On the Capacity Enlargement of Gaussian Broadcast Channels with Passive Noisy Feedback

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Abstract

It is well known that the capacity region of an average transmit power constrained Gaussian Broadcast Channel (GBC) with independent noise realizations at the receivers is enlarged by the presence of causal noiseless feedback. Capacity region enlargement is also known to be possible by using only passive noisy feedback, when the GBC has identical noise variances at the receivers. The last fact remains true even when the feedback noise variance is very high, and available only from one of the receivers. While such capacity enlargements are feasible for several other feedback models in the Gaussian BC setting, it is also known that feedback does not change the capacity region for physically degraded broadcast channels. In this paper, we consider a two user GBC with independent noise realizations at the receivers, where the feedback links from the receivers are corrupted by independent additive Gaussian noise processes. We investigate the set of four noise variances, two forward and two feedback, for which no capacity enlargement is possible. A sharp characterization of this region is derived, i.e., any quadruple outside the presented region will lead to a capacity enlargement, whereas quadruples inside will leave the capacity region unchanged. Our results lead to the conclusion that when the forward noise variances are different, too noisy a feedback from one of the receivers alone is not always beneficial.

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for enlarging the capacity region, be it from the stronger user or the weaker one, in sharp contrast to the case of equal forward noise variances.

Index Terms

Gaussian Broadcast Channel, Noisy Feedback, Capacity Region, Feedback Coding, Capacity Enlargement.

I. INTRODUCTION

Feedback does not increase the capacity of a memoryless point-to-point channel, a result which goes back to C. E. Shannon [1]. However, feedback has a positive impact in simplifying coding schemes and boosting error exponents [2]. With the discovery of capacity regions for several multiuser models in the ’70s and ’80s, it was of significant interest to find the impact of feedback on these models. In the absence of feedback, identifying suitable auxiliary variables which can lead to single letter expressions for the rate-region turned out to be a key step in discovering the capacity region of a degraded Broadcast Channel (BC) [3]. Notice that, without any feedback, one can turn a stochastically degraded BC to an equivalent physical degraded setup, as only the marginal distributions to the individual receivers matter in decay of the error probability with blocklength. When perfect causal feedback is available in a two user BC, a single letter characterization of the capacity region in terms of one auxiliary variable is obtained by El Gamal in [4] and [5], where the physical degradation assumption is crucial. More specifically, [4] and [5] respectively show that the capacity regions of a physically degraded BC and a physically degraded Gaussian BC (GBC) are unchanged by the presence of causal feedback. In contrast, Dueck [6] demonstrated a BC for which rate pairs outside its no-feedback capacity region can be attained using feedback. For a stochastically degraded two user GBC, Ozarow and Leung developed a feedback coding scheme to show that perfect noiseless feedback from both the receivers enlarges the capacity region [7], when there is no physical degradation. It was later shown that perfect noiseless feedback from the stronger receiver was sufficient to enlarge the capacity region [8]. The optimism of capacity enlargement using feedback did carry over to a variety of models. Recent works considered BCs with noisy feedback [9], [10], [11] and rate limited feedback [12]. In fact, for a two user GBC with equal receiver noise variances, passive noisy feedback from any one of the receivers enlarges the capacity region, even when the feedback noise is of very high variance [10].
It was shown in [13] that noisy feedback always enlarges the capacity region of a Gaussian Multiple Access Channel (MAC), a fact which remains true with the availability of feedback to only one of the transmitters. Furthermore, a MAC-BC duality while employing linear feedback coding schemes with noiseless feedback is known [14], allowing many of the MAC results to be relevant for the BC as well. In another related result, [15] showed that the gains due to feedback can be potentially unbounded in correlated noise channels. In summary, capacity enlargement for Gaussian BCs using noisy feedback turned out to be true for several models which are not physically degraded. That this is not always the case when the receivers have different noise variances is shown in the present paper. The results here significantly expand some of the initial results in the conference versions [16], [17], which considered one sided feedback from the stronger receiver. The main interest here is in identifying the set of four noise variances, corresponding to two forward noise processes and two feedback noise processes, such that the capacity region differs from that without feedback. We wish to highlight the following aspects of the paper:

- noisy feedback from both the receivers are considered.
- the exact threshold behaviour is characterized, i.e., any set of four noise variances can be classified based on whether the capacity region of the GBC with feedback, having these parameters, is enlarged or not.
- that too noisy a feedback from the weaker receiver of a two user GBC does not enlarge the capacity region was hitherto unknown.

A. Notations:

For a positive integer \( n \), we use \( U^n \) to denote the tuple \( U_1, U_2, \ldots, U_n \). The diagonal square matrix of size \( n \times n \) is denoted by \( \text{diag}(\bar{d}) \), where the diagonal elements are given by the vector \( \bar{d} \). We write \( U \sim \mathcal{N}(\mu, K) \) to denote a random vector \( U \) having a Multivariate Gaussian Distribution with covariance matrix \( K \). The acronyms LHS and RHS stand for Left Hand Side and Right Hand Side respectively, of the mathematical equation in consideration. Logarithms are taken with respect to base 2.

B. Paper Organization:

The organization of the paper is as follows. We introduce the two user scalar GBC with passive noisy feedback from both the receivers in the next section, and present our objectives
and the main result. Before we proceed to the detailed proof, we take a detour in Section III to present results on the utility of feedback in some related channel models. Then, in order to prove the main result, we first develop a converse argument in Section IV, to show there are regimes of possible noise variances (four parameters, two forward and two feedback) where the capacity region remains the same as that without feedback. The boundary of the regime thus characterized is then shown to be sharp, by constructing an achievable region which strictly enlarges the no feedback capacity for noise parameters exterior to the regime. This is presented in Section V. Finally, Section VI concludes the paper.

II. MODEL AND RESULTS

Consider a memoryless two user scalar GBC, as shown in Fig. 1. Assume independent memoryless noisy feedback links from both receivers to the transmitter. In this setup, $X$ refers to the signal transmitted by the sender, and the additive noise processes $(Z_1, Z_2)$ in the forward links are distributed according to $\mathcal{N}(0, \text{diag}(\sigma_1^2, \sigma_2^2))$. Unless otherwise stated, we will assume

$$\sigma_1^2 \leq \sigma_2^2, \quad (1)$$

i.e., receiver 2’s outputs are more noisy than receiver 1’s. The passive causal feedback links are corrupted by additive noise $(Z_{fb1}, Z_{fb2}) \sim \mathcal{N}(0, \text{diag}(\sigma_{fb1}^2, \sigma_{fb2}^2))$, independent of the forward noise processes.

A pair of messages $(W_1, W_2)$, independently and uniformly chosen from $\{1, \cdots, 2^{nR_1}\} \times \{1, \cdots, 2^{nR_2}\}$ is to be conveyed in $n$ channel uses of the GBC. The encoder chooses the symbol

![Diagram](image-url)
\(X_i\) at time \(i \in \{1, \ldots, n\}\) as a function of both the messages as well as causal noisy feedback from both receivers:

\[
X_i = g_i(W_1, W_2, Y_i^{i-1} + Z_{i-1}^{i-1} + Z_{i-1}^{i-1}).
\] (2)

The transmissions are constrained to meet an average power constraint \(P\), i.e. \(\sum_{i=1}^{n} E|X_i|^2 \leq nP\). After every block of \(n\) transmissions, each receiver attempts to decode the message intended to it. Let \(P_e(n)\) denote the average error probability that any of the two receivers makes a decoding error. We are interested in the capacity region \(C_{\text{noisy-fb}}\) of this model. Using standard definitions [2], the capacity region can be concisely defined as follows.

