SOME TECHNIQUES IN DENSITY ESTIMATION

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Abstract. Density estimation is an interdisciplinary topic at the intersection of statistics, theoretical computer science and machine learning. We review some old and new techniques for bounding sample complexity of estimating densities of continuous distributions, focusing on the class of mixtures of Gaussians and its subclasses.

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1. Introduction

These days unsupervised learning is very popular due to the amount of available unlabeled data. The general goal in unsupervised learning is to find structure in the data. This ‘structure’ can be the clusters in the data, the principal components of the data, or the intrinsic dimension of the data, and so on. Distribution learning (also known as density estimation) is the task of explicitly estimating the distribution underlying the data, which can then be explored to find the desired structure, or to generate new data. We mention two examples.

The first example, taken from [17], is anomaly detection: interpreting X-ray images (mammograms) for detecting breast cancer. In this case the training data consists of normal (non-cancerous) images; a probability density function \( \mu : \mathbb{R}^d \rightarrow \mathbb{R} \) is learned from the data. When a new input \( x' \) is presented to the system, a high value for \( \mu(x') \) indicates a normal image, while a low value indicates a novel input, which might be characteristic of an abnormality; the patient is then referred to a clinician for further examination. The second example, taken from [16], is synthesis and sampling, or generative models: in many cases we would like to generate new examples that are similar to those in the training data, e.g., in media applications, where it can be expensive or boring for an artist to generate large volumes...
of content by hand. Given the training data, the algorithm estimates a probability density function \( \mu : \mathbb{R}^d \to \mathbb{R} \) that models the data, and then generates new examples according to this distribution. For example, video games can automatically generate (random but reasonable) textures for large objects or landscapes, rather than requiring an artist to manually colour each pixel.

For supervised learning (in particular, classification problems), there are by now a variety of mathematical tools to understand the hardness of the problem (VC-dimension, Rademacher complexity, covering numbers, margins, etc., see [1, 21]). We lack such a satisfactory mathematical understanding in the case of unsupervised learning (in particular, distribution learning): determining the sample complexity of learning with respect to a general class of distributions is an open problem (see [11, Open Problem 15.1]).

More specifically, distribution learning refers to the following task: given data generated from an unknown target probability distribution \( \mu \), find a distribution \( \hat{\mu} \) that is ‘close’ to \( \mu \). To define this problem more precisely, one needs to specify:

1. What is assumed about the target distribution? This question is more pertinent than ever in this era of large high-dimensional data sets. Typically one assumes the target belongs to some class of distributions, or it is close to some distribution in that class.
2. What does ‘close’ mean? There are various statistical measures of closeness, e.g., the Kullback-Leibler divergence, \( L_1 \) and \( L_2 \) distances (see [10, Chapter 5] for a discussion).
3. How is data sampled from the distribution? One usually assumes access to i.i.d. data, but in some settings other models such as Markov Chain-based sampling may be more appropriate.

Once the above questions are answered, we have a well defined problem, for which we can propose algorithms. Such an algorithm is evaluated using two metrics: (i) the sample complexity, i.e., the number of samples needed to guarantee a small error, and (ii) the computational complexity, or the running time of the algorithm.

In this survey, we assume the target class consists of mixtures of Gaussians in high dimensions, or is a subclass of this class. We focus on the \( L_1 \) distance as the measure of closeness, and we assume i.i.d. sampling. Our goal is to give bounds for the sample complexity for distribution learning, or density estimation. We shall use these two phrases interchangeably here; distribution learning is usually used in the computer science/machine learning community and is a broader term, whereas density estimation is usually used in the statistics literature (see [11, Section 2] for a discussion).

The reason for this choice is that recently this problem has attracted much attention again, and many results have been proved during the last few years (see [2, 3, 6, 13, 22]).

The literature on density estimation is vast and we have not tried to be comprehensive. We shall just review some techniques that have been particularly successful in proving rigorous bounds for sample complexity of learning mixtures of Gaussians. The reader is referred to [11] for a broader, recent survey. For a general, well written introduction to density estimation, read [10]. This survey is based on the papers [2, 3, 6, 22]; the reader is referred to the original papers for full proofs. Most of the material in Section 3 also appear in [10].
In Section 2, we set up our notation. In each subsequent section we present one technique and demonstrate it by showing a bound on the sample complexity of learning a particular class of distributions. Concluding remarks appear in Section 8.

2. The formal framework

A distribution learning method or density estimation method is an algorithm that takes as input an i.i.d. sample generated from a distribution \( g \), and outputs (a description) of a distribution \( \hat{g} \) as an estimation for \( g \). Furthermore, we assume that \( g \) belongs to some known class \( \mathcal{F} \) of distributions, but \( \hat{g} \) is not required to belong to \( \mathcal{F} \) (if it does, then the method is called a ‘proper’ learner).

We only consider continuous distributions in this survey, and so we identify a ‘probability distribution’ by its ‘probability density function.’ Let \( Z \) be an arbitrary set, and let \( f_1 \) and \( f_2 \) be two distributions defined over the Borel \( \sigma \)-algebra \( B \subseteq 2^Z \).

The total variation distance between \( f_1 \) and \( f_2 \) is defined by

\[
\|f_1 - f_2\|_{TV} = \sup_{B \in B} |f_1(B) - f_2(B)| = \sup_{B \in B} \left| \int_B f_1 - \int_B f_2 \right| = \frac{1}{2} \|f_1 - f_2\|_1,
\]

where \( \|f\|_1 := \int_Z |f(x)|dx \) is the \( L_1 \) norm of \( f \). In the following definitions, \( \mathcal{F} \) is a class of probability distributions, and \( g \) is an arbitrary distribution. The total variation distance and the \( L_1 \) distance are within constant factor of each other, and we generally do not worry about constants in this survey, so we will use them interchangeably, except when a confusion might occur.

We write \( X \sim g \) to denote the random variable (or random vector) \( X \) has distribution \( g \), and we write \( S \sim g^m \) to mean that \( S \) is an i.i.d. sample of size \( m \) generated from \( g \).

**Definition 1** (\( \varepsilon \)-approximation, \( \varepsilon \)-close). A distribution \( \hat{g} \) is an \( \varepsilon \)-approximation for \( g \), or is \( \varepsilon \)-close to \( g \), if \( \|\hat{g} - g\|_1 \leq \varepsilon \).

