HILBERT FUNCTIONS OF SOCLE IDEALS

HOANG LE TRUONG AND HOANG NGOC YEN

Abstract. In this paper, we explore a relationship between Hilbert functions and the irreducible decompositions of ideals in local rings. Applications are given to characterize the regularity, Gorensteinness, Cohen-Macaulayness and sequentially Cohen-Macaulayness of local rings.

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1. Introduction

Let $I$ be an ideal of a Noetherian local ring $(R, \mathfrak{m})$ such that $R/I$ is Artinian. The socle of $I$ is the ideal which is defined as $I : \mathfrak{m}$. It is the unique largest ideal $J$ of $R$ with $J\mathfrak{m} \subseteq I$. Moreover, it is also the unique largest submodule of module $R/I$ which has the structure of a module over the residue field $k = R/\mathfrak{m}R$. Therefore $\ell_R(I : \mathfrak{m}/I) = \dim_k(I : \mathfrak{m}/I)$ is the minimal number of socle generators of $I$, where $\ell_R(*)$ stands for the length. The minimal number of socle generators of modules are as important as the minimal number of generators of the modules, to which they are (in some sense) dual, however, in general, they are much harder to find. For a deeper discussion of socle ideals we refer the reader to [1], [2], [13], [20].

The minimal number of socle generators of modules is in relative to the irreducible decompositions of modules. Irreducible ideals were already presented in the famous proof of Noether that ideals in Noetherian commutative rings have a primary decomposition. She firstly observed that the Noetherian property implied every ideal $\mathfrak{a}$ of $R$ can be expressed as an irredundant intersection of irreducible ideals of $R$ and the number of irreducible ideals appearing in such an expression depends only on $\mathfrak{a}$ and not on the expression. Let us call the number $\mathcal{N}(\mathfrak{a})$ of irreducible ideals of $\mathfrak{a}$ that appear in an irredundant irreducible decomposition of $\mathfrak{a}$ the index of reducibility of $\mathfrak{a}$. Remember

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that, in the case in which \( a = I \), \( \mathcal{N}(I) = \ell_R([I :_R m]/I) \) and so the index of reducibility of \( I \) is also the minimal number of socle generators of \( I \).

The minimal number of socle generators is closely related to the ideas in the theory of Gorenstein rings. Northcott and Rees [13, Theorem 3] proved that for all parameter ideals \( q \), the minimal number of socle generators of \( q \) is 1 then \( R \) is Gorenstein. In 1957 D. G. Northcott [17, Theorem 3] proved that for parameter ideals \( q \) in a Cohen-Macaulay local ring \( R \), the minimal number of socle generators of \( q \) is constant and independent of the choice of \( q \). However, this property of constant the minimal number of socle generators for parameter ideals does not characterize Cohen-Macaulay rings. The example of a non-Cohen-Macaulay local ring \( R \) with \( \mathcal{N}(q) = 2 \) for every parameter ideal \( q \) was firstly given in 1964 by S. Endo and M. Narita [11]. In 1984 S. Goto and N. Suzuki [14] explored, for a given finitely generated \( R \)-module \( M \), the supremum \( \sup_q \mathcal{N}(q) \), where \( q \) runs through parameter ideals of \( R \) and showed that the supremum is finite, when \( R \) is a generalized Cohen-Macaulay module. Compared with the case of rings and modules with finite local cohomologies, the general case is much more complicated and difficult to treat. No standard induction techniques work. However, from this point of view, a natural question is how to characterize Cohen-Macaulayness of rings in term of the minimal number of socle generators.

On the other hand, a very interesting and important numerical invariant of a graded finitely generated \( S \)-module \( M \) is its Hilbert function. Suppose that \( S = \bigoplus_{n \geq 0} S_n \) is positively graded and \( S_0 \) is an Artinian local ring. The Hilbert function of \( S \) is \( H_S(n) = \ell_{S_0}(S_i) \) for \( i \in \mathbb{N} \). Hilbert's insight was that \( H_S \) is determined by finitely many of its values. He proved that there exists a polynomial (called the Hilbert polynomial) \( h_S(t) \in \mathbb{Q}[t] \) such that \( H_S(t) = h_S(t) \) for \( t \gg 0 \).

Now let \( \text{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1} \) and call it the associated graded ring of \( I \). Put \( S = (0) :_{\text{gr}_I(R)} \mathfrak{m} \), where \( \mathfrak{m} = m/I \oplus \bigoplus_{n \geq 1} I^n/I^{n+1} \). Then in special cases \( (n_0(m) > 1) \), we see that \( \ell_R(S_n) = \ell([I^{n+1} : m]/I^{n+1}) \) for all large enough \( n \). Therefore the function of the minimal number of socle generators of \( I^n \) on \( n \) become polynomial when large enough \( n \). In general there exists a polynomial \( p_I(n) \) of degree \( d - 1 \) with rational coefficients such that

\[
\mathcal{N}(I^{n+1}; R) = \ell_R([I^{n+1} : R m]/I^{n+1}) = p_I(n)
\]

for all large enough \( n \). Then, there are integers \( f_i(I) \) such that

\[
p_I(n) = \sum_{i=0}^{d-1} (-1)^i f_i(I) \binom{n + d - 1 - i}{d - 1 - i}.
\]

These integers \( f_i(I) \) are called the Noetherian coefficients of \( I \). In particular, the leading coefficient \( f_0(I) \) is called the irreducible multiplicity of \( I \), by first author. It was shown that the index of reducibility of parameter ideals can be used to characterize the Cohen-Macaulayness of local rings. From this point of view, we explore this notions in this paper, where we apply it to characterize the regularity, Gorensteinness, Cohen-Macaulayness and sequentially Cohen-Macaulayness of local rings.

Now let us recall the definition of the Hilbert-Samuel polynomial of \( I \). It is well known that the Hilbert-Samuel function \( \ell_R(R/I^{n+1}) \) become the polynomial which is
called Hilbert-Samuel polynomial
\[ \ell_R(R/I^{n+1}) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n+d-i}{d-i}. \]
for all large enough \( n \). These integers \( e_i(I) \) are called the Hilbert coefficients of \( I \). In the particular case, the leading coefficient \( e_0(I) \) is said to be the multiplicity of \( I \). In [16], Nagata gave a characterization of the regularity of local rings in term of the multiplicity of the maximal ideal. The multiplicity of the maximal ideal is 1 if and only if \( R \) is regular, provided that \( R \) is unmixed, that is \( \dim R/p = d \) for all \( p \in \text{Ass}(\hat{R}) \). Therefore it is a natural question to ask whether one may establish a similar correspondence between the regularity of local rings and the irreducible multiplicity of the maximal ideal. The following result is given an answer of this question.

**Theorem 1.1.** Assume that \( R \) is unmixed. Then \( R \) is regular if and only if \( f_0(m) = 1 \).

Recall \( e_1(I) \) is called by Vasconselos (27) the Chern coefficient of \( I \). Then the following result give a relationship between the Chern coefficient and the irreducible multiplicity.

**Proposition 1.2.** Assume that \( R \) is unmixed. Then for all parameter ideals \( q \subseteq m^2 \), we have
\[ e_1(q : m) - e_1(q) \leq f_0(q). \]

In [26] the first author showed to characterize the Gorensteinness of rings in term of the Chern coefficient of socle parameter ideals. A local ring \( R \) is Gorenstein iff \( e_1(q : m) - e_1(q) \leq 1 \) for all parameter ideals \( q \subseteq m^2 \), provided \( R \) is unmixed. From this point of view, a natural question is how to characterize Gorensteinness of rings in term of the irreducible multiplicity of parameter ideals. The following result is given an answer of this question.

**Theorem 1.3.** Assume that \( R \) is unmixed. Then \( R \) is Gorenstein if and only if \( f_0(q) = 1 \), for all parameter ideals \( q \subseteq m^2 \).

We denote by \( r(R) = \ell_R(\text{Ext}^d_R(R/m, R)) \) the Cohen-Macaulay type. The first author in [26] proved that if \( R \) is unmixed then \( R \) is Cohen-Macaulay iff \( e_1(q : m) - e_1(q) \leq r(R) \) for all parameter ideals \( q \subseteq m^2 \). From this point of view, as in Theorem 1.3 we shall show the following result which is an answer of above question.

**Theorem 1.4.** Assume that \( R \) is unmixed. Then \( R \) is Cohen-Macaulay if and only if \( f_0(q) = r(R) \), for all parameter ideals \( q \subseteq m^2 \).

A natural question from Theorem 1.3 and 1.4 is what happen if \( R \) is not unmixed. To sate the answer of this question, let us fix our notation and terminology. Let \( M \) be a finitely generated \( R \)-module with finite Krull dimension, say \( d = \dim_R M \). A filtration
\[ \mathcal{D} : D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = M \]
of \( R \)-submodules of \( M \) is called the dimension filtration of \( M \), if for all \( 1 \leq i \leq \ell \), \( D_{i-1} \) is the largest \( R \)-submodule of \( D_i \) with \( \dim_R D_{i-1} < \dim_R D_i \), where \( \dim_R (0) = -\infty \) for convention. We say that \( M \) is a sequentially Cohen-Macaulay \( R \)-module, if \( C_i = D_i/D_{i-1} \) is a Cohen-Macaulay \( R \)-module (necessarily with \( \dim_R C_i = \dim_R D_i \)) for all
1 \leq i \leq \ell$ (21 [22]). Hence $M$ is a sequentially Cohen–Macaulay $R$–module with $\ell = 1$ if and only if $M$ is a Cohen–Macaulay $R$–module with $\dim R/p = \dim p M$ for every $p \in \Ass_R M$. We say that $R$ is a sequentially Cohen–Macaulay ring, if $\dim R < \infty$ and $R$ is a sequentially Cohen–Macaulay module over itself.

Let $x = x_1, x_2, \ldots, x_d$ be a system of parameters of $M$. Then $x$ is said to be distinguished, if $(x_j \mid d < j \leq d)D_i = (0)$ for all $1 \leq i \leq \ell$, where $d_i = \dim_R D_i$. A parameter ideal $q$ of $M$ is called distinguished, if there exists a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of $M$ such that $q = (x_1, x_2, \ldots, x_d)$. Therefore, if $M$ is a Cohen–Macaulay $R$–module, every parameter ideal of $M$ is distinguished. Let $\Lambda(M) = \{\dim R L \mid L$ is an $R$–submodule of $M, L \neq (0)\}$.

With this notation the main results of this paper are summarized into the following, which gives a complete generalization of the results in the Cohen–Macaulay case to those of sequentially Cohen–Macaulay rings.

**Theorem 1.5.** Assume that $R$ is a homomorphic image of a Cohen–Macaulay local ring. Then the following statements are equivalent.

(i) $R$ is sequentially Cohen–Macaulay.

(ii) There exists an integer $n$ such that for all good parameter ideals $q \subseteq m^n$ and $2 \leq j \in \Lambda(R)$, we have

$$r_j(R) \geq (-1)^{d-j}(e_{d-j+1}(q : m) - e_{d-j+1}(q)).$$

(iii) There exists an integer $n$ such that for all distinguished parameter ideals $q \subseteq m^n$ and $2 \leq j \in \Lambda(R)$, we have

$$r_j(R) \geq (-1)^{d-j}f_{d-j}(q; R).$$

Later we will give some applications of these results. First, as an immediate consequence of our main result, the assumption of Theorem 1.1, 1.3 and 1.4 are essential. The necessary condition of the following result was proved by N. T. Cuong, P. H. Quy and first author ([9, Corollary 5.3]).

