Spectral and Transport Properties of $D$-Wave Superconductors With Strong Impurities

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One of the remarkable features of disordered $d$-wave superconductors is strong sensitivity of long range properties to the microscopic realization of the disorder potential. Particularly rich phenomenology is observed for the – experimentally relevant – case of dilute distributions of isolated impurity centers. Building on earlier diagrammatic analyses, the present paper derives and analyses a low energy effective field theory of this system. Specifically, the results of previous diagrammatic $T$-matrix approaches are extended into the perturbatively inaccessible low energy regimes, and the long range (thermal) transport behaviour of the system is discussed. It turns out that in the extreme case of a half-filled tight binding band and infinitely strong impurities (impurities at the unitary limit), the system is in a delocalized phase.

1. INTRODUCTION

Disorder in the cuprates is commonly attributed to the presence of dilute distributions of strong local scatterers – e.g. $Zn$-atoms replacing $Cu$ – immersed into the two-dimensional matrices of $CuO_2$ planes $[1]$. While for conventional superconductors details of the microscopic realization of the disorder potential are of little, if any concern, the situation with anomalous superconductors is different. In fact, microscopic aspects of the impurity scattering have proven responsible for many of the notorious difficulties encountered in developing quantitative theories of disordered quasi-particles in $d$-wave superconductors:

First, the spectral characteristics of $d$-wave superconductors polluted by dilute strong scatterers – commonly dubbed ‘unitary scatterers’ – differ strikingly from those of systems with continuously distributed disorder: While in the latter case, the spectral density of quasi-particles categorically vanishes upon approaching zero energy $[3]$, it may diverge for unitary disorder $[1]$. This is demonstration of the fact $[3]$ that the standard paradigm of “insensitivity of long range properties of fermion systems to details of the disorder” is essentially violated in anomalous superconductors.

Second, conventional diagrammatic perturbation theory rules out as a tool to quantitatively address the low energy regime of the system. Based on self-consistent $T$-matrix (SCTA) approximation schemes, early perturbative approaches to the problem $[3]$, $[4]$ did not take the existence of infrared-singular diagram classes $[3]$, $[4]$ into account. These diagrams, similar in appearance to the diffusons and cooperons of normal metals, reflect the existence of a Goldstone mode linked to the particle-hole exchange symmetry of the system. For not too small energies, they can be brought under control by summing up weak localization type corrections to the Gorkov Green functions $[3]$, $[4]$. However, for low energies these corrections proliferate and perturbation theory produces unphysically divergent results. For the same reason, diagrammatic theory cannot be applied to assess the transport behaviour of the quasi-particle system.

In contrast, field theoretical approaches to the problem $[3]$, $[4]$ provide the means for controlled inclusion of all low-energy soft modes. However, third, the specific case of dilute impurity distributions, commonly referred to as Poisson distributed disorder, has been out of reach of previous field theoretical formulations. (This is because the standard construction route of field theories of disordered fermions is based on models of continuously, mostly Gaussian distributed disorder.)

To surmount these difficulties, a number of alternative approaches to the problem have been developed. Building on an explicit representation of the realization-specific Gorkov Green function, Pepin and Lee (PL) $[4]$ found the low energy single particle density of states (DoS) $\nu(\varepsilon)$ to be power-law divergent, $\nu(\varepsilon) \sim 1/|\varepsilon|^{\nu}$, while this result was obtained for the specific case of a half filled band, PL argue that the band center singularity should be insensitive to not too strong variations of the chemical potential. (Considering applications to the cuprates, the stability issue is of prime relevance: At half filling, or zero doping, the Cuprates are in an anti-ferromagnetic phase, i.e. the very formation of a superconducting phase requires a finite amount of dopants. This implies that strong sensitivity of the band center singularity to deviations off half filling would impose severe limitations on the practical relevance of the result.)

However, this latter statement is at variance with numerical $[4]$ and analytical $[4]$ findings indicating that the band center singularity represents a highly fragile phenomenon. As shown by Yashenkin et al. (YAGHK) $[4]$ the appearance of a zero energy divergence is tightly linked to the formation of a second class of soft modes, generated by the nesting symmetry of the half filled band and existing in parallel to the standard particle-hole modes. Within perturbation theory, these modes
lead to the formation of positive logarithmic (but not power law, like in PL) corrections to the low energy DoS. Further, even slight deviations from the combined limit (unitarity/half filling) generate a mass gap for the anomalous soft modes and induce a crossover to the negative logarithmic corrections characteristic for the standard particle-hole modes.

While the perturbative analysis reveals the existence of an anomalous phase in the half filled system, it cannot reliably predict the low-energy profile of the DoS, the critical properties of the anomalous phase, and its transport properties. To shed more light on the low energy phase of the Poisson disordered system in general, Chamon and Mudry [11] suggested a duality transformation relating the limit of unitary scattering to the opposite extreme of Gaussian, or Born disorder. The structure of this transformation was suggested on the basis of variational equations controlling the high-energy regime of the system. However, since the notorious soft modes do not at all enter the analysis of Chamon and Mudry, the extensibility of the transformation into the fluctuation dominated low energy regime remains speculative.

Building on these previous analyses, it is the objective of the present paper to describe the Poisson-disordered $d$-wave superconductor by field-theoretical methods. The generalized formalism will then be applied to explore spectral and localization properties of the system, at and away from half filling. We begin our analysis with a brief review of the general field theory approach to the $d$-wave superconductor, a qualitative discussion of the extension to pointlike disorder, and a summary of the results the generalized model predicts for spectral and transport properties of the system (section V). The next two sections contain technical details of the construction, first for generic pointlike disorder (section VI), then for the unitary/half-filled limit (section VII). We conclude in section VIII.

II. QUALITATIVE DISCUSSION AND RESULTS

As with normal systems, the properties of disordered superconductors are to a large extend determined by symmetries. Specifically, a time-reversal invariant spin-singlet superconductor obeys the two relations,

$$C : \hat{H} = -\sigma_3 \hat{H}^T \sigma_2, \quad \mathcal{T} : \hat{H} = \hat{H}^T,$$

where $\hat{H}$ is the Gorkov Hamilton operator and $\sigma_i$ are Pauli-matrices acting in particle-hole space. The first of these relations expresses the general particle-hole conjugation symmetry of the spin-singlet superconductor while the second is time reversal. In some sense the symmetry class identified through [11], termed class CI in Ref. [8], plays a role analogous to the standard orthogonal symmetry class of disordered normal systems.

For a conventional superconductor, the presence of a large quasi-particle excitation gap implies that the symmetries [11] only marginally affect low-energy, or long-range system properties. However, in a gapless environment, as realized, e.g., in the $d$-wave superconductor, the situation is different. Here, the symmetry $C$ entails the formation of long-lived particle-hole excitations: A particle and a hole with relative energy difference $\epsilon$ interfere to form a slowly fluctuating composite mode $\Pi(\epsilon)$. As $\epsilon \to 0$, the coherence between the particle and the hole amplitude becomes perfect, implying infrared singular, or massless behaviour of $\Pi(\epsilon)$.

The manifestation of these modes in observable low-energy properties can efficiently be explored by field integral methods. A fermion-replica field theory describing disordered superconductors of class CI was first derived by Oppermann [12,13] and later rediscovered, then within the context of $d$-wave superconductivity, by Seuthil and Fisher [8]. Supersymmetric implementations of the theory, needed to explore the perturbatively inaccessible infrared region, have been introduced in Refs. [8].

