Spin-Hamilton Operator, Graviton-Photon Coupling and an
Eigenvalue Problem

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Abstract We solve exactly the eigenvalue problem for a spin Hamilton operator describing graviton-photon coupling. Entanglement of the eigenstates are also studied. Other spin-coupled Hamilton operators involving spin-1 and spin-2 are also investigated and compared.

1 Introduction

We study the eigenvalue problem of a spin Hamilton operator given by photon-graviton coupling. Furthermore spin Hamilton operators for spin-$\frac{1}{2}$-spin-2-spin-$\frac{1}{2}$ coupling and spin-$\frac{1}{2}$-spin-1-spin-$\frac{1}{2}$ couplings are also discussed.

The photon $\gamma$ is a spin-1 particle assumed without rest mass [1] and is described by the traceless hermitian $3 \times 3$ spin matrices

\[
\begin{align*}
p_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & p_2 &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, & p_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\end{align*}
\]

with $x \leftrightarrow 1$, $y \leftrightarrow 2$, $z \leftrightarrow 3$. The eigenvalues of these matrices are +1, 0, −1. The normalized eigenvectors for $p_1$ are

\[
\begin{align*}
u_{1,1} &= \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, & u_{1,0} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, & u_{1,-1} &= \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.
\end{align*}
\]

The normalized eigenvectors for $p_2$ are

\[
\begin{align*}
u_{2,1} &= \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2}i \\ -1 \end{pmatrix}, & u_{2,0} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, & u_{2,-1} &= \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2}i \\ -1 \end{pmatrix}.
\end{align*}
\]
The normalized eigenvectors for $p_3$ is the standard basis denoted by $u_{3,1}$, $u_{3,0}$, $u_{3,-1}$. We have the well-known commutation relations $[p_1, p_2] = ip_3$, $[p_2, p_3] = ip_1$, $[p_3, p_1] = ip_2$. The graviton $g$ is a spin-2 particle assumed without rest mass [1] and described by the traceless hermitian $5 \times 5$ spin matrices

$$g_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & \sqrt{6}/2 & 0 & 0 \\
0 & \sqrt{6}/2 & 0 & \sqrt{6}/2 & 0 \\
0 & 0 & \sqrt{6}/2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
g_2 = i \begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & -\sqrt{6}/2 & 0 & 0 \\
0 & \sqrt{6}/2 & 0 & -\sqrt{6}/2 & 0 \\
0 & 0 & \sqrt{6}/2 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
g_3 = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -2 \\
\end{pmatrix}.$$

The eigenvalues of these matrices are $+2$, $+1$, $0$, $-1$, $-2$. The normalized eigenvectors for $g_1$ are

$$v_{1,-2} = \frac{1}{4} \begin{pmatrix} 1 \\ -2 \\ \sqrt{6} \\ -2 \\ 1 \end{pmatrix},
v_{1,2} = \frac{1}{4} \begin{pmatrix} 1 \\ 2 \\ \sqrt{6} \\ 2 \\ 1 \end{pmatrix},$$

$$v_{1,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix},
v_{1,1} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix},
v_{1,0} = \frac{\sqrt{3}}{\sqrt{8}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2/3} \\ 0 \\ 1 \end{pmatrix}.$$

The normalized eigenvectors for $g_2$ are

$$v_{2,-2} = \frac{1}{4} \begin{pmatrix} 1 \\ -2i \\ -\sqrt{6} \\ 2i \\ 1 \end{pmatrix},
v_{2,2} = \frac{1}{4} \begin{pmatrix} 1 \\ 2i \\ -\sqrt{6} \\ -2i \\ 1 \end{pmatrix},$$

$$v_{2,-1} = \frac{1}{2} \begin{pmatrix} 1 \\ -i \\ 0 \\ -i \\ -1 \end{pmatrix},
v_{2,1} = \frac{1}{2} \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ -1 \end{pmatrix},
v_{2,0} = \frac{\sqrt{3}}{\sqrt{8}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2/3} \\ 0 \\ 1 \end{pmatrix}.$$
The normalized eigenvectors for $g_3$ are the standard basis $e_j$ ($j = 1, \ldots, 5$) with $v_{3,2} = e_1$, $v_{3,1} = e_2$, $v_{3,0} = e_3$, $v_{3,-1} = e_4$, $v_{3,-2} = e_5$. We have the well-known commutation relations $[g_1, g_2] = ig_3$, $[g_2, g_3] = ig_1$, $[g_3, g_1] = ig_2$.

We investigate the eigenvalue problem for the Hamilton operator of the coupled photon-graviton spin system

$$\hat{K} \equiv \frac{\hat{H}}{\hbar \omega} = p_1 \otimes g_1 \otimes p_1 + p_2 \otimes g_2 \otimes p_2 + p_3 \otimes g_3 \otimes p_3$$

where $\otimes$ denotes the Kronecker product [2]. The Hamilton operator $\hat{K}$ acts in the Hilbert space $\mathbb{C}^{45}$. Note that $\hat{K}$ is a hermitian matrix and thus the eigenvalues are real. Since the trace of the matrices $p_1$, $p_2$, $p_3$, $g_1$, $g_2$, $g_3$ vanish, we find that $\text{tr}(\hat{K}) = 0$. Consequently the sum of the 45 eigenvalues is 0. We also study the entanglement of the the non-degenerate eigenvectors utilizing the Schmidt decomposition [3, 4, 5, 6, 7, 8].

