INTRINSIC DIFFERENTIAL GEOMETRY AND THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS

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Abstract. We give a new proof of the existence of nontrivial quasimeromorphic mappings on a smooth Riemannian manifold, using solely the intrinsic geometry of the manifold.

1. Introduction and Background

The existence of quasimeromorphic (qm) mappings on \( C^\infty \)-Riemannian manifolds without boundary is due to Peltonen \[16\], and represents a generalization of previous results of Tukia \[26\] and Martio-Srebro \[13\]. In \[18\] we have extended Peltonen’s result to include manifolds with boundary and of lower differentiability class. A further generalization to certain classes of orbifolds was given in \[19\], \[21\].

The essential ingredient in all the results above is construction of a thick (or fat) “chessboard triangulation” (i.e. such that two given \( n \)-simplices having a \((n-1)\)-dimensional face in common will have opposite orientations), each of its simplices being then quasiconformally mapped on the unit sphere \( S^n \) using the classical Alexander method \[1\].

Recall that thick triangulations are defined as follows:

**Definition 1.1.** Let \( \tau \subset \mathbb{R}^n \); \( 0 \leq k \leq n \) be a \( k \)-dimensional simplex. The **thickness** \( \varphi \) of \( \tau \) is defined as being:

\[
\varphi = \varphi(\tau) = \inf_{\sigma < \tau} \frac{\text{Vol}_j(\sigma)}{\text{diam}_j \sigma}.
\]

The infimum is taken over all the faces of \( \tau \), \( \sigma < \tau \), and \( \text{Vol}_j(\sigma) \) and \( \text{diam}_j \sigma \) stand for the Euclidian \( j \)-volume and the diameter of \( \sigma \) respectively. (If \( \text{dim} \sigma = 0 \), then \( \text{Vol}_0(\sigma) = 1 \), by convention.)

A simplex \( \tau \) is \( \varphi_0 \)-thick, for some \( \varphi_0 > 0 \), if \( \varphi(\tau) \geq \varphi_0 \). A triangulation (of a submanifold of \( \mathbb{R}^n \)) \( T = \{\sigma_i\}_{i \in I} \) is \( \varphi_0 \)-thick if all its simplices are \( \varphi_0 \)-thick. A triangulation \( T = \{\sigma_i\}_{i \in I} \) is thick if there exists \( \varphi_0 \geq 0 \) such that all its simplices are \( \varphi_0 \)-thick.
The definition above is the one introduced in [9]. For some different, yet equivalent definitions of thickness, see [5], [6], [14], [16], [26].

Note that in our generalizations [18], [19] we have used Peltone’s result, in conjunction with methods of Munkres [14] and Cheeger et al. [9] to obtain the desired thick triangulation.

The method of proof employed in [16] is based upon extrinsic Differential Geometric considerations. More precisely, the idea of the proof is as follows: Start by isometrically embedding the $n$-dimensional, complete, Riemannian manifold $M^n$ into $\mathbb{R}^\nu$, for some large enough $\nu$. (The existence of such an embedding dimension “$\nu$” is guaranteed by Nash’s Embedding Theorem [15].)

Then one constructs an exhaustion of $M^n$ by a sequence of compact manifolds $\{M_i\}_{i \in \mathbb{N}}$.

To control the size of these compact manifolds and that of the “pasting zones” between them (as well as the density of the set of vertices of the triangulation to be constructed), one makes appeal to two geometric features, namely the osculatory (or tubular) radius and the connectivity radius, who are defined as follows:

**Definition 1.2.** (1) $S^{\nu-1}(x, \rho)$ is an osculatory sphere at $x \in M^n$ iff:

- (a) $S^{\nu-1}(x, \rho)$ is tangent at $x$;
- (b) $\mathbb{B}^{\nu}(x, \rho) \cap M^n = \emptyset$.

