HIGHLY SYMMETRIC FUNDAMENTAL CELLS FOR LATTICES IN $\mathbb{R}^2$ AND $\mathbb{R}^3$

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Abstract. The fundamental cell of a lattice $\Gamma$ in $\mathbb{R}^d$ is the fundamental domain $\mathbb{R}^d/\Gamma$, viewed as a compact subset of $\mathbb{R}^d$. It is shown that most lattices $\Gamma$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess fundamental cells $F$ having more symmetries than the point group $P(\Gamma)$, i.e., the group $P(\Gamma) \subset O(d)$ fixing $\Gamma$. In particular, $P(\Gamma)$ is a subgroup of the symmetry group $S(F)$ of $F$ of index 2 in these cases. The exceptions are rhombic lattices in the plane case and cubic lattices in the three-dimensional case.

1. Introduction

The question inspiring this work was “Given a group $G$ acting on some space $X$. How much symmetries can the fundamental domain $X/G$ have, compared to $G$?”. Symmetry of $G$ can be understood as symmetry of the Cayley graph of $G$, or symmetry of some canonical embedding of $G$ in $X$. Symmetry in $X$ can be formulated if $X$ is a metric group, for instance. We will specify a particular instance of this question here and provide some answers.

A lattice in $\mathbb{R}^d$ is the $\mathbb{Z}$-span of $d$ linearly independent vectors in $\mathbb{R}^d$. The point group $P(\Gamma)$ of a lattice $\Gamma$ in $\mathbb{R}^d$ is the set of Euclidean motions fixing both $\Gamma$ and the origin. In other words, $P(\Gamma) \subset O(d)$ is the set of orthogonal maps fixing $\Gamma$. It is clear that each lattice $\Gamma$ has a fundamental cell having $P(\Gamma)$ as its symmetry group, see Lemma 1.4. (For more detailed definitions see below.) For instance, consider the square lattice $\mathbb{Z}^2$ in the plane $\mathbb{R}^2$. Its point group $P(\mathbb{Z}^2)$ is the dihedral group $D_4$ of order eight, containing rotations by $0$, $\pi/2$, $\pi$, $3\pi/2$, together with four reflections. One possible fundamental cell of $\mathbb{Z}^2$ is a unit square, centred in 0. Clearly $P(\mathbb{Z}^2)$ is the symmetry group of this unit square as well.

In this paper we show that most lattices in $\mathbb{R}^2$ and $\mathbb{R}^3$ possess fundamental cells with more symmetry than the point group of the lattice. In general, these fundamental cells will be neither simply connected, nor will their interior be connected. Some of these cells are of fractal appearance. The two main results are the following.

**Theorem 1.1.** Let $\Gamma \subset \mathbb{R}^2$ be a lattice with point group $P(\Gamma)$, such that $\Gamma$ is not a rhombic lattice. Then there is a compact fundamental cell $F$ of $\Gamma$ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.

**Theorem 1.2.** Let $\Gamma \subset \mathbb{R}^3$ be a lattice with point group $P(\Gamma)$, such that $\Gamma$ is not a cubic lattice. Then there is a compact fundamental cell $F$ of $\Gamma$ with symmetry group $S(F)$ such that $P(\Gamma)$ is a subgroup of $S(F)$ of index $[S(F) : P(\Gamma)] = 2$.

In the remainder of this section the necessary definitions and notations are introduced. Section 2 is dedicated to the proof of Theorem 1.1. Section 3 contains the proof of Theorem 1.2. Section 4 contains some remarks and further questions.
**Notation:** $C_n$ denotes the cyclic group of order $n$. $D_n$ denotes the dihedral group of order $2n$. $O(d)$ is the orthogonal group over $\mathbb{R}^d$. $O(d)$ can be identified with the group of Euclidean motions (i.e., isometries of $\mathbb{R}^d$, including reflections) fixing the origin. The closure of a set $A \subset \mathbb{R}^d$ is denoted by $\text{cl}(A)$. For any set $X \subset \mathbb{R}^d$, let $S(X)$ denote the symmetry group of $X$; that is, the set of all Euclidean motions (including reflections and translations) $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ with $\varphi(X) = X$. The cone centred in $x$ spanned by $m$ vectors $v_1, \ldots, v_m \in \mathbb{R}^d$ is defined by

$$\text{cone}(x; v_1, \ldots, v_m) = \{ x + \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \geq 0 \}.$$ 

A sum $A + B$, where $A, B \subset \mathbb{R}^d$, always means the Minkowski sum

$$A + B = \{ a + b \mid a \in A, b \in B \}.$$ 

The line segment with endpoints $x, y \in \mathbb{R}^d$ is denoted by $[x, y]$. A lattice in $\mathbb{R}^d$ is a discrete cocompact subgroup of $\mathbb{R}^d$. Any lattice in $\mathbb{R}^d$ can be written as $\langle b_1, \ldots, b_d \rangle \mathbb{Z}$, where $b_1, \ldots, b_d$ span $\mathbb{R}^d$. Such a set $b_1, \ldots, b_d$ is called a basis of the lattice. A basis of a given lattice is not unique.

