LOGARITHMIC SINGULARITIES OF SOLUTIONS TO NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

HIDETOSHI TAHARA AND HIDESHI YAMANE

Abstract. We construct a family of singular solutions to some nonlinear partial differential equations which have resonances in the sense of a paper due to T. Kobayashi. The leading term of a solution in our family contains a logarithm, possibly multiplied by a monomial. As an application, we study nonlinear wave equations with quadratic nonlinearities. The proof is by the reduction to a Fuchsian equation with singular coefficients.

Introduction

In this paper, we study singular solutions to nonlinear partial differential equations with holomorphic (or real-analytic) coefficients. The solutions to be constructed shall be singular along a noncharacteristic hypersurface. This phenomenon presents a striking contrast to linear theory. Probably, the most well-known example in this direction is the KdV equation:

\[ u_{ttt} - 6uu_x + u_x = 0 \quad (t, x \in \mathbb{C}). \]

The surface \( t = 0 \) is noncharacteristic but (0.1) has solutions of the form

\[ u = \frac{2}{t^2} + gt^2 + h t^4 - \frac{1}{24}g_x t^5 + \ldots, \]

where \( g = g(x) \) and \( h = h(x) \) are arbitrary functions. Note that many solutions have been obtained in the form of Laurent series for some integrable PDEs (a useful reference is [1]).

In [7], Kichenassamy and Srinivasan introduced an expansion of a generalized form in order to solve PDEs with polynomial nonlinearities. In their paper, the solutions behave asymptotically

\[ u(t, x) \sim u_0(x)t^\nu \quad (as \ t \to 0), \]

where \( \nu \) is a rational number. The remainder term may contain logarithms. Besides this general result, specific cases are dealt with in [5], [6] and [8]: in these papers, it is proved that the Liouville equation \( \Box u = e^u \) and Einstein’s vacuum equations admit solutions led by logarithmic terms.

2000 Mathematics Subject Classification. Primary 35A20 ; Secondary 35L70 .

Key words and phrases. singular solutions, Fuchsian equations, logarithmic singularities, nonlinear wave equations.

This research was partially supported by Grant-in-Aid for Scientific Research (No.16540169, No.17540182), Japan Society for the Promotion of Science. Parts of this work has been done during the authors’ stay at Wuhan University. They thank Professor Chen Hua for hospitality and fruitful discussions.
On the other hand, in \[9\], Kobayashi considered a certain kind of nonlinear PDEs, mainly those with polynomial nonlinearities, and constructed solutions of the form

\[ u(t, x) = t^{\sigma_c} \sum_{k=0}^{\infty} u_k(x) t^{k/p}, \]

where \( \sigma_c \in \mathbb{Q}, p \in \mathbb{N}^* = \{1, 2, \ldots\} \). The exponent \( \sigma_c \), which is called the characteristic exponent, is determined by the nonlinear term of the equation and Kobayashi imposed a kind of generalized nonresonance condition on it. In particular, he assumed \( \sigma_c \neq 0, 1, 2, \ldots, m - 2 \), where \( m \) is the order of the equation. In the present paper, the authors shall deal with these excluded cases and construct solutions with a logarithm in the leading term. If \( \sigma_c = l \in \{0, 1, 2, \ldots, m - 2\} \), then the asymptotic behavior of the solutions is

\[(0.2) \quad u(t, x) \sim a(x) t^l \log t \quad (\text{as } t \to 0)\]

and the remainder term involves an arbitrary holomorphic function in \( x \). Note that the case where \( \sigma_c = 0 \) has already been treated in \[14\] in a different formulation.

This result about nonlinear wave equations shall be improved in Part 1, \( \S \).

Note that Tahara extended Kobayashi's result for first-order equations with entire nonlinearities in \[11\] and \[12\].

All the above mentioned authors employ the method of Fuchsian Reduction: the leading terms can be found by formal calculation and the remainder terms are obtained by solving nonlinear Fuchsian equations. Here the word "Fuchsian" contains some ambiguity, because there are many versions of the notions of Fuchsian or related equations. One has to choose a suitable version on each occasion. In the present work, we employ still another version, i.e. equations with singular coefficients.

The organization of the present paper is as follows: In Part 1 we shall study an equation of the form

\[ \partial_t^m u = f \left(t, x, (\partial_t^j \partial_x^\alpha u) \right), \]

where \( f \) is holomorphic (real-analytic) in its arguments, and construct solutions with the asymptotic behavior \( u(t, x) \sim a(x) t^l \log t \), \( l \in \{0, 1, \ldots, m - 2\} \). We shall explain how this equation is reduced to a Fuchsian equation with singular coefficients:

\[ (t\partial_t)^m u = F \left(t, x, ((t\partial_t)^j \partial_x^\alpha u) \right). \]

Here \( F(t, x, Z) \) is singular at \( t = 0 \). The latter equation shall be solved in Part 2 by using the techniques developed in \[13\]. The characteristic exponents can be arbitrary.

Part 1. Logarithmic singularities

1. Main result

Let \((t, x) = (t, x_1, \ldots, x_n) \in \mathbb{C} \times \mathbb{C}^n \), let \( m \in \mathbb{N}^* \) be fixed and set: \( I_m = \{(j, \alpha) \in \mathbb{N} \times \mathbb{N}^n; j + |\alpha| \leq m \text{ and } j < m\} \), \( N = \) the cardinal of \( I_m \), and \( U = \{(j, \alpha)\}_{(j, \alpha) \in I_m} \in \mathbb{C}^N \). We set \( \partial_t = \partial/\partial t, \partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n} \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \).

We study nonlinear PDEs of the form

\[(1.1) \quad \partial_t^m u = f \left(t, x, (\partial_t^j \partial_x^\alpha u)_{(j, \alpha) \in I_m} \right). \]
Here $f(t, x, U)$ is holomorphic in $\{(t, x) \in \mathbb{C}_t \times \mathbb{C}_x^n; |t| < r_0, |x| < R_0\} \times \mathbb{C}_U^r$, where $r_0$ and $R_0$ are positive constants. Note that Kobayashi assumed that $f$ was a polynomial in $U$. We shall deal with several infinite sums closely related to $f$. Their convergence follows from the fact that $f(t, x, U)$ is entire in $U$ (see Proposition 2.2 for example).

We may write

$$f(t, x, U) = \sum_{\mu \in \mathcal{M}} f_\mu(t, x) U^\mu, \quad \mu = (\mu_{j,\alpha})_{(j,\alpha) \in I_m}, \quad U^\mu = \prod_{(j,\alpha) \in I_m} U^{\mu_{j,\alpha}}$$

for some subset $\mathcal{M}$ of $\mathbb{N}^N$, the set of $\mathbb{N}$-valued functions on $I_m$. We assume that $f_\mu(t, x)$ does not vanish identically if $\mu \in \mathcal{M}$ (then $\mathcal{M}$ is unique).

