Supersymmetric Unified Models

Lecture given at the KOSEF-JSPS Winter School, Recent Developments in Particle and Nuclear Theory
February 21- March 2, 1996,
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1. Motivations for Supersymmetry

A. Gauge Hierarchy

1. Standard model
   All the available experimental data at low energies ($E < 100$ GeV) can be adequately described by the standard model with $SU(3) \times SU(2) \times U(1)$ gauge group. The three different gauge coupling constants originates from the three different interactions, namely, strong, weak and electromagnetic interactions. The standard model has many parameters which have to be measured by experiments. There are also other conceptually unsatisfactory points as well. For instance, the electric charge is found to be quantized in nature, but this phenomena is just an accident in the standard model.

2. Grand unified theories
   The three interactions described by the three different gauge groups can be truly unified into a single gauge group if we choose a simple gauge group to describe all three interactions. This is realized by the grand unified theories proposed by Georgi and Glashow. The grand unified theories achieved at least two good points:
   - Because of simple gauge group, the electromagnetic charge is now quantized.
   - Since it unifies all three couplings at high energies, it gives one constraint for three couplings. Therefore it predicts the Weinberg angle $\theta_W$. The prediction with a simplest possibility was found to be not very far from the experimental data. On the other hand, the unification energy $M_G$ is now very large compared to the electroweak mass scale $M_W$.

\[
\frac{M_W^2}{M_G^2} \approx \left( \frac{10^2}{10^{16}} \right)^2 \approx 10^{-28} \tag{1.1}
\]

3. Gravity
   Even if one do not accept the grand unified theories, one is sure to accept the existence of gravitational interactions. The mass scale of the gravitational interactions is given by the Planck mass $M_{Pl}$

\[
\frac{M_W^2}{M_{Pl}^2} \approx \left( \frac{10^2}{10^{19}} \right)^2 \approx 10^{-34} \tag{1.2}
\]

Now we have a problem of how to explain these extremely small ratios between the mass squared $M_W^2$ to the fundamental mass squared $M_{Pl}^2$ or $M_G^2$ in eq.(1.1) or eq.(1.2). This problem is called the gauge hierarchy problem.

B. Higgs Scalar

When we say explain, we mean that it should be given a symmetry reason. This principle is called the naturalness hypothesis. More precisely, the system should acquire higher symmetry as we let the small parameter going to zero. The examples of the enhanced symmetry corresponding to the small mass parameter are

\[
m_{J=1/2} \to 0 \leftrightarrow \text{Chiral symmetry} \tag{1.3}
\]

\[
m_{J=1} \to 0 \leftrightarrow \text{Local gauge symmetry}
\]

The electroweak mass scale $M_W$ originates from the vacuum expectation value $v$ of the Higgs field. The scale of $v$ in turn comes from the quadratic term of the higgs potential, namely the (negative) mass squared of the Higgs scalar $\varphi$. Therefore we need to give symmetry reasons for the vanishing Higgs scalar mass in order to explain the gauge hierarchy problem.

Classically the vanishing mass for scalar field does give rise to an enhanced symmetry called scale invariance. However, it is well-known that the scale invariance cannot be maintained quantum mechanically. Therefore we have only two options to explain the gauge hierarchy problem.
II. INTRODUCTION TO SUPERSYMMETRY

A. Spinors

1. Convention

Metric ($\eta_{\mu\nu} = \eta_{\mu\nu}^{\text{Wess-Bagger}} = -\eta_{\mu\nu}^{\text{Bjorken-Drell}}$)

\[ \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

(2.1)

\[ P^\mu = (P^0, P^1, P^2, P^3), \]  

(2.2)

\[ P_\mu = (-P^0, P^1, P^2, P^3) \]  

(2.3)

\[ P \cdot Q = P^\mu \eta_{\mu\nu} Q^\nu \]

\[ = -P^0 Q^0 + P^1 Q^1 + P^2 Q^2 + P^3 Q^3 \]  

(2.4)

\[ \gamma \text{ matrix} \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = -2 \eta_{\mu\nu} \quad \]  

(2.5)

Chiral $\gamma$ matrix

\[ \gamma_5 = \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \]

\[ = \gamma_5^{\text{Bjorken-Drell}} = i\gamma_5^{\text{Wess-Bagger}} \]  

(2.6)

2. Bilinear Covariants of Majorana Spinors

\[ \bar{\psi}_1 \psi_2 = \bar{\psi}_2 \psi_1, \]  

(2.19)

\[ \bar{\psi}_1 \gamma^\mu \psi_2 = -\bar{\psi}_2 \gamma^\mu \psi_1, \]  

(2.20)

\[ \bar{\psi}_1 \gamma^\mu \gamma^\nu \psi_2 = -\bar{\psi}_2 \gamma^\mu \gamma^\nu \psi_1, \]  

(2.21)

\[ \bar{\psi}_1 \gamma_5 \gamma^\mu \psi_2 = \bar{\psi}_2 \gamma_5 \gamma^\mu \psi_1, \]  