All these models were derived for Gaussian distributed disorder. By manipulations similar to those employed in the construction of the standard $\sigma$-models for systems of Wigner-Dyson symmetry, the microscopic model was mapped onto an action functional

$$Z = \int DT \exp(-S[T]),$$

where $T$ is a field taking values in the group $\text{Sp}(2r)$ ($r$: number of replicas) for the replica-implementations, and in the supergroup $\text{OSp}(2|2)$ for supersymmetry. (For a precise definition of this group, see Eqs. (24) and (27) below.)

As with the standard $\sigma$-models, the structure of the action of this field theory is fixed by a few macroscopic system parameters. Specifically, for a 2$d$ gapless superconductor with diffusive quasi-particle dynamics, the (bare) action has the form

$$S[T] = \frac{\pi \nu_0}{8} \int d^2 r \left[ -D \text{str} (\partial T \partial T^{-1}) - 2i \epsilon \text{str} (T + T^{-1}) \right],$$

where $D$ is the diffusion constant and $\nu_0$ the DoS computed in the self-consistent Born approximation (SCBA). Importantly, the symmetry of the fields $T$ which in turn is fixed by the physical symmetries [11] - does not permit other contributions to the low energy action. In other words, the action is completely fixed by the value of the two constants $D$ and $\nu_0$. Notice that for the $d$-wave superconductor, and on the level of the bare theory,

$$D\nu_0 = \frac{1}{\pi^2} \frac{t^2 + \Delta^2}{t \Delta} \equiv 4g_s,$$

where $\Delta$ and $t$ are the order parameter and hopping strength, respectively and $g_s$ is the spin conductance. I.e.
the bare, or Drude spin conductance does not depend on disorder concentration [8].

Starting from the action (3), spectral and transport properties of the quasi-particles can conveniently be explored. However, before turning to the phenomenology of the system let us briefly discuss what changes with the theory (3) if the disorder is Poissonian distributed. Certainly, the fundamental symmetries (4) are insensitive to the specifics of the disorder distribution. Further, the quasi-particle dynamics on intermediate time scales, larger than the scattering time but smaller than any scale related to quasi-particle localization, will continue to be diffusive. This leaves us with two principal possibilities: Either there are some fundamental reasons that exclude diffusive. This leaves us with two principal possibilities: Either there are some fundamental reasons that exclude the existence of a local effective action of the Poisson disordered system. Or, a low-energy action exists and is bound to have the same structure as the action (3).

The result of a technical derivation detailed in section 111 will be that the second scenario is realized, i.e. that the low-energy phase of the Poisson disordered system is again described by (3). The only difference to the Gaussian case is that the value of the constant \( v_0 \) is now set by the SCTA rather than by the SCBA density of states.

That both Gaussian and the Poissonian disorder map onto the field theory (3) implies a number of predictions on the long range behaviour of the system:

▷ First, the quasi-particle DoS obtains as

\[
\nu(\epsilon) = \frac{v_0}{8} \text{Re} (\text{str} ((T + T^{-1})\mathcal{P}))_{T},
\]

where \( \langle \ldots \rangle = \int \mathcal{D}T e^{-S[T]} \) stands for functional averaging and \( \mathcal{P} \) is a constant projector matrix [14]. In the most basic approximation to the functional integral, fluctuations of the fields \( T \) are neglected and one obtains \( \nu(\epsilon) = v_0 \) in agreement with the SCBA/SCTA diagrammatic result. Going beyond this approximation,

▷ a one-loop integration over fluctuations of the fields \( T \) leads to a negative correction term [3],

\[
\delta \nu(\epsilon) \sim -\frac{1}{g_s} \ln \left( \Lambda \sqrt{\frac{g_s}{\epsilon v_0}} \right),
\]

where \( \Lambda \sim a^{-1} \) is an ultraviolet cutoff and \( a \) the lattice spacing. Eq. (3) is in agreement with the results obtained by extended diagrammatic series summations [10], i.e. analyses accounting for the presence of singular ladder diagrams. Notice that the magnitude of the quantum correction depends on the (disorder independent!) scale \( g_s(t/\Delta) \). The smaller the anisotropy \( t/\Delta > 1 \), the more important become quantum fluctuations. As concerns the ‘mass parameter’ \( \nu_0 \) truncating the logarithm, there is an important difference between the Gaussian and the Poissonian case. While for Gaussian disorder, \( \nu_0 \) increases quadratically with the strength of the impurity potential, \( \nu \), it decreases as \( \nu^{-1} \) for strong pointlike scatterers. (For quantitative expressions, see section 111 below.), i.e. the small parameters of the perturbative approaches to the two limits scale reciprocallly with the impurity strength, which is in agreement with the ‘duality’ hypothesis of Ref. 11.

▷ For systems of finite size \( L \) and energies smaller than the Thouless energy \( \epsilon = \frac{D}{L \pi^2} \), spatial fluctuations of the fields \( T \) are effectively frozen out. As with normal systems, a non-perturbative integration over the spatially constant zero mode \( T \) obtains results compatible with RMT behaviour. Specifically, the low-energy spectral density scales as

\[
\nu(\epsilon) \sim \left| \frac{\epsilon}{\Delta} \right|^\beta.
\]

where \( \Delta = (v_0 L^2)^{-1} \) is the single particle level spacing.

▷ Turning to transport, it has been shown [5] that under renormalization, the conductance scales from its disorder-independent bare value to zero, with a \( \beta \)-function \( \beta = 1/(4\pi^2 g_s^2) \). This implies that for large systems quasi-particles should be localized on the scale of some localization length \( \xi \sim \exp(4\pi^2 g_s) \).

In the thermodynamic limit the system is a (spin) insulator. Similarly,

▷ in the limit \( L \gg \xi \), mechanisms of quantum localization and spectral repulsion conspire to manufacture a spectral gap at zero energy: Consider a quasi-particle at energies \( \epsilon < D/\xi^2 \), corresponding to time scales larger than the time needed to diffusively explore a localization volume \( \sim \xi^2 \). Confined to stay within the localization volume, the quasi-particle will exhibit ergodic behaviour, i.e. the dynamics will become random matrix like. One thus expects [3] that

\[
\nu(\epsilon) = \left| \frac{\epsilon}{\Delta_\xi} \right|,
\]

where \( \Delta_\xi = (v_0 \xi^2)^{-1} \) is the level spacing of a single localization volume. (While for the two-dimensional superconductor Eq. (8) has the status of a conjecture, similar predictions for quasi one-dimensional quantum wires of non-Wigner-Dyson symmetry were confirmed by transfer matrix methods [13,16].)

Summarizing, we find that for both Gaussian and Poisson disorder the system scales to a spin-insulator phase. While the DoS’s obtained for the two types of disorder are qualitatively similar, the extension of regimes with
different spectral properties scales reciprocally with the disorder strength.

While these results apply to the case of generic disorder, there is one particular point in the phase diagram where drastically different behaviour is observed: In the combined limit half-filling/unitary scattering, the zero energy DoS exhibits a power law divergence \[ \frac{1}{\epsilon \sim |\epsilon|^\nu} \] In YAGHK the peculiar properties of this limit, henceforth referred to as the 'unitary limit' for brevity, were related to the formation of a second class of soft modes, existing in parallel to the particle-hole modes discussed above. It was shown that that perturbative inclusion of these modes leads to positive logarithmic corrections to the DoS. However, as with the generic particle-hole modes discussed above, no safe conclusions on the profile of the DoS can be drawn on the basis of pure perturbation theory.

