Commensurations of the outer automorphism group of a universal Coxeter group

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Abstract

This paper studies the rigidity properties of the abstract commensurator of the outer automorphism group of a universal Coxeter group of rank $n$, which is the free product $W_n$ of $n$ copies of $\mathbb{Z}/2\mathbb{Z}$. We prove that for $n \geq 5$ the natural map $\text{Out}(W_n) \to \text{Comm}(\text{Out}(W_n))$ is an isomorphism and that every isomorphism between finite index subgroups of $\text{Out}(W_n)$ is given by a conjugation by an element of $\text{Out}(W_n)$.

1 Introduction

Given a group $G$, the abstract commensurator of $G$, denoted by $\text{Comm}(G)$, is the group of equivalence classes of isomorphisms between finite index subgroups of $G$. Two such isomorphisms are equivalent if they agree on some common finite index subgroup of their domain. Note that every automorphism of $G$ induces an element of $\text{Comm}(G)$, and in particular the action of $G$ on itself by global conjugation gives a homomorphism $G \to \text{Comm}(G)$.

The abstract commensurator of $G$ captures a notion of symmetry for the group that is weaker than its group of automorphisms. For instance, the abstract commensurator of $\mathbb{Z}^m$ is isomorphic to $\text{GL}(m, \mathbb{Q})$ while the abstract commensurator of a nonabelian free group is not finitely generated (see [BB]). However, some groups satisfy strong rigidity properties and the group $\text{Comm}(G)$ is then not much larger than $\text{Aut}(G)$ or $G$ itself. For instance, the Mostow-Prasad-Margulis rigidity theorem and Margulis arithmeticity theorem (see for instance [Zim]) imply that if $\Gamma$ is an irreducible lattice in a connected semisimple Lie group $G$ with trivial center and no compact factor, and if $G \neq \text{PSL}(2, \mathbb{R})$, then $\Gamma$ is a finite index subgroup of $\text{Comm}(\Gamma)$ if and only if $\Gamma$ is not arithmetic, otherwise $\text{Comm}(\Gamma)$ is dense in $G$. In the case of the extended mapping class group of a connected orientable closed surface $S_g$ of genus $g$ at least 3, we have an even stronger result due to Ivanov [Iva] since the natural homomorphism $\text{Mod}^\pm(S_g) \to \text{Comm}(\text{Mod}^\pm(S_g))$ is an isomorphism. This result also extends to the case of the mapping class group of
a connected orientable surface with genus equal to 2 and with at least two boundary components. In the context of the outer automorphism group of a free group $F_N$ of rank $N$, Farb and Handel ([FH]) proved that, for $N \geq 4$, the natural map from $\text{Out}(F_N)$ to $\text{Comm}(\text{Out}(F_N))$ is an isomorphism and that every isomorphism between two finite index subgroups of $\text{Out}(F_N)$ extends to an inner automorphism of $\text{Out}(F_N)$. This result was later extended by Horbez and Wade ([HW]) to the case $N = 3$ using a more geometric approach. Their techniques also enabled them to compute the abstract commensurator of many interesting subgroups of $\text{Out}(F_N)$, like its Torelli subgroup. These rigidity results have been extended to other groups, such as handlebody groups ([Hen]) and big mapping class groups ([BDR]).

In this article, we are interested in the outer automorphism group of a universal Coxeter group. Let $n$ be an integer greater than 1. Let $F = \mathbb{Z}/2\mathbb{Z}$ be a cyclic group of order 2 and $W_n = \ast_n F$ be a universal Coxeter group of rank $n$, that is a free product of $n$ copies of $F$. We prove the following theorem.

**Theorem 1.1.** Let $n \geq 5$. The natural homomorphism $$\text{Out}(W_n) \to \text{Comm}(\text{Out}(W_n))$$ is an isomorphism.

The group $\text{Out}(W_2)$ is finite and the group $\text{Out}(W_3)$ is isomorphic to $\text{PGL}(2, \mathbb{Z})$. This gives an almost complete classification except for $n = 4$, where our proof for $n \geq 5$ cannot be immediately adapted to this case as $\text{Out}(W_4)$ does not contain any direct product of two nonabelian free groups. Hence the case $n = 4$ remains open. Theorem 1.1 is a major improvement of [Gue1, Théorème 1.1] which states that, for $n \geq 5$, the only automorphisms of $\text{Out}(W_n)$ are the global conjugations. In turn, Theorem 1.1 implies that every isomorphism between two finite index subgroups of $\text{Out}(W_n)$ is given by a conjugation by an element of $\text{Out}(W_n)$. The proof of the present Theorem 1.1 significantly differs from the one of [Gue1, Théorème 1.1] since the proof of [Gue1, Théorème 1.1] is based on the study of torsion subgroups of $\text{Out}(W_n)$, whereas $\text{Out}(W_n)$ is virtually torsion free (see [GL3, Corollary 5.5]).

We now sketch our proof of Theorem 1.1. It is inspired by the proof of the similar result in the context of $\text{Out}(F_N)$ given by Horbez and Wade ([HW]). However, their proof relies extensively on the possibility of writing a free group as an HNN extension, which is not possible in a universal Coxeter group. Instead, we use the fact that $W_n$ can be written as a free product $W_n = A \ast B$, where $B$ is a finite abelian subgroup of $W_n$. Following a strategy that dates back to Ivanov’s work ([Iva]), we study the action of $\text{Out}(W_n)$ on various graphs which are rigid, that is, every graph automorphism is induced by an element of $\text{Out}(W_n)$. These graphs include the spine $K_n$ of the Outer space of $W_n$ as defined by Guirardel and Levitt in [GL3], generalizing Culler and Vogtmann’s Outer space of a free group ([CV]), or the free splitting graph $\overline{K}_n$ of $W_n$ (see [Gue2, Theorem 1.1 and 1.2] and Section 2.2 for definitions). The proof of Theorem 1.1 relies on the action of $\text{Out}(W_n)$ on a subset of the vertices of $\overline{K}_n$, called the set of $W_k$-stars. Let $k \in \{0, \ldots, n - 1\}$. A $W_k$-star is a free splitting $S$ of $W_n$ such that the underlying
graph of the induced graph of groups \(W_n/S\) is a tree with \(n - k\) edges, such that the degree of one of the vertices, called the center, is equal to \(n - k\), and such that the group associated with the center is isomorphic to \(W_k\) and the groups associated with the leaves are all isomorphic to \(F\). The \(W_k\)-stars are the analogue for \(W_n\) of the roses in the Outer space of a free group. They play a significant role in the proof of other rigidity results for \(\text{Out}(W_n)\) (see [Gue1, Gue2]).

This allows us to introduce a graph called the \emph{graph of one-edge compatible} \(W_{n-2}\)-stars, and denoted by \(X_n\). It is defined as follows: vertices are \(W_n\)-equivariant homeomorphism classes of \(W_{n-2}\)-stars, where two vertices \(S\) and \(S'\) are adjacent if there exist \(S \in S\) and \(S' \in S'\) such that \(S\) refines \(S'\) or conversely. We first show that every graph automorphism of \(X_n\) induces a graph automorphism of \(X'_n\) and that the induced map \(\text{Aut}(X_n) \to \text{Aut}(X'_n)\) is injective. Using the rigidity of \(X'_n\) (see Theorem [34]), we show that any graph automorphism of \(X_n\) is induced by an element of \(\text{Out}(W_n)\).

We then show that every commensuration \(f\) of \(\text{Out}(W_n)\) induces a graph automorphism of \(X_n\). Once we have that result, a general argument (see Proposition [21]) gives the isomorphism between \(\text{Out}(W_n)\) and \(\text{Comm}(\text{Out}(W_n))\). In order to construct such a homomorphism \(\text{Comm}(\text{Out}(W_n)) \to \text{Aut}(X_n)\), we first give an algebraic characterisation of the stabilizers of equivalence classes of \(W_{n-2}\)-stars. The characterization relies on the examination of maximal abelian subgroups of \(\text{Out}(W_n)\) and of direct products of nonabelian free groups in \(\text{Out}(W_n)\). In particular, we prove (see Theorem [51]), using the action of \(\text{Out}(W_n)\) on a simplicial complex called the \emph{free factor complex} of \(W_n\), the following result.

\textbf{Theorem 1.2.} Let \(n \geq 5\). The natural homomorphism

\[
\text{Out}(W_n) \to \text{Aut}(X_n)
\]

is an isomorphism.

Our proof of Theorem 1.2 requires the rigidity of another graph, called the \emph{graph of} \(W_n\)-\emph{stars}, and denoted by \(X'_n\). It is the graph whose vertices are the \(W_n\)-equivariant homeomorphism classes of \(W_{n-2}\)-stars with \(k\) varying in \(\{0, \ldots, n - 2\}\), where two vertices \(S\) and \(S'\) are adjacent if there exist \(S \in S\) and \(S' \in S'\) such that \(S\) refines \(S'\) or conversely. We first show that every graph automorphism of \(X'_n\) induces a graph automorphism of \(X'_n\) and that the induced map \(\text{Aut}(X_n) \to \text{Aut}(X'_n)\) is injective. Using the rigidity of \(X'_n\) (see Theorem [34]), we show that any graph automorphism of \(X'_n\) is induced by an element of \(\text{Out}(W_n)\).

One example of such a maximal direct product of nonabelian free subgroups of \(\text{Out}(W_n)\) is the following one. Let \(W_n = \langle x_1, \ldots, x_n \rangle\) be a standard generating set for \(W_n\) and let \(W = \langle x_1, x_2, x_3 \rangle\). For every \(i \geq 4\) and every \(w \in W\), let \(F_{i,w}\) be the automorphism which fixes \(x_j\) for every \(j \neq i\) and which sends \(x_i\) to \(wx_iw^{-1}\). Let \([F_{i,w}]\) be the outer automorphism class of \(F_{i,w}\) and let \(H_i = \langle [F_{i,w}] \rangle_{w \in W}\). Then the group \(\langle H_i \rangle_{i \geq 4}\) is a subgroup of \(\text{Out}(W_n)\) isomorphic to a direct product of \(n - 3\) nonabelian free groups.
The complete characterisation of stabilizers of equivalence classes of $W_{n-2}$-stars being quite technical, we do not give the complete statement in the introduction (see Propositions 6.9 and 7.7). However, we remark that this characterisation relies on the following key points: the fact that stabilizers of equivalence classes of $W_{n-2}$-stars contain a maximal free abelian subgroup and the fact that it contains a direct product of $n - 3$ nonabelian free groups. The characterisation also features a study of the group of twists of a $W_{n-2}$-star, which is a direct product of two virtually nonabelian free groups by a result of Levitt ([Lev]) and such that each of which has finite index in the centralizer in $\text{Out}(W_n)$ of the other.

This characterisation being preserved by commensurations of $\text{Out}(W_n)$, it induces a homomorphism from $\text{Comm}(\text{Out}(W_n))$ to the group $\text{Bij}(VX_n)$ of bijections of the set of vertices of $X_n$. In order to show that this map extends to the edge set of $X_n$, we also present an algebraic characterisation of compatibility of $W_{n-2}$-stars, which is essentially based on the fact that if the intersection of stabilizers of equivalence classes of $W_{n-1}$-stars contains a maximal abelian subgroup of $\text{Out}(W_n)$, then the $W_{n-1}$-stars are pairwise compatible (see Propositions 6.10 and 8.1). We deduce that the map $\text{Comm}(\text{Out}(W_n)) \to \text{Bij}(VX_n)$ extends to a map $\text{Comm}(\text{Out}(W_n)) \to \text{Aut}(X_n)$, which completes our proof.

Finally, we prove in the appendix the rigidity of another natural graph endowed with an $\text{Out}(W_n)$-action, called the graph of $W_{n-1}$-stars. It is the graph whose vertices are $W_n$-equivariant homeomorphism classes of $W_{n-1}$-stars, where two vertices $S$ and $S'$ are adjacent if there exist $S \in S$ and $S' \in S'$ such that $S$ and $S'$ have a common refinement. This graph arises naturally in the study of $\text{Out}(W_n)$ and its action on the free splitting graph $K_n$ as it is isomorphic to the full subgraph of $K_n$ whose vertices are the equivalence classes of $W_k$-stars, with $k$ varying in $\{0, \ldots, n - 1\}$. This gives another geometric rigid model for $\text{Out}(W_n)$ (see Theorem A.1).

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2 Preliminaries

2.1 Commensurations

Let $G$ be a group. The abstract commensurator of $G$, denoted by $\text{Comm}(G)$, is the group whose elements are the equivalence classes of isomorphisms between finite index subgroups of $G$ for the following equivalence relation. Two isomorphisms between finite index subgroups $f : H_1 \to H_2$ and $f' : H'_1 \to H'_2$ are equivalent if they agree on some common finite index subgroup $H$ of their domains. If $f$ is an isomorphism between finite index subgroups, we denote by $[f]$ the equivalence class of $f$. The identity of $\text{Comm}(G)$ is the equivalence class of the identity map on $G$. Let $[f], [f'] \in \text{Comm}(G)$, and let $f : H_1 \to H_2$ and $f' : H'_1 \to H'_2$ be representatives. The composition law $[f] \cdot [f']$ is given by $[f] \cdot [f'] = \{f \circ f' \}_{f^{-1}(H_1) \cap H'_2}$. Note that if $H$ is a finite index subgroup of $G$, then the natural map $\text{Comm}(G) \to \text{Comm}(H)$ obtained by restriction is an isomorphism.
Two subgroups $G_1$ and $G_2$ in $G$ are commensurable if $G_1 \cap G_2$ has finite index in both $G_1$ and $G_2$. Being commensurable is an equivalence relation. If $H$ is a subgroup of $G$, we will denote by $[H]$ its commensurability class in $G$. The group $\text{Comm}(G)$ acts on the set of all commensurability classes as follows. Let $[H]$ be the commensurability class of a subgroup $H$. Let $[f] \in \text{Comm}(G)$ and let $f : H_1 \to H_2$ be a representative of $[f]$. Then we define $[f] \cdot [H] = [f(H \cap H_1)]$.

The next result, due to Horbez and Wade, gives a sufficient condition for $\text{Comm}(G)$ to be rigid. It comes from ideas due to Ivanov when studying mapping class groups (see [Iva]). It requires the existence of a graph on which $G$ acts by graph automorphisms.

**Proposition 2.1.** [HW, Proposition 1.1] Let $G$ be a group. Let $X$ be a graph with no edge-loops, no multiple edges between pairs of vertices and such that $G$ acts on $X$ by graph automorphisms. Let $\text{Aut}(X)$ be the group of graph automorphisms of $X$. Assume that:

1. the natural homomorphism $G \to \text{Aut}(X)$ is an isomorphism,
2. given two distinct vertices $v$ and $w$ of $X$, the groups $\text{Stab}_G(v)$ and $\text{Stab}_G(w)$ are not commensurable in $G$,
3. the sets $\mathcal{I} = \{[\text{Stab}_G(v)] \mid v \in V_X\}$ and $\mathcal{J} = \{[[\text{Stab}_G(v)], [\text{Stab}_G(w)]] \mid vw \in E_X\}$ are $\text{Comm}(G)$-invariant (in the latter case with respect to the diagonal action).

Then any isomorphism $f : H_1 \to H_2$ between finite index subgroups of $G$ is given by the conjugation by an element of $G$ and the natural map $G \to \text{Comm}(G)$ is an isomorphism.

### 2.2 Free splittings and free factor systems of $W_n$

Let $n$ be an integer greater than 1. Let $F = \mathbb{Z}/2\mathbb{Z}$ be a cyclic group of order 2 and $W_n = \ast_n F$ be a universal Coxeter group of rank $n$. A **splitting** of $W_n$ is a minimal, simplicial $W_n$-action on a simplicial tree $S$ such that:

1. The finite graph $W_n \setminus S$ is not empty and not reduced to a point.
2. Vertices of $S$ with trivial stabilizer have degree at least 3.

Here minimal means that $W_n$ does not preserve any proper subtree of $S$. A splitting $S$ of $W_n$ is free if all edge stabilizers are trivial. A splitting $S'$ is a blow-up, or equivalently a refinement, of a splitting $S$ if $S$ is obtained from $S'$ by collapsing some edge orbits in $S'$. Two splittings are compatible if they have a common refinement. We define an equivalence class in the set of free splittings, where two splittings $S$ and $S'$ are equivalent if there exists a $W_n$-equivariant homeomorphism between them.

A **free factor system** of $W_n$ is a set $\mathcal{F}$ of conjugacy classes of subgroups of $W_n$ which arises as the set of all conjugacy classes of nontrivial point stabilizers in some (nontrivial) free splitting of $W_n$. Equivalently, there exist $k \in \mathbb{N} - \{0, 1\}$ and $[A_1], \ldots, [A_k]$
conjugacy classes of nontrivial, proper subgroups of \( W_n \) such that \( W_n = A_1 \cdots A_k \) and \( \mathcal{F} = \{ [A_1], \ldots, [A_k] \} \). The free factor system is \textit{sporadic} if \( k = 2 \), and \textit{nonsporadic} otherwise. The set of all free factor systems of \( W_n \) has a natural partial order, where \( \mathcal{F} \leq \mathcal{F}' \) if every factor of \( \mathcal{F} \) is conjugate into one of the factors of \( \mathcal{F}' \). Remark that if \( \{ x_1, \ldots, x_n \} \) is a standard generating set of \( W_n \), then for every free factor system \( \mathcal{F} \) of \( W_n \) and every \( i \in \{ 1, \ldots, n \} \), there exists \([A] \in \mathcal{F}\) such that \( x_i \) is conjugate into \( A \). In other words, the free factor system \( \{ [x_1], \ldots, [x_n] \} \) is a minimum for the partial order on the set of free factor systems of \( W_n \).

Let \( \mathcal{F} \) be a free factor system of \( W_n \). We denote by \( \text{Out}(W_n, \mathcal{F}) \) the subgroup of \( \text{Out}(W_n) \) consisting of all outer automorphisms that preserve all the conjugacy classes of subgroups in \( \mathcal{F} \). If \( \mathcal{F} = \{ [A_1], \ldots, [A_k] \} \), we denote by \( \text{Out}(W_n, \mathcal{F}(i)) \) the subgroup of \( \text{Out}(W_n, \mathcal{F}) \) consisting of all outer automorphisms which have a representative whose restriction to each \( A_i \) with \( i \in \{ 1, \ldots, k \} \) is a global conjugation by some \( g_i \in W_n \).

A \( (W_n, \mathcal{F}) \)-tree is an \( \mathbb{R} \)-tree equipped with a \( W_n \)-action by isometries and such that every subgroup of \( W_n \) whose conjugacy class belongs to \( \mathcal{F} \) is elliptic. A \textit{free splitting of} \( W_n \) \textit{relative to} \( \mathcal{F} \) is a free splitting of \( W_n \) such that every free factor in \( \mathcal{F} \) is elliptic. A \textit{free factor of} \( (W_n, \mathcal{F}) \) is a subgroup of \( W_n \) which arises as a point stabilizer in a free splitting of \( W_n \) relative to \( \mathcal{F} \). A free factor of \( (W_n, \mathcal{F}) \) is \textit{proper} if it is nontrivial, not equal to \( W_n \) and not conjugate to an element of \( \mathcal{F} \). An element \( g \in W_n \) is \( \mathcal{F} \)-\textit{peripheral} (or simply \textit{peripheral} if there is no ambiguity) if it is conjugate into one of the subgroups of \( \mathcal{F} \), and \( \mathcal{F} \)-\textit{nonperipheral} otherwise. In particular, for every free factor system \( \mathcal{F} \) of \( W_n \), and every element \( x \in W_n \) appearing in a standard generating set of \( W_n \), we see that \( x \) is \( \mathcal{F} \)-peripheral.

### 2.3 The Outer space of \((W_n, \mathcal{F})\)

We recall the definition of the \textit{unprojectivised Outer space of} \((W_n, \mathcal{F})\), denoted by \( \mathcal{O}(W_n, \mathcal{F}) \) and introduced by Guirardel and Levitt in [GL3]. It is the set of all \((W_n, \mathcal{F})\)-equivariant isometry classes \( \mathcal{S} \) of metric trees with a nontrivial action of \( W_n \), with trivial arc stabilizers and such that a subgroup is elliptic if and only if it is peripheral. The set \( \mathcal{O}(W_n, \mathcal{F}) \) is equipped with the Gromov-Hausdorff equivariant topology introduced in [Pau]. The \textit{projectivised Outer space of} \((W_n, \mathcal{F})\), denoted by \( \mathbb{P}\mathcal{O}(W_n, \mathcal{F}) \), is defined as the space of homothety classes of trees in \( \mathcal{O}(W_n, \mathcal{F}) \). The spaces \( \mathcal{O}(W_n, \mathcal{F}) \) and \( \mathbb{P}\mathcal{O}(W_n, \mathcal{F}) \) come equipped with a right action of \( \text{Out}(W_n, \mathcal{F}) \) given by precomposition of the actions.

The space \( \mathbb{P}\mathcal{O}(W_n, \mathcal{F}) \) has a natural structure of a simplicial complex with missing faces. Indeed, every element \( \mathcal{S} \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F}) \) defines an open simplex as follows. Let \( S \) be a representative of \( \mathcal{S} \) such that the sum of the edge lengths of \( W_n \backslash S \) is equal to 1. We associate an open simplex by varying the lengths of the edges, so that the sum of the edge lengths is still equal to 1. A homothety class \( \mathcal{S}' \in \mathbb{P}\mathcal{O}(W_n, \mathcal{F}) \) of a splitting \( \mathcal{S}' \) defines a codimension 1 face of the simplex associated with \( \mathcal{S} \) if we can obtain \( \mathcal{S}' \) from some representative \( S \) of \( \mathcal{S} \) by contracting one orbit of edges in \( S \).
The closure $\overline{O(W_n, F)}$ of Outer space in the space of all isometry classes of minimal nontrivial $W_n$-actions on $\mathbb{R}$-trees, equipped with the Gromov-Hausdorff equivariant topology, was identified in [Hor] with the space of all very small $(W_n, F)$-trees, which are the $(W_n, F)$-trees whose arc stabilizers are either trivial, or cyclic, root-closed and nonperipheral, and whose tripod stabilizers are trivial. The space $\mathbb{P}\overline{O(W_n, F)}$ equipped with the quotient topology is compact (see [Hor] Theorem 1)).

We recall the definition of a simplicial complex on which the space $\mathbb{P}\overline{O(W_n, F)}$ retracts Out$(W_n, F)$-equivariantly, called the spine of Outer space of $(W_n, F)$ and denoted by $K(W_n, F)$. It is the flag complex whose vertices are the $W_n$-equivariant homeomorphism classes $S$ of free splittings relative to $F$ with the property that, if $S \in S$, then all elliptic subgroups in $S$ are peripheral. Two vertices $S$ and $S'$ in $K(W_n, F)$ are linked by an edge if there exist $S \in S$ and $S' \in S'$ such that $R$ refines $S'$ or conversely. There is an embedding $K : K(W_n, F) \rightarrow \mathbb{P}\overline{O(W_n, F)}$ whose image is the barycentric spine of $\mathbb{P}\overline{O(W_n, F)}$. We will from now on identify $K(W_n, F)$ with $F(K(W_n, F))$. If $F$ consists of exactly $n$ copies of $F$, we simply write $K_n$ for $K(W_n, F)$. In this case the dimension of the simplicial complex $K_n$ is $n - 2$. Indeed, if $S$ is an equivalence class of a free splitting $S$ in $K_n$ such that the number of edges of $W_n \setminus S$ is minimal, then the number of edges in $W_n \setminus S$ is equal to $n - 1$. If $S$ is an equivalence class of a free splitting $S$ in $K_n$ such that the number of edges of $W_n \setminus S$ is maximal, then $W_n \setminus S$ has $n$ leaves and every vertex of $W_n \setminus S$ that is not a leaf has degree equal to 3. As $S$ is a tree, this shows that the number of edges in $W_n \setminus S$ is equal to $2n - 3$. Since, every splitting $S$ of $K_n$ collapes onto a splitting $S'$ such that $W_n \setminus S'$ has $n - 1$ edges, we see that the dimension of $K_n$ is equal to $2n - 3 - (n - 1) = n - 2$.

**Proposition 2.2.** Let $n \geq 3$. The virtual cohomological dimension of Out$(W_n)$ is equal to $n - 2$. In particular, the maximal rank of a free abelian subgroup of Out$(W_n)$ is equal to $n - 2$.

**Proof.** The group Out$(W_n)$ acts cocompactly on $K_n$ with finite stabilizers. Since the dimension of $K_n$ is equal to $n - 2$, since the Outer space $\mathbb{P}\overline{O(W_n)}$ is contractible (see [GL3, Theorem 4.2]) and since $\mathbb{P}\overline{O(W_n)}$ retracts Out$(W_n)$-equivariantly on $K_n$, we see that the virtual cohomological dimension of Out$(W_n)$ is at most equal to $n - 2$.

Conversely, let $\{x_1, \ldots, x_n\}$ be a standard generating set of $W_n$. For $i \in \{3, \ldots, n\}$, let $F_i$ be the automorphism sending $x_i$ to $x_1x_2x_i x_1$ and, for every $j \neq i$, fixing $x_j$, and let $[F_i]$ be its image in Out$(W_n)$. Remark that, for distinct $i, j \in \{3, \ldots, n\}$, the automorphisms $F_i$ and $F_j$ have disjoint support, hence they commute. Since, for every $i \in \{1, \ldots, n\}$, the outer automorphism $[F_i]$ has infinite order, the group $(\langle [F_i] \rangle)_{i \geq 3}$ is isomorphic to $\mathbb{Z}^{n-2.}$ This shows that the virtual cohomological dimension of Out$(W_n)$ is at least $n - 2$ and this concludes the proof. □

The free splitting graph of $W_n$, denoted by $\overline{K}_n$, is the following graph. The vertices of $\overline{K}_n$ are the $W_n$-equivariant homeomorphism classes of free splittings. Two distinct equivalence classes $S$ and $S'$ are joined by an edge in $\overline{K}_n$ if there exist $S \in S$ and $S' \in S'$ such that $S$ refines $S'$ or conversely. The free splitting graph of $W_n$ is the 1-skeleton
of the closure of $K_n$ in the space of free splittings of $W_n$. The group $\text{Aut}(W_n)$ acts on $\overline{K_n}$ on the right by precomposition of the action. As $\text{Inn}(W_n)$ acts trivially on $\overline{K_n}$, the action of $\text{Aut}(W_n)$ induces an action of $\text{Out}(W_n)$ on $\overline{K_n}$.

### 2.4 The free factor graph of $(W_n, F)$

Let $F$ be a free factor system of $W_n$. We now define a Gromov hyperbolic graph on which $\text{Out}(W_n, F)$ acts by isometries. The free factor graph relative to $F$, denoted by $\text{FF}(W_n, F)$, is the following graph. Its vertices are the $W_n$-equivariant homeomorphism classes of free splittings of $W_n$ relative to $F$. Two equivalence classes $S$ and $S'$ are joined by an edge if there exist $s \in S$ and $s' \in S'$ such that $S$ and $S'$ are compatible or share a common nonperipheral elliptic element. The free factor graph is always hyperbolic (see [BE, HM, GH2]). The next proposition is due to Guirardel and Horbez. Here, if $H$ is a subgroup of $\text{Out}(W_n)$ and if $F$ is a free factor system of $W_n$, we say that $F$ is $H$-periodic if there exists a finite index subgroup $H'$ of $H$ such that $H'(F) = F$.

