Parity Effect in a Small Superconducting Particle

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(January 20, 2022)

Abstract

Matveev and Larkin calculated the parity effect on the ground state energy of a small superconducting particle in the regimes where the mean level spacing $\delta$ is either large or small compared to the bulk gap $\Delta$. We perform a numerical calculation which extends their results to intermediate values of $\delta/\Delta$.

I. INTRODUCTION

Black, Ralph, and Tinkham performed tunneling experiments on nanometer scale particles of aluminum at temperatures $T \approx 50$ mK well below the superconducting transition temperature $T_c \sim$ 1 K. In some particles they found a superconducting energy gap larger than the level spacing. This gap was slightly enhanced from the bulk gap. In smaller particle, the level spacing exceeded the bulk gap and no evidence of superconductivity was found. One may then ask whether all evidence of superconductivity is extinguished in these smaller grains.

Matveev and Larkin addressed this question within a BCS model

$$H = \sum_{j,\sigma} \varepsilon_j c_j^{\dagger} c_{j\sigma} - g \sum_{|\varepsilon_j|,|\varepsilon_j'|<\omega_d} c_{j\uparrow}^{\dagger} c_{j'\downarrow} c_{j'\uparrow} c_{j\downarrow}.$$ (1)

Letting $E_g^{(m)}$ be the ground state energy for the system with $m$ electrons, they defined the parity effect parameter

$$\Delta_p = E_g^{(2n+1)} - \frac{1}{2} \left( E_g^{(2n)} + E_g^{(2n+2)} \right).$$ (2)

This parameter measures the degree to which the odd parity ground states have a higher energy than the even ones. Matveev and Larkin studied the limit $g \downarrow 0$, $\omega_d \to \infty$, at fixed mean level spacing $\delta$ and at fixed value of the bulk gap

$$\Delta \equiv 2\omega_d e^{-\delta/\omega_d}.$$ (3)

They found the asymptotic results

$$\Delta_p/\Delta \sim 1 - \frac{\delta/\Delta}{2}, \quad \delta/\Delta << 1$$ (4a)
$$\Delta_p/\Delta \sim \frac{\delta/\Delta}{2 \ln(\delta/\Delta)}, \quad \delta/\Delta >> 1$$ (4b)
A minimum was therefore expected in \( \Delta P/\Delta \) for \( \delta/\Delta \) near 1.

We perform a numerical calculation for \( \Delta P/\Delta \) with constant level spacing \( \delta \), and locate the position and value of the minimum. We also verify the asymptotic limits of \( \Delta P/\Delta \) are given by (4a) and (4b).

II. METHOD

We consider a collection of states \( j \in J = \{-n, \ldots, n\} \) about the Fermi surface and study the effective Hamiltonian for these states

\[
\tilde{H} = \delta \sum j c^+_j \sigma_j - \tilde{g} \sum c^+_j c^+_j c_j c_j^\uparrow.
\]  

The effective coupling \( \tilde{g} \) is given in perturbation theory by

\[
\tilde{g} = \frac{\delta}{\ln(a_{n,\mu}(2n+1)\delta/\Delta)}
\]

where the constant \( a_{n,\mu} \approx 1 \) for \( n > 2 \) and \( |\mu| < \delta \). We nevertheless use the exact \( a_{n,\mu} \) in this calculation.

We are interested in the ground state energy of \( \tilde{H} \) for both odd and even electron number. In the even case the ground state has no singly-occupied level and can therefore be mapped onto the ground state of the spin Hamiltonian

\[
\tilde{H} = \delta \sum j \sigma_j^z = -1 ; \quad \tilde{H}^{(2n+1)} = 0 ; \quad \tilde{H}^{(2n+2)} = 1
\]

for \( j \in J \), where the spin operators are defined by

\[
\begin{align*}
\sigma_j^z &= c_j^\uparrow c_j^\downarrow + c_j^\downarrow c_j^\uparrow - 1 \\
\sigma_j^+ &= c_j^\uparrow c_j^\downarrow \\
\sigma_j^- &= c_j^\downarrow c_j^\uparrow
\end{align*}
\]

The odd case is similar except for one singly-occupied level \( (j = 0) \) at the Fermi surface. This level is inert and can be ignored for our purposes. We therefore use the effective Hamiltonian (8) with \( j \in J - \{0\} \) in the odd case.

