EVOLUTION OF ONE-POINT DISTRIBUTIONS
FROM GAUSSIAN INITIAL FLUCTUATIONS

Lev Kofman,¹ ² ³ Edmund Bertschinger,⁴ James M. Gelb,⁴ ⁵
Adi Nusser,⁶ ⁷ and Avishai Dekel,⁶ ⁸

Accepted: The Astrophysical Journal 1994, 420, January 1

¹ CIAR Cosmology program and CITA, University of Toronto, Toronto, ON M5S 1A7, Canada
² On leave of absence from Tartu Astrophysical Observatory, Estonia EE-2444
³ Institute for Astronomy, University of Hawaii, 2680 Woodlawn Dr., Honolulu, HI 96822
⁴ Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139
⁵ Fermilab, MS 209, P.O. Box 500, Batavia, IL 60510
⁶ Racah Institute of Physics, The Hebrew University of Jerusalem, 91904, Israel.
⁷ Department of Astronomy, University of California, Berkeley, CA 94720
⁸ Institut d’Astrophysique and Observatoire de Paris, France
ABSTRACT

We study the quasilinear evolution of the one-point probability density functions (PDFs) of the smoothed density and velocity fields in a cosmological gravitating system beginning with Gaussian initial fluctuations. Our analytic results are based on the Zel’dovich approximation and laminar flow. A numerical analysis extends the results into the multistreaming regime using the smoothed fields of a CDM N-body simulation. We find that the PDF of velocity, both Lagrangian and Eulerian, remains Gaussian under the laminar Zel’dovich approximation, and it is almost indistinguishable from Gaussian in the simulations. The PDF of mass density deviates from a normal distribution early in the quasilinear regime and it develops a shape remarkably similar to a lognormal distribution with one parameter, the $\sigma$ density fluctuation. Applying these results to currently available data we find that the PDFs of the velocity and density fields, as recovered by the POTENT procedure from observed velocities assuming $\Omega = 1$, or as deduced from a redshift survey of IRAS galaxies assuming that galaxies trace mass, are consistent with Gaussian initial fluctuations.

Subject headings: cosmology — dark matter — galaxies: clustering — galaxies: formation
1. INTRODUCTION

The “standard” model for the formation of large-scale structure is based on gravitational instability of small initial fluctuations in the energy density. These are assumed to originate from quantum fluctuations that were stretched to large comoving scales during the inflation phase (see Efstathiou 1990 for a review). The fluctuations are assumed to be a random field, i.e. a set of random variables, one for each point in space, which is fully specified by the $m$-point joint probability density functions (hereafter PDFs; cf. Monin and Yaglom 1971). The time evolution of the PDFs for $m \geq 2$ may be sensitive to the nature of the dark matter (e.g. being baryonic or nonbaryonic, hot or cold; cf. Trimble 1987 for a review). For example, the effect of the dark matter on the two-point correlation function (or its Fourier transform, the power spectrum), which is a moment of the $m = 2$ PDF, is well known. But the one-point PDF is not explicitly sensitive to the nature of the dark matter, at least in the linear regime. This makes it a useful statistic for relating the present fluctuations to the initial conditions independently of the nature of the dark matter.

The natural choice for the density field is here, as in many other physical systems, a Gaussian random field, where the one-point probability distribution of the density fluctuation field, $\delta \equiv \delta \rho / \rho$, is normal with zero mean:

$$P(\delta) = \frac{1}{(2\pi \sigma^2)^{1/2}} e^{-\delta^2 / 2\sigma^2},$$

and the joint probabilities are multivariate normal distributions. (Note that we use $P$ to denote the probability density, or frequency function, and not the cumulative probability.) In the Gaussian case all the moments are determined by the variance $\sigma$ and the Fourier components of the density field have random phases. In the linear regime, the density fluctuations and the three components of the peculiar velocity field, $\mathbf{v}$, are related linearly:

$$\delta \propto -\nabla \cdot \mathbf{v},$$

so each of the velocity components is a Gaussian random field too. The one-point PDFs of density and the three velocity components are all independent.

