Padé Estimate of QCD’s Infrared Boundary

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Abstract

A mass scale $\mu_c$ representing the boundary between effective theories of strong interactions and perturbative QCD (infrared and ultraviolet regimes) exists when the $\beta$-function is characterized by a simple pole. This behaviour, which is known to occur in $N = 1$ supersymmetric Yang-Mills theories, leads to an infrared attractor in the evolution of the coupling constant, with the mass scale $\mu_c$ of the attractor providing a natural boundary between the infrared and ultraviolet regimes. It is demonstrated that $[2|2]$, $[3|1]$ and $[1|3]$ Padé-approximant versions of the three-flavour ($n_f = 3$) QCD $\beta$-function each contain a simple pole corresponding to such an infrared attractor. All three approximants, separately considered, are seen to lead to nearly equivalent estimates for the mass scale $\mu_c$.

Although QCD is well-understood both qualitatively and quantitatively as a perturbative theory for the strong interactions, we have a surprisingly limited amount of information about the infrared boundary of its perturbative domain. Such a boundary (which is also anticipated by arguments in which hadrons and quarks are dual field variables for weak and strong QCD) must exist to separate effective strong interaction theories (e.g., chiral perturbation theory, linear sigma model, etc.) from the higher-momentum region characterised by the perturbative quantum field theory of quarks and gluons. Consequently, one can argue that the infrared behaviour of the perturbative QCD coupling should not be characterised by smooth evolution to an infrared-stable fixed point, but rather by behaviour that would clearly separate the infrared domain of effective strong-interaction theories from perturbative QCD. Indeed, there exists both lattice and analytical, and Padé-approximant corroborations for the idea that an infrared-stable fixed point does not occur within QCD unless the number of active fermion flavours contributing to the evolution of the QCD coupling is substantially larger than three.

A clear demarcation between the infrared and ultraviolet regions would occur if the $n_f = 3$ QCD $\beta$-function were characterised not by a positive zero corresponding to an infrared-stable fixed point, but rather by a positive pole, an infrared-attractive singularity in the $\beta$-function occurring at a momentum scale which necessarily corresponds to a lower bound on the domain of the running QCD coupling constant (assumed here to be real). Such behaviour, schematically presented in Fig. 1, is known to characterise the exact $\beta$-function for $N = 1$ supersymmetric Yang-Mills theory in the NSVZ renormalisation scheme, a “supersymmetric gluodynamics” whose $\beta$-function pole forms an infrared-attractive terminal point for coupling-constant evolution within the theory’s asymptotically-free

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phase \[1\]. There exists evidence from Padé-approximant methods that similar dynamics may also characterise the infrared region of QCD \[4\]. In the present note, we discuss whether such methods can provide information as to the actual mass scale which separates the infrared and ultraviolet domains of strong interaction physics, as well as the strong-interaction couplant magnitude characterising this mass scale.

Weighted asymptotic Padé-approximant (WAPAP) methods have been utilised to estimate the \(n_f\)-dependence of the five-loop contribution (\(\beta_4\)) to the QCD \(\beta\)-function, defined here for \(\alpha_s(\mu)/\pi\) to be

\[
\mu^2 \frac{d\alpha}{d\mu^2} = -\sum_{k=0}^{\infty} \beta_k x^{k+2} = -\beta_0 x^2 \left(1 + R_1 x + R_2 x^2 + R_3 x^3 + R_4 x^4 + \ldots\right) . \tag{1}
\]

For \(n_f = 3\), the known \(\overline{\text{MS}}\) \(\beta\)-function coefficients in \(1\) are \[1\]

\[
\beta_0 = \frac{9}{4} , \quad \beta_1 = 4 , \quad \beta_2 = \frac{3863}{384} , \quad \beta_3 = 47.2280 . \tag{2}
\]

The WAPAP estimate of the \(n_f\)-dependence of the five-loop term (inclusive of quadratic-Casimir contributions to \(\beta_3\)) is found to be \[1\]

\[
\beta_4 \cong \frac{1}{4^5} \left(7.59 \times 10^{-5} - 2.19 \times 10^5 n_f + 2.05 \times 10^4 n_f^2 - 49.8 n_f^3 - 1.84 n_f^4\right) \xrightarrow{n_f \to 3} 278 , \tag{3}
\]

\[
R_4 = \frac{\beta_4}{\beta_0} \xrightarrow{n_f \to 3} 124 . \tag{4}
\]

