Capital Ownership under Market Incompleteness: Does it matter?∗

EVA CARCELES POVEDA† AND DANIELE COEN-PIRANI‡

November 6, 2007

Abstract. Many recent papers in macroeconomics have studied the implications of models with household heterogeneity and incomplete financial markets under the assumption that households own the stock of physical capital and undertake the intertemporal investment decisions. In these models, production exhibits constant returns to scale, households maximize expected discounted utility and firms rent capital and labor from households to maximize period by period profits. This paper considers the case in which infinitely lived firms, rather than households, make the intertemporal investment decisions. Under this assumption, it shows that there exists an objective function for firms that results in the same equilibrium allocation as in the standard setting with one period lived firms. The objective requires that firms maximize their asset value, which is defined as the discounted value of future cash flows using present value processes that do not allow for arbitrage opportunities.

Keywords: Incomplete Markets, Firm Objectives, Shareholder Disagreement.

JEL Classification: D52, E44, G12, L20

1. Introduction

Following Bewley (1977, 1986), an extensive macroeconomic literature has studied the quantitative implications of models with heterogeneous agents and incomplete financial markets. Among others, Aiyagari (1994), Huggett (1997), Storesletten, Telmer and Yaron (2001, 2004, 2007) and Krusell and Smith (1997, 1998) have analyzed the effects of such a framework on the aggregate saving rate, the shape of the wealth distribution, the asset returns, and the welfare costs of business cycles.

An important assumption shared by the previous models is that households are the owners of the physical capital stock and undertake the intertemporal investment decisions. Further, firms simply rent capital and labor from the households to maximize profits on a period by period basis. Whereas this assumption is innocuous in a complete markets setting, things are very different when markets are incomplete. In this last case, when firms make the intertemporal investment decisions, shareholder disagreement may result in equilibrium, and the standard objective of value maximization is no longer well defined. This is due to the fact that the available markets do not provide sufficient information to value future profits unambiguously.

∗We are very grateful to Herakles Polemarchakis, Yair Tauman, Tom Muench and Harald Uhlig for fruitful discussions on the topic. The paper has also benefited from comments of conference participants at the Meetings of the Society for the Advancement in Economic Theory, the NBER Summer Institute, the Econometric Society and the Conference on Computational Economics and Finance, as well as seminar participants at Arizona State, Atlanta Fed, Washington Board of Governors, CEMFI, Duke, Goethe University at Frankfurt, Nova de Lisboa, North Carolina State, Rochester, University of Bilbao and University of Marne la Vallee.

†Correspondence: Department of Economics, State University of New York, Stony Brook NY-11794-4384 and IAE-CSIC, Campus UAB, 08193 Bellaterra, Barcelona. Email: ecarcelespov@notes.cc.sunysb.edu. Webpage: http://ms.cc.sunysb.edu/~ecarcelespov/.

‡Correspondence: Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15217. Email: coenp@andrew.cmu.edu. Webpage: http://www.andrew.cmu.edu/user/coenp/res.htm.
In this paper, we characterize the competitive equilibrium for the class of incomplete markets models typically studied in the macroeconomic literature under the assumption that firms decide on the level of investment. In particular, we show that there exists a natural generalization of the concept of value maximization to an incomplete markets context that gives rise to the same equilibrium allocation as in the economies cited above, in which firms solve static optimization problems. Hence, the competitive equilibrium allocations are the same regardless of whether the stock of physical capital is owned and accumulated by households, who rent it to firms, or by firms who, as infinitely-lived entities owned by shareholders, make the intertemporal investment decisions.

The framework for our analysis is an infinite horizon economy with one good, aggregate uncertainty, idiosyncratic risk, and general portfolio restrictions. Markets are incomplete and firms are assumed to make the investment decision in order to achieve “value” maximization. Under incomplete markets, value maximization is defined as the objective that discounts cash flows with present value processes that are consistent with security prices, in the sense that they satisfy a no arbitrage relation between security prices and their payoffs. Following the general equilibrium literature (see for example DeMarzo (1988, 1993) and Duffie and Shafer (1986b)), we first define a value maximizing competitive equilibrium as an allocation where both households and firms optimize, and the objective of the latter is value maximization. We then show that a value maximizing competitive equilibrium is also an equilibrium in the standard setting where firms maximize period by period profits. Conversely, an equilibrium in the standard setting is a value maximizing equilibrium in an economy where infinitely lived firms make the intertemporal investment decisions. One of the important implications of these results is that the equivalence of equilibrium allocations holds regardless of the particular present value process used by value maximizing firms to discount future profits.

Several remarks are worth noting. First, our main characterization result is derived under constant returns to scale in production and expected discounted utility for the households, two key assumptions that are typically satisfied in the macroeconomic literature mentioned above. The latter guarantees convergence of present value sums and it rules out stock price bubbles in equilibrium for all present value processes that are consistent with security prices. Further, in the absence of price bubbles, the assumption of constant returns to scale guarantees that the capital stock that is chosen by a value maximizing firm is equal to its stock market value. It turns out that this is also the level of capital that households would choose if they were making the intertemporal investment decision in a setting with one period lived firms. Thus, if aggregate capital is the same, the consumption opportunity set for the households is the same across the two economies, providing a justification for the equivalence of the equilibrium allocations in the versions of the model with one period lived and value maximizing firms.

Second, the set of consistent present value processes is a singleton under complete markets, while there may exist a continuum of present value processes if markets are incomplete, potentially leading to a continuum of value maximizing equilibria. However, using our main characterization result, we can establish that a value maximizing equilibrium will be invariant with respect to the particular discount factor of the firm if the equilibrium with one period lived firms is unique.

Third, our competitive equilibrium concept does not take into account the ownership structure of the firm. In the paper, we define a value maximizing competitive equilibrium with shareholder agreement as one where the valuation of future cash flows of the firm coincides with the valuation of shareholders in the control group. Using this definition, we show that only the shareholders who are unconstrained in the stock market every period agree with a value maximizing production plan. Note that this is due to the fact that their marginal rates of substitution (or valuation of future profits) belong to the set of consistent
present value processes. Therefore, they agree with the valuation of the firm. In contrast to this, our main characterization result is independent of whether shareholders are constrained or not in the asset market, an appealing feature of our approach.

Last, whereas value maximization is a natural objective, others are clearly possible. Under different firm objectives, however, the equivalence of allocations with the standard setting is not guaranteed. In other words, otherwise similar incomplete markets models might give rise to different quantitative implications than the ones established in the macroeconomic literature mentioned earlier. As an example, authors like Aiyagari (1994), Huggett (1997) or Krusell and Smith (1997, 1998) have shown that imperfect risk sharing in the standard setting can lead to an increase in the aggregate capital due to precautionary savings. In contrast, aggregate household wealth, which is equal to the stock market value of firms, need not be the same as the aggregate capital stock when firms do not maximize their market value. In other words, precautionary savings are not necessarily be reflected in the aggregate capital stock in the absence of value maximization. This is illustrated by Carceles-Poveda (2007), who studies the quantitative implications of alternative firm objective functions in a two agent model with incomplete markets. In such a framework, it is shown that the aggregate capital stock is very sensitive to the assumption on firm’s objectives. In addition, it would be interesting to study further implications of firm objectives on issues such as the wealth distribution. Furthermore, a different setup where this could be important are the standard overlapping generation models with production, where social security typically crowds out aggregate capital when households make the investment decisions. In such a setting, if firms decide on the investment level but do not maximize their value, they will not necessarily invest less when social security is introduced, potentially alleviating the crowding out effect. These are important issues that we leave for further research.

Finally, our work is related to the general equilibrium literature with incomplete markets and production in a multiperiod context. In this literature, the concept of value maximization that we have adopted has been used before by DeMarzo (1988, 1993) and Duffie and Shafer (1986b). The former demonstrates the validity of the Modigliani-Miller theorem while the latter study the issue of shareholder disagreement and show the existence of equilibria. Differently from these authors, who focus exclusively on firms as intertemporal decision-makers, we study the relationship between the allocations obtained in settings in which firms accumulate physical capital and the allocations obtained in the standard macroeconomic setting in which firms solve static decision problems. Therefore, our work can be viewed as establishing a link between this general equilibrium literature and the quantitative macro-finance literature mentioned in the opening paragraph.

The rest of the paper is organized as follows. The following section presents the model and section three discusses the main equivalence results. These are further discussed in section four. Section five summarizes and concludes.

2. The Model Economies

In this section, we first introduce a common general environment and then present the two different model economies. The first economy is the one typically considered in the macroeconomic literature, where households own the stock of physical capital and make

---

1 Carceles-Poveda and Coen-Pirani (2006) discuss the issue of shareholder disagreement in a similar context. However, they focus on the preferences of shareholders with respect to the investment decision of the firm without postulating a particular firm objective. In addition, Duffie and Shafer (1986b) show that shareholders generically disagree with value maximization in a more general setting.

2 For a survey of this literature see Imrohoroglu, Imrohoroglu and Joines (1999).

3 In a multiperiod context with incomplete markets, authors such as Hernandez and Santos (1996), Levine (1989), Magill and Quinzii (1994a, 1994b), and Levine and Zame (1996) have established the existence of an equilibrium in exchange economies.
intertemporal investment decisions, whereas the representative firm simply rents capital and labor from the households to maximize profits on a period by period basis. In this sense, the firm can be considered as being static or short lived. Second, we consider the case in which the firm is the owner of the stock of physical capital. Here, the firm is dynamic and is assumed to undertake all the intertemporal investment decisions.\footnote{Whereas the analysis assumes the existence of a representative firm (or a large number of identical firms) and no external financing of the investment level, our results can also be extended to the cases where firms are heterogeneous and where investment is financed with external funds. This will be further discussed in a later section.}

2.1. The General Environment. We consider an infinite horizon economy with aggregate uncertainty, idiosyncratic income shocks and sequential trading. Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Further, the resolution of uncertainty is represented by an information structure or event-tree $N$. Each node or date-state $s^t \in N$, summarizing the history of the environment through and including date $t$, has a finite number $S(s^t)$ of immediate successors. We use the notation $s^t|s^r$ with $r \geq t$ to indicate that node $s^r$ belongs to the sub-tree with root $s^t$. Further, with the exception of the unique root node $s^0$ dated at $t = 0$, each node has a unique predecessor dated at $t - 1$, which we denote by $s^{t-1}$. The probability of date-event $s^t$ at period zero is denoted by $\pi(s^t)$, with $\pi(s^0) = 1$, since the initial realization $s^0$ is given. In addition, $\pi(s^t|s^r)$ denotes the probability of $s^t$ given $s^r$. Throughout the text, we let $x = \{x(s^t)\}_{s^t \in N}$.

**Technology.** At each node $s^t \in N$, there exists a spot market for a single consumption good $y(s^t)$, produced with the following aggregate technology:

$$y(s^t) = f(z(s^t), k(s^{t-1}), n(s^t))$$

(1)

where $k(s^{t-1}) \in \mathbb{R}_+$ and $n(s^t) \in \mathbb{R}_+$ denote the aggregate physical capital and labor, $z(s^t)$ is an aggregate productivity shock, and the initial stock of capital $k(s^{-1}) \in \mathbb{R}_{++}$ is given. We make the following assumptions.