We expand $f_\mu(t, x)$ in $t$:

$$f_\mu(t, x) = t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu, k}(x) t^k.$$  

We assume that $f_{\mu, 0}(x)$ does not vanish identically. Summing up, we have

$$f(t, x, U) = \sum_{\mu \in \mathcal{M}} f_\mu(t, x) U^\mu = \sum_{\mu \in \mathcal{M}} \left( t^{k_\mu} \sum_{k=0}^{\infty} f_{\mu, k}(x) t^k \right) U^\mu.$$  

We set

$$|\mu| = \sum_{(j,\alpha) \in I_m} \mu_{j,\alpha}, \quad \gamma(\mu) = \sum_{(j,\alpha) \in I_m} j\mu_{j,\alpha}.$$  

Lemma 1.1. For any integer $l$, we have

$$t^{\gamma(\mu) - l|\mu|} U^\mu = t^{\gamma(\mu) - l|\mu|} \prod_{(j,\alpha) \in I_m} U^{\mu_{j,\alpha}} = \prod_{(j,\alpha) \in I_m} (t^{j-1} U^{j,\alpha})^{\mu_{j,\alpha}}.$$

We assume the following:

(A0) $\sup_{\mu \in \mathcal{M}, |\mu| \geq 2} \frac{\gamma(\mu) - m - k_\mu}{|\mu| - 1} = l \in \{0, 1, 2, \ldots, m - 2\}.$

Note that the left hand side is the characteristic exponent $\sigma_c$ in [9] (It is proved in [9] that $\sigma_c < m - 1$ holds if the supremum is attained by some $\mu.$)

(A1) $\mathcal{M}_0 = \{ \mu \in \mathcal{M}; |\mu| \geq 2, (\gamma(\mu) - m - k_\mu)/(|\mu| - 1) = l \}$ is non-empty: i.e. the supremum in (A0) is attained.

(A2) If $\mu \in \mathcal{M}_0$ and $\mu_{j,\alpha} \neq 0$, then $j \geq l + 1$ and $\alpha = 0$.

(A3) For a sufficiently small positive constant $C > 0$, we have

$$m - l + k_\mu - \gamma(\mu) + l|\mu| \geq C \sum_{(j,\alpha) \in I_m} \mu_{j,\alpha}$$

for any $\mu \in \mathcal{M} \setminus \mathcal{M}_0$. (This is trivial if $f$ is a polynomial by Lemma 2.2 below.)

Example 1.2. The prototype is the ODE

$$u^{(m)} = t^k \{u^{(m-1)}\}^2, \quad l = m - k - 2.$$

This equation is satisfied if $u^{(m-1)} = C_1 t^{k-1}$, where $C_1$ is a suitable constant. In this case, we have $u \sim C_2 t^l \log t$ as $t \to 0$, where $C_2$ is another constant.
We shall construct solutions to (1.1) which behave like (0.2). In order to give a precise statement, we introduce a function class \( \tilde{O}_+ \).

We use the following notation:

- \( \mathcal{R}(\mathbb{C} \setminus \{0\}) \), the universal covering space of \( \mathbb{C} \setminus \{0\} \),
- \( S_\theta = \{ t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) : \arg t < \theta \} \), a sector in \( \mathcal{R}(\mathbb{C} \setminus \{0\}) \),
- \( S(\varepsilon(y)) = \{ t \in \mathcal{R}(\mathbb{C} \setminus \{0\}) : 0 < |t| < \varepsilon(\arg t) \} \), where \( \varepsilon(y) \) is a positive-valued continuous function on \( \mathbb{R}_y \),
- \( D_r = \{ x = (x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| < r \text{ for } i = 1, \ldots, n \} \).

**Definition 1.3.** \( \tilde{O}_+ \) denotes the set of all \( v(t, x) \) satisfying the following two conditions:

i) \( v(t, x) \) is a holomorphic function on \( S(\varepsilon(y)) \times D_r \) for some positive-valued continuous function \( \varepsilon(y) \) on \( \mathbb{R}_y \) and \( r > 0 \).

ii) there is an \( a > 0 \) such that for any \( \tilde{r} \in ]0, r[ \) and \( \theta > 0 \) we have

\[
\max_{x \in D_{\tilde{r}}} |v(t, x)| = O(|t|^a) \quad (as \ t \to 0 \text{ in } S_\theta).
\]

Our main result is the following:

**Theorem 1.4.** Assume (A0)–(A3) and set \( \beta_{j, l} = (-1)^{j-l-1}l!(j-l)! \) for \( j \geq l+1 \). Let \( A = a(x) \) be a solution to

\[
(1.3) \quad \sum_{\mu \in \mathcal{M}_0} f_{\mu, 0}(x) \left( \prod_{j=l+1}^{m-1} \beta_{j, l}^{\mu_j, 0} \right) A^{\mu|\mu|-1} = \beta_{m, l}.
\]

Then, for any holomorphic function \( b(x) \) in a neighborhood of \( x = 0 \), there exists a function \( v(t, x) \in \tilde{O}_+ \) such that

\[
A(t) = a(x) t^l \log t + t^l b(x) + t^l v(t, x) = t^l \{ a(x) t^l \log t + b(x) + v(t, x) \}
\]

is a solution to (1.1).

The convergence of the sum in the left hand side of (1.3) shall be proved in Proposition 2.2. Theorem 1.4 itself shall be proved in \( \S 2 \).

We give some examples below. Note that the possibilities are \( 0 \leq l \leq m - 2 \). See (3) for an example of the case \( m = 2, l = 0 \).

**Example 1.5.** \([l = m - k - 2; |\mu| = 2; \gamma(\mu) = 2(m - 1)]\)

\[
\partial_t^m u = t^k (\partial_t^{m-1} u)^2 + t^k (\partial_t^3 \partial_x u)^2,
\]

where \( j \leq m - 1, |\alpha| \leq m - j, 2(m - 1) - k \geq 2j - k' \). This is just a PDE version of the ODE explained in Example 1.2. The second term on the right hand side is a perturbation. Hence the set \( \mathcal{M}_0 \) consists of a single element and for the only \( \mu \in \mathcal{M}_0 \), we have \( |\mu| = 2, \gamma(\mu) = 2(m - 1) \). This is what is briefly stated after the semicolon between the square brackets. We shall employ the same shorthand notation in the following examples.

**Example 1.6.** \([m = 3, l = 0; \gamma(\mu) = 3 + k]\)

\[
\partial_t^3 u = t^k (\partial_t^3 u)^2 (\partial_t u)^{3+k-2q}u^b + t^k (\partial_t^3 u)^p (\partial_t u)^q (\partial_x^2 u)^r,
\]

where \( 2p + q - k' - 3 < 0, |\alpha| \leq 3 \). If the numerator \( \gamma(\mu) - m - k_\mu \) vanishes in (A0), the ratio also does and \( |\mu| \) in the denominator is irrelevant. It makes it easy to construct examples of the case \( l = 0 \).
Example 1.7. \([m = 3, \ l = 1; \ |\mu| = 3, \ \gamma(\mu) = 6]\)
\[
\partial_t^3 u = t(\partial_t^2 u)^3 + t^k(\partial_t^1 \partial_z^2 u)^3,
\]
where \(3j - k \leq 4, \ |\alpha| \leq 3 - j\).

Example 1.8. \([m = 4, \ l = 0; \ \gamma(\mu) = k + 4]\)
\[
\partial_t^4 = t^k(\partial_t^3 u)^a(\partial_t^2 u)^b(\partial_t u)^c + t^k(\partial_t^3 u)^p(\partial_t^2 u)^q(\partial_t u)^r(\partial_z^s u)^s,
\]
where \(3p + 2q + r - k' - 4 < 0, \ |\alpha| \leq 4\).