Below we will show that the emergence of a new class of soft modes is indicative of critical behaviour. The system at unitarity and \( \epsilon = 0 \) sits on a critical line (parameterized through the ratio \( t/\Delta \)). One has to emphasize that this fixed line lies outside the phase diagram of 'real' high \( T_c \) materials; cuprates at half filling are anti-ferromagnets rather than superconductors. However, in the vicinity of the critical regime spectral and transport observables will exhibit anomalous scaling behaviour traces of which may (or may not) be observable in real systems.

But what, then, is the origin of the exotic behaviour? In YAGHK the formation of the anomalous soft modes was traced back to the nesting symmetry of the half filled band. The \( 2 \times 2 \) matrix Gorkov Hamiltonian function of the clean system obeys the symmetry

\[
\mathcal{N} : \hat{H}_0(k) = -\hat{H}_0(k + Q),
\]

where \( Q = (\pi, \pi) \) is the nesting vector and a square lattice with unit lattice spacing is understood. Notice that what matters in the description of spectral/transport properties is not the Hamiltonian but the Green function \( \tilde{G} = (\epsilon + \mu \sigma_3 - \hat{H})^{-1} \). Specifically, (i) the middle of the band supports a delocalized phase and (ii) the DoS diverges as a power law \( \nu(\epsilon) \sim |\epsilon|^{-\kappa} \) with some non-universal parameter \( \kappa \), upon approaching the band center. These phenomena reflect the fact that the \( \epsilon = 0 \) system with \( \mathcal{N} \)-symmetry belongs to a non-Wigner-Dyson symmetry class, viz. class AIII in the classification of Ref. [10]. Breaking this symmetry, finite values of the parameters \( \epsilon \) or \( \mu \) induce a crossover to the Wigner-Dyson unitary class (or class A), i.e. the symmetry class of generic electronic systems with broken time reversal invariance.

To obtain these results Gade described the normal metal sublattice system in terms of a (boson replica) effective field theory, which was then subjected to a renormalization group analysis. For future reference we note that this theory has a supersymmetric extension defined through \( Z = \int DT \exp(-S[T]) \) and

\[
S[T] = \frac{\pi \nu_0}{8} \int d^2 r \left[ -D \text{str}(\partial T T^{-1}) + 2i \epsilon \text{str}(T + T^{-1}) \right] + c \int d^2 r \left[ \text{str}(T T^{-1}) \right]^2,
\]

where \( c \) is a non-universal disorder-dependent constant and the fields \( T \) take values in the supergroup GL(1|1) (i.e. the group of invertible two-dimensional supermatrices.)

While Gade obtained the field theory [14] (or rather its boson replica analog) for the specific model of electron hopping on a bipartite lattice, it is now understood that there is a more general way of looking at the problem (cf., e.g., the discussion in Ref. [14]). Field theories like [14] generally describe the long range behaviour of Hamiltonians of symmetry class AIII. A Hamiltonian belongs to this symmetry class if it can be represented in a block form,

\[
\hat{H} = \begin{pmatrix} Z^1 & Z^2 \\ Z^2 & Z^1 \end{pmatrix},
\]
where $Z$ are arbitrary matrices. (Indeed, the symmetry (1) states that the Hamiltonian assumes a block off-diagonal form when represented in a sublattice basis.)

The generalized interpretation helps to understand what happens with the sublattice system when the superconductive aspects of the problem are switched on. Indeed, it is straightforward to show (cf. Appendix A) that (i) the superconductor sublattice Hamiltonian can be transformed to a representation (2) and (ii) that there are no other symmetries in the problem. This implies that the superconductor with $\mathcal{N}$-symmetry is again described by the low-energy effective action (1), a claim to be verified by explicit construction in section IV. There is one difference to the previously discussed case, viz. that the fields $T$ now take values in the larger group GL(2|2) (i.e. extra degrees of freedom are needed to accommodate the superconductor symmetries (1).) However, as far as the long range properties of the system are concerned, an extension of the field space to GL(n|n) (here $n = 2$) is inconsequential: As shown by renormalization group methods in Ref. [18],

$\triangleright$ for zero excitation energy, $\epsilon = 0$, the system is critical.

$\triangleright$ Upon approaching zero energy, the DoS diverges as

$$\nu(\epsilon) \sim \frac{e^{-\kappa \sqrt{-\ln|\epsilon|}}}{|\epsilon|},$$

where $\kappa$ is some non-universal, disorder dependent scale that cannot be reliably determined from the RG analysis. For sufficiently small energies, $\epsilon < \exp(-\kappa^2)$, the DoS scales in a near power-law manner, $\nu(\epsilon) \sim |\epsilon|^{-1}$.

$\triangleright$ The conductance does not renormalize, i.e. the system is a (thermal) metal.

$\triangleright$ All three, finite values of $\epsilon$, the chemical potential $\mu$, and deviations off unitarity, $v^{-1} > 0$, represent relevant perturbations that drive the system towards the generic superconductor symmetry class CI.

While the first three of the characteristics listed above can directly be inferred from Gade’s analysis, the last needs some clarification. As shown in section IV, deviations off the unitary limit lead to the appearance of an extra operator

$$S_{sb} = \Gamma \int d^2r \text{str} \left( TT^* + (TT^*)^{-1} \right),$$

(14)

to be added to (1). Here, $\Gamma = \Gamma(\mu, v^{-1})$ is a symmetry breaking parameter whose explicit dependence on $\mu$ and $v^{-1}$ is given in Eqs. (13) and (10) below. Further, $X^s$ denotes a kind of generalized matrix transposition. More important than the explicit definition of this operation (see Eq. (26) below) is that finite values of $\Gamma$ enforce $T^* = T^{-1}$ which is the defining relation of the orthosymplectic group OSp(2|2) $\subset$ GL(2|2). In other words, the action $S_{sb}$ induces a crossover between the extended symmetry at the unitary limit (AIII) and the generic symmetry class CI [21]. When embedded into an RG analysis, the operator $S_{sb}$ represents a relevant perturbation with scaling dimension 2 (by elementary power counting.) I.e. deviations off the unitary limit rapidly increase under the RG operation and drive the system towards the phase of the generic superconductor.

![FIG. 1. Qualitative diagram of regimes with different behaviour of the DoS $\nu(\epsilon)$. Horizontal axis: energy, $\epsilon$; vertical axis: $\mathcal{N}$-breaking by finite chemical potentials $\mu$ and/or non-infinite scattering strengths $v^{-1}$. On the abscissa the system is in class AIII, everywhere else in CI. For large values of $\epsilon$ the DoS is well described by SCTA. The energy below which quantum corrections begin to play a role, increases with the anisotropy parameter $\Delta/t$. Depending on whether the product $\epsilon \nu_0$ is smaller or larger than the symmetry breaking parameter $\Gamma(\mu, v^{-1})$ (the crossover region indicated by the solid line), quantum corrections to the DoS are negative or positive. At the origin of the diagram $(\Gamma, \epsilon) = (0, 0)$, the system is critical and the DoS diverges.

