Orbifolds, Penrose Limits and Supersymmetry Enhancement

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Abstract

We consider supersymmetric PP-wave limits for different $\mathcal{N} = 1$ orbifold geometries of the five sphere $S^5$ and the five dimensional Einstein manifold $T^{1,1}$. As there are several interesting ways to take the Penrose limits, the PP-wave geometry can be either maximal supersymmetric $\mathcal{N} = 4$ or half-maximal supersymmetric $\mathcal{N} = 2$. We discuss in detail the cases $AdS_5 \times S^5/\mathbb{Z}_3$, $AdS_5 \times S^5/(\mathbb{Z}_m \times \mathbb{Z}_n)$ and $AdS_5 \times T^{1,1}/(\mathbb{Z}_m \times \mathbb{Z}_n)$ and we identify the gauge invariant operators which correspond to stringy excitations for the different limits.
1 Introduction

The duality between open strings and closed strings has been explored extensively over the last years. One important example is the AdS/CFT conjecture between the $\mathcal{N} = 4$ field theory and type IIB strings on $\text{AdS}_5 \times S^5$ \cite{1,2,3}. The conjecture has been generalized to orbifolds of $S^5$ \cite{4,5} and to conifolds \cite{6}. The supergravity limit of the string has been mainly considered so far because of the difficulties in quantizing strings in the presence of RR-fluxes. On the other hand, another maximal supersymmetric background, the pp-wave, has been discussed recently in \cite{7} and string theory on the pp-wave is an exactly solvable model, where one can identify all the string oscillators \cite{8}.

The pp-wave solution appear as a Penrose limit of the $\text{AdS}_5 \times S^5$ solution \cite{9} so it can be used to obtain information about the AdS/CFT correspondence. The authors of \cite{9} have extended the AdS/CFT conjecture to the case of strings moving on a pp-wave backgrounds where the corresponding field theory operators are the ones with high R charge and in this case the field theory describe not only the supergravity but also the full closed string theory.

The idea of \cite{1} has been extended in many directions \cite{10,19,15,16,17,20,21,35}. The direction we are pursuing in this work was initiated in a series of papers \cite{15,16,17,20,21,35} and involves geometries more complicated than the $S^5$. Especially interesting are the cases of orbifolds of $S^5$ or conifolds where one can take two kinds of pp-wave limits, one which preserves the supersymmetry and the other one which enlarges the supersymmetry. As discussed in \cite{3}, if one takes the Penrose limit on directions orthogonal to the orbifolding direction, then we expect to get the same amount of supersymmetry, but a Penrose limit along the orbifolding direction will get an increase of supersymmetry. One example of the second type was described in \cite{15,16,17} for the case of D3 branes at a conifold singularity, where the Penrose limit gives a maximal supersymmetric solution. In this case we expect a supersymmetry enhancement in field theory, from $\mathcal{N} = 1$ to $\mathcal{N} = 4$ and the relevant $\mathcal{N} = 1$ multiplets which give rise to an $\mathcal{N} = 4$ multiplet have been identified. In \cite{20,21} a similar discussion has been developed for the supersymmetry enhancement from $\mathcal{N} = 2$ to $\mathcal{N} = 4$ in the case of $S^5/Z_k$.

In the present work we study the supersymmetry enhancement in the Penrose limit for several examples of orbifolds. As only the infinitesimal neighborhood of the null geodesic is probed in the pp-wave limit, the orbifold action disappears unless it is considered locally around the null geodesic. In other words, the orbifolding action also changes in the pp-wave limit. Thus, in general, it is not possible to build duals to string oscillators in the Penrose limit from gauge invariant operators of the original orbifold theory. We have found that, in the Penrose limit, one needs to consider operators from
the covering space of the original space. We also comment on anomalous dimensions and correlation function for the orbifold theories and on the interpretation as a limit of a DLCQ theory with the light-cone momentum $p^+$ fixed.

In section 2 we will describe examples of $\mathcal{N} = 1$ orbifolds of $S^5$. The first model is $S^5/Z_3$ whose Penrose limit was outlined in [16] for which we describe the string/field theory matching. As a second example we consider different boostings for the $S^5/(\mathbb{Z}_k \times \mathbb{Z}_l)$ orbifold which can give an enlargement of supersymmetry from $\mathcal{N} = 1$ to $\mathcal{N} = 2$ or $\mathcal{N} = 4$. In section 3 we consider Penrose limits of $T^{1,1}/(\mathbb{Z}_k \times \mathbb{Z}_l)$ along the fixed circles of the quotienting action.

2 $\mathcal{N} = 1$ orbifolds of $S^5$

2.1 Review of the $AdS_5 \times S^5$ result

We start with a brief review of the result of [9], pointing out the features which we expect to get from the orbifold discussion.

Consider $AdS_5 \times S^5$ where the anti-de-Sitter space $AdS_5$ is represented as a universal covering of a hyperboloid of radius $R$ in the flat $\mathbb{R}^{2,4}$ and a sphere $S^5$ of radius $R$ in the flat space $\mathbb{R}^{0,6}$. One may regard the $AdS_5$ (resp. $S^5$) as a foliation of a time-like direction and a three sphere $\Omega_3$. (resp. a circle parameterized by $\psi$ and a three sphere $\Omega_3'$.) Then the induced metric on $AdS_5 \times S^5$ becomes

$$ds^2 = R^2 \left[ -dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3'^2 \right].$$

(1)

One now considers the pp-limit by boosting along the $\psi$ direction around $\rho = 0$. The metric in this limit can be obtained by taking $R \to \infty$ after introducing coordinates

$$x^+ = \frac{1}{2}(t + \psi), x^- = \frac{R^2}{2}(t - \psi)$$

(2)

and rescaling $\rho = r/R, \theta = y/R$ as follows:

$$ds^2 = -4dx^+dx^- - (r \cdot r + y \cdot y)dx^{+2} + dy^2 + dr^2$$

(3)

where $y$ and $r$ parameterize points on $\mathbb{R}^4$. Only the components of the RR 5-form $F$ with a plus index survive in this limit.

2
The energy is given by \( E = i \partial_t \) and the angular momentum in the direction \( \psi \) is \( J = -i \partial_\psi \) and the latter is seen as a generator that rotates a 2-plane inside the original \( \mathbb{R}^6 \).

In terms of the dual \( \mathcal{N} = 4 \) theory, the energy \( E \) is related to the conformal weight \( \Delta \) and the angular momentum to the R-charge. As discussed in [9], the relation between the oscillations of the string in the pp-wave geometry (3) and the field theory quantities is

\[
(\Delta - J)_n = \sqrt{1 + \frac{4\pi g N n^2}{J^2}}
\]

where \( N \) stands for the rank of the gauge theory and \( g \) is the string coupling constant. The vacuum has \( \Delta - J = 0 \).

In the \( \mathcal{N} = 4 \) field theory, the interpretation of the string vacuum and of the string oscillators is made in terms of the gauge invariant operators. Consider the \( \mathcal{N} = 4 \) multiplet in terms of a triplet of \( \mathcal{N} = 1 \) multiplets, denoted by \( Z, Y^1, Y^2 \), the dimension of each field being 1. The complex field \( Z \) is on the directions whose rotation generator is \( J \), so the value of \( J \) for the field \( Z \) is 1, therefore for the field \( Z \) we have \( \Delta - J = 0 \). The other fields, \( Y^1, Y^2 \) (and their complex conjugates \( \bar{Y}^1, \bar{Y}^2 \)) have \( J = 0 \) and \( \Delta - J = 1 \).

