BEM solutions to exponentially variable coefficient Helmholtz equation of anisotropic media

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Abstract. Boundary value problems (BVPs) governed by a Helmholtz type equation for anisotropic exponentially graded media are solved using Boundary Element Method (BEM). The variable coefficient governing equation is transformed to a constant coefficient equation which is then transformed to a boundary integral equation. The results show the convergence, consistency, and accuracy of the BEM solutions.

1. Introduction
Several types of constant coefficient equations have been solved using BEM (see for examples [1, 2, 3, 4]). But in general this is not the case for variable coefficient equation. There is some progress in using BEM to solve several types of variable coefficient governing equations (see for examples [5, 6, 7, 8, 9, 10, 11])

In this paper we will consider an exponentially variable coefficient Helmholtz equation of the form
\[
\frac{\partial}{\partial \xi_i} \left( \theta_{ij}(\xi_1, \xi_2) \frac{\partial \nu}{\partial \xi_j} \right) + \beta^2(\xi_1, \xi_2) \nu(\xi_1, \xi_2) = 0
\] (1)

where the coefficients \(\theta_{ij}\) and \(\beta^2\) depend on \(\xi_1\) and \(\xi_2\) and the repeated summation convention (summing from 1 to 2) is employed.

The matrix of coefficients \(\theta_{ij}\) is a real symmetric positive definite matrix so that equation (1) is a second order elliptic partial differential equation. Therefore equation (1) may be written explicitly as

\[
\frac{\partial}{\partial \xi_1} \left( \theta_{11} \frac{\partial \nu}{\partial \xi_1} \right) + 2 \frac{\partial}{\partial \xi_1} \left( \theta_{12} \frac{\partial \nu}{\partial \xi_2} \right) + \frac{\partial}{\partial \xi_2} \left( \theta_{22} \frac{\partial \nu}{\partial \xi_2} \right) + \beta^2 \nu = 0
\]

Further, the coefficients \(\theta_{ij}\) and \(\beta\) are required to be twice differentiable functions of the two independent variables \(\xi_1\) and \(\xi_2\). The analysis is specially relevant to an anisotropic medium but it equally applies to isotropic media. For isotropy, the coefficients in (1) take the form \(\theta_{11} = \theta_{22}\) and \(\theta_{12} = 0\) and use of these equations in the following analysis immediately yields the corresponding results for an isotropic medium.

Acoustic problems (when \(\beta^2 > 0\), and antiplane strain in elastostatics and plane thermostatic problems (when \(\beta^2 = 0\)) are among the areas for which (1) is usually used as the model especially for anisotropic inhomogeneous media. Works on Helmholtz equation have been done previously by Barucq et. al. in [12] and Loeffler et. al. in [13]. But the works are limited for isotropic and/or homogeneous media where the governing equation is a special case of the equation (1).
The technique of transforming (1) to constant coefficient equations will again be used for obtaining a boundary integral equation for the solution of (1).

2. The boundary value problem
Referred to a Cartesian frame \(O\xi_1\xi_2\) a solution to (1) is sought which is valid in a region \(\Omega\) in \(R^2\) with boundary \(\partial\Omega\) which consists of a finite number of piecewise smooth closed curves. On \(\partial\Omega_1\) the dependent variable \(\nu(\xi) (\xi = (\xi_1, \xi_2))\) is specified and on \(\partial\Omega_2\)

\[P(\xi) = \theta_{ij} (\partial \nu / \partial \xi_j) n_i\]  

(2)
is specified where \(\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2\) and \(n = (n_1, n_2)\) denotes the outward pointing normal to \(\partial\Omega\).

3. The boundary integral equation
The boundary integral equation is derived by transforming the variable coefficient equation (1) to a constant coefficient equation. The coefficients \(\theta_{ij}\) and \(\beta\) are required to take the form

\[\theta_{ij}(\xi) = \overline{\theta}_{ij} g(\xi)\]  

(3)

\[\beta^2(\xi) = \overline{\beta}^2 g(\xi)\]  

(4)

where the \(\overline{\theta}_{ij}\) and \(\overline{\beta}\) are constants and \(g\) is an exponential function of \(\xi\). Use of (3) and (4) and in (1) yields

\[\overline{\theta}_{ij} \frac{\partial}{\partial \xi_i} \left( g \frac{\partial \nu}{\partial \xi_j} \right) + \overline{\beta}^2 g \nu = 0\]  

(5)

Let

\[\nu(\xi) = g^{-1/2}(\xi) \psi(\xi)\]  

