Constant field strengths on $T^{2n}$

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Abstract

We analyse field strength configurations in $U(N)$ Yang-Mills theory on $T^{2n}$ that are diagonal and constant, extending early work of Van Baal on $T^4$. The spectrum of fluctuations is determined and the eigenfunctions are given explicitly in terms of theta functions on tori. We show the relevance of the analysis to higher dimensional D-branes and discuss applications of the results in string theory.

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1 Introduction

Constant field strengths were studied as solutions to $U(N)$ Yang-Mills on $T^4$ fifteen years ago in an attempt to get a handle on the mechanism of confinement \cite{1}. The spectrum of fluctuations around the background field configuration was explicitly determined by Van Baal in terms of theta functions on the four torus \cite{2}. These results were later used in the context of string theory in, amongst others (\cite{3}, \cite{4}) to compare the spectrum of open strings ending on D-branes to the predictions from Yang-Mills theory and the non-abelian Dirac-Born-Infeld action. The number of massless torons also played a crucial role in the black hole entropy counting in \cite{5}.

We extend the analysis of Van Baal to $U(N)$ Yang-Mills solutions with constant and diagonal field strength on general even dimensional tori. We will show how this analysis applies to the low-energy effective action for D-branes in a flat background and with small field strengths. Specifically, the results are relevant for higher dimensional D-branes wrapped on even dimensional tori, e.g. D6-branes wrapped around a six torus. In the Yang-Mills theory, we derive the conditions on the field strengths to have a stable configuration. We examine the criterion to have massless bosonic modes and give explicitly the full set of fluctuations in terms of theta functions on higher tori. In the dimensionally reduced SYM theory relevant to D-branes the spectrum of fermionic fluctuations on $T^{2n}$ can also be determined and index theory is used to check the number of fermionic massless modes.

The article starts out with dimensionally reducing $D = 9 + 1 \, \mathcal{N} = 1$ $U(N)$ Yang Mills to the low-energy 'non-relativistic' D-brane action. We delimit the gauge field backgrounds we will study. Next, we compactify the theory on an even dimensional torus and concentrate on the dependence of the fluctuations on the internal coordinates. Here we extend the analysis of Van Baal straightforwardly to higher tori. Moreover we derive the spectrum of scalar and fermionic fluctuations in the given background and discuss the zeromodes of the configurations in detail. Finally, we situate some of the applications in string theory in the context of our systematic analysis and indicate new applications.
2 Reduced Action

Van Baal studied a constant diagonal field strength configuration for $U(N)$ Yang-Mills on a four torus [2]. We will later look at $U(N)$ Yang-Mills on a general even dimensional torus, but we will first show how this fits in the Yang-Mills approximation to D-brane actions. We will closely follow the analysis and notation of [2] and [5] in the next few sections. The bosonic part of the $D = 9 + 1 \mathcal{N} = 1 U(N)$ Yang Mills action can be written in terms of the ten dimensional field strength $G_{\mu\nu}$ ($\mu = 0, \ldots, 9$),

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i [B_\mu, B_\nu]$$

as

$$S_{9+1} = -\frac{1}{4} \int d^{10}x \text{Tr} G_{\mu\nu}^2.$$ 

Reduced to $p + 1$ dimensions the action becomes ($\alpha = 0, \ldots, p$ and $m = p + 1, \ldots, 9$):

$$S_{p+1} = -\frac{1}{4} \int d^{p+1}x \text{Tr} \left( G_{\alpha\beta}^2 - [\phi_m, \phi_n]^2 + 2(\partial_\alpha \phi_m + i [B_\alpha, \phi_m])^2 \right) \quad (1)$$

where all fields depend only on the coordinates $x^\beta$ and the scalars and gauge fields are:

$$\phi_m = B_m(x^\beta)$$
$$B_\alpha = B_\alpha(x^\beta). \quad (2)$$

The dimensionally reduced gauge transformations read:

$$B \to \Omega B_\alpha \Omega^{-1} - i \Omega \partial_\alpha \Omega^{-1}$$
$$\phi \to \Omega \phi_m \Omega^{-1}, \quad (3)$$

with $\Omega \in U(N)$.

3 Background and fluctuations

In this section we determine the action for the fluctuations around a general diagonal and constant background field strength $G^0$. We fix notations as
follows:

\[ B_\alpha = B_\alpha^0 + A_\alpha \]

\[ G_{\alpha\beta}^0 = \partial_\alpha B_\beta^0 - \partial_\beta B_\alpha^0 + i[B_\alpha^0, B_\beta^0] \]

\[ D_\alpha = \partial_\alpha + i[B_\alpha^0, .] \]

\[ G_{\alpha\beta} = G_{\alpha\beta}^0 + F_{\alpha\beta} \]

\[ F_{\alpha\beta} = D_\alpha A_\beta - D_\beta A_\alpha + i[A_\alpha, A_\beta] \] (4)

where the gauge field fluctuations are denoted \( A \), with corresponding field strength \( F \), and we have defined a background covariant derivative \( D \). We choose the background gauge fixing condition \( D_\alpha A^\alpha = 0 \). When the background gauge field and field strength are diagonal and constant (solving the classical equations of motion), we find the following action for the fluctuations:

\[
S = -\frac{1}{4} \int d^{p+1}x Tr(G_{\alpha\beta}^0)^2 + -2A_\alpha D^2 A_\alpha - 4iA_\alpha [G_{\alpha\beta}^0, A_\beta] \\
-2\phi_m D^2 \phi_m + 2i(D_\alpha A_\beta - D_\beta A_\alpha) [A_\alpha, A_\beta] + 4i\phi_m D_\alpha [\phi_m, A_\alpha] \\
- [A_\alpha, A_\beta]^2 - 2[A_\alpha, \phi_m]^2 - [\phi_m, \phi_n]^2 (5)
\]