We can proceed to compare the stringy results with the field theory results, the string vacuum is given by \( \text{Tr}[Z^J] \) and the stringy oscillators are given by inserting \( Y^1, Y^2, \bar{Y}^1, \bar{Y}^2 \), i.e. the operators:

\[
\text{Tr}[Z^{J-1}]Y^i, \; \text{Tr}[Z^{J-1}]{\bar{Y}}^i, \; i = 1, 2
\]

We can also have gauge invariant operators \( \text{Tr}[Z^{J-1}]{\bar{Z}}, \; \text{Tr}[Z^{J-2}]Y^1{\bar{Y}}^2 \), etc, but in [9] arguments have been given that such operators will get infinite mass.

### 2.2 String Oscillators in the pp-limit of the \( AdS_5 \times S^5/\mathbb{Z}_3 \)

The geometry \( AdS_5 \times S^5/\mathbb{Z}_3 \) is obtained as a near horizon geometry of \( N \) D3 branes placed at a \( \mathbb{C}^3/\mathbb{Z}_3 \) orbifold. The generator \( g \) of \( \mathbb{Z}_3 \) acts on \( \mathbb{C}^3 \) by

\[
g \cdot (z_1, z_2, z_3) \rightarrow (\omega z_1, \omega z_2, \omega z_3), \quad \omega^3 = 1.
\]

We consider the boosting along the direction of the orbifolding which has been studied in [10]. We need to consider a metric for the 3-fold covering of \( S^5 \). As [13], it is convenient...
to consider $S^5$ as a Hopf fibration over $\mathbf{CP}^2$. The metric could be written as:

$$ds^2 = (3d\psi + A)^2 + d_{\mathbf{CP}^2}^2$$

(7)

where $dA/2$ gives the Kähler class of $\mathbf{CP}^2$. As $\psi$ ranges from 0 to $2\pi$, we get a 3-fold of $S^5$. More generally, we may take an orbifold theory on $\mathbf{C}^3/\mathbf{Z}_m$ where the generator $g$ of $\mathbf{Z}_m$ acts on $\mathbf{C}^3$ by

$$g \cdot (z_1, z_2, z_3) \rightarrow (\omega^{a_1} z_1, \omega^{a_2} z_2, \omega^{a_3} z_3), \omega^m = 1, \quad a_1 + a_2 + a_3 = 0 \pmod{m}, a_i > 0.$$ 

(8)

Then the $S^5$ is a Hopf fibration over a weighted projective space $\mathbf{CP}(a_1, a_2, a_3)$. As long as the null geodesic does not lie over the singular locus of the weighted projective space $\mathbf{CP}(a_1, a_2, a_3)$, there will no change in the argument.

We now choose the null coordinates as:

$$x^+ = \frac{1}{2} (t + \frac{1}{3} \psi)$$
$$x^- = \frac{R^2}{2} (t - \frac{1}{3} \psi)$$

(9)

In the limit $R \rightarrow \infty$ and after rescaling the transversal direction $\mathbf{CP}^2$, we obtain the maximally supersymmetric pp-wave metric (3) as in (10). The light-cone momenta can be written in terms of conformal weight $\Delta$ and the angular momentum $J = -i\partial_\psi$:

$$2p^- = i\partial_{x^+} = i(\partial_t + 3\partial_\psi) = \Delta - 3J$$
$$2R^2 p^+ = i\partial_{x^-} = i(\partial_t - 3\partial_\psi) = \Delta + 3J$$

(10)

Before we describe the duality string/field theory in the Penrose limit, we recall the results of (4, 5) concerning the field theory on D3 branes at $\mathbf{C}^3/\mathbf{Z}_3$ singularities. By starting with $3N$ D3 branes in the covering space of $\mathbf{C}^3/\mathbf{Z}_3$ orbifold, the $SU(3N)$ gauge group is broken to $SU(N)^3$ by orbifold action on the Chan-Paton factors and there are three fields in the bifundamental representation for each pair of gauge groups, denoted by $X_i, Y_i, Z_i, i = 1, 2, 3$ (they come as $3N \times N$ blocks inside each $3N \times 3N$ matrices $X, Y, Z$ describing the transversal motion of the D-branes). The surviving KK modes are of the form (11):

$$\text{Tr}(X_i^{m_1} Y_i^{m_2} Z_{i+2}^{m_3}), \quad m_1 + m_2 + m_3 = 0 \pmod{3}, \quad i = 1, 2, 3 \pmod{3}$$

(11)

The quiver gauge theories have a quantum $\mathbf{Z}_3$ symmetry and the surviving KK modes have to be invariant under it. In the Penrose limit, the effect of the $\mathbf{Z}_3$ action on the transversal direction to the boosting direction disappears as the string probes an infinitesimally small neighborhood of the boosting circle parameterized by $\psi$. In the quantum vacua, the $\mathbf{Z}_3$ action remains along the boosting direction as we see in (11).
In the orbifold theory \( S^5/\mathbb{Z}_3 \), the global symmetry \( SO(6) \approx SU(4) \) is broken up into \( U(1) \times \mathbb{Z}_3 \). Before the limit, the Hopf fibration is non-trivial, so even if the \( \mathbb{Z}_3 \) acts only along the Hopf fiber, this does not imply the breaking of global \( SO(6) \) isometry. In the pp-limit, the fibration becomes trivial and it breaks the global symmetry \( SO(6) \) to \( SO(4) \times SO(2) \), with the \( SO(2) \) being in the boosting direction and \( SO(4) \) in the transverse directions.

To describe the string/field theory duality, we denote by \( Z \) the boosted direction and by \( X, Y \) the transverse direction where the orbifold does not act so \( X, Y \) do not enter in a gauge invariant form. The action of \( \mathbb{Z}_3 \) orbifold is only on the Hopf fiber parameterized by \( Z \). We identify the scalar field along the Hopf fiber as \( Z = Z_1 Z_2 Z_3 \) where \( Z_i \) are the above fields in the bifundamental representation of \( SU(N)_i \times SU(N)_{i+1} \), \( i = 1, 2, 3 \). The field \( Z \) is in the adjoint representation of \( SU(N) \) and has angular momentum on the \( U(1) \) direction equal to 3. The fields \( X, Y \) are also in the adjoint representation of the same \( SU(N) \) and together with \( Z \) they form an \( \mathcal{N} = 4 \) multiplet.

The vacuum of the string in the presence of the \( \mathbb{Z}_3 \) is

\[ \frac{1}{\sqrt{3JN^{3J/2}}} \text{Tr}[Z^J] \]  

(12)

The first excited states are obtained by insertions of \( X, Y, \bar{X}, \bar{Y} \), for string in the pp-wave background these states being obtained by acting with a single oscillator on the ground states. Because there are eight bosonic zero modes oscillators, we expect to find eight bosonic states with \( \Delta - 3J = 1 \). They are

\[ \text{Tr}[Z^J X], \text{Tr}[Z^J \bar{X}] \text{ or } \text{Tr}[Z^J Y], \text{Tr}[Z^J \bar{Y}] \]  

(13)

and the ones with the covariant derivative

\[ \text{Tr}[Z^J D_\mu Z] \]  

(14)

The non-supergravity modes are obtained by acting with creation operators which imply the introduction of a position dependent phase, besides the above insertions [9].

