EXTREMES OF GAUSSIAN CHAOS PROCESSES WITH TREND

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Abstract: Let $X(t) = (X_1(t), \ldots, X_d(t)), t \in [0, S]$ be a Gaussian vector process and let $g(x), x \in \mathbb{R}^d$ be a continuous homogeneous function. In this paper we are concerned with the exact tail asymptotics of the chaos process $g(X(t)) + h(t), t \in [0, S]$ with trend function $h$. Both scenarios $X(t)$ is locally-stationary and $X(t)$ is non-stationary are considered. Important examples include the product of Gaussian processes and chi-processes.

Keywords: Gaussian chaos; Gaussian vector processes; asymptotic methods; Pickands constant.

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1. Introduction

Let $X(t) = (X_1(t), \ldots, X_d(t)), t \in [0, S], d \geq 1$ be an $\mathbb{R}^d$-valued centered Gaussian vector process with continuous sample paths and let $g(x), x \in \mathbb{R}^d$ be a homogeneous function of order $p > 0$, i.e., $g(cx) = c^p g(x), p \in (0, \infty)$ for any $c > 0, x \in \mathbb{R}^d$. Adopting the terminology of [1], we shall refer to $Y(t), t \in [0, S]$ as the Gaussian chaos process of $X$ with respect to $g$ defined by

$$Y(t) = g(X(t)), \quad t \in [0, S].$$

Throughout in the following we shall assume that $g$ is not-negative, i.e., $g(x_0) > 0$ for some $x_0 \in \mathbb{R}^d$.

The tail asymptotics of the supremum of Gaussian chaos processes has been recently derived in [1, 2], see also [3] for a simpler case. Specifically, if $X_i$'s are centered stationary and independent Gaussian processes with unit variance and common correlation function $r$ satisfying

$$r(t) = 1 - a|t|^\alpha + o(|t|^\alpha), \quad t \to 0, \quad a > 0, \quad \alpha \in (0, 2], \quad r(t) < 1, \quad t \in (0, T],$$

then by [1] (under some restrictions on $g$)

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} Y(t) > u \right\} \sim \mathcal{H}_\alpha a^{1/\alpha} T(u/\tilde{g})^{2/\alpha} \mathbb{P} \{ Y(0) > u \}, \quad u \to \infty,$$

where $\tilde{g} = \max_{e \in S_{d-1}} g(e)$ with $S_{d-1}$ the unit sphere on $\mathbb{R}^d$ and $\mathcal{H}_\alpha =: \mathcal{H}_\alpha^0$ the Pickands constant defined for any $\alpha, \delta$ positive by

$$\mathcal{H}_\alpha^\delta = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in \delta Z \cap [0, T]} e^{\sqrt{2} \alpha B_{\alpha}(t) - |t|^{\alpha}} \right\},$$

where $B_{\alpha}(t), t \in \mathbb{R}$ is a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0, 1]$ and we interpret $\delta Z$ as $\mathbb{R}$ if $\delta = 0$.

In this paper, we are interested in the tail asymptotics of supremum of the Gaussian chaos process $Y(t) = g(X(t))$ where $X_i$'s are some general non-stationary Gaussian processes considering further a trend function $h$, i.e., we shall investigate

$$p_h(u) := \mathbb{P} \left\{ \sup_{t \in [0, T]} (Y(t) + h(t)) > u \right\}$$

as $u \to \infty$. The non-stationary case treated here is quite different from the stationary one already dealt with in the aforementioned reference. Since we allow for a trend function $h$, our results are new even for $X_i$'s being stationary.

The main challenges when dealing with the Gaussian chaos process $Y$ is that the Slepian inequality (see [4] and [5]) does not hold in general. In the particular case of chi-square processes, using the duality of the norms, the
problem in question can be related to that of supremum of a Gaussian random field (see e.g., [4, 6]) and thus Slepian inequality or some modifications of it (see [7, 8]) can still be used.

Organisation of the rest of the paper: In Section 2 we show our main results and some examples. Following are the proofs and some useful lemmas in Section 3 and Section 4, respectively.

2. Main results

Next, we introduce some restriction on $g$, assuming first that

$$\{x : g(x) > 0, x \in \mathbb{R}^d\} \neq \emptyset.$$  \hfill (3)

Consider spherical coordinates $x = (r, \varphi)$, with $\varphi = (\varphi_1, \ldots, \varphi_{d-1}) \in \Pi_{d-1} = [0, \pi)^{d-2} \times [0, 2\pi)$, being the angular coordinates of $v$, and with $r = |v|$. The Jacobian of the mapping is given by

$$J(r, \varphi) = r^{d-1} \sin^{d-2} \varphi_1 \cdots \sin \varphi_{d-2}.$$  \hfill (4)

Write below $g(\varphi) = g(x/|x|)$ and without loss of generalities set

$$\hat{g} := \max_{\varphi \in \Pi_{d-1}} g(\varphi) = \max_{e \in S_{d-1}} g(e),$$

$$M := \{e \in S_{d-1} : g(e) = 1\}, \quad M_\varphi := \{\varphi \in \Pi_{d-1} : g(\varphi) = 1\}.$$  

In the following we shall assume the following condition:

**Condition 2.1.** There exists an $\varepsilon > 0$ such that $g(\varphi)$ is twice continuously differentiable in the neighbourhood

$$M_\varphi(\varepsilon) := \{\varphi \in \Pi_{d-1} : g(\varphi) > \hat{g} - \varepsilon\} \supseteq M_\varphi.$$

Further we consider two different structures of the $M$:

(i) $M$ consists of a finite number of points, $m = 0$, and $|\det g''(\varphi)| > 0$ for every $\varphi \in M_\varphi(\varepsilon)$, where

$$g''(\varphi) := \left[\frac{\partial^2 g(\varphi)}{\partial \varphi_i \partial \varphi_l}\right]_{i,l=1,\ldots,d-1}$$

is the Hessian matrix of $g(\varphi)$.

(ii) $M$ is a smooth $m$-dimensional manifold, $1 \leq m \leq d-2$, and the rank of the matrix $g''(\varphi)$ equals $d-1-m$ for all $\varphi \in M_\varphi(\varepsilon)$.

Next through this paper, we always assume that $g$ satisfied Condition 2.1.

Since $h(t), t \in [0, T]$ is continuous and for $\hat{g} \neq 1$ and $u$ large

$$\mathbb{P}\left\{\sup_{t \in [0,T]} (Y(t) + h(t)) > u\right\} = \mathbb{P}\left\{\sup_{t \in [0,T]} \left(\frac{1}{\hat{g}} g(X(t)) \lor 0\right) + \frac{1}{\hat{g}} b(t)\right\} > \frac{u}{\hat{g}},$$

and $g_1(x) := \frac{1}{\hat{g}} g(x) \lor 0$ with $\max_{e \in S_{d-1}} g_1(e) = 1$ satisfies Condition 2.1 if $g$ satisfies it.

Then next without loss of generalities, hereafter we shall assume that $g$ is non-negative and $\hat{g} = 1$.

2.1. Non-stationary cases. In this section, we assume that $X_i(t), i = 1, \ldots, d$, are i.i.d. centered Gaussian processes with variance function $\sigma^2(t)$ and correlation function $r(s, t)$. Further, we assume that $\sigma(t)$ attains its maximum at point $t_0 \in [0, T]$ over $[0, T]$ with

$$\sigma(t) = 1 - b|t - t_0|^\beta, \quad t \to t_0, b, \beta > 0,$$

and

$$r(t, s) = 1 - a|t - s|^{\alpha} + o(|t - s|^{\alpha}), \quad t, s \to t_0, a > 0, \alpha \in (0, 2].$$  \hfill (6)

$$r(t, s) = 1 - a|t - s|^{\alpha} + o(|t - s|^{\alpha}), \quad t, s \to t_0, a > 0, \alpha \in (0, 2].$$  \hfill (7)
Moreover, for a continuous function \( h(t), t \in [0, T] \), we assume that

\[
(8) \quad h(t_0) - h(t) \sim c|t - t_0|^\gamma, \quad t \to t_0,
\]

where \( c \geq 0 \) and \( \gamma > 0 \) are some positive constants.

In order to avoid unnecessary notation we shall assume in the following that \( t_0 = 0 \) or \( t_0 = T \). In the non-stationary case, another important constant appearing in the asymptotics of supremum of Gaussian processes is the Piterbarg constant defined for some set \( E \subset \mathbb{R} \) and \( a > 0, \alpha \in (0, 2], \delta > 0 \) by

\[
\mathcal{P}_{\alpha, a}^E f = \mathbb{E} \left\{ \sup_{t \in E} e^{r(t)}(t - a|t| - f(t)) \right\}
\]

see [9–17] for various properties of \( \mathcal{H}_\alpha \) and \( \mathcal{P}_{\alpha, a}^f \).

**Theorem 2.2.** Suppose that \( X \) satisfies (6) and (7) and the continuous function \( h \) satisfies (8). Further assume that \( g \) is homogeneous of order \( p \in (0, \infty) \) satisfying (3) and set \( \alpha^* = \alpha p \) and \( \beta^* = \min(\beta_p, \frac{2p}{T - p})I_{p < 2} + \beta_p I_{p \geq 2} \). Then we have

\[
p_h(u) \sim C_{t_0} u^{(\frac{1}{p} - \frac{1}{2})} + \mathbb{P} \{ Y(t_0) > u - h(t_0) \},
\]

where

\[
(9) \quad C_{t_0} = \begin{cases}
\mathcal{H}_\alpha a^\frac{1}{p} \int_0^\infty e^{-f(t)} dt, & \text{if } \beta^* > \alpha^*, \\
\mathcal{P}_{\alpha, a}^f [0, \infty), & \text{if } \beta^* = \alpha^*, \\
1, & \text{if } \beta^* < \alpha^*,
\end{cases}
\]

with \( f(t) = \frac{1}{p} t^\gamma I_{(\beta^* - \frac{p}{2})}, b t^\gamma I_{(\beta^* = \beta_p)}, t \geq 0 \).

**Remarks 2.3.** i) In Theorem 2.2, as can be seen by its proof, when \( p \in [2, \infty) \) the results still hold without (8). Further, we notice that

\[
\mathbb{P} \{ Y(t_0) > u - h(t_0) \} \sim \mathbb{P} \{ Y(t_0) > u \} \begin{cases}
eq e^{h(t_0)}, & \text{if } p = 2, \\
1, & \text{if } p > 2.
\end{cases}
\]

ii) In Theorem 2.2, setting \( h(t_0) = 0 \) and \( c = 0 \), we retrieve the result for \( h(t) = 0, t \in [0, T] \).

iii) If \( t_0 \in (0, T) \), then the above results hold with \( 2\mathcal{H}_\alpha \) instead of \( \mathcal{H}_\alpha \) and \( \mathcal{P}_{\alpha, a}^f (-\infty, \infty) \) instead of \( \mathcal{P}_{\alpha, a}^f [0, \infty) \). The same comments hold for all our results below.

### 2.2. Locally-stationary cases.

In this section, we assume that \( X_i(t), i = 1, \ldots, d \), are i.i.d. centered stationary Gaussian processes with unit variance function and covariance function \( r(t) \) satisfying satisfying

\[
(10) \quad r(t, t + s) = 1 - a(t)|s|^{\alpha} + o(|s|^{\alpha}), \quad s \to 0, \quad \alpha \in (0, 2],
\]

where \( a(t) \) are positive continuous function on \([0, T]\). Further assume

\[
(11) \quad r(t, t + s) < 1, \quad \forall \ t, t + s \in [0, T] \quad \text{and} \quad s \neq 0.
\]

Write below \( h_m \) for the maximum of a continuous function \( h(t), t \in [0, T] \) and define

\[
H := \{ s \in [0, T] : h(s) = h_m \}.
\]

If \( h_m \) is attained as some point \( t_0 \in [0, T] \) it is important to know the behaviour of \( h(t_0) - h(t) \) for \( t \) close to \( t_0 \). Specifically, we shall assume that (8) is satisfied for some \( c > 0 \), and \( \gamma > 0 \).
Theorem 2.4. Suppose that Gaussian vector process $X(t)$ with continuous sample paths satisfies (10) and (11), a homogeneous function $g(x)$ with order $p > 0$ satisfies (3), and $h(t)$ is a continuous function. Further, set $\alpha^* = \alpha p$, and $\beta^* = \frac{2p}{p-1}$.

(1) If $h(t) \equiv 0$, then as $u \to \infty$

$$p_h(u) \sim H_\alpha \int_0^T (a(t))^{1/\alpha} dt u^{\alpha^*} \mathbb{P} \{Y(0) > u\}.$$  

(2) For $p \in (0, 2)$, if $H = \{t_0\}$ and (8) holds for some $c > 0$, $\gamma > 0$, then we have as $u \to \infty$

$$p_h(u) \sim C_{t_0} u^{(\alpha^*/p)} \mathbb{P} \{Y(0) > u - h_m\},$$

where $C_{t_0}$ is the same as in (9) with $f(t) = \frac{t}{p} = 1$ and $a = a(t_0)$.