However, observed correlations in the distribution of galaxies and clusters on scales
≥ 20 h^{-1}\text{Mpc} (e.g. Maddox et al. 1990; Efstathiou et al. 1990; Bahcall 1988; Olivier et al. 1993), and the coherence of their velocities on large scales (e.g., Lynden-Bell et al. 1988), have motivated a consideration of non-Gaussian initial fluctuations. This is because these observed correlations are in excess of the predictions of the “standard” CDM model when normalized to fit the distribution of galaxies on smaller scales. This model assumes Gaussian, adiabatic initial fluctuations with a scale-free spectrum and cold dark matter dominating an Ω = 1, Λ = 0 Friedmann universe.

Theoretically, it has been shown that the general inflation picture still permits a wide variety of non-Gaussian fluctuations within the standard gravitational instability theory (e.g. Linde 1990; Kofman 1991a). Non-Gaussian perturbations also arise in other scenarios, e.g. where the fluctuations originate from topological defects, such as cosmic strings (see Bertschinger 1989 for a review) or textures (Turok 1991), or from non-gravitational cosmic explosions (see Ostriker 1988 for a review). The statistical nature of the initial fluctuations is therefore a basic distinguishing feature between major competing theories.

Several recent observations of the large-scale fields allow determinations of the PDFs of density and velocity based on their spatial distributions in increasingly large volumes. Are these fields consistent with Gaussian initial fluctuations? Can they be used to reject this or other hypotheses? In order to be able to answer these questions we first need to study how the PDFs evolve in time under gravity. During linear evolution, when all Fourier components evolve at the same rate, the PDFs do not change form. However, nonlinear evolution can introduce strong non-Gaussian features.

Weinberg and Cole (1992) have compared the effects of initial non-Gaussianity with nonlinear evolution from Gaussian initial conditions using a series of N-body simulations. They found that, while nonlinear evolution produces non-Gaussian density distributions which may smear out the initial conditions, some features of the initial conditions are preserved, enabling one to use the present PDF for distinguishing certain models of non-Gaussian initial conditions from Gaussian ones.
In this paper we study the mildly nonlinear evolution of the PDFs from Gaussian initial conditions. In §2 we present formal expressions for the weakly nonlinear one-point density and velocity PDFs in terms of the initial PDFs for general initial conditions. We evaluate these expressions for an initial Gaussian random field and then approximate the evolution using the Zel’doovich formalism in the limit of laminar flow. Then, in §3, we use a “standard” CDM high-resolution N-body simulation to test the approximation and extend the results into the multistreaming regime. Our results are applied in §4 to two derivations of the smoothed velocity and density fields in our cosmological neighborhood: the POTENT analysis of the observed radial peculiar velocities of galaxies (Dekel et al. 1990; Bertschinger et al. 1990) and the fluctuation fields deduced from a redshift survey of IRAS galaxies (Strauss et al. 1992). Our analysis and results are discussed in §5.

2. PDF EVOLUTION IN THE LAMINAR ZEL’DOVICH APPROXIMATION

2.1. Lagrangian vs. Eulerian PDFs

To avoid confusion, we introduce the basic concepts and our notations methodically and in detail. Consider a large comoving volume $V (\to \infty)$ in the space of comoving positions $x$, which contains a total mass $M$. Assume that at time $t = 0$ the mass is distributed uniformly in space among particles of identical, infinitesimal masses $dm (\to 0)$, at initial (Lagrangian) comoving positions $q$. Each particle is identified from then on by its Lagrangian position $q$. Let the Eulerian position of particle $q$ at time $t$ be $x(q,t)$. If the mapping from $q$ to $x$ is one-to-one, we call the flow laminar or single-stream. If it is many-to-one, i.e., if more than one $q$ arrives at the same $x$ at a fixed time $t$, we call the flow nonlaminar or multistream.

