DISCRETE SCHRODINGER EQUATION AND ILL-POSEDNESS
FOR THE EULER EQUATION

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(Communicated by Walter Strauss)

ABSTRACT. We consider the 2D Euler equation with periodic boundary conditions in a family of Banach spaces based on the Fourier coefficients, and show that it is ill-posed in the sense that ‘norm inflation’ occurs. The proof is based on the observation that the evolution of certain perturbations of the ‘Kolmogorov flow’ given in velocity by

\[ U(x, y) = \begin{pmatrix} \cos y \\ 0 \end{pmatrix} \]

can be well approximated by the linear Schrödinger equation, at least for a short period of time.

1. Introduction. We consider the 2D Euler equation in vorticity formulation on the torus:

\[ \partial_t \Omega + U \cdot \nabla \Omega = 0, \quad U = \nabla \Delta^{-1} \Omega, \quad x \in \mathbb{T}^2 \]

and we study its local well-posedness in different topologies. Without loss of generality, we will restrict ourselves to mean zero functions on \( \mathbb{T}^2 \), and define the norms

\[ \| \Omega \|_{X^s_p} : = \| |k|^s \cdot \hat{\Omega}(k) \|_{L^p(\mathbb{Z}^2)}, \]

where the Fourier transform is defined by

\[ \hat{f}(k) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x)e^{-ik \cdot x} dx. \]

Our main result is the following

**Theorem 1.1.** Let \( s > 2, \ 1 < p \leq \infty \) and \( p \neq 2 \). There exists a sequence of times \( t_n \to 0 \) and a sequence of initial data \( \Omega_{0,n} \in C^\infty(\mathbb{T}^2) \) with associated solution \( \Omega_n \in C^\infty(\mathbb{T}^2 \times [0, \infty)) \) such that

\[ \| \Omega_{0,n} \|_{X^s_p} \to 0 \quad \text{and} \quad \| \Omega_n(t_n) \|_{X^s_p} \to +\infty. \]

2010 Mathematics Subject Classification. Primary: 35Q31, 76D03; Secondary: 37B55.

Key words and phrases. The Euler equation, ill-posedness, norm inflation, discrete Schrödinger equation, Kolmogorov flow.

The second author is partly funded by the Sloan foundation and by the NSF grant DMS-1362940.

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Thus the flow of (1) is ill-posed in any neighborhood of 0 in $X^s_p$.

**Remark 1.** For $s > 1$, (1) is locally well-posed in $H^s$, thus the above result is false when $p = 2$ and another way to interpret Theorem 1.1 is that for $p > 1$, $s > 2$, the Euler equation (1) is well-posed in $X^s_p$ if and only if $p = 2$.

Let us briefly comment on the norms introduced above. The regularity, $s > 2$, is natural to make sense of the equation when $p = \infty$; for lower values of $p$, this could be lowered, but for simplicity, we did not pursue this since we consider the result strongest for large $s$. As for the summability exponent, the case $p = 2$ simply corresponds to classical Sobolev spaces $H^s$, and the global well-posedness of 2D Euler and Navier-Stokes equations with vorticity in $H^s$ ($s > 1$) is well-known; this goes back to the work of Ebin and Marsden [4] (on compact manifolds) and Kato [7, 8] on $\mathbb{R}^2$. In the case $p = \infty$, $X^\infty_p$ has a geometric flavor, as the unit ball consists of functions whose Fourier coefficients lie in an infinite-dimensional rectangle. Firmly based on this geometric fact, Mattingly and Sinai [9] gave a simple proof of the well-posedness for 2D Navier-Stokes equation by first showing existence and uniqueness in the space $L^\infty([0, T] : X^\infty_p)$ for $s$ large enough. Incidentally, the result of Cheskidov and Shvydkoy [3] can be adapted to show that for the Euler equation, there are initial data from $X^\infty_p$ whose solutions cannot be contained in $C^0([0, T] : X^\infty_p)$ (again for $s$ large enough), but this method fails for $p < \infty$. Hence the spaces based on other values of $p$ can be viewed as interpolation/extrapolation spaces, which avoid certain pathologies of the case $p = \infty$ and it seems natural to ask the question of well-posedness in them.

We refer to [6] for a use of similar norms for a Schrödinger-type equation. There is a large literature on the well-posedness problem of the Euler equation (see [2] for instance), but we only mention a few. Cheskidov-Shvydkoy [3] work in Besov spaces based on $\ell^\infty$ and show existence of initial data whose solutions cannot be continuous in the same norm. While the initial data are explicit, it is unclear how their data evolve, since the proof is based on a contradiction argument. In particular, their proof does not show ‘norm inflation’. Nevertheless, this work has partially motivated ours; indeed, their ill-posedness is due to the fast and non-uniform oscillations in time of certain Fourier modes, a behavior which our approximate solutions explicitly show. On the other hand, the papers of Bourgain and Li [1, 2], and Elgindi and Masmoudi [5] consider $\ell^2$-based spaces (and are thus beyond the scope of our analysis), but rely on the fact that the Riesz transform fails to be bounded on $L^\infty$, while all our spaces are trivially invariant by the Riesz transform (in $x$). In these works, the initial data (and the evolution) are not so explicitly given. We refer to the aforementioned works for the precise notion of well-posedness/ill-posedness as well as a more extensive list of references on the topic.

Our approach is to study flows which are small perturbations of a ‘Kolmogorov state’, given in vorticity by

$$\tilde{\Omega}(x, y) = \sin y,$$

and to show that the linear Schrödinger equation is somehow embedded in the Euler equation, even for arbitrarily short times (see Lemma 1.3 below). Here, a key point

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1Importantly, functions of finite support are not dense in $\ell^\infty$, and one can use this to show that the system of infinite harmonic oscillators $\ddot{x}_k + k^2x_k = 0$ ($k \geq 1$) is ill-posed in $C^0_t\ell^\infty$, a fact closely connected with the negative results in spaces based on $\ell^\infty$. Note however that this system does not exhibit ‘norm inflation’.
is that as we place the initial perturbation further away in the frequency plane, we get to follow the Schrödinger dynamics for a longer period of time. From this, we deduce easily that only the norms which remain controlled along a Schrödinger evolution lead to local well-posedness, thus forcing \( p = 2 \). Incidentally, we have shown that the stationary solution \( \hat{\Omega} \) is highly unstable in the spaces \( X^s_p \) with \( s > 2 \) and \( 1 < p \leq \infty \) with \( p \neq 2 \).

