Numerical and Approximate Solutions for Two-Dimensional Hyperbolic Telegraph Equation via Wavelet Matrices

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Abstract In this paper, we present the Legendre wavelet operational matrix method (LWOMM) to find the numerical solution of two-dimensional hyperbolic telegraph equations (HTE) with appropriate initial and boundary conditions. The Legendre wavelets series with unknown coefficients have been used for approximating the solution in both of the spatial and temporal variables. The basic idea for discretizing two-dimensional HTE is based on differentiation and integration of operational matrices. By implementing LWOMM on HTE, HTE is transformed into algebraic generalized Sylvester equation. Numerical experiments are provided to illustrate the accuracy and efficiency of the presented numerical scheme. Comparisons of numerical results associated with the proposed method with some of the existing numerical methods confirm that the method is easy, accurate and fast experimentally. Moreover, we have investigated the convergence analysis of multidimensional Legendre wavelet approximation.

Keywords Telegraph equation · Legendre wavelets · Operational matrices · Kronecker multiplications · Algebraic generalized Sylvester equation · BICGSTAB method

Applications The study of the electric signal in dispersive wave propagation, a transmission line, pulsating blood flow in arteries and random motion of bugs along a hedge are among a host of physical and biological phenomena which can be described by the telegraph partial differential equation (TPDE). So, the TPDE and its solutions are important in many areas of applications.

1 Introduction

Partial differential equations (PDEs) are of widespread interest because of their connection with phenomena in the physical world. Nowadays most of the physical problems are described in the form of mathematical models consisting of PDEs. PDEs are observed in many fields of engineering and applied sciences. Among these PDEs, hyperbolic PDEs play an important role in several areas of engineering and applied sciences. The propagation of signal (digital and analog) through media, the propagation of electromagnetic waves in the earth-ionosphere waveguide [1], mechanical wave [2], an ecological and cosmological phenomena are modeled using hyperbolic PDEs [3]. Recently, many methodologies have been investigated to find the numerical solution of the telegraph equation due to their universal applications in the area of applied mathematics (see [4–6]). Goldstien derived the one-dimensional telegraph equation with probabilistic argument. He proved that a particle which moves forward and backward direction with speed c satisfied hyperbolic one-dimensional telegraph equation (7). Hyperbolic telegraph equations are commonly used in wave phenomena [8] and also wave propagation of electric signal in [9]. And also, an effort has been made for the extended result for two-dimensional case of random planar motion ([10–12]). Specially in two dimensional hyperbolic PDEs such as telegraph equations in real-world applications, we should impose some boundary limitations on the two-dimensional space variable. Neglecting spatial dimension in multidimensional can affect the accuracy of the model for describing the
Chemical and physical events. Therefore, multidimensional PDEs are considered to model and simulate the aforementioned events. Among the PDEs model, parabolic PDEs model describes a phenomena with the help of some physical laws, but in some of the model, it may be better modeled by hyperbolic PDEs. However, few articles are devoted to the implementation of the analytical methods for solving the telegraph equation with boundary conditions [13]. Also, in different circumstance, direct symbolic differentiation, and integration, analytical schemes make them time-consuming. This is one more disadvantage of the analytical method for solving multidimensional PDEs. Because of these type of disadvantages in PDEs, the robust and efficient tool should compute the numerical solution of the problems.

In this article, we consider the more general form of two dimensional hyperbolic telegraph equation as follows

\[
\frac{\partial^2 \Phi(\eta, \xi, t)}{\partial t^2} + 2\lambda_1 \frac{\partial \Phi(\eta, \xi, t)}{\partial t} + \lambda_2^2 \Phi(\eta, \xi, t) = \frac{\partial^2 \Phi(\eta, \xi, t)}{\partial \eta^2} + \frac{\partial^2 \Phi(\eta, \xi, t)}{\partial \xi^2} + F(\eta, \xi, t), \quad (\eta, \xi, t) \in \Omega \times (0, T],
\]

where \( \Omega = [0, 1] \times [0, 1] \times [0, 1] \) and \( t \in (0, T] \).

The initial conditions and the Dirichlet boundary conditions are

\[
\begin{align*}
\Phi(\eta, \xi, 0) &= f_1(\eta, \xi), \\
\Phi(\eta, \xi, 0) &= f_2(\eta, \xi),
\end{align*}
\]

and

\[
\begin{align*}
\Phi(0, \xi, t) &= f_3(\xi, t), \quad \Phi(1, \xi, t) = f_4(\xi, t), \\
\Phi(\eta, 0, t) &= f_5(\eta, t), \quad \Phi(\eta, 1, t) = f_6(\eta, t). \quad (\eta, \xi, t) \in \lambda_1 \times (0, T]
\end{align*}
\]

, respectively, where \( \lambda_1 \) and \( \lambda_2 \) are constants.

Various numerical methods have been developed to solve hyperbolic PDEs, and the above telegraph Eqs. (1–3) have been considered by some researchers for numerical solutions. Dehghan and Ghesmati [14] have used two meshless methods namely meshless local Petrov-Galerkin (MLPG) and meshless local weak-strong (MLWS) to solve two-dimensional telegraph Eqs. (1–3). A semi-discretization method which is unconditionally stable has been proposed by Gao and Chi [15]. Mohanty developed three-level unconditionally stable schemes based on the finite difference ( [16, 17]). In [18], Yousefi and Dehghan used He’s variational iteration approach to solve one-dimensional telegraph equation numerically. However, In [19], authors used compact finite difference approximation for space derivative. In [20], Lakestani and Saray developed operational matrix approach based on interpolating scaling functions to solve the telegraph equation. Shokri and Dehghan explored radial basis function in [21], a meshless method in [22] and a meshfree technique in [23]. Moreover, Saadatmandi and Dehghan [24] approximate the solution in terms of shifted Chebyshev polynomials to get the numerical solution. In [25] authors proposed Taylor matrix method for the numerical solution of telegraph equation. Also, in [26] authors implement Chebyshev wavelet method to solve one-dimensional telegraph equation numerically. In the progress of solution of telegraph equation, various meshless techniques have been developed. Dehgan and Mohebbi established a higher-order implicit collocation method in [27]. An unconditionally stable implicit scheme has been presented by Mohanty et al. [28]. Ding and Zhang [29] developed a fourth-order compact finite difference scheme. Jiwari et al. [30] proposed a numerical technique based on polynomial differential quadrature method (PDQM). Mittal and Bhatia proposed cubic B-spline collocation method in [31] and differential quadrature method based on modified B-spline with space discretization in [32]. Recently, an operational matrix approach based on Bernoulli polynomials is proposed by Singh et al. [33]. The methods based on operational matrices have proved to be very effective. The main advantage of using operational matrices is the sparsity of the operational matrices. In the numerical analysis, operational matrices based on approximation technique provide a powerful technique for approximating solutions of partial differential equations which arise from mathematical modeling (for instance see [34–41]). The motivation and philosophy behind the operational matrix approach are that it has some characteristic as follows:

(i) It reduces singularities from the proposed mathematical problems in an easy way.
(ii) It does not only simplify the proposed problem but also speeds up the computation.
(iii) It is transformed the PDEs into the algebraic system.
(iv) The method is computer-oriented, thus solving higher-order PDEs becomes a matter of dimension increasing.
(v) The solution is convergent, even though the size of increment may be large.

The basic idea of an operational matrix technique is as follows:

(a) The unknown function or its derivatives with respect to time(or space) in the given PDEs are approximated...
by linear combinations of the orthonormal basis functions and truncating them up to optimal levels.

(b) In this article, the operational matrix approximation converts the main problem into simple algebraic equations whose solutions can be obtained using Sylvester’s approach that gives approximate solutions for PDEs.