**Definition 1:** The capacity region \(C_{\text{noisy-fb}}\) is the convex closure of the set of all nonnegative rate pairs \((R_1, R_2)\) such that there exists a sequence of encoder-decoder pairs with \(P_e(n) \to 0\) as \(n \to \infty\).

Let \(C_{\text{wo-fb}}\) refer to the capacity region when there are no feedback links present, well known to be achieved by superposition coding [18]. \(C_{\text{wo-fb}}\) is the collection of nonnegative rate pairs \((R_1, R_2)\) such that

\[
R_1 \leq \frac{1}{2} \log \left(1 + \frac{\theta P}{\sigma_1^2}\right),
\] (3a)

\[
R_2 \leq \frac{1}{2} \log \left(1 + \frac{(1 - \theta) P}{\sigma_2^2}\right),
\] (3b)

for some \(\theta \in [0, 1]\). By ignoring feedback, it is easy to see

\[
C_{\text{wo-fb}} \subseteq C_{\text{noisy-fb}}.
\] (4)

For equal noise variances, i.e. \(\sigma_1^2 = \sigma_2^2\), Venkatramanan and Pradhan [10] showed that the inclusion in (4) is strict, irrespective of the feedback noise variances. In fact, it is shown that noisy feedback from even one of the two receivers always enlarges the capacity region. This brings the following question to the fore: For what set of parameters \((\sigma_1^2, \sigma_2^2, \sigma_{i1}^2, \sigma_{i2}^2)\) in \((\mathbb{R}^+ \cup \{\infty\})^4\) will the inclusion in (4) stays strict? Alternately, are there regimes of noise variances where feedback is rendered futile in enlarging the capacity region? Our main result is the following.

**Theorem 1:** For \(\sigma_1^2 \leq \sigma_2^2\), the relation \(C_{\text{wo-fb}} = C_{\text{noisy-fb}}\) holds if and only if

\[
\frac{\sigma_2^2}{\sigma_1^2} \geq \left(\frac{\sigma_1^2}{\sigma_{i1}^2} + 1\right) \left(\frac{\sigma_2^2}{\sigma_{i2}^2} + 1\right).
\] (5)

**Proof:** The proof involves two main steps. One is a converse argument to show that the capacity region is unchanged if the condition in (5) is met. The second part requires an achievable
communication scheme, which operates at rates outside the no feedback capacity region when (5) is not met. These details will be provided in the next two sections.

The following corollaries can be obtained by simple algebraic manipulations.

**Corollary 2 (One-sided feedback):** Consider a GBC with $\sigma_1^2 \leq \sigma_2^2$. When feedback is available only from receiver 2, i.e. $\sigma_{fb1}^2 = \infty$, the relation $C_{wo-fb} = C_{noisy-fb}^*$ holds if and only if

$$\sigma_{fb2}^2 \geq \sigma_2^2 \cdot \left( \frac{\sigma_2^2}{\sigma_1^2} - 1 \right)^{-1}.$$  \hspace{1cm} (6)

On the other hand, if feedback is only available from receiver 1, i.e. $\sigma_{fb2}^2 = \infty$, then the relation $C_{wo-fb} = C_{noisy-fb}^*$ holds if and only if

$$\sigma_{fb1}^2 \geq \sigma_1^2 \cdot \left( \frac{\sigma_1^2}{\sigma_2^2} - 1 \right)^{-1}.$$  \hspace{1cm} (7)

Above corollary can be rephrased to show that one-sided feedback from receiver 1 increases capacity if and only if $\sigma_{fb1}^2 \leq \frac{\sigma_1^2}{\sigma_{fb1}^2} \left( \frac{\sigma_1^2}{\sigma_{fb1}^2} + 1 \right)$. Similarly, one-sided feedback from receiver 2 increases capacity if and only if $\sigma_{fb2}^2 > \frac{\sigma_2^2}{\sigma_{fb2}^2} \sigma_{fb2}^2/\sigma_2^2 + 1$.

Fig. 2 depicts the regimes of noise-variances where capacity is enlarged with one-sided feedback from receiver 1. The figure also allows for $\sigma_1^2 > \sigma_2^2$. In this case, the desired result is obtained by swapping indices 1 and 2 in above corollary. The figure illustrates that for $\sigma_1^2 = \sigma_2^2$, one-sided feedback from receiver 1 is always helpful, no matter how noisy. Otherwise, if $\sigma_1^2 \neq \sigma_2^2$, then one-sided feedback from receiver 1 becomes useless when the feedback noise-variance exceeds a certain threshold. Similar statements hold also for one-sided feedback from receiver 2. In fact, the corresponding plot can be obtained from Fig. 2 by mirroring the plot on the $\sigma_1^2 = \sigma_2^2$ line.

Theorem 1 further provides the following corollary on the usefulness of feedback.

**Corollary 3:** If $\sigma_1^2 = \sigma_2^2$, feedback enlarges the capacity region whenever $\sigma_{fb1}^2 < \infty$ or $\sigma_{fb2}^2 < \infty$.

On the contrary, if $\sigma_1^2 \neq \sigma_2^2$ there are finite feedback noise variances $\sigma_{fb1}^2 < \infty$ and $\sigma_{fb2}^2 < \infty$ such that even feedback from both users does not enlarge capacity.

**Proof:** The proof follows by noting that for $\sigma_1^2 = \sigma_2^2$, the LHS of (5) evaluates to 1 and is thus dominated by its RHS whenever $\sigma_{fb1}^2$ or $\sigma_{fb2}^2$ are finite. For $\sigma_1^2 < \sigma_2^2$, the LHS of (5) exceeds 1 and it is possible to find finite $\sigma_{fb1}^2$ and $\sigma_{fb2}^2$ such that the RHS of (5) does not exceed its LHS.  \hspace{1cm} ■
Fig. 2: The patterned region indicates where one-sided feedback from receiver 1 enlarges capacity, and the shaded region indicates where capacity remains unchanged.

III. MORE ON THE USEFULNESS OF FEEDBACK

In this section we consider two related BC setups where the feedback links to the transmitter do not increase the capacity region. The results here may be of interest on their own. Moreover, Corollary 2 can be proved using the first result that we present in this section, stated as Theorem 4 below. However, the results and tools presented in this section seem not enough to prove the converse to our more general result presented in Theorem 1.

A. A Vector-BC with Partial Feedback

We start with a slight extension of El Gamal’s negative result on the usefulness of feedback for physically degraded BCs, to vector channels with partial feedback. Consider a general (not necessarily Gaussian) BC where the first receiver observes $Y_1 = (\tilde{Y}_1, \hat{Y}_1)$, and the second receiver observes $Y_2 = (\tilde{Y}_2, \hat{Y}_2)$. Let the transition law of the BC be described by $P_{\tilde{Y}_1, \hat{Y}_1, \tilde{Y}_2, \hat{Y}_2|X}$. Feedback is only from outputs $\tilde{Y}_1$ and $\tilde{Y}_2$ to the transmitter, and can be noisy or perfect. The following negative result holds.

