TYPICAL STRUCTURE OF SPARSE EXPONENTIAL RANDOM GRAPH MODELS

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Abstract. We consider general Exponential Random Graph Models (ERGMs) where the sufficient statistics are functions of homomorphism counts for a fixed collection of simple graphs $F_k$. Whereas previous work has shown a degeneracy phenomenon in dense ERGMs, we show this can be cured by raising the sufficient statistics to a fractional power. We rigorously establish the naive mean-field approximation for the partition function of the corresponding Gibbs measures, and in case of “ferromagnetic” models with vanishing edge density show that typical samples resemble a typical Erdős–Rényi graph with a planted clique and/or a planted complete bipartite graph of appropriate sizes. We establish such behavior also for the conditional structure of the Erdős–Rényi graph in the large deviations regime for excess $F_k$-homomorphism counts. These structural results are obtained by combining quantitative large deviation principles, established in previous works, with a novel stability form of a result of [5] on the asymptotic solution for the associated entropic variational problem. A technical ingredient of independent interest is a stability form of Finner’s generalized Hölder inequality.

1. Introduction

With $[n] := \{1, \ldots, n\}$, let $\mathcal{G}_n$ be the set of symmetric $\{0, 1\}$-valued functions on $[n]^2$ with zero diagonal, i.e.

$$G : [n]^2 \to \{0, 1\}, \ (i,j) \mapsto G_{i,j}, \ \text{with} \ G_{i,j} = G_{j,i}, \ G_{i,i} = 0 \ \forall i,j \in [n].$$

Elements of $\mathcal{G}_n$ are naturally identified with simple graphs (undirected and loop-free with at most one edge between any pair of vertices) over the labeled vertex set $[n]$, with $G_{i,j} = 1$ when $i,j$ are joined by an edge. We are concerned with the case that $n$ is large or tending to infinity.

An exponential random graph model (ERGM) is a probability measure on $\mathcal{G}_n$ with density of the form

$$\frac{1}{Z_n(\alpha, \beta)} \exp \left( n^2 H(G; \beta) - \alpha e(G) \right), \ G \in \mathcal{G}_n$$

for parameters $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^m$, where $Z_n(\alpha, \beta)$ is the normalizing constant, or partition function, $e(G) := \sum_{i \leq j} G_{i,j}$ is the total number of edges in $G$, and the Hamiltonian $H(G; \beta)$ is of the form

$$H(G; \beta) = \sum_{k=1}^m \beta_k f_k(G)$$

for a fixed collection of functions $f_k : \mathcal{G}_n \to \mathbb{R}$. The $f_k$ are typically taken as graph functionals that can be estimated through sampling, such as the frequency of a particular subgraph.

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The parameter $\alpha$ controls the sparsity of typical samples from the ERGM, with large positive/negative values of $\alpha$ leading to sparse/dense graphs, respectively; in physical terms $\alpha$ plays the role of the strength of an external field. (We could have included $\alpha e(G)$ as one of the terms in the Hamiltonian (1.3), but it will be convenient to keep this term separate.)

ERGMs generalize the Erdős–Rényi (or binomial) random graph. Indeed, with

$$\alpha = \log \frac{1-p}{p}$$

(1.4)

for $p \in (0,1)$ and the Hamiltonian set to zero, the partition function is $Z_\alpha = (1-p)^{-\binom{n}{2}}$, and the density (1.2) is then $p^{\alpha(G)}(1-p)^{\binom{n}{2}-\alpha(G)}$. In general, under the parametrization (1.4) one can view the ERGM (1.2) as the tilt of the Erdős–Rényi($p$) distribution by the function $\exp(n^2H(\cdot;\beta))$. Indeed, this is the perspective we will take later in the article – see (1.12) – as it will be convenient for importing results on the large deviation theory of Erdős–Rényi graphs.

We further note that our results cover a more general class of distributions than in (1.2)–(1.3), and that unlike in (1.2), in (1.12) we scale the Hamiltonian by a factor $r_{n,p}$ different from $n^2$, which will be important for treating sparse graphs (for which $p = o(1)$). However, for the present discussion we focus on distributions of the form (1.2).

Apart from the Erdős–Rényi model, perhaps the best-known non-trivial ERGM is the edge-triangle model, with Hamiltonian

$$H(G;\beta) = \beta t(K_3,G) = \beta \frac{1}{n^3} \sum_{i_1,i_2,i_3=1}^n G_{i_1,i_2}G_{i_2,i_3}G_{i_3,i_1}.$$

(1.5)

With the normalization by $n^3$, $t(K_3,G)$ is the probability that three independently and uniformly sampled vertices of $G$ form a triangle. Thus, typical samples should have more (resp. fewer) triangles than an Erdős–Rényi graph when $\beta$ is taken to be positive (resp. negative).

1.1. Previous works. ERGMs were introduced and developed in the statistics and social sciences literature in the 80s and 90s [28, 33, 48]; see [23, 24] for a survey of the subsequent vast literature. The motivation was to develop a parametric class of distributions on graphs that could be fit to social networks via parameter estimation. A key feature of social networks is transitivity – that friends of friends are more likely to be friends – a feature that is not present in typical Erdős–Rényi graphs. In particular, it was hoped that transitivity would arise by reweighting the Erdős–Rényi distribution to promote triangles, as in the edge-triangle model (1.5) with $\beta > 0$.

The general form of the ERGMs (1.2)–(1.3) is appealing as they are exponential families, and the separable form of the Hamiltonian means that the functions $f_k(G)$ and edge density $n^{-2}e(G)$ are sufficient statistics for the model parameters $\beta_k$ and $\alpha$, respectively. For Bayesian inference and maximum likelihood estimation of the parameters it is important to have an accurate approximation for the partition function $Z_n(\alpha,\beta)$, which has often been obtained via Markov chain Monte Carlo sampling schemes. Sampling is also used to understand the typical structure of ERGMs.

Problems with various aspects of this program were noted empirically from early on, ranging from the inability of ERGMs to fit realistic social networks, to the inability of sampling algorithms to converge in a reasonable time [31, 44, 47]. With regards to the former, it was observed that in large regimes of the parameter space, typical samples exhibit no transitivity. Moreover, in some regimes ERGMs seem to concentrate in neighborhoods of a small number
of graphs with trivial structure — such as the empty graph and the complete graph — a phenomenon known as degeneracy. We refer to [45] for discussion of these and other problems, as well as some proposals to circumvent them.

More recent mathematically rigorous works have helped to clarify these issues. An important work of Bhamidi, Bressler and Sly [4] considered the case that the functions \( f_k \) in (1.3) are densities \( t(F_k, G) \) of a fixed collection of graphs \( F_k, k = 1, \ldots, m \), that is

\[
H(G; \beta) = \sum_{k=1}^{m} \beta_k t(F_k, G) \tag{1.6}
\]

generalizing the edge-triangle model (1.5) (see (1.8) below for the general definition of \( t(F, G) \)). For the case that the model is in the “ferromagnetic” parameter regime with all \( \beta_k > 0 \), they are able to characterize a “low-temperature” parameter regime where local MCMC sampling schemes take exponential time to converge; in the complementary high-temperature regime where sampling algorithms have polynomial convergence time, typical samples exhibit the structure of an Erdős–Rényi graph, in particular lacking the transitivity property.

Another major development on models of the form (1.6) was made in work of Chatterjee and Diaconis [10], where they applied the large deviation theory of [11] for the Erdős–Rényi graph to rigorously establish a variational approach to estimating the partition function known as the naïve mean-field (NMF) approximation from statistical physics. They also show that in the ferromagnetic regime, ERGMs are close to a mixture of Erdős–Rényi graphs — that is, with the parameter \( p \) sampled from some distribution. For the case of the edge-triangle model (1.5), they rigorously establish a degeneracy phenomenon wherein for large positive values of \( \alpha \), as \( \beta \) increases through a critical threshold \( \beta_\alpha(\alpha) \geq 0 \) the expected edge density jumps from nearly zero to nearly one; see [10, Theorem 5.1].

The works [4,10] focus on ERGMs with Hamiltonians having the special form (1.6), and with parameters \( \alpha, \beta_k \) fixed independent of \( n \), which means that samples are typically dense, i.e. with constant edge density. One further work on dense ERGMs of particular relevance to the present work is that of Lubetzky and Zhao [38], who had the insight to consider a modified edge-\( F \) model, with Hamiltonian of the form

\[
H(G; \beta) = \beta t(F, G)^\gamma \tag{1.7}
\]

for a fractional power \( \gamma \in (0, 1) \). In particular, they show that for \( \Delta \)-regular \( F \) and \( \alpha \) above a certain threshold depending on \( \Delta \), taking \( \gamma \in (0, \frac{\Delta}{\alpha(F)}) \) cures the degeneracy phenomenon established in [10], in the sense that there exists an open interval of values of \( \beta \) for which a typical sample from the ERGM does not look like an Erdős–Rényi graph with high probability. However, they left open the problem of determining what a typical sample does look like. The structure of such models in the limit as \( (\alpha, \beta) \) tend to infinity along rays was studied in [18], extending the work of [49] for the case \( \gamma = 1 \).

Starting with the work [9] of Chatterjee and the second author, the recent development of quantitative approaches to large deviations for nonlinear functions on product spaces — such as subgraph counts in Erdős–Rényi graphs — has opened up the analysis of ERGMs in the sparse regime where \( \alpha \) and \( \beta \) depend on \( n \). We refer to [8] for an introduction to this recent and rapidly developing area. In particular, the works [9,19] established the NMF approximation for the partition function under some growth conditions on the parameters. In [21], building on the nonlinear large deviations theory from [19,20], Eldan and Gross showed that under some growth conditions on \( \alpha, \beta \), models of the form (1.6) are close to low-complexity mixtures of stochastic block models, with barycenters close to critical points of the NMF free energy. We also mention that the correlation between fixed edges in sparse ERGMs with negative
were studied in [50]. Mixing properties of the Glauber dynamics were used to establish concentration inequalities and CLTs in [29].

In the language of statistical physics, ERGMs are grand canonical ensembles, and we mention there has been a long line of works on the structure of the corresponding microcanonical ensembles, which are graphs drawn uniformly under hard constraints on subgraph counts; see [34, 35, 41–43].

1.2. Generalized ERGMs. In the present work we apply results from [15, 16] on a quantitative large deviations theory for the Erdős–Rényi graph to extend the NMF approximation to sparser ERGMs than in previous works, and to establish the typical structure of samples. Our setup is more general than the separable form of (1.6).

For sparse models we need to introduce some scalings in the model (1.2)–(1.3). Generalizing (1.1), for a set $S$, an $S$-weighted graph over $[n]$ is a symmetric $S$-valued function on $[n]^2$ with zero diagonal. Thus, elements of $\mathcal{G}_n$ are $\{0,1\}$-weighted graphs. For an $\mathbb{R}$-weighted graph $X$ over $[n]$ and a fixed graph $F = (V(F),E(F))$, we define the homomorphism density of $F$ in $X$ by

$$t(F,X) := \frac{1}{n^{\nu(F)}} \sum_{\phi : V(F) \to [n]} \prod_{(u,v) \in E(F)} X_{\phi(u),\phi(v)}.$$  \hspace{1cm} (1.8)

For the case that $X$ is a $\{0,1\}$-weighted graph, $t(F,X)$ is the probability that a uniform random mapping of the vertices of $F$ into $[n]$ is a graph homomorphism from $F$ to the graph associated with $X$—that is, maps edges onto edges.

Fixing integers $\Delta \geq 2, m \geq 1$, a sequence $\bar{F} = (F_1, \ldots, F_m)$ of graphs of max-degree $\Delta$, and a function $h : \mathbb{R}^m \to \mathbb{R}$, we define a Hamiltonian function on the space of $\mathbb{R}$-weighted graphs by

$$H(X) := h\left(t(F_1,X), \ldots, t(F_m,X)\right).$$  \hspace{1cm} (1.9)

We make the following assumptions on $h$.

**Assumption 1.1.** The function $h : \mathbb{R}^m \to \mathbb{R}$ in (1.9) is continuous, coordinate-wise nondecreasing, and satisfies the growth condition

$$h(\bar{x}) = o_{\|\bar{x}\| \to \infty} \left( \sum_{k=1}^{m} x_k^{\Delta/e(F_k)} \right).$$  \hspace{1cm} (1.10)

Letting $p \in (0,1)$ (possibly depending on $n$), the large deviation rate parameter is defined

$$r = r_{n,p} := n^2 p^\Delta \log(1/p).$$  \hspace{1cm} (1.11)

Now with $G = G_{n,p} \in \mathcal{G}_n$ denoting an Erdős–Rényi($p$) graph, we define a measure $\nu^H_{n,p}$ on $\mathcal{G}_n$ with density

$$\nu^H_{n,p}(G) = \exp(r_{n,p} H(G/p) - \Lambda^H_{n,p}), \hspace{0.5cm} G \in \mathcal{G}_n$$  \hspace{1cm} (1.12)

where we denote the log-moment generating function

$$\Lambda^H_{n,p} := \log \mathbb{E} \exp(r_{n,p} H(G_{n,p}/p)).$$  \hspace{1cm} (1.13)

A sample from $\nu^H_{n,p}$ is denoted by $G^H_{n,p}$ (thus $G_{n,p} \overset{d}{=} G^0_{n,p}$).

To connect our notation with the the normalizing constant $Z_n$ for ERGMs discussed in Section 1.1, with $\alpha = \log \frac{1}{p} \mathbb{E}$ we can alternatively express the density (1.12) in the form

$$\frac{1}{Z} \exp \left( r_{n,p} H(G/p) - \alpha \mathbb{E}(G) \right)$$  \hspace{1cm} (1.14)
(as in (1.2) but with $r_{n,p}$ in place of $n^2$ and $G$ scaled by $1/p$). Then the normalizing factor \( \Lambda_{n,p}^H \) in (1.12) is related to the free energy $\log Z$ by

$$
\Lambda_{n,p}^H = \log Z + \left( \frac{n}{2} \right) \log (1 - p).
$$

(1.15)

We note that (under mild conditions on $h$) $\Lambda_{n,p}^H$ is of order $r_{n,p}$, which is of lower order than the second term on the right hand side above when $p \ll 1$. Hence, in the sparse case, our results on $\Lambda_{n,p}^H$ below provide asymptotics for the nontrivial sub-leading order of the free energy.

As an example, under our scalings the edge-triangle model (with some choice of $h$) has density proportional to

$$
\exp \left( r_{n,p} \cdot h(t(K_3, G/p)) - \log \left( \frac{1 - p}{p} \right) e(G) \right)
$$

(1.16)

$$
= \exp \left( n^2 p^2 \log (1/p) \cdot h \left( \frac{1}{n^3 p^3} \sum_{i, i_2, i_3 = 1}^n G_{i, i_2} G_{i_2, i_3} G_{i_3, i_1} \right) - \log \left( \frac{1 - p}{p} \right) \sum_{i<j} G_{i,j} \right).
$$

In Corollary 1.12 we determine the typical structure of samples from this model when $h = \beta f$ for a parameter $\beta > 0$ and a fixed function $f$. Following the insight of [38] (see (1.7)) it will be crucial to impose the growth condition (1.10), which translates to $x^{-2/3} f(x) \to 0$ as $x \to \infty$.

1.3. The NMF approximation. Under the definition (1.12) we have that $\nu_{n,p}^0$ is the Erdős–Rényi($p$) measure on $\mathcal{G}_n$. The Gibbs (or Donsker–Varadhan) variational principle states that

$$
\Lambda_{n,p}^H = \sup_{\mu} \left\{ r_{n,p} \mathbb{E}_{G \sim \mu} H(G/p) - D(\mu||\nu_{n,p}^0) \right\}
$$

(1.17)

where the supremum is taken over all probability measures on $\mathcal{G}_n$, and $D(\cdot||\cdot)$ is the relative entropy. While this yields a formula for the normalizing constant of ERGMs via (1.15), it is not a computationally feasible alternative to sampling since $\mu$ ranges over a set of dimension exponential in $n^2$.

The NMF approximation posits that the supremum in (1.17) is approximately attained on the subset of product probability measures on $\mathcal{G}_n$, which are parametrized by the $\binom{n}{2}$-dimensional cube of $[0, 1]$-weighted graphs

$$
Q_n := \{ Q : |n|^2 \to [0, 1] : Q_{i,j} = Q_{j,i}, Q_{i,j} = 0 \forall i, j \in [n]\}.
$$

Indeed, product probability measures on $\mathcal{G}_n$ are of the form $\mu_Q$, the distribution of an inhomogeneous Erdős–Rényi graph that independently includes edges $\{i,j\}$ with probability $Q_{i,j}$.

In our setting the NMF approximation takes the form

$$
\Lambda_{n,p}^H \approx \sup_{Q \in Q_n} \left\{ r_{n,p} H(Q/p) - \sum_{i<j} I_p(Q_{i,j}) \right\} =: \Psi_{n,p}^H
$$

(1.18)

where we use the common notation

$$
I_p(q) := D(\text{Ber}(q)||\text{Ber}(p)) = q \log \frac{q}{p} + (1 - q) \log \frac{1 - q}{1 - p}
$$

(1.19)

for the relative entropy of the Bernoulli($q$) law on $\{0, 1\}$ with respect to the Bernoulli($p$) law.

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Note we have made the additional step of taking the expectation $\mathbb{E}_{G \sim \mu_{\Psi}^H} H(G/p)$ under $H$; the difference turns out to be negligible in our setting. We follow [5, 9] by taking $\Psi_{n,p}^H$ as the definition of the NMF approximation.
While (1.18) is not always true for Gibbs measures (as is notably the case for spin glass models), our first result shows it is a good approximation in our setting of generalized ERGMs, under some conditions on \( p \) and \( h \). See Section 1.8 for our conventions on asymptotic notation.

**Proposition 1.2.** Assume \( n^{-1/(\Delta+1)} \ll p \leq 1 \) and that \( h \) satisfies Assumption 1.1. Then
\[
\Lambda_{n,p}^H = \Psi_{n,p}^H + o(r_{n,p}).
\]

**Remark 1.3.** Proposition 1.2 extends to \( n^{-1/\Delta} \ll p \leq 1 \), when every vertex of degree \( \Delta \) in \( F_k \), \( k \in [m] \), is in an isolated star. Using results from [15] we could also allow \( p \) as small as \((\log n)^C n^{-1/\Delta}\) for a sufficiently large constant \( C \) if every \( F_k \) is a cycle (so \( \Delta = 2 \) in this case). From [16] we also have a matching LDP lower bound in Proposition 3.3, and by relying on the latter result one can eliminate the restriction to non-decreasing \( h(\cdot) \) in Proposition 1.2. We skip the proofs of such refinements here.

In the case that \( p = o(1) \), our next result says that (1.18) can be further reduced to a two-dimensional variational problem. It involves a certain function \( T_F : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0} \) associated to a graph \( F \) that was identified in the work [5] on the upper tail problem for \( t(F,G_{n,p}) \). Recall the independence polynomial of a graph \( F \) is defined as
\[
P_F(x) = 1 + \sum_{\emptyset \neq U \in \mathcal{I}(F)} x^{|U|}
\] where the sum runs over the non-empty independent sets in \( \mathcal{V}(F) \). Letting \( F^* \) denote the induced subgraph of \( F \) on the vertices of maximal degree, we set
\[
T_F(a,b) := P_{F^*}(b) + a^{v(F)/2} 1(F \text{ is regular}).
\]
Define
\[
\psi_{F,h} := \sup_{a,b \geq 0} \left\{ h(T_{F_1}(a,b), \ldots, T_{F_n}(a,b)) - \frac{1}{2} a - b \right\}.
\]

**Theorem 1.4.** Assume \( n^{-1/(\Delta+1)} \ll p \ll 1 \) and that \( h \) satisfies Assumption 1.1. Then
\[
\lim_{n \to \infty} \frac{1}{r_{n,p}} \Lambda_{n,p}^H = \psi_{F,h}.
\]

### 1.4. Typical structure for sparse ERGMs and conditioned Erdős–Rényi graphs.
Proposition 1.2 and Theorem 1.4 follow in a relatively straightforward way from recent results on joint upper tails for homomorphism densities in Erdős–Rényi graphs [5, 16] via Varadhan lemma-type arguments. Now we state our main results, which determine the typical structure of samples – both from the generalized ERGMs (1.12), as well as the Erdős–Rényi graph conditioned on a joint upper-tail event for homomorphism densities. In both cases, samples concentrate around a two-parameter family of weighted graphs identified in [5]. This is the reason for the two-dimensional reduction of Theorem 1.4 – indeed, the functions \( T_F(a,b) \) (appropriately rescaled) from (1.21) give the behavior of the homomorphism density functionals \( t(F,\cdot/p) \) on this two-parameter family.

For \( \xi > 0 \) and \( I, J \subset [n] \), let \( \mathcal{G}^{I,J}_n(\xi) \) be the set of \( G \in \mathcal{G}_n \) satisfying
\[
\sum_{i,j \in I} G_{i,j} \geq |I|^2 - 2\xi n^2 p^\Delta \quad \text{and} \quad \sum_{i \in I, j \in J^c} G_{i,j} \geq |J|(n - |J|) - \xi n^2 p^\Delta,
\]
and for \( a, b \geq 0 \), set
\[
\mathcal{G}^1_n(a,b,\xi) := \bigcup_{I,J \subset [n] \text{ disjoint}} \mathcal{G}^{I,J}_n(\xi).
\]
Thus, elements of $G_n^1(a, b, \xi)$ correspond to graphs containing an almost-clique over the vertex set $I$ and an almost-biclique across the vertex bipartition $(J, J^c)$, with $I, J$ of appropriate size. Note that for $n^{-\Delta} \ll p \ll 1$ and $a, b > 0$ fixed independent of $n$ we have $1 \ll |J| \ll |I| \ll n$. (In particular it is only a matter of convenience to insist that $I, J$ be disjoint, as the number of edges in $I \times J$ is $o(p^2n^2)$ and hence negligible.)