(2) Let $X \subset M^n$. The number $\omega = \omega_X = \sup \{\rho > 0 \mid S^{\nu-1}(x, \rho) \text{ osculatory at any } x \in X\}$ is called the maximal osculatory (tubular) radius at $X$.

where tangentiality generalizes in a straightforward manner the classical notion defined for surfaces in $\mathbb{R}^3$:

**Definition 1.3.** $S^{\nu-1}(x, r)$ is tangent to $M^n$ at $x \in M^n$ iff there exists $S^{\nu}(x, r) \subset S^{\nu-1}(x, r)$, such that $T_x(S^{\nu}(x, r)) \equiv T_x(M^n)$.

(Here $\mathbb{B}^{\nu}(x, r) = \{y \in \mathbb{R}^\nu \mid d_{eucl} < r\}$; $S^{\nu-1}(x, r) = \partial \mathbb{B}^{\nu}(x, r)$.)

Note that there exists an osculatory sphere at any point of $M^n$ (see, e.g. [16]).

**Definition 1.4.** Let $U \subset M^n, U \neq \emptyset$, be a relatively compact set, and let $T = \bigcup_{x \in U} \sigma(x, \omega_U)$. The number $\kappa_U = \max \{r \mid \sigma^n(x, r) \text{ is connected for all } s \leq \omega_U, x \in T\}$, is called the maximal connectivity radius at $U$.

(Here and $\sigma^n(x, r) = M^n \cap \mathbb{B}^{\nu}(x, r)$.)

These geometric features help us assure that the manifold does not “turn on itself too fast”, piercing a simplex of the (future) triangulation. Moreover, they are interrelated through the following inequality (see [16], Lemma 3.1):

$$\omega_U \leq \frac{\sqrt{3}}{3} \kappa_U.$$

It follows, therefore, that to obtain a vertex set of the required density, one can employ estimates that are solely functions of $\omega_U$. 


Obviously, this construction is basically extrinsic, since it essentially uses the Nash embedding and because the geometric features that control the density of the vertices and the thickness of the simplices are also extrinsic (see above). Moreover, computing the osculatory and connectivity radii is very difficult. Even computing the principal curvatures of the Nash embedding by solving the specific Gauss Equation is highly problematic. (See [21] for a discussion of these aspects and also [22] for their applicative side.)

We have recently given in [23] a simpler proof of the existence of thick triangulations on manifolds (and hence of $qm$-mappings), where by “simpler” we mean that it mainly uses tools of Elementary Differential Topology. However, this proof still requires the embedding of $M^n$ into some $\mathbb{R}^N$, for $N$ large enough, hence it is still partially extrinsic in nature. More important, most of the geometric information regarding the manifold is (evidently) lost or hard to retrieve when using the Differential Topology approach. However, in many cases the manifold comes not merely endowed with a Riemannian metric, but also with some more concrete information on its geometry, usually in the form of bounds for curvatures, volume and diameter. It is therefore useful to have a construction that uses this geometric data. It is the goal of this paper to produce precisely such a construction, which we present in the next section. Finally, for the sake of completeness, in the last section we remind the reader how a quasimeromorphic mapping is obtained once a thick triangulation is constructed.

2. The construction

As in Peltonen’s construction, the idea of the proof is to use the basic fact that $M^n$ is $\sigma$-compact, i.e. it admits an exhaustion by compact submanifolds $\{M_i\}_i$ (see, e.g. [21]). This is a standard fact for metrizable manifolds. However, it is conceivable that the “cutting surfaces” $N_{ij}, \bigcup_{j=1}^k N_{ij} = \partial M_j$, are merely $C^0$, so even the existence of a triangulation for these hypersurfaces is not always assured, hence a fortiori that of smooth triangulations. (See. e.g. [25] for a brief review of the results regarding the existence of triangulations).

