REGULARIZED DETERMINANTS OF THE LAPLACIAN FOR COFINITE KLEINIAN GROUPS WITH FINITE-DIMENSIONAL UNITARY REPRESENTATIONS

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ABSTRACT. For cofinite Kleinian groups (or equivalently, finite-volume three-dimensional hyperbolic orbifolds) with finite-dimensional unitary representations, we evaluate the regularized determinant of the Laplacian using W. Müller’s regularization. We give an explicit formula relating the determinant to the Selberg zeta-function.

1. INTRODUCTION

The regularized determinant of the Laplacian has been well studied on Riemann surfaces. In the case of compact Riemann surfaces, D’Hoker and Phong [DP86], and Sarnak [Sar87] related the regularized determinant to the Selberg zeta-function.

For non-cocompact cofinite Fuchsian groups (or equivalently, finite-area non-compact Riemann surfaces with elliptic fixed points) Venkov, Kalinin, and Faddeev [VKF73] defined a regularized determinant for the Laplacian $\Delta$ and related the determinant to the Selberg zeta-function. They regularized the trace of the resolvent kernel using the theory of Krein’s spectral shift function [Kre53, BK62, Yaf92].

Efrat [Efr88, Efr91] defined a regularized determinant for cofinite torsion-free Fuchsian groups with singular characters, and related it to the Selberg zeta-function. His regularization was essentially based on the Selberg trace formula. Efrat’s paper gave rise to an interesting question: Can the regularized determinant be defined cleanly in terms of general operator theory? In the compact case, the answer is yes. Here zeta-regularization is defined in terms of the heat kernel, which is of trace-class. In the non-compact case the heat kernel is not even Hilbert-Schmidt.

W. Müller [Mul98, Mul83, Mul87, Mul92] applied Krein’s theory to define a regularization of the determinant, a relative determinant $\det(H, H_0)$ for two self-adjoint operators $H, H_0$, satisfying $\text{tr} \left( e^{-Ht} - e^{-H_0t} \right) < \infty$.

Müller’s regularization can be used for elliptic operators on non-compact manifolds. In [Mul92], Müller evaluates his determinant for the case of the Laplacian for finite-area surfaces with hyperbolic ends (a class of surfaces that includes Riemann surfaces), and relates the determinant to Efrat’s regularization (and hence to the Selberg zeta-function).

In [Par05] J. Park studies a closely related problem. He studies eta-invariants of Dirac operators, and relates the regularized determinant of the Dirac Laplacian to the Selberg zeta-function for odd-dimensional hyperbolic manifolds with cusps. Park also uses the regularized determinant to extract information about Selberg zeta-function.
Regularized determinants have also been evaluated in the case of infinite volume Riemann surfaces, by Borthwick, Judge, and Perry [BJP].

In this paper, we evaluate Müller’s relative determinant of $\Delta$ for the case of finite-volume three-dimensional hyperbolic orbifolds with finite-dimensional unitary representations. Or in other words the Laplacian acting on the Hilbert space of $\chi$–automorphic ($\chi$ is a finite-dimensional unitary representation) functions on hyperbolic three-space. We relate the determinant to the Selberg zeta-function using the appropriate version of the Selberg trace formula (proved previously in [Fri05a, Fri05b]).

We remark that zeta-regularization of determinants has found application in quantum field theory, in the works of Dowker and Critchley [DC76], Hawking [Haw77], Elizalde et al. [EOR+94], and Bytsenko, Cognola and Zerbini [BCZ97].

**Main Results.** Next we define some of the basic notions needed to state our main results. A Kleinian group is a discrete subgroup of $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \pm I$. Each element of $\text{PSL}(2, \mathbb{C})$ is identified with a Möbius transformation, and has a well-known action on hyperbolic three-space $\mathbb{H}^3$ and on its boundary at infinity —the Riemann sphere $\mathbb{P}^1$ (see [EGM98, Section 1.1]). A Kleinian group is cofinite iff it has a fundamental domain $F \subseteq \mathbb{H}^3$ of finite hyperbolic volume.

We use the following coordinate system for hyperbolic three-space, $\mathbb{H}^3 \equiv \{(x, y, r) \in \mathbb{R}^3 | r > 0\} \equiv \{z + rj \in \mathbb{R}^3 | r > 0\}$, with the hyperbolic metric
$$ds^2 \equiv \frac{dx^2 + dy^2 + dr^2}{r^2},$$
and volume form
$$dv \equiv \frac{dx \, dy \, dz}{r^3}.$$

The Laplace-Beltrami operator is defined by
$$\Delta \equiv -r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) + r \frac{\partial}{\partial r},$$
and it acts on the space of smooth functions $f : \mathbb{H}^3 \mapsto V$, where $V$ is a finite-dimensional complex vector space with inner-product $\langle \cdot, \cdot \rangle_V$.

Suppose that $\Gamma$ is a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$ ($\text{Rep}(\Gamma, V)$ is the space of finite-dimensional unitary representations of $\Gamma$ in $V$). Then the Hilbert space of $\chi$–automorphic measurable functions is defined by
$$\mathcal{H}(\Gamma, \chi) \equiv \{f : \mathbb{H}^3 \rightarrow V | f(\gamma P) = \chi(\gamma)f(P) \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \text{and} \langle f, f \rangle \equiv \int_{\mathbb{F}} \langle f(P), f(P) \rangle_V \, dv(P) < \infty\}.$$

Here $\mathbb{F}$ is a fundamental domain for $\Gamma$ in $\mathbb{H}^3$, and $\langle \cdot, \cdot \rangle_V$ is the inner product on $V$. Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Next we briefly describe the motivation for the functional regularized determinant. Let $f(s) = \sum'_{m \in \mathcal{D}} \lambda_m^{-s}$ be a sum over the non-zero eigenvalues of $\Delta$. Then formally
$$f'(0) = \frac{d}{ds} f(s) \bigg|_{s=0} = -\left( \sum'_{m \in \mathcal{D}} \log(\lambda) \lambda_m^{-s} \right) \bigg|_{s=0} = -\left( \sum'_{m \in \mathcal{D}} \log(\lambda) \right),$$
and\[
e^{-f'(0)} = \prod_{\lambda_m \neq 0} \lambda_m.
\]
With this formal calculation in mind, one can think of \(e^{-f'(0)}\) as the regularized determinant. Now, typically, \(f(0)\) does not even converge, but \(f(s)\) does converge for \(\text{Re}(s)\) sufficiently large. Analytic continuation gives a possible value for \(f'(0)\).

The formal argument above works well when the orbifold in question is compact. In the non-compact case we compare \(\Delta\) with another self adjoint operator, \(\Delta_0\), the self-adjoint extension of the operator\[
\sum_{i=1}^{k_\infty} \left( -r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} \right) : \bigoplus_{i=1}^{k_\infty} C^\infty_0 ([Y, \infty)) \rightarrow \bigoplus_{i=1}^{k_\infty} L^2([Y, \infty), r^{-3}dr)
\]
with respect to Dirichlet boundary conditions ( \(\{ f \in C^\infty_0 ([Y, \infty)) \mid f(Y) = 0 \}\)). See \[3.3\] for the definitions of the notations used above. Define the projection (onto the constant Fourier coefficient)\[
p_0 : H(\Gamma, \chi) \rightarrow \bigoplus_{i=1}^{k_\infty} L^2([Y, \infty), r^{-3}dr)
\]
by\[
p_0[f](r) = \frac{1}{|P|} \int_P f(x, y, r) \, dxdy \quad \text{for} \quad r \geq Y.
\]
Once again, see \[3.3\] for the definitions of the notations used above.

The analogue of \(f(s)\) for non-compact spaces, following Müller, is the relative zeta-function\[
\zeta(s, \Delta, \Delta_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \right) - \text{dim ker} \Delta \right) \, dt.
\]
Here \(\text{Re}(s) > 2\). Note that in order for the integral above to converge, we need to know the asymptotics of\[
\text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \right) - \text{dim ker} \Delta
\]
at both \(t = 0\) and \(t = \infty\). These asymptotics are given in Lemma \[4.3\].

Our main article of interest is the regularized characteristic polynomial, \(\det(\Delta - (1-s^2))\), which we call the regularized determinant. For \(\text{Re}(s) > 2\) define\[
H(w, s) = H(w, s, \Delta, \Delta_0) = \frac{1}{\Gamma(w)} \int_0^\infty t^{w-1} \text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \right) e^{t(1-s^2)} \, dt,
\]
and following \[Sar87\], we define\[
\det(\Delta - (1-s^2)) = e^{-\frac{dH}{dw}(0, s)}.
\]

Our main results are:

**Theorem.** Let \(\Gamma\) be a cofinite Kleinian group with one cusp at infinity, and let \(\chi \in \text{Rep}(\Gamma, V)\). Then there exists constants \(C_2, C_3, D_1\), depending on \(\Gamma\) and \(\chi\) (they are explicitly determined in \[5\]) such that\[
\log \det(\Delta - (1-s^2)) = \log Z(s, \Gamma, \chi) + s \left( k(\Gamma, \chi) \log(Y) + C_1 \right) + \frac{l_\infty}{[\Gamma : \Gamma_\infty]} \log(\Gamma(s+1) + \Omega(s) - \frac{C_2}{2} \log s - \frac{2}{3} C_3 s^3 - D_1).
\]
Here $Z(s, \Gamma, \chi)$ is the Selberg zeta-function (see \S 5.1), $\Omega(s)$ is a meromorphic function (see Equation 5.3). The constant $Y$ comes about from the decomposition of $\mathcal{F} = \mathcal{F}_Y \cup \mathcal{F}^Y$ into a compact set $\mathcal{F}_Y$ and a noncompact cusp sector $\mathcal{F}^Y$ (see \S 3.1 for more details).

The rest of the notation is defined in \S 5.

**Corollary.** Let $\Gamma$ be a cofinite torsion-free Kleinian group with one cusp at infinity, and let $\chi \in \text{Rep}(\Gamma, V)$ be a regular character. Then

$$\det (\Delta -(1-s^2)) = Z(s, \Gamma, \chi) \exp \left( -s \frac{\text{vol} (\Gamma \setminus \mathbb{H}^3)}{6\pi} + s L(\Lambda_\infty, \psi) \right).$$

The constant $L(\Lambda_\infty, \psi)$ comes about from regularity at a cusp, and its value is computed using Kronecker's second limit formula. It can be realized explicitly using the Siegel function $g_{v,u}(\tau)$, namely

$$L(\Lambda, \psi) = \frac{-2\pi y}{y} \log |g_{v,u}(\tau)|.$$ 

See \S 3.1.1 for more details.

