Construction of Recurrent Fractal Interpolation Surfaces with Function Scaling Factors and Estimation of Box-counting Dimension on Rectangular Grids

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Abstract

We consider a construction of recurrent fractal interpolation surfaces with function vertical scaling factors and estimation of their box-counting dimension. A recurrent fractal interpolation surface (RFIS) is an attractor of a recurrent iterated function system (RIFS) which is a graph of bivariate interpolation function. For any given data set on rectangular grids, we construct general recurrent iterated function systems with function vertical scaling factors and prove the existence of bivariate functions whose graph are attractors of the above constructed RIFSs. Finally, we estimate lower and upper bounds for the box-counting dimension of the constructed RFISs.

Keywords: Recurrent Iterated Functions System (RIFS), Fractal surface, Fractal Interpolation function (FIF), Box-counting dimension

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1 Introduction

Fractal interpolation surfaces (FISs), fractal sets which are graphs of bivariate fractal interpolation functions, are being widely used in approximation theory, computer graphics, image compression, metallurgy, physics, geography, geology and so on. (See [1, 4, 9, 12, 14, 21, 22].) Barnsley [2] defined a fractal function as a function whose graph is an attractor of an iterated function system (IFS) and such fractal functions and fractal interpolation are widely studied to construct fractal curves and surfaces in many papers ([1, 6, 7, 8, 9, 11, 13, 14, 15, 16, 17, 18, 22]).

One classical method of construction of fractal surfaces is to use fractal curves [9, 14, 17, 18, 20]. This approach is useful in the case when data sets (measurement data) are given only on the boundaries of the domain. The history and recent developments of this approach are described in [20].

Another (direct) method of construction of fractal surfaces is to use bivariate fractal interpolation functions. This method is useful in the case when data sets are given on the mesh-points of the whole domain. Massopust [15] provided a construction of self-affine fractal interpolation surfaces with data set on triangular domain, where the interpolation points (data points) on the boundary are assumed coplanar. This results was generalized to allow more general boundary data and domains in [11].

Many authors studied construction methods of FISs with the data set given on rectangular grids ([6, 7, 8, 13, 16]). In [8] for such data sets on a rectangular grid that interpolation points on the boundary are collinear, a construction of FISs which are attractors of IFSs was provided. This was generalized in Malysz [13], where IFS was constructed by using constant vertical scaling factors, linear
contraction transformations of domain and quadratic polynomials. In [16] the authors allowed an arbitrary data set and constructed IFS using function scaling factors and Lipschitz transformations of domain, and estimated lower and upper bounds for the box-counting dimensions of the constructed surfaces. This estimation for box-counting dimension is improved in [19](See [10], too.)

A recurrent iterated functions system (RIFS) defined in [3] is generalization of IFS and in [7] they suggested a construction of recurrent fractal interpolation surfaces (RFISs) using RIFS. This is a more flexible method of constructing fractal surfaces than using IFS and applied to the image compression ([21]). Bouboulis and Dalla [6] provide a general construction of recurrent fractal interpolation functions (RFIFs) on \( \mathbb{R}^N \) by RIFS. In [16] they used domain contraction transformations and constant scaling factors.

This paper is a continuation and extension of [16] and [19] where they used IFS and function vertical scaling factors. We present a flexible construction of RFISs by RIFSs with function vertical scaling factors and estimate lower and upper bounds for the box-counting dimensions of the constructed surfaces.

The remainder of the article is organized as follows: The section 2 describes construction of recurrent fractal interpolation surfaces on the rectangular grids and gives an example. In the section 3 we estimate upper and lower bounds of the box-counting dimension of RFISs constructed in section 2. We refer to [3] [5] [20] for necessary preliminaries on RIFS.

2 Construction of Recurrent Fractal Interpolation Surfaces

In this section, we construct recurrent fractal interpolation surfaces with a data set on rectangular grids. Let a data set on the rectangular grid be given by

\[
P = \{(x_i, y_j, z_{ij}) \in \mathbb{R}^3; \ i = 0, 1, \ldots, n; \ j = 0, 1, \ldots, m\},
\]

\[
(x_0 < x_1 < \ldots < x_n; \ y_0 < y_1 < \ldots < y_m).
\]

Let denote \( N = n \cdot m; \ N_{nm} = \{1, \ldots, n\} \times \{1, \ldots, m\} \)

\[
I_i = [x_{i-1}, x_i], \ J_j = [y_{j-1}, y_j], \ E = [x_0, x_n] \times [y_0, y_m], \ E_{ij} = I_i \times J_j, \ (i, j) \in N_{nm}
\]

and \( E_{ij} \) are called regions. \( N \) is the set of all positive integers.