Nowadays, wavelets have been discussed in numerous zones of logical and designing angles. As another reason for speaking to capacities, some consider it as a system for time-recurrence investigation, and others consider it another numerical subject. Every one of them is right since "wavelets" is a flexible apparatus with rich scientific substance and extraordinary potential for applications. As this subject is still amidst quick improvement, it is certainly too soon to give a unified presentation. The subject of wavelets has had a spot in the core of engineering, science and mathematics. Wavelet is an energizing new technique for taking care of troublesome issues in engineering, mathematics and physics, with present-day applications as differing as wave proliferation, data compression, image processing, signal processing, computer graphics. As the contribution of wavelets by Chebyshev, Bernoulli and Legendre wavelets (for instant see [35–38]) based solution of partial differential equations has been obtained. Favorable circumstances of wavelets bases over operational matrix strategy have prompted enormous application in science and engineering. The exact solution proves the accuracy and efficiency of wavelet operational matrix methods with the good agreement of mathematical results. Likewise, the wavelet operational matrix strategy is simple, efficient and delivers extremely precise numerical outcomes with an impressively modest number of basis functions and hence reduces computational exertion. Additionally, the technique is easy to apply for multidimensional problems.

So, we transform Eqs. (1–3) into its equivalent construction of integro-PDEs which consists of both initial and boundary conditions, and therefore, we can solve then by using the operational matrices of integration and differentiation of Legendre wavelets together with the completeness of these wavelets, the integro-PDEs reduce to the system of algebraic Sylvester equation. Hence, we can achieve the solution of Eqs. (1–3) in terms of Legendre wavelets on solving the Sylvester equation by generalized biconjugate gradient stabilized method (BICGSTAB) (i.e., robust Krylov subspace iterative method [33]).

The outline of this paper is as follows. In Sect. 3, we explain an introduction to the Multidimensional Legendre wavelet, function approximation and convergence of approximations. In Sect. 4, we constructed operational matrices based on Legendre wavelet for multidimensional functional approximation with Kronecker multidimensional. In Sect. 5, we proposed numerical method for solution. We employ some literature problems for showing the ability of the new technique in the current investigation in Sect. 6. Finally, a conclusion is given in Sect. 7.

2 Preliminaries: Construction of Basis Functions and Their Properties

In this section, the properties of the Legendre wavelet and their associative operational matrices are reviewed. Further, using the Kronecker multiplication, we extend the one-dimensional operational matrix to multi-dimensions. One can point out many applications of Legendre wavelet in the numerical solution of partial differential equations (PDEs) like: Schrödinger equation [42], Poisson equation in [43], etc. Properties of Legendre wavelet are discussed in [36].

2.1 Legendre Wavelet [36]

Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter \( a \) and the translation parameter \( b \) vary continuously, we have the following family of continuous wavelets:

\[
\Psi_{a,b}(\eta) = |a|^{-\frac{n}{2}}\Psi\left(\frac{\eta - b}{a}\right), \quad a, b \in \mathbb{R}, \quad a \neq 0.
\]

If the parameter \( a \) and \( b \) are restricted to the discrete values as \( a = a_0^k, b = nb_0a_0^k \), \( a_0 > 1, b_0 > 0 \), \( n \) and \( k \) are positive integers, from the above Eq. we have the following family of discrete wavelets:

\[
\Psi_{k,n}(\eta) = |a_0|^{-\frac{n}{2}}\Psi(a_0^\eta - nb_0),
\]

where \( \Psi_{k,n}(\eta) \) form a wavelet basis for \( L^2(\mathbb{R}) \). In particular, when \( a_0 = 2 \) and \( b_0 = 1 \) then \( \Psi_{k,n}(\eta) \) form an orthonormal basis.

Legendre wavelet \( \Psi_{n,m}(\eta) = \Psi(k, \hat{n}, m, \eta) \) have four arguments \( \hat{n} = 2n - 1, n = 1, 2, \ldots, 2^{k-1}, k \in \mathbb{Z}^+, m \) is the order of Legendre polynomials, and \( \eta \) is normalized time. They are defined on \([0, 1)\) ([36]):
\[\Psi_{n,m}(\eta) = \Psi(k, n, m, \eta) = \begin{cases} \sqrt{m + \frac{1}{2} 2^{k} p_{m}(2^{k} \eta - \hat{n})}, & \text{if } \frac{n - 1}{2^k} \leq \eta < \frac{n + 1}{2^k}; \\ 0, & \text{if otherwise} \end{cases} \]

where the coefficient \(\sqrt{m + \frac{1}{2}}\) is for orthonormality.

### 2.2 Multi-dimensional Legendre Wavelet

Two-dimensional Legendre wavelet are expressed as product of one-dimensional Legendre wavelet as follows:

\[\Psi_{n,m,m'}(\eta, \zeta) = \Psi_{n,m}(\eta) \Psi_{n',m'}(\zeta)\]

Three-dimensional Legendre wavelet are expressed as product of one-dimensional Legendre wavelet as follows:

\[\Psi_{n,m,m',n''}(\eta, \zeta, t) = \Psi_{n,m}(\eta) \Psi_{n',m'}(\zeta) \Psi_{n'',m''}(t)\]

where

\[\Psi_{n,m}(\eta) = \sqrt{m + \frac{1}{2} 2^{k} p_{m}(2^{k} \eta - \hat{n})}, \quad \Psi_{n',m'}(\zeta) = \sqrt{m' + \frac{1}{2} 2^{k'} p_{m'}(2^{k'} \zeta - \hat{n'})}, \quad \Psi_{n'',m''}(t) = \sqrt{m'' + \frac{1}{2} 2^{k''} p_{m''}(2^{k''} t - \hat{n''})}\]

and

\[m = 0, 1, 2, ..., M - 1, M \in Z^+ \cup \{0\};\]
\[m' = 0, 1, 2, ..., M' - 1, M' \in Z^+ \cup \{0\};\]
\[m'' = 0, 1, 2, ..., M'' - 1, M'' \in Z^+ \cup \{0\};\]
\[\hat{n} = 2n - 1, \hat{n'} = 2n' - 1, \hat{n''} = 2n'' - 1,\]
\[n = 1, 2, 3, ..., 2^{k-1}, n' = 1, 2, 3, ..., 2^{k'-1}, n'' = 1, 2, 3, ..., 2^{k''-1} \]

where also \(Z^+\) is positive integer, and \(p_{m}, p_{m'}\) and \(p_{m''}\) are Legendre polynomial of order \(m, m'\) and \(m''\), respectively, which are defined over the interval \([0, 1]\) and also three-dimensional Legendre wavelet are orthonormal set over \(\Omega = [0, 1] \times [0, 1] \times [0, 1]\).

### 2.3 Function Approximation

Suppose that \(f(\eta)\) is an arbitrary function in \(L^2([0, 1])\), then it is approximated as follows:

\[f(\eta) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm}\Psi_{nm}(\eta) \quad (10)\]

If the infinite series (10) is truncated for \(m = M - 1\), then approximation of the series (10) is represented as in the following form

\[f(\eta) = \sum_{n=1}^{M-1} \sum_{m=1}^{M-1} f_{nm}\Psi_{nm}(\eta) = F^T_{M-1}\Psi(\eta). \quad (11)\]

where \(\Psi(\eta) = [\Psi_{10}(\eta), \Psi_{11}(\eta), ..., \Psi_{1(M-1)}, ..., \Psi_{2^{k-1}-1,1}(\eta)]^T\) and \(F^T_{M-1}\) is \(1 \times (2^{k-1}(M-1))\) vector given by:

\[F^T_{M-1} = \begin{bmatrix} \int_{0}^{1} f(\eta)\Psi_{10}(\eta)d\eta \\ \int_{0}^{1} f(\eta)\Psi_{11}(\eta)d\eta \\ \vdots \\ \int_{0}^{1} f(\eta)\Psi_{1(M-1)}(\eta)d\eta \\ \int_{0}^{1} f(\eta)\Psi_{2^{k-1}0}(\eta)d\eta \\ \int_{0}^{1} f(\eta)\Psi_{2^{k-1}1}(\eta)d\eta \\ \vdots \\ \int_{0}^{1} f(\eta)\Psi_{2^{k-1}(M-1)}(\eta)d\eta \end{bmatrix} \]