2 Spectrum

To find an estimate for the eigenvalue we can utilize the inequality

$$|\lambda| \leq \max_{1 \leq j \leq n} \sum_{\ell=1}^{n} |a_{j\ell}|$$

which is valid for any eigenvalue of an $n \times n$ matrix $A$. For the Hamilton operator $\hat{K} = (k_{j\ell})$ we find

$$\max_{1 \leq j \leq 45} \sum_{\ell=1}^{45} |k_{j\ell}| = 4\sqrt{3}.$$ 

A numerical study to find the eigenvalues of the Hamilton operator $\hat{K}$ using the eigenvalue packages of R [9] and Octave [10] provides the hint that 0 (7 times degenerate), 1 and $-1$ (each 6 times degenerate) and 2 and $-2$ (each 6 times degenerate) are eigenvalues. Now we calculate symbolically the characteristic polynomial $\det(\hat{K} - \lambda I_{45})$. The eigenvalues given above can now be used to reduce the characteristic polynomial and we finally arrive at the following 45 eigenvalues ordered from smallest to largest

$$\lambda_{1,2,3} = -\frac{\sqrt{33} + 9}{\sqrt{2}}, \text{ (3 times)}$$
The eigenvalues are symmetric around 0. Only the eigenvalues $\sqrt{3}$ and $-\sqrt{3}$ are not degenerate. The normalized eigenvectors $w_j \ (j = 1, \ldots, 45)$ are pairwise orthogonal and thus form an orthonormal basis in the Hilbert space $\mathbb{C}^{45}$.

Owing to the degeneracies of most of the eigenvalues the Hamilton operator $K$ admits symmetries, i.e.

$$P^T \hat{K} P = \hat{K}$$

where $P$ is a $45 \times 45$ permutation matrix. One of them is the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \otimes I_5 \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $I_5$ is the $5 \times 5$ identity matrix and the first matrix in the Kronecker product is the $3 \times 3$ NOT-gate, i.e. the $3 \times 3$ matrix with all 1’s in the counter-diagonal and 0 otherwise. We have $U_{NOT} = U_{NOT}^*$ and $\hat{K} = P^* \hat{K} P$.

All symmetries of $\hat{K}$ are given by

$$U \begin{pmatrix} \bigoplus_{j=1}^{11} V_j \end{pmatrix} U^*$$

and $V_1, V_2, V_3$ and $V_4$ are unitary $6 \times 6$ matrices, $V_5, V_6, V_7, V_8$ are unitary $3 \times 3$ matrices, $V_9$ and $V_{10}$ are $1 \times 1$ and unitary and $V_{11}$ is a $7 \times 7$ unitary matrix, and
\[ U = \sum_{j=1}^{6} w_{\lambda_{4},j} e_j + \sum_{j=1}^{6} w_{\lambda_{14},j} e_{j+6} + \sum_{j=1}^{6} w_{\lambda_{27},j} e_{j+12} + \sum_{j=1}^{6} w_{\lambda_{37},j} e_{j+18} \]

\[ + \sum_{j=1}^{3} w_{\lambda_{1},j} e_{j+24} + \sum_{j=1}^{3} w_{\lambda_{11},j} e_{j+27} + \sum_{j=1}^{3} w_{\lambda_{33},j} e_{j+30} + \sum_{j=1}^{3} w_{\lambda_{43},j} e_{j+33} \]

\[ + w_{\lambda_{16}} e_{j+36} + w_{\lambda_{36}} e_{j+37} + \sum_{j=1}^{7} w_{\lambda_{20},j} e_{j+38} \]

where \( w_{\lambda_{k},j} \) are the elements of an orthonormal basis for each eigenvalue \( \lambda_k \) of \( \hat{K} \), and \{ \( e_1, \ldots, e_{45} \) \} denotes the standard basis in \( \mathbb{C}^{45} \).

### 3 Entanglement

The normalized eigenvectors of \( \hat{K} \) are elements of the Hilbert space \( \mathbb{C}^{45} \) and form an orthonormal basis in \( \mathbb{C}^{45} \). Now \( 45 = 9 \cdot 5 = 3 \cdot 5 \cdot 3 \). The normalized eigenvectors are pairwise orthogonal and form an orthonormal basis in \( \mathbb{C}^{45} \). We ask the question whether the eigenvectors can be written as the Kronecker product of vectors in \( \mathbb{C}^{9} \) and vectors in \( \mathbb{C}^{5} \). For the vector space \( \mathbb{C}^{9} \) we could ask again whether the vector can be written as a Kronecker product of two vectors in \( \mathbb{C}^{3} \).

