Colouring of plane graphs with unique maximal colours on faces

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Abstract

The Four Colour Theorem asserts that the vertices of every plane graph can be properly coloured with four colors. Fabrici and Göring conjectured the following stronger statement to also hold: the vertices of every plane graph can be properly coloured with the numbers 1, ..., 4 in such a way that every face contains a unique vertex coloured with the maximal color appearing on that face. They proved that every plane graph has such a colouring with the numbers 1, ..., 6. We prove that every plane graph has such a colouring with the numbers 1, ..., 5 and we also prove the list variant of the statement for lists of sizes seven.

1 Introduction

A lot of research in graph theory was sparked by the problem of four colors posed by Francis Guthrie in 1852. It took more than 125 years until the problem was resolved by Appel and Haken [1] and the conjectured statement became known as the Four Colour Theorem. A refined proof of the Four Colour Theorem was given by Robertson, Seymour, Sanders and Thomas [4]. Our work is motivated by a conjecture of Fabrici and Göring [2] which, if true, would strengthen the Four Colour Theorem.

Conjecture 1.1. (Fabrici and Göring [2, Conjecture 9]) Every plane graph has a proper colouring using the number 1, 2, 3 and 4 such that every face contains a unique vertex coloured with the maximal color appearing on that face.

We will refer to a colouring of this kind as to a capital colouring, i.e., a capital colouring is proper vertex colouring using integers such that every face contains a unique vertex coloured with the maximal color appearing on that face. The name comes from the fact that every face (region) has a unique vertex (capital) with the maximal color. The capital chromatic number $\chi_C(G)$ of a graph $G$ is the smallest $k$ such that there exists a capital colouring using 1, ..., $k$.

Note that Conjecture 1.1 holds for triangulations since any proper colouring of a triangulation has the required properties. Fabrici and Göring [2] proved that every plane graph has a capital colouring using colors 1, ..., 6. We prove a stronger result that every plane graph has a capital colouring using colors 1, ..., 5.

Theorem 1.2. If $G$ is a plane graph then $\chi_C(G) \leq 5$.

In addition, we consider the list version of capital colourings and we show that if each vertex of a plane graph is assigned a list of seven integers, then there exists a capital colouring assigning each vertex a color from its list. Throughout this paper, a plane graph is a loopless graph embedded in the plane that may contain parallel edges which may (but need not) form bigons.

2 Unique maximum 5-colouring

We start by recalling an auxiliary Lemma 2.1 from [2].
Lemma 2.1. (Fabrici and Göring [2, Lemma 6]) Let $G$ be a plane graph with no parallel edges, let $xy \in E(G)$ be an edge of $G$ incident with the outer face, and let $c \in \{\text{black, blue}\}$. There is a 3-vertex-colouring of $G$ with colors red, blue and black such that

1. vertex $x$ has colour $c$,  
2. vertex $y$ is black,  
3. each edge is incident with at most one blue vertex,  
4. no vertex incident with the outer fact is red,  
5. each inner face is incident with at most one red vertex, and  
6. each inner face that is not incident with a red vertex is incident with exactly one blue vertex.

The proof of Theorem 1.2 uses a stronger version of Lemma 2.1. The version differs by adding the condition that all triangles contain at least one blue or red vertex.

Lemma 2.2. Let $G$ be a plane graph without bigons, let $xy \in E(G)$ be an edge of $G$ incident with the outer face, and let $c \in \{\text{black, blue}\}$. There is a 3-vertex-colouring of $G$ with colors red, blue and black such that

1. vertex $x$ has colour $c$,  
2. vertex $y$ is black,  
3. each edge is incident with at most one blue vertex,  
4. no vertex incident with the outer fact is red,  
5. each inner face is incident with at most one red vertex,  
6. each inner face that is not incident with a red vertex is incident with exactly one blue vertex, and  
7. each triangle contains at least one vertex that is not black.

Proof. We proceed by induction on the number of vertices. Let $xy \in E(G)$ be an edge of $G$ incident with the outer face and $c \in \{\text{black, blue}\}$. If $G$ has no separating cycle of length two or three, then Lemma 2.1 yields the statement. Assume there are separating cycles of length two and consider an inner most separating cycle of length two $C$. Use $u_1$ and $u_2$ for the vertices of this cycle. Let $G_1$ being the graph contained outside $C$ and $G_2$ be the graph contained inside $C$.