We note in passing that studying other equilibriums, one can realize other equations by the same mechanism; for a simple example, with \( \hat{\Omega}(x,y) = \cos y \), one can get the linear wave equation.

In our opinion, the main interests of the present work are threefold: (i) it shows a strong version of ill-posedness, even for \( p = \infty \), (ii) the observation that \( p = 2 \) is an isolated index for local well-posedness for the Euler equation in \( X^s_p \), (iii) it exhibits solutions to linear dispersive equations inside solutions of the Euler equation. It would be interesting to know if one can also exhibit a classical nonlinear dispersive equation.

1.1. Notations. We define the left and right discrete difference operators on sequences \( \{a_p\}_{p \in \mathbb{Z}} \),

\[
\nabla^L_h : (\nabla^L_h c)_p = h^{-1}(c_p - c_{p-1}), \quad \nabla^R_h : (\nabla^R_h c)_p = h^{-1}(c_{p+1} - c_p),
\]

\[
\Delta_h = \nabla^L_h \nabla^R_h = \nabla^R_h \nabla^L_h : (\Delta_h c)_p = h^{-2}(c_{p+1} - 2c_p + c_{p-1})
\]

and we notice the integration by parts formula:

\[
\sum_{p \in \mathbb{Z}} a_p (\nabla^R_h b)_p = - \sum_{p \in \mathbb{Z}} (\nabla^L_h a)_p b_p.
\]

We will work in the Fourier space and we introduce norms on sequence spaces. Sequences indexed by \( \mathbb{Z} \) will often be denoted by lower case letter, while sequence indexed by \( \mathbb{Z}^2 \) will be denoted by capitalized letters. We define

\[
\|c\|_{h^{-s}(\mathbb{Z}^d)} := \left( \sum_{k \in \mathbb{Z}^d} \left[ 1 + |k|^2 \right]^s |c_k|^2 \right)^{\frac{1}{2}}, \quad \|c\|_{x^s(\mathbb{Z}^d)} := \left( \sum_{k \in \mathbb{Z}^d} \left[ 1 + |k|^2 \right]^s \sum_{\nu \geq s} |c_k|^\nu \right)^{\frac{1}{\nu}}.
\]

When the summation domain is \( \mathbb{Z} \), we will often omit it.

We write \( A \lesssim B \) if there exists an absolute constant \( C > 0 \) such that \( A \leq CB \). Similarly, we write \( A \simeq B \) if \( C_1 B \leq A \leq C_2 B \) for some constants \( C_1, C_2 \). If such a constant depends on some parameters, we will write it explicitly by using a subscript, thus \( A \lesssim_s B \) means that the constant \( C = C(s) \) depends on the parameter \( s \).

1.2. Main reduction. In Fourier space, \([1]\) becomes

\[
\partial_t \hat{\Omega}(k_1, k_2, t) = - \sum_{0 \neq (l_1, l_2) \in \mathbb{Z}^2} \frac{(-l_2, l_1) \cdot (k_1, k_2)}{l_1^2 + l_2^2} \hat{\Omega}(l_1, l_2, t) \hat{\Omega}(k_1 - l_1, k_2 - l_2, t)
\]

We first need a lemma to control our ansatz.

**Lemma 1.2.** Fix \( \varphi \in C^\infty_c((-1,1)) \) and \( 0 < h \ll 1 \) and let \( \sigma \) solve

\[
\partial_t \sigma = i \Delta_h \sigma, \quad \sigma_p(0) = h^\frac{1}{2} \varphi(hp),
\]

then \( \sigma \) is defined for all times. In addition, \( \sigma \) remains close to its continuous analogue: defining \( c \in C^\infty(\mathbb{R} \times \mathbb{R}) \) to be the solution of

\[
\partial_t c = i \partial_{xx} c, \quad c(x,0) = \varphi(x),
\]
and defining its sample by
\[ c_p(t) := h^2 c(hp, t), \]
we have
\[ \|\sigma - c\|_{\mathcal{H}^s} \leq C_{s, \varphi, \epsilon} \cdot h^2 \cdot t(1 + t)^s \cdot \left(\frac{1 + t}{h}\right)^s \]
with a constant \( C_{s, \varphi, \epsilon} > 0 \) depending only on \( s, \varphi \) and \( \epsilon \). In addition, there exist constants \( C_{s, \varphi}^1, C_{s, \varphi}^2 > 0 \) depending only on \( s \) and \( \varphi \) such that so long as \( C_{s, \varphi}^1 \leq t \leq C_{s, \varphi}^2 \cdot h^{-1} \) if \( q \geq 2 \), we have
\[ \|\sigma\|_{\mathcal{H}^s} \simeq_{s, \varphi, \epsilon} \left(\frac{1 + t}{h}\right)^s, \quad \|x^p\|_{\mathcal{H}^s} \simeq_{s, \varphi, \epsilon} \left(\frac{1 + t}{h}\right)^{s + \frac{1}{q} - \frac{1}{2}}. \]

**Remark 2.** Note that by conservation of mass, we also have for all time
\[ \|\sigma(t)\|_{\mathcal{H}(Z)} \lesssim 1. \]

Next, given parameters \( h, k \) and a profile \( \varphi \in C^\infty_c(-1, 1) \), we build an approximate solution \( A \) as follows:
\[ A(0, 1, t) = A(0, -1, t) = i, \]
\[ A(k, p, t) = A(-k, p, t) = \eta \cdot e^{2i(k-1/k)t} \chi(p/k)\sigma_p(\tau t) \]
and all other modes equal to 0, where \( \eta > 0 \) is a normalization constant to be determined later, \( \chi \in C^\infty_c(-2, 2) \) is a bump function satisfying \( \chi(x) = 1 \) for \( |x| \leq 1 \), \( \tau := (k - 1/k)h^2 \) and \( \sigma \) is the solution of \( (4) \) given by Lemma 1.2.

