The Analytical Methods for Modelling the Two-Dimensional in the Open Water Circuits Development

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Abstract. The basic system of non-stationary two-dimensional open water circuit motion equations, which does not contradict the previously known system for a stationary circuit, is obtained.

Introduction
The modern hydraulic structures’ design is unthinkable without a comprehensive consideration of the flow’s dynamic properties acting on them. The continuously increasing sizes and reliability requirements for feeding channels, free-flow pipes, open spillways require, in particular, the in-depth specific properties’ knowledge of the open streams with high speeds. The classical theory of one-dimensional open flows cannot give the proper answers to many questions put forward by the hydraulic engineering practice. A two-dimensional planned flow is a flow in an open channel, which surface width is several times greater than the depth, the bottom relief is smooth and the curvature of the jets in the plan is not large.

At the origins of the two-dimensional planned open currents theory are the scientists V.M. Makkaveev, N. M. Vernadsky, N. T. Meleschenko, G.I. Sukhomel, S.I. Numerov, A.T. Ippen, R.T. Knapp. Both in Russian and in foreign works, the potential flows in a horizontal channel are mainly considered, which made it possible to widely use the well-known gas-hydraulic analogy and the associated method of characteristics. The results obtained in the above-mentioned studies are fundamental and retain their significance nowadays.

The theory under consideration was further developed in the works of S. N. Numerov, who extended the calculation method by the characteristics to the case of vortex motion, and also took into account the friction forces for the first time and gave a numerical calculation method. In addition, S. N. Numerov solved some questions of the two-dimensional theory as applied to the calm planned flows. A detailed study of the problem of spreading a stormy stream on the basis of a two-dimensional theory is the subject of a number of articles and the dissertation by I. A. Sherenkov. He developed an original method for calculating the potential flow by characteristics; as well as the numerical method that takes into account friction and bottom slope.
In the works of a number of foreign authors (A. Ippen, X. Rouse, D. Harleman, D. Dauson, R. Knapp), the main provisions of the two-dimensional potential flows’ theory were used to construct the methods for calculating the transition sections of fast currents. They also conducted a number of experimental studies.

**Derivation of the equations of motion of stationary flow**

The aim of the work is to derive the non-stationary two-dimensional motion equations in terms of open water circuits. As noted in the works [1, 2] the study of potential two-dimensional in terms of open water circuits has a certain theoretical and important practical value. The theoretical significance of this model lies in the possibility of a phased expansion of the model to solve the practical problems. At the first stage, the main laws of a simplified model of potential circuit are revealed. Further, the model, based on the analysis of the obtained solutions of the problem at the first stage, is complicated to the possibility of practical use of the simulation results. The practical value of the model lies in the possibility of designers using the hydraulic structures to use the results of both the first stage of solving the problem and the next stages, taking into account the resistance forces to circuit.

This is possible in a number of cases of circuit in short sections - sudden expansion or narrowing of the channel, where the forces of resistance to the circuit from the outlet channel can be neglected in comparison with the forces of inertia and gravity and the potential can be accepted [1, 2].

In [1], the concept of a two-dimensional model in terms of open water circuit is given. However, the method of characteristics and analytical methods for calculating the circuit parameters are developed as applied to the stationary potential circuits.

First, we describe the analytical method for calculating the potential steady circuits [3, 4, 5] using the plane of the speed hodograph $\Gamma(\tau, \Theta)$, where $\tau$ - is the particle velocity function; $\Theta$ - is the angle of inclination of the fluid particle velocity vector to the longitudinal axis of circuit symmetry; $\tau, \Theta$ - are the independent parameters and outline ways to expand it to unsteady circuits.

The system of motion equations of the circuit in the plane of the travel time curve has the form:

$$
\begin{cases}
\frac{\partial \phi}{\partial \tau} = \frac{h_0}{2H_0} \frac{3\tau - 1}{\tau(1 - \tau)^2} \frac{\partial \phi}{\partial \theta} \\
\frac{\partial \phi}{\partial \theta} = \frac{2h_0}{H_0} \frac{\tau}{1 - \tau} \frac{\partial \phi}{\partial \tau}
\end{cases}
$$

(1)

where $\tau = \frac{V^2}{2gH_0}$; $V$ - is the local circuit rate; $H_0$ - denotes the constant in the integral of D. Bernoulli; $\Theta$ - defines the tilt angle of the velocity vector $\vec{V}$ liquid particles to the longitudinal axis of circuit symmetry «OX»; $g$ - is the gravity acceleration; $\phi(\tau, \Theta), \phi(\tau, \Theta)$ - denote the potential function and stream function (dependent variables) in a system of linear partial differential equations.

The relationship between the physical region of the circuit and the plane of the travel time curve is determined by the formula [3]:

$$
dz = dx + idy = \left( d\phi + i \frac{h_0}{h} d\phi \right) e^{i\Theta} V.
$$

(2)

where $i$ – is the complex unit; $e$ – is the base of the natural logarithm; $h$ – is the local circuit depth.

