Anti-magic Labeling of Cartoon Flowers and Consecutive Wounded Flowers

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Abstract. An anti-magic labeling of a finite simple undirected graph $G$ is a bijection from the set of edges to the set of integers $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums are pairwise distinct, where the vertex sum at one vertex is the sum of labels of all edges incident to such vertex. A graph is called anti-magic if it admits an anti-magic labeling. In this paper, we established an anti-magic labeling of the cartoon flowers. We further define a new subclass of the cartoon flower called the consecutive wounded cartoon flower and then established an anti-magic labeling of the consecutive wounded cartoon flowers. Keywords: Anti-magic, Anti-magic labeling, Cartoon flower, Wounded cartoon flower.

1. Introduction

In this paper, we consider only simple undirected finite graphs. Let $G$ be a simple undirected finite graph with $V(G)$ and $E(G)$ to be the set of vertices and the set of edges of the graph $G$, respectively.

The study of the graph labeling was introduced by Sedlacek [4]. An excellent survey on graph labeling can be found in Gallian [1]. The problem of anti-magic labeling of graphs was introduced by Hartsfield and Ringe [2]. The formal definition of anti-magic is given below.

Definition 1.1. A bijective function $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ is said to be labeling of graph $G$. The vertex sum function associated with a labeling $f$ of graph $G$ is a function $S : V(G) \to \mathbb{N}$ defined as

$$S(u) = \sum_{v \in N(u)} f(uv).$$

That is, $S(u)$ is the sum of labels of the edges incident with $u$. A labeling $f$ of a graph $G$ is said to be anti-magic if the vertex sum function associated with $f$ is an injective function. A graph having an anti-magic labeling is said to be anti-magic.

In this paper, we denote $S_e = \{u \in V(G) | S(u) \text{ is even}\}$ and $S_o = \{u \in V(G) | S(u) \text{ is odd}\}$. Then $V(G) = S_o \cup S_e$ and $S_o \cap S_e = \phi$. Note that an odd value is not equal to any even value. Therefore, to show that $S$ is injective on $V(G)$, it is enough to show that $S$ is injective on $S_e$ and $S_o$. 
The authors [3] introduced new classes of graph called cartoon flower and wounded cartoon flower. They investigated E-super vertex magic labeling for these graphs. The following definitions of cartoon flower and wounded cartoon flower are presented in [3].

**Definition 1.2.** An $n$-cartoon flower graph is a graph, denoted by $CF(n)$, with the set of vertices and set of edges as follows:

$$V(CF(n)) = \{v_1, \ldots, v_n, c_1, \ldots, c_n\}$$

and

$$E(CF(n)) = \{v_iv_{i+1} | 1 \leq i < n\} \cup \{v_nv_1\}
\cup \{c_iv_i, c_iv_{i+1} | 1 \leq i < n\} \cup \{c_nv_n, c_nv_1\}.$$ 

Note that the set $\{v_iv_{i+1} | 1 \leq i < n\} \cup \{v_nv_1\}$ forms an $n$-cycle. Any vertex and any edge of this $n$-cycle are referred to as a ring vertex and a ring edge, respectively. Any vertex from the set $\{c_1, \ldots, c_n\}$ is called a petal corner and any edge from the set $\{v_ic_i, v_{i+1}c_i, 1 \leq i < n\} \cup \{v_nc_n, v_1c_n\}$ is called a petal edge. The graph $CF(n)$ is of order $2n$ and of size $3n$. Since $n = 3$ is the smallest value for which the cartoon flower is defined, in this paper, we assume that $n \geq 3$.

**Definition 1.3.** A wounded $n$-cartoon flower with $t$-petal graph is a graph, denoted by $WF(n, t)$, obtained by adding $t$ petals to an $n$-cycle.

In other words, $WF(n, t)$ is the graph obtained by plucking $(n - t)$ petals from a cartoon flower $CF(n)$. Note that the graph $WF(n, t)$ is of order $n + t$ and of size $n + 2t$. We define a new subclass of wounded cartoon flowers called consecutive wounded flower as follows:

**Definition 1.4.** A consecutive wounded $n$-cartoon flower with $t$-petal graph is a graph, denoted by $CWF(n, t)$, obtained by adding $t$ consecutive petals to an $n$-cycle.

