Global Well-posedness for the Three Dimensional Muskat Problem in the Critical Sobolev Space

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Abstract

We prove that the three dimensional stable Muskat problem is globally well-posed in the critical Sobolev space $\dot{H}^2 \cap \dot{W}^{1,\infty}$ provided that the semi-norm $\|f_0\|_{\dot{H}^2}$ is small enough. Consequently, this allows the Lipschitz semi-norm to be arbitrarily large. The proof is based on a new formulation of the three dimensional Muskat problem that allows for the capture at the hidden oscillatory nature of the problem. The latter formulation allows to prove the $\dot{H}^2 a\,priori$ estimates. In the literature, all the known global existence results for the three dimensional Muskat problem are for small slopes (less than 1). This is the first arbitrary large slope theorem for the three dimensional stable Muskat problem.

1. Introduction

In this article, we study the three dimensional Muskat problem which models the dynamics of two incompressible and immiscible fluids with different densities and viscosities separated by a porous media (see [49,50]). This problem, initiated by Morris Muskat in the early ’30, has appeared in the first time in the study of science of geophysics mainly for petroleum engineering applications [51]. His main contributions has been to introduce a mathematical concepts to the knowledge of flow of oil and gas in sands. Since then, many other applications such as in civil engineering or in modern biology have been studied (see for example [45]). Since the fluids are immiscible and separated by a porous media, they therefore lie in two different time dependent domains. Set $\Omega_1(t)$ and $\Omega_2(t)$ these two different fluid regions. We assume that $\rho_i$ is the density of the fluid in the moving region $\Omega_i(t)$ and that the two fluids have the same viscosity (see for example [38] for the viscosity jump case). The velocity $v_i$ in the fluid domain $\Omega_i(t)$ for $i = 1, 2$, is given by the following so-called Darcy’s [33] law:

$$\frac{\mu}{\kappa} v_i = (0, 0, g \rho_i) - \nabla P_i,$$ (1.1)
there $g$ is the gravity, $\kappa$ is the permeability of the porous media, $\mu$ is the viscosity. Since $g$, $\kappa$ and $\mu$ are fixed constants, without loss for generality, we may assume that there are all equal to 1 for simplicity. The second identity means that the two fluids are incompressible. Recall that $P_i$ is the pressure on the different fluid domains, while on the interface $\partial \Omega_1(t) = \partial \Omega_2(t)$ the pressure are equal that is $P_1 = P_2$. Lastly, since the density $\rho_i$ is transported by the flow, it obeys the following equation:

$$\partial_t \rho_i + v_i \cdot \nabla \rho_i = 0. \quad (1.3)$$

The coupling of equations (1.1), (1.2), (1.3) is the incompressible porous media equation [50]. Note that all those physical quantities namely $v_i$, $\rho_i$, $P_i$ are functions of $(x, t) \in \mathbb{R}^3 \times [0, \infty)$. In particular, since the two fluids have different densities, $\rho_i$ is a step function, that is,

$$\rho(x, t) = \rho_1 \mathbb{I}_{\Omega_1(t)}(x) + \rho_2 \mathbb{I}_{\Omega_2(t)}(x).$$

This problem is analogous to the so-called Hele–Shaw equation [22, 43, 44]. We refer to [23, 37, 41, 55] for a complete picture of this analogy and to [2, 3] for some recent mathematical developments on this equation and related models.

Since $\rho_1 \neq \rho_2$, we may assume that $\rho_1 < \rho_2$. In that case, the word “stable” Muskat problem means that $\Omega_2(t)$ corresponds to the heavier fluid domain which lies below $\Omega_1(t)$ which is the lighter fluid domain. This physical structure is preserved for any time as long as the interface is a graph of a regular enough function and this is the case as long as the Rayleigh–Taylor condition is satisfied (see [55]). Indeed, a common assumption when studying the moving fluid domains is to parametrize the interface as being the graph of a sufficiently regular function. In this case the Rayleigh–Taylor simplifies to $\rho_2 - \rho_1 > 0$. By using classical tools from potential theory, it was shown in [24] that the interface obeys a nice contour equation which is both nonlocal (unlike its Eulerian version) and nonlinear. This formulation gives a closed equation which is fully determined only by the dynamics of the interface itself. The dynamics of this moving interface is a function $f$ which depends of the position $x \in \mathbb{R}^2$ and time $t \geq 0$. This gives rise to an evolution equation which is called the Muskat problem. We shall further assume that we are dealing with an interface which is flat at infinity and that there is no surface tension.

In this paper we shall focus on the three dimensional case. The three dimensional Muskat problem reads as

$$(\mathcal{M}_1) : \begin{cases} f_i(t, x) = \frac{\rho}{2\pi} P.V. \int \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} \frac{1}{(1 + \Delta_y^2)^{3/2}} \, dy, & (x, t) \in \mathbb{R}^2 \times [0, T] \\ f(0, x) = f_0(x), \end{cases}$$

where $\rho = \rho_2 - \rho_1 > 0$ and the operator $\Delta_y f(x, t) = \frac{f(x,t) - f(x - y, t)}{|y|}$. Note that the $p.v.$ is mainly needed when $y$ approaches 0, some models have been studied taking into account this fact (see for example [42]). Local existence for this equation in subcritical spaces either in two dimensional or three dimensional
has been studied in several articles. Local existence in the Sobolev space in $H^k$, $k \geq 3$ and illposedness results in the unstable regime have been shown in [24]. In [18], Cheng, Granero-Bellinchón and Shkoller proved local well-posedness in $H^2$ provided the norm $H^{3/2+\epsilon}, \epsilon \in (0, 1/2)$ is small enough. In [21], Constantin, Gancedo, Schydkoy and Vicol were able to prove that the Muskat problem is locally-well posed in $\dot{W}^{2,p}, p > 1$. They also proved a regularity criteria in terms of the uniform continuity of the bounded slope (see also [41] where a very weak regularity criteria is proved). The later result has been recently extended in [1] to the three dimensional case and to the wider class of subcritical Sobolev spaces $W^{s,p}$ where $s \in (1+1/p, 2)$ and $p \in (1, \infty)$. In [48], Matioc proved local-wellposedness in the subcritical Sobolev space $H^{3/2+\epsilon}, \epsilon \in (0, 1/2)$. By using a purely paradifferential approach, Nguyen and Pausader [52] were able to prove that the Muskat problem is locally-well posed in $H^s, s > 1 + d/2$ regardless of the characteristic of the fluids.

In the two dimensional case, the homogeneous version of the result in [52] has been obtained by Alazard and the second author [4] using a paralinearization formula of the Muskat equation [4]. The latter allows one to identify the most important terms in the study of the Cauchy problem.

Similarly, up to an integration by parts (see [24]), the three dimensional Muskat problem may be written as

$$(\mathcal{M}_1) : \begin{cases} f_t(t, x) = \frac{\rho}{2\pi} P.V. \frac{\nabla f(x) \cdot y - (f(x, t) - f(x - y, t))}{(1 + \Delta_y^2)^{3/2}} \frac{dy}{|y|^3}, \\ f(0, x) = f_0(x). \end{cases}$$

The latter formulation is well adapted when dealing with the Cauchy problem for the Muskat equation with data in the Lipschitz class. Indeed, it has been used for instance in the recent work by Cameron [13] to prove global regularity for small slopes for the three dimensional Muskat problem. In addition, being a physically relevant quantity when dealing with the geometry of the moving interface, the Lipschitz semi-norm is also a fundamental quantity in the Muskat problem (see the survey [37,41]).

Importantly, the Muskat equation has a scaling. Namely, if $f$ is a solution to three dimensional Muskat problem with initial data $f_0$ so does the whole family $\lambda^{-1} f(\lambda x, \lambda t)$ with initial data $\lambda^{-1} f_0(\lambda x)$, where $\lambda > 0$. Recall that a space is called critical if its norm (or semi-norm) is left invariant by the scaling of the equation. In the case of the three dimensional Muskat problem, it is not difficult to observe that the Lipschitz space, the Wiener space studied in [19], the homogeneous Sobolev space $H^2$ or the homogeneous Besov space $B^{1,\infty}_{\infty,\infty}$ are examples of critical spaces for the three dimensional Muskat problem. To get a first idea of the structure of the equation a classical idea consists in linearizing around the trivial solution. By doing so, one may check that the equation reduces to

$$\partial_t f(x, t) = \frac{\rho}{2\pi} \Lambda f,$$

where, in two dimensional,

$$\Lambda f(x, t) = \frac{P.V.}{\pi} \int \frac{f(x, t) - f(x - y, t)}{|y|^3} dy.$$
This linearization shows that one needs $\rho > 0$ in order to ensure existence of a local solution to the “half” heat equation.

The Cauchy problem for equation $\mathcal{M}_1$ in the critical setting is delicate, even if one assumes smallness of the initial data. Indeed, the Muskat problem is not a fully parabolic PDE since regular enough solutions may blow-up as it has been show by Castro, Córdoba, Gancedo and Fefferman in [14, 15]. Indeed, they proved that there exists a class of smooth initial data which fails to be $C^4$ regular after sometimes and after a later time becomes a non-graph (see also [40]). The instability of the Cauchy problem associated to regular enough initial data is also very well described in a series of papers by Córdoba, Gómez-Serrano and Zlatoš [27, 28]. They were able to show some special dynamical scenarios are possible for example solutions passing from stable regime to unstable regime and finally go back to stable regime. Another kind of singularity are the so-called splash singularity (the curve self intersect in a point) or splat singularity (the curve self intersect in set a of Lebesgue measure $> 0$) while its regularity is preserved. For the Muskat problem, both splash [39] and splat singularities [26, 30] have been ruled out. In the one phase Muskat problem problem splash singularities are possible as it was shown by Castro, Córdoba, Fefferman, Gancedo and López-Fernández in [17]. These kind of singularities have been shown to exist or ruled out for water waves and related fluid equations (see [16, 31, 32, 36, 39]). Note that the Muskat can be seen as the “parabolic” version of the water-waves equations (see for example [41]).

All the singularity results known require initial data which are sufficiently regular and with sufficiently high slope. Global existence results for very small slopes have been obtained by Constantin and Pugh [30] or Escher-Matioc [35] they were able to ruled out turnover scenario. Actually, if one assumes that the initial data is sufficiently small in the critical Lipschitz space $\dot{W}^{1,\infty}$, then the Muskat problem turns out to be more stable. More precisely, there is a maximum principle for the slope [25] in the sense that, if the Lipschitz semi-norm is initially smaller than 1 so do the solutions for all time. In [20], Constantin, Córdoba, Gancedo, and Rodriguez-Piazza and Strain were able to prove that if the initial data is at least $H^3$ (to ensure local existence [24]) and if the initial data is smaller than 1/3 in the Lipschitz class, then the three dimensional Muskat problem is globally well-posed. We refer also to [53] where decay estimates are obtained. Recently, Cameron [13] was able to construct global unique solution for initial data $\|\nabla_x f_0\|_{L^\infty} < \frac{1}{\sqrt{5}}$. The unique solution can be unbounded provided that it grows sublinearly. However, unlike his result in the two dimensional case [12], the main results in the three dimensional case deals with small slopes only.

While arbitrary large slope results have been shown to exist globally for the two dimensional Muskat problem in,

- Deng, Lei, and Lin [34] (under a monotonicity assumption)
- Cameron [12] (under the condition that $\sup f_0'(x) \times \sup -f_0'(y) < 1$)
- Córdoba and the second author [29] (small data in the critical $\dot{H}^{3/2}$ space),

no large solutions in Lipschitz are known to exist for the three dimensional Muskat problem. In terms of the geometry of the interface, the condition of very small slopes ($< 1$) is quite restrictive.
The aim of this article is to show that the three dimensional Muskat problem is globally well-posed for any large initial data in Lipschitz. Indeed, we shall only assume smallness in the critical $\dot{H}^2$ semi-norm. So the slope can be arbitrarily large, this is the first result of large slope solutions for the three dimensional Muskat problem.

Besides being mathematically challenging to prove global results without any smallness assumption on the Lipschitz semi-norm, it is also physically relevant since it would show that the interface can be highly oscillating in an arbitrarily short time. This is obviously impossible to observe if the slope is small. Also, allowing the slope to be arbitrarily large shows that there exist solutions which can be arbitrarily close to the turnover phenomena observed by Castro, Córdoba, Fefferman, Gancedo and López-Fernández in [17]) but without never reaching it.

When dealing with the Cauchy problem for data in the critical $\dot{H}^2$ space, both aforementioned formulations give rise to severe difficulties to close the a priori estimates for the most singular terms. This motivate the introduction of a new formulation to treat the Cauchy problem (1.4) for initial data in $\dot{H}^2$. The idea behind this new formulation in terms of oscillatory integral was pioneered in an article by Córdoba and the second author [29] were they studied the Cauchy problem for two dimensional Muskat equation with regular enough data and small $\dot{H}^{3/2}$ semi-norm. However, the three dimensional case (two dimensional interface) is not only more nonlinear than the two dimensional case (one dimensional interface) but also more technical because of the fact that one has to deal with directional derivatives. The fact that the rational function in $\Delta_y f$ appearing in the Muskat equation cannot be seen as the restriction of the Fourier transform of some well chosen $L^1$ function (in the same spirit as [29]) generates some technical difficulties. Also, one of the most difficult term is $S_{2,2}$. This is mainly because one looses the nice symmetry of the Hilbert transform which gives rise to nice controlled commutators in the case of the two dimensional Muskat problem [29]. In higher dimension, we get the Riesz transforms but due to the fact that the critical space becomes $\dot{H}^2$ it seems not possible to get some nice cancellations and symmetries. One has to guess which decompositions will give the desired control to close the energy estimates.

2. Main Result

**Theorem 2.1.** Let $F(x) = C(1 + x^2)^{-3/2}$, where $C > 0$ is a fixed constant. For any initial data $f_0 \in \dot{H}^2 \cap \dot{H}^{1,\infty}$ with $\|f_0\|_{\dot{H}^2} < F(\|f_0\|_{\dot{H}^{1,\infty}})$ small enough, then, there exists a unique global solution $f$ to the three dimensional Muskat problem such that $f \in L^\infty([0, T], \dot{H}^2 \cap \dot{H}^{1,\infty}) \cap L^2([0, T], \dot{H}^{5/2})$ for all $T > 0$.

**Remark 1** This theorem allows the interface to be arbitrarily large in $\dot{W}^{1,\infty}$ which is the first result of this kind in the three dimensional case. Note that the smallness is only assumed on the critical $\dot{H}^2$ Sobolev semi-norm. Also, this theorem is fully dealing with the critical setting in the sense that both the initial data and the smallness lie in critical spaces.
Remark 2 The proof of the \textit{a priori} estimates in $\dot{H}^2$ is based on a series of decomposition of the terms together with estimates on homogeneous Besov spaces. This $\dot{H}^2$ control shows that there is a regularizing effect of order $L^2\dot{H}^{5/2}$. The control of the slope by means of the $L^2\dot{H}^{5/2}$ semi-norm is obtained thanks to a combination of the study of the evolution of the extrema (justified thanks to Rademacher’s theorem) together with Besov estimates.

Remark 3 When performing $\dot{H}^2$ \textit{a priori} estimates we know that the dissipation will be of fractional order. This amount to take fractional derivatives into the nonlinear term. One would need to use multilinear estimates of singular integral operators together with estimates of composition functions [11]. This may lead to tedious computations. However, the strategy to get the $\dot{H}^2$ \textit{a priori} estimates presented in this paper avoid this difficulty.

Remark 4 When performing $\dot{H}^2$ \textit{a priori} estimates, the dissipation one hope for is of fractional order. This amount to take fractional derivatives into the nonlinear term. One would need to use multilinear estimates of singular integral operators together with estimates of composition functions [11]. This may lead to tedious computations. However, the strategy to get the $\dot{H}^2$ \textit{a priori} estimates presented in this paper avoid this difficulty.

Remark 5 During the review process of this paper, several other existence results in the critical case appeared in two dimensional and three dimensional. In particular, Alazard and Hung proved the well-posedness in two dimensional [5–7] in the critical Sobolev space $\dot{H}^{3/2}$ and in three dimensional [8] in the critical space $\dot{H}^2 \cap \tilde{W}^{1,\infty}$. Besides giving another proof of our main result, they also obtained local well-posedness.

The plan of the paper is the following: in the next section, we shall introduce a new formulation of the three dimensional Muskat problem in terms of oscillatory integrals. In the second section, we shall give the definition of the functional spaces together with notations of some operators that will be used throughout the article. The third section, which is the central part of the article, is devoted to the proof of the $\dot{H}^2$ \textit{a priori} estimates. The fourth section contains the Sobolev energy inequality. The fifth section is the control of the slope together with a bootstrap argument to close the estimates with respect to critical quantities only. The sixth and last section is the proof of the uniqueness.

3. A New Formulation of the Three Dimensional Muskat Problem

Let us recall that the Muskat equation in $\mathbb{R}^3$ in the stable case and when the interface is parametrized as a graph is given by the following two dimensional equation:

\[
(M_1) : \begin{cases} f_t(t, x) = \frac{\rho}{2\pi} P.V. \int \nabla_x \Delta_y f \frac{y}{|y|^2} \frac{1}{(1 + \Delta_y^2 f)^{3/2}} \ dy \\
 f(0, x) = f_0(x). \end{cases}
\]
In this section we shall prove the following Proposition which gives an equivalent formulation of the three dimensional Muskat in terms of oscillatory integrals.

**Proposition 3.1.** Consider the following Cauchy problem:

\[
(M_2) : \begin{cases}
  f_t(t, x) = \frac{\rho}{2\pi} P.V. \int \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k \cos(k \Delta_y f)} \, dk \, dy, \\
  f(0, x) = f_0(x).
\end{cases}
\]

Then,

\[(M_1) \iff (M_2).\]

**Proof of Proposition 3.1.** One may easily check that, for any \(x \in \mathbb{R}\),

\[
\frac{1}{1 + x^2} = \int_0^\infty e^k \cos(kx) \, dk \text{ and } \cos(\arctan(x)) = \frac{1}{\sqrt{1 + x^2}}.
\]

Hence, for any \(x \in \mathbb{R}\),

\[
\frac{1}{(1 + x^2)^{3/2}} = \int_0^\infty e^k \cos(kx) \cos(\arctan(x)) \, dk.
\]

In particular, for \(x = \Delta_y f\), one gets the identity

\[
\frac{1}{(1 + \Delta_y^2 f)^{3/2}} = \int_0^\infty e^k \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \, dk.
\]

Hence \(M_1 \iff M_2\).

\[\square\]

### 4. Functional Setting and Notations

As usually, for \(s > 0\), \(\dot{H}^s\) denotes the homogeneous Sobolev space endowed with the semi-norm

\[\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}.\]

The definition of the homogeneous Besov spaces that we shall use have been introduced by Besov in [10]. Let \((p, q, s) \in [1, \infty]^2 \times \mathbb{R}^2\), a tempered distribution \(f\) (we assume that its Fourier transform is locally integrable near 0) belongs to the homogeneous Besov space \(\dot{B}^s_{p,q}(\mathbb{R}^2)\) if and only if the following semi-norm is finite:

\[\|f\|_{\dot{B}^s_{p,q}} = \left\| \frac{1}{|y|^s} \int_{\mathbb{R}^2} |\hat{f}(y)|^p \left(1_{[0,1]}(s) \delta_y f + 1_{[1,2]}(s) (\delta_y f + \bar{\delta}_y f) \right) \, dy \right\|_{L^q(\mathbb{R}^2, |y|^{-2} \, dy)} < \infty.\]

Here \(\delta_y f(x) = f(x) - f(x - y)\) and \(\bar{\delta}_y f(x) = f(x) - f(x + y)\).

We have the following embedding between homogeneous Besov spaces (see for example [9,47,54]): for all \((p_1, p_2, r) \in [1, \infty]^3\) such that \(p_1 \leq p_2\) and \(q_1 \leq q_2\) we have

\[\dot{B}^s_{p_1,r}(\mathbb{R}^2) \hookrightarrow \dot{B}^s_{p_2,r}(\mathbb{R}^2).\]
Here \((s_1, s_2) \in \mathbb{R}^2\) are such that \(s_1 + \frac{2}{p_2} = s_2 + \frac{2}{p_1}\). We also have, for all \((p_1, s_1) \in [2, \infty] \times \mathbb{R}\),

\[
\dot{\mathcal{B}}^{s_1}_{p_1, r_1}(\mathbb{R}^2) \hookrightarrow \dot{\mathcal{B}}^{s_2}_{p_1, r_2}(\mathbb{R}^2),
\]

for all \((r_1, r_2) \in ]1, \infty]^2\) such that \(r_1 \leq r_2\).