(A.1) The technology shock follows a stationary (Markov) process with state space $S_z = \{z_m : m \in M_z, z_m \in [z, \bar{z}], \}$, where $M_z$ is a finite set of integers, $0 < \underline{z} < \bar{z} < +\infty$, and the initial realization $z(s^0)$ is given.

(A.2) Given $z$, the production function $f(z, \cdot, \cdot) : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+$ is continuously differentiable on the interior of its domain, strictly increasing, strictly quasiconcave, and homogeneous of degree one in the two arguments. We also assume that $f(z, 0, n) = 0$, $f_k(z, k, n) > 0$ and $f_n(z, k, n) > 0$ for all $k > 0$ and $n > 0$. Further, $\lim_{k \rightarrow 0} f_k(z, k, n) = \infty$ and $\lim_{k \rightarrow \infty} f_k(z, k, n) = 0$ for all $n > 0$.

The previous two assumptions are standard in the macroeconomics literature. Assumption (A.1) models the technology shock as a stationary Markov chain. It is important to note that our results only require that the shock takes a finite number of positive values, and we can therefore relax the Markov assumption. Further, assumption (A.2) imposes standard conditions on the production process. In particular, the homogeneity assumption implies that $f(z, k, n) = f_k(z, k, n)k + f_n(z, k, n)n$ via Euler’s theorem. As we will see later, this last property is crucial to obtain our results.

The aggregate capital stock depreciates at the rate $\delta \in (0, 1)$, and we denote the total supply of goods available from production at $s^t$ including undepreciated capital by:

$$F(z(s^t), k(s^{t-1}), n(s^t)) = f(z(s^t), k(s^{t-1}), n(s^t)) + (1 - \delta)k(s^{t-1}).$$

(2)
Financial Markets. At each date-state $s^t$, there exist spot markets for a finite number $L$ of securities. The first is a claim to productive activity that is indexed by $l = 1$. The rest are financial assets whose returns are denominated in units of the consumption good.

A security $l \in L$ traded at $s^t$ is defined by its current price $q^t(l^t) \in \mathbb{R}_+$ and by the payoffs it promises to deliver at future nodes. Holding a portfolio of securities $a(s^{t-1}) \in \mathbb{R}^L$ at the end of period $t-1$ entitles the owner to a one period payoff of $R(s^t)' = [d(s^t) + q^t(s^t)']a(s^{t-1})$ if date-state $s^t$ is realized, where $q(s^t) = (q^1(s^t), ..., q^L(s^t))'$ and $d(s^t) = (d^1(s^t), ..., d^L(s^t))$ are the vectors of prices and dividends respectively.

A security traded at $s^t$ is of finite maturity if there exists a date $T$ such that $R^t(s^t|s^t) = 0$ for all $s^t|s^t$ with $r \geq T$. Otherwise, the security is infinitely lived. Further, security markets are one period complete at node $s^t$ if the rank of the matrix defined by $[R(s^{t+1})|s^{t+1}|s^t]$, where one row corresponds to $R(s^{t+1})'$ for each node $s^{t+1}|s^t$, is equal to $S(s^t)$. Markets are complete if they are one period complete at every date-state. We make the following assumptions.

(A.3) For all $s^t \in N$, $d(s^t) \in \mathbb{R}_+$ and there exist some $\varepsilon > 0$ such that $d^1(s^t) \geq \varepsilon$.

(A.4) $L \leq S(s^t)$ at all $s^t \in N$.

Assumption (A.3) requires that dividends are nonnegative. This is consistent with the fact that free disposal of securities implies nonnegative security prices. In addition, it imposes the additional restriction that dividends on the productive claim are bounded away from zero at each node. As we will show later, this rules out stock price bubbles in equilibrium.\footnote{Note that we are abstracting from securities that pay bundles of other securities and from trade in different securities at different date-states, although this can be easily incorporated at the expense of additional notation. See Santos and Woodford (1997) or Hernandez and Santos (1996).}

Clearly, a necessary condition for markets to be complete is that $L \geq S(s^t)$ at all $s^t \in N$. On the other hand, since we are particularly interested in the case where markets are incomplete, assumption (A.4) limits the number of available assets at each node.

No Arbitrage Pricing. The security price process $q$ is arbitrage free at $s^t$ if there does not exist a portfolio $a(s^t) \in \mathbb{R}^L$ such that $R(s^{t+1})'a(s^t) \geq 0$ for all $s^{t+1}|s^t$ and $q(s^t)'a(s^t) \leq 0$, with at least one strict inequality. In other words, arbitrage free prices have to be such that it is not possible to construct a portfolio with non-positive value and nonnegative payoffs at every successor node. While this must be the case in equilibrium, the presence of no arbitrage at date-state $s^t$ implies the existence of positive present value prices $\lambda(s^t) > 0$ and $\{\lambda(s^{t+1})\}$ with $\lambda(s^{t+1}) > 0$ for each $s^{t+1}|s^t$, such that:

$$ q(s^t)' = \sum_{s^{t+1}|s^t} \frac{\lambda(s^{t+1})}{\lambda(s^t)} R(s^{t+1})' $$

(3)

Given $(q, d)$, the absence of arbitrage at each date-state $s^t$ allows us to define processes $\lambda = \{\lambda(s^t)\}_{s^t \in N}$ for the entire information structure such that the previous no arbitrage equation holds. In what follows, we denote the set of such processes for the sub-tree with root $s^t$ by $Q_{s^t}(q, d)$. Further, we define $\lambda^s_{t+1} = \lambda(s^r)/\lambda(s^t)$ for $s^r|s^t$. Note that the present value ratios $\lambda^s_{t+1}$ that are consistent with security prices are uniquely determined by (3) if markets are complete. On the other hand, the number of linearly independent equations is not sufficient to uniquely determine the ratios when markets are incomplete.

(5) Note that we are abstracting from securities that pay bundles of other securities and from trade in different securities at different date-states, although this can be easily incorporated at the expense of additional notation. See Santos and Woodford (1997) or Hernandez and Santos (1996).

\footnote{Since the dividends on the productive claim are positive in a neighbourhood of the non stochastic steady state, one can actually impose restrictions on the process of the exogenous technology shock and on the initial capital stock $k(s^{-1})$ such that the restriction on $d_1(s^t)$ is satisfied at each node. For example, one could assume that the variance of the exogenous technology shock is sufficiently small, while the initial capital is sufficiently close to the steady state value.}
The previous present value prices can be used to evaluate future streams of consumption goods. In particular, for a non-negative stream $x$ that specifies $x(s^t) \in \mathbb{R}_+$ for all $s^t \in N$, the present value at $s^t$ of the subsequent stream with respect to some $\lambda \in Q_{sl}(q,d)$ can be defined as:

$$v_x(s^t, \lambda) = \sum_{r=1}^{\infty} \sum_{s^{t+r} \mid s^t} \lambda^{s^{t+r}} x(s^{t+r})$$

Similarly, we can define the fundamental value $v_{dl}(s^t, \lambda)$ of security $l$ with respect to some $\lambda \in Q_{sl}(q,d)$. In addition, using some algebra, the bubble component of the security with respect to $\lambda \in Q_{sl}(q,d)$ can be expressed as:

$$\sigma^l(s^t, \lambda) = q^l(s^t) - v_{dl}(s^t, \lambda) = \lim_{r \to \infty} \sum_{s^{t+r} \mid s^t} \lambda^{s^{t+r}} q^l(s^{t+r})$$

As shown by Santos and Woodford (1987), if a security price is non-negative, its fundamental value $v_{dl}(s^t, \lambda)$ satisfies $0 \leq v_{dl}(s^t, \lambda) \leq q^l(s^t)$ for all $\lambda \in Q_{sl}(q,d)$. Further, whereas the fundamental value need not be the same for all state prices satisfying equation (3), the authors show that it must lie between the finite bounds:

$$v_{dl}(s^t, \lambda) = \inf_{\lambda} v_{dl}(s^t, \lambda) \text{ and } \sup_{\lambda} v_{dl}(s^t, \lambda).$$

Clearly, there exists no bubble for $l \in L$ if $v_{dl}(s^t, \lambda) = \sup_{\lambda} v_{dl}(s^t, \lambda) = q^l(s^t)$. In this case, the fundamental value is uniquely defined for all $\lambda \in Q_{sl}(q,d)$.

**Households.** The economy is populated by a countable set of infinitely lived households $I$. Households’ preferences $\succsim = (\succsim_i)_{i \in I}$ over consumption plans $c_i$ satisfy the following assumption.

(A.5) For every $i \in I$, $\succsim_i$ can be represented by the following function:

$$U_i(c_i) = \sum_{l=0}^{\infty} \sum_{s^t} \beta^l_i \pi(s^t) u_i(c_i(s^t))$$

where $\beta_i \in (0, 1)$ is the individual discount factor, and the period utility function $u_i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable in the interior of its domain, with $\lim_{c_i \to 0} u'_i(c_i) = \infty$ and $\lim_{c_i \to \infty} u'_i(c_i) = 0$.

The class of preferences in assumption (A.5) is standard in the macroeconomic literature. It is important to note that this class of preferences satisfies the property that there is a “uniform lower bound on the impatience” of each agent. This last property, which is stated formally in the appendix, has been assumed by several authors studying infinite horizon exchange economies with incomplete markets, such as Santos and Woodford (1997), Magill and Quinzi (1994a, 1994b), Hernandez and Santos (1996) and Levine and Zame (1996). In essence, it requires that at each node $s^t$ an agent is willing to give up a fraction of his future consumption after node $s^t$ in exchange for a multiple of the current aggregate endowment. Further, the fraction of future consumption that each agent is willing to give up (or the degree of impatience), is uniform across all nodes and feasible consumption plans. As we will see later, this property is crucial to establish the absence of price bubbles in the present setup.

Each household $i \in I$ enters the markets at $t = 0$ with a finite initial endowment $a_i^l(s^{-1})$ of each security, whose sum across households determines the net supply of the security at each node, which we denote by $A^l = \sum_{i \in I} a_i^l(s^{-1})$. Without loss of generality, the supply of the productive claim and of the rest of securities is normalized to one and zero respectively.

---

The results follow from Propositions 2.1 and 2.2 in Santos and Woodford (1997), which can be directly applied to our setup.
and we let $A = (A^1, \ldots, A^L)'$. At each date-state $s^t \in N$, households are also endowed with one unit of time that is entirely allocated to labor, and which they can transform into $\epsilon_i(s^t)$ efficiency labor units that will be used to produce output in exchange of wages. Given this, the labor income of the household at $s^t$ is given by $w_i(s^t) = w(s^t)\epsilon_i(s^t)$, where $w(s^t)$ is the fraction of output allocated to labor payments. We make the following assumptions.

(A.6) For all $i \in I$, $a_i(s^{-1}) \in \mathbb{R}_+$.

(A.7) The labor income shock $\epsilon_i$ follows a stationary (Markov) process with state space $S_\epsilon = \{\epsilon_{im} : m \in M_\epsilon, \epsilon_{im} \in [\underline{\epsilon}, \overline{\epsilon}]\}$, where $M_\epsilon$ is a finite set of integers, $0 < \underline{\epsilon} < \overline{\epsilon} < 1$, and the initial realization $\epsilon_i(s^0)$ is given.

