Some associative submanifolds of the squashed 7-sphere

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Abstract

The squashed 7-sphere $S^7$ is a 7-sphere with an Einstein metric given by the canonical variation and its cone $\mathbb{R}^8 - \{0\}$ has full holonomy $\text{Spin}(7)$. There is a canonical calibrating 4-form $\Phi$ on $\mathbb{R}^8 - \{0\}$. A minimal 3-submanifold in $S^7$ is called associative if its cone is calibrated by $\Phi$.

In this paper, we classify two types of fundamental associative submanifolds in the squashed $S^7$. One is obtained by the intersection with a 4-plane and the other is homogeneous. Then we study their infinitesimal associative deformations and explicitly show that all of them are integrable.

1 Introduction

A Riemannian 7-manifold $(Y, g)$ is called a nearly parallel $G_2$-manifold if its cone $(C(Y), \overline{g}) = (\mathbb{R}_{>0} \times Y, dr^2 + r^2 g)$ has holonomy contained in $\text{Spin}(7)$. The existence of such a structure is equivalent to that of a spin structure with a real Killing spinor (II), which is also used in supergravity and superstring theory in physics. There is a canonical calibrating 4-form $\Phi$ on $C(Y)$. A 3-submanifold $M$ in $Y$ is called associative if its cone $C(M)$ is Cayley, i.e. it is calibrated by $\Phi$.

By definition, Sasaki-Einstein manifolds, especially 3-Sasakian manifolds, admit nearly parallel $G_2$-structures. Moreover, every compact 3-Sasakian 7-manifold admits a second nearly parallel $G_2$-structure whose cone metric has full holonomy $\text{Spin}(7)$ (II). The 7-sphere $S^7$ with this second nearly parallel $G_2$-structure is called the squashed $S^7$.

Associative submanifolds in the standard $S^7$ were studied by Lotay [7]. In this paper, we study some fundamental associative submanifolds in the squashed $S^7$ and compare the properties. The most basic examples of associative submanifolds in the squashed $S^7$ are fibers of the Hopf fibration $\pi : S^7 \to S^4$. The Hopf lifts of $I'_1$-holomorphic curves in $\mathbb{C}P^3$ are also associative in the squashed $S^7$ (Proposition 4.3), where $I'_1$ is an almost complex structure on $\mathbb{C}P^3$ given by (4.3).

First, we classify associative submanifolds obtained by the intersection with a 4-plane.

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Theorem 1.1. Let $V^4 \subset \mathbb{R}^8$ be a 4-plane. Suppose that $V^4 \cap S^7$ is associative in the squashed $S^7$. Then up to the $\text{Sp}(1)\text{Sp}(2)$-action, $V$ is either

$$V_1 = \{(z_1, z_2, 0, 0) \in \mathbb{C}^4\}, \text{ which is a } \mathbb{H}-\text{plane, or}$$

$$V_2 = \{(z_1, 0, z_3, 0) \in \mathbb{C}^4\}, \text{ which is a horizontal } I_1\text{-plane.}$$

In other words, the space $\mathcal{M}$ of 4-planes whose intersections with $S^7$ are associative is described as

$$\mathcal{M} = \text{Sp}(1)\text{Sp}(2)/K_1 \sqcup \text{Sp}(1)\text{Sp}(2)/K_2,$$

where $K_1 = \text{Sp}(1)(\text{Sp}(1) \times \text{Sp}(1))$, and $K_2 = \text{U}(1)\text{U}(2)$.

Remark 1.2. Thus $\mathcal{M}$ consists of two connected components, while the corresponding space in the standard $S^7$ is a homogeneous space $\text{Spin}(7)/K$, where $K = \text{SU}(2)^3/\mathbb{Z}_2$ (10).

Both $V_j \cap S^7 (j = 1, 2)$ are totally geodesic submanifolds in the squashed $S^7$. Actually, we should classify totally geodesic associative submanifolds, but it would be difficult because the squashed $S^7$ is neither a space of the constant curvature nor a symmetric space. It is just a homogeneous space $\text{Sp}(1)\text{Sp}(2)/\text{Sp}(1)\text{Sp}(1)$.

Next, we classify homogeneous associative submanifolds.

Theorem 1.3. Let $A$ be a connected associative 3-fold in the squashed $S^7 \subset \mathbb{C}^4$ which is the orbit of a closed Lie subgroup of $\text{Sp}(1)\text{Sp}(2)$. Then, up to the $\text{Sp}(1)\text{Sp}(2)$-action, $A$ is one of the following.

1. $L_1 = V_1 \cap S^7$,
2. $L_2 = V_2 \cap S^7$,
3. $A_1 = T^3 \cdot \frac{1}{2}(1, 1, 1, i) \cong T^3$, where $T^3$-action is given by (6.1),
4. $A_2 = \text{SU}(2) \cdot \frac{1}{2}(1, 0, 0, 0) \cong \text{SU}(2)/\mathbb{Z}_3$, where $\text{SU}(2)$-action is given by (6.7),
5. $A_3 = \text{SU}(2) \cdot \frac{1}{2}(0, 0, 1, 0) \cong \text{SU}(2)$, where $\text{SU}(2)$-action is given by (6.9).

Remark 1.4. Since $T^3$ in (5.1) and $\text{SU}(2)$ in (6.9) are contained in $\text{SU}(4) \subset \text{Spin}(7)$ by an appropriate change of coordinates, we obtain the similar orbits $A_1, A_2,$ and $A_3$ as in the standard $S^7$ case (7). However, since $G_2$ is not contained in $\text{Sp}(1)\text{Sp}(2)$, there are no corresponding associative orbits in the squashed $S^7$ to Lagrangian (totally real) submanifolds in $S^6$ classified by (9).

Remark 1.5. The examples $A_1, A_2,$ and $A_3$ are Hopf lifts of $I_1'$-holomorphic curves in $\mathbb{C}P^3$, where $I_1'$ is an almost complex structure on $\mathbb{C}P^3$ given by (4.3).

In particular, $A_2$ (resp. $A_3$) is a Hopf lift of a horizontal holomorphic curve (resp. a null-torsion $I_1'$-holomorphic curve defined in Definition 7.15) in $\mathbb{C}P^3$.

Thus, unfortunately, we cannot find homogeneous examples which do not arise from other geometries as in the standard $S^7$ case. It is a further problem to find an associative submanifold which is not congruent to the fiber of $S^7 \rightarrow S^4$ or the Hopf lift of an $I_1'$-holomorphic curve in $\mathbb{C}P^3$ by the $\text{Sp}(1)\text{Sp}(2)$-action.

However, by virtue of this property, we can explain their associative deformations.
Theorem 1.6. The associative deformations of $L_1, L_2,$ and $A_1$ are trivial, i.e. all the associative deformations come from the $\text{Sp}(1)\text{Sp}(2)$-action, while $A_2$ and $A_3$ have nontrivial associative deformations.

All the associative deformations of $A_2$ consist of deformations of $p_1(A_2)$ as a horizontal holomorphic curve, i.e. those from the $\text{PGL}(4, \mathbb{C})$-action on $\mathbb{C}P^3$ via the Hopf lift, and those from actions of $j, k \in \text{Sp}(1)$, where $p_1 : S^7 \to \mathbb{C}P^3$ is a projection.

All the associative deformations of $A_3$ consist of deformations of $p_1(A_3)$ as a null-torsion holomorphic curve, and those from actions of $j, k \in \text{Sp}(1)$.

Remark 1.7. The deformations of the associative submanifolds in the standard $S^7$ are studied by the author ([6]). We could not explain the deformation space of the associative submanifold corresponding to $A_3$, which did not arise from other known geometries. However, in the squashed $S^7$ case, the associative deformations of $A_3$ are explained by the property in Remark 1.5. We use the one-to-one correspondence between null-torsion $J^I$-holomorphic curves and horizontal holomorphic curves in $\mathbb{C}P^3$ ([12]).

This paper is organized as follows. In Section 2, we review the fundamental facts of $G_2$ and $\text{Spin}(7)$ geometry. In Section 3, we review the canonical variation and summarize some useful equations. In Section 4, we apply it to the 7-sphere $S^7$ and describe the nearly parallel $G_2$-structure on the squashed $S^7$ explicitly. Then we give basic examples of associative submanifolds in the squashed $S^7$.

In Section 5, we prove Theorem 1.1 by choosing a “good” frame by $\text{Sp}(1)\text{Sp}(2)$-action. In Section 6, we prove Theorem 1.3 as an analogue of [7], [9]. In Section 7, we prove Theorem 1.6 by using the representation theory as [6], [10].

2 Preliminaries

2.1 $G_2$ and Spin(7) geometry

Definition 2.1. Define a 3-form $\phi_0$ on $\mathbb{R}^7$ by

$$\phi_0 = dx_{123} + dx_1(dx_{45} + dx_{67}) + dx_2(dx_{46} - dx_{57}) - dx_3(dx_{47} + dx_{56}),$$

where $(x_1, \ldots, x_7)$ is the standard coordinate on $\mathbb{R}^7$ and wedge signs are omitted. The Hodge dual of $\phi_0$ is given by

$$*\phi_0 = dx_{4567} + dx_{23}(dx_{67} + dx_{45}) + dx_{13}(dx_{57} - dx_{46}) - dx_{12}(dx_{56} + dx_{47}).$$

Decompose $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ and denote by $x_0$ the coordinate on $\mathbb{R}$. Define a self-dual 4-form $\Phi_0$ on $\mathbb{R}^8$ by

$$\Phi_0 = dx_0 \wedge \phi_0 + *\phi_0.$$ 

If we identify $\mathbb{R}^8 \simeq \mathbb{C}^4$ via $\mathbb{R}^8 \ni (x_0, \ldots, x_7) \mapsto (x_0 + ix_1, x_2 + ix_3, x_4 + ix_5, x_6 + ix_7) =: (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, then $\Phi_0$ is described as

$$\Phi_0 = \frac{1}{2} \omega_0 \wedge \omega_0 + \text{Re} \Omega_0,$$

where $\omega_0 = \frac{i}{4} \sum_{j=1}^{4} dz_j$ and $\Omega_0 = dz_{1234}$ are the standard Kähler form and the holomorphic volume form on $\mathbb{C}^4$, respectively.
The stabilizers of $\varphi_0$ and $\Phi_0$ are the exceptional Lie group $G_2$ and Spin(7), respectively:

$$G_2 = \{ g \in GL(7, \mathbb{R}); g^* \varphi_0 = \varphi_0 \}, \quad \text{Spin}(7) = \{ g \in GL(8, \mathbb{R}); g^* \Phi_0 = \Phi_0 \}.$$ 

The Lie group $G_2$ fixes the standard metric $g_0 = \sum_{i=1}^{7} (dx_i)^2$ and the orientation on $\mathbb{R}^7$. They are uniquely determined by $\varphi_0$ via

$$6g_0(v_1, v_2)\text{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0, \quad (2.1)$$

where $\text{vol}_{g_0}$ is a volume form of $g_0$, $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

Similarly, Spin(7) fixes the standard metric $h_0 = \sum_{i=0}^{7} (dx_i)^2$ and the orientation on $\mathbb{R}^8$. They are uniquely determined by $\Phi_0$ via

$$\Phi_0^2 = 14\text{vol}_{h_0}, \quad (i(w_2)i(w_1))\Phi_0^2 \wedge \Phi_0 = 6\|w_1 \wedge w_2\|^2 \text{vol}_{h_0}, \quad (2.2)$$

where $\text{vol}_{h_0}$ is a volume form of $h_0$, and $w_i \in T(\mathbb{R}^8)$.

**Definition 2.2.** Let $Y$ be an oriented 7-manifold and $\varphi$ a 3-form on $Y$. A 3-form $\varphi$ is called a $G_2$-structure on $Y$ if for each $y \in Y$, there exists an oriented isomorphism between $T_yY$ and $\mathbb{R}^7$ identifying $\varphi_y$ with $\varphi_0$. From [22], $\varphi$ induces the metric $g$ and the volume form on $Y$. A $G_2$-structure $\varphi$ is said to be **nearly parallel** if $d\varphi = 4 \ast \varphi$. We call a manifold with a nearly parallel $G_2$-structure a **nearly parallel $G_2$-manifold** for short. A $G_2$-structure $\varphi$ is called **torsion-free** if $d\varphi = 0, d \ast \varphi = 0$.

Let $X$ be an oriented 8-manifold and $\Phi$ a 4-form on $X$. A 4-form $\Phi$ is called a Spin(7)-structure on $X$ if for each $x \in X$, there exists an oriented isomorphism between $T_xX$ and $\mathbb{R}^8$ identifying $\Phi_x$ with $\Phi_0$. From [22], $\Phi$ induces the metric $h$ and the volume form on $X$. A Spin(7)-structure $\Phi$ is called **torsion-free** if $d\Phi = 0$.

**Lemma 2.3.** [17] A $G_2$-structure $\varphi$ is torsion-free if and only if $\text{Hol}(g) \subset G_2$. A Spin(7)-structure $\Phi$ is torsion-free if and only if $\text{Hol}(h) \subset \text{Spin}(7)$.

**Lemma 2.4.** The 3-form $\varphi$ is a nearly parallel $G_2$-structure if and only if its Riemannian cone $C(Y) = \mathbb{R}_{>0} \times Y$ admits a torsion-free Spin(7)-structure $\Phi = r^3dr \wedge \varphi + r^4 \ast \varphi$ with the induced cone metric $\mathcal{F} = dr^2 + r^2g$.

Next, we give a summary of the facts about the submanifolds. Let $Y$ be a manifold with a $G_2$-structure $\varphi$ and the induced metric $g$.

**Lemma 2.5.** [8] For every oriented $k$-dimensional subspace $V^k \subset T_pY$ ($\forall p \in Y, k = 3, 4$), we have $\varphi|_{V^3} \leq \text{vol}_{V^3}, \ast \varphi|_{V^4} \leq \text{vol}_{V^4}$. An oriented 3-submanifold $L^3 \subset Y$ is called **associative** if $\varphi|_{TL^3} = \text{vol}_{L^3}$. An oriented 4-submanifold $L^4$ is called **coassociative** if $\ast \varphi|_{TL^4} = \text{vol}_{L^4}$.

**Lemma 2.6.** [8] An oriented 3-submanifold $L^3$ is associative if and only if $\ast \varphi(v_1, v_2, v_3, \cdot) = 0$ for any $v_j \in TL^3$. An oriented 4-submanifold $L^4$ is coassociative if and only if $\ast \varphi|_{TL^4} = 0$.

**Remark 2.7.** Define the cross product $\times : TY \times TY \to TY$ by

$$g(u \times v, w) = \varphi(u, v, w)$$

for $u, v, w \in TY$. When $L^3$ is associative, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ satisfying $e_3 = e_1 \times e_2$ at any point in $L^3$.

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Definition 2.8. Let $X$ be a manifold with a Spin(7)-structure $\Phi$. Then for every oriented 4-dimensional subspace $W \subset T_x X (\forall x \in X)$, we have $\Phi|_W \leq \text{vol}_W$. An oriented 4-submanifold $N \subset X$ is called \textbf{Cayley} if $\Phi|_{TN} = \text{vol}_N$.