Let $\rho(x,t)$ be the mass density at position $x$ at $t$. We may treat $\rho$ as a random field in Eulerian space: $\rho$ is drawn at random from a probability density $P(\rho)$ at each position. We can imagine an ensemble of density fields where, at each position, $P(\rho)d\rho$ is the probability that $\rho$ is in the range $(\rho, \rho + d\rho)$.

Define $\rho(q,t)$ over Lagrangian space to be $\rho[x(q,t), t]$, the density in the $x$ position of
particle $q$ at time $t$. (Note that in the multistream case, more than one Lagrangian point $q$ may correspond to the same density $\rho$.) This is a random field over Lagrangian space: $\rho$ is drawn at random from a probability density $Q(\rho)$ for any randomly chosen particle. We again have in mind an ensemble of realizations where, for each particle, $Q(\rho)d\rho$ is the probability that the particle resides in a region of density in the range $(\rho, \rho + d\rho)$. The difference between $P$ and $Q$ is that the probability measure is based on volume for $P$ and on mass (i.e., Lagrangian volume) for $Q$.

The PDFs for the one-dimensional components of the velocity, $P(v)$ and $Q(v)$, are defined analogously. The distributions of $v$ are assumed to be isotropic, i.e., the marginal distributions of each component are identical, $P(v_x) = P(v_y) = P(v_z)$.

A general relation between the Lagrangian and Eulerian PDFs of density, which will be useful later, is

$$P(\rho) = \bar{\rho} \frac{1}{\rho} Q(\rho),$$

where $\bar{\rho} \equiv \int \rho P(\rho)d\rho$ is the mean density. One simple way to prove this is by using the alternative, spatial interpretation of the distributions. Assuming ergodicity in Lagrangian space [that $Q(\rho)$ is independent of $q$], one can replace the distribution over the ensemble of random realizations at a given $q$ with the distribution over Lagrangian space in one realization, so $Q(\rho)d\rho$ is also the fraction of mass which resides in regions where $\rho$ is in the specified range. The total mass with that property is then $MQ(\rho)d\rho$. Assuming ergodicity in Eulerian space [that $P(\rho)$ is independent of $x$, which, by the way, follows from the ergodicity in Lagrangian space when the mapping between the spaces is one-to-one and onto], one can replace the ensemble distribution with the spatial distribution such that $P(\rho)d\rho$ can also be interpreted as the fraction of volume in which $\rho$ is in the specified range. The total volume with that property is $VP(\rho)d\rho$.

Using this interpretation involving the total mass of the particles that reside in regions of a given density and the corresponding volume occupied by this mass, it is clear that $MQ(\rho)d\rho = \rho VP(\rho)d\rho$, which implies Eq. (2) with $\bar{\rho} = M/V$. 

6
2.2. On Multistreaming

Our aim in this paper is to calculate the mildly-nonlinear PDFs of density and velocity at time $t$, given the distributions at an initial time. Consider first each particle on its own. Imagine it to be a mass element $dm$ initially spread uniformly inside an infinitesimal volume $d^3q$. Let its volume at time $t$ be $d^3x_q$, where the explicit mapping from Lagrangian to Eulerian space, $x(q,t)$, is provided by the dynamics. Assuming mass conservation one can write for each element separately $\varrho(q,t)d^3x_q = \bar{\rho}d^3q$, so

$$\varrho(q,t) = \bar{\rho} \frac{\| \partial x \|}{\| \partial q \|}, \quad (3)$$

where the double vertical bars denote the Jacobian determinant. We call this density of an individual mass element (or “a stream”), $\varrho$, the “single-stream” density. It is not necessarily the same as the true, “total” density $\rho$ used in \S\,2.1. This is because the mapping $x(q)$ is not necessarily one-to-one. The mass elements may overlap in certain places, i.e. particles from different $q$’s can cross in the same $x$ at $t$. The total density $\rho$ at position $x$ is the sum of the contributions $\varrho_i$ from all the streams that arrive at $x$ at that time. For a laminar flow, $\varrho = \rho$.

