A TALE OF TWO SURFACES

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ABSTRACT. We point out a link between two surfaces which have appeared recently in the literature: the surface of cuboids and the Schoen surface. Both give rise to a surface with $q = 4$, whose canonical map is 2-to-1 onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

1. INTRODUCTION

The aim of this note is to point out a link between two surfaces which have appeared recently in the literature: the surface of cuboids (ST, vL) and the surface (actually a family of surfaces) discovered by Schoen [S]. We will show that both surfaces give rise to a surface $X$ with $q = 4$, whose canonical map is 2-to-1 onto a complete intersection of 4 quadrics $\Sigma \subset \mathbb{P}^6$ with 48 nodes. In the first case ($\S 2$) $X$ is a quotient $(C \times C')/\mathbb{Z}/2$, where $C$ and $C'$ are genus 5 curves with a free action of $(\mathbb{Z}/2)^2$. In the second case ($\S 3$), $X$ is a double étale cover of the Schoen surface.

When the canonical map of a surface $X$ of general type has degree $>1$ onto a surface, that surface either has $p_g = 0$ or is itself canonically embedded ([B1], Th. 3.1). Our surfaces $X$ provide one more example of the latter case, which is rather exceptional (see [CPT] for a list of the examples known so far).

2. THE SURFACE OF CUBOIDS AND ITS DEFORMATIONS

In $\mathbb{P}^4$, with coordinates $(x, y; u, v, w)$, we consider the curve $C$ given by

$$u^2 = a(x, y), \quad v^2 = b(x, y), \quad w^2 = c(x, y)$$

where $a, b, c$ are quadratic forms in $x, y$. We assume that the zeros $\{p'_a, p'_b, p'_c, p''_a, p''_b, p''_c\}$ of $a, b, c$ form a set $Z \subset \mathbb{P}^1$ of 6 distinct points. Then $C$ is a smooth curve of genus 5, canonically embedded. It is preserved by the group $\Gamma_+ \cong (\mathbb{Z}/2)^3$ which acts on $\mathbb{P}^4$ by changing the signs of $u, v, w$. Let $\Gamma \subset \Gamma_+$ be the subgroup (isomorphic to $(\mathbb{Z}/2)^2$) which changes an even number of signs. It acts freely on $C$, so the quotient curve $B := C/\Gamma$ has genus 2. The subring of $\Gamma$-invariant elements in $H^0(C, K_C)$ is generated by $x, y$ and $z := uvw$, with the relation $z^2 = abc$; thus $B$ is the double cover of $\mathbb{P}^1$ branched along $Z$.

Let $JB_2$ be the group of 2-torsion line bundles on $B$ (isomorphic to $(\mathbb{Z}/2)^4$). The $\Gamma$-covering $\pi: C \to B$ corresponds to a subgroup $\cong (\mathbb{Z}/2)^2$ of $JB_2$, namely the kernel of $\pi^*: JB \to JC$. Since the divisor $\pi^*(p'_a + p''_a)$ is cut out on $C$ by the canonical divisor $u = 0$, we have $\pi^*(p'_a - p''_a) \sim 0$, and similarly for $b$ and $c$; thus $\ker \pi^* = \{0, p'_a - p''_a, p'_b - p''_b, p'_c - p''_c\}$. This is a Lagrangian subgroup of $JB_2$ for the Weil pairing [M2]; conversely, any Lagrangian subgroup of $JB_2$ is of that form. Thus the
curves $C$ we are considering are exactly the $(\mathbb{Z}/2)^2$-étale covers of a curve $B$ of genus 2 associated to a Lagrangian subgroup of $JB_2$. In particular they form a 3-dimensional family.

The group $\Gamma+/\Gamma \cong \mathbb{Z}/2$ acts on $B = C/\Gamma$ through the hyperelliptic involution, so $C/\Gamma+$ is isomorphic to $\mathbb{P}^1$.

**Proposition 1.** Let $C, C'$ be two genus 5 curves of type (1), and let $X$ be the quotient of $C \times C'$ by the diagonal action of $\Gamma \cong (\mathbb{Z}/2)^2$.