We finally discuss how these findings compare to the results of previous analyses, notably PL and YAGHK. To leading order, the divergence of the DoS, (13), agrees with the result of PL. There are, however, some discrepancies on the level of logarithmic corrections to powerlaw scaling (PL find $\nu(\epsilon) \sim 1/|\epsilon| |\ln(|\epsilon|)|^2$) whose origin we do not understand. The positive logarithmic corrections to the SCTA-DoS found in YAGHK are perturbative precursors of the full powerlaw divergence. As a new result, we find that transport in the band center is metallic. It is tempting to interpret this metallic behaviour as a consequence of resonant tunneling transport in the narrow impurity band created by unitary scattering centers. However, this picture is purely speculative. Finally, we disagree with the assertion of PL, that the divergence
survives generalization to finite values of $\mu$. Using RG language, we rather find that $\mu$ acts like a (dynamically increasing) mass scale terminating the RG flow for small enough $\epsilon$. More specifically, one expects (cf. Fig. 1) that for large values of $\epsilon$ (i) the DoS will be given by the SCTA result. Perturbative quantum corrections to the large energy asymptotics (ii) scale logarithmically with $\epsilon$ and inversely proportional to the anisotropy parameter $t/\Delta \gg 1$. In an intermediate energy regime, RG summation over all quantum interference contributions leads to the increasing DoS profile [13]. Finally, (iii) below a crossover scale set by $\hbar c/\Gamma(\mu, v^{-1}) = 1$, the DoS follows the decreasing profile of the superconductor of class C1. Non-monotonous behaviour of this type has indeed been observed numerically in YAGHK. Whether or not traces of the intermediate regime (ii), or even of the scaling law [13], might be observable experimentally cannot be decided by the present approach. (Mentioning experiment, one should also notice that quasi-particle interactions act as an additional ingredient spoiling sublattice symmetries. I.e. the Hartree-Fock potential seen by the quasi-particles is local in real space and, therefore, violates [14].)

This concludes our qualitative discussion. Starting from a microscopic lattice Hamiltonian, the subject of the next two sections will be the construction the field theories (3) and (11). Basically, this will amount to an embedding of SCTA-summation techniques into a field integral approach. (Readers not interested in technicalities may skip these sections.)

III. CONSTRUCTION OF THE FIELD THEORY

Consider the quasi-particle Hamiltonian of a two-dimensional lattice $d$-wave superconductor,

$$
\hat{H} = \sum_{ij} \Psi_i^\dagger \left( H_{ij}^0 + H_{ij}^{\text{dis}} \right) \Psi_j,
$$

$$
\hat{H}_{ij}^0 = \xi_{ij} \sigma_3 + \Delta_{ij} \sigma_1,
$$

$$
\hat{H}_{ij}^{\text{dis}} = V_{ij}^p \delta_{ij} \sigma_3 + V_{ij}^{g2} \sigma_3 + V_{ij}^{g1} \sigma_1,
$$

where $\xi_{ij} = t_{ij} - \mu$,

$$
\Psi_i = \left( \begin{array}{c} \psi_1^i \\ \psi_2^i \\ \psi_3^i \\ \psi_4^i \end{array} \right),
$$

is a spinor comprising a spin-up hole and a spin-down particle, and $\sigma_i$ are Pauli matrices acting in particle-hole space. The clean part of the Hamiltonian, $\hat{H}^0$, contains the chemical potential, $\mu$, a nearest neighbour hopping matrix element, $t_{ij}$, and the $d$-wave order parameter, $\Delta_{ij}$. The Fourier components of these operators are given by

$$
\xi(k) \equiv t(\cos k_x + \cos k_y) - \mu,
$$

$$
\Delta(k) = \Delta(\cos k_x - \cos k_y),
$$

where $t$ and $\Delta$ define the strength of hopping and order parameter, respectively.

The stochastic part of the Hamiltonian, $\hat{H}^{\text{dis}}$, accounts for two different types of disorder. First, the potential generated by $N$ randomly placed impurities is represented through

$$
V_i^p = \sum_{j=1}^N v(\mathbf{r}_i - \mathbf{r}_j),
$$

where $\{\mathbf{r}_j\}$ are the coordinates of the impurity centers.

Configurational averaging over $V_i^p$,

$$
\langle \cdots \rangle_p \equiv \prod_{j=1}^N \frac{1}{L^2} \int d^2 R_j \langle \cdots \rangle,
$$

will amount an independent, or Poissonian average over impurity coordinates.

Second, the random operators $V^{ga}_i, a = 1, 3$ model the presence of residual statistical fluctuations superimposed on the hopping and the order parameter amplitudes, respectively. For convenience, we assume these amplitudes to be Gaussian distributed with zero mean and variance,

$$
\langle V^{ga}_i V^{ga'}_j \rangle_g = g \delta_{ii'} \delta_{jj'} \delta^{aa'}.
$$

The motivation behind introducing Gaussian bond disorder in addition to the Poisson impurity system is not only to make the modeling more 'realistic'. Below we will employ the Gaussian disorder average as a vehicle to introduce the soft fields $T$. Averaging over the impurity system will then be done in an after-step [22]. (Notice that for the purposes of the construction, the variance $g$ can be assumed to be infinitesimally weak.)

To prepare the construction of a field theory of this system, we change to an 'off-diagonal' representation, where the role of the Pauli matrix $\sigma_3$ is taken over by $\sigma_1$. This is done by a unitary transformation $\hat{H} \to \exp(-i\frac{\pi}{4}\sigma_1) \hat{H} \exp(i\frac{\pi}{4}\sigma_1)$, after which

$$
\hat{H}^0 = \hat{\xi} \sigma_2 + \Delta \sigma_1,
$$

$$
\hat{H}^{\text{dis}} = (V_i^p + V^{g2})\sigma_3 + V^{g1}\sigma_1.
$$

The new representation changes the appearance of the basic symmetries [4], viz.

$$
C : \hat{H} = -\sigma_2 \hat{H}^T \sigma_2, \quad \mathcal{T} : \hat{H} = \sigma_1 \hat{H}^T \sigma_1.
$$

We next introduce a supersymmetric partition function

$$
Z[J] = \int \mathcal{D}(\bar{\psi}, \psi) \exp \left( i\bar{\psi} \left[ \epsilon - \hat{\mathcal{H}} + \hat{J} \right] \psi \right),
$$

from which disorder averaged matrix elements of the retarded Gorkov Green function can be obtained by differentiation with respect to elements of the source matrix $\hat{J}$. In [21], $\Im \epsilon > 0$ and

$$
\psi = \left( \begin{array}{c} \psi_1^b \\ \psi_2^b \\ \psi_3^b \\ \psi_4^b \end{array} \right)
$$
is a four-component superfield whose complex commuting (anticommuting) components \( \psi^b \) (\( \psi^\dagger \)) carry a particle-hole structure like in (1). While convergence of the integral requires \( \bar{\psi}^b = \psi^b \), the Grassmann fields \( \bar{\psi}^\dagger \) and \( \psi^\dagger \) are independent. For transparency, the lattice index summation is suppressed in the notation. Similarly, the source field \( \hat{J} \) will be set to zero in much of our further discussion.