(3) For $p \in [2, \infty)$, then we have as $u \to \infty$

$$p_h(u) \sim H_\alpha \int_0^T (a(t))^{1/\alpha} e^{\frac{h(t)}{2} \mathbb{P} \{Y(0) > u - h_m\}}.$$  

Remarks 2.5. i) If $H$ consists of $n$ discrete points, say $t_1, \ldots, t_n$, then as mentioned in [4] the tail of the supremum is easily obtained assuming that for each $t_i$ the assumptions of Theorem 2.4 statement i) hold, implying that

$$p_h(u) \sim \left( \sum_{j=1}^n C_{t_j} \right) u^{(\alpha^*/p)} \mathbb{P} \{Y(0) > u - h_m\}. $$

ii) If $H = [A, B] \subset [0, T]$ with $0 \leq A < B \leq T$, then as $u \to \infty$

$$p_h(u) \sim H_\alpha \int_A^B (a(t))^{1/\alpha} dt u^{\alpha^*} \mathbb{P} \{Y(0) > u - h_m\},$$

which is proven in Appendix.

iii) In Theorem 2.4, we investigate that if $p > 2$, then $h$ does not contribute to the asymptotics.

We present next some examples of Gaussian chaos processes.

Example 2.6. i) ($L^p$ norm process) For $g(x) = ||x||_p^p$, $p > 0$ with and

$$||x||_p = \left\{ \begin{array}{ll}
(\sum_{i=1}^d |x_i|^p)^{1/p}, & \rho \in [1, \infty), \\
\max(|x_1|, \ldots, |x_d|), & \rho = \infty,
\end{array} \right.$$  

we have Theorem 2.2 and Theorem 2.4 hold with order $p > 0$.

ii) (Product of several i.i.d Gaussian processes) For $g(x) = \Pi_{i=1}^d x_i$, we have Theorem 2.2 and Theorem 2.4 hold with order $p = d$.

iii) (Maximum of several i.i.d Gaussian processes) For $g(x) = \max_{1 \leq i \leq d} x_i$, we have Theorem 2.2 and Theorem 2.4 hold with order $p = 1$.

3. Proofs

We first give several preliminary lemmas, which play an important role in the proof of Theorem 2.2 and Theorem 2.4. The proof of lemmas are shown in Appendix. In the following we set

$$\xi_{f,u}(t) = (\xi_{1,f,u}(t), \ldots, \xi_{d,f,u}(t))$$

where $\xi_{i,f,u}(t) = \frac{\xi_i(u^{-2/\alpha + t_0})}{1 + u^{-2/\alpha - f_i(t)}}$, where $f_i$’s given function and $\xi(t) = (\xi_1(t), \ldots, \xi_d(t))$ is a centered Gaussian vector process.
Lemma 3.1. Let $X(t) = (X_1(t), \ldots, X_d(t))$, $t \in [0, T], d \geq 1$ be a centered continuous vector process with independent marginals which have unit variances and correlation functions satisfying (11). If $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function such that for some $p > 0$
\[ |g(x)| \leq \|x\|^p, \quad \forall x \in \mathbb{R}^d, \]
where $\|\cdot\|$ is the Euclidean norm and then for $0 < t_1 < t_2 < t_3$ and $u$ large enough
\[ \mathbb{P}\left\{ \sup_{t \in [0, t_1]} Y(t) > u, \sup_{t \in [t_2, t_3]} Y(t) > u \right\} \leq 2\Psi\left( \frac{2u^p - D}{\sqrt{4 - \delta}} \right), \]
where $D$ is a constant. Further, by (57) we have
\[ \mathbb{P}\left\{ \sup_{t \in [0, t_1]} Y(t) > u, \sup_{t \in [t_2, t_3]} Y(t) > u \right\} = o(\mathbb{P}\{Y(0) > u\}), \; u \rightarrow \infty. \]

Lemma 3.2. Let $\xi(t) = (\xi_1(t), \ldots, \xi_d(t))$, $t \in \mathbb{R}, d \geq 1$ be a centered continuous vector process with independent marginals which have unit variances and correlation functions satisfying (10). Further we assume that $g(x), x \in \mathbb{R}^d, d \geq 1,$ is a non-negative homogeneous function satisfying Condition 2.1 with order 1. Set $a = a(t_0), t_0 \in \mathbb{R}$ and $K_u$ a family of index sets and $u_k$ satisfying that
\[ \lim_{u \rightarrow \infty} \sup_{k \in K_u} \left| \frac{u_k}{u} - 1 \right| = 0. \]
If $f_i, i \leq d$ are a continuous functions vanishing at 0, then we have that for some constants $S_1, S_2 \geq 0$ and $S_1 + S_2 > 0$
\[ \lim_{u \rightarrow \infty} \sup_{k \in K_u} \left| \frac{\sup_{t \in [0, u]} g(\xi_{f, u}(t)) > u_k}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} - \mathbb{P}f_{a, a}[-S_1, S_2] \right| = 0. \]
If $\lim_{u \rightarrow \infty} \sup_{k \in K_u} |ku^{-2/\alpha}| \leq T$ some small enough $T > 0$, we have for some positive constant $S$ that when $u$ large enough
\[ H_\alpha[0, (a - \varepsilon T)^{1/\alpha}S] \leq \frac{\sup_{t \in [0, S]} g(\xi(u^{-2/\alpha}t + ku^{-2/\alpha}S + t_0)) > u_k}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} \leq H_\alpha[0, (a + \varepsilon T)^{1/\alpha}S], \]
holds for any $k \in K_u$ where $\varepsilon T \rightarrow 0, a T \rightarrow 0$. Specially, if $T = 0$, we have
\[ \lim_{u \rightarrow \infty} \sup_{k \in K_u} \left| \frac{\sup_{t \in [0, u]} g(\xi(u^{-2/\alpha}t + ku^{-2/\alpha}S + t_0)) > u_k}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} - H_\alpha[0, a^{1/\alpha}S] \right| = 0. \]

Lemma 3.3. Let the Gaussian vector process $\xi(t), t \in \mathbb{R}$ with independent marginals which have common correlation function $r(t)$ satisfying (10) and the homogeneous function $g(x)$ satisfies Condition 2.1 with order 1. Further, for some $t_0 \in [0, T], \alpha = a(t_0)$, and let $K_u$ be a family of countable index sets such that for given positive constants $u_k, k \in K_u$ we have
\[ \lim_{u \rightarrow \infty} \sup_{k \in K_u} \left| \frac{u_k}{u} - 1 \right| = 0. \]
If $\varepsilon_0$ is such that for all $t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]$
\[ \frac{a}{2} |t - t_0|^\alpha \leq 1 - r(t) \leq 2a |t - t_0|^\alpha, \]
then we can find a constant $C$ such that for all $S > 0$ and $T_2 - T_1 > S$ we have
\[ \lim_{u \rightarrow \infty} \sup_{k \in K_u} \frac{\mathbb{P}\{A_1(u_k), A_2(u_k)\}}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} \leq C \exp \left( \frac{a}{8} |T_2 - T_1 - S|^\alpha \right), \]
where $A_i(u_k) = \left\{ \sup_{t \in [T, T+S]} g(\xi(u^{-2/a}(t + kS) + t_0)) > u_k \right\}$, $i = 1, 2$, $a = \inf_{t \in [t_0-c_0, t_0+c_0]} a(t)$ and

$$\lim_{a \to \infty} \sup_{k \in K_a} \left| u^{-2/a} k \right| \leq \varepsilon_0.$$  

In the following proofs, $Q_i, i \in \mathbb{N}$ denote some positive constants which can be different from line to line. Further, in the proofs of Theorem 2.2 and Theorem 2.4, we denote for some sets $\Delta_1, \Delta_2 \subseteq \mathbb{R}$

$$P_u(\Delta_1) = \mathbb{P} \left\{ \sup_{t \in [\Delta_1]} (Y(t) + h(t)) > u \right\}, \quad P_u(\Delta_1, \Delta_2) = \mathbb{P} \left\{ \sup_{t \in [\Delta_1]} (Y(t) + h(t)) > u, \sup_{t \in [\Delta_2]} (Y(t) + h(t)) > u \right\},$$

$$Q_u(\Delta_1) = \mathbb{P} \left\{ \sup_{t \in [\Delta_1]} (Y(t) + h(t)) > u \right\}, \quad Q_u(\Delta_1, \Delta_2) = \mathbb{P} \left\{ \sup_{t \in [\Delta_1]} (Y(t) + h(t)) > u, \sup_{t \in [\Delta_2]} (Y(t) + h(t)) > u \right\}.$$

**Proof of Theorem 2.2:** We present first the proof for $t_0 = 0$. Without loss of generalities we shall assume that $h(0) = 0$.

For all $u$ large, let $\Delta(u) = [0, \delta(u)]$, where

$$\delta(u) = \left( \frac{(\ln u)^q}{u} \right)^{2/b}, \quad q > \max\left( \frac{p}{2}, \frac{p}{2 - p} \right).$$

It follows that for $\theta > 0$ small enough

$$P_u(\Delta(u)) \leq P_u([0, T]) \leq P_u([0, T]) + P_u([\delta(u), \theta]) + P_u([\theta, T]).$$

We first give upper bounds of $P_u([\delta(u), \theta])$ and $P_u([\theta, T])$ which finally imply that as $u \to \infty$

$$P_u([\delta(u), \theta]) = o(P_u(\Delta(u))), \quad P_u([\theta, T]) = o(P_u(\Delta(u))).$$

Set $\overline{X}(t) = (\overline{X}_1(t), \ldots, \overline{X}_d(t))$ with $\overline{X}_i(t) = \frac{X(t)}{\sigma(t)}, i \leq d$ and define

$$h^* = \max_{t \in [0, T]} h(t), \quad \sigma_0 = \sup_{t \in [0, T]} \sigma(t) < 1.$$  

Then by Borell inequality

$$P_u([\theta, T]) \leq \mathbb{P} \left\{ \sup_{t \in [\theta, T]} Y(t) > u - h^* \right\} \leq \mathbb{P} \left\{ \sup_{t \in [\theta, T]} g(\overline{X}(t)) > \frac{u - h^*)^{1/p}}{\sigma_0} \right\} \leq \exp \left( \frac{-(u-h^*)^{1/p}}{\sigma_0} \right)^2$$

$$= \mathbb{E} \left( \sup_{t \in [T, \Delta]} (Y(t), z) \right), \quad u \to \infty,$$

where $Q_0 := \mathbb{E} \left\{ \sup_{t \in [\theta, T]} (\overline{X}(t), z) \right\}$. By (6) and (8), we know that for some $\varepsilon, \varepsilon_1 \in (0, 1)$

$$\frac{(u-h(t))^{1/p}}{\sigma(t)} \geq u^{1/p} + c \left( \frac{1 - \epsilon}{u} \right) |t|^\gamma (1 + (1 - \varepsilon)b |t|^\beta) \geq u^{1/p} \left( 1 + \frac{c(1 - \epsilon)}{u} |t|^\gamma + (1 - \varepsilon)b |t|^\beta \right),$$

and

$$\frac{(u-h(t))^{1/p}}{\sigma(t)} \leq u^{1/p} + c \left( \frac{1 + \epsilon_1}{u} \right) |t|^\gamma (1 + (1 + \varepsilon_1)b |t|^\beta) \leq u^{1/p} \left( 1 + \frac{c(1 + \epsilon_1)}{u} |t|^\gamma + (1 + \varepsilon) b |t|^\beta \right)$$

holds for $t \in [0, \theta]$, then

$$\inf_{t \in [\delta(u), \theta]} \frac{(u - h(t))^{2/p}}{\sigma^2(t)} \geq \inf_{t \in [\delta(u), \theta]} u^{2/p} \left( 1 + \frac{c(1 - \varepsilon)}{u} |t|^\gamma + (1 - \varepsilon)b |t|^\beta \right)^2$$

$$\geq u^{-2/p} + \mathbb{E} \left( \sup_{t \in [\delta(u), \theta]} \frac{a(t)^2}{t^{2/p}} \right).$$

By (7), we have that

$$\mathbb{E} \left\{ \left( (\overline{X}(t))^2 \right) \right\} = 1$$

and

$$\mathbb{E} \left\{ \left( (\overline{X}(t), z) - (\overline{X}(s), z') \right)^2 \right\} \leq 2 \mathbb{E} \left\{ \left( (\overline{X}(t), z) - (\overline{X}(s), z) \right)^2 \right\} + 2 \mathbb{E} \left\{ \left( (\overline{X}(s), z) - (\overline{X}(s), z') \right)^2 \right\}.$$
\begin{align*}
\leq 2\mathbb{E}\left\{(\mathbf{X}(t) - \mathbf{X}(s), z)^2\right\} + 2\mathbb{E}\left\{(\mathbf{X}(s), z - z')^2\right\} \\
\leq Q_2|s - t|^{\alpha} + Q_3 \sum_{i=1}^{d} |z_i - z'_i|^{2} \leq Q_4 \left(|s - t|^{\alpha} + \sum_{i=1}^{d} |z_i - z'_i|^{\alpha}\right)
\end{align*}
holds for \(s, t \in [0, \theta]\) and \(z, z' \in \mathbb{S}_{d-1}\). Then it follows from [4][Theorem 8.1] that
\begin{align*}
P_u \left(\{\delta_u, \theta\} \right) \leq \mathbb{P}\left\{\sup_{t \in [\delta(u), \theta]} g(X(t)) \geq \inf_{t \in [\delta(u), \theta]} \frac{u - h(t)}{\sigma^2(t)}\right\} \leq \mathbb{P}\left\{\sup_{t \in [\delta(u), \theta]} X(t) > \inf_{t \in [\delta(u), \theta]} \frac{(u - h(t))^{1/p}}{\sigma(t)}\right\} \\
\leq Q_5 u^{\frac{2(d+1)}{\alpha}} \Psi \left(\inf_{t \in [\delta(u), \theta]} \frac{(u - h(t))^{1/p}}{\sigma(t)}\right) = o\left(\mathbb{P}\{Y(0) > u\}\right), \ u \to \infty,
\end{align*}
where in the last equation we use the fact that
\[\inf_{t \in [\delta(u), \theta]} \frac{(u - h(t))^{1/p}}{\sigma(t)} - u^{2/p} \geq Q_1 (\ln u)^{\frac{2(d+1)}{p} \epsilon^{2/p}} \to \infty, \ u \to \infty.\]

