On a relation between stochastic integration and geometric measure theory

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Abstract

Two problems are addressed for the path of certain stochastic processes: a) do they define currents? b) are these currents of a classical type? A general answer to question a) is given for processes like semimartingales or with Lyons-Zheng structure. As to question b), it is shown that Hölder continuous paths with exponent $\gamma > 1/2$ define integral flat chains.

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1 Introduction

This work starts from two motivations. The first one is the intuition by one of the authors and T. Lyons that interesting relations could exist between the theory of currents and the theory of rough paths (see [11] and [15] as basic references on these topics). The second motivation comes from recent investigations

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on stochastic models of vortex filaments in turbulent fluids, where the vorticity field is rather singular and a description by means of current looks promising (see [6], [7], [8] and the previous works [2], [10], [14]). The present work provides only some initial results on these two problems.

We address the following two questions:

a) do the paths of some stochastic processes define 1-currents?

b) are these currents of a classical type?

The definition of random current is given in the next section, but roughly speaking our subject of investigation are the Stratonovich stochastic integrals of the form

$$S(\varphi) = \int_0^T \langle \varphi (X_t), \circ dX_t \rangle$$

where \((X_t)_{t \in [0,T]}\) is a stochastic process in \(\mathbb{R}^d\) and \(\varphi\) is a 1-form on \(\mathbb{R}^d\). We give results also for Itô integrals, but the Stratonovich ones do not depend on the parametrization and therefore are more natural in a geometrical context. Question a) amounts to ask whether the mapping \(\varphi \mapsto S(\varphi)\) has a pathwise meaning and is (pathwise) linear continuous in some topology. Question b) looks more deep, and concerns the classification of \(S\) in terms of classical types of currents.

Concerning problem a) we deal with the following claim:

all semimartingales and Lyons-Zheng processes define 1-currents.

The qualitative part of this result is a by-product of known theories even if it was not explicitly stated in the language of currents; our contribution here is a precise estimate of the Sobolev regularity of such currents. The topology on \(\varphi\) that we are able to consider is the \(H^s\) topology for \(s > d/2\) or \(s > d/2 + 1\) depending on some conditions. Our result on problem a) is essentially of analytic nature, with little geometrical content, in contrast to problem b).

Problem a) has been considered in the literature (not with the language of currents). Among others, two very relevant results are provided by the integration theories of Young [23] and Lyons [13]. In the theory of Young, \((X_t)\) is a deterministic function having an Hölder continuous regularity with exponent \(\gamma > 1/2\). For such (and more general) curves we are even able to solve the more difficult problem b), as described below. The outstanding (and complex) theory of Lyons on integration of rough paths solves question a) in a variety of cases. Restricting such theory to continuous semimartingales, it provides pathwise integration with continuous dependence on \(\varphi\) in a class like \(C^{1,\varepsilon}\) (\(\varepsilon\)-Hölder continuous derivatives). Our results are not exactly comparable because of the different topologies on \(\varphi\), but we can at least say that our results are much less deep than those of Lyons, just more straightforward. A more abstract result which implies that semimartingales define 1-currents is a general theorem of
Minlos' approach on linear random functionals in nuclear spaces. Compared to that, the advantages of our approach is that it gives a precise Sobolev regularity of these currents, which is of interest in application to fluid dynamics, see \[8\].

Concerning problem b), the most common class of 1-currents considered in the literature on geometric measure theory are the integer multiplicity (i.m.) rectifiable 1-currents. This class is certainly not suitable to incorporate the paths of the usual stochastic processes since such paths commonly have infinite 1-dimensional Hausdorff measure. One of the other most important classes of currents are the integral flat chains. We concentrate our investigation on this class. In the 1-dimensional case, an integral flat chain \( S \) is the boundary of an i.m. rectifiable 2-current \( T \), up to an additional i.m. rectifiable 1-current \( R \):

\[
S(\varphi) = \partial T(\varphi) + R(\varphi).
\]

In very loose terms, an integral flat 1-chain is (a piece of) the boundary of a quite good surface, a surface that in particular has finite 2-dimensional Hausdorff measure. In intuitive geometrical terms we try to understand whether a very irregular curve as the path of a common stochastic process may be seen as the boundary of a surface with finite 2-dimensional Hausdorff measure.

The Hausdorff dimension of typical paths of Brownian motion is already 2, so it is very difficult to expect a positive answer in such a case (a small room is left open by the fact that the correct gauge function for Brownian paths is \( r^2 \log r \) in dimension \( d \geq 3 \) and slightly better in smaller dimension). On the contrary, we can prove that the class of integral flat 1-chains includes the paths of many processes just a little more regular than semimartingales. We prove that curves of class \( C^\gamma \) with \( \gamma > 1/2 \) define integral flat chains.

These classes include for instance the typical paths of fractional Brownian motion with Hurst parameter \( H = \gamma \).

The fact that such irregular paths are boundaries of rectifiable 2-currents may open interesting future investigations, like minimal surfaces with prescribed irregular boundary, homotopy theory in such class, and others. A direct consequence is the existence of the concept of integral flat distance between two stochastic processes with paths \( C^\gamma, \gamma > 1/2 \): it is a more geometrical notion of distance with respect to the usual analytic topologies, independent for instance from the parametrization, and we hope it may be useful in the future.

From the viewpoint of integration theory, our result on problem b) gives a new definition of integral for processes with \( C^\gamma \) paths, \( \gamma > 1/2 \). By a different method such integrals have been introduced before by Young. It is interesting to notice that Young’s definition employs a telescopic series; the 2-current \( T \) used here vaguely looks like a continuous version of such series.

For processes like Brownian motion we have not found a classical notion of current that may fit. The class of integral flat 1-chains is presumably not suitable, and at least we are sure that the surfaces we consider in this paper have infinite 2-dimensional Hausdorff measure for Brownian paths, see the last section. However the technique we use to define the integral can be generalized...
to provide an integration theory for curves in classes $C^\gamma$ with $\gamma > 1/3$ which delivers results analogous to that of T. Lyons [15]. These results will be reported elsewhere [12].

Since we address this work to different audiences, we sometimes repeat for the reader’s convenience some definition and facts that are very elementary in one of the two theories (stochastic analysis and geometric measure theory). We apologize with expert readers for the style that may look redundant in a number of places.

2 Preliminaries on deterministic and stochastic currents

2.1 Deterministic currents

In this subsection we briefly recall a few definitions of geometric measure theory used in the sequel. More informations can be found in [3], [18], [21], [11].

We shall denote the Euclidean norm and scalar product in $\mathbb{R}^d$ ($d$ is fixed throughout the paper) by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ respectively. We shall also need the space $\Lambda_2\mathbb{R}^d$ of 2-vectors, its dual space $\Lambda^2\mathbb{R}^d$ of 2-covectors, and the duality between them, still denoted by $\langle \cdot, \cdot \rangle$. If we represent vectors and covectors in the standard basis (the summations are extented from 1 to $d$)

$$v = \sum_{i<j} v^{ij} e_i \wedge e_j, \quad w = \sum_{i<j} w_{ij} e^i \wedge e^j$$

then

$$\langle w, v \rangle = \sum_{i<j} w_{ij} v^{ij}.$$ 

Notationally, we prefer to write the covectors in the first argument of $\langle \cdot, \cdot \rangle$ since later on the notation for stochastic integrals is more natural. The norm of 2-vectors and 2-covectors is also defined as

$$|v|^2 = \sum_{i<j} (v^{ij})^2, \quad |w|^2 = \sum_{i<j} (w_{ij})^2.$$ 

Notice that for all $v_1, v_2 \in \mathbb{R}^d$ and $w^1, w^2 \in \mathbb{R}^d$, the duality between the 2-vector

$$v_1 \wedge v_2 = \sum_{i<j} \left[ (v_1)_i (v_2)_j - (v_1)_j (v_2)_i \right] e_i \wedge e_j$$

and the 2-covector $w^1 \wedge w^2$ similarly defined, is given by

$$\langle w^1 \wedge w^2, v_1 \wedge v_2 \rangle = \det \left( \langle v_i, w^j \rangle \right).$$

Similarly we have

$$|v_1 \wedge v_2|^2 = \det \left( \langle v_i, v_j \rangle \right) = |v_1|^2 |v_2|^2 - \langle v_1, v_2 \rangle^2.$$
Let $D^k$ be the space of all infinitely differentiable and compactly supported $k$-forms on $\mathbb{R}^d$. A \textit{k-dimensional current} is a linear continuous functional on $D^k$. We denote by $D_k$ the space of $k$-currents. In this paper we are only interested in the cases $k = 1, 2$, so we recall a few corresponding notations. Let $\varphi$ (resp. $\psi$) be an element of $D^1$ (resp. $D^2$). We shall write them as

$$\varphi = \sum_{i=1}^{d} \varphi_i \, dx^i, \quad \psi = \sum_{i<j} \psi_{ij} \, dx^i \wedge dx^j$$

where $\varphi_i$ and $\psi_{ij}$ are infinitely differentiable and compactly supported functions on $\mathbb{R}^d$. Typical examples of 1-currents are those induced by regular curves $(X_t)_{t \in [0,T]}$ in $\mathbb{R}^d$:

$$\varphi \mapsto S(\varphi) := \int_0^T \langle \varphi(X_t), \dot{X}_t \rangle \, dt$$

while typical examples of 2-currents are given by regular surfaces $(f(t,s))_{(t,s) \in A}$ in $\mathbb{R}^d$, where $A$ is a Borel set in $\mathbb{R}^2$:

$$\psi \mapsto T(\psi) := \int_A \left \langle \psi \circ f, \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} \right \rangle \, dtds$$

Notice that the $k$-dimensional Hausdorff measure $\mathcal{H}^k$ allows one to rewrite the previous examples in a more compact and intrinsic way. Assume for instance that $(X_t)_{t \in [0,T]}$ and $(f(t,s))_{(t,s) \in A}$ are injective. Then

$$S(\varphi) = \int_\mathcal{M} \langle \varphi, \xi \rangle \, d\mathcal{H}^1, \quad T(\psi) = \int_\mathcal{M} \langle \psi, \xi \rangle \, d\mathcal{H}^2$$

where in the first example $\mathcal{M}$ is the support of the curve $(X_t)_{t \in [0,T]}$ and $\xi$ is the unit tangent vector with the orientation given by $(X_t)_{t \in [0,T]}$, while in the second example $\mathcal{M}$ is the support of the surface $(f(t,s))_{(t,s) \in A}$ and $\xi$ is the unit 2-vector

$$\xi = \left | \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} \right |^{-1} \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s}$$

defining an orientation of the tangent manifold. Here

$$\left | \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} \right | = \sqrt{\left | \frac{\partial f}{\partial t} \right |^2 \left | \frac{\partial f}{\partial s} \right |^2 - \left \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right \rangle^2}.$$ 