To show that one can obtain (by “cutting along”) smooth hypersurfaces, we briefly review the main idea of the proof of the $\sigma$-compactness of $M^n$ (for the full details, see, for example [24]): Starting from an arbitrary base point $x_0 \in M^n$, one considers the interval $I = I(x_0) = \{r > 0 \mid \beta^n(x_0, r) \text{ is compact}\}; \beta^n(x, r) = \exp_x(\mathbb{B}^n(0, r))$, where $\exp_x$ denotes the exponential map: $\exp_x : T_x(M^n) \to M^n$, and where $\mathbb{B}^n(0, r) \subset T_x(M^n), \mathbb{B}^n(0, r) = \{y \in \mathbb{R}^n \mid d_{eucl}(y, 0) < r\}$. If $I = \mathbb{R}$, then $M^n = \bigcup_{i=0}^\infty \beta^n(x, i)$, hence $\sigma$-compact. If $I \neq \mathbb{R}$, one constructs the compacts sets $M_i, M_0 = \{x_0\}, M_{i+1} = \bigcup_{y \in M_i} \beta^n(y, r(y))$, where $r(y) = \frac{1}{\gamma} \sup\{r \in I(y)\}$. Then it can be shown that $M^n = \bigcup_{i \geq 0} M_i$, i.e. $M^n$ is $\sigma$-compact.

The smoothness of the surfaces $N_{ij}$ now follows from Wolter’s result [28] regarding the 2-differentiability of the cut locus of the exponential map.
We shall construct thick triangulations of $M_i$ and $N_{ij}$ of thickness $\varphi_1 = \varphi_1(n)$ and $\varphi_2 = \varphi_2(n - 1)$, respectively. We can then apply repeatedly the “mashing” technique developed in [19], for collars of $N_{ij}$ in $M_i$ and $M_{i+1}$, $j \geq 0$, rendering a triangulation of $M^n$, of uniform thickness $\varphi = \varphi(n)$ (see [19], [9]).

Up to this point, our construction is practically identical to that we used in [23]. However, to produce the fat triangulations of $M_i$ and $N_{ij}$, we shall employ, as stated before, methods of Intrinsic Differential Geometry, instead of (rather than) the ones of Differential Topology we applied in [23].

We start by noting that the manifolds $M_i$, and $N_{ij}$, $i, j \in \mathbb{N}$ are compact, hence the have bounded sectional curvatures (see, e.g. [4]) and diameters. Let $k_i, k_{ij}$ and $K_i, K_{ij}$ denote the lower bound, respective the upper bound, for the sectional curvatures, and let $D_i, D_{ij}$ denote the upper bound of the diameter of $M_i$ and $N_{ij}$, respectively.

Therefore, for each of these manifolds, we can make avail of a triangulation method that, according to [4], was developed, yet not published, by Karcher, but which, to the best of our knowledge, appeared for the first time in [11]. (The same method was applied by Weinstein [27], to obtain a similar result in even dimension.)

The idea is to use so called efficient packings:

Definition 2.1. Let $p_1, \ldots, p_{n_0}$ be points $\in M^n$, satisfying the following conditions:

1. The set $\{p_1, \ldots, p_{n_0}\}$ is an $\varepsilon$-net on $M^n$, i.e. the balls $\beta^n(p_k, \varepsilon)$, $k = 1, \ldots, n_0$ cover $M^n$;
2. The balls (in the intrinsic metric of $M^n$) $\beta^n(p_k, \varepsilon/2)$ are pairwise disjoint.

Then the set $\{p_1, \ldots, p_{n_0}\}$ is called a minimal $\varepsilon$-net and the packing with the balls $\beta^n(p_k, \varepsilon/2)$, $k = 1, \ldots, n_0$, is called an efficient packing. The set $\{(k, l) \vert k = 1, \ldots, n_0 \text{ and } \beta^n(p_k, \varepsilon) \cap \beta^n(p_l, \varepsilon) \neq \emptyset\}$ is called the intersection pattern of the minimal $\varepsilon$-net (of the efficient packing).