It will be convenient to expand the fluctuations in a Lie algebra basis for \( U(n) \), namely, \( (T_i)_{ab} = \delta_{ia}\delta_{ib} \) and \( (e_{ij})_{ab} = \delta_{ia}\delta_{jb} \) for \( i \neq j \):

\[
A_\alpha = a^i_\alpha T_i + b_{ij}^{\alpha} e_{ij} \\
\phi_m = c^i_m T_i + d_{ij}^{m} e_{ij} (6)
\]

From the reality properties of the gauge fields and the scalars we find that \( a \) and \( c \) are real, and that \( b_{ij}^{\alpha} = b_{ij}^{\alpha*} \) and \( d_{ij}^{m} = d_{ij}^{m*} \) are complex.

### 4 Compactification

Now we will focus on our main interest. Consider \( 2n \) spatial dimensions of the D-brane to be wrapped on a torus of dimension \( 2n \) with radii \( R_\hat{\alpha} = \frac{L_\hat{\alpha}}{2\pi} \), where \( (\hat{\alpha} = 1, \ldots, 2n) \). In what follows, we only consider non-trivial field strengths in these directions. Magnetic flux quantization\(^1\) and the fact that

\(^1\)We restrict to configurations where the branes are wrapped only once around, for example, the odd cycles. Generalizing this to wrapping more than once these cycles changes the quantization condition. See for instance [4] [5].
the background field strength is diagonal implies then that we can write our background in terms of the integers $n^i_{\hat{\alpha}\hat{\beta}}$:

$$G^0_{\hat{\alpha}\hat{\beta}} = 2\pi \frac{n^i_{\hat{\alpha}\hat{\beta}}}{L_{\hat{\alpha}} L_{\hat{\beta}}} T_i.$$  (7)

We can choose the background gauge field to be:

$$B^0_{\hat{\alpha}} = -\pi \frac{n^i_{\hat{\alpha}\hat{\beta}} x^\beta}{L_{\hat{\alpha}} L_{\hat{\beta}}} T_i.$$  (8)

Next, we substitute this form of the background into the action (5) and concentrate on the terms quadratic in the fluctuations. Since we will analyse the spectrum of small fluctuations around the background, we neglect the interactions between the fluctuations from now on. The quadratic action is:

$$S^{(2)} = \frac{-1}{2} \int d^{p+1}x (a^i_\alpha M_0 a^i_\alpha + c^i_m M_0 c^i_m + 2 \sum_{i<j} (b_{ij}^\alpha (M_{ij} \delta_{\alpha\beta} - 4\pi i J_{ij}^\alpha) b_{ij}^\beta + d_{ij}^\alpha M_{ij} d_{ij}^\beta))$$  (9)

where we have defined $J$, a measure for the difference in field strength in sector $i$ and $j$ (on brane $i$ and brane $j$), and the mass operators $M_0$ and $M_{ij}$:

$$J_{ij}^{\hat{\alpha} \hat{\beta}} = \frac{n^i_{\hat{\alpha}\hat{\beta}} - n^j_{\hat{\alpha}\hat{\beta}}}{L_{\hat{\alpha}} L_{\hat{\beta}}}$$

$$M_0 = -\partial_\alpha^2$$

$$M_{ij} = \left(\frac{\partial_\beta}{l}\right)^2 - \left(\frac{\partial_{n+1}}{l}\right)^2 - \ldots - \left(\frac{\partial_{\beta}}{l}\right)^2 - (\pi J_{ij}^\alpha x^\beta)^2.$$  (10)

5 The mass operators on $T^{2n}$

From now on we will concentrate on the dependence of the fluctuations on the coordinates of the torus, the internal coordinates $x^{\hat{\alpha}}$. Specifically, we consider the problem of diagonalizing the mass operators as operators on $T^{2n}$. As usual, this gives information on the spectrum of modes living in the non-compact space. The quadratic action (9) can be analysed in each sector $ij$ separately. In the following we will leave out the indices $i$ and $j$ to
simplify notation. We note that the analysis of the bosonic fluctuations in the following sections is valid for general $n$.

It will turn out that the mass operator (10) can be written in a simple form. Transforming coordinates with an element of $O(2n)$, we can bring the difference in field strengths in sector $ij$ in the standard form:

$$J_{\alpha\beta} = \begin{pmatrix} 0 & f_1 & 0 & 0 & 0 \\ -f_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & f_n & 0 \\ 0 & 0 & 0 & -f_n & 0 \end{pmatrix}$$  \hspace{1cm} (11)$$

where $f_1 \geq f_2 \geq \ldots \geq f_n \geq 0$.