To fully account for both symmetries (1), it is necessary to subject the functional integral to a doubling, or 'charge conjugation' operation. Using that

\[
\bar{\psi} \hat{H} \psi = \frac{1}{2} \left( \bar{\psi} \hat{H} \psi + \psi^T \sigma_3^b \hat{H}^T \bar{\psi}^T \right),
\]

the functional integral can be re-written as,

\[
Z[0] = \int \mathcal{D} \Psi \exp \left( \frac{i}{2} \Psi \left[ \epsilon \sigma^c_3 - \hat{H} \right] \bar{\Psi} \right),
\]

where the eight-component fields

\[
\Psi = \left( \begin{array}{c} \psi \\ \sigma_2 \bar{\psi}^T \end{array} \right), \quad \bar{\Psi} = \left( \bar{\psi}, -\psi^T \sigma_2 \otimes \sigma_3^b \right).
\]

Here, the matrices \( \sigma_i \), \( \sigma_3^b \), and \( \sigma_3^c \) are Pauli matrices acting in particle-hole, boson-fermion, and in the newly introduced charge-conjugation space, respectively. The fields \( \Psi \) and \( \bar{\Psi} \) are related to each other by

\[
\Psi = \Psi^T \sigma_2 \otimes \tau, \quad \bar{\Psi} = -\sigma_2 \otimes \tau \bar{\Psi}^T,
\]

where

\[
\tau = E_1^{bf} \otimes (i \sigma_2^c) + E_2^{bf} \otimes \sigma_1^c
\]

and \( E_{ij}^{bf} \) is a matrix acting in bf-space and containing zeros everywhere save for a unity at position \( ij \). Fortunately, all we need to know about the \( \Psi \)'s is encapsulated by (23), i.e. there will be no need to explicitly unravel the entangled eight component structure of these fields.

Continuing with the construction, we next average the functional integral over Gaussian disorder and decouple the resulting \( \Psi \)-interaction by means of a Hubbard-Stratonovich transformation:

\[
Z_{\text{bf}}[0] \equiv \langle Z[0] \rangle_{\text{G}} = \int \mathcal{D} \left( Q_0, Q_3 \right) e^{-\frac{1}{2} \sum_{ij} \text{str} \left( (Q_0^2, Q_3^2, Q_3^T Q_0, Q_0^T Q_3) \right) \times}
\]

\[
\times \int \mathcal{D} \Psi \exp \left( \frac{i}{2} \Psi \left[ \epsilon \sigma^c_3 + 4 \hat{Q}_0 \bar{\Psi}^T \sigma_3 + -\hat{H}^p \right] \bar{\Psi} \right) = \int \mathcal{D} \left( Q_0, Q_3 \right) e^{-\frac{1}{2} \sum_{ij} \text{str} \left( (Q_0^2, Q_3^2, Q_3^T Q_0, Q_0^T Q_3) \right) \times}
\]

\[
\times \exp \left( \frac{1}{2} \text{str} \ln \left( \epsilon \sigma^c_3 + 4 \hat{Q}_0 \bar{\Psi}^T \sigma_3 - \hat{H}^p \right) \right),
\]

where \( \hat{H}^p \mid_{\Psi=0} \) contains only Poisson disorder and \( Q_{a,ij}, a = 0,3 \) are four component supermatrices living on the non-directed links \( (ij) \) of the lattice and acting in bf and cc space. Finally,

\[
\langle \hat{Q}_a \rangle_{ij} \equiv \delta_{ij} \frac{1}{N} \sum_{i \in N_i} Q_{a,ij},
\]

where \( N_i \) is the set of four nearest neighbours of site \( i \).

We next subject the functional integral to a stationary phase approximation. Temporarily neglecting both the energy increment \( \epsilon \) and the impurity potential one finds that the variational equations \( \frac{\delta S}{\delta \sigma} = 0 \) are solved by the matrix-diagonal and spatially uniform configuration \( (Q_0, Q_3) = i \frac{\kappa(\sigma^c_3)}{2}(\sigma^c_3, 0) \). Here, \( \kappa(g) \) is some function of the disorder strength which vanishes monotonously in the limit \( g \sim 0 \).

As usual with non-linear \( \sigma \)-models, the diagonal configuration is but one representative of an entire manifold of solutions of the mean field equation. To identify this manifold, we imagine our diagonal configuration substituted into the second line of (23) and explore what happens when the \( \Psi \)-field is subjected to a transformation (matrix structure in ph-space),

\[
\Psi \rightarrow \left( \begin{array}{cc} T_1 & 0 \\ 0 & T_2 \end{array} \right) \Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} \left( \begin{array}{cc} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{array} \right).
\]

Here, \( T_1, \ldots, T_2 \) are spatially constant supermatrices of dimension four subject to the condition (to ensure compatibility with (23))

\[
\tau^{-1} T_i^T \tau = T_i^s = T_i.
\]

Remembering that \( \hat{H} \) is off-diagonal in ph-space, we find that the transformation leaves the Hamiltonian invariant provided that

\[
T_i = T_i^s = T_i^{-1},
\]

which is the defining relation of the supergroup OSp(2|2).

We thus find that, notwithstanding the presence of the Poisson impurity potential in \( \hat{H}^p \), the action has the continuous group OSp(2|2) \( \otimes \) OSp(2|2) as an invariance manifold. While these transformations commute with the Hamiltonian, they change the diagonal saddle point configurations according to

\[
i \kappa \left( \begin{array}{cc} \sigma^c_3 & \sigma^c_3 \\ \sigma^c_3 & \sigma^c_3 \end{array} \right) \rightarrow i \kappa \left( \begin{array}{cc} T_2^{-1} \sigma^c_3 T_1 & 0 \\ 0 & T_1^{-1} \sigma^c_3 T_2 \end{array} \right).
\]

This equation identifies the set of all field configurations \( T \equiv T_2^{-1} \sigma^c_3 T_1 \in \text{OSp}(2|2) \) as the Goldstone mode manifold of the model. For fixed \( T_2 \), the factor \( T_2^{-1} \sigma^c_3 \in \text{OSp}(2|2) \) can be absorbed in \( T_1 \), i.e. the Goldstone mode is isomorphic to a single copy of Osp(2|2) and can be parameterized by the representative \( T \) freely running through this group. Finally, disregarding massive fluctuations (i.e. field-configurations non-commutative with \( \hat{H} \)) and using that the weight factor
$Z_g = \int DT \exp \left( -\frac{1}{2} \text{str} \ln \left( \epsilon_{\sigma \beta}^{cc} + \left[ \begin{array}{cc} i\kappa T & -\hat{H}_1^p \\ 0 & i\kappa T \end{array} \right] \right) \right),$ 

as an effective functional capturing the low energy degrees of freedom of the problem.

We next have to face up to the analysis of the pointlike disorder included in $H^p$. More specifically, the object we need to compute is $Z_{gp} = \langle Z_g \rangle_p$, expanded in slow fluctuations of the soft field $T$. Trading the average over a fixed number of impurities for a grand canonical average, it is possible to formulate an exact representation of the averaged functional $\bar{Z}_{gp}$ [11]. However, transcendental in the fields $T$, the action of the grand canonical functional does not appear to be a particularly inviting starting point for subsequent low-energy expansions.