Efficient packings have the following important properties, which we list below (for proofs see [11]):

Lemma 2.2. There exists $n_1 = n_1(n, k_i, D_i)$, such that if $\{p_1, \ldots, p_{n_0}\}$ is an $\varepsilon$-net on $M^n$, then $n_0 \leq n_1$.

Lemma 2.3. There exists $n_2 = n_2(n, k_i, D_i)$, such that for any $x \in M^n$, $\vert\{k \vert k = 1, \ldots, n_0 \text{ and } \beta^n(x, \varepsilon) \cap \beta^n(p_k, \varepsilon) \neq \emptyset\}\vert \leq n_2$, for any minimal $\varepsilon$-net $\{p_1, \ldots, p_{n_0}\}$.

Lemma 2.4. Let $M^n$, $\mathfrak{M}^n$, be manifolds having the same bounds $k_i$ and $D_i$ (see above) and let $\{p_1, \ldots, p_{n_0}\}$ and $\{q_1, \ldots, q_{n_0}\}$ be minimal $\varepsilon$-nets with the same intersection pattern, on $M^n$, $\mathfrak{M}^n$, respectively. Then there exists a constant $n_3 = n_3(n, k_i, D_i, K_i)$, such that if $d(q_i, q_j) < K_i \cdot \varepsilon$, then $d(q_i, q_j) < n_3 \cdot \varepsilon$. 
Such an efficient packing is always possible on a closed, connected Riemannian manifold and, by using the properties above, one can construct a simplicial complex having as vertices the centers of the balls $\beta^n(p_k, \varepsilon)$. (Edges are connecting the centers of adjacent balls; further edges being added to ensure the cell complex obtained is triangulated to obtain a simplicial complex.)

**Remark 2.5.** Let $M^n, \mathcal{W}^n$, be manifolds having the same bounds $k, D$ and $v$, where $v$ denotes the lower bound for volume. There exists an $\varepsilon = \varepsilon(k, D, v) > 0$, such that any two minimal $\varepsilon$-nets on $M^n, \mathcal{W}^n$ with the same intersection pattern, are homeomorphic. Moreover, given $k, D$ and $v$ as above, the number of such homotopy types is finite (see [11]).

One can ensure that the triangulation will be convex and that its simplices are convex, by choosing $\varepsilon = \text{ConvRad}(M^n)$, where the *convexity radius* ConvRad($M^n$) is defined as follows:

**Definition 2.6.** Let $M^n$ be a Riemannian manifold. The **convexity radius** of $M^n$ is defined as $\inf \{ r > 0 | \beta^n(x, r) \text{ is convex}, \text{ for all } x \in M^n \}$. This follows from the fact that $\beta^n(x, \text{ConvRad}(M^n)) \subset \beta^n(x, \text{InjRad}(M^n))$, (since ConvRad($M^n$) = $\frac{1}{2}$InjRad($M^n$) – see, e.g. [4]). Here InjRad($M^n$) denotes the *injectivity radius*:

**Definition 2.7.** Let $M^n$ be a Riemannian manifold. The **injectivity radius** of $M^n$ is defined as: $\text{InjRad}(M^n) = \inf x \in M^n | \text{Inj}(x) = \sup \{ r | \exp_x|_{B^n(x, r)} \text{ is a diffeomorphism} \}$. Note that by a classical result of Cheeger [7], there is a universal positive lower bound for InjRad($M_i$) in terms of $k_i, D_i$ and $v_i$, where $v_i$ is the lower bound for the volume of $M_i$. It is precisely this result (and similar ones – see also the discussion below) that make the triangulation exposed above a simple and practical one, at least in many cases.

The same method of triangulation can be applied to the manifolds $N_{ij}$. The triangulations thus obtained can be “thickened” by applying the techniques of [9] or [20]. Then one can “mash” and “thicken” the triangulations of $M_i$ and $N_{ij}$, to obtain using the method of [18]. Applying this process inductively for all the elements of the exhaustion $\{M_i\}_{i \in \mathbb{N}}$, one obtains a uniformly thick triangulation of $M^n$, thus concluding the announced alternative proof of Peltonen’s result:

**THEOREM 2.8.** Let $M^n, n \geq 2$, be complete, connected, $C^\infty$ Riemannian manifold. Then $M^n$ admits a (uniformly) thick triangulation.

