C\) (note that nonexpansiveness is strictly subject to the underlying norm). The FPP and \(w\)-FPP are equivalent under reflexivity and these properties have been extensively studied in the framework of Banach spaces for the last 60 years, during which multiple and robust connections were displayed linking this area of Metric Fixed Point Theory with Geometry and Renorming Theory in Banach spaces. Although the first positive results for the existence of fixed points for nonexpansive mappings date back to 1965 \cite{25}, this theory is very far from being complete and there are still many interesting open problems left and some unsolved long-standing conjectures. In fact, it was long believed that all Banach spaces fulfilling the FPP had to be reflexive. In 2008 P.K. Lin proved that this statement was untrue by providing the sequence space \(\ell_1\) with an equivalent norm that let it have the FPP \cite{28}. This automatically meant that the fixed point property could be extended beyond reflexivity (see also \cite{6, 8, 14, 17, 18, 19, 29}). In fact, as a consequence of some results included in the paper, we will show that this is the case in some particular classes of Variable Lebesgue Spaces, where neither reflexivity nor any renorming argument are needed for establishing the FPP.

The organization of the article is the following:

In the second section we develop some preliminary results concerning modular function spaces and mainly related to the class of variable Lebesgue spaces, the reflexivity condition and the order continuity of the Luxemburg norm, which is the standard norm for VLSs.

The third section focuses on the proper study of the \(w\)-FPP. A complete description of those VLSs satisfying the \(w\)-FPP is obtained in terms of the exponent function \(p(\cdot)\). Due to Alspach’s example \cite{2}, who proved that \(L^1[0, 1]\) fails to have the \(w\)-FPP, we know that classic Lebesgue spaces \(L^p(\Omega)\) have the \(w\)-FPP if and only if they are reflexive. We can assert that this equivalence does not longer hold in our variable setting. Furthermore, we
establish the equivalence between the \( w \)-FPP and the absence of an isometric copy of \( L^1[0,1] \) when restricted to the family of VLSs.

One significant breakthrough in Metric Fixed Point Theory supporting the still open question as to whether “FPP is implied by reflexivity” is due to B. Maurey in 1980. Despite the fact that the Lebesgue space \( L^1[0,1] \) fails to have the \( w \)-FPP \cite{2}, B. Maurey proved that every closed reflexive subspace of \( L^1[0,1] \) does have the FPP \cite{31}. Almost two decades later, P. Dowling and C. Lennard \cite{15} obtained a converse statement: every closed subspace of \( L^1[0,1] \) with the FPP is reflexive. Hence, the FPP is completely characterized by reflexivity within the family of closed subspaces of \( L^1[0,1] \).

In the fourth section we prove that Maurey’s result can be literally extended to (nonreflexive) VLSs with order continuous Luxemburg norm, that is, every closed reflexive subspace of an order continuous VLS satisfies the FPP. Surprisingly, P. Dowling and C. Lennard’s converse does not hold in the variable framework, that is, under some slightly weak assumptions, we can prove that for every nonreflexive VLS there exists a further closed nonreflexive subspace that does verify the FPP. Near-infinity concentrated norms defined in \cite{8} will become an essential tool in our drive and will lead us to discover that Nakano spaces \( \ell^{p_n} \) when \( (p_n) \subset (1, +\infty) \) and \( \lim_n p_n = 1 \) endowed with the Luxemburg norm are nonreflexive Banach spaces that do have the FPP (see Corollary \cite{17} and comments afterwards). Note that, up to this stage, no classic nonreflexive Banach space endowed with its standard norm was known to have the FPP.

Finally, it is well-known that the failure of the FPP can be linked with the existence of an asymptotically isometric copy of \( \ell_1 \) (see \cite{15} \cite{16} and references therein). By using the subsequence splitting lemma for Banach lattices \cite{40}, in the last section of the paper we analyze the failure of asymptotically isometric copies of \( \ell_1 \) in VLSs, unless the trivial case is considered (when \( \ell_1 \) is already contained isometrically in \( L^{p(\cdot)}(\Omega) \)). Some open questions and new insights sparked by the previous results are also displayed. We would like to remark that the study of the fixed point properties in VLSs enables us to highlight, yet again, that the variable counterpart of the \( L^p \)-spaces exhibits a much richer and heterogeneous structure giving rise to a variety of new problems and lines of research that do not occur for the classic Lebesgue spaces.

2. Preliminaries and reflexivity in Variable Lebesgue spaces

**Definition 2.1.** Let \( \mathcal{X} \) be an arbitrary vector space.

(a) A functional \( \rho : \mathcal{X} \to [0, \infty] \) is called a convex modular if for \( x, y \in \mathcal{X} \):

(i) \( \rho(x) = 0 \) if and only if \( x = 0 \);

(ii) \( \rho(\alpha x) = \rho(x) \) for every scalar \( \alpha \) with \( |\alpha| = 1 \);

(iii) \( \rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y) \) if \( \alpha + \beta = 1 \) and \( \alpha, \beta \geq 0 \).

(b) A modular \( \rho \) defines a corresponding modular space, i.e. the vector space \( \mathcal{X}_\rho \) given by \( \{ x \in \mathcal{X} : \rho(x/\lambda) < \infty \text{ for some } \lambda > 0 \} \).
Given a vector space $\mathcal{X}$ with a convex modular $\rho$, the formula
\[ \|x\| = \inf \left\{ \alpha > 0 : \rho \left( \frac{x}{\alpha} \right) \leq 1 \right\} \quad \text{for} \ x \in \mathcal{X}_\rho, \]
defines a norm which is frequently called the Luxemburg norm and $\mathcal{X}_\rho$ endowed with this norm is a Banach space.

Throughout this paper, $(\Omega, \Sigma, \mu)$ will be a $\sigma$-finite measure space. We will always assume that the measure is complete. Let $p : \Omega \to [1, +\infty]$ be a measurable function and we consider the vector space $\mathcal{X}$ of all measurable functions $g : \Omega \to \mathbb{R}$. Define the modular
\[ \rho(g) := \int_{\Omega_f} |g(t)|^p d\mu + \text{ess sup}_{p^{-1}(\{+\infty\})} |g(t)|, \]
where $\Omega_f := \{ t \in \Omega : p(t) < +\infty \}$. Alongside with $\Omega_f$ and $p^{-1}(\{+\infty\})$, we will also distinguish the sets $p^{-1}(\{1\})$ and $\Omega^* = \Omega \setminus p^{-1}(\{1, +\infty\})$.

The Variable Lebesgue Space (VLS) $L^{p(\cdot)}(\Omega)$ is defined as the modular space endowed with the Luxemburg norm associated to the modular $\rho$ defined above. It is well-known that $L^{p(\cdot)}(\Omega)$ is a Banach function lattice whose geometry is strongly attached to the behaviour of the exponent function $p(\cdot)$. Note that Lebesgue spaces $L^p(\Omega)$ are particular examples of this construction just by considering the constant function $p(t) = p$ for all $t \in \Omega$.

Following the usual notation, given a measurable set $E \subset \Omega$, we define
\[ p_-(E) := \text{ess inf}_{t \in E} p(t), \quad p_+(E) := \text{ess sup}_{t \in E} p(t). \]
If $E = \Omega$ we just denote $p_- := p_-(\Omega)$ and $p_+ := p_+(\Omega)$.

A modular space $\mathcal{X}_\rho$ is said to satisfy the $\Delta_2$-condition if there exists $M > 0$ such that $\rho(2f) \leq M\rho(f)$ for every $f \in \mathcal{X}_\rho$. It is easy to prove that $L^{p(\cdot)}(\Omega)$ satisfies the $\Delta_2$-condition if $p_+(\Omega_f) < \infty$ (see [9, Proposition 2.14]). Moreover, in this case $\rho(g) < +\infty$ for every $g \in L^{p(\cdot)}(\Omega)$.

A Banach lattice $X$ is said to have an order continuous norm if every monotone order bounded sequence is convergent. Using Lebesgue’s Dominated Convergence Theorem, it is not difficult to prove that the Luxemburg norm of a VLS is order continuous whenever $p_+(\Omega_f) < +\infty$ and $p^{-1}(\{\infty\})$ is the union of at most a null set and finitely many atoms. The following properties relating the modular and the Luxemburg norm in VLSs will be used through this paper.

