INTERPOLATION BETWEEN HILBERT SPACES

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Abstract. This note comprises a synthesis of certain results in the theory of exact interpolation between Hilbert spaces. In particular, we examine various characterizations of interpolation spaces and their relations to a number of results in operator-theory and in function-theory.

1. Interpolation theoretic notions

1.1. Interpolation norms. When $X, Y$ are normed spaces, we use the symbol $L(X; Y)$ to denote the totality of bounded linear maps $T : X \to Y$ with the operator norm

$$\| T \|_{L(X; Y)} = \sup \{ \| Tx \|_Y : \| x \|_X \leq 1 \}.$$  

When $X = Y$ we simply write $L(X)$.

Consider a pair of Hilbert spaces $H = (H_0, H_1)$ which is regular in the sense that $H_0 \cap H_1$ is dense in $H_0$ as well as in $H_1$. We assume that the pair is compatible, i.e., both $H_i$ are embedded in some common Hausdorff topological vector space $M$.

We define the $K$-functional \((1)\) for the couple $H$ by

$$K(t, x) = K(t, x; \overline{H}) = \inf_{x = x_0 + x_1} \{ \| x_0 \|_0^2 + t \| x_1 \|_1^2 \}, \quad t > 0, x \in M.$$  

The sum of the spaces $H_0$ and $H_1$ is defined to be the space consisting of all $x \in M$ such that the quantity $\| x \|_\Sigma := K(1, x)$ is finite; we denote this space by the symbols

$$\Sigma = \Sigma(\overline{H}) = H_0 + H_1.$$  

We shall soon see that $\Sigma$ is a Hilbert space (see Lemma 1.1). The intersection

$$\Delta = \Delta(\overline{H}) = H_0 \cap H_1$$

is a Hilbert space under the norm $\| x \|_\Delta^2 := \| x \|_0^2 + \| x \|_1^2$.

A map $T : \Sigma(\overline{H}) \to \Sigma(\overline{K})$ is called a couple map from $\overline{H}$ to $\overline{K}$ if the restriction of $T$ to $H_i$ maps $H_i$ boundedly into $K_i$ for $i = 0, 1$. We use the notations $T \in L(\overline{H}; \overline{K})$ or $T : \overline{H} \to \overline{K}$ to denote that $T$ is a couple map. It is easy to check that $L(\overline{H}; \overline{K})$ as a Banach space, when equipped with the norm

$$\| T \|_{L(\overline{H}; \overline{K})} := \max_{j=0,1} \{ \| T \|_{L(H_j; K_j)} \}. \quad (1.1)$$

If $\| T \|_{L(\overline{H}; \overline{K})} \leq 1$ we speak of a contraction from $\overline{H}$ to $\overline{K}$.

A Banach space $X$ such that $\Delta \subset X \subset \Sigma$ (continuous inclusions) is called intermediate with respect to the pair $\overline{H}$.

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1 More precisely, this is the quadratic version of the classical Peetre $K$-functional.
Let $X$, $Y$ be intermediate spaces with respect to couples $\mathcal{H}$, $\mathcal{K}$, respectively. We say that $X$, $Y$ are (relative) interpolation spaces if there is a constant $C$ such that $T : \mathcal{H} \to \mathcal{K}$ implies that $T : X \to Y$ and

\[(1.2) \quad \| T \|_{\mathcal{L}(X;Y)} \leq C \| T \|_{\mathcal{L}(\mathcal{H};\mathcal{K})}.\]

In the case when $C = 1$ we speak about exact interpolation. When $\mathcal{H} = \mathcal{K}$ and $X = Y$ we simply say that $X$ is an (exact) interpolation space with respect to $\mathcal{H}$.

Let $H$ be a suitable function of two positive variables and $X$, $Y$ spaces intermediate to the couples $\mathcal{H}$, $\mathcal{K}$, respectively. We say that the spaces $X$, $Y$ are of type $H$ (relative to $\mathcal{H}$, $\mathcal{K}$) if for any positive numbers $M_0$, $M_1$ we have

\[(1.3) \quad \| T \|_{\mathcal{L}(\mathcal{H};\mathcal{K})} \leq M_i, \quad i = 0, 1 \quad \text{implies} \quad \| T \|_{\mathcal{L}(X;Y)} \leq H(M_0, M_1).\]

The case $H(x, y) = \max\{x, y\}$ corresponds to exact interpolation, while $H(x, y) = x^{1-\theta} y^\theta$ corresponds to the convexity estimate

\[(1.4) \quad \| T \|_{\mathcal{L}(X;Y)} \leq \| T \|_{\mathcal{L}(\mathcal{H};\mathcal{K})} \| T \|_{\mathcal{L}(\mathcal{H};\mathcal{K})}^{\theta}.\]

In the situation of (1.4), one says that the interpolation spaces $X$, $Y$ are of exponent $\theta$ with respect to the couples $\mathcal{H}$, $\mathcal{K}$.

1.2. $K$-spaces. Given a regular Hilbert couple $\mathcal{H}$ and a positive Radon measure $\varrho$ on the compactified half-line $[0, \infty]$ we define an intermediate quadratic norm by

\[(1.5) \quad \| x \|_{*}^2 = \| x \|_{\varrho}^2 = \int_{[0, \infty]} \left(1 + t^{-1}\right) K(t, x; \mathcal{H}) \, d\varrho(t).\]

Here the integrand $k(t) = \left(1 + t^{-1}\right) K(t, x)$ is defined at the points 0 and $\infty$ by $k(0) = \| x \|_0^2$ and $k(\infty) = \| x \|_{\infty}^2$; we shall write $\mathcal{H}_{*}$ or $\mathcal{H}_{\varrho}$ for the Hilbert space defined by the norm (1.5).

Let $T \in \mathcal{L}(\mathcal{H};\mathcal{K})$ and suppose that $\| T \|_{\mathcal{L}(\mathcal{H};\mathcal{K})} \leq M_i$; then

\[(1.6) \quad K(t, T x; \mathcal{K}) \leq M_i^2 K \left(M_i^2 t/M_i^2, x; \mathcal{H}\right), \quad x \in \Sigma.\]

In particular, $M_i \leq 1$ for $i = 0, 1$ implies $\| T x \|_{\mathcal{K}_{\varrho}} \leq \| x \|_{\mathcal{H}_{\varrho}}$ for all $x \in \mathcal{H}_{\varrho}$. It follows that the spaces $\mathcal{H}_{\varrho}$, $\mathcal{K}_{\varrho}$ are exact interpolation spaces with respect to $\mathcal{H}$, $\mathcal{K}$.

**Geometric interpolation.** When the measure $\varrho$ is given by

\[d\varrho(t) = c_{\varrho} \frac{t^{-\theta}}{1 + t} \, dt, \quad c_{\varrho} = \frac{\pi}{\sin \theta \pi}, \quad 0 < \theta < 1,\]

we denote the norm (1.5) by

\[(1.7) \quad \| x \|_{\varrho}^2 := c_{\varrho} \int_0^\infty t^{-\theta} K(t, x) \frac{dt}{t}.\]

The corresponding space $\mathcal{H}_{\varrho}$ is easily seen to be of exponent $\theta$ with respect to $\mathcal{H}$.

In §3.1, we will recognize $\mathcal{H}_{\varrho}$ as the geometric interpolation space which has been studied independently by several authors, see [27, 40, 25].
1.3. Pick functions. Let $H$ be a regular Hilbert couple. The squared norm $\|x\|^2_1$ is a densely defined quadratic form in $H_0$, which we represent as

$$\|x\|^2_1 = \langle Ax, x \rangle_0 = \|A^{1/2}x\|^2_0$$

where $A$ is a densely defined, positive, injective (perhaps unbounded) operator in $H_0$. The domain of the positive square-root $A^{1/2}$ is $\Delta$.

Lemma 1.1. We have in terms of the functional calculus in $H_0$

$$K(t, x) = \left\langle \frac{tA}{1+tA} x, x \right\rangle_0, \quad t > 0. \tag{1.8}$$

In the formula (1.8), we have identified the bounded operator $\frac{tA}{1+tA}$ with its extension to $H_0$.

Proof. Fix $x \in \Delta$. By a straightforward convexity argument, there is a unique decomposition $x = x_{0,t} + x_{1,t}$ which is optimal in the sense that

$$K(t, x) = \|x_{0,t}\|^2_0 + t \|x_{1,t}\|^2_1. \tag{1.9}$$

It follows that $x_{i,t} \in \Delta$ for $i = 0, 1$. Moreover, for all $y \in \Delta$ we have

$$\frac{d}{de}\{\|x_{0,t} + ey\|^2_0 + t \|x_{1,t} - ey\|^2_1\}_{e=0} = 0,$$

i.e.,

$$\langle A^{-1/2}x_{0,t} - tA^{1/2}x_{1,t}, A^{1/2}y \rangle_0 = 0, \quad y \in \Delta.$$  

By regularity, we conclude that $A^{-1/2}x_{0,t} = tA^{1/2}x_{1,t}$, whence

$$x_{0,t} = \frac{tA}{1+tA} x \quad \text{and} \quad x_{1,t} = \frac{1}{1+tA} x. \tag{1.10}$$

(Note that the operators in (1.10) extend to bounded operators on $H_0$.) Inserting the relations (1.10) into (1.9), one finishes the proof of the lemma. \qed

Now fix a positive Radon measure $\varrho$ on $[0, \infty]$. The norm in the space $H_\varrho$ (see (1.5)) can be written

$$\|x\|^2_\varrho = \langle h(A)x, x \rangle_0, \tag{1.11}$$

where

$$h(\lambda) = \int_{[0, \infty]} \frac{(1+t)\lambda}{1+t\lambda} d\varrho(t). \tag{1.12}$$

The class of functions representable in this form for some positive Radon measure $\varrho$ is the class $P'$ of Pick functions, positive and regular on $\mathbb{R}_+$. Notice that for the definition (1.11) to make sense, we just need $h$ to be defined on $\sigma(A) \setminus \{0\}$, where $\sigma(A)$ is the spectrum of $A$. (The value $h(0)$ is irrelevant since $A$ is injective).

A calculus exercise shows that for the space $H_\theta$ (see (1.7)) we have

$$\|x\|^2_\theta = \langle A^\theta x, x \rangle_0. \tag{1.13}$$
1.4. Quadratic interpolation norms. Let $\mathcal{H}_*$ be any quadratic intermediate space relative to $\overline{\mathcal{H}}$. We write $$\| x \|_*^2 = \langle Bx, x \rangle_0$$ where $B$ is a positive injective operator in $\mathcal{H}_0$ (the domain of $B^{1/2}$ is $\Delta$).

For a map $T \in \mathcal{L}(\overline{\mathcal{H}})$ we shall often use the simplified notations $$\| T \| = \| T \|_{\mathcal{L}(\mathcal{H}_0)} \quad , \quad \| T \|_A = \| T \|_{\mathcal{L}(\mathcal{H}_1)} \quad , \quad \| T \|_B = \| T \|_{\mathcal{L}(\mathcal{H}_*)}.$$ The reader can check the identities $$\| T \|_A = \| A^{1/2}TA^{-1/2} \| \quad \text{and} \quad \| T \|_B = \| B^{1/2}TB^{-1/2} \|.$$ We shall refer to the following lemma as Donoghue’s lemma, cf. [14, Lemma 1].

**Lemma 1.2.** If $\mathcal{H}_*$ is exact interpolation with respect to $\overline{\mathcal{H}}$, then $B$ commutes with every projection which commutes with $A$ and $B = h(A)$ where $h$ is some positive Borel function on $\sigma(A)$.

**Proof.** For an orthogonal projection $E$ on $\mathcal{H}_0$, the condition $\| E \|_A \leq 1$ is equivalent to that $EAE \leq A$, i.e., that $E$ commute with $A$. The hypothesis that $\mathcal{H}_*$ be exact interpolation thus implies that every spectral projection of $A$ commutes with $B$. It now follows from von Neumann’s bicommutator theorem that $B = h(A)$ for some positive Borel function $h$ on $\sigma(A)$. \qed

In view of the lemma, the characterization of the exact quadratic interpolation norms of a given type $H$ reduces to the characterization of functions $h : \sigma(A) \to \mathbb{R}_+$ such that for all $T \in \mathcal{L}(\overline{\mathcal{H}})$

\begin{equation}
(1.14) \quad \| T \| \leq M_0 \quad \text{and} \quad \| T \|_A \leq M_1 \quad \Rightarrow \quad \| T \|_{h(A)} \leq H(M_0, M_1),
\end{equation}

or alternatively,

\begin{equation}
(1.15) \quad T^*T \leq M_0^2 \quad \text{and} \quad T^*AT \leq M_1^2 A \quad \Rightarrow \quad T^*h(A)T \leq H(M_0, M_1)^2 h(A).
\end{equation}

The set of functions $h : \sigma(A) \to \mathbb{R}_+$ satisfying these equivalent conditions forms a convex cone $C_{H,A};$ its elements are called interpolation functions of type $H$ relative to $A$. In the case when $H(x,y) = \max\{x,y\}$ we simply write $C_A$ for $C_{H,A}$ and speak of exact interpolation functions relative to $A$.

1.5. **Exact Calderón pairs and the $K$-property.** Given two intermediate normed spaces $Y$, $X$ relative to $\overline{\mathcal{H}}$, we say that they are (relatively) exact $K$-monotonic if the conditions $$x^0 \in X \quad \text{and} \quad K(t,y^0; \overline{\mathcal{H}}) \leq K(t,x^0; \overline{\mathcal{H}}), \quad t > 0$$ implies that $$y^0 \in Y \quad \text{and} \quad \| y^0 \|_Y \leq \| x^0 \|_X.$$

It is easy to see that exact $K$-monotonicity implies exact interpolation.

**Proof of this.** If $\| T \|_{\mathcal{L}(\overline{\mathcal{H}}, \overline{\mathcal{K}})} \leq 1$ then $\forall x, t$: $K(t,Tx; \overline{\mathcal{H}}) \leq K(t,x; \overline{\mathcal{K}})$ whence $\| Tx \|_Y \leq \| x \|_X$, by exact $K$-monotonicity. Hence $\| T \|_{\mathcal{L}(X,Y)} \leq 1$. \qed
Two pairs \( \mathcal{H}, \mathcal{K} \) are called exact relative Calderón pairs if any two exact interpolation (Banach-) spaces \( Y, X \) are exact \( K \)-monotonic. Thus, with respect to to exact Calderón pairs, exact interpolation is equivalent to exact \( K \)-monotonicity. The term "Calderón pair" was coined after thorough investigation of A. P. Calderón’s study of the pair \( (L_1, L_\infty) \), see [10] and [11].

In our present discussion, it is not convenient to work directly with the definition of exact Calderón pairs. Instead, we shall use the following, closely related notion.

Let \( x, y \in \Sigma(\mathcal{K}) \), then letting \( t, \) gives that \( \Sigma(\mathcal{K}) \), \( t > 0 \), then \( \Sigma(\mathcal{K}) \). We have shown that \( Y, X \) are exact \( K \)-monotonic.

In the diagonal case \( \mathcal{H} = \mathcal{K} \), we simply say that \( \mathcal{H} \) is an exact Calderón couple if for intermediate spaces \( Y, X \), the property of being exact interpolation is equivalent to being exact \( K \)-monotonic. Likewise, we say that \( \mathcal{H} \) has the \( K \)-property if the pair of couples \( \mathcal{H}, \mathcal{H} \) has that property.

Remark 1.4. For an operator \( T : \mathcal{K} \to \mathcal{H} \) to be a contraction, it is necessary and sufficient that

\[
K(t, x; \mathcal{H}) \leq K(t, x; \mathcal{K}), \quad x \in \Sigma(\mathcal{K}), \quad t > 0.
\]

Indeed, the necessity is immediate. To prove the sufficiency it suffices to observe that letting \( t \to \infty \) in (1.17) gives \( \|Tt\|_0 \leq \|tx\|_0 \), and dividing (1.17) by \( t \), and then letting \( t \to 0 \), gives that \( \|T \|_1 \leq \|x\|_1 \).

2. Mapping Properties of Hilbert Couples

2.1. Main results. We shall elaborate on the following main result from [2].

Theorem I. Any pair of regular Hilbert couples \( \mathcal{H}, \mathcal{K} \) has the relative \( K \)-property.

Before we come to the proof of Theorem I, we note some consequences of it. We first have the following corollary, which shows that a strong form of the \( K \)-property is true.

Corollary 2.1. Let \( \mathcal{H} \) be a regular Hilbert couple and \( x^0, y^0 \in \Sigma \) elements such that

\[
K(t, y^0) \leq M_0^2 K(M_1^2 t/M_0^2, x^0), \quad t > 0.
\]

Then

(i) There exists a map \( T \in \mathcal{L}(\mathcal{H}) \) such that \( T x^0 = y^0 \) and \( \|T \|_{\mathcal{L}(\mathcal{H})} \leq M_i, \)

for \( i = 0, 1 \).

