13-Moment System with Global Hyperbolicity for Quantum Gas

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Abstract

We point out that the quantum Grad’s 13-moment system [13] is lack of global hyperbolicity, and even worse, the thermodynamic equilibrium is not an interior point of the hyperbolicity region of the system. To remedy this problem, by fully considering Grad’s expansion, we split the expansion into the equilibrium part and the non-equilibrium part, and propose a regularization for the system with the help of the new theory developed in [4, 7]. This provides us a new model which is hyperbolic for all admissible thermodynamic states, and meanwhile preserves the approximate accuracy of the original system. It should be noted that this procedure is not a trivial application of the theory in [4, 7].

Keywords: quantum gas, quantum Boltzmann equation, Grad’s 13-moment system, global hyperbolicity, regularization.

1 Introduction

The behavior of dilute quantum gas can be modeled by the quantum Boltzmann equation (QBE), which is also known as Uehling-Uhlenbeck equation [12]. Studies on the quantum kinetic equation may promote the understanding of sorts of quantum effects, for example, Bose-Einstein condensation (BEC) [5]. However, the development of quantum kinetic theory lags far behind the gas kinetic theory. Due to the complexity of the distribution function, it is quite different to derive quantum hydrodynamic equations from the QBE. The progress in the study of gas kinetic theory should be illustrative to the study of quantum kinetic theory. Recently, Yano has extended the well-known Grad’s 13-moment system in gas kinetic theory to the quantum case to obtain the quantum Grad’s 13-moment system [13], which is a new development and application of Grad’s moment method in quantum kinetic theory.

As is well known in gas kinetic theory, there are a number of defects of Grad’s 13-moment system, one of which is that Grad’s 13-moment system are not globally hyperbolic. Precisely, for 1D flow, the hyperbolicity can only be obtained near the thermodynamic equilibrium [10]. And for 3D case, there is no neighborhood of thermodynamic equilibrium...
contained in the hyperbolicity region\cite{3}. In this paper, we investigate the hyperbolicity of the quantum Grad’s 13-moment system. By studying the Jacobian of the convection part of the quantum Grad’s 13-moment system, we find that the quantum Grad’s 13-moment system shares the same defect as the classical Grad’s 13-moment system. For 1D flow, the hyperbolicity of the quantum Grad’s 13-moment system can always be obtained near the equilibrium, and the hyperbolicity region depends on the equilibrium state due to the quantum effects, while for the classical Grad’s 13-moment system, the hyperbolicity region only depends on the non-equilibrium part. For 3D case, the system is not hyperbolic in any neighborhood of the equilibrium; that is to say, the equilibrium is on the boundary of the hyperbolicity region.

The loss of hyperbolicity directly breaks the well-posedness of the partial differential equations, thus it is hard to apply the quantum Grad’s 13-moment model to practical problems. In gas kinetic theory, by investigating the coefficient matrix of the moment system, a globally hyperbolic regularization has been proposed to settle the loss of hyperbolicity of Grad’s moment system \cite{1,2}. In \cite{4,7}, it was revealed that the essential of the regularization is to treat the time and space derivatives in the same way, then the authors extended the regularization to a framework on moment model reduction from kinetic equations to globally hyperbolic hydrodynamic system. Using the theory therein, one of the tasks in this paper is to deduce a globally hyperbolic hydrodynamic model from the quantum Boltzmann equation. Unfortunately, if one applies the framework \cite{4,7} in a trivial way, the resulting system would be not compatible with Grad’s 13-moment system due to the particular formation of the quantum thermodynamic equilibrium. Precisely, the linearized equations of the resulting system at the equilibrium is different from those of Grad’s 13-moment system, and the first step of Maxwellian iteration\cite{9} of the resulting system fails to give the correct NSF law.

Since the equilibrium plays an essential role in the quantum Grad’s expansion, we split the expansion into two parts: the equilibrium part and the non-equilibrium part. By applying the framework \cite{4,7} only on the non-equilibrium part, we obtained a new regularized 13-moment system, which is proved to be hyperbolic for any admissible state. The linearized equations of the regularized 13-moment model are the same as those of the quantum Grad’s 13-moment system. Hence, the regularized 13-moment model differs from the quantum Grad’s 13-moment model by only some high-order terms, and thus the new system preserves most of the merits of the original model. As an example of these merits, we show that the first step of Maxwellian iteration\cite{9} of the new model also yields the correct NSF law. Moreover, we bring a new observation on the globally hyperbolic regularization that the key point of the regularization is to split the convection operator into the product of the multiplying velocity operator and the space derivative operator.

The layout of this paper is as follows. In Sect. 2, an overview of the quantum Boltzmann equation and the quantum Grad’s 13-moment system are given. In Sect. 3, the hyperbolicity of the quantum Grad’s 13-moment system is studied in details both for 1D and 3D case. The hyperbolic regularization for the quantum Grad’s 13-moment system and the regularized 13-moment system are then proposed and discussed in Sect. 4. The difference between the regularized system and the original system is discussed. Furthermore, the hyperbolicity and the physical properties of the regularized 13-moment system are also studied. At last, a conclusion is presented in Sect. 5.
2 Grad’s 13-Moment System for QBE

Denoting the distribution function of quantum gas particles by \( f(t, x, v) \), which describes the probability density to find a particle at space point \( x \) and time \( t \) with velocity \( v \), we have the quantum Boltzmann equation, the well-known Uehling-Uhlenbeck equation \([12]\)

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f),
\]

where the collision term is defined as

\[
Q(f, f) = \int_{\mathbb{R}^3} \int_{0}^{2\pi} \int_{0}^{\pi} \left[ (1 - \theta f)(1 - \theta f^*_s)f f^*_s - (1 - \theta f')(1 - \theta f'_s)f f'_s \right] g\sigma \sin \chi \, d\chi \, d\epsilon \, dv_s.
\]

Here \( f, f', f_s \) and \( f'_s \) are the shorthand notations for \( f(t, x, v), f(t, x, v'), f(t, x, v_s) \) and \( f(t, x, v'_s) \). \((v, v_s) \) and \((v', v'_s) \) are the velocities before and after collision. \( \epsilon \) is the deflection angle, \( \chi \) is the scattering angle, \( g = |v - v_s| \) and \( \sigma \) is the differential cross section. \( \theta = 1, 0, -1 \) correspond to Fermion, classical gas and Boson, respectively. Although Fermion and Boson are major considerations of the paper, almost all the results are also valid for classical gases.

Similar to the classical Boltzmann equation, the quantum collision term conserves mass, momentum and energy, i.e.

\[
\int_{\mathbb{R}} Q(f, f) \, dv = 0, \quad \int_{\mathbb{R}} Q(f, f)v_i \, dv = 0, \quad i = 1, 2, 3, \quad \int_{\mathbb{R}} Q(f, f)|v|^2 \, dv = 0,
\]

where \( 1, v_i(i = 1, 2, 3), |v|^2 \) are called collision invariants. If we define the density \( \rho \), the macroscopic velocity \( u \) and the pressure \( p \) as

\[
\rho = \frac{m}{\hat{h}^3} \int_{\mathbb{R}^3} f \, dv, \quad \rho u = \frac{m}{\hat{h}^3} \int_{\mathbb{R}^3} f v \, dv, \quad p = \frac{m}{3 \hat{h}^3} \int_{\mathbb{R}^3} f |v - u|^2 \, dv,
\]

where \( \hat{h} = h/m \), and \( m \) is the mass of the particle, \( h \) is Planck’s constant, and define the pressure tensor \( p_{ij} \) and the heat flux \( q_i \) as

\[
p_{ij} = \frac{m}{\hat{h}^3} \int_{\mathbb{R}^3} f c_i c_j \, dv, \quad i, j = 1, 2, 3, \quad q_i = \frac{1}{2} \frac{m}{\hat{h}^3} \int_{\mathbb{R}^3} f c_i |c|^2 \, dv, \quad i = 1, 2, 3,
\]

where \( c_i = v_i - u_i, i = 1, 2, 3 \), then

\[
\frac{d\rho}{dt} + \rho \frac{\partial u_d}{\partial x_d} = 0, \quad (6a)
\]

\[
\rho \frac{\partial u_i}{\partial t} + \frac{\partial p_{id}}{\partial x_d} = 0, \quad i = 1, 2, 3, \quad (6b)
\]

\[
\frac{dp}{dt} + \frac{2}{3} p_{id} \frac{\partial u_i}{\partial x_d} + p \frac{\partial u_d}{\partial x_d} + \frac{2}{3} \frac{\partial q_d}{\partial x_d} = 0, \quad (6c)
\]

where \( \frac{dp}{dt} = \frac{\partial p}{\partial t} + u_d \frac{\partial p}{\partial x_d} \). The thermodynamic equilibrium is

\[
f_{eq} = \frac{1}{\sqrt{\pi} \sigma} \exp \left( \frac{|v - u|^2}{2\sigma^2} \right) + \theta,
\]

where \( \theta \) is Planck’s constant, and define the heat flux \( q_i \) as

\[
q_i = \frac{1}{2} \frac{m}{\hat{h}^3} \int_{\mathbb{R}^3} f c_i |c|^2 \, dv, \quad i = 1, 2, 3.
\]
where $z$ and $RT$ is related to $\rho$ and $p$ as
\[
\rho = \frac{m}{h^3} \sqrt{2\pi RT} \frac{1}{2} \frac{L_i z_i}{2}, \quad p = \frac{m}{h^3} \sqrt{2\pi RT}^{3/2} RT L_i z_i, \tag{8}
\]
and $L_i := \theta L_i \theta(-\theta z)$ is the polylogarithm. For the special case $\theta = 0$, let $L_i = z$.