We also define another neighborhood of the graphs with an almost-clique at $(I, J^c)$. Given $I, J \subset [n]$ disjoint, define the associated weighted clique-hub graph $Q^{I,J} \in \mathbb{Q}_n$, with

$$Q_{i,j}^{I,J} = p + (1 - p)\left[ 1_{i,j \in I} + 1_{(i,j) \in J \times J^c} \right].$$

(We informally refer to $J$ as a “hub” as it is possible to move between any two vertices $i, j$ using two edges of weight 1 by passing through a vertex of $J$.) For $a, b \geq 0$ let

$$Q_n(a, b) = \{ Q^{I,J} : |I| = \lfloor (ap)^{1/2}n \rfloor, |J| = \lfloor bp^{1/2}n \rfloor \}$$

and for $\xi > 0$ let

$$\mathcal{Q}_n^2(a, b, \xi) := \bigcup_{Q \in Q_n(a, b)} \left\{ G \in \mathcal{G}_n : \|G - Q\|_{2\to2} < \xi np^{1/2} \right\}$$

where with slight abuse we extend the spectral operator norm for matrices to $\mathbb{R}$-weighted graphs $X$, that is

$$\|X\|_{2\to2} = \sup_{0 \neq u, v : \|u\|_{\ell^2([n])} = \|v\|_{\ell^2([n])}} \frac{\sum_{i,j=1}^n X_{i,j}u(i)v(j)}{\|u\|_{\ell^2([n])} \|v\|_{\ell^2([n])}}.$$

We note how the spectral norm enforces control on edge discrepancies: Consider any $Q^{I,J} \in Q_n(a, b)$ and $G \in \mathcal{G}_n$ with $\|G - Q\|_{2\to2} < \xi np^{1/2}$. For $A, B \subset [n]$ let

$$e_G(A, B) := \sum_{i \in A, j \in B} G_{i,j}$$

be the number of edges in $G$ with one end in $A$ and the other in $B$ (counting edges contained in $A \cap B$ twice). Then taking $u, v$ of the form $\pm 1_A, 1_B$ in (1.29), we have that for either $A, B \subset I$ or $A \subset J, B \subset J^c$,

$$0 \leq 1 - \frac{e_G(A, B)}{|A||B|} < \xi \left( \frac{n^2p^\Delta}{|A||B|} \right)^{1/2}$$

and for all $A, B \subset J^c$ such that at least one of $A, B$ lies in $I^c$,

$$\left| \frac{e_G(A, B)}{p|A||B|} - 1 \right| < \xi \left( \frac{n^2p^{\Delta-2}}{|A||B|} \right)^{1/2}.$$

Comparing with (1.24), we see that elements of $\mathcal{Q}_n^2(a, b, \xi)$ not only have an almost-clique and almost-hub of appropriate size, but also look uniformly like Erdős–Rényi graphs outside of $I, J$.

In the following we use the parameter

$$\Delta_* = \Delta_*(F) := \frac{1}{2} \max_{1 \leq k \leq m} \max_{\{u, v\} \in E(F_k)} \left\{ \deg_{F_k}(u) + \deg_{F_k}(v) \right\},$$

where $\deg_{F_k}(u) = |\{u' \in V(F) : \{u, u'\} \in E(F)\}|$ is the degree of $u$ in $F_k$. Note that $\Delta + 1 \leq 2\Delta_* \leq 2\Delta$, with the lower bound holding when each $F_k$ is a $\Delta$-armed star, and the upper bound when each $F_k$ is $\Delta$-regular. We denote the set of optimizers in (1.22) by

$$\text{Opt}(\psi) := \left\{ (a, b) \in \mathbb{R}_+^2 : h(T_{F_1}(a, b), \ldots, T_{F_m}(a, b)) - \frac{1}{2}a - b = \psi_{E, h} \right\}.$$
(Note this set depends on $F$ and $h$, but we suppress this from the notation.)

**Theorem 1.5.** Assume that the graphs $F_1, \ldots, F_m$ are connected and that $h$ satisfies Assumption 1.1. Then for any $\xi > 0$ there exists $\eta_0 > 0$ depending only on $F, h$ and $\xi$ such that the following hold:

(a) If $n^{-1/(\Delta+1)} \ll p \ll 1$, then for all $n$ sufficiently large,

$$
\mathbb{P}\left( G_{n,p}^H \in \bigcup_{(a,b) \in \text{Opt}(\psi)} G_1^1(a,b,\xi) \right) \geq 1 - \exp(-\eta_0 r_{n,p}).
$$

(b) If $n^{-1} \log n \ll p^{2\Delta^*} \ll 1$, then for all $n$ sufficiently large,

$$
\mathbb{P}\left( G_{n,p}^H \in \bigcup_{(a,b) \in \text{Opt}(\psi)} G_2^2(a,b,\xi) \right) \geq 1 - \exp(-\eta_0 r_{n,p}).
$$

We deduce Theorem 1.5 from the next result concerning the structure of an Erdős–Rényi graph $G_{n,p}$ conditioned on a joint upper tail event for homomorphism densities. For given $\bar{z} \in \mathbb{R}_{\geq 0}^m$, denote the joint superlevel set

$$
\mathcal{U}_p(F, \bar{z}) := \{ Q \in Q_n : t(F_k, Q/p) \geq 1 + s_k \ \forall \ k \in [m] \}.
$$

Analogous to (1.18), the NMF approximation for joint upper tail probabilities states that $-\log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F, \bar{z}))$ is approximately given by

$$
\Phi_{n,p}(F, \bar{z}) := \inf_{Q \in Q_n} \left\{ \sum_{i<j} I_p(Q_{i,j}) : Q \in \mathcal{U}_p(F, \bar{z}) \right\}.
$$

This and closely related asymptotics were established in several recent works under various hypotheses on $p$ and $F$ [1, 2, 9, 11, 15, 16, 19, 32]. Parallel works [5, 6, 39] have shown that the optimization problem $\Phi_{n,p}(F, \bar{z})$ over the $\binom{n}{2}$-dimensional cube $Q_n$ asymptotically reduces (after normalization by $r_{n,p}$) to the following optimization problem over the plane:

$$
\phi_{F}(\bar{z}) := \inf_{a,b \geq 0} \left\{ \frac{1}{2} a + b : T_{F_k}(a,b) \geq 1 + s_k \ \forall \ k \in [m] \right\}.
$$

An easy computation shows that the probability that $G_{n,p}$ lies in an appropriate neighborhood of $Q_n(a,b)$ (such as $G_1^1(a,b,\xi)$ or $G_2^2(a,b,\xi)$) is roughly $\exp(-\frac{1}{2}a+b)r_{n,p}$, and moreover that $t(F_k, Q/p) \sim T_{F_k}(a,b)$ for $Q \in Q_n(a,b)$. The asymptotic

$$
\log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F, \bar{z})) \sim -\phi_{F}(\bar{z}) r_{n,p}
$$

established in previous works thus suggests that the joint upper tail event roughly coincides with the event that $G_{n,p}$ lies in a neighborhood of $Q_n(a,b)$ for some $(a,b)$ attaining the infimum in (1.37). Note that such a conditional structure result does not follow from (1.38) since the tail probability is only determined to leading order in the exponent.

The following establishes such conditional structure results for joint upper tail events under decay conditions on $p$. We denote the set of optimizers in (1.37) by $\text{Opt}(\phi_{F}, \bar{z})$ (suppressing the dependence on $F$).

**Theorem 1.6.** Suppose $F_1, \ldots, F_m$ are connected. For any fixed $\xi > 0$ and $\bar{z} \in \mathbb{R}_{\geq 0}^m$ there exists $\eta_1 = \eta_1(F, \bar{z}, \xi) > 0$ such that the following hold:

(a) If $n^{-1/(\Delta+1)} \ll p \ll 1$, then for all $n$ sufficiently large,

$$
\mathbb{P}\left( G_{n,p} \in \bigcup_{(a,b) \in \text{Opt}(\phi_{F}, \bar{z})} G_1^1(a,b,\xi) \mid G_{n,p} \in \mathcal{U}_p(F, \bar{z}) \right) \geq 1 - \exp(-\eta_1 r_{n,p}).
$$

(b) Parallel works \[1, 2, 9, 11, 15, 16, 19, 32\].
(b) If further $n^{-1} \log n \ll p^{2\Delta_*} \ll 1$, then for all $n$ sufficiently large,

\[
P \left( G_{n,p} \in \bigcup_{(a,b) \in \text{Opt}(\psi)} G_n^2(a,b,\xi) \right| G_{n,p} \in \mathcal{U}_p(\mathcal{F},\mathcal{S}) \right) \geq 1 - \exp(-\eta_1 r_{n,p}) .
\]

**Remark 1.7.** We note that for $m = 1$ the work [32] established (1.39) in an essentially optimal range of $p$ for the case of a clique $F_1$. However, in view of [2] one expects to find conditionally on $\mathcal{U}_p(\mathcal{F},\mathcal{S})$ different structures for bipartite $\Delta$-regular $F_1$ and $p = o(n^{-1/\Delta})$ (and the same thus applies for typical samples from the ERGM-s corresponding to such such $F_1$ and $p$).

The following gives some information on the relation between the optimization problems $\psi_{\mathcal{F},h}, \phi_{\mathcal{F}}$ in (1.22),(1.37), and on their sets of optimizers $\text{Opt}(\psi), \text{Opt}(\phi; \mathcal{S})$. The proof is given in Appendix B.

**Proposition 1.8.**

(a) We have that $\phi_{\mathcal{F}}$ is continuous and non-decreasing in each argument, with $\phi_{\mathcal{F}}(0) = 0$ and

\[
\phi_{\mathcal{F}}(s) \geq \sum_{k=1}^{m} s^k / e(F_k) \quad \forall s \in \mathbb{R}_{\geq 0}^m : \|s\|_{\infty} \geq C(F) \quad (1.41)
\]

for a sufficiently large constant $C(F) > 0$.

(b) For any $s \in \mathbb{R}_{\geq 0}^m$, the set $\text{Opt}(\phi; s) \subset \mathbb{R}_{\geq 0}^m$ of optimizers in (1.37) is a non-empty, finite set of points on the closed line segment $\{(a,b)_\mathcal{S} : \frac{1}{2}a + b = \phi_{\mathcal{F}}(s), a \geq 0, b \geq 0\}$.

(c) For $h$ satisfying Assumption 1.1, we have the duality relation

\[
\psi_{\mathcal{F},h} = \sup_{s \in \mathbb{R}_{\geq 0}^m} \left\{ h(1 + s) - \phi_{\mathcal{F}}(s) \right\} .
\]

The supremum is attained on a nonempty bounded set $S^* \subset \mathbb{R}_{\geq 0}^m$.

(d) For $h$ satisfying Assumption 1.1, we have

\[
\text{Opt}(\psi) = \bigcup_{s \in S^*} \text{Opt}(\phi; s) ,
\]

and in particular that $\text{Opt}(\psi)$ is a nonempty bounded subset of $\mathbb{R}_{\geq 0}^m$.

**Remark 1.9.** For certain choices of $h$ we can have that $S^*$, and hence $\text{Opt}(\psi)$, are uncountable, though for generic choices they will be finite sets, such as when $h(x) = \sum \beta_k (x_k - a_k)^\gamma_k$ for constants $\beta_k > 0, a_k \in \mathbb{R}$ and $\gamma_k \in (0, \Delta / e(F_k))$.

### 1.5. Edge-F models

In this subsection we specialize the measures (1.12) to the case that the Hamiltonian of (1.9) takes the form

\[
H(X) = H(X; \beta) = \beta f(t(F, X))
\]

for fixed connected graph $F$, $\beta > 0$ and $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ that is continuous, and strictly increasing and differentiable on $(1, \infty)$, with $f(x) = o_{x \to \infty}(x^{\Delta / e(F)})$. We let $G_{n,p,\beta}$ denote a sample from the corresponding generalized ERGM $\mu_n^{H_{\beta}}$.

In this case it turns out that the optimizers in (1.22) lie on the $a$ or $b$ axis, as was shown to be the case for the large deviation optimization problem (1.37) in [5]. Indeed, the main result in [5] states that for $n^{-1/\Delta} \ll p \ll 1$ we have

\[
\phi_{\mathcal{F}}(s) = \begin{cases} 
P_F^{-1}(1 + s) & F \text{ is irregular} \\
\min \left\{ \frac{1}{2} s^{2 / e(F)}, P_F^{-1}(1 + s) \right\} & F \text{ is } \Delta\text{-regular}.
\end{cases}
\]
In the latter case, the minimum is given by the first expression \( \frac{1}{2} s^{2/\nu(F)} \) for \( s \in [s_c(F), \infty) \) and the second expression \( P^{-1}_F(1+s) \) for \( s \in [0, s_c(F)] \), where \( s_c(F) \) is the unique positive solution of the equation \( \frac{1}{2} s^{2/\nu(F)} = P^{-1}_F(1+s) \).

Let

\[
U(\beta, s) = \beta f(1+s) - \phi_F(s)
\]

be the objective function in (1.42). When \( F \) is regular we further denote the restrictions of \( U(\beta, \cdot) \) to \([0, s_c(F)]\) and \([s_c(F), \infty)\) by

\[
U_{\text{hub}}(\beta, s) = \beta f(1+s) - P^{-1}_F(1+s), \quad s \in [0, s_c(F)],
\]

\[
U_{\text{clique}}(\beta, s) = \beta f(1+s) - \frac{1}{2} s^{2/\nu(F)}, \quad s \in [s_c(F), \infty).
\]

The following result about the typical structure of Edge-F models is proved in Section 5.

**Proposition 1.10.** In the above setting, the following hold:

Case of \( F \) irregular. Suppose that \( f \) is such that for all \( \beta \geq 0 \), \( U(\beta, \cdot) \) attains its maximum on \( \mathbb{R}_{\geq 0} \) at a unique point \( s^*(\beta) \). Then for every \( \beta, \xi > 0 \) there exists \( c = c(\beta, \xi, F, f) > 0 \) such that when \( n^{-1/(\Delta+1)} \ll p \ll 1 \), we have

\[
\mathbb{P}(G_{n,p,\beta} \in G^1_n(0, b_*(\beta), \xi)) \geq 1 - \exp(-cr_{n,p})
\]

for all \( n \) sufficiently large, where \( b_*(\beta) = P^{-1}_F(1+s^*(\beta)) \).

Case of \( F \) regular. Suppose \( f \) is such that for all \( \beta \geq 0 \), \( U_{\text{hub}}(\beta, \cdot) \) and \( U_{\text{clique}}(\beta, \cdot) \) attain their maxima at unique points \( s^*_{\text{clique}}(\beta) \) and \( s^*_{\text{hub}}(\beta) \) in their respective domains. Then there exists \( \beta_{c}(F, f) > 0 \) such that for any fixed \( \beta, \xi > 0 \) there exists \( c = c(\beta, \xi, F, f) > 0 \) such that the following holds when \( n^{-1/(\Delta+1)} \ll p \ll 1 \). If \( \beta < \beta_{c}(F, f) \), then for all \( n \) sufficiently large,

\[
\mathbb{P}(G_{n,p,\beta} \in G^1_n(0, b_*(\beta), \xi)) \geq 1 - \exp(-cr_{n,p})
\]

while if \( \beta > \beta_{c}(F, f) \), then for all \( n \) sufficiently large,

\[
\mathbb{P}(G_{n,p,\beta} \in G^1_n(a_*(\beta), 0, \xi)) \geq 1 - \exp(-cr_{n,p}),
\]

where \( a_*(\beta) = s^*_{\text{clique}}(\beta)^{2/\nu(F)} \) and \( b_*(\beta) = P^{-1}_F(1+s^*_{\text{hub}}(\beta)) \).

Moreover, the same conclusions hold in all cases with \( G^2_n \) in place of \( G^1_n \) when \( n^{-1} \log n \ll p^{2\Delta(F)} \ll 1 \).

We note in passing that \( s^*(\beta) > 0 \) if and only if \( \beta > \beta_o := \inf_{s>0} \{ \phi_F(s)/(f(1+s) - f(1)) \} \).

**Remark 1.11** (Absence of degeneracy). Recall the degeneracy phenomenon established in [10] for the version (1.5) of the edge-triangle model, wherein there exists \( \beta_* = \beta_*(\alpha) \geq 0 \) such that typical samples transition from almost-empty to almost-full as \( \beta \) increases through \( \beta_* \). The sparse setting of Proposition 1.10 corresponds to the limiting case \( \alpha \to -\infty \), and without (1.10) we have such a degeneracy transition at \( \beta_* = 0 \). Indeed, for \( \Delta \)-regular \( F \), \( \phi_F(s) \) grows like \( s^{2/\nu(F)} = s^{\Delta/\nu(F)} \) when \( s \) is large. Thus, taking \( f \) to be linear as in (1.5), we have that the optimizing value of \( s \) for (1.45) is zero for \( \beta = 0 \) and \( +\infty \) as soon as \( \beta > 0 \).

The following result gives a more explicit version of Proposition 1.10 for the case of \( F = K_3 \) and a specific choice of \( f \). The computations are given in Section 5.

**Corollary 1.12** (The edge-triangle model). With hypotheses as in Proposition 1.10, take \( F = K_3 \) and

\[
f(x) = (x-1)^{1/3}.
\]
Assume $n^{-1/3} \ll p \ll 1$. For any fixed $\beta, \xi > 0$ there exists $c(\beta, \xi) > 0$ such that if $\beta < 16/9$, then for all $n$ sufficiently large,

$$
P\left( G_{n,p,\beta} \in \mathcal{G}^1_n(0, \frac{1}{3} \beta^{3/2}, \xi) \right) \geq 1 - \exp(-cr_{n,p})$$

while if $\beta > 16/9$, then for all $n$ sufficiently large,

$$
P\left( G_{n,p,\beta} \in \mathcal{G}^1_n(\beta^2, 0, \xi) \right) \geq 1 - \exp(-cr_{n,p}).$$

The same conclusions hold with $\mathcal{G}^2_n$ in place of $\mathcal{G}^1_n$ when $n^{-1/4} \log^{1/4} n \ll p \ll 1$.

### 1.6. Stability form of Finner’s inequality

Our proof of Theorem 1.6 builds on analysis in [5] of the entropic optimization problem (4.4). That work makes use of a Brascamp–Lieb-type inequality for product measure spaces due to Finner [27], restated below for the case of product probability spaces.

We note that the stability of other special cases of the Brascamp–Lieb inequalities, such as the Riesz–Sobolev inequality, as well as “reverse” Brascamp–Lieb inequalities such as the Brunn–Minkowski inequality, have been a subject of recent interest – see [12, 13, 25, 26] and references therein.

In the following we consider a finite set $V$ and a set system $\mathcal{A}$ over $V$ – that is, a finite collection of subsets $A \subseteq V$, allowing repetitions (thus $\mathcal{A}$ is in general a multiset). We assume $\emptyset \notin \mathcal{A}$. Say two elements $u, v \in V$ are equivalent if for every $A \in \mathcal{A}$, either $\{u, v\} \subseteq A$ or $\{u, v\} \subseteq V \setminus A$, and let $\mathcal{B}$ denote the partition of $V$ into equivalence classes. Thus, $\mathcal{B}$ is the smallest partition of $V$ so that every element of $\mathcal{A}$ can be expressed as a union of elements of $\mathcal{B}$. To each $v \in V$ we associate a probability space $(\Omega_v, \mu_v)$. For nonempty $A \subseteq V$ we write $\Omega_A := \prod_{v \in A} \Omega_v$, $\mu_A := \otimes_{v \in A} \mu_v$, and let $\pi_A : \Omega_V \to \Omega_A$ denote the associate coordinate projection mapping. We abbreviate $(\Omega, \mu) = (\Omega_V, \mu_V)$.

**Theorem 1.13 (Finner’s inequality).** In the above setting, let $\Lambda = (\lambda_A)_{A \in \mathcal{A}}$ be a collection of positive weights such that $\sum_{A \ni v} \lambda_A \leq 1$ for each $v \in V$. Suppose $(f_A)_{A \in \mathcal{A}}$ is a collection of functions $f_A : \Omega_A \to \mathbb{R}_{\geq 0}$ such that $\int f_A d\mu_A \leq 1$ for all $A \in \mathcal{A}$. Then

$$
\int_{\Omega_A} \prod_{A \in \mathcal{A}} f_A^A \circ \pi_A \, d\mu \leq 1. \quad (1.48)
$$

The case that $\mathcal{A}$ consists of two copies of $V$ is Hölder’s inequality. The case that $\mathcal{A}$ consists of all subsets of $V$ of a fixed size was obtained earlier by Calderón [7].

In [27] Finner also shows that equality holds in (1.48) if and only if there are functions $h_{A, B} : \Omega_B \to \mathbb{R}_{\geq 0}$ for each $A \in \mathcal{A}$ and $\mathcal{B} \supset B \subseteq A$, such that $f_A = \otimes_{B \subseteq A} h_{A, B}$, with $h_{A, B}$ and $h_{A', B}$ $\mu_B$-almost-surely equal up to a constant multiple $K_{A, A', B} > 0$ whenever $B \subseteq A \cap A'$.

For the proofs of Theorems 1.5 and 1.6 we make use the following stability version of Theorem 1.13, which is a robust statement of the case for equality.

**Theorem 1.14 (Stability version of Finner’s inequality).** With hypotheses as in Theorem 1.13, suppose

$$1 - \varepsilon \leq \int_{\Omega_A} \prod_{A \in \mathcal{A}} f_A^A \circ \pi_A \, d\mu \quad (1.49)$$

for some $\varepsilon \geq 0$. Then there is a collection of functions $(h_B)_{B \in \mathcal{B}}$ with $h_B : \Omega_B \to \mathbb{R}_{\geq 0}$ and $\int h_B d\mu_B = 1$ for each $B \in \mathcal{B}$, such that for every $A \in \mathcal{A}$,

$$\left\| f_A - \otimes_{B \subseteq A} h_B \right\|_{L_1(\Omega_B)} \lesssim \varepsilon^c \quad (1.50)$$
where \( c > 0 \) and the implicit constant depend only on \(|V|, A\) and \( \Lambda \).

**Remark 1.15.** Note it follows from the theorem statement that \( \|h_B - 1\|_{L^1(\Omega)} \leq e^c \) for any \( B \in B \) such that \( \sum_{A \supseteq B} \lambda_A < 1 \). Indeed, if such a set \( B \) exists, then we can add a copy of \( B \) to \( A \), taking \( f_B = 1 \) and \( \lambda_B = 1 - \sum_{A \supseteq B} \lambda_A \).

**Remark 1.16.** It would be interesting to determine the optimal exponent \( c \) in (1.50) depending on the structure of the set system \( A \). It is not hard to see that \( c = 1/2 \) is optimal for Hölder’s inequality, as well as the generalized Hölder inequality (see Remark A.2). Inspection of the proof shows that we can also take \( c = 1/2 \) in our application to Theorem 1.6, where \( A \) is the edge set of a simple graph. However, in general the proof gives a smaller value of \( c \). We mention that the work [22] obtains a similar stability result for the special case of the Uniform Cover Inequality, which concerns the case that the sets in \( A \) are all distinct and \(|\{A \in A : v \in A\}| \equiv d\) for some \( d \), and where the \( f_A \) are taken to be indicators of bodies (open sets with compact closure). (The Uniform Cover Inequality is a generalization of the Loomis–Whitney inequality.) Notably, in that setting they obtain an approximation as in (1.50) with \( c = 1 \), which is optimal under their hypotheses.

### 1.7. Discussion and future directions

Theorem 1.5 shows that sparse ERGMs can exhibit non-trivial structure different from Erdős–Rényi graphs, provided the Hamiltonian satisfies a growth condition. In particular, the clique and hub structures of Theorem 1.5 introduce some amount of transitivity, and cure the most severe forms of the degeneracy phenomenon studied in [10]. However, the class of clique-hub graphs is still unlikely to be rich enough to provide useful models for social networks, which has been the main motivation for these models.