**Definition 2** (density estimation method, sample complexity). A density estimation method for \( \mathcal{F} \) has sample complexity \( m_\mathcal{F}(\varepsilon, \delta) \) if, for any distribution \( g \in \mathcal{F} \) and any \( \varepsilon, \delta \in (0, 1) \), given \( \varepsilon, \delta \), and an i.i.d. sample of size \( m_\mathcal{F}(\varepsilon, \delta) \) from \( g \), with probability at least \( 1 - \delta \) outputs an \( \varepsilon \)-approximation of \( g \).

In the machine learning literature, such a density estimation method for class \( \mathcal{F} \) is called a ‘PAC distribution learning method for \( \mathcal{F} \) in the realizable setting,’ or an ‘\( \mathcal{F} \)-leaner,’ with sample complexity \( m_\mathcal{F}(\varepsilon, \delta) \). We also say we can ‘learn class \( \mathcal{F} \) with \( m_\mathcal{F}(\varepsilon, \delta) \) samples’. Typically the dependence on \( \delta \) is logarithmic and hence non-crucial, and sometimes we may just say the sample complexity is \( m_\mathcal{F}(\varepsilon) \), ignoring its dependence on \( \delta \) (this means we take, e.g., \( \delta = 1/3 \)). Note that the sample complexity should not depend on the specific underlying probability distribution \( g \), but should uniformly hold for all \( g \in \mathcal{F} \). This uniform notion of learning is sometimes called minimax density estimation in the statistics literature.

Let

\[
\Delta_k := \left\{ (w_1, \ldots, w_k) : w_i \geq 0, \sum w_i = 1 \right\}
\]

denote the \( k \)-dimensional simplex.
Table 1. Results covered in this survey. The top 5 rows are upper 
bounds, and the bottom 5 rows are lower bounds.

| Distribution family | Bound on sample complexity | Reference | Section |
|---------------------|----------------------------|-----------|---------|
| \( \mathcal{G}_{d,1} \) | \( O(d^2/\varepsilon^2) \) | \[3\] | \[3\] |
| \( \mathcal{A}_{d,1} \) | \( O(d/\varepsilon^2) \) | \[3\] | \[3\] |
| \( \mathcal{G}_{1,k} = \mathcal{A}_{1,k} \) | \( \tilde{O}(k/\varepsilon^2) \) | \[6\] | \[6\] |
| \( \mathcal{G}_{d,k} \) | \( \tilde{O}(kd^2/\varepsilon^4) \) | \[3\] | \[5\] |
| \( \mathcal{A}_{d,k} \) | \( \tilde{O}(kd/\varepsilon^2) \) | \[2\] | \[6\] |
| \( \mathcal{G}_{d,1} \) | \( \Omega(d/\varepsilon^2) \) | \[22\] | \[7\] |
| \( \mathcal{A}_{d,1} \) | \( \Omega(d/\varepsilon^2) \) | \[22\] | \[7\] |
| \( \mathcal{G}_{1,k} = \mathcal{A}_{1,k} \) | \( \Omega(k/\varepsilon^2) \) | \[22\] | \[7\] |
| \( \mathcal{G}_{d,k} \) | \( \Omega(kd/\varepsilon^2) \) | \[22\] | \[7\] |
| \( \mathcal{A}_{d,k} \) | \( \Omega(kd/\varepsilon^2) \) | \[22\] | \[7\] |

Definition 3 \((k\text{-mix}(\mathcal{F}))\). The class of \(k\text{-mixtures}\) of \(\mathcal{F}\), written \(k\text{-mix}(\mathcal{F})\), is 
defined as
\[
k\text{-mix}(\mathcal{F}) := \left\{ \sum_{i=1}^{k} w_i f_i : (w_1, \ldots, w_k) \in \Delta_k, f_1, \ldots, f_k \in \mathcal{F} \right\}.
\]

A one-dimensional Gaussian random variable with mean \(\mu\) and variance \(\sigma^2\) is denoted by \(\mathcal{N}(\mu, \sigma^2)\). Let \(d\) be a positive number, denoting the dimension. A 
dimensional Gaussian with mean \(\mu \in \mathbb{R}^d\) and (positive semidefinite) covariance 
matrix \(\Sigma \in \mathbb{R}^{d \times d}\) is a probability distribution over \(\mathbb{R}^d\), denoted 
\(\mathcal{N}(\mu, \Sigma)\), with probability density function
\[
\mathcal{N}(\mu, \Sigma)(x) := \exp\left(- (x - \mu)^T \Sigma^{-1} (x - \mu) / 2 \right) / \sqrt{2\pi} \det \Sigma.
\]

A random variable with density \(\mathcal{N}(\mu, \Sigma)\) is denoted by \(\mathcal{N}(\mu, \Sigma)\). Let \(\mathcal{G}_{d,1}\) denote 
the class of \(d\)-dimensional Gaussian distributions, and let \(\mathcal{G}_{d,k}\) denote the class of 
k-mixtures of \(d\)-dimensional Gaussian distributions: \(\mathcal{G}_{d,k} := k\text{-mix}(\mathcal{G}_{d,1})\).

If \(\Sigma\) is a diagonal matrix, then \(\mathcal{N}(\mu, \Sigma)\) is called an \(\text{axis-aligned}\) Gaussian, since 
in this case the eigenspace of \(\Sigma\) coincide with the standard basis. Let \(\mathcal{A}_{d,1}\) denote 
the class of \(d\)-dimensional axis-aligned Gaussian distributions, and let \(\mathcal{A}_{d,k} := k\text{-mix}(\mathcal{A}_{d,1})\).

We demonstrate some of the techniques used in density estimation by proving 
the upper and lower bounds in Table 1. For proving the upper bounds, we provide 
a density estimation method. For the lower bounds, we show that \(\text{any} \) density 
estimation method for the corresponding class must use at least the given number 
of samples. Throughout the survey, \(\tilde{O}\) allows for logarithmic factors, and \([n] := \{1, \ldots, n\} \).
3. Sample complexity upper bounds via VC-dimension

The VC-dimension of a set system, first introduced by Vapnik and Chervonenkis, has applications in diverse areas such as graph theory, discrete geometry [20], and the theory of empirical processes [23], and is known to precisely capture the sample complexity of learning in the setting of binary classification [1]. In this section we show it can also be used to give upper bounds for sample complexity of density estimation. The methods of this section have been developed in [10], which is the first place where VC-dimension is used for bounding the sample complexity of density estimation. The main results of this section appear in [3].