In other words, $CWF(n, t)$ is the graph obtained by plucking $(n - t)$ consecutive petals from a cartoon flower $CF(n)$.

2. Cartoon Flower Graph

In this section, we show that the cartoon flower graphs $CF(n)$ are anti-magic for all $n \geq 3$.

**Theorem 2.1.** If $n \geq 3$ is an odd integer, then the cartoon flower graph $CF(n)$ is anti-magic.

**Proof.** Define a bijective function $f : E(CF(n)) \to \{1, 2, \ldots, 3n\}$ as follows:

$$f(c_iv_i) = 2i - 1 \quad \text{for} \quad 1 \leq i \leq n$$
$$f(c_iv_{i+1}) = 2i \quad \text{for} \quad 1 \leq i \leq n - 1$$
$$f(c_nv_1) = 2n$$
$$f(v_iv_{i+1}) = 2n + i \quad \text{for} \quad 1 \leq i \leq n - 1$$
$$f(v_nv_1) = 3n.$$ 

Then the vertex sum function is given as

$$S(c_i) = 4i - 1 \quad \text{for} \quad 1 \leq i \leq n,$$
$$S(v_1) = 7n + 2,$$
$$S(v_1) = 4n + 6i - 4 \quad \text{for} \quad 2 \leq i \leq n.$$ 

Since $n$ is odd, we get $S_o = \{c_1, \ldots, c_n, v_1\}$ and $S_e = \{v_2, \ldots, v_n\}$. Since $Max\{S(c_i) | 1 \leq i \leq n\} = 4n - 1 < 7n + 2 = S(v_1)$, we have $S(c_i) < S(v_1)$ for $1 \leq i \leq n$. Since $S(c_1) < S(c_2) < \cdots < S(c_n) < S(v_1)$, the vertex sum function $S$ is injective on $S_o$. Since $S$ is strictly increasing function on $S_c$, the vertex sum function $S$ is injective on $S_c$. Therefore, the cartoon flower graph $CF(n)$ is anti-magic. \hfill \qed
Theorem 2.2. If \( n \geq 4 \) is an even integer, then the cartoon flower graph \( CF(n) \) is anti-magic.

Proof. Define a bijective function \( f : E(CF(n)) \to \{1, 2, \ldots, 3n\} \) as follows:

\[
\begin{align*}
 f(c_iv_i) & = 2i - 1 \quad \text{for } 1 \leq i \leq n \\
 f(c_iv_{i+1}) & = 2i \quad \text{for } 1 \leq i \leq n - 1 \\
 f(c_nv_1) & = 2n \\
 f(v_2v_3) & = 2n + 1 \\
 f(v_1v_2) & = 2n + 2 \\
 f(v_{i}v_{i+1}) & = 2n + i \quad \text{for } 3 \leq i \leq n - 1 \\
 f(v_nv_1) & = 3n.
\end{align*}
\]

Then the vertex sum function is given as

\[
\begin{align*}
 S(c_i) & = 4i - 1 \quad \text{for } 1 \leq i \leq n \\
 S(v_1) & = 7n + 3 \\
 S(v_3) & = 4n + 13 \\
 S(v_i) & = 4n + 6i - 4 \quad \text{for } 4 \leq i \leq n, i = 2.
\end{align*}
\]

The labeling of the graph is very similar to the case when \( n \) is odd. The only difference is that labeling of the first ring edge is switched with labeling of the second ring edge. Thus, the vertex sum function is almost similar to the case when \( n \) is odd except \( S(v_1) \) and \( S(v_3) \). Since \( n \) is even, we get \( S_o = \{c_1, \ldots, c_n, v_1, v_3\} \) and \( S_e = \{v_2, v_4, \ldots, v_n\} \). In this case, \( \text{Max}\{S(c_i)\} = 4n - 1 < 4n + 13 = S(v_3) \) and \( S(v_3) < S(v_1) \). Therefore, we have \( S(c_i) < S(c_j) < S(v_3) < S(v_1) \) for \( 1 \leq i < j \leq n \). Also, \( S \) is strictly increasing function on \( S_e \). The vertex sum function \( S \) is injective on \( S_o \) and on \( S_e \). Hence the cartoon flower graph \( CF(n) \) is anti-magic.

\[\square\]

Figure 1. Labeling of \( CF(n) \)

The labelings of the cartoon flower graph for \( n \)-odd and \( n \)-even are shown in the Fig. 1(a) and Fig. 1(b), respectively. Combining Theorem 2.1 and Theorem 2.2, we get the following result.