Example 1.9. \([m = 4, \ l = 1; \ |\mu| = 3, \ \gamma(\mu) = 6]\)
\[
\partial_t^4 u = (\partial_t^3 u)^2 u + \partial_t^2 u \cdot \partial_h u \cdot \partial_z u,
\]
where \(|\alpha| \leq 4\).

Example 1.10. \([m = 4, \ l = 2; \ |\mu| = k + 2, \ \gamma(\mu) = 3(k + 2)]\)
\[
\partial_t^4 u = t^k(\partial_t^3 u)^{k+2} + t^k(\partial_t^1 \partial_z^2 u)^{k+2},
\]
where \(k' \geq k, \ 0 \leq j \leq 2, \ |\alpha| \leq 4 - j\).

Remark 1.11. We constructed solutions with the growth order \(|u| = O(|t^l \log t|)\) under the conditions (A0)–(A3). If \(u\) is a solution with the growth order \(|u| = O(|t^l | \log t|)\), \(l' > l\), then we have \(|u| = O(|t^{(l+l')/2}|)\), \((l+l')/2 > l\) and a result in [9] implies that it can be extended as a holomorphic solution up to some neighborhood of the origin.

2. Reduction to a Fuchsian Equation

In this section, we reduce the equation (1.1) to a Fuchsian equation with singular coefficients. The latter shall be the topic of Part 2.

The assumption (A0) is equivalent to the following:
\[
m - l = \sup_{\mu \in \mathcal{M}, |\mu| \geq 2} (\gamma(\mu) - l|\mu| - k_\mu).
\]

Then we have
\[
m - l = \sup_{\mu \in \mathcal{M}} (\gamma(\mu) - l|\mu| - k_\mu),
\]
because \(\gamma(\mu) - l|\mu| - k_\mu\) is smaller than \(m - l\) if \(|\mu| = 0, 1\). Therefore,

**Lemma 2.1.** We have
\[
m - l + k_\mu = \gamma(\mu) - l|\mu| \quad \text{if} \quad \mu \in \mathcal{M}_0,
\]
\[
m - l + k_\mu > \gamma(\mu) - l|\mu| \quad \text{if} \quad \mu \in \mathcal{M} \setminus \mathcal{M}_0.
\]

**Proposition 2.2.** The sum in the left hand side of (1.1) is an entire function in \(A \in \mathbb{C}\) and hence has at most one exceptional value in the sense of Picard’s theorem in the value distribution theory of complex analysis.

**Proof.** By Lemma 2.1 we have
\[
\sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-l} W_{j,0}^{\rho_j,0} = r^{m-l} \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) r^{k_\mu} \prod_{j=l+1}^{m-l} (W_{j,0}/r^{j-l})^{\mu_j,0}
\]
for any \(W = (W_{j,0})_{(j,0)}\) and \(r > 0\). Except for the factor \(r^{m-l}\), the sum in the right hand side is nothing but a partial sum of (1.2), evaluated at \(t = r, U = W/r^{j-l}\) by...
Lemma 2.4. Hence it is convergent if \( r > 0 \) is sufficiently small and so is the left hand side. Set \( W_{j,0} = \beta_{j,1} A \), then we have

\[
\sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} W_{j,0}^{\mu_j,0} = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \left( \prod_{j=l+1}^{m-1} \beta_{j,1}^{\mu_j,0} \right) A^{[\mu]}.
\]

This sum is convergent for any \( A \). \( \square \)

We set \([\rho; j] = \Gamma(\rho + 1)/\Gamma(\rho - j + 1) = \rho(\rho - 1)(\rho - 2) \ldots (\rho - j + 1)\).

Note that \([\rho; j] = 0\) if \( j - \rho \) is a positive integer. We define the sequence \( \{b_{j,l}\}_j \) by

\[
b_{0,l} = 0, \quad b_{j+1,l} = [l; j] + (l - j)b_{j,l}.
\]

Then we have

**Lemma 2.3.**

\[
\frac{\partial^j_t}{t^j} (t^j \log t) = \begin{cases} t^{j-l} \{[l; j] \log t + b_{j,l}\} & (j \leq l), \\ \beta_{j,l} t^{j-l} & (j \geq l + 1). \end{cases}
\]

We introduce new unknown functions \( a(x) \) and \( v(t, x) \) by setting

\[
u(t, x) = a(x) t^l \log t + t^l b(x) + t^l v(t, x).
\]

Here the function \( b(x) \) is arbitrary.

By using \( t^{j-l} \partial^j_t \) \( = |t \partial_t + l; j| \) and \( t^l \in \text{Ker} \partial^{l+1}_t \), we obtain

**Lemma 2.4.**

\[
t^{j-l} \partial^j_t \partial^\alpha_x u = \begin{cases} \partial^\alpha_x a(x) \{[l; j] \log t + b_{j,l}\} + |t \partial_t + l; j| \partial^\alpha_x (b + v) & (j \leq l), \\ \beta_{j,l} \partial^\alpha_x a(x) + |t \partial_t + l; j| \partial^\alpha_x v & (j \geq l + 1). \end{cases}
\]

Let us calculate the right hand side of (11). We have

\[
f\left(t, x, (\partial^j_t \partial^\alpha_x u)_{(j, \alpha) \in I_m}\right) = \sum_{\mu \in \mathcal{M}} \left( \sum_{k=0}^{\infty} f_{\mu,k}(x) t^{k+1} \right) \prod_{(j, \alpha) \in I_m} (\partial^j_t \partial^\alpha_x u)^{\mu_j,\alpha}.
\]

We extract the terms of the smallest weight and set

\[
S = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) t^{k_\mu} \prod_{(j, \alpha) \in I_m} (\partial^j_t \partial^\alpha_x u)^{\mu_j,\alpha}.
\]

Then by (A2), we have

\[
S = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) t^{k_\mu} \prod_{j=l+1}^{m-1} (\partial^j_t u)^{\mu_j,0}
\]

and it is free of logarithms. It is a partial sum of \( f(t, x, U) \) and its convergence is obvious. Note that

\[
m - l + k_\mu = \gamma(\mu) - l |\mu| = \sum_{j=l+1}^{m-1} (j - l) \mu_j
\]
for \( \mu \in \mathcal{M}_0 \). Hence by Lemma 1.1 we have

\[
t^{m-l}S = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} (t^{j-l}\partial_t^j u)^{\mu_{j,0}} = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} \{\beta_{j,0}a(x) + [t\partial_t + l; j]v\}^{\mu_{j,0}}.
\]

This quantity consists of terms of weight 0. All the remaining parts of \( t^{m-l}f \) consists of terms of positive weight (the weight of \( \log t \) is 0).

By binomial expansion, we obtain

\[
t^{m-l}S = T_0 + T_1 + T_2,
\]

where

\[
T_0 = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=l+1}^{m-1} \{\beta_{j,0}a(x)\}^{\mu_{j,0}} = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \left( \prod_{j=l+1}^{m-1} \beta_{j,0}^{\mu_{j,0}} \right) a(x)^{|\mu|},
\]

\[
T_1 = \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \prod_{j=1}^{m-1} \left( \prod_{\substack{i \neq j \geq l+1}} \{\beta_{i,0}a(x)\}^{\mu_{i,0}} \right) \mu_{j,0} \{\beta_{j,0}a(x)\}^{\mu_{j,0}-1} [t\partial_t + l; j]v
\]

\[
= \sum_{\mu \in \mathcal{M}_0} f_{\mu,0}(x) \sum_{j=l+1}^{m-1} \left( \prod_{i \neq j} \beta_{i,0}^{\mu_{i,0}} \right) \mu_{j,0} \beta_{j,0}^{\mu_{j,0}-1} a(x)^{|\mu|-1} [t\partial_t + l; j]v,
\]

\[
T_2 = \text{a polynomial in } (t\partial_t)^kv \quad (k = 0, 1, \ldots, m - 1) \text{ free of terms of degree } \leq 1.
\]

Note that \( T_0 \) is free of \( v \) and its derivatives.

