Stabilization of the critical and subcritical semilinear inhomogeneous and anisotropic elastic wave equation

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Abstract We prove exponential decay of the critical and subcritical semilinear inhomogeneous and anisotropic elastic wave equation with locally distributed damping on bounded domain. One novelty compared to previous results, is to give a checkable condition of the inhomogeneous and anisotropic medias. Another novelty is to establish a framework to study the stability of the damped semilinear inhomogeneous and anisotropic elastic wave equation, which is hard to apply Carleman estimates to deal with. We develop the Morawetz estimates and the compactness-uniqueness arguments for the semilinear elastic wave equation to prove the unique continuation, observability inequality and stabilization result.

It is pointing that our proof is different from the classical method (See Dehman et al.[15], Joly et al.[26] and Zuazua [59]), which succeeds for the subcritical semilinear wave equation and fails for the critical semilinear wave equation.

Keywords inhomogeneous and anisotropic elastic wave equation, critical and subcritical nonlinearity, exponential stabilization, morawetz estimates

Mathematics Subject Classification 35L51,74E05,74E10,93D15,93D20

1 Some Notations

Let $O$ be the origin of $\mathbb{R}^n (n \geq 3)$ and

$$r(x) = |x|, \quad x \in \mathbb{R}^n \quad (1.1)$$

be the standard distance function of $\mathbb{R}^n$. Moreover, let $\langle \cdot , \cdot \rangle$, div, $\nabla$, $\Delta$ and $I_n = (\delta_{i,j})_{n \times n}$ be the standard inner product of $\mathbb{R}^n$, the standard divergence operator of $\mathbb{R}^n$, the standard gradient operator of $\mathbb{R}^n$, the standard Laplace operator of $\mathbb{R}^n$ and the unit matrix.

Let $(a_{ijkl})_{n \times n \times n \times n} (x)$ be a smooth tensor function defined on $\mathbb{R}^n$ satisfying

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x), \quad (1.2)$$

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for any \( x \in \mathbb{R}^n \) and any \( 1 \leq i, j, k, l \leq n \), and the ellipticity condition
\[
\alpha \sum_{i,j=1}^{n} \varepsilon_{ij} \varepsilon_{ij} \leq \sum_{i,j,k,l=1}^{n} a_{ijkl}(x) \varepsilon_{ij} \varepsilon_{kl} \leq \beta \sum_{i,j=1}^{n} \varepsilon_{ij} \varepsilon_{ij}, \quad x \in \mathbb{R}^n, \tag{1.3}
\]
for every symmetric tensor \( (\varepsilon_{ij})_{n \times n} \), where \( \alpha, \beta \) are positive constants.

Let \( u(x,t) = (u_1(x,t), ..., u_n(x,t)) : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}^n \) be a function. Denote
\[
\frac{\partial u_i}{\partial x_j}, \quad \frac{\partial u_i}{\partial t}, \quad \frac{\partial^2 u_i}{\partial t^2}, \tag{1.4}
\]
and
\[
\varepsilon_{ij}(u) = \frac{1}{2}(u_{ij} + u_{ji}), \quad \varepsilon_{ij,k}(u) = \frac{\partial \varepsilon_{ij}(u)}{\partial x_k}, \tag{1.5}
\]
for any \( x \in \mathbb{R}^n \) and any \( 1 \leq i, j, k \leq n \). For any \( x \in \mathbb{R}^n \) and any \( 1 \leq i, j, k, l \leq n \), we define
\[
\sigma_{ij}(u) = \sum_{k,l=1}^{n} a_{ijkl}(x) \varepsilon_{kl}(u). \tag{1.6}
\]
Denote
\[
\nabla u = (\nabla u_1, ..., \nabla u_n) = (u_{i,j})_{n \times n}, \quad |\nabla u|^2 = \sum_{i=1}^{n} |\nabla u_i|^2, \tag{1.7}
\]
\[
\sigma(u) = (\sigma_1(u), ..., \sigma_n(u)) = (\sigma_{ij}(u))_{n \times n}, \tag{1.9}
\]
and
\[
\varepsilon(u) = (\varepsilon_{ij}(u))_{n \times n}. \tag{1.10}
\]
Denote
\[
B(h) = \left\{ x \mid |x| \leq h \right\}, \quad \forall h > 0. \tag{1.11}
\]
Let \( S(r) \) be the sphere in \( \mathbb{R}^n \) with radius \( r \). Then
\[
\langle X, \frac{\partial}{\partial r} \rangle = 0, \quad \text{for} \quad X \in S(r)_x, x \in \mathbb{R}^n \setminus O. \tag{1.12}
\]

2 Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth compact boundary \( \Gamma \) and let \( \nu(x) = (\nu_1(x), ..., \nu_n(x)) \) be the unit normal vector outside \( \Omega \) for \( x \in \Gamma \).

It is assumed that \( \Gamma = \Gamma_0 \cup \Gamma_1 \), where \( \Gamma_0, \Gamma_1 \subset \Gamma, \Gamma_0 \cap \Gamma_1 = \emptyset, \Gamma_0 \neq \emptyset \) and
\[
\frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma_0 \quad \text{and} \quad \frac{\partial r}{\partial \nu} \geq 0, \quad x \in \Gamma_1. \tag{2.1}
\]

Let \( \omega \) be an open subset of \( \Omega \) such that
\[
\omega \supset \bigcup_{x \in \Gamma_1} \left\{ y \in \Omega \mid |y - x| < \xi \right\}, \tag{2.2}
\]
for some \( \xi > 0 \).