(6)

so that (5) may be written in the form

\[\overline{\theta}_{ij} \frac{\partial}{\partial \xi_i} \left[ g \left( g^{-1/2} \psi \right) \frac{\partial \psi}{\partial \xi_j} \right] + \overline{\beta}^2 g^{1/2} \psi = 0\]  

(7)

That is

\[\overline{\theta}_{ij} \left[ \left( \frac{1}{4} g^{-3/2} \partial g / \partial \xi_i \partial \xi_j - \frac{1}{2} g^{-1/2} \partial^2 g / \partial \xi_i \partial \xi_j \right) \psi + g^{1/2} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} \right] + \overline{\beta}^2 g^{1/2} \psi = 0\]  

(7)

Use of the identity

\[\frac{\partial^2 g^{1/2}}{\partial \xi_i \partial \xi_j} = -\frac{1}{4} g^{-3/2} \partial g / \partial \xi_i \partial \xi_j + \frac{1}{2} g^{-1/2} \frac{\partial^2 g}{\partial \xi_i \partial \xi_j}\]permits (7) to be written in the form

\[g^{1/2} \overline{\theta}_{ij} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} - \psi \overline{\theta}_{ij} \frac{\partial^2 g^{1/2}}{\partial \xi_i \partial \xi_j} + \overline{\beta}^2 g^{1/2} \psi = 0\]  

(8)

If we further restrict the function \(g(\xi)\) to take the exponential form

\[g(\xi) = \left[ A \exp(\alpha_m \xi_m) \right]^2 \]  

\[\overline{\theta}_{ij} \alpha_i \alpha_j = \overline{\beta}^2\]  

(9)

where \(\alpha_m\) are constant, then
Substitution (10) into (8) implies a constant coefficients equation

$$\bar{\theta}_{ij} \frac{\partial^2 \psi}{\partial \xi_i \partial \xi_j} = 0$$  \hspace{1cm} (11)

Also, substitution of (3) and (6) into (2) gives

$$P = -P_g \psi + P_\psi g^{1/2}$$  \hspace{1cm} (12)

where

$$P_g (\xi) = \bar{\theta}_{ij} \frac{\partial g^{1/2}}{\partial \xi_j} n_i \quad P_\psi (\xi) = \bar{\theta}_{ij} \frac{\partial \psi}{\partial \xi_j} n_i$$

A boundary integral equation for the solution of (11) is given in the form

$$\kappa (\xi_0) \psi (\xi_0) = \int_{\partial \Omega} [\Gamma (\xi, \xi_0) \psi (\xi) - \Phi (\xi, \xi_0) P_\psi (\xi)] \, ds (\xi)$$  \hspace{1cm} (13)

where \( \xi_0 = (a, b) \), \( \kappa = 0 \) if \( (a, b) \notin \Omega \cup \partial \Omega \), \( \kappa = 1 \) if \( (a, b) \in \Omega \), \( \kappa = \frac{1}{2} \) if \( (a, b) \in \partial \Omega \) and \( \partial \Omega \) has a continuously turning tangent at \((a, b)\).

The so called fundamental solution \( \Phi \) in (13) is any solution of the equation

$$\bar{\theta}_{ij} \frac{\partial^2 \Phi}{\partial \xi_i \partial \xi_j} = \delta (\xi - \xi_0)$$

and the \( \Gamma \) is given by

$$\Gamma (\xi, \xi_0) = \bar{\theta}_{ij} \frac{\partial \Phi (\xi, \xi_0)}{\partial \xi_j} n_i$$

where \( \delta \) is the Dirac delta function. Following Azis [14], for two-dimensional problems \( \Phi \) and \( \Gamma \) are given by

$$\Phi (\xi, \xi_0) = \frac{\mathcal{K}}{2\pi} \ln R \quad \Gamma (\xi, \xi_0) = \frac{\mathcal{K}}{2\pi} \frac{1}{R} \bar{\theta}_{ij} \frac{\partial R}{\partial \xi_j} n_i$$  \hspace{1cm} (14)

where

$$\begin{align*}
\mathcal{K} &= \frac{\dot{\tau}}{\zeta} \\
\zeta &= \left[ \bar{\theta}_{11} + 2 \bar{\theta}_{12} \dot{\tau} + \bar{\theta}_{22} \left( \dot{\tau}^2 + \ddot{\tau}^2 \right) \right] / 2 \\
R &= \sqrt{(\dot{\xi}_1 - \dot{a})^2 + (\ddot{\xi}_2 - \ddot{b})^2} \\
\dot{\xi}_1 &= \xi_1 + \dot{\tau} \xi_2 \\
\dot{a} &= a + \dot{\tau} b \\
\dot{\xi}_2 &= \ddot{\tau} \xi_2 \\
\dot{b} &= \ddot{\tau} b
\end{align*}$$