1) $X$ is a minimal surface of general type with $q = 4$, $p_g = 7$, $K^2 = 32$.

2) The involution $i_X$ of $X$ defined by the action of $\Gamma+/\Gamma \cong \mathbb{Z}/2$ has 48 fixed points. The canonical map $\varphi_X : X \to \mathbb{P}^6$ factors through $i_X$, and induces an isomorphism of $X/i_X$ onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

**Proof:** The computation of the numerical invariants of $X$ is straightforward.

Let us denote by $(x', y'; u', v', w')$ the coordinates on $C'$, and by $a', b', c'$ the corresponding quadratic forms. A basis of the space $H^0(X, K_X) = (H^0(C, K_C) \otimes H^0(C', K_{C'}))^\Gamma$ is given by the elements

$$X = x \otimes x', \quad Y = x \otimes y', \quad Z = y \otimes x', \quad T = y \otimes y', \quad U = u \otimes u', \quad V = v \otimes v', \quad W = w \otimes w'.$$

They satisfy the relations

$$XT - YZ = 0, \quad U^2 = A(X, Y, Z, T), \quad V^2 = B(X, Y, Z, T), \quad W^2 = C(X, Y, Z, T),$$

where $A, B, C$ are quadratic forms satisfying $A(X, Y, Z, T) = a(x, y) \otimes a(x', y')$ and the analogous relations for $B$ and $C$.

Let $\Sigma$ be the surface defined by these 4 quadratic forms, and let $\varphi : X \to \Sigma$ be the induced map. We have $\varphi \circ i_X = \varphi$, so $\varphi$ induces a map $\tilde{\varphi}$ from $X/i_X = (C \times C')/\Gamma+$ into $\Sigma$. We consider the commutative diagram

$$
\begin{array}{ccc}
(C \times C')/\Gamma+ & \xrightarrow{\varphi} & \Sigma \\
p \downarrow & & q \\
Q \cong \mathbb{P}^1 \times \mathbb{P}^1 & &
\end{array}
$$

where $p : (C \times C')/\Gamma+ \to (C/\Gamma+) \times (C'/\Gamma+)$ is the quotient map by $\Gamma+$, and $q$ the projection $(X, Y, Z, T; U, V, W) \mapsto (X, Y, Z, T)$. The group $(\mathbb{Z}/2)^3$ acts on $\Sigma$ by changing the signs of $(U, V, W)$; then $\tilde{\varphi}$ is an equivariant map of $(\mathbb{Z}/2)^3$-coverings, hence an isomorphism.

It remains to show that $i_X$ has 48 fixed points. These fixed points are the images (mod. $\Gamma$) of the points of $C \times C'$ fixed by one of the elements of $\Gamma+ \times \Gamma$. Such an element changes the sign of one of the coordinates $\ell = u, v$ or $w$, hence fixes the 64 points $(m, m')$ of $C \times C'$ with $\ell(m) = \ell(m') = 0$. This gives $(3 \times 64)/4 = 48$ fixed points in $X$.

**Example.** Let us take for $C$ and $C'$ the curve

$$u^2 = xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$

The set of zeros of $a, b, c$ is $\{0, \infty, \pm 1, \pm i\}$, so the genus 2 curve $B$ is given by $z^2 = x(x^4 - 1)$. 

We get for $\Sigma$ the following equations:

$$XT = YZ = U^2, \quad V^2 = X^2 - Y^2 - Z^2 + T^2, \quad W^2 = X^2 + Y^2 + Z^2 + T^2;$$

or, after the linear change of variables $X = x + t, T = t - x, Y = y + iz, Z = y - iz, U = u, V = 2v, W = 2w$:

$$t^2 = x^2 + y^2 + z^2, \quad u^2 = y^2 + z^2, \quad v^2 = x^2 + z^2, \quad w^2 = x^2 + y^2.$$