Next, we give the asymptotic of \(P_u (\Delta(u))\), as \(u \to \infty\). Set for any \(S > 0\),
\begin{align*}
I_k(u) &= [ku^{-2/\alpha^*} S, (k + 1)u^{-2/\alpha^*} S], \ k \in \mathbb{N}, \ N(u) = \left[\left(\ln u\right)^{\frac{2d}{p}} u^{\frac{1}{2\alpha^*} - \frac{1}{2p}} S^{-1}\right]. \\
G_{u, + \epsilon}(k) &= u^{1/p} \left(1 + \frac{c(1 + \epsilon)}{\alpha^*}\right) \left|\left[(k + 1)u^{-2/\alpha^*} S\right]^{\gamma} + (1 + \epsilon)b \left|\left[(k + 1)u^{-2/\alpha^*} S\right]^{\alpha}\right|\right), \\
G_{u, - \epsilon}(k) &= u^{1/p} \left(1 + \frac{c(1 - \epsilon)}{\alpha^*}\right) \left|\left[ku^{-2/\alpha^*} S\right]^{\gamma} + (1 - \epsilon)b \left|\left[ku^{-2/\alpha^*} S\right]^{\alpha}\right|\right).
\end{align*}

**Case 1:** \(\beta^* > \alpha^*\). For \(u\) large enough we have
\begin{equation}
\sum_{k=0}^{N(u)} P_u (I_k(u)) \geq P_u (\Delta(u)) \geq \sum_{k=0}^{N(u)-1} P_u (I_k(u)) - \sum_{i=1}^{2} \Lambda_i(u),
\end{equation}
where
\[\Lambda_1(u) = \sum_{k=0}^{N(u)} P_u (I_k(u), I_{k+1}(u)), \ \Lambda_2(u) = \sum_{0 \leq k, l \leq N(u), l \geq k+2} P_u (I_k(u), I_l(u)).\]
Set \(\tilde{g}(x) = g^{1/p}(x)\), then \(\tilde{g}(x)\) is homogeneous function with order 1 and satisfies Condition 2.1. Then in light of Lemma 3.2 and (24), we have that for some \(\epsilon \in [0, 1]\),
\begin{align*}
\sum_{k=0}^{N(u)} P_u (I_k(u)) \leq \sum_{k=0}^{N(u)} \mathbb{P}\left\{\sup_{t \in I_k(u)} \tilde{g}(X(t)) > G_{u, - \epsilon}(k)\right\} \\
= \sum_{k=0}^{N(u)} \mathbb{P}\left\{\sup_{t \in [\delta(u), \theta]} \tilde{g}(X(u^{-2/\alpha^*} t + ku^{-2/\alpha^*} S)) > G_{u, - \epsilon}(k)\right\} \\
\sim h_0 H_{\alpha}[0, a^{1/\alpha} S] u^{\frac{m-3}{p}} \sum_{k=0}^{N(u)} \exp \left(-\frac{1}{2} (G_{u, - \epsilon}(k))^2\right) \\
\sim h_0 H_{\alpha}[0, a^{1/\alpha} S] u^{\frac{m-3}{p}} \exp \left(-\frac{u^2}{2}\right) \\
\times \sum_{k=0}^{N(u)} \exp \left(-\frac{c}{p}(1 - \epsilon - \epsilon) u^{2/p} |k S u^{-\frac{1}{\alpha^*}}|^\gamma - b(1 - \epsilon - \epsilon) u^{2/p} |k S u^{-\frac{1}{\alpha^*}}|^\beta\right) \\
\sim h_0 H_{\alpha}[0, a^{1/\alpha} S] u^{\frac{m-3}{p}} \exp \left(-\frac{u^2}{2}\right) \sum_{k=0}^{N(u)} \exp \left(-(1 - \epsilon - \epsilon) f(u^\frac{2}{\alpha^*} |k S u^{-\frac{1}{\alpha^*}}|^\beta)\right) \\
\sim P \{Y(0) > u\} a^{1/\alpha} H_{\alpha}[0, a^{1/\alpha} S] u^{\frac{m-3}{p}} \int_0^\infty \exp \left(-(1 - \epsilon - \epsilon) f(t)\right) dt \\
\sim P \{Y(0) > u\} a^{1/\alpha} H_{\alpha} u^{\frac{m-3}{p}} \int_0^\infty e^{-f(t)} dt,
\end{align*}
(27)
as \( u \to \infty, S \to \infty, \varepsilon \to 0, \epsilon \to 0 \) where \( f(t) = \frac{c}{p} |t|^{\gamma} I_{\{\beta^* = \frac{\gamma + \gamma}{\gamma + \epsilon}\}} + b |t|^{\beta} I_{\{\beta^* = \beta_p\}} \).

Similarly, we derive that as \( u \to \infty, S \to \infty, \)

\[
\sum_{k=0}^{N(u)-1} \mathcal{P}_u \{ I_k(u) \} \geq \mathbb{P} \{ Y(0) > u \} a^{1/\alpha} H_\alpha u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} \int_0^\infty e^{-f(t)} dt.
\]

Moreover,

\[
\Lambda_1(u) \leq \sum_{k=0}^{N(u)} (\mathcal{P}_u \{ I_k(u) \} + \mathcal{P}_u \{ I_{k+1}(u) \} - \mathcal{P}_u \{ (I_k(u) \cup I_{k+1}(u)) \})
\]

\[
\leq \sum_{k=0}^{N(u)} \left( \mathbb{P} \left\{ \sup_{t \in I_k(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(k) \right\} + \mathbb{P} \left\{ \sup_{t \in I_{k+1}(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(k) \right\} - \mathbb{P} \left\{ \sup_{t \in (I_k(u) \cup I_{k+1}(u))} \tilde{g}(X(t)) > \tilde{\mathcal{G}}_{u,-\varepsilon}(k) \right\} \right)
\]

\[
\leq h_0 \left( 2H_\alpha [0, a^{1/\alpha} S] - H_\alpha [0, 2a^{1/\alpha} S] \right) u^{\frac{\omega}{\alpha - 1}} \sum_{k=0}^{N(u)} \exp \left\{ - \frac{1}{2} \left( \tilde{\mathcal{G}}_{u,-\varepsilon}(k) \right)^2 \right\}
\]

\[
\geq \frac{2H_\alpha [0, a^{1/\alpha} S] - H_\alpha [0, 2a^{1/\alpha} S]}{S} \int_0^\infty \exp \left\{ -(1 - \varepsilon - \epsilon) f(t) \right\} dt \frac{u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} \mathbb{P} \{ Y(0) > u \}}{u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} \mathbb{P} \{ Y(0) > u \}}.
\]

where \( \tilde{\mathcal{G}}_{u,-\varepsilon}(k) = \min(\mathcal{G}_{u,-\varepsilon}(k), \mathcal{G}_{u,-\varepsilon}(k + 1)). \) By Lemma 3.3, we have

\[
\Lambda_2(u) \leq \sum_{0 \leq k, l \leq N(u), l \geq k + 2} \mathbb{P} \left\{ \sup_{t \in I_k(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(k), \sup_{t \in I_l(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(l) \right\}
\]

\[
\leq \sum_{0 \leq k \leq N(u)} \sum_{l=k+2}^{N(u)} \mathbb{P} \left\{ \sup_{t \in I_k(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(k), \sup_{t \in I_l(u)} \tilde{g}(X(t)) > \mathcal{G}_{u,-\varepsilon}(l) \right\}
\]

\[
\leq Q_6 \left( \sum_{k=0}^{N(u)} \mathbb{P} \{ \tilde{g}(X(0)) > \mathcal{G}_{u,-\varepsilon}(k) \} \right)^\infty \sum_{l=2}^\infty \exp \left\{ -(lS)^{\alpha}/8 \right\}
\]

\[
\leq Q_7 \mathbb{P} \{ Y(0) > u \} \frac{u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} S}{\mathbb{P} \{ Y(0) > u \}} \sum_{l=0}^\infty \exp \left\{ -(lS)^{\alpha}/8 \right\}
\]

\[
\Lambda_2(u) = o \left( u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} \mathbb{P} \{ Y(0) > u \} \right), \ u \to \infty, S \to \infty.
\]

Combing (27)-(30) with (26), we obtain

\[
\mathcal{P}_u \{ \Delta(u) \} \sim \mathbb{P} \{ Y(0) > u \} a^{1/\alpha} H_\alpha u^{\frac{\omega}{\alpha} - \frac{\alpha}{\alpha + \epsilon}} \int_0^\infty \exp \left\{ -f(t) \right\} dt, \ u \to \infty.
\]

**Case 2**: \( \beta^* = \alpha^* \). We consider that for \( u \) large enough

\[
\mathcal{P}_u \{ I_0(u) \} \leq \mathcal{P}_u \{ \Delta(u) \} \leq \sum_{k=0}^{N(u)} \mathcal{P}_u \{ I_k(u) \}.
\]

Using Lemma 3.2 and (25), we have that for some small \( \epsilon \in (0, 1) \)

\[
\mathcal{P}_u \{ I_0(u) \} \geq \mathbb{P} \left\{ \sup_{t \in [0,S]} \left( Y(tu^{-2/\alpha^*}) + h(tu^{-2/\alpha^*}) \right) > u \right\}
\]

\[
\geq \mathbb{P} \left\{ \sup_{t \in [0,S]} \frac{\tilde{g}(X(tu^{-2/\alpha^*}))}{1 + \frac{c(1+\epsilon)}{a} |tu^{-2/\alpha^*}|^\gamma + (1+\epsilon)b |tu^{-2/\alpha^*}|^\beta} > u^{1/p} \right\}
\]

\[
\geq \mathbb{P} \left\{ \sup_{t \in [0,S]} \frac{\tilde{g}(X(tu^{-2/\alpha^*}))}{1 + (1+\epsilon + \epsilon) u^{-2/\alpha^*} f(t)} > u^{1/p} \right\}
\]
Thus we complete the proof.

Moreover, by Lemma 3.2
\[
\sum_{k=1}^{N(u)} P_u(I_k(u)) \leq \sum_{k=1}^{N(u)} \mathbb{P}\left\{ \sup_{t \in I_k(u)} \tilde{g}(\tilde{X}(t)) > G_{u,-\varepsilon}(k) \right\}
\]
\[
\sim h_0 H_\alpha[0, a^{1/\alpha} S] u^m \sum_{k=1}^{N(u)} \exp \left( -\frac{1}{2} (G_{u,-\varepsilon}(k))^2 \right)
\]
\[
\leq h_0 H_\alpha[0, a^{1/\alpha} S] u^m \exp \left( -\frac{u^2}{2} \right)
\]
\[
\times \sum_{k=1}^{N(u)} \exp \left( -(1 - \varepsilon - \varepsilon) f \left( u^{2/\alpha^*} k S u^{-\frac{2}{\alpha}} \right) \right)
\]
\[
\leq \mathbb{P}\{ Y(0) > u \} H_\alpha[0, a^{1/\alpha} S] \sum_{k=1}^{\infty} \exp (-Q_8(k S)^\gamma)
\]
\[
\sim Q_9 \mathbb{P}\{ Y(0) > u \} H_\alpha a^{1/\alpha} S \exp (-Q_{10} S^\gamma)
\]
\[
= o(\mathbb{P}\{ Y(0) > u \}) \quad u \to \infty, \quad S \to \infty.
\]

Inserting (34) and (33) into (32), we have

\[
\mathcal{P}_u(\Delta(u)) \sim \mathcal{P}_{\alpha,a}[0, \infty) \mathbb{P}\{ Y(0) > u \}, \quad u \to \infty.
\]

**Case 3:** \( \beta^* < \alpha^* \).

\[
\mathcal{P}_u(\Delta(u)) \geq \mathbb{P}\{ Y(0) > u \}.
\]

For any \( \varepsilon_2 \in (0, 1) \), \( \Delta(u) \subseteq [0, u^{-2/\alpha^*} \varepsilon_2] \) when \( u \) large enough. Then as \( u \to \infty, \: \varepsilon_2 \to 0 \)
\[
\mathcal{P}_u(\Delta(u)) \leq \mathbb{P}\left\{ \sup_{t \in [0, u^{-2/\alpha^*} \varepsilon_2]} \tilde{g}(\tilde{X}(t)) > u^{1/p} \right\} \sim H_\alpha[0, a^{1/\alpha} \varepsilon_2] \mathbb{P}\{ Y(0) > u \} \sim \mathbb{P}\{ Y(0) > u \}.
\]

Together with (36), we get

\[
\mathcal{P}_u(\Delta(u)) \sim \mathbb{P}\{ Y(0) > u \}.
\]

Further, (22) are derived according to (31)-(37).