On the other hand, Theorem 1.5 at least demonstrates a clique phase for ERGMs. The appearance of cliques is a common feature of social networks that is not exhibited by other models for real-world networks, such as the preferential attachment model, which are locally tree-like [51]. Based on recent works on large deviations for random graphs, it seems likely that extensions of the models considered here incorporating degree constraints would exhibit more involved dense structures with multiple cliques [6, 30]. Furthermore, richer structures may result from considering Hamiltonians that depend on induced homomorphism densities\(^2\), as suggested by recent work on the upper tail for induced 4-cycle counts in the Erdős–Rényi graph [14]. The NMF approximation was established for the upper tail of induced subgraph counts in the Erdős–Rényi hypergraph in [16] in a certain range of \( p \), but apart from the aforementioned work [14] on 4-cycles, the structure of optimizers for the NMF variational problem remains open in general.

In this work we consider the ferromagnetic regime with \( h \) in (1.9) nondecreasing, or alternatively, taking \( \beta_R > 0 \) in (1.3). The NMF approximation could be extended to the case of decreasing \( h \) using results from [16] on joint lower tails for homomorphism counts in \( G_{n,p} \); see also [36] for a result for the case \( m = 1 \) in an optimal sparsity range. For the more delicate structural result of Theorem 1.5, the first step of providing an asymptotic solution to the lower-tail large deviations variational problem – analogous to the result of [5] for the upper tail – remains open.

Finally, we mention that the large deviations results we import from [16] were developed in the more general setting of \( r \)-uniform hypergraphs, and would permit extensions of Proposition 1.2 to exponential random hypergraph models, which have been proposed in [3, 46] to

\(^2\)An induced graph homomorphism from \( F \) to \( G \) is a mapping of the vertices of \( F \) into the vertices of \( G \) such that edges of \( F \) are mapped onto edges of \( G \) and non-edges of \( F \) are mapped onto non-edges of \( G \).
model multiway interactions in social networks. As with the problem of allowing $h$ to be decreasing, the extension of the structure result of Theorem 1.5 requires an analysis of the NMF optimization problem, which has been done for hypergraphs in only a few cases [37,38,40].

1.8. Notational conventions. We generally write $\mathbb{R}_+$ for $(0, \infty)$ and $\mathbb{R}_{\geq 0}$ for $[0, \infty)$. For $J \subset [n]$ we write $J^c$ for $[n] \setminus J$.

We use the following standard asymptotic notation. For a nonnegative real quantity $g$ and a parameter (or vector of parameters) $q$ we write $O_q(g)$ to denote a real quantity $f$ such that $|f| \leq C(q)g$ for some finite constant $C(q)$ depending only on $q$. We also write $f \lesssim_q g$, $g \gtrsim_q f$ and $g = \Omega_q(f)$ to say that $f = O_q(g)$, and $f \sim_q g$ to mean $f \lesssim_q g \lesssim_q f$. When there is no subscript it means the implied constant is universal, unless noted otherwise. We use $C, c, c_0$, etc. to denote positive, finite constants whose value may change from line to line, assumed to be universal unless dependence on parameters is indicated.

For $g$ depending on an asymptotic parameter $\varepsilon$ we write $o_{\varepsilon \to \varepsilon_0}(g)$ to denote a real quantity $f$ depending on $\varepsilon$ such that $\lim_{\varepsilon \to \varepsilon_0} f/g = 0$. In most of the article the asymptotic parameter is $n$, tending to $\infty$ through the nonnegative integers, and we suppress the limit from the subscript in this case. The exception is Section 2 where the asymptotic parameter is generally $p \in (0, 1)$ tending to zero, and we similarly suppress the subscript $p \to 0$ there ($n$ does not appear in that section). We often write $f \ll g$, $g \gg f$ to mean $f = o(g)$, and $f \sim g$ to mean $f/g \to 1$.

All graphs in the article are simple – that is, with undirected edges and no self loops. For a graph $F$ we use $V(F), E(F)$ to denote its sets of vertices and edges, respectively, and write $v(F) = |V(F)|, e(F) = |E(F)|$. By abuse of notation we extend the edge-counting function to $G_n$ (see (1.1)) as $e(G) := \sum_{1 \leq i < j \leq n} G_{i,j}$. With $I_p$ as in (1.19) we often use the shorthand notations

$$I_p(Q) := \sum_{i \leq j} I_p(Q_{i,j}) \quad \text{for} \quad Q \in \mathcal{Q}_n, \quad \text{and} \quad I_p(\mathcal{E}) := \inf_{Q \in \mathcal{E}} I_p(Q) \quad \text{for} \quad \mathcal{E} \subseteq \mathcal{Q}_n. \quad (1.51)$$

1.9. Organization of the paper. The core of the proofs of our main results concerning the typical structure of ERGMs (Theorem 1.5) and conditioned Erdős–Rényi graphs (Theorem 1.6) is a stability analysis for an entropic optimization problem on graphon space, which we carry out in Section 2; see Proposition 2.2 for the statement of the result, with Theorem 1.14 is key for this stability analysis that also uses our understanding from Proposition 1.8 of the optimizing set. In Section 3 we combine Proposition 2.2 with the quantitative large deviation results of Propositions 3.2 and 3.3 due to [15,16] to establish Theorem 1.6 on the conditional structure of Erdős–Rényi graphs. In Section 4 we establish our main results on ERGMs, namely Proposition 1.2, Theorem 1.4 and Theorem 1.5 using corresponding large deviation results for Erdős–Rényi graphs (where the non-asymptotic upper-tail estimate of Proposition C.1 allows for an a-priori truncation to a compact set). Proposition 1.10 and Corollary 1.12 on edge-F models are proved in Section 5. The appendices contain the proofs of the stability version of Finner’s inequality (Theorem 1.14), Proposition 1.8, and a non-asymptotic upper-tail estimate of the correct shape for homomorphism densities (Proposition C.1).

2. Stability for the upper-tail entropic optimization problem

Throughout this section the asymptotic notation $o(1)$ and $\sim$ is with respect to the limit $p \to 0$ unless indicated otherwise (the asymptotic parameter $n$ makes no appearance here).

As was shown in [5,6], the infimum in the upper-tails NMF optimization problem (1.36) over the $\binom{n}{2}$-dimensional domain $\mathcal{Q}_n$ is asymptotically attained by matrices having off-diagonal
entries in \( \{p, 1\} \), taking value 1 on the edge sets of a clique and complete bipartite graph of appropriate sizes, effectively reducing (1.36) to the two-dimensional problem (1.37). In this section we prove Proposition 2.2 below, showing that near-optimizers for (1.36) are close to such “clique-hub” matrices, a key step towards proving Theorems 1.5 and 1.6. Following [5, 6], we establish this stability in the broader setting of an infinite-dimensional optimization problem over the space of graphons, whose definition we now recall.

We denote the Lebesgue measure on \([0, 1]^d\) by \( | \cdot | \). All integrals are understood to be with respect to Lebesgue measure unless otherwise indicated. For the Lebesgue spaces \( L_q([0, 1]^d) \) with \( d = 1, 2 \) we write \( \|g\|_q \) for the \( L_q \)-norm for \( q \geq 1 \). For a set \( E \subset [0, 1]^d \) we take \( E^c \) to mean \([0, 1]^d \setminus E\).

A \textit{graphon} is a symmetric measurable function \( g : [0, 1]^2 \to [0, 1] \), and an \textit{asymmetric graphon} is a measurable function \( g : [0, 1]^2 \to [0, 1] \) with no symmetry constraint. We denote the space of graphons by \( W \). Given a partition \( \mathcal{P} \) of \([0, 1] \) into finitely many measurable sets, we let \( \mathcal{W}_\mathcal{P} \) denote the subspace of graphons that are a.e. constant on sets \( S \times T \) with \( S, T \in \mathcal{P} \).

For a graph \( F \) we define the homomorphism counting functional on symmetric measurable functions \( f : [0, 1]^2 \to \mathbb{R}_+ \) by

\[
t(F, f) := \int_{[0, 1]^{V(F)}} \prod_{e \in E(F)} f(x_e) dx \tag{2.1}
\]

where \( x_e := (x_u, x_v) \) for \(\{u, v\} \in E(F)\). For \( F \) bipartite with ordered bipartition \((U, W)\) the above definition extends unambiguously to asymmetric functions, with \( x_e := (x_u, x_w) \) for \( u \in U, w \in W \).

For \( s \in \mathbb{R}_+^m \), \( p \in (0, 1) \), and \( F = (F_1, \ldots, F_m) \) a fixed sequence of distinct, connected graphs of maximum degree \( \Delta \geq 2 \), the graphon upper-tail entropic optimization problem — referred to hereafter as the \textit{graphon problem} — is defined

\[
\omega(F, s, p) := \inf_{g \in \mathcal{W}} \left\{ \frac{1}{2} \int p \circ g : t(F_k, g/p) \geq 1 + s_k \quad \forall 1 \leq k \leq m \right\}. \tag{2.2}
\]

The following result, extracted from an argument in [6] (which builds upon [5]) shows that in the sparse limit \( p \to 0 \), the infinite-dimensional graphon problem (2.2) reduces to the 2-dimensional problem (1.37).

**Theorem 2.1** (Solution of the graphon problem). \textit{In the above setup, for fixed \( F, s \) we have}

\[
\omega(F, s, p) \sim \phi(F, s)p^\Delta \log(1/p). \tag{2.3}
\]

The following class of \textit{clique-hub} graphon witnesses the upper bound in (2.3): for disjoint measurable sets \( S, T \subset [0, 1] \) we denote the graphon

\[
g_{S,T} = p + (1-p) \left[ \chi_{S \times S} + \chi_{T \times T} + \chi_{S^c \times T^c} \right] \tag{2.4}
\]

and for \( a, b > 0 \) let

\[
\mathcal{W}(a, b) := \{ g_{S,T} : |S| = (ap^\Delta)^{1/2}, |T| = bp^\Delta \}. \tag{2.5}
\]

Note these are the graphon analogues of \( Q^{I,J} \) and \( Q_m(a, b) \) from (1.26), (1.27).

Recall our notation \( \text{Opt}(\phi; s) \) for the set of optimizers for \( \phi(F; s) \). As noted in [6], for any \( (a, b) \in \text{Opt}(\phi; s) \) and \( g_{S,T} \in \mathcal{W}(a, b) \) we have that \( t(F_k, g_{S,T}/p) \geq 1 + s_k \) for every \( 1 \leq k \leq m \), and

\[
\frac{1}{2} \int p \circ g_{S,T} \sim \phi(F, s)p^\Delta \log(1/p), \tag{2.6}
\]
so the graphons in \( W(a, b) \) attain the upper bound \( \omega(F, s, p) \leq \phi_F(s) p^\Delta \log(1/p) \). The following stability result shows that any near-optimizer for \( \omega(F, s, p) \) must be close to an element of \( W(a_*, b_*) \) for some \( (a_*, b_*) \in \text{Opt}(\phi; s) \).

**Proposition 2.2 (Stability for the graphon problem).** Let \( F = (F_1, \ldots, F_m) \) be a sequence of graphs as above and let \( s \in \mathbb{R}^m_+ \) and \( \eta > 0 \). There exist \( c_0(F) > 0 \) and \( p_0(F, s, \eta) > 0 \) such that the following holds for all \( 0 < p \leq p_0 \). For any graphon \( g \) satisfying

\[
t(F_k, g/p) \geq 1 + s_k - \eta \quad \forall 1 \leq k \leq m
\]  

(2.7)

and

\[
\frac{1}{2} \int I_p \circ g \leq (\phi_F(s) + \eta) p^\Delta \log(1/p),
\]

(2.8)

there exist \( (a_*, b_*) \in \text{Opt}(\phi; s) \) and \( g_{S,T} \in W(a_*, b_*) \) such that

\[
\|g - g_{S,T}\|_2 \leq E_{\pm} \eta^{\alpha(E)} p^{\Delta/2}.
\]  

(2.9)

Moreover, if \( g \in W_P \) for some finite partition \( P \) of \( [0, 1] \) then we may take \( g_{S,T} \in W_P \cap W(a'_*, b'_*) \) for some \( a'_*, b'_* \) such that \( |a'_* - a_*|, |b'_* - b_*| \leq E_{\pm} \eta^{\alpha(E)} \).

**Remark 2.3.** The conclusion for the case \( g \in W_P \) is needed for the proof of Theorem 1.6, where we take \( P \) to be the partition of \( [0, 1] \) into intervals of length \( 1/n \) in order to pass from graphons to weighted graphs over \( [n] \).

**Remark 2.4.** By (2.6), (2.8) and (2.9) we have \( \int I_p \circ g_{S,T} - I_p \circ g \leq E_{\pm} \eta^{\alpha(E)} p^{\Delta} \log(1/p) \), so that (2.2) is a stability-type strengthening of Theorem 2.1. Indeed, setting \( E = \{g_{S,T} = 1\} \), since \( I_p(p) \leq I_p(1) \) it suffices to show that for any \( g \in W \),

\[
\int_E (1 - g)^2 \leq \varepsilon^2 p^\Delta \quad \Rightarrow \quad \frac{1}{I_p(1)} \int_E (I_p(1) - I_p \circ g) \leq E_{\pm} (\varepsilon + o(1)) p^\Delta
\]

which, as \( I_p(x)/I_p(1) \geq x - o(1) \) and \( |E| = O_{E_{\pm}}(p^\Delta) \), follows by Cauchy–Schwarz.

2.1. **Preliminary lemmas.** Before commencing with the proof of Proposition 2.2 we collect a few lemmas. The first is a stability result for the set of optimizers of the 2-dimensional problem (1.37). In the sequel we abbreviate \( T_k := T_{F_k} \). For \( s \in \mathbb{R}^m_+ \) and \( \eta > 0 \) let

\[
R(s, \eta) = \left\{ (a, b) \in \mathbb{R}^2_+ : \frac{1}{2} a + b \leq \phi_F(s) + \eta \right\} \cap \bigcap_{k=1}^m \left\{ (a, b) : T_k(a, b) \geq 1 + s_k - \eta \right\}.
\]  

(2.10)

In the following we write \( B_q(r) \) for the open ball in \( \mathbb{R}^2 \) of radius \( r \) centered at \( q \), and use sumset notation \( R + S = \{ r + s : r \in R, s \in S \} \).

**Lemma 2.5 (Stability for the planar problem).** For each \( s \in \mathbb{R}^m_+ \) and \( \eta > 0 \) we have that

\[
\text{Opt}(\phi; s) \subset R(s, \eta) \subset \text{Opt}(\phi; s) + B(0,0)(\varepsilon_\eta),
\]

for some \( \varepsilon_\eta = O_{E_{\pm}}(\eta) \).

**Proof.** With \( \eta \mapsto R(s, \eta) \) non-decreasing and \( R(s, 0) = \text{Opt}(\phi; s) \), the first containment is obvious. Further, by the compactness of \( R(s, \eta) \) and continuity of all functions of \( (a, b) \) in its definition, any collection \( q_0 \in R(s, \eta) \) must have a limit point \( q_0 \in R(s, 0) \), implying the second containment for some \( \varepsilon_\eta \to 0 \). With \( \text{Opt}(\phi; s) \) a finite set (see Proposition 1.8(b)), it remains only to show that for fixed \( q = (a, b) \in \text{Opt}(\phi; s) \) and small \( \varepsilon > 0 \) the set \( R(s, \eta) \cap B_q(\varepsilon) \) has diameter \( O_{E_{\pm}}(\eta) \). To this end, fixing such \( q \), it is argued in the proof of Proposition 1.8(b) (see Appendix B) that \( q \) must be a point of non-smoothness on the boundary of \( \bigcap_k \{ T_k \geq 1 + s_k \} \) where the linear function \( T_0(a, b) := \frac{1}{2} a + b \) of slope \( m_0 = -\frac{1}{2} \) is minimized (incorporating
hereafter the constraint of being in $\mathbb{R}^2$ via $s_{m+1} = s_{m+2} = -1$, $T_{m+1} = a$ and $T_{m+2} = b$). As such, at least two constraints, $k_L$ and $k_R$ in $[m+2]$, must hold with equality at $q$, where the corresponding curves intersect transversally with slopes $-\infty \leq m_{k_L} < m_0 < m_{k_R} \leq 0$ (and the strict inequalities here are due to the strict convexity of $T_k, k \leq m$). Setting

$$S(\eta, \varepsilon) := B_q(\varepsilon) \cap \{T_{k_L} \geq T_{k_L}(q) - \eta\} \cap \{T_{k_R} \geq T_{k_R}(q) - \eta\} \cap \{T_0 \leq T_0(q) + \eta\},$$

clearly $R(\mathcal{N}, \eta) \cap B_q(\varepsilon) \subseteq S(\eta, \varepsilon)$. Further, when $k_L \in [m]$ or $k_R \in [m]$, applying the mean-value theorem for the corresponding function of smooth gradient of norm $\gtrsim L_{\mathcal{N}, \eta} 1$, yields that

$$S(\eta, \varepsilon) \subset \tilde{S}(2\eta, \varepsilon)$$

for all $\varepsilon \leq \varepsilon_0(F, g, q)$, where $\tilde{S}(\eta, \varepsilon)$ is defined as $S(\eta, \varepsilon)$ except for replacing $T_{k_L}$ and $T_{k_R}$ by the corresponding linearizations around $q$ of slopes

$$m_{k_L} \vee O_{\mathcal{L}, g}(\varepsilon^{-1}) + O_{\mathcal{L}, g}(\varepsilon) < m_0 < m_{k_R} - O_{\mathcal{L}, g}(\varepsilon).$$

In particular, $\tilde{S}(2\eta, \varepsilon)$ is contained within a closed triangle, whose interior point $q$ is of distance $O_{\mathcal{L}, g}(\eta)$ from all three sides, thereby having a diameter $O_{\mathcal{L}, g}(\eta)$, as claimed. 

We next recall a result from [5] identifying dominant terms in an expansion for $t(F, g/p)$ in terms of subgraphs of $F$ and a decomposition of the support of $g$. Letting $f = g - p$, we can expand

$$t(F, g/p) = 1 + \sum_{F' \subseteq F} N(F', F)t(F', f/p)$$

(2.11)

where the sum runs over nonempty subgraphs $F'$ of $F$ (up to isomorphism), and $N(F', F)$ is the number of subgraphs of $F'$ isomorphic to $F'$. It is shown in [5, Corollary 6.2] that for $g \geq p$ satisfying

$$\int I_p \circ g \leq Kp^\Delta \log(1/p)$$

(2.12)

for some $K = O(1)$, the only non-negligible terms in (2.11) are for $F' = F$, as well as $F' = F^U$ for some $U \in \mathcal{I}(F^*)$, where $F^U$ denotes the induced subgraph of $F$ on $U \cup \mathcal{N}_F(U)$, with $\mathcal{N}_F(U)$ the vertex neighborhood of $U$ in $F$. The expansion is further refined based on a decomposition of $f$ that we now recall. For $d > 0$ we denote

$$D_d(g) := \left\{ x \in [0, 1] : \int_0^1 \max(g(x, y) - p, 0)dy \geq d \right\}$$

(2.13)

(note the integrand is $f$ if $g \geq p$, but we do not assume this in general). We abbreviate

$$\tilde{f} := f_{\chi D_d(g) \times D_d(g)^c}, \quad \tilde{f} := f_{\chi D_d(g) \times D_d(g)^c}.$$
Proof. Fixing $k \in [m]$, the expression on the RHS of (2.15), with the sum restricted to $F_k^U \neq F_k$, $f$ replacing both $\tilde{f}$ and $\tilde{f}'$, and with a qualitative error term $o(1)$, is precisely [5, Cor. 6.2] (where the factor $N(F_k^U, F_k)$ of subgraphs of $F_k$ isomorphic to $F_k^U$, appearing explicitly in [5], is embedded into such number of repetitions in our enumeration over ordered bipartitions of $V(F_k^*)$). Upon confirming that $F_k^U \neq F_k$ must be irregular when $F_k$ is connected (for any $U \in \mathcal{I}(F_k^*)$ non-empty), one has that $t(F_k^U, f/p) = t(F_k^U, \tilde{f}/p) + o(1)$ for some $p^{1/3} \ll d \ll 1$ (thanks to [5, Prop. 6.5(a)]). In case that $F_k$ is regular and non-bipartite, we have the additional term $t(F_k, f/p) = t(F_k, \tilde{f}/p) + o(1)$ by [5, Prop. 6.5(b)], whereas for $F_k$ regular and bipartite, we must add to the latter expression the terms $t(F_k^U, \tilde{f}/p)$ for the two possible ways for selecting each bipartition $U$ of $F_k$ (see [5, Prop. 6.5(c)]). We thus recovered the expression on the RHS in (2.15), but with a qualitative error term $o(1)$ and with such a qualitative upper bound on $d$. Further inspection of the relevant proofs yields the explicit bounds we have stated. Indeed, the proof of [5, Cor. 6.2] in fact yields a quantitative error of the form $O_{\mathcal{F}_k,K}(p^\kappa)$ for some $\kappa(F_k) > 0$. Likewise, for $F_k$ irregular, [5, Prop. 6.5(a)] and its proof show that we may take $\varepsilon = 0$, $d = p^{1/4}$, and $\kappa_0 = \kappa_0(F_k)$ depending only on $F_k$. For $F_k$ regular and non-bipartite, [5, Prop. 6.5(b)] and its proof show that we may take $\varepsilon = 0$, and $\kappa_0 = \kappa_0(F_k)$ sufficiently small depending only on $F_k$. Finally, for a single $F_k$, the proof of [5, Prop. 6.5(c)] gives an error of the form $\varepsilon + p^{\kappa_0}$ for some $d = d(g) \leq p^{\kappa_0}$ and $\kappa_0 = 2^{-\mathcal{O}_{L,K}(1/\varepsilon)}$ where $L = O_{\mathcal{F}_k}(1)$ is the length of the longest cycle in $F_k$; to get uniformity over $F_1, \ldots, F_m$ we can take $L = \max_k \{v(F_k)\}$. \hfill $\Box$

Remark 2.7. We note that by optimizing the parameter $\varepsilon$ we can replace $\varepsilon + p^{\kappa_0(\varepsilon)}$ with $O_{\mathcal{F}_k,K}(1/\log(1/p)).$

Next, we need a stability version of a quadratic approximation used in [5, 39] for the function $I_p : [0, 1] \to \mathbb{R}_{\geq 0}$. In place of $I_p$, it will be convenient to work with the function

\[ J_p(x) = \frac{I_p(p + x)}{\log(1/p)}, \quad x \in [-p, 1 - p]. \tag{2.16} \]

Lemma 2.8 (Estimates on $J_p$). For any $p \in (0, 1)$ and $x \in [-p, 0]$,

\[ J_p(x) \geq \frac{x^2}{2p \log(1/p)}. \tag{2.17} \]

Moreover, there exists a constant $c > 0$ such that for any $0 < p \leq c$ and $x \in [0, 1 - p]$,

\[ J_p(x) - x^2 \geq \min(x^2, (1 - p - x)^2). \tag{2.18} \]

The key point is that we only have $J_p(x) \approx x^2$ near 0 and $1 - p$, and (2.18) can hence be viewed as a stability version of the inequality $J_p(x) \geq x^2$ used in [5, 39]. This will allow us to deduce that near-optimizers of the graphon problem (2.2) are well approximated by functions taking values in $\{p, 1\}$.