(2.22)

\[ \bar{\psi}_1 \gamma_5 \psi_2 = \bar{\psi}_2 \gamma_5 \psi_1 \]  

(2.23)

If $\psi_1 = \psi_2 \to \bar{\psi} \gamma^\mu \psi = \bar{\psi} \gamma^\mu \gamma^\nu \psi = 0$  

(2.24)
3. Derivative of Grassmann Number

\[ \partial \frac{\partial \psi_\beta}{\partial \psi_\alpha} = \delta_\alpha^\beta, \]  
(2.25)

\[ \partial \frac{\partial \bar{\psi}_\beta}{\partial \psi_\alpha} = \delta_\alpha^\beta, \]  
(2.26)

\[ \partial \frac{\partial \bar{\psi}_\beta}{\partial \psi_\alpha} = (C^{-1})_\alpha^\beta, \]  
(2.27)

\[ \partial \frac{\partial \psi_\beta}{\partial \psi_\alpha} = (C)_\alpha^\beta \]  
(2.28)

\[ \partial \frac{\partial \psi_\beta}{\partial \bar{\psi}_\alpha} = -(C)_{\alpha}^\beta \frac{\partial}{\partial \psi_\beta} \]  
(2.29)

\[ \bar{\epsilon} \frac{\partial}{\partial \bar{\theta}} = - \frac{\partial}{\partial \bar{\theta}} \epsilon \]  
(2.31)

4. Weyl Basis and Two Component Spinor

Weyl basis of \( \gamma \) matrix

\[ \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]  
(2.32)

\[ \gamma_j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \]  
(2.33)

\[ \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]  
(2.34)

\[ C = -i \gamma_2 \gamma_0 = \begin{pmatrix} i \sigma^2 & 0 \\ 0 & i \sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon_{\alpha\beta} \end{pmatrix} \]  
(2.35)

Two component spinor notation

\[ \psi = \begin{pmatrix} \xi_\alpha \\ \eta^{*\alpha} \end{pmatrix}, \]  
(2.36)

\[ \epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma \]  
(2.37)

\[ \bar{\psi} = \begin{pmatrix} (\eta^{*\alpha})^* \\ (\xi_\alpha)^* \end{pmatrix} = \begin{pmatrix} \eta^{*\gamma} \\ \xi^{*\alpha} \end{pmatrix} \]  
(2.38)

\[ \psi_+ = \begin{pmatrix} 0 \\ \eta^{*\alpha} \end{pmatrix} \quad \psi_- = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix} \]  
(2.39)

5. Majorana spinor in the Weyl basis

\[ \psi = \begin{pmatrix} \xi_\alpha \\ \epsilon^{\dot{\alpha}} \xi^{*\dot{\alpha}} \end{pmatrix} \]  
(2.40)

\[ \psi^c \equiv C \psi^T = \begin{pmatrix} \epsilon_{\alpha\beta} \eta^{\beta} \\ \eta_\alpha \end{pmatrix} \]  
(2.41)

\[ \xi_\alpha \equiv \epsilon^{\alpha\beta} \xi_\beta, \quad \eta_\alpha \equiv \epsilon_{\alpha\beta} \eta^\beta \]  
(2.42)

\[ \bar{\epsilon} \frac{\partial}{\partial \bar{\theta}} = - \frac{\partial}{\partial \bar{\theta}} \epsilon \]  
(2.43)

\[ \theta_{\mp \alpha} \theta_{\pm \beta} = -\frac{1}{2} \theta_{\pm \alpha} \theta_{\mp \beta} \left( \frac{1 + \gamma_5}{2} \right)_{\alpha\beta} \]  
(2.44)

5. Fierz Identity for Chiral Spinor

\[ \theta \Sigma^\beta_{\pm \alpha} \theta^\pm = -\frac{1}{2} \theta \Sigma^\beta_{\mp \alpha} \left( \frac{1 + \gamma_5}{2} \right) \]  
(2.45)

B. Supertransformation

1. Superfield

Distinction between bosons and fermions by \( \theta \rightarrow x^\mu, \theta \) as coordinates in superspace

Superfield = field in superspace \( \rightarrow \) 16 component fields

\[ \Phi(x, \theta) = C(x) + \bar{\theta} \dot{\psi}(x) - \frac{1}{2} \bar{\theta} \theta N(x) - \frac{i}{2} \bar{\theta} \gamma_5 \theta M(x) \]  
\[ -\frac{1}{2} \bar{\theta} \gamma^\mu \gamma_5 \theta v_\mu(x) + i \bar{\theta} \theta \gamma_5 \lambda(x) + \frac{1}{4} (\bar{\theta} \theta)^2 D(x) \]  
(2.46)