**Lemma 2.2.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure, $p : \Omega \to [1, +\infty]$ be an exponent function.

a) If $g \in L^{p(\cdot)}(\Omega), \ g \neq 0$, then $\rho \left( \frac{g}{\|g\|} \right) \leq 1$. Additionally, $\|g\| \leq \rho(g)$ when $\|g\| \geq 1$.

b) Assume that $p_+(\Omega_f) < \infty$ and $g \in L^{p(\cdot)}(\Omega)$. Then:
b.i) If $a \geq 1$, $a \rho(g) \leq \rho(ag) \leq a^{p_{+(\Omega_f)}} \rho(g)$.

b.ii) If $0 < a < 1$, $a^{p_{+(\Omega_f)}} \rho(f) \leq \rho(af) \leq a \rho(f)$.

c) Assume $p_{+(\Omega_f)} \lt \infty$ and $(g_n)$ is a sequence in $L^{p(\cdot)}(\Omega)$. Then:
   
c.i) $\lim_n \|g_n\| = 1$ if and only if $\lim_n \rho(g_n) = 1$.
   
c.ii) $\lim_n \|g_n\| = 0$ if and only if $\lim_n \rho(g_n) = 0$.

Proof. Assertion a) is proved in [9, Proposition 2.21]. Assertions b.i) and b.ii) are consequences of the convexity and the definition of the modular and assertions c.i) and c.ii) can be easily proved using the inequalities in a) and b).

When the measure space $(\Omega, \Sigma, \mu)$ is purely atomic, the exponent function $p(\cdot)$ can be considered as a sequence $(p_n)_n \subset [1, +\infty]$. The corresponding VLS is denoted by $\ell^{p_n}$ and they are usually known in the literature as a particular class of Musielaz-Orlicz sequence spaces or simply as Nakano spaces [35] (note that the modular considered in [35] is $\rho$).

When the norm fails to be order continuous, it is a general fact in the theory of Banach function lattices the existence of an isomorphic copy of $\ell_{\infty}$ [32, Corollary 2.4.3] (see also [30, Proposition 4.2]). In particular every nonreflexive function lattice contains an isomorphic copy of $\ell_{\infty}$ [32, Theorem 2.4.2]. In the specific case of the family of Musielak-Orlicz spaces, it was proved in [22] (for nonatomic $\sigma$-finite measures) and in [24] (for purely atomic measures) that, in absence of the $\Delta_2$-condition, there is an isometric copy of $\ell_{\infty}$ when the Luxemburg norm is considered. This stronger result is essential when is to be applied to the analysis of the fixed point property, since having an isomorphic copy of $\ell_{\infty}$ does not exempt a Banach space from satisfying the $w$-FPP (see [12]). Although variable Lebesgue spaces lay within the scope of the Musielak-Orlicz class, on the sake of completeness, we next include a proof of when such an isometric copy of $\ell_{\infty}$ can be found in the proper context of this article and including measures that may have both atomic and nonatomic parts.

**Theorem 2.3.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $p : \Omega \to [1, +\infty)$ be a measurable function. If $p_{+} = +\infty$, the Banach space $L^{p(\cdot)}(\Omega)$ contains an isometric copy of $\ell_{\infty}$. Consequently, under these assumptions, $L^{p(\cdot)}(\Omega)$ contains an isometric copy of every separable Banach space.

Proof. Since $p_{+} = +\infty$, either there exists a sequence of atoms $\{m_n\}$ such that $p(m_n) \to +\infty$ or there exists $M$ such that the set $p^{-1}((M, +\infty))$ does not contain any atom. In any case, we can find a real sequence $\{p_n\} \uparrow +\infty$ such that $\mu(p^{-1}([p_n, p_n+1])) > 0$ and $(1 + \frac{1}{n})^{p_n} > 2^n$. Let
$S_n \subset p^{-1}([p_n, p_{n+1}))$ such that $0 < \mu(S_n) < +\infty$. Hence $S_n \cap S_m = \emptyset$ if $n \neq m$. Denote by $\{r_n\}$ the increasing sequence formed by all prime numbers greater than 1. Note that if $t \in S_{r_n}$ then $p(t) \geq p_{r_n} \geq p_j$ for all $n, j \in \mathbb{N}$. For every $n \in \mathbb{N}$ we define the function

$$f_n(t) := \sum_{j=1}^{\infty} x_{n,j}^{1/p(t)} \chi_{S_{r_n}}(t), \quad \text{where} \quad x_{n,j} = \frac{1}{2^{n+1+j} \mu(S_{r_n})} \quad \forall j \in \mathbb{N}.$$ 

By construction $ho(f_n) = \frac{1}{2^{n+1}}$, which implies that $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$. Let $\lambda > 1$ and choose $j_0$ such that $1 + \frac{1}{j} < \lambda$ for $j \geq j_0$. We have:

$$\rho(\lambda f_n) = \sum_{j=1}^{\infty} \int_{S_{r_n}} x_{n,j} \lambda^{p(t)} d\mu \geq \sum_{j=j_0}^{\infty} \int_{S_{r_n}} x_{n,j} \left(1 + \frac{1}{j}\right)^{p_j} d\mu \geq \sum_{j=j_0}^{\infty} x_{n,j} 2^j 2^{p_j} \mu(S_{r_n}) = \sum_{j=j_0}^{\infty} 2^j \frac{1}{2^{n+1+j}} = +\infty.$$ 

The previous arguments prove that $\|f_n\| = 1$ for every $n \in \mathbb{N}$. Likewise, it can be checked that $\|\sum_{n=1}^{\infty} f_n\| = 1$. At this stage, it is not difficult to conclude that the sequence $(f_n)$ spans an isometric copy of $\ell_\infty$ in $L^p(\cdot)$ (see for instance [23, Theorem 1]). The statement of the theorem is complete due to the fact that $\ell_\infty$ contains an isometric copy of every separable Banach space [1, Theorem 2.5.7].

A complete analysis of the reflexivity condition for variable Lebesgue spaces was studied for nonatomic measures in [30] and for purely atomic measures in [39]. In fact, for the nonatomic case the following characterization was obtained:

**Theorem 2.4.** [30, Theorem 3.3] Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite nonatomic measure space. The following conditions are all equivalent:

(a) $1 < p_- \leq p_+ < \infty$.
(b) $L^{p_+}(\Omega)$ is uniformly convex.
(c) $L^{p_-}(\Omega)$ is reflexive.

The nonatomic assumption in Theorem 2.4 is used by the authors exclusively in the proof of “(c) implies (a)”. The proof of “(a) implies (b)” holds for every $\sigma$-finite measure space. Actually, Theorem 2.4 does not entirely hold when the measure contains atoms, since reflexivity can be obtained in absence of uniform convexity: consider the purely atomic case $\ell^{p_n}$ for $p_1 = p_2 = 1$ and $p_n = 2$ for $n > 2$. The VLS space $\ell^{p_n}$ fails to be uniformly convex, since it contains $\ell_1(2)$ isometrically, but it is reflexive since it is isomorphic to $\ell_2$. As reflexivity will be at the core of many of our next results, we first aim to achieve a complete characterization of reflexivity for VLSs including all $\sigma$-finite measures:
Theorem 2.5. Let \((\Omega, \Sigma, \mu)\) be an arbitrary \(\sigma\)-finite measure space and let \(p : \Omega \to [1, +\infty)\) be a measurable function. The following conditions are equivalent:

i) \(L^{p(\cdot)}(\Omega)\) is reflexive.

ii) \(L^{p(\cdot)}(\Omega)\) contains no isomorphic copy of \(\ell_1\).

iii) Let \(\Omega^* := \Omega \setminus p^{-1}(\{1, +\infty\})\). Then \(1 < p_-(\Omega^*) \leq p_+(\Omega^*) < +\infty\) and \(p^{-1}(\{1, +\infty\})\) is essentially formed by finitely many atoms at most.

Proof. i) implies ii) is straightforward. Let us prove ii) implies iii): If \(p^{-1}(\{1, +\infty\})\) contains infinitely many atoms, either \(\ell_1\) or \(\ell_\infty\) would be isometrically embedded in \(L^{p(\cdot)}(\Omega)\). In any case, \(L^{p(\cdot)}(\Omega)\) would contain an isometric copy of \(\ell_1\). If \(p^{-1}(\{1, +\infty\})\) contains a nonatomic set with positive measure we would arrive at the same conclusion, since either \(L^1[0,1]\) or \(L^\infty([0,1])\) would be isometrically embedded into \(L^{p(\cdot)}(\Omega)\). If \(p_+(\Omega^*) = +\infty\), we would obtain an isometric copy of \(\ell_1\) in \(L^{p(\cdot)}(\Omega^*)\) in view of Theorem 2.3 and, obviously, in \(L^{p(\cdot)}(\Omega)\). Finally, if \(p_-(\Omega^*) = 1\), we will later prove in Theorem 1.6 that it is possible to find a subspace within \(L^{p(\cdot)}(\Omega)\) which is hereditarily \(\ell_1\) (and therefore, it contains \(\ell_1\) isomorphically).