**Remark 2.9.** As already noted in the introduction, the result above can be extended to manifolds with boundary, of low differentiability class (see [18]) and to certain types of orbifolds (see [19], [21]).

**Remark 2.10.** Instead of using the method of estimating convexity radii for the manifolds int$M_i$ and their boundary components $N_{ij}$, we could have used the estimates for the convexity radii of $N_{ij}$ using the methods of [3]. However,
this would have provided more difficult. In addition, the hierarchical approach adopted here is the classical one of [5].

Remark 2.11. The same basic method of triangulation as employed herein may be applied by considering bounds on the Ricci curvature of the manifolds $M_i$ and $N_{ij}$, $i, j \in \mathbb{N}$. This relaxation allows us to apply this technique to manifolds for which less geometric control is possible.

We conclude this section by reviewing the advantages and disadvantages of the triangulation method introduced above, as compared to that of [16]. As we have already noted above, an immediate advantage stems from the fact that, by using solely the intrinsic geometry of the manifold, this approach does not necessitate the cumbersome Nash embedding technique, that results in a quasi-impossible computation of the curvatures required in Peltone’s construction. But, perhaps, the main advantage resides in the fact that universal lower bounds can be computed for the injectivity radius (hence also for the convexity radius). Besides Cheeger’s classical result mentioned above, many other such theorems for compact manifolds exist – see [4] for a plethora of relevant theorems. In addition, as we mentioned in Remark 2.10, a lower bound for injectivity radius can be determined also for manifolds with boundary. Moreover, as noted in Remark 2.11 such bounds can be attained in terms of the Ricci curvature, further extending the class of the manifolds for which our method may be easily applied. It should also be noted that, by a theorem of Maeda [12], for certain types of noncompact manifolds a universal lower bound also exist, more precisely for (noncompact) manifolds with sectional curvatures $K$ satisfying $0 < K \leq K_0$, the following inequality holds: $\text{InjRad}(M_i) \geq \pi / \sqrt{K_0}$. Moreover, if $n = 2$ and if $M^n$ is homeomorphic to $\mathbb{R}^2$, then the same lower bound is achieved even under the weaker assumption that $0 \leq K \leq K_0$.

The main disadvantage, as compared to [16], of the approach adopted herein, resides in the lack of control of the curvatures of the “cutting” surfaces $N_{ij}$. In consequence, possible drastic changes in sectional curvatures may occur, thence in injectivity radii and, implicitly, in the size of the simplices, when passing from $M_i$ to $N_{ij}$. (As a typical case for this kind of behavior, consider a “crumpled” closed ball in $\mathbb{R}^3$. Then its interior has the trivial Euclidean geometry of the ambient space, whence sectional curvatures $\equiv 0$, while the sectional (i.e. Gaussian) curvature of the boundary may attain arbitrarily large values of $|K|$.)

However, it is possible to smoothen the Riemannian metric of $M^n$, to obtain a metric having a sectional curvatures bound, while remaining arbitrarily close to the original metric (see, e.g. [8], [29]). (We should note here, in conjunction with Remark 2.11 that results regarding the smoothing of sectional curvature, under weaker Ricci curvatures bounds, also exist (see, e.g. [10], [2], [17]).
3. THE EXISTENCE OF QUASIMEROMORPHIC MAPPINGS

We begin this section by reminding the reader the definition of quasimeromorphic mappings:

**Definition 3.1.** Let $M^n, N^n$ be oriented, Riemannian $n$-manifolds.

1. $f : M^n \to N^n$ is called quasiregular (qr) iff
   a. $f$ is locally Lipschitz (and thus differentiable a.e.);
   b. $0 < |f'(x)|^n \leq K J_f(x)$, for any $x \in M^n$;
   where $f'(x)$ denotes the formal derivative of $f$ at $x$, $|f'(x)| = \sup_{|h|=1} |f'(x)h|$, and where $J_f(x) = \det f'(x)$;

2. quasimeromorphic (qm) iff $N^n = \mathbb{S}^n$, where $\mathbb{S}^n$ is usually identified with $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ endowed with the spherical metric.