2. Supertransformation

\[ \delta \theta = \epsilon, \quad \delta x^\mu = -i \bar{\epsilon} \gamma^\mu \theta \]  
(2.47)

\[ \delta \Phi(x, \theta) = \bar{\epsilon} \left( \frac{\partial}{\partial \bar{\theta}} - i \gamma^\mu \theta \frac{\partial}{\partial x^\mu} \right) \Phi(x, \theta) \]  
(2.48)
3. Supersymmetry Algebra

\[ [\Phi, [\bar{\epsilon}_1 Q, \bar{Q}\epsilon_2]] = [\Phi, [\bar{\epsilon}_1 Q, \bar{Q}\epsilon_2]] \]
\[ = [[\Phi, \bar{\epsilon}_1 Q], \bar{Q}\epsilon_2] - [[\Phi, \bar{Q}\epsilon_2], \bar{\epsilon}_1 Q] \]
\[ = (\delta(\epsilon_2)) (\delta(\epsilon_1)) (\Phi - (\delta(\epsilon_1)) (\delta(\epsilon_2)) \Phi) \]
\[ = \left[ \left( \frac{\partial}{\partial \theta} + i \bar{\epsilon}_2 \gamma^\mu \partial_\mu \right) \right] \epsilon_2 \bar{\epsilon}_1 \left( \frac{\partial}{\partial \theta} - i \gamma^\nu \partial_\nu \right) \Phi(x, \theta) \]
\[ = 2\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \left( -i \partial_\mu \Phi(x, \theta) \right) \]
\[ = 2\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \left( \Phi(x, \theta), P_\mu \right) \]
\[ \text{(2.49)} \]

Supersymmetry algebra

\[ \{ Q_\alpha, \bar{Q}_\beta \} = 2(\gamma^\mu)_{\alpha\beta} P_\mu \]
\[ \text{(2.50)} \]
\[ \{ Q_\alpha, Q_\beta \} = -2(\gamma^\mu C)_{\alpha\beta} P_\mu \]
\[ \text{(2.51)} \]

Other commutation relations

\[ [Q, P_\mu] = 0, \]
\[ \text{(2.52)} \]
\[ [Q_\alpha, J^{\mu\nu}] = i\frac{1}{2}(\gamma^{\mu\nu})_{\alpha\beta} Q_\beta \]
\[ \text{(2.53)} \]
\[ [P_\mu, J_\nu] = 0, \]
\[ \text{(2.54)} \]
\[ [P_\mu, J^{\lambda\nu}] = -i(\eta^{\nu\lambda} P^\mu - \eta^{\mu\lambda} P^\nu) \]
\[ \text{(2.55)} \]
\[ [J^{\mu\nu}, J^{\lambda\rho}] = -i(\eta^{\mu\rho} J^{\nu\lambda} + \eta^{\nu\lambda} J^{\mu\rho} - \eta^{\mu\nu} J^{\lambda\rho} - \eta^{\nu\lambda} J^{\mu\rho}) \]
\[ \text{(2.56)} \]

Characteristic features of supersymmetry

1. Involving anticommutators
2. Spacetime symmetry

C. Unitary Representation

Unitary Representation of Supersymmetry Algebra

\[ \rightarrow \text{Physical Particle Content} \]

1. Massive case

- Representation of the Poincaré group
  - Diagonalize \( P^\mu \)
  - Standard frame \( P^\mu = (M, 0, 0, 0) \)
  - Little group = Stability group of \( (M, 0, 0, 0) = SO(3) \)

Angular momentum \( j, z \) component \( m \)

2. Representation of \( Q \) by combining \( (j, m) \)

\[ [Q, P_\mu] = 0 \]
\[ \text{(2.57)} \]

\( P^\mu \) can be diagonalized

\[ [Q_\alpha, J^{\mu\nu}] = i\frac{1}{2}(\gamma^{\mu\nu})_{\alpha\beta} Q_\beta \]
\[ \text{(2.58)} \]

\( Q \) changes \( j \) and \( m \) by \( \pm \frac{1}{2} \)

\[ \{ Q^{-1} Q_{\beta} \} = \{ Q_{\alpha}, Q_{\beta} \} = 0 \]
\[ \text{(2.59)} \]

2 kinds of “fermions”

\[ \overline{Q^{-\alpha}}, \quad \alpha = 1, 2 \] (annihilation operator)
\[ \overline{Q_{\alpha}}, \quad \alpha = 1, 2 \] (creation operator)

Suppose \( \overline{Q^{-\alpha}} | j > = 0, \alpha = 1, 2 \)

\[ \begin{pmatrix} | j > & Q^{-1} | j > \\ Q^{-2} | j > & Q^{-1} Q^{-2} | j > \end{pmatrix} = \begin{pmatrix} j - \frac{1}{2} \\ j + \frac{1}{2} \end{pmatrix} \]
\[ \text{(2.61)} \]

(a) \( j = 0 \) case \( \Rightarrow \) Chiral scalar multiplet

| spin \( j \) | field | degree of freedom |
|--------|------|------------------|
| 0      | two real scalar | 2                |
| 1/2    | a Majorana spinor | 2                |

(b) \( j = \frac{1}{2} \) case \( \Rightarrow \) Vector multiplet

| spin \( j \) | field | degree of freedom |
|--------|------|------------------|
| 0      | a real scalar | 1                |
| 1/2    | 2 Majorana spinor | 4                |
| 1      | a real vector | 3                |

2. Massless case

Standard frame \( P^\mu = (P, 0, 0, P) \)