Theorem 2.3. The cartoon flower graph \( CF(n) \) is anti-magic for all \( n \geq 3 \).
3. Consecutive Wounded Cartoon Flower Graph

In this section, we show that the consecutive wounded cartoon flower graphs $CWF(n)$ are anti-magic for all $n \geq 3$ and $0 \leq t \leq n - 1$.

**Theorem 3.1.** Let $n \geq 3$. The (consecutive) wounded cartoon flower $CWF(n, 0)$ is anti-magic.

**Proof.** Theorem follows from the facts that any $n$-cycle is anti-magic for $n \geq 3$ (see [2]) and $CWF(n, 0)$ is an $n$-cycle. \qed

**Theorem 3.2.** Let $n \geq 3$. The (consecutive) wounded cartoon flower $CWF(n, 1)$ is anti-magic.

**Proof.** Define a bijective function $f : E(CWF(n, 1)) \to \{1, 2, \ldots, n + 2\}$ as follows:

\[
\begin{align*}
    f(c_1v_1) &= n + 2 \\
    f(c_1v_2) &= n + 1 \\
    f(c_2v_2) &= n \\
    f(v_i v_{i+1}) &= i - 1 \quad \text{for} \quad 2 \leq i \leq n - 1 \\
    f(v_n v_1) &= n - 1.
\end{align*}
\]

Then the vertex sum function is given as
\[
\begin{align*}
    S(v_1) &= 3n + 1 \\
    S(v_2) &= 2n + 2 \\
    S(v_i) &= 2i - 3 \quad \text{for} \quad 3 \leq i \leq n \\
    S(c_1) &= 2n + 3.
\end{align*}
\]

Since $\max\{S(v_i) | 3 \leq i \leq n\} = 2n - 3$, we get $2n - 3 < 2n + 2 < 2n + 3 < 3n + 1$. Thus,
\[
S(v_i) < S(v_j) < S(v_2) < S(c_1) < S(v_1) \quad \text{for} \quad 3 \leq i < j \leq n.
\]

Therefore, the vertex sum function $S$ is injective function on $V(CWF(n, 1))$. Thus, $CWF(n, 1)$ is anti-magic. \qed

**Theorem 3.3.** Let $n \geq 3$. The (consecutive) wounded cartoon flower $CWF(n, n - 1)$ is anti-magic.

**Proof.** We prove this result by considering the following two cases.

**Case (i)** $n$ is even.

Define a bijective function $f : E(CWF(n, n - 1)) \to \{1, 2, \ldots, 3n - 2\}$ as follows:

\[
\begin{align*}
    f(c_i v_i) &= 2i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(c_i v_{i+1}) &= 2i \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(v_i v_{i+1}) &= 2n + i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(v_n v_1) &= 2n - 1.
\end{align*}
\]

Then the vertex sum function is given as
\[
\begin{align*}
    S(v_1) &= 4n \\
    S(v_i) &= 4n + 6i - 6 \quad \text{for} \quad 2 \leq i \leq n - 1 \\
    S(v_n) &= 7n - 5 \\
    S(c_i) &= 4i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1.
\end{align*}
\]

Since $n$ is even, $S_o = \{v_n, c_1, \ldots, c_{n-1}\}$ and $S_e = \{v_1, \ldots, v_{n-1}\}$. Since $\max\{S(c_i) | 1 \leq i \leq n - 1\} = 4n - 5 < 7n - 5 = S(v_n)$, we have $S(c_i) < S(c_j) < S(v_n)$ for $1 \leq i < j \leq n - 1$. Therefore, $S$ is injective on $S_o$. Since $\min\{S(v_i) | 2 \leq i \leq n - 1\} = 4n < 4n + 6 = S(v_1)$, we have $S(v_1) < S(v_i) < S(v_3)$ for $2 \leq i < j \leq n - 1$. Therefore, $S$ is injective on $S_e$. \qed
Case (ii) \( n \) is odd.