Next, we introduce a complex structure on the torus as follows:

$$z = (z_1, z_2, \ldots, z_n) = \frac{1}{\sqrt{2}}(x_1 - ix_2, \ldots, x_{2n-1} - ix_{2n})$$

$$A = (A_{z_1}, A_{z_2}, \ldots, A_{z_n})$$

$$= \frac{1}{\sqrt{2}}(A_1 + iA_2, \ldots, A_{2n-1} + iA_{2n})$$

$$(\partial_{z_1}, \ldots, \partial_{z_n}) = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2}, \ldots, \partial_{x_{2n-1}} + i\partial_{x_{2n}})$$  \hspace{1cm} (12)$$

Further, we define the positive hermitian form $H$,

$$H(z, w) = 2(z_1 f_1 \bar{w}_1 + \ldots + z_n f_n \bar{w}_n)$$

$$= w^h h z$$

$$h = 2 \text{diag}(f_1, \ldots, f_n),$$  \hspace{1cm} (13)$$

and the creation and annihilation operators

$$a_k = \frac{1}{i} \left( \frac{\partial}{\partial z_k} + \pi f_k \bar{z}_k \right)$$

$$a_k^\dagger = \frac{1}{i} \left( \frac{\partial}{\partial z_k} - \pi f_k \bar{z}_k \right),$$  \hspace{1cm} (15)$$

where $(k = 1, \ldots, n)$. Thus we can write a nice expression for the relevant non-trivial mass operator $M_{ij}$ (11):

$$M = \{a_k, a_k^\dagger\}$$  \hspace{1cm} (16)$$

Before we can diagonalize the mass operator, it is crucial to discuss the boundary conditions the eigenfunctions have to satisfy. They encode the topological data of the background gauge field.
6 Boundary conditions

We recall that the background field strength and gauge fields were given by:

\[ G^0_{\alpha\beta} = \frac{2\pi}{L_\alpha L_\beta} n^i_{\alpha\beta} T_i, \]
\[ B^0_\alpha = -\frac{\pi}{L_\alpha L_\beta} n^i_{\alpha\beta} x^\beta T_i. \]  

The transition functions \( \Omega \) of the gauge bundle over the torus have to satisfy

\[ B^0_\alpha(x^\beta + L_\beta) = \Omega_{\beta} B^0_\alpha(x^\beta) \Omega^{-1}_{\beta} - i \Omega_{\beta} \partial_\alpha \Omega^{-1}_{\beta} \]
\[ = B^0_\alpha(x^\beta) - \frac{\pi}{L_\alpha} n^i_{\alpha\beta} T_i. \]  

We choose them to be

\[ \Omega_\alpha = \exp \left(-\pi i n^i_{\alpha\beta} x^\beta T_i / L_\beta \right). \]  

The boundary conditions following from the background gauge field induced transition functions read, using (3) and (6):

\[ a^i_\alpha(x^\beta + L_\beta) = a^i_\alpha(x^\beta) \]
\[ b^{ij}_\alpha(x^\beta + L_\beta) = \exp \left(-\pi i n^i_{\alpha\gamma} x^\gamma / L_\alpha \right) b^{ij}_\alpha(x^\beta) \]
\[ c^i_m(x^\beta + L_\beta) = c^i_m(x^\beta) \]
\[ d^{ij}_m(x^\beta + L_\beta) = \exp \left(-\pi i n^i_{\beta\gamma} x^\gamma / L_\beta \right) d^{ij}_m(x^\beta). \]  

7 Spectrum and eigen functions

After the preliminary work of writing the non-trivial mass operator \( M \) in a harmonic oscillator form in terms of complex coordinates (16), and discussing the boundary conditions the fluctuations have to satisfy (20), we determine the spectrum and the eigenfunctions. We only discuss in detail the non-trivial case of off-diagonal modes. Moreover, the difference between gauge field off-diagonal modes and scalar off-diagonal modes is a mere constant in the eigenvalues, so we can treat them in one go. We follow the analysis of [2] and [5]. (See [6] for an early mathematical treatment.) The ground state \( \chi_0 \)
of the mass operator in the off-diagonal sector we take to satisfy the usual conditions

\[ 0 = a_k \chi_0 = \frac{1}{i} \left( \frac{\partial}{\partial z_k} + \frac{\pi}{2} h_{kl} \bar{z}_l \right) \chi_0, \quad (21) \]

and it has to obey the boundary conditions for the off-diagonal sector

\[ \chi_0(x^\hat{\beta} + L_{\hat{\beta}}) = e^{-\pi i n^{ij}_{\hat{\beta}, \gamma} x^\gamma / L_\gamma} \chi_0(x^\hat{\beta}). \quad (22) \]