**Corollary.** Let $\Gamma$ be a cocompact Kleinian group, $\chi \in \text{Rep}(\Gamma, V)$. Then

$$\det (\Delta -(1-s^2)) = Z(s, \Gamma, \chi) \exp \left( -s \frac{\text{vol} (\Gamma \setminus \mathbb{H}^3)}{6\pi} + s C_E \right),$$

where

$$C_E = \sum_{\{R\}\text{nice}} \frac{\text{tr}_V \chi(R) \log N(T_0)}{4 |E(R)| \sin^2 \left( \frac{\pi}{m(R)} \right)}.$$ 

The constant $C_E$ is related to the non-cuspidal elliptic elements of $\Gamma$. See \S 3.1.2 for more details.

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2. General Definition of the Relative Zeta-function

In this section (following [Müller98]) we state some basic facts concerning Krein’s spectral shift function, and show how they lead to the definition of the general relative zeta-function. Later, in \S 4 we specialize to $\Delta$ on $\mathcal{H}(\Gamma, \chi)$. For more details on the spectral shift function see [Kre53, BK62, Yaf92].

We first establish some notation. For $B$, a self-adjoint operator on $\mathcal{H}$, $\sigma(B)$ and $\sigma_{\text{ess}}(B)$ are the spectrum and essential spectrum of $B$ respectively.

Let $A$ and $A_0$, be bounded self-adjoint operators in $\mathcal{H}$. Suppose that $V \equiv A - A_0$ is of trace-class. Let $R_0(z) = (A_0 - z)^{-1}$ be the resolvent $A_0$. The spectral shift function of $A$ and $A_0$

$$\xi(\lambda) = \xi(\lambda; A, A_0) = \pi^{-1} \lim_{\epsilon \to 0} \arg \det (1 + V R_0(\lambda + i \epsilon)), \quad (2.1)$$

exists for a.e. \( \lambda \in \mathbb{R} \), is real-valued, and belongs to \( L^1(\mathbb{R}) \). The determinant in (2.1) is the Fredholm determinant. In addition,
\[
\text{tr}(A - A_0) = \int_{\mathbb{R}} \xi(\lambda) \, d\lambda, \quad \|\xi\| \leq \|A - A_0\|_1,
\]
where \( \|\cdot\|_1 \) is the trace norm.

The theory of the spectral shift function is reminiscent of the Selberg trace formula. Let
\[
\mathcal{G} = \left\{ \phi : \mathbb{R} \to \mathbb{R} \mid \phi \in L^1 \text{ and } \int_{\mathbb{R}} |\hat{\phi}(p)| (1 + |p|) \, dp < \infty \right\}.
\]
Then for every \( \phi \in \mathcal{G} \), \( \phi(A) - \phi(A_0) \) is a trace class operator and
\[
\text{tr}(\phi(A) - \phi(A_0)) = \int_{\mathbb{R}} \phi'(\lambda) \xi(\lambda) \, d\lambda.
\]

Part 3 of the next lemma will allow us to define the relative zeta function for \( H, H_0 \).

**Lemma 2.1.** [Mü98, Page 315] Let \( H, H_0 \) be two non-negative self-adjoint operators in \( \mathcal{H} \) and assume that \( e^{-tH} - e^{-tH_0} \) is a trace class operator for \( t > 0 \). Then there exists a unique real valued locally integrable function \( \xi(\lambda) = \xi(\lambda; H, H_0) \) on \( \mathbb{R} \) such that for each \( t > 0 \), \( e^{-t\lambda} \xi(\lambda) \in L^1(\mathbb{R}) \) and the following conditions hold:

1. \( \text{tr}(e^{-tH} - e^{-tH_0}) = -t \int_{0}^{\infty} e^{-t\lambda} \xi(\lambda) \, d\lambda. \)
2. For every \( \phi \in \mathcal{G} \), \( \phi(H) - \phi(H_0) \) is a trace class operator and
\[
\text{tr}(\phi(H) - \phi(H_0)) = \int_{\mathbb{R}} \phi'(\lambda) \xi(\lambda) \, d\lambda.
\]
3. In addition, suppose \( \sigma_{\text{ess}}(H_0) \subset [c, \infty) \), where \( c > 0 \); then \( \ker H \) and \( \ker H_0 \) are both finite-dimensional, and there exists \( c_1 > 0 \) such that
\[
\text{tr}(e^{-tH} - e^{-tH_0}) = \dim \ker H - \dim \ker H_0 + O(e^{-c_1 t})
\]
as \( t \to \infty \).

Let \( h = \dim \ker H - \dim \ker H_0 \). Then it follows from Lemma 2.1 that for \( \text{Re}(s) > 0 \), the integral
\[
\int_{0}^{\infty} t^{s-1} \left( \text{tr} \left( e^{-tH} - e^{-tH_0} \right) - h \right) \, dt
\]
converges absolutely.

**Definition 2.2.** [Mü98, Page 317] Suppose that \( \sigma_{\text{ess}}(H_0) \subset [c, \infty) \), where \( c > 0 \). Then for \( \text{Re}(s) > 0 \), the relative zeta-function of \( H \) and \( H_0 \), \( \zeta(s; H, H_0) \) is defined by
\[
\zeta(s; H, H_0) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} \left( \text{tr} \left( e^{-tH} - e^{-tH_0} \right) - h \right) \, dt
\]

### 3. The Operators \( \Delta, \Delta_0 \) and Their Heat Kernels

The main goal of this section is to give an explicit formula for
\[
\text{tr}(e^{-\Delta t} - e^{-\Delta_0 t}).
\]
3.1. **Notation.** Before we can define $\Lambda_0$ we must establish some notation (see [Fri05a, Fri05b] for more details). In order to simplify our notation (and to make our paper more readable) we present our results under the assumption:

**Assumption 3.1.** The cofinite Kleinian group $\Gamma$ has only one class of cusps at $\zeta = \infty \in \mathbb{P}$, and $\chi \in \text{Rep}(\Gamma, V)$. The lattice associated with $\zeta = \infty$ is

$$\Lambda_\infty = \mathbb{Z} \oplus \mathbb{Z} \tau_\alpha, \quad \text{Im}(\tau_\alpha) > 0.$$ 

Let $\Gamma_\infty < \Gamma$ denote the stabilizer subgroup of the cusp at infinity $\zeta = \infty$,

$$\Gamma_\infty \equiv \{ \gamma \in \Gamma \mid \gamma(\infty) = \infty \},$$

and let $\Gamma'_\infty$ be the maximal torsion-free parabolic subgroup of $\Gamma_\infty$. By definition (of a cusp), $\Gamma'_\infty$ is a free abelian group of rank two. The possible values for the index of $[\Gamma_\infty : \Gamma'_\infty]$ are $1, 2, 3, 4, \text{ and } 6$. See [EGM98].

The subgroup $\Gamma'_\infty$ is canonically isomorphic to a lattice $\Lambda_\infty = \mathbb{Z} \oplus \mathbb{Z} \tau_\alpha$. Without loss of generality we can assume that $\text{Im}(\tau_\alpha) > 0$. Let $\epsilon$ be root of unity of order $[\Gamma_\infty : \Gamma'_\infty]$. Then

$$\Gamma'_\infty = \left\{ \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \mid b \in \Lambda_\infty \right\},$$

$$\Gamma_\infty = \left\{ \left( \begin{array}{cc} \epsilon^n & \epsilon^n b \\ 0 & \epsilon^{-n} \end{array} \right) \mid b \in \Lambda_\infty, n = 0, \ldots, [\Gamma_\infty : \Gamma'_\infty] \right\} / \{ \pm I \}.$$

Let $\mathcal{P} \subset \mathbb{C}$ be a fundamental domain for the action of $\Gamma_\infty$ on $\mathbb{C}$, and let $\mathcal{P}'$ be the fundamental parallelogram with base point at the origin for the lattice $\Lambda_\infty$. For $Y > 0$ set

$$F^Y \equiv F(Y) \equiv \{ z + rj \mid z \in \mathcal{P}, r \geq Y \}.$$ 

Then for $Y$ sufficiently large, there exists a compact set $\mathcal{F}_Y$, disjoint from $F^Y$, so that $\mathcal{F} \equiv \mathcal{F}_Y \cup F^Y$ is a fundamental domain for $\Gamma$.

Define the *singular* space by

$$V_\infty \equiv \{ v \in V \mid \chi(\gamma)v = v, \quad \forall \gamma \in \Gamma_\infty \},$$

and the *almost singular* space

$$V'_\infty \equiv \{ v \in V \mid \chi(\gamma)v = v, \quad \forall \gamma \in \Gamma'_\infty \}.$$ 

If $\dim V_\infty > 0$, then $\chi$ is called *singular* with index of singularity $k(\Gamma, \chi) \equiv k_\infty \equiv \dim V_\infty$. If $\dim V_\infty = 0$, then $\chi$ is called *regular*. Set $l_\infty \equiv \dim V'_\infty$. Let $P_\infty$ denote the orthogonal projection

$$P_\infty : V \mapsto V_\infty.$$ 

Fix an orthonormal basis $\{ v_i \}_{i=1}^{k_\infty}$ for $V_\infty$. For $P \in \mathbb{H}^3$, $\text{Re}(s) > 1$, and $i = 1 \ldots k_\infty$, we define the *Eisenstein series* by

$$E_i(P, s) \equiv E(P, s, i, \Gamma, \chi) \equiv \sum_{M \in \Gamma_\infty \setminus \Gamma} (r(MP))^{1+s} \chi(M)^* v_i.$$

\footnote{The set $\mathcal{P}$ is a euclidean polygon.}
The series $E_i(P, s)$ converges uniformly and absolutely on compact subsets of $\{\Re(s) > 1\} \times \mathbb{H}^3$ to a $\chi$-automorphic function that satisfies
$$\Delta E(\cdot, s, \alpha, v) = \lambda E(\cdot, s, \alpha, v), \quad \lambda = 1 - s^2,$$
and admits a meromorphic continuation to the whole complex plane [Fri05a].

For $P = z + rj, P' = z' + r'j \in \mathbb{H}^3$ set
$$\delta(P, P') \equiv |z - z'|^2 + r^2 + r'^2.$$
It follows that $\delta(P, P') = \cosh(d(P, P'))$, where $d$ denotes the hyperbolic distance in $\mathbb{H}^3$. Next, for $k \in S \equiv S((1, \infty))$ a Schwartz-class function, define $K(P, Q)$ by
$$K(P, Q) = k(\delta(P, Q)).$$
The function $K(P, Q)$ is called a point-pair invariant. Set
$$K_\Gamma(P, Q) \equiv \sum_{\gamma \in \Gamma} \chi(\gamma) K(P, \gamma Q).$$
The decay properties of the function $k$ guarantee that the series above converges absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$ [EGM98, Theorem 6.4.1]. The function $K_\Gamma(P, Q)$ is the kernel of a bounded operator $K : \mathcal{H}(\Gamma, \chi) \mapsto \mathcal{H}(\Gamma, \chi)$.