**Example 2.1.** Let \( R_0, R_1, \varepsilon_0 \) be positive constants such that \( R_1 > R_0, \varepsilon_0 < R_1 - R_0 \). An example can be given by \( \Omega = B(R_1) \setminus B(R_0), \omega = B(R_1) \setminus B(R_0 + \varepsilon_0), \Gamma_0 = S(R_0) \) and \( \Gamma_1 = S(R_1) \).
We consider the following system.

\[
\begin{cases}
  u_{tt} - \text{div} \sigma(u) + a(x)u_t + f(u) = 0 & (x, t) \in \Omega \times (0, +\infty), \\
  u = 0 & (x, t) \in \Gamma_0 \times (0, +\infty), \\
  \sigma(u)u^T = 0 & (x, t) \in \Gamma_1 \times (0, +\infty), \\
  u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & x \in \Omega,
\end{cases}
\]  

(2.3)

where \(a(x) \in C^1(\mathbb{R}^n)\) is a nonnegative function and

\[
f(u) = (|u_1|^{p_1-1}u_1, ..., |u_n|^{p_n-1}u_n),
\]

(2.4)

where

\[
1 < p_i \leq \frac{n + 2}{n - 2} \quad \text{for} \quad 1 \leq i \leq n.
\]

(2.5)

The energy of the system (2.3) is defined by

\[
E(t) = \frac{1}{2} \int_\Omega \left( |u_t|^2 + \sigma(u) \circ \varepsilon(u) + 2F(u) \right) dx,
\]

(2.6)

where

\[
F(u) = \sum_{i=1}^{n} \frac{1}{p_i + 1} |u_i|^{p_i+1},
\]

(2.7)

and \(\circ\) is defined by

\[
B \circ D = \sum_{i,j=1}^{n} b_{ij}d_{ji}.
\]

(2.8)

for real matrixes \(B = (b_{ij})_{n \times n}\) and \(D = (d_{ij})_{n \times n}\).

**Remark 2.1.** The system (2.3) can be rewritten as for \(1 \leq i \leq n\),

\[
\begin{cases}
  u_{iti} - \sum_{j=1}^{n} \sigma_{ij,j}(u) + a(x)u_{it} + |u_i|^{p_i-1}u_i = 0 & (x, t) \in \Omega \times (0, +\infty), \\
  u_i |_{\Gamma_0} = 0 & t \in (0, +\infty), \\
  \sum_{j=1}^{n} \sigma_{ij}(u)v_j |_{\Gamma_1} = 0 & t \in (0, +\infty), \\
  u_i(x, 0) = u_{0i}(x), \quad u_{i,t}(x, 0) = u_{1i}(x) & x \in \Omega,
\end{cases}
\]

(2.9)

Then \(E(t)\) can be rewritten as

\[
E(t) = \frac{1}{2} \int_\Omega \left( \sum_{i=1}^{n} u_{iti}^2 + \sum_{i,j=1}^{n} \sigma_{ij}(u)\varepsilon_{ij}(u) + \sum_{i=1}^{n} \frac{2}{p_i + 1} |u_i|^{p_i+1} \right) dx.
\]

(2.10)

There are a wealth of literatures on the controllability and stabilization of the elastic wave equation. For homogeneous isotropic elastic wave equation, see [2 4 29 30]. For homogeneous nonisotropic elastic wave equation, see [3 7 8 13 21 31 60 55 56 60]. For the inhomogeneous elastic wave equation, see [9 34 37 42 51].

There exist few literature on the stabilization of the semilinear elastic wave equation. Stabilization of the subcritical semilinear wave equation has been fully studied. See [13 14 15 26 32 59 60]. Microlocal analysis given by [4 17] are the main methods to deal with the stabilization of the semilinear wave equation. However, microlocal analysis doesn’t work for the critical semilinear wave equation. As is known, the Morawetz estimate is a simple and effective tool to deal with the...
energy estimate on hyperbolic PDEs. See [33, 36, 39, 40, 57, 58, 60]. Therefore, we develop the Morawetz estimates and the compactness-uniqueness arguments to try to prove the stabilization of the critical and subcritical semilinear inhomogeneous elastic wave equation.

It is pointing that the (elastic) wave equation with Dirichlet/Neumann boundary condition has special physical meaning, see [16, 28, 38, 35].

The following assumption is the main assumption.

**Assumption (A)** There exists constant \( \delta > 0 \) such that for any \( x \in \Omega \) and for every symmetric tensor \( (\varepsilon_{ij})_{n \times n} \),

\[
\sum_{ijkl=1}^{n} \left( (1 - \delta)\alpha_{ijkl} - \frac{r}{2} \frac{\partial \alpha_{ijkl}}{\partial r} \right) \varepsilon_{ij} \varepsilon_{kl} \geq 0. \tag{2.11}
\]

**Remark 2.2.** We don’t know whether the condition \((2.11)\) is necessary. However from a view of inhomogeneous and anisotropic wave equation:

\[
\begin{cases}
    u_{tt} - \text{div} A(x) \nabla u = 0 & (x, t) \in \Omega \times (0, +\infty), \\
    u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) & x \in \Omega,
\end{cases} \tag{2.12}
\]

the condition \((2.11)\) may be a general condition.

Similar with the condition \((2.11)\), we give the following condition for the inhomogeneous and anisotropic wave equation. There exists a constant \( \delta > 0 \) such that

\[
\langle \left( (1 - \delta)A(x) - \frac{r}{2} \frac{\partial A(x)}{\partial r} \right) X, X \rangle \geq 0 \quad \text{for} \quad X \in \mathbb{R}^3, \ x \in \overline{\Omega}. \tag{2.13}
\]

Let \( G(x) = A^{-1}(x) \). Let \( x \in \mathbb{R}^n, X, Y \in \mathbb{R}^n \) and \( Y = G(x)X \). We deduce that

\[
Y^T \left( \lambda A(x) - \frac{r}{2} \frac{\partial A(x)}{\partial r} \right) Y \\
= \langle G(x) \left( \lambda A(x) - \frac{r}{2} \frac{\partial A(x)}{\partial r} \right) G(x)X, X \rangle \\
= \langle \left( \lambda G(x) + \frac{r}{2} \frac{\partial (G(x))}{\partial r} \right) X, X \rangle, \tag{2.14}
\]

where \( \lambda \) is a constant. It follows from Lemma 3.3 and Lemma 3.4 in [43] that the condition \((2.13)\) is almost equivalent to GCC (geometric control condition).