In this paper, we focus on 13-moment system, where 13 moments are $\rho, u_i, p_{ij}$ and $q_{ij}$, $i, j = 1, 2, 3$. Defining
\[
\sigma_{ij} = p_{ij} - p \delta_{ij}, \quad q_{ijk} = \frac{m}{h^3} \int_{\mathbb{R}^3} f c_i c_j c_k \, dv, \quad \Delta_{ij} = \frac{m}{h^3} \int_{\mathbb{R}^3} f c_i c_j |c|^2 \, dv, \tag{9}
\]
we have
\[
\frac{d\sigma_{ij}}{dt} + \sigma_{ij} \frac{\partial u_i}{\partial x_d} + 2 p_d q_{ij} \frac{\partial u_j}{\partial x_d} + \frac{2}{3} \frac{\partial q_{ijd}}{\partial x_d} = Q^{(2)}_{ij}, \quad i, j = 1, 2, 3, \tag{10a}
\]
\[
\frac{d q_{ij}}{dt} + 5 \frac{p}{2} \frac{d u_j}{dt} + \sigma_{ij} \frac{d u_j}{dt} + 2 q_{ij} \frac{\partial u_i}{\partial x_d} + q_{ijd} \frac{\partial u_j}{\partial x_d} + \frac{1}{2} \frac{\partial \Delta_{ijd}}{\partial x_d} = Q^{(3)}_{ij}, \quad i = 1, 2, 3, \tag{10b}
\]
where $Q^{(2)}_{ij} = \int_{\mathbb{R}^3} c_i c_j Q(f, f) \, dv$ and $Q^{(3)}_{ij} = \frac{1}{2} \int_{\mathbb{R}^3} c_i |c|^2 Q(f, f) \, dv$. Particularly, for the quantum Bhatnagar-Gross-Krook model\cite{11},
\[
\hat{Q}^{(2)}_{BGK,ij} = -\frac{1}{7} \sigma_{ij}, \quad \hat{Q}^{(3)}_{BGK,i} = -\frac{2}{7} q_i. \tag{11}
\]
Eqs. (6) and (10) are the 13-moment system, with $q_{ijk}$ and $\Delta_{ij}$ are undetermined. In\cite{13}, Yano extended Grad’s expansion\cite{8} into the quantum case to obtain the quantum Grad’s 13-moment system. Actually, Yano gave two kinds of expansions\cite{13}:
\[
f_{G13}^{H1} = f_{eq} \left(1 + \frac{\sigma_{ij}}{2p} \frac{L_i z_i}{2} \right) \left(\frac{c_i c_j}{RT} - \delta_{ij} \frac{L_i z_i}{2} \right) + \frac{\mathfrak{B} - 1}{5pRT} c_i \left(\frac{|c|^2}{RT} - \frac{5}{2} \frac{L_i z_i}{2} \right), \tag{11}
\]
\[
f_{G13}^{H2} = f_{eq} + f_{eq} \left(1 - \theta f_{eq} \right) \left(\frac{\sigma_{ij}}{2p} \frac{c_i c_j}{RT} - \delta_{ij} \frac{|c|^2}{3RT} \right) + \frac{\mathfrak{B}_2 - 1}{5pRT} c_i \left(\frac{|c|^2}{RT} - \frac{5}{2} \frac{L_i z_i}{2} \right), \tag{12}
\]
where $\mathfrak{B} = 7/2(L_i z_i / L_i z_i) - 5/2(L_i z_i / L_i z_i)^2$, and $\mathfrak{B}_2 = 7/2(L_i z_i / L_i z_i) - 5/2(L_i z_i / L_i z_i)^2$. Both of the expansions degenerates into Grad’s expansion\cite{8} for classical gases. In\cite{13}, the expansion (11) was used to derive the quantum Grad’s 13-moment system, and the expansion (12) to assist the analysis of Grad’s moment system (essentially for Chapman-Enskog expansion\cite{5} and collision term\cite{11}). However, for the quantum Boltzmann equation, these two different expansions would result in different moment systems. Substituting the expansion (11) or (12) into (9) yields the expression of $q_{ijk}$ and $\Delta_{ij}$ directly. For the expansion (11), the moment closure is
\[
q_{ijk} = \frac{2}{5} \left(\delta_{ij} q_k + \delta_{ik} q_j + \delta_{kj} q_i \right), \tag{13}
\]
\[
\Delta_{ij} = \frac{m}{h^3} \sqrt{2\pi RT}^{3} RT^2 L_i z_i \left(5 \delta_{ij} q + 7 \frac{\sigma_{ij}}{p} \frac{L_i z_i}{2} \right). \tag{13}
\]
For the expansion (12), the moment closure is
\[
q_{ijk} = \frac{2}{5} \left(\delta_{ij} q_k + \delta_{ik} q_j + \delta_{kj} q_i \right), \tag{14}
\]
\[
\Delta_{ij} = \frac{m}{h^3} \sqrt{2\pi RT}^{3} RT^2 L_i z_i \left(5 \delta_{ij} q + 7 \frac{\sigma_{ij}}{p} \right). \tag{14}
\]
Both the system (6) and (10) with the moment closure (13) or (14) are Grad’s 13-moment system for the quantum Boltzmann equation.

In this paper, we follow the work in [13], and only consider the case with the moment closure (13). All the deductions below can be extended to the moment system with the moment closure (14) without essential difficulties.

3 Hyperbolicity of Grad’s 13-Moment System

As is well known, loss of hyperbolicity, which leads to the loss of well-posedness of the model, is a severe drawback of Grad’s 13-moment system[10]. In [3], the authors pointed out that the thermodynamic equilibrium is not an interior point of the hyperbolicity region of Grad’s 13-moment system. Therefore, any small perturbation of the equilibrium state may lead to the loss of hyperbolicity, thus the existence of the solution of Grad’s 13-moment system is hardly achieved. In this section, we investigate the hyperbolicity of the quantum Grad’s 13-moment system with the moment closure (13), and the results can be extended to the system with the moment closure (14) with routine calculations. Let us first define the hyperbolicity as follows.

**Definition 1 (Hyperbolicity).** A system of first order quasi-linear partial differential equations

\[
\frac{\partial w}{\partial t} + A_d(w) \frac{\partial w}{\partial x_d} = 0
\]

is called hyperbolic in some region \( \Omega \) if and only if any linear combination of \( A_d(w) \) is diagonalizable with real eigenvalues for all \( w \in \Omega \).

3.1 Hyperbolicity for 1D case

For 1D case, the quantum Grad’s 13-moment system reduces to a smaller system containing only five equations, which can be obtained by setting \( u_2 = u_3 = 0 \), \( p_{12} = p_{12} = p_{23} = 0 \), \( q_2 = q_3 = 0 \) and \( p_{22} = p_{33} \). Denoting \( w_5 = (\rho, u_1, p_{11}, q_1, p) \), one has

\[
\frac{\partial w_5}{\partial t} + A_5 \frac{\partial w_5}{\partial x_1} = Q_5,
\]

where \( Q_5 = (0, 0, Q^{(2)}_{11}, Q^{(3)}_{1}, 0)^T \), and

\[
A_5 = \begin{pmatrix}
    u_1 & \rho & 0 & 0 & 0 \\
    0 & u_1 & \frac{1}{\rho} & 0 & 0 \\
    0 & 3p_{11} & u_1 & 6/5 & 0 \\
    -a_1 & \frac{16}{5}q_1 & a_2 & u_1 & a_3 \\
    0 & p + \frac{2}{3}p_{11} & 0 & 2/3 & u_1
\end{pmatrix},
\]
Eq. (19) has five distinct zeros, which read

\[ a_1 = \frac{5pRTb}{2\rho} \left( \frac{7L_i^2 L_i}{2} - \frac{5L_i}{2} \right) + \frac{7\sigma_{11}RTb}{2\rho} \left( \frac{L_i^2 L_i}{2} - \frac{5L_i}{2} \right), \]

\[ a_2 = \frac{7RTL_i^2}{2L_i} - \frac{3p}{2\rho} - \frac{p_{11}}{\rho}, \]

\[ a_3 = \frac{5RT}{2} \left( 1 + b \right) \frac{L_i^2}{L_i} - \frac{3b}{2} \frac{L_i}{L_i} \left( 1 - \frac{L_i}{L_i^2} \right) \]

\[ + \frac{7RT}{2} \left( \frac{\sigma_{11}b}{p} - 1 \right) \frac{L_i^2}{L_i} - \frac{3b}{2} \frac{L_i}{L_i} \frac{\sigma_{11}}{\rho} \left( 1 - \frac{L_i}{L_i^2} \right). \]

Here \( b = \frac{2}{5-3L_i^2 - L_i} \). The characteristic polynomial of \( A_5 \) can be directly calculated as

\[ |\lambda I - A_5| = \frac{1}{75} (\lambda - u_1) \left( 75(\lambda - u_1)^4 - \left( 90a_2 + 50a_3 + 225\frac{p_{11}}{\rho} \right) (\lambda - u_1)^2 + 90 \left( a_1 + a_3 \frac{\sigma_{11}}{\rho} \right) + 288\frac{q_1}{\rho} (\lambda - u_1) \right). \]  