The fundamental domain of a lattice $\Gamma$ is $\mathbb{R}^d/\Gamma$. By a fundamental cell we denote the closure of some geometric embedding of a fundamental domain into $\mathbb{R}^d$. For instance, a fundamental domain of $\mathbb{Z}^d$ is the $d$-torus, and a fundamental cell of the lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$ is the $d$-dimensional unit cube $[0, 1]^d$. A particular fundamental cell of a lattice $\Gamma = \langle b_1, \ldots, b_d \rangle \mathbb{Z}$ is the fundamental parallelepiped $[0, b_1] + \ldots + [0, b_d]$. Note that for any fundamental cell $F$ of a lattice $\Gamma$, $\{ F + g \mid g \in \Gamma \}$ is a tiling of $\mathbb{R}^d$. A tiling of $\mathbb{R}^d$ is a packing of $\mathbb{R}^d$ which is also a covering of $\mathbb{R}^d$. In other words, a tiling is a covering of $\mathbb{R}^d$ by pairwise non-overlapping compact sets $T_i$. Two compact sets are non-overlapping if their interiors are disjoint.

Trivially, the symmetry group $S(\Gamma)$ of any lattice contains a subgroup isomorphic to $\Gamma$, namely, the group of all translations by elements of $\Gamma$. The subgroup $P(\Gamma) = S(\Gamma)/\Gamma$ is called point group of $\Gamma$. For lattices in $\mathbb{R}^d$, one has:

$$S(\Gamma) = P(\Gamma) \rtimes \Gamma.$$ 

The following fact is usually called the crystallographic restriction (see for instance [3], Section 4.5).

**Proposition 1.3.** Rotations fixing a lattice in $\mathbb{R}^2$ or $\mathbb{R}^3$ are either 2-fold, 3-fold, 4-fold or 6-fold.

The Voronoi cell of a lattice point $x$ in $\mathbb{R}^d$ is the set of points in $\mathbb{R}^d$ whose distance to $x$ is not greater than their distance to any other lattice point. It is easy to see that for any lattice $\Gamma \subset \mathbb{R}^d$ the closed Voronoi cell $V = V(0)$ of 0 is a fundamental cell of $\Gamma$ with $S(V) = P(\Gamma)$. We formulate this as a lemma.

**Lemma 1.4.** If $\Gamma$ is a lattice in $\mathbb{R}^d$ then $\Gamma$ has a fundamental cell $F$ such that $S(F) = P(\Gamma)$.

We will use orbifold notation to denote planar symmetry groups in the sequel, compare [4]. For instance, $*442$ denotes the symmetry group $S(\mathbb{Z}^2)$ of the square lattice $\mathbb{Z}^2$, and $*432$ denotes the symmetry group of the cube. For a translation of orbifold notation into your favourite notation, see [3] or [18]. In principle we can denote cyclic groups $C_n$ and dihedral groups $D_n$ in orbifold notation, too. Since the sign for $C_n$—regarded as the symmetry group
of some object in the plane—is just \( n \) in orbifold notation, we will rather use the former abbreviation for the sake of clarity.

2. Dimension 2

From the theory of Coxeter groups [4, 5, 11] we know the list of all finite groups of Euclidean motions. Thus we know that each finite group of Euclidean motions in the plane is either \( C_n \) or \( D_n \). By the crystallographic restriction (Proposition 1.3) there are just 10 candidates for such groups being point groups of a planar lattice, namely

\[ C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6. \]

Note that \( C_2 \) and \( D_1 \) are equal as abstract groups, since there is only one group of order two up to isomorphisms. But since we are dealing with groups of Euclidean motions, we will use the convention that a cyclic group \( C_n \) contains rotations only, and a dihedral group \( D_n \) contains \( n \) rotations (including the identity) and \( n \) reflections. The fact that each planar lattice is fixed under a rotation through \( \pi \) about the origin implies that \( C_1, C_3, D_1 \) and \( D_3 \) cannot be point groups of any planar lattice. Some further thought yields the following result.

**Proposition 2.1.** If \( \Gamma \) is a lattice in \( \mathbb{R}^2 \), then \( P(\Gamma) \in \{C_2, D_2, D_4, D_6\} \), and \( S(\Gamma) \in \{\ast 632, \ast 442, \ast 2222, 2 \ast 22, 2222\} \).

This result is well known. Nevertheless, since we are not aware of a decent reference, we will sketch the proof here.