In this section we show an upper bound of $O(d^2/\varepsilon^2)$ for learning $\mathcal{G}_{d,1}$, and an upper bound of $O(d/\varepsilon^2)$ for learning $\mathcal{A}_{d,1}$. The plan is to first connect the sample complexity of learning an arbitrary class $F$ to the VC-dimension of a class of a related set system, called the Yatracos class of $F$ (this is done in Theorem 10), and then provide upper bounds on VC-dimension of this set system. Let $Z$ be an arbitrary set, which will be the domain of our probability distributions.

**Definition 4 (VC-dimension).** Let $H$ be a family of subsets of a set $Z$. We say a set $A \subseteq Z$ is shattered if, for any $B \subseteq A$, there exists some $C \in H$ such that $A \cap C = B$. The VC-dimension of $H$, denoted $\text{VC-dim}(H)$, is the size of the largest shattered set.

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**Definition 5 ($A$-distance).** Let $A \subseteq 2^Z$, and let $p$ and $q$ be two probability distributions over $Z$. The $A$-distance between $p$ and $q$ is defined as

$$||p - q||_A := \sup_{A \in A} |p(A) - q(A)|.$$ 

**Definition 6 (empirical distribution).** Let $S = (x_i)_{i=1}^m$ be a sequence of members of $Z$. The empirical distribution corresponding to this sequence is defined by $\hat{p}_S(x) = \frac{1}{m} \sum_{i=1}^m 1\{x = x_i\}$.

The following lemma is a well known refinement of the uniform convergence theorem of [24], due to Talagrand [23]: we use the wording of [1, Theorem 4.9].

**Lemma 7** (uniform convergence theorem). Let $p$ be a probability distribution over $Z$. Let $A \subseteq 2^Z$ and let $v$ be the VC-dimension of $A$. Then, there exist universal positive constants $c_1, c_2, c_3$ such that

$$\Pr_{S \sim p^m} \{|p - \hat{p}_S|_A \geq \varepsilon\} \leq \exp(c_1 + c_2 v - c_3 m \varepsilon^2).$$

**Definition 8** (Yatracos class). For a class $F$ of functions from $Z$ to $\mathbb{R}$, the associated Yatracos class is the family of subsets of $Z$ defined as

$$\mathcal{Y}(F) := \{x \in Z : f_1(x) \geq f_2(x) \text{ for some } f_1, f_2 \in F\}.$$ 

Observe that if $f, g \in F$ then $||f - g||_{TV} = ||f - g||_{\mathcal{Y}(F)}$. To see this, let $A := \{x : f(x) \geq g(x)\} \in \mathcal{Y}(F)$, and observe that

$$||f - g||_{TV} = \frac{1}{2} ||f - g||_1 = \frac{1}{2} \int_A |f(x) - g(x)| dx = \int_A (f(x) - g(x)) dx = \int_A f(x) dx - \int_A g(x) dx = |f(A) - g(A)| \leq ||f - g||_{\mathcal{Y}(F)}.$$
The other direction, namely \(\|f - g\|_{TV} \geq \|f - g\|_{\mathcal{Y}(\mathcal{F})}\), follows from the definition of the total variation distance.

**Definition 9** (empirical Yatracos minimizer). Let \(\mathcal{F}\) be a class of distributions over domain \(Z\). The empirical Yatracos minimizer is a function \(L^\mathcal{F}: \bigcup_{m=1}^{\infty} X^m \to \mathcal{F}\) defined as

\[
L^\mathcal{F}(S) = \arg\min_{q \in \mathcal{F}} \|q - \hat{p}_S\|_{\mathcal{Y}(\mathcal{F})}.
\]

If the argmin is not unique, we may choose one arbitrarily.

**Theorem 10** (density estimation via empirical Yatracos minimizer). Let \(\mathcal{F}\) be a class of probability distributions, and let \(S \sim p^m\), where \(p \in \mathcal{F}\). Then, with probability at least \(1 - \delta\) we have

\[
\|p - L^\mathcal{F}(S)\|_{TV} \leq c \sqrt{\frac{v + \log \frac{1}{\delta}}{m}}
\]

where \(v\) is VC-dimension of \(\mathcal{Y}(\mathcal{F})\), and \(c\) is a universal constant.

We remark that variants of this result, without explicit dependence on the failure probability, is proved implicitly in [10] and also appears explicitly in [12, Lemma 6].

**Proof.** We have

\[
\|p - L^\mathcal{F}(S)\|_{TV} = \|L^\mathcal{F}(S) - p\|_{\mathcal{Y}(\mathcal{F})} \leq \|L^\mathcal{F}(S) - \hat{p}_S\|_{\mathcal{Y}(\mathcal{F})} + \|\hat{p}_S - p\|_{\mathcal{Y}(\mathcal{F})}
\]

\[
\leq 2\|p - \hat{p}_S\|_{\mathcal{Y}(\mathcal{F})} \leq c \sqrt{\frac{v + \log \frac{1}{\delta}}{m}}.
\]

The equality is because \(L^\mathcal{F}(S)\), \(p \in \mathcal{F}\). The first inequality is the triangle inequality. The second inequality is because \(L^\mathcal{F}(S)\) is the empirical minimizer of the \(\mathcal{Y}(\mathcal{F})\)-distance. The third inequality holds by Lemma 7 with probability \(\geq 1 - \delta\). \(\square\)

**Corollary 11.** For any class \(\mathcal{F}\), the sample complexity for learning \(\mathcal{F}\) is bounded by \(O((\text{VC-dim}(\mathcal{Y}(\mathcal{F}))) + \log(1/\delta)/\varepsilon^2)\).

In view of Corollary [11] to prove the sample complexity bounds for \(\mathcal{G}_{d,1}\) and \(\mathcal{A}_{d,1}\), it remains to show upper bounds on the VC-dimensions of the Yatracos classes \(\mathcal{Y}(\mathcal{G}_{d,1})\) and \(\mathcal{Y}(\mathcal{A}_{d,1})\). We provide the proof for general Gaussians only; the proof for axis-aligned Gaussians is very similar.