![Figure 1: Cycle C.](image)

$G_2$ is not empty as $C$ is a separating cycle. Next apply the induction assumption on the graph $G_1 + u_1u_2$ (i.e. there is only one of the edges of $C$ present). Then use $xy$ and $c$ to get a 3-colouring of $G_1 + u_1u_2$ with the properties as desired in the statement. As $u_1u_2$ is an edge, there are four possibilities for the colours of the vertices $u_1$ and $u_2$: one vertex is red and the other blue, one vertex is red and the other black, one vertex is blue and the other black, or both vertices are black. We now consider these four cases.
One vertex red and one vertex blue. Assume that $u_1$ is red. Consider the induced graph on $V(G_2) \cup \{u_2\}$. Then if $u_2$ is joined to a vertex in $G_2$ take an edge $u_2v$ on the outer face. If $u_2$ is not joined to a vertex in $G_2$ take a vertex $v$ on the outer face of $G_2$, and add edge $u_2v$. Then apply the inductive assumption to the constructed graph with $u_2$ as $x$ with colour blue and $v$ as $y$ coloured black. The constructed 3-colouring matches up with the one of $G_1 + u_1u_2$ and gives the desired 3-colouring.

One vertex red and one vertex black. Suppose that $u_1$ is red. Consider the induced graph on $V(G_2) \cup \{u_2\}$. Then if $u_2$ is joined to a vertex in $G_2$ take an edge $u_2v$ on the outer face. If $u_2$ is not joined to a vertex in $G_2$ take a vertex $v$ on the outer face of $G_2$, and add an edge $u_2v$. Then apply the inductive assumption to the constructed graph with $u_2$ as $x$ with colour black and $v$ as $y$ coloured black. The constructed 3-colouring matches up with the one on $G_1 + u_1u_2$ and gives the desired 3-colouring.

One vertex blue and one vertex black. Say that $u_1$ is blue. Since $C$ does not bound a face, the graph $G_2 + u_1u_2$ contains another vertex on its outer face. Let $v$ be this vertex. Apply the inductive assumption to the graph $(G_2 + u_1u_2)\backslash\{v\}$ with $u_1$ as $x$ coloured blue and $u_2$ as $y$ coloured black. Then colour $v$ red to get the required 3-colouring on $G_2 \cup C$. The constructed 3-colouring matches up with the one on $G_1 + u_1u_2$ and gives the desired 3-colouring.

Both vertices are black. Look at the face bounded by $u_1u_2$ that is removed in $G_2 + u_1u_2$. This face is not a bigon so we know there is another vertex $v$ on this face which is not $u_1$ or $u_2$. Consider the graph $(G_2 + u_1u_2)\backslash\{v\}$. Then apply the inductive assumption with $u_1$ as $x$ with colour black and $u_2$ as $y$ coloured black. Then colour $v$ red to get the required 3-colouring on $G_2 \cup C$. The constructed 3-colouring matches up with the one on $G_1 + u_1u_2$ and gives the desired 3-colouring.

Assume there are no separating cycles of length two but $G$ has separating triangles. Let $T$ be an innermost triangle. Use $t_1$, $t_2$ and $t_3$ for the vertices of this triangle. Let $G_1$ be the graph contained outside $T$ and $G_2$ be the graph contained inside $T$.

![Figure 3: Triangle T.](image)

$G_2$ is not empty as $T$ is a separating triangle. Apply the induction assumption on the graph $G_1 \cup T$ with $xy$ and $c$ to get a 3-colouring of $G_1 \cup T$ with the properties as desired in the statement. As $T$ bounds a face in $G_1 \cup T$, some of its vertices must be coloured with blue or red and there are three possibilities: one vertex is red, blue and black, one vertex is red and two are black, or one vertex is blue and two are black. We now consider these three cases.
Theorem 3.2. Apply the inductive assumption on the graph induced by $V(G_2) \cup \{t_2, t_3\}$ with $t_2$ as $x$ to be coloured blue, which is the colour $c$, and $t_3$ as $y$ to be black. The 3-colourings on $G_1 \cup T$ and the graph induced by $V(G_2) \cup \{t_2, t_3\}$ match up and give us the required 3-colouring of $G$ having the desired properties.