Let \( l \geq 2 \) \((l \in \mathbb{N})\). In the rectangle \( E \) we choose rectangles \( \hat{E}_k (k = 1, \ldots, l) \) which consist of some regions, and call \( \hat{E}_k \) domains. Then \( \hat{E}_k = \hat{I}_k \times \hat{J}_k, k = 1, \ldots, l \), where \( \hat{I}_k \) and \( \hat{J}_k \) are closed intervals on the \( x \) and \( y \) axes, respectively. The end points of \( \hat{I}_k \) \((k \in \{1, \ldots, l\})\) coincide with some end points of intervals \( I_i, (i = 1, \ldots, n) \). If the indices of the start point and the end point of \( \hat{I}_k \) are respectively denoted by \( s_x(k), e_x(k) \), then two mappings \( s_x: \{1, \ldots, l\} \to \{1, \ldots, n\}, e_x: \{1, \ldots, l\} \to \{1, \ldots, n\} \) are well defined. For \( \hat{J}_k \), the two mappings \( s_y: \{1, \ldots, l\} \to \{1, \ldots, m\}, e_y: \{1, \ldots, l\} \to \{1, \ldots, m\} \) are defined similarly and \( \hat{I}_k, \hat{J}_k \) are respectively represented by

\[
\hat{I}_k = [x_{s_x(k)}, x_{e_x(k)}], \quad \hat{J}_k = [y_{s_y(k)}, y_{e_y(k)}].
\]

Assume that \( e_x(k) - s_x(k) \geq 2, e_y(k) - s_y(k) \geq 2, k = 1, \ldots, l \). This means that the intervals \( \hat{I}_k, \hat{J}_k \) include at least 2 small intervals \( I_i, J_j \) and the domain \( \hat{E}_k \) includes \([e_x(k) - s_x(k) - 1, e_x(k) - s_x(k) + 1] \times [e_y(k) - s_y(k) - 1, e_y(k) - s_y(k) + 1]\) regions.

To each region \( E_{ij} \), we relate a domain \( \hat{E}_k \). This correspondence is represented by a map \( \gamma: N_{nm} \to \{1, \ldots, l\} \). Throughout this paper we fix a map \( \gamma \) and denote \( k = \gamma(i, j) \).

Let \( L_{x,ij}: [x_{s_x(k)}, x_{e_x(k)}] \to [x_{i-1}, x_i] \) and \( L_{y,ij}: [y_{s_y(k)}, y_{e_y(k)}] \to [y_{j-1}, y_j] \), \((i, j) \in N_{nm}\) be contraction homeomorphisms. These mappings respectively map end points of \( \hat{I}_k, \hat{J}_k \) into end points of the intervals \( I_i, J_j \), that is

\[
L_{x,ij}(\{x_{s_x(k)}, x_{e_x(k)}\}) = \{x_{i-1}, x_i\}, \quad L_{y,ij}(\{y_{s_y(k)}, y_{e_y(k)}\}) = \{y_{j-1}, y_j\}.
\]
These mappings can be easily constructed as \[7\].

Let \( s_{ij} : E_{ij} \to \mathbb{R} \) be contraction mappings on regions such that \( 0 < |s_{ij}(x, y)| \leq s < 1 \), which are called \textit{vertical scaling factors}. Let \( Q_{ij} : \tilde{E}_k \to \mathbb{R} \) be Lipschitz mappings. We define mappings \( L_{ij} : \tilde{E}_k \to E_{ij} \) and \( F_{ij} : \tilde{E}_k \times \mathbb{R} \to \mathbb{R} \) as follows, respectively:

\[
L_{ij}(x, y) = (L_{x,ij}(x), L_{y,ij}(y)), \quad F_{ij}(x, y, z) = s_{ij}(L_{ij}(x, y))z + Q_{ij}(x, y).
\]

Now we define transformations \( W_{ij} : \tilde{E}_k \times \mathbb{R} \to E_{ij} \times \mathbb{R} (i = 1, \ldots, n, j = 1, \ldots, m) \) by

\[
W_{ij}(x, y, z) = (L_{ij}(x, y), F_{ij}(x, y, z)).
\]

Here \( s_{ij} \) are taken as free unknown functions.

For construction of RFIS, the \textit{following condition} for \( W_{ij} \) is \textit{important}: there exists at least one continuous function \( g : E \to \mathbb{R} \) interpolating the given data set \( P \) such that

\[
F_{ij}(x, y, g(x, y)) = g(L_{ij}(x, y)), \quad \alpha \in \{ s_x(k), e_x(k) \}, \ y \in [y_{s_x(k)}, y_{e_x(k)}], \quad \beta \in \{ s_y(k), e_y(k) \}.
\]

The transformations \( W_{ij} \) satisfying \( [3] \) and \( [4] \) can be constructed as follows, for example.

**Example 1.** Select one Lipschitz continuous function \( g_0 \) interpolating the data set \( P \) and let \( Q_{ij}(x, y) = g_0(L_{ij}(x, y)) - s_{ij}(L_{ij}(x, y)) \cdot g_0(x, y), \ (x, y) \in \tilde{E}_k \), then

\[
F_{ij}(x, y, z) = s_{ij}(L_{ij}(x, y)) \cdot (z - g_0(x, y)) + g_0(L_{ij}(x, y)), \ (x, y) \in \tilde{E}_k.
\]

Then the \( W_{ij} \) given by \( [2] \) with these \( F_{ij} \) are just the needed transformations. That is why \( g_0 \) and the function \( h \) coincided with \( g_0 \) in \( \partial E_{ij} \) satisfy the conditions \( [3] \) and \( [4] \).