**Proof.** We consider the distinct possibilities of properties of basis vectors of \( \Gamma \). First, if \( \Gamma \) has a basis of two orthogonal vectors of equal length, this yields (up to similarity) the square lattice \( \mathbb{Z}^2 \), with point group \( D_4 \) and symmetry group \( \ast 442 \). Second, if \( \Gamma \) has a basis of two vectors of equal length with angle \( \pi/3 \), this yields (up to similarity) the hexagonal lattice \( A_2 = ((1,0)^T, (1/2, \sqrt{3}/2)^T) \mathbb{Z} \), with point group \( D_6 \) and symmetry group \( \ast 632 \). Third, if \( \Gamma \) has a basis of two vectors of equal length, but neither with angle \( \pi/3 \) nor \( \pi/2 \) nor \( 2\pi/3 \), then \( \Gamma \) is called **rhombic lattice** and has point group \( D_2 \) and symmetry group \( 2 \ast 22 \). A planar lattice which has orthogonal basis vectors of different length (but not of equal length) is called **rectangular lattice**. It has also point group \( D_2 \), but its symmetry group is \( \ast 2222 \). In particular, the entire symmetry group of a rhombic lattice is not isomorphic to the entire symmetry group of a rectangular lattice, even though their point groups agree. (Compare [13, p 210.]) All other lattices are called **oblique lattices** and have point group \( C_2 \), and symmetry group \( 2222 \). \( \square \)

Regarding the five cases above in connection with Lemma 1.4 one obtains that one possible fundamental cell of the hexagonal (square, rectangular, rhombic, oblique) lattice is a regular hexagon (square, rectangle, hexagon with \( D_2 \) symmetry, hexagon with \( C_2 \) symmetry). We will proof Theorem 1.1 by considering four out of these five cases. The first case—the square lattice—is due to V. Elser [7]. To the knowledge of the author his proof has not been published anywhere, so we give a detailed proof here.

**Proposition 2.2 (Elser).** The square lattice \( \mathbb{Z}^2 \) has a fundamental cell \( F_\Box \) with \( S(F_\Box) = D_8 \).
Figure 1. The first step of the construction of a fundamental cell for the square lattice with eight-fold symmetry. The shaded octagon is $F_0$.

Proof. The point group of the square lattice $\mathbb{Z}^2$ is $D_4$. The claim is proved by constructing a fundamental cell $F_\square$ of $\mathbb{Z}^2$ with symmetry group $D_8$.

Consider a packing of $\mathbb{R}^2$ by regular octagons, such that each $x \in \mathbb{Z}^2$ is the centre of some octagon. Let all octagons be of edge length $\ell := \sqrt{2} - 1$ and non-overlapping, see Figure 1. The packing looks like the Archimedean tiling $4.8^2$ by octagons and squares, where the squares are the holes of the packing. The intersection of two distinct octagons is either empty, or a full edge. Denote this octagon packing by $P_0$.

Let the octagon centred in 0 be distinguished from the others, for instance, by colouring this octagon red and all other octagons white. Let $F_0$ denote the red octagon. Now consider a further packing by smaller octagons of edge length $\ell_0 := \sqrt{2} - 1$, such that each vertex of a large octagon is the centre of some small octagon (see Figure 2). The packing of $\mathbb{R}^2$ by these small octagons is denoted by $P_1$.

Proceed with each small octagon $O$ in $P_1$ according to the following rules: If the centre of $O$ is the vertex of two white octagons in $P_0$, then $O$ is coloured white. If the centre of $O$ is the vertex of one white and one red octagon in $P_0$, then we divide $O$ as follows: Let $e_i := \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, $e_1 := 2^{-1/2} \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$, $e_2 := 2^{-1/2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right)$, $e_3 := -e_i \quad (0 \leq i \leq 4)$.

For convenience, let $e_8 := e_0$. Denote the centre of $O$ by $x$. $O$ is divided into eight pieces $O_1, \ldots, O_8$ by the cones $\text{cone}(x; e_i, e_{i+1})$:

$$O_i = O \cap \text{cone}(x; e_i, e_{i+1}) \quad (0 \leq i \leq 7).$$

Now, since $O$ is centred on the vertex of two large octagons, exactly two of the eight points $x + \ell(e_i + e_{i+1})$ are centres of large octagons in $P_0$. (see Figure 2). Note, that if $x + \ell(e_i + e_{i+1}), x + \ell(e_j + e_{j+1})$ are those points, then $|i - j| = 3$. Thus the following rule is well-defined.

If $x + \ell(e_0 + e_1)$ (or $x + \ell(e_2 + e_3)$, or $x + \ell(e_4 + e_5)$, or $x + \ell(e_6 + e_7)$) is the centre of a red octagon, then $O_0, O_2, O_4, O_6$ are coloured red, $O_1, O_3, O_5, O_7$ are coloured white. Analogously, if $x + \ell(e_0 + e_1)$ (or $x + \ell(e_2 + e_3)$, or $x + \ell(e_4 + e_5)$, or $x + \ell(e_6 + e_7)$) is the centre of a white octagon, then $O_0, O_2, O_4, O_6$ are coloured white, $O_1, O_3, O_5, O_7$ are coloured red. Denote the union of all red pieces of small octagons by $R_1$. Now let $F_1 := \text{cl}(F_0 \setminus P_1 \cup R_1)$. 