(ii) If \( x^0 \in X \) where \( X \) is an interpolation space of type \( H \), then

\[
\|y^0\|_X \leq H(M_0, M_1) \|x^0\|_X.
\]
Proof: (i) Introduce a new couple $\overline{\mathcal{K}}$ by letting $\|x\|_{\mathcal{K}_i} = M_i \|x\|_{\mathcal{H}_i}$. The relation (2.1) then says that

$$K(t, y^0; \overline{\mathcal{H}}) \leq K(t, x^0; \overline{\mathcal{K}}), \quad t > 0.$$ 

By Theorem I there is a contraction $T: \mathcal{K} \to \mathcal{H}$ such that $Tx^0 = y^0$. It now suffices to note that $\|T\|_{\mathcal{L}(\mathcal{H}_i)} = M_i \|T\|_{\mathcal{L}(\mathcal{K}_i; \mathcal{H}_i)}$; (ii) then follows from Lemma 1.3. \hfill \Box

We next mention some equivalent versions of Theorem I, which uses the families of functionals $K_p$ and $E_p$ defined (for $p \geq 1$ and $t, s > 0$) via

$$K_p(t) = K_p(t, x) = K_p(t, x; \overline{\mathcal{H}}) = \inf_{x = x_0 + x_1} \{ \|x_0\|_0^p + t \|x_1\|_1^p \}$$

$$E_p(s) = E_p(s, x) = E_p(s, x; \overline{\mathcal{H}}) = \inf_{\|x_0\|_0^p \leq s} \{ \|x - x_0\|_1^p \}.$$ 

Note that $K = K_2$ and that $E_p(s) = E_1(s^{1/p})^p$; the $E$-functionals are used in approximation theory. One has that $E_p$ is decreasing and convex on $\mathbb{R}_+$ and that $K_p(t) = \inf_{s > 0} \{ s + tE_p(s) \}$, which means that $K_p$ is a kind of Legendre transform $E_p$. The inverse Legendre transformation takes the form

$$E_p(s) = \sup_{t > 0} \left\{ \frac{K_p(t)}{t} - s \right\}.$$ 

It is now immediate that, for all $x \in \Sigma(\mathcal{K})$ and $y \in \Sigma(\mathcal{H})$, we have

$$K_p(t, y) \leq K_p(t, x), \quad t > 0 \iff E_p(s, y) \leq E_p(s, x), \quad s > 0.$$ 

Since moreover $E_p(s) = E_2(s^{2/p})^{p/2}$, the conditions in (2.3) are equivalent to that $K(t, y) \leq K(t, x)$ for all $t > 0$. We have shown the following result.

**Corollary 2.2.** In Theorem I, one can substitute the $K$-functional for any of the functionals $K_p$ or $E_p$.

Define an exact interpolation norm $\|\cdot\|_{q,p}$ relative to $\overline{\mathcal{H}}$ by

$$\|x\|_{q,p} = \left( \int_{[0, \infty]} (1 + t^{-1}) K_p(t, x) d\varrho(t) \right)^{1/p}$$

where $\varrho$ is a positive Radon measure on $[0, \infty]$. This norm is non-quadratic when $p \neq 2$, but is of course equivalent to the quadratic norm corresponding to $p = 2$.

2.2. **Reduction to the diagonal case.** It is not hard to reduce the discussion of Theorem I to a diagonal situation.

**Lemma 2.3.** If the $K$-property holds for regular Hilbert couples in the diagonal case $\overline{\mathcal{H}} = \overline{\mathcal{K}}$, then it holds in general.

**Proof.** Fix elements $y^0 \in \Sigma(\mathcal{H}_1)$ and $x^0 \in \Sigma(\mathcal{K}_1)$ such that the inequality (1.16) holds. We must construct a map $T: \mathcal{K} \to \mathcal{H}$ such that $Tx^0 = y^0$ and $\|T\| \leq 1$.

To do this, we form the direct sum $\overline{\mathcal{S}} = (\mathcal{H}_0 \oplus \mathcal{K}_0, \mathcal{H}_1 \oplus \mathcal{K}_1)$. It is clear that $S_0 + S_1 = (\mathcal{H}_0 + \mathcal{K}_0, \mathcal{H}_1 + \mathcal{K}_1)$, and that

$$K(t, x \oplus y; \overline{\mathcal{S}}) = K(t, x; \overline{\mathcal{H}}) + K(t, y, \overline{\mathcal{K}}).$$

Then

$$K(t, 0 \oplus y^0; \overline{\mathcal{S}}) \leq K(t, x^0 \oplus 0; \overline{\mathcal{S}}).$$
Hence assuming that the couple \( S \) has the \( K \)-property, we can assert the existence of a map \( S \in L(S) \) such that \( S(x^0 \oplus 0) = 0 \oplus y^0 \) and \( \| S \| \leq 1 \). Letting \( P : S_0 + S_1 \to K_0 + K_1 \) be the orthogonal projection, the assignment \( Tx = PS(x \oplus 0) \) now defines a map such that \( Tx^0 = y^0 \) and \( \| T \|_{L(H, K)} \leq 1 \). □

2.3. The principal case. The core content of Theorem I is contained in the following statement.

Theorem 2.4. Suppose that a regular Hilbert couple \( \overline{H} \) is finite dimensional and that all eigenvalues of the corresponding operator \( A \) are of unit multiplicity. Then \( H \) has the \( K \)-property.

We shall settle for proving Lemma 2.4 in this section, postponing to Section 5 the general case of Theorem I.

To prepare for the proof, we write the eigenvalues \( \lambda_i \) of \( A \) in increasing order,
\[
\sigma(A) = \{\lambda_i\}_{i=1}^{n} \quad \text{where} \quad 0 < \lambda_1 < \cdots < \lambda_n.
\]
Let \( e_i \) be corresponding eigenvectors of unit length for the norm of \( H_0 \). Then for a vector \( x = \sum x_i e_i \) we have
\[
\| x \|_0^2 = \sum_{i=1}^{n} |x_i|^2 \quad \text{and} \quad \| x \|_1^2 = \sum_{i=1}^{n} \lambda_i |x_i|^2.
\]
Working in the coordinate system \( (e_i) \), the couple \( \overline{H} \) becomes identified with the \( n \)-dimensional weighted \( \ell_2 \) couple \( \overline{\ell}_2^n(\lambda) := (\ell_2^n, \ell_2^n(\lambda)) \),
where we write \( \lambda \) for the sequence \( (\lambda_i)_{i=1}^{n} \).

We will henceforth identify a vector \( x = \sum x_i e_i \) with the point \( x = (x_i)_{i=1}^{n} \) in \( \mathbb{C}^n \); accordingly, the space \( L(\ell_2^n) \) is identified with the \( C^* \)-algebra \( M_n(\mathbb{C}) \) of complex \( n \times n \) matrices.

It will be convenient to reparametrize the \( K \)-functional for the couple \( \overline{\ell}_2^n(\lambda) \) and write
\[
(2.4) \quad k_\lambda(t, x) := K \left( \frac{1}{t}, x; \overline{\ell}_2^n(\lambda) \right).
\]
By Lemma 1.1 we have
\[
(2.5) \quad k_\lambda(t, x) = \sum_{i=1}^{n} \frac{\lambda_i}{t + \lambda_i} |x_i|^2, \quad x \in \mathbb{C}^n.
\]

2.4. Basic reductions. To prove that the couple \( \overline{\ell}_2^n(\lambda) \) has the \( K \)-property, we introduce an auxiliary parameter \( \rho > 1 \). The exact value of \( \rho \) will change meaning during the course of the argument, the main point being that it can be chosen arbitrarily close to 1.

Initially, we pick any \( \rho > 1 \) such that \( \rho \lambda_i < \lambda_{i+1} \) for all \( i \); we assume also that we are given two elements \( x^0, y^0 \in \mathbb{C}^n \) such that
\[
(2.6) \quad k_\lambda(t, y^0) < \frac{1}{\rho} k_\lambda(t, x^0), \quad t \geq 0.
\]
We must construct a matrix \( T \in M_n(\mathbb{C}) \) such that
\[
(2.7) \quad Tx^0 = y^0 \quad \text{and} \quad k_\lambda(t, Tx) \leq k_\lambda(t, x), \quad x \in \mathbb{C}^n, \quad t > 0.
\]
Define $\tilde{x}^0 = ([x^0_i])^T$ and $\tilde{y}^0 = ([y^0_i])^T$ and suppose that

$$k_{\lambda}(t, \tilde{y}^0) < \frac{1}{\rho} k_{\lambda}(t, \tilde{x}^0), \quad t \geq 0.$$  

Suppose that we can find an operator $T_0 \in M_n(\mathbb{C})$ such that $T_0 \tilde{x}^0 = \tilde{y}^0$ and $k_{\lambda}(t, T_0 x) < k_{\lambda}(t, x)$ for all $x \in \mathbb{C}^n$ and $t > 0$. Writing $x_k^0 = e^{i\theta_k} \tilde{x}^0_k$ and $y_k^0 = e^{i\varphi_k} \tilde{y}^0_k$ where $\theta_k, \varphi_k \in \mathbb{R}$, we then have $Tx^0 = y^0$ and $k_{\lambda}(t, Tx) < k_{\lambda}(t, x)$ where

$$T = \text{diag}(e^{i\varphi_k}) T_0 \text{diag}(e^{-i\theta_k}).$$

Replacing $x^0, y^0$ by $\tilde{x}^0, \tilde{y}^0$ we can thus assume that the coordinates $x^0_i$ and $y^0_i$ are non-negative; replacing them by small perturbations if necessary, we can assume that they are strictly positive, at the expense of slightly diminishing the number $\rho$.

Now put $\beta_i = \lambda_i$ and $\alpha_i = \rho \lambda_i$. Our assumption on $\rho$ means that

$$0 < \beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n.$$  

Using the explicit expression for the $K$-functional, it is plain to check that

$$k_{\beta}(t, x) \leq k_{\alpha}(t, x) \leq \rho k_{\beta}(t, x), \quad x \in \mathbb{C}^n, t \geq 0.$$  

Our assumption (2.6) therefore implies that

$$k_{\alpha}(t, y^0) < k_{\beta}(t, x^0), \quad t \geq 0.$$  

We shall verify the existence of a matrix $T = T_\rho = T_{\rho, x^0, y^0}$ such that

$$Tx^0 = y^0 \quad \text{and} \quad k_{\alpha}(t, Tx) \leq k_{\beta}(t, x), \quad x \in \mathbb{C}^n, t > 0.$$  

It is clear by compactness that, as $\rho \downarrow 1$, the corresponding matrices $T_\rho$ will cluster at some point $T$ satisfying $Tx^0 = y^0$ and $\|T\|_K(\mathfrak{P}) \leq 1$. (See Remark 1.4.)

In conclusion, the proof of Theorem 2.4 will be complete when we can construct a matrix $T$ satisfying (2.9) with $\rho$ arbitrarily close to 1.

2.5. Construction of $T$. Let $\mathcal{P}_k$ denote the linear space of complex polynomials of degree at most $k$. We shall use the polynomials

$$L_{\alpha}(t) = \prod_{i=1}^{n} (t + \alpha_i), \quad L_{\beta}(t) = \prod_{i=1}^{n} (t + \beta_i),$$

and the product $L = L_{\alpha} L_{\beta}$. Notice that

$$L'(-\alpha_i) < 0, \quad L'(-\beta_i) > 0.$$  

Recalling the formula (2.5), it is clear that we can define a real polynomial $P \in \mathcal{P}_{2n-1}$ by

$$\frac{P(t)}{L(t)} = k_{\beta}(t, x^0) - k_{\alpha}(t, y^0).$$

Clearly $P(t) > 0$ when $t \geq 0$. Moreover, a consideration of the residues at the poles of the right hand member shows that $P$ is uniquely defined by the values

$$P(-\beta_i) = (x^0_i)^2 \beta_i L'(-\beta_i), \quad P(-\alpha_i) = -(y^0_i)^2 \alpha_i L'(-\alpha_i).$$  

Combining with (2.10), we conclude that

$$P(-\alpha_i) > 0 \quad \text{and} \quad P(-\beta_i) > 0.$$
Perturbing the problem slightly, it is clear that we can assume that $P$ has exact degree $2n - 1$, and that all zeros of $P$ have multiplicity 1. (We here diminish the value of $\rho > 1$ somewhat, if necessary.)

Now, $P$ has $2n - 1$ simple zeros, which we split according to

$$P^{-1}(0) = \{-r_i\}_{i=1}^{2m-1} \cup \{-c_i,-\overline{c_i}\}_{i=1}^{n-m},$$

where the $r_i$ are positive and the $c_i$ are non-real, and chosen to have positive imaginary parts. The following is the key observation.

**Lemma 2.5.** We have that

$$L'(-\beta_i)P(-\beta_i) > 0 \quad \text{and} \quad L'(-\alpha_i)P(-\alpha_i) < 0$$

and there is a splitting $\{r_i\}_{i=1}^{2m-1} = \{\delta_i\}_{i=1}^{m} \cup \{\gamma_i\}_{i=1}^{m-1}$ such that

$$L(\delta_j)P'(\delta_j) > 0 \quad \text{and} \quad L(\gamma_k)P'(\gamma_k) < 0.$$

**Proof.** The inequalities (2.14) follow immediately from (2.13) and (2.10). It remains to prove (2.15).

Let $-h$ denote the leftmost real zero of the polynomial $LP$ (of degree $4n - 1$). We claim that $P(-h) = 0$. If this were not the case, we would have $h = \alpha_n$. Since the degree of $P$ is odd, $P(-t)$ is negative for large values of $t$, and so $P(-\alpha_n) < 0$ contradicting (2.13). We have shown that $P(-h) = 0$. Since all zeros of $LP$ have multiplicity 1, we have $(LP)'(-h) \neq 0$, whence

$$LP(-h)P'(-h) = (LP)'(-h) > 0.$$

We write $\delta_m = h$ and put $P_*(t) = P(t)/(t + \delta_m)$. Since $t + \delta_m > 0$ for $t \in \{-\alpha_i,-\beta_i\}_i^n$, we have by (2.13) that for all $i$

$$P_*(-\alpha_i) > 0 \quad \text{and} \quad P_*(-\beta_i) > 0.$$

Denote by $\{-r_i^*\}_{i=1}^{2m-2}$ the real zeros of $P_*$ since the degree of $LP_*$ is even and the polynomial $(LP_*)'$ has alternating signs in the set $\{-\alpha_i,-\beta_i\}_i^n \cup \{-r_i^*\}_{i=1}^{2m-2}$, we can split the zeros of $P_*$ as $\{-\delta_i,-\gamma_i\}_{i=1}^{m-1}$, where

$$L(-\delta_i)P'_*(\delta_i) > 0 \quad \text{and} \quad L(-\gamma_i)P'_*(\gamma_i) < 0.$$

Since $P'(r_j^*) = (\delta_m - r_j^*)P_*(r_j^*)$ and $\delta_m > r_j^*$, the signs of $P'(r_j^*)$ and $P'_*(r_j^*)$ are equal, proving (2.15). \hfill \square

Recall that $\{-c_i\}_{i=1}^{n-m}$ denote the zeros of $P$ such that $\text{Im} c_i > 0$. We put (with the convention that an empty product equals 1)

$$L_\delta(t) = \prod_{i=1}^{m}(t + \delta_i) \quad \text{and} \quad L_\gamma(t) = \prod_{i=1}^{m-1}(t + \gamma_i) \quad \text{and} \quad L_c(t) = \prod_{i=1}^{n-m}(t + c_i).$$

We define a linear map $F : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^{n+m-1}$ in the following way. First define a subspace $U \subset P_{2n-1}$ by

$$U = \{ L_\delta q : q \in P_{n+m-1} \}.$$

Notice that $U$ has dimension $n + m - 1$ and that $P \in U$; in fact $P = aL_\delta L_\gamma L_\delta$ where $a$ is the leading coefficient and the $*$-operation is defined by $L^*(z) = \overline{L(z)}$. 

For a polynomial \( Q \in U \) we have

\[
\frac{|Q(t)|^2}{L(t)P(t)} = \sum_{i=1}^{n} |x_i|^2 \frac{\beta_i}{t + \beta_i} + \sum_{i=1}^{n} |x_i'|^2 \frac{\delta_i}{t + \delta_i} - \sum_{i=1}^{n} |y_i|^2 \frac{\alpha_i}{t + \alpha_i} - \sum_{i=1}^{m-1} |y_i'|^2 \frac{\gamma_i}{t + \gamma_i},
\]

where, for definiteness,

\[
\begin{align*}
(2.18) & \quad x_i = \frac{Q(-\beta_i)}{\sqrt{\beta_i L(-\beta_i)P(-\beta_i)}}; \quad x_i' = \frac{Q(-\delta_i)}{\sqrt{\delta_i L(-\delta_i)P(-\delta_i)}} \\
(2.19) & \quad y_i = \frac{Q(-\alpha_i)}{\sqrt{-\alpha_i L(-\alpha_i)P(-\alpha_i)}}; \quad y_i' = \frac{Q(-\gamma_i)}{\sqrt{-\gamma_i L(-\gamma_i)P(-\gamma_i)}}.
\end{align*}
\]

The identities in (2.18) give rise to a linear map

\[
(2.20) \quad M : \mathbb{C}^n \oplus \mathbb{C}^m \to U ; \quad [x; x'] \mapsto Q.
\]

We can similarly regard (2.19) as a linear map

\[
(2.21) \quad N : U \to \mathbb{C}^n \oplus \mathbb{C}^{m-1} ; \quad Q \mapsto [y; y'].
\]

Our desired map \( F \) is defined as the composite

\[
F = NM : \mathbb{C}^n \oplus \mathbb{C}^m \to \mathbb{C}^n \oplus \mathbb{C}^{m-1} ; \quad [x; x'] \mapsto [y; y'].
\]

Notice that if \( Q = M [x; x'] \) and \( [y; y'] = F [x; x'] \) then (2.17) means that

\[
k_{\beta \oplus \delta} (t, [x; x']) - k_{\alpha \oplus \gamma} (t, F [x; x']) = \frac{|Q(t)|^2}{L(t)P(t)} \geq 0, \quad t \geq 0.
\]

This implies that \( F \) is a contraction from \( \ell_2^{n+m}(\beta \oplus \delta) \) to \( \ell_2^{n+m-1}(\alpha \oplus \gamma) \).

We now define \( T \) as a "compression" of \( F \). Namely, let \( E : \mathbb{C}^n \oplus \mathbb{C}^{m-1} \to \mathbb{C}^n \) be the projection onto the first \( n \) coordinates, and define an operator \( T \) on \( \mathbb{C}^n \) by

\[
Tx = E F [x; 0], \quad x \in \mathbb{C}^n.
\]

Taking \( Q = P \) in (2.17) we see that \( Tx^0 = y^0 \). Moreover,

\[
k_{\beta} (t, x) - k_{\alpha} (t, Tx) = \sum_{i=1}^{n} |x_i|^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^{n} |y_i|^2 \frac{\alpha_i}{t + \alpha_i} \\
\geq \sum_{i=1}^{n} |x_i|^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^{n} |y_i|^2 \frac{\alpha_i}{t + \alpha_i} - \sum_{j=1}^{m-1} |y_i'|^2 \frac{\gamma_i}{t + \gamma_i} \\
= k_{\beta \oplus \delta} (t, [x; 0]) - k_{\alpha \oplus \gamma} (t, F [x; 0]) = \frac{|Q(t)|^2}{L(t)P(t)}.
\]

Since the right hand side is non-negative, we have shown that

\[
k_{\alpha} (t, Tx) \leq k_{\beta} (t, x), \quad t > 0, \ x \in \mathbb{C}^n,
\]

as desired. The proof of Theorem 2.4 is finished. q.e.d.
2.6. **Real scalars.** Theorem 2.4 holds also in the case of Euclidean spaces over the real scalar field. To see this, assume without loss of generality that the vectors $x^0, y^0 \in \mathbb{C}^n$ have real entries (still satisfying $k_\lambda(t, y^0) \leq k_\lambda(t, x^0)$ for all $t > 0$).