The function $k$ leads to two other useful function: $h$, the Selberg–Harish-Chandra transform of $k$; and $g$, the Fourier transform of $h$. Explicitly:

\begin{equation}
(3.3) \quad h(\lambda) = h(1 - s^2) \equiv \frac{\pi}{s} \int_1^\infty k \left( t + \frac{1}{t} \right) (t^s - t^{-s}) \left( t - \frac{1}{t} \right) \frac{dt}{t}, \quad \lambda = 1 - s^2,
\end{equation}
and for $r \in \mathbb{R}$ set
$$g(r) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + x^2) e^{-ixr} dx.$$

For $v, w \in V$ let $v \otimes w$ be the linear operator in $V$ defined by $v \otimes w(x) = \langle x, w \rangle v$. An immediate application of the Spectral Decomposition Theorem ([Fri05a, Fri05b], and the Selberg–Harish-Chandra Transform yields (see [EGM98, Equation 6.4.10, page 278]):

**Lemma 3.2.** Let $k \in S$ and $h : \mathbb{C} \to \mathbb{C}$ be the Selberg–Harish-Chandra Transform of $k$. Then

\begin{equation}
(3.4) \quad K_\Gamma(P, Q) = \sum_{m \in D} h(\lambda_m) e_m(P) \otimes e_m(Q)
\end{equation}
$$+ \frac{1}{4\pi} \sum_{l=1}^{k} \frac{1}{|P|} \int_{\mathbb{R}} h \left( 1 + x^2 \right) E_l(P, ix) \otimes \overline{E_l(Q, ix)} dt,$$
where $|P|$ denotes the euclidian area of $P \subset \mathbb{C}$. The sum and integrals converge absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$.

We conclude this section with some notation that will be needed to state our main result.

\footnote{The space of smooth functions $k : [1, \infty) \to \mathbb{C}$ that satisfy $\lim_{x \to \infty} x^n k^{(m)}(x) = 0$ for all $n, m \in \mathbb{N}_{\geq 0}$.}
3.1.1. Regular representations and Siegel's theta function. Recall that $\Lambda_\infty = \mathbb{Z} \oplus \mathbb{Z} \tau \subset \mathbb{C}$ with $\text{Im}(\tau) > 0$. It follows that $\chi$ restricted to (the abelian group) $\Lambda_\infty$ diagonalizes into characters $\psi_l$ for $l = 1, \ldots, l_\infty$, and the identity character for $l = l_\infty + 1, \ldots, \dim V$. For each $\psi_l, u_l, v_l \in \mathbb{R}$ are not both integers, satisfying $\psi_l(1) = e^{2\pi i u_l}$ and $\psi_l(\tau) = e^{2\pi i v_l}$. We define

$$L(\Lambda_\infty, \psi_l) = \frac{-2\pi}{y} \log |g_{-v_l, u_l}(\tau)|,$$

where $g_{a_1, a_2}$ is the Siegel function,

$$g_{a_1, a_2}(\tau) = -q_{\tau}^{-1/2}B_2(a_1) e^{2\pi i a_2(a_1 - 1)/2} \prod_{n=1}^\infty (1 - q_n^a q_n)(1 - q_n^{-a}q_n).$$

$B_2(X) = X^2 - X + 1/6, q_\tau = e^{2\pi i \tau}, q_n = e^{2\pi i \tau},$ and $z = a_1 \tau + a_2$.

3.1.2. Non-cuspidal elliptic elements. Let $\{R\}_\text{nce}$ be a set of representatives of the non-cuspidal elliptic elements of $\Gamma$, the elliptic elements that do not fix a cusp ($\infty$ under Assumption 3.1). Following [EGM98, Definition 5.3.2], the Elliptic number of $\Gamma$ is

$$\sum_{(R)\text{nce}} \text{tr}_V \chi(R) \frac{N(T_0)}{4|\mathcal{E}(R)| \sin^2\left(\frac{2\pi}{m(R)}\right)}.$$  

For a fixed representative $R$, $N(T_0)$ is the minimal norm of a hyperbolic or loxodromic element of the centralizer $\mathcal{C}(R)$. The element $R$ is understood to be a $k$-th power of a primitive non cuspidal elliptic element $R_0 \in \mathcal{C}(R)$ describing a hyperbolic rotation around the fixed axis of $R$ with minimal rotation angle $\frac{2\pi}{m(R)}$. Further, $\mathcal{E}(R)$ is the maximal finite subgroup contained in $\mathcal{C}(R)$.

3.1.3. Cuspidal elliptic elements. Denote by $\mathcal{CE}$ set of elements of $\Gamma$ which are $\Gamma$-conjugate to an element of $\Gamma_\infty \setminus \Gamma_\infty' = \{ \gamma \in \Gamma_\infty \mid \gamma \text{ is not parabolic nor the identity element} \}$. We fix representatives of conjugacy classes of $\mathcal{CE}$, $g_1, \ldots, g_4$ that have the form

$$(3.5) \quad g_i = \begin{pmatrix} \epsilon_i & \epsilon_i \omega_i \\ 0 & (\epsilon_i)^{-1} \end{pmatrix}.$$  

Let $\mathcal{C}(g)$ denote the centralizer in $\Gamma$ of an element $g \in \mathcal{CE}$. In addition, let $\{p_i, \infty\}$ be the set of fixed points in $\mathbb{P}$ of the element $g_i$. Since $g_i$ is a cuspidal elliptic element it follows that $p_i$ is a cusp of $\Gamma$ (see [EGM98] page 52). Hence by Assumption 3.1 there is an element $\gamma_i \in \Gamma$ with $\gamma_i \infty = p_i$. Let $\epsilon_i$ is the lower left hand (matrix) entry of $\gamma_i$.

3.2. The Heat Kernel of $\Delta$ as a Poincaré Series. The heat kernel for $\Delta$ on $\mathbb{H}^3$ is a function

$u : \mathbb{H}^3 \times \mathbb{H}^3 \times (0, \infty) \mapsto \mathbb{R}$

satisfying

$$e^{-\Delta t} f(P) = \int_{\mathbb{H}^3} u(P, Q, t) f(Q) \, dv(P),$$

\footnote{One-dimensional unitary representations.}  

\footnote{There are only finitely many distinct conjugacy classes of elliptic elements in a cofinite Kleinian group.}  

\footnote{We abuse notation and allow $\Delta$ to represent both the self-adjoint operator on $\mathcal{H}(\Gamma, \chi)$ and the standard differential operator on smooth functions of $\mathbb{H}^3$.}
for all $f$ in the domain of the self-adjoint operator $e^{-\Delta t}$. It is a classical result [Dav89] that

$$u(P, Q, t) = u(\rho, t) = (4\pi t)^{-3/2} \frac{\rho}{\sinh \rho} \exp \left( \frac{-\rho^2}{4t} \right), \quad \text{where } \rho = d(P, Q).$$

In order to apply the theory of point pair invariants, we need to find a Schwartz-class function $k_t$ so that

$$k_t(\delta(P, Q)) = k_t(\cosh(d(P, Q))) = u(P, Q, t).$$

Using the formula:

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}), \quad \text{for } x \geq 1,$$

we obtain

$$k_t(x) = \frac{e^{-t}}{(4\pi t)^{3/2}} \frac{\ln(x + \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}} \exp \left( \frac{-\ln(x + \sqrt{x^2 - 1})^2}{4t} \right).$$

Observe that as $x \to 1^+$, $k_t(x) \to \frac{e^{-t}}{(4\pi t)^{3/2}}$, and that $k_t \in \mathcal{S}$. We have:

**Lemma 3.3.** Let

$$K_\Gamma(P, Q, t, \chi) = \sum_{\gamma \in \Gamma} \chi(\gamma) k_t(\delta(P, \gamma Q)).$$

Then $K_\Gamma(P, Q, t, \chi)$ is the heat kernel for $\Delta$ on the Hilbert space $\mathcal{H}(\Gamma, \chi)$.

It follows from Equation (3.3) and Lemma 3.2 that $h(x) = e^{-tx}$. By definition

$$g(r) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-(1+x^2)t} e^{-ixr} dx = \frac{\exp(-t)}{\sqrt{4\pi t}} \exp \left( -\frac{r^2}{4t} \right).$$

By applying Lemma 3.2 we obtain the spectral expansion of $K_\Gamma$:

$$K_\Gamma(P, Q, t, \chi) = \sum_{m \in \mathcal{D}} e^{-\lambda_m t} e_m(P) \otimes e_m(Q)$$

$$+ \frac{1}{4\pi} \sum_{i=1}^{k_\infty} \frac{1}{|P|} \int_{\mathbb{R}} \exp \left( -(1+x^2)t \right) E_i(P, ix) \otimes \overline{E_i(Q, ix)} dx.$$

### 3.3. The Operator $\Delta_0$ and its Heat Kernel

For $\mathcal{F} \equiv \mathcal{F}_Y \cup \mathcal{F}^Y$, let $\Delta_0$ be the self-adjoint extension of the operator

$$\sum_{i=1}^{k_\infty} \left( -r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} \right): \bigoplus_{i=1}^{k_\infty} C_0^\infty([0, Y]) \to \bigoplus_{i=1}^{k_\infty} L^2([0, Y), r^{-3} dr)$$

with respect to Dirichlet boundary conditions ( $\{ f \in C_0^\infty([0, Y]) \mid f(Y) = 0 \}$). Note that $\Delta_0$ depends on $Y$. It is understood that $\Delta_0$ acts componentwise with respect to the basis for $V_\infty$ fixed in [3.1]. The operator $\Delta_0$ can be thought of as a Laplacian operator in its own right. In fact $\Delta_0$ is closely related to the restriction of $\Delta$ to $\mathcal{F}^Y$. Define

$$p_0: \mathcal{H}(\Gamma, \chi) \to \bigoplus_{i=1}^{k_\infty} L^2([0, Y), r^{-3} dr)$$

\(^6\text{The limit follows from applying l'Hôpital's rule to either (3.6) or (3.7). The fact that the singularity cancels out is one advantage to working the heat kernel instead of the resolvent kernel, which has a singularity and must be iterated.}\)
by
\[
p_{0}[f](r) = \frac{1}{|P|} \int_{P} P_{\infty} f(x, y, r) \, dx \, dy \quad \text{for } r \geq Y.
\]

**Lemma 3.4.** For all \( f \) in the domain of \( \Delta \),
\[
p_{0}[\Delta f] = \Delta_{0} p_{0}[f].
\]

**Proof.** The proof follows from [ECM98, Pages 236-237] and the definition of \( p_{0} \). \( \square \)

For \( F = F_{Y} \cup F_{Y}', \ t > 0, \ P = z + rj, \ P' = z' + r'j \in F \) define

\[
k(P, P', t) \equiv \begin{cases} P_{\infty} \frac{rr_{j}e^{-t}}{|P|\sqrt{4\pi t}} \left[ \exp \left( -\frac{\log(r/r')^{2}}{4t} \right) - \exp \left( -\frac{(\log(r/r')-2\log(Y))^{2}}{4t} \right) \right] & \text{for } P, P' \in F_{Y}, \\
0 & \text{else}
\end{cases}
\]

The classical theory of the Heat Equation, for the half line, tells us that \( k \) is the heat kernel of \( \Delta_{0} \). In other words:

**Lemma 3.5.** For all \( f \in H(\Gamma, \chi) \),
\[
e^{-\Delta_{0} t} p_{0}[f] = \int_{F} k(\cdot, P', t)f(P') \, dv(P').
\]

### 3.4. The Regularized Trace.
In this section we will prove that the regularized heat kernel
\[
e^{-\Delta t} - e^{-\Delta_{0} t} p_{0}
\]
is of trace-class. Then using ideas from the standard proof of the Selberg trace formula, we will evaluate the trace explicitly. But before we can proceed, we need Theorem 3.6. Its proof is based on the classical Poisson Summation Formula, and it is proved in Appendix A.