On the other hand, we have

\[
t^{m-l}\partial_t^m u = \beta_{\mu,0}a(x) + [t\partial_t + l; m]v.
\]

We multiply the left and right hand sides of (1.1) by \( t^{m-l} \). If \( a(x) \) is determined by (1.3), the function \( u = at^l \log t + t^l b + t^l v \) is a solution to (1.1) if and only if \( v \) is a solution to the equation below:

\[
(2.3) \quad [t\partial_t + l; m]v - T_1 = t^{m-l}f - T_0 - T_1.
\]

Set

\[
\delta(\mu) = m - l + k_\mu - \gamma(\mu) + l|\mu|, \quad |\mu| = \sum_{(j, a) \leq m} \mu_{j, a}.
\]

Since \( \delta(\mu) \) is an integer, Lemma 2.1 implies that \( \delta(\mu) = 0 \) for \( \mu \in \mathcal{M}_0 \) and that \( \delta(\mu) \geq 1 \) for \( \mu \in \mathcal{M} \setminus \mathcal{M}_0 \). Moreover, (A3) can be written in a simple form:

\[
\delta(\mu) \geq C|\mu|, \quad \mu \in \mathcal{M} \setminus \mathcal{M}_0.
\]

These two estimates imply that

\[
(2.4) \quad \delta(\mu) = \delta(\mu)/3 + \delta(\mu)/3 + \delta(\mu)/3 \geq 1/3 + \delta(\mu)/3 + C|\mu|/3
\]

holds for any \( \mu \in \mathcal{M} \setminus \mathcal{M}_0 \).
By the way, we have
\[
(2.5) \quad t^{m-l} f(t, x, U) = t^{m-l} \sum_{\mu \in \mathcal{M}} \left( t^{k_0} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^k \right) U^\mu 
= \sum_{\mu \in \mathcal{M}} \left( \sum_{k=0}^{\infty} f_{\mu,k}(x) t^{\delta(\mu)+k} \right) \prod_{(j,\alpha) \in I_m} (t^{j-l} U_{j,\alpha})^{\mu_{j,\alpha}}.
\]

**Proposition 2.5.** The equation (2.3) satisfies the conditions \(C_1\) and \(C_2\) in Part 2.
Here our unknown function \(v(t, x)\) plays the role of \(u\) in Part 2.

**Proof.** Set
\[
a^{(\alpha)} = \partial_{x}^\alpha a(x), \quad b^{(\alpha)} = \partial_{x}^\alpha b(x),
\]
\[
W_{j,\alpha} = t^{j-l} U_{j,\alpha} - a^{(\alpha)} t^{j-l} \partial_x^j (t^l \log t) - [l; j] b^{(\alpha)},
\]
then \(W_{j,\alpha}\) corresponds to \(t^{j-l} \partial_x^j (t^l v) = [t \partial_t + l; j] \partial_x^j v(t, x)\). Note that \([t \partial_t + l; j] v\}_{j=0, \ldots, m-1}\) is equivalent to \([t \partial_t]^j v\}_{j=0, \ldots, m-1}\), the latter being used in Part 2.
They are transformed into each other by the action of a lower triangular matrix whose diagonal elements are all 1.

For brevity, we set
\[
\tilde{W}_{j,\alpha} = W_{j,\alpha} + a^{(\alpha)} t^{j-l} \partial_x^j (t^l \log t) + [l; j] b^{(\alpha)} (t^{j-l} U_{j,\alpha})
= \begin{cases} W_{j,\alpha} + a^{(\alpha)} [l; j] \log t + b_{j,l} & (j \leq l), \\
W_{j,\alpha} + b_{j,l} a^{(\alpha)} & (j \geq l+1).
\end{cases}
\]
Here we have used Lemma 2.3 and the fact that \([t; j] = 0\) if \(j \geq l + 1\). By (2.5) we have
\[
\{t^{m-l} f(t, x, U) - T_0 - T_1\} - T_2 = t^{m-l} f(t, x, U) - t^{m-l} S = I_1 + I_2,
\]
where
\[
I_1 = \sum_{\mu \in \mathcal{M}_0} \sum_{k \geq 1} f_{\mu,k}(x) t^k \prod_{j=l+1}^{m-1} \tilde{W}_{j,0}^{\mu_{j,0}},
\]
\[
I_2 = \sum_{\mu \notin \mathcal{M}_0} \sum_{k=0}^{\infty} f_{\mu,k}(x) t^{\delta(\mu)+k} \prod_{(j,\alpha) \in I_m} \tilde{W}_{j,\alpha}^{\mu_{j,\alpha}} \prod_{(j,\alpha) \in I_m} \tilde{W}_{j,\alpha}^{\mu_{j,\alpha}}.
\]
The convergence of \(I_1\) can be proved by the method of Proposition 2.2. If \(|t| \leq r\), then the following estimates hold:
\[
|I_1| \leq |t|^{m-l-1} \sum_{\mu \in \mathcal{M}_0} \sum_{k \geq 1} |f_{\mu,k}(x)| t^k \prod_{j=l+1}^{m-1} (|\tilde{W}_{j,0}| r^{j-1})^{\mu_{j,0}}.
\]
We see that \(I_1\) and its derivatives in \(W\) are of order \(O(|t|)\).

Next let us consider \(I_2\). The trivial fact \(\delta(\mu) > 0, \mu \notin \mathcal{M}_0\) helps, to be sure, but it is not good enough. If \(j \leq l\), the quantity \(\tilde{W}_{j,\alpha}\) contains a logarithm, whose unboundedness is the greatest obstacle. We overcome it by assuming (A3). The trick is the following fact: if \(C > 0\), then \(t^C \tilde{W}_{j,\alpha}\) is bounded as \(t \to 0\).
If \(|t| < r < 1\), the inequality (2.4) implies that
\[
|\delta^{|\mu|}| \leq |t|^{1/3} \times (r^{1/3})^{|\mu|} \times (|t|^{C/3})^{|\mu_j,\alpha|}
\]
\[
= |t|^{1/3} p(m-l+\kappa_\mu)/3 \times \prod_{(j,\alpha) \in I_m} \left( \frac{|t|^{C/3}}{r^{(j-l)/3}} \right)^{\mu_j,\alpha} \times \prod_{(j,\alpha) \in I_m, j \geq l+1} \left( \frac{1}{r^{(j-l)/3}} \right)^{\mu_j,\alpha}.
\]
Therefore
\[
|I_2| \leq |t|^{1/3} \sum_{\mu \notin M_0} \sum_{k=0}^\infty |f_{\mu,k}(x)| (r^{1/3})^{m-l+\kappa_\mu} |t|^k
\]
\[
\times \prod_{(j,\alpha) \in I_m} \left( r^{(l-j)/3} |t|^{C/3} |W_{j,\alpha}| \right)^{\mu_j,\alpha} \times \prod_{(j,\alpha) \in I_m, j \geq l+1} \left( r^{(l-j)/3} |W_{j,\alpha}| \right)^{\mu_j,\alpha}.
\]
If \(r > 0\) is sufficiently small, then \(I_2\) is convergent in \(|t| < r < r^{1/3}\). We see that \(I_2\) and its derivatives in \(W\) are of order \(O(|t|^{1/3})\).

\[\square\]

End of Proof of Theorem 1.4

Theorem 1.4 follows from Proposition 2.3 and Theorem 1.1 of Part 2.