**Proposition 2.3.** [GH2, Theorem 5.1] Let $n \geq 3$ and let $F$ be a nonsporadic free factor system of $W_n$. Let $H$ be a subgroup of $\text{Out}(W_n, F)$ which acts on $\text{FF}(W_n, F)$ with bounded orbits. Then there exists an $H$-periodic free factor system $F'$ such that $F \subseteq F'$ and $F \neq F'$.

The Gromov boundary of $\text{FF}(W_n, F)$ has been described in terms of relatively arational trees (see the work of Reynolds [Rev] for the definition of an arational tree in the context of a free group, the work of Bestvina-Reynolds and Hamenstädt ([BR, Ham] for the description of the boundary in the case of a free group, and the work of Guirardel-Horbez [GH2] in the case of a free product). A $(W_n, F)$-tree $T$ is arational if no proper $(W_n, F)$-free factor acts elliptically on $T$ and, for every proper $(W_n, F)$-free factor $A$, the $A$-minimal invariant subtree of $T$ (that is the union of the axes of the loxodromic elements of $A$ for the action of $W_n$ on $T$, see [CM] Proposition 3.1]) is a simplicial $A$-tree in which every nontrivial point stabilizer can be conjugated into one of the subgroups of $F$. We equip each arational $(W_n, F)$-tree with the observers’ topology: this is the topology on a tree $T$ such that a basis of open sets is given by the connected components of the complements of points in $T$. We equip the set of arational $(W_n, F)$-trees with an equivalence relation, where two arational $(W_n, F)$-trees are equivalent if they are $W_n$-equivariantly homeomorphic with the observers’ topology.

**Theorem 2.4.** [GH2, Theorem 3.4] Let $n \geq 3$. Let $F$ be a nonsporadic free factor system of $W_n$. The Gromov boundary of $\text{FF}(W_n, F)$ is $\text{Out}(W_n, F)$-equivariantly homeomorphic to the space of all equivalence classes of arational $(W_n, F)$-trees.

Given $T \in \overline{\text{O}(W_n, F)}$, let $[T]$ be the homothety class of $T$. The homothetic stabilizer $\text{Stab}([T])$ is the stabilizer of $[T]$ for the action of $\text{Out}(W_n, F)$ on $\overline{\text{O}(W_n, F)}$. Equivalently, $\Phi \in \text{Out}(W_n, F)$ lies in $\text{Stab}([T])$ if there exists a lift $\tilde{\Phi} \in \text{Aut}(W_n, F)$ of $\Phi$ and a homothety $I_{\Phi}: T \to T$ such that, for all $g \in W_n$ and $x \in T$, we have $I_{\Phi}(gx) = \tilde{\Phi}(g)I_{\Phi}(x)$. The scaling factor of $I_{\Phi}$ does not depend on the choice of a representative of $\Phi$, and we denote it by $\lambda_T(\Phi)$. This gives a homomorphism
\[
\text{Stab}([T]) \rightarrow \mathbb{R}^*_+ \\
\Phi \rightarrow \lambda_T(\Phi).
\]

The kernel of this morphism is called the isometric stabilizer of \( T \) and is denoted by \( \text{Stab}^0(T) \). It is the stabilizer of \( T \) for the action of \( \text{Out}(W_n,F) \) on \( \overline{\text{O}(W_n,F)} \).

**Lemma 2.5.** [GH2, Lemma 6.1] Let \( n \geq 3 \). Let \( F \) be a nonsporadic free factor system of \( W_n \). For every \( T \in \overline{\text{O}(W_n,F)} \), the image of the morphism \( \lambda_T \) is a cyclic (maybe trivial) subgroup of \( \mathbb{R}^*_+ \).

**Lemma 2.6.** [GH2, Proposition 13.5] Let \( n \geq 3 \). Let \( F \) be a nonsporadic free factor system of \( W_n \), and let \( H \) be a subgroup of \( \text{Out}(W_n,F) \). If \( H \) fixes a point in \( \overline{\text{v}_x \cdot FF(W_n,F)} \), then \( H \) has a finite-index subgroup that fixes the homothety class of an arational \((W_n,F)\)-tree.

Finally, we state a proposition due to Guirardel and Horbez concerning the isometric stabilizer of an arational tree.

**Proposition 2.7.** [GH2, Proposition 6.5] Let \( n \geq 3 \). Let \( F \) be a nonsporadic free factor system of \( W_n \), and let \( T \) be an arational \((W_n,F)\)-tree. Let \( H \) be a subgroup of \( \text{Out}(W_n,F) \) which is virtually contained in \( \text{Stab}^0(T) \). Then \( H \) has a finite index subgroup \( H' \) which fixes infinitely many \((W_n,F)\)-free splittings, and in particular \( H \) fixes the conjugacy class of a proper \((W_n,F)\)-free factor.

Note that the statement of Proposition 2.7 in [GH2] only mentions that \( H' \) fixes one \((W_n,F)\)-free splitting, but the proof uses an arbitrary free splitting of \( W_n \), so that one can construct infinitely many pairwise distinct free splittings fixed by \( H' \) by varying the chosen free splitting of \( W_n \).

### 2.5 Groups of twists

Let \( S \) be a splitting of \( W_n \), let \( v \in VS \), let \( e \) be a half-edge incident to \( v \), and let \( z \) be an element of \( C_{G_v}(G_e) \). We define the **twist by \( z \) around \( e \)** to be the automorphism \( D_{e,z} \) of \( W_n \) defined as follows (see [Lev]). Let \( \overline{S} \) be the splitting obtained from \( S \) by collapsing all the half-edges of \( S \) outside of the orbit of the initial half edge of \( e \). Then \( \overline{S} \) is a tree. Let \( \overline{e} \) be the image of \( e \) in \( \overline{S} \) and let \( \overline{v} \) be the image of \( v \) in \( \overline{S} \). Let \( \overline{v} \) be the endpoint of \( \overline{e} \) distinct from \( \overline{v} \). The automorphism \( D_{e,z} \) is defined to be the unique automorphism that acts as the identity on \( G_{\overline{e}} \) and as conjugation by \( z \) on \( G_{\overline{v}} \). The element \( z \) is called the **twistor of \( D_{e,z} \)**. It is well-defined up to composing on the right by an element of \( C_{W_n}(G_{\overline{e}}) \cap C_{G_v}(G_e) \). The **group of twists of \( S \)** is the subgroup of \( \text{Out}(W_n) \) generated by all twists around half-edges of \( S \).

We now give a description of the stabilizer of a point in \( \overline{\text{K}_n} \) due to Levitt. If \( S \in V \overline{\text{K}_n} \), we denote by \( \text{Stab}(S) \) the stabilizer of \( S \) under the action of \( \text{Out}(W_n) \). Let \( S \) be a representative of \( S \). We denote by \( \text{Stab}^0(S) \) the subgroup of \( \text{Stab}(S) \) consisting of all elements \( F \in \text{Out}(W_n) \) such that the graph automorphism induced by \( F \) on \( W_n \backslash S \) is the identity.
Proposition 2.8. [Lev, Propositions 2.2, 3.1 and 4.2] Let \( n \geq 4 \) and \( S \in \mathcal{S}_n \). Let \( S \) be a representative of \( S \) and let \( v_1, \ldots, v_k \) be the vertices of \( W_n \backslash S \) with nontrivial associated groups. For \( i \in \{1, \ldots, k\} \), let \( G_i \) be the group associated with \( v_i \).

1. The group \( \text{Stab}^0(S) \) fits in an exact sequence

\[
1 \to \mathcal{T} \to \text{Stab}^0(S) \to \prod_{i=1}^k \text{Out}(G_i) \to 1,
\]

where \( \mathcal{T} \) is the group of twists of \( S \).

2. The group \( \text{Stab}^0(S) \) is isomorphic to

\[
\prod_{i=1}^k G_i^{\deg(v_i)-1} \rtimes \text{Aut}(G_i),
\]

where \( \text{Aut}(G_i) \) acts on \( G_i^{\deg(v_i)-1} \) diagonally.

3. The group of twists \( \mathcal{T} \) of \( S \) is isomorphic to

\[
\mathcal{T} \cong \bigoplus_{i=1}^k G_i^{\deg(v_i)}/Z(G_i),
\]

where the center \( Z(G_i) \) of \( G_i \) is embedded diagonally in \( G_i^{\deg(v_i)} \).

Remark 2.9. In [Lev, Proposition 2.2], Levitt shows that the kernel of the natural homomorphism \( \text{Stab}^0(S) \to \prod_{i=1}^k \text{Out}(G_i) \) given by the action on the vertex groups is generated by bitwists. Since edge stabilizers are trivial, the group of bitwists is equal to the group of twists. More generally (see [Lev, Proposition 2.3]), if the outer automorphism group of every edge stabilizer is finite (in particular, if edge stabilizers are isomorphic to \( \mathbb{Z} \) or to \( F \)) then the group of twists is a finite index subgroup of the group of bitwists.

Finally, if the centralizer in \( W_n \) of an edge stabilizer is trivial, then the group of bitwists about this edge is trivial. Therefore, if the edge stabilizer is not cyclic, then the group of bitwists about this edge is trivial. In all cases, we see that, for every equivalence class \( S \) of a splitting \( S \) of \( W_n \), the group of twists of \( S \) is a finite index subgroup of the group of bitwists of \( W_n \).

We establish one last fact about twists about edges whose centralizer is cyclic (see [CL1, Lemma 5.3] for a similar statement in the context of the outer automorphism group of a nonabelian free group).

Lemma 2.10. Let \( n \geq 3 \) and let \( S \) be the equivalence class of a splitting \( S \). Suppose that there exists an edge \( e \) of \( S \) with cyclic stabilizer and let \( D \) be the outer automorphism class of a twist about \( e \). Let \( H_S \) be the subgroup of \( \text{Stab}^0(S) \) which induces the identity on the edge stabilizer \( G_e \) of \( e \). Then \( D \) is central in \( H_S \).

In particular, \( \text{Stab}^0(S) \) has a finite index subgroup \( H_S \) such that \( D \) is central in \( H_S \).
Theorem 3.3.

We introduce in this section a graph, \( \text{Out}(W_n) \), on which \( \text{Out}(W_n) \) acts by simplicial automorphisms. We prove that this graph is a rigid geometric model for \( \text{Out}(W_n) \). The proof relies on the study of the rigidity of an additional graph on which \( \text{Out}(W_n) \) acts, the graph of \( W_k \)-stars, to be defined after Theorem 3.3.

\[ 3 \] Geometric rigidity in the graph of \( W_k \)-stars

We start by defining \( W_k \)-stars, which are the main splittings of interest in this article.

Definition 3.1. Let \( n \geq 3 \), and let \( k \geq 1 \) be an integer.

1. A free splitting \( S \) is a \( k \)-edge free splitting if \( W_n \setminus S \) has exactly \( k \) edges.

2. Suppose that \( 0 \leq k \leq n - 2 \). A \( W_k \)-star is an \((n - k)\)-edge free splitting such that:

   - the underlying graph of \( W_n \setminus S \) has \( n - k + 1 \) vertices and one of them, called the center of \( W_n \setminus S \), has degree exactly \( n - k \),
   - the group associated with the center of \( W_n \setminus S \) is isomorphic to \( W_k \) (we use the convention that \( W_0 = \{1\} \) and that \( W_1 = F \)),
   - the group associated with any leaf of \( W_n \setminus S \) is isomorphic to \( F \).

3. A \( W_{n-1} \)-star is a one-edge free splitting \( S \) such that one of the vertex groups of \( W_n \setminus S \) is isomorphic to \( W_{n-1} \) while the other vertex group is isomorphic to \( F \).

Note that, in [Gue2], a \( W_{n-1} \)-star is called an \( F \)-one-edge free splitting. Using Proposition 2.8 (2), we see that, if \( k \in \{0, \ldots, n - 2\} \), and if \( S \) is the equivalence class of a \( W_k \)-star, then the group \( \text{Stab}^0(S) \) is isomorphic to \( F^{n-k-1} \rtimes \text{Aut}(W_k) \).

Note that, if \( S \) is a \( W_k \)-star with \( k \in \{0, \ldots, n - 2\} \) and \( S' \) is a splitting on which \( S \) collapses, then there exists \( \ell \in \{k, \ldots, n - 1\} \) such that \( S' \) is a \( W_\ell \)-star. In particular, for every \( k \in \{0, \ldots, n - 2\} \), if \( S \) is a \( W_k \)-star, then every one-edge free splitting on which \( S \) collapses is a \( W_{n-1} \)-star. A similar statement is also true for refinements of \( W_k \)-stars (see Lemma 3.3).

3.1 Rigidity of the graph of \( W_n \)-stars

We introduce in this section a graph, the graph of one-edge compatible \( W_{n-2} \)-stars, on which \( \text{Out}(W_n) \) acts by simplicial automorphisms. We prove that this graph is a rigid geometric model for \( \text{Out}(W_n) \). The proof relies on the study of the rigidity of an additional graph on which \( \text{Out}(W_n) \) acts, the graph of \( W_k \)-stars, to be defined after Theorem 3.3.
Definition 3.2. (1) The graph of $W_{n-2}$-stars, denoted by $\tilde{X}_n$, is the graph whose vertices are the $W_n$-equivariant homeomorphism classes of $W_{n-2}$-stars, where two equivalence classes $S$ and $S'$ are joined by an edge if there exist $S \in S$ and $S' \in S'$ such that $S$ and $S'$ are compatible.

(2) The graph of one-edge compatible $W_{n-2}$-stars, denoted by $X_n$, is the graph whose vertices are the $W_n$-equivariant homeomorphism classes of $W_{n-2}$-stars where two equivalence classes $S$ and $S'$ are joined by an edge if there exist $S \in S$ and $S' \in S'$ such that $S$ and $S'$ have a common refinement which is a $W_{n-3}$-star.

Note that the adjacency in the graph $X_n$ is equivalent to having both a common collapse (which is a $W_{n-1}$-star) and a common refinement. The graph $X_n$ is a subgraph of $\tilde{X}_n$. The group $\text{Aut}(W_n)$ acts on $\tilde{X}_n$ and $X_n$ by precomposition of the action. As $\text{Inn}(W_n)$ acts trivially on $X_n$, the action of Aut($W_n$) induces an action of Out($W_n$). We denote by Aut($X_n$) the group of graph automorphisms of $X_n$. In Section 3.2, we prove the following theorem.

Theorem 3.3. Let $n \geq 5$. The natural homomorphism

$$\text{Out}(W_n) \to \text{Aut}(X_n)$$

is an isomorphism.

In order to prove this theorem, we take advantage of the action of Out($W_n$) on another graph, namely the graph of $W_k$-stars, denoted by $X'_n$. The vertices of this graph are the $W_n$-equivariant homeomorphism classes of $W_k$-stars, with $k$ varying in $\{0, \ldots, n - 2\}$. Two equivalence classes $S$ and $S'$ are joined by an edge if there exist $S \in S$ and $S' \in S'$ such that $S$ refines $S'$ or conversely. Note that we have a natural embedding $X'_n \hookrightarrow K_n$. We identify from now on $X'_n$ with its image in $K_n$. In this section, we prove the following theorem.

Theorem 3.4. Let $n \geq 5$. The natural homomorphism

$$\text{Out}(W_n) \to \text{Aut}(X'_n)$$

is an isomorphism.

Theorem 3.4 relies on the fact that $X'_n$ contains a rigid subgraph, namely the graph of $\{0\}$-stars and $F$-stars, and denoted by $L_n$. The vertices of this graph are the $W_n$-equivariant homeomorphism classes of $\{0\}$-stars and $F$-stars. Two equivalence classes $S$ and $S'$ are joined by an edge if there exist $S \in S$ and $S' \in S'$ such that $S$ refines $S'$ or conversely.

We recall the following theorem.

Theorem 3.5. [Gue3, Theorem 3.1, Corollary 3.2] Let $n \geq 4$. Let $f$ be an automorphism of $L_n$ preserving the set of $\{0\}$-stars and the set of $F$-stars. Then $f$ is induced by the action of a unique element $\gamma$ of Out($W_n$). In particular, for every $n \geq 5$, the natural homomorphism

$$\text{Out}(W_n) \to \text{Aut}(L_n)$$

is an isomorphism.
The strategy in order to prove Theorem 3.4 is to show that every automorphism of $X'_n$ preserves $L_n$ and that the natural map $\text{Aut}(X'_n) \to \text{Aut}(L_n)$ is injective.

**Remark 3.6.** Using the same techniques, we may prove that the graph of $W_{n-1}$-stars is rigid. This is done in the appendix (see Theorem A.1).

First we recall a theorem due to Scott and Swarup.

**Theorem 3.7.** [SS, Theorem 2.5] Let $n \geq 4$. Any set $\{S_1, \ldots, S_k\}$ of pairwise nonequivalent, pairwise compatible, one-edge free splittings of $W_n$ has a unique refinement $S$ such that $W_n \setminus S$ has exactly $k$ edges. Moreover, the equivalence class of $S$ only depends on the equivalence classes of $S_1, \ldots, S_k$. If $S$ is a free splitting such that $W_n \setminus S$ has exactly $k$ edges, then $S$ refines exactly $k$ pairwise nonequivalent one-edge free splittings.

We also need the following lemma concerning refinements of $W_k$-stars.

**Lemma 3.8.** Let $k, \ell \in \{0, \ldots, n-2\}$ and let $S$ and $S'$ be respectively a $W_k$-star and a $W_\ell$-star. If $S$ and $S'$ have a common refinement, then there exists $j \in \{0, \ldots, n-2\}$ and a $W_j$-star $S''$ which refines both $S$ and $S'$. Moreover, $S''$ can be chosen such that $S''$ is a refinement of $S$ and $S'$ with the minimal number of orbits of edges.

**Proof.** Let $S_1, \ldots, S_{n-k}$ be $n-k$ $W_{n-1}$-stars onto which $S$ collapses and let $S'_1, \ldots, S'_{n-\ell}$ be $n-\ell$ $W_{n-1}$-stars onto which $S'$ collapses. Then the set $\{S_1, \ldots, S_{n-k}, S'_1, \ldots, S'_{n-\ell}\}$ is a set of pairwise compatible $W_{n-1}$-stars. For every $s \in \{1, \ldots, n-k\}$ and every $t \in \{1, \ldots, n-\ell\}$, let $S_s$ be the equivalence class of $S_s$ and $S'_t$ be the equivalence class of $S'_t$. Let $n-j = |\{S_1, \ldots, S_{n-k}, S'_1, \ldots, S'_{n-\ell}\}|$. By Theorem 3.7 there exists a free splitting $S''$ with $n-j$ edges which refines every $W_{n-1}$-star of the set $\{S_1, \ldots, S_{n-k}, S'_1, \ldots, S'_{n-\ell}\}$. But, as $F$ is freely indecomposable, a common refinement of two $W_{n-1}$-stars $U$ and $U'$ is obtained from $U$ by blowing-up an edge at the vertex of $W_n \setminus U$ whose associated group is isomorphic to $W_{n-1}$. Since $U'$ is also a $W_{n-1}$-star, this common refinement has two orbits of edges and the two corresponding leaves have a stabilizer isomorphic to $F$, hence it is a $W_{n-2}$-star. The same argument shows that, if $U_0$ is a $W_{n-1}$-star and if $U_1$ is a $W_k$-star with $k \in \{1, \ldots, n-1\}$ compatible with $U_0$, then a common refinement of $U_0$ and $U_1$ with a minimal number of orbits of edges is either a $W_k$-star (if the equivalence classes of $U_0$ and $U_1$ are adjacent in $\overline{K_n}$) or a $W_k$-star. Therefore, by induction on $i \in \{1, \ldots, n-\ell\}$, we see that a common refinement of $\{S_1, \ldots, S_{n-k}, S'_1, \ldots, S'_{n-\ell}\}$ with the minimal number of orbits of edges is a $W_j$-star. This shows that $S''$ is a $W_j$-star. This concludes the proof.

Lemma 3.8 implies that the set of $W_k$-stars with $k$ varying in $\{0, \ldots, n-1\}$ is closed under taking collapse and taking refinement with a minimal number of orbits of edges.

**Lemma 3.9.** Let $n \geq 5$. For every $f \in \text{Aut}(X'_n)$, we have $f(L_n) = L_n$. Moreover, if $f|_{L_n} = \text{id}_{L_n}$, then $f = \text{id}_{X'_n}$.

**Proof.** Let $f \in \text{Aut}(X'_n)$. The fact that $f(L_n) = L_n$ follows from the fact that vertices of $K_0 \cap X'_n$ in $X'_n$ are characterized by the fact that they are the vertices with finite valence. The proof is identical to the proof of [Gue2 Proposition 5.1].
Now suppose that \( f|_{L_n} = \text{id}_{L_n} \) and let \( S \) be the equivalence class of a \( W_{n-2} \)-star \( S \). Let us prove that \( f(S) = S \). Let \( \{x_1, \ldots, x_n\} \) be a standard generating set of \( W_n \) such that the free factor decomposition of \( W_n \) induced by \( S \) is

\[
W_n = \langle x_1 \rangle \ast \langle x_2, \ldots, x_{n-1} \rangle \ast \langle x_n \rangle.
\]

Let \( \mathcal{X} \) be the equivalence class of the \( F \)-star \( X \) depicted in Figure 1.

![Figure 1: The \( F \)-stars \( X \) (on the left) and \( \mathcal{X}' \) (on the right) of the proof of Lemma 3.9](image)

We see that \( S \) and \( \mathcal{X} \) are adjacent in \( X_n' \). Therefore, as \( f(\mathcal{X}) = \mathcal{X} \), we see that \( f(S) \) and \( \mathcal{X} \) are adjacent in \( X_n' \).

Let \( S' \) be the equivalence class of a \( W_{n-2} \)-star adjacent to \( \mathcal{X} \) and distinct from \( S \). Then, as \( \mathcal{X} \) and \( S' \) are adjacent, there exist distinct \( i, j \in \{1, \ldots, n\} \) and a representative \( S' \) of \( S' \) such that the free factor decomposition of \( W_n \) induced by \( S' \) is

\[
W_n = \langle x_1 \rangle \ast \langle x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n \rangle \ast \langle x_j \rangle.
\]

Since \( S \neq S' \), we may suppose that \( i \notin \{1, n\} \). But then \( S \) is adjacent to the equivalence class \( \mathcal{X}' \) of the \( F \)-star \( X' \) depicted in Figure 1 whereas \( S' \) is not adjacent to \( \mathcal{X}' \). Since \( f(\mathcal{X}') = \mathcal{X}' \), this shows that \( f(S) \neq S' \).

Finally, let \( k \in \{2, \ldots, n-3\} \) and let \( S^{(2)} \) be the equivalence class of a \( W_k \)-star \( S^{(2)} \) which is adjacent to \( \mathcal{X} \). We prove that \( f(S) \neq S^{(2)} \). Since \( k \leq n-3 \), the underlying graph of \( W_n \setminus S^{(2)} \) has at least 3 edges. Therefore, there exists \( i \notin \{1, n\} \) and a leaf \( v \) of the underlying graph of \( W_n \setminus S^{(2)} \) such that the preimage by the marking of \( W_n \setminus S^{(2)} \) of the generator of the group associated with \( v \) is \( x_i \). But then the equivalence class \( S^{(2)} \) is not adjacent to the equivalence class \( \mathcal{X}' \) of the \( F \)-star \( X' \) depicted in Figure 1. As \( S \) is adjacent to \( \mathcal{X}' \) and as \( f(\mathcal{X}') = \mathcal{X}' \), we see that \( f(S) \neq S^{(2)} \). Therefore, \( f(S) = S \).

The above paragraphs show that \( f \) fixes pointwise the set of equivalence classes of \( W_{n-2} \)-stars. Let \( k \in \{2, \ldots, n-3\} \) and let \( T \) be the equivalence class of a \( W_k \)-star \( T \). By Theorem 3.7, the equivalence class \( T \) is uniquely determined by the set of \( W_{n-1} \)-stars on which \( T \) collapses. Since two distinct equivalence classes of \( W_{n-2} \)-stars are adjacent in \( \overline{K_n} \) to distinct pairs of equivalence classes of \( W_{n-1} \)-stars, the equivalence class \( T \) is uniquely determined by the set of \( W_{n-2} \)-stars on which it collapses. Since \( f \) fixes pointwise the set of equivalence classes of \( W_{n-2} \)-stars, we see that \( f(T) = T \). Hence \( f = \text{id}_{X_n'} \). This concludes the proof.

**Proof of Theorem 3.4** Let \( n \geq 5 \). We first prove the injectivity. The homomorphism \( \text{Out}(W_n) \to \text{Aut}(L_n) \) is injective by Theorem 3.5. Moreover, the homomorphism
Figure 2: Two triangles in $X_n$, one corresponding to a $W_{n-3}$-star (on the left) and one corresponding to a $W_{n-4}$-star (on the right).

Out($W_n) \to \text{Aut}(L_n)$ factors through Out($W_n) \to \text{Aut}(X'_n) \to \text{Aut}(L_n)$. We therefore deduce the injectivity of Out($W_n) \to \text{Aut}(X'_n)$. We now prove the surjectivity. Let $f \in \text{Aut}(X'_n)$. By Lemma 3.9, we have a homomorphism $\Phi: \text{Aut}(X'_n) \to \text{Aut}(L_n)$ defined by restriction. By Theorem 3.5, the automorphism $\Phi(f)$ is induced by an element $\gamma \in \text{Out}(W_n)$. Since the homomorphism $\Phi$ is injective by Lemma 3.9, $f$ is induced by $\gamma$. This concludes the proof. \hfill \Box

3.2 Rigidity of the graph of one-edge compatible $W_{n-2}$-stars

In this section, we prove Theorem 3.8. In order to do so, we construct an injective homomorphism $\text{Aut}(X_n) \to \text{Aut}(X'_n)$. First, we need to show some technical results concerning the graph $X_n$. Indeed, let $\Delta$ be a triangle (that is, a cycle of length 3) in $X_n$, and let $S_1$, $S_2$ and $S_3$ be the vertices of this triangle. By Theorem 3.7, for every $i \in \{1, 2, 3\}$, there exists $S_i \in S$ such that $S_1$, $S_2$, and $S_3$ have a common refinement $S$, and we suppose that $S$ has the minimal number of orbits of edges among the common refinements of $S_1$, $S_2$, and $S_3$. Since $S_1$, $S_2$, and $S_3$ are $W_{n-2}$-stars, there exists $k \in \{0, \ldots, n-3\}$ such that $S$ is a $W_k$-star. By definition of the adjacency in $X_n$, the splitting $S$ is either a $W_{n-3}$-star or a $W_{n-4}$-star (see Figure 2). Our first result shows that we can distinguish these two types of triangles.