**Theorem 3.6.** For \( P = z + rj, \ P' = z' + r'j \in F_{Y} \),
\[
K_{F}(P, P', t, \chi) = P_{\infty} \frac{rr_{j}e^{-t}}{|P|\sqrt{4\pi t}} \exp \left( -\frac{\log^{2}(r/r')}{4t} \right) + O(1).
\]

**Theorem 3.7.** The operator
\[
e^{-\Delta t} - e^{-\Delta_{0} t} p_{0}
\]
is of trace-class.

**Proof.** The proof is based on the decay properties of Theorem 3.6 and Equation 3.9 and a clever trick, using the semi-group properties of the heat kernels, of Deift-Simon. We follow [Müll83, Page 259] and [Par05, Prop. 2.1]. First note that by Theorem 3.6 and Equation 3.9 \( e^{-\Delta t} - e^{-\Delta_{0} t} p_{0} \) is Hilbert-Schmidt.

Next write \( e^{-\Delta t} - e^{-\Delta_{0} t} p_{0} \) as
\[
(3.10) \quad e^{-\Delta t} \left( e^{-\Delta_{0} t} - e^{-\Delta_{0} t} p_{0} \right) + \left( e^{-\Delta_{0} t} - e^{-\Delta_{0} t} p_{0} \right) e^{-\Delta_{0} t}.
\]
where \( \tau = t/2 \). Next choose a function \( f \in C^\infty(\mathcal{F}) \) so that \( 0 < f \leq 1, f(P) = 1 \) if \( P \in \mathcal{F} \), \( F(z + rj) = (Y/r)^{1/4} \) if \( P = z + rj \in \mathcal{F}' \). Let \( m_f \) be the multiplication operator by \( f \). Now rewrite (3.10) once again as

\[
e^{-\Delta \tau} m_f^{-1} (e^{-\Delta \tau} - e^{-\Delta_0 \tau} p_0) + (e^{-\Delta \tau} - e^{-\Delta_0 \tau} p_0) m_f^{-1} m_f e^{-\Delta_0 \tau}.
\]

The idea is to borrow \( r^{-1/4} \) from \( e^{-\Delta \tau} - e^{-\Delta_0 \tau} p_0 \) and lend it to \( e^{-\Delta \tau} \) and \( e^{-\Delta_0 \tau} \). It follows from Theorem 3.6 and Equation 3.9 that each of the operators \( e^{-\Delta \tau} m_f, m_f^{-1} (e^{-\Delta \tau} - e^{-\Delta_0 \tau} p_0), (e^{-\Delta \tau} - e^{-\Delta_0 \tau} p_0) m_f^{-1} \) and \( m_f e^{-\Delta_0 \tau} \) is Hilbert-Schmidt. Hence \( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \) is of trace-class. For more details, see [Mu83] Page 259.

Now we can apply standard Selberg theory to explicitly evaluate the integral trace

\[
\int_{\mathcal{F}} (\text{tr}_V K_V(P, P, t) - \text{tr}_V k(P, P, t)) \, dv(P).
\]

Let \( \mathcal{S}(s) \) be the scattering matrix of \( \Delta \). That is the matrix formed from the constant terms of the Fourier coefficients of the Eisenstein Series. Let \( \phi(s) \) be the determinant of the scattering matrix (see [Fri05a, Fri05b] for more details).

**Theorem 3.8.**

\[
\text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \right) = \int_{\mathcal{F}} (\text{tr}_V K_V(P, P, t) - \text{tr}_V k(P, P, t)) \, dv(P) = \\
\sum_{m \in D} e^{-\lambda m t} - \frac{1}{4\pi} \int_{\mathbb{R}} \exp \left( -(1 + x^2) t \right) \frac{\phi'(ix)}{\phi(i)} \, dx \\
+ \frac{1}{4} e^{-t} \text{tr} \mathcal{S}(0) + \frac{e^{-t}}{\sqrt{4 \pi t}} k(\Gamma, \chi) \log Y + \frac{e^{-t}}{4} k(\Gamma, \chi).
\]

**Proof.** Since \( f(P) \equiv \text{tr}_V K_V(P, P, t, \chi) - \text{tr}_V k(P, P, t) \in L^1(\mathcal{F}) \), we can rewrite the integral above as

\[
(3.11) \quad \lim_{A \to \infty} \int_{\mathcal{F}_A} (\text{tr}_V K_V(P, P, t, \chi) - \text{tr}_V k(P, P, t)) \, dv(P) = \\
\lim_{A \to \infty} \left( \int_{\mathcal{F}_A} \text{tr}_V K_V(P, P, t, \chi) \, dv(P) - \int_{\mathcal{F}_A} \text{tr}_V k(P, P, t) \, dv(P) \right).
\]

A standard application of the Maaß—Selberg relations (see [Fri05a], [ECM98] Page 305, or [Ven82] Pages 67-70) gives us

\[
\int_{\mathcal{F}_A} \text{tr}_V K_V(P, P, t, \chi) \, dv(P) = \\
\log(A) k(\Gamma, \chi) \frac{e^{-t}}{\sqrt{4 \pi t}} + \sum_{m \in D} e^{-\lambda m t} - \frac{1}{4\pi} \int_{\mathbb{R}} \exp \left( -(1 + x^2) t \right) \frac{\phi'(ix)}{\phi(i)} \, dx + \frac{1}{4} e^{-t} \text{tr} \mathcal{S}(0) + o(1).
\]
A straightforward calculation shows that

\[
\int_{F \setminus F_A} \text{tr} V k(P, P, t) \, dv(P) = k(\Gamma, \chi) \int_{F \setminus F_A} \frac{r^2 e^{-t}}{|P| \sqrt{4\pi t}} \left( 1 - \exp \left( \frac{-(\log(r) - \log(Y))^2}{t} \right) \right) \, dv(P) =
\]

\[
k(\Gamma, \chi) \int_{A} \int_{Y} \int_{P} \frac{r^2 e^{-t}}{|P| \sqrt{4\pi t}} \frac{dxdydr}{r^3}
\]

\[
= k(\Gamma, \chi) \int_{A} \int_{Y} \int_{P} \frac{r^2 e^{-t}}{|P| \sqrt{4\pi t}} \frac{dxdydr}{r^3}
\]

\[
\log(A) k(\Gamma, \chi) \frac{e^{-t}}{\sqrt{4\pi t}} - \frac{e^{-t}}{\sqrt{4\pi t}} k(\Gamma, \chi) \log Y - \frac{e^{-t}}{4} k(\Gamma, \chi) + o(1) .
\]

The term \( \frac{e^{-t}}{4} k(\Gamma, \chi) \) is obtained using a simple \( u \)-substitution,

\[
u = \frac{\log(r) - \log(Y)}{\sqrt{t}} .
\]

4. The Relative Zeta-Function and Relative Determinant

Now that we have an explicit formula for the trace

\[
\text{tr} (e^{-\Delta t} - e^{-\Delta_0 t} p_0)
\]

(Theorem 3.8), we can proceed and give an explicit evaluation of the relative zeta-function.

Since \( \sigma(\Delta_0) = \sigma_{\text{ess}}(\Delta) = [1, \infty) \) when the representation \( \chi \) is singular,

\[
q_\chi \equiv \dim \ker \Delta - \dim \ker \Delta_0 = \dim \ker \Delta .
\]

Following Müller we define the relative zeta-function \( \zeta(s, \Delta, \Delta_0) \) by

\[
(4.1) \quad \zeta(s, \Delta, \Delta_0) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \text{tr} (e^{-\Delta t} - e^{-\Delta_0 t} p_0) - q_\chi \right) \, dt .
\]

Note that in order for the integral above to converge, we need to know the asymptotics of

\[
\text{tr} (e^{-\Delta t} - e^{-\Delta_0 t} p_0) - q_\chi
\]

at both \( t = 0 \) and \( t = \infty \). These asymptotics are given in Lemma 4.3.

Since \( \text{vol}(F) < \infty \), it follows that [EGM98 Theorem 3.6.4]

\[
q_\chi = \begin{cases} 
\dim V & \text{if } \chi \text{ is trivial} \\
0 & \text{else} .
\end{cases}
\]

**Theorem 4.1.** For \( \text{Re}(s) > 2 \)

\[
\zeta(s, \Delta, \Delta_0) = \sum_{m \in D} \lambda_m^{-s} - \frac{1}{4\pi} \int_\mathbb{R} (1 + x^2)^{-s} \frac{\phi'(ix)}{\phi(ix)} \, dx + \frac{1}{4} (\text{tr} \, \mathcal{G}(0) + k(\Gamma, \chi))
\]

\[
+ \frac{k(\Gamma, \chi)}{\sqrt{4\pi}} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \log Y .
\]

**Proof.** The proof follows from the standard properties of the Mellin transform, Theorem 3.8 and Lemma 4.3. Note that \( q_\chi \) cancels out with any of the terms coming from the zero eigenvalues of \( \Delta \). \( \square \)
4.1. Asymptotics of the Heat Kernel. The main tool that allows us to study the asymptotic behavior (near \( t = 0 \) and \( t = \infty \)) of the regularized heat kernel is the Selberg trace formula for the case of a cofinite Kleinian group with finite-dimensional unitary representations [Fri05a,Fri05b].

