ON SOME HYPERCOMPLEX 4-DIMENSIONAL LIE GROUPS OF
CONSTANT SCALAR CURVATURE

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ABSTRACT. In this paper we study sectional curvature of invariant hyper-Hermitian metrics on simply connected 4-dimensional real Lie groups admitting invariant hypercomplex structure. We give the Levi-Civita connections and explicit formulas for computing sectional curvatures of these metrics and show that all these spaces have constant scalar curvature. We also show that they are flat or they have only non-negative or non-positive sectional curvature.

1. Introduction

Hypercomplex structures on 4n-dimensional manifolds are of interesting structures in mathematics which have many applications in theoretical physics. For example these structures appear in supersymmetric sigma model [5]. Gibbons, Papadopoulos and Stelle showed that the geometry of the moduli space of a class of black holes in five dimensional is hyper-Kähler [3]. Therefore it is important to study hypercomplex spaces. In this paper we consider invariant hyper-Hermitian metrics on simply connected 4-dimensional real Lie groups admitting invariant hypercomplex structure. These spaces classified by M. L. Barberis (see [1]). One of important quantities which associate to Riemannian manifold is sectional curvature. In this article after obtaining the Levi-Civita connections of these Riemannian spaces we give explicit formulas for computing sectional curvatures of these manifolds. Then by using these formulas we show that these spaces have constant scalar curvature.

2. Preliminaries

Definition 2.1. Suppose that $M$ is a $4n$-dimensional manifold. Also let $J_i, i = 1, 2, 3$, be three fiberwise endomorphism of $TM$ such that

\begin{align}
J_1J_2 &= -J_2J_1 = J_3, \\
J_i^2 &= -Id_{TM}, & i &= 1, 2, 3, \\
N_i &= 0, & i &= 1, 2, 3,
\end{align}

where $N_i$ is the Nijenhuis tensor (torsion) corresponding to $J_i$ defined as follows:

\begin{equation}
N_i(X, Y) = [J_iX, J_iY] - [X, J_iY] - J_i([X, J_iY] + [J_iX, Y]),
\end{equation}

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for all vector fields $X, Y$ on $M$. The family $\mathcal{H} = \{J_i\}_{i=1,2,3}$ is a hypercomplex structure on $M$.

In fact a hypercomplex structure on a $4n$-dimensional manifold $M$ is a family $\mathcal{H} = \{J_i\}_{i=1,2,3}$ of complex structures on $M$ satisfying in the relation (2.1) (since an almost complex structure is a complex structure if and only if it has no torsion, see [4] page 124.).

**Definition 2.2.** A Riemannian metric $g$ on a hypercomplex manifold $(M, \mathcal{H})$ is called hyper-Hermitian if $g(J_iX, J_iY) = g(X, Y)$, for all vector fields $X, Y$ on $M$ and $i = 1, 2, 3$.

**Definition 2.3.** A hypercomplex structure $\mathcal{H} = \{J_i\}_{i=1,2,3}$ on a Lie group $G$ is said to be (left) invariant if for any $a \in G$,

$$J_i = Tl_a \circ J_i \circ Tl_{a^{-1}},$$

where $Tl_a$ is the differential function of the left translation $l_a$.

From now we suppose that $G$ is a simply connected 4-dimensional real Lie group. Suppose that $g$ is a left invariant Riemannian metric on a Lie group $G$ with Lie algebra $\mathfrak{g}$, then the Levi-Civita connection of $g$ is defined by the following relation

$$2g(\nabla_U V, W) = g([U, V], W) - g([V, W], U) + g([W, U], V),$$

for any $U, V, W \in \mathfrak{g}$, where $<,>$ is the inner product induced by $g$ on $\mathfrak{g}$.

### 3. Sectional and Scalar Curvatures

In this section we compute the Levi-Civita connections and the sectional curvatures of left invariant hyper-Hermitian metrics on (left) invariant hypercomplex 4-dimensional simply connected Lie groups. Then we compute the scalar curvatures of these spaces and show that these spaces are of constant scalar curvature. We mention that these spaces are complete because they are homogeneous Riemannian manifolds. M. L. Barberis has classified these Lie groups in [1].