Proof. For (2.17), letting $L_p(x) = I_p(x) - (x - p)^2/2p$, we have $L_p(p) = L''_p(p) = 0$ and $L''_p(x) = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{p} > 0$ for $x \in [0, p]$. Thus, $L_p > 0$ on $[0, p]$, which yields (2.17).

Turning the bound (2.18), let $c_0 \in (0, 1/2)$ be a constant to be taken sufficiently small. From [5, Lemma 4.3] we have that for $p \ll x \leq 1 - p$,

\[ J_p(x) \sim x \frac{\log(x/p)}{\log(1/p)}. \]
In particular, for \(x \in [c_0, 1 - c_0]\),
\[
J_p(x) \geq x - o(1)
\]  
which easily yields the claim in this case. From [5, Lemma 4.4] we have
\[
J_p(x) \geq (x/x_0)^2 J_p(x_0)
\]  
for any \(0 \leq x \leq x_0 \leq 1/2\), provided \(p\) is at most a sufficiently small constant. Applying this with \(x_0 = c_0\), combined with (2.19) at \(x = c_0\), yields \(J_p(x) \gtrsim x^2/c_0\), which gives the claim for the range \(x \in (0, c_0]\) assuming \(c_0\) is sufficiently small.

Now for the range \(y := 1 - p - x \in [0, c_0]\), setting \(K_p(y) := J_p(1 - p - y) - (1 - p - y)^2\), it suffices to show
\[
K_p(y) \gtrsim y^2
\]  
for \(0 \leq y \leq c_0\). Since
\[
J_p(1 - p - y) = 1 - y + \frac{1}{I_p(y)} \left( I(1 - y) + y \log \frac{1}{1 - p} \right) \geq 1 - y - \frac{1}{I_p(1)} \left( y \log \frac{1}{y} + \log \frac{1}{1 - y} \right)
\]
we have
\[
K_p(y) \geq 2p + y - \frac{y}{I_p(1)} \left( \log \frac{1}{y} + O(1) \right) - O((p + y)^2).
\]  
For \(p^{3/4} \leq y \leq c_0\) this yields \(K_p(y) \gtrsim y\), giving (2.20).

Now set \(y = tp\) for \(t \leq p^{-1/4}\). From (2.21) we get
\[
K_p(y) \geq (2 - o(1))p + \frac{y}{I_p(1)} (\log t - O(1))
\]
giving \(K_p(y) \gtrsim y/I_p(1)\) for \(t \geq C\) for a sufficiently large constant \(C > 0\). Since \(I_p(1) \leq (1/p)^{1/10} \leq y^{-4/30}\), say, this yields (2.20) for \(Cp \leq y \leq p^{3/4}\). If \(1 \leq t \leq 1\) then the RHS above is at least \((2 - o(1))p \gtrsim y\), so we have established (2.20) for the range \(p \leq y \leq c_0\).

For \(y \leq p\), i.e. \(t \leq 1\), we have
\[
K_p(y) \geq (2 - o(1))p - y \frac{\log(1/t)}{I_p(1)}.
\]
For \(p^2 \leq y \leq p\) the RHS above is at least \((1 - o(1))p \gtrsim y\), whereas for \(y \leq p^2\) we have \(\log(1/t) \leq \log(1/y) \leq (1/y)^{1/10}\), say, giving a lower bound of \((2 - o(1))p - y^{9/10} = (2 - o(1))p \gtrsim y^{1/2}\). \(\square\)

Finally, we need the following elementary fact, which we state for general product probability spaces (still abbreviating \(\| \cdot \|_q := \| \cdot \|_{L_q(\mu^{\otimes 2})}\) as we do for the Lebesgue spaces).

**Lemma 2.9.** Let \((\Omega, \mathcal{F}, \mu)\) be a probability space and let \(E\) be a subset of \(\Omega^2\), measurable under the product \(\sigma\)-algebra, that is asymmetric under interchanging of the coordinates. Suppose there is a measurable function \(h : [0, 1] \to \mathbb{R}_{\geq 0}\) such that \(\|\chi_E - h \otimes h\|_q < \epsilon \mu(E)^{1/q}\) for some \(q \geq 1\). Then there is a measurable set \(S \subset [0, 1]\) such that \(\|\chi_E - \chi_{S \times S}\|_q \lesssim_q \epsilon^{1/2} \mu(E)^{1/q}\).

**Proof.** We may assume \(\epsilon\) is sufficiently small depending on \(q\) (for otherwise we take \(S = \emptyset\)). For \(k \in \mathbb{Z}\) let \(h_k = h \chi_{S_k}\) with \(S_k = \{2^{k-1} \leq h < 2^k\}\). We take \(S := S_{-1} \cup S_0 \cup S_1\). Our aim is to show
\[
\mu(E \Delta S^2) \lesssim_q \epsilon^{1/2} \mu(E)\]  
(2.22)
First, by our assumption,
\[
\epsilon^q \mu(E) > \|\chi_E - h \otimes h\|_q^q \geq \int_{S \setminus E} h^q \otimes h^q d\mu^{\otimes 2} \gtrsim_q \mu(S^2 \setminus E).
\]
It remains to show that \( \mu(E \setminus S^2) \leq q \varepsilon^{q/2} \mu(E) \). On any \( S_k \times S_{\ell} \) with \( k + \ell > 2 \) we have \( h_k \otimes h_{\ell} - \chi_E \geq 1 \), and hence \( h_k \otimes h_{\ell} - \chi_E \in [2^{k+\ell-3}, 2^{k+\ell}) \). Thus
\[
\varepsilon^q \mu(E) > \sum_{k+\ell \geq 2} \int_{S_k \times S_{\ell}} (h \otimes h - \chi_E)^q d\mu^{\otimes 2} \geq \sum_{k+\ell \geq 2} q^{(k+\ell-3)q} \mu(S_k) \mu(S_{\ell}) \geq \mu\left(E \cap \bigcup_{k+\ell \geq 2} S_k \times S_{\ell}\right).
\]
From the second inequality above we moreover have
\[
\mu(S_k) \leq q \varepsilon^{q/2} 2^{-kq} \mu(E)^{1/2}
\] (2.23)
for all \( k \geq 2 \). Now for \( k + \ell < 0 \), \( \chi_E - h \otimes h \in [\frac{1}{2}, 1] \) on \( E \cap (S_k \times S_{\ell}) \), so
\[
\varepsilon^q \mu(E) > \sum_{k+\ell < 0} \int_{E \cap (S_k \times S_{\ell})} (\chi_E - h \otimes h)^q d\mu^{\otimes 2} \gtrsim_q \mu\left(E \cap \bigcup_{k+\ell < 0} S_k \times S_{\ell}\right).
\]
By symmetry, to establish (2.22) it now suffices to show
\[
\sum_{k \geq 2, k+\ell = i} \mu(S_k \times S_{\ell}) \lesssim_q \varepsilon^{q/2} \mu(E)
\] (2.24)
for each \( i \in \{0, 1, 2\} \). Fixing such an \( i \), from (2.23) we have that
\[
\sum_{k \geq 2, k+\ell = i} \mu(S_k \times S_{\ell}) \lesssim_q \varepsilon^{q/2} \mu(E)^{1/2} \sum_{k \geq 2} 2^{-kq} \mu(S_{i-k}).
\]
Now since
\[
\mu(E)^{1/2} \asymp_q \|h\|_q \gtrsim_q \sum_{k \in \mathbb{Z}} 2^{-kq} \mu(S_k) \asymp_q \sum_{k \in \mathbb{Z}} 2^{-kq} \mu(S_{i-k})
\]
for any \( i = O(1) \) we obtain (2.24) and hence the claim. \( \square \)

2.2. Proof of Proposition 2.2. Let \( E, \underline{\xi}, \eta \) and \( g \) be as in the statement of the proposition. We may assume \( \eta \) is sufficiently small depending on \( \underline{E} \) and \( \underline{\xi} \), and that \( p \) is sufficiently small depending on \( E, \underline{\xi} \) and \( \eta \). Setting \( f = g - p \), we have
\[
\frac{1}{2} \int J_p \circ f \leq (\phi_{\underline{E} \underline{\xi}}(\underline{\xi}) + \eta) p^\Delta \lesssim_{\underline{E}, \underline{\xi}} p^\Delta.
\] (2.25)
In particular (2.12) holds with \( K = O_{\underline{E}, \underline{\xi}}(1) \). Letting \( d \in (0, 1) \) to be chosen below depending on \( g, p, \eta, E \) and \( \underline{\xi} \), with \( \widehat{f}, \tilde{f} \) as in (2.14) we set
\[
a_g := p^{-\Delta} \int J_p \circ \widehat{f}, \quad b_g := p^{-\Delta} \int J_p \circ \tilde{f}.
\] (2.26)
Since \( \widehat{f} \) and \( \tilde{f} \) have disjoint supports, from (2.25) we have
\[
\frac{1}{2} a_g + b_g \leq \phi_{\underline{E} \underline{\xi}}(\underline{\xi}) + \eta.
\] (2.27)
For brevity, we encapsulate the two cases that \( g \in \mathcal{W}_p \) or \( g \in \mathcal{W} \) is a general graphon by using the following convention: by *measurable* we will mean Lebesgue measurable in the general case, whereas for the case that \( g \in \mathcal{W}_p \) we take “measurable” to mean measurable under the finite \( \sigma \)-algebra generated by \( \mathcal{P} \), or the product \( \sigma \)-algebra generated by \( \mathcal{P} \times \mathcal{P} \) as the case may be. Thus, our goal is to locate measurable sets \( S, T \) of appropriate size such that \( g \) is well approximated by \( g_{S,T} \).

The remainder of the proof is divided into the following five steps. Steps 1–4 establish the proposition under the extra assumption that
\[
g \geq p \quad \text{a.e.}
\] (2.28)
i.e. \( f \geq 0 \) a.e.

**Step 1:** Show that \( T_k(a_g, b_g) \geq 1 + s_k - O(\eta) \) for each \( k \), which, together with (2.27), gives that \((a_g, b_g)\) is an approximate extremizer of (1.37) (this step is a restatement of the arguments from [5, 6]).

**Step 2:** Deduce that the bounds in Step 1 actually hold with approximate equality.

**Step 3:** Using the approximate equality in the applications of Finner’s inequality (in (2.30) and (2.31)), apply Theorem 1.14 to deduce that \( \hat{f} \) and \( \tilde{f} \) are well approximated by tensor products of univariate functions.

**Step 4:** From approximate equality in the passage from the relative entropy functional \( J_p \) to \( L_\Delta \)-norms (see (2.33)), together with Lemmas 2.8 and 2.9, deduce that \( \hat{f} \) and \( \tilde{f} \) are well approximated by indicators of product sets \((\chi_{S \times S} \text{ and } \chi_{T \times T^c})\), respectively, concluding the proof under the additional assumption (2.28).

**Step 5:** Remove the assumption (2.28).

**Step 1.** Let \( k \in [m] \) be arbitrary. Applying Lemma 2.6 with \( \varepsilon = \eta/2 \) we have that for \( \hat{f}(x, y) = \hat{f}(x, y) + \tilde{f}(y, x) \), and all \( p \) sufficiently small depending on \( \eta \),

\[
1 + s_k - \eta \leq 1 + \eta + \sum_{\emptyset \neq U \in \mathcal{I}(F_k)} t(F_k^U, \hat{f}/p) + 1(F_k \text{ regular}) \cdot \left(t(F_k, \hat{f}/p) + 1(F_k \text{ bipartite}) \cdot t(F_k, \tilde{f}/p)\right). \tag{2.29}
\]

Next, from Theorem 1.13 we have

\[
t(F_k, \hat{f}/p) \leq \|\hat{f}/p\|_{\Delta}^{\delta(F_k)}, \quad t(F_k, \tilde{f}/p) \leq \|\tilde{f}/p\|_{\Delta}^{\delta(F_k)}, \tag{2.30}
\]

and

\[
t(F_k^U, \hat{f}/p) \leq \|\hat{f}/p\|_{\Delta}^{|U|} \quad \forall U \in \mathcal{I}(F_k^*). \tag{2.31}
\]

Substituting these bounds in (2.29), we get

\[
1 + s_k \leq 2\eta + P_{F_k}(\|\hat{f}/p\|_{\Delta}^{\delta}) + 1(F_k \text{ regular})\|\hat{f}/p\|_{\Delta}^{\delta(F_k)} \leq 2\eta + T_k(\|\hat{f}/p\|_{\Delta}^{\delta}, \|\tilde{f}/p\|_{\Delta}^{\delta}), \tag{2.32}
\]

where the second line follows from the definition of \( T_k \) and noting that \( e(F_k) = \Delta v(F_k)/2 \) when \( F_k \) is regular. Finally, from Lemma 2.8,

\[
\|\hat{f}/p\|_{\Delta}^{\delta} \leq p^{-\Delta} \|\hat{f}/2\|_2 \leq a_g \quad \text{and} \quad \|\tilde{f}/p\|_{\Delta}^{\delta} \leq p^{-\Delta} \|\tilde{f}/2\|_2 \leq b_g \tag{2.33}
\]

(using only that the RHS of (2.18) is nonnegative). Combining with (2.32) and by the monotonicity of \( T_k \) we get

\[
T_k(a_g, b_g) \geq 1 + s_k - 2\eta. \tag{2.34}
\]

In the notation of (2.10), the bounds (2.27) and (2.34) say that

\[
(a_g, b_g) \in R(s, 2\eta). \tag{2.35}
\]

From Lemma 2.5, assuming \( \eta \leq \eta_0(\underline{\Delta}) \) we have

\[
(a_g, b_g) \in B_{q_*}(O_{E_{\underline{\Delta}}}(\eta)) \tag{2.36}
\]
for some \( q_* = (a_*, b_*) \in \text{Opt}(\phi; s) \), and moreover that \( (a_g, b_g) \) is separated by distance \( \geq E_{\Delta} \) from all other elements of the finite set \( \text{Opt}(\phi; s) \) (see Proposition 1.8(b)). From (2.27), (2.32)–(2.33) and the monotonicity of \( T_k \) we similarly conclude that
\[
p^{-\Delta}(\|\tilde{f}\|_r^r, \|\tilde{f}\|_r^r) \in B_{\eta_p}\left(O_{E_{\Delta}}(\eta)\right), \quad r = 2, \Delta.
\]

**Step 2.** Since \( q_* \in \text{Opt}(\phi; s) \) it follows that \( T_k(q_*) = 1 + s_k \) for at least one value of \( k \in [m] \). Let \( K^* \subset [m] \) denote the set of such \( k \). Since \( T_k \) is locally Lipschitz it follows that for \( k \in K^* \),
\[
T_k(a_g, b_g) \leq 1 + s_k + O_{E_{\Delta}}(\eta) .
\]
Hence, for \( k \in K^* \), all of the bounds (2.29)–(2.33) hold with equality up to an additive error of \( O_{E_{\Delta}}(\eta) \). In particular,
\[
\|\tilde{f}/p\|^{\Delta[U]} - t(F^U_k, \tilde{f}/p) \lesssim E_{\Delta} \eta \quad \forall k \in K^* \quad \forall U \in \mathcal{I}(F^*_k)
\]
and
\[
b_g p^\Delta - \|\tilde{f}\|_2^2 \lesssim E_{\Delta} \eta p^\Delta ,
\]
and if \( F_k \) is regular for some \( k \in K^* \), then
\[
\|\tilde{f}/p\|^{e(F_k)} - t(F_k, \tilde{f}/p) \lesssim E_{\Delta} \eta
\]
and
\[
a_g p^\Delta - \|\tilde{f}\|_2^2 \lesssim E_{\Delta} \eta p^\Delta.
\]
If \( F_k \) is irregular for every \( k \in K^* \), it follows that \( a_* = 0 \) and we have from (2.36), (2.33) that
\[
\|\tilde{f}\|^{\Delta} \leq \|\tilde{f}\|_2^2 \leq a_g p^\Delta = O_{E_{\Delta}}(\eta p^\Delta).
\]
In the remainder of the proof we combine the above estimates with lemmas from Section 2.1 and Theorem 1.14 to locate the measurable sets \( S \) and \( T \).

**Step 3.** Using Theorem 1.14 we obtain the following consequence of (2.39) and (2.41). Here and in the remainder of this section, for functions \( f_1, f_2 : [0, 1]^d \to \mathbb{R}_{\geq 0} \) \((d = 1 \text{ or } 2)\), we write
\[
f_1 \approx_r f_2
\]
to mean \( \|f_1 - f_2\|_r^r \lesssim E_{\Delta} \eta c p^\Delta \) for some \( c = c(E) > 0 \) depending only on \( E \).

**Claim 2.10.** There exist measurable \( \hat{h}, \tilde{h} : [0, 1] \to \mathbb{R}_{\geq 0} \) supported on \( D_d(g)^c \) and \( D_d(g) \), respectively, with \( \|\hat{h}\|_{\Delta} = \|\tilde{h}\|_{\Delta} = 1 \), such that
\[
\tilde{f}^\Delta \approx_1 \|\tilde{f}\|^{\Delta}(\hat{h} \otimes \hat{h})^\Delta
\]
and
\[
\tilde{f}^\Delta \approx_1 \|\tilde{f}\|^{\Delta}(\tilde{h} \otimes \chi_{[0,1]}).
\]

**Proof.** Since \( s_k > 0 \) for all \( k \) we have that \( \phi_{E_{\Delta}}(s) > 0 \) and so \( (a_*, b_*) \neq (0, 0) \). If \( a_* = 0 \) then (2.44) holds trivially by (2.43) and the triangle inequality (for arbitrary \( \hat{h} \)). The same reasoning gives \( \|\tilde{f}\|^{\Delta} = O_{E_{\Delta}}(\eta p^\Delta) \) and hence (2.45) in the case that \( b_* = 0 \).

If \( b_* > 0 \), then from (2.37) it follows that for any \( k \in K^* \) we have (assuming \( \eta \) is sufficiently small) that \( \|\tilde{f}\|^{\Delta} \gtrsim E_{\Delta} p \). Then for any nonempty \( U \in \mathcal{I}(F^*_k) \), from dividing through by \( \|\tilde{f}/p\|^{\Delta[U]} \gtrsim E_{\Delta} 1 \) in (2.39) we get
\[
t(F^U_k, \tilde{f}/p) \geq 1 - O_{E_{\Delta}}(\eta).
\]
Taking $U$ to be any singleton $\{u\}$, we apply Theorem 1.14 with $V = V(F^{[u]}_k)$, $A = E(F^{[u]}_k)$, $f_{\{u,v\}}(x_u,x_v) = \tilde{f}(x_u,x_v)/\|\tilde{f}\|_\Delta$ for each $\{u,v\} \in A$, and weights $\lambda_A \equiv 1/\Delta$, to obtain $\hat{h} : [0,1] \to \mathbb{R}_{\geq 0}$ supported on $D_d(g)$ with $\|\hat{h}\|_\Delta = 1$, such that (2.45) holds. Here we have used Remark 1.15 and the fact that in $F^{[v]}_k$, the sum of weights on any vertex other than $v$ is $1/\Delta < 1$, to take the second factor of the tensor product to be $\chi_{[0,1]}$.

If $a_\ast > 0$ then we must have that $F_k$ is regular for some $k \in K^\ast$, and by similar lines as above we obtain $\hat{h} : [0,1] \to \mathbb{R}_{\geq 0}$ supported on $D_d(g)$ such that (2.44) holds. (Theorem 1.14 initially provides an approximation with some $\hat{h}_1 \otimes \hat{h}_2$ in place of $\hat{h} \otimes \hat{h}$, but from the symmetry of $\tilde{f}$ and the triangle inequality it quickly follows that $\hat{h}_1$ and $\hat{h}_2$ are themselves close in $L_1$, so that we may take a single function $\hat{h}$.)

**Step 4.** We now combine Claim 2.10 with (2.40), (2.42), (2.43) to deduce the following, which immediately yields the claimed approximation (2.9) and concludes the proof of Proposition 2.2 under the added assumption (2.28).

**Claim 2.11.** There are measurable sets $T \subset D_d(g)$ and $S \subset D_d(g)^c$ such that

$$\tilde{f} \approx_2 (1-p)\chi_{T \times [0,1]} \quad \text{and} \quad \tilde{f} \approx_2 (1-p)\chi_{S \times S}. \quad (2.46)$$

If $F_k$ is irregular for each $k \in K^\ast$ then we can take $S = \emptyset$.

Indeed, we get the claimed approximation (2.9) by taking $S,T$ as in the above claim, noting that they have the claimed measure by (2.37) (note also that modifications of $g$ on $T \times T$ have negligible impact since $|T|^2 = O_{E_d}(p^{2\Delta}))$).

**Proof.** We begin with the first approximation in (2.46). Considering $k \in K^\ast$, we get from (2.40) and a straightforward application of Lemma 2.8 the existence of a measurable set $\tilde{E} \subset D_d(g) \times D_d(g)^c$ such that

$$\tilde{f} \approx_2 (1-p)\chi_{\tilde{E}}. \quad (2.47)$$

Next, note that

$$\|\tilde{f}^\Delta - (1-p)^\Delta \chi_{\tilde{E}}\|_2^2 = \int_{\tilde{E}} \tilde{f}^\Delta + \int_{\tilde{E}}^c \left( (1-p)^\Delta - \tilde{f}^\Delta \right)^2 \leq \int_{\tilde{E}} \tilde{f}^2 + \Delta^2 \int_{\tilde{E}}^c (1-p - \tilde{f})^2 <_{\Delta} \|\tilde{f} - (1-p)\chi_{\tilde{E}}\|_2^2,$$

so from (2.47) we have

$$\tilde{f}^\Delta \approx_{\Delta} (1-p)^\Delta \chi_{\tilde{E}}. \quad (2.48)$$

Denote

$$\tilde{f}_1 : [0,1] \to [0,1-p], \quad \tilde{f}_1(x) = \|\tilde{f}(x,\cdot)\|_\Delta,$$

which is supported on $D_d(g)$. From (2.45) it easily follows that

$$\tilde{f}^\Delta \approx_1 \tilde{f}_1^\Delta \otimes \chi_{[0,1]}.$$

Since both sides are bounded by 1 we have $\tilde{f}^\Delta \approx_2 \tilde{f}_1^\Delta \otimes \chi_{[0,1]}$. Together with (2.48) and the triangle inequality we get that

$$\chi_{\tilde{E}} \approx_2 h_1 \otimes \chi_{[0,1]} \quad (2.49)$$

where $h_1 := \tilde{f}^\Delta / (1-p)^\Delta$. Now, by (2.47) and the triangle inequality, it suffices to show that $\chi_{\tilde{E}} \approx_2 \chi_{T \times [0,1]}$ for $T = \{x \in [0,1] : |\tilde{E}_x| > 1/2\}$, where $\tilde{E}_x := \{y \in [0,1] : (x,y) \in \tilde{E}\}$. By Fubini’s theorem and (2.49), this in turn will follow from showing

$$h_1 \approx_2 \chi_T. \quad (2.50)$$
To show this, note that
\[
\|\chi_{\tilde{E}} - h_1 \otimes \chi_{[0,1]}\|_2^2 = \int_{\tilde{E}^c} h_1^2 \otimes \chi_{[0,1]} + \int_\tilde{E} (1 - h_1 \otimes \chi_{[0,1]})^2
\]
\[
= \int_0^1 (1 - |\tilde{E}_x|)h_1(x)^2 + \int_0^1 |\tilde{E}_x|(1 - h_1(x))^2
\]
\[
\geq \frac{1}{2} \int_T h_1(x)^2 + \frac{1}{2} \int_T (1 - h_1(x))^2 = \frac{1}{2} \|h_1 - \chi_T\|^2_2.
\]
Combining the above with (2.49) we obtain (2.50), hence that \( \tilde{f} \approx_2 (1 - p)\chi_{T \times [0,1]} \), as stated.