(17)

Introducing the dimensionless quantity, \( \hat{\lambda} = (\lambda - u_1)/\sqrt{RT} \), we reduce \( |\lambda I - A_5| = 0 \) into

\[ \hat{\lambda} \left[ 75\hat{\lambda}^4 RT^2 - \left( 90a_2 + 50a_3 + 225\frac{p_{11}}{\rho} \right) \hat{\lambda}^2 RT + 90 \left( a_1 + a_3 \frac{\sigma_{11}}{\rho} \right) + 288\frac{q_1}{\rho} \hat{\lambda} \sqrt{RT} \right] = 0. \]

(18)

For the special case \( \sigma_{11} = 0, q_1 = 0 \), i.e. the thermodynamic equilibrium state, \( \hat{\lambda} \) reduces into

\[ \hat{\lambda} \left[ 75\hat{\lambda}^4 RT^2 - \left( 90a_2 + 50a_3 + 225\frac{P}{\rho} \right) \hat{\lambda}^2 RT + 90a_1 \right] = 0. \]

(19)

Since

\[ 90a_1 > 0, \quad 90a_2 + 50a_3 + 225\frac{P}{\rho} > 0, \]

and it is easy to check

\[ \Delta_{G5} := \left( 90a_2 + 50a_3 + 225\frac{P}{\rho} \right)^2 - 4 \cdot 75RT^2 \cdot 90a_1 > 0, \]

Eq. (19) has five distinct zeros, which read

\[ \hat{\lambda}_{1,5} = \pm \sqrt{\frac{90a_2 + 50a_3 + 225\frac{P}{\rho} + \sqrt{\Delta_{G5}}}{150RT^2}}, \quad \hat{\lambda}_3 = 0, \]

\[ \hat{\lambda}_{2,4} = \pm \sqrt{\frac{90a_2 + 50a_3 + 225\frac{P}{\rho} - \sqrt{\Delta_{G5}}}{150RT^2}}. \]

(20)

In this case, \( A_5 \) has five distinct eigenvalues as \( \lambda_i = u_1 + \sqrt{RT} \hat{\lambda}_i, i = 1, 2, \ldots, 5 \). If \( \sigma_{11}/p \) and \( q_1/(p\sqrt{RT}) \) are small enough, the zeros of (18) are still real and separable. Hence, the matrix \( A_5 \) has no multiple eigenvalue around the equilibrium, which means the system (15) is hyperbolic around the equilibrium. Moreover, the hyperbolicity of (15) depends
the pressure tensor $\sigma$ for some given $q$ is symmetric on the heat flux $\theta$ region of (15). Figs. 1 and 2 give the hyperbolicity regions of the system (15) with $f$ we mirror the distribution function $f$ with respect to $\xi_1$ and $\xi_2$ respectively, and the red point corresponds to the equilibrium. Then only on $\zeta$, $\sigma_{11}/p$ and $q_1/(p\sqrt{RT})$. For a given $\zeta$, we can directly plot the hyperbolicity region of (15). Figs. 1 and 2 give the hyperbolicity regions of the system (15) with $\theta = -1, 1$ for some given $\zeta$, respectively. It is reminded that $\sigma_{11}/p \in (-1, 2)$ due to the fact that the pressure tensor $p_{ij}$ is positive definite, and $q_1/(p\sqrt{RT}) \in \mathbb{R}$.

Similar to the Grad’s 13-moment system for classical gasses [10], for the quantum case, the thermodynamic equilibrium is an interior point of the hyperbolicity region, but the system is not hyperbolic for all admissible $\sigma_{11}$ and $q_1$. Additionally, as shown in Figs. 1 and 2 the hyperbolicity region expands as $\zeta$ increasing for Boson, while the hyperbolicity region shrinks as $\zeta$ increasing for Fermion. Therefore, the hyperbolicity region depends on $\zeta$. However, for the classical Grad’s 13-moment system, the hyperbolicity region does not depend on the equilibrium. It is the performance of the quantum effect. It is well-known that for classical gas the sum of two equilibrium distribution with the same mean velocity and temperature is still an equilibrium, which does not hold any more for quantum gas due to the quantum effect. And as the fugacity decreases, the quantum effect weakens, which corresponds $\text{Li}_s/\zeta \to 1$ as $\zeta \to 0$ (for classical gas, $\text{Li}_s/\zeta = 1$) in mathematics. For Fermion, the dimensionless coefficient of $\frac{q_1}{p\sqrt{RT}}$ is $\frac{\text{Li}_{3/2}}{\text{Li}_{1/2}}$, which increases as the fugacity $\zeta$ increasing. So for given $\frac{q_1}{p\sqrt{RT}}$ the perturbation raised by the heat flux turns stronger as $\zeta$ increasing, which results in the hyperbolicity region shrinks. Analogously, for Boson, the hyperbolicity region expands as $\zeta$ increasing due to $\frac{\text{Li}_{3/2}}{\text{Li}_{1/2}}$ decreasing. Moreover, the hyperbolicity region is symmetric on the heat flux $q_1$ but not symmetric on the stress $\sigma_{11}$, since $q_1$ is an odd order moment of $f$ with respect to $\xi_1$ while $\sigma_{11}$ is an even order moment. Precisely, if we mirror the distribution function $f$ on the $\xi_1$ direction, i.e $f(\xi_1, \xi_2, \xi_3) \to f(-\xi_1, \xi_2, \xi_3)$,
then $q_1$ switches its sign, i.e $q_1 \rightarrow -q_1$ and $\sigma_{11}$ keeps unchanged.

### 3.2 Hyperbolicity for 3D case

As pointed out above, the quantum Grad’s 13-moment system for 1D flow is hyperbolic around the thermodynamic equilibrium, but not hyperbolic for all admissible states. Intuitively, for 3D case, the model would also be hyperbolic around the equilibrium. However, in [3], the authors revealed the opposite fact, and pointed out that the classical Grad’s 13-moment system is not hyperbolic even around the equilibrium. In the following, we investigate the hyperbolicity of the quantum Grad’s 13-moment system for 3D case.

![Figure 3: A cross section of diagonalizable region of $A_1$ in (21) with $\zeta = 0.5$ for $\theta = -1$ (Boson), 0 (Classical gas) and 1 (Fermion). The blue part and the yellow part denote the diagonalizable region and the non-diagonalizable region, respectively, and the red point corresponds to the equilibrium.](image)

Let $w = (\rho, u_1, u_2, u_3, p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}, q_1, q_2, q_3)^T$, then the quantum Grad’s 13-moment system can be written as

$$\frac{\partial w}{\partial t} + A_d \frac{\partial w}{\partial x_d} = Q,$$

where $Q = (0, 0, 0, 0, Q^{(2)}_{11}, Q^{(2)}_{12}, Q^{(2)}_{13}, Q^{(2)}_{22}, Q^{(2)}_{23}, Q^{(2)}_{33}, Q^{(3)}_{11}, Q^{(3)}_{22}, Q^{(3)}_{33}, Q^{(2)}, Q^{(3)})^T$, and $A_d$ can be obtained directly from [6] and [10].

If $w$ represents the thermodynamic equilibrium, i.e.

$$p_{ij} = p \delta_{ij}, \quad q_i = 0, \quad i, j = 1, 2, 3,$$

then the characteristic polynomial of $A_1$ can be directly calculated as

$$|\lambda I - A_1| = (RT)^{13/2} \tilde{\lambda}^5 \left( \tilde{\lambda}^2 - \frac{7Li_2}{2} \lambda \right)^2 \left( \tilde{\lambda}^4 - c_1 \tilde{\lambda}^2 + c_0 \right),$$

where $\tilde{\lambda} = (\lambda - u_1)/\sqrt{RT}$, and $c_0$ and $c_1$ and quantities depending on $\zeta$, and satisfying $c_0 > 0$ and $c_1 > 4c_0$ (see Appendix A for their concrete forms). Hence, in this case, the
matrix $A_1$ has 13 real eigenvalues $\lambda_i = u_1 + \sqrt{RT} \hat{\lambda}_i$, $i = 1, \cdots, 13$ with

$$
\hat{\lambda}_{1,13} = \pm \frac{c_1 + \sqrt{c_1^2 - 4c_0}}{2}, \quad \hat{\lambda}_{2,3,11,12} = \pm \frac{7L_i}{5L_i},
$$

$$
\hat{\lambda}_{4,10} = \pm \frac{c_1 - \sqrt{c_1^2 - 4c_0}}{2}, \quad \hat{\lambda}_{5,6,7,8,9} = 0.
$$

Furthermore, we point out that $A_1$ and any linear combination of $A_d$ are diagonalizable. The proof will be given in Sect. 4.3.3 and Sect. 4.3.4. Next, we will show that if $w$ is perturbed around the equilibrium, $A_1$ may be not diagonalizable anymore. Consider the case

$$
p_{11} = p_{22} = p_{33} = p, \quad p_{13} = p_{23} = 0, \quad q_1 = q_2 = q_3 = 0, \quad p_{12} = \epsilon p.
$$

The thermodynamic equilibrium corresponds to $\epsilon = 0$. The characteristic polynomial of $A_1$ is

$$
|\lambda I - A_1| = \frac{RT^{13/2}}{125} \lambda^3 \left(5\lambda^2 - \frac{7L_i}{L_i}\right) g(\hat{\lambda}),
$$