For classes \(\mathcal{F}\) and \(\mathcal{G}\) of functions from \(Z\) to \(\mathbb{R}\), let

\[
\text{NN}(\mathcal{G}) := \{\{x : f(x) \geq 0\} \text{ for some } f \in \mathcal{G}\} \subseteq 2^Z,
\]

and

\[
\Delta \mathcal{F} := \{f_1 - f_2 : f_1, f_2 \in \mathcal{F}\},
\]

and notice that

\[
\mathcal{Y}(\mathcal{F}) = \text{NN}(\Delta \mathcal{F}).
\]

We upper bound the VC-dimension of \(\text{NN}(\Delta \mathcal{G}_{d,1})\) via the following well known result in statistical learning theory, which first appeared in this form in [15] Theorem 7.2 (see [10] Lemma 4.2] for a historical discussion).

**Theorem 12** (VC-dimension of vector spaces). Let \(\mathcal{F}\) be an \(n\)-dimensional vector space of real-valued functions. Then \(\text{VC-dim}(\text{NN}(\mathcal{F})) = n\).
Now let $h$ be the indicator function for an arbitrary element in $\text{NN}(f_1 - f_2)$, where $f_1, f_2 \in \mathcal{G}_{d,1}$. Then $h$ is a $\{0, 1\}$-valued function and we have:

$$h(x) = \mathbb{1}\{\mathcal{N}(\mu_1, \Sigma_1)(x) > \mathcal{N}(\mu_2, \Sigma_2)(x)\}$$

$$= \mathbb{1}\{\alpha_1 \exp\left(-\frac{1}{2}(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1)\right) > \alpha_2 \exp\left(-\frac{1}{2}(x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2)\right)\}$$

$$= \mathbb{1}\{(x - \mu_1)^T \Sigma_1^{-1}(x - \mu_1) > (x - \mu_2)^T \Sigma_2^{-1}(x - \mu_2)\}.$$

The inner expression is a quadratic form, and the linear dimension of all quadratic functions is $O(d^2)$. Hence, by Theorem 12 we have $\text{VC-dim}(\mathcal{Y}(\mathcal{G}_{d,1})) = O(d^2)$. Combined with Corollary 11 this gives a sample complexity upper bound of $O(d^2/\varepsilon^2)$ for learning $\mathcal{G}_{d,1}$, which is the main result of this section. An $O(d/\varepsilon^2)$ upper bound for $\mathcal{A}_{d,1}$ can be proved similarly.

The problem with extending these results to mixtures of Gaussians is that it is not easy to bound the VC-dimension of the Yatracos class of the family of mixtures of Gaussians. It is an intriguing open problem whether $\text{VC-dim}(\mathcal{Y}(\mathcal{G}_{d,k})) = \tilde{O}(k \text{VC-dim}(\mathcal{Y}(\mathcal{G}_{d,1})))$. One can also ask a more ambitious question: is it true that for any class $\mathcal{F}$ of distributions, $\text{VC-dim}(\mathcal{Y}(k-\text{mix}(\mathcal{F}))) = O(k \text{VC-dim}(\mathcal{Y}(\mathcal{F})))$?

We believe the answer to this latter question is no, but this is yet to be disproved.

The answers to these questions are unknown, so new ideas are required for density estimation of mixtures, described next. First, in Section 4 we discuss the 1-dimensional case, and we discuss the high-dimensional case in Sections 5 and 6.

### 4. Sample complexity upper bounds via piecewise polynomials

In this section we give an $\tilde{O}(k/\varepsilon^2)$ upper bound for learning the class $\mathcal{G}_{1,k}$. This was proved in [6], which also gives a polynomial time algorithm for density estimation of this class. The main idea is to approximate a Gaussian with a piecewise polynomial function. For positive integers $D, t$, let $\mathcal{P}_{t,D}$ denote the class of density functions that are piecewise polynomials with at most $t$ pieces, where each piece is a polynomial of degree at most $D$. First, we give a sample complexity upper bound for learning $\mathcal{P}_{t,D}$ using the ideas from Section 3.

We need to bound $\text{VC-dim}(\mathcal{Y}(\mathcal{P}_{t,D})) = \text{VC-dim}(\mathcal{NN}(\Delta \mathcal{P}_{t,D}))$. Note that $\Delta \mathcal{P}_{t,D} = \mathcal{P}_{t,D}$. And since any $p \in \mathcal{P}_{t,D}$ has at most $tD$ roots and is continuous, any element in $\mathcal{NN}(\mathcal{P}_{t,D})$ is a union of at most $tD$ intervals. The VC-dimension of the class of unions of at most $k$ intervals can be easily seen to be $O(k)$. This gives $\text{VC-dim}(\mathcal{Y}(\mathcal{P}_{t,D})) = O(tD)$, hence by Corollary 11 the sample complexity of learning $\mathcal{P}_{t,D}$ is $O(tD/\varepsilon^2)$.

For any $g \in \mathcal{G}_{1,1}$ there exists $p \in \mathcal{P}_{3, O(\log(1/\varepsilon))}$ with $\|g - p\|_1 \leq \varepsilon$. (This is obtained by taking the Taylor polynomial for the main body of the Gaussian, and taking the zero polynomial for the two tails, see [6] for the details.) Also, $k-\text{mix}(\mathcal{P}_{t,D}) \subseteq \mathcal{P}_{kt,D}$. This implies that, for any $g \in \mathcal{G}_{1,k}$ there exists $p \in \mathcal{P}_{3k, O(\log(1/\varepsilon))}$ with $\|g - p\|_1 \leq \varepsilon$.

Let $g$ be the target distribution. Now, consider the empirical Yatracos minimizer (see Definition 9 for the class $\mathcal{P}_{3k, O(\log(1/\varepsilon))}$). Given samples from $g$, the minimizer ‘imagines’ the samples are coming from $p$, and after taking $O(k \log(1/\varepsilon)/\varepsilon^2)$ samples, outputs an estimate $\tilde{p}$ such that $\|\tilde{p} - p\|_1 \leq \varepsilon$. Then, the triangle inequality gives

$$\|\tilde{p} - g\|_1 \leq \|\tilde{p} - p\|_1 + \|\tilde{p} - g\| \leq 2\varepsilon,$$

as required.
There is an issue with the above argument; our proof for Theorem 1 assumed the samples are from a distribution in the known class of distributions \( P_{3k, O(\log(1/\varepsilon))} \) in this case, whereas in this case, they are not. However, one can amend the argument (by applying two careful triangle inequalities) to show that, if the samples are coming from a distribution \( g \) that is not necessarily in \( F \), then with high probability the empirical Yatrocas minimizer outputs a distribution \( L^F(S) \) satisfying:

\[
\|g - L^F(S)\|_{TV} \leq 3 \inf_{p \in F} \|p - g\|_{TV} + c \sqrt{\frac{\log \frac{1}{\delta} + \frac{1}{m}}{m}},
\]

which will be \( O(\varepsilon) \) in our case, as required (see [3] for the proof of (4.1)). Such a result is called agnostic learning, since it does not assume the target belongs to the known class, but only assumes it can be approximated well by some element of the class.