By Theorem 2.4 we can find a (complex) contraction $T$ of $\ell^2 \mathbb{C}$ such that $Tx^0 = y^0$. It is clear that the operator $T^* \mathbb{R} = T(\bar{x})$ satisfies those same conditions. Replacing $T$ by $\frac{1}{2}(T + T^*)$ we obtain a real matrix $T \in M_n(\mathbb{R})$, which is a contraction of $\ell^2 \mathbb{R}$ and maps $x^0$ to $y^0$. \(\square\)

2.7. **Explicit representations.** We here deduce an explicit representation for the operator $T$ constructed above.

Let $x^0$ and $y^0$ be two non-negative vectors such that

$$k_\lambda(t, y^0) \leq k_\lambda(t, x^0), \quad t > 0.$$ 

For small $\rho > 0$ we perturb $x^0, y^0$ slightly to vectors $\tilde{x}^0, \tilde{y}^0$ which satisfy the conditions imposed the previous subsections. We can then construct a matrix $T = T_\rho$ such that

$$T\tilde{x}^0 = \tilde{y}^0 \quad \text{and} \quad k_\alpha(t, Tx) \leq k_\beta(t, x), \quad t > 0, x \in \mathbb{C}^n,$$

where $\beta = \rho \lambda$ and $\alpha = \rho \lambda$. As $\rho \to 0$, $\tilde{x}^0, \tilde{y}^0$ approaches $1, x^0$, resp. $y^0$, it is clear that any cluster point $T$ of the set of contractions $T_\rho$ will satisfy

$$Tx^0 = y^0 \quad \text{and} \quad k_\lambda(t, Tx) \leq k_\lambda(t, x), \quad t > 0, x \in \mathbb{C}^n.$$

**Theorem 2.6.** The matrix $T = T_\rho = (\tau_{ilk})^n_{k=1}$ where

$$\tau_{ilk} = \text{Re} \left[ \frac{1}{\alpha_i - \beta_k} \frac{\beta_k L_\delta(-\alpha_i)L_{\bar{c}}(-\alpha_i)L_\alpha(-\beta_k)}{\alpha_i L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)L_\alpha(-\alpha_i)} \right] \quad (2.23)$$

satisfies (2.22).

**Proof.** The range of the map $\mathbb{C}^n \to U, x \mapsto M[x; 0]$ (see 2.20) is precisely the $n$-dimensional subspace

$$V := L_\delta L_\epsilon \cdot \mathcal{P}_{n-1} \cup \{L_\delta L_\epsilon R; R \in \mathcal{P}_{n-1}\} \subset U.$$

We introduce a basis $(Q_k)^n_{k=1}$ for $V$ by

$$Q_k(t) = \frac{L_\delta(t)L_{\bar{c}}(t)L_\beta(t)}{t + \beta_k} \frac{\sqrt{\beta_k L_\delta(-\beta_k)L_\alpha(-\beta_k)}}{L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)L_\alpha(-\beta_k)}.$$

Then

$$\frac{Q_k(-\beta_i)}{\sqrt{\beta_i L_\delta(-\beta_i)L_\alpha(-\beta_i)}} = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

Denoting by $(e_i)$ the canonical basis in $\mathbb{C}^n$ and using (2.18), (2.19) we get

$$\tau_{ik} = (T e_k)_i = \frac{Q_k(-\alpha_i)}{\sqrt{\alpha_i L_\delta(-\alpha_i)L_\alpha(-\alpha_i)}} \frac{\beta_k L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)}{L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)L_\alpha(-\beta_k)} \left( \frac{\beta_k L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)}{L_\delta(-\beta_k)L_{\bar{c}}(-\beta_k)L_\alpha(-\beta_k)} \right)^{1/2}.$$

Inserting the expressions (2.12) for $P(-\alpha_i)$ and $P(-\beta_k)$ and taking real parts (see the remarks in §2.6), we obtain the formula (2.23). \(\square\)
Remark 2.7. It is easy to see that, if we pick all matrix-elements real, some elements \( \tau_{ik} \) of the matrix \( T \) in (2.23) will be negative, even while the numbers \( x_i^0 \) and \( y_i^0 \) are positive. It was proved in [2], Theorem 2.3, that this is necessarily so. Indeed, one there constructs an example of a five-dimensional couple \( \ell_2^5(\lambda) \) and two vectors \( x^0, y^0 \in \mathbb{R}^5 \) having non-negative entries such that no contraction \( T = (\tau_{ik})_{i,k=1}^5 \) on \( \ell_2^5(\lambda) \) having all matrix entries \( \tau_{ik} \geq 0 \) can satisfy \( T x^0 = y^0 \). On the other hand, if one settles for using a matrix with \( \| T \| \leq \sqrt{2} \), then it is possible to find one with only non-negative matrix entries. Indeed, such a matrix was used by Sedaev [35], see also [39].

2.8. On sharpness of the norm-bounds. We shall show that if \( m < n \) (i.e. if the polynomial \( P \) has at least one non-real zero), then the norm \( \| T \|_{\mathcal{L}(\mathcal{H}_i)} \) of the contraction \( T \) constructed above is very close to 1 for \( i = 0, 1 \).

We first claim that \( \| T \|_{\mathcal{L}(\mathcal{H}_0)} = 1 \). To see this, we notice that if \( m < n \), then there is a non-trivial polynomial \( Q^{(1)} \) in the space \( V \) (see (2.24)) which vanishes at the points \( 0, \gamma_1, \ldots, \gamma_m \). If \( x_i^{(1)} \) and \( y_i^{(1)} \) are defined by the formulas (2.18) and (2.19) (while \( (x_i^{(1)})' = (y_i^{(1)})' = 0 \)), we then have \( T x_i^{(1)} = y_i^{(1)} \) and

\[
k_\beta(t, x^{(1)}) - k_\alpha(t, y^{(1)}) = \frac{|Q^{(1)}_i(t)|^2}{L(t)P(t)}, \quad t > 0.
\]

Choosing \( t = 0 \) we conclude that \( \| x^{(1)} \|_{\ell_2^2}^2 - \| T x^{(1)} \|_{\ell_2^2}^2 = 0 \), whence \( \| T \|_{\mathcal{L}(\mathcal{H}_0)} \geq 1 \), proving our claim.

Similarly, the condition \( m < n \) implies the existence of a polynomial \( Q^{(2)} \in V \) of degree at most \( n + m - 2 \) vanishing at the points \( \gamma_1, \ldots, \gamma_{m-1} \). Constructing vectors \( x^{(2)}, y^{(2)} \) via (2.18) and (2.19) we will have \( T x^{(2)} = y^{(2)} \) and

\[
k_\beta(t, x^{(2)}) - k_\alpha(t, y^{(2)}) = \frac{|Q^{(2)}_i(t)|^2}{L(t)P(t)}, \quad t > 0.
\]

Multiplying this relation by \( t \) and then sending \( t \to \infty \), we find that \( \| x^{(2)} \|_{\ell_2^2(\beta)}^2 - \| T x^{(2)} \|_{\ell_2^2(\alpha)}^2 = 0 \), which implies \( \| T \|_{\mathcal{L}(\mathcal{H}_i)} \geq \rho^{-1/2} \).

2.9. A remark on weighted \( \ell_p \)-couples. As far as we are aware, if \( 1 < p < \infty \) and \( p \neq 2 \), it is still an open question wether the couple \( \ell_2^p(\lambda) = (\ell_2^p, \ell_2^p(\lambda)) \) is an exact Calderón couple or not. (When \( p = 1 \) or \( p = \infty \) it is exact Calderón; see [36] for the case \( p = 1 \); the case \( p = \infty \) is essentially just the Hahn-Banach theorem.)

It is well-known, and easy to prove, that the \( K_p \)-functional (see (2.2)) corresponding to the couple \( \ell_2^p(\lambda) \) is given by the explicit formula

\[
K_p(\lambda; \ell_2^p(\lambda)) = \sum_{i=1}^n |x_i|^p \frac{t\lambda_i}{(1 + (t\lambda_i)^{1-p})^{p-1}}.
\]

It was proved by Sedaev [35] (cf. [39]) that if \( K_p(\lambda; \ell_2^p(\lambda)) \leq K_p(\lambda; \ell_2^p(\lambda)) \) for all \( t > 0 \) then there is \( T : \ell_2^p(\lambda) \to \ell_2^p(\lambda) \) of norm at most \( 2^{1/p'} \) such that \( T x^0 = y^0 \). (Here \( p' \) is the exponent conjugate to \( p \).)
Although our present estimates are particular for the case $p = 2$, our construction still shows that, if we re-define $P(t)$ to be the polynomial

$$\frac{P(t)}{L(t)} = \sum_{i=1}^{n} \frac{(\tilde{x}^0_i)^p \beta_i}{t + \beta_i} - \sum_{i=1}^{n} \frac{(\tilde{y}^0_i)^p \alpha_i}{t + \alpha_i},$$

then the matrix $T$ defined by

$$\tau_{ik} = \text{Re} \left[ \frac{1}{\alpha_i - \beta_k} \frac{(\tilde{x}^0_i)^{p-1} \beta_k L_\delta(-\alpha_i) L_c(-\alpha_i) L_\alpha(-\beta_k)}{(\tilde{y}^0_i)^{p-1} \alpha_i L_\delta(-\beta_k) L_c(-\beta_k) L_\alpha(-\alpha_i)} \right]$$

will satisfy $T \tilde{x}^0 = \tilde{y}^0$, at least, provided that $P(t) > 0$ when $t \geq 0$. (Here $L_\delta$ and $L_c$ are constructed from the zeros of $P$ as in the case $p = 2$.)

The matrix (2.26) differs from those used by Sedaev [35] and Sparr [39]. Indeed the matrices from [35, 39] have non-negative entries, while this is not so for the matrices (2.26). It seems to be an interesting problem to estimate the norm $\|T\|_{\ell(np)}(\lambda)$ for the matrix (2.26), when $p \neq 2$. The motivation for this type of question is somewhat elaborated in §6.7, but we shall not discuss it further here.

2.10. A comparison with Löwner’s matrix. In this subsection, we briefly explain how our our matrix $T$ is related to the matrix used by Löwner [26] in his original work on monotone matrix functions. (2)

We shall presently display four kinds of partial isometries; Löwner’s matrix will be recognized as one of them. In all cases, operators with the required properties can alternatively be found using the more general construction in Theorem 2.4. The following discussion was inspired by the earlier work of Sparr [38], who seems to have been the first to note that Löwner’s matrix could be constructed in a similar way.

In this subsection, scalars are assumed to be real. In particular, when we write "$\ell^2_n$" we mean the (real) Euclidean $n$-dimensional space.

Suppose that two vectors $x^0, y^0 \in \mathbb{R}^n$ satisfy

$$k_\lambda(t, y^0) \leq k_\lambda(t, x^0), \quad t > 0.$$  

Let

$$L_\lambda(t) = \prod_{i=1}^{n} (t + \lambda_i),$$

and let $P \in \mathcal{P}_{n-1}$ be the polynomial fulfilling

$$\frac{P(t)}{L_\lambda(t)} = k_\lambda(t, x^0) - k_\lambda(t, y^0) = \sum_{i=1}^{n} \frac{\lambda_i}{t + \lambda_i} \left[ (x^0_i)^2 - (y^0_i)^2 \right].$$

By assumption, $P(t) \geq 0$ for $t \geq 0$. Moreover, $P$ is uniquely determined by the $n$ conditions

$$P(-\lambda_i) = \frac{(x^0_i)^2 - (y^0_i)^2}{\lambda_i L_\lambda^\prime(-\lambda_i)}.$$  

Let $u_1, v_1, u_2, v_2, \ldots$ denote the canonical basis of $\ell^2_n$ and let

$$\ell^2_n = O \oplus E$$

---

2By "Löwner’s matrix", we mean the unitary matrix denoted "$V$" in Donoghue’s book [12], on p. 71. A more explicit construction of this matrix is found in [26], where it is called "$T$".
be the corresponding splitting, i.e.,

\[ O = \text{span} \{ u_i \} \quad , \quad E = \text{span} \{ v_i \}. \]

Notice that

\[ \dim O = \lfloor (n-1)/2 \rfloor + 1 \quad , \quad \dim E = \lfloor (n-2)/2 \rfloor + 1, \]

where \( \lfloor x \rfloor \) is the integer part of a real number \( x \).

We shall construct matrices \( T \in M_n(\mathbb{R}) \) such that

\[ (2.27) \quad Tx^0 = y^0 \quad \text{and} \quad k_\lambda(t, Tx) \leq k_\lambda(t, x), \quad t > 0, \ x \in \mathbb{R}^n, \]

for some choice of signs.

\[ \text{Case 1.} \quad \text{Suppose that} \quad \lambda_i = \xi_i \quad \text{when} \quad i \text{ is odd and} \quad \lambda_i = \eta_i \quad \text{when} \quad i \text{ is even.} \]

We also write

\[ L_\xi(t) = \prod_{i \text{ odd}} (t + \xi_i) \quad , \quad L_\eta(t) = \prod_{i \text{ even}} (t + \eta_i), \]

and write \( L = L_\xi L_\eta. \) Notice that \( L'_\lambda(-\xi_i) > 0 \) and \( L'_\lambda(-\eta_i) < 0. \)

To proceed with the solution, we rename the \( \lambda_i \) as \( \lambda_i = \xi_i \) when \( i \) is odd and \( \lambda_i = \eta_i \) when \( i \) is even. We also write

\[ q(t)^2 \quad L_\lambda(t) = \sum_{k \text{ odd}} \frac{\xi_k}{t+\xi_k} \left( x_k^0 \right)^2 - \sum_{k \text{ even}} \frac{\eta_k}{t+\eta_k} \left( y_k^0 \right)^2, \]

where

\[ (2.28) \quad x_k^0 = \frac{\epsilon_k q(-\xi_k)}{\sqrt{\xi_k L'_\lambda(-\xi_k)}} \quad , \quad y_k^0 = \frac{\zeta_k q(-\eta_k)}{\sqrt{-\eta_k L'_\lambda(-\eta_k)}} \]

for some choice of signs \( \epsilon_k, \zeta_k \in \{ \pm 1 \}. \)

By (2.28) are defined linear maps

\[ O \to \mathcal{P}_{(n-1)/2}(\mathbb{R}) : \ x \mapsto Q \quad ; \quad \mathcal{P}_{(n-1)/2}(\mathbb{R}) \to E : \ Q \mapsto y. \]

The composition is a linear map

\[ T_0 : O \to E : \ x \mapsto y. \]

We now define \( T \in M_n(\mathbb{R}) \) by

\[ T : O + E \to O + E : \ [x; v] \mapsto [0; T_0 x]. \]

Then clearly \( Tx^0 = y^0 \) and

\[ k_\lambda(t, [x; v]) - k_\lambda(t, T[x; v]) \]

\[ \geq k_\xi(t, x) - k_\eta(t, T_0 x) \]

\[ = \frac{Q(t)^2}{L_\lambda(t)} \geq 0, \quad t > 0, \ x \in O, \ v \in E. \]
We have verified (2.27) in case 1. A computation similar to the one in the proof of Theorem 2.6 shows that, with respect to the bases \( u_k \) and \( v_i \),

\[
(T_0)_{ik} = \frac{\varepsilon_k \zeta_i}{\xi_k - \eta_i} \left( L_{-\xi_i} \left( \xi_k L'_{-\xi_k} L_{-\eta_i} \right) \right)^{1/2}.
\]

Notice that, multiplying (2.29) by \( t \), then letting \( t \to \infty \) implies that

\[
\sum_{k \text{ odd}} x_k^2 \xi_k - \sum_{i \text{ even}} (T_0 x_i)_{k}^2 \eta_i = 0.
\]

This means that \( T \) is a partial isometry from \( O \) to \( E \) with respect to the norm of \( \ell_2^O(\lambda) \).

**Case 2.** Now assume that \( P(t) = tq(t)^2 \), \( q \in \mathcal{P}_{(n-2)/2}(\mathbb{R}) \), \( x^0 \in E \), and \( y^0 \in O \). Then

\[
\frac{tq(t)^2}{L_\lambda(t)} = -\sum_{i \text{ odd}} (y_i^0)^2 \frac{\xi_i}{t + \xi_i} + \sum_{k \text{ even}} \frac{\eta_k}{t + \eta_k} (x_k^0)^2,
\]

where

\[
y_i^0 = \frac{\varepsilon_i' q(-\xi_i)}{\sqrt{L'_\lambda(-\xi_i)}} \quad , \quad x_k^0 = \frac{-\zeta_k' q(-\eta_k)}{\sqrt{-L'_\lambda(-\eta_k)}}
\]

for some \( \varepsilon_i', \zeta_k' \in \{ \pm 1 \} \).

By (2.30) are defined linear maps

\[ E \to \mathcal{P}_{(n-2)/2}(\mathbb{R}) : x \mapsto Q : \mathcal{P}_{(n-2)/2}(\mathbb{R}) \to O : Q \mapsto y. \]

We denote their composite by

\[ T_1 : E \to O : x \mapsto y. \]

Define \( T \in M_n(\mathbb{R}) \) by

\[ T : O \oplus E \to O \oplus E : [u; x] \mapsto [T_1 x; 0]. \]

We then have

\[
-k_\lambda (t, T[u; x]) + k_\lambda (t, [u; x]) \geq -k_\xi (t, T_1 x) + k_\eta (t, x) = \frac{tQ(t)^2}{L_\lambda(t)} \geq 0, \quad t > 0, \quad u \in O, \quad x \in E,
\]

and (2.27) is verified also in case 2.