Remark 2.6. There is an arbitrary function \(b(x)\) in the family of solutions in Theorem 1.4. In some cases, there may be more: as is stated in Remark 4.2 of Part 2, the equation (2.3) may admit a family of solutions involving one or more arbitrary functions in \(x\).

3. NONLINEAR WAVE EQUATION

We can relax the condition imposed in 1.4. Although we formulate our result in the complex domain, it is trivial that an analogous result holds in the real-analytic category.

We consider
\[
(3.1) \quad \Box u(s, y) = g(s, y; u, \partial_s u, \nabla_y u)
\]
in an open set of \(C^{n+1} = C_s \times C^n_y\). Here \(\Box = \partial^2 / \partial s^2 - \sum_{j=1}^{n} \partial^2 / \partial y_j^2\), \(\nabla_y u = (\partial u / \partial y_1, \ldots, \partial u / \partial y_n)\). We assume that \(g(s, y; z, \sigma, \eta)\) is a holomorphic function in all its arguments and is entire in \((z, \sigma, \eta)\). Moreover we assume that it is a polynomial of degree 2 in \((\sigma, \eta)\). Its homogeneous part of degree 2 is denoted by \(g_2\).

Let \(\psi(y)\) be a holomorphic function with
\[
(3.2) \quad 1 - |\nabla_y \psi(y)|^2 \neq 0,
\]
where \(\nabla_y \psi(y) = (\psi_1(y), \ldots, \psi_n(y))\), \(\psi_1(y) = \partial \psi(y) / \partial y_1 (i = 1, 2, \ldots, n)\). Moreover, assume that
\[
(3.3) \quad g_2(\psi(y), y; 0, 1, -\nabla_y \psi(y)) \neq 0.
\]
Note that this assumption corresponds to \(k_\mu = 0\), where \(k_\mu\) is as in 1.1.

**Theorem 3.1.** Assume (3.2) and (3.3). Then, in a neighborhood of the hypersurface \(s = \psi(y)\), there exists a family of solutions \(u(s, y)\) to (3.1) with the asymptotic behavior
\[
u(s, y) \sim -\frac{1 - |\nabla_y \psi(y)|^2}{g_2(\psi(y), y; 0, 1, -\nabla_y \psi(y))} \log(s - \psi(y)).
Here the remainder term involves an arbitrary holomorphic function on \( \{ s = \psi(y) \} \).

Proof. Set

\[ t = s - \psi(y), \quad x = y, \quad \Psi = 1 - |\nabla_y \psi(y)|^2 (\neq 0). \]

Then, as is proved in [5], we have

\[ \partial_s = \partial_t, \quad \partial_{y_i} = -\psi_i \partial_t + \partial_{x_i}, \]

\[ \Box = \Box_{s,y} = \Psi \partial_t^2 + 2 \sum_{i=1}^{n} \psi_i \partial_{x_i} \partial_t + (\Delta_y \psi) \partial_t - \Delta_x. \]

In a neighborhood of \( t = s - \psi(y) = 0 \), the original equation (3.1) becomes

\[ \partial_t^2 u = (\text{linear part}) + \Psi^{-1} g_2 \left( \psi(x), x; 0, 1, -\nabla_y \psi(y) \right) (\partial_t u)^2. \]

When we expand the right hand side in a power series in \( t \), we find the term

\[ k_\mu = 0, \quad \gamma(\mu) = 2, \quad f_{\mu,0}(x) = \Psi^{-1} g_2 \left( \psi(x), x; 0, 1, -\nabla_y \psi(y) \right). \]

It corresponds to \( \mu \) with \( \mu_{1,0} = 2, \mu_{j,0} = 0 \) (otherwise). For this \( \mu \), we have

\[ f_{\mu,0}(x) A = -1. \]

Theorem 1.4 enables us to construct a solution in a neighborhood of each point on the hypersurface. In spite of Remark 2.6, these solutions overlap, if \( b(x) \) is fixed, because they are constructed in the same way, i.e. by Proposition 5.2 (6.5) and (6.6).

\[ \Box \]

Part 2. Nonlinear Fuchsian equations with singular coefficients

We shall generalize the result of [13] to the case where the equations have singular coefficients at \( t = 0 \). We employ the same notation as in Part 1. Two more sets have to be introduced:

- \( S_\theta(\delta) = \{ t \in S_\theta : 0 < |t| < \delta \} \) a sectorial domain in \( \mathcal{R}(\mathbb{C} \setminus \{0\}) \),
- \( S_\theta(\varepsilon(y)) = S_\theta \cap S(\varepsilon(y)) \).

4. AN EXISTENCE THEOREM

We consider

\[ (t \partial_t)^m u = F \left( t, x, \left( (t \partial_t)^j \partial_x^a u \right)_{(j,a) \in I_m} \right) \tag{4.1} \]

with the unknown function \( u = u(t, x) \). Here the function \( F \) is allowed to be singular at \( t = 0 \). Typically, it may involve powers of \( \log t \). More precisely, we assume:

- \( C_1 \) \( F(t, x, Z) \) is a holomorphic function in \( (t, x, Z), Z = (Z_{j,a})_{(j,a) \in I_m} \in \mathbb{C}^N \) on \( S(\varepsilon(y)) \times D_{R_0} \times \{|Z| < L\} \) for a positive-valued continuous function \( \varepsilon(y) \) and constants \( R_0 > 0, L > 0 \).
there exist a constant $s > 0$ and holomorphic functions $c_j(x)$ $(0 \leq j \leq m-1)$ on $D_{R_0}$ such that for any $\theta > 0$, $(j, \alpha) \in I_m$ and $(i, \beta) \in I_m$ we have
\[ \sup_{x \in D_{R_0}} |F(t, x, 0)| = O(|t|^s) \quad \text{(as $S_\theta \ni t \to 0$)}, \]
\[ \sup_{x \in D_{R_0}} \left| \frac{\partial F}{\partial Z_{j,0}}(t, x, 0) - c_j(x) \right| = O(|t|^s) \quad \text{(as $S_\theta \ni t \to 0$)}, \]
\[ \sup_{x \in D_{R_0}} \left| \frac{\partial F}{\partial Z_{j,\alpha}}(t, x, 0) \right| = O(|t|^s) \quad \text{(as $S_\theta \ni t \to 0$) if $|\alpha| > 0$}, \]
\[ \sup_{x \in D_{R_0}, |Z| < L} \left| \frac{\partial^2 F}{\partial Z_{j,\alpha} \partial Z_{k,\beta}}(t, x, Z) \right| = O(1) \quad \text{(as $S_\theta \ni t \to 0$)}. \]

Then we have:

**Theorem 4.1.** Assume the conditions $C_1$) and $C_2$). Then, the equation (4.1) has a solution $u(t, x)$ in the class $\hat{O}_+$. 

**Remark 4.2.** The solution of (4.1) in $\hat{O}_+$ is not necessarily unique. There may be a family of solutions involving one or more arbitrary functions in $x$. See [13].