Define a bijective function \( f : E(CWF(n, n - 1)) \rightarrow \{1, 2, \ldots, 3n - 2\} \) as follows:

\[
\begin{align*}
    f(c_iv_i) &= 2i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(c_iv_{i+1}) &= 2i \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(v_iv_{i+1}) &= 2n + i - 2 \quad \text{for} \quad 1 \leq i \leq n - 1 \\
    f(v_nv_1) &= 3n - 2.
\end{align*}
\]

Then the vertex sum function is given as

\[
\begin{align*}
    S(v_1) &= 5n - 2 \\
    S(v_i) &= 4n + 6i - 8 \quad \text{for} \quad 2 \leq i \leq n - 1 \\
    S(v_n) &= 8n - 7 \\
    S(c_i) &= 4i - 1 \quad \text{for} \quad 1 \leq i \leq n - 1.
\end{align*}
\]

In this case, \( S_o = \{v_1, v_n, c_1, \ldots, c_{n-1}\} \) and \( S_e = \{v_2, \ldots, v_{n-1}\} \). Since \( 5 < 3n \), \( S(v_1) = 5n - 2 < 8n - 7 = S(v_n) \). Since \( \max\{S(c_i) | 1 \leq i \leq n - 1\} = 4n - 5 < 5n - 2 = S(v_1) \), we get \( S(c_j) < S(c_j) < S(v_1) < S(v_n) \) for \( 1 \leq i < j \leq n - 1 \). Therefore, \( S \) is injective on \( S_o \).

Since \( S(v_i) < S(v_j) \) for \( 2 \leq i < j \leq n - 1 \). Therefore, \( S \) is injective on \( S_e \).

\[\square\]

**Figure 2.** Labeling of \( CWF(n, n - 1) \)

The labelings of the wounded cartoon flower graph \( CWF(n, n - 1) \) for \( n \)-even and \( n \)-odd are shown in the Fig. 2(a) and Fig. 2(b), respectively. Note that for \( t = 0, 1, n - 1 \), the wounded cartoon flower is the same as the consecutive wounded cartoon flower. That is, \( WF(n, t) = CWF(n, t) \) for \( t = 0, 1, n - 1 \).

**Theorem 3.4.** Let \( n \geq 3 \) and \( 2 \leq t \leq n - 2 \). The consecutive wounded cartoon flower \( WF(n, t) \) is anti-magic.

**Proof.** We prove this result by considering the following four cases.
Case (i) \( n \) and \( t \) both are even.

Define a bijective function \( f : E(CWF(n, t)) \to \{1, 2, \ldots, n + 2t\} \) as follows:

\[
\begin{align*}
f(c_i v_i) &= n - t + 2i - 1 \quad \text{for} \ 1 \leq i \leq t \\
f(c_i v_{i+1}) &= n - t + 2i \quad \text{for} \ 1 \leq i \leq t \\
f(v_{i+1} v_{i+1}) &= n + t + i \quad \text{for} \ 1 \leq i \leq t \\
f(v_{t+i} v_{t+i+1}) &= i \quad \text{for} \ 1 \leq i \leq n - t - 1 \\
f(v_n v_1) &= n - t.
\end{align*}
\]

Then the vertex sum function is given as

\[
\begin{align*}
S(v_1) &= 3n - t + 2 \\
S(v_i) &= 4n + 6i - 4 \quad \text{for} \ 2 \leq i \leq t \\
S(v_{i+1}) &= 2n + 3t + 1 \\
S(v_{i+i}) &= 2i - 1 \quad \text{for} \ 2 \leq i \leq n - t \\
S(c_i) &= 2n - 2t + 4i - 1 \quad \text{for} \ 1 \leq i \leq t.
\end{align*}
\]

In this case, \( S_o = \{v_{t+1}, \ldots, v_n, c_1, \ldots, c_t\} \) and \( S_e = \{v_1, \ldots, v_t\} \). Since \( S(v_{t+i}) < S(v_{t+j}) \) for \( 2 \leq i < j \leq n - t \) and \( S(c_k) < S(c_l) \) for \( 1 \leq k < l \leq t \), we get \( \max \{S(v_{t+i})\} \leq 2 \leq k \leq l \leq t \). Since \( S(v_{t+j}) < S(c_k) \) for \( 2 \leq j \leq n - t, 1 \leq k \leq t \). Furthermore, \( \max \{S(c_i)\} \leq 2 \leq i \leq n \). Thus, \( S(v_{t+i}) < S(v_{t+j}) < S(c_k) < S(c_l) < S(v_{t+1}) \) implies \( S(c_l) < S(v_{t+1}) \) for \( 1 \leq l \leq t \). Thus,

\[
S(v_{t+i}) < S(v_{t+j}) < S(c_k) < S(c_l) < S(v_{t+1})
\]

for \( 2 \leq i < j \leq n - t, 1 \leq k < l \leq t \).