**Remark 1.** Then the mappings \( L_{ij} \) map the vortices of the domains \( \tilde{E}_k \) into the vortices of the regions \( E_{ij} \). That is, for \( \alpha \in \{ s_x(k), e_x(k) \}, \ \beta \in \{ s_y(k), e_y(k) \} \), we have

\[
L_{ij}(x_{\alpha}, y_{\beta}) = (x_{\alpha}, y_{\beta}) \text{ (where } a \in \{i-1,i\}, \ b \in \{j-1,j\}).
\]

And from the conditions \( [3] \) and \( [4] \), the transformations \( W_{ij} \) map the data points (in \( P \)) given on the vortices of the domains into the data points given on the vortices of the regions, that is, for \( \alpha \in \{ s_x(k), e_x(k) \}, \ \beta \in \{ s_y(k), e_y(k) \} \), we have

\[
F_{ij}(x_{\alpha}, y_{\beta}, z_{\alpha\beta}) = z_{ab} \text{ (where } L_{ij}(x_{\alpha}, y_{\beta}) = (x_{\alpha}, y_{\beta}) \text{, } a \in \{i-1,i\}, \ b \in \{j-1,j\}).
\]

We sometimes denote \( L_{ij}, W_{ij} \) by \( L_{ij,k}, W_{ij,k} \) (\( k = \gamma(i,j) \)) explicitly pointing their domains. We denote Lipschitz (or contraction) constant of Lipschitz (or contraction) mapping \( f \) by \( L_f (c_f) \) in what follows.

We define a distance \( \rho_0 \) in \( \mathbb{R}^3 \) for \( \theta > 0 \) by

\[
\rho_0((x, y, z), (x', y', z')) = |x - x'| + |y - y'| + \theta|z - z'|, \ (x, y, z), (x', y', z') \in \mathbb{R}^3
\]

as \( [8] \). Let

\[
\bar{c}_L = \max\{ c_{L,ij} | (i, j) \in N_{nm} \}, \quad \bar{L}_Q = \max\{ L_{Q,ij,k} | (i, j) \in N_{nm} \}.
\]

If \( 0 < \theta < (1 - \bar{c}_L)/\bar{L}_Q \), then the distance \( \rho_0 \) is equivalent to the Euclidean metric on \( \mathbb{R}^3 \) and \( W_{ij}, i = 1, \ldots, n, j = 1, \ldots, m \) are contraction mappings with respect to the distance \( \rho_0 \) (see \( [6], [7], [8] \)).
We define a row-stochastic matrix \( M = (p_{st})^{N}_{s,t=1} \) by
\[
p_{st} = \begin{cases} 
\frac{1}{a_s}, & E_{s-1}(s) \subset \tilde{E}_{\gamma(\tau^{-1}(t))}, \\
0, & E_{s-1}(s) \not\subset \tilde{E}_{\gamma(\tau^{-1}(t))}.
\end{cases}
\]
Here the mapping \( \tau: N_{nm} \to \{1, \ldots, N\} \) is the bijection defined by \( \tau(i,j) = i + (j-1)n \) and for every fixed \( s = 1, \ldots, N \), the number \( a_s \) indicates the number of elements of the set \( \{ t \in \{1, \ldots, N\} | E_{s-1}(s) \subset \tilde{E}_{\gamma(\tau^{-1}(t))} \} \). In other words, \( a_s \) is the number of non-zero elements in \( s \)-th row of the above row-stochastic matrix \( M \). This means that \( p_{st} \) is positive iff there exists a transformation \( L_{ij} \) that maps \( s \)-th row into \( t \)-th region (7).

Now we assume that \( M \) is irreducible and define the recurrent iterated function system (RIFS) corresponding to the given data set \( P \) by \( \{\mathbf{R}^2; M; W_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\} \). Its connection matrix \( C = (c_{st})_{N \times N} \) is given as follows. (Then \( C \) is also irreducible.)
\[
c_{st} = \begin{cases} 
1, & p_{ts} > 0, \\
0, & p_{ts} = 0.
\end{cases}
\]
We denote the attractor of the RIFS constructed above by \( \mathcal{A} \). The following theorem shows that \( \mathcal{A} \) is a recurrent fractal surface.

**Theorem 1** There exists an interpolation function \( f: E \to \mathbf{R} \) of the data set \( P \) whose graph is the attractor \( \mathcal{A} \) of the RIFS constructed above.