**Theorem 4:** Consider a BC of transition law $P_{\tilde{Y}_1, \hat{Y}_1, \tilde{Y}_2, \hat{Y}_2|X}$. If there exists a conditional law $P_{\tilde{Y}_2, \hat{Y}_2|\tilde{Y}_1, \hat{Y}_1}$ such that the joint law

$$P_{\tilde{Y}_1, \hat{Y}_1, \tilde{Y}_2, \hat{Y}_2|X} \triangleq P_{\tilde{Y}_2, \hat{Y}_2|\tilde{Y}_1, \hat{Y}_1} \cdot P_{\tilde{Y}_1, \hat{Y}_1|X}$$

(8)
satisfies

\[ P'_{Y_1|X,Y_1,Y_2} = P_{Y_1|X,Y_1,Y_2}, \quad (9a) \]
\[ P'_{Y_2|X,Y_1,Y_2} = P_{Y_2|X,Y_1,Y_2}, \quad (9b) \]
\[ P'_{Y_1,Y_2|X} = P_{Y_1,Y_2|X}, \quad (9c) \]

then causal feedback of the two outputs \( \tilde{Y}_1 \) and \( \tilde{Y}_2 \) to the transmitter does not enlarge the capacity region.

**Proof:** Consider the BC of transition law \( P'_{Y_1,Y_1,Y_2,Y_2|X} \) as defined in (8). This channel is indeed physically degraded because its law satisfies the Markov chain \( X \rightarrow (\tilde{Y}_1, \tilde{Y}_1) \rightarrow (\tilde{Y}_2, \tilde{Y}_2) \).

Moreover, by Assumptions (9), under any encoding strategy, and both in the presence and absence of feedback, the joint law of the tuples \((W_1, \tilde{Y}_1^n, \hat{Y}_1^n)\) and \((W_2, \tilde{Y}_2^n, \hat{Y}_2^n)\) is the same in both the original BC \( P_{Y_1,Y_1,Y_2,Y_2|X} \) and the physically degraded version \( P'_{Y_1,Y_1,Y_2,Y_2|X} \).

Since the error probability at each receiver only depends on its own observations, but not on the observations at the other receiver, the capacity regions of the original and the physically degraded BCs coincide.

On any physically degraded BC, the capacity region, even with full causal feedback, remains the same as that without feedback [4], [5]. By the above observations, the same must also hold for our original BC, thus concluding the proof.

### B. Feedback from the Weak Receiver to all Terminals

Recall the stochastically degraded GBC in Section II, where in particular \( \sigma_1^2 \leq \sigma_2^2 \). Let us assume one-sided noisy feedback from receiver 2 to the transmitter. In addition, suppose that the same feedback values are observed at receiver 1 as well. So, receiver 1 observes its own channel outputs \( Y_1^n \) as well as the noisy feedback outputs

\[ Y_{c}^n \triangleq Y_2^n + Z_{fb2}^n, \quad (10) \]

which are causally observed also at the transmitter. The model is illustrated in Fig. 3a. We denote its capacity region by \( C_{\text{weak-fb}} \). Notice that the described setup is physically degraded only if the feedback noise \( Z_{fb2}^n \) is deterministic and thus \( \sigma_{fb2}^2 = 0 \). Interestingly, Ozarow considered this degraded setup in [7] to derive an outer bound to the GBC capacity region with perfect feedback.

The following theorem is the main result of this subsection and determines the feedback capacity region \( C_{\text{weak-fb}} \). Interestingly, the result shows that the feedback link to the transmitter has no effect on the capacity region of the described model. In other words, once receiver 1
learns the feedback signal from receiver 2, the capacity region does not depend on whether also the transmitter observes the feedback or not.

**Theorem 5:** The capacity region $C^{\text{weak}}_{\text{FB}}$ is the same as that without feedback to the transmitter, and is characterized as:

$$
C^{\text{weak}}_{\text{FB}} = \bigcup_{\theta \in [0,1]} \left\{ (R_1, R_2) : R_1 \leq \log \left( 1 + \frac{\theta P}{\sigma^2_{\text{comb}}} \right) \text{ and } R_2 \leq \log \left( 1 + \frac{(1-\theta) P + \sigma^2_z}{\theta P + \sigma^2_z} \right) \right\},
$$

where

$$
\sigma^2_{\text{comb}} \triangleq \frac{\sigma^2_z}{\sigma^2_1 + \sigma^2_z + \sigma^2_{\text{fb}2}}.
$$

**Proof:** Since $C^{\text{weak}}_{\text{FB}}$ is same as the capacity region without feedback, its achievability follows from superposition coding and maximum ratio combining at receiver 1.

The key steps in the converse are to show that the capacity region $C^{\text{weak}}_{\text{FB}}$ is included in the capacity region of the augmented BC in Fig. 3b, and to invoke Theorem 4 to show that feedback does not increase the capacity of this augmented BC. The final step is then to argue that the capacity region without feedback of the augmented BC coincides with the RHS of (11).

Consider the augmented BC depicted in Fig. 3b. Receiver 1 observes outputs $(Y^n_1, Y^n_c)$ and receiver 2 observes outputs $(Y^n_a, Y^n_c)$, where

$$
Y^n_a \triangleq Y^n_2 - \alpha Z^n_{\text{fb}2} = X^n + Z^n_2 - \alpha Z^n_{\text{fb}2} \triangleq Z^n_a
$$

and

$$
\alpha \triangleq \frac{\sigma^2_z}{\sigma^2_{\text{fb}2}}.
$$

The transmitter observes the feedback output $Y_{c,i}$ after each channel use $i$. The capacity region $C^{\text{noisy}}_{\text{FB aug}}$ of this augmented BC with noisy feedback includes the original capacity region $C^{\text{weak}}_{\text{FB}}$, i.e.

$$
C^{\text{weak}}_{\text{FB}} \subseteq C^{\text{noisy}}_{\text{FB aug}}
$$

because receiver 1 in the augmented BC observes the same outputs as in the original BC, and receiver 2 can compute $Y^n_2$ from its observations $(Y^n_c, Y^n_a)$, see (10) and (13).

Notice next that feedback does not increase the capacity of the augmented BC, because this BC satisfies the assumptions in Theorem 4. To see this, specialize the theorem to $\hat{Y}_1 = Y_c$, $\hat{Y}_1 = Y_1$, $\hat{Y}_2 = Y_c$, and $\hat{Y}_2 = Y_a$. Now set $P'_{Y_2|\hat{Y}_1, \hat{Y}_1} = P_{Y_2|\hat{Y}_1, \hat{Y}_1} = P'_{Y_2|\hat{Y}_1}$ (i.e. $\hat{Y}_1 = Y_2$ under both $P$ and $P'$), and choose $P'_{Y_2|\hat{Y}_1, \hat{Y}_2} = P'_{Y_2|\hat{Y}_1}$ in such a way that

$$
\hat{Y}_2 = \hat{Y}_1 + \hat{Z}_2
$$
for \( \hat{Z}_2 \) a zero-mean Gaussian random variable of variance \( \sigma_2^2 + \alpha^2 \sigma_{fb2}^2 - \sigma_1^2 \) and independent of all other random variables. This is possible since \( \sigma_2^2 \geq \sigma_1^2 \).