Note that this theorem is essential in the proof of Theorem 4.4.

### 5. Some preparatory discussion

Before the proof of Theorem 4.1, let us present some preparatory discussion. For a function $\phi(x)$ on $D_r$, we define the norm $\|\phi\|_r$ by
\[ \|\phi\|_r = \sup_{x \in D_r} |\phi(x)|. \]

Let $\varepsilon(y)$ be a positive-valued continuous function on $\mathbb{R}_y$. We say that $\varepsilon(y)$ is decreasing in $|y|$ if the following condition holds: $|y_1| \leq |y_2|$ implies $\varepsilon(y_1) \geq \varepsilon(y_2)$.

**Definition 5.1.** (1) For $d \geq 0$ and $\theta > 0$, we denote by $\hat{O}_d(S_\theta(\varepsilon(y)) \times D_R)$ the set of all the holomorphic functions on $S_\theta(\varepsilon(y)) \times D_R$ that satisfy the following estimate: for any $0 < r < R$ there is a constant $C > 0$ such that
\[ |u(t, x)| \leq C|t|^d \quad \text{on} \quad S_\theta(\varepsilon(y)) \times D_r. \]

(2) We set $\hat{O}_d(S(\varepsilon(y)) \times D_R) = \bigcap_{\theta > 0} \hat{O}_d(S_\theta(\varepsilon(y)) \times D_R)$.

Let $m \in \mathbb{N}^*$ and $c_j(x)$ $(j = 0, 1, \ldots, m-1)$ be as in [4]. Set
\[ C(\lambda, x) = \lambda^m - c_{m-1}(x)\lambda^{m-1} - \cdots - c_1(x)\lambda - c_0(x), \]
and denote by $\lambda_1(x), \ldots, \lambda_m(x)$ the roots of $C(\lambda, x) = 0$ in $\lambda$, and let us consider the following equation:
\[ C(t\partial_t, x)v = g(t, x). \]

**Proposition 5.2.** Let $a > 0$. Suppose that
\[ \{a, 2a, 3a, \ldots\} \cap \{\Re \lambda_1(0), \ldots, \Re \lambda_2(0)\} = \emptyset \]
and that $\varepsilon(y)$ is decreasing in $|y|$. Then we can take a sufficiently small $R_1 > 0$ satisfying the following properties $(\ast)_k$ and $(\sharp)_k$ for $k = 1, 2, \ldots$.
\((\ast)_k\): For any \(g(t,x) \in \hat{\Theta}_{\alpha k}(S(\varepsilon(y)) \times D_{R_1})\), the equation \(5.2\) has a solution \(v(t,x) \in \hat{\Theta}_{\alpha k}(S(\varepsilon(y)) \times D_{R_1})\).

\((\sharp)_k\): Moreover, if \(g(t,x)\) satisfies
\[
\|g(t,x)\|_r \leq C|t|^{\alpha k} \text{ on } S_\theta(\varepsilon(y))
\]
for some \(0 < r < R_1\), \(C > 0\) and \(\theta > 0\), we have the estimate
\[
\|(t\partial_t)^j v(t)\|_r \leq \frac{M_0}{k^{m-j}} C|t|^{\alpha k} \text{ on } S_\theta(\varepsilon(y)) \text{ for } j = 0, 1, \ldots, m - 1,
\]
where the constant \(M_0 > 0\) is independent of \(k\), \(g(t,x)\), \(r\) and \(j\).

**Proof.** This proposition can be proved by the same argument as in the proof of Lemma 6 in [13]. We can choose a suitable path of integration because of the assumption that \(\varepsilon(y)\) is decreasing. \(\square\)

**Lemma 5.3** (Nagumo’s Lemma). If \(\phi(x)\) is a holomorphic function on \(D_R\) and if
\[
\|\phi\|_r \leq \frac{C}{(R-r)^b} \text{ for any } 0 < r < R
\]
holds for some \(C \geq 0\) and \(b \geq 0\), then we have
\[
\left\| \frac{\partial \phi}{\partial x_i} \right\|_r \leq \frac{e(b+1)C}{(R-r)^{b+1}} \text{ for any } 0 < r < R \text{ and } i = 1, \ldots, n.
\]

**Proof.** See Nagumo [10] or Lemma 5.1.3 of Hörmander [4]. \(\square\)

6. **Proof of Theorem 5.1**

Assume the conditions \(C_1\) and \(C_2\). Then, by expanding \(F(t,x,Z)\) in \(Z\), our equation \(4.1\) is written in the form
\[
C(t\partial_t,x)u = a(t,x) + \sum_{(j,\alpha) \in I_m} b_{j,\alpha}(t,x)(t\partial_t)^j \partial_x^\alpha u
\]
\[
+ \sum_{|\nu| \geq 2} g_\nu(t,x) \prod_{(j,\alpha) \in I_m} [(t\partial_t)^j \partial_x^\alpha u]^{\nu_{j,\alpha}},
\]
where \(\nu = (\nu_{j,\alpha})_{(j,\alpha) \in I_m} \in \mathbb{N}^N\) and \(|\nu| = \sum_{(j,\alpha) \in I_m} \nu_{j,\alpha} \geq 2\). The coefficients \(a(t,x), b_{j,\alpha}(t,x)\) and \(g_\nu(t,x)\) are all holomorphic functions on \(S(\varepsilon(y)) \times D_{R_0}\) with suitable growth order to be specified below. By replacing \(\varepsilon(y)\) if necessary, we may suppose that \(0 < \varepsilon(y) \leq 1\) and that \(\varepsilon(y)\) is decreasing in \(|y|\).

Let us construct a formal solution. By taking \(a > 0\) suitably we may suppose that \(0 < a \leq s\) and
\[
\{a, 2a, 3a, \ldots\} \cap \{\Re \lambda_1(0), \ldots, \Re \lambda_m(0)\} = \emptyset
\]
hold. By Proposition 5.2 we have such an \(R_1 > 0\) that the properties \((\ast)_k\) and \((\sharp)_k\) are valid for \(k = 1, 2, \ldots\). Since \(R_1 > 0\) can be very small, we may assume that \(a(t,x), b_{j,\alpha}(t,x)\) and \(g_\nu(t,x)\) have the following properties:

i) \(a(t,x) \in \hat{\Theta}_a(S(\varepsilon(y) \times D_{R_1}))\),

ii) \(b_{j,\alpha}(t,x) \in \hat{\Theta}_a(S(\varepsilon(y) \times D_{R_1}))\) for \((j,\alpha) \in I_m, |\alpha| > 0\),

iii) \(g_\nu(t,x) \in \hat{\Theta}_0(S(\varepsilon(y) \times D_{R_1}))\) for \(|\nu| \geq 2\).
We shall construct a formal solution of (6.1) in the form

\[ u(t, x) = \sum_{k \geq 1} u_k(t, x), \quad u_k(t, x) \in \mathcal{O}_{ak}(S(\varepsilon(y) \times D_{R_1})). \]

Let us decompose our equation (6.1). We have formally

\[ \sum_{k \geq 1} C(t\partial, x)u_k = a(t, x) + \sum_{k \geq 1} b_{j,\alpha}(t, x)(t\partial)^2 \partial_x^\alpha u_k \]

\[ + \sum_{|\nu| \geq 2} g_{\nu}(t, x) \prod_{(j,\alpha) \in I_m} \left[ \sum_{k \geq 1} (t\partial)^j \partial_x^\alpha u_k \right]^{\nu_{j,\alpha}}. \]