**Theorem 1.6.** $R$ is Gorenstein if and only if for all parameter ideals $q \subseteq m^2$ and $n \gg 0$, we have

$$N(q^{n+1}; R) = \binom{n + d - 1}{d - 1}.$$ 

In [9] Theorem 5.2, N. T. Cuong, P. H. Quy and first author showed that $R$ is Cohen–Macaulay if and only if for all parameter ideals $q \subseteq m^2$ and $n \geq 0$, we have $N(q^{n+1}; R) = r_d(R)\left(\binom{n + d - 1}{d - 1}\right)$. Note that the condition of Hilbert function $N(q^{n+1}; R)$, holding true for all $n \geq 0$, is necessary to their proof. The result of Theorem 5.2 in [9] was actually covered in the following result, but in view of the importance of the following result we changed the condition from Hilbert function to Hilbert polynomial.

**Theorem 1.7.** $R$ is Cohen–Macaulay if and only if for all parameter ideals $q \subseteq m^2$ and $n \gg 0$, we have

$$N(q^{n+1}; R) = r_d(R)\binom{n + d - 1}{d - 1}.$$
Let us explain how this paper is organized. Section 2 is devoted to a brief survey on dimension filtrations, the notion of Goto sequences and the existence of Goto sequences. The computation of the minimal number of socle generators of a special parameter ideal of sequentially Cohen-Macaulay ring is well understood in section 3. In section 4, our aim now is to establish a characterization of sequentially Cohen-Macaulay rings in term of the Hilbert coefficients of the socle of distinguished parameter ideals, which is a part of Theorem 1.5. Continuing our discussion in section 4, the section 5 will give the complete proof of Theorem 1.5. In last section, we are going to discuss the characterizations of the regularity, Gorensteinness and Cohen-Macaulayness of local rings.

2. Goto sequences

Let $R$ be a commutative Noetherian ring, which is not assumed to be a local ring. Let $M$ be a finitely generated $R$-module with finite Krull dimension, say $d = \dim_R M$. We put

$$\text{Assh}_R M = \{ p \in \text{Supp}_R M \mid \dim R/p = d \}.$$ 

Then

$$\text{Assh}_R M \subseteq \text{Min}_R M \subseteq \text{Ass}_R M.$$ 

Let $\Lambda(M) = \{ \dim_R L \mid L \text{ is an } R\text{-submodule of } M, L \neq (0) \}$. We then have

$$\Lambda(M) = \{ \dim R/p \mid p \in \text{Ass}_R M \}.$$ 

We put $\ell = \sharp \Lambda(M)$ and number the elements $\{ d_i \}_{1 \leq i \leq \ell}$ of $\Lambda(M)$ so that

$$0 \leq d_1 < d_2 < \cdots < d_\ell = d.$$ 

Then because the base ring $R$ is Noetherian, for each $1 \leq i \leq \ell$ the $R$-module $M$ contains the largest $R$-submodule $D_i$ with $\dim_R D_i = d_i$. Therefore, letting $D_0 = (0)$, we have the filtration

$$\mathcal{D} : D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = M$$

of $R$-submodules of $M$, which we call the dimension filtration of $M$. The notion of dimension filtration was firstly given by P. Schenzel [21]. Our notion of dimension filtration is a little different from that of [6, 21], but throughout this paper let us utilize the above definition. It is standard to check that $\{ D_j \}_{0 \leq j \leq i}$ (resp. $\{ D_j/D_i \}_{i \leq j \leq \ell}$) is the dimension filtration of $D_i$ (resp. $M/D_i$) for every $1 \leq i \leq \ell$. We put $C_i = D_i/D_{i-1}$ for $1 \leq i \leq \ell$.

We note two characterizations of the dimension filtration. Let

$$0 = \bigcap_{p \in \text{Ass}_R M} M(p)$$

be a primary decomposition of $(0)$ in $M$, where $M(p)$ is an $R$-submodule of $M$ with $\text{Ass}_R M/M(p) = \{ p \}$ for each $p \in \text{Ass}_R M$. We then have the following.

**Proposition 2.1** ([21 Proposition 2.2, Corollary 2.3]). The following assertions hold true.

1. $D_i = \bigcap_{p \in \text{Ass}_R M, \ dim R/p \geq d_{i+1}} M(p)$ for all $0 \leq i < \ell$.
2. Let $1 \leq i \leq \ell$. Then $\text{Ass}_R C_i = \{ p \in \text{Ass}_R M \mid \dim R/p = d_i \}$ and $\text{Ass}_R D_i = \{ p \in \text{Ass}_R M \mid \dim R/p \leq d_i \}$. 

(3) Ass$_R M/D_i = \{p \in \text{Ass}_R M \mid \dim R/p \geq d_{i+1}\}$ for all $1 \leq i < \ell$.

We now assume that $R$ is a local ring with maximal ideal $m$ and let $M$ be a finitely generated $R$-module with $d = \dim_R R \geq 1$ and $D = \{D_i\}_{0 \leq i \leq \ell}$ the dimension filtration. Let $\underline{x} = x_1, x_2, \ldots, x_d$ be a system of parameters of $M$. Then $\underline{x}$ is said to be distinguished, if

$$(x_j \mid d_i < j \leq d)D_i = (0)$$

for all $1 \leq i \leq \ell$, where $d_i = \dim_R D_i$. A parameter ideal $q$ of $M$ is called distinguished, if there exists a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of $M$ such that $q = (x_1, x_2, \ldots, x_d)$. Therefore, if $M$ is a Cohen-Macaulay $R$-module, every parameter ideal of $M$ is distinguished. Distinguished system of parameters exist and if $x_1, x_2, \ldots, x_d$ is a distinguished system of parameters of $M$, then $x_1^{n_1}, x_2^{n_2}, \ldots, x_d^{n_d}$ is also a distinguished system of parameters of $M$ for all integers $n_j \geq 1$.

**Settings 2.2.** Let $\underline{x} = x_1, x_2, \ldots, x_s$ be a system of elements of $R$ and $q_j$ denote the ideal generated by $x_1, \ldots, x_j$ for all $j = 1, \ldots, s$.

**Definition 2.3.** A system $\underline{x}$ of elements of $R$ is called Goto sequence on $M$, if for all $0 \leq j \leq s - 1$ and $0 \leq i \leq \ell$, we have the following

1. Ass($C_i/q_j C_i$) $\subseteq$ Assh($C_i/q_j C_i$) $\cup \{m\}$,
2. $x_j D_i = 0$ if $d_i < j \leq d_{i+1}$,
3. $q_{j-1} : x_j = H^n_\text{m}(M/q_{j-1} M)$ and $x_j \notin q$ for all $p \in \text{Ass}(M/q_{j-1} M) - \{m\}$.

At first glance, the definition of normal does not seem very intuitive. Once we enter the world of sequences, however, we will see that Goto sequence has a very nice interpretation and properties. We will also see that Goto sequence is useful for many inductive proofs in the next sections. Before we can give some properties of this sequence, we first need to reformulate the notion of $d$-sequences. The sequence $x_1, x_2, \ldots, x_s$ of elements of $R$ is called a $d$-sequence on $M$ if

$$q_j M : x_{i+1} x_j = q_i M : x_j$$

for all $0 \leq i < j \leq s$. The concept of a $d$-sequence is given by Huneke \[15\] and it plays an important role in the theory of Blow up algebra, e.g. Ress algebra. In the following lemma, we will give some properties of Goto sequences that will be used in the next sections when we study the Hilbert coefficients and Noetherian coefficients.

**Lemma 2.4.** Let $\underline{x} = x_1, x_2, \ldots, x_s$ form a Goto sequence on $M$. Then we have

1. $\underline{x}$ is part of a system of parameters of $M$.
2. $\underline{x}$ is a $d$-sequence.
3. If $d = s$ then $\underline{x}$ is a distinguished system of parameters of $M$.
4. $x_{i+1}, \ldots, x_s$ is also a Goto sequence on $M/q_i M$.

**Proof.** As an immediate consequence of the definitions we have the first assertion and the third assertion. The second assertion is followed from (vii) of \[23\, Theorem 1.1].

**Lemma 2.5.** Let $R$ be a homomorphic image of a Cohen-Macaulay local ring. Assume that system $\underline{x} = x_1, x_2, \ldots, x_d$ of parameters form a Goto sequence on $M$. Let $N$ denote the unmixed component of $M/q_{d-2} M$ and $d \geq 2$. If $M/N$ is Cohen-Macaulay, so is also $M/D_{\ell-1}$.
Moreover, since \( \text{Ass}(\text{M}/xM) \) is a finite set. Assume that \( x \in R \) is a Goto sequence of length one on \( M \). Let \( N \) denote the unmixed component of \( M/xM \) and \( d \geq 3 \). If \( \text{H}_m^i(M/N) = 0 \) for \( i \leq d - 2 \) then \( \text{H}_m^i(C_\ell) = 0 \) for \( i \leq d - 1 \).

For a submodule \( N \) of \( M \), we denote \( \overline{N} = (N + xM)/xM \) the submodule of \( M/xM \). Since \( x \) is a Goto sequence of length one on \( M \), \( \text{Ass}(C_\ell/xC_\ell) \subseteq \text{Assh}(C_\ell/xC_\ell) \cup \{m\} \).

Thus \( H_1^{M/xM} = \text{Ann}(\mathfrak{m}) \). For a submodule \( I \) of \( M \), we denote \( \overline{I} = \text{I}/\mathfrak{m} \) a Cohen-Macaulay module, \( H_m^i(M/\overline{I}) = 0 \) for all \( 0 < i < d - 1 \). Therefore, we derive from the exact sequence

\[
0 \to M/\overline{D}_{\ell-1} \xrightarrow{\delta} M/\overline{D}_{\ell-1} \to M/\overline{D}_{\ell-1} + xM \to 0
\]

the following exact sequence:

\[
0 \to H_m^0(M/\overline{D}_{\ell-1} + xM) \to H_m^1(M/\overline{D}_{\ell-1}) \xrightarrow{\delta} H_m^1(M/\overline{D}_{\ell-1}) \to 0.
\]

Thus \( H_m^1(M/\overline{D}_{\ell-1}) = 0 \), and so \( N/\overline{D}_{\ell-1} = H_m^0(M/\overline{D}_{\ell-1} + xM) = 0 \). Hence \( N = \overline{D}_{\ell-1} \).

Moreover, since \( x \) is \( C_\ell = M/\overline{D}_{\ell-1}\)-regular and \( C_\ell /xC_\ell \cong \overline{M}/\overline{D}_{\ell-1} = \overline{M}/N \) a Cohen-Macaulay module, \( C_\ell \) is a Cohen-Macaulay module.

We now denote \( r_j(M) = \ell_R((0):H_m^j(M) \mathfrak{m}) \) for all \( j \in \mathbb{Z} \).

**Definition 2.6.** A system \( \underline{r} \) of elements of \( R \) is called **Goto sequence of type I** on \( M \), if we have

\[
r_{d-j}(M/q_jM) \leq r_{d-j-1}(M/q_{j+1}M),
\]

for all \( 0 \leq j \leq s - 1 \).

Now, we explore the existence of Goto sequence of type I. We have divided the proof of the existence of Goto sequence into sequence of lemmas. First, we begin with the following result of S. Goto and Y. Nakamura [12].