Lemma 2.9. Let $(Y, \varphi, g)$ be a nearly parallel $G_2$-manifold and $L \subset Y$ be an oriented 3-submanifold. By Lemma 2.4, $C(Y)$ is a manifold with a torsion-free Spin(7)-structure $\Phi$. Then $L \subset Y$ is associative if and only if $C(L) \subset C(Y)$ is Cayley.

Lemma 2.10. \cite{7} There are no coassociative submanifolds of a nearly parallel $G_2$-manifold $(Y, \varphi, g)$.

Proof. If $L$ is a coassociative submanifold, we have $\varphi|_{TL} = 0$, which implies that $4\text{vol}_L = 4 \star \varphi|_{TL} = d\varphi|_{TL} = 0$. This is a contradiction. \qed

3 Canonical Variation

3.1 Riemannian submersion

We give a summary of Chapter 9 of \cite{2}. Let $(M, g)$ and $(B, h)$ be Riemannian manifolds and suppose that there exists a Riemannian submersion $\pi: (M, g) \to (B, h)$. Decompose the tangent bundle $TM = V \oplus H$, where a vertical distribution $V$ is a vector subbundle tangent to the fibers $\pi: M \to B$, and a horizontal distribution $H$ is the orthogonal complement bundle of $V$. Denote by $\nabla$ the Levi-Civita connection of $g$.

Definition 3.1. Define $(1,2)$-tensors $A, T \in C^\infty(M, \otimes^2 T^*M \otimes TM)$ by

$$A_E F = (\nabla_{E\uparrow} F \downarrow)\uparrow + (\nabla_{E\downarrow} F \uparrow)\downarrow, T_E F = (\nabla_{E\perp} F \perp)\uparrow + (\nabla_{E\uparrow} F \downarrow)\perp,$$

for $E, F \in \mathfrak{X}(M)$, where $\uparrow : TM \to H$ and $\perp : TM \to V$ are projections.

Remark 3.2. The distribution $H$ is involutive if and only if $A \equiv 0$. The fibers of $\pi: M \to B$ are totally geodesic if and only if $T \equiv 0$.

In the following, we suppose that $T \equiv 0$.

Lemma 3.3. Let $X, Y$ be the horizontal vector fields, $U, V$ be the vertical vector fields, and $E, F$ be any vector fields on $M$. We have

$$A_U X = 0, A_U V = 0, A_X U = (\nabla_X U)\uparrow, A_X Y = (\nabla_X Y)\downarrow,$$

$$A_Y Y = -A_Y X, A_X Y = \frac{1}{2}[X, Y]\downarrow, g(A_X E, F) = -g(E, A_X F),$$

which implies that

$$\nabla_{U\uparrow} V = (\nabla_{U\downarrow} V)\downarrow, \nabla_{U\downarrow} X = (\nabla_{U\uparrow} X)\uparrow, \nabla_X U = (\nabla_X U)\downarrow + A_X U, \nabla_X Y = A_X Y + (\nabla_X Y)\uparrow.$$
Remark 3.4. Usually, we set $t = 1$ for simplicity. However, we introduce a parameter $t$ to define the nearly parallel $G_2$-structure. See Proposition 4.3.

Denote by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}$. Set $(1,2)$-tensors $\tilde{A}$ and $\tilde{T}$ as in Definition 3.1.

Remark 3.5. The assumption $T \equiv 0$ implies that $\tilde{T} \equiv 0$ for all $s, t > 0$.

Under the canonical variation, the tensor $A$ in Definition 3.1 and the Levi-Civita connection are changed as follows.

Lemma 3.6. Let $X,Y$ be the horizontal vector fields, and $U,V$ be the vertical vector fields on $M$. We have

\[
\tilde{\nabla}_X Y = A_X Y, \quad \tilde{\nabla}_X U = \tilde{T}_X U, \\
\tilde{\nabla}_U V = \nabla_U V, \quad \tilde{\nabla}_U X = \nabla_U X, \\
\tilde{\nabla}_U X = \frac{s^2}{t^2} (\nabla_U X)^\top + \left(1 - \frac{s^2}{t^2}\right) [U, X]^\top.
\]

This lemma implies the following useful equation.

Lemma 3.7. For $E_1, E_2 \in \mathfrak{X}(M)$, we have

\[
\tilde{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 = \left(-1 + \frac{s^2}{t^2}\right) (A_{E_1} E_2^\perp + A_{E_2} E_1^\perp).
\]

4 Nearly parallel $G_2$-structure on the squashed $S^7$

The standard $S^7$ admits a canonical nearly parallel $G_2$-structure. By the canonical variation, we obtain the second nearly parallel $G_2$-structure on $S^7$ (Proposition 4.3). First, we review a 3-Sasakian structure on $S^7$.

4.1 3-Sasakian structure on $S^7$

Consider the following Lie groups:

\[
\text{Sp}(1) = \{a_1 + a_2 j \in \mathbb{H}; a_i \in \mathbb{C}, |a_1|^2 + |a_2|^2 = 1\}, \\
\text{Sp}(2) = \{g \in \text{GL}(2, \mathbb{H}); g \text{ preserves the metric on } \mathbb{H}^2\} \\
= \{g \in \text{U}(4); g J g = J\} \\
= \{(u, J u, v, J v); u, v \in \mathbb{C}^4, |u| = |v| = 1, \langle v, u \rangle_C = \langle v, J u \rangle_C = 0\},
\]

where $J = \begin{pmatrix} J' & 0 \\ 0 & J' \end{pmatrix}$, $J' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $\langle \cdot, \cdot \rangle_C : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}$ is a standard Hermitian metric on $\mathbb{C}^4$.

Let $\text{Sp}(1) \times \text{Sp}(2)$ act on $\mathbb{H}^2$ by

\[
(q, A) \cdot (q_1, q_2) = q(q_1, q_2)^{\top} A,
\]
where \((q, A) \in \text{Sp}(1) \times \text{Sp}(2), (q_1, q_2) \in \mathbb{H}^2\). Via the identification \(\mathbb{C}^4 \ni (z_1, \ldots, z_4) \mapsto (z_1 + z_2j, z_3 + z_4j) \in \mathbb{H}^2\), the \text{Sp(1)}-action on \(\mathbb{C}^4\) is described as

\[
(a_1 + a_2j) \cdot u = a_1u + a_2j\overline{u},
\]

where \(u \in \mathbb{C}^4\), and \(\text{Sp}(2) \subset U(4)\) acts on \(\mathbb{C}^4\) canonically. By definition, the \text{Sp(1)}-action commutes with the \text{Sp(2)}-action.

The actions of \(i, j, k \in \text{Sp}(1)\) induce complex structures \(I_1, I_2, I_3\) on \(\mathbb{C}^4\), respectively, and hence induce the 3-Sasakian structure \(\{(\Phi_i, \xi_i, \eta_i, g)\}_{i=1, 2, 3}\) on \(S^7\), where \(g\) is a standard metric on \(S^7\), and a 1-form \(\eta_i \in \Omega^1(S^7)\), and a (1, 1)-tensor \(\Phi_i \in C^\infty(S^7, \text{End}(TS^7))\) are defined by

\[
(\xi_i)_2 = -I_i(z) (z \in \mathbb{C}^4), \ \eta_i = g(\xi_i, \cdot), \ \Phi_i = \begin{cases} I_i & \text{on Ker}\eta_i \\ 0 & \text{on } \mathbb{R}\xi_i. \end{cases}
\]

Note that the following conditions are satisfied:

\[
\Phi_{i+2} = \Phi_i \circ \Phi_{i+1} - \eta_{i+1} \otimes \xi_i = -\Phi_{i+1} \circ \Phi_i + \eta_i \otimes \xi_{i+1},
\]

\[
\xi_{i+2} = \Phi_i(\xi_{i+1}) = -\Phi_{i+1}(\xi_i), \ \eta_{i+2} = \eta_i \otimes \Phi_{i+1} = -\eta_{i+1} \circ \Phi_i,
\]

where \(i \in \mathbb{Z}/3\). These tensors are described explicitly as follows.

**Lemma 4.1.**

\[
\xi_1 = -i(z_1, z_2, z_3, z_4), \ \xi_2 = i(\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3), \ \xi_3 = i(\overline{z}_2, -\overline{z}_1, \overline{z}_4, -\overline{z}_3),
\]

\[
\eta_1 = \text{Im} \left( \sum_{j=1}^4 z_j d\overline{z}_j \right), \ \eta_2 + i\eta_3 = -z_1 dz_2 + z_2 dz_3 - z_3 dz_4 + z_4 dz_3,
\]

\[
d\eta_1 = -i \sum_{j=1}^4 dz_j = -2g(\Phi_1(\cdot, \cdot), d(\eta_2 + i\eta_3) = -2(dz_{12} + dz_{34}).
\]

### 4.2 Second nearly parallel \(G_2\)-structure on \(S^7\)

Applying the canonical variation to a Riemannian submersion \(\pi : S^7 \to S^4 = \mathbb{H}P^1\), we obtain the second nearly parallel \(G_2\)-structure \((\tilde{\varphi}, \tilde{g})\) on \(S^7\). Denote by \(\omega_i = \frac{1}{2} d\eta_i = \frac{1}{2} dp_i((\cdot)^\top, (\cdot)^\top) \in \Omega^2(S^7)\) the covariant differentiation of \(\frac{1}{2}\eta_i\), where \(\top : TS^7 \to T^*S^7\) is a canonical projection. In other words, we have

\[
\omega_1 = d\eta_1 + \eta_{23}, \ \omega_2 = d\eta_2 + \eta_{31}, \ \omega_3 = d\eta_3 + \eta_{12}.
\]

**Remark 4.2.** We have \(\omega_i = -g(\Phi_i(\cdot)^\top, (\cdot)^\top)\). Take any unit vector \(X_0 \in \mathcal{H}\) and set \(X_i = \Phi_i(X_0)(i = 1, 2, 3)\). Denote by \(\{X^i\}\) the dual of \(\{X_i\}\). Then we have

\[
\omega_1 = -(X^{01} + X^{23}), \ \omega_2 = -(X^{02} + X^{31}), \ \omega_3 = -(X^{03} + X^{12}).
\]

**Proposition 4.3.** Define the Riemannian metric \(\tilde{g}\), a 3-form \(\tilde{\varphi} \in \Omega^3(S^7)\),
and the 4-form \( \ast \hat{\varphi} \in \Omega^4(S^7) \) on \( S^7 \) by

\[
\tilde{g}|_{V \times V} = \left( \frac{3}{5} \right)^2 g|_{V \times V}, \quad \tilde{g}|_{H \times H} = \left( \frac{3}{\sqrt{5}} \right)^2 g|_{V \times V}, \quad \tilde{g}|_{H \times V} = 0,
\]

\[
\hat{\varphi} = \frac{27}{25} \left( \frac{1}{5} \eta_{123} + \sum_{i=1}^{3} \eta_i \wedge \omega_i \right),
\]

\[
\ast \hat{\varphi} = \frac{27}{25} \left( \frac{1}{2} \sum_{i=1}^{3} \omega_i^2 + \frac{3}{5} (\eta_{23} \wedge \omega_1 + \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3) \right).
\]

Then \((\hat{\varphi}, \tilde{g})\) is a nearly parallel \(G_2\)-structure with \(\text{Hol}(\tilde{g}) = \text{Spin}(7)\) and \(\ast \hat{\varphi}\) is a Hodge dual of \(\hat{\varphi}\) with respect to \(\tilde{g}\). We call \((S^7, \hat{\varphi}, \tilde{g})\) the **squashed** \(S^7\).

Outline of the proof. Set

\[
\tilde{g}|_{V \times V} = s^2 g|_{V \times V}, \quad \tilde{g}|_{H \times H} = t^2 g|_{V \times V}, \quad \tilde{g}|_{H \times V} = 0, \quad \hat{\varphi} = s^3 \eta_{123} + st^2 \sum_{i=1}^{3} \eta_i \wedge \omega_i,
\]

for \(s, t > 0\). We find \(s, t > 0\) satisfying \(d \hat{\varphi} = 4 \ast \hat{\varphi}\). Setting \(G_1 = s^3 \eta_{123}, G_2 = st^2 \sum_{i=1}^{3} \eta_i \wedge \omega_i\), we have

\[
\ast G_1 = \frac{t^4}{6} \sum_{i=1}^{3} \omega_i^2, \quad \ast G_2 = s^2 t^2 (\eta_{23} \wedge \omega_1 + \eta_{31} \wedge \omega_2 + \eta_{12} \wedge \omega_3),
\]

\[
d(\eta_{123}) = \frac{2}{s^2 t^2} \ast G_2, \quad d \left( \sum_{i=1}^{3} \eta_i \wedge \omega_i \right) = \frac{12}{t^4} \ast G_1 + \frac{2}{s^2 t^2} \ast G_2.
\]

Then we see that \(d \hat{\varphi} = \frac{12}{5} \ast G_1 + (\frac{2}{5} + \frac{2}{5}) \ast G_2\), and hence \(d \hat{\varphi} = 4 \ast \hat{\varphi}\) is equivalent to \(s = 5/3, t = 3/\sqrt{5}\). The metric \(\tilde{g}\) is not Sasaki-Einstein, and hence satisfies \(\text{Hol}(\tilde{g}) = \text{Spin}(7)\) by the classification of the dimensions of the spaces of real Killing spinors.

**Remark 4.4.** Proposition 4.3 is valid for any compact 3-Sasakian manifolds. The metric \(\tilde{g}\) is Einstein if and only if \(s = t\) or \(s = t/\sqrt{5}\).

Since \(\eta_1 = \text{Im}(z \bar{\pi})\), \(\eta_2 + i \eta_3 = -d^z \cdot Jz\), where \(z = \{z_1, z_2, z_3, z_4\}\), \(\text{Sp}(2)\) preserves \(\eta_j (j = 1, 2, 3)\). For \(g = a_1 + a_2 j \in \text{Sp}(1)\), we have \((q^* \eta_1, q^* \eta_2, q^* \eta_3) = (\eta_1, \eta_2, \eta_3)^T M_q\), where \(M_q \in \text{SO}(3)\) is described as

\[
M_q = \begin{pmatrix}
|a_1|^2 - |a_2|^2 & 2 \text{Im}(a_1 \bar{\pi}_2) & 2 \text{Re}(a_1 \bar{\pi}_2) \\
2 \text{Im}(a_1 a_2) & \text{Re}(a_1^2 + a_2^2) & \text{Im}(-a_1^2 + a_2^2) \\
-2 \text{Re}(a_1 a_2) & \text{Im}(a_1^2 + a_2^2) & \text{Re}(a_1^2 - a_2^2)
\end{pmatrix}.
\]

Hence we see that \(\text{Sp}(2)\) and \(\text{Sp}(1)\) preserve \(g|_{H \times H}, g|_{V \times V}, \tilde{g}\) and \(\hat{\varphi}\). In fact, we have the following.