A computation shows that, with respect to the bases \( v_k \) and \( u_i \),

\[
(T_1)_{ik} = \frac{\varepsilon_k' \zeta_i}{\eta_k - \xi_i} \frac{L_{-\xi_i}(-\eta_k) L'_{-\xi_k} L_{-\eta_i}(-\eta_k)}{L'_{-\xi_k} L_{-\eta_i}(-\eta_k)} \left( \frac{L_{-\xi_i}(-\eta_k) L'_{-\xi_k} L_{-\eta_i}(-\eta_k)}{L'_{-\xi_k} L_{-\eta_i}(-\eta_k)} \right)^{1/2}.
\]

Inserting \( t = 0 \) in (2.31) we find that

\[
-\sum_{i \text{ odd}} (T_1 x_i)^2 + \sum_{k \text{ even}} (x_k)^2 = 0,
\]

i.e., \( T \) is a partial isometry from \( E \) to \( O \) with respect to the norm of \( \ell_2^O \).

In the case of even \( n \), the matrix \( T_1 \) coincides with Löwner’s matrix.
3. Quadratic interpolation spaces

3.1. A classification of quadratic interpolation spaces. Recall that an intermediate space \( X \) with respect to \( H \) is said to be of type \( H \) if \( \| T \|_{\mathcal{L}(H_i)} \leq M_i \) for \( i = 0, 1 \) implies that \( \| T \|_{\mathcal{L}(X)} \leq H(M_0, M_1) \). We shall henceforth make a mild restriction, and assume that \( H \) be homogeneous of degree one. This means that we can write

\[
H(s, t)^2 = s^2 H(t^2/s^2)
\]

for some function \( H \) of one positive variable. In this situation, we will say that \( X \) is of type \( H \). The definition is chosen so that the estimates \( \| T \|_{\mathcal{L}(H_i)}^2 \leq M_i \) for \( i = 0, 1 \) imply \( \| T \|_{\mathcal{L}(X)} \leq M_0 H(M_1/M_0) \).

In the following we will make the standing assumptions: \( H \) is an increasing, continuous, and positive function on \( \mathbb{R}_+ \) with \( H(1) = 1 \) and \( H(t) \leq \max\{1, t\} \).

Notice that our assumptions imply that all spaces of type \( H \) are exact interpolation. Note also that \( H(t) = t^\theta \) corresponds to geometric interpolation of exponent \( \theta \).

Suppose now that \( \overline{H} \) is a regular Hilbert couple and that \( \mathcal{H}_s \) is an exact interpolation space with corresponding operator \( B \). By Donoghue’s lemma, we have that \( B = h(A) \) for some positive Borel function \( h \) on \( \sigma(A) \).

The statement that \( \mathcal{H}_s \) is intermediate relative to \( \overline{H} \) is equivalent to that

\[
c_1 \frac{A}{1+A} \leq B \leq c_2(1 + A)
\]

for some positive numbers \( c_1 \) and \( c_2 \).

Let us momentarily assume that \( \mathcal{H}_0 \) be separable. (This restriction is removed in Remark 3.1.) We can then define the scalar-valued spectral measure of \( A \),

\[
\nu_A(\omega) = \sum 2^{-k} \langle E(\omega)e_k, e_k \rangle_0
\]

where \( E \) is the spectral measure of \( A \), \( \{e_k; k = 1, 2, \ldots\} \) is an orthonormal basis for \( \mathcal{H}_0 \), and \( \omega \) is a Borel set. Then, for Borel functions \( h_0, h_1 \) on \( \sigma(A) \), one has that \( h_1 = h_2 \) almost everywhere with respect to \( \nu_A \) if and only if \( h_1(A) = h_2(A) \).

Note that the regularity of \( \overline{H} \) means that \( \nu_A(\{0\}) = 0 \).

**Theorem II.** If \( \mathcal{H}_s \) is of type \( H \) with respect to \( \overline{H} \), then \( B = h(A) \) where the function \( h \) can be modified on a null-set with respect to \( \nu_A \) so that

\[
h(\lambda)/h(\mu) \leq H(\lambda/\mu), \quad \lambda, \mu \in \sigma(A) \setminus \{0\}.
\]

**Proof.** Fix a (large) compact subset \( K \subset \sigma(A) \cap \mathbb{R}_+ \) and put \( \mathcal{H}_0 = \mathcal{H}_1 = E_K(\mathcal{H}_0) \) where \( E \) is the spectral measure of \( A \), and the norms are defined by restriction,

\[
\| x \|_{\mathcal{H}_i} = \| x \|_{\mathcal{H}}, \quad \| x \|_{\mathcal{H}_i} = \| x \|_{\mathcal{H}_s}, \quad x \in E_K(\mathcal{H}_0).
\]

It is clear that the operator \( A' \) corresponding to \( \overline{H}' \) is the compression of \( A \) to \( \mathcal{H}_0 \) and likewise the operator \( B' \) corresponding to \( \mathcal{H}_s' \) is the compression of \( B \) to \( \mathcal{H}_0 \). Moreover, \( \mathcal{H}_s' \) is of interpolation type \( H \) with respect to \( \overline{H}' \) and the operator \( B' = (h|_K)(A') \). For this reason, and since the compact set \( K \) is arbitrary, it clearly suffices to prove the statement with \( \overline{H} \) replaced by \( \overline{H}' \). Then \( A \) is bounded above and below. Moreover, by (3.2), also \( B \) is bounded above and below.
Let $c < 1$ be a positive number such that $\sigma(A) \subset (c, c^{-1})$. For a fixed $\varepsilon > 0$ with $\varepsilon < c/2$ we set

$$E_\varepsilon = \sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon)$$

and consider the functions

$$m_\varepsilon(\lambda) = \text{ess inf}_{E_\varepsilon} h, \quad M_\varepsilon(\lambda) = \text{ess sup}_{E_\varepsilon} h,$$

the essential inf and sup being taken with respect to $\nu_A$.

Now fix a small positive number $\varepsilon'$ and two unit vectors $e_\lambda, e_\mu$ supported by $E_\lambda, E_\mu$ respectively, such that

$$\|e_\lambda\|_*^2 \geq M_\varepsilon(\lambda) - \varepsilon', \quad \|e_\mu\|_*^2 \leq m_\varepsilon(\mu) + \varepsilon'.$$

Now fix $\lambda, \mu \in \sigma(A)$ and let $Tx = \langle x, e_\mu \rangle_0 e_\lambda$. Then

$$\|Tx\|_0^2 = |\langle x, e_\mu \rangle_0|^2 \|e_\lambda\|_1^2 \leq \frac{1}{(\mu - \varepsilon)^2} \|\langle x, e_\mu \rangle_1\|^2 (\lambda + \varepsilon)$$

$$\leq \frac{(\mu + \varepsilon)(\lambda + \varepsilon)}{(\mu - \varepsilon)^2} \|x\|_1^2.$$ 

Likewise,

$$\|Tx\|_0^2 \leq |\langle x, e_\mu \rangle_0|^2 \leq \|x\|_0^2,$$

so $\|T\| \leq 1$ and $\|T\|_A^2 \leq \alpha_{\mu, \lambda, \varepsilon}$ where $\alpha_{\mu, \lambda, \varepsilon} = \frac{\mu \mu + \varepsilon(\lambda + \varepsilon)}{(\mu - \varepsilon)^2}$.

Since $H_\varepsilon$ is of type $H$, we conclude that

$$\|T\|_B^2 \leq H(\alpha_{\mu, \lambda, \varepsilon}),$$

whence

$$M_\varepsilon(\lambda) - \varepsilon' \leq \|e_\lambda\|_*^2 = \|Te_\mu\|_*^2 \leq H(\alpha_{\mu, \lambda, \varepsilon}) \|e_\mu\|_*^2$$

$$\leq H(\alpha_{\mu, \lambda, \varepsilon}) (m_\varepsilon(\mu) + \varepsilon').$$

In particular, since $\varepsilon'$ was arbitrary, and $m_\varepsilon(\lambda) \leq \|e_\lambda\|_*^2 \leq \|B\|$, we find that

$$M_\varepsilon(\lambda) - m_\varepsilon(\lambda) \leq [H(\alpha_{\mu, \lambda, \varepsilon}) - 1] \|B\|.$$ 

By assumption, $H$ is continuous and $H(1) = 1$. Hence, as $\varepsilon \downarrow 0$, the functions $M_\varepsilon(\lambda)$ diminish monotonically, converging uniformly to a function $h_*(\lambda)$ which is also the uniform limit of the family $m_\varepsilon(\lambda)$. It is clear that $h_*$ is continuous, and since $m_\varepsilon \leq h_* \leq M_\varepsilon$, we have $h_* = h$ almost everywhere with respect to $\nu_A$. The relation (3.3) now follows for $h = h_*$ by letting $\varepsilon$ and $\varepsilon'$ tend to zero in (3.4). \qed

A partial converse to Theorem II is found below, see Theorem 6.3.

Remark 3.1. (The non-separable case.) Now consider the case when $H_0$ is non-separable. (By regularity this means that also $H_1$ and $H_\varepsilon$ are non-separable.)

First assume that the operator $A$ is bounded. Let $H'_0$ be a separable reducing subspace for $A$ such that the restriction $A'$ of $A$ to $H'_0$ has the same spectrum as $A$. The space $H'_0$ reduces $B$ by Donoghue’s lemma; by Theorem II the restriction $B'$ of $B$ to $H'_0$ satisfies $B' = h'(A')$ for some continuous function $h'$ satisfying (3.3) on $\sigma(A)$. Let $H''_0$ be any other separable reducing subspace, where (as before) $B'' = h''(A'')$. Then $H'_0 \oplus H''_0$ is a separable reducing subspace on which $B = h(A)$ for some third continuous function $h$ on $\sigma(A)$. Then $h(A') \oplus h(A'') = h'(A') \oplus h''(A'')$ and by continuity we must have $h = h' = h''$ on $\sigma(A)$. The function $h$ thus satisfies $B = h(A)$ as well as the estimate (3.3).
The same reasoning as above shows that the spectral projection of \( A \) corresponding to the spectral set \([0, n] \cap \sigma(A), n = 1, 2, \ldots\) is an intermediate Hilbert space. We shall here prove the following equivalent version of Donoghue's result (see \[2, 3\]).

3.3. Donoghue's theorem. The exact quadratic interpolation spaces relative to a Hilbert couple were characterized by Donoghue in the paper \[13\]. We shall here prove the following equivalent version of Donoghue's result (see \[2, 3\]).

**Theorem III.** An intermediate Hilbert space \( \mathcal{H}_* \) relative to \( \overline{\mathcal{H}} \) is an exact interpolation space if and only if there is a positive radon measure \( \varrho \) on \([0, \infty)\) such that

\[
\| x \|_*^2 = \int_{[0, \infty]} (1 + t^{-1}) K(t, x; \overline{\mathcal{H}}) \, d\varrho(t).
\]

Equivalently, \( \mathcal{H}_* \) is exact interpolation relative to \( \overline{\mathcal{H}} \) if and only if the corresponding operator \( B \) can be represented as \( B = h(A) \) for some function \( h \in \mathcal{P} \).

The statements that all norms of the given form are exact quadratic interpolation norms has already been shown (see §1.2). There remains to prove that there are no others.
Donoghue’s original formulation of the result, as well as other equivalent forms of the theorem, are found in Section 6 below. Our present approach follows [2] and is based on K-monotonicity.

Remark 3.4. The condition that \( H_\ast \) be exact interpolation with respect to \( \overline{H} \) means that \( H_\ast \) is of type \( H \) where \( H(t) = \max \{1, t\} \). In view of Theorem II (and Remark 3.1), this means that we can represent \( B = h(A) \) where \( h \) is quasi-concave on \( \sigma(A) \setminus \{0\} \).

\[
(3.5) \quad h(\lambda) \leq h(\mu) \max \{1, \lambda/\mu\}, \quad \lambda, \mu \in \sigma(A) \setminus \{0\}.
\]

In particular, \( h \) is locally Lipschitzian on \( \sigma(A) \cap R_+ \).

Remark 3.5. A related result concerning non-exact quadratic interpolation was proved by Ovchinnikov [30] using Donoghue’s theorem. Cf. also [4].

3.4. The proof for simple finite-dimensional couples. Similar to our approach to Calderón’s problem, our strategy is to reduce Theorem III to a case of "simple couples".

Theorem 3.6. Assume that \( H_0 = H_0 = C^n \) as sets and that all eigenvalues \((\lambda_i)_1^n\) of the corresponding operator \( A \) are of unit multiplicity. Consider a third Hermitian norm \( \|x\|^2 = \langle Bx, x \rangle_0 \) on \( C^n \). Then \( H_\ast \) is exact interpolation with respect to \( \overline{H} \) if and only if \( B = h(A) \) where \( h \in P' \).

Remark 3.7. The lemma says that the class of functions \( h \) on \( \sigma(A) \) satisfying

\[
(3.6) \quad T^*T \leq 1, \quad T^*AT \leq A \Rightarrow T^*h(A)T \leq h(A), \quad (T \in M_n(C))
\]

is precisely the set \( P'|\sigma(A) \) of restrictions of \( P' \)-functions to \( \sigma(A) \). In this way, the condition (3.6) provides an operator-theoretic solution to the interpolation problem by positive Pick functions on a finite subset of \( R_+ \).

Proof of Theorem 3.6. We already know that the spaces \( H_\ast \) of the asserted form are exact interpolation relative to \( \overline{H} \) (see subsections 1.2 and 1.3).

Now let \( H_\ast \) be any exact quadratic interpolation space. By Donoghue’s lemma and the argument in §2.3, we can for an appropriate positive sequence \( \lambda = (\lambda_i)_1^n \) identify \( \overline{H} = \ell_2^n(\lambda) \), \( A = \text{diag}(\lambda_i) \), and \( B = h(A) \) where \( h \) is some positive function defined on \( \sigma(A) = (\lambda_i)_1^n \).

Our assumption is that \( \ell_2^n(h(\lambda)) \) is exact interpolation relative to \( \ell_2^n(\lambda) \). We must prove that \( h \in P'|\sigma(A) \). To this end, write

\[
k_{\lambda_i}(t) = \frac{(1+t)\lambda_i}{1+t\lambda_i},
\]

and recall that (see Lemma 1.1)

\[
K(t, x; \ell_2^n(\lambda)) = (1+t^{-1})^{-1} \sum_1^n |x_i|^2 k_{\lambda_i}(t).
\]

Let us denote by \( C \) the algebra of continuous complex functions on \([0, \infty]\) with the supremum norm \( \|u\|_\infty = \sup_{t>0} |u(t)| \). Let \( V \subset C \) be the linear span of the \( k_{\lambda_i} \) for \( i = 1, \ldots, n \). We define a positive functional \( \phi \) on \( V \) by

\[
\phi(\sum_1^n a_i k_{\lambda_i}) = \sum_1^n a_i h(\lambda_i).
\]
We claim that $\phi$ is a positive functional, i.e., if $u \in V$ and $u(t) \geq 0$ for all $t > 0$, then $\phi(u) \geq 0$.

To prove this let $u = \sum_{i=1}^{n} a_i k_{\lambda_i}$ be non-negative on $\mathbb{R}_+$ and write $a_i = |x_i|^2 - |y_i|^2$ for some $x, y \in \mathbb{C}^n$. The condition that $u \geq 0$ means that

$$(1 + t^{-1}) K (t, x; \ell_2^n(\lambda)) = \sum_{i=1}^{n} |x_i|^2 k_{\lambda_i}(t)$$

(3.7)

$$\geq \sum_{i=1}^{n} |y_i|^2 k_{\lambda_i}(t)$$

$$= (1 + t^{-1}) K (t, y; \ell_2^n(\lambda)), \quad t > 0.$$ 

Since $\ell_2^n(\lambda)$ is an exact Calderón couple (by Theorem 2.4), the space $\ell_2^n(h(\lambda))$ is exact $K$-monotonic. In other words, (3.7) implies that

$$\| x \|_{\ell_2^n(h(\lambda))} \geq \| y \|_{\ell_2^n(h(\lambda))},$$

i.e.,

$$\phi(u) = \sum_{i=1}^{n} (|x_i|^2 - |y_i|^2) h(\lambda_i) \geq 0.$$ 

The asserted positivity of $\phi$ is thereby proved.

Replacing $\lambda_i$ by $c \lambda_i$ for a suitable positive constant $c$ we can without losing generality assume that $1 \in \sigma(A)$, i.e., that the unit $1(x) \equiv 1$ of the $C^*$-algebra $C$ belongs to $V$. The positivity of $\phi$ then ensures that

$$\| \phi \| = \sup_{u \in V; \|u\|_\infty \leq 1} |\phi(u)| = \phi(1).$$

Let $\Phi$ be a Hahn-Banach extension of $\phi$ to $C$ and note that

$$\| \Phi \| = \| \phi \| = \phi(1) = \Phi(1).$$

This means that $\Phi$ is a positive functional on $C$ (cf. [29], §3.3). By the Riesz representation theorem there is thus a positive Radon measure $\varrho$ on $[0, \infty]$ such that

$$\Phi(u) = \int_{[0, \infty]} u(t) d\varrho(t), \quad u \in C.$$ 

In particular

$$h(\lambda_i) = \phi(k_{\lambda_i}) = \Phi(k_{\lambda_i}) = \int_{[0, \infty]} \frac{(1 + t)\lambda_i}{1 + t\lambda_i} d\varrho(t), \quad i = 1, \ldots, n.$$ 

We have shown that $h$ is the restriction to $\sigma(A)$ of a function of class $P^\prime$. 

3.5. The proof of Donoghue’s theorem. We here prove Theorem III in full generality.

We remind the reader that if $S \subset \mathbb{R}_+$ is a subset, we write $P^\prime|S$ for the convex cone of restrictions of $P^\prime$-functions to $S$. We first collect some simple facts about this cone.

Lemma 3.8. (i) The class $P^\prime|S$ is closed under pointwise convergence.