The mapping also determines the comoving “single-stream” velocity of particle $q$ at time $t$: $\vartheta(q,t) = dx(q,t)/dt$. (Note that this coordinate velocity must be multiplied by the expansion scale factor to give the proper peculiar velocity.) The actual “total” velocity $v$ at $x$ is the mass-weighted average of the velocities of all the particles that cross there at $t$. The difference between $\vartheta$ and $v$ is equivalent to that between the velocity of a molecule and the fluid velocity of a gas of molecules.

Given the PDFs of the initial fluctuations, a specific mapping $x(q,t)$ is used below (\S\,2.3 and \S\,2.4) to compute the Lagrangian PDFs of the single-stream quantities $\varrho$ and $\vartheta$ at time $t$, which we denote $Q(\varrho)$ and $Q(\vartheta)$ (where $\vartheta$ is one component of $\vartheta$). In the case of laminar flow, the single-stream quantities are equal to the total ones so the single-stream PDFs are equal to the PDFs of the total quantities, $Q(\rho)$ and $Q(v)$. The desired Eulerian $P(\rho)$ can then be extracted using Eq. (2) and $P(v)$ can be extracted in a similar way.
The analytic results of this paper are limited to this laminar-flow approximation. It is, however, worthwhile to continue the analysis a bit further in the general framework of multistreaming in order to better understand the nature of the approximation and to predict its range of validity.

Consider the general case in which multistreaming may be present. Each Eulerian volume element $d^3x$ may then contain several Lagrangian volume elements $d^3q_i$, with $i$ labeling the streams. The Eulerian and Lagrangian volumes are related by $d^3x = J(q_i)d^3q_i$, where $J$ is the Jacobian determinant appearing in equation (3). We can define a total single-stream volume by integrating over all mass, $V \equiv \int J(q) d^3q$. In general, $V > V$ because Eulerian volumes with multiple streams are counted more than once. The Eulerian single-stream PDF $P(\rho)$ is now defined as the probability that a point randomly selected from $V$ has single-stream density in the range $(\rho, \rho + d\rho)$ (and similarly for $\vartheta$). The difference between $P$ and $P$ is that the probability measure is based on $V$ in the first case and $V$ in the second. Using ergodicity, $P(\rho)$ is also the fraction of $V$ that has single-stream density in the appropriate range. Using mass conservation in analogy with Equation (2), one then obtains

$$P(\rho) = \bar{\rho} Q(\rho) ,$$

where $\bar{\rho} \equiv \int \rho P(\rho) d\rho$ is the mean “Eulerian” single-stream density, not to be confused with the total $\bar{\rho} = \int P(\rho) d\rho$ when multistreaming is important.

An indicator for the degree of multistreaming is given by the mean number of streams at an Eulerian point,

$$N_s \equiv \frac{V}{V} = \frac{\bar{\rho}}{\bar{\rho}} .$$

This parameter can be computed from $Q(\rho)$ by

$$N_s = \int \frac{\bar{\rho}}{\rho} Q(\rho) d\rho ,$$

where the total $\bar{\rho}$ is determined by the initial conditions and it changes in time only due to the expansion of the universe, $\bar{\rho} \propto a^{-3}$ with $a(t)$ the expansion factor. Equation (6)
follows from the normalization of $\mathcal{P}$ as a PDF, $\int \mathcal{P}(\varrho)d\varrho = 1$, combined with relation (4) with $\bar{\varrho}$ replaced by $\bar{\rho}/N_s$ according to the definition (5).

Normally, $N_s = 1$ at early times and $N_s$ remains close to unity as long as the flow is mostly laminar. If the mapping $\mathbf{x}(\mathbf{q}, t)$ is continuous, i.e., it is onto the Eulerian space such that no empty regions are formed, $N_s \geq 1$. Eventually it grows to $N_s \gg 1$ in the severe multistreaming regime. Therefore, $N_s(t)$ can serve as an indicator for the deviation from laminar flow. We will estimate $N_s(t)$ analytically below using the Zel’dovich approximation to provide the mapping $\mathbf{x}(\mathbf{q}, t)$.