Below we will adopt a different and less rigorous averaging scheme. The basic idea will be to employ elements of the standard approximations implied in the analysis of dilute systems of scatterers, notably the self-consistent $T$-matrix approximation, in the expansion of the effective action. To be more specific, what we want to do is expand the logarithm appearing in the action of $Z_g$ in (a) the 'mass' brought about by finite energy arguments $\epsilon$ and (b) the finite action due to spatial fluctuations of the fields. The expansion of the logarithm in these perturbations, for a fixed realization of the impurity potential, leads to expressions like

$$Z = \int DT \left\langle \int d^2r_1 O_1(T(r_1)) \int d^2r_2 O_2(T(r_2)) \ldots \right\rangle_p,$$

where $O_i(T)$ symbolically stands for a local operator depending on the fields $T$ and a continuum approximation is understood. (Here, the attribute 'local' means locality on the scale of the elastic mean free path set by the scattering potential.) To make further progress with this expression, we rely on one central assumption: For a typical configuration of coordinates $(r_1, r_2, \ldots)$, the spacings $|r_j - r_i|$ will be much larger than the correlation range $\xi$ of the impurity potential. For such configurations, the average can be assumed uncorrelated, i.e.

$$\langle O_1(T(r_1))O_2(T(r_2)) \ldots \rangle_p |_{r_i - r_j \gg \xi} = \langle O_1(T(r_1)) \rangle_p \langle O_2(T(r_2)) \rangle_p.$$

In a small subset of the integration volume, one or several integration variables come close to each other and averaging over impurity coordinates leads to residual correlations. This leads to the generation of composite operators, i.e. local operators of higher order in gradients and/or energies $\epsilon$. Relying on the fact that operators of this type will carry negative scaling dimension, we neglect statistical correlations and approximate the average by

$$Z \approx \int DT \int d^2r_1 \langle O_1(T(r_1)) \rangle_p \int d^2r_2 \langle O_2(T(r_2)) \rangle_p \ldots,$$

which amounts to replacing the average of the functional by an average of the action:

$$Z_{gp} \approx \int DT \times \exp \left( -\frac{1}{2} \text{str} \ln \left( \epsilon_{\sigma \beta}^{cc} + \left[ \begin{array}{cc} i\kappa T & -\hat{H}_1^p \\ 0 & i\kappa T \end{array} \right] \right) \right) \langle \ldots \rangle_p.$$

We will re-assess the validity of this approximation below.

The rest of the derivations is straightforward. We begin by computing the action $S_\epsilon$ corresponding to finite values of the energy parameter $\epsilon$. Expanding the action to first order in $\epsilon$ and using that for sufficiently slow fluctuations the fields $T$ and the Hamiltonian $\hat{H}_T$ are approximately commutative, one obtains

$$S_\epsilon = -\frac{\epsilon}{2} \text{str} \left( T\hat{G}_{11} + T^{-1}\hat{G}_{22} \right),$$

where $\hat{G} = \langle \hat{G} \rangle_p$, 

$$\hat{G} = \left( \begin{array}{cc} i\kappa & -\hat{H}_1^p \\ -\hat{H}_2^p & i\kappa \end{array} \right)^{-1} \tag{29}$$

is the disordered Gorkov Green function and indices are in ph-space.

a) \centerline{\includegraphics{fig2a.png}}

b) \centerline{\includegraphics{fig2b.png}}

FIG. 2. a) Diagrammatic representation of the SCTA series for the single particle Green function. b) The SCTA diffusion mode.

Borrowing from previous analyses [8], we compute the averaged superconductor Green function $\bar{G}$ by SCTA summation techniques [23]. The basic presumption underlying the SCTA (For a more comprehensive discussion of the approach and its application to the $d$-wave superconductor see, e.g., Ref. [24]) is that for decreasing impurity concentration, $n_\downarrow$, correlations between scattering processes off different impurities do not affect the structure of the averaged Green function. Neglecting any amount of residual inter-impurity correlation, the SCTA approximates the self energy, $\hat{\Sigma}$, of the Green function by the set of diagrams depicted in Fig. 3. Summation over these contributions leads to

$$\hat{\Sigma} = n_\downarrow \hat{T},$$
where the $T$-matrix (not to be confused with the slow fields $T$ of the effective action approach) is defined through the Dyson equation
\[ \hat{T} = v\sigma_2 + vg\sigma_2\hat{T} \Rightarrow \hat{T} = (v^{-1}\sigma_2 - g)^{-1}. \] (30)

Here, $g \equiv \sum_k \hat{G}(k)$,
\[ \hat{G}(k) = \left[i\kappa - \xi(k)\sigma_2 - \Delta(k)\sigma_1 - \hat{\Sigma}\right]^{-1}, \] and we anticipate that for a short range scattering potential $v(k) \equiv v = \text{const.}$, the self energy is momentum independent. Expanding in Pauli matrices, $\hat{\Sigma} = \Sigma = g_0 + g_2\sigma_2$ assumes the form
\[ g_0 = \Sigma_0\Xi, \quad g_2 = (\mu - \Sigma_2)\Xi, \] (32)
where
\[ \Xi = \sum_k \frac{1}{-\Sigma_0^2 + \xi(k)^2 + \Delta(k)^2}. \] Linearizing the dispersion $H_0(k)$ around the four nodal points of the Fermi surface of the $d$-wave superconductor one finds
\[ \Xi \simeq \frac{8}{\pi\Delta} \ln \left( \frac{\Lambda}{|\Sigma_0|} \right), \] (33)
where $\Lambda \gg |\Sigma_0|$ is a cutoff measuring the extent of the 'linearization volume' in $k$-space. With (30), and
\[ \hat{T} = T_0 + T_2\sigma_2, \]
we finally arrive at the self consistency equation
\[ T_0 = \frac{g_0}{-g_0 + (v^{-1} - g_2)^2}, \] \[ T_2 = \frac{v^{-1} - g_2}{-g_0 + (v^{-1} - g_2)^2}, \] (34)
where $g_{0,2}$ depend on $T_i$ through (32). There are two limiting cases in which the solution of these equations is straightforward: In the \textit{Born limit}, $v^{-1} - g_2 \gg |g_0|$, \[ T_0 \xrightarrow{\text{Born}} v^2g_0, \quad T_2 \xrightarrow{\text{Born}} v, \]
i.e. $T_0$ is given by the familiar first order Born scattering rate while $T_2$ is the averaged potential background. (Here we have neglected the chemical potential and the real part of the self energy for simplicity.) In the opposite, \textit{unitary limit}, $v^{-1} - g_2 \rightarrow 0$,
\[ T_0 \xrightarrow{\text{unitary}} -g_0^{-1}, \quad T_2 \xrightarrow{\text{unitary}} 0. \]

With these results in place, we turn back to the analysis of $S_\epsilon$. Expressing the trace over the average Green function through (32) and (33) one obtains
\[ S_\epsilon = \frac{\epsilon}{2} \int d^2r \int d^2k |\hat{G}(1) + T^{-1}(r)\hat{G}(2)|^2 = -\frac{i\pi\epsilon v_0}{4} \int d^2r \str(T + T^{-1}), \]
where
\[ v_0 = \frac{1}{\pi} \Im \str(g) = \frac{8|\Sigma_0|}{\pi^2\Lambda} \ln \left( \frac{\Lambda}{|\Sigma_0|} \right) \] (36)
is the SSTA density of states at zero energy.

