Research Article

Applications of Magnetohydrodynamic Couple Stress Fluid Flow between Two Parallel Plates with Three Different Kernels

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In this paper, we investigate the implementations of newly introduced nonlocal differential operators as convolution of power law, exponential decay law, and the generalized Mittag-Leffler law with fractal derivative in fluid dynamics. The new operators are referred as fractal-fractional differential operators. The governing equations for the problem are constructed with the fractal-fractional differential operators. We present the stability analysis and the error analysis.

1. Introduction

Magnetohydrodynamics (MHD) deals with the study of the motion of electrically conducting fluids in the presence of the magnetic field. MHD flow has significant importance applications between infinite parallel plates in various areas such as geophysical, astrophysical, and metallurgical processing, MHD generators, pumps, geothermal reservoirs, polymer technology, and mineral industries [1–6]. In last few decades, fractional calculus has taken much interest in many fields [7, 8]. There are many definitions for the fractional derivative operators, and among them are Caputo-Fabrizio (CF) [9] and Atangana and Baleanu (AB) [10] definitions of fractional derivatives with a nonlocal and nonsingular kernels having all the characteristics of the old definitions [7, 11–23]. Farman et al. [24] have analyzed the numerical solution of SEIR Epidemic model of measles with noninteger time fractional derivatives by using the Laplace Adomian decomposition method. Ghanbari and Djilali [25] have taken mathematical analysis of a fractional-order predator-prey model with prey social behavior and infection developed in predator population. Ghanbari and Atangana [26, 27] have given the new edge detecting techniques based on fractional derivatives with nonlocal and nonsingular kernels. Recently, another idea of differentiation has been proposed by Atanagna [28].

We organize our manuscript as follows. We present the main definitions in Section 2. We construct the problem formulation in Section 3. We present the analysis of the model with the power law kernel in Section 4. We give the analysis of the model with the exponential decay kernel in Section 5. We discuss the analysis of the model with the Mittag-Leffler kernel in Section 6. We present the error analysis in Section 7. We give the conclusion in the last section.

2. Preliminaries

Definition 1. Assume that $g(\cdot)$ is a continuous function in the $(c_{11},d_{11})$ and fractal differentiable on $(c_{11},d_{11})$ with order $\eta$ then the fractal-fractional derivative of $g$ of order $\lambda$ in Riemann-Liouville sense with power law kernel is...
where
\[ \frac{dg(x)}{dx^\lambda} = \lim_{\epsilon \to 0} \frac{g(x) - g(x - \epsilon x^{1-\lambda})}{\epsilon x^{1-\lambda}}. \] (2)

**Definition 2.** Assume that \( g(z) \) is a continuous function in the \((c_{11}, d_{11})\) and fractal differentiable on \((c_{11}, d_{11})\) with order \( \eta \) then the fractal-fractional derivative of \( g \) of order \( \lambda \) in Riemann-Liouville sense with the exponential decay kernel is introduced as \([29]\)
\[ \mathcal{D}_t^\lambda g(z) = \frac{M(\eta)}{(1-\eta) \Gamma(1-\lambda)} \int_b^\infty g(x) \exp \left(-\frac{\eta}{1-\eta} (z-x)^\lambda \right) dx, \quad 0 < \eta, \lambda \leq 1. \] (3)

**Definition 3.** Assume that \( g(z) \) is a continuous function in the \((c_{11}, d_{11})\) and fractal differentiable on \((c_{11}, d_{11})\) with order \( \eta \) then the fractal-fractional derivative of \( g \) of order \( \lambda \) in Riemann-Liouville sense with the generalized Mittag-Leffler kernel is introduced as \([29]\)
\[ \mathcal{D}_t^\lambda g(z) = \frac{AB(\eta)}{(1-\eta) \Gamma(1-\lambda)} \int_b^\infty g(x) E_\lambda \left(-\frac{\eta}{1-\eta} (z-x)^\lambda \right) dx, \quad 0 < \eta, \lambda \leq 1. \] (4)

### 3. Problem Formulation

We consider
\[ \frac{\partial u_{i1}(\xi_{i1}, \tau_{i1})}{\partial \tau_{i1}} = \mu \frac{\partial^4 u_{i1}(\xi_{i1}, \tau_{i1})}{\partial \xi_{i1}^4} - \eta \frac{\partial^4 u_{i1}(\xi_{i1}, \tau_{i1})}{\partial \xi_{i1}^4} - \sigma B_{0i}^2 \frac{\partial^2 u_{i1}(\xi_{i1}, \tau_{i1})}{\partial \xi_{i1}^2}, \] (5)
\[ u_{i1}(0, \tau_{i1}) = 0, \quad u_{i1}(d, \tau_{i1}) = 0, \quad 0 \leq \xi_{i1} \leq h, \] (6)
\[ \frac{\partial^2 u_{i1}(\xi_{i1}, \tau_{i1})}{\partial \xi_{i1}^2} = 0, \text{ at } \xi_{i1} = 0 \text{ and } \xi_{i1} = h \text{ for any } \tau_{i1} > 0, \] (7)
\[ v = \frac{u_{i1}}{U_0}, \quad t = \frac{\tau_{i1} U_0}{h}, \quad y = \frac{\xi_{i1}}{h}, \] (8)
into Eqs. (5)-(8), and we obtain
\[ \frac{\partial^2 v(y, t)}{\partial t} = \frac{1}{\mathrm{Re}} \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \frac{\partial^3 v(y, t)}{\partial y^4} - M_{11} v(y, t) \right), \] (9)
\[ v(y, 0) = 0, \quad v(0, t) = 0, \quad v(1, t) = 1, \] (10)
\[ \frac{\partial^2 v(y, t)}{\partial y^2} = 0, \text{ at } y = 0 \text{ and } y = 1 \text{ for any } t > 0, \] (11)