**Proof** Let \( C(E) \) be the following set:
\[
C(E) = \{ h \in C^0(E) | h \text{ interpolates the data set } P \text{ and satisfies (3), (4)} \}.
\]
Then \( C(E) \) is not empty set and complete metric space with respect to norm \( || \cdot ||_\infty \). When \( g \in C(E) \), if we define a function \( Tg \) on \( E \) by
\[
(Tg)(x, y) = F_{ij}(L_{ij}^{-1}(x, y), g(L_{ij}^{-1}(x, y))), \quad (x, y) \in E_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m,
\]
then \( Tg \in C(E) \). In fact, for any fixed \( (i, j) \in N_{nm} \) and for any \( \alpha \in \{ s_{\alpha}(k), e_{\alpha}(k) \} \) \((k = \gamma(i, j)), \)
\[
(Tg)(L_{ij}(x, y), y) = F_{ij}(x_\alpha, y, g(x_\alpha, y)) = g(L_{ij}(x_\alpha, y), y \in [y_{\alpha}(k), y_{\gamma}(k)],
\]
\[
(Tg)(L_{ij}(x, y), y) = F_{ij}(x, y_\alpha, g(x, y_\alpha)) = g(L_{ij}(x, y_\alpha), x \in [x_{\alpha}(k), x_{\gamma}(k)].
\]
This means that \( Tg = g \) on the segments \( \{(x, y); y \in [y_0, y_m]\}, \{(x, y); x \in [x_0, x_n]\}, i = 0, 1, \ldots, n, j = 0, 1, \ldots, m \). Thus, we have
\[
F_{ij}(x_\alpha, y, (Tg)(x_\alpha, y)) = F_{ij}(x_\alpha, y, g(x_\alpha, y)) = g(L_{ij}(x_\alpha, y)) = (Tg)(L_{ij}(x_\alpha, y)).
\]
Thus \( Tg \) satisfies (3). Similarly, we can prove that \( Tg \) satisfies (4).

Therefore, the operator \( T: C(E) \to C(E) \) is well defined and the operator \( T \) is contractive from the assumption \( |s_{ij}| \leq s < 1 \). Hence, the operator \( T \) has a unique fixed point \( f \in C(E) \), which is presented by
\[
f(x, y) = F_{ij}(L_{ij}^{-1}(x, y), f(L_{ij}^{-1}(x, y))), i = 1, \ldots, n; j = 1, \ldots, m.
\]
Thus, for the graph \( Gr(f) \) of the function \( f \) we obtain
\[
Gr(f) = \bigcup_{s=1}^{N} \bigcup_{t \in \Lambda(s)} W_{\tau^{-1}(s)}(Gr(f)|_{E_{s-1}(s)}).
\]
Here $\Lambda(s) = \{ t \in \{1, \cdots, N\} | p_{ts} > 0 \}, s = 1, \ldots, N$. This means that $Gr(f)$ is the attractor of the RIFS constructed above. The uniqueness of the attractor implies $A = Gr(f)$. (QED)

**Remark 2.** Experiments often shows that if $0 < |s_{ij}| < 1$ outside a set with zero measure, the attractor of the RIFS constructed above becomes a recurrent fractal surface.

**Example 2.** A data set is given by the following table 1, the graph of which is shown in the figure 1. Let $s_{ij}(x,y) = \sin(10x^2 + 10y^2)$ (see the figure 2). Let $g_0(x,y)$ be the Lagrangean interpolation function. Then the attractor of the RIFS constructed in the example 1 on page 3 by (2) and (5) is drawn in the figure 3.

| X  | Y  | 0  | 50 | 100 | 150 | 200 |
|----|----|----|----|-----|-----|-----|
| 0  | 35 | 42 | 76 | 61  | 44  |     |
| 50 | 43 | 28 | 88 | 83  | 33  |     |
| 100| 78 | 84 | 58 | 33  | 25  |     |
| 150| 68 | 33 | 73 | 86  | 77  |     |
| 200| 47 | 29 | 88 | 43  | 54  |     |

Table 1. Data set in example 1.
Figure 2: Vertical scaling factor $s(x, y) = \sin(10x^2 + 10y^2)$.

Figure 3: Recurrent Fractal Interpolation Surface constructed from the dataset in table 1.
3 Box-counting Dimensions of Recurrent Fractal Surfaces

In this section we estimate upper and lower bounds of the box-counting dimension of the attractor \( \mathcal{A} \) of RFIS constructed in the previous section.

There exists a bi-Lipschitz homeomorphism which maps a rectangle \([0, 1] \times [0, t]\) \((t > 0)\) to any rectangle in \( \mathbb{R}^2 \) and box-counting dimension is invariant under bi-Lipschitz homeomorphisms. Thus we can assume that \( E = [0, 1] \times [0, m/n] \) and the end points of the regions and domains satisfy the following conditions.

\[
x_{i+1} - x_i = y_{j+1} - y_j = \frac{1}{n}, \quad x_{e_v(k)} - x_{s_v(k)} = y_{e_v(k)} - y_{s_v(k)} = \frac{a}{n},
\]

\(i = 0, 1, \ldots, n - 1, \quad j = 0, 1, \ldots, m - 1, \quad a \in \mathbb{N}, \quad k = 1, \ldots, l.\)

Then there are exactly \( a^2 \) regions in every domain.

For \( r > 0 \), we define a set \( \mathcal{B}_r \) of cubes as follows:

\[
\mathcal{B}_r = \left\{ \left[ \frac{u - 1}{a^r}, \frac{u}{a^r} \right] \times \left[ \frac{v - 1}{a^r}, \frac{v}{a^r} \right] : u, v \in \mathbb{N}, \quad b \in \mathbb{R} \right\}.
\]

Let denote the smallest number of cubes in \( \mathcal{B}_r \) necessary to cover \( \mathcal{A} \) by \( N(\frac{1}{a^r}) \) and the smallest number of \( \frac{1}{a^r} \)-mesh cubes that cover \( \mathcal{A} \) by \( N'(\frac{1}{a^r}) \). We can easily see that

\[
N'(\frac{1}{a^r}) \leq N(\frac{1}{a^r}) \leq 8 \cdot N'(\frac{1}{a^r}),
\]

which allows us to use \( N(\frac{1}{a^r}) \) to estimate the box-counting dimension of \( \mathcal{A} \).

