FORMALITY AND KONTSEVICH–DUFLO TYPE THEOREMS FOR LIE PAIRS

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Dedicated to the memory of our friend and colleague Krzysztof Wysocki

ABSTRACT. Kontsevich’s formality theorem states that there exists an $L_\infty$ quasi-isomorphism from the dgla $T_{\text{poly}}^\bullet(M)$ of polyvector fields on a smooth manifold $M$ to the dgla $D_{\text{poly}}^\bullet(M)$ of polydifferential operators on $M$, which extends the classical Hochschild–Kostant–Rosenberg map. In this paper, we extend Kontsevich’s formality theorem to Lie pairs, a framework which includes a range of diverse geometric contexts such as complex manifolds, foliations, and $g$-manifolds (that is, manifolds endowed with an action of a Lie algebra $g$). The spaces $\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R T_{\text{poly}}^\bullet)$ and $\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R D_{\text{poly}}^\bullet)$ associated with a Lie pair $(L, A)$ each carry an $L_\infty$-algebra structure canonical up to $L_\infty$ quasi-isomorphism. These two spaces serve as replacements for the spaces of polyvector fields and polydifferential operators, respectively. Their corresponding cohomology groups $H_{\text{CE}}^\bullet(A, T_{\text{poly}}^\bullet)$ and $H_{\text{CE}}^\bullet(A, D_{\text{poly}}^\bullet)$ are Gerstenhaber algebras. We establish the following formality theorem for Lie pairs: there exists an $L_\infty$ quasi isomorphism from $\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R T_{\text{poly}}^\bullet)$ to $\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R D_{\text{poly}}^\bullet)$ whose first Taylor coefficient is equal to $\text{hkr}(\text{td}_{L/A})^\frac{1}{2}$. Here $(\text{td}_{L/A})^\frac{1}{2}$ acts on $\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R T_{\text{poly}}^\bullet)$ by contraction. As a consequence, we prove a Kontsevich–Duflo type theorem for Lie pairs: the Hochschild–Kostant–Rosenberg map twisted by the square root of the Todd class of the Lie pair $(L, A)$ is an isomorphism of Gerstenhaber algebras from $H_{\text{CE}}^\bullet(A, T_{\text{poly}}^\bullet)$ to $H_{\text{CE}}^\bullet(A, D_{\text{poly}}^\bullet)$. As applications, we establish formality theorems and Kontsevich–Duflo type theorems for complex manifolds, foliations, and $g$-manifolds. In the case of complex manifolds, we recover the Kontsevich–Duflo theorem of complex geometry.

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Introduction

In the late 1990’s, Kontsevich revolutionized the field of deformation quantization with his formality theorem: there exists an $L_\infty$ quasi-isomorphism from the dgla $T_{poly}^\bullet(M)$ of polyvector fields on a smooth manifold $M$ to the dgla $D_{poly}^\bullet(M)$ of polydifferential operators on $M$ extending the classical Hochschild–Kostant–Rosenberg map. Indeed, the formality theorem implies the existence of deformation quantizations for every smooth Poisson manifold [22, 46, 14]. In his paper [22], Kontsevich gave an explicit formula for the formality quasi-isomorphism in the case $M = \mathbb{R}^d$ and then outlined how the result can be generalized to arbitrary smooth manifolds. Later, Dolgushev gave a detailed proof of the globalization to arbitrary smooth manifolds of Kontsevich’s formality quasi-isomorphism for $\mathbb{R}^d$ based on Fedosov’s patching technique [14, 18].

In this paper, we extend Kontsevich’s formality theorem to Lie pairs, a framework which includes a wide range of diverse geometric contexts including complex manifolds, foliations, and $g$-manifolds. By a Lie pair $(L, A)$, we mean an inclusion $A \hookrightarrow L$ of Lie $k$-algebroids over a smooth manifold $M$. (Throughout the paper, we use the symbol $k$ to denote either of the fields $\mathbb{R}$ and $\mathbb{C}$.) Recall that a Lie $k$-algebroid is a $k$-vector bundle $L \to M$, whose space of sections is endowed with a Lie bracket $[\cdot, \cdot]$, together with a bundle map $\rho : L \to TM \otimes_k k$ called anchor such that $\rho : \Gamma(L) \to \mathfrak{X}(M) \otimes k$ is a morphism of Lie algebras and $[X, fY] = f[X, Y] + (\rho(X)f)Y$ for all $X, Y \in \Gamma(L)$ and $f \in C^\infty(M, k)$. A $k$-vector bundle $L \to M$ is a Lie algebroid if and only if $\Gamma(L)$ is a Lie–Rinehart algebra [41] over the commutative ring $C^\infty(M, k)$. Lie pairs arise naturally in a number of subdisciplines of mathematics such as complex geometry, foliation theory, and Lie theory. A complex manifold $X$ determines a Lie pair (over $\mathbb{C}$): $L = TX \otimes \mathbb{C}$ and $A = T^{0,1}X$. A foliation on a smooth manifold $M$ determines a Lie pair (over $\mathbb{R}$): $L = TM$ and $A$ is the integrable distribution on $M$ tangent to the foliation. A manifold equipped with an action of a Lie algebra $g$ gives rise to a Lie pair in a natural way (see [38, Example 5.5] and [29, 28]).

Given a Lie pair $(L, A)$, the quotient $L/A$ is naturally an $A$-module. When $L$ is the tangent bundle to a manifold $M$ and $A$ is an integrable distribution on $M$, the $A$-action on $L/A$ is given by the Bott connection [5]. The spaces $\text{tot}(\Gamma(\mathcal{A}^V) \otimes_R T_{poly}^\bullet)$ and $\text{tot}(\Gamma(\mathcal{A}^V) \otimes_R D_{poly}^\bullet)$ associated with a Lie pair $(L, A)$ serve as replacements for the spaces of polyvector fields and polydifferential operators respectively. Each
Denoting the algebra of smooth functions on the manifold $M$ can thus consider the complex of cohomology groups $\Delta : H^\bullet(A) \rightarrow H^\bullet(A)$ and the Hochschild cohomology group $HH^\bullet(A)$. For instance, for the Lie pair $\Gamma(A \cdot A^\vee) \otimes_R T^\bullet_{\text{poly}}$, we set $\mathcal{T}^\bullet_{\text{poly}} = \Gamma(L/A)$ for $k \geq 0$, $\mathcal{T}^{-1}_{\text{poly}} = R$, and $\mathcal{T}^k_{\text{poly}} = \bigoplus_{k=1}^\infty \mathcal{T}^k_{\text{poly}}$. The Bott $A$-connection on $L/A$ makes every $\mathcal{T}^k_{\text{poly}}$ an $A$-module. We can thus consider the complex of $A$-modules with trivial differential

$$0 \rightarrow \mathcal{T}^{k-1}_{\text{poly}} \rightarrow \mathcal{T}^k_{\text{poly}} \rightarrow \mathcal{T}^{k+1}_{\text{poly}} \rightarrow \cdots$$

Its Chevalley–Eilenberg hypercohomology cochain complex is denoted $\text{tot} (\Gamma(A \cdot A^\vee) \otimes_R T^\bullet_{\text{poly}})$. Similarly, we set $\mathcal{D}^\bullet_{\text{poly}} = \bigoplus_{k=1}^\infty \mathcal{D}^k_{\text{poly}}$ where $\mathcal{D}^{k-1}_{\text{poly}} = R$, $\mathcal{D}^0_{\text{poly}} = \frac{I(U(L))}{I(U(L))}$ and $\mathcal{D}^k_{\text{poly}}$ with $k \geq 1$ is the tensor product $\mathcal{D}^0_{\text{poly}} \otimes_R \cdots \otimes_R \mathcal{D}^0_{\text{poly}}$ of $(k + 1)$ copies of the left $R$-module $\mathcal{D}^0_{\text{poly}}$. Multiplication in $\mathcal{U}(L)$ from the left by elements of $\Gamma(A)$ induces an $A$-module structure on the quotient $\frac{I(U(L))}{I(U(L))}$. This action of $A$ on $\mathcal{D}^0_{\text{poly}}$ extends naturally to an action of $A$ on $\mathcal{D}^k_{\text{poly}}$ for each $k$. In fact, $\mathcal{D}^0_{\text{poly}}$ is a cocommutative coassociative coalgebra over $R$ whose comultiplication $\Delta : \mathcal{D}^0_{\text{poly}} \rightarrow \mathcal{D}^0_{\text{poly}} \otimes_R \mathcal{D}^0_{\text{poly}}$ is a morphism of $A$-modules. Therefore, the Hochschild complex

$$0 \rightarrow \mathcal{D}^{k-1}_{\text{poly}} \rightarrow \mathcal{D}^k_{\text{poly}} \rightarrow \mathcal{D}^{k+1}_{\text{poly}} \rightarrow \cdots$$

determined by the comultiplication $\Delta : \mathcal{D}^0_{\text{poly}} \rightarrow \mathcal{D}^0_{\text{poly}} \otimes_R \mathcal{D}^0_{\text{poly}}$ is a complex of $A$-modules. Its Chevalley–Eilenberg hypercohomology complex is denoted $\text{tot} (\Gamma(A \cdot A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$. For instance, for the Lie pair $L = TX \otimes \mathbb{C}$ and $A = T^0_{\mathbb{C}^\cdot}$ stemming from a complex manifold $X$, the pair of spaces $\text{tot} (\Gamma(A \cdot A^\vee) \otimes_R T^\bullet_{\text{poly}})$ and $\text{tot} (\Gamma(A \cdot A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$ are precisely the standard dglas $(\Omega^0(T^\bullet_{\text{poly}}(X)), \partial)$ and $(\Omega^0(T^\bullet_{\text{poly}}(X)), \partial + d_{\text{poly}})$. The corresponding Chevalley–Eilenberg hypercohomology groups $\mathbb{H}^\bullet_{\text{CE}}(A, T^\bullet_{\text{poly}})$ and $\mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ are isomorphic to the sheaf cohomology group $\mathbb{H}^\bullet(X, \Lambda^\bullet T_X)$ and the Hochschild cohomology group $HH^\bullet(X)$, respectively.

The skew-symmetric extension of the natural inclusion $\Gamma(L/A) \rightarrow \mathcal{D}^0_{\text{poly}}$ to the complex of $A$-modules $T^\bullet_{\text{poly}}$ yields a morphism of $A$-modules $hkr : T^\bullet_{\text{poly}} \rightarrow \mathcal{D}^\bullet_{\text{poly}}$. The induced morphism of Chevalley–Eilenberg hypercohomology cochain complexes $hkr : \text{tot} (\Gamma(A \cdot A^\vee) \otimes_R T^\bullet_{\text{poly}}) \rightarrow \text{tot} (\Gamma(A \cdot A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$, which is also called Hochschild–Kostant–Rosenberg map, is actually a quasi-isomorphism. It is thus natural to ask whether $hkr$ can be extended to an $L_\infty$ quasi-isomorphism analogous to Kontsevich’s formality quasi-isomorphism for smooth manifolds. The answer is negative in general and the reason is quite simple. For a smooth manifold $M$, the Hochschild–Kostant–Rosenberg map induces an isomorphism of Lie algebras (in fact an isomorphism of Gerstenhaber algebras) from the polynomials fields $T^\bullet_{\text{poly}}(M)$ on $M$ equipped with the Schouten bracket to the Hochschild cohomology $H^\bullet((\mathcal{D}^\bullet_{\text{poly}}(M), d_{\text{poly}}))$ equipped with the Gerstenhaber bracket. However, for a Lie pair $(L, A)$, the morphism in cohomology $hkr : \mathbb{H}^\bullet_{\text{CE}}(A, T^\bullet_{\text{poly}}) \rightarrow \mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ induces by the Hochschild–Kostant–Rosenberg map no longer preserves neither the Lie algebra nor the associative algebra structures. The Hochschild–Kostant–Rosenberg map $hkr$ must indeed be modified; it must be tweaked by the square root of the Todd cocycle of the Lie pair.

The Atiyah class of a Lie pair $(L, A)$ was introduced and studied by Chen–Stiénon–Xu in [12]. It captures the obstruction to the existence of ‘compatible’ $L$-connections on $L/A$ extending the Bott $A$-representation. The Atiyah class of Lie pairs is a simultaneous extension of both the classical Atiyah class of holomorphic vector bundles [1] and the Molino class of foliations [39]. As was first observed for holomorphic vector bundles by Kapranov [21], the Atiyah class of Lie pairs is the source of homotopy Lie algebras [12, 25, 26]. Let us briefly recall its definition. Given a Lie pair $(L, A)$ with quotient $B = L/A$, choose an $L$-connection $\nabla$ on $B$ extending the Bott $A$-representation. The curvature of $\nabla$ induces a section $R^\nabla_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \text{End}(L/A))$, which is a Chevalley–Eilenberg 1-cocycle for the Lie algebroid $A$ with values in the $A$-module
A ⊗ \text{End}(L/A). Its cohomology class \( \alpha_{L/A} \in H^1_{CE}(A, A ⊗ \text{End}(L/A)) \) does not depend on the choice of \( L \)-connection \( \nabla \) and is called Atiyah class of the Lie pair \((L, A)\).

We can assign a Todd cocycle — defined in terms of the Atiyah cocycle — to each Lie pair \((L, A)\) in the exact same way the Todd cocycle of a complex manifold is derived from its Atiyah cocycle. The Todd cocycle of a Lie pair \((L, A)\) is the Chevalley-Eilenberg cocycle

\[
\text{td}_{L/A} = \det \left( \frac{R_{1,1}^\nabla}{1 - e^{-R_{1,1}^\nabla}} \right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp). \tag{1}
\]

Its cohomology class \( \text{Td}_{L/A} \in \bigoplus_{k=0} H^k_{CE}(A, \Lambda^k A^\perp) \) is the Todd class of the Lie pair \((L, A)\). See Section 1.2 for details.

The main goal of this paper is to establish the following formality theorem for Lie pairs: There exists an \( L_\infty \) quasi isomorphism from \( \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R T^\text{poly}_\bullet) \) to \( \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R D^\text{poly}_\bullet) \) whose first Taylor coefficient is equal to \( \text{hkr} \circ (\text{td}_{L/A}^\nabla)^\frac{1}{2}, \) with \((\text{td}_{L/A}^\nabla)^\frac{1}{2} \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp) \) acting on \( \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R T^\text{poly}_\bullet) \) by contractions. See Theorem 2.1.

As an immediate consequence, we obtain the following Kontsevich–Duflo type theorem for Lie pairs: Given a Lie pair \((L, A)\), the map \( \text{hkr} \circ \text{td}_{L/A}^\frac{1}{2} : \mathcal{H}_{CE}(\mathfrak{a}, T_{poly}^\mathfrak{a}) \to \mathcal{H}_{CE}(A, D^\text{poly}_\mathfrak{a}) \) is an isomorphism of Gerstenhaber algebras. See Theorem 2.2.

Our result is very much inspired by Kontsevich’s seminal work [22], in which it is highlighted that the classical Duflo theorem is one of many consequences of the formality construction. For every Lie algebra \( \mathfrak{g} \), the symmetrization map \( \text{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \) is an isomorphism of \( \mathfrak{g} \)-modules called Poincaré–Birkhoff–Witt isomorphism. The induced isomorphism \( \text{pbw} : S(\mathfrak{g})^\mathfrak{g} \to \mathcal{U}(\mathfrak{g})^\mathfrak{g} \) between subspaces of \( \mathfrak{g} \)-invariants does not intertwine the obvious multiplications on \( S(\mathfrak{g})^\mathfrak{g} \) and \( \mathcal{U}(\mathfrak{g})^\mathfrak{g} \). However, it can be modified so as to become an isomorphism of associative algebras. The Duflo element \( J \in \tilde{S}(\mathfrak{g}^\vee) \) of a Lie algebra \( \mathfrak{g} \) is the formal polynomial on \( \mathfrak{g} \) defined by \( J(x) = \det \left( \frac{1-e^{-\text{ad}_x}}{\text{ad}_x} \right) \), for all \( x \in \mathfrak{g} \). Considered as a translation-invariant formal differential operator on \( \mathfrak{g}^\vee \), the square root of the Duflo element defines a transformation \( J^\frac{1}{2} : S(\mathfrak{g}) \to S(\mathfrak{g}) \). A remarkable theorem due to Duflo [16] asserts that the composition \( \text{pbw} \circ J^\frac{1}{2} : S(\mathfrak{g})^\mathfrak{g} \to \mathcal{U}(\mathfrak{g})^\mathfrak{g} \) is an isomorphism of associative algebras. Duflo’s theorem generalizes a fundamental result of Harish-Chandra regarding the center of the universal enveloping algebra of a semi-simple Lie algebra. Duflo’s original proof was based on deep and sophisticated techniques of representation theory including Kirillov’s orbit method. As an application of his formality construction, Kontsevich proposed a new proof of Duflo’s theorem by means of the induced associative algebra structure on tangent cohomology at a Maurer–Cartan element. Indeed, Kontsevich’s approach [22] has led to an extension of Duflo’s theorem: For every finite dimensional Lie algebra \( \mathfrak{g} \), the map \( \text{pbw} \circ J^\frac{1}{2} : \mathcal{H}_{CE}(\mathfrak{g}, S(\mathfrak{g})) \to \mathcal{H}_{CE}(\mathfrak{g}, \mathcal{U}(\mathfrak{g})) \) is an isomorphism of graded associative algebras. The classical Duflo theorem is the isomorphism of the cohomologies in degree 0. A detailed proof of the above extended Duflo theorem was given by Pevzner–Torossian [40] (see also [31, 32]).

Kontsevich discovered a similar phenomenon in complex geometry. Recall that the Hochschild cohomology groups \( HH^\bullet(X) \) of a complex manifold \( X \) are defined as the groups \( \text{Ext}^\bullet_{\mathcal{O}_X} \left( \mathcal{O}_X, \mathcal{O}_X \right) \) [9]. Gerstenhaber–Shack [19] derived an isomorphism of cohomology groups \( \text{hkr} : \mathcal{H}^\bullet(\mathcal{O}_X, \Lambda^* \mathcal{T}_X) \to HH^\bullet(X) \) from the classical Hochschild–Kostant–Rosenberg map. This isomorphism fails to intertwine the multiplications in both cohomologies but can be tweaked so as to produce an isomorphism of associative algebras. More precisely, Kontsevich [22] obtained the following theorem: The composition \( \text{hkr} \circ (\text{td}_X)^\frac{1}{2} : \mathcal{H}^\bullet(\mathcal{O}_X, \Lambda^* \mathcal{T}_X) \to HH^\bullet(X), \) were the symbol \( \text{td}_X \) denotes the Todd class of the complex manifold \( X \), is an isomorphism of associative algebras. The multiplications on \( \mathcal{H}^\bullet(\mathcal{O}_X, \Lambda^* \mathcal{T}_X) \) and \( HH^\bullet(X) \) are respectively the wedge product and the Yoneda product. Calaque–Van den Bergh [8] wrote a detailed proof of Kontsevich’s theorem, and
showed that the map $\text{hkr} \circ (\text{Td}_X)^{\frac{1}{2}}$ actually preserves the Gerstenhaber algebra structures on both cohomologies. We refer the reader to [15, 47] for a related result.

Hence Kontsevich’s formality revealed a hidden connection between two areas of mathematics: complex geometry and Lie theory. The mysterious and surprising similarity between the Todd class of a Lie algebra — two seemingly unrelated objects — led to further exciting developments. In 1998, Shoikhet [42] announced the so called Kontsevich–Shoikhet theorem (Theorem 2.17), which explains the deep ties between Lie theory and complex geometry and provides a unified framework for their study. The theorem states that a formula of Duflo type holds for every dg manifold $\mathbb{R}^m|\Lambda^n$, $Q$. See [7] for a detailed proof.

Our approach is inspired by Dolgushev’s proof of Kontsevich’s global formality theorem for smooth manifolds [14] and relies heavily on the Fedosov dg Lie algebroid constructed by two of the authors in [45] (and independently by Batakidis–Voglaire in the special case of matched pairs [4]). Roughly speaking, a Fedosov dg Lie algebroid associated with a Lie pair $(L, A)$ is a dg Lie algebra whose associated spaces of polyvector fields and polydifferential operators are homotopy equivalent to $\text{tot}(T^\Lambda L, \mathcal{D}_\text{poly})$ and $\text{tot}(\Gamma(L^\vee) \otimes R \mathcal{D}_\text{poly})$, respectively (in a style reminiscent of Dolgushev’s Fedosov resolutions [14]). More precisely, having chosen some additional geometric data, one can endow the graded manifold $M = L[1] \oplus L/A$ with a structure of dg manifold $(M, Q)$ quasi-isomorphic to $(A[1], d_A)$. We call any such a dg manifold $(M, Q)$ a ‘Fedosov dg manifold associated with the Lie pair $(L, A)$.’ The Fedosov dg Lie algebroid $F$ is a certain dg Lie subalgebroid of the tangent dg Lie algebroid $T_M$ of the Fedosov dg manifold $(M, Q)$. In other words, $F$ is the dg Lie algebroid encoding a certain dg foliation of $(M, Q)$. Since a Lie algebroid can be thought of as an extension of the tangent bundle of a manifold, the notions of polyvector fields and polydifferential operators admit extensions to a context of a Lie algebroid and these each carry a natural dgla structure [49, 50]. Likewise, the notions of polyvector fields and polydifferential operators can be extended in an appropriate sense to the context of a dg Lie algebroid, yielding two dgla $T^\Lambda_{poly}$ and $\mathcal{D}^\vee_{poly}$ whose corresponding cohomology groups are naturally Gerstenhaber algebras. The “polyvector fields” and “polydifferential operators” associated to the Fedosov dg Lie algebroid $F$ can be viewed geometrically as polyvector fields and polydifferential operators tangent to the dg foliation on the Fedosov dg manifold $(M, Q)$. In fact, one can identify the “polyvector fields” and “polydifferential operators” on $F$ by $\text{tot}(\Gamma(L^\vee) \otimes_R \mathcal{D}^\vee_{poly})$, respectively, where $\mathcal{D}^\vee_{poly}$ denotes the formal polyvector fields and $\mathcal{D}^\vee_{poly}$ the formal polydifferential operators tangent to the fibers of the vector bundle $L/A \to M$.

By applying Kontsevich formality theorem fiberwisely to $F \to M$, we prove that there exists an $L_\infty$ quasi-isomorphism

$$\Phi : (\Gamma(L^\vee) \otimes R \mathcal{D}^\vee_{poly}, [Q, -], [-, -]) \to (\Gamma(L^\vee) \otimes R \mathcal{D}^\vee_{poly}, [Q, -] + d_{\mathcal{D}}\mathcal{D}, [-, -]).$$

This $L_\infty$ quasi-isomorphism $\Phi$ is in fact a sequence of maps $(\Phi_n)_{n=1}^{\infty}$ — its ‘Taylor coefficients’ — the first amongst which is a quasi-isomorphism of cochain complexes

$$\Phi_1 : \Gamma(L^\vee) \otimes_R \mathcal{D}^\vee_{poly} \to \Gamma(L^\vee) \otimes_R \mathcal{D}^\vee_{poly}.$$ 

The latter induces an isomorphism of Lie algebras on the level of cohomologies. A standard argument of Kontsevich, Manchon–Torossian, and Mochizuki [22, 31, 32, 37] suffices to prove that $\Phi_1$ intertwines the associative multiplications carried by the cohomologies as well. Hence, in cohomology, $\Phi_1$ really is an isomorphism of Gerstenhaber algebras.

Next, we apply the Kontsevich–Shoikhet theorem (Theorem 2.17) in order to prove that $\Phi_1$ is essentially the fiberwise HKR map enhanced by the Todd class of the Fedosov dg Lie algebroid. More precisely, we prove that $\Phi_1 : \Gamma(L^\vee) \otimes_R \mathcal{D}^\vee_{poly} \to \Gamma(L^\vee) \otimes_R \mathcal{D}^\vee_{poly}$ is the composition

$$\Phi_1 = \text{hkr} \circ (\text{td}_F)^{\frac{1}{2}}.$$
of the natural extension hkr of the fiberwise Hochschild–Kostant–Rosenberg map \( \mathcal{F}_\text{poly} \to \mathcal{F}_\text{poly} \) and the action on \( \Gamma(\Lambda^* L_Y) \otimes_R \mathcal{F}_\text{poly} \) (by contraction) of the Todd cocycle \( \tilde{\omega}^\text{can} \) of the Fedosov dg Lie algebroid \( \mathcal{F} \) associated with the canonical connection defined by Equation (9).

Then our main theorem essentially follows from a careful combination of the above results together with various \( L_\infty \) quasi-isomorphisms. Our approach was largely influenced and indeed relies on several standard techniques pioneered by Kontsevich in his seminal paper [22] and expounded at greater length in subsequent literature [42, 8, 7]. However, we emphasize the role of Fedosov dg Lie algebroids as it sheds new light on and indeed provides transparent understanding of Kontsevich’s global formality theorem and, in particular, the Kontsevich–Duflo phenomenon.

In Section 3, we apply our results to a number of interesting classes of examples of Lie pairs, namely those arising from complex manifolds, from regular foliations, and from \( \mathfrak{g} \)-manifolds. In each case, we obtain a formality theorem and a Kontsevich–Duflo type theorem. In the case of lie pairs stemming from complex manifolds, we recover the Kontsevich–Duflo theorem of complex geometry [22, 8]. As far as we know, the formality and Kontsevich–Duflo type theorems obtained for geometric situations such as foliations and \( \mathfrak{g} \)-manifolds are new. In the future, we plan to investigate the implications of the formality theorem in deformation quantization, in particular for the special instances of Lie pairs listed above.

**Terminology and notations.**

**Natural numbers.** We use the symbol \( \mathbb{N} \) to denote the set of positive integers and the symbol \( \mathbb{N}_0 \) for the set of nonnegative integers.

**Field \( k \) and ring \( R \).** We use the symbol \( k \) to denote the field of either real or complex numbers. The symbol \( R \) always denotes the algebra of smooth functions on \( M \) with values in \( k \).

**Tensor products.** For any two \( R \)-modules \( P \) and \( Q \), we write \( P \otimes_R Q \) to denote the tensor product of \( P \) and \( Q \) as \( R \)-modules and \( P \otimes Q \) to denote the tensor product of \( P \) and \( Q \) regarded as \( k \)-modules.

**Completed symmetric algebra.** Given a module \( \mathcal{M} \) over a ring, the symbol \( \hat{S}(\mathcal{M}) \) denotes the \( m \)-adic completion of the symmetric algebra \( S(\mathcal{M}) \), where \( m \) is the ideal of \( S(\mathcal{M}) \) generated by \( \mathcal{M} \).

**Duality pairing.** For every vector bundle \( E \to M \), we define a duality pairing
\[
\Gamma(\hat{S}(E^\vee)) \times \Gamma(S(E)) \to R
\]
by
\[
\langle \nu_1 \otimes \cdots \otimes \nu_p | v_1 \otimes \cdots \otimes v_q \rangle = \begin{cases} \sum_{\sigma \in S_p} \prod_{k=1}^p \langle \nu_k | v_{\sigma(k)} \rangle & \text{if } p = q, \\ 0 & \text{otherwise}. \end{cases}
\]

**Multi-indices.** Let \( E \to M \) be a smooth vector bundle of finite rank \( r \), let \( (\partial_i)_{i \in \{1, \ldots, r\}} \) be a local frame of \( E \) and let \( (\chi_j)_{j \in \{1, \ldots, r\}} \) be the dual local frame of \( E^\vee \). Thus, we have \( \langle \chi_i | \partial_j \rangle = \delta_{i,j} \). Given a multi-index \( I = (I_1, I_2, \ldots, I_r) \in \mathbb{N}_0^r \), we adopt the following multi-index notations:
\[
I! = I_1! \cdot I_2! \cdots I_r! \quad |I| = I_1 + I_2 + \cdots + I_r
\]
\[
\partial^I = \partial_1 \circ \cdots \circ \partial_1 \circ \partial_2 \circ \cdots \circ \partial_2 \circ \cdots \circ \partial_r \circ \cdots \circ \partial_r
\]
\[
\chi^I = \chi_1 \circ \cdots \circ \chi_1 \circ \chi_2 \circ \cdots \circ \chi_2 \circ \cdots \circ \chi_r \circ \cdots \circ \chi_r
\]
We use the symbol $e_k$ to denote the multi-index all of whose components are equal to 0 except for the $k$-th which is equal to 1. Thus $\chi^{e_k} = \chi_k$.

**Shuffles.** A $(p, q)$-shuffle is a permutation $\sigma$ of the set $\{1, 2, \ldots, p+q\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)$. The symbol $\mathfrak{S}_p^q$ denotes the set of $(p, q)$-shuffles.

**Graduation shift.** Given a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$, $V[i]$ denotes the graded vector space obtained by shifting the grading on $V$ according to the rule $(V[i])^{(k)} = V^{(i+k)}$. Accordingly, if $E = \bigoplus_{k \in \mathbb{Z}} E^{(k)}$ is a graded vector bundle over $M$, $E[i]$ denotes the graded vector bundle obtained by shifting the degree in the fibers of $E$ according to the above rule.

**Koszul sign.** The Koszul sign $\text{sgn}(\sigma; v_1, \ldots, v_n)$ of a permutation $\sigma$ of homogeneous vectors $v_1, v_2, \ldots, v_n$ of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is determined by the equality

$$v_{\sigma(1)} \odot v_{\sigma(2)} \odot \cdots \odot v_{\sigma(n)} = \text{sgn}(\sigma; v_1, \ldots, v_n) \cdot v_1 \odot v_2 \odot \cdots \odot v_n$$

in the graded commutative algebra $S(V)$.

**$L_\infty$ algebra.** An $L_\infty[1]$ algebra $[33, 24, 44, 11]$ is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ endowed with a sequence $(Q_k)_{k=1}^\infty$ of linear maps $Q_k : S^k(V) \rightarrow V[1]$ satisfying the generalized Jacobi identities

$$\sum_{p+q=n} \sum_{\sigma \in \mathfrak{S}_p^q} \text{sgn}(\sigma; v_1, \ldots, v_n) \cdot Q_{1+q}(Q_p(v_{\sigma(1)}, \ldots, v_{\sigma(p)}), v_{\sigma(p+1)}, \ldots, v_n) = 0$$

for each $n \in \mathbb{N}$ and for all homogeneous vectors $v_1, v_2, \ldots, v_n \in V$. In particular, the first map $Q_1$ is a coboundary operator on $V$.

Alternatively, an $L_\infty[1]$ algebra is a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ together with a (k-linear) coderivation $Q$ of degree +1 of the symmetric coalgebra $S(V)$ satisfying the properties $Q \circ Q = 0$ and $Q(1) = 0$.

This coderivation $Q$ determines the sequence of multibrackets $(Q_k)_{k=1}^\infty$ through commutative diagrams

$$S(V) \xrightarrow{Q} S(V)[1]$$

$$S^k(V) \xrightarrow{Q_k} S^1(V)[1].$$

Dualizing the coalgebra $S(V)$, we obtain the algebra $\hat{S}(V^\vee)$, which can be thought of as the algebra of functions on the graded manifold $V[-1]$. The derivation of the algebra $\hat{S}(V^\vee)$ dual to the coderivation $Q$ of $S(V)$ can be regard as a homological vector field on $V[-1]$.

A $\mathbb{Z}$-graded vector space $V$ is an $L_\infty$ algebra if and only if $V[1]$ is an $L_\infty[1]$ algebra.