Little group = Stability group of \( (P, 0, 0, P) \)

\[ E_2 = (J^{12}, J^{01} + iJ^{23}, J^{20} + iJ^{13}) \]
\[ \text{(2.62)} \]

Representation label by

\[ J^2 = j(j + 1) \quad \text{and} \quad \text{helicity} \ J^{12} = \pm j \]
\[ \text{(2.63)} \]

\[ \{ Q_\alpha, \overline{Q}_\beta \} = 2((\gamma_0 + \gamma_3)_{\alpha\beta} P = 4P \]
\[ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
\[ \text{(2.64)} \]
 helicities gravitino multi. θγ of fields (2
 multiplet multiplet name of

But If Φ(x,θ)

Multiplet — Key ingredient to construct field theories.

One should find a constraint consistent with supersymmetry. Therefore this constraint is not consistent with Supersymmetry. Defined Covariant derivative

\[ D_αΦ(x,θ) = \left( -\frac{∂}{∂θ_α} - i(\bar{θ}_γ∂_μθ_α) \right)Φ(x,θ) \]  

D_α satisfies the same algebra as Q_α

2. Chiral Projected Covariant Derivative

\[ D_±α = \frac{∂}{∂θ_±α} + i(\gamma^μ∂_μθ_±)α \]  

Negative chiral scalar superfield

\[ D_+αΦ(x,θ) = 0 \]  

Define

\[ z^μ ≡ x^μ + \frac{i}{2}(θ^γγ_5θ_γ) = x^μ - iθ_γγ_5θ_γ = x^μ + i\bar{θ}_γγ_5θ_± \]  

Then

\[ D_+αz^μ = \left( \frac{∂}{∂θ_+α} + i(\gamma^μ∂_μθ_-)α \right) (x^μ - iθ_γγ_5θ_γ) = i(\gamma^μθ_- - i(γ^μθ_-)α = 0 \]  

\[ D_+αθ_- = 0, \]  

\[ D_+αθ_+ = -\left( \frac{1 + γ_5}{2}C \right)αβ ≠ 0 \]  

Changing variables (x,θ_+,θ_-) → (z,θ_+,θ_-)

\[ D_+αΦ(x,θ) = 0 \implies Φ = Φ(z,θ_-) \]  

Namely, Φ is independent of θ_+ if z is fixed.

Let us denote negative chiral scalar field as Φ_-(z,θ_-)

Negative chiral scalar field can be used as a representation space of supersymmetry \(|Q, D = 0)\)
3. Properties of chiral scalar superfield

\[ \Phi_-(z, \theta_-) = e^{A_+(\theta_+)} \theta_+ \gamma_5 \Phi_-(x, \theta_-) \]  
\[ = \left( A_-(z) + \sqrt{2} \bar{\theta}_- \psi_-(z) + \bar{\theta}_- \theta_- F_-(z) \right) \]
\[ = e^{\frac{i}{2} \bar{\theta}_- \gamma_5} \left( A_-(x) + \sqrt{2} \bar{\theta}_- \psi_-(x) + \bar{\theta}_- \theta_- F_-(x) \right) \]

Chiral scalar field is complex

Degree of freedom of component fields

| fields       | real or complex spin | off-shell d.o.f. | on-shell d.o.f. |
|--------------|----------------------|------------------|-----------------|
| \( A_-(x) \) | complex scalar       | 2                | 2               |
| \( \psi_-(x) \) | 2-comp. spinor     | 4                | 2               |
| \( F_-(x) \) | complex aux. scalar | 2                | 0               |

\( \psi \) obeys the Dirac equation. On-shell d.o.f. is counted by \((\text{Dirac})^2 = \text{Klein-Gordon}\). 

Product of chiral scalar superfields \( \Phi_1^- \) and \( \Phi_2^- \)

\[ \Phi_1^-(z, \theta_-) \Phi_2^-(z, \theta_-) \]
\[ = (A_1^-(z) + \sqrt{2}\bar{\theta}_- \psi_1^-(z) + \bar{\theta}_- \theta_- F_1^-(z)) \]
\[ \times (A_2^-(z) + \sqrt{2}\bar{\theta}_- \psi_2^-(z) + \bar{\theta}_- \theta_- F_2^-(z)) \]
\[ = A_1^1 A_2^1 + \sqrt{2}\bar{\theta}_+ (A_1^2 \psi_2^1 + \psi_2^1 A_2^2) \]
\[ + \bar{\theta}_- \theta_- (F_1^- A_2^2 + A_1^2 F_2^-) + 2\sqrt{2} \psi_1^+ \bar{\theta}_+ \psi_2^2 \]
\[ = A_1^1 A_2^2 + \sqrt{2}\bar{\theta}_+ (A_2^1 \psi_1^2 + \psi_1^2 A_1^1) \]
\[ + \bar{\theta}_+ \theta_- (F_2^- A_1^2 + A_2^2 F_1^-) - (\psi_1^1 \bar{\psi}_1^2) \]  
\[ = \frac{1}{(\theta_+^1 \bar{\psi}_1^2)} ((\psi_1^1) \theta_-) (\bar{\psi}_1^2 \theta_-) \]  
\[ = \frac{1}{2} (\bar{\theta}_+ \theta_-) (\psi_1^1 C^{-1} \psi_2^2) \]  
\[ \text{(2.85)} \]