This estimate may be used to construct non-trivial \(N + M = 4 [N|M]\)-Padé approximants that reproduce the known coefficients \(R_1-R_3\) and the estimate (4) for \(R_4\) within the \(\beta\)-function series of (1):

\[
\beta^{[2][2]}(x) = \frac{9}{4} x^2 \left[\frac{1 - 5.4983 x - 1.9720 x^2}{1 - 7.2762 x + 6.4923 x^2}\right] \tag{5}
\]

\[
\beta^{[1][3]}(x) = \frac{9}{4} x^2 \left[\frac{1 - 5.7397 x}{1 - 7.5174 x + 8.8933 x^2 - 3.1896 x^3}\right] \tag{6}
\]

\[
\beta^{[3][1]}(x) = \frac{9}{4} x^2 \left[\frac{1 - 4.1155 x - 6.0058 x^2 - 5.3589 x^3}{1 - 5.8932 x}\right] \tag{7}
\]

All three approximants above exhibit a positive pole that precedes any positive zeroes, consistent with infrared dynamics analogous to those of NSVZ supersymmetric gluodynamics, as discussed above \[1\]. This first positive pole, which corresponds to the couplant magnitude at the infrared-boundary momentum scale \(\mu_c\), is surprisingly comparable for all three approximants:

\[
[2][2] : \quad x(\mu_c) = 0.160 \quad , \tag{8}
\]

\[
[1][3] : \quad x(\mu_c) = 0.162 \quad ,
\]

\[
[3][1] : \quad x(\mu_c) = 0.170 .
\]

The fact that all three approximants exhibit a positive pole of nearly equivalent magnitude (which precedes any positive zeroes) is indicative that such a pole is not likely to be an artefact defect pole \[1\], but rather a manifestation of a true pole within the underlying all-orders \(\beta\)-function. It is to be noted that the pole couplant magnitude, as estimated in \(1\), is itself sufficiently small for perturbative physics to remain viable near the infrared boundary. This behaviour is very different from an infrared-slavery scenario in which the QCD couplant grows non-perturbatively large in the vicinity of a deep-infrared Landau singularity.

Similar consistency of poles obtained from differing Padé approximants to the perturbative \(\beta\)-function has already been shown to characterise supersymmetric gluodynamics in both NSVZ and in DRED renormalization

\[1\] The fact that a positive pole is always found to precede any positive zeroes for all three approximants has already been established \[1\] for arbitrary values of \(R_4\).
schemes [1]. We reiterate that the pole in the former of these two schemes is known to occur from the all-orders β-function expression, whether derived via NSVZ instanton calculus [3] or via imposition of the Adler-Bardeen theorem upon the supermultiplet structure of the theory [3, 12].

One can utilise the infrared-attractive couplant values [8] in order to obtain separate estimates of the infrared boundary μc for each approximant considered. For specific n_f = 3 [N|M] approximant versions of the QCD β-function, one finds that

\[ \mu_c^{[N|M]} = m_\pi \exp \left[ \frac{1}{2} \int_{\mu_c}^{\Omega} \frac{dx}{\beta^{[N|M]}(x)} \right] . \]  

(9)

We utilise the approximants [3, 4] within the integrand of (9), as well as the respective values [8] for [2|2], [1|3], and [3|1] approximant values of \( x(\mu_c) \) for the upper bound of integration. For the lower bound of integration, we assume \( \alpha_s^{\overline{MS}}(\mu_c) = \pi x(\mu_c) = 0.33 \pm 0.02 \), consistent with recent analyses [13]. We then obtain via (9) the following values for the infrared-boundary momentum scale:

\[
\begin{align*}
\mu_c^{[2|2]} &= 1.14 \pm 0.11 \text{ GeV} \\
\mu_c^{[1|3]} &= 1.13 \pm 0.11 \text{ GeV} \\
\mu_c^{[3|1]} &= 1.09 \pm 0.11 \text{ GeV}
\end{align*}
\]  

(10)

These results not only exhibit remarkable consistency with each other, but also support the identification of perturbative QCD’s infrared boundary with a momentum scale at (or somewhat above) the mass scale characterising nucleons and the vector meson octet. This picture is quite different from the usual one of an perturbative QCD’s infrared boundary with a momentum scale at (or somewhat above) the mass scale characterising nucleons and the vector meson octet. This picture is quite different from the usual one of an