**$L_\infty$ morphism.** Let $V$ and $W$ be two $L_\infty[1]$ algebras. An $L_\infty$ morphism from $V$ to $W$ is a morphism $\Phi : S(V) \rightarrow S(W)$ of coalgebras over $\mathbb{k}$ intertwining the coderivations of $S(V)$ and $S(W)$ and satisfying $\Phi(1) = 0$. Every $L_\infty$ morphism $\Phi$ is entirely determined by its sequence of ‘Taylor coefficients’ $(\Phi_k)_{k=1}^\infty$, the $k$-th of which is the composition

$$S^k(V) \xrightarrow{\Phi_k} S(V) \xrightarrow{\Phi} S(W) \xrightarrow{Q_1} S^1(W),$$

and its first Taylor coefficient $\Phi_1$ is a chain map from the cochain complex $(V, Q_1)$ to the chain complex $(W, Q'_1)$. An $L_\infty$ quasi-isomorphism is an $L_\infty$ morphism whose first Taylor coefficient happens to be a quasi-isomorphism of cochain complexes.
The definitions of $L_\infty$ algebras and $L_\infty$ morphisms can also be formulated in terms of exterior tensor algebras, which are isomorphic to the symmetric tensor algebras via the décalage map. Later in the paper, we adopt the exterior algebra point of view.

**Lie algebroid.** In this paper ‘Lie algebroid’ always means ‘Lie $k$-algebroid’ unless specified otherwise.

**Contraction.** Let $(C, \partial)$ and $(K, d)$ be two cochain complexes. A contraction of $(K, d)$ onto $(C, \partial)$ consists of a pair of chain maps $\tau : C \to K$ and $\sigma : K \to C$ together with a chain homotopy operator $h : K \to K[-1]$ satisfying

\[
\sigma \tau = \text{id}_C; \quad \text{id}_K - \tau \sigma = dh + hd; \\
\sigma h = 0; \quad h^2 = 0; \quad \text{and} \quad h\tau = 0.
\]

We symbolize such a contraction by a diagram

\[
(C, \partial) \xrightarrow{\tau} (K, d) \xleftarrow{\sigma} h.
\]

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1. Preliminaries

1.1. Connections and representations for Lie algebroids. Let $M$ be a smooth manifold, let $L \to M$ be a Lie $k$-algebroid with anchor map $\rho : L \to T_M \otimes_k k$, and let $E \to M$ be a vector bundle over $k$. The algebra of smooth functions on $M$ with values in $k$ will be denoted $R$.

The traditional description of a (linear) $L$-connection on $E$ is in terms of a covariant derivative

\[
\Gamma(L) \times \Gamma(E) \to \Gamma(E) : (l, e) \mapsto \nabla_l e
\]

characterized by the following two properties:

\[
\nabla_{f e} = f \cdot \nabla_l e, \tag{2}
\]

\[
\nabla_l (f \cdot e) = \rho(l) f \cdot e + f \cdot \nabla_l e, \tag{3}
\]

for all $l \in \Gamma(L)$, $e \in \Gamma(E)$, and $f \in R$.

**Remark 1.1.** A covariant derivative $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ induces a covariant derivative $\nabla : \Gamma(L) \times \Gamma(S(E)) \to \Gamma(S(E))$ through the relation

\[
\nabla_l (e_1 \odot \cdots \odot e_n) = \sum_{k=1}^n e_1 \odot \cdots \odot \nabla_l e_k \odot \cdots \odot e_n,
\]

for all $l \in \Gamma(L)$ and $e_1, \ldots, e_n \in \Gamma(E)$.

**Remark 1.2.** A covariant derivative $\nabla : \Gamma(L) \times \Gamma(S(E)) \to \Gamma(S(E))$ induces a covariant derivative $\nabla : \Gamma(L) \times \Gamma(\tilde{S}(E^\vee)) \to \Gamma(\tilde{S}(E^\vee))$ through the relation

\[
\rho(l) \langle \sigma | s \rangle = \langle \nabla_l \sigma | s \rangle + \langle \sigma | \nabla_l s \rangle
\]

for all $l \in \Gamma(L)$, $s \in \Gamma(S(E))$, and $\sigma \in \Gamma(\tilde{S}(E^\vee))$. 
A representation of a Lie algebroid $L$ on a vector bundle $E \to M$ is a flat $L$-connection $\nabla$ on $E$, i.e. a covariant derivative $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$ satisfying
\[
\nabla_l \nabla_le - \nabla_l \nabla_e = \nabla_{[l_1,l_2]}e,
\]
for all $l_1, l_2 \in \Gamma(L)$ and $e \in \Gamma(E)$. A vector bundle endowed with a representation of the Lie algebroid $L$ is called an $L$-module. More generally, given a left $R$-module $\mathcal{M}$, by an infinitesimal action of $L$ on $\mathcal{M}$, we mean a $k$-bilinear map $\nabla : \Gamma(L) \times \mathcal{M} \to \mathcal{M}$, $(l, e) \mapsto \nabla_le$ satisfying Equations (2), (3), and (4). In other words, $\nabla$ is a representation of the Lie–Rinehart algebra $(\Gamma(L), R)$ \cite{41}.

**Example 1.3.** Let $(L, A)$ be a Lie pair, i.e. an inclusion $A \hookrightarrow L$ of Lie algebroids. The Bott representation of $A$ on the quotient $L/A$ is the flat connection defined by
\[
\nabla^\text{Bott}_a q(l) = q([a, l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L),
\]
where $q$ denotes the canonical projection $L \to L/A$. Thus the quotient $L/A$ of a Lie pair $(L, A)$ is an $A$-module.

Let $L$ be a Lie algebroid over a smooth manifold $M$, and $R$ be the algebra of smooth functions on $M$ valued in $k$. The Chevalley–Eilenberg differential
\[
d_L : \Gamma(\Lambda^k L^\vee) \to \Gamma(\Lambda^{k+1} L^\vee)
\]
defined by
\[
(d_L \omega)(l_0, l_1, \ldots, l_k) = \sum_{i=0}^n (-1)^i \rho(l_i)(\omega(l_0, \ldots, \hat{l_i}, \ldots, l_k)) + \sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \ldots, \hat{l_i}, \ldots, \hat{l_j}, \ldots, l_k)
\]
and the exterior product make $\bigoplus_{k \geq 0} \Gamma(\Lambda^k L^\vee)$ into a differential graded commutative $R$-algebra.

Given a Lie algebroid $L$ of rank $n$, and an $L$-connection $\nabla$ on a vector bundle $E \to M$, the covariant differential is the operator
\[
d_L^\nabla : \Gamma(\Lambda^k L^\vee \otimes E) \to \Gamma(\Lambda^{k+1} L^\vee \otimes E)
\]
that takes a section $\omega \otimes e$ of $\Lambda^k L^\vee \otimes E$ to
\[
d_L^\nabla (\omega \otimes e) = (d_L \omega) \otimes e + \sum_{j=1}^n (\nu_j \wedge \omega) \otimes \nabla_{v_j} e,
\]
where $v_1, v_2, \ldots, v_n$ and $\nu_1, \nu_2, \ldots, \nu_n$ are any pair of dual local frames for the vector bundles $L$ and $L^\vee$. If the connection $\nabla$ is flat, then $d_L^\nabla$ is a coboundary operator: $d_L^\nabla \circ d_L^\nabla = 0$.

Let $L$ be a Lie algebroid. The symbol $\mathcal{L}$ denotes the abelian category of left modules over $\mathcal{U}(L)$. Its bounded derived category is denoted by $D^+(\mathcal{L})$. The Chevalley–Eilenberg cohomology in degree $k$ of a complex of $\mathcal{U}(L)$-modules $\mathcal{E}^\bullet$ is
\[
\mathbb{H}_{\text{CE}}^k(L, \mathcal{E}^\bullet) := \text{Hom}_{D^+(\mathcal{L})}(R, \mathcal{E}^\bullet[k]).
\]
It is computed as the total cohomology in degree $k$ of the double complex

\[ \cdots \to \Gamma(\Lambda^{p-1}L^\vee) \otimes_R E^{q+1} \xrightarrow{id \otimes (−1)^{p−1}d_E} \Gamma(\Lambda^p L^\vee) \otimes_R E^{q+1} \xrightarrow{d_E^r} \Gamma(\Lambda^{p+1}L^\vee) \otimes_R E^{q+1} \to \cdots \]

Here we follow Koszul’s sign convention: $id \otimes d_E(\omega \otimes e) = (−1)^p \omega \otimes d_E e$, for all $\omega \in \Gamma(\Lambda^p L^\vee)$ and $e \in E^\bullet$.

1.2. Atiyah class and Todd class of a Lie pair. Let $(L, A)$ be a pair of Lie algebroids over $k$. Consider the short exact sequence of vector bundles

\[ 0 \to A \xrightarrow{i} L \xrightarrow{q} L/A \to 0. \]

Given $L$-connection $\nabla$ on $L/A$, we define a bundle map $T^\nabla : \Lambda^2 L \to L/A$ by

\[ T^\nabla(x, y) = \nabla_x q(y) - \nabla_y q(x) - q([x, y]), \quad \forall x, y \in \Gamma(L). \]

An $L$-connection $\nabla$ on $L/A$ is said to extend the Bott $A$-connection on $L/A$ (see Example 1.3) if

\[ \nabla_{i(a)} q(l) = \nabla^\text{Bott}_a q(l) = q([i(a), l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L). \]

**Lemma 1.4.** The following assertions are equivalent:

1. The $L$-connection $\nabla$ on $L/A$ extends the Bott $A$-connection on $L/A$.
2. For all $a \in \Gamma(A)$ and $l \in \Gamma(L)$, we have $T^\nabla(i(a), l) = 0$.
3. There exists a unique bundle map $\beta^\text{\nabla} : \Lambda^2(L/A) \to L/A$ such that the diagram

\[
\begin{array}{ccc}
\Lambda^2 L & \xrightarrow{T^\nabla} & L/A \\
\downarrow q & & \\
\Lambda^2(L/A) & \xrightarrow{\beta^\text{\nabla}} & \\
\end{array}
\]

commutes.

Hence a torsion-free $L$-connections on $L/A$ is necessarily an extension of the Bott $A$-connection.

**Proof.** Since $q \circ i = 0$, we have

\[ 0 = T^\nabla(i(a), l) = \nabla_{i(a)} q(l) - q([i(a), l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L). \]

An $L$-connection $\nabla$ on $B$ is said to be torsion-free if $T^\nabla = 0$ (and hence $\beta^\text{\nabla} = 0$).

**Lemma 1.5.** Given a Lie pair $(L, A)$, there exist torsion-free $L$-connections on $L/A$. 

The Todd cocycle of a Lie pair

Lemma 1.7. The following lemma will be needed later on. Choosing a splitting \( i \circ p + j \circ q = \text{id}_L \) of the short exact sequence

\[
0 \longrightarrow A \xrightarrow{\iota} L \xrightarrow{\pi} B \longrightarrow 0 \tag{5}
\]

and set

\[
\nabla'_l b = q([i \circ p(l), j(b)]) + \nabla_{jq(l)} b.
\]

Finally, obtain a torsion-free connection \( \nabla'' : \Gamma(L) \times \Gamma(L/A) \to \Gamma(L/A) \) from \( \nabla' \) by setting

\[
\nabla''_l b = \nabla'_l b - \frac{1}{2} g^{\nabla'}(q(l), b).
\]

Given a Lie pair \( L, A \) with quotient \( B = L/A \), let \( \nabla \) be an \( L \)-connection on \( B \) extending the Bott \( A \)-representation. The curvature of \( \nabla \) is the bundle map \( R^\nabla : \Lambda^2 L \to \text{End}(L/A) \) defined by

\[
R^\nabla(x, y) = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]}, \quad \forall x, y \in \Gamma(L).
\]

Since \( L/A \) is an \( A \)-module, its restriction to \( \Lambda^2 A \) vanishes. Hence the curvature induces a section \( R^\nabla_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \text{End}(L/A)) \) or, equivalently, a bundle map \( R^\nabla_{1,1} : A \otimes (L/A) \to \text{End}(L/A) \) given by

\[
R^\nabla_{1,1}(a; q(l)) = R^\nabla(a, l) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{a,l}, \quad \forall a \in \Gamma(A), l \in \Gamma(L).
\]

Proposition 1.6 ([12]).

1. The section \( R^\nabla_{1,1} \in \Gamma(A^\vee \otimes A^\perp \otimes \text{End}(L/A)) \) is a 1-cocycle for the Lie algebroid \( A \) with values in the \( A \)-module \( A^\perp \otimes \text{End}(L/A) \).

2. The cohomology class \( \alpha_{L/A} \in H^1_{CE}(A, A^\perp \otimes \text{End}(L/A)) \) of the 1-cocycle \( R^\nabla_{1,1} \) does not depend on the choice of \( L \)-connections extending the Bott \( A \)-action.

We call \( R^\nabla_{1,1} \) the Atiyah cocycle associated with the \( L \)-connection \( \nabla \). Its cohomology class

\[
\alpha_{L/A} \in H^1_{CE}(A, A^\perp \otimes \text{End}(L/A)) = H^1_{CE}(A, B^\vee \otimes \text{End}(B))
\]

called the Atiyah class of the Lie pair.

Choosing a splitting \( i \circ p + j \circ q = \text{id}_L \) of the short exact sequence (5), we can identify \( \Lambda^2 L^\vee \) with the Whitney sum \( \Lambda^2 A^\vee \oplus (A^\vee \otimes B^\vee) \oplus \Lambda^2 B^\vee \).

The following lemma will be needed later on.

Lemma 1.7. Under the identification above, the curvature of \( \nabla \) decomposes as

\[
R^\nabla = R^\nabla_{1,1} + R^\nabla_{0,2}
\]

where \( R^\nabla_{1,1} \in \Gamma(\Lambda^2 L^\vee \otimes \text{End}(B)) \) denotes the skew-symmetrization of \( R^\nabla_{1,1} \in \Gamma(A^\vee \otimes B^\vee \otimes \text{End}(B)) \), and \( R^\nabla_{0,2} : \Lambda^2 L \to \text{End}(B) \) is the bundle map defined by

\[
R^\nabla_{0,2}(x, y) = R^\nabla(j \circ q(x), j \circ q(y)), \quad \forall x, y \in \Gamma(L).
\]

The Todd cocycle of a Lie pair \( (L, A) \) is the Chevalley-Eilenberg cocycle

\[
\nabla^\text{td}_{L/A} = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp),
\]

\[
\nabla^\text{td}_{L/A} = \det \left( \frac{R^\nabla_{1,1}}{e^{2R^\nabla_{1,1}} - e^{-2R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0} \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp).
\]
The Todd class of a Lie pair \((L, A)\) is the cohomology class
\[
Td_{L/A} = \det \left( \frac{\alpha_{L/A}}{1 - e^{-\alpha_{L/A}}} \right) \in \bigoplus_{k=0} H^k_{\text{CE}}(A, \Lambda^k A^\perp),
\]
\[
{\widetilde{Td}}_{L/A} = \det \left( \frac{\alpha_{L/A}}{e^2 \alpha_{L/A} - e^{-2\alpha_{L/A}}} \right) \in \bigoplus_{k=0} H^k_{\text{CE}}(A, \Lambda^k A^\perp).
\]

In the particular case of the Lie pair comprised of the Lie \(\mathbb{C}\)-algebroids \(L = T_X \otimes \mathbb{C}\) and \(A = T^{0,1}_X\) associated with a complex manifold \(X\), the quotient of the pair is \(T^{1,0}_X\) and the Atiyah class and the Todd class of the pair are the classical Atiyah class of \(T_X\) and the classical Todd class of the complex manifold \(X\).

### 1.3. Polydifferential operators.

The universal enveloping algebra \(\mathcal{U}(L)\) of the Lie algebroid \(L\) is a coalgebra over \(R\) [50]. Its comultiplication
\[
\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \otimes_R \mathcal{U}(L)
\]
is characterized by the identities
\[
\Delta(1) = 1 \otimes 1;
\]
\[
\Delta(x) = 1 \otimes x + x \otimes 1, \quad \forall x \in \Gamma(L);
\]
\[
\Delta(u \cdot v) = \Delta(u) \cdot \Delta(v), \quad \forall u, v \in \mathcal{U}(L),
\]
where \(1 \in R\) denotes the constant function on \(M\) with value 1 while the symbol \(\cdot\) denotes the multiplication in \(\mathcal{U}(L)\). We refer the reader to [50] for the precise meaning of the last equation above. Explicitly, we have
\[
\Delta(l_1 \cdot l_2 \cdots \cdot l_n) = 1 \otimes (l_1 \cdot l_2 \cdots \cdot l_n) + \sum_{p+q=n} \sum_{\sigma \in S_p} (l_{\sigma(1)} \cdots \cdot l_{\sigma(p)}) \otimes (l_{\sigma(p+1)} \cdots \cdot l_{\sigma(n)}) + (l_1 \cdot l_2 \cdots \cdot l_n) \otimes 1,
\]
for all \(l_1, \ldots, l_n \in \Gamma(L)\).

Let \((L, A)\) be a pair of Lie algebroids over \(\mathbb{k}\). Writing \(\mathcal{U}(L)\Gamma(A)\) for the left ideal of \(\mathcal{U}(L)\) generated by \(\Gamma(A)\), the quotient \(\mathcal{U}(L)\Gamma(A) / \mathcal{U}(L)\Gamma(A)\) is automatically an \(R\)-coalgebra since
\[
\Delta(\mathcal{U}(L) \Gamma(A)) \subseteq \mathcal{U}(L) \otimes_R (\mathcal{U}(L) \Gamma(A)) + (\mathcal{U}(L) \Gamma(A)) \otimes_R \mathcal{U}(L).
\]

Let \(\mathcal{D}^{-1}_{\text{poly}}\) denote the algebra of smooth functions on the manifold \(M\), let \(\mathcal{D}_0^0_{\text{poly}}\) denote the left \(\mathcal{U}(A)\)-module \(\frac{\mathcal{U}(L)}{\mathcal{U}(L) \Gamma(A)}\), let \(\mathcal{D}_0^k_{\text{poly}}\) denote the tensor product \(\mathcal{D}_0^0_{\text{poly}} \otimes_R \cdots \otimes_R \mathcal{D}_0^0_{\text{poly}}\) of \((k + 1)\) copies of the left \(R\)-module \(\mathcal{D}_0^0_{\text{poly}}\), and set \(\mathcal{D}_k_{\text{poly}} = \bigoplus_{k=0}^{\infty} \mathcal{D}_k^k_{\text{poly}}\). Since \(\mathcal{U}(A)\) is a Hopfalgobroid, it follows that for each \(k \geq -1\), \(\mathcal{D}_k_{\text{poly}}\) is also naturally a \(\mathcal{U}(A)\)-module [50].

#### Lemma 1.8.

The \(\mathcal{U}(A)\)-module \(\mathcal{D}_0^0_{\text{poly}}\) is a cocommutative coassociative coalgebra over \(R\) whose comultiplication \(\Delta : \mathcal{D}_0^0_{\text{poly}} \to \mathcal{D}_0^0_{\text{poly}} \otimes_R \mathcal{D}_0^0_{\text{poly}}\) is a morphism of \(\mathcal{U}(A)\)-modules.

Since the comultiplication \(\Delta\) is coassociative, the Hochschild operator \(d_{\mathcal{H}} : \mathcal{D}_{k+1}^k_{\text{poly}} \to \mathcal{D}_k^k_{\text{poly}}\) defined by
\[
d_{\mathcal{H}}(u_1 \otimes \cdots \otimes u_k) = 1 \otimes u_1 \otimes \cdots \otimes u_k + \sum_{i=1}^k (-1)^i u_1 \otimes \cdots \otimes \Delta(u_i) \otimes \cdots \otimes u_k
\]
\[
+ (-1)^{k+1} u_1 \otimes \cdots \otimes u_k \otimes 1,
\]
for all \(u_1, u_2, \ldots, u_k \in \mathcal{D}_0^0_{\text{poly}}\), is a coboundary operator, i.e. \(d_{\mathcal{H}}^2 = 0\).
Moreover, \( d_{\text{gr}} : \mathcal{D}^{-1}_{\text{poly}} \to \mathcal{D}^0_{\text{poly}} \) is a morphism of \( \mathcal{U}(A) \)-modules, since the comultiplication \( \Delta : \mathcal{D}^0_{\text{poly}} \to \mathcal{D}^0_{\text{poly}} \otimes_R \mathcal{D}^0_{\text{poly}} \) is a morphism of \( \mathcal{U}(A) \)-modules. Therefore, the Hochschild complex
\[
0 \longrightarrow \mathcal{D}^{-1}_{\text{poly}} \overset{d_{\text{gr}}}{\longrightarrow} \mathcal{D}^0_{\text{poly}} \overset{d_{\text{gr}}}{\longrightarrow} \mathcal{D}^1_{\text{poly}} \overset{d_{\text{gr}}}{\longrightarrow} \mathcal{D}^2_{\text{poly}} \overset{d_{\text{gr}}}{\longrightarrow} \cdots
\]
is a complex of \( \mathcal{U}(A) \)-modules.

The Chevalley–Eilenberg cohomology in degree \( k \) of the Hochschild complex of the pair \((L, A)\), which is defined as
\[
\mathbb{H}^k_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}}) := \text{Hom}_{D^+(\mathfrak{g})}(R, \mathcal{D}^\bullet_{\text{poly}}[k]),
\]
can be computed as the degree \( k \) hypercohomology of the double complex
\[
\begin{array}{cccc}
\vdots & & & \\
\text{id} \otimes d_{\text{gr}} & \uparrow & \text{id} \otimes d_{\text{gr}} & \\
\Gamma(\Lambda^0 A^\vee) \otimes_R \mathcal{D}^0_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \Gamma(\Lambda^1 A^\vee) \otimes_R \mathcal{D}^0_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \Gamma(\Lambda^2 A^\vee) \otimes_R \mathcal{D}^0_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \cdots \\
\text{id} \otimes d_{\text{gr}} & \uparrow & \text{id} \otimes d_{\text{gr}} & & & & \\
\Gamma(\Lambda^0 A^\vee) \otimes_R \mathcal{D}^{-1}_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \Gamma(\Lambda^1 A^\vee) \otimes_R \mathcal{D}^{-1}_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \Gamma(\Lambda^2 A^\vee) \otimes_R \mathcal{D}^{-1}_{\text{poly}} & \overset{d^{\mathcal{D}}_A}{\longrightarrow} & \cdots
\end{array}
\]

The coboundary operator \( d^{\mathcal{D}}_A : \Gamma(\Lambda^p A^\vee) \otimes \mathcal{D}^q_{\text{poly}} \to \Gamma(\Lambda^{p+1} A^\vee) \otimes \mathcal{D}^q_{\text{poly}} \) is defined by
\[
d^{\mathcal{D}}_A(\omega \otimes u_0 \otimes \cdots \otimes u_q) = (d_A \omega) \otimes u_0 \otimes \cdots \otimes u_q + \sum_{j=1}^{\text{rk}(A)} \sum_{k=0}^{q} (\alpha_j \wedge \omega) \otimes u_0 \otimes \cdots \otimes u_{k-1} \otimes \alpha_j \cdot u_k \otimes u_{k+1} \otimes \cdots \otimes u_q,
\]
for all \( \omega \in \Gamma(\Lambda^p A^\vee) \) and \( u_1, u_2, \ldots, u_k \in \mathcal{D}^0_{\text{poly}} \). Here \( (\alpha_i)_{i \in \{1, \ldots, r\}} \) designates any local frame of \( A \) and \( (\alpha_j)_{j \in \{1, \ldots, r\}} \) the corresponding dual local frame of \( A^\vee \).

**Lemma 1.9.** For any Lie pair \((L, A)\), the Hochschild cohomology \( \mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}}) \) is an associative algebra, whose multiplication stems from the tensor product of left \( R \)-modules \( \otimes_R \) in \( \mathcal{D}^\bullet_{\text{poly}} \).

**Remark 1.10.** Since, unlike the universal enveloping algebra \( \mathcal{U}(L) \) of a Lie algebroid \( L \), the space \( \mathcal{D}^0_{\text{poly}} \) is not a Hopf algebroid, the Gerstenhaber bracket \( (20) \) does not extend to \( \mathcal{D}^\bullet_{\text{poly}} \) (and \( \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \)). Therefore, unlike \( \mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{U}(L) \otimes_R \mathcal{D}^\bullet_{\text{poly}}) \), the Hochschild cohomology \( \mathbb{H}^\bullet_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}}) \) does not, a priori, admit a Gerstenhaber algebra structure. However, it turns out that \( \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \) does actually admit an \( L_\infty \) structure and that its cohomology admits a Gerstenhaber algebra structure. These arise from what we call a Fedosov dg Lie algebroid associated with the Lie pair \((L, A)\) — see Corollary 4.19.

### 1.4. Polyvector fields

Given a Lie pair \((L, A)\), let \( \mathcal{T}^{-1}_{\text{poly}} \) denote the algebra \( R \) of smooth functions on the manifold \( M \), set \( \mathcal{T}^k_{\text{poly}} := \Gamma(\Lambda^{k+1}(L/A)) \) for \( k \geq 0 \), and consider \( \mathcal{T}^\bullet_{\text{poly}} = \bigoplus_{k=-1}^\infty \mathcal{T}^k_{\text{poly}} \) as a complex of \( \mathcal{U}(A) \)-modules with trivial differential:
\[
0 \longrightarrow \mathcal{T}^{-1}_{\text{poly}} \overset{0}{\longrightarrow} \mathcal{T}^0_{\text{poly}} \overset{0}{\longrightarrow} \mathcal{T}^1_{\text{poly}} \overset{0}{\longrightarrow} \mathcal{T}^2_{\text{poly}} \overset{0}{\longrightarrow} \cdots
\]

Its Chevalley–Eilenberg cohomology in degree \( k \)
\[
\mathbb{H}^k_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}}) := \text{Hom}_{D^+(\mathfrak{g})}(R, \mathcal{T}^\bullet_{\text{poly}}[k])
\]
is computed as the degree $k$ hypercohomology of the double complex

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \\
\Gamma(\Lambda^0 A^\vee) \otimes_R T^0_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \Gamma(\Lambda^1 A^\vee) \otimes_R T^0_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \Gamma(\Lambda^2 A^\vee) \otimes_R T^0_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \cdots \\
0 & 0 & 0 & \\
\Gamma(\Lambda^0 A^\vee) \otimes_R T^{-1}_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \Gamma(\Lambda^1 A^\vee) \otimes_R T^{-1}_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \Gamma(\Lambda^2 A^\vee) \otimes_R T^{-1}_{\text{poly}} & \xrightarrow{d_{\text{Bott}}} & \cdots 
\end{array}
\]  

(7)

The coboundary operator $d^\text{Bott}_A : \Gamma(\Lambda^p A^\vee) \otimes T^q_{\text{poly}} \to \Gamma(\Lambda^{p+1} A^\vee) \otimes T^q_{\text{poly}}$ is defined by

\[
d^\text{Bott}_A(\omega \otimes b_0 \wedge \cdots \wedge b_q) = (d_A \omega) \otimes b_0 \wedge \cdots \wedge b_q + \sum_{j=1}^{\text{rk}(A)} \sum_{k=0}^{q} (\alpha_j \wedge \omega) \otimes b_0 \wedge \cdots \wedge b_{k-1} \wedge \nabla^\text{Bott}_{a_j} b_k \wedge b_{k+1} \wedge \cdots \wedge b_q,
\]

for all $\omega \in \Gamma(\Lambda^p A^\vee)$ and $b_0, b_1, \ldots, b_q \in \Gamma(L/A)$. Here $(a_i)_{i \in \{1, \ldots, r\}}$ designates any local frame of $A$ and $(\alpha_j)_{j \in \{1, \ldots, r\}}$ the corresponding dual local frame of $A^\vee$.

In fact, the coboundary operator $d^\text{Bott}_A$ extends to an $L_\infty$ structure on $\Gamma(\Lambda^\bullet A^\vee) \otimes T^\bullet_{\text{poly}}$ — see Corollary 4.14.

Again, a priori, $\mathbb{H}^\bullet_{\text{CE}}(A, T^\bullet_{\text{poly}})$ is only a vector space. The following lemma, however, is obvious.

**Lemma 1.11.** For any Lie pair $(L, A)$, $\mathbb{H}^\bullet_{\text{CE}}(A, T^\bullet_{\text{poly}})$ is an associative algebra, whose multiplication stems from the wedge product on $T^\bullet_{\text{poly}}$.

1.5. **Hochschild–Kostant–Rosenberg isomorphism.** The natural inclusion $\Gamma(L/A) \hookrightarrow D^0_{\text{poly}}$ extends to a morphism of complexes of $\mathcal{U}(A)$-modules

$$\text{hkr} : T^\bullet_{\text{poly}} \to D^\bullet_{\text{poly}}$$

by skew-symmetrization:

$$\text{hkr}(b_1 \wedge \cdots \wedge b_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{\sigma(1)} \otimes b_{\sigma(2)} \otimes \cdots \otimes b_{\sigma(n)}, \quad \forall b_1, \ldots, b_n \in \Gamma(L/A).$$

Furthermore, hkr induces a morphism of double complexes:

**Lemma 1.12.** The map

$$\text{id} \otimes \text{hkr} : (\Gamma(\Lambda^\bullet A^\vee) \otimes_R T^\bullet_{\text{poly}}, d^\text{Bott}_A, 0) \to (\Gamma(\Lambda^\bullet A^\vee) \otimes_R D^\bullet_{\text{poly}}, d_A^\text{Bott}, \pm \text{id} \otimes d_{\text{poly}})$$

is a morphism of double complexes from (7) to (6).

The induced map between total cohomologies is called *Hochschild–Kostant–Rosenberg map*. Abusing notations, we will denote it hkr instead of id $\otimes$ hkr.

**Proposition 1.13.** For any Lie pair $(L, A)$, the Hochschild–Kostant–Rosenberg map

$$\text{hkr} : \mathbb{H}^\bullet_{\text{CE}}(A, T^\bullet_{\text{poly}}) \to \mathbb{H}^\bullet_{\text{CE}}(A, D^\bullet_{\text{poly}})$$

is an isomorphism.

Proposition 1.13 can be proved by a spectral sequence argument. Repeating the argument in [22, Theorem 4.10], one can prove the following lemma:
Lemma 1.14. For each \( p \), the map

\[
\text{id} \otimes \text{hkr} : \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{T}_{\text{poly}}^\bullet, 0 \right) \to \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, \pm \text{id} \otimes d_{\mathcal{H}} \right)
\]

is a quasi-isomorphism.