where \( M_{11} = \sigma B_{0i}^2 d^2 / \mu \) is the magnetic field parameter, and \( \mathrm{Re} = \rho U_0 d / \mu \) is the Reynold number and \( h^2 = \eta / \mu \). We demonstrate the geometry of the physical model in Figure 1.

### 4. Solution of the Problem with the Power Law Kernel

We take into consideration the Eq. (10) with fractal-fractional differential operator using Definition 1 of power law kernel as
\[ \mathcal{D}_t^\lambda v(y, t) = \frac{1}{\mathrm{Re}} \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \frac{\partial^3 v(y, t)}{\partial y^4} - M_{11} v(y, t) \right), \] (12)
\[ \frac{1}{\Gamma(1 - \alpha_i)} \int_0^t \frac{d}{d\lambda} v(y, \lambda) (t - \lambda)^{-\alpha_i} d\lambda = \frac{\beta_{i1}}{\mathrm{Re}} \frac{1}{\Gamma(1 - \alpha_i)} \int_0^t \left( \frac{\partial^2 v(y, \lambda)}{\partial y^2} - \frac{\partial^3 v(y, \lambda)}{\partial y^4} - M_{11} v(y, \lambda) \right) (t - \lambda)^{\alpha_i - 1} d\lambda. \] (13)

The, we get
\[ v(y, \lambda) = \frac{\beta_{i1}}{\mathrm{Re}} \frac{1}{\Gamma(1 - \alpha_i)} \int_0^t F(y, \lambda) (t - \lambda)^{\alpha_i - 1} d\lambda. \] (14)

For simplicity, we take
\[ F(y, \lambda) = \beta_{i1} \lambda^{\alpha_i - 1} \left( \frac{\partial^2 v(y, \lambda)}{\partial y^2} - \frac{\partial^3 v(y, \lambda)}{\partial y^4} - M_{11} v(y, \lambda) \right), \] (15)
\[ v(y, \lambda) = \frac{1}{\mathrm{Re}} \frac{1}{\Gamma(1 - \alpha_i)} \int_0^t F(y, \lambda) (t - \lambda)^{\alpha_i - 1} d\lambda. \]

We discretize this equation at \((y_j, t = t_{n+1})\) and get
\[ v(y_j, t_{n+1}) = \frac{1}{\mathrm{Re}} \frac{1}{\Gamma(1 - \alpha_i)} \int_0^{t_{n+1}} F(y, \lambda) (t_{n+1} - \lambda)^{\alpha_i - 1} d\lambda, \]
\[ v(y_j, t_{n+1}) = \frac{1}{\mathrm{Re}} \frac{1}{\Gamma(1 - \alpha_i)} \sum_{j=0}^{n} F(y_j, \lambda) (t_{n+1} - \lambda)^{\alpha_i - 1} d\lambda. \] (16)

We apply the two-step Lagrange polynomial as
\[ p_j(\lambda) = \frac{\lambda - t_{j+1}}{t_j - t_{j+1}} F(y_j, t_j) - \frac{\lambda - t_j}{t_{j+1} - t_j} F(y_j, t_{j+1}). \] (17)
Thus, we will get
\[ v(\gamma, t_{\alpha}) = \frac{1}{\gamma(\alpha)} \sum_{\gamma} p(\lambda) (t_{\alpha} - \lambda)^{-\gamma+1} d\lambda \]
\[ = \sum_{\gamma} \left[ \hat{h}^{\alpha} F(y, t_{\alpha}) \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \right. \]
\[ - \left. (n-j)^{\alpha} (n-j+2+2a) \right] \left( \hat{h}^{\alpha} F(y, t_{\alpha+2}) \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \right) \]
\[ = \sum_{\gamma} \left[ \hat{h}^{\alpha} F(y, t_{\beta}) \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \right. \]
\[ - \left. (n-j)^{\alpha} (n-j+2+2a) \right] \left( \hat{h}^{\alpha} F(y, t_{\beta+2}) \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \right) \]
\[ = \sum_{\gamma} \left[ \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right. \]
\[ - \left. (n-j)^{\alpha} (n-j+2+2a) \right] \left( \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right) \]
\[ \times \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \left( (n+1-j)^{\alpha} (n-j+2+2a) \right) \]
(18)

We have
\[ F(y, t_{\beta}) = \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \]
(19)

Then, we will obtain
\[ v(\gamma, t_{\alpha}) = \frac{1}{\gamma(\alpha)} \sum_{\gamma} p(\lambda) (t_{\alpha} - \lambda)^{-\gamma+1} d\lambda \]
\[ = \sum_{\gamma} \left[ \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right. \]
\[ - \left. (n-j)^{\alpha} (n-j+2+2a) \right] \left( \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right) \]
\[ \times \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \left( (n+1-j)^{\alpha} (n-j+2+2a) \right) \]
(20)