These two examples can be generalized in the notion of \textit{integer multiplicity} (i.m. in the sequel) \textit{rectifiable k-current}: it is a $k$-current $T$ of the form ($\phi \in D^k$)

$$T(\phi) = \int_\mathcal{M} \langle \phi, \xi \rangle \, \theta \, d\mathcal{H}^k$$
where $M$ is a $k$-rectifiable set in $\mathbb{R}^d$, $\theta : M \to \mathbb{R}$ is integer valued and $\mathcal{H}^k|_M$ integrable, and $\xi$ is a $\mathcal{H}^k$-measurable unitary $k$-vector field on $M$, with $\xi(x) \in T_xM$ for $\mathcal{H}^k|_M$-a.e. $x$ (compared to the most general definitions, here we have restricted ourselves to sets $M$ with $\mathcal{H}^k(M) < \infty$). We recall that a $k$-rectifiable set $M$ in $\mathbb{R}^d$ is a $\mathcal{H}^k$-measurable set with $\mathcal{H}^k(M) < \infty$ that can be decomposed as

$$M = N \cup \bigcup_i f_i(A_i)$$

where $N$ is a set with $\mathcal{H}^k(N) = 0$, $A_1, A_2, \ldots$ are Borel sets of $\mathbb{R}^k$, and $f_1, f_2, \ldots$ are Lipschitz continuous maps $f_i : A_i \to \mathbb{R}^d$. One can prove that there exists such a decomposition with $C^1$ mappings $f_i$ and disjoint images $f_1(A_1), f_2(A_2)$, and so on. The $k$-rectifiable sets, although admit quite wild singularities and intersections, have a number of properties similar to those of $C^1$ $k$-manifolds.

For instance, a $k$-rectifiable set $M$ has finite $k$-dimensional Hausdorff measure; and there is a set $N' \subset M$ with $\mathcal{H}^k(N') = 0$ such that for all $x \in M \setminus N'$ a tangent space $T_xM$ can be defined and a measurable orientation can be chosen (i.e. a unitary $k$-vector field $\xi$ as above).

Finally, we recall the notion of integral flat chain. Given a $k$-current $T$, it is defined a $(k-1)$-current $\partial T$, the boundary of $T$, by the identity $\partial T(\phi) = T(d\phi)$, where $\phi \in D^{k-1}$ and $d\phi$ is its differential. For instance, in the case $k = 2$, we have

$$\phi = \sum_i \phi_i x^i, \quad d\phi = \sum_{i<j} \left( \frac{\partial \phi_j}{\partial x_i} - \frac{\partial \phi_i}{\partial x_j} \right) x^i \wedge x^j.$$

An integral flat $(k-1)$-chain $S$ is a $(k-1)$-current of the form

$$S(\phi) = \partial T(\phi) + R(\phi)$$

where $T$ is a i.m. rectifiable $k$-current and $R$ is a i.m. rectifiable $(k-1)$-current. In general, integral flat $(k-1)$-chain are not necessarily i.m. rectifiable $(k-1)$-currents: the boundary of a i.m. rectifiable $k$-current need not to be rectifiable. These non-rectifiable boundaries are exactly the places where we can find the irregular paths of certain stochastic processes. Here is a trivial example of this fact for 1-dimensional stochastic processes (unfortunately not very illuminating about the difficulties arising in dimension $d > 1$).

**Example 1** Let $(X_t)_{t \in [0,T]}$ be a continuous real valued function, for instance any typical path of a 1-dimensional stochastic process. Let $\lambda$ be a number smaller than the minimum of $(X_t)$. Consider the subgraph of $(X_t)$ above $\lambda$, i.e. the set

$$\mathcal{M} = \{(t,x) \in \mathbb{R}^2 : 0 \leq t \leq T, \lambda \leq x \leq X_t\}.$$

It defines the i.m. rectifiable 2-current in $\mathbb{R}^2$ given by

$$T(\psi) := \int_{\mathcal{M}} \langle \psi, e_1 \wedge e_2 \rangle \, d\mathcal{H}^2 = \int_{\mathcal{M}} \psi_{tx} \, d\mathcal{H}^2$$
where $\psi = \psi_t \, dt \wedge dx$. Topologically, the boundary of $M$ is $(X_t)$ plus three segments, so intuitively we see that $(X_t)$ defines an integral flat 1-chain. At rigorous ground, denoting 1-forms as

$$\varphi = \varphi_t \, dt + \varphi_x \, dx$$

the boundary of $T$ is given by

$$\partial T (\varphi) = T (d\varphi) = \int_0^T \left( \int_\lambda^{X_t} \left( \frac{\partial \varphi_x}{\partial t} - \frac{\partial \varphi_t}{\partial x} \right) \, dx \right) \, dt.$$

Assume now that on $(X_t)$ we can perform rules of calculus of classical type (as for Lipschitz functions or semimartingales by Stratonovich stochastic calculus). Since the final result of this example is trivial, we do not insist on the rigorous details. Then, setting

$$\Phi_x (t,x) = \int_\lambda^x \varphi_x (t,y) \, dy,$$

we have

$$\partial T (\varphi) = \int_0^T \frac{\partial}{\partial t} \left[ \Phi_x (t,X_t) - \Phi_x (t,\lambda) \right] \, dt - \int_0^T \varphi_x (t,X_t) \circ dX_t + \int_0^T \left( - \varphi_t (t,X_t) + \varphi_t (t,\lambda) \right) \, dt$$

(we have used $\circ$ to denote the suitable kind of integration required by the previous computations) and therefore

$$\int_0^T \langle \varphi, \circ (dt,dX_t) \rangle = \int_0^T \varphi_x (t,X_t) \circ dX_t + \int_0^T \varphi_t (t,X_t) \, dt = T (d\varphi) + \int_\lambda^{X_t} \varphi_x (T,y) \, dy - \int_\lambda^{X_0} \varphi_x (0,y) \, dy + \int_0^T \varphi_t (t,\lambda) \, dt.$$

Several comments are in order on this example. When $(X_t)$ is Brownian motion, it provides an example of i.m. rectifiable 2-current $T$ with a boundary that is not an i.m. rectifiable 1-current. It also shows that integral flat 1-chains may be the right objects to describe stochastic integrals. Finally, restricted to dimension 1 it solves problems a) and b) posed in the introduction in complete generality. However, with respect to problem a) this success is just a more complicate version of the well known definition (see for instance [3])

$$\int_0^T \varphi_x (t,X_t) \circ dX_t = \Phi_x (T,X_T) - \Phi_x (0,X_0) - \int_0^T \frac{\partial \Phi_x}{\partial t} (t,X_t) \, dt.$$

In other words, it is well known that in dimension one Stratonovich stochastic integration can be defined pathwise (also Itô integration can be performed pathwise when the quadratic variation is finite, see [4]). The analysis in dimension $d > 1$ is completely different.
2.2 Stochastic currents

Let \((X_t)_{t \in [0,T]}\) be a continuous semimartingale with values in \(\mathbb{R}^d\). To each smooth 1-form \(\varphi\) on \(\mathbb{R}^d\) we may associate the random variable

\[
S(\varphi) := \int_0^T \langle \varphi(X_t), \odot dX_t \rangle
\]

where the integral is understood in the sense of Stratonovich. The mapping \(\varphi \mapsto S(\varphi)\) is continuous with respect to the convergence in probability. Motivated by this basic example it looks reasonable to give the following definition.

**Definition 2** Given a complete probability space \((\Omega, \mathcal{A}, P)\), a stochastic \(k\)-current is a continuous linear mapping from the space \(\mathcal{D}^k\) to the space \(L^0(\Omega)\) of real valued random variables on \((\Omega, \mathcal{A}, P)\), endowed with the convergence in probability.

The usual classes of stochastic processes considered in stochastic analysis give rise to stochastic 1-currents: semimartingales, Lyons-Zheng processes, processes with finite \(p\) variation (for suitable \(p\)), fractional Brownian motion (for suitable Hurst parameter), certain Dirichlet processes.

At this level it is difficult to see an interesting relation with classical geometric measure theory. The link arises if we try to understand the previous stochastic integrals in a pathwise sense. So we introduce the following definition.

**Definition 3** We say that the stochastic \(k\)-current \(\varphi \mapsto S(\varphi)\) has a pathwise realization if there exists a measurable mapping

\(\omega \mapsto S(\omega)\)

from \((\Omega, \mathcal{A}, P)\) to the space \(\mathcal{D}_k\) of deterministic currents (endowed with the natural topology of distributions), such that

\[
[S(\varphi)](\omega) = [S(\omega)](\varphi) \quad \text{for } P\text{-a.e. } \omega \in \Omega.
\]

In terms of these definitions we may reformulate the two problems of the introduction as follows:

a) given a stochastic process \((X_t)_{t \in [0,T]}\), does there exist a pathwise realization \(S(\omega)\) of the associated stochastic 1-current \(S(\varphi)\) defined by (1)?

b) can we classify \(S(\omega)\) in terms of classical currents, for \(P\)-a.e. \(\omega\)?

The existence of a pathwise realization is a difficult problem. The difficulty is described in the remark at the beginning of the next section in terms of selection of representatives in the equivalence classes of stochastic integrals. In the theory of stochastic processes this is the problem of existence of a continuous...
modification of a given random field \((X_\varphi(\omega))\). Here the parameter of the field is \(\varphi \in D^k\), so the parameter space is infinite dimensional and well-known criteria like the Kolmogorov regularity theorem do not apply (some generalizations are known in the literature but their effective use is very limited). The problem of existence of a continuous modification of a random field with infinite dimensional parameter space has been studied and some general ideas have been developed, but usually it is better to find out ad hoc methods, as we shall do. We just recall now two general criteria from [22] (see also [5]).

**Lemma 4** Let \(\varphi \mapsto S(\varphi)\) be a linear continuous mapping from a separable Banach space \(E\) to \(L^0(\Omega)\) (with the convergence in probability). Assume that there exists a random variable \(C(\omega)\) such that for all given \(\varphi \in E\) we have

\[
|S(\varphi)(\omega)| \leq C(\omega) \|\varphi\|_E \quad \text{for } P\text{-a.e. } \omega \in \Omega.
\]

Then there exists a measurable mapping \(\omega \mapsto S(\omega)\) from \((\Omega, \mathcal{A}, P)\) to the dual \(E'\) such that for all given \(\varphi \in E\) we have \(S(\varphi)\) (hence \(S(\omega)\) is a pathwise realization of \(S(\varphi)\)).