These are the equations of the surface of cuboids, studied in [ST], [vL]. It encodes the relations in a cuboid (= rectangular box) between the sides $x, y, z$, the diagonals of the faces $u, v, w$, and the big diagonal $t$. Thus the surface of cuboids belongs to a 6-dimensional family of intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

**Remark 1.** The surfaces $X$ fit into a tower of $(\mathbb{Z}/2)^2$-étale coverings:

$$C \times C' \longrightarrow X \xrightarrow{r} B \times B'.$$

The abelian covering $r$ is the pull back of a $(\mathbb{Z}/2)^2$-étale covering of $JB \times JB'$:

$$\begin{array}{c}
X \xleftarrow{i_X} A \\
| \quad \downarrow \quad | \\
B \times B' \xleftarrow{i} JB \times JB'.
\end{array}$$

The abelian variety $A$ is the Albanese variety of $X$, and $\alpha$ is the Albanese map. Since the quotient $X/i_X$ is regular, $i_X$ acts as $(-1)$ on the space $H^0(X, \Omega^1_X)$; therefore if we choose $\alpha$ so that it maps a fixed point of $i_X$ to 0, $i_X$ is induced by $(-1_A)$.

### 3. The Schoen surface

The Schoen surfaces $S$ have been defined in [S], and studied in [CMR]. A Schoen surface $S$ is contained in its Albanese variety $A$; it has the following properties:

a) $K^2_S = 16, \quad p_g = 5, \quad q = 4$ (hence $\chi(O_S) = 2$);

b) The canonical map $\varphi_S : S \rightarrow \mathbb{P}^4$ factors through an involution $i_S$ with 40 fixed points, and induces an isomorphism of $S/i_S$ onto the complete intersection of a quadric and a quartic in $\mathbb{P}^4$ with 40 nodes [CMR].

Since $S/i_S$ is a regular surface, $i_S$ acts as $(-1)$ on the space $H^0(S, \Omega^1_S)$. Therefore if we choose the Albanese embedding $S \hookrightarrow A$ so that it maps a fixed point of $i_S$ to 0, $i_S$ is induced by the involution $(-1_A)$.

Let $\ell$ be a line bundle of order 2 on $A$; we denote by $\pi : B \rightarrow A$ the corresponding étale double cover, and put $X := \pi^{-1}(S)$. The restriction of $\ell$ to $S$, which we will still denote by $\ell$, is nontrivial (because the restriction map $\text{Pic}^0(A) \rightarrow \text{Pic}^0(S)$ is an isomorphism), hence $X$ is connected.

**Proposition 2.** $X$ is a minimal surface of general type with $q = 4, \quad p_g = 7, \quad K^2_X = 32$. 
Proof: The formulas $K_X^2 = 32$ and $\chi(O_X) = 4$ are immediate; we must prove $q(X) = 4$, that is, $H^1(S, \ell) = 0$.

By construction the Schoen surfaces fit into a flat family over the unit disk $\Delta$:

$$
\begin{array}{ccc}
S & \xrightarrow{\sim} & A \\
\downarrow & & \downarrow \\
\Delta & & \\
\end{array}
$$

where:

- $A/\Delta$ is a smooth family of abelian varieties;
- at a point $z \neq 0$ of $\Delta$, $S_z$ is a Schoen surface, and $S_z \hookrightarrow A_z$ is the Albanese embedding;
- $A_0 = JC \times JC$ for a genus 2 curve $C$; $S_0$ is the union of $JC$ embedded diagonally in $JC \times JC$, and of $C \times C \subset JC \times JC$ we choose an Abel-Jacobi embedding $C \subset JC$). These two components intersect transversally along the diagonal $C \subset C \times C$.