**Lemma 2.7.** Let \( R \) be a homomorphic image of a Cohen-Macaulay local ring and assume that \( \text{Ass}(R) \subseteq \text{Assh}(R) \cup \{\mathfrak{m}\} \). Then

\[
\mathcal{F} = \{\mathfrak{p} \in \text{Spec}(R) \mid \text{ht}_R(\mathfrak{p}) > 1 = \text{depth}(R_p)\}
\]

is a finite set.

The next proposition shows the existence of a special element which is useful for the existence of Goto sequence.

**Proposition 2.8.** Let \( R \) be a homomorphic image of a Cohen-Macaulay local ring and \( I \) an \( \mathfrak{m}\)-primary ideal of \( R \). Assume that \( \mathcal{F} = \{M_i\}_{i=0} \) is a finite filtration of submodules of \( M \) such that \( \text{Ass} L_i \subseteq \text{Assh} L_i \cup \{\mathfrak{m}\} \), where \( L_i = M_i/M_{i-1} \). Then there exists an element \( x \in I \) satisfies the following conditions

1. \( \text{Ass}(L_i/x^n L_i) \subseteq \text{Assh}(L_i/x^n L_i) \cup \{\mathfrak{m}\}, \) for all \( i = 0, \ldots, \ell - 1 \).
2. \( x \notin \mathfrak{p}, \) for all \( \mathfrak{p} \in \text{Ass}(M) - \{\mathfrak{m}\}. \)
3. \( (0):_{L_i} x \in H_m^0(L_i) \) and \( (0):_{M} x \in H_m^0(M), \) for all \( i = 0, \ldots, \ell - 1 \).

**Proof.** Set \( I_i = \text{Ann}(L_i) \), and \( R_i = R/I_i \), then \( \text{Ass}(R_i) \subseteq \text{Assh}(R_i) \cup \{\mathfrak{m}\} \) and \( \text{dim} R/I_i < \text{dim} R/I_{i+1} \) for all \( i = 0, \ldots, s - 1 \). Moreover, we have

\[
\text{Ass}(R_i) = \text{Ass}(L_i) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \in \text{Ass}(M) \} \text{ and } \text{dim} R/\mathfrak{p} = \text{dim} R/I_i = d_i \cup \{\mathfrak{m}\}.
\]
Set
\[ F_i = \{ p \in \text{Spec}(R) \mid I_i \subseteq p \text{ and } \text{ht}_{R_i}(p/I_i) > 1 = \text{depth}((L_i)_p) \}. \]

By Lemma 2.7 and the fact \( \text{Ass}(L_i) \subseteq \text{Assh}(L_i) \cup \{ m \} \), we see that the set
\[ \{ p \in \text{Spec}(R_i) \mid \text{ht}_{R_i}(p) > 1 = \text{depth}((L_i)_p) \} \]
is finite, and so that \( F_i \) are a finite set for all \( i = 1, \ldots, \ell \). Put \( F = \text{Ass}(M) \cup \bigcup_{i=1}^{\ell} F_i \backslash \{ m \} \).

By the Prime Avoidance Theorem, we can choose \( y \in I \) such that \( y \not\in \bigcup_{p \in F} p \) and \( \dim M_i/yM_i = \dim M_i - 1 \) for all \( i = 1, \ldots, \ell \). On the other hand, we can choose an integer \( n_0 \) such that \((0) :_M y^n = (0) :_M y^{n_0} \) and \((0) :_{L_i} y^n = (0) :_{L_i} y^{n_0} \), for all \( n \geq n_0 \) and \( i = 1, \ldots, \ell \). Put \( x = y^{n_0+1} \). Then we have \( x \not\in \bigcup_{p \in F} p \) and \((0) :_{L_i} x^2 = (0) :_{L_i} x \) for all \( i = 1, \ldots, \ell \). Now we show that \( x \) have the conditions as required.

First let us prove the condition (1). To this end, consider \( p \in \text{Ass}(N_i/xN_i) \) with \( p \not\neq m \). Then we have \( \text{depth}(L_i/xL_i)_p = 0 \). Hence \( \text{depth}(L_i)_p = 1 \). It implies that \( \text{ht}_{R_i}(p) = 1 \), since \( p \not\in F_i \). By the assumption \( R_i \) is a catenary ring, therefore
\[ \dim R/p = \dim R - \text{ht}_{R_i}(p) = \dim R_i/xR_i = \dim L_i/xL_i. \]

Hence \( p \in \text{Ass}(L_i/xL_i) \).

Since the condition (2) is trivial, it remains to prove the condition (3). Take \( p \in \text{Ass}(R)(0) :_{L_i} x \) with \( p \neq m \). Hence \((0) :_{L_i} x)_p = (0) \) and this is a contradiction. It implies that \( (0) :_{N_i} x \) is finite length. Since \( (0) :_{L_i} x^2 = (0) :_{L_i} x \), we have \( (0) :_{L_i} x = H^0_m(L_i) \).

It follows from the following exact sequence
\[ 0 \to M_{i-1} \to M_i \to L_i \to 0 \]
and \( x H^0_m(L_i) = 0 \) for all \( i = 1, \ldots, \ell \) that the following sequence
\[ 0 \to H^0_m(M_{i-1}) \to H^0_m(M_i) \to H^0_m(L_i) \]
and \( 0 \to (0) :_{M_{i-1}} x \to (0) :_{M_i} x \to (0) :_{L_i} x \)
are exact. By induction and \( (0) :_M x = (0) :_M x^2 \), we have \( (0) :_M x = H^0_m(M) \) and this completes the proof. \( \square \)

The existence of Goto sequence is establishe by our next Corollary.

**Corollary 2.9.** Assume that \( R \) is a homomorphic image of a Cohen-Macaulay local ring and \( I \) an \( m \)-primary ideal of \( R \). Then there exists a system \( x = x_1, x_2, \ldots, x_s \) of elements of \( I \) such that \( x \) is a Goto sequence on \( M \).

**Proof.** We prove this by induction on \( s \), the case in which \( s = 1 \) having been dealt with in Lemma 2.8. So we suppose that \( s = j \geq 2 \) and that the result has been proved for smaller values of \( s \). Suppose that \( d_i < j \leq d_i+1 \) for some \( i \). We see immediately from this induction hypothesis that
\[ \text{Ass}(N_i/q_{j-1}N_i) \subseteq \text{Assh}(N_{j-1}/q_{j-1}N_i) \cup \{ m \}, \]
where \( q_{j-1} = (x_1, \ldots, x_{j-1}) \), for all \( i = 0, \ldots, \ell-1 \). Moreover the sequence \( x_1, x_2, \ldots, x_d \) is a system of parameters of \( D_i \). Therefore \( \text{Ann}(D_i) + q_{j-1} \) is \( m \)-primary ideals. So that, by Lemma 2.8 there exists an element \( x_j \in I \cap \text{Ann}(D_i) \), as required. This completes the inductive step, and the proof. \( \square \)
Let $q = (x_1, x_2, \ldots, x_d)$ be a parameter ideal in $R$ and let $M$ be an $R$-module. For each integer $n \geq 1$ we denote by $x^n$ the sequence $x_1^n, x_2^n, \ldots, x_d^n$. Let $K^\bullet(x^n)$ be the Koszul complex of $R$ generated by the sequence $x^n$ and let $H^\bullet(x^n; M) = H^\bullet(\text{Hom}_R(K^\bullet(x^n), M))$ be the Koszul cohomology module of $M$. Then for every $p \in \mathbb{Z}$ the family $\{H^p(x^n; M)\}_{n \geq 1}$ naturally forms an inductive system of $R$-modules, whose limit

$$H^p_q = \lim_{n \to \infty} H^p(x^n; M)$$

is isomorphic to the local cohomology module

$$H^p_m(M) = \lim_{n \to \infty} \text{Ext}^p_R(R/m^n, M)$$

For each $n \geq 1$ and $p \in \mathbb{Z}$ let $\phi^{p,n}_{x,M} : H^p(x^n; M) \to H^p_m(M)$ denote the canonical homomorphism into the inductive limit.

**Definition 2.10** ([13] Lemma 3.12). Let $R$ be a Noetherian local ring with the maximal ideal $m$ and $\dim R = d \geq 1$. Let $M$ be a finitely generated $R$-module. Then there exists an integer $n_0$ such that for all systems of parameters $\underline{x} = x_1, \ldots, x_d$ for $R$ contained in $m^{n_0}$ and for all $p \in \mathbb{Z}$, the canonical homomorphisms

$$\phi^{p,1}_{\underline{x}, M} : H^p(\underline{x}, M) \to H^p_m(M)$$

into the inductive limit are surjective on the socles. The least integer $n_0$ with this property is called a Goto number of $R$-module $M$ and denote by $g(M)$.

With this notation we have the following result.

**Lemma 2.11** ([14], Lemma 1.7). Let $M$ be a finitely generated $R$-module and $x$ an $M$-regular element and $\underline{x} = x_1, \ldots, x_r$ be a system of elements in $R$ with $x_1 = x$. Then there exists a splitting exact sequence for each $p \in \mathbb{Z}$,

$$0 \to H^p(\underline{x}, M) \to H^p(\underline{x}, M/xM) \to H^{p+1}(\underline{x}, M) \to 0.$$  

**Lemma 2.12.** Let $M$ be a finitely generated $R$-module. Assume that $x$ is an $M$-regular element of $M$ such that $x \in m^{g(M)}$. Then we have

$$g(M/xM) \leq g(M),$$

and

$$r_i(M) \leq r_{i-1}(M/xM)$$

for all $i \in \mathbb{Z}$.

**Proof.** Let $x_2, \ldots, x_d$ be a system of parameters of module $M/xM$ such that $x_i \in m^{g(M)}$. Put $\underline{x} = x_1, x_2, \ldots, x_d$ and $q = (\underline{x})$, where $x_1 = x$. Since $x \in m^{g(M)}$, we have $q \subseteq m^{g(M)}$. By the definition of Goto number, we have the canonical homomorphism

$$H^i(\underline{x}, M) \to H^i_m(M)$$

into the inductive limit are surjective on the socles, for each $i \in \mathbb{Z}$. By the regularity of $x = x_1$ on $M$, it follows from the following sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$
that there are induced the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^i(x; M) & \rightarrow & H^i(x; M/xM) & \rightarrow & H^{i+1}(x; M) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow & \\
& & H^i_m(M) & \rightarrow & H^i_m(M/xM) & \rightarrow & H^{i+1}_m(M) & & \\
\end{array}
\]

commutes, for all \( i \in \mathbb{Z} \). It follows from the above commutative diagrams and Lemma 2.11 that after applying the functor \( \text{Hom}(k, \ast) \), we obtain the commutative diagram

\[
\begin{array}{cccccc}
\text{Hom}(k, H^i(x; M/xM)) & \rightarrow & \text{Hom}(k, H^{i+1}(x; M)) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \\
\text{Hom}(k, H^i_m(M/xM)) & \rightarrow & \text{Hom}(k, H^{i+1}_m(M)) & & \\
\end{array}
\]

for all \( i \in \mathbb{Z} \). Since the map \( \text{Hom}(k, H^{i+1}(x; M)) \rightarrow \text{Hom}(k, H^{i+1}_m(M)) \) is surjective, so is the map \( \text{Hom}(k, H^i_m(M/xM)) \rightarrow \text{Hom}(k, H^{i+1}_m(M)) \). Therefore the map \( \text{Hom}(k, H^i(x; M/xM)) \rightarrow \text{Hom}(k, H^i_m(M/xM)) \) is surjective and \( r_i(M) \leq r_{i-1}(M/xM) \) for all \( i \in \mathbb{Z} \). Thus for all systems \( x \) of parameters of module \( M/xM \) such that \( x_i \in \mathfrak{m}^g(M) \), we have the map \( \text{Hom}(k, H^i(x; M/xM)) \rightarrow \text{Hom}(k, H^i_m(M/xM)) \) is surjective for all \( i \in \mathbb{Z} \). Hence we have we have

\[
g(M/xM) \leq g(M),
\]

as required.