**Proof of Proposition 1.13.** Consider the spectral sequences associated with the filtrations

\[
F^k \left( \bigoplus_{p \geq 0, q \geq -1} \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{T}_{\text{poly}}^q \right) \right) = \bigoplus_{p \geq k, q \geq -1} \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{T}_{\text{poly}}^q \right)
\]

\[
F^k \left( \bigoplus_{p \geq 0, q \geq -1} \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{D}_{\text{poly}}^q \right) \right) = \bigoplus_{p \geq k, q \geq -1} \left( \Gamma(\Lambda^p \mathcal{A}^V) \otimes_R \mathcal{D}_{\text{poly}}^q \right)
\]

on the double complexes (7) and (6). The map induced by \( \text{id} \otimes \text{hkr} \) between the \( E_0 \)-terms of these two spectral sequences is precisely the quasi-isomorphism of Lemma 1.14. Therefore \( \text{id} \otimes \text{hkr} \) induces isomorphism between the \( E_k \)-terms of the two spectral sequences for each \( k \) larger than or equal to 1. Since both filtrations are complete and exhaustive, it follows from the Eilenberg–Moore comparison theorem that \( \text{hkr} : \mathbb{H}_{\text{CE}}^\bullet(A, \mathcal{T}_{\text{poly}}^\bullet) \to \mathbb{H}_{\text{CE}}^\bullet(A, \mathcal{D}_{\text{poly}}^\bullet) \) is an isomorphism. \( \square \)

1.6. Atiyah and Todd cocycles/classes of a dg Lie algebroid. A \( \mathbb{Z} \)-graded manifold \( \mathcal{M} \) with base manifold \( M \) is a sheaf \( \mathcal{R} \) of \( \mathbb{Z} \)-graded, graded-commutative algebras \( \{ \mathcal{R}_U \mid U \subset M \text{ open} \} \) over \( M \) isomorphic to \( C^\infty(U) \otimes S(V^v) \) for sufficiently small open subsets \( U \) of \( M \). Here \( S(V^v) \) denotes the graded algebra of formal polynomials on some \( \mathbb{Z} \)-graded vector space \( V \). The underlying local \( \mathbb{Z} \)-graded manifold is normally denoted \( U \times V_{\text{formal}} \). By \( C^\infty(\mathcal{M}) \), we denote the \( \mathbb{Z} \)-graded, graded-commutative algebra of global sections of \( \mathcal{R} \). A dg manifold is a \( \mathbb{Z} \)-graded manifold endowed with a homological vector field, i.e. a vector field \( Q \) of degree +1 satisfying \( [Q, Q] = 0 \). For instance, if \( A \) is a Lie \( \mathbb{K} \)-algebroid, then \( A[1] \) is a dg manifold with the Chevalley–Eilenberg differential \( d_{\text{CE}} \) as homological vector field. According to Vařantb [48], there is a bijection between Lie algebroid structures on a vector bundle \( A \to M \) and homological vector fields on the \( \mathbb{Z} \)-graded manifold \( A[1] \).

A dg vector bundle \([35, 23]\) is a vector bundle in the category of dg manifolds. Given a vector bundle \( \mathcal{E} \to \mathcal{M} \) of graded manifolds, its space of sections, denoted \( \Gamma(\mathcal{E}) \), is defined to be \( \bigoplus_{j \in \mathbb{Z}} \Gamma(\mathcal{E})_j \), where \( \Gamma(\mathcal{E})_j \) consists of sections of degree \( j \), i.e. maps \( s \in \text{Hom}(\mathcal{M}, \mathcal{E}[-j]) \) such that \( (\pi[-j]) \circ s = \text{id}_\mathcal{M} \). Here \( \pi[-j] : \mathcal{E}[-j] \to \mathcal{M} \) is the natural map induced by \( \pi \); see [35] for more details. When \( \mathcal{E} \to \mathcal{M} \) is a dg vector bundle, the homological vector fields \( Q^\mathcal{E} \) and \( Q^\mathcal{M} \) on \( \mathcal{E} \) and \( \mathcal{M} \) naturally determine an operator \( Q \) of degree +1 on \( \Gamma(\mathcal{E}) \), making \( \Gamma(\mathcal{E}) \) a dg module over the dg algebra \( C^\infty(\mathcal{M}) \). Indeed, \( \mathcal{E} \to \mathcal{M} \) is a graded vector bundle if and only if “the vector field \( Q^\mathcal{E} \) projects onto \( Q^\mathcal{M} \)” i.e.

\[
Q^\mathcal{E}(\pi^*(f)) = \pi^*(Q^\mathcal{M}(f)), \quad \forall f \in C^\infty(\mathcal{M})
\]

and “the flow of \( Q^\mathcal{E} \) preserves the linear structure of the fibers of \( \pi \)” i.e. the submodule \( \Gamma(\mathcal{E}^V) \) of \( C^\infty(\mathcal{E}) \) comprised of all smooth functions on \( \mathcal{E} \) “linear along the fibers of \( \pi \)” is stable under the derivation \( Q^\mathcal{E} \). The restriction of \( Q^\mathcal{E} \) to \( \Gamma(\mathcal{E}^V) \) determines an operator \( Q \) on \( \Gamma(\mathcal{E}) \) through the relation

\[
Q^\mathcal{M}(\langle \zeta | e \rangle) = \langle Q^\mathcal{E}(\zeta) | e \rangle + (-1)^{|\zeta|} \langle \zeta | Q^\mathcal{E}(e) \rangle, \quad \forall \zeta \in \Gamma(\mathcal{E}^V), e \in \Gamma(\mathcal{E}).
\]

Since \( \pi^*(C^\infty(\mathcal{M})) \) and \( \Gamma(\mathcal{E}^V) \) together generate the algebra \( C^\infty(\mathcal{E}) \) multiplicatively, knowledge of the vector field \( Q^\mathcal{M} \) and the operator \( Q \) suffices to recover the homological vector field \( Q^\mathcal{E} \).

In this case, the degree +1 operator \( Q \) on \( \Gamma(\mathcal{E}) \) gives rise to a cochain complex

\[
\cdots \to \Gamma(\mathcal{E})_1 \xrightarrow{Q} \Gamma(\mathcal{E})_{i+1} \to \cdots,
\]

whose cohomology group will be denoted by \( H^\bullet(\Gamma(\mathcal{E}), Q) \).
A dg Lie algebroid is a Lie algebroid object in the category of dg manifolds. For more details, we refer the reader to [35, 34], where dg Lie algebroids are called $Q$-algebroids. It is simple to see that if $\mathcal{M}$ is a dg manifold, then $T_{\mathcal{M}}$ is naturally a dg Lie algebroid.

The notion of Atiyah class of dg Lie algebroids was introduced and studied by Mehta–Stiénon–Xu [36]. It extends the notion of Atiyah class of a dg manifold, which was first investigated by Shoikhet [42] in relation with Kontsevich’s formality theorem and Duflo’s formula.

Let $\mathcal{A}$ be a dg Lie algebroid with anchor $\rho: \mathcal{A} \to T_{\mathcal{M}}$. An $\mathcal{A}$-connection on $\mathcal{A}$ is a map

$$\nabla : \Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$$

of degree 0 satisfying

$$\nabla_{fX}Y = f\nabla_XY$$

$$\nabla_X(fY) = (\rho(X)f)Y + (-1)^{|X||f|}f\nabla_XY$$

for all $f \in C^\infty(\mathcal{M})$ and all (homogeneous) $X, Y \in \Gamma(\mathcal{A})$. The notation $|\cdot|$ is used to denote the degree of the argument. Here the degree of $\nabla$ is its degree as a map from $\Gamma(\mathcal{A}) \otimes \Gamma(\mathcal{A})$ to $\Gamma(\mathcal{A})$. More precisely, by saying that $\nabla$ is a map of degree 0 we mean that $|\nabla_XY| = |X| + |Y|$ for every pair of homogeneous elements $X$ and $Y$. Connections always exist since the standard partition of unity argument holds in the context of graded manifolds.

Given a dg Lie algebroid $\mathcal{A}$, the associated operator $Q$ of degree $+1$ on $\Gamma(\mathcal{A})$, and an $\mathcal{A}$-connection $\nabla$ on $\mathcal{A}$, one defines a bundle map $At_{\mathcal{A}}^\nabla : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ of degree $+1$ by

$$At_{\mathcal{A}}^\nabla(X, Y) := Q(\nabla_XY) - \nabla_{Q(X)}Y - (-1)^{|X||Y|}\nabla_X(Q(Y)), \quad \forall X, Y \in \Gamma(\mathcal{A}).$$

Alternatively, we may think of $At_{\mathcal{A}}^\nabla$ as a section of degree $+1$ in $\Gamma(\mathcal{A}^\vee \otimes \text{End} \mathcal{A})$. It is immediate that $Q(At_{\mathcal{A}}^\nabla) = 0$. Since $Q^2 = 0$, we may thus regard $At_{\mathcal{A}}^\nabla$ as a 1-cocycle in the cochain complex $(\Gamma(\mathcal{A}^\vee \otimes \text{End} \mathcal{A}), Q)$.

**Definition 1.15.** The 1-cocycle $At_{\mathcal{A}}^\nabla \in Z^1(\Gamma(\mathcal{A}^\vee \otimes \text{End} \mathcal{A}), Q)$ is called the Atiyah 1-cocycle of the dg Lie algebroid $\mathcal{A}$ with respect to the $\mathcal{A}$-connection $\nabla$ on $\mathcal{A}$.

It is simple to check that its cohomology class $\alpha_{\mathcal{A}} := [At_{\mathcal{A}}^\nabla] \in H^1(\Gamma(\mathcal{A}^\vee \otimes \text{End} \mathcal{A}), Q)$ is independent of the choice of the connection $\nabla$. The class $\alpha_{\mathcal{A}}$ is called the Atiyah class of the dg Lie algebroid $\mathcal{A}$. It is the obstruction class to the existence of a dg compatible $\mathcal{A}$-connection on $\mathcal{A}$. (See [36] for more details.)

The **Todd cocycle** $\text{td}_{\mathcal{A}}$ (or $\tilde{\text{td}}_{\mathcal{A}}$) and **Todd class** $\text{Td}_A$ (or $\tilde{\text{Td}}_A$) of a dg Lie algebroid $\mathcal{A}$ are defined as follows:

$$\text{td}_{\mathcal{A}} := \text{Ber} \left( \frac{At_{\mathcal{A}}^\nabla}{1 - e^{-At_{\mathcal{A}}^\nabla}} \right) \in \prod_{k \geq 0} \Gamma(\Lambda^k \mathcal{A}^\vee)_k,$$

$$\tilde{\text{td}}_{\mathcal{A}} := \text{Ber} \left( \frac{At_{\mathcal{A}}^\nabla}{e^{\frac{1}{2} At_{\mathcal{A}}^\nabla} - e^{-\frac{1}{2} At_{\mathcal{A}}^\nabla}} \right) \in \prod_{k \geq 0} \Gamma(\Lambda^k \mathcal{A}^\vee)_k,$$

$$\text{Td}_A := \text{Ber} \left( \frac{\alpha_{\mathcal{A}}}{1 - e^{-\alpha_{\mathcal{A}}}} \right) \in \prod_{k \geq 0} H^k(\Gamma(\Lambda^k \mathcal{A}^\vee), Q),$$

$$\tilde{\text{Td}}_A := \text{Ber} \left( \frac{\alpha_{\mathcal{A}}}{e^{\frac{1}{2} \alpha_{\mathcal{A}}} - e^{-\frac{1}{2} \alpha_{\mathcal{A}}}} \right) \in \prod_{k \geq 0} H^k(\Gamma(\Lambda^k \mathcal{A}^\vee), Q),$$

where $\Lambda^k A$ denotes the dg vector bundle $S^k(\mathcal{A}^\vee[-1])[k] \to \mathcal{M}$. The definition of the Berezinian $\text{Ber}$ can be found in [13]. It is known that $\text{Td}_A$ and $\tilde{\text{Td}}_A$ can be expressed in terms of the scalar Atiyah classes $c_k := \frac{1}{k!}(\frac{i}{2\pi})^k \text{str} \alpha_{\mathcal{A}}^k \in H^k(\Gamma(\Lambda^k \mathcal{A}^\vee), Q)$, where $\text{str} : \text{End} \mathcal{A} \to C^\infty(\mathcal{M})$ denotes the supertrace and $\text{str} \alpha_{\mathcal{A}}^k \in \Gamma(\Lambda^k \mathcal{A}^\vee)$ since $\alpha_{\mathcal{A}}^k \in \Gamma(\Lambda^k \mathcal{A}^\vee) \otimes C^\infty(\mathcal{M}) \text{End} \mathcal{A}$. 


Example 1.16. [36, Example 3.4] Consider the tangent dg Lie algebroid $T_M$ of a dg manifold $M = (V, Q)$, where $V = \mathbb{R}^m \times V_{formal}$ for some finite dimensional $\mathbb{Z}$-graded vector space $V$ over $k$. The algebra of functions on the graded manifold $V$ is $C^\infty(V) = C^\infty(\mathbb{R}^m) \otimes \tilde{S}(V^\vee)$. Let $(z_1, \ldots, z_N)$ be a choice of coordinate functions on $V$. Writing the homological vector field $Q$ as $Q = \sum_k Q_k \frac{\partial}{\partial z_k}$, the Atiyah 1-cocycle associated with the trivial connection $\nabla$ on $M$ is the determinant and the Todd cocycle $\lambda$.

Indeed, $\det$ and $\lambda$ are local frames for $M$ and $L$, respectively.

$$\lambda = \det \left( \frac{\partial}{\partial z_i} \cdot \frac{\partial}{\partial z_j} \right) = (-1)^{|z_i|+|z_j|} \sum_k \frac{\partial^2 Q_k}{\partial z_i \partial z_j} \frac{\partial}{\partial z_k}.$$  

Hence the Atiyah 1-cocycle $\lambda$ captures the second- and higher-order information contained in the homological vector field.

In this paper, we are particularly interested in an important class of dg Lie algebroids, namely the Fedosov dg Lie algebroids associated with a Lie pair $(L, A)$. See the Appendix or [45] for more details.

1.7. Atiyah and Todd cocycles/classes of the Fedosov dg Lie algebroid. Let $(L, A)$ be a Lie pair over a smooth manifold $M$. Each choice of (1) a splitting of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and (2) a torsion-free $L$-connection on $B$ determines a homological vector field $Q$ on the graded manifold $M = L[1] \oplus B$ — see Theorem 4.7 in the Appendix. Any such dg manifold $(M, Q)$ is called a Fedosov dg manifold associated with the Lie pair $(L, A)$. The pullback $F \to M$ of the canonical projection $M \to M$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_M \to M$ — see Proposition 4.9 in the Appendix. Any such dg Lie algebroid $F \to M$ is called a Fedosov dg Lie algebroid associated with the Lie pair $(L, A)$.

Since $F \to M$ is both the pullback of the vector bundle $B \to M$ through the canonical map $M \to M$ and a vector subbundle of $T_M \to M$, we have the inclusions

$$\Gamma(B) \hookrightarrow C^\infty(M) \otimes C^\infty(M) \Gamma(B) \xrightarrow{\otimes} \Gamma(F \to M) \hookrightarrow \mathfrak{X}(M)$$

Indeed, $\mathfrak{X}(F \to M)$ is the $C^\infty(M)$-submodule of $\mathfrak{X}(M)$ generated by $\Gamma(B)$. In particular, if $\partial_1, \ldots, \partial_r$ is a local frame for $B$ and $\chi_1, \ldots, \chi_r$ is the dual local frame for $B^\vee$, then $\partial_k$ is the vector field $\frac{\partial}{\partial \chi_k}$ on $M$, i.e. the derivation $\lambda \otimes \chi^M \mapsto \lambda \otimes M_k \chi^{M-e_k}$ of $C^\infty(M) \cong \Gamma(\Lambda L^\vee \otimes \tilde{S}(B^\vee))$.

There exists a canonical $F$-connection on $F$ characterized by the relation

$$\nabla^\text{can}_b \xi = 0, \quad \forall b, c \in \Gamma(B).$$

Definition 1.17. The Atiyah 1-cocycle $\text{At}_F \in Z^1(\Gamma(M; F^\vee \otimes \text{End} F), Q)$ corresponding to the canonical connection $\nabla^\text{can}$ is called the canonical Atiyah 1-cocycle.

Since $\Gamma(M; F^\vee \otimes \text{End} F)$ can be identified with $\Gamma(\Lambda^1 L^\vee \otimes \tilde{S}B^\vee \otimes B^\vee \otimes \text{End} B)$, the canonical Atiyah cocycle is essentially the tensor product of an endomorphism of $B$ and an element of $\Gamma(\Lambda^1 L^\vee \otimes \tilde{S}B^\vee \otimes \Lambda^1 B^\vee)$. Therefore, the Berezinian appearing in the expression for the Todd cocycle of $\nabla^\text{can}$ is simply the classical determinant and the canonical Todd cocycle is

$$\text{td}^\text{can}_F := \det \left( \frac{\text{At}_F}{1 - e^{-\text{At}_F}} \right) \in \prod_{k \geq 0} \Gamma(\Lambda^k L^\vee \otimes \tilde{S}B^\vee \otimes \Lambda^k B^\vee).$$  

Similarly,

$$\tilde{\text{td}}^\text{can}_F := \det \left( \frac{\text{At}_F}{e^{\frac{1}{2} \text{At}_F} - e^{-\frac{1}{2} \text{At}_F}} \right) \in \prod_{k \geq 0} \Gamma(\Lambda^k L^\vee \otimes \tilde{S}B^\vee \otimes \Lambda^k B^\vee).$$  

Lemma 1.18. Given any local frame $\partial_1, \ldots, \partial_r$ for $B$, the canonical Atiyah $1$-cocycle $A^\text{can}_F : F \otimes F \to F$ of the Fedosov dg Lie algebroid $F \to M$ admits the local expression
\begin{equation}
A^\text{can}_F(\hat{\partial}_i, \hat{\partial}_j) = \sum_{k=1}^r \hat{\partial}_i(\hat{\partial}_j f_k) \cdot \hat{\partial}_k,
\end{equation}
where the functions $f_k \in C^\infty(M)$ are the components of the vector field $X^\nabla = \sum_{k=1}^r f_k \cdot \hat{\partial}_k$ of Theorem 4.7 relative to the chosen frame.

Proof. Recall that the coboundary operator $\delta$ relative to the chosen frame.

Proposition 1.19. The quasi-isomorphism
\[\sigma_\nabla : \left(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R T_{\text{poly}}^{r,s}(B, LQ)\right) \to \left(\Gamma(\Lambda^\bullet A^\nabla) \otimes_R T_{\text{poly}}^{r,s}(A, d_A^\text{Bott})\right)\]
of Proposition 4.11 takes $A^\text{can}_F$, $td^\text{can}_F$, and $\tilde{td}^\text{can}_F$ to $R^\nabla_{1,1}$, $td^\nabla_{L/A}$, and $\tilde{td}^\nabla_{L/A}$, respectively:
\[\sigma_\nabla(A^\text{can}_F) = R^\nabla_{1,1}, \quad \sigma_\nabla(td^\text{can}_F) = td^\nabla_{L/A}, \quad \sigma_\nabla(\tilde{td}^\text{can}_F) = \tilde{td}^\nabla_{L/A}.\]

Proof. It suffices to prove the first statement; the second and third statements are immediate consequences of the first. Lemma 1.18, together with the natural identification of $\Gamma(M; F^\nabla \otimes \text{End} F)_1$ with $C^\infty(M) \otimes C^\infty(M)$, $\Gamma(B^\nabla \otimes B^\nabla \otimes B)$, implies that
\[A^\text{can}_F = \sum_{i,j,k=1}^r \hat{\partial}_i(\hat{\partial}_j f_k) \otimes (\chi_i \otimes \chi_j \otimes \hat{\partial}_k),\]
where $\hat{\partial}_i(\hat{\partial}_j f_k) \in C^\infty(M) = \Gamma(L^\nabla \otimes \hat{S}(B^\nabla))$ and $\chi_i \otimes \chi_j \otimes \hat{\partial}_k \in \Gamma(B^\nabla \otimes B^\nabla \otimes B)$.

According to Theorem 4.7, we have $X^\nabla = \sum_{i=1}^2 \sum_{t=0}^\infty f_i^{(t)}(\hat{\partial}_k)$ with $f_i^{(t)} \in \Gamma(L^\nabla \otimes S^t(B^\nabla))$. Moreover,
\[X_2 = \sum_{k=1}^r f_k^{(2)} \otimes \hat{\partial}_k\]
decomposes as the sum of $h_k(R^\nabla_{0,2}) = \sum_k f_k^{(1,1)} \otimes \hat{\partial}_k$ and $h_k(R^\nabla_{0,2}) = \sum_k f_k^{(0,2)} \otimes \hat{\partial}_k$, where $f_k^{(1,1)} \in \Gamma(p^\nabla(A^\nabla) \otimes S^2B^\nabla)$ and $f_k^{(0,2)} \in \Gamma(q^\nabla(B^\nabla) \otimes S^2B^\nabla)$. 

Since $\hat{d}_i(\hat{d}_j f_k^{(t)}) \in \Gamma(L^t \otimes S^{t-2}B^\vee)$, we have $\sigma(\hat{d}_i(\hat{d}_j f_k^{(t)})) = 0$ for $t \geq 3$. Since $\hat{d}_i(\hat{d}_j f_k^{(0,2)}) \in \Gamma(q^\top(B^\vee))$, we also have $\sigma(\hat{d}_i(\hat{d}_j f_k^{(0,2)})) = 0$.

Therefore, we obtain

$$\sigma_2(\text{At}^{\text{can}}_\mathcal{F}) = \sigma_2 \left( \sum_{i,j,k=1}^r \hat{d}_i(\hat{d}_j f_k) \otimes (\chi_i \otimes \chi_j \otimes \partial_k) \right)$$

$$= \sum_{i,j,k=1}^r \sigma \left( \sum_{t=2}^\infty \hat{d}_i(\hat{d}_j f_k^{(t)}) \right) \otimes (\chi_i \otimes \chi_j \otimes \partial_k)$$

$$= \sum_{i,j,k=1}^r \hat{d}_i(\hat{d}_j f_k^{(1,1)}) \otimes (\chi_i \otimes \chi_j \otimes \partial_k)$$

$$= 2 \sum_{k=1}^r f_k^{(1,1)} \otimes \partial_k$$

$$= 2 h_{i_2}(R_{1,1}^{\triangledown})$$

Corollary 1.20. (1) The isomorphism $\sigma_2 : H^1(\Gamma(\mathcal{M}; F^\vee \otimes \text{End} \mathcal{F}), \mathbb{Q}) \to H^1_{\text{CE}}(A, B^\vee \otimes \text{End} B)$ induced by the quasi-isomorphism of Proposition 4.11 (for $r = 2$ and $s = 1$) takes the Atiyah class $\alpha_\mathcal{F}$ of the Fedosov dg Lie algebroid $\mathcal{F}$ to the Atiyah class $\alpha_{L/A}$ of the Lie pair $(L, A)$:

$$\sigma_2(\alpha_\mathcal{F}) = \alpha_{L/A}.$$

(2) The isomorphism $\sigma_2 : H^k(\Gamma(\mathcal{M}; \Lambda^k F^\vee), \mathbb{Q}) \to H^k_{\text{CE}}(A, \Lambda^k B^\vee)$ induced by the quasi-isomorphism of Theorem 4.14 takes the Todd class of the Fedosov dg Lie algebroid $\mathcal{F}$ to the Todd class of the Lie pair $(L, A)$:

$$\sigma_2(\text{Td}_\mathcal{F}) = \text{Td}_{L/A}, \quad \sigma_2(\text{Td}_\mathcal{F}) = \text{Td}_{L/A}.$$

2. Formality theorem for Lie pairs

2.1. Statements of main theorems. We are ready to state the main theorems of the paper. Let $(L, A)$ be a Lie pair. According to Corollary 4.14 and Corollary 4.19, $\text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}})$ and $\text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$ are $L_\infty$ algebras with $d_A^{\text{Batt}}$ and $d_A^H + d_{\mathcal{F}}$ as their respective unary brackets. Moreover, their cohomologies $\mathbb{H}^*_{\text{CE}}(A, \mathcal{T}^\bullet_{\text{poly}})$ and $\mathbb{H}^*_{\text{CE}}(A, \mathcal{D}^\bullet_{\text{poly}})$ carry canonical Gerstenhaber algebra structures. The main result of the paper is the following

**Theorem 2.1** (Formality theorem for Lie pairs). Let $(L, A)$ be a Lie pair. Assume the associated graded vector spaces $\text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}})$ and $\text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$ are endowed with their inherited $L_\infty$ algebras — see Corollaries 4.14 and 4.19. Then, there exists an $L_\infty$ quasi-isomorphism

$$\mathcal{I} : \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}) \to \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$$

with first Taylor coefficient $\mathcal{I}_1 : \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}) \to \text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}})$ satisfying the following two properties:

(1) $\mathcal{I}_1$ preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;

(2) $\mathcal{I}_1 = \text{hkr} \circ (\text{td}_{L/A}^{\triangledown})^{\frac{1}{2}},$ where $(\text{td}_{L/A}^{\triangledown})^{\frac{1}{2}} \in \bigoplus_{k=0}^\infty \Gamma(\Lambda^k A^\vee \otimes \Lambda^k A^\perp)$ acts on $\text{tot}(\Gamma(\Lambda^* A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}})$ by contraction.

As an immediate consequence, we have the following
Theorem 2.2 (Kontsevich-Duflo type theorem for Lie pairs). Given a Lie pair \((L, A)\), the map
\[
hkr \circ \text{Td}_{L/A}^{\frac{1}{2}} : \mathbb{H}_{\mathcal{C}E}^\bullet(A, \mathcal{T}_{\text{poly}}) \to \mathbb{H}_{\mathcal{C}E}^\bullet(A, \mathcal{D}_{\text{poly}}^\bullet)
\]
is an isomorphism of Gerstenhaber algebras — the square root of the Todd class
\[
\text{Td}_{L/A}^{\frac{1}{2}} \in \bigoplus_{k=0} \mathcal{H}_{\mathcal{C}E}^k(A, \Lambda^k A^\perp)
\]
acts on \(\mathbb{H}_{\mathcal{C}E}^\bullet(A, \mathcal{T}_{\text{poly}})\) by contraction.

To prove Theorem 2.1, our method is to use Fedosov dg Lie algebroids \(\mathcal{F} \to \mathcal{M}\) [45] associated to Lie pairs (see the Appendix for details). This is a dg Lie algebroid over the dg manifold \((\mathcal{M}, Q)\), where \(\mathcal{M} = \mathcal{L}[1] \oplus L/A\) and \(Q\) is a homological vector field on \(\mathcal{M}\), called the Fedosov differential. More precisely, a Fedosov dg Lie algebroid \(\mathcal{F}\) is a dg Lie subalgebroid of the tangent dg Lie algebroid \(T_M\) of the Fedosov dg manifold \((\mathcal{M}, Q)\). In other words, \(\mathcal{F}\) is the dg Lie algebroid encoding a dg foliation of \((\mathcal{M}, Q)\).

Since a Lie algebroid can be thought of as an extension of the tangent bundle of a manifold, the notions of polyvector fields and polydifferential operators admit extensions to the context of a Lie algebroid and these each carry a natural dgla structure [49, 50]. Likewise, the notions of polyvector fields and polydifferential operators can be extended in an appropriate sense to the context of a dg Lie algebroid. The “polyvector fields” and “polydifferential operators” associated to a Fedosov dg Lie algebroid \(\mathcal{F}\) can be viewed geometrically as polyvector fields and polydifferential operators tangent to the dg foliation on the Fedosov dg manifold \((\mathcal{M}, Q)\). In fact, one can identify the dglas of “polyvector fields” and “polydifferential operators” on \(\mathcal{F}\) to \((\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet, [Q, -])\) and \((\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, [[Q, -]])\), respectively, where \(\mathcal{P}_{\text{poly}}^\bullet\) denotes the formal polyvector fields and \(\mathcal{D}_{\text{poly}}^\bullet\) the formal polydifferential operators tangent to the fibers of the vector bundle \(L/A \to M\).

In fact, according to Corollary 4.14 and Corollary 4.19, the \(L_\infty\) structures on \(\otimes (\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet)\) and \(\otimes (\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet)\) are indeed obtained by the homotopy transfer from the dgla structures on \((\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet, [Q, -])\) and \((\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, [[Q, -]])\), respectively (see [45]). Therefore, as a key step, we apply Kontsevich formality theorem to the Fedosov dg Lie algebroid \(\mathcal{F}\) and establish the following

Theorem 2.3. There exists an \(L_\infty\) quasi-isomorphism
\[
\Psi : (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet, [Q, -], [[Q, -]]) \to (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, [[Q, -] + d_{\mathcal{F}}, [[Q, -]]])
\]
from the dgla of “polyvector fields” on \(\mathcal{F}\) to the dgla of “polydifferential operators” on \(\mathcal{F}\) with first Taylor coefficient
\[
\Psi_1 : (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet) \to (\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet)
\]
satisfying the following two properties:

1. \(\Psi_1\) preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;
2. \(\Psi_1 = \text{hkr} \circ (\text{td}^{\text{can}}_{\mathcal{F}})^{\frac{1}{2}}\), where \(\text{hkr}\) denotes the natural extension of the fiberwise Hochschild–Kostant–Rosenberg map \(\mathcal{P}_{\text{poly}}^\bullet \to \mathcal{D}_{\text{poly}}^\bullet\) and \((\text{td}^{\text{can}}_{\mathcal{F}})^{\frac{1}{2}}\) the action of the square root of the canonical Todd cocycle \(\text{td}^{\text{can}}_{\mathcal{F}}\) of the Fedosov dg Lie algebroid \(\mathcal{F}\) on \(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{P}_{\text{poly}}^\bullet\) by contraction.

2.2. Kontsevich formality morphism for Lie pairs.

2.2.1. Tangent \(L_\infty\) algebras. Let \(\mathfrak{g}\) and \(\mathfrak{g}'\) be two \(L_\infty\) algebras and let \(Q\) and \(Q'\) denote the corresponding homological vector fields on the associated dg manifolds \(\mathfrak{g}[1]\) and \(\mathfrak{g}'[1]\). An \(L_\infty\) morphism \(\mathcal{U} : \mathfrak{g} \to \mathfrak{g}'\) is, by definition, a morphism of dg manifolds \(\mathfrak{g}[1] \to \mathfrak{g}'[1]\), which means that the homomorphism of algebras \(\mathcal{U}^* : \mathcal{C}^\infty(\mathfrak{g}'[1]) \to \mathcal{C}^\infty(\mathfrak{g}[1])\) intertwines the derivations: \(\mathcal{U}^* \circ Q' = Q \circ \mathcal{U}^*\). Such an \(L_\infty\) morphism \(\mathcal{U}\) is entirely determined by its so-called “Taylor coefficients,” which are a sequence \((\mathcal{U}_n)_{n=1,2,...}\) of morphisms of graded vector spaces
\[
\mathcal{U}_n : \Lambda^n \mathfrak{g} \to \mathfrak{g}'[1 - n].
\]
A Maurer–Cartan (MC) element of an $L_\infty$ algebra $(\mathfrak{g}, Q)$ is an element $\omega \in \mathfrak{g}_1$ (of degree 1) satisfying
\[
\sum_{j=1}^{\infty} \frac{1}{j!} Q_j(\omega^j) = 0,
\]
where $\omega^j = \omega \wedge \cdots \wedge \omega \in \Lambda^j \mathfrak{g}$. In particular, the MC elements $\omega$ of a dgla satisfy the classical Maurer–Cartan equation
\[
d\omega + \frac{1}{2} [\omega, \omega] = 0.
\]
Given a MC element $\omega$ of an $L_\infty$ structure $Q$ on a graded vector space $\mathfrak{g}$, there is a new $L_\infty$ algebra structure $Q_\omega$ on $\mathfrak{g}$ called tangent $L_\infty$ algebra $[22]$; the Taylor coefficients of $Q_\omega$ satisfy
\[
(Q_\omega)_n(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} Q_{n+j}(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n \mathfrak{g}.
\]
In general, the convergence of the summations above is an issue that has to be addressed. However, if $\mathfrak{g}$ is a dgla with differential $d$ and Lie bracket $[-,-]$, the sums are finite and the tangent $L_\infty$ algebra is again a dgla with the same bracket but with the modified differential $d_\omega = d + [\omega, -]$.