(ii) If $S$ is finite and if $\lambda = (\lambda_i)_{i=1}^{\infty}$ is an enumeration of the points of $S$ then $h$ belongs to $P^\prime|S$ if and only if $\ell_2^n(h(\lambda))$ is exact interpolation with respect to the pair $\ell_2^n(\lambda)$. 

□
(iii) If S is infinite, then a continuous function h on S belongs to $P'|S$ if and only if h ∈ $P'|\Lambda$ for every finite subset $\Lambda \subset S$.

Proof. (i) Let $h_n$ be a sequence in $P'$ converging pointwise on S and fix $\lambda \in S$. It is clear that the boundedness of the numbers $h_n(\lambda)$ is equivalent to boundedness of the total masses of the corresponding measures $\rho_n$ on the compact set $[0, \infty]$. It now suffices to apply Helly’s selection theorem.

(ii) This is Theorem 3.6.

(iii) Let $\Lambda_n$ be an increasing sequence of finite subsets of S whose union is dense. Let $h_n = h|\Lambda_n$ where h is continuous on S. If $h_n \in P'|\Lambda_n$ for all n then the sequence $h_n$ converges pointwise on $\cup \Lambda_n$ to h. By part (i) we then have $h \in P'|\sigma(A)$. □

We can now finish the proof of Donoghue’s theorem (Theorem III).

Let $H_n$ be exact interpolation with respect to $\overline{H}$ and represent the corresponding operator as $B = h(A)$ where h satisfies (3.3). By the remarks after Theorem III, the function h is locally Lipschitzian.

In view of Lemma 3.8 we shall be done when we have proved that $\ell_2^2(h(\lambda))$ is exact interpolation with respect to $\overline{\ell_2^2}(\lambda)$ for all sequences $\lambda = (\lambda_i)_i \subset \sigma(A)$ of distinct points. Let us arrange the sequences in increasing order: $0 < \lambda_1 < \cdots < \lambda_n$.

Fix $\varepsilon > 0$, $\varepsilon < \min\{\varepsilon, \lambda_1, 1/\lambda_n\}$ and let $E_i = [\lambda_i - \varepsilon, \lambda_i + \varepsilon] \cap \sigma(A)$; we assume that $\varepsilon$ is sufficiently small that the $E_i$ be disjoint. Let $M = \cup_i E_i$. We can assume that $h$ has Lipschitz constant at most 1 on $M$.

Let $\mathcal{M}$ be the reducing subspace of $H_0$ corresponding to the spectral set $\mathcal{M}$, and let $\tilde{A}$ be the compression of $A$ to $\mathcal{M}$. We define a function $g$ on $\mathcal{M}$ by $g(\lambda) = \lambda_i$ on $E_i$. Then $|g(\lambda) - \lambda| < \varepsilon$ on $\sigma(\tilde{A})$, so

$$(3.8) \quad \|\tilde{A} - g(\tilde{A})\| \leq \varepsilon, \quad \|h(\tilde{A}) - h(g(\tilde{A}))\| \leq \varepsilon.$$ 

Lemma 3.9. Suppose that $A', A'' \in \mathcal{L}(\mathcal{M})$ satisfy $A' \geq \delta > 0$ and $\|A' - A''\| \leq \varepsilon$. Then $\|T\|_{A'} \leq \sqrt{1 + 2\varepsilon/\delta} \max\{\|T\|, \|T\|_{A'}\}$ for all $T \in \mathcal{L}(\mathcal{M})$.

Proof. By definition, $\|T\|_{A'}$ is the smallest number $C > 0$ such that $T^* A'T \leq C^2 A'$. Thus

$$T^* A''T = T^*(A'' - A')T + T^* A'T \leq \|T\|^2 \varepsilon + \|T\|^2_{A'} (A'' + (A' - A'')) \leq 2\varepsilon \max\{\|T\|^2, \|T\|_{A'}^2\} + \|T\|^2_{A'} \quad \|A''\| \leq \max\{\|T\|^2, \|T\|_{A'}^2\} (1 + 2\varepsilon/\delta) A''.$$ 

We can find $\delta > 0$ such that the operators $\tilde{A}$, $g(\tilde{A})$, $h(\tilde{A})$, and $h(g(\tilde{A}))$ are $\geq \delta$. Then by repeated use of Lemma 3.9,

$$\|T\|_{h(g(\tilde{A}))} \leq \sqrt{1 + 2\varepsilon/\delta} \max\{\|T\|, \|T\|_{h(\tilde{A})}\} \leq \sqrt{1 + 2\varepsilon/\delta} \max\{\|T\|, \|T\|_{\tilde{A}}\} \leq (1 + 2\varepsilon/\delta) \max\{\|T\|, \|T\|_{g(\tilde{A})}\}, \quad T \in \mathcal{L}(\mathcal{M}).$$

Let $e_i$ be a unit vector supported by the spectral set $E_i$ and define a space $\mathcal{V} \subset \mathcal{M}$ to be the n-dimensional space spanned by the $e_i$. Let $A_0$ be the compression of
g(\Delta) to \mathcal{V}; then

\begin{equation}
\| T \|_{h(A_0)} \leq (1 + 2\varepsilon/\delta) \max \{ \| T \|, \| T \|_{A_0} \}, \quad T \in \mathcal{L}(\mathcal{V}).
\end{equation}

Identifying \mathcal{V} with \ell^n_2 and \mathcal{A}_0 with the matrix \text{diag}(\lambda_i), we see that (3.9) is independent of \varepsilon. Letting \varepsilon diminish to 0 in (3.9) now gives that \ell^n_2(h(\lambda)) is exact interpolation with respect to \overline{\mathcal{F}}_2^\lambda. In view of Lemma 3.8, this finishes the proof of Theorem III. q.e.d.

4. Classes of matrix functions

In this section, we discuss the basic properties of interpolation functions: in particular, the relation to the well-known classes of monotone matrix functions. We refer to the books [12] and [34] for further reading on the latter classes.

4.1. Interpolation and matrix monotone functions. Let \( A_1 \) and \( A_2 \) be positive operators in \( \ell_2^n \) (\( n = \infty \) is admitted). Suppose that \( A_1 \leq A_2 \) and form the following operators on \( \ell_2^n \oplus \ell_2^n \),

\[
T_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} A_2 & 0 \\ 0 & A_1 \end{pmatrix}.
\]

It is then easy to see that \( T_0^*T_0 \leq 1 \) and that \( T_0^*AT_0 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \leq A \).

Now assume that a function \( h \) on \( \sigma(A) \) belongs to the class \( C_A \) defined in §1.4, i.e., that \( h \) satisfies

\begin{equation}
T^*T \leq 1, \quad T^*AT \leq A \implies T^*h(A)T \leq h(A),
\end{equation}

where \( T \) denotes an operator on \( \ell_2^{2n} \).

We then have \( T_0^*h(A)T_0 \leq h(A) \), or

\[
\begin{pmatrix} h(A_1) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} h(A_2) & 0 \\ 0 & h(A_1) \end{pmatrix}.
\]

In particular, we find that \( h(A_1) \leq h(A_2) \). We have shown that (under the assumptions above)

\begin{equation}
A_1 \leq A_2 \implies h(A_1) \leq h(A_2).
\end{equation}

We now change our point of view slightly. Given a positive integer \( n \), we let \( C_n \) denote the convex of positive functions \( h \) on \( \mathbb{R}_+ \) such that (4.1) holds for all positive operators \( A \) on \( \ell_2^n \) and all \( T \in \mathcal{L}(\ell_2^n) \).

Similarly, we let \( P_n^* \) denote the class of all positive functions \( h \) on \( \mathbb{R}_+ \) such that \( h(A_1) \leq h(A_2) \) whenever \( A_1, A_2 \) are positive operators on \( \ell_2^n \) such that \( A_1 \leq A_2 \). We refer to \( P_n^* \) as the cone of positive functions monotone of order \( n \) on \( \mathbb{R}_+ \).

We have shown above that \( C_{2n} \subset P_n^* \).

In the other direction, assume that \( h \in P_n^* \). Let \( A, T \) be bounded operators on \( \ell_2^n \) with \( A \geq 0, T^*T \leq 1 \) and \( T^*AT \leq A \). Assume also that \( h \) is continuous. We will use the following lemma due to Hansen [19]. We recall the proof for completeness.

**Lemma 4.1.** ([19]) \( T^*h(A)T \leq h(T^*AT) \).

**Proof.** Put \( S = (1 - TT^*)^{1/2} \) and \( R = (1 - T^*T)^{1/2} \) and consider the \( 2n \times 2n \) matrix

\[
U = \begin{pmatrix} T & S \\ R & -T^* \end{pmatrix}, \quad X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.
\]
It is well-known, and easy to check, that $U$ is unitary and that
\[ U^* X U = \begin{pmatrix} T^* A T & T^* A S \\ SAT & SAS \end{pmatrix}. \]

Next fix a number $\varepsilon > 0$, a constant $\lambda > 0$ (to be fixed), and form the matrix
\[ Y = \begin{pmatrix} T^* A T + \varepsilon & 0 \\ 0 & 2\lambda \end{pmatrix} \]
which, provided that we choose $\lambda \geq \|SAS\|$, satisfies
\[ Y - U^* X U = \begin{pmatrix} \varepsilon & -T^* A S \\ -S A T & 2\lambda - SAS \end{pmatrix} \geq \begin{pmatrix} \varepsilon & D \\ D^* & \lambda \end{pmatrix}, \]
where we have written $D = -T^* A S$.

If we now also choose $\lambda$ so that $\lambda \geq \|D\|^2/\varepsilon$, then we obtain for all $\xi, \eta \in \mathbb{C}^n$
that
\[ \left\langle \begin{pmatrix} \varepsilon \\ D^* \\ \lambda \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle = \varepsilon \|\xi\|^2 + \langle D \eta, \xi \rangle + \langle D^* \xi, \eta \rangle + \lambda \|\eta\|^2 \]
\[ \geq \varepsilon \|\xi\|^2 - 2\|D\| \|\xi\| \|\eta\| + \lambda \|\eta\|^2 \geq 0. \]
Hence $U^* X U \leq Y$ and as a consequence $U^* h(X) U = h(U^* X U) \leq h(Y)$, since $h$ is matrix monotone of order $2n$. The last inequality means that
\[ \begin{pmatrix} T^* h(A) T & T^* h(A) S \\ Sh(A) T & Sh(A) S \end{pmatrix} \leq \begin{pmatrix} h(T^* A T + \varepsilon) & 0 \\ 0 & h(2\lambda) \end{pmatrix}, \]
so in particular $T^* h(A) T \leq h(T^* A T + \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, and since $h$ is assumed to be continuous, we conclude the lemma.

We now continue our discussion. Assuming that $T^* T \leq 1$ and $T^* A T \leq A$, and that $h \in P'_{2n}$ is continuous, we have $h(T^* A T) \leq h(A)$ [since $h \in P'_{n}$], so $T^* h(A) T \leq h(A)$ by Lemma 4.1. We conclude that $h \in C_n$.

To prove that $P'_{2n} \subset C_n$, we need to remove the continuity assumption on $h$ made above. This is completely standard: let $\varphi$ be a smooth positive function on $\mathbb{R}_+$ such that $\int_0^\infty \varphi(t) \, dt = 1$, and define a sequence $h_k$ by $h_k(\lambda) = k^{-1} \int_0^\infty \varphi(\lambda^k t) \, dt$. The class $P'_{2n}$ is a convex cone, closed under pointwise convergence [12], so the functions $h_1, h_2, \ldots$ are of class $P'_{2n}$. They are furthermore continuous, so by the argument above, they are of class $C_n$. By Lemma 3.8, the cone $C_n$ is also closed under pointwise convergence, so $h = \lim h_n \in C_n$.

To summarize, we have the inclusions $C_{2n} \subset P'_{n}, P'_{2n} \subset C_n$, and also $C_{n+1} \subset C_n, P'_{n+1} \subset P'_{n}$. In view of Theorem III, we have the identity $\cap_1^\infty C_n = P'$. The inclusions above now imply the following result, sometimes known as "Löwner’s theorem on matrix monotone functions".

**Theorem 4.2.** We have $\cap_1^\infty P'_{n} = \cap_1^\infty C_n = P'$.

The identity $\cap_1^\infty P'_{n} = P'$ says that a positive function $h$ is monotone of all finite orders if and only it is of class $P'$. The somewhat less precise fact that $P'_{\infty} = P'$ is interpreted as that the class of *operator monotone functions* coincides with $P'$.

The identity $C_{\infty} = P'$ is, except for notation, contained in the work of Foiaş and Lions, from [17]. See §6.4.

Note that the inclusion $P'_{2n} \subset C_n$ shows that a matrix monotone functions of order $2n$ can be interpolated by a positive Pick function at $n$ points. Results of a
similar nature, where it is shown, in addition, that an interpolating Pick function can be taken rational of a certain degree, are discussed, for example, in Donoghue’s book [13, Chapter XIII] or (more relevant in the present connection) in the paper [14].

It seems somewhat inaccurate to refer to the identity \( \cap \mathbb{P}_n = P' \) as "Löwner’s theorem", since Löwner discusses more subtle results concerning matrix monotone functions of a given finite order \( n \). In spite of this, it is common nowadays to let "Löwner’s theorem" refer to this identity.

4.2. More on the cone \( C_A \). Donoghue’s precise description of the cone \( C_A \) used the following properties of that cone.

**Theorem 4.3.** For a positive function \( h \) on \( \sigma(A) \) we define two positive functions \( \hat{h} \) and \( h^* \) on \( \sigma(A^{-1}) \) by \( h(\lambda) = \lambda h(1/\lambda) \) and \( h^*(\lambda) = 1/h(1/\lambda) \). Then the following conditions are equivalent,

(i) \( h \in C_A \),
(ii) \( \hat{h} \in C_{A^{-1}} \),
(iii) \( h^* \in C_{A^{-1}} \).

**Proof.** Let \( \mathcal{H}_* \) be a quadratic intermediate space relative to a regular Hilbert couple \( \overline{H} \); let \( B = h(A) \) be the corresponding operator. It is clear that \( \mathcal{H}_* \) is exact interpolation relative to \( \overline{H} \) if and only if \( \mathcal{H}_* \) is exact interpolation relative to the reverse couple \( \overline{H}^* = (\mathcal{H}_1, \mathcal{H}_0) \). The latter couple has corresponding operator \( A^{-1} \) and it is clear that the identity \( \| \|x\|_2^2 = \langle h(A)x, x \rangle_0 \) is equivalent to that \( \| \|x\|_2^2 = \langle A^{-1}\hat{h}(A^{-1})x, x \rangle_1 \). We have shown the equivalence of (i) and (ii).

Next let \( \overline{H}_*^* = (\mathcal{H}_0^*, \mathcal{H}_1^*) \) be the dual couple, where we identify \( \mathcal{H}_0^* = \mathcal{H}_0 \). With this identification, \( \mathcal{H}_1^* \) becomes associated to the norm \( \| \|x\|_{\overline{H}_1^*}^2 = \langle A^{-1}x, x \rangle_0 \) and \( \mathcal{H}_*^* \) is associated with \( \| \|x\|_{\overline{H}_*^*}^2 = \langle B^{-1}x, x \rangle_0 \). It remains to note that \( \mathcal{H}_* \) is exact interpolation relative to \( \overline{H} \) if and only if \( \mathcal{H}_*^* \) is exact interpolation relative to \( \overline{H}_*^* \), proving the equivalence of (i) and (iii).

Combining with Theorem III, one obtains alternative proofs of the interpolation theorems for \( P' \)-functions discussed by Donoghue in the paper [14].

**Remark 4.4.** The exact quadratic interpolation spaces which are fixed by the duality, i.e., which satisfy \( \mathcal{H}_*^* = \mathcal{H}_* \), correspond precisely to the class of \( P' \)-functions which are self-dual: \( h^* = h \). This class was characterized by Hansen in the paper [20].

4.3. Matrix concavity. A function \( h \) on \( \mathbb{R}_+ \) is called matrix concave of order \( n \) if we have Jensen’s inequality

\[ \lambda h(A_1) + (1 - \lambda)h(A_2) \leq h(\lambda A_1 + (1 - \lambda)A_2) \]

for all positive \( n \times n \) matrices \( A_1, A_2 \), and all numbers \( \lambda \in [0, 1] \). Let us denote by \( \Gamma_n \) the convex cone of positive concave functions of order \( n \) on \( \mathbb{R}_+ \). The fact that \( \cap_1^{\infty} \Gamma_n = P' \) follows from the theorem of Kraus [23]. Following [2] we now give an alternative proof of this fact.

**Proposition 4.5.** For all \( n \) we have the inclusion \( C_{3n} \subset \Gamma_n \subset P'_n \). In particular \( \cap_1^{\infty} \Gamma_n = P' \).
Proof: Assume first that \( h \in C_{3n} \) and pick two positive matrices \( A_1 \) and \( A_2 \). Define \( A_3 = (1 - \lambda)A_1 + \lambda A_2 \) where \( \lambda \in [0,1] \) is given, and define matrices \( A \) and \( T \) of order \( 3n \) by

\[
A = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & \sqrt{1-\lambda} & 0 \\ \sqrt{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

It is clear that \( T^*T \leq 1 \) and

\[
T^*AT = \begin{pmatrix} A_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq A,
\]

so, since \( h \in C_{3n} \), we have \( T^*h(A)T \leq h(A) \), or

\[
\begin{pmatrix} (1 - \lambda)h(A_1) + \lambda h(A_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leq \begin{pmatrix} h(A_3) & 0 & 0 \\ 0 & h(A_1) & 0 \\ 0 & 0 & h(A_2) \end{pmatrix}.
\]

Comparing the matrices in the upper left corners, we find that \( h \in \Gamma_n \).

Assume now that \( h \in \Gamma_n \), and take positive definite matrices \( A_1, A_2 \) of order \( n \) with \( A_1 \leq A_2 \). Also pick \( \lambda \in (0,1) \). Then \( \lambda A_2 = \lambda A_1 + (1 - \lambda)A_3 \) where \( A_3 = \lambda(1 - \lambda)^{-1}(A_2 - A_1) \). By matrix concavity, we then have

\[
h(\lambda A_2) \geq \lambda h(A_1) + (1 - \lambda)h(A_3) \geq \lambda h(A_1),
\]

where we used non-negativity to deduce the last inequality. Being concave, \( h \) is certainly continuous. Letting \( \lambda \uparrow 1 \) one thus finds that \( h(A_1) \leq h(A_2) \). We have shown that \( h \in P_n^z \).