(24)

where $\hat{\lambda} = (\lambda - u_1)/\sqrt{RT}$, and

$$
g(x) = 25x^4 + c_4 x^3 + c_3 x^2 + c_2 x
$$

$$
-14c^2 \frac{L_i^2 \left(14L_i^2 L_i^2 L_i^2 + 35L_i^2 L_i^2 L_i^2 - 80L_i L_i^2 L_i^2 + 35L_i^4\right)}{L_i^2 L_i^3 \left(5L_i L_i^2 - 3L_i L_i^2\right)},
$$

(25)

where $c_2, c_3, c_4$ are some quantities depending on $\xi$ and $\epsilon$ (see Appendix A for its concrete forms). If $\epsilon \neq 0$, then the constant term of $g(x)$

$$
-14c^2 \frac{L_i^2 \left(14L_i^2 L_i^2 L_i^2 + 35L_i^2 L_i^2 L_i^2 - 80L_i L_i^2 L_i^2 + 35L_i^4\right)}{L_i^2 L_i^3 \left(5L_i L_i^2 - 3L_i L_i^2\right)} < 0.
$$

So $g(x)$ has at least one negative root, which indicates $A_1$ has at least two complex eigenvalues, i.e. $A_1$ is not real diagonalizable. Consequently, for arbitrarily small $\epsilon \neq 0$, the system (21) is not hyperbolic. The thermodynamic equilibrium is not an interior point of the hyperbolicity region of the quantum Grad’s 13-moment system. Whether $A_1$ is diagonalizable depends on several variables, and we cut a cross section of the region crossing the thermodynamic equilibrium. For a given $\xi$, set $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$, $q_2 = q_3 = 0$, and plot the cross sections in Fig. 3. It is reminded that since $\sigma_{12}$ is an odd order moment of $f$ with respect to $\xi_1$, the diagonalizable regions are symmetric on both $q_1$ and $\sigma_{12}$. One can observe that the equilibrium is just on the boundary of the real diagonalizable region of $A_1$. Therefore, the quantum Grad’s 13-moment system is not hyperbolic in any neighborhood of the equilibrium. Without the hyperbolicity in a neighborhood of the equilibrium, the well-posedness of the quantum Grad’s 13-moment system is lost even if the phase density is extremely close to the equilibrium. Although in [13], the author claimed that the quantum Grad’s 13-moment system possesses a lot of good properties, this fatal drawback tremendously limits the practical application of the system.
4 Regularization of the 13-Moment system

In this section, we try to regularize the quantum Grad’s 13-moment system for the QBE to a globally hyperbolic model. Hyperbolicity is a critical condition for the existence and stability of the solution of a model. However, it is not enough to guarantee the model is a good approximation of the QBE. Here we concern two criteria:

1. the linearized equations of the regularized system are same as those of the quantum Grad’s 13-moment system;
2. contains the correct NSF law.

Since the equilibrium plays an essential role in Grad’s expansion, it is expected that at the equilibrium, the regularized system shares the same wave structures, i.e. the coefficients matrix has the eigenvalues \((23)\) and possesses same eigenvectors as those of Grad’s 13-moment system, which is equivalent to the first criterion. The second criterion is natural because Grad’s moment system contains the NSF law.

In [4, 7], the authors proposed a framework on moment model reduction from kinetic equations to globally hyperbolic hydrodynamic system, by restudied their previous work [1, 2]. A natural way is to directly apply the framework on the quantum Grad’s 13-moment system to obtain a globally hyperbolic system. However, the resulting system fails to satisfy the two criteria. Given the important role of the equilibrium in Grad’s expansion, we split the expansion into the equilibrium part and the non-equilibrium part, and only apply the framework on the non-equilibrium. The resulting system is globally hyperbolic and also consistent with the two criteria. Further study shows the essential of the regularization is decoupling the derivative operator and the multiplying velocity operator.

4.1 Revisit of Grad’s 13-moment system

Let us examine the quantum Grad’s 13-moment system in further details at first. Given \(z, u\) and \(RT\), we have a weight function denoted as

\[
\omega_{[z,u,RT]}(v) = \frac{1}{z-1} \exp\left(\frac{|v-u|^2}{2RT}\right) + \theta. \tag{26}
\]

Then we denote \(H\) be the weighted \(L^2\) function space with the inner product

\[
\langle g, h \rangle_{[z,u,RT]} = \frac{m}{h^3} \int_{\mathbb{R}^3} gh \omega_{[z,u,RT]} dv. \tag{27}
\]

Let us consider the distribution functions satisfying \((1)\) in the Hilbert space \(H\). Define a set of basis functions

\[
\phi^{(0)} = \frac{1}{\rho} \omega_{[z,u,RT]}(v), \quad \phi^{(2)}_{ij} = \omega_{[z,u,RT]} \left( \frac{1}{2p} \frac{Li_2}{Li_3} \left( \frac{c_i c_j}{RT} - \delta_{ij} \frac{Li_3}{Li_2} \right) \right),
\]

\[
\phi^{(1)}_i = \omega_{[z,u,RT]} \left( \frac{c_i}{p} \right), \quad \phi^{(3)}_i = \omega_{[z,u,RT]} \left( \frac{3}{5pRT} \frac{c_i}{RT} - \frac{5}{4} \frac{Li_3}{Li_2} \right).
\]
where \( c_i = v_i - u_i, \rho = \frac{m}{h^3} \sqrt{2 \pi RT} \lambda^{3} \), \( p = \frac{m}{h^3} \sqrt{2 \pi RT} \lambda^{3} \), and \( \mathcal{B} = 7/2(\lambda^{3}/\lambda_{\tau}^{3}) - 5/2(\lambda_{\tau}^{3}/\lambda_{\tau}^{2}) \). Let the sub-space of \( \mathbb{H} \), spanned by \( \phi(0), \phi(1), \phi(2), \) and \( \phi(3) \), to be denoted as
\[
\mathbb{H}[^{\mathbb{H}}] \text{span} \left\{ \phi(0), \phi(1), \phi(2), \phi(3) \right\}.
\]
(28)
Since the basis functions are product of the weight function and polynomials, we have
\[
\mathbb{H}[^{\mathbb{H}}] = \text{span} \left\{ \phi(0), \phi(1), \phi(2), \phi(3) \right\}.
\]
(29)
Let \( \mathcal{P} \) to be the natural projection from \( \mathbb{H} \) to \( \mathbb{H}[^{\mathbb{H}}] \) defined as
\[
\mathbb{H} \rightarrow \mathbb{H}[^{\mathbb{H}}]
\]
\[
\mathcal{P} : \ g \rightarrow \mathcal{G} = a^{(0)} + a^{(1)} + a^{(2)} + a^{(3)}
\]
(30)
where the coefficients \( a^{(0)}, a^{(1)}, a^{(2)} \) and \( a^{(3)} \) are defined as
\[
a^{(0)} = p(g\phi(0))^\mathbb{H}, \quad a^{(1)} = p(g\phi(1))^\mathbb{H},
\]
\[
a^{(2)} = \frac{2p\lambda^{2}}{\lambda_{\tau}^{2}} (g\phi(2))^\mathbb{H} - b\delta_{ij}, \quad a^{(3)} = 5p(\mathcal{B})^\mathbb{H} (g\phi(3))^\mathbb{H},
\]
(31)
where \( b = (1-b)^{\frac{2m}{h^3}} \int_{\mathbb{R}^3} g \left( \frac{1}{3} |c|^2 - \lambda_{\tau}^{2} \right) \) \( dv \), and \( b = \frac{2}{5-3\lambda_{\tau}^{2}/(\lambda_{\tau}^{2})} \). Obviously, the projection \( \mathcal{P} \) is an orthogonal projection on \( \mathbb{H} \) and is identical on \( \mathbb{H}[^{\mathbb{H}}] \), i.e., \( \mathcal{P} \mathcal{G} = \mathcal{G} \) for \( \forall \mathcal{G} \in \mathbb{H}[^{\mathbb{H}}] \). We recall that \( \mathcal{P}, \mathbb{H}, \mathcal{B}, \) and \( \mathbb{R} \) are given parameters, and the projection \( \mathcal{P} \) is dependent on these parameters. Particularly, if \( \mathcal{P}, \mathbb{H}, \mathcal{B}, \) and \( \mathbb{R} \) are defined as
\[
\int_{\mathbb{R}^3} g \ dv = \sqrt{2 \pi RT \lambda_{\tau}^{3}}, \quad \int_{\mathbb{R}^3} g v_i \ dv = \sqrt{2 \pi RT \lambda_{\tau}^{3}} v_i, \quad \int_{\mathbb{R}^3} g |c|^2 \ dv = 3\sqrt{2 \pi RT \lambda_{\tau}^{3}} RT \lambda_{\tau}^{2},
\]
and denoted by \( \mathcal{P}, \mathbb{H}, \mathcal{B}, \mathbb{R} \) and \( \mathbb{R} = \mathbb{R}(g) \), then we have
\[
a^{(0)} = \rho, \quad a^{(1)} = 0, \quad a^{(2)} = \sigma_{ij}, \quad a^{(3)} = \Delta_{ij}.
\]
(32)
Then for a distribution \( f \in \mathbb{H} \), if \( \mathcal{P} = \mathcal{P}(f), \mathbb{H} = \mathbb{H}(f) \) and \( \mathcal{R} = \mathcal{R}(f) \), \( \mathcal{P} \mathcal{F} \) is Grad’s expansion \( \mathcal{P} \mathcal{F} \), i.e, \( f_{G_{13}} = \mathcal{P} \mathcal{F} \). Therefore, we need only to study the distribution in the space \( \mathbb{H}[^{\mathbb{H}}] \) in the following.