We use the symbol $T_\omega \mathfrak{g}$ to distinguish the tangent $L_\infty$ algebra at $\omega$ from the original $L_\infty$ algebra $\mathfrak{g}$.

Given an $L_\infty$ morphism of dglas $U : \mathfrak{g} \to \mathfrak{g}'$ and a MC element $\omega$ of $\mathfrak{g}$, consider the element $U(\omega)$ of $\mathfrak{g}'$ defined by
\[
U(\omega) = \sum_{j=1}^{\infty} \frac{1}{j!} U_j(\omega^j)
\]
assuming the summation converges. Then $U(\omega)$ is a Maurer–Cartan element of $\mathfrak{g}'$ and therefore both $T_\omega \mathfrak{g}$ and $T_{U(\omega)} \mathfrak{g}'$ are dglas. There is a tangent $L_\infty$ morphism
\[
U_\omega : T_\omega \mathfrak{g} \to T_{U(\omega)} \mathfrak{g}'
\]
defined through the relations
\[
(U_\omega)_n(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} U_{n+j}(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n \mathfrak{g}.
\]
Provided the summations in the r.h.s. of Equation (12) converge, $U_\omega$ is indeed a well defined $L_\infty$ morphism. One may read [22, 51] for more details.

2.2.2. Kontsevich formality morphism for $\mathbb{k}^d$. In this section, we briefly recall the definition of Kontsevich’s formality morphism for $\mathbb{k}^d$ (where $\mathbb{k}$ is either $\mathbb{R}$ or $\mathbb{C}$), which we need later on. For more details, the reader may want to refer to Kontsevich’s original paper [22].

Kontsevich’s formality morphism is an $L_\infty$ quasi-isomorphism
\[
U : \mathcal{T}_\text{poly}(\mathbb{k}^d) \to \mathcal{D}_\text{poly}(\mathbb{k}^d)
\]
between the two dglas $\mathcal{T}_\text{poly}(\mathbb{k}^d)$ and $\mathcal{D}_\text{poly}(\mathbb{k}^d)$. Its ‘Taylor coefficients’ are of the form
\[
U_n = \sum_{m \geq 0} \sum_{\Gamma \in \mathcal{G}_{n,m}} W_\Gamma U_\Gamma,
\]
where $\mathcal{G}_{n,m}$ denotes the set of admissible graphs of type $(n, m)$, $W_\Gamma$ is a number called Kontsevich weight of the graph $\Gamma$, and $U_\Gamma$ is a map which assembles $n$ polyvector fields into a single polydifferential operator in a way determined by the graph $\Gamma$.

We will now describe $\mathcal{G}_{n,m}$, $U_\Gamma$, and $W_\Gamma$ successively.
Admissible graphs. A directed graph \( \Gamma \) is a pair of (finite) sets \( V_\Gamma \) and \( E_\Gamma \) together with two maps \( s, t : E_\Gamma \to V_\Gamma \). The elements of \( V_\Gamma \) are called vertices. The elements of \( E_\Gamma \) are called edges. Each edge \( e \in E_\Gamma \) starts at its source \( s(e) \in V_\Gamma \) and ends at its target \( t(e) \in V_\Gamma \). Given a vertex \( v \in V_\Gamma \), we use the symbol \( \text{Out}(v) \) to denote the set \( s^{-1}(v) \) of all edges starting at \( v \) and we use the symbol \( \text{In}(v) \) to denote the set \( t^{-1}(v) \) of all edges ending at \( v \).

An admissible graph of type \((n, m)\) is a directed graph \( \Gamma = (V_\Gamma, E_\Gamma) \) with labels on its vertices and edges satisfying the following requirements.

1. The set of vertices is partitioned into two subsets: \( V_\Gamma = V_\Gamma^1 \sqcup V_\Gamma^2 \). The elements of \( V_\Gamma^1 \) are called vertices of the first type or aerial vertices. The elements of \( V_\Gamma^2 \) are called vertices of the second type or terrestrial vertices.
2. For all \( e \in E_\Gamma \), \( s(e) \in V_\Gamma^1 \).
3. For all \( e \in E_\Gamma \), \( s(e) \neq t(e) \).
4. No two edges have the same source and the same target.
5. The aerial vertices are labelled by the symbols \( 1, 2, 3, \ldots, n \) while the terrestrial vertices are labelled by the symbols \( 1, 2, 3, \ldots, m \).
6. For every vertex \( k \in V_\Gamma^1 \) of the first type, the elements of \( \text{Out}(k) \) are labelled by the symbols \( e_1^k, e_2^k, e_3^k, \ldots \).

Assembling a polydifferential operator from polyvector fields according to an admissible graph. Fix an admissible graph \( \Gamma \in \mathcal{G}_{n, m} \). Each choice of a vertex \( v \in V_\Gamma \) and a map \( I : E_\Gamma \to \{1, \ldots, d\} \) determines a constant differential operator

\[
D_I^v := \prod_{e \in \text{In}(v)} \frac{\partial}{\partial x_{I(e)}}
\]

on \( \mathbb{k}^d \). Furthermore, each choice of an aerial vertex \( k \in V_\Gamma^1 \) and a map \( I : E_\Gamma \to \{1, \ldots, d\} \) determines a map

\[
\mathcal{T}_{\text{poly}}(\mathbb{k}^d) \ni \gamma \mapsto \gamma^{I(\text{Out}(k))} \in C^\infty(\mathbb{k}^d)
\]

through the relation

\[
\gamma^{I(\text{Out}(k))} := (dx_{I(e_1^k)} \otimes \cdots \otimes dx_{I(\text{Out}(k))})|_\text{alt}(\gamma).
\]

The bundle map \( \text{alt} : \Lambda^* T_{\mathbb{k}^d} \to \bigotimes^* T_{\mathbb{k}^d} \) is the antisymmetrization

\[
\xi_1 \wedge \cdots \wedge \xi_r \xrightarrow{\text{alt}} \sum_{\sigma \in S_r} \text{sgn}(\sigma) \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(r)}.
\]

The admissible graph \( \Gamma \) with \( n \) aerial and \( m \) terrestrial vertices is a recipe for assembling \( n \) polyvector fields \( \gamma_1, \gamma_2, \ldots, \gamma_n \) on \( \mathbb{k}^d \) into an \( m \)-differential operator \( \mathcal{B}_\Gamma(\gamma_1, \ldots, \gamma_n) \) on \( \mathbb{k}^d \), which is defined by

\[
(f_1, \ldots, f_m) \mapsto \mathcal{B}_\Gamma(\gamma_1, \ldots, \gamma_n) \sum_{l : E_\Gamma \to \{1, \ldots, d\}} \left( \prod_{k=1}^n D_I^k \left( \gamma_k^{I(\text{Out}(k))} \right) \right) \left( \prod_{l=1}^m D_I^l (f_l) \right),
\]

for all \( f_1, \ldots, f_m \in C^\infty(\mathbb{k}^d) \). We note that \( \gamma^{I(\text{Out}(k))} = 0 \) if \( \gamma \in \mathcal{T}_{\text{poly}}(\mathbb{k}^d) \) with \( r+1 \neq |\text{Out}(k)| \). Therefore, \( \mathcal{B}_\Gamma(\gamma_1, \ldots, \gamma_n) = 0 \) if \( \gamma_1, \ldots, \gamma_n \) are homogeneous elements of \( \mathcal{T}_{\text{poly}}(\mathbb{k}^d) \) and \( |\gamma_1| + \cdots + |\gamma_n| + n \neq |E_\Gamma| \).

Configuration spaces and their compactifications. The Kontsevich weights are obtained from integrals over compactified configuration spaces.

Let \( \mathbb{H}^+ = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) denote the hyperbolic plane and let \( \mathbb{H}^+ \) denote its closure in \( \mathbb{C} \). The group \( G_2 := \mathbb{R}^+ \times \mathbb{R} = \{ z \mapsto az + b \mid a, b \in \mathbb{R}, \ a > 0 \} \) acts on the configuration space

\[
\text{Conf}_{n,m}^+ := \{ (z_1, \ldots, z_n, q_1, \ldots, q_m) \in (\mathbb{H}^+)^n \times \mathbb{R}^m \mid z_i \neq z_j \text{ if } i \neq j ; q_1 < \cdots < q_m \}.
\]
The quotient $C_{n,m}^+ = \text{Conf}^+_{n,m} / G_2$ is a manifold of dimension $2n + m - 2$. Fixing $z_1$ at $i = \sqrt{-1}$, we may identify $C_{n,m}^+$ with an open subset of $\mathbb{C}^{n-1} \times \mathbb{R}^m$ and transfer the standard orientation of the affine space to $C_{n,m}^+$.

We now proceed with the compactification of $C_{n,m}^+$.

Let $\text{Conf}_n$ be the space of configurations of $n$ distinct points $z_1, z_2, \ldots, z_n$ in $\mathbb{C}$. The group $G_3 := \mathbb{R}^+ \ltimes \mathbb{C}$ acts on $\text{Conf}_n$ by dilations and translations. The quotient $C_n := \text{Conf}_n / G_3$ is a manifold which we embed into $(S^1)^{n(n-1)} \times (\mathbb{R}P^2)^{(n(n-1)(n-2))}$ by recording all possible angles $\arg(z_i - z_j)$ and homogeneous coordinate triples $|z_i - z_j| : |z_j - z_k| : |z_k - z_i|$.

Since $C_{n,m}^+$ is itself embedded into $C_{2n+m}$ by the map

$$(z_1, \ldots, z_n, q_1, \ldots, q_m) \mapsto (z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n, q_1, \ldots, q_m),$$

we obtain an embedding

$$C_{n,m}^+ \hookrightarrow C_{2n+m} \hookrightarrow (S^1)^{N_1} \times (\mathbb{R}P^2)^{N_2}$$

with $N_1 = (2n+m)(2n+m-1)$ and $N_2 = (2n+m)(2n+m-1)(2n+m-2)$. The desired compactification of $C_{n,m}^+$ is the closure $\overline{C_{n,m}^+}$ of the image of the above embedding.

**Kontsevich weight of an admissible graph.** Consider the hyperbolic angle function $\varphi : \mathbb{H}^+ \times \mathbb{H}^+ \to S^1$ defined by $\varphi(z, w) = \frac{1}{2\pi} \arg \left( \frac{z - w}{z \cdot w} \right)$.

Given an admissible graph $\Gamma \in \mathcal{G}_{n,m}$, define a function $\varphi_e : C_{n,m}^+ \to S^1$ for each edge $e \in E_{\Gamma}$ by

$$\varphi_e(z_1, \ldots, z_n, z_1, \ldots, z_m) = \varphi(z_{s(e)}, z_{t(e)})$$

and a differential form $\kappa_\Gamma$ of degree $|E_{\Gamma}|$ on $C_{n,m}^+$ by

$$\kappa_\Gamma = \bigwedge_{e \in E_{\Gamma}} d\varphi_e,$$

where $d\varphi_e$ denotes the pullback of the standard volume on $S^1$ through $\varphi_e$. In the exterior product, the 1-forms are multiplied according to the lexicographic order $e_1^1, e_1^2, \ldots, e_2^1, e_2^2, \ldots$ of the edges of the graph. Note that $\kappa_\Gamma$ extends smoothly to $C_{n,m}^+$. Integrating $\kappa_\Gamma$ over the (oriented) compactified configuration space, we obtain the Kontsevich weight of the graph $\Gamma$:

$$W_{\Gamma} = \prod_{k=1}^{n} \frac{1}{|\text{Out}(k)|!} \int_{C_{n,m}^+} \kappa_\Gamma.$$

Obviously, the Kontsevich weight of a graph $\Gamma \in \mathcal{G}_{n,m}$ is zero if $|E_{\Gamma}| / \dim(C_{n,m}^+) = 2n + m - 2$.

**Kontsevich formality theorem for $k^d$.** For all graphs $\Gamma \in \mathcal{G}_{n,m}$ and all homogeneous polyvector fields $\gamma_1, \ldots, \gamma_n$ on $k^d$, we know that $\mathcal{Y}_\Gamma(\gamma_1, \ldots, \gamma_n)$ is a homogeneous element of degree $m - 1$ in $\mathcal{D}^\text{poly}(k^d)$ and that $W_{\Gamma} \mathcal{Y}_\Gamma(\gamma_1, \ldots, \gamma_n) \neq 0$ only when $|\gamma_1| + \cdots + |\gamma_n| + n = |E_{\Gamma}| = 2n + m - 2$. It follows that $\mathcal{Y}_n : \bigotimes^n \mathcal{T}^\text{poly}(k^d) \to \mathcal{D}^\text{poly}(k^d)$ is a map of degree $1 - n$.

Consider the case $n = 1$. There are $m!$ distinct graphs $\Gamma \in \mathcal{G}_{1,m}$ satisfying the property $|E_{\Gamma}| = 2 \cdot 1 + m - 2 = m$. In each such graph, each one of the $m$ terrestrial vertices is the target of a single edge starting from the unique aerial vertex. Any two such graphs only differ by the labelling of the $m$ edges. Moreover, all such graphs have the same weight $W_{\Gamma} = (m!)^{-2}$. It follows that the ‘first Taylor coefficient’ $\mathcal{Y}_1$ of the formality map is precisely the Hochschild–Kostant–Rosenberg map:

$$\mathcal{T}^\text{poly}(k^d) \xrightarrow{\mathcal{Y}_1 = \text{hkr}} \mathcal{D}^\text{poly}(k^d).$$
Theorem 2.4 (Kontsevich formality theorem [22]). The maps \((\mathcal{U}_n)_{n=1}^\infty\) defined above are the ‘Taylor coefficients’ of an \(L_\infty\) quasi-isomorphism

\[
\mathcal{U} : \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}^d) \to \mathcal{D}_{\text{poly}}^\bullet(\mathbb{k}^d)
\]

satisfying the following additional properties.

(1) The first Taylor coefficient of \(\mathcal{U}\) is the Hochschild–Kostant–Rosenberg map \(\text{hkr}\).
(2) The formality morphism \(\mathcal{U}\) is \(GL(\mathbb{k}^d)\)-equivariant.
(3) For all \(n \geq 2\) and \(\xi_1, \cdots, \xi_n \in \mathcal{T}_{\text{poly}}^0(\mathbb{k}^d)\), we have

\[
\mathcal{U}_n(\xi_1, \cdots, \xi_n) = 0.
\]

(4) Provided \(\xi\) is a linear vector field on \(\mathbb{k}^d\) and \(n \geq 2\), we have

\[
\mathcal{U}_n(\xi, \eta_2, \cdots, \eta_n) = 0
\]

for all \(\eta_2, \cdots, \eta_n \in \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}^d)\).

Furthermore, the formality morphism \(\mathcal{U}\) can be defined for \(\mathbb{k}^d_{\text{formal}}\) as well.

Remark 2.5. With suitable sign corrections, Kontsevich’s formality theorem was generalized to \(\mathbb{Z}\)-graded manifolds by Cattaneo–Felder [10]. Later, these sign corrections were given a simple operadic explanation.

2.2.3. Fiberwise formality map. Let \((L, A)\) be a Lie pair over a smooth manifold \(M\). As before, set \(R = C^\infty(M)\). The quotient \(B = L/A\) is a vector bundle over \(M\) whose fibers are all (noncanonically) isomorphic to \(\mathbb{k}^d\). Next we apply Kontsevich’s formality theorem (essentially) fiberwisely to a Fedosov dg Lie algebroid. See Section 4.2 for the construction of Fedosov dg Lie algebroids.

Since the formality morphism \(\mathcal{U} : \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}}) \to \mathcal{D}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}})\) is \(GL(\mathbb{k}^d)\)-equivariant (see Theorem 2.4 (2)), there exist \(R\)-linear maps

\[
\mathcal{U}_n^f : \Lambda^n \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}}) \to \mathcal{D}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}})[1 - n]
\]

whose restrictions to each fiber of \(B \to M\) coincide with the Taylor coefficients

\[
\mathcal{U}_n : \Lambda^n \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}}) \to \mathcal{D}_{\text{poly}}^\bullet(\mathbb{k}^d_{\text{formal}})[1 - n]
\]

of \(\mathcal{U}\). Extending them \(\Gamma(\Lambda^n L^\vee)\)-linearly, we obtain \(R\)-linear maps

\[
\mathcal{U}_n^f : \Lambda^n \text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet) \to \text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet)[1 - n].
\]

Consider the difference

\[
\omega = Q - d_{\mathcal{U}}^L \in \Gamma(L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^0 \subset \mathcal{T}_{\text{poly}}^0(L[1] \oplus B)
\]

of the derivations \(Q = d_{\mathcal{U}}^L = -\delta + d_{\mathcal{U}}^L + X^\vee\) and \(d_{\mathcal{U}}^L\) of the algebra

\[
\Gamma(AL^\vee \otimes \hat{S}(B^\vee)) = C^\infty(L[1] \oplus B)
\]

appearing in Theorem 4.7. We do not claim that \(\omega\) is a MC element for any dgl structure on \(\text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet)\). Nevertheless, we define a sequence \((\Phi_n)_{n=1,2,\ldots}\) of \(R\)-linear maps

\[
\Phi_n : \Lambda^n \left( \text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet) \right) \to \text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet)[1 - n]
\]

by

\[
\Phi_n(\gamma) = \sum_{j=1}^\infty \mathcal{U}_{n+j}^f(\omega^j \wedge \gamma), \quad \forall \gamma \in \Lambda^n \left( \text{tot}(\Gamma(\Lambda^n L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet) \right).
\]

Lemma 2.6. The maps \((\Phi_n)_{n=1}^\infty\) are well defined.
Proof. Suppose $\gamma_k \in \Gamma(\Lambda^k L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^k$ for $k \in \{1, 2, \cdots, n\}$. Since
\[
\mathcal{U}_{n+j} : \Lambda^{n+j} \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}}) \rightarrow \mathcal{D}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})
\]
is a map of degree $1 - (n + j)$ and $\omega \in \Gamma(L^\vee) \otimes \mathcal{T}_0^\bullet$, we have
\[
\mathcal{U}_{n+j}^f(\omega \wedge \cdots \wedge \omega \wedge \gamma_1 \wedge \cdots \wedge \gamma_n) \in \Gamma(\Lambda^1 L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^{r_1 + \cdots + r_n + 1 - (n + j)}.
\]
As $j$ increases, $r_1 + \cdots + r_n + 1 - (n + j)$ eventually becomes smaller than $-1$ forcing $\mathcal{U}_{n+j}^f(\omega \wedge \cdots \wedge \omega \wedge \gamma_1 \wedge \cdots \wedge \gamma_n)$ to vanish. Therefore, only finitely many of the terms of $\Phi_n(\gamma_1 \wedge \cdots \wedge \gamma_n)$ are not zero. □

Although $\omega$ is not a MC element, the maps $(\Phi_n)_{n=1}^\infty$ still define an $L_\infty$ morphism.

**Proposition 2.7.** The maps $(\Phi_n)_{n=1}^\infty$ are the Taylor coefficients of an $L_\infty$ morphism of dglas
\[
\Phi : (\text{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet), \{Q, -\}, \{[-, -]\}) \rightarrow (\text{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet), \{Q, -\} + d_{\mathcal{M}}, \{[-, -]\})
\]

We will need the following well known lemma.

**Lemma 2.8.** Let $(C, \tilde{d})$ be a cdga, and $\mathcal{U} : (g, \tilde{d}, [-, -]) \rightarrow (g', \tilde{d}', [-, -])$ be an $L_\infty$ morphism of dglas

1. Then $(C \otimes g, \tilde{d} \otimes \text{id} + \text{id} \otimes \tilde{d}, [-, -])$ and $(C \otimes g', \tilde{d} \otimes \text{id} + \text{id} \otimes \tilde{d}', [-, -])$ are dglas
2. and the $C$-linear extension of $\mathcal{U}$
\[
\mathcal{U} : (C \otimes g, \tilde{d} \otimes \text{id} + \text{id} \otimes \tilde{d}, [-, -]) \rightarrow (C \otimes g', \tilde{d} \otimes \text{id} + \text{id} \otimes \tilde{d}', [-, -])
\]
is an $L_\infty$ morphism of dglas.

**Proof of Proposition 2.7.** Choosing a local trivialization $B|_U \cong U \times \mathbb{k}^d$ of the vector bundle $B$ over an open subset $U$ of $M$ yields identifications
\[
(\text{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet))|_U \cong \Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}}),
\]
\[
(\text{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet))|_U \cong \Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}}).
\]
According to Lemma 2.8, the restrictions to $U$ of the maps $(\mathcal{U}_{n+j}^f)_{n=1,2,\cdots}$ constructed earlier are the Taylor coefficients of an $L_\infty$ morphism of dglas
\[
(\text{tot}(\Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})), d_L \otimes \text{id}_{\mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})}, [-, -])
\]
\[
\mathcal{U}^f
\]
\[
(\text{tot}(\Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})), d_L \otimes \text{id}_{\mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})} + \text{id} \otimes d_{\mathcal{M}}, [-, -]).
\]
In the chosen local trivialization $B|_U \cong U \times \mathbb{k}^d$ of the vector bundle $B$ over the open subset $U$ of $M$, we may compare $Q$ with the derivation $d_L \otimes \text{id}_{\mathbb{k}[x_1, \cdots, x_d]}$ of the algebra
\[
\Gamma(U; \Lambda^\bullet L^\vee \otimes \tilde{SB}^\vee) \cong \Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathbb{k}[x_1, \cdots, x_d].
\]
Since $Q^2 = 0$, their difference
\[
\varpi = Q - d_L \otimes \text{id} \in \Gamma(U; L^\vee) \otimes_k \mathcal{T}^0_{\text{poly}}(\mathbb{k}_{\text{formal}})
\]
is a MC element of the dga
\[
(\text{tot}(\Gamma(\Lambda^\bullet L^\vee) \otimes_R \mathcal{T}_{\text{poly}}^\bullet))|_U \cong (\text{tot}(\Gamma(U; \Lambda^\bullet L^\vee) \otimes_k \mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})))
\]
endowed with the differential $d_L \otimes \text{id}_{\mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})}$ and the $\Gamma(U; L^\vee)$-multilinear extension of the Schouten bracket on $\mathcal{T}_{\text{poly}}^\bullet(\mathbb{k}_{\text{formal}})$. 
Since
\[
\mathcal{U}_U^f(\varpi) = \sum_{j=1}^{\infty} \frac{1}{j!} \mathcal{U}_j^f(\varpi^j) \quad \text{by Equation (11)}
\]
\[
= \mathcal{U}_1^f(\varpi) \quad \text{by Theorem 2.4 (3)}
\]
\[
= \text{hkr}(\varpi) \quad \text{by Theorem 2.4 (1)}
\]
\[
= \varpi,
\]
we obtain the tangent \(L_\infty\) morphism of \(\mathcal{U}_U^f\) at \(\varpi\):
\[
\mathcal{U}_U^f: \left( \text{tot}(\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{P}_{\text{poly}})|_U, Q, [-,-] \right) \rightarrow \left( \text{tot}(\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{P}_{\text{poly}})|_U, Q + d\mathcal{X}, [-,-] \right).
\]
Adapting the argument used for \(\Phi\) earlier, one can show that the tangent \(L_\infty\) morphism \(\mathcal{U}_U^f\) is well defined.

Since the map \(\Phi_n\) depends only locally on its arguments, we may consider its restriction to the open subset \(U\) of \(M\). We claim that the \(n\)-th Taylor coefficient of the \(L_\infty\) morphism \(\mathcal{U}_U^f\) is the restriction of \(\Phi_n\) to \(U\).

Indeed, one easily checks that \(\omega - \varpi\) is (the tensor product of a section of \(L^\vee\) over \(U\) with) a linear vertical vector field on \(k_d\text{formal}\) and it then follows from Theorem 2.4 (4) that, for all \(\gamma \in \Lambda^n(\text{tot}(\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{P}_{\text{poly}})|_U)\),
\[
\Phi_n(\gamma) = \sum_{j=1}^{\infty} \mathcal{U}_{n+j}^f(\varpi^j \wedge \gamma) = \sum_{j=1}^{\infty} \mathcal{U}_{n+j}^f(\varpi^j \wedge \gamma) = (\mathcal{U}_{U,\varpi}^f)_n(\gamma).
\]
This shows that \((\Phi_n)_{n=1,2,\cdot\cdot\cdot}\) is the sequence of Taylor coefficients of an \(L_\infty\) morphism
\[
\Phi: \left( \text{tot}(\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{P}_{\text{poly}}), [Q,-], [-,-] \right) \rightarrow \left( \text{tot}(\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{P}_{\text{poly}}), [Q,-] + d\mathcal{X}, [-,-] \right)
\]
defined globally on \(M\).

Our construction of the quasi-isomorphism \(\Phi\) is essentially the same as the one given by Dolgushev in [14] except that we define its Taylor coefficients \(\Phi_n\) globally from the get-go rather than by gluing local data.

2.3. **Algebraic homomorphism property.** In this section, we sketch a proof why \(\Phi_1\) is a morphism of associative algebras up to homotopy. For more details, the reader may want to consult [31, 32, 37, 22, 8].

2.3.1. **Kontsevich’s eye.** The compactified configuration space \(\overline{C}_{2,0}^+\), which is customarily called ‘Kontsevich’s eye,’ is represented in Figure 1. Its boundary admits the following decomposition in strata:

\[
\partial(\overline{C}_{2,0}^+) = C_{1,0} \sqcup C_{1,1} \sqcup C_{1,1} \sqcup C_{0,2}.
\]

The stratum \(C_{1,0}\) — the pupil of the eye — is reached when the two aerial vertices \(z_1\) and \(z_2\) merge. The first copy of \(C_{1,1}\) — the upper eyelid — is reached when the aerial vertex \(z_1\) approaches the real line. The second copy of \(C_{1,1}\) — the lower eyelid — is reached when the aerial vertex \(z_2\) approaches the real line. The stratum \(C_{0,2}\) is made of two points — the corners of the eye. The left corner is reached when the vertices \(z_1\) and \(z_2\)
each approach a distinct point of the real line simultaneously and $z_1$ is the leftmost of the two points. The right corner is reached when the vertices $z_1$ and $z_2$ each approach a distinct point of the real line simultaneously and $z_1$ is the rightmost of the two points.

2.3.2. Vanishing lemma. Given a configuration space $C_{n,m}^+$ with $n \geq 2$, consider the projection $\pi : C_{n,m}^+ \to C_{2,0}^+$, which forgets all but the first two of the $n$ aerial points in $\mathbb{H}^+$ and all $m$ points on the real line. More precisely, consider its continuous extension $\pi : \overline{C_{n,m}^+} \to \overline{C_{2,0}^+}$ to the compactified configuration spaces.

Now choose a smooth path $\xi : [0,1] \to \overline{C_{2,0}^+}$ starting from a point on the inner boundary of Kontsevich’s eye and ending at the right corner. The inverse image of $\xi([0,1])$ under $\pi$ in $\overline{C_{n,m}^+}$ is a compact subspace denoted $Z_{n,m}$. Kontsevich assigns a weight

$$\tilde{W}_\Gamma = \prod_{k=1}^n \frac{1}{|\text{Out}(\Gamma)|!} \int_{Z_{n,m}} \hat{j}(\kappa_{\Gamma})$$

to each admissible graph $\Gamma \in \mathcal{G}_{n,m}$. The symbol $j$ denotes the embedding of $Z_{n,m}$ into $\overline{C_{n,m}^+}$. Since $\dim Z_{n,m} = 2n + m - 3$ and $\kappa_{\Gamma}$ is an $|E|\text{ form}$, the weight $\tilde{W}_\Gamma$ is zero unless $|E| = 2n + m - 3$.

The following vanishing lemma is analogous to Theorem 2.4 (4).

**Lemma 2.9.** If $\xi \in T_{\text{poly}}^0(\mathbb{R}^d)$ is a vector field linear on $\mathbb{R}^d$ and $\Gamma \in \mathcal{G}_{n,m}$ is an admissible graph with $n \geq 3$, then

$$\tilde{W}_\Gamma \hat{\mathcal{Z}}_{\Gamma}(\eta_1, \ldots, \eta_{n-1}, \xi) = 0$$

for all $\eta_1, \ldots, \eta_{n-1} \in T_{\text{poly}}^\ast(\mathbb{R}^d)$.

All ingredients of the proof can be found in Kontsevich’s original paper [22].

Recall the hyperbolic angle function $\varphi : \mathbb{H}^+ \times \mathbb{H}^+ \to S^1$ defined by $\varphi(z, w) = \frac{1}{2\pi} \arg\left(\frac{z-w}{\bar{z}-\bar{w}}\right)$. Given a point $z_0$ of $\mathbb{H}^+$, let $d\varphi(z, z_0)$ denote the pullback to $\mathbb{H}^+$ of the standard volume form on $S^1$ through the function $\mathbb{H}^+ \ni z \mapsto \varphi(z, z_0) \in S^1$. Likewise let $d\varphi(z_0, z)$ denote the pullback to $\mathbb{H}^+$ of the standard volume form on $S^1$ through the function $\mathbb{H}^+ \ni z \mapsto \varphi(z_0, z) \in S^1$.

**Lemma 2.10** ([22, Lemmas 7.3, 7.4, and 7.5]).

1. For every pair of distinct points $z_1$ and $z_2$ in $\mathbb{H}^+$, we have

$$\int_{z \in \mathbb{H}^+ \setminus \{z_1, z_2\}} d\varphi(z_1, z) \wedge d\varphi(z, z_2) = 0.$$

2. For every pair of points $z_1 \in \mathbb{H}^+$ and $z_2 \in \mathbb{R} = \partial(\mathbb{H}^+)$, we have

$$\int_{z \in \mathbb{H}^+ \setminus \{z_1\}} d\varphi(z_1, z) \wedge d\varphi(z, z_2) = 0.$$

3. For every point $z_0$ in $\mathbb{H}^+$, we have

$$\int_{z \in \mathbb{H}^+ \setminus \{z_0\}} d\varphi(z_0, z) \wedge d\varphi(z, z_0) = 0.$$

**Proof of Lemma 2.9.** Since $\xi \in T_{\text{poly}}^0(\mathbb{R}^d)$, we have $\xi^{I(\text{Out}(n))} = 0$ for all maps $I : E_\Gamma \to \{1, \ldots, d\}$ unless $|\text{Out}(n)| = 1$. Moreover, if $|\text{In}(n)| > 1$, the order of the differential operator $D_{\Gamma}^\alpha$ is at least two, no matter which map $I : E_\Gamma \to \{1, \ldots, d\}$ is considered, and the function $D_{\Gamma}^\alpha \xi^{I(\text{Out}(n))}$ vanishes as $\xi$ is linear. Hence $\hat{\mathcal{Z}}_{\Gamma}(\eta_1, \ldots, \eta_{n-1}, \xi) = 0$ unless $|\text{Out}(n)| = 1$ and $|\text{In}(n)| \leq 1$. We may thus assume without loss of generality that $\text{Out}(n) = \{e_n\}$ and $|\text{In}(n)| \in \{0, 1\}$.
Let’s assume for now that \( \text{Out}(n) = \{ e^1_n \} \) and \( \text{In}(n) = \{ e' \} \) — we will treat the other case later. Consider the graph \( \Delta \in \mathcal{G}_{n-1,m} \) obtained from \( \Gamma \in \mathcal{G}_{n,m} \) by removing the \( n \)-th aerial vertex \( n \) and all the edges starting or ending at it. We have
\[
\kappa_\Gamma = \pm d\varphi_{e'} \wedge d\varphi_{e_1} \wedge F_n^* (\kappa_\Delta),
\]
where \( F_n : C^+_{n,m} \to C^+_{n-1,m} \) is the projection which forgets the \( n \)-th aerial point \( z_n \) of a configuration \((z_1, \ldots, z_n; q_1, \ldots, q_m)\). Making use of Fubini’s theorem, we obtain
\[
\widetilde{W}_\Gamma = \int_{Z_{n,m}} j^*(\kappa_\Gamma) = \pm \int_{Z_{n,m}} j^* (d\varphi_{e'} \wedge d\varphi_{e_1} \wedge F_n^* (\kappa_\Delta)) = \pm \int_{F_n(Z_{n,m})} f \cdot j^* (\kappa_\Delta),
\]
where \( f \) denotes the function on \( C^+_{n,m} \) obtained by integration of the 2-form \( d\varphi_{e'} \wedge d\varphi_{e_1} \) along the fibers of \( F_n : C^+_{n,m} \to C^+_{n-1,m} \). Since
\[
f(z_1, \ldots, z_{n-1}; z_1, \ldots, z_m) = \int_{z_n \in \mathbb{H}^+ \setminus \{ z_1, \ldots, z_{n-1} \}} d\varphi(z_{s(e')}, z_n) \wedge d\varphi(z_n, z_{t(e_1)}),
\]
it follows from Lemma 2.10 that \( \widetilde{W}_\Gamma = 0 \).