Note that the colouring scheme in the last step can in principle produce octagons of three different kinds: purely red octagons (all its vertices are red), purely white octagons (all its vertices are white) and octagons of mixed colour (then its vertices are alternating red-white-red-white...)

Let us proceed in an analogous manner. Let $O^{(n)}$ be the octagon of edge length $(\sqrt{2} - 1)^n$. Place translates of $O^{(n)}$ on all vertices of the octagons of the previous packing $P_{n-1}$. Denote this octagon packing by $P_n$. (Figure 3 indicates the third step of this construction.) Note that the centre of each of these octagons $O^{(n)} + x$ are vertices of either one or two larger octagons in $P_{n-1}$. If the centre of $O^{(n)} + x$ is vertex of only one larger octagon, $O^{(n)} + x$ inherits its colour from this vertex. Similarly, if the centre of $O^{(n)} + x$ is vertex of two larger octagons, and these two vertices possess the same colour, $O^{(n)} + x$ inherits its colour from these two vertices. If the centre of $O^{(n)} + x$ is both a white vertex (of one larger octagon in $P_{n-1}$) and at the same a red vertex (of another larger octagon in $P_{n-1}$), we cut $O^{(n)} + x$ into eight pieces $O_0^{(n)}, \ldots, O_7^{(n)}$ precisely as above. Then exactly two of the eight points $x + (\sqrt{2} - 1)^n(e_i + e_{i+1})$ are centres of some octagons in $P_{n-1}$. As above, if $x + (\sqrt{2} - 1)^n(e_0 + e_1)$, or $x + (\sqrt{2} - 1)^n(e_2 + e_3)$, or $x + (\sqrt{2} - 1)^n(e_4 + e_5)$, or $x + (\sqrt{2} - 1)^n(e_6 + e_7)$ is the centre of a red octagon (white octagon), then $O_0^{(n)}, O_2^{(n)}, O_4^{(n)}, O_6^{(n)}$ are coloured red (white), $O_1^{(n)}, O_3^{(n)}, O_5^{(n)}, O_7^{(n)}$ become white (red). Denote the union of all red pieces of small octagons by $R_n$. Now let $F_n := \text{cl}((F_{n-1} \setminus P_n) \cup R_n)$.

By construction, each $F_n$ has $D_8$-symmetry. Let $F_\square$ be the closure of the limit of the sequence $F_n$ for $n \to \infty$. Then $F_\square$ has $D_8$-symmetry as well. It remains to show that $F_\square$ is a well-defined compact set, and that $F_\square$ is a fundamental cell for the square lattice. The latter is equivalent to saying that $\{F + x \mid x \in \mathbb{Z}^2\}$ is a tiling of $\mathbb{R}^2$.

Let us postpone the question of the compactness of $F_\square$. Clearly, in each step, $\mathbb{Z}^2 + F_n$ is a packing, where the holes occupy $c \cdot (\sqrt{2} - 1)^n$ of the plane, where $c = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2} = 0.1715728 \ldots$ is the portion of the plane occupied by the holes in the packing $P_0$. Thus the portion of the plane occupied by holes tends to zero in the packing $P_n$ with growing $n$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The second step of the construction of an eight-fold symmetric fundamental cell for the square lattice. The shaded part is $F_1$.}
\end{figure}
Moreover, in each step, $F_n$ and $x + F_n$ are non-overlapping ($x \in \mathbb{Z}^2$). This can be seen as follows: By construction, the red parts and the white parts are distinct, thus their closures are non-overlapping. We can repeat the construction with any other octagon coloured red, instead of the octagon at 0. Again, this new fundamental cell $(F + e_1, \text{say})$ does not overlap with any white parts. In particular, it does not overlap with $F$. Consequently, $F$ does not overlap with any of its neighbours $x + F, (x \in \Gamma, x \neq 0)$. Thus the limit of $P_n = \mathbb{Z}^2 + F_n$ is a tiling of the plane, assumed $F_{\square}$ is a compact set.

We are left with showing that $F_{\square}$ is compact. Since we defined $F_{\square}$ as the closure of $\lim_{n \to \infty} F_n$, it suffices to show that $F_{\square}$ is bounded. This follows from the construction: The diameter of

![Figure 3](image1.png)

Figure 3. The third step of the construction of an eight-fold symmetric fundamental cell for the square lattice

![Figure 4](image2.png)

Figure 4. The fundamental cell $F_{\square}$ (black) of $\mathbb{Z}^2$ and some of its copies, illustrating how they form a tiling.
Figure 5. The first two iterates of the construction of the 12-fold fundamental cell $F_\triangle$ of the hexagonal lattice $A_2$ (left and middle), and a higher iterate (right).