Now generalised (11) series as follows

\[f(\eta) = \sum_{n=1}^{2^{k-1}-1} \sum_{m=1}^{2^{k-1}-1} \sum_{m'=1}^{2^{k-1}M'} f_{nmn'm''}\Psi_{nmn'm''}(\eta, \zeta) = F^T_{M'}\Psi(\eta, \zeta), \quad (12)\]

where \(\Psi(\eta, \zeta) = \Psi(\eta) \otimes \Psi(\zeta)\) (For the numerical solution, we used the concept of Kronecker product (\(\otimes\)) [36] is \(2^{k-1}2^{k-1}MM \times 1\), vector given as follows:
\[ \Psi = \left[ \Psi_{1010}, \ldots, \Psi_{10(\ell-1)}, \Psi_{1020}, \ldots, \Psi_{102(\ell-1)}, \ldots \right], \]
\[ \Psi_{102^{\ell-1}(\ell-1)}, \ldots, \Psi_{1(\ell-1)2^{\ell-1}(\ell-1)}, \Psi_{1(\ell-1)12^{\ell-1}(\ell-1)}, \Psi_{1(\ell-1)2^{\ell}(\ell-1)}, \ldots, \Psi_{1(\ell-1)(\ell-1)^{\ell-1}(\ell-1)}, \Psi_{2(\ell-1)}2^{\ell-1}(\ell-1), \ldots, \Psi_{2^{\ell-1}(\ell-1)(\ell-1)^{\ell-1}(\ell-1)}  \right]^T, \]
and
\[ f(\eta, \xi, t) = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \sum_{\ell=0}^{M-1} \sum_{m'\neq m} \sum_{n'\neq n} f_{\ell mn'm'n'} \Psi_{\ell mn'm'n'}(\eta) \]
\[ = F^T \Psi(\eta, \xi, t), \]
where \( \Psi(\eta, \xi, t) = \Psi(\eta) \otimes \Psi(\xi) \otimes \Psi(t) \)
are \(2^{k-1}2^{k'}-1\) two-dimensional MM'MM'' \times 1, vector given as follows:
\[ F^T = \left[ \begin{array}{c} 
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{10001}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{10010}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{10101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{11001}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{11101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{12001}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{12101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{20001}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{20101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{21001}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{21101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{22001}(\eta, \xi, t) d\eta d\xi dt \\
\vdots \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{22101}(\eta, \xi, t) d\eta d\xi dt \\
\int_{0}^{1} \int_{0}^{1} f(\eta, \xi, t) \Psi_{22201}(\eta, \xi, t) d\eta d\xi dt \\
\end{array} \right] \]
\[ T \]

**Theorem 1** The series \( \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m'=0}^{\infty} f_{\ell mn'm'n'} \Psi_{\ell mn'm'n'}(\eta, \xi, t) \) is three-dimensional Legendre wavelets which is defined in Eq. (6), uniformly converges to a continuous function \( f(\eta, \xi, t) \).

**Proof** Let \( H^1(\mathcal{A}_1) \) be the Hilbert space and \( \Psi(\eta, \xi, t) \) is defined in Eq. (6) forms an orthonormal basis. So for fixed \( k, k' \) and \( k'' \)
\[ f(\eta, \xi, t) = \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \sum_{m'=0}^{M''-1} f_{mn'm'} \Psi_{mn'm'}(\eta, \xi, t), \]
where
\[ f_{nn'nm'} = \int_0^1 \int_0^1 f(\eta, \xi, t) \psi_{nn'nm'}(\eta, \xi, t) d\eta d\xi dt, \]
\[ = \langle f(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \rangle. \]  
(18)

Now, truncate series (18) up to \(N\) level as follows
\[ f(\eta, \xi, t) = \sum_{m=0}^N \sum_{m'=0}^N \sum_{m''=0}^N f_{nn'nm'} \psi_{nn'nm'}(\eta, \xi, t) = S_N \text{(say)}. \]  
(19)

Now,
\[ (f(\eta, \xi, t), S_N) = \left( f(\eta, \xi, t), \sum_{m=0}^N \sum_{m'=0}^N \sum_{m''=0}^N f_{nn'nm'} \psi_{nn'nm'}(\eta, \xi, t) \right) \]
\[ = \sum_{m=0}^N \sum_{m'=0}^N \sum_{m''=0}^N f_{nn'nm'} \langle f(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \rangle. \]  
(20)

From Eq. (19) and (20)
\[ \langle s - f(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \rangle \]
\[ = \langle s, \psi_{nn'nm'}(\eta, \xi, t) \rangle - \langle f(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \rangle, \]
\[ = \left( \lim_{N \to \infty} S_N, \psi_{nn'nm'}(\eta, \xi, t) \right) - \langle f(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \rangle, \]
\[ = \lim_{N \to \infty} \langle S_N, \psi_{nn'nm'}(\eta, \xi, t) \rangle - f_{nn'nm'}, \]
\[ = \lim_{N \to \infty} \left( \sum_{m=0}^N \sum_{m'=0}^N \sum_{m''=0}^N f_{nn'nm'} \psi_{nn'nm'}(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \right) - f_{nn'nm'}, \]
\[ = \lim_{N \to \infty} f_{nn'nm'} \left( \psi_{nn'nm'}(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t), \psi_{nn'nm'}(\eta, \xi, t) \right) - f_{nn'nm'}, \]
\[ = f_{nn'nm'} - f_{nn'nm'} \quad \text{(using orthonormality of Legendre wavelets)}, \]
\[ = 0. \]  
(25)

Hence, \( s = f(\eta, \xi, t) \).

Thus, the series \( \sum_{m=0}^\infty \sum_{m'=0}^\infty \sum_{m''=0}^\infty f_{nn'nm'n'm''n''} \psi_{nn'nm'n'm''n''}(\eta, \xi, t) \) converge uniformly to \( f(\eta, \xi, t) \).

Hence, the theorem. \( \square \)

Now, we claim that
\[ \langle f(\eta, \xi, t), S_N \rangle = \sum_{m=0}^N \sum_{m'=0}^N \sum_{m''=0}^N |f_{nn'nm'}|^2 \text{for} N \geq N'. \]  
(22)

Since

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3 Operational Matrices

Let \( \Psi(t) = [\Psi_0(t), \Psi_1(t), ..., \Psi_N(t)]^T \) be the basis functions. Then,

\[
\frac{d}{dt} \begin{bmatrix} \Psi_0(t) \\ \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{bmatrix} \approx \begin{bmatrix} F & 0 & 0 & \cdots & 0 \\ 0 & F & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & F \end{bmatrix}_{2^{l-1}M \times 2^{l-1}M} \begin{bmatrix} \Psi_0(t) \\ \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{bmatrix}
\]

= \( D_t \Psi^T(t) \),

(26)

in which \( F \) is \( M \times M \) matrix with entries

\[
F_{rs} = \begin{cases} 
2^l \sqrt{(2r-1)(2s-1)}, & \text{if } r = 2, 3, \ldots, M, \\
0, & \text{otherwise,}
\end{cases}
\]

where, \( s = 1, 2, \ldots, r - 1, (r + s) \text{ odd}; \)

\[
\text{if otherwise.}
\]

And

\[
\int_0^t \begin{bmatrix} \Psi_0(t') \\ \Psi_1(t') \\ \vdots \\ \Psi_N(t') \end{bmatrix} dt' \approx \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 & \cdots & 0 \\ -\sqrt{3} & 0 & \sqrt{3} & 0 & \cdots & 0 \\ 0 & -\sqrt{5} & 0 & \sqrt{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{2M-3} \\ 0 & 0 & 0 & 0 & \cdots & \frac{\sqrt{2M-1}}{(2M-1)\sqrt{2M-3}} \end{bmatrix}_{N \times N} \begin{bmatrix} \Psi_0(t) \\ \Psi_1(t) \\ \vdots \\ \Psi_N(t) \end{bmatrix} = I_t \Psi^T(t)
\]

Here, \( I_t \) is \( 2^{k-1}M \times 2^{k-1}M \) matrix.