The differential equation is immediately solved in terms of the hermitian form \( H \) and a general holomorphic function \( f \),

\[ \chi_0(z) = e^{-\frac{\pi i}{2} H(z, z)} f(z), \quad (23) \]

but the treatment of the boundary conditions is more involved. The boundary condition for the fluctuation \( \chi_0 \) implies a non-trivial boundary condition for the holomorphic function \( f(z) \). It will be convenient to introduce some extra machinery to write these boundary conditions in terms of objects well-known in the mathematical literature \[7\] on theta functions \[4\]. In terms of the hermitian form \( H(z, w) \), we define an antisymmetric form \( E(z, w) \):

\[
H(z(x), w(y)) = x^\alpha (-J^2)^{\frac{1}{2}} y^\beta + ix^\alpha J_{\alpha \beta} y^\beta \\
E(z, w) = \text{Im} H(z, w) = \frac{1}{2i} (H(z, w) - H(w, z)) = x^\alpha J_{\alpha \beta} y^\beta. \quad (24)
\]

We introduce the notation

\[ q = (q_1, \ldots, q_n) \equiv \frac{1}{\sqrt{2}} (m_1 L_1 - i m_2 L_2, \ldots, m_{2n-1} L_{2n-1} - i m_{2n} L_{2n}), \quad (25) \]

to write the second degree bicharacter \( \alpha(q) \) in the simple form:

\[ \alpha(q) = e^{\pi i \sum_{\alpha, \beta} m_{\alpha} n_{\alpha \beta} m_{\beta}}. \quad (26) \]

These objects make it easy to write down the boundary condition for the fluctuations \( \chi \) for windings around the torus for any number of times in different directions:

\[ \chi_0(z + q) = \chi_0(z) e^{-\pi i E(q, z) \alpha(q)}. \quad (27) \]
The boundary conditions the holomorphic function $f$ satisfies can then finally be written in terms of the hermitian form $H$ and the second degree bicharacter $\alpha$:

$$f(z + q) = f(z)\alpha(q)e^{\pi H(z, q) + \pi H(q, q)}$$  \hspace{1cm} (28)

Now comes the pay-off for introducing the appropriate mathematical machinery. These holomorphic functions $f$ are theta functions on $T^{2n}$\cite{7}. They span a vectorspace of dimension $|Pf(n_{\alpha\beta})|$. This space of theta functions is the space of ground state fluctuations around the given gauge field background. They can be written down explicitly and we do so in the appendix.

The higher modes are given by acting with the creation operators on the ground state. They automatically satisfy the boundary conditions. It is clear from (16) then that the spectrum of off-diagonal scalar field fluctuations is given by the harmonic oscillator formula:

$$\lambda = 2\pi \left( \sum_{i=1}^{n} (2m_i + 1)f_i \right)$$  \hspace{1cm} (29)

and after a further trivial diagonalisation (compare (11)) for the off-diagonal gauge fields we get the shifted spectra:

$$\lambda_k^\pm = 2\pi \left( \sum_{i=1}^{n} (2m_i + 1)f_i \pm 2f_k \right)$$  \hspace{1cm} (30)

8 Summary

We summarize the spectrum and eigenfunctions for diagonal, off-diagonal and gauge field and scalar fluctuations in the following table. We use the notations $e^{(z_k)} = \frac{1}{\sqrt{2}}(0,0,\ldots,1,-i,\ldots,0,0)$ for the eigenvectors of $J$\cite{11}, $V$ for the volume of the torus, $i,j \in \{1,\ldots,N\}; ij \in \{1,\ldots,N(N-1)\}; k \in \{1,\ldots,n\}; p \in \mathbb{Z}^4; m \in \mathbb{N}^n; r_i \in \{0,\ldots,e_i-1\}$. All of these notations are straightforward except for the components $r_i$ and $e_i$ for which we refer to the appendix. Suffice it to remark that the degeneracy of the off-diagonal fluctuations is given by $Pf(n_{ij}) = \prod_i e_i$ in sector $ij$, and the space of theta functions is indexed by $r$. We have moreover:

$$\phi^{p,i} = \frac{1}{\sqrt{V}}e^{2\pi ip_i/T_i}$$

$$\phi^{m,r,ij} = (a_1^\dagger)^{m_1} \ldots (a_n^\dagger)^{m_n}/\sqrt{m_1! \ldots m_n!} \chi r \epsilon_{ij}$$  \hspace{1cm} (31)
Eigenfunctions and eigenvalues

|                      | Scalar Fluctuations | Eigenvalues                        |
|----------------------|---------------------|------------------------------------|
| diagonal             | \( c_i^j T_i \)     | \( \sum_{\alpha} \left( \frac{2\pi p_{\alpha}}{L_{\alpha}} \right)^2 \) |
| off-diagonal         | \( d_i^j e_{ij} \)  | \( 2\pi (\sum_{i=1}^{n} (2m_i + 1) f_i) \) |

Gauge field Fluctuations

|                      | Eigenvalues                        |
|----------------------|------------------------------------|
| diagonal             | \( a^j_{\alpha} T_i \)             | \( \sum_{\alpha} \left( \frac{2\pi p_{\alpha}}{L_{\alpha}} \right)^2 \) |
| off-diagonal         | \( b_{\alpha} e_{ij} \)            |                                                                 |

\( f_i \neq 0 \quad \phi_i^{m,r,ij} \quad 2\pi (\sum_{i=1}^{n} (2m_i + 1) f_i) \)

\( f_i \neq 0 \quad e^{i\phi_i^{m,r,ij}} \quad 2\pi (\sum_{i=1}^{n} (2m_i + 1) f_i - 2f_k) \)

\( f_i \neq 0 \quad e^{i\phi_i^{m,r,ij}} \quad 2\pi (\sum_{i=1}^{n} (2m_i + 1) f_i + 2f_k) \)

We only catalogued the case where all \( f^{ij} \) are different from zero and \( H \) is non-degenerate. The eigenvalues and eigenfunctions can also be classified easily in the other cases.