The results [3] and (10) are, of course, sensitive to input information. An across-the-board 150 MeV decrease from (10) in the estimated range of \( \mu_c \) is seen to occur if we choose the three-flavour threshold at \( \mu_3 = m_\pi (\mu_c) \cong 1.25 \text{ GeV} \). More significantly, the set of values (10) for \( \mu_c \) is WAPAP estimate (3,4) for the infrared terminus of the couplant evolution within the asymptotically free phase of n_f = 3 QCD are consistent with both the applicability of controllably-perturbative QCD down to O(1 GeV) momentum scales, as well as the necessity for alternative descriptions (e.g. effective Lagrangians and hadronic field variables) to characterise sub-GeV (or sub-4πf_π(2) strong-interaction physics.

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To test the uniformity of the infrared-boundary mass scale obtained via different approximants, let us choose R_4 for each approximant so as to ensure the occurrence of a positive pole at \( \alpha_s (\mu_c) = \pi/4 \) (i.e., at \( x = 1/4 \)). This particular choice is motivated both as the critical value of the strong coupling for chiral symmetry breaking [17] as well as by the couplant value characterising Nambu-Jona-Lasinio and linear-sigma-model approaches toward the generation of a dynamical quark mass [3].

The appropriate set of approximants for the n_f = 3 β-function series

\[ \mu^2 \frac{dx}{d\mu^2} = -\frac{9}{4} x^2 \left[ 1 + \frac{16}{9} x + \frac{3863}{864} x^2 + 20.9902 x^3 + R_4 x^4 + \ldots \right] \]  

(11)

Such is the case for a prior asymptotic Padé-approximant prediction of the four-loop term [3,4], \( \beta_3^{pred} \) = \{ 23600 − 6400n_f + 350n_f^2 + 1.499n_f^3 \}/256, whose polynomial coefficients are in good term-by-term agreement with those of the subsequent exact calculation [3,4].

\[ \beta_3^{true} = (29243.9 - 6946.30n_f + 405.089n_f^2 + 1.499n_f^3) / 256. \]  

When n_f = 3, however, the estimated value \( \beta_3^{true} = 29.65 \) differs from \( \beta_3^{true} = 47.22 \) by a relative-error (37%) whose magnitude is a factor of two or more larger than that of the relative error characterising each estimated polynomial coefficient (19%, 7.9%, and 14%, respectively).
Table 1: Values of $\mu_c^{[N|M]}$, as obtained via (14), for $[N|M]$ approximants to the $n_f = 3$ QCD $\beta$-function. The (unknown) five-loop contribution $R_4$ to the perturbative series (11) is chosen (see (15)) to ensure a pole at $x (\mu_c) = 1/4$ for all three Padé-approximant cases.

| $\mu_c^{[2]}$ | 785 MeV | 874 MeV | 960 MeV |
| $\mu_c^{[1]}$ | 778 MeV | 867 MeV | 952 MeV |
| $\mu_c^{[3]}$ | 775 MeV | 863 MeV | 948 MeV |

With $R_4$ arbitrary is

$$\beta^{[2]}(x) = -\frac{9}{4} x^2 \frac{[1 + (7.19456 - 0.102610 R_4) x + (-11.3292 + 0.0756438 R_4) x^2]}{[1 + (5.41678 - 0.102610 R_4) x + (-25.4301 + 0.258062 R_4) x^2]}, \quad (12)$$

$$\beta^{[1]}(x) = -\frac{9}{4} x^2 \frac{[1 + (4.03067 - 0.0933552 R_4) x + (-11.6367 + 0.165965 R_4) x^2 + (-18.3242 + 0.122349 R_4) x^3]}{[1 + (5.80845 - 0.0933552 R_4) x]}. \quad (13)$$

$$\beta^{[3]}(x) = -\frac{9}{4} x^2 \frac{[1 + (1.77778 - 0.0476412 R_4) x + (4.447106 - 0.0846954 R_4) x^2 + (20.9902 - 0.213007 R_4) x^3]}{[1 - 0.0476412 R_4 x]}. \quad (14)$$

The first positive pole for all three approximants is seen to precede any positive zeroes and is clearly dependent on the value of $R_4$. One can easily show that all three approximants acquire a positive pole at $x = 1/4$ for very similar values of $R_4$:

$$[2|2]: \quad R_4 = 80.307$$
$$[1|3]: \quad R_4 = 89.925$$
$$[3|1]: \quad R_4 = 83.961$$

Given $\alpha_s (m_\tau)$ and $x (\mu_c) = 1/4$, as ensured by these respective choices for $R_4$, one can utilise (14) to estimate corresponding mass scales $\mu_c$ for the three approximants. In Table 1, such values are obtained via couplant evolution from three different choices for $\alpha_s (m_\tau)$ in the range $\alpha_s (m_\tau) = 0.33 \pm 0.02$ (13). In all these estimates, three-flavour coupling constant evolution is assumed to be valid below $m_\tau$.