**Example 2.2.** Let

\[
a_{ijkl}(x) = \lambda(x)\delta_{ij}\delta_{kl} + \mu(x)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad 1 \leq i, j, k, l \leq n, \quad x \in \mathbb{R}^n, \tag{2.15}
\]

where \( \lambda(x), \mu(x) \in C^\infty(\mathbb{R}^n) \) satisfy

\[
0 < \alpha \leq \mu(x) \leq \beta \quad \text{and} \quad 0 < \alpha \leq \lambda(x) + 2\mu(x) \leq \beta, \quad x \in \mathbb{R}^n. \tag{2.16}
\]

Assume that there exists constant \( 0 < \delta \leq 1 \) such that

\[
(1 - \delta)\mu(x) - \frac{r}{2} \frac{\partial \mu(x)}{\partial r} \geq 0, \quad x \in \Omega, \tag{2.17}
\]

and

\[
(1 - \delta)(\lambda(x) + 2\mu(x)) - \frac{r}{2} \frac{\partial (\lambda(x) + 2\mu(x))}{\partial r} \geq 0, \quad x \in \Omega. \tag{2.18}
\]
Then for any $x \in \Omega$, 
\[
\begin{align*}
\sum_{ijkl=1}^{n} \left( (1 - \delta)a_{ijkl}(x) - \frac{r}{2} \frac{\partial a_{ijkl}}{\partial r} \right) \varepsilon_{ij} \varepsilon_{kl} \\
= \left( (1 - \delta)\lambda(x) - \frac{r}{2} \frac{\partial \lambda(x)}{\partial r} \right) \left( \sum_{i=1}^{n} \varepsilon_{ii} \right)^{2} \\
+ 2 \left( (1 - \delta)\mu(x) - \frac{r}{2} \frac{\partial \mu(x)}{\partial r} \right) \sum_{i,j=1}^{n} \varepsilon_{ij} \varepsilon_{ij} \\
\geq 0.
\end{align*}
\] (2.19)

Well-posedness of the subcritical semilinear wave equation has been studied by [10, 19, 20, 21, 25, 45] and well-posedness of the critical semilinear wave equation has been studied by [5, 21, 22, 23, 27, 47, 48, 49, 50]. There exists similar results for the nonlinear elastic wave equation. See [1, 46, 61]. It is pointing that well-posedness of the critical semilinear wave equation on bounded domain with Dirichlet boundary condition or Neumann boundary condition has been proved by [11, 12]. However, well-posedness of the critical semilinear wave equation on Riemannian manifold or with variable coefficients is still an open problem. As far as we know, the well-posedness of critical semilinear wave equation on Riemannian manifold or with variable coefficients is so hard that there exists no noteworthy study recently. Since we are mainly interested in stabilization of the system (2.3), we assume the following condition hold throughout the paper.

Denote 
\[
H^{1}_{\Gamma_0}(\Omega) = \{ w \in H^1(\Omega), \quad w|_{\Gamma_0} = 0 \}. \tag{2.20}
\]

**Assumption (S)** Let $E_0 > 0$ be a constant. For any $E(0) \leq E_0$, there exists a unique solution of the system (2.3) such that 
\[
(u, u_t) \in C \left( [0, +\infty), \left( H^1_{\Gamma_0}(\Omega) \right)^n \times (L^2(\Omega))^n \right). \tag{2.21}
\]

**Remark 2.3.** If $E_0$ is sufficiently small, the above condition is equivalent to the global existence of the system (2.3) with small initial data.

**Theorem 2.1.** Let Assumption (A) hold true. Then there exists positive constants $C_1, C_2$, which are dependent on $E_0$ given by (2.21), such that 
\[
E(t) \leq C_1e^{-C_2t}E(0), \quad \forall t > 0. \tag{2.22}
\]
3 Key Lemmas

Lemma 3.1. Suppose that \(u(x,t)\) solves the system (2.3). Let \(H = \phi(x)x = \phi(x) \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\), where \(\phi \in C^1(\mathbb{R}^n)\) is a nonnegative function. Then

\[
\int_0^T \int_{\Gamma} \left( H(u) \sigma(u) u^T \right) d\Gamma dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u) \right) H : \nu d\Gamma dt \\
\geq \int_0^T \int_{\Omega} u_t (H(u)) T d\Gamma + \delta \int_0^T \int_{\Omega} \phi(x) \sigma(u) \circ \varepsilon(u) dx dt \\
- C \int_0^T \int_{\Omega} |\nabla \phi| |\nabla u|^2 dx dt + \int_0^T \int_{\Omega} a(x) u_t (H(u)) T dx dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u) \right) \text{div} H dx dt,
\]

(3.1)

Moreover, assume that \(P \in C^1(\mathbb{R}^n) : \mathbb{R}^n \mapsto \mathbb{R}^1\) is a real function. Then

\[
\int_0^T \int_{\Omega} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - \sum_{i=1}^{n} |u_i|^{p_i+1} \right) P dx dt \\
\geq \int_0^T \int_{\Omega} \left( |u_i|^2 - P u_i u^T dx \right) + C \int_0^T \int_{\Omega} |\nabla P| |u| |\nabla u| dx dt - \int_0^T \int_{\Gamma} P u \sigma(u) u^T d\Gamma dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega} \left( P a(x) |u|^2 dx \right),
\]

(3.2)

where \(C\) depends on \(\alpha, \beta\), given by (1.3).