We claim that this gives an approximate solution; more precisely,

**Lemma 1.3.** Given \( k \gg 1, 0 < 1/k \ll h \ll 1, T > 0 \) such that
\[ kh^3 T \leq 1 \]
and a nonnegative, nonzero function \( \varphi \in C^\infty_c(-1, 1) \), let \( A \) be the function defined in \( (9) \). Assume also that \( \tau \geq \tau_0(\varphi) \gg 1 \) is large enough. Using \( (8) \), we have that
\[ \|A\|_{L^{\infty}(0, T; L^2(Z^2))} \lesssim 1 + \eta k^s \]
for any \( s \geq 0 \), and for some constant \( C_{s, \varphi}^1 > 0 \),
\[ \|A(0)\|_{L^2(Z^2)} \simeq_{s, \varphi} 1 + \eta k^s h^{\frac{1}{2} - \frac{s}{4}}, \]
\[ \|A(t)\|_{L^2(Z^2)} \simeq_{s, \varphi} 1 + \eta k^s (h/(1 + \tau t))^{\frac{1}{4} - \frac{s}{2}}, \quad C_{s, \varphi}^1 \tau^{-1} \leq t \leq T. \]
Moreover, the error \( E(A) = E \) defined by
\[ E(k_1, k_2, t) := \partial_t A(k_1, k_2, t) + \sum_{0 \neq (l_1, l_2) \in \mathbb{Z}^2} \frac{(-l_2, l_1) \cdot (k_1, k_2)}{l_1^2 + l_2^2} A(l_1, l_2, t) A(k_1 - l_1, k_2 - l_2, t) \]
satisfies
\[ \|E\|_{L^{\infty}(0, T; L^2(Z^2))} \lesssim_{s, \epsilon} \eta k^s \cdot \left[ \eta k^{1+\epsilon} [\theta^{s+4+\epsilon} + \theta^{2+\epsilon}] + h\theta^s + k^{-1}\theta^2 \right] \]
with \( \theta := \left(\frac{1 + t}{h}\right) \) for any \( s \geq 0 \) and \( \epsilon > 0 \).

The following stability Lemma will let us to produce an exact solution close to \( A \):
Lemma 1.4. For \( s > 2 \) and \( B \in C([0, T] : H^{s+1}) \), we define
\[
E(B) := \partial_t B + \nabla^2 \Delta^{-1} B \cdot \nabla B,
\]
and assume that \( B \) satisfies
\[
\|B\|_{L^\infty([0, T]; H^s)} \leq \beta_0,
\]
\[
\|B\|_{L^\infty([0, T]; H^{s+1})} \leq \beta_1,
\]
\[
\|E(B)\|_{L^1([0, T]; H^s)} \leq \epsilon.
\]
Let \( W \) be the solution of (1) with initial data \( W(0) = B(0) \). There exists a constant \( C_s > 0 \) such that, if we have
\[
\epsilon (1 + \beta_1 T \exp(2\beta_0 T)) \cdot \exp(2\beta_0 T) \leq C_s \beta_0,
\]
then
\[
\|W - B\|_{C^0([0, T]; H^s)} \lesssim \epsilon (1 + \beta_1) \int_0^T \exp(2\beta_0 t) dt \cdot \exp(2\beta_0 T).
\]

The key observation in Lemma 1.4 is that one only needs \( \beta_0 T + \epsilon \beta_1 T \ll 1 \) to have a good approximate solution.

Finally, combining these results, we may prove Theorem 1.1.

Proof of Theorem 1.1. We first treat the case \( 2 < p \leq \infty \). We observe that, using the scaling
\[
\Omega \to \Omega^\lambda, \quad \Omega^\lambda(x, t) = \lambda \Omega(x, \lambda t),
\]
which trivially changes the norm of the initial data, and using the time reversibility nature of our equation, it suffice to prove the following:

Claim b. There exists a sequence of solutions \( \Omega'_n \) and a sequence of times \( t'_n \to 0 \) satisfying that
\[
\|\Omega'_n(0)\|_{X^s_p} \to \infty, \quad \|\Omega'_n(t'_n)\|_{X^s_p} \lesssim 1.
\]

Indeed, one may then set \( \lambda_n := \max\{\|\Omega'_n(0)\|_{X^s_p}, (t'_n)^{-1/2}) \to 0 \) and
\[
\Omega_n(x, t) := \lambda_n \Omega'_n(x, t'_n - \lambda_n t).
\]

To prove Claim b, we fix a nonzero, nonnegative \( \varphi \in C_c(-1, 1) \), a sequence \( k \to +\infty \), and for fixed \( k \), we let
\[
k = \hat{h}^{-3}, \quad \tau \approx \hat{h}^{-1}, \quad t_k = \hat{h}^{2\alpha}, \quad \eta k \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} = 1, \quad \theta \leq \hat{h}^{1+2\alpha}
\]
for \( 1/3 < \alpha < 1/2 \). With \( \sigma \) defined as in (4), we may produce \( A = A_k \) by (9) which satisfies
\[
\|A(0)\|_{X^s_p(\mathbb{Z}^2)} \approx 1 + \eta k \hat{h}^{\frac{1}{2} - \frac{1}{p}} = 1 + \eta k \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} \cdot \hat{h}^{1+\frac{1}{2}}(\theta^{1+2\alpha}) \to +\infty,
\]
\[
\|A(t_k)\|_{X^s_p(\mathbb{Z}^2)} \lesssim 1 + \eta k \hat{h}^{\frac{1}{2} - \frac{1}{p}}(1 + \tau t_k)^{-\frac{1}{2}} \lesssim 1 + \eta k \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} \lesssim 1,
\]
\[
\|A\|_{L^\infty([0, t_k]; H^s(\mathbb{Z}^2))} \lesssim 1 + \eta k \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} \cdot \hat{h}^{\alpha - 1} k \hat{h}^{-s},
\]
and finally
\[
\|E(A)\|_{L^1([0, t_k]; H^s(\mathbb{Z}^2))}
\]
\[
\lesssim t_k \cdot \eta \hat{h}^{\frac{1}{2} - \frac{1}{p}} \left( \eta k \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} + \hat{h}^{1+\alpha + \epsilon + s} + \hat{h}^{1+2\alpha} \right)
\]
\[
\lesssim t_k \cdot \left( \eta \hat{h}^{2(1-\alpha)(\frac{1}{2} - \frac{1}{p})} + \hat{h}^{1+\alpha + \epsilon + s} + \hat{h}^{1+2\alpha} \right)
\]
\[
\lesssim k^{-1} \hat{h}^{2\alpha}
\]
\[ \beta_0 = h^{\alpha - 1}, \quad \beta_1 = k \cdot h^{\alpha - 1}, \quad \varepsilon = k^{-1} k^{2 \alpha}, \quad \beta_0 t_k = h^{3 \alpha - 1} \to 0, \quad \varepsilon \beta_1 = h^{3 \alpha - 1} \to 0 \]