Unfortunately, the idea of piecewise polynomial approximation cannot be extended to higher dimensions, because to approximate a high-dimensional Gaussians, one needs a piecewise polynomial with either the degree or the number of pieces being exponential in the dimension. The ideas for extending the bounds to higher dimensions are quite different and are described next.

5. A generic upper bound for mixtures

We consider a more general problem in this section. Assume that we have a method to learn an arbitrary class \( F \). Does this mean that we can learn \( k\text{-mix}(F) \)? And if so, what is the sample complexity of this task? We give an affirmative answer to the first question, and provide a bound for sample complexity of learning \( k\text{-mix}(F) \). As an application of this general result, we give an upper bound for the case of mixtures of Gaussians in high-dimensions. This section is based on [3].

**Theorem 13** (sample complexity of learning mixtures). Assume that \( F \) can be learned with sample complexity \( m_{F}(\varepsilon, \delta) = \lambda(\varepsilon, \delta)/\varepsilon^\alpha \) for some \( \alpha \geq 1 \) and some function \( \lambda(\varepsilon, \delta) = \Omega(\log(1/\delta)) \). Then there exists a density estimation method for \( k\text{-mix}(F) \) requiring \( O\left( k \log k \cdot m_{F}(\varepsilon, \delta)/\varepsilon^2 \right) \) samples.

One may wonder about tightness of this theorem. In Theorem 2 in [22], it is shown that if \( F \) is the class of spherical Gaussians, we have \( m_{k\text{-mix}(F)}(\varepsilon, \delta) = \Omega(k m_{F}(\varepsilon, \delta)/k) \), therefore, the factor of \( k \) is necessary in general. However, it is not clear whether the additional factor of \( \log k/\varepsilon^2 \) in the theorem is tight.

If we apply this theorem to the class \( F = G_{d,1} \), which has sample complexity \( O(d^2/\varepsilon^2) \) as proved in Section 3 we immediately obtain an upper bound of \( O(kd^2/\varepsilon^4) \) for the sample complexity of learning \( G_{d,k} \), and a sample complexity upper bound of \( O(kd/\varepsilon^4) \) for \( A_{d,k} \).

We now give a sketch of the proof of Theorem 13. Suppose the target distribution is \( g = \sum_{i=1}^k w_i g_i \), where each \( g_i \in F \). The \( w_i \) are called the mixing weights, and the \( g_i \) are called the components. Consider a die with \( k \) faces, such that when you roll it, the \( i \)th face has probability \( w_i \) of coming. To generate a point according to \( g \), one can roll this die, and if the \( i \)th face comes, generate a point according to distribution \( g_i \). So, any i.i.d. sample generated from \( g \) can be coloured with \( k \) colours, such that almost a \( w_i \) fraction of points have colour \( i \), and the points with colour \( i \) are i.i.d. distributed as \( g_i \).
Now, if the colouring was given to the algorithm, there was a clear way to proceed: estimate each of the \( g_i \) using the \( F \)-learner, and estimate \( w_i \) by the proportion of points with colour \( i \), and then output the resulting mixture. The issue is that the colouring is not given to the algorithm. But, in principle, it can do an exhaustive search over all possible colourings, and ‘choose the best one.’

More precisely, the algorithm has two main steps. In the first step we generate a finite set of ‘candidate distributions,’ such that at least one of them is \( \varepsilon \)-close to \( g \) in \( L_1 \) distance. These candidates are of the form \( \sum_{i=1}^{k} \hat{w}_i \hat{G}_i \), where the \( \hat{G}_i \)’s are extracted from samples and are estimates for the real components \( G_i \), and the \( \hat{w}_i \)’s come from a fixed discretization of \( \Delta_k \), and are estimates for the real mixing weights \( w_i \). In the second step, we take lots of additional samples and use the following result to choose the best one among them, giving a distribution that is \( O(\varepsilon) \)-close to \( g \).

The following theorem provides an algorithm that chooses the almost-best one among a finite set of candidate distributions. It follows from [10, Theorem 6.3] and a standard Chernoff bound.

**Theorem 14** (handpicking from a finite set of candidates). Suppose we are given \( M \) candidate distributions \( f_1, \ldots, f_M \) and we have access to i.i.d. samples from an unknown distribution \( g \). Then there exists an algorithm that given the \( f_i \)’s and \( \varepsilon > 0 \), takes \( \log(3M^2/\delta)/2\varepsilon^2 \) samples from \( g \), and with probability \( \geq 1 - \delta \) outputs an index \( j \in [M] \) such that

\[
\|f_j - g\|_1 \leq 3 \min_{i \in [M]} \|f_i - g\|_1 + 4\varepsilon.
\]

We now analyze the sample complexity of our proposed algorithm. First consider the simpler case that all mixing weights are equal to \( 1/k \). To estimate \( g \) within distance \( \varepsilon \), it suffices to estimate each \( g_i \) within distance \( \varepsilon \). Therefore, we need \( Ckm_{\mathcal{F}}(\varepsilon, \delta/k) \) total data points from \( g \), with some large constant \( C \), so that we get \( \Omega(m_{\mathcal{F}}(\varepsilon, \delta/k)) \) samples from each \( g_i \) with probability \( \geq 1 - \delta \). For each fixed way of colouring these \( Ckm_{\mathcal{F}}(\varepsilon, \delta/k) \) data points with \( k \) colours, we provide the points of each colour to the \( F \)-learner, and get an estimate \( \hat{g}_i \), and then we add \( \sum_{i=1}^{k} \frac{1}{k} \hat{g}_i \) to the set of candidate distributions (recall that we have assumed the mixture weights are \( 1/k \)). Hence, the total number of candidate distributions is \( M = k^{Ckm_{\mathcal{F}}(\varepsilon, \delta/k)} = \exp(Ckm_{\mathcal{F}}(\varepsilon, \delta/k) \log k) \).