Therefore, \( S \) is injective on \( S_o \).
Since \( S(v_1) = 3n - t + 2 < 4n + 8 = S(v_2) \), we get \( S(v_1) < S(v_i) < S(v_j) \) for \( 2 \leq i < j \leq t \).
Therefore, \( S \) is injective on \( S_e \).

Case (ii) \( n \) is odd and \( t \) is even.

Define a bijective function \( f : E(CWF(n, t)) \to \{1, 2, \ldots, n + 2t\} \) as follows:

\[
\begin{align*}
f(c_i v_i) &= n - t + 2i - 1 \quad \text{for} \ 1 \leq i \leq t \\
f(c_i v_{i+1}) &= n - t + 2i \quad \text{for} \ 1 \leq i \leq t \\
f(v_{i+1} v_{i+1}) &= n + t + i \quad \text{for} \ 1 \leq i \leq t \\
f(v_{t+i} v_{t+i+1}) &= i \quad \text{for} \ 1 \leq i \leq n - t - 1 \\
f(v_n v_1) &= n - t - 1.
\end{align*}
\]

Then the vertex sum function is given as

\[
\begin{align*}
S(v_1) &= 3n - t + 1 \\
S(v_i) &= 4n + 6i - 4 \quad \text{for} \ 2 \leq i \leq t \\
S(v_{t+1}) &= 3n + 2t \\
S(v_{t+2}) &= n - t + 1 \\
S(v_{t+i}) &= 2i - 3 \quad \text{for} \ 3 \leq i \leq n - t \\
S(c_i) &= 2n - 2t + 4i - 1 \quad \text{for} \ 1 \leq i \leq t.
\end{align*}
\]

Thus, \( S_o = \{v_{t+1}, v_{t+3}, v_{t+5}, \ldots, v_n, c_1, \ldots, c_t\} \) and \( S_e = \{v_1, \ldots, v_t, v_{t+2}\} \). Since \( S(v_{t+i}) < S(v_{t+j}) \) for \( 3 \leq i < j \leq n - t \) and \( S(c_k) < S(c_l) \) for \( 1 \leq k < l \leq t \), we get
Max\{S(v_{t+j})|3 \leq j \leq n-t\} = 2n - 2t - 3 < 2n - 2t + 3 = Min\{S(c_k)|1 \leq k \leq t\}

Since \(S(c_t) = 2n + 2t - 1 < 3n + 2t = S(v_{t+1})\), we get

\[S(v_{t+i}) < S(v_{t+j}) < S(c_k) < S(c_t) < S(v_{t+1})\]

for \(3 \leq i < j \leq n-t, 1 \leq k \leq t\).

Therefore, \(S\) is injective on \(S_o\).

Since \(S(v_{t+2}) < S(v_1)\) and \(Min\{S(v_i)|2 \leq i \leq t\} = 4n + 8\), we get \(S(v_1) < S(v_i)\) for \(2 \leq i \leq t\). Thus,

\[S(v_{t+2}) < S(v_1) < S(v_i) < S(v_j)\] for \(2 \leq i < j \leq t\).

Therefore, \(S\) is injective on \(S_e\).

**Figure 3.** Labeling of \(CWF(n, t)\) for \(t\)-even

**Case (iii)** \(n\) is even and \(t\) is odd.

Define a bijective function \(f : E(CWF(n, t)) \rightarrow \{1, 2, \ldots, n+2t\}\) as follows:

\[
\begin{align*}
  f(c_iv_i) &= n - t + 2i - 1 & \text{for } 1 \leq i \leq t - 1 \\
  f(c_iv_{i+1}) &= n - t + 2i & \text{for } 1 \leq i \leq t - 1 \\
  f(c_tv_i) &= n + 2t - 1 \\
  f(c_tv_{t+1}) &= n + 2t \\
  f(v_iv_{t+1}) &= n + t + i - 2 & \text{for } 1 \leq i \leq t - 1 \\
  f(v_tv_{t+1}) &= n + 2t - 2 \\
  f(v_{t+1}v_{t+2}) &= n - t \\
  f(v_{t+i}v_{t+i+1}) &= i - 1 & \text{for } 2 \leq i \leq n - t - 1 \\
  f(v_nv_1) &= n - t - 1.
\end{align*}
\]
Then the vertex sum function is given as

