Reliability modeling of gear system considering strength degradation

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Abstract: The traditional reliability models of gear system were usually developed without considering strength degradation of gear tooth. Although some analysis methods considered strength degradation, but they ignored the uncertainty of strength degradation that is due to the random load. In order to consider the statistically dependent failure between the gears and the uncertainty of gear strength in the strength degradation process, the reliability models of gear system are developed based on the theory of stress - strength interference. The failure relationships between teeth are expressed by the minimum order statistics. The uncertainty of the strength degradation is expressed by the distribution of material parameter. Using the Monte Carlo simulation, the proposed model is validated. Taking a planetary gear system for a case study, the system reliabilities with strength degradation are analyzed. The results show that the strength degradation has a little influence on the gear system reliability at the early stage and it has a great influence on the gear system reliability at the middle and late stages.

1. Introduction

Gear is critical component of the mechanical transmission equipment. Its reliability has a great influence on the life and property, for instance, wind turbine gearbox and aircraft engine\cite{1-2}.

Lots of investigations have been carried out on the reliability modeling and estimation of gear systems. In some reliability analyses, a gear is simplified as a tooth\cite{3-4}. Some studies consider a tooth as an independent failure component in the reliability analyses of gear systems\cite{5-6}. Other studies\cite{7-8} propose the reliability analysis methods of gear systems with statistically dependent failures between teeth, which are more reasonable than these methods considering the statistically independent failures of gears\cite{9-11}, but these methods are either too complicate to apply for the reliability analyses of gear systems, or take too much simulation time. With respect to the fatigue reliability analyses of gear systems, some reliability models and analysis methods were developed\cite{12-13}. They calculated the fatigue life and reliability based on the S-N curve with the certainty probability, but the uncertainty of the residual strength was ignored.

Based on the above studies, the reliability models of gear system considering the uncertainties of load and strength degradation are proposed. The theory of order statistics is used to express the failure relationships between teeth, which can obtain high calculation efficiency and simplify the complicate modeling process considering dependent failure of tooth-pairs. A strength degradation model is used to
consider the strength degradation of gear material in the reliability model. According to the total probability theorem, the reliability model of gear system considering the uncertainty of the strength degradation is developed based on the stress - strength interference theory.

2. Reliability of gear system

2.1 Reliability of a gear tooth

During the operation of gear-pairs, the tooth roots are subjected to the random bending stresses. If the bending stress can be described as a random variable, it will be able to be expressed by using a probability density function (pdf). Supposing \( f(\delta) \) denotes the pdf of tooth-root strength \( \delta \). The bending stress \( s \) can be calculated by the torque load \( T \). Let \( f(T) \) denotes the pdf of torque load \( T \). According to the stress – strength interference theory, the reliability of a tooth root under the stress action for one time can be calculated as

\[
R_T = \int_0^\infty f(T) \int_0^{\delta(T)} f(\delta) d\delta dT
\]

When a tooth is subjected to \( n \) times of torque load action under the variable amplitude torque load history \( T \), the reliability of the tooth equals to the probability of the tooth surviving under the maximum load of \( n \) torque load samples. Because the load history \( T \) is random variable, the maximum load of \( n \) torque load samples also is random variable. Its pdf and reliability can be expressed as Eq. (2) according to the theory of maximum order statistics.

\[
\begin{align*}
&\{ h^{(n)}(T) = n \cdot \left[ F(T) \right]^{n-1} f(T) \\
&R_T = \int_0^\infty h^{(n)}(T) \int_{s(T)}^{\delta} f(\delta) d\delta dT
\end{align*}
\]

Where \( F(T) \) is the cumulative distribution function (cdf) of the maximum extreme value of \( n \) torque load samples.

2.2 Reliability of a gear

The failure of a tooth will cause the failure of the gear. Therefore, a gear can be considered as a series system composed of its teeth. When a gear is subjected to one time of torque load action, the reliability of this gear equals to the probability of the tooth surviving under the maximum load of \( n \) torque load samples. Because of the randomness of minimum strength of its teeth, for a gear with \( z \) teeth, the pdf of minimum strength of its teeth equals to the pdf of minimum order statistics of its teeth strengths. The pdf can be expressed as

\[
g^{(z)}(\delta) = z \cdot [1 - F(\delta)]^{z-1} f(\delta)
\]

Then, using Eq. (3) to replace \( f(\delta) \) in Eq. (2), the reliability of a gear with \( z \) teeth under \( n \) times of torque load action can be expressed as

\[
R_{gear} = \int_0^\infty h^{(n)}(T) \int_{s(T)}^{\delta} g^{(z)}(\delta) d\delta dT
\]