Lemma 3.10. Let $n \geq 5$. Let $S_1$, $S_2$, and $S_3$ be three equivalence classes of $W_{n-2}$-stars which are pairwise adjacent in $X_n$. Let $S_1$, $S_2$, and $S_3$ be representatives of $S_1$, $S_2$, and $S_3$ which have a common refinement $S$. Suppose that $S$ is the refinement of $S_1$, $S_2$, and $S_3$ which has the minimal number of orbit of edges. Then $S$ is a $W_{n-4}$-star if and only if there exists an equivalence class $S_4$ of a $W_{n-2}$-star $S_4$ distinct from $S_1$, $S_2$, and $S_3$ such that, for every $i \in \{1, 2, 3\}$, the equivalence classes $S_i$ and $S_4$ are adjacent in $X_n$.

Proof. Suppose first that $S$ is a $W_{n-4}$-star. Let $\{x_1, \ldots, x_n\}$ be a standard generating set of $W_n$ such that the free factor decomposition of $W_n$ induced by $S$ is

$W_n = \langle x_1 \rangle \ast \langle x_2 \rangle \ast \langle x_3 \rangle \ast \langle x_4 \rangle \ast \langle x_5, \ldots, x_n \rangle$.
Since being adjacent in $X_n$ is equivalent to having a common refinement which is a $W_{n-3}$-star and having a common collapse which is a $W_{n-1}$-star, the $W_{n-2}$-stars $S_1$ and $S_2$ share a common collapse $S'$ which is a $W_{n-1}$-star. Let $S'$ be the equivalence class of $S'$. We claim that there exists an orbit of edges $E$ in $S_3$ such that the splitting obtained from $S_3$ by collapsing every orbit of edges of $S_3$ except $E$ is in $S'$. Indeed, suppose towards a contradiction that this is not the case. Then, as for every $i \in \{1, 2\}$, the equivalence classes $S_i$ and $S_3$ are adjacent in $X_n$, we see that, for every $i \in \{1, 2\}$, the splittings $S_i$ and $S_3$ share a common collapse onto a $W_{n-1}$-star $S'$. Recall that we supposed that there does not exist an orbit of edges $E$ in $S_3$ such that the splitting obtained from $S_3$ by collapsing every orbit of edges of $S_3$ except $E$ is in $S'$. This implies that for every $i \in \{1, 2\}$, the equivalence class $S'_i$ of $S'_i$ is distinct from $S'$. Since $S_1$ and $S_2$ are $W_{n-2}$-stars, they collapse onto exactly 2 distinct $W_{n-1}$-stars. Therefore, for every $i \in \{1, 2\}$, the equivalence classes $S'_1$ and $S'_2$ are the two equivalence classes of $W_{n-1}$-stars onto which $S_i$ collapses. It follows that a common refinement of $S'_1$, $S'_2$ and $S'$ is also a common refinement of $S_1$, $S_2$ and $S_3$. But a common refinement of $S'_1$, $S'_2$ and $S'_3$ is a $W_{n-3}$-star. This contradicts the fact that $S$ has the minimal number of edges among common refinements of $S_1$, $S_2$ and $S_3$. Thus $S_3$ collapses onto a $W_{n-1}$-star in the equivalence class $S'$. Let $j \in \{1, \ldots, 4\}$ be such that the free factor decomposition of $W_n$ induced by $S'$ is:

$$W_n = \langle x_j \rangle \ast \langle x_1, \ldots, \hat{x}_j, \ldots, x_n \rangle.$$ 

Let $S_4$ be the equivalence class of the $W_{n-2}$-star $S_4$ whose induced free factor decomposition is:

$$W_n = \langle x_j \rangle \ast \langle x_1, \ldots, \hat{x}_j, \ldots, x_n \rangle \ast \langle x_5 \rangle.$$ 

Then, for every $i \in \{1, 2, 3\}$, the equivalence classes $S_i$ and $S_4$ are adjacent in $X_n$.

Conversely, suppose that $S$ is a $W_{n-3}$-star. Let $\{x_1, \ldots, x_n\}$ be a standard generating set of $W_n$ such that the free factor decomposition of $W_n$ induced by $S$ is

$$W_n = \langle x_1 \rangle \ast \langle x_2 \rangle \ast \langle x_3 \rangle \ast \langle x_4, \ldots, x_n \rangle.$$ 

Then, up to reordering, we may suppose that, for every $i \in \{1, 2, 3\}$ the free factor decomposition of $W_n$ induced by $S_i$ is:

$$W_n = \langle x_i \rangle \ast \langle x_{i+1} \rangle \ast \langle x_1, \ldots, \hat{x}_i, x_{i+1}, \ldots, x_n \rangle,$$

where, for $i = 3$, the index $i + 1$ is taken modulo 3. Let $S'$ be the equivalence class of a $W_{n-2}$-star $S'$ adjacent to $S_1$ in $X_n$ and distinct from $S_2$ and $S_3$. Then, up to changing the representative $S'$, there exists $j \in \{1, 2\}$ such that $S'$ collapses onto the $W_{n-1}$-star whose associated free factor decomposition is:

$$W_n = \langle x_j \rangle \ast \langle x_1, \ldots, \hat{x}_j, \ldots, x_n \rangle.$$ 

If $j = 1$, then, as $S'$ is distinct from $S_1$ and $S_3$, we see that $S'$ is not adjacent to $S_2$ in $X_n$. If $j = 2$, then, as $S'$ is distinct from $S_1$ and $S_2$, we see that $S'$ is not adjacent to $S_3$ in $X_n$. In both cases, we see that there exists $i \in \{2, 3\}$ such that $S'$ is not adjacent to $S_i$. This concludes the proof. □
Corollary 3.11. Let \( n \geq 5 \). Let \( k \geq 4 \) and let \( S_1, \ldots, S_k \) be \( k \) equivalence classes of \( W_{n-2} \)-stars which are pairwise adjacent in \( X_n \). For \( i \in \{1, \ldots, k\} \), let \( S_i \) be a representative of \( S_i \). Let \( S \) be a refinement of \( S_1, \ldots, S_k \) whose number of orbits of edges is minimal. Then \( S \) is a \( W_{n-k-1} \)-star.

Proof. For every distinct \( i, j \in \{1, \ldots, k\} \), the equivalence classes \( S_i \) and \( S_j \) are adjacent in \( X_n \). Hence, for every distinct \( i, j \in \{1, \ldots, k\} \), there exists a common refinement of \( S_i \) and \( S_j \) which is a \( W_{n-3} \)-star. This implies that, for every \( p \in \{1, \ldots, k\} \) and for every \( i_1, \ldots, i_p \in \{1, \ldots, k\} \), a common refinement of \( S_{i_1}, \ldots, S_{i_p} \) is obtained from a common refinement of \( S_{i_1}, \ldots, S_{i_{p-1}} \) whose number of orbits of edges is minimal by adding at most one orbit of edges. We claim that a common refinement of \( S_{i_1}, \ldots, S_{i_p} \) whose number of orbits of edges is minimal has exactly \( p + 1 \) orbits of edges. Indeed, otherwise there would exist \( i, j, \ell \in \{1, \ldots, k\} \) pairwise distinct such that a \( W_{n-3} \)-star which refines both \( S_i \) and \( S_j \) also refines \( S_\ell \). This is not possible by Lemma 3.10 since \( k \geq 4 \). This proves the claim. Taking \( p = k \) concludes the proof of the lemma.

Proposition 3.12. Let \( n \geq 5 \). There exists a \( \text{Out}(W_n) \)-equivariant injective homomorphism \( \Phi : \text{Aut}(X_n) \to \text{Aut}(X'_n) \).

![Figure 3: The construction of the map \( \text{Aut}(X_n) \to \text{Aut}(X'_n) \).](image)

Proof. We first explicit a map \( \Phi : \text{Aut}(X_n) \to \text{Bij}(VX'_n) \). Let \( f \in \text{Aut}(X_n) \). Let \( k \in \{0, \ldots, n-2\} \) and let \( S \) be the equivalence class of a \( W_k \)-star \( S \). If \( k = n-2 \), then we set \( \Phi(f)(S) = f(S) \). If \( k \leq n-3 \), let \( S_0 \) be a \( W_{n-1} \)-star refined by \( S \). Let \( S_1, \ldots, S_{n-k-1} \) be the \( W_{n-2} \)-stars such that, for every \( i \in \{1, \ldots, n-k-1\} \), \( S \) refines \( S_i \) and \( S_i \) refines \( S_0 \) (see Figure 3). For every \( i \in \{1, \ldots, n-k-1\} \), let \( S_i \) be the equivalence class of \( S_i \), and let \( T_i \) be a representative of \( f(S_i) \). By Corollary 3.11 if \( n-k-1 \geq 4 \), the \( W_{n-2} \)-stars \( T_1, \ldots, T_{n-k-1} \) are refined by a \( W_k \)-star \( T' \). This \( W_k \)-star is unique up to \( W_n \)-equivariant homeomorphism by Theorem 3.7. In the case where \( k = n-3 \), we have \( n-k-1 = 2 \) and, since \( f(S_1) \) and \( f(S_2) \) are adjacent in \( X_n \), the splittings \( T_1 \) and \( T_2 \) are refined by a \( W_{n-3} \)-star \( T' \) and it is unique up to \( W_n \)-equivariant homeomorphism by Theorem 3.7. Finally, when \( k = n-4 \), Lemma 3.10 implies that a common refinement of \( T_1, T_2 \) and \( T_3 \) with the minimal number of orbits of edges is a \( W_{n-4} \)-star \( T' \) and it is unique up to \( W_n \)-equivariant homeomorphism by Theorem 3.7. In all cases, let \( T' \) be the equivalence class of \( T' \). We set \( \Phi(f)(S) = T' \).

We now prove that \( \Phi \) is well-defined. Let \( k \in \{0, \ldots, n-2\} \) and let \( S \) be the equivalence class of a \( W_k \)-star \( S \). Let \( S_0 \) and \( S'_0 \) be two distinct \( W_{n-1} \)-stars onto which \( S \) collapses and let \( S_0 \) and \( S'_0 \) be their equivalence classes. Let \( S_1, \ldots, S_{n-k-1} \) be the
$W_{n-2}$-stars such that, for every $i \in \{1, \ldots, n-k-1\}$, $S$ refines $S_i$ and $S_i$ refines $S_0$ and let $S'_1$, $\ldots$, $S'_{n-k-1}$ be the $W_{n-2}$-stars such that, for every $i \in \{1, \ldots, n-k-1\}$, $S$ refines $S'_i$ and $S'_i$ refines $S'_0$. For $i \in \{1, \ldots, n-k-1\}$, let $S_i$ be the equivalence class of $S_i$ and let $S'_i$ be the equivalence class of $S'_i$. For every $i \in \{1, \ldots, n-k-1\}$, let $T_i$ be a representative of $f(S_i)$ and let $T'_i$ be a representative of $f(S'_i)$. Let $T$ be a $W_k$-star which refines $T_1, \ldots, T_{n-k-1}$ and let $T'$ be a $W_k$-star which refines $T'_1, \ldots, T'_{n-k-1}$. Finally, let $T$ be the equivalence class of $T$ and let $T'$ be the equivalence class of $T'$. We claim that $T = T'$. Indeed, we first remark that there exist $i, j \in \{1, \ldots, n-k-1\}$ such that $S_i = S'_j$; it is the equivalence class of the $W_{n-2}$-star which refines both $S_i$ and $S'_0$. Up to reordering, we may suppose that $i = j = 1$, that $S_1 = S'_1$ and that $T_1 = T'_1$. Therefore, both $T$ and $T'$ collapse onto $T_1$.

Let $U_2, \ldots, U_{n-k-1}$ be the $W_{n-3}$-stars such that, for every $j \in \{2, \ldots, n-k-1\}$, the $W_{n-3}$-star $U_j$ refines $S_1$ and $U_j$ is refined by $S$. For every $j \in \{2, \ldots, n-k-1\}$ there exist $\ell, \ell' \in \{2, \ldots, n-k-1\}$ such that $S_\ell$ and $S_{\ell'}$ are refined by $U_j$. Therefore, the application $g: \{2, \ldots, n-k-1\} \rightarrow \{2, \ldots, n-k-1\}$ sending $\ell$ to $\ell'$ is a bijection. Thus, we may suppose that $g$ is the identity, that is, we may suppose that $\ell = \ell'$. It follows that for every $j \in \{2, \ldots, n-k-1\}$, the equivalence class of the $W_{n-3}$-star which refines $S_1$ and $S_j$ is the same one as the equivalence class of the $W_{n-3}$-star which refines $S_1$ and $S'_j$. Therefore, for every $i \in \{2, \ldots, n-k-1\}$, the set $\{S_1, S_i, S'_i\}$ defines a triangle in $X_n$ which corresponds to the equivalence class of a $W_{n-3}$-star. By Lemma 3.10, for every $i \in \{2, \ldots, n-k-1\}$, the set $\{f(S_i), f(S'_i), f(S'_i)\}$ defines a triangle in $X_n$ which corresponds to the equivalence class of a $W_{n-3}$-star. Thus, up to changing the representative $T'_i$, for every $i \in \{1, \ldots, n-k-1\}$, the $W_{n-3}$-star which refines $T_1$ and $T_i$ is the same one as the $W_{n-3}$-star which refines $T_1$ and $T'_i$. As $T$ and $T'$ are characterized by the set of equivalence classes of $W_{n-3}$-stars which collapses onto $T_1$ and on which $T$ and $T'$ collapse, we see that $T = T'$. Therefore, the map $\Phi(f): VX_n \rightarrow VX_n$ is well-defined. As $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \text{id}$, we see that $\Phi(f)$ is a bijection.

We now prove that the application $\Phi: \text{Aut}(X_n) \rightarrow \text{Bij}(VX_n)$ induces a monomorphism $\bar{\Phi}: \text{Aut}(X_n) \rightarrow \text{Aut}(X'_n)$. Let $f \in \text{Aut}(X_n)$ and let us prove that $\bar{\Phi}(f)$ preserves $EX_n'$. Let $S, S'$ be adjacent vertices in $X_n'$. Up to exchanging the roles of $S$ and $S'$, we may suppose that there exist $S \in S$ and $S' \in S'$ such that $S'$ collapses onto $S$. Let $k, \ell \in \{1, \ldots, n-2\}$ be such that $S$ is a $W_k$-star and $S'$ is a $W_{k-\ell}$-star. Let $S_0$ be a $W_{n-1}$-star such that $S$ refines $S_0$. Let $S_1, \ldots, S_{n-k-1}$ be the $W_{n-2}$-stars such that, for every $i \in \{1, \ldots, n-k-1\}$, $S$ refines $S_i$ and $S_i$ refines $S_0$. As $S'$ refines $S$, there exist $\ell$ $W_{n-2}$-stars $S_{n-k}, S_{n-k+\ell-1}$ such that the $W_{n-2}$-stars $S_1, \ldots, S_{n-k+\ell-1}$ are the $n-k+\ell-1$ $W_{n-2}$-stars which collapse onto $S_0$ and which are refined by $S'$. For every $i \in \{1, \ldots, n-k+\ell-1\}$, let $S_i$ be the equivalence class of $S_i$. By definition of $\Phi(f)$, there exist a representative $T$ of $\Phi(f)(S)$ and representatives $T_1, \ldots, T_{n-k-1}$ of $f(S_1), \ldots, f(S_{n-k-1})$ such that $T$ is a common refinement of $T_1, \ldots, T_{n-k-1}$. Moreover, there exist a representative $T'$ of $\Phi(f)(S')$ and representatives $T_{n-k}, \ldots, T_{n-k+\ell-1}$ of $f(S_{n-k}), \ldots, f(S_{n-k+\ell-1})$ such that $T'$ is a common refinement of $f(S'_1), \ldots, f(S'_{n-k+\ell-1})$. As $\{f(S'_1), \ldots, f(S'_{n-k+\ell-1})\}$ is a subset of $\{f(S'_1), \ldots, f(S_{n-k+\ell-1})\}$, we see that $f(S)$ and $f(S')$ are adjacent. This shows that the
Proof. The underlying graph 
free splitting by Lemma 4.2. Theorem 4.1. Since 
unique vertex group. Moreover, for every 
v ∈ L, there exist k ∈ {0, . . . , n − 1} and

4 The group of twists of a Wn−1-star

In this section, we study the centralizers in Out(Wn) of twists about a Wn−1-star. We first need some preliminary results about stabilizers of free factors of Wn isomorphic to Wn−1.

Let \{x1, . . . , xn\} be a standard generating set of Wn. For distinct i, j ∈ {1, . . . , n}, let σij : Wn → Wn be the automorphism sending xj to xixjxi and, for k ≠ j, fixing xk. For distinct i, j ∈ {1, . . . , n}, let (i j) be the automorphism of Wn switching xixj and, for k ≠ i, j, fixing xk. The following theorem is due to Mühlherr.

Theorem 4.1. [Mühl Theorem B] Let n ≥ 2. The set \{σij | i ≠ j\} ∪ {(i j) | i ≠ j} is a generating set of Aut(Wn).

We now introduce a finite index subgroup of Out(Wn) which will be used throughout the remainder of this paper. For every i, j ∈ {1, . . . , n} distinct, both σij and (i j) preserve the set of conjugacy classes \{[x1], . . . , [xn]\}. Since \{σij | i ≠ j\} ∪ {(i j) | i ≠ j} generates Aut(Wn) by Theorem 4.1, we see that we have a well-defined homomorphism Out(Wn) → Bij([x1], . . . , [xn]). Let Cn be the kernel of this homomorphism. The group Cn has finite index in Out(Wn). We will mostly work in Cn from now on because of the following lemma.

Lemma 4.2. Let n ≥ 3 and let f ∈ Cn. Suppose that f fixes the equivalence class S of a free splitting S. Then the graph automorphism of the underlying graph of Wn\S induced by f is the identity. Therefore we have StabCn(S) = Stab0Cn(S).

Proof. The underlying graph Wn\S of Wn\S is a tree. Moreover, since S is a free splitting, if L is the set of leaves of Wn\S, then the set \{[Gv] | v ∈ L\} is a free factor system of Wn. Note that, as \{[x1], . . . , [xn]\} is a free factor system of Wn which is minimal for inclusion, for every i ∈ {1, . . . , n}, there exists one \(v \in VS\) such that \(x_i \in G_v\). Since S is a free splitting, for every i ∈ {1, . . . , n}, the element xi is contained in a unique vertex group. Moreover, for every v ∈ L, there exist k ∈ {0, . . . , n − 1} and
\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\} such that \(G_v\) is isomorphic to \(W_k\) and \([x_{i_1}] \cap G_v, \ldots, [x_{i_k}] \cap G_v\) is a free factor system of \(G_v\). As \(f \in C_n\), and as \(f\) fixes \(S\), it follows that, for every \(v \in L\), we have \(f([G_v]) = [G_v]\). Hence the graph automorphism \(\hat{f}\) of \(W_n/\sim\) induced by \(f\) acts as the identity on \(L\). As any graph automorphism of a finite tree is determined by its action on the set of leaves, it follows that \(\hat{f} = \text{id}\). This concludes the proof. 

\begin{remark}

The subgroup \(C_n\) of \(\text{Out}(W_n)\) is our (weak) analogue of the subgroup \(\text{IA}_N(\mathbb{Z}/3\mathbb{Z})\) of \(\text{Out}(F_N)\), which is defined as the kernel of the natural homomorphism \(\text{Out}(F_N) \to \text{GL}(N, \mathbb{Z}/3\mathbb{Z})\). Indeed, the group \(\text{IA}_N(\mathbb{Z}/3\mathbb{Z})\) satisfies a statement similar to Lemma 4.2, but it has the additional property that if \(\phi \in \text{IA}_N(\mathbb{Z}/3\mathbb{Z})\) has a periodic orbit in the free splitting graph of \(F_N\), then the cardinality of this orbit is equal to 1. In the context of \(C_n\), we do not know if \(C_n\) contains a torsion free finite index subgroup which satisfies this property.
\end{remark}

The next lemma relates the stabilizer of a free factor of \(W_n\) isomorphic to \(W_{n-1}\) and the stabilizer of a \(W_{n-1}\)-star.

\begin{lemma}

Let \(n \geq 3\). Let \(A\) be a free factor of \(W_n\) isomorphic to \(W_{n-1}\). Then, up to \(W_{n-1}\)-equivariant homeomorphism, there exists a unique free splitting \(S\) in which \(A\) is elliptic. In particular, if \(f \in \text{Out}(W_n)\) is such that \(f([A]) = [A]\), then \(f\) fixes the equivalence class of \(S\).
\end{lemma}

\begin{proof}

By definition of a free factor, there exists a free splitting \(S\) of \(W_n\) such that \(A\) is elliptic in \(S\). This proves the existence. We now prove the uniqueness statement. We may assume that \(\{x_1, \ldots, x_{n-1}\}\) is a standard generating set of \(A\) and \(x_n \in W_n\) is such that 

\[W_n = A \ast \langle x_n \rangle.\]

Then, the free factor system \(\mathcal{F} = \{[A], \langle x_n \rangle\}\) is a sporadic free factor system which contains \([A]\). Let \(\mathcal{F}'\) be a free factor system of \(W_n\) which contains \([A]\). Since the free factor system \(\{\langle x_1 \rangle, \ldots, \langle x_n \rangle\}\) is the minimal element of the set of free factor systems of \(W_n\), we see that there exists \([B] \in \mathcal{F}'\) such that \(x_n \in B\). As \(\mathcal{F}'\) contains \([A]\) and as \(W_n = A \ast \langle x_n \rangle\), it follows that \(W_n = A \ast B\) and that \(B \subseteq \langle x_n \rangle\). Therefore \([B] = \langle x_n \rangle\) and \(\mathcal{F}' = \{[A], \langle x_n \rangle\}\). We deduce that \(\mathcal{F}\) is the unique nontrivial free factor system which contains \([A]\). But the spine \(K(W_n, \mathcal{F})\) of the Outer space relative to \(\mathcal{F}\) is reduced to a point, i.e. it is reduced to a unique equivalence class of free splittings. This proves the uniqueness statement.
\end{proof}

\begin{remark}

In the context of \(\text{Out}(F_N)\), the analogue of the splitting given by Lemma 4.4 is the following one. Let \([A]\) be the conjugacy class of a free factor of \(F_N\) isomorphic to \(F_{N-1}\). Then the canonical splitting associated with \(A\) is the splitting corresponding to the HNN extension \(F_N = A\ast\) over the trivial group. However, there does not exist a natural choice (up to conjugacy) of an element \(g \in F_N\) such that \([A], [g]\) is a free factor system of \(F_N\).
\end{remark}

Let \(S\) be a splitting with exactly one orbit of edges, whose stabilizer is root-closed and isomorphic to \(\mathbb{Z}\). Then the group of twists of \(S\) is isomorphic to \(\mathbb{Z}\) by a result of
Let $w \in W_n$ be the root-closed element of infinite order. Let $x \in W_n$ be such that $W_n = A \ast \langle x \rangle$. Let $S$ be the equivalence class of a splitting $S$ whose associated amalgamated decomposition of $W_n$ is the following:

$$W_n = A \ast \langle w \rangle \langle \langle w \rangle * \langle x \rangle \rangle.$$  

Let $D$ be a nontrivial twist about $S$. Let $\mathcal{R}$ be the equivalence class of a free splitting $R$ of $W_n$ such that $D(\mathcal{R}) = \mathcal{R}$. Let $\mathcal{R}'$ and $S'$ be metric representatives of $R$ and $S$, let $\mathcal{R}'$ and $S'$ be their $W_n$-equivariant isometry classes and let $[\mathcal{R}']$ and $[S']$ be their homothety classes.

1. In $\mathbb{PO}(W_n)$, there exists an increasing function $\psi: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\lim_{n \rightarrow \infty} D^{\psi(n)}([\mathcal{R}']) = [S'].$$  

2. The splittings $S$ and $R$ are compatible.

**Proof.** We prove the first part. As $\mathbb{PO}(W_n)$ is compact, up to passing to a subsequence, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \in (\mathbb{R}_{+}^{*})^\mathbb{N}$ and an $W_n$-equivariant isometry class $\mathcal{T}$ of an $\mathbb{R}$-tree $T$ such that

$$\lim_{n \rightarrow \infty} \lambda_n D^n(\mathcal{R'}) = \mathcal{T}.$$  

Since translation length functions are continuous for the Gromov-Hausdorff topology (see [Pau]), for every $g \in W_n$, we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_T,$$

where $\|g\|_T$ is the translation length of $g$ in $T$. Hence, for every $g \in W_n$, the limit $\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \|g\|_T$ is finite. But as $D$ has infinite order, we have $\lim_{n \rightarrow \infty} \lambda_n = 0$. As there exists a representative $\phi \in \text{Aut}(W_n)$ of $D$ such that $\phi_A = \text{id}_A$, for every $g \in A$, we have:

$$\lim_{n \rightarrow \infty} \lambda_n \|g\|_{D^n(\mathcal{R}')} = \lim_{n \rightarrow \infty} \lambda_n \|g\|_{\mathcal{R}'} = 0.$$  

Hence every element of $A$ fixes a point in $T$. As $A$ is finitely generated, this implies that $A$ fixes a point in $T$ (see [Ser] 1.6.5 Corollary 2]). Similarly, we see that $\langle w \rangle * \langle x \rangle$ fixes a point in $T$. Let $U$ be the free splitting of $W_n$ associated with the free factor decomposition $W_n = A \ast \langle x \rangle$. Let $v_0$ be the vertex of $U$ fixed by $A$, let $v_1$ be the vertex fixed by $x$ and let $v_2$ be the vertex fixed by $w_0xw_0^{-1}$. Let $e_1$ be the edge between $v_0$ and $v_1$ and $e_2$ be the edge between $v_0$ and $v_2$. The arguments above show that we have a canonical $W_n$-equivariant morphism from $U$ to $T$. This morphism is obtained by a fold of the
edges $e_1$ and $e_2$ of $U$ and this fold is extended $W_n$-equivariantly. Since $w$ is root-closed, there is no other edge of $U$ that can be folded as otherwise the stabilizer of an edge of $T$ would not be cyclic. Therefore the $\mathbb{R}$-tree $T$ is simplicial and the decomposition of $W_n$ associated with $W_n \setminus T$ is

$$W_n = A \ast \langle w \rangle \langle \langle w \rangle \ast \langle x \rangle \rangle.$$ 

Hence $T = S'$ and the first statement follows.

Let us prove the second statement. For every $n \in \mathbb{N}$, the equivalence classes $\lambda_n D^n(\mathcal{R})$ and $\mathcal{R}$ have compatible representatives. But as $\lim_{n \to \infty} \lambda_n D^n(\mathcal{R}) = S$, it follows from [GLT Corollary A.12] that, in the limit, the splittings $S$ and $R$ are compatible.