**Lemma 4.5.** The automorphism group of the squashed \((S^7, \hat{\varphi}, \tilde{g})\) is \(\text{Sp}(1)\text{Sp}(2) = \text{Sp}(1) \times \text{Sp}(2)/\{\pm(1, 1)\}\).

**Remark 4.6.** In this paper, we often consider the subgroup of \(\text{Sp}(1)\text{Sp}(2)\). If there may be some confusion, denoting \(\text{Sp}(1) = \text{Sp}(1)_L\) and \(\text{Sp}(2) = \text{Sp}(2)_R\), we distinguish subgroups of \(\text{Sp}(1)\text{Sp}(2) = \text{Sp}(1)_L\text{Sp}(2)_R\).
Lemma 4.7. For any $E_1, E_2 \in \mathfrak{X}(S^7)$, we have
\[ \check{g}(E_1, E_2) = -\frac{36}{25} \sum_{j=1}^{3} \eta_j(E_1)\eta_j(E_2) + \frac{9}{5}g(E_1, E_2), \]
\[ \check{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 = \frac{4}{5} \Theta(E_1, E_2), \]
where $\Theta \in C^\infty(S^7, \otimes^2 T^*S^7)$ is defined by
\[ \Theta(E_1, E_2) = \sum_{i=1}^{3} (\eta_i(E_1)\Phi_i(E_2) + \eta_i(E_2)\Phi_i(E_1)). \]

Proof. The first equation is proved easily and we omit the proof. Set $(s,t) = (3/5, 3/\sqrt{15})$ in Lemma 3.7. Since $AxU = -\sum_{i=1}^{3} \eta_i(U)\Phi_i(X)$ for a horizontal vector $X$ and a vertical vector $U$, we have
\[ \check{\nabla}_{E_1} E_2 - \nabla_{E_1} E_2 = \frac{4}{5} \sum_{i=1}^{3} (\eta_i(E_1)\Phi_i(E_2^\top) + \eta_i(E_2)\Phi_i(E_1^\top)). \]

We easily see that the right hand side is equal to $\frac{4}{5} \Theta(E_1, E_2)$. □

4.3 Associative submanifolds of the squashed $S^7$

By the definition of $\tilde{\varphi}$ in Proposition 4.3, we see the following.

Remark 4.8. There are no horizontal associative submanifolds, i.e. there are no associative submanifolds whose tangent spaces are contained in $\mathcal{H}$.

Let $\pi : S^7 \rightarrow S^4$, $p_1 : S^7 \rightarrow \mathbb{C}P^3$, and $p_2 : \mathbb{C}P^3 \rightarrow S^4$ be canonical projections satisfying $\pi = p_2 \circ p_1$. Denote by $\mathcal{V}$ and $\mathcal{H}$ be the distributions of $\mathbb{C}P^3$ induced by $\mathcal{V}$ and $\mathcal{H}$, respectively. In other words, $\mathcal{V}$ is a vector subbundle of $T\mathbb{C}P^3$ tangent to the fibers $p_2$, and $\mathcal{H}$ is the orthogonal complement bundle of $\mathcal{V}$. Define the almost complex structure $I_1'$ on $\mathbb{C}P^3$ by
\[ I_1'|_{\mathcal{V}} = -I_1|_{\mathcal{V}}, I_1'|_{\mathcal{H}} = I_1|_{\mathcal{H}}. \]

(4.3)

The almost complex structure $I_1'$ is never integrable, and defines the nearly Kähler structure on $\mathbb{C}P^3$.

Proposition 4.9. Let $\Sigma \subset \mathbb{C}P^3$ be an $I_1'$-holomorphic curve. Then the Hopf lift $p_{1}^{-1}(\Sigma) \subset S^7$ of $\Sigma$ is associative in the squashed $S^7$.

Proof. Use the notation of Remark 4.2 and Proposition 4.3. Setting $\tilde{\eta}_1 = (3/5)\eta_1$ and $\tilde{X}^i = (3/\sqrt{15})X^i$, we have
\[ \tilde{\varphi} = \tilde{\eta}_1(\tilde{\eta}_23 - \tilde{X}^{12} - \tilde{X}^{23}) - \tilde{\eta}_2(\tilde{X}^{02} + \tilde{X}^{31}) - \tilde{\eta}_3(\tilde{X}^{03} + \tilde{X}^{12}). \]

Then we obtain $\tilde{\eta}_23 - \tilde{X}^{12} - \tilde{X}^{23} = -\tilde{G}(I_1', \cdot)$, where $\tilde{G} = \tilde{\eta}_2 \otimes \tilde{\eta}_2 + \tilde{\eta}_3 \otimes \tilde{\eta}_3 + \sum_{j=0}^{3} \tilde{X}^j \otimes \tilde{X}^j$, which gives the proof. □

Remark 4.10. The intersection of $\mathcal{H}$-plane and $S^7$, which is of the form $\pi^{-1}(\ast')$, is associative. If $\Sigma \subset \mathbb{C}P^3$ is a horizontal $I_1'$-holomorphic curve, where we call the curve $\Sigma$ horizontal if $T\Sigma \subset \mathcal{H}$, $\Sigma \subset \mathbb{C}P^3$ is also an $I_1'$-holomorphic curve.
5 Classification of Cayley planes

In this section, we prove Theorem 1.1. Let $V^4 \subset \mathbb{R}^8$ be a 4-plane. We classify the associative submanifolds of the form $V \cap S^7$ by choosing a “good” frame of $V$ by the Sp(1)Sp(2)-action to consider the associative condition.

Suppose that $V$ is spanned by $e_0, \cdots, e_3$ ($e_i \in \mathbb{C}^4 = \mathbb{R}^8$). Since Sp(1)Sp(2) acts transitively on $S^7$, we may assume that $e_0 = \langle 1, 0, 0, 0 \rangle$.

The stabilizer of Sp(1)Sp(2) at $e_0$ is diffeomorphic to Sp(1)Sp(1), which acts on $S^7 \subset \mathbb{H}^2 = \mathbb{R}^8$. Thus we may assume that $e_1 = \langle c_i, 0, s, 0 \rangle$,

for $c, s \geq 0, c^2 + s^2 = 1$. Since $\{[(z, z)]; z \in U(1)\} \subset Sp(1)Sp(1)$ fixes $e_1$, by sweeping out the first entry, we may assume that $e_2 = \langle 0, A_2, A_3 + iB_3, A_4 + iB_4 \rangle$,

for $A_j, B_k \in \mathbb{R}, A_2 \geq 0$.

Lemma 5.1. We have

$$\frac{5}{3} (e_1 \times e_2)_{e_0} = \langle 5B_3, 5A_4, (-A_2c + 5B_4)i, -B_3c + A_3ci, (-B_4c - A_2s) + A_4ci \rangle.$$ 

Thus denoting by $e_3$ the left-hand side, we see that span$_{\mathbb{R}} \{e_1, e_2, e_3\} \subset T_{e_0}S^7$ is associative. We deduce the condition by calculating $*\tilde{\phi}(e_1, e_3, \cdot)_{e_i} = 0$ in the following cases:

1) $c > 0, A_2 > 0, (2) c > 0, A_2 = 0, (3) c = 0$.

Lemma 5.2. In the case (1), the condition $*\tilde{\phi}(e_0, e_2, e_3, \cdot)_{e_i} = 0$ is equivalent to

(i) $s = 0$,

(ii) $s \neq 0, A_3 = B_3 = 0, c^2 - 3s^2 = 0$, or

(iii) $s \neq 0, A_3 = B_3 = 0, c(A_4^2 + 3A_4^2 + 3B_4^2) - 2sA_2B_4 = 0$.

We abbreviate the case that (1) and (ii) hold as the case (1)-(ii) in the following.

Lemma 5.3. In the case (1)-(ii) or (1)-(iii), by normalizing $e_2$, we may assume that $A_2^2 + A_4^2 + B_4^2 = 1$. Then $*\tilde{\phi}(e_0, e_1, e_3, \cdot)_{e_2} = 0$ is equivalent to

(a) $A_4 = B_4 = 0$,

(b) $A_4 = 0, A_2^2 - 3B_4^2 = 0$, or

(c) $A_4 = 0, (c^2 + 3s^2)A_2 - 2csB_4 = 0$. 

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Proof of Lemma 5.1. At $e_0$, we have
\[ \xi_1 = \iota(-i, 0, 0, 0), \; \xi_2 = \iota(0, -1, 0, 0), \; \xi_3 = \iota(0, -i, 0, 0). \]
Setting $X_0 = \iota(0, 0, 1, 0)$, we see $X_0 \in \mathcal{H}_{e_0}$. Then $X_i = \Phi_i(X_0)(i = 1, 2, 3)$ is described as
\[ X_1 = \iota(0, 0, i, 0), \; X_2 = \iota(0, 0, 0, 1), \; X_3 = \iota(0, 0, 0, i), \]
and we have
\[ e_1 = -c\xi_1 + sX_1, e_2 = -A_2\xi_2 + A_4X_0 + B_1X_1 + A_4X_2 + B_4X_3. \]
By the definition of $\varphi$ in Proposition 4.3, we obtain
\[ \varphi(e_1, e_2, \cdot)e_0 = \frac{27}{125}c(A_2\eta_3 + 5A_3X^4 - 5B_3X^0 + 5A_4X^3 - 5B_4X^2) \]
\[ + \frac{27}{25}s(-A_2X^2 - B_3\eta_1 - A_4\eta_2 - B_4\eta_3). \]
Since $g = \frac{9}{25}\sum_{l=1}^{3}\eta_l + \frac{9}{5}\sum_{n=0}^{3}X^n$, we obtain the lemma.

Proof of Lemma 5.2. As in the proof of Lemma 5.1, we have at $e_1$
\[ \xi_1 = \iota(c, 0, -is, 0), \; \xi_2 = \iota(0, ic, 0, -s), \; \xi_3 = \iota(0, -c, 0, -is). \]
Setting $X_0 = \iota(0, is, 0, c) \in \mathcal{H}_{e_1}, X_i = \Phi_i(X_0)(i = 1, 2, 3)$ is described as
\[ X_1 = \iota(0, -s, 0, ic), \; X_2 = \iota(is, 0, -c, 0), \; X_3 = \iota(-s, 0, -ic, 0). \]
Then by a direct computation, $*\varphi(e_0, e_2, e_3, \cdot)e_1 = 0$ is equivalent to
\[ 4s(c^2 - 3s^2)(cA_2^2 + 3cA_4^2 + 3cB_1^2 - 2sA_2B_4) = 0, \quad (5.1) \]
\[ s \left( \begin{array}{cc} c(-2s^2 + c^2) & -2s^3 \\ 3sc & 3s^2 + c^2 \end{array} \right) \left( \begin{array}{c} A_3A_4 \\ A_2B_3 \\ B_3B_4 \end{array} \right) = 0, \quad (5.2) \]
\[ s \left( \begin{array}{cc} sc & c^2 - 2s^2 \\ -3s & -3s^2 + c^2 \end{array} \right) \left( \begin{array}{c} A_2A_3 \\ A_3B_4 \\ B_3A_4 \end{array} \right) = 0. \quad (5.3) \]
It is clear that $s = 0$ is a solution of (5.1), (5.2) and (5.3). We may assume that $s \neq 0$. From (5.2) and (5.3), we have
\[ (A_3A_4, A_2B_3, B_3B_4) = k(-c^2 + 5s^2, 5sc, c^2), (A_2A_3, A_3B_4, B_3A_4) = l(5s, c, c), \]
for $k,l \in \mathbb{R}$. Since $A_3A_4B_3B_4 = -k^2c^2(c^2 + 5s^2) = l^2c^2$, we obtain $k = l = 0$. The assumption $A_2 > 0$ gives $A_3 = B_3 = 0$.

Proof of Lemma 5.3. As in the proof of Lemma 5.1, we have at $e_1$
\[ \xi_1 = \iota(0, -iA_2, 0, -iA_4), \; \xi_2 = \iota(A_2, 0, A_4, -iB_4, 0), \; \xi_3 = \iota(iA_2, 0, B_4 + iA_4, 0). \]
Setting $X_0 = \iota(A_4 + iB_4, 0, -A_2, 0) \in \mathcal{H}_{e_2}, X_i = \Phi_i(X_0)(i = 1, 2, 3)$ is described as
\[ X_1 = \iota(-B_4 + iA_4, 0, -iA_2, 0), \; X_2 = \iota(0, A_4 - iB_4, 0, -A_2), \; X_3 = \iota(0, B_4 + iA_4, 0, -iA_2). \]
Then by a direct computation, \[ *\tilde{\varphi}(e_0, e_1, e_3, \cdot_c)_{c^2} = 0 \] is equivalent to

\[
A_4 \{ cA_2(cA_2^2 - 2sA_2B_4 - 3cA_4^2 - 3cB_4^2) \\
+ 6B_4s(-3sA_2B_4 + 2cA_4^2 + 2cB_4^2) \} = 0, \tag{5.4}
\]

\[
(c^2 + 3s^2)A_2^2B_4 - 2csA_2^2B_4^2 + 3(3s^2 - c^2)A_2A_4^2 - 6csA_4^1 + 6csB_4^1 = 0, \tag{5.5}
\]

\[
sA_2A_4(cA_2^2 - 2sA_2B_4 + 3cA_4^2 + 3cB_4^2) = 0. \tag{5.6}
\]

Suppose that \( A_4 \neq 0 \) for a contradiction. Then (5.6) implies that

\[
cA_2^2 - 2sA_2B_4 + 3cA_4^2 + 3cB_4^2 = 0. \tag{5.7}
\]

Eliminating \( A_4^2 \) and \( B_4^2 \) from (5.4), we have \( 2A_2(cA_2 - 5sB_4)(cA_2 + sB_4) = 0 \). However, the left hand side of (5.7) is greater than 0 when \( B_4 = \frac{c}{s}A_2 \) or \(-\frac{c}{s}A_2 \). Thus we have \( A_4 = 0 \).