Table 1 shows striking uniformity in the values for the infrared boundary mass scale $\mu_c$ obtained via three distinct Padé-approximants. Moreover, the three approximants generate values for $\mu_c$ very near the $\rho$-meson mass when the lowest value for $\alpha_s (m_\tau)$ is chosen.

Indeed, Figure 2 demonstrates that the infrared boundary for all three approximants is nearly equivalent even if $R_4$ is allowed to vary arbitrarily. The corresponding mass scale $\mu_c$, as plotted in Figure 2, is obtained for each approximant by substituting (14) into the integrand of (11), with (11)'s upper bound of integration identified with the first positive poles of (12, 14). For a given choice of $R_4$, Figure 2 shows that the infrared-boundary mass scales characterising all three approximants are surprisingly close. The $O(600 \text{ MeV})$ lower bound evident from the figure for all three approximants is particularly striking. Different Padé approximants to the $n_f = 3$ QCD $\beta$-function thus appear to be quite consistent in the infrared dynamics they predict, suggestive that similar pole-driven dynamics may characterise the infrared boundary of QCD itself.

The above results have all been obtained in the $\overline{\text{MS}}$ scheme, an explicitly gauge-invariant renormalization procedure for which the $\beta$-function has been explicitly calculated to four-loop order, thereby facilitating our use of

When $R_4$ becomes negative, the [3|1] approximant version of the $n_f = 3$ $\beta$-function no longer exhibits a positive pole (or any positive zeroes, as would be the case with an infrared stable fixed point), but still exhibits a Landau singularity extractable from (11) provided the upper bound of integration in (11) is replaced by infinity. The [3|1] approximant curve in Figure 2 displays the mass scale for this Landau singularity when $R_4$ is negative. The figure shows this mass scale to be consistent with the mass scales characterising the simple poles within the [2|2]- and [1|3]-approximant versions of the $\beta$-function for the same negative values of $R_4$. 

\[\alpha_s (m_\tau) = 0.31 \quad \alpha_s (m_\tau) = 0.33 \quad \alpha_s (m_\tau) = 0.35\]
Padé approximants to the series (1). Moreover, perturbative QCD phenomenology is more readily available (and more likely to be corroborated) in MS than in other renormalization schemes. The question remains, however, as to whether the infrared boundary we observe is a peculiarity of the MS scheme we employ. In eq. (1), the coefficients \( R_2, R_3, R_4, \ldots \) (corresponding to \( \beta_k \) with \( k \geq 2 \)) are all scheme-dependent, and therefore negotiable within the context of formal perturbative QCD. Indeed, a QCD renormalization scheme proposed by ’t Hooft in which \( \beta_k = 0 \) for \( k \geq 2 \) is guaranteed by construction to be free of \( \beta \)-function poles.

Recent work has developed a procedure by which the leading renormalization-scheme dependence (i.e., explicit dependence on \( R_2 \)) can be eliminated from a perturbative QCD result. This approach leads ultimately to \( \beta \)-functions characterized by the remaining renormalization-scheme parameters (i.e., \( R_3, \ldots \)). A reasonable set of Padé approximants to \( \beta \)-functions extracted by this procedure is shown in to exhibit poles of a magnitude \( (0.29 \lesssim x \lesssim 0.33) \) sufficiently large for chiral symmetry breaking at the infrared boundary of QCD.