To derive the system (1), the authors used the following dependencies:

- D. Bernoulli integral valid for two-dimensional in terms of the potential stationary water circuits
\[ V^2 \frac{1}{2g} + h = H_0; \quad (3) \]

- the potential condition circuit

\[ \frac{\partial \varphi}{\partial x} = u_x; \quad \frac{\partial \varphi}{\partial y} = u_y, \quad (4) \]

where \( \varphi = \varphi(x, y) \) is the potential function in the plane \( \Phi(x, y) \);

- the streamline conditions

\[ \frac{\partial \Phi}{\partial x} = -\frac{h}{h_0} u_y; \quad \frac{\partial \Phi}{\partial y} = \frac{h}{h_0} u_x, \quad (5) \]

following from the continuity equation circuit, we get:

\[ \frac{\partial (u_x h)}{\partial x} + \frac{\partial (u_y h)}{\partial y} = 0. \quad (6) \]

In [3, 4, 5], a method for determining the analytical solutions of system (1) was shown, and on their basis a method for solving boundary value problems for two-dimensional circuits in terms of potential stationary circuits was developed.

*The unsteady circuits motion equations’ derivation*

For the non-stationary water circuits, according to the theory in [1, 6], the following system of two-dimensional motion differential equations in terms of circuit is valid (excluding the circuit resistance forces):

\[
\begin{align*}
\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + g \frac{\partial h}{\partial x} &= 0; \\
\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} + g \frac{\partial h}{\partial y} &= 0, \\
\end{align*}
\]

(7)

where for potential circuits the condition of irrotational motion is valid

\[ \frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}; \quad u_x = \frac{\partial \varphi}{\partial x}; \quad u_y = \frac{\partial \varphi}{\partial y}. \]

Then the system of equations of circuit motion (7) is simplified to the form:

\[
\begin{align*}
\frac{\partial}{\partial x} \left[ \frac{\partial \varphi}{\partial t} + \frac{u_x^2 + u_y^2}{2} + gh \right] &= 0; \\
\frac{\partial}{\partial y} \left[ \frac{\partial \varphi}{\partial t} + \frac{u_x^2 + u_y^2}{2} + gh \right] &= 0 = 0, \\
\end{align*}
\]

(8)

From the system of equations (8) we conclude the existence of the integral:
\[
\frac{1}{g} \frac{\partial \phi}{\partial t} + \frac{V^2}{2g} + h = f(t),
\]  

(9)

where \( f(t) \) - the function of time is the same for the entire region of the circuit.

Equation (9) is the Cauchy-Lagrange integral, valid for open non-stationary two-dimensional in terms of potential circuits [7].

In this case, the continuity equation will look like [6]:

\[
\frac{\partial h}{\partial t} + \frac{\partial (u_x h)}{\partial x} + \frac{\partial (u_y h)}{\partial y} = 0,
\]  

(10)

From (9), (10) in the particular case for the stationary circuit of a water stream, the equations follow:

\[
\frac{V^2}{2g} + h = H_0, \quad \frac{\partial (u_x h)}{\partial x} + \frac{\partial (u_y h)}{\partial y} = 0,
\]  

(11)

widely known in the literature [5-7].

The system of equations:

\[
\begin{align*}
\frac{1}{g} \frac{\partial \phi}{\partial t} + \frac{\partial (u_x h)}{\partial x} + \frac{\partial (u_y h)}{\partial y} + h &= f(t); \\
\frac{\partial h}{\partial t} + \frac{\partial (u_x h)}{\partial x} + \frac{\partial (u_y h)}{\partial y} &= 0
\end{align*}
\]  

(12)

is a closed system of partial differential equations with respect to unknown functions: \( \phi = \phi(t, x, y), h = h(t, x, y) \), where \( x, y \) - are the independent variables (point coordinates); \( t \) - denotes time; \( f(t) \) - is the known function for the entire region of the circuit, determined from the initial and boundary conditions in the region of the circuit «G».

Note that the system (12) is valid in the physical region of the circuit \( \Phi(x, y) \) and in the particular case of a stationary two-dimensional in terms of potential circuit is transformed to a system of equations:

\[
\begin{align*}
\frac{\partial}{\partial x} (u_x h) + \frac{\partial}{\partial y} (u_y h) &= 0; \\
\frac{V^2}{2g} + h &= H_0; \\
V^2 &= u_x^2 + u_y^2; \\
u_x &= \frac{\partial \phi}{\partial x}; \quad u_y = \frac{\partial \phi}{\partial y},
\end{align*}
\]  

(13)
In view of equalities (3), (4), (5), the system (13) is reduced to the form:

\[
\begin{align*}
\frac{V^2}{2g} + h &= H_0; \\
V^2 &= u_x^2 + u_y^2; \\
u_x &= \frac{\partial \phi}{\partial x}; \\
u_y &= \frac{\partial \phi}{\partial y}; \\
\frac{\partial \phi}{\partial x} &= -\frac{h}{h_0} u_y; \\
\frac{\partial \phi}{\partial y} &= \frac{h}{h_0} u_x,
\end{align*}
\]

(14)

where \(\phi(x, y)\) and \(\phi(x, y)\) - accordingly, are the potential function and the stream function in the physical plane of the circuit.

So, the goal set in the work is achieved. In further works, boundary-value problems of the two-dimensional dynamics in terms of open unsteady water circuits will be formulated.

The analytical ways to solve the practical problems in the circuit of non-stationary potential two-dimensional circuits are associated with the possibility of mutual transition between the plane \(\Phi(x, y)\) and plane \(\Gamma(\tau, \theta)\) the hodograph of speed and obtaining a system of equations similar to the system (1), as well as obtaining its analytical solutions similar to those in [3, 4] for a stationary model.

**Summary**

The basic motion equations system of non-stationary and potential, two-dimensional in terms of the open water circuit, which does not contradict the previously known system for the stationary circuit, is obtained. The system of equations (12) can be used by the researchers for two-dimensional water circuits.

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