For a set \( D \subset \mathbb{R}^2 \), we define the maximum variation of a function \( f \) on \( D \) as follows:

\[
R_f[D] = \sup \{|f(x_2, y_2) - f(x_1, y_1)| : (x_1, y_1), (x_2, y_2) \in D\}.
\]

**Lemma 1** Let \( D \) be a rectangle in \( \mathbb{R}^2 \) and \( W : D \times \mathbb{R} \to D \times \mathbb{R} \) the transformation of the form

\[
W \left( \begin{array}{c} x \\ y \\ z \end{array} \right) = \left( \begin{array}{c} L(x, y) \\ F(x, y, z) \end{array} \right) = \left( \begin{array}{c} L(x, y) \\ s(L(x, y))z + Q(x, y) \end{array} \right).
\]

Here \( Q \) is Lipschitz function with the Lipschitz constants \( \mathcal{L}_Q \), \( L \) is the domain contraction transformation (defined just like \( \mathcal{L}_f \) in the section 2) with contraction factor \( \mathcal{L}_D \) and \( s(x, y) \) is a contraction function with \( |s(x, y)| < 1 \). Then for any continuous function \( f : D \to \mathbb{R} \), we have

\[
R_{f(L^{-1}, f \circ L^{-1})}[L(D)] \leq \bar{s}R_f[D] + \text{diam}(D)(c_s \bar{f} + L_Q).
\]

Here \( \text{diam}(D) \) is a diameter of the set \( D \), \( \bar{s} = \max_D |s(x, y)| \), \( c_s \) is a contraction factor of \( s(x, y) \), \( \bar{f} = \max_D |f(x, y)| \).

**Proof.** For \( (x, y), (x', y') \in L(D) \), let denote \( (\tilde{x}, \tilde{y}) = L^{-1}(x, y), (\tilde{x}', \tilde{y}') = L^{-1}(x', y') \in D \).

Then we have

\[
|F(L^{-1}, f \circ L^{-1})(x, y) - F(L^{-1}, f \circ L^{-1})(x', y')| =
\]

\[
= |F(L^{-1}(x, y), f \circ L^{-1}(x, y)) - F(L^{-1}(x', y'), f \circ L^{-1}(x', y'))| =
\]

\[
= |s(x, y)f(\tilde{x}, \tilde{y}) + Q(\tilde{x}, \tilde{y}) - s(x', y')f(\tilde{x}', \tilde{y}') - Q(\tilde{x}', \tilde{y}')| =
\]

\[
= |s(x, y)f(\tilde{x}, \tilde{y}) - s(x, y)f(\tilde{x}', \tilde{y}') + s(x, y)f(\tilde{x}', \tilde{y}') - s(x', y')f(\tilde{x}', \tilde{y}') + Q(\tilde{x}, \tilde{y}) - Q(\tilde{x}', \tilde{y}')| \leq \bar{s}R_f[D] + \mathcal{L}_D(d(x, y), (x', y'))f + \mathcal{L}_Q(\tilde{x}, \tilde{y}), (\tilde{x}', \tilde{y}').
\]

\[
\leq \bar{s}R_f[D] + \text{diam}(D)(c_s \bar{f} + L_Q). \quad \text{(QED)}
\]
For $N \times N$ matrix $U = (u_{ij})$, $V = (v_{ij})$ we define the relation $" < "$ by

$$U < V \iff u_{ij} < v_{ij}, \; i, j = 1, 2, \ldots, N$$

Points of a set $B$ in $\mathbb{R}^3$ are said to be $x$ (or $y$)-collinear if all the points of the set $B$ with the same $x$ (or $y$) coordinates lies on one line.

**Theorem 2** Let the function $f : E \rightarrow \mathbb{R}$ be the interpolation function constructed in Theorem 1. Let $\mathbf{S}$ and $\mathbf{S}_\tau$ be $N \times N$ diagonal matrices

$$\mathbf{S} = \text{diag}(s_{\tau-1(1)}, \ldots, s_{\tau-1(N)}), \quad \mathbf{S}_\tau = \text{diag}(\xi_{\tau-1(1)}, \ldots, \xi_{\tau-1(N)})$$

where $s_{\tau-1}(i, j) = s_{ij} = \max_{E_{ij}} |s_{ij}(x, y)|$, $\xi_{\tau-1}(i, j) = \xi_{ij} = \min_{E_{ij}} |s_{ij}(x, y)|$, $i \in \{1, \ldots, N\}$, $t = \tau(i, j)$. If there exists a domain $\bar{E}_{k_0}$ such that the interpolation points of $P \cap (\bar{E}_{k_0} \times \mathbb{R})$ are not $x$-collinear or not $y$-collinear, then the box-counting dimension $\dim_B A$ of the attractor $A$ is estimated as follows:

1) If $\bar{\lambda} > a$, then

$$1 + \log_a \bar{\lambda} \leq \dim_B A \leq 1 + \log_a \bar{\lambda}.$$  

2) If $\bar{\lambda} \leq a$, then

$$\dim_B A = 2.$$  

Here $\bar{\lambda} = \rho(\mathbf{SC})$ and $\bar{\lambda} = \rho(\mathbf{SC})$ are spectral radii of the irreducible matrices $\mathbf{SC}$ and $\mathbf{SC}$, respectively.