Then the left-hand sides of (5.3) and (5.6) vanish, and that of (5.5) is equal to \( B_4(A_2^2 - 3B_4^2)(c^2 + 3s^2)A_2 - 2csB_4 \), hence the proof is done. \( \square \)

**Proof of Theorem 1.1.** From Lemma 5.2 and 5.3, we consider the following cases:

**Case (1)-(i)** By the \( \text{Sp}(1) \)-action, we may assume that \( B_3 = A_4 = B_4 = 0 \). Normalizing \( e_2 \), we may assume \( A_2^2 + A_3^2 = 1 \). Then as in the proof of Lemma 5.2, \( *\tilde{\varphi}(e_0, e_1, e_3, \cdot_c)_{c_2} = 0 \) is equivalent to \( A_3(A_2^2 - A_3^2) = 0 \). Hence we have

\[
(c, s, A_2, A_3, B_3, A_4, B_4) = (1, 0, 1, 0, 0, 0, 0), \tag{5.8}
\]

\[
\left( 1, 0, \frac{\sqrt{3}}{2}, \pm \frac{1}{2}, 0, 0, 0 \right). \tag{5.9}
\]

**Case (1)-(ii)-(a)** By normalizing \( e_2 \), we have \( A_2 = 1 \). Then we see

\[
(c, s, A_2, A_3, B_3, A_4, B_4) = \left( \frac{\sqrt{3}}{2}, \frac{1}{2}, 1, 0, 0, 0, 0 \right). \tag{5.10}
\]

**In case (1)-(ii)-(b), (1)-(iii)-(c), and (1)-(iii)-(b),** we have the following solutions:

\[
(c, s, A_2, A_3, B_3, A_4, B_4) = \left( \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0, \pm \frac{1}{2} \right), \tag{5.11}
\]

\[
\left( \frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{\sqrt{3}}{2} \right), \tag{5.12}
\]

\[
\left( \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0, \frac{1}{2} \right). \tag{5.13}
\]

**In case (1)-(iii)-(a) and (1)-(iii)-(c),** we have no solutions.

The solution (5.8) corresponds to the \( H \)-plane. The planes corresponding to (5.10), (5.11), (5.12), and (5.13) are congruent up to the \( \text{Sp}(1)\text{Sp}(2) \)-action to that of (5.3), which is not associative at \((e_0 + e_1)/\sqrt{2} \), \((e_2 + e_3)/\sqrt{2} \), \((e_2 - e_3)/\sqrt{2} \), \((e_0 + e_1)/\sqrt{2} \) \( (e_0 + e_1)/\sqrt{2} \) $\neq 0$. 

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Case (2) We may assume that the first and the second entries of $e_3$ are zero. Hence we have $B_3s = A_3s = B_4s = 0$. If $s \neq 0$, we obtain the plane $V_2$. If $s = 0$, the corresponding plane is congruent up to Sp(2)-action to $V_2$.

Case (3) We may assume that the first and the second entries of $e_2$ and $e_3$ are zero. However, this implies that $e_3 = 0$, which is a contradiction. □

6 Classification of homogeneous associative submanifolds

In this section, we prove Theorem 1.3. First, we classify compact Lie subgroups of Sp(1)Sp(2) which have 3-dimensional orbits. Let $G$ be a compact connected Lie subgroup of Sp(1)Sp(2). Suppose that $G$ has a 3-dimensional orbit $A$. Since $G$ acts on $A$ as an isometry group, $\dim G \leq 3 \cdot (3 + 1)/2 = 6$ and $\dim G \neq 5$. (see [13], Chapter IV, Theorem 9.1). We only have to consider the Lie algebra $g \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ of $G$.

6.1 Case $\dim g = 3$

Suppose that $\dim g = 3$. By the classification of the compact Lie algebras, $g$ is isomorphic to $\mathfrak{su}(2)$ or $\mathfrak{t}^3$, where $\mathfrak{t}^3$ is a Lie algebra of the 3-torus $T^3$. The case $g = \mathfrak{t}^3$ corresponds to the inclusion $T^3 \hookrightarrow U(1)Sp(2) \subset U(4)$ given by

$$(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) \mapsto \text{diag}(e^{i(\alpha+\beta)}, e^{i(\alpha-\beta)}, e^{i(\alpha+\gamma)}, e^{i(\alpha-\gamma)}),$$

which is a maximal torus of Sp(1)Sp(2) and induces the $T^3$-action on $S^7$. Define the basis $\{F_1, F_2, F_3\}$ of the Lie algebra $\mathfrak{t}^3 \approx \mathbb{R}^3$ of $T^3$ by

$$F_1 = (1, 0, 0), F_2 = (0, 1, 0), F_3 = (0, 0, 1).$$

Via the inclusion $\mathfrak{t}^3 \hookrightarrow \mathfrak{u}(1) \oplus \mathfrak{sp}(2)$, $F_1, F_2, F_3$ correspond to

$$\begin{pmatrix} i & i & -i \\ -i & 0 & 0 \\ -i & 0 & -i \end{pmatrix},$$

respectively.

When $g = \mathfrak{su}(2)$, we see that $\mathfrak{su}(2) = \mathfrak{sp}(1)_L$ or $\mathfrak{su}(2) \subset \mathfrak{sp}(2)_R$. Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)_R$. Recall that any representation of the compact Lie group SU(2) is completely reducible and the dimension of the real irreducible representation of SU(2) is of the form $4k, 2l - 1(k, l \geq 1)$. Thus we have 3 types of inclusions $\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(5)$ given by

$$\begin{align*}
\mathfrak{su}(2) &= \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(5), \\
\mathfrak{su}(2) &\hookrightarrow \mathfrak{so}(4) \hookrightarrow \mathfrak{so}(5), \\
\mathfrak{su}(2) &\hookrightarrow \mathfrak{so}(5): \text{irreducibly}.
\end{align*}$$

The identification $\mathfrak{sp}(2) = \mathfrak{so}(5)$ induces three types of inclusions $SU(2) \hookrightarrow \text{Sp}(2)$. Hence we have the following four types of inclusions $SU(2) \hookrightarrow \text{Sp}(1)\text{Sp}(2)$.

1. $SU(2) = \text{Sp}(1)_L$ acting on $S^7$ by $(4.1)$.,
2. The inclusion $\text{SU}(2) \hookrightarrow \text{Sp}(2)$ given by

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mapsto \begin{pmatrix} \pi & -b \\ b & \pi \end{pmatrix},$$

which induces the $\text{SU}(2)$-action on $S^7$. Define the basis $\{E_1, E_2, E_3\}$ of the Lie algebra $\text{su}(2)$ of $\text{SU}(2)$ satisfying $[E_i, E_{i+1}] = 2E_{i+2}$ ($i \in \mathbb{Z}/3$) by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, E_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$ (6.4)

Via this inclusion $\text{su}(2) \hookrightarrow \text{sp}(2)$, $E_1, E_2, E_3$ correspond to

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -i & i \\ i & -i \end{pmatrix}, \begin{pmatrix} -i & i \\ i & -i \end{pmatrix},$$

respectively.

3. The inclusion $\text{SU}(2) \hookrightarrow \text{Sp}(2)$ given by

$$A \mapsto \begin{pmatrix} A & O_2 \\ O_2 & I_2 \end{pmatrix}.$$ (6.5)

Via this inclusion $\text{su}(2) \hookrightarrow \text{sp}(2)$, $E_1, E_2, E_3$ correspond to

$$\begin{pmatrix} -1 & 1 \\ 0 & O_2 \end{pmatrix}, \begin{pmatrix} i & O_2 \\ O_2 & i \end{pmatrix}, \begin{pmatrix} i & -i \\ O_2 & O_2 \end{pmatrix},$$ (6.6)

respectively.

4. The inclusion $\text{SU}(2) \hookrightarrow \text{Sp}(2)$ given by

$$\begin{pmatrix} a & -b \\ b & \pi \end{pmatrix} \mapsto \begin{pmatrix} a^3 & -\bar{a}^3 \\ \bar{a}^3 & a^3 \end{pmatrix} - \begin{pmatrix} \sqrt{3}ab^2 & a^3 b \\ a^3 b & -\sqrt{3}ab^2 \end{pmatrix} - \begin{pmatrix} \pi |a|^2 - 2|b|^2 \\ -\bar{b}(2|a|^2 - |b|^2) \end{pmatrix}.$$ (6.9)

which induces the $\text{SU}(2)$-action on $S^7$. This action is an irreducible representation of $\text{SU}(2)$ on $\mathbb{C}^4$. This is the induced action of $\text{SU}(2)$ on $V_3 = \mathbb{C}^4$ from the standard action on $\mathbb{C}^2$, where we use the notation of Lemma 7.6.

Via $\text{su}(2) \hookrightarrow \text{sp}(2)$, $E_1, E_2, E_3$ correspond to

$$\begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ -\sqrt{3} & -2 \end{pmatrix}, \begin{pmatrix} \sqrt{3}i & \sqrt{3i} \\ \sqrt{3i} & 2i \end{pmatrix}, \begin{pmatrix} 3i & -3i \\ -3i & -i \end{pmatrix},$$ (6.10)

respectively.
6.2 Case dim $\mathfrak{g} = 4$

By the classification of the compact Lie algebras, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2) \oplus \mathbb{R}$. Since the inclusions $\mathfrak{su}(2) \hookrightarrow \mathfrak{sp}(1) \oplus \mathfrak{sp}(2)$ are classified, we have to find the 1-dimensional Lie subalgebra which commute with $\mathfrak{su}(2)$. Set

$$Z(\mathfrak{su}(2)) = \{X \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(2); [X, Y] = 0 \text{ for any } Y \in \mathfrak{su}(2)\}.$$

First consider the case $\mathfrak{su}(2) = \mathfrak{sp}(1)_L$. Then we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(2)_R$. Take any 1-dimensional subspace $\mathfrak{k} \subset \mathfrak{sp}(2)_R$ and suppose that $G$ is the Lie subgroup of $\text{Sp}(1)\text{Sp}(2)$ whose Lie algebra is $\mathfrak{su}(2) \oplus \mathfrak{k}$. Since the $\text{Sp}(1)_L$-action commutes with the $\text{Sp}(2)_R$-action, the $G$-orbit through $p \in S^7$ should be contained in $\text{Sp}(1) \cdot p$ so that it is 3-dimensional. Thus this case is reduced to that of $(6.1)$.

Next, suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from $(6.3)$. In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(2)_{1L} \oplus (R\text{diag}(i, -i, i, -i))_R$. The Lie subgroup $G \subset \text{Sp}(2)$ whose Lie algebra is $(\mathfrak{su}(2) \oplus R\text{diag}(i, -i, i, -i))_R$ is $U(2)$ whose restriction to $\text{SU}(2)$ is given by $(6.3)$. This $U(2)$ action has the same orbits as the $\text{SU}(2)$-action. The new 3-dimensional orbits do not appear from $\mathfrak{sp}(1)_L$, and this case is reduced to that of $(6.3)$.

Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from $(6.7)$. In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(1)_L \oplus \left( O_2 \quad \mathfrak{su}(2) \right)_R$. This case is also reduced to that of $(6.7)$ in the same way.

Suppose that $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$ is induced from $(6.9)$. In this case, we have $Z(\mathfrak{su}(2)) = \mathfrak{sp}(1)_L$. This case is also reduced to that of $(6.9)$ in the same way.

6.3 Case dim $\mathfrak{g} = 6$

By the classification of the compact Lie algebras, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2) \oplus t^3$ or $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. When $\mathfrak{g} \cong \mathfrak{su}(2) \oplus t^3$, we have $\mathfrak{g} \cong t^3_1 \oplus (\mathfrak{su}(2) \oplus t^3)_R$. Since there are no 2-dimensional commutative Lie subalgebras of $\mathfrak{sp}(2)$ which commute with $\mathfrak{su}(2)$ by Section 6.2, this case does not occur.

When $\mathfrak{g} \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have $G = \text{Sp}(1)_L \cdot \text{SU}(2)_R$ or $\left( \begin{array}{cc} \text{SU}(2) & \mathfrak{su}(2) \\ \mathfrak{su}(2) & \mathfrak{su}(2) \end{array} \right)_R$, which reduces to the case above.

Thus we only have to consider the orbits of $(6.1)$, $(6.4)$, $(6.7)$, and $(6.9)$.

6.4 $T^3$-orbits

We classify associative submanifolds which are orbits of $T^3$ acting on $S^7$ as $(6.1)$.

Proposition 6.1. Up to the $\text{Sp}(1)\text{Sp}(2)$-action, $T^3 \cdot \frac{1}{2} \ell(1, 1, 1, i)$ is the unique associative submanifold in the squashed $S^7$ which is an orbit of the $T^3$-action.

Remark 6.2. The associative orbit $A_1 = T^3 \cdot \frac{1}{2} \ell(1, 1, 1, i)$ is the Hopf lift of a $T_1'$-holomorphic curve in $\mathbb{C}P^3$, where $T_1'$ is defined by $(13)$. We have

$$A_1 = \left\{ (z_1, z_2, z_3, z_4) \in S^7; \begin{array}{c} |z_1| = |z_2| = |z_3| = |z_4|, \\ \text{Re}(z_1 z_2 z_3 z_4) = 0, \text{Im}(z_1 z_2 z_3 z_4) < 0 \end{array} \right\}.$$
which is a special Legendrian given in \[6\] via \(\tilde{\{z_1, z_2, z_3, z_4\}} \mapsto \{z_1, z_2, \overline{z}_3, \overline{z}_4\}\). The inclusion \([6,1]\) induces the metric \(\frac{1}{5}(F^1)^2 + \frac{27}{50}(F^2)^2 + \frac{27}{50}(F^3)^2\), where \(\{F^i\}\) is the dual of \(\{F_1\}\).

Proof. Fix \(p_0 = \{z_1, z_2, z_3, z_4\} \in S^7\) and set \(A = T^3 \cdot p_0\). Then the tangent space \(T_{p_0}A\) is spanned by the vectors \(F_1^*\) generated by \(F_i\) in \([6,2]\):

\[(F_1^*)_p = i'(z_1, z_2, z_3, z_4) = -\xi_1, (F_2^*)_p = i'(z_1, -z_2, 0, 0), (F_3^*)_p = i'(0, 0, z_3, -z_4).

By Lemma 6.3, we consider the condition \(\tilde{\psi}(F_1^*, F_2^*, F_3^*, T_{p_0}S^7) = 0\). We easily see that \(-i(F_1^*)^* \psi = (3^4/5^3)\text{Im}((\eta_2 - i\eta_3) \wedge d(\eta_2 + i\eta_3))\). From Lemma 4.1 we have

\[
\begin{align*}
(\eta_2 + i\eta_3)(F_2^*) &= 2iz_1z_2, (\eta_2 + i\eta_3)(F_1^*) = 2iz_3z_4, \\
d(\eta_2 + i\eta_3)(F_3^*), &= -2id(z_1z_2), d(\eta_2 + i\eta_3)(F_3^*), = -2id(z_3z_4),
\end{align*}
\]

which implies that the condition \(\tilde{\psi}(F_1^*, F_2^*, F_3^*, T_{p_0}S^7) = 0\) is equivalent to \(d(\text{Im}(z_1z_2\overline{z}_3\overline{z}_4)) = 0\). The restriction of this form to \(T^3 S^7\) is given by \(d(\text{Im}(z_1z_2\overline{z}_3\overline{z}_4)) - d(\text{Im}(z_1z_2\overline{z}_3\overline{z}_4))(\frac{\partial}{\partial r}) = \text{Re}(\sum_{j=1}^4 \xi_j dz_j)\), where \(\frac{\partial}{\partial r}\) is a position vector, and

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{pmatrix} = \begin{pmatrix}
-i z_2 \overline{z}_3 \overline{z}_4 \\
-i z_2 \overline{z}_3 \overline{z}_4 \\
i z_1 z_2 \overline{z}_4 \\
i z_1 z_2 \overline{z}_4
\end{pmatrix} - 4\text{Im}(z_1z_2\overline{z}_3\overline{z}_4) \begin{pmatrix}
\overline{z}_1 \\
\overline{z}_2 \\
\overline{z}_3 \\
\overline{z}_4
\end{pmatrix}.
\]

Thus we see that the condition \(\tilde{\psi}(F_1^*, F_2^*, F_3^*, T_{p_0}S^7) = 0\) is equivalent to \(\xi_j(p_0) = 0\) on \(S^7\). We see that \(S^7 = T^3 \cdot \Sigma\). Hence we may assume that \(p_0 \in \Sigma\) and \(x_1, x_2, x_3 \neq 0\) so that \(T^3 \cdot p_0\) is 3-dimensional. Then we can solve \(\xi_j = 0\) easily to obtain

\[
x_1 = x_2 = x_3 = 1/2, x_4 = 0, y_4 = \pm 1/2.
\]

The \(T^3\)-orbit through \(\{1, 1, 1, 1\}/2\) is mapped to that through \(\{1, 1, 1, -1\}/2\) by

\[
\begin{pmatrix}
I_2 & 0 \\
0 & K
\end{pmatrix} \in \text{Sp}(2), \quad K = \begin{pmatrix}
0 & -i \\
-i & 0
\end{pmatrix},
\]

and we obtain the statement.

\[
\Box
\]

### 6.5 SU(2)-orbits

We consider the SU(2)-orbits of \([4,1], [6,4], [6,7], \) or \([6,9]\). First, we introduce the useful lemma to study associative orbits.