We shall use this criterium in the next section. It is not very powerful since its assumption is a pathwise estimate (it is almost a tautology). The proof is elementary and can be found in the above mentioned references. More interesting is the following criterium since it is based on an assumption in mean square. However, in the next section we shall not use it directly but an ad hoc argument based on Fourier transform, that looks more flexible. We sketch the proof (contained in [22], [5]), for comparison with the method of the next section.

**Lemma 5** Let \(\varphi \mapsto S(\varphi)\) be a linear continuous mapping from a separable Hilbert space \(H\) to \(L^2(\Omega)\). Assume that it is Hilbert-Schmidt: for some complete orthonormal system \(\{e_i\}\) in \(H\) we have

\[
\sum_{i=1}^{\infty} \|S(e_i)\|_{L^2(\Omega)}^2 < \infty.
\]

Then there exists a measurable mapping \(\omega \mapsto S(\omega)\) from \((\Omega, \mathcal{A}, P)\) to \(H\), i.e. a random vector of \(H\), such that for all given \(\varphi \in H\) we have \(\|\varphi\|_{L^2(\Omega)}^2 = C(\omega) \|\varphi\|_H^2\).

**Proof.** Schwartz inequality gives us

\[
|S(\varphi)(\omega)|^2 = \left| \sum_{i=1}^{\infty} S(e_i)(\omega) \langle \varphi, e_i \rangle_H \right|^2 
\leq \sum_{i=1}^{\infty} |S(e_i)(\omega)|^2 \sum_{i=1}^{\infty} \langle \varphi, e_i \rangle_H^2 
= C(\omega) \|\varphi\|_H^2.
\]

The non negative (a priori possibly infinite) r.v. \(C(\omega)\) has finite mean by assumption, then it finite a.s. So we may apply the first lemma. \(\Box\)
3 Paths of semimartingales and Lyons-Zheng processes define 1-currents

3.1 The case of semimartingales

In the sequel we tacitly assume that processes are defined on a complete probability space \((\Omega, \mathcal{A}, P)\), with expectation denoted by \(E\). We also assume that a standard filtration \(\mathcal{F} = (\mathcal{F}_t)\) is given, so that concepts like martingale or adaptedness are referred to this filtration.

For the definition of semimartingale and corresponding integrals, see [19], [13] or many other references. We just recall a few facts directly used below.

A continuous semimartingale \((X_t)_{t \in [0,T]}\) is the sum of a continuous local martingale \((M_t)\) and a continuous adapted process of bounded variation \((V_t)\). The decomposition is unique. Given a continuous adapted process \((Y_t)\) in \(\mathbb{R}^d\), the Itô integral
\[
\int_0^T \langle Y_t, dX_t \rangle := \int_0^T \langle Y_t, dM_t \rangle + \int_0^T \langle Y_t, dV_t \rangle
\]
and similarly for the Stratonovich integral \(\int_0^T \langle Y_t, \circ dX_t \rangle\), where now \((Y_t)\) is assumed to be either a continuous semimartingale or \(Y_t = \varphi(X_t)\) with the 1-form \(\varphi\) is of class \(C^1\) (one can unify these two cases with the language of Dirichlet processes). The previous integrations in \(dV_t\) are classical pathwise integrations in the Riemann-Stieltjes sense, while the integrals with respect to \((M_t)\) are the following limits in probability (they exist under the previous assumptions on \((Y_t)\)):
\[
\int_0^T \langle Y_t, dM_t \rangle := \text{P-lim}_{n \to \infty} \sum_{t_i \in \pi_n} \langle Y_{t_i}, M_{t_{i+1}} - M_{t_i} \rangle
\]
\[
\int_0^T \langle Y_t, \circ dM_t \rangle := \text{P-lim}_{n \to \infty} \sum_{t_i \in \pi_n} \left( \frac{Y_{t_{i+1}} + Y_{t_i}}{2}, M_{t_{i+1}} - M_{t_i} \right)
\]
where \(\pi_n\) is any sequence of partitions of \([0, T]\) converging to zero.

**Remark 6** These integrals are \(P\)-equivalence classes. The evaluation at a given \(\omega\), namely the pathwise integration, is a priori meaningless. Given \(\varphi\) one may of course take a representative in the equivalence class and have a meaning for all \(\omega\), but an arbitrariness choice of the representative cannot give us any good property (even the linearity) of the mapping \(\varphi \mapsto S(\varphi)\) evaluated at single points \(\omega\). The existence of a pathwise realization means that it is possible to choose representatives in such a way that the mapping \(\varphi \mapsto S(\varphi)\), evaluated at almost every given point \(\omega\), is linear and continuous.

Given two continuous semimartingales \((X_t)\) and \((Y_t)\) in \(\mathbb{R}^d\), one can define
the quadratic covariation processes between their components \([X^\alpha, Y^\beta]\) as

\[
[X^\alpha, Y^\beta]_t := \lim_{n \to \infty} \sum_{t_i \in \pi_n, t_i \leq t} \langle X^\alpha_{t_{i+1}}, Y^\beta_{t_{i+1}} - X^\alpha_{t_i}, Y^\beta_{t_i} \rangle.
\]

One has \([X^\alpha, Y^\beta]\) as

\[
M^X_{\alpha}X^\alpha_{t}, M^Y_{\beta}Y^\beta_{t}\]

where \(M^X_{\alpha}\) and \(M^Y_{\beta}\) are the martingale parts of \((X_t)\) and \((Y_t)\) (the bounded variation terms do not contribute to the quadratic variation). The relation between Stratonovich and Itô integral is now

\[
\int_0^T \langle Y_t, \circ dX_t \rangle = \int_0^T \langle Y_t, dX_t \rangle + 1/2 \sum_{\alpha=1}^d [Y^\alpha, X^\alpha]_T
\]

as one may easily check by means of the finite sums. Similar facts hold true when \(Y_t = \varphi(X_t)\), with the additional formula:

\[
[X^\alpha, \varphi_{\beta}(X)]_T = \int_0^T \sum_{\delta=1}^d \frac{\partial \varphi_{\beta}}{\partial x_\delta}(X_t) d \left[ M^\delta, M^\alpha \right]_t
\]

where \((M_t)\) is the martingale part of \((X_t)\).

Finally, we recall the Burkholder-Davis-Gundy inequality. For all \(p \geq 1\), there exists a constant \(C_p > 0\) such that

\[
E \left[ \left( \int_0^T Y_t dM^\delta_t \right)^{2p} \right] \leq C_p E \left[ \left( \int_0^T |Y_t|^2 dM^\delta_t \right)^{p} \right]
\]

where \((Y_t)\) is any continuous adapted scalar process.

Let us come to our results. Denote the Fourier transform of \(\varphi(x)\) by \(\hat{\varphi}(k)\):

\[
\hat{\varphi}(k) := \int_{\mathbb{R}^d} e^{-i(k,x)} \varphi(x) \, dx.
\]

The following simple lemma will be our key ingredient and we guess it may be useful in other contexts. It is inspired by the vision of stochastic integrals as currents (i.e. as generalized random fields, so that it is not strange to perform their Fourier transform) and by the computations of [7].

**Lemma 7** Let \((M_t)\) be an \(L^2\)-bounded continuous martingale \((E[|M_t|^2] < \infty \text{ for } t \in [0,T])\), \((X_t)\) be a continuous adapted process, and \(\varphi\) be in \(D^1\). Then

\[
\int_0^T \langle \varphi(X_t), dM_t \rangle = \int_{\mathbb{R}^d} \langle \hat{\varphi}(k), Z_k \rangle \, dk \quad P\text{-a.s.}
\]

where

\[
Z_k := \int_0^T e^{-i(k,X_t)} dM_t.
\]
A similar result holds true for \((V_t)\) in place of \((M_t)\), when \(\|V\|_{\text{var}}^2 \in L^1(\Omega)\). Moreover, a similar result holds true for the Stratonovich integral when \((X_t)\) is a semimartingale where the martingale and bounded variation parts satisfy the same integrability assumptions of \((M_t)\) and \((V_t)\) (so in particular for \(X = M + V\)).

**Proof.** We give the proof only in the first case, since the others are entirely similar. First notice that the mapping \((\omega, k) \mapsto Z_k(\omega)\) is measurable (by the formula for the Itô integral as limit of finite sums) and the function \(\langle \hat{\phi}(k), Z_k \rangle\) is jointly integrable in \((\omega, k)\), so the right-hand-side of (3) is well defined. Indeed

\[
E[|Z_k|^2] \leq C_d E \int_0^T [M]_t = C_d E [M]_T < \infty \tag{4}
\]

(the last inequality is due to the assumption on \((M_t)\) and Corollary 1.25, Ch. IV of [19]) so \(E[Z_k] \leq (E[Z_k^2])^{1/2} \leq (C_d E [M]_T)^{1/2}\) and therefore

\[
E \left[ \int_{\mathbb{R}^d} |\langle \hat{\phi}(k), Z_k \rangle| \, dk \right] \leq \int_{\mathbb{R}^d} |\hat{\phi}(k)| \, E[|Z_k|] \, dk 
\leq \sqrt{C_d E [M]_T} \int_{\mathbb{R}^d} |\hat{\phi}(k)| \, dk
\]

where the last integral converges because of the decay properties of \(|\hat{\phi}(k)|\).