Throughout the article, we shall use the following notations:

- \(\Delta_y f(x, t) = \frac{f(x, t) - f(x - y, t)}{|y|}\)
- \(\delta_y f(x, t) = f(x, t) - f(x - y, t)\)
- \(\tilde{\Delta}_y f(x, t) = \frac{f(x, t) - f(x + y, t)}{|y|}\)
- \(\tilde{\delta}_y f(x, t) = f(x, t) - f(x + y, t)\)
- \(S_y f(x, t) = \frac{2f(x, t) - f(x - y, t) - f(x + y, t)}{|y|}\)
- \(s_y f(x, t) = 2f(x, t) - f(x - y, t) - f(x + y, t)\)
- \(D_y f(x, t) = \frac{f(x + y, t) - f(x - y, t)}{|y|}\)
- \(d_y f(x, t) = f(x + y, t) - f(x - y, t)\)

For the sake of readability, we shall not write the time dependence.

The notation \(\nabla_i\) will denote the gradient vector with respect to the variable \(i \in \mathbb{R}^2\).

The operator \(\Delta\) will always mean the classical Laplacian with respect to \(x\).

As well, \(A \lesssim B\) means that there exists a fixed constant \(C > 0\) such that \(A \leq CB\).

5. A Priori Estimates in \(H^2\)

We shall use an energy method. That is, we shall do \(H^2\) a priori estimates which allows us to get enough compactness to pass to the limit in a regularized equation. Without loss of generality we may assume that \(\rho = 2\pi\). To prove the existence of solution, one classically needs to introduced a regularized equation. For simplicity, we shall use the following regularized Muskat equation which was introduced by Alazard and Hung (see [6]). It should be noted that one can choose another regularization whose viscosity term is linear in the parameter since the artificial viscous term preserve the Lipschitz semi-norm since it has the good sign.

Let \(\phi > 0\) be a smooth bounded even function whose integral over \(\mathbb{R}^2\) is 1 and such that \(\phi(x) = 1\) in \(B_1\) (the ball of radius 1 centered at the origin) and 0 outside \(B_2\). Let \(\epsilon \in (0, 1]\) and set \(\phi_\epsilon(x) := \epsilon^{-1}\phi(\epsilon^{-1}|x|)\). Then

\[
(M_\epsilon) :\begin{cases}
\partial_t f_\epsilon(t, x) = \int \nabla_x \Delta_y f_\epsilon \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f_\epsilon)) \\
\int_0^\infty e^{-k} \cos(k \Delta_y f_\epsilon)(1 - \phi_\epsilon(y)) \, dk \, dy + |\log(\epsilon)|^{-1} \Delta f_\epsilon \\
f_\epsilon(0, x) = f_0(x) \ast \phi_\epsilon(x).
\end{cases}
\]
Then, using Section 2.6 in [6] we know that for all \( \epsilon \in (0, 1] \) and all data in \( H^2(\mathbb{R}^2) \) the regularized Muskat equation admits a unique global solution \( f_\epsilon \in C^1([0, \infty), H^\infty(\mathbb{R}^2)) \). The aim will be to prove that the associated solution to the Cauchy problem \( (\mathcal{M}_\epsilon) \) will converge (as \( \epsilon \) goes to 0) in some Banach spaces. The strong compactness in \( (L^2 L^2)_{loc} \) will be obtained in the usual way thanks to the Rellich compactness theorem (see for example [46]). In the sequel we assume that the solution is from this regularized equation but we will omit to write the parameter \( \epsilon \). For the passage to the limit, we refer the reader to [6] since the arguments are similar.

The main effort will be devoted to the proof of the \( a \ priori \) estimates in the critical space \( \dot{H}^2(\mathbb{R}^2) \). By taking the Laplacian of the Muskat equation and multiplying by \( \Delta f \) and finally integrating in the space variable, one finds that

\[
\frac{1}{2} \partial_t \|f\|_{\dot{H}^2}^2 = \int \Delta f \Delta \left( \int \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \right) \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \right) dy \right) dx
\]

Then, by using classical formulas for the differential operator \( \Delta \), we find that

\[
\frac{1}{2} \partial_t \|f\|_{\dot{H}^2}^2 = \int \Delta f \int \nabla_x \Delta_y \Delta f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \, dy \, dx
\]

\[+ 2 \int \Delta f \int \Delta_y \Delta f \cdot \frac{y}{|y|^2} \cdot \nabla_x \left( \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \, dy \, dx
\]

\[+ \int \Delta f \int \Delta_y \nabla_x f \cdot \frac{y}{|y|^2} \Delta \left( \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \, dy \, dx,
\]

and hence, we obtain

\[
\frac{1}{2} \partial_t \|f\|_{\dot{H}^2}^2 = \int \Delta f \int \nabla_x \Delta_y \Delta f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \, dy \, dx
\]

\[:= S(\text{most singular term})
\]

\[+ 2 \int \Delta f \int \Delta_y \Delta f \cdot \frac{y}{|y|^2} \cdot \nabla_x \left( \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \, dy \, dx
\]

\[+ 2 \int \Delta f \int \Delta_y \Delta f \cdot \frac{y}{|y|^2} \cdot \cos(\arctan(\Delta_y f)) \nabla_x \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \right) \, dy \, dx
\]

\[+ \int \Delta f \int \Delta_y \nabla_x f \cdot \frac{y}{|y|^2} \Delta \left( \cos(\arctan(\Delta_y f)) \right) \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) dk \right) \, dy \, dx
\]
\[
\int_0^\infty e^{-k \cos(k\Delta_y f)} \, dk \, dy \, dx \\
+ 2 \int \Delta f \int \Delta_y \nabla_x f, \frac{y}{|y|^2} \nabla_x \left( \cos(\arctan(\Delta_y f)) \right) \cdot \nabla_x \left( \frac{\int_0^\infty e^{-k \cos(k\Delta_y f)} \, dk}{\frac{y}{|y|^2}} \cos(\arctan(\Delta_y f)) \Delta 
\right) \left( \int_0^\infty e^{-k \cos(k\Delta_y f)} \, dk \right) \, dy \, dx \\
+ \int \Delta f \int \Delta_y \nabla_x f, \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \Delta 
\right) \left( \int_0^\infty e^{-k \cos(k\Delta_y f)} \, dk \right) \, dy \\
:= S + \sum_{i=1}^5 T_i
\]

Our aim will be to control $\frac{1}{2} \partial_t \|f\|^2_{H^2}$, we shall actually prove that $\frac{1}{2} \partial_t \|f\|^2_{H^2} < 0$ if the $H^2$ is sufficiently small and the Lipschitz semi-norm does not blow-up. This, combining with the control of the Lipschitz semi-norm will give the main result by using a bootstrap argument.

6. Estimates of the Less Singular Term: $T = \sum_{i=1}^5 T_i$

To estimate the less singular terms, one does not have to introduce any symmetrizations since the spatial derivatives will be well balanced. Indeed, these terms come from the differentiation of the oscillatory parts. More precisely, we will prove the following Lemma:

**Lemma 6.1.** The less singular terms can be estimated as follows

\[
\sum_{i=1}^5 T_i \lesssim \|f\|^2_{H^{5/2}} \left( \|f\|_{H^2} + \|f\|^2_{H^2} \right)
\]  

(6.1)

**Proof of Lemma 6.1.** The estimates of this terms do not require to use technical decompositions since it would be easy to balance the regularity in $x$ and in $y$. We shall estimate each $T_i$ for $i = 1, \ldots, 5$ separately.

6.1. Estimate of $T_1$

We start by estimating $T_1$, that is,

\[
T_1 = 2 \int \Delta f \int \Delta_y \Delta f, \frac{y}{|y|^2} \nabla_x \left( \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k \cos(k\Delta_y f)} \, dk \, dy \, dx \\
\lesssim 2 \int |\Delta f| \int |\Delta_y \Delta f| \frac{1}{|y|} \left| \nabla_x \left( \cos(\arctan(\Delta_y f)) \right) \right| \, dy \, dx.
\]
Then, since an easy computation gives \(|\nabla_x \left(\cos(\arctan(\Delta_y f))\right)| \lesssim |\nabla_x \Delta_y f|\) one finds that

\[
T_1 \lesssim \int |\Delta f| \int |\Delta_y \Delta f| \frac{1}{|y|} |\nabla_x \Delta_y f| \, dy \, dx \\
\lesssim \|f\|_{\dot{H}^2} \int \frac{\|\Delta \delta_y f\|_{L^2}}{|y|^{3/2}} \frac{\|\nabla_x \delta_y f\|_{L^\infty}}{|y|^{3/2}} \, dy \\
\lesssim \|f\|_{\dot{H}^2} \|\Delta f\|_{B_{2,2}^{1/2}} \|\nabla_x f\|_{B_{2,2}^{1/2}} \\
\lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}
\]

(6.2)

6.2. Estimate of \(T_2\)

Recall that

\[
T_2 = 2 \int \Delta f \int \Delta_y f \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \nabla_x \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \right) \, dy \, dx \\
\lesssim \int |\Delta f| \int |\Delta_y \Delta f| \frac{1}{|y|^2} |\nabla_x \cos(k \Delta_y f)| \, dk \, dy \, dx
\]

Using that \(|\nabla_x \cos(k \Delta_y f)| \lesssim |\nabla_x \Delta_y f|\), one finds

\[
T_2 \lesssim \int |\Delta f| \int |\Delta_y \Delta f| \frac{1}{|y|} |\nabla_x \Delta_y f| \, dy \, dx,
\]

which is the same estimate as (6.2), so we conclude as in the estimate of \(T_2\), that is,

\[
T_2 \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.
\]

6.3. Estimate of \(T_3\)

We have

\[
T_3 = \int \Delta f \int \Delta_y \nabla_x f \frac{y}{|y|^2} \Delta \left( \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \, dx,
\]

so that

\[
T_3 \lesssim \int |\Delta f| \int \frac{|\Delta_y \nabla_x f|}{|y|} \left| \Delta \left( \cos(\arctan(\Delta_y f)) \right) \right| \, dy \, dx
\]

Then, an easy estimate on \(\Delta \left( \cos(\arctan(\Delta_y f)) \right)\) gives that

\[
T_3 \lesssim \int |\Delta f| \int \frac{|\Delta_y \nabla_x f|}{|y|} |\Delta_y \Delta f| \, dy \, dx + \int |\Delta f| \int \frac{|\Delta_y \nabla_x f|}{|y|} |\Delta_y \nabla_x f|^2 \, dy \, dx
\]
Using the same step as (6.2) one may estimate the first term in the right hand side as $T_1$. For the second term, we observe that

$$
\int |\Delta f| \int \frac{|\Delta y \nabla_x f|}{|y|} \frac{|\Delta y \nabla_x f|^2}{|y|^4} \ dy \ dx \lesssim \|f\|_{H^2} \int \frac{\|\nabla_y \delta_y f\|^3}{|y|^6} \ dy \\
\lesssim \|f\|_{H^2} \|f\|_{B^5_{6,3}^3} \tag{6.3}
$$

Then, using that $\dot{H}^{7/3} \hookrightarrow \dot{B}^{5/3}_{6,3}$, we find that

$$
\int |\Delta f| \int \frac{|\Delta y \nabla_x f|}{|y|} \frac{|\Delta y \nabla_x f|^2}{|y|^4} \ dy \ dx \lesssim \|f\|_{H^2} \|f\|_{\dot{H}^{7/3}}^3,
$$

and finally, using that $\dot{H}^{7/3} = \left[\dot{H}^2, \dot{H}^{5/2}\right]^{1/3}_{3/5}$, we find that

$$
\int |\Delta f| \int \frac{|\Delta y \nabla_x f|}{|y|} \frac{|\Delta y \nabla_x f|^2}{|y|^4} \ dy \ dx \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}^2.
$$

Hence,

$$
T_3 \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}^2.
$$

### 6.4. Estimate of $T_4$

We have

$$
T_4 = 2 \int \Delta f \int \Delta_y \nabla_x f \frac{y}{|y|^2} \nabla_x \left(\cos(\arctan(\Delta_y f))\right) \cdot \nabla_x \left(\int_0^\infty e^{-k} \cos(k\Delta_y f) \ dk\right) \ dy \ dx.
$$

Therefore,

$$
T_4 \lesssim \int |\Delta f| \int |\Delta_y \nabla_x f| \frac{1}{|y|} |\nabla_x \left(\cos(\arctan(\Delta_y f))\right)| \left|\nabla_x \left(\int_0^\infty e^{-k} \cos(k\Delta_y f) \ dk\right)\right| \ dy \ dx.
$$

Using that

$$
|\nabla_x \left(\cos(\arctan(\Delta_y f))\right)| \lesssim |\Delta_y \nabla_x f|,
$$

and that

$$
\left|\nabla_x \left(\int_0^\infty e^{-k} \cos(k\Delta_y f) \ dk\right)\right| \lesssim |\Delta_y \nabla_x f|,
$$

one finds that

$$
T_4 \lesssim \|f\|_{H^2} \int \frac{\|\nabla_y \delta_y f\|^3}{|y|^4} \ dy \lesssim \|f\|_{H^2} \|f\|_{B^5_{6,3}^3}^3.
$$

This is the same estimate as (6.3), hence following exactly the same step as the control of $T_2$, we finally find that

$$
T_4 \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}^2.
$$
6.5. Estimate of $T_5$

We write

$$T_5 = \int \Delta f \int \Delta_y \nabla_x f \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \Delta \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \right) \, dy \, dx \leq \int |\Delta f| \int \frac{|\Delta_y \nabla_x f|}{|y|} \left| \Delta \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \right) \right| \, dy \, dx.$$

Using the fact that $|\Delta \left( \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \right)| \lesssim |\Delta_y \nabla_x f|^2 + |\Delta_y \Delta f|$ we find that

$$T_5 \lesssim \|f\|_{\dot{H}^2} \left( \int \frac{\|\nabla_x \delta_y f\|_{L^6}^3}{|y|^4} \, dy + \int \frac{\|\nabla_x \delta_y f\|_{L^\infty} \|\Delta \delta_y f\|_{L^2}}{|y|^{3/2}} \, dy \right).$$

Hence, following the same steps as (6.2) and (6.3), one finds that

$$T_5 \lesssim \|f\|_{\dot{H}^{5/2}}^2 \left( \|f\|_{\dot{H}^2}^2 + \|f\|_{\dot{H}^2}^2 \right).$$

We have therefore obtained that all the less singular terms $T_i$ for any $i = 1, \ldots, 5$ are controlled as follows:

$$\sum_{i=1}^5 T_i \lesssim \|f\|_{\dot{H}^{5/2}}^2 \left( \|f\|_{\dot{H}^2}^2 + \|f\|_{\dot{H}^2}^2 \right). \tag{6.4}$$

This ends the estimates of the less singular term and the proof of Lemma 8.4 is complete.

In the next section, we shall estimate the more singular term. The analysis of the singular term requires much more effort, the first part consist in symmetrizing in a tricky way.

7. Symmetrization and Useful Identities

Throughout the article, we shall need to use some identities involving second finite differences. We collect all those identities in the next lemma.

**Lemma 7.1.** Set $K_y f := \frac{1}{1+\Delta_y^2 f}$ and $\tilde{K}_y f := \frac{1}{1+\Delta_y^2 f}$. The following equalities hold:

$$\nabla_y \{ \arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f) \} = -\frac{1}{2} S_y f D_y f K_y f \tilde{K}_y f \nabla_y D_y f$$

$$+ \frac{1}{2} \left( K_y f + \tilde{K}_y f \right) \nabla_y S_y f. \tag{7.1}$$

Analogously,

$$\nabla_y \{ \arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f) \} = -\frac{1}{2} S_y f D_y f K_y f \tilde{K}_y f \nabla_y S_y f$$

$$+ \frac{1}{2} \left( K_y f + \tilde{K}_y f \right) \nabla_y D_y f. \tag{7.2}$$
Proof of Lemma 7.1. Set \( A(x) := \nabla_y \{ \arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f) \} \). One may write that

\[
A(x) = \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} - \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} = 0
\]

\[
= \nabla_y \Delta_y f \frac{(\Delta_y f + \tilde{\Delta}_y f)(\Delta_y f - \Delta_y f)}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)} + \frac{\nabla_y S_y f}{1 + \Delta_y^2 f}
\]

\[
= -\nabla_y \Delta_y f \frac{S_y f D_y f}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)} + \frac{\nabla_y S_y f}{1 + \Delta_y^2 f}.
\] (7.3)

On the other hand,

\[
A(x) = \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} - \frac{\nabla_y \tilde{\Delta}_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \tilde{\Delta}_y f}{1 + \Delta_y^2 f} = 0
\]

\[
= \frac{\nabla_y S_y f}{1 + \Delta_y^2 f} + \nabla_y \tilde{\Delta}_y f \left( \frac{1}{1 + \Delta_y^2 f} - \frac{1}{1 + \Delta_y^2 f} \right)
\]

\[
= \frac{\nabla_y S_y f}{1 + \Delta_y^2 f} + \nabla_y \tilde{\Delta}_y f \frac{S_y f D_y f}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)}.
\] (7.4)

Therefore, by combining (7.3) and (7.4) one gets (7.2). Analogously, set \( B(x) := \nabla_y \{ \arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f) \} \), then we write that

\[
B(x) = \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} - \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} - \frac{\nabla_y \Delta_y g}{1 + \Delta_y^2 g} + \frac{\nabla_y \Delta_y g}{1 + \Delta_y^2 g} - \frac{\nabla_y \Delta_y g}{1 + \Delta_y^2 g} = 0
\]

\[
= \nabla_y \Delta_y f \left( \frac{1}{1 + \Delta_y^2 f} - \frac{1}{1 + \Delta_y^2 f} \right) + \nabla_y \Delta_y f - \nabla_y \tilde{\Delta}_y f
\]

\[
= -\nabla_y \Delta_y f \frac{(\Delta_y f + \tilde{\Delta}_y f)(\Delta_y f - \Delta_y f)}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)} + \frac{\nabla_y \Delta_y f - \nabla_y \tilde{\Delta}_y f}{1 + \Delta_y^2 f}
\]

\[
= -\nabla_y \Delta_y f \frac{S_y f D_y f}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)} + \frac{\nabla_y D_y f}{1 + \Delta_y^2 f}.
\] (7.5)

On the other hand we may write that
Lemma 7.2. We have that

\[ B(x) = \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} - \frac{\nabla_y \Delta_y f - \nabla_y \Delta_y f}{1 + \Delta_y^2 f} + \frac{\nabla_y \Delta_y f}{1 + \Delta_y^2 f} \]

Hence, combining (7.6) and (7.7), we get (7.2).

We shall need to compute gradients with respect to \( y \) of the operators \( S_y \) and \( D_y \). The following lemma collects the main identities that we shall use.