**Theorem 4.2. (Selberg trace formula)** Let \( \Gamma \) be a cofinite Kleinian group with one cusp at infinity, \( \chi \in \text{Rep}(\Gamma, V) \), \( h \) be a holomorphic function on \( \{ s \in \mathbb{C} \mid |\text{Im}(s)| < 2 + \delta \} \) for some \( \delta > 0 \), satisfying \( h(1+z^2) = O(1+|z|^2)^{3/2-\epsilon} \) as \( |z| \to \infty \), and let

\[
g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2)e^{-tx} dt.
\]

Then

\[
\begin{align*}
\sum_{m \in D} h(\lambda_m) - \frac{1}{4\pi} \int_{\mathbb{R}} h(1+t^2)\frac{d}{dt}(it) dt &= \frac{\text{vol}(\Gamma \setminus \mathbb{H}^3)}{4\pi^2} \dim_C V \int_{\mathbb{R}} h(1+t^2)t^2 dt \\
&\quad + \sum_{\{R\}_{nec}} \frac{\text{tr}_V \chi(R)(g(0)) \log N(T_0)}{4|\text{E}(R)| \sin^2 \left( \frac{\pi}{m(R)} \right)} + \sum_{\{T\}_{loch}} \frac{\text{tr}_V \chi(T)(g(0) \log N(T))}{|\text{E}(T)| |a(T) - a(T)^{-1}|^2} \log N(T_0) \\
&\quad - \frac{\text{tr}(\mathbb{S}(0))h(1)}{4} \\
&\quad + \sum_{i=1}^d \frac{\text{tr} \chi(g_i)}{|C(g_i)|} \left( \frac{2g(0) \log |c_i|}{1-|c_i|^2} + \frac{1}{1-|c_i|^2} \int_0^\infty g(x) \sinh x \cosh x - 1 + \frac{1-|c_i|^2}{2} dx \right) \\
&\quad + \frac{L}{|\Gamma_\infty : \Gamma'_\infty|} \left( g(0) \frac{h(1)}{4} + g(0) \left( \frac{\eta_\infty}{2} - \gamma \right) - \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2)(1+it) dt \right) \\
&\quad + \frac{g(0)}{|\Gamma_\infty : \Gamma'_\infty|} \sum_{l=1}^n L(\Lambda_\infty, \psi_l).
\end{align*}
\]

Here \( \{\lambda_m\}_{m \in D} \) are the eigenvalues of \( \Delta \) counted with multiplicity. The summation with respect to \( \{R\}_{nec} \) extends over the finitely many \( \Gamma \)-conjugacy classes of the non cuspidal elliptic elements (elliptic elements that do not fix a cusp) \( R \in \Gamma \), and for such a class \( N(T_0) \) is the minimal norm of a hyperbolic or loxodromic element of the centralizer \( C(R) \). The element \( R \) is understood to be a \( k \)-th power of a primitive non cuspidal elliptic element \( R_0 \in C(R) \) describing a hyperbolic rotation around the fixed axis of \( R \) with minimal rotation angle \( 2\pi \frac{2m(R)}{m(R)} \). Further, \( \mathbb{E}(R) \) is the maximal finite subgroup contained in \( C(R) \). The summation with respect to \( \{T\}_{loch} \) extends over the \( \Gamma \)-conjugacy classes of hyperbolic or loxodromic elements of \( \Gamma \), \( T_0 \) denotes a primitive hyperbolic or loxodromic element associated with \( T \). The element \( T \) is conjugate in \( \text{PSL}(2, \mathbb{C}) \) to the transformation described by the diagonal matrix with diagonal entries \( a(T), a(T)^{-1} \) with \( |a(T)| > 1 \), and \( N(T) = |a(T)|^2 \).

For \( s \in \mathbb{C} \), \( \mathbb{S}(s) \) is a \( k(\Gamma, \chi) \times k(\Gamma, \chi) \) matrix-valued meromorphic function, called the scattering matrix of \( \Delta \), and \( \phi(s) = \det \mathbb{S}(s) \). The elements \( g_i \) are complete representatives for the conjugacy classes of \( \{ \gamma \in \Gamma_\infty \mid \gamma \text{ is not parabolic nor the identity element } \} \), \( C(g_i) \) is the order of the centralizer of \( \Gamma \) of the element \( g_i \). The numbers \( c_i \in \mathbb{C} \) are constants depending on the \( g_i \), respectively (see [3.1.3]). Finally \( \gamma \) is Euler’s constant, and \( \eta_\infty \) is the analogue of Euler’s constant for the lattice \( \Lambda_\infty \subset \mathbb{R}^2 \). The term \( L(\Lambda_\infty, \psi_l) \) is defined in [3.1.1] See [Fri05a,Fri05b,EFGM98] for more details.

Next, using the Selberg trace formula, we study the regularized heat kernel. We have
Lemma 4.3. Let $\theta(t) \equiv \text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} \right)$. Then there exists constants $a, b, c, d$ so that

$$\theta(t) = at^{-\frac{3}{2}} + b(\log t)t^{-\frac{1}{2}} + ct^{-\frac{1}{2}} + d + O(\sqrt{t} \log t) \text{ as } t \to 0^+,$$

and there exists a positive constant $c > 0$ so that $\theta(t) - q_\chi = O(e^{-ct})$ as $t \to \infty$.

Proof. Our argument is analogous to [Efr88, Efr91, Proposition 1]. The asymptotics as $t \to \infty$ follows immediately from the spectral trace formula (Theorem 3.8).

An application of the Selberg trace formula for the pair of functions

$$h(z) = \exp(-zt), \quad g(r) = \frac{\exp(-t)}{\sqrt{4\pi t}} \exp\left(-\frac{r^2}{4t}\right),$$

yields (on applying the left side of the Selberg trace formula) the nontrivial terms from the trace of the heat kernel in Theorem 3.8, namely

$$\sum_{m \in D} e^{-t\lambda_m} - \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(1+x^2)} \frac{\phi'}{\phi}(ix) dx.$$

It remains for us to estimate each term on the right as $t \to 0^+$.

We start with the loxodromic sum

$$\exp(-t) \sum_{(T)_{\text{lox}}} \text{tr} V \chi(T) \exp\left(-\frac{4\log N(T)}{4t}\right) \log N(T_0).$$

Since $N(T) > 1$ for all loxodromic $T$, the sum decays to zero as $t \to 0^+$.

Next note that $g(0) = \frac{\exp(-t)}{\sqrt{4\pi t}^2}$ and $h(1) = e^{-t}$. The finite sum involving the non-cuspidal elliptic terms is easily estimated,

$$\exp(-t) \sum_{(R)_{\text{nc}}} \text{tr} V \chi(R) \log N(T_0) \frac{4|\mathcal{E}(R)|}{\sin^2 m(R)} = O(t^{-\frac{1}{2}}),$$

and so are all the other terms with only $g(0)$ or $h(1)$.

Next we must estimate

$$\exp(-t) \int_0^\infty \exp\left(-\frac{x^2}{4t}\right) \frac{\sinh x}{\cosh x - 1 + \frac{|1-x^2|}{2}} dx.$$

An elementary $u-$substitution with $u = \frac{x}{\sqrt{2}}$ shows that the integral above decays to zero (exponentially fast) as $t \to 0^+$.

The next integral can be calculated explicitly:

$$\frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{4\pi^2} \text{dim}_C V \int_{\mathbb{R}} \exp(-t(1+x^2))x^2 dx = \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{4\pi^2} (\text{dim}_C V) \sqrt{\pi e^{-t}} \frac{\Gamma'(1+ix)}{\Gamma(1+ix)} dx.$$

The last integral

$$\int_{\mathbb{R}} e^{-(1+x^2)t} \frac{\Gamma'(1+ix)}{\Gamma(1+ix)} dx$$
Lemma 4.4. The following hold:

By applying the Mellin Transform to Theorem 3.8 we obtain, for formulas \[GR65\]:

\[ -2t ie^{-t} \int_{-\infty}^{\infty} xe^{-tx^2} \log \Gamma(1 + ix) \, dx. \]

Using Sterling's formula

\[ \log \Gamma(1 + ix) = \left( \frac{1}{2} + ix \right) \log(1 + ix) - (1 + ix) + \log \sqrt{2\pi} + O(\frac{1}{x}) \text{ as } x \to \infty, \]

it follows that for constants \( \beta', \gamma', \delta' \)

\[ \int_{\mathbb{R}} e^{-(1+x^2)t} \frac{\Gamma'(1+ix)}{\Gamma(1+ix)} \, dx = \beta' \frac{\log t}{\sqrt{t}} + \frac{\gamma'}{\sqrt{t}} + \delta' + O(\sqrt{t} \log t) \text{ as } t \to 0+. \]

See \[Efr88, Efr91\] Proposition 1 for more details. \(\square\)

4.2. The Regularized Determinant. In this section we define \(\det(\Delta - (1 - s^2))\). For \(Re(s) > 2\) define

\[ H(w, s) \equiv H(w, s, \Delta, \Delta_0) = \frac{1}{\Gamma(w)} \int_{0}^{\infty} t^{w-1} \left( \text{tr} \left( e^{-\Delta t} - e^{-\Delta_0 t} p_0 \right) \right) e^{t(1-s^2)} \, dt. \]

By applying the Mellin Transform to Theorem 3.8 we obtain, for \(Re(s) > 2\),

\[ H(w, s) = \sum_{\lambda \in \mathcal{D}} (\lambda - (1 - s^2))^{-w} - \frac{1}{4\pi} \int_{\mathbb{R}} (x^2 + s^2)^{-w} \frac{\partial}{\partial s} \frac{\Gamma'(1+ix)}{\Gamma(1+ix)} \, dx + s^{2w-1} \frac{1}{4} \left( \text{tr} \, \mathfrak{S}(0) + k(\Gamma, \chi) \right) \]

\[ + s^{-(2w-1)} \frac{k(\Gamma, \chi)}{\sqrt{4\pi}} \frac{\Gamma(w - 1/2)}{\Gamma(w)} \log Y. \]

In order to define \(\det(\Delta - (1 - s^2))\), we will need to know that \(H(w, s)\) is regular at \(w = 0\).

Lemma 4.4. The following hold:

1. For fixed \(s > 2\), \(H(w, s)\) is regular at \(w = 0\).
2. \(\frac{\partial H}{\partial w}(0, s) \sim a(s^2 - 1)^\frac{s}{2} + 2\sqrt{\pi} b(s^2 - 1)^\frac{s}{2} \left( \log(s^2 - 1) + (\gamma + \log(4) - 2) \right)\)

\[ - 2\sqrt{\pi} c(s^2 - 1)^\frac{s}{2} - d \log(s^2 - 1) \text{ as } s \to \infty, \]

where \(\gamma\) is Euler's constant, and the constants \(a, b, c, d\) are from Lemma 4.3.