We now show that at least one of the candidate distributions is \( O(\varepsilon) \)-close to the target. Consider the colouring that assigns points to components correctly. Then the \( F \)-learner would provide us with \((\hat{g}_i)^k_{i=1} \) that each is \( \varepsilon \)-close to the corresponding \( g_i \) with probability \( \geq 1 - \delta/k \). So, by the union bound, they are simultaneously close, with probability \( \geq 1 - \delta \). Thus, when we apply the algorithm of Theorem 14 with probability \( \geq 1 - \delta \) it provides us with one of the candidate distributions that is \( 7\varepsilon \)-close to the target. The total sample complexity of the whole algorithm is thus

\[
O(km_{\mathcal{F}}(\varepsilon, \delta/k)) + O(\log M/\varepsilon^2) = O \left( k \log k \cdot m_{\mathcal{F}}(\varepsilon, \frac{\delta}{3k}) / \varepsilon^2 \right),
\]

as required.

The general case of arbitrary mixing weights brings two challenges: first, we do not know the weights, and so we also do an exhaustive search over a finite fine grid on the simplex \( \Delta_k \) to make sure that at least one of the candidate distributions
also gets the weights right; it turns out that this does not increase the sample complexity by more than a constant factor. The more important problem is that, for components with very small weight, we may not get enough samples if we take a total of $C_{km}(\varepsilon, \delta/k)$ samples from the mixture. The solution is to have different precision for different components: from small-weight components we will have fewer data points, so we will estimate them with larger error; this is compensated by the fact that their weight is small, so the effect of this error in the total estimation error can be controlled. Here is the place that, for the error controlling calculations to work out, we need the technical conditions in the theorem, namely that $m_{F}(\varepsilon, \delta) = \lambda(F, \delta)/\varepsilon^\alpha$ for some $\alpha \geq 1$ and that $\lambda(F, \delta) = \Omega(\log(1/\delta))$. See [3] for the details.

6. Sample complexity upper bounds via compression schemes

The method of previous section would give a sample complexity upper bound of $\tilde{O}(kd/\varepsilon^4)$ for learning $A_{d,k}$. In this section, which is based on [2], we improve this to $\tilde{O}(kd/\varepsilon^2)$ using a technique called ‘compression.’ As usual, let $F$ be a class of distributions over a domain $Z$.

**Definition 15 (distribution decoder).** A distribution decoder for $F$ is a deterministic function $J: \bigcup_{n=0}^{\infty} Z^n \times \bigcup_{n=0}^{\infty} \{0,1\}^n \rightarrow F$, which takes a finite sequence of elements of $Z$ and a finite sequence of bits, and outputs a member of $F$.

**Definition 16 (distribution compression scheme).** Let $\tau, t, m: (0, 1) \rightarrow \mathbb{Z}_{\geq 0}$ be functions. We say $F$ admits $(\tau, t, m)$-compression if there exists a decoder $J$ for $F$ such that for any distribution $g \in F$ the following holds:

For any $\varepsilon \in (0, 1)$, if $S \sim g_{m(\varepsilon)}$, then with probability at least $2/3$, there exists a sequence $L$ of at most $\tau(\varepsilon)$ elements of $S$, and a sequence $B$ of at most $t(\varepsilon)$ bits, such that $\|J(L, B) - g\|_1 \leq \varepsilon$.

Essentially, the definition asserts that with high probability, there should be a (small) subset of $S$ and some (small number of) additional bits, from which $g$ can be reconstructed, or ‘decoded.’ We say that the distribution $g$ is ‘encoded’ with $L$ and $B$, and in general we would like to have a compression scheme of a small size, for a reason that will be clarified soon.

**Remark 17.** In the above definition we required the probability of existence of $L$ and $B$ to be at least $2/3$, but one can boost this probability to $1 - \delta$ by generating a sample of size $m(\varepsilon) \log(1/\delta)$.

We next establish a connection between compression and learning, and also show some properties of compression schemes. The proofs can be found in [2].

**Lemma 18 (compression implies learning).** Suppose $F$ admits $(\tau, t, m)$-compression. Let $\tau'(\varepsilon) := \tau(\varepsilon/6) + t(\varepsilon/6)$. Then $F$ can be learned using

$$O\left(m\left(\frac{\varepsilon}{6}\right) \log \frac{1}{\delta} + \tau'(\varepsilon) \log\left(m\left(\frac{\varepsilon}{6}\right) \log(1/\delta) + \log(1/\delta)\right)\right) = \tilde{O}\left(m\left(\frac{\varepsilon}{6}\right) + \tau'(\varepsilon) \frac{\varepsilon}{\varepsilon^2}\right)$$

samples.

The proof resembles that for Theorem [13] perform an exhaustive search over all possibilities for the ‘defining sequences’ $L, B$ to generate some candidates; one of
Theorem 23. The class bound on the sample complexity of learning mixtures of axis-aligned Gaussians.

Proof. By Lemma 22, \( G_{1,1} \) admits \((O(1), 0, O(1/\varepsilon))\)-compression. By Lemma 19 the class \( A_{d,1} = G_{1,1}^d \) admits \((O(d), O(d), O((d \log d)/\varepsilon))\)-compression. Then by Lemma 20 the class \( k\text{-mix}(A_{d,1}) = A_{d,k} \) admits \((O(kd), O(kd+k \log (k/\varepsilon)), O((kd \log k \log d/\varepsilon))\)-compression. Applying Lemma 18 gives the theorem.

7. A LOWER BOUND VIA FANO’S INEQUALITY

In the previous sections we give several techniques for upper bounding the sample complexity of density estimation. This survey would feel incomplete if we do not discuss at least one technique for proving lower bounds. Note that each of our upper bounds holds uniformly over a class: the sample complexity does not depend...
on the specific distribution. Similarly, the lower bound we discuss in this section also holds for a class of distribution rather than for a specific distribution. Such a bound is called a minimax lower bound in the statistics literature, and a worst-case lower bound in the computer science literature.

In this section, which is based on [22], we give a sample complexity lower bound of $\Omega(\frac{kd}{\epsilon^2})$ for $A_{d,k}$, which will imply a similar lower bound for $G_{d,k}$, since $A_{d,k} \subseteq G_{d,k}$. That is, we show that any density estimation method that learns the class $A_{d,k}$ uniformly, in the sense of Definition 2, must have a sample complexity of $\Omega(\frac{kd}{\epsilon^2})$. This shows the density estimation method described in the previous section has optimal sample complexity, up to logarithmic factors.