**Lemma 7.2.** We have that

\[ D_y f = \frac{y}{|y|} \left( \int_0^1 \nabla (f(x + (r-1)y) + f(x - (r-1)y) - 2f(x)) \, dr \right) + 2\nabla f \cdot \frac{y}{|y|}. \quad (7.8) \]

Moreover,

\[ y \cdot \nabla_y D_y f = \frac{1}{|y|} \int_0^1 y \cdot s_{r-1} \nabla_x f \, dr + \frac{y}{|y|} \nabla_y s_y f \quad (7.9) \]

\[ y \cdot \nabla_y S_y f = \frac{1}{|y|} \cdot s_y f(x) + \frac{y}{|y|} \nabla_y \delta_y f - \frac{y}{|y|} \nabla_y \delta_y f. \quad (7.10) \]

**Proof of Lemma 7.2.** In order to prove (7.8), we first recall that since we have \( D_y f = \frac{1}{|y|}(f(x + y) - f(x - y)) \) one may readily check that

\[ D_y f = \frac{1}{|y|} \int_0^1 (\nabla f(x + (r-1)y) \cdot y + \nabla f(x - (r-1)y) \cdot y - 2\nabla f \cdot y) \, dr + 2\nabla f \cdot \frac{y}{|y|}. \]

The proof of (7.8) is obtained as follows. First, we write that

\[ \nabla_y D_y f = \nabla_y \left( \frac{1}{|y|} \cdot f(x + y) - f(x - y) \right) + \frac{1}{|y|} \nabla_y (f(x + y) - f(x - y)) \]

\[ = \nabla_y \left( \frac{1}{|y|} \cdot f(x + y) - f(x - y) \right) + \frac{1}{|y|} \nabla_x f(x + y) + \nabla_x f(x - y) \]

\[ = \nabla_y \left( \frac{1}{|y|} \right) \int_0^1 \nabla f(x + (r-1)y) \cdot y + \nabla f(x - (r-1)y) \cdot y - 2\nabla f(x) \cdot y \, dr \]

\[ + \frac{\nabla_x f(x + y) + \nabla_x f(x - y)}{|y|} + \nabla_y \left( \frac{2}{|y|} \right) (\nabla f(x) \cdot y). \]

Using that \( \nabla_y \frac{1}{|y|} = -\frac{1}{|y|^2} \) and recalling that \( s_y \) denotes the second finite difference operator, we immediately find that
\[ y.\nabla_y D_y f = -\frac{1}{|y|} \int_0^1 (\nabla f(x + (r-1)y) \cdot y + (\nabla f(x - (r-1)y) \cdot y) - 2\nabla f(x) \cdot y \, dr \\
+ \frac{\nabla_x f(x + y) + \nabla_x f(x - y) - 2\nabla_x f(x)}{|y|} \cdot y \]
\[ = \frac{1}{|y|} \int_0^1 y.s(r)\nabla_x f \, dr + \frac{y.\nabla_x s_y f}{|y|}, \]
which is the desired identity (7.9). Let us prove (7.10). We observe that
\[ \nabla_y S_y f = -\nabla_y \frac{1}{|y|} (f(x + y) + f(x - y) - 2f(x)) - \frac{1}{|y|} \nabla_y f(x + y) + \frac{1}{|y|} \nabla_y f(x - y) \]
\[ = -\nabla_y \frac{1}{|y|} (f(x + y) + f(x - y) - 2f(x)) - \frac{1}{|y|} \nabla_x f(x + y) + \frac{1}{|y|} \nabla_x f(x - y) \]
\[ = -\nabla_y \frac{1}{|y|} (f(x + y) + f(x - y) - 2f(x)) - \frac{1}{|y|} \nabla_x(f(x + y) - f(x)) \]
\[ + \frac{1}{|y|} \nabla_x(f(x - y) - f(x)) \]
\[ = s_y f \nabla_y \frac{1}{|y|} + \frac{1}{|y|} \nabla_x \delta_y f - \frac{1}{|y|} \nabla_x \delta_y f \]
\[ = s_y f \nabla_y \frac{1}{|y|} - \nabla_x D_y f. \]
Hence,
\[ \nabla_y S_y f = s_y f \nabla_y \frac{1}{|y|} - \nabla_x D_y f. \quad (7.11) \]
Therefore,
\[ y.\nabla_y S_y f = \frac{1}{|y|} s_y f(x) + \frac{y}{|y|} \nabla_x \delta_y f - \frac{y}{|y|} \nabla_x \delta_y f, \]
which is the wanted identity (7.10). This ends the proof of Lemma 7.1. \[ \square \]

Finally, we state an easy lemma that will be systematically used throughout the article.

**Lemma 7.3.** Letting \( r > 0 \), we have that
\[ \left| \nabla. \frac{x}{|x|^r} \right| \lesssim \frac{1}{|x|^r}. \quad (7.12) \]

**Proof of Lemma 7.3.** A direct computation leads to the estimate. \[ \square \]

**8. Estimates of the Most Singular Term:** \( S = \sum_{i=1}^4 S_i \)

In order to control the most singular terms, that is when the Laplacian operator falls onto the non-oscillatory term, one has to balance the regularity in both \( x \) and \( y \). This is mainly because of the fact that if we only balance the derivatives in the spatial variable then this amounts to control terms whose regularity in Sobolev or Besov spaces are higher than 1. Recall that controlling such terms require to have second finite order difference. The main goal of the next Lemma is to force the appearance of these terms, in other words, we need to symmetrize the terms.
8.1. Algebraic decomposition of the most singular term: $S$

Set $D_y f := \Delta_y f - \bar{\Delta}_y f$ and $S_y f := \Delta_y f + \bar{\Delta}_y f$. We shall prove the following Lemma:

Lemma 8.1. (symmetrization of the singular term) We have the following decomposition:

$$S = \frac{1}{2} \int \nabla_x \Delta_y f. \frac{y}{|y|^2} \sin \left( \frac{1}{2} \left( \arctan(\Delta_y f) + \arctan(\bar{\Delta}_y f) \right) \right)$$

$$\times \sin \left( \frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) dk dy$$

$$+ \int \nabla_x \Delta_y f. \frac{y}{|y|^2} \sin \left( \frac{1}{2} \left( \arctan(\Delta_y f) + \arctan(\bar{\Delta}_y f) \right) \right)$$

$$\times \sin \left( \frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) dk dy$$

$$+ \frac{1}{2} \int \nabla_x \Delta_y f. \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f) + \arctan(\bar{\Delta}_y f) \right) \right)$$

$$\times \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) dk dy$$

$$+ \frac{1}{2} \int \nabla_x \Delta_y f. \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f) + \arctan(\bar{\Delta}_y f) \right) \right)$$

$$\times \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \left( \cos \left( \frac{k}{2} D_y f \right) \cos \left( \frac{k}{2} S_y f \right) \right) dk dy$$

$$:= \sum_{i=1}^{4} S_i$$

Proof of Lemma 8.2. We start by symmetrizing the non-oscillatory part, that is, we write

$$S = \int \left( \Delta \nabla_x \Delta_y f - \tilde{\Delta} \nabla_x \tilde{\Delta}_y f \right). \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) dk dy$$

$$- \int \nabla_x \Delta_y f. \frac{y}{|y|^2} \cos(\arctan(\tilde{\Delta}_y f)) \int_0^\infty e^{-k} \cos(k \tilde{\Delta}_y f) dk dy.$$

Then, by doing a change of variable ($y \rightarrow -y$), one finds that
Then, one observes that

\[
S = \int (\Delta \nabla_x \Delta_y f - \Delta \nabla_x \tilde{\Delta}_y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \\
- \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \\
- \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \\
- \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy
\]

Noticing that the last term is nothing but \(-S(t)\), one finds that

\[
S = \frac{1}{2} \int (\nabla_x \Delta \Delta_y f - \nabla_x \tilde{\Delta}_y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \\
- \frac{1}{2} \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) - \cos(k \Delta_y f) \, dk \, dy \\
- \frac{1}{2} \int \nabla_x \Delta \tilde{\Delta}_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\tilde{\Delta}_y f)) - \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy
\]

Then, one observes that

\[
S = \frac{1}{2} \int (\nabla_x \Delta \Delta_y f - \nabla_x \tilde{\Delta}_y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \\
- \frac{1}{2} \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) + \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) - \cos(k \Delta_y f) \, dk \, dy \\
+ \frac{1}{2} \int \nabla_x (\Delta \Delta_y f - \Delta \tilde{\Delta}_y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \\
= -C_2
\]
Then, by noticing that the third and fourth terms cancel out, one finds that
\[ -\frac{1}{2} \int \nabla_x (\Delta \tilde{\Delta} y f - \Delta \Delta_y f) \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta} y f)) - \cos(\arctan(\Delta y f))) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \tilde{\Delta} y f)) \, dk \, dy \]
\[ \frac{-1}{2} \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta} y f)) - \cos(\arctan(\Delta y f))) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \tilde{\Delta} y f)) \, dk \, dy \]
\[ + \frac{1}{2} \int \nabla_x \Delta \Delta_y f \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta} y f)) - \cos(\arctan(\Delta y f))) \int_0^\infty e^{-k} \cos(k \tilde{\Delta} y f) \, dk \, dy \]

Therefore,
\[ S = \frac{1}{2} \int (\nabla_x \Delta \Delta_y f - \nabla_x \Delta \tilde{\Delta} y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta y f)) \int_0^\infty e^{-k} \cos(k \Delta y f) \, dk \, dy \, dx \]

\[ -\frac{1}{4} \int \Delta f \int \nabla_x \Delta \tilde{\Delta} y f \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\tilde{\Delta} y f)) + \cos(\arctan(\Delta y f)) \right) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \tilde{\Delta} y f)) \, dk \, dy \]
\[ + \frac{1}{4} \int \nabla_x (\Delta \tilde{\Delta} y f - \Delta \Delta_y f) \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta y f)) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \tilde{\Delta} y f)) \, dk \, dy \]
\[ -\frac{1}{4} \int \Delta \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta} y f)) - \cos(\arctan(\Delta y f))) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \tilde{\Delta} y f)) \, dk \, dy \]

Then, by noticing that the third and fourth terms cancel out, one finds that
\[ S = \frac{1}{2} \int \nabla_x \Delta \tilde{\Delta} y f - \nabla_x \Delta \tilde{\Delta} y f \cdot \frac{y}{|y|^2} \cos(\arctan(\Delta y f)) \int_0^\infty e^{-k} \cos(k \Delta y f) \, dk \, dy \]

\[ -\frac{1}{4} \int \nabla_x \Delta \tilde{\Delta} y f \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\tilde{\Delta} y f)) + \cos(\arctan(\Delta y f)) \right) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \tilde{\Delta} y f)) \, dk \, dy \]
\[ -\frac{1}{4} \int \Delta \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta} y f)) - \cos(\arctan(\Delta y f))) \]
\[ \times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \tilde{\Delta} y f)) \, dk \, dy \]

Then, one observes that the first term, namely,
Hence, to prove this identity, the idea is to try to symmetrize the integral (8.1). To this end, one writes that

\[
L := \frac{1}{2} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} \cos(\arctan(\Delta y f)) \int_0^\infty e^{-k} \cos(k \Delta y f) \, dk \, dy,
\]

may be rewritten as

\[
L = \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) - \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \Delta y f)) \, dk \, dy
\]

\[
+ \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) + \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \Delta y f)) \, dk \, dy.
\]

To prove this identity, the idea is to try to symmetrize the integral (8.1). To this end, one writes that

\[
L = \frac{1}{2} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) - \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} \cos(k \Delta y f) \, dk \, dy
\]

\[
+ \frac{1}{2} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} \cos(\arctan(\Delta y f))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \Delta y f)) \, dk \, dy
\]

\[
- \frac{1}{2} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} \cos(\arctan(\Delta y f)) \int_0^\infty e^{-k} \cos(k \Delta y f) \, dk \, dy.
\]

Noticing that the last integral is equal to \(-L\) one may symmetrize and find that

\[
L = \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) - \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \Delta y f)) \, dk \, dy
\]

\[
+ \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) + \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) + \cos(k \Delta y f)) \, dk \, dy.
\]

Hence,

\[
S = \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) - \cos(\arctan(\Delta y f)))
\]

\[
\times \int_0^\infty e^{-k} (\cos(k \Delta y f) - \cos(k \Delta y f)) \, dk \, dy
\]

\[
+ \frac{1}{8} \int (\nabla_x \Delta y f - \nabla_x \Delta y f, \frac{y}{|y|^2} (\cos(\arctan(\Delta y f)) + \cos(\arctan(\Delta y f)))
\]
Finally, by denoting $D_y f = \Delta_y f - \tilde{\Delta}_y f$ and $S_y f = \Delta_y f + \tilde{\Delta}_y f$ along with the use of trigonometry identities, we obtain the desired decomposition:

$$S = \frac{1}{2} \int \nabla_x \Delta y f \cdot \frac{y}{|y|^2} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \times \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \int \nabla_x \Delta y f \cdot \frac{y}{|y|^2} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \times \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \int \nabla_x \Delta y f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \times \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \frac{1}{2} \int \nabla_x \Delta y f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) (\cos \left( \frac{k}{2} (S_y f) \right) \, dk \, dy$$

$$:= \sum_{i=1}^4 \sigma_i(t).$$

This ends the proof the Lemma 8.1

In the sequel we shall use the notation $S_i := (\Delta f, \sigma_i)$

8.2. Estimate of $S_1$.

In this subsection, we are going to prove the following control of $S_1$. The main idea will be to transfer the regularity from the singular non-oscillatory term onto the oscillatory terms by using the regularity in $x$ and then write it in terms of $y$. In this subsection, we are going to prove the following estimate for $S_1$:

**Lemma 8.2.** The term $S_1$ is estimated as follows:

$$S_1 = \sum_{i=1}^7 S_{1,i} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}$$

(8.3)
Proof of Lemma 8.2. To estimate $S_1$ one first needs to integrate by parts

$$S_1 = \frac{1}{2} \int \Delta f \int \frac{\nabla_x (\delta_y \Delta f - \delta_y \Delta f)}{|y|} \cdot \frac{y}{|y|^2} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.$$ 

In order to make appear the more favorable second finite order differences it suffices to observe for instance that $\nabla_x (\delta_y f - \delta_y f) = -\nabla_x (\delta_y f + \delta_y f)$. Hence, we may integrate by parts (in $y$) and we find that

$$S_1 = \frac{1}{2} \int \Delta f \int \Delta s_y f \nabla_y \left( \frac{y}{|y|^2} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \right)$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.$$ 

(8.4)

Then, by using the identities (7.2) and (7.2), we find that

$$S_1 = \frac{1}{2} \int \Delta f \int \Delta s_y f \left( \nabla_y \frac{y}{|y|^2} \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) S_y f D_y f K_y f K_y f \cdot \nabla_y D_y f \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) (K_y f + K_y f) y \cdot \nabla_y S_y f \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) S_y f D_y f K_y f K_y f \cdot \nabla_y S_y f \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) (K_y f + K_y f) y \cdot \nabla_y D_y f \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)$$

$$\cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) y \cdot \nabla_y S_y f$$

$$\cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty k e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx$$

$$+ \frac{1}{4} \int \Delta f \int \Delta s_y f \left( \frac{y}{|y|^2} \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) y \cdot \nabla_y D_y f$$

$$\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) y \cdot \nabla_y D_y f \int_0^\infty k e^{-k} \sin \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right)$$

$$\, dk \, dy \, dx.$$
\[ S_{1,i} := \sum_i S_{1,i}. \]

### 8.2.1. Estimate of \( S_{1,1} \)

We have that

\[
S_{1,1} = \frac{1}{8} \int \frac{\Delta f}{|y|^3} \left( \gamma_y \sum_{i} S_{1,i} \right) \sin \left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\Delta_y f) \right) \sin \left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.
\]

Since \( \left| \nabla \cdot \frac{\gamma_y}{|y|^3} \right| \lesssim \frac{1}{|y|^3} \), we find that

\[
S_{1,1} \lesssim \frac{\Gamma(2)}{4} \left\| f \right\|_{\dot{H}^2} \left\| \Delta f \right\|_{L^2} \left\| s_y f \right\|_{L^\infty} \, dy \lesssim \frac{1}{2} \left\| f \right\|_{\dot{H}^2} \left\| \Delta f \right\|_{B^1_{2,2}} \left\| f \right\|_{B^3_{\infty,2}}.
\]

Then, since \( \dot{H}^{3/2} \hookrightarrow B^{3/2}_{\infty,2} \) one finds that

\[
S_{1,1} \lesssim \left\| f \right\|_{\dot{H}^{3/2}}^2 \left\| f \right\|_{\dot{H}^2}.
\]

### 8.2.2. Estimate of \( S_{1,2} \)

We have that

\[
S_{1,2} = \frac{1}{8} \int \frac{\Delta f}{|y|^3} \frac{s_y f}{|y|^3} \left( \gamma_y \sum_{i} S_{1,i} \right) \sin \left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\Delta_y f) \right) \sin \left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.
\]  

It is not really difficult to observe that

\[
S_{1,2} \lesssim \left| \Delta f \right| \left| \gamma_y \sum_{i} S_{1,i} \right| \sin \left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\Delta_y f) \right) \sin \left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.
\]

From the inequality \( \left| \frac{a^2 - b^2}{(1 + a^2)(1 + b^2)} \right| \leq 2 \) valid for any \((a, b) \in \mathbb{R}^2\) one gets that

\[
\left| \frac{S_y f D_y f}{(1 + \Delta_y^2 f)(1 + \Delta_y^2 f)} \right| \leq 2,
\]

which means that

\[
\left| S_y f D_y f \right| \leq 2.
\]
Therefore,

\[ S_{1,2} \lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^3} |y.\nabla_y D_y f| \, dy \, dx. \quad (8.8) \]

Then, using equality (7.9) and a classical scaling argument

\[
S_{1,2} \lesssim \|f\|_\dot{H}^2 \int_0^1 \left( \int \frac{\|s_y \Delta f\|_{L^2}^2}{|y|^3} \right)^{1/2} \left( \int \frac{\|s_{(r-1)y} \nabla_x f\|_{L^\infty}^2}{|y|^3} \, dy \right)^{1/2} dr \\
+ \|f\|_\dot{H}^1 \left( \int \frac{\|s_y \Delta f\|_{L^2}^2}{|y|^3} \, dy \right)^{1/2} \left( \frac{\|\nabla_x s_y f\|_{L^\infty}^2}{|y|^3} \, dy \right)^{1/2}.
\]

Hence,

\[
S_{1,2} \lesssim \|\Delta f\|_{L^2} \left( \int \frac{\|s_y \Delta f\|_{L^2}^2}{|y|^3} \right)^{1/2} \left( \int \frac{\|s_{(r-1)y} \nabla_x f\|_{L^\infty}^2}{|y|^3} \, dy \right)^{1/2} dr \\
+ \|\Delta f\|_{L^2} \left( \int \frac{\|s_y \Delta f\|_{L^2}^2}{|y|^3} \, dy \right)^{1/2} \left( \frac{\|\nabla_x s_y f\|_{L^\infty}^2}{|y|^3} \, dy \right)^{1/2}.
\]

Thus, that one finally finds that

\[
S_{1,2} \lesssim \|f\|_\dot{H}^2 \|\Delta f\|_\dot{H}^1/2 \|\nabla_x f\|_{\dot{B}^{1/2}_{\infty,2}} \\
\lesssim \|f\|_\dot{H}^2 \|s_y f\|_\dot{H}^2.
\]

8.2.3. Estimate of \( S_{1,3} \)

We have that

\[
S_{1,3} = -\frac{1}{8} \int \Delta f \int \frac{\Delta s_y f}{|y|^3} (K_y f + \tilde{K}_y f) \, y.\nabla_y S_y f \\
\times \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \\
\times \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx.
\]

Using that

\[
\left| \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \right| \leq \frac{\Gamma(2)}{2} |S_y f|,
\]

together with the bound

\[ |K_y f + \tilde{K}_y f| \leq 2, \quad (8.9) \]

one has that
\[
S_{1, 3} \lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^3} |y \nabla_y S_y f| \, dy \, dx.
\]

Now, we use the identity (7.10), that is,
\[
y \cdot \nabla_y S_y f = \frac{1}{|y|} s_y f(x) + \frac{y}{|y|} \nabla_s \delta_y f - \frac{y}{|y|} \nabla_x \delta_y f,
\]
so that
\[
S_{1, 3} \lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^3} |y \cdot \nabla_y S_y f| \, dy \, dx
\]
\[
\lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^4} |s_y f| \, dy \, dx
\]
\[
+ \int |\Delta f| \int \frac{|\Delta f|}{|y|^3} \left( |\nabla_x \delta_y f| + |\nabla_x \delta_y f| \right) \, dy \, dx
\]
\[
\lesssim \|f\|_{\dot{H}^2} \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^4} \|s_y f\|_{L^\infty} \, dy
\]
\[
+ \|f\|_{\dot{H}^2} \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \left( \|\nabla_x \delta_y f\|_{L^\infty} + \|\nabla_x \delta_y f\|_{L^\infty} \right) \, dy
\]
\[
\lesssim \|f\|_{\dot{H}^2} \int \frac{\|s_y \Delta f\|_{L^2}}{|y|^4} \|s_y f\|_{L^\infty} \, dy
\]
\[
+ \|f\|_{\dot{H}^2} \left( \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \, dy \int \frac{\|\nabla_x \delta_y f\|_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\]
\[
+ \|f\|_{\dot{H}^2} \left( \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \, dy \int \frac{\|\nabla_x \delta_y f\|_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\]
\[
\lesssim \|f\|_{\dot{H}^2} \left( \int \frac{\|s_y \Delta f\|_{L^2}}{|y|^3} \, dy \int \frac{\|s_y f\|_{L^\infty}}{|y|^5} \, dy \right)^{1/2}
\]
\[
+ \|f\|_{\dot{H}^2} \left( \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \, dy \int \frac{\|\nabla_x \delta_y f\|_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\]
\[
+ \|f\|_{\dot{H}^2} \left( \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \, dy \int \frac{\|\nabla_x \delta_y f\|_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\]
\[
\lesssim \|f\|_{\dot{H}^2} \left( \|\Delta f\|_{\dot{H}^{1/2}} \|f\|_{\dot{B}^{3/2}_{\infty, 2}} + \|\Delta f\|_{\dot{H}^{1/2}} \|\nabla_x f\|_{\dot{B}^{1/2}_{\infty, 2}} \right). \tag{8.10}
\]