Assumption (A.6) guarantees that the supply of each security is non-negative. Further, assumption (A.7) models the labor income shock as a discrete state Markov chain. As before, our results only require that the shock takes a finite number of positive values, and we can therefore relax the Markov assumption. The aggregate and idiosyncratic shocks could potentially be correlated, and we denote their joint transition matrix by $\Pi$ in what follows.

At each node $s^t$, household $i \in I$ chooses consumption $c_i(s^t) \in \mathbb{R}_+$ and a portfolio of securities $a_i(s^t) \in \mathbb{R}^L$ subject to the following constraints:

\[ c_i(s^t) + q(s^t)'a_i(s^t) \leq \omega_i(s^t) \tag{5} \]
\[ \omega_i(s^{t+1}) = w_i(s^{t+1}) + R(s^{t+1})'a_i(s^t) \tag{6} \]
\[ q(s^t)'a_i(s^t) \geq B_i(s^t). \tag{7} \]

Equation (5) is the standard budget constraint with sequential markets and equation (6) is the law of motion of the individual wealth $\omega_i(s^t)$. At $t = 0$, equation (6) takes the same form with $\omega_i(s^0) = q'(s^0)a_i(s^{-1}) + d^1(s^0)a_i^1(s^{-1}) + w_i(s^0)$, where we have used the fact that $d^1(s^0) = 0$ for $l \geq 2$. Finally, to avoid Ponzi schemes, equation (7) imposes a finite limit of $B_i(s^t)$ on the total amount that households can borrow at every node.\footnote{Alternatively, one could impose constraints on the individual asset holdings, since market clearing implies that in every asset market and every node there exists at least one household that is unconstrained.}

A possible trading restriction that one can impose is the present value constraint, which is effectively never binding at any finite date. In particular, it is the tightest borrowing limit such that the portfolio holdings satisfy the budget constraint with $c_i(s^t) \in \mathbb{R}_+$ for all $s^t \in N$, and wealth is always non-negative after a finite date. As shown by Santos and Woodford (1997), this constraint can be formally specified as:

\[ B_i(s^t) = -v_{w_i}(s^t, \lambda) \text{ where } v_{w_i}(s^t, \lambda) = \inf_{\omega \in Q^{(w_i)}(s^t, \lambda)} v_{w_i}(s^t, \lambda) \tag{8} \]

In essence, the restriction implies that households can borrow at most the lowest present value of their individual endowments in order to be solvent. The two production economies are described in what follows.

2.2. The $k$-economy. In the $k$-economy, we make the usual assumption in the macrofinance literature, implying that households are the owners of the physical capital stock and make the inter-temporal investment decision. In this case, the problem of the firm is particularly simple. At each date-state $s^t$, after observing the realization of the productivity shock $z$, the firm chooses capital and labor to maximize period profits. Thus, it solves a sequence of static problems:

\[ \max_{k, n} F(z(s^t), k(s^{t-1}, n(s^t))) - w(s^t)n(s^t) - r(s^t)k(s^{t-1}) \tag{9} \]
leading to the following necessary and sufficient first order conditions:

\[
w(s^t) = F_a(z(s^t), k(s^{t-1}), n(s^t)) = f_a(z(s^t), k(s^{t-1}), n(s^t)) \tag{10}
\]

\[
r(s^t) = F_k(z(s^t), k(s^{t-1}), n(s^t)) = f_k(z(s^t), k(s^{t-1}), n(s^t)) + 1 - \delta \tag{11}
\]

where \(w(s^t) \in \mathbb{R}_+\) and \(r(s^t) \in \mathbb{R}_+\) are the competitively determined wage and gross capital rental rates respectively. Further, each household \(i \in I\) maximizes the preferences in (A.5) subject to the following constraints:

\[
c_i(s^t) + k_i(s^t) + \sum_{l \geq 2} q^l(s^t) a^l_i(s^t) \leq \omega_i(s^t) \tag{12}
\]

\[
\omega_i(s^{t+1}) = w_i(s^{t+1}) + r(s^{t+1}) k_i(s^t) + \sum_{l \geq 2} R^l(s^{t+1}) a^l_i(s^t) \tag{13}
\]

\[
k_i(s^t) + \sum_{l \geq 2} q^l(s^t) a^l_i(s^t) \geq B_i(s^t) \tag{14}
\]

In the previous equations, \(k_i(s^t)\) is the amount of physical capital held by the household at the end of period \(t\), illustrating the fact that households make the inter-temporal investment decision. If we denote by \(k_i(s^{-1})\) and \(a_i(s^{-1})\) the initial asset holdings of \(i\) at \(t = 0\), the period zero budget constraint takes the same form with \(\omega_i(s^0) = w_i(s^0) + r(s^0) k_i(s^{-1}) + q(s^0) a_i(s^{-1})\).

A \(k\)-economy is specified by a set of preferences \(\succ\), a transition matrix \(\Pi\), initial values \((k_0, a_0, \omega_0) = \{k(s^{-1}), (a_i(s^{-1}), e_i(s^0))\}_{i \in I}, z(s^0)\), security processes \(d^a = (d^l)_{l \geq 2}\) and borrowing limits \(B = (B_i)_{i \in I}\), where \(a^l_i(s^{-1}) = k_i(s^{-1})/k(s^{-1})\) represents the initial endowment of capital shares of household \(i \in I\). A \(k\)-economy is therefore described by \(E_k = \{\succ, (k_0, a_0, \omega_0, \Pi, d^a, B)\}\).

**Definition 2.1.** The vector of processes \(\{(c_i, k_i, (a^l_i)_{l \geq 2}) \in I, (q^l)_{l \geq 2}, (w, r)\}\) is a CE for \(E_k = \{\succ, (k_0, a_0, \omega_0, \Pi, d^a, B)\}\) if (i) for each \(i \in I\) and for each \(s^1 \in N\), \(\{c_i, k_i, (a^l_i)_{l \geq 2}\}\) is optimal under the preferences \(\succ\) given \((q^l)_{l \geq 2}, (w, r), (k_0, a_0, \omega_0, \Pi, d^a)\) and \(B\) (ii) \((w, r)\) satisfies the firm’s optimality conditions (iii) all markets clear, i.e., for all \(s^t \in N\), \(n(s^t) = \sum_{i \in I} c_i(s^t)\), \(\sum_{i \in I} k_i(s^t) = k(s^t)\), \(\sum_{i \in I} a^l_i(s^t) = A^l\) for \(l \geq 2\) and \(\sum_{i \in I} c_i(s^t) = w_i(c(s^t)) n(s^t) + d^l(s^t) A^l\).

Before discussing the framework with dynamic firms, it is important to note that the constraints faced by the household sector in the \(k\)-economy can be directly mapped into the framework of the general environment if we define the shares of physical capital held by household \(i\) at date-state \(s^t\) as \(A^l_i(s^t) = k_i(s^t)/k(s^t)\). With this normalization, the total supply of shares is positive and equal to \(A^1 = 1\). Further, \(q^1(s^t) = k(s^t), R^1(s^{t+1}) = r(s^{t+1})k(s^t)\) and \(d^1(s^t) = r(s^t)k(s^{t-1}) - k(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) = N_I(s^t)\).

**The e-Economy.** In the e-economy, we assume that the firm owns the entire stock of capital and undertakes the inter-temporal investment decision by solving a dynamic optimization problem. Further, households are entitled to the future dividend payments through their ownership of a perfectly divisible equity share in the firm that is traded at price \(q^1(s^t)\).

At each node \(s^t\), households maximize the preferences in (A.5) subject to constraints (5)-(7). Further, the firm produces output, pays wages to the total labor employed and decides on the amount of investment. Investment is entirely financed with retained earnings, and the residual of gross profits (output net of labor payments) and investment is paid out as dividends to the firm equity owners, i.e.,

\[
d^1(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - w(s^t)n(s^t) - k(s^t) = N_I(s^t) \tag{15}
\]
where $N_f(s^t)$ is the net cash flow of the firm. Unfortunately, the definition of an appropriate firm objective is more complicated than before, since the standard approach, that firms maximize their share value, is not well specified under market incompleteness. The reason is that the available markets do not provide sufficient information to value future dividend streams unambiguously. To see this, consider the case of effective complete markets and let $m_{r,t}^{s+r} s^t$ be the $t+r$-period ahead pricing kernel. Note that $m_{r,t}^{s+r}$ represents the period $t$ price of one unit of time $t+r$ consumption, contingent on the economy being at date-state $s^{t+r} | s^t$. Since all the shareholders will agree on the pricing kernel under complete markets, the objective of the firm at date-state $s^t$ can then be naturally specified as follows:

$$\text{Max}_{k,n} \sum_{r=0}^{\infty} \sum_{s^{t+r}} m_{r,t}^{s+r} N_f(s^{t+r}) \text{ where } m_{r,t}^{s+r} = \beta_i \frac{\pi(s^{t+r} | s^t) u'_i(c_i(s^{t+r}))}{u'_i(c_i(s^t))} \text{ for all } i \in I$$

As usual when markets are complete, the firm maximizes the present discounted value of its net cash flows, using as a discount factor the unique present value process of its shareholders, which is also the only element of the set $Q_s(q,d)$. In addition, both the agents and the firm value future output in each state identically, and all shareholders will therefore agree with the investment choice made by the firm. On the other hand, since a unique present value process that is consistent with market prices is not necessarily available under market incompleteness, the previous objective is no longer well defined, and shareholder disagreement may result in equilibrium.\(^9\)

In what follows, we briefly discuss some of the approaches that have been proposed in the literature following the seminal paper of Diamond (1967). Moreover, a discussion of shareholder disagreement in the present setup is provided in a later section.\(^10\)

**Value Maximization.** As noted by DeMarzo (1993), a natural generalization of the previous Arrow Debreu firm objective to an incomplete markets setup is to require firms to maximize the value of their output according to some consistent present value prices, in the sense that they satisfy the no arbitrage condition in (3). The two period value maximizing firm objective postulated by the author can therefore be expressed in our multi-period setup as follows:

$$U_f(N_f) = \text{Max}_{k,n} \sum_{r=0}^{\infty} \sum_{s^{t+r}} \lambda_{r,t}^{s+r} N_f(s^{t+r}) \text{ for some } \lambda \in Q_s(q,d)$$

This approach has also been followed by DeMarzo (1988) and Duffie and Shafer (1986b), who study the validity of the Modigliani Miller theorem and the existence of equilibrium and shareholder agreement in a general incomplete markets context. As noted by the authors, one could alternatively assume that the firm maximizes its share price according to some valuation function that assigns a price process to a given stream of cash flows. Further, as long as this valuation does not predict security prices that allow for arbitrage opportunities, there exist some positive present value prices $\lambda \in Q_s(q,d)$ such that the valuation conjectured by the firm is equal to the objective function above.

The optimization problem under value maximization can be characterized by the follow-
ing necessary and sufficient first order conditions:

$$w(s^t) = f_n(z(s^t), k(s^{t-1}), n(s^t))$$  \hspace{1cm} (17)

$$1 = \sum_{s^{t+1}|s^t} \lambda^{s^{t+1}}_t [f_k(z(s^{t+1}), k(s^t), n(s^{t+1})) + 1 - \delta]$$  \hspace{1cm} (18)

The first equation determines the equilibrium aggregate wage rate. Further, the second equation determines the optimal production plan and it illustrates that the inter-temporal investment decision in this economy is made by the firm.