and, letting \( \Omega'_k \) be the solution of \( (1) \) with initial data \( \hat{\Omega}'_k(0) = A_k(0) \), we see that
\[
\| \hat{\Omega}'_k - A_k \|_{L^\infty([0, t_k], x^s_p(\mathbb{Z}^2))} \leq \| \hat{\Omega}'_k - A_k \|_{L^\infty([0, t_k], h^s(\mathbb{Z}^2))} \to 0
\]
and therefore \( \Omega'_k \) satisfies **Claim b.**

We now treat the case \( 1 < p < 2 \). This time, we will construct \( \Omega'_n \) and \( t'_n \to 0 \) in a way that
\[
\| \Omega'_n(0) \|_{X^s_p} \lesssim 1, \quad \| \Omega'_n(t'_n) \|_{X^s_p} \to +\infty \tag{14}
\]
and again a simple scaling argument will establish the result.

We let \( \delta = 1/2p \) and fix \( \alpha > 0 \) a small parameter such that
\[ 2\delta + 5s\alpha \leq 1. \]

This time, we set
\[ h = k^{-2\alpha}, \quad t_k = k^{5\alpha - 1}, \quad \tau \simeq k^{1 - 4\alpha}, \quad \eta k^s h^{\frac{1}{2} - \frac{\delta}{2}} = 1, \quad \theta \leq k^{3\alpha - 1} \]
and again using \( \sigma \) in \( (4) \), we may produce \( A = A_k \) by \( (9) \) which satisfies
\[
\| A(0) \|_{X^s_p(\mathbb{Z}^2)} \simeq 1,
\| A(t_k) \|_{X^s_p(\mathbb{Z}^2)} \simeq 1 + \eta k^s h^{\frac{1}{2} - \frac{\delta}{2}} \left( 1 + \tau t_k \right)^{-\left( \frac{1}{2} - \frac{\delta}{2} \right)} \gtrsim k^{-\alpha \left( \frac{1}{2} - \frac{\delta}{2} \right)} \to +\infty,
\| A \|_{L^\infty([0, t_k], h^s(\mathbb{Z}^2))} \lesssim 1 + \eta k^s \lesssim 1 + k^{\frac{s}{2} - \frac{\delta}{2}},
\]
and
\[
\| E(A) \|_{L^1([0, t_k], h^{s + 2\delta}(\mathbb{Z}^2))} \lesssim t_k \cdot \eta k^{2\delta} \left[ \eta k^{1+\varepsilon} \left( k^{(3\alpha - 1)(4 + s + \varepsilon) + k^{(3\alpha - 1)(2+\varepsilon)}} + h k^{s(3\alpha - 1)} + k^{-1} k^{2(3\alpha - 1)} \right) \right] \lesssim t_k \cdot k^{2\delta} k^{-2} k^{3\alpha} \lesssim k^{-2}
\]
according to Lemma [1.3].

We may now control the difference between \( A \) and the actual solution in \( L^\infty([0, t_k], h^{s + 2\delta}(\mathbb{Z}^2)) \) by using Lemma [1.4] with
\[ \beta_0 = k^{2\delta}, \quad \beta_1 = k \cdot k^{2\delta}, \quad \varepsilon = k^{-2}, \quad \beta_0 t_k = k^{5\alpha + 2\delta - 1} \to 0, \quad \varepsilon \beta_1 = k^{2\delta - 1} \to 0 \]
and we deduce that
\[
\| \hat{\Omega}_k - A \|_{L^\infty([0, t_k], x^s_p(\mathbb{Z}^2))} \lesssim \| \hat{\Omega}_k - A \|_{L^\infty([0, t_k], h^{s + 2\delta}(\mathbb{Z}^2))} \to 0,
\]
which establishes \( (14) \). \( \Box \)

**Remark 3.** Inspecting the proof, the problem when \( p = 1 \) comes form the fact that the growth in the approximate solution is provided in terms of \( \tau t_k \simeq kh^2 t_k \to +\infty \), while to control the difference between the approximate solution and the real solution via Sobolev inequality, we need to control the \( h^{s+s'} \)-norm, for \( s' > 2(1/p - 1/2) \) which leads to use stability in \( h^{s+s'} \). Then \( \beta_0 \simeq h^{s'} \cdot h^{1/2} \) has to satisfy \( \beta_0 t_k = k^{s'} h^{\frac{1}{2} t_k} \to 0 \). When \( s' \geq 1 \), this is no longer covered by our simple analysis.
2. Discrete Schrödinger equations.