Finally, we note that if \( t_0 \in (0, T) \) and \( t_0 = T \), the proof is the same with as above by simply replacing \( \Delta(u) \) by \( \Delta(u) = [-\delta(u), \delta(u)] \) and \( \Delta(u) = [-\delta(u), 0] \), respectively. Thus we complete the proof. \( \square \)

**Proof of Theorem 2.4:** (1) For any \( \theta > 0 \) and \( S > 0 \), set \( \alpha^* = \alpha p \)

\[
I_k(\theta) = [k\theta, (k + 1)\theta], \quad a_k = a(k \theta) \quad k \in \mathbb{N}, \quad N(\theta) = \left\lceil \frac{T}{\theta} \right\rceil,
\]
\[
J_k^u(u) = \left[ k \theta + lu^{-2/\alpha^*} S, k \theta + (l + 1)u^{-2/\alpha^*} S \right], \quad M(u) = \left\lceil \frac{\theta u^{2/\alpha^*}}{S} \right\rceil.
\]

We have
\[
\sum_{k=0}^{N(\theta) - 1} \sum_{l=0}^{M(u) - 1} \mathcal{Q}_u(J_k^u(u)) - \sum_{i=1}^{4} A_i(u) \leq \mathcal{Q}_u([0, T]) \leq \sum_{k=0}^{N(\theta)} \mathcal{Q}_u(I_k(\theta)) \leq \sum_{k=0}^{N(\theta)} \sum_{l=0}^{M(u)} \mathcal{Q}_u(J_k^u(u)) \quad i = 1, 2, 3, 4,
\]

where
\[
A_i(u) = \sum_{(k_1, l_1, k_2, l_2) \in \mathcal{L}_i} \mathcal{Q}_u(J_{k_1}^{l_1}(u), J_{k_2}^{l_2}(u)), \quad i = 1, 2, 3, 4,
\]
with
\[
\mathcal{L}_1 = \{0 \leq k_1 = k_2 \leq N(\theta) - 1, 0 \leq l_1 + 1 = l_2 \leq M(u) - 1\},
\]
\[
\mathcal{L}_2 = \{0 \leq k_1 + 1 = k_2 \leq N(\theta) - 1, l_1 = M(u), l_2 = 0\},
\]
\[
\mathcal{L}_3 = \{0 \leq k_1 + 1 < k_2 \leq N(\theta) - 1, 0 \leq l_1, l_2 \leq M(u) - 1\},
\]
\[
\mathcal{L}_4 = \{0 \leq k_1 \leq k_2 \leq N(\theta) - 1, k_2 - k_1 \leq 1, 0 \leq l_1, l_2 \leq M(u) - 1\} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2).
\]

By Lemma 3.2,
\[
\sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \mathcal{Q}_u \left( J^k_l(u) \right) \right) = \sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \mathbb{P} \left( \sup_{t \in [0, S]} Y(k\theta + lu^{-2/\alpha^*}S + u^{-2/\alpha^*}t) > u \right) \right)
\]
\[
= \sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \mathbb{P} \left( \sup_{t \in [0, S]} \tilde{g}(X(k\theta + lu^{-2/\alpha^*}S + u^{-2/\alpha^*}t)) > u^{1/p} \right) \right)
\]
\[
\leq \sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \mathbb{P} \left( Y(0) > u \right) \right)
\]
\[
\sim \left(\sum_{k=0}^{N(\theta) - 1} (a_k + \varepsilon_\theta)^{\frac{1}{\alpha}} \right) \mathcal{H}_\alpha u^{2/\alpha^*} \mathbb{P} \left( Y(0) > u \right)
\]
\[
\sim \int_0^T (a(t))^{1/\alpha} dt u^{-2/\alpha^*} \mathcal{H}_\alpha \mathbb{P} \left( Y(0) > u \right), \ u \to \infty, \ S \to \infty, \ \theta \to 0.
\]

where \(\tilde{g}(x) = g^{1/p}(x)\). Similarly,
\[
\sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \mathcal{Q}_u \left( J^k_l(u) \right) \right) \geq \int_0^T (a(t))^{1/\alpha} dt u^{-2/\alpha^*} \mathcal{H}_\alpha \mathbb{P} \left( Y(0) > u \right), \ u \to \infty, \ S \to \infty, \ \theta \to 0.
\]

Now we find an upper bound for \(\sum_{i=1}^4 \mathcal{A}_i(u)\), by Lemma 3.2
\[
\mathcal{A}_1(u) = \sum_{k=0}^{N(\theta) - 1} \left(\sum_{l=0}^{M(u) - 1} \left(\mathcal{Q}_u \left( J^k_l(u) \right) + \mathcal{Q}_u \left( J^k_{l+1}(u) \right) - \mathcal{Q}_u \left( J^k_l(u) \cup J^k_{l+1}(u) \right) \right) \right)
\]
\[
\sim \sum_{k=0}^{N(\theta) - 1} \left(\mathcal{H}_\alpha [0, (a_k + \varepsilon_\theta)^{\frac{1}{\alpha^*}}S] + \mathcal{H}_\alpha [0, (a_k + \varepsilon_\theta)^{\frac{1}{\alpha^*}}S] - \mathcal{H}_\alpha [0, 2(a_k - \varepsilon_\theta)^{\frac{1}{\alpha^*}}S] \right) \sum_{l=0}^{M(u) - 1} \mathbb{P} \left( Y(0) > u \right)
\]
\[
\leq \mathcal{Q}_1 \left(\sum_{k=0}^{N(\theta) - 1} \left( (a_k + \varepsilon_\theta)^{\frac{1}{\alpha^*}} - (a_k - \varepsilon_\theta)^{\frac{1}{\alpha^*}} \right) \theta \right) u^{2/\alpha^*} \mathbb{P} \left( Y(0) > u \right)
\]
\[
= o \left( u^{2/\alpha^*} \mathbb{P} \left( Y(0) > u \right) \right), \ u \to \infty, \ S \to \infty, \ \theta \to 0.
\]

Similarly,
\[
\mathcal{A}_2(u) = \sum_{k=0}^{N(\theta) - 1} \mathbb{P} \left( J^k_{M(u) - 1}(u), J^k_{M(u) - 1}(u) \right)
\]
\[
\leq \sum_{k=0}^{N(\theta) - 1} \mathbb{P} \left( \sup_{t \in [0, 2S]} Y((k + 1)\theta - u^{-2/\alpha^*}t) > u, \sup_{t \in [0, 2S]} Y((k + 1)\theta + u^{-2/\alpha^*}t) > u \right)
\]
\[
= \sum_{k=0}^{N(\theta) - 1} \left(\mathbb{P} \left( \sup_{t \in [0, 2S]} Y((k + 1)\theta - u^{-2/\alpha^*}t) > u \right) + \mathbb{P} \left( \sup_{t \in [0, 2S]} Y((k + 1)\theta + u^{-2/\alpha^*}t) > u \right) \right)
\]
\[
- \mathbb{P} \left( \sup_{t \in [-2S, 2S]} Y((k + 1)\theta - u^{-2/\alpha^*}t) > u \right).
\]
Finally by Lemma 3.3 for $u$ large enough and $\theta$ small enough

$$A_4(u) \leq \sum_{k=0}^{N(\theta)-1} \left( \sum_{l=0}^{2M(u)} \sum_{i=2}^{2M(u)} \mathcal{Q}_u \left( J^k_l(u), J^k_{l+1}(u) \right) \right)$$

$$\leq \sum_{k=0}^{N(\theta)-1} 2M(u) \sum_{l=0}^{M(u)} \mathbb{P}\left\{ Y(0) > u \right\} \left( \sum_{i=1}^{\infty} \mathcal{Q}_4 \exp \left( -\mathcal{Q}_5 |iS|^\alpha \right) \right)$$

$$\leq \mathcal{Q}_6 \frac{T}{S} u^{-2/\alpha} \mathbb{P}\left\{ Y(0) > u \right\} \left( \sum_{i=1}^{\infty} \exp \left( -\mathcal{Q}_5 |iS|^\alpha \right) \right)$$

$$= o \left( u^{2/\alpha} \mathbb{P}\left\{ Y(0) > u \right\} \right), \quad u \to \infty, \quad S \to \infty, \quad \theta \to 0.$$
and by (1)
\[ P_u([0, T]) \leq \sum_{k=0}^{N(\theta)} \mathcal{P}_u(I_k) \leq \sum_{k=0}^{N(\theta)} \mathbb{P} \left\{ \sup_{t \in I_k} Y(t) > u - M_1^1(k) \right\} \]
\[ \sim \sum_{k=0}^{N(\theta)} a_k^k (u - M_1^1(k)) \frac{1}{k} \mathcal{H}_a \mathbb{P} \left\{ Y(0) > u - M_1^1(k) \right\} \]
\[ \sim u^{\frac{1}{\alpha}} \mathcal{H}_a \mathbb{P} \left\{ Y(0) > u \right\} \int_0^T (a(t))^{\frac{1}{\alpha}} e^{\frac{h(t)}{2}} dt, \; u \to \infty, \; \theta \to 0. \]

Similarly,
\[ \sum_{k=0}^{N(\theta) - 1} \mathcal{P}_u(I_k) \geq \sum_{k=0}^{N(\theta) - 1} \mathbb{P} \left\{ \sup_{t \in I_k} Y(t) > u - M_2^2(k) \right\} \]
\[ \sim u^{\frac{1}{\alpha}} \mathcal{H}_a \mathbb{P} \left\{ Y(0) > u \right\} \int_0^T (a(t))^{\frac{1}{\alpha}} e^{\frac{h(t)}{2}} dt, \; u \to \infty, \; \theta \to 0. \]

Further, we have
\[ \Lambda_1 \leq \sum_{k=0}^{N(\theta)} \left( \mathcal{P}_u(I_k) + \mathcal{P}_u(I_{k+1}) - \mathcal{P}_u(I_k \cup I_{k+1}) \right) \]
\[ \leq \sum_{k=0}^{N(\theta)} \left( \mathbb{P} \left\{ \sup_{t \in I_k} Y(t) > u - \tilde{M}_1^1(k) \right\} + \mathbb{P} \left\{ \sup_{t \in I_{k+1}} Y(t) > u - \tilde{M}_1^1(k) \right\} - \mathbb{P} \left\{ \sup_{t \in I_k \cup I_{k+1}} Y(t) > u - \tilde{M}_1^1(k) \right\} \right) \]
\[ \sim \sum_{k=0}^{N(\theta)} \left( a_k^{1/\alpha} + a_{k+1}^{1/\alpha} - 2a_k^{1/\alpha} \right) \theta u^{1/\alpha} e^{\frac{h(t)}{2}} \mathbb{P} \{ Y(0) > u \} = o \left( u^{1/\alpha} \mathbb{P} \{ Y(0) > u \} \right), \; u \to \infty, \; \theta \to 0, \]

where \( \tilde{M}_1^1(k) = \max(M_1^1(k), M_1^1(k + 1)). \)

Set \( h_m = \sup_{t \in [0, T]} h(t) \) and for any \( \theta > 0 \)
\[ \mathbb{E} \left\{ X_i(t)X_i(s) \right\} = r(|t - s|) \leq 1 - \delta(\theta) \]
for \( (s, t) \in I_k \times I_j, j > k + 1 \) where \( \delta(\theta) > 0 \) is a constant related to \( \theta \). Then by Lemma 3.1
\[ \Lambda_2 = \sum_{k=0}^{N(\theta)} \mathcal{P}_u(I_k \cap I_j) \leq \sum_{k=0}^{N(\theta)} \sum_{j > k + 1} \mathbb{P} \left\{ \sup_{t \in I_k} Y(t) > u - h_m, \sup_{t \in I_j} Y(t) > u - h_m \right\} \]
\[ \leq \sum_{k=0}^{N(\theta)} \sum_{j > k + 1} \mathbb{P} \left\{ \frac{2(u - h_m)\delta - \Phi_2}{\sqrt{4 - \delta(\theta)}} \right\} = o \left( \mathbb{P} \{ Y(0) > u \} \right), \; u \to \infty, \; \theta \to 0. \]

Thus, we have
\[ P_u([0, T]) \sim \int_0^T (a(t))^{\frac{1}{\alpha}} e^{\frac{h(t)}{2}} dt \mathcal{H}_a u^{\frac{1}{\alpha}} \mathbb{P} \{ Y(0) > u \}, \; u \to \infty. \]

If \( p \in (2, \infty) \), set \( M_1 = \inf_{t \in [0, T]} h(t) \) and \( M_2 = \sup_{t \in [0, T]} h(t) \). Since \( h(t) \) is a continuous function, we have
\[ -\infty < M_1 \leq M_2 < \infty. \]
Further, since when \( p \in (2, \infty) \),
\[ \mathbb{P} \{ Y(0) > u + \Phi_2 \} \sim \mathbb{P} \{ Y(0) > u \} \]
hold for any \( \Phi_2 \in \mathbb{R} \). Hence, by (1)
\[ P_u([0, T]) \geq \mathbb{P} \left\{ \sup_{t \in [0, T]} Y(t) > u - M_1 \right\} \sim \int_0^T (a(t))^{\frac{1}{\alpha}} dt u^{\frac{1}{2}} \mathcal{H}_a \mathbb{P} \{ Y(0) > u \}, \; u \to \infty, \]
and
\[ P_u([0,T]) \leq \mathbb{P} \left\{ \sup_{t \in [0,T]} Y(t) > u - M_2 \right\} \sim \int_0^T (a(t))^{1/\alpha} dt \alpha^{2/\alpha} \mathcal{H}_a \mathbb{P} \{ Y(0) > u \} , \ u \to \infty. \]