**Proof.** Proof of (1). We simply denote the maximum variance $R_f[\bar{E}_{k(i, j)}]$ by $R_{ij}$. Let denote $\frac{1}{a}$ by $\varepsilon_r$. Then $r \rightarrow \infty \Leftrightarrow \varepsilon_r \rightarrow 0$.

After applying once each $W_{ij} = W_{ij, k}$ ($k = \gamma(i, j)$) to the interpolation points in the domain $\bar{E}_{k_0}$, we have $(a + 1)^2$ new image points of interpolation points in the region $E_{ij}$. According to the hypothesis, the interpolation points lying inside the domain $\bar{E}_{k_0}$ are not $x$-collinear or not $y$-collinear and the $(a + 1)^2$ image points in the region $E_{i_0, j_0}$ ($k_0 = \gamma(i_0, j_0)$) are not $x$-collinear or not $y$-collinear. On the other hand the connection matrix $C$ is irreducible and thus the region $E_{i_0, j_0}$ is mapped into arbitrary regions $E_{ij}$ by applying the appropriately selected transformations from $\{W_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$ several times. So in each region $E_{ij}$ there exist the $(a + 1)^2$ image points of interpolation points which are not $x$-collinear or not $y$-collinear. Therefore, in each region $E_{ij}$ there are at least 3 image points of interpolation points which are not collinear and the maximum vertical distance computed only with respect to the z-axis from one of the 3 points to the line through other 2 points is greater than 0 (17). The maximum value is called a height and denote by $H_{ij}$.

Ont the other hand, by Lemma 1 on each region $E_{ij}$ we have

$$R_f[E_{ij}] \leq s_{ij} R_f[\bar{E}_{\gamma(i, j)}] + \frac{a}{n} b.$$  

where $b = 2^2 (c_s f + L_Q)$.

We define non negative vectors $\mathbf{h}_1, \mathbf{r}, \mathbf{u}_1$ and $\mathbf{i}$ as follows:

$$\mathbf{h}_1 = \begin{pmatrix} H_{\tau-1(1)} \\ \vdots \\ H_{\tau-1(N)} \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} s_{\tau-1(1)} R_{\tau-1(1)} \\ \vdots \\ s_{\tau-1(N)} R_{\tau-1(N)} \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad \mathbf{u}_1 = \mathbf{r} + \frac{a}{n} b \mathbf{i}.$$  

Since $A$ is the graph of a continuous function defined on $E$, the smallest number of cubes in $B_r$ necessary to cover $(E_{ij} \times \mathbb{R}) \cap A$ is greater than the smallest number of cubes in $B_r$ necessary to
cover vertical line with the length $H_{ij}$ and less than the smallest number of cubes in $B_r$ necessary to cover the rectangular parallelepiped $E_{ij} \times \int_{ij}$, where

$$f_{ij} = \min_{E_{ij}} f(x, y), \quad \bar{f}_{ij} = \max_{E_{ij}} f(x, y).$$

Therefore (in the bellow $[d]$ is the integer part of $d \in \mathbb{R}$),

$$\sum_{i=1}^{n} \sum_{j=1}^{m} [H_{ij} \varepsilon^{-1}_r] \leq N(\varepsilon_r) \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \left\lfloor (s_{ij} R_{ij} + \frac{a}{n} b) \varepsilon^{-1}_r \right\rfloor + 1 \right) \left( \left\lceil \frac{\varepsilon^{-1}_r}{n} \right\rceil + 1 \right)^2,$$

$$\sum_{i=1}^{N} (H_{\tau^{-1}(i)} \varepsilon^{-1}_r) - N \leq N(\varepsilon_r) \leq \sum_{i=1}^{N} \left( (s_{\tau^{-1}(i)} R_{\tau^{-1}(i)} + \frac{a}{n} b) \varepsilon^{-1}_r + 1 \right) \left( \left\lceil \frac{\varepsilon^{-1}_r}{n} \right\rceil + 1 \right)^2$$

and thus if we denote $\Phi(a) = a_1 + \cdots + a_N$ for $a = (a_1, \ldots, a_N)$, then we have

$$\Phi(h_1 \varepsilon^{-1}_r) - N \leq N(\varepsilon_r) \leq \Phi(u_1 \varepsilon^{-1}_r + i) \left( \frac{\varepsilon^{-1}_r}{n} + 1 \right)^2,$$

where $r$ is selected so large that $\frac{1}{N} > \varepsilon_r$.