Because we discuss the \( \mathbb{Z}_3 \) orbifold, we do not have a DLCQ limit as in [20, 21], which holds only for \( \mathbb{Z}_n \) with large \( n \). Therefore, if we make the identification of the radius of the \( x^- \) direction as in [20, 21]:

\[ \frac{\pi R^2}{n} = 2\pi R_- \]  

(15)

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1 This set of \( X, Y, Z \) is different from the original complex coordinates of \( \mathbb{C}^3 \) in [11]. But a change of complex structures we may identify them as complex coordinates of the infinitesimal neighborhood of the boosting circle.
where $R^2$ is approximatively $N$ (the rank of the gauge group), we see that when $n$ is small, the radius $R_-$ of the $x^-$ direction is infinite, so we are not allowed to use a Discrete Light Cone Quantization. There is no winding mode discussion for the $\mathbb{Z}_3$ orbifold and the insertions corresponding to the non-supergravity modes are identical to the ones of [9].

An interesting case of supersymmetry enhancement was treated in [20, 21] for $\mathcal{N} = 2$ orbifolds $S^5/\mathbb{Z}_n$. By boosting along the non-fixed directions of the orbifold, one gets a maximal $\mathcal{N} = 4$ theory. One interesting related development would be to consider the supersymmetry enhancement when D3 branes probe backgrounds of D7/O7 planes [31, 32]. The Penrose limit in the fixed direction (orthogonal to O7) was considered in [35] but the discussion of Penrose limits on the non-fixed directions still remains to be discussed. One step further in this direction would be to consider the Penrose limit for the case when D3 branes probe geometries with orthogonal D7 branes as in [32, 33, 34].

### 2.3 $\mathbb{Z}_m \times \mathbb{Z}_n$ Orbifolds of $S^5$

In this subsection we consider the geometry $AdS_5 \times S^5/(\mathbb{Z}_m \times \mathbb{Z}_n)$ which is the near-horizon limit of the D3 branes placed at the tip of $C^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$ [4]. The coordinates of $C^3$ are $z_1, z_2, z_3$ and the generators $g_m, g_n$ of $\mathbb{Z}_m, \mathbb{Z}_n$ act on $(z_1, z_2, z_3)$ as

\begin{align}
g_m & : (z_1, z_2, z_3) \rightarrow (e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2, z_3) \\
g_n & : (z_1, z_2, z_3) \rightarrow (e^{2\pi i/n}z_1, z_2, e^{-2\pi i/n}z_3).
\end{align}

The singular points in the quotient are points left invariant under elements of the discrete group. The complex curve $z_1 = z_2 = 0$, parameterized by $z_3$, is invariant under the $\mathbb{Z}_m$ and becomes a curve of $A_{m-1}$ singularities, the complex curve $z_1 = z_3 = 0$, parameterized by $z_2$, is invariant under the $\mathbb{Z}_n$ and becomes a curve of $A_{n-1}$ singularities and the complex curve $z_2 = z_3 = 0$, parameterized by $z_1$, is invariant under the $\mathbb{Z}_r, r = \gcd(m, n)$ and becomes a curve of $A_{r-1}$ singularities.

The field theory on D3 branes at $C^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$ singularity is $\mathcal{N} = 1$ theory with gauge group $\prod_{i=1}^m \prod_{j=1}^n SU(N)_{(i,j)}$ and chiral bifundamentals [24, 25]. The gauge invariant operators are

\begin{align}
\text{Tr}H_{(i,j)(i+1,j)}D_{(i+1,j)(i,j-1)}V_{(i,j-1)(i,j)}
\end{align}

where $H_{(i,j)(i+1,j)}$ are in the bifundamental representation of $SU(N)_{(i,j)} \times SU(N)_{(i+1,j)}$, $V_{(i,j)(i,j+1)}$ are in the bifundamental representation of $SU(N)_{(i,j)} \times SU(N)_{(i,j+1)}$ and

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2 This model was also discussed in [20]
$D_{(i+1,j+1)(i,j)}$ are in the bifundamental representation of $SU(N)_{(i+1,j+1)} \times SU(N)_{(i,j)}$. If D3 branes move to the points of $A_{m-1}$, (resp. $A_{n-1}$ or $A_{r-1}$) singularities described above, the field theory on the D3 branes becomes $\mathcal{N} = 2$ with gauge group $SU(N)^m$ (resp. $SU(N)^n$ or $SU(N)^r$). Hence there are flat directions in the $\mathcal{N} = 1$ theory which connect it to an $\mathcal{N} = 2$ theory.

In the $S^5/(\mathbb{Z}_m \times \mathbb{Z}_n)$ geometry, there are many interesting directions along which we can consider the boosting and the amount of the supersymmetry enhancement will depend on both the direction and the locality of the trajectories. We now classify the different possibilities:

**Case 1.** Boosting in the direction of the $\mathbb{Z}_m$ orbifolding (the same discussion holds for the direction of the $\mathbb{Z}_n$ or $\mathbb{Z}_r$ orbifolding).

We understand by the direction of $\mathbb{Z}_m$ orbifolding a $U(1)$ direction where $\mathbb{Z}_m$ is embedded in. For this purpose, it is convenient to consider $S^5$ as a foliation of the $S^3$ in $\mathbb{C}^2$ with coordinates $z_1, z_2$ and the $S^1$ in $\mathbb{C}^1$ with coordinates $z_3$. Furthermore, we consider $S^3$ as a Hopf fibration over $\mathbb{C}P^1$ after changing the complex structure $z_2$ to $\bar{z}_2$ and the $\mathbb{Z}_3$ will locally act along the Hopf fiber. From this geometric description of $S^5$, we obtain the metric for the $AdS_5 \times S^5$ as:

\[
\begin{align*}
    dS^2_{AdS} &= R^2 (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2) \\
    ds^2_{S^5} &= R^2 [d\theta^2 + \sin^2 \theta d\Omega^2_{S^1} + \cos^2 \theta \left[(d\tau + (\cos \chi - 1)d\phi)^2 + (d\chi^2 + \sin^2 \chi d\phi^2)\right]]
\end{align*}
\]  

(19)

where $\tau$ is the coordinate for the fiber direction and $d\chi^2 + \sin^2 \chi d\phi^2$ is the metric for the base $\mathbb{C}P^1$ in the Hopf fibration of $S^3$. As in the previous section, we need to consider an $mn$-fold cover of $S^5$. As the string probes only an infinitesimal neighborhood of the boosting direction, the action on the transverse directions to the Hopf fiber is irrelevant. For simplicity we take an $m$-fold covering of the $S^5$ where the $S^3$ part of the metric changes to

\[
(m \ d\tau + (\cos \chi - 1)d\phi)^2 + (d\chi^2 + \sin^2 \chi d\phi^2)\]

(20)

We choose the null coordinates as

\[
x^+ = \frac{1}{2}(t + \frac{\tau}{m}), \quad x^- = \frac{R^2}{2}(t - \frac{\tau}{m})
\]