**Lemma 6.3.** ([4] Lemma 5.6.) Let \((V, \rho)\) be an orthogonal representation of \(\text{SU}(2)\), \(\langle \cdot, \cdot \rangle\) be an \(\text{SU}(2)\)-invariant inner product on \(V\), and \(S_1 \subset V\) be the unit sphere. Let \(M = \text{SU}(2) \cdot p\) be a 3-dimensional orbit through \(p \in S_1\). Define the function \(\lambda_j : M \to \mathbb{R} (j = 1, 2, 3)\) by

\[
\lambda_j = ((\rho_*(E_j))^*, (\rho_*(E_j))^*) |_M,
\]

where \(E_j\) are the fixed vectors of \(\rho\). Then \(\lambda_j \in \mathbb{R}\) and \(\lambda_j(M) \subset \mathbb{R}\) is a non-constant function.
where \( \{ E_i \} \) is a basis of \( \text{su}(2) \) satisfying \( [E_i, E_{i+1}] = 2E_{i+2}(i \in \mathbb{Z}/3) \) and \((\rho_{*}(E_j))^\ast \) is a vector field on \( V \) generated by \( \rho_{*}(E_j) \in \mathfrak{gl}(V) \). Denote by \( \{ E^j \} \) the dual 1-form on \( M \) of \( \{(\rho_{*}(E_j))^\ast \} \). Then there exists \( g \in \text{SU}(2) \), the induced metric \( \langle \cdot, \cdot \rangle_{M} \) is described as

\[
\langle \cdot, \cdot \rangle_{M} = \sum_{j=1}^{3} \lambda_j(E^{j})^{2},
\]

\((6.11)\)

at \( g \cdot p \in M \). Moreover, \( (M, \langle \cdot, \cdot \rangle_{M}) \) is a space of constant curvature \( k \) if and only if \( \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{k} \).

**Remark 6.4.** ([9] Remark 5.4.) There exists \( g' \in \text{SU}(2) \) satisfying \((6.11)\) and \( \lambda_1(g') = \lambda_a(g), \lambda_2(g') = \lambda_b(g), \lambda_3(g') = \lambda_c(g) \), where \( \{ a, b, c \} \) is any permutation of \( \{ 1, 2, 3 \} \). Thus we can “permute” the \( \lambda_j \).

6.5.1 SU(2)-orbits 1

If an SU(2)-action is given by \((6.1)\), the orbit is the intersection of a quaternionic plane and \( S^7 \), which is an obvious totally geodesic associative submanifold.

6.5.2 SU(2)-orbits 2

Consider the SU(2)-action given by \((6.3)\). Let \( A \) be an SU(2)-orbit through \( p_0 = \langle z_1, z_2, z_3, z_4 \rangle \). Then the tangent space to \( A \) at \( p_0 \) is spanned by the vectors \( E_i^* \) generated by \( E_i \) in \((6.3)\):

\[
(E_1^*)_{p_0} = \langle z_3, z_4, -z_1, -z_2 \rangle, (E_2^*)_{p_0} = \langle -z_3, z_4, -z_1, z_2 \rangle, (E_3^*)_{p_0} = \langle -z_1, z_2, z_3, -z_4 \rangle.
\]

We easily see that \( g(E_i^*, E_j^*)_{p_0} = \delta_{ij} \), where \( g \) is the standard metric on \( S^7 \). Then from Lemma \((6.3)\) \( A \) is a constant curvature 1 submanifold of \((S^7, g)\). Thus \( A \) is of the form \( V \cap S^7 \), where \( V \subset \mathbb{R}^8 \) is a 4-plane. These associative submanifolds are classified by Theorem \((6.1)\).

6.5.3 SU(2)-orbits 3

Consider the SU(2)-action given by \((6.7)\). Let \( A \) be an SU(2)-orbit through \( p_0 = \langle z_1, z_2, z_3, z_4 \rangle \). By the SU(2)-action, we may assume that \( p_0 = \langle x_1, 0, z_3, z_4 \rangle \) where \( x_1 > 0, z_3, z_4 \in \mathbb{C} \). Then the tangent space to \( A \) at \( p_0 \) is spanned by the vectors \( E_i^* \) generated by \( E_i \) in \((6.3)\):

\[
(E_1^*)_{p_0} = \langle 0, -x_1, 0, 0 \rangle, (E_2^*)_{p_0} = \langle 0, i x_1, 0, 0 \rangle, (E_3^*)_{p_0} = \langle i x_1, 0, 0, 0 \rangle.
\]

We compute

\[
(\eta_i(E_j^*)) = \begin{pmatrix} 0 & 0 & -x_1^2 \\ -x_1^2 & 0 & 0 \\ 0 & -x_1^2 & 0 \end{pmatrix}, \begin{pmatrix} \text{Im}(dz_2) \\ \text{Re}(dz_2) \\ \text{Re}(dz_1) \end{pmatrix}, \begin{pmatrix} x_1^2 & 0 & 0 \\ 0 & 0 & -x_1^2 \\ 0 & x_1^2 & 0 \end{pmatrix}, \begin{pmatrix} -dz_1 \\ idz_1 \\ -idz_2 \end{pmatrix},
\]

\[
(i(E_i^*)d\eta) = 2x_1 \begin{pmatrix} \text{Im}(dz_2) \\ \text{Re}(dz_2) \\ \text{Re}(dz_1) \end{pmatrix}, (i(E_i^*)d(\eta_2 + i \eta_3)) = 2x_1 \begin{pmatrix} -dz_1 \\ idz_1 \\ -idz_2 \end{pmatrix},
\]

\[
\sum_{i=1}^{3} d\eta_i(E_1^*, E_2^*, E_3^*, \cdot) = 12x_1^3 dx_1, d(\eta_{123}) = 2x_1^5 dx_1.
\]
Since \( \ast \tilde{\varphi} = \frac{25}{2} (15 + 16x_1^2) \), we obtain \( \ast \tilde{\varphi}(E_1^3, E_2^3, \cdot) = \frac{25}{2} x_1^2 (15 + 16x_1^2) \). The restriction of \( dx_1 \) to \( TS^7 \) is given by

\[
dx_1 - dx_1 \left( r \frac{\partial}{\partial r} \right) \frac{dr}{r} = dx_1 - x_1 \left( x_1 dx_1 + \Re(z_3 dz_3 + z_4 dz_4) \right),
\]

where \( r \frac{\partial}{\partial r} \) is a position vector and \( \frac{dr}{r} \) is its dual. This implies that \( \ast \tilde{\varphi}(E_1, E_2, E_3, \cdot)|_{x_1 S^7} = 0 \) is equivalent to \( x_1 = 1, z_3 = z_4 = 0 \), and the resulting associative submanifold is \( \{(z_1, z_2, 0, 0) \in \mathbb{C}^4; |z_1|^2 + |z_2|^2 = 1\} \).

### 6.4 SU(2)-orbits

For the SU(2)-action given by (6.9), we obtain the following.

**Proposition 6.5.** Let \( A \) be an associative submanifold in the squashed \( S^7 \) which is an orbit of the SU(2)-action given in (6.9). Then up to the \( \text{Sp}(1) \text{Sp}(2) \)-action,

\[
A = A_2 := \text{SU}(2) \cdot \ast (1, 0, 0, 0) \quad \text{or} \quad A_3 := \text{SU}(2) \cdot \ast (0, 0, 1, 0).
\]

**Remark 6.6.** The associative orbit \( A_2 \) is the Hopf lift of a horizontal holomorphic curve

\[
\{[a^3 : b^3 : \sqrt{3}a^2b : \sqrt{3}a^2b] \in \mathbb{C}P^3 ; a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1\}
\]

in \( \mathbb{C}P^3 \). This is a degree 3 \( \mathbb{C}P^1 \) in \( \mathbb{C}P^3 \) of the constant curvature called the Veronese curve. The associative orbit \( A_3 \) is the Hopf lift of a null-torsion \( I \)-holomorphic curve in \( \mathbb{C}P^3 \), which is defined in Definition 7.15. The inclusion (6.9) induces \( \tilde{g}|A_2 = \frac{25}{2} (5E_1^3)^2 + 5(E_2^3)^2 + 3(E_3^3)^2 \) and \( \tilde{g}|A_3 = \frac{25}{2} (19E_1^3)^2 + 19(E_2^3)^2 + (E_3^3)^2 \), where we use the notation of Lemma 6.3.

**Remark 6.7.** Set \( A_2(a, b) := \text{SU}(2) \cdot \ast (a, b, 0, 0) \) and \( A_3(a, b) := \text{SU}(2) \cdot \ast (0, a, b) \) for \( a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \). Then by the action of \( a + bj \in \text{Sp}(1) \text{L} \), \( A_j \) is congruent to \( A_j(a, b)(j = 2, 3) \). Via \( \ast (z_1, z_2, z_3, z_4) \mapsto \ast (z_1, z_4, z_3, z_2) \), \( A_2(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) \) is special Legendrian given by (8).

**Proof of Proposition 6.5.** Let \( A \) be an SU(2)-orbit through \( p_0 = \ast (z_1, z_2, z_3, z_4) \).

Then the tangent space to \( A \) at \( p_0 \) is spanned by the vectors \( E_i^* \) generated by \( E_i \) in (6.5):

\[
(E_1^*)_{p_0} = \ast (\sqrt{3}z_4, -\sqrt{3}z_3, \sqrt{3}z_2 - 2z_4, -\sqrt{3}z_1 + 2z_3),
\]
\[
(E_2^*)_{p_0} = \ast (\sqrt{3}iz_4, \sqrt{3}iz_3, \sqrt{3}iz_2 + 2iz_4, \sqrt{3}iz_1 + 2iz_3),
\]
\[
(E_3^*)_{p_0} = \ast (3iz_1, -3iz_2, -i3z_3, iz_4).
\]

Since \( \text{SU}(2) \subset \text{Sp}(2) \)-action preserves \( \eta_j \), we have \( L_{E_i^*} \eta_j = d\eta_j(E_i^*, \cdot) + d\eta_j(E_i^*) = 0 \). Then by the equation \([E_j^*, E_{j+1}^*] = -2E_{j+2}^*(j \in \mathbb{Z}/3) \), we have

\[
\sum_{i=1}^{3} (d\eta_i)^2(E_1^*, E_2^*, E_3^*, \cdot) = 2 \sum_{i,j=1}^{3} d(\eta_i E_j^*)^2,
\]
\[
d(\eta_{123})(E_1^*, E_2^*, E_3^*, \cdot) = -d(\eta_{123}(E_1^*, E_2^*, E_3^*)).
\]
We compute
\[
\eta_1(E_2^+) + i\eta_1(E_2^-) = -2\sqrt{3}(\overline{z_1}z_4 + z_2\overline{z_3}) - 4z_3z_4,
\]
\[
\eta_1(E_3^+) = -3|z_1|^2 + 3|z_2|^2 + |z_3|^2 - |z_4|^2,
\]
\[
(n_2 + in_3)(E_2^+) = 2\sqrt{3}(z_1z_3 + z_2z_4) - 2(z_3^2 + z_4^2),
\]
\[
(n_2 + in_3)(E_2^-) = 2\sqrt{3}(-z_1z_3 + z_2z_4) + 2i(-z_3^2 + z_4^2),
\]
\[
(n_2 + in_3)(E_2^+) = 6iz_1z_2 - 2iz_3z_4.
\]
Then we have \(\sum_{i,j=1}^3 \eta(E_j)^2 = 9\) and \(\sum_{i=1}^3 (d\eta)^2(E_1, E_2, E_3, \cdots) = 0\) by (6.12).
Since \(*\tilde{\varphi} = \frac{27}{\sqrt{3}} \sum_{i=1}^3 (d\eta)^2 + \frac{1}{8}d(\eta_1z_3))\) the condition \(*\tilde{\varphi}(E_1, E_2, E_3, \cdots) = 0\) is equivalent to
\[
d(\text{det } M) = 0,
\]
where \(M = (\eta(E_j^+))\).

Now, we use Lemma 6.3. We may assume that \(\{E_1^+, E_2^-, E_3^+\}\) are mutually orthogonal at \(p_0 = q(z_1, z_2, z_3, z_4)\) with respect to \(g\). Then we have
\[
z_1\overline{z}_4 - \overline{z}_2z_3 = 0, \text{Im}(z_1\overline{z}_3 + \overline{z}_2z_4) = 0. \tag{6.13}
\]
Setting
\[
\lambda_1 = |E_1^+|^2 = 4(|z_1|^2 + |z_2|^2) - 4\sqrt{3}\text{Re}(z_1\overline{z}_3 + \overline{z}_2z_4) + 3,
\]
\[
\lambda_2 = |E_2^-|^2 = 4(|z_1|^2 + |z_2|^2) + 4\sqrt{3}\text{Re}(z_1\overline{z}_3 + \overline{z}_2z_4) + 3,
\]
\[
\lambda_3 = |E_3^+|^2 = 8(|z_1|^2 + |z_2|^2) + 1.
\]
We consider the following two cases as the proof of Lemma 5.7 in [4]:
(1) all of the \(\lambda_j\) are distinct, (2) at least two of the \(\lambda_j\) are equal.

Consider the case (1). Since we can permute the \(\lambda_j\) by Remark 6.3, we may assume that \(\lambda_3 < \lambda_1 < \lambda_2\). The inequality \(\lambda_1 < \lambda_2\) implies that \(\text{Re}(z_1\overline{z}_3 + \overline{z}_2z_4) > 0\). Thus we have \((z_1, z_2), (z_3, z_4) \neq 0\). From (6.13), there exists \(\mu \in \mathbb{R}\) satisfying
\[
z_3 = \mu z_1, z_4 = \mu z_2. \tag{6.14}
\]
Note that \(\lambda_3 < \lambda_1\) is equivalent to \(\mu > \sqrt{3}\). Moreover, since the \(\text{Sp}(1)_L\)-action commutes the \(\text{Sp}(2)_R\)-action and \(q(z_1, z_2, \mu z_1, \mu z_2)\) is mapped to \(\sqrt{\mu^2 + 1}q(1, 0, \mu, 0)\) by \((\overline{z}_1 - \overline{z}_2)/\sqrt{|z_1|^2 + |z_2|^2} \in \text{Sp}(1)_L\), we may assume that \(p_0 = \frac{1}{\sqrt{\mu^2 + 1}}q(1, 0, \mu, 0)\).