Turning to the second approximation in (2.46), in view of (2.43) we may assume henceforth the existence of some regular \( F_k, k \in K^* \). In that case, we get from (2.42) and Lemma 2.8 that there exists a measurable set \( \tilde{E} \subset D_d(g)^c \times D_d(g)^c \) such that
\[
\tilde{f} \approx_2 (1 - p)\chi_{\tilde{E}}.
\]
Consequently, by the triangle inequality we only need to show \( \chi_{\tilde{E}} \approx_2 \chi_{S \times S} \) for some \( S \subseteq D(g)^c \). Reasoning as we did for \( \tilde{E} \), we deduce from (2.44) in Claim 2.10 that
\[
\chi_{\tilde{E}} \approx_2 h \otimes h
\]
where \( h = \tilde{f}_{\Delta} / ((1 - p)\|\tilde{f}\|_\Delta) \Delta/2 \), with \( \tilde{f}_x(x) := \|\tilde{f}(x, \cdot)\|_\Delta \). Finally, our claim that \( \chi_{\tilde{E}} \approx_2 \chi_{S \times S} \) now follows from Lemma 2.9.

**Step 5.** It only remains to remove the extra assumption (2.28). For general \( g \) as in the statement of the proposition, let \( f = g - p, g_+ = \max(g, p) \) and \( f_+ = g_+ - p = \max(g - p, 0) \). Note that \( D_d(g) = D_d(g_+) \), so that \( \tilde{f} \) and \( \tilde{f}_+ \) are both supported on \( D_d(g)^c \times D_d(g)^c \), and \( \tilde{f}, \tilde{f}_+ \) are supported on \( D_d(g) \times D_d(g)^c \). By the proof for the case \( g \geq p \) we have \((a_{g_+}, b_{g_+}) \in B_{q_+}(O_{E_{\Delta}}(\eta))\) for some \( q_+ = (a_+, b_+) \in \text{Opt}(\phi; \xi) \) as in (2.36), and that (2.9) holds with \( g_+ \) in place of \( g \). By the monotonicity of \( I_p \) on \([0, p]\) we have
\[
a_{g_+} \leq a_g, \quad b_{g_+} \leq b_g
\]
and
\[
1 + s_k - \eta \leq t(F_k, g/p) \leq t(F_k, g_+/p) \leq T_k(a_{g_+}, b_{g_+}) + O_{E_{\Delta}}(\eta)
\]
for every \( k \in [m] \). From this it follows that \((a_g, b_g) \in B_{q_+}(O_{E_{\Delta}}(\eta))\). Hence,
\[
0 \leq a_g - a_{g_+} \leq E_{\Delta} \eta, \quad 0 \leq b_g - b_{g_+} \leq E_{\Delta} \eta.
\]
Applying (2.17), we get that
\[
\|g - g_+\|_2^2 \leq E_{\Delta} \eta p^{\Delta + 1} I_p(1)
\]
and (2.9) holds for \( g \) by the same bound for \( g_+ \) and the triangle inequality.

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6, establishing the conditional structure of Erdős–Rényi graphs on joint upper-tail events for homomorphism densities.
3.1. Quantitative LDPs and counting lemmas. Recall that an \( \mathbb{R} \)-weighted graph is a symmetric function \( X : [n]^2 \to \mathbb{R} \) that is zero on the diagonal, and that \( \mathcal{G}_n \) and \( \mathcal{Q}_n \) denote the sets of \( \{0,1\} \)-weighted graphs and \( [0,1] \)-weighted graphs, respectively. Recall also from (1.12) that \( \nu_{n,p}^0 \) is the Erdős–Rényi(p) measure on \( \mathcal{G}_n \). In this section we write \( \mathbf{G} \) for a sample from \( \nu_{n,p}^0 \).

**Definition 3.1** (Upper-LDP). Given a collection \( \mathbb{K} = (\mathcal{K}_G)_{G \in \mathcal{G}_n} \) of closed convex sets \( \mathcal{K}_G \subseteq \mathcal{Q}_n \) with \( G \in \mathcal{K}_G \) for each \( G \), for a set \( \mathcal{E} \subseteq \mathcal{G}_n \) we define the \( \mathbb{K} \)-neighborhood of \( \mathcal{E} \) as

\[
(\mathcal{E})_{\mathbb{K}} := \bigcup_{G \in \mathcal{E}} \mathcal{K}_G \subseteq \mathcal{Q}_n.
\]

(Note that while \( \mathcal{E} \) is a subset of the discrete cube \( \mathcal{G}_n \), its \( \mathbb{K} \)-neighborhood is a subset of the solid cube \( \mathcal{Q}_n \).) For a lower semi-continuous function \( J : \mathcal{Q}_n \to [0, \infty] \) and \( r_*, r_{\text{ME}} > 0 \), we say that a probability measure \( \mu \) on \( \mathcal{G}_n \) satisfies a quantitative upper-LDP under \( \mathbb{K} \), with rate function \( J \), cutoff rate \( r_* \) and metric entropy rate \( r_{\text{ME}} \), if for any \( \mathcal{E} \subseteq \mathcal{G}_n \),

\[
\log \mu(\mathcal{E}) \leq -\min \left( r_*, \inf \{ J(Q) : Q \in (\mathcal{E})_{\mathbb{K}} \} - r_{\text{ME}} \right). \tag{3.1}
\]

Our quantitative large deviation are formulated in terms of two different norms on the space of \( \mathbb{R} \)-weighted graphs. The first is the spectral norm (1.29), for which we have the following result from [15].

**Proposition 3.2** (Large deviations: spectral norm).

(a) (Upper-LDP). Let \( K_0, K_1 \geq 1 \) and \( \delta \in (n^{-100}, 1] \), and assume

\[
np^\delta \geq \frac{K_0 \log n}{\delta^2}. \tag{3.2}
\]

For each \( G \in \mathcal{G}_n \) let \( \mathcal{K}_{G}^{2-\delta/2}(\delta) = \{ Q \in \mathcal{Q}_n : \|Q - G\|_{2-\delta/2} \leq \delta \np^{\delta/2} \} \). Then \( \nu_{n,p}^0 \) satisfies an upper-LDP under \( \mathbb{K}_{2-\delta/2}(\delta) = (\mathcal{K}_{G}^{2-\delta/2}(\delta))_{G \in \mathcal{G}_n} \), with rate function \( I_p \), and

\[
r_* \gtrsim K_1 r_{n,p}, \quad r_{\text{ME}} \lesssim K_0^{-1} r_{n,p}. \tag{3.3}
\]

(b) (Counting lemma). Let \( p \in (0,1) \), \( L \geq 1 \) and \( \varepsilon > 0 \) be arbitrary. Suppose \( \mathcal{K} \subseteq \mathcal{Q}_n \) is a convex set of diameter at most \( \varepsilon \np^{\delta/2 \bullet} \) in the spectral norm, and that for every induced strict subgraph \( F' \subset F \) there exists \( Q \in \mathcal{K} \) such that

\[
t(F', Q/p) \leq L. \tag{3.4}
\]

Then for every \( F' \subset F \) and \( Q_1, Q_2 \in \mathcal{K} \),

\[
|t(F', Q_1/p) - t(F', Q_2/p)| \lesssim_{F,L} F. \varepsilon. \tag{3.5}
\]

**Proof.** Part (b) follows from [15, Prop. 3.7]. For part (a) we apply [15, Prop. 3.4] with \( K = K_1 \), \( \delta_0 = c\delta^{\delta/2} / \delta^2 \) for a sufficiently small constant \( c > 0 \), and taking there \( k = K_1 c e^{-2 \delta^{\delta/2} \log(1/p)} \).

With \( N \) the cardinality of the relevant net in [15, Prop. 3.4], we have as in [15, (3.11)] that \( \log N \lesssim K_1 \delta^{-2} n \log n \log(1/p) = O(K_1 K_0^{-1} r_{n,p}) \) (with the latter equality due to our assumption (3.2)), yielding, as claimed, the RHs of (3.3).

Our second quantitative large deviation result is stated in terms of a variant of the cut norm introduced in [16] (a special case of a general class of norms for the study of p-sparse
Part (a) is the specialization of the Proof. where we write \( \delta < \xi/2 \) any \( \kappa \). We have following results from Proposition 3.3 (Large deviations: B*-norm).

(a) (Upper-LDP). For a sufficiently large absolute constant \( C_0 > 0 \), let \( K_0 \geq C_0 \), \( K_1 \geq 1 \), and \( \delta \in (0, 1) \), and assume

\[
np^{\Delta+1} \geq \frac{K_0 \log n}{\delta^2(1 \vee \log(1/p))}.
\]

For each \( G \in \mathcal{G}_n \) let \( \mathcal{K}_G^\ast(\delta) \) be the convex hull of \( \{ G' \in \mathcal{G}_n : \| G - G' \|_{\mathcal{B}}^\ast \leq \delta p \} \). Then \( \nu_{n, p}^0 \) satisfies an upper-LDP with respect to \( \mathcal{K}_* \) \( \mathcal{K}_*^\ast(\delta) = (\mathcal{K}_G^\ast(\delta))_{G \in \mathcal{G}_n} \) with rate function \( I_p \) and \( r_\ast, r_{ME} \) satisfying (3.3).

(b) (Counting lemma). Let \( p \in (0, 1) \), \( L \geq 1 \) and \( \varepsilon > 0 \) be arbitrary. Suppose \( \mathcal{E} \subseteq \mathcal{G}_n \) has diameter at most \( \varepsilon p \) in the B*-norm, and that there exists \( G_0 \in \mathcal{E} \) such that

\[
t(F', G_0/p) \leq L
\]

for every proper subgraph \( F' \subseteq F \). Then for every \( Q_1, Q_2 \) in the convex hull of \( \mathcal{E} \),

\[
|t(F, Q_1/p) - t(F, Q_2/p)| \lesssim F \varepsilon.
\]

Proof. Part (a) is the specialization of [16, Theorem 3.1(a)] for the case of graphs (2-uniform hypergraphs) with the B*-norm (3.6) and growth parameter \( w_{n, p}(\mathcal{B}) = np^{\Delta+1} \), see [16, Exmp. 2.6 & (2.16)], and taking \( \kappa = K_1 p^\Delta \). Part (b) is [16, Theorem 2.10].

3.2. Proof of Theorem 1.6. First take \( s = 0 \). Since \( \text{Opt}(\phi, 0) = \{(0, 0)\} \) and \( \mathcal{G}_n^1(0, 0, \xi) = \mathcal{G}_n \) for any \( \xi > 0 \), part (a) then trivially holds. For part (b), note that \( \mathcal{Q}_n(0, 0) = \{ p \} \), where we abusively write \( p \) for the element of \( \mathcal{Q}_n \) that is identically \( p \) off the diagonal. Hence for any \( \delta < \xi/2 \),

\[
(\mathcal{G}_n^2(0, 0, \xi)^c)_{\mathcal{E}2\rightarrow2}(\delta) \subset \{ Q \in \mathcal{Q}_n : \| Q - p \|_{2\rightarrow2} \geq \frac{1}{2} \xi np^{\Delta/2} \}.
\]

For any element \( Q \) of the RHS we have

\[
\xi^2 n^2 p^\Delta \lesssim \| Q - p \|_{2\rightarrow2}^2 \leq \| Q - p \|_{HS}^2 = 2 \sum_{i<j} (Q_{ij} - p)^2
\]

where we write \( \| X \|_{HS} = (\sum_{i,j=1}^n X_{i,j}^2)^{1/2} \) for the Hilbert–Schmidt norm on \( \mathbb{R} \)-weighted graphs (viewed as kernels of operators on \( l^2([n]) \)). Further, the RHS of (3.11) is \( \lesssim I_p(Q)/\log(1/p) \) (as \( J_p(x) \geq x^2 \), see Lemma 2.8 or [39]). Consequently,

\[
I_p((\mathcal{G}_n^2(0, 0, \xi)^c)_{\mathcal{E}2\rightarrow2}(\delta)) \gtrsim \xi^2 r_{n, p}.
\]
Taking $\varepsilon_1 = c\xi^2$ for a sufficiently small constant $c > 0$, from (3.12) and (3.13) we deduce that
\[
\log \mathbb{P}(G \in G^2_n(0,0,\xi^c)) \leq - \min \left\{ r_{n,p}, \ I_p \left( \left( G^2_n(0,0,\xi^c) \right) |_{K_{2^{\alpha_2}(\delta)}} \right) - \varepsilon_1 r_{n,p} \right\}. \tag{3.13}
\]
for all $n$ sufficiently large and some absolute constant $c' > 0$. Since $\mathbb{P}(G \in U_p(E,0)) \gtrsim 1$, the claim follows.

Now take $s \neq 0$. We may assume WLOG that $s_k > 0$ for each $k$. Indeed, otherwise consider the restriction $(E^+, s^+)$ of $E, s$ to the indices with $s_k > 0$, denoting by $E_0$ the complementary part of $E$. Then, setting
\[
E_u := \bigcup_{(a,b) \in \text{Opt}(\phi_\xi)} G^a_n(a,b,\xi), \quad u = 1, 2
\]
we have that
\[
\mathbb{P}(G \notin E_u | G \in U_p(E, s)) = \frac{\mathbb{P}(G \in E_u \cap U_p(E, s))}{\mathbb{P}(G \in U_p(E, s))} \leq \frac{\mathbb{P}(G \in E_u \cap U_p(E^+, s^+))}{\mathbb{P}(G \in U_p(E^+, s^+)) \mathbb{P}(G \in U_p(E_0, 0))}
\]
where in the numerator we used that $U_p(E, s) \subseteq U_p(E^+, s^+)$, and in the denominator we applied the FKG inequality. Clearly $\mathbb{P}(G \in U_p(E_0, 0)) \gtrsim 1$, so as claimed, it suffices to fix hereafter $E = E^+$ and $s$ with $s_k > 0$ for all $k$.

Now, from [16, Lemma 7.2] and the union bound, there exists $L = L(E, s)$ finite such that for all $n$ large enough,
\[
\mathbb{P}(G \in L_{\leq}(E, L)) \geq 1 - e^{-(\phi_E(s)+1)r_{n,p}}, \tag{3.15}
\]
where
\[
L_{\leq}(E, L) := \bigcap_{k \in [n]} \bigcap_{F \subseteq F_k} \{ Q \in \mathcal{Q}_n : t(F, Q/p) \leq L \}.
\]
Fixing $\xi > 0$ and such $L$, for $u = 1, 2$ consider the sets
\[
E'_u = E'_u(s, \xi) = G_n \cap L_{\leq}(E, L) \cap U_p(E, s) \cap E^c
\]
with $E_u$ as in (3.14). In view of (3.15), it suffices to show that for $u = 1, 2$,
\[
\limsup_{n \to \infty} r_{n,p}^{-1} \log \mathbb{P}(G \in E'_u) < -\phi_E(s). \tag{3.17}
\]

We first establish (3.17) for the case $u = 2$, which gives part (b) of the theorem. Setting $\delta := \varepsilon_1 p^{\Delta_1 - \Delta/2}$ as before, the main step is to show
\[
I_p((E'_2)_{K_{2^{\alpha_2}(\delta)}}) > (\phi_E(s) + \eta)r_{n,p}. \tag{3.18}
\]
for $\varepsilon_1, \eta > 0$ sufficiently small depending on $E, s, \xi$. Indeed, granted (3.18), we have from Proposition 3.2(a) applied with $K_1 = C(1 + \phi_E(s))$ and $K_0 = CK_1/\eta^2$ for a sufficiently large constant $C < \infty$, that for all $n$ sufficiently large,
\[
\log \mathbb{P}(G \in E'_2) \leq - \min \left\{ (\phi_E(s) + 1)r_{n,p}, \ I_p((E'_2)_{K_{2^{\alpha_2}(\delta)}}) - \frac{1}{2} \eta r_{n,p} \right\} \leq - (\phi_E(s) + \frac{1}{2} \eta)r_{n,p}
\]
and (3.17) follows.
Turning to the proof of (3.18), let \( \mathcal{P} \) be the partition of \([0, 1]\) into intervals of length \(1/n\), such that \( \mathcal{Q}_n \) embeds in \( \mathcal{W}_\mathcal{P} \) via

\[
Q \mapsto g_Q, \quad g_Q(x, y) = Q_{\lfloor xn \rfloor, \lfloor yn \rfloor}.
\]  

(3.19)

For \( \varepsilon > 0 \) let

\[
\mathcal{Q}_n'(s, \varepsilon) = \bigcup_{(a, b) \in \text{Opt}(\phi; s)} \bigcup_{\|(a', b') - (a, b)\|_\infty \leq \varepsilon} \mathcal{Q}_n(a', b')
\]

(3.20)

and let \( \mathcal{W}'(s, \varepsilon) \subset \mathcal{W}_\mathcal{P} \) denote the corresponding set of graphons via the identification (3.19). Taking hereafter \( \varepsilon_1 < \xi/2, \) since \( \Delta_* \geq \Delta/2, \) for any \( Q \in (\mathcal{E}_2')_{K_{2 \rightarrow 2}(\delta)} \) there exists \( G \in \mathcal{E}_2' \) with

\[
\|Q - G\|_{2 \rightarrow 2} \leq \varepsilon_1 n p^{\Delta_\ast} \leq \frac{1}{2} \xi n p^{\Delta/2}.
\]

As \( G \) does not lie in the set \( \mathcal{E}_u \) from (3.14), it then follows by the triangle inequality for the spectral norm that

\[
\|Q - Q_{I,J}\|_{2 \rightarrow 2} \geq \frac{1}{2} \xi n p^{\Delta/2} \quad \forall Q_{I,J} \in \mathcal{Q}_n'(s, 0)
\]

and consequently, that for any \( Q \in (\mathcal{E}_2')_{K_{2 \rightarrow 2}(\delta)} \)

\[
\|g_Q - g_{Q_{I,J}}\|_2 = \frac{1}{n} \|Q - Q_{I,J}\|_{HS} \geq \frac{1}{n} \|Q - Q_{I,J}\|_{2 \rightarrow 2} \geq \frac{1}{2} \xi n p^{\Delta/2} \quad \forall Q_{I,J} \in \mathcal{Q}_n'(s, 0).
\]

(3.21)

Further, as seen in the proof of (4.17) below, for \( c = \frac{1}{32} \) and any \( g \in \mathcal{W}'(s, c\xi^2) \) there exists \( Q_{I,J} \in \mathcal{Q}_n'(s, 0) \) such that

\[
\|g - g_{Q_{I,J}}\|_2 \leq \frac{1}{4} \xi n p^{\Delta/2}.
\]

Hence, by the triangle inequality, for any \( Q \) satisfying (3.21),

\[
\|g_Q - g\|_2 \geq \frac{1}{4} \xi n p^{\Delta/2} \quad \forall g \in \mathcal{W}'(s, c\xi^2).
\]

(3.22)

Next, for \( Q \in (\mathcal{E}_2')_{K_{2 \rightarrow 2}(\delta)} \) and the corresponding \( G \in \mathcal{E}_2' \) we have from Proposition 3.2(b), that for some finite \( C = C(\mathcal{E}, s) \geq 2 \)

\[
t(F_k, Q/p) \geq t(F_k, G/p) - O_E(L\varepsilon_1) \geq 1 + s_k - C\varepsilon_1, \quad \forall k \in [m].
\]

(3.23)

Choosing \( \varepsilon_1 < \eta/C \) and \( \eta > 0 \) small enough so that \( \eta^{c_0} < c_1 \min(c\xi^2, \xi/4) \), for the positive \( c_0 = c_0(\mathcal{E}) \) of Proposition 2.2 and some \( c_1 = c_1(\mathcal{E}, s) > 0 \) sufficiently small, it follows from (3.22), (3.23) and Proposition 2.2 (in the contrapositive), that

\[
\frac{1}{n^2} I_p(Q) = \frac{1}{2} \int I_p \circ g_Q > (\phi_{\mathcal{E}, s}(\xi) + \eta)p^\Delta \log(1/p)
\]

for all \( Q \in (\mathcal{E}_2')_{K_{2 \rightarrow 2}(\delta)} \). Having thus obtained (3.18) and thereby (3.17) for \( u = 2 \), this concludes the proof of part (b) of Theorem 1.6.