Proof of Lemma 1.2. The difference \( d := \sigma - c \) satisfies
\[
(\partial_t - i\Delta) d = e, \quad d(0) = 0,
\]
where
\[
e(p, t) = i\hbar^{\frac{1}{2}} \left( \frac{\epsilon(hp + \hbar, t) - 2\epsilon(hp, t) + \epsilon(hp - \hbar, t)}{\hbar^2} - \partial_{xx} \epsilon(hp, t) \right),
\]
and a Taylor expansion gives that
\[
|e(p, t)| \lesssim \hbar^2 \cdot \hbar^{\frac{1}{2}} \cdot \sup_{[h(p-1), h(p+1)]} |\partial_x^{4+\alpha} \epsilon(\cdot, t)|.
\]
Similarly, we get
\[
|\nabla_\alpha \epsilon(p, t)| \lesssim \hbar^2 \cdot \hbar^{\frac{1}{2}} \cdot \sup_{[h(p-\alpha-1), h(p+\alpha+1)]} |\partial_x^{4+\alpha} \epsilon(\cdot, t)|,
\]
where \( \nabla_\alpha \) denotes some \( \alpha \) copies of \( \nabla_\hbar \) and \( \nabla_\hbar^L \).

Using the Fraunhofer formula for \( \epsilon \) in (24), we may apply Lemma 2.1 with \( \kappa = \hbar^2 \), \( \alpha_* \geq s + 1 \) to deduce (6).

We then obtain (7) from (6) and (22), (23). To see this, we first note that
\[
\|c\|_{x^q} \sim_{s, \varphi} \left( \frac{1 + t}{\hbar} \right)^{s + \frac{1}{2} - \frac{1}{q}}.
\]
Indeed, from the expression for \( \epsilon \) in (22), we see directly that the main term gives above estimate, while the contribution from the remainder term \( r \) is smaller by a factor of \( 1 / (1 + t) \) up to a constant depending only on \( s \) and \( \varphi \), and hence this contribution can be made arbitrarily small by choosing \( C_{s, \varphi}^1 \) large if necessary.

Therefore, for \( 2 \leq q \leq \infty \),
\[
\|\sigma\|_{x^q} - \|c\|_{x^q} \lesssim \|\sigma - c\|_{x^q} \lesssim \|\sigma - c\|_{h^s} \lesssim \left( \frac{1 + t}{\hbar} \right)^{s + \frac{1}{2} - \frac{1}{q}} \cdot \left( \frac{1 + t}{\hbar} \right)^{\frac{1}{2} - \frac{1}{q}} \hbar^2 (1 + t)^{1 + \epsilon} \leq \frac{1}{2} \|c\|_{x^q}.
\]
while for \( 1 \leq q \leq 2 \), using Hölder’s inequality,
\[
\|\sigma\|_{x^q} - \|c\|_{x^q} \lesssim \|\sigma - c\|_{h^{s+\frac{1}{2} - \frac{1}{q}}} \lesssim \left( \frac{1 + t}{\hbar} \right)^{s + \frac{1}{2} - \frac{1}{q}} \cdot \left( \frac{1 + t}{\hbar} \right)^{\frac{1}{2} - \frac{1}{q}} \hbar^2 (1 + t)^{1 + \epsilon} \leq \frac{1}{2} \|c\|_{x^q},
\]
by choosing a smaller constant \( C_{s, \varphi}^2 \) if necessary.

Lemma 2.1. Let \( d \) be the solution of (15) with error \( e \) satisfying
\[
|\nabla_\hbar^\alpha e_p(t)| \leq \kappa \cdot \frac{\hbar^\frac{1}{2}}{\sqrt{1 + \frac{t}{(hp/1)^2} + 1}}, \quad 0 \leq \alpha \leq \alpha_*.\]
for \( t \geq 0 \) and \( \alpha_\star \in \mathbb{N} \), where \( \nabla^p_h \) means any application of \( \alpha \) copies of \( \nabla^R_h \), \( \nabla^k_h \). Then, for all integer \( 0 \leq s \leq \alpha_\star - 1 \) and small \( \epsilon > 0 \), there holds that

\[
\| \{ (hp)^{s} d_p(t) \} \|_{L^{s}_{t}} \lesssim_{\alpha_\star, \epsilon} \kappa \cdot t (1 + t)^{s + \epsilon}.
\]

Proof of Lemma 2.1. We may assume \( \kappa = 1 \). For \( Nh \geq 1 \), we observe that

\[
\| (\nabla^R_h)^{\alpha} e \|_{L^{\infty}((j \geq N))} \leq C \left[ 1 + Nh/(1 + t)^{1 - 2\alpha_\star} \right]^1.
\]

for some constant \( C = C(\alpha) \).

We pick a monotonically increasing function \( \chi \in C^\infty[0, \infty) \) which satisfy \( \chi \leq 1 \) and \( \chi'(x) \leq 2\chi(2x) \). Then we define the quantities

\[
E^p_N(t)^2 := \| (\nabla^R_h)^p d \|_{L^2}\|^2,
\]

\[
E_{\geq N}(t)^2 := \sum_{j \geq 0} \chi^2(\tfrac{j}{N}) \left\| (\nabla^R_h)^p d_j \right\|^2, \quad E_{\leq -N}(t)^2 := \sum_{j \leq 0} \chi^2(\tfrac{j}{N}) \left\| (\nabla^R_h)^p d_j \right\|^2.
\]

Applying the time derivative,

\[
\partial_t (E_{\geq N}(t)^2) = \sum_j \chi^2(\tfrac{j}{N}) \partial_t \| (\nabla^R_h)^p d_j \|^2 = I + II
\]

\[
I = -2 \sum_j \chi^2(\tfrac{j}{N}) \Im \left\{ (\nabla^R_h)^p d_j \overline{(\nabla^R_h)^p d_j} \right\}
\]

\[
II = 2 \sum_j \chi^2(\tfrac{j}{N}) \Re \left\{ (\nabla^R_h)^p e_j \overline{(\nabla^R_h)^p d_j} \right\}.
\]