Finally, we turn our attention to the situation where \( \text{Out}(n) = \{ e^1_n \} \) and \( \text{In}(n) = \emptyset \). We start with making two observations.

1. For all \( e \in E_\Gamma \) with \( e \neq e^1_n \), the aerial vertex \( n \) is neither the source nor the target of the edge \( e \) and, consequently, the function \( \varphi_e : C^+_{n,m} \to S^1 \) is constant along the fibers of the projection \( F_n : C^+_{n,m} \to C^+_{n-1,m} \) which forgets the \( n \)-th aerial point \( z_n \) of a configuration \((z_1, \ldots, z_n; q_1, \ldots, q_m)\).

2. Each fiber of the projection \( F_n : C^+_{n,m} \to C^+_{n-1,m} \) is diffeomorphic to \( \mathbb{H}^+ \) punctured at \( n-1 \) points and is foliated by its intersections with the level sets of the function \( \varphi_{e_1} : C^+_{n,m} \to S^1 \). In other words, \( C^+_{n,m} \) is foliated by the fibers of \( F_n \), which are themselves foliated by curves along which the function \( \varphi_{e_1} \) is constant. Obviously, the subspace \( Z_{n,m} \) of \( C^+_{n,m} \) is a union of such curves.

It follows from these observations that \( Z_{n,m} \) is foliated by curves along which all functions \( \varphi_e \) for all edges \( e \in E_\Gamma \) are constant. Therefore, the component of the form \( j^*(A_{e \in E_\Gamma} d\varphi_e) \) of degree \( \dim(Z_{n,m}) = 2n + m - 3 \) vanishes. Hence \( \widetilde{W}_\Gamma = \int_{Z_{n,m}} j^*(\kappa_\Gamma) = 0 \).

2.3.3. Homotopy operator. For every admissible graph \( \Gamma \in \mathcal{G}_{n,m} \), the operator
\[
\mathcal{U}_\Gamma : T^\bullet_{\text{poly}} (k_{\text{formal}})^{\otimes n} \to \mathcal{D}_\text{poly}^{m-1} (k_{\text{formal}})
\]
defined by Equation (14) is \( GL_d(k) \)-equivariant. Therefore there exists an \( R \)-linear map
\[
\mathcal{U}_\Gamma^f : (\mathcal{D}_\text{poly}^\bullet)^{\otimes n} \to \mathcal{D}_\text{poly}^{m-1}
\]
whose restrictions to each fiber of \( B \to M \) coincide with \( \mathcal{U}_\Gamma \). Extending the latter \( \Gamma(\Lambda^* L^\vee) \)-multilinearily, we obtain an \( R \)-linear operator
\[
\mathcal{U}_\Gamma^f : (\Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{D}_\text{poly}^\bullet)^{\otimes n} \to \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{D}_\text{poly}^{m-1}.
\]

Using the maps \( \mathcal{U}_\Gamma^f \), the weights \( \widetilde{W}_\Gamma \), and the difference
\[
\omega = Q - d^\Sigma_L \in \Gamma(L^\vee) \otimes_R \mathcal{D}_\text{poly}^0
\]
of the derivations \( Q \) and \( d^\Sigma_L \) appearing in Theorem 4.7, we define an operator
\[
H : (\Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{D}_\text{poly}^\bullet) \times (\Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{D}_\text{poly}^\bullet) \to \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{D}_\text{poly}^\bullet.
\]
We may thus rewrite the r.h.s. of Equation (15) as the finite sum
\[
H(\alpha, \beta) = \sum_{n \geq 0} \frac{1}{n!} \sum_{m \geq 0} (-1)^{m+1} \tilde{W}_\Gamma \mathcal{U}_\Gamma^f (\alpha, \beta, \omega, \ldots, \omega). 
\]

**Lemma 2.11.** The operator $H$ is well defined.

**Proof.** It suffices to prove that the infinite sum in the r.h.s. of Equation (15) converges for pairs of homogeneous elements $\alpha \in \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^a$ and $\beta \in \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^b$.

We know that, for all $\Gamma \in \mathcal{G}_{2+n,m}$,
- $\mathcal{U}_\Gamma^f (\alpha, \beta, \omega, \ldots, \omega) = 0$ unless $|E_\Gamma| = (a + 1) + (b + 1) + n(0 + 1)$
- and $\tilde{W}_\Gamma = 0$ unless $|E_\Gamma| = \dim(Z_{2+n,m})$.

Therefore, for all $\Gamma \in \mathcal{G}_{2+n,m}$, we have $\tilde{W}_\Gamma \mathcal{U}_\Gamma^f (\alpha, \beta, \omega, \ldots, \omega) = 0$ unless
\[
a + b + n + 2 = |E_\Gamma| = 2n + m + 1.
\]

We may thus rewrite the r.h.s. of Equation (15) as the finite sum
\[
(-1)^{a+b} \sum_{n=0}^{a+b+1} \frac{(-1)^n}{n!} \sum_{\Gamma \in \mathcal{G}_{2+n, a+b+1-n}} \tilde{W}_\Gamma \mathcal{U}_\Gamma^f (\alpha, \beta, \omega, \ldots, \omega). \quad \Box
\]

**Proposition 2.12.** For all $\alpha \in (\Gamma(\Lambda^* L) \otimes \mathcal{P}_\text{poly}^a)$ and $\beta \in (\Gamma(\Lambda^* L) \otimes \mathcal{P}_\text{poly}^b)$, we have
\[
\Phi_1(\alpha \cup \beta) - \Phi_1(\alpha) \cup \Phi_1(\beta) = (Q + d_{\mathfrak{m}})(H(\alpha, \beta)) - H(Q(\alpha), \beta) + (-1)^a H(\alpha, Q(\beta)).
\]

**Sketch of proof.** Recall from the proof of Theorem 2.7 that, though $\omega$ is not a Maurer–Cartan element, its restriction to any open subset of the manifold $M$ over which the vector bundle $B$ is trivial is equal to the sum of a Maurer–Cartan element and a linear vector field. It follows from Lemma 2.9, the definitions and locality of $\Phi_1$ and $H$ that, for the purpose of this proof, $\omega$ may be treated as if it were a Maurer–Cartan element. The rest of the proof is then virtually identical to a difficult computation due to Manchon and Torossian [31, 32, Théorème 4.6] — see also Mochizuki’s work [37, Equation 56]. There is only one significant difference with [31, 32, Théorème 4.6]: their Poisson bivector $h_\gamma$ must be replaced by our vector field $\omega$. This is responsible for the discrepancy in the number of edges of the admissible graphs appearing here and in [31, 32, Théorème 4.6]. \quad \Box

**Remark 2.13.** The operator $H$ defined above, which is implicit in [31, 32, Théorème 4.6] and [37, Equation 56], was made explicit in [8, Proposition 9.1].

2.4. **Explicit formula for $\Phi_1$.** Consider the first ‘Taylor coefficient’
\[
\text{tot} \left( \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^a (B) \right) \xrightarrow{\Phi_1} \text{tot} \left( \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^b (B) \right)
\]

of the $L_\infty$ morphism $\Phi_1$ constructed in Section 2.2.3.

In this section, we prove the following

**Proposition 2.14.** The map $\Phi_1 : \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^a \rightarrow \Gamma(\Lambda^* L^\vee) \otimes_R \mathcal{P}_\text{poly}^b$ is the modification of the Hochschild–Kostant–Rosenberg map by (the square root of) the canonical Todd cocycle:
\[
\Phi_1 = \text{hkr} \circ (t d_{\mathcal{F}})^{\frac{1}{2}}.
\]
Suppose that an open subset $U$ of $M$ diffeomorphic to $\mathbb{R}^m$ is the domain of a coordinate chart of $M$ over which the vector bundles $L$ and $B$ are trivial. The algebra of functions of the graded manifold $\mathcal{V} = \mathbb{R}^m \otimes \hat{k}^r[l]_0$ obtained by restriction of the Fedosov dg manifold $\mathcal{M}$ to the support $U$ is

$$C^\infty(\mathcal{V}) = C^\infty(\mathbb{R}^m) \otimes \hat{S}(\{\hat{k}^r \oplus \hat{k}^l[1]\})^\vee.$$ 

Here $m$ is the dimension of the manifold $M$ while $r$ is the rank of the vector bundle $B$ and $l$ is the rank of the vector bundle $L$.

There are natural injections

$$C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0 \rightarrow \mathcal{X}(\mathbb{R}^m \times \hat{k}^0[l]_0) \rightarrow \mathcal{X}(\mathcal{V}).$$

The restriction of the Fedosov homological vector field $Q \in \mathcal{X}(\mathcal{M})$ to $U$ is the sum

$$Q = d_L + \varpi$$

of $d_L \in \mathcal{X}(\mathbb{R}^m \times \hat{k}^0[l]_0)$ and $\varpi \in C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0$ as observed in Section 2.2.3.

**Lemma 2.15.** The sum $Q = Q_1 + Q_2$ of two vector fields $Q_1 \in \mathcal{X}(\mathbb{R}^m \times \hat{k}^0[l]_0) \subset \mathcal{X}(\mathcal{V})$ and $Q_2 \in C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0 \subset \mathcal{X}(\mathcal{V})$ is a homological vector field on $\mathcal{V} = \mathbb{R}^m \otimes \hat{k}^r[l]_0$ if and only if (1) $Q_1$ is a homological vector field on $\mathcal{V} = \mathbb{R}^m \otimes \hat{k}^r[l]_0$ (i.e. $Q_1$ is of degree $+1$ and $Q_1^2 = 0$) and (2) $Q_2$ satisfies the Maurer–Cartan equation $L_{Q_2} Q_2 + \frac{1}{2} [Q_2, Q_2] = 0$.

Moreover, in this case, $(C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0), Q_1)$ is a cdga and $Q_2$ is a MC element in $C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0$ endowed with its $L_\infty$ algebra structure determined by $Q_1$.

According to Vaǐntrob [48], Condition (1) in Lemma 2.15 above means that the trivial vector bundle $\mathbb{R}^m \times \hat{k}^l \rightarrow \mathbb{R}^m$ carries a Lie algebroid structure.

It follows from Lemma 2.15 that $(C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0), d_L)$ is a cdga and $\varpi$ is a Maurer–Cartan element of the dgla $C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0$ determined by the differential $d_L \otimes \text{id}_{\mathcal{T}^0_{\text{poly}} \hat{k}^r[l]_0} = L_{d_L}$ and the restriction of the Schouten bracket in $\mathcal{T}^0_{\text{poly}} \mathcal{V}$.

In Section 2.2.3, we proved that $\Phi_1$ depends only locally on its arguments and that its restriction to $U$ is

$$C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} (\hat{k}^r[l]_0) \xrightarrow{(\hat{\mathcal{U}}_{\mathcal{V}}, \odot)} C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{D}^0_{\text{poly}} (\hat{k}^r[l]_0),$$

the first Taylor coefficient of the tangent $L_\infty$ morphism to $\mathcal{U}_{\mathcal{V}}$ at the Maurer–Cartan element $\varpi$.

Therefore, to establish Proposition 2.14, it suffices to prove that

$$(\hat{\mathcal{U}}_{\mathcal{V}, \varpi})_1 = \text{id} \circ (\text{td} \Phi_1)^{\frac{1}{2}}$$

in every coordinate chart $U$ of $M$ over which the vector bundles $B$ and $L$ are trivial.

Note that the dglas $C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{T}^0_{\text{poly}} (\hat{k}^r[l]_0)$ and $C^\infty(\mathbb{R}^m \times \hat{k}^0[l]_0) \otimes \mathcal{D}^0_{\text{poly}} (\hat{k}^r[l]_0)$, the restrictions of $\Gamma(\mathcal{T}^0_{\text{poly}} \mathcal{V}) \otimes_R \mathcal{T}^0_{\text{poly}}$ and $\Gamma(\mathcal{T}^0_{\text{poly}} \mathcal{V}) \otimes_R \mathcal{D}^0_{\text{poly}}$ over $U$, are dg Lie subalgebras of $(\mathcal{T}^0_{\text{poly}} \mathcal{V}, L_{Q_1}, [\cdot, \cdot])$ and $(\mathcal{D}^0_{\text{poly}} \mathcal{V}, d_{\mathcal{V}}, [\cdot, \cdot])$, respectively.

Now, consider the Kontsevich formality $L_\infty$ quasi-isomorphism

$$(\mathcal{T}^0_{\text{poly}} \mathcal{V}, 0, [-,-]) \xrightarrow{\mathcal{U}_{\mathcal{V}}} (\mathcal{D}^0_{\text{poly}} \mathcal{V}, d_{\mathcal{V}}, [-,-])$$
devised for $\mathcal{V} = \mathbb{R}^m \times \mathbb{k}_{\text{formal}}$ in [10, Appendix].

Since $[Q, Q] = 0$, the vector field $Q \in T^0_{\text{poly}} \mathcal{V}$ is a Maurer–Cartan element of the dgla $T^\bullet_{\text{poly}} \mathcal{V}$ and we can consider the tangent $L_\infty$ morphism $\mathcal{Y}_Q$ defined per Equation (12). Since $Q$ is a vector field, it follows from Theorem 2.4 (4) that $\mathcal{Y}^V(Q) = Q \in T^0_{\text{poly}} \mathcal{V}$, where $\mathcal{Y}^V(Q)$ is given by the graded version of Equation (11) as in [10]. Hence we obtain the $L_\infty$ quasi-isomorphism

$$(T^\bullet_{\text{poly}} \mathcal{V}, [Q, -], [-, -]) \xrightarrow{\mathcal{Y}^V_Q} (D^\bullet_{\text{poly}} \mathcal{V}, d, [Q, -], [[-], [-]]).$$

**Lemma 2.16.** In the category of cochain complexes of $k$-modules, the diagram

$$\xymatrix{ (T^\bullet_{\text{poly}} \mathcal{V}, [Q, -]) \ar[r]^-{(\mathcal{Y}^V_Q)_1} & (D^\bullet_{\text{poly}} \mathcal{V}, d, [Q, -]) \ar[r]^-{\alpha} & (\bigwedge C^\infty(\mathbb{R}^m \times \mathbb{k}^{0|l}_{\text{formal}}) \otimes T^\bullet \mathbb{k}_r_{\text{formal}})_{\mathfrak{I}} \ar[r]^-{\beta} & \bigwedge C^\infty(\mathbb{R}^m \times \mathbb{k}^{0|l}_{\text{formal}}) \otimes D^\bullet \mathbb{k}_r_{\text{formal}} \ar[l]^-{\mathfrak{J}} }$$

is commutative.

**Proof.** Let $\gamma \in C^\infty(\mathbb{R}^m \times \mathbb{k}^{0|l}_{\text{formal}}) \otimes T^k \mathbb{k}_r_{\text{formal}}$ be a $(k + 1)$-vector field. It follows from Equations (12), (13), and (14) that

$$(\mathcal{Y}^V_Q)_1(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{Y}^V_{1+j}(Q \wedge \cdots \wedge Q \wedge \gamma)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m \geq 0} \sum_{\Gamma \in \mathcal{J}_{1+j,m}} W_\Gamma \mathcal{Y}^V_{1+j}(Q \wedge \cdots \wedge Q \wedge \gamma)$$

and

$$(\mathcal{Y}^f_{\mathfrak{U}, \mathfrak{w}})_1(\gamma) = \sum_{j=0}^{\infty} \frac{1}{j!} (\mathcal{Y}^f_{\mathfrak{U}, \mathfrak{w}})_{1+j}(\mathfrak{w} \wedge \cdots \wedge \mathfrak{w} \wedge \gamma)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \mathcal{Y}^V_{1+j}(\mathfrak{w} \wedge \cdots \wedge \mathfrak{w} \wedge \gamma)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{m \geq 0} \sum_{\Gamma \in \mathcal{J}_{1+j,m}} W_\Gamma \mathcal{Y}^V_{1+j}(\mathfrak{w} \wedge \cdots \wedge \mathfrak{w} \wedge \gamma).$$

Therefore, since $Q = d_L + \mathfrak{w}$, it suffices to prove that, provided $X_1, X_2, \ldots, X_j \in \{d_L, \mathfrak{w}\}$ and $X_p = d_L$ for at least one $p \in \{1, 2, \ldots, j\}$, the expression $W_\Gamma \mathcal{Y}^V_{1+j}(X_1 \wedge X_2 \wedge \cdots \wedge X_j \wedge \gamma)$ vanishes for all $\Gamma \in \mathcal{J}_{1+j,m}$.

Given a graph $\Gamma \in \mathcal{J}_{1+j,m}$, we know that

- $W_\Gamma = 0$ if $|E_\Gamma| \neq 2j + m$;
- $\mathcal{Y}_\Gamma(X_1, \ldots, X_j, \gamma) = 0$ if $|E_\Gamma| \neq j + k + 1$;
- and $\mathcal{Y}_\Gamma(X_1, \ldots, X_j, \gamma) = 0$ if the number of edges starting from the $(j + 1)$-th aerial vertex is different from $k + 1$ (since $\gamma$ is a $(k + 1)$-vector field).

Therefore, if $\Gamma \in \mathcal{J}_{1+j,m}$, we have $W_\Gamma \mathcal{Y}_\Gamma(X_1, \ldots, X_j, \gamma) = 0$ unless $|\text{Out}(v_{j+1})| = k + 1 = j + m$. In other words, since $(j + 1) + m$ is the total number of vertices of the graph $\Gamma$, we have $W_\Gamma \mathcal{Y}_\Gamma(X_1, \ldots, X_j, \gamma) = 0$ unless a single edge runs from the $(j + 1)$-th aerial vertex of $\Gamma$ to each one of the other $j + m$ vertices of $\Gamma$.

However, if an edge $e'$ of $\Gamma$ starts at the $(j + 1)$-th aerial vertex $v_{j+1}$ (the aerial vertex corresponding to $\gamma$) and ends at $v_p$ (the aerial vertex corresponding to $X_p = d_L$), the factor $D_{\Gamma}^{v_p}(X_p^{\text{Out}(p)})$ appearing in each
term of the expansion (14) of $\mathcal{H}(X_1, \ldots, X_j, \gamma)$ must vanish. Indeed, $D_3^p$ is a composition of one or more partial derivatives w.r.t. the coordinates on $k[x]_r^{\text{formal}}$ containing $\frac{\partial}{\partial x_i(x')}$. The square root of the Todd cocycle is $\rho$.

The proof is complete. \hfill $\Box$

The following theorem was first announced by Shoikhet in [42]. We refer the interested reader to [43, 7] for more details.

**Theorem 2.17** (Kontsevich–Shoikhet [42]). The first ‘Taylor coefficient’ $(\mathcal{U}_Q^V)_1 : \mathcal{T}^\bullet_{\text{poly}} V \to \mathcal{D}^\bullet_{\text{poly}} V$ of the tangent $L_\infty$ quasi-isomorphism $\mathcal{U}_Q^V$ is the modification

$$ (\mathcal{U}_Q^V)_1 = \text{hkr} \circ (\text{td}_{T_V}^\text{trivial})^{\frac{1}{2}} $$

of the Hochschild–Kostant–Rosenberg map by (the square root of) the Todd cocycle $\text{td}_{T_V}^\text{trivial} \in \prod_{k=0}^\infty \Omega^k(V)$ of the dg manifold $(V, Q)$ associated with the trivial connection as in Example 1.16. The Todd cocycle acts on $\mathcal{T}^\bullet_{\text{poly}} V$ by contraction.

**Lemma 2.18.** For all $s \in \mathbb{N}$, we have

$$ \text{tr} \left( (\text{At}_{\mathcal{F}[U]}^\text{can})^s \right) = \mathcal{F}^\top \text{str} \left( (\text{At}_{T_V}^\text{trivial})^s \right). $$

**Proof.** Let $U \xrightarrow{(x_1, \ldots, x_m)} \mathbb{R}^m$ be a local chart of $\mathcal{M}$.

Let $\partial_1, \ldots, \partial_r$ be a local frame for $B \to M$ over $U$ and let $\chi_1, \ldots, \chi_r$ be the dual local frame for $B^\vee \to M$. Likewise, let $\eta_1, \ldots, \eta_l$ be a local frame for $L \to M$ over $U$ and let $\lambda_1, \ldots, \lambda_l$ be the dual local frame for $L^\vee \to M$ with the degree shift: $|\lambda_j| = 1$.

The restrictions to $U$ of the anchor map $\rho : L \to \mathcal{T}_M$, the Lie bracket on $\Gamma(L)$, the bundle map $q : L \to B$, and the $L$-connection $\nabla$ on $B$ admit local expressions

$$ \rho(\eta_j) = \sum_{i=1}^m \rho_{ij} \frac{\partial}{\partial x_i} \quad [\eta_i, \eta_j] = \sum_{k=1}^l c_{ij}^k \eta_k $$

$$ q(\eta_j) = \sum_{i=1}^r q_{ij} \partial_i \quad \nabla_{\eta_i} \partial_j = \sum_{k=1}^r \Gamma_{ij}^k \partial_k $$

where $\rho_{ij}$, $c_{ij}^k$, $q_{ij}$, and $\Gamma_{ij}^k$ are functions of the coordinates $x_1, \ldots, x_m$.

Then $(z_1, \ldots, z_{m+l+r}) = (x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_l, \chi_1, \ldots, \chi_r)$ are coordinates on $\mathcal{V} = \mathbb{R}^m \times k[x]_r^{\text{formal}}$ whose degrees are

$$ |x_i| = 0, \quad |\lambda_j| = 1, \quad |\chi_k| = 0, $$

for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, l\}$, and $k \in \{1, \ldots, r\}$. The homological vector field on $\mathcal{V}$ is the sum $Q = -\delta + d_L^\nabla + X^\nabla$ of

$$ \delta = \sum_{k=1}^r \sum_{j=1}^l q_{kj} \lambda_j^j \frac{\partial}{\partial \chi_k}, $$

$$ d_L^\nabla = \sum_{j=1}^m \sum_{i=1}^l \rho_{ji} \lambda_i^i \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{i,j,k=1}^l c_{ij}^k \lambda_i^j \lambda_j^k \frac{\partial}{\partial \lambda_k} - \sum_{i=1}^l \sum_{j=1}^r \Gamma_{ij}^k \lambda_i^j \chi_j^k \frac{\partial}{\partial \chi_k}, $$

where $\delta$ is the differential, $d_L^\nabla$ is the divergence of the homological vector field on $\mathcal{V}$, and $X^\nabla$ is the homological vector field on $\mathcal{V}$.
and

\[ X^\nabla = \sum_{k=1}^{r} f_k \frac{\partial}{\partial \chi_k}, \]

where the functions \( f_k \in C^\infty(V) \) are linear in the \( \lambda \)-coordinates and have trivial jet of order 1 w.r.t. the \( \chi \)-coordinates as per Theorem 4.7.

Let \( \hat{z}_1, \ldots, \hat{z}_{m+l+r} \) be the local frame of \( T_V^\vee \) dual to \( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{m+l+r}} \). This local frame is essentially \( dz_1, \ldots, dz_{m+l+r} \) but they have different degrees: \( |\hat{z}_i| = |dz_i| - 1 = |z_i| \). It follows from Lemma 1.18 that

\[ \text{At}^{\text{can}}_{T^\vee|U} = \sum_{i,j,k=1}^{r} \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j} \hat{z}_i \otimes \left( \hat{z}_j \otimes \frac{\partial}{\partial z_k} \right) \]  

(16)

and from Example 1.16 that the Atiyah 1-cocycle of the dg Lie algebroid \( T_V \) associated with the trivial connection \( \nabla^\text{trivial} \frac{\partial}{\partial z_j} = 0 \) is

\[ \text{At}^{\text{trivial}}_{T_V} = \sum_{i,j,k=1}^{m+l+r} (-1)^{|z_i|+|z_j|} \frac{\partial^2 (Q(z_k))}{\partial z_i \partial z_j} \hat{z}_i \otimes \left( \hat{z}_j \otimes \frac{\partial}{\partial z_k} \right). \]

Then, we have

\[ (\mathcal{G}^\top \otimes \text{id})(\text{At}^{\text{trivial}}_{T_V}) = \sum_{i,j,k=1}^{r} \sum_{i,j,k=1}^{r} (-1)^{|z_i|} \frac{\partial^2 (Q(z_k))}{\partial \chi_i \partial \chi_j} \hat{z}_i \otimes \left( \hat{z}_j \otimes \frac{\partial}{\partial z_k} \right). \]

Since \( Q(x_k) = \sum_{i=1}^{l} \rho_{ki} \lambda_i \) and the functions \( \rho_{ki} \) depend on the \( x \)-coordinates only, we have

\[ \frac{\partial^2 (Q(x_k))}{\partial \chi_i \partial \chi_j} = 0. \]

Since \( Q(\lambda_k) = -\frac{1}{2} \sum_{i,j=1}^{l} \epsilon_{ij} \lambda_i \lambda_j \) and the functions \( \epsilon_{ij} \) depend on the \( x \)-coordinates only, we have

\[ \frac{\partial^2 (Q(\lambda_k))}{\partial \chi_i \partial \chi_j} = 0. \]

Since \( Q(\chi_k) = -\sum_{j=1}^{l} q_{kj} \lambda_j - \sum_{i,j} \Gamma^i_{ij} \lambda_i \chi_j + f_k \), the functions \( q_{kj} \) and \( \Gamma^i_{jk} \) depend on the \( x \)-coordinates only, we have

\[ \frac{\partial^2 (Q(\chi_k))}{\partial \chi_i \partial \chi_j} = \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j}. \]

Therefore, the matrix representation of \( (\mathcal{G}^\top \otimes \text{id})(\text{At}^{\text{trivial}}_{T^\vee|U}) \) with respect to the frame \( \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{m+l+r}} \right) \) is

\[ (\mathcal{G}^\top \otimes \text{id})(\text{At}^{\text{trivial}}_{T^\vee|U}) = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & \frac{\partial^2 f_k}{\partial \chi_i \partial \chi_j} \hat{z}_i \end{bmatrix}. \]  

(17)

It follows immediately from Equations (16) and (17) that the image of \( \text{At}^{\text{can}}_{T^\vee|U} \) under the composition

\[ \mathcal{F}^\vee \otimes \text{End}(\mathcal{F}) \xrightarrow{\left(\ldots\right)^s} \Lambda^s \mathcal{F}^\vee \otimes \text{End}(\mathcal{F}) \xrightarrow{\text{id} \otimes \text{tr}} \Lambda^s \mathcal{F}^\vee \]

and the image of \( (\mathcal{G}^\top \otimes \text{id})(\text{At}^{\text{trivial}}_{T^\vee|U}) \) under the composition

\[ \mathcal{F}^\vee|U \otimes \text{End}(T_V) \xrightarrow{\left(\ldots\right)^s} \Lambda^s \mathcal{F}^\vee|U \otimes \text{End}(T_V) \xrightarrow{\text{id} \otimes \text{str}} \Lambda^s \mathcal{F}^\vee|U \]
are equal:
\[(\text{id} \otimes \text{tr})(\text{At}_{\mathcal{T}}^{\text{can}}) = (\text{id} \otimes \text{str})(\mathcal{S}^{\text{T}} \otimes \text{id})(\text{At}_{\mathcal{T}}^{\text{triv}})).\]

Since the diagram
\[
\begin{array}{ccc}
T_{\mathcal{V}} \otimes \text{End}(T_{\mathcal{V}}) & \xrightarrow{(\ldots)^{*}} & \Lambda^{s}T_{\mathcal{V}} \otimes \text{End}(T_{\mathcal{V}}) \\
\mathcal{F}_{U}^{\mathcal{V}} \otimes \text{End}(T_{\mathcal{V}}) & \xrightarrow{(\ldots)^{*}} & \Lambda^{s}\mathcal{F}_{U}^{\mathcal{V}} \otimes \text{End}(T_{\mathcal{V}})
\end{array}
\]
commutes, we obtain the desired conclusion:
\[(\text{id} \otimes \text{tr})(\text{At}_{\mathcal{T}}^{\text{can}}) = \mathcal{S}^{\text{T}}(\text{id} \otimes \text{str})(\text{At}_{\mathcal{T}}^{\text{triv}})).\]

Since the Todd cocycle can be expressed in terms of scalar Atiyah cocycles \(\frac{1}{2\pi}(\ldots)^{s} \text{str}(\text{At}^{s})\), we have the following immediate corollary.

**Corollary 2.19.** The diagram
\[
\begin{array}{ccc}
\mathcal{T}_{\mathcal{poly}}(\mathcal{V}) & \xrightarrow{\tilde{\text{td}}_{\mathcal{V}}^{\text{triv}}} & \mathcal{T}_{\mathcal{poly}}(\mathcal{V}) \\
C^{\infty}(\mathbb{R}^{m} \times \mathfrak{k}^{0}_{\text{formal}}) \otimes \mathcal{T}_{\mathcal{poly}^{\text{triv}}} & \xrightarrow{\tilde{\text{td}}_{\mathcal{V}}^{\text{can}}} & C^{\infty}(\mathbb{R}^{m} \times \mathfrak{k}^{0}_{\text{formal}}) \otimes \mathcal{T}_{\mathcal{poly}^{\text{triv}}} \\
\mathcal{F}_{U}^{\mathcal{V}} & \xrightarrow{\tilde{\text{td}}_{\mathcal{V}}^{\text{triv}}} & \mathcal{F}_{U}^{\mathcal{V}}
\end{array}
\]
commutes.

Here \(\tilde{\text{td}}_{\mathcal{V}}^{\text{can}} \in \prod_{k=0}^{\infty} C^{\infty}(\mathbb{R}^{m} \times \mathfrak{k}^{0}_{\text{formal}}) \otimes \Omega^{k}(\mathfrak{k}^{0}_{\text{formal}})\) is the Todd cocycle of the restriction to \(U\) of the Fedosov Lie algebroid \(\mathcal{F}\) associated with the canonical connection while \(\tilde{\text{td}}_{\mathcal{V}}^{\text{triv}} \in \prod_{k=0}^{\infty} \Omega^{k}(\mathcal{V})\) is the Todd cocycle of the dg manifold \((\mathcal{V}, Q)\) associated with the trivial connection as in Example 1.16.

The Todd cocycles act by contraction on the spaces of polyvector fields.