We define
\[ A^{\alpha}_{n,j} = \hat{h}^{\alpha} \beta^{\alpha+1}(1) K_{\alpha+1}(\alpha+2)^{2} \]
\[ B^{\alpha}_{n,j} = \hat{h}^{\alpha} \beta^{\alpha+1}(1) M_{\alpha+1}(\alpha+2)^{2} \]
\[ C_{n,j} = \hat{h}^{\alpha} \beta^{\alpha+1}(1) M_{\alpha+1}(\alpha+2)^{2} \]
\[ D^{\alpha}_{n,j} = \hat{h}^{\alpha} \beta^{\alpha+1}(1) Y_{\alpha+1}(\alpha+2)^{2} \]
(21)

Then, we get
\[ v(\gamma, t_{\alpha}) = \sum_{\gamma} A^{\alpha}_{n,j} \left( \gamma + 2 \gamma + \gamma + 1 \right) - B^{\alpha}_{n,j} \]
\[ = \sum_{\gamma} \left[ \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right. \]
\[ - \left. (n-j)^{\alpha} (n-j+2+2a) \right] \left( \hat{h}^{\alpha} \beta^{\alpha+1}(1) \left( \Gamma_{\alpha+1} \right)^2 \right) \]
\[ \times \left( (n+1-j)^{\alpha} (n-j+2+a) \right) \left( (n+1-j)^{\alpha} (n-j+2+2a) \right) \]
(22)

We choose \( \varepsilon_{\alpha} = \delta_{1} \exp (i k m \gamma) \). Then, we have
\[ \delta_{\alpha} \exp (i k m \gamma) = A^{\alpha}_{n,j} \left[ \delta_{1} \exp (i k m \gamma) \right] + B^{\alpha}_{n,j} \exp (i k m \gamma) + C_{n,j} \exp (i k m \gamma) \]
\[ + D^{\alpha}_{n,j} \exp (i k m \gamma) \]
(23)

After simplification, we obtain
\[ \delta_{\alpha} = A^{\alpha}_{n,j} \left[ \delta_{1} \exp (i k m \gamma) \right] - B^{\alpha}_{n,j} \exp (i k m \gamma) + C_{n,j} \exp (i k m \gamma) \]
\[ + D^{\alpha}_{n,j} \exp (i k m \gamma) \]
(24)
When $n > 1$, we have

$$
\delta_{n+1} = \sum_{j=0}^{n} \left( A_{n,j} \delta_j (\text{exp}(ik_m\Delta y) - 2\delta_j + \delta_j \text{exp}(-ik_m\Delta y)) - B_{n,j} \delta_j (\text{exp}(2ik_m\Delta y) - 2\delta_j + \delta_j \text{exp}(-2ik_m\Delta y)) - C_{n,j} \delta_j \right) + \sum_{j=n+1}^{n} \left( \delta_{j-1} A_{n,j-1} + B_{n,j-1} \delta_{j-1} \text{exp}(ik_m\Delta y) + \delta_{j-1} \text{exp}(-ik_m\Delta y) \right) - C_{n,j-1} \delta_{j-1},
$$

(28)

Then, we get

$$
\delta_{n+1} = \sum_{j=0}^{n} \delta_j \left( -4 \sin^2 \left( \frac{k_m\Delta y}{2} \right) A_{n,j} + 4B_{n,j} \right) + \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j} \right) + \sum_{j=n+1}^{n} \left( \delta_{j-1} A_{n,j-1} + B_{n,j-1} \delta_{j-1} \right) \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j-1} \right).
$$

(29)

We assume that for all $n \geq 1$, $|\delta_n| < |\delta_0|$. We want to prove that $|\delta_{n+1}/\delta_0| < 1$. However,

$$
|\delta_{n+1}| \leq \sum_{j=0}^{n} |\delta_j| \left( -4 \sin^2 \left( \frac{k_m\Delta y}{2} \right) A_{n,j} + 4B_{n,j} \right) + \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j} \right) + \sum_{j=n+1}^{n} \left( \delta_{j-1} A_{n,j-1} + B_{n,j-1} \delta_{j-1} \right) \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j-1} \right).
$$

(30)

By induction hypothesis for all $n \geq 1$, $|\delta_n| < |\delta_0|$, we have

$$
|\delta_{n+1}| < |\delta_0| \left( \sum_{j=0}^{n} \left( -4 \sin^2 \left( \frac{k_m\Delta y}{2} \right) A_{n,j} + 4B_{n,j} \right) + \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j} \right) \right) + \sum_{j=n+1}^{n} \left( \delta_{j-1} A_{n,j-1} + B_{n,j-1} \delta_{j-1} \right) \left( \sin^2 \left( \frac{k_m\Delta y}{2} \right) - C_{n,j-1} \right).
$$

(31)

This inequality is true for all $m$. Thus, we reach

$$
|\delta_{n+1}| < |\delta_0| \left( \sum_{j=0}^{n} \left( -4A_{n,j} + 2B_{n,j} - C_{n,j} \right) \right) + \sum_{j=n+1}^{n} \left( -4A_{n,j-1} + 2B_{n,j-1} - C_{n,j-1} \right).
$$

(32)