We claim it suffices to give a lower bound of $\Omega(\frac{d}{\epsilon^2})$ for $A_{d,1}$. For, consider a mixture of axis-aligned Gaussians whose components are extremely far away, such that the total variation distance between any two components is very close to 1. To learn the mixture distribution, one needs to learn each component. But each data point can help in learning one of the $k$ components. Since for learning any of the components one needs $\Omega(\frac{d}{\epsilon^2})$ samples, one will need $\Omega(\frac{kd}{\epsilon^2})$ samples to learn the mixture. Some nontrivial work has to be done to make this intuitive argument rigorous, but we omit that, and focus on proving the lower bound of $\Omega(\frac{d}{\epsilon^2})$ for $A_{d,1}$.

We will need the definition of the Kullback-Leibler divergence (KL-divergence, also called the relative entropy) between two distributions.

**Definition 24 (Kullback-Leibler divergence).** Let $f$ and $g$ be densities over domain $Z$. Their KL-divergence is defined as

$$\text{KL}(f \parallel g) = \int_Z f(x) \log \frac{f(x)}{g(x)} \, dx.$$ 

The KL-divergence is a measure of closeness between distributions. It is always non-negative, and is zero if and only if the two distributions are equal almost everywhere. However, it is not a metric, since it is not symmetric, and it can be $+\infty$.

The proof of the following lemma, which is called the ‘generalized Fano’s inequality,’ uses Fano’s inequality from information theory ([7 Theorem 2.10.1]). It was first proved in [9, page 77]. We write here a slightly stronger version, which appears in [25, Lemma 3].

**Lemma 25 (generalized Fano’s inequality).** Suppose we have $M$ distributions $f_1, \ldots, f_M$ with

$$\text{KL}(f_i \parallel f_j) \leq \beta \text{ and } \|f_i - f_j\|_1 \geq \alpha \quad \forall i \neq j \in [M].$$

Consider any density estimation method that gets $n$ i.i.d. samples from some $f_i$, and outputs an estimate $\hat{f}$ (the method does not know $i$). For each $i$, define $e_i$ as follows: assume the method receives samples from $f_i$, and outputs $f$. Then $e_i := \mathbb{E}\|f_i - \hat{f}\|_1$. Then, we have

$$\max_i e_i \geq \alpha(\log M - n\beta + \log 2)/(2 \log M).$$

This immediately leads to the following sample complexity lower bound for learning a class $\mathcal{F}$. 

Corollary 26. Suppose for all small enough $\varepsilon > 0$ there exist $M$ densities $f_1, \ldots, f_M \in \mathcal{F}$ with

$$\text{KL}(f_i \parallel f_j) = O(\varepsilon^2) \quad \text{and} \quad \|f_i - f_j\|_1 = \Omega(\varepsilon) \quad \forall i \neq j \in [M].$$

Then the sample complexity of learning $\mathcal{F}$ is $\Omega((\log M)/\varepsilon^2)$.

Let $M := 2^{d/5}$. To prove a lower bound of $\Omega(d/\varepsilon^2)$ for $\mathcal{A}_{d,1}$, we will build $M$ densities $f_1, \ldots, f_M \in \mathcal{A}_{d,1}$ satisfying the conditions of the corollary.

By the Gilbert-Varshamov bound in coding theory, there exists $2^{d/5}$ elements in $\{0, 1\}^d$ such that any two of them differ in at least $d/5$ components. (To see this, note that the size of a Hamming ball of radius $d/5$ is

$$\sum_{j=0}^{d/5} \binom{d}{j} \leq \left(\frac{ed}{d/5}\right)^{d/5} = (5e)^{d/5} < 2^{d/5};$$

hence one can start from an empty set $S$, then add elements from $\{0, 1\}^d$ one by one to $S$, and delete the Hamming ball of each added element. So long as $S$ has less than $2^{d/5}$ elements, the number of deleted elements is less than $2^{d/5} \times 2^{d/5} = 2^d$, so there are still undeleted elements in $\{0, 1\}^d$, and one still can add more elements to $S$.) Call them $\mu_1, \ldots, \mu_M$, and let $f_i := \mathcal{N}(\mu_i \varepsilon / \sqrt{d}, I_d)$. The densities $f_1, \ldots, f_M$ are Gaussians with identity covariance matrix, with their means are chosen carefully at vertices of $d$-dimensional hypercube with side length $\varepsilon / \sqrt{d}$.

The KL-divergence between two general Gaussians $\mathcal{N}(\mu, \Sigma)$ and $\mathcal{N}(\mu', \Sigma')$ is given by (see, e.g., [14, Section 9])

$$\text{KL}(\mathcal{N}(\mu, \Sigma) \parallel \mathcal{N}(\mu', \Sigma')) = \frac{1}{2} \left( \text{Tr}(\Sigma^{-1} \Sigma' - I) + (\mu - \mu')^T \Sigma^{-1} (\mu - \mu') - \ln \left( \frac{\det(\Sigma')}{\det(\Sigma)} \right) \right),$$

thus, for $i, j \in [M]$ we get

$$\text{KL}(f_i \parallel f_j) = \text{KL}(\mathcal{N}(\mu_i \varepsilon / \sqrt{d}, I_d), \mathcal{N}(\mu_j \varepsilon / \sqrt{d}, I_d))$$

$$= \frac{1}{2} \left( ||\mu_i - \mu_j||_2^2 / d \right) \leq \frac{1}{2} \left( d \varepsilon^2 / d \right) = \varepsilon^2 / 2,$$

as required.

Fix distinct $i$ and $j$. We now lower bound the $L_1$ distance between $f_i$ and $f_j$. Let $X_i \sim f_i$ and $X_j \sim f_j$. Any two $\mu_i$ and $\mu_j$ differ in at least $d/5$ coordinates. Fix such $d/5$ coordinates, and, without loss of generality, assume that in $\mu_i$ these coordinates are 0, and they are 1 in $\mu_j$. If we project $X_i$ onto one such coordinate, we get an $\mathcal{N}(0, 1)$ random variable, so the sum over these coordinates of $X_i$ has distribution $\mathcal{N}(0, d/5)$. Similarly, if we project $X_j$ onto one such coordinate, we get an $\mathcal{N}(\varepsilon / \sqrt{d}, 1)$ random variable, so the sum over these coordinates of $X_j$ has distribution $\mathcal{N}(\varepsilon / \sqrt{d}, d/8)$. The total variation distance between $\mathcal{N}(0, d/8)$ and $\mathcal{N}(\varepsilon / \sqrt{d}, d/8)$ equals the total variation distance between $\mathcal{N}(0, 1)$ and $\mathcal{N}(\sqrt{d} \varepsilon, 1)$, which is $\Omega(\varepsilon)$ (see Lemma 27 below). Hence, the total variation distance between $f_i$ and $f_j$ is also $\Omega(\varepsilon)$, as required.