By using classical Besov embeddings, one finally finds that
\[
S_{1, 3} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.
\]
8.2.4. Estimate of $S_{1,4}$  
We have, using the identity (7.9), together with the bound (8.7), that
\[
S_{1,4} = \frac{1}{8} \int \Delta f \int \frac{\Delta y f}{|y|^2} \left( S_y f D_y f K_y f \tilde{K}_y f + y.\nabla_y S_y f \right.
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)
\cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right)
\left. - \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \right)
\lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^2} \left| y.\nabla_y S_y f \right| \, dy \, dx.
\]
One observes that this last estimate is exactly the same as (8.10)
\[
S_{1,2} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|_{\dot{H}^2}.
\]

8.2.5. Estimate of $S_{1,5}$  
We have that
\[
S_{1,5} = -\frac{1}{8} \int \Delta f \int \frac{\Delta y f}{|y|^2} \left( K_y f + \tilde{K}_y f \right) y.\nabla_y D_y f \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)
\cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right)
\left. - \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \right)
\lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^2} \left| y.\nabla_y S_y f \right| \, dy \, dx.
\]
One notices that this last estimate is exactly the same as (8.10). Therefore, one directly infers that
\[
S_{1,5} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|_{\dot{H}^2}.
\]

8.2.6. Estimate of $S_{1,6}$  
Recall that
\[
S_{1,6} = \frac{1}{4} \int \Delta f \int \frac{s_y \Delta f}{|y|^3} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right)
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right)
\left. - \int_0^\infty k e^{-k} \cos \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \right)
\lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^3} \left| y.\nabla_y S_y f \right| \, dy \, dx.
\]
Then, it sufficient to notice that this term may be estimated by means of (8.8), so that
\[
S_{1,6} \lesssim \| f \|^2_{\dot{H}^{3/2}} \| f \|_{\dot{H}^2}.
\]
8.2.7. Estimate of $S_{1,7}$

\[
S_{1,7} = \frac{1}{4} \int \Delta f \int \frac{\Delta_y f}{|y|^3} \sin\left(\frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))\right) \\
\sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) y \cdot \nabla_y D_y f \\
\int_0^\infty k e^{-k} \sin\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) dk dy dx
\]

\[
\lesssim \int |\Delta f| \int \frac{|\Delta_y f|}{|y|^2} |y \cdot \nabla_y f| dy dx.
\]

Then, the analysis done for (8.8) allows one to get the same control as $S_{1,4}$, that is,

\[
S_{1,7} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.
\]

Finally, collecting all the estimates, we obtain that

\[
S_1 = \sum_{i=1}^{7} S_{1,i} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.
\] (8.11)

This ends the proof of Lemma 8.2 \hfill \square

8.3. Estimate of $S_2$

The term $S_2$ will be decomposed into several terms which involve the second finite order differences. The goal will be to prove the following estimate:

**Lemma 8.3.** The term $S_2$ is controlled as follows:

\[
S_2 \lesssim \|f\|_{\dot{H}^{5/2}}^2 (\|f\|_{\dot{H}^2} + \|f\|_{\dot{H}^2}^2).
\]

**Proof of Lemma 8.3.** Recall that

\[
S_2 = \int \Delta f \int \nabla_x \Delta_y f \cdot \frac{y}{|y|^2} \sin\left(\frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))\right) \\
\sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) dk dy dx.
\]

This term is too singular, we cannot estimate it directly. The idea is to try to balance the regularity in the space variable. More precisely, we write that

\[
S_2 = \int \Delta f \int \nabla_x \Delta f \cdot \frac{y}{|y|^2} \sin\left(\frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))\right) \\
\times \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) dk dy dx
\]
\[- \int \Delta f \int \nabla_x \Delta f (x - y) \frac{y}{|y|^2} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \\
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( k \frac{S_y f}{2} \right) \cos \left( k \frac{D_y f}{2} \right) \, dk \, dy \, dx.\]

Using that \( \Delta f \nabla_x \Delta f = \frac{1}{2} \nabla_x (\Delta f)^2 \), we may integrate by parts in \( x \) and get that

\[
S_2 = -\frac{1}{2} \int \Delta f \int \Delta \delta_y f \frac{y}{|y|^3} \cdot \nabla_x \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \\
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( k \frac{S_y f}{2} \right) \cos \left( k \frac{D_y f}{2} \right) \right) \, dx \, dy \, dk \\
- \frac{1}{2} \int \Delta f \int \nabla_x \Delta f (x - y) \frac{y}{|y|^3} \cdot \nabla_x \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \\
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( k \frac{S_y f}{2} \right) \cos \left( k \frac{D_y f}{2} \right) \right) \, dx \, dy \, dk \\
- \frac{1}{2} \int \Delta f \int \nabla_x \Delta f (x - y) \frac{y}{|y|^3} \cdot \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \\
\sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( k \frac{S_y f}{2} \right) \cos \left( k \frac{D_y f}{2} \right) \right) \, dx \, dy \, dk \\
= S_{2,1} + S_{2,2} + S_{2,3}. \tag{8.12}
\]

**8.3.1. Estimate of \( S_{2,1} \)** In order to control \( S_{2,1} \), one observes that, by setting

\[
T(f) := \nabla_x \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( k \frac{S_y f}{2} \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \cos \left( k \frac{D_y f}{2} \right) \, dk \, dy \right), \tag{8.13}
\]

one may easily notice that

\[
|T(f)| \lesssim \frac{\left| \nabla_x (f(x) - f(x \pm y)) \right|}{|y|} |R(f)|, \tag{8.14}
\]

where the operator \( R(f) \) is uniformly bounded by a fixed constant.

Now, set \( \delta_y^\pm f := f(x) - f(x \pm y) \), then
The following inequality holds: 

\[ S_{2,1} \lesssim \| \Delta f \|_{L^4} \int \frac{\| \Delta \delta_y f \|_{L^2}}{|y|^{3/2}} \frac{\| \nabla \delta_y f \|_{L^4}}{|y|^{3/2}} \, dy \]
\[ \lesssim \| f \|_{\dot{H}^2} \| \Delta f \|_{\dot{B}^{1/2}_{2,2}} \| \nabla f \|_{\dot{B}_{2,2}^{1/2}} \]
\[ \lesssim \| f \|_{\dot{H}^5/2} \| f \|_{\dot{H}^2}, \]

where we used the Sobolev embedding $\dot{H}^{1/2} \hookrightarrow L^4$ and the fact that $\dot{H}^1 \hookrightarrow \dot{B}_{4,2}^{1/2}$.

### 8.3.2. Estimate of $S_{2,2}$

Recall that

\[ S_{2,2} = -\frac{1}{2} \int \Delta f \int \Delta f (x - y) \frac{y}{|y|^3} \nabla_x \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \right) \times \]
\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \right) \, dk \, dy \, dx \]
\[ = -\frac{1}{2} \int \Delta f \int \Delta f (x - y) \frac{y}{|y|^3} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \nabla_x \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \right) \, dk \, dy \, dx \]
\[ = S_{2,2,1} + S_{2,2,2}. \]

- **Estimate of $S_{2,2,1}$**

To estimate the term $S_{2,2,1}$, it is not difficult to see that an estimate of the kind (8.14) does not work anymore. One needs to find a slightly more refined inequality. More precisely, we shall use

**Lemma 8.4.** The following inequality holds:

\[ |S_{2,2,1}| \lesssim \int |\Delta f| \int \frac{|\Delta f(x - y)|}{|y|^2} \left| \frac{\nabla_x (f(x) - f(x \pm y))}{|y|} \right| |S_y f| \, dx \, dy. \]

(8.15)

**Proof of Lemma 8.4.** Using twice the mean value theorem for instance, we have that \( |\sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \right) \| \leq |S_y f| \) then, if the derivative hits on one of the terms

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \right) \, dk \, dy, \]

it will be easily controlled by \( \left| \frac{\nabla_x (f(x) - f(x \pm y))}{|y|} \right| \), which proves that (8.15) holds. \( \square \)
Using Lemma 8.15 along with Sobolev embedding, we may estimate $S_{2,2,1}$ as follows:

$$|S_{2,2,1}| \lesssim \|\Delta f\|_{L^4} \|\Delta f\|_{L^2} \int \|\nabla_x (f(x) - f(x \pm y))\|_{L^4} \|s_y f\|_{L^\infty} \frac{dy}{|y|^{3/2}} \frac{dx}{|y|^{5/2}}$$

$$\lesssim \|f\|_{H^{5/2}} \|\nabla_x f\|_{B_{4,2}^{5/2}} \|\nabla_x f\|_{B_{\infty,2}^{5/2}}$$

$$\lesssim \|f\|_{H^{5/2}} \|\nabla_x f\|_{B_{4,2}^{5/2}} \|\nabla_x f\|_{B_{\infty,2}^{5/2}}$$

$$\lesssim \|f\|_{H^{5/2}} \|\nabla_x f\|_{B_{4,2}^{5/2}}$$

(8.16)

**Estimate of $S_{2,2,2}$**

We now estimate the more delicate term $S_{2,2,2}$, namely,

$$S_{2,2,2} = -\frac{1}{2} \int \Delta f \int \Delta f(x-y) \frac{y}{|y|^3} \cdot \nabla_x \left( \sin\left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f) \right) \right)$$

$$\sin\left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f) \right) \int_0^\infty e^{-k} \cos\left( \frac{k}{2} S_y f \right) \cos\left( \frac{k}{2} D_y f \right) dk \ dy \ dx.$$

To do thus, we shall use the fact that

$$\nabla_x \left( \sin\left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f) \right) \right) = \frac{1}{2} \left( \frac{\nabla_x \Delta_y f}{1 + \Delta^2_y f} + \frac{\nabla_x \tilde{\Delta}_y f}{1 + \tilde{\Delta}^2_y f} \right)$$

$$\times \cos\left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f) \right),$$

together with the fact that

$$\frac{\nabla_x \Delta_y f}{1 + \Delta^2_y f} + \frac{\nabla_x \tilde{\Delta}_y f}{1 + \tilde{\Delta}^2_y f} = \nabla_x S_y f + \nabla_x D_y f$$

$$\frac{\nabla_x S_y f}{(1 + \Delta^2_y f)(1 + \tilde{\Delta}^2_y f)} + \frac{\nabla_x D_y f}{(1 + \Delta^2_y f)(1 + \tilde{\Delta}^2_y f)}.$$

(8.17)

Hence, this decomposition gives rise to two terms, which are

$$S_{2,2,2} = -\frac{1}{2} \int \Delta f \int \Delta f(x-y) \frac{y}{|y|^3} \cdot \nabla_x D_y f \frac{S_y f \ D_y f}{(1 + \Delta^2_y f)(1 + \tilde{\Delta}^2_y f)}$$

$$\sin\left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f) \right) \int_0^\infty e^{-k} \cos\left( \frac{k}{2} S_y f \right) \cos\left( \frac{k}{2} D_y f \right) dk \ dy.$$

$$-\frac{1}{2} \int \Delta f \int \Delta f(x-y) \frac{y}{|y|^3} \cdot \frac{\nabla_x S_y f}{1 + \Delta^2_y f}$$

$$\sin\left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f) \right) \int_0^\infty e^{-k} \cos\left( \frac{k}{2} S_y f \right) \cos\left( \frac{k}{2} D_y f \right) dk \ dy$$

$$= S_{2,2,2,1} + S_{2,2,2,2}.$$

The analysis of the first term of this last equality can be done by means of the Lemma 8.4. Indeed, since

$$\frac{|D_y f|}{(1 + \Delta^2_y f)(1 + \tilde{\Delta}^2_y f)} < 1 \text{ and since we have that } |\nabla_x D_y f| \lesssim \frac{\nabla_x f(x) - f(x \pm y)}{|y|}.$$
The part involving the term $\frac{\nabla_x S_y f}{1 + \Delta_y f}$ in equation (8.17) is more delicate. The full term corresponds to $\mathcal{S}_{2,2,2}$. One shall use another strategy since there is an obvious lack of regularity. The idea is to try to balance the derivatives. Since the rational function in $\Delta_y f$ is not regular enough, one has to make appear oscillatory terms in order to avoid regularity issues. More precisely, we have

**Lemma 8.5.** The term $\mathcal{S}_{2,2,2}$ may be rewritten as follows:

$$\mathcal{S}_{2,2,2} = \frac{1}{4} \int \Delta f \int (\Delta f(x - y) - \Delta f(x + y)) \frac{y}{|y|^3} \nabla_x S_y f$$

$$\times \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2} S_y f)$$

$$\sin(\frac{\gamma}{2} D_y f) \int_0^\infty e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dk \, dy \, dx$$

$$- \frac{1}{4} \int \Delta f \int (\Delta f(x - y) + \Delta f(x + y)) \frac{y}{|y|^3} \nabla_x S_y f$$

$$\times \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-\gamma} (\cos(\gamma \Delta_y f) + \cos(\gamma \tilde{\Delta}_y f))$$

$$\int_0^\infty e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dk \, dy \, dx.$$

**Proof of Lemma 8.5.** First, recall that

$$\mathcal{S}_{2,2,2} = -\frac{1}{2} \int \Delta f \int (\Delta f(x - y) - \Delta f(x + y)) \frac{y}{|y|^3} \nabla_x S_y f$$

$$\sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dk \, dy.$$
\[
\int_0^\infty e^{-\gamma} \cos(\gamma \Delta_y f) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx.
\]

By doing the change of variable \(y \mapsto -y\), one observes that the last term is equal to \(-S_{2,2,2,2}\) and that the two first terms may be symmetrized. More precisely, we find that

\[
S_{2,2,2,2} = -\frac{1}{8} \Delta f \int (\Delta f (x - y) - \Delta f (x + y)) \frac{y}{|y|^3} \nabla_y S_y f \\
\times \sin\left(\frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \right) \int_0^\infty e^{-\gamma} \cos(\gamma \Delta_y f) - \cos(\gamma \Delta_y f) \, dy \, dk \, dy \, dx
\]

\[
-\frac{1}{4} \Delta f \int (\Delta f (x - y) + \Delta f (x + y)) \frac{y}{|y|^3} \nabla_y S_y f \\
\times \sin\left(\frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \right) \int_0^\infty e^{-\gamma} \cos(\gamma \Delta_y f) + \cos(\gamma \Delta_y f) \, dy \, dk \, dy \, dx
\]

This ends the proof of Lemma 8.4 \( \Box \)

Then, by using classical trigonometry formula and the fact that

\[
\Delta(f(x - y) - f(x + y)) = -\nabla_y \nabla_x S_y f
\]

and

\[
\Delta(f(x - y) + f(x + y)) = \nabla_y \nabla_x (f(x) - f(x - y) + f(x + y) - f(x)) = \nabla_y \nabla_x (\delta_y f - \bar{\delta}_y f),
\]

one may write that,

\[
S_{2,2,2,2} = -\frac{1}{4} \Delta f \int (\nabla_y \nabla_x S_y f) \frac{y}{|y|^3} \nabla_x S_y f \sin\left(\frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \right) \\
\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx
\]

\[
-\frac{1}{2} \Delta f \int (\nabla_y \nabla_x d_y f) \frac{y}{|y|^3} \nabla_x S_y f \sin\left(\frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \right) \\
\int_0^\infty e^{-\gamma} \cos\left(\frac{\gamma}{2} S_y f\right) \cos\left(\frac{\gamma}{2} D_y f\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx.
\]

Hence, one finds that

\[
S_{2,2,2,2} = -\frac{1}{4} \Delta f \int \nabla_y \nabla_x (s_y f + 2d_y f) \frac{y}{|y|^3} \nabla_x S_y f \\
\times \sin\left(\frac{1}{2} \left( \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \right)
\]
It is quite obvious that there is a lack of regularity $S_{2,2,2,2}$, indeed one needs to use the regularity in $y$ and therefore it is convenient by integrating by parts (with respect to $y$). The idea is to put more oscillation in the non-oscillatory term by adding the artificial term $2dy f$. One finds that

$$S_{2,2,2,2} = \frac{1}{4} \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right)$$

$$\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right)$$

$$\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx.$$

$$\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$

$$\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx$$

$$+ \frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x \nabla_y S_y f$$

$$\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$

$$\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x S_y f \int_0^\infty \gamma e^{-\gamma} \nabla_y D_y f$$

$$\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$

$$\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x S_y f \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right)$$

$$\nabla_y (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f)) \cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$

$$\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$

$$\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x S_y f, \nabla_y S_y f$$

$$\times \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right)$$
\[ \times \int_0^\infty ke^{-k \sin(\frac{k}{2}S_y f)} \cos(\frac{k}{2}D_y f) \, dy \, dk \, dy \, dx \]

\[-\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x S_y f \nabla_y f \]

\[ \times \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2}S_y f) \sin(\frac{\gamma}{2}D_y f) \sin(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \]

\[ \int_0^\infty ke^{-k \cos(\frac{k}{2}S_y f)} \sin(\frac{k}{2}D_y f) \, dy \, dk \, dy \, dx \]

\[ = \sum_i S_{2,2,2,i}. \]

One can now start estimating \( S_{2,2,2,i} \), \( i = 1, \ldots, 7 \).