Supertransformation for an “infinitesimal” \( \epsilon \)

\[ \delta z^\mu = \delta x^\mu + \frac{i}{2} \epsilon \left( \bar{\psi} \gamma^\mu \gamma_5 \theta \right) \]
\[ = -\bar{\psi} \gamma_5 \theta + \bar{\psi} \gamma^\mu \gamma_5 \theta \]
\[ = -2i \bar{\psi} \gamma_5 \mu \theta_- \]  
\[ \text{(2.87)} \]

\[ \delta \Phi_-(z, \theta_-) = \left( \frac{\delta z^\mu}{\partial z^\mu} + \frac{\partial}{\partial \theta_-} \right) \Phi_-(z, \theta_-) \]
\[ = \left( -2i \bar{\psi} \gamma_5 \mu \theta_- + \frac{\partial}{\partial \theta_-} \right) \Phi_-(z, \theta_-) \]
\[ \times (A_-(z) + \sqrt{2} \bar{\theta}_- \psi_-(z) + \bar{\theta}_- \theta_- F_-(z)) \]
\[ = \sqrt{2} \bar{\psi} \gamma_5 \psi_- + 2 \bar{\psi} \gamma_5 \theta_- F_- - 2 \bar{\psi} \gamma_5 \theta_- \partial \mu A_- \]
\[ - 2 \sqrt{2} \bar{\psi} \gamma_5 \mu \theta_- \bar{\theta}_+ \psi_- \]
\[ = \sqrt{2} \bar{\psi} \gamma_5 \psi_- + \sqrt{2} \bar{\theta}_+ \bar{\theta}_- \psi_- + i \gamma^\mu \epsilon \bar{\psi} \gamma_5 \theta_- F_- \]
\[ + \bar{\theta}_+ \bar{\theta}_- \sqrt{2} \bar{\psi} \gamma_5 \mu \theta_- \]  
\[ \text{(2.88)} \]

Therefore

\[ \delta A_- = \sqrt{2} \bar{\theta}_- \psi_- \]
\[ \delta \psi_- = \sqrt{2} (\epsilon_- F_- + i \gamma^\mu \epsilon_- \partial \mu A_-) \]
\[ \delta F_- = i \sqrt{2} \bar{\psi} \gamma_5 \mu \theta_- \psi_- \]  
\[ \text{(2.89)} \]

4. Positive Chiral Scalar Field

\[ D_- \Phi_+ = 0, \]  
\[ \text{(2.90)} \]
\[ z^\mu = x^\mu - \frac{i}{2} \bar{\theta}_+ \gamma_5 \gamma_\mu \theta_- \]  
\[ \text{(2.91)} \]
\[ \Phi_+ = \Phi_+ (z^+, \theta_+) \]
\[ = A_+ (z^+) + \sqrt{2} \bar{\theta}_+ \psi_+ (z^+) + \bar{\theta}_+ \theta_+ F_+ (z^+) \]  
\[ \text{(2.92)} \]

Product of positive chiral and negative chiral scalar fields is a general superfield (without a definite chirality)

Complex conjugation changes the chirality

\[ (\Phi_-(z, \theta_-))^* = e^{\frac{i}{2} \bar{\theta}_- \gamma_5 \gamma_\mu \theta_-} \]
\[ \times (A_+^*(x) + \sqrt{2} \bar{\theta}_+ \psi_+^c (x) + \bar{\theta}_+ \theta_+ F_+^c (x)) \]  
\[ \text{(2.93)} \]

E. Supersymmetric Field Theory

1. Lagrangian with Chiral Scalar Fields

Lagrangian invariant under supersymmetry transformation up to a total divergence:

1. Two possibilities

(a) \( D \)-term of general superfield \( \Phi \)

\[ [\Phi]_D = \frac{1}{8} (\bar{D} D)^2 \Phi \]  
\[ \text{(2.94)} \]

(b) \( F_- \)-term of chiral scalar superfield \( \Phi_{\pm} \)

\[ [\Phi_{\pm}]_F = -\frac{1}{4} \bar{D} D \Phi_{\pm} \]  
\[ \text{(2.95)} \]

2. Dimensional analysis

\[ [\theta_\alpha] = L^{\frac{5}{2}} = M^{\frac{5}{2}}, \]
\[ [D_\alpha] = \left[ \bar{D}_\alpha \right] = M^{\frac{5}{2}} \]  
\[ \text{(2.96)} \]
3. Renormalizable Lagrangian (in 4-dimension) operators with dimension \( \leq 4 \).

(a) D-type:
\[
(\mathcal{D}D)^2 \cdot \Phi_1, \Phi_2 \tag{2.99}
\]
Dimension \([\mathcal{D}D] = M^2\)

(b) F-type:
\[
(\mathcal{D}D)(a\Phi_1 + b\Phi_1 \Phi_2 + c\Phi_1 \Phi_2 \Phi_3) \tag{2.100}
\]
Since \(\mathcal{D}D\) has dimension \(M^1\), up to third order polynomials of chiral scalar superfields of one chirality are renormalizable.