After applying $W_{ij}$ twice, in each region $E_{ij}$ we have $a^2$ new small squares of side $\frac{1}{an}$, which are mapped by the transformation $W_{ij}$ from the regions $E_{ij'}$ lying inside the domain $\bar{E}_k = \bar{E}_{\gamma(i,j)}$. And since segments parallel to $z$-axis are mapped to those parallel to $z$-axis, for each region $E_{ij}$ the height on these new small squares is not less than $s_{ij} \cdot H$, where $H$ is the height on the original region $E_{ij'}$ contained in domain $\bar{E}_k$. Therefore, the sum of maximum variances of $f$ on $a^2$ small squares of side $\frac{1}{an}$ contained in the region $E_{ij}$ is not greater than $\tau(i,j)$-th coordinate of the vector $u_2 = \mathcal{S} \mathcal{C} u_1 + \frac{a^2}{n} bi$ and the sum of the heights is not less than $\tau(i,j)$-th coordinate of the vector $h_2 = \mathcal{S} \mathcal{C} h_1$. So we have

$$\Phi(h_2 \varepsilon^{-1}_r) - a^2 N \leq N(\varepsilon_r) \leq \Phi(u_2 \varepsilon^{-1}_r + a^2 i) \left( \frac{\varepsilon^{-1}_r}{n} + 1 \right)^2$$

where $\frac{1}{an} > \varepsilon_r$.

By induction we get the following conclusion: if we take $k$ such that

$$\frac{a \varepsilon_r}{a^k - 1} > \varepsilon_r \iff r - \log_\frac{a}{n} n + 1 > k \geq r - \log_\frac{a}{n} n$$

and apply $k$ times the transformations $\{ W_{ij} : i = 1, \ldots, n; j = 1, \ldots, m \}$, then we get $a^{2(k-1)}$ small squares of side $\frac{1}{a^k - 1}$ contained in each region $E_{ij}$ and

$$\Phi(h_k \varepsilon^{-1}_r) - a^{2(k-1)} N \leq N(\varepsilon_r) \leq \Phi(u_k \varepsilon^{-1}_r + a^{2(k-1)} i) \left( \frac{\varepsilon^{-1}_r}{a^k - 1} \right)^2$$

(6)

where

$$u_k = \mathcal{S} \mathcal{C} u_{k-1} + \frac{a^k}{n} bi, \quad h_k = \mathcal{S} \mathcal{C} h_{k-1}.$$

Then we have

$$u_k = (\mathcal{S} \mathcal{C})^{k-1} r + (\mathcal{S} \mathcal{C})^{k-1} \frac{a}{n} bi + (\mathcal{S} \mathcal{C})^{k-2} \frac{a^2}{n} bi + \ldots + (\mathcal{S} \mathcal{C}) \frac{a^{k-1}}{n} bi + \frac{a^k}{n} bi,$$

$$h_k = (\mathcal{S} \mathcal{C})^{(k-1)} h_1.$$
Since $\mathcal{S}C$, $\mathcal{S}$ are non-negative irreducible matrix, from Frobenius’s theorem (see [5] [20]) there are strictly positive eigenvectors $e$, $\bar{e}$ of $\mathcal{S}C$, $\mathcal{S}$ which correspond to eigenvalues $\bar{\lambda} = \rho(\mathcal{S}C)$, $\lambda = \rho(\mathcal{S})$ of $\mathcal{S}C$, $\mathcal{S}$ and we can choose $e$, $\bar{e}$ so that

$$0 < e < \mathbf{1}, \quad r \leq \bar{e}, \quad b \leq n\bar{e}$$

Then by (6), we have

$$N(\varepsilon_r) \leq \Phi \left( u_k \varepsilon_r^{-1} + a^{2(k-1)}r \right) \left( a^{\frac{1}{k-1}n} + 1 \right)^2$$

$$\leq \Phi \left( u_k \varepsilon_r^{-1} + a^{2(k-1)}r \right) (a + 1)^2$$

$$\leq \Phi \left( (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} + (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} \right) + (\mathcal{S}C)^{k-2} \varepsilon_r^{-1} + \ldots$$

$$+ (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} + a^{2(k-1)}r \right) (a + 1)^2$$

$$\leq \Phi \left( (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} + (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} + (\mathcal{S}C)^{k-2} \varepsilon_r^{-1} + \ldots$$

$$+ (\mathcal{S}C)^{k-1} \varepsilon_r^{-1} + a^{2(k-1)}r \right) (a + 1)^2$$

$$\leq \left\{ \lambda^{k-1} \Phi(\bar{e}) e_r^{-1} + \lambda^{k-1} \Phi(\bar{e}) e_r^{-1} + \lambda^{k-2} \Phi(\bar{e}) a^2 e_r^{-1} + \ldots$$

$$+ \lambda \Phi(\bar{e}) a^{k-1} e_r^{-1} + \Phi(\bar{e}) a^2 e_r^{-1} + a^{2(k-1)} \Phi(\bar{e}) \right\} (a + 1)^2$$

$$\leq \left\{ \lambda^{r-1} \mu e_r^{-1} + \lambda^{r-1} \mu e_r^{-1} + \lambda^{r-1} \mu e_r^{-1} + \lambda^{r-1} \mu e_r^{-1} + \ldots$$

$$+ \lambda \mu e_r^{-1} + a^2 e_r^{-1} + a^{2(r-\nu)} \right\} (a + 1)^2.$$ (7)

where $\nu = \log_a n$, $\mu = \Phi(\bar{e})$.