We need to show that $|\delta_{n+1}/\delta_0| < 1$. Thus, we reach

$$
\left( \sum_{j=0}^{n} \left( -4A_{n,j} + 2B_{n,j} - C_{n,j} \right) \right) + \sum_{j=n+1}^{n} \left( -4A_{n,j-1} + 2B_{n,j-1} - C_{n,j-1} \right) < 1.
$$

(33)
5. Solution of the Problem with the Exponential Decay Kernel

We consider Eq. (10) with fractal-fractional differential operator using Definition 2 of exponential decay kernel as

\[
F_{FE}^\alpha D_{t}^\beta \psi (y,t) = \frac{1}{\Re} \left( \frac{\partial^2 \psi (y,t)}{\partial y^2} - \frac{\partial^2 \psi (y,t)}{\partial y^4} - M_1 \psi (y,t) \right),
\]

\[
M(\alpha_1) \int_{0}^{t} \psi (y,\lambda) \exp \left( - \frac{\alpha_1}{(1 - \alpha_1)} (t - \lambda) \right) d\lambda
= \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \left( \frac{\partial^2 \psi (y,t)}{\partial y^2} - \frac{\partial^2 \psi (y,t)}{\partial y^4} - M_1 \psi (y,t) \right).
\]

For simplicity, we define

\[
F(y,t, v(y,t)) = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \left( \frac{\partial^2 \psi (y,t)}{\partial y^2} - \frac{\partial^2 \psi (y,t)}{\partial y^4} - M_1 \psi (y,t) \right).
\]

Then, we reach

\[
v(y,\lambda) = \frac{1 - \alpha_1}{M(\alpha_1)} F(y,t, v(y,t)) + \frac{\alpha_1}{M(\alpha_1)} \int_{0}^{t} F(y,\lambda, v(\lambda)) d\lambda.
\]

We discretize Eq. (36) at \((y_j, t_{n+1})\) and \((y_j, t_n)\) as

\[
v_{i}^{n+1} = \frac{1 - \alpha_1}{M(\alpha_1)} F(y_j, t_{n+1}, v_i^{n+1}) + \frac{\alpha_1}{M(\alpha_1)} \int_{t_n}^{t_{n+1}} F(y_j, \lambda, v(\lambda)) d\lambda,
\]

\[
v_{i}^{n} = \frac{1 - \alpha_1}{M(\alpha_1)} F(y_j, t_n, v_i^{n-1}) + \frac{\alpha_1}{M(\alpha_1)} \int_{0}^{t_n} F(y_j, \lambda, v(\lambda)) d\lambda.
\]

Then, we obtain

\[
v_{i}^{n+1} = v_{i}^{n} + \frac{1 - \alpha_1}{M(\alpha_1)} \left( F(y_j, t_{n+1}, v_i^{n}) - F(y_j, t_{n+1}, v_i^{n-1}) \right)
+ \frac{\alpha_1}{M(\alpha_1)} \int_{t_n}^{t_{n+1}} F(y_j, \lambda, v(\lambda)) d\lambda,
\]

\[
v_{i}^{n} = v_{i}^{n} + \frac{1 - \alpha_1}{M(\alpha_1)} \left( F(y_j, t_n, v_i^{n}) - F(y_j, t_n, v_i^{n-1}) \right)
+ \frac{\alpha_1}{M(\alpha_1)} \left( \frac{3h}{2} F(y_j, t_n, v_i^{n}) - \frac{h}{2} F(y_j, t_{n-1}, v_i^{n-1}) \right),
\]

where

\[
F(y_j, t_n, v(y_j, t_n)) = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \left( \frac{v_{i}^{n+1} - 2v_i^{n} + v_i^{n-1}}{(\Delta y)^2} - \frac{v_{i}^{n+1} - 4v_i^{n+1} + 6v_i^{n} - 4v_i^{n-1} + v_i^{n-2}}{(\Delta y)^4} - M_1 v_i^{n} \right).
\]

Thus, we acquire

\[
\psi_{i}^{n+1} = \psi_{i}^{n} + \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \left( \frac{v_{i}^{n+1} - 2v_i^{n} + v_i^{n-1}}{(\Delta y)^2} - \frac{v_{i}^{n+1} - 4v_i^{n+1} + 6v_i^{n} - 4v_i^{n-1} + v_i^{n-2}}{(\Delta y)^4} - M_1 v_i^{n} \right).
\]

\[
\frac{3(1 - \alpha_1) + 3h_1}{2M(\alpha_1)} \psi_{i}^{n+1} - \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \left( \frac{v_{i}^{n+1} - 2v_i^{n} + v_i^{n-1}}{(\Delta y)^2} - \frac{v_{i}^{n+1} - 4v_i^{n+1} + 6v_i^{n} - 4v_i^{n-1} + v_i^{n-2}}{(\Delta y)^4} - M_1 v_i^{n} \right)
\]

\[
\frac{2(1 - \alpha_1) + h_1}{2M(\alpha_1)} \psi_{i}^{n+1} = \psi_{i}^{n+1} + \left( \frac{1 - \alpha_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \right) F(y_j, t_{n+1}, v_i^{n+1}).
\]