**Lemma 4.7.** Let $n \geq 3$ and let $S$ be the equivalence class of a $W_{n-1}$-star $S$. Let $T$ be the group of twists of $S$ and let $f \in T$ be an element of infinite order. Let $\mathcal{R}$ be the equivalence class of a $W_{n-1}$-star $R$ such that $f(\mathcal{R}) = \mathcal{R}$. Then $S$ and $R$ are compatible.

**Proof.** Let

$$W_n = A \ast \langle x_n \rangle$$

be a free factor decomposition of $W_n$ associated with $S$ and let $z_f \in A$ be the twistor of $f$. Let $z$ be a root-closed element of $A$ such that there exists $m \geq 1$ with $z^m = z_f$. Let $h \in T$ be the twist about $z$. We see that $h^m = f$. Let $S'$ be the splitting associated with the following amalgamated decomposition of $W_n$:

$$W_n = A \ast \langle z \rangle \langle \langle x_n \rangle \ast \langle z \rangle \rangle.$$ 

Let $S'$ be the equivalence class of $S'$. Let $T'$ be the group of twists of $S'$. Since $A$ is isomorphic to $W_{n-1}$ and since $z$ is root-closed, we see that $C_A(z) = \langle z \rangle$. Therefore $T'$ is isomorphic to $\mathbb{Z}$ and a generator of $T'$ is $h$. As $f(\mathcal{R}) = \mathcal{R}$, Lemma [1.6] implies that $S'$ and $R$ are compatible. Let $U$ be a common refinement of $S'$ and $R$ whose number of orbits of edges is minimal. Since both $S'$ and $R$ are one-edge splittings and are different, the splitting $U$ has 2 orbits of edges. It follows that $W_n \setminus U$ is obtained from $W_n \setminus S'$ by blowing-up an edge at one of the two vertices of $W_n \setminus S'$. Let $\tilde{v}$ be the vertex of $S'$ whose stabilizer is $A$ and let $v$ be its image in $W_n \setminus S'$. Let $\tilde{w}$ be the vertex of $S'$ fixed by $\langle x_n \rangle \ast \langle z \rangle$ and let $w$ be its image in $W_n \setminus S'$.

**Claim.** Either $\mathcal{S} = \mathcal{R}$ or the splitting $W_n \setminus U$ is obtained from $W_n \setminus S'$ by blowing-up an edge at $v$.

**Proof.** Suppose that $W_n \setminus U$ is obtained from $W_n \setminus S'$ by blowing-up an edge at $w$. Then, since the group $G_w$ associated with $w$ is $\langle x_n \rangle \ast \langle z \rangle$ and since $z$ must fix an edge of $U$, we see that a free splitting of $G_w$ such that $z$ fixes a vertex is a $(G_w, \langle \langle z \rangle, \langle x_n \rangle \rangle)$-free splitting. But $(G_w, \langle \langle z \rangle, \langle x_n \rangle \rangle)$ has exactly one such equivalence class of one-edge free splitting: the one with vertex stabilizers conjugated with $\langle z \rangle$ and $\langle x_n \rangle$. This implies that $\mathcal{R} = S$. The claim follows.

Suppose that $\mathcal{R} \neq S$. The claim implies that the amalgamated decomposition of $W_n$ associated with $U$ is

$$W_n = B \ast C \ast \langle z \rangle \langle \langle z \rangle \ast \langle x_n \rangle \rangle,$$

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where $B$ and $C$ are free factors of $W_n$ such that $A = B \ast C$ and $z \in C$. Let $U'$ be a refinement of $U$ whose associated amalgamated decomposition of $W_n$ is:

\[ W_n = B \ast C \ast \langle z \rangle \ast \langle x_n \rangle, \]

that is, $z$ and $x_n$ fix distinct points in $U'$. Then, since $A = B \ast C$, the splitting $U'$ is a refinement of $S$. This concludes the proof. \hfill \Box

**Proposition 4.8.** Let $n \geq 3$. Let $S$ be a $W_{n-1}$-star and let $f \in \text{Out}(W_n)$ be a twist about the unique edge of $W_n \setminus S$. Let $g \in C_n$ be such that $g \in C_{\text{Out}(W_n)}(f)$. Then $g(S) = S$.

**Proof.** Let

\[ W_n = \langle x_1, \ldots, x_{n-1} \rangle \ast \langle x_n \rangle \]

be the free factor decomposition associated with $S$ and let $S$ be the equivalence class of $S$. By Lemma 4.4 in order to prove that $g(S) = S$, it suffices to show that $g$ preserves the conjugacy class of $A = \langle x_1, \ldots, x_{n-1} \rangle$. Let $\tilde{f}$ be a representative of $f$ such that $\tilde{f}|_A = \text{id}_A$. Let $\tilde{g}$ be a representative of $g$. Suppose towards a contradiction that $\tilde{g}$ does not preserve the conjugacy class of $A$. By hypothesis, there exists $I \in \text{Inn}(W_n)$ such that $\tilde{f} \circ \tilde{g} = I \circ \tilde{g} \circ \tilde{f}$. Thus,

\[ \tilde{f} \circ \tilde{g}(A) = I \circ \tilde{g} \circ \tilde{f}(A) = I \circ \tilde{g}(A). \]

Therefore, $f$ preserves the conjugacy class of $\tilde{g}(A)$. By Lemma 4.4 $f$ fixes the unique equivalence class $R$ of the $W_{n-1}$-star $R$ associated with $\tilde{g}(A)$. By Lemma 4.7, the splittings $S$ and $R$ are compatible. Since we suppose that $\tilde{g}(A) \notin [A]$, there exists a common refinement $S'$ of $S$ and $R$ which is a $W_{n-2}$-star. Thus, there exists $y_n \in W_n$ such that the free factor decomposition associated with $S'$ is

\[ W_n = \langle x_n \rangle \ast B \ast \langle y_n \rangle, \]

where $B$ is such that $A = B \ast \langle y_n \rangle$ and $B \ast \langle x_n \rangle$ is a conjugate of $\tilde{g}(A)$. Up to changing the representative $\tilde{g}(A)$, we may suppose that $\tilde{g}(A) = B \ast \langle x_n \rangle$. This implies that $x_n \in \tilde{g}(A)$, that is $\tilde{g}^{-1}(x_n) \in A$. But, since $A = \langle x_1, \ldots, x_{n-1} \rangle$, we see that $[\tilde{g}^{-1}(x_n)] \in \{[x_1], \ldots, [x_{n-1}]\}$. This contradicts the fact that $g \in C_n$. \hfill \Box

Combining Lemma 4.7 and Proposition 4.8 we have the following corollary.

**Corollary 4.9.** Let $n \geq 3$. Let $S$ and $R$ be two distinct $W_n$-equivariant homeomorphism classes of two $W_{n-1}$-stars $S$ and $R$. Let $f$ and $g$ be twists about respectively $S$ and $R$ such that $f$ and $g$ commute. Then $S$ and $R$ are compatible.

**Proof.** Let $k \geq 1$ be such that $g^k \in C_n$. By Proposition 4.8 since $g^k$ and $f$ commute, we have $g^k(S) = S$. Since $g^k$ is a twist about $R$, by Lemma 4.7, we have that $S$ and $R$ are compatible. \hfill \Box

Let $S$ be the equivalence class of a $W_{n-1}$-star $S$ and let

\[ W_n = \langle x_1, \ldots, x_{n-1} \rangle \ast \langle x_n \rangle \]

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be the free factor decomposition of $W_n$ associated with $S$. Let $A = \langle x_1, \ldots, x_{n-1} \rangle$. Let $f \in \text{Stab}_{\text{Out}(W_n)}(S)$. Then any representative of $f$ sends $A$ to a conjugate of itself. Let $\tilde{f}$ be a representative of $f$ such that $\tilde{f}(A) = A$. Since the vertices in $S$ fixed by $A$ and $x_n$ are adjacent, and since the stabilizer of every vertex in $S$ adjacent to the vertex fixed by $A$ is a conjugate of $\langle x_n \rangle$ by an element of $A$, we see that $\tilde{f}'(x_n) = xx_nx^{-1}$ with $x \in A$. Therefore, there exists a representative $\tilde{f}$ of $f$ such that $\tilde{f}(A) = A$ and $\tilde{f}(x_n) = x_n$. The automorphism $\tilde{f}$ is the unique representative of $f$ such that $\tilde{f}(A) = A$ and $\tilde{f}(x_n) = x_n$.

We have a similar result for $W_{n-2}$-stars. Indeed, let $S'$ be the equivalence class of a $W_{n-2}$-star $S'$ and let

$$W_n = \langle x_1 \rangle * \langle x_2, \ldots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of $W_n$ induced by $S'$ and let $A = \langle x_2, \ldots, x_{n-1} \rangle$. Let $\text{Stab}_{\text{Out}(W_n)}(S)$. Since the centralizer in $W_n$ of $x_n$ is $\langle x_n \rangle$ and since $A$ is malnormal in $W_n$, we see that $\tilde{f} = \tilde{f}'$. Hence $\tilde{f}(A) = A$, and, since $A$ is malnormal, we see that $g \in A$. Therefore, $f \in T'$, which concludes the proof.

**Lemma 4.11.** Let $n \geq 4$. Let $S$ be the $W_n$-equivariant homeomorphism class of a $W_{n-1}$-star $S$. Let $T$ be the group of twists of $S$. Let $S'$ be the $W_n$-equivariant homeomorphism class of a $W_{n-2}$-star $S'$ which refines $S$. Let $e$ be the edge of $W_n \setminus S'$ such that a representative of $S$ is obtained from $W_n \setminus S'$ by collapsing the edge distinct from $e$. Let $T'$ be the group of twists of $S'$ about the edge $e$. Then $T \cap \text{Stab}_{C_n}(S') \subseteq T'$.

**Proof.** Let

$$W_n = \langle x_1 \rangle * \langle x_2, \ldots, x_{n-1} \rangle * \langle x_n \rangle$$

be the free factor decomposition of $W_n$ induced by $S'$ and let $A = \langle x_2, \ldots, x_{n-1} \rangle$. Let $\text{Stab}_{\text{Out}(W_n)}(S)$. Since the centralizer in $W_n$ of $x_n$ is $\langle x_n \rangle$ and since $A$ is malnormal in $W_n$, we see that $\tilde{f} = \tilde{f}'$. Hence $\tilde{f}(A) = A$, and, since $A$ is malnormal, we see that $g \in A$. Therefore, $f \in T'$, which concludes the proof.
Proof. By Proposition 2.3 (2), the group \( \text{Stab}(S) \) is isomorphic to \( \text{Aut}(A) \). The isomorphism \( \text{Stab}(S) \to \text{Aut}(A) \) is defined by sending \( f \in \text{Stab}(S) \) to its representative \( \tilde{f} \) such that \( \tilde{f}(A) = A \) and \( \tilde{f}(x_n) = x_n \). In particular, for every \( h_1, h_2 \in \text{Out}(W_n) \cap \text{Stab}(S) \), we see that \( h_1 \) and \( h_2 \) commute if and only if there exist representatives \( \tilde{h}_1 \) and \( \tilde{h}_2 \) of \( h_1 \) and \( h_2 \) respectively such that \( \tilde{h}_1(A) = A, \tilde{h}_2(A) = A, \tilde{h}_1(x_n) = \tilde{h}_2(x_n) = x_n \) and \( \tilde{h}_1 \circ \tilde{h}_2 = \tilde{h}_2 \circ \tilde{h}_1 \). Moreover, Proposition 2.8 identifies the group of twists \( T \) with the group \( \text{Inn}(A) \).

For \( a \in A \), let \( \text{ad}_a \) be the inner automorphism of \( A \) induced by \( a \). Since, for every \( h \in \text{Aut}(A) \) and every \( a \in A \), we have \( h \text{ad}_a h^{-1} = \text{ad}_{h(a)} \), we see that \( h \) commutes with \( \text{ad}_a \) if and only if \( h(a) = a \). Hence \( g \in C_{\text{Out}(W_n)}(\langle f \rangle) \) if and only if \( g(z_f) = z_f \).

5 Direct products of nonabelian free groups in \( \text{Out}(W_n) \)

Following [HW, Section 6], we define the product rank of a group \( H \), denoted by \( \text{rk}_{\text{prod}}(H) \), to be the maximal integer \( k \) such that a direct product of \( k \) nonabelian free groups embeds in \( H \). Note that, if \( H' \) is a finite index subgroup of \( H \), then \( \text{rk}_{\text{prod}}(H') = \text{rk}_{\text{prod}}(H) \). Moreover, if \( \phi : H \to Z \) is a homomorphism, then \( \text{rk}_{\text{prod}}(\ker(\phi)) = \text{rk}_{\text{prod}}(H) \).

The aim of this section is to prove the following theorem.

Theorem 5.1. (1) For every \( n \geq 3 \), we have \( \text{rk}_{\text{prod}}(\text{Aut}(W_n)) = n - 2 \).

(2) For every \( n \geq 4 \), we have \( \text{rk}_{\text{prod}}(\text{Out}(W_n)) = n - 3 \).

(3) Suppose that \( n \geq 5 \). If \( H \) is a subgroup of \( \text{Out}(W_n) \) isomorphic to a direct product of \( n - 3 \) nonabelian free groups, then \( H \) has a subgroup \( H' \) isomorphic to a direct product of \( n - 3 \) nonabelian free groups which virtually fixes the \( W_{n-1} \)-equivariant homeomorphism class of any one-edge free splitting that is not a \( W_{n-1} \)-star.

We first recall an estimate regarding product ranks and group extensions due to Horbez and Wade.

Lemma 5.2. [HW, Lemma 6.3] Let \( 1 \to N \to G \to Q \to 1 \) be a short exact sequence of groups. Then \( \text{rk}_{\text{prod}}(G) \leq \text{rk}_{\text{prod}}(N) + \text{rk}_{\text{prod}}(Q) \).

In order to compute the product rank of \( \text{Out}(W_n) \), we take advantage of its action on the Gromov hyperbolic free factor complex. We recall a general result concerning actions of direct products on a hyperbolic space.

Lemma 5.3. [HW, Proposition 4.2, Lemma 4.4] Let \( X \) be a Gromov hyperbolic space, and let \( H \) be a group acting by isometries on \( X \). Assume that \( H \) contains a normal subgroup \( K \) isomorphic to a direct product \( K = \prod_{i=1}^{k} K_i \).

If there exists \( j \in \{1, \ldots, k\} \) such that \( K_j \) contains a loxodromic element, then \( \prod_{i \neq j} K_i \) has a finite orbit in \( \partial_X X \).

If there exist two distinct \( i, j \in \{1, \ldots, k\} \) such that both \( K_i \) and \( K_j \) contain a loxodromic element, then \( H \) has a finite orbit in \( \partial_X X \).

If, for every \( j \in \{1, \ldots, k\} \), the group \( K_j \) does not contain a loxodromic element, then either \( K \) has a finite orbit in \( \partial_X X \) or \( H \) has bounded orbits in \( X \).
We will also use a theorem due to Guirardel and Horbez which assigns to every nonempty collection of free splittings whose elementwise stabilizer is infinite a canonical (not necessarily free) splitting.

**Theorem 5.4.** [GHS] Let $n \geq 3$. There exists an $\text{Out}(W_n)$-equivariant map which assigns to every nonempty collection $C$ of free splittings of $W_n$ whose elementwise $\text{Out}(W_n)$-stabilizer is infinite, a nontrivial splitting $U_C$ of $W_n$ whose set of vertices $VU_C$ has a $W_n$-invariant partition $VU_C = V_1 \sqcup V_2$ with the following properties:

1. For every vertex $v \in V_1$, the following holds:
   
   (a) either some edge incident on $v$ has trivial stabilizer, or the set of stabilizers of edges incident on $v$ induces a nontrivial free factor system of the vertex stabilizer $G_v$,
   
   (b) there exists a finite index subgroup $H_0$ of the elementwise stabilizer of the collection $C$ such that every outer automorphism in $H_0$ has a representative in $\text{Aut}(W_n)$ which restricts to the identity on $G_v$.

2. The collection of all conjugacy classes of stabilizers of vertices in $V_2$ is a free factor system of $W_n$. $\square$

**Proof of Theorem 5.1.** The proof is inspired by [HW] Theorem 6.1] due to Horbez and Wade and [HHW] Theorem 4.3] due to Hensel, Horbez and Wade.

We first prove that if $n \geq 4$, then $\text{rk}_{prod}(\text{Out}(W_n)) \geq n - 3$ and that, if $n \geq 3$, then $\text{rk}_{prod}(\text{Aut}(W_n)) \geq n - 2$. Pick a standard generating set $\{x_1, \ldots, x_n\}$ of $W_n$. Then the group $H$ generated by $\{x_1x_2, x_2x_3\}$ is a nonabelian free group (see [Müh] Theorem A).

Suppose first that $n \geq 4$. For $i \in \{4, \ldots, n\}$ and $h \in H$, let $F_{i,h}$ be the automorphism sending $x_i$ to $hx_ih^{-1}$ and, for $j \neq i$, fixing $x_j$. Then, for every distinct $i, j \in \{4, \ldots, n\}$ and for every $g, h \in H$, the automorphisms $F_{i,g}$ and $F_{j,h}$ commute, giving a direct product of $n - 3$ nonabelian free groups in $\text{Out}(W_n)$. Moreover, for every $g, h \in H$, and every $i \in \{4, \ldots, n\}$, the inner automorphism $\text{ad}_h$ commutes with $F_{i,h}$, which yields a direct product of $n - 2$ nonabelian free groups in $\text{Aut}(W_n)$. In the case where $n = 3$, the group $\text{Aut}(W_3)$ contains the subgroup $\langle \text{ad}_h \rangle_{h \in H}$, which is a nonabelian free group.

We now prove that, if $n \geq 3$, then $\text{rk}_{prod}(\text{Aut}(W_n)) \leq n - 2$, if $n = 3$, then $\text{rk}_{prod}(\text{Out}(W_n)) = 1$ and if $n \geq 4$, then $\text{rk}_{prod}(\text{Out}(W_n)) \leq n - 3$. The proof is by induction on $n$. The base case where $n = 3$ follows from the fact that the group $\text{Aut}(W_3)$ is isomorphic to $\text{Aut}(F_2)$ (see [Var] Lemma 2.3]) and the fact that the group $\text{Aut}(F_2)$ does not contain a direct product of two nonabelian free groups (see [HW] Lemma 6.2]). Moreover, by [Gue1] Proposition 2.2], the group $\text{Out}(W_3)$ is isomorphic to $\text{PGL}(2,\mathbb{Z})$ which is virtually free.

Let $k \geq \max\{n - 3, 2\}$ and let $H = H_1 \times H_1 \times \ldots \times H_k$ be a subgroup of $\text{Out}(W_n)$ isomorphic to a direct product of $k$ nonabelian free groups. Note that $k = n - 3$ if $n \geq 5$ and $k = 2$ if $n = 4$. We prove that there exists a subgroup $K$ of $H$ isomorphic to a direct product of $k$ nonabelian free groups which virtually fixes a one-edge free splitting of $W_n$. Let $\mathcal{F}$ be a maximal $H$-periodic free factor system. If $\mathcal{F}$ is sporadic, then $H$ virtually
fixes a one-edge free splitting, so we are done. Therefore, we may suppose that \( F \) is nonsporadic. As \( F \) is supposed to be maximal, by Proposition 2.8 the group \( H \) acts on \( \text{FF}(W_n, F) \) with unbounded orbits. Lemma 5.3 implies that, after possibly reordering the factors, the group \( H' = H_1 \times H_2 \times \ldots \times H_{k-1} \) has a finite orbit in \( \hat{v} \text{FF}(W_n, F) \).

By Lemma 2.8 the group \( H' \) virtually fixes the homothety class \([T]\) of an arational \((W_n, F)\)-tree \( T \).

Let \( H_0 \) be a normal subgroup of finite index in \( H' \) that is contained in \( \text{Stab}([T]) \).

**Claim.** The group \( H \) contains a subgroup isomorphic to a direct product of \( k \) nonabelian free groups, which fixes the equivalence class of a one-edge free splitting.

**Proof.** By Lemma 2.5 the morphism \( \lambda_T|_{H_0} \) from \( H_0 \) to \( \mathbb{R}_+^* \) given by the scaling factor has cyclic image. As \( H_0 \) contains a direct product of \( k - 1 \) nonabelian free groups, so does \( P = \ker(\lambda_T|_{H_0}) \) (see the beginning of Section 5). As \( P \) is contained in the isometric stabilizer of \( T \), Proposition 2.7 implies that \( P \) contains a finite index subgroup \( P_0 \) which fixes infinitely many \((W_n, F)\)-free splittings.

Let \( C \) be the (nonempty) collection of all \((W_n, F)\)-free splittings fixed by the infinite group \( P_0 \), let \( U_C \) be the splitting provided by Theorem 5.4 and let \( U_C \) be its equivalence class. Since \( P_0 \) commutes with \( H_k \), the equivalence class \( U_C \) is \((P_0 \times H_k)\)-invariant.

Suppose first that the splitting \( U_C \) contains an edge \( e \in EU_C \) with trivial stabilizer. Let \( U' \) be the splitting obtained from \( U_C \) by collapsing every edge of \( U_C \) that is not contained in the orbit of \( e \), and let \( U' \) be its equivalence class. Then \( U' \) is the equivalence class of a one-edge free splitting virtually fixed by \( P_0 \times H_k \). Since \( P_0 \) contains a direct product of \( k - 1 \) nonabelian free groups, the claim follows.

Thus, we can suppose that all edge stabilizers of \( U_C \) are nontrivial. We show that this leads to a contradiction. Let \( VU_C = V_1 \cup V_2 \) be the partition of \( VU_C \) given by Theorem 5.4. Let \( P' \) be a finite index subgroup of \( P_0 \) which acts trivially on the quotient \( W_n \setminus U_C \). We claim that the intersection of \( P' \) with the group of twists of \( U_C \) is trivial. Indeed, let \( e \) be an half-edge of \( U_C \). As every subgroup of \( W_n \) with nontrivial centralizer is cyclic, if the edge stabilizer \( G_e \) of \( e \) is not cyclic, the group of twists around this half-edge is trivial. Thus, half-edges with nontrivial group of twists have cyclic stabilizers. But twists about edges with cyclic stabilizers are central in a finite index subgroup of \( \text{Stab}^0(U_C) \) by Lemma 2.10. As the center of every finite index subgroup of \( P' \) is trivial, we see that the intersection of \( P' \) with the group of twists is trivial. By Remark 2.9 up to passing to a further finite index subgroup of \( P' \), we may suppose that the intersection of \( P' \) with the group of bitwists is trivial.

By Proposition 2.8 (1) and Remark 2.9 the fact that the intersection of \( P' \) with the group of bitwists is trivial implies that we have an injective homomorphism

\[
\lambda \colon P' \to \prod_{v \in W_n \setminus U_C} \text{Out}(G_v).
\]

By Theorem 5.3 (1)(b), for every vertex \( v \in V_1 \), the homomorphism \( \lambda \colon P' \to \text{Out}(G_v) \) has finite image. Therefore, up to passing to a finite index subgroup of \( P' \), we have an
injective map

\[ P' \to \prod_{v \in W_n \setminus V_2} \text{Out}(G_v). \]

By Theorem \ref{thm:main}(2), for every \( v \in V_2 \), the vertex stabilizer \( G_v \) is an element of a free factor system of \( W_n \). Therefore, there exists \( k \) such that \( G_v \) is isomorphic to \( W_k \). By Lemma \ref{lem:factor} we have:

\[ n - 4 \leq k - 1 = \prod_{v \in W_n \setminus V_2} \text{rk}(P') \leq \sum_{v \in W_n \setminus V_2} \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Out}(G_v)). \]

By induction, we see that, if \( |W_n \setminus V_2| \geq 2 \), then

\[ \sum_{v \in W_n \setminus V_2} \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Out}(G_v)) \leq n - 6, \]

which leads to a contradiction. Thus \( |W_n \setminus V_2| = 1 \). Let \( v \in W_n \setminus V_2 \). Then there exists \( \ell \in \{1, \ldots, n - 1\} \) such that \( G_v \) is isomorphic to \( W_{\ell} \). If \( \ell \leq n - 2 \), then

\[ \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Out}(G_v)) \leq n - 5, \]

which leads to a contradiction. If \( \ell = n - 1 \), then the free factor system \( F \) contains a free factor isomorphic to \( W_{n-1} \) and is therefore a sporadic free factor system, which leads to a contradiction.

Therefore, we see that there exists a subgroup \( K \) of \( H \) isomorphic to a direct product of \( k \) nonabelian free groups such that \( K \) fixes the \( W_n \)-equivariant homeomorphism class of a one-edge-free splitting \( S \). We now prove that \( S \) is the equivalence class of a \( W_{n-1} \)-star. Let \( S \) be a representative of \( \mathcal{S} \), let \( v_1 \) and \( v_2 \) be the vertices of the underlying graph of \( W_n \setminus S \) and, for \( i \in \{1, 2\} \), let \( k_i \) be such that \( W_{k_i} \) is isomorphic to \( G_{v_i} \). Let \( K_0 \) be the finite index subgroup of \( K \) which acts as the identity on \( W_n \setminus S \). Then \( K_0 \subseteq \text{Stab}^0(S) \). By Proposition \ref{prop:factor}(2), the group \( \text{Stab}^0(S) \) is isomorphic to \( \text{Aut}(W_{k_1}) \times \text{Aut}(W_{k_2}) \). Suppose towards a contradiction that, for every \( i \in \{1, 2\} \), we have that \( k_i \neq 1 \). Suppose first that, for every \( i \in \{1, 2\} \), we have \( k_i \geq 3 \). Then, by Lemma \ref{lem:factor}, we see that:

\[ k = \prod_{v \in W_n \setminus V_2} \text{rk}(K_0) \leq \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Aut}(W_{k_1})) + \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Aut}(W_{k_2})) \leq k_1 - 2 + k_2 - 2 = n - 4, \]

where the second inequality comes from the induction hypothesis. If there exists \( i \in \{1, 2\} \) such that \( k_i = 2 \), then, as \( \text{Aut}(W_2) \) is virtually cyclic (it is isomorphic to \( W_2 \) by \ref{thm:main}(2)), we see that:

\[ k = \prod_{v \in W_n \setminus V_2} \text{rk}(K_0) \leq \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Aut}(W_{k_1})) + \prod_{v \in W_n \setminus V_2} \text{rk}(\text{Aut}(W_{k_2})) \leq k_1 - 2 \leq n - 4. \]

In both cases, we have a contradiction as \( k \geq n - 3 \) when \( k \geq 5 \) and \( k = n - 2 \) when \( n = 4 \). Thus, there exists \( i \in \{1, 2\} \) such that \( k_i = 1 \). This shows that \( S \) is a \( W_{n-1} \)-star. In particular, when \( k = n - 3 \), that is, when \( n \geq 5 \), this proves Theorem \ref{thm:main}(3).
Since $K_0 \subseteq \text{Stab}^0(S)$, Proposition 2.8 (2) implies that

$$k = \text{rk}_{\text{prod}}(K_0) \leq \text{rk}_{\text{prod}}(\text{Aut}(W_{n-1})) = n - 1 - 2 = n - 3.$$ 

When $n = 4$, then $k = 2 = n - 2$. Therefore, we have a contradiction in this case. This shows that, for all $n \geq 4$, the product rank of $\text{Out}(W_n)$ is equal to $n - 3$. This concludes the proof of Theorem 5.1 (2).