Let us prove iii) implies i): We split \(\Omega = \Omega_a \cup \Omega_b\); \(\Omega_a, \Omega_b\) being the purely atomic and the nonatomic part of \(\Omega\) respectively. From the assumptions, we have that \(1 < p_-(\Omega_b) \leq p_+(\Omega_b) < +\infty\) and from Theorem 2.4 we know that the variable Lebesgue space \(L^{p(\cdot)}(\Omega^*)\) is uniformly convex. From iii) we also know that there are some integers \(0 \leq r_1 \leq r_2\) such that \(\Omega_a \cap p^{-1}(\{1\}) = \{t_1, \ldots, t_{r_1}\}\) and \(\Omega_a \cap p^{-1}(\{+\infty\}) = \{t_{r_1+1}, \ldots, t_{r_2}\}\). Set \(r := r_2 - r_1 \geq 0\). Thus we can write

\[
\rho(g) = \int_{\Omega_b} |g(t)|^{p(t)} dt + \sum_{i=1}^{r_1} |g(t_i)| + \sup_{r_1 < i \leq r_2} |g(t_i)| \quad \forall g \in L^{p(\cdot)}(\Omega).
\]

We aim to prove that \(L^{p(\cdot)}(\Omega)\) can be renormed to be uniformly convex and therefore it is reflexive. In order to do that, we define the measurable function \(\tilde{p} : \Omega \to (1, +\infty)\) given by \(\tilde{p}(t) = 2\) if \(t \in \Omega_a \cap p^{-1}(\{1, +\infty\})\) and \(\tilde{p}(t) = p(t)\) otherwise. We denote by \(\| \cdot \|_{\tilde{p}}\) the Luxemburg norm rising from the modular

\[
\tilde{\rho}(g) = \int_{\Omega} |g(t)|^{\tilde{p}(t)} dt = \int_{\Omega_b} |g(t)|^{p(t)} dt + \sum_{i=1}^{r_2} |g(t_i)|^2.
\]

We next check that \(\frac{1}{r_1+r_2+2}\|g\| \leq \|g\|_{\tilde{p}} \leq \max\{1, r\}\|g\|\) for all \(g \in L^{p(\cdot)}(\Omega)\):

Assume that \(\|g\| = 1\). From Lemma 2.2a we know that \(\rho(g) \leq 1\), which in particular implies that \(\|g(t_i)| \leq 1\) for all \(1 \leq i \leq r_2\). This gives \(\tilde{\rho}(g) \leq \max\{1, r\}\rho(g) \leq \max\{1, r\}\) and \(\tilde{\rho} \left( \frac{g}{\max\{1, r\}} \right) \leq \frac{1}{\max\{1, r\} \tilde{\rho}(g)} \leq 1\) yielding to \(\|g\|_{\tilde{p}} \leq \max\{1, r\}\|g\|\) for all \(g \in L^{p(\cdot)}(\Omega)\). Assume now that \(\|g\|_{\tilde{p}} = 1\) and therefore \(\tilde{\rho}(g) \leq 1\) and \(\|g(t_i)| \leq 1\) for all \(1 \leq i \leq r_2\). Hence, \(\rho(g) \leq
\[ \tilde{\rho}(g) + r_1 + 1 \leq r_1 + 2. \] By convexity, we have \[ \rho \left( \frac{g}{r_1 + 2} \right) \leq \frac{1}{r_1 + 2} \rho(g) \leq 1 \] and \[ \|g\| \leq r_1 + 2. \]

Hence, we have obtained that \( L^{\tilde{p}(\cdot)}(\Omega) \) is isomorphic to \( L^{\tilde{p}(\cdot)}(\Omega) \) which is in turn uniformly convex, since \( 1 < \tilde{p}_- \leq \tilde{p}_+ < +\infty \) (see remark after Theorem 2.4) and this concludes the proof.

Finally, we would like to recall that some fixed point results have already appeared for VLSs when they are considered modular spaces and focusing the notion of nonexpansivity with respect to the modular \( \rho(\cdot) \) defined by (2.1) [3, 4, 5]. In the next sections our goal is completely different and addresses toward the analysis of the fixed point property when nonexpansiveness is measured with respect to the Luxemburg norm and the potential connections linking the geometry and reflexivity of the underlying variable space.

3. Weak Fixed Point Property in Variable Lebesgue Spaces

In this section we will obtain a characterization of the w-FPP in variable Lebesgue spaces in terms of the variable exponent function \( p(\cdot) \) and whether or not \( L_1[0,1] \) can be isometrically embedded in \( L^{p(\cdot)}(\Omega) \). In particular, we will exhibit that there are some VLSs with the w-FPP which are not reflexive, in sharp contrast to the classic \( L^p(\Omega) \) spaces, where \( L^p(\Omega) \) has the w-FPP if and only if \( L^{p(\cdot)}(\Omega) \) is reflexive.

We will start with two technical lemmas, the first of which is just a measure theory result likely well-known. We include the proof for the sake of completeness.

**Lemma 3.1.** Let \((\Omega, \Sigma, \mu)\) be a \( \sigma \)-finite measure space and \( (f_n) \) be a bounded sequence in \( L_1(\mu) \). Then the following statement holds: For almost every \( t \in \Omega \), the scalar sequence \( \{f_n(t)\}_n \) has at least one finite accumulation point.

**Proof.** Since the measure is \( \sigma \)-finite we can assume that \( \Omega = \bigcup_{s=1}^{\infty} \Omega_s \) with \( \mu(\Omega_s) < \infty \). If the previous statement is proved for finite measures it follows for \( \sigma \)-finite measures. Hence, without loss of generality, we can assume that the measure is finite. Let \( I := \{ t \in \Omega : \lim_n |f_n(t)| = +\infty \} \). It is clear the scalar sequence \( \{f_n(t)\} \) has an accumulation point if and only if \( t \in \Omega \setminus I \).

We will prove that \( \mu(I) = 0 \):

Fix a real number \( a > 0 \). For \( n \in \mathbb{N} \), set \( A_n(a) = \{ t \in I : |f_n(t)| < a \} \subset I \). Since the sequence \( \{\chi_{A_n(a)}\}_n \) converges to zero pointwise and \( \chi_{A_n(a)}(t) \leq a \chi_\Omega(t) \) for all \( t \in \Omega \), using Lebesgue’s Dominated Convergence Theorem, we have \( \lim_n \mu(A_n(a)) = 0 \). For every \( k \in \mathbb{N} \), choosing \( a = 2^k \) and repeating the process successively, we can find a subsequence \( (n_k) \) such that

\[ \mu(\{ t \in I : |f_{n_k}(t)| < 2^k \}) \leq \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}. \]
Take $M = \sup_n \|f_n\|_1$. By Chebychev inequality

$$
\mu(\{t \in I : |f_{n_k}(t)| \geq 2^k\}) \leq \mu(\{t \in \Omega : |f_{n_k}(t)| \geq 2^k\}) \leq \frac{M}{2^k}.
$$

Hence, for every $k \in \mathbb{N}$, we have $I = \{t \in I : |f_{n_k}(t)| < 2^k\} \cup \{t \in I : |f_{n_k}(t)| \geq 2^k\}$, which implies that $\mu(I) \leq \frac{M+1}{2^k}$. Taking limits when $k$ goes to infinity we finally deduce that $\mu(I) = 0$.

We recall that a sequence $(x_n) \subset L^{p(\cdot)}(\Omega)$ is said to be $\rho$-bounded if $\sup_n \rho(x_n) < +\infty$ where $\rho(\cdot)$ is the modular defined by (2.1).

**Lemma 3.2.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and assume that the exponent function $p(\cdot)$ verifies $1 < p(t) < \infty$ a.e. Let $u, v \in L^{p(\cdot)}(\Omega)$. Assume that there exists a $\rho$-bounded sequence $(x_n)$ in $L^{p(\cdot)}(\Omega)$ verifying

(3.1)
$$
\lim_n \int_\Omega \left( |x_n(t) - u(t)|^{p(t)} + |x_n(t) - v(t)|^{p(t)} - 2 \left| x_n(t) - \frac{u(t) + v(t)}{2} \right|^{p(t)} \right) d\mu = 0.
$$

then $u = v$ a.e.