Formally the result (3) is a consequence of the heuristic formula

\[
\int_0^T \langle \varphi (X_t) , dM_t \rangle = \int_0^T \langle \varphi (x) , f (x) \rangle \, dx
\]

where

\[
f (x) := \int_0^T \delta (x - X_t) \, dM_t.
\]

Let \(p_\varepsilon (x)\) denote the heat kernel \((2\pi \varepsilon)^{-d/2} \exp(-|x|^2/(2\varepsilon))\) and let

\[
f_\varepsilon (x) := \int_0^T p_\varepsilon (x - X_t) \, dM_t.
\]

Then, by stochastic Fubini theorem, [19] p. 167,

\[
\int_{\mathbb{R}^d} \langle \varphi (x) , f_\varepsilon (x) \rangle \, dx = \int_0^T \left\langle \int_{\mathbb{R}^d} \varphi (x) \, p_\varepsilon (x - X_t) \, dx \right\rangle \, dM_t.
\]

Hence, by Parseval theorem (we exchange the order for comparison with (3) to be proved)

\[
\int_0^T \langle \varphi_\varepsilon (X_t) , dM_t \rangle = \int_{\mathbb{R}^d} \left\langle \hat{\phi}(k) , \hat{f}_\varepsilon (k) \right\rangle \, dk \tag{5}
\]

where \(\varphi_\varepsilon = p_\varepsilon * \varphi\). We also have

\[
\hat{f}_\varepsilon (k) = \int_0^T e^{i(k, X_t)} \hat{p}_\varepsilon (k) \, dM_t.
\]
Since \( \hat{p}_\varepsilon (k) \to 1 \) uniformly on compact sets of \( k \), as \( \varepsilon \to 0 \), and \( \varphi_\varepsilon (X_t) \) converges to \( \varphi (X_t) \) uniformly in \( t \), \( \mathbb{P} \)-a.s., and the convergence is dominated by a constant, we have that \( \hat{f}_\varepsilon (k) \) and \( \int_0^T \langle \varphi_\varepsilon (X_t), dM_t \rangle \) converge to \( Z_k \) for all \( k \) and to \( \int_0^T \langle \varphi (X_t), dM_t \rangle \) respectively, in mean square. Therefore, first, the l.h.s. of (4) converges to the one of (3) in mean square. As to the r.h.s.,

\[
E \left[ \int_{\mathbb{R}^d} \left| \langle \hat{\varphi} (k), \hat{f}_\varepsilon (k) - Z_k \rangle \right| dk \right] \leq \int_{\mathbb{R}^d} |\hat{\varphi} (k)| E \left[ \left| \hat{f}_\varepsilon (k) - Z_k \right| \right] dk \\
\leq \int_{\mathbb{R}^d} |\hat{\varphi} (k)| \sqrt{C_d E \left[ \left| \hat{f}_\varepsilon (k) - Z_k \right|^2 \right]} \, dk
\]

The term \( E[|\hat{f}_\varepsilon (k) - Z_k|^2] \) converges to zero for every \( k \), and is bounded by a constant (it is easily proved as (4)). From the decay properties of \( |\hat{\varphi} (k)| \) we deduce that \( \langle \hat{\varphi} (k), \hat{f}_\varepsilon (k) - Z_k \rangle \) converges to zero in \( L^1 \) with respect to \( (\omega, k) \) so \( \int_{\mathbb{R}^d} \langle \hat{\varphi} (k), \hat{f}_\varepsilon (k) \rangle \, dk \) converges to \( \int_{\mathbb{R}^d} \langle \hat{\varphi} (k), Z_k \rangle \, dk \) in \( L^1 \) with respect to \( \omega \). This completes the proof of (3).

\[ \square \]

**Theorem 8** Let \( (X_t) \) be a semimartingale in \( \mathbb{R}^d \) of the form \( X_t = M_t + V_t \) as above. Consider the stochastic 1-current \( S (\varphi) \) defined by the Stratonovich integral

\[
S (\varphi) = \int_0^T \langle \varphi (X_t), \circ dX_t \rangle
\]

and the stochastic 1-current \( I (\varphi) \) defined by the Itô integral

\[
I (\varphi) = \int_0^T \langle \varphi (X_t), dX_t \rangle.
\]

Then \( \varphi \mapsto S (\varphi) \) has a pathwise realization \( S \), with

\[
S (\omega) \in H^{-s-1} (\mathbb{R}^d, \mathbb{R}^d) \quad \mathbb{P} \text{-a.s.}
\]

for all \( s > \frac{d}{2} \), and \( \varphi \mapsto I (\varphi) \) has a pathwise realization \( I \), with

\[
I (\omega) \in H^{-s} (\mathbb{R}^d, \mathbb{R}^d) \quad \mathbb{P} \text{-a.s.}
\]

If in addition

\[
[M^i, M^j] = 0 \text{ for } i \neq j \text{ and } [M^i] = m_i \text{ for all } i
\]

(6)

and for some increasing process \( (m_t) \), then

\[
S (\omega) \in H^{-s} (\mathbb{R}^d, \mathbb{R}^d) \quad \mathbb{P} \text{-a.s.}
\]

(the same result holds true for reversible semimartingales, see the next theorem). Moreover, if \( (M_t) \) is a square integrable martingale and \( \|V\|_{\text{var}}^2 \in L^1 (\Omega) \), then

\[
S (\cdot) \in L^2 (\Omega, H^{-s-1} (\mathbb{R}^d, \mathbb{R}^d))
\]
\[ I(\cdot) \in L^2(\Omega, H^{-s}(\mathbb{R}^d, \mathbb{R}^d)) \]
and under the assumption \([2]\)
\[ S(\cdot) \in L^2(\Omega, H^{-s}(\mathbb{R}^d, \mathbb{R}^d)). \]
Finally, except for the result under assumption \([1]\), the same results hold true for the Itô integral
\[ \tilde{I}(\varphi) = \int_0^T \langle \varphi(\tilde{X}_t), dX_t \rangle \]
and the analogous Stratonovich integral, when \((\tilde{X}_t)\) is another semimartingale in \(\mathbb{R}^d\) (with integrability assumptions similar to those of \((X_t)\) for the last results on summability).

**Proof. Step 1** (localized problem and basic estimates; Itô integral). Let \(\tau_n^M\) be a sequence of stopping times that localizes \((M_t)\). Let \(\tau_n^{M[L]}\) be the one defined as
\[ \tau_n^{M[L]} = \inf \{ t \geq 0 : [M]_t \geq n \} \]
when this set is non empty, \(\tau_n^{M[L]} = T\) otherwise. Similarly, let \(\tau_n^V\) be defined as
\[ \tau_n^V = \inf \{ t \geq 0 : \|V\| \geq n \}. \]
when this set is non empty, \(\tau_n^V = T\) otherwise. Finally, let \(\tau_n\) be defined as \(\tau_n = \tau_n^M \wedge \tau_n^{M[L]} \wedge \tau_n^V\). It localizes \((M_t)\) by Doob’s stopping theorem, so \((M_t^{(n)})\) defined as
\[ M_t^{(n)} = M_{t \wedge \tau_n} \]
is a martingale, and in addition \([M^{(n)}]_t \leq n\). Moreover, setting \(V_t^{(n)} := V_{t \wedge \tau_n}\), we have \(\|V\| \leq n\). Let us set \(X_t^{(n)} := M_t^{(n)} + V_t^{(n)}\) and introduce the stochastic current
\[ I_n(\varphi) := \int_0^T \langle \varphi(X_t^{(n)}), dX_t^{(n)} \rangle. \]
By the previous lemma we have
\[ I_n(\varphi) = \int_{\mathbb{R}^d} \langle \hat{\varphi}(k), Z_k^{(n)} \rangle \, dk \]
where
\[ Z_k^{(n)} := \int_0^T e^{i(k,X_t^{(n)})} dX_t^{(n)}. \]
On these “Fourier coefficients” we have the estimate
\[ E \left[ |Z_k^{(n)}|^2 \right] \leq 2E \left[ \int_0^T e^{i(k,X_t^{(n)})} dM_t^{(n)} \right]^2 + 2E \left[ \int_0^T e^{i(k,X_t^{(n)})} dV_t^{(n)} \right]^2 \]
\[ \leq C_d E \int_0^T \left[ M^{(n)} \right]_t + C_d E \left[ \left( \int_0^T d\|V^{(n)}\|_t \right)^2 \right] \]
\[ = C_d E \left[ M^{(n)} \right]_T + C_d E \left[ V^{(n)} \right]_T \leq 2C_d n. \]
We now have

\[
|I_n(\varphi)| \leq \left( \int_{\mathbb{R}^d} \frac{|Z_{k}^{(n)}|^2}{(1 + |k|^2)^s} \, dk \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |\hat{\varphi}(k)|^2 (1 + |k|^2)^s \, dk \right)^{\frac{1}{2}} \leq C(\omega) \|\varphi\|_{H^s}
\]

where

\[
E \left[ |C|^2 \right] = E \int_{\mathbb{R}^d} \frac{|Z_{k}^{(n)}|^2}{(1 + |k|^2)^s} \, dk \leq 2C_d n \int_{\mathbb{R}^d} \frac{1}{(1 + |k|^2)^s} \, dk < \infty
\]

for \( s > d/2 \). Therefore \( C(\omega) \) is finite \( P \)-a.s. and the lemma above applies with \( E = H^s \left( \mathbb{R}^d, \mathbb{R}^d \right) \). We have proved that the stochastic current \( \varphi \mapsto I_n(\varphi) \) has a pathwise realization \( I_n(\omega) \).

**Step 2** (Stratonovich integral, general case). Consider now the stochastic current \( \varphi \mapsto S_n(\varphi) \) defined as

\[
S_n(\varphi) := \int_{0}^{T} \langle \varphi(X_t^{(n)}), \circ dX_t^{(n)} \rangle.
\]

By the relation between Stratonovich and Itô integrals it is intuitively clear that we should have the same result with one more derivative of \( \varphi \), i.e. the topology \( H^{s+1} \left( \mathbb{R}^d, \mathbb{R}^d \right) \) on \( \varphi \). Let us prove the result. We have

\[
S_n(\varphi) = \int_{\mathbb{R}^d} \langle \hat{\varphi}(k), Z_k^{(n)} \rangle \, dk
\]

where now we set

\[
Z_k^{(n)} := \int_{0}^{T} e^{i(k,X_t^{(n)})} \circ dX_t^{(n)}.
\]

We have

\[
Z_k^{(n)} = \int_{0}^{T} e^{i(k,X_t^{(n)})} \circ dX_t^{(n)} + \frac{1}{2} \sum_{\alpha=1}^{d} k_{\alpha} \int_{0}^{T} e^{i(k,X_t^{(n)})} \circ [X_t^{(n)\alpha}, X_t^{(n)\beta}]_t
\]

so that, by the estimate of the previous step,

\[
E \left[ |Z_k^{(n)}|^2 \right] \leq 4C_d n + C_d' |k|^2 \sum_{\alpha,\beta=1}^{d} E \left[ \left| \int_{0}^{T} e^{i(k,X_t^{(n)})} \circ [X_t^{(n)\alpha}, X_t^{(n)\beta}]_t \right|^2 \right].
\]

Recall that \( [X^{(n)\alpha}, X^{(n)\beta}]_t \) is a bounded variation function and is given by

\[
\frac{1}{4} [X^{(n)\alpha} + X^{(n)\beta}]_t - \frac{1}{4} [X^{(n)\alpha} - X^{(n)\beta}]_t
\]

that provides the decomposition as difference of non decreasing functions. Each of them can be controlled by \( [X^{(n)\alpha}]_t \) and \( [X^{(n)\beta}]_t \). Therefore we finally have, for a new constant,

\[
E \left[ |Z_k^{(n)}|^2 \right] \leq C_d n (1 + |k|^2).
\]

(7)
Repeating the argument of the previous step we get

$$|S_n(\varphi)| \leq C(\omega) \| \varphi \|_{H^{s+1}}$$

with $C(\omega)$ a.s. finite. This proves that $\varphi \mapsto S_n(\varphi)$ has a pathwise realization $S_n(\omega)$, continuous in the $H^{s+1}$-topology.