It remains to prove that, if $n \geq 4$, we have $\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \leq n - 2$. We have the following short exact sequence

$$1 \rightarrow W_n \rightarrow \text{Aut}(W_n) \rightarrow \text{Out}(W_n) \rightarrow 1.$$ 

By Lemma 5.2 as $W_n$ is virtually free, we see that

$$\text{rk}_{\text{prod}}(\text{Aut}(W_n)) \leq \text{rk}_{\text{prod}}(W_n) + \text{rk}_{\text{prod}}(\text{Out}(W_n)) = 1 + n - 3 = n - 2.$$ 

This concludes the proof of Theorem 5.1 (1).

\[\square\]

6 Subgroups of stabilizers of $W_{n-1}$-stars

In the next two sections, we prove an algebraic characterisation of stabilizers of equivalence classes of $W_{n-2}$-stars. In this section, we take advantage of properties satisfied by stabilizers of equivalence classes of $W_{n-2}$-stars which are sufficiently rigid to show that a subgroup $H$ of $\text{Out}(W_n)$ which satisfies these properties virtually fixes a $W_{n-1}$-star. In the next section, we will take advantage of the fact that stabilizers of equivalence classes of compatible $W_{n-2}$-stars have large intersections to give a characterisation of stabilizers of equivalence classes of $W_{n-2}$-stars.

Let $\Gamma$ be a finite index subgroup of the group $C_n$ (defined after Theorem 4.1). We introduce the following algebraic property for a subgroup $H \subseteq \Gamma$.

(PWn−2) The group $H$ satisfies the following three properties:

1. The group $H$ contains a normal subgroup isomorphic to a direct product $K_1 \times K_2$ of two normal subgroups such that each one contains a nonabelian finitely generated normal free subgroups of finite index and such that for every $i \in \{1, 2\}$, for every nontrivial normal subgroup $P$ of a finite index subgroup $K_i'$ of $K_i$, and for every finite index subgroup $P'$ of $P$, the group $C_{C_{K_i'}}(P')$ contains $K_{i+1}$ as a finite index subgroup (where indices are taken modulo 2).

2. The group $H$ contains a direct product of $n - 3$ nonabelian free groups.

3. The group $H$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$.
Remark 6.1. (1) Notice that property $(P_{W_{n-2}})$ is closed under taking finite index subgroups.
(2) Hypothesis $(P_{W_{n-2}})$ (1) implies that, if for every $i \in \{1, 2\}$, the group $P_i$ is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of $K_i$, the centralizer in $C_n$ of $P_1 \times P_2$ is finite.

We first prove that the stabilizer in $\Gamma$ of the equivalence class of a $W_{n-2}$-star satisfies $(P_{W_{n-2}})$. We then show that a group satisfying $(P_{W_{n-2}})$ virtually fixes the equivalence class of a $W_{n-1}$-star.

6.1 Properties of $\mathcal{Z}_{RC}$-factors

In order to prove that the stabilizer in $\Gamma$ of the equivalence class of a $W_{n-2}$-star satisfies $(P_{W_{n-2}})$, we first need some background concerning $\mathcal{Z}_{RC}$-splittings. Let $G$ be a finitely generated group. A $\mathcal{Z}_{RC}$-splitting of $G$ is a splitting of $G$ such that every edge stabilizer is either trivial or isomorphic to $\mathcal{Z}$ and root-closed. A $\mathcal{Z}_{RC}$-factor of $G$ is a subgroup of $G$ which arises as a vertex stabilizer of a $\mathcal{Z}_{RC}$-splitting of $G$. Note that since edge stabilizers are root-closed, so are the vertex stabilizers. We outline here some properties of $\mathcal{Z}_{RC}$-factors (see e.g. [HW, Proposition 7.3]).

Proposition 6.2. Let $n \geq 3$. The $\mathcal{Z}_{RC}$-factors of $W_n$ satisfy the following properties.

(1) Let $H$ be a finitely generated subgroup of $W_n$ which is not virtually cyclic. There exists $g \in H$ which is not contained in any proper $\mathcal{Z}_{RC}$-factor of $H$.

(2) There exists $C \in \mathbb{N}^*$ such that, for every strictly ascending chain $G_1 \subset \ldots \subset G_k$ of $\mathcal{Z}_{RC}$-factors of $W_n$, one has $k \leq C$.

(3) If a subgroup $K$ of $W_n$ is not contained in any proper $\mathcal{Z}_{RC}$-factor of $W_n$ and if $P$ is either a finite index subgroup of $K$ or a nontrivial normal subgroup of $K$, then $P$ is not contained in any proper $\mathcal{Z}_{RC}$-factor of $W_n$.

(4) A subgroup $K$ of $W_n$ is contained in a proper $\mathcal{Z}_{RC}$-factor of $W_n$ if and only if every element of $K$ is contained in a proper $\mathcal{Z}_{RC}$-factor of $W_n$.

Proof. The first assertion is a consequence of [GeH, Lemma 4.3] due to Genevois and Horbez.

For the second assertion, let $G_1 \subset \ldots \subset G_k$ be a sequence of strictly ascending $\mathcal{Z}_{RC}$-factors. Then, since $\mathcal{Z}_{RC}$-factors are root-closed, for every $i \geq 3$ the group $G_i$ is not cyclic. Thus, as we want an upper bound on the number of subgroups of such a sequence, we may suppose that for every $i \in \{1, \ldots, n\}$, the group $G_i$ is not cyclic. We claim that, for every $i \in \{1, \ldots, k\}$, there exists $\phi_i \in \text{Aut}(W_n)$ such that $\text{Fix}(\phi_i) = G_i$. Indeed, let $S_i$ be a $\mathcal{Z}_{RC}$-splitting of $W_n$ such that there exists $v \in VS_i$ whose stabilizer is equal to $G_i$. Up to collapsing edges, we may suppose that every vertex of $S_i$ has nontrivial stabilizer. Let $e_1, \ldots, e_k$ be the edges with origin $v$. Let $F \subset \{e_1, \ldots, e_k\}$ be the subset made of all edges with nontrivial stabilizer. By the definition of a $\mathcal{Z}_{RC}$-splitting, for every $e_s \in F$, the group $G_{e_s}$ is cyclic. For every $e_s \in F$, let $z_s$ be a generator of $G_{e_s}$. For
every $e_{i'} \in \{e_1, \ldots, e_\ell\} - F$, let $z_{i'} \in G_1$ be such that, if $w_{i'}$ is the endpoint of $e_{i'}$, distinct from $v$, we have $z_{i'}G_{w_{i'}}z_{i'}^{-1} \neq G_{w_{i'}}$. Let $\phi_i = D_{e_1,z_1} \circ \cdots \circ D_{e_\ell,z_\ell}$ be a multitwist about every edge with origin $v$. Then, as the centralizer of an infinite cyclic subgroup of $W_n$ is infinite cyclic, we have $\text{Fix}(\phi_i) = G_i$. Therefore, in order to prove the second assertion, it suffices to prove that there exists $C \in \mathbb{N}^+$ such that for every strictly ascending chain $\text{Fix}(\phi_1) \subseteq \cdots \subseteq \text{Fix}(\phi_k)$ of fixed points sets of automorphisms of $W_n$, one has $k \leq C$.

Let $\{x_1, \ldots, x_n\}$ be a standard generating set of $W_n$. By [Müh] Theorem A] the kernel $K'$ of the homomorphism $W_n \to F$ which, for every $i \in \{1, \ldots, n\}$, sends $x_i$ to the nontrivial element of $F$ is a nonabelian free group on $n - 1$ generators. Remark that $K'$ does not depend on the choice of the basis since, for every element $x$ of order 2, there exists $i \in \{1, \ldots, n\}$ and $g \in W_n$ such that $x = gx_ig^{-1}$. Moreover, $K'$ is a characteristic subgroup of index 2 of $W_n$ and the natural homomorphism $\Phi: \text{Aut}(W_n) \to \text{Aut}(K')$ given by restriction is injective. Then $\text{Fix}(\Phi(\phi_1)) \subseteq \cdots \subseteq \text{Fix}(\Phi(\phi_k))$ is an ascending chain of fixed points sets.

**Claim.** For every $i \in \{2, \ldots, k - 1\}$, the set $\{\text{Fix}(\Phi(\phi_{i-1})), \text{Fix}(\Phi(\phi_i)), \text{Fix}(\Phi(\phi_{i+1}))\}$ contains at least 2 elements.

**Proof.** Suppose towards a contradiction that

$$|\{\text{Fix}(\Phi(\phi_{i-1})), \text{Fix}(\Phi(\phi_i)), \text{Fix}(\Phi(\phi_{i+1}))\}| = 1.$$ 

As $\text{Fix}(\phi_{i-1}) \subset \text{Fix}(\phi_i)$ and $\text{Fix}(\Phi(\phi_{i-1})) = \text{Fix}(\Phi(\phi_i))$, there exists $a \in W_n - K'$ such that $\phi_i(a) = a$ and $\phi_{i-1}(a) \neq a$. Since the index of $K'$ is equal to 2, we see that $\phi_{i-1}(a^2) = a^2$. Therefore, $\phi_{i-1}(a^2) = a^2$ and $\phi_{i-1}(a)$ is a square root of $a^2$. If $a^2$ has infinite order, its only square root is $a$. This implies that $\phi_{i-1}(a) = a$, a contradiction. Thus we can assume that $a$ has order 2 and, up to changing the basis $\{x_1, \ldots, x_n\}$, we may suppose that $a = x_1$.

As the index of $K'$ is equal to 2, we have $W_n = K' \sqcup x_1K'$. Let $x \in \text{Fix}(\phi_{i+1}) - K'$. Then there exists $y \in K'$ such that $x = x_1y$. As $x_1 \in \text{Fix}(\phi_i)$ and $\text{Fix}(\phi_i) \subset \text{Fix}(\phi_{i+1})$, we have that $\phi_{i+1}(x_1) = x_1$. Hence $\phi_{i+1}(y) = y$. As $y \in K'$ and $\text{Fix}(\Phi(\phi_i)) = \text{Fix}(\Phi(\phi_{i+1}))$, we see that $\phi_i(y) = y$ and $\phi_i(x) = \phi_i(x_1y) = x_1y = x$. Therefore we have that $\text{Fix}(\phi_i) = \text{Fix}(\phi_{i+1})$, which is a contradiction. The claim follows.

The claim implies that the length of the strictly ascending chain associated with $\text{Fix}(\Phi(\phi_1)) \subseteq \cdots \subseteq \text{Fix}(\Phi(\phi_k))$ is at least equal to $\frac{2}{3}$. But any strictly ascending chain of fixed subgroups in a free group on $n - 1$ generators as length at most $2(n - 1)$ (see [Müh], Theorem 4.1). Therefore, there exists $C$ which depends only on $n$ such that $k \leq C$.

The second assertion of Proposition 6.2 follows.

We now prove the third assertion. Let $P$ and $K$ be as in Proposition 6.2 (3). If $K$ is a virtually infinite cyclic group, then $K$ is either isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2$. Let $a$ be a generator of the subgroup of $K$ isomorphic to $\mathbb{Z}$ and root-closed in $K$. Since $\langle a \rangle$ is a finite index subgroup of $K$ and since $K$ is not contained in any proper $\mathbb{Z}_{RC}$-factor of $W_n$, then neither is $a$. Remark that any nontrivial normal subgroup of $K$ intersects the subgroup $\langle a \rangle$ non trivially. Therefore, if $P$ is contained in a proper $\mathbb{Z}_{RC}$-factor of $W_n$,
then $a$ is elliptic in a $Z_{RC}$-splitting. This contradicts the fact that $a$ is not contained in any proper $Z_{RC}$-factor of $W_n$.

So we can assume that $K$ is not virtually cyclic. As every finite index subgroup contains a nontrivial normal subgroup of $K$, we may assume that $P$ is a nontrivial normal subgroup of $K$. Notice that $P$ is necessarily noncyclic. Suppose towards a contradiction that $P$ is contained in a $Z_{RC}$-factor. Then there exists a $Z_{RC}$-splitting $S$ of $W_n$ such that $P$ is elliptic in $S$. Since edge stabilizers are cyclic, the group $P$ fixes a unique vertex $x$ of $S$. But, as $P$ is normal in $K$, for every $k \in K$, we have that $kx$ is also fixed by $P$, hence we have $kx = x$. Therefore, $x$ is fixed by $K$, which contradicts the fact that $K$ is not contained in any proper $Z_{RC}$-factor.

We finally prove Proposition 6.2 (4). Suppose that $K$ is contained in a proper $Z_{RC}$-factor. Then it is clear that every element of $K$ is contained in a proper $Z_{RC}$-factor.

Conversely, assume that $K$ is not contained in any proper $Z_{RC}$-factor of $W_n$. Let us prove that there exists $g \in K$ such that $g$ is not contained in any proper $Z_{RC}$-factor. By Proposition 6.2 (2), there exists a bound on the length of an increasing chain of $Z_{RC}$-factors of $W_n$. Therefore, the group $K$ contains a finitely generated subgroup $K'$ which is not contained in any proper $Z_{RC}$-factor. By Proposition 6.2 (1), there exists $g \in K'$ such that $g$ is not contained in a proper $Z_{RC}$-factor of $K'$. Let $S$ be a $Z_{RC}$-splitting of $W_n$. As $K'$ is not contained in any proper $Z_{RC}$-factor of $W_n$, the group $K'$ has a well-defined, nontrivial minimal subtree $S_{K'}$ with respect to the action of $K'$ on $S$. As $S$ is a $Z_{RC}$-splitting of $W_n$, the splitting $S_{K'}$ is a $Z_{RC}$-splitting of $K'$. Since $g$ is not contained in any proper $Z_{RC}$-factor of $K'$, it follows that $g$ is a hyperbolic isometry of $S_{K'}$ and is not elliptic in $S$. As $S$ is arbitrary, it follows that $g$ is not contained in any $Z_{RC}$-factor of $W_n$.  

Proper $Z_{RC}$-factors appear naturally when studying stabilizers of conjugacy classes of elements as shown by the following theorem. Recall that, if $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a finite family of finitely generated subgroups of $W_n$, the group $\text{Out}(W_n, \mathcal{H})$ is the subgroup of $\text{Out}(W_n)$ consisting of all outer automorphisms $\phi \in \text{Out}(W_n)$ such that, for every $i \in \{1, \ldots, k\}$, there exists a representative $\tilde{\phi}_i \in \text{Aut}(W_n)$ of $\phi$ such that $\tilde{\phi}_i(H_i) = H_i$ and $\tilde{\phi}_i|_{H_i} = \text{id}_{H_i}$.

**Theorem 6.3.** [GL2, Theorem 7.14] Let $n \geq 3$ and let $g \in W_n$. Then the subgroup $\text{Out}(W_n, \langle g \rangle)$ of outer automorphisms which preserve $\langle g \rangle$ up to conjugacy is infinite if and only if $g$ is contained in a proper $Z_{RC}$-factor of $W_n$.

More generally, if $\mathcal{H}$ is a finite family of finitely generated subgroups, then the group $\text{Out}(W_n, \mathcal{H})$ is infinite if and only if there exists a nontrivial $Z_{RC}$-splitting $S$ of $W_n$ such that every subgroup of $\mathcal{H}$ fixes a vertex of $S$.  

6.2 Stabilizers of $W_{n-2}$-stars satisfy $(P_{W_{n-2}})$

**Lemma 6.4.** Let $n \geq 5$ and let $\Gamma$ be a finite index subgroup of $C_n$. Let $S$ be the equivalence class of a $W_{n-2}$-star $S$. Let $e_1$ and $e_2$ be the two edges of $W_{n-2}\backslash S$ and, for $i \in \{1, 2\}$, let $T_i$ be the group of twists about $e_i$ in $\text{Stab}_\Gamma(S)$. Let $i \in \{1, 2\}$, let $T_i$ be a finite index subgroup
of $T'_i$ and let $P'$ be a finite index subgroup of a nontrivial normal subgroup of $T_i$. Then for every finite index subgroup $P_0$ of $P'$, the group $P_0$ fixes exactly one equivalence class of $W_{n-2}$-stars.

**Proof.** Let

$$W_n = \langle x_1 \rangle \ast \langle x_3, \ldots, x_n \rangle \ast \langle x_2 \rangle$$

be a free factor decomposition associated with $W_n \setminus S$ and $A = \langle x_3, \ldots, x_n \rangle$. Up to exchanging the roles of $e_1$ and $e_2$, we may suppose that $P'$ is contained in the group of twists of the equivalence class of the $W_{n-1}$-star $S_1$ whose associated free factor decomposition of $W_n$ is, up to global conjugation:

$$W_n = \langle x_1 \rangle \ast \langle x_2, x_3, \ldots, x_n \rangle.$$

Let $B = \langle x_2, x_3, \ldots, x_n \rangle$ and let $S_1$ be the equivalence class of $S_1$. Finally, let $S_2$ be the equivalence class of the $W_{n-1}$-star $S_2$ whose associated free factor decomposition of $W_n$ is, up to global conjugation:

$$W_n = \langle x_2 \rangle \ast \langle x_1, x_3, \ldots, x_n \rangle.$$

Let $C = \langle x_1, x_3, \ldots, x_n \rangle = A \ast \langle x_1 \rangle$.

We claim that the only equivalence classes of $W_{n-1}$-stars fixed by any finite index subgroup of $P'$ are $S_1$ and $S_2$. Indeed, fix $i \in \{1, 2\}$. The group $T_i$ is isomorphic to a finite index subgroup $N$ of $W_{n-2}$. By Proposition 6.2 (3) applied with $K = W_{n-2}$ and $P = N$, as $n \geq 5$, the group $N$ is not contained in any proper $\mathcal{Z}_{RC}$-free factor of $W_{n-2}$. By Proposition 6.2 (4), there exists $g \in N$ such that $W_{n-2}$ is freely indecomposable relative to $g$. Hence there exists $g \in A$ such that $A$ is freely indecomposable relative to $g$ and $P'$ contains the twist about $e_1$ whose twistor is $g$. Note that this twist can be seen as a twist about the $W_{n-1}$-star $S_1$. Let $S'_i$ be the equivalence class of the one-edge cyclic splitting $S'_i$ whose associated amalgamated decomposition of $W_n$ is, up to global conjugation:

$$W_n = \langle \langle x_1 \rangle \ast \langle g \rangle \rangle \ast \langle g \rangle \ast \langle x_1 \rangle \ast \langle g \rangle \rangle \ast \langle g \rangle \ast \langle x_1 \rangle \ast \langle g \rangle \rangle B.$$

Let $S_3$ be the equivalence class of a $W_{n-1}$-star $S_3$ fixed by some finite index subgroup of $P'$ and distinct from $S_1$. Let

$$W_n = \langle g \rangle \ast D$$

be the free factor decomposition associated with $S_3$. We claim that $S_3 = S_2$. As $P'$ contains the twist about $g$, by Lemma 4.7, the splitting $S_3$ is compatible with $S'_i$. Let $U$ be a two-edge refinement of $S'_i$ and $S_3$. Then $U$ is obtained from $S_3$ by blowing-up an edge at vertices whose stabilizers are conjugate to $D$. Moreover, $U$ is obtained from $S'_i$ by blowing-up an edge at vertices whose stabilizers are conjugate to $\langle x_1 \rangle \ast \langle g \rangle$. But, the second case can only occur when $S_3 = S_1$ (see the claim in the proof of Lemma 4.7). Therefore, we may suppose that $U$ is obtained from $S'_i$ by blowing up an edge at vertices whose stabilizers are conjugate to $B$. Thus, up to applying a global conjugation, we may assume
that $\langle x_1 \rangle \ast \langle g \rangle \subseteq D$. But, as $g$ is not contained in any proper $Z_{RC}$-factor of $A$ and as $A \cap D$ is a free factor of $A$, we see that $A \cap D = A$. Hence $A \ast \langle x_1 \rangle \subseteq D$, and, as $A \ast \langle x_1 \rangle$ is isomorphic to $W_{n-1}$, we have in fact $A \ast \langle x_1 \rangle = D$. It follows that $C = D$ and, by Lemma 4.4, we see that $S_2 = S_3$. Thus the only equivalence classes of $W_{n-1}$-stars fixed by finite index subgroups of $P'$ are $S_1$ and $S_2$.

Therefore the only equivalence classes of $W_{n-2}$-stars fixed by finite index subgroups of $P'$ are the equivalence classes of the $W_{n-2}$-stars which refine $S_1$ and $S_2$. As $S_1$ and $S_2$ are refined by a unique (up to $W_n$-equivariant homeomorphism) $W_{n-2}$-star by Theorem 3.7, we conclude that $S$ is the only equivalence class of $W_{n-2}$-star fixed by finite index subgroups of $P'$. This completes the proof.

**Proposition 6.5.** Let $n \geq 5$ and let $\Gamma$ be a finite index subgroup of $C_n$. Let $S$ be the equivalence class of a $W_{n-2}$-star $S$. Then $\text{Stab}_\Gamma(S)$ satisfies $(P_{W_{n-2}})$. Moreover, we can choose for the subgroup $K_1 \times K_2$ of Property $(P_{W_{n-2}})$ (1) the direct product of the groups of twists of $S$ about the two edges of $S$.

**Proof.** The fact that $\text{Stab}_\Gamma(S)$ satisfies $(P_{W_{n-2}})$ (2) follows from the fact that $\text{Stab}_\Gamma(S)$ contains the stabilizer in $\Gamma$ of the equivalence class of a $W_3$-star obtained from $S$ by blowing-up $n-5$ edges at the center of $W_n \setminus S$. Indeed, Proposition 2.8 (3) ensures that the group of twists of a $W_3$-star is isomorphic to a direct product of $n-3$ copies of $W_3$.

The fact that $\text{Stab}_\Gamma(S)$ satisfies $(P_{W_{n-2}})$ (3) follows from the fact that $\text{Stab}_\Gamma(S)$ contains the stabilizer in $\Gamma$ of the equivalence class of a $W_2$-star obtained from $S$ by blowing-up $n-4$ edges at the center of $W_n \setminus S$. Indeed the group of twists of a $W_2$-star is isomorphic to a direct product of $n-2$ copies of $W_2$ by Proposition 2.8 (3).

Let us now prove that $\text{Stab}_\Gamma(S)$ satisfies $(P_{W_{n-2}})$ (1). Let $T'$ be the group of twists of $S$ and let $T = T' \cap \Gamma$. The group $T$ is normal in $\text{Stab}_\Gamma(S)$ since $\Gamma \subseteq C_n$. By Proposition 2.8 (3), the group $T'$ is isomorphic to $T'_1 \times T'_2$, where, for $i \in \{1, 2\}$, $T'_i$ is the group of twists in $\text{Out}(W_n)$ about one edge of $W_n \setminus S$. For $i \in \{1, 2\}$, let $T_i = T'_i \cap \Gamma$. For every $i \in \{1, 2\}$, the group $T_i$ is a normal subgroup of $\text{Stab}_\Gamma(S)$ and the group $T_1 \times T_2$ is a normal subgroup of $\text{Stab}_\Gamma(S)$. Let $T_1^{(2)}$ be a finite index subgroup of $T_1$ and let $P'$ be a finite index subgroup of a nontrivial normal subgroup of $T_1^{(2)}$. We prove that the centralizer of $P'$ in $\Gamma$ contains $T_2$ as a finite index subgroup. This will conclude the proof of the proposition by symmetry of $T_1$ and $T_2$. By Lemma 6.4 the equivalence class $S$ is the only equivalence class of $W_{n-2}$-star fixed by every finite index subgroup of $P'$. Hence $C_\Gamma(P')$ fixes $S$.

Let $H$ be a finite index subgroup of $C_\Gamma(P')$ which fixes $S$. Let

$$W_n = \langle x_1 \rangle \ast \langle x_3, \ldots, x_n \rangle \ast \langle x_2 \rangle$$

be a free factor decomposition associated with $W_n \setminus S$ and $A = \langle x_3, \ldots, x_n \rangle$. By Proposition 2.8 (1), the kernel of the natural homomorphism $H \to \text{Out}(A)$ is isomorphic to $H \cap T$. We claim that the image of $H$ in $\text{Out}(A)$ is finite. Indeed, as $P'$ is a finite index subgroup of a nontrivial normal subgroup of a finite index subgroup of $T_1$ and as $T_1$ is isomorphic to a finite index subgroup of $W_{n-2}$, we see that $P'$ is isomorphic to a
finite index subgroup \(N\) of a nontrivial normal subgroup of a finite index subgroup of \(W_{n-2}\). By Proposition 6.2 (3), \(N\) is not contained in any proper \(Z_{RC}\)-factor of \(W_{n-2}\). By Proposition 6.2 (4), there exists \(g \in N\) such that \(g\) is not contained in any proper \(Z_{RC}\)-factor of \(W_{n-2}\). Thus, there exists \(g \in A\) such that \(g\) is not contained in any proper \(Z_{RC}\)-factor of \(A\) and the twist about \(g\) is contained in \(P'\). As \(H\) commutes with the twist about \(g\), Lemma 4.11 implies that \(H\) preserves the conjugacy class of \(g\). Hence, by Theorem 6.3, the image of \(H\) in \(\text{Out}(A)\) is finite.

Thus, \(H \cap T\) has finite index in \(H\) and in \(C\Gamma(P')\). But, as \(H\) commutes with \(P' \subseteq T_1\), and as \(T_1\) is virtually a nonabelian free group, the intersection \(H \cap T\) has finite index in \(H \cap T\), hence has finite index in \(C\Gamma(P')\). This completes the proof.  \(\square\)

### 6.3 Groups satisfying \((P_{W_{n-2}})\) and stabilizers of \(W_{n-1}\)-stars

We prove in this section that if \(H\) is a subgroup of \(\text{Out}(W_n)\) which satisfies \((P_{W_{n-2}})\), then \(H\) virtually fixes the equivalence class of a \(W_{n-1}\)-star. We first recall a general lemma.

**Lemma 6.6.** Let \(G\) be a group and let \(N\) be a finitely generated normal subgroup of \(G\). Let \(n \in \mathbb{N}^+\).

(1) There exist only finitely many subgroups of \(N\) of index equal to \(n\).

(2) For every finite index subgroup \(N'\) of \(N\) there exists a finite index subgroup \(G'\) of \(G\) such that \(N'\) is a normal subgroup of \(G'\).