The approximation given in Sect. 3 for numerical method is extended for the higher dimension. Two variables functions, namely \( f_3(\xi, t), f_4(\xi, t), f_5(\eta, t) \) and \( f_6(\eta, t) \) in \( L^2([0, 1] \times [0, 1]) \) is approximated as:

where,

\[
\begin{cases} 
f_3(\xi, t) \approx \sum_{n=1}^{2^{l-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{l-1}} \sum_{m'=0}^{M-1} f_{3n,m'\eta} \Psi_m(\xi) \Psi_n(t) = \Psi^T(\xi)F_3(\eta), \\
f_4(\xi, t) \approx \sum_{n=1}^{2^{l-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{l-1}} \sum_{m'=0}^{M-1} f_{4n,m'\eta} \Psi_m(\xi) \Psi_n(t) = \Psi^T(\xi)F_4(\eta), \\
f_5(\eta, t) \approx \sum_{n=1}^{2^{l-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{l-1}} \sum_{m'=0}^{M-1} f_{5n,m'\eta} \Psi_m(\eta) \Psi_n(t) = \Psi^T(\eta)F_5(\eta), \\
f_6(\eta, t) \approx \sum_{n=1}^{2^{l-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{l-1}} \sum_{m'=0}^{M-1} f_{6n,m'\eta} \Psi_m(\eta) \Psi_n(t) = \Psi^T(\eta)F_6(\eta), 
\end{cases}
\]

(27)
\[ F_3 = \begin{bmatrix}
 f_{1010}^3 & f_{1011}^3 & \cdots & f_{101(M^* - 1)}^3 & f_{111(M^* - 1)}^3 & \cdots & f_{1(M^* - 1)(M^* - 1)}^3 \\
 f_{200}^3 & f_{201}^3 & \cdots & f_{202(M^* - 1)}^3 & f_{212(M^* - 1)}^3 & \cdots & f_{2(M^* - 1)(2(M^* - 1))}^3 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 f_{2^{k-1}02^{k-1}0}^3 & f_{2^{k-1}02^{k-1}1}^3 & \cdots & f_{2^{k-1}02^{k-1}(M^* - 1)}^3 & f_{2^{k-1}12^{k-1}(M^* - 1)}^3 & \cdots & f_{2^{k-1}(M^* - 1)(2^{k-1}(M^* - 1))}^3 
\end{bmatrix} \]

\[ F_4 = \begin{bmatrix}
 f_{1010}^4 & f_{1011}^4 & \cdots & f_{101(M^* - 1)}^4 & f_{111(M^* - 1)}^4 & \cdots & f_{1(M^* - 1)(M^* - 1)}^4 \\
 f_{200}^4 & f_{201}^4 & \cdots & f_{202(M^* - 1)}^4 & f_{212(M^* - 1)}^4 & \cdots & f_{2(M^* - 1)(2(M^* - 1))}^4 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 f_{2^{k-1}02^{k-1}0}^4 & f_{2^{k-1}02^{k-1}1}^4 & \cdots & f_{2^{k-1}02^{k-1}(M^* - 1)}^4 & f_{2^{k-1}12^{k-1}(M^* - 1)}^4 & \cdots & f_{2^{k-1}(M^* - 1)(2^{k-1}(M^* - 1))}^4 
\end{bmatrix} \]

\[ F_5 = \begin{bmatrix}
 f_{1010}^5 & f_{1011}^5 & \cdots & f_{101(M^* - 1)}^5 & f_{111(M^* - 1)}^5 & \cdots & f_{1(M^* - 1)(M^* - 1)}^5 \\
 f_{200}^5 & f_{201}^5 & \cdots & f_{202(M^* - 1)}^5 & f_{212(M^* - 1)}^5 & \cdots & f_{2(M^* - 1)(2(M^* - 1))}^5 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 f_{2^{k-1}02^{k-1}0}^5 & f_{2^{k-1}02^{k-1}1}^5 & \cdots & f_{2^{k-1}02^{k-1}(M^* - 1)}^5 & f_{2^{k-1}12^{k-1}(M^* - 1)}^5 & \cdots & f_{2^{k-1}(M^* - 1)(2^{k-1}(M^* - 1))}^5 
\end{bmatrix} \]

\[ F_6 = \begin{bmatrix}
 f_{1010}^6 & f_{1011}^6 & \cdots & f_{101(M^* - 1)}^6 & f_{111(M^* - 1)}^6 & \cdots & f_{1(M^* - 1)(M^* - 1)}^6 \\
 f_{200}^6 & f_{201}^6 & \cdots & f_{202(M^* - 1)}^6 & f_{212(M^* - 1)}^6 & \cdots & f_{2(M^* - 1)(2(M^* - 1))}^6 \\
 \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
 f_{2^{k-1}02^{k-1}0}^6 & f_{2^{k-1}02^{k-1}1}^6 & \cdots & f_{2^{k-1}02^{k-1}(M^* - 1)}^6 & f_{2^{k-1}12^{k-1}(M^* - 1)}^6 & \cdots & f_{2^{k-1}(M^* - 1)(2^{k-1}(M^* - 1))}^6 
\end{bmatrix} \]

Lemma 1: Let \( \lambda_i = \int_0^1 \Psi^T(\eta)d\eta \) and \( \lambda_i I_L = \lambda_i I_L^T \) are the \( 1 \times 2^{k-1}M \) vectors, then the following relations hold:

(i) \( \Psi^T(\xi) = \Psi^T(\eta, \xi)P \),

(ii) \( \eta \Psi^T(\xi) = \Psi^T(\eta, \xi)P \),

(iii) \( \eta_i I_L \otimes \Psi^T(\xi) = \Psi^T(\eta, \xi)Q \),

(iv) \( \xi(\Psi^T(\eta)I \otimes \lambda_i I_B) = \Psi^T(\eta, \xi)Q \),

(v) \( \Psi^T(\eta) = \Psi^T(\eta, \xi)R \),

(vi) \( \xi \Psi^T(\eta) = \Psi^T(\eta, \xi)R \).

Proof: Let \( \Psi(\eta) = [\Psi_{10}(\eta), \Psi_{11}(\eta), \ldots, \Psi_{1(M^* - 1)}(\eta), \ldots, \Psi_{2^{k-1}0}(\eta), \Psi_{2^{k-1}1}(\eta), \ldots, \Psi_{2^{k-1}(M^* - 1)}(\eta)]^T \) be the one dimensional Legendre wavelet basis of \( L^2[0, 1] \). Then \( \eta \Psi^T(\xi) \) is written as

\[ \eta \Psi^T(\xi) = \eta \otimes \Psi^T(\xi) \]  \hspace{1cm} (28)

Let \( \eta = g(\eta) \in L^2[0, 1] \). Then \( g(\eta) \) is approximated in terms of Legendre wavelet basis functions as

\[ g(\eta) = \eta \approx \sum_{n=1}^{M-1} \sum_{m=0}^{M-1} g_{nm} \Psi_{nm}(\eta) = A^T \Psi(\eta) \]  \hspace{1cm} (29)

where \( A = [a_{10}, a_{11}, \ldots, a_{1(M^* - 1)}, \ldots, a_{2^{k-1}0}, a_{2^{k-1}1}, \ldots, a_{2^{k-1}(M^* - 1)}]^T \) and the coefficients of \( A \) are calculated by the formula

\[ a_i = \int_0^1 \eta \Psi_i(\eta)d\eta \]  \hspace{1cm} (28)

hence from Eq.(18)

\[ \eta \Psi^T(\xi) = \Psi^T(\eta)A \otimes \Psi^T(\xi) = (\Psi^T(\eta) \otimes \Psi^T(\xi))(A \otimes I_{N+1}) \]  \hspace{1cm} (29)

where \( P = A \otimes I \).

Similarly, we can write relation (iii) as

\[ \eta(\lambda_i I_l \otimes I \Psi^T(\xi)) = (\eta \lambda_i I_l) \otimes I \Psi^T(\xi) \]
Since, $\eta \lambda_1 I_L$ is $1 \times 2^{k-1} M$ vector and each element of this vector is a function of $\eta$ so, we can make similar argument as given in the proof of (ii).

Hence,

\[
(\eta \lambda_1 I_L) \otimes \Psi^T(\xi) = (\Psi^T(\eta) \otimes \Psi^T(\xi))(B^T \otimes I),
\]

\[
= \Psi^T(\eta, \xi) P,
\]

where $P = (B^T \otimes I_{N+1})$ and

\[
B = \begin{bmatrix}
c_{10} & c_{11} & \cdots & c_{1(M-1)} \\
c_{20} & c_{21} & \cdots & c_{2(M-1)} \\
\vdots & \vdots & \ddots & \vdots \\
c_{2^{k-1}0} & c_{2^{k-1}1} & \cdots & c_{2^{k-1}(M-1)}
\end{bmatrix} = [c_{nm}]_{2^{k-1}M \times 2^{k-1}M}.
\]

The coefficients $c_{nm}$ are calculated as follows

\[
c_{nm} = \int_0^1 [(\eta \lambda_1 I_L) \Psi_m(\eta)] d\eta.
\]

Note: (a) All other matrices $\mathcal{O}$, $R$ and $\mathcal{R}$ are calculated in the similar manner.