Therefore, the equation (6.1) is satisfied if \{u_k(t, x) ; k = 1, 2, \ldots\} is determined by the following recurrent family of equations (6.5) and (6.6):

\[ C(t\partial, x)u_1 = a(t, x) \]

and for \( k \geq 2, \)

\[ C(t\partial, x)u_k = \sum_{(j,\alpha) \in I_m} b_{j,\alpha}(t, x)(t\partial)^j \partial_x^\alpha u_{k-1} \]

\[ + \sum_{2 \leq |\nu| \leq k} g_{\nu}(t, x) \sum_{|k(\nu)| = k} \prod_{(j,\alpha) \in I_m} \prod_{l=1}^{\nu_{j,\alpha}} (t\partial)^j \partial_x^\alpha u_{kj_{j,\alpha}(l)}, \]

where

\[ k_{j,\alpha}(l) \in \mathbb{N}^*, \]

\[ k(\nu) = \{(k_{j,\alpha}(l)) ; (j,\alpha) \in I_m, 1 \leq l \leq \nu_{j,\alpha}\}, \]

\[ |k(\nu)| = \sum_{(j,\alpha) \in I_m} (k_{j,\alpha}(1) + \cdots + k_{j,\alpha}(\nu_{j,\alpha})). \]

It should be remarked that in the right hand side of (6.6) only the terms \( u_1, \ldots, u_{k-1} \)

and their derivatives appear. Thus, by applying Proposition 5.2 to (6.5) and (6.6) (\( k \geq 2 \)) inductively on \( k \) we can obtain a solution \{u_k(t, x) ; k = 1, 2, \ldots\} of the recurrent family (6.5) and (6.6) (for \( k \geq 2 \)) such that

\[ (t\partial)^j \partial_x^\alpha u_k \in \mathcal{O}_{ak}(S(\varepsilon(y) \times D_{R_1})) \quad \text{for} \quad (j,\alpha) \in I_m, k = 1, 2, \ldots. \]

This proves

**Proposition 6.1.** *In the above situation, we can construct

\[ u(t, x) = \sum_{k \geq 1} u_k(t, x) \]

with the condition (6.7) which solves the equation (6.1) formally in the sense that \{u_k(t, x) ; k = 1, 2, \ldots\} satisfies (6.5) and (6.6) (for \( k \geq 2 \)).

Set \( f_1(t) = a(t, x) \) and for \( k \geq 2 \)

\[ f_k(t) = \sum_{(j,\alpha) \in I_m} b_{j,\alpha}(t, x)(t\partial)^j \partial_x^\alpha u_{k-1} \]

\[ + \sum_{2 \leq |\nu| \leq k} g_{\nu}(t, x) \sum_{|k(\nu)| = k} \prod_{(j,\alpha) \in I_m} \prod_{l=1}^{\nu_{j,\alpha}} (t\partial)^j \partial_x^\alpha u_{kj_{j,\alpha}(l)}. \]
Then, by (4) of Proposition 5.2 we see also

**Proposition 6.2.** In Proposition 6.1 we have the following additional property: if \( f_k \) satisfies

\[
\|f_k(t)\|_r \leq C|t|^{\alpha k} \quad \text{on} \quad S_\theta(\varepsilon(y))
\]

for some \( 0 < r < R_1, \ C > 0 \) and \( \theta > 0 \), we have the estimate

\[
\|(t\partial_t)^j u_k(t)\|_r \leq \frac{M_\theta}{k^{m-j}} C|t|^{\alpha k} \quad \text{on} \quad S_\theta(\varepsilon(y)) \quad \text{for} \quad j = 0, 1, \ldots, m - 1.
\]

Next, let us prove the convergence of the formal solution (6.8). Our aim is to show that (6.8) gives an \( \tilde{O} \)-solution of (6.1).

Take any \( \theta > 0 \) and \( 0 < R < R_1 \) (with \( 0 < R \leq 1 \)) and fix them. To show our aim it is sufficient to prove that the formal solution (6.8) is convergent in \( \tilde{O}(S_\theta(\delta) \times D_{R/2}) \) for some \( \delta > 0 \).

By the assumption there exist constants \( B_{j,\alpha} \geq 0 \) \((j, \alpha) \in I_m\) and \( G_\nu \geq 0 \) \[((|\nu| \geq 2)\) satisfying the following properties:

i) \(|b_{j,\alpha}(t, x)| \leq B_{j,\alpha}|t|^{\alpha} \quad \text{on} \quad S_\theta(\varepsilon(y)) \times D_R,

ii) \(|g_{\nu}(t, x)| \leq G_\nu \quad \text{on} \quad S_\theta(\varepsilon(y)) \times D_R,

iii) \( \sum_{|\nu| \geq 2} G_\nu Z^\nu \) is convergent in a neighborhood of \( Z = 0 \in \mathbb{C}^N \).

By (6.7) we have \((t\partial_t)^j \partial^\nu_x u_1 \in \tilde{O}(\alpha(S(\varepsilon(y) \times D_{R_1})) \) for any \((j, \alpha) \in I_m\); therefore we can take a constant \( A_1 \geq 0 \) such that

\[
\|(t\partial_t)^j \partial^\nu_x u_1(t)\|_r \leq A_1|t|^{\alpha} \quad \text{on} \quad S_\theta(\varepsilon(y)), \quad (j, \alpha) \in I_m.
\]

Set \( \beta = (em)^m \). Using these \( A_1, \ \beta, B_{j,\alpha} \) and \( G_\nu \), we consider the following holomorphic functional equation with respect to \( Y \):

\[
Y = A_1 z + M_\theta \left[ \sum_{(j, \alpha) \in I_m} \frac{B_{j,\alpha} z^{J Y}}{(R - r)^m} + \sum_{|\nu| \geq 2} \frac{G_\nu}{(R - r)^{m|\nu| - 1}} (\beta Y)^{|\nu|} \right],
\]

where \( r \) is a parameter with \( 0 < r < R \).

By the implicit function theorem we see that the equation (6.13) has a unique holomorphic solution \( Y(z) \) with \( Y(0) = 0 \) in a neighborhood of \( z = 0 \). If we expand this into

\[
Y(z) = \sum_{k \geq 1} Y_k z^k,
\]

we see that the coefficients \( Y_k \ (k = 1, 2, \ldots) \) are determined uniquely by the following recurrence formulas:

\[
Y_1 = A_1
\]

and for \( k \geq 2 \),

\[
Y_k = M_\theta \left[ \sum_{(j, \alpha) \in I_m} \frac{B_{j,\alpha} \beta Y_{k-1}}{(R - r)^m} + \sum_{2 \leq |\nu| \leq k} \frac{G_\nu}{(R - r)^{m|\nu| - 1}} \prod_{l=1}^{\nu_j,\alpha} \prod_{i=1}^{\nu_j,\alpha} \beta Y_{k,l,i} \right].
\]
Moreover, by induction on \( k \) we see that each \( Y_k \) has the form
\( Y_k = \frac{C_k}{(R-r)^{m(k-1)}}, \quad k = 1, 2, \ldots, \)
where \( C_1 = A_1 \) and \( C_k \geq 0 \) \((k \geq 2)\) are constants independent of the parameter \( r \).