- **Estimate of \( S_{2,2,2,1} \)**
  We start by estimating \( S_{2,2,2,1} \). Using Lemma 7.3 together with the fact that \( \dot{H}^{3/2} \hookrightarrow \dot{B}_{4,2}^{1/2} \), and by using that the kernel is not that singular one finds that

  \[ S_{2,2,2,1} = \frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \nabla_y \left( \frac{y}{|y|^3} \right) \nabla_x S_y f \]

  \[ \times \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2}S_y f) \sin(\frac{\gamma}{2}D_y f) \sin(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \]

  \[ \int_0^\infty e^{-k} \cos(\frac{k}{2}S_y f) \cos(\frac{k}{2}D_y f) \, dy \, dk \, dy \, dx \]

  \[ \lesssim \| \Delta f \|_{L^2} \int \frac{\| \nabla_x \delta_{\Delta_y f} \|_{L^\infty} \| \nabla_x S_y f \|_{L^2}}{|y|^{3/2}} \| \nabla_y S_y f \|_{L^2} \| \nabla_x S_y f \|_{L^2} \]

  \[ \lesssim \| f \|_{\dot{H}^{3/2}} \| f \|_{\dot{B}_{4,2}^{1/2}} \| f \|_{\dot{H}^{5/2}} \]

  \[ \lesssim \| f \|_{\dot{H}^{5/2}}^2 \| f \|_{\dot{H}^{2}} \]  \hspace{1cm} (8.18)  

- **Estimate of \( S_{2,2,2,2} \)**
  Using identity (7.11), one finds that

  \[ S_{2,2,2,2} = \frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x S_y f \]

  \[ \times \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2}S_y f) \sin(\frac{\gamma}{2}D_y f) \sin(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \]

  \[ \times \int_0^\infty e^{-k} \cos(\frac{k}{2}S_y f) \cos(\frac{k}{2}D_y f) \, dy \, dk \, dy \, dx \]

Then, we write that

\[ S_{2,2,2,2} = \frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x \left( \nabla_y \left( \frac{1}{|y|} \right) s_y f \right) \]

\[ \times \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2}S_y f) \sin(\frac{\gamma}{2}D_y f) \sin(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \]
\[
\times \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx \\
- \frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x \cdot (\nabla_x D_y f) \\
\times \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f))\right) \\
\times \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx \\
= S_{2,2,2,2,1} + S_{2,2,2,2,2}.
\]

One observes that the estimate of \( S_{2,2,2,2,1} \) is similar to \( S_{2,2,2,2,1} \) (see (8.18)). Indeed, we have that

\[
S_{2,2,2,2,1} \lesssim \|\Delta f\|_{L^2} \int \frac{\|\nabla_x \delta_{\gamma} f\|_{L^\infty} \|\nabla_x S_y f\|_{L^2}}{|y|^{3/2}} \, dy,
\]

hence,

\[
S_{2,2,2,2,1} \lesssim \|f\|^2_{\dot{H}^{5/2}} \|f\|_{\dot{H}^2}. \quad (8.19)
\]

As for \( S_{2,2,2,2,2} \), using Sobolev embedding and that \( \dot{H}^1 \hookrightarrow B^{1/2}_{4,2} \), we find that

\[
S_{2,2,2,2,2} \lesssim \|\Delta f\|_{L^4} \int \frac{\|\nabla_x \delta_{\gamma} f\|_{L^4} \|\Delta \delta_{\gamma} f\|_{L^2}}{|y|^{3/2}} \, dy \\
\lesssim \|\Delta f\|_{L^4} \|f\|_{B^{3/2}_{4,2}} \|f\|_{\dot{H}^{5/2}} \\
\lesssim \|f\|^2_{\dot{H}^{5/2}} \|f\|_{\dot{H}^2} \quad (8.20)
\]

Hence combining (8.19) and (8.20), one finds that

\[
S_{2,2,2,2,2} \lesssim \|f\|^2_{\dot{H}^{5/2}} \|f\|_{\dot{H}^2}.
\]

\* \textbf{Estimate of } \( S_{2,2,2,2,3} \)

First, recall that

\[
S_{2,2,2,2,3} = \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x S_y f \int_0^\infty \gamma e^{-\gamma} \nabla_y S_y f \\
\times \cos\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f))\right) \\
\times \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx.
\]

We split this term using identity (7.11), and we bound all the other oscillatory by 1, we find that

\[
S_{2,2,2,2,3} = \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f), \frac{y}{|y|^3} \nabla_x S_y f \nabla_y \left(\frac{1}{y}\right) s_y f \\
\times \int_0^\infty \gamma e^{-\gamma} \cos\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f))\right)
\]
\[ \int_0^{\infty} e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dy \, dk \, dx \]
\[-\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2 d_y f) \cdot \frac{\gamma}{|y|^3} \nabla_x S_y f \nabla_x D_y f \int_0^{\infty} \gamma e^{-\gamma} \cos(\frac{\gamma}{2} S_y f) \]
\[\times \sin(\frac{\gamma}{2} D_y f) \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))) \]
\[\times \int_0^{\infty} e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dy \, dk \, dx \]
\[= S_{2,2,2,3,1} + S_{2,2,2,3,2}. \]

For the term \( S_{2,2,2,2,3,1}, \) it suffices to use that \( \dot{H}^{\eta + 1/2} \hookrightarrow \dot{B}^k_{4,4} \) for \( \eta = 3/2 \) and \( \eta = 2, \) hence

\[ S_{2,2,2,2,3,1} = \frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2 d_y f) \cdot \frac{\gamma}{|y|^3} \nabla_x S_y f \nabla_x D_y f \int_0^{\infty} \gamma \]
\[\times e^{-\gamma} \cos(\frac{\gamma}{2} S_y f) \sin(\frac{\gamma}{2} D_y f) \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))) \]
\[\times \int_0^{\infty} e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dy \, dk \, dx \]
\[\lesssim \| \Delta f \|_{L^2} \left\| \frac{\nabla_x \delta^\pm f}{|y|} \frac{\| \nabla_x S_y f \|_{L^4}}{|y|^{3/2}} \frac{\| \nabla_x D_y f \|_{L^4}}{|y|^{3/2}} \right\|_{L^\infty} \]
\[\lesssim \| \Delta f \|_{L^2} \left\| f \right\|_{\dot{B}^{1/2}_{\infty,2}} \left( \int \left\| \frac{\nabla_x \delta^\pm f}{|y|^{4}} \frac{\| \nabla_x S_y f \|_{L^4}}{|y|^{3/2}} \right\|_{L^4} \frac{\| \nabla_x D_y f \|_{L^4}}{|y|^{3/2}} \right)^{1/4} \]
\[\lesssim \| f \|_{\dot{H}^1} \| f \|_{\dot{B}^{1/2}_{\infty,2}} \left\| f \right\|_{\dot{B}^{1/2}_{4,4}} \]
\[\lesssim \| f \|^2_{\dot{H}^{1/2}} \| f \|^2_{\dot{H}^2}. \quad (8.22) \]

In order to estimate \( S_{2,2,2,2,3,2}, \) we use that \( \dot{B}^{5/3}_{6,3} \hookrightarrow \dot{H}^{7/3} = [\dot{H}^{5/2}, \dot{H}^2]_{\frac{1}{2}, \frac{1}{2}}, \) so one finds that

\[ S_{2,2,2,2,3,2} = -\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2 d_y f) \cdot \frac{\gamma}{|y|^3} \nabla_x S_y f \nabla_x D_y f \]
\[\times \int_0^{\infty} \gamma e^{-\gamma} \cos(\frac{\gamma}{2} S_y f) \sin(\frac{\gamma}{2} D_y f) \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))) \]
\[\times \int_0^{\infty} e^{-k} \cos(\frac{k}{2} S_y f) \cos(\frac{k}{2} D_y f) \, dy \, dk \, dx \]
\[\lesssim \| \Delta f \|_{L^2} \left\| \frac{\nabla_x \delta^\pm f}{|y|^4} \frac{\| \nabla_x S_y f \|_{L^6}}{|y|^6} \right\|_{L^\infty} \]
\[\lesssim \| f \|_{\dot{H}^1} \| f \|^3_{\dot{B}^{7/3}_{6,3}} \]
\[\lesssim \| f \|^2_{\dot{H}^{1/2}} \| f \|^2_{\dot{H}^2}. \quad (8.24) \]

Hence, combining (8.21) and (8.23), we find that

\[ S_{2,2,2,2,3} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|^2_{\dot{H}^2}. \]
• Estimate of $S_{2,2,2,4}$

Using the identity (7.9), we may decompose $S_{2,2,2,4}$ as follows:

\[ S_{2,2,2,4} = \frac{1}{8} \int \Delta f \int \nabla_s (s_y f + 2d_y f) \frac{y}{|y|^3} \nabla_s S_y f \int_0^\infty \gamma e^{-\gamma} \]  
\[ \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) dy \]  
\[ = \frac{1}{8} \int \Delta f \int \nabla_s (s_y f + 2d_y f) \frac{1}{|y|^3} \nabla_s S_y f \frac{1}{|y|} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \]  
\[ \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) dy \]  
\[ = S_{2,2,2,4,1} + S_{2,2,2,4,2}. \]  

In order to estimate $S_{2,2,2,4,1}$, one uses an easy scaling argument for the integral in $r$, so that

\[ S_{2,2,2,4,1} = \frac{1}{8} \int \Delta f \int \nabla_s (s_y f + 2d_y f) \frac{1}{|y|^3} \nabla_s S_y f \frac{1}{|y|} \]  
\[ \int_0^1 \int_0^\infty y \cdot s_{(r-1)\gamma} \nabla_s f \gamma e^{-\gamma} \sin \left( \frac{y}{2} S_y f \right) \cos \left( \frac{y}{2} D_y f \right) \]  
\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \]  
\[ \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) dr \]  
\[ \lesssim \| f \|_{\dot{H}^2} \int \frac{\| \Delta_y \nabla_s f \|_{L^6}^3}{|y|^4} dy \]  
\[ \lesssim \| f \|_{\dot{H}^2} \| f \|_{B_{4,3}^{1/3}}^3 \]  
\[ \lesssim \| f \|_{\dot{H}^{5/2}}^2 \| f \|_{\dot{H}^2}^2. \]  

where we used again that $B_{3,6}^{5/3} \leftrightarrow \dot{H}^{7/3} = [\dot{H}^{5/2}, \dot{H}^2]_{\frac{1}{2}, 1}$. 

The estimate of $S_{2,2,2,4,2}$ is relatively easy, indeed, it suffices to observe that it is as regular as $S_{2,2,2,4,1}$. More precisely, we have that
\[ S_{2,2,2,4,2} \lesssim \| \Delta f \|_{L^2}^2 \int \frac{\| s_y \nabla_x f \|_{L^4}^2 \| s_y \nabla_x f \|_{L^6}}{|y|^{3/2}} \, dy \]
\[ \lesssim \| \Delta f \|_{L^2} \int \frac{\| \delta_y \nabla_y f \|_{L^6}^3}{|y|^4} \, dy \]
\[ \lesssim \| f \|_{H^{3/2}}^2 \| f \|_{H^2}^2. \]

- **Estimate of** \( S_{2,2,2,2,5} \)

Recall that
\[ S_{2,2,2,2,5} = -\frac{1}{4} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x S_y f \]
\[ \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f \right) \sin\left(\frac{\gamma}{2} D_y f \right) \nabla_y (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f)) \]
\[ \cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right) \]
\[ \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f \right) \cos\left(\frac{k}{2} D_y f \right) \, dy \, dk \, dy. \]

Using formula (7.2), we may decompose \( S_{2,2,2,2,5} \) as follows:
\[ S_{2,2,2,2,5,1} = -\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x S_y f \]
\[ \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f \right) \sin\left(\frac{\gamma}{2} D_y f \right) S_y f D_y f K_y f \bar{K}_y f \nabla_y S_y f \]
\[ \cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right) \]
\[ \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f \right) \cos\left(\frac{k}{2} D_y f \right) \, dy \, dk \, dy \]
\[ -\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x S_y f \]
\[ \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f \right) \sin\left(\frac{\gamma}{2} D_y f \right) (K_y f + \bar{K}_y f) \nabla_y D_y f \]
\[ \cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\bar{\Delta}_y f))\right) \]
\[ \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f \right) \cos\left(\frac{k}{2} D_y f \right) \, dy \, dk \, dy \]
\[ = S_{2,2,2,2,5,1} + S_{2,2,2,2,5,2}. \]

By means of inequality (8.7), one may write that
\[ S_{2,2,2,2,5,1} \lesssim \int |\Delta f| \int \frac{[\delta_y \nabla_y f] \| s_y \nabla_x f \|_{L^4}}{|y|^{3/2}} \| s_y \nabla_x f \|_{L^6} |y \nabla_y S_y f | \, dy. \tag{8.31} \]

Then, using the formula (7.10), one immediately finds that
\[ S_{2,2,2,5,1} \lesssim \int |\Delta f| \int \frac{|\delta_y^\pm \nabla_x f|}{|y|^{3/2}} \frac{|s_y \nabla_x f|}{|y|^{3/2}} |s_y f(x)| \, dy \, dx \\
+ \int |\Delta f| \int \frac{|\delta_y^\pm \nabla_x f|}{|y|^{3/2}} \frac{|s_y \nabla_x f|}{|y|^{3/2}} |\nabla_x \delta_y^\pm f| \, dy \, dx \\
\lesssim \|\Delta f\|_{L^2} \left( \int \frac{\|\nabla_x \delta_y^\pm f\|_{L^4}}{|y|^{3/2}} \frac{\|\nabla_x s_y f\|_{L^4}}{|y|^{3/2}} \frac{\|s_y f\|_{L^\infty}}{|y|^{5/2}} \, dy \right) \quad (8.32) \\
+ \int \frac{\|\delta_y^\pm \nabla_x f\|_{L^6}^3}{|y|^4} \, dy \quad (8.33) \]

To control (8.32), one may follow the same steps as (8.21) and (8.29), and therefore,
\[ S_{2,2,2,2,5,1} \lesssim \|f\|_{\bar{H}^{5/2}}^2 \|f\|_{H^2}^2. \quad (8.34) \]

As for \( S_{2,2,2,5,2} \), using that \(|K_y f + \tilde{K}_y f| \leq 2\), we may write that
\[ S_{2,2,2,5,2} \lesssim \int |\Delta f| \int \frac{|s_y \nabla_x f|^2}{|y|^4} |y, \nabla_y D_y f| \, dy \, dx. \]

Using formula (7.9), one finds that
\[ S_{2,2,2,5,2} \lesssim \int |\Delta f| \int \frac{|\delta_y^\pm \nabla_x f|}{|y|^{1/2}} \frac{|s_y \nabla_x f|}{|y|^{1/2}} \int_0^1 \left( |s_{(r-1)y} \nabla_x f| + |\nabla_x s_y f| \right) \, dr \, dy \, dx \\
\lesssim \|\Delta f\|_{L^2} \int \frac{\|\delta_y^\pm \nabla_x f\|_{L^6}^3}{|y|^4} \, dy, \]

where we use the same steps as in (8.29), and hence we have that
\[ S_{2,2,2,2,5} \lesssim \|f\|_{\bar{H}^{5/2}}^2 \|f\|_{H^2}^2. \]

- **Estimate of \( S_{2,2,2,2,6} \)**

Using identity (7.10), one finds that
\[ S_{2,2,2,2,6} = -\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \cdot \frac{y}{|y|^3} \nabla_x s_y f \cdot \nabla_y s_y f \\
\int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f\right) \sin\left(\frac{\gamma}{2} D_y f\right) \\
\sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \\
\int_0^\infty k e^{-k} \sin\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dk \, dy \, dx \\
\lesssim \int |\Delta f| \int \frac{|\delta_y^\pm \nabla_x f|}{|y|} \frac{|s_y \nabla_x f|}{|y|^{3/2}} |y, \nabla_y s_y f| \, dy \, dx. \]
Therefore, following the same steps as in (8.31), we obtain the same control as (8.34), that is,

\[ S_{2,2,2,6} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|^2_{\dot{H}^2}. \]

It remains to estimate the last term, that is,

\[ S_{2,2,2,2,7} = -\frac{1}{8} \int \Delta f \int \nabla_x (s_y f + 2d_y f) \frac{y}{|y|^3} \nabla_x S_y f \nabla_y D_y f \]

\[ \int_0^\infty e^{-\gamma} \sin\left(\frac{\gamma}{2} S_y f \right) \sin\left(\frac{\gamma}{2} D_y f \right) \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))\right) \]

\[ \int_0^\infty k e^{-k} \cos\left(\frac{k}{2} S_y f \right) \sin\left(\frac{k}{2} D_y f \right) d\gamma \, dk \, dy \, dx. \]

Up to some bounded harmless terms, \( S_{2,2,2,2,7} \) is analogous to \( S_{2,2,2,2,4} \) (see (8.25)), and therefore we may directly conclude that

\[ S_{2,2,2,2,7} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|^2_{\dot{H}^2}. \]

Finally, we have obtained that

\[ S_{2,2,2,2} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|^2_{\dot{H}^2}. \]

Hence, combining all the previous estimates, we conclude that

\[ S_{2,2} \lesssim \| f \|^2_{\dot{H}^{5/2}} \| f \|^2_{\dot{H}^2}. \]

### 8.3.3. Estimate of \( S_{3,3} \)

It remains to estimate \( S_{3,3} \), we have that

\[ S_{3,3} = -\int \Delta f \int \nabla_x \Delta f (x - y) \frac{y}{|y|^3} \sin\left(\frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta y f))\right) \]

\[ \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f \right) \cos\left(\frac{k}{2} D_y f \right) dk \, dy. \]

Unlike \( S_{2,1} \) and \( S_{2,2} \) there are no derivative in \( x \) in the oscillatory terms, so it cannot be treated in the same way as these terms. It is rather clear that the term \( \nabla_x \Delta f (x - y) \) is quite problematic. We would need a term of the kind \( f (x - y) - f (x) = -\delta_y f \) in stead of \( f (x - y) \). By using the fact that \( \Delta \nabla_x f (x - y) = -\Delta \nabla_y \delta_y f \), one may integrate by parts in \( y \) and obtain a kind of regularization of this term. More precisely, we have, by integrating by parts in \( y \), that

\[ S_{2,3} = -\int \Delta f \int \Delta \delta_y f \nabla_y \left( \frac{y}{|y|^3} \sin\left(\frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta y f))\right) \right) \]

\[ \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f \right) \cos\left(\frac{k}{2} D_y f \right) dk \, dy. \]

This term may be controlled exactly the same way as \( S_1 \) in (8.4). Indeed, the operator \( s_y \) in \( \Delta \delta_y f \) may be replaced by \( \Delta \delta_y f \). This is because of the fact that even if we would like to use the maximal regularity of \( \Delta s_y f \) the operator \( s_y \) would not
be helpful. Recall that $\dot{H}^{5/2}$ is the maximale regularity one can afford. Hence, if we replace $\Delta_s f$ by $\Delta \delta f$ it will give the same outcome. Moreover, the action of the differential operator $\nabla_y$, when one integrates by parts will give rise to the same terms up to some harmless bounded functions (essentially trigonometric functions and Gamma functions evaluated in special values). Therefore, we have the same control as the first term, namely,

$$S_{2,3} \lesssim \| f \|_{\dot{H}^{5/2}}^2 \| f \|_{\dot{H}^2}.$$  

(8.35)

Hence, we have proved that

$$S_2 \lesssim \| f \|_{\dot{H}^{5/2}}^2 \left( \| f \|_{\dot{H}^2} + \| f \|_{\dot{H}^2}^2 \right).$$  

(8.36)

Therefore, the proof of Lemma 8.3 is complete.

8.4. Estimate of $S_3$

The estimate of $S_3$ is analogous to $S_2$ that is the following Lemma holds. The term $S_2$ will be decomposed into several terms which involve the second finite order differences. The goal will be to prove

**Lemma 8.6.** The term $S_3$ is estimated as follows:

$$S_3 \lesssim \| f \|_{\dot{H}^{5/2}}^2 (\| f \|_{\dot{H}^2} + \| f \|_{\dot{H}^2}^2).$$

**Proof of Lemma 8.6.** Recall that we have that

$$S_3 = \int \nabla_x \Delta_y \nabla f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f + \arctan(\Delta_x f)) \right) \times \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f - \arctan(\Delta_x f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy. $$

If we do the change of variable $y \leftarrow -y$, then

$$S_3 = \int \nabla_x \Delta_y \nabla f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f + \arctan(\Delta_x f)) \right) \times \cos \left( \frac{1}{2} \left( \arctan(\Delta_y f - \arctan(\Delta_x f)) \right) \int_0^\infty e^{-k} \sin \left( \frac{k}{2} S_y f \right) \sin \left( \frac{k}{2} D_y f \right) \, dk \, dy. $$

Recall also that $S_3$ is

$$S_3 = \int \nabla_x \Delta \Delta_x f \cdot \frac{y}{|y|^2} \sin \left( \frac{1}{2} \left( \arctan(\Delta_x f + \arctan(\Delta_y f)) \right) \times \sin \left( \frac{1}{2} \left( \arctan(\Delta_x f - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy. $$

It is clear that they are equal up to interchanging the role of the sine and cosine functions. The role played by the oscillatory terms (that is all terms involving cosine and sine) in the estimate of $S_2$ was not important since we finally estimated these terms by 1. Also, one notice that importantly, $S_3$ and $S_2$ have the same symmetry
properties, that is, they are left invariant by the transformation $y \rightarrow -y$. Hence we may directly follow the same steps as the control of $S_2$ for the term $S_3$. We deduce that

$$S_3 \lesssim \|f\|^2_{\dot{H}^{5/2}} \left( \|f\|_{\dot{H}^2} + \|f\|^2_{\dot{H}^2} \right),$$

(8.37)

which is the desired estimated. 

8.5. Estimate of $S_4$

This term is fundamental in the sense that it contains the dissipation term. In order to extract the diffusive term, we have not only to linearize the oscillatory integrals but also to keep track of the directional derivative in the singular integral. Recall that

$$S_4 = \frac{1}{2} \int \Delta f \int \nabla_x \Delta D_y f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) + \arctan (\tilde{\Delta}_y f) \right) \right) \times \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) - \arctan (\tilde{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \cos \left( \frac{k}{2} (S_y f) \right) \, dk \, dy \, dx.$$

We are going to prove the following Lemma:

**Lemma 8.7.** The term $S_4$ is controlled as follows:

$$S_4 \lesssim -\frac{1}{2} \frac{1}{(1 + K(t)^2)^{3/2}} \|f\|^2_{\dot{H}^{5/2}} + \|f\|^2_{\dot{H}^2} \|f\|_{\dot{H}^2}.$$ 

(8.38)

Here, $K(t) = \sup_{x \in \mathbb{R}^2} |\nabla_x f|_{L^\infty}(t)$.