The $e$-economy is specified by a set of preferences $\zeta$, initial values $(k_0, a_0, z_0, \epsilon_0) = \{k(s^{-1}), (a_i(s^{-1}), \epsilon_i(s^0))\}_{i \in I}, z(s^0)$, a transition matrix $\Pi$, security processes $d^a = (d^a_l)_{l \geq 2}$ and limits $B = (B_i)_{i \in I}$. The $e$-economy is then described by $E_e = \{\zeta, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}.$

**Definition 2.2.** The vector of processes $\{(c_i, a_i)\}_{i \in I}, q, w, k$ is a value maximizing CE (VM CE) for $E_e = \{\zeta, (k_0, a_0, z_0, \epsilon_0), \Pi, d^a, B\}$ if (i) for each $i \in I$ and for each $s^t \in N$, $(c_i, a_i)$ is optimal under the preferences $\zeta$ given $(q, w), (k_0, a_0, z_0, \epsilon_0), \Pi, d^a$ and $B$ (ii) $(w, k)$ satisfies the firm’s optimality conditions for some $\lambda \in Q_s(q, d)$ (iii) all markets clear, i.e., for all $s^t \in N$, $n(s^t) = \sum_{i \in I} a_i(s^t), \sum_{i \in I} a_i(s^t) = A^i$ for $l \in L$ and $\sum_{i \in I} c_i(s^t) = w(s^t) n(s^t) + d(s^t) A$.

Several remarks are worth noting. First, the previous equilibrium definition implies that the set of allowable present value processes $Q_s(q, d)$ that the firm can use to discount its net cash flows has to satisfy a fixed point problem in the following sense. When the firm discounts profits with some $\lambda$ that belongs to the set of admissible present value prices $Q_s(q, d)$, its production choice $k(\lambda)$ generates a new asset structure $(q(\lambda), d(\lambda))$ and a new set of admissible present value prices $Q_s(q(\lambda), d(\lambda))$ to which the original $\lambda$ has to belong. Thus, if we define a mapping from the admissible set of present value prices to the set of present value prices that it generates, the equilibrium set of discount factors can be seen as a fixed point of this mapping. Moreover, if the set satisfying the previous fixed point problem is not single valued, the presence of incomplete financial markets might generate indeterminacy of equilibria with respect to the firm discount factor (see Duffie and Shafer (1986b)). Second, since the state process $\lambda$ can be interpreted as the discount factor used by the firm to value future net cash flows, value maximization will generate shareholder disagreement if $\lambda$ does not agree with the valuation of the controllers of the firm.

Following the seminal paper of Diamond (1967), several authors have proposed alternative criteria concerning the discount factor of the firm. In a two period context, Dreze (1974) has proposed as a discount factor a weighted average of the marginal rates of substitution of the different shareholders, with the weights reflecting the final holdings of shares. Grossmann and Hart (1979) have extended this idea to a multi-period setting, arguing that the weights should reflect the initial allocation of shares among shareholders. Dreze (1985) and DeMarzo (1993) have introduced a control mechanism based on majority voting to decide among alternative production plans, implying that the firm should discount future profits using some weighted average of the marginal rates of substitution of the controllers of the firm. In addition, other authors, such as Radner (1972a), Sandmo (1972), Sondermann (1974) or Leland (1972), have simply assumed the existence of a utility function for the firm, defined exogenously over profits. In the present paper, we focus on value maximization, since this is the objective that will yield the same allocations as in the standard setting with one period lived firms.

**Competitive Equilibrium with Shareholder Agreement.** It is important to note that the equilibrium concept in definition 2.2 does not take into account the relationship between the investment decision of the firm and its ownership structure. In other words, the
investment decisions can potentially be made without taking into account the preferences of the shareholders.

To address this issue, let \( I^e (s^t) \subseteq I \) be the subset of shareholders that have control over the firm at \( s^t \). We can now extend the previous equilibrium definition to an equilibrium concept with shareholder agreement, in the sense that all shareholders in the control group support the production plan chosen by the firm. To do this, we should replace condition (ii) in Definition 2.2 with the following condition: (ii)' \((w,k)\) satisfies the firm’s optimality conditions for some \( \lambda \in Q_s(q,d) \) that coincides with the valuation of future cash flows of all \( i \in I^e (s^t) \). If this condition is satisfied, a production plan that is unilaterally chosen by the value maximizing firm also maximizes the utility of the shareholders in the control group. The issue of shareholder disagreement is briefly analyzed in section four.

3. Equivalence of the Production Allocations

This section shows the equivalence of the set of equilibria in the two production economies under value maximization. Throughout the section, we assume that the assumptions of Section 2 are satisfied. We start by stating several results that we will use to prove the main theorems, and relegate most of the proofs to the appendix. To distinguish the allocations, the caret bearing variables always denote \( k \)-economy allocations.

**Proposition 3.1** Consider a CE for \( E_k \) or \( E_e \). For each node \( s^t \in N \) and for each security \( l \in L \) traded at \( s^t \) that is either (i) of finite maturity or (ii) in positive supply, we have that \( q^t (s^t) = v_{s^t} (s^t, \lambda) \) for all \( \lambda \in Q_{s^t}(q,d) \). In addition, \( \lim_{r \to -\infty} \sum_{s^t+r|s^t} \lambda^{s^t+r}_k k(s^{t+r}) = 0 \) for all \( \lambda \in Q_{s^t}(q,d) \).

Proposition 3.1 establishes the absence of price bubbles for securities that are of finite maturity or in positive supply. Further, it implies that the discounted value of the aggregate capital converges to zero as time goes to infinity. These results can be established by showing that our preference assumptions imply that bubbles cannot exist for any asset that is either (i) of finite maturity or (ii) in positive supply and for any present value process \( \lambda \in Q_{s^t}(q,d) \) such that the present value of the aggregate labor endowment \( v_{w_k}(s^t, \lambda) \) is finite when this state price process is used. This part directly follows from Santos and Woodford (1997), and we only include the proof for completeness. Second, we show that the presence of trade in a claim to productive activity implies that the present value of the aggregate labor endowment is finite for any consistent present value process. To do this, we rely on the definition of the productive dividend payments. Clearly, the previous results imply that bubbles cannot exist for any asset that is finitely lived or in positive supply. The next lemma shows that the aggregate capital stock chosen by a value maximizing firm in the absence of price bubbles is equal to the ex-dividend firm value \( q^1 \).

**Lemma 3.1** If the \( e \)-economy firm discounts its net cash flows with some \( \lambda \in Q_{s^t}(q,d) \), the equilibrium investment plan satisfies \( k(s^t) = q^1(s^t) = v_{N_{s^t}}(s^t, \lambda) \) for all \( s^t \in N \).

To prove the lemma, recall that the first order conditions of the firm’s problem under value maximization imply that:

\[
1 = \sum_{s^{t+1} | s^t} \lambda^{s^{t+1}}_t [f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta]
\]

\(^{11}\)As briefly discussed by Santos and Woodford (1997), who study the existence of price bubbles in exchange economies, bubbles can be ruled out in the presence of a claim to productive activity if one assumes that \( d^1(s^t) \geq \phi w_e(s^t) \) for some \( \phi > 0 \), where \( w_e \) are the total resources in the economy (or the sum of dividends and labor payments). In the present setup, we assume instead that \( d^1(s^t) \) is bounded away from zero. Further, we provide a direct proof of the absence of bubbles that just relies on the definition of the productive dividend payments.
where \( \lambda \in Q_s(q, d) \). Multiplying the previous expression with \( k(s^t) \), adding and subtracting \( k(s^{t+1}) \) on the right hand side, and using the homogeneity condition of the production function, we obtain:

\[
k(s^t) = \sum_{s^t+1} \lambda_t^{s^t+1} [N_f(s^{t+1}) + k(s^{t+1})]
\]

Further, substituting iteratively for \( k(s^{t+r}) \) for \( 1 \leq r \leq T \), we have that:

\[
k(s^t) = \sum_{r=1}^{T} \sum_{s^t+r} \lambda_t^{s^t+r} [N_f(s^{t+r})] + \sum_{s^t+T} \lambda_t^{s^t+T} k(s^{t+T})
\]

The first term on the right-hand side of the previous equation has a well defined limit, and the second term converges to zero as \( T \) goes to infinity by proposition 3.1. Thus, taking limits of the previous equation as \( T \) goes to infinity, the aggregate capital stock can be expressed as:

\[
k(s^t) = \sum_{r=1}^{\infty} \sum_{s^t+r} \lambda_t^{s^t+r} [N_f(s^{t+r})] = v_{N_f}(s^t, \lambda)
\]

On the other hand, the ex-dividend firm value is equal to the value of equity \( q^1(s^t) \), whose dividends are given by \( d^1(s^t) = N_f(s^t) \) at all \( s^t \in N \). Further, since equity is in positive supply, proposition 3.1 implies that \( q^1(s^t) = v_{N_f}(s^t, \lambda) \) for all \( \lambda \in Q_s(q, d) \), establishing the result.

The result of the previous lemma is crucial to establish the equivalence of the equilibrium allocations in the two production economies. In particular, as shown by lemma 3.2 below, it implies that the set of budget feasible allocations is the same across the two production economies as long as they are characterized by the same preferences, initial values, transition matrices for the shocks, financial asset structure, portfolio constraints and production plans.

**Lemma 3.2** Consider optimal allocations in the \( k \)- and \( e \)-economies. Further, assume that \((k_0, a_0, z_0, e_0), (d^t, q^t)_{t \geq 2} \), \( B \) and \( k \) are the same. If the firm in the \( e \)-economy has a value maximizing objective, the set of budget feasible allocations is the same in the two production economies.

Lemma 3.2 shows that households in the \( k \)-economy can achieve the same consumption allocation as in the \( e \)-economy (and vice-versa) by choosing the same portfolio of assets \((a_i^t)_{t \geq 2}\) and a physical capital investment \( k_i \) that is equal to the total equity investment \( q^1(a^t) \) they would choose in the \( e \)-economy. If the production plan \( k \) is the same across the two production economies, this asset choice generates the same financial wealth, implying that a consumption plan that is feasible in one economy is also feasible in the other. As before, the result relies on the homogeneity of the production function, since it requires that \( k(s^t) = q^1(s^t) \). We are now ready to state our main results.

**Theorem 3.1** Let \( \{(\xi_i, a_i, \zeta_i, q_i, w, k_i)_{t \geq 1}, Q_s(q, d), \Pi, d^t, B\} \) be a VM CE for \( E_a = \{(\xi_i, (k_0, a_0, z_0, e_0), (d^t, q^t)_{t \geq 2}, w, k_i)_{t \geq 1}, (q^t)_{t \geq 2}, (w, k_i) \} \) is a CE for \( E_k = \{(\xi_i, (k_0, a_0, z_0, e_0), (d^t, q^t)_{t \geq 2}, w, k_i)_{t \geq 1}, (q^t)_{t \geq 2}, (w, k_i) \} \) in particular, \( k_i(s^t) = q^1(s^t)a^t_i(s^t) \) and \( \zeta_i(s^t) = R_k(s^t)/q^1(s^t) \) for all \( s^t \in N \).