**Corollary 2.13.** Let \( M \) be a finitely generated \( R \)-module with \( \dim M \geq 2 \). Then there exists an integer \( n \) such that for all parameter elements \( x \in \mathfrak{m}^n \), we have

\[
r_d(M) \leq r_{d-1}(M/xM).
\]

**Proof.** Since \( \dim M \geq 2 \) and \( x \) is a parameter element of \( M \), we have \( H^d_m(M) = H^d_m(M/H^0_m(M)) \) and \( H^{d-1}_m(M/xM) = H^{d-1}_m(M/xM + H^0_m(M)) \). Therefore we have been working under the assumption that depth \( M \geq 0 \). Then by Lemma 2.12 we have

\[
r_d(M) \leq r_{d-1}(M/xM),
\]

and the proof is complete.

Addition, the existence of Goto sequence of type I is established by our next Proposition.

**Proposition 2.14.** Assume that \( R \) is a homomorphic image of a Cohen-Macaulay local ring and \( I \) an \( \mathfrak{m} \)-primary ideal of \( R \). Then there exists a system \( x = x_1, x_2, \ldots, x_s \) of elements of \( I \) such that \( x \) is a Goto sequence of type I on \( M \).

**Proof.** We shall now show the our result by induction on \( s \). In the case in which \( s = 1 \) there is nothing to prove, because of the Lemma 2.8. So we suppose, inductively, that \( s = j > 1 \) and the results have both been proved for smaller values of \( s \). Suppose that \( d_i < j \leq d_{i+1} \) for some \( i \). By induction we have system \( x_1, \ldots, x_{j-1} \) of \( R \) such that satisfies the following conditions
(1) \( \text{Ass}(N_i/q_{j-1}N_i) \subseteq \text{Assh}(N_j/q_{j-1}N_i) \cup \{m\} \), where \( q_{j-1} = (x_1, \ldots, x_{j-1}) \), for all \( i = 0, \ldots, \ell - 1 \).

(2) The sequence \( x_1, x_2, \ldots, x_d \) is a system of parameters of \( D_i \).

Let \( \overline{R} = R/q_{j-1} \overline{M} = M/q_{j-1}M \) \( n = m/q_{j-1} \). It follows from Corollary 2.13 that there exists an integer \( n \) such that for all \( x \in m^n \) we have \( r_{d-j+1}(\overline{M}) \leq r_{d-j}(M/xM) \). Put \( J = (\text{Ann}(D_i) + q_{j-1}) \cap I \cap m^n \). Then \( \overline{fR} \) is an \( n \)-primary ideal of \( \overline{R} \). By Lemma 2.8 we can choose \( x_{j+1} \in \text{Ann}(D_i) \cap I \cap m^n \), as required. With this observation, we can complete the inductive step and the proof. \( \square \)

3. Socle polynomial

In this section, we introduce the notion of Noetherian coefficients and the its computation in the sequentially Cohen-Macaulay cases. Recall, we say that an \( R \)-submodule \( N \) of \( M \) is irreducible if \( N \) is not written as the intersection of two larger \( R \)-submodules of \( M \). Every \( R \)-submodule \( N \) of \( M \) can be expressed as an irredundant intersection of irreducible \( R \)-submodules of \( M \) and the number of irreducible \( R \)-submodules appearing in such an expression depends only on \( N \) and not on the expression. Let us call, for each \( m \)-primary ideal \( I \) of \( M \), the number \( N(I; M) \) of irreducible \( R \)-submodules of \( M \) that appearing in an irredundant irreducible decomposition of \( IM \) the index of reducibility of \( M \) with respect to \( I \). Remember that

\[
N(I; M) = \ell_R([IM :_M m]/IM).
\]

Moreover, by Proposition 2.1 [9], it is well known that there exists a polynomial \( p_{I,M}(n) \) of degree \( d - 1 \) with rational coefficients such that

\[
N(I^{n+1}; M) = \ell_R([I^{n+1}M :_M m]/I^{n+1}M) = p_{I,M}(n)
\]

for all large enough \( n \). Then, there are integers \( f_i(I; M) \) such that

\[
p_{I,M}(n) = \sum_{i=0}^{d-1} (-1)^i f_i(I; M) \binom{n + d - 1 - i}{d - 1 - i}.
\]

These integers \( f_i(I; M) \) are called the Noetherian coefficients of \( M \) with respect to \( I \). In particular, the leading coefficient \( f_0(I; M) \) is called the irreducible multiplicity of \( M \) with respect to \( I \). When \( M = R \), we abbreviate \( f_0(I; M) \) to \( f_0(I) \). The following result will be necessary in the computation of the Noetherian coefficients of distinguished parameter ideals.

**Lemma 3.1.** Let \( N \) be a submodule of \( M \) such that \( M/N \) is Cohen-Macaulay and \( \dim N < \dim M \). Assume that \( q \) is a parameter ideal generated by \( x_1, \ldots, x_d \) such that

\[
[N + qM] :_M m = N + [qM :_M m],
\]

Let \( 0 \leq s \leq d \) and \( b = (x_1, \ldots, x_s) \). Then we have

\[
[q^nM + N + b] :_M m = [q^nM :_M m] + N + b,
\]

for all \( n \geq 0 \).
Proof. We put $\mathcal{M} = M/bM + N$ and we denote $\text{gr}_q(\mathcal{M}) = \bigoplus_{n \geq 0} q^n \mathcal{M}/q^{n+1} \mathcal{M}$. Since $\mathcal{M}$ is a Cohen-Macaulay $R$-module and $q$ is a parameter ideal of $R$-module $\mathcal{M}$, sequence $x_{s+1}, \ldots, x_d$ is an $\mathcal{M}$-regular. Since $\mathcal{M}$ is Cohen-Macaulay, we have a natural isomorphism of graded modules

$$\text{gr}_q(\mathcal{M}) \cong (\mathcal{M}/q\mathcal{M}[T_{s+1}, \ldots, T_d])_{n+1},$$

where $T_{s+1}, \ldots, T_d$ are indeterminates. This deduces $R$-isomorphisms on graded parts

$$q^n \mathcal{M}/q^{n+1} \mathcal{M} \to (\mathcal{M}/q\mathcal{M}[T_{s+1}, \ldots, T_d])_{n+1} \cong \mathcal{M}/q\mathcal{M}(n+1)$$

for all $n \geq 0$. On the other hand, since $q$ is a parameter ideal of a Cohen-Macaulay modules $\mathcal{M}$, $q^{n+1} \mathcal{M} : m \subseteq q^{n+1} \mathcal{M} : q = q^n \mathcal{M}$. It follows that $q^{n+1} \mathcal{M} : m = q^n \mathcal{M}(q\mathcal{M} : M m)$. So we have

$$[q^{n+1}M + N + bM] : m = q^n([qM + N + b] : m) + N + b$$

because $b \subseteq q$. Since $[N + qM] : M m = N + [qM : M m]$, therefore we have

$$[q^{n+1}M + N + bM] : m \subseteq q^n(qM : m) + N + bM \subseteq q^{n+1}M : m + N + bM.$$

Thus $[q^{n+1}M + N + bM] : m = q^{n+1}M : m + N + bM$. Hence

$$[q^nM + N + bM] : M m = [q^nM : M m] + N + bM,$$

for all $n \geq 0$.

The notion of a sequentially Cohen-Macaulay module was introduced firstly by Stanley [22] for the graded case and in [21] for the local case.

Definition 3.2 ([21] [22]). Let $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of $M$. We say that $M$ is a sequentially Cohen-Macaulay $R$-module, if $C_i$ is a Cohen-Macaulay $R$-module for all $1 \leq i \leq \ell$, where $C_i = D_i/D_{i-1}$. We say that $R$ is a sequentially Cohen-Macaulay ring, if $\text{dim } R < \infty$ and $R$ is a sequentially Cohen-Macaulay module over itself.

We maintain the following settings.

Settings 3.3. Let $M$ be a sequentially Cohen-Macaulay $R$-module, $d = \text{dim } M \geq 1$, and $\mathcal{D} = \{D_i\}_{0 \leq i \leq \ell}$ the dimension filtration. We put $N = D_{\ell-1}$, $L = M/D_{\ell-1}$ and choose a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of $M$ such that

$$N(q; M) = \sum_{j \in \mathbb{Z}} r_j(M).$$

where $q = (x_1, x_2, \ldots, x_d)$.

Fact 3.4. (See [8]) The following assertions hold true.

(1) Module $N$ is sequentially Cohen-Macaulay and $L$ is Cohen-Macaulay.

(2) We have $[N + qM] : M m = N + [qM : M m]$.

(3) The parameter ideal $q$ is also a distinguished parameter ideal of $N$ such that

$$N(q; N) = \sum_{j \in \mathbb{Z}} r_j(N).$$
(4) Let $M = R$. If $e_0(m; R) > 1$ or $q \subseteq m^2$ then we have $I^2 = qI$, where $I = q : m$.

**Proposition 3.5.** We have

$$
\mathcal{N}(q^{n+1}; M) = \sum_{i=1}^{d} r_i(M) \binom{n+i-1}{i-1} + r_0(M)
$$

for all $n \geq 1$.

**Proof.** We denote $\text{gr}_q(L) = \bigoplus_{n \geq 0} q^n L/q^{n+1} L$. Since $L$ is a Cohen-Macaulay $R$-module and $q$ is a parameter ideal of $R$-module $L$, sequence $x_1, \ldots, x_d$ is an $L$-regular. Since $L$ is Cohen-Macaulay, we have a natural isomorphism of graded modules

$$
\text{gr}_q(L) = \bigoplus_{n \geq 0} q^n L/q^{n+1} L \to L/qL[T_1, \ldots, T_d],
$$

where $T_1, \ldots, T_d$ are indeterminates. This deduces $R$-isomorphisms on graded parts

$$
q^n L/q^{n+1} L \to (L/qL[T_1, \ldots, T_d])_n \cong L/qL^{(n+d-1)}
$$

for all $n \geq 0$. On the other hand, since $q$ is a parameter ideal of a Cohen-Macaulay modules $L$, $q^{n+1}L : m \subseteq q^{n+1}L : q = q^nL$. It follows from $\ell(qL : m/qL) = r_d(M)$ that

$$
\ell(q^{n+1}L : m/q^{n+1}L) = \ell(qL : m/qL) \binom{n+d-1}{d-1} = r_d(M) \binom{n+d-1}{d-1}.
$$

Since the parameter ideal $q$ is good and $L$ is Cohen-Macaulay, the following exact sequence

$$
0 \to N \to M \to L \to 0
$$

induces the following exact sequence

$$
0 \to N/q^{n+1}N \to M/q^{n+1}M \to L/q^{n+1}L \to 0.
$$

It follows from $[q^{n+1}M + N] : m = q^{n+1}M : m + N$, by the Fact 3.4 and lemma 3.1, that by applying $\text{Hom}_R(k, *)$, we obtain the following exact sequence

$$
0 \to \text{Hom}_R(k, N/q^{n+1}N) \to \text{Hom}_R(k, M/q^{n+1}M) \to \text{Hom}_R(k, L/q^{n+1}L) \to 0
$$

Therefore we get that

$$
\ell_R([q^{n+1}M : m]/q^{n+1}M) = \ell_R([q^{n+1}N : m]/q^{n+1}N) + \ell_R([q^{n+1}L : m]/q^{n+1}L).
$$

A simple inductive argument therefore shows that

$$
\mathcal{N}(q^{n+1}; M) = \ell([q^n M : m]/q^n M) = \sum_{i=1}^{d} r_i(M) \binom{n+i-1}{i-1} + r_0(M)
$$

for all $n \geq 1$. Thus the proof is complete.