9 Stability and supersymmetry

In this section we discuss stability, supersymmetry and the occurrence of massless bosonic modes in our configurations. From the general classification we find that there are no tachyonic modes when \( f_2 + f_3 + \ldots + f_n \geq f_1 \) (for all \( i, j \)). If this condition is satisfied, stability of the gauge field configuration is insured, at quadratic level. On \( T^4 \) the condition for stability implies that the field strength is self-dual \((f_1 = f_2)\). (Recall that we still have \( f_1 \geq f_2 \geq \ldots \geq f_n \geq 0 \). The condition for stability is much more loose when there are more than two non-trivial field strength components. Massless modes for the gauge fields occur when the equality \( f_2 + f_3 + \ldots + f_n = f_1 \) is satisfied. For higher tori they appear only in a complex combination of the gauge field components \( A_{1,2} \). On \( T^4 \), stable configurations automatically have massless modes and they occur for the gauge field components \( A_{1,2} \) and \( A_{3,4} \). For higher dimensions a stable configuration does not necessarily have bosonic massless modes.
Turning back to our starting point, we can regard the theory we study (for \( n \leq 4 \)) as dimensionally reduced Super Yang-Mills theory. The supersymmetry variation of the adjoint fermions in our background with trivial scalars then reads:

\[ \delta \psi = F_{\hat{\alpha}\hat{\beta}} \gamma^{\hat{\alpha}\hat{\beta}} \epsilon. \]  

(32)

It is clear that if \( f_2 + f_3 + \ldots + f_n = f_1 \), the following conditions project onto the preserved supersymmetry parameter:

\[ -\gamma_{12} \epsilon = \gamma_{34} \epsilon = \ldots = \gamma_{2n-12n} \epsilon. \]

(33)

There are two other possibilities to preserve supersymmetry when \( n = 4 \), namely if we have \( f_1 + f_4 = f_2 + f_3 \) or if \( f_1 = f_2, f_3 = f_4 \). The following projection conditions yield the preserved supersymmetry:

\[ -\gamma_{12} \epsilon = \gamma_{34} \epsilon = \gamma_{56} \epsilon = -\gamma_{78} \epsilon, \]

(34)

and

\[ -\gamma_{12} \epsilon = \gamma_{34} \epsilon \]
\[ \gamma_{56} \epsilon = -\gamma_{78} \epsilon \]

(35)

respectively. These last configurations do not have any bosonic massless modes. In general, each projection condition halves the number of supersymmetries. Note that for the four torus the condition for stability, namely self-duality, coincides with the condition for preservation of supersymmetry. For higher tori this is not the case. The space of stable gauge field configurations is for higher tori much larger than the space of supersymmetric gauge field configurations.

10 Fermionic spectrum

Up till now we ignored the fermions in our reduction of Super Yang-Mills. In this section we determine the spectrum of the fermionic fluctuations in the theory reduced to \( p + 1 \) dimensions, compactified on \( T^{2n} \), and in the

\(^{2}\)We do not consider the case where one of the field strengths vanishes, although it is easily incorporated in our framework.
background gauge field configuration \([\mathbb{T}^4]\). It will turn out that the analysis is simple once the bosonic case has been treated in detail.

The ten dimensional Majorana-Weyl fermions are in the adjoint representation and can be decomposed as follows:

\[ \psi = \psi^iT_i + \psi^{kl}e_{kl}, \quad (36) \]

where \(\psi^{kl} = \psi^{lk}\) and \(k \neq l\). The relevant Dirac equation for the fermionic modes is easily derived from the fermionic part of the Yang-Mills action quadratic in the fluctuations. As for the bosonic case, we can analyse the spectrum of the Dirac operator on \(T^{2n}\) to find the mass spectrum in the non-compact directions. To that end, we analyse the equation \([\mathbb{T}^3]\):

\[ (i^n\gamma_{1\ldots 2n})\gamma_{\dot{\alpha}}(\frac{\partial_{\dot{\alpha}}}{i} - \pi J^{kl}_{\dot{\alpha}\beta}x^{\dot{\beta}})\psi^{kl} = \mu_0\psi^{kl} \quad (37) \]