Actually, Grad’s moment system is derived by
\[
\frac{m}{h^3} \int_{\mathbb{R}^3} \psi \frac{\partial f_{G_{13}}}{\partial t} \ dv + \frac{m}{h^3} \int_{\mathbb{R}^3} \psi v_i \frac{\partial f_{G_{13}}}{\partial x_i} \ dv = \frac{m}{h^3} \int_{\mathbb{R}^3} \psi Q(f_{G_{13}}, f_{G_{13}}) \ dv,
\]
(33)
where \( \psi = 1, v_i, v_i v_j, v_i |v|^2 \). Noticing \( f_{G_{13}} = \mathcal{P} f \), Eq. (33) is equivalent to
\[
\left( \phi, \frac{\partial f}{\partial t} \right)^{[^{\mathbb{H}}, \mathcal{B}]_{\mathbb{R}}} + \left( \phi, v_i \frac{\partial f}{\partial x_i} \right)^{[^{\mathbb{H}}, \mathcal{B}]_{\mathbb{R}}} = \left( \phi, Q(Pf, Pf) \right)^{[^{\mathbb{H}}, \mathcal{B}]_{\mathbb{R}}}, \quad \phi \in \mathbb{H}[^{\mathbb{H}}].
\]
(34)
Since \( \mathcal{P} \) is orthogonal, Grad’s 13-moment system is essentially equivalent to
\[
\mathcal{P} \frac{\partial f}{\partial t} + \mathcal{P} v_i \frac{\partial f}{\partial x_i} = \mathcal{P} Q(Pf, Pf).
\]
(35)
For the time derivative part, direct calculation yields
\[
P \frac{\partial P f}{\partial t} = \frac{\partial p}{\partial t} \phi_i^{(0)} + \rho \frac{\partial u_i}{\partial t} \phi_i^{(1)} + \left( \frac{\partial \sigma_{ij}}{\partial t} + \delta_{ij} b \left( \frac{\partial p}{\partial t} - \frac{\partial \rho}{\partial t} \right) \right) \phi_i^{(2)} + \left( \frac{\partial q_i}{\partial t} + \sigma_{ij} \frac{\partial u_j}{\partial t} + 5 \left( p - \rho RT \frac{L_i^2}{L_j^2} \right) \frac{\partial u_i}{\partial t} \right) \phi_i^{(3)}. \tag{36}
\]

We remark that the upper equation is also valid for the derivatives of \( x_d \). Next, we calculate the convection term:
\[
P v_d \frac{\partial P f}{\partial x_d} = P(u_d + c_d) \frac{\partial P f}{\partial x_d} = u_d P \frac{\partial P f}{\partial x_d} + P c_d \frac{\partial P f}{\partial x_d}. \tag{37}
\]

Direct calculation and simplification yield
\[
P v_d \frac{\partial P f}{\partial x_d} = u_d \frac{\partial P f}{\partial x_d} + \rho \frac{\partial u_d}{\partial x_d} \phi_i^{(0)} + \frac{\partial p_d}{\partial x_d} \phi_i^{(1)} + \left( \sigma_{ij} \frac{\partial u_i}{\partial x_d} + 2 p_d \frac{\partial q_i}{\partial x_d} \frac{\partial \sigma_{ij}}{\partial x_d} + \frac{4 \partial q_i}{\partial x_d} \right) \phi_i^{(2)} + \left( \frac{7}{5} \frac{\partial u_d}{\partial x_d} + \frac{7}{5} q_d \frac{\partial \sigma_{ij}}{\partial x_d} + \frac{2}{5} q_k \frac{\partial u_k}{\partial x_d} + \frac{1}{2} \frac{\partial \Delta_{id}}{\partial x_d} - \frac{5}{2} \frac{RT}{L_i^2} \frac{L_i^2}{L_j^2} \frac{\partial p_d}{\partial x_d} \right) \phi_i^{(3)}. \tag{38}
\]

Because of (3) and the definitions of \( Q_{ij}^{(2)} \) and \( Q_i^{(3)} \), the collision part can be written as
\[
P Q(f, f) = Q_{ij}^{(2)} \phi_{ij}^{(2)} + Q_i^{(3)} \phi_i^{(3)} \tag{39}
\]

and \( Q_i^{(2)} = 0 \).

Collecting (36), (38), (39), and matching the coefficients of \( \phi_i^{(0)}, \phi_i^{(1)}, \phi_{ij}^{(2)} \) and \( \phi_i^{(3)} \), we obtain the moment system

\begin{align*}
\frac{d\rho}{dt} + \rho \frac{\partial u_d}{\partial x_d} &= 0, \tag{40a} \\
\rho \frac{\partial u_i}{\partial t} + \frac{\partial p_d}{\partial x_d} &= 0, \quad i = 1, 2, 3, \tag{40b} \\
\frac{d\sigma_{ij}}{dt} + \sigma_{ij} \frac{\partial u_d}{\partial x_d} + 2 p_d \frac{\partial q_i}{\partial x_d} + 4 \frac{\partial q_i}{\partial x_d} &= 0, \tag{40c} \\
\frac{dq_i}{dt} + \frac{5}{2} \left( p - RT \frac{L_i^2}{L_j^2} \right) \frac{d}{dt} + \sigma_{ij} \frac{d}{dt} + 7 \frac{5}{5} q_i \frac{d}{dx_d} + 7 \frac{5}{5} \frac{d}{dx_d} + \frac{2}{5} q_k \frac{d}{dx_i} \tag{40d} \\
\frac{1}{2} \frac{\partial \Delta_{id}}{\partial x_d} - \frac{5}{2} \frac{RT}{L_i^2} \frac{L_i^2}{L_j^2} \frac{\partial p_d}{\partial x_d} &= Q_i^{(3)},
\end{align*}

which is exactly the same as the quantum Grad’s 13-moment system in Sect. 2.

The projection \( \mathcal{P} \) is an alternative representation of Grad’s expansion (11), and Eq. (35) is an equivalent form of (33). The new formula helps us understand the relationship between the QBE and Grad’s moment system, and is also useful in the following part of this section.
4.2 A trivial regularized 13-moment system

In Ref. [7], the authors pointed out that the underlying reason due to the loss of hyperbolicity of Grad’s moment system is that Grad’s moment method treats the time derivative and space derivatives in different ways. Precisely, we may see this point by comparison of the QBE (1) and (35) in details. In (35), the corresponding time derivative operator \( \frac{\partial}{\partial t} \) in (1), and \( P \partial_x \) is derived from the convection operator \( v \frac{\partial}{\partial x} \) in (1).

Notice that \( v \frac{\partial}{\partial x} \) can be split into the multiplication by velocity and the spatial derivative operator, while the term \( P \partial_x \) has no similar splitting anymore. Hence, in (35), the time and space derivatives are treated in different ways. The authors of [7] suggest split the convection operator \( P v \frac{\partial}{\partial x} \) as \( P v \partial_x P \frac{\partial}{\partial x} \), which indicates the equivalent equation of the moment system turns to

\[
P \frac{\partial P f}{\partial t} + P v_\alpha \frac{\partial P f}{\partial x_\alpha} = P Q(f, f).
\] (41)

The corresponding moment system is guaranteed to be globally hyperbolic by Thm. 1 in [7] or Thm. 4.1 in [4].

Since (36) is also valid for space derivatives, direct calculation yields

\[
P v_\alpha \frac{\partial P f}{\partial x_\alpha} = u_d \frac{\partial P f}{\partial x_\alpha} + \rho \frac{\partial u_d}{\partial x_\alpha} \phi + \frac{\partial \rho_d}{\partial x_\alpha} \phi(1) + \frac{\partial \rho_d}{\partial x_\alpha} \phi(2) + \frac{\partial \rho_d}{\partial x_\alpha} \phi(3) + \frac{\partial \rho_d}{\partial x_\alpha} \phi(3).
\] (42)

Collecting (36), (12) and (39), and matching the coefficients of \( \phi(0), \phi(1), \phi(2) \) and \( \phi(3) \), we obtain the regularized 13-moment system:

\[
\begin{align*}
\frac{d\rho}{dt} + \rho \frac{d \mu_d}{dx_d} &= 0, \quad \text{(43a)}
\end{align*}
\]

\[
\begin{align*}
\rho \frac{d \mu_i}{dt} + \rho \frac{d \mu_d}{dx_d} &= 0, \quad i = 1, 2, 3, \quad \text{(43b)}
\end{align*}
\]

\[
\begin{align*}
\frac{d \sigma_{ij}}{dt} + 2 \rho \frac{d \mu_j}{dx_j} + \frac{4}{5} \frac{d q_{ij}}{dx_j} + \frac{4}{5} \frac{d q_{id}}{dx_d} + \frac{d \sigma_{ij}}{dx_j} + \delta_{ij} b \left( \frac{d \rho}{dt} - \frac{d \rho_d}{dx_d} \right) + \frac{2}{3} \left( p k_d \frac{d u_k}{dx_d} + \frac{d q_d}{dx_d} \right) = Q^{(2)}_{ij},
\end{align*}
\] (43c)

\[
\begin{align*}
\frac{d q_{ij}}{dt} + \frac{5}{2} \left( p - \rho RT \frac{L_i}{L_i^2} \frac{d u_j}{dt} + \sigma_{ij} \frac{d u_j}{dt} + RT \left( \frac{7 L_i^2}{2 L_i^2} - \frac{3 L_i}{2 L_i^2} \right) \frac{d \sigma_{id}}{dx_d} \right) = Q^{(3)}_{ij},
\end{align*}
\] (43d)
Comparing the quantum Grad’s 13-moment system (40) and the regularized 13-moment system (43), one can observe that only the governing equations of $\sigma_{ij}$ and $q_i$ are different, and the two moment systems share the same governing equations of $\rho$, $u_i$, and $p$.