Proof. First, we multiply the elastic wave equations in (2.3) by \((H(u))^T\) and integrate over
\[\Omega \times (0, T).\] Note that

\[
\sigma(u) \odot (\nabla (H(u))) = \sum_{i,j,m=1}^{n} \sigma_{ij}(u) (\phi(x)x_{m}u_{i,m})_j \\
= \phi(x) \sum_{i,j,m=1}^{n} \sigma_{ij}(u) (x_{m}u_{i,m})_j + \sum_{i,j,m=1}^{n} \sigma_{ij}(u) \frac{\partial \phi}{\partial x_j} x_{m}u_{i,m} \\
= \phi(x) \left( \sum_{i,j=1}^{n} \sigma_{ij}(u)u_{i,j} + \sum_{i,j,m=1}^{n} \sigma_{ij}(u)x_{m}u_{i,j,m} \right) \\
+ \sum_{i,j,m=1}^{n} \sigma_{ij}(u) \frac{\partial \phi}{\partial x_j} x_{m}u_{i,m} \\
= \phi(x) \left( \sum_{i,j=1}^{n} \sigma_{ij}(u)\varepsilon_{ij}(u) + \sum_{i,j,k,l,m=1}^{n} a_{ijkl}(x)\varepsilon_{kl}(u)x_{m}\varepsilon_{ij,m}(u) \right) \\
+ \sum_{i,j,m=1}^{n} \sigma_{ij}(u) \frac{\partial \phi}{\partial x_j} x_{m}u_{i,m} \\
= \phi(x) \left( \sum_{i,j=1}^{n} \sigma_{ij}(u)\varepsilon_{ij}(u) - \frac{r}{2} \frac{\partial}{\partial r} \left( \sum_{i,j=1}^{n} \sigma_{ij}(u)\varepsilon_{ij}(u) \right) - \frac{r}{2} \sum_{i,j,k,l=1}^{n} \frac{\partial a_{ijkl}(x)}{\partial r} \varepsilon_{ij}(u)\varepsilon_{kl}(u) \right) \\
+ \sum_{i,j,m=1}^{n} \sigma_{ij}(u) \frac{\partial \phi}{\partial x_j} x_{m}u_{i,m} \\
= \phi(x) \sum_{i,j,k,l=1}^{n} \left( a_{ijkl} - \frac{r}{2} \frac{\partial a_{ijkl}}{\partial r} \right) \varepsilon_{ij}(u)\varepsilon_{kl}(u) + \frac{1}{2} H \left( \sum_{i,j=1}^{n} \sigma_{ij}(u)\varepsilon_{ij}(u) \right) \\
+ \sum_{i,j,m=1}^{n} \sigma_{ij}(u) \frac{\partial \phi}{\partial x_j} x_{m}u_{i,m} \tag{3.3}
\]

Hence

\[
\sigma(u) \odot (\nabla (H(u))) \geq \delta \phi(x)\sigma(u) \odot \varepsilon(u) - Cr|\nabla \phi||\nabla u|^2 \\
+ \frac{1}{2} H (\sigma(u) \odot \varepsilon(u)) \\
= \delta \phi(x)\sigma(u) \odot \varepsilon(u) - Cr|\nabla \phi||\nabla u|^2 \\
+ \frac{1}{2} \text{div} (\sigma(u) \odot \varepsilon(u)H) - \frac{1}{2} \text{div} H (\sigma(u) \odot \varepsilon(u)), \tag{3.4}
\]
Therefore
\begin{align*}
0 &= (u_{tt} - \nabla \sigma(u) + a(x)u_t + f(u)) (H(u))^T \\
&= \left( (u_t (H(u))^T)_t - \frac{1}{2} \nabla (u_t^2 H) + \frac{1}{2} u_t^2 \nabla H \right) \\
&\quad - \left( \nabla \left( (\sigma(u) (H(u))^T) - \sigma(u) \odot (\nabla (H(u))) \right) + a(x)g(u_t) (H(u))^T + \nabla (F(u)) \\
&\quad - \frac{1}{2} \nabla (u_t^2 H) + \frac{1}{2} u_t^2 \nabla H \\
&\quad + a(x)u_t (H(u))^T + F(u) \nabla H - F(u) \nabla H \\
&\quad \geq (u_t (H(u))^T)_t + \delta \phi(x) \sigma(u) \odot \varepsilon(u) - C \varepsilon \nabla \phi \nabla u^2 \\
&\quad - \nabla \left( \sigma(u) (H(u))^T \right) + \frac{1}{2} (|u_t|^2 - \sigma(u) \odot \varepsilon(u) - 2F(u)) \nabla H \\
&\quad + \frac{1}{2} \nabla (u_t^2 H) - \frac{1}{2} \nabla (u_t^2 H) + \frac{1}{2} u_t^2 \nabla H \\
&\quad + a(x)u_t (H(u))^T. \tag{3.5}
\end{align*}

In addition, we multiply the wave equation in (2.3) by $Pu$, and integrate over $\Omega \times (0,T)$. Note that
\begin{align*}
0 &= (u_{tt} - \nabla \sigma(u) + a(x)u_t + f(u)) Pu^T \\
&= \left( (Pu_{tt}u^T)_t - P|u_t|^2 \right) \\
&\quad - \left( \nabla (Pu_t u^T) - Pu \sigma(u) \odot \varepsilon(u) - \sigma(u) \odot (\nabla P) u \right) \\
&\quad + \frac{1}{2} (Pa(x)|u|^2) + P \sum_{i=1}^n |u_t|^{p_t+1} \\
&= (Pu_{tt}u) - \nabla (\sigma(u) Pu^T) + \sigma(u) \odot (\nabla P) u + \frac{1}{2} (Pa(x)|u|^2) \tag{3.6}
\end{align*}