4. General Lagrangian with a single chiral scalar field
\[
L = L_{\text{kin}} + L_{\text{int}}. \tag{2.101}
\]

\[
L_{\text{kin}} = \frac{1}{32}(\mathcal{D}D)^2\Phi^+ \Phi._- - \frac{1}{4}\partial^2 A^+ A_- - \frac{1}{2}\partial_\mu A^+ \partial^\mu A_- + \frac{1}{4}A^+ \partial^2 A_-
+ F^+ F_- + \frac{1}{2}\bar{\psi} i \gamma^\mu \partial_\mu \psi_- - \frac{1}{2}\bar{\psi} i \gamma^\mu \partial_\mu \psi_-
= -\partial_\mu A^+ \partial^\mu A_- + \bar{\psi} i \gamma^\mu \partial_\mu \psi_- + F^+ F_- + \text{total derivatives} \tag{2.102}
\]

Elimination of auxiliary fields \(F\) from \(L\)
Euler eq. for \(F_\mu\)
\[
F^+_\mu + \sqrt{2} f A^+_\mu + m A_- = 0 \tag{2.104}
\]

\[
L \rightarrow -\partial_\mu A^+ \partial^\mu A_- + \frac{1}{2}\bar{\psi} i \gamma^\mu \partial_\mu \psi_- - \frac{m}{2}\bar{\psi} \psi
- \left(\sqrt{2} f (\bar{\psi}^- c \psi_+) A_- + \text{h.c.}\right)
- \left|\sqrt{2} f A^+ + m A_- \right|^2 \tag{2.105}
\]

\(m\) = mass of a Majorana spinor \(\psi\) and a complex scalar \(A\)
\(f\) = Yukawa coupling and \(|A^2|^2\) coupling

5. Feynman diagram calculation is facilitated by superfield perturbation
\[
-\frac{1}{4}\mathcal{D}D \approx \frac{1}{2}d\theta_1 d\theta_2 \equiv d^2 \theta \tag{2.106}
\]

\[
\frac{1}{32}(\mathcal{D}D)^4 \approx \frac{1}{4}d\theta_1 d\theta_2 d\theta_3 d\theta_4 \equiv d^4 \theta \tag{2.107}
\]

1. Gauge Transformation
Ordinary local gauge transformation
\[
\psi(x) \rightarrow e^{-i\Lambda^a(x)T^a} \psi(x) \tag{2.108}
\]

Supersymmetric extension
\[
\begin{align*}
\text{matter} & \quad \text{chiral scalar superfield} \\
\psi(x) & \quad \Phi_-(x, \theta) \tag{2.109}
\end{align*}
\]

\(x\)-dependent gauge function \(\Lambda(x)\) is generalized to a chiral scalar superfield \(\Lambda_-(x, \theta)\)
\[
\Lambda(x) \rightarrow \Lambda_-(x, \theta) \tag{2.110}
\]

Supersymmetric local gauge transformation
\[
\Phi_-(x, \theta) \rightarrow \exp(-i\Lambda^a(x, \theta)T^a)\Phi_-(x, \theta) \tag{2.111}
\]

using gauge function superfield with the same chirality
\[
\Phi^+_\mu(x, \theta) \rightarrow \Phi^+_\mu(x, \theta) \exp(i\Lambda^a(x, \theta)T^a) \tag{2.112}
\]

2. Gauge Invariant Kinetic Term for Matter Fields
(a) A General Superfield for Gauge Boson and Gaugino
\[
e^{2gV^a T^a} \tag{2.113}
\]

Gauge transformation
\[
e^{2gV^a T^a} \rightarrow e^{-i\Lambda^a(x, \theta)T^a} e^{2gV^a T^a} e^{i\Lambda^a T^a} \tag{2.114}
\]

(b) Kinetic Term for a Chiral Scalar Field \(\Phi_-\)
\[
L_{\text{kin}} = \frac{1}{32}(\mathcal{D}D)^2(\Phi^+_\mu e^{2gV^a T^a} \Phi^-_\mu) \tag{2.115}
\]
is gauge invariant
The general superfield \(V^a\) is dimensionless and real
\[
V^a = V^a \tag{2.116}
\]