**Step 3** (Stratonovich integral, under assumption $\text{[3]}$). With the notations of step 2, where we drop $n$ for simplicity of notations, we decompose $Z_k$ in the direction of $k$ and its orthogonal by means of a suitable projection $p_k$:

$$Z_k = \frac{k}{|k|^2} \langle Z_k, k \rangle + p_k Z_k.$$

We have (using Itô formula in the first line $[19],[13]$, and the relation between Stratonovich and Itô integrals in the second one)

$$\langle Z_k, k \rangle = \int_0^T e^{i(k \cdot X_t)} \circ d \langle k, X_t \rangle = -ie^{i(k \cdot X_T)} + ie^{i(k \cdot X_0)}$$

$$Z_k^\beta = \int_0^T e^{ik \cdot X_t} dX_t^\beta + \frac{i\beta k \cdot X_T}{2} \int_0^T e^{ik \cdot X_t} dm_t$$

by the assumption on the covariation, and therefore

$$p_k Z_k = \int_0^T e^{ik \cdot X_t} d(p_k X_t).$$

Summarizing we have

$$Z_k = \frac{-ik}{|k|^2} \left( e^{i(k \cdot X_T)} - e^{i(k \cdot X_0)} \right) + \int_0^T e^{ik \cdot X_t} d(p_k M_t) + \int_0^T e^{ik \cdot X_t} d(p_k V_t).$$

It is now easy, with estimates similar to those above, to prove that

$$E \left[ |Z_k|^2 \right] \leq C d n.$$

Notice that under the additional assumption $\text{[3]}$ we do not have the factor $(1 + |k|^2)$ which appears in the estimate $\text{[3]}$. Therefore $\varphi \mapsto S_n(\varphi)$ has a pathwise realization $S_n(\omega)$, continuous in the $H^s$-topology.

**Step 4** (conclusion). Consider for instance the case of the Stratonovich integral (the other is similar). By the locality property of stochastic integrals, $[13]$ proposition 2.11 of Ch. IV, on the event $\Omega_n = \{ \tau_n = T \}$ we have $S_n(\varphi) = S(\varphi)$ a.s., for any given $\varphi$. The sequence $(\Omega_n)$ increases to $\Omega$. Define $S(\omega)$ as

$$S(\omega) = S_n(\omega) \text{ for all } \omega \in \Omega_n.$$ 

The definition is a.s. correct because $S_{n+1}(\omega) = S_n(\omega)$ a.s. on $\Omega_n$. Indeed, given $\varphi$, we have

$$S_{n+1}(\omega) \varphi = S_{n+1}(\varphi)(\omega) \text{ for a.e. } \omega \in \Omega.$$
hence by the locality property
\[ S_{n+1}(\omega) \varphi = S(\varphi)(\omega) \] for a.e. \( \omega \in \Omega_{n+1} \)
and similarly
\[ S_n(\omega) \varphi = S(\varphi)(\omega) \] for a.e. \( \omega \in \Omega_n \).
This proves that \( S_{n+1}(\omega) = S_n(\omega) \) a.s. on \( \Omega_n \), that the definition is correct and that
\[ S(\omega) \varphi = S(\varphi)(\omega) \] for a.e. \( \omega \in \Omega \).
Therefore we have found a pathwise realization of \( S \). Finally, the estimates in mean value and the generalization to \((\tilde{X}_t)\) can be obtained just by inspection in the previous arguments and inequalities. \( \square \)

**Remark 9** The previous result and proof is very related to \cite{7}, although the aim is different. The proof is also related to the one of the last lemma of the previous section.

**Remark 10** Condition (6) is fulfilled for instance by the \( d \)-dimensional Brownian motion \((W_t)\) since \( [W^i, W^j]_t = \delta_{ij}t \).

**Remark 11** The result under assumption (6) is slightly surprising. Indeed, recall that \( H^s \) is embedded into the space of continuous 1-forms, and \( H^{s+1} \) into the continuously differentiable ones. Therefore the results in the general cases have a correspondence with the fact that Itô integrals are defined when \( \varphi \) is continuous and Stratonovich one when \( \varphi \) is continuously differentiable. But the result under assumption (6) says that we have a well defined Stratonovich integral, even pathwise defined, for all functions \( \varphi \in H^s \), that are not necessarily continuously differentiable. This result should be compared more carefully with results on stochastic integration obtained when the processes have densities with respect to the Lebesgue measure. We do not stress this direction here.

### 3.2 The case of process with Lyons-Zheng structure

The concept of process with Lyons-Zheng structure has been introduced and studied by \cite{16}, \cite{17}, \cite{20}, among other references. We follow the presentation of \cite{20}. We say that \((X_t)_{t \in [0,T]}\) is a Lyons-Zheng process if it has the form
\[ X_t = M^{(1)}_t + M^{(2)}_t + V_t \]
where \( (M^{(1)}_t) \) is a continuous local martingale with respect to a filtration \( \{\mathcal{F}_t\} \), \( (M^{(2)}_t) \), defined as \( \hat{M}^{(2)}_t = M^{(2)}_{T^{-1}} \), is a continuous local martingale with respect to a filtration \( \{\mathcal{H}_t\} \), \( (V_t) \) is a bounded variation process, \( (X_t) \) is adapted to \( \{\mathcal{F}_t\} \) and \( (\hat{X}_t) \) is adapted to \( \{\mathcal{H}_t\} \), and finally we have
\[ [M^{(1)} - M^{(2)}] = 0 \] for all \( \alpha = 1, \ldots, d \). (8)
These processes arise in the theory of Dirichlet forms and relevant examples are the reversible semimartingales. For these processes it is possible to define stochastic integrals in the sense of Stratonovich, taking advantage of cancellations coming from assumption (8). For all continuous 1-forms \( \varphi \) we have

\[
\int_0^T \langle \varphi(X_t), \circ dX_t \rangle = \int_0^T \langle \varphi(X_t), dM^{(1)}_t \rangle - \int_0^T \langle \varphi(\hat{X}_t), d\hat{M}^{(2)}_t \rangle + \int_0^T \langle \varphi(X_t), dV_t \rangle \tag{9}
\]

where the first two integrals on the right-hand-side are usual Itô integrals. Indeed, arguing a little bit formally (one has to repeat the computations on finite sums and for a regularized \( \varphi \) and prove the final result taking the limit in probability), we have

\[
\int_0^T \langle \varphi(X_t), \circ dX_t \rangle = \int_0^T \langle \varphi(X_t), \circ dM^{(1)}_t \rangle + \int_0^T \langle \varphi(X_t), \circ dM^{(2)}_t \rangle + \int_0^T \langle \varphi(X_t), dV_t \rangle \]

Moreover

\[
\int_0^T \langle \varphi(X_t), \circ dM^{(1)}_t \rangle = \sum_{a=1}^d \left( \int_0^T \varphi_a(X_t) dM^{(1)a}_t + \frac{1}{2} [\varphi_a(X), M^{(1)a}]_T \right)
\]

\[
\int_0^T \langle \varphi(\hat{X}_t), \circ d\hat{M}^{(2)}_t \rangle = \sum_{a=1}^d \left( \int_0^T \varphi_a(\hat{X}_t) d\hat{M}^{(2)a}_t + \frac{1}{2} [\varphi_a(\hat{X}), \hat{M}^{(2)a}]_T \right)
\]

\[
[\varphi_a(\hat{X}), \hat{M}^{(2)a}]_T = [\varphi_a(X), M^{(2)a}]_T
\]

and finally

\[
[\varphi_a(X), M^{(1)a}]_T - [\varphi_a(\hat{X}), \hat{M}^{(2)a}]_T = [\varphi_a(X), M^{(1)a} - M^{(2)a}]_T = 0
\]

by (8). This proves (9).

**Theorem 12** Let \( (X_t) \) be a continuous Lyons-Zheng process of the form

\[
X_t = M^{(1)}_t + M^{(2)}_t + V_t
\]

as above. Then the stochastic 1-current \( \varphi \mapsto S(\varphi) \) defined by (4) has a pathwise realization \( S(\omega) \), with

\[
S(\omega) \in H^{-s}(\mathbb{R}^d, \mathbb{R}^d) \quad P\text{-a.s.}
\]
for all $s > \frac{d}{2}$. If in addition

$$\left[ M^{(1)} \right]_T, \left[ M^{(2)} \right]_T, \|V\|_{\text{var}}^2 \in L^1(\Omega),$$

then we also have the integrability property

$$S(.) \in L^2(\Omega, H^{-s}(\mathbb{R}^d, \mathbb{R}^d)).$$

In particular these results hold true for reversible semimartingales.

**Proof.** Because of (1), it is sufficient to prove the result for each one of the three addenda separately. For the last one it is true by ordinary integral calculus (recall that $H^s(\mathbb{R}^d, \mathbb{R}^d)$ is continuously embedded into the space of continuous 1-forms), while for the first two it is a consequence of the last claim of the previous theorem on semimartingales. The proof is complete. \qed

4 \hspace{1em} \gamma\text{-Hölder curves (}\gamma > 1/2\text{) define integral flat chains}

The results of this section do not require or involve any stochastic structure of $(X_t)_{t \in [0,T]}$, which therefore is supposed to be a deterministic function. The results are of course applicable to stochastic processes, just path by path (it is also easy to check that all the quantities constructed below depend measurably on the random parameter).

Our approach gives a formula for the stochastic integral in terms of double random integrals that seems to be new and could be used in contexts like the stochastic analysis of fractional Brownian motion.