(b) If we take $\eta = 1$ then the proof of (i) is same as (i). Similarly, the proof of (iv), (v) and (vi) is same as (i), (ii) and (iii), respectively. \hfill \Box

Lemma 2 If $D_n = I_{2^{k-1}M \times 2^{k-1}M} \otimes D_0, I_{\xi} = I_{2^{k-1}M \times 2^{k-1}M} \otimes I_0, I_{\xi} = I_{2^{k-1}M \times 2^{k-1}M}$, and $I_n = I_{2^{k-1}M \times 2^{k-1}M} \otimes I_0$, where $I$ denotes the identity matrix then we have

(i) \[ \frac{\partial^2 \Psi(\eta, \xi)}{\partial \eta^2} = D_2^2 \Psi(\eta, \xi), \]

(ii) \[ \frac{\partial^2 \Psi(\eta, \xi)}{\partial \xi^2} = D_2^2 \Psi(\eta, \xi), \]

(iii) \[ \int_0^\infty \Psi(\eta, \xi) d\xi^2 \approx \int_0^\infty \Psi(\eta, \xi) d\xi^2 \approx (I_0)^2 \Psi(\eta, \xi), \]

(iv) \[ \int_0^\infty \Psi(\eta, \xi) d\xi^2 \approx \int_0^\infty \Psi(\eta, \xi) d\xi^2 \approx (I_0)^2 \Psi(\eta, \xi), \]

(v) \[ \frac{\partial^2 \Psi(\eta, \xi)}{\partial \eta \partial \xi} = \Psi(\eta, \xi) = D_1^2 \Psi(\eta, \xi), \]

(vi) \[ \frac{\partial^2 \Psi(\eta, \xi)}{\partial \eta \partial \xi} = \Psi(\eta, \xi) = D_1^2 \Psi(\eta, \xi), \]

(vii) \[ \frac{\partial^2 \Psi(\eta, \xi)}{\partial \eta \partial \xi} = \Psi(\eta, \xi) = D_1^2 \Psi(\eta, \xi), \]

Proof In Sect. 4, we can see the proof which is summarized here. \hfill \Box

4 Numerical Method of Solution

To find numerical solution of Eq.(1)-(3), we will use all the initial and boundary conditions on (1) then (1) will convert into partial integro-differential equation (PIDE). For this purpose, we rewrite Eq.(1) as follows

\[
\frac{\partial^2 \Phi(\eta, \xi, t)}{\partial \eta^2} = \frac{\partial^2 \Phi(\eta, \xi, t)}{\partial t^2} + 2 \lambda_1 \frac{\partial \Phi(\eta, \xi, t)}{\partial t} + \lambda_2^2 \Phi(\eta, \xi, t) - \frac{\partial^2 \Phi(\eta, \xi, t)}{\partial \xi^2} - F(\eta, \xi, t).
\]

Integrating Eq.(31) in the interval $[0, \eta]$, we obtain

\[
\Phi(\eta, \xi, t) = \Phi(0, \xi, t) + \int_0^\eta \Phi(\eta', \xi, t) + 2 \lambda_1 \Phi(\eta', \xi, t) + \lambda_2^2 \Phi(\eta', \xi, t) - \Phi(\xi, \xi', t) - F(\eta', \xi, t) d\eta'.
\]

Further integrating Eq.(32) in the interval $[0, \eta]$, we get

\[
\Phi(\eta, \xi, t) = \Phi(0, \xi, t) + \eta \Phi(0, \xi, t) - \Phi(\xi, \xi', t) - F(\eta', \xi, t) d\eta'.
\]

Put $\eta = 1$ in Eq.(33), we get

\[
\Phi(0, \xi, t) = \Phi(1, \xi, t) - \Phi(\xi, \xi', t) - F(\eta', \xi, t) d\eta'.
\]

Substituting the value of $\Phi(0, \xi, t)$ from Eq.(34) to (33), we get

\[
\Phi(\eta, \xi, t) = \eta \Phi(\xi, \xi', t) + (1 - \eta) \Phi(0, \xi, t) - \eta \int_0^\eta \int_0^\eta (\Phi(\eta', \xi, t) + 2 \lambda_1 \Phi(\eta', \xi, t) + \lambda_2^2 \Phi(\eta', \xi, t) - \Phi(\xi, \xi', t) - F(\eta', \xi, t) d\eta') d\eta d\eta'.
\]

To use boundary conditions in $\xi$, we integrate $\Phi(\xi, \eta, t)$ from 0 to $\xi$, we obtain

\[
\int_0^\xi \Phi(\xi, \eta, t) d\xi = \Phi(\xi, \eta, t) - \Phi(0, \eta, t).
\]
\[ \int_0^\xi \int_0^\xi \Phi_{\xi\xi}(\eta, \xi, t) d\xi' d\xi'' = \Phi(\eta, \xi, t) - \Phi(\eta, 0, t) - \xi \Phi(\eta, 0, t), \]

or

\[ \Phi(\eta, \xi, t) = f_3(\eta, t) + \xi \Phi_\xi(\eta, 0, t) + \int_0^\xi \int_0^\xi \Phi_{\xi\xi}(\eta, \xi', \eta, t) d\xi' d\xi''. \]  

(37)

To use the condition \( \Phi(\eta, 1, t) = f_6(\eta, t) \), put \( \xi = 1 \) in Eq. (37), we get

\[ \Phi_\xi(\eta, 0, t) = f_6(\eta, t) - f_5(\eta, t) - \int_0^{1} \int_0^\xi \Phi_{\xi\xi}(\eta, \xi', \eta, t) d\xi' d\xi, \]

(38)

substitute the value of \( \Phi_\xi(\eta, 0, t) \) obtained by Eq. (38) into (37), we obtain

\[ \Phi(\eta, \xi, t) = f_6(\eta, t) + (1 - \xi) f_5(\eta, t) - \xi \int_0^1 \int_0^\xi \Phi_{\xi\xi}(\eta, \xi', t) d\xi' d\xi + \int_0^\xi \int_0^\xi \Phi_{\xi\xi}(\eta, \xi', t) d\xi' d\xi'. \]

(39)

Now, our goal is to use the initial conditions (2). For this purpose, we use same process which is used to use the boundary conditions in \( \xi \) i.e., integrate \( \Phi_\eta(\eta, \xi, t) \) two times from 0 to \( t \) and obtain

\[ \Phi(\eta, \xi, t) = f_1(\eta, \xi) + f_2(\eta, \xi) + \int_0^t \int_0^\xi \Phi_\eta(\eta, \xi, t') dt' d\xi. \]

The above Eq. is rewritten as

\[ \Phi(\eta, \xi, t) = f(\eta, \xi, t) + \int_0^t \int_0^\xi \Phi_\eta(\eta, \xi, t') dt' d\xi \]  

(40)

where \( f(\eta, \xi, t) = f_1(\eta, \xi) + tf_2(\eta, \xi). \)

Since, we obtain the original Eq. (1) with boundary conditions \( \Phi(0, \xi, t) = f_3(\xi, t) \) and \( \Phi(1, \xi, t) = f_4(\xi, t) \) by differentiating two times Eq. (35) and replacing \( \eta \) by 0 and 1 in Eq. (35), respectively. Including the boundary conditions \( \Phi(\eta, 0, t) = h_1(\eta, t) \), \( \Phi(\eta, 1, t) = h_2(\eta, t) \) and initial conditions \( \Phi(\eta, \xi, 0) = f_1(\eta, \xi) \), \( \Phi_\xi(\eta, \xi, 0) = f_2(\eta, \xi) \) calculated by replacing \( \xi \) by 0 and 1 in Eq. (39) and \( t \) by 0 and 1 in Eq. (40), respectively. Hence, the Eq. (35), (39) and (40) are the equivalent formulation of the proposed problem (1)-(3). To solve these PIDE, all the known and unknown functions are approximated in terms of basis functions. The known function of three variables \( F(\eta, \xi, t) \) are approximated in terms of basis functions as follows

\[ F(\eta, \xi, t) \approx \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \sum_{k=1}^{N_3} \Phi_{apx}(\eta) \Psi_{Bn}(n) \Psi_{Cm}(m) \Psi_{Dk}(k), \]