Next we check the regularized 13-moment system by the two criteria proposed at the beginning of this section.

### 4.2.1 Linearized equations at the equilibrium

The primary idea of Grad’s moment method is that assuming the distribution function is not far from the equilibrium, and approximating the distribution by polynomials with the equilibrium as the weight function. Hence, it is expected that at the equilibrium, the regularized moment system is a high-order approximation of Grad’s moment system. So the linearized equations of the regularized moment system and Grad’s moment system must be same. For a given equilibrium $\mathbf{w}^0 = \mathbf{w}_{eq}$, assume $\mathbf{w} = \mathbf{w}^0 + \epsilon \mathbf{\hat{w}}$, where $\epsilon$ is a small quantity. The linearized equations of the quasi-linear equations

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x} = \mathbf{Q}$$

is

$$\frac{\partial \mathbf{\hat{w}}}{\partial t} + \mathbf{A}(\mathbf{w}^0) \frac{\partial \mathbf{\hat{w}}}{\partial x} = \frac{1}{\epsilon} \mathbf{Q}_i^{(3)}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + u_0^d \frac{\partial}{\partial x_d}$, $\text{Li}_s = -\theta \text{Li}_s(-\theta^0)$, and $\mathbf{Q}_i^{(3)}$ denote the first order parts and $Q_i^{(3)}$. The linearized equations of (43d) is

$$\frac{d\hat{q}_i}{dt} + \frac{5}{2} p_0^0 \left( 1 - \frac{\text{Li}_3^2 \text{Li}_3^2}{\text{Li}_2^2} \right) \frac{d\hat{u}_i}{dt} + \left( \frac{7\text{Li}_2^2}{2\text{Li}_2^2} - \frac{5\text{Li}_2^2}{2\text{Li}_2^2} \right) \frac{R T^0_0}{\text{Li}_2^2} \frac{\partial \hat{\sigma}_{id}}{\partial x_d}$$

$$+ \frac{5}{2} p_0^0 \left( RT^0_0 \left( 1 - \frac{\text{Li}_3^2 \text{Li}_3^2}{\text{Li}_2^2} \right) \frac{\partial \hat{\rho}}{\partial x_i} + \frac{\text{Li}_2^2}{\text{Li}_2^2} \frac{\partial R T^0_0}{\partial x_i} \right) = \frac{1}{\epsilon} \mathbf{Q}_i^{(3)}$$

Easy to find the linearized equations (40d) and (43d) are different. (It is remarked that the linearized equations of $\rho$, $u_i$, and $\sigma_{ij}$ of the two moment system are same.) Hence, the linearized equations of Grad’s 13-moment system and the regularized-13 moment system are different, which indicates the regularized 13-moment system (43) is not a good approximation of the QBE.

### 4.2.2 NSF law

In Chapmann-Enskog expansion, the equilibrium corresponds to the 0-th order expansion. For the first-order Chapmann-Enskog expansion, the major term of the regularized 13-moment system (43) is expected to be same as that of the quantum Grad’s 13-moment system, which indicates the regularization only modifies some high-order terms of the quantum Grad’s 13-moment system in the sense of Chapmann-Enskog expansion.
For the QBGK collision, the first step of Maxwellian iteration \cite{9} of the regularized 13-moment system (43) yields the Navier-Stokes-Fourier law as

\[
\sigma_{ij}^{(1)} = -2\tau_p \partial u_{(i} \frac{\partial}{\partial x_{j})},
\]

\[
q_i^{(1)} = -\frac{5}{2}\rho \left( 1 - \frac{Li_2^2}{Li_2^2} \right) \left( -\frac{1}{\rho} \right) \frac{\partial p}{\partial x_i} - \frac{5}{2}RT \left( \frac{7Li_7^2 - 3Li_5^2}{5 - 3Li_7^2 Li_5^2} \right) \left( \frac{\partial p}{\partial x_i} - \frac{p}{\rho} \frac{\partial \rho}{\partial x_i} \right),
\]

\[
\neq -\frac{5}{2}\tau_p \left( \frac{7Li_7^2}{2Li_5^2} - \frac{5Li_5^2}{2Li_3^2} \right) \frac{\partial RT}{\partial x_i}.
\]

Therefore, the regularized 13-moment system can not give the correct Fourier law. In this sense, the regularized 13-moment system is also not a proper approximation of the QBE.

4.3 Regularized 13-moment system

In Sect. 3 we have pointed out that the quantum Grad’s 13 moment system is not hyperbolic even around the equilibrium. And in the upper subsection, we proposed a regularized 13-moment system based on the framework in \cite{4,7}. However, the moment system does not satisfy the two criteria, i.e. it fails to share the same linearized equation of Grad’s 13-moment system and can not give the correct NSF law. In this subsection, we focus on a new regularized moment system, which not only is hyperbolic but also satisfies the two criteria.

The quantum Grad’s 13-moment system is essentially equivalent to (35), i.e \( \mathcal{P} \frac{\partial \mathcal{P} f}{\partial t} + \mathcal{P}_v \frac{\partial \mathcal{P} f}{\partial x_d} = \mathcal{P}Q(\mathcal{P} f, \mathcal{P} f) \), and it is not globally hyperbolic but satisfies the two criteria.

The regularized 13-moment system (43) is essentially equivalent to (41), i.e \( \mathcal{P} \frac{\partial \mathcal{P} f}{\partial t} + \mathcal{P}_v \frac{\partial \mathcal{P} f}{\partial x_d} = \mathcal{P}Q(\mathcal{P} f, \mathcal{P} f) \), and it is globally hyperbolic but does not satisfy the two criteria. As is pointed out in Sect. 4.2 the equilibrium plays an essential role in Grad’s moment method, and is also essential in the two criteria. So we split the expansion (11) into two parts: the equilibrium part and the non-equilibrium part

\[
\mathcal{P} f = f_{eq} + (\mathcal{P} f - f_{eq}),
\]

and use the treatment in (35) to deal with the equilibrium part, and the treatment in (41) to deal with the non-equilibrium part:

\[
\mathcal{P} \frac{\partial f_{eq}}{\partial t} + \mathcal{P}_v \frac{\partial f_{eq}}{\partial x_d} + \mathcal{P} \frac{\partial \mathcal{P} f - f_{eq}}{\partial t} + \mathcal{P}_v \mathcal{P} \frac{\partial \mathcal{P} f - f_{eq}}{\partial x_d} = \mathcal{P}Q(\mathcal{P} f, \mathcal{P} f).
\]  

The upper equation is equivalent to

\[
\mathcal{P} \frac{\partial \mathcal{P} f}{\partial t} + \mathcal{P}_v \frac{\partial f_{eq}}{\partial x_d} + \mathcal{P}_v \mathcal{P} \frac{\partial \mathcal{P} f - f_{eq}}{\partial x_d} = \mathcal{P}Q(\mathcal{P} f, \mathcal{P} f).
\]
Direct calculation yields

\[
P \xi_d \frac{\partial f_{eq}}{\partial x_d} = u_d P \frac{\partial f_{eq}}{\partial x_d} + \rho \frac{\partial u_d}{\partial x_d} \phi_i^{(0)} + \frac{\partial p}{\partial x_d} \phi_i^{(1)}
\]

\[
+ \left( 2p \frac{\partial u_d}{\partial x_d} + \frac{2 \delta_{ij} b}{3} \rho \frac{\partial u_d}{\partial x_d} \right) \phi_i^{(2)} + \frac{5p}{2} \frac{\partial RT}{\partial x_d} \phi_i^{(3)}
\]

(48)

and

\[
P \xi_d \frac{\partial P f - f_{eq}}{\partial x_d} = u_d P \frac{\partial P f - f_{eq}}{\partial x_d} + \frac{\partial \sigma_{id}}{\partial x_d} \phi_i^{(1)}
\]

\[
+ \left( \frac{4}{5} \frac{\partial q_i}{\partial x_d} + \frac{4}{5} \sigma_{k(i} \frac{\partial u_{k)}}{\partial x_d} \right) \phi_i^{(2)}
\]

\[
+ RT \left( \frac{7 \text{Li}_2}{2 \text{Li}_2} - \frac{5 \text{Li}_2}{2 \text{Li}_2} \right) \frac{\partial \sigma_{id}}{\partial x_d} \phi_i^{(3)}
\]

(49)

Collecting (36), (48), (49) and (39), and matching the coefficients of \( \phi^{(0)}_i, \phi^{(1)}_i, \phi^{(2)}_{ij} \) and \( \phi^{(3)}_i \), we obtain the new regularized 13-moment system:

\[
\frac{dp}{dt} + \rho \frac{\partial u_d}{\partial x_d} = 0,
\]