(21)

and consider a scaling limit $R \to \infty$ around $\theta = \chi = 0$ with

\[
\rho = \frac{r}{R}, \quad \theta = \frac{u}{R}, \quad \chi = \frac{v}{R}
\]

(22)
In this limit, the metric becomes

\[ ds^2 = dr^2 + r^2 d\Omega_3^2 - r^2 dx^+ dx^- - 2 dx^+ dx^- \]
\[ + du^2 + u^2 d\Omega_3^2 - 2 dx^+ dx^- - v^2 dx^+ d\phi - u^2 dx^+ d\phi - v^2 d\phi^2 \]
\[ = -4 dx^+ dx^- - (r^2 + u^2) dx^+ \]
\[ + dr^2 + r^2 d\Omega_3^2 + du^2 + u^2 d\Omega_3^2 + dv^2 + v^2 d\phi^2 - v^2 dx^+ d\phi \]
\[ = -4 dx^+ dx^- - (r^2 + u^2 + v^2) dx^+ \]
\[ + dr^2 + r^2 d\Omega_3^2 + du^2 + u^2 d\Omega_3^2 + dv^2 + v^2 d\phi^2 \]

(23)

where \( \phi' = \phi - 1/2x^+ \).

After changing to the rectangular coordinate system, one may rewrite this as

\[ ds^2 = -4 dx^+ dx^- - (r^2 + u^2 + v^2) dx^+^2 + dr^2 + du^2 + dv^2 \]

(24)

The pp wave has a natural decomposition of the \( R^8 \) transverse space into \( R^4 \times R^2 \times R^2 \) where \( R^4 \) is parameterized by \( r \) and the \( R^2 \times R^2 \) by \( u \) and \( v \), respectively. The covariantly constant flux of the R-R field is on the \((x^+,r)\) and \((x^+,u,v)\). In this geometry, the light cone momenta are:

\[ 2p^--i\partial_+ = i(\partial_t + m\partial_r) = \Delta - mJ \]
\[ 2R^2p^+ = i\partial_- = i(\partial_t - m\partial_r) = \Delta + mJ \]

(25)

The effective angular momentum in the boosting direction is \( mJ \) and this is the quantity which should be large in the Penrose limit of the AdS/CFT correspondence. Therefore we have two options, the first one being to consider a discrete orbifold group \( Z_m \) with a very large \( m \) and finite \( J \) and the second a discrete orbifold group \( Z_m \) with finite and with a large value for \( J \) [20, 21].

In the field theory, the supersymmetry is enlarged from \( N = 1 \) to \( N = 4 \) and the corresponding global symmetries are \( SO(2) \) in the boosted direction and \( SO(4) \) in the transverse direction to the boosting. To identify the gauge invariant operators, we need to use the fact that we boost along the direction of the \( Z_m \) orbifolding and the rest of the space is invariant. The direction of the boosting is denoted by \( Z \) and, as in the previous subsection, we denote the transverse coordinates to the boosting by \( X \) and \( Y \). In the terms of the fields of the \( N = 1 \) theory, \( Z \) should be in a gauge invariant form and is written as a product \( Z = \prod_{i=1}^m Z_i \) where \( Z_i \) are either \( H_{(i,j)(i+1,j)} \) for fixed \( j \), \( D_{(i+1,j+1)(i,j)} \) for fixed \( j \) or \( V_{(i,j)(i+1,j+1)} \) for fixed \( i \). The above fields are in the bifundamental representation of \( SU(N)_i \times SU(N)_{i+1} \), the field \( Z \) transforms in the adjoint representation of the group \( SU(N) \). Together with the scalar fields denoting the transverse direction, \( X \) and \( Y \), they form an \( N = 4 \) multiplet.
The coupling of the $SU(N)$ gauge theory is of order $g_{YM}^2 = g_s m$ and the effective 't Hooft parameter is $g_{YM}^2 N / g_s m^2$ which is finite being of the order of $g_s N m / R^4$ which is finite. Therefore we can treat the $SU(N)$ gauge theory perturbatively.

We can now proceed to describe the gauge invariant operators corresponding to the stringy ground state and excitations. The gauge invariant operators $\text{Tr}(Z^J)$ have angular momentum $mJ$ in the boosted direction due to the action of $Z_m$, and this corresponds to the vacuum of the string theory. To describe the excitations, we need to consider the two cases discussed above, i.e. when $m$ is either small or large.

For the case of small $m$ and large $J$, the first level eight bosonic zero mode oscillators are

$$\text{Tr}(Z^J X), \text{Tr}(Z^J Y), \text{Tr}(Z^J X), \text{Tr}(Z^J Y)$$

(26)

together with $\text{Tr}(Z^J D_{\mu} Z)$. In this case $x^-$ is not compact as is was for the $S^5 / \mathbb{Z}_3$ case discussed in the previous section. The insertions of $X, Y, \tilde{X}, \tilde{Y}$ should be made as $\text{Tr}(Z^J X Z^{J-l})$, etc. The non-supergravity oscillations are obtained by introducing extra phases in the above operators.

More interesting is the case when $m$ is very large and the light cone is compact with radius $\frac{\pi R^2}{m}$, the light cone momentum being quantized $2p^+ = \frac{m}{R}$. The string theory has a matrix string description which mimics the one of the flat space as pointed out in [20, 36]. In [20] string propagation in DLCQ pp-wave has been considered and the states were labeled by two quantum numbers, the first being the DLCQ momentum $k$ and the second being the winding number $m$ in the $x^-$ direction.

The vacuum corresponds to $\text{Tr}(Z^J)$ which has $2p^+ = \frac{m}{R}$ and zero winding number. As $J$ is finite, we can consider $J = 1$. The insertions of the fields $X, Y, \tilde{X}, \tilde{Y}$ should now be made into the trace of the string of $Z_i$ fields. To do this, we also need to consider the splitting of the matrices $X, Y$ into $m N \times N$ blocks, each one being inserted in $m$ different positions and then a summation over the position is required to ensure gauge invariance.

In terms of the original $\mathcal{N} = 1$ theory, if we chose $Z_i$ to be the fields $H_{i,j}(i+1,j)$ for fixed $j$, then the fields $X$ and $Y$ are build by $m N \times N$ blocks which can be either $D_{i+1,j+1}(i,j)$ for fixed $i$ (we denote these by $X_i$) or $V_{i,j}(i,j+1)$ for fixed $i$ (we denote these by $Y_i$). By choosing $Z_i$ transform in the bifundamental representation of $SU(N)_i \times SU(N)_{i+1}$, it results that $X_i$ transform in the bifundamental of $SU(N)_{i+1} \times SU(N)_i$ and $Y_i$ are in the adjoint representation of $SU(N)_i$. Therefore the fields $X_i$ should be inserted in between $Z_i$ and $Z_{i+1}$ and the fields $Y_i$ should be inserted in between $Z_{i-1}$ and $Z_i$. The first oscillators with zero winding number will then be

$$\sum_{i=1}^{m} \text{Tr}(Z_1 Z_2 \cdots Z_i X_i Z_{i-1} \cdots Z_m), \sum_{i=1}^{m} \text{Tr}(Z_1 Z_2 \cdots Z_{i-1} Y_i Z_i \cdots Z_m),$$

(27)
and
\[
\sum_{i=1}^{m} \text{Tr}(Z_1 Z_2 \cdots Z_{i-1} \bar{X}_i Z_{i+1} \cdots Z_m), \sum_{i=1}^{m} \text{Tr}(Z_1 Z_2 \cdots Z_{i-1} \bar{Y}_i Z_{i+1} \cdots Z_m),
\]
where the summation over \( i \) ensures the gauge invariance. The states which have winding numbers are built with an additional factor \( e^{\frac{2\pi i}{m}} \) in the above formulas.