$F_0$ (a regular octagon with edge length one) is $h := 2\sqrt{1 - \sqrt{2}/2}$. In each step, the diameter of $F_n$ grows by $h(\sqrt{2} - 1)^n$. Thus the diameter of $F_\square$ is

(1) \[ \text{diam}(F) = \sum_{n \geq 0} h(\sqrt{2} - 1)^n = \frac{h}{1 - (\sqrt{2} - 1)} = \sqrt{2} + \sqrt{2} = 1.847759\ldots \]

In particular, $F_\square$ is bounded.

Proposition 2.3. The hexagonal lattice $A_2$ has a compact fundamental cell $F_\triangle$ with $S(F_\triangle) = D_{12}$.

Proof (sketch). The proof is very similar to the proof of Proposition 2.2. One starts with a packing $Q_0$ by regular dodecagons, each one centred on a point of the hexagonal lattice $A_2$, leaving triangular holes (corresponding to the Archimedean tiling $3.12^2$ by triangles and dodecagons). Then one places smaller dodecagons on the vertices of the larger ones, either white, red or mixed, compare Figure 5. Iteration yields a fundamental cell $F_\triangle$ of $A_2$ with $S(F_\triangle) = D_{12}$, see Figure 5, right. Proving that $F_\triangle$ is a fundamental cell is completely analogous to the proof of Proposition 2.2. The diameter of $F_\triangle$ is

(2) \[ \text{diam}(F_\triangle) = \frac{2}{\sqrt{3}} \sum_{n \geq 0} (2 - \sqrt{3})^n = \frac{1}{3}(3 + \sqrt{3}) = 1.57735\ldots \]

Interestingly, the set $F_\triangle$ appears in an entirely different context in [1].

Proof (of Theorem 1.1). We consider the four cases when $\Gamma$ is a square lattice, a hexagonal lattice, a rectangular lattice and an oblique lattice.

Case 1: $\Gamma = \mathbb{Z}^2$: This is Proposition 2.2. We have $S(F_\square) = D_8$ and $P(\Gamma) = D_4$.

Case 2: $\Gamma = A_2$: This is Proposition 2.3. We have $S(F_\triangle) = D_{12}$ and $P(\Gamma) = D_6$. 
Case 3: Let $\Gamma$ be an oblique lattice. Without loss of generality one basis of $\Gamma$ is $b_1 = (\frac{x}{z}, 0)$, $b_2 = (\frac{y}{z}, \frac{z}{z})$, $z \neq 0$. (A canonical fundamental cell is the parallelogram $F' = [0, b_1] + [0, b_2]$.) Then, let $F$ be the rectangle with vertices $(\frac{x}{0}, 0)$, $(x, 0)$, $(x, z)$, $(0, z)$, see Figure 6.

It is easy to see that $\Gamma + F = \mathbb{R}^2$, and the copies of $F$ do not overlap. Thus $F$ is a fundamental cell of $\Gamma$. We have $S(F) = D_2$ and $P(\Gamma) = C_2$.

Case 4: Let $\Gamma$ be a rectangular lattice, with basis $(a, 0)$, $(0, b)$. In a similar way as in the proof of Proposition 2.2 we will construct a fundamental domain with $D_4$-symmetry, whereas $P(\Gamma) = D_2$.

Without loss of generality, let $a > b$. First, consider a square packing $P_0$ with squares of edge length $b$, where each $x \in \Gamma$ is the centre of some square in $P_0$ (see Figure 7). For later purposes, let $r_0 = a$, $r_1 = b$, and $v_1 = \lfloor \frac{a}{b} \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less or equal to $x$. Distinguish the square with centre 0 from the others, say, by colouring it black and the other squares grey. Let $F_0$ denote the black square.

In the following it is convenient to distinguish between a ‘step’ of the procedure and an ‘iteration’ of the procedure. A step consists of one or more iterations. An iteration means placing squares of some edge length $\ell$ on all vertices of the squares in the current packing $P_{m,j}$, yielding a packing $P_{m,j+1}$. The next iteration in the same step places squares on all vertices of squares in $P_{m,j+1}$, and so on. These new squares have the same edge length $\ell$ as in the last iteration (see Figure 8). If such an iteration is no longer possible without producing overlaps, one goes over to the next step. In the next step we place squares with an an edge length strictly smaller than $\ell$ (see Figure 7). Several iterations—using squares of the same edge length—are considered as one single step of the procedure.