Applying Cauchy-Schwartz, and using (17), we directly see that

\[
|I| \leq 2E_{\geq N}(t)^2 \| (\nabla^R_h)^p e \|_{L^2((j \geq N/2))} \leq C \cdot E_{\geq N}(t)^2 \left[ 1 + (Nh/(1 + t))^{1 - 2\alpha_\star} \right], \tag{18}
\]

while using (2), the mean-value theorem and the fact that \( \chi'(x) \leq 2\chi(2x) \), we get

\[
I = 2 \sum_j \Im \left\{ (\nabla^R_h)^p d_j \overline{(\nabla^R_h)^p d_j} \right\} \frac{1}{Nh} \left( \chi^2(\tfrac{j + 1}{N}) - \chi^2(\tfrac{j}{N}) \right)
\]

\[
\leq 2 \sum_j \left\| (\nabla^R_h)^p d_j \overline{(\nabla^R_h)^p d_j} \right\| \frac{1}{Nh} \chi'(\tfrac{j + \xi}{N}) 2\chi(\tfrac{j}{N}) \quad \text{(for some} \ 0 \leq \xi \leq 1 \text{)}
\]

\[
\leq \frac{8}{Nh} \sum_j \left\| (\nabla^R_h)^p d_j \overline{(\nabla^R_h)^p d_j} \right\| \chi(\tfrac{j + \xi}{N}) \chi(\tfrac{j}{N}).
\]

Combining (18) and (19), we conclude that

\[
(Nh)\partial_t E_{\geq N}(t) \leq C \left( E_{\geq N/2}(t)^2 + Nh [1 + Nh/(1 + t)]^{\frac{1}{2} - 2\alpha_\star} \right)
\]

and similarly,

\[
\partial_t E^{\alpha} \leq \| (\nabla^R_h)^p e \|_{L^2} \leq 10.
\]

Since \( E^{\alpha}(0) = 0 \), we deduce that \( E^{\alpha}(t) \leq 10t \).

Observe now that, as long as \( p \leq \alpha_\star \), there holds that

\[
(t/Nh)^p + 1/(Nh)^{\alpha_\star} \gtrsim [1 + Nh/(1 + t)]^{-\alpha_\star - 1}.
\]
Therefore, since \( E^{\alpha}(0) = 0 \), using (20), we successively find that
\[
E^{\alpha_*}(t) \leq 10t,
\]
\[
E^{\alpha_*-1}(t) \lesssim t \left[ t/Nh + (Nh)^{-\alpha_*} \right],
\]
\[
E^{\alpha_*-p}(t) \lesssim t \left[ (t/Nh)^p + (Nh)^{-\alpha_*} \right], \quad 0 \leq p \leq \alpha_*.
\]
Similarly, we find that
\[
E^{\alpha_*-p}(t) \lesssim t \left[ (t/Nh)^p + (Nh)^{-\alpha_*} \right], \quad 0 \leq p \leq \alpha_*.
\]
In particular, for \( N \geq \hbar^{-1} \) and for integer \( 0 \leq \alpha \leq \alpha_* \),
\[
\|d_j\|_{L^2(\{\xi_j \geq N\})} \leq C_{\alpha_*} t \left( \frac{1 + t}{N\hbar} \right)^{\alpha}
\]
and by convexity, we may extend this for real \( 0 \leq \alpha \). Letting \( N \) range over dyadic integers bigger than \( \hbar^{-1} \), and applying the above estimate for \( \alpha = s + \epsilon \), we find that
\[
\|d_j\|_{h^{-1}(2)} \leq \hbar^{-s}\|d_j\|_{L^2(\{\xi_j \leq 2\hbar^{-1}\})} + \sum_{N \geq \hbar^{-1}} N^{s}\|d_j\|_{L^2(\{\xi_j \geq N\})}
\]
\[
\leq C \left( \hbar^{-s} + \sum_{N \geq \hbar^{-1}} N^{-t}(1 + t)^{s+\epsilon}\hbar^{-s} \right) \leq C_{\epsilon} t(1 + t)^{s+\epsilon}\hbar^{-s}
\]
so that we finally obtain (16).

Finally, we recall the well-known Fraunhofer formula:

**Lemma 2.2.** Assume that \( \varphi \in C^\infty_c(-1,1) \) and let \( p, w \in \mathbb{N} \). Then we have the Fraunhofer formula for \( \xi \), the solution of (17) with initial data \( \varphi \):
\[
\xi(x,t) = \frac{e^{iyt^2}}{(4\pi it)^{\frac{1}{2}}}(\hat{\varphi}(\frac{x}{2t})) + v(x,t),
\]
with estimates
\[
|\partial^s_x v(x,t)| \leq C_{p,w}(\varphi) \left[ \frac{1}{\sqrt{1+t}} \right]^{w}, \quad t \geq 1
\]
\[
|\partial^s_x \xi(x,t)| \leq C_{p,w}(\varphi) \left[ \frac{1}{\sqrt{1+t}} \right]^{w}.
\]

**Proof of Lemma 2.2.** If \( 0 \leq t \leq 1 \), the bound for \( \xi \) follows directly from computing \( \mathcal{F}^{-1}e^{it|\xi|^2}\hat{\varphi} \). Assume now that \( t \geq 1 \). Using the explicit integral kernel for the Schrödinger equation, we observe that
\[
v(x,t) = \frac{e^{iyt^2}}{(4\pi it)^{\frac{1}{2}}} \int_{[-1,1]} e^{-i\frac{\varphi}{2t}} \cdot \varphi(y)(e^{iyt^2} - 1)dy
\]
and since, on the support of integration
\[
|e^{iyt^2} - 1| \leq \frac{2}{1+t}, \quad -i\frac{x}{2t}e^{-i\frac{\varphi}{2t}} = \partial_y(e^{-i\frac{\varphi}{2t}}),
\]
integrations by parts give the result.
3. Approximate solutions to Euler.

Proof of Lemma 1.3. For convenience we set
\[ A(k, p, t) = \eta e^{2i(k-1/k)t} \chi(p/k) \sigma_p(\tau t) = \eta \cdot b_p(t), \]
so that for all time, we have the energy bound \( \sum_p |b_p(t)|^2 \lesssim 1 \). Since \( \{b_p\} \) is supported on \( |p| \leq 2k \), we deduce
\[ \|A\|_{L^\infty((0,T);H^s)} \lesssim 1 + \eta k^s \]
for any \( s \geq 0 \) and \( T > 0 \). We observe that \( \tau th \leq 1 \), and therefore, the bounds on \( A \) follow directly from (7) since
\[ ||\sigma||_{\ell^s(\mathbb{Z})} - ||\sigma||_{\ell^s(\mathbb{Z})} \lesssim \kappa^{-1} ||\sigma||_{\ell^s} \lesssim ||\sigma||_{\ell^s} \left( \frac{1 + \tau t}{kh} \right) \lesssim \tau^{-1} ||\sigma||_{\ell^s}. \]
Thus, for \( \tau \) large enough, the two norms are equivalent.