In this form, the stringy operators have an expansion which is similar to the Kaluza Klein expansion of a generic field of five dimensional theory reduced on a circle used in [22, 23] to conjecture the deconstruction of a five dimensional theory for large \( m \) quiver theories in four dimensions. Our \( S^5/(Z_m \times Z_n) \) model should actually be related to a \( (1,1) \) theory in six dimensions [23], but we expect to get a five dimensions theory as long as we boost along the orbifolding directions. The two directions needed to deconstruct a six dimensional theory are obtained in different boosting, one discussed in this subsection and the other discussed in the next subsection.

The conclusion is that a fast moving particle in the \( \tau \) direction reduces the gauge group to \( SU(N) \) and enhances the supersymmetry from \( \mathcal{N} = 1 \) to \( \mathcal{N} = 4 \).

**Case 2.** Boosting in the direction of the fixed locus of the \( Z_m \) orbifolding (the same discussion holds for the \( Z_n \) or \( Z_r \) orbifolding)

We take the same form of the metric as in [19], we parameterize the angle of \( S^1 \) by \( \psi \), the phase of \( z_3 \) and we boost along the \( \psi \) direction. Since \( Z_n \) acts on \( z_3 \), we take an \( n \)-fold covering of \( S^5 \) replacing \( \psi \) by \( n\psi \) in the metric. We introduce the null coordinates

\[
x^+ = \frac{1}{2}(t + \frac{\psi}{n}), \quad x^- = \frac{R^2}{2}(t - \frac{\psi}{n})
\]

and consider a scaling limit \( R \to \infty \) around \( \theta = \pi/2 \) with

\[
\rho = \frac{r}{R}, \quad \theta - \frac{\pi}{2} = \frac{u}{R}.
\]

The computation is essentially the same as in [19], the transversal \( S^3 \) part of the metric \([ (d\tau + (\cos \chi - 1)d\phi)^2 + (d\chi^2 + \sin^2 \chi d\phi^2) \] is left intact in this limit and hence the \( Z_m \) action remains.

We now denote the scalar field parameterizing the boosted direction by \( z_3 = Z \) and the scalar fields parameterizing the transverse directions by \( z_1 = X, z_2 = Y \). The \( Z_m \) discrete group acts now on \( X, Y, Z \) as

\[
X \to e^{2\pi i/m}X, \quad Y \to e^{-2\pi i/m}Y, \quad Z \to Z
\]
and there is also an action of $\mathbb{Z}_n$ discrete group on the boosting direction:

$$
Z \to e^{-2\pi i/n} Z
$$

Because of the latter action, the field $Z$ should enter at the power $n$ and this is obtained if we consider that $Z$ is a product of the $\mathcal{N} = 1$ fields $V_{(ij)(i+1)j}$ for fixed $i$. We introduce the notation

$$
Z^n = V_{(i,j)(i+1)j} V_{(i+1,j)j+1} \cdots V_{(i+n-1)(i,j+n)}
$$

where $j = j + n \pmod{n}$. The field $Z^n$ is in the adjoint representation of $SU(N)_{i,j}$ for fixed $i, j$. For future use, we also introduce the notation $Z_j = V_{(i,j)(i+1)j}$.

In this case the field theory after the boosting becomes $\mathcal{N} = 2 \prod_{i=1}^{m} SU(N)_{i,j}$, the gauge coupling constants of the gauge groups are of order $g_{YM}^2 = g_s n$ and the effective 't Hooft parameters are $\frac{g_{YM}^2 N}{m^2}$ which are finite being of the order of $g_s N m / R^4$.

Because of the $\mathbb{Z}_m$ projection, the field $Z$ is actually promoted to a $mN \times mN$ matrix, with $mN \times N$ blocks, each block being in the adjoint representation of an $SU(N)_{i,j}$. Together with the corresponding vectors of $SU(N)_{i,j}$, they form $\mathcal{N} = 2$ multiplets. The effective angular momentum in the boosting direction for $\text{Tr} Z^J$ being $nJ$, we again have two choices, one when $n$ is small and the other when $n$ is big.

Consider first the case when $n$ is small. The vacuum of the string theory corresponds to the $\mathbb{Z}_m$ invariant operators:

$$
\frac{1}{\sqrt{mJ}} \text{Tr}[S^q Z^n J] \tag{34}
$$

where $S = (1, e^{2\pi i/m}, \ldots, e^{2\pi i(m-1)/m})$ denotes the $q-th$ twisted sector. The oscillations of the string belong to the untwisted modes which are of the type

$$
\text{Tr}[S^q Z^n J D_\mu(Z)] \tag{35}
$$

and

$$
\text{Tr}[S^q Z^n J \chi] \tag{36}
$$

where $D_\mu$ is the covariant derivative and $\chi$ is the supersymmetric partner of the scalar $Z$. The scalar fields $X$ and $Y$ are now $mN \times mN$ matrices with $mN \times N$ extra diagonal blocks denoted by $X_i$ and $Y_i$, each one transforming in the bifundamental representation of the group $SU(N)_{i,j} \times SU(N)_{i+1,j}$. For the twisted sectors we need to consider states built with oscillators with a fractional moding. These are obtained by multiplying with $X$ and $Y$ which are acted upon by the $\mathbb{Z}_m$ group, together with a position independent phase factor $e^{2\pi i n(q)}$ when inserting $X, Y$ and $e^{2\pi i n(-q)}$ for insertions of $\bar{X}, \bar{Y}$.
The discussion changes when \( m \) is very large. In this case we have a compact light cone with radius \( \frac{\pi R^2}{n} \) and the light cone momentum is quantized \( 2p^+ = \frac{m}{R} \). The vacuum and the oscillations of the string belonging to the untwisted modes are the same as before, but we have a change in the definition of the oscillations of the twisted sectors. The insertions of \( X \) and \( Y \) should be now made between \( Z_{j-1} \) and \( Z_j \). To do this, we have to consider all the blocks \( X_i, Y_i, i = 1, \ldots, m \) as \( nN \times nN \) matrices and to split each one of them into \( n \) diagonal \( N \times N \) blocks denoted by \( X_{ij}, Y_{ij}, i = 1, \ldots, n \) for fixed \( i \). The insertions will then be

\[
\sum_{j=1}^{n} \text{Tr}(Z_1 \cdots Z_{j-1}X_{ij}Z_j \cdots Z_n) \tag{37}
\]

or

\[
\sum_{j=1}^{n} \text{Tr}(Z_1 \cdots Z_{j-1}Y_{ij}Z_j \cdots Z_n) \tag{38}
\]

where \( j \) denotes the insertion and \( i \) denotes the twisted sector. The winding modes are obtained by using the same formulas with an extra \( e^{i\frac{2\pi j}{m}} \) factor.