In the first step, the first iteration is placing squares of edge length $r_1 = b$ on each vertex of the squares in $P_0$. Denote the collection of these new squares by $P_{1,1}$. There are two possibilities: A small square $S \in P_{1,1}$ is centred on two coincident vertices of grey squares of $P_0$. Then $S$ is coloured grey. Or the small square $S$ is centred on the vertex of a grey square $W$ which coincides with a vertex of a black square $G$. Then $S$ gets two colours: One quarter of $S$, namely, $S \cap W$, is coloured grey. The opposite quarter of $S$ is also coloured grey. The two remaining quarters, $S \cap G$ and its opposite, are coloured black (see Figure 7). Denote the union of the black pieces (black squares or black parts of squares) of $P_{1,1}$ by $B_{1,1}$. Let $F_{1,1} := \text{cl}(F_0 \setminus P_{1,1}) \cup B$.

If $v_1 > 1$, do further iterations (placing squares of edge length
$r_1 = b$ on the vertices in $P_{1,1}$ and so on), altogether $v_1$ times. In each iteration we get a packing $P_{1,m}$ and a set $F_{1,m} := \text{cl}((F_{1,m-1} \setminus P_{1,m}) \cup B_{1,m})$, where $B_{1,m}$ always denotes the union of all black pieces (black squares or black parts of squares) in $P_{1,m}$. After $v_1$ iterations we cannot proceed by placing squares of edge length $b$ on vertices in $P_{1,v_1}$ without producing overlaps. Thus we proceed with the second step.

In the $n+1$-th step ($n \geq 1$), let $v_n = \left\lfloor \frac{r_{n-1}}{r_n} \right\rfloor$. Furthermore, let $r_{n+1} = r_{n-1} - v_n r_n$. Place squares of edge length $r_{n+1}$ on the vertices of all squares in $P_{n,1}$. Do $v_n$ iterations, to obtain the packing $P_{n,v_n}$, following the same rules as above (with 1 replaced by $n$ appropriately).

Note that after step $n - 1$ we are left with a packing $\Gamma + F_{n,v_n}$, which has rectangular holes of size $r_n \times r_{n+1}$. Note also that the sequence $(r_j)_j$ is—by definition of the $r_j$—the output of the Euclidean algorithm applied to $a, b$.

If $\frac{a}{b} \in \mathbb{Q}$ this yields a fundamental cell $F := F_{n,v_n}$ after finitely many steps. By construction, $F$ is a fundamental cell of $\Gamma$, and it has the desired symmetry group, namely, $D_4$. Clearly it is compact, since it is the finite union of compact sets.

![Figure 7](image_url)

**Figure 7.** A rectangular lattice and its Voronoi cells (left), the first three steps of the construction of a fundamental cell with $D_4$-symmetry (right).

![Figure 8](image_url)

**Figure 8.** The last two iterations of the construction of the $D_4$-symmetric fundamental cell of a lattice with basis vectors of length 5 and 8. These two iterations correspond to one ‘step’ in the proof of Theorem 1.1.
If \( \frac{a}{b} \notin \mathbb{Q} \) then the construction needs infinitely many steps, and we get a fundamental cell \( F \) in the limit. As above, \( F \) is closed and \( D_4 \)-symmetric by construction. \( F \) is compact, since \( F \) is bounded. \( F \) is bounded since the sequence \( (r_j)_j \) is the output of the Euclidean algorithm. The length of \( F \) in direction \( \left( \begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) \) is

\[
a + b + r_2 + \cdots = \sum_{i=0}^{\infty} r_i.
\]

It is known that the worst case—i.e., the slowest convergence of the partial fractions in the continued fraction expansion—is attained if \( a/b = \frac{1}{2}(\sqrt{5} + 1) =: \tau \). Here “slow convergence” holds in a pretty strong sense, i.e., \( \frac{1}{a} \sum_{i=0}^{\infty} r_i \) attains its maximal value for \( a/b = \frac{1}{2}(\sqrt{5} + 1) \), see Hurwitz’ theorem \cite[Theorem 193]{10}.

Hence the length of \( F \) in direction \( \left( \begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) \) is maximal for \( a/b = \tau \). Then we obtain \( r_i = a\tau^{-i} \), and the length of \( F \) in direction \( \left( \begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) \) is

\[
a \sum_{i=0}^{\infty} r_i = a \sum_{i=0}^{\infty} \tau^{-i} = a \frac{1}{1 - \tau^{-1}} = a\tau^2.
\]

Since \( F \) has the symmetry of a square, the diameter of \( F \) is at most \( \sqrt{2}a\tau^2 \).

It is not obvious how to devise some similar general construction of a fundamental cell for any rhombic lattice.