The line bundle $\ell$ extends to a line bundle $L$ of order 2 on $A$. Let $\ell_0$ be the restriction of $L$ to $S_0$; we want to compute $H^1(S_0, \ell_0)$. We have an exact sequence

$$(2) \quad 0 \to \ell_0 \to \ell_0|JC \oplus \ell_0|C \times C \to \ell_0|C \to 0.$$  

The line bundle $L_0$ on $JC \times JC$ can be written $\alpha \boxtimes \beta$, where $\alpha$ and $\beta$ are 2-torsion line bundles on $JC$, not both trivial; we use the same letters to denote their restriction to $C$. The cohomology exact sequence associated to $L$ gives

$$
H^0(JC, \alpha \boxtimes \beta) \oplus H^0(C \times C, \alpha \boxtimes \beta) \to H^0(C, \alpha \boxtimes \beta) \to H^1(S_0, \ell_0) \xrightarrow{w} H^1(JC, \alpha \boxtimes \beta) \oplus H^1(C \times C, \alpha \boxtimes \beta) \to H^1(C, \alpha \boxtimes \beta).
$$

The restriction map $H^0(JC, \alpha \boxtimes \beta) \to H^0(C, \alpha \boxtimes \beta)$ is surjective, so $w$ is injective. If $\alpha$ and $\beta$ are nontrivial, $H^1(C \times C, \alpha \boxtimes \beta)$ is zero, and the restriction map $H^1(JC, \alpha \boxtimes \beta) \to H^1(C, \alpha \boxtimes \beta)$ is injective, so $H^1(S_0, \ell_0) = 0$. If, say, $\beta$ is trivial, $H^1(JC, \alpha)$ is zero and the map $H^1(C \times C, \text{pr}_1^*\alpha) \to H^1(C, \alpha)$ is bijective, hence $H^1(S_0, \ell_0) = 0$ again.

By semi-continuity this implies $H^1(S_z, L_z) = 0$ for $z$ general in $\Delta$, or equivalently $q(S_z) = q(S_z) = 4$, where $\tilde{S} \to S$ is the étale double covering defined by $L$. But $q$ is a topological invariant, so this holds for all $z \neq 0$ in $\Delta$, hence $H^1(S, \ell) = 0$.

The surface $X$ has a natural action of $(\mathbb{Z}/2)^2$, given by the involution $i_X$ induced by $(-1_B)$ and the involution $\tau$ associated to the double covering $X \to S$, which is induced by a translation of $B$. We want to determine how these involutions act on $H^0(X, K_X)$. The decomposition of $H^0(X, K_X)$ into eigenspaces for $\tau$ is

$$
H^0(X, K_X) \cong H^0(S, K_S) \oplus H^0(S, K_S \otimes \ell).
$$
By property b) above, \( i_S \) acts trivially on \( H^0(S, K_S) \). It remains to study how it acts on \( H^0(S, K_S \otimes \ell) \), or equivalently on \( H^2(S, \ell) \). To define this action we choose the isomorphism \( u : (−1)^{j(\ell)} \varphi H^p(S, \ell) \rightarrow \ell \) over \( A \) such that \( u(0) = 1 \), and we consider the involutions \( H^p(i_S, u) : H^p(S, \ell) \rightarrow H^p(S, i_S^2 \ell) \rightarrow H^p(S, \ell) \).

**Proposition 3.** There exist line bundles \( \ell \) of order 2 on \( A \) for which \( i_S \) acts trivially on \( H^2(S, \ell) \). In that case \( i_X \) has 48 fixed points.

**Proof:** We will denote by \( A_2 \) and \( \hat{A}_2 \) the 2-torsion subgroups of \( A \) and its dual abelian variety \( \hat{A} \), and similarly for \( B \). The fixed point set of \( i_S \) is \( A_2 \cap S \), and that of \( i_X \) is \( B_2 \cap X \).

We apply the holomorphic Lefschetz formula to the automorphism \( i_S \) of \( S \) and the \( i_S \)-linearization \( u : i_S^2 \ell \rightarrow \ell \):

\[
\sum_p (-1)^p \text{Tr} H^p(i_S, u) = \frac{1}{4} \sum_{a \in A_2 \cap S} u(a).
\]

(At a point \( a \) of \( A_2 \), \( u(a) : \ell_a \rightarrow \ell_a \) is the multiplication by a scalar, which we still denote \( u(a) \).)