Given the single-stream PDFs, what are the desired total PDFs in the presence of multistreaming? The result for the density may be written

$$P(\rho) = \frac{1}{N_s} \mathcal{P}\left(\frac{\rho}{N_s}\right) + \delta P(\rho),$$

(7)

where $\delta P(\rho)$ has vanishing zeroth and first moments so that $P(\rho)$ maintains the proper normalization and has the correct mean, $\bar{\rho}$. The correction $\delta P$ is induced by caustics of the mapping $\mathbf{x}(\mathbf{q})$. An example of this behavior is provided by the gravitational microlensing problem, whose mathematics corresponds to the two dimensional Zel’dovich approximation extended beyond caustic formation. There the probability distribution function $P(A)$ of magnification $A$ (the analogue of density here) has the caustic-induced feature $\delta P(A)$ for large $A$ (Nityananda and Ostriker 1984). The same effect is expected in three-dimensional dynamics, whether exact or given by the Zel’dovich approximation.

As a rough approximation we may use equation (7) with $\delta P = 0$, which should be valid while $N_s(t)$ is close to unity. However, this is certainly not an exact solution in general and it is not even guaranteed to be a reasonable approximation. For an exact solution one has to sum over a combination of single-stream probabilities under the constraint that the single-stream densities (or velocities) sum up (or average) to the given total density (or velocity). The true PDF can be schematically written as

$$P(\rho) = \sum_{N=1}^{\infty} p(N) \tilde{P}\left(\sum_{i=1}^{N} \varrho_i = \rho\right),$$

(8)
where \( p(N) \) is the probability that there are \( N \) streams at a point (with mean \( N_s \)) and \( \tilde{P} \) is the joint probability density for \( N \) streams at one point to sum up to a total density \( \rho \), which is some function of the single-stream PDFs under the constraint. This nontrivial calculation is beyond the scope of this paper.

### 2.3. Velocity PDF in the Zel’dovich Approximation

In any isotropic cosmology, there is a statistical symmetry between positive and negative peculiar velocities (in particular \( \langle v \rangle = 0 \)), which must persist in the nonlinear regime as well. Therefore, nonlinear deviations of an initially Gaussian velocity distribution are subtle; they are reflected only in the fourth order irreducible moment of the one-point distribution — the kurtosis — or in higher even moments, which characterize features such as the sharpness of the peak and the extent of the tail of the distribution. (Recall that we are considering the PDF for one of the velocity components, e.g., \( v_x \).) A priori, even such deviations could be large, but we show below that they are in fact remarkably small in quasilinear gravitating systems.

Let us assume that the mapping from Lagrangian space to Eulerian space is given by the Zel’dovich approximation (Zel’dovich 1970),

\[
x(q, t) = q + D(t)\psi(q),
\]

where \( D(t) \) is a universal function of time \([D(t) \propto a(t) \text{ in a spatially flat, pressureless universe}]\). The comoving peculiar velocity of particle \( q \) is then

\[
\dot{\vartheta}(q, t) \equiv \dot{x}_q = \dot{D}\psi,
\]

and the single-stream density is given by (Eq. 3)

\[
\vartheta(q, t) = \frac{\bar{\rho}}{\| I + D_2\psi \|},
\]

where \( I \) is the unit tensor. For the total density \( \rho(x, t) \) one should sum equation (11) over all streams \( q \) at \( x \).
It is easy to see that the Lagrangian PDF of velocities, $Q(\vartheta)$, is *time-invariant* under the Zel’dovich approximation aside from a simple scaling of $\vartheta$ in time (Eq. 10). For an initially Gaussian PDF we thus have for any of the three components of the velocity, at any time $t$ as long as the Zel’dovich approximation is valid,

$$Q(\vartheta) = \frac{1}{[2\pi \sigma_\vartheta^2(t)]^{1/2}} \exp \left[ -\frac{\vartheta^2}{2\sigma_\vartheta^2(t)} \right], \quad \sigma_\vartheta^2(t) = \dot{D}(t) \langle |\psi|^2 \rangle .$$

Note that the time-invariance of the form of the PDF holds regardless of the form of the initial PDF.