We need the following result in next section.

**Lemma 3.6.** We have $q^{n+1}M : m = q^n(qM : m)$ for all $n \geq 0$. 


Proof. Since $L$ is Cohen-Macaulay and $q$ is a parameter ideal of $L$, we have $q^n M \cap N = q^n N$ and

$q^{n+1} M : m \subseteq [q^{n+1} M + N] : m = q^n(qM : m) + N,$

for all $n \geq 0$, because of the Fact 3.4 and the Lemma 3.1. Let $a \in q^{n+1} M : m$ and we write $a = b + c$ for $b \in q^n(qM : m)$ and $c \in N$. Then $mc = m(a - b) \in q^{n+1} M \cap N = q^n+1 N$. Thus $c \in q^{n+1} N : m$. Therefore $q^{n+1} M : m \subseteq q^n(qM : m) + q^{n+1} N : m$. Hence

$q^{n+1} M : m = q^n(qM : m) + q^{n+1} N : m$

Since $N$ is sequentially Cohen-Macaulay and $q$ is a good parameter idea of $N$, by the induction on $\ell$, we have $q^{\ell+1} N : m = q^n[qN : m]$. Therefore we have

$q^{n+1} M : m = q^n(qM : m),$

as required. 

\[ \square \]

We close this section with the following, which is the main result of [23] of the first author.

**Theorem 3.7 ([23 Theorem 1.1]).** There exists an integer $n \gg 0$ such that for every distinguished parameter ideals $q$ of $M$ contained in $m^n$, one has the equality

\[ N(q; M) = \sum_{j \in \mathbb{Z}} \ell_R((0) :_{H^j_d(M)} m). \]

4. **Hilbert Coefficients of Socle Ideals**

The purpose of this section is to give a characterization of sequentially Cohen-Macaulay rings in term of the Hilbert coefficients of the socle of distinguished parameter ideals. To discuss this, we need the concept of Hilbert coefficients.

Let $I$ be a $m$-primary ideal of a Noetherian local ring $(R, m)$. The associated graded ring $gr_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ is a standard graded ring with $[gr_I(R)]_0 = R/I$ Artinian. Let $M$ be a finitely generated $R$-module of dimension $d$. Therefore the associated graded module $gr_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M$ of $I$ with respect to $M$ is a finitely generated graded $gr_I(R)$-module. The Hilbert-Samuel function of $M$ with respect to $I$ is

\[ H(n) = \ell_R(M/I^{n+1} M) = \sum_{i=0}^{n} \ell_R(I^i M/I^{i+1} M), \]

where $\ell_R(*)$ stands for the length. For sufficiently large $n$, the Hilbert-Samuel function of $M$ with respect to $I$ $H(n)$ is of polynomial type,

\[ \ell_R(M/I^{n+1} M) = \sum_{i=0}^{d} (-1)^i e_i(I, M) \binom{n + d - i}{d - i}. \]

These integers $e_i(I, M)$ are called the Hilbert coefficients of $M$ with respect to $I$. In the particular case, the leading coefficient $e_0(I, M)$ is said to be the multiplicity of $M$ with respect to $I$ and $e_1(I, M)$ is called by Vasconcelos([27]) the Chern coefficient of $I$ with respect to $M$. When $M = R$, we abbreviate $e_i(I, M)$ to $e_i(I)$ for all $i = 1, \ldots, s$. 

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**Settings 4.1.** Assume that $R$ is a homomorphic image of a Cohen-Macaulay local ring. Let $D = \{a_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of $R$ with $\dim a_i = d_i$. We put $S = R/a_{\ell-1}$ and choose a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of $R$. Put $q = (x_1, x_2, \ldots, x_d)$, $b = (x_{d_{\ell-1}+1}, \ldots, x_d)$ and $I = q : m$.

In fact, the following property serves to characterize sequentially Cohen-Macaulay rings, as we will show in this section.

**Proposition 4.2.** Assume that $R$ is sequentially Cohen-Macaulay and

$$\mathcal{N}(q; R) = \sum_{j \in \mathbb{Z}} r_j(R).$$

Then we have

$$e_j(I) - e_j(q) = f_{j-1}(q; R) = \begin{cases} (-1)^{d-1}(r_1(R) + r_0(R)), & \text{if } j = d, \\ (-1)^{j-1}r_{d-j+1}(R) & \text{otherwise,} \end{cases}$$

if $e_0(m; R) > 1$ or $q \subseteq m^2$.

**Proof.** By Fact 3.4 (4), we have $I^2 = qI$ and so $I^{n+1} = q^nI$ for all $n \geq 1$. It follows from Lemma 3.6 that

$$\ell(R/q^{n+1}) - \ell(R/I^{n+1}) = \ell((q^nI)/q^{n+1}) = \ell((q^n(q : m))/q^{n+1}) = \ell((q^{n+1} : m)/q^{n+1})$$

for all $n \geq 0$. Since $I^2 = qI$, we have $e_0(q) = e_0(I)$. Therefore we have

$$e_j(I) - e_j(q) = f_{j-1}(q; R)$$

By Theorem 3.5 we have

$$e_j(I) - e_j(q) = f_{j-1}(q; R) = \begin{cases} (-1)^{d-1}(r_1(R) + r_0(R)), & \text{if } j = d, \\ (-1)^{j-1}r_{d-j+1}(R) & \text{otherwise.} \end{cases}$$

□

**Corollary 4.3.** Suppose that $M$ is a sequentially Cohen-Macaulay $R$-module. Then there exists an integer $n \gg 0$ such that for every distinguished parameter ideals $q$ of $M$ contained in $m^n$, one has the equality

$$e_j(I) - e_j(q) = f_{j-1}(q; R) = \begin{cases} (-1)^{d-1}(r_1(R) + r_0(R)), & \text{if } j = d, \\ (-1)^{j-1}r_{d-j+1}(R) & \text{otherwise.} \end{cases}$$

**Proof.** This is now immediate from Proposition 4.2 and Theorem 3.7. □

**Lemma 4.4.** Assume that $S$ is Cohen-Macaulay and

$$[q + a_{\ell-1}] : m = q : m + a_{\ell-1}.$$ 

Then

$$e_j(I; R/b) - e_j(q; R/b) = \begin{cases} (-1)^s((e_{s+j}(I; R) - e_{s+j}(q; R))) + r_d(R) & \text{if } j = 1, \\ (-1)^s(e_{s+j}(I; R) - e_{s+j}(q; R)) & \text{if } j \geq 2, \end{cases}$$

where $s = d - d_{\ell-1}$.

**Proof.** Since $[q + a_{\ell-1}] : m = q : m + a_{\ell-1}$, we have $IS = qS : mS$. 

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Claim 1. \((I^n + b) \cap a_{\ell-1} = I^n \cap a_{\ell-1}\) for all \(n\).

**Proof.** Since \(q \subseteq m^2\) and \(S\) is Cohen-Macaulay, we have \((IS)^2 = (qS)(IS)\) by [3] Theorem 3.7, so that \(gr_{rS}(S)\) is a Cohen-Macaulay ring. Therefore, we have

\[ I^n S : x_d = I^{n-1} S \]

for all \(n \in \mathbb{Z}\). Consequently, \((I^n + a_{\ell-1}) : x_d = I^{n-1} + a_{\ell-1}\).

Let \(a \in (I^n + (x_d)) \cap a_{\ell-1}\). Write \(a = b + x_d c\) for \(b \in I^n\) and \(c \in R\). Then \(a \in (I^n + a_{\ell-1}) : x_d = I^{n-1} + a_{\ell-1}\). Thus since \(q\) is a distinguish parameter ideal, we have \(x_d c \in x_d I^{n-1} + x_d a_{\ell-1} \subseteq I^n\). Therefore \(a \in I^n \cap a_{\ell-1}\). Hence \(I^n \cap a_{\ell-1} = (I^n + (x_d)) \cap a_{\ell-1}\).

By induction, we have

\[ I^n \cap a_{\ell-1} = (I^n + (x_d)) \cap a_{\ell-1} = \ldots = (I^n + b) \cap a_{\ell-1}, \]

as required. \(\square\)

It follows from the above claim and the following exact sequences

\[ 0 \rightarrow a_{\ell-1} / I^n \cap a_{\ell-1} \rightarrow R/I^n \rightarrow S/I^n S \rightarrow 0 \]

for all \(n \geq 0\) and

\[ 0 \rightarrow a_{\ell-1} / (b + I^n) \cap a_{\ell-1} \rightarrow R/b + I^n \rightarrow S/(b + I^n) S \rightarrow 0 \]

for all \(n \geq 0\), that we have

\[ \ell(R/I^n) - \ell(S/I^n S) = \ell(R/b + I^n) - \ell(S/(b + I^n) S) \]

Since \(S\) is Cohen-Macaulay, by Lemma [4,2] we have

\[ \ell(S/I^n S) = e_0(I; S) \binom{n + d}{d} - r_d(S) \binom{n + d - 1}{d - 1} \]

\[ \ell(S/(b + I^n) S) = e_0(I; S) \binom{n + d_{\ell-1}}{d_{\ell-1}} - r_d(S) \binom{n + d_{\ell-1} - 1}{d_{\ell-1} - 1} \]

for all \(n \geq 0\). Consequently, it follows on comparing the coefficients of the polynomials in the above equality that

\[ e_j(I; R/b) = \begin{cases} (-1)^j e_{s+1}(I; R) + r_d(S) & \text{if } j = 1, \\ (-1)^j e_{s+j}(I; R) & \text{if } j \geq 2 \end{cases} \]

Similarly, we have

\[ e_j(q; R/b) = (-1)^j e_{s+j}(I; R) \]

for all \(1 \leq j \leq d_{\ell-1}\). It follows that

\[ e_j(I; R/b) - e_j(q; R/b) = \begin{cases} (-1)^j (e_{s+j}(I; R) - e_{s+j}(q; R)) + r_d(R) & \text{if } j = 1, \\ (-1)^j (e_{s+j}(I; R) - e_{s+j}(q; R)) & \text{if } j \geq 2 \end{cases} \]

\(\square\)

**Proposition 4.5.** Assume that \(d \geq 2\) and there exists an integer \(n\) such that for all distinguish parameter ideals \(q \subseteq m^n\) we have

\[ e_1(I) - e_1(q) \leq r_d(R). \]

Then \(S\) is Cohen-Macaulay.
Proof. In the case in which \(e_0(m; R) = 1\), we have \(e_0(m; S) = 1\), because \(\dim \mathfrak{a}_\ell - 1 < \dim R\). And so the result in this case follows from \(S\) is unmixed and Theorem 40.6 in [16]. Thus we suppose henceforth in this proof that \(e_0(m; R) > 1\).