**Proof of Lemma 8.7.** In order to linearize we use the fact that $\cos(x) - 1 = -2 \sin^2 \left( \frac{x}{2} \right)$ twice, hence we may write

$$S_4 = \frac{1}{2} \int \Delta f \int \nabla_x \Delta D_y f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) + \arctan (\tilde{\Delta}_y f) \right) \right) \times \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) - \arctan (\tilde{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \cos \left( \frac{k}{2} (S_y f) \right) \, dk \, dy \, dx$$

$$- \int \Delta f \int \nabla_x \Delta D_y f \cdot \frac{y}{|y|^2} \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) + \arctan (\tilde{\Delta}_y f) \right) \right) \cos \left( \frac{1}{2} \left( \arctan (\Delta_y f) - \arctan (\tilde{\Delta}_y f) \right) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right)$$

$$\sin^2 \left( \frac{k}{4} (S_y f) \right) \, dk \, dy \, dx$$

$$= -\int \Delta f \int \nabla_x \Delta D_y f \cdot \frac{y}{|y|^2} \sin^2 \left( \frac{1}{4} \left( \arctan (\Delta_y f) + \arctan (\tilde{\Delta}_y f) \right) \right)$$
\[
\cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \times \int_0^\infty e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
- \int \Delta f \int \nabla_x \Delta f \cdot \frac{y}{|y|^2} \cos(\frac{1}{2}(\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))) \times \\
\cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-k} \cos(\frac{k}{2}(D_y f)) \\
\sin^2(\frac{k}{4}(S_y f)) \, dk \, dy \, dx \\
+ \frac{1}{2} \int \Delta f \int \nabla_x \Delta f \cdot \frac{y}{|y|^2} \cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \\
\int_0^\infty e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
= S_{4,1} + S_{4,2} + S_{4,3}.
\]

In the sequel, we are going to estimate each of the \( S_{4,i} \) and with a special attention on the term \( S_{4,3} \) which contains the elliptic component, the other terms being remainders. One the main difficulty in estimating the term \( S_{4,3} \) is to have estimate of the singular integral which does not depend on the direction.

### 8.5.1. Estimate of \( S_{4,1} \)

By integrating by parts, we find that

\[
S_{4,1} = \int \Delta f \int \frac{\Delta s_y f}{|y|^3} \sin^2(\frac{1}{4}(\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))) \times \\
\cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
- \int \Delta f \int \frac{\Delta s_y f}{|y|^3} y \cdot \nabla_y (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \\
\times \sin(\frac{1}{2}(\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))) \\
\cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
+ \int \Delta f \int \frac{\Delta s_y f}{|y|^3} \sin^2(\frac{1}{4}(\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))) \\
\times y \cdot \nabla_y (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)) \times \\
\sin((\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) \int_0^\infty e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
\times \cos(\frac{1}{2}(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))) y \cdot \nabla_y (D_y f) \\
\times \int_0^\infty k e^{-k} (\cos(\frac{k}{2}(D_y f))) \, dk \, dy \, dx \\
= S_{4,1,1} + S_{4,1,2} + S_{4,1,3} + S_{4,1,4}.
\]
In order to estimate $S_{4,1,1}$ we use the embedding $\dot{H}^{5/2} \hookrightarrow \dot{B}^{3/2}_{\infty,2}$, hence we get

$$S_{4,1,1} \lesssim \|f\|_{\dot{H}^2} \int \frac{\|\Delta s_y f\|_{L^2}}{|y|^3} \frac{\|s_y f\|_{L^\infty}}{|y|} \, dy$$

$$\lesssim \|f\|_{\dot{H}^2} \|\Delta f\|_{\dot{B}^{1/2}_{2,2}} \|f\|_{\dot{B}^{3/2}_{\infty,2}}$$

$$\lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.$$

The estimate of $S_{4,1,2}$ is not difficult since, it suffices, for instance, to use the formula (7.2), so we get that

$$S_{4,1,2} \lesssim \int |\Delta f| \int \frac{|s_y \Delta f|}{|y|^3} |y, \nabla_y D_y f| |S_y f D_y f| K_y f \tilde{K}_y f| \, dy \, dx.$$  

Using the same step as (8.5), we finally find that

$$S_{4,1,2} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.$$

It is not difficult to check that for $i = 2, 3, 4$, we have that

$$S_{4,1,i} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.$$

8.5.2. Estimate of $S_{4,2}$  

Recall that

$$S_{4,2} = -\int \Delta f \int \nabla_x \Delta D_y f, \frac{y}{|y|^2} \cos\left(\frac{1}{2} \left(\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f)\right)\right) \times$$

$$\cos\left(\frac{1}{2} \left(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)\right)\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} (D_y f)\right)$$

$$\sin^2\left(\frac{k}{4} (S_y f)\right) \, dk \, dy \, dx.$$  

By integration by parts, it is easy to estimate that

$$S_{4,2} \lesssim \|f\|_{\dot{H}^{5/2}}^2 \|f\|_{\dot{H}^2}.$$

8.5.3. Estimate of $S_{4,3}$  

Recall that

$$S_{4,3} = \frac{1}{2} \int \Delta f \int \nabla_x \Delta D_y f, \frac{y}{|y|^2} \cos\left(\frac{1}{2} \left(\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f)\right)\right)$$

$$\times \int_0^\infty e^{-k} \cos\left(\frac{k}{2} (D_y f)\right) \, dk \, dy \, dx. \quad (8.39)$$

This term is absolutely fundamental since it plays a central role in the analysis of the Cauchy problem in the critical Sobolev space. Indeed, it contains the competition between the elliptic term and the diffusion. Of course, to see this competition one has to go through the term via the actions of “symmetrization” operators giving
rise to sub-principal terms and the wanted ellipticity versus dissipative term. More 
precisely, one start by noticing that

$$S_{4,3} = -\frac{1}{2} \int \Delta f \int \frac{\Delta \delta_y f}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) - \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \times \int_0^\infty e^{-k \cos(k \frac{y}{|y|}, \nabla f(x))} \, dk \, dy \, dx \]

$$= -\frac{1}{2} \int \Delta f \int \frac{\Delta \delta_y f}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) + \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \times \int_0^\infty e^{-k \cos(k \frac{y}{|y|}, \nabla f(x))} \, dk \, dy \, dx. \]

Then, we write that

$$S_{4,3} = -\frac{1}{2} \int \Delta f \int \frac{\Delta \delta_y f}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) - \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \times \int_0^\infty e^{-k \cos(k \frac{y}{|y|}, \nabla f(x))} \, dk \, dy \, dx \]

$$- \frac{1}{4} \int \Delta f \int \frac{\Delta \delta_y f}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) + \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \int_0^\infty e^{-k \left( \cos(k \frac{y}{|y|}, \nabla f(x)) - \cos(k \frac{y}{|y|}, \nabla f(x - y)) \right)} \, dk \, dy \, dx \]

$$- \frac{1}{4} \int \Delta f \int \frac{\Delta \delta_y f}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) + \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \int_0^\infty e^{-k \left( \cos(k \frac{y}{|y|}, \nabla f(x)) + \cos(k \frac{y}{|y|}, \nabla f(x - y)) \right)} \, dk \, dy \, dx \]

$$= S_{4,3,1} + S_{4,3,2} + S_{4,3,3}. \]

We first remark that $S_{4,3,i}$, for $i = 1, 2$ are easy to control. Indeed, it suffices to see that

$$S_{4,3,i} \lesssim \|f\|_{\dot{H}^2} \int \frac{\|\delta_y \Delta f\|_{L^2}}{|y|^{3/2}} \frac{\|\nabla f\|_{L^\infty}}{|y|^{3/2}} \, dy$$

$$\lesssim \|f\|_{\dot{H}^2} \|f\|_{\dot{H}^5/2} \|f\|_{\dot{H}^3/2}^{3/2} \lesssim \|f\|_{\dot{H}^5/2} \|f\|_{\dot{H}^2}$$

As for $S_{4,3,3}$, we need to extract the dissipation via several symmetrizations. More precisely, one writes that

$$S_{4,3,3} = -\frac{1}{8} \int \int \frac{|\Delta \delta_y f|^2}{|y|^3} \left( \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x))) + \cos(\text{arctan}(\frac{y}{|y|}, \nabla f(x - y))) \right)$$

\[ \int_0^\infty e^{-k \left( \cos(k \frac{y}{|y|}, \nabla f(x)) + \cos(k \frac{y}{|y|}, \nabla f(x - y)) \right)} \, dk \, dy \, dx \]

$$= -\frac{1}{8} \int \int \frac{|\Delta \delta_y f|^2}{|y|^3} \left( \frac{1}{\sqrt{1 + (\frac{y}{|y|}, \nabla f(x))^2}} + \frac{1}{\sqrt{1 + (\frac{y}{|y|}, \nabla f(x - y))^2}} \right)$$

\[ \times \left( \frac{1}{\sqrt{1 + (\frac{y}{|y|}, \nabla f(x))^2}} + \frac{1}{\sqrt{1 + (\frac{y}{|y|}, \nabla f(x - y))^2}} \right) \, dx \, dy \]
\[
\begin{aligned}
&= \frac{1}{8} \int \int \frac{|\Delta \delta_y f|^2}{|y|^3} \\
&\quad \left( -\frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{3/2} + \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \frac{1}{3/2} \\
&\quad -2 \sqrt{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \right) \, dk \, dy \\
&= \frac{1}{8} \int \int \frac{|\Delta \delta_y f|^2}{|y|^3} \\
&\quad \times \left( -4 + \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{3/2} + \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \frac{1}{3/2} \\
&\quad -2 \sqrt{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \right) \, dy \, dx \\
&\leq -\frac{1}{2} \| f \|^2_{H^{5/2}} + \frac{1}{8} \int \int \frac{|\Delta \delta_y f|^2}{|y|^3} \\
&\quad \times \left( 4 - \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{3/2} + \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \frac{1}{3/2} \\
&\quad -2 \sqrt{1 + (\frac{y}{|y|} \cdot \nabla f(x-y))^2} \frac{1}{1 + (\frac{y}{|y|} \cdot \nabla f(x))^2} \right) \, dy \, dx \\
&\leq -\frac{1}{2} \| f \|^2_{H^{5/2}} + \frac{1}{8} \| f \|^2_{H^{5/2}} \left( 4 - \frac{1}{(1 + K^2)^{3/2}} \right). \\
&\equiv -\frac{1}{2} \| f \|^2_{H^{5/2}} + \frac{1}{8} \| f \|^2_{H^{5/2}} \left( 1 - \frac{1}{(1 + K^2)^{3/2}} \right).
\end{aligned}
\]

Hence, one finally finds that

\[ S_{4,3} \leq -\frac{1}{2} \| f \|^2_{H^{5/2}} + \frac{1}{8} \| f \|^2_{H^{5/2}} \left( 1 - \frac{1}{(1 + K^2)^{3/2}} \right), \quad (8.40) \]

where \( K(t) = \sup_{x \in \mathbb{R}^2} |\nabla_x f| L^\infty (t). \)

Hence, we have proved that

\[ S_4 \lesssim -\frac{1}{2} \| f \|^2_{H^{5/2}} + \| f \|^2_{H^{5/2}} \| f \|_{\dot{H}^2}. \quad (8.41) \]

This end the proof of the estimate of \( S_4. \)

\[ \square \]

**Remark 6** It is crucial to note that estimate (8.40) above shows the parabolic character of the Muskat problem whenever the slope does not blow-up. Indeed, when \( K(t) \to \infty, \) the regularizing effect disappears (as was also observed in the two dimensional case [29]).
9. Sobolev Energy Inequality

Combining all the estimates obtained in Lemmas 6.4, 8.2, 8.3, 8.6 and 8.7, we have proved that
\[
\frac{1}{2} \partial_t \| f \|_{\dot{H}^2}^2 + \frac{1}{2(1 + K(t)^2)^{3/2}} \| f \|_{\dot{H}^{5/2}}^2 \lesssim \| f \|_{\dot{H}^{5/2}}^2 \left( \| f \|_{\dot{H}^2} + \| f \|_{\dot{H}^2}^2 \right). \tag{9.1}
\]
Integrating in time \( s \in [0, T] \), one finds that
\[
\| f(\cdot, t) \|_{\dot{H}^2}^2 + \frac{1}{(1 + K^2)^{3/2}} \int_0^T \| f \|_{\dot{H}^{5/2}}^2 \, ds \lesssim \| f_0 \|_{\dot{H}^2}^2 + \int_0^T \| f \|_{\dot{H}^{5/2}}^2 \left( \| f \|_{\dot{H}^2} + \| f \|_{\dot{H}^2}^2 \right) \, ds,
\]
where \( K = \sup_{t \geq 0} \sup_{x \in \mathbb{R}^2} |\nabla_x f(x, t)|. \)
\( \Box \)

10. Slope Control and Uniform Bound Using Control of Slope

In this section we show how to control \( \| \nabla f \|_{L^\infty} \) in terms of critical Sobolev norms. More precisely, we have the following Lemma:

**Lemma 10.1.** Let \( f \) be a solution to the three dimensional Muskat equation with initial data \( f_0 \in \dot{H}^2 \cap \dot{W}^{1,\infty} \), then one has the following control of the Lipschitz semi-norm
\[
\| \nabla f \|_{L^\infty}^2(t) \leq \| \nabla f_0 \|_{L^\infty}^2 + \int_0^t \| f \|_{\dot{H}^{5/2}}^2(s) \, ds. \tag{10.1}
\]

**Proof of Lemma 10.1.** Recall that the Muskat problem can be written as follows:
\[
\partial_t f(t, x) = P.V. \int \nabla f(x) \cdot y - (f(x) - f(x - y)) \frac{dy}{|y|^3} \frac{1}{(1 + \Delta_x^2 f(x))^{3/2}}. \tag{10.2}
\]
By taking one derivative in equation (10.2) one finds that
\[
\partial_j f(t, x) = \nabla \partial_j f(x) \cdot P.V. \int \left[ \frac{\partial_j f(x)}{|y|^3} - \frac{\Delta_y f(x)}{|y|} \right] \frac{dy}{(1 + \Delta_x^2 f(x))^{3/2}} - \frac{\partial_j f(x)}{|y|} \Delta_y \partial_j f(x) \frac{dy}{(1 + \Delta_x^2 f(x))^{3/2}}.
\]
Set \( M(t) = \sup_{x \in \mathbb{R}^2} \partial_j f(x, t) \). Since we are considering a regular solution for example \( f(t, \cdot) \in C^2 \), we have that \( M(t) = \sup_{x \in \mathbb{R}^2} \partial_j f(x, t) = \partial_j f(x, t) \) and that \( M'(t) = \partial_j f_t(x, t) \) are differentiable almost every time \( t \) (thanks to Rademacher’s theorem). By evaluating the above evolution equation at \( x = x_t \) one finds that the first term on
the right is zero and the second has a sign. Omitting to write the \( p.v. \), for simplicity, we find that

\[
M'(t) \leq -3 \int \frac{\nabla f(x_t) \cdot \frac{y}{|y|} - \Delta_y f(x_t)}{|y|} dy - \frac{\Delta_y f(x_t)}{(1 + \Delta^2 f(x_t))^{3/2}} dy
\]

\[
2 \left( \int \| \nabla f(x_t) \cdot \frac{y}{|y|} - \Delta_y f(x_t) \|_{L^\infty} \| \partial_y \delta_y f(x_t) \|_{L^\infty} \frac{dy}{|y|^2} \right)^{1/2} \left( \int \| \partial_y \delta_y f(x_t) \|_{L^\infty}^2 \frac{dy}{|y|^3} \right)^{1/2}.
\]

Hence,

\[
M'(t) \lesssim \| \nabla f \|_{L^\infty}^2 \lesssim \| f \|_{H^{5/2}}^2.
\]

Analogously, the same holds for the evolution of the minimum \( m(t) \), so that, by integrating in time,

\[
\| \nabla f \|_{L^\infty}^2(t) \leq \| \nabla f_0 \|_{L^\infty}^2 + \int_0^t \| f \|_{H^{5/2}}^2(s) ds.
\]

From the Sobolev energy inequality of the previous section, we have that

\[
\partial_t \| f \|_{H^2}^2(t) + \frac{\| f \|_{H^{5/2}}^2}{(1 + \| \nabla f \|_{L^\infty}^2)^{3/2}} \leq C \| f \|_{H^{5/2}}^2 \left( \| f \|_{H^2}(t) + \| f \|_{H^2}^2(t) \right),
\]

where \( C > 0 \) is a fixed constant. Since \((1 + x^2 + D(t))^{-3/2} \leq (1 + x^2)^{-3/2}\) for any \( D(t) \geq 0 \), from inequality (10.3), we obtain that

\[
\partial_t \| f \|_{H^2}^2 + \frac{\| f \|_{H^{5/2}}^2}{(1 + \| \nabla f_0 \|_{L^\infty}^2 + D(t))^{3/2}} \leq C \| f \|_{H^{5/2}}^2 \left( \| f \|_{H^2}(t) + \| f \|_{H^2}^2(t) \right),
\]

where

\[
D(t) = \int_0^t \| f \|_{H^{5/2}}^2(s) ds, \quad \text{with} \quad D(0) = 0.
\]

We consider the smallness conditions (to get control of the \( L^2 H^{5/2} \) semi-norm) for \( \| f_0 \|_{H^2} \) given by

\[
C(\| f_0 \|_{H^2} + \| f_0 \|_{H^2}^2) < \frac{1}{(2 + \| \nabla f_0 \|_{L^\infty}^2)^{3/2}},
\]

together with

\[
\frac{\| f_0 \|_{H^2}^2(2 + \| \nabla f_0 \|_{L^\infty}^2)^{3/2}}{1 - C(\| f_0 \|_{H^2} + \| f_0 \|_{H^2}^2)(2 + \| \nabla f_0 \|_{L^\infty}^2)^{3/2}} < 1.
\]
Therefore, after a short amount of time,
\[ \partial_t \| f \|_{H^2}^2 < 0, \] together with \( D(t) < 1. \)

By integrating in time,
\[ \| f \|_{H^2}^2(t) + \left( \frac{1}{(2 + \| \nabla f_0 \|_{L^\infty}^2)^{3/2}} - C \left( \| f_0 \|_{H^2} + \| f_0 \|_{H^2}^2 \right) \right) D(t) \leq \| f_0 \|_{H^2}^2, \]
so that by a bootstrapping the argument, we are able to find the above identity for all time \( t > 0 \) so that
\[ \| f \|_{H^2}^2(t) \leq \| f_0 \|_{H^2}^2, \] together with \( D(t) < 1. \)

Assuming that there exists a first time \( t^* \) such that \( D(t^*) = 1 \), gives a contradiction. This ends the proof of Lemma 10.1. \( \Box \)

11. Uniqueness

We are going to prove the following Lemma which will imply the uniqueness:

**Lemma 11.1.** Let \( f \) and \( g \) be two solutions to the three dimensional Muskat equation with the same initial data and such that they belong to the space \( C([0, T], W^{1, \infty} \cap H^2) \cap L^2([0, T], H^2). \) Set \( U := f - g, \) then \( U \) verifies the Grönwall’s type inequality
\[ \| U \|_{L^\infty H^1} \leq \| U_0 \|_{H^1} \exp \left( c(K) \left( \| f \|_{L^\infty H^2}^2 + \| g \|_{L^\infty H^2}^2 \right) \left( \| f \|_{L^2 H^2}^2 + \| g \|_{L^2 H^2}^2 \right) \right), \]
where \( K \) is the space-time Lipschitz semi-norm.