Set \(v = q(-\mu, 0, 1, 0) \in T_{p_0}S^7\). Then we compute
\[
M_{p_0} = \frac{1}{\mu^2 + 1} \begin{pmatrix}
0 & 0 & \mu^2 - 3 \\
2\mu(-\mu + \sqrt{3}) & 0 & 0 \\
0 & -2\mu(\mu + \sqrt{3}) & 0 
\end{pmatrix},
\]
\[
(v(M))_{p_0} = \frac{1}{\sqrt{\mu^2 + 1}} \begin{pmatrix}
0 & 0 & 8\mu \\
-2(\sqrt{3}\mu - 1)(\mu + \sqrt{3}) & 0 & 0 \\
0 & 2(\sqrt{3}\mu + 1)(\mu - \sqrt{3}) & 0 
\end{pmatrix}.
\]
Thus we have no associative SU(2)-orbits in the case (1).

Next, consider the case (2). We may assume that $\lambda_1 = \lambda_2$ by Remark 6.4. Then we have $\Re(z_1\overline{z}_3 + z_2\overline{z}_4) = 0$, and (6.13) implies that

$$z_1\overline{z}_4 - \overline{z}_2z_3 = 0, z_1\overline{z}_3 + \overline{z}_2z_4 = 0.$$ 

Thus,

$$z_1z_2\overline{z}_3\overline{z}_4 = |z_2z_3|^2 = -|z_2z_4|^2 = 0, \overline{z}_1\overline{z}_2z_3z_4 = |z_1z_4|^2 = -|z_1z_3|^2 = 0.$$ 

We deduce that either $z_1 = z_2 = 0$ or $z_3 = z_4 = 0$. Since $(z_1, z_2, 0, 0)$ (resp. $(0, 0, z_3, z_4)$) is mapped to $(1, 0, 0, 0)$ (resp. $(0, 0, 1, 0)$) by $\overline{z}_1 - z_2j$ (resp. $\overline{z}_3 - z_4j$) $\in \text{Sp}(1)_2$, we only have to consider at $p_0 = (1, 0, 0, 0)$ or $(0, 0, 1, 0)$.

At $p_0 = (1, 0, 0, 0)$, we have

$$E^*_1 = \{0, 0, 0, -\sqrt{3}\}, E^*_2 = \{0, 0, 0, \sqrt{3}\}, E^*_3 = \{3i, 0, 0, 0\} = -3\xi_1,$$ 

which are also orthogonal to each other with respect to $\tilde{g}$ and $\varphi(E^*_1, E^*_2, E^*_3) = -243/25 = -|E^*_1|_g|E^*_2|_g|E^*_3|_g$. At $p_0 = (0, 0, 1, 0)$, we have

$$E^*_1 = \{0, -\sqrt{3}, 0, 2\}, E^*_2 = \{0, \sqrt{3}, 0, 2i\}, E^*_3 = \{0, 0, -i, 0\} = \xi_1,$$ 

which are also orthogonal to each other with respect to $\tilde{g}$ and $\varphi(E^*_1, E^*_2, E^*_3) = 3^{3/2} \cdot 19/5^3 = |E^*_1|_g|E^*_2|_g|E^*_3|_g$. Thus we see that both SU(2)-orbits are associative.

\[\square\]

7 Deformations of homogeneous associative submanifolds

We study the deformations of homogeneous associative submanifolds in the squashed $S^7$. We apply the same method of [6] in the standard $S^7$.

**Proposition 7.1.** [6] Let $(Y, \varphi, g)$ be a nearly parallel $G_2$-manifold, and $A^3 \subset Y$ be an associative submanifold. Denote by $\nu$ the normal bundle of $A$ in $Y$ and by $\nabla^{\perp-A}$ the connection on $\nu$ induced by the Levi-Civita connection $\nabla$ of $(Y, g)$.

Taking any local orthonormal frame $\{e_1, e_2, e_3\}$ of $TA$, define the operator $D : C^\infty(A, \nu) \to C^\infty(A, \nu)$ by

$$D\psi := \sum_{i=1}^{3} e_i \times \nabla^{\perp-A}_{e_i} \psi.$$ 

Then the vector space of all infinitesimal associative deformations of $A^3 \hookrightarrow Y$ is identified with \{ $\psi \in C^\infty(A, \nu)$; $D\psi = -\psi$ \}.

Thus to compute the dimensions of the infinitesimal deformation spaces, we only have to know $\nabla^{\perp-A}$ and $\times$. The next lemma is useful for the computation.
Lemma 7.2. Let \( \{e_1, e_2, e_3\} \) be the local oriented orthonormal frame of \( TA \) satisfying \( e_3 = e_1 \times e_2 \). Take the local normal vector field \( V_i \) with \( |V_i| = 1 \).

Set \( V_2 = e_1 \times V_1, V_3 = e_2 \times V_1, V_4 = -e_3 \times V_1 \). Then \( \{V_1, V_2, V_3, V_4\} \) is a local orthonormal frame of \( \nu \) satisfying \( \varphi = e^{123} + e^1(V^{12} + V^{34}) + e^2(V^{13} + V^{42}) - e^3(V^{14} + V^{23}) \), where \( \{e^i, V^i\} \) is a dual coframe of \( \{e_i, V_j\} \). In particular, we have

\[
(e_i \times V_j) = \begin{pmatrix}
  V_3 & -V_2 & V_4 & -V_1 \\
  -V_3 & V_2 & -V_1 & V_4 \\
  -V_4 & V_3 & V_1 & -V_2 \\
  V_4 & -V_3 & -V_2 & V_1
\end{pmatrix}.
\]

Lemma 7.3. \( \Box \) For any \( X, u, v \in \mathfrak{X}(A), \eta \in C^\infty(A, \nu) \), we have

\[
\nabla_X^{\perp, A}(u \times \eta) = (\nabla_X^{\perp, A}u) \times \eta + u \times (\nabla_X^{\perp, A}\eta) - (\chi(X, u, \eta))^{\perp, A},
\]

where \( \chi(X, u, \eta) = X \times (u \times \eta) + g(X, u)\eta \) and \( \nabla \colon \text{TY} \to \text{TA} \) and \( \perp \colon \text{TA} \to \nu \) are projections.

We can compute \( \nabla_{e_i}^{\perp, A} V_j \) from \( \nabla_{e_i}^{\perp, A} e_j \) and \( \nabla_{e_i}^{\perp, A} V_1 \) by Lemma 7.3

Lemma 7.4. Denote \( \nabla_{e_i}^{\perp, A} e_j = \sum_{k=1}^3 \Gamma_{ijk}^k e_k \) and \( \nabla_{e_i}^{\perp, A} V_1 = \sum_{j=2}^3 K_{ij} V_j \). Then we have for \( i = 1, 2, 3 \)

\[
\begin{align*}
\nabla_{e_1}^{\perp, A} V_2 &= -\kappa_{12} V_1 + (\Gamma_{11}^2 - \kappa_{14} - \delta_{13}) V_3 + (-\Gamma_{11}^3 + \kappa_{14} + \delta_{12}) V_4, \\
\nabla_{e_1}^{\perp, A} V_3 &= -\kappa_{13} V_1 + (\Gamma_{11}^3 - \kappa_{14} - \delta_{13}) V_2 + (-\Gamma_{11}^2 + \kappa_{14} + \delta_{12}) V_4, \\
\nabla_{e_1}^{\perp, A} V_4 &= -\kappa_{14} V_1 + (\Gamma_{12}^3 - \kappa_{14} - \delta_{13}) V_2 + (-\Gamma_{12}^2 + \kappa_{14} + \delta_{12}) V_3.
\end{align*}
\]

By the definition of the Levi-Civita connection, we have the following.

Lemma 7.5. Suppose that \( A \) is a Lie group \( G \) and \( \{e_i\}_{i=1,2,3} \) are left invariant vector fields. Denoting \( [e_i, e_j] = \sum_{k=1}^3 \Gamma_{ijk}^k e_k \ (\Gamma_{ijk}^k \in \mathbb{R}) \), we have

\[
\nabla_{e_i}^{\perp, A} e_j = \frac{1}{2} \sum_{k=1}^3 \left( \epsilon_{ijk}^k - \epsilon_{ij}^l - \epsilon_{ik}^l \right) e_k.
\]

7.1 Computations on SU(2)

For the convenience of the computation, we summarize formulas on SU(2). Define the basis \( \{E_1, E_2, E_3\} \) of su(2) as (6.5).

Lemma 7.6. Let \( V_n \) be a \( \mathbb{C} \)-vector space of all complex homogeneous polynomials with two variables \( z_1, z_2 \) of degree \( n(n \geq 0) \) and define the representation \( \rho_n : \text{SU}(2) \to \text{GL}(V_n) \) as

\[
\left( \rho_n \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right) f(z_1, z_2) = f \left( \begin{pmatrix} z_1, z_2 \end{pmatrix} \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right).
\]

Define the Hermitian inner product \( \langle , \rangle \) of \( V_n \) such that

\[
\langle v_k^{(n)} \rangle = \frac{1}{\sqrt{k! (n-k)!}} z_1^{n-k} z_2^k \quad 0 \leq k \leq n
\]
is a unitary basis of $V_n$. Denoting by $\hat{\SU(2)}$ the set of all equivalence classes of finite dimensional irreducible representations of $\SU(2)$, we know that $\SU(2) = \{ (V_n, \rho_n) ; n \geq 0 \}$. Then every $\mathbb{C}$-valued continuous function on $\SU(2)$ is uniformly approximated by the $\mathbb{C}$-linear combination of the following functions:

$$\{ (\rho_n(\cdot)v^{(n)}_i, v^{(n)}_j) ; n \geq 0, 0 \leq i, j \leq n \},$$

which are mutually orthogonal with respect to the $L_2$ inner product.

By a direct computation, we see the following.

**Lemma 7.7.** Identify $X \in \mathfrak{su}(2)$ with the left invariant differential operator on $\SU(2)$. For $u = \sum_{l=0}^{n} C^{(n)}_l \in V_n$, set

$$u^* = \sum_{l=0}^{n} (-1)^n i\mathbf{e}_l u^{(n)} \in V_n.$$

Then for any $n \geq 0$, $0 \leq k, l \leq n$, $v, u \in V_n, X \in \mathfrak{su}(2)$, we have

$$X(\rho_n(\cdot)v, u) = \langle \rho_n(\cdot) d\rho_n(X) v, u \rangle, \quad (d\rho_n(X)v)(z_1, z_2) = \left( \frac{\partial v}{\partial z_1}, \frac{\partial v}{\partial z_2} \right) X \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right),$$

$$\langle \rho_n(\cdot)v^{(n)}_k, u \rangle = (-1)^k \langle \rho_n(\cdot) v^{(n)}_{-k}, u^* \rangle,$$

$$(-iE_1 + E_2)(\rho_n(\cdot)v^{(n)}_k, u) = \begin{cases} 2i\sqrt{(k+1)(n-k)}(\rho_n(\cdot)v^{(n)}_{k+1}, u), & (k < n) \\ 0, & (k = n) \end{cases}$$

$$(iE_1 + E_2)(\rho_n(\cdot)v^{(n)}_k, u) = \begin{cases} 2i\sqrt{(n-k+1)(n-k)}(\rho_n(\cdot)v^{(n)}_{k-1}, u), & (k > 0) \\ 0, & (k = 0) \end{cases}$$

$$iE_3(\rho_n(\cdot)v^{(n)}_k, u) = (-n+2k)(\rho_n(\cdot)v^{(n)}_k, u).$$

**Lemma 7.8.** Suppose that $\{e_1, e_2, e_3\} = \{pe_1, pe_2, qe_3\} (0 \neq p, q \in \mathbb{R})$ is an oriented orthonormal basis of $\mathfrak{su}(2)$ for some metric and orientation. Define the differential operator $D_{\lambda, \mu} : C^\infty(\SU(2), \mathbb{R}^4) \to C^\infty(\SU(2), \mathbb{R}^4)$ by

$$D_{\lambda, \mu} \left( \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \right) = \left( \begin{array}{cccc} 0 & -e_1 & -e_2 & e_3 \\ e_1 & 0 & e_3 & e_2 \\ e_2 & -e_3 & 0 & -e_1 \\ -e_3 & -e_2 & e_1 & 0 \end{array} \right) + \left( \begin{array}{cc} \lambda & \mu \\ \mu & \lambda \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \right),$$

(7.1)

for $\lambda, \mu \in \mathbb{R}$. Setting $\Psi_1 = \psi_1 + i\psi_4, \Psi_2 = \psi_2 - i\psi_3, D_{\lambda, \mu}$ is described as

$$D_{\lambda, \mu} \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) = \left( \begin{array}{c} -ie_3 \\ e_1 - ie_2 \\ ie_3 \end{array} \right) + \left( \begin{array}{cc} \lambda & \mu \\ \mu & \lambda \end{array} \right) \left( \begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right).$$

Setting $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$. Then $D_{\lambda, \mu} \psi = \alpha \psi (\alpha \in \mathbb{R})$ is equivalent to

$$(-ie_3 + \lambda - \alpha) \Psi_1 - (e_1 + ie_2) \Psi_2 = 0,$$

$$e_1 - ie_2) \Psi_1 + (ie_3 + \nu - \alpha) \Psi_2 = 0.$$

(7.2)

(7.3)
These equations imply that $\Gamma_{p,q,\lambda,\mu,\alpha} \Psi_2 = 0$, where $\Gamma_{p,q,\lambda,\mu,\alpha}$ is defined by

$$\Gamma_{p,q,\lambda,\mu,\alpha} = \Delta_+ + \left( \mu - \lambda + 2q - \frac{2p^2}{q} \right) ie_3 + (-2q + \lambda - \alpha)(-\mu + \alpha),$$

(7.4)

where $\Delta_+ = -\sum_{i=1}^{3} e_i^2$ is a Laplacian on SU(2). Especially, for any $n \geq 0$, $0 \leq k \leq n$, $u \in V_n$, we have

$$\Delta_+ \langle \rho_n(\cdot)\nu_k^{(n)}, u \rangle = \left\{ (-p^2 + q^2)(n-2k)^2 + p^2(n^2 + 2n) \right\} \langle \rho_n(\cdot)\nu_k^{(n)}, u \rangle,$$

(7.5)

$$\Gamma_{p,q,\lambda,\mu,\alpha} \langle \rho_n(\cdot)\nu_k^{(n)}, u \rangle = \left\{ (-p^2 + q^2)(n-2k)^2 + p^2(n^2 + 2n) - (q(-\mu + \lambda) + 2(p^2 - q^2))(n-2k) \right. \left. + (-2q + \lambda - \alpha)(-\mu + \alpha) \right\} \langle \rho_n(\cdot)\nu_k^{(n)}, u \rangle.$$  

(7.6)

**Remark 7.9.** In the case of SU(2)/Γ for some finite subgroup Γ, we may consider the Γ equivariant solutions of (7.2) and (7.3).