The smallest number $K$ that satisfies condition (b) above is called the outer dilatation of $f$.

Before proceeding further, we need the following technical lemma (for its proof, see [13], [16]):

**Lemma 3.2.** Let $T$ be a fat triangulation of $M^n \subset \mathbb{R}^N$, and let $\tau, \sigma \in T$, $\tau = (p_0, \ldots, p_n)$, $\sigma = (q_0, \ldots, q_n)$; and denote $|\tau| = \tau \cup \text{int } \tau$. Then there exists an orientation-preserving homeomorphism $h = h_\tau : |\tau| \to \hat{\mathbb{R}}^n$ such that:

1. $h(|\tau|) = |\sigma|$, if $\det(p_0, \ldots, p_n) > 0$ and $h(|\tau|) = \hat{\mathbb{R}}^n \setminus |\sigma|$, if $\det(p_0, \ldots, p_n) < 0$.
2. $h(p_i) = q_i$, $i = 0, \ldots, n$.
3. $h|_{|\tau|}$ is a PL homeomorphism.
4. $h|_{\text{int } |\sigma|}$ is quasiconformal.

We can now prove the existence of qm-mappings on open Riemannian manifolds:

**Theorem 3.3.** Let $M^n, n \geq 2$, be a connected, complete, oriented $C^\infty$ Riemannian manifold. Then there exists a non-constant quasimeromorphic mapping $f : M^n \to \hat{\mathbb{R}}^n$.

**Proof** Let $T$ be the thick triangulation provided by Theorem 2.8. Furthermore, by performing a barycentric type subdivision before starting the fattening process of the triangulation given by Theorem 2.8 ensure that all the simplices of the triangulation satisfy the condition that every $(n-2)$-face is bi incident to an even number of $n$-simplices. Let $f : M^n \to \hat{\mathbb{R}}^n$ be defined by: $f|_{|\sigma|} = h_\sigma$, where $h$ is a homeomorphism constructed in the lemma above. Then $f$ is a local homeomorphism on the $(n-1)$-skeleton of $T$ too, while its branching set $B_f$ is the $(n-2)$-skeleton of $T$. By its construction $f$ is quasiregular. Moreover, given the uniform fatness of the triangulation $T$, the
dilatation of $f$ depends only on the dimension $n$.

\[\square\]

\textbf{Remark 3.4.} Again, this result may be extended to include manifolds with boundary, of low differentiability class (see \cite{18}) and to certain types of orbifolds (see \cite{19}, \cite{21}).