Finally we have
\[\mathcal{S} \circ (\mathcal{W}_{U, \varnothing})_{1} = (\mathcal{W}_{Q}^{\mathcal{V}})_{1} \circ \mathcal{S} \quad \text{(by Lemma 2.16)}\]
\[= \text{hkr} \circ (\tilde{\text{td}}_{\mathcal{T}}^{\text{triv}})^{\frac{1}{2}} \circ \mathcal{S} \quad \text{(by Theorem 2.17)}\]
\[= \text{hkr} \circ \mathcal{S} \circ (\tilde{\text{td}}_{\mathcal{T}}^{\text{can}})^{\frac{1}{2}} \quad \text{(by Corollary 2.19)}\]
\[= \mathcal{S} \circ \text{hkr} \circ (\tilde{\text{td}}_{\mathcal{T}}^{\text{can}})^{\frac{1}{2}}\]

Therefore, in every coordinate chart \(U\) of \(M\) over which the vector bundles \(B\) and \(L\) are trivial, we have
\[(\mathcal{W}_{U, \varnothing})_{1} = \text{hkr} \circ (\tilde{\text{td}}_{\mathcal{T}}^{\text{can}})^{\frac{1}{2}}.\]

The proof of Proposition 2.14 is complete.

2.5. **Proof of Theorem 2.3.** The difference between Equations (9) and (8) is the factor \(e^{\frac{1}{2}} \text{tr} \text{At}_{\mathcal{T}}^{\text{can}}\). We start with considering \(\text{tr} \text{At}_{\mathcal{T}}^{\text{can}} \in \Gamma(\mathcal{T}_{\mathcal{V}})_{1}\).

**Lemma 2.20.** We have
\[\text{tr} \text{At}_{\mathcal{T}}^{\text{can}} = d_{\mathcal{T}}(\text{div} X^{\nabla}),\]
where \(\text{div} X^{\nabla} \in C^{\infty}(M) = \Gamma(L^{\mathcal{V}} \otimes \hat{B}^{\nabla})\) is the divergence of the formal vertical vector field \(X^{\nabla}\). (Since the vector field \(X^{\nabla}\) is tangent to the fibers and the fibers are vector spaces, we do not need to specify a volume
form in order to make sense of the divergence of $X^\nabla$. Indeed, on a vector space, the divergence is canonically defined.) More explicitly, $\text{div } X^\nabla = \sum_k \partial_k f_k$.

**Proof.** By Equation (10),

$$\text{tr } \text{At}^\text{can}_\mathcal{F} = \sum_{i,k=1}^r \hat{\partial}_i (\hat{\partial}_k f_k) \chi_i = d_\mathcal{F}(\text{div } X^\nabla).$$

**Lemma 2.21.** Let $\mathcal{A} \to \mathcal{M}$ be a dg Lie algebroid. Let $\mathcal{Q}$ denote the endomorphism of $\Gamma(\mathcal{A})$ encoding the dg structure and let $d_\mathcal{A}$ denote the Chevalley–Eilenberg differential. If $\xi \in \Gamma(\mathcal{A}^\nabla)$ satisfies $d_\mathcal{A} = 0$ and $\mathcal{Q} = 0$, then the contraction with $\xi$ is a derivation of the differential Gerstenhaber algebra $(\Gamma(\Lambda^\bullet \mathcal{A}), [-, -], \mathcal{Q})$.

Applying Lemma 2.21 to the Fedosov dg Lie algebroid $\mathcal{F}$ and the section $\text{tr } (\text{At}^\text{can}_\mathcal{F})$ of $\mathcal{F}^\nabla$ and noting that $\Gamma(\Lambda^\bullet \mathcal{F}) \cong \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)$ and $\mathcal{Q} = [\mathcal{Q}, -]$, we obtain

**Corollary 2.22.**

1. The contraction by $\text{tr } \text{At}^\text{can}_\mathcal{F}$ is a derivation of the differential Gerstenhaber algebra

$$\left( \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet), [\mathcal{Q}, -] \right).$$

2. The contraction by $e^{\frac{1}{2} \text{tr } \text{At}^\text{can}_\mathcal{F}}$ is an automorphism of the differential Gerstenhaber algebra

$$\left( \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet), [\mathcal{Q}, -] \right).$$

The composition $\Psi = \Phi \circ e^{\frac{1}{2} \text{tr } \text{At}^\text{can}_\mathcal{F}}$ of the $L_\infty$ quasi-isomorphism

$$\Phi : \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) \to \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)$$

constructed in Proposition 2.7 of $\Phi$ with the contraction operator $e^{\frac{1}{2} \text{tr } \text{At}^\text{can}_\mathcal{F}}$, which is an automorphism of the dgla $\text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)$ according to Corollary 2.22, is an $L_\infty$ quasi-isomorphism $\Psi$ from $\text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)$ to $\text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)$. Its first Taylor coefficient is

$$\Psi_1 = \Phi_1 \circ e^{\frac{1}{2} \text{tr } \text{At}^\text{can}_\mathcal{F}} = \text{hkr } (\text{td}^\text{can}_\mathcal{F})^{\frac{1}{2}} \circ e^{\frac{1}{2} \text{tr } \text{At}^\text{can}_\mathcal{F}} = \text{hkr } (\text{td}^\text{can}_\mathcal{F})^{\frac{1}{2}}$$

— we used Proposition 2.14 in the second equality. The proof of Theorem 2.3 is thus complete.

### 2.6. Proof of Theorem 2.1

A straightforward computation yields the following lemma.

**Lemma 2.23.** The following diagram commutes.

$$\begin{array}{ccc}
\text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) & \xrightarrow{\text{hkr}} & \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) \\
\sigma_\mathcal{F} & & \sigma_\mathcal{F} \\
\text{tot}(\Gamma(\Lambda^\bullet A^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) & \xrightarrow{\text{hkr}} & \text{tot}(\Gamma(\Lambda^\bullet A^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)
\end{array}$$

The following result is an immediate consequence of Proposition 1.19.

**Corollary 2.24.** The following diagram commutes.

$$\begin{array}{ccc}
\text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) & \xrightarrow{(\text{td}^\text{can}_\mathcal{F})^{\frac{1}{2}}} & \text{tot}(\Gamma(\Lambda^\bullet L^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) \\
\sigma_\mathcal{F} & & \sigma_\mathcal{F} \\
\text{tot}(\Gamma(\Lambda^\bullet A^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet) & \xrightarrow{(\text{td}^\nabla_{\Lambda^\bullet})^{\frac{1}{2}}} & \text{tot}(\Gamma(\Lambda^\bullet A^\nabla) \otimes_R \mathcal{D}_\text{poly}^\bullet)
\end{array}$$
According to Theorem 4.12, we have a contraction
\[
\left( \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}} \right), q^A_{\text{Bott}} \right) \xrightarrow{\tilde{\tau}_2} \left( \text{tot} \left( \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}(B) \right), L_Q \right) \xrightarrow{h_2}.
\]

The r.h.s. \( \text{tot} \left( \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}(B) \right) \) is a dgla while the l.h.s. \( \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}} \right) \) inherits an \( L_\infty \)-structure from the dgla structure of the r.h.s. by homotopy transfer.

**Lemma 2.25** (Homotopy transfer of \( L_\infty \) structures [2, Theorem 1.9]). Let \((C, \partial)\) and \((K, d)\) be two cochain complexes and let

\[
\begin{bmatrix}
(C, \partial) \\
(K, d)
\end{bmatrix} \xrightarrow{\sigma} h
\]

be a contraction of \((K, d)\) onto \((C, \partial)\). Given an \( L_\infty \) algebra structure on \( K \) (with \( d \) as unary bracket), there exists a ‘transferred’ \( L_\infty \)-algebra structure on \( C \) and a pair of \( L_\infty \) quasi-isomorphisms \( T : C \to K \) and \( \Sigma : K \to C \) having the chain maps \( \tau \) and \( \sigma \) as respective first Taylor coefficients.

Lemma 2.25 asserts the existence of an \( L_\infty \) quasi-isomorphism \( T_\sharp \) having \( \tilde{\tau}_2 \) as first Taylor coefficient.

According to Theorem 4.18, there is also a contraction
\[
\left( \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \right), d_A^L + d_{\mathcal{F}} \right) \xrightarrow{\tilde{\tau}_2} \left( \text{tot} \left( \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \right), [Q + m, \cdot] \right) \xrightarrow{h_2}.
\]

Again, the l.h.s. inherits an \( L_\infty \)-structure from the dgla structure of the r.h.s. by homotopy transfer. Lemma 2.25 asserts the existence of an \( L_\infty \) quasi-isomorphism \( \Sigma_\sharp \) having \( \sigma_\sharp \) as first Taylor coefficient.

Consider the \( L_\infty \) quasi-isomorphism
\[
\mathcal{I} : \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \right) \to \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \right)
\]

obtained as the composition
\[
\mathcal{I} = \Sigma_\sharp \circ \psi \circ \tilde{T}_\sharp
\]

of the \( L_\infty \) quasi-isomorphism \( \psi : \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}} \to \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}} \) of Theorem 2.3 with the \( L_\infty \) quasi-isomorphisms \( \tilde{T}_\sharp \) and \( \Sigma_\sharp \):
\[
\begin{array}{ccc}
\text{tot} \left( \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}(B) \right) & \xrightarrow{\psi} & \text{tot} \left( \Gamma(L^\bullet L^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}}(B) \right) \\
\tilde{T}_\sharp & \downarrow & \downarrow \Sigma_\sharp \\
\text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{T}^\bullet_{\text{poly}} \right) & \xrightarrow{\mathcal{I}_1 = (\Sigma_\sharp)_{1} \circ \psi_{1} \circ (\tilde{T}_\sharp)_{1}} & \text{tot} \left( \Gamma(L^\bullet A^\vee) \otimes_R \mathcal{D}^\bullet_{\text{poly}} \right).
\end{array}
\]

Its first Taylor coefficient is
\[
\mathcal{I}_1 = (\Sigma_\sharp)_{1} \circ \psi_{1} \circ (\tilde{T}_\sharp)_{1} = \sigma_\sharp \circ (hkr \circ (td_{\mathcal{F}}^{\text{can}})^{\frac{1}{2}}) \circ \tilde{\tau}_2 \quad \text{(by Theorem 2.3)}
\]
\[
= hkr \circ \sigma_\sharp \circ (td_{\mathcal{F}}^{\text{can}})^{\frac{1}{2}} \circ \tilde{\tau}_2 \quad \text{(by Lemma 2.23)}
\]
\[
= hkr \circ (td_{L/A}^{\nabla})^{\frac{1}{2}} \circ \sigma_\sharp \circ \tilde{\tau}_2 \quad \text{(by Corollary 2.24)}
\]
\[
= hkr \circ (td_{L/A}^{\nabla})^{\frac{1}{2}}
\]

This concludes the proof of Theorem 2.1.
3. Applications

3.1. Complex manifolds. Let $X$ be a complex manifold. Then $T_X \otimes \mathbb{C} \cong T_X^{0,1} \bowtie T_X^{1,0}$ is a matched pair of Lie algebroids (see Section 4.1 or [38] for the definition of matched pairs). Hence $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is a Lie pair with quotient $T_X^{1,0}$. Its Bott representation is the flat $T_X^{0,1}$-connection on $T_X^{1,0}$ which encodes the holomorphic vector bundle structure of $T_X^{1,0}$; the sections of $T_X^{1,0}$ which are flat w.r.t. the $T_X^{0,1}$-connection are precisely the holomorphic sections of $T_X^{1,0}$. It is simple to see that the Bott $A$-connection on $L/A$, which is a flat $T_X^{0,1}$-connection on $T_X^{1,0}$, coincides exactly with the one making $T_X^{1,0}$ into a holomorphic vector bundle over $X$. In other words, the Chevalley–Eilenberg differential associated with the Bott representation of the Lie pair $(T_X \otimes \mathbb{C}, T_X^{0,1})$ is the Dolbeault operator

$$
\bar{\partial} : \Omega^{0,\bullet}(T_X^{1,0}) \to \Omega^{0,\bullet+1}(T_X^{1,0}).
$$

3.1.1. Atiyah and Todd classes of complex manifolds. A torsion-free $T_X \otimes \mathbb{C}$-connection $\nabla$ on $T_X^{1,0}$ extending the Bott $T_X^{0,1}$-connection is necessarily the sum $\nabla = \bar{\partial} + \nabla^{1,0}$ — more precisely $d\nabla = \bar{\partial} + d\nabla^{1,0}$ — of the Dolbeault operator and a torsion-free $T_X^{0,1}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$, i.e. a $\mathbb{C}$-bilinear map $\nabla^{1,0} : \Gamma(T_X^{1,0}) \times \Gamma(T_X^{1,0}) \to \Gamma(T_X^{1,0})$ satisfying the usual connection axioms and the condition

$$
\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \Gamma(T_X^{1,0}).
$$

The Atiyah cocycle associated with such a connection $\nabla$ is the element $R_{1,1}^\nabla \in \Omega^{1,1}(\text{End}(T_X^{1,0}))$ defined by

$$
R_{1,1}^\nabla(a; b) = \nabla_a \nabla_b - \nabla_b \nabla_a - \nabla_{[a, b]}, \quad \forall a \in \Gamma(T_X^{1,1}), b \in \Gamma(T_X^{1,0}).
$$

Its cohomology class $\alpha_X \in H^{1,1}(X, \text{End}(T_X^{1,0}))$ is independent of the choice of the connection $\nabla$ and is precisely the Atiyah class of the complex manifold $X$.

The Todd cocycle associated with the connection $\nabla$ is

$$
\text{td}_X = \det \left( \frac{R_{1,1}^\nabla}{1 - e^{-R_{1,1}^\nabla}} \right) \in \bigoplus_{k=0}^{\infty} \Omega^k(\Lambda^k(T_X^{0,1})^\vee) \cong \bigoplus_{k=0}^{\infty} \Omega^{k,k}(X).
$$

Its cohomology class

$$
\text{Td}_X = \det \left( \frac{\alpha_X}{1 - e^{-\alpha_X}} \right) \in \bigoplus_{k=0}^{\infty} H^k(X, \Lambda^k(T_X^{0,1})^\vee).
$$

is independent of the choice of the connection $\nabla$ and is called the Todd class of the complex manifold $X$.

3.1.2. Polyvector fields and polydifferential operators on complex manifolds. Since $T_X^{1,0} \bowtie T_X^{1,0}$ is a matched pair, it follows from Corollary 4.14 that $\Omega^{0,\bullet}(X, T_{\text{poly}}(X))$ is a differential Gerstenhaber algebra with the Dolbeault operator $\bar{\partial} : \Omega^{0,\bullet}(X, T_{\text{poly}}(X)) \to \Omega^{0,\bullet+1}(X, T_{\text{poly}}(X))$ as differential, the wedge product as associative multiplication and the natural extension (see Equation (21))

$$
[\xi_1 \otimes b_1, \xi_2 \otimes b_2] = \xi_1 \wedge \xi_2 \otimes [b_1, b_2] + \xi_1 \wedge \nabla^{\text{Bott}}_b \xi_2 \otimes b_2 - \nabla^{\text{Bott}}_b \xi_1 \wedge \xi_2 \otimes b_1,
$$

$$
\forall \xi_1, \xi_2 \in \Omega^{0,\bullet}(X), \quad b_1, b_2 \in \Gamma(T_X^{1,0}),
$$

of the Lie bracket on $\Gamma(T_X^{1,0})$ as graded Lie bracket.

Similarly, $\Omega^{0,\bullet}(X, D_{\text{poly}}(X))$ is a dgla and its cohomology is a Gerstenhaber algebra (Corollary 4.19). Here the differential is $\bar{\partial} + \text{id} \otimes d_{\mathcal{F}}$, where $\bar{\partial} : \Omega^{0,\bullet}(X, D_{\text{poly}}(X)) \to \Omega^{0,\bullet+1}(X, D_{\text{poly}}(X))$ is the Dolbeault operator, while the associative multiplication and the graded Lie bracket are given by Proposition 4.4.
Note that \((\Omega^0(\mathcal{X}, T_{\text{poly}}^\bullet(\mathcal{X})), \bar{\partial})\) is the Dolbeault resolution of the complex of sheaves

\[
0 \to \mathcal{O}_X \xrightarrow{0} \Theta_X \xrightarrow{0} \Lambda^2 \Theta_X \xrightarrow{0} \Lambda^3 \Theta_X \to \cdots
\]

of holomorphic polyvector fields over \(X\), while \((\Omega^0(\mathcal{X}, \mathcal{D}_{\text{poly}}^\bullet(\mathcal{X})), \bar{\partial} + \text{id} \otimes d_H)\) is the Dolbeault resolution of the complex of sheaves

\[
0 \to \mathcal{O}_X \to \mathcal{D}_X \xrightarrow{d_{\mathcal{D}}} \mathcal{D}_X^2 \xrightarrow{d_{\mathcal{D}}} \mathcal{D}_X^3 \to \cdots
\]

of holomorphic polydifferential operators over \(X\).

As a result, for the Lie pair \((\mathcal{L} = T_X \otimes \mathbb{C}, \mathcal{A} = T_{X}^{0,1})\), the cohomology

\[
\mathbb{H}^\bullet_{\text{CE}}(\mathcal{A}, T^\bullet_{\text{poly}}(\mathcal{X})) = \mathbb{H}((\Omega^0(\mathcal{X}, T_{\text{poly}}^\bullet(\mathcal{X})), \bar{\partial})
\]

is isomorphic to the sheaf cohomology \(\mathbb{H}^\bullet_{\text{sheaf}}(\mathcal{X}, \Lambda^\bullet \Theta_X)\) while the cohomology

\[
\mathbb{H}^\bullet_{\text{CE}}(\mathcal{A}, D^\bullet_{\text{poly}}) = \mathbb{H}^\bullet(\Omega^0(\mathcal{X}, D_{\text{poly}}^\bullet(\mathcal{X})), \bar{\partial} + \text{id} \otimes d_H)
\]

is isomorphic to the Hochschild cohomology \(HH^\bullet(X) \cong \text{Ext}^\bullet_{\mathcal{O}_X \times \mathcal{X}}(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta})\) (see [9]) of the complex manifold \(X\).

### 3.1.3. Formality theorem for complex manifolds

Theorems 2.1 and 2.2 imply the following two theorems.

**Theorem 3.1** (Formality theorem for complex manifolds). Let \(X\) be a complex manifold. Choose a torsion-free \(T_{X}^{1,0}\)-connection \(\nabla_{1,0}\) on \(T_{X}^{1,0}\). There exists an \(L_\infty\) quasi isomorphism \(I_1: \Omega^0_{\text{poly}}(\mathcal{X}) \to \Omega^0_{\text{poly}}(\mathcal{X})\) with first Taylor coefficient \(I_1\) satisfying the following two properties:

- \(I_1\) preserves the associative algebra structures up to homotopy;
- \(I_1 = \text{hkr}(\text{td}_{\mathcal{X}}^{1,0})^{1/2}\), where the square root of the Todd cocycle \(\text{td}_{\mathcal{X}}^{1,0} = \bigoplus_{k=0} \Omega^{k,k}(X)\) acts on \(\Omega^0_{\text{poly}}(\mathcal{X})\) by contraction.

**Theorem 3.2** (Kontsevich-Duflo theorem for complex manifolds). For every complex manifold \(X\), the composition

\[
\text{hkr}(\text{td}_{\mathcal{X}}^{1,0})^{1/2} : \mathbb{H}^\bullet_{\text{sheaf}}(\mathcal{X}, \Lambda^\bullet T_X) \to HH^\bullet(X)
\]

is an isomorphism of Gerstenhaber algebras. It is understood that the square root of the Todd class

\[
(\text{td}_{\mathcal{X}}^{1,0}) \in \bigoplus_{k=0} \Omega^{k,k}(X) \cong H^k_{\text{sheaf}}(X, \Omega_X^k)
\]

acts on \(\mathbb{H}^\bullet_{\text{sheaf}}(\mathcal{X}, \Lambda^\bullet T_X)\) by contraction.

The Kontsevich-Duflo theorem for complex manifolds is due to Kontsevich [22] (for associative algebra structures). See [8] for a detailed proof including Gerstenhaber algebra structures.

### 3.2. Lie algebra pairs

A Lie algebra pair is a Lie pair \((\mathfrak{g}, \mathfrak{h})\), where \(\mathfrak{g}\) is a finite-dimensional Lie algebra and \(\mathfrak{h}\) is a Lie subalgebra of \(\mathfrak{g}\).
3.2.1. Atiyah and Todd classes of Lie algebra pairs. A \( g \)-connection on \( g/h \) is simply a bilinear map \( \nabla : g \times g/h \to g/h \). Its torsion is the linear map \( T^\nabla : \Lambda^2 g \to g/h \) defined by

\[
T^\nabla(X,Y) = \nabla_X q(Y) - \nabla_Y q(X) - q([X,Y]), \quad \forall X,Y \in g.
\]

The map \( q : g \to g/h \) is the canonical projection.

Let \( \nabla \) be a \( g \)-connection on \( g/h \) which extends the Bott \( h \)-connection: \( \nabla^\text{Bott}_a q(l) = q([a,l]) \), for all \( a \in h \) and \( l \in g \). The Atiyah cocycle associated with \( \nabla \) is the bilinear map \( R^\nabla_{1,1} : h \otimes g/h \to \text{End}(g/h) \)

\[
R^\nabla_{1,1}(a; q(l)) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}, \quad \forall a \in h, \ l \in g.
\]

According to Proposition 1.6, the element \( R^\nabla_{1,1} \in h^\vee \otimes h^\perp \otimes \text{End}(g/h) \) is a Chevalley–Eilenberg 1-cocycle for the Lie algebra \( h \) with values in the \( h \)-module \( h^\perp \otimes \text{End}(g/h) \). Its cohomology class \( \alpha_{g/h} \in H^1_{\text{CE}}(h,h^\perp \otimes \text{End}(g/h)) \) is independent of the choice of \( g \)-connection \( \nabla \) and is called the Atiyah class of the Lie algebra pair \( (g,h) \).

The Todd class of the Lie algebra pair \( (g,h) \) is the corresponding Chevalley-Eilenberg cohomology class

\[
\text{Td}_{g/h} = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0} \Lambda^k h^\vee \otimes \Lambda^k h^\perp.
\]

3.2.2. Polyvector fields and polydifferential operators on Lie algebra pairs. For a Lie algebra pair \( (g,h) \), it follows from Corollary 4.14 and Corollary 4.19 that both \( \text{tot} (\Lambda^\bullet h^\vee \otimes \Lambda^{\bullet+1}(g/h)) \) and \( \text{tot} (\Lambda^\bullet h^\vee \otimes \left( \frac{\mathcal{U}(g)}{\mathcal{U}(g) \cdot h} \right)^{\otimes \bullet+1} \) carry \( L_\infty \) algebra structures unique up to \( L_\infty \)-quasi-isomorphisms. (Whenever \( g = h \gg m \) is a matched pair, \( \frac{\mathcal{U}(g)}{\mathcal{U}(g) \cdot h} \) is isomorphic to \( \mathcal{U}(m) \) and the two \( L_\infty \) algebras above are actually differential graded Lie algebras.)

The quotient \( g/h \) of a Lie algebra pair \( (g,h) \) is an \( h \)-module with the action

\[
a \cdot q(l) = \nabla^\text{Bott}_a q(l) = q([a,l]), \quad \forall a \in h, \ l \in g.
\]

Again, \( q : g \to g/h \) is the canonical projection. This action extends by the Leibniz rule to an \( h \)-action on \( \mathcal{D}^\text{poly}_\bullet = \Lambda^{\bullet+1}(g/h) \). Let \( d^\text{Bott}_h : \Lambda^p h^\vee \otimes \Lambda^{q+1}(g/h) \to \Lambda^{p+1} h^\vee \otimes \Lambda^{q+1}(g/h) \) be the corresponding Chevalley–Eilenberg differential. According to Corollary 4.14, the space \( \text{tot} (\Lambda^\bullet h^\vee \otimes \Lambda^{\bullet+1}(g/h)) \) carries an \( L_\infty \) algebra structure, unique up to \( L_\infty \) quasi-isomorphism, with \( d^\text{Bott}_h \) as first bracket. Furthermore, when endowed with the wedge product, the hypercohomology \( \mathbb{H}^\text{CE}_\bullet (h,\Lambda^{\bullet+1}(g/h)) \) becomes a Gerstenhaber algebra.

Similarly, the Lie algebra \( h \) acts on \( \mathcal{D}^\text{poly}_0 = \frac{\mathcal{U}(g)}{\mathcal{U}(g) \cdot h} \) by left multiplication and henceforth on \( \mathcal{D}_\text{poly}^\bullet = \left( \frac{\mathcal{U}(g)}{\mathcal{U}(g) \cdot h} \right)^{\otimes \bullet+1} \) as well. The Chevalley–Eilenberg differential associated with this action is denoted

\[
d^A_h : \Lambda^p h^\vee \otimes \mathcal{D}^q_\text{poly} \to \Lambda^{p+1} h^\vee \otimes \mathcal{D}^q_\text{poly},
\]

Meanwhile, the Hochschild differential \( d_\text{HH} : \mathcal{D}^q_\text{poly} \to \mathcal{D}^{q+1}_\text{poly} \) extends to

\[
id \otimes d_\text{HH} : \Lambda^p h^\vee \otimes \mathcal{D}^q_\text{poly} \to \Lambda^p h^\vee \otimes \mathcal{D}^{q+1}_\text{poly}.
\]
By Corollary 4.19, the graded vector space \( \text{tot}(\Lambda^\bullet h^\vee \otimes \left( \frac{U(g)}{U(g) \cdot h} \right)^{\otimes \bullet+1}) \) carries an \( L_\infty \) algebra structure, unique up to quasi-isomorphism, with \( d_4^L + \text{id} \otimes d_{g^\vee} \) as first bracket. When endowed with the cup product, the corresponding hypercohomology \( H^\bullet_{\mathbb{C}E}(h, \left( \frac{U(g)}{U(g) \cdot h} \right)^{\otimes \bullet+1}) \) becomes a Gerstenhaber algebra.

The above \( L_\infty \) algebra structures depend on the choice of a splitting of the short exact sequence \( 0 \to h \to g \to g/h \to 0 \) and a torsion-free \( g \)-connection on \( g/h \). However, different choices induce quasi-isomorphic \( L_\infty \) algebras. Moreover, the first ‘Taylor coefficient’ of the \( L_\infty \) quasi-isomorphism is the identity map. Therefore, the Gerstenhaber algebra structures inherited by the cohomologies are in fact canonical [45, 3].

The natural map induced by skew-symmetrization (see Section 1.5)

\[
\text{hkr} : \text{tot}(\Lambda^\bullet h^\vee \otimes \Lambda^{\bullet+1}(g/h)) \to \text{tot}(\Lambda^\bullet h^\vee \otimes \left( \frac{U(g)}{U(g) \cdot h} \right)^{\otimes \bullet+1})
\]

is a quasi-isomorphism of cochain complexes.

3.2.3. Formality theorem for Lie algebra pairs. Theorems 2.1 and 2.2 imply the following corollaries.

**Theorem 3.3** (Formality theorem for Lie algebra pairs). Let \((g, h)\) be a Lie algebra pair. Given a splitting of the short exact sequence \( 0 \to h \to g \to g/h \to 0 \) and a torsion-free \( g \)-connection \( \nabla \) on \( g/h \), there exists an \( L_\infty \) quasi-isomorphism

\[
\mathcal{I} : \text{tot}(\Lambda^\bullet h^\vee \otimes \Lambda^{\bullet+1}(g/h)) \to \text{tot}(\Lambda^\bullet h^\vee \otimes \left( \frac{U(g)}{U(g) \cdot h} \right)^{\otimes \bullet+1})
\]

with first ‘Taylor coefficient’ \( \mathcal{I}_1 \) satisfying the following two properties:

1. \( \mathcal{I}_1 \) preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;
2. \( \mathcal{I}_1 = \text{hkr} \circ (td_{g/h}^\nabla)^{\frac{1}{2}} \), where

\[
(td_{g/h}^\nabla)^{\frac{1}{2}} \in \bigoplus_{k=0}^\infty \Lambda^k h^\vee \otimes \Lambda^k h^\perp = \bigoplus_{k=0}^\infty \Lambda^k h^\vee \otimes \Lambda^k (g/h)^\vee
\]

acts on \( \text{tot}(\Lambda^\bullet h^\vee \otimes \Lambda^{\bullet+1}(g/h)) \) by contraction.

**Theorem 3.4** (Kontsevich-Duflo type theorem for Lie algebra pairs). Given a Lie algebra pair \((g, h)\), the map

\[
\text{hkr} \circ \text{Td}_{g/h}^{\frac{1}{2}} : \mathbb{H}_{\mathbb{C}E}^\bullet(h, \Lambda^{\bullet+1}(g/h)) \xrightarrow{\sim} \mathbb{H}_{\mathbb{C}E}^\bullet(h, \left( \frac{U(g)}{U(g) \cdot h} \right)^{\otimes \bullet+1})
\]

is an isomorphism of Gerstenhaber algebras. It is understood that the square root \( \text{Td}_{g/h}^{\frac{1}{2}} \) of the Todd class \( \text{Td}_{g/h} \) acts on \( \mathbb{H}_{\mathbb{C}E}^\bullet(h, \Lambda^{\bullet+1}(g/h)) \) by contraction.

3.3. \( g \)-manifolds. In this section, we consider the formality theorem for a \( g \)-manifold, i.e. a smooth manifold with a Lie algebra action (see [27] for more details). Let \( M \) be a \( g \)-manifold with infinitesimal action \( g \ni a \mapsto \hat{a} \in \mathfrak{X}(M) \). Every \( g \)-manifold \( M \) determines in a canonical way a matched pair of Lie algebroids \((g \ltimes M) \ltimes T_M\) (see e.g. [38, Example 5.5] or [29]). The notation \( g \ltimes M \) refers to the transformation Lie algebroid arising from the infinitesimal \( g \)-action on \( M \). Therefore, we can form a Lie pair \((L, A)\), where \( L = (g \ltimes M) \ltimes T_M \) and \( A = g \ltimes M \). In this case, the quotient \( L/A \) is isomorphic to \( T_M \) and the Bott \( A \)-connection on \( L/A \) is the map

\[
\nabla^{\text{Bott}} : C^\infty(M, g) \otimes \mathfrak{X}(M) \to \mathfrak{X}(M)
\]

defined by

\[
\nabla^{\text{Bott}}_f X = f \cdot [\hat{a}, X],
\]

for all \( a \in g \), \( f \in C^\infty(M) \), and \( X \in \mathfrak{X}(M) \).
3.3.1. Atiyah and Todd classes of \( g \)-manifolds. It is not difficult to that, for the Lie pair constituted of the Lie algebroid \( L = (g \ltimes M) \rightthistack T_M \) and its Lie subalgebroid \( A = g \ltimes M \), a choice of \( L \)-connection on \( L/A \) extending the Bott \( A \)-connection is essentially a choice of affine connection on \( M \). Moreover, the torsion of the \( L \)-connection on \( L/A \) reduces to the torsion of the corresponding affine connection on \( M \).

Given an affine connection \( \nabla \) on \( M \), the Atiyah 1-cocycle associated with \( \nabla \) is the map

\[ R^\nabla_{1,1} : g \times \mathfrak{X}(M) \to \text{End}_\mathbb{R} \mathfrak{X}(M) \]

defined by

\[ R^\nabla_{1,1}(a, X) = \mathcal{L}_a \circ \nabla_X - \nabla_X \circ \mathcal{L}_a - \nabla_{\mathcal{L}_a X}, \]

for all \( a \in g \) and \( X \in \mathfrak{X}(M) \).