The equality (3.2) follows from Green’s formula. ∎

**Lemma 3.2.** Let $u(x,t)$ solve the system (2.3). Then
\begin{equation}
E(t)\bigg|_0^T = - \int_0^T \int_\Omega a(x)|u_t|^2 dx dt, \tag{3.7}
\end{equation}
which implies $E(t)$ is decreasing.
Proof. Multiplying the equation in (2.3) by $u_t$, and integrating over $\Omega \times (0, T)$, the equality (3.7) follows from Green’s formula. $\square$

Proposition 3.1. Let Assumption (A) hold true. Then there exists $T_0 \geq 0$ such that for any $T > T_0$, the only solution $(u, u_t) \in C([0, T], (H^1(\Omega))^n \times (L^2(\Omega))^n)$ to the system

\[
\begin{aligned}
    &u_{tt} - \text{div} \sigma(u) + f(u) = 0 \quad (x, t) \in \Omega \times (0, T), \\
    &u|_{\Gamma_0} = 0 \quad t \in (0, +\infty), \\
    &\sigma(u)\nu|_{\Gamma_1} = 0 \quad t \in (0, +\infty), \\
    &u_t = 0 \quad (x, t) \in \omega \times (0, T),
\end{aligned}
\]

where $f(u)$ is given by (2.4), is the trivial one $u \equiv 0$.

Proof Let $a(x) \equiv 0$, it follows from (3.7) that

$$E(t) = E(0), \quad t > 0. \tag{3.9}$$

Let $H = x$ and $a(x) \equiv 0$. It follows from (3.1) that

$$
\begin{aligned}
    &\int_0^T \int_\Omega (H(u)\sigma(u)u^T) \, d\Gamma dt + \frac{1}{2} \int_0^T \int_{\partial\Omega} (|u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) \, H \cdot \nu \, d\Gamma dt \\
    \geq &\int_0^T \int_\Omega u_t (H(u))^T \, dx |_0^T + \delta \int_0^T \int_\Omega \phi(x)\sigma(u) \circ \varepsilon(u) \, dx dt \\
    &- C \int_0^T \int_\Omega r|\nabla \phi||\nabla u|^2 \, dx dt \\
    &+ \frac{1}{2} \int_0^T \int_\Omega (|u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) \, \text{div} H \, dx dt \\
    &= \int_\Omega u_t (H(u))^T \, dx |_0^T + \delta \int_0^T \int_\Omega \phi(x)\sigma(u) \circ \varepsilon(u) \, dx dt \\
    &- C \int_0^T \int_\Omega r|\nabla \phi||\nabla u|^2 \, dx dt \\
    &+ \frac{n}{2} \int_0^T \int_\Omega \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - \sum_{i=1}^n |u_i|^{p_i+1} \right) \, dx dt \\
    &+ \int_0^T \int_\Omega \sum_{i=1}^n \frac{(p_i - 1)n}{2(p_i + 1)} |u_i|^{p_i+1} \, dx dt. \tag{3.10}
\end{aligned}
$$

Denote

$$
\delta_c = \min_{1 \leq i \leq n} \left\{ \delta_i \left( \frac{(p_i - 1)n}{2(p_i + 1)} \right) \right\}. \tag{3.11}
$$

Let $P = \frac{n - \delta_c}{2}$ and $a(x) \equiv 0$. Substituting the formula (3.2) into the formula (3.10), we obtain

$$
\begin{aligned}
    &\int_0^T \int_\Omega (H(u)\sigma(u)u^T) \, d\Gamma dt + \frac{1}{2} \int_0^T \int_{\partial\Omega} (|u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) \, H \cdot \nu \, d\Gamma dt \\
    \geq &\int_\Omega u_t (H(u) + Pu)^T \, dx |_0^T \\
    &+ \frac{\delta_c}{2} \int_0^T \int_\Omega (|u_i|^2 + \sigma(u) \circ \varepsilon(u) + 2F(u)) \, dx dt. \tag{3.12}
\end{aligned}
$$

Note that $u|_{\Gamma_0} = 0$, then for $1 \leq i, j, m \leq n$,

$$
    u_{i,m} \nu_j = u_{i,jm} \nu = u_{i,jm}, \quad x \in \Gamma_0. \tag{3.13}
$$
Hence

\[ H(u)\sigma(u)\nu^T = \sum_{i,j,m=1}^n x_m u_{i,m} \sigma_{ij}(u) \nu_j \]

\[ \quad = \sum_{i,j,m=1}^n u_{i,j} \sigma_{ij}(u) x_m \nu_m \]

\[ \quad = \sum_{i,j,m=1}^n \varepsilon_{ij}(u) \sigma_{ij}(u) x_m \nu_m \]

\[ \quad = \sigma(u) \circ \varepsilon(u)(H \cdot \nu), \quad x \in \Gamma_0. \]  

(3.14)

With \( u_t = \sigma(u)\nu = 0, \quad x \in \Gamma_1 \),

(3.15)

and

\[ \frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma_0 \quad \text{and} \quad \frac{\partial r}{\partial \nu} \geq 0, \quad x \in \Gamma_1, \]