**Proof.** Assertion (1) is well known, we only prove assertion (2). Let \(N'\) be a subgroup of \(N\) of index \(n\) and let \(g \in G\). As \(N\) is a normal subgroup of \(G\), the automorphism \(\text{ad}_g : G \to G\) induces an automorphism \(\text{ad}_g|_N : N \to N\) by restriction. Therefore, \(\text{ad}_g\) permutes the subgroups of index \(n\) in \(N\). Since there exists a finite number of subgroups of index \(n\) in \(N\) by the first assertion, we see that there exists a finite index subgroup \(G'\) of \(G\) such that, for every \(g \in G'\), we have \(\text{ad}_g(N') = N'\). Therefore \(N'\) is a normal subgroup of \(G'\). This concludes the proof.  \(\square\)

**Lemma 6.7.** Let \(n \geq 5\). Let \(H\) be a subgroup of \(C_n\) satisfying \((P_{W_{n-2}})\). Let \(K_1 \times K_2\) be a normal subgroup of \(H\) given by \((P_{W_{n-2}})\) (1). Then one of the following holds.

(1) For every \(i \in \{1, 2\}\), the group \(K_i\) does not virtually fix the equivalence class of a free splitting.

(2) The group \(H\) virtually fixes the equivalence class of a one-edge free splitting.

**Proof.** Suppose that there exists \(i \in \{1, 2\}\) such that \(K_i\) virtually fixes the equivalence class of a free splitting. Up to reordering, we may assume that \(i = 1\). Let \(K_1'\) be a finite index subgroup of \(K_1\) which fixes the equivalence class of a free splitting, and let \(C\) be the set of all equivalence classes of free splittings fixed by \(K_1'\). Since \(K_1\) is a finitely generated normal subgroup of \(H\), by Lemma 6.6 (2), there exists a finite index subgroup
H₀ of H such that \( K'_1 \) is a normal subgroup of \( H₀ \). In particular, the set \( C \) is preserved by \( H₀ \).

Suppose first that the set \( C \) is finite. Then the set \( C \) is virtually fixed pointwise by \( H₀ \). Hence the group \( H \) virtually fixes the equivalence class of a free splitting.

So we may assume that the set \( C \) is infinite. Let \( U_C \) be the splitting provided by Theorem 5.4 and let \( \mathcal{U}_C \) be its equivalence class. By the equivariance property in Theorem 5.4, the equivalence class \( \mathcal{U}_C \) is \( H₀ \)-invariant. Suppose first that the splitting \( U_C \) contains an edge \( e ∈ EU_C \) with trivial stabilizer. Let \( U' \) be the splitting obtained from \( U_C \) by collapsing every edge of \( U_C \) that are not contained in the orbit of \( e \), and let \( \mathcal{U}' \) be its equivalence class. Then \( \mathcal{U}' \) is the equivalence class of a one-edge free splitting virtually fixed by \( H \).

Thus, we may assume that all edge stabilizers of \( U_C \) are nontrivial. We show that this leads to a contradiction. Let \( H' \) be the subgroup of finite index in \( H₀ \) which acts trivially on \( W_n \setminus U_C \). We claim that the intersection of \( H' \) with the group of twists of \( U_C \) is finite. Indeed, let \( e \) be a half-edge of \( U_C \). As \( W_n \) is virtually free, if the edge stabilizer \( G_e \) of \( e \) is not cyclic, the group of twists about this half-edge is trivial. Thus, as we suppose that all edge stabilizers are nontrivial, half-edges with nontrivial group of twists have cyclic stabilizers. But by Lemma 2.10 twists about edges with cyclic stabilizers are central in a finite index subgroup of \( \text{Stab}^0(U_C) \). Note that Remark 6.1(2) implies that the center of every finite index subgroup of \( H' \) is finite. Therefore the intersection of \( H' \) with the group of twists is finite. By Remark 2.9 the intersection of \( H' \) with the group of bitwists is finite. Thus, up to passing to a finite index subgroup, we may suppose that the map

\[
H' \to \prod_{v ∈ V(W_n \backslash U_C)} \text{Out}(G_v)
\]

given by the action on the vertex groups is injective.

Let \( VU_C = V₁ \sqcup V₂ \) be the partition of \( VU_C \) given by Theorem 5.4 and, for every \( i \in \{1, 2\} \), let \( H_i \) be the subgroup of \( H' \) made of all automorphisms whose image in \( \prod_{v ∈ W_n \setminus V_i} \text{Out}(G_v) \) is trivial. Then \( H₁ \) and \( H₂ \) centralize each other and, by Theorem 5.4(1)(b), the group \( H₁ \cap K'_1 \) is a finite index subgroup of \( K'_1 \). Thus \( H₂ \) centralizes a finite index subgroup of \( K'_1 \). We prove that \( \text{rk}_\text{prod}(H₂) ≥ 2 \), which will contradict the fact that the centralizer of every finite index subgroup of \( K'_1 \) is virtually free.

By Theorem 5.4(2), the set of all conjugacy classes of groups \( G_v \), with \( v ∈ V₂ \) is a free factor system of \( W_n \). In particular, for every \( v ∈ V₂ \), there exists \( k_v ∈ \{0, \ldots, n - 1\} \) such that \( G_v \) is isomorphic to \( W_{k_v} \). Suppose first that \( |W_n \setminus V₂| \geq 3 \). In this case, by Theorem 5.4(2) and since \( \text{rk}_\text{prod}(\text{Out}(W₃)) = 1 \) and \( \text{rk}_\text{prod}(\text{Out}(W₂)) = 0 \), for all \( v ∈ V₂ \), we have \( \text{rk}_\text{prod}(\text{Out}(W_{k_v})) ≤ k_v - 2 \). Hence

\[
\text{rk}_\text{prod} \left( \prod_{v ∈ W_n \setminus V₂} \text{Out}(G_v) \right) ≤ n - 6.
\]

Since \( \text{rk}_\text{prod}(H') = n - 3 \), using Lemma 5.2 we see that \( \text{rk}_\text{prod}(H₂) ≥ 3 \). This leads to a contradiction. Suppose now that \( |W_n \setminus V₂| = 2 \) and let \( v₁, v₂ ∈ W_n \setminus V₂ \) be distinct. Then
for every \( i \in \{1, 2\} \) there exists \( k_i \in \{1, \ldots, n - 1\} \) such that \( G_{v_i} \) is isomorphic to \( W_{k_i} \). If \( W_n = W_{k_1} \ast W_{k_2} \), then the group \( H' \) virtually fixes the equivalence class of the one-edge free splitting determined by this free factor decomposition of \( W_n \). So we may assume that \( W_n \neq W_{k_1} \ast W_{k_2} \). This implies that \( k_1 + k_2 \leq n - 1 \). Hence

\[
\text{rk}_{\text{prod}} \left( \prod_{v \in W_n \setminus V_2} \text{Out}(G_v) \right) \leq n - 5.
\]

Since \( \text{rk}_{\text{prod}}(H') = n - 3 \), using Lemma 6.2, we see that \( \text{rk}_{\text{prod}}(H_2) \geq 2 \). This leads to a contradiction. Suppose now that \( k \in \{1, \ldots, n - 1\} \) such that \( G_v \) is isomorphic to \( W_k \). Suppose first that \( k \leq n - 2 \). Then by Theorem 5.1 (2), and since \( \text{rk}_{\text{prod}}(\text{Out}(W_3)) = 1 \), \( \text{rk}_{\text{prod}}(\text{Out}(W_1)) = 0 \), and \( \text{rk}_{\text{prod}}(\text{Out}(W_2)) = 0 \), if \( n \neq 5 \), we have

\[
\text{rk}_{\text{prod}}(\text{Out}(W_k)) \leq n - 5.
\]

Thus, by Lemma 6.2, we see that \( \text{rk}_{\text{prod}}(H_2) \geq 2 \). When \( n = 5 \), the case where \( k = 3 \) and \( \text{rk}_{\text{prod}}(\text{Out}(W_k)) = 1 = n - 4 \) can occur. But by Property \((P_{W_{n-2}})\) (3), the group \( H' \) contains a subgroup isomorphic to \( \mathbb{Z}^3 \). Since \( \text{Out}(W_3) \) is virtually free, the group \( H_2 \) contains a subgroup isomorphic to \( \mathbb{Z}^2 \). This contradicts the fact that the centralizer of every finite index subgroup of \( K_1 \) is virtually nonabelian free. Hence we have \( k = n - 1 \). But then, by Lemma 4.3, the group \( H' \) (and hence the group \( H \)) virtually fixes the equivalence class of a \( W_{n-1} \)-star. This concludes the proof. \( \square \)

**Lemma 6.8.** Let \( n \geq 5 \). Let \( \mathcal{F} \) be a nonsporadic free factor system. Let \( H \) be a subgroup of \( C_n \cap \text{Out}(W_n, \mathcal{F}) \) containing a direct product of \( n - 3 \) nonabelian free groups. Then \( H \) cannot contain a finite index subgroup which fixes the homothety class of a \((W_n, \mathcal{F})\) - arational tree.

**Proof.** Suppose towards a contradiction that \( H \) has a finite index subgroup which fixes the equivalence class of a \((W_n, \mathcal{F})\)-arational tree. Up to passing to a finite index subgroup, we may suppose that \( H \) itself fixes the homothety class of a \((W_n, \mathcal{F})\)-arational tree. By Lemma 2.5 there exists a homomorphism from \( H \) to \( \mathbb{Z} \) whose kernel \( K' \) is exactly the isometric stabilizer of a \((W_n, \mathcal{F})\)-arational tree. Note that \( K' \) contains a direct product of \( n - 3 \) nonabelian free groups as it is the kernel of a homomorphism from \( H \) to \( \mathbb{Z} \). By Proposition 2.7 there exists a finite index subgroup \( K \) of \( K' \) such that \( K \) fixes infinitely many equivalence classes of free splittings. Let \( \mathcal{C} \) be the collection of all equivalence classes of free splittings fixed by \( K \).

We claim that \( \mathcal{C} \) is in fact finite, which will lead to a contradiction. Since \( K \subseteq C_n \), Lemma 4.2 implies that if \( \mathcal{S} \) is the equivalence class of a free splitting \( S \) fixed by \( K \), then the group \( K \) fixes the equivalence class of every one-edge free splitting onto which \( S \) collapses. By Theorem 6.7, if \( \mathcal{S} \) is the equivalence class of a free splitting \( S \), then \( S \) is determined by the finite set of equivalence classes of one-edge free splittings onto which \( S \) collapses. Therefore, it suffices to show that \( K \) can only fix finitely many equivalence
classes of one-edge free splittings. Let $S$ be the equivalence class of a one-edge free splitting fixed by $K$. Since $K$ contains a direct product of $n - 3$ nonabelian free groups, Theorem 5.1 (3) implies that $S$ is a $W_{n-1}$-star. Let

$$W_n = \langle x_1, \ldots, x_{n-1} \rangle \ast \langle x_n \rangle$$

be a free factor decomposition associated with $S$ and let $A = \langle x_1, \ldots, x_{n-1} \rangle$. By Proposition 2.8 (1), the kernel of the natural homomorphism $K \to \text{Out}(A)$ is the intersection of $K$ with the group of twists $T$ of $S$. By Theorem 5.1 (2), the product rank of $\text{Out}(A)$ is equal to $n - 4$. Since $K$ contains a direct product of $n - 3$ nonabelian free groups, we see that $K \cap T$ is infinite. Therefore, for every equivalence class $S$ of a $W_{n-1}$-star $S$ fixed by $K$, the group $K$ contains an infinite twist about $S$.

Let $S$ and $S'$ be two distinct equivalence classes of $W_{n-1}$-stars fixed by $K$. Let $S$ be a representative of $S$ and let $S'$ be a representative of $S'$. We claim that $S$ and $S'$ are compatible. Indeed, by the above, there exists $f \in K$ of infinite order such that $f$ is a twist about $S$. Since $f$ fixes $S'$, Lemma 6.7 implies that $S$ and $S'$ are compatible. Therefore, for every distinct equivalence classes $S$ and $S'$ of one-edge free splittings fixed by $K$, there exist $S \in S$ and $S' \in S'$ such that $S$ and $S'$ are compatible. By Theorem 5.1 this is only possible when $C$ is finite. This leads to a contradiction since $K$ must fix infinitely many equivalence classes of free splittings. This concludes the proof. 

**Proposition 6.9.** Let $n \geq 5$. Let $H$ be a subgroup of $C_n$ satisfying $(P_{W_{n-2}})$. Then $H$ virtually fixes the equivalence class of a $W_{n-1}$-star.

**Proof.** The proof is inspired by [HW] Proposition 8.2 and [HHW] Proposition 6.5. We prove that $H$ virtually fixes the equivalence class of a one-edge free splitting. Since $H$ contains a direct product of $n - 3$ nonabelian free groups, we will then conclude by Theorem 6.1 (3). Suppose towards a contradiction that $H$ does not virtually fix the equivalence class of a one-edge free splitting. Let $F$ be a maximal $H$-periodic free factor system. We can assume that $F$ is nonsporadic otherwise $H$ virtually fixes the equivalence class of a one-edge free splitting and we are done. As $F$ is maximal, by Proposition 2.3 the group $H$ acts with unbounded orbits on $\text{FF}(W_n, F)$.

Let $K_1 \times K_2$ be a normal subgroup of $H$ given by $(P_{W_{n-2}})$ (1). Suppose first that $K_1 \times K_2$ does not contain a loxodromic element on $\text{FF}(W_n, F)$. As $H$ has unbounded orbits on $\text{FF}(W_n, F)$, Lemma 5.3 implies that $K_1 \times K_2$ has a finite orbit in $\hat{\text{c}}_2\text{FF}(W_n, F)$.

By Lemma 2.9 there exists a finite index subgroup $K'_1 \times K'_2$ of $K_1 \times K_2$ such that $K'_1 \times K'_2$ fixes the homothety class of an arational $(W_n, F)$-tree $T$. Since $K_1 \times K_2$ does not contain a loxodromic element, $K'_1 \times K'_2$ fixes $T$ up to isometry, not just homothety (see e.g. [GH2] Proposition 6.2). By Proposition 2.7 the group $K'_1 \times K'_2$ virtually fixes infinitely many equivalence classes of $(W_n, F)$-free splittings. By Lemma 6.7 the group $H$ virtually fixes the equivalence class of a one-edge free splitting of $W_n$.

So we may suppose that there exists a loxodromic element $\Phi \in K_1 \times K_2$. First suppose that there exists a unique $i \in \{1, 2\}$ such that the group $K_i$ contains a loxodromic element $\Phi_i$. We may assume, up to reordering, that only $K_2$ contains a loxodromic element $\Phi$. 

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Lemma 6.8. edge free splitting $S$. If $H$ is a nonabelian free group, then the group $H$ equivalency class of a group $S$ commute, by Corollary 4.9. Let $K$ be a normal subgroup $H$ of $S$. By Proposition 2.8, given by the action on the vertex group contains an element of infinite order. Let $W$ virtually stabilize the equivalence class of a group $H$. By Lemma 5.3, the whole group $K$ is virtually fixes a point in $\partial_\infty \mathcal{F}(W_1, \mathcal{F})$. By Lemma 6.8, the group $H$ virtually fixes the equivalence class of a one-edge-free splitting of $W_n$.

Now suppose that for every $i \in \{1, 2\}$, the group $K_i$ contains a loxodromic element. By Lemma 5.3, the whole group $H$ virtually fixes a point in $\partial_\infty \mathcal{F}(W_n, \mathcal{F})$. By Lemma 2.6, the group $H$ virtually fixes the homothety class of a oneENSIONAL TREE. By Proposition 2.7, the group $G$ relative to $W_n$ isomorphic to $\mathcal{F}$. By Lemma 2.6, the group $H$ of $\mathcal{F}$, by Proposition 6.10, the group $H$ contains a subgroup $S$ which virtually stabilizes the equivalence class of a free splitting relative to $\mathcal{F}$. By Lemma 6.4, the group $H$ virtually fixes the equivalence class of a one-edge-free splitting of $W_n$.

Therefore, in all cases, the group $H$ virtually fixes the equivalence class $S$ of a one-edge-free splitting $S$. By Theorem 5.1 (3), since $H$ contains a direct product of $n - 3$ nonabelian free groups, the group $H$ virtually fixes the equivalence class of a $W_{n-1}$-star.

We now prove a proposition which gives a sufficient condition for equivalence classes of $W_{n-1}$-stars provided by Proposition 6.9 to be compatible.

**Proposition 6.10.** Let $n \geq 5$ and let $H$ be a subgroup of $C_n$ of finite index. Let $k \in \mathbb{N}^*$ and let $H_1, \ldots, H_k$ be subgroups of $H$ which satisfy $(P_{k-2})$ and such that the intersection $\bigcap_{i=1}^k H_i$ contains a subgroup $H$ isomorphic to $\mathbb{Z}^{n-2}$. For $i \in \{1, \ldots, k\}$, let $S_i$ be the equivalence class of a $W_{n-1}$-star $S_i$ which is virtually fixed by $H_i$. Then, for every $i, j \in \{1, \ldots, k\}$, the $W_{n-1}$-stars $S_i$ and $S_j$ are compatible.

**Proof.** Let $i, j \in \{1, \ldots, k\}$ be distinct integers. Let $H'$ be a finite index subgroup of $H$ contained in $\text{Stab}(S_i) \cap \text{Stab}(S_j)$. Let $A_i$ and $A_j$ be the vertex groups isomorphic to $W_{n-1}$ of respectively $W_n \setminus S_i$ and $W_n \setminus S_j$ (well defined up to conjugation). By Proposition 2.2, the rank of a maximal abelian subgroup of $\text{Out}(W_{n-1})$ is equal to $n - 3$. Therefore, the kernel of the homomorphisms $H' \to \text{Out}(A_i)$ and $H' \to \text{Out}(A_j)$ given by the action on the vertex group contains an element of infinite order. Let $f_i \in \ker (H' \to \text{Out}(A_i))$ and $f_j \in \ker (H' \to \text{Out}(A_j))$ be infinite order elements. By Proposition 2.8 (1), $f_i$ and $f_j$ are twists about respectively $S_i$ and $S_j$. As $f_i$ and $f_j$ commute, by Corollary 4.9, $S_i$ and $S_j$ are compatible. This concludes the proof.

### 7 Algebraic characterization of stabilizers of $W_{n-2}$-stars

In this section, we give an algebraic characterization of stabilizers of $W_{n-2}$-stars. By the previous section, we know that groups which satisfy $(P_{W_{n-2}})$ virtually stabilize equivalence classes of $W_{n-1}$-stars, and we have given an algebraic criterion to show that these $W_{n-1}$-stars are compatible. In order to prove that a group $H$ which satisfies $(P_{W_{n-2}})$ virtually stabilizes the equivalence class of a $W_{n-2}$-star, we study the intersection of a normal subgroup $K_1 \times K_2$ of $H$ given by $(P_{W_{n-2}})$ (1) with the group of twists of the equivalence class of a $W_{n-1}$-star virtually fixed by $H$. 

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7.1 Groups of twists in groups satisfying \((P_{W_{n-2}})\)

We start this section with a lemma which gives a sufficient condition for a group \(H\) satisfying \((P_{W_{n-2}})\) to be the stabilizer of a \(W_{n-2}\)-star.

**Lemma 7.1.** Let \(n \geq 5\) and let \(\Gamma\) be a subgroup of finite index of \(C_n\). Let \(H\) be a subgroup of \(\Gamma\) which satisfies \((P_{W_{n-2}})\) and let \(K_1 \times K_2\) be a normal subgroup of \(H\) given by \((P_{W_{n-2}})\) (1). Let \(S_1\) be the equivalence class of a \(W_{n-1}\)-star \(S_1\) virtually fixed by \(H\) and let \(T_1\) be the group of twists of \(S_1\).

Suppose that \(T_1 \cap K_1\) is infinite and that there exists an equivalence class \(S_2\) of a \(W_{n-1}\)-star \(S_2\) such that the intersection of \(K_2\) with the group of twists \(T_2\) of \(S_2\) is infinite. Then \(S_1\) and \(S_2\) are compatible and \(H\) virtually fixes the equivalence class \(S\) of the \(W_{n-2}\)-star which refines \(S_1\) and \(S_2\). Moreover, \(S\) is the unique equivalence class of a \(W_{n-2}\)-star virtually fixed by \(H\). Finally, the groups \(T_1 \cap \text{Stab}_\Gamma(S)\) and \(K_1\) (resp. \(T_2 \cap \text{Stab}_\Gamma(S)\) and \(K_2\)) are commensurable.

**Proof.** For \(i \in \{1, 2\}\), let \(f_i \in T_i \cap K_i\) be of infinite order. First remark that, as \(f_1\) and \(f_2\) generate a free abelian group of order 2, we have \(T_1 \neq T_2\) because the group of twists of a \(W_{n-1}\)-star is virtually a nonabelian free group. Hence we have \(S_1 \not= S_2\). As \(K_1\) commutes with \(f_2\), Proposition 4.8 shows that \(K_1\) fixes \(S_2\). As \(K_1\) contains a twist of \(S_1\), Lemma 4.1 shows that \(S_1\) and \(S_2\) are compatible.

Let \(S\) be a \(W_{n-2}\)-star which refines \(S_1\) and \(S_2\), let \(S\) be its equivalence class and let \(T\) be the group of twists of \(S\) in \(\Gamma\). Then \(T\) contains a finite index normal subgroup isomorphic to \(K_1^{S_1} \times K_2^{S_2}\), where \(K_1^{S_1}\) and \(K_2^{S_2}\) are virtually nonabelian free groups. By Proposition 4.8 we can choose \(K_1^{S_1} \times K_2^{S_2}\) such that \(K_1^{S_1} \times K_2^{S_2}\) is a group satisfying Property \((P_{W_{n-2}})\) (1). Moreover, up to reordering, \(K_1^{S_1} \subseteq T_1\) and \(K_2^{S_2} \subseteq T_2\). Since \(K_1\) fixes both \(S_1\) and \(S_2\), we see that \(K_1\) fixes \(S\). Therefore, by Proposition 4.8 (1), we have a homomorphism \(\Phi: K_1 \to \text{Out}(W_{n-2})\) whose kernel is exactly \(K_1 \cap T\). By Lemma 4.10 we see that \(T_1 \cap \text{Stab}_\Gamma(S) \cap K_1^{S_1}\) is a finite index subgroup of \(T_1 \cap \text{Stab}_\Gamma(S)\). As \(K_1 \cap T_1\) is infinite, so is \(K_1 \cap K_1^{S_1}\). Let \(P = \ker(\Phi) \cap K_1^{S_1} = K_1 \cap K_1^{S_1}\). Then, since \(K_1 \subseteq C_n\), the group \(K_1^{S_1} \cap K_1\) is a normal subgroup of \(K_1\). Therefore \(P\) is a nontrivial normal subgroup of \(K_1\). By Property \((P_{W_{n-2}})\) (1), we see that \(K_2\) is a finite index subgroup of \(C_\Gamma(P)\). But \(P\) is centralized by \(K_2^{S_2}\) since \(P \subseteq K_1^{S_1}\). Hence \(K_2^{S_2} \cap K_2\) is a finite index subgroup of \(K_2^{S_2}\). As \(K_1^{S_1}\) and \(K_2^{S_2}\) are compatible, \(K_1^{S_1} \cap K_2\) is a finite index subgroup of the centralizer of \(K_2^{S_2}\) by Property \((P_{W_{n-2}})\) (1), and as \(K_1\) is a finite index subgroup of the centralizer of \(K_2\), we see that \(K_1^{S_1} \cap K_1\) has finite index in \(K_1\) and therefore \(P\) has finite index in \(K_1\). Let

\[
W_n = \langle x_1 \rangle \ast \langle x_3, \ldots, x_n \rangle \ast \langle x_2 \rangle
\]

be the free factor decomposition of \(W_n\) induced by \(S\) and let \(A = \langle x_3, \ldots, x_n \rangle\). Then, up to reordering, for every \(f \in P\), there exists \(z_f \in A\) and a representative \(F\) of \(f\) such that \(F\) sends \(x_1\) to \(z_f x_1 z_f^{-1}\), and, for every \(i \neq 1\), fixes \(x_i\).

**Claim.** The only equivalence classes of \(W_{n-1}\)-stars which are virtually fixed by \(K_1\) are \(S_1\) and \(S_2\).
Proof. Let $S_3$ be the equivalence class of a $W_{n-1}$-star $S_3$ virtually fixed by $K_1$. Suppose towards a contradiction that $S_3$ is distinct from both $S_1$ and $S_2$. Let $K'_1 = K_1 \cap \text{Stab}_F(S_3)$ and $P' = P \cap \text{Stab}_F(S_3)$. Then, as $P$ is an infinite subgroup of the group of twists of $S_1$, and as $P'$ is a finite index subgroup of $P$, we see that $P'$ is an infinite subgroup of the group of twists of $S_1$. By Lemma 4.7, we see that $S_1$ and $S_3$ are compatible. Let $S'$ be a $W_{n-2}$-star that refines $S_1$ and $S_3$ and let $S'$ be its equivalence class. Let

$$W_n = \langle y_1 \rangle \ast \langle y_3, \ldots, y_n \rangle \ast \langle y_2 \rangle$$

be the free factor decomposition of $W_n$ induced by $S'$ and let $B = \langle y_3, \ldots, y_n \rangle$. Since $S$ is a refinement of $S_1$, we may suppose that $B * \langle y_2 \rangle = A * \langle x_2 \rangle$ and that $y_1$ is a conjugate of $x_1$ by an element of $B * \langle y_2 \rangle$. Up to applying a global conjugation, we may also suppose that $y_1 = x_1$ and that $B * \langle y_2 \rangle = A * \langle x_2 \rangle$.

Let $T'$ be the group of twists of $S'$. Then $T'$ contains a finite index normal subgroup isomorphic to $P'_1 \times P'_2$, where both $P'_1$ and $P'_2$ are virtually nonabelian free subgroups of $T'$ which correspond to the groups of twists about the two edges of $W_n \setminus S'$. Then, as $P'$ is a group of twists of $S_1$, and as $P'$ fixes $S'$, by Lemma 4.10 up to reordering, the group $P'$ is contained in $P'_1$.

Let $f' \in P'_1$, let $F'$ be the representative of $f'$ which acts as the identity on $B * \langle y_2 \rangle$ and let $z_{f'} \in B$ be the twistor of $F'$. Then $F'$ acts as the identity on $A * \langle x_2 \rangle$ and $F'(x_1) = z_{f'}x_1z_{f'}^{-1}$. Recall that for every $\psi \in P'$, there exists a unique $z_{\psi} \in A$ and a unique representative $\Psi$ of $\psi$ such that $\Psi$ sends $x_1$ to $z_{\psi}x_1z_{\psi}^{-1}$, and, for every $i \neq 1$, fixes $x_i$. Thus, a necessary condition for $f'$ to be in $P'$ is that $z_{f'} \in A \cap B$.