Under this choice, (9a) holds since \( \hat{Y}_1 = X + Z_1 \) under both \( P \) and \( P' \), with \( Z_1 \perp \perp (Z_2, Z_{fb2}) \). The assumption (9b) also holds, since the pair \((Z_a, Z_2 + Z_{fb2})\) has independent Gaussian entries and it has the same distribution as \((Z_1 + \hat{Z}_2, Z_2 + Z_{fb2})\). Since \( \hat{Y}_1 = \bar{Y}_2 = Y_c \), the condition (9c) holds as well. Thus, we can employ Theorem 4 to conclude that feedback does not increase the capacity of the augmented BC, i.e.,

\[
C_{aug}^{noisy-fb} = C_{aug}^{wo-fb},
\]

where \( C_{aug}^{wo-fb} \) denotes the capacity region of the augmented BC without feedback. The capacity region without feedback of the augmented BC is obtained from the well-known capacity region of a Gaussian vector BC:

\[
C_{aug}^{wo-fb} = \bigcup_{\theta \in [0,1]} \left\{ (R_1, R_2) : R_1 \leq \log \left( 1 + \frac{\theta P}{\sigma_{comb}^2} \right) \text{ and } R_2 \leq \log \left( 1 + \frac{(1 - \theta)P}{\theta P + \sigma_2^2} \right) \right\},
\]

where \( \sigma_{comb}^2 \) is defined in (12). Notice that here we used the fact that receiver 2 can compute \( Y_2 \) from the pair \((Y_a, Y_c)\) and that, due to the Markov chain \( X \rightarrow Y_2 \rightarrow (Y_a, Y_c) \), the pair \((Y_a, Y_c)\) is redundant in view of \( Y_2 \).

Combining (15), (17) and (18) establishes the desired converse proof.

\[
\text{(a) Original BC} \quad \text{(b) Augmented BC}
\]

Fig. 3: Original and Augmented BCs

IV. OUTER BOUNDS FOR GBC WITH NOISY FEEDBACK (CONVERSE TO THEOREM 1)

Before we embark on proving the converse, notice that there is no obvious physical degradation in our GBC model with noisy feedback shown in Fig. 1. In addition, Theorem 4 cannot be applied
as such. Interestingly, under the condition in (5), an outer bound to the capacity region can be constructed by equipping each receiver with suitable information about the noise processes in the model. That the proposed outer bound turns out achievable even without feedback clinches the deal, further underlining the novelty of the proposed bound. While some steps in our proof effectively employ the degradation between different variables, these are somewhat implicit in the manipulations. To make the exposition gradual, we consider a related problem first, that of feedback only from the strong receiver, where Theorem 4 turns out useful.

A. Noisy Feedback from Strong Receiver Alone

With feedback only from the stronger receiver in a GBC, let us devise the converse for the second part of Corollary 2. The model is depicted in Fig. 4a. We have to show that for $\sigma_{fb2}^2 = \infty$ and

$$\frac{\sigma_1^2}{\sigma_{fb1}^2} \leq \frac{\sigma_2^2}{\sigma_1^2} - 1,$$

feedback from the stronger receiver (receiver 1) does not enlarge capacity.

Let us first construct an augmented BC in which receiver 1 observes $(Y_1^n + Z_{fb1}^n, Y_b^n)$, where

$$Y_b^n := Y_1^n - \alpha Z_{fb1}^n = X^n + Z_{fb1}^n - \alpha Z_{fb1}^n,$$

for

$$\alpha \triangleq \frac{\sigma_1^2}{\sigma_{fb}^2}.$$ (21)

As before, the transmitter observes the feedback signal $Y_{1,i} + Z_{fb1,i}$ after each channel use $i$. The capacity region $C_{aug}^{noisy-fb}$ of this augmented BC, which is depicted in Fig. 4b, includes the capacity region $C^{noisy-fb}$ of our original BC:

$$C^{noisy-fb} \subseteq C_{aug}^{noisy-fb}.$$ (22)

This is immediate since $(1 + \alpha)Y_1 = \alpha(Y_1 + Z_{fb1}) + Y_b$, and thus receiver 1 can compute $Y_1^n$ from $Y_1^n + Z_{fb1}^n$ and $Y_b^n$. We next argue that the augmented BC satisfies the assumptions in Theorem 4 and thus feedback does not enlarge its capacity region. To this end, specialize Theorem 4 by identifying $\hat{Y}_1 = Y_1 + Z_{fb1}$, $\hat{Y}_1 = Y_b$, $\hat{Y}_2 = \emptyset$, and $\hat{Y}_2 = Y_2$, and then choose $P_{\hat{Y}_2|\hat{Y}_1,\hat{Y}_1}'$ such that under this new law

$$\hat{Y}_2 = \hat{Y}_1 + \hat{Z}_2,$$ (23)
with \( \hat{Z}_2 \) a zero-mean Gaussian random variable of variance \( \sigma_2^2 - \alpha^2 \sigma_{fb1}^2 - \sigma_1^2 > 0 \), and independent of all other random variables. Clearly, (9a) and (9c) are immediate under this choice. The choice of parameter \( \alpha \) in (21) ensures that the condition (9b) is also met. To see this, notice that \( Z_2 \) is independent of \( Z_1 + Z_{fb1} \) under \( P \), whereas \( Z_b + \hat{Z}_2 \) is independent of \( Z_1 + Z_{fb1} \) under \( P' \). Thus, the noise sequences \( Z_2^n \) and \( Z_b^n + \hat{Z}_2^n \) are both independent of the feedback noise \( (Z_1^n + Z_{fb1}^n) \) in the respective models. Also, notice that the random variables \( Z_2 \) and \( Z_b + \hat{Z}_2 \) are identically distributed. Since all the required assumptions are met, we can employ Theorem 4 to conclude that feedback does not increase the capacity of the augmented BC, and thus

\[
C_{aug}^{\text{noisy-fb}} = C_{wo-fb}.
\]  

The capacity region without feedback of the augmented BC is obtained from the well-known capacity region of a Gaussian vector BC [2]. Combining (22) and (24) establishes the desired converse.

![Diagram](attachment:image.png)

(a) Original BC  
(b) Augmented BC

Fig. 4: Original and Augmented BCs

B. Noisy Feedback from both Receivers

Observe that feeding back the signal \( Y_2 \) will allow the transmissions to depend on \( Z_2 + Z_{fb2} \). Thus, there is no obvious stochastic degradation between \( Y_2 \) and any signal derived from the observed symbols at the strong receiver. Nevertheless, the dependence between \( Z_2 \) and the transmitted symbols can be decoupled by careful conditioning and subsequent manipulations, which enable the identification of a suitable degradation structure as shown below.
By Fano’s inequality [2], after ignoring the $o(n)$ terms,

$$nR_1 = I(W_1; Y^n_1|W_2)$$

$$\leq I(W_1; Y^n_1, Z^n_{fb1}, Z^n_{2,1} + Z^n_{fb2}|W_2)$$

$$= \sum_{i=1}^{n} I(W_1; Y_1i|W_2, Y_{2,i}^{i-1}, Z_{2,i}^{i-1}, Z_{fb1}^{i-1} + Z_{fb2}^{i-1})$$

$$+ I(W_1; Z_{fb1,i}, Z_{2,i} + Z_{fb2,i}|W_2, Z_{fb1}^{i-1}, Z_{2}^{i-1} + Z_{fb2}^{i-1}, Y_1i)$$

$$= \sum_{i=1}^{n} I(W_1; Y_1i|W_2, Y_{2,i}^{i-1}, Z_{2,i}^{i-1}, Z_{fb1}^{i-1} + Z_{fb2}^{i-1})$$

$$\leq \sum_{i=1}^{n} \{ h(Y_1i|U_i, X_{1,i}^{i-1} + Z_{1,i}^{i-1}, Z_{fb1}^{i-1}) - h(Z_{1,i}) \}$$

Let us now expand the second term as

$$= \sum_{i=1}^{n} h(Y_1i|U_i, X_{1,i}^{i-1} + Z_{2,i}^{i-1} - \gamma Z_{fb1}^{i-1}) - h(Z_{1,i}). \quad (25)$$