**Proof of Lemma 11.1.** Let \( f \) and \( g \) be two solutions of the three dimensional Muskat equation with the same initial data. Then \( U \) verifies that
\[ \partial_t U = \int \Delta_y \nabla_y U \cdot \frac{y}{|y|^2} \int_0^\infty e^{-k} \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \, dk \, dy \]
\[ + \int \Delta_y \nabla_y g \cdot \frac{y}{|y|^2} \int_0^\infty e^{-k} \left[ \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \right. \]
\[ \left. - \cos(k \Delta_y g) \cos(\arctan(\Delta_y g)) \right] \, dk \, dy \]

We shall do estimates in \( H^1 \) on \( U. \) That is, we dot multiply the gradient of the evolution equation with \( \nabla U \) and integrate with respect to the space variable. We obtain
\[ \frac{1}{2} \partial_t \| U \|_{H^1}^2 = \int \nabla U \cdot \nabla \left( \Delta_y \nabla U \cdot \frac{y}{|y|^2} \int_0^\infty e^{-k} \cos(k \Delta_y f) \right. \]
\[ \left. \cos(\arctan(\Delta_y f)) \right) \, dk \, dy \]
\[ + \int \nabla U \cdot \nabla \left( \Delta_y \nabla g \cdot \frac{y}{|y|^2} \right. \]
\[ \times \int_0^\infty e^{-k} \left[ \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \right. \]
\[ \left. - \cos(k \Delta_y g) \cos(\arctan(\Delta_y g)) \right] \, dk \, dy \]
\[ = A_1 + A_2. \]
11.1. Estimate of $A_1$

We first notice that the most singular term is when the gradient hits $\Delta_y \nabla U$ and a reminder which corresponds to the term where the gradient hits the oscillatory integrals. We have that

$$A_1 = \int \nabla U \nabla \left( \int \Delta_y \nabla U \frac{y}{|y|^2} \int_0^\infty e^{-k} \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \, dk \, dy \, dx \right)$$

$$= \int \nabla U \nabla \left( \int \Delta_y \nabla U \frac{y}{|y|^2} \int_0^\infty e^{-k} \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \, dk \, dy \, dx \right)$$

$$+ \int \nabla U \nabla \left( \Delta_y \nabla U \frac{y}{|y|^2} \int_0^\infty e^{-k} \nabla \left( \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \right) \, dk \, dy \, dx \right)$$

$$= A_{1,1} + A_{1,2}.$$

We start by estimating the more singular term, that is $A_{1,1}$. We use the a priori estimates in $\tilde{H}^2$ obtained previously and replace the first two $\Delta f$ by $\nabla U$. We immediately find that

$$A_{1,1} = \int \nabla U \nabla \Delta_y \nabla U \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos(k \Delta_y f) \, dk \, dy \, dx .$$

Again, since the first Laplacian operator $\Delta$ does not play any role in the proof of the Lemma 8.2. We may replace it by the nabla operator $\nabla$ and get that

$$A_{1,1} = \frac{1}{2} \int \nabla U \nabla \nabla \Delta_y \nabla U \frac{y}{|y|^2} \sin\left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times$$

$$\sin\left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \sin\left( \frac{k}{2} S_y f \right) \sin\left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \int \nabla U \nabla \Delta_y \nabla U \frac{y}{|y|^2} \cos\left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times$$

$$\cos\left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \sin\left( \frac{k}{2} S_y f \right) \sin\left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \int \nabla U \nabla \Delta_y \nabla U \frac{y}{|y|^2} \sin\left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times$$

$$\sin\left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos\left( \frac{k}{2} S_y f \right) \cos\left( \frac{k}{2} D_y f \right) \, dk \, dy$$

$$+ \frac{1}{2} \int \nabla U \nabla \nabla \Delta_y \nabla U \frac{y}{|y|^2} \cos\left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k}$$

$$\cos\left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \cos\left( \frac{k}{2} (D_y f) \right) \cos\left( \frac{k}{2} (S_y f) \right) \, dk \, dy$$

$$:= \sum_{i=1}^{4} A_{1,1,i}(f).$$

To estimates $A_{1,1,1}$, we integrate by parts in $y$ and then estimate. The first term is when we differentiate the kernel, that is,
\[ A_{1,1.1} = \frac{1}{2} \int \nabla U \int (s_y \nabla U) \left( \nabla \frac{y}{|y|^3} \right) \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta_y} f)) \right) \] 

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta_y} f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \]

\[ + \frac{1}{2} \int \nabla U \int (s_y \nabla U) \frac{1}{|y|^2} \nabla_y \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta_y} f)) \right) \right) \]

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta_y} f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \]

\[ = A_{1,1.1.1} + A_{1,1.1.2} \]

Since \[ \nabla \frac{y}{|y|^3} \lesssim \frac{1}{|y|^4} \], we find that,

\[ A_{1,1.1.1} \lesssim \|U\|_{L^2}^2 \int \frac{\|s_y f\|^2_{L^\infty}}{|y|^5} \, dy \]

\[ \lesssim \|\nabla U\|_{L^2}^2 \|f\|_{B^{3/2}}^2 \]

\[ \lesssim \|U\|_{H^1}^2 \|f\|_{H^{5/2}}^2 \]

Otherwise, we would differentiate one of the oscillatory terms. In this case, we use Holder \((L^2 - L^2 - L^\infty - L^\infty)\) where one of the \(L^\infty\) norm will necessary be in a term of order \(\nabla_x \delta_\alpha f\) and the other one in any of the \(s_\alpha f\). Thus, one finds that

\[ A_{1,1.1.2} \lesssim \|\nabla U\|_{L^2}^2 \int \frac{\|s_y f\|^2_{L^\infty}}{|y|^4} \, dy \]

\[ \lesssim \|\nabla U\|_{L^2}^2 \left( \int \frac{\|s_y f\|^2_{L^\infty}}{|y|^5} \, dy \int \frac{\|\delta_y \nabla f\|^2_{L^\infty}}{|y|^3} \, dy \right)^{1/2} \]

\[ \lesssim \|\nabla U\|_{L^2}^2 \|f\|_{B^{3/2}} \|\nabla f\|_{B^{1/2}}^2 \]

\[ \lesssim \|U\|_{H^1}^2 \|f\|_{H^{5/2}}^2 \]

**11.1.1. Estimate of \(A_{1,2}\)** Now, we estimates \(A_{1,2}\). We use the decomposition previously proved (see (8.12)). We analogously find that

\[ A_{1,2.1} = -\frac{1}{2} \int \nabla U \int \delta_y \nabla y \frac{y}{|y|^3} \cdot \nabla \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta_y} f)) \right) \right) \times \]

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta_y} f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \]

\[ - \frac{1}{2} \int \nabla U \int \nabla U(x - y) \frac{y}{|y|^3} \cdot \nabla \left( \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta_y} f)) \right) \right) \times \]

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta_y} f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \]

\[ - \int \nabla U \int \nabla U(x - y) \cdot \frac{y}{|y|^3} \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\tilde{\Delta_y} f)) \right) \times \]

\[ \sin \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta_y} f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \]

\[ = A_{1,2.1} + A_{1,2.2} + A_{1,2.3}. \]
To estimate $A_{1,1,2,1}$, we write
\[
A_{1,1,2,1} \lesssim \|\nabla U\|_{L^2}^2 \int \frac{\|\delta_y \nabla U\|_{L^2} \|\nabla \delta_y f\|_{L^\infty}}{|y|^{3/2}} \, dy
\lesssim \|\nabla U\|_{\dot{H}^2} \|\nabla U\|_{\dot{B}^{1/2}_{2,2}} \|\nabla f\|_{\dot{B}^{1/2}_{2,2}}
\lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^{5/2}},
\]
where $\delta_y f := f(x) - f(x \pm y)$.
The estimate of $A_{1,1,2,2}$ is derived by using the decomposition
\[
A_{1,1,2,2} = -\frac{1}{2} \int \nabla U \int \nabla U(x - y) \frac{y}{|y|^3} \nabla_x \left( \sin \left( \frac{1}{2} \arctan(\Delta_y f) \right) + \arctan(\Delta_y f)) \times \sin \left( \frac{1}{2} \arctan(\Delta_y f) - \arctan(\Delta_y f) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \right)
\]
\[
= -\frac{1}{2} \int \nabla U \int \nabla U(x - y) \frac{y}{|y|^3} \cdot \nabla_x \left( \sin \left( \frac{1}{2} \arctan(\Delta_y f) \right) - \arctan(\Delta_y f)) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} S_y f \right) \cos \left( \frac{k}{2} D_y f \right) \, dk \, dy \, dx \right)
\]
\[
= A_{1,1,2,2,1} + A_{1,1,2,2,2}.
\]
$A_{1,1,2,2,1}$ is easy to estimate, indeed, we have that
\[
A_{1,1,2,2,1} \lesssim \|\nabla U\|_{L^2}^2 \int \frac{\|s_y f\|_{L^\infty} \|\delta_y \nabla f\|_{L^\infty}}{|y|^4} \, dy
\lesssim \|\nabla U\|_{L^2}^2 \left( \int \frac{\|s_y f\|_{L^\infty}^2}{|y|^4} \, dy \int \frac{\|\delta_y \nabla f\|_{L^\infty}^2}{|y|^3} \, dy \right)^{1/2}
\lesssim \|\nabla U\|_{L^2}^2 \|f\|_{\dot{B}^{3/2}_{2,2}} \|\nabla f\|_{\dot{B}^{1/2}_{2,2}}
\lesssim \|U\|_{\dot{H}^1} \|f\|_{\dot{H}^{5/2}}.
\]
As for $A_{1,1,2,2,2}$, we use the fact that $\nabla_x \left( \sin \left( \frac{1}{2} \arctan(\Delta_y f) + \arctan(\Delta_y f) \right) \right)$ may be written as follows (see (8.17)):
\[
\frac{\nabla_x \Delta_y f}{1 + \Delta_y f} + \frac{\nabla_x \Delta_y f}{1 + \Delta_y f} = \frac{\nabla_x s_y f}{1 + \Delta_y f} + \frac{\nabla_x D_y f}{1 + \Delta_y f} \frac{s_y f}{1 + \Delta_y f} \frac{D_y f}{1 + \Delta_y f}.
\]
The latter identity gives two terms which we call $A_{1,1,2,2,2,1}$ and $A_{1,1,2,2,2,2}$, namely,
\[
A_{1,1,2,2,2,1} = -\frac{1}{2} \int \nabla U \int \nabla U (x-y) \frac{y}{|y|^3} \cdot \frac{\nabla_x S_y f}{1 + \Delta^2_y f} \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))) \int_0^\infty e^{-k} \cos(k/2S_y f) \cos(k/2D_y f) \, dk \, dy \, dx
\]
and
\[
A_{1,1,2,2,2,2} = -\frac{1}{2} \int \nabla U \int \nabla U (x-y) \frac{y}{|y|^3} \cdot \frac{\nabla_x D_y f - S_y f}{(1 + \Delta^2_y f)(1 + \Delta^2_y f)} \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))) \int_0^\infty e^{-k} \cos(k/2S_y f) \cos(k/2D_y f) \, dk \, dy \, dx.
\]

To estimate $A_{1,1,2,2,2,1}$ we use the following lemma, whose proof is completely analogous to Lemma 8.5:

**Lemma 11.2.** The term $A_{1,1,2,2,2,2}$ may be rewritten as
\[
A_{1,1,2,2,2,2} = \frac{1}{4} \int \nabla U \int (\nabla U (x-y) - \nabla U (x+y)) \frac{y}{|y|^3} \cdot \nabla_x S_y f \\
\times \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))) \int_0^\infty e^{-\gamma} \sin(\frac{\gamma}{2} S_y f) \\
\sin(\frac{\gamma}{2} D_y f) \int_0^\infty e^{-k} \cos(k/2S_y f) \cos(k/2D_y f) \, dy \, dk \, dy \, dx
\]
\[
- \frac{1}{2} \int \nabla U \int \nabla U (x-y) \frac{y}{|y|^3} \cdot \nabla_x S_y f \sin(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta y f))) \times \int_0^\infty e^{-\gamma} \cos(\frac{\gamma}{2} S_y f) \cos(\frac{\gamma}{2} D_y f) \\
\int_0^\infty e^{-k} \cos(k/2S_y f) \cos(k/2D_y f) \, dy \, dk \, dy \, dx
\]
\[
= A_{1,1,2,2,2,2,1} + A_{1,1,2,2,2,2,2}.
\]

**Proof of Lemma 11.2.** It suffices to follow the same steps as the proof of Lemma 8.5.

The first term $A_{1,1,2,2,2,1}$ is easy to estimated, indeed, it suffices to observe that
\[
A_{1,1,2,2,2,1} \lesssim \|\nabla U\|^2_{L^2} \int \frac{\|s_y f\|_{L^\infty} \|\delta_y \nabla f\|_{L^\infty}}{|y|^4} \, dy
\]
\[
\lesssim \|\nabla U\|^2_{L^2} \left( \int \frac{\|s_y f\|^2_{L^\infty}}{|y|^5} \, dy \int \frac{\|\delta_y \nabla f\|^2_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\]
\[
\lesssim \|\nabla U\|^2_{L^2} \|f\|_{B^{3/2}_{\infty,2}} \|\nabla f\|_{B^{1/2}_{\infty,2}}
\]
\[
\lesssim \|U\|^2_{H^{3/2}} \|f\|^2_{H^{3/2}}.
\]
As for the second one, we first observe that by using the change of variables \( y \to -y \), one obtains that

\[
\mathcal{A}_{1.1,2,2,2,2} = -\frac{1}{4} \int \nabla U \int \nabla (U(x + y)) \frac{y}{|y|^3} \nabla_x S_y f \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \\
\times \int_0^\infty e^{-\gamma} \cos\left(\frac{\gamma}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \\
\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx
\]

Then, by using the fact that \( \nabla (U(x - y) + U(x + y)) = \nabla_s s_y U \), we may integrate by parts in \( y \) to find that

\[
\mathcal{A}_{1.1,2,2,2,2} = -\frac{1}{4} \int \nabla U \int \nabla (U(x + y)) \frac{y}{|y|^3} \nabla_x S_y f \sin\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \\
\times \int_0^\infty e^{-\gamma} \cos\left(\frac{\gamma}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \\
\int_0^\infty e^{-k} \cos\left(\frac{k}{2} S_y f\right) \cos\left(\frac{k}{2} D_y f\right) \, dy \, dk \, dx
\]

\[
\lesssim \|\nabla U\|_{L^2} \left( \frac{\|s_y U\|_{L^\infty} \|\nabla s_y f\|_{L^2}}{|y|^{3/2}} \, dy + \int \frac{\|s_y U\|_{L^\infty} \|\Delta s_y f\|_{L^2}}{|y|^{3/2}} \, dy \right)
\]

\[
\lesssim \|U\|_{\dot{H}^1} \left( \|U\|_{B_{\infty,2}^{1/2}} \|\nabla f\|_{B_{2,2}^{3/2}} + \|U\|_{\dot{H}_{\infty,2}^{1/2}} \|\Delta f\|_{B_{2,2}^{1/2}} \right)
\]

\[
\lesssim \|U\|_{\dot{H}^1} \|\dot{U}\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^{5/2}} + \|U\|_{\dot{H}^1}^2 \|f\|_{\dot{H}^{5/2}}^2
\]

The control of \( \mathcal{A}_{1.1,3} \) is the same as one of \( \mathcal{A}_{1.1,2} \) since they are the equal up to interchanging the role of one sine and cosine (they are just bounded by 1 in all the steps). Hence, we have that

\[
\mathcal{A}_{1.3} \lesssim \|U\|_{\dot{H}^1} \|\dot{U}\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^{5/2}} + \|U\|_{\dot{H}^1}^2 \|f\|_{\dot{H}^{5/2}}^2
\]

The last term, that \( \mathcal{A}_{1.4} \) contains the dissipation. More precisely, we have that

\[
\mathcal{A}_{1.1,4} = \frac{1}{2} \int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \cos\left(\delta (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))\right) \int_0^\infty e^{-k} \\
\cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \cos\left(\frac{k}{2} \left(D_y f\right)\right) \cos\left(\frac{k}{2} \left(S_y f\right)\right) \, dk \, dy \, dx
\]

We start by linearizing

\[
\mathcal{A}_{1.1,4} = \frac{1}{2} \int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \cos\left(\delta (\arctan(\Delta_y f) + \arctan(\tilde{\Delta}_y f))\right) \times \\
\cos\left(\frac{1}{2} (\arctan(\Delta_y f) - \arctan(\tilde{\Delta}_y f))\right) \int_0^\infty e^{-k} \cos\left(\frac{k}{2} \left(D_y f\right)\right) \, dk \, dy \, dx
\]
\[ -\int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \] 
\[ \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \] 
\[ \sin^2 \left( \frac{k}{4} (S_y f) \right) \, dk \, dy \, dx \]
\[ = -\int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \sin^2 \left( \frac{1}{4} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \] 
\[ \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \times \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ -\int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \cos \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \] 
\[ \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \] 
\[ \sin^2 \left( \frac{k}{4} (S_y f) \right) \, dk \, dy \, dx \]
\[ + \frac{1}{2} \int \nabla U \int \nabla D_y \nabla U \frac{y}{|y|^2} \sin \left( \frac{1}{4} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \] 
\[ \times \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ = A_{1,1,4,1} + A_{1,1,4,2} + A_{1,1,4,3}. \]

To estimate \(A_{1,1,4,1}\), we balance the derivative in \(x\) by using again the fact that \(\delta_y \nabla_x d_y = -\delta_y \nabla_y s_y\), then integrating by parts in \(y\) gives that
\[ A_{1,1,4,1} = \int \nabla U \int \frac{\nabla s_y U}{|y|^3} \sin^2 \left( \frac{1}{4} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \times \] 
\[ \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ -\int \nabla U \int \frac{\nabla s_y U}{|y|^3} y. \nabla_y (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \] 
\[ \times \sin \left( \frac{1}{2} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \]
\[ \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ + \int \nabla U \int \frac{\nabla s_y U}{|y|^3} \sin^2 \left( \frac{1}{4} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \] 
\[ \times y. \nabla_y (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \times \] 
\[ \sin \left( (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \cos \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ + \frac{1}{2} \int \nabla U \int \frac{\nabla s_y U}{|y|^3} \sin^2 \left( \frac{1}{4} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \] 
\[ \times \cos \left( \frac{1}{2} (\arctan(\Delta_y f) - \arctan(\Delta_y f)) \right) y. \nabla_y (D_y f) \] 
\[ \times \int_0^\infty k e^{-k} \sin \left( \frac{k}{2} (D_y f) \right) \, dk \, dy \, dx \]
\[ A_{1,1,4,1,1} + A_{1,1,4,1,2} + A_{1,1,4,1,3} + A_{1,1,4,1,4}. \]

The estimate of \( A_{1,1,4,1,1} \) is easy, indeed, it suffices to write that
\[
A_{1,1,4,1,1} \lesssim \|\nabla U\|_{L^2}^2 \int \frac{\|s_y f\|_{L^\infty}}{|y|^4} \, dy
\lesssim \|U\|_{\dot{H}^1}^2 \|f\|_{\dot{H}^{5/2}}^2.
\]

Using Lemma (7.1) (7.10) and (7.9) for \( i = 2, 3, 4 \) one has that
\[
A_{1,1,4,1,i} \lesssim \|\nabla U\|_{L^2}^2 \int \frac{\|s_y f\|_{L^\infty} \|\delta^\pm f\|_{L^\infty}}{|y|^4} \, dy
\lesssim \|\nabla U\|_{L^2}^2 \left( \int \frac{\|s_y f\|_{L^\infty}}{|y|^5} \, dy \int \frac{\|\delta^\pm f\|_{L^\infty}}{|y|^3} \, dy \right)^{1/2}
\lesssim \|\nabla U\|_{L^2}^2 \|f\|_{B^{3/2}_{\infty,2}} \|\nabla f\|_{B^{1/2}_{\infty,2}}
\lesssim \|U\|_{\dot{H}^1}^2 \|f\|_{\dot{H}^{5/2}}^2.
\]

Then, notice that \( A_{1,1,4,1} \) and \( A_{1,4,2} \) have exactly the same regularity in the sense that the terms \( \sin^2 \left( \frac{1}{4} (\arctan(\Delta_y f) + \arctan(\Delta_y f)) \right) \) and \( \sin^2 \left( \frac{k}{4} (S_y f) \right) \) have the same regularity. Indeed they are both bounded by \( c|S_y f|^2 \) where \( c > 0 \) is a constant. Hence, we conclude that
\[
A_{1,1,4,2} \lesssim \|U\|_{\dot{H}^1}^2 \|f\|_{\dot{H}^{5/2}}^2.
\]