Theorem 3.1 asserts that a value maximizing equilibrium in the \( e \)-economy is also an equilibrium in a \( k \)-economy with the same characteristics. The argument of the proof is very simple. We first note that the \( k \)-economy aggregate capital stock \( k \) is equal to the value of the firm \( q^1 \) in the \( e \)-economy, which is in turn equal to the \( e \)-economy capital stock \( k \) by lemma 3.1. Given this, the returns on labor and capital defined by \( (w, k_i) \) satisfy the firm’s optimality conditions in the \( k \)-economy, and lemma 3.2 implies that an optimal household
allocation in the e-economy is also optimal in the k-economy. Finally, market clearing in the latter economy follows from market clearing in the first. Theorem 3.2 below states that the reverse is also true.

**Theorem 3.2** Let \( \{ (\hat{c}_i, \hat{k}_i, (\hat{d}_i)_{t \geq 1}) \}_{i \in I}, (\hat{q}_i)_{t \geq 2}, \hat{w}, \hat{r} \} \) be a CE for the economy specified by \( E_k = \{ \hat{z}, (\hat{k}_0, \hat{a}_0, \hat{z}_0, \epsilon_0), \Pi, \hat{\omega}, \hat{B} \} \). Then, there exist processes for \( a^1, \) and \( q^1 \) such that 
\[
\{ (\hat{c}_i, a^1_i, (\hat{d}_i)_{t \geq 1}) \}_{i \in I}, q^1, (\hat{q}_i)_{t \geq 2}, \hat{w}, \hat{k}_i \}
\]
is a VM CE for \( E_e = \{ \hat{z}, (\hat{k}_0, \hat{a}_0, \hat{z}_0, \epsilon_0), \Pi, \hat{\omega}, \hat{B} \} \).

In particular, \( a^1_i(s^t) = \frac{\hat{k}_i(s^t)}{k(s^t)} \) and \( q^1(s^t) = \hat{k}(s^t) = \sum_{i \in I} \hat{k}_i(s^t) = \hat{q}^1(s^t) \) for all \( s^t \in N \).

Theorem 3.2 can be proved using similar arguments. In particular, since the aggregate capital is the same in the two economies, the fact that \( q^1 = \hat{q}^1 \) implies that both economies have the same asset structure and the same set of consistent present value prices. Given this, the k-economy aggregate wage rate \( \hat{w} \) and aggregate capital stock \( \hat{k} \), which satisfies the no arbitrage pricing condition in (3), satisfy the firm’s optimality conditions in the e-economy for some \( \lambda \in Q_{s^t}(q, d) \). Finally, lemma 3.2 implies that an optimal household allocation in the k-economy is also optimal in the e-economy, and market clearing in the latter economy directly follows from market clearing in the first.

The previous two theorems imply that value maximization leads to the same dimension of the set of equilibria in the two production economies. On the other hand, the equilibrium in the e-economy might depend on the particular firm discount factor \( \lambda \in Q_{s^t}(q, d) \) (see Duffie and Shafer (1986b)). As stated by the following theorem, however, if the equilibrium with one period lived firms exists and is unique, the equilibrium under dynamic firms is independent from the discount factor of the firm.

**Theorem 3.3** If a CE in the k-economy exists and is unique, the e-economy VM CE is invariant with respect to the firm discount factor \( \lambda \in Q_{s^t}(q, d) \).

The proof of this theorem follows from the previous two. In particular, theorems 3.1 and 3.2 establish that the set of equilibria has the same dimension in the two production economies. Thus, if we consider two equilibria in the e-economy that just differ in the discount factor chosen by the firm, the theorems show that they are also equilibria in the k-economy. On the other hand, if the k-economy equilibrium is unique, the two e-economy equilibria will clearly result in the same allocations.

### 4. Discussion

Theorems 3.1 and 3.2 extend to an incomplete markets setting with general portfolio restrictions the result, well known under complete markets, that capital ownership is irrelevant under value maximization. In other words, equilibria are the same whether the agents own the capital and rent it to the firm, or whether the firm, considered as an infinitely lived corporation that is owned by the agents as shareholders, holds the capital stock. Several remarks are worth noting.

First, in addition to the assumption of constant returns to scale in production, the results require the standard assumption that agents are impatient, which implies convergence of the present value calculations in equilibrium. To get some intuition for the previous findings and for why these assumptions are needed, consider the investment decision in the two production economies. In a k-economy with portfolio restrictions, the aggregate capital stock is determined by the unconstrained households, whose marginal rate of substitution between periods \( t \) and \( t + 1 \) is denoted by \( m_{s^{t+1}} \). If we substitute for \( r(s^{t+1}) \) and multiply by \( k(s^t) \) the first order condition of an unconstrained household, it follows that the capital stock in this economy has to satisfy the following Euler condition:

\[
k(s^t) = \sum_{s^{t+1} \mid s^t} m_{s^{t+1}} \left[ F_k(z(s^{t+1}), k(s^t), n(s^{t+1}))k(s^t) \right] . \tag{19}
\]
Similarly, consider an analogous $e$-economy where the investment decision is made by a value maximizing firm. In the absence of price bubbles, the homogeneity of the production function implies that the stock of a value maximizing firm is equal to the aggregate capital stock, $k(s^t) = q^1(s^t)$, as shown by Lemma 3.1. Given this, the aggregate investment plan in the $e$-economy has to satisfy the following Euler condition for some $\bar{x} \in Q_{e}(q, d)$:

$$k(s^t) = \sum_{s^t+1|s^t} X_{t}^{s^t+1} \left[ d^1(s^{t+1}) + q^1(s^{t+1}) \right] = q^1(s^t) \tag{20}$$

In essence, the results of section three can be seen as the equivalence of equations (19) and (20). In this case, a value maximizing firm in the $e$-economy will choose the same capital stock as if households were making the inter-temporal investment decision in the $k$-economy, and viceversa. To see that our assumptions guarantee that this is the case, consider the process $m \in Q_{s^e}(q, d)$, where $m_{t}^{s^{t+1}}$ is defined by the marginal rate of substitution of the particular household that is unconstrained at each node between $s^t$ and $s^{t+1}|s^t$, and which may or may not be the same household every period. First, the absence of price bubbles implies that we can substitute $\bar{x}$ with $m$ in equation (20). This is due to the fact that $q^1(s^t)$ is equal to its fundamental value $v_{m^t}(s^t, \lambda)$, which is uniquely defined for all $\lambda \in Q_{s^e}(q, d)$ and in particular for $m \in Q_{s^e}(q, d)$. Second, under constant returns to scale, Euler’s Theorem implies that $d^1(s^t) + q^1(s^t) = F_k(z(s^{t+1}), k(s^t), n(s^{t+1}))k(s^t)$. It therefore follows that equations equations (19) and (20) are equivalent. In turn, this also implies that both economies will have the same budget sets (Lemma 3.2) and the same equilibrium allocations (Theorems 3.1 and 3.2). On the other hand, since the assumptions needed are relatively standard in the macro-finance literature, our results can be applied to a wide class of economies. Two important extensions are briefly discussed in what follows.

Second, theorems 3.1-3.3 can be easily extended to economies with (i) external finance and (ii) heterogeneous firms. In the first case, we assume that firms in the two production settings can rise capital by issuing different assets. In the second case, we assume that firms differ in their productivity process. In addition, the $e$-economy firms can also differ in their discount factors as long as they belong to the set of consistent present value prices, which is common across firms. Using similar arguments to the ones in Proposition 3.1, price bubbles can be ruled out in these two cases. In the presence of external financing, we can show that the ex-dividend firm value in the $e$-economy, which is equal to the market value of the assets in its capital structure, is equal to the economy wide capital stock. In the economy with heterogeneous firms, we can show that each firm $j$ will set its investment level to its market value, $k^j(s^t) = q^j(s^t) = v_{\lambda^j}(s^t, \lambda^j)$ where $\lambda^j \in Q_{s^e}(q, d)$. Given this, the results of Lemma 3.2 and Theorems 3.1-3.3 follow through in both cases.\footnote{The proofs of these results can be found in the technical appendix accompanying the paper, which can be provided by the authors upon request.}

Third, in spite of the fact that a value maximizing firm in the $e$-economy chooses the same production plan as if households were making the intertemporal investment decision, this does not necessarily imply that the plan is unanimously approved by the shareholders that belong to the control group $I^c(s^t) \subseteq I$ of the firm. In particular, a plan that is chosen by a value maximizing firm will only be approved by the shareholders if the discount factor $\lambda$ coincides with the future cash flow valuation of the controllers of the firm. On the other hand, equation (20) implies that the valuation of the firm only coincides with the valuation of future profits of shareholders that are unconstrained every period. To see this, note that
the equation can be rewritten as:

$$k(s^t) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \bar{\lambda}^s_{s^{t+r}} d^t (s^{t+r}) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} m^s_{s^{t+r}} d^t (s^{t+r})$$  \tag{21}

The first equality represents that fundamental value of equity with respect to \( \bar{\lambda} \). Further, since this is uniquely defined for all consistent present value processes, the fact that \( m \in Q_{s^t} (q, d) \) implies that \( v_{s^t} (s^t, \bar{\lambda}) = v_{s^t} (s^t, m) \). Here, it is important to note that only the marginal rate of substitution of a shareholder that is unconstrained every period belongs to the set of consistent present value prices \( Q_{s^t} (q, d) \). Given this, only if the shareholders that belong to the control group of the firm are unconstrained, unanimity will obtain with respect to value maximization. As an example, this would happen if the controllers of the firm were subject to the present value constraint in (8), since it is effectively never binding.

This result relates to the unanimity findings in Carceles-Poveda and Coen-Pirani (2006), where the preferences of the shareholders with respect to investment alternatives are derived explicitly without postulating any objective for the firm. In particular, it is shown that, with a constant returns to scale technology, all the shareholders of a firm agree on setting \( k(s^t) = q(s^t) \) as long as the borrowing constraints are not binding. Clearly, this implies that a value maximizing investment plan maximizes the utility of the unconstrained shareholders. In contrast, a value maximizing equilibrium with shareholder agreement might not exist if portfolio restrictions are effectively binding.