(3.16)

we obtain

\[ \int_0^T \int_{\Gamma} ((H(u) + Pu)\sigma(u)\nu^T) d\Gamma dt \]

\[ + \frac{1}{2} \int_0^T \int_{\Gamma} |u_t|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) H \cdot \nu d\Gamma dt \]

\[ = \int_0^T \int_{\Gamma_0} ((H(u) + Pu)\sigma(u)\nu^T) d\Gamma dt \]

\[ + \frac{1}{2} \int_0^T \int_{\Gamma_0} |u_t|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) H \cdot \nu d\Gamma dt \]

\[ + \int_0^T \int_{\Gamma_1} ((H(u) + Pu)\sigma(u)\nu^T) d\Gamma dt \]

\[ + \frac{1}{2} \int_0^T \int_{\Gamma_1} |u_t|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u)) H \cdot \nu d\Gamma dt \]

\[ = \frac{1}{2} \int_0^T \int_{\Gamma_0} \sigma(u) \circ \varepsilon(u)(H \cdot \nu) d\Gamma dt \]

\[ - \frac{1}{2} \int_0^T \int_{\Gamma_1} (\sigma(u) \circ \varepsilon(u) + 2F(u)) H \cdot \nu d\Gamma dt \]

\[ \leq 0. \]  

(3.17)

It follows from the Korn’s inequality with Dirichlet boundary conditions \( \text{[41]} \) that

\[ \int_\Omega |\nabla u|^2 dx \]

\[ \leq C \int_\Omega \varepsilon(u) \varepsilon(u) dx \]

\[ \leq C \int_\Omega \sigma(u) \varepsilon(u) dx. \]  

(3.18)

Substituting (3.17) into (3.12), we obtain

\[ \int_0^T E(t) dt \leq CE(0), \]  

(3.19)
which implies
\[(T - C)E(0) \leq 0.\] (3.20)

The assertion holds true. □

By a similar proof with Proposition 3.1, the following proposition holds.

**Proposition 3.2.** Let Assumption (A) hold true. Then there exists \(T_0 \geq 0\) such that for any \(T > T_0\), the only solution \((u, u_t) \in C([0, T], (H^1(\Omega))^n \times (L^2(\Omega))^n)\) to the system
\[
\begin{aligned}
u_{tt} - \text{div} \sigma(u) &= 0 \quad (x, t) \in \Omega \times (0, T), \\
u|_{\Gamma_0} &= 0 \quad t \in (0, +\infty), \\
\sigma(u)\nu|_{\Gamma_1} &= 0 \quad t \in (0, +\infty), \\
u_t &= 0 \quad (x, t) \in \omega \times (0, T),
\end{aligned}
\] (3.21)
is the trivial one \(u \equiv 0\).

4 Proofs of Theorem 2.1

**Lemma 4.1.** Let Assumption(A) hold true. Let \(u\) solve the system (2.3). Then there exists a positive constant \(C\) such that
\[
\int_0^T E(t)dt \leq C \int_0^T \int_{\Omega} a(x)|u_t|^2dxdt + C \int_0^T \int_{\Omega} |u|^2dxdt
\] (4.1)
for sufficiently large \(T\).
**Proof.** It follows from classical Korn’s inequality [14] that
\[
\int_{\Omega} |\nabla u|^2dx \
\leq C \int_{\Omega} (|u|^2 + \epsilon(u) \ast \epsilon(u))dx,
\] (4.2)
and the Korn’s inequality with Dirichlet boundary conditions [11] that
\[
\int_{\Omega} |\nabla u|^2dx \
\leq C \int_{\Omega} \epsilon(u) \ast \epsilon(u)dx \
\leq C \int_{\Omega} \sigma(u) \ast \epsilon(u)dx.
\] (4.3)

Let \(\overline{\omega} \subset \Omega\) be an bounded open set with smooth boundary such that
\[
\Gamma_1 \subset \overline{\omega} \quad \text{and} \quad (\overline{\omega} \setminus \Gamma_1) \subset \omega.
\] (4.4)
Let \(\phi \in C^\infty(\mathbb{R}^n)\) be a nonnegative function such that
\[
\phi = 1, x \in \Omega \setminus \omega \quad \text{and} \quad \phi = 0, x \in \overline{\omega}.
\] (4.5)
Let
\[
H = \phi(x)x.
\] (4.6)
It follows from (3.1) that
\[
\int_0^T \int_{\Omega} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u) \right) \mathbf{H} \cdot \mathbf{v} \, dt + \frac{1}{2} \int_0^T \int_{\Omega} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - 2F(u) \right) H \cdot \mathbf{v} \, dt 
\geq \int_\Omega u_i (H(u))^T dx \bigg|_0^T + \delta \int_\Omega \int_\Omega (\phi(x) \sigma(u) \circ \varepsilon(u) dx dt
\]

\[
- C \int_0^T \int_{\Omega} r|\nabla| \nabla u|^2 dx dt + \int_0^T \int_{\Omega} a(x) u_i (H(u))^T dx dt
\]

\[
+ \frac{1}{2} \int_0^T \int_{\Omega} \left( |u_i|^2 - \sigma(u) \circ \varepsilon(u) - \sum_{i=1}^n |u_i|^{p_i+1} \right) \text{div} H dx dt
\]

\[
+ \int_0^T \int_{\Omega} \sum_{i=1}^n \frac{(p_i - 1) \text{div} H}{2(p_i + 1)} |u_i|^{p_i+1} dx dt. \quad (4.7)
\]

Note that
\[
\text{div} H = n, \quad x \in \Omega \setminus \omega. \quad (4.8)
\]

Denote
\[
\delta_c = \min_{1 \leq i \leq n} \left\{ \delta, \frac{(p_i - 1)n}{2(p_i + 1)} \right\}, \quad (4.9)
\]

Let \( P = \frac{1}{2} \left( \text{div} H - \phi_0 \right) \), substituting (3.2) into (3.1), with (4.2) and (4.3), we obtain
\[
\int_0^T \int_{\Omega \setminus \omega} \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) + 2F(u) \right) dx dt
\]

\[
\leq C(E(0) + E(T)) + C \int_0^T \int_{\Omega \setminus \omega} \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) + \sum_{i=1}^n |u_i|^{p_i+1} \right) dx dt
\]

\[
+ C \int_0^T \int_{\Omega} a(x) \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) \right) dx dt
\]

\[
+ \int_0^T \int_{\Omega} \left( C_c |u|^2 + \epsilon \sigma(u) \circ \varepsilon(u) \right) dx dt. \quad (4.10)
\]