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\textbf{References}

[1] J. W. Alexander, \textit{Note on Riemann spaces}. Bull. Amer. Math. Soc. \textbf{26} (1920) 370-372.
[2] M. Anderson, \textit{Convergence and rigidity of manifolds under Ricci curvature bounds}. Invent. Math. \textbf{102} (1992), 429-445.
[3] S. B. Alexander, I. D. Berg and R. L. Bishop, \textit{Geometric curvature bounds in Riemannian manifolds with boundary}. Trans. Amer. Math. Soc. \textbf{339} (1993), 2, 703-716.
[4] M. Berger, \textit{A Panoramic View of Riemannian Geometry}. Springer-Verlag, Berlin, 2003.
[5] S. S. Cairns, \textit{On the triangulation of regular loci}. Ann. of Math. \textbf{35} (1934), 579-587.
[6] S. S. Cairns, \textit{Polyhedral approximation to regular loci}. Ann. of Math. \textbf{37} (1936), 409-419.
[7] J. Cheeger, \textit{Finiteness theorems for Riemannian manifolds}. Amer. J. Math. \textbf{92} (1970), 61-74.
[8] J. Cheeger and M. Gromov, \textit{Chopping Riemannian manifolds}. J. Differential Geom. \textbf{52} (1991), 85-94.
[9] J. Cheeger, W. M"{u}ller, and R. Schrader, \textit{On the Curvature of Piecewise Flat Spaces}. Comm. Math. Phys. \textbf{92} (1984), 405-454.
[10] X. Dai, G. Wei and R. Ye, \textit{Smoothing Riemannian metrics with Ricci curvature bounds}. Manu. Math. \textbf{90} (1996), 49-61.
[11] K. Grove and P. Petersen, \textit{Bounding homotopy types by geometry}. Ann. of Math. \textbf{128} (1988), 195-206.
[12] M. Maeda, \textit{On the Injectivity Radius of Noncompact Riemannian Manifolds}. Proc. Japan Acad. \textbf{50} (1974), 148-151.
[13] O. Martio and U. Srebro, \textit{On the existence of automorphic quasimeromorphic mappings in $R^n$}. Ann. Acad. Sci. Fenn., Series I Math. \textbf{3} (1977), 123-130.
[14] J. R. Munkres, \textit{Elementary Differential Topology}. (rev. ed.) Princeton University Press, Princeton, N.J., 1966.
[15] J. Nash, \textit{The embedding problem for Riemannian manifolds}. Ann. of Math. \textbf{63} (1956), 20-63.
[16] K. Peltonen, \textit{On the existence of quasiregular mappings}. Ann. Acad. Sci. Fenn., Series I Math., Dissertationes, 1992.
[17] P. Petersen, G. Wei and R. Ye, \textit{Controlled geometry via smoothing}. Comment. Math. Helv. \textbf{74} (1999) 345-363.
[18] E. Saucan, \textit{Note on a theorem of Munkres}. Mediterr. j. math., \textbf{2} (2005), 2, 215 - 229.
[19] E. Saucan, \textit{The Existence of Quasimeromorphic Mappings}. Ann. Acad. Sci. Fenn., Series A I Math., \textbf{31} (2006), 131-142.
[20] E. Saucan, \textit{The Existence of Quasimeromorphic Mappings in Dimension 3}. Conform. Geom. Dyn., \textbf{10} (2006), 21-40.
[21] E. Saucan, \textit{Remarks on the Existence of Quasimeromorphic Mappings}. Contemporary Math. \textbf{455} (2008), 325-331.
[22] E. Saucan, E. Appleboim and Y. Y. Zeevi, Sampling and Reconstruction of Surfaces and Higher Dimensional Manifolds. J. of Math. Im. and Vision. 30 (2008), 1, 105-123.
[23] E. Saucan and M. Katchalski, The existence of thick triangulations – an “elementary” proof. preprint (arXiv:0812.0456v1 [math.GT]), 2008.
[24] M. Spivak, A comprehensive Introduction to Differential Geometry, volume I. Publish or Perish, Boston, MA, 1970.
[25] W. Thurston, Three-Dimensional Geometry and Topology, Vol.1. (Edited by S. Levy). Princeton University Press, Princeton, N.J. 1997.
[26] P. Tukia, Automorphic Quasimeromorphic Mappings for Torsionless Hyperbolic Groups. Ann. Acad. Sci. Fenn. 10 (1985), 545-560.
[27] A. Weinstein, A. On the homotopy type of positively-pinched manifolds, Archiv. der Math. 18 (1967), 523-524.
[28] F.-E. Wolter, Cut Loci in Bordered and Unbordered Riemannian Manifolds. Ph.D. Thesis, 1985.
[29] D. Yang, Convergence of Riemannian manifolds with integral bounds on curvature I. Ann. Scient. Ec. Norm. Sup. 25 (1992), 77-105.

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