Proof. The proof is a standard exercise using the Mellin transform, Lemma 4.3 and the following formulas \[GR65\]:

\[ \frac{1}{\Gamma(w)} \int_{0}^{\infty} t^{-\epsilon} e^{t(1-s^2)} t^{w-1} \, dt = \frac{1}{\Gamma(w)} (s^2 - 1)^{-w-\epsilon} \Gamma(\epsilon), \quad \epsilon = 0, \frac{1}{2}, \frac{3}{2} \]

\[ \frac{1}{\Gamma(w)} \int_{0}^{\infty} \frac{\log t}{\sqrt{t}} e^{t(1-s^2)} t^{w-1} \, dt = \frac{\Gamma(w - 1/2)}{\Gamma(w)} (s^2 - 1)^{1/2 - w} \left( \Psi(w - 1/2) - \log(s^2 - 1) \right). \]

Here \(\Psi(z)\) is the logarithmic derivative of \(\Gamma(z)\). Regularity follows from the fact that \(\frac{1}{\Gamma(w)}\) vanishes at \(w = 0\). See \[Efr88, Efr91\] Prop 2 and 3 for more details. \(\square\)
Next, for Re(s) > 2, define the regularized determinant by

\[ \det(\Delta -(1-s^2)) = e^{-\mathcal{H}_{\chi}(0,s)}. \]

Our main result, Theorem \ref{thm:main_result}, will give the meromorphic continuation to Re(s) ≤ 2.

5. Selberg’s zeta-function and the regularized determinant

5.1. The Definition of the Selberg zeta-function. In this section we define the Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations. For more details see [Fri05b]. Suppose \( T \in \Gamma \) is loxodromic (we consider hyperbolic elements as loxodromic elements). Then \( T \) is conjugate in \( \text{PSL}(2, \mathbb{C}) \) to a unique element of the form

\[ D(T) = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix} \]

such that \( a(T) \in \mathbb{C} \) has \( |a(T)| > 1 \). Let \( N(T) \) denote the norm of \( T \), defined by

\[ N(T) \equiv |a(T)|^2, \]

and let by \( C(T) \) denote the centralizer of \( T \) in \( \Gamma \). There exists a (primitive) loxodromic element \( T_0 \), and a finite cyclic elliptic subgroup \( \mathcal{E}(T) \) of order \( m(T) \), generated by an element \( E_T \) such that

\[ C(T) = \langle T_0 \rangle \times \mathcal{E}(T). \]

Here \( \langle T_0 \rangle = \{ T_0^n \mid n \in \mathbb{Z} \} \). Next, Let \( t_1, \ldots, t_n \), and \( t_1', \ldots, t_n' \) denote the eigenvalues of \( \chi(T_0) \) and \( \chi(E_T) \) respectively. The elliptic element \( E_T \) is conjugate in \( \text{PSL}(2, \mathbb{C}) \) to an element of the form

\[ \begin{pmatrix} \zeta(T_0) & 0 \\ 0 & \zeta(T_0)^{-1} \end{pmatrix}, \]

where here \( \zeta(T_0) \) is a primitive \( 2m(T) \)-th root of unity.

For Re(s) > 1 the Selberg zeta-function \( Z(s, \Gamma, \chi) \) is defined by

\[ Z(s, \Gamma, \chi) \equiv \prod_{\{T_0\} \in R} \prod_{j=1}^{\dim V} \prod_{l,k \geq 0 \atop c(T,j,l,k) = 1} \left( 1 - t_j a(T_0)^{-2k} a(T_0)^{-2l} N(T_0)^{s-1} \right). \]

Here the product with respect to \( T_0 \) extends over a maximal reduced system \( R \) of \( \Gamma \)-conjugacy classes of primitive loxodromic elements of \( \Gamma \). The system \( R \) is called reduced if no two of its elements have representatives with the same centralizer. The function \( c(T,j,l,k) \) is defined by

\[ c(T,j,l,k) = t_j' \zeta(T_0)^{2l} \zeta(T_0)^{-2k}. \]

\[ \text{See [EGM95b] section 5.4 for more details.} \]
5.2. The Relationship Between the Selberg Zeta-Function and the Regularized Determinant.
One way to study the Selberg zeta-function is to apply the Selberg trace formula to the pair of functions,
\[
h(w) = \frac{1}{s^2 + w - 1} - \frac{1}{B^2 + w - 1} \quad \text{and} \quad g(x) = \frac{1}{2s} e^{-s|x|} - \frac{1}{2B} e^{-B|x|},
\]
where \(1 < \text{Re}(s) < \text{Re}(B)\). Let \(Z(s) \equiv Z(s, \Gamma, \chi)\) be the Selberg zeta-function under Assumption 3.1. We have [Fri05b].

Lemma 5.1.
\[
\frac{1}{2s} \frac{Z'(s)}{Z(s)} - \frac{1}{2B} \frac{Z'(B)}{Z(B)} = \frac{1}{2s} \sum_{(T) \cos} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T)|a(T) - a(T)^{-1}|^2} N(T)^{-s} - \frac{1}{2B} \sum_{(T) \cos} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T)|a(T) - a(T)^{-1}|^2} N(T)^{-B}
\]
\[
= \sum_{n \in L} \left( \frac{1}{\lambda_n - (1 - s^2)} - \frac{1}{\lambda_n - (1 - B^2)} \right) - \frac{1}{4\pi} \int_{\mathbb{R}} \left( \frac{1}{s^2 + x^2} - \frac{1}{B^2 + x^2} \right) \frac{d}{d\phi}(ix) \, dx
\]
\[
+ \frac{l}{2\pi[\Gamma_\infty : \Gamma_\infty']} \int_{\mathbb{R}} \left( \frac{1}{s^2 + x^2} - \frac{1}{B^2 + x^2} \right) \frac{\Gamma'(1 + ix)}{\Gamma(1 + ix)} \, dx + \frac{\text{tr} \mathcal{G}(0)}{4s^2} - \frac{\text{tr} \mathcal{G}(0)}{4B^2}
\]
\[
- \sum_{i=1}^{l} \frac{\text{tr} \chi(g_i)}{|C(g_i)||1 - e^{-2s}|^2} \int_{0}^{\infty} \left( e^{-sx} - \frac{e^{-Bx}}{2s} \right) \frac{\sinh x}{\cosh x - 1 + \frac{|1 - e^{-2s}|^2}{2}} \, dx
\]
\[
- \left( \frac{1}{2s} - \frac{1}{2B} \right) \sum_{(R) \cos} \frac{\text{tr} \chi(R) \log N(T_0)}{4[\Gamma_\infty : \Gamma_\infty']^2} \frac{\text{vol} \Gamma \setminus \mathbb{H}^3 \dim V}{4\pi} \frac{\log x}{x}
\]
\[
- \left( \frac{1}{2s} - \frac{1}{2B} \right) \sum_{i=1}^{l} \frac{2 \text{tr} \chi(g_i) \log \frac{|c_i|}{|1 - e^{-2s}|^2}}{|C(g_i)||1 - e^{-2s}|^2} \left( \frac{1}{2s} - \frac{1}{2B} \right) \frac{1}{[\Gamma_\infty : \Gamma_\infty']} \left( \log \frac{\eta_{\infty}}{2} - \gamma \right) + \sum_{t=l_{\infty} + 1}^{n} L(\Lambda_\infty, \psi_t).
\]

Recall that
\[
H(w, s) = \sum_{m \in D} ((\lambda_m - 1) + s^2)^{-w} \frac{1}{4\pi} \int_{\mathbb{R}} (x^2 + s^2)^{-w} \frac{d}{d\phi}(ix) \, dx + s^{-2w} \frac{1}{4} (\text{tr} \mathcal{G}(0) + k(\Gamma, \chi))
\]
\[
+ s^{(1-2w)} \frac{k(\Gamma, \chi)}{\Gamma(w-1/2)} \frac{\Gamma(w-1/2)}{\Gamma(w)} \log Y.
\]

Applying the following elementary equations:
\[
- \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{ds} \left( \frac{\partial}{\partial w} (u + s^2)^{-w} \bigg|_{w=0} \right) \right) = - \frac{\partial}{\partial w} \left( \frac{1}{2s} \frac{d}{ds} (u + s^2)^{-w} \bigg|_{w=0} \right) = \frac{-2s}{(u + s^2)^2},
\]
\[
- \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{ds} \left( \frac{\partial}{\partial w} s^{(1-2w)} \frac{\Gamma(w-1/2)}{\Gamma(w)} \bigg|_{w=0} \right) \right) = - \frac{\sqrt{\pi}}{s^2}.
\]
to $H(w, s)$, we see that

$$
- \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{dw} H(w, s) \bigg|_{w=0} \right) = - \frac{\partial}{\partial w} \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{dw} H(w, s) \right) \bigg|_{w=0} = \sum_{m \in \mathbb{D}} \frac{-2s}{(\lambda_m - 1) + s^2}^2 - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{-2s - \phi'(ix)}{(x^2 + s^2)^2} \phi(x) dx - \frac{k(\Gamma, \chi)}{2s^2} \log Y.
$$

**Caution.** Differentiation through the sum and integral is justified by regularity at $w = 0$ (Lemma 4.4). Instead of differentiating first with respect to $w$ at $w = 0$, we switch the order of differentiation, differentiate with respect to $s$, and restrict $w$ so that $w > 2$ (where the sum and integral converge uniformly). Finally we differentiate with respect to $w$, and using analytic continuation (and uniqueness of analytic continuation), we obtain (5.2).

For $\text{Re}(s) > 0$ set

$$
\Omega(s) = - \sum_{i=1}^{l} \frac{\text{tr} \chi(g_i)}{|C(g_i)||1 - \epsilon_i^2|^2} \int_0^{\infty} e^{-s x} \sinh x \left( \frac{1}{\cosh x - 1 + \frac{1 - \epsilon_i^2}{2}} \right) dx, \text{ for } \epsilon_i \neq 1,
$$