The following lemma guarantees that \( Y(z) \) can be used as a majorant series of the formal solution \( (6.8) \).

**Proposition 6.3.** For any \( k = 1, 2, \ldots \) we have
\( (6.17) \quad \| (t\partial_t)^j \partial_x^\alpha u_k(t) \|_r \leq \beta Y_k(r)|t|^a \) \( \text{on} \ S_\theta(\varepsilon(y)) \)
for any \( 0 < r < R \) and \((j, \alpha) \in I_m\).

**Proof.** By the definition of \( A_1 \) in \( (6.12) \) we have
\[ \| (t\partial_t)^j \partial_x^\alpha u_1(t) \|_r \leq A_1|t|^a = Y_1|t|^a \leq \beta Y_1|t|^a \text{ on } S_\theta(\varepsilon(y)). \]
This proves \( (6.17) \) for \( k = 1 \).

Let us show the general case by induction on \( k \). Suppose that \( k \geq 2 \) and that \( (6.17) \) has already been proved for \( u_1, \ldots, u_{k-1} \). Then, by \( (6.9) \) we have
\[ \| f_k(t) \|_r \leq \sum_{(j, \alpha) \in I_m} B_{j,\alpha} |t|^a \times \beta Y_{k-1}|t|^{a(k-1)} \]
\[ + \sum_{2 \leq |\nu| \leq k} G_\nu \prod_{|k(\nu)|=k, (j, \alpha) \in I_m} \beta Y_{k_{j,\alpha}(t)}|t|^{\nu_{j,\alpha}(t)} \]
\[ = |t|^a \left( \sum_{(j, \alpha) \in I_m} B_{j,\alpha} \beta Y_{k-1} + \sum_{2 \leq |\nu| \leq k} G_\nu \prod_{|k(\nu)|=k, (j, \alpha) \in I_m} \beta Y_{k_{j,\alpha}(t)} \right). \]
Therefore, by comparing this with \( (6.15) \) and by using \( 1/(R-r) > 1 \) we have
\[ \| f_k(t) \|_r \leq \frac{(R-r)^m}{M_\theta} Y_k |t|^a = \frac{C_k}{M_\theta(R-r)^{m(k-2)}} |t|^a \text{ on } S_\theta(\varepsilon(y)) \]
for any \( 0 < r < R \). Hence, by Proposition 6.2 we have
\( (6.18) \quad \| (t\partial_t)^j \partial_x^\alpha u_k(t) \|_r \leq \frac{C_k}{(R-r)^{m(k-2)}} |t|^a \text{ on } S_\theta(\varepsilon(y)) \)
for any \( 0 < r < R \) and \( j = 0, 1, \ldots, m-1 \). By applying Lemma 6.3 (Nagumo’s lemma) to this estimate we have
\[ \| (t\partial_t)^j \partial_x^\alpha u_k \|_r \leq \frac{1}{k^{m-\nu}} \frac{\{m(k-2)+1\} \cdots \{m(k-2)+|\alpha|\} |e|^{|\alpha|} C_k |t|^{a(k-1)}}{(R-r)^{m(k-2)+|\alpha|}} |t|^a \]
\[ \leq \frac{1}{k^{m-\nu}} \frac{m^{\nu} |e|^{|\alpha|} C_k}{(R-r)^{m(k-2)+|\alpha|}} |t|^a \]
\[ \leq \frac{\beta C_k}{(R-r)^{m(k-2)+|\alpha|}} |t|^a \leq \frac{\beta C_k}{(R-r)^{m(k-2)+|\alpha|}} Y_k |t|^a = \beta Y_k |t|^a \]
on \( S_\theta(\varepsilon(y)) \) for any \( 0 < r < R \) and \((j, \alpha) \in I_m\); this proves \( (6.17) \).

Thus, we have proved Proposition 6.3. \( \square \)
Lastly, let us complete the proof of Theorem 4.1. Set $r = R/2$ and fix it. Since $Y(z) = \sum_{k \geq 1} Y_k(r)z^k$ is convergent, we can take a small constant $\delta > 0$ so that $C = \sum_{k \geq 1} \beta Y_k(r)\delta^k < \infty$ holds. Then, for any $(t, x) \in S_0(\delta) \times D_{R/2}$ we have

$$\sum_{k \geq 1} |u_k(t, x)| \leq \sum_{k \geq 1} \|u_k(t)\|_{R/2} \leq \sum_{k \geq 1} \beta Y_k |t|^\alpha \delta^k \leq \sum_{k \geq 1} \beta Y_k \delta^k |t|^\alpha \delta^k \leq C|t|^\alpha \delta^k .$$

This proves that the formal solution (6.8) is convergent in $\tilde{O}_a(S_0(\delta) \times D_{R/2})$.

Thus, we have proved Theorem 4.1.

REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, London Mathematical Society Lecture Note Series 149, Cambridge University Press, 1996.

[2] R. Gérard and H. Tahara, Solutions holomorphes et singulières d’Équations aux dérivées partielles singulières non linéaires, Publ. Res. Inst. Math. Sci., 29(1993), 121-151.

[3] R. Gérard and H. Tahara, Singular nonlinear partial differential equations, Vieweg, 1996.

[4] L. Hörmander, Linear partial differential operators, Springer, 1963.

[5] S. Kichenassamy and W. Littman, Blow-up surfaces for nonlinear wave equations, I, Comm. Partial Differential Equations, 18(3&4)(1993), 431-452.

[6] S. Kichenassamy and W. Littman, Blow-up surfaces for nonlinear wave equations, II, Comm. Partial Differential Equations, 18(11)(1993), 1869-1899.

[7] S. Kichenassamy and G. K. Srinivasan, The structure of WTC expansions and applications, J. Phys. A, 28(1995), 1977-2004.

[8] S. Kichenassamy and A. D. Rendall, Analytic description of singularities in Gowdy spacetimes, Classical Quantum Gravity, 15(1998), 1339-1355.

[9] T. Kobayashi, Singular solutions and prolongation of holomorphic solutions to nonlinear differential equations, Publ. Inst. Math. Sci., 34(1998), 43-63.

[10] M. Nagumo, Über das Anfangswertproblem Parteller Differentialgleichungen, Japan. J. Math., 18 (1941), 41-47.

[11] H. Tahara, On the singularities of solutions of nonlinear partial differential equations in the complex domain, Microlocal Analysis and Complex Fourier Analysis, World Sci. Publ., Hackensack, NJ (2002), 273-283.

[12] H. Tahara, On the singularities of solutions of nonlinear partial differential equations in the complex domain, II, Differential equations & asymptotic theory in mathematical physics, Ser. Anal., World Sci. Publ., Hackensack, NJ (2004), 343-354.

[13] H. Tahara and H. Yamazawa, Structure of solutions of nonlinear partial differential equations of Gerard-Tahara type, Publ. Res. Inst. Math. Sci., 41 (2005), 339-373.

[14] H. Yamane, Nonlinear wave equations and singular solutions, to appear in Proc. Amer. Math. Soc., arXiv: math.AP/0601308

Department of mathematics, Sophia University, Kioicho, Chiyoda-ku, Tokyo 102-8554, JAPAN
E-mail address: h-tahara@hoffman.cc.sophia.ac.jp

Department of Physics, Kwansei Gakuin University, Gakuen 2-1, Sanda, Hyougo 669-1337, JAPAN
E-mail address: yamane@ksc.kwansei.ac.jp