(41)

where \( \Psi_T(\eta, \xi) = \Psi_T(\eta) \otimes \Psi_T(\xi) \), and \( F \) is matrix calculated in similar manner as \( F \) in Sect. 4. Similarly, we approximate \( f(\eta, \xi, t) \) as follows

\[ f(\eta, \xi, t) \approx \Psi_T(\eta, \xi) F(\eta, \xi, t). \]

(42)

Similarly, the unknown function \( \Psi(\eta, \xi, t) \) is approximated in terms of basis functions as

\[ \Phi(\eta, \xi, t) \approx \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \sum_{k=1}^{N_3} \Phi_{apx}(n) \Psi_{Bn}(n) \Psi_{Cm}(m) \Psi_{Dk}(k), \]

(43)

where

\[ \Phi_{apx} = \begin{bmatrix}
\Psi_{101010} & \Psi_{202020} & \cdots & \Psi_{102020} \\
\Psi_{101011} & \Psi_{202020} & \cdots & \Psi_{202020} \\
\vdots & \vdots & \ddots & \cdots \\
\Psi_{10101(M^p-1)} & \Psi_{20202(M^p-1)} & \cdots & \Psi_{20202(M^p-1)} \\
\Psi_{10111(M^p-1)} & \Psi_{20212(M^p-1)} & \cdots & \Psi_{20212(M^p-1)} \\
\vdots & \vdots & \ddots & \cdots \\
\Psi_{101(M^p-1)1(M^p-1)} & \Psi_{2012(M^p-1)2(M^p-1)} & \cdots & \Psi_{2012(M^p-1)2(M^p-1)} \\
\Psi_{111(M^p-1)1(M^p-1)} & \Psi_{212(M^p-1)2(M^p-1)} & \cdots & \Psi_{212(M^p-1)2(M^p-1)} \\
\vdots & \vdots & \ddots & \cdots \\
\Psi_{1(M^p-1)1(M^p-1)1(M^p-1)} & \Psi_{2(M^p-1)2(M^p-1)2(M^p-1)} & \cdots & \Psi_{2(M^p-1)2(M^p-1)2(M^p-1)} \\
\end{bmatrix}^{T} \]
where \( \Phi_{\text{apx}} \) is \((2^{k-1}M2^{k-1}M') \times 2^{k-1}M'' \) matrix. It is to be noted that we have to calculate the unknown matrix \( \Phi_{\text{apx}} \).

Now, we substitute the approximated value of \( F(\eta, \xi, t), \Phi(\eta, \xi, t), f_1(\xi, t) \) and \( f_2(\xi, t) \) which are given by Eq. (41)-(43) and (27) in the right hand side of Eq. (35). After using the operational matrices of differentiation and integration and Lemma 2, Eq. (35) reduces to

\[
\Phi(\eta, \xi, t) \approx \xi \eta \xi^{\Psi T}(\eta) (F_6 - F_5) \Psi(t) + \eta \xi^{\Psi T}(\eta) F_5 \Psi(t) - \xi (\Psi^T \otimes x \xi I_n) (D^{\xi}_x)^2 \chi \Psi(t) + \eta \xi^{2 \Psi T}(\eta) (D^{\xi}_x)^2 \chi \Psi(t).
\]

By using relation (iv),(v) and (vi) of Lemma (1), above Eq. is written as

\[
\Phi(\eta, \xi, t) = \eta \xi^{\Psi T}(\eta) Y \Psi(t),
\]

where

\[
Y = \overline{R}(F_6 - F_5) + RF_5 + \left((I^T_1)^2 - \overline{Q}\right)(D^2_\xi)^2 X.
\]

In the next step, Eq.(49) is taken as an approximation for \( \Phi(\eta, \xi, t) \). Substituting the approximated values of \( f(\eta, \xi, t), \Phi(\eta, \xi, t) \) together with operational matrices of differentiation and integration in the right hand side of Eq.(40), we get

\[
\Phi(\eta, \xi, t) = \eta \xi^{\Psi T}(\eta) Y \Psi(t) + \eta \xi^{2 \Psi T}(\eta) Y D^2_\xi \chi \Psi(t).
\]

On combining above approximation \( \Phi(\eta, \xi, t) \) with Eq.(43), we get

\[
\Psi^T(\eta, \xi) \Phi_{\text{apx}} \Psi(t) = \eta \xi^{\Psi T}(\eta) F \Psi(t) + \eta \xi^{2 \Psi T}(\eta) \chi D^2 \chi \Psi(t),
\]

\[
\chi = \overline{R}(F_6 - F_5) + RF_5 + \left((I^T_1)^2 - \overline{Q}\right)(D^2_\xi)^2 \overline{X}
\]

\[
+ \left((I^T_1)^2 - \overline{Q}\right)(D^2_\xi)^2 \chi \Psi(t) + \eta \xi^{2 \Psi T}(\eta) \chi D^2 \chi \Psi(t).
\]

Eq.(50) is reduced into the following matrix Eq.

\[
\Phi_{\text{apx}} + X_1 \Phi_{\text{apx}} Y_1 + X_2 \Phi_{\text{apx}} Y_2 = Z
\]

which is in the form of Sylvester Eq., where

\[
X_1 = \eta \xi^{2 \Psi T}(\eta)(D^2_\xi)^2 \left((I^T_1)^2 - \overline{Q}\right),
\]

\[
Y_1 = -(D^2_\xi + 2 \xi I_x) D \chi D + \xi^2 (D^2_\xi)^2 \chi D^2 \chi,
\]

\[
X_2 = \eta \xi^{2 \Psi T}(\eta)(D^2_\xi)^2 \left((I^T_1)^2 - \overline{Q}\right) (D^2_\xi)^2 \chi D^2 \chi,
\]

\[
Y_2 = D^2 \chi D^2 \chi,
\]

and
\[ Z = F + RF_3 D^2_t I^2_x + \mathcal{R}(F_6 - F_3)D^2_t I^2_x + \left((I^2_x)^2 - \mathcal{O}\right)(D^2_t)^2 \]
\[
(\mathcal{P} - F_4 + F_3) + PF_3 - \left((I^2_x)^2 - \mathcal{O}\right)F \right)D^2_t I^1_x.
\]

Now, Eq.(51) is re-written as follows
\[
[I + X_1(Y^1_t \otimes I) + X_2(Y^2_t \otimes I)]\Phi_{apx} = Z,
\]
where \( I \) is the identity matrix.

Finally, we get the system of equations in the form of Eq.(52) which is known as the Sylvester equation. Bartels proposed numerical scheme to solve Sylvester equation ([44]). In this article, Sylvester equation (51) is solved for \( \Phi_{apx} \) by robust Krylov subspace iterative method (i.e., generalized BICGSTAB, [45]) and hence by Eq.(52), we get the approximate solution of Eq. (1-3) in terms of Legendre wavelet at small number of basis functions.

5 Numerical Experiments

In this section, we provide some numerical examples to analyze the applicability of proposed LWOMM for two dimensional hyperbolic telegraph equation (1). For the application of proposed method, we have included three literature examples and examine the applicability and effectiveness of the new method (Numerical simulations have been done with the help of MATLAB). Since the proposed method is a numerical technique based on the Legendre wavelet function, we can reach to the exact solutions if the solutions of the considered hyperbolic telegraph equations are in polynomial forms. Moreover, presented method produces spectral accuracy for dealing with two-dimensional hyperbolic telegraph equation which has exact solutions in non-polynomial forms. We take \( k = k' = k'' = 3, M = M' = M'' = 4, 5, 6, 7 \) and \( t = 1 \) also computed results that have been shown in the Table in terms of relative errors, \( L^2 \) and \( L^\infty \). The results are compared with Mittal and Bhatia [32] and found better accuracy by Legendre wavelet at small number of basis function. In order to illustrate the performance of the LWOMM, we have considered the maximum absolute error, which is denoted by \( e_n \). Also, in some figures the history of error between the analytical and numerical solutions, which is computed by the proposed LWOMM, is represented as \( e_n = \Phi - \Phi_{apx} \). Further, to assess the performance of the method LWOMM, we computed \( L^2 \) and \( L^\infty \) norms errors.