**Proof.** Let $u, v \in L^{p(\cdot)}(\Omega)$ and assume that there exists a $\rho$-bounded sequence $(x_n)$ in $L^{p(\cdot)}(\Omega)$ verifying that the limit in (3.1) is null. Under these conditions, we are going to find a subset $B \subset \Omega$ with $\mu(B) = 0$ such that $u(t) = v(t)$ for all $t \in \Omega \setminus B$.

Note that the $\rho$-boundedness of the sequence $(x_n)$ implies that the sequence $(h_n)$ defined by $h_n(t) = |x_n(t)|^{p(t)}$ is a bounded sequence in $L_1(\Omega)$. Using Lemma 3.1, we can assume that for almost every $t \in \Omega$, the scalar sequence $\{x_n(t)^{p(t)}\}_n$ has a finite accumulation point, and so does the scalar sequence $\{x_n(t)\}_n$ for almost every $t \in \Omega$.

For all $n \in \mathbb{N}$ define the function

(3.2)
$$
g_n(t) := |x_n(t) - u(t)|^{p(t)} + |x_n(t) - v(t)|^{p(t)} - 2 \left| x_n(t) - \frac{u(t) + v(t)}{2} \right|^{p(t)}.
$$

Note that $g_n \geq 0$ by convexity and, by assumption, $\lim_n \int_\Omega g_n(t) d\mu = 0$. Extracting a subsequence, denoted again by $(g_n)$, we can assume that $\lim_n g_n(t) = 0$ for almost every $t \in \Omega$. Thus, we can assume that there exists some $B \subset \Omega$ with $\mu(B) = 0$ such that for all $t \in \Omega \setminus B$ we have that $\lim_n g_n(t) = 0$ and there exists a subsequence $(n'_k)$ (depending on $t$) such that $\lim_k x_{n'_k}(t) = \alpha_t$, where $\alpha_t$ is a finite scalar. Hence, for every $t \in \Omega \setminus B$, taking limit when $k$ goes to infinite over the subsequence $(n'_k)$ in (3.2) we obtain

(3.3)
$$
|\alpha_t - u(t)|^{p(t)} + |\alpha_t - v(t)|^{p(t)} - 2 \left| \alpha_t - \frac{u(t) + v(t)}{2} \right|^{p(t)} = 0.
$$
We have concluded that for all \( t \in \Omega \setminus B \), the equation (3.3) holds. The strict convexity of the function \( s \to s^{p(t)} \) (since \( p(t) > 1 \)) implies that \( u(t) = v(t) \) for \( t \in \Omega \setminus B \) as we wanted to prove.

\[ \square \]

We recall that a Banach space \( X \) is said to have weak normal structure (w-NS) if for every convex weakly compact subset \( C \) with \( \text{diam}(C) > 0 \), there exists some \( x_0 \in C \) such that \( \sup\{\|x_0 - y\| : y \in C\} < \text{diam}(C) \). The notion of normal structure was initially defined by Brodskii and Milman in 1948 [7] and W. Kirk established its relationship with the existence of fixed points for nonexpansive mappings: Every Banach space with w-NS satisfies the w-FPP [25].

The main theorem of this section is the following:

**Theorem 3.3.** Let \( (\Omega, \Sigma, \mu) \) be a \( \sigma \)-finite measure space and \( p : \Omega \to [1, +\infty] \) be a measurable function. As usual, let us denote by \( \Omega_f = \{t \in \Omega : p(t) < +\infty\} \). The following conditions are all equivalent:

1. \( L^p(\Omega_f) \) satisfies the weak normal structure.
2. \( L^p(\Omega_f) \) satisfies the w-FPP.
3. \( L^p(\Omega_f) \) does not contain isometrically \( L^1[0,1] \).
4. \( p_+(\Omega_f) < +\infty \), \( p^{-1}(\{+\infty\}) \) contains finitely many atoms at most and every measurable atomless subset of \( p^{-1}([1, +\infty]) \) is negligible.

**Proof.** We already know that 1) \( \Rightarrow \) 2) and 2) \( \Rightarrow \) 3) from [25] and [2]. Clearly 3) \( \Rightarrow \) 4). Indeed, if \( p_+(\Omega_f) = +\infty \) we can find an isometric copy of \( L^1[0,1] \) from Theorem 2.3. If \( p^{-1}(\{+\infty\}) \) contains infinitely many atoms, we have an isometric copy of \( \ell_\infty \) and hence an isometric copy of \( L^1[0,1] \). Finally, if there is a measurable atomless subset contained in \( p^{-1}([1, +\infty]) \) with positive measure, once more \( L^p(\Omega_f) \) contains an isometric copy of \( L^1[0,1] \).

Finally, let us prove that 4) \( \Rightarrow \) 1): Set \( F := F_1 \cup F_\infty \) where by \( F_1, F_\infty \) we denote the set of atoms in \( p^{-1}(\{1\}) \) and \( p^{-1}(\{+\infty\}) \) respectively. In view of 4), the cardinal of \( F_\infty \) is finite so \( L^p(\Omega_f) \) has the Schur property (since it is isomorphic to \( \ell_1 \)).

Assume, by contrary, that \( L^p(\Omega_f) \) fails to have weakly normal structure. Standard arguments imply that \( L^p(\Omega_f) \) contains a weakly null diametral sequence (see for instance [20] Lemma 4.1), that is, a sequence \( (y_n) \) with \( \text{diam}(\{y_n : n \in \mathbb{N}\}) = 1 \) and such that \( \lim_n d(y_n, \co \{y_1, ..., y_{n-1}\}) = 1 \). In particular \( \lim_n \|y_n - y\| = 1 \) for all \( y \in \co(\{y_n\}) \).

We select \( u, v \in \co(\{y_n\}) \) that will be fixed in what follows.

Note that we can write \( y_n = z_n + x_n \), where \( z_n(t) = y_n(t) \) if \( t \in F \) and zero otherwise, while \( x_n = y_n - z_n \). It is clear that \( (z_n) \) and \( (x_n) \) are weakly null sequences, \( \text{supp}(z_n) \subseteq F \) and \( \text{supp}(x_n) \subseteq \Omega \setminus F \) for all \( n \in \mathbb{N} \). From the Schur property, the sequence \( (z_n) \) is norm convergent, \( \lim_n \|x_n - y_n\| = 0 \) and this implies that