Following Proposition 1.6, we prove the following

**Proposition 3.5.**

1. The Atiyah cocycle \( R^\nabla_{1,1} \in g^\vee \otimes \Gamma(T^\nabla_M \otimes \text{End} T_M) \) is a Chevalley–Eilenberg 1-cocycle of the \( g \)-module \( \Gamma(T^\nabla_M \otimes \text{End} T_M) \).

2. The cohomology class \( \alpha_{M/\emptyset} \in H^1_{\text{CE}}(g, \Gamma(T^\nabla_M \otimes \text{End} T_M)) \) of the 1-cocycle \( R^\nabla_{1,1} \) does not depend on the choice of connection \( \nabla \).

The cohomology class \( \alpha_{M/\emptyset} \) is called the Atiyah class of the \( g \)-manifold \( M \). It is the obstruction class to the existence of a \( g \)-invariant connection on \( M \), i.e. an affine connection \( \nabla \) on \( M \) satisfying

\[ [\hat{a}, \nabla_X Y] = \nabla_{[\hat{a}, X]} Y + \nabla_X [\hat{a}, Y] \]

for all \( a \in g \) and \( X, Y \in \mathfrak{X}(M) \). Note that if \( g \) is a compact Lie algebra, \( \alpha_{M/\emptyset} \) vanishes since \( g \)-invariant connections always exist.

In the context of \( g \)-manifolds, the Todd cocycle of a \( g \)-manifold \( M \) is the Chevalley–Eilenberg cocycle

\[ \text{td}^\nabla_M = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0} \Lambda^k g^\vee \otimes \Omega^k(M). \]

Its corresponding Chevalley–Eilenberg cohomology class is the Todd class

\[ \text{Td}_{M/\emptyset} = \det \left( \frac{\alpha_{M/\emptyset}}{1 - e^{-\alpha_{M/\emptyset}}} \right) \in \bigoplus_{k=0} H^k_{\text{CE}}(g, \Omega^k(M)). \]

The spaces \( \Omega^k(M) \), with \( k \geq 0 \), are endowed with their natural \( g \)-module structures. Since the Lie algebra \( g \) is finite dimensional, the above expression for the Todd class \( \text{Td}_{M/\emptyset} \) reduces to a finite sum.

3.3.2. Polyvector fields and polydifferential operators on \( g \)-manifolds. The space of polyvector fields and the space of polydifferential operators on the Lie pair \( (g \ltimes M) \rightthistack T_M, g \ltimes M \) are naturally isomorphic to \( \text{tot} \left( \Lambda^\bullet g^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet(M) \right) \) and \( \text{tot} \left( \Lambda^\bullet g^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet(M) \right) \), respectively. Here \( \mathcal{T}_{\text{poly}}^\bullet(M) \) denotes the space of ordinary polyvector fields on \( M \), while \( \mathcal{D}_{\text{poly}}^\bullet(M) \) denotes the space of ordinary polydifferential operators on \( M \). Since \( (g \ltimes M) \rightthistack T_M \) is a matched pair, it follows from Proposition 4.4 that both \( \text{tot} \left( \Lambda^\bullet g^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet(M) \right) \) and \( \text{tot} \left( \Lambda^\bullet g^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet(M) \right) \) are dglas.

We proceed to describe these dglia structures. The \( g \)-action on \( M \) and the Schouten bracket together determine a \( g \)-module structure on \( \mathcal{T}_{\text{poly}}^k \) for each \( k \geq -1 \):

\[ a \cdot \gamma = [\hat{a}, \gamma] \quad \forall \ a \in g, \ \gamma \in \mathcal{T}_{\text{poly}}^k(M). \]

Therefore, the complex with trivial differential

\[ \cdots \to \mathcal{T}_{\text{poly}}^k(M) \overset{0}{\to} \mathcal{T}_{\text{poly}}^{k+1}(M) \to \cdots \]
is a complex of $\mathfrak{g}$-modules and we obtain the differential Gerstenhaber algebra
\[
\left( \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet (M)), d_{\text{CE}} + 0, [-,-], \wedge \right),
\]
whose graded Lie bracket and product are defined by
\[
[\alpha \otimes \mathcal{X}, \beta \otimes \mathcal{Y}] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [\mathcal{X}, \mathcal{Y}]
\]
and
\[
(\alpha \otimes \mathcal{X}) \wedge (\beta \otimes \mathcal{Y}) = (-1)^{q_1 p_2} (\alpha \wedge \beta) \otimes (\mathcal{X} \wedge \mathcal{Y}),
\]
for all $\alpha \otimes \mathcal{X} \in \Lambda^{p_1} \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^{q_1} (M)$ and $\beta \otimes \mathcal{Y} \in \Lambda^{p_2} \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^{q_2} (M)$.
Likewise, the $\mathfrak{g}$-action on $M$ and the Gerstenhaber bracket together determine a $\mathfrak{g}$-module structure on $\mathcal{D}_{\text{poly}}^\bullet$:
\[
a \cdot \mu = [\bar{a}, \mu] \quad \forall a \in \mathfrak{g}, \mu \in \mathcal{D}_{\text{poly}}^\bullet (M).
\]
Since the Gerstenhaber bracket satisfies the graded Jacobi identity, the infinitesimal $\mathfrak{g}$-action on $\mathcal{D}_{\text{poly}}^\bullet (M)$ is compatible with the Hochschild differential. Consequently, the Hochschild complex
\[
\cdots \to \mathcal{D}_\text{poly}^k (M) \xrightarrow{d_{\text{CE}}} \mathcal{D}_\text{poly}^{k+1} (M) \to \cdots
\]
is a complex of $\mathfrak{g}$-modules. Next, we endow $\Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet (M)$ with the differential $d_{\text{CE}} + d_{\mathfrak{g}}$, the cup product $\cup$, and the Gerstenhaber bracket $[-,-]$ defined by
\[
(\alpha \otimes \xi) \cup (\beta \otimes \eta) = (-1)^{q_1 p_2} (\alpha \wedge \beta) \otimes (\xi \cup \eta)
\]
\[
[\alpha \otimes \xi, \beta \otimes \eta] = (-1)^{q_1 p_2} \alpha \wedge \beta \otimes [\xi, \eta]
\]
for all $\alpha \otimes \xi \in \Lambda^{p_1} \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^{q_1} (M)$ and $\beta \otimes \eta \in \Lambda^{p_2} \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^{q_2} (M)$. It follows from Proposition 4.4 that
\[
\left( \text{tot}(\Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet (M)), d_{\text{CE}} + \text{id} \otimes d_{\mathfrak{g}}, [-,-] \right)
\]
is a dgla whose cohomology $\mathbb{H}_{\text{CE}} (\mathfrak{g}, \mathcal{D}_{\text{poly}}^\bullet (M))$, endowed with the cup product and the Gerstenhaber bracket, is a Gerstenhaber algebra.

The $\Lambda^\bullet \mathfrak{g}$-linear extension $\text{hkr} : \Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet (M) \to \Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet (M)$ of the classical HKR map of the smooth manifold $M$ is a quasi-isomorphism of cochain complexes but does not preserve the Lie structures on cohomologies.

3.3.3. Formality theorem for $\mathfrak{g}$-manifolds. Theorem 2.1 and Theorem 2.2 imply the following:

**Theorem 3.6** (Formality theorem for $\mathfrak{g}$-manifolds). Given a $\mathfrak{g}$-manifold $M$ and an affine torsion-free connection $\nabla$ on $M$, there exists an $L_\infty$ quasi-isomorphism $I$ from the dgla $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet (M) \right)$ to the dgla $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{D}_{\text{poly}}^\bullet (M) \right)$ with first ‘Taylor coefficient’ $I_1$ satisfying the following two properties:

1. $I_1$ is, up to homotopy, an isomorphism of associative algebras (and hence induces an isomorphism of associative algebras of the cohomologies);
2. $I_1 = \text{hkr} \circ \left( \text{td}_{M/\mathfrak{g}} \right)^{1/2}$, where $(\text{td}_{M/\mathfrak{g}})^{1/2} \in \bigoplus_{k=0}^{\infty} \Lambda^k \mathfrak{g}^\vee \otimes \Omega^k (M)$ acts on $\text{tot} \left( \Lambda^\bullet \mathfrak{g}^\vee \otimes_k \mathcal{T}_{\text{poly}}^\bullet (M) \right)$ by contraction.

**Theorem 3.7** (Kontsevich–Duflo type theorem for $\mathfrak{g}$-manifolds). Given a $\mathfrak{g}$-manifold $M$, the map
\[
\text{hkr} \circ \text{td}_{M/\mathfrak{g}}^{1/2} : \mathbb{H}_{\text{CE}}^\bullet (\mathfrak{g}, \mathcal{T}_{\text{poly}}^\bullet (M)) \xrightarrow{\cong} \mathbb{H}_{\text{CE}}^\bullet (\mathfrak{g}, \mathcal{D}_{\text{poly}}^\bullet (M))
\]
is an isomorphism of Gerstenhaber algebras. It is understood that the square root $\text{td}_{M/\mathfrak{g}}^{1/2}$ of the Todd class $\text{td}_{M/\mathfrak{g}} \in \bigoplus_{k=0}^{\infty} H^k_{\text{CE}} (\mathfrak{g}, \Omega^k (M))$ acts on $\mathbb{H}_{\text{CE}}^\bullet (\mathfrak{g}, \mathcal{T}_{\text{poly}}^\bullet (M))$ by contraction.
3.4. **Foliations.** Let $\mathcal{F}$ be a regular foliation of a smooth manifold $M$. The tangent bundle of $\mathcal{F}$ is a subbundle of $T_M$, denoted $T_\mathcal{F}$, whose sections are closed under the Lie bracket of vector fields. Therefore, $(T_M, T_\mathcal{F})$ is a Lie pair. Its quotient $N_\mathcal{F} = T_M/T_\mathcal{F}$ is the normal bundle of the foliation $\mathcal{F}$. We have the short exact sequence of vector bundles

$$0 \to T_\mathcal{F} \to T_M \xrightarrow{\phi} N_\mathcal{F} \to 0.$$

The Bott $T_\mathcal{F}$-connection on $N_\mathcal{F}$ is defined by

$$\nabla^\text{Bott}_a q(l) = q([a, l]), \quad \forall a \in \Gamma(T_\mathcal{F}), \ l \in \mathfrak{X}(M).$$

The Chevalley–Eilenberg Lie algebroid cohomology $H^*_CE(T_\mathcal{F}, \mathfrak{M})$ with coefficients in a $T_\mathcal{F}$-module $\mathfrak{M}$ coincides exactly with the leafwise de Rham cohomology $H^*_\text{dR}(\mathcal{F}, \mathfrak{M})$ of the foliation $\mathcal{F}$ with coefficients in the module $\mathfrak{M}$.

3.4.1. **Atiyah and Todd classes of foliations.** Let $\nabla$ be a $T_M$-connection on $N_\mathcal{F}$ extending the Bott $T_\mathcal{F}$-connection. The torsion of $\nabla$ is the bundle map $T^\nabla : \Lambda^2 T_M \to N_\mathcal{F}$ defined by

$$T^\nabla(X, Y) = \nabla_X q(Y) - \nabla_Y q(X) - q([X, Y]),$$

for all vector fields $X$ and $Y$ on $M$. The Atiyah cocycle associated with $\nabla$ is the bundle map $R^\nabla_{1,1} : T_\mathcal{F} \otimes N_\mathcal{F} \to \text{End}(N_\mathcal{F})$ — or the corresponding section of $T^\mathcal{F}_\mathcal{F} \otimes T^\perp_\mathcal{F} \otimes \text{End}(N_\mathcal{F})$ — defined by

$$R^\nabla_{1,1}(a; q(l)) = \nabla_a \nabla_1 - \nabla_1 \nabla_a - \nabla_{[a, l]}, \quad \forall a \in \Gamma(T_\mathcal{F}), \ l \in \Gamma(T_M).$$

According to Proposition 1.6, $R^\nabla_{1,1} \in \Gamma(T^\mathcal{F}_\mathcal{F} \otimes T^\perp_\mathcal{F} \otimes \text{End}(N_\mathcal{F}))$ is a leafwise de Rham closed 1-form with values in the $T_\mathcal{F}$-module $T^\mathcal{F}_\mathcal{F} \otimes \text{End}(N_\mathcal{F})$. Its cohomology class $\alpha_\mathcal{F} \in H^1_{\text{dR}}(\mathcal{F}, T^\mathcal{F}_\mathcal{F} \otimes \text{End}(N_\mathcal{F}))$ is independent of the choice of $T_M$-connection $\nabla$ extending the Bott $T_\mathcal{F}$-connection and is called the Atiyah class of the foliation $\mathcal{F}$. It is precisely the invariant of the foliation that was first introduced by Molino [39].

The Todd cocycle of the foliation $\mathcal{F}$ associated with the connection $\nabla$ is the leafwise closed form

$$\text{td}^\nabla = \det \left( \frac{R^\nabla_{1,1}}{1 - e^{-R^\nabla_{1,1}}} \right) \in \bigoplus_{k=0}^{\infty} \Gamma(\Lambda^k T^\mathcal{F}_\mathcal{F} \otimes \Lambda^k T^\perp_\mathcal{F}).$$

The Todd class of the foliation $\mathcal{F}$ is the corresponding cohomology class

$$\text{Td}_\mathcal{F} = \det \left( \frac{\alpha_\mathcal{F}}{1 - e^{-\alpha_\mathcal{F}}} \right) \in \bigoplus_{k=0}^{\infty} H^k_{\text{dR}}(\mathcal{F}, \Lambda^k T^\mathcal{F}_\mathcal{F}).$$

3.4.2. **Transversal polyvector fields and transversal polydifferential operators on foliations.** It follows from Corollary 4.14 and Corollary 4.19 applied to the Lie pair $(T_M, T_\mathcal{F})$ that both $\text{tot} (\Gamma(\Lambda^k T^\mathcal{F}_\mathcal{F}) \otimes_R \mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F}))$ and $\text{tot} (\Gamma(\Lambda^k T^\mathcal{F}_\mathcal{F}) \otimes_R \mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F}))$ can be endowed with $L_\infty$ algebra structures, unique up to $L_\infty$ quasi-isomorphisms. Here $\mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F}) = \Gamma(\Lambda^0 T^\mathcal{F}_\mathcal{F})$ can be considered as the space of polyvector fields transversal to $\mathcal{F}$. The first bracket on $\text{tot} (\Gamma(\Lambda^k T^\mathcal{F}_\mathcal{F}) \otimes_R \mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F}))$ is the leafwise de Rham differential $d_{\text{dR}}$ with values in $\mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F})$. Similarly, $\mathcal{D}^\bullet_{\text{poly}}(N_\mathcal{F}) = \bigoplus_{k=1}^{\infty} \mathcal{D}^k_{\text{poly}}(N_\mathcal{F})$ can be considered as the space of polydifferential operators transversal to $\mathcal{F}$. Here $\mathcal{D}^k_{\text{poly}}(N_\mathcal{F})$ denotes the algebra of $R$-smooth functions on the manifold $M$, $\mathcal{D}^0_{\text{poly}}(N_\mathcal{F})$ denotes the left $R$-module $\mathcal{U}(T_M)/\mathcal{U}(T_\mathcal{F}) \cong \mathcal{D}^0_{\text{poly}}(M)/\mathcal{D}^0_{\text{poly}}(T_\mathcal{F})$ of ‘transverse differential operators,’ and $\mathcal{D}^k_{\text{poly}}(N_\mathcal{F})$ denotes the tensor product $\mathcal{D}^0_{\text{poly}}(N_\mathcal{F}) \otimes_R \cdots \otimes_R \mathcal{D}^0_{\text{poly}}(N_\mathcal{F})$ of $(k+1)$ copies of the left $R$-module $\mathcal{D}^0_{\text{poly}}(N_\mathcal{F})$. (If there existed a foliation $\mathcal{F}'$ transverse to $\mathcal{F}$, the space $\mathcal{D}^0_{\text{poly}}(N_\mathcal{F})$ would be isomorphic to $\mathcal{U}(T_\mathcal{F})$, the space of differential operators in the direction of $\mathcal{F}'$.)

For every $k \geq 0$, $\mathcal{D}^k_{\text{poly}}(N_\mathcal{F})$ is naturally a left $\mathcal{U}(T_\mathcal{F})$-module and we can consider the associated leafwise de Rham differential

$$d_{\text{dR}} : \Gamma(\Lambda^k T^\mathcal{F}_\mathcal{F}) \otimes_R \mathcal{D}^k_{\text{poly}}(N_\mathcal{F}) \to \Gamma(\Lambda^{k+1} T^\mathcal{F}_\mathcal{F}) \otimes_R \mathcal{D}^k_{\text{poly}}(N_\mathcal{F}).$$
Since \( \frac{U(T_M)}{U(T_M) \cdot T_F} \) is a coalgebra over \( R \) with an associative comultiplication

\[
\Delta : \frac{U(T_M)}{U(T_M) \cdot T_F} \to \frac{U(T_M)}{U(T_M) \cdot T_F} \otimes_R \frac{U(T_M)}{U(T_M) \cdot T_F},
\]

there is a Hochschild differential

\[
d_{\mathcal{F}} : \mathcal{D}_\text{poly}^k(N_F) \to \mathcal{D}_\text{poly}^{k+1}(N_F),
\]

which extends to a \( \Gamma(\Lambda^* T^\vee_F) \)-graded linear operator of degree +1 on \( \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right) \) still denoted \( d_{\mathcal{F}} \) by abuse of notation. The first bracket of the \( L_\infty \) algebra structure on \( \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right) \) is \( d_{\mathcal{F}} + d_{\mathcal{F}} \).

The \( L_\infty \) structures on \( \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right) \) and \( \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right) \) depend on the choice of a splitting of the short exact sequence \( 0 \to T_F \to T_M \to N_F \to 0 \) and a torsion-free \( T_M \)-connection on \( N_F \), extending the Bott \( T_F \)-connection [45]. However, different choices induce quasi-isomorphic \( L_\infty \) algebra structures. Moreover, the first ‘Taylor coefficient’ of the \( L_\infty \) quasi-isomorphism is the identity map. Therefore, the resulting Gerstenhaber algebra structures on the cohomologies \( \mathbb{H}^*_{dR}(F, \mathcal{T}_\text{poly}(N_F)) \) and \( \mathbb{H}^*_{dR}(F, \mathcal{D}_\text{poly}^* (N_F)) \) are indeed canonical [3].

According to Section 1.5, skew-symmetrization induces a quasi-isomorphism of cochain complexes

\[
hkr : \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{T}_\text{poly}^* (N_F) \right) \to \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right).
\]

3.4.3. Formality theorem for foliations. Theorem 2.1 and Theorem 2.2 imply the following

**Theorem 3.8** (Formality theorem for foliations). Let \( F \) be a regular foliation on a smooth manifold \( M \). Given a splitting of the short exact sequence \( 0 \to T_F \to T_M \to N_F \to 0 \) and a torsion-free \( T_M \)-connection \( \nabla \) on \( N_F \), there exists an \( L_\infty \) quasi-isomorphism

\[
\mathcal{I} : \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{T}_\text{poly}^* (N_F) \right) \to \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{D}_\text{poly}^* (N_F) \right)
\]

with first ‘Taylor coefficient’ \( \mathcal{I}_1 \) satisfying the following two properties:

1. \( \mathcal{I}_1 \) preserves the associative algebra structures (wedge and cup product, respectively) up to homotopy;
2. \( \mathcal{I}_1 = hkr \circ (\text{td}_F)^\frac{1}{2} \), where \( (\text{td}_F)^\frac{1}{2} \in \bigoplus_{k=0}^\infty \Gamma(\Lambda^k T_F^\vee \otimes \Lambda^k T_F^\perp) \) acts on \( \text{tot} \left( \Gamma(\Lambda^* T^\vee_F) \otimes_R \mathcal{T}_\text{poly}^* (N_F) \right) \) by contraction.

**Theorem 3.9** (Kontsevich-Duflo type theorem for foliations). Given a regular foliation \( F \) on a smooth manifold \( M \), the map

\[
hkr \circ \text{td}_F^\frac{1}{2} : \mathbb{H}^*_{dR}(F, \mathcal{T}_\text{poly}^* (N_F)) \xrightarrow{\mathcal{I}} \mathbb{H}^*_{dR}(F, \mathcal{D}_\text{poly}^* (N_F))
\]

is an isomorphism of Gerstenhaber algebras. It is understood that the square root \( \text{td}_F^\frac{1}{2} \) of the Todd class \( \text{td}_F \in \bigoplus_{k=0}^\infty H^k_{dR}(F, \Lambda^k T_F^\perp) \) acts on \( \mathbb{H}^*_{dR}(F, \mathcal{T}_\text{poly}^* (N_F)) \) by contraction.

4. Appendix: Fedosov dg Lie algebroids

In this section, we recall basic ingredients needed to establish our main result (Theorem 2.1) in Section 2. For details, we refer the interested reader to [45].
4.1. DGLAs associated to dg Lie algebroids. One can make sense of polyvector fields and polydifferential operators for a dg Lie algebroid just as one does for ordinary Lie algebroids. Both give rise to dglas and their cohomology groups are in fact Gerstenhaber algebras. More precisely, a $k$-vector field on a dg Lie algebroid $L \to M$ is a section of the vector bundle $\Lambda^k L \to M$ while a $k$-differential operator is an element of $U(L) \otimes^k$, the tensor product (as left $C^\infty(M)$-modules) of $k$ copies of the universal enveloping algebra $U(L)$.

It is clear that the differential $Q : \Gamma(L) \to \Gamma(L)$ and the homological vector field $Q : C^\infty(M) \to C^\infty(M)$ extend naturally to a degree $+1$ differential $Q : \Gamma(\Lambda^k+1 L) \to \Gamma(\Lambda^k+1 L)$ and the the Lie algebroid structure on $L$ yields a Schouten bracket

\[ [-, -] : \Gamma(\Lambda^{u+1} L) \otimes \Gamma(\Lambda^{u+1} L) \to \Gamma(\Lambda^{u+u+1} L). \]

**Proposition 4.1.** Let $L$ be a dg Lie algebroid over $M$.

1. When endowed with the differential $Q$, the wedge product, and the Schouten bracket, the space of 'polyvector fields' $\Gamma(\Lambda^{*+1} L)$ is a differential Gerstenhaber algebra — whence a dga.
2. When endowed with the wedge product and the Schouten bracket, the cohomology $H^* (\Gamma(\Lambda^{*+1} L), Q)$ is a Gerstenhaber algebra.

Adapting the definition given for Lie algebroids, one can define the universal enveloping algebra of a dg Lie algebroid. The universal enveloping algebra of a dg Lie algebroid $L \to M$ is a dg Hopf algebroid $\mathcal{U}(L)$ over the dga $C^\infty(M)$. For each $k \geq 0$, the dg structure on the dg Lie algebroid $L \to M$ determines a differential $Q : \mathcal{U}(L)^\otimes k+1 \to \mathcal{U}(L)^\otimes k+1$ of degree $+1$. A Hochschild coboundary differential

\[ d_{\mathcal{H}} : \mathcal{U}(L)^\otimes k \to \mathcal{U}(L)^\otimes k+1 \]

and Gerstenhaber bracket

\[ [-, -] : \mathcal{U}(L)^\otimes u+1 \otimes \mathcal{U}(L)^\otimes v+1 \to \mathcal{U}(L)^\otimes u+v+1 \quad (18) \]

can be defined by the following explicit algebraic expressions:

\[ d_{\mathcal{H}}(u_1 \otimes \cdots \otimes u_k) = 1 \otimes u_1 \otimes \cdots \otimes u_k + \sum_{i=1}^k (-1)^i u_1 \otimes \cdots \otimes \Delta(u_i) \otimes \cdots \otimes u_k + (-1)^{k+1} u_1 \otimes \cdots \otimes u_k \otimes 1, \quad (19) \]

and

\[ [\phi, \psi] = \phi \star \psi - (-1)^{uv} \psi \star \phi \in \mathcal{U}(L)^\otimes u+v+1, \quad (20) \]

where $\phi \star \psi \in \mathcal{U}(L)^\otimes u+v+1$ is defined by

\[ \phi \star \psi = \sum_{k=0}^u (-1)^{kv} d_0 \otimes_R \cdots \otimes_R d_{k-1} \otimes_R (\Delta^v d_k) \cdot \psi \otimes_R d_{k+1} \otimes_R \cdots \otimes_R d_u \]

if $\phi = d_0 \cdot d_1 \cdots d_u$ for some $d_0, d_1, \ldots, d_u \in \mathcal{U}(L)$.

We refer the reader to [50] for the precise meaning of the product $(\Delta^v d_k) \cdot \psi$ in $\mathcal{U}(L)^\otimes u+1$ appearing in the last equation above.

**Proposition 4.2.** Let $L$ be a dg Lie algebroid over $M$.

1. When endowed with the differential $Q + d_{\mathcal{H}}$ and the Gerstenhaber bracket (18), $\mathcal{U}(L)^\otimes *+1$ is a dga.
2. When endowed with the cup product (i.e. the tensor product $\otimes C^\infty(M)$) and the Gerstenhaber bracket, the Hochschild cohomology $H^* (\mathcal{U}(L)^\otimes *+1, Q + d_{\mathcal{H}})$, is a Gerstenhaber algebra.
Recall that, for a Lie pair \((L, A)\), if a splitting \(j : B (= L/A) \to L\) of the short exact sequence \(0 \to A \to L \to B \to 0\) is given, whose image \(j(B)\) happens to be a Lie subalgebroid of \(L\), then \(A\) and \(B\) are said to form a \textit{matched pair of Lie algebroids} — see [38] for more details. In such a situation, we write \(L = A \bowtie B\) to highlight that \(A\) and \(B\) play symmetric roles as a pair of complementary Lie subalgebroids of the Lie algebroid \(L\).

**Lemma 4.3.** If \(A \bowtie B\) is a matched pair of Lie algebroids, then \((A[1] \oplus B, d^Bott_A)\) is a dg Lie algebroid over \((A[1], d_A)\).

**Proof.** A classical result of Mackenzie [30] asserts that, if \(A \bowtie B\) is a matched pair of Lie algebroids over a smooth manifold \(M\), then

\[
\begin{array}{c}
A \bowtie B \to B \\
\downarrow & \downarrow \\
A & \to M
\end{array}
\]

is a double Lie algebroid. Moreover, Gracia-Saz and Mehta [20] proved that, given a double Lie algebroid

\[
\begin{array}{c}
D \to B \\
\downarrow & \\
A & \to M
\end{array}
\]

the graded vector bundle \(D[1] \to A[1]\) is automatically a differential graded Lie algebroid. \(\square\)

Here the dg manifold structures on \((A[1] \oplus B, d^Bott_A)\) and \((A[1], d_A)\) result from the Lie algebroid structures on \(A \oplus B \to B\) and \(A \to M\), respectively. In what follows, denote by \(B\) the dg manifold \(A[1] \oplus B\). The space of sections of \(B \to A[1]\) can be naturally identified with \(\Gamma(\Lambda^\bullet A^\vee \otimes B)\). The bracket on \(\Gamma(\Lambda^\bullet A^\vee \otimes B)\) is defined in terms of the Bott representation of \(B\) on \(\Lambda A^\vee\) by

\[
[\xi_1 \otimes b_1, \xi_2 \otimes b_2] = \xi_1 \wedge \xi_2 \otimes [b_1, b_2] + \xi_1 \wedge \nabla^Bott_{b_1} \xi_2 \otimes b_2 - \nabla^Bott_{b_2} \xi_1 \otimes b_1
\]

for all \(\xi_1, \xi_2 \in \Gamma(\Lambda^\bullet A^\vee)\) and \(b_1, b_2 \in \Gamma(B)\), while the anchor map \(\Gamma(\Lambda^\bullet A^\vee \otimes B) \xrightarrow{\bar{\rho}} \text{Der}(\Lambda^\bullet A^\vee)\) is defined by

\[
\bar{\rho}(\xi \otimes b)(\eta) = \xi \wedge \nabla^Bott_b \eta,
\]

for all \(\xi, \eta \in \Gamma(\Lambda^\bullet A^\vee)\) and \(b \in \Gamma(B)\). Finally, the differential on the space of sections of \(B \to A[1]\) is simply the Chevalley–Eilenberg differential \(d^Bott_A : \Gamma(\Lambda^\bullet A^\vee \otimes B) \to \Gamma(\Lambda^{\bullet+1} A^\vee \otimes B)\), corresponding to the Bott representation of \(A\) on \(B\).

According to Proposition 4.1, the dg Lie algebroid \(B\) induces a differential Gerstenhaber algebra structure on \(\Gamma(\Lambda^{\bullet+1} B) \cong \Gamma(\Lambda^\bullet A^\vee \otimes \Lambda^{\bullet+1} B)\). Its differential is the Chevalley–Eilenberg differential

\[
d^Bott_A : \Gamma(\Lambda^\bullet A^\vee \otimes \Lambda^{\bullet+1} B) \to \Gamma(\Lambda^{\bullet+1} A^\vee \otimes \Lambda^{\bullet+1} B),
\]

corresponding to the Bott representation of \(A\) on \(\Lambda B\) and its Lie bracket is the Schouten bracket of the dg Lie algebroid \(B \to A[1]\) — essentially the extension of Equations (21) and (22) by the graded Leibniz rule.

Next, consider the universal enveloping algebra \(\mathcal{U}(B)\) of the dg Lie algebroid \(B\), which is a dg Hopf algebroid over \((\Gamma(\Lambda^\bullet A^\vee), d_A)\). It is clear that \(\mathcal{U}(B) \cong \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)\), and \(\mathcal{U}(B)^{\otimes k+1} \cong \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes k+1}\). Under this identification, the differential \(Q : \mathcal{U}(B)^{\otimes k+1} \to \mathcal{U}(B)^{\otimes k+1}\) becomes the Chevalley–Eilenberg differential

\[
d^Q_A : \Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes k+1} \to \Gamma(\Lambda^{\bullet+1} A^\vee) \otimes_R \mathcal{U}(B)^{\otimes k+1}.
\]

Here the \(A\)-module structure on \(\mathcal{U}(B)\) follows from the canonical identification of \(\mathcal{U}(B)\) with \(\mathcal{U}(L)/(\mathcal{U}(L)\Gamma(L))\) — the Lie algebroid \(A\) acts on the latter by multiplication from the left — and extends to an \(A\)-module structure on \(\mathcal{U}(B)^{\otimes k+1}\) in the natural way. As a consequence, the total differential \(Q + d_{\mathcal{U}(B)}\) on \(\mathcal{U}(B)^{\otimes k+1}\) coincides with \(d^A_L + d_{\mathcal{U}(B)}\) on \(\text{tot}(\Gamma(\Lambda^\bullet A^\vee) \otimes_R \mathcal{U}(B)^{\otimes k+1})\).

The following proposition summarizes the discussion above:

**Proposition 4.4.** Suppose \((A, B)\) is a matched pair of Lie algebroids.
(1) When endowed with the differential $d_{A}^{Bott}$ as in (23) and the Schouten bracket defined by Equations (21)-(22), $\Gamma(\Lambda^{*}A^{\vee} \otimes \Lambda^{*+1}B)$ is a differential Gerstenhaber algebra, whence a dgla.