\[
\begin{align*}
S(v_1) &= 3n - t - 1 \\
S(v_i) &= 4n + 6i - 8 \quad \text{for } 2 \leq i \leq t - 1 \\
S(v_t) &= 4n + 7t - 8 \\
S(v_{t+1}) &= 3n + 3t - 2 \\
S(v_{t+2}) &= n - t + 1 \\
S(v_{i+1}) &= 2i - 3 \quad \text{for } 3 \leq i \leq n - t \\
S(c_i) &= 2n - 2t + 4i - 1 \quad \text{for } 1 \leq i \leq t - 1 \\
S(c_t) &= 2n + 4t - 1.
\end{align*}
\]

In this case, \( S_o = \{v_t, v_{t+1}, v_{t+3}, \ldots, v_n, c_1, \ldots, c_t\} \) and \( S_e = \{v_1, \ldots, v_{t-1}, v_{t+2}\} \). Clearly, \( S(v_{i+1}) < S(v_{i+j}) \) for \( 3 \leq i < j \leq n - t \) and \( S(c_k) < S(c_l) \) for \( 1 \leq k < l \leq t - 1 \). Since \( S(v_n) = 2n - 2t - 3 < 2n - 2t + 3 = S(c_1) \), we get

\[
S(v_{i+j}) < S(v_{i+j+1}) < S(v_{i+1}) < S(v_i) < S(v_{i+2}) \quad \text{for } 3 \leq i < j \leq n - t, 1 \leq k < l \leq t - 1.
\]

Since \( S(c_{t-1}) = 2n + 2t - 5 < 2n + 4t - 1 = S(c_t) \), we get \( S(c_i) < S(c_l) \) for \( 1 \leq i \leq t - 1 \). Using the inequality \( t < n - 1 \), we get \( 2n + 4t - 1 < 3n + 3t - 2 \). Hence \( S(c_i) < S(v_{t+1}) \).

Further, since \( n \geq 3, 3n + 3t - 2 < 4n + 7t - 8 \). Thus, \( S(v_{t+1}) < S(v_t) \). Therefore,

\[
S(v_{i+j}) < S(v_{i+j+1}) < S(v_k) < S(c_i) < S(v_{i+1}) < S(v_i) \quad \text{for } 3 \leq i < j \leq n - t, 1 \leq k < l \leq t - 1.
\]

Hence \( S \) is injective on \( S_o \).

Since \( S(v_{t+2}) = n - t + 1 < 3n - t - 1 = S(v_1) \) and \( S(v_1) = 3n - t - 1 < 4n + 4 = S(v_2) \).

Thus,

\[
S(v_{i+2}) < S(v_i) < S(v_j) \quad \text{for } 2 \leq i < j \leq t - 1.
\]

Hence \( S \) is injective on \( S_e \).

**Case (iv)** \( n \) and \( t \) both are odd.

Define a bijective function \( f : E(CWF(n, t)) \to \{1, 2, \ldots, n + 2t\} \) as follows:

\[
\begin{align*}
f(c_{i+1}) &= n - t + 2i + 1 \quad \text{for } 1 \leq i \leq t - 2 \\
f(c_{i+1}) &= n - t + 2i \quad \text{for } 1 \leq i \leq t - 2 \\
f(c_{i-1}v_i) &= n + 2t + 1 \\
f(c_{i-1}v_i) &= n + 2t \\
f(c_{i-1}v_i) &= n + t - 3 \\
f(c_{i-1}v_i) &= n + t - 2 \\
f(v_i v_{i+1}) &= n + t + i - 2 \quad \text{for } 1 \leq i \leq t - 2 \\
f(v_{i-1}v_i) &= n + 2t - 2 \\
f(v_{i-1}v_i) &= n + 2t - 3 \\
f(v_{i+1}v_{i+2}) &= i \quad \text{for } 1 \leq i \leq n - t - 1 \\
f(v_n v_1) &= n - t.
\end{align*}
\]