Hence
\[
\int_0^T \int_{\Omega \setminus \omega} \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) + 2F(u) \right) dx dt
\]

\[
\leq C(E(0) + E(T)) + C \int_0^T \int_{\Omega} a(x) \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) + \sum_{i=1}^n |u_i|^{p_i+1} \right) dx dt
\]

\[
+ C \int_0^T \int_{\Omega} |u|^2 dx dt. \quad (4.11)
\]

Therefore
\[
\int_0^T \int_{\Omega} \left( u_i^2 + \sigma(u) \circ \varepsilon(u) + 2F(u) \right) dx dt
\]

\[
\leq C(E(0) + E(T)) + C \int_0^T \int_{\Omega} a(x) \left( |u_i|^2 + \sigma(u) \circ \varepsilon(u) + \sum_{i=1}^n |u_i|^{p_i+1} \right) dx dt
\]

\[
+ C \int_0^T \int_{\Omega} |u|^2 dx dt. \quad (4.12)
\]
Let \( P = a(x) \) in the equality (3.2), we obtain
\[
\int_0^T \int_\Omega a(x) (\sigma(u) \otimes \varepsilon(u) + 2F(u)) \, dxdt 
\leq C(E(0) + E(T)) + C \int_0^T \int_\Omega a(x) (|\mathbf{u}_t|^2 + |\mathbf{u}|^2) \, dxdt 
+ C \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt 
+ \int_0^T \int_\Omega (C_\varepsilon |\mathbf{u}|^2 + \varepsilon \sigma(u) \otimes \varepsilon(u)) \, dxdt.
\] (4.13)

With (4.12), we obtain
\[
\int_0^T E(t) \, dt \leq C(E(0) + E(T)) + C \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt 
+ C \int_0^T \int_\Omega |\mathbf{u}|^2 \, dxdt.
\] (4.14)

It follows from (3.7) that
\[
C(E(0) + E(T)) = 2CE(T) + C \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt.
\] (4.15)

Note that \( E(t) \) is decreasing, then, for \( T \geq 4C \)
\[
\int_0^T E(t) \, dt \leq C \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt + C \int_0^T \int_\Omega |\mathbf{u}|^2 \, dxdt.
\] (4.16)

□

**Lemma 4.2.** Let Assumption (A) hold true. Let \( u(x, t) \) solve the system (2.3). Then for any \( E(0) \leq E_0 \),
\[
\int_0^T E(t) \, dt \leq C(E_0, T) \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt,
\] (4.17)

for sufficiently large \( T \).

**Proof.** We apply compactness-uniqueness arguments to prove the conclusion. It follows from (4.1) that
\[
\int_0^T E(t) \, dt \leq C \int_0^T \int_\Omega a(x)|\mathbf{u}_t|^2 \, dxdt + C \int_0^T \int_\Omega |\mathbf{u}|^2 \, dxdt.
\] (4.18)

Then, if the estimate (4.17) doesn’t hold true, there exist \( \{u_k\}_{k=1}^\infty \) such that
\[
E_k(0) \leq E_0,
\] (4.19)
where
\[
E_k(t) = \frac{1}{2} \int_\Omega (|\mathbf{u}_{k,t}|^2 + \sigma(u_k) \otimes \varepsilon(u_k) + 2F(u_k)),
\] (4.20)
and
\[
\int_0^T \int_\Omega |\mathbf{u}_k|^2 \, dxdt \geq k \int_0^T \int_\Omega a(x)|\mathbf{u}_{k,t}|^2 \, dxdt.
\] (4.21)
With (3.7), we have
\[ E_k(t) \leq E_0, \quad 0 \leq t \leq T. \] (4.22)
and
\[ \int_0^T E_k(t) dt \leq T E_0. \] (4.23)
Therefore, there exists \( \hat{u}_0 \) and a subset of \( \{ u_k \}_{k=1}^{\infty} \), still denoted by \( \{ u_k \}_{k=1}^{\infty} \), such that
\[ u_k \rightharpoonup \hat{u}_0 \text{ weakly in } (H^1(\Omega \times (0, T)))^n, \] (4.24)
and
\[ u_k \rightarrow \hat{u}_0 \text{ strongly in } (L^2(\Omega \times (0, T)))^n. \] (4.25)

Case a:
\[ \int_0^T \int_\Omega |\hat{u}_0|^2 dx dt > 0. \] (4.26)
Denote
\[ q_i = \frac{2n}{(n-2)p_i}, \quad q_i^* = \frac{q_i}{q_i - 1}. \] (4.27)
for \( 1 \leq i \leq n \). Since \( 1 < p_i \leq \frac{n+2}{n-2} \), then
\[ \frac{2n}{n+2} \leq q_i, q_i^* \leq \frac{2n}{n-2}. \] (4.28)
Note that
\[ \frac{1}{q_i} + \frac{1}{q_i^*} = 1, \] (4.29)
Then, \( L^{q_i^*}(\Omega) \) is the dual space of \( L^{q_i}(\Omega) \).