Thus we complete the proof. \( \square \)

4. Appendix

4.1. Appendix A. First we give the proof of (12).

**Proof of (12):** Here we use \( P_u(\cdot) \) the same as in (20). We consider the case \( 0 < A < B < T \). First by (1) of Theorem 2.4, we have as \( u \to \infty \)
\[ P_u([A,B]) = \mathbb{P} \left\{ \sup_{t \in [A,B]} Y(t) > u - h_m \right\} \sim \int_A^B (a(t))^{1/\alpha} dt \alpha^{2/\alpha} \mathbb{P} \{ Y(0) > u - h_m \} . \]

Set \( \Delta_\varepsilon = [A - \varepsilon, B + \varepsilon] \cap [0,T] \) for some \( \varepsilon > 0 \), then we have
\[ P_u([A,B]) \leq P_u([0,T]) \leq P_u([\Delta_\varepsilon]) + P_u([0,T] \setminus \Delta_\varepsilon). \]

Since \( h(t) \) is a continuous function and we have \( h_\varepsilon := \sup_{t \in [0,T] \setminus \Delta_\varepsilon} h(t) < h_m \), then by (1) of Theorem 2.4
\[ P_u([0,T] \setminus \Delta_\varepsilon) \leq \mathbb{P} \left\{ \sup_{t \in \Delta_\varepsilon} Y(t) > u - h_\varepsilon \right\} \sim \mathcal{O}_1\varepsilon^{2/\alpha} \mathbb{P} \{ Y(0) > u - h_\varepsilon \} \]
\[ = o\left( \varepsilon^{2/\alpha} \mathbb{P} \{ Y(0) > u - h_m \} \right) , \ u \to \infty, \ \varepsilon \to 0. \]

Further, we have by (1) of Theorem 2.4
\[ P_u([\Delta_\varepsilon]) \leq \mathbb{P} \left\{ \sup_{t \in \Delta_\varepsilon} Y(t) > u - h_m \right\} \leq \int_{A-\varepsilon}^{B+\varepsilon} (a(t))^{1/\alpha} dt \alpha^{2/\alpha} \mathbb{P} \{ Y(0) > u - h_m \} \]
\[ \sim \int_A^B (a(t))^{1/\alpha} dt \alpha^{2/\alpha} \mathbb{P} \{ Y(0) > u - h_m \} , \ u \to \infty, \ \varepsilon \to 0. \]

Hence the claims follow. \( \square \)

**Proof of Example 2.6:** Notice that we just need to prove that \( g(x) \) satisfies Condition 2.1.

i) By the twice continuously differentiable of \( g \), it easily to prove that Condition 2.1 is satisfied, except case \( \rho = 2 \). For the case \( \rho = 2 \), we have \( \hat{g} = 1 \) and \( \mathcal{M} \) is the whole unit sphere \( S_{d-1} \) (which is a manifold of dimension \( d-1 \)), and according to [6], we know that Theorem 2.2 and Theorem 2.4 hold. A similar detail analysis can be found in [18][Example 1].

ii) For \( g(x) = \Pi_{i=1}^d x_i \), we have \( p = d \) and further \( \hat{g} = \frac{1}{d^{1/2}} \) since
\[ \mathcal{M} = \left\{ \left( \pm 1/\sqrt{d}, \ldots, \pm 1/\sqrt{d} \right) \text{ with even number of negative coordinates} \right\}, \]
which consists of \( 2^{d-1} \) points (the product \( x_1, \ldots, x_d \) should be positive). Further, we introduce the spherical coordinates
\[ v_1 = \cos \varphi_1, \]
\[ v_2 = \sin \varphi_1 \cos \varphi_2, \]
\[ \ldots \]
\[ v_{d-1} = \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{d-2} \cos \varphi_{d-1}, \]
\[ v_d = \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{d-2} \sin \varphi_{d-1}, \]
with \( \varphi \in \Pi_{d-1} = [0, \pi)^{d-2} \times [0, 2\pi) \) and

\[
g(\varphi) = \sin^{d-1} \varphi_1, \ldots, \sin \varphi_{d-1} \cos \varphi_1, \ldots, \cos \varphi_{d-1}.
\]

For instance, at the point \( \left( 1/\sqrt{d}, \ldots, 1/\sqrt{d} \right) \) we have \( \cos \varphi_i = \sqrt{1/d-i+1} \) and \( \sin \varphi_i = \sqrt{d-i+1} \). Further calculation shows that for any \( \varphi \in \mathcal{M}_\varphi \)

\[
g''_{\varphi,\varphi_i}(\varphi) = -2g(\varphi)(d - i + 1) = \frac{2(d - i + 1)}{d^{d/2}}, \quad g''_{\varphi,\varphi_j}(\varphi) = 0, \text{ for } i \neq j,
\]

which leads \( |\text{det}g''(\varphi)| = 2^{d-i}d!d^{(d-1)/2} > 0 \). By the continuity of \( |\text{det}g''(\varphi)| \) over some \( \mathcal{M}_\varphi(\varepsilon) \), we know that Condition 2.1 is satisfied.

i) For \( g(x) = \max_{1 \leq i \leq d} x_i \), we have \( p = 1 \) and further \( \hat{g} = 1 \) since \( \mathcal{M} \) consists of \( d \) points \((0, \ldots, 0, 1, 0, \ldots, 0)\). For instance, around the point \( v = (0, \ldots, 0, 1) \) over unit sphere \( S_{d-1} \) we have that

\[
g(\varphi) = \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{d-2} \sin \varphi_{d-1}
\]

and \( \varphi_i = \frac{\pi}{2}, i = 1, \ldots, d \). It is clear that

\[
g''_{\varphi,\varphi_i}(\varphi) = -g(\varphi) = -1, \quad g''_{\varphi,\varphi_j}(\varphi) = 0, \text{ for } i \neq j,
\]

which leads \( |\text{det}g''(\varphi)| = 1 > 0 \). By the continuity of \( |\text{det}g''(\varphi)| \) over some \( \mathcal{M}_\varphi(\varepsilon) \), we know that Condition 2.1 is satisfied.

Next we give the proofs of Lemma 3.1, Lemma 3.2 and Lemma 3.3. In the following proofs, \( \mathcal{Q}_i, i \in \mathbb{N} \) denote some positive constants which can be different from line by line.

**Proof of Lemma 3.1:** By (11) and the continuity of \( r(s, t) \), for some \( \delta > 0 \) we have

\[
\mathbb{E} \{ X_i(t)X_i(s) \} = r(s, t) \leq 1 - \frac{\delta}{2}, \quad i = 1, 2, \ldots, d,
\]

holds for any \( (s, t) \in [0, t_1] \times [t_2, t_3] \). Set \( Z(t, v, s, w) = \langle X(t), v \rangle + \langle X(s), w \rangle \) where \( v, w \in S_{d-1} \) with \( S_{d-1} \) the unit sphere in \( \mathbb{R}^d \) with respect to Euclidean norm \( \|\cdot\| \). Since \( Z(t, v, s, w) \) is a center Gaussian fields, we have further

\[
\text{Var} \left( Z(t, v, s, w) \right) = 2 + 2r(s, t) \left( \sum_{i=1}^{d} v_i w_i \right) \leq 2 + r(s, t) \left( \sum_{i=1}^{d} v_i^2 + \sum_{i=1}^{d} w_i^2 \right)
\]

\[
= 2 + 2r(s, t) \leq 4 - \delta
\]

for any \( (t, v, s, w) \in [0, t_1] \times S_{d-1} \times [t_2, t_3] \times S_{d-1} \). By Borell inequality (see e.g., [7, 19])

\[
\mathbb{P} \left\{ \sup_{t \in [0, t_1]} Y(t) > u, \sup_{t \in [t_2, t_3]} Y(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, t_1]} |Y(t)| > u, \sup_{t \in [t_2, t_3]} |Y(t)| > u \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{t \in [0, t_1]} |X(t)| > u, \sup_{t \in [t_2, t_3]} |X(t)| > u \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{(t, v) \in [0, t_1] \times S_{d-1}} \langle X(t), v \rangle > u, \sup_{(s, w) \in [t_2, t_3] \times S_{d-1}} \langle X(s), w \rangle > u \right\}
\]

\[
\leq \mathbb{P} \left\{ \sup_{(t, v, s, w) \in [0, t_1] \times S_{d-1} \times [t_2, t_3] \times S_{d-1}} Z(t, v, s, w) > 2u \right\}
\]

\[
\leq 2\Psi \left( \frac{2u - D}{\sqrt{4 - \delta}} \right),
\]

where \( D \) is some constant such that

\[
\mathbb{P} \left\{ \sup_{(t, v, s, w) \in [0, t_1] \times S_{d-1} \times [t_2, t_3] \times S_{d-1}} Z(t, v, s, w) > D \right\} \leq \frac{1}{2},
\]

hence the first claim follows. The second claim follows for (57). \( \square \)
Before giving the proof of Lemma 3.2, we remark that from [1][Corollary 4] it follows that only arbitrary small vicinity in the sphere of the maximum point set gives contribution to the asymptotic behavior of probabilities of (2). Therefore we can change \( g(\varphi) \) outside of \( \mathcal{M}_\varphi(\varepsilon) \) is such a way that first,

\[
g(\varphi) \geq 1 - \varepsilon, \varphi \in \Pi_{d-1},
\]

keeping the same asymptotic behavior of the probability in question; and second, having \( g(\varphi) \) twice continuously differentiable for all \( \varphi \in \Pi_{d-1} \).

**Proof of Lemma 3.2:** i) First we prove (15) and assume for simplicity that \( f_i = f, i \leq d \). For any \( W > 0 \) and all \( u \) large (set \( Z_u(t) = \frac{g(\xi(t))}{1 + u^{-1}g(\xi(t))} \)) and write simple \( p(x) \) instead of \( p(x) \)

\[
P \left\{ \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right\} = \int_{-\infty}^{\infty} \frac{1}{p(g(\xi(t)))} \left( \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right) \left| \frac{\partial}{\partial x} \right| p(x) \, dx
\]

\[
= u_k^{-1} \int_{-\infty}^{\infty} \frac{1}{p(g(\xi(t)))} \left( \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right) \left| \frac{\partial}{\partial x} \right| p(x) \, dx
\]

\[
= u_k^{-1} \int_{-W}^{W} \frac{1}{p(g(\xi(t)))} \left( \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right) \left| \frac{\partial}{\partial x} \right| p(x) \, dx
\]

\[
= u_k^{-1} \int_{-W}^{W} \left\{ \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right\} \left( \sup_{t \in [-S_1, S_2]} g(\xi(t)) = u_k, y \right) p(u_k, y) \, dy
\]

\[
+ u_k^{-1} \int_{-W}^{W} \left\{ \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right\} \left( \sup_{t \in [-S_1, S_2]} g(\xi(t)) = u_k, y \right) p(u_k, y) \, dy
\]

\[
= I_1(u) + I_2(u) + I_3(u),
\]

where \( u_k, y = u_k - u_k^{-1}y \) and \( p(x) \) is the density function of \( g(\xi(t)) \) which is showed in Theorem 4.2.

Since

\[
\mathcal{P}_{a, a}^{f(t)}[-S_1, S_2] = \mathbb{E} \left\{ \exp \left( \sup_{t \in [-S_1, S_2]} \eta(t) \right) \right\} = \int_{-\infty}^{\infty} e^y \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \, dy,
\]

where \( \eta(t) := \sqrt{2a} B_\alpha(t) - a \alpha |t| - f(t) \), we have

\[
\sup_{k \in K_u} \left| \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \right\} - \mathcal{P}_{a, a}^{f(t)}[-S_1, S_2] \right|
\]

\[
\leq \sup_{k \in K_u} \left| \int_{-W}^{W} e^y \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \, dy \right|
\]

\[
+ \sum_{\sup_{k \in K_u} \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \geq W} e^y \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \, dy
\]

\[
\leq \sup_{k \in K_u} \left| \sum_{\sup_{k \in K_u} \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \geq W} e^y \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \, dy \right|
\]