By Proposition 2.14 there exists a Goto sequence \(x_1, \ldots, x_{d-2}\) of type I in \(m^n\). Let \(q_{d-2} = (x_1, \ldots, x_{d-2})\) and \(A = R/q_{d-2}\) and let \(N\) denote the unmixed component of \(A\). Then \(A/N\) is a generalized Cohen-Macaulay ring since \(\dim A/N = 2\) and \(A/N\) is unmixed. Therefore there exists an integer \(r_0 > n\) such that for all parameters \(x \in m^{n_0}\), we have \(r_1(A/(x) + N) = r_1(A/N) + r_2(A/N)\). Suppose that \(d_{i_0} < d - 2\) for some \(i_0\). Then \(\text{Ann}(a_{i_0}) + q_{d-2}\) is an \(m\)-primary ideal of \(R\). Then we can choose \(x_{d-1} \in m^{n_0} \cap \text{Ann}(a_{i_0})\) as in Proposition 2.8.

Let \(q_{d-1} = (q_{d-2}, x_{d-1})\) and \(B = R/q_{d-1}\). Since \(\dim B = 1\), \(B\) is sequentially Cohen-Macaulay ring. It follows from \(\text{Ann} a_{i-1} + q_{d-1}\) is an \(m\)-primary ideal of \(R\) and Corollary 4.3, we have choose \(x_d \in \text{Ann} a_{i-1}\) such that

\[
e_1(x_d B : B mB; B) - e_1(x_d B; B) = r_1(B).
\]

Since \(e(m; R) > 1\), by Proposition 2.3 in [13], we get that \(mI^n = m q^n\) for all \(n\). Therefore \(I^n \subseteq q^n : m\) for all \(n\). Put \(G = \bigoplus_{n \geq 0} q^n / q^{n+1}\) and \(M = m / q \oplus \bigoplus_{n \geq 1} q^n / q^{n+1}\). Then we have \((0 : M)_n = (q^n : m) / q^{n+1}\) for all \(n\). We have \((0 : M)_n = (q^{n+1} : m) / q^{n+1}\) for all large \(n\). Since \(x_1, \ldots, x_d\) is a Goto sequence, by Lemma 2.4, \(x_1, x_2, \ldots, x_d\) is a \(d\)-sequence. Therefore, \(x_1\) is a superficial element of \(R\) with respect to \(q\). And so, we have \(q^{n+1} : x_1 = q^n\) for all \(n \geq 0\). It follows from the exact sequence

\[
0 \to \frac{I^n \cap (q^{n+1} : x_1)}{q^n} \to \frac{I^n}{q^n} \to \frac{I^{n+1}}{q^{n+1}} \to x_1I^n + q^{n+1} \to 0
\]

and

\[
0 \to \frac{I^{n+1} \cap (x_1 + q^{n+1})}{x_1I^n + q^{n+1}} \to \frac{I^{n+1}}{x_1I^n + q^{n+1}} \to \frac{I^{n+1} + (x_1)}{q^{n+1} + (x_1)} \to 0
\]

for all \(n \geq 0\) that

\[
\ell\left(\frac{I^{n+1}}{q^{n+1}}\right) - \ell\left(\frac{I^n}{q^n}\right) = \ell\left(\frac{I^{n+1}}{x_1I^n + q^{n+1}}\right) \geq \ell\left(\frac{I^{n+1} + (x_1)}{q^{n+1} + (x_1)}\right).
\]

Consequently, we have

\[
e_1(I; R) - e_1(q; R) \geq e_1(qR : mR; R) - e_1(qR; R)
\]

because \(\ell(R/q^{n+1}) - \ell(R/I^{n+1}) = \ell(I^{n+1}/q^{n+1})\) for all \(n \geq 0\). Proceed inductively, we have

\[
e_1(I; R) - e_1(q; R) \geq e_1(x_d B : B mB; B) - e_1(x_d B; B) = r_1(B),
\]

because of the choice of \(x_d\). However, since \(x_1, \ldots, x_d\) is a Goto sequence, by Lemma 2.4 \(q\) is a distinguish parameter ideal and \(r_2(A) \geq r_d(R)\). By hypothesis, \(r_d(R) \geq e_1(I; R) - e_1(q; R)\). Therefore, \(r_2(A) \geq r_1(B)\). On the other hand, it follows from the exact sequence

\[
0 \to N \to A \to A/N \to 0
\]

and \(x_{d-1}\) is a regular of \(A/N\) that \(r_2(A/N) = r_2(A)\) and

\[
0 \to N/x_{d-1}N \to B \to A/(x_{d-1}) + N \to 0.
\]
Corollary 4.6. For all integers $n$ there exists a parameter ideal $q \subseteq m^n$, we have
\[ r(R) \leq e_1(I; R) - e_1(q; R), \]
where $I = q : m$.

The observation in Proposition 4.5 and Lemma 4.4 provides a clue to a characterization of sequentially Cohen-Macaulay rings in terms of Hilbert coefficients of socle parameter ideals.

Proposition 4.7. Assume that there exists an integer $n$ such that for all distinguished parameter ideals $q \subseteq m^n$ we have
\[ r_j(R) \geq (-1)^{d-j}(e_{d-j+1}(I) - e_{d-j+1}(q)), \]
for all $2 \leq j \in \Lambda(R)$. Then $R$ is sequentially Cohen-Macaulay.

Proof. We use induction on the dimensional $d$ of $R$. In the case in which $\dim R = 1$, it is clear that $R$ is sequentially Cohen-Macaulay. Suppose that $\dim R > 1$ and that our assertion holds true for $\dim R - 1$. Recall that $a_{\ell-1}$ is the unmixed component of $R$. Therefore, by the Prime Avoidance Theorem, we can choose the part of a system $x_{d_{\ell-1}+1}, \ldots, x_d$ of parameters of $R$ such that $b \subseteq m^n$ and $b \cap a_{\ell-1} = 0$, where $b = (x_{d_{\ell-1}+1}, \ldots, x_d)$. Consequently, $(a_i + b)/b = a_i$ for all $i = 0, \ldots, \ell - 1$, and so $\Lambda(R) - \{d\} \subseteq \Lambda(R/b)$. On the other hand, since $d \in \Lambda(R)$, we obtain $e_1(I; R) - e_1(q) \leq r_d(R)$ for all distinguished parameter ideals $q \subseteq m^n$. By Proposition 4.5, $S$ is Cohen-Macaulay. It follows from the exact sequence
\[ 0 \to a_{\ell-1} \to R \to S \to 0 \]
that the sequence
\[ 0 \to a_{\ell-1} \to R/b \to S/bS \to 0 \]
is exact, and so $\Lambda(R/b) = \Lambda(R) - \{d\}$.

Now let $x_1, \ldots, x_{d_{\ell-1}}$ be a distinguished system of parameters of $R/b$. We show that $x_1, \ldots, x_d$ is a distinguished system of parameters of $R$. Indeed, let $d_i + 1 \leq j \leq d_{i+1}$ for some $i \neq \ell - 1$. Since $d_i \in \Lambda(R/b)$, $R/b$ contain the largest ideal $c_i$ with $\dim c_i = d_i$. Therefore $(a_i + b)/b \subseteq c_i$. Since $x_j c_i = 0$, we obtain $x_j a_i \subseteq b \cap a_i \subseteq b \cap a_{\ell-1} = 0$. Hence $x_j a_i = 0$ for all $d_i + 1 \leq j \leq d_{i+1} + 1$ and $i < \ell - 1$. Hence system $x_1, \ldots, x_d$ of parameters is distinguished.

Put $q = (x_1, \ldots, x_{d_{\ell-1}}, b)$ and assume that $q \subseteq m^n$. It follows from Lemma 4.4 that
\[
e_j(I; R/b) - e_j(q; R/b) = \begin{cases} (-1)^s((e_{s+j}(I; R) - e_{s+j}(q; R)) + r_d(R)) & \text{if } j = 1, \\ (-1)^s(e_{s+j}(I; R) - e_{s+j}(q; R)) & \text{if } j \geq 2, \end{cases} \]

Since $\dim N/x_{d-1}N = 0$, we have $H^0_m(B) = H^1_m(A/(x_{d-1}) + N)$, and so
\[
r_1(B) = r_1(A/(x_{d-1}) + N) = r_1(A/N) + r_2(A/N) = r_1(A/N) + r_2(A),
\]
because of the choice of $x_d$. Therefore we have $r_1(A/N) = 0$, and so $r_1(S/q_{d-2}S) = 0$. Hence $S$ is Cohen-Macaulay, because of Lemma 2.5, and the proof is complete. □

The next corollary is now immediate.
where $s = d - d_{i-1}$. However, it follows from $S$ is Cohen-Macaulay and the exact sequence

$$0 \rightarrow a_{i-1} \rightarrow R/b \rightarrow S/bS \rightarrow 0$$

that the following sequence

$$0 \rightarrow H^{d_{i-1}}_m(a_{i-1}) \rightarrow H^{d_{i-1}}_m(R/b) \rightarrow H^{d_{i-1}}_m(S/bS) \rightarrow 0$$

is exact. Moreover we have $H^{d_{i-1}}_m(a_{i-1}) \cong H^{d_{i-1}}_m(R/b)$ and $H^{d_{i-1}}_m(R) \cong H^{d_{i-1}}_m(a_{i-1}) \cong H^{d_{i-1}}_m(R/b)$ for all $i < d_{i-1}$. Thus we have $r_i(R) = r_i(R/b)$ for all $i < d_{i-1}$ and

$$r_{d_{i-1}}(R/b) = r_{d_{i-1}}(R) + r_{d_{i-1}}(S/bS) = r_{d_{i-1}}(R) + r_d(S).$$

Therefore since $s + d_{i-1} - j = d - j$ we have

$$r_j(R/b) = r_j(R) \geq (-1)^{d-j}(e_{d-j+1}(I; R) - e_{d-j+1}(q; R))$$

$$= (-1)^{d_{i-1}-j}(e_{d_{i-1}-j+1}(I; R/b) - e_{d_{i-1}-j}(q; R/b))$$

for all $2 \leq j \in \Lambda(R/b) - \{d_{i-1}\}$. Moreover, we have

$$r_{d_{i-1}}(R/b) = r_{d_{i-1}}(R) + r_d(S)$$

$$\geq (-1)^{s+1}(e_{s+1}(I; R) - e_{s+1}(q; R)) + r_d(S)$$

$$= e_1(I; R/b) - e_1(q; R/b)$$

Consequently, $r_{d_{i-1}-j+1}(R/b) \geq (-1)^{j+1}(e_j(I; R/b) - e_j(q; R/b))$ for all distinguished parameter ideals $q \subseteq m^n$ of $R/b$ and $2 \leq j \in \Lambda(R/b)$. By the induction hypothesis, $R/b$ is sequentially Cohen-Macaulay and so is also $R$. This completes the inductive step, and the proof.

Now, we can provide a characterization of sequentially Cohen-Macaulay rings.

**Theorem 4.8.** The following statements are equivalent.

(i) $R$ is sequentially Cohen-Macaulay.

(ii) There exists an integer $n$ such that for all distinguish parameter ideals $q \subseteq m^n$ and $0 \leq i \leq d - 1$, we have

$$r_{d-i+1}(R) = (-1)^{i+1}(e_i(I) - e_i(q)),$$

where $I = q : m$.

(iii) There exists an integer $n$ such that for all good parameter ideals $q \subseteq m^n$ and $2 \leq j \in \Lambda(R)$, we have

$$r_j(R) \geq (-1)^{d-j}(e_{d-j+1}(I) - e_{d-j+1}(q)),$$

where $I = q : m$.