But somewhat surprising is the fact that for Gaussian initial fluctuations and under the Zel’dovich approximation the Eulerian PDF of single-stream velocities is in fact *equal* to the corresponding Lagrangian PDF at all times,

$$P(\vartheta) = Q(\vartheta),$$

so it is also time-invariant!

To prove this, return to the interpretation of ensemble distribution, and let $Q(\vartheta, \varrho)$ be the bivariate Lagrangian probability density for $\vartheta$ and $\varrho$. Then, by Eq (4), the corresponding joint probability density in Eulerian space is $P(\vartheta, \varrho) = (\partial/\partial \varrho)Q(\vartheta, \varrho)$, so one can write the Eulerian PDF for velocities as

$$P(\vartheta) = \int \frac{\partial}{\partial \varrho} Q(\vartheta, \varrho) d\varrho .$$

Eq. (13) follows if $\vartheta$ and $\varrho$ are statistically independent so that the bivariate distribution factors into the product of univariate distributions,

$$Q(\vartheta, \varrho) = Q(\vartheta) Q(\varrho) .$$

In the Zel’dovich approximation (Eqs. 10 and 11) at a fixed time the velocity of a particle is a function of $\psi$ only, while the density at a particle is a function of $(\partial \psi/\partial q)$ only. Thus, if $\psi$ and its spatial derivatives are statistically independent, then $Q(\vartheta, \varrho)$ can be
separated as in Eq. (15). This condition is automatically met for a Gaussian random field: the covariance \( \langle \psi_i \partial \psi_j / \partial q_k \rangle \) vanishes by isotropy when \( \psi \) and \( \partial \psi / \partial \mathbf{q} \) are evaluated at the same position. Vanishing covariance implies statistical independence for a bivariate normal distribution. (Equation 15 could be valid for other random fields as well but this issue is beyond the scope of this paper; we therefore restrict the following discussion to initially Gaussian fields.)

The integral over \( \varrho \) (in Eq. 14) can now be performed for a fixed \( \vartheta \), yielding by Eqs (4) and the normalization of \( P \)

\[
P(\vartheta) = Q(\vartheta) \int \frac{\varrho}{\bar{\varrho}} Q(\varrho) d\varrho = Q(\vartheta) \int P(\varrho) d\varrho = Q(\vartheta) ,
\]

proving Eq. (13). Hence, since \( Q(\vartheta) \) is time invariant under the Zel’’dovich approximation, \( P(\vartheta) \) is time invariant too.

Note that the above invariance is valid only for the one-point PDF. The velocity field does not remain Gaussian, in the sense that the distribution of velocities at different points in space is not expected to remain a multivariate normal distribution.

We conclude that, as long as orbit crossing is negligible, \( P(v) = P(\vartheta) \) is time invariant [aside from the trivial time-dependence \( \sigma_\vartheta(t) \)] and should remain Gaussian. In the case of multistreaming, \( Q(v) \) and \( P(v) \) are not necessarily time invariant because, for example, the probability for a given number of streams at a point (Eq. 8) varies with time. Also, too far into the multistreaming regime the Zel’dovich approximation itself breaks down. We test below (§3) the validity of this result in the presence of multistreaming using the smoothed velocity field in an N-body simulation.