For a further discussion of classes of convex matrix functions and their relations to monotonicity, we refer to the paper [21].

4.4. Interpolation functions of two variables. In this section, we briefly discuss a class of interpolation functions of two matrix variables. The following discussion is not in any way conclusive, but we hope that it can be of use for a future investigation.

Let \( H_1 \) and \( H_2 \) be Hilbert spaces. One turns \( H_1 \otimes H_2 \) into a Hilbert space by defining the inner product on elementary tensors via \( \langle x_1 \otimes x_2, x_1' \otimes x_2' \rangle := \langle x_1, x_1' \rangle_{H_1} \cdot \langle x_2, x_2' \rangle_{H_2} \) (then extend via sesqui-linearity.) Similarly, if \( T_i \) are operators on \( H_i \), the tensor product \( T_1 \otimes T_2 \) is defined on elementary tensors via \( (T_1 \otimes T_2)(x_1 \otimes x_2) = T_1 x_1 \otimes T_2 x_2 \). It is then easy to see that if \( A_i \) are positive operators on \( H_i \) for \( i = 1, 2 \), then \( A_1 \otimes A_2 \geq 0 \) as an operator on the tensor product. Furthermore, we have \( A_1 \otimes A_2 \leq A_1' \otimes A_2' \) if \( A_i \leq A_i' \) for \( i = 1, 2 \).

Given two positive definite matrices \( A_i \) of orders \( n_i \) and a function \( h \) on \( \sigma(A_1) \times \sigma(A_2) \), we define a matrix \( h(A_1, A_2) \) by

\[
h(A_1, A_2) = \sum_{(\lambda_1, \lambda_2) \in \sigma(A_1) \times \sigma(A_2)} h(\lambda_1, \lambda_2) E^1_{\lambda_1} \otimes E^2_{\lambda_2}
\]

where \( E^j \) is the spectral resolution of the matrix \( A_j \).

We shall say that \( h \) gives rise to exact interpolation relative to \((A_1, A_2)\), and write \( h \in C_{A_1, A_2} \), if the condition

\[
T_j^* T_j \leq 1, \quad T_j^* A_j T_j \leq A_j, \quad j = 1, 2
\]

for
implies

\begin{equation}
\begin{aligned}
&h(A_1, A_2) + (T_1 \otimes T_2)^* h(A_1, A_2)(T_1 \otimes T_2) \\
&\quad - (T_1 \otimes 1)^* h(A_1, A_2)(T_1 \otimes 1) - (1 \otimes T_2)^* h(A_1, A_2)(1 \otimes T_2) \geq 0.
\end{aligned}
\end{equation}

Taking $T_1 = T_2 = 0$ we see that $h \geq 0$ for all $h \in C_{A_1, A_2}$. It is also clear that $C_{A_1, A_2}$ is a convex cone closed under pointwise convergence on the finite set $\sigma(A_1) \times \sigma(A_2)$.

If $h = h_1 \otimes h_2$ is an elementary tensor where $h_j \in C_{A_j}$ is a function of one variable, then (4.3) implies $T_j^* h_j(A_j) T_j \leq h_j(A_j)$, whence $(h_1(A_1) - T_1^* h_1(A_1) T_1) \otimes (h_2(A_2) - T_2^* h_2(A_2) T_2) \geq 0$, which implies (4.4). We have shown that $C_{A_1} \otimes C_{A_2} \subset C_{A_1, A_2}$.

Since for each $t \geq 0$ the $P'$-function $\lambda \mapsto \frac{1 + t \lambda}{1 + tA}$ is of class $C_{A_1}$, we infer that every function representable in the form

\begin{equation}
h(\lambda_1, \lambda_2) = \int \int_{[0, \infty]^2} \frac{(1 + t_1)\lambda_1 (1 + t_2)\lambda_2}{1 + t_1\lambda_1} \frac{d\varrho(t_1, t_2)}{1 + t_2\lambda_2}
\end{equation}

with some positive Radon measure $\varrho$ on $[0, \infty]^2$ is in the class $C_{A_1, A_2}$.

We shall say that a function $h$ on $\sigma(A_1) \times \sigma(A_2)$ has the separate interpolation-property if for each fixed $b \in \sigma(A_2)$ the function $\lambda_1 \mapsto h(\lambda_1, b)$ is of class $C_{A_1}$, and a similar statement holds for all functions $\lambda_2 \mapsto h(a, \lambda_2)$.

**Lemma 4.6.** Each function of class $C_{A_1, A_2}$ has the separate interpolation-property.

**Proof.** Let $T_2 = 0$ and take an arbitrary $T_1$ with $T_1^* T_1 \leq 1$ and $T_1^* A_1 T_1 \leq A_1$. By hypothesis,

$$(T_1 \otimes 1)^* h(A_1, A_2)(T_1 \otimes 1) \leq h(A_1, A_2).$$

Fix an eigenvalue $b$ of $A_2$ and let $y$ be a corresponding normalized eigenvector. Then for all $x \in H_1$ we have $\langle h(A_1, A_2) x \otimes y, x \otimes y \rangle = \langle h(A_1, b) x, x \rangle_{H_1}$ and $\langle (T_1 \otimes 1)^* h(A_1, A_2)(T_1 \otimes 1) x \otimes y, x \otimes y \rangle = \langle T_1^* h(A_1, b) T_1 x, x \rangle_{H_1}$, so

$$\langle T_1^* h(A_1, b) T_1 x, x \rangle_{H_1} \leq \langle h(A_1, b) x, x \rangle_{H_1}.$$

The functions $h(a, \lambda_2)$ can be treated similarly. \hfill $\square$

**Example.** The function $h(\lambda_1, \lambda_2) = (\lambda_1 + \lambda_2)^{1/2}$ clearly has the separate interpolation-property for all $A_1, A_2$. However, it is not representable in the form (4.5). Indeed, $\text{Re} \{h(i, i) - h(-i, i)\} = 1$ while it is easy to check that $\text{Re} \{h(\lambda_1, \lambda_2) - h(\bar{\lambda_1}, \bar{\lambda_2})\} \leq 0$ whenever $\text{Im } \lambda_1, \text{Im } \lambda_2 > 0$ and $h$ is of the form (4.5).

Let us say that a function $h(\lambda_1, \lambda_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ is an interpolation function (of two variables) if $h \in C_{A_1, A_2}$ for all $A_1, A_2$. Lemma 4.6 implies that interpolation functions are separately real-analytic in $\mathbb{R}_+ \times \mathbb{R}_+$ and that the functions $h(a, \cdot)$ and $h(\cdot, b)$ are of class $P'$ (cf. Theorem III).

The above notion of interpolation function is close to Korányi’s definition of monotone matrix function of two variables: $f(\lambda_1, \lambda_2)$ is matrix monotone in a rectangle $I = I_1 \times I_2$ ($I_1, I_2$ intervals in $\mathbb{R}$) if $A_1 \leq A_1'$ (with spectra in $I_1$) and $A_2 \leq A_2'$ (with spectra in $I_2$) implies

$$f(A_1', A_2') - f(A_1', A_2) - f(A_1, A_2') + f(A_1, A_2) \geq 0.$$

**Lemma 4.7.** Each interpolation function is matrix monotone in $\mathbb{R}_+ \times \mathbb{R}_+$.
Proof. Let $0 < A_i \leq A'_i$ and put $\tilde{A}_i = \begin{pmatrix} A'_i & 0 \\ 0 & A_i \end{pmatrix}$, $T_i = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Since $T_i^* \tilde{A}_i T_i \leq \tilde{A}_i$, an interpolation function $h$ will satisfy the interpolation inequality (4.4) with $A_i$ replaced by $\tilde{A}_i$. Applying this inequality to vectors of the form $\left( \begin{array}{c} x^1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} x^2 \\ 0 \end{array} \right)$ we readily obtain
\[
\langle h(A'_1, A'_2)x_1 \otimes x_2, x_1 \otimes x_2 \rangle - \langle h(A_1, A'_2)x_1 \otimes x_2, x_1 \otimes x_2 \rangle \\
- \langle h(A'_1, A_2)x_1 \otimes x_2, x_1 \otimes x_2 \rangle + \langle h(A_1, A_2)x_1 \otimes x_2, x_1 \otimes x_2 \rangle \geq 0.
\]
The same result obtains with $x_1 \otimes x_2$ replaced by a sum $x_1 \otimes x_2 + x'_1 \otimes x'_2 + \ldots$, i.e., $h$ is matrix monotone. \qed

Remark 4.8. Assume that $f$ is of the form $f(\lambda_1, \lambda_2) = g_1(\lambda_1) + g_2(\lambda_2)$. Then $f$ is matrix monotone for all $g_1, g_2$ and $f$ is an interpolation function if and only if $g_1, g_2 \in P'$. In order to disregard "trivial" monotone functions of the above type, Korányi [22] imposed the normalizing assumption (a) $f(\lambda_1, 0) = f(0, \lambda_2) = 0$ for all $\lambda_1, \lambda_2$.

It follows from Lemma 4.7 and the proof of [22, Theorem 4] that, if $h$ is a $C^2$-smooth interpolation function, then the function
\[
k(x_1, x_2; y_1, y_2) = \frac{h(x_1, x_2) - h(x_1, y_2) - h(y_1, x_2) + h(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}
\]
is positive definite in the sense that $\sum_m \sum_n k(x_m, y_n; x_n, y_m) \alpha_m \bar{\alpha}_n \geq 0$ for all finite sequences of positive numbers $x_j, y_k$ and all complex numbers $\alpha_l$. (The proof uses Löwner’s matrix.) Korányi uses essentially this positive definiteness condition (and condition (a) in the remark above) to deduce an integral representation formula for $h$ as an integral of products of Pick functions. See Theorem 3 in [22]. However, in contrast to our situation, Korányi considers functions monotone on the rectangle $(-1, 1) \times (-1, 1)$, so this last result can not be immediately applied. (It easily implies local representation formulas, valid in finite rectangles, but these representations do not appear to be very natural from our point of view.)

This is not the right place to attempt to extend Korányi’s methods to functions on $\mathbb{R}_+ \times \mathbb{R}_+$; it would seem more appropriate to give a more direct characterization of the classes $C_{A_1, A_2}$ or of the class of interpolation functions. At present, I do not know if there is an interpolation function which is not representable in the form (4.5) and I will here settle for asking the question whether or not this is the case.

5. PROOF OF THE $K$-PROPERTY

In this section we extend the result of Theorem 2.4 to obtain the full proof of Theorem I. The discussion is in principle not hard, but it does require some care to keep track of both norms when reducing to a finite-dimensional case.

Recall first that, by Lemma 2.3, it suffices to consider the diagonal case $\mathcal{H} = \mathcal{K}$.

To prove Theorem I we fix a regular Hilbert couple $\overline{\mathcal{H}}$; we must prove that it has the $K$-property (see §1.5). By Theorem 2.4, we know that this is true if $\mathcal{H}$ is finite dimensional and the associated operator only has eigenvalues of unit multiplicity.

We shall use a weak* type compactness result ([2]). To formulate it, let $\mathcal{L}(\overline{\mathcal{H}})$ be the unit ball in the space $\mathcal{L}(\overline{\mathcal{H}})$. Moreover, let $\Sigma_t$ be the sum $\mathcal{H}_0 + \mathcal{H}_1$ normed
by \( \| x \|_{\Sigma}^2 := K(t, x) \). Note that \( \| \cdot \|_{\Sigma} \) is an equivalent norm on \( \Sigma \) and that \( \Sigma_1 = \Sigma \) isometrically. We denote by \( L_1(\Sigma_t) \) the unit ball in the space \( L(\Sigma_t) \).

In view of Remark 1.4, one has the identity
\[
(5.1) \quad L_1(\overline{H}) = \bigcap_{t \in \mathbb{R}_+} L_1(\Sigma_t).
\]

We shall use this to define a compact topology on \( L_1(\overline{H}) \).

**Lemma 5.1.** The subset \( L_1(\overline{H}) \subset L_1(\Sigma) \) is compact relative to the weak operator topology inherited from \( L(\Sigma) \).

Recall that the weak operator topology on \( L(H) \) is the weakest topology such that for some \( \rho > 0 \), a net \( T_i \) converges to the limit \( T \) if the inner product \( \langle T_i x, y \rangle_H \) converges to \( \langle Tx, y \rangle_H \) for all \( x, y \in H \).

**Proof of Lemma 5.1.** The weak operator topology coincides on the unit ball \( L_1(\Sigma) \) with the weak*-topology, which is compact, due to Alaoglu's theorem (see [29], Chap. 4 for details). It is clear that for a fixed \( t > 0 \), the subset \( L_1(\Sigma_t) \cap L_1(\Sigma_t) \) is weak operator closed in \( L_1(\Sigma) \); hence it is also compact. In view of (5.1), the set \( L_1(\overline{H}) \) is an intersection of compact sets. Hence the set \( L_1(\overline{H}) \) is itself compact, provided that we endow it with the subspace topology inherited from \( L_1(\Sigma) \). \( \square \)

Denote by \( P_n \) the projections \( P_n = E_{\sigma(A) \cap [n^{-1}, n]} \) on \( \mathcal{H}_0 \) where \( E \) is the spectral resolution of \( A \) and \( n = 1, 2, 3, \ldots \). Consider the couple
\[
\overline{H}^{(n)} = (P_n(\mathcal{H}_0), P_n(\mathcal{H}_1)),
\]
the associated operator of which is the compression \( A_n \) of \( A \) to the subspace \( P_n(\mathcal{H}_0) \). Note that the norms in the couple \( \overline{H}^{(n)} \) are equivalent, i.e., the associated operator \( A_n \) is bounded above and below.

We shall need two lemmas.

**Lemma 5.2.** If \( \overline{H}^{(n)} \) has the \( K \)-property for all \( n \), then so does \( \overline{H} \).

**Proof.** Note that \( \| P_n \|_{L(\overline{H})} = 1 \) for all \( n \), and that \( P_n \to 1 \) as \( n \to \infty \) relative to the strong operator topology on \( L(\Sigma) \). Suppose that \( x^0, y^0 \in \Sigma \) are elements such that, for some \( \rho > 1 \),
\[
(5.2) \quad K(t, y^0) \leq \frac{1}{\rho} K(t, x^0), \quad t > 0.
\]

Then \( K(t, P_n y^0) \leq K(t, y^0) < \rho^{-1} K(t, x^0) \). Moreover, the identity \( K(t, P_n y^0) = \left\langle \frac{1}{t + \tau A_n} P_n y^0, P_n y^0 \right\rangle \) shows that we have an estimate of the form \( K(t, P_n y^0) \leq C_n \min \{1, t\} \) for \( t > 0 \) and large enough \( C_n \) (this follows since \( A_n \) is bounded above and below).

The functions \( K(t, P_m x^0) \) increase monotonically, converging uniformly on compact subsets of \( \mathbb{R}_+ \) to \( K(t, x^0) \) when \( m \to \infty \). By concavity of the function \( t \mapsto K(t, P_m x^0) \) we will then have
\[
(5.3) \quad K(t, P_n y^0) \leq \frac{1}{\tilde{\rho}} K(t, P_m x^0), \quad t \in \mathbb{R}_+,
\]
provided that \( m \) is sufficiently large, where \( \tilde{\rho} \) is any number in the interval \( 1 < \tilde{\rho} < \rho \).
Indeed, let \( A = \lim_{t \to \infty} K(t, P_n y^0) \) and \( B = \lim_{t \to 0} K(t, P_n y^0) / t \). Take points \( t_0 < t_1 \) such that \( K(t, P_n y^0) \geq A / \rho' \) when \( t \geq t_1 \) and \( K(t, P_n y^0) / t \leq B \rho' \) when \( t \leq t_0 \). Here \( \rho' \) is some number in the interval \( 1 < \rho' < \rho \).

Next use (5.2) to choose \( m \) large enough that \( K(t, P_m x^0) > \rho K(t, P_n y^0) \) for all \( t \in [t_0,t_1] \). Then \( K(t, P_m x^0) > (\rho / \rho') K(t, P_n y^0) \) for \( t = t_1 \), hence for all \( t \geq t_1 \), and \( K(t, P_m x^0) / t > (\rho / \rho') K(t, P_n y_0) / t \) for \( t = t_0 \) and hence also when \( t < t_0 \). Choosing \( \rho' = \rho / \rho \) now establishes (5.3).

Put \( N = \max \{m,n\} \). If \( \overline{\mathcal{H}_N} \) has the \( K \)-property, we can find a map \( T_{nm} \in \mathcal{L}_1(\overline{\mathcal{H}}) \) such that \( T_{nm} P_m x^0 = P_n y^0 \). (Define \( T_{mn} = 0 \) on the orthogonal complement of \( P_N(H_0) \) in \( \Sigma \).) In view of Lemma 5.1, the maps \( T_{nm} \) must cluster at some point \( T \in \mathcal{L}_1(\overline{\mathcal{H}}) \). It is clear that \( T x^0 = y^0 \). Since \( \rho > 1 \) was arbitrary, we have shown that \( \overline{\mathcal{H}} \) has the \( K \)-property.

**Lemma 5.3.** Given \( x^0, y^0 \in \mathcal{H}_0(n) \) and a number \( \epsilon > 0 \) there exists a positive integer \( n \) and a finite-dimensional couple \( \mathcal{V} \subset \overline{H(n)} \) such that \( x^0, y^0 \in \mathcal{V}_0 + \mathcal{V}_1 \) and
\[
(1 - \epsilon) K(t, x; \mathcal{H}) \leq K(t, x; \mathcal{V}) \leq (1 + \epsilon) K(t, x; \mathcal{H}), \quad t > 0, \ x \in \mathcal{V}_0 + \mathcal{V}_1.
\]
Moreover, \( \mathcal{V} \) can be chosen so that all eigenvalues of the associated operator \( A_{\mathcal{V}} \) are of unit multiplicity.