Example 1 In this example, we consider the HTE Eq.(1) in the domain \( (\eta, \xi, t) \in \Omega \) and
\[ F(\eta, \xi, t) = (-2\lambda_1 + \lambda_2^2 - 1)\exp(-t) \sinh \eta \sinh \xi. \]
The initial and the Dirichlet boundary conditions are given by
\[
\begin{align*}
\Phi(\eta, \xi, 0) &= \sinh(\eta) \sinh(\xi), \\
\Phi_t(\eta, \xi, 0) &= -\sinh(\eta) \sinh(\xi)
\end{align*}
\]
and
\[
\begin{align*}
\Phi(0, \xi, t) &= 0, \quad \Phi(1, \xi, t) = \exp(-t) \sinh(1) \sinh(\xi) \\
\Phi(\eta, 0, t) &= 0, \quad \Phi(\eta, 1, t) = \exp(-t) \sinh(\eta) \sinh(1)
\end{align*}
\]
, respectively.

The exact solution is given by
\[ \Phi(\eta, \xi, t) = \exp(-t) \sinh(\eta) \sinh(\xi). \]
In this example, we have taken \( \lambda_1 = 10, \lambda_2 = 5 \) and solved for \( N = 4, 5, 6 \) and \( 7 \).

Example 2 In this example, we consider the HTE Eq.(1) in the domain \( (\eta, \xi, t) \in \Omega \) with \( \lambda_1 = \lambda_2 = 1 \) and
\[ F(\eta, \xi, t) = 2 \cos(t) \sin(\eta) \sinh(\xi) - 2 \sin(t) \sin(\eta) \sin(\xi). \]
The initial and boundary conditions are given by
\[
\begin{align*}
\Phi(\eta, \xi, 0) &= \sin(\eta) \sin(\xi), \\
\Phi_\xi(\eta, \xi, 0) &= 0
\end{align*}
\]
and
\[
\begin{align*}
\Phi(0, \xi, t) &= 0, \quad \Phi(1, \xi, t) = \cos(t) \sin(1) \sin(\xi) \\
\Phi(\eta, 0, t) &= 0, \quad \Phi(\eta, 1, t) = \cos(t) \sin(\eta) \sin(1)
\end{align*}
\]
, respectively.

The exact solution is given by
\[ \Phi(\eta, \xi, t) = \cos t \sin(\eta) \sin(\xi). \]

Example 3 In this example, we consider the HTE Eq.(1) in the domain \( (\eta, \xi, t) \in \Omega \) with \( \lambda_1 = 10, \lambda_2 = 5 \) and
\[ F(\eta, \xi, t) = 22 \cos(t) \sin(\eta) \sinh(\xi) - 20 \sin(t) \sinh(\eta) \sinh(\xi). \]
The initial and Dirichlet boundary conditions are given below
\[
\begin{align*}
\Phi(\eta, \xi, 0) &= \sinh(\eta) \sinh(\xi), \\
\Phi_\xi(\eta, \xi, 0) &= 0
\end{align*}
\]
and
\[
\begin{align*}
\Phi(0, \xi, t) &= 0, \quad \Phi(1, \xi, t) = \cos(t) \sin(1) \sinh(\xi) \\
\Phi(\eta, 0, t) &= 0, \quad \Phi(\eta, 1, t) = \cos(t) \sinh(\eta) \sinh(1)
\end{align*}
\]
, respectively.

The exact solution is given by
\[ \Phi(\eta, \xi, t) = \cos t \sinh(\eta) \sinh(\xi). \]
\[
\Phi(\eta, \xi, t) = \cos t \sinh(\eta) \sinh(\xi).
\]

**Example 4** In this example, we consider the HTE Eq. (1) in the domain \((\eta, \xi, t) \in \Omega\) with \(\lambda_1 = 0.5, \lambda_2 = 1\) and \(F(\eta, \xi, t) = e^{-t}(\eta\xi(\eta - 1)(\xi - 1)(2 + 4(\lambda_1 - 1)t) + t^2(-2\xi(\xi - 1) + \eta(\xi(1 - 2\lambda_1 + \lambda_2^2)(1 - \xi) - 2) + \eta^2(\xi(-1 + 2\lambda_1 - \lambda_2^2)(1 - \xi) - 2))\). The initial and Dirichlet boundary conditions are given below.

### Table 1 Absolute error of LWOMM at \(e_N(\eta, \xi, 1)\) for values of \(N = 4, 5, 6, 7\) of Example 1

| \((\eta, \xi)\) | \(N = 4\) | \(N = 5\) | \(N = 6\) | \(N = 7\) |
|-----------------|---------|---------|---------|---------|
| (0.0,0.0)       | 9.8020e-16 | 1.0604e-14 | 5.8261e-14 | 4.6324e-13 |
| (0.1,0.1)       | 7.5943e-05 | 2.4407e-04 | 1.0070e-04 | 7.5510e-05 |
| (0.2,0.2)       | 2.4400e-04 | 9.3702e-04 | 3.7520e-04 | 2.8222e-04 |
| (0.3,0.3)       | 3.1769e-04 | 2.0000e-03 | 7.4580e-04 | 5.5994e-04 |
| (0.4,0.4)       | 1.4792e-05 | 3.1000e-03 | 1.1000e-04 | 8.2115e-04 |
| (0.5,0.5)       | 1.0000e-03 | 4.2000e-03 | 1.3000e-03 | 9.7446e-04 |
| (0.6,0.6)       | 3.2000e-03 | 5.0000e-03 | 1.3000e-03 | 9.5378e-04 |
| (0.7,0.7)       | 6.7000e-02 | 5.2000e-03 | 9.6017e-04 | 7.4814e-04 |
| (0.8,0.8)       | 1.2100e-02 | 4.8000e-03 | 3.9005e-04 | 4.2101e-04 |
| (0.9,0.9)       | 1.9200e-02 | 3.8000e-03 | 2.5801e-04 | 9.4096e-05 |
| (1.0,1.0)       | 2.8100e-02 | 2.5000e-03 | 6.6554e-04 | 1.6198e-04 |

### Table 2 Norm error of LWOMM for values of \(N = 4, 5, 6, 7\) of Example 1

| Norm | \(N = 5\) | \(N = 4\) | \(N = 6\) | \(N = 7\) |
|------|---------|---------|---------|---------|
| \(l^2\) | 2.3900e-02 | 1.1300e-02 | 2.6000e-03 | 2.1000e-03 | 1.7174e-04 |
| \(l^\infty\) | 1.9200e-02 | 5.2000e-03 | 1.3000e-03 | 9.7446e-04 | 5.6395e-04 |