**Proof.** (1) $\Rightarrow$ (2) This is now immediate from Theorem 3.7 and Proposition 4.2.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) This now immediate from Proposition 4.7.

$\square$
5. Noetherian coefficients

Continuing our discussion in last section, we will see how the sequentially Cohen-Macaulayness of rings is relation to the Noetherian coefficients of the socle of distinguished parameter ideals. To discuss it, we need a relationship between Chern coefficient and the irreducible multiplicity.

**Settings 5.1.** Assume that $R$ is a homomorphic image of a Cohen-Macaulay local ring. Let $D = \{a_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of $R$ with $dim a_i = d_i$. We put $S = R/a_{\ell-1}$ and choose a distinguished system $x_1, x_2, \ldots, x_d$ of parameters of $R$. Put $q = (x_1, x_2, \ldots, x_d)$, $b = (x_{d_{\ell-1}+1}, \ldots, x_d)$ and $I = q : m$.

**Lemma 5.2.** Assume that $e_0(m; R) > 1$. Then for all parameter ideal $q$, we have

$$e_1(I; R) - e_1(q; R) \leq f_0(R),$$

where $I = q : m$.

**Proof.** Since $e_0(m; R) > 1$, by Proposition 2.3 in [13], we get that $m^n = mq^n$ for all $n$. Therefore $I^n \subseteq q^n : m$ for all $n$. Consequence, we obtain

$$\ell(R/q^{n+1}) - \ell(R/I^{n+1}) = \ell(I^{n+1}/q^{n+1}) \leq \ell((q^{n+1} : m)/q^{n+1}).$$

But this means that $e_1(I; R) - e_1(q; R) \leq f_0(q; R)$. \hfill $\square$

**Proposition 5.3.** Assume that $R$ is unmixed. Then for all parameter ideals $q \subseteq m^2$, we have

$$e_1(q : m) - e_1(q) \leq f_0(q).$$

**Proof.** Our result in the case in which $e_0(m; R) > 1$ is immediate from Lemma 5.2. Thus we suppose henceforth in this proof that $e_0(m; R) = 1$. It follows from $R$ is unmixed and Theorem 40.6 in [16] that $R$ is Cohen-Macaulay. Since $R$ is unmixed, every parameter ideals $q$ are distinguished. Therefore by Theorem 3.7 there exists an integer $n$ such that for all parameter ideals $q$, we have

$$\mathcal{N}(q; R) = \sum_{j \in \mathbb{Z}} r_j(R).$$

It follows from Proposition 4.2 we have

$$e_1(q : m) - e_1(q) = f_0(q; R),$$

and the proof is complete. \hfill $\square$

**Corollary 5.4.** Assume that $d \geq 2$ and there exists an integer $n$ such that for all distinguish parameter ideals $q \subseteq m^n$ we have

$$f_0(q; R) \leq r_d(R).$$

Then $S$ is Cohen-Macaulay.

**Proof.** In the case in which $e_0(m; R) = 1$, we have $e_0(m; S) = 1$, because $dim a_{\ell-1} < dim R$. And so the result in this case follows from $S$ is unmixed and Theorem 40.6 in [16]. Thus we suppose henceforth in this proof that $e_0(m; R) > 1$. By Lemma 5.2 for all distinguish parameter ideals $q \subseteq m^n$ we have

$$e_1(I; R) - e_1(q; R) \leq f_0(R) \leq r_d(R).$$

It follows from Proposition 4.3 that $S$ is Cohen-Macaulay. \hfill $\square$
The next corollary is now immediate.

**Corollary 5.5.** For all integers $n$ there exists a parameter ideal $q \subseteq m^n$, we have

$$r_d(R) \leq f_0(q; R).$$

**Lemma 5.6.** Assume that $S$ is Cohen-Macaulay and

$$[q + a_{\ell-1}] : m = q : m + a_{\ell-1}.$$

Then

$$f_j(q; R/b) = \begin{cases} (-1)^sf_{s+j}(q; R) + r_d(R) & \text{if } j = 0, \\ (-1)^sf_{s+j}(q; R) & \text{if } j \geq 1, \end{cases}$$

where $s = d - d_{\ell-1}$.

**Proof.** Since $S$ is Cohen-Macaulay and $q$ is a parameter ideal of $S$, we have

$$0 \to a_{\ell-1}/q^na_{\ell-1} \to R/q^n \to S/q^nS \to 0.$$

It follows from $[q^n + a_{\ell-1}] : m = q^n : m + a_{\ell-1}$, by the lemma 3.1 that by applying $\text{Hom}_R(k, R)$, we obtain the following exact sequence

$$0 \to q^n a_{\ell-1} : m \to q^n : m \to q^n S : m \to 0.$$

Since $S$ is Cohen-Macaulay and $b$ is an ideal generated by a part system of parameters of $S$, we have

$$0 \to a_{\ell-1}/q^na_{\ell-1} \to R/q^n \to S/q^nS \to 0.$$

It follows from $[q^n + a_{\ell-1} + b] : m = q^n : m + a_{\ell-1} + b$, by the lemma 3.1 that by applying $\text{Hom}_R(k, R)$, we obtain the following exact sequence

$$0 \to q^n a_{\ell-1} : m \to (q^n + b) : m \to (q^n S + bS) : m \to 0.$$

From the above exact sequences, we have

$$\ell\left(\frac{q^n : m}{q^n}\right) - \ell\left(\frac{q^n S : m}{q^n}\right) = \ell\left(\frac{q^n + b : m}{q^n + b}\right) - \ell\left(\frac{(q^n S + bS) : m}{q^n S + bS}\right).$$

Since $S$ is Cohen-Macaulay, by Proposition 3.5 we have $\ell(\frac{q^{n+1}S : m}{q^{n+1}S + bS}) = r_d(S)^{(n+1)_{d-1}}$ and $\ell\left(\frac{q^{n+1}S + bS : m}{q^{n+1}S + bS}\right) = r_d(S)^{(n+1)_{d-1}}$. Therefore since $d_{\ell-1} < d$ and $r_d(S) = r_d(R)$, we have

$$f_j(q; R/b) = \begin{cases} (-1)^sf_{s+j}(q; R) + r_d(R) & \text{if } j = 0, \\ (-1)^sf_{s+j}(q; R) & \text{if } j \geq 1, \end{cases}$$

where $s = d - d_{\ell-1}$. \qed

**Proposition 5.7.** Assume that there exists an integer $n$ such that for all distinguished parameter ideals $q \subseteq m^n$ and $2 \leq j \in \Lambda(R)$, we have

$$(-1)^{d-j}f_{d-j}(q; R) \leq r_j(R).$$

Then $R$ is sequentially Cohen-Macaulay.
Proof. We argue by induction on the dimensional $d$ of $R$, the result being clear in the case in which $d = 1$. Suppose, inductively, that $d > 1$ and the result has been proved for smaller values of $d$.

Recall that $a_{d-1}$ is the unmixed component of $R$. Therefore, by the Prime Avoidance Theorem, we can choose the part of a system $x_{d_{\ell-1}+1}, \ldots, x_d$ of parameters of $R$ such that $b \subseteq m^n$ and $b \cap a_{d-1} = 0$, where $b = (x_{d_{\ell-1}+1}, \ldots, x_d)$. Consequently, $(a_i + b)/b = a_i$ for all $i = 0, \ldots, \ell-1$, and so $\Lambda(R) - \{d\} \subseteq \Lambda(R/b)$. On the other hand, since $d \in \Lambda(R)$, we obtain $f_0(q; R) \leq r_d(R)$ for all distinguished parameter ideals $q \subseteq m^n$. By Corollary 5.4, $S$ is Cohen-Macaulay. It follows from the exact sequence

$$0 \to a_{d-1} \to R \to S \to 0$$

that the sequence

$$0 \to a_{d-1} \to R/b \to S/bS \to 0$$

is exact, and so $\Lambda(R/b) = \Lambda(R) - \{d\}$.

Now let $x_1, \ldots, x_{d_{\ell-1}}$ be a distinguished system of parameters of $R/b$. We show that $x_1, \ldots, x_d$ is a distinguished system of parameters of $R$. Indeed, let $d_i + 1 \leq j \leq d_{i+1}$ for some $i \neq \ell - 1$. Since $d_i \in \Lambda(R/b)$, $R/b$ contain the largest ideal $c_i$ with dim $c_i = d_i$. Therefore $(a_i + b)/b \subseteq c_i$. Since $x_j c_i = 0$, we obtain $x_j a_i \subseteq b \cap a_i \subseteq b \cap a_{d-1} = 0$. Hence $x_j a_i = 0$ for all $d_i + 1 \leq j \leq d_{i+1} + 1$ and $i < \ell - 1$. Hence system $x_1, \ldots, x_d$ of parameters is distinguished.

Put $q = (x_1, \ldots, x_{d_{\ell-1}}, b)$ and assume that $q \subseteq m^n$. It follows from Lemma 4.4 that 

$$f_j(q; R/b) = \begin{cases} (-1)^i f_{s+j}(q; R) + r_d(R) & \text{if } j = 0, \\ (-1)^i f_{s+j}(q; R) & \text{if } j \geq 1, \end{cases}$$

where $s = d - d_{\ell-1}$. However, it follows from $S$ is Cohen-Macaulay and the exact sequence

$$0 \to a_{d-1} \to R/b \to S/bS \to 0$$

that the following sequence

$$0 \to H^{d_{\ell-1}}_m(a_{d-1}) \to H^{d_{\ell-1}}_m(R/b) \to H^{d_{\ell-1}}_m(S/bS) \to 0$$

is exact. Moreover we have $H^{d_{\ell-1}}_m(R) \cong H^{d_{\ell-1}}_m(a_{d-1})$ and $H^{i}_m(R) \cong H^{i}_m(a_{d-1}) \cong H^{i}_m(R/b)$ for all $i < d_{\ell-1}$. Thus we have $r_i(R) = r_i(R/b)$ for all $i < d_{\ell-1}$ and

$$r_{d_{\ell-1}}(R/b) = r_{d_{\ell-1}}(R) + r_{d_{\ell-1}}(S/bS) = r_{d_{\ell-1}}(R) + r_d(S).$$

Therefore since $s + d_{\ell-1} - j = d - j$ we have

$$r_j(R/b) = r_j(R) \geq (-1)^{d-j} f_{d-j}(q; R) = (-1)^{d_{\ell-1} - j} f_{d_{\ell-1} - j}(q; R/b)$$

for all $2 \leq j \in \Lambda(R/b) - \{d_{\ell-1}\}$. Moreover, we have

$$r_{d_{\ell-1}}(R/b) = r_{d_{\ell-1}}(R) + r_d(S) \geq (-1)^s f_s(q; R) + r_d(S) = f_0(q; R/b).$$

Consequently,

$$(-1)^{d_{\ell-1} - j} f_{d_{\ell-1} - j}(q; R/b) \leq r_j(R/b).$$

for all distinguished parameter ideals $q \subseteq m^n$ of $R/b$ and $2 \leq j \in \Lambda(R/b)$. Application of the inductive hypothesis to the ring $R/b$ shows that $R/b$ is sequentially Cohen-Macaulay and so is also $R$. This completes the inductive step, and the proof.

\[\square\]
We close this section with the following, which will be used in our discussion of regular local rings in the next section.

**Theorem 5.8.** The following statements are equivalent.

(i) \( R \) is a sequentially Cohen-Macaulay \( R \)-module.