We next consider the action $S_H$ due to spatial fluctuations of the fields $T$. Turning back to (28) and temporarily setting $\epsilon = 0$ we use the cyclic invariance of the trace, to re-write the action as
\[ S_{H=0}[\bar{T}] = \frac{1}{2} \left\langle \str \ln \left( -\hat{H}_{12}^0 - \hat{H}_{21}^0 \right) \right\rangle_p. \]
(Notice that the impurity potential, diagonal in real space, commutes with the fields.) Due to $[\hat{H}_{12}^0, T] = -i\nu_{12}(k) \cdot \nabla T + O(\partial^2 T)$, where $\nu_{12}(k) \equiv \nabla_k H_{12}$, the commutator between $H_0$ and $T$ contains derivatives of the slow fields and, therefore, counts as a small object. Expansion of the action to second order (The first order contribution vanishes by symmetry.) in the commutator leads to
\[ S_H = \frac{1}{4} \left\langle \str \left[ \bar{G}_{12}(T^{-1}\bar{v}_{12} \cdot \nabla T) \bar{G}_{21}(T^{-1}\bar{v}_{12} \cdot \nabla T) \right] \right\rangle_p. \]

In the limit of negligibly small real space correlation of the potential (assumed throughout), $v(k) \equiv v_0 = \text{const.}$, vertex corrections to $S_H$ vanish, i.e.
\[ S_H = \frac{1}{4} \left\langle \str \left[ \bar{G}_{12}(T^{-1}\bar{v}_{12} \cdot \nabla T) \bar{G}_{21}(T^{-1}\bar{v}_{12} \cdot \nabla T) \right] \right\rangle_p. \]

At this stage, the further evaluation of $S_H$ seems straightforward: One should substitute (31), adopt the linear ('nodal') approximation previously used in (33), and do the momentum integral. However, there is a subtle problem with this strategy. Straightforward substitution of the linearized dispersion leads to expression of the type $\cdot \cdot \cdot 0 \cdot \cdot \cdot$, where '0' is due to the $k$-space symmetry of the integral and '0' due to ultraviolet divergences.

There are different ways to get around this problem. Abandoning the nodal approximation one may attempt to do the full momentum integral over the Brillouin zone. This integral is UV benign (We are working on a lattice!) but cumbersome. Alternatively, one may stick to the nodal approximation and implement one of several UV regularization schemes available for fermions with linear dispersion. Choosing the second route, and following a regularization procedure introduced in Ref. [3], we obtain
$S_\Pi[T] = -\frac{D\nu_0 \pi}{8} \int d^2 r \str (\partial T \partial T^{-1}), \quad (37)$

where the – disorder independent – coupling constant is related to the parameters $t$ and $\Delta$ through (4). Combination of Eqs. (36) and (38) finally yields the effective low energy action (3) discussed in the introduction.

At this stage, it is instructive to make explicit contact with the perturbative analysis of YAGHK. To this end, we expand the fields as $T = \exp(W) = I + W + W^2/2 + \ldots$, where $W \in \mathfrak{osp}(2|2)$ are group generators of OSp(2|2). Expansion of the action to second order in $W$ then obtains

$S[W] = \frac{1}{2\pi} \int d^2 r \str \left( W (-D\partial^2 - 2i\kappa) W \right) + \ldots, \quad (38)$

where the kernel $\Pi^{-1} \equiv -D\partial^2 + 2i\kappa$ is the inverse of the SCTA particle-hole diffusion mode depicted in Fig. 2 b) and derived diagrammatically in YAGHK. (That the quadratic action correctly describes diffusion on the background of the isolated impurity centers is evidence for the validity of the statistical assumptions underlying the derivation of the action.) To compute the first order quantum interference corrections to the DoS, one expands the functional expectation value $\hat{\Pi}$ to second order in $W$ and integrates. This produces the logarithmically divergent results of YAGHK. For a list of more sophisticated schemes of evaluating the field theory, we refer back to the discussion of $[\square]$.

**IV. THE HALF-FILLED BAND**

Building on the analysis of the previous section, we next consider the unitary limit (by our abuse of language, the combined limit half filling/infinitely strong scattering.) To understand what will change, let us go back to the level of the prototypical field theory $[\square]$ and consider the bi-linear form

$\hat{\Psi} \hat{H}_0 \hat{\Psi}$.

As discussed in section $[\square]$, the clean Hamiltonian $\hat{H}_0$ possesses the symmetries $C, T, O$ and the sublattice symmetry $N$. As we are aiming to describe a situation where not only $C$ and $T$, but also $N$ is left intact, let us explore how this symmetry will change the internal structure of the field theory.

As discussed in section $[\square]$, the symmetry $N$ is linked to the existence of two nested sublattices $A$ and $B$. A superconductor Hamiltonian describing hopping between these lattices can be symbolically represented as

$\hat{H}_0 = \hat{h}_1 \otimes \sigma_1 + \hat{h}_2 \otimes \sigma_2, \quad (39)$

where the lattice Hamiltonians $\hat{h}_i$ obey $[\hat{h}_i, \sigma^a_3] = 0$, $i = 1, 2$, and $\sigma^a_3$ is defined in $[\square]$. Likewise, the fields $\hat{\Psi}$ and $\hat{\Psi}$ carry a two-component structure in sub-lattice space. The important point now is that for a theory globally compatible with $N$, the internal symmetry is larger than that discussed in the previous section. Indeed, it it straightforward to verify that $\hat{\Psi} \hat{H}_0 \hat{\Psi}$ remains invariant under transformations

$\Psi \rightarrow \hat{T} \Psi, \quad (40)$

where the matrix structure in sublattice and particle-hole space is defined through $(A_p, A_h, B_p, B_h)$ and $T_{1,2}$ are two independent matrices drawn from $\text{GL}(2|2) \supset \text{OSp}(2|2)$. In other words, the symmetry of the theory has grown from OSp(2|2) × OSp(2|2) to the larger manifold GL(2|2) × GL(2|2).

Having understood that, we proceed to explore (i) the structure of the fields entering the functional integral, (ii) the form of the low energy action, (iii) the role of finite chemical potentials $\mu$, and, most importantly, (iv) why the symmetry $N$ survives generalization to unitary disorder.

Temporarily ignoring the Poisson disorder, we notice that, as before, the diagonal configuration $\kappa_\sigma \sigma^c_3 \hat{T}$ solves the saddle point equation. Transformation by a smoothly fluctuating field of the structure $[\square]$, then leads to

\[ i\kappa T \sigma^c_3 \hat{T} \equiv i\kappa \begin{bmatrix} T & T^s & T^{s-1} \\ \end{bmatrix} \]

which identifies $T \equiv T_1 \sigma^c_3 T_2^s \in \text{GL}(2|2)$ as the soft field of the theory and answers (i).

Turning to (ii), we note that with the results of the previous section in place, the low energy structure of the action can be fixed by symmetry considerations, plus a few algebraic manipulations. Rather than explicitly expanding in slow fluctuations of the field $T$, we use that the low-energy action $S[T]$ we are looking for must collapse to $[\square]$ if the symmetry $N$ is broken. This condition alone implies that the action of the system is given by $[\square]$. Indeed, for fields drawn from the subset OSp(2|2) ⊂ GL(2|2) (i.e. the transformation group of the non-$N$-symmetric model) $S[T]$ reduces to the action derived in the previous section. Further, $S[T]$ contains all two-derivative operators compatible with the gross symmetries of the system. The only coupling constant that cannot be fixed by simple symmetry reasoning is $c$. However, for the purposes of the present analysis, the bare value of this constant is inessential.