(50a)

\[
\rho \frac{du_i}{dt} + \frac{\partial p_{id}}{\partial x_d} = 0, \quad i = 1, 2, 3,
\]

(50b)

\[
\frac{d \sigma_{ij}}{dt} + 2 \rho \frac{\partial u_d}{\partial x_d} + \frac{2}{5} \frac{\partial q_i}{\partial x_d} + \frac{4}{5} \sigma_{d(i} \frac{\partial u_{k)}}{\partial x_d}
\]

\[
+ \frac{\delta_{ij} b}{3} \left( \frac{\partial p}{\partial x_d} - \rho \frac{dp}{dt} + \frac{2}{3} \left( \sigma_{d(k} \frac{\partial u_{k)}}{\partial x_d} + \frac{\partial q_d}{\partial x_d} \right) \right) = Q^{(2)}_{ij},
\]

(50c)

\[
\frac{d q_i}{dt} + \frac{5}{2} \left( p - \rho RT \text{Li}_2 \right) \frac{du_i}{dt} + \sigma_{ij} \frac{du_j}{dt} + \frac{5}{2} \frac{\partial RT}{\partial x_i} \phi_i^{(3)}
\]

\[
+ RT \left( \frac{7 \text{Li}_2}{2 \text{Li}_2} - \frac{5 \text{Li}_2}{2 \text{Li}_2} \right) \frac{\partial \sigma_{id}}{\partial x_d} = Q^{(3)}_{i}.
\]

(50d)

Similar as that for (43), comparing the quantum Grad’s 13-moment system (40) and the regularized 13-moment system (50), one can observe that only the governing equations of \( \sigma_{ij} \) and \( q_i \) are different, and the two moment systems share the same governing equations of \( \rho, u_i \) and \( p \).

Next, we check the hyperbolicity and the two criteria of the moment system (50).

4.3.1 Linearized equations at the equilibrium

Following the method in Sect. 4.2.1, we linearize the quantum Grad’s 13-moment system (40) and the regularized 13-moment system (50). Direct calculation yields that both the linearized equation of (40) and (50) are

\[
\frac{d \rho}{dt} + \rho \frac{\partial u_d}{\partial x_d} = 0,
\]

(51a)
\[ \rho^0 \frac{d\hat{u}_i}{dt} + \frac{\partial \hat{p}_d}{\partial x_d} = 0, \quad i = 1, 2, 3, \quad (51b) \]

\[ \frac{d\hat{\sigma}_{ij}}{dt} + 2p^0 \frac{\partial \hat{v}_{(i)}}{\partial x_j} + \frac{4}{5} \frac{\partial \hat{q}_{(i)}}{\partial x_j} = 0, \quad i = 1, 2, 3, \quad (51c) \]

\[ \frac{d\hat{q}_i}{dt} + \frac{5}{2} p^0 \left( 1 - \frac{\text{Li}_3}{\text{Li}_7^2} \right) \frac{d\hat{u}_i}{dt} + \frac{5}{2} \frac{\partial \hat{R}T^0 \hat{q}_d}{\partial x_d} = 0, \quad i = 1, 2, 3, \quad (51d) \]

where \( \frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \), \( \text{Li}_s = -\theta \text{Li}_s(-\theta^3) \), and \( \hat{Q}^{(2)}_{ij} \) and \( \hat{Q}^{(3)}_i \) denote the first order parts of \( Q_{ij}^{(2)} \) and \( Q_i^{(3)} \), respectively. Hence, the regularized 13-moment system (50) satisfy the first criterion.

### 4.3.2 NSF law

Similar as that in Sect. 4.2.2, we derive the NSF law from the regularized 13-moment system (50). For the QBGK collision, the first step of Maxwellian iteration [9] of the regularized 13-moment system (50) yields the Navier-Stokes-Fourier law as

\[ \sigma_{ij}^{(1)} = -2\tau p \frac{\partial u_{(i)}}{\partial x_j}, \]

\[ q_i^{(1)} = -\frac{5}{2} \left( p - \rho \text{RT} \frac{\text{Li}_3}{\text{Li}_7^2} \right) \left( -\frac{1}{p} \right) \frac{\partial \rho}{\partial x_i} - \frac{5}{2} \text{RT} \frac{\text{Li}_3}{\text{Li}_7^2} \frac{\partial \text{RT}}{\partial x_i} \]

This is the correct NSF law, same as that of the quantum Grad’s 13-moment system [13].

### 4.3.3 Hyperbolicity

In this subsection, we study the hyperbolicity of the regularized 13-moment system (50). Since \( \sigma_{ij} = p_{ij} - \frac{\delta_{ij}}{3} p_{kk} \) and \( p = \frac{1}{3} p_{kk} \), denote

\[ w = (\rho, u_1, u_2, u_3, p_{11}, p_{12}, p_{13}, p_{22}, p_{23}, p_{33}, q_1, q_2, q_3)^T, \]

then the regularized 13-moment system (50) can be written as

\[ D \frac{dw}{dt} + M_d \frac{\partial w}{\partial x_d} = Q, \quad (53) \]
where $Q$ is the same as that in (21), and

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0

-\frac{\rho}{b} & 0 & 0 & 0 & \frac{b+1}{3} & 0 & \frac{b-1}{3} & 0 & \frac{b-1}{3} & 0

0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0

-\frac{\rho}{b} & 0 & 0 & 0 & \frac{b-1}{3} & 0 & \frac{b+2}{3} & 0 & \frac{b-1}{3} & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0

-\frac{\rho}{b} & 0 & 0 & 0 & \frac{b-1}{3} & 0 & \frac{b-2}{3} & 0 & \frac{b-2}{3} & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},

(54)
$$

where

$$
\mathcal{D} = \frac{5}{2} \left(1 - \frac{\text{Li}_3^2}{\text{Li}_3^2}\right).
$$

Matrices $M_d$ can be obtained from (50), and particularly, $M_1$ is

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

-\frac{\rho}{b} & 0 & 0 & 0 & 1 + m_2 & 0 & 0 & m_2 & 0 & m_2

0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0

0 & 2m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & m_3 \cdot \frac{2}{3} m_1 & 0 & 0 & m_3 - \frac{1}{3} m_1 & 0 & m_3 - \frac{1}{3} m_1

0 & 0 & 0 & 0 & 0 & m_1 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & m_1 & 0 & 0 & 0
\end{pmatrix},

(55)
$$

$$
m_1 = \frac{\text{Li}_3^2}{\text{Li}_3^2} RT,
m_2 = \frac{1}{2} \left(1 - \frac{\text{Li}_3^2}{\text{Li}_3^2}\right),
m_3 = \frac{5}{6b} \frac{\text{Li}_3^2}{\text{Li}_3^2} - \frac{3}{\text{Li}_3^2} \left(2 \frac{\text{Li}_3^2}{\text{Li}_3^2} - 3 \left(1 - \frac{\text{Li}_3^2}{\text{Li}_3^2}\right)\right).
$$

If we rewrite the system (50) into the quasi-linear form as

$$
\frac{\partial w}{\partial t} + A_d^R \frac{\partial w}{\partial x_d} = Q,
$$

(56)

then $A_d^R = D^{-1} (M_d + u_d I) D$.

**Theorem 1.** The regularized 13-moment system (56) is globally hyperbolic.

We use the technique in [3] to prove the theorem. Firstly, we present a lemma about matrix and its eigenvalues without proof.
Lemma 1. For a square matrix $A \in \mathbb{R}^{n \times n}$, denote $\lambda_i, i = 1, \cdots, k, (k \leq n)$ by the all distinct eigenvalues of $A$, and $p(A) = \prod_{i=1}^{k} (A - \lambda_i I)$, then $A$ is real diagonalizable if and only if $\lambda_i$ are all real and $p(A) = 0$.

Proof of Thm. [7]. Since both the QBE and the Hilbert space $\mathbb{H}[\mathbb{u}, \mathbb{RT}]$ are Galilean invariant, the system (56) is Galilean invariant. We just need to prove $A_1^{R}$ is real diagonalizable, which is equivalent to $M_1$ is real diagonalizable. Direct calculation yields the characteristic polynomial of $M_1$ is

$$|\lambda I - M_1| = \lambda^5 \left( \lambda^2 - \frac{7Li_2}{5Li_1^2} \mathbb{RT} \right)^2 \left( \lambda^4 - c_1 \mathbb{RT} \lambda^2 + c_0 (\mathbb{RT})^2 \right), \quad (57)$$

where $c_0$ and $c_1$ are same as that in (22). Easy to check all the eigenvalues of $M_1$ are real and all zeros of $\lambda^4 - c_1 \mathbb{RT} \lambda^2 + c_0 (\mathbb{RT})^2$ are nonzero and distinct (all the eigenvalues of $M_1$ will be studied numerically in the later of this subsection.)

- Case 1: $\frac{7Li_2}{5Li_1^2}$ is not a zero of $x^2 - c_1 x + c_0$. Let

$$p(M_1) = M_1 \left( M_1^2 - \frac{7Li_2}{5Li_1^2} \mathbb{RT} \right) \left( M_1^4 - c_1 \mathbb{RT} M_1^2 + c_0 (\mathbb{RT})^2 \right). \quad (58)$$

With the help of Maple, we can directly check $p(M_1) = 0$. Using Lem. 1, we prove that $M_1$ is diagonalizable.