Last, whereas value maximization is a natural objective, others are clearly possible. In the absence of value maximization, however, the equivalence of allocations result might not hold. To see this, consider a firm with the general objective function:

$$Max_{\{k, n\}} \sum_{r=0}^{\infty} \sum_{s^{t+r}} \lambda^s_{s^{t+r}} u_f (N_f (s^{t+r})),$$

where \( u_f \) is the period utility function and \( \lambda_f \) is the firm’s discount factor. It is easy to show that the optimality condition determining aggregate capital implies that:

$$k(s^t) = \sum_{r=1}^{\infty} \sum_{s^{t+r}|s^t} \lambda^s_{s^{t+r}} \frac{u_f (N_f (s^{t+r}))}{u_f (N_f (s^t))} N_f (s^{t+r})$$

This equation reflects that the aggregate capital stock \( k(s^t) \) under a general objective will not be equal to the value of the firm \( q^1 (s^t) \) unless \( u_f \) is linear and \( \lambda_f \in Q_{s^t} (q, d) \), corresponding to value maximization. Further, if \( k(s^t) \neq q^1 (s^t) \), Lemma 3.1 implies that the set of feasible allocations in the economy with infinitely lived firms is not the same as in the standard setting, potentially leading to different equilibrium allocations. In sum, in the absence of value maximization, otherwise similar incomplete market models might lead to different quantitative implications with respect to the ones established in the macroeconomic literature.\(^\text{13}\)

\(^\text{13}\) An example of such a case is the objective postulated by Grossmann and Hart (1979), assuming that firms discount profits with a weighted average of the marginal rates of substitution of the different shareholders. Since this discount factor does not belong to the set of consistent present value processes, the equilibrium allocations differ from the ones in the standard setting (see Carceles-Poveda (2007) for a quantitative evaluation of the effects of different firm objectives).
5. Summary

This paper characterizes the competitive equilibrium in a class of incomplete market economies where firms make the intertemporal investment decision and households are subject to general portfolio restrictions. In particular, it shows that there is a specific objective function for firms that yields the same equilibrium as the economies recently considered in the macroeconomic literature. Under such objective, capital ownership is irrelevant, in the sense that the equilibrium allocations are the same regardless of whether consumers or firms own the stock of physical capital in the economy. Thus, the paper extends to an incomplete markets setting the standard result about the irrelevance of capital ownership obtained under complete markets.

Specifically, it is shown that, if firms undertake the intertemporal investment decision to maximize their market value, defined as the present value of future profits using as a discount factor present value processes that do not allow for arbitrage opportunities, the equilibrium allocations are the same as in the standard setup where households make the investment decision and firms rent capital and labor to maximize profits on a period by period basis. The result is derived under constant returns to scale in production and expected discounted utility for the households. Since these two assumptions are relatively standard in the macrofinance literature, the result can be applied to a relatively large class of models that have been used in the literature for quantitative analysis.

We conclude by noting that the implications of models with incomplete markets and dynamic firms in the absence of value maximization might differ from the standard setup with static firms. A quantitative assessment of some of these implications is an important issue which we leave to future research.
APPENDIX

Proof of proposition 3.1

(a) We first show that, if there exists a state prices process \( \lambda \in Q_{s^t}(q, d) \) such that the present value of the aggregate labor income \( v_{\omega_n}(s^t, \lambda) \) is finite for every date-state \( s^t \), then \( q^t(s^r) = v_{d^t}(s^r, \lambda) \) for all \( s^r|s^t \) with \( r \geq t \) and each security traded at date-state \( s^t \) that is either (i) of finite maturity or (ii) in positive net supply. To prove this, note that the preferences defined by \( \succeq \) and satisfying assumption (A.5) have the following property, which is labelled a sufficient degree of impatience in the general equilibrium literature. For each \( i \in I \), there exists a \( 0 \leq \gamma_i < 1 \) such that for any date state \( s^t \in N \),
\[ (c^t_i(s^t), c_i(s^t) + w_c(s^t), \gamma c^t_i(s^t)) > (c^t_i(s^t), c_i(s^t), c^t_i(s^t)) \] (1)
for all consumption plans satisfying \( c^t_i(s^t) \leq w_c(s^t) \) at each \( s^r \in N \) and all \( \gamma \geq \gamma_i \). Here, \( \succ_i \) denotes strict preference, \( c^t_i(s^t) \) denotes the consumption coordinates at all nodes other than the sub-tree nodes \( s^r \in N \) such that \( s^r|s^t \), and \( c^t_i(s^t) \) denotes the consumption coordinates at the nodes \( s^r \in N \) such that \( s^r|s^t \) and \( r > t \). Given this, if the plan \((c_i, a_i)\) is optimal at \( q \), we have that, for all \( s^t \):
\[ (1 - \gamma_i)q(s^t')a_i(s^t') \leq w_c(s^t) \] (2)

To see that equation (2) is true, suppose that \((1 - \gamma_i)q(s^t')a_i(s^t') > w_c(s^t)\) for some \( s^t \). House-\(hold \) \( i \) could then choose the alternative plan \((\tilde{c}_i, \tilde{a}_i)\):
\[ (\tilde{c}_i(s^t), \tilde{c}_i(s^t), \tilde{c}^t_i(s^t)) = (c^t_i(s^t), c_i(s^t) + w_c(s^t), \gamma c^t_i(s^t)) \]
\[ (\tilde{a}_i(s^t), \tilde{a}_i(s^t), \tilde{a}^t_i(s^t)) = (a^t_i(s^t), \gamma_i a_i(s^t), \gamma_i a^t_i(s^t)) \]
which is feasible and would be preferred to \((c_i, a_i)\) by equation (1), contradicting the fact that \((c_i, a_i)\) is optimal. Given this, equation (2) must hold. Next, we show that, for all \( s^t \):
\[ n \sum_{r=1}^{\infty} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})c_i(s^{t+r}) \geq \sum_{r=1}^{\infty} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})w_i(s^{t+r}) + \lambda(s^t)q(s^t')a_i(s^t) \] (3)

To see that this is the case, we can multiply the date-state \( s^t \) budget constraint of consumer \( i \), satisfied with equality for each date-state given our assumptions on preferences, with some \( \lambda \in Q_{s^t}(q, d) \) for which \( v_{\omega_n}(s^t, \lambda) < +\infty \). Further, summing over all date-states \( s^{t+r} \), with dates \( 1 \leq r \leq T \), we obtain:
\[ n \sum_{r=1}^{T} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})c_i(s^{t+r}) + \sum_{s^{t+r}|s^t} \lambda(s^{t+T})q(s^{t+T})a_i(s^{t+T}) \]
\[ = \sum_{r=1}^{T} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})w_i(s^{t+r}) + \lambda(s^t)q(s^t')a_i(s^t) \]
Substituting equation (2) and taking the limit of the previous equation as \( T \) goes to infinity, we have that:
\[ \lim_{T \to \infty} \sum_{r=1}^{T} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})c_i(s^{t+r}) + \lim_{T \to \infty} (1 - \gamma_i)^{-1} \sum_{s^{t+T}|s^t} \lambda(s^{t+T})w_c(s^{t+T}) \]
\[ \geq \lim_{T \to \infty} \sum_{r=1}^{T} \sum_{s^t|s^{t+r}} \lambda(s^{t+r})w_i(s^{t+r}) + \lambda(s^t)q(s^t')a_i(s^t) \]
Since \( v_{w_n}(s^t, \lambda) < +\infty \) by assumption, it follows that \( v_{w_n}(s^t, \lambda) < +\infty \) for all \( i \), and the right hand side of the previous equation has a finite limit equal to \( \lambda(s^t) v_{w_n}(s^t, \lambda) < +\infty \). Since \( w_n(s^t) + d(s^t)A = w_c(s^t) \) and \( v_{dA}(s^t, \lambda) \leq q(s^t)' A < +\infty \), \( v_{w_n}(s^t, \lambda) < +\infty \) also implies that \( v_{w_n}(s^t, \lambda) < +\infty \), and it follows that \( v_c(s^t, \lambda) < +\infty \) for all \( i \in I \). Given this, the first term on the left hand side of the previous equation also has a well defined and finite limit equal to \( \lambda(s^t) v_{c_i}(s^t, \lambda) < +\infty \). Finally, since \( v_{w_n}(s^t, \lambda) < +\infty \) and \( w_c(s^t+T) \geq 0 \) for all date-states \( s^t+T \mid s^t \), we have that \( \lim_{T \to -\infty} (1 - \gamma_T)^{-1} \sum_{s^t+T \mid s^t} \lambda(s^t+T)w_c(s^t+T) = 0 \), which establishes the inequality in equation (3). Summing the inequality over households, we obtain:

\[
\sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda(s^{t+r})c(s^{t+r}) \geq \sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda(s^{t+r})w_n(s^{t+r}) + \lambda(s^t)q(s^t)' A
\]

Finally, substituting for \( c(s^t) = w(s^t)n(s^t) + d(s^t)A \), we have that:

\[
\sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda(s^{t+r})d(s^{t+r})A \geq q(s^t)' A
\]

On the other hand, the fact that \( v_{d^t}(s^t, \lambda) \leq q^t(s^t) \) for all \( l \in L \) implies that

\[
\sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda(s^{t+r})d(s^{t+r})A \leq q(s^t)' A
\]

Therefore, \( \sigma'(s^t, \lambda)A = 0 \), where \( \sigma'(s^t, \lambda) = (\sigma^1(s^t, \lambda), \ldots, \sigma^L(s^t, \lambda))' \), and \( \sigma'(s^t, \lambda) = 0 \) if \( A^t \in \mathbb{R}_{++} \). Note that, for finite maturity securities, the result directly follows from their definition.

(b) We now show that \( v_{w_n}(s^t, \lambda) < +\infty \) and \( \lim_{r \to -\infty} \sum_{s^t \mid s^t} \lambda_r(s^{t+r}) = 0 \) for all \( s^t \) and all \( \lambda \in Q_{s^t}(q, d) \). To prove this, note that, in equilibrium,

\[
d^1(s^t) = F(z(s^t), k(s^t-1), n(s^t)) - w(s^t)n(s^t) - k(s^t) \geq 0
\]

Given that \( v_{d^t}(s^t, \lambda) \leq q^t(s^t) \) for all \( s^t \in N \) and all \( \lambda \in Q_{s^t}(q, d) \), we have that, for all \( \lambda \in Q_{s^t}(q, d) \):

\[
\sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda_r(s^{t+r})[F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - w(s^{t+r})n(s^{t+r}) - k(s^{t+r})] < +\infty
\]

Suppose now that \( v_{w_n}(s^t, \lambda) = +\infty \) for some \( \lambda \in Q_{s^t}(q, d) \). Since the previous inequality holds for every \( \lambda \in Q_{s^t}(q, d) \), this would imply that:

\[
\sum_{r=1}^{\infty} \sum_{s^t \mid s^t} \lambda_r(s^{t+r})[F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - k(s^{t+r})] = +\infty
\]

On the other hand, equation (7) implies that the equity dividends can be expressed as a fraction \( \phi(s^t) \) of output net of investment, i.e., \( d^1(s^t) = \phi(s^t)x(s^t) \), where \( x(s^t) = [F(z(s^t), k(s^t-1), n(s^t)) - k(s^t)] \). Let \( \phi = \inf_{s^t} \phi(s^t) > 0 \), where the last inequality follows from the fact that the productive dividend payments are bounded away from zero. Given this, we have that:

\[
d^1(s^t) \geq \phi[F(z(s^t), k(s^t-1), n(s^t)) - k(s^t)]
\]
implying that
\[
\phi \sum_{r=1}^{\infty} \sum_{s^{t+r} \mid s^t} \lambda^s [F(z(s^{t+r}), k(s^{t+r-1}), n(s^{t+r})) - k(s^{t+r})] \leq v_{d1}(s, \lambda) < +\infty
\]

which contradicts equation (9). Therefore, it follows that \( v_{w_n}(s^t, \lambda) < +\infty \) for all \( \lambda \in Q_d(q, d) \). Finally, to see that
\[
\lim_{n \to -\infty} \sum_{s^{t+r} \mid s^t} \lambda^s k(s^{t+r}) = 0
\]
for all \( \lambda \in Q_d(q, d) \), note that this directly follows from (a) in the \( k \)-economy. Further, to show that this is also the case in the \( e \)-economy, note that \( w(s^t)n(s^t) + d^1(s^t) + k(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) \).