We next ask, (iii), how finite values of the chemical potential $\mu$ affect the picture. For finite $\mu$, and still ignoring
the presence of the Poissonian disorder, the intermediate action assumes the form (cf. the fourth line in Eq. (23))

\[
\text{str} \ln \left( i \kappa T \sigma_3 \tilde{T} + \mu \sigma_1 - \tilde{H}_0 \right).
\]

One verifies that the isolated Pauli matrix \( \sigma_1 \) is incompatible with the full set of transformations encapsulated in \( \tilde{T} \); using the language of (12), only transformations with \( T^a = T^{-1} \rightarrow T \in \text{OSp}(2|2) \) commute with \( \sigma_1 \). To quantitatively describe the symmetry crossover, one expands the action to second order in the symmetry breaking parameter \( \mu \). (The first order term vanishes after integration over the eigenvalues of \( \tilde{H}_0 \).) This obtains the symmetry breaking operator (14), with coupling

\[
\Gamma \equiv \Gamma_1 \sim \frac{\mu^2}{t \Delta}.
\]  

We next employ the \( T \)-matrix equations (44) and (45) to explore the invariance properties of the theory under transformations \( T \). Due to \( T \tilde{H}_0 T^{-1} = \tilde{H}_0 \), the non-disordered Green function \( \tilde{G}_0[T] \) transforms as \( \tilde{G}_0[T] = \tilde{T}^{-1} \tilde{G}_0[1] \tilde{T}^{-1} \). Had the disordered Green function the same property, the theory would be invariant under the full set of transformations \( T \). From Eq. (44) we see that invariance of the Green function is ensured if \( \Sigma_i[T] = T \Sigma_i[1] T \). But if the Green function was invariant, we would have

\[
\Sigma_i[T] = \left[ v_i^{-1} \sigma_2 - \tilde{T}^{-1} \tilde{G}[1] u_i T^{-1} \right]^{-1}
\]

\[
= T \left[ v_i^{-1} \tilde{T} \sigma_2 T - \tilde{G}[1] u_i \right]^{-1} T.
\]

The last line shows that for generic values of the impurity strength, the self energy operator does not have the required invariance properties. Only in the limit of an infinitely strong potential \( v_i \rightarrow \infty \), the offending term \( v_i^{-1} \sigma_2 \sigma_2 T \) drops out and the action is invariant under the full set of transformations \( T \). This is the unitary limit where the system is described by the soft action (11). (Notice that for transformations drawn from the subgroup OSp(2|2) (i.e. the symmetry group of the non-N-symmetric superconductor) \( v_i^{-1} \tilde{T} \sigma_2 T = v_i^{-1} \sigma_2 \), i.e. invariance under the reduced set of transformations holds regardless of the impurity strength.)

For scattering close to the unitary limit, \( v \gg 1 \), an expansion of the action to second order in \( v^{-1} \) obtains, again, the operator (14), this time with coupling

\[
\Gamma \equiv \Gamma_2 \sim \frac{n_i}{(v_0 v)^2}.
\]  

The operator (14), with coupling \( \Gamma = \Gamma_1 + \Gamma_2 \), describes the breaking of AI III symmetry by finite chemical potentials and/or deviations off unitarity. An expansion of (14) to second order in the field generators \( W \) (cf. Eq. (38)) shows that for finite \( (\mu, v^{-1}) \) the soft mode (the ‘dimension’) induced by the sublattice symmetry contains two mass terms, derived and discussed before by YAGKH. Whether or not traces of the AI III-type soft modes remain visible in the long range behaviour of transport and spectral observables depends on the relative value \( \Gamma/(v_0 \kappa) \). If this parameter is small, the mass of the superconductor soft modes is mainly set by deviations off zero energy and the (relatively small) additional breaking of the sublattice symmetry is inessential. In the opposite case, the sublattice symmetry is strongly broken and the system effectively in class CI.

V. SUMMARY

This concludes our field-theoretical analysis of \( d \)-wave superconductors with pointlike scatterers. We have
shown that in all but one point of a phase plane defined through impurity strength and filling factor, respectively, the system behaves similar to d-wave superconductors with generic, non-pointlike disorder. The DoS vanishes linearly upon approaching zero energy and eigenstates are localized (i.e. the system is a thermal insulator.)

There exists one point in the phase diagram – infinitely strong disorder and half filling – where phenomenology of strikingly different type is realized: The DoS diverges as

$$\nu(\epsilon) \sim |\epsilon|^{-1} f(\ln|\epsilon|),$$

where \(f(\ln|\epsilon|)\) stands for logarithmic corrections to power law scaling [4], and zero-energy eigenstates are localized. Apart from deviations in the function \(f\) – whose origin remains obscure – the spectral profile found in the present analysis agrees with that of Ref. [4]. The identification of a metallic phase is a new result.

We do not agree with the statement [4] that the peculiar behaviour realized at the critical point should be by and large insensitive to detuning of the parameters chemical potential/impurity strength. The question whether traces of critical behaviour – e.g. regions where the DoS increases upon lowering the energy – might be observable under ‘realistic conditions’ is not for the present approach to decide. However, given that the very formation of a superconducting phase requires a finite concentration of dopands (i.e. deviations off half-filling), and that residual quasiparticle interactions spoil the sublattice symmetry, it seems likely that the ‘real’ superconductor will display behaviour characteristic for the generic symmetry class.

Note added: Shortly before completion of this manuscript Ref. [25] appeared. The authors of that reference discuss structure and renormalization of superconductor \(\sigma\)-models with sublattice symmetry (including the case of broken time reversal invariance.) Where the same symmetry classes are considered, the results of Ref. [25] and the present paper agree.

It is a pleasure to acknowledge discussions with C. Mudry, B. D. Simons, and M. Zirnbauer.

### APPENDIX A: BLOCK REPRESENTATION OF THE SUPERCONDUCTOR WITH NESTING SYMMETRY

We show that the Hamiltonian \(\hat{H}\) of the superconductor with nesting symmetry can be transformed to a block off-diagonal representation. Due to the symmetry (10), \(\hat{H}\) can be represented through (12), where the matrix structure is in sublattice space. The superconductor symmetries (1) further imply that the block-matrices \(Z\) obey

\[
Z = Z^*, \quad \sigma_y Z \sigma_y = -Z,
\]

i.e. \(\hat{H}\) has the form

\[
\hat{H} = \begin{pmatrix}
    h_2 & h_1 \\
    h^*_1 & -h^*_2
\end{pmatrix},
\]

where the sub-structure of the blocks is in ph-space and the operators \(h_1, h_2\) are real. A unitary transformation in ph-space brings \(\hat{H}\) into the form

\[
\hat{H} = \begin{pmatrix}
    \tilde{Z} & \tilde{Z}^* \\
    \tilde{Z}^T & \tilde{Z}^T
\end{pmatrix},
\]

where we have defined the complex operator \(\tilde{Z} = h_1 - ih_2\). Finally, changing the order of blocks, \(\hat{H}\) can be rewritten as

\[
\hat{H} = \begin{pmatrix}
    \tilde{Z} & \tilde{Z}^* \\
    \tilde{Z}^T & \tilde{Z}^T
\end{pmatrix}.
\]

(Apart from an inessential transposition in the lower left hand block,) these are two identical copies of a class AIII-Hamiltonian. The replicated appearance of two AIII-Hamiltonians implies that our field theory takes values in the doubled field manifold GL(2|2) (instead of GL(1|1) for just one Hamiltonian.)
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[22] Of course, the structure of the model must be insensitive to the way by which its low energy degrees of freedom are introduced. E.g., avoiding the introduction of Gaussian disorder, one might equally well construct the theory by averaging over a window of chemical potentials. Technically, however, it is more convenient to implement the disorder-route.
[23] In the present approach \( \hat{G} \) merely determines the short-scale fluctuation behaviour of the fields \( T \). I.e. in spite of the perturbative treatment of \( \hat{G} \), the essential information stored in the model is left intact.
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