(see 3.1.3 for the definition of $\epsilon_i$ and $g_i$). Using the equation

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{(s^2 + x^2)^2} \Gamma(1 + ix) dx = \frac{1}{s} \Gamma(1 + s),$$

it follows that

$$
\frac{d}{ds} \left( \frac{1}{2s} Z'(s) \right) = \sum_{n \in D} \frac{-2s}{(\lambda_n - 1 + s^2)^2} - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{-2s - \phi'(ix)}{(s^2 + x^2)^2} \phi(x) dx
$$

$$
+ \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{ds} \Omega(s) \right) + \frac{1}{2s^2} \sum_{(R) \text{nc}} \text{tr} \chi(R) \log N(T) \left( \frac{\Gamma'(s + 1)}{\Gamma(s + 1)} \right) - \frac{1}{16s^2} \sum_{(R) \text{nc}} \text{tr} \Gamma(0) \left( \frac{\Gamma'(s + 1)}{\Gamma(s + 1)} \right)
$$

$$
+ \frac{1}{2s^2} \left[ \sum_{i=1}^{l} \frac{2 \text{tr} \chi(g_i) \log |c_i|}{|C(g_i)||1 - \epsilon_i^2|^2} + \frac{1}{\Gamma_{\infty} : \Gamma'_{\infty}} \left( \frac{\eta_{\infty}}{2} - \gamma \right) + \sum_{l=1}_{l_{\infty} + 1} L(A_{\infty}, \psi) \right]
$$

$$
+ \frac{\text{vol}(\Gamma \setminus \mathbb{H}^3) \dim V}{4\pi}.
$$

Simplifying, and recalling the definition of $\det(\Delta - (1 - s^2))$ we arrive at

$$
\frac{d}{ds} \left( \frac{1}{2s} Z'(s) \right) = \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{ds} \left( \log(\det(\Delta - (1 - s^2))) \right) \right) + \frac{k(\Gamma, \chi)}{2s^2} \log Y - \frac{d}{ds} \frac{1}{2s} \left( \frac{1}{\Gamma_{\infty} : \Gamma'_{\infty}} \Gamma'(s + 1) \right)
$$

$$
- \frac{d}{ds} \left( \frac{1}{2s} \frac{d}{ds} \Omega(s) \right) + \frac{1}{2s^2} C_1 - \frac{1}{2s^3} C_2 + C_3,
$$

where $C_1, C_2, C_3$ are constants.
where $C_1, C_2, C_3$ are easily read off from (5.4). Next, integrating twice, we obtain:

$$
\log Z(s) = \log \det (\Delta - (1 - s^2)) - (s)k(\Gamma, \chi) \log Y - \frac{l_{\infty}}{[\Gamma_\infty : \Gamma_\infty']} \log \Gamma(s + 1)
- \Omega(s) - sC_1 + \frac{C_2}{2} \log s + \frac{2}{3} C_3 s^3 + D_1 + s^2 D_2,
$$

where $D_1, D_2$ are constants of integration. They can be determined by letting $s \to \infty$, and applying Lemma 4.4. More specifically, Lemma 4.4 tells us the asymptotic growth of $\log \det (\Delta - (1 - s^2))$ as $s \to \infty$. Noting that

$$
\lim_{s \to \infty} \log Z(s) = 0, \quad \lim_{s \to \infty} \Omega(s) = 0,
$$

and applying Sterling’s formula we obtain:

$$
D_1 = \gamma + \log(4) - 2 + \frac{l_{\infty}}{[\Gamma_\infty : \Gamma_\infty']} \log \sqrt{2\pi}, \quad D_2 = 0.
$$

We have proved

**Theorem 5.2.** For $\Re(s) > 2$,

$$
\log \det (\Delta - (1 - s^2)) = \log Z(s, \Gamma, \chi) + (s)k(\Gamma, \chi) \log(Y) + C_1
+ \frac{l_{\infty}}{[\Gamma_\infty : \Gamma_\infty']} \log \Gamma(s + 1) + \Omega(s) - \frac{C_2}{2} \log s - \frac{2}{3} C_3 s^3 - D_1,
$$

where

$$
C_1 = \sum_{\{R\} \in \mathcal{E}} \text{tr} \chi(R) \log N(T_0) \frac{1}{4|\mathcal{E}(R)|} \sin^2 \left( \frac{\pi k}{\text{disc}(R)} \right)
+ \left[ \sum_{i=1}^l \frac{2 \text{tr} \chi(g_i) \log |c_i|}{|C(g_i)|} \left[ 1 - \epsilon_i^2 \right] + \frac{1}{[\Gamma_\infty : \Gamma_\infty']} \left( l_{\infty} \left( \frac{\eta_{\infty}}{2} - \gamma \right) + \sum_{l=l_{\infty}+1}^n \frac{\text{tr} \chi(\psi_l)}{\text{vol} (\Gamma \setminus \mathbb{H}^3)} \text{dim} V \right) \right],
$$

$$
C_2 = \left( \text{tr} \chi(0) - \frac{l_{\infty}}{[\Gamma_\infty : \Gamma_\infty']} \right),
$$

$$
C_3 = \frac{\text{vol} (\Gamma \setminus \mathbb{H}^3) \text{dim} V}{4\pi},
$$

and

$$
D_1 = \gamma + \log(4) - 2 + \frac{l_{\infty}}{[\Gamma_\infty : \Gamma_\infty']} \log \sqrt{2\pi}.
$$

**Corollary 5.3.** Let $\Gamma$ be torsion-free with one cusp at $\infty$, and let $\chi$ be a regular character (a one-dimensional unitary representation). Then

$$
\det (\Delta - (1 - s^2)) = Z(s, \Gamma, \chi) \exp \left( -s^3 \frac{\text{vol} (\Gamma \setminus \mathbb{H}^3)}{6\pi} + sL(\Lambda_\infty, \psi) \right).
$$

**Corollary 5.4.** Let $\Gamma$ be cocompact, and let $\chi$ be a regular character. Then

$$
\det (\Delta - (1 - s^2)) = Z(s, \Gamma, \chi) \exp \left( -s^3 \frac{\text{vol} (\Gamma \setminus \mathbb{H}^3)}{6\pi} + sC \right),
$$
where
\[ C = \sum_{\{ R \}_{\text{ne}}} \text{tr} V \chi(R) \frac{\log N(T_0)}{4|E(R)| \sin^2 \left( \frac{\pi k}{m(R)} \right)}. \]

**APPENDIX A. PROOF OF THEOREM 3.6**

In this section we prove Theorem 3.6. That is for \( P = z + rj, P' = z' + r'j \in \mathcal{F} \), we show that
\[ K_{\Gamma}(P, P', t, \chi) = P_{\infty} \exp \left( \frac{-t}{|P| \sqrt{4\pi t}} \right) \exp \left( -\frac{\log^2 (r/r')}{4t} \right) + O(1). \]

As usual we are under Assumption 3.1.

The first step is to split up \( K_{\Gamma}(P, P', t, \chi) \) as
\[ K_{\Gamma}(P, P', t, \chi) = \sum_{\gamma \in \Gamma} \chi(\gamma) k_t(\delta(P, \gamma P')). \]

From [EGM98, Equation 4.5.9 or Lemma 6.4.2], it follows that
\[ \sum_{\gamma \in \Gamma \setminus \Gamma_{\infty}} |k_t(\delta(P, \gamma P'))| = O(1) \text{ as } r \to \infty. \]

Hence
\[ \sum_{\gamma \in \Gamma \setminus \Gamma_{\infty}} \chi(\gamma) k_t(\delta(P, \gamma P')) = O(1) \text{ as } r \to \infty. \]

It remains to estimate
\[ f_{\infty}(P, P') = \sum_{\gamma \in \Gamma_{\infty}} \chi(\gamma) k_t(\delta(P, \gamma P')). \]

The subgroup \( \Gamma_{\infty} \) is not an abelian group. So in general we can not diagonalize the unitary representation, \( \chi \) restricted to \( \Gamma_{\infty} \), into unitary characters. However, we have the following lemma which is almost as good as diagonalizing \( \chi \) (see [Fri05b, Lemma 2.4]).

**Lemma A.1.** Let \( \infty \) be the one cusp of \( \Gamma \) (Assumption 3.1). Then there exist \( E, R, S \in \Gamma_{\infty} \) with the following properties:

1. \( \Gamma_{\infty} = \{ E^k R^i S^j \mid 0 \leq k < m, i, j \in \mathbb{Z} \} \). Here \( R, S \) are parabolic elements with \( R(P) = P + 1 \) and \( S(P) = P + \tau \) (here \( \Lambda_{\infty} = \mathbb{Z} \oplus \mathbb{Z} \tau \)) for all \( P \in \mathbb{H}^3 \), and \( E \) is elliptic of order \( m \).
2. \( \Gamma'_{\infty} = \{ R^i S^j \mid i, j \in \mathbb{Z} \} \).
3. The elements \( R \) and \( S \) commute, but the group \( \Gamma_{\infty} \) is not abelian when \( m > 1 \).
4. If in addition, \( m > 1 \), then \( \chi(E) \) maps \( V_{\infty}' \) onto itself. Furthermore, there exists a basis of \( V_{\infty}' \) so that \( \chi(E)|_{V_{\infty}'} \) is diagonal.

The notation used in the lemma above is explained in §3.1.

Next we split \( f_{\infty}(P, P') \) into two sums,
\[ f_{\infty}(P, P') = \sum_{\gamma \in \Gamma_{\infty}} P_{\infty} \chi(\gamma) k_t(\delta(P, \gamma P')) + \sum_{\gamma \in \Gamma_{\infty}} (I_V - P_{\infty}) \chi(\gamma) k_t(\delta(P, \gamma P')). \]
Lemma A.2 (Poisson Summation Formula). Let \( f : \mathbb{R}^2 \to \mathbb{C} \) be a Schwartz-class function, and let \( \Lambda \) be a two-dimensional lattice in \( \mathbb{R}^2 = \mathbb{C} \). Then

\[
\sum_{\omega \in \Lambda} f(\omega) = \frac{1}{|\Lambda|} \sum_{\omega \in \Lambda^0} \hat{f}(\omega).
\]

Here \( \Lambda^0 \) is the dual lattice to \( \Lambda \), and \( \hat{f} \) is the Fourier transform of \( f \),

\[
\hat{f}(z) = \int_{\mathbb{R}^2} f(u) e^{-2\pi i \langle u, z \rangle} \, du,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard real inner product on \( \mathbb{C} = \mathbb{R}^2 \).

Lemma A.3.

\[
\sum_{\gamma \in \Gamma_\infty} P_\infty \chi(\gamma) k_t(\delta(P, \gamma P')) = P_\infty \exp\left(- \frac{\log^2(r/{r'})}{4t} \right) + O(1).
\]

Proof. It follows from the definition of singularity that for \( \gamma \in \Gamma_\infty \), \( P_\infty \chi(\gamma) = P_\infty \). Thus

\[
\sum_{\gamma \in \Gamma_\infty} P_\infty \chi(\gamma) k_t(\delta(P, \gamma P')) = P_\infty \sum_{\gamma \in \Gamma_\infty} k_t(\delta(P, \gamma P')).
\]

Using Part 1 of Lemma A.1, we can rewrite the above sum as

\[
P_\infty \sum_{k=0}^m \sum_{i,j \in \mathbb{Z}^2} k_t(\delta(P, E^k R^i S^j P')) = P_\infty \sum_{k=0}^m \sum_{i,j \in \mathbb{Z}^2} k_t(\delta(E^{-k} P, R^i S^j P')).
\]

The last equality follows because \( \delta \) is a point-pair invariant. Once again applying Lemma A.1 and the definition of \( \delta \) we can write the above sum as

\[
P_\infty \sum_{k=0}^m \sum_{\omega \in \Lambda_\infty} f_k(\omega),
\]

where

\[
f_k(u) = k_t \left( \frac{|z - z(E^k P') + u|^2 + r^2 + r'^2}{2rr'} \right).
\]

We have used\(^8\) the fact that \( r(E^k P) = r(P) = r \). Next we apply the Poisson summation to obtain

\[
\sum_{\omega \in \Lambda_\infty} f_k(\omega) = |\Lambda_\infty|^{-1} \hat{f}_k(0) + |\Lambda_\infty|^{-1} \sum_{\omega \in \Lambda_\infty \setminus \omega \neq 0} \hat{f}_k(\omega).
\]

A straightforward computation\(^{\text{[EGM98, Lemma 3.5.5]}}\) shows\(^9\) that

\[
\hat{f}_k(0) = rr' g \left( \log \left( \frac{r}{r'} \right) \right)
\]

where

\[
g(x) = \frac{\exp(-t)}{\sqrt{4\pi t}} \exp \left( -\frac{x^2}{4t} \right).
\]

---

\(^8\) We abuse notation here and let \( r, z \) represent both coordinates in \( \mathbb{H}^3 \) and coordinate functions.