The conclusion is that a fast moving particle moving in the \( \psi \) direction reduces the gauge group to \( \prod_{i=1}^{m} SU(N)_i \) and enhances the supersymmetry from \( \mathcal{N} = 1 \) to \( \mathcal{N} = 2 \).

**Case 3.** Boosting in a general direction which is neither Case 1 nor Case 2.

In this case both discrete groups \( \mathbb{Z}_m \) or \( \mathbb{Z}_n \) are on the direction of the boosting and the string probes only a small strip along this direction, therefore there is no orbifold action on the scalar fields and the result is a maximal supersymmetric Penrose limit. Because we do not have any orbifold projection on the three scalar fields \( Z, X, Y \), the situation is similar to moving the D3 brane from the tip of the \( \mathbb{Z}_m \times \mathbb{Z}_n \) orbifold in the bulk, when the supersymmetry is changed from \( \mathcal{N} = 1 \) to \( \mathcal{N} = 4 \). The string/field theory duality then reduces to the one of subsection 2.1. There is no change in the angular momentum on the boosted directions due to the orbifolding.

We have identified several boosting directions which imply an enlargement of supersymmetry. Three directions are along the \( \mathbb{Z}_m, \mathbb{Z}_n \) or \( \mathbb{Z}_r \) orbifolding which give maximal supersymmetric pp-limits, three directions are along the fixed loci of \( \mathbb{Z}_m, \mathbb{Z}_n \) or \( \mathbb{Z}_r \) orbifolds which give \( \mathcal{N} = 2 \) supersymmetry and an infinite number of boosting directions are along a general direction which would give \( \mathcal{N} = 4 \).

The discussion is different for large \( m, n \) as compared to the case of small \( m, n \). For the first case we get compact a compact light cone and this can be used to describe the pp-wave as the limit of a DLCQ theory with fixed \( p^+ \). In terms of the choice of boosting,
we get a specific circle so we get a two dimensional torus when both \( m \) and \( n \) are large. These two directions are the ones used by [23] to describe the deconstruction of the six dimensional (1,1) theories.

### 2.4 Correlation Functions and Supersymmetry Enhancement

In [11], a detailed analysis has been made for the anomalous dimensions and three point functions for the chiral and almost chiral operators introduced by [9] (see also [12] for similar discussion). In particular, the authors of [11] have identified the parameter \( g_2 = J^2/N \) as the genus counting parameter in the free Yang-Mills theory, such the correlation function of \( \text{Tr} \bar{Z}^J \) and \( \text{Tr} Z J \) has a contribution \( JN^J g_2^h \) from the genus \( h \) Feynman diagram.

We want to see what happens for the pp-wave limits of orbifold theories. To do this, we start from the observation that the correlation functions for the orbifold theories coincide with those of \( \mathcal{N} = 4 \) theory, modulo the rescaling of the gauge coupling constant, as observed in [13] with string theory methods and [14] by using field theory methods. If we consider the case \( S^5/\mathbb{Z}_n \), in the \( \mathcal{N} = 2 \) theory we have a factor of \( 1/n \) in front of the correlations functions. After the Penrose limit on the non-fixed direction, we go from the orbifolds theory to the covering space and therefore the factor \( 1/n \) disappears. The correlation functions for the orbifold theories are then expected to have a similar expansion in genus as in the \( S^5 \) case. It would be interesting to show this in detail, by analogous computations to [13, 14].

### 3 The \( \mathcal{N} = 1 \) orbifolds of \( T^{1,1} \)

The case of D3 branes at the conifold or at orbifolds of the conifold has been discussed extensively in the literature [8, 26, 27, 29]. The conifold is a three dimensional hypersurface singularity in \( \mathbb{C}^4 \) defined by:

\[
  z_1 z_2 - z_3 z_4 = 0
\]

which is a metric cone over the 5-dimensional Einstein manifold \( T^{1,1} = SU(2) \times SU(2)/U(1) \). The conifold can be realized as a holomorphic quotient of \( \mathbb{C}^4 \) by the \( \mathbb{C}^* \) action given by [8]

\[
  (A_1, A_2, B_1, B_2) \rightarrow (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2).
\]
The map
\[ z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1 \]
provides an isomorphism between these two representations of the conifold. The horizon \( T^{11} \) can be identified with \(|A_1|^2 + |A_2|^2 = |B_1|^2 + |B_2|^2 = 1\) quotient by an \( U(1) \) action induced by \((40)\). Following \([28]\), we can parameterize \( A_i, B_i \) in terms of Euler angles of \( SU(2) \times SU(2) \):
\[
\begin{align*}
A_1 &= \cos \frac{\theta}{2} \exp \frac{i}{2}(\psi + \phi_1) \\
A_2 &= \sin \frac{\theta}{2} \exp \frac{i}{2}(\psi - \phi_1) \\
B_1 &= \cos \frac{\theta}{2} \exp \frac{i}{2}(\psi + \phi_2) \\
B_2 &= \sin \frac{\theta}{2} \exp \frac{i}{2}(\psi - \phi_2),
\end{align*}
\]
and the \( U(1) \) is diagonally embedded in \( SU(2) \times SU(2) \). After taking a further quotient by the remaining \( U(1) \) factor of \( SU(2) \times SU(2) \), we obtain a product of two projective spaces \( \mathbb{CP}_1 \times \mathbb{CP}_2 \) and may identify the parameters \( \theta, \phi_i \) with the spherical coordinates of \( \mathbb{CP}_i \) for \( i = 1, 2 \). Now \( T^{11} \) is a \( U(1) \) fibration over \( \mathbb{CP}_1 \times \mathbb{CP}_2 \) and the \( U(1) \) fiber can be parameterized by \( \psi := \psi_1 + \psi_2 \). The Einstein metric on \( T^{11} \) of radius \( R \) is
\[
ds^2_{T^{11}} = R^2 \left( \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) \right),
\]
(43)

Consider an orbifold theory of the conifold where the discrete group \( \mathbb{Z}_m \times \mathbb{Z}_n \) acts on \( A_i, B_j \) by
\[
(A_1, A_2, B_1, B_2) \to (e^{-2\pi i/m} A_1, A_2, e^{2\pi i/m} B_1, B_2),
\]
(44)
and
\[
(A_1, A_2, B_1, B_2) \to (e^{-2\pi i/n} A_1, A_2, B_1, e^{2\pi i/n} B_2).
\]
(45)

The action \((44)\) descends to the horizon \( T^{11} \) and yields two fixed circles \(|A_2|^2 = |B_2|^2 = 1, A_1 = B_1 = 0 \) (mod \( U(1) \)) and \(|A_1|^2 = |B_1|^2 = 1, A_2 = B_2 = 0 \) (mod \( U(1) \)) \([24]\). Similarly, the action \((45)\) yields two fixed circles \(|A_2|^2 = |B_2|^2 = 1, A_1 = B_1 = 0 \) (mod \( U(1) \)) and \(|A_1|^2 = |B_1|^2 = 1, A_2 = B_2 = 0 \) (mod \( U(1) \)). The horizon \( T^{11}/(\mathbb{Z}_m \times \mathbb{Z}_n) \) is singular along these circles, having an \( A_{m-1} \) singularity along the first two circles and an \( A_{n-1} \) singularity along the last two circles. The discrete group \( \mathbb{Z}_m \times \mathbb{Z}_n \) breaks the \( SU(2) \times SU(2) \) part of the isometry group \( SU(2) \times SU(2) \times U(1) \) of \( T^{11} \) and the \( U(1) \) part remains as the global \( R \) symmetry.