2.3 Reliability of gear system

For a gear system composed of its gears, the failure of each gear is caused by the torque load action of input shaft. The failures of its gears should not be considered as statistical independence. For a gear system with \( m \) gears, the probability \( P[s(T)] \) that the minimum bending strength of each gear is greater than the bending stress under the torque load \( T \) can be expressed as

\[
P[s(T)] = \prod_{i=1}^{m} \int_{s(T)}^{\delta} g^{(z)}(\delta) d\delta
\]

Therefore, according to the load – strength interference theory, the reliability of a gear system with \( m \) gears can be expressed as

\[
R_{sys} = \int_0^\infty h^{(n)}(T) P[s(T)] dT = \int_0^\infty h^{(n)}(T) \prod_{i=1}^{m} \int_{s(T)}^{\delta} g^{(z)}(\delta) d\delta dT
\]

2.4 Reliability of gear system considering the strength degradation

The strength degradation rules of material include the power function rule and logarithm rule et al\cite{14-15}.
In this study, the residual strength of component (tooth-root) is expressed as
\[ \delta(n) = \delta_0 [1 - D(n)]^a \]  
(7)

Where \( \delta_0 \) is the initial strength, \( D(n) \) is the cumulative damage under the cycle load action, \( a \) is the material parameter.

The cumulative damage can be calculated according to the Palmgren - Miner linear damage equation as
\[ D = \sum_{j=1}^{k} \frac{n_j}{N_j} \]  
(8)

Where \( n_j \) and \( N_j \) are respectively the stress cycle number and the stress cycle number of failures under the stress level \( s_j \).

According to the S-N curve equation, Eq. (8) can be expressed as
\[ D = \sum_{j=1}^{k} \frac{n_j}{N_j} = \frac{n}{c} \sum_{j=1}^{k} s_m f(s) \Delta s \approx \frac{n}{c} \int_{s=0}^{+\infty} s^m f(s) ds \]  
(9)

Where \( m \) and \( C \) are the material parameters, \( n \) is the number of load cycles with respect to a gear.

Then, the residual strength Eq. (7) can be expressed as
\[ \delta(n) = \delta_0 [1 - \frac{n}{c} \int_{s=0}^{+\infty} s^m f(s) ds]^a \]  
(10)

For a group of specimens tested under different stress levels, the individual specimen has the same percentile in the fatigue life distributions of different stress levels. Therefore, for a same specimen, its damage under different stress levels can be calculated according to the same S-N curve. The parameters \( m \) and \( C \) are variables for the S-N curves with different reliabilities. From Eq. (11), it can be found that the relationship between \( m \) and \( \ln C \) is linear. The parameters \( m \) and \( \ln C \) of the Reference [18] P-S-N curves with different reliabilities are shown in Table 1.

\[ m_R \cdot \ln S + \ln N = \ln C_R \]  
(11)

| Table 1 Parameters \( m_R \) and \( \ln C_R \) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( R \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 0.95 |
| \( m_R \) | 19.14 | 18.83 | 18.58 | 18.35 | 18.10 | 17.82 | 17.47 | 17.47 | 16.99 | 16.17 |
| \( \ln C_R \) | 140.76 | 138.49 | 136.74 | 135.07 | 133.32 | 131.32 | 128.86 | 125.50 | 119.83 | 114.32 |

By the linear regression of the parameters of P-S-N curves (using the regression Eq. (12)), the fitting result is shown in Fig. 1. The goodness of fit is good.

\[ \ln C_R = A + B \cdot m_R \]  
(12)

\[ \ln C = 6.9779 + 6.9822 \cdot m_s \]

Fig. 1 Regression curve of the parameters \( m \) and \( \ln C \)

Then, for the S-N curves of material with any reliability can be expressed using the parameters \( m_R \) or \( \ln C_R \). If using \( f(m_R) \) denotes the pdf of \( m_R \), the reliability of any S-N curve can be determined as
\[ R_{SN} = 1 - \sum f(m_R) \Delta m_R \approx \int_{m_R}^{\infty} f(m_R) dm_R \]  

(13)

Where \( \Delta m_R \) is the discrete interval.

From Eq. (11), the distribution type of \( m_R \) is consistent with that of the fatigue lives. For the test of gear bending fatigue, the Normal distribution, Lognormal distribution and two-parameter Weibull distribution are usually used for the test of the life distribution[19]. The weighted least square method[20] is used for the distribution fitting of \( m_R \). The statistical parameters of the distribution of \( m_R \) are listed in Table 2. The goodness of fit with respect to the two-parameters Weibull distribution is better than others, which shows the good agreement with that of fatigue lives.

| Variable | Distribution | Mean or Scale parameter | Std or Shape parameter | R-square |
|----------|--------------|-------------------------|------------------------|----------|
| \( m_R \) | Normal       | 17.6757                 | 1.3545                 | 0.9447   |
|          | Log-normal   | 2.8708                  | 0.0792                 | 0.9309   |
|          | Weibull      | 18.1611                 | 15.9498                | 0.9768   |