Two black vertices and one red. Assume $t_1$ is red, and the other two vertices are black. The inductive assumption is applied on the graph induced by $V(G_2) \cup \{t_2, t_3\}$ again with $t_2t_3$ being $xy$ to be both coloured black. The 3-colourings on $G_1 \cup T$ and the graph induced by $V(G_2) \cup \{t_2, t_3\}$ match up on $t_2t_3$ and give us a 3-colouring as described in the statement of Lemma 2.2.

Proof of Theorem 1.2. If $G$ has bigons replaces these with a single edge, then a capital colouring of the altered graph is a capital colouring of the original graph. So without loss of generality assume $G$ has no bigons. Choose a vertex $v \in V(G)$ on the outer face and then apply Lemma 2.2 to the graph $G \setminus v$. Then let $v$ be coloured red to get a 3-colouring. Note that each face has either exactly one red vertex, or no red vertex and exactly one blue vertex. Moreover, every triangle contains at least one red or blue vertex. Let $H$ be the induced subgraph of $G$ formed by the black vertices. As $H$ is triangle-free, by Grötzsch theorem, there exists a proper 3-colouring of it using $\{1, 2, 3\}$. Then assign blue vertices the colour 4 and red vertices the colour 5. The constructed 5-colouring is proper and has a unique maximal colour on each face from the construction.

3 List Colouring

In this section we will present an upper bound for the capital list colouring of a plane graph $G$.

Definition 3.1. A list assignment is a function $L : V(G) \to \mathcal{P}(\mathbb{N})$. A graph $G$ has a capital $L$-colouring if it has a capital colouring $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$. We say that a graph $G$ is capital $k$-choosable if there is a capital $L$-colouring for all list assignments with $|L(v)| \geq k$ for all $v \in V(G)$. The minimum $k$ such that $G$ is capital $k$-choosable is denoted by $\chi^L_c(G)$.

We will prove an upper bound of seven on $\chi^L_c(G)$ for any plane graph $G$.

Theorem 3.2. If $G$ is a plane graph then $\chi^L_c(G) \leq 7$.

The proof of Theorem 3.2 shall use a discharging argument. We assume Theorem 3.2 is false and take $G$ to be a minimum counter-example with a list assignment $L$, such that $G$ has the minimum number of
vertices, the minimum number of 2-faces and the maximal number of edges (in this order).

Before examining properties of a minimum counter-example, we introduce some notation. A vertex of degree \( d \) is called a \( d \)-vertex and a \( \geq d \)-vertex is a vertex of degree at least \( d \). A \( d \)-face is a face incident with exactly \( d \) edges and \( \geq d \)-face is a face incident with at least \( d \) edges. If \( f \) is a face, then we write \( c(f) \) to be the maximal colour of \( f \) under a colouring \( c \).

### 3.1 Reducible Configurations

In this subsection, we will explore properties of a minimum counter-example \( G \) with list colouring \( L \).

**Lemma 3.3.** Let \( G \) be a minimum counter-example.

1. \( G \) is 2-connected
2. For all vertices \( v \in V(G) \), if \( v \) is adjacent to \( k \) vertices and \( l \) faces of size at least four then \( k + l \geq 7 \).
3. Each vertex of \( G \) is a \( \geq 4 \)-vertex
4. Each face of \( G \) is a \( \geq 3 \)-face
5. No two 3-faces share a 4-vertex

**Proof.**

1. If \( G \) is disconnected, then it has different components \( G_1 \) and \( G_2 \). Pick a vertex on the face shared by \( G_1 \) and \( G_2 \) for both \( G_1 \) and \( G_2 \) and then add an edge between the two. By minimality of \( G \) we can find an \( L \)-colouring. This colouring is also a colouring of the original graph.