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References

[1] N. Seiberg and E. Witten, Nucl. Phys. B 426 (1994) 19 and B 431 (1994) 484.
[2] Y. Iwasaki, K. Kanaya, S. Sakai, T. Yoshié, Phys. Rev. Lett. 69 (1992) 21.
[3] T. Banks and A. Zaks, Nucl. Phys. B 196 (1982) 189;
   T. Appelquist, J. Terning and L.C.R. Wijewardhana, Phys. Rev. Lett. 77 (1996) 1214;
   V.A. Miransky and K. Yamawaki, Phys. Rev. D 55 (1997) 5051 and Erratum 56 (1997) 3768;
   M. Velkovsky and E. Shuryak, Phys. Lett. B437 (1998) 398;
   T. Appelquist, A. Ratnaweera, J. Terning and L.C.R. Wijewardhana, Phys. Rev. D58 (1998) 105017;
   E. Gardi and M. Karliner, Nucl. Phys. B529 (1998) 383;
   E. Gardi, G. Grunberg and M. Karliner, JHEP 07 (1998) 007;
   V. A. Miransky, Phys. Rev. D59 (1999) 105003;
   E. Gardi and G. Grunberg, JHEP 9903 (1999) 024.
[4] F.A. Chishtie, V. Elias, V.A. Miransky, T.G. Steele, Prog. Theor. Phys. 104 (2000) 603.
[5] V. Elias, T.G. Steele, F. Chishtie, R. Migneron and K. Sprague, Phys. Rev. D 58 (1998) 116007;
   F.N. Ndili, hep-ph/0011116 (to appear Phys. Rev. D).
[6] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, Nucl. Phys. B229 (1983) 381.
[7] I.I. Kogan and M. Shifman, Phys. Rev. Lett. 75 (1995) 2085.
[8] T. van Ritbergen, J.A.M. Vermassen and S.A. Larin, Phys. Lett. B400 (1997) 379.
[9] J. Ellis, I. Jack, D.R.T. Jones, M. Karliner and M.A. Samuel, Phys. Rev. D 57 (1998) 2665.
[10] G. Baker, P. Graves-Morris, Padé Approximants [Vol. 13 of Encyclopedia of Mathematics and its Applications] (Addison-Wesley, Reading, MA, 1981) pp. 48–57.
[11] V. Elias, J. Phys. G 27 (2001) 217.
[12] D.R.T. Jones, Phys. Lett. B 123 (1983) 45.
[13] ALEPH Collaboration (R. Barate et al.), Eur. Phys. J. C 4 (1998) 409;
   G. Cvetic and T. Lee, hep-ph/0101297;
   C.J. Maxwell and A. Mirjalili, hep-ph/0103169.

Curiously, the existence and approximate size of the \( n_f \) threshold for \( \beta \)-function zeroes (i.e., infrared-stable fixed points) in the ’t Hooft scheme is corroborated by Padé-approximants constructed from the known terms of the MS \( \beta \)-function.
[14] Particle Data Group (D.E. Groom et al), Eur. Phys. J. C 15 (2000) 1.

[15] A. Manohar and H. Georgi, Nucl. Phys. B 234 (1984) 189.

[16] J. Ellis, M. Karliner, and M. A. Samuel, Phys. Lett. B 400 (1997) 176.

[17] P. Fomin and V.A. Miransky, Phys. Lett. B 64 (1976) 166;
    P.I. Fomin, V.P. Gusynin, V.A. Miransky, and Yu. A. Sitenko, Riv. Nuovo Cim. 6 (1983) 1;
    K. Higashijima, Phys. Rev. D 29 (1984) 1228.

[18] V. Elias and M. D. Scadron, Phys. Rev. Lett. 53 (1984) 1129;
    L. R. Baboukhadia, V. Elias, and M. D. Scadron, J. Phys. G 23 (1997) 1065.

[19] G. ’t Hooft, Recent Developments in Gauge Theories, [Vol. 59 of NATO Advanced Study Institute Series B: Physics, ed. G. ’t Hooft et al.], (Plenum New York, 1980).

[20] G. Cvetic, Phys. Lett. B 486 (2000) 100.
Figure 1: Qualitative behaviour of the running coupling $\alpha(\mu)$ in an asymptotically-free theory with an infrared attractor devolving from a simple pole in the $\beta$-function.
Figure 2: Dependence of the infrared-boundary scale $\mu_c$ on the five-loop $\beta$-function coefficient $R_4 = \beta_4/\beta_0$, based upon $n_f = 3$ evolution of the couplant from an initial value of $\alpha_s(m_\tau) = 0.33$. We only plot values of $R_4$ less than the WAPAP estimate (4); if $R_4$ is allowed to increase much past this value, the first positive poles of (12\,–\,14) are soon seen to be reached at values of $\mu$ exceeding the four-flavour threshold $\mu_t$. 