Let $(X_t)_{t \in [0,T]}$ be a $\gamma$-Hölder continuous function with values in $\mathbb{R}^d$, with $\gamma > 1/2$. We first analyse its mollifications. Let $\eta$ be a real-valued piece-wise $C^1$ function on $\mathbb{R}$, with compact support in $B_1(0)$, non-negative, such that $\int \eta(t) \, dt = 1$. For all $\alpha \in (0, 1]$ we set

$$\eta_\alpha(t) = \frac{1}{\alpha} \eta \left( \frac{t}{\alpha} \right)$$

so that $\int \eta_\alpha(t) \, dt = 1$. We also set

$$A = [0, T] \times (0, 1]$$

$$X_{t,\alpha} = (\eta_\alpha * \tilde{X})_t = \int \eta_\alpha(t-s) \tilde{X}_s \, ds, \quad (t, \alpha) \in A$$

where $(\tilde{X}_t)_{t \in [-1,T+1]}$ is any $C^\gamma$ extension of $X_t$. Therefore $(X_{t,\alpha})_{t \in A}$ is a $C^1$ mapping from $A$ to $\mathbb{R}^d$ (notice that $\alpha = 0$ is excluded). We shall write $(X_t)$ and $(X_{t,\alpha})$ for shortness.
We have the following formulae and estimates:

\[
\frac{\partial X_{t,\alpha}}{\partial t} = \int \frac{\partial \eta_{t,\alpha}}{\partial t} (t-s) \tilde{X}_s \, ds = \left( \frac{\partial \eta_{t,\alpha}}{\partial t} * \tilde{X} \right)_t
\]

\[
= \int \frac{\partial \eta_{t,\alpha}}{\partial t} (t-s) (\tilde{X}_s - \tilde{X}_t) \, ds
\]

\[
\frac{\partial X_{t,\alpha}}{\partial \alpha} = \int \frac{\partial \eta_{t,\alpha}}{\partial \alpha} (t-s) \tilde{X}_s \, ds = \left( \frac{\partial \eta_{t,\alpha}}{\partial \alpha} * \tilde{X} \right)_t
\]

\[
= \int \frac{\partial \eta_{t,\alpha}}{\partial \alpha} (t-s) (\tilde{X}_s - \tilde{X}_t) \, ds
\]

\[
\frac{\partial \eta_{t,\alpha}}{\partial t} (t) = -\frac{1}{\alpha^2} \eta' \left( \frac{t}{\alpha} \right), \quad \frac{\partial \eta_{t,\alpha}}{\partial \alpha} (t) = -\frac{1}{\alpha^2} \eta \left( \frac{t}{\alpha} \right) - \frac{t}{\alpha^2} \eta'' \left( \frac{t}{\alpha} \right)
\]

\[
\left| \frac{\partial X_{t,\alpha}}{\partial t} \right| \leq C \alpha^{-1}, \quad \left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right| \leq C \alpha^{-1}
\]

because

\[
\left| \frac{\partial X_{t,\alpha}}{\partial t} \right| = \left| \int \frac{\partial \eta_{t,\alpha}}{\partial t} (t-s) (\tilde{X}_s - \tilde{X}_t) \, ds \right|
\]

\[
\leq C \alpha^{-2} \int \left| \eta' \left( \frac{t-s}{\alpha} \right) \right| \left| t-s \right| \gamma \, ds
\]

\[
= C \alpha^{-1} \int \left| \eta' (r) \right| \left| r \right| \gamma \, dr \quad \text{with} \quad r = \frac{t-s}{\alpha}
\]

\[
\left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right| = \left| \int \frac{\partial \eta_{t,\alpha}}{\partial \alpha} (t-s) (\tilde{X}_s - \tilde{X}_t) \, ds \right|
\]

\[
\leq C \alpha^{-2} \left| \int \left| \eta \left( \frac{t-s}{\alpha} \right) + \frac{t-s}{\alpha} \eta' \left( \frac{t-s}{\alpha} \right) \right| \left| t-s \right| \gamma \, ds \right|
\]

\[
= C \alpha^{-1} \left| \int \left| \eta (r) + r \eta' (r) \right| \left| r \right| \gamma \, dr \right| \quad \text{with} \quad r = \frac{t-s}{\alpha}
\]

The next theorem is perhaps the main result of this paper. It states that any $\gamma$-Hölder continuous curve in $\mathbb{R}^d$, with $\gamma > 1/2$, defines an integral flat chain, denoted by $\int_0^T \langle \varphi (X_t), \circ dX_t \rangle$ and gives a formula in terms of a double integral. For curves with such regularity there is no distinction between Stratonovich and Itô integrals (the quadratic variation is easily proved to be zero), so we keep the notation of Stratonovich integral since we believe it should be the appropriate one in case the following theorem will have some kind of generalization to semimartingales in the future. Notice that the following result is not entirely trivial a priori since $(X_t)$ may have infinite mass and Hausdorff dimension $\gamma^{-1}$.
Theorem 13 Assume that \((X_t)\) is a \(\gamma\)-H"older continuous curve in \(\mathbb{R}^d\), with \(\gamma > 1/2\). Then:

a) \((X_t,\alpha)\) defines the following i.m. rectifiable 2-current \(T\):

\[
T(\psi) = \int_A \left\langle \psi(X_t,\alpha), \frac{\partial X_{t,\alpha}}{\partial t} \wedge \frac{\partial X_{t,\alpha}}{\partial \alpha} \right\rangle \, dt \, d\alpha \\
= \int_A \sum_{i<j} \psi_{ij}(X_{t,\alpha}) \left( \frac{\partial X_{i,\alpha}^j}{\partial t} \frac{\partial X_{i,\alpha}}{\partial \alpha} - \frac{\partial X_{i,\alpha}^i}{\partial t} \frac{\partial X_{i,\alpha}}{\partial \alpha} \right) \, dt \, d\alpha
\]

for all continuous 2-forms \(\psi\) on \(\mathbb{R}^d\) represented as

\[
\sum_{i<j} \psi_{ij}(x) \, dx^i \wedge dx^j.
\]

b) for every continuously differentiable 1-form \(\varphi\), the following limit

\[
\lim_{\alpha \to 0} \int_0^T \langle \varphi(X_t,\alpha), dX_t,\alpha \rangle
\]

exists (the integral is understood as a classical Riemann integral) and will be denoted by \(\int_0^T \langle \varphi(X_t), \circ dX_t \rangle\).

c) The mapping \(\varphi \mapsto \int_0^T \langle \varphi(X_t), \circ dX_t \rangle\) is the integral flat chain (of degree one) given by

\[
\int_0^T \langle \varphi(X_t), \circ dX_t \rangle = T(d\varphi) + \int_0^T \left\langle \varphi(X_{t,1}), \frac{\partial X_{t,1}}{\partial t} \right\rangle \, dt \\
- \int_0^1 \left\langle \varphi(X_{t,0}), \frac{\partial X_{t,0}}{\partial \alpha} \right\rangle \, d\alpha + \int_0^1 \left\langle \varphi(X_{0,\alpha}), \frac{\partial X_{0,\alpha}}{\partial \alpha} \right\rangle \, d\alpha
\]

(10)

for all continuously differentiable 1-forms \(\varphi\) on \(\mathbb{R}^d\) represented as \(\sum_{i=1}^d \varphi_i(y) \, dy^i\) and where

\[
T(d\varphi) = \int_A \sum_{i,j=1}^d \varphi_i(X_{t,\alpha}) \left( \frac{\partial X_{t,\alpha}^j}{\partial t} \frac{\partial X_{t,\alpha}^i}{\partial \alpha} - \frac{\partial X_{t,\alpha}^i}{\partial t} \frac{\partial X_{t,\alpha}^j}{\partial \alpha} \right) \, dt \, d\alpha.
\]

Proof. Step 1 (The support \(\mathcal{M}\) of \(T\) is a 2-rectifiable set). Let us introduce the function \(f : A \to \mathbb{R}^d\) defined as \(f(t,\alpha) = X_{t,\alpha}\) and let \(\mathcal{M} = f(A)\). We have

\[
Df = \left( \begin{array}{c} \frac{\partial X_{t,\alpha}}{\partial t} \\ \frac{\partial X_{t,\alpha}}{\partial \alpha} \end{array} \right)
\]

and

\[
(Df)^* Df = \left( \begin{array}{c} \left| \frac{\partial X_{t,\alpha}}{\partial t} \right|^2 \\ \left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right|^2 \\ \left\langle \frac{\partial X_{t,\alpha}}{\partial t}, \frac{\partial X_{t,\alpha}}{\partial \alpha} \right\rangle^2 \\ \left\langle \frac{\partial X_{t,\alpha}}{\partial \alpha}, \frac{\partial X_{t,\alpha}}{\partial t} \right\rangle^2 \end{array} \right)
\]
\[ J_f = \sqrt{\det (Df)^* Df} \]
\[ = \sqrt{\left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right|^2 \left| \frac{\partial X_{t,\alpha}}{\partial t} \right|^2 - \left\langle \frac{\partial X_{t,\alpha}}{\partial t}, \frac{\partial X_{t,\alpha}}{\partial \alpha} \right\rangle^2} \].

By the estimates of the previous subsection we have
\[ J_f \leq C\alpha^{2(\gamma-1)} \]
which is integrable on \( A \):
\[ \int_A J_f \, dt \, d\alpha < \infty. \]

This is the basic property that will imply the final result.

Let \( A_n \) be defined as
\[ A_n = \left\{ (t, \alpha) \in \mathbb{R}^2 : t \in [0, T], \alpha \in \left( \frac{1}{n+1}, 1 \right) \right\}. \]

We have \( \mathcal{M} = \bigcup_n f(A_n) \) where \( f|A_n \) are Lipschitz functions. In particular (see [11] p. 75), \( f(A_n) \) are \( \mathcal{H}^2 \)-measurable sets, and so it is \( \mathcal{M} \). Since \( f|A_n \) is Lipschitz, the area formula gives us
\[ \int_{f(A_n)} \mathcal{H}^0 \left( f^{-1}(x) \cap A_n \right) \, d\mathcal{H}^2(x) = \int_{A_n} J_f \, dt \, d\alpha \leq \int_A J_f \, dt \, d\alpha < \infty. \] (11)

On one side it follows that
\[ \mathcal{H}^2(f(A_n)) = \int_{f(A_n)} d\mathcal{H}^2(x) \leq \int_A J_f \, dt \, d\alpha < \infty \]
for all \( n \), and therefore, by the continuity of \( \mathcal{H}^2 \),
\[ \mathcal{H}^2(\mathcal{M}) < \infty. \]

This completes the proof that \( \mathcal{M} \) is a 2-rectifiable set in \( \mathbb{R}^d \).