By [4] [Theorem 8.1] and the fact that for \( s, t \geq 0 \)

\[
\mathbb{E} \left\{ (B_\alpha(t) - B_\alpha(s))^2 \right\} = |t - s|^\alpha,
\]

we have for any \( y > 0 \)

\[
\mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \leq 2 \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \sqrt{2a} B_\alpha(t) > y \right\} \leq 2 \mathbb{P} \left\{ \sup_{t \in [0, \max(S_1, S_2)]} B_\alpha(t) > \frac{y}{\sqrt{2a}} \right\}
\]
where by \((16)\), we have
\[
\mathbb{P}\{g(\xi(t_0)) > u_k + u_k^{-1}W\} \to 0, \quad u \to \infty, \quad W \to \infty,
\]
Next, using \((57)\) and \((14)\), we have
\[
\sup_{k \in K_u} \mathbb{P}\{g(\xi(t_0)) > u_k\} \leq \sup_{k \in K_u} \frac{\mathbb{P}\{g(\xi(t_0)) > u_k + u_k^{-1}W\}}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} \to 0, \quad u \to \infty, \quad W \to \infty,
\]
where we use the fact that
\[
(u_k + u_k^{-1}W)^2 = u_k^2 + 2W + u_k^{-2}W^2.
\]
By \((14)\), \((56)\) and \((57)\) for \(\varepsilon_k < \frac{1}{2}\)
\[
\sup_{k \in K_u, -W < y < u_k^{-2\varepsilon_k}} \left| \frac{u_k^{-1}p_{(u_k,y)}}{\mathbb{P}\{g(\xi(t_0)) > u_k\} e^y} - 1 \right|
= \sup_{k \in K_u, -W < y < u_k^{-2\varepsilon_k}} \left| \frac{u_k^{-1}h_0(1 + R(u_k,y)) (u_k,y)^m e^{-\frac{1}{2}u_k,y}}{\mathbb{P}\{g(\xi(t_0)) > u_k\} e^y} - 1 \right|
\]
\[
= \sup_{k \in K_u, -W < y < u_k^{-2\varepsilon_k}} \left| \frac{(1 + R(u_k,y)) (1 - u_k^{-2}y)^2 e^{-\frac{y^2}{2u_k}}}{1 + R_1(u_k)} - 1 \right| \to 0, \quad u \to \infty,
\]
where \(R(x) \to 0, R_1(x) \to 0, x \to \infty, h_0\) is the same as in Theorem 4.2, and we use the fact that
\[
(u_k,y)^2 = u_k^2 (1 - u_k^{-2}y)^2 = u_k^2 - 2y + u_k^{-2}y^2,
\]
and
\[
(u_k,y)^m = u_k^m (1 - u_k^{-2}y)^m.
\]
Setting
\[
P_{u_k}(y) := \mathbb{P}\{\sup_{t \in [-S_1,S_2]} Z_u(t) > u_k | g(\xi(t_0)) = u_k,y\} \leq 1,
\]
we have
\[
\sup_{k \in K_u} \left| \frac{I_1(u)}{\mathbb{P}\{g(\xi(t_0)) > u_k\} e^y} - \int_{-W}^W e^y \mathbb{P}\{\sup_{t \in [-S_1,S_2]} \eta(t) > y\} \right| dy
\]
\[
\leq \sup_{k \in K_u} \left| \int_{-W}^W P_{u_k}(y) \frac{u_k^{-1}p_{(u_k,y)}}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} dy - \int_{-W}^W e^y P_{u_k}(y) dy \right|
+ \sup_{k \in K_u} \int_{-W}^W e^y P_{u_k}(y) - e^y \mathbb{P}\{\sup_{t \in [-S_1,S_2]} \eta(t) > y\} \right| dy
\]
\[
\leq \sup_{k \in K_u} e^W \int_{-W}^W \left| \frac{u_k^{-1}p_{(u_k,y)}}{\mathbb{P}\{g(\xi(t_0)) > u_k\}} - 1 \right| dy + \sup_{k \in K_u} e^W \int_{-W}^W \left| P_{u_k}(y) - \mathbb{P}\{\sup_{t \in [-S_1,S_2]} \eta(t) > y\} \right| dy,
\]
where by \((40)\), the first term goes to zero as \(u \to \infty\).

Thus we just need to prove
\[
\lim_{u \to \infty} \sup_{k \in K_u, y \in [-W,W]} \left| P_{u_k}(y) - \mathbb{P}\{\sup_{t \in [-S_1,S_2]} \eta(t) > y\} \right| = 0
\]
and
\[
\lim_{u \to \infty} \sup_{k \in K_u} \frac{I_2(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} = 0,
\]
which will be dealt with in the following **Step 1** and **Step 2**, respectively.

**Step 1**: Using the idea of Piterbarg, we use $e \in \mathbb{S}_{d-1}$ as the spherical coordinates of $v$ with
\[
\mathcal{V} := \{ v : g(v) = 1, v \in \mathbb{R}^d \} = \{ v : v = v(e) = |v| e, e \in \mathbb{S}_{d-1}, g(v) = 1 \},
\]
and write
\[
\{ g(\xi(t_0)) = u_{k,y} \} =: \bigcup_{e \in \mathbb{S}_{d-1}} U_e,
\]
where $U_e = \{(u_{k,y})^{-1} \xi(t_0) = v(e)\}$. Then
\[
(42) \quad P_{u_k}(y) = \int_{\mathbb{S}_{d-1}} f_k(e) \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} Z_u(t) > u_k \big| U_e \right\} de
\]
where $de$ is the elementary volume in $\mathbb{S}_{d-1}$, $f_k$ is the conditional probability density of the Gaussian vector $(u_{k,y})^{-1} \xi(t_0)$ given $g((u_{k,y})^{-1} \xi(t_0)) = 1$ induced on $\mathbb{S}_{d-1}$.

We have
\[
\mathbb{E} \left\{ \xi_i(u^{-2\alpha}t + t_0) | U_e \right\} = \mathbb{E} \left\{ \xi_i(u^{-2\alpha}t + t_0) \big| \xi_i(t_0) = u_{k,y}v_i \right\} = \frac{r(u^{-2\alpha}|t|)u_{k,y}v_i}{1 + u^{-2}f(t)} =: m_i^u(t),
\]
i = 1, \ldots, d, where $v = v(e) = (v_1, \ldots, v_d)$. Further calculating the conditional variance we have
\[
\text{Var} \left( \frac{\xi_i(u^{-2\alpha}s + t_0)}{1 + u^{-2}f(t)} \big| U_e \right) = \text{Var} \left( \frac{\xi_i(u^{-2\alpha}s + t_0)}{1 + bu^{-2}f(s)} - \frac{\xi_i(u^{-2\alpha}t + t_0)}{1 + bu^{-2}f(t)} \right) - \left( \frac{r(u^{-2\alpha}|s|)}{1 + u^{-2}f(s)} - \frac{r(u^{-2\alpha}|t|)}{1 + u^{-2}f(t)} \right)^2
\]
\[
= : V_u(s, t).
\]
Let us recall that
\[
u_{k,y} = u_k \left( 1 - yu_k^{-2} \right),
\]
which combined with (10) leads to
\[
m_i^u(t) = u_k v_i \left( 1 - (f(t) + a|t|^\alpha) u^{-2} - yu_k^{-2} + u^{-2}p_1(u, k, t, y) \right)
\]
\[
V_u(s, t) = 2au^{-2}(|t - s|^\alpha + p_2(u, s, t)),
\]
where $p_1(u, k, t, y) \to 0$, as $u \to \infty$ uniformly in $t, y, k$ and $p_2(u, s, t) \to 0$, as $u \to \infty$ uniformly in $s, t$. Let us introduce $d$ independent Gaussian processes $X_i^u(t), i = 1, 2, \ldots, d$ with zero means, continuous trajectories with $t \in [-S_1, S_2]$, and covariance functions equal to the respective conditional covariance function of $\frac{\xi_i(u^{-2\alpha}s + t_0)}{1 + u^{-2}f(t)}$.

Now write
\[
P_{u_k}(y) = \int_{\mathbb{S}_{d-1}} f_k(e) \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} g(\mathcal{X}_u(t) + m_u(t)) > u_k \right\} de,
\]
with $\mathcal{X}_u(t) = (X_i^u(t), i = 1, \ldots, d)$ and $m_u(t) = (m_i^u(t), i = 1, \ldots, d)$.

Notice also that $g(v) = 1$, and thus
\[
|v| = |v| g(e)(g(e))^{-1} = g(v)(g(e))^{-1} = (g(e))^{-1}.
\]
Now let us consider the probability
\[
P_{u_k, e}(y) := \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} g(\mathcal{X}_u(t) + m_u(t)) > u_k \right\} = \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} g(e)|\mathcal{X}_u(t) + m_u(t)| > u_k \right\}
\]
\[
\begin{align*}
\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \left( \|X_u(t) + m_u(t)\|^2 - (g(e))^{-2}u_k^2 \right) > 0 \right\} \\
\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \left( \|X_u(t) + m_u(t)\|^2 - |v|^2u_k^2 \right) > 0 \right\} \\
\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \left( \|X_u(t) + m_u(t) - vu_k\|^2 + 2 \langle X_u(t) + m_u(t) - vu_k, vu_k \rangle \right) > 0 \right\} \\
\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \zeta_u(t) > 0 \right\},
\end{align*}
\]

where \( \zeta_u(t) := \langle X_u(t) + m_u(t) - vu_k, vu_k \rangle + \frac{1}{2} \|X_u(t) + m_u(t) - vu_k\|^2 \).

As in [1] we obtain

\[
P_{u_k, e}(y) \leq Q_1 \exp(-Q_2y^2) + Q_3 \exp(-Q_4y),
\]

where \( Q_4 > 1 \), and further

\[
(45) \quad P_{u_k}(y) \leq Q_5 \exp(-Q_2y^2) + Q_6 \exp(-Q_4y).
\]

By (14), (43) and (44), as \( u \to \infty \)

\[
\mathbb{E}\{\zeta_u(t)\} \to -(f(t) + a|t|^\alpha)|v|^2 - y|v|^2,
\]

holds uniformly for all \( t \in [-S_1, S_2] \) and \( k \in K_u \), and moreover

\[
\text{Var}(\zeta_u(t) - \zeta_u(s)) \to 2a|t - s|^\alpha|v|^2,
\]

holds uniformly for all \( s, t \in [-S_1, S_2] \) and \( k \in K_u \). Given \( B_u(t), \ B^*_u(t), \ i = 1, 2, \ldots, d \) independent fractional Brownian motion with the same Hurst index \( \alpha/2 \in (0, 1] \) we can write

\[
\begin{align*}
\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \sum_{i=1}^{d} \sqrt{2a}B^*_u(t)v_i - (f(t) + a|t|^\alpha)|v|^2 - y|v|^2 > 0 \right\} \\
= \mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \sqrt{2a}B^*_u(t)|v| - (f(t) + a|t|^\alpha)|v|^2 > y|v|^2 \right\} \\
= \mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \sqrt{\frac{2a}{(g(e))^2}}B^*_u(t) - \frac{1}{(g(e))^2}f(t) - \frac{a}{(g(e))^2}|t|^\alpha > y(g(e))^{-2} \right\} =: P_e(y).
\end{align*}
\]

Consequently,

\[
\lim_{u \to \infty} \sup_{k \in K_u, y \in [-W, W]} |P_{u_k, e}(y) - P_e(y)| = 0.
\]

By [1], we know that there exist a density function \( j(e) \) on \( \mathcal{M} \) such that

\[
\lim_{u \to \infty} \sup_{k \in K_u} \left| \int_{S_{d-1}} f_k(e)P_{u_k, e}(y)de - \int_{\mathcal{M}} j(e)P_e(y)de \right| = 0,
\]

and

\[
\int_{\mathcal{M}} j(e)P_e(y)de = \int_{\mathcal{M}} j(e)\mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} = \mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\},
\]

where we used the fact that \( g(e) = 1 \) for \( e \in \mathcal{M} \) and \( \int_{\mathcal{M}} j(e)de = 1 \). Hence we have

\[
(46) \quad \lim_{u \to \infty} \sup_{k \in K_u, y \in [-W, W]} \left| P_{u_k}(y) - \mathbb{P}\left\{ \sup_{t \in [-S_1, S_2]} \eta(t) > y \right\} \right| = 0.
\]
Step 2: For \( \varepsilon_1 < \frac{1}{2} \), we have (below we set \( U_u(t) := g(\xi(u^{-\alpha/2}t + t_0)) \))

\[
I_2(u) \leq u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) = u_{k,y} \right\} p(u_k, y) dy
\]

\[
= u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) = u_{k,y} \right\} p(u_k, y) dy
\]

\[
+ u_k^{-1} \int_{u_k^{-1}}^{\infty} \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) = u_{k,y} \right\} p(u_k, y) dy
\]

\[
= u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) = u_{k,y} \right\} p(u_k, y) dy
\]

\[
+ \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) \leq u_k - u_k^{1-\varepsilon_1} \right\}
\]

\[
= : J_1(u) + J_2(u).
\]

Then we use the similar argumentation in Step 1 to deal with

\[
P_{u_k}(y) = \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} U_u(t) > u_k, g(\xi(t_0)) = u_{k,y} \right\}, \ y \in [W, u_k^{2-\varepsilon_1}]
\]

and we get

\[
\lim_{u \to \infty} \sup_{k \in K_u, y \in [W, u_k^{2-\varepsilon_1}]} \left| P_{u_k}(y) - \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta_1(t) > y \right\} \right| = 0,
\]

where \( \eta_1(t) = \sqrt{2\alpha B_\alpha(t) - a |t|^\alpha} \). Then by (40) and similarly to (38)

\[
\sup_{k \in K_u} \frac{J_1(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \leq Q_7 \int_W \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} \eta_1(t) > y \right\} e^y dy
\]

\[
\leq Q_8 \int_W e^y y^{2/\alpha} \Psi \left( \frac{y}{\sqrt{2\alpha \max(S_1, S_2)}} \right) dy \to 0, \ W \to \infty.
\]

(47)

Now let us proceed the analysis of \( J_2(u) \). We have

\[
J_2(u) \leq \mathbb{P} \left\{ \sup_{t \in [-u^{-2/\alpha S_1, u^{-2/\alpha S_2}]} (g(\xi(t + t_0)) - g(\xi(t_0))) > u_k^{1-\varepsilon_1} \right\}.
\]