In terms of Euler angles of \( SU(2) \times SU(2) \), the discrete group \( \mathbb{Z}_m \times \mathbb{Z}_n \) action is given by
\[
\begin{align*}
(\psi_1, \phi_1, \psi_2, \phi_2) &\to (\psi_1 - 2\pi i/m, \phi_1 - 2\pi i/m, \psi_2 + 2\pi i/m, \phi_2 + 2\pi i/m) \\
(\psi_1, \phi_1, \psi_2, \phi_2) &\to (\psi_1 - 2\pi i/n, \phi_1 - 2\pi i/n, \psi_2 + 2\pi i/n, \phi_2 - 2\pi i/n)
\end{align*}
\]
(46)
What we see from the above equations is that the coordinate of the $U(1)$ fiber ($\psi = \phi_1 + \phi_2$) is left invariant under the action of $\mathbb{Z}_m \times \mathbb{Z}_n$, as should be in order to preserve the $\mathcal{N} = 1$ supersymmetry.

Now we study the Penrose limits of $\text{AdS}_5 \times \mathbb{T}^1/\mathbb{Z}_m \times \mathbb{Z}_n$. The metric for $\text{AdS}_5 \times \mathbb{T}^{1,1}$ is

$$ds^2 = R^2[-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + \frac{1}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)]$$ (47)

The Penrose limit for the conifold has been studied in [15, 16, 17]. As in the previous section, there are many directions of boosting. We want to study the boosting along the fixed locus of the discrete group action. Consider first the boosting along the circle $|A_1|^2 = |B_1|^2 = 1, A_2 = B_2 = 0 \pmod{U(1)}$ which is a fixed locus of $\mathbb{Z}_m$ action. In terms of the parameters used in (43), this is located at $\theta_1 = \theta_2 = 0$ and can be parameterized by $\psi + \phi_1 + \phi_2$. Because of the action of $\mathbb{Z}_n$, we are actually dealing with an $n$-covering of $\mathbb{T}^{1,1}$ and the metric of $\mathbb{T}^{1,1}$ changes into

$$\frac{n^2}{9}(d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)]$$ (48)

We introduce the null coordinates

$$x^+ = \frac{1}{2}\left(t + \frac{1}{3n}(\psi + \phi_1 + \phi_2)\right)$$
$$x^- = \frac{R^2}{2}\left(t - \frac{1}{3n}(\psi + \phi_1 + \phi_2)\right)$$ (49)

and consider a scaling limit $R \to \infty$ around $\theta_1 = \theta_2 = 0$ with

$$\rho = \frac{r}{R}, \quad \theta_i = \frac{\sqrt{6}}{R}\xi_i, \quad i = 1, 2$$ (50)

and in the limit $R \to \infty$, the metric becomes:

$$ds^2 = -4dx^+ dx^- - r^2 dx^{+2} + dr^2 + r^2 d\Omega_3^2 + \sum_{i=1,2}(d\xi_i^2 + \xi_i^2 d\phi_i^2 - 2\xi_i^2 d\phi_i dx^+)$$ (51)

where $w = (\xi_1 e^{i(\phi_1 - x^+)}, \xi_2 e^{i(\phi_2 - x^+)}).$

The same discussion can be extended to the other fixed circles $A_1 = B_1 = 0, A_1 = B_2 = 0$ and $A_2 = B_1 = 0$, where the boosting is again on the direction $\psi + \phi_1 + \phi_2$, 

15
but around $\theta_1 = \theta_2 = \pi$, $\theta_1 = \pi, \theta_2 = 0$ and $\theta_1 = 0, \theta_2 = \pi$, respectively. The Penrose limit will be identical with (51) after redefining $\mathbf{w}$ appropriately. The transverse space $\mathbf{R}^8$ decomposes into a product of $\mathbf{R}^4$ which is in the $r^i$ directions and $\mathbf{C}^2$ whose coordinates is given by $w_1$ and $w_2$. We now investigate the effect of the orbifolding on the geometry of the pp-limit. Note that if we project the conifold (39) in $\mathbf{C}^4$ to $\mathbf{C}^3$ by $(z_1, z_2, z_3, z_4) \rightarrow (z_1, z_3, z_4)$, we can identify the the boosting direction of $\mathbf{T}^{1,1}$ with the angular direction of $z_1$ which is parameterized by $1/2(\psi + \phi_1 + \phi_2)$ as in (41) and (42), and the transversal space $\mathbf{C}^2$ can be parameterized by $z_3$ and $z_4$. On the pp-limit, $\mathbb{Z}_n$ acts on the boosting direction as

$$z_1 \rightarrow e^{-2\pi i/n} z_1 \quad (52)$$

and on the transversal direction trivially, and on the other hand, $\mathbb{Z}_m$ acts on the transversal direction as

$$(z_3, z_4) \rightarrow (e^{-2\pi i/m} z_3, e^{2\pi i/m} z_4) \quad (53)$$

and acts trivially on the boosting direction $z_1$ which is along the circle of boosting. In terms of the coordinate of the boosting direction, there is an $\mathbb{Z}_n$ action on $\tilde{\psi} = \psi + \phi_1 + \phi_2$ as

$$\tilde{\psi} \rightarrow \tilde{\psi} - \frac{\pi i}{n}. \quad (54)$$

We now identify the field theories gauge invariant operators which are dual to the strings modes on the above pp-wave geometry. The transverse space is $\mathbf{S}^5/\mathbb{Z}_n$ or $\mathbf{S}^5/\mathbb{Z}_m$, so the field theory is $\mathcal{N} = 2$, $\prod_{i=1}^m SU(N)_i$ or $\prod_{i=1}^m SU(N)_i$. Before the boosting the field theory is $\mathcal{N} = 1$

$$\prod_{i=1}^m \prod_{j=1}^n SU(N)_{ij} \times \prod_{i=1}^m \prod_{j=1}^n SU(N)'_{ij}. \quad (55)$$

and there are bifundamental fields $(A_1)_{i,j;i,j}$ in $SU(N)_{i,j} \times SU(N)'_{i,j}$, $(A_2)_{i+1,j+1;i,j}$ in $SU(N)_{i+1,j+1} \times SU(N)'_{i,j}$, $(B_1)_{i,j;i,j+1}$ in $SU(N)_{i,j} \times SU(N)_{i,j+1}$ and $(B_2)_{i,j;i+1,j}$ in $SU(N)'_{i,j} \times SU(N)'_{i,j}$, the products of $A_i, B_i$, which enter in the definitions of $z_1, z_3, z_4$ are

$$(A_1B_1)_{i,j;i,j+1} \text{ in } SU(N)_{i,j} \times SU(N)_{i,j+1}, (A_1B_2)_{i,j;i+1,j} \text{ in } SU(N)_{i,j} \times SU(N)_{i+1,j}, \text{ and } (A_2B_1)_{i+1,j+1;i,j} \text{ in } SU(N)_{i+1,j+1} \times SU(N)_{i,j+1}.$$
superfields, modulo F- and D- flatness condition \[29\]. They are products of the form

\[ A_i B_j A_i B_j \cdots A_{i+m} B_{j+n} \], symmetrized in \( A_i \) and \( B_j \). For fixed \( i \), a particular example of a chiral primary involving only \( A_1 \) and \( B_1 \) fields is:

\[
\text{Tr}((A_1)_{i,j}(B_1)_{i,j+1}(A_1)_{i,j+1}(B_1)_{i,j+1}) \cdots (A_1)_{i,j+n-1}(B_1)_{i,j+n-1})
\]

where \( j+n = j \pmod{n} \) so the trace is taken over the adjoint representation of \( SU(N)_{i,j} \). The R-charges of the fields \( A_i, B_i \) are not changed by the quotienting so the R-charge of the gauge invariant operator \((56)\) is \( n \).