**Proof.** It is straightforward to derive (7.2) and (7.3). Since $[e_1, e_2] = \frac{2q^2}{q} e_3$, $[e_2, e_3] = 2q e_1$, $[e_3, e_1] = 2qe_2$, we have $(e_1 - ie_2)ie_3 = (ie_3 + 2q)(e_1 - ie_2)$. Applying $(e_1 - ie_2)$ to (7.2), we obtain

$$(-ie_3 - 2q + \lambda - \alpha)(e_1 - ie_2)\Psi_1 + \left( -e_1^2 - e_2^2 - \frac{2p^2}{q}ie_3 \right) \Psi_2 = 0.$$  

(7.7)

Eliminating $\Psi_1$ from (7.2) by (7.3) gives (7.4). From Lemma 7.7, we obtain (7.6).  

**7.2 The case L₁**

Let SU(2) = Sp(1) act on $S^7$ as (7.1). Then $L_1$ is the SU(2)-orbit through $p_0 = \{1,0,0,0\}$. Identifying SU(2) ⊃ $\left( \begin{array}{cc} a & -b \\ b & \pi \end{array} \right) \mapsto a - b \in \text{Sp}(1)$, the vector fields $E_i^*$ generated by $E_i \in \mathfrak{su}(2)(i = 1, 2, 3)$ in (7.5) are described as

$$E_1^* = \{1,0,0,0\} = -\xi_2, E_2^* = \{0, i, 0, 0\} = -\xi_3, E_3^* = \{i, 0, 0, 0\} = -\xi_1,$$

at $p_0$, which induces the orthonormal basis $\{e_1, e_2, e_3\} = 5/3\{E_1, E_2, -E_3\}$ of $\mathfrak{su}(2)$.

Set $v_1 = \sqrt{\frac{5}{2}}(0,0,1,0) \in \nu_{p_0}$, which is horizontal and $|v_1| = 1$. Denote $X_0 = \{0,0,1,0\}$, which is horizontal at $p_0$ and $X_i = \Phi_i(X_0)(i = 1, 2, 3)$. By the definition of $\tilde{\varphi}$ in Proposition 4.3, the vectors $v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as $\{v_1, v_2, v_3, v_4\} = \tilde{\varphi}(X_0, X_2, X_3, X_1)$. Define the vector field $V_i$ on $L_1$ by $(V_i)_{p_0} = g_*v_i (g \in \text{SU}(2))$, we obtain the following by Lemma 4.7 and Lemma 7.3:

$$\left( \begin{array}{c} \nabla_{v_1} V_1 \\ \nabla_{v_2} V_1 \\ \nabla_{v_3} V_1 \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} V_2 \\ V_3 \\ -V_4 \end{array} \right), \left( \nabla_{v_i}^* v_j \right) = \frac{5}{3} \left( \begin{array}{ccc} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{array} \right).$$  

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This computation and Lemma 7.3 give the following.

\[
(\mathring{\nabla}_e V_j) = \frac{1}{3} \begin{pmatrix}
V_2 & -V_1 & V_4 & -V_3 \\
V_3 & -V_4 & -V_1 & V_2 \\
-V_4 & -V_3 & V_2 & V_1
\end{pmatrix}.
\]

Then by the trivialization of \(\nu\) via \(\{V_1, V_2, V_3, V_4\}\), we have \(D = D_{-1, -1}\), where \(D_{\lambda, \mu}\) is defined in (7.4). Using the notations of Lemma 7.8, we see that \(\Psi_2\) is constant, and hence \(\Phi_1\) is constant.

Thus we obtain \(\dim \mathbb{R}\{\psi \in C^\infty(L_1, \nu); D\psi = -\psi\} = 4\). Since \(\text{Sp}(1)\text{Sp}(2)\) induces \(4(= \dim \mathbb{R}\text{Sp}(1)\text{Sp}(2)/\text{Sp}(1)\times\text{Sp}(1))\)-dimensional associative deformations of \(L_1\), we obtain the following.

**Proposition 7.10.** The associative deformations of \(L_1\) is trivial. Its deformation space is \(\text{Sp}(1)\text{Sp}(2)/\text{Sp}(1)\times\text{Sp}(1)) = \mathbb{H}P^1 = S^4\). The associative deformations of \(L_1\) are the deformations of fibers of \(\pi : S^7 \to S^4\) parametrized by the base space \(S^4\).

### 7.3 The case \(L_2\)

Let \(\text{SU}(2)\) act on \(S^7\) by \(\mathbf{6}_A\). Then \(L_2\) is the \(\text{SU}(2)\)-orbit through \(p_0 = (1, 0, 0, 0)\). By \(\mathbf{6}_A\), the vector fields \(E_i^*\) generated by \(E_i \in \mathfrak{su}(2)\) in \(\mathbf{6}_A\) are described as

\[E_1^* = 1(0, 0, -1, 0), E_2^* = 1(0, 0, -i, 0), E_3^* = 1(-i, 0, 0, 0) = \xi_1,\]

and satisfy \(\bar{\nabla}(E_1^*, E_2^*, E_3^*) = -27/25 < 0\) at \(p_0\). Then we obtain the induced oriented orthonormal basis \(\{e_1, e_2, e_3\} = \{\sqrt{\frac{5}{3}} E_1, \sqrt{\frac{5}{3}} E_2, -\frac{2}{5} E_3\}\) of \(\mathfrak{su}(2)\).

Set \(v_1 = \frac{\sqrt{5}}{3}(0, 1, 0, 0) = -\frac{\sqrt{2}}{2} \xi_2\), which satisfies \(|v_1|_3 = 1\). Denote \(X_0 = 1(0, 0, 1, 0)\), which is horizontal at \(p_0\) and \(X_i = \Phi_0(X_0)(i = 1, 2, 3)\). Since \(\{e_1, e_2, e_3\} = \{-\frac{\sqrt{5}}{3} X_0, -\frac{\sqrt{5}}{3} X_1, \xi_1\}\), vectors \(v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1\) are described as \(\{v_1, v_2, v_3, v_4\} = \{-\frac{\sqrt{5}}{3} \xi_2, \frac{\sqrt{5}}{3} X_2, -\frac{\sqrt{5}}{3} X_3, -\frac{2}{5} \xi_3\}\). Define the vector field \(V_i\) on \(L_2\) by \((V_i)_{g, p_0} = g(v_i(g \in \text{SU}(2))\). As in the case \(L_1\), we obtain

\[
(\mathring{\nabla}_e V_j) = \frac{1}{3} \begin{pmatrix}
-V_2 & V_1 & -V_4 & V_3 \\
-V_3 & V_4 & V_1 & -V_2 \\
-5V_4 & -V_3 & V_2 & 5V_1
\end{pmatrix}.
\]

Then by the trivialization of \(\nu\) via \(\{V_1, V_2, V_3, V_4\}\), we have \(D = D_{-1, -1, -1}\), where \(D_{\lambda, \mu, \nu}\) is defined in (7.4). Setting \((p, q, \lambda, \mu, \nu) = (\frac{\sqrt{5}}{3}, -\frac{2}{5}, 1, \frac{1}{3}, 1)\) in (7.0), we see that

\[\Psi_2 = (\rho_2(\cdot)v_1(2)^{\nu}, u)\]

for \(u \in V_2\). Since \(\ker(e_1 - i e_2) \cap \ker(i e_3) = \mathbb{C}\), (7.2) and (7.3) imply that

\[\Psi_1 = -\sqrt{\frac{10}{3}} \langle \rho_2(\cdot)v_2(2)^{\nu}, u \rangle + C\]

for \(C \in \mathbb{C}\). Thus we obtain \(\dim \mathbb{R}\{\psi \in C^\infty(L_2, \nu); D\psi = -\psi\} = 8\). Since \(\text{Sp}(1)\text{Sp}(2)\) induces \(8 (= \dim \mathbb{R}\text{Sp}(1)\text{Sp}(2)/U(1)U(2))\)-dimensional associative deformations of \(L_2\), we obtain the following.

**Proposition 7.11.** The associative deformations of \(L_2\) is trivial. Its deformation space is \(\text{Sp}(1)\text{Sp}(2)/U(1)U(2)\).
7.4 The case A₁

Let $T^3$ act on $S^7$ by $(5.1)$. Then $A₁$ is the $T^3$-orbit through $p₀ = \frac{1}{2}(1, 1, 1, i)$. By $(6.3)$, the vector fields $F'_i$ generated by $F_i(i = 1, 2, 3)$ in $(6.2)$ are described as

\[ F'_1 = \frac{1}{2}(i, i, i, -1) = -\xi_t, \quad F'_2 = \frac{1}{2}(i, -i, 0, 0), \quad F'_3 = \frac{1}{2}(0, 0, i, 1), \]

and satisfy $\varphi(F'_1, F'_2, F'_3) = -81/250 < 0$ at $p₀$. Then we obtain the induced oriented orthonormal basis $\{e₁, e₂, e₃\} = \{\frac{2}{3}F₁, \frac{5\sqrt{6}}{9}F₂, \frac{-5\sqrt{6}}{9}F_3\}$ of $t^3$.

Set $v₁ = \sqrt{\frac{5}{6}}((-1, -1, 1, i)$, which is horizontal at $p₀$ and $|v₁|_{\gamma} = 1$. Denote $X₀ = \frac{1}{2}((-1, -1, 1, i)$, which is horizontal at $p₀$ and $X_i = \Phi_i(X₀)(i = 1, 2, 3)$. Since

\[ e₁ = -\frac{5}{3}ξ₁, \quad e₂ = \frac{5\sqrt{6}}{18}(ξ₃ + X₃), \quad e₃ = \frac{5\sqrt{6}}{18}(ξ₂ - X₂), \]

vectors $v₂ = e₁ \times v₁, e₃ = e₂ \times v₁, e₄ = -e₃ \times v₁$ are described as

\[ \{v₁, v₂, v₃, v₄\} = \left\{ \frac{\sqrt{5}}{3}X₀, \frac{2\sqrt{3}}{3}X₁, \sqrt{\frac{30}{18}}(-X₃ + 5ξ₃), \sqrt{\frac{30}{18}}(X₂ + 5ξ₂) \right\}. \]

Define the vector field $V_i$ on $T^3$ by $(V_i)_{g \cdot p₀} = g \cdot v_i (g \in T^3)$. As in the case $L₁$, we obtain

\[ (\nabla_{e₁} V_j) = \frac{1}{9} \begin{pmatrix} 3V₂ & -3V₁ & -12V₃ & 12V₄ \\ -2V₃ & 7V₄ & 2V₁ & -7V₂ \\ 2V₄ & 7V₃ & -7V₂ & -2V₁ \end{pmatrix}. \]

Then by the trivialization of $\nu$ via $\{V₁, V₂, V₃, V₄\}$, we have

\[ D = \begin{pmatrix} 0 & -e₁ & -e₂ & e₃ \\ e₁ & 0 & e₃ & e₂ \\ e₂ & -e₃ & 0 & -e₁ \\ -e₃ & -e₂ & e₁ & 0 \end{pmatrix} + \frac{1}{9} \begin{pmatrix} 1 & 11 & 21 \\ 11 & 21 & \end{pmatrix}. \]

Suppose $Dψ = -ψ$, where $ψ = (ψ₁, ψ₂, ψ₃, ψ₄)$ and $ψ_i \in C^\infty(T^3)$. Eliminating $ψ₂$ by $ψ₂ = \frac{9}{20}(e₁(ψ₁) + e₃(ψ₃) + e₂(ψ₄))$, we obtain

\[ \begin{pmatrix} \frac{9}{20}e₁e₃ + e₂ \end{pmatrix} ψ₁ + \begin{pmatrix} \frac{9}{20}e₁e₃ + e₂ \end{pmatrix} ψ₃ + \begin{pmatrix} \frac{9}{20}e₂e₃ + e₁ \end{pmatrix} ψ₄ = 0, \quad (7.8) \]

\[ \begin{pmatrix} \frac{9}{20}e₁e₃ - e₂ \end{pmatrix} ψ₁ + \begin{pmatrix} \frac{9}{20}e₁e₃ + e₂ \end{pmatrix} ψ₃ + \begin{pmatrix} \frac{9}{20}e₂e₃ - e₁ \end{pmatrix} ψ₄ = 0, \quad (7.9) \]

\[ \begin{pmatrix} \frac{9}{20}e₁e₃ - e₂ \end{pmatrix} ψ₁ + \begin{pmatrix} \frac{9}{20}e₁e₃ + e₂ \end{pmatrix} ψ₃ + \begin{pmatrix} \frac{10}{3} + \frac{9}{20}e₂ \end{pmatrix} ψ₄ = 0. \quad (7.10) \]

Define the smooth function $f_γ \in C^\infty(T^3, \mathbb{C})$ $(γ = (γ₁, γ₂, γ₃) \in \mathbb{Z}^3)$ on $T^3 \cong (\mathbb{R}/2\pi\mathbb{Z})^3$ by $f_γ(θ₁, θ₂, θ₃) = \exp(i \sum_{i=1}^{3} γ_i θ_i)$. Identifying $e_i \in t^3$ with the left invariant differential operator on $T^3$, we have

\[ e₁(f_γ) = \frac{5}{3}γ₁if_γ, \quad e₂(f_γ) = \frac{5\sqrt{6}}{9}γ₂if_γ, \quad e₃(f_γ) = -\frac{5\sqrt{6}}{9}γ₃if_γ. \]
By a Fourier series expansion, set
\[ \psi_1 = \sum_{\gamma \in \mathbb{C}} C_{\gamma} f_{\gamma}, \psi_2 = \sum_{\gamma \in \mathbb{C}} D_{\gamma} f_{\gamma}, \psi_3 = \sum_{\gamma \in \mathbb{C}} E_{\gamma} f_{\gamma}, \]
where \( C_{\gamma}, D_{\gamma}, E_{\gamma} \in \mathbb{C}. \) Then (6.8), (6.9), and (6.10) are equivalent to \( M_{\gamma} \psi(C_{\gamma}, D_{\gamma}, E_{\gamma}) = 0, \)
where
\[ M_{\gamma} = \begin{pmatrix} 8 - 9\gamma_1^2 & 3\sqrt{6}\gamma_1\gamma_3 - 4\sqrt{6}\gamma_2i & -3\sqrt{6}\gamma_1\gamma_2 - 4\sqrt{6}\gamma_3i \\ 3\sqrt{6}\gamma_1\gamma_3 + 4\sqrt{6}\gamma_2i & -6\gamma_3^2 + 24 & 6\gamma_2\gamma_3 - 12\gamma_1i \\ -3\sqrt{6}\gamma_1\gamma_2 + 4\sqrt{6}\gamma_3i & 6\gamma_2\gamma_3 + 12\gamma_1i & -6\gamma_3^2 + 24 \end{pmatrix}. \]
To obtain a nontrivial solution \( \psi(C_{\gamma}, D_{\gamma}, E_{\gamma}) \neq 0, \)
\[ \det M_{\gamma} = 16 \left\{ (9\gamma_1^2 + 6\gamma_2^2 + 6\gamma_3^2 - 22) + 4(12(\gamma_2^2 + \gamma_3^2) - 49) \right\} \]
must vanish. We see that \( \det M_{\gamma} = 0 \) if and only if
\[ (\gamma_1, \gamma_2, \gamma_3) = \pm (2, 0, 0), \pm (0, 2, 0), \pm (0, 0, 2), \pm (0, 1, 1), \pm (0, 1, -1). \] (7.11)

For each \( \gamma \) in (7.11), we can check \( \dim \ker M_{\gamma} = 1. \) Moreover, we have \( C_{\gamma} = D_{-\gamma}, D_{\gamma} = D_{-\gamma}, \) and \( E_{\gamma} = E_{-\gamma} \) so that every \( \psi_i \) is \( R \)-valued. Hence we obtain \( \dim \ker \{ \psi \in C^\infty(A_1, \nu); D\psi = -\psi \} = 10. \) Since \( \text{Sp}(1)\text{Sp}(2) \) induces 10(= \( \dim \text{sp}(1)\text{sp}(2)/T^3 \))-dimensional associative deformations of \( A_1, \) we obtain the following.