\[
\lim_n \|x_n - y\| = 1 \quad \forall y \in \co(\{y_n\}).
\]
The above condition implies that
\[
\lim_n \|x_n - u\| = \lim_n \|x_n - v\| = \lim_n \left\| x_n - \frac{u + v}{2} \right\| = 1.
\]
From the assumption \(p_+(\Omega_f) < +\infty\) and Lemma 2.2.c.i we infer that
\[
\lim \rho(x_n - u) = \lim \rho(x_n - v) = \lim \rho \left( x_n - \frac{u + v}{2} \right) = 1
\]
and consequently
\[
\lim_n \left[ \rho(x_n - u) + \rho(x_n - v) - 2\rho \left( x_n - \frac{u + v}{2} \right) \right] = 0.
\]
Define \(u_F(t) = u(t)\) if \(t \in F\), zero otherwise and \(u_0 = u - u_F\). Analogously we define \(v_F\) and \(v_0\). Thus \(u = u_0 + u_F\), \(v = v_0 + v_F\), where \(u_0, v_0 \in L^{p(\cdot)}(\Omega \setminus F)\) and \(u_F, v_F \in L^{p(\cdot)}(F)\). If we denote by \(\rho_0(g) := \int_{\Omega \setminus F} |g|^{p(t)} d\mu\) and \(\rho_F(g) := \rho(g) - \rho_0(g)\) for \(g \in L^{p(\cdot)}(\Omega)\), we have
\[
\rho(x_n - u) = \rho_0(x_n - u_0) + \rho_F(u_F)
\]
and a similar decomposition is obtained for \(\rho(x_n - v)\) and \(\rho \left( x_n - \frac{u + v}{2} \right)\).
Condition (3.4) is now translated to \(A_1 + A_2 = 0\), where
\[
A_1 := \rho_F(u_F) + \rho_F(v_F) - 2\rho_F \left( \frac{u_F + v_F}{2} \right)
\]
and
\[
A_2 := \lim_n \left[ \rho_0(x_n - u_0) + \rho_0(x_n - v_0) - 2\rho_0 \left( x_n - \frac{u_0 + v_0}{2} \right) \right].
\]
By convexity both \(A_1, A_2 \geq 0\), so we have that \(A_1 = A_2 = 0\). Consequently
\[
\lim_n \int_{\Omega \setminus F} \left( |x_n(t) - u_0(t)|^{p(t)} + |x_n(t) - v_0(t)|^{p(t)} - 2 \left| x_n(t) - \frac{u_0(t) + v_0(t)}{2} \right|^{p(t)} \right) d\mu = 0.
\]
Due to the fact that \(p_+(\Omega_f) < +\infty\) and the remaining conditions in 4), we have \(\sup_n \rho_0(x_n) < +\infty\). Furthermore, \(1 < p(t) < +\infty\) a.e. in \(\Omega \setminus F\). Consequently, for the vectors \(u, v \in \co \{y_n\}\) chosen beforehand, we know of the existence of a \(\rho\)-bounded sequence \((x_n)\) such that (3.5) holds.
Applying Lemma 3.2 for the set \(\Omega \setminus F\), we deduce that \(u_0 = v_0\) e.c.t. \(\Omega \setminus F\) and \(u(t) = v(t)\) a.e. in \(\Omega \setminus F\).
Due to the arbitrariness of the vectors \(u, v \in \co \{y_n\}\), we can deduce that for all \(n, m\) we have that \(y_n = y_m\) a.e. in \(\Omega \setminus F\). Since \((y_n)\) is a weakly null sequence, \(y_n(t) = 0\) a.e. in \(\Omega \setminus F\) and \(y_n \in L^{p(\cdot)}(F)\), which has the Schur property. Consequently \(\lim_n \|y_n\| = 0\) in contradiction with the fact that \((y_n)\) is diametral.
At this stage we would like to highlight that the absence of an isometric copy of $L^1[0,1]$ is a necessary condition for having the $w$-FPP. As it was proved in Theorem 3.3, it turns out to be an equivalence for the family of VLSs. Furthermore, according to Theorem 3.3, we can find plenty of examples of nonreflexive VLSs that still have the $w$-FPP. For instance, the Banach function space $L^{1+x}([0,1])$ is one of these spaces. Notice that this is not possible for classic Lebesgue spaces, where $L^p(\Omega)$ has the $w$-FPP if and only if it is reflexive. In the particular case of a purely atomic measure, sufficient conditions implying the $w$-FPP had been studied previously in [10, 13] (see also [27, Chapter 12] and references therein).

4. Fixed point property and reflexivity: Maurey’s result and its converse in Variable Lebesgue Spaces

As it was mentioned in the Introduction, one of the most relevant results backing the long-standing conjecture “reflexivity implies FPP” was published by B. Maurey in 1980 [31]: every closed reflexive subspace of $L^1[0,1]$ has the FPP. This together with P. Dowling and C. Lennard’s converse [15] lead to the following characterization:

**Theorem 4.1.** [31] [15] Let $X$ be a closed subspace of $L^1[0,1]$. Then $X$ is reflexive if and only if it has the FPP.

Since every $\sigma$-finite $L^1$-space is isometric to a probability space, Theorem 4.1 easily applies to the $\sigma$-finite case $L^1(\mu)$. The natural question that rises straight away is whether one or the two implications in Theorem 4.1 may still hold in our variable setting. In this section we will prove that Maurey’s result extends to (nonreflexive) VLSs, whereas P. Dowling and C. Lennard’s converse is no longer true.

**Theorem 4.2.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $p_+ < +\infty$. Let $X$ be a reflexive subspace of $L^{p(\cdot)}(\Omega)$. Then, $X$ satisfies the FPP.

**Proof.** Let $X$ be a reflexive subspace of $L^{p(\cdot)}(\Omega)$ with $p_+ < \infty$. If $L^{p(\cdot)}(\Omega)$ has the $w$-FPP, we are done. Thus according to Theorem 3.3, we can assume that $\mu(p^{-1}(\{1\})) > 0$.

Denote $\Omega_1 := p^{-1}(\{1\})$, $\Omega_2 := \Omega \setminus \Omega_1$. Define $Y := \{ f \in X : f|_{\Omega_2} = 0 \text{ a.e.} \}$. Note that $Y$ is a closed subspace (possible empty) of $X$ and therefore $Y$ is reflexive. Furthermore, $Y$ is embedded isometrically in $L^1(\Omega_1)$.

By contradiction, let us assume that $X$ fails to have the FPP. Standard arguments show that there exist a convex weakly compact subset $K \subset X$ and $T : K \to K$ nonexpansive without fixed points. Furthermore, we can assume that $K$ is minimal $T$-invariant, $0 \in K$, $\text{diam}(K) = 1$ and there exists a weakly-null sequence $\{x_n\}$ in $K$ which is an approximate fixed point sequence. As a consequence of Goebel-Karlovitz Lemma [20, page 124], $\lim_n \|x_n - x\| = 1$ for all $x \in K$. Fix some $u, v \in K$. Proceeding as in the
proof of Theorem 3.3 we can assume that for \( i = 1, 2 \):
\[
\lim_n \int_{\Omega_i} \left( |x_n(t) - u(t)|^{p(t)} + |x_n(t) - v(t)|^{p(t)} - 2 \left| x_n(t) - \frac{u(t) + v(t)}{2} \right|^{p(t)} \right) \, d\mu = 0.
\]

In particular, applying Lemma 3.2 to \( \Omega_2 \), where \( 1 < p(t) < \infty \) a.e., we deduce that \( u_{\chi_{\Omega_2}} = v_{\chi_{\Omega_2}} \) a.e. Since \( 0 \in K \), \( u_{\chi_{\Omega_2}} = 0 \) a.e. for all \( u \in K \).
Thus, \( K \) is a convex weakly compact of \( Y \), which has the FPP according to Maurey [31]. This implies that \( K \) is a singleton (since it is minimal) in contradiction to the fact that \( \text{diam}(K) = 1 \).

\[\Box\]

The second part of this section is dedicated to study the converse of Maurey’s result. Surprisingly, we are going to prove that, under certain conditions over the function \( p(\cdot) \), every nonreflexive \( L^{p(\cdot)}(\Omega) \) contains a further nonreflexive Banach space fulfilling the FPP, in sharp contrast with the \( L^1[0, 1] \)-case and bringing to light new intrinsic features of variable Lebesgue spaces that are not shared by their classic counterparts.

In order to do that, we introduce the concept of near-infinity concentrated norm defined in [8] for Banach spaces with a Schauder basis. Recall that if \( \{e_n\} \) is a Schauder basis for a Banach space \( X \), we denote by \( \text{supp} \{x\} = \{n \in \mathbb{N} : x(n) \neq 0\} \), \( Q_k(x) = \sum_{n=k}^{\infty} x(n)e_n \) and \( P_k(x) = \sum_{n=1}^{k} x(n)e_n \), where \( x = \sum_{n=1}^{\infty} x(n)e_n \in X \). The norm is said to be premonotone for the basis \( \{e_n\} \) when \( \|Q_k\| \leq 1 \) for every \( k \in \mathbb{N} \). For \( k \in \mathbb{N} \) and \( x \in X \), we say that \( k \leq x \) if \( k \leq \min\{\text{supp} \{x\}\} \).

**Definition 4.3.** [8] Let \((X, \| \cdot \|)\) be a Banach space with a Schauder basis \( \{e_n\} \). The norm is said to be near-infinity concentrated (n.i.c.) if it has the following properties:

1. The norm is sequentially separating [6], that is, for every \( \epsilon > 0 \) there exists some \( k \in \mathbb{N} \) such that
   \[
   \|x\| + \limsup_n \|x_n\| \leq (1 + \epsilon) \limsup_n \|x + x_n\|
   \]
   whenever \( k \leq x \) and \( \{x_n\} \) is a block basic sequence of \( \{e_n\} \).

2. The norm is premonotone.

3. There exist some \( R_0 > 5 \) and \( M \in (0, 1) \) such that for every \( k \in \mathbb{N} \), we can find a function \( F_k : (0, \infty) \to [0, \infty) \) satisfying the following conditions:
   
   (3.a) \( \lim_{\lambda \to 0^+} \frac{F_k(\lambda)}{\lambda} \leq \frac{M}{R_0} \).
   