For simplicity, we let

\[
A_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{N_{a_1}}{\mu(\Delta y)}, \quad B_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{N_{a_1}}{\mu(\Delta y)},
\]

\[
C_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{M_1 N_{a_1}}{\mu(\Delta y)}, \quad D_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{N_{a_1}}{\mu(\Delta y)},
\]

\[
E_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{M_1 N_{a_1}}{\mu(\Delta y)}, \quad F_{n,a,\beta_1} = \frac{\beta_1}{\Re} \frac{\mu_{n-1}}{\mu_r} \frac{M_1 N_{a_1}}{\mu(\Delta y)}.
\]

We choose \(h_1 = 4(\varepsilon_{n+1} - \varepsilon_{n})\). Therefore, we reach

\[
\delta_{n+1} = \delta_{n} - C_{n,\beta_1} \delta_{n} + F_{n,a,\beta_1} \delta_{n} + A_{n,a,\beta_1} \left( \delta_n \exp \left( i k_{m} y \Delta y \right) \right)
\]

\[
- D_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - E_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right)
\]

\[
\delta_{n+1} = \delta_{n} - 4D_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - 4E_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]

After simplification, we obtain

\[
\delta_{n+1} = \delta_{n} - C_{n,\beta_1} \delta_{n} + F_{n,a,\beta_1} \delta_{n} + A_{n,a,\beta_1} \left( \delta_n \exp \left( i k_{m} y \Delta y \right) \right)
\]

\[
- D_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - E_{n,a,\beta_1} \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]

\[
\delta_{n+1} = \left( \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right) \right) - \delta_{n} \exp \left( i k_{m} (y - \Delta y) \right)
\]
Thus, we have
\[
\delta_{n+1} = \delta_0 \left( 1 - 4A_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{n,n+1} \right) + \delta_n. \tag{44}
\]

For \( n = 0 \), we get
\[
\delta_1 = \delta_0 \left( 1 - 4A_{0,n} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{0,n} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{0,n} \right). \tag{45}
\]

The \( |\delta_1/\delta_0| < 1 \) implies
\[
\left| 1 - 4A_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{n,n+1} \right| < 1. \tag{46}
\]

This is true for all \( m \). Thus, we get
\[
\left| 1 - 4A_{0,n} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{0,n} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{0,n} \right| < 1. \tag{47}
\]

We assume that \( |\delta_1/\delta_0| < 1 \). Thus, we need to show \( |\delta_{n+1}/\delta_n| < 1 \).

Thus, we have
\[
|\delta_{n+1}| = |\delta_0 | \left( 1 - 4A_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{n,n+1} \right) \]
\[\leq |\delta_n | \left( 1 - 4A_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{n,n+1} \right). \tag{48}
\]

Thus, we obtain
\[
\left| 1 - 4A_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) + 4B_{n,n+1} \sin^2 \left( \frac{k_m \Delta y}{2} \right) - C_{n,n+1} \right| < 1. \tag{49}
\]

This inequality is true for all \( m \). Thus, we get
\[
\left| \left( 1 - 4A_{n,n+1} + 20B_{n,n+1} - C_{n,n+1} \right) \right| < 1. \tag{50}
\]

6. Solution of the Problem with the Generalized Mittag-Leffler Kernel

We take into consideration the Eq. (10) with fractal-fractional differential operator using Definition 3 of Mittag-Leffler kernel as
\[
\int_0^t D_t^{\alpha_i} v(y, t) = \frac{1}{\text{Re} \left( \alpha_i \right)} \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \frac{\partial^4 v(y, t)}{\partial y^4} - M_1 v(y, t) \right),
\]
\[
\frac{A B \alpha_i}{1 - \alpha_i} \left\{ \frac{1}{\text{Re} \left( \alpha_i \right)} \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \frac{\partial^4 v(y, t)}{\partial y^4} - M_1 v(y, t) \right) \right\} \right|_0^t F(y, \lambda, v(y, \lambda)) d\lambda. \tag{51}
\]

For simplicity, we define
\[
F(y, v, t) = \frac{\alpha_i}{\text{Re} \left( \alpha_i \right)} \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \frac{\partial^4 v(y, t)}{\partial y^4} - M_1 v(y, t) \right). \tag{52}
\]

Then, we get
\[
v(y, \lambda) = \frac{1}{\alpha_i} F(y, t, v(y, t)) + \frac{\alpha_i}{\text{Re} \left( \alpha_i \right)} \int_0^t F(y, \lambda, v(y, \lambda)) d\lambda. \tag{53}
\]

We discretize above Eq. (53) at \( (y, t_{n+1}) \) as
\[
v_{i+1}^{n+1} = \frac{1}{\alpha_i} F(y_{i+1}, v_{i+1}^{n+1}) + \frac{\alpha_i}{\text{Re} \left( \alpha_i \right)} \sum_{j=0}^{n+1} \left[ \frac{h^n F(y_{i+1}, v_{i+1}^{n+1})}{\text{Re} \left( \alpha_i \right)} \right] \left[ (n + 1 - j)^n (n + 1 - j + 1) \right]. \tag{54}
\]

Then, we obtain
\[
v_{i+1}^{n+1} - \frac{1}{\alpha_i} F(y_{i+1}, v_{i+1}^{n+1}) + \frac{\alpha_i}{\text{Re} \left( \alpha_i \right)} \sum_{j=0}^{n+1} \left[ \frac{h^n F(y_{i+1}, v_{i+1}^{n+1})}{\text{Re} \left( \alpha_i \right)} \right] \left[ (n + 1 - j)^n (n + 1 - j + 1) \right]. \tag{55}
\]