**Proposition 7.12.** The associative deformations of \( A_1 \) is trivial. Its deformation space is \( \text{Sp}(1)\text{Sp}(2)/T^3. \)

### 7.5 The case \( A_2 \)

Let \( \text{SU}(2) \) act on \( S^7 \) by (6.9). Then \( A_2 \) is the \( \text{SU}(2) \)-orbit through \( p_0 = (1, 0, 0, 0). \) By (6.15), \( \{ e_1, e_2, e_3 \} = \{ \frac{\sqrt{2}}{3} E_1, \frac{\sqrt{3}}{3} E_2, -\frac{\sqrt{3}}{3} E_3 \} \) is the induced oriented orthonormal basis of \( \text{su}(2), \) where \( E_i \in \text{su}(2)(i = 1, 2, 3) \) is defined in (6.9).

Set \( v_1 = \frac{2}{3}(0, 1, 0, 0), \) which satisfies \( |v_1|_g = 1. \) Denote \( X_0 = \Phi(q)(0, 0, 0, 1), \) which is horizontal at \( p_0 \) and \( X_i = \Phi(q)(0, 1, 2, 3). \) Since
\[ e_1 = -\frac{\sqrt{5}}{3} X_0, e_2 = \frac{\sqrt{5}}{3} X_1, e_3 = -\frac{5}{3} \xi_1, \]

vectors \( v_2 = e_1 \times v_1, v_3 = e_2 \times v_1, v_4 = -e_3 \times v_1 \) are described as
\[ \{ v_1, v_2, v_3, v_4 \} = \left\{ \frac{5}{3} \xi_2, \frac{\sqrt{5}}{3} X_2, \frac{\sqrt{5}}{3} X_3, \frac{5}{3} \xi_3 \right\}. \]

Define the vector field \( V_i \) in the neighborhood of \( p_0 \) of \( A_2 \) by \( (V_i)_g p_0 = g(v_i (g \in \text{SU}(2)). \) As in the case \( L_1, \) we obtain
\[ (\nabla_{e_i} V_j) = \frac{1}{9} \begin{pmatrix} -3V_2 & 3V_1 & -3V_4 & 3V_3 \\ -3V_4 & 3V_4 & 3V_1 & -3V_2 \\ -15V_4 & 17V_3 & -17V_2 & 15V_1 \end{pmatrix}. \]
Then by the local trivialization of $\nu$ via \{\(V_1, V_2, V_3, V_4\)\}, we have \(D = D_{-1, 23/9}\), where \(D_{\lambda, \mu}\) is defined in (7.4). Setting \((p, q, \lambda, \mu, \alpha) = (\frac{\sqrt{5}}{5}, -\frac{5}{5}, -1, \frac{23}{3}, -1)\) in (7.4), we see that

\[
\Psi_2 = \langle p_0(\cdot)v_6^{(6)}, u \rangle
\]

for \(u \in V_2\). Since \(\ker(e_1 - ie_2) \cap \ker(ie_3) = \mathbb{C}\), (7.2) and (7.3) imply that

\[
\Psi_1 = -\frac{\sqrt{10}}{5}(p_0(\cdot)v_6^{(6)}, u) + C
\]

for \(C \in \mathbb{C}\). These solutions are \(\mathbb{Z}_3\)-equivariant, and hence we obtain \(\dim_\mathbb{C} \{\psi \in C^\infty(A_2, \nu); D\psi = -\psi\} = 16\). Since Sp(1)Sp(2) induces \(9(= \dim_\mathbb{R}\text{Sp(1)Sp(2)/U(1)}SU(2))\)-dimensional associative deformations of \(A_2\), \(A_2\) can have at most 7-dimensional family of nontrivial associative deformations. In fact, we obtain the following.

**Proposition 7.13.** All associative deformations of \(A_2\) are induced by the Sp(1)Sp(2)-action and by the \(\text{PSp}(2, \mathbb{C})\)-action on \(\mathbb{C}P^3\) via the Hopf lift. In other words, All the associative deformations of \(A_2\) are given by the following.

- the \(\text{PSp}(2, \mathbb{C})\)-action on \(\mathbb{C}P^3\) via the Hopf lift, which corresponds to the deformation of \(p_1(A_2)\) as a horizontal holomorphic curve, where \(p_1 : S^7 \to \mathbb{C}P^3\) is a projection,

- the action generated by \(j, k \in \text{Sp}(1)\).

Note that \(\text{PSp}(2, \mathbb{C})\) acts on \(\mathbb{C}P^3\) as the group of biholomorphic maps which preserve the horizontal distribution \[8, 10\].

**Proof.** First description is an analogue of \[10, 6\] and we omit the proof. The second description follows from the next lemma.

**Lemma 7.14.** The subgroup of \(\text{PSp}(2, \mathbb{C})\) which preserves \(p_1(A_2)\) is isomorphic to \(\text{PSL}(2, \mathbb{C})\). Thus the deformation space of \(p_1(A_2)\) as a holomorphic curve is \(\text{PSp}(2, \mathbb{C})/\text{PSL}(2, \mathbb{C})\), which is 14-dimensional.

**Proof.** The inclusion \(\text{SU}(2) \hookrightarrow \text{Sp}(2)\) of \[6.9\] is canonically extended to \(\text{GL}(2, \mathbb{C}) \hookrightarrow \text{GL}(4, \mathbb{C})\):

\[
(g_{ij}) \mapsto \begin{pmatrix}
    g_{11}^3 & g_{12}^3 & \sqrt{3}g_{11}g_{12}^2 & \sqrt{3}g_{11}^2g_{12} \\
    g_{21}^3 & g_{22}^3 & \sqrt{3}g_{21}g_{22}^2 & \sqrt{3}g_{21}^2g_{22} \\
    \sqrt{3}g_{11}g_{21} & \sqrt{3}g_{12}g_{21} & g_{22}(g_{11}g_{22} + 2g_{12}g_{21}) & g_{21}(2g_{11}g_{22} + g_{12}g_{21}) \\
    \sqrt{3}g_{11}^2g_{21} & \sqrt{3}g_{12}^2g_{21} & g_{12}(2g_{11}g_{22} + g_{12}g_{21}) & g_{11}(g_{11}g_{22} + 2g_{12}g_{21})
\end{pmatrix},
\]

which is the group of biholomorphic maps which preserve \(p_1(A_2)\). We can check that \(\text{GL}(2, \mathbb{C}) \cap \text{Sp}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})\), and hence we obtain the proof.

### 7.6 The case \(A_3\)

Let \(\text{SU}(2)\) act on \(S^7\) by \[6.9\]. Then \(A_3\) is the \(\text{SU}(2)\)-orbit through \(p_0 = \langle 0, 0, 1, 0 \rangle\). By \[6.10\], \(\{e_1, e_2, e_3\} = \{\frac{5\sqrt{5}}{5}E_1, \frac{5\sqrt{5}}{5}E_2, \frac{5}{5}E_3\}\) is the induced oriented orthonormal basis of \(\text{su}(2)\), where \(E_i \in \text{su}(2)(i = 1, 2, 3)\) is defined in \[6.4\].
Set $v_1 = \frac{\sqrt{5}}{19}(1,0,0,0) \in \nu_{p_0}$, which is horizontal at $p_0$ and $|v_1|_g = 1$. Denote $X_0 = \{1,0,0,0\}$, which is horizontal at $p_0$ and $X_i = \Phi_i(X_0)$ ($i = 1, 2, 3$). Since
\begin{equation}
eq \frac{5\sqrt{19}}{57}(2\xi_2 + \sqrt{3}X_2), e_2 = \frac{5\sqrt{19}}{57}(-2\xi_3 + \sqrt{3}X_3), e_3 = \frac{5}{3}\xi_1, \tag{7.12}
\end{equation}

vectors $e_2 = e_1 \times v_1, e_3 = e_2 \times v_1, v_4 = -e_3 \times v_1$ are described as
\begin{align*}
\{v_1, v_2, v_3, v_4\} = \left\{ \frac{\sqrt{5}}{3}X_0, \frac{\sqrt{195}}{57}(-5\sqrt{3}\xi_2 + 2X_0), \frac{\sqrt{195}}{57}(5\sqrt{3}\xi_3 + 2X_3), \frac{\sqrt{5}}{3}X_1 \right\}.
\end{align*}
Define the vector field $V_i$ on SU(2) by $(V_i)_{g \cdot p_0} = g \cdot v_i (g \in \text{SU}(2))$. As in the case $L_1$, we obtain
\begin{align*}
(\nabla_{e_i}^A V_j) = \frac{1}{57} \begin{pmatrix} -31V_2 & 31V_1 & -31V_4 & 31V_3 \\ -31V_3 & 31V_4 & 31V_1 & -31V_2 \\ 361V_4 & 31V_3 & 31V_4 & -31V_3 \\ 119V_3 & 119V_2 & -31V_4 & 361V_1 \end{pmatrix}.
\end{align*}
Then by the local trivialization of $e$ via $\{V_1, V_2, V_3, V_4\}$, we have $D = D_{141/19,-1}$, where $D_{\lambda, \mu}$ is defined in (7.1). Setting $(\lambda, \mu, \alpha) = (\frac{\sqrt{19}}{57}, \frac{5}{3}, 4, 1, -1, -1)$ in (7.6), we see that
\begin{align*}
\Psi_2 = \langle \rho_6(\cdot)v_4^{(6)}, u \rangle + C
\end{align*}
for $u \in V_2, C \in \mathbb{C}$. Since ker$(ie_3 - \frac{462}{19}) = \{0\}$, (7.2) and (7.3) imply that
\begin{align*}
\Psi_1 = \frac{-19\sqrt{190}}{10} \langle \rho_6(\cdot)v_5^{(6)}, u \rangle.
\end{align*}
Thus we obtain dim$_R \{\psi \in C^\infty(A_3, \nu); D\psi = -\psi\} = 16$. Since Sp(1)Sp(2) induces 10(= dim$_R \text{Sp}(1)\text{Sp}(2)/\text{SU}(2)$)-dimensional associative deformations of $A_3, A_3$ can have at most 6-dimensional family of nontrivial associative deformations.

The associative deformation space of $A_3$ is explained by a one-to-one correspondence between null-torsion $I_1$-holomorphic curves and horizontal holomorphic curves in $\mathbb{C}P^3$ (12).

Decompose $T\mathbb{C}P^3 = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is a vector bundle tangent to the fibers of $p_2 : \mathbb{C}P^3 \to S^4$, and $\mathcal{H}$ is its orthogonal complement bundle of $\mathcal{V}$. Define a map $P : \mathcal{H} \to \mathbb{C}P^3$ by $P(v) = [\tilde{v}], \tilde{v} \in \mathcal{H} \subset TS^4$ is a horizontal lift of $v$ with respect to $p_1 : S^7 \to \mathbb{C}P^3$ and we identify $\tilde{v}$ with a vector in $\mathbb{C}^4$.

Let $pr_{\mathcal{H}} : T\mathbb{C}P^3 \to \mathcal{H}$ be a canonical projection and $\Sigma \subset \mathbb{C}P^3$ be an $I_1$-holomorphic curve with $pr_{\mathcal{H}}(T\Sigma) \neq 0$. Then there exist a holomorphic line bundle $L \subset \mathcal{H}|\Sigma$ such that $pr_{\mathcal{H}}(T\Sigma) \subset L$. If $pr_{\mathcal{H}}$ is nowhere vanishing on $\Sigma$, $L = pr_{\mathcal{H}}(T\Sigma)$. Denote by $L^\perp \subset \mathcal{H}|\Sigma$ the orthonormal complement bundle of $L$ and set $\Sigma = P(L^\perp - \{0\})$.

Definition 7.15. A non-vertical $I_1$-holomorphic curve $\Sigma$ is called null-torsion if $\Sigma$ is a horizontal holomorphic curve.

Proposition 7.16. There is a one-to-one correspondence between null-torsion $I_1$-holomorphic curves and horizontal holomorphic curves via $\Sigma \mapsto \tilde{\Sigma}$. 28
Since $p_1(A_3)$ is a image of $\mathbb{C}P^1$, it is a null-torsion ([12]). We see the following.

**Lemma 7.17.** By Proposition 7.16, $p_1(A_3)$ corresponds to $p_1(A_2)$.

**Proof.** Since $pr_H$ is nowhere vanishing on $p_1(A_3)$, $L = pr_H(T(p_1(A_3)))$. By (7.12), $T_{p_1(p_0)}(p_1(A_3))$ is a projection of the subspace of $T_{p_0}S^7$ spanned by $-2\xi_2 - \sqrt{3}X_2$ and $-2\xi_3 + \sqrt{3}X_3$. Thus the vector bundle $L^\perp H$ over $A_3$ whose fiber at $g \cdot p_0 (g \in SU(2))$ is spanned by $g^* X_0$ and $g^* X_1$ satisfies $(p_1)_*(L^\perp H) = L^\perp H$, which implies that

$$p_1(A_3) = [L^\perp H - \{0\}] = \{ [g^*(1,0,0,0)] \in \mathbb{C}P^3; g \in SU(2) \} = p_1(A_2).$$

□

**Remark 7.18.** We easily see that $\widehat{p_1(A_1)} = p_1(A_1)$, and hence $p_1(A_1)$ is not null-torsion.

Since the deformation space of $p_1(A_2)$ as a horizontal holomorphic curve is 14-dimensional by Proposition 7.13, we obtain the following result.

**Proposition 7.19.** All the associative deformations of $A_3$ are given by the following.

- the Hopf lift of null-torsion $I'_1$-holomorphic curves, which correspond to horizontal holomorphic curves obtained by deforming $p_1(A_2)$ by the $\text{PSp}(2, \mathbb{C})$-action on $\mathbb{C}P^3$ by Proposition 7.16.

- the action generated by $j, k \in \text{Sp}(1)$.

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