Now turn to prove part (a), namely (3.17) for \( u = 1 \). Since \( \mathcal{G}_n(a, b, \xi) = \mathcal{G}_n \) if \( \max(\mathcal{F}, b) \leq \xi \), we may assume that \( \max(\mathcal{F}, b) > \xi \) for every \( (a, b) \in \text{Opt}(\phi; s) \). The main step is to show

\[
I_p((\mathcal{E}_1')_{K_1(\delta_1)}) > (\phi_{\mathcal{E}, s}(\xi) + \eta)r_{n,p}
\]

(3.24)

for some \( \delta_1, \eta > 0 \) sufficiently small depending on \( \mathcal{E}, s, \xi \). Indeed, since \( np^{\Delta+1} \log(1/p) \gg \log n \), we can then apply Proposition 3.3(a) to conclude exactly as we did for part (b).
Now for any \( Q \in K^*_G(\delta_1) \) and \( G \in \mathcal{E}'_1 \) we get by Proposition 3.3(b) that (3.23) holds with \( \delta_1 \) replacing \( \varepsilon_1 \). Thus, following the preceding proof of part (b), we will arrive at (3.24) upon showing that analogously with (3.21) we have that
\[
\|Q - Q^{I,J}\|_{\text{HS}}^2 \geq \frac{\xi}{2} n^2 p^\Delta \quad \forall Q^{I,J} \in \mathcal{Q}_n(\varepsilon,0) .
\] (3.25)

Fixing \( Q^{I,J} \in \mathcal{Q}_n(\varepsilon,0) \), note that \( \|Q - G\|_{\text{HS}}^2 \leq \delta p \) for some \( G \in \mathcal{E}'_1 \), so in particular
\[
|\sum_{i \in I} (Q_{ij} - G_{ij})| \leq \delta p \max(|I|, np^{\Delta-1})^2 \leq (a \lor 1)\delta n^2 p^{\Delta+1} \leq \xi n^2 p^\Delta
\]
for all \( n \) large enough (as \( a \leq 2\phi_F(\varepsilon) \), \( \delta \leq 1 \) and \( p \ll 1 \)). As \( G \notin \mathcal{G}^{I,J}_n(\xi) \), it thus follows that
\[
2\xi n^2 p^\Delta \leq \sum_{i \in I} (1 - G_{ij}) \leq \xi n^2 p^\Delta + \sum_{i \in I} (1 - Q_{ij}) .
\]
Consequently, by Cauchy–Schwarz,
\[
(\xi n^2 p^\Delta)^2 \leq \left[ \sum_{i \in I} (1 - Q_{ij}) \right]^2 \leq |I|^2 \left[ \sum_{i \in I} (1 - Q_{ij})^2 \right] \leq an^2 p^\Delta \left[ \sum_{i \in I} (1 - Q_{ij})^2 \right] ,
\]
yielding the lower bound
\[
\|Q - Q^{I,J}\|_{\text{HS}}^2 \geq \sum_{i \in I} (1 - Q_{ij})^2 \geq \frac{\xi^2}{a} n^2 p^\Delta .
\] (3.26)

Arguing as above on \( J \times [n] \setminus J \), we similarly get for any \( \delta \leq \xi/4 \) and all \( n \) large enough
\[
|\sum_{J \times [n] \setminus J} (Q_{ij} - G_{ij})| \leq \delta p n(|J| + np^{\Delta-1}) \leq \frac{\xi}{2} n^2 p^\Delta
\]
(as \( b \leq \phi_F(\varepsilon) \) and \( p \ll 1 \)). Since \( G \notin \mathcal{G}^{I,J}_n(\xi) \) it then follows that
\[
\left( \frac{\xi}{2} n^2 p^\Delta \right)^2 \leq \left[ \sum_{J \times [n] \setminus J} (1 - Q_{ij}) \right]^2 \leq n|J| \left[ \sum_{J \times [n] \setminus J} (1 - Q_{ij})^2 \right] \leq \frac{b}{2} n^2 p^\Delta \|Q - Q^{I,J}\|_{\text{HS}} .
\]
Combining this with (3.26) we find that
\[
\|Q - Q^{I,J}\|_{\text{HS}}^2 \geq \frac{\xi^2}{2b \lor a} n^2 p^\Delta \geq \frac{\xi}{2} n^2 p^\Delta
\]
(by our assumption that \( \frac{\xi}{2} \lor b > \xi \)). With (3.25) established, this concludes the proof. \( \square \)

4. Proofs of results for ERGMs

In this section we establish Proposition 1.2, Theorem 1.4 and Theorem 1.5. For the last result we rely on Theorem 1.6, which we already proved in Section 3.

We may assume wlog that \( h(1) = 0 \). Hereafter, \( r = r_{n,p} \) of (1.11). Recalling (1.21), as in Section 2 we abbreviate \( T_k := T_{F_k}, k \leq m \), and also re-index these \( m \) graphs so that \( F_k \) is regular if and only if \( k \leq m' \) for some \( m' \in [m] \).
4.1. Proof of Proposition 1.2. We begin with a lemma showing that the limited growth of $h(\cdot)$ allows for truncating the tails of $H(G_{n,p}/p)$.

**Lemma 4.1.** Assume $1 \geq p \gg n^{-1/(\Delta+1)}$. Then, for $h(\cdot)$ satisfying (1.10), and $H(\cdot)$ of (1.9),

$$\limsup_{L \to \infty} \limsup_{n \to \infty} r^{-1} \log \mathbb{E} \left[ e^{rH(G_{n,p}/p)} \sum_{k=1}^{m} \mathbb{1}(t(F_k, G_{n,p}/p) \geq L) \right] = -\infty. \tag{4.1}$$

**Proof.** Since $t(F_k, G_{n,p}/p) \leq p^{-e(F_k)}$ are uniformly bounded when $p \geq 1$ we may assume hereafter WLOG that $p \ll 1$. Further, since $r = r_{n,p} \to \infty$ for the range of $p(n)$ we consider, in view of (1.9) and (1.10) it suffices for (4.1) to show that for some $\eta > 0$ and any $k \leq m$,

$$\limsup_{n \to \infty} r^{-1} \log \mathbb{E} \left[ \exp(\eta r t(F_k, G_{n,p}/p)^{\Delta/e(F_k)}) \right] < \infty$$

(combine [17, Lemma 4.3.8] with Hölder’s inequality). This in turn follows from the uniform large deviations upper bound

$$\limsup_{u, n \to \infty} \frac{1}{u^{\Delta/e(F_k)} r_{n,p}} \log \mathbb{P}(t(F_k, G_{n,p}/p) \geq 1 + u) < 0 \tag{4.2}$$

which we derive in Proposition C.1 for $1 \gg p \gg n^{-1/(\Delta+1)}$. \hfill \Box

We turn to the proof of Proposition 1.2, starting with the lower bound $\Lambda_{n,p}^{H} \geq \Psi_{n,p}^{H} - o(r_{n,p})$. To this end, fixing positive integers $K, \zeta^{-1}$, we consider the finite $\zeta$-mesh $J_\zeta = (\{\zeta, 2\zeta, \ldots, K\})^m$ of $[0, K]^m$. By the monotonicity of $x \mapsto h(x)$, we have that for any $s \geq \zeta 1$

$$\Lambda_{n,p} \geq r \cdot h((1 - \zeta)1 + s) + \log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F, s - \zeta 1)). \tag{4.3}$$

Next, recall the entropic optimization problem for joint upper tails of homomorphism counts

$$\Phi_{n,p}(F, s) := \mathbb{1}_{\mathcal{U}_p(F, s)}, \tag{4.4}$$

in terms of $\mathcal{U}_p(F, s)$ of (1.35) (here we use the notation (1.51)). From [16, Prop. 9.1] we have that for all $n$ large enough,

$$\inf_{s \in J_\zeta} \left\{ \log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F, s') - \zeta 1)) + (1 + \zeta)\Phi_{n,p}(F, s') \right\} \geq -o(r_{n,p}). \tag{4.5}$$

Since $h(\cdot)$ is uniformly continuous on $[0, K]^m$, for any $\eta > 0$ and $\zeta = \zeta(\eta)$ small enough,

$$h(1 + s) \geq (1 + \zeta)h(1 + 2\zeta)1 + s) - \eta, \quad \forall s \in [0, K]^m. \tag{4.6}$$

Setting for $K$ finite, the non-negative

$$\Psi_{n,p}^{(K)} := \sup_{s \in \mathbb{R}_+^n, \|s\| \leq K} \left\{ r_{n,p}h(1 + s) - \Phi_{n,p}(F, s) \right\}, \tag{4.7}$$

we get by (4.3)–(4.6) and the monotonicity of $\Phi_{n,p}(F, \cdot)$ that for any $\eta > 0$ and such $\zeta = \zeta(\eta)$,

$$\Lambda_{n,p}^{H} \geq \sup_{s \in J_\zeta, s \in [s_k, \infty]} \left\{ r_{n,p}h((1 - \zeta)1 + s') - (1 + \zeta)\Phi_{n,p}(F, s') \right\} - o(r_{n,p})$$

$$\geq (1 + \zeta) \sup_{s \in \mathbb{R}_+^n, \|s\| \leq K} \left\{ r_{n,p} \sup_{s' \in J_\zeta, s_k \leq s_k} \left\{ h((1 + \zeta)1 + s') - \Phi_{n,p}(F, s) \right\} - \eta r_{n,p} - o(r_{n,p}) \right\}$$

$$\geq \Psi_{n,p}^{(K)} - \eta r_{n,p} - o(r_{n,p}). \tag{4.8}$$
Hereafter, let \((\tilde{g})_+\) denote the projection of \(g \in \mathbb{R}^m\) onto \(\mathbb{R}_{\geq 0}^m\). With \(h(\cdot)\) non-decreasing in each argument and \(\Phi_{n,p}(F, \tilde{g}) = \Phi_{n,p}(F, (\tilde{g})_+)\), decomposing the supremum in (1.18) according to \(s_k := t(F_k, Q/p) - 1\), yields that
\[
\Phi_{n,p}(F, \tilde{g}) = \sup_{\tilde{g} \in \mathbb{R}_{\geq 0}^m} \left\{ r_{n,p} h(1 + \tilde{g}) - \Phi_{n,p}(F, \tilde{g}) \right\}.
\]
By definition, see (4.4),
\[
\Phi_{n,p}(F, \tilde{g}) \geq \Phi_{n,p}(F_k, s_k), \quad \forall k \in [m].
\]
Recalling (1.10), we have from [16, Lemma 7.2], that for some \(\eta = \eta(F) > 0\),
\[
\Phi_{n,p}(F_k, s_k) \geq \eta m r_{n,p} (s^{\Delta/e(F_k)} - 1), \quad \forall s \geq 0, k \in [m].
\]
Consequently, for any \(\tilde{g} \in \mathbb{R}_{\geq 0}^m\),
\[
\Phi_{n,p}(F, \tilde{g}) \geq \eta r_{n,p} \left( \sum_{k=1}^{m} s_k^{\Delta/e(F_k)} - m \right).
\]
In view of the growth condition (1.10) for \(h(\cdot)\), the above bound implies that the supremum in (4.9) is attained at \(\tilde{g}_s\) uniformly bounded in \(n\), thereby matching (for some fixed \(K\) and all \(n\) large), the expression \(\Psi_{n,p}^{(K)}\) of (4.7). Having \(\eta\) in (4.8) arbitrarily small thus yields the lower bound \(\Lambda_{n,p}^H \geq \Psi_{n,p}^H - o(r_{n,p})\).

Turning to prove the upper bound \(\Lambda_{n,p}^H \leq \Psi_{n,p}^H + o(r_{n,p})\), in view of Lemma 4.1 it suffices to show that for \(r = r_{n,p}\), any \(K \in \mathbb{N}, \eta > 0\) and all \(n\) sufficiently large (depending on \(F, h, K\) and \(\eta\)),
\[
\log \mathbb{E} \left[ e^{rH(G_{n,p}/p)} \mathbb{1}(\max_{k \leq m} t(F_k, G_{n,p}/p) \leq K) \right] \leq \Psi_{n,p}^H + 2\eta r.
\]
Fixing \(K, \zeta^{-1} \in \mathbb{N}\) let \(J^*_\zeta\) denote the finite \(\zeta\)-mesh analogous to \(J^*_\zeta\), except for including now also \(-\zeta\) and 0 as possible values for each coordinate \(s_k\). We apply the upper bound of [16, Thm. 1.1] at any \(\tilde{g} \in J^*_\zeta\) for the restriction \((F^+, \tilde{g}^+)\) of \((F, \tilde{g})\) to those \(k \in [m]\) where \(s_k > 0\). Doing so, the monotonicity of \(h(\cdot)\) yields the following bound on the LHS of (4.11):
\[
\log \left[ \sum_{\tilde{g} \in J^*_\zeta} e^{r h((1 + \zeta)^{1 + \tilde{g}})} \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F^+, \tilde{g}^+)) \right]
\leq \log |J^*_\zeta| + \max_{\tilde{g} \in J^*_\zeta} \left\{ r h((1 + \zeta) 1 + \tilde{g}) + \log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F^+, \tilde{g}^+)) \right\}
\leq o(r) + \max_{\tilde{g} \in J^*_\zeta} \left\{ r h((1 + \zeta) 1 + \tilde{g}) - (1 + \zeta)^{-1}\Phi_{n,p}(F^+, \tilde{g}^+ - 1) \right\}.
\]
For \(\tilde{g} \in J^*_\zeta\), the vector \(\tilde{g}^+ - \zeta 1\) consists of all the non-negative coordinates of \(\tilde{g} - \zeta 1\), from which it follows that \(\Phi_{n,p}(F^+, \tilde{g}^+ - 1) = \Phi_{n,p}(F, (\tilde{g} - \zeta 1)_+)\). Plugging this in (4.12) and considering \(\zeta\) small enough that (4.6) holds, yields (4.11) and thereby completes the proof. \(\square\)

4.2. Proof of Theorem 1.4. From Proposition 1.2 it suffices to show that
\[
\frac{1}{r_{n,p}} \Psi_{n,p}^H \rightarrow \psi_{F,h}.
\]
Recall from [6, Prop. 1.10] that for any \(n^{-1/\Delta} \ll p \ll 1\),
\[
\lim_{n \rightarrow \infty} \frac{\Phi_{n,p}(F, \tilde{g})}{r_{n,p}} = \phi_{F}(\tilde{g}), \quad \forall \tilde{g} \in \mathbb{R}_{\geq 0}^m.
\]
Considering (4.9), (1.42) and (4.14), it thus remains only to show that

$$\limsup_{n \to \infty} \sup_{s \in \mathbb{R}_{\geq 0}^2} \left\{ h(1 + s) - \frac{\Phi_{n,p}(F_{n,s})}{r_{n,p}} \right\} \leq \psi_{F,h} = \sup_{s \in \mathbb{R}_{\geq 0}^2} \left\{ h(1 + s) - \phi_F(s) \right\}. \quad (4.15)$$

In the course of proving Proposition 1.2 we saw that the supremum on the LHS is attained at some $s_n$, which are uniformly bounded in $n$. Hence, by the continuity of $h$ we obtain (4.15) and thereby complete the proof, upon showing that for any sequence $s_n \to s_\infty$

$$\liminf_{n \to \infty} \left\{ \frac{\Phi_{n,p}(F_{n,s_n})}{r_{n,p}} \right\} \geq \phi_F(s_\infty).$$

Now, by the monotonicity of $s \mapsto \Phi_{n,p}(F_{n,s})$ we can replace $s_n$ with $(1 - \varepsilon)s_\infty$. Having done so, we use (4.14) and conclude upon taking $\varepsilon \to 0$ (relying on the continuity of $\phi_F(\cdot)$).

### 4.3. Proof of Theorem 1.5.

We proceed to prove Theorem 1.5 using Theorem 1.6, by combining the argument used when proving Proposition 1.2 with the following containment property.

**Lemma 4.2.** For any $\xi > 0$ and $K$ finite there exist $\varepsilon' > 0$ and $n_0 < \infty$, such that

$$\bigcup_{|a - a'| \leq \varepsilon', |b - b'| \leq \varepsilon'} G_n^u(a, b, \xi) \subset G_n^u(a', b', 2\xi), \quad \forall n \geq n_0, \, \|(a', b')\|_\infty \leq K. \quad (4.16)$$

**Proof.** For both $u = 1$ and $u = 2$, we can and shall assume WLOG that $I = [1, |I|]$ and $J = [n - |J|, n]$ are disjoint intervals, setting the corresponding object to have $I' = [1, |I'|]$ and $J' = [n - |J'|, n]$.

In case $u = 2$, our claim (4.16) follows in view of (1.28) and the triangle inequality, from

$$\left\| Q - Q' \right\|_{2 \to 2} \leq \xi np^{\Delta/2}. \quad (4.17)$$

Indeed, with $|Q_{ij} - Q'_{ij}| \leq 1$ for all $ij$, $Q = Q^{I,J}$ and $Q' = Q^{I',J'}$, clearly $\left\| Q - Q' \right\|_{2 \to 2} \leq \left\| Z \right\|_{2 \to 2}$, where $Z_{ij} = 1_{\{Q_{ij} \neq Q'_{ij}\}}$. It is easy to check that $Z$ consists here of a reversed-$L$ shape of dimensions $\max(|I|, |I'|) \times |I| - |I'|$ and a disjoint cross-with-hole shape of dimensions $|J| - |J'| \times n$. With the $2 \to 2$ operator norm of each part of $Z$ thus bounded by the geometric mean of its two dimensions, we arrive at

$$\left\| Q - Q' \right\|_{2 \to 2} \leq (\sqrt{|a - a'|} + \sqrt{|b - b'|})np^{\Delta/2} \leq 2\varepsilon' np^{\Delta/2} \leq 2\varepsilon np^{\Delta/2},$$

from which we get (4.17) when $\varepsilon' \leq \xi^2/4$.

When $u = 1$ it suffices for (4.16) to show that $G_n^{I,J}(\xi) \subset G_n^{I',J'}(2\xi)$. That is, having at least $|I|^2/2 - \xi n^2p^\Delta$ edges in $G[I]$ and at least $|J|(n - |J|) - \xi n^2p^\Delta$ edges in $G[J, J']$ yields at least $|I'|^2/2 - 2\xi n^2p^\Delta$ edges in $G[I']$ and at least $|J'|(n - |J'|) - 2\xi n^2p^\Delta$ edges in $G[J', J'']$. This holds universally for any graph $G$, provided that

$$|I|^2 - (|I|^2 - |I'|^2)_+ \geq |I'|^2 - \xi n^2p^\Delta.$$

$$|J|(n - |J|) - (|J| - |J'|)_+(n - |J|) \geq |J'|(n - |J'|) - \xi n^2p^\Delta.$$

With both $|J|, |J'|$ being $o(n)$ (since $p \ll 1$), we arrive at the simpler sufficient conditions

$$\xi > (a' - a)_+ \quad \text{and} \quad \xi > (b' - b)_+. \quad (4.18)$$

That is, any $\varepsilon' < \xi$ will do to complete the proof. \(\square\)
By Theorem 1.4, parts (a) and (b) of Theorem 1.5 amount to having for \( u = 1, 2 \), respectively, and for any fixed \( \xi > 0 \),
\[
\limsup_{n \to \infty} r^{-1} \log \mathbb{E} \left[ e^{r \mathbf{H}(G_{n,p})} \mathbf{1}(\Gamma_n^u(\xi)) \right] < \psi_{F,h},
\]
where
\[
\Gamma_n^u(\xi) := \bigcap_{(a',b') \in \text{Opt}(\psi)} (G_n^{u}(a', b', 2\xi))^c, \quad u = 1, 2.
\]

Proceeding as in the proof of (4.11), while intersecting all the events there with \( \Gamma_n^u(\xi) \), we find analogously to (4.12), that for any fixed \( \xi > 0 \) and \( \eta > 0 \), the LHS of (4.18) is at most
\[
2\eta + \max_{\xi \in \mathcal{J}^*} \left\{ h(1+(\mathbf{s})_+) - \phi_{F,h}(\mathbf{s}^+, \xi) \right\},
\]
for some \( \zeta = \zeta(\eta) > 0 \), where
\[
\phi_{F,h}(\mathbf{s}, \xi) := -\limsup_{n \to \infty} r^{-1} \log \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F,h) \cap \Gamma_n(\xi)).
\]

Fixing hereafter \( \xi > 0 \), upon taking \( \eta \to 0 \) we get (4.18) provided that the function
\[
f_\xi(\mathbf{s}) := \psi_{F,h} + \phi_{F,h}(\mathbf{s}^+, \xi) - h(1+\mathbf{s})
\]
is bounded away from zero over \( \mathbf{s} \in \mathbb{R}^m_{\geq 0} \). To this end, consider the continuous mapping \( T : \mathbb{R}^2 \to \mathbb{R}^m_{\geq 0} \) given in terms of \( T_k = T_{F_k} \) of (1.21), by
\[
T(a, b) := \{T_1(a, b) - 1, T_2(a, b) - 1, \ldots, T_m(a, b) - 1\}.
\]

With \( h \) non-decreasing in each argument, recall from (1.37) that for any \( (a, b) \in \text{Opt}(\psi; \mathbf{s}) \),
\[
f_0(\mathbf{s}) = \psi_{F,h} + \frac{1}{2}a + b - h(1+\mathbf{s}) \geq \psi_{F,h} + \frac{1}{2}a + b - h(1+T(a, b)).
\]

As seen while proving Proposition 1.2, thanks to the growth condition (1.10), the non-negative continuous function on the RHS of (4.21) diverges when \( \|(a, b)\| \to \infty \), and it is therefore bounded away from zero on the complement of any small neighborhood of its bounded set \( \text{Opt}(\psi) \) of global minimizers. Consequently, for any \( \varepsilon > 0 \), the function \( f_0(\cdot) \) is bounded away from zero on the complement of
\[
\mathbb{B}_\varepsilon := \{ \mathbf{s} \in \mathbb{R}^m_{\geq 0} : \max_{(a,b) \in \text{Opt}(\psi; \mathbf{s})} \min_{(a',b') \in \text{Opt}(\psi)} \{|a-a'| + |b-b'|\} \leq \varepsilon \}.
\]

It follows from (1.37) that \( \phi_{F,h}(\mathbf{s}) = \phi_{F,h}(\mathbf{s}^+) \) for any \( \mathbf{s} \in \mathbb{R}^m_{\geq 0} \), with the same set of optimal \( (a, b) \) in both variational problems. Hence, \( f_\xi(\cdot) \geq f_0(\cdot) \) and we shall complete the proof upon showing that
\[
\liminf_{\varepsilon \downarrow 0} \{ f_\xi(\mathbf{s}) \} > 0.
\]

Turning to this task, combining Theorem 1.6(a) with the upper bound of [16, Thm. 1.1] and (4.14), we have some \( \eta(\mathbf{s}) > 0 \), such that for \( u = 1 \),
\[
\limsup_{n \to \infty} r^{-1} \log \mathbb{P}(G_{n,p} \in \Gamma_n^u(F,h, \mathbf{s}^+, \xi)) \leq -\phi_{F,h}(\mathbf{s}^+) - 2\eta(\mathbf{s}), \quad \forall \mathbf{s} \in \mathbb{B}_1,
\]
where
\[
\Gamma_n^u(F,h, \mathbf{s}, \xi) := \mathcal{U}_p(F,h, \mathbf{s}) \cap \bigcap_{(a,b) \in \text{Opt}(\psi; \mathbf{s})} (G_n^{u}(a, b, \xi))^c, \quad u = 1, 2.
\]
We similarly get in the setting of Theorem 1.6(b), that (4.24) holds with \( u = 2 \). Now, parsing the definitions (4.19), (4.22) and (4.25), we deduce from Lemma 4.2 that for some \( \varepsilon = \varepsilon(\xi) > 0 \) small enough, if \( s \in B_\varepsilon \) then for all \( n \) large enough,

\[
U_n(\xi, \xi^+) \cap \Gamma_n(\xi) \subset \Gamma_n(\xi, \xi^+),
\]

Comparing (4.20) with (4.24), the preceding containment implies that throughout \( B_\varepsilon \),

\[
f_\xi(\mathbf{z}) \geq \psi_{\mathbf{z}, h} + \phi_{\mathbf{z}}(\mathbf{z}) + 2\eta(\mathbf{z}) - h(1 + \mathbf{z}),
\]

yielding by the continuity of \( \phi_{\mathbf{z}}(\cdot) - h(1 + \cdot) \), that for some \( \zeta(\mathbf{z}) > 0 \) and any \( \mathbf{z} \in B_\varepsilon \),

\[
\inf_{\|\mathbf{z} - \mathbf{z'}\|_\infty < \zeta(\mathbf{z})} \{f_\xi(\mathbf{z}')\} \geq \eta(\mathbf{z}) > 0.
\]

Since \( T(a, b) \geq \mathbf{z} \) coordinate-wise for any \( (a, b) \in \text{Opt}(\phi; \mathbf{z}) \), we deduce from the boundedness \( \text{Opt}(\psi) \) that \( B_\varepsilon \) is a bounded, hence pre-compact subset of \( \mathbb{R}_{\geq 0}^m \). Apply (4.26) for a finite cover of \( B_\varepsilon \) by \( \| \cdot \|_\infty \)-balls of centers \( \mathbf{z} \) and radii \( \zeta(\mathbf{z}) \), to arrive at (4.23), thus completing the proof. \( \square \)

5. **Edge-F models: Proofs of Proposition 1.10 and Corollary 1.12**

To lighten notation, we suppress the dependence of \( \phi_F, \nu(F), \psi(F), \psi_F, \psi_{F, b} \) on \( F \) throughout this section and write \( \psi(\beta) := \psi(\beta) \). Thus,

\[
\text{Opt}(\psi) = \{(a, b) : \beta f(T_F(a, b)) - \frac{1}{2} a - b = \psi(\beta)\}.
\]

From Theorem 1.5, to obtain Proposition 1.10 it suffices to prove the following.