But as $A$ and $B$ are free factors of $W_n$, the group $A \cap B$ is a free factor of $B$. To see this, let $U$ be a free splitting of $W_n$ such that $A$ is a vertex stabilizer of $U$ and let $U_B$ be the minimal subtree of $B$ in $U$. Then, as $U$ is a free splitting of $W_n$, we see that $U_B$ is a free splitting of $B$. But then, as $A$ is a vertex stabilizer in $U$, we see that $A \cap B$ is a vertex stabilizer in $U_B$. Therefore, $A \cap B$ is a free factor of $B$. Thus one can find a $W_{n-3}$-star $S(2)$ which refines $S'$ and such that, for every $f' \in P'$, the twistor $z_{f'}$ fixes a vertex of $S(2)$. Indeed, one can equivariantly blow-up an edge $e$ at the vertex of $S'$ whose stabilizer is $B$ such that the stabilizer of one of the endpoints of $e$ is a subgroup $C$ isomorphic to $W_{n-3}$ with $A \cap B \subseteq C$. Therefore we may also assume that $S(2)$ is a $W_{n-3}$-star. Let $S(2)$ be the equivalence class of $S(2)$. By Proposition 2.8 (3), the group of twists of $S(2)$ is isomorphic to a direct product $W_{n-3}^3$ of three infinite groups, where each factor is a group of twists about an edge of $W_n \setminus S(2)$. This implies that $P'$ is contained in exactly one of the three factors isomorphic to $W_{n-3}$. It follows that the centralizer of $P'$ contains two elements which generates a free abelian group of order 2. This contradicts the fact that the centralizer of $P'$ is virtually a nonabelian free group by $(P_{W_{n-2}})$ (1). The claim follows.

The claim above then implies, as $K_1$ is a normal subgroup of $H$, that $H$ virtually fixes $S_2$. As $H$ virtually fixes $S_1$, we see that $H$ virtually fixes the equivalence class $S$. Moreover, the above claim shows that $S$ is the unique equivalence class of a $W_{n-2}$-star virtually fixed by $K_1$, and hence virtually fixed by $H$. 

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We finally prove that $K_1$ and $T_1 \cap \text{Stab}_\Gamma(S)$ (resp. $K_2$ and $T_2 \cap \text{Stab}_\Gamma(S)$) are commensurable. By Lemma 7.1, for every $i \in \{1, 2\}$ we see that $K_i^{S_i} \cap T_i \cap \text{Stab}_\Gamma(S)$ is a finite index subgroup of $T_i \cap \text{Stab}_\Gamma(S)$. Moreover, for every $i \in \{1, 2\}$ and every $f \in K_i^{S_i}$, the twist $f$ of $S$ is also a twist of $S_i$. Hence we have $K_i^{S_i} \subseteq T_i \cap \text{Stab}_\Gamma(S)$. Therefore, for every $i \in \{1, 2\}$, the groups $K_i^{S_i}$ and $T_i \cap \text{Stab}_\Gamma(S)$ are commensurable. Hence it suffices to show that, for every $i \in \{1, 2\}$, the groups $K_i$ and $K_i^{S_i}$ are commensurable.

Recall that $K_2^{S_2} \cap K_2$ is a finite index subgroup of $K_2^{S_2}$ and that $K_1^{S_1} \cap K_1$ has finite index in $K_1$. Since $H$ virtually fixes $S$, and since $K_2^{S_2}$ is a normal subgroup of $\text{Stab}_\Gamma(S)$, we see that $K_2^{S_2} \cap K_2$ is a normal subgroup of a finite index subgroup of $K_2$. We know that $K_2^{S_2} \cap K_2$ commutes with $K_1^{S_1}$ because $K_1^{S_1}$ and $K_2^{S_2}$ commute with each other. Thus, by Property $(P_{W_{n-2}})$ (1) applied to $K_1 \times K_2$, the centralizer of $K_2^{S_2} \cap K_2$ contains $K_1$ as a finite index subgroup. This shows that $K_1 \cap K_1^{S_1}$ is a finite index subgroup of $K_1^{S_1}$. Hence $K_1$ and $K_1^{S_1}$ are commensurable. By Property $(P_{W_{n-2}})$ (1) applied to $K_1^{S_1} \times K_2^{S_2}$, the centralizer of a finite index subgroup of $K_1^{S_1}$ contains $K_2^{S_2}$ as a finite index subgroup. Moreover, the centralizer of a finite index subgroup of $K_1$ contains $K_2$ as a finite index subgroup. Hence the centralizer of $K_1 \cap K_1^{S_1}$ contains both $K_2$ and $K_2^{S_2}$ as finite index subgroups. Thus $K_2$ and $K_2^{S_2}$ are commensurable. This completes the proof of Lemma 7.1.

Lemma 7.1 suggests that in order to show that a group $H$ which satisfies $(P_{W_{n-2}})$ is in fact virtually the stabilizer of the equivalence class of a $W_{n-2}$-star, it suffices to study the intersection of $H$ with groups of twists. A first step towards such a result is the following lemma.

Lemma 7.2. Let $n \geq 5$ and let $\Gamma$ be a subgroup of $C_n$ of finite index. Let $H$ be a subgroup of $\Gamma$ satisfying $(P_{W_{n-2}})$ and let $K_1 \times K_2$ be a normal subgroup of $H$ given by $(P_{W_{n-2}})$ (1). Let $S$ be the equivalence class of a $W_{n-1}$-star $S$ virtually fixed by $H$ and let $T$ be the group of twists of $S$ contained in $\Gamma$.

There exists a unique $i \in \{1, 2\}$ such that $K_i \cap T$ is infinite. Moreover, $H \cap T \cap K_i$ has finite index in $H \cap T$.

Proof. Up to passing to a finite index subgroup of $H$, we may suppose that $H$ fixes $S$. The uniqueness assertion follows from the fact that $T$ is virtually a nonabelian free group and that $K_1 \times K_2$ is a direct product. Therefore, up to reordering, we may suppose that $K_1 \cap T$ is finite. So there exists a finite index subgroup $K'_1$ of $K_1$ such that $K'_1 \cap T$ is trivial. Since $K_1$ is a finitely generated normal subgroup of $H$, Lemma 7.1 implies that there exists a finite index subgroup $H'$ of $H$ such that $K'_1$ is a normal subgroup of $H'$. Therefore, we may suppose that $K_1 \cap T$ is trivial. By Proposition 2.2 (1), the natural homomorphism $K_1 \to \text{Out}(W_{n-1})$ given by the action on the vertex groups is injective. We claim that $H \cap T$ is infinite. Indeed, consider the natural homomorphism $\Phi: H \to \text{Out}(W_{n-1})$. By Proposition 2.2 the rank of a maximal free abelian subgroup of $\text{Out}(W_{n-1})$ is equal to $n-3$. As $H$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$ by $(P_{W_{n-2}})$ (3), the kernel of $H \to \text{Out}(W_{n-1})$ is infinite. But, by Proposition 2.2 (1), this is precisely $H \cap T$. Therefore, $H \cap T$ is infinite.
We now prove that \( H \cap T \cap K_2 \) has finite index in \( H \cap T \). This will conclude the proof as \( H \cap T \) is infinite. Let \( K = \Phi^{-1}(\Phi(K_2)) \). Note that \( H \cap T \subseteq K \). Then, as \( K_2 \) is normal in \( H \), we see that \( K \) is a normal subgroup of \( H \) which contains \( H \cap T \) and \( K_2 \). We claim that \( K \cap K_1 \) is finite. Indeed, suppose towards a contradiction that there exists \( f \in K \cap K_1 \) of infinite order. Then, as the homomorphism
\[
\Phi|_{K_1} : K_1 \to \text{Out}(W_{n-1})
\]
is injective, the element \( \Phi(f) \) has infinite order. By definition of \( K \), we see that \( \Phi(f) \in \Phi(K_1) \cap \Phi(K_2) \). But, as the homomorphism \( \Phi|_{K_1} : K_1 \to \text{Out}(W_{n-1}) \) is injective, and as \( K_1 \) is virtually a nonabelian free group, there exists \( g \in K_1 \) of infinite order such that \( \Phi(g) \) does not commute with \( \Phi(f) \). Since \( \Phi(f) \in \Phi(K_2) \) this contradicts the fact that \( K_1 \) and \( K_2 \) commute with each other. Hence \( K \cap K_1 \) is finite.

The groups \( K \) and \( K_1 \) are two normal subgroups of \( H \) with finite intersection. Let \( K^{(2)}_1 \) be a finite index normal subgroup of \( K_1 \) such that \( K \cap K^{(2)}_1 = \{1\} \). Since \( K_1 \) is finitely generated, by Lemma [6.6] (2), there exists a finite index subgroup \( H^{(2)} \) of \( H \) such that \( K^{(2)}_1 \) is a normal subgroup of \( H^{(2)} \). Hence \( K^{(2)}_1 \) and \( K \cap H^{(2)} \) are normal subgroups of \( H^{(2)} \) with trivial intersection. Therefore, \( K \cap H^{(2)} \subseteq C_1(K^{(2)}_1) \). But, Property \((P_{W_{n-2}})\) (1) implies that \( K \) and \( K_2 \) are commensurable. Since \( K \) contains \( H \cap T \), we see that \( K_2 \cap T \) and \( H \cap T \) are commensurable. This concludes the proof. \( \square \)

### 7.2 Groups satisfying \((P_{W_{n-2}})\) and stabilizers of \( W_{n-2}\)-stars

In this section we prove that a subgroup of \( C_n \) which satisfies \((P_{W_{n-2}})\) virtually fixes the equivalence class of a \( W_{n-2}\)-star. We first recall a theorem due to Guirardel and Levitt which provides a canonical splitting for a relative one-ended group (recall that a group \( G \) is one-ended relative to a family of subgroups \( \mathcal{H} \) if \( G \) does not have a one-edge splitting with finite edge stabilizers such that every subgroup of \( \mathcal{H} \) fixes a point).

**Theorem 7.3.** ([GL1], Theorem 9.14] Let \( G \) be a hyperbolic group and let \( \mathcal{H} \) be a family of subgroups such that \( G \) is one-ended relative to \( \mathcal{H} \). There exists a JSJ splitting \( T \) such that:

1. **Every edge stabilizer is virtually infinite cyclic.**
2. **For every \( H \in \mathcal{H} \), the group \( H \) is elliptic in \( T \).**
3. **The tree \( T \) is invariant under all automorphisms of \( G \) preserving \( \mathcal{H} \). Moreover, \( T \) is compatible with every splitting \( S \) with virtually cyclic edge stabilizers and such that for every \( H \in \mathcal{H} \), the group \( H \) is elliptic in \( S \).**

We also need some results about splittings over virtually cyclic groups, whose generalization to virtually free groups is due to Cashen.

**Theorem 7.4.** ([C4], Theorem 1.2] Let \( G_1 \) and \( G_2 \) be finitely generated virtually nonabelian free groups, and let \( C \) be a virtually cyclic group which is a proper subgroup of both \( G_1 \) and \( G_2 \). Then \( G_1 \ast_C G_2 \) is virtually a nonabelian free group if and only if there exists
which is impossible when

This proves

Hence from the fact that

Thus, for every

Therefore, for every

There exists

Therefore, there exist

Finally, the third point is a direct consequence of the inequality 

which is impossible when 

This concludes the proof.

\[ \Box \]
In the next lemma, we will use the notion of the abelian rank of a group $G$. The abelian rank of $G$ is the rank of a maximal free abelian subgroup of $G$. It is closed under taking finite index subgroups.

**Lemma 7.6.** Let $n \geq 4$ and let $S$ be a one-edge splitting of $W_n$ whose edge stabilizers are isomorphic to $W_2$. Let $S$ be its equivalence class. Let $v_1$ and $v_2$ be adjacent vertices of $S$ and let $e$ be the edge between $v_1$ and $v_2$. Suppose that the abelian rank of the image of the natural homomorphism

$$
\Phi: \text{Stab}(S) \to \text{Out}(G_{v_1}, G_e) \times \text{Out}(G_{v_2}, G_e)
$$

is equal to the abelian rank of $\text{Out}(W_n)$. For every $i \in \{1, 2\}$, let

$$
\Phi_i: \text{Stab}(S) \to \text{Out}(G_{v_i}, G_e)
$$

be the natural homomorphism given by the action on the vertex group.

1. For every $i \in \{1, 2\}$, the abelian rank of $\Phi_i(\text{Stab}(S))$ is equal to the abelian rank of $\text{Out}(G_{v_i})$.

2. For every $i \in \{1, 2\}$, there exists a refinement of $S$ by blowing-up a one-edge $Z_{RC}$-splitting of $G_{v_i}$ at $v_i$.

**Proof.** (1) By Corollary 7.3 (2), for every $i \in \{1, 2\}$, there exists $k_i \in \mathbb{N}^+$ such that $G_{v_i}$ is isomorphic to $W_{k_i}$. Moreover, we have $k_1 + k_2 = n + 2$ and, for every $i \in \{1, 2\}$, we have $k_i \geq 3$. By Proposition 2.2, for every $i \in \{1, 2\}$, the abelian rank of $\text{Out}(G_{v_i})$ is equal to $k_i - 2$ and the abelian rank of $\text{Out}(W_n)$ is equal to $n - 2$. By the assumption of the lemma, the abelian rank of $\Phi(\text{Stab}(S))$ is hence equal to $n - 2$. This implies that

$$n - 2 \leq k_1 - 2 + k_2 - 2 = n + 2 - 4 = n - 2.$$

Therefore, we conclude that for every $i \in \{1, 2\}$, the abelian rank of $\Phi_i(\text{Stab}(S))$ is equal to the abelian rank of $\text{Out}(G_{v_i})$. This proves Assertion (1).

(2) Let $i \in \{1, 2\}$. By the first assertion the group $\text{Out}(G_{v_i}, G_e)$ is infinite. Since $G_e$ is isomorphic to $W_2$, the group $\text{Out}(G_e)$ is finite. Therefore, the group $\text{Out}(G_{v_i}, G_e^{t_i})$ is infinite. By Theorem 6.3, the group $G_{v_i}$ has a $Z_{RC}$ splitting $U$ such that $G_e$ fixes exactly one vertex. Let $S'$ be the splitting obtained from $S$ by blowing-up $U$ at $v_i$ and by attaching $e$ to the vertex of $U$ fixed by $G_e$. Then $S'$ satisfies Assertion (2). This concludes the proof. \qed

**Proposition 7.7.** Let $n \geq 5$ and let $\Gamma$ be a finite index subgroup of $C_n$. Let $H$ be a subgroup of $\Gamma$ which satisfies $(P_{W_{n-2}})$. Then $H$ virtually stabilizes the equivalence class of a $W_{n-2}$-star. Moreover, this equivalence class is unique.

**Proof.** By Proposition 6.9, the group $H$ virtually fixes the equivalence class $S$ of a $W_{n-1}$-star $S$. Let

$$W_n = A * \langle x_n \rangle$$

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be the free factor decomposition of $W_n$ induced by $S$. Up to passing to a finite index subgroup, we may suppose that $H$ fixes $S$. Let $T$ be the group of twists of $S$ contained in $\Gamma$. By Proposition 2.8 (2), the group $\text{Stab}(S)$ is isomorphic to $\text{Aut}(A)$ and the group of twists of $S$ is identified with the inner automorphism group of $A$.

Let $K_1 \times K_2$ be a normal subgroup of $H$ given by Property $(P_{W_{n-2}})$ (1). By Lemma 2.2 up to exchanging the roles of $K_1$ and $K_2$, we may assume that $K_1 \cap T$ is infinite, that $H \cap T \cap K_1$ is a finite index subgroup of $H \cap T$ and that $K_2 \cap T$ is finite. Up to passing to a finite index subgroup, we may assume that $K_2 \cap T = \{1\}$. In particular, the natural homomorphism $\phi : K_2 \to \text{Out}(A)$ is injective. Let $K \subseteq A$ be the group of twistors associated with twists contained in $K_1$. Note that to every splitting $S_0$ of $A$ such that $K$ fixes a unique vertex of $S_0$, one can deduce a splitting $S'_0$ of $W_n$ such that $K_1 \cap T$ fixes a point of $S'_0$. Indeed, by blowing-up the splitting $S_0$ at the vertex $v$ of $S$ whose associated group is $A$, and by attaching the edges of $S$ adjacent to $v$ to the vertex fixed by $K$, we obtain a splitting $S'_0$ of $W_n$ such that $K$ fixes a point of $S'_0$. Let $S'_0$ be the equivalence class of $S'_0$. We claim that the group $K_1 \cap T$ fixes $S'_0$. Indeed, let $e_0$ be the edge of $S'_0$ adjacent to the vertex $v_0$ fixed by $K$ and the vertex fixed by $\langle x_n \rangle$. Since the stabilizer of $e_0$ is trivial, Proposition 2.8 implies that the group of twistors about $e_0$ at the vertex $v_0$ contains all the twists whose twistor is an element of $K$. Hence $K_1 \cap T$ fixes $S'_0$.

We now construct a one-edge free splitting $S_0$ of $A$ such that $K$ fixes a vertex of $S_0$. By the above discussion, this will give a two-edge free splitting of $W_n$ such that $K$ fixes a vertex of this splitting which is not a leaf and whose equivalence class is fixed by $K_1 \cap T$. Moreover, we prove at the same time that there does not exist a free splitting of $W_n$ with at least 3 orbits of edges and such that $K$ fixes a vertex of this splitting. We distinguish between three cases, according to whether $A$ is one-ended relative to $K$ and according to the edge stabilizers of a splitting of $A$ relative to $K$.

**Case 1.** There exists a free splitting $S_0$ of $A$ such that $K$ fixes a vertex of $S_0$.

In particular, the corresponding splitting $S'_0$ of $W_n$ constructed above is a free splitting of $W_n$. We claim that the splitting $S'_0$ has two orbits of edges. Indeed, suppose that $S'_0$ has $k$ orbits of edges, with $k \geq 3$. Then, $S'_0$ is obtained from $S$ by blowing-up at least two orbits of edges at $v$. Therefore, the group of twistors $K$ is contained in a free factor $B$ of $W_n$ isomorphic to $W_{n-3}$. Let $B'$ be a free factor of $W_n$ isomorphic to $W_2$ such that

$$W_n = \langle x_n \rangle \ast B \ast B'$$

and let $R$ be the free splitting associated with this decomposition. Then the equivalence class $\mathcal{R}$ of $R$ is a free splitting of $W_n$ fixed by $K_1 \cap T$. But by Proposition 2.8 (3), the group of twists of $\mathcal{R}$ is isomorphic to $B \times B \times W_2$. Moreover, the group $K_1 \cap T$ is contained in one of the factors of $B \times B \times W_2$ isomorphic to $B$. Therefore, the centralizer of $K_1 \cap T$ contains a free abelian group of rank 2. Since $K_1 \cap T$ is a normal subgroup of $K_1$, this contradicts the fact that the centralizer of $K_1 \cap T$ is virtually a nonabelian free group by Property $(P_{W_{n-2}})$ (1). Therefore, the splitting $S'_0$ is a two-edge free splitting.

**Case 2** There exists a splitting $S_0$ of $A$ such that $K$ fixes a vertex of $S_0$ and such that one of the edge stabilizers of $S_0$ is finite.
Let $S'_0$ be the corresponding splitting of $W_n$ constructed in the above discussion. If $S_0$ has an edge $e'$ with trivial stabilizer, then by collapsing every orbit of edges of $S_0$ except the one containing $e'$, we obtain a splitting $S_1$ of $A$ such that $K$ fixes a vertex of $K$. Then the corresponding splitting $S'_1$ of $W_n$ is a free splitting. Thus, we can apply Case 1.

Therefore, we may assume that every edge stabilizer of $S_0$ is infinite or a nontrivial finite subgroup of $W_n$. By collapsing every edge of $S_0$ with infinite stabilizer and by collapsing all but one orbit of edges with finite edge stabilizer, we may suppose that $S_0$ is a one-edge splitting such that every edge stabilizer of $S_0$ is a nontrivial finite subgroup of $W_n$. Every finite subgroup of $W_n$ is isomorphic to $F$ and is in fact a free factor of $W_n$. We claim that we can construct a splitting $X_0$ of $A$ which contains an edge with trivial stabilizer and such that $K$ fixes a vertex of $X_0$. Indeed, let $x_0$ be the vertex of $S_0$ fixed by $K$, let $f_0$ be an edge adjacent to $x_0$ and let $x_1$ be the vertex of $f_0$ distinct from $v_0$. Let $G_{x_0}$ be the stabilizer of $x_0$, let $G_{x_1}$ be the stabilizer of $x_1$ and let $G_{f_0}$ be the stabilizer of $f_0$. Note that, since there does not exist HNN extensions in $W_n$, the groups $G_{x_0}$ and $G_{x_1}$ are not conjugate in $W_n$. The group $G_{f_0}$ is a free factor of both $G_{x_0}$ and $G_{x_1}$. Thus there exists a free factor $A'$ of $G_{x_1}$ such that $G_{x_1} = G_{f_0} * A'$. Let $U$ be the splitting of $A$ such that the underlying tree of $W_n \backslash U$ is the same one as the underlying tree of $W_n \backslash S_0$, such that the stabilizer of every vertex which is not in the orbit of $x_1$ is the same one as the stabilizer of the corresponding vertex in $S_0$ and the stabilizer of $x_1$ is $A'$. Then the edge $f_0$ has trivial stabilizer in $U$ and $K$ fixes a vertex of $U$. This proves the claim. Therefore Case 2 is a consequence of Case 1.

**Case 3** The group $A$ is one-ended relative to $K$.

By Theorem 7.3 there exists a canonical splitting $S_0$ of $A$ whose edge stabilizers are virtually infinite cyclic, such that $K$ fixes a point of $S_0$ and such that every automorphism of $A$ preserving $K$ fixes the equivalence class of $S_0$. Up to collapsing some orbits of edges, we may suppose that $S_0$ has exactly one orbit of edges and that $S_0$ is fixed by a finite index subgroup of the group of automorphisms of $A$ preserving $K$. Let $S'_0$ be the corresponding splitting of $W_n$, and let $S'_0$ be its equivalence class. Recall that the group $K_1 \cap T$ is a normal subgroup of $\text{Inn}(A)$. Hence the group $H$ viewed as a subset of $\text{Aut}(A)$ preserves $K$. Thus $H$ has a finite index subgroup $H'$ which preserves $S'_0$.

Let $v_0$ be the vertex of $S'_0$ fixed by $K$, let $e_0$ be the edge of $S'_0$ between $v_0$ and the point fixed by $\langle x_n \rangle$, let $e$ be an edge adjacent to $e_0$ which is not in the orbit of $e_0$ and let $w_0$ be the endpoint of $e$ distinct from $v_0$. By construction, the stabilizer of every edge of $S'_0$ which is not in the orbit of $e_0$ is virtually cyclic, that is it is either isomorphic to $Z$ or to $W_2$. By Lemma 2.10, a twist about an edge whose stabilizer is isomorphic to $Z$ is central in a finite index subgroup of $\text{Stab}_{\text{Out}(W_n)}(S'_0)$. Since any finite index subgroup of $H$ has finite center by Remark 6.4 (2), we see that the stabilizer of every edge of $S'_0$ which is not in the orbit of $e_0$ is isomorphic to $W_2$. In particular, the stabilizer of $e$ is isomorphic to $W_2$. Therefore, Remark 2.9 implies that the group of bitwists about every edge of $S'_0$ which is not in the orbit of $e_0$ is trivial. Thus, the group of bitwists $T_0$ of $S'_0$ is reduced to the group of twists about $e_0$. Since the group of twists about $e_0$ is virtually a nonabelian free group, the abelian rank of $T_0$ is equal to 1. Moreover,
by Corollary \[\text{Corollary 7.3}\] (2), there exist \(k_{v_0}, k_{w_0} \in \mathbb{N}^*\) such that the groups \(G_{v_0}\) and \(G_{w_0}\) are isomorphic to \(W_{k_{v_0}}\) and \(W_{k_{w_0}}\), with \(3 \leq k_{v_0}, k_{w_0} \leq n-2\) and \(k_{v_0} + k_{w_0} = n-1+2 = n+1\).

By Proposition \[\text{Proposition 2.3}\] and Remark \[\text{Remark 2.9}\] we have a natural homomorphism

\[
\Psi : H' \to \text{Out}(G_{v_0}) \times \text{Out}(G_{w_0})
\]

whose image is contained in \(\text{Out}(G_{v_0}, \{K, G_e\}) \times \text{Out}(G_{w_0}, G_e)\) and whose kernel is \(T_0 \cap H'\). Since \(H'\) contains a subgroup isomorphic to \(\mathbb{Z}^{n-2}\) by the third part of Property \(P_{W_{n-2}}\) and since \(T_0\) is virtually a nonabelian free group, we see that the abelian rank of \(\text{Out}(G_{v_0}, \{K, G_e\}) \times \text{Out}(G_{w_0}, G_e)\) is at least equal to \(n-3\). Recall that, by Proposition \[\text{Proposition 2.3}\] for every \(m \geq 3\), the abelian rank of \(\text{Out}(W_m)\) is equal to \(m-2\). Since \(k_{v_0} + k_{w_0} = n+1\) and since the abelian rank of a direct product is the sum of the abelian ranks of the factors, we see that the abelian rank of \(\text{Out}(G_{v_0}, \{K, G_e\}) \times \text{Out}(G_{w_0}, G_e)\) is at most equal to \(n-3\). Therefore, the abelian rank of \(\text{Out}(G_{v_0}, \{K, G_e\}) \times \text{Out}(G_{w_0}, G_e)\) is equal to \(n-3\). Thus the group \(\text{Out}(G_{v_0}, \{K, G_e\}) \times \text{Out}(G_{w_0}, G_e)\) contains a free abelian subgroup whose rank is equal to the abelian rank of \(\text{Out}(A)\). Lemma \[\text{Lemma 7.6}\] (1) then shows that the abelian rank of \(\text{Out}(G_{v_0}, \{K, G_e\})\) is equal to the abelian rank of \(\text{Out}(G_{v_0})\). Moreover, Lemma \[\text{Lemma 7.6}\] (2) implies that there exists a refinement \(S'_1\) of \(S_0\) by blowing-up a one-edge \(Z_{RC}\)-splitting \(U_1\) of \(G_{v_0}\) at \(v_0\). Let \(S'_1\) be the equivalence class of \(S'_1\). Note that, since the group of twists about the edge \(e_0\) of \(S_0\) is contained in the group of twists of \(S'_1\), the group \(K_1 \cap T\) fixes \(S'_1\). Note also that the stabilizer of the vertex of \(S'_1\) fixed by \(K\) is equal to \(G_{v_0}\). If \(U_1\) is a free splitting, then \(S'_1\) is a splitting of \(W_n\) with one trivial edge stabilizer and such that \(K\) fixes a point of \(S'_1\). In this case, after collapsing every orbit of edges of \(S'_1\) with nontrivial stabilizer, we can apply Case 1 to conclude.

So we may assume that \(S'_1\) has an edge stabilizer isomorphic to \(\mathbb{Z}\). By Lemma \[\text{Lemma 1.1}\] the twist \(D\) about this edge is central in a finite index subgroup of \(\text{Stab}(S'_1)\). Hence \(D\) commutes with a finite index subgroup of \(K_1 \cap T\).