2. Suppose \( G \) has a vertex \( v \in V(G) \) such that \( k + l \leq 6 \). Let \( v_1, \ldots, v_k \) be the neighbours of \( v \) in the cyclic order. Remove \( v \) and add edges \( v_1v_2, v_2v_3, \ldots v_{k-1}v_k \) and \( v_kv_1 \). By minimality of \( G \) we can colour the remaining graph from the lists \( L \). Then assign a colour to \( v \) from its list that is not assigned to its neighbours and that is not the maximal colour on any of the incident faces. Since there are at most \( k + l \) such colours, there is a colour in \( L(v) \) that can be assigned to \( v \). On the faces containing \( v \), the maximal colour of the face is either on the vertex that had the maximal colour in the modified graph or on \( v \). Therefore \( G \) has a capital \( L \)-colouring.

3. This follows from Part 2 as \( l \leq k \) for any vertex.

4. If \( f \) is a 2-face remove one of the edges. By minimality of \( G \), we can find a capital \( L \)-colouring. This colouring is still a capital colouring of the original graph.

5. Follows directly from Part 2.

We now look at a 4-face sharing an edge with a 3-face.

**Proposition 3.4.** In a minimum counter-example \( G \), no 3-face and 4-face can share an edge joining two 4-vertices.

**Proof.** Assume Proposition 3.4 is false and there exists a minimum counter-example \( G \) with such a configuration. Let the 3-face be \( v_1v_2v_3 \) and the 4-face \( v_1v_2v_5v_4 \) with \( v_1 \) and \( v_2 \) being 4-vertices. Let \( v_7 \) be the remaining vertex connected to \( v_1 \), \( f_1 \) the face bounded partially by \( v_3v_1v_7 \) and \( f_2 \) the face partially bounded by \( v_7v_1v_4 \). Let \( v_6 \) be the remaining vertex connected to \( v_2 \), \( f_3 \) the face partially bounded by \( v_5v_2v_6 \) and \( f_4 \) the face partially bounded by \( v_6v_2v_3 \). See Figure 3.
Let $H$ be the graph $G \setminus \{v_1, v_2\}$ but with the additional edges $v_4v_7$, $v_7v_3$, $v_3v_6$ and $v_6v_5$. Let $f_1'$ be the new face partially bounded by $v_3v_7$, $f_2'$ by $v_7v_4$, $f_3'$ by $v_5v_6$ and $f_4'$ by $v_6v_3$. Then by the minimality of $G$, $H$ has an $L$-colouring $c$. As $v_4$ and $v_5$ are adjacent by symmetry we can assume $c(v_4) > c(v_5)$. Colour $v_2$ from $L(v_2)$ by a colour different from $c(v_3), c(v_4), c(v_5), c(v_6), c(f_1')$ and $c(f_2')$; call this colour $c(v_2)$. Then colour $v_1$ from $L(v_1)$ by a colour different from $c(v_2), c(v_3), c(v_4), c(v_5), c(f_1')$ and $c(f_2')$. The resulting colouring is a capital $L$-colouring. Indeed, the maximal colour on the face $v_1v_2v_3v_4$ is the color of either $v_1, v_2$ or $v_5$. Therefore a capital $L$-colouring of $G$ exists contradicting that $G$ is a minimal counter-example.

The proofs of Proposition 3.5 and 3.6 use the same idea as in the proof of Proposition 3.4.

**Proposition 3.5.** In a minimum counter-example $G$, no 3-face and 4-face can share an edge joining a 4-vertex and a 5-vertex incident to three 3-faces.

**Proof.** Let $v_1$ be a 5-vertex and let its neighbours be $v_2, v_3, v_4, v_5$ and $v_6$ in the cyclic order with $v_1v_2v_3$ being a 3-face. Let $v_1v_2v_7v_6$ be a 4-face and $v_2$ a 4-vertex with $v_8$ being the neighbour of $v_2$ different from $v_1, v_3$ and $v_7$. Let $f_1$ be the face partially bounded by $v_7v_2v_8$ and $f_2$ by $v_8v_2v_3$. Let $f_3$ be the second $\geq 4$-face adjacent to $v_1$. Let vertices $v_k$ and $v_{k+1}$ be neighbours of $v_1$ that are on face $f_3$. One of these configurations is depicted in Figure 4.