where \( \dot{\tau} \) and \( \ddot{\tau} \) are respectively the real and the positive imaginary parts of the complex root \( \tau \) of the quadratic

$$\bar{\theta}_{11} + 2 \bar{\theta}_{12} \tau + \bar{\theta}_{22} \tau^2 = 0$$
The derivatives $\partial R/\partial \xi_j$ needed for the calculation of the $\Gamma$ in (14) are given by

$$
\frac{\partial R}{\partial \xi_1} = \frac{1}{R} \left( \dot{\xi}_1 - \dot{\alpha} \right)
$$

$$
\frac{\partial R}{\partial \xi_2} = \ddot{\tau} \left[ \frac{1}{R} \left( \dot{\xi}_1 - \dot{\alpha} \right) \right] + \dddot{\tau} \left[ \frac{1}{R} \left( \dot{\xi}_2 - \dot{\beta} \right) \right]
$$

Use of (6) and (12) in (13) yields

$$
\kappa (\xi_0) \frac{g^{1/2} (\xi_0)}{2} (\xi_0) \nu (\xi_0) = \int_{\partial \Omega} \left\{ \left[ \frac{g^{1/2} (\xi)}{\Gamma (\xi, \xi_0) - P (\xi) \Phi (\xi, \xi_0)} \right] \nu (\xi)
- \left[ \frac{g^{-1/2} (\xi) \Phi (\xi, \xi_0)}{P (\xi)} \right] \right\} ds (\xi)
$$

This equation provides a boundary integral equation for determining $\nu$ and $P$ at all points of $\Omega$.

4. Numerical examples

In order to show the appropriateness of the BEM and the validity of the analysis used above for deriving the boundary integral equation (15), some particular boundary value problems will be solved. The integrals in equation (15) are evaluated numerically using the Bode’s quadrature (see Abramowitz and Stegun [15]).

4.1. Problem 1

The purpose of this example is to show the convergence and accuracy of the BEM. So we will consider a problem with analytical solution. The constant coefficients $\overline{\theta}_{ij}$ and $\overline{\beta}_2$ are taken to be

$$
\overline{\theta}_{11} = 1, \overline{\theta}_{12} = 0.5, \overline{\theta}_{22} = 1
$$

$$
\overline{\beta}^2 = 7.9375
$$

so that the inhomogeneity function $g(\xi)$ satisfying (9) is

$$
g(\xi) = [2 \exp (0.5 \xi_1 + 0.75 \xi_2)]^2
$$

Plot of $g(\xi)$ is shown in Figure 1. The analytical solution is

$$
\nu(\xi) = \frac{3.5 (1 + 1.5 \xi_1 + 2.5 \xi_2)}{2 \exp (0.5 \xi_1 + 0.75 \xi_2)}
$$

The geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 2.

The results are shown in Table 1. The BEM solution converges to the analytical solution as the number of segments increases. The accuracy of the BEM solutions lie on the fourth figure of the decimal.

4.2. Problem 2

We consider a problem for a homogeneous isotropic medium with $g(\xi) = 9, \overline{\theta}_{11} = \overline{\theta}_{22} = 1, \overline{\theta}_{12} = 0$ and $\overline{\beta} = 0$. The geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 3. As shown in Figure 3 the boundary conditions are symmetrical about the axes $\xi_1 = 0.5$. Table 2 shows the results of the BEM solution using 80, 160, 320 and 640 segments of equal length. As expected, the results converge as the number of segments increases and also they are symmetrical about the axes $\xi_1 = 0.5$. 

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Figure 1. An exponential inhomogeneity function $g(\xi) = [2 \exp (0.5\xi_1 + 0.75\xi_2)]^2$

Figure 2. The geometry of Problem 1

Figure 3. The geometry of Problem 2 and Problem 3
### Table 1. BEM and analytical solutions for Problem 1