   (3.b) \( \forall z \in X \) with \( \|z\| \leq R_0 \), \( \text{supp} \{z\} \subset [1, k] \) and for every bounded coordinate-null sequence \( \{x_n\} \subset X \) with \( \liminf_n \|x_n\| \geq 1 \) we have:
   
   \[
   \limsup_n \|x_n + \lambda z\| \leq \limsup_n \|x_n\| + F_k(\lambda)\|z\| \quad \forall \lambda \in (0, \infty).
   \]

The main result in [8] is the following:
Theorem 4.4. Let \((X, \| \cdot \|)\) be a Banach space with a boundedly complete Schauder basis. If the norm \(\| \cdot \|\) is n.i.c., then \((X, \| \cdot \|)\) has the FPP.

We will make use of the following technical lemma:

Lemma 4.5. [\(\) Lemma 3.3] Let \((X, \| \cdot \|)\) be a Banach space with a Schauder basis \(\{e_n\}\). The norm \(\| \cdot \|\) is sequentially separating if and only if

\[
\lim_{n} \inf \{ \lim_{k} \sup \| x + x_n \| \} = 2,
\]

where the infimum is taken over all \(x \in X\) such that \(x = \sum_{i \geq k} x(i)e_i\) with \(\|x\| = 1\) and all normalized block basic sequences \((x_n) \subset X\).

We recall that a Banach space \(X\) is said to be hereditarily \(\ell_1\) if every closed subspace contains a further subspace which is isomorphic to \(\ell_1\). Now the main result is the following:

Theorem 4.6. Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space, \(p : \Omega \to [1, +\infty]\) be a measurable function. If \(p_\gamma(\Omega \setminus p^{-1}(\{1\})) = 1\), then \(L^p(\Omega)\) contains a closed subspace with the FPP which is hereditarily \(\ell_1\), and therefore nonreflexive.

Proof. From the assumptions, we can find a decreasing sequence \((\gamma_n)\) in \((1, +\infty)\) such that \(\lim_n \gamma_n = 1\) and \(A_n := \{ t \in \Omega : \gamma_n < p(t) \leq \gamma_n - 1\}\) has positive measure. Let \(f_n\) be a normalized function in \(L^p(\Omega)\) with \(\supp(f_n) \subset A_n\) for every \(n \in \mathbb{N}\). We next prove that the closed subspace \(X\) generated by the sequence \(\{f_n\}\) is nonreflexive and satisfies the FPP:

Firstly, note that the sequence \(\{f_n\}\) is a Schauder basis for \(X\) which is boundedly complete (see for instance [\(\) Definition 3.2.8]), since the function \(p(\cdot)\) is bounded in the union of all subsets \(A_n\). Furthermore, the norm is premonotone. We now check conditions 1) and 3) in Definition 1.3.

In order to prove (1) we use Lemma 4.5. Fix \(k \in \mathbb{N}\) and let \(x = \sum_{i \geq k} x(i)f_i\) with \(\|x\| = 1\). Let \((x_n)\) be a block basic sequence in \(X\). Without loss of generality we can assume that \(\max \supp(x) < \min \supp(x_n)\) in regards to their coordinates respect to the basis \((f_n)\). Note that if \(t \in \supp(x) \cup \supp(x_n)\) then \(p(t) < \gamma_{k-1}\). Besides, \(p(x) = p(x_n) = 1\) for all \(n \in \mathbb{N}\) since they are normalized vectors (see Lemma 2.2.4.c.i applied to a constant sequence and taking in mind that the exponent function \(p(\cdot)\) is bounded from above in the union of all the subsets \(A_n\)). For \(r := 2^{\gamma_{k-1}}\) we have:

\[
\rho \left( \frac{x + x_n}{r} \right) = \rho \left( \frac{x}{r} \right) + \rho \left( \frac{x_n}{r} \right)
= \int_{\supp(x) \gamma_{k-1}} \left( \frac{1}{r} \right)^{p(t)} |x(t)|^{p(t)} d\mu + \int_{\supp(x_n) \gamma_{k-1}} \left( \frac{1}{r} \right)^{p(t)} |x_n(t)|^{p(t)} d\mu
\geq \left( \frac{1}{r} \right) \rho(x) + \left( \frac{1}{r} \right) \rho(x_n) = 1.
\]
Therefore, \( \limsup_n \|x + x_n\| \geq 2^{-1} \). Taking limits when \( k \) goes to infinity we deduce that the norm is sequentially separating. This in particular implies that \( X \) is nonreflexive, since it is hereditarily \( \ell_1 \) [Corollary 7.7].

We next prove condition (3) in Definition 4.3. Take any \( R_0 > 0 \) and \( k \in \mathbb{N} \). Let \( (x_n) \) be a block basic sequence with \( \liminf_n \|x_n\| \geq 1 \) and \( z \in X \) with \( z = \sum_{i=1}^k z(i)f_i \) and \( \|z\| \leq R_0 \). We can assume, without loss of generality, that \( x_n = \sum_{i=k+1}^{\infty} x_n(i)f_i \) and \( \|x_n\| \geq 1 \) for all \( n \in \mathbb{N} \). We start by proving:

(4.1) \[ \|x_n + \lambda z\| \leq \|x_n\| + \lambda \gamma^k \|z\| \quad \forall \lambda \leq R_0^{-1}. \]

Indeed, note that \( p(t) > \gamma_k \) for all \( t \in \text{supp}(z) \) and \( \lambda \|z\| \leq 1 \leq \|x_n\| + \lambda \gamma^k \|z\| \) when \( \lambda \leq R_0^{-1} \). Furthermore, \( (\|x_n\| + \lambda \gamma^k \|z\|)^{\gamma_k^{-1}} \geq 1 \). This implies that:

\[
\rho \left( \frac{x_n + \lambda z}{\|x_n\| + \lambda \gamma^k \|z\|} \right) = \rho \left( \frac{\|x_n\|}{\|x_n\| + \lambda \gamma^k \|z\|} \left( \frac{x_n}{\|x_n\|} \right) \right) + \rho \left( \frac{\lambda \|z\|}{\|x_n\| + \lambda \gamma^k \|z\|} \left( \frac{z}{\|z\|} \right) \right) \\
\leq \frac{\|x_n\|}{\|x_n\| + \lambda \gamma^k \|z\|} \rho \left( \frac{x_n}{\|x_n\|} \right) + \left( \frac{\lambda \|z\|}{\|x_n\| + \lambda \gamma^k \|z\|} \right)^{\gamma_k} \rho \left( \frac{\lambda \|z\|}{\|x_n\| + \lambda \gamma^k \|z\|} \left( \frac{z}{\|z\|} \right) \right) \\
= \frac{\|x_n\|}{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}} + \left( \frac{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}}{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}} \right)^{\gamma_k} \\
\leq \frac{\|x_n\|}{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}} + \left( \frac{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}}{\|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k}} \right)^{\gamma_k} \\
= 1.
\]

Therefore, if \( \lambda \leq R_0^{-1} \):

\[
\limsup_n \|x_n + \lambda z\| \leq \limsup_n \|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k} = \limsup_n \|x_n\| + \lambda \gamma^k \|z\|^{\gamma_k-1} \|z\| \\
\leq \limsup_n \|x_n\| + \lambda \gamma^k R_0^{\gamma_k-1} \|z\|.
\]

Thus, we can consider \( F_k(\lambda) = \lambda \) if \( \lambda > R_0^{-1} \) and \( F_k(\lambda) = \lambda \gamma^k R_0^{\gamma_k-1} \) otherwise. Now it is clear that \( \lim_{\lambda \to 0} \frac{F_k(\lambda)}{\lambda} = 0 \) and this shows that the norm on \( X \) is n.i.c. Consequently \( X \) verifies the FPP as we wanted to prove.

Likewise, it can proved that the Luxemburg norm in Musielak-Orlicz sequence spaces \( \ell^p\) is near-infinity concentrated when the sequence \( \{p_n\} \subset (1, +\infty) \) and \( \lim_n p_n = 1 \). This drives us to discover a family of classic sequence Banach spaces which are nonreflexive and enjoy the FPP without enduring any renorming process (and non-isomorphic to \( \ell_1 \)):

**Corollary 4.7.** Let \( \{p_n\} \) be a sequence in \( (1, +\infty) \) with \( \lim_n p_n = 1 \). Then the sequence Musielak-Orlicz space \( \ell^{p_n} \) endowed with the Luxemburg norm enjoys the FPP.