### Table 3 Absolute error of LWOMM at \(e_N(\eta, \xi, 1)\) for values of \(N = 4, 5, 6, 7\) of Example 2

| \((\eta, \xi)\) | \(N = 4\) | \(N = 5\) | \(N = 6\) | \(N = 7\) |
|-----------------|---------|---------|---------|---------|
| (0.0,0.0)       | 6.6368e-14 | 1.0240e-15 | 3.9761e-18 | 3.8272e-18 |
| (0.1,0.1)       | 2.7356e-04 | 7.1939e-05 | 2.9464e-04 | 1.2611e-05 |
| (0.2,0.2)       | 1.1000e-03 | 2.8673e-04 | 1.1185e-04 | 3.7931e-05 |
| (0.3,0.3)       | 2.4000e-03 | 6.1143e-04 | 2.3039e-04 | 5.6171e-05 |
| (0.4,0.4)       | 4.1000e-03 | 9.8017e-04 | 3.6060e-04 | 5.3182e-05 |
| (0.5,0.5)       | 6.3000e-03 | 1.3000e-03 | 4.7503e-04 | 2.6537e-05 |
| (0.6,0.6)       | 8.8000e-03 | 1.6000e-03 | 5.5010e-04 | 1.3745e-05 |
| (0.7,0.7)       | 1.1800e-02 | 1.8000e-03 | 5.7469e-04 | 5.1243e-05 |
| (0.8,0.8)       | 1.5300e-02 | 2.0000e-03 | 5.6041e-04 | 7.5205e-05 |
| (0.9,0.9)       | 1.9600e-02 | 2.7000e-03 | 5.5311e-04 | 9.9891e-05 |
| (1.0,1.0)       | 2.5100e-02 | 4.4000e-03 | 6.4481e-04 | 1.3731e-05 |
### Table 4
Norm error of LWOMM for values of $N = 4, 5, 6, 7$ of Example 2

| Norm | $N = 4$ | $N = 5$ | $N = 6$ | $N = 7$ | Mittal and Bhatia [32] |
|------|---------|---------|---------|---------|------------------------|
| $l^2$ | 3.9100e-02 | 6.3000e-03 | 1.4000e-03 | 2.1077e-04 | 9.8870e-05 |
| $l^\infty$ | 2.5100e-02 | 4.4000e-03 | 6.4481e-03 | 1.3731e-04 | 2.4964e-04 |

### Table 5
Absolute error of LWOMM at $e_N(\eta_i, \zeta, 1)$ for values of $N = 4, 5, 6, 7$ of Example 3

| $(\eta, \zeta)$ | $N = 4$ | $N = 5$ | $N = 6$ | $N = 7$ |
|----------------|---------|---------|---------|---------|
| $(0.0,0.0)$ | 6.7126e-14 | 1.0492e-15 | 8.3206e-17 | 1.0593e-18 |
| $(0.1,0.1)$ | 6.3835e-05 | 9.9583e-06 | 3.3025e-04 | 1.0929e-04 |
| $(0.2,0.2)$ | 3.1414e-04 | 3.1003e-05 | 1.2000e-03 | 4.0642e-04 |
| $(0.3,0.3)$ | 9.3733e-04 | 4.3952e-05 | 1.3000e-03 | 8.0493e-04 |
| $(0.4,0.4)$ | 2.2000e-03 | 2.0484e-05 | 3.4000e-03 | 1.2000e-03 |
| $(0.5,0.5)$ | 4.5000e-03 | 7.3449e-05 | 4.1000e-03 | 1.4000e-03 |
| $(0.6,0.6)$ | 8.1000e-03 | 3.0522e-04 | 4.1000e-03 | 1.3000e-03 |
| $(0.7,0.7)$ | 1.3600e-02 | 7.8217e-04 | 3.4000e-03 | 9.6647e-04 |
| $(0.8,0.8)$ | 2.1300e-02 | 1.7000e-03 | 2.1000e-03 | 3.4296e-04 |
| $(0.9,0.9)$ | 3.1600e-02 | 3.4000e-03 | 6.7019e-04 | 3.1452e-04 |
| $(1.0,1.0)$ | 4.4700e-02 | 6.4000e-03 | 5.0796e-04 | 6.0130e-04 |

### Table 6
Norm error of LWOMM for values of $N = 4, 5, 6, 7$ of Example 3

| Norm | $N = 4$ | $N = 5$ | $N = 6$ | $N = 7$ | Mittal and Bhatia [32] |
|------|---------|---------|---------|---------|------------------------|
| $l^2$ | 6.0900e-02 | 7.5000e-03 | 1.0500e-03 | 2.7000e-03 | 1.6144e-03 |
| $l^\infty$ | 4.4700e-02 | 6.4000e-03 | 6.7000e-04 | 1.4000e-04 | 3.8000e-04 |

### Table 7
Absolute error of LWOMM at $e_N(\eta_i, \zeta, 1)$ for values of $N = 4, 5, 6, 7$ of Example 4

| $(\eta, \zeta)$ | $N = 4$ | $N = 5$ | $N = 6$ | $N = 7$ |
|----------------|---------|---------|---------|---------|
| $(0.0,0.0)$ | 4.3600e-17 | 8.1914e-18 | 1.0515e-16 | 2.1374e-19 |
| $(0.1,0.1)$ | 1.6000e-03 | 6.7923e-04 | 2.4650e-04 | 1.7372e-04 |
| $(0.2,0.2)$ | 5.5000e-03 | 2.3000e-03 | 8.3644e-04 | 5.8749e-04 |
| $(0.3,0.3)$ | 9.8000e-03 | 4.2000e-03 | 1.5000e-03 | 1.1000e-03 |
| $(0.4,0.4)$ | 1.2900e-03 | 5.6000e-03 | 2.0000e-03 | 1.4000e-03 |
| $(0.5,0.5)$ | 1.4000e-03 | 6.1000e-03 | 2.2000e-03 | 1.6000e-03 |
| $(0.6,0.6)$ | 1.2900e-03 | 5.6000e-03 | 2.0000e-03 | 1.4000e-03 |
| $(0.7,0.7)$ | 9.8000e-03 | 4.2000e-03 | 1.5000e-03 | 1.1000e-03 |
| $(0.8,0.8)$ | 5.5000e-03 | 2.3000e-03 | 8.3644e-04 | 5.8749e-04 |
| $(0.9,0.9)$ | 1.6000e-03 | 6.7923e-04 | 2.4650e-04 | 1.7372e-04 |
| $(1.0,1.0)$ | 3.8062e-15 | 8.1914e-18 | 7.1569e-16 | 2.1418e-19 |
Table 8 Norm error of LWOMM for values of $N = 4, 5, 6, 7$ of Example 4

| Norm     | $N = 4$        | $N = 5$        | $N = 6$        | $N = 7$        |
|----------|----------------|----------------|----------------|----------------|
| $l^2$    | 1.6200e-02     | 1.2100e-02     | 4.3000e-03     | 3.1000e-03     |
| $l^\infty$ | 9.8000e-03     | 6.1000e-03     | 2.2000e-03     | 1.6000e-03     |

Fig. 1 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 1 at $N = 4$

Fig. 2 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 1 at $N = 5$

Fig. 3 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 1 at $N = 6$

Fig. 4 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 1 at $N = 7$
\[ \begin{align*}
U(0, n; 0) &= 0, \\
U(1, n; 0) &= 0, \\
U(0, 0; t) &= 0, \\
U(1, 0; t) &= 0,
\end{align*} \]

respectively.

The exact solution is given by

\[ U(0; n; t) = \frac{g}{C_0} \frac{g^2}{C_0} t^2 e^{-t}. \]

### 6 Conclusion

The telegraph equations are of special interest as they are used to understand various physical and complex phenomena. Two-dimensional hyperbolic telegraph equations (HTE) are usually difficult to solve analytically. In this paper, Legendre wavelet matrix method (LWOMM) has been developed to solve the two-dimensional HTE with Dirichlet boundary conditions. For this purpose, we
constructed operational matrices based on Legendre wavelet for integration and differentiation for the solution of HTE. After implementing of LWOMM on HTE, HTE converted into algebraic generalized Sylvester equation which is solved by BICGSTAB method. The results of the numerical solution of Eq. (1) for a small number of Legendre wavelet basis functions are illustrated in the form of Tables 1, 2, 3, 4, 5, 6, 7 and 8 and the Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16. Also, we have comprised propose method pointwise errors with the some existing method [32] for different values of \( k_1; k_2; k; M; M' \) and \( M'' \). Numerical results of absolute errors, \( l^2 \) and \( l^\infty \) errors are provided in Tables 1, 3, 5 and 7 and Table 2, 4, 6 and 8, respectively.

Fig. 9 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 3 at \( N = 4 \)

Fig. 10 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 3 at \( N = 5 \)

Fig. 11 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 3 at \( N = 6 \)

Fig. 12 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 3 at \( N = 7 \)

Fig. 13 The spatio-temporal distribution of an error between the exact solution and its numerical solution utilising the LWOMM for Example 4 at \( N = 4 \)
theoretical convergence analysis for the approximate solution is provided. So, by using convergence analysis, absolute error, $l^2$ error, $l^\infty$ error, error tables, and error graphs, we conclude that the proposed method is easy, gives better accuracy, and is efficient at a small number of Legendre wavelet basis.

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