Let $G$ be a Lie group as above with Lie algebra $\mathfrak{g}$. She has shown that $g$ is either Abelian or isomorphic to one of the following Lie algebras:

1. $[Y, Z] = W, [Z, W] = Y, [W, Y] = Z, X$ central,
2. $[X, Z] = X, [Y, Z] = Y, [X, W] = Y, [Y, W] = -X,$
3. $[X, Y] = Y, [X, Z] = Z, [X, W] = W,$
4. $[X, Y] = Y, [X, Z] = \frac{1}{2}Z, [X, W] = \frac{1}{2}W, [Z, W] = \frac{1}{2}Y,$

where $\{X, Y, Z, W\}$ is an orthonormal basis.

All above Lie groups are diffeomorphic to $\mathbb{R}^4$ unless case (1) which is diffeomorphic to $\mathbb{R} \times S^3$ (see [1] and [2]).

Now we compute the Levi-Civita connection and sectional curvature of any case separately. In Abelian case obviously $(G, g)$ is flat therefore we consider the other cases.
Case 1. A direct computation, by using formula (2.6) shows that for Levi-Civita connection of this case we have:

\[
\nabla X X = 0, \quad \nabla X Y = 0, \quad \nabla X Z = 0, \quad \nabla X W = 0, \\
\text{(3.1)}
\]

\[
\nabla Y X = 0, \quad \nabla Y Y = 0, \quad \nabla Y Z = \frac{1}{2} W, \quad \nabla Y W = -\frac{1}{2} Z, \\
\nabla Z X = 0, \quad \nabla Z Y = -\frac{1}{2} W, \quad \nabla Z Z = 0, \quad \nabla Z W = \frac{1}{2} Y, \\
\nabla W X = 0, \quad \nabla W Y = \frac{1}{2} Z, \quad \nabla W Z = -\frac{1}{2} Y, \quad \nabla W W = 0.
\]

Now by using Levi-Civita connection for curvature tensor we have:

\[
\mathbf{R}(Y, Z) Y = -\mathbf{R}(Z, W) W = -\frac{1}{4} Z, \\
\mathbf{R}(Y, W) W = \mathbf{R}(Y, Z) Z = \frac{1}{4} Y, \\
\mathbf{R}(Z, W) Z = \mathbf{R}(Y, W) Y = -\frac{1}{4} W, \\
\text{(3.2)}
\]

and in other cases \( \mathbf{R} = 0 \).

Let \( U = aX + bY + cZ + dW \) and \( V = \tilde{a}X + \tilde{b}Y + \tilde{c}Z + \tilde{d}W \) be two arbitrary vectors in \( \mathfrak{g} \) then we have:

\[
\mathbf{R}(V, U) U = -\frac{1}{4} \{(bc - \tilde{c}\tilde{b})(cY - bZ) + (bd - \tilde{d}\tilde{b})(dY - bW) + (cd - \tilde{d}\tilde{c})(dZ - cW)\}. \\
\text{(3.3)}
\]

Now assume that \( \{U, V\} \) is an orthonormal set then for sectional curvature \( K(U, V) \) we have:

\[
K(U, V) = \frac{1}{4} ((bc - \tilde{c}\tilde{b})^2 + (bd - \tilde{d}\tilde{b})^2 + (cd - \tilde{d}\tilde{c})^2) \geq 0. \\
\text{(3.4)}
\]

The last equation shows that in this case \( (G, g) \) is of non-negative sectional curvature.

Now consider the orthonormal basis \( \{X, Y, Z, W\} \). By using above formula for sectional curvature we can show that the scalar curvature \( S \) of this space is \( S = \frac{3}{2} \).