Given this, we can use the same arguments as above to show that for some \( \phi > 0 \), we have that \( w(s^t)n(s^t) \geq \phi[F(z(s^t), k(s^{t-1}), n(s^t))] \).

This clearly implies that the first infinite sum in equation (9) is finite for every \( \lambda \in Q_d(q, d) \). Therefore, for the total sum to be finite, it must be the case that:
\[
\sum_{r=1}^{\infty} \sum_{s^{t+r} \mid s^t} \lambda^s [k(s^{t+r})] < +\infty
\]

implying that \( \lim_{r \to -\infty} \sum_{s^{t+r} \mid s^t} \lambda^s k(s^{t+r}) = 0 \) for every \( \lambda \in Q_d(q, d) \).

**Proof of lemma 3.2**

To prove the lemma, let \( \widehat{F}_i(s^t) \) and \( F_i(s^t) \) be the set of budget feasible allocations at \( s^t \) in the two production economies. Note first that \( \widehat{c}_i(s^t) \in \widehat{F}_i(s^t) \) if there exists a set of portfolio strategies \( \{ \widehat{k}_i, (\widehat{a}^i_l)_{l \geq 1} \}_{i \in I} \) such that, for all \( s^t \in N \) and all \( i \in I \):

\[
\widehat{c}_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{q}^i_l(s^t) \widehat{a}^i_l(s^t) \leq \widehat{\omega}_i(s^t)
\]

\[
\widehat{\omega}_i(s^{t+1}) = \widehat{\omega}_i(s^{t+1}) + [f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + 1 - \delta] \widehat{k}_i(s^t) + \sum_{l \geq 2} \widehat{R}_i(s^{t+1}) \widehat{a}^i_l(s^t)
\]

where we have substituted for the equilibrium values of \( \widehat{F}(s^{t+1}) = f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + 1 - \delta \).

Similarly, \( c_i(s^t) \in F_i(s^t) \) if there exists a set of portfolio strategies \( (a^i_l)_{l \geq 1} \) such that, for all \( s^t \) and all \( i \in I \):

\[
c_i(s^t) + q^1(s^t)a^i_1(s^t) + \sum_{l \geq 2} q^1(s^t)a^i_l(s^t) \leq \omega_i(s^t)
\]

\[
\omega_i(s^{t+1}) = w_i(s^{t+1}) + [f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta] q^1(s^t)a^i_1(s^t) + \sum_{l \geq 2} R_i(s^{t+1})a^i_l(s^t)
\]

where we have used homogeneity of the production function and the fact that \( q^1(s^t) = k(s^t) \) by lemma 3.1, implying that:

\[
q^1(s^{t+1}) + d^1(s^{t+1}) = (f_k(z(s^{t+1}), k(s^t), n(s^t)) + 1 - \delta) k(s^t)
\]

Let \( \widehat{c}_i(s^t) \in \widehat{F}_i(s^t) \) and assume that the hypothesis of the lemma are satisfied. We now show that a plan setting \( c_i(s^t) = \widehat{c}_i(s^t) \) at each node is feasible in the \( e \)-economy. To see this, consider
any date-state \( s^t \in N \). If \( \omega_i(s^t) = \widehat{\omega}_i(s^t) \), households can choose the portfolio \( a_i^l(s^t) = \widehat{a}_i^l(s^t) \) for \( l \geq 2 \) and \( q^1(s^t) a_i^1(s^t) = \widehat{k}_i(s^t) \), implying that:

\[
\widehat{c}_i(s^t) + q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) = \widehat{c}_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} q^l(s^t) \widehat{a}_i^l(s^t) = \widehat{\omega}_i(s^t) = \omega_i(s^t)
\]

\[
q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) = \widehat{k}_i(s^t) + \sum_{l \geq 2} q^l(s^t) \widehat{a}_i^l(s^t) \geq \widehat{B}_i(s^t) = B_i(s^t)
\]

Further, if household \( i \in I \) chooses this portfolio, his wealth at the beginning of next period will be equal to:

\[
\omega_i(s^{t+1}) = \widehat{\omega}_i(s^{t+1}) + [f_k(z(s^{t+1}), \widehat{k}(s^t), \widehat{n}(s^t)) + (1 - \delta)\widehat{k}_i(s^t) + \sum_{l \geq 2} R^l(s^{t+1}) \widehat{a}_i^l(s^t) = \widehat{\omega}_i(s^{t+1})
\]

where we have used the fact that \( w_i(s^{t+1}) = \widehat{w}_i(s^{t+1}) \) and \( n(s^t) = \widehat{n}(s^t) \). Therefore, \( c_i(s^{t+1}) = \widehat{c}_i(s^{t+1}) \) is also feasible in the \( \epsilon \)-economy at date-state \( s^{t+1}|s^t \). Finally, if the initial values are the same, implying that \( k(s^{-1}) a_i^1(s^{-1}) = \widehat{k}_i(s^{-1}) \), the period zero wealth of household \( i \in I \) in the \( \epsilon \)-economy is given by:

\[
\omega_i(s^0) = \sum_{l \geq 2} q^l(s^0) a_i^l(s^{-1}) + \omega_i(s^0) + [f_k(z(s^0), k(s^{-1}), n(s^0)) + (1 - \delta)k(s^{-1})] a_i^1(s^{-1})
\]

\[
= \sum_{l \geq 2} q^l(s^0) \widehat{a}_i^l(s^{-1}) + \widehat{w}_i(s^0) + [f_k(z(s^0), \widehat{k}(s^{-1}), \widehat{n}(s^0)) + (1 - \delta)\widehat{k}_i(s^{-1}) = \widehat{\omega}_i(s^0)
\]

Since \( \omega_i(s^0) = \widehat{\omega}_i(s^0) \), it follows that \( c_i(s^0) = \widehat{c}_i(s^0) \) is feasible, implying that \( \widehat{c}_i(s^t) \in F_i(s^t) \) at all \( s^t \in N \). Conversely, assume that \( c_i(s^t) \in F_i(s^t) \) and consider any date-state \( s^t \in N \). If \( \widehat{\omega}_i(s^t) = \omega_i(s^t) \), households in the \( \epsilon \)-economy can choose the portfolio \( \widehat{a}_i^l(s^t) = a_i^l(s^t) \) for \( l \geq 2 \) and \( \widehat{k}_i(s^t) = q^1(s^t) a_i^1(s^t) \), achieving the same consumption allocation as in the \( \epsilon \)-economy at date-state \( s^t \), since:

\[
c_i(s^t) + \widehat{k}_i(s^t) + \sum_{l \geq 2} q^l(s^t) \widehat{a}_i^l(s^t) = c_i(s^t) + q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} q^l(s^t) a_i^l(s^t) \leq \omega_i(s^t) = \widehat{\omega}_i(s^t)
\]

\[
\widehat{k}_i(s^t) + q^1(s^t) a_i^1(s^t) = q^1(s^t) a_i^1(s^t) + q^1(s^t) a_i^1(s^t) \geq B_i(s^t) = \widehat{B}_i(s^t)
\]

Further, since \( \widehat{w}_i(s^{t+1}) = w_i(s^{t+1}) \) and \( \widehat{n}(s^t) = n(s^t) \), this will lead to the same wealth next period, i.e.,

\[
\widehat{\omega}_i(s^{t+1}) = w_i(s^{t+1}) + [f_k(z(s^{t+1}), k(s^t), n(s^0)) + (1 - \delta)q^1(s^t) a_i^1(s^t) + \sum_{l \geq 2} R^l(s^{t+1}) a_i^l(s^t)
\]

Since \( \widehat{\omega}_i(s^{t+1}) = \omega_i(s^{t+1}) \), we again have that \( c_i(s^{t+1}) = \widehat{c}_i(s^{t+1}) \) is feasible in the \( \epsilon \)-economy at date state \( s^{t+1}|s^t \). Finally, since \( \widehat{\omega}_i(s^0) = \omega_i(s^0) \), it follows that \( c_i(s^0) = \widehat{c}_i(s^0) \) is feasible, and \( c_i(s^t) \in F_i(s^t) \) at all nodes.

**Proof of Theorem 3.1.**

Let \( \{(c_i, a_i)_{i \in I}, q, w, k\} \) be a VM CE for \( E_k = \{Z, (k_0, a_0, z_0, e_0), \Pi, d^a, B\} \). To show that \( \{(c_i, \widehat{k}_i, (a_i^l)_{l \geq 2}), (q^1)_{l \geq 2}, w, \widehat{\rho}\} \) with \( \widehat{\rho}(s^t) = R^1(s^t)/q^1(s^{t-1}) \) and \( \widehat{k}_i(s^t) = q^1(s^t) a_i^1(s^t) \) for all \( s^t \in N \) is a CE for \( E_k = \{Z, (k_0, a_0, z_0, e_0), \Pi, d^a, B\} \), note first that the aggregate capital in the \( \epsilon \)-economy is given by:

\[
\widehat{k}(s^t) = \sum_{i \in I} \widehat{k}_i(s^t) = \sum_{i \in I} q^1(s^t) a_i^1(s^t) = q^1(s^t) = k(s^t)
\]
where the last equality holds by lemma 3.1. Further, we have used the fact that $a_i^l$ generates market clearing in the $c$-economy. Given this, the two factor prices:

$$w(s^t) = f_n(z(s^t), k(s^{t-1}), n(s^t))$$

$$\hat{r}(s^t) = R^l(s^t)/q^l(s^{t-1}) = (d^l(s^t) + k(s^t))/k(s^{t-1}) = f_k(z(s^t), k(s^{t-1}), n(s^t)) + 1 - \delta$$

satisfy the firm’s optimality conditions in the $k$-economy, where we have substituted for the labor market clearing conditions and have again used lemma 3.1. Second, since $(k_0, a_0, v_0, \epsilon_0), \Pi, d^l, B$, $(q^l)_{t \geq 2}$ and $k$ are the same across the two production economies, lemma 3.2 implies that $F_i(s^t) = F_i(s^t)$ for all $i \in I$ and all $s^t \in N$. Thus, the fact that $c_i$ is optimal for each $i \in I$ in the $c$-economy implies that it is also optimal for each $i \in I$ in the $k$-economy. In addition, the portfolio strategies achieving this allocation, given by $\hat{a}_i^l(s^t) = a_i^l(s^t)$ for $l \geq 2$ and $\hat{k}_i(s^t) = q^i(s^t)a^i(s^t)$, are optimal.

To see that they satisfy the portfolio constraints, note that:

$$\sum_{l \geq 2} \hat{q}^i(s^t)\hat{a}_i^l(s^t) + \hat{k}_i(s^t) = \sum_{l \geq 2} q^i(s^t)a^i_l(s^t) + q^i(s^t)a^i(s^t) \geq B_i(s^t) = \hat{B}_i(s^t)$$

Finally, the fact that $(c_i, a_i)$ generates market clearing in the $c$-economy implies that the allocations still clear the markets in the $k$-economy. To see this, note that:

$$\sum_{i \in I} \hat{c}_i(s^t) = \sum_{i \in I} c_i(s^t) = F(z(s^t), k(s^{t-1}), n(s^t)) - k(s^t) + \sum_{l \geq 2} d^l(s^t)A_l = w_c(s^t)$$

$$\sum_{i \in I} \hat{a}_i^l(s^t) = \sum_{i \in I} a_i^l(s^t) = A^l = \hat{A}^l \text{ for } l \geq 2$$

$$\sum_{i \in I} \hat{k}_i(s^t) = \sum_{i \in I} q^i(s^t)a^i(s^t) = q^i(s^t) = k(s^t) = \hat{k}(s^t)$$

This establishes the result.