Let \( a \in A_2 \). By [MII, property iv] p. 304, we have \( u(a) = (−1)^{\langle a, \ell \rangle} \), where \( \langle , \rangle : A_2 \times \hat{A}_2 \rightarrow \mathbb{Z}/2 \) is the canonical pairing. On the other hand, dualizing the exact sequence of \((\mathbb{Z}/2)\)-vector spaces

\[
0 \rightarrow (\mathbb{Z}/2)\ell \rightarrow \hat{A}_2 \rightarrow B_2
\]

and using the canonical pairings we get an exact sequence

\[
B_2 \rightarrow A_2 \xrightarrow{\langle , \ell \rangle} \mathbb{Z}/2 \rightarrow 0.
\]

Thus \( u(a) = 1 \) if \( a \in \pi(B_2) \), and \( u(a) = −1 \) otherwise. For \( i = 0 \) or 1, let \( f_i \) be the number of points \( a \in A_2 \cap S \) with \( \langle a, \ell \rangle = i \). The right hand side of the Lefschetz formula is \( \frac{1}{4}(f_0 − f_1) \).

We have \( H^0(S, \ell) = H^1(S, \ell) = 0 \) (Proposition 2), hence \( \dim H^2(S, \ell) = \chi(O_S) = 2 \). Thus the left hand side is \( \text{Tr} H^2(i_S, u) \in \{2, 0, −2\} \). Since \( f_0 + f_1 = 40 \) this gives \( f_1 \in \{16, 20, 24\} \); the case \( f_0 = 24 \) corresponds to \( H^2(i_S, u) = 1 \). Moreover the number of fixed points of \( i_X \) is \( \#(B_2 \cap X) = 2f_0 \). Thus the Proposition will follow if we can find \( \ell \) in \( \hat{A}_2 \) with \( f_0 = 24 \).

Put \( F := A_2 \cap S \). Consider the homomorphism \( \hat{A}_2 \rightarrow (\mathbb{Z}/2)^F \) given by \( \ell \mapsto (\langle a, \ell \rangle)_{a \in F} \). For \( \ell \neq 0 \), the weight of the element \( j(\ell) \) of \( (\mathbb{Z}/2)^F \) (that is, the number of its nonzero coordinates) is \( f_1 \), which belongs to \( \{16, 20, 24\} \). Therefore \( j \) is injective; its image is a 8-dimensional vector subspace of \( (\mathbb{Z}/2)^F \), that is, a linear code, such that the weight of any nonzero vector belongs to \( \{16, 20, 24\} \). A simple linear algebra lemma ([B2], lemma 1) shows that a code in \((\mathbb{Z}/2)^{40}\) of dimension \( \geq 7 \) contains elements of weight \( < 20 \); thus there exist elements \( \ell \) in \( \hat{A}_2 \) with \( f_1 = 16 \), hence \( f_0 = 24 \).

From now on we choose \( \ell \) so that \( i_S \) acts as trivially on \( H^2(S, \ell) \). Thus \( i_X \) acts trivially on \( H^0(X, K_X) \) and has 48 fixed points.

**Proposition 4.** The canonical map \( \varphi_X : X \rightarrow \mathbb{P}^6 \) factors through \( i_X \), and induces an isomorphism of \( X/i_X \) onto a complete intersection of 4 quadrics in \( \mathbb{P}^6 \) with 48 nodes.
**Proof**: Since \( i_X \) acts as trivially on \( H^0(X, K_X) \), we have a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_X} & \Sigma \\
\downarrow{\pi} & & \downarrow{p} \\
S & \xrightarrow{\varphi_S} & \Xi
\end{array}
\]

where \( \varphi_X \) and \( \varphi_S \) are the canonical maps, \( \Sigma \) and \( \Xi \) their images, \( p \) the projection corresponding to the injection \( H^0(S, K_S) \to H^0(X, K_X) \), \( p_{\Xi} \) its restriction to \( \Sigma \).