We thus define the Hamiltonians \( \tilde{H}^{(2n)}, \tilde{H}^{(2n+1)}, \) and \( \tilde{H}^{(2n+2)} \) by (8) and the following constraints:

\[
\begin{align*}
\tilde{H}^{(2n)} : \quad \sum \sigma_j^z &= -1 ; \quad j \in J ; \quad \mu^{(2n)} = -\delta/2 \\
\tilde{H}^{(2n+1)} : \quad \sum \sigma_j^z &= 0 ; \quad j \in J - \{0\} ; \quad \mu^{(2n+1)} = 0 \\
\tilde{H}^{(2n+2)} : \quad \sum \sigma_j^z &= 1 ; \quad j \in J ; \quad \mu^{(2n+2)} = \delta/2
\end{align*}
\]
The couplings entering the \( \tilde{H}^{(m)} \) are \( \tilde{g}_{even} \) for \( m = 2n \), \( 2n + 2 \) and \( \tilde{g}_{odd} \) for \( m = 2n + 1 \). These are given by expression (7). As \( n \) increases, the ground state energy of \( \tilde{H}^{(m)} \) approaches that of the original Hamiltonian \( H \) with \( m \) electrons, up to a constant which is the same for all \( m \).

Since the Hamiltonians \( \tilde{H}^{(2n)} \) and \( \tilde{H}^{(2n+2)} \) are related by particle-hole symmetry \( \varphi \), which sends \( \sigma^z \to -\sigma^z \) and \( \sigma^\pm \to \sigma^\mp \),

\[
\varphi \tilde{H}^{(2n+2)} \varphi = \tilde{H}^{(2n)} - \tilde{g}_{even},
\]

we need calculate only the ground state energies \( E_g^{(2n)} \) and \( E_g^{(2n+1)} \). We do this by exact diagonalization and form the combination

\[
\Delta_P = E_g^{(2n+1)} - \frac{1}{2} \left( E_g^{(2n)} + E_g^{(2n+2)} \right)
\]

\[
= E_g^{(2n+1)} - E_g^{(2n)} + \frac{\tilde{g}_{even}}{2}
\]

III. RESULTS

In Figure 1 we plot the results for intermediate values of \( \delta/\Delta \). The curves from top to bottom are the results for \( n = 3, 4, 5, \) and 6, respectively, and the straight line is the strong coupling asymptote. We find excellent convergence in the region near the minimum by \( n = 6 \). For constant level distribution, we find the minimum \( \Delta_P/\Delta \approx 0.7 \) occurs at \( \delta/\Delta \approx 0.9 \).

In Figure 2 we show the agreement with the weak coupling asymptote, where \( \Delta_p(\text{asymptotic}) \) is given by expression (11).

Note. As this manuscript was being completed, there appeared a manuscript by Mastelone et al. which obtains similar results.

ACKNOWLEDGEMENTS

We are grateful to D. C. Ralph and M. Tinkham for helpful conversations. This work was supported by the NSF through Grant No. DMR 94-16910, and through the Harvard Materials Research Science and Engineering Center, Grant No. DMR 94-00396.
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FIG. 1. The curves from the top to bottom are for $n = 3, 4, 5,$ and 6. They approach the strong coupling asymptote indicated by the straight line.
FIG. 2. The curves from top to bottom are for \( n = 3, 4, 5, \) and 6. All approach the weak coupling limit \( \Delta_p(\text{asymptotic}) \) given by expression (4b).