**Proof of Theorem 3.2**

Let $\left\{\left(\hat{c}_i, \hat{k}_i, (\hat{a}_i^l)_{l \geq 2}, \hat{\omega}, \hat{r}\right)\right\}_{i \in I}$ be a CE for $E_k = \left\{\gamma, (\hat{k}_0, \hat{a}_0, \gamma_0, \epsilon_0), \Pi, \hat{d}^l, \hat{B}\right\}$. We now show that $\left\{\left(\hat{c}_i, a_i^l, (\hat{a}_i^l)_{l \geq 2}, q^i, (\hat{q}^i)_{l \geq 2}, \hat{\omega}, \hat{r}\right)\right\}$ with $a_i^l(s^t) = \hat{k}_i(s^t)/\hat{k}(s^t)$ and $q^i(s^t) = \hat{k}(s^t)$ is a VE CE for $E_c = \left\{\gamma, (\hat{k}_0, \hat{a}_0, \gamma_0, \epsilon_0), \Pi, \hat{d}^l, \hat{B}\right\}$.

To prove this, note first that the absence of arbitrage implies that the aggregate capital stock in the $k$-economy satisfies the following condition:

$$\hat{k}(s^t) = \sum_{s^{t+1}|s^t} \hat{\lambda}_{s^{t+1}}^l \left[ f_k(z(s^{t+1}), \hat{k}(s^t), \hat{n}(s^t))\hat{k}(s^t) + (1 - \delta)\hat{k}(s^t) \right]$$

for some $\hat{\lambda} \in Q_{s^t}(\hat{q}, \hat{d})$, where we have substituted for:

$$\hat{R}^l(s^t) = \hat{r}(s^t)\hat{k}(s^{t-1}) = f_k(z(s^t), \hat{k}(s^{t-1}), \hat{n}(s^t))\hat{k}(s^{t-1}) + (1 - \delta)\hat{k}(s^{t-1})$$

Since $q^i(s^t) = \hat{q}^i(s^t)$ and $d^l(s^t) = \hat{d}^l(s^t)$ due to the fact that the aggregate capital stock is the same in the two economies, it follows that $Q_{s^t}(q, d) = Q_{s^t}(\hat{q}, \hat{d})$. Therefore, the following values of $k(s^t)$ and $w(s^t)$ satisfy the firm’s optimality conditions in the $c$-economy for some $\lambda \in Q_{s^t}(q, d)$:

$$k(s^t) = \hat{k}(s^t) = \sum_{s^{t+1}|s^t} \hat{\lambda}_{s^{t+1}}^l \left[ f_k(z(s^{t+1}), \hat{k}(s^t), \hat{n}(s^t))\hat{k}(s^t) + (1 - \delta)\hat{k}(s^t) \right]$$

21
\[ w(s^t) = \hat{w}(s^t) = f_n(z(s^t), \hat{k}(s^{t-1}), \hat{n}(s^t)) \]

Second, since \((\hat{k}_0, \hat{a}_0, z_0, \epsilon_0), \Pi, \hat{\delta}, \hat{\beta}, \hat{\alpha}, \hat{k}\) are the same in the two economies, lemma 3.2 implies that \(F_i(s^t) = \tilde{F}_i(s^t)\) for all \(i \in I\) and all \(s^t \in N\). Therefore, since \(\hat{a}_i\) is optimal for each \(i \in I\) in the \(k\)-economy, it is also optimal for each \(i \in I\) in the \(e\)-economy. In addition, this also implies that the portfolio strategies achieving this allocation \(a_i^t(s^t) = \hat{a}_i^t(s^t)\) for \(l \geq 2\) and \(a_i^t(s^t) = \hat{k}_i(s^t)/\hat{k}(s^t)\), implying that \(q^1(s^t)a_i^1(s^t) = \hat{k}_i(s^t)\), are optimal, and they also satisfy the portfolio constraints, since:

\[
\sum_{l \geq 2} q^l(s^t)a_i^t(s^t) + q^1(s^t)a_i^1(s^t) = \sum_{l \geq 2} \hat{q}^l(s^t)\hat{a}_i^l(s^t) + \hat{k}_i(s^t) \geq \tilde{B}_i(s^t) = B_i(s^t)
\]

Finally, the fact that \(\{\hat{c}_i, k_i, (\hat{a}_i)_{l \geq 2}\}\) generate market clearing in the \(k\)-economy implies that the allocations also clear the markets in the \(e\)-economy. To see this note that:

\[
\sum_{i \in I} c_i(s^t) = \sum_{i \in I} \hat{c}_i(s^t) = F(z(s^t), \hat{k}(s^{t-1}), \hat{n}(s^t)) - \hat{k}(s^t) + \sum_{l \geq 2} \hat{\delta}^l(s^t)A^l = \tilde{w}_c(s^t) = w_c(s^t)
\]

\[
\sum_{l \geq 2} a_i^l(s^t) = \sum_{i \in I} \hat{a}_i^l(s^t) = \hat{A}^l = A^l \text{ for } l \geq 2
\]

\[
\sum_{i \in I} a_i^1(s^t) = \sum_{i \in I} \hat{k}_i(s^t)/\hat{k}(s^t) = 1 = A^1
\]

This establishes the result. \(\blacksquare\)
REFERENCES

Aiyagari D. R. (1994): Uninsured Idiosyncratic Risk and Aggregate Saving, *The Quarterly Journal of Economics*, 109 (3), 659-684.

Angeletos. G. M., (2007): Uninsured Idiosyncratic Investment Risk and Aggregate Saving, *Review of Economic Dynamics* 10 (1).

Angeletos. G. M. and Calvet L.(2006): Idiosyncratic Production Risk, Growth and the Business Cycle, *NBER Working Paper* 9764, *Journal of Monetary Economics* 53:6.

Angeletos. G. M. and Calvet L. (2005), “Incomplete Market Dynamics in a Neoclassical Growth Economy”, *Journal of Mathematical Economics* (41), 407-438.

Bewley, T. F. (1986): Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers, in Hildebrand, W. and A. Mas-Collel (eds.), *Contributions of Mathematical Economics in Honor of Gerard Debreu*, Amsterdam: North-Holland, 79-102.

Bewley, T. F. (1977): The Permanent Income Hypothesis: A Theoretical Formulation, *Journal of Economic Theory*, 16 (2), 252-292.

Carceles-Poveda E. (2007): Asset Prices and Business Cycles under Market Incompleteness, Unpublished manuscript.

Carceles-Poveda E. and Coen-Pirani, D., (2006): Shareholders Unanimity with Incomplete Markets, Unpublished manuscript.

DeMarzo, P.M. (1993): Majority Voting and Corporate Control: The Rule of the Dominant Shareholder, *Review of Economic Studies*, 60, 713-734.

DeMarzo, P.M. (1988): An Extension of the Modigliani-Miller Theorem to Stochastic Economies with Incomplete Markets and Interdependent Securities, *Journal of Economic Theory*, 45, 353-369.

Diamond, P. A. (1967): The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty, *American Economic Review*, 57, 759-76.

Drèze, J. H. (1985): (Uncertainty and) the Firm in General Equilibrium Theory, *Economic Journal*, 95, 1-20.

Drèze, J. H. (1974): Investment under Private Ownership: Optimality, Equilibrium and Stability, in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by Drèze, J., NY, Macmillan.

Duffie, D., and W. Shaffer (1986b), Equilibrium and the Role of the Firm in Incomplete Markets, Unpublished manuscript.

Grossmann, S. J. and O. D. Hart (1979): A Theory of Competitive Equilibrium in Stock Market Economies, *Econometrica*, 47 (2).

Grossmann, S. J., and J., E. Stiglitz (1980): Stockholder Unanimity in Making Production and Financial Decisions, *The Quarterly Journal of Economics*, 94 (3), 543-566.

Grossmann, S. J., and J., E. Stiglitz (1977), On Value Maximization and Alternative Objectives for the Firm, *The Journal of Finance*, 32 (2), 389-402.

Hernandez, A., and M. Santos (1996), Competitive Equilibria for Infinite Horizon Economies, *Journal of Economic Theory*, 71, 102-130.

Huggett, M., (1997): The One-Sector Growth Model with Idiosyncratic Shocks: Steady States and Dynamics, *Journal of Monetary Economics*, 39, 385- 403.

Imrohoroglu, A., Imrohoroglu. S, and D. Joines (1999), Computational Models of Social Security, in R. Marimon and A. Scott, eds., *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, 221-237.

Krusell, P. and A. Smith (1998): Income and Wealth Heterogeneity in the Macroeconomy, *Journal of Political Economy*, 106 (5).

Krusell, P. and A. Smith (1997): Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns, *Macroeconomic Dynamics*, 1 (2), 387-422.

Leland, H., E. (1972): Theory of the Firm Facing Uncertain Demand, *The American Economic Review*, 279-291.
Levine, D., (1989): Infinite Horizon Equilibrium with Incomplete Markets, *Journal of Mathematical Economics*, 18, 357-376.

Levine, D., and W. Zame (1996): Debt Constraints and Equilibrium in Infinite Horizon Equilibrium with Incomplete Markets, *Journal of Mathematical Economics*, 26, 103-131.

Magill, M. and M. Quinzii (1994a): Incomplete Markets over an Infinite Horizon, Long Lived Securities and Speculative Bubbles, *Journal of Mathematical Economics*, 26, 133-170.

Magill, M. and M. Quinzii (1994b): Infinite Horizon Incomplete Markets, *Econometrica*, 62 (4), 853-888.

Radner, R. (1972a): New ideas in pure theory, Problems in the Theory of Markets under Uncertainty, *American Economic Review*, 454-460.

Sandmo, A. (1972): On the Theory of the Competitive Firm Under Price Uncertainty, *The American Economic Review*, 65-73.

Santos, M., and M. Woodford (1997): Rational Asset Pricing Bubbles, *Econometrica*, 65, (1), 19-57.

Sondermann, D. (1974): Temporary Competitive Equilibrium Under Uncertainty, in *Allocation under Uncertainty: Equilibrium and Optimality*, ed. by Drèze, J., NY, Macmillan.

Storesletten, K., Telmer, C., and A. Yaron (2004): Consumption and Risk Sharing over the Life Cycle, *Journal of Monetary Economics*, 51, 609-633.

Storesletten, K., Telmer, C., and A. Yaron (2001): The Welfare Costs of Business Cycles Revisited: Finite Lives and Cyclical Variation in Idiosyncratic Risk, *European Economic Review*, 45 (7), 1311-1339.

Storesletten, K., Telmer, C., and A. Yaron (2007): Asset Pricing with Idiosyncratic Risk and Overlapping Generations, *Review of Economic Dynamics*, forthcoming.