Here (a) follows since $(Z_{fb1,i}, Z_{2,i} + Z_{fb2,i})$ is independent of $(W_1, W_2, Z_{fb1}^{i-1}, Z_{2,i}^{i-1} + Z_{fb2}^{i-1}, Y_1i)$. In (b), we took $U_i = (W_2, Z_{2,i}^{i-1} + Z_{fb2,i})$, and used the fact that $(W_1, U_i, Y_{1,i}^{i-1}, Z_{fb1}^{i-1})$ determines the transmitted symbol $X_i$. Clearly, given $X_i$, the remaining uncertainty in $Y_{1,i}$ is only due to $Z_{1,i}$, which is independent of $(W_1, U_i, Y_{1,i}^{i-1}, Z_{fb1}^{i-1})$. The constant in (c) above was taken as $\gamma \triangleq \frac{\sigma_f^2}{\sigma_{fb1}^2}$, thus making $(U_i, X_{1,i}^{i-1} + Z_{2,i}^{i-1} - \gamma Z_{fb1}^{i-1})$ independent of $Z_{1,i} + Z_{fb1,i}$, which forms the encoder’s information about the noise process at receiver 1. For the weaker receiver, again by applying Fano’s inequality and ignoring the $o(n)$ terms,

$$nR_2 \leq h(Y^n_2) - h(Y^n_2|W_2) \quad \text{(26)}$$

$$\leq \frac{n}{2} \log 2\pi e(P + \sigma_f^2) - h(Y^n_2|W_2). \quad \text{(27)}$$

Let us now expand the second term as

$$h(Y^n_2|W_2) = \sum_{i=1}^{n} h(Y_2i|W_2, Y_2^{i-1})$$

$$\geq \sum_{i=1}^{n} h(Y_2i|W_2, X_{1,i}^{i-1} + Z_{2,i}^{i-1} + Z_{fb2}^{i-1})$$

$$= \sum_{i=1}^{n} h(Y_2i|U_i, X_{1,i}^{i-1} + Z_{2,i}^{i-1} - \beta(Z_{2,i}^{i-1} + Z_{fb2}^{i-1}))$$

$$= \sum_{i=1}^{n} h(Y_2i|U_i, X_{1,i}^{i-1} + Z_{2,i}^{i-1}), \quad \text{(28)}$$
where $\beta \triangleq \frac{\sigma^2}{\sigma_t^2 + \sigma_{n2}^2}$, and $\tilde{Z}_2 \triangleq \frac{\sigma_{fb2}^2}{\sigma_{n2}^2 + \sigma_2^2} Z_2 - \frac{\sigma_2^2}{\sigma_{n2}^2 + \sigma_2^2} Z_{fb2}$ is independent of $Z_2 + Z_{fb2}$, the latter being the encoder’s information about the noise process at receiver 2. Observe that since $(\tilde{Z}_2, Z_1 - \gamma Z_{fb1})$ is independent of the encoders’ feedback information $(Z_1 + Z_{fb1}, Z_2 + Z_{fb2})$, the former has no effect on the transmitted symbols. Therefore, by denoting $\tilde{Y}_2 = X + \tilde{Z}_2$,

$$h(Y_{1i}|U_i, X^{i-1} + Z_1^{i-1} - \gamma Z_{fb1}^{i-1}) \leq h(Y_{1i}|U_i, \tilde{Y}_2^{i-1})$$

(29)

remains true using data processing theorem, as long as

$$\text{Var}(Z_1 - \gamma Z_{fb1}) \leq \text{Var}(\tilde{Z}_2).$$

(30)

Notice that (30) is equivalent to

$$\frac{\sigma_2^2}{\sigma_1^2} \geq \left( \frac{\sigma_1^2}{\sigma_{fb1}^2} + 1 \right) \left( \frac{\sigma_2^2}{\sigma_{fb2}^2} + 1 \right).$$

(31)

Under the above condition, (25) yields

$$nR_1 \leq \sum_{i=1}^n h(Y_{1i}|U_i, V_i) - h(Z_{1i}),$$

(32)

where we defined $V_i$ as $\tilde{Y}_2^{i-1}$. Also, from (26),

$$nR_2 \leq \sum_{i=1}^n h(Y_{2i}) - h(Y_{2i}|U_i, V_i).$$

(33)

Notice that $(U_i, V_i) \to X_i \to (Y_{1i}, Y_{2i})$, and we can now obtain single letter rate expressions using one auxiliary random variable $\tilde{U} = (U, V)$. The optimality of Gaussian auxiliary variable can then be proved along the lines of [19], by incorporating feedback as in [20]. However, we proceed through a more standard route, by applying the following version of EPI (similar to [5]) to connect (32) and (33).

**Lemma 6:**

$$2 \frac{\sigma_2^2}{\sigma_1^2} \sum_{i=1}^n h(Y_{1i}|U_i, V_i) \geq 2 \frac{\sigma_2^2}{\sigma_1^2} \sum_{i=1}^n h(Y_{1i}|U_i, V_i) + 2 \pi e (\sigma_2^2 - \sigma_1^2).$$

(34)

**Proof:** The proof is presented in Appendix A.

The remaining part of the proof is more routine. Since $\frac{n}{2} \log 2 \pi e \sigma_2^2 \geq h(Y_2^n|W_2) \geq h(Z_2^n)$, we can take

$$h(Y_2^n|W_2) = \frac{n}{2} \log (2 \pi e (\sigma_2^2 + \theta P)),$$

(35)

for some $\theta \in [0, 1]$. Using (34) and the fact that conditioning reduces entropy,

$$\sum_{i=1}^n h(Y_{1i}|U_i, V_i) \leq \frac{n}{2} \log (2 \pi e (\sigma_1^2 + \theta P)).$$

(36)
Using the above two formulas in (27) and (32), we get for some $\theta \in [0, 1],$

\begin{align*}
R_2 &\leq \frac{1}{2} \log \left( 1 + \frac{(1 - \theta)P}{\theta P + \sigma_2^2} \right) \tag{37} \\
R_1 &\leq \frac{1}{2} \log \left( 1 + \frac{\theta P}{\sigma_1^2} \right). \tag{38}
\end{align*}

This completes the proof of the converse part of Theorem 1.

V. ZERO-FORCING ACHIEVABLE SCHEMES FOR THE GBC WITH NOISY FB

(DIRECT PART TO THEOREM 1)

In order to complete the proof of Theorem 1, we now show that a rate pair outside $C^{\text{wo-fb}}$ is achievable if the condition in (5) is violated by the given tuple $(\sigma_1^2, \sigma_2^2, \sigma_{fb1}^2, \sigma_{fb2}^2)$ of noise variances. While several feedback coding schemes are available in literature, the main difficulty is in having tractable rate expressions which can show the required enlargement. Rate regions incorporating noisy feedback are typically stated in terms of the intersections of several hyperplanes, and are thus difficult to express in suitable functional forms for comparison [9], [10]. Some simplifications are possible, for example, when $\sigma_1^2 = \sigma_2^2$, the rate region proposed in [10] is shown to achieve rate-pairs outside $C^{\text{wo-fb}}$, by suitable substitution of auxiliary coding variables, and thereby simplifying the expressions. However extending this to find the set of noise variances for which an enlargement becomes possible seems difficult in general.