3. Gauge Transformation

1. Gauge transformation in components
\(\text{U}(1)\) case
\[
V \rightarrow V + \frac{i}{2g}(\Lambda_- - \Lambda'^- ) \tag{2.117}
\]
In terms of components
\[ V(x, \theta) \equiv C(x) + i \theta^+ \chi_-(x) - i \theta^- \chi_+(x) \]
\[ + \frac{i}{2} \theta^+ \theta_-(M + iN) - \frac{i}{2} \theta^- \theta_+(M - iN) \]
\[ - \frac{1}{2} \gamma^\nu \theta_+ v_\mu(x) \]
\[ + i \theta^+ \theta_-(\lambda_+ + \frac{i}{2} \gamma^\mu \partial_\mu \lambda_-) \]
\[ - i \theta^- \theta_+ \lambda_+ \gamma^\mu \partial_\mu \lambda_+ \]
\[ + \frac{1}{2} \theta^+ \theta_+ (D + \frac{1}{2} \partial^2 C) \] (2.118)

\[ C \rightarrow C + \frac{i}{2g}(A_- - A_+) \]
\[ \chi_- \rightarrow \chi_- + \sqrt{2} \frac{1}{2g}\psi_- , \]
\[ \chi_+ \rightarrow \chi_+ + \sqrt{2} \frac{1}{2g}(\psi_-)^c \]
\[ M \rightarrow M + \frac{1}{2g}(F_- + F_+) , \]
\[ N \rightarrow N + \frac{i}{2g}(F_- - F_+) \] (2.119)

\[ C, \chi, M, N \text{ can be gauge away} \]
\[ v^\mu \rightarrow v^\mu + \frac{1}{2g} \partial^\mu (A_- + A_+) \] (2.120)

\[ v^\mu \text{ is an ordinary gauge field} \]
\[ \lambda \rightarrow \lambda \quad D \rightarrow D \] (2.121)

\[ \lambda, D \text{ are gauge invariant.} \]

2. Wess-Zumino gauge

Eliminate \( C, \chi, M, N \) by choosing \( A_- \)
\[ V_{WZ} = - \theta^+ \gamma^\mu \partial_\mu v_\mu(x) + i \theta^+ \theta_+ \theta^- \lambda_+ \]
\[ - i \theta^- \theta^- \lambda_+ + \frac{1}{2} \theta^+ \theta_+ \theta_+ D(x) \] (2.122)
\[ = - \frac{1}{2} \theta^+ \gamma^\mu \gamma_5 \theta v_\mu(x) + i \theta \theta \theta \theta \gamma_5 \lambda + \frac{1}{4} (\theta \theta)^2 D(x) \]

\[ \text{Wess-Zumino gauge is not manifestly supersymmetric.} \]

However, particle content is most easily seen.

4. Supersymmetric Gauge Field Strength

Gauge field strength = gauge covariant building block
\[ \lambda^a(x) \text{ is the gauge covariant field with lowest dimension} \]

\[ \text{Derivative} \rightarrow D_{\pm \alpha} \]
\[ W_{-\alpha} = \frac{-1}{8g} (D_{-\alpha}D_+) \left( e^{-2gV^\tau T^a} D_{-\alpha} e^{2gV^\tau T^a} \right) \] (2.123)

The first suffix denotes negative chiral projection for the index \( \alpha \).

The second suffix denotes negative chiral superfield.
\[ D_+ W_{-\alpha} = 0 \] (2.124)

\[ W_{-\alpha} \text{ is gauge covariant} \]
\[ W_{-\alpha} \rightarrow e^{-iA_+^\alpha T^a} W_{-\alpha} e^{iA_-^\alpha T^a} \] (2.125)

Similarly positive chiral field strength is given by
\[ W_{+\alpha} = \frac{-1}{8g} (D_+D_-) (e^{2gV^\tau T^a} D_{+\alpha} e^{-2gV^\tau T^a} ) \] (2.126)

Kinetic term for vector superfield is given by
\[ L_{\text{gauge}} = \frac{-1}{16} D_{+\alpha}D_- \left( W_{+\alpha} W_{-\alpha}^\dagger \right) + \text{h.c.} \] (2.127)

In the Wess-Zumino gauge
\[ W_{-\alpha} = e^{2\tau^a \partial_\mu \gamma_5 \theta} \times \left[ i \lambda_- \left( D + \frac{i}{2} \gamma^\mu \gamma_\nu v_{\mu\nu} \right) \theta_- + \bar{\theta}_- \gamma_\mu \gamma_\nu \lambda_- \right] \] (2.128)
\[ v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig[v_\mu, v_\nu] \] (2.129)
\[ \nabla_\mu \lambda_- = \partial_\mu \lambda_- + ig[v_\mu, \lambda_-] \] (2.130)

\[ L_{\text{gauge}} = \frac{1}{2} \lambda_- (i\gamma^\mu \nabla_\mu \lambda_-)^a - \frac{1}{8} \varepsilon_{\mu\nu}^{\sigma\tau} v_{\mu\nu} v_{\sigma\tau} + \frac{1}{4} D^a D^a \]
\[ + \frac{i}{16} \varepsilon_{\mu\nu\rho\sigma} v_{\mu\nu} v_{\rho\sigma} + \text{h.c.} \] (2.131)
\[ = \frac{1}{2} \lambda_- (i\gamma^\mu \nabla_\mu \lambda)^a - \frac{1}{4} \varepsilon_{\mu\nu}^{\sigma\tau} v_{\mu\nu} v_{\sigma\tau} + \frac{1}{2} D^a D^a \]

\[ D^a \text{ is an auxiliary field} \]