Since

\[
g(x + y) - g(x) = \int_0^1 \langle \nabla g(x + hy), y \rangle dh,
\]

and

\[
|g(x + y) - g(x)| \leq g_1 |y|,
\]

where \( g_1 = \max_{|u| = 1} |\nabla g(u)| \). Hence, denoting \( \Delta \xi(t) = \xi(t + t_0) - \xi(t_0) \), we get that

\[
\sup_{t \in [-u^{-2/\alpha S_1, u^{-2/\alpha S_2}]} (g(\xi(t + t_0)) - g(\xi(t_0))) \leq g_1 \sup_{t \in [-u^{-2/\alpha S_1, u^{-2/\alpha S_2}]} |\Delta \xi(t)|.
\]

Further, since by (14) for \( \varepsilon > 0 \) when \( a \) large enough

\[
(48) \quad \inf_{k \in K_u} u_k \geq (1 - \varepsilon)u,
\]

then we have

\[
\mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} (U_u(t) - g(\xi(t_0))) > u_k^{1-\varepsilon_1} \right\} \leq \mathbb{P} \left\{ \sup_{t \in [-u^{-2/\alpha S_1, u^{-2/\alpha S_2}]} (g(\xi(t + t_0)) - g(\xi(t_0))) > u_k^{1-\varepsilon_1} \right\}
\]
where we used the fact that \( Q_\tau = (1 - \varepsilon)^{1-\varepsilon} q_\tau^{-1}(\max(S_1, S_2))^{-\alpha/2} \). Now consider the vector process

\[ X_1(t) := t^{-\alpha/2} \Delta \xi(t), \quad t \in [-u^{-2/\alpha} S_1, u^{-2/\alpha} S_2], \]

which tends weakly to zero, hence

\[ \mathbb{P} \left\{ \sup_{t \in [-S_1, S_2]} (U_n(t) - g(\xi(t))) > u_k^{1-\varepsilon} \right\} \leq \mathbb{P} \left\{ \sup_{t \in [-u^{-2/\alpha} S_1, u^{-2/\alpha} S_2]} G(X_1(t)) > Q_\tau u^{2-\varepsilon} \right\}, \]

where \( G(x) = |x|, \ x \in \mathbb{R}^d \), is a homogeneous function of order 1. Now from Theorem 4.1 it follows that the probability \( J_2(u) \) is exponentially smaller than the probability \( I_1(u) \).

ii) Next, we prove (16). By (10), for \( \varepsilon_T \in (0, a) \) we can find \( T \) small enough such that

\[ (a - \varepsilon_T) \leq a(\varepsilon_T) \leq (a + \varepsilon_T), \quad (a - \varepsilon_T) |t - s|^a \leq 1 - r(s, t) \leq (a + \varepsilon_T) |t - s|^a \]

holds for \( s, t \in [t_0 - T, t_0 + T] \). Set

\[ \xi'(u, k)(t) = \xi(u^{-2/\alpha} (a(ku^{-2/\alpha} S + t_0))^{-1/\alpha} t + ku^{-2/\alpha} S + t_0), \quad t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]. \]

Then similar to i), we have for any \( W > 0 \) and all \( u \) large

\[ \mathbb{P} \left\{ \sup_{t \in [0, S]} g(\xi'(u, k)(t)) > u_k \right\} = \int_{-\infty}^{\infty} \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k \left| g(\xi'(u, k)(0)) = x \right. \right\} p(x) dx \]

\[ = u_k^{-1} \int_{-\infty}^{\infty} \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k \left| g(\xi'(u, k)(0)) = u_k \right. \right\} p(u_k, y) dy \]

\[ = u_k^{-1} \int_{-W}^{W} \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k \left| g(\xi'(u, k)(0)) = u_k \right. \right\} p(u_k, y) dy \]

\[ + \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k, \ g(\xi'(u, k)(0)) \leq u_k - u_k^{-1} W \right\} \]

\[ + \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k, \ g(\xi'(u, k)(0)) > u_k + u_k^{-1} W \right\} \]

\[ =: I'_1(u) + I'_2(u) + I'_3(u), \]

where we used the fact that

\[ p(x) = p(y)(x), \quad x \in \mathbb{R}. \]

By

\[ I'_3(u) \leq \mathbb{P} \left\{ g(\xi'(u, k)(0)) > u_k + u_k^{-1} W \right\} = \mathbb{P} \left\{ g(\xi(t_0)) > u_k + u_k^{-1} W \right\}, \]

we know that as in (39)

\[ \sup_{k \in K_u} \frac{I'_3(u)}{\mathbb{P} \left\{ g(\xi(t_0)) > u_k \right\}} \leq \sup_{k \in K_u} \frac{\mathbb{P} \left\{ g(\xi(t_0)) > u_k + u_k^{-1} W \right\}}{\mathbb{P} \left\{ g(\xi(t_0)) > u_k \right\}} \rightarrow 0, \ u \rightarrow \infty, W \rightarrow \infty. \]

Next we consider \( I'_1(u) \). Setting

\[ P'_{u_k}(y) = \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha} S + t_0))^{1/\alpha} S]} g(\xi'(u, k)(t)) > u_k \left| g(\xi'(u, k)(0)) = u_k \right. \right\}, \]
we have by (49)

\[ P'_{u_k}(y) \geq P \left\{ \sup_{t \in [0, (a-\varepsilon_T)^{1/\alpha} S]} g(\xi'_{u,k}(t)) > u_k \big| g(\xi'_{u,k}(0)) = u_{k,y} \right\}, \tag{50} \]

and

\[ P'_{u_k}(y) \leq P \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} g(\xi'_{u,k}(t)) > u_k \big| g(\xi'_{u,k}(0)) = u_{k,y} \right\}. \tag{51} \]

Similar to the proof of i), we analyse

\[ I^+_u(W) = u_k^{-1} \int_{-W}^{W} \mathbb{P} \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} g(\xi'_{u,k}(t)) > u_k \big| g(\xi'_{u,k}(0)) = u_{k,y} \right\} p(u_{k,y}) dy \]

and

\[ I^-_u(W) = u_k^{-1} \int_{-W}^{W} \mathbb{P} \left\{ \sup_{t \in [0, (a-\varepsilon_T)^{1/\alpha} S]} g(\xi'_{u,k}(t)) > u_k \big| g(\xi'_{u,k}(0)) = u_{k,y} \right\} p(u_{k,y}) dy \]

for some constant \( W > 0 \). Next we use the similar arguments as in ii) to analyse \( I^+_u(W) \) with \( \eta_2(t) = \sqrt{2} B_\alpha(t) - |t|^{\alpha} \). Since for \( t \in [0, (a+\varepsilon_T)^{1/\alpha} S] \) and all \( k \in K_u \)

\[ \text{Cov}(\xi'_{u,k,i}(t), \xi'_{u,k,i}(0)) \sim 1 - a(ku^{-2}/\alpha + t_0) \left| u^{-2/\alpha} (a(ku^{-2}/\alpha + t_0))^{-1/\alpha} t \right|^{\alpha} = 1 - u^{-2} |t|^{\alpha}, \]

as \( u \to \infty \), we replace (43) and (44) with

\[ m_k(t) = u_k^2 (1 - |t|^{\alpha} u^{-2} - y u_{k,-2} + u^{-2} \rho_1(u, k, t, y)), \quad V_u(s,t) = 2 u^{-2} (|t-s|^{\alpha} + \rho_2(u, s, t)). \]

Further for \( \zeta_u(t) := \langle \mathcal{X}_u(t) + m_u(t) - v u_k, v u_k \rangle + \frac{1}{2} |\mathcal{X}_u(t) + m_u(t) - v u_k|^2 \) where \( \mathcal{X}_u(t) \) is Gaussian vector process where \( \mathcal{X}_u(t), i = 1, 2, \ldots, d \) have zero means, continuous trajectories with \( t \in [0, (a+\varepsilon_T)^{1/\alpha} S] \), and covariance functions equal to the respective conditional covariance function of \( \xi'_{u,k,i}(t) \), then

\[ \mathbb{E} \{ \zeta_u(t) \} \to -|t|^{\alpha} |\mathbf{v}|^2 - y |\mathbf{v}|^2, \]

holds uniformly for all \( t \in [0, (a+\varepsilon_T)^{1/\alpha} S], y \in [-W, W] \) and \( k \in K_u \) and

\[ \text{Var} (\zeta_u(t) - \zeta_u(s)) \to 2 |t-s|^{\alpha} |\mathbf{v}|^2, \]

holds uniformly for all \( s, t \in [0, (a+\varepsilon_T)^{1/\alpha} S], y \in [-W, W] \) and \( k \in K_u \). Then

\[ \mathbb{P} \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} \sqrt{2} B_\alpha(t) - |t|^{\alpha} |\mathbf{v}|^2 - y |\mathbf{v}|^2 > 0 \right\} = \mathbb{P} \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} \sqrt{2} B_\alpha(t) - |t|^{\alpha} |\mathbf{v}|^2 > y |\mathbf{v}|^2 \right\} \]

\[ = \mathbb{P} \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} \sqrt{2} \frac{2}{(g(e))^2 - g(e)} \right\} \]

and similarly we analyse \( I^-_u(W) \) with

\[ P_-^e(y) := \mathbb{P} \left\{ \sup_{t \in [0, (a-\varepsilon_T)^{1/\alpha} S]} \sqrt{2} \frac{2}{(g(e))^2 - g(e)} B_\alpha(t) - \frac{1}{(g(e))^2} |t|^{\alpha} > y (g(e))^{-2} \right\}. \]

Thus we have that for \( u \) large enough

\[ \mathbb{P} \left\{ \sup_{t \in [0, (a-\varepsilon_T)^{1/\alpha} S]} \eta_2(t) > y \right\} \leq P'_{u_k}(y) \leq \mathbb{P} \left\{ \sup_{t \in [0, (a+\varepsilon_T)^{1/\alpha} S]} \eta_2(t) > y \right\} \]

holds for all \( k \in K_u \) and \( y \in [-W, W] \), which combining with (40) drives that for \( u \) large enough and any \( W > 0 \)

\[ \mathcal{H}_\alpha[0, (a-\varepsilon_T)^{1/\alpha} S] \leq \frac{I^+_u(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \leq \mathcal{H}_\alpha[0, (a+\varepsilon_T)^{1/\alpha} S]. \]
holds for any $k \in K_u$.

Now we consider $I_2(u)$. Similar to Step 2 of i), for $\varepsilon_1 < \frac{1}{2}$, we have

$$I_2(u) \leq u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k \bigg| g(\xi'_{(u,k)}(0)) = u_{k,y} \right\} p(u_{k,y}) dy$$

$$= u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k \bigg| g(\xi'_{(u,k)}(0)) = u_{k,y} \right\} p(u_{k,y}) dy$$

$$+ u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k \bigg| g(\xi'_{(u,k)}(0)) = u_{k,y} \right\} p(u_{k,y}) dy$$

$$= u_k^{-1} \int_W \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k \bigg| g(\xi'_{(u,k)}(0)) = u_{k,y} \right\} p(u_{k,y}) dy$$

$$+ \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k, \ g(\xi'_{(u,k)}(0)) \leq u_k - u_k^{-1/\alpha_1} \right\}$$

$$=: J'_1(u) + J'_2(u).$$

Then we use the similar argumentation in the former step to deal with

$$P'_{u_k}(y) = \mathbb{P} \left\{ \sup_{t \in [0,(a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k \bigg| g(\xi'_{(u,k)}(0)) = u_{k,y} \right\}, \ y \in [W, u_k^{-2/\alpha}],$$

and we get

$$\lim_{u \to \infty} \sup_{k \in K_u, y \in [W, u_k^{-2/\alpha}]} P'_{u_k}(y) \leq \mathbb{P} \left\{ \sup_{t \in [0, (a + \varepsilon T)]^{1/\alpha}} \eta_2(t) > y \right\}.$$

With (40) and similarly to (38), we derive that

$$\sup_{k \in K_u} \mathbb{P} \left\{ g(\xi(t_0)) > u_k \right\} \leq Q_{11} \int_W \mathbb{P} \left\{ \sup_{t \in [0, (a + \varepsilon T)]^{1/\alpha}} \eta_2(t) > y \right\} e^y dy$$

$$\leq Q_{12} \int_W e^y y^{2/\alpha} \Phi \left( \frac{y}{\sqrt{2(a + \varepsilon T)^{1/\alpha}}} \right) dy \to 0, \ W \to \infty.$$

Since

$$J'_2(u) = \mathbb{P} \left\{ \sup_{t \in [0, (a(ku^{-2/\alpha}S+t_0))]^{1/\alpha}} g(\xi'_{(u,k)}(t)) > u_k, \ g(\xi'_{(u,k)}(0)) \leq u_k - u_k^{-1/\alpha_1} \right\}$$

$$\leq \mathbb{P} \left\{ \sup_{t \in [0,\infty]} g(\xi(u^{-2/\alpha}t + ku^{-2/\alpha}S + t_0)) > u_k, \ g(ku^{-2/\alpha}S + t_0) \leq u_k - u_k^{-1/\alpha_1} \right\},$$

then using the same argument for $J'_2(u)$ as in Step 2 of i) with $t_0$ replaced by $ku^{-2/\alpha}S + t_0$, we obtain that

$$\lim_{u \to \infty} \sup_{k \in K_u} \mathbb{P} \left\{ g(\xi(t_0)) > u_k \right\} = 0.$$

Thus we finish the proof of (16).

If we let $T \to 0$ in (16), then $\varepsilon_T \to 0$ and (17) follows.