**Proof.** Indeed, consider the standard Schauder basis \( \{e_n\} \) in \( \ell^{p_n} \). Take \( A_n = \{n\} \) for every \( n \in \mathbb{N} \), the variable function \( p(n) = p_n \) with \( \lim_n p_n = 1 \). Now we are done just mimicking the same steps as in the proof of Theorem 4.6.
Note that, as far as the authors are concerned, the Musielak-Orlicz spaces $\ell^{p_n}$ with the Luxemburg norm are the first known nonreflexive Banach spaces satisfying the FPP without having to undergo any renorming process. We would like to point out that, directly referring to [41], in [6] it was quoted that: “$\ell^{p_n}$ with $\lim_n p_n = 1$ endowed with the Luxemburg norm fails the FPP because it contains an asymptotically isometric copy of $\ell_1$” (see definition and its consequences in the next section). Corollary 4.7 shows that this is not possible. In fact, after a careful reading, the authors of this manuscript strongly believe that there is a misunderstanding between the Luxemburg norm and the Orlicz norm in [41].

Finally, we conclude this section with the following corollary:

**Corollary 4.8.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $p : \Omega \to [1, +\infty]$ be a measurable function such that $L^{p(\cdot)}(\Omega)$ is nonreflexive. Assume that one of the following conditions holds:

a) $L^{p(\cdot)}(\Omega)$ contains an isometric copy of $\ell_\infty$,

b) $p^{-1}(\{1\})$ is essentially formed by finitely many atoms at most, or

c) $L^{p(\cdot)}(\Omega)$ does not contain isometrically $\ell_1$.

Then $L^{p(\cdot)}(\Omega)$ contains a further nonreflexive closed subspace with the FPP.

**Proof.** If $L^{p(\cdot)}(\Omega)$ contains isometrically $\ell_\infty$, in particular it contains isometrically the Musielak-Orlicz space of Corollary 4.7. If $L^{p(\cdot)}(\Omega)$ is nonreflexive and b) or c) holds, using Theorem 2.5 iii), either we can find an isometric copy of $\ell_\infty$ or $p_- (\Omega \setminus p^{-1}(\{1\})) = 1$ and we can apply Theorem 4.6. □

5. **The failure of asymptotically isometric copies of $\ell_1$ in Variable Lebesgue Spaces**

The failure of the FPP is strongly connected to the existence of asymptotically isometric copies of $\ell_1$, notion that was first defined by Hagler [21] and revitalized several years later by P. Dowling and C. Lennard [15]. We recall that a Banach space $X$ is said to contain an asymptotically isometric copy (a.i.c.) of $\ell_1$ if there exist a sequence $(z_n) \subset X$ and some sequence $(\epsilon_n) \subset (0, 1)$ with $\lim_n \epsilon_n = 0$ such that

$$\sum_{n=1}^{\infty} (1 - \epsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n z_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

for all $(t_n) \in \ell_1$. It was proved in [15] (see also [16]) that:

1) If $(z_n)$ spans an a.i.c. of $\ell_1$, then the closed span $[z_n]$ fails the FPP. Consequently, every Banach space containing an asymptotically isometric copy of $\ell_1$ fails to have the FPP.
2) Every nonreflexive closed subspace of $L^1[0, 1]$ contains an asymptotically isometric copy of $\ell_1$.

Theorem 4.6 shows that the verbatim translation of the assertion 2) above does not follow when $L^1[0, 1]$ is replaced by a nonreflexive VLS. From Corollary 4.8 we know that every nonreflexive $L^p(\Omega)$ contains a further nonreflexive subspace with the FPP whenever it does not contain $\ell_1$ isometrically. In this latter case, we wonder whether the whole space $L^p(\Omega)$ could satisfy the FPP. If an a.i.c. of $\ell_1$ were found in $L^p(\Omega)$, the answer would be negative by assertion 1) above. A natural question arises: Assume that $L^p(\Omega)$ does not contain $\ell_1$ isometrically. Can $L^p(\Omega)$ contain an a.i.c. of $\ell_1$? In this last section we prove that the absence of an a.i.c. of $\ell_1$ is, in fact, the general rule in nonreflexive VLS. To achieve our goals we need to introduce the subsequence splitting property and some preliminary results (see for instance [40]).

**Definition 5.1.** A Banach function space $X$ is said to have the subsequence splitting property (SSP) if for every bounded sequence $(f_n) \subset X$ there is a subsequence $(f_{nk})$ and sequences $(g_k)$, $(h_k)$ with $|g_k| \wedge |h_k| = 0$ and $f_{nk} = g_k + h_k$ for all $k \in \mathbb{N}$ such that

i) The sequence $(g_k)$ is equi-integrable in $X$.

ii) The $h_k$’s are pairwise disjoint.

In the framework of a Banach lattice with an order continuous norm and a weak unit, the SSP was completely characterized in [40, Theorem 2.5]:

**Theorem 5.2.** [40] Let $X$ be a Banach lattice with order continuous norm and weak unit. Then $X$ has the SSP if and only if $\ell_\infty$’s are not equi-normably embedded into $X$ (see Definition 2.4 in [40]).

As a consequence we can deduce:

**Corollary 5.3.** Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and $p : \Omega \to [1, +\infty)$ be a measurable function. Then $L^p(\Omega)$ verifies the SSP when $L^p(\Omega)$ contains no isometric copy of $\ell_\infty$.

**Proof.** Since $(\Omega, \sigma, \mu)$ is $\sigma$-finite, it is not difficult to find $g > 0$ a.e. and therefore $g$ is a weak unit. The absence of isometric copies of $\ell_\infty$ implies that $P_+ := p_+(\Omega_f) < +\infty$, $p^{-1}(\{+\infty\})$ is essentially formed by finitely many atoms at the most and the norm is order continuous. We next prove that $\ell_\infty$’s cannot be equi-normably embedded in $L^p(\Omega_f)$ (which automatically implies the same assertion for $L^p(\Omega)$ and the SSP by Theorem 5.2):

Let $\epsilon > 0$ and assume that for every $n \in \mathbb{N}$ we can find $\{g_1, \ldots, g_n\}$ disjointly supported functions in $L^p(\Omega_f)$ with $\|g_i\| = 1$ for $1 \leq i \leq n$ and $\|\sum_{i=1}^n g_i\| \leq 1 + \epsilon$. Using Lemma 2.2.c.i) we know that $\rho(g_i) = 1$ for $1 \leq i \leq n$ and consequently:

$$1 \geq \rho \left( \frac{\sum_{i=1}^n g_i}{1 + \epsilon} \right) = \sum_{i=1}^n \rho \left( \frac{g_i}{1 + \epsilon} \right) \geq \sum_{i=1}^n \left( \frac{1}{1 + \epsilon} \right)^{P_+} \rho(g_i) = \frac{n}{(1 + \epsilon)^{P_+}},$$
which implies that \( n \leq (1 + \epsilon)^{P_+} \) which is not possible.

Note that the following stability properties for asymptotically isometric copies of \( \ell_1 \) are easily obtained from the inequalities in (5.1):

i) Every subsequence of an a.i.c. of \( \ell_1 \) and every absolutely convex block basic sequence of an a.i.c. of \( \ell_1 \) span again an a.i.c. of \( \ell_1 \).

ii) If \((z_n)\) spans an a.i.c. of \( \ell_1 \), \(\|u_n\| \leq 1\) and \(\lim_n \|z_n - u_n\| = 0\), removing finitely many terms if necessary, \((u_n)\) spans an a.i.c. of \( \ell_1 \).

Finally, the main theorem of this section aims to show the lack of asymptotically isometric copies of \( \ell_1 \) in VLSs unless the trivial case is considered:

**Theorem 5.4.** Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and \(p: \Omega \to [1, +\infty]\) be a measurable function. The following are equivalent:

a) \(L^{p}(\Omega)\) contains an isometric copy of \( \ell_1 \).

b) \(L^{p}(\Omega)\) contains an asymptotically isometric copy of \( \ell_1 \).

**Proof.** a) implies b) is straightforward. Let us prove b) implies a): Assume \(L^{p}(\Omega)\) has a sequence \((f_n)\) spanning an a.i.c. of \( \ell_1 \). Denote, as usually, \(\Omega_f = \{t \in \Omega : p(t) < \infty\}\) and we set \(p^{-1}(\{1, +\infty\}) = A \cup B\) where \(A\) is the atomic part and \(B\) a nonempty set.