Lemma 27. Let $\varepsilon \in [0, 1]$. Then,

$$\|\mathcal{N}(0, 1) - \mathcal{N}(\varepsilon, 1)\|_1 \geq \varepsilon / 5.$$
Proof.

\[ \|N(0,1) - N(\varepsilon, 1)\|_1 = \frac{2}{\sqrt{2\pi}} \int_{\varepsilon/2}^{\infty} e^{-(x-\varepsilon)^2/2} - e^{-x^2/2} \, dx \]

\[ = \frac{2}{\sqrt{2\pi}} \int_{\varepsilon/2}^{\infty} e^{-x^2/2} \left( e^{-\varepsilon^2/2 + x\varepsilon} - 1 \right) \, dx \]

\[ \geq \frac{2}{\sqrt{2\pi}} \int_{\varepsilon/2}^{\infty} e^{-x^2/2} \left( -\varepsilon^2/2 + x\varepsilon \right) \, dx \]

\[ = \frac{-\varepsilon^2}{\sqrt{2\pi}} \int_{\varepsilon/2}^{\infty} e^{-x^2/2} \, dx + \frac{2\varepsilon}{\sqrt{2\pi}} \int_{\varepsilon/2}^{\infty} e^{-x^2/2} \, dx \]

\[ = -\varepsilon^2 \Pr[N(0,1) \geq \varepsilon/2] + \frac{2\varepsilon}{\sqrt{2\pi}} e^{-\varepsilon^2/2} \]

\[ = \varepsilon \left( \frac{2\varepsilon - \varepsilon^2}{\sqrt{2\pi}} - \varepsilon \Pr[N(0,1) \geq \varepsilon/2] \right) \geq \varepsilon/5, \]

since \( \Pr[N(0,1) \geq \varepsilon/2] \leq 1/2 \) and \( \varepsilon \in [0,1] \), completing the proof. \( \square \)

We mentioned here just one lower bound technique based on Fano’s inequality. There are at least two other methods for proving lower bounds, see \([25]\).

8. Concluding remarks

Characterizing the sample complexity of a class. An insight from supervised learning theory is that the sample complexity of learning a class (of concepts, functions, or distributions) must be proportional to the intrinsic dimension of the class divided by \( \varepsilon^2 \), where \( \varepsilon \) is the error tolerance. (We would expect this dimension to be equal the number of parameters needed to describe an object in that class using its ‘natural’ parametrization, or the ‘degrees of freedom’ of an object in the class.) One challenge in learning theory is to formally define this dimension. For the case of binary classification, the intrinsic dimension is captured by the VC-dimension of the concept class (see \([24,5]\)).

In the case of distribution learning/density estimation, a candidate for quantifying the dimension of a class, is the VC-dimension of its Yatrocas class. However, it is not hard to come up with examples where this VC-dimension is infinite while the class can be learned with finite samples. A second candidate is using covering numbers. This method also does not work; for instance, the class of Gaussians do not have a finite covering number, yet their sample complexity is finite.

We showed in Section 6 that if a class of distribution has a compression scheme of size \( D \), then its sample complexity is \( D/\varepsilon^2 \), up to logarithmic factors. Hence, the size of the smallest compression is one candidate for capturing the intrinsic dimension of a distribution class. As we have shown here, for mixtures of axis-aligned Gaussians, the smallest compression size captures the sample complexity up to logarithmic factors (and it is also equal to the number of parameters needed to describe a distribution in the class using the mean and the covariance matrix). It is an intriguing open question whether distribution learning and compressibility are equivalent (such a statement has recently been shown to be true in the setting of binary classification \([19]\)).

Robust density estimation, or agnostic learning. We have assumed that the target belongs to the prescribed class of distributions. In practice, this is rarely
the case. Fortunately, it turns out that all the methods we discussed here can be extended to the case where the target is not necessarily in the class, but is close (in total variation distance) to some member of this class, albeit with some loss in the approximation error. An example of a guarantee that can be given in this setting is Equation (4.1). We refer the reader to the relevant papers for details.

**Discrete distributions and covering numbers.** We have focused on continuous distributions in this survey. Learning discrete distributions is a whole other world and many exciting techniques have been developed for those classes. One important technique is finding an $\varepsilon$-cover for the target class and then applying Theorem 14. See [11] and the references therein.

**Kernel density estimation.** A very popular method for density estimation in practice is kernel density estimation (see, e.g., [10, Chapter 9]). The few proven convergence rate results for this method, require certain smoothness assumptions on the class of densities (e.g., [10, Theorem 9.5]). The class of Gaussians is not universally Lipschitz and do not satisfy these assumptions, so these results do not apply. Indeed we believe that any kernel-based method for learning a high dimensional Gaussian must have sample complexity exponential in the dimension, but this is yet to be proved.

**Variants of the problem.** There are many natural and important variants of the problems presented above. What if instead of the $L_1$ distance, we consider the KL-divergence, or the $L_2$ distance, as the measure of closeness? The $L_1$ results do not carry over immediately, and several new ideas are needed. Also, in practice the i.i.d. assumption is not realistic, and an interesting direction is to extend the above results to settings where some correlation among the input data points is possible.

**Computational complexity.** Designing efficient algorithms for distribution learning is crucial for practical applications; the sample complexity does not always capture the computational difficulty. While we have totally ignored computational issues in this survey for brevity, a whole complexity theory can be developed around the task of distribution learning: which classes are ‘easy’ to learn, and which ones are hard? We refer the reader to [13] and the references therein.

**Parameter estimation.** In parameter estimation, which has been greatly popular in the theoretical computer science community recently (see, e.g., [8, 4, 18]), the goal is to identify the ‘parameters’ of the target distribution, for example, the mixing weights and the parameters of the Gaussians, up to a desired accuracy. Parameter estimation is a more difficult problem than density estimation, and any algorithm for parameter estimation requires some separability assumptions for the target Gaussians, whereas for density estimation no such assumption is needed. E.g., consider the case that $k = 2$ and the two components are identical; then there is no way to learn their mixing weights.

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