**Proof of Lemma 3.3:** We use several results and arguments from ii) of the proof of Lemma 3.2. Set

$$\xi'_{(u,k)}(t) = \xi(u^{-2/\alpha}(a(ku^{-2/\alpha}S + t_0))^{-1/\alpha} t + ku^{-2/\alpha}S + t_0), \ t \in \mathbb{R}, \ a_{u,k} = (a(ku^{-2/\alpha}S + t_0))^{1/\alpha},$$

$$A'_i(u_k) = \left\{ \sup_{t \in [a_{u,k}T_i, a_{u,k}(T_i + S)]} g(\xi'_{(u,k)}(t)) > u_k \right\}, \ i = 1, 2.$$
For any $W > 0$ and all $u$ large
\[
P \{ A_1(u_k), A_2(u_k) \} = \int_{-\infty}^{\infty} \mathbb{P} \left\{ A'_1(u_k), A'_2(u_k) \middle| g(\xi'_{(u_k)}(0)) = x \right\} p(x) dx
\]
\[= u_k^{-1} \int_{-\infty}^{\infty} \mathbb{P} \left\{ A'_1(u_k), A'_2(u_k) \middle| g(\xi'_{(u_k)}(0)) = u_k \right\} p(u_k, y) dy
\]
\[= u_k^{-1} \int_{-W}^{W} \mathbb{P} \left\{ A'_1(u_k), A'_2(u_k) \middle| g(\xi'_{(u_k)}(0)) = u_k \right\} p(u_k, y) dy
\]
\[+ \mathbb{P} \left\{ A'_1(u_k), A'_2(u_k), g(\xi'_{(u_k)}(0)) \leq u_k - u_k^{-1} W \right\}
\]
\[+ \mathbb{P} \left\{ A'_1(u_k), A'_2(u_k), g(\xi'_{(u_k)}(0)) > u_k + u_k^{-1} W \right\}
\]
\[=: I_1(u) + I_2(u) + I_3(u),
\]
where $u_k, y = u_k - u_k^{-1} y$ and $p(x)$ is the density function of $g(\xi(t_0))$ which is showed in Theorem 4.2. Similar to the proof of Lemma 3.2 ii), we know
\[
\sup_{k \in K_u} \frac{I_2(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \leq \sup_{k \in K_u} \frac{\mathbb{P} \left\{ A'_1(u_k), g(\xi'_{(u_k)}(0)) \leq u_k - u_k^{-1} W \right\}}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \to 0, \ u \to \infty.
\]
Further, as in (39)
\[
\sup_{k \in K_u} \frac{I_3(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \leq \sup_{k \in K_u} \frac{\mathbb{P} \{ g(\xi(t_0)) \leq u_k - u_k^{-1} W \}}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}} \to 0, \ u \to \infty, \ W \to \infty,
\]
then we can choose $W$ large enough such that $\frac{I_3(u)}{\mathbb{P} \{ g(\xi(t_0)) > u_k \}}$ less that $\mathbb{Q}_1 \exp \left( -\frac{a}{\bar{a}} |T_2 - T_1 - S|^{\alpha} \right)$. Set below
\[
\eta(t) = \sqrt{2} B_{\alpha}(t) - \frac{1}{\gamma} - \frac{a(t)}{\bar{a}}, \quad \bar{p}_{uk}(y, z) = \mathbb{P} \left\{ A'_1(u_k, z), A'_2(u_k, z) \middle| g(\xi'_{(u_k)}(0)) = u_k \right\},
\]
\[
\bar{A}_i'(u_k, z) = \sup_{t \in [zT_i, z(T_i + S)]} g(\xi'_{(u_k)}(t)) > u_k, \quad z \in [a, \bar{a}], \quad i = 1, 2,
\]
and $\underline{a} = \inf_{t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]} a(t)$ and $\bar{a} = \sup_{t \in [t_0 - \varepsilon_0, t_0 + \varepsilon_0]} a(t)$. First note that
\[
I_3(u) \leq \sup_{z \in [\underline{a}, \bar{a}]} \frac{u_k^{-1}}{\bar{P}_{uk}(y, z)} \int_{-W}^{W} \bar{p}_{uk}(y, z) p(u_k, y) dy.
\]
Similarly to the arguments as (41) in proof of Lemma 3.2 we have that for $u$ large enough
\[
\sup_{k \in K_u} \frac{u_k^{-1} \int_{-W}^{W} \bar{p}_{uk}(y, z) p(u_k, y) dy}{\mathbb{E} \mathbb{P} \{ g(\xi(t_0)) > u_k \}} - \int_{-W}^{W} e^{y \mathbb{P} \{ g(\xi(t_0)) > u_k \}} \left\{ \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right\} \left( e^{y \mathbb{P} \{ g(\xi(t_0)) > u_k \}} - 1 \right) dy
\]
\[\leq \sup_{k \in K_u} \left( e^{W} \int_{-W}^{W} \left| \frac{u_k^{-1} p(u_k, y)}{\mathbb{E} \mathbb{P} \{ g(\xi(t_0)) > u_k \}} - 1 \right| \right) dy
\]
\[+ \sup_{k \in K_u} \left( e^{W} \int_{-W}^{W} \left| \bar{P}_{uk}(y, z) - \mathbb{P} \left\{ \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right\} \right| \right) dy,
\]
where the first term goes to zero as $u \to \infty$ by (40).

By similar arguments as in Step 2 of the Lemma 3.2 we have for any $(y, z) \in [-W, W] \times [\underline{a}, \bar{a}]
\[
c_u(y, z) := \sup_{k \in K_u} \left| \bar{P}_{uk}(y, z) - \mathbb{P} \left\{ \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right\} \right| \to 0, \ u \to \infty.
\]
By the well-known Severini-Egorov theorem, for any $\varepsilon > 0, W > 0$ the convergence
\[
c_u(y, z) \to 0, \ u \to \infty
is uniform for \((y, z) \in (\mathbb{R}^n \times [\tilde{a}, \tilde{b}]) \setminus K_\varepsilon\) with \(K_\varepsilon\) a measurable set with Lebesgue measure not exceeding \(\varepsilon\). Hence, we can write

\[
\lim_{u \to \infty} \sup_{k \in K_u} \sup_{(y, z) \in ((-W, W) \times [\tilde{a}, \tilde{b}]) \setminus K_\varepsilon} \left| \bar{P}_{u_k}(y, z) - \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right) \right\} \right| = 0.
\]

Since \(\varepsilon > 0\) can be chosen arbitrary small, we obtain

\[
\lim_{u \to \infty} \sup_{k \in K_u} \sup_{z \in [\tilde{a}, \tilde{b}]} e^{uW} \int_{-W}^{W} \left| \bar{P}_{u_k}(y, z) - \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right) \right\} \right| dy = 0.
\]

Thus we have as \(u \to \infty\)

(51) \[
\sup_{k \in K_u} \sup_{z \in [\tilde{a}, \tilde{b}]} \frac{1}{\Psi(u_k)} \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > u_k \right) \right\} \leq \mathbb{Q}_2 \exp \left( -\frac{\alpha}{8} |T_2 - T_1 - S|^\alpha \right).
\]

Further, we have for \(W > 0\)

\[
\mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > u_k \right) \right\}
= \frac{1}{\sqrt{2\pi u_k}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u_k - y)^2} \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > u_k \right) \right\} dy
= \frac{1}{\sqrt{2\pi u_k}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u_k - y)^2} \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > u_k \right) \right\} dy
\geq \Psi(u_k) \int_{-W}^{W} e^{uy} \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > y \right) \right\} dy,
\]

where

\[\xi_{(u,k),1}^{\prime}(t) = u_k \left( \xi_{(u,k),1}(t) - u_k \right) + y|\xi_{(u,k),1}(0)| = u_k y.\]

Thus we have

(52) \[
\sup_{k \in K_u} \sup_{z \in [\tilde{a}, \tilde{b}]} \int_{-W}^{W} e^{uy} \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{(u,k),1}^{\prime}(t) > y \right) \right\} dy \leq \mathbb{Q}_2 \exp \left( -\frac{\alpha}{8} |T_2 - T_1 - S|^\alpha \right).
\]

From condition (10), we obtain uniformly with respect to \(t \in [zT_1, z(T_1 + S)] \cup [zT_2, z(T_2 + S)], z \in [\tilde{a}, \tilde{b}], y \in [-W, W]\) and \(k \in K_u\) that

(53) \[
\mathbb{E} \left\{ \xi_{u,k}^{\prime}(t) \right\} \to -|t|^\alpha, \quad u \to \infty,
\]

and also for any \(t, t' \in [zT_1, z(T_1 + S)] \cup [zT_2, z(T_2 + S)], z \in [\tilde{a}, \tilde{b}], y \in \mathbb{R}\) and \(k \in K_u\)

(54) \[
\text{Var} \left( \xi_{u,k}^{\prime}(t) - \xi_{u,k}^{\prime}(t') \right) \to 2|t - t'|^\alpha, \quad u \to \infty.
\]

Consequently, since supremum is a continuous functional, for any \(y \in [-W, W]\), we have

\[
\lim_{u \to \infty} \sup_{k \in K_u} \sup_{y \in [-W, W], z \in [\tilde{a}, \tilde{b}]} \left| \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \xi_{u,k}^{\prime}(t) > y \right) \right\} - \mathbb{P} \left\{ \bigcap_{i=1,2} \left( \sup_{t \in [zT_i, z(T_i + S)]} \eta(t) > y \right) \right\} \right| = 0.
\]
The above equality implies
\[
\lim_{u \to \infty} \sup_{y \in K} \left| \int_{-w}^{w} e^{y/P} \left\{ \prod_{i=1,2} \left( \sup_{t \in [z,T] \cap (T+S)} \xi^{y}_{u,k}(t) > y \right) \right\} dy - \int_{-w}^{w} e^{y/P} \left\{ \prod_{i=1,2} \left( \sup_{t \in [z,T] \cap (T+S)} \eta(t) > y \right) \right\} dy \right| = 0,
\]
which combined with (50), (51) and (52) yields
\[
(55) \quad P\{g(\xi(t_0)) > u_k\} \leq Q_2 \exp \left( -\frac{a}{8} |T_2 - T_1 - S|^\alpha \right)
\]
establishing the proof. □

4.2. Appendix B. Below we state two results which are used in our proofs.

**Theorem 4.1.** Let \( X(t), t \in [0, T] \), be a centered vector Gaussian process taking valued in \( \mathbb{R}^n \) with continuous trajectories and \( G(x) \), \( x \in \mathbb{R}^n \) is such that \( |G(x)| \leq G|x|^p \) for some \( G, p > 0 \) and all \( x \). Assume that the covariance matrix
\[
R_t = \mathbb{E} \{ X(t) X(t)^\top \}
\]
is uniformly non-degenerated in \([0, T] \). Assume also that for some \( \Gamma, \gamma > 0 \),
\[
\mathbb{E} \left\{ |X(t) - X(s)|^2 \right\} \leq \Gamma |t - s|^\gamma.
\]
Denote
\[
m(T) = \min_{t \in [0, T], |e| = 1} \left| R_t^{-1/2} e \right|.
\]
Then there exist constants \( \Gamma_1 \) and \( \gamma_1 \) such that for all \( u \geq 1 \),
\[
P \left\{ \sup_{t \in [0, T]} G(X(t)) > u \right\} \leq \Gamma_1 u^{-\gamma_1} \exp \left( -\frac{m^2(S) u^{2/\beta}}{2G^{2/\beta}} \right).
\]
This theorem is derived from [1][Corollary 2].

Next, we give the theorem about asymptotic expansions for probability density and tail probability of the Gaussian random chaos in \([18, 21, 22] \).

Denote by \( g_{d-1-m}(\varphi) \) any non-singular \((d - 1 - m)\)-sub-matrix of \( g''(\varphi) \) in (5) and \( J(r, \varphi) \) the same as in (4).

With \( \varphi \) fixed, determinants of all such sub-matrices are the same as can be seen by applying suitable orthogonal transformations.

**Theorem 4.2.** If \( g(\varphi) \) satisfies Condition 2.1 and the probability density \( p_g(\xi)(x) \) of \( g(\xi) \) exists, then the following asymptotic relations holds,
\[
\begin{align*}
(56) \quad p_g(\xi)(x) &= \frac{1}{p \bar{g}} \left( \frac{x}{\bar{g}} \right)^{m-1} \exp \left( -\frac{x^2}{2g_0^2} \right) (1 + o(1)), \\
(57) \quad \mathbb{P} \{ g(\xi) > x \} &= h_0 \left( \frac{x}{\bar{g}} \right)^{m-1} \exp \left( -\frac{x^2}{2g_0^2} \right) (1 + o(1)),
\end{align*}
\]
as \( x \to \infty \), where for the case (i) in Condition 2.1
\[
h_0 := \frac{1}{\sqrt{2\pi}} \left( \frac{p \bar{g}}{2} \right)^{d-1/2} \sum_{\varphi \in \mathcal{M}_\varphi} \frac{J(1, \varphi)}{\sqrt{|\det g''(\varphi)|}};
\]
and for the case (ii) in Condition 2.1
\[
h_0 := \frac{1}{(2\pi)^{(m+1)/2} p \bar{g}^{d-m/2}} \int_{\mathcal{M}_\varphi} \frac{J(1, \varphi)}{\sqrt{|\det g''_{d-1-m}(\varphi)|}} dV_\varphi,
\]
with \( dV_\varphi \) the elementary volume in \( \mathcal{M}_\varphi \).
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