We turn to the task of estimating the error \( E(k_1, k_2, t) \). We begin by splitting it according to the \( k_1 \)-coordinate, since it takes only finitely many values \( 0, \pm k, \) and \( \pm 2k \). In addition, since the solution is real, it suffices to treat the cases \( k_1 = 0, \) and \( 2k \). Let us write \( E = e^0 + e^k + e^{-k} + e^{2k} + e^{-2k} \), where \( e_{k_1}^k(t) = E(k_1, p, t) \). The bound (11) will follow from the estimates (25), (26) and (29) below.

Beginning with \( e^0 \), we compute
\[ e^0_p(t) = kpn^2 \cdot \sum_{|r| \leq 2k, |p-r| \leq 2k} \frac{(p-r)^2 - r^2}{(k^2 + r^2)(k^2 + (p-r)^2)} \chi_f(r) \chi_f \left( \frac{p-r}{k} \right) \sigma_r(\tau t) \sigma_{p-r}(\tau t), \]
and is zero unless \( |p| \leq 4k \). Therefore, we see that
\[ |e^0_p(t)| \lesssim \eta^2 \sum_r \left( |r^3 \sigma_r(\tau t)||\sigma_{p-r}(\tau t)| + |\sigma_r(\tau t)||r(p-r)^3 \sigma_{p-r}(\tau t)| \right) \]
and therefore
\[ ||e^0_p(t)||_{h^s} \lesssim \eta^2 \sum_r \left( |r^3 \sigma_r(\tau t)||\sigma_{p-r}(\tau t)| + |\sigma_r(\tau t)||r(p-r)^3 \sigma_{p-r}(\tau t)| \right) \lesssim \frac{\eta^2}{k^3} \left( \frac{1 + \tau t}{h} \right)^{s+4+\epsilon}. \] \hspace{1cm} (25)

Next,
\[ e^{2k}_p(t) = \eta^2 \sum_{|r| \leq 2k, |p-r| \leq 2k} \frac{k(p-2r)}{k^2 + r^2} \chi_f(r) \chi_f \left( \frac{p-r}{k} \right) \sigma_r(\tau t) \sigma_{p-r}(\tau t), \]
and again, this is zero unless \( |p| \leq 4k \). Again, we see that
\[ |e^{2k}_p(t)| \leq \frac{\eta^2}{k} \sum_r \left( |r \sigma_r(\tau t)||\sigma_{p-r}(\tau t)| + |\sigma_r(\tau t)||r(p-r) \sigma_{p-r}(\tau t)| \right) \]
and therefore
\[ ||e^{2k}_p(t)||_{h^s} \lesssim \frac{\eta}{k} \cdot k^s \cdot ||\sigma(\tau t)||_{h^{2s}} \cdot ||\sigma(\tau t)||_{\ell^2} \lesssim \frac{\eta}{k} \cdot k^s \cdot \left( \frac{1 + \tau t}{h} \right)^{2+\epsilon}. \] \hspace{1cm} (26)

We now compute
\[ e^{k}_p(t) = e^{2i(k-1/k)t} \eta \chi \left( \frac{p}{k} \right) \left\{ \tau \partial_{t} \sigma_p - i(k - \frac{1}{k}) h^2 (\Delta_p \sigma_p \right) \right\} + f^{(1)}_p + f^{(2)}_p, \]
where
\[ f^{(1)}_p := i k \eta e^{2i(k-1/k)t} \left( \sigma_{p-1} - (\chi \left( \frac{p-1}{k} \right) - \chi \left( \frac{p}{k} \right)) - \sigma_{p+1} (\chi \left( \frac{p+1}{k} \right) - \chi \left( \frac{p}{k} \right)) \right) \]
and
We consider

and

In conclusion, from (27) and (28) we deduce

\[
|\sigma_{p+1}|(\frac{p+1}{k}) - 2\chi(\frac{k}{p}) + \chi(\frac{p-1}{k})])],
\]

We start with estimating

\[
f_p^{(1)} = i \kappa \eta e^{2i(k-1/k)t} \left[ (\sigma_{p-1} - \sigma_{p+1}) \left( \frac{p}{k} \right) - \chi \left( \frac{p-1}{k} \right) \right]
\]

with

\[
|f_p^{(1)}| \leq C_X \cdot \kappa \eta \left[ \frac{1}{k} |\sigma_{p-1} - \sigma_{p+1}| + \frac{1}{k^2} |\sigma_{p+1}| \right] \cdot \text{1}_{\{|p| \geq k/2\}}, \quad C_X = \|\chi\|_{L^\infty} + \|\chi''\|_{L^\infty}
\]

and therefore

\[
\|f^{(1)}\|_{L^2} \leq C \kappa k^{-s} \left[ |\sigma_{p+1} - \sigma_{p-1}| h^s + k^{-1} |\sigma_{p+1}| h^s \right] \lesssim \eta \left[ \frac{1 + \tau t}{h k} \right]^s.
\]

Similarly,

\[
|f_p^{(2)}| \leq \eta |\sigma_{p+1}| \left[ \frac{1}{k} \right] \frac{k}{k^2 + (p+1)^2} + \frac{1}{k} \left[ \chi \left( \frac{p+1}{k} \right) - \chi \left( \frac{p}{k} \right) \right]
\]

and therefore

\[
\|f^{(2)}\|_{L^2} \leq \eta \left[ \frac{2C}{k^{s+2}} \|\sigma\|_{h^s} + \frac{2}{k^2} \|\sigma\|_{h^s} \right] \leq \frac{C \eta}{k} \left[ \frac{1 + \tau t}{h k} \right]^s + \left( \frac{1 + \tau t}{h k} \right)^2.
\]