We now prove that the centralizer of the group \(K_1 \cap T\) contains a free abelian group of rank 2. This will contradict Property \(P_{W_{n-2}}\) (1). Since \(W_{k_{v_0}}\) is a hyperbolic group, and since \(G_{v_0}\) is one-ended relative to \(K\) and \(G_e\), one may apply Theorem \[\text{Theorem 7.7}\] to \((G_{v_0}, \{K, G_e\})\) to obtain a canonical one-edge splitting \(U_2\) of \(G_{v_0}\) such that both \(K\) and \(G_e\) fixes a vertex. The edge stabilizers of \(U_2\) are all virtually infinite cyclic. Let \(S'_2\) be the refinement of \(S'_1\) obtained by blowing-up \(U_2\) at the vertex of \(S'_1\) fixed by \(K\) and by attaching the edges according to their fixed points in \(U_2\). Let \(S'_2\) be the equivalence class of \(S'_2\).

Suppose first that one of the edge stabilizer of \(U_2\) is isomorphic to \(\mathbb{Z}\). Note that this case always happen when \(k_{v_0} = 3\) by Corollary \[\text{Corollary 7.5}\] (3). Then \(S'_2\) contains two edges in distinct orbits whose edge stabilizers are isomorphic to \(\mathbb{Z}\). By Lemma \[\text{Lemma 2.10}\] the group \(\text{Stab}(S'_2)\) has a finite index subgroup which contains a central subgroup isomorphic to \(\mathbb{Z}^2\). In particular, the centralizer of a finite index subgroup of \(K_1 \cap T\) contains a free abelian group of rank 2. This contradicts Property \(P_{W_{n-2}}\) (1).

So we may suppose that \(k_{v_0} \geq 4\) and that every edge stabilizer of \(U_2\) is isomorphic to \(W_2\). By Remark \[\text{Remark 2.9}\] the group of bitwists of the equivalence class of \(U_2\) trivial. Since \(U_2\) is canonical, \(U_2\) is fixed by a finite index subgroup \(H_2\) of \(\text{Out}(G_{v_0}, \{K, G_e\})\). Consider
the natural homomorphism

$$\Psi_2: H_2 \to \prod_{x \in G_{v_0}\setminus U_2} \text{Out}(G_x)$$

given by the action on the vertex groups. By Proposition 2.8 and Remark 2.9, the kernel of this homomorphism is the group of bitwists, which is trivial. Recall that the abelian rank of \(\text{Out}(G_{v_0}, \{K, G_e\})\) is equal to the abelian rank of \(\text{Out}(G_{v_0})\). Therefore the abelian rank of the image of \(\Psi_2\) is equal to the abelian rank of \(\text{Out}(G_{v_0})\). By Lemma 7.6 (2), the splitting \(U_2\) has a refinement \(U_3\) obtained by blowing-up a \(Z_{RC}\)-splitting at a vertex of \(U_2\) which is not in the orbit of the vertex fixed by \(K\). This gives a refinement \(S'_3\) of \(S'_1\) by blowing-up \(U_3\) at the vertex of \(S'_1\) fixed by \(K\). Let \(S'_3\) be the equivalence class of \(S'_3\). Note that, as \(K\) fixes a point in \(S'_3\), and as \(\langle x_n \rangle\) is adjacent to the vertex fixed by \(K\), we see that \(K_1 \cap T\) is contained in the group of twists of \(S'_3\). If \(U_3\) has a trivial edge stabilizer, then, after collapsing every orbit of edges in \(S'_3\) with nontrivial edge stabilizer, one can apply Case 1 to conclude. Otherwise, \(U_3\) has an edge stabilizer isomorphic to \(Z\). In this case, we see that \(S'_3\) contains two edges in distinct orbits whose edge stabilizers are isomorphic to \(Z\). By Lemma 2.10, the group \(\text{Stab}(S'_3)\) contains a finite index subgroup with a central subgroup isomorphic to \(Z^2\). In particular, the centralizer of a finite index subgroup of \(K_1 \cap T\) contains a free abelian group of rank 2. This contradicts Property \((P_{W_{n-2}})\) (1). Therefore, there exists a refinement \(S'_0\) of \(S'_0\) such that \(K\) fixes a vertex of \(S'_0\) and \(S'_0\) has an edge with trivial edge stabilizer. After collapsing every orbit of edges of \(S'_0\) with nontrivial stabilizer, we can apply Case 1 to conclude. The conclusion in Case 3 follows.

Therefore, we have constructed a free splitting \(S'_0\) of \(W_n\) which is a two-edge free splitting fixed by \(K_1 \cap T\). Moreover, the construction of the splitting is such that the vertex of the underlying graph of \(W_n\setminus S'_0\) whose associated group contains \(K\) is not a leaf. We now prove that \(S'_0\) is a \(W_{n-2}\)-star. Let \(C\) be the vertex stabilizer of \(S'_0\) containing \(K\), and let \(C'\) be a vertex stabilizer of \(S'_0\) which is not a conjugate of \(C\) nor \(\langle x_n \rangle\). Then \(C'\) is the vertex group of a leaf of the underlying graph of \(W_n\setminus S'_0\). By Proposition 2.8 (3), the group of twists of \(S'_0\) is isomorphic to \(C \times C' / Z(C')\). Since the centralizer of \(K \cap T_1\) is virtually a nonabelian free group by Property \((P_{W_{n-2}})\) (1), we conclude that \(C' / Z(C')\) is finite. Hence \(C'\) is isomorphic to \(F\) and \(S'_0\) is a \(W_{n-2}\)-star.

We now prove that \(H\) virtually fixes \(S'_0\). By Proposition 2.8 (3), the group of twists of \(S'_0\) is isomorphic to \(W_{n-2} \times W_{n-2}\). By Lemma 4.10, the group \(K_1 \cap T\) is contained in one of the factors isomorphic to \(W_{n-2}\) of the group of twists of \(S'_0\). Therefore, \(K_1 \cap T\) is centralized by the other factor of the group of twists of \(S'_0\). Since the centralizer of \(K_1 \cap T\) contains \(K_2\) as a finite index subgroup, the group \(K_2\) contains a twist \(f\) of infinite order about the edge \(e\) of \(S'_0\) which does not collapse onto \(S\). This twist is a twist about a \(W_{n-1}\)-star obtained from \(S'_0\) by collapsing the orbit of edges which does not contain \(e\). By Lemma 7.1, the group \(H\) virtually fixes \(S'_0\). Moreover, \(K_1\) is commensurable with \(T \cap \text{Stab}(S'_0)\), that is \(K_1\) is commensurable with the group of twists about one edge of \(S'_0\). Lemma 6.8 then implies that \(K_1\) virtually fixes a unique equivalence class of \(W_{n-2}\)-stars.

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Therefore, since $K_1$ is a normal subgroup of $H$, we see that $H$ virtually fixes a unique equivalence class of $W_{n-2}$-stars. This concludes the proof.

**Proposition 7.8.** Let $n \geq 5$ and let $\Gamma$ be a finite index subgroup of $C_n$. Let $\Psi \in \text{Comm}(\Gamma)$. Then for every equivalence class $S$ of $W_{n-2}$-stars, there exists a unique equivalence class $S'$ of $W_{n-2}$-stars such that $\Psi([\text{Stab}_\Gamma(S)]) = [\text{Stab}_\Gamma(S')]$.

**Proof.** The uniqueness statement follows from Lemma 6.4 which shows that the stabilizer in finite index subgroups of $\text{Out}(W_n)$ of two distinct equivalence classes of $W_{n-2}$-stars are not commensurable.

We now prove the existence statement. Let $f: \Gamma_1 \to \Gamma_2$ be an isomorphism between finite index subgroups of $\Gamma$ that represents $\Psi$. By Proposition 6.5, the group $\text{Stab}_{\Gamma_1}(S)$ satisfies $(P_{W_{n-2}})$. As $f$ is an isomorphism, we deduce that $f(\text{Stab}_{\Gamma_1}(S))$ also satisfies $(P_{W_{n-2}})$. Proposition 7.7 implies that there exists a unique equivalence class of $W_{n-2}$-stars $S'$ such that $f(\text{Stab}_{\Gamma_1}(S)) \subseteq \text{Stab}_{\Gamma_2}(S')$, where the inclusion holds up to a finite index subgroup. Applying the same argument with $f^{-1}$, we see that there exists an equivalence class $S''$ of a $W_{n-2}$-star such that

$$\text{Stab}_{\Gamma_1}(S) \subseteq f^{-1}(\text{Stab}_{\Gamma_2}(S')) \subseteq \text{Stab}_{\Gamma_1}(S''),$$

where the inclusion holds up to a finite index subgroup. Lemma 6.4 then implies that $S$ is the unique equivalence class of $W_{n-2}$-stars virtually fixed by $\text{Stab}_{\Gamma_1}(S)$. Therefore, we see that $S = S''$ and we have equality everywhere. This completes the proof.

8 Algebraic characterization of compatibility of $W_{n-2}$-stars and conclusion

8.1 Algebraic characterization of compatibility of $W_{n-2}$-stars

In this section, we give an algebraic characterization of the fact that two equivalence classes of $W_{n-2}$-stars have both a common collapse and a common refinement. This will imply that $\text{Comm}(\text{Out}(W_n))$ preserves the set of pairs of commensurability classes of stabilizers of adjacent pairs in the graph $X_n$ introduced in Definition 3.2 (2).

Let $n \geq 5$ and let $\Gamma$ be a finite index subgroup of $C_n$. We consider the following properties of a pair $(H_1, H_2)$ of subgroups of $\Gamma$.

(P$_{\text{comp}}$) The pair $(H_1, H_2)$ satisfies the following properties.

1. For every $i \in \{1, 2\}$, the group $H_i$ satisfies $(P_{W_{n-2}})$.

2. For every normal subgroups $K_1^{(1)} \times K_2^{(1)}$ of $H_1$ and $K_1^{(2)} \times K_2^{(2)}$ of $H_2$ given by $(P_{W_{n-2}})$ (1), there exist $i, j \in \{1, 2\}$ such that $K_i^{(1)} \cap K_j^{(2)}$ is infinite.

3. The group $H_1 \cap H_2$ contains a subgroup isomorphic to $\mathbb{Z}^{n-2}$. 

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Proposition 8.1. Let \( n \geq 5 \) and let \( \Gamma \) be a finite index subgroup of \( \Gamma_n \). Let \( S_1 \) and \( S_2 \) be two distinct equivalence classes of \( W_{n-2} \)-stars \( S_1 \) and \( S_2 \) and, for every \( i \in \{1, 2\} \), let \( H_i = \text{Stab}_\Gamma(S_i) \). Then \( S_1 \) and \( S_2 \) have a refinement \( S \) which is a \( W_{n-3} \)-star if and only if \( (H_1, H_2) \) satisfies Property (\( P_{\text{comp}} \)).

Proof. We first assume that \( S_1 \) and \( S_2 \) have a common refinement \( S \) which is a \( W_{n-3} \)-star. Let \( S \) be the equivalence class of \( S \). Let us prove that \( (H_1, H_2) \) satisfies \( (P_{\text{comp}}) \). By Proposition 6.5 for every \( i \in \{1, 2\} \), the group \( H_i \) satisfies \( (P_{W_{n-2}}) \). This proves that the pair \( (H_1, H_2) \) satisfies \( (P_{\text{comp}}) \).

Let us check Property \( (P_{\text{comp}}) \) (2). For every \( i \in \{1, 2\} \), let \( T_1^{(i)} \times T_2^{(i)} \) be the group of twists of \( S_i \) and let \( K_1^{(i)} = T_1^{(i)} \cap \Gamma \) and \( K_2^{(i)} = T_2^{(i)} \cap \Gamma \). By Proposition 6.3 for every \( i \in \{1, 2\} \), the group \( K_1^{(i)} \times K_2^{(i)} \) satisfies \( (P_{W_{n-2}}) \) (1) and Lemma 6.1 implies that every normal subgroup of \( H_i \) given by \( (P_{W_{n-2}}) \) (1) is commensurable with \( K_1^{(i)} \times K_2^{(i)} \). Thus it suffices to check \( (P_{\text{comp}}) \) (2) for \( K_1^{(1)} \times K_1^{(2)} \) and \( K_1^{(2)} \times K_2^{(2)} \). The group of twists of \( S \) is isomorphic to a direct product \( A_1 \times A_2 \times A_3 \) of three copies of \( W_{n-3} \). Since \( n \geq 5 \), we have \( n - 3 \geq 2 \) and \( W_{n-3} \) is infinite. Since \( S \) is a common refinement of \( S_1 \) and \( S_2 \) and since \( S \) has 3 orbits of edges there exists a \( W_{n-1} \)-star \( S_0 \) which is a common collapse of \( S_1 \) and \( S_2 \). Moreover, there exists \( k \in \{1, 2, 3\} \) such that \( A_k \) is contained in the group of twists of \( S_0 \). Therefore, for every \( i \in \{1, 2\} \), there exists \( j \in \{1, 2\} \) such that the group \( A_k \) is contained in \( T_j^{(i)} \). Thus, there exist \( i, j \in \{1, 2\} \) such that \( A_k \cap \Gamma \leq K_1^{(i)} \cap K_2^{(j)} \). In particular, \( K_1^{(i)} \cap K_2^{(j)} \) is infinite. This shows \( (P_{\text{comp}}) \) (2).

Finally, since \( n \geq 5 \), the \( W_{n-2} \)-stars \( S_1 \) and \( S_2 \) have a common refinement which is a \( W_2 \)-star (take any \( W_2 \)-star which refines \( S \)). Since the group of twists of a \( W_2 \)-star contains a subgroup isomorphic to \( \mathbb{Z}^{n-2} \) by Proposition 2.8 (3), this shows \( (P_{\text{comp}}) \) (3).

Conversely, suppose that \( (H_1, H_2) \) satisfies \( (P_{\text{comp}}) \). For \( i \in \{1, 2\} \), let \( K_1^{(i)} \times K_2^{(i)} \) be the direct product of the groups of twists in \( \Gamma \) about the two edges of \( S_i \). Then for every \( i \in \{1, 2\} \), the group \( (H_i \cap K_1^{(1)}) \times (H_i \cap K_2^{(1)}) \) satisfies \( (P_{W_{n-2}}) \) (1) by Proposition 6.5. Hence Property \( (P_{\text{comp}}) \) (2) implies that there exists \( i, j \in \{1, 2\} \) such that

\[
\left( H_1 \cap K_1^{(1)} \right) \cap \left( H_2 \cap K_2^{(2)} \right)
\]

is infinite. For \( i \in \{1, 2\} \), let \( S_1^{(i)} \) and \( S_2^{(i)} \) be the two distinct \( W_{n-1} \)-stars on which \( S_i \) collapses. By Proposition 6.10 since \( H_1 \cap H_2 \) fixes pointwise the set \( \{S_1^{(1)}, S_1^{(2)}, S_2^{(1)}, S_2^{(2)}\} \), and since \( H_1 \cap H_2 \) contains a subgroup isomorphic to \( \mathbb{Z}^{n-2} \) by \( (P_{\text{comp}}) \) (3), the \( W_{n-1} \)-stars \( S_1^{(1)}, S_2^{(1)}, S_1^{(2)}, S_2^{(2)} \) are pairwise compatible. Hence \( S_1 \) and \( S_2 \) have a common refinement \( S \) which is either a \( W_{n-3} \)-star or a \( W_{n-4} \)-star. Since the groups of twists of \( S_1 \) and \( S_2 \) have infinite intersection, the refinement \( S \) cannot be a \( W_{n-4} \)-star since otherwise the \( W_{n-1} \)-stars \( S_1^{(1)}, S_2^{(1)}, S_1^{(2)}, S_2^{(2)} \) would be pairwise nonequivalent and hence their groups of twists would have trivial intersection. Thus \( S \) is a \( W_{n-3} \)-star. This concludes the proof.

\[\square\]
8.2 Conclusion

In this last section, we complete the proof of our main theorem.

**Theorem 8.2.** Let \( n \geq 5 \) and let \( \Gamma \) be a finite index subgroup of \( C_n \). Then any isomorphism \( f : H_1 \rightarrow H_2 \) between two finite index subgroups of \( \Gamma \) is given by conjugation by an element of \( \text{Out}(W_n) \) and the natural map:

\[
\text{Out}(W_n) \rightarrow \text{Comm}(\text{Out}(W_n))
\]

is an isomorphism.

**Proof.** Suppose that \( S \) and \( S' \) are two distinct equivalence classes of \( W_{n-2} \)-stars. Then \( \text{Stab}_\Gamma(S) \) and \( \text{Stab}_\Gamma(S') \) are not commensurable by Lemma 6.4. Proposition 7.8 shows that the collection \( I \) of all commensurability classes of \( \Gamma \)-stabilizers of equivalence classes of \( W_{n-2} \)-stars is \( \text{Comm}(\Gamma) \)-invariant. Proposition 8.1 shows that the collection \( J \) of all pairs \( \text{pr}_1 \text{Stab}_\Gamma(S), \text{pr}_2 \text{Stab}_\Gamma(S') \) is also \( \text{Comm}(\Gamma) \)-invariant. Since the natural homomorphism \( \text{Out}(W_n) \rightarrow \text{Aut}(X_n) \) is an isomorphism by Theorem 3.3, the conclusion follows from Proposition 2.1 and the fact that \( \text{Comm}(\Gamma) \) is isomorphic to \( \text{Comm}(\text{Out}(W_n)) \) since \( \Gamma \) has finite index in \( \text{Out}(W_n) \).

\[\square\]

### A Rigidity of the graph of \( W_{n-1} \)-stars

The *graph of \( W_{n-1} \)-stars*, denoted by \( Y_n \), is the graph whose vertices are the \( W_n \)-equivariant homeomorphism classes of \( W_{n-1} \)-stars, where two equivalence classes \( S \) and \( S' \) are joined by an edge if there exist \( S \in S \) and \( S' \in S' \) such that \( S \) and \( S' \) are compatible. This graph arises naturally in the study of \( \text{Out}(W_n) \) as it is isomorphic to the full subgraph of the free splitting graph \( K_n \) of \( W_n \) whose vertices are equivalence classes of \( W_k \)-stars, with \( k \) varying in \( \{0, \ldots, n-1\} \). As \( \text{Aut}(W_n) \) acts on \( K_n \) by precomposition of the marking, we have an induced action of \( \text{Aut}(W_n) \) on \( Y_n \). As \( \text{Inn}(W_n) \) acts trivially on \( Y_n \), the action of \( \text{Aut}(W_n) \) induces an action of \( \text{Out}(W_n) \). We denote by \( \text{Aut}(Y_n) \) the group of graph automorphisms of \( Y_n \). In this section we prove the following theorem.

**Theorem A.1.** Let \( n \geq 4 \). The natural homomorphism

\[
\text{Out}(W_n) \rightarrow \text{Aut}(Y_n)
\]

is an isomorphism.

In order to prove this theorem, we take advantage of the action of \( \text{Out}(W_n) \) on the graph of \( \{0\} \)-stars and \( F \)-stars \( L_n \). The strategy in order to prove Theorem A.1 is to construct an injective homomorphism \( \Phi: \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n) \) such that every automorphism in the image preserves the set of \( \{0\} \)-stars and the set of \( F \)-stars.

The homomorphism \( \Phi: \text{Aut}(Y_n) \rightarrow \text{Aut}(L_n) \) is defined as follows. Let \( f \in \text{Aut}(Y_n) \). Let \( S \) be the equivalence class of a \( \{0\} \)-star and let \( S' \) be a representative of \( S \). By Theorem 8.7 there exist exactly \( n \) \( W_{n-1} \)-stars \( S_1, \ldots, S_n \) refined by \( S \). Moreover, these

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$W_{n-1}$-stars are pairwise compatible. For $i \in \{1, \ldots, n\}$, let $S_i$ be the equivalence class of $S_i$. Since $f$ is an automorphism of $Y_n$, $f(S_1), \ldots, f(S_n)$ are pairwise adjacent in $Y_n$. Let $S_1', \ldots, S_n'$ be representatives of respectively $f(S_1), \ldots, f(S_n)$ that are pairwise compatible. Then Theorem 3.7 implies that there exists a unique common refinement $S'$ of $S_1', \ldots, S_n'$ with exactly $n$ edges. Since, for every $i \in \{1, \ldots, n\}$, the splitting $S_i'$ is a $W_{n-1}$-star, the splitting $S'$ is necessarily a $\{0\}$-star. Let $S'$ be the equivalence class of $S'$. We then define $\Phi(f)(S) = S'$. If $T$ is an $F$-star, we define $\Phi(f)(T)$ similarly.

Lemma A.2. Let $n \geq 4$. Let $f \in \text{Aut}(Y_n)$. Let $\Phi(f)$ be as above.

1. The map $\Phi(f) : VL_n \to VL_n$ induces a graph automorphism $\tilde{\Phi}(f) : L_n \to L_n$.

2. If $\tilde{\Phi}(f) = \text{id}_{L_n}$, then $f = \text{id}_{Y_n}$.

Proof. We prove the first statement. As $\Phi(f) \circ \Phi(f^{-1}) = \Phi(f \circ f^{-1}) = \text{id}$, we see that $\Phi(f)$ is a bijection. Let $S$ be the equivalence class of a $\{0\}$-star and let $T$ be the equivalence class of an $F$-star. Suppose that $S$ and $T$ are adjacent in $L_n$. We prove that $\Phi(f)(S)$ and $\Phi(f)(T)$ are adjacent in $L_n$. Applying the same result to $f^{-1}$, this will prove that $S$ and $T$ are adjacent in $L_n$ if and only if $\Phi(f)(S)$ and $\Phi(f)(T)$ are adjacent in $L_n$, and this will conclude the proof. Let $S$ and $T$ be representatives of respectively $S$ and $T$. Let $S_1, \ldots, S_n$ be the $n$ $W_{n-1}$-stars refined by $S$, and let $T_1, \ldots, T_{n-1}$ be the $n-1$ $W_{n-1}$-stars refined by $T$. As $S$ refines $T$, and as $S$ refines exactly $n$ $W_{n-1}$-stars by Theorem 3.7, up to reordering, we can suppose that, for every $i \in \{1, \ldots, n-1\}$, we have $S_i = T_i$. For $i \in \{1, \ldots, n\}$, let $S_i$ be the equivalence class of $S_i$, and let $S'_i$ be a representative of $\Phi(f)(S_i)$ such that for distinct $i, j \in \{1, \ldots, n\}$, $S_i$ and $S_j$ are compatible. Then, by Theorem 3.7 a representative $T'$ of $\Phi(f)(T)$ is the unique (up to $W_n$-equivariant homomorphism) $F$-star such that, for every $j \in \{1, \ldots, n-1\}$, $T'$ is compatible with $S'_j$. Moreover, a representative $S'$ of $\Phi(f)(S)$ is the unique $\{0\}$-star such that, for every $i \in \{1, \ldots, n\}$, $S'$ is compatible with $S'_i$. For $i \in \{1, \ldots, n\}$, let $x_i$ be the preimage by the marking of $W_n \setminus S'_i$ (well defined up to global conjugation) of the generator of the vertex group isomorphic to $F$ (which exists since $S'_i$ is a $W_{n-1}$-star). Then the preimages by the marking of $W_n \setminus T'$ of the generators of the groups associated with the $n-1$ leaves of the underlying graph of $W_n \setminus T'$ are $x_1, \ldots, x_{n-1}$ and the preimage by the marking of $W_n \setminus T'$ of the generator of the group associated with the center of the underlying graph of $W_n \setminus T'$ is $x_n$. Moreover, the preimages by the marking of $W_n \setminus S'$ of the generators of the groups associated with the $n$ leaves of the underlying graph of $W_n \setminus S'$ are $x_1, \ldots, x_n$. Let $v_n$ be the leaf of the underlying graph of $W_n \setminus S'$ such that the preimage by the marking of $W_n \setminus S'$ of the generator of the group associated with $v_n$ is $x_n$. Then $T'$ is obtained from $S'$ by contracting the edge adjacent to $v_n$. Thus $\Phi(f)(S)$ and $\Phi(f)(T)$ are adjacent in $L_n$.

The proof of the second statement is identical to the proof of [Gue2, Lemma 5.4]. We add the proof for completeness as the statement of [Gue2, Lemma 5.4] is about automorphisms of $\mathcal{T}_n$. Let $S \in VY_n$ and let $S$ be a representative of $S$. We prove that $f(S) = S$. Let

$$W_n = \langle x_1, \ldots, x_{n-1} \rangle * \langle x_n \rangle$$

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be the free factor decomposition of $W_n$ induced by $S$. Let $S'$ be a representative of $f(S)$. Let $\mathcal{X}$ be the equivalence class of the $F$-star $X$ represented in Figure 4 on the left.

![Figure 4: The $F$-stars $X$ and $X'$ of the proof of Lemma A.2](image)

Since $\Phi(f)(\mathcal{X}) = \mathcal{X}$, the free splitting $S'$ is a $W_{n-1}$-star obtained from $X$ by collapsing $n-1$ edges. But if $T$ is a $W_{n-1}$-star obtained from $X$ by collapsing $n-1$ edges, then there exists $i \in \{1, \ldots, n\}$ such that the free factor decomposition of $W_n$ induced by $T$ is

$$W_n = \langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle \ast \langle x_i \rangle.$$

For $i \in \{1, \ldots, n\}$, we will denote by $T_i$ the $W_{n-1}$-star with associated free factor decomposition $\langle x_1, \ldots, \hat{x}_i, \ldots, x_n \rangle \ast \langle x_i \rangle$, and by $T_i$ its equivalence class. For $i \neq n$, the free splitting $T_i$ is a collapse of the $F$-star $X'$ depicted in Figure 4 on the right, whereas $S$ is not a collapse of $X'$.

Let $\mathcal{X}'$ be the equivalence class of $X'$. Since $\Phi(f)(\mathcal{X}') = \mathcal{X}'$, there does not exist a representative of $f(S)$ that is obtained from a representative of $\mathcal{X}'$ by collapsing a forest. Thus, for all $i \neq n$, we have $f(S) \neq T_i$. Therefore, as $S = T_n$, we conclude that $f(S) = S$. □

Proof of Theorem A.1
Let $n \geq 4$. We first prove injectivity. The homomorphism $\text{Out}(W_n) \to \text{Aut}(L_n)$ is injective by Theorem 3.5. Moreover, the homomorphism $\text{Out}(W_n) \to \text{Aut}(Y_n)$ factors through $\text{Out}(W_n) \to \text{Aut}(Y_n) \to \text{Aut}(L_n)$. Therefore we deduce the injectivity of $\text{Out}(W_n) \to \text{Aut}(Y_n)$. We now prove surjectivity. Let $f \in \text{Aut}(Y_n)$. By Lemma A.2 (1), we have a homomorphism $\Phi : \text{Aut}(Y_n) \to \text{Aut}(L_n)$ whose image consists in automorphisms preserving the set of $\{0\}$-stars and the set of $F$-stars. By Theorem 3.5 the automorphism $\Phi(f)$ is induced by an element $\gamma \in \text{Out}(W_n)$. Since the homomorphism $\text{Aut}(Y_n) \to \text{Aut}(L_n)$ is injective by Lemma A.2 (2), $f$ is induced by $\gamma$. This concludes the proof. □

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