- Case 2: $\frac{7Li_2}{5Li_1^2}$ is a zero of $x^2 - c_1 x + c_0$. We note that this case only occur for Fermion with $\gamma \approx 11.69$. Let

$$p(M_1) = M_1 \left( M_1^4 - c_1 \mathbb{RT} M_1^2 + c_0 (\mathbb{RT})^2 \right). \quad (59)$$

With the help of Maple, we can also check $p(M_1) = 0$. Using Lem. 1, we prove that $M_1$ is diagonalizable.

This completes the proof. \hfill \Box

In the proof, we get the characteristic polynomials of the matrix $M_1$ in (57), and all the eigenvalues of the regularized 13-moment system are given in (23). In Figs. 4 and 5, we present the positive eigenvalues of $M_1$ as the fugacity varies. For Fermion, due to Pauli exclusion principle, the high-speed particles increase faster than the low-speed particles as the fugacity increasing, thus the propagation speed increase. Even though, there exists an upper bound for the propagation speed for any $\gamma$. It is worth to remind that for $\gamma \approx 11.69$, there is a point of intersection in the right figure of Fig. 5. Analogously, for Boson, more particles are staying on the ground states that the propagation speeds decrease as the fugacity decreasing.

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1Maple is a trademark of Waterloo Maple Inc.
4.3.4 Discussion

We have verified the regularized 13-moment system (50) is not only hyperbolic but also satisfies the two criteria. Here we explore the connotation of the model.

For the QBE, \( \frac{\partial}{\partial t} \) is the time derivatives operator, and \( v_d \frac{\partial}{\partial x_d} \) is the convection operator. The convection operator can be split into the product of the multiplying velocity operator \( v_d \) and the space derivative operator \( \frac{\partial}{\partial x_d} \). For Grad’s moment method (35), \( \mathcal{P} \frac{\partial}{\partial t} \) and \( \mathcal{P} v_d \frac{\partial}{\partial x_d} \) are the corresponding time derivative operator and convection operator, respectively. But the convection operator cannot be split into the product of two operator anymore. If we rewrite the quantum Grad’s 13-moment system (40) as

\[
D \frac{\partial w}{\partial t} + B_d \frac{\partial w}{\partial x_d} = Q, \tag{60}
\]

where \( B_d = DA_d - u_d I \), then comparing it with (1) and (35), we can find the following corresponding relation:

\[
f \rightarrow \mathcal{P} f \leftrightarrow w, \quad \frac{\partial}{\partial t} \rightarrow \mathcal{P} \frac{\partial}{\partial t} \leftrightarrow D \frac{\partial}{\partial t}, \quad v_d \frac{\partial}{\partial x_d} \rightarrow \mathcal{P} v_d \frac{\partial}{\partial x_d} \leftrightarrow B_d \frac{\partial}{\partial x_d}. \tag{61}
\]
In other words, $D \frac{\partial}{\partial t}$ and $B_d \frac{\partial}{\partial x_d}$ are the corresponding time derivative operator and convection operator, respectively.

Analogously, for the regularized 13-moment system (50), comparing (53) with (1), we can find the following corresponding relation:

$$f \rightarrow w, \quad \frac{\partial}{\partial s} \rightarrow D \frac{\partial}{\partial s}, \quad s = t, x_d, \quad v_d \rightarrow M_d, \quad v_d \frac{\partial}{\partial x_d} \rightarrow M_d D \frac{\partial}{\partial x_d}. \quad (62)$$

In this case, the convection operator $M_d D \frac{\partial}{\partial x_d}$ can be split in the product of the multiplying velocity operator $M_d$ and the space derivative operator $D \frac{\partial}{\partial x_d}$, which is similar as that for the QBE. In this sense, (41) is a nature approximation of the QBE. And the key point of the regularization is to split the convection operator into the product of the multiplying velocity operator and the space derivative operator.

Since $M_d$ is a limit of the multiplying velocity operator $v_d$ on the Hilbert space $H^{[\mathbf{u},RT]}$, the matrix $M_d$ is expected to dependent only on $\mathbf{z}$, $u_d$ and $RT$, i.e. the equilibrium variables. Actually, it is true due to (55). That is to say, $M_d(w) = M_d(w_{eq})$, where $w_{eq} = (\rho, u_1, u_2, u_3, p, 0, 0, p, 0, 0, 0, 0)$ represent the equilibrium state.

In Sect. 4.3.1, we pointed out the linearized equations of the regularized 13-moment system (50) and Grad’s 13-moment system (40) are same. Hence, $A_d(w_{eq}) = A_d^R(w_{eq})$, i.e $B_d(w_{eq}) = M_d(w_{eq})D(w_{eq})$. Let $M_d^f = B_dD^{-1}$, then $M_d^f(w) \neq M_d^f(w_{eq})$, which indicates $M_d^f$ is not a limit of the multiplying velocity operator $v_d$ on the Hilbert space $H^{[\mathbf{u},RT]}$. Given $M_d(w) = M_d(w_{eq})$, we have $M_d(w) = M_d^f(w_{eq})$. It indicates the matrices $M_d$ can be also calculated by $M_d = B_d(w_{eq})D^{-1}(w_{eq})$.

Meanwhile, since $A_d(w_{eq}) = A_d^R(w_{eq})$, Thm. 1 indicates the system (21) is hyperbolic on the equilibrium.

On the other hand, the hyperbolicity yields the upper bound on the propagation speed of the regularized moment system while the QBE allows infinity propagation speed. Actually, deterministic methods for the QBE, for example, the discrete velocity method and the spectral method, also has the upper bound on the propagation speed. How to choose the upper bound is an important issue for the deterministic methods. If the upper bound is greater enough, the deterministic methods work well. For the regularized 13-moment system, the upper bound is fixed, so the application of the regularized system is limited. However, on the other hand, we can increase the upper bound by using more moments. In [2], the authors proposed an arbitrary order regularized moment method, the upper bound of the propagation speed of which can be as greater enough as the number of the moments increasing. The regularization proposed in this subsection can be also extended to derive arbitrary order regularized moment method. Hence, the upper bound of the propagation speed can be enlarged by using more moments if necessary.

5 Conclusion

We study the hyperbolicity of the quantum Grad’s 13-moment system. It is found that the model is not hyperbolic even around the thermodynamic equilibrium. Applying the moment model reduction framework in [7] in a trivial way to the quantum Boltzmann equation, we can not obtain a good model, which is hyperbolic and satisfies the two criteria. By further studying the framework in [7] and the quantum Grad’s 13-moment
system, we propose a regularization for the model and obtain a globally hyperbolic 13-moment system. We are expecting that the regularized moment system is helpful to understand the quantum effects and to develop the moment method in quantum kinetic theory.

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**Appendix**

A **Coefficients of (22) and (25)**

The coefficients in (22) and (25) can be directly obtained by calculating the determinant of the matrix \( A_1 \). Here we list the coefficients in (22) and (25) as following.

\[
\begin{align*}
 c_0 &= \frac{3 \left( 7Li_1^2 Li_2^2 - 5Li_2^4 \right)}{5Li_1^2 Li_2^2 - 3Li_3^2}, \\
 c_1 &= \frac{140Li_1^4 Li_3^4 + 175Li_1^4 Li_5^2 - 84Li_1^2 Li_2^2 Li_5^2 - 75Li_1^2 Li_2^4 Li_5^2}{15Li_1^2 \left( 5Li_1^2 Li_2^2 - 3Li_3^2 \right)}, \\
 c_2 &= \frac{1}{Li_1^2 Li_2^2 Li_3^2 \left( 5Li_1^2 Li_2^2 - 3Li_3^2 \right)} \left[ -735Li_1^3 Li_2^3 Li_3^2 + 560 \epsilon^2 Li_1^2 Li_2^2 Li_3^2 \left( Li_1^2 Li_2^2 - Li_3^2 \right) \\
&\quad + 294Li_1^2 Li_2^2 Li_3^2 Li_5^2 \left( \epsilon^2 Li_1^2 Li_3^2 - \left( \frac{17\epsilon^2}{7} - \frac{25}{14} \right) Li_2^2 \right) \\
&\quad + \epsilon^2 Li_1^2 Li_2^2 Li_3^2 \left( 196Li_1^2 Li_2^2 - 210Li_1^2 Li_2^2 Li_3^2 + 450Li_1^2 Li_2^2 \right) \right], \\
 c_3 &= \frac{1}{3Li_1^2 Li_2^2 Li_3^2 \left( 5Li_1^2 Li_2^2 - 3Li_3^2 \right)} \left[ -588Li_1^4 Li_2^2 + 720 \epsilon^2 Li_1^2 Li_2^2 Li_3^2 \\
&\quad + Li_2^3 \left( -525 Li_1^2 Li_2^2 Li_3^2 + 1575 Li_3^3 \right) \\
&\quad + Li_2^2 \left( \left( -432 \epsilon^2 - 1125 \right) Li_1^2 Li_2^2 + 1225 Li_1 Li_2^4 \right) Li_2^2 + 980 Li_1 Li_2^2 Li_3^2 \right], \\
 c_4 &= \frac{-1225 Li_1 Li_2^2 + 375 Li_1 Li_3 Li_2^2}{3Li_2^2 \left( 5Li_1 Li_2^2 - 3Li_3^2 \right)}. 
\end{align*}
\]
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