**Proof.** Let \( A_n \) be the operator associated with the couple \( \overline{H(n)} \); thus \( 1/n \leq A_n \leq n \).

Take \( \eta > 0 \) and let \( \{A_i\}_1^N \) be a finite subset of \( \sigma(A_n) \) such that \( \sigma(A_n) \subset \bigcup_1^N E_i \)
where \( E_i = (A_i - \eta/2, A_i + \eta/2) \). We define a Borel function \( w : \sigma(A_n) \to \sigma(A_n) \) by \( w(\lambda) = \lambda_i \) on \( E_i \cap \sigma(A_n) \); then \( \|w(A_n) - A_n\|_{\mathcal{L}(H_0)} \leq \eta \).

Let \( k_t(\lambda) = \frac{\lambda}{(1 + t)^{\lambda}} \). It is easy to check that the Lipschitz constant of the restriction \( k_t | \sigma(A_n) \) is bounded above by \( C_1 \min \{1,t\} \) where \( C_1 = C_1(n) \) is independent of \( t \). Hence
\[
\|k_t(w(A_n)) - k_t(A_n)\|_{\mathcal{L}(H_0)} \leq C_1 \eta \min \{1,t\}.
\]
It follows readily that
\[
|\langle k_t(w(A_n)) - k_t(A_n)\rangle x, x\rangle_0| \leq C_1 \eta \min \{1,t\} \|x\|_0^2, \quad x \in P_n(H_0).
\]
Now let \( c > 0 \) be such that \( A \geq c \). The elementary inequality \( k_t(c) \geq (1/2) \min \{1,ct\} \) shows that
\[
\langle k_t(A_n) x, x\rangle_0 \geq C_2 \min \{1,t\} \|x\|_0^2, \quad x \in P_n(H_0),
\]
where \( C_2 = (1/2) \min \{1,c\} \). Combining these estimates, we deduce that
\[
|\langle k_t(w(A_n)) x, x\rangle_0 - \langle k_t(A_n) x, x\rangle_0| \leq C_3 \eta \langle k_t(A_n) x, x\rangle_0, \quad x \in P_n(H_0)
\]
for some suitable constant \( C_3 = C_3(n) \).

Now pick unit vectors \( e_i, f_i \) supported by the spectral sets \( E_i \cap \sigma(A_n) \) such that \( x^0 \) and \( y^0 \) belong to the space \( W \) spanned by \( \{e_i, f_i\}_1^N \). Put \( W_0 = W_1 = W \) and define norms on those spaces by
\[
\|x\|_{W_0} = \|x\|_{H_0}, \quad \|x\|_{W_1} = \langle w(A) x, x\rangle_{H_0}.
\]
The operator associated to \( \overline{W} \) is then the compression of \( w(A_n) \) to \( W_0 \), i.e.,
\[
\|x\|^2_{W_1} = \langle A_{\overline{W}} x, x\rangle_{W_0} = \langle w(A_n) x, x\rangle_{H_0}, \quad x \in W.
\]
Let \( \epsilon = 2C_3 \eta \) and observe that, by (5.5)
\[
|K(t, x; \overline{W}) - K(t, x; \mathcal{H})| \leq (\epsilon/2)K(t, x; \mathcal{H}), \quad f \in \mathcal{W}.
\]
The eigenvalues of $A_{\mathcal{V}}$ typically have multiplicity 2. To obtain unit multiplicity, we perturb $A_{\mathcal{V}}$ slightly to a positive matrix $A_{\tilde{\mathcal{V}}}$ such that $\| A_{\mathcal{V}} - A_{\tilde{\mathcal{V}}} \|_{\mathcal{L}(H_0)} < \epsilon/2C_3$.

Let $\mathcal{V}$ be the couple associated to $A_{\mathcal{V}}$, i.e., put $\mathcal{V}_i = \mathcal{W}$ for $i = 0, 1$ and

$$\| x \|_{\mathcal{V}_0} = \| x \|_{\mathcal{W}_0} \quad \text{and} \quad \| x \|_{\mathcal{V}_1}^2 = \langle A_{\mathcal{V}} x, x \rangle_{\mathcal{V}_0}.$$ 

It is then straightforward to check that

$$\| x \|_{\mathcal{V}_0} = \| x \|_{\mathcal{W}_0} \quad \text{and} \quad \| x \|_{\mathcal{V}_1}^2 = \langle A_{\mathcal{V}} x, x \rangle_{\mathcal{V}_0}.$$ 

Combining this with the estimate (5.6), one finishes the proof of the lemma. \qed

**Proof of Theorem I.** Given two elements $x^0, y^0 \in \Sigma$ as in (5.2) we write $x^n = P_n(x^0)$ and $y^n = P_n(y^0)$. By the proof of Lemma 5.2 we then have $K(t, y^n) \leq \tilde{\rho}^{-1} K(t, x^n)$ for large enough $n$, where $\tilde{\rho}$ is any given number in the interval $(1, \rho)$.

We then use Lemma 5.3 to choose a finite-dimensional sub-couple $\mathcal{V} \subset \mathcal{H}^{(n)}$ such that

$$K(t, y^n; \mathcal{V}) \leq (1 + \epsilon)K(t, y^n; \mathcal{H}) \leq \tilde{\rho}^{-1} K(t, x^n; \mathcal{V}) + \epsilon (K(t, x^n; \mathcal{V}) + K(t, y^n; \mathcal{H})).$$

Here $\epsilon > 0$ is at our disposal.

Choosing $\epsilon$ sufficiently small, we can arrange that

(5.7) \quad $K(t, y^n; \mathcal{V}) \leq K(t, x^n; \mathcal{V}), \quad t > 0.$

By Theorem 2.4, the condition (5.7) implies the existence of an operator $T' \in \mathcal{L}_1(\mathcal{V})$ such that $T' x^n = y^n$. Considering the canonical inclusion and projection

$$I : \Sigma(\mathcal{V}) \rightarrow \Sigma(\mathcal{H}) \quad \text{and} \quad \Pi : \Sigma(\mathcal{H}) \rightarrow \Sigma(\mathcal{V}),$$

we have, by virtue of Lemma 5.3,

$$\| I \|_{\mathcal{L}(\mathcal{V}; \mathcal{H})} \leq (1 - \epsilon)^{-1} \quad \text{and} \quad \| \Pi \|_{\mathcal{L}(\mathcal{H}; \mathcal{V})} \leq 1 + \epsilon.$$

Now let $T = T_{\epsilon} := IT'\Pi \in \mathcal{L}(\mathcal{H}^{(n)})$. Then $\| T \| \leq \frac{1 + \epsilon}{1 - \epsilon}$ and $T x^n = y^n$. As $\epsilon \downarrow 0$ the operators $T_{\epsilon}$ will cluster at some point $T \in \mathcal{L}_1(\mathcal{H}^{(n)})$ such that $T x^n = y^n$ (cf. Lemma 5.1).

We have shown that $\mathcal{H}^{(n)}$ has the $K$-property. In view of Lemma 5.2, this implies that $\mathcal{H}$ has the same property. The proof of Theorem I is therefore complete. \qed

6. **Representations of interpolation functions**

6.1. **Quadratic interpolation methods.** Let us say that an interpolation method defined on regular Hilbert couples taking values in Hilbert spaces is a quadratic interpolation method. (Donoghue [13] used the same phrase in a somewhat wider sense, allowing the methods to be defined on non-regular Hilbert couples as well.)

If $F$ is an exact quadratic interpolation method, and $\mathcal{H}$ a Hilbert couple, then by Donoghue’s theorem III there exists a positive Radon measure $\varrho$ on $[0, \infty]$ such that $F(H) = H_{\varrho}$, where the latter space is defined by the familiar norm $\| x \|_{\varrho}^2 = \int_{[0, \infty]} (1 + t^{-1}) K(t, x) \, d\varrho(t)$.

A priori, the measure $\varrho$ could depend not only on $F$ but also on the particular $\mathcal{H}$. That $\varrho$ is independent of $\mathcal{H}$ can be realized in the following way. Let $\mathcal{H}$ be a regular Hilbert couple such that every positive rational number is an eigenvalue of
We have shown that replacing \( \lambda \) by \( \tilde{A} \) the associated operator. Let \( B' \) be the operator associated to the exact quadratic interpolation space \( F(\overline{H}) \). There is then a clearly unique \( P'-\)function \( h \) on \( \sigma (A') \) such that \( B' = h(A') \), viz. there is a unique positive Radon measure \( \rho \) on \([0, \infty] \) such that \( F(\overline{H}) = \mathcal{H}'_{\rho} \) (see §1.2 for the notation).

If \( \overline{H} \) is any regular Hilbert couple, we can form the direct sum \( \overline{S} = \overline{H} \oplus \overline{H} \). Denote by \( \tilde{A} \) the corresponding operator and let \( B = \tilde{h}(A) \) be the operator corresponding to the exact quadratic interpolation space \( F(\overline{S}) \). Then \( \tilde{h}(A) = \tilde{h}(A') \oplus \tilde{h}(A) = h(A') \oplus h(A) \). This means that \( \tilde{h}(A') = h(A') \), i.e. \( \tilde{h} = h \). In particular, the operator \( B \) corresponding to the exact interpolation space \( F(\overline{H}) \) is equal to \( h(A) \).

We have shown that \( F(\overline{H}) = \mathcal{H}_e \). We emphasize our conclusion with the following theorem.

**Theorem 6.1.** There is a one-to-one correspondence \( \rho \mapsto F \) between positive Radon measures and exact quadratic interpolation methods.

We will shortly see that Theorem 6.1 is equivalent to the theorem of Foiaş and Lions [17]. As we remarked above, a more general version of the theorem, admitting non-regular Hilbert couples, is found in Donoghue’s paper [13].

### 6.2. Interpolation type and reiteration

In this subsection, we prove some general facts concerning quadratic interpolation methods; we shall mostly follow Fan [15].

Fix a function \( h \in P' \) of the form

\[
    h(\lambda) = \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} \, d\rho(t).
\]

It will be convenient to write \( \overline{H}_h \) for the corresponding exact interpolation space \( \mathcal{H}_h \). Thus, we shall denote

\[
    \|x\|_h^2 = \langle h(A)x, x \rangle_0 = \int_{[0, \infty]} (1 + t^{-1}) K(t, x) \, d\rho(t).
\]

More generally, we shall use the same notation when \( h \) is any quasi-concave function on \( \mathbb{R}_+ \); then \( \overline{H}_h \) is a quadratic interpolation space, but not necessarily exact.

Recall that, given a function \( H \) of one variable, we say that \( \mathcal{H}_+ \) is of type \( H \) with respect to \( \overline{H} \) if \( \|T\|_{L(\mathcal{H}_+)} \leq M_0 \) implies \( \|T\|_{L(\mathcal{H}_+)} \leq K_0 H(M_1/M_0) \).

We shall say that a quasi-concave function \( h \) on \( \mathbb{R}_+ \) is of type \( H \) if \( \overline{H}_h \) is of type \( H \) relative to any regular Hilbert couple \( \overline{H} \). The following result somewhat generalizes Theorem 3.2. The class of functions of type \( H \) clearly forms a convex cone.

**Theorem 6.2.** Let \( h \) be of type \( H \), where (i) \( H(1) = 1 \) and \( H(t) \leq \max\{1, t\} \), and (ii) \( H \) has left and right derivatives \( \theta_{\pm} = H'(1\pm) \) at the point 1, where \( \theta_- \leq \theta_+ \). Then for any positive constant \( c \),

\[
    (6.1) \quad \min\{\lambda^{\theta_-}, \lambda^{\theta_+}\} \leq \frac{h(c\lambda)}{h(c)} \leq \max\{\lambda^{\theta_-}, \lambda^{\theta_+}\}, \quad \lambda \in \mathbb{R}_+.
\]

In particular, if \( H(t) \) is differentiable at \( t = 1 \) and \( H'(1) = \theta \), then \( h(\lambda) = \lambda^\theta \), \( \lambda \in \mathbb{R}_+ \).

**Proof.** Replacing \( A \) by \( cA \), it is easy to see that if \( h \) is of type \( H \), then so is \( h_c(t) = h(ct)/h(c) \). Fix \( \mu > 0 \) and consider the function \( h_0(t) = h_c(\mu t)/h_c(\mu) \). By
Theorem II, we have \( h_0(t) \leq H(t) \) for all \( t \). Furthermore \( h_0(1) = H(1) = 1 \) by (i).

Since \( h_0 \) is differentiable, the assumption (ii) now gives \( \theta_1 \leq h'_0(1) \leq \theta_2 \), or

\[
\theta_1 \leq \frac{\mu h'_0(\mu)}{h_0(\mu)} \leq \theta_2.
\]

Dividing through by \( \mu \) and integrating over the interval \([1, \lambda]\), one now verifies the inequalities in (6.1). \( \square \)

The following result provides a partial converse to Theorem II.

**Theorem 6.3.** ([15]) Let \( h \in P' \) and set \( H(t) = \sup_{s \geq 0} h(st)/h(s) \). Then \( h \) is of type \( H \).

**Proof.** Let \( T \in \mathcal{L} (\overline{\mathcal{H}}) \) be a non-zero operator; put \( M_j = \| T \| \mathcal{L}(\mathcal{H}_j) \) and \( M = M_1/M_0 \). We then have (by Lemma 1.1)

\[
\| Tx \|^2_h = \int_{[0, \infty]} (1 + t^{-1}) K(t, Tx) \, dg(t) \\
\quad \leq M_0 \int_{[0, \infty]} (1 + t^{-1}) K(tM, x) \, dg(t) \\
\quad = M_0 \int_{[0, \infty]} \left< \frac{(1 + t)MA}{1 + tMA}, x, x \right>_0 \, dg(t) \\
\quad = M_0 \langle h(MA) x, x \rangle_0.
\]

Letting \( E \) be the spectral resolution of \( A \), we have

\[
\langle h(MA) x, x \rangle_0 = \int_0^\infty h(M\lambda) \, d \langle E\lambda x, x \rangle_0.
\]

Since \( h(M\lambda)/h(\lambda) \leq H(M) \), we conclude that

\[
\| Tx \|^2_h \leq M_0 H(M) \int_0^\infty h(\lambda) \, d \langle E\lambda x, x \rangle_0 = M_0 H(M) \| x \|^2_h,
\]

which finishes the proof. \( \square \)

Given a function \( h \) of a positive variable, we define a new function \( \tilde{h} \) by

\[
\tilde{h}(s, t) = s \, h(t/s).
\]

The following reiteration theorem is due to Fan.

**Theorem 6.4.** ([15]) Let \( h, h_0, h_1 \in P' \), and \( \varphi(\lambda) = \tilde{h}(h_0(\lambda), h_1(\lambda)) \). Then \( \overline{\mathcal{H}}_\varphi = (\overline{\mathcal{H}}_{h_0}, \overline{\mathcal{H}}_{h_1})_h \) with equal norms. Moreover, \( \overline{\mathcal{H}}_\varphi \) is an exact interpolation space relative to \( \overline{\mathcal{H}} \).

**Proof.** Let \( \overline{\mathcal{H}}' \) denote the couple \( (\overline{\mathcal{H}}_{h_0}, \overline{\mathcal{H}}_{h_1}) \). The corresponding operator \( A' \) then obeys

\[
\| x \|_{\overline{\mathcal{H}}_{h_1}} = \| (A')^{1/2} x \|_{\overline{\mathcal{H}}_0'} = \| \varphi_0(A)^{1/2} (A')^{1/2} x \|_0, \quad x \in \Delta(\overline{\mathcal{H}}').
\]

On the other hand, \( \| x \|_{\overline{\mathcal{H}}_{h_1}} = \| \varphi_1(A)^{1/2} x \|_0 \), so

\[
(A')^{1/2} x = \varphi_0(A)^{-1/2} \varphi_1(A)^{1/2} x, \quad x \in \Delta(\overline{\mathcal{H}}').
\]
We have shown that $A' = \varphi_0(A)^{-1} \varphi_1(A)$, whence (by Lemma 1.1)
\[
K(t, x; \mathcal{H}) = \left\langle \frac{t\varphi_0(A)^{-1} \varphi_1(A) - x}{1 + t\varphi_0(A)^{-1} \varphi_1(A)} , x \right\rangle_{\mathcal{H}_0'}, \quad (6.2)
\]
\[
= \left\langle \frac{t\varphi_1(A)}{1 + t\varphi_0(A)^{-1} \varphi_1(A)} , x \right\rangle_{\mathcal{H}_0'}.
\]

Now let the function $h \in P'$ be given by
\[
h(\lambda) = \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} \, d\vartheta(t),
\]
and note that the function $\varphi = \tilde{h}(h_0, h_1)$ is given by
\[
\varphi(\lambda) = \int_{[0, \infty]} \frac{(1 + t)h_1(\lambda)}{1 + th_1(\lambda)/h_0(\lambda)} \, d\vartheta(t).
\]

Combining with (6.2), we find that
\[
\|x\|_{\mathcal{H}_h}^2 = \int_{[0, \infty]} (1 + t^{-1}) K(t, x; \mathcal{H}) \, d\vartheta(t)
\]
\[
= \int_0^\infty \left[ \int_{[0, \infty]} \frac{(1 + t)h_1(\lambda)}{1 + th_1(\lambda)/h_0(\lambda)} \, d\vartheta(t) \right] \, d\langle \{E_\lambda x, x\}, 0 \rangle = \|x\|_{\mathcal{H}_\varphi}^2.
\]

This finishes the proof of the theorem. \hfill \Box

Combining with Donoghue’s theorem III, one obtains the following, purely function-theoretic corollary. Curiously, we are not aware of a proof which does not use interpolation theory.

**Corollary 6.5.** (15) Suppose that $h \in P'$ and that $h_0, h_1 \in P'|F$, where $F$ is some closed subset of $\mathbb{R}_+$. Then the function $\varphi = \tilde{h}(h_0, h_1)$ is also of class $P'|F$.

6.3. **Donoghue’s representation.** Let $\mathcal{H}$ be a regular Hilbert couple. In Donoghue’s setting, the principal object is the space $\Delta = \mathcal{H}_0 \cap \mathcal{H}_1$ normed by $\|x\|_\Delta^2 = \|x\|_0^2 + \|x\|_1^2$. In the following, all involutions are understood to be taken with respect to the norm of $\Delta$.