We concentrate on the non-trivial off-diagonal components in sectors \(kl\). The standard trick to find the spectrum of the fermionic mass operator is to square it:

\[ \left( \gamma_{\dot{\alpha}}(\frac{\partial_{\dot{\alpha}}}{i} - \pi J^{kl}_{\dot{\alpha}\beta}x^{\dot{\beta}}) \right)^2 = \sum_k \{a_k, a_k^\dagger\} - 2\pi if_1\gamma_{12} - \cdots - 2\pi fn\gamma_{2n-1, 2n} \]

where we used \([\mathbb{T}^3]\). It is then easy to determine the spectrum of the fermionic mass operator by projecting onto eigenspinors of \(\gamma_{2k-1, 2k}\):

\[ \lambda = \sum_k 2\pi(2m_k + 1)f_k \pm 2\pi f_1 \pm \cdots \pm 2\pi fn. \quad (38) \]

Implicitly, we have made use of the fact that the off-diagonal fermions satisfy the same boundary conditions as the bosons\([\mathbb{T}^3]\). Drawing on the results in the previous sections, the eigenfunctions can then also easily be determined. Remark that the fermions all have a certain helicity associated to the magnetic fields in the directions 12, 34, \ldots and each component of the magnetic field is responsible for a Zeeman splitting of the energy levels.

\[ ^3 \text{The appearance of the chirality operator is to ensure that the non-compact and the compact part of the Dirac operator commute. It does not play a crucial role in our analysis since it doesn’t change the Dirac algebra \([\mathbb{T}^3]\).} \]

\[ ^4 \text{We do not want to break more supersymmetry by adding non-trivial monodromies on top of the ones induced by the background.} \]
11 Zeromodes and supersymmetry

11.1 Fermionic zeromodes

In this section we take a closer look at the bosonic and fermionic zeromodes that often play a crucial role in applications. We start by describing the fermionic zeromodes in greater detail. It is clear from the analysis in the previous section that only the following projected spinor has zeromodes:

$$\psi_{++...}^{kl} = (1 + i\gamma_1\gamma_2)(1 + i\gamma_3\gamma_4)\cdots(1 + i\gamma_{2n-1}\gamma_{2n})\frac{\psi^{kl}}{2^n}. \quad (39)$$

The signs of the projection operators reverse for sector $lk$, since the field strengths are opposite in that sector. Explicitly, the differential equations and boundary conditions in sectors $kl$ and $lk$ read:

$$a_i\psi_{++...}^{kl} = 0 \quad (40)$$
$$a_i^*\psi_{--...}^{lk} = 0 \quad (41)$$
$$\psi^{kl}(x^\beta + L_{\beta}) = e^{-\pi i n_{kl}^0 x^\gamma / L_{\gamma}}\psi^{kl}(x^\beta) \quad (42)$$

From these equations we determine the total number of massless fermionic modes, namely $\frac{16}{2^n}\sum_{k<l} Pf(n^{kl}_{\alpha\beta})$ complex zeromodes. In certain cases, there is a shorter route to get the number of fermionic zeromodes, via index theory. As a preliminary, we indicate the $2n$ dimensional chirality of the spinors. It is clear that:

$$\begin{align*}
(1 - i^n\gamma_1 \cdots \gamma_{2n})\psi_{++...}^{kl} &= 0 \\
(1 - (-i)^n\gamma_1 \cdots \gamma_{2n})\psi_{--...}^{lk} &= 0,
\end{align*} \quad (43, 44)$$

such that, for $n = 2k$ even, the zeromodes have the same chirality in the two sectors, while for $n = 2k + 1$ odd, they have opposite chirality. Note that this strokes with $(41)$, $(42)$ and the fact that the complex conjugate representation of the Weyl representation is the original one for $SO(4k)$, and of different chirality for $SO(4k + 2)$.

11.2 Index theory check on number of zeromodes

Using index theory, we can learn the difference in number of massless fermionic modes of positive and negative chirality. If the fermionic zeromodes all
have the same chirality, index theory predicts the total number of fermionic zeromodes. We will use the index theorem for the twisted spin complex on a flat manifold for the adjoint representation \[\text{[12]}\] :

\[
\text{index}(\Delta_\pm \times (adj), D) = \int_M \text{ch}(F_{adj}) = \nu_+ - \nu_-
\]

(45)

where \(\text{ch}(F_{adj}) = \sum_j \text{Tr} F_{adj}^j \) is the Chern class evaluated in the adjoint representation and \(\nu_+\) is the number of positive, negative chirality zeromodes. In the background \([\text{[17]}]\) the integral is easily evaluated on \(T^{2n}\):

\[
\nu_+ - \nu_- = \sum_{kl} Pf(n_{\hat{a}\hat{\beta}}^{kl}).
\]

(46)

For \(d = 4k + 2\) the sum over all sectors is zero, the contributions from sector \((kl)\) (strings going one way) cancelling the contribution from sector \((lk)\) (strings going the other way). Indeed, from the analysis in the previous section we know that the number of zeromodes of positive chirality equals the number of zeromodes of negative chirality in this case. For \(d = 4k\) we find, in our conventions, only zeromodes of positive chirality, and index theory counts \(\sum_{kl} Pf(n_{\hat{a}\hat{\beta}}^{kl})\) complex fermionic zeromodes in \(d = 4k\). Taking into account the multiplicity of the zeromodes originating in ten dimensions, we find \(\frac{16}{2} \sum_{k<l} Pf(n_{\hat{a}\hat{\beta}}^{kl})\) non-constant complex spinor zeromode components, as before. This straightforwardly extends the well-known results in four dimensions \([\text{[13]}]\).