The dissipation comes from the term \( A_{1,1,4,3} \) which is analogous to the term \( S_{4,3} \) (see (8.39)). By replacing the first two \( \Delta f \in S_{4,3} \) by \( \nabla U \) we immediately find that
\[
A_{1,1,4,3} = -\frac{1}{2} \int \nabla U \int \frac{\nabla \delta_y U}{|y|^3} \left( \cos((\arctan(\frac{y}{|y|}) \nabla f(x)))
- \cos((\arctan(\frac{y}{|y|}) \nabla f(x - y))) \right)
\times \int_0^\infty e^{-k} \cos(k \frac{y}{|y|} \nabla f(x)) \, dk \, dy \, dx
- \frac{1}{4} \int \nabla U \int \frac{\nabla \delta_y U}{|y|^3} \left( \cos((\arctan(\frac{y}{|y|}) \nabla f(x)))
+ \cos((\arctan(\frac{y}{|y|}) \nabla f(x - y))) \right)
\int_0^\infty e^{-k} \left( \cos(k \frac{y}{|y|} \nabla f(x)) - \cos(k \frac{y}{|y|} \nabla f(x - y)) \right) \, dk \, dy \, dx
- \frac{1}{4} \int \nabla U \int \frac{\nabla \delta_y U}{|y|^3} \left( \cos((\arctan(\frac{y}{|y|}) \nabla f(x)))
+ \cos((\arctan(\frac{y}{|y|}) \nabla f(x - y))) \right)
\int_0^\infty e^{-k} \left( \cos(k \frac{y}{|y|} \nabla f(x)) - \cos(k \frac{y}{|y|} \nabla f(x - y)) \right) \, dk \, dy \, dx
\]
\[
+ \cos((\arctan(\frac{y}{|y|}) \nabla f(x - y)))
\int_0^\infty e^{-k} \left( \cos(k \frac{y}{|y|} \nabla f(x)) - \cos(k \frac{y}{|y|} \nabla f(x - y)) \right) \, dk \, dy \, dx.
\]
\[
\int_0^\infty e^{-k} \left( \cos(k \frac{y}{|y|}, \nabla f(x)) + \cos(k \frac{y}{|y|}, \nabla f(x - y)) \right) \, dk \, dy \, dx
\]

\[= \mathcal{A}_{1,1,4,3,1} + \mathcal{A}_{1,1,4,3,2} + \mathcal{A}_{1,1,4,3,3}.\]

The first two terms are easy to control, indeed, one has, for \( i = 1, 2 \), that

\[\mathcal{A}_{1,1,4,3,i} \lesssim \|U\|_{\dot{H}^1} \int \frac{\|\delta_y \nabla U\|^2_{L_2} \|\delta_y \nabla f\|_{L_\infty}}{|y|^{3/2}} \, dy \lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^{3/2,2}} \lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^2}.\]

The term \( \mathcal{A}_{1,1,4,3,3} \) is the dissipative term. Following the same step as \( \mathcal{S}_{4,3,3} \), one finds that

\[\mathcal{A}_{1,1,4,3,3} = \frac{1}{8} \int \int \frac{|
abla \delta_y U|^2}{|y|^3} \times \left( -4 + 4 - \frac{1}{1 + \left( \frac{y}{|y|} \cdot \nabla f(x - y) \right)^2} \right) \, dy \, dx \]

\[= \frac{1}{2} \|U\|^2_{\dot{H}^{3/2}} + \frac{1}{8} \int \int \frac{|
abla \delta_y U|^2}{|y|^3} \times \left( 4 - \frac{1}{1 + \left( \frac{y}{|y|} \cdot \nabla f(x - y) \right)^2} \right) \, dy \, dx \]

\[\lesssim \frac{1}{2} \|U\|^2_{\dot{H}^{3/2}} + \frac{1}{2} \|U\|^2_{\dot{H}^{3/2}} \left( 1 - \frac{1}{1 + K^2} \right).\]

Now we estimate \( \mathcal{A}_{1,2} \), that is,

\[\mathcal{A}_{1,2} = \int \nabla U \cdot \left( \Delta_y \nabla U \cdot \frac{y}{|y|^2} \right) \int_0^\infty e^{-k} \nabla \left( \cos(k \Delta_y f) \cos(\arctan(\Delta_y f)) \right) \, dk \, dy \, dx.\]

Note that, it suffices to treat the case where the gradient hits \( \cos(k \Delta_y f) \) since the other term is analogous. Then,

\[\mathcal{A}_{1,2} \lesssim \|\nabla U\|_{L_2} \int \frac{\|\delta_y U\|^2_{L_2} \|\delta_y f\|_{L_\infty}}{|y|^{3/2}} \, dy \lesssim \|U\|_{\dot{H}^1} \left( \int \frac{\|\delta_y U\|^2_{L_2}}{|y|^3} \, dy \right)^{1/2} \left( \int \frac{\|\delta_y f\|^2_{L_\infty}}{|y|^3} \, dy \right)^{1/2} \lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}} \|f\|_{\dot{H}^{5/2}}.\]
11.2. Estimate of $A_2$

For $A_2$, introduce the operator

$$S(f, g) := \int \nabla U \nabla \left( \int \Delta_y \nabla f, \frac{y}{|y|^2} \cos(\arctan(\Delta_y g)) \right) \int_0^\infty e^{-k} \cos(k \Delta_y g) \, dk \, dy \, dx$$

One easily notices that we have $A_2 = S(g, f) - S(g, g)$ and therefore, as a direct application of the Lemma 8.2, we may write that

$$S(g, f) - S(g, g)$$

$$= \frac{1}{8} \int \nabla U \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) - \cos(\arctan(\tilde{\Delta}_y f)) \right)$$

$$\times \int_0^\infty e^{-k} (\cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f)) \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \nabla U \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y g)) - \cos(\arctan(\tilde{\Delta}_y g)) \right)$$

$$\times \int_0^\infty e^{-k} (\cos(k \Delta_y g) - \cos(k \tilde{\Delta}_y g)) \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \nabla U \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) + \cos(\arctan(\tilde{\Delta}_y f)) \right)$$

$$\int_0^\infty e^{-k} (\cos(k \Delta_y f) + \cos(k \tilde{\Delta}_y f)) \, dk \, dy \, dx$$

$$- \frac{1}{8} \int \nabla U \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y g)) + \cos(\arctan(\tilde{\Delta}_y g)) \right)$$

$$\times \int_0^\infty e^{-k} (\cos(k \Delta_y g) + \cos(k \tilde{\Delta}_y g)) \, dk \, dy \, dx$$

$$- \frac{1}{4} \int \nabla U \nabla \left( \Delta_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y f)) + \cos(\arctan(\tilde{\Delta}_y f)) \right)$$

$$\int_0^\infty e^{-k} (\cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f)) \, dk \, dy \, dx$$

$$+ \frac{1}{4} \int \nabla U \nabla \left( \Delta_y \nabla g, \frac{y}{|y|^2} \cos(\arctan(\Delta_y g)) + \cos(\arctan(\tilde{\Delta}_y g)) \right)$$

$$\int_0^\infty e^{-k} (\cos(k \Delta_y g) - \cos(k \tilde{\Delta}_y g)) \, dk \, dy \, dx$$

$$= \sum_{i=1}^8 A_{2,1,i}.$$
We shall consider $\mathcal{A}_{2,1,i}$ and $\mathcal{A}_{2,1,i+1}$ for $i = 1\ldots7$ and find some nice cancellations. More precisely, we write that

\[
\mathcal{A}_{2,1,1} = \frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\tilde{\Delta}_y f)) \right) \times \int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx
\]

\[
= \frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\tilde{\Delta}_y f)) \right)
\]

\[
+ \cos(\arctan(\tilde{\Delta}_y g)) - \cos(\arctan(\Delta_y f)) \right) \int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx
\]

\[
+ \frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y g)) - \cos(\arctan(\tilde{\Delta}_y g)) \right)
\]

\[
\times \int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx.
\]

On the other hand, for $\mathcal{A}_{2,1,2}$, we may write that

\[
\mathcal{A}_{2,1,2} = -\frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y g)) - \cos(\arctan(\tilde{\Delta}_y g)) \right)
\]

\[
\times \int_0^\infty e^{-k} \left( \cos(k \Delta_y g) - \cos(k \Delta_y f) + \cos(k \tilde{\Delta}_y g) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx
\]

\[-\frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y g)) - \cos(\arctan(\tilde{\Delta}_y g)) \right)
\]

\[
\times \int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx.
\]

Hence, noticing that the second term in $\mathcal{A}_{2,1,1}$ and $\mathcal{A}_{2,1,2}$ cancels out, one finds that

\[
\mathcal{A}_{2,1,1} + \mathcal{A}_{2,1,2} = \frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f))
\]

\[
- \cos(\arctan(\Delta_y g)) + \cos(\arctan(\tilde{\Delta}_y g)) - \cos(\arctan(\tilde{\Delta}_y f)) \right)
\]

\[
\int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx
\]

\[-\frac{1}{8} \int \nabla u \cdot \nabla \left( \int \Delta_y \nabla g - \tilde{\Delta}_y \nabla g \right) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y g))
\]

\[
- \cos(\arctan(\tilde{\Delta}_y g)) \right) \int_0^\infty e^{-k} \left( \cos(k \Delta_y f) - \cos(k \tilde{\Delta}_y f) \right) \, dk \, dy \, dx.
\]

Now, we may estimate $\mathcal{A}_{2,1,1} + \mathcal{A}_{2,1,2}$. By integrating by parts and by using (7.1) (7.10) and (7.9) together with the mean value theorem and classical Besov embeddings, one finds that
\[ A_{2,1,1} + A_{2,1,2} \lesssim \| U \|_{H^1} \int \frac{\| g \|_{L^2} \| U \|_{L^\infty}}{|y|^{3/2}} \, dy + \| U \|_{H^1} \int \frac{\| \nabla g \|_{L^\infty}}{|y|^{3/2}} \, dy + \| U \|_{H^1} \int \frac{\| \nabla \delta_y f + \nabla \delta_y g \|_{L^4}}{|y|^{5/4}} \, dy \]

Then, we consider \( A_{2,1,3} + A_{2,1,4} \), we have that

\[ A_{2,1,3} = \frac{1}{8} \int \nabla U \nabla \int (\Delta_y \nabla g - \nabla \Delta_y g) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f)) + \cos(\arctan(\Delta_y f)) \right) \]

\[ = \frac{1}{8} \int \nabla U \nabla \int (\Delta_y \nabla g - \nabla \Delta_y g) \cdot \frac{y}{|y|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\Delta_y f)) \right) \]

\[ \cdots \]

Using the fact that \( \dot{H}^{3/2} = [\dot{H}^2, \dot{H}^{5/2}]_{1/2} \), one finally gets that

\[ A_{2,1,1} + A_{2,1,2} \lesssim \| U \|_{H^1} \| U \|_{H^3/2} \left( \| g \|_{H^{5/2}} + \| g \|_{H^{5/2}} \| \dot{U} \|_{H^2} \right) + \| g \|_{H^2} \| g \|_{H^{5/2}} \| f \|_{H^{5/2}} \). \]
The last term cancels out $A_{2.1.4}$, and therefore,

$$A_{2.1.3} + A_{2.1.4} = \frac{1}{8} \int \nabla U \cdot \nabla \int (\Delta_y \nabla g - \nabla \tilde{A}_y g) \cdot \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\Delta_y g)) \\
- \cos(\arctan(\tilde{A}_y g)) + \cos(\arctan(\tilde{A}_y f)) \right) \int_0^\infty e^{-k} (\cos(k \tilde{A}_y f) + \cos(k \Delta_y f)) \, dk \, dy \, dx$$

$$+ \frac{1}{8} \int \nabla U \cdot \nabla \int (\Delta_y \nabla g - \nabla \tilde{A}_y g) \cdot \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\Delta_y g)) + \cos(\arctan(\tilde{A}_y g)) \right) \int_0^\infty e^{-k} (\cos(k \tilde{A}_y f) - \cos(k \Delta_y f)) \, dk \, dy \, dx.$$ 

This has the same structure as the term $A_{2.1.1} + A_{2.1.2}$, and by integrating by parts and following the same steps. We easily get that

$$A_{2.1.3} + A_{2.1.4} \lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}}$$

$$\left( \|g\|_{\dot{H}^{5/2}} + \|g\|_{\dot{H}^{5/2}} \|g\|_{\dot{H}^2} + \|g\|_{\dot{H}^{1/2}}^2 \|g\|_{\dot{H}^{5/2}} \|f\|_{\dot{H}^2}^2 \|f\|_{\dot{H}^{5/2}} \right).$$

We now estimate $A_{2.1.5} + A_{2.1.6}$, we first notice that

$$A_{2.1.5} = -\frac{1}{4} \int \nabla U \cdot \nabla \int \tilde{A}_y \nabla g, \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\tilde{A}_y f)) + \cos(\arctan(\Delta_y f)) \right)$$

$$\times \int_0^\infty e^{-k} (\cos(k \tilde{A}_y f) - \cos(k \Delta_y f)) \, dk \, dy \, dx$$

$$= -\frac{1}{4} \int \nabla U \cdot \nabla \int \tilde{A}_y \nabla g, \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\Delta_y f)) - \cos(\arctan(\tilde{A}_y g)) \\
- \cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y f)) \right) \times$$

$$\int_0^\infty e^{-k} (\cos(k \tilde{A}_y f) - \cos(k \Delta_y f)) \, dk \, dy \, dx$$

$$- \frac{1}{4} \int \nabla U \cdot \nabla \int \tilde{A}_y \nabla g, \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y g)) \right)$$

$$\times \int_0^\infty e^{-k} (\cos(k \tilde{A}_y f) - \cos(k \Delta_y f)) \, dk \, dy \, dx.$$ 

On the other hand, we have that

$$A_{2.1.6} = \frac{1}{4} \int \nabla U \cdot \nabla \int \tilde{A}_y \nabla g, \frac{\mathbf{y}}{|\mathbf{y}|^2} \left( \cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y g)) \right) \times$$

$$\int_0^\infty e^{-k} (\cos(k \tilde{A}_y g) - \cos(k \Delta_y g)) \, dk \, dy \, dx$$
\[ \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y g) - \cos(k \Delta_y f)) \, dk \, dy \, dx \]
\[ + \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y g) - \cos(k \Delta_y f)) \, dk \, dy \, dx. \]

One notices that the last two terms in \( \mathcal{A}_{2,1,5} \) and \( \mathcal{A}_{2,1,6} \) cancel out. Hence,

\[ \mathcal{A}_{2,1,5} + \mathcal{A}_{2,1,6} = -\frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} \]
\[ (\cos(\arctan(\Delta_y f)) - \cos(\arctan(\Delta_y g)) \]
\[ - \cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y f))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y f) - \cos(k \Delta_y f)) \, dk \, dy \, dx \]
\[ + \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y g) - \cos(k \Delta_y f)) \, dk \, dy \, dx. \]

Here again we integrate by parts and notice that this term can be estimated in a similar manner as to \( \mathcal{A}_{2,1,1} + \mathcal{A}_{2,1,2} \), and therefore,

\[ \mathcal{A}_{2,1,5} + \mathcal{A}_{2,1,6} \lesssim \| U \|_{\dot{H}^{1/2}} \| U \|_{\dot{H}^{3/2}} \]
\[ \left( \| g \|_{\dot{H}^{3/2}}^2 + \| g \|_{\dot{H}^{3/2}} \| g \|_{\dot{H}^{3/2}} + \| g \|_{\dot{H}^{3/2}} \| g \|_{\dot{H}^{3/2}} + \| f \|_{\dot{H}^{3/2}} \| f \|_{\dot{H}^{3/2}} \right). \]

It remains to estimate \( \mathcal{A}_{2,1,7} + \mathcal{A}_{2,1,8} \). To do that, one first writes that

\[ \mathcal{A}_{2,1,7} = -\frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} \]
\[ (\cos(\arctan(\Delta_y f)) - \cos(\arctan(\Delta_y g)) \]
\[ + \cos(\arctan(\Delta_y g)) - \cos(\arctan(\Delta_y f))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y f) + \cos(k \Delta_y f)) \, dk \, dy \, dx \]
\[ + \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\Delta_y g)) - \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k \Delta_y f) + \cos(k \Delta_y f)) \, dk \, dy \, dx. \]

Then, we may rewrite \( \mathcal{A}_{2,1,8} \) as
\[ A_{2,1,8} = \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta}_y g)) - \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k\Delta_y g) - \cos(k\Delta_y f) - \cos(k\Delta_y g)) + \cos(k\tilde{\Delta}_y g) \) dk dy dx \]
\[ - \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta}_y g)) - \cos(\arctan(\Delta_y g))) \times \]
\[ \int_0^\infty e^{-k} (\cos(k\Delta_y f) + \cos(k\Delta_y f)) ) dk dy dx. \]

Then, we again notice that the last terms cancel out and find that

\[ A_{2,1,7} + A_{2,1,8} = -\frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} \]
\[ \left( \cos(\arctan(\tilde{\Delta}_y f)) - \cos(\arctan(\tilde{\Delta}_y g)) \right) \]
\[ - \cos(\arctan(\Delta_y g)) + \cos(\arctan(\Delta_y f)) \] \[ \times \int_0^\infty e^{-k} (\cos(k\tilde{\Delta}_y f) - \cos(k\Delta_y f)) ) dk dy dx \]
\[ + \frac{1}{4} \int \nabla U \cdot \nabla \int \Delta_y \nabla g \cdot \frac{y}{|y|^2} (\cos(\arctan(\tilde{\Delta}_y g)) \]
\[ - \cos(\arctan(\Delta_y g))) \times \int_0^\infty e^{-k} (\cos(k\Delta_y g) - \cos(k\Delta_y f) ) \]
\[ - \cos(k\Delta_y g)) + \cos(k\tilde{\Delta}_y g)) ) dk dy dx. \]

By integrating by parts, we again notice that this term is similar to \( A_{2,1,1} + A_{2,1,2} \), hence we have that

\[ A_{2,1,7} + A_{2,1,8} \lesssim \|U\|_{\dot{H}^1} \|U\|_{\dot{H}^{3/2}} \]
\[ \left( \|g\|_{\dot{H}^{5/2}} + \|g\|_{\dot{H}^{5/2}} \|g\|_{\dot{H}^2} + \|g\|_{\dot{H}^{5/2}} \|g\|_{\dot{H}^{5/2}} \|f\|_{\dot{H}^{5/2}} \|f\|_{\dot{H}^{5/2}} \right). \]

Finally, \[ A_2 \lesssim C(K) \|U\|^2_{\dot{H}^1} \|g\|^2_{\dot{H}^{5/2}} + \frac{1}{100(1 + K^2)^{3/2}} \|U\|^2_{\dot{H}^{3/2}} \]
\[ + C(K) \left( \sup_{t \in [0, T]} \|g\|_{\dot{H}^2} \right)^2 \|U\|^2_{\dot{H}^1} \|g\|^2_{\dot{H}^{5/2}} + \frac{1}{100(1 + K^2)^{3/2}} \|U\|^2_{\dot{H}^{3/2}} \]
\[ + C(K) \left( \sup_{t \in [0, T]} \|g\|_{\dot{H}^2} \right) \left( \sup_{t \in [0, T]} \|f\|_{\dot{H}^2} \right) \|U\|^2_{\dot{H}^1} \|g\|_{\dot{H}^{5/2}} \|f\|_{\dot{H}^{5/2}} \]
\[ + \frac{1}{100(1 + K^2)^{3/2}} \|U\|^2_{\dot{H}^{3/2}}. \]

Combining the latter inequality with the estimate obtained for \( A_1 \), one finally finds that
\[ \frac{1}{2} \partial_t \| \nabla U \|_{L^2}^2 + \frac{1}{2(1 + K(t)^2)^{3/2}} \| U \|_{\dot{H}^{3/2}}^2 \]
\[ \lesssim C(K) \| \nabla U \|_{L^2}^2 \left( \left( \sup_{t \in [0, T]} \| f \|_{\dot{H}^{2}} \right)^2 + \left( \sup_{t \in [0, T]} \| g \|_{\dot{H}^{2}} \right)^2 \right) \times \left( \| g \|_{H^{5/2}}^2 + \| f \|_{H^{5/2}}^2 \right). \]

Hence, integrating in time \( s \in [0, T] \) and using Grönwall’s inequality, one finally concludes that
\[ \sup_{t \in [0, T]} \| U(t) \|_{\dot{H}^{1}}^2 \leq \| U_0 \|_{\dot{H}^{1}}^2 \exp \left( C(K) \left( \left( \sup_{t \in [0, T]} \| f \|_{\dot{H}^{2}} \right)^2 + \left( \sup_{t \in [0, T]} \| g \|_{\dot{H}^{2}} \right)^2 \right) \right) \times \int_0^T \left( \| g(s) \|_{H^{5/2}}^2 + \| f(s) \|_{H^{5/2}}^2 \right) ds. \]

This readily gives uniqueness. \( \square \)

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Data Availability The authors declare that data sharing not applicable to this article, as no datasets were generated or analysed during the study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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