Note that
\[ H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega). \] (4.30)
Therefore, it follows from (4.22) that for \( 1 \leq i \leq n \)
\[ \{|u_{ki}|^{p_i-1}u_{ki}\} \text{ are bounded in } L^\infty([0,T],L^{q_i}(\Omega)). \] (4.31)
Then
\[ \{|u_{ki}|^{p_i-1}u_{ki}\} \text{ are bounded in } L^{q_i}(\Omega \times (0,T)). \] (4.32)
Hence for \( 1 \leq i \leq n \)
\[ |u_{ki}|^{p_i-1}u_{ki} \rightharpoonup |\hat{u}_{0i}|^{p_i-1}\hat{u}_{0i} \text{ weakly in } L^{q_i}(\Omega \times (0,T)). \] (4.33)
It follows from (4.21) that
\[ a(x)\hat{u}_{0t} = 0 \quad (x,t) \in \mathbb{R}^n \times (0,T). \] (4.34)
Therefore, with (4.24) and (4.33), we obtain
\[
\begin{cases}
\hat{u}_{0t} - \text{div } \sigma(\hat{u}_0) + f(\hat{u}_0) = 0 & (x,t) \in \Omega \times (0,T), \\
\hat{u}_{0|\Gamma_0} = 0 & t \in (0, +\infty), \\
\sigma(\hat{u}_0)\nu|_{\Gamma_1} = 0 & t \in (0, +\infty), \\
\hat{u}_{0t} = 0 & (x,t) \in \omega \times (0,T),
\end{cases}
\] (4.35)
where \( f(\hat{u}_0) \) is given by (2.4). It follows from Proposition 3.1 that
\[
\hat{u}_0 \equiv 0, \quad (x, t) \in \Omega \times (0, T),
\] (4.36)
which contradicts (4.26).

**Case b:**
\[
\hat{u}_0 \equiv 0 \quad \text{on} \quad \Omega \times (0, T).
\] (4.37)

Denote
\[
v_k = u_k / \sqrt{c_k} \quad \text{for} \quad k \geq 1,
\] (4.38)
where
\[
c_k = \int_0^T \int_{\Omega} |u_k|^2 \, dx \, dt.
\] (4.39)

Then \( v_k \) satisfies for \( 1 \leq i \leq n \),
\[
v_{k,tt} - \text{div} \sigma(v_k) + a(x)v_{k,t} + f(u_k) \sqrt{c_k} = 0 \quad (x, t) \in \Omega \times (0, T),
\] (4.40)
and
\[
\int_0^T \int_{\Omega} |v_k|^2 \, dx \, dt = 1.
\] (4.41)

It follows from (4.21) that
\[
1 \geq k \int_0^T \int_{\Omega} a(x)v_{k,t}^2 \, dx \, dt.
\] (4.42)

Therefore, It follows from (4.18) that
\[
\hat{E}_k(0) \leq 1 + \frac{1}{k} \leq 2,
\] (4.43)
where
\[
\hat{E}_k(t) = \frac{1}{2} \int_{\Omega} \left( |v_{k,t}|^2 + \sigma(v_k) \cdot \varepsilon(u_k) + \sum_{i=1}^{n} \frac{2}{p_i - 1} |u_{ki}|^{p_i - 1} |v_{ki}|^2 \right).
\] (4.44)

Hence, there exists \( v_0 \) and a subset of \( \{v_k\}_{k=1}^{\infty} \), still denoted by \( \{v_k\}_{k=1}^{\infty} \), such that
\[
v_k \rightharpoonup v_0 \quad \text{weakly in} \quad (H^1(\Omega \times (0, T)))^n,
\] (4.45)
and
\[
v_k \rightarrow v_0 \quad \text{strongly in} \quad (L^2(\Omega \times (0, T)))^n.
\] (4.46)

It follows from (3.7), (4.38) and (4.43) that
\[
\hat{E}_k(t) \leq \hat{E}_k(0) \leq 2, \quad \forall 0 \leq t \leq T.
\] (4.47)

Let \( q_i, q_i^* \) be given by (4.27). Note that
\[
H^1(\Omega) \hookrightarrow L^\frac{2n}{n-2} (\Omega).
\] (4.48)

Therefore, it follows from (4.47) that for \( 1 \leq i \leq n \)
\[
\int_0^T \int_{\Omega} \sum_{i=1}^{n} |u_{ki}|^{p_i - 1} v_{ki} \, dx \, dt
\]
\[
= \int_0^T \int_{\Omega} \sum_{k=1}^{n} c_k \frac{q_i(p_i - 1)}{2} |v_{ki}|^{\frac{2n}{n-2}} \, dx \, dt
\]
\[
\leq \sum_{k=1}^{n} \frac{q_i(p_i - 1)}{2} C(T).
\] (4.49)
With (4.37) and (4.39), we obtain
\[
\lim_{k \to +\infty} \int_0^T \int_{\Omega} \sum_{i=1}^n \left( |u_{ki}|^{p_i-1} v_{ki} \right)^{q_i} dx dt = 0.
\] (4.50)

It follows from (4.42) that
\[
a(x)v_{0t} = 0 \quad (x, t) \in \mathbb{R}^n \times (0, T).
\] (4.51)

Therefore, it follows from (4.40) and (4.50) that
\[
\begin{aligned}
\begin{cases}
v_{0t} - \text{div} \sigma(v_0) = 0 & (x, t) \in \Omega \times (0, T), \\
v_0|_{\Gamma_0} = 0 & t \in (0, +\infty), \\
\sigma(v_0)v|_{\Gamma_1} = 0 & t \in (0, +\infty), \\
v_{0t} = 0 & (x, t) \in \omega \times (0, T),
\end{cases}
\end{aligned}
\] (4.52)

Then it follows from Proposition 3.2 that
\[
v_0 = 0, \quad x \in \Omega,
\] (4.53)

which contradicts (4.41). □

**Proof of Theorem 2.1** It follows from (3.7) and (4.17) that, for sufficiently large \( T \),
\[
TE(T) \leq \int_0^T E(t) dt \leq C(E_0, T) \int_0^T \int_{\Omega} a(x)|u_t|^2 dx dt
\leq C(E_0, T) (E(0) - E(T)).
\] (4.54)

Hence,
\[
E(T) \leq \frac{C(E_0, T)}{C(E_0, T) + T} E(0).
\] (4.55)

The estimate (2.22) holds. □

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