**Step 2** \( (T \) is an i.m. rectifiable 2-current). Again from [11] we deduce that the function \( N(f,A,) : \mathcal{M} \to \mathbb{R} \) defined as \( N(f,A,x) = \mathcal{H}^0(f^{-1}(x)) \) is \( \mathcal{H}^2[\mathcal{M}] \)-a.s. finite and integer valued, and also \( \mathcal{H}^2[\mathcal{M}] \)-integrable. Indeed, by the monotone convergence theorem
\[ \int_{\mathcal{M}} N(f,A,x) \, d\mathcal{H}^2(x) = \lim_{n \to \infty} \int_{f(A_n)} \mathcal{H}^0 \left( f^{-1}(x) \cap A_n \right) \, d\mathcal{H}^2(x) \] (12)
\[ \leq \int_A J_f \, dt \, d\alpha < \infty. \]

In particular, it follows that the following vector field is well defined \( \mathcal{H}^2[\mathcal{M}] \)-a.s. since it is given by a finite sum:
\[ v(x) = \sum_{(t,\alpha) \in f^{-1}(x)} \frac{\zeta_{t,\alpha}}{|\zeta_{t,\alpha}|} \]
with
\[ \zeta_{t,\alpha} := \left( \frac{\partial X_{t,\alpha}}{\partial t} \right) \wedge \left( \frac{\partial X_{t,\alpha}}{\partial \alpha} \right). \]

Let us introduce the 2-current \( T_n \) defined as
\[ T_n (\psi) := \int_{A_n} \langle \psi (X_{t,\alpha}), \zeta_{t,\alpha} \rangle \ dtd\alpha. \]

Since \( f|A_n \) is Lipschitz, from the change of variable formula (\[1\] p. 75) we have
\[
T_n (\psi) = \int_{f(A_n)} \left\langle \psi (X_{t,\alpha}), \frac{\zeta_{t,\alpha}}{|\zeta_{t,\alpha}|} \right\rangle \ J_f \ dtd\alpha \\
= \int_{f(A_n)} \sum_{(t,\alpha) \in f^{-1}(x) \cap A_n} \left\langle \psi (X_{t,\alpha}), \frac{\zeta_{t,\alpha}}{|\zeta_{t,\alpha}|} \right\rangle \ d\mathcal{H}^2 (x) \\
= \int_{f(A_n)} \langle \psi (x), v_n (x) \rangle \ d\mathcal{H}^2 (x)
\]
where \( v_n (x) \) is defined as
\[ v_n (x) := \sum_{(t,\alpha) \in f^{-1}(x) \cap A_n} \frac{\zeta_{t,\alpha}}{|\zeta_{t,\alpha}|}. \]

Since \( f(A_n) \) is a 2-rectifiable set, the density \( \Theta^2 (f(A_n), x) \) is equal to 1 for \( \mathcal{H}^2 [f(A_n)] \)-a.e. \( x \) (see \[3\], th. 3.2.19), hence the cone of approximate tangent vectors of \( f(A_n) \) at such \( x \) is the plane identified by any one of the 2-vectors \( \zeta_{t,\alpha}/|\zeta_{t,\alpha}| \). These 2-vectors are unitary, hence they coincide up to the constant \( \pm 1 \). Therefore there exist an \( \mathcal{H}^2 \)-measurable unitary 2-vector field \( \xi_n \) on \( f(A_n) \), with \( \xi_n (x) \in T_x f(A_n) \) for \( \mathcal{H}^2 [f(A_n)] \)-a.e. \( x \), and an integer valued and \( \mathcal{H}^2 [f(A_n)] \) integrable (by (12)) function \( \theta_n : f(A_n) \rightarrow \mathbb{R} \), such that \( v_n = \xi_n \theta_n \) \( \mathcal{H}^2 [f(A_n)] \)-a.s. This proves the (well known) fact that \( T_n \) is a i.m. rectifiable 2-current:
\[ T_n (\psi) = \int_{f(A_n)} \langle \psi, \xi_n \rangle \theta_n \ d\mathcal{H}^2. \] (13)

Let us now decompose \( \mathcal{M} \) into the sets \( \mathcal{N}_n = f (A - A_n) \) and \( \mathcal{M}_n = f (A_n) \setminus \mathcal{N}_n \). We have
\[
\mathcal{H}^2 (\mathcal{N}_n) = \int_{A \setminus A_n} d\mathcal{H}^2 (x) \\
\leq \int_{f(A \setminus A_n)} \mathcal{H}^0 (f^{-1} (x) \cap (A \setminus A_n)) \ d\mathcal{H}^2 (x) \\
= \int_{A \setminus A_n} J_f \ dtd\alpha \rightarrow 0 \ \text{as} \ n \rightarrow \infty.
\]
In a sense, the subset $M_n$ of $M$ is a stable part when $n$ increases (it is not visited any more by $f$), and the remainder $N_n$ becomes negligible. We have, for all $m > n$,

$$T_n M = T_x f (A_m) = T_x f (A_n) \text{ for a.e. } x \in M_n. \quad (14)$$

Indeed, arguing again on the density ([3], th. 3.2.19), for a.e. $x \in M_n$ the three tangent planes (that exist) are the only part of the corresponding cones of approximate tangent vectors (a priori one bigger than the other), so they coincide. We also have (directly from the definitions)

$$v_m (x) = v_n (x) \text{ for a.e. } x \in M_n.$$ 

Choosing $\theta_n$ above with non negative values (it is always possible), we deduce

$$\xi_m (x) = \xi_n (x) \text{ and } \theta_m (x) = \theta_n (x) \text{ for a.e. } x \in M_n.$$ 

It is therefore well defined an $H^2$-measurable unitary 2-vector field $\xi$ on $M$, with (by (14)) $\xi (x) \in T_x M$ for $H^2 [M]$-a.e. $x$, and an integer valued and $H^2 [M]$ integrable (again by (12)) function $\theta : M \to \mathbb{R}$, such that

$$\xi (x) = \xi_n (x) \text{ and } \theta (x) = \theta_n (x) \text{ for a.e. } x \in M_n.$$ 

From (13) we have

$$T_n (\psi) = \int_{M_n} \langle \psi, \xi \rangle \theta \, dH^2 + \int_{N_n} \langle \psi, v_n \rangle \, dH^2.$$ 

Since $\int_A J_f \, dtd\alpha < \infty$, $T_n (\psi) \to T (\psi)$ as $n \to \infty$. By monotone convergence, the second term converges to $\int_M \langle \psi, \xi \rangle \theta \, dH^2$. Finally,

$$\left| \int_{N_n} \langle \psi, v_n \rangle \, dH^2 \right| \leq \int_{N_n} \| \psi \|_\infty \| v (x) \| \, dH^2 (x)$$

$$\leq \| \psi \|_\infty \int_{N_n} N (f, A, x) \, dH^2 (x)$$

that converges to zero by (12). Therefore in the limit we obtain

$$T (\psi) = \int_M \langle \psi, \xi \rangle \theta \, dH^2$$

so $T$ is an i.m. rectifiable 2-current.

**Step 3** (conclusion). The proof of b) and of formula (10) is elementary: first one writes a formula like (14) for classical integrals on $A_\alpha = [0, T] \times (\alpha, 1]$ (the classical Stokes formula), then pass to the limit. The last three integrals (boundary terms) in (10) are well defined classical integrals and define i.m. rectifiable 1-currents. Therefore (10) tells us that $\int_0^T \langle \varphi (X_t), \cdot dX_t \rangle$ is an integral flat chain (recall that $T (d\varphi) = \partial T (\varphi)$). The proof is complete. $\square$

A priori the integral flat chain $\int_0^T \langle \varphi (X_t), \cdot dX_t \rangle$ defined above may depend on the choice of $\eta$ and of the extension $\tilde{X}$. This is not the case, as the following proposition asserts.
Proposition 14  a) If \((X_t^{(n)})\) is a sequence of functions in \(C^\gamma\), with \(\gamma > 1/2\), that converges to \((X_t) \in C^\gamma\) in the \(C^\gamma\) seminorm and in \(C^0\), then for every continuously differentiable 1-form \(\varphi\) we have
\[
\int_0^T \langle \varphi(X_t^{(n)}), \od X_t^{(n)} \rangle \to \int_0^T \langle \varphi(X_t), \od X_t \rangle
\]
where the \(C^\gamma\)-extensions of \((X_t^{(n)})\) and \((X_t)\) are arbitrarily chosen and the same \(\eta\) is taken.

b) If \((X_t)\) is continuously differentiable, \(\int_0^T \langle \varphi(X_t), \od X_t \rangle\) defined above coincides with the usual Riemann integral, for all \(\eta\) and \(C^\gamma\)-extension of \((X_t)\).

c) If \((X_t)\) is in \(C^\gamma\) with \(\gamma > 1/2\), the definition of the integral flat chain \(\int_0^T \langle \varphi(X_t), \od X_t \rangle\) is independent of the choice of \(\eta\) and of the \(C^\gamma\)-extension of \((X_t)\).

Proof.  a) This follows from the explicit representation formula for the integral \(\int_0^T \langle \varphi(X_t^{(n)}), \od X_t^{(n)} \rangle\) and \(\int_0^T \langle \varphi(X_t), \od X_t \rangle\), after some tedious but elementary estimates similar to those used above.

b) It is just Stokes formula.

c) It follows from a) and b) by taking as \((X_t^{(n)})\) a sequence of \(C^1\) approximations of \((X_t)\).

□

Remark 15  For processes with strong fluctuations, like the fractional Brownian motion, the following fact is plausible: in \(d = 2\) the set \(f(A)\) contains a lot of superpositions; in \(d \geq 3\) there are only selfintersections in \(f(A)\), and their set has 2-dimensional Hausdorff measure (in other words, \(N(f,A,x) = 1, H^2|\mathcal{M}-a.s.)\).