**Proposition 5.1.** With hypotheses as in Proposition 1.10, if \( F \) is irregular, then \( \text{Opt}(\psi) = \{(0, b_*(\beta))\} \). If \( F \) is regular, then there exists \( \beta_c > 0 \) (depending only of \( F, f \)) such that for \( \beta < \beta_c \) we have \( \text{Opt}(\psi) = \{(0, b_*(\beta))\} \), while if \( \beta > \beta_c \), then \( \text{Opt}(\psi) = \{(a_*(\beta), 0)\} \).

For the proof we need two lemmas.

**Lemma 5.2.** Let \( g, h \) be real-valued functions on a closed (possibly infinite) interval \( I \), and assume \( g \) is strictly increasing. For \( \beta \in \mathbb{R} \) and \( s \in I \) let

\[
U(\beta, s) = \beta g(s) - h(s).
\]

Then for any \( \beta_1 < \beta_2 \) and any maximizers \( s_1, s_2 \in I \) for \( U(\beta_1, \cdot) \) and \( U(\beta_2, \cdot) \) respectively, we have \( s_2 \geq s_1 \).

**Proof.** Suppose toward a contradiction that \( s_2 < s_1 \). Then

\[
U(\beta_2, s_2) = \beta_1 g(s_2) - h(s_2) + (\beta_2 - \beta_1)g(s_2)
\leq \beta_1 g(s_1) - h(s_1) + (\beta_2 - \beta_1)g(s_2)
\leq \beta_1 g(s_1) - h(s_1) + (\beta_2 - \beta_1)g(s_1)
= \beta_2 g(s_1) - h(s_1) \leq U(\beta_2, s_2),
\]

a contradiction. \( \square \)

For the function \( U(\beta, s) \) of (1.45), we let \( S^*(\beta) \) denote the set of maximizers for \( U(\beta, \cdot) \) in \( \mathbb{R}_{\geq 0} \). From Proposition 1.8(c) we have that \( S^*(\beta) \neq \emptyset \).

**Lemma 5.3.** Assume \( F \) is \( \Delta \)-regular. Then for all \( \beta \geq 0 \) we have that \( s \notin S^*(\beta) \).
Proof. Since \( \beta f(1 + \cdot) \) is continuous and differentiable on \( \mathbb{R}_+ \) with \( \phi(s) \) the minimum of two differentiable functions that are equal only at \( s = 0 \) and at \( s = s_c > 0 \), it suffices to verify that

\[
\lim_{s \uparrow s_c} \phi'(s) > \lim_{s \downarrow s_c} \phi'(s).
\]

Indeed, it then follows that \( s_c \) cannot be a local maximum for \( \beta f(1 + \cdot) - \phi(\cdot) \). The above amounts to showing that

\[
\frac{1}{\nu} s_c^{2/\nu - 1} < \frac{1}{P_F'(\frac{1}{2} s_c^{2/\nu})}, \tag{5.2}
\]

where we denote \( b_0(s) := P_F^{-1}(1 + s) \). Writing \( P_F(b) = 1 + \nu b + R(b) \), where \( R(b) \) collects all terms of degree at least 2, we have for \( b > 0 \) that

\[
P_F'(b) = \nu + R'(b) = \nu + \sum_{U \subset I: |U| \geq 2} |U| b^{r-1} \leq \nu \left( 1 + \frac{R(b)}{2b} \right),
\]

where we applied the bound \( \epsilon/\Delta = \nu/2 \) on the size of an independent set in \( F \). Now since

\[
1 + s_c = P_F(b_0(s_c)) = 1 + \nu b_0(s_c) + R(b_0(s_c)),
\]

when combining with the previous display, we have that

\[
P_F'(b_0(s_c)) \leq \nu \left( 1 + s_c^{1-2/\nu} - \frac{\nu}{2} \right) < \nu s_c^{1-2/\nu}
\]

(using here that \( \nu > 2 \) since \( \Delta \geq 2 \)). This rearranges to give (5.2) and the claim follows. \( \square \)

Proof of Proposition 5.1. Write

\[
a_0(s) = s^{2/\nu}, \quad b_0(s) = P_F^{-1}(1 + s). \tag{5.3}
\]

It is shown in [5] that for \( F \) irregular and any \( s \geq 0 \), or \( F \) regular and \( s \in [0, s_c) \), we have

\[
\text{Opt}(\phi; s) = \{(0, b_0(s))\} \tag{5.4}
\]

while if \( F \) is regular and \( s > s_c \), then

\[
\text{Opt}(\phi; s) = \{(a_0(s), 0)\}. \tag{5.5}
\]

In the case that \( F \) is irregular the proposition then immediately follows from Proposition 1.8(d), with \( b_\star(\beta) = b_0(s_\star(\beta)) \).

Assume now that \( F \) is regular. Recalling the notation (1.45), from Proposition 1.8(d) and (5.4)–(5.5) is suffices to show there exists \( \beta_c = \beta_c > 0 \) such that

\[
\beta \in [0, \beta_c) \Rightarrow S_\star(\beta) = \{s_\star_{\text{hub}}(\beta)\} \subset (0, s_c)
\]

and

\[
\beta \in (\beta_c, \infty) \Rightarrow S_\star(\beta) = \{s_\star_{\text{clique}}(\beta)\} \subset (s_c, \infty). \tag{5.7}
\]

From Lemma 5.2 applied to \( U_{\text{hub}} \) (taking \( f(1 + \cdot) \) for \( g \) and \( P_F^{-1}(1 + \cdot) \) for \( h \), along with the assumption that \( s_{\text{hub}}^*(\beta) \) is the unique maximizer of \( U_{\text{hub}}(\beta, \cdot) \) for all \( \beta \geq 0 \), it follows that \( s_{\text{hub}}^* : \mathbb{R}_+ \to [0, s_c] \) is continuous and non-decreasing. Indeed, the monotonicity is a direct consequence of the lemma, and from this it follows that any point \( \beta_0 \) of discontinuity of \( s_{\text{hub}}^* \) must be a jump discontinuity. However, from the joint continuity of \( U_{\text{hub}}(\cdot, \cdot) \) we would then have that the left and right limits \( \lim_{\beta \downarrow \beta_0} s_{\text{hub}}^*(\beta) \) and \( \lim_{\beta \uparrow \beta_0} s_{\text{hub}}^*(\beta) \) would both be maximizers for \( U_{\text{hub}}(\beta_0, \cdot) \), which contradicts the uniqueness assumption. By the same reasoning we get that \( s_{\text{clique}}^* : \mathbb{R}_+ \to [s_c, \infty) \) is continuous and non-decreasing.
Clearly \( S^*(0) = \{0\} = \{s^*_{\text{hub}}(0)\}. \) Moreover, since the global maximum of \( U(\beta, \cdot) \) is at least the maximum over \([0, s_c]\) and \([s_c, \infty)\) respectively, we have

\[
S^*(\beta) \cap [0, s_c) \neq \emptyset \Rightarrow S^*(\beta) \cap [0, s_c) = \{s^*_{\text{hub}}(\beta)\}
\]

and similarly

\[
S^*(\beta) \cap (s_c, \infty) \neq \emptyset \Rightarrow S^*(\beta) \cap (s_c, \infty) = \{s^*_{\text{clique}}(\beta)\}
\]

We set

\[
\beta_c := \sup \{\beta \geq 0 : S^*(\beta) \cap [0, s_c) \neq \emptyset\}.
\]

Note that since \( \phi \) and its derivative are bounded on \([0, s_c]\), there exists \( B < \infty \) such that \( s^*_{\text{hub}}(\beta) = s_c \) for all \( \beta \geq B \). From Lemma 5.3 this implies \( \beta_c < \infty \). Since \( S^*(\beta) \neq \emptyset \) for all \( \beta \geq 0 \) we conclude that \( S^*(\beta) = \{s^*_{\text{clique}}(\beta)\} \) for all \( \beta > \beta_c \), and from Lemma 5.3 we have \( s^*_{\text{clique}}(\beta) \in (s_c, \infty) \) for all such \( \beta \), giving (5.7).

To argue that \( S^*(\beta) = \{s^*_{\text{hub}}(\beta)\} \) for all \( \beta < \beta_c \), suppose towards a contradiction that \( S^*(\beta) \cap (s_c, \infty) \neq \emptyset \) for some \( \beta < \beta_c \). By definition this means that \( S^*(\beta) \) has nonempty intersection with both \([0, s_c]\) and \((s_c, \infty)\). But from Lemma 5.2 it follows that \( S^*(\beta') \subset (s_c, \infty) \) for all \( \beta' > \beta \), since the minimal element of \( S^*(\beta') \) bounds the maximal element of \( S^*(\beta) \). We thus obtain a contradiction, so \( S^*(\beta) = \{s^*_{\text{hub}}(\beta)\} \) for all \( \beta < \beta_c \). It only remains to note that from Lemma 5.3 if follows that \( s^*_{\text{hub}}(\beta) \in (0, s_c) \) for such \( \beta \), which gives (5.6) and completes the proof.

**Proof of Corollary 1.12.** For \( F = K_3 \) we have \( v(F) = 3 \) and \( P_F(x) = 1 + 3x \), and hence

\[
\phi(s) = \min \left\{ \frac{1}{2} s^{2/3}, \frac{1}{3}s \right\} = \begin{cases} 
\frac{1}{2} s^{2/3} & s \in [0, \frac{27}{8}] \\
\frac{1}{3}s & s \in \left[\frac{27}{8}, \infty\right).
\end{cases}
\]

From Proposition 1.10, it suffices to verify that

(a) \( a^*(\beta) = s^*_{\text{clique}}(\beta)^{2/3} = \beta^2 \);

(b) \( b^*(\beta) = P_{K_3}^{-1}(1 + s^*_{\text{hub}}(\beta)) = \frac{1}{3} \beta^{3/2} \); and

(c) \( s \mapsto U(\beta, s) = \beta s^{1/3} - \phi(s) \) achieves its global maximum in \([0, \frac{27}{8}]\) when \( 0 \leq \beta < 16/9 \), and in \((\frac{27}{8}, \infty)\) when \( \beta > 16/9 \).

For (a), one merely verifies that \( U_{\text{clique}}(\beta, s) = \beta s^{1/3} - \frac{1}{2} s^{2/3} \) is maximized at \( s^*_{\text{clique}}(\beta) = \beta^3 \).

Similarly, for (b) one verifies that \( U_{\text{hub}}(\beta, s) = \beta s^{1/3} - \frac{3}{2} s \) is maximized at \( s^*_{\text{hub}}(\beta) = \beta^{3/2} \).

Finally, for (c) one verifies that \( U_{\text{hub}}(\beta, \beta^{3/2}) > U_{\text{clique}}(\beta, \beta^3) \) if and only if \( \beta < 16/9 \), where for \( 0 \leq \beta < 16/9 \), the global maximum of \( U(\beta, \cdot) \) is attained at \( s^*_{\text{hub}}(\beta) = \beta^{3/2} \in [0, 27/8) \), while for \( \beta > 16/9 \), the global maximum is attained at \( s^*_{\text{clique}}(\beta) = \beta^3 \in (27/8, \infty) \).

**Appendix A. Stability of Finner’s inequality**

In this appendix we prove Theorem 1.14. In what follows we abuse notation by writing e.g. \( \prod_A f_A \) instead of \( \prod_A f_A \circ \pi_A \) with \( \pi_A : \Omega \to \Omega_A \) the coordinate mapping projection. We further use \( \| \cdot \|_q \) for the \( L_q \) norms, whenever the underlying space is clear from the context.

**A.1. Stability of Hölder’s inequality.** We shall prove Theorem 1.14 by induction on \( n \) (following Finner’s argument for the case \( \varepsilon = 0 \), i.e. characterizing the case for equality), relying on the following stability property of Hölder’s inequality.
Lemma A.1. For any $\lambda \in (0, 1)$, $\varepsilon \in [0, 1]$ and $g : \Omega \to \mathbb{R}_{\geq 0}$ on a probability space $(\Omega, \nu)$,

$$
\int g d\nu \leq 1, \quad 1 - \varepsilon \leq \int g^\lambda d\nu \implies \|g - 1\|_1 \leq 2 \bar{C}_\lambda \varepsilon^{1/2}
$$

(A.1)

where $\bar{C}_\lambda := \sqrt{\frac{2}{\lambda(1 - \lambda)}}$.

Remark A.2. The case of $g = 1 \pm \bar{C}_\lambda \varepsilon^{1/2}$ and $\nu$ the Bernoulli(1/2) measure shows that the bound (A.1) is optimal up to a factor 2 for small $\varepsilon$.

Proof. Note that $\varphi(x) := 1 - (1 + x)^\lambda + \lambda x - \frac{(1 - \lambda)}{2} x^2 1_{\{x < 0\}}$ is non-negative on $[-1, \infty)$ (indeed, $\varphi(0) = \varphi'(0) = 0$ and $\varphi''(x) \geq 0$ both on $[-1, 0]$ and on $\mathbb{R}_{\geq 0}$). In particular, $\int \varphi(\hat{g}) d\nu \geq 0$ for $\hat{g} := g - 1 : \Omega \to [-1, \infty)$. Our assumptions that $\int \hat{g} d\nu \leq 0$ and

$$
\varepsilon \geq \int (1 - (1 + \hat{g})^\lambda) d\nu = \int \varphi(\hat{g}) d\nu - \lambda \int \hat{g} d\nu + \frac{\lambda(1 - \lambda)}{2} \int \hat{g}^2 d\nu,
$$

thus yield that

$$
\bar{C}_\lambda^2 \varepsilon \geq \int_{\hat{g} < 0} \hat{g}^2 d\nu \geq \left( \int_{\hat{g} < 0} |\hat{g}| d\nu \right)^2 \geq \frac{1}{4} \|\hat{g}\|^2_1
$$

as claimed (for the last inequality, note that $\int |\hat{g}| d\nu = 2 \int_{\hat{g} < 0} |\hat{g}| d\nu + \int \hat{g} d\nu$). \qed

Using Lemma A.1 we deduce the following stability of the (classical) generalized Hölder inequality.

Proposition A.3 (Stability of the generalized Hölder inequality). Suppose $f_i \geq 0$ on a probability space $(\Omega, \mu)$ are such that $\int f_i d\mu \leq 1$. Then, for any $m \geq 2$, $\lambda_i > 0$ such that $\sum_i \lambda_i \leq 1$ and $\varepsilon \in [0, 1]$,

$$
1 - \varepsilon \leq \prod_{i=1}^m f_i^{\lambda_i} d\mu \implies \|f_k - f_\ell\|_1 \leq C(\lambda_k, \lambda_\ell) \varepsilon^{1/2} \quad \forall k, \ell \in [m],
$$

(A.2)

where $C(\lambda, \lambda') = (\lambda + \lambda')^{-1/2} C_\lambda(\lambda + \lambda')$ and $C_\lambda = 2 \bar{C}_\lambda + 1/(\lambda \wedge (1 - \lambda))$.

Proof. Set the strictly positive $p_i = 1/\lambda_i$ for $i \neq k, \ell$, with $q = 1/(\lambda_k + \lambda_\ell) \geq 1$ and $r = 1/(\sum_i \lambda_i) \geq 1$, so that $1/r = 1/q + \sum_{i \neq k, \ell} 1/p_i$. It then follows from the LHS of (A.2) and the generalized Hölder inequality, that for $\lambda := \lambda_k q$,

$$
1 - q \varepsilon \leq (1 - \varepsilon)^q \leq (\prod_{i=1}^m f_i^{\lambda_i})^q \leq (\prod_{i=1}^m f_i^{\lambda_i})^q \leq \left( \int f_i^{1-\lambda_k} f_i^{\lambda_k} d\mu \right) (\prod_{i \neq k, \ell} \|f_i^{\lambda_i}\|_{p_i})^q.
$$

Since by assumption $\|f_i^{\lambda_i}\|_{p_i} = (\int f_i d\mu)^{1/p_i} \leq 1$, we have thus reduced the LHS of (A.2) to

$$
1 - \varepsilon' \leq \int f^{1-\lambda_k} h^{\lambda_k} d\mu = \|f\|_1 \int g^\lambda d\nu \leq \int g^\lambda d\nu,
$$

(A.3)

for $\varepsilon' = q \varepsilon \wedge 1$, $f = f_\ell$, $h = f_k$, $g = (h/f) 1_{\{f > 0\}}$ and the probability measure $\nu = \frac{f}{\|f\|_1} d\mu$.

Further,

$$
\|f\|_1 \int g d\nu = \|h\|_1 - \int_{\{f = 0\}} h d\mu \leq \|h\|_1,
$$

(A.4)

so in case $\|f_k\|_1 \leq \|f_\ell\|_1$, we deduce from (A.1) that

$$
\int_{\{f > 0\}} |h - f| d\mu = \|f\|_1 \int |g - 1| d\nu \leq 2 \bar{C}_\lambda \sqrt{\varepsilon'}.
$$

(A.5)
In addition, combining (A.3), (A.4) and Jensen’s inequality, we arrive at

\[ 1 - \varepsilon' \leq \|f\|_1 \int g^\lambda d\nu \leq \|f\|_1^{1-\lambda} (\|f\|_1 \int g d\nu)^\lambda = \|f\|_1^{1-\lambda} (\|h\|_1 - \int_{\{f=0\}} h d\mu)^\lambda. \]

Consequently, as \( \lambda, \varepsilon' \leq 1, \)

\[ \int_{\{f=0\}} |h-f| d\mu = \int_{\{f=0\}} h d\mu \leq 1 - (1 - \varepsilon')^{1/\lambda} \leq \frac{\varepsilon'}{\lambda} \leq \frac{\sqrt{\varepsilon'}}{\lambda}, \tag{A.6} \]

which, together with (A.5), results with \( \|f_k - f_\ell\|_1 \leq C_\lambda \sqrt{\varepsilon'}. \) The same applies when \( \|f_\ell\|_1 < \|f_k\|_1, \) except for exchanging the roles of \( f \) and \( h \) by mapping \( \lambda \mapsto 1 - \lambda. \)

**Remark A.4.** From the proof of Proposition A.3, when \( \int f_i d\mu \) is constant in \( i, \) the result improves to \( C_\lambda = 2(C_\lambda + 1) \) and consequently, to \( C(\lambda, \lambda') \leq 6(\lambda \wedge \lambda')^{-1/2}, \) by choosing the roles of \( f \) and \( h \) according to whether \( \lambda > 1/2 \) or not.

Proposition A.3 at \( \lambda_i = r/p_i, \) where \( \sum_i 1/p_i = 1/r, \) amounts to

\[ \|g_i\|_{p_i} = 1, \quad 1 - \varepsilon \leq \left\| \prod_i g_i \right\|_r^r \quad \Rightarrow \quad \int \|g_k|^p - |g_k|^{p_i} d\mu \leq C(\lambda_k, \lambda_i) \sqrt{\varepsilon}. \tag{A.7} \]

**A.2. Proof of Theorem 1.14.** Clearly we may assume that \( \mathcal{A} \) covers \( V. \) Without loss of generality we may further assume \( \mathcal{B} = \{\{v\}\}_{v \in V}. \) We may also assume

\[ \sum_{A \ni v} \lambda_A = 1 \quad \forall \, v \in V. \tag{A.8} \]

Indeed, for any \( v \) for which this does not hold, we can add \( \{v\} \) to the set system \( \mathcal{A}, \) setting \( \lambda_{\{v\}} = 1 - \sum_{A \ni v} \lambda_A \) and \( f_{\{v\}} = 1. \)

We proceed by induction on \( n := |V|, \) showing that for any set \( V \) of size \( n \) and \( \mathcal{A}, (f_A)_{A \in \mathcal{A}} \) and \( \mathcal{A} \) as in the theorem statement, there exist functions \( h_v : \Omega_v \rightarrow \mathbb{R}_{\geq 0} \) with \( \int h_v d\mu_v = 1 \) for each \( v \in V \) such that

\[ \|f_A - h_A\|_{L_1(\Omega_v)} \leq C_n \varepsilon^n \tag{A.9} \]

for constants \( C_n(\mathcal{A}, \Lambda) > 0 \) and \( c_n(\mathcal{A}, \Lambda) \in (0, \frac{1}{2}) \) to be determined, where \( h_A := \bigotimes_{v \in A} h_v. \)

For the base case \( n = 1, \) the \( f_A \) are functions over a common space \( (\Omega, \mu), \) so we can apply Proposition A.3 to conclude \( \|f_A - f_{A'}\|_1 \lesssim \varepsilon^{1/2} \) for all \( A, A' \in \mathcal{A}. \) Now from our hypothesis (1.49) and Theorem 1.13 we have

\[ 1 - \varepsilon \leq \prod_A f_A^{\lambda_A} d\mu \leq \prod_A \left( \int f_A d\mu \right)^{\lambda_A} \leq \left( \int f_{A_0} d\mu \right)^{\lambda_{A_0}} \]

for each \( A_0 \in \mathcal{A}. \) Fixing an arbitrary \( A_0, \) by Lemma A.1 one has for \( h \equiv 1 \) that \( \|h - f_{A_0}\|_1 \lesssim \varepsilon^{1/2}, \) and (A.9) follows by the triangle inequality, with \( c_1 = 1/2 \) and some \( C_1 \) sufficiently large depending on \( \Lambda. \)

Consider now the case that \( |V| = n \geq 2 \) and that the theorem statement holds for any \( V \) with \( |V| < n. \) Denote \( \mathcal{A}_n = \{A \in \mathcal{A} : |A| = n\}. \) The case \( \mathcal{A} = \mathcal{A}_n \) is handled exactly as in the case \( n = 1 \) so we assume \( \mathcal{A} \neq \mathcal{A}_n. \)

Consider first the case that \( \mathcal{A}_n = \emptyset. \) For \( v \in V \) we denote the contracted set system

\[ \mathcal{A}^{(v)} = \{A \setminus \{v\} : A \in \mathcal{A}\} \tag{A.10} \]

over \( V \setminus \{v\} \) (retaining repeats). Define functions

\[ g_{A,(v)} = \int f_A d\mu : \Omega_{A\setminus\{v\}} \rightarrow \mathbb{R}_{\geq 0}, \quad v \in A. \tag{A.11} \]
By (1.49) and the generalized Hölder inequality,
\[
1 - \varepsilon \leq \int \prod_{A \notin v} f_A^{\Lambda} \left( \int \prod_{A \equiv v} f_A^{\Lambda} \, d\mu_v \right) \, d\mu_{V \setminus \{v\}} \leq \int \prod_{A \notin v} f_A^{\Lambda} \prod_{A \equiv v} g_A^{\Lambda} \, d\mu_{V \setminus \{v\}}.
\]

Since \( \int g_{A,(v)} \, d\mu_{A \setminus \{v\}} = \int f_A \, d\mu_A \leq 1 \) for all \( A \ni v \), we can apply the induction hypothesis, with set system \( A^{(v)} \) over \( V \setminus \{v\} \), to obtain functions \( h_u^{(v)} : \Omega_u \to \mathbb{R}_{\geq 0} \) for each \( u \in V \setminus \{v\} \) with \( \int h_u^{(v)} \, d\mu_u = 1 \), such that
\[
\| f_A - h_A^{(v)} \|_{L^1(\Omega_A)} \leq C(v)\varepsilon^{c(v)}, \quad \forall A \ni v, \quad (A.12)
\]
\[
\| g_A^{(v)} - h_A^{(v)} \|_{L^1(\Omega_{A \setminus \{v\}})} \leq C(v)\varepsilon^{c(v)}, \quad \forall A \ni v, \quad (A.13)
\]
where \( h_A^{(v)} := \bigotimes_{u \in A} h_u^{(v)} \) and \( C(v) := C_{n-1}(A^{(v)}, \Lambda^{(v)}), \; c(v) := c_{n-1}(A^{(v)}, \Lambda^{(v)}) \). (Here \( \Lambda^{(v)} \) is the collection of weights for \( A^{(v)} \) inherited from \( \Lambda \) under the contraction (A.10).)