On the other hands, since $(\mathcal{S}C)_{ij} \leq (\mathcal{S}C)_{ij}$ for $(i, j) \in N_{nm}$, from Frobenius’s theorem we have

$$\lambda \leq \bar{\lambda}.$$ If $\lambda > a$, then $1 > \frac{\lambda}{a} \geq \frac{2}{k}$ and thus we obtain

$$N(\varepsilon_r) \leq \lambda^{r-\nu} \mu e_r^{-1} \left[ 1 + a + \frac{a^2}{\lambda} + \ldots + \frac{a^{r-\nu+1}}{\lambda^{r-\nu}} + \frac{a^{r-2\nu} N}{\lambda^{r-\nu} \mu} \right] (a + 1)^2$$

$$= \lambda^{r-\nu} \mu e_r^{-1} \left[ 1 + a + \frac{1 - (a/\lambda)^{r-\nu+1}}{1 - a/\lambda} + \frac{a^{r-2\nu} N}{\lambda^{r-\nu} \mu} \right] (a + 1)^2.$$

Let denote

$$\delta(r) = \lambda^{r-\nu} \mu \left[ 1 + a + \frac{1 - (a/\lambda)^{r-\nu+1}}{1 - a/\lambda} + \frac{a^{r-2\nu} N}{\lambda^{r-\nu} \mu} \right] (a + 1)^2.$$

Then $\delta(r) > 0$ and

$$\frac{\log N(\varepsilon_r)}{\log \varepsilon_r} \leq 1 + \log_a \lambda + \frac{1}{2} \log_a \delta(r),$$

thus we have

$$\dim B = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \leq 1 + \log_a \bar{\lambda}.$$ (8)

By (6), we have

$$N(\varepsilon_r) \geq \Phi(\mathbf{h}_k e_r^{-1}) - a^{2(k-1)} N = \Phi((\mathcal{S}C)^{k-1} \mathbf{h}_k e_r^{-1}) - a^{2(k-1)} N$$

$$\geq \Phi((\mathcal{S}C)^{k-1} \varepsilon_r^{-1}) - a^{2(k-1)} N = \Lambda^{k-1} \mu e_r^{-1} - a^{2(k-1)} N$$

$$\geq \lambda^{r-\nu} \mu e_r^{-1} - a^{r-\nu} N e_r^{-1}$$

$$= e_r^{-1} \lambda^{r-\nu} \left( \mu - \frac{a^{r-2\nu} N}{\lambda^{r-\nu} \mu} \right).$$
where $\mu = \Phi(e)$. Since $\lambda > a$, there is $r_0$ such that

$$
\eta(r) := \lambda^{-\nu-1} \left( \frac{\mu - a^{r-2\nu} N}{\lambda^{r-\nu-1}} \right), \quad \text{for any } r > r_0.
$$

Therefore for $r > r_0$ we have

$$
\frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \geq 1 + \frac{1}{r} \log_a \eta(r) \quad \text{and thus}
$$

$$
\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \geq 1 + \log_a \lambda. \tag{9}
$$

By (8) and (9), if $\lambda > a$, then we get

$$
1 + \log_a \lambda \leq \dim_B A \leq 1 + \log_a \lambda. \tag{10}
$$

Proof of 2). If $\overline{\lambda} \leq a$, then by (7)

$$
N(\varepsilon_r) \leq \{ \lambda^{-\nu} \mu \varepsilon_r^{-1} + \lambda^{r-\nu} a \mu \varepsilon_r^{-1} + \lambda^{r-\nu-1} a^2 \mu \varepsilon_r^{-1} + \ldots + \lambda a^{r-\nu} \mu \varepsilon_r^{-1} + a^{r-\nu} \mu \varepsilon_r^{-1} + a^{2(r-\nu)} N \} (a + 1)^2
$$

$$
\leq \{ a^{r-\nu} \mu \varepsilon_r^{-1} + a^{r-\nu} \mu \varepsilon_r^{-1} + a^{r-\nu+1} \mu \varepsilon_r^{-1} + \ldots + a^{r-\nu+1} \mu \varepsilon_r^{-1} + a^{r-\nu+1} \mu \varepsilon_r^{-1} + a^{r-\nu+1} \mu \varepsilon_r^{-1} + a^{r-\nu+1} \mu \varepsilon_r^{-1} + a^{r-\nu} N \} (a + 1)^2
$$

$$
\leq \varepsilon_r^{-2} \{ (a^{r-\nu} \mu + (r - \nu + 1) a^{r-\nu+1} \mu + a^{r-2\nu} N \} (a + 1)^2.
$$

Hence, we have

$$
\dim_B A = \lim_{\varepsilon_r \to 0} \frac{\log N(\varepsilon_r)}{-\log \varepsilon_r} \leq 2 + \frac{1}{r} \log_a \{ (a^{r-\nu} \mu + (r - \nu + 1) a^{r-\nu+1} \mu + a^{r-2\nu} N \} (a + 1)^2.
$$

On the other hands, since $A$ is the surface in $\mathbb{R}^3$, we have $\dim_B A \geq 2$. Hence $\dim_B A = 2$. (QED)

Remark 3. In the case where $s_{ij}(x, y) = s_{ij}(\text{constant})$, if $\overline{\lambda} = \lambda > a$, then $\dim_B A = 1 + \log_a \lambda$. This is the estimation of Box-counting dimension of RFISs in [7].

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