Then, combining Eqs. (6), (10), (12) and (13), according to the total probability theorem, the reliability model of gear system considering strength degradation can be expressed as

\[ R_{sys} = \int_0^\infty f(m_R) \int_0^\infty h^{(n)}(T) \prod_{i=1}^m \int_{s_i(T)}^{\infty} g_i^{(2)}(\delta(\delta_0)) d\delta(\delta_0) dT dm_R \]  

(14)

3. Analysis and discussion of an example

Taking a planetary gear system for the case study, it is shown in Fig. 2, which contains three planetary gears. The parameters of its gears are listed in Table 3.

| Gear      | Tooth Numbers | Normal module(mm) | Pressure angle (degree) | Helix angle(degree) | Face width(mm) | Profile shift coefficient |
|-----------|---------------|-------------------|-------------------------|---------------------|-----------------|--------------------------|
| Sun       | 30            | 12                | 20                      | 15                  | 220             | -0.33                    |
| Planetary | 57            | 12                | 20                      | 15                  | 220             | 0.38                     |
| Ring      | 144           | 12                | 20                      | 15                  | 220             | -0.43                    |
| Gear-1    | 81            | 14                | 20                      | 15                  | 110             | 0.52                     |
| Gear-2    | 25            | 14                | 20                      | 15                  | 110             | 0.44                     |
| Gear-3    | 46            | 22                | 20                      | 15                  | 50              | 0.36                     |
| Gear-4    | 29            | 22                | 20                      | 15                  | 50              | -0.23                    |

Fig. 2 Planetary gear system

In order to validate the proposed reliability model (Eq. (14)), Monte Carlo simulation is used for the comparison with the proposed method. Because Monte Carlo simulation is inefficient, the gear system composed of the sun gear and the planetary gears is used to validate the proposed model. The procedure of the Monte Carlo simulation is shown in Fig. 3. In this case study, it is assumed that the initial strength of the gear-tooth root follows the Gaussian distribution, \( \delta_0 \sim N(1000,100^2) \) and the torque load of input shaft follows the Gaussian distribution, \( T \sim N(1150,115^2) \). P-S-N curves for this
study are from Table 1. The relationship between $m_R$ and $\ln C_R$ is shown in Fig. 1. It is assumed that the parameter $a$ of the degradation rule equals to 0.3. The reliabilities obtained from the proposed method and Monte Carlo simulation are shown in Fig. 4.

![Fig. 3 Procedure of Monte Carlo simulation](image)

![Fig. 4 Comparison between the proposed method and Monte Carlo simulation](image)
From Fig. 4, it can be seen that the reliabilities estimated by the proposed model are consistent with the results obtained by using the Monte Carlo simulation. The relative errors are very little.

The reliabilities of the planetary gear system with parameter $a=0.3$ are shown in Fig. 5.

From Fig. 5, the reliabilities considering the strength degradation are less than the results without considering the degradation strength. With the increase of the number of cycles, the residual strength and the reliability of the gear system both decrease gradually, which shows the consistence with fatigue physics.

When the strength degradation curve is deterministic (i.e. $m_R$ and $\ln C_R$ are deterministic), then, Eq. (14) is degenerated into Eq. (15). It is assumed that $m_R$ equals to 16.5, 17, 17.5 and 18 respectively, $\ln C_R$ is determined according to the regression function (shown in Fig. 1). The reliability estimated by Eq. (15) is shown in Fig. 6.

$$R_{sys} = \int_0^\infty h^{(n)}(T) \prod_{i=1}^m \int_{s_i(T)}^{\infty} g_i^{(2)}(d[\delta(\delta_0)]) d\delta(\delta_0) dT$$ (15)

From Fig. 6, it can be seen that different $m_R$ values have great influences on the gear system reliability. Different $S-N$ curves used in the strength degradation will result in different paths of strength degradation. The reliabilities obtained by using different $m_R$ values are either over-estimated or un-
der-estimated. With the increase of the number of cycles, the relative errors also increase. The reliability obtained by using the random \( m_R \) is between the reliabilities using the deterministic \( m_R \) value. Due to the randomness of load and strength, it is hard to determine the residual strength. Therefore, considering the strength degradation as randomness is more reasonable than using the deterministic parameter in the strength degradation process.

4. Conclusion

According to the total probability theorem and the theory of order statistics, the reliability models considering the uncertainty of strength degradation and load and statistically independent failure are developed. The reliability of the gear system estimated by the proposed model decreases with the decrease of residual strength, which shows the correct fatigue failure physics. Different \( m_R \) values have great influences on the gear system reliability. In unknown condition of the strength degradation parameters, using the proposed reliability models can effectively estimate the reliabilities of gear systems. The strength degradation has a great influence on the reliability of the gear system at the middle and late stages, which is relative to the early stage of life.

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