We have
\[
F(y_{i+1}, v_{i+1}^{n+1}) = \beta_i h^{4i-1} \left( \frac{\partial^{2i} v_{i+1}^{n+1}}{\partial y^{2i}} + \frac{\partial^{4i} v_{i+1}^{n+1}}{\partial y^{4i}} - M_1 v_{i+1}^{n+1} \right). \tag{56}
\]
Then, we will obtain

\[
\psi_{n+1}^{v_i} = \beta_i (1 - \alpha_i) \Gamma(\beta_i)^{-1} \psi_{n}^{v_i} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i)) \frac{a_i}{Re AB(\alpha_i)} \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i))). \tag{57}
\]

For simplicity, we let

\[
M_{ai} = \frac{\beta_i t_j^{\beta_i-1} (1 - \alpha_i)}{Re (\Delta y)^2 AB(\alpha_i)}, \quad K_{ai} = \frac{\beta_i t_j^{\beta_i-1} M_{ij} (1 - \alpha_i)}{Re AB(\alpha_i)}, \\
T_{ai} = \frac{\beta_i t_j^{\beta_i-1} (1 - \alpha_i)}{Re (\Delta y)^2 AB(\alpha_i)}, \\
A_{a_{i},b_{i}} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} K_{ai}}{I(a_i + 2) Re (\Delta y)^2 AB(\alpha_i)}, \\
B_{a_{i},b_{i}} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} K_{ai}}{I(a_i + 2) Re (\Delta y)^2 AB(\alpha_i)}, \\
C_{a_{i},b_{i}} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} M_{ij} K_{ai}}{I(a_i + 2) Re AB(\alpha_i)}, \\
A_{a_{i},b_{i}}^{n,j} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} M_{ij} K_{ai}}{I(a_i + 2) Re (\Delta y)^2 AB(\alpha_i)}, \\
B_{a_{i},b_{i}}^{n,j} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} M_{ij} K_{ai}}{I(a_i + 2) Re (\Delta y)^2 AB(\alpha_i)}, \\
C_{a_{i},b_{i}}^{n,j} = \frac{h_{a_i}^{\varepsilon} \beta_i t_j^{\beta_i-1} M_{ij} K_{ai}}{I(a_i + 2) Re AB(\alpha_i)}, \\
K_{a_{i}}^{n,j} = ((n + j)^\varepsilon - (n - j)^\varepsilon (n - j + 2 \alpha_i)), \\
y_{a_{i}}^{n,j} = (n + j)^\varepsilon - (n - j)^\varepsilon (n - j + 1 + \alpha_i)). \tag{58}
\]

Then, we get

\[
\psi_{n+1}^{v_i} = M_{ai} \psi_{n}^{v_i} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i)) \frac{a_i}{Re AB(\alpha_i)} \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i))). \tag{59}
\]

We choose \( \varepsilon = \delta_n \exp (i \kappa_n y) \). Then, we acquire

\[
\delta_n = \exp (i \kappa_n y) - \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i)) \frac{a_i}{Re AB(\alpha_i)} \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i))). \tag{57}
\]

Then, we get

\[
\delta_{n+1} = M_{ai} \delta_n \exp (i \kappa_n y) - \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} + \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i)) \frac{a_i}{Re AB(\alpha_i)} \sum_{j=1}^{n} \frac{\alpha_i^{\delta_j} \beta_i^{\delta_j} \psi_{n-j+2\alpha_i}^{v_i} - 4v_i^{\delta_j} + 6v_i^{\delta_j} - 4v_i^{\delta_j} + v_i^{\delta_j} - M_{ij} \psi_{n}^{v_i}}{(\Delta y)^j} \times (n + j)^\varepsilon - (n - j)^\varepsilon (n + j + 2\alpha_i))). \tag{57}
\]

We prove by induction. For \( n = 0 \), we have
\[ \delta_1 = M_n [\delta_0 \exp (ik_n \Delta y) - 2\delta_0 + \delta_0 \exp (ik_n \Delta y)] - K_n \delta_0 - T_n (\delta_0 \exp (2ik_n \Delta y)) - 4\delta_0 \exp (ik_n \Delta y) + 6\delta_0 - 4\delta_0 \exp (ik_n \Delta y) + \delta_0 \exp (-2ik_n \Delta y)] + A^{n,0}_{0,i} [\delta_0 \exp (ik_n \Delta y) - 2\delta_0 + \delta_0 \exp (-ik_n \Delta y)] - B^{n,0}_{0,i} \left( \delta_0 \exp (2ik_n \Delta y) - \delta_0 \exp (ik_n \Delta y) + 6\delta_0 - 4\delta_0 \exp (ik_n \Delta y) + \delta_0 \exp (-2ik_n \Delta y) \right) - C^{n,0}_{0,i} \delta_0. \]

(62)