Remark 16  Recall the spaces \(W^{s,p}\) as defined for instance in [1]. Let \(W^{1/2,2}_C\) be the set of all continuous functions \((X_t)_{t \in [0, T]}\) taking values in \(\mathbb{R}^d\) such that
\[
\int_0^T \int_0^T \frac{|X_t - X_s|^2}{|t - s|^2} dt ds + \int_0^T \frac{|X_T - X_s|}{|T - s|} ds + \int_0^T \frac{|X_0 - X_s|}{|s|} ds < \infty.
\]
We easily have
\[
\bigcup_{\gamma > 1/2} C^\gamma([0, T]) \subset W^{1/2,2}_C
\]
The results of the previous section extends to the class \(W^{1/2,2}_C\): one can prove that
\[
\int_0^T \int_0^1 \left( \left| \frac{\partial X_{t,\alpha}}{\partial t} \right|^2 + \left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right|^2 \right) dt d\alpha < \infty
\]
\[
\int_0^1 \left( \left| \frac{\partial X_{T,\alpha}}{\partial \alpha} \right| + \left| \frac{\partial X_{T,\alpha}}{\partial \alpha} \right| \right) d\alpha < \infty;
\]
and therefore
\[ \int_A J_f \, dtd\alpha < \infty. \]

The extension to \( W^{1/2,2}_C \) of Theorem 1 can be interpreted as a result of trace theory in Sobolev spaces. We think that the constructive proof we outline is simple and straightforward enough to dispense with abstract arguments, which in addition would require a careful investigation of the boundary terms at \( t = 0, T \). Moreover as it will be shown in [12] the explicit computations we make give some insights on the difficulties of extending the result to more irregular paths (e.g. samples of Brownian motion).

**Remark 17** Given two curves \((X_t)\) and \((Y_t)\) of class \(C^\gamma\), \(\gamma > 1/2\) (the same can be said for the class \(W^{1/2,2}_C\), see remark 16 above), we may now define the integral flat distance \(d_F(X,Y)\) between them as the infimum of the numbers
\[ M(T) + M(R) \]
over all i.m. rectifiable 2-currents \(T\) and i.m. rectifiable 1-currents \(R\) such that
\[ \int_0^T \langle \varphi(X_t), \circ dX_t \rangle - \int_0^T \langle \varphi(Y_t), \circ dY_t \rangle = \partial T + R \]
(by the previous theorem, at least one pair \((T, R)\) exists). Here \(M(.)\) denotes the mass of the corresponding current (see [12]). In a sense, this distance is more geometric than the usual ones employed in stochastic analysis, it is independent of the parametrization, and thus we hope it may have applications in future researches.

### 4.1 A \(d+1\) dimensional variant

Geometrically it may appear unpleasant that the support of the rectifiable current \((X_{t,\alpha})\) may have very complex overlaps (especially in dimension 2). There are many ways to associate to \((X_t)\) a piece of boundary of a rectifiable 2-current, some of them giving us surfaces with a nicer appearance. An easy way is to see \((X_t)\) in \(\mathbb{R}^{d+1}\), namely to consider the function
\[ t \mapsto Y_{t,\alpha} := (t, X_t). \]
We conjecture that in some relevant examples this mapping is injective.

This modification has the additional advantage that it defines integrals over 1-forms depending also on \(t\). In other words, we define integrals of the form
\[ \int_0^T \langle \varphi(t, X_t), \circ dX_t \rangle. \]

**Theorem 18** Assume that \((X_t)\) is a \(\gamma\)-Hölder continuous curve in \(\mathbb{R}^d\) with \(\gamma > 1/2\). Then:
a) \((Y_{t,\alpha})\) defines the following integer multiplicity rectifiable 2-current \(\tilde{T}\) in \(\mathbb{R}^{d+1}\):

\[
\tilde{T}(\psi) = \int_A \langle \psi(t, X_{t,\alpha}), \left(1, \frac{\partial X_{t,\alpha}}{\partial t} \right) \rangle \wedge \left(0, \frac{\partial X_{t,\alpha}}{\partial \alpha} \right) \rangle \, dt \, d\alpha
\]

\[
= \int_A \sum_{i,j=1}^d \psi_{ij}(t, X_{t,\alpha}) \left(\frac{\partial X_{i,\alpha}}{\partial \alpha} \frac{\partial X_{j,\alpha}}{\partial t} - \frac{\partial X_{i,\alpha}}{\partial t} \frac{\partial X_{j,\alpha}}{\partial \alpha} \right) \, dt \, d\alpha
\]

for all continuous 2-forms \(\psi\) on \(\mathbb{R}^{d+1}\) represented as

\[
\sum_{i,j=1}^d \psi_{ij}(t, y) \, dy^j \wedge dy^i + \sum_{j=1}^d \psi_{0j}(t, y) \, dy^j \wedge dt + \sum_{i=1}^d \psi_{i0}(t, y) \, dt \wedge dy^i.
\]

b) for every continuously differentiable 1-form \(\phi\), the following limit

\[
\lim_{\alpha \to 0} \int_0^T \langle \phi(t, X_{t,\alpha}), \circ dX_{t,\alpha} \rangle
\]

exists (the integral is understood as a classical Riemann integral) and will be denoted by \(\int_0^T \langle \phi(t, X_t), \circ dX_t \rangle\).

The proof is similar to that of Theorem 13 in the previous subsection. In this case we define

\[
\tilde{f}(t, \alpha) = (t, X_{t,\alpha}) = (t, (\eta_\alpha \ast X)_t)
\]

and we use the formulae

\[
D\tilde{f} = \left( \begin{array}{cc} \frac{1}{\partial t} & 0 \\ \frac{\partial X_{t,\alpha}}{\partial \alpha} & \frac{\partial X_{t,\alpha}}{\partial \alpha} \end{array} \right)
\]

\[
(D\tilde{f})^* D\tilde{f} = \left( \begin{array}{cc} 1 + \left(\frac{\partial X_{t,\alpha}}{\partial \alpha}\right)^2 & \langle \frac{\partial X_{t,\alpha}}{\partial t}, \frac{\partial X_{t,\alpha}}{\partial \alpha} \rangle \\ \langle \frac{\partial X_{t,\alpha}}{\partial t}, \frac{\partial X_{t,\alpha}}{\partial \alpha} \rangle & \left(\frac{\partial X_{t,\alpha}}{\partial \alpha}\right)^2 \end{array} \right)
\]

for all 1-forms \(\phi\) on \(\mathbb{R}^d\) represented as \(\sum_{i=1}^d \phi_i(t, y) \, dy^i + \phi_0(t, y) \, dt\).
\[ J_\tilde{f} = \sqrt{\text{det} \left( D \tilde{f} \right)^*} D \tilde{f} \]
\[ = \sqrt{\left| \frac{\partial X_{t,\alpha}}{\partial \alpha} \right|^2 \left( 1 + \left| \frac{\partial X_{t,\alpha}}{\partial t} \right|^2 \right) - \left\langle \frac{\partial X_{t,\alpha}}{\partial t}, \frac{\partial X_{t,\alpha}}{\partial \alpha} \right\rangle^2}. \]

Moreover, the analog of Proposition 14 holds true.

### 4.2 Remarks for Brownian curves

The results of the previous section do not apply to the typical paths of a Brownian motion \((W_t)\) in \(\mathbb{R}^d\) and presumably it is not possible to modify the argument and prove that such paths are integral flat currents. To clarify this point we report below a negative result on the summability of \(J_f\) in the case of Brownian motion.

**Lemma 19** We have identities in law (for every \(\lambda > 0\)) between the following processes, in the range \(t \geq 1, \alpha \in (0, 1):\)

\[
\left( \frac{\partial W_{t,\alpha}}{\partial t} (\lambda t, \lambda \alpha), \frac{\partial W_{t,\alpha}}{\partial \alpha} (\lambda t, \lambda \alpha) \right) \leq \frac{1}{\sqrt{\lambda}} \left( \frac{\partial W_{t,\alpha}}{\partial t} (t, \alpha), \frac{\partial W_{t,\alpha}}{\partial \alpha} (t, \alpha) \right)
\]

\[ J_f (\lambda t, \lambda \alpha) \leq \frac{1}{\lambda} J_f (t, \alpha). \]

Moreover, the law of these processes do not change by a time shift.

**Proof.** Recall that the law of the process \((W_{\lambda t})\) is equal to the law of \((\sqrt{\lambda} W_t)\). Then

\[
\frac{\partial W_{t,\alpha}}{\partial t} (\lambda t, \lambda \alpha) = \int \frac{\partial \eta_\alpha}{\partial t} (\lambda (t-s), \lambda \alpha) W_{\lambda s} ds
\]

\[ = - \int \frac{1}{(\lambda \alpha)^2} \eta_\alpha \left( \frac{t-s}{\alpha} \right) W_{\lambda s} \lambda^2 ds = \frac{1}{\sqrt{\lambda}} \frac{\partial W_{t,\alpha}}{\partial t} (t, \alpha). \]

The computation for \(\frac{\partial W_{t,\alpha}}{\partial \alpha}\) is similar. To give a complete proof one has to repeat the previous computation jointly on the two previous random variables and jointly at generic points \((t_1, \alpha_1), \ldots, (t_n, \alpha_n)\), which is just more lengthy.

The property for \(J_f\) is a direct consequence of the previous result. The invariance under time shift is due to the representation (obtained by integration by parts)

\[
\frac{\partial W_{t,\alpha}}{\partial t} = \int \eta_\alpha (t-s) dW_s
\]

\[
\frac{\partial W_{t,\alpha}}{\partial \alpha} = - \int \eta_\alpha (t-s) \frac{t-s}{\alpha} dW_s
\]

along with the stationarity of the increments of the Brownian motion. \(\square\)
Proposition 20  Over the set $(t, \alpha) \in [1, 2] \times (0, 1]$ we have
\[
E \int_{[1,2] \times (0,1]} J_f \, dt \, d\alpha = +\infty.
\]

Proof. Split $[1, 2] \times (0, 1]$ in strips $S_n = [1, 2] \times (2^{-n-1}, 2^{-n}]$, $n = 0, 1, \ldots$
Split each strip in squares:
\[
S_n = \bigcup_{k=0}^{2^n+1-1} S_{n,k} \text{ with } S_{n,k} = \left[1 + \frac{k}{2^{n+1}}, 1 + \frac{k+1}{2^{n+1}}\right] \times \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right).
\]
Set
\[
A_{n,k} = \int_{S_{n,k}} J_f \, dt \, d\alpha.
\]
The lemma implies that for every $n \geq 1$ the r.v. $A_{n,0}, \ldots, A_{n,2^n+1-1}$ have the same law, which is also equal to the law of $\frac{1}{2^n} A_{n-1,0}, \ldots, 1/2^n A_{n,2^n-1}$. Therefore, setting $a_n = E[A_{n,k}]$ (it may be infinite and does not depend on $k$), we have $a_n = a_{n-1}/2$ and (the integrand is positive so we may interchange the operations)
\[
E \int_{S_n} J_f \, dt \, d\alpha = E \sum_{k=0}^{2^n+1-1} A_{n,k} = \sum_{k=0}^{2^n+1-1} a_n = 2^n a_{n-1} = \ldots = 2^n a_0.
\]
The claim of the lemma follows at once. \hfill \Box

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