Having obtained the family of functions \( \{ h_u^{(v)} : u, v \in V, u \neq v \} \) satisfying (A.12)--(A.13), we now fix arbitrary distinct \( w, z \in V \) and take
\[
h_u := \begin{cases} h_u^{(v)} & u \neq w \\
 z & u = w. \end{cases} \quad (A.14)
\]

It only remains to verify (A.9), i.e. that
\[
\| f_A - h_w^{(v)} \otimes h_A^{(v)} \|_{L^1(\Omega_A)} \leq C_n\varepsilon^{c_n} \quad (A.15)
\]
for appropriate \( C_n, c_n \) and each \( A \in \mathcal{A} \) containing \( w \) (for all other \( A \) the claim is immediate from (A.12), taking \( C_n \geq C(w) \) and \( c_n \leq c(w) \)).

We first claim that for any \( A \in \mathcal{A}, u \in A \) and \( v \notin A \),
\[
\| h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{v\}})} \leq C(u)\varepsilon^{c(u)} + C(v)\varepsilon^{c(v)}. \quad (A.16)
\]
Indeed, from the triangle inequality and (A.13) the LHS above is bounded by
\[
\| g_{A,(u)} - h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{v\}})} + C(u)\varepsilon^{c(u)}
\]
and we can express the first term above as
\[
\left\| \int (f_A - h_A^{(v)}) \, d\mu_u \right\|_{L^1(\Omega_{A \setminus \{u\}})} \leq \| f_A - h_A^{(v)} \|_{L^1(\Omega_A)} \leq C(v)\varepsilon^{c(v)}
\]
where we applied Minkowski’s inequality and (A.12).

We now establish (A.15). For the case that \( w \in A \) and \( z \notin A \) this follows for any \( C_n \geq 2C(z) + C(w), \; c_n \leq c(z) \wedge c(w) \) from the triangle inequality, (A.12) and (A.16) with \( v = z \) and \( u = w \). Now assume \( \{w, z\} \subseteq A \). Under our assumption that \( \mathcal{A}_n = \emptyset \) we can select an arbitrary \( v \in V \setminus A \) and bound the LHS of (A.15) by
\[
\| f_A - h_A^{(v)} \|_{L^1(\Omega_A)} + \| h_w^{(v)} - h_w^{(v)} \otimes h_A^{(v)} \|_{L^1(\Omega_A)} + \| (h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)}) h_A^{(v)} \|_{L^1(\Omega_A)}
\]
\[
\leq C(v)\varepsilon^{c(v)} + \| h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{v\}})} + \| h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{v\}})}.
\]

The second term on the RHS is at most \( C(z)\varepsilon^{c(z)} + C(w)\varepsilon^{c(w)} \) from (A.16). For the third term,
\[
\| h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{u\}})} = \left\| \int (h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)}) \, d\mu_{A \setminus \{u, z\}} \right\|_{L^1(\Omega_{A \setminus \{u\}})} \leq \| h_{A \setminus \{u\}}^{(v)} - h_{A \setminus \{u\}}^{(v)} \|_{L^1(\Omega_{A \setminus \{u\}})}
\]
which is at most \( C(v) e^{c(v)} + C(z) e^{c(z)} \) by (A.16) (if \( A = \{w, z\} \) then \( A \setminus \{z\} = \{w\} \), trivially yielding the same bound). Altogether we conclude in case \( A_n = \emptyset \), that (A.9) holds for any

\[
C_n \geq C_n(A, \Lambda) := 5 \max \{C(u) : u \in V\}, \quad c_n \leq c_n'(A, \Lambda) := \min \{c(u) : u \in V\}. \tag{A.17}
\]

Finally, suppose \( \emptyset \neq A_n \neq A \). Put \( \lambda_A := \sum_{A \in A_n} \lambda_A < 1 \) and 
\( F := \prod_{\{A: |A|<n\}} f_A^{X_A} \) with \( \lambda_A^\prime := \lambda_A/(1 - \lambda_A) \). Then, from (1.49) and the generalized Hölder inequality,

\[
1 - \varepsilon \leq \int F^{1 - \lambda_A} \prod_{\{A: |A| = n\}} f_A^{X_A} d\mu_V \leq \left( \int F d\mu_V \right)^{1 - \lambda_A}. \tag{A.18}
\]

Thus,

\[
1 - \frac{\varepsilon}{1 - \lambda_A} \leq (1 - \varepsilon)^{1/(1 - \lambda_A)} \int F d\mu_V = \int \prod_{\{A: |A|<n\}} f_A^{X_A} d\mu_V.
\]

Applying the result for the case that \( A_n = \emptyset \), with \( A' := A \setminus A_n \) in place of \( A \) and \( \Lambda' := (\lambda_A^\prime)_{A \in A'} \) in place of \( \Lambda \), we obtain \( (h_v)_v \in V \) such that

\[
\|f_A - h_A\|_{L_1(\Omega_A)} \leq C_n(A', \Lambda') \left( \frac{\varepsilon}{1 - \lambda_A} \right)^{c_n'(A', \Lambda')} \quad \forall A \in A'. \tag{A.19}
\]

Assuming \( C_n \geq C_n'(A', \Lambda')(1 - \lambda_A)^{-c_n'(A', \Lambda')} \) and \( c_n \leq c_n'(A', \Lambda') \), it only remains to establish (A.9) for \( A \in A_n \). Now from Proposition A.3 and the first inequality in (A.18) it follows that

\[
\|f_A - F\|_{L_1(\Omega_V)} \leq \varepsilon/2 \quad \forall A : |A| = n, \tag{A.20}
\]

so by the triangle inequality and taking \( C_n \) larger, if necessary, it suffices to show

\[
\|F - h_V\|_{L_1(\Omega_V)} \leq C_n \varepsilon^n \tag{A.21}
\]

for possibly adjusted values of \( C_n, c_n \). We obtain this by expanding the difference as a telescoping sum over \( A \in A' \) and applying (A.19). Enumerating the elements of \( A' \) as \( A_j, 1 \leq j \leq m \), we have

\[
F - h_V = \prod_{\{A: |A|<n\}} f_A^{X_A} - \prod_{\{A: |A|<n\}} h_A^{X_A} = \sum_{i=1}^{m} (-1)^i (f_{A_i} - h_{A_i}^{X_A}) \prod_{j<i} h_{A_j}^{X_{A_j}} \prod_{j>i} h_{A_j}^{X_{A_j}}
\]

(note we used (A.8) in the first equality). Taking \( L_1 \)-norms on both sides and applying the triangle inequality and Finner’s inequality (for \( A' \) and \( \Lambda' \)), we obtain

\[
\|F - h_V\|_{L_1(\Omega_V)} \leq \sum_{i=1}^{m} \left( \int f_{A_i}^{X_{A_i}} - h_{A_i}^{X_{A_i}} \prod_{j<i} h_{A_j}^{X_{A_j}} \prod_{j>i} h_{A_j}^{X_{A_j}} d\mu_V \right)^{X_{A_i}}
\]

\[
\leq \sum_{i=1}^{m} \left( \int f_{A_i} - h_{A_i} \prod_{j<i} h_{A_j} \prod_{j>i} h_{A_j} d\mu_V \right)^{X_{A_i}}
\]

(\( \begin{align*}
(\text{using the elementary bound } |x - y|^p & \leq |x|^p - |y|^p \text{ for } x, y \geq 0, p \geq 1, \text{ in the last step). The claim now follows by substituting the bounds (A.19) and taking } \n C_n \geq \sum_{i=1}^{m} \left[ C_n'(A', \Lambda')(1 - \lambda_A)^{-c_n'(A', \Lambda')} \right] X_{A_i}, \quad c_n \leq c_n'(A', \Lambda') \cdot \min \{X_{A_i} \}. \end{align*} \)
\]
Appendix B. Proof of Proposition 1.8

Proof of Part (a). All but the last claim (1.41) are immediate from the fact that for each $k$, $T_k$ is continuous, non-decreasing and unbounded, with $T_k(0,0) = 0$. Now for (1.41), since $\phi_F(s) \geq \phi_F(s_k)$ for each $k \in [m]$, it suffices to establish the case $m = 1$. We claim that for all $a, b \geq 0$ with $a + b \geq 1$,
\[
T_F(a, b) \preceq_F (a + b) \epsilon(F)/\Delta. \quad (B.1)
\]
Indeed, $\epsilon(F)/\Delta$ is an upper bound for the size of any independent set in $F^*$, and hence for the degree of $P_{F^*}$, and in the case that $F$ is regular we have $v(F)/2 = \epsilon(F)/\Delta$. Now let $C = C(F) > 0$ to be taken sufficiently large. Then for arbitrary $s \geq C$, if $a, b \geq 0$ are such that $T_F(a, b) \geq 1 + s$, then taking $C$ sufficiently large it follows that $a + b \geq 1$, and from (B.1) we get that $\frac{1}{2} a + b \geq F s^{\Delta/\epsilon(F)}$. The claim follows.

Proof of Part (b). For the case that $s = 0$ we have that Opt$(\phi; 0)$ is the singleton set $\{(0, 0)\}$ and the claim follows.

Assume now that $s \neq 0$. By throwing out redundant constraints $T_k \geq 1$ we may assume wlog that $s_k > 0$ for each $k$, while re-indexing $F_k$ (as in Section 4), so that $F_k$ is regular if and only if $k \leq m'$ for some $m' \in [m]$. Setting $a_k^*(s_k) = s_k^2/\sqrt{\epsilon(F_k)}$, $b_k^*(s_k) = P_k^{-1}(1 + s_k)$, note that for any $k \leq m'$ the curve $\Gamma_k(s_k) = \{(a, b) : T_k(a, b) = 1 + s_k \cap \mathbb{R}^2_{\geq 0}\}$ is a smooth arc of positive curvature with endpoints $(a_k^*, 0)$ and $(0, b_k^*)$. For $k > m'$ we have that $\Gamma_k(s_k) = \{(a, b) : a \in \mathbb{R}_{\geq 0} \}$ is a horizontal ray with endpoint on the $b$-axis. The infimum of the increasing linear function $(a, b) \mapsto \frac{1}{2} a + b$ over $R := \{T_k(a, b) \geq 1 + s_k\}$ must be attained at some finite point on the boundary of $R$. The boundary consists of a finite connected union of smooth curves overlapping only at their endpoints: the ray $\Gamma^\prime_{\text{vert}} := \{(0, b) : b \geq b_0\}$ with $b_0 = \max_k b_k^*(s_k)$, a connected infinite subset $\Gamma^\prime_{\text{horiz}}$ of the ray $\{(a, b) : a \geq 0\}$ with $b^* = 0 \vee \max_{k > m'} b_k^*(s_k) \leq b_0$, and a (possibly empty) finite union of sub-arcs $\Gamma^\prime_{\alpha}$ of the bounded curves $\Gamma_k(s_k), k \leq m'$. (In particular, if $m' = 0$ then $b^* = b_0$ and $R$ is the axis-aligned quadrant $\{(a, b) : a \geq 0, b \geq b_0\}$, so the infimum is attained at the single point $(0, b_0)$.) One easily sees that for each $\alpha$, $\inf \{\frac{1}{2} a + b : (a, b) \in \Gamma^\prime_{\alpha}\}$ cannot be achieved on the interior of $\Gamma^\prime_{\alpha}$. Indeed, $(a, b) \mapsto \frac{1}{2} a + b$ is strictly monotone on $\Gamma^\prime_{\text{vert}} \cup \Gamma^\prime_{\text{horiz}}$, and since it is linear it can only have a local maximum on the interior of any of the $\Gamma^\prime_{\alpha}$ (being a subset of a level curve of one of the strictly convex functions $T_k, k \leq m'$). Thus, the infimum can only be achieved at one of the finitely-many intersection points of the curves $\Gamma^\prime$.

Proof of Part (c). We abbreviate $\psi := \psi_{F, h}$ and denote the RHS of (1.42) by $\psi'$. For $a, b \geq 0$ we hereafter denote
\[
\underline{s}(a, b) := (T_1(a, b) - 1, \ldots, T_m(a, b) - 1).
\]
For $s, s' \in \mathbb{R}^m_{\geq 0}$ we understand $s \geq s'$ to mean $s_k \geq s'_k$ for each $k \in [m]$.

We first argue $S^*$ is nonempty and bounded. Indeed, this follows from the continuity of $h$, part (a) and the assumption (1.10).

Now to show $\psi' \leq \psi$, since $S^*$ is nonempty we may fix an arbitrary $s' \in \mathbb{R}^m_{\geq 0}$ such that $\psi' = h(1 + s') - \phi_F(s')$. We have
\[
\psi' = h(1 + s') - \phi_F(s') = \sup_{a, b \geq 0} \{h(1 + s') - \frac{1}{2} a - b : \underline{s}(a, b) \geq s'\}
\]
\[
\leq \sup_{a, b \geq 0} \{h(1 + \underline{s}(a, b)) - \frac{1}{2} a - b\} = \psi
\]
where for the inequality we used the assumption that $h$ is monotone.
To see that $\psi \leq \psi'$ (which in fact holds under no assumptions on $h$), letting $a, b \geq 0$ be arbitrary, we have

$$h(1 + \underline{s}(a, b)) - \frac{1}{2}a - b \leq h(1 + \underline{s}(a, b)) - \phi_F(\underline{s}(a, b)) \leq \psi'$$

and the claim follows upon taking the supremum over $a, b$ on the LHS. 

Proof of Part (d). For the containment $\supseteq$, from parts (b) and (c) we may fix arbitrary $\underline{s} \in S^*$ and $(a, b) \in \text{Opt}(\phi; \underline{s})$. Thus, $\phi_F(\underline{s}) = \frac{1}{2}a + b$, and $\underline{s}(a, b) \geq \underline{s}$. Then we have

$$h(1 + \underline{s}(a, b)) - \frac{1}{2}a - b = h(1 + \underline{s}(a, b)) - \phi_F(\underline{s})$$

$$\geq h(1 + \underline{s}) - \phi_F(\underline{s}) = \psi \geq h(1 + \underline{s}(a, b)) - \frac{1}{2}a - b$$

where in the first inequality we used the monotonicity assumption, and in the last we used the formula (1.42) established in (c). Thus, the inequalities in fact hold with equality, and hence $(a, b) \in \text{Opt}(\psi)$.

For the containment $\subseteq$, note that from (b), (c) and the containment $\supseteq$ just established, it follows that $\text{Opt}(\psi)$ is nonempty (this can also be seen directly following similar reasoning as in the proof of (a)). Thus, we may fix an arbitrary element $(a, b) \in \text{Opt}(\psi)$. We claim that $\underline{s}(a, b) \in S^*$ and $(a, b) \in \text{Opt}(\phi; \underline{s}(a, b))$, from which the result follows. Indeed,

$$\psi = h(1 + \underline{s}(a, b)) - \frac{1}{2}a - b \leq h(1 + \underline{s}(a, b)) - \phi_F(\underline{s}(a, b)) \leq \psi$$

where in the final bound we used the relation (1.42) established in (c). Thus, equality holds throughout, and from equality in the first bound we get that $(a, b) \in \text{Opt}(\phi; \underline{s}(a, b))$, while equality in the second bound implies $\underline{s}(a, b) \in S^*$. By (c) the set $S^*$ is pre-compact, so $\phi_F(\cdot)$ is bounded on $S^*$. Hence, by (b) and the preceding containment, $\text{Opt}(\psi)$ is also bounded. 

## Appendix C. Order of the upper tail

### Proposition C.1

For any graph $H$ of max degree $\Delta$, there exist finite $C(H)$ and positive $c(H)$ such that if $p \in (0, 1/2]$ and $np^{\Delta + 1} \geq C(H)$, then for any $s \geq 2$,

$$\log \mathbb{P}(t(H, G_{n,p}/p) \geq s) \leq \begin{cases} -c(H)s^{\Delta/e(H)}p^{\Delta - 2}\log(1/p), & \Delta \geq 2, \\ -c(H)s^{\Delta/e(H)}p^{\Delta - 2}\log s, & \Delta = 1. \end{cases} \tag{C.1}$$

### Remark C.2

Using [16, Thms. 2.10 and 3.1], the same proof below yields the corresponding bound in the case that $G_{n,p}$ is the $r$-uniform Erdős–Rényi hyper-graph, for any $r \geq 2$, any $r$-graph $H$, assuming $np^{\Delta'(H)} \geq C(H)$, with $\Delta'(H)$ as defined in [16].

### Proof

We argue by induction on $e(H)$, having as induction hypothesis that the bound (C.1) holds with $F$ and $\Delta(F)$ in place of $H$ and $\Delta(H)$, for all graphs $F$ with $e(F) < e(H)$.

The claim in the case $\Delta = 1$ (which includes the base case $e(H) = 1$) follows from a standard tail bound for the binomial distribution, along with the fact that $t(H_1 \cup H_2, X) = t(H_1, X)t(H_2, X)$ for $H$ a disjoint union of two graphs $H_1, H_2$. (In particular we get $c(H) = c/e(H)$ in this case.)

Assume now that $\Delta \geq 2$. To establish (C.1) for $H$ we use the union bound,

$$\mathbb{P}(G_{n,p} \in \mathcal{U}_p(H, s)) \leq \mathbb{P}(G_{n,p} \in \mathcal{U}_p(H, s) \cap \mathcal{L}_{\leq}(H, s)) + \sum_{F \subseteq H} \mathbb{P}(G_{n,p} \in \mathcal{U}_p(F, s)), \tag{C.2}$$
where
\[ \mathcal{L}_c(H,s) := \bigcap_{F \subseteq H} \{ Q \in \mathcal{Q}_n : t(F,Q/p) \leq s \}. \]

As \( t(H,Q/p) \leq p^{-e(H)} \) for any \( Q \in \mathcal{Q}_n \), it suffices to consider only
\[ s \leq p^{-e(H)}. \tag{C.3} \]

Since (3.9) applies for any \( E \subset \mathcal{G}_n \) intersecting \( \mathcal{L}_c(H,L) \), upon taking \( \delta = \delta(H) > 0 \) small we get by (3.10) of Proposition 3.3(b) that
\[ (U_p(H,s) \cap \mathcal{L}_c(H,s))_{E^{\star}(\delta)} \subseteq U_p(H,s/2). \]

Consequently, see (4.10),
\[ I_p \left( (U_p(H,s) \cap \mathcal{L}_c(H,s))_{E^{\star}(\delta)} \right) \geq s^{\Delta/e(H)} K_{n,p}. \]

We thus get the bound on the RHS of (C.1) for the first term in (C.2) (and some small \( c(H) > 0 \)), as a consequence of the upper-LDP of Proposition 3.3(a) at \( K_1 = c(H)s^{\Delta/e(H)} \). Indeed, (3.8) is satisfied for the assumed range of \( p \) once we set
\[ C(H) \geq 2K_0(\Delta + 1) \delta^{-2}. \]

For the remaining terms in (C.2), taking \( c(H) \leq \min\{c(F) : F \subseteq H\} \), we have by the induction hypothesis, for any \( F \subseteq H \) with \( \Delta(F) \geq 2 \),
\[
\log \mathbb{P}(t(F,G_{n,p}/p) \geq s) \leq -c(F)(s^{1/e(F)} p)^{\Delta(F)} n^2 \log(1/p) \\
\leq -c(H)(s^{1/e(H)} p)^{\Delta(H)} n^2 \log(1/p), \tag{C.4}
\]

since \( e(F) < e(H) \) (we only need \( e(F) \leq e(H) \), \( sp^{e(H)} \leq 1 \) (see (C.3)) and \( \Delta(F) \leq \Delta(H) \).

For the case that \( F \subseteq H \) with \( \Delta(F) = 1 \), we get from the case \( \Delta = 1 \) of (C.1) that
\[
\log \mathbb{P}(t(F,G_{n,p}/p) \geq s) \leq -c(F)s^{1/e(F)} pn^2 \log s. \tag{C.5}
\]

The RHS of (C.4) increases in \( \Delta(H) \geq 2 \), thereby (C.5) yields (C.4) whenever
\[ s^{1/e(F) - 2/e(H)} \geq p \log(1/p). \tag{C.6}
\]

Note that (C.6) trivially holds if \( e(F) \leq e(H)/2 \), while otherwise we get from (C.3) upon recalling that \( e(H) \geq e(F) + 1 \), that
\[ s^{1/e(F) - 2/e(H)} \geq p^{-e(H)(1/e(F) - 2/e(H))} = p^{2 - e(H)/e(F)} \geq p^{1 - 1/e(F)} \geq p \log(1/p). \]

We have thus established (C.6) and thereby (C.4) for any \( F \subseteq H \). Plugging this back in (C.2) completes our induction step. \( \square \)

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