3. Dimension 3

Similar to the proof of Theorem \[\ref{thm:1.1}\] the proof of Theorem \[\ref{thm:1.2}\] consists of considering all possible cases. Fortunately, we can utilise Theorem \[\ref{thm:1.1}\] to cover most cases: we can use thickened, three-dimensional versions of the plane fundamental cells. In \( \mathbb{R}^3 \) there are 32 finite groups of Euclidean motions obeying the crystallographic restriction in Proposition \[\ref{prop:1.3}\], see \[\ref{6}\], Section 15.6. Only seven of them occur as point groups of lattices. Table \[\ref{tab:3}\] summarises the situation: The second column contains the name of the lattice, more precisely: the name of the family of lattices with a common symmetry group (the names as being used in crystallography). The third column contains the point group of the lattice in orbifold notation, the fourth column contains the order of the point group. The last column indicates the two-dimensional fundamental cell of Theorem \[\ref{thm:1.1}\] which yields a three-dimensional fundamental cell \( F \) for the current three-dimensional lattice, and the order \( |S(F)| \) in parentheses.

Since the list of finite groups of Euclidean motions in \( \mathbb{R}^3 \) is known, we know that there is no such group containing the group \(*432\) as a subgroup of index 2. (The only candidates—the ones of order 96—are the (non-primitive) groups \( C_{96}, D_{48} \) and \( C_2 \times D_{24} \), regarded as symmetry groups of solids in \( \mathbb{R}^3 \).) The corresponding lattices are the so-called cubic lattices: the primitive cubic lattice \( \mathbb{Z}^3 \), the body centred cubic lattice \( \mathbb{Z}^3 \cup (\mathbb{Z}^3 + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T) \) (bcc) and the face centred cubic lattice (fcc). So we cannot expect to find fundamental cells for these three cubic lattices with more symmetry than their point group \(*432\).

**Proof (of Theorem \[\ref{thm:1.2}\])**. We consider 6 cases (numbers 4-14 in Table \[\ref{tab:3}\] and Figure \[\ref{fig:9}\] identified if they have equal point groups). This will yield the entries of the last column of Table \[\ref{tab:3}\]
Figure 9. Illustrations of the 14 types of lattices in $\mathbb{R}^3$. The shaded nodes indicate how the lattices consist of layers of two-dimensional lattices. Angles omitted in the figure are assumed to be $\pi/2$ (or $\pi/3$ in 4). Edges labelled with equal letters are of equal length. The image is taken from [17] and only slightly modified.

which shows the name of the two-dimensional fundamental cell used, and the order of the symmetry group of the corresponding three-dimensional fundamental cell (in parentheses).

| Nr | Name                        | Point group | Order | 2-dim fundamental cell |
|----|-----------------------------|-------------|-------|------------------------|
| 1  | $\mathbb{Z}_3^3$           | $*432$      | 48    | —                      |
| 2  | body centred cubic          | $*432$      | 48    | —                      |
| 3  | face centred cubic          | $*432$      | 48    | —                      |
| 4  | Hexagonal                   | $*622$      | 24    | 12fold (48)            |
| 5  | Tetragonal primitive        | $*422$      | 16    | 8fold (32)             |
| 6  | Tetragonal body-centred     | $*422$      | 16    | 8fold (32)             |
| 7  | Rhombohedral                | 2 $\star$ 3 | 12    | 6fold (24) / 12fold(48) |
| 8  | Orthorhombic primitive      | $*222$      | 8     | 4fold (16)             |
| 9  | Orthorhombic base-centred   | $*222$      | 8     | 4fold (16)             |
| 10 | Orthorhombic body-centred   | $*222$      | 8     | 4fold (16)             |
| 11 | Orthorhombic face-centred   | $*222$      | 8     | 4fold (16)             |
| 12 | Monoclinic primitive        | 2$\star$    | 4     | 2fold (8) / 4fold(16)  |
| 13 | Monoclinic base-centred     | 2$\star$    | 4     | 2fold (8) / 4fold(16)  |
| 14 | Triclinic primitive         | 2$\star$    | 2     | mon.(4) / 2fold (8) / 4fold (16) |

Table 1. The 14 types of lattices with respect to their point groups.
Case 1: Hexagonal (4). The lattice consists of (equidistant) layers of hexagonal lattices. Attaching a thickened version of the fundamental cell $F$—say, $F := F \times [0, \ell]$, where $\ell$ is the distance of two adjacent layers—to each lattice point yields a tiling of $\mathbb{R}^3$. The symmetry group $S(F)$ of $F$ is $*122$, the new symmetries coming from rotating $F$ along an axis which is parallel to the layers of hexagonal lattices about $\pi$ (“turning $F$ upside down”).

Case 2: Tetragonal (5-6). These lattices consist of equidistant layers of square lattices. Thus we can use the thickened version of the fundamental cell $F$ of the square lattice, its symmetry being $*82$, having order 32.