### 11.3 Supersymmetry and massless modes

The number of massless fermionic and massless bosonic modes differs in supersymmetric configurations, and at first sight it is difficult to see how they form a representation of supersymmetry when it is partially unbroken. Nevertheless, they do. We discuss this slightly puzzling feature in this subsection. The unbroken supersymmetry transformations rules are given by the dimensionally reduced formulae of \(\mathcal{N} = 1\) SYM in ten dimensions. In ten dimensions the formulae read:

\[
\delta A_\mu^a = \frac{i}{2} \bar{\epsilon} \gamma_\mu \psi^a
\]

(47)

\[
\delta \psi^a = -\frac{1}{4} F^a_{\mu\nu} \gamma^{\mu\nu} \epsilon
\]

(48)
Consider first the case where the unbroken supersymmetry is given by

\[- \gamma_{12} \epsilon = \gamma_{34} \epsilon = \ldots = \gamma_{2n-12n} \epsilon. \quad (49)\]

Starting out with a fermionic zeromode satisfying

\[\psi^{kl}_{++...+} = (1 + i \gamma_1 \gamma_2) (1 + i \gamma_3 \gamma_4) \ldots (1 + i \gamma_{2n-1} \gamma_{2n}) \frac{\psi^{kl}}{2^n}, \quad (50)\]

we easily see from (47), (49), (50) and \(\mu = m\) that it will never transform into a massless scalar. That is consistent with the spectrum we found earlier. When \(n \geq 3\), we find only non-trivial gauge field components in the 1 and 2 direction. For gauge field components in the other directions, the projection condition on the parameter \(\epsilon\) and the zeromode \(\psi\) make sure that the variation vanishes. This is again consistent with what we found earlier. It can easily be checked that on \(T^4\) a similar analysis yields complex bosonic zeromodes in directions 1, 2 and 3, 4. Moreover for the cases where the supersymmetry parameter satisfies \(\gamma_{12} \epsilon = -\gamma_{34} \epsilon\) and \(-\gamma_{56} \epsilon = \gamma_{78} \epsilon\) we find that no bosonic zeromodes are generated, as expected. Finally, it can be checked using the formulae for the bosonic zeromodes that the transformation rule (48) is always trivial (at the order we are working). Thus, the analysis of fermionic and bosonic zeromodes is perfectly consistent with supersymmetry, as it should be. Of course, one could perform a similar consistency check for higher modes in the spectrum.

\section{Applications and conclusion}

The theory we have been discussing is the low-energy theory of D-branes in a trivial background and with only small (‘non-relativistic’) gauge fields excited. Indeed, string theory should be seen as the high-energy completion of the theories we studied. Nevertheless, the approximation to string theory we discussed was often used to understand results in string theory. We will briefly review some of the applications of the results that we obtained that have already been made in the literature and that can be coherently presented and extended in our framework.

In [5] the fluctuation spectrum on \(T^4\) in the Yang-Mills approximation was compared to the string theory fluctuation spectrum and the role of the (non-abelian) Dirac-Born-Infeld action in resolving the discrepancy was clarified. The precise form of the non-abelian Dirac-Born-Infeld action remained...
unclear. Our analysis could be useful for further studying fluctuation spectra from the different points of view along the lines of [4], for higher branes on higher tori. Note also that the explicit form for a tachyonic fluctuation was used in [4] to discuss tachyon condensation intuitively. That discussion is now easily extended to higher branes.

The condition for the background gauge fields to preserve supersymmetry is familiar in D-brane physics, especially in its T-dual form. To repeat this well-known point, it is sufficient to give an archetypical example. Consider the following configuration: a pair of D8-branes compactified on $T^8$ with a constant field strength on the first D8-brane $F_{12} \geq F_{34} \geq F_{56} \geq F_{78} \geq 0$. When $F_{12} = F_{34} + F_{56} + F_{78}$, supersymmetry is conserved. T-dualize this configuration over directions 2, 4, 6, 8 to obtain a pair of D4-branes at angles. The angles are given by the following formulae:

\[ F_{12} = \tan \phi_1, \quad F_{34} = \tan \phi_2, \quad F_{56} = \tan \phi_3, \quad F_{78} = \tan \phi_4. \]

Working at small angle, or taking into account the modifications the Born-Infeld action induces in the Yang-Mills theory in the spirit of [4], we find that the condition coincides with the well-known one for rotated branes [10] [11]. Note though that for the case $n = 4$ we saw a physical distinction between the case where $f_1 = f_2 + f_3 + f_4$ and $f_1 + f_4 = f_2 + f_3$. The difference between the two cases is probably related to the mechanism of the creation of a string in the D0-D8 system (or D4-D4 system), but we do not pursue this here.