We express the norms in the spaces $\mathcal{H}_i$ as
\[
\|x\|_0^2 = \langle Hx, x \rangle_\Delta \quad \text{and} \quad \|x\|_1^2 = \langle (1 - H)x, x \rangle_\Delta,
\]
where $H$ is a bounded positive operator on $\Delta$, $0 \leq H \leq 1$. The regularity of $\mathcal{H}$ means that neither $0$, nor $1$ is an eigenvalue of $H$.

To an arbitrary quadratic intermediate space $\mathcal{H}_*$ there corresponds a bounded positive injective operator $K$ on $\Delta$ such that
\[
\|x\|_*^2 = \langle Kx, x \rangle_\Delta.
\]

It is then easy to see that $\mathcal{H}_*$ is exact interpolation if and only if, for bounded operators $T$ on $\Delta$, the conditions $T^*HT \leq H$ and $T^*(1 - H)T \leq 1 - H$ imply $T^*KT \leq K$. It is straightforward to check that the relations between $H$, $K$ and the operators $A$, $B$ used in the previous sections are:

\[
H = \frac{1}{1 + A} , \quad A = \frac{1 - H}{H} , \quad K = \frac{B}{1 + A} , \quad B = \frac{K}{H} .
\]

(6.3)

(It follows from the proof of Lemma 1.2 that $H$ and $K$ commute.)
By Theorem III we know that $\mathcal{H}_s$ is an exact interpolation space if and only if $B = h(A)$ for some $h \in P'$. By (6.3), this is equivalent to that $K = k(H)$ where

$$k(H) = \frac{h(A)}{1 + A} = H h \left( \frac{1 - H}{H} \right).$$

In its turn, this means that

$$k(\lambda) = \lambda \int_{[0, \infty]} \frac{(1 + t)(1 - \lambda)/\lambda}{1 + t(1 - \lambda)/\lambda} d\rho(t)$$

$$= \int_{[0, \infty]} \frac{(1 + t)\lambda(1 - \lambda)}{\lambda + t(1 - \lambda)} d\rho(t), \quad \lambda \in \sigma(H),$$

where $\rho$ is a suitable Radon measure. Applying the change of variables $s = 1/(1+t)$ and defining a positive Radon measure $\mu$ on $[0, 1]$ by $d\mu(s) = d\rho(t)$, we arrive at the expression

$$(6.4) \quad k(\lambda) = \int_{0}^{1} \frac{\lambda(1 - \lambda)}{(1 - s)(1 - \lambda) + s \lambda} d\mu(s), \quad \lambda \in \sigma(H),$$

which gives the representation exact quadratic interpolation spaces originally used by Donoghue in [13].

6.4. J-methods and the Foiaş-Lions theorem. We define the (quadratic) $J$-functional relative to a regular Hilbert couple $\mathcal{H}$ by

$$J(t,x) = J(t,x; \mathcal{H}) = \|x\|_{0}^{2} + t \|x\|_{1}^{2}, \quad t > 0, x \in \Delta(\mathcal{H}).$$

Note that $J(t,x)^{1/2}$ is an equivalent norm on $\Delta$ and that $J(1,x) = \|x\|_{\Delta}^{2}$.

Given a positive Radon measure $\nu$ on $[0, \infty]$, we define a Hilbert space $J_{\nu}(\mathcal{H})$ as the set of all elements $x \in \Sigma(\mathcal{H})$ such that there exists a measurable function $u : [0, \infty) \to \Delta$ such that

$$(6.5) \quad x = \int_{[0, \infty]} u(t) \, d\nu(t) \quad \text{(convergence in } \Sigma)$$

and

$$(6.6) \quad \int_{[0, \infty]} \frac{J(t,u(t))}{1 + t} \, d\nu(t) < \infty.$$

The norm in the space $J_{\nu}(\mathcal{H})$ is defined by

$$(6.7) \quad \|x\|_{J_{\nu}}^{2} = \inf_{u} \int_{[0, \infty]} \frac{J(t,u(t))}{1 + t} \, d\nu(t)$$

over all $u$ satisfying (6.5) and (6.6).

The space (6.7) were (with different notation) introduced by Foiaş and Lions in the paper [17], where it was shown that there is a unique minimizer $u(t)$ of the problem (6.7), namely

$$(6.8) \quad u(t) = \varphi_{t}(A)x \quad \text{where} \quad \varphi_{t}(\lambda) = \frac{1 + t}{1 + t\lambda} \left( \int_{[0, \infty]} \frac{1 + s}{1 + s\lambda} \, d\nu(s) \right)^{-1}.$$

Inserting this expression for $u$ into (6.7), one finds that

$$\|x\|_{J_{\nu}}^{2} = \langle h(A)x, x \rangle_{0}.$$
where
\begin{equation}
(6.9) \quad h(\lambda)^{-1} = \int_{[0, \infty]} \frac{1 + t}{1 + t\lambda} d\nu(t).
\end{equation}

It is easy to verify that the class of functions representable in the form (6.9) for some positive Radon measure \( \nu \) coincides with the class \( P' \). We have thus arrived at the following result.

**Theorem 6.6.** Every exact quadratic interpolation space \( \mathcal{H}_* \) can be represented isometrically in the form \( \mathcal{H}_* = J_\nu(H) \) for some positive Radon measure \( \nu \) on \([0, \infty]\). Conversely, any space of this form is an exact quadratic interpolation space.

In the original paper [17], Foiaş and Lions proved the less precise statement that each exact quadratic interpolation method \( F \) can be represented as \( F = J_\nu \) for some positive Radon measure \( \nu \).

6.5. **The relation between the \( K \)- and \( J \)-representations.** The assignment \( K_\theta = J_\nu \) gives rise to a non-trivial bijection \( \varrho \mapsto \nu \) of the set of positive Radon measures on \([0, \infty]\). In this bijection, \( \varrho \) and \( \nu \) are in correspondence if and only if
\[ \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\varrho(t) = \left( \int_{[0, \infty]} \frac{1 + t}{1 + t\lambda} d\nu(t) \right)^{-1}. \]

As an example, let us consider the geometric interpolation space (where \( c_\theta = \pi / \sin(\pi \theta) \))
\[ \| x \|^2_\theta = \langle A^\theta x, x \rangle_0 = c_\theta \int_0^\infty t^{-\theta} K(t, x) \frac{dt}{t}. \]

The measure \( \varrho \) corresponding to this method is \( d\varrho(t) = \frac{c_\theta t^{-\theta} dt}{1 + t} \). On the other hand, it is easy to check that
\[ \lambda^\theta = \left( \int_0^\infty \frac{1 + t}{1 + t\lambda} d\nu(t) \right)^{-1} \quad \text{where} \quad d\nu(t) = \frac{c_\theta t^\theta dt}{1 + t}. \]

We leave it to the reader to check that the norm in \( \mathcal{H}_\theta \) is the infimum of the expression
\[ c_\theta \int_0^\infty t^\theta J(t, u(t)) \frac{dt}{t} \]
over all representations
\[ x = \int_0^\infty u(t) \frac{dt}{t}. \]

We have arrived at the Hilbert space version of Peetre’s \( J \)-method of exponent \( \theta \). The identity \( J_{\varrho_\theta} = K_{\theta \nu} \) can now be recognized as a sharp (isometric) Hilbert space version of the equivalence theorem of Peetre, which says that the standard \( K_\theta \) and \( J_{\theta \nu} \)-methods give rise to equivalent norms on the category of Banach couples (see [7]).

The problem of determining the pairs \( \varrho, \nu \) having the property that the \( K_\nu \) and \( J_{\nu} \) methods give equivalent norms was studied by Fan in [15, Section 3].
6.6. Other representations. As we have seen in the preceding subsections, using the space \( \mathcal{H}_0 \) to express all involutions and inner products leads to a description of the exact quadratic interpolation spaces in terms of the class \( P' \). If we instead use the space \( \Delta \) as the basic object, we get Donoghue’s representation for interpolation functions. Similarly, one can proceed from any fixed interpolation space \( \mathcal{H}_* \) to obtain a different representation of interpolation functions.

6.7. On interpolation methods of power \( p \). Fix a number \( p, 1 < p < \infty \). We shall write \( L_p = L_p(X, \mathcal{A}, \mu) \) for the usual \( L_p \)-space associated with an arbitrary but fixed \((\sigma\text{-finite})\ measure \( \mu \) on a measure space \((X, \mathcal{A})\). Given a positive measurable weight function \( w \), we write \( L_p(w) \) for the space normed by

\[
\| f \|^p_{L_p(w)} = \int_X |f(x)|^p w(x) \, d\mu(x).
\]

We shall write \( \mathcal{I}_p(w) = (L_p, L_p(w)) \) for the corresponding weighted \( L_p \) couple. Note that the conditions imposed mean precisely that \( \mathcal{I}_p(w) \) be separable and regular.

Let us say that an exact interpolation functor \( F \) defined on the totality of separable, regular weighted \( L_p \)-couples and taking values in the class of weighted \( L_p \)-spaces is of power \( p \).

Define, for a positive Radon measure \( \varrho \) on \([0, \infty]\), an exact interpolation functor \( F = K_p(p) \) by the definition

\[
\| f \|^p_{F(\mathcal{I}_p(w))} := \int_{[0, \infty]} (1 + t^{-\frac{1}{p-1}})^{p-1} K_p(t, f; \mathcal{I}_p(w)) \, d\varrho(t).
\]

We contend that \( F \) is of power \( p \).

Indeed, it is easy to verify that

\[
K_p(t, f; \mathcal{I}_p(w)) = \int_X |f(x)|^p \frac{tw(x)}{(1 + (tw(x))^p)} d\mu(x),
\]

so Fubini’s theorem gives that

\[
\| f \|^p_{F(\mathcal{I}_p(w))} = \int_X |f(x)|^p h(w(x)) \, d\mu(x),
\]

where

\[
h(\lambda) = \int_{[0, \infty]} (1 + t^{\frac{1}{p-1}})^{p-1} \lambda \, d\varrho(t), \quad \lambda \in w(X).
\]

We have shown that \( F(\mathcal{I}_p(w)) = L_p(h(w)) \), so \( F \) is indeed of power \( p \).

Let us denote by \( \mathcal{K}(p) \) the totality of positive functions \( h \) on \( \mathbb{R}_+ \) representable in the form (6.10) for some positive Radon measure \( \varrho \) on \([0, \infty]\).

Further, let \( \mathcal{I}(p) \) denote the class of all (exact) interpolation functions of power \( p \), i.e., those positive functions \( h \) on \( \mathbb{R}_+ \) having the property that for each weighted \( L_p \) couple \( \mathcal{I}_p(w) \) and each bounded operator \( T \) on \( \mathcal{I}_p(w) \), it holds that \( T \) is bounded on \( L_p(h(w)) \) and

\[
\| T \|_{L_p(h(w))} \leq \| T \|_{\mathcal{I}_p(w)}.
\]

The class \( \mathcal{I}(p) \) is in a sense the natural candidate for the class of "operator monotone functions on \( L_p \)-spaces". The class \( \mathcal{I}(p) \) clearly forms a convex cone; it was shown by Peetre [31] that this cone is contained in the class of concave positive functions on \( \mathbb{R}_+ \) (with equality if \( p = 1 \)).
We have shown that $\mathcal{K}(p) \subset \mathcal{I}(p)$. By Theorem 6.1, we know that equality holds when $p = 2$. For other values of $p$ it does not seem to be known whether the class $\mathcal{K}(p)$ exhausts the class $\mathcal{I}(p)$, but one can show that we would have $\mathcal{K}(p) = \mathcal{I}(p)$ provided that each finite-dimensional $L_p$-couple $\mathcal{T}_{\nu}(\lambda)$ has the $K_p$-property (or equivalently, the $K$-property, see (2.2)). Naturally, the latter problem (about the $K_p$-property) also seems to be open, but some comments on it are found in Remark 2.9.

Let $\nu$ be a positive Radon measure on $[0, \infty]$. In [17], Foiaş and Lions introduced a method, which we will denote by $F = J_\nu(p)$ in the following way. Define the $J_\nu$-functional by

$$J_\nu(t; f; \mathcal{T}_\nu(\lambda)) = \| f \|_0^p + t \| f \|_1^p, \quad f \in \Delta, \ t > 0.$$ 

We then define an intermediate norm by

$$\| f \|_{p,F(\mathcal{T}_\nu(\lambda))} := \inf \int_{[0,\infty]} (1 + t) - \frac{1}{p-1} J_\nu(t; u(t); \mathcal{T}_\nu(\lambda)) \, d\nu(t),$$

where the infimum is taken over all representations

$$f = \int_{[0,\infty]} u(t) \, d\nu(t)$$

with convergence in $\Sigma$. It is straightforward to see that the method $F$ so defined is exact; in [17] it is moreover shown that it is of power $p$. More precisely, it is there proved that

$$\| f \|_{p,F(\mathcal{T}_\nu(\lambda))} = \int_X |f(x)|^p h(w(x)) \, d\mu(x),$$

where

$$h(\lambda) - \frac{1}{p} = \int_{[0,\infty]} \frac{(1 + t) \frac{1}{p}}{(1 + t\lambda)^{\frac{1}{p}}} \, d\nu(t), \quad \lambda \in w(X).$$

Let us denote by $\mathcal{J}(p)$ the totality of functions $h$ representable in the form (6.11). We thus have that $\mathcal{J}(p) \subset \mathcal{I}(p)$. In view of our preceding remarks, we conclude that if all weighted $L_p$-couples have the $K_p$ property, then necessarily $\mathcal{J}(p) \subset \mathcal{K}(p)$. Note that $\mathcal{J}(2) = \mathcal{K}(2)$ by Theorem 6.6.

**Appendix: The Complex Method is Quadratic**

Let $S = \{ z \in \mathbb{C}; 0 \leq \text{Re} \, z \leq 1 \}$. Fix a Hilbert couple $\mathcal{H}$ and let $\mathcal{F}$ be the set of functions $S \to \Sigma$ which are bounded and continuous in $S$, analytic in the interior of $S$, and which maps the line $j + i \mathbb{R}$ into $\mathcal{H}_j$ for $j = 0, 1$. Fix $0 < \theta < 1$. The norm in the complex interpolation space $C_\theta(\mathcal{H})$ is defined by

$$(*) \quad \| x \|_{C_\theta(\mathcal{H})} = \inf \{ \| f \|_p; f(\theta) = x \}. $$

Let $\mathcal{P}$ denote the set of polynomials $f = \sum_{i=1}^N a_i z^i$ where $a_i \in \Delta$. We endow $\mathcal{P}$ with the inner product

$$\langle f, g \rangle_{M_0} = \sum_{j=0,1} \int_{\mathbb{R}} \langle f(j + it), g(j + it) \rangle_j P_j(\theta, t) \, dt,$$

where $\{P_0, P_1\}$ is the Poisson kernel for $S$,

$$P_j(\theta, t) = \frac{e^{-\pi t} \sin \theta \pi}{\sin^2 \theta \pi + (\cos \theta \pi - (-1)^j e^{-\pi t})^2}.$$
Let $M_\theta$ be the completion of $\mathcal{P}$ with this inner product. It is easy to see that the elements of $M_\theta$ are analytic in the interior of $S$, and that evaluation map $f \mapsto f(\theta)$ is continuous on $M_\theta$. Let $N_\theta$ be the kernel of this functional and define a Hilbert space $\mathcal{H}_\theta$ by

$$\mathcal{H}_\theta = M_\theta/N_\theta.$$ 

We denote the norm in $\mathcal{H}_\theta$ by $\| \cdot \|_{\theta}$.

**Proposition A.1.** $C_\theta(\overline{\mathcal{F}}) = \mathcal{H}_\theta$ with equality of norms.

**Proof.** Let $f \in \mathcal{F}$. By the Calderón lemma in [7, Lemma 4.3.2], we have the estimate

$$\log \| f(\theta) \|_{C_\theta(\overline{\mathcal{F}})} \leq \sum_{j=0,1} \int_\mathbb{R} \log \| f(j + it) \|_j P_j(\theta, t) \, dt.$$ 

Applying Jensen’s inequality, this gives that

$$\| f(\theta) \|_{C_\theta(\overline{\mathcal{F}})} \leq \left( \sum_{j=0,1} \int_\mathbb{R} \| f(j + it) \|_j^2 P_j(\theta, t) \, dt \right)^{1/2} = \| f \|_{M_\theta}.$$ 

Hence $\mathcal{H}_\theta \subset C_\theta(\overline{\mathcal{F}})$ and $\| \cdot \|_{C_\theta(\overline{\mathcal{F}})} \leq \| \cdot \|_{\theta}$. On the other hand, for $f \in \mathcal{P}$ one has the estimates

$$\| f(\theta) \|_{\theta} \leq \| f \|_{M_\theta} \leq \sup \{ \| f(j + it) \|_j ; t \in \mathbb{R}, j = 0, 1 \} = \| f \|_{\mathcal{F}},$$

whence $C_\theta(\overline{\mathcal{F}}) \subset \mathcal{H}_\theta$ and $\| \cdot \|_{C_\theta(\overline{\mathcal{F}})} \geq \| \cdot \|_{\theta}$. \hfill \qed

It is well-known that the method $C_\theta$ is of exponent $\theta$ (see e.g. [7]). We have shown that $C_\theta$ is an exact quadratic interpolation method of exponent $\theta$.

**Complex interpolation with derivatives.** In [15, pp. 421-422], Fan considers the more general complex interpolation method $C_{\theta(n)}$ for the $n$th derivative. This means that in (*), one consider representations $x = \frac{1}{n!} f^{(n)}(\theta)$ where $f \in \mathcal{F}$; the complex method $C_\theta$ is thus the special case $C_{\theta(0)}$. It is shown in [15] that, for $n \geq 1$, the $C_{\theta(n)}$-method is represented, up to equivalence of norms, by the quasi-power function $h(\lambda) = \lambda^\theta / (1 + \frac{\theta(1-\theta)}{n} |\log \lambda|)^n$. The complex method with derivatives was introduced by Schechter [37]; for more details on that method, we refer to the list of references in [15].

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