Case 3: Rhombohedral (7). This lattice consists of equidistant layers of the hexagonal lattice $A_2$. So we can either use a thickened fundamental cell of $A_2$ with $D_6$-symmetry, yielding a three-dimensional fundamental cell $F$ with $S(F) = *62$, $|S(F)| = 24$ and index $[S(F) : P(\Gamma)] = 2$. Or we can use a thickened version of $F$ as in case 1, yielding a fundamental cell with $S(F) = *122$, $|S(F)| = 48$ and index $[S(F) : P(\Gamma)] = 4$.

Case 4: Orthorhombic (8-11). These lattices consist of equidistant layers of rectangular lattices. Thus we can use the thickened version of the fundamental cell of the rectangular lattice, its symmetry group being $*42$ of order 16.

Case 5: Monoclinic (12-13). These two lattices also consist of equidistant layers of rectangular lattices. Thus we can reason as in the preceding case.

Case 6: Triclinic (14). This lattice—or rather: these lattices—consist of layers of oblique lattices. We may use a right prism over a parallelogram as a fundamental cell, or a skew prism over a rectangle, or even a cuboid (erected on the rectangles of Figure 6). This yields symmetry groups $*2$ of the fundamental cell of order 4, or $*22$ of order 8, or $*222$ of order 16, respectively.

4. Conclusions and Outlook

The above results can imply several further questions. Here we mention a few we assume to be of possible interest.

4.1. Rhombic lattices. The reason that we excluded cubic lattices in Theorem 1.2 is that there exist no fundamental cells $F$ for the cubic lattices $\mathbb{Z}^3$, bcc or fcc such that $S(F)$ contains $P(\Gamma)$ ($\Gamma \in \{\mathbb{Z}^3, \text{bcc}, \text{fcc}\}$) as a proper subgroup of finite index. This is just because there are no such groups $S(F) \subset O(3)$. There still may be fundamental cells which have more symmetries in the sense that $|S(F)| > |P(\Gamma)|$ (but the author doubts it).

The reason that we excluded rhombic lattices in Theorem 1.1 is that we were not able to find a general construction for fundamental cells $F$ for any rhombic lattice $\Gamma$ such that $S(F)$ is larger than $P(\Gamma)$. One can construct such fundamental cells for several particular cases, but a general construction seems hard to obtain. If one tries to use the same idea as in the other non-obtuse cases—start with a packing of polygons of higher symmetry (octagon,
dodecagon, square) and refine—one runs into problems because the packings are in general not vertex-to-vertex from some point on.

4.2. **Even more symmetry.** Are there fundamental cells $F$ with $[S(F) : P(G)] > 2$? We have found a few ones: In the case of oblique plane lattices there is a rectangular fundamental cell with $[S(F) : P(G)] = 4$. In the case of the triclinic primitive lattice there is a cuboidal fundamental cell with $[S(F) : P(G)] = 8$. What is the maximal value of the index $[S(F) : P(G)]$ in $\mathbb{R}^d$ ($d \geq 2$)?

4.3. **Higher Dimensions.** The results in the present paper have been obtained by considering all different classes of lattices with respect to their symmetry group. There are 5 such classes in $\mathbb{R}^2$, 14 such classes in $\mathbb{R}^3$, 64 such classes in $\mathbb{R}^4$, 189 such classes in $\mathbb{R}^5$ and 826 such classes in $\mathbb{R}^6$ [2] [8] [12] [15] [16]. At some point it seems desirable to find more general arguments than case-by-case considerations. However, it is very likely that in higher dimensions there are several lattices $\Gamma$ with fundamental cells $F$ such that $P(\Gamma)$ is a proper subgroup of $S(F)$.

4.4. **Non-Euclidean spaces.** The constructions used in this paper work also in spherical or hyperbolic spaces, using spherical or hyperbolic regular $n$-gons. The fact that these $n$-gons are not similar to each other on different length scales does not matter. All we need is that there are edge-to-edge packings by regular $n$-gons on different length scales.

4.5. **Fractal Dimension.** The fundamental cells $F_\square$ of the square lattice, $F_\triangle$ of the hexagonal lattice and those of the rectangular lattices with incommensurate basis lengths are of fractal appearance. It might be possible to compute the Hausdorff dimensions of the boundaries of these cells, as well as other fractal dimensions, like the box-counting dimension or the affinity dimension [9], see also [14] and references therein. The two latter dimensions are particularly easy to compute if one finds an iterated function system (IFS) generating the fractal under consideration, see [14]. Up to the knowledge of the author, no IFS for $F_\square$ or $F_\triangle$ or the fundamental cells of rectangular lattices are known yet.

4.6. **Alternative Constructions.** The constructions used in this paper can be altered in many ways. For instance, there are other ways to partition the octagons, dodecagons and squares into two regions of different colours than the one used in the proof of Theorem 1.1. All that is required is to keep the mirror symmetry of the partition, and take care that no overlaps occur. One possibility is just to interchange the colours.

**Acknowledgements**

The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 247029.

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