Assume, by contradiction, that \(L^{p}(\Omega)\) contains no isometric copy of \( \ell_1 \). This automatically implies that \(p_+(\Omega_f) < +\infty\), \(A\) is formed by finitely many atoms (possibly empty) and \(\mu(B) = 0\). In view of these assumptions we can assume that the Luxemburg norm is order continuous and \(f_n = f_n\chi_{\Omega \setminus B}\) in \(L^{p}(\Omega)\). We split the proof in several steps:

Step 1: We can assume that the a.i.c. \((f_n)\) is pairwise disjoint and \(\text{supp} (f_n) \subset \Omega \setminus p^{-1}(\{1, +\infty\})\) for all \(n \in \mathbb{N}\) since \(p^{-1}(\{1, +\infty\}) = A \cup B\), \(A\) is a collection of a finite number of atoms and \(\mu(B) = 0\): From Corollary 5.3 \(L^{p}(\Omega)\) has the SSP and without loss of generality we can assume that \(f_n = g_n + h_n\) where \(|g_n| \wedge |h_n| = 0\), the \(h_n\)'s are pairwise disjoint, \((g_n)\) is equi-integrable and therefore relatively weakly compact (see [32] Definition 3.6.1 and Proposition 3.6.5). Taking a further subsequence, we can assume that \(\{g_n\}\) is weakly convergent and so the sequence \(g'_n = \frac{g_{n+1} - g_{n+2}}{2}\) is weakly convergent to zero. Using Mazur's Theorem, there exist \(p_1 \leq q_1 < p_2 \leq q_2 < \cdots\) and a nonnegative sequence \(\{\lambda_i\}\) such that \(\sum_{i=p_n}^{q_n} \lambda_i = 1\) and the sequence \(\{G_n := \sum_{i=p_n}^{q_n} \lambda_i g'_i\}\) is norm-null convergent.

Define \(F_n := \sum_{i=p_n}^{q_n} \lambda_i \frac{f_{n+1} - f_{n+2}}{2}\) and \(H_n = \sum_{i=p_n}^{q_n} \lambda_i \frac{h_{n+1} - h_{n+2}}{2}\) so \(F_n = H_n + G_n\) and \(\lim_n \|F_n - H_n\| = \lim_n \|G_n\| = 0\). From i) and ii) above, \(\{F_n\}\) spans an a.i.c. of \( \ell_1 \) and so does \(\{H_n\}\), which is pairwise disjoint.

Step 2: We can suppose that \(p_-(\text{supp} (f_n)) > 1\) for all \(n \in \mathbb{N}\):
Take a nonincreasing sequence \((\gamma_k)_k \subset (1, +\infty)\) with \(\lim_k \gamma_k = 1\). Fix some \(n \in \mathbb{N}\). Since \(\supp(f_n) \subset \Omega \setminus p^{-1}(\{1, +\infty\})\) for all \(n \in \mathbb{N}\), the sequence 
\[
    f_n \chi_{p^{-1}(\{1, \gamma_k\})} \rightarrow_k 0 \quad \text{a.e.} \quad t \in \Omega
\]
and therefore \(\lim_k \rho(f_n \chi_{p^{-1}(\{1, \gamma_k\})}) = 0\). Since \(p_+(\Omega_f) < +\infty\), from Lemma 2.2\(c.ii\) we have \(\lim_k \|f_n \chi_{p^{-1}(\{1, \gamma_k\})}\| = 0\). Thus, for all \(n \in \mathbb{N}\) we can consider some \(k_n \in \mathbb{N}\) such that 
\[
    \lim_n \|f_n - f_n \chi_{p^{-1}(\gamma_{k_n}, \infty)}\| = 0.
\]
Define \(h_n := f_n \chi_{p^{-1}(\gamma_{k_n}, \infty)}\). Now the sequence \((h_n)\) is pairwise disjoint, spans an a.i.c. of \(\ell_1\) and \(p_-(\supp(h_n)) \geq \gamma_{k_n} > 1\).

Once that Steps 1 and 2 have been proved, we can assume that \(L^{p(\cdot)}(\Omega)\) contains a pairwise disjoint sequence \((f_n)\) with \(p_n := p_-(A_n) > 1\), where \(A_n := \supp(f_n)\) for all \(n \in \mathbb{N}\) and spanning an a.i.c. of \(\ell_1\). Let \((\epsilon_n)\) be the null sequence verifying the inequalities \((5.1)\) for \((f_n)\) and take a subsequence \((\epsilon_{n_k})\) such that \(\epsilon_{n_k} < \frac{1}{2^k}\). Defining \(g_k = f_{n_k}, \eta_k = \epsilon_{n_k}\), the sequence \((g_k)\) spans an a.i.c. of \(\ell_1\) and \(\lim_k k\eta_k = 0\). Hence, we can assume that the sequences \((f_n)\) and \((\epsilon_n)\) obtained in Step 2 additionally satisfy \(\lim_n n\epsilon_n = 0\).

In particular, since \(p_1 > 1\), there exists some \(n_0 \in \mathbb{N}\) such that:

\[
    n^{-\frac{1-p_1}{2}} + n\epsilon_n < 1 - \epsilon_1 \quad \text{for all } n \geq n_0.
\]

Consider the function \(h : [1, +\infty) \rightarrow [0, +\infty)\) by \(h(t) = t^{(p_1-1)}(t-1),\) which is strictly increasing in \([1, +\infty), h(1) = 0\) and \(\lim_{t \rightarrow +\infty} h(t) = +\infty\). Thus \(h\) is bijective and bicontinuous. We claim that:

\[
    \left\| \frac{f_1}{n} + f_n \right\| \leq h^{-1}\left( \frac{1}{n^{p_1}} \right) \quad \text{for all } n \in \mathbb{N}.
\]

Indeed, fix some \(n \in \mathbb{N}\). We rename \(s := h^{-1}\left( \frac{1}{n^{p_1}} \right) > 1\). Then:

\[
    \rho\left( \frac{f_1}{n} + f_n \right) = \int_{A_n} \left( \frac{1}{ns} \right)^{p(t)} |f_1(t)|^{p(t)} d\mu + \int_{A_n} \left( \frac{1}{s} \right)^{p(t)} |f_n(t)|^{p(t)} d\mu
\]

\[\leq \left( \frac{1}{ns} \right)^{p_1} \rho(f_1) + \frac{1}{s} \rho(f_n) \leq \left( \frac{1}{ns} \right)^{p_1} + \frac{1}{s} = 1\]

and the claim is proved. From the left-hand side of the inequalities \((5.1)\) and using \((5.2)\), for all \(n \geq n_0\) we have that:

\[
    h^{-1}\left( \frac{1}{n^{p_1}} \right) \geq \frac{1-\epsilon_1}{n} + 1 - \epsilon_n \geq n^{-\frac{1-p_1}{2}} + 1.
\]

Since the function \(h\) is strictly increasing, the above implies that:

\[
    h\left( n^{-\frac{p_1+1}{2}} + 1 \right) = \left( n^{-\frac{p_1+1}{2}} + 1 \right)^{p_1-1} n^{-\frac{p_1+1}{2}} \leq \frac{1}{n^{p_1}}
\]

and:

\[
    \left( n^{-\frac{p_1+1}{2}} + 1 \right)^{p_1-1} \leq n^{\frac{1-p_1}{2}}.
\]
Taking limits when $n$ goes to infinity, we arrive at the inequality $1 \leq 0$, which shows that $L^{p(\cdot)}(\Omega)$ cannot have an a.i.c. of $\ell_1$ in the absence of isometric copies of $\ell_1$ and the proof is complete. □

It is uncertain for the authors whether every nonreflexive $L^{p(\cdot)}(\Omega)$ without an isometric copy of $\ell_1$ may satisfy the FPP. What can at once be deduced from Theorem 5.4 is that, in order to assert the failure of the FPP, either a precise counterexample would need to be found or new alternative techniques would need to emerge, since asymptotically isometric copies of $\ell_1$ do not play any relevant role in contrast to the $L^1$-case.

A seemingly much easier problem does not have a precise answer yet: Consider a purely atomic $\sigma$-finite measure space and a bounded sequence $(p_n) \subset [1, +\infty)$ with $\lim\inf_n p_n = 1$ (which implies that it is nonreflexive). Does the Musielak-Orlicz space $\ell^{p_n}$ with the Luxemburg norm have the FPP? (We only know that the answer is positive when $\lim_n p_n = 1$ due to Corollary 4.7).

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