![Figure 5: Configuration of Proposition 3.4 and reduction.](image)

Let $H$ be the graph $G \setminus \{v_1, v_2\}$ with edges $v_6v_8, v_3v_8$ and $v_kv_{k+1}$ added. Let $f_1'$ be the new face in $H$ partially bounded by $v_7v_8$, $f_2'$ by $v_3v_8$ and $f_3'$ by $v_6v_{k+1}$. By the minimality of $G$ there is a $L$-colouring $c$ of $H$. As $v_6$ and $v_7$ are adjacent, they have different colours examine two cases:

**Case** $c(v_6) > c(v_7)$. Colour $v_2$ by a colour from $L(v_2)$ different from $c(v_3), c(v_6), c(v_7)$ and $c(f_2')$ call it $c(v_2)$. Colour $v_1$ a colour from $L(v_1)$ different from $c(v_2), c(v_3), c(v_4)$ and $c(f_1')$. This is a capital $L$-colouring as on the 4-face $v_1v_2v_7v_8$ the maximal colour is that of either $v_1, v_2$ or $v_6$.

**Case** $c(v_7) > c(v_6)$. Colour $v_1$ by a colour from $L(v_1)$ different from $c(v_3), c(v_4)$, $c(v_5)$, $c(v_6)$ and $c(f_3')$ call it $c(v_1)$. Colour $v_2$ a colour from $L(v_2)$ different from $c(v_1), c(v_3), c(v_7)$ and $c(f_1)'$. This is a capital $L$-colouring as on the 4-face $v_1v_2v_7v_8$ the maximal colour is that of either $v_1, v_2$ or $v_7$.

Therefore $G$ has an $L$-colouring.

**Proposition 3.6.** In a minimum counter example $G$, no 3-face and 4-face can share an edge joining a 4-vertex and a 6-vertex incident with five 3-faces.
Proof. Let $v_1$ be a 6-vertex and let its neighbours be $v_2$, $v_3$, $v_4$, $v_5$, $v_6$ and $v_7$ in the cyclic order. Let $v_1v_2v_3$, $v_1v_3v_4$, $v_1v_4v_5$, $v_1v_5v_6$, $v_1v_6v_7$ be 3-faces. Let $v_1v_2v_8v_7$ be a 4-face and $v_2$ is a 4-vertex. The remaining neighbour of $v_2$ is $v_9$. Let $f_1$ be the face partially bounded by $v_8v_2v_9$ and $f_2$ by $v_1v_2v_9$.

![Figure 7: Configuration of Proposition 3.6 and reduction.](image)

Let $H$ be the induced graph on $G\setminus\{v_1, v_2\}$ but with the added edges $v_8v_9$ and $v_7v_8$. Then by the minimality of $G$, $H$ has an $L$-colouring, $c$. Then as $v_7$ and $v_8$ are adjacent, they have different colours. We consider two cases:

Case $c(v_8) > c(v_7)$. Colour $v_1$ by a colour from $L(v_1)$ different from $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$, $c(v_7)$ and $c(v_8)$, call it $c(v_1)$. Colour $v_2$ a colour from $L(v_2)$ different from $c(v_1)$, $c(v_3)$, $c(v_7)$, $c(v_8)$, $c(f'_1)$ and $c(f'_2)$). This is a capital $L$-colouring of $G$ as in the 4-face $v_1v_2v_8v_7$ the maximal colour is that of either $v_1$, $v_2$ or $v_8$.

Case $c(v_7) > c(v_8)$. Colour $v_2$ by a colour from $L(v_2)$ different from $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$, $c(v_7)$ and $c(f'_2)$ and call it $c(v_2)$. Then colour $v_1$ a colour from $L(v_1)$ different from $c(v_2)$, $c(v_3)$, $c(v_4)$, $c(v_5)$, $c(v_6)$ and $c(v_7)$. This is a capital $L$-colouring of $G$ as in the 4-face $v_1v_2v_8v_7$ the maximal colour is that of either $v_1$, $v_2$ or $v_7$.

So there is a $L$-colouring of $G$. \qed

### 3.2 Discharging

The existence of a minimal counter-example will be disproved by the discharging method. The initial charge of each $d(f)$-face $f \in F(G)$ is $d(f) - 4$, and the initial charge of each $d(v)$-vertex $v \in V(G)$ is $d(v) - 4$. By Euler’s formula the total amount of charge is

$$
\sum_{v \in V(G)} d(v) - 4 + \sum_{f \in F(G)} d(f) - 4 = 2|E(G)| - 4|V(G)| + 2|E(G)| - 4|F(G)| = -8.
$$

The initial charge is redistributed by the following rules.