Case 2. Similar to case 1 by using formula (2.6) we have

\[
\nabla X X = -Z, \quad \nabla X Y = 0, \quad \nabla X Z = X, \quad \nabla X W = 0, \\
\text{(3.5)}
\]

\[
\nabla Y X = 0, \quad \nabla Y Y = -Z, \quad \nabla Y Z = Y, \quad \nabla Y W = 0, \\
\nabla Z X = 0, \quad \nabla Z Y = 0, \quad \nabla Z Z = 0, \quad \nabla Z W = 0, \\
\nabla W X = -Y, \quad \nabla W Y = X, \quad \nabla W Z = 0, \quad \nabla W W = 0.
\]

The curvature tensor of the above connection is as follows:

\[
\mathbf{R}(X, Y) X = -\mathbf{R}(Y, Z) Z = Y, \\
\text{(3.6)}
\]

\[
\mathbf{R}(X, Y) Y = \mathbf{R}(X, Z) Z = -X, \\
\mathbf{R}(X, Z) X = \mathbf{R}(Y, Z) Y = Z,
\]
and in other cases $R = 0$. In this case for any $U$ and $V$ we have:

$$R(V, U)U = -\{(a\tilde{b} - b\tilde{a})(aY - bX) + (a\tilde{c} - c\tilde{a})(aZ - cX) + (b\tilde{c} - c\tilde{b})(bZ - cY)\},$$

and for an orthonormal set $\{U, V\}$ the sectional curvature $K(U, V)$ is

$$K(U, V) = -\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2\} \leq 0,$$

which shows that in the case 2 $(G, g)$ is of non-positive sectional curvature.

Now the formula of sectional curvature shows that the scalar curvature is of the form $S = -6$.

**Case 3.** If we repeat the computations for case 3 then we have:

$$\nabla_X X = 0, \quad \nabla_X Y = 0, \quad \nabla_X Z = 0, \quad \nabla_X W = 0,$$

$$\nabla_Y X = -Y, \quad \nabla_Y Y = X, \quad \nabla_Y Z = 0, \quad \nabla_Y W = 0,$$

$$\nabla_Z X = -Z, \quad \nabla_Z Y = 0, \quad \nabla_Z Z = X, \quad \nabla_Z W = 0,$$

$$\nabla_W X = -W, \quad \nabla_W Y = 0, \quad \nabla_W Z = 0, \quad \nabla_W W = X,$$

and therefore for $R$ we have

$$R(X, Y)X = -R(Y, Z)Z = -R(Y, W)W = Y,$$

$$R(X, Y)Y = R(X, Z)Z = R(X, W)W = -X,$$

$$R(X, Z)X = R(Y, Z)Y = -R(Z, W)W = Z,$$

$$R(X, W)X = R(Y, W)Y = R(Z, W)Z = W,$$

and in other cases $R = 0$. Then for any $U$ and $V$ we have:

$$R(V, U)U = -\{(a\tilde{b} - b\tilde{a})(aY - bX) + (a\tilde{c} - c\tilde{a})(aZ - cX) + (b\tilde{c} - c\tilde{b})(bZ - cY)$$

$$+ (a\tilde{d} - d\tilde{a})(aW - dX) + (b\tilde{d} - d\tilde{b})(bW - dY) + (c\tilde{d} - d\tilde{c})(cW - dZ)\}.$$

Hence for an orthonormal set $\{U, V\}$ the sectional curvature $K(U, V)$ is as follows:

$$K(U, V) = -\{(a\tilde{b} - b\tilde{a})^2 + (a\tilde{c} - c\tilde{a})^2 + (b\tilde{c} - c\tilde{b})^2$$

$$+ (a\tilde{d} - d\tilde{a})^2 + (b\tilde{d} - d\tilde{b})^2 + (c\tilde{d} - d\tilde{c})^2\} \leq 0.$$

Therefore $(G, g)$ is of non-positive sectional curvature.

Similar to above cases we can obtain that the scalar curvature is of the form $S = -12$.

**Case 4.** Similar to aforementioned cases we can obtain $\nabla$ and $R$ as follows:

$$\nabla_X X = 0, \quad \nabla_X Y = 0, \quad \nabla_X Z = 0, \quad \nabla_X W = 0,$$

$$\nabla_Y X = -Y, \quad \nabla_Y Y = X, \quad \nabla_Y Z = -\frac{1}{4}W, \quad \nabla_Y W = \frac{1}{4}Z,$$

$$\nabla_Z X = -\frac{1}{2}Z, \quad \nabla_Z Y = -\frac{1}{4}W, \quad \nabla_Z Z = \frac{1}{2}X, \quad \nabla_Z W = \frac{1}{4}Y,$$