Then, we get

\[ \delta_1 = \delta_0 M_n [\exp (ik_n \Delta y) - 2 + \exp (-ik_n \Delta y)] - K_n \delta_0 - \delta_0 T_n (\exp (2ik_n \Delta y)) - 4 \exp (ik_n \Delta y) + 6 - 4 \exp (-ik_n \Delta y) + \exp (-2ik_n \Delta y)] + \delta_0 A^{n,0}_{0,i} [\exp (ik_n \Delta y) - 2 + \exp (-ik_n \Delta y)] - \delta_0 B^{n,0}_{0,i} \exp (2ik_n \Delta y) - 4 \exp (ik_n \Delta y) + 6 - 4 \exp (ik_n \Delta y) + \exp (-2ik_n \Delta y)] - \delta_0 C^{n,0}_{0,i} \delta_0. \]

(63)

Thus, we reach

\[ \delta_1 = \delta_0 M_n \left[ -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) - K_n \delta_0 + 4i\delta_0 T_n \left( \sin \left( \frac{k_n \Delta y}{2} \right) + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right) \right] + \delta_0 \left[ -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) A^{n,0}_{0,i} + 4 \sin \left( \frac{k_n \Delta y}{2} \right) + \sin \left( \frac{k_n \Delta y}{2} \right) \right] - C^{n,0}_{0,i} \delta_0. \]

(64)

After simplification, we obtain

\[ \delta_1 = \delta_0 \left( -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) \left( M_n + A^{n,0}_{0,i} \right) - K_n - C^{n,0}_{0,i} \right) \left[ + \left( T_n + B^{n,0}_{0,i} \right) \left( \sin \left( \frac{k_n \Delta y}{2} \right) + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right) \right]. \]

(65)

We should show \( |\delta_1/\delta_0| < 1 \). Therefore, we have

\[ \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) \left( M_n + A^{n,0}_{0,i} \right) - K_n - C^{n,0}_{0,i} \right| < 1. \]

(66)

\[ \left| + \left( T_n + B^{n,0}_{0,i} \right) \left( \sin \left( \frac{k_n \Delta y}{2} \right) + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right) \right| < 1. \]

(67)

Since we have for all \( m \), we obtain

\[ \left| -4 \left( M_n + A^{n,0}_{0,i} \right) - K_n - C^{n,0}_{0,i} + 20 \left( T_n + B^{n,0}_{0,i} \right) \right| < 1. \]

(68)

When \( n > 1 \), we have

\[ \delta_{n+1} = M_n \left[ \delta_0 \exp (ik_n \Delta y) - 2\delta_0 + \delta_0 \exp (-ik_n \Delta y) \right] - K_n \delta_0 + T_n \left( \delta_0 \exp (2ik_n \Delta y) \right) - 4\delta_0 \exp (ik_n \Delta y) + 6\delta_0 - 4\delta_0 \exp (-ik_n \Delta y) + \delta_0 \exp (-2ik_n \Delta y)] + A^{n,0}_{0,i} \left[ \delta_0 \exp (ik_n \Delta y) - 2\delta_0 + \delta_0 \exp (-ik_n \Delta y) \right] - \delta_0 B^{n,0}_{0,i} \left( \delta_0 \exp (2ik_n \Delta y) - \delta_0 \exp (ik_n \Delta y) + 6\delta_0 - 4\delta_0 \exp (ik_n \Delta y) + \delta_0 \exp (-2ik_n \Delta y) \right) - \delta_0 C^{n,0}_{0,i} \delta_0. \]

(69)

We suppose that for all \( n \geq 1 \), \( |\delta_n/\delta_0| < 1 \). We want to prove that \( |\delta_{n+1}/\delta_0| < 1 \). However,

\[ |\delta_{n+1}| \leq |\delta_0| \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) M_n + K_n \right| + \sum_{j=1}^{n} |\delta_j| \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) A^{n,0}_{0,i} + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right. \]

(70)

By induction hypothesis for all \( n \geq 1 \), \( |\delta_n| < |\delta_0| \), we have

\[ |\delta_{n+1}| \leq |\delta_0| \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) M_n + K_n \right| + \sum_{j=1}^{n} |\delta_j| \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) A^{n,0}_{0,i} + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right. \]

(71)

This inequality is true for all \( m \). Thus, we reach

\[ |\delta_{n+1}| \leq 4M_n + K_n \left| -4 \sin^2 \left( \frac{k_n \Delta y}{2} \right) A^{n,0}_{0,i} + 4 \sin \left( \frac{k_n \Delta y}{2} \right) \right. \]

(72)
We need to show that $|\delta_{n+1}/\delta_0| < 1$. Thus, we get

$$
\left| -4M_{n} + K_n \right| + \sum_{j=0}^{n} \left( -4A^{n,\beta_1}_{n,j} + 20B^{n,\beta_1}_{n,j} - C^{n,\beta_1}_{n,j} \right) \\
+ \left( 4A^{n,\beta_1}_{n,j+1} + 20B^{n,\beta_1}_{n,j} - C^{n,\beta_1}_{n,j+1} \right) < 1.
$$

(72)

7. Error Analysis

In this section, we will consider the error analysis.

$$
\mathcal{P}_{\mu}^{\alpha,\beta_1} = \frac{1}{\Re \mathcal{I}(\alpha)} \int_0^t \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \partial^4 v(y, t) - M_{11} v(y, t) \right).
$$

(73)

Then, we get

$$
v(y, t) = \frac{\beta_1}{\Re \mathcal{I}(\alpha)} \int_0^t \left( \frac{\partial^2 v(y, t)}{\partial y^2} - \partial^4 v(y, t) - M_{11} v(y, t) \right)(t - \lambda)^{n-1} d\lambda.
$$