In conclusion, from (27) and (28) we deduce

\[
\|e^k_{p}\|_{h^s} \lesssim k^s \|e^k_{p}\|_{L^2} \lesssim \eta k^s \cdot \left[ \frac{1 + \tau t}{h k} \right]^s + \frac{1}{k} \left( \frac{1 + \tau t}{h k} \right)^2.
\]

\[\square\]

**Proof of Lemma [L4]**. We consider \( D = W - B \), which satisfies the equation

\[
\partial_t D + \nabla^\perp \Delta^{-1} W \cdot \nabla D = -E - \nabla^\perp \Delta^{-1} D \cdot \nabla B.
\]

We first estimate the \( H^{s-1} \)-norm of \( D \). We will use the following two simple estimates (direct in Fourier space)

\[
\|\nabla^{\alpha} (FG) - F \nabla^{\alpha} G\|_{L^2} \lesssim \|F\|_{H^{s+1}} \|G\|_{H^{s-1}},
\]

\[
\|\nabla^{\alpha} (FG)\|_{L^2} \lesssim \|F\|_{H^s} \|G\|_{H^s}, \quad \alpha > 1
\]

where \( \nabla^{\alpha} \) corresponds to multiplication of the Fourier coefficients by \(|k|^\alpha \). Letting

\[
D_\alpha = |\nabla|^{\alpha} D, \quad B_\alpha = |\nabla|^{\alpha} B, \quad E_\alpha = |\nabla|^{\alpha} E
\]
we deduce that
\[ \partial_t D_\alpha + \nabla \cdot \Delta^{-1} W \cdot \nabla D_\alpha = -E_\alpha + \mathcal{E}_\alpha, \]
\[ \mathcal{E}_\alpha = -|\nabla|^\alpha (\nabla \cdot D \cdot B) \]
\[ + |\nabla|^\alpha (\nabla \cdot W \cdot \nabla D) - \nabla \Delta^{-1} W \cdot \nabla D_\alpha, \tag{31} \]
and from (30) we obtain that
\[ \|\mathcal{E}_{s-1}\|_{L^2} \lesssim (\|D\|_{H^{s-1}} + \|B\|_{H^s})\|D\|_{H^{s-1}}. \]

Multiplying both sides of the first line of (31) by \( D \) and integrating over \( T^2 \) gives
\[ \partial_t \|D\|_{H^{s-1}} \lesssim (\|D\|_{H^{s-1}} + \|B\|_{H^s}) \cdot \|D\|_{H^{s-1}} + \|E\|_{H^{s-1}}. \tag{32} \]

Since \( D = 0 \) at \( t = 0 \), we may take the maximal time \( 0 < T_1 \leq T \) where \( \|D\|_{H^{s-1}} \leq \beta_0 \) on \([0, T_1]\). Then for \( 0 < t \leq T_1 \), we have the bound
\[ \partial_t \|D\|_{H^{s-1}} \lesssim 2\beta_0 \|D\|_{H^{s-1}} + \|E\|_{H^{s-1}}, \]
and by Gronwall’s inequality we obtain on \( 0 < t \leq T_1 \) the bound
\[ \|D(t)\|_{H^{s-1}} \leq \int_0^t \exp(C_s \beta_0 (t - t')) \|E(t')\|_{H^{s-1}} dt' \leq \exp(C_s \beta_0 t) \varepsilon, \tag{33} \]
but from our assumption we deduce \( \|D(T_1)\|_{H^{s-1}} < \beta_0 \). Hence, we conclude that the estimate (33) is valid for \( 0 < t \leq T \).

Next, we estimate \( D \) in \( H^s \). We proceed as before except that we decompose
\[ \mathcal{E}_s = -\nabla \Delta^{-1} D \cdot \nabla B_\alpha + \mathcal{E}_s', \]
\[ \mathcal{E}_s' = -\sum (|\nabla|^\alpha (\nabla \cdot D \cdot B) - \nabla \Delta^{-1} D \cdot \nabla B_\alpha) \]
\[ + |\nabla|^\alpha (\nabla \Delta^{-1} W \cdot \nabla D) - \nabla \Delta^{-1} W \cdot \nabla D_\alpha. \]

Using (30) again,
\[ \|\nabla \Delta^{-1} D \cdot \nabla B_\alpha\|_{L^2} \lesssim \|D\|_{H^s} \|B\|_{H^{s+1}}, \]
\[ \|\mathcal{E}_s'\|_{L^2} \lesssim (\|D\|_{H^s} + \|B\|_{H^s}) \|D\|_{H^s} \]
and again, multiplying by \( D_\alpha \) and integrating, we conclude that
\[ \partial_t \|D\|_{H^s} \lesssim (\|D\|_{H^s} + \|B\|_{H^s}) \cdot \|D\|_{H^s} + \|D\|_{H^{s-1}} \|B\|_{H^{s+1}} + \|E\|_{H^s}. \tag{34} \]

We plug in the bound we have obtained for \( \|D\|_{H^{s-1}} \) and restrict to the maximal time interval \([0, T_2]\) with \( T_2 \leq T \) on which \( \|D\|_{H^s} \leq \beta_0 \). Then on this time interval, we obtain
\[ \partial_t \|D\|_{H^s} \lesssim 2\beta_0 \|D\|_{H^s} + \varepsilon \beta_1 \exp(2\beta_0 t) + \|E\|_{H^s}. \]
Gronwall’s inequality gives
\[ \|D(t)\|_{H^s} \lesssim \exp(2\beta_0 t) \int_0^t \beta_1 \varepsilon \exp(2\beta_0 t') + \|E(t')\|_{H^s} dt' \]
\[ \lesssim \varepsilon (1 + \beta_1 \int_0^t \exp(2\beta_0 t') dt') \exp(2\beta_0 t), \tag{35} \]
for \( 0 < t \leq T_2 \), and with our assumptions, we can show that \( T_2 = T \) as previously.

**Acknowledgments.** We thank T. Elgindi and Ya. Sinai for helpful discussions.
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Received November 2015; revised September 2016.

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