Then the vertex sum function is given as

\[
\begin{align*}
S(v_1) &= 3n - t \\
S(v_i) &= 4n + 6i - 8 \quad \text{for } 2 \leq i \leq t - 2 \\
S(v_{t-1}) &= 4n + 7t - 11 \\
S(v_t) &= 4n + 7t - 8 \\
S(v_{t+1}) &= 2n + 3t - 4 \\
S(v_{t+2}) &= 2i - 1 \quad \text{for } 2 \leq i \leq n - t \\
S(c_i) &= 2n - 2t + 4i - 1 \quad \text{for } 1 \leq i \leq t - 2 \\
S(c_{t-1}) &= 2n + 4t - 1 \\
S(c_t) &= 2n + 2t - 5.
\end{align*}
\]
In this case, $S_o = \{v_t, \ldots, v_n, c_1, \ldots, c_t\}$ and $S_e = \{v_1, \ldots, v_{t-1}\}$. Note that $S(v_{t+i}) < S(v_{t+j})$ for $2 \leq i < j \leq n-t$ and $S(c_k) < S(c_l)$ for $1 \leq k < l \leq t-2$. Since $\text{Max}\{S(v_{t+j})|2 \leq j \leq n-t\} = 2n - 2t - 1 < 2n - 2t + 3 = \text{Min}\{S(c_k)|1 \leq k \leq t-2\}$, we get

$$S(v_{t+j}) < S(c_k) \text{ for } 2 \leq j \leq n-t, 1 \leq k \leq t-2.$$  

$$\text{Max}\{S(c_l)|1 \leq l \leq t-2\} = 2n + 2t - 9 < 2n + 2t - 5 = S(c_t).$$  

Thus,

$S(c_l) < S(c_t) \text{ for } 1 \leq l \leq t-2.$

Since $S(c_t) = 2n + 2t - 5 < 2n + 3t - 4 = S(v_{t+1}) < 2n + 4t - 1 = S(c_{t-1}) < 4n + 7t - 8 = S(v_t)$, we get

$$S(v_{t+i}) < S(v_{t+j}) < S(c_k) < S(c_l) < S(v_{t+1}) < S(v_{t-1}) < S(v_t)$$  

for $2 \leq i < j \leq n-t, 1 \leq k < l \leq t-2.$

Hence $S$ is injective on $S_o$.

Since $S(v_1) = 3n - t < 4n + 4 = S(v_2)$, we get $S(v_1) < S(v_i) < S(v_j)$ for $2 \leq i < j \leq t-2$.  

$S(v_{t-2}) = 4n + 6t - 20 < 4n + 7t - 11 = S(v_{t-1})$. Thus $S(v_i) < S(v_{t-1})$ for $2 \leq i \leq t-2$. Therefore,

$$S(v_1) < S(v_i) < S(v_j) < S(v_{t-1}) \text{ for } 2 \leq i < j \leq t-2.$$  

Hence $S$ is injective on $S_e$.

Figure 4. Labeling of $CW\text{F}(n, t)$ for $t$-odd

For $t$-even, the labelings of the wounded cartoon flower graph $CW\text{F}(n, t)$ for $n$-even and $n$-odd are shown in the Fig. 3(a) and Fig. 3(b), respectively. For $t$-odd, the labelings of the wounded cartoon flower graph $CW\text{F}(n, t)$ for $n$-even and $n$-odd are shown in the Fig. 4(a) and Fig. 4(b), respectively.
4. Conclusions
In this paper, a new subclass of the cartoon flower graph, the consecutive wounded cartoon flower, was defined. The cartoon flower of any size is anti-magic. Furthermore, the graph obtained by plucking any number of consecutive petals from a cartoon flower is also anti-magic.

For future work, we would like to examine the statement: “The wounded cartoon flower of any size is anti-magic.” If the statement is true, it would open the door to finding the anti-magic labeling of WF(n, t) for all possible values of n and t and for all possible positions of plucking the petals. If the statement is not true, it would open the door to finding the number and positions of petals to be plucked such that the graph is not anti-magic. Moreover, this graph would be a counter example for the conjecture presented by Hartsfield and Ringel [2] which states that all the connected graphs except $K_2$ are anti-magic.

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