$$\nabla_W X = -\frac{1}{2}W, \quad \nabla_W Y = \frac{1}{4}Z, \quad \nabla_W Z = -\frac{1}{4}Y, \quad \nabla_W W = \frac{1}{2}X,$$
\(-R(X, Y)Y = -4R(X, Z)Z = -4R(X, W)W = 8R(Y, Z)W = -8R(Y, W)Z = -4R(Z, W)Y = X,\)
\(R(X, Y)X = -8R(X, Z)W = 8R(X, W)Z = -\frac{16}{7}R(Y, Z)Z = -\frac{16}{7}R(Y, W)W = 4R(Z, W)X = Y,\)
\(-4R(X, Y)W = 4R(X, Z)X = -8R(X, W)Y = \frac{16}{7}R(Y, Z)Y = 8R(Y, W)X = -\frac{16}{7}R(Z, W)W = Z,\)
\(4R(X, Y)Z = 8R(X, Z)Y = 4R(X, W)X = -8R(Y, Z)X = \frac{16}{7}R(Y, W)Y = \frac{16}{7}R(Z, W)Z = W,\)

and in other cases \(R = 0.\) For any \(U\) and \(V\) we have:

\[
R(V, U)U = -(\tilde{a} - b\tilde{a})(aY - bX - \frac{d}{4}Z + \frac{c}{4}W) + (a\tilde{c} - c\tilde{a})(-\frac{c}{4}X - \frac{d}{8}Y + \frac{a}{4}Z + \frac{b}{8}W)
\]
\[
+ (a\tilde{d} - d\tilde{a})(-\frac{d}{4}X + \frac{c}{8}Y - \frac{b}{8}Z + \frac{a}{4}W) + (b\tilde{c} - c\tilde{b})(\frac{d}{8}X - \frac{7c}{16}Y + \frac{7b}{16}Z - \frac{a}{8}W)
\]
\[
+ (b\tilde{d} - d\tilde{b})(-\frac{c}{8}X - \frac{7d}{16}Y + \frac{a}{8}Z + \frac{7b}{16}W) + (cd - d\tilde{c})(-\frac{b}{4}X + \frac{a}{4}Y - \frac{7d}{16}Z + \frac{7c}{16}W).\]

The above equation shows that for an orthonormal set \(\{U, V\}\) the sectional curvature \(K(U, V)\) is as follows:

\[
K(U, V) = -((\tilde{a} - b\tilde{a} + \frac{1}{4}(cd - d\tilde{c}))^2 + \frac{1}{4}(a\tilde{c} - c\tilde{a} + \frac{1}{2}(b\tilde{d} - d\tilde{b}))^2)
\]
\[
+ \frac{1}{4}(a\tilde{d} - d\tilde{a} - \frac{1}{2}(b\tilde{c} - c\tilde{b}))^2 + \frac{3}{8}((b\tilde{c} - c\tilde{b})^2 + (b\tilde{d} - d\tilde{b})^2 + (cd - d\tilde{c})^2) \leq 0.
\]

Therefore \((G, g)\) is of non-positive sectional curvature.

In this case the scalar curvature is \(S = \frac{-45}{8}.\)

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\textbf{REFERENCES}

1. M. L. Barberis, \textit{Hypercomplex Structures on Four-Dimensional Lie Groups}, Proceeding of the American Mathematical Society \textbf{125}(4), (1997) 1043-1054.

2. M. L. Barberis, \textit{Hyper-Kahler Metrics Conformal to Left Invariant Metrics on Four-Dimensional Lie Groups}, Mathematical Physics, Analysis and Geometry \textbf{6}, (2003) 1-8.

3. G.W. Gibbons, G. Papadopoulos and K.S. Stelle, \textit{HKT and OKT geometries on soliton black hole moduli spaces}, Nucl. Phys. B \textbf{508}, (1997) 623-658.

4. S. Kobayashi and K. Nomizu, \textit{Foundations of Differential Geometry}, VOL.2, (INTERSCIENCE PUBLISHERS) (1969).

5. Y. S. Poon, \textit{Examples of Hyper-Kähler Connections with Torsion}, Vienna, Preprint ESI \textbf{770}, (1999) 1-7.

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