(ii) There exists an integer \( n \) such that for all distinguished parameter ideals \( q \subseteq m^n \) and \( j = 0, \ldots, d - 2 \), we have

\[
(-1)^j f_j(q; R) = r_{d-j}(R).
\]

(iii) There exists an integer \( n \) such that for all distinguished parameter ideals \( q \subseteq m^n \) and \( 2 \leq j \in \Lambda(R) \), we have

\[
(-1)^{d-j} f_{d-j}(q; R) \leq r_j(R).
\]

**Proof.** (1) \( \Rightarrow \) (2) This is now immediate from Theorem 3.7 and Proposition 3.5.

(2) \( \Rightarrow \) (3) This is obvious.

(3) \( \Rightarrow \) (1) This now immediate from Proposition 5.7.

\[\square\]

### 6. Chern coefficients

In this section, we are going to discuss the characterizations of the regularity, Gorensteinness and Cohen-Macaulayness of local rings. Probably the most important applications of Theorem 5.8 and 4.8 can be summarized in this section.

**Settings 6.1.** Assume that \( R \) is a homomorphic image of a Cohen-Macaulay local ring. Let \( D = \{a_i\}_{0 \leq i \leq \ell} \) be the dimension filtration of \( R \) with \( \dim a_i = d_i \).

**Theorem 6.2.** \( R \) is Gorenstein if and only if for all parameter ideals \( q \subseteq m^2 \) and \( n \gg 0 \), we have

\[
\mathcal{N}(q^{n+1}; R) = \binom{n+d-1}{d-1}.
\]

**Proof.** **Only if:** Since \( R \) is Gorenstein, by Proposition 3.5, we have

\[
\mathcal{N}(q^{n+1}; R) = \sum_{i=1}^{d} r_i(R) \binom{n+i-1}{i-1} + r_0(R)
\]

for all \( n \geq 1 \). On the other hand, we have \( r_i(R) = 0 \) for all \( i < d \) and \( r_d(R) = 1 \), because \( R \) is Gorenstein. Hence

\[
\mathcal{N}(q^{n+1}; R) = \binom{n+d-1}{d-1}.
\]

for all \( n \geq 1 \).

**If:** By the hypothesis, for all distinguished parameter ideals \( q \subseteq m^2 \) we have

\[
(-1)^j f_j(q; R) \leq r_{d-j}(R).
\]

It follows from Theorem 5.8 that \( R \) is sequentially Cohen-Macaulay. By Theorem 3.7, there exists a distinguished parameter ideals \( q \subseteq m^2 \) such that \( \mathcal{N}(q; R) = \sum_{j \in \mathbb{Z}} r_j(R) \).
Apply the Proposition 3.5 to this equation to obtain that

\[ N(q^{n+1}; R) = \sum_{i=1}^{d} r_i(R) \binom{n + i - 1}{i - 1} + r_0(R). \]

Since \( N(q^{n+1}; R) = \binom{n + d - 1}{d - 1} \), we have \( r_i(R) = 0 \) for all \( i \leq d - 1 \) and \( r_d(R) = 1 \). Hence \( R \) is Gorenstein.

In [9, Theorem 5.2], N. T. Cuong, P. H. Quy and first author showed that \( R \) is Cohen-Macaulay if and only if for all parameter ideals \( q \subseteq m^2 \) and \( n \geq 0 \), we have

\[ N(q^{n+1}; R) = r_d(R) \binom{n + d - 1}{d - 1}. \]

Note that the condition of Hilbert function \( N(q^{n+1}; R) \), holding true for all \( n \geq 0 \), is necessary to their proof. The result of Theorem 5.2 in [9] was actually covered in the following result, but in view of the importance of the following result we changed the condition from Hilbert function to Hilbert polynomial.

**Theorem 6.3.** \( R \) is Cohen-Macaulay if and only if for all parameter ideals \( q \subseteq m^2 \) and \( n \gg 0 \), we have

\[ N(q^{n+1}; R) = r_d(R) \binom{n + d - 1}{d - 1}. \]

**Proof.** Only if: Since \( R \) is Cohen-Macaulay, by Proposition 3.5 we have

\[ N(q^{n+1}; R) = \sum_{i=1}^{d} r_i(R) \binom{n + i - 1}{i - 1} + r_0(R) \]

for all \( n \geq 1 \). On the other hand, we have \( r_i(R) = 0 \) for all \( i < d \), because \( R \) is Cohen-Macaulay. Hence

\[ N(q^{n+1}; R) = r_d(R) \binom{n + d - 1}{d - 1}. \]

for all \( n \geq 1 \).

If: By the hypothesis, for all distinguished parameter ideals \( q \subseteq m^2 \) we have

\[ (-1)^j f_j(q; R) \leq r_{d-i}(R). \]

It follows from Theorem 5.8 that \( R \) is sequentially Cohen-Macaulay. By Theorem 3.7 there exists a distinguished parameter ideals \( q \subseteq m^2 \) such that \( N(q; R) = \sum_{j \in \mathbb{Z}} r_j(R) \).

Apply the Proposition 3.5 to this equation to obtain that

\[ N(q^{n+1}; R) = \sum_{i=1}^{d} r_i(R) \binom{n + i - 1}{i - 1} + r_0(R). \]

Since \( N(q^{n+1}; R) = r_d(R) \binom{n + d - 1}{d - 1} \), we have \( r_i(R) = 0 \) for all \( i \leq d - 1 \). Hence \( R \) is Cohen-Macaulay.

In the sequel, we shall only use the following.

**Settings 6.4.** Let \( R \) be a Noetherian local ring with maximal ideal \( m \), \( d = \dim R \geq 2 \).

Assume that \( R \) is unmixed, that is \( \dim \hat{R}/p = d \) for all \( p \in \text{Ass}(\hat{R}) \).

**Theorem 6.5.** \( R \) is regular if and only if \( f_0(m) = 1 \).
Proof. Since $R$ is regular, $R$ is Cohen-Macaulay and $\mathfrak{m}$ is a parameter ideal of $R$. Since $R$ is unmixed, $\mathfrak{m}$ is a distinguished parameter ideal of $R$. Thus by Theorem 5.8 we have $f_0(\mathfrak{m}; R) = r_d(R)$. But $r_d(R) = 1$ because $R$ is regular. Hence $f_0(\mathfrak{m}; R) = 1$.

Conversely, since $R$ is unmixed, we have depth($R$) > 0. Therefore there exists an integer $n_0$ such that $\mathfrak{m}^{n_0+1}: \mathfrak{m} = \mathfrak{m}^n$ for all $n \geq n_0$. It follows that $e_0(\mathfrak{m}; R) = f_0(\mathfrak{m}; R) = 1$. Since $R$ is unmixed, by Theorem 40.6 in [16], $R$ is regular.

□

Theorem 5.8 and 4.8 have a very important corollary. One obvious consequence is the following

Theorem 6.6. The following statements are equivalent.

(1) $R$ is Gorenstein.
(2) For all parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^2$, we have $f_0(\mathfrak{q}; R) = 1$.
(3) For all parameter ideals $\mathfrak{q} \subseteq \mathfrak{m}^2$, we have $e_1(I) - e_1(\mathfrak{q}) \leq 1,$

where $I = \mathfrak{q} : \mathfrak{m}$.

Proof. (1) ⇒ (2) Since $R$ is Gorenstein, $R$ is Cohen-Macaulay. Let $\mathfrak{q}$ be a parameter ideal of $R$. Then we have $\mathcal{N}(\mathfrak{q}; R) = \sum \ell_R((0) : \mathfrak{m}^{n_0}(\mathfrak{q}) \mathfrak{m})$. Since $R$ is unmixed, parameter ideal $\mathfrak{q}$ is distinguished. Then by Proposition 3.5 we have $f_0(\mathfrak{q}; R) = r_d(R)$. But $r_d(R) = 1$ because $R$ is Gorenstein. Hence $f_0(\mathfrak{q}; R) = 1$.

(2) ⇒ (3) Our result in the case in which $e_0(\mathfrak{m}; R) > 1$ is immediate from Lemma 5.2. Thus we suppose henceforth in this proof that $e_0(\mathfrak{m}; R) = 1$. It follows from $R$ is unmixed and Theorem 40.6 in [16] that $R$ is Cohen-Macaulay. Since $R$ is unmixed, every parameter ideals $\mathfrak{q}$ are distinguished. Therefore by Theorem 3.7 there exists an integer $n$ such that for all parameter ideals $\mathfrak{q}$, we have $\mathcal{N}(\mathfrak{q}; R) = \sum_{j \in \mathbb{Z}} r_j(R)$.

It follows from Proposition 4.2 we have $e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}; R) = 1$.

(3) ⇒ (1) Since $e_1(I) - e_1(\mathfrak{q}) \leq 1$ for all parameter ideals $\mathfrak{q}$, by Proposition 4.5 $R$ is Cohen-Macaulay. Since $R$ is unmixed, every parameter ideals $\mathfrak{q}$ are distinguished. Therefore by Theorem 3.7 there exists an integer $n$ such that for all parameter ideals $\mathfrak{q}$, we have $\mathcal{N}(\mathfrak{q}; R) = \sum_{j \in \mathbb{Z}} r_j(R)$. It follows from Proposition 4.2 we have $r_d(R) = e_1(I) - e_1(\mathfrak{q}) = 1$.

Hence $R$ is Gorenstein.

□

Our next result establishes some absolutely fundamental facts about Cohen-Macaulay rings.

Theorem 6.7. The following statements are equivalent.
(1) $R$ is Cohen-Macaulay.
(2) For all parameter ideals $q \subseteq m^2$, we have
\[ f_0(q; R) = r_d(R). \]
(3) For all parameter ideals $q \subseteq m^2$, we have
\[ e_1(I; R) - e_1(q; R) \leq r_d(R), \]
where $I = q : m$.

Proof. (1) $\Rightarrow$ (2) Let $q$ be a parameter ideal of $R$. Then we have $N(q; R) = \sum_{j \in \mathbb{Z}} \ell_R((0) : \text{H}_m^j(R)m)$, because $R$ is Cohen-Macaulay. Since $R$ is unmixed, parameter ideal $q$ is distinguished. Then by Proposition 3.2, we have $f_0(q; R) = r_d(R)$.

(2) $\Rightarrow$ (3) In the case in which $e_0(m; R) > 1$ there is nothing to prove, because of Lemma 5.2 and so we suppose that $e_0(m; R) = 1$. Then by Theorem 40.6 in [16], $R$ is Cohen-Macaulay because $R$ is unmixed. Let $q$ be a parameter ideal of $R$ such that $q \subseteq m^2$ and put $I = q : m$. Then we have
\[ N(q; R) = \sum_{j \in \mathbb{Z}} \ell_R((0) : \text{H}_m^j(R)m), \]

since $R$ is Cohen-Macaulay. It follows from Proposition 4.2 and $q \subseteq m^2$, we have
\[ e_1(I) - e_1(q) = f_0(q; R) = r_d(R). \]

(3) $\Rightarrow$ (1) Since $e_1(I) - e_1(q) \leq r_d(R)$ for all parameter ideals $q \subseteq m^2$, by Proposition 4.3, $R$ is Cohen-Macaulay, as required.

\[ \square \]

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Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road 10307 Hanoi Vietnam

E-mail address: hltruong@math.ac.vn

The Department of Mathematics, Thai Nguyen University of Education. 20 Luong Ngoc Quyen Street, Thai Nguyen City, Thai Nguyen Province, Viet Nam.

E-mail address: hnyen91@gmail.com