(2) When endowed with the wedge product and the Schouten bracket, the cohomology $\mathbb{H}_{CE}^{*}(A, \Lambda^{*+1}B)$ is a Gerstenhaber algebra.

(3) When endowed with the differential $d_{A}^{d} + d_{\varphi}$ (see (19) and (24)) and the Gerstenhaber bracket, $(\text{tot}(\Gamma(\Lambda^{*}A^{\vee}) \otimes R U(B)^{\otimes*+1})$ is a Gerstenhaber algebra.

(4) When endowed with the cup product and the Gerstenhaber bracket, the Hochschild cohomology $\mathbb{H}_{CE}^{*}(A, U(B)^{\otimes*+1})$, i.e. the cohomology of the complex $(\text{tot}(\Lambda^{*}A^{\vee} \otimes R U(B)^{\otimes*+1}, d_{A}^{d} + d_{\varphi})$, is a Gerstenhaber algebra.

**Remark 4.5.** Note that the Gerstenhaber bracket on $\text{tot}(\Gamma(\Lambda^{*}A^{\vee}) \otimes R U(B)^{\otimes*+1})$ is not the obvious extension of the Gerstenhaber bracket on $U(B)^{\otimes*+1}$ obtained by tensoring with the commutative associative algebra $\Gamma(\Lambda^{*}A^{\vee})$. In fact, to write down an explicit formula — which is quite involved — one needs to use the Bott representation of $B$ on $\Gamma(\Lambda^{*}A^{\vee})$.

### 4.2. Fedosov dg Lie algebroids

Let $(L, A)$ be a Lie pair. We use the symbols $B$ to denote the quotient vector bundle $L/A$ and $r$ to denote its rank.

Consider the endomorphism $\delta$ of the vector bundle $\Lambda^{*}L^{\vee} \otimes \hat{S}B^{\vee}$ defined by

$$\delta(\omega \otimes \chi^{J}) = \sum_{m=1}^{r} (q^{\top}(\chi_{m}) \wedge \omega) \otimes J_{m} \chi^{J-e_{m}},$$

for all $\omega \in \Lambda L^{\vee}$ and $J \in \mathbb{N}^{r}$. Here $\{\chi_{k}\}_{k=1}^{r}$ denotes an arbitrary local frame for the vector bundle $B^{\vee}$, the symbol $e_{m}$ denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$ having its single nonzero entry in $m$-th position, and

$$\chi^{J} = \chi_{1} \cdots \chi_{1} \cdots \chi_{2} \cdots \chi_{2} \cdots \chi_{r} \cdots \chi_{r} \text{ having its single nonzero entry in } m \text{-th position},$$

if $J = (J_{1}, J_{2}, \ldots, J_{r})$.

The operator $\delta$ is a derivation of degree $+1$ of the graded commutative algebra $\Gamma(\Lambda^{*}L^{\vee} \otimes \hat{S}B^{\vee})$ and satisfies $\delta^{2} = 0$. The resulting cochain complex

$$\cdots \rightarrow \Lambda^{n-1}L^{\vee} \otimes \hat{S}B^{\vee} \rightarrow \Lambda^{n}L^{\vee} \otimes \hat{S}B^{\vee} \rightarrow \Lambda^{n+1}L^{\vee} \otimes \hat{S}B^{\vee} \rightarrow \cdots$$

deformation retracts onto the trivial complex

$$\cdots \rightarrow \Lambda^{n-1}A^{\vee} \rightarrow \Lambda^{n}A^{\vee} \rightarrow \Lambda^{n+1}A^{\vee} \rightarrow \cdots$$

Indeed, for every choice of splitting $i \circ p + j \circ q = \text{id}_{L}$ of the short exact sequence

$$0 \rightarrow A \xrightarrow{i} L \xrightarrow{q} B \rightarrow 0 \quad (25)$$

and its dual

$$0 \rightarrow B^{\vee} \xrightarrow{q^{\top}} L^{\vee} \xrightarrow{i^{\top}} A^{\vee} \rightarrow 0 ,$$

the chain maps

$$\sigma : \Lambda^{*}L^{\vee} \otimes \hat{S}B^{\vee} \rightarrow \Lambda^{*}A^{\vee}$$

and

$$\tau : \Lambda^{*}A^{\vee} \rightarrow \Lambda^{*}L^{\vee} \otimes \hat{S}B^{\vee}$$
respectively defined by
\[ \sigma(\omega \otimes \chi^J) = \begin{cases} 
\omega \otimes \chi^J & \text{if } v = 0 \text{ and } |J| = 0 \\
0 & \text{otherwise}, 
\end{cases} \]
for all \( \omega \in p^\top(A^u A^v) \otimes q^\top(\Lambda^v B^v) \), and
\[ \tau(\alpha) = p^\top(\alpha) \otimes 1, \]
for all \( \alpha \in \Lambda^\bullet(A^v) \), satisfy
\[ \sigma \tau = \text{id} \quad \text{and} \quad \text{id} - \tau \sigma = h\delta + \delta h, \]
where the homotopy operator
\[ h : \Lambda^\bullet L^v \otimes \hat{S}B^v \to \Lambda^{\bullet-1}L^v \otimes \hat{S}B^v \]
is defined by
\[ h(\omega \otimes \chi^J) = \begin{cases} 
\frac{1}{v + |J|} \sum_{k=1}^r (\iota_{\partial_k} \omega) \otimes \chi^{J + \epsilon_k} & \text{if } v > 1 \\
0 & \text{if } v = 0 
\end{cases} \]
for all \( \omega \in p^\top(\Lambda^u A^v) \otimes q^\top(\Lambda^v B^v) \). Here \( \{\partial_k\}_{k=1}^r \) denotes the local frame for \( B \) dual to \( \{\chi_k\}_{k=1}^r \). Note that the operator \( h \) is not a derivation of the algebra \( \Gamma(\Lambda^\bullet L^v \otimes \hat{S}B^v) \). Also, we note that \( h\tau = 0, \sigma h = 0, \) and \( h^2 = 0 \).

**Lemma 4.6.** Let \( (L, A) \) be a Lie pair and let \( \nabla \) be an \( L \)-connection on \( B \) extending the Bott \( A \)-connection. The torsion \( T^\nabla \) of \( \nabla \) vanishes (see Proposition 1.4) if and only if \( \delta d^\nabla_L + d^\nabla_L \delta = 0 \).

Consider the four maps \( \delta_2, \sigma_2, h_2, \) and \( \tau_2 \)
\[ \Gamma(\Lambda^\bullet B) \xleftarrow{\sigma_2} \Gamma(\Lambda^\bullet L^v \otimes \hat{S}B^v \otimes B) \xrightarrow{\delta_2} \Gamma(\Lambda^\bullet+1 L^v \otimes \hat{S}B^v \otimes B) \]
defined by
\[ \delta_2(\omega \otimes \sigma \otimes b) = \delta(\omega \otimes \sigma) \otimes b, \quad \sigma_2(\omega \otimes \sigma \otimes b) = \sigma(\omega \otimes \sigma) \otimes b, \]
\[ h_2(\omega \otimes \sigma \otimes b) = h(\omega \otimes \sigma) \otimes b, \quad \tau_2(\alpha \otimes b) = \tau(\alpha) \otimes b, \]
for all \( \alpha \in \Gamma(\Lambda A^v), \omega \in \Gamma(\Lambda L^v), \sigma \in \Gamma(\hat{S}B^v), \) and \( b \in \Gamma(B) \).

**Theorem 4.7 ([45]).** Let \( (L, A) \) be a Lie pair with quotient \( B = L/A \). We interpret the sections of the bundle \( L^v \otimes \hat{S}B^v \otimes B \) as derivations of the algebra \( \Gamma(\Lambda^\bullet L^v \otimes \hat{S}B^v) \) in the natural way. Given a splitting of the short exact sequence (25) and a torsion-free \( L \)-connection \( \nabla \) on \( B \), there exists a unique derivation
\[ X^\nabla \in \Gamma(\Lambda L^v \otimes \hat{S}B^v \otimes B), \]
satisfying \( h_2(X^\nabla) = 0 \) and such that the derivation \( Q : \Gamma(\Lambda^\bullet L^v \otimes \hat{S}B^v) \to \Gamma(\Lambda^{\bullet+1} L^v \otimes \hat{S}B^v) \) defined by
\[ Q = -\delta + d^\nabla_L + X^\nabla \]
satisfies \( Q^2 = 0 \). Moreover, writing \( X_k \) for the component of \( X^\nabla \) in \( L^v \otimes S^k B^v \otimes B \), we have \( X^\nabla = \sum_{k=2}^{\infty} X_k \) with
\[ X_2 = h_2(R^\nabla) = h_2(\hat{R}^\nabla_{1,1}) + h_2(R^\nabla_{0,2}). \]
As a consequence, \( (M = L[1] \oplus B, Q = -\delta + d^\nabla_L + X^\nabla) \) is a dg manifold, which we call a Fedosov dg manifold associated with the Lie pair \( (L, A) \).
Sketch of proof. Suppose there exists such an $X^\nabla$ and consider its decomposition $X^\nabla = \sum_{k=2}^{\infty} X_k$, where $X_k \in \Gamma(L^\vee \otimes 2k^{\nabla} B^\vee \otimes B)$. Then $D = -\delta + d_{\nabla}^L + X_2 + X_{\geq 3}$ with $X_{\geq 3} = \sum_{k=3}^{\infty} X_k$ and 

$$D^2 = \delta^2 - (\delta d_{\nabla}^L + d_{\nabla}^L \delta) + \{d_{\nabla}^L, d_{\nabla}^L - \delta X_2 - X_2 \delta\} + \{d_{\nabla}^L X^\nabla + X^\nabla d_{\nabla}^L + X^\nabla 2 - \delta X_{\geq 3} - X_{\geq 3} \delta\}$$

$$= \delta^2 - [\delta, d_{\nabla}^L] + \{R^\nabla - [\delta, X_2]\} + \{[d_{\nabla}^L + \frac{1}{2} X^\nabla, X^\nabla] - [\delta, X_{\geq 3}]\}.$$

For degree reasons, the requirement $D^2 = 0$ is equivalent to the pair of equations:

$$[\delta, X_2] = R^\nabla \quad \text{and} \quad [\delta, X_{\geq 3}] = [d_{\nabla}^L + \frac{1}{2} X^\nabla, X^\nabla].$$

Note that $\sigma_2(X_2) = 0$ and $\sigma_2(X_{\geq 3}) = 0$, since $X_2, X_{\geq 3} \in \Gamma(L^\vee \otimes 2^{\nabla} B^\vee \otimes B)$, and also that $h_2(X_2) = 0$ and $h_2(X_{\geq 3}) = 0$, as $h_2(X^\nabla) = 0$. Since $\delta_2 h_2 + h_2 \delta_2 = \text{id} - \tau_2 \sigma_2$, we obtain $h_2 \delta_2(X_2) = X_2$ and $h_2 \delta_2(X_{\geq 3}) = X_{\geq 3}$. It follows that

$$X_2 = h_2 \delta_2(X_2) = h_2([\delta, X_2]) = h_2(R^\nabla)$$

while

$$X_{\geq 3} = h_2 \delta_2(X_{\geq 3}) = h_2([\delta, X_{\geq 3}]) = h_2[d_{\nabla}^L + \frac{1}{2} X^\nabla, X^\nabla].$$

Projecting the latter equation onto $\Gamma(L^\vee \otimes 2^{\nabla + 1} B^\vee \otimes B)$, we obtain

$$X_{k+1} = h_2 \left( d_{\nabla}^L \circ X_k + X_k \circ d_{\nabla}^L + \sum_{p+q=k+1 \atop 2 \leq p, q \leq k-1} X_p \circ X_q \right), \quad \text{for } k \geq 2,$$

which shows that the higher terms of $X^\nabla = \sum_{k=2}^{\infty} X_k$ can be computed iteratively starting from $X_2 = h_2(R^\nabla)$. The derivation $X^\nabla$ is thus uniquely determined by the torsion-free connection $\nabla$. \hfill \Box

The Fedosov dg manifold $(\mathcal{M}, Q)$ of Theorem 4.7 was also obtained independently by Batakidis–Voglaire [4] in the case of matched pairs.

Remark 4.8. When $L$ is the tangent bundle to a smooth manifold and $A$ is its trivial subbundle of rank 0, Theorem 4.7 reduces to a classical theorem of Emmrich–Weinstein [17] (see also [14]). In the particular case of the Lie pair comprised of the complex Lie algebroids $L = TL_X \otimes \mathbb{C}$ and $A = T^{0,1}X$ associated with a complex manifold $X$, Theorem 4.7 reduces to Theorem 5.9 in [6].

The identification of $C^\infty(M)$ with the subalgebra $\Gamma(\bigwedge^0 L^\vee \otimes S^0(B^\vee))$ of $C^\infty(M) = \Gamma(\bigwedge^0 L^\vee \otimes \hat{S}(B^\vee))$ determines a surjective submersion $\mathcal{M} \rightarrow M$. Let $\mathcal{F} \rightarrow \mathcal{M}$ denote the pullback of the vector bundle $B \rightarrow M$ through $\mathcal{M} \rightarrow M$. It is a graded vector bundle whose total space $\mathcal{F}$ is the graded manifold with support $M$ associated with the graded vector bundle $L[1] \otimes B \otimes B \rightarrow M$. Its space of sections $\Gamma(\mathcal{F} \rightarrow \mathcal{M})$ is canonically identified with $C^\infty(M) \otimes_{C^\infty(M)} \Gamma(B) = \Gamma(L^\vee \otimes \hat{S}(B^\vee) \otimes B)$. It is naturally a vector subbundle of $T_M \rightarrow M$; the inclusion $\Gamma(\mathcal{F} \rightarrow \mathcal{M}) \hookrightarrow \mathfrak{X}(\mathcal{M})$ takes the section $(\lambda \otimes \hat{\chi}^j) \otimes \partial_k \in C^\infty(M) \otimes_{C^\infty(M)} \Gamma(B)$ of the vector bundle $\mathcal{F} \rightarrow \mathcal{M}$ to the derivation $\mu \otimes \hat{\chi}^j \mapsto \lambda \wedge \mu \otimes M_k \hat{\chi}^{j+M-\epsilon_k}$ of $C^\infty(M)$.

Proposition 4.9. The pullback $\mathcal{F} \rightarrow \mathcal{M}$ of the vector bundle $B \rightarrow M$ to the Fedosov dg manifold $(\mathcal{M}, Q)$ is a dg Lie subalgebroid of the tangent dg Lie algebroid $T_M \rightarrow \mathcal{M}$.

In other words, $\mathcal{F}$ is a dg foliation of the dg manifold $(\mathcal{M}, Q)$. Each such dg Lie algebroid $\mathcal{F} \rightarrow \mathcal{M}$ is called a Fedosov dg Lie algebroid associated with the Lie pair $(L, A)$. 
4.3. **Dolgushev–Fedosov type quasi-isomorphisms on** $T^\bullet_{\text{poly}}$ **and** $T^\bullet_{\text{poly}}$. Below we describe an extension of Dolgushev–Fedosov type quasi-isomorphisms [14] to the context of Lie pairs. Actually, a stronger result holds: the quasi-isomorphisms are contractions.

Set $T^{r,s}_{\text{poly}} \colonequals \Gamma((B^r)^\otimes \otimes R B^s)$ and let $\mathcal{F}^{r,s}_{\text{poly}}(B)$ denote the space of formal vertical tensors of type $(r, s)$ on the vector bundle $B \to M$, i.e.

$$\mathcal{F}^{r,s}_{\text{poly}}(B) = \Gamma(\hat{S}(B^r)) \otimes_R T^{r,s}_{\text{poly}}.$$  

It is simple to see that

$$\Gamma(M; (\mathcal{F}^r)^\otimes \otimes \mathcal{F}^s) \cong \Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}_{\text{poly}}^{r,s}(B) \cong \Gamma(\Lambda^r L^\vee \otimes \hat{S} B^r) \otimes_R T^{r,s}_{\text{poly}}.$$  

Since $Q$ is a homological vector field on the graded manifold $M = L[1] \otimes B$, the Lie derivative $L_Q$ is a coboundary operator on the space $T^{r,s}_M$ of tensors of type $(r, s)$ on $M$. The Lie derivative $L_Q$ stabilizes the subspaces of tensors of type $(r, s)$ “tangent to the dg Lie subalgebroid $\mathcal{F}$ of $T_M$.”

**Lemma 4.10.** The subspace $\Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}_{\text{poly}}^{r,s}(B)$ of $T^{r,s}_M$ is stable under $L_Q$.

By $\sigma_2$, we denote the map $\sigma \otimes \text{id}$:

$$\Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}_{\text{poly}}^{r,s}(B) \cong \Gamma(\Lambda^r L^\vee \otimes \hat{S} B^r) \otimes_R T^{r,s}_{\text{poly}} \xrightarrow{\sigma \otimes \text{id}} \Gamma(\Lambda^r A^\vee) \otimes_R T^{r,s}_{\text{poly}}.$$  

We have the following Dolgushev–Fedosov type quasi-isomorphism [14].

**Proposition 4.11 ([45]).** For each type $(r, s)$, the chain map

$$\left( \Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}_{\text{poly}}^{r,s}(B), L_Q \right) \xrightarrow{\sigma_2} \left( \Gamma(\Lambda^r A^\vee) \otimes_R T^{r,s}_{\text{poly}}, d_A^\text{Bott} \right)$$

is a quasi-isomorphism.

Indeed, a stronger result was proved in [45]. We only need this result for polyvector fields.

Set $T^k_{\text{poly}} \colonequals \Gamma(\Lambda^{k+1} B)$ and let $\mathcal{F}^k_{\text{poly}}(B)$ denote the space of formal vertical $(k + 1)$-vector fields on $B$, i.e.

$$\mathcal{F}^k_{\text{poly}}(B) = \Gamma(\hat{S}(B^r)) \otimes_R T^k_{\text{poly}}.$$  

Note that $T^k_{\text{poly}} \subset T^0_{\text{poly}}$ and $\mathcal{F}^k_{\text{poly}}(B) \subset \mathcal{F}^0_{\text{poly}}$. Then

$$\Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}^k_{\text{poly}}(B) \cong \Gamma(\Lambda^r L^\vee \otimes \hat{S} B^r) \otimes_R T^k_{\text{poly}}.$$  

Denote by $\sigma_2$ the map

$$\Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}^k_{\text{poly}}(B) \cong \Gamma(\Lambda^r L^\vee \otimes \hat{S} B^r) \otimes_R T^k_{\text{poly}} \xrightarrow{\sigma \otimes \text{id}} \Gamma(\Lambda^r A^\vee) \otimes_R T^k_{\text{poly}}.$$  

**Theorem 4.12 ([45]).** There exists a contraction

$$\left( \text{tot} \left( \Gamma(\Lambda^r A^\vee) \otimes_R T^\bullet_{\text{poly}}, d_A^\text{Bott} \right), \right) \xrightarrow{\sigma_2} \left( \text{tot} \left( \Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}^\bullet_{\text{poly}}(B), L_Q \right) \right) \xrightarrow{\bar{h}_2}$$

Consider the Fedosov dg Lie algebroid $\mathcal{F} \to M$ of Section 4.2. It is clear that

$$\Gamma(M; \Lambda^k \mathcal{F}) \cong \Gamma(\Lambda^r L^\vee) \otimes_R \mathcal{F}^k_{\text{poly}}(B).$$  

Applying Proposition 4.1 to the dg Lie subalgebroid $\mathcal{F}$ of $T_M$, we obtain
Proposition 4.13.  

(1) Since the subspace $\Gamma(\Lambda^* L^\vee) \otimes_R T^*_\text{poly}(B)$ of the space $T^k_{\text{poly}}(\mathcal{M})$ of $(k+1)$-vector fields on $\mathcal{M} = L[1] \oplus B$ is stable under $L_Q$, we obtain a cochain complex

$$\cdots \longrightarrow \Gamma(\Lambda^u L^\vee) \otimes_R T^*_\text{poly} \overset{L_Q}{\longrightarrow} \Gamma(\Lambda^{u+1} L^\vee) \otimes_R T^*_\text{poly} \longrightarrow \cdots$$

for each $k \geq -1$.

(2) The total complex $\left( \text{tot} \left( \Gamma(\Lambda^* L^\vee) \otimes_R T^*_\text{poly}(B) \right), L_Q \right)$ is a differential Gerstenhaber algebra, whence a dgla.

It follows from the homotopy transfer theorem for $L_\infty$ algebras (see Lemma 2.25) applied to the contraction $\sigma_z$ that the dgla structure carried by $\text{tot} \left( \Gamma(\Lambda^* L^\vee) \otimes_R T^*_\text{poly}(B) \right)$ determines an $L_\infty$ algebra structure on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ are also associative, we have $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ is indeed canonical. Moreover, when the Lie pair happens to be a matched pair, the transferred $L_\infty$ algebra structure on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ is precisely the dgla structure described in Proposition 4.4.

Corollary 4.14 ([45]). Given a Lie pair $(L, A)$, each choice of a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and of a torsion-free $L$-connection $\nabla$ on $B$ determines

(1) an $L_\infty$-algebra structure on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ with the operator $d^\text{Bott}_A$ as unary bracket

(2) and a Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, T^*_\text{poly})$, the cohomology of the complex $\left( \text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right), d^\text{Bott}_A \right)$.

A priori, the $L_\infty$ algebra structure on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ in Corollary 4.14 is not canonical; it depends on a choice of quasi-isomorphic ‘Dolgushev–Fedosov type’ replacement for the complex $\left( \text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right), d^\text{Bott}_A \right)$ via a Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$. The construction of the Fedosov differential involves the choice of a torsion-free connection $\nabla : \Gamma(L) \times \Gamma(B) \to \Gamma(B)$ and a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$. However, one can prove that different choices yield isomorphic $L_\infty$ algebra structures on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ [3]. Therefore, the induced Gerstenhaber algebra structure on $\mathbb{H}^\bullet_{\text{CE}}(A, T^*_\text{poly})$ is indeed canonical. Moreover, when the Lie pair happens to be a matched pair, the transferred $L_\infty$ algebra structure on $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ is precisely the dgla structure described in Proposition 4.4.

Proposition 4.15. Under the hypotheses of Corollary 4.14 and the additional assumption that $j(B)$ is a Lie subalgebroid of $L$ — i.e., $L = A \bowtie B$ is a matched pair — the $L_\infty$ algebra $\text{tot} \left( \Gamma(\Lambda^* A^\vee) \otimes_R T^*_\text{poly} \right)$ and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, T^*_\text{poly})$ of Corollary 4.14 coincide respectively with the dgla $\Gamma(\Lambda^* A^\vee \otimes \Lambda^{*-1} B)$ and the Gerstenhaber algebra $\mathbb{H}^\bullet_{\text{CE}}(A, \Lambda^{*-1} B)$ of Proposition 4.4.

4.4. Dolgushev–Fedosov type quasi-isomorphism on $D^\bullet_{\text{poly}}$. Let $C^n := \text{Hom}_C(\mathcal{C} \otimes \mathcal{C}^{n+1}, \mathcal{C})$ denote the space of Hochschild $n$-cochains of the algebra $\mathcal{C} := C^\infty(L[1] \oplus B)$. The Gerstenhaber bracket of two cochains $\phi \in C^n$ and $\psi \in C^m$ is the cochain

$$[[\phi, \psi]] = \phi \star \psi - (-1)^{uv} \psi \star \phi \in C^{n+u}$$

where $\phi \star \psi \in C^{n+u+v}$ is defined by

$$(\phi \star \psi)(a_0 \otimes a_1 \otimes \cdots \otimes a_{u+v}) = \sum_{k=0}^{u} (-1)^{ku} \phi(a_0 \otimes \cdots \otimes a_{k-1} \otimes \psi(a_k \otimes \cdots \otimes a_{k+v}) \otimes a_{k+1+v} \otimes \cdots \otimes a_{u+v}),$$

for all $a_0, a_1, \ldots, a_{u+v} \in \mathcal{C}$. The Gerstenhaber bracket satisfies the graded Jacobi identity. Since the multiplication $m$ in $C^\infty(L[1] \oplus B)$ is associative, we have $[m, m] = 0$ and the standard Hochschild coboundary operator $[m, -]$ turns $C^\bullet$ into a cochain complex.

The space $D^\bullet_{\text{poly}}(L[1] \oplus B)$ of polydifferential operators on $L[1] \oplus B$ is a subspace of $C^\bullet$ closed under the Gerstenhaber bracket. Note that $Q \in \mathcal{X}(L[1] \oplus B) \subset D^0_{\text{poly}}(L[1] \oplus B)$ and $m \in D^1_{\text{poly}}(L[1] \oplus B)$. 
Lemma 4.16. We have $[Q + m, -]^2 = 0$.

Proof. We have $[m, m] = 0$ since the multiplication $m$ is associative, $[Q, m] = 0$ since $Q$ is a derivation of $m$, and $[Q, Q] = 0$ since $Q$ is a homological vector field. The conclusion follows from the Jacobi identity. □

Let $\mathcal{D}_{\text{poly}}^k$ denote the space of formal vertical $(k+1)$-polydifferential operators on the vector bundle $B$, and let $\mathfrak{g}_{\text{poly}} = \bigoplus_{k=1}^{\infty} \mathcal{D}_{\text{poly}}^k$. Set $\mathcal{S} = \Gamma(\hat{S}(B^\vee))$. There exists a canonical isomorphism

$$\Gamma(\hat{S}(B^\vee) \otimes S(B) \otimes \cdots \otimes S(B)) \xrightarrow{\varphi} \mathcal{D}_{\text{poly}}^{k+1}.$$ 

To $\chi^I \otimes \partial_{j_0} \otimes \cdots \otimes \partial_{j_k} \in \Gamma(\hat{S}(B^\vee) \otimes S(B) \otimes \cdots \otimes S(B))$, the isomorphism $\varphi$ associates the polydifferential operator

$$\mathcal{S} \otimes \mathcal{D}_{\text{poly}}^{k+1} \ni \chi^I \otimes \partial_{j_0} \otimes \cdots \otimes \partial_{j_k} \mapsto \chi^I \cdot \partial_{j_0}(\chi^{M_0}) \cdots \partial_{j_k}(\chi^{M_k}) \in \mathcal{S}.$$ 

The algebra of functions $C^\infty(L[1] \oplus B)$ is a module over its subalgebra $\Gamma(\Lambda^*L^\vee) \cong \Gamma(\Lambda^*L^\vee \otimes S^0(B^\vee))$. The subspace of $D_{\text{poly}}^\bullet(L[1] \oplus B)$ comprised of all $\Gamma(\Lambda^*L^\vee)$-multilinear polydifferential operators is easily identified to $\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}$. It is simple to see that the universal enveloping algebra $\mathcal{U}(\mathcal{F})$ of the Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$ is naturally identified with $\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^0$, which is a dg Hopf algebroid over $C^\infty(\mathcal{M}) \cong \Gamma(\Lambda^*L^\vee \otimes \hat{S}B^\vee)$. Moreover, $\mathcal{U}(\mathcal{F})$ is a dg Hopf subalgebroid of $D_{\text{poly}}^0(L[1] \oplus B)$. Note that

$$\mathcal{U}(\mathcal{F}) \otimes \mathcal{D}_{\text{poly}}^{k+1} \cong \Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^k.$$ 

Thus, as a consequence of Proposition 4.2, we have the following

Proposition 4.17. (1) The subspace $\Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet(L[1] \oplus B)$ is stable under the Hochschild coboundary operator $[Q + m, -]$.

(2) The triple $(\text{tot } \Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet), [Q + m, -], [[,]]$ is a dgl.a.

(3) The cohomology group $H^\bullet \left( \text{tot } \Gamma(\Lambda^*L^\vee) \otimes_R \mathcal{D}_{\text{poly}}^\bullet, [Q + m, -] \right)$ is a Gerstenhaber algebra.

Now consider the map

$$\sigma_5 : \Gamma(\Lambda^uL^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u \to \Gamma(\Lambda^uA^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u$$

defined by the commutative diagram

$$\begin{array}{ccc}
\Gamma(\Lambda^uL^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u & \xrightarrow{\sigma_5} & \Gamma(\Lambda^uA^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u \\
\text{id} \otimes \varphi \equiv & \sigma_5 \otimes \varphi & \sigma_5 \otimes \varphi \\
\Gamma(\Lambda^uL^\vee \otimes \hat{S}B^\vee \otimes (SB)^{\otimes u+1}) & \xrightarrow{\sigma_5 \otimes \varphi} & \Gamma(\Lambda^uA^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u.
\end{array}$$

The map $\sigma_5$ is a quasi-isomorphism of Dolgushev–Fedosov type similar to the classical Fedosov resolution of the polydifferential operators on a smooth manifold obtained by Dolgushev [14].

Theorem 4.18 ([45]). We have a construction

$$\left( \text{tot } (\Gamma(\Lambda^uA^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u), d_A^u + d_{\mathcal{F}} \right) \xrightarrow{h_5} \left( \text{tot } (\Gamma(\Lambda^uL^\vee) \otimes_R \mathcal{D}_{\text{poly}}^u), [Q + m, -] \right) \xrightarrow{h_5}$$

Corollary 4.19 ([45]). Given a Lie pair $(L, A)$, each choice of a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$ and of a torsion-free $L$-connection $\nabla$ on $B$ determines
(1) an $L\infty$-algebra structure on $\text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right)$ with the operator $d_A^L + d_{\mathcal{F}}$ as unary bracket

(2) and a Gerstenhaber algebra structure on $\mathbb{H}_{\text{CE}}^\bullet(\Lambda, D^\bullet_{\text{poly}})$, the cohomology of the complex

$$\left( \text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right), d_A^L + d_{\mathcal{F}} \right).$$

A priori, the $L\infty$-algebra structure on $\text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right)$ in Corollary 4.19 is not canonical; it depends on a choice of quasi-isomorphic ‘Dolgushev–Fedosov type’ replacement for the complex $\left( \text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right), d_A^L + d_{\mathcal{F}} \right)$ via a Fedosov dg Lie algebroid $\mathcal{F} \to \mathcal{M}$. The construction of the Fedosov differential involves the choice of a torsion-free connection $\nabla : \Gamma(L) \times \Gamma(B) \to \Gamma(B)$ and a splitting $j : B \to L$ of the short exact sequence of vector bundles $0 \to A \to L \to B \to 0$. However, one can prove that different choices yield isomorphic $L\infty$ algebra structures on $\text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right)$ [3]. Therefore, the induced Gerstenhaber algebra structure on $\mathbb{H}_{\text{CE}}^\bullet(A, D^\bullet_{\text{poly}})$ is indeed canonical. Moreover, when the Lie pair happens to be a matched pair, the transferred $L\infty$ algebra structure on $\text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right)$ is precisely the dgla structure described in Proposition 4.4.

**Proposition 4.20.** Under the hypotheses of Corollary 4.19 and the additional assumption that $j(B)$ is a Lie subalgebroid of $L$ — i.e. $L = A \ltimes B$ is a matched pair — the $L\infty$ algebra $\text{tot} \left( \Gamma(\Lambda^\bullet A^V) \otimes_R D^\bullet_{\text{poly}} \right)$ and the Gerstenhaber algebra $\mathbb{H}_{\text{CE}}^\bullet(A, D^\bullet_{\text{poly}})$ of Corollary 4.19 coincide respectively with the dgla $\Gamma(\Lambda^\bullet A^V) \otimes_R \mathcal{U}(B)^{n+1}$ and the Gerstenhaber algebra $\mathbb{H}_{\text{CE}}^\bullet(A, \mathcal{U}(B)^{n+1})$ of Proposition 4.4.

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