A. A SIMPLE LINEAR-FEEDBACK CODING SCHEME

We will employ linear feedback coding schemes, where the noise realization from each receiver after an odd numbered transmission instant, perceived through the noisy feedback link, is linearly combined and sent along with new symbols in the very next instant. Thus the transmitted signal only depends on feedback during even channel uses, and the feedback values need to be stored for just one instant at the receiver. The fresh symbols at each instant, which are linearly combined with feedback, are generated and conveyed to all parties, as in standard random coding arguments [2]. Specifically, we construct two independent Gaussian codebooks to convey the two messages $W_1$ and $W_2$ to the respective users. Let $U$ denote the codeword symbols to the first receiver and
V denote the codeword symbols to the second receiver. In order to convey the symbols \((u_i, v_i)\) chosen from the codebooks, the transmitter sends,

\[
X_{2i-1} = u_i + v_i \tag{39}
\]

\[
X_{2i} = \sqrt{\alpha} \left[ u_i - v_i + \beta_1(Z_{1,2i-1} + Z_{fb1,2i-1}) \right. \\
\left. + \beta_2(Z_{2,2i-1} + Z_{fb2,2i-1}) \right] \tag{40}
\]

where \(\alpha, \beta_1\) and \(\beta_2\) are appropriate real valued parameters. Let \(\hat{Z}_{j,i}\) be the MMSE estimate of \(Z_{j,2i-1} + Z_{fbj,2i-1}\), for \(j = 1, 2\). Then,

\[
\hat{Z}_{j,i} \triangleq \frac{\sigma_j^2}{\sigma_j^2 + \sigma_{fbj}^2} (Z_{j,2i-1} + Z_{fbj,2i-1}), \tag{41}
\]

which has variance \(\hat{\sigma}_j^2 \triangleq \frac{\sigma_j^4}{\sigma_j^2 + \sigma_{fbj}^2}\). Defining

\[
\gamma_j \triangleq \frac{\beta_j \sigma_j^2}{\sigma_j^2 + \sigma_{fbj}^2}, \quad j = 1, 2, \tag{42}
\]

we can rewrite (40) as

\[
X_{2i} = \sqrt{\alpha} (u_i - v_i + \gamma_1 \hat{Z}_1 + \gamma_2 \hat{Z}_2). \tag{43}
\]

To facilitate random coding arguments, let us choose the distribution \(p(u, v)\) according to \((U, V) \sim \mathcal{N}(0, \text{diag}(\theta P, (1 - \theta) P))\) for some \(\theta' \in [0, 1]\). This choice of \((U, V)\) ensures that the average power constraint is met over the transmissions \(X_{2i-1}\) at odd time instants. To ensure the same for \(X_{2i}\), we choose

\[
\frac{1}{\alpha} = 1 + \frac{\gamma_1^2 \hat{\sigma}_1^2}{P} + \frac{\gamma_2^2 \hat{\sigma}_2^2}{P}. \tag{44}
\]

The following operations are performed at the two receivers for decoding the respective messages.

**Receiver 1:** In two consecutive instants, the observations are

\[
Y_{1,2i-1} = u_i + v_i + Z_{1,2i-1} \quad \tag{45}
\]

\[
Y_{1,2i} = \sqrt{\alpha}(u_i - v_i + \gamma_1 \hat{Z}_1 + \gamma_2 \hat{Z}_2) + Z_{1,2i}. \quad \tag{46}
\]

A simple zero forcing is achieved by computing

\[
S_{1,i} \triangleq Y_{1,2i-1} + \frac{Y_{1,2i}}{\sqrt{\alpha}} = 2u_i + \gamma_1 \hat{Z}_1 + \gamma_2 \hat{Z}_2 + Z_{1,2i-1} + \frac{Z_{1,2i}}{\sqrt{\alpha}}. \quad \tag{47}
\]
**Receiver 2:** Here also we do zero forcing. On observing
\[ Y_{2,2i-1} = u_i + v_i + Z_{2,2i-1} \]  
\[ Y_{2,2i} = \sqrt{\alpha}(u_i - v_i + \gamma_1 \hat{Z}_1 + \gamma_2 \hat{Z}_2) + Z_{2,2i} \]
in two consecutive instants, the receiver computes
\[ S_{2,i} \triangleq Y_{2,2i} - \frac{Y_{2,2i-1}}{\sqrt{\alpha}} = 2v_i - \gamma_1 \hat{Z}_1 - \gamma_2 \hat{Z}_2 + Z_{2,2i-1} - \frac{Z_{2,2i}}{\sqrt{\alpha}}. \]

Each receiver \( j \in \{1, 2\} \) will attempt to decode its intended message \( W_j \) based on the symbols \( S_{j,1}, \ldots, S_{j,n} \) where \( n \) is the codeword length in each codebook.

**Analysis of error of probability:** Notice that the zero forcing performed above creates an equivalent point-to-point channel to each receiver, with no interference from the other user’s symbols. However, the transmissions take place over a blocklength of \( 2n \) instants now. Therefore, standard random coding arguments [18] imply that the rate pair \((\tilde{R}_1, \tilde{R}_2)\) is achievable, where
\[
\tilde{R}_1 = \frac{1}{4} \log \left( 1 + \frac{4\theta'P}{\sigma_1^2 (1 + \frac{1}{\alpha}) + (\gamma_1^2 + 2\gamma_1)\sigma_1^2 + \gamma_2^2 \sigma_2^2} \right) \quad (51)
\]
\[
\tilde{R}_2 = \frac{1}{4} \log \left( 1 + \frac{4(1 - \theta')P}{\sigma_2^2 (1 + \frac{1}{\alpha}) + (\gamma_2^2 - 2\gamma_2)\sigma_2^2 + \gamma_1^2 \sigma_1^2} \right), \quad (52)
\]
with \( \alpha \) given by (44), and \((\theta', \gamma_1, \gamma_2)\) being arbitrary real tuples satisfying \( \theta' \in [0, 1] \) and \( \gamma_1, \gamma_2 \geq 0 \). To exemplify the utility of the proposed scheme, consider the case with \( \sigma_1^2 = \sigma_2^2 \), and \( \gamma_2 = 0 \), i.e., no feedback from the second receiver. Then, taking \( \theta' = 0.5 \) and \( \gamma_1 = 0 \) will recover the equal rate point on the no-feedback capacity region. However, some minimal algebra suffices to show that small negative values of \( \gamma_1 \) will cause the region given by (51) to include rate-pairs outside the no-feedback capacity region. Therefore, this scheme almost immediately suggests a capacity enlargement using passive noisy feedback in a GBC with \( \sigma_1^2 = \sigma_2^2 \). Notice the remarkable simplicity when compared to the schemes in [8], [10], however, the latter ones can achieve superior rate-regions.

Generalizing the above idea to different noise variances at the receivers needs more analytical effort. To keep things tractable, we first show that one can obtain rate points outside \( C^{\text{wo-fb}} \) at low average transmit powers, also known as the wideband regime.
B. Improving on $C^{\text{wo--fb}}$ at Low Powers

Recall our assumption that $\sigma_1^2 \leq \sigma_2^2$, and notice that the Pareto optimal rate-pairs on the boundary of $C^{\text{wo--fb}}$ given in (3) can be indexed by the continuous parameter $\theta \in [0, 1]$. We fix a suitable $\theta \in [0, 1]$ and show that for small enough $P$ the corresponding Pareto optimal rate pair in (3) is dominated by the achievable rate pair $(\tilde{R}_1, \tilde{R}_2)$ in (51) for some appropriate choice of $(\theta', \gamma_1, \gamma_2)$.