\(^9\) \text{[EGM98, Lemma 6.4.2]} \) seems to have an extra factor of “2.”
Noting that \( \hat{f}_k(0) \) is independent of \( k \), and that
\[
\frac{m}{|\Lambda_\infty|} = \frac{|E|}{|\Lambda_\infty|} = \frac{[\Gamma_\infty : \Gamma'_\infty]}{|\Lambda_\infty|} = \frac{1}{|P|}
\]
we recover the leading term of the lemma. To conclude we show that
\[
\sum_{\omega \in \Lambda_\infty, \omega \neq 0} \hat{f}_k(\omega) = O(1).
\]
Since \( f_k \) is smooth, \( \hat{f}_k \) is of rapid decay, hence [EGM98, Lemma 6.4.2] (A.1)
\[
\hat{f}_k(v) = O((rr')^{-N}|v|^{-2N}) \quad \text{for any } N > 0.
\]
The lemma now follows. \( \square \)

Equation (A.1) will be used to show that
\[
\sum_{\gamma \in \Gamma_\infty} (I_V - P_\infty) \chi(\gamma)k_t(\delta(P, \gamma P')) = O(1) \quad \text{as } r \to \infty.
\]

In order to proceed, we need to understand \( I_V - P_\infty \) as projection operator on the subspace \( V_\infty^\perp \).

First, decompose \( V = V_\infty \oplus V'_\infty \oplus V_b \).

By Lemma [A.1] the unitary representation \( \chi \) restricted to \( \Gamma_\infty \) can be diagonalized in a block matrix form with respect to the decomposition \( V = V_\infty \oplus V'_\infty \oplus V_b \).

Hence we can write \( I_V - P_\infty = P_a + P_b \), where \( P_a \) is the orthogonal projection onto \( V'_\infty \) and \( P_b \) is the orthogonal projection onto \( V_b \).

Lemma A.4.
\[
\sum_{\gamma \in \Gamma_\infty} (I_V - P_\infty) \chi(\gamma)k_t(\delta(P, \gamma P')) = O(1).
\]

Proof. We first show that
\[
\sum_{\gamma \in \Gamma_\infty} P_a \chi(\gamma)k_t(\delta(P, \gamma P')) = O(1).
\]
The proof is almost identical to the proof of Lemma [A.3] except that the term corresponding to \( \omega = 0 \) is zero. Indeed,
\[
P_a \sum_{k=0}^m \sum_{i,j \in \mathbb{Z}^2} \chi(E^k R^i S^j)k_t(\delta(P, E^k R^i S^j P')) = \left( \sum_{k=0}^m \chi(E)^k \right) P_a \sum_{i,j \in \mathbb{Z}^2} k_t(\delta(P, E^k R^i S^j P')),
\]
where \( \chi(E) \) is a diagonal matrix with each element on the diagonal a finite root of unity not equal to one. Since the order of each root of unity divides \( m \) we must have
\[
\sum_{k=0}^m \chi(E)^k = 0.
\]
Hence the “constant term” (the term corresponding to \( \omega = 0 \) cancels out).

Next, to show that
\[
\sum_{\gamma \in \Gamma_\infty} P_b \chi(\gamma)k_t(\delta(P, \gamma P'))
\]
is bounded it suffices to estimate the lattice sum
\[(A.2) \quad P_b \sum_{l,j \in \mathbb{Z}^2} k_t(\delta(P; R^1 S^j P')).\]

Since $R$ and $S$ commute, we can diagonalize $\chi(\Gamma'_{\infty})$ restricted to $V^b_{\infty}$. Hence we can assume that $\chi$ is a lattice character of the form
\[\chi(R^1 S^l) = \exp(2\pi i (lR + nS)).\]

Now we can rewrite (A.2) as
\[\sum_{l,n \in \mathbb{Z}^2} \exp(2\pi i (lR + nS))k_t(\delta(z + rj, l + n\tau(z' + r'j))).\]

By unraveling the definition of $P_b$, it follows that at least one of $\theta_R, \theta_S$ is not an integer. By applying the Poisson summation formula to the function
\[f_1(w, v) = \exp(2\pi i (w\theta_R + v\theta_S))k_t(\delta(z + rj, w + v\tau(z' + r'j))),\]
we obtain
\[\sum_{l,n \in \mathbb{Z}^2} f_1(l, n) = \sum_{l,n \in \mathbb{Z}^2} \hat{f}_1(l, n).\]

However the exponential factor $\exp(2\pi i (l\theta_R + n\theta_S))$ shifts the Fourier transform of the function $f_2(w, v) = k_t(\delta(z + rj, w + v\tau(z' + r'j)))$, and wipes out the unbounded “constant term” $\hat{f}_2(0, 0)$. In other words, if we applied the Poisson summation formula to $f_2$, we would see, as we did with $f_k$, that
\[\sum_{l,n \in \mathbb{Z}^2} f_2(l, n) = \hat{f}_2(0, 0) + \sum_{l,n \in \mathbb{Z}^2 \atop (l,n) \neq (0,0)} \hat{f}_2(l, n).\]

The latter sum decays, while the first term would not. The effect of multiplying $\exp(2\pi i (w\theta_R + v\theta_S))$ is to shift the sum away from the integers. That is
\[\sum_{l,n \in \mathbb{Z}^2} \hat{f}_1(l, n) = \sum_{l,n \in \mathbb{Z}^2} f_2(l + \alpha, n + \beta),\]
where
\[(0,0) \notin \mathbb{Z}^2 + (\alpha, \beta).\]

Hence we can apply (A.1) to conclude the lemma.
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REFERENCES

[BCZ97] Andrei A. Bytsenko, Guido Cognola, and Sergio Zerbini, *Determinant of the Laplacian on a non-compact three-dimensional hyperbolic manifold with finite volume*, J. Phys. A 30 (1997), no. 10, 3543–3552.

[BJP] David Borthwick, Chris Judge, and Peter Perry, *Determinants of Laplacians and isopolar metrics on surfaces of infinite area*, Duke Math. J., to appear.

[BK62] M. S. Birman and M. G. Kreĭn, *On the theory of wave operators and scattering operators*, Dokl. Akad. Nauk SSSR 144 (1962), 475–478. MR MR0139007 (25 #2447)

[Dav89] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. MR MR990239 (90c:35123)

[DC76] J. S. Dowker and Raymond Critchley, *Scalar effective Lagrangian in de Sitter space*, Phys. Rev. D (3) 13 (1976), no. 2, 224–234. MR MR0449343 (56 #7648)

[DP86] Eric D’Hoker and D. H. Phong, *On determinants of Laplacians on Riemann surfaces*, Comm. Math. Phys. 104 (1986), no. 4, 537–545.

[Efr91] Isaac Efrat, *Determinants of Laplacians on surfaces of finite volume*, Comm. Math. Phys. 119 (1988), no. 3, 443–451.

[Efr91] ———, Erratum: "Determinants of Laplacians on surfaces of finite volume" [Comm. Math. Phys. 119 (1988), no. 3, 443–451; MR 90c:58184], Comm. Math. Phys. 138 (1991), no. 3, 607.

[EGM98] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998, Harmonic analysis and number theory.

[EOR+94] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko, and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific Publishing Co. Inc., River Edge, NJ, 1994. MR MR1346490 (96m:81156)

[Fri05a] Joshua S. Friedman, *The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations*, Ph.D. thesis, Stony Brook University, 2005, http://arxiv.org/abs/math.NT/0612807.

[Fri05b] ———, *The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations*, Math. Z. 250 (2005), no. 4, 939–965. MR MR2180383

[GK65] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Fourth edition prepared by Ju. V. Geronimus and M. Ju. Ceĭtlin. Translated from the Russian by Scripta Technica, Inc. Translation edited by Alan Jeffrey, Academic Press, New York, 1965.

[Haw77] S. W. Hawking, *Zeta function regularization of path integrals in curved spacetime*, Comm. Math. Phys. 55 (1977), no. 2, 133–148. MR MR0524257 (58 #25823)

[Kre53] M. G. Kreĭn, *On the trace formula in perturbation theory*, Mat. Sbornik N.S. 33(75) (1953), 597–626.

[Müll83] Werner Müller, *Spectral theory for Riemannian manifolds with cusps and a related trace formula*, Math. Nachr. 111 (1983), 197–288.

[Müll87] ———, *Manifolds with cusps of rank one*, Lecture Notes in Mathematics, vol. 1244, Springer-Verlag, Berlin, 1987, Spectral theory and $L^2$-index theorem.

[Müll92] ———, *Spectral geometry and scattering theory for certain complete surfaces of finite volume*, Invent. Math. 109 (1992), no. 2, 265–305.

[Müll98] ———, *Relative zeta functions, relative determinants and scattering theory*, Comm. Math. Phys. 192 (1998), no. 2, 309–347.

[Par05] Jinsung Park, *Eta invariants and regularized determinants for odd dimensional hyperbolic manifolds with cusps*, Amer. J. Math. 127 (2005), no. 3, 493–534.

[Sar97] Peter Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. 110 (1987), no. 1, 113–120.

[Ven82] A. B. Venkov, *Spectral theory of automorphic functions*, Proc. Steklov Inst. Math. (1982), no. 4(153), ix+163 pp. (1983), A translation of Trudy Mat. Inst. Steklov. 153 (1981).

[VKF73] A. B. Venkov, V. L. Kalinin, and L. D. Faddeev, *A nonarithmetic derivation of the Selberg trace formula*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 37 (1973), 5–42, Differential geometry, Lie groups and mechanics. MR MR0506043 (58 #21949)

[Yaf92] D. R. Yafaev, *Mathematical scattering theory*, Translations of Mathematical Monographs, vol. 105, American Mathematical Society, Providence, RI, 1992, General theory, Translated from the Russian by J. R. Schulenberger. MR MR1180965 (94f:47012)
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