| (\(\xi_1, \xi_2\)) | \(\nu\) | \(\partial\nu/\partial\xi_1\) | \(\partial\nu/\partial\xi_2\) | \(\nu\) | \(\partial\nu/\partial\xi_1\) | \(\partial\nu/\partial\xi_2\) |
|-------------------|---------|-----------------|-----------------|---------|-----------------|-----------------|
| \((.1,.5)\)       | 2.7465  | .3416           | .8050           | 2.7462  | .3425           | .8029           |
| \((.3,.5)\)       | 2.7956  | .1545           | .4956           | 2.7953  | .1550           | .4937           |
| \((.5,.5)\)       | 2.8105  | .0002           | .2375           | 2.8103  | .0002           | .2358           |
| \((.7,.5)\)       | 2.7618  | -.1258          | -.0233          | 2.7614  | -.2286          | -.1533          |
| \((.9,.5)\)       | 2.5269  | .6338           | 1.2742          | 2.5279  | .6331           | 1.2689          |
| \((.5,.1)\)       | 2.7203  | .2732           | .6857           | 2.7205  | .2727           | .6828           |
| \((.5,.3)\)       | 2.7227  | -.2020          | -.0987          | 2.8223  | -.2018          | -.0998          |
| \((.5,.7)\)       | 2.7769  | -.3477          | -.3460          | 2.7763  | -.3473          | -.3463          |

**BEM 160 segments**

**BEM 320 segments**

| \((.1,.5)\)       | 2.7460  | .3428           | .8019           | 2.7458  | .3432           | .8009           |
| \((.3,.5)\)       | 2.7952  | .1551           | .4927           | 2.7951  | .1553           | .4917           |
| \((.5,.5)\)       | 2.8102  | .0001           | .2350           | 2.8101  | .0000           | .2342           |
| \((.7,.5)\)       | 2.7971  | -.1268          | .0217           | 2.7970  | -.1271          | .0212           |
| \((.9,.5)\)       | 2.7611  | -.2294          | -.1533          | 2.7609  | -.2301          | -.1534          |
| \((.5,.1)\)       | 2.5284  | .6327           | 1.2667          | 2.5288  | .6322           | 1.2644          |
| \((.5,.3)\)       | 2.7206  | .2724           | .6815           | 2.7207  | .2721           | .6802           |
| \((.5,.7)\)       | 2.8220  | -.2017          | -.1003          | 2.8218  | -.2016          | -.1008          |
| \((.5,.9)\)       | 2.7760  | -.3471          | -.3467          | 2.7757  | -.3470          | -.3470          |

**BEM 640 segments**

**Analytical**

| \((.1,.5)\)       | 2.7460  | .3428           | .8019           | 2.7458  | .3432           | .8009           |
| \((.3,.5)\)       | 2.7952  | .1551           | .4927           | 2.7951  | .1553           | .4917           |
| \((.5,.5)\)       | 2.8102  | .0001           | .2350           | 2.8101  | .0000           | .2342           |
| \((.7,.5)\)       | 2.7971  | -.1268          | .0217           | 2.7970  | -.1271          | .0212           |
| \((.9,.5)\)       | 2.7611  | -.2294          | -.1533          | 2.7609  | -.2301          | -.1534          |
| \((.5,.1)\)       | 2.5284  | .6327           | 1.2667          | 2.5288  | .6322           | 1.2644          |
| \((.5,.3)\)       | 2.7206  | .2724           | .6815           | 2.7207  | .2721           | .6802           |
| \((.5,.7)\)       | 2.8220  | -.2017          | -.1003          | 2.8218  | -.2016          | -.1008          |
| \((.5,.9)\)       | 2.7760  | -.3471          | -.3467          | 2.7757  | -.3470          | -.3470          |

### Table 2. BEM solution for Problem 2

| (\(\xi_1, \xi_2\)) | \(\nu\) | \(\partial\nu/\partial\xi_1\) | \(\partial\nu/\partial\xi_2\) | \(\nu\) | \(\partial\nu/\partial\xi_1\) | \(\partial\nu/\partial\xi_2\) |
|-------------------|---------|-----------------|-----------------|---------|-----------------|-----------------|
| \((.1,.5)\)       | .5221   | .2153           | .9230           | .5222   | .2155           | .9222           |
| \((.3,.5)\)       | .5576   | .1288           | .8031           | .5578   | .1288           | .8020           |
| \((.5,.5)\)       | .5708   | -.0000          | .7607           | .5710   | .0000           | .7595           |
| \((.7,.5)\)       | .5576   | -.1288          | .8031           | .5578   | -.1288          | .8020           |
| \((.9,.5)\)       | .5221   | -.2154          | .9230           | .5222   | -.2155          | .9222           |
| \((.5,.1)\)       | .3472   | .0000           | .2866           | .3479   | .0000           | .2855           |
| \((.5,.3)\)       | .4353   | -.0000          | .5773           | .4357   | .0000           | .5761           |
| \((.5,.7)\)       | .7339   | .0000           | .0y8588         | .7339   | .0000           | .8575           |
| \((.5,.9)\)       | .9106   | .0000           | .9005           | .9103   | .0000           | .8990           |