Consider the eigenvectors belonging to the non-degenerate eigenvalues \( \pm \sqrt{3} \) (page 6). We find the Schmidt decompositions

\[
\sqrt{24(2 - \pm \sqrt{3})} w_{T, \pm \sqrt{3}}^T \]

\[
= P_{\text{GB}} \left( 1, 0, 0, 0, 0, 0, 0, 0, 1 \right) \otimes \left( 0, 1, 0, (\pm \sqrt{3} - 2)i, 0 \right) \\
+ P_{\text{GB}} \left( 0, 0, 1, 0, 0, 0, 1, 0, 0 \right) \otimes \left( 0, (2 - \pm \sqrt{3})i, 0, -1, 0 \right) \\
+ (1 - \pm \sqrt{3})(i + 1) P_{\text{GB}} \left( 0, 0, 0, 0, 1, 0, 0, 0, 0 \right) \otimes \left( -1, 0, 0, 0, 1 \right)
\]

(\( P_{\text{GB}} \) is the permutation matrix which rearranges the photon-photon-graviton state as a photon-graviton-photon state) which expresses the (photon pair) – (graviton) entanglement,

\[
\sqrt{24(2 - \pm \sqrt{3})} w_{T, \pm \sqrt{3}}^T \\
= \left( 0, (\pm \sqrt{3} - 1)(i + 1), 0 \right) \otimes \left( 0, 1, 0, 0, 0, 0, 0, 0, 0 \right)
\]

(\( \otimes \) denotes the Kronecker product.)
Eigenvectors for $\sqrt{3}$ and $-\sqrt{3}$. 
\[+ (1, 0, (2 - \pm \sqrt{3})i, 0) \otimes (0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0)\]
\[+ ((2 - \pm \sqrt{3})i, 0, 1) \otimes (0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0)\]

which expresses the (first photon) – (graviton - second photon) entanglement, and

\[
\sqrt{24(2 - \pm \sqrt{3})w^T_{\pm \sqrt{3}}}
\]
\[= (0, 1, 0, 0, 0, 0, 0, 0, 0, -1, 0) \otimes (1, 0, (2 - \pm \sqrt{3})i)\]
\[+ (0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0) \otimes ((\pm \sqrt{3} - 2)i, 0, -1)\]
\[+ (0, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0) \otimes (0, (\pm \sqrt{3} - 1)(1 + i), 0)\]

which expresses the (first photon - graviton) – (second photon) entanglement.

### 4 Other Spin Couplings

For completeness we also provide the eigenvalues for spin-$\frac{1}{2}$-spin-2-spin-$\frac{1}{2}$ and spin-$\frac{1}{2}$-spin-1-spin-$\frac{1}{2}$ couplings. The spin matrices $s_1$, $s_2$, $s_3$ for spin-$\frac{1}{2}$ are given by

\[
s_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

with the commutation relations $[s_1, s_2] = is_3$, $[s_2, s_3] = is_1$, $[s_3, s_1] = is_2$. We consider the spin-Hamilton operators

\[
\hat{K}_{ge} = \frac{\hat{H}_{ge}}{\hbar \omega} = s_1 \otimes g_1 \otimes s_1 + s_2 \otimes g_2 \otimes s_2 + s_3 \otimes g_3 \otimes s_3
\]

and

\[
\hat{K}_{pe} = \frac{\hat{H}_{pe}}{\hbar \omega} = s_1 \otimes p_1 \otimes s_1 + s_2 \otimes p_2 \otimes s_2 + s_3 \otimes p_3 \otimes s_3.
\]

The Hamilton operators $\hat{K}_{ge}$ for the spin-$\frac{1}{2}$-spin-2-spin-$\frac{1}{2}$ coupling is a hermitian $20 \times 20$ matrix with trace equal to 0. The eigenvalues are

\[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0\]

each with multiplicity 4. The Hamilton operators $\hat{K}_{pe}$ for the spin-$\frac{1}{2}$-spin-1-spin-$\frac{1}{2}$ coupling is a hermitian $12 \times 12$ matrix with trace equal to 0. The eigenvalues are

\[-\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}, 0\]
each with multiplicity 4. Thus we see that the eigenvalues of the Hamilton operator $\hat{K}_{pe}$ are also eigenvalues of $\hat{K}_{ge}$ with the same multiplicity and $\hat{K}_{ge}$ has two additional eigenvalues $-\sqrt{3}/2$ and $\sqrt{3}/2$ which are twice the eigenvalues $-\sqrt{3}/4$ and $\sqrt{3}/4$, respectively. The coupled photon-graviton Hamilton operator admits the non-degenerate eigenvalues $\sqrt{3}$ and $-\sqrt{3}$ which are twice the eigenvalues $\sqrt{3}/2$, $-\sqrt{3}/2$, respectively.

5 Conclusion

We considered a spin Hamilton for graviton-photon coupling. The eigenvectors of the associated Hamilton operator, for non-degenerate eigenvalues, yield pairwise entangled systems. Entanglement in the eigenspaces for non-degenerate eigenvalues has not been investigated. Also the dynamics of the entanglement under this Hamilton operator can still be investigated. The eigenvalues for two other related spin-coupled Hamilton operators have been provided and compared.

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