Consider an arbitrary quadruple $(\theta, \theta', \gamma_1, \gamma_2)$ and define
\[
\zeta \triangleq \gamma_1^2 \hat{\sigma}_1^2 + \gamma_2^2 \hat{\sigma}_2^2.\tag{53}
\]
By (51), the pair $(\tilde{R}_1, \tilde{R}_2)$ (for the parameters $\theta', \gamma_1, \gamma_2$) dominates the pair $(R_1, R_2)$ (for $\theta$) if
\[
1 + \frac{4\theta' P}{\sigma_1^2 \left(1 + \frac{1}{2}\right) + 2\gamma_1 \hat{\sigma}_1^2 + \zeta} \geq \left(1 + \frac{\theta P}{\sigma_1^2}\right)^2, \tag{54}
\]
\[
1 + \frac{4(1-\theta') P}{\sigma_2^2 \left(1 + \frac{1}{2}\right) - 2\gamma_2 \hat{\sigma}_2^2 + \zeta} > \left(1 + \frac{(1-\theta) P}{\theta P + \sigma_2^2}\right)^2, \tag{55}
\]
or equivalently,
\[
\frac{4\theta'}{\sigma_1^2 \left(1 + \frac{1}{2}\right) + 2\gamma_1 \hat{\sigma}_1^2 + \zeta} \geq \frac{2\theta}{\sigma_1^2} + \frac{\theta^2 P}{\sigma_1^4}, \tag{56}
\]
\[
\frac{4(1-\theta')}{\sigma_2^2 \left(1 + \frac{1}{2}\right) - 2\gamma_2 \hat{\sigma}_2^2 + \zeta} > \frac{2(1-\theta)}{\theta P + \sigma_2^2} + \frac{(1-\theta)^2 P}{(\theta P + \sigma_2^2)^2}. \tag{57}
\]
Let us change the variables from $(\gamma_1, \gamma_2, \theta')$ to $(a_1, a_2, \mu)$ by defining
\[
a_j \triangleq (-1)^j \gamma_j \hat{\sigma}_j, \quad j = 1, 2 \tag{58}
\]
\[
\mu \triangleq \frac{1}{P} \left(\frac{\theta'}{\theta} - 1\right). \tag{59}
\]
While there are no restrictions on the parameters $(a_1, a_2)$, the parameter $\mu$ needs to lie in the interval $[-\frac{1}{P}, \frac{1}{P}(\frac{1}{\theta} - 1)]$ so that $0 \leq \theta' \leq 1$. Notice that the described interval for possible $\mu$ can be made to include any desired real value by choosing $P$ sufficiently small.

Using the transformations (58) – (59) on (56) – (57), we get
\[
\frac{4(1 + \mu P)}{(a_1^2 + a_2^2)(P^2 + P \sigma_1^2) + 2\sigma_1^2 - 2a_1 \sigma_1 P} \geq \frac{2}{\sigma_1^2} + \frac{\theta P}{\sigma_1^4}, \tag{60}
\]
\[
\frac{4(1 - \theta(1 + \mu P))}{(a_1^2 + a_2^2)(P^2 + P \sigma_2^2) + 2\sigma_2^2 - 2a_2 \sigma_2 P} > \frac{2(1-\theta)}{\theta P + \sigma_2^2} + \frac{(1-\theta)^2 P}{(\theta P + \sigma_2^2)^2}. \tag{61}
\]
Clearly, the above expressions hold with equality at $P = 0$. Therefore, if the pair of derivatives on the LHS dominates the corresponding RHS derivatives as $P \to 0^+$, we are done, as this
shows that the required capacity enlargement is possible at low enough powers. Differentiating with respect to $P$ and setting $P = 0$, we get the conditions

$$2 \mu \sigma_1^2 \geq \theta + (a_1^2 + a_2^2) \sigma_1^2 - 2a_1 \sigma_1 \hat{\sigma}_1$$  \hspace{1cm} (62)

$$(1 - \theta)(2a_2 \sigma_2 - (a_1^2 + a_2^2) \sigma_2^2) > (1 - \theta)(1 - 3\theta) + 2\theta \mu \sigma_2^2.$$  \hspace{1cm} (63)

The first of the above two equations can be guaranteed by choosing

$$\mu \triangleq \frac{\theta}{2\sigma_1^2} + \frac{a_1^2 + a_2^2}{2} - \frac{a_1 \sigma_1}{\sigma_1^2},$$  \hspace{1cm} (64)

whereas the second equation, on substitution of (64), will yield

$$a_1^2 + a_2^2 < \frac{2a_1 \theta}{\sqrt{\sigma_1^2 + \sigma_{fb1}^2}} + \frac{2a_2 (1 - \theta)}{\sqrt{\sigma_2^2 + \sigma_{fb2}^2}} - \frac{\theta^2}{\sigma_1^2} - \frac{(1 - \theta)(1 - 3\theta)}{\sigma_2^2}.$$  \hspace{1cm} (65)

Notice that above choice of $\mu$ does not depend on $P$ and thus lies in the desired interval for all sufficiently small values of $P > 0$. Since $a_1, a_2$ are free parameters, we can choose them as

$$a_1 = \frac{\theta}{\sqrt{\sigma_1^2 + \sigma_{fb1}^2}},$$

$$a_2 = \frac{1 - \theta}{\sqrt{\sigma_2^2 + \sigma_{fb2}^2}}.$$

By substituting this into (65), we need to verify

$$\frac{\theta^2}{\sigma_1^2} + \frac{(1 - \theta)(1 - 3\theta)}{\sigma_2^2} - \frac{\theta^2}{\sigma_1^2 + \sigma_{fb1}^2} - \frac{(1 - \theta)^2}{\sigma_2^2 + \sigma_{fb2}^2} < 0.$$  \hspace{1cm} (66)

for some value of $\theta \in [0, 1]$. Defining $x = \frac{\theta}{1 - \theta}$, one can equivalently check if $g(x) < 0$ for some value of $x > 0$, where

$$g(x) \triangleq \frac{x \sigma_{fb1}^2}{\sigma_1^2 (\sigma_1^2 + \sigma_{fb1}^2)} + \frac{\sigma_{fb2}^2}{x \sigma_2^2 (\sigma_2^2 + \sigma_{fb2}^2)} - \frac{2}{\sigma_2^2}.$$  

In fact, the function $g(x)$ is minimized for $x > 0$ by

$$x^* = \sqrt{\left(1 + \frac{\sigma_{fb1}^2}{\sigma_1^2}ight) \left(\frac{\sigma_2^2}{\sigma_{fb2}^2}ight)}.$$  \hspace{1cm} (67)

Now the condition for $g(x^*) < 0$ can be seen to be equivalent to

$$\frac{\sigma_2^2}{\sigma_1^2} < \left(\frac{\sigma_1^2}{\sigma_{fb1}^2} + 1\right) \left(\frac{\sigma_2^2}{\sigma_{fb2}^2} + 1\right),$$  \hspace{1cm} (68)

which is the complement of condition (5) given in Theorem 1.