**Rule V5** A 5-vertex incident with at most two 3-faces shall give each 3-face 1/2 units of charge. A 5-vertex incident with three 3-faces shall give each 3-face 1/3 units of charge.

**Rule V6** A 6-vertex incident with at most four 3-faces shall give each 3-face 1/2 units of charge. A 6-vertex incident with five 3-faces shall give each 3-face 1/3 units of charge.

**Rule V7** A 7-vertex incident with at most six 3-faces shall give each 3-face 1/2 units of charge. A 7-vertex incident with seven 3-faces shall give each 3-face 1/3 units of charge.

**Rule V8** A $\geq 8$-vertex will give every 3-face 1/2 units of charge.

**Rule E1** A $\geq 5$-face will give 1/2 units of charge to every 3-face adjacent via an edge joining two 4-vertices.

**Rule E2** A $\geq 5$-face will give 1/6 units of charge to every 3-face adjacent via an edge joining a 4-vertex and a $\geq 5$-vertex.

Rules V5-8 do not allow a vertex to give out more charge than it started with, therefore any vertex after the application of the rules has non-negative charge. Also note that 4-faces are unaffected by the rules so they keep zero charge.
Lemma 3.7. Every \( \geq 5 \)-face after the rules have been applied has non-negative charge.

Proof. Consider a \( \geq 5 \)-face, let \( v_1, v_2 \) and \( v_3 \) be 3 consecutive vertices on its boundary. If Rule E1 applies to the edge \( v_1v_2 \), then the other face partially bounded by \( v_2v_3 \) is a \( \geq 4 \)-face by Lemma 3.3 Part 5. If the edge \( v_1v_2 \) uses Rule E2 with \( v_2 \) being the 4-vertex, then the other face containing \( v_2v_3 \) is a \( \geq 4 \)-face. If the edge \( v_1v_2 \) uses Rule E2 with \( v_1 \) being the 4-vertex, then Rule E2 can apply with respect to \( v_2v_3 \). Therefore the \( \geq 5 \)-face sends out through any two consecutive edges at most 1/2 units of charge. Therefore, \( \geq 6 \)-faces have non-negative charge after discharging.

Let \( f \) be a \( 5 \)-face \( v_1v_2v_3v_4v_5 \). Note that the initial charge of \( f \) is 1. If no edge on the boundary of \( f \) uses Rule E1, then it gives out at most 5/6 units of charge. Suppose the \( v_1v_2 \) uses Rule E1, then \( v_1 \) and \( v_2 \) are 4-vertices. By Lemma 3.3 Part 5 the other faces containing \( v_2v_3 \) and \( v_1v_5 \) are \( \geq 4 \)-faces, so no rule applies with respect to them. Since at most 1/2 units of charge is sent through \( v_3v_4 \) and \( v_4v_5 \) by the argument in the above paragraph, \( 5 \)-faces have non-negative charge after the rules are applied.

It remains to consider \( 3 \)-faces.

Lemma 3.8. Every \( 3 \)-face after discharging has non-negative charge.

Proof. We distinguish cases based on how many 4-vertices are incident to a \( 3 \)-face \( f \). Recall that the initial charge of \( f \) is -1.

Three 4-vertices By Proposition 3.4 each edge of the \( 3 \)-face is incident to a \( \geq 5 \)-face. Therefore by Rule E1 the face \( f \) receives 1/2 unit of charge from each of these three \( \geq 5 \)-faces. So its final charge after discharging is 1/2.

Two 4-vertices From Proposition 3.4 the edge of the \( 3 \)-face that is joining the two 4-vertices is also contained in \( \geq 5 \)-face. By Lemma 3.3 Part 5, the other two faces incident with \( f \) are \( \geq 4 \)-faces. Then there are three possibilities regarding the remaining vertex which we call \( v \): \( v \) is either \( \geq 6 \)-vertex, a \( 5 \)-vertex incident with two or less \( 3 \)-faces, or a \( 5 \)-vertex incident with three \( 3 \)-faces.