We now relate the field theory R-charge with the other \( U(1) \) charges that appear in the field theory and geometry. In the geometry we have two rotation charges for the \( U(1) \times U(1) \) isometry group which are denoted by \( J_1 \) and \( J_2 \) and they are related to the Cartan generators of the \( SU(2) \times SU(2) \) global symmetry of the dual superconformal field theory by \[15, 16\]:

\[
J_a = -i \frac{\partial}{\partial \phi_a} |_{x^\pm} = -i \frac{\partial}{\partial \phi_a} |_{t,\psi} + i \frac{\partial}{\partial \psi} |_{t,\phi} = Q_a - \frac{1}{2} R, \quad a = 1, 2
\]

Because of the \( Z_n \) action on the fixed circle the quotiented conifold, the above relation becomes:

\[
nJ_a = nQ_a - \frac{R}{2}
\]

We use the convention that \( A_1 \) has \( Q_1 = \frac{1}{2} \) and \( B_1 \) has \( Q_2 = \frac{1}{2} \). In \[15, 16, 17\], the vacuum of the string theory has been identified with the state \( J_1 = J_2 = 0 \) and the first oscillations of the strings with \( J_1 = \pm 1, J_2 = 0 \) and \( J_1 = 0, J_2 = \pm 1 \).

Consider now the boosting along \( z_1 \) direction and we want to identify the gauge invariant operators which correspond to the string theory ground state and first oscillation modes. In the case of the conifold, the ground state \( J_1 = J_2 = 0 \) was identified with the the gauge theory operators: \[15, 16, 17\]:

\[
\text{Tr}(A_1 B_1)^l,
\]

the first oscillations \( J_1 = -1, J_2 = 0 \) or \( J_1 = 0, J_2 = -1 \) were identified with multiplication by

\[ A_1 B_2 \text{ or } A_2 B_1 \]

and the first oscillations \( J_1 = 1, J_2 = 0 \) or \( J_1 = 0, J_2 = 1 \) were identified with multiplication by

\[ A_1 \bar{A}_2 \text{ or } B_2 B_1 \]
where \( A_i, B_i \) are all \( N \times N \) matrices. \( A_1 \bar{A}_2 \) or \( \bar{B}_2 B_1 \) came from the semi-conserved currents of the \( SU(2) \) groups and were introduced in [30]. When there is a quotient action on the \( SU(2) \) groups, \( A_1 \bar{A}_2 \) or \( \bar{B}_2 B_1 \) are not invariant so they do not appear in the spectrum. Because the supersymmetry in Penrose limit is \( \mathcal{N} = 2 \), we do not need the semi-conserved currents to build \( \mathcal{N} = 2 \) multiplets and we only need \( A_1 B_2 \) and \( A_2 B_1 \) in order to build the field theory duals to the twisted sectors of the string theory.

For the quotiented conifold, the matrix \( A_1 B_1 \) is promoted to a \( m \times m \) matrix which splits into \( m \times N \) diagonal matrices in the adjoint representation \( SU(N)_{i,j} \), \( i = 1, \ldots, m \), for fixed \( j \). The matrices \( A_1 B_2 \) and \( A_2 B_1 \) become \( m \times N \) matrices which also split into \( m \) extra-diagonal \( N \times N \) and each block corresponds to fields transforming in the bifundamental representation of \( SU(N)_{i,j} \times SU(N)_{i+1,j} \). The boosted direction is acted upon by the discrete group \( \mathbb{Z}_m \) so the invariant quantity is a product as in (56) with \( n \) copies of \( A_1 \) and \( n \) copies of \( B_1 \), of the form

\[
(A_1)_{i;j;i,j}(B_1)_{i;j+1;j+1} \cdots (A_1)_{i;j+n-1;j+n-1}(B_1)_{i;j+n-1;j+n},
\]

which is indeed in the adjoint representation of \( SU(N)_{i,j} \). Denoting this by \( (A_1 B_1)^n \), we see that the equation (58) implies that it has \( J_1 = J_2 = 0 \) and it is the ground state of the string. The vector field for all \( SU(N)_{i,j} \) with fixed \( j \), together with the field \( [(A_1 B_1)^n]_i \) form an \( \mathcal{N} = 2 \) multiplet. The ground state is given by \( m \) mutually orthogonal \( \mathbb{Z}_m \) invariant single trace operators

\[
\text{Tr}[S^q (A_1 B_1)^n J]
\]

where \( S \) is defined as \( S = (1, e^{2\pi i/m}, \ldots, e^{2\pi i(m-1)/m}) \) denotes the \( q - th \) twisted sector.

The first level untwisted sectors are built with derivatives and descendants of \( (A_1 B_1)^n \) and are of the form:

\[
\text{Tr}[S^q (A_1 B_1)^n D_\mu (A_1 B_1)^n]
\]

and

\[
\text{Tr}[S^q (A_1 B_1)^n J \chi]
\]

where \( D_\mu \) is the covariant derivative and \( \chi \) is the supersymmetric partner of the scalar \( (A_1 B_1)^n \).

The first level twisted sectors are written with insertions of \( A_1 B_2 \) and \( A_2 B_1 \), which are acted upon by \( \mathbb{Z}_m \) but are invariant under \( \mathbb{Z}_n \). They have zero angular momentum in the boosted direction so they are used to build first level string oscillations. The discussion is similar to the one of [19].
As the effective angular momentum of the string states is $nJ$, we again have the choice of choosing $n$ to be either small or large. For the case of large $n$, the insertions of $A_1B_2$ and $A_2B_1$ should be made between different $(A_1)_{i,j}i,j(B_1)_{i,j}i,j+1$. The Penrose limits of quotiented conifold will then be the limit of a DLCQ theory with constant $p^+$. 

4 Conclusions

In this paper we studied the Penrose limits of different $\mathcal{N} = 1$ orbifold geometries of $S^5$ and $T^{11}$ which lead to supersymmetric PP-wave backgrounds with enlarged supersymmetry. We have considered the gauge invariant chiral operators in the different Penrose limits and we have identified the string oscillations in terms of the gauge invariant operators. We discussed the different choices for the rank of the quotient groups.

Acknowledgments

We would like to thank Sunil Mukhi for correspondence, Keshav Dasgupta for comments on the manuscript and especially Gianguido Dall’Agata for several important discussions.

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