5. Gauge Interaction in the Wess-Zumino Gauge
\[ L_{\text{kin.of} \Phi_-} = - (\nabla_\mu A_-)^a \nabla_\mu A_- + \bar{\psi}_- i\gamma^\mu \nabla_\mu \psi_- \]
\[ + F^T F_- + i \sqrt{2g} (A_+^T \psi_- \lambda_a^c - \bar{\lambda}_c^a \psi_+ T^a A_-) \]
\[ + g A_+^T D^a T^a A_- \] (2.132)
\[ \nabla_\mu A_- = \partial_\mu A_- + ig\alpha T^a A_- \] (2.133)
Eliminating $D$ by Euler eq. from $L_{gauge} + L_{kin}$

$$D^a + gA^a T^a A_+ = 0 \quad (2.134)$$

$$\frac{1}{2}D^a D^a + gA^a T^a A_+ = -\frac{1}{2}\sum_a g^2 |A^a T^a A_+|^2 \quad (2.135)$$

This is the D-term of the scalar potential

$U(1)$ $\xi$-term (Fayet-Iliopoulos term)

$$L_\xi = \frac{1}{16}(\bar{D}D)^2 \xi V = \xi D \quad (2.136)$$

$$[\xi] = M^2 \quad (2.137)$$

III. SUPERSYMMETRIC $SU(3) \times SU(2) \times U(1)$ MODEL

A. Yukawa Coupling

1. Nonsupersymmetric Standard Model

In the nonsupersymmetric $SU(2) \times U(1)$ model, we have left-handed quark doublet $q_j$, the right-handed $u$-type quark $u_{Ri}$ and $d$-type quark $d_{Ri}$, left-handed lepton doublet $l_j$, the right-handed electron $e_{Ri}$, together with Higgs doublets. We shall denote the generation index by lower suffixes $i,j, \cdots$. We also denote the Higgs doublets to give the masses to the $u$-type ($d$-type) quark as $\varphi_u$ ($\varphi_d$). We can write down the Yukawa interaction between quarks, leptons and Higgs fields in terms of the Yukawa couplings $f$ as

$$L_{Yukawa} = -\frac{1}{4} \bar{D}W(\Phi) + h.c. \quad (3.1)$$

where

$$W = f_{ij} U^c_i H_u^T \varepsilon q_j + f_{ij} D^c_i H_d^T \varepsilon d_j + f_{ij} E^c_i H_d^T \varepsilon l_j \quad (3.2)$$

and we denoted the negative chiral scalar superfield by capital letters and the charge conjugate of the positive chiral scalar superfield in terms of the upper suffix $c$.

B. Particle Content

Now we find that we need at least a pair of Higgs doublet superfield, we will list the minimal particle content of the supersymmetric standard model. We shall use the convention for the $U(1)$ charge $Q$ as

$$Q = I_3 + Y \quad (3.3)$$

Let us note that the Higgsino (chiral fermions associated with the Higgs scalar) in general introduces the anomaly in gauge currents. The simplest way out of such anomaly problem is to introduce the Higgsino doublet in pairs. Then the anomaly coming from $\tilde{\varphi}_u$ and $\tilde{\varphi}_d$ always cancel each other. This is another reason to introduce pair of Higgs doublet superfield $H_u$ and $H_d$.

In the nonsupersymmetric model, we can choose the Higgs doublet $\varphi_u$ and $\varphi_d$ to be the complex conjugate of each other

$$\varphi_u = \varepsilon \cdot \varphi_d^* \quad (3.4)$$

This is the choice in the minimal standard model.
\[ J = 1 \quad J = 1/2 \quad J = 0 \quad I \quad Y \quad SU(3) \]

| Gauge fields | \( G^\mu \) | \( g^\mu \) | \( W^\mu \) | \( W^\nu \) | \( B^\mu \) | \( B^\nu \) |
|--------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Higgs field  | \( \tilde{\varphi}_u \) | \( \varphi_u \) | \( 1/2 \) | \( 1/2 \) |
| \( H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix} \) | \( \tilde{\varphi}_d \) | \( \varphi_d \) | \( 1/2 \) | \( -1/2 \) |
| Quark field  | \( \tilde{q}_i \) | \( q_i \) | \( 1/2 \) | \( 1/2 \) | | \( 3 \) |
| \( Q_l = \begin{pmatrix} U^c_i \\ D^c_l \end{pmatrix} \) | \( \tilde{u}^c_i \) | \( u^c_i \) | \( 0 \) | \( 3^* \) |
| \( U^c_i \) | \( \tilde{d}^c_l \) | \( d^c_l \) | \( 1/2 \) | \( -1/2 \) | | \( 3^* \) |
| Lepton field | \( \tilde{l}_i \) | \( l_i \) | \( 1/2 \) | \( -1/2 \) | | | |
| \( L_i = \begin{pmatrix} N^c_i \\ E^c_i \end{pmatrix} \) | \( \tilde{e}^c_i \) | \( e^c_i \) | \( 0 \) | \( 1 \) | | | |
| \( E^c_i \) | \( \tilde{\nu}^c_i \) | \( \nu^c_i \) | \( 0 \) | \( 0 \) | | | |

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