The vertex \( v \) is a \( \geq 6 \)-vertex. Note that the number of \( 3 \)-faces incident with \( v \) is at most \( d(v) - 2 \). So by Rules V6-8, the face receives 1/2 unit of charge from \( v \). It also receives 1/2 unit of charge from the \( \geq 5 \)-face containing the other two vertices of \( f \) by Rule E1. Therefore the face \( f \) has non-negative charge after discharging.

The vertex \( v \) is a \( 5 \)-vertex incident with two or less \( 3 \)-faces. By Rule V5 the \( 3 \)-face receives 1/2 unit of charge from \( v \). The \( 3 \)-face also receives 1/2 unit of charge from the \( \geq 5 \)-face containing the other two vertices of \( f \) by Rule E1. Therefore after discharging the face \( f \) has non-negative charge.

The vertex \( v \) is a \( 5 \)-vertex incident with three \( 3 \)-faces. By Proposition 3.3 the faces incident with \( v \) that share an edge with the face \( f \) are \( \geq 5 \)-faces. By Rule V5 the face \( f \) receives 1/3 units of charge from each of the two \( \geq 5 \)-faces sharing an edge containing \( v \) by Rule E2. Finally, the face \( f \) receives 1/2 unit of charge from the \( \geq 5 \)-face sharing the edge not containing \( v \). Therefore after discharging the faces has 1/6 units of charge.

Figure 8: Two 2-vertex cases.
A single 4-vertex. Let v be one of the two \( \geq 5 \)-vertices. The edge between v and the 4-vertex is contained in the 3-face \( f \) and a \( \geq 4 \)-face by Lemma 3.3 Part 5. Therefore the number of 3-faces incident to v is at most \( d(v) - 1 \). Then one of five cases happens with respect to v: v is \( \geq 7 \)-vertex, v is a 6-vertex incident with four or less 3-faces, v is a 5-vertex with incident with two or less 3-faces, v is a 6-vertex incident with five 3-faces, or v is a 5-vertex incident with three 3-faces.

The vertex v is \( \geq 7 \)-vertex, v is a 6-vertex incident with four or less 3-faces or v is a 5-vertex with incident with two or less 3-faces. In all these cases the face \( f \) receives at least 1/2 units of charge from v by Rule V5-8.

The vertex v is a 6-vertex incident with five 3-faces. By Proposition 3.6 the edge between v and the 4-vertex is contained in a \( \geq 5 \)-face. By Rule E2, the 3-face receives 1/6 units of charge from the \( \geq 5 \)-face. By Rule V6, the 3-face also receives 1/3 units of charge from v.

The vertex v is a 5-vertex incident with three 3-faces. By Proposition 3.5 the edge between v and the 4-vertex is adjacent to a \( \geq 5 \)-face. By Rule E2, the 3-face receives 1/6 units of charge from the \( \geq 5 \)-face. By Rule V6, the 3-face receives 1/3 units of charge from v.

In each of the cases the face receives at least 1/2 unit of charge together from the vertex v and the edge containing v and the 4-vertex. So, the 3-face \( f \) has non-negative charge after discharging.

No 4-vertex. By Rules V5-8 the face receives at least 1/3 unit of charge from each vertex it contains. Therefore the 3-face has non-negative charge after discharging.

Proof of Theorem 3.2. From Lemma 3.8 we have that every vertex and face after discharging has non-negative charge. This contradicts the amount of charge in the system. So, there is no counter-example to Theorem 3.2.

4 Conclusion

In the case of ordinary colouring, it is known that every plane graph is 5-choosable [5]. The bound is tight as shown by Voigt [6]. We have tried to construct an example of a plane graph with lists of sizes five such that there is no capital colouring assigning each vertex a color from its list. However, we have not managed to have done so, which led us to propose the following.

Conjecture 4.1. If each vertex of a plane graph is assigned a list of five integers, then there exists a capital colouring assigning each vertex a color from its list.

5 Acknowledgements

I would like to thank Dan Král’ for supervising me over summer in the Undergraduate Research Support Scheme at Warwick University and the university for funding such projects. I would also like to thank Zdeněk Dvořák and Ondřej Pangrác at Charles University for helping me whilst I visited their
university as well as the university itself for the support it offered for my trip. Thanks to Tomáš Kaiser and the University of West Bohemia for help and support on my trip. Finally, I would like to thank Igor Fabrici for his comments on the presentation.

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