(74)

For simplicity, we take

$$
F(y, \lambda) = \beta_1 \lambda^{n-1} \left( \frac{\partial^2 v(y, \lambda)}{\partial y^2} - \partial^4 v(y, \lambda) - M_{11} v(y, \lambda) \right).
$$

(75)

Then, we have

$$
v(y, t) = \frac{1}{\Re \mathcal{I}(\alpha)} \int_0^t F(y, \lambda)(t - \lambda)^{n-1} d\lambda.
$$

(76)

At $y = y_{n+1}$, we get

$$
v^{n+1} = \frac{1}{\Re \mathcal{I}(\alpha)} \int_0^t F(y, \lambda)(t - \lambda)^{n-1} d\lambda
$$

$$
= \frac{1}{\Re \mathcal{I}(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \mathcal{P}_{\mu}^{\alpha,\beta_1}(y, \lambda)(t - \lambda)^{n-1} d\lambda
$$

$$
= \frac{1}{\Re \mathcal{I}(\alpha)} \int_0^t F(y, \lambda)(t - \lambda)^{n-1} d\lambda + \left( \frac{\lambda - t_j}{2t_{j+1} - t_j} \right)^{n-1} \partial F(y, \lambda) \big|_{t_j}
$$

$$
= \frac{1}{\Re \mathcal{I}(\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \mathcal{P}_{\mu}^{\alpha,\beta_1}(y, \lambda)(t - \lambda)^{n-1} d\lambda + \left( \frac{\lambda - t_j}{2t_{j+1} - t_j} \right)^{n-1} \partial F(y, \lambda) \big|_{t_j}
$$

(77)

Then, we have

$$
\mathcal{R}_{n+1} = \frac{1}{\Re \mathcal{I}(\alpha)} \sum_{j=0}^{n} \left( \frac{\lambda - t_j}{2t_{j+1} - t_j} \right)^{n-1} \partial F(y, \lambda) \big|_{t_j}
$$

$$
\cdot (t_j - \lambda)^{n-1} d\lambda
$$

(78)

We have

$$
\sup_{\mathcal{C}(y, \lambda) \in \mathcal{I}(\alpha)} \left| \frac{\partial^2 v(y, \lambda)}{\partial y^2} - \partial^4 v(y, \lambda) - M_{11} v(y, \lambda) \right|
$$

$$
< \mathcal{R}_{n+1} \left( \beta_1 - 1 \right) \beta_1 \lambda^{n-1} \left( \frac{\partial^2 v(y, \lambda)}{\partial y^2} + \frac{\partial^4 v(y, \lambda)}{\partial y^4} + \left| M_{11} v(y, \lambda) \right| \right)
$$

$$
+ 2 \mathcal{R}_{n+1} \left( \beta_1 - 1 \right) \lambda^{n-1} \left( \frac{\partial^2 v(y, \lambda)}{\partial y^2} + \frac{\partial^4 v(y, \lambda)}{\partial y^4} + \left| M_{11} v(y, \lambda) \right| \right)
$$

(79)
Therefore, we acquire
\[
|R^h_n| < \frac{C_1C_2}{F(a_1)} \left[ |(\beta_1 - 1)(\beta_1 - 2)|A_1 \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right] + |M_{11}||v|_{\infty}
\]
\[+ 2|\beta_1 - 1|A_1 \left( \frac{\partial \nu}{\partial y} \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right) + |M_{11}||v|_{\infty}\]
\[+ A_3 \left( \frac{\partial \nu}{\partial y} \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right) + |M_{11}||v|_{\infty} \sum_{j=1}^{n} (t_{n+1} - \lambda)^{a_i-1} d\lambda,
\]
where
\[
\frac{1}{F(a_1)} \sum_{j=1}^{n} \int_{t_j}^{t_{n+1}} (t_{n+1} - \lambda)^{a_i-1} d\lambda = \frac{(\Delta t)^{a_i}(n+1)^{a_i}}{F(1+a_i)}.
\]

Therefore, we obtain
\[
|R^h_n| < \frac{C_1C_2}{F(a_1)} \left[ |(\beta_1 - 1)(\beta_1 - 2)|A_1 \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right] + |M_{11}||v|_{\infty}
\]
\[+ 2|\beta_1 - 1|A_1 \left( \frac{\partial \nu}{\partial y} \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right) + |M_{11}||v|_{\infty}\]
\[+ A_3 \left( \frac{\partial \nu}{\partial y} \left( \frac{\partial \nu}{\partial y} \right)_{\infty} + \frac{\partial \nu}{\partial y} \right) + |M_{11}||v|_{\infty} \sum_{j=1}^{n} (\Delta t)^{a_i}(n+1)^{a_i} \frac{1}{F(1+a_i)}
\]
(82)

**Remark 4.** Error analysis with exponential decay kernel and Mittag-Leffler kernel can be obtained likewise. Therefore, we misplaced the error analysis for them.

### 8. Conclusion

In this paper, we investigated the fractional MHD incompressible couple stress fluid flow between two parallel plates. We discussed the discretization and the stability analysis for three different kernels. Additionally, we discussed the error analysis of the model in details.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflict of interest.

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