Putting it all together, we have shown a capacity enlargement at sufficiently small powers, when at least one of the passive feedback links is not too noisy, as implied by (68).
C. Improving on $C^{\text{wo--fb}}$ at All Powers

We now show that any capacity enlargement at low powers naturally extends to a capacity enlargement at arbitrary transmit powers. This can be shown by message splitting and bootstrapping the low power achievable scheme. Consider a rate-pair in which user-1 demands a small enough positive rate $R_1$. Let us split the message $W_2$ for user 2 into 2 sub-messages $W_{2,1}$ and $W_{2,2}$ of respective rates $R_{2,1}$ and $R_{2,2}$. Consider a small positive $\epsilon$, and appropriate parameters $\theta' \in [0, 1]$ and $\gamma_1, \gamma_2 > 0$. The message $W_{2,2}$ is conveyed to receiver 2 using a simple point-to-point scheme of power $P - \epsilon$. We use the variables $U \sim \mathcal{N}(0, \theta' \epsilon)$ and $V_1 \sim \mathcal{N}(0, (1 - \theta') \epsilon)$ to denote the code symbols employed in the above linear feedback code construction to encode $(W_1, W_{2,1})$, whereas the variable $V_2 \sim \mathcal{N}(0, P - \epsilon)$ denotes the symbol employed in the point-to-point code encoding $W_{2,2}$. All codebooks are generated independently, and we employ standard random coding arguments to find the error probability. Over two successive channel uses, the sender then transmits

$$X_{2i-1} = u_i + v_{1,i} + v_{2,2i-1}$$

$$X_{2i} = \sqrt{\alpha}(u_i - v_{1,i} + \gamma_1 \hat{Z}_{1,2i-1} + \gamma_2 \hat{Z}_{2,2i-1}) + v_{2,2i}.$$  \hspace{1cm} (69)

Similar to (43) and (44), the parameters $\alpha, \gamma_1$, and $\gamma_2$ are chosen so that the sum $\sqrt{\alpha}(u_i - v_{1,i} + \gamma_1 \hat{Z}_{1,2i-1} + \gamma_2 \hat{Z}_{2,2i-1})$ satisfies the average power constraint of $\epsilon$.

Receiver 2 infers its desired messages via the following steps:

- It decodes message $W_{2,2}$ by treating everything as noise. Clearly the transmissions in successive instants are independent, and we can treat the rest of the transmissions as memoryless Gaussian noise sequences while decoding the $V_2$ codewords.
- After inferring $W_{2,2}$, it subtracts the $V_2$ codeword to obtain a more clean BC with equivalent average transmit power $\epsilon$, and it applies the zero forcing decoding scheme as in (50).

Receiver 1 decodes in a similar way: it first decodes messages $W_{2,2}$ (even though this message is not intended for it), subtracts the $V_2$ codeword, and applies the zero forcing decoding scheme described in (47).

We show that for appropriate choices of $\epsilon, \theta', \gamma_1, \gamma_2$, the proposed scheme achieves a rate-point outside the no-feedback capacity region $C^{\text{wo--fb}}$. We first notice that $R_1$ and $R_{2,1}$ can be chosen as the rates $\tilde{R}_1$ and $\tilde{R}_2$ in (51) and (52), however with the average power $P$ replaced by $\epsilon$. We
have shown in the previous subsection that for sufficiently small \( \epsilon > 0 \) there exist choices of \( \theta', \gamma_1, \gamma_2 \) and \( \theta \) such that

\[
R_1 = \frac{1}{2} \log_2 \left( 1 + \frac{\theta \epsilon}{\sigma_1^2} \right),
\]

(70)

\[
R_{2,1} > \frac{1}{2} \log_2 \left( 1 + \frac{(1 - \theta) \epsilon}{\theta \epsilon + \sigma_2^2} \right).
\]

(71)

We continue with such a sufficiently small value of \( \epsilon \). Since Message \( W_{2,2} \) can be transmitted at a rate

\[
R_{2,2} = \frac{1}{2} \log_2 \left( 1 + \frac{P - \epsilon}{\epsilon + \sigma_2^2} \right),
\]

(72)

by introducing \( \theta'' \triangleq \frac{\theta P}{P} \), we conclude that the rate pair

\[
R_1 = \frac{1}{2} \log_2 \left( 1 + \frac{\theta'' P}{\sigma_1^2} \right),
\]

(73)

\[
R_2 = R_{2,1} + R_{2,2}
\]

\[
> \frac{1}{2} \log_2 \left( 1 + \frac{(1 - \theta) \epsilon}{\theta \epsilon + \sigma_2^2} \right) + \frac{1}{2} \log_2 \left( 1 + \frac{P - \epsilon}{\epsilon + \sigma_2^2} \right) = \frac{1}{2} \log_2 \left( 1 + \frac{(1 - \theta'') P}{\theta'' P + \sigma_2^2} \right).
\]

(74)

is achievable. Observing that this rate-pair lies outside \( C_{wo-fb} \) is sufficient to conclude the proof.

In short, we used the enlargement for small values of power, and allotted all the remaining power for transmissions to the weak receiver. This allowed us to strictly improve the no-feedback capacity region at high powers as well.

VI. Conclusion

We have analyzed the effect of passive noisy feedback in enlarging the capacity region of a Gaussian broadcast channel. Interestingly, too much noise in both the feedback links does not lead to any enlargement in the asymmetric user case, in sharp contrast with the case of identical channel transition laws to the receivers. For the latter case, it is known that any noise of finite variance in the feedback link is beneficial for capacity enlargement. While we have characterized the regime of noise variances for which no enlargement occurs, our achievable scheme shows an enlargement outside this proposed regime, thus making the characterization sharp. While showing an enlargement, however small, was sufficient for our purposes here, there is still some way to go for finding the actual capacity region with enlargement. More sophisticated coding schemes than the ones proposed here may be required to achieve that.

The genie aided converse constructions, and associated ideas, seem to be beneficial in analyzing other multiuser systems with feedback. This is currently under consideration.
APPENDIX A

PROOF OF LEMMA

Proof: The proof is very similar to that in [5]. For the sake of completeness, here we repeat the arguments, which proceeds by induction on $n$. For $n = 1$, the inequality follows from entropy power inequality [2, pg. 22] since we may write

$$h(Y_{21}|U_1, V_1) = h(Y_{11} + \tilde{Z}_1|U_1, V_1),$$

where $\tilde{Z}_1 \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$. Notice that $\tilde{Z}_i$ is independent of the transmitted symbols, messages and other noise processes. Similarly, because $Z_{1m}$ and $Z_{2m}$ are both Gaussian of variances $\text{Var}(Z_{1m}) \leq \text{Var}(Z_{2m})$ and independent of $(U_m, V_m, X^m)$, we can write for any integer $m \geq 1$:

$$h(Y_{2m}|U_m, V_m) = h(Y_{1m} + \tilde{Z}_m|U_m, V_m).$$ \hfill (75)

for some zero-mean Gaussian $\tilde{Z}_m$ of variance $\text{Var}(\tilde{Z}_m) = \sigma_2^2 - \sigma_1^2$ independent of $(Y_{1m}, U_m, V_m)$.

Now, assume that (34) is true for $n = m - 1$. By the conditional EPI [2],

$$2^{2h(Y_{2m}|U_m, V_m)} \geq 2^{2h(Y_{1m}|U_m, V_m)} + 2^{2h(\tilde{Z}_m)}.$$

i.e.,

$$2h(Y_{2m}|U_m, V_m)) \geq \log \left(2^{2h(Y_{1m}|U_m, V_m)} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right).$$

Therefore,

$$\frac{2}{m} \sum_{i=1}^{m} h(Y_{2i}|U_i, V_i) = \frac{m-1}{m} \left( h(Y_{2i}|U_i, V_i) + \frac{2}{m} h(Y_{2m}|U_m, V_m) \right)$$

$$\geq \frac{m-1}{m} \log \left(2^{2h(Y_{1m}|U_m, V_m)} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right)$$

$$+ \frac{1}{m} \log \left(2^{2h(Y_{1m}|U_m, V_m)} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right)$$

$$\geq \log \left(2^{\frac{2}{m} \sum_{i=1}^{m} h(Y_{1i}|U_i, V_i)} + 2\pi e(\sigma_2^2 - \sigma_1^2)\right).$$

where (a) follows from the induction hypothesis and the EPI above, and (b) follows from convexity of $\log(2^u + v)$ in $u$ for $v \geq 0$. 

$\blacksquare$
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