The map \( \varphi_S \circ \pi : X \to \Xi \) gives the quotient of \( X \) by the action of \((\mathbb{Z}/2)^2\). Since \( \tau \) acts non-trivially on \( H^0(X, K_X) \), \( \varphi_X \) identifies \( \Sigma \) with the quotient \( X/i_X \). Thus all the maps in the left hand square of the above diagram are double coverings, étale outside finitely many points. In particular, since \( K_X^2 = 32 \), we have \( \deg \Sigma = 16 \).

We choose bases \((x_0, \ldots, x_4)\) and \((u, v)\) of the \((+1)\) and \((-1)\)-eigenspaces in \( H^0(X, K_X) \) with respect to \( \tau \). The elements \( u^2, uv, v^2 \) of \( H^0(X, K_X^{\otimes 2}) \) are invariant under \( \tau \) and \( i_X \), therefore they are pull-back of \( i_S \)-invariant forms in \( H^0(S, K_S^{\otimes 2}) \). Such a form comes from an element of \( H^0(\Xi, \mathcal{O}_\Xi(2)) \), hence from an element of \( H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \). Thus we have

\[ u^2 = a(x) \quad uv = b(x) \quad v^2 = c(x) \]

where \( a, b, c \) are quadratic forms in \( x_0, \ldots, x_4 \). Moreover the irreducible quadric \( Q \) containing \( \Xi \) is defined by a quadratic form \( q(x) \) which vanishes on \( \Sigma \).

Thus \( \Sigma \) is contained in the subvariety \( V \) of \( \mathbb{P}^6 \) defined by these 4 quadratic forms. If \( V \) is a surface, it has degree 16 and therefore is equal to \( \Sigma \). Thus it suffices to prove that the morphism \( p_V : V \to Q \) induced by the projection \( p \) is not surjective.

Assume that \( p_V \) is surjective; it has degree 2, and we have a cartesian diagram

\[
\begin{array}{ccc}
\Sigma & \xrightarrow{p_{\Xi}} & V \\
\downarrow{p_{\Xi}} & & \downarrow{p_V} \\
\Xi & \xrightarrow{p_V} & Q
\end{array}
\]

The variety \( V \) is irreducible: otherwise \( \Sigma \) is contained in one of its component, which maps birationally to \( Q \), and \( p_{\Xi} \) has degree 1, a contradiction. Since \( Q \setminus \text{Sing}(Q) \) is simply connected, \( p_V \) is branched along a surface \( R \subset Q \). Since \( \Xi \) is an ample divisor in \( Q \) (cut out by a quartic equation), it meets \( R \) along a curve, and \( p_{\Xi} \) is branched along that curve, a contradiction.

**Remark 2**. It follows that \( \Xi = p(\Sigma) \) is defined by the equations \( q(x) = b(x)^2 - a(x)c(x) = 0 \). The 40 nodes of \( \Xi \) break into two sets: the 16 points in \( \mathbb{P}^4 \) defined by \( a(x) = b(x) = c(x) = q(x) = 0 \) are the images by \( p_{\Xi} \) of smooth points of \( \Sigma \) fixed by the involution induced by \( \tau \); \( p_{\Xi} \) is étale over the other 24 nodes of \( \Xi \), giving rise to the 48 nodes of \( \Sigma \).

**Remark 3**. The two families of surfaces \( X \) that we have constructed are different; in fact, a surface \( X_1 \) of the first family is not even homeomorphic to a surface \( X_2 \) of the second one. Indeed \( X_1 \) admits an irrational genus 2 pencil \( X \to B \), and this is a topological property [1]. But for a general member
$X_2$ of the second family, the Albanese variety of the corresponding Schoen surface is simple [S], so its double cover $\text{Alb}(X_2)$ is also simple; therefore $X_2$ cannot have an irrational pencil of genus 2.

It follows that the corresponding surfaces $\Sigma$ belong to two different connected components of the moduli space of complete intersections of 4 quadrics in $\mathbb{P}^6$ with an even set of 48 nodes.

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