**BEM 80 segments**

**BEM 160 segments**

| \((.1,.5)\)       | .5223   | .2156           | .9218           | .5224   | .2157           | .9217           |
| \((.3,.5)\)       | .5579   | .1289           | .8014           | .5579   | .1289           | .8011           |
| \((.5,.5)\)       | .5711   | -.0000          | .7589           | .5711   | -.0000          | .7586           |
| \((.7,.5)\)       | .5579   | -.1289          | .8014           | .5579   | -.1289          | .8011           |
| \((.9,.5)\)       | .5223   | -.2156          | .9218           | .5224   | -.2157          | .9217           |
| \((.5,.1)\)       | .3482   | .0000           | .2851           | .3484   | .0000           | .2849           |
| \((.5,.3)\)       | .4360   | -.0000          | .5754           | .4361   | .0000           | .5751           |
| \((.5,.7)\)       | .7338   | -.0000          | .8568           | .7338   | -.0000          | .8565           |
| \((.5,.9)\)       | .9101   | -.0000          | .8983           | .9100   | -.0000          | .8979           |

**BEM 320 segments**

**BEM 640 segments**
Table 3. BEM solution for Problem 3

| $(\xi_1, \xi_2)$ | $\nu$ | $\partial\nu/\partial\xi_1$ | $\partial\nu/\partial\xi_2$ | $\nu$ | $\partial\nu/\partial\xi_1$ | $\partial\nu/\partial\xi_2$ |
|-----------------|------|-----------------|-----------------|------|-----------------|-----------------|
| $(.1,.5)$        | .5865| .7426           | .8937           | .5868| .7426           | .8944           |
| $(.3,.5)$        | .6880| .2918           | .7437           | .6882| .2911           | .7424           |
| $(.5,.5)$        | .7097| -.0621          | .7072           | .7099| -.0623          | .7056           |
| $(.7,.5)$        | .6669| -.3593          | .7808           | .6671| -.3591          | .7793           |
| $(.9,.5)$        | .5681| .6237           | .9248           | .5682| .6242           | .9247           |
| $(.5,.1)$        | .4720| -.0866          | .3585           | .4728| -.0867          | .3570           |
| $(.5,.3)$        | .5738| .0814           | .6252           | .5742| .0815           | .6235           |
| $(.5,.7)$        | .8471| -.0381          | .6460           | .8470| -.0383          | .6445           |
| $(.5,.9)$        | .9603| -.0110          | .4673           | .9599| -.0117          | .4672           |
| $(.1,.5)$        | .5872| .7416           | .8938           | .5874| .7413           | .8936           |
| $(.3,.5)$        | .6885| .2905           | .7414           | .6885| .2902           | .7410           |
| $(.5,.5)$        | .7100| -.0625          | .7047           | .7100| -.0626          | .7043           |
| $(.7,.5)$        | .6672| -.3591          | .7785           | .6672| -.3591          | .7781           |
| $(.9,.5)$        | .5684| .6241           | .9243           | .5684| .6240           | .9240           |
| $(.5,.1)$        | .4733| -.0872          | .3564           | .4735| -.0872          | .3562           |
| $(.5,.3)$        | .5745| .0819           | .6226           | .5747| .0819           | .6222           |
| $(.5,.7)$        | .8469| -.0385          | .6435           | .8469| -.0386          | .6431           |
| $(.5,.9)$        | .9597| -.0123          | .4665           | .9596| -.0123          | .4661           |

4.3. Problem 3

Now we consider a problem for an exponentially graded isotropic material with $g(\xi) = [2 \exp (0.5\xi_1 + 0.75\xi_2)]^2$, $\bar{\sigma}_{11} = \bar{\sigma}_{22} = 1$, $\bar{\sigma}_{12} = 0$ and $\beta^2 = 0.8125$. Again, the geometry of the region $\Omega$ and the boundary conditions are as depicted in Figure 3. Table 3 shows the results of the BEM solution using 80, 160, 320 and 640 segments of equal length. As expected, the results converge as the number of segments increases but they are not symmetrical anymore as the material is not homogeneous.

5. Conclusion

The Helmholtz type governing equation (1) is used for modelling physical problems such as acoustic problems (when $\beta^2 > 0$), and antiplane strain in elastostatics and plane thermostatic problems (when $\beta^2 = 0$). The boundary integral equation (15) is derived from this governing equation (1) and then from (15) a BEM is constructed for calculation of numerical solutions to the problems for anisotropic exponentially graded media. The results show that the BEM solution gives a convergence, consistency, and accuracy. Therefore the results also prove that the analysis used for deriving the boundary integral equation (15) is valid. Together with its ease in implementation, it may be concluded that BEM is a useful numerical method for solving such kind of problems.

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