As noted before, on higher tori, the space of stable gauge field configurations is much larger than the space of supersymmetric configurations. The same applies therefore to all kinds of D-brane constructions in gauge theories. The results on $T^4$ for the Yang-Mills theory were extensively discussed in for instance [3] [4]. Stability of some special configurations on $T^6$ and $T^8$ was used in [3] to adhere zero-branes to six- and eight-branes in Matrix theory. It is now obvious that these are points in a larger parameter space of stable configurations. Note though that we only proved stability up to quadratic order in the fluctuations [9]. It may be useful to recall that for instance the black hole with only D0- and D6-brane charge, which has a corresponding quadratically stable representation in the gauge theory on the D6-brane, is in fact metastable [9] [10].

Another motivation for our work can be found in the following problem. Consider the only regular four dimensional supersymmetric black hole that is solely made out of D-branes [14]. Specifically, dualize to the configuration

\[ F_{12} = \tan \phi_1, \quad F_{34} = \tan \phi_2, \quad F_{56} = \tan \phi_3, \quad F_{78} = \tan \phi_4. \]

We ignore some constant factors.
where the compact part carries D6-D2-D2-D2 brane charges. The entropy for this black hole, calculated in supergravity, was microscopically accounted for in [7] [8] up to a constant factor which it would be interesting to determine. We can now easily find supersymmetric configurations in the gauge theory living on the D6-branes that represent such a black hole. Next, we can use the results derived in this article to calculate the dimension of the moduli space of these configurations by counting massless states. Although in principle this program as suggested in [7] looks sound, we have as yet not been able to make it work. Finally, we indicate that the analysis of strings stretching between branes in Matrix theory is analogous (see for instance [17]) and we believe that our systematic treatment could be of practical use in that context too.

To sum up, we have determined the spectrum and explicit eigenfunctions of the fluctuations around constant and diagonal field strengths in $U(N)$ Yang-Mills on an even dimensional torus, extending earlier work [2] [5] on $T^4$. We discussed supersymmetry and stability of these configurations, and the counting of zeromodes. The analysis yields a systematic framework for applications in string theory.

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**Appendix: Explicit eigenfunctions**

To find the explicit form of the theta functions we need some more machinery [7]. We go to a canonical (Frobenius) basis of the lattice using an $SL(2n, \mathbb{Z})$ transformation:

\[
 n = \begin{pmatrix}
 0 & 0 & e_1 & 0 \\
 0 & 0 & \ldots & \ldots \\
 0 & 0 & 0 & e_n \\
 -e_1 & 0 & 0 & 0 \\
 \ldots & \ldots & 0 & 0 \\
 0 & -e_n & 0 & 0 \\
\end{pmatrix}
\]  
(51)
where the \( e_i \) are positive integers, each \( e_i \) dividing the next \( e_{i+1} \). We choose a \( \mathbf{C} \)-basis as follows:

\[
z = \sum_{i=1}^n \frac{\xi_{(n+1)}}{e_1} + \ldots + \frac{\xi_{(2n)}}{e_n} = U \tilde{z}
\]  

(52)

where \( \xi^* \) denotes the canonical basis of the lattice. By \( X \) we denote the real vector space generated by \( \xi_{n+1}, \ldots, \xi_{2n} \). Since \( \text{Im} H = E \) is zero on \( X \times X \), we find that \( \tilde{h} = U^\dagger h U \) is real and symmetric. We define then the symmetric \( C \)-bilinear form \( S \) uniquely associated with \( H \)

\[
S(z, w) = -\tilde{w}^T \tilde{h} \tilde{z}
\]

(53)

and the quasi-hermitian form \( Q \)

\[
Q(z, w) = H(z, w) + S(z, w).
\]

(54)

Moreover, we need the period matrix \( \tau \) defined in terms of the quasi-hermitian form \( Q \):

\[
\tau_{ij} \equiv \frac{i}{2} Q(\xi^{(i)}, \xi^{(j)}).
\]

(55)

Further definitions are required, namely the bicharacter \( e^{\pi i B(q, q)} \) with respect to the canonical basis \( \xi \):

\[
q = \tilde{q}_\alpha \xi^*
\]

\[
B(q, q) = \sum_{i=1}^n e_i \tilde{q}_i \tilde{q}_{i+n}
\]

(56)

and \( \beta(q) \), encoding the characters \( m \) and \( l \) of \( \alpha \):

\[
\alpha(q) = e^{\pi i B(q, q)} \beta(q)
\]

\[
\beta(q) = e^{2\pi i \sum_{i=1}^n (m_k \tilde{q}_{i+n} - l_k \tilde{q}_i)}
\]

(57)

The thetafunctions are then explicitly given by:

\[
\theta(z) = \sum_{0 \leq r_i < |e_i|} d_r \theta_r(z)
\]

\[
\theta_r(z) = \sum_{p \in \mathbb{Z}^n} e^{\pi i (p+e^{-1}(m+r), \tau(p+e^{-1}(m+r))+2\pi i (p+e^{-1}(m+r)).(\tilde{z}+l)}
\]

(58)

where \( e_1 | e_2 | e_3 \ldots | e_n \) and \( e \) is diagonal.
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