2.4. Density PDF in the Zel’dovich Approximation

The density PDF, contrary to the velocity PDF, is strongly affected by nonlinear effects. For one thing, a symmetric distribution of small density fluctuations (with \( \langle \delta \rangle = 0 \)) develops an asymmetry because the positive fluctuations can grow to any large value while the negative fluctuations are limited by definition to \( \delta \geq -1 \ (\rho \geq 0) \). This eventually
results in a sharp drop in $Q(\delta)$ and in $P(\delta)$ toward $\delta = -1$. Another important effect is
due to the fact that positive $\delta$’s are typically associated with collapse, and therefore tend
to occupy smaller volumes at later times, while negative $\delta$’s typically occur in ‘voids’ which expand in time. This tends to shift $P(\delta)$ from positive to negative values. Finally, the
formation of pancakes with high densities produces an extended tail for $Q(\delta)$ and $P(\delta)$ at large $\delta$’s.

The key of this section is the computation of the single-stream Lagrangian PDF $Q(\rho, t)$ of an initially Gaussian density field that has evolved under the Zel’dovich mapping. The somewhat elaborate calculation can be summarized as follows (cf. Kofman 1991b). Define

$$\tilde{\rho} \equiv \rho/\bar{\rho}.$$ 

Based on continuity, we write the Zel’dovich density (11) as

$$\tilde{\rho}(q, t) = |\nu_1(q, t)|^{-1}, \quad \nu_1 \equiv (1 - D\lambda_1)(1 - D\lambda_2)(1 - D\lambda_3),$$

(17)

with $\lambda_i$ the eigenvalues of the deformation tensor $-\partial \psi_i/\partial q_j$, provided as initial conditions. The absolute value in the denominator allows one to continue using this expression even after the particle has passed through a caustic (where $\nu_1 = 0$). For convenience we can write $\nu_1$ as a cubic in $D(t)$,

$$\nu_1 = 1 - D\mu_1 + D^2\mu_2 - D^3\mu_3,$$

(18)

$$\mu_1 \equiv \lambda_1 + \lambda_2 + \lambda_3, \quad \mu_2 \equiv \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad \mu_3 \equiv \lambda_1\lambda_2\lambda_3.$$

The crucial input into the desired calculation is the joint probability density for the eigenvalues in a Gaussian field, which has been computed by Doroshkevich (1970) to be:

$$Q_\lambda(\lambda_1, \lambda_2, \lambda_3) = \frac{5^{5/2} 27}{8\pi \sigma_{in}^6} \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} \exp \left[ \frac{-1}{\sigma_{in}^2} (3\mu_1^2 - 7.5\mu_2) \right],$$

(19)

with $\sigma_{in}$ equaling the variance of $\tilde{\rho}$ at some initial time when the field is Gaussian. Equation (19) now allows us to determine the probability density for $\tilde{\rho}$ as a function of the $\lambda_i$’s through equation (17).
The calculation becomes easier if we replace the eigenvalues \( \lambda_i \) by the more convenient variables
\[
\nu_1, \quad \nu_2 \equiv D^2 \mu_2 - D^3 \mu_3, \quad \nu_3 \equiv D^3 \mu_3 ,
\]
the first of which is simply related to the desired \( \tilde{\varrho} \) (Eq. 17). Given this transformation, the joint PDF of the new variables can be expressed in terms of the PDF of the old variables:
\[
Q_\nu(\nu_1, \nu_2, \nu_3) = Q_\lambda(\lambda_1, \lambda_2, \lambda_3) \left| \frac{\partial(\nu_1, \nu_2, \nu_3)}{\partial(\lambda_1, \lambda_2, \lambda_3)} \right|^{-1}.
\]
Then, the PDF of \( \nu_1 \) is given simply by double integration,
\[
Q_{\nu_1}(\nu_1) = \int d\nu_2 \int d\nu_3 Q_\nu(\nu_1, \nu_2, \nu_3)
\]
over the appropriate range in the \((\nu_1, \nu_2, \nu_3)\) space. The desired PDF of relative density, \( \tilde{\varrho} = |\nu_1|^{-1} \), is then given by
\[
Q(\tilde{\varrho}) = \tilde{\varrho}^{-2}[Q_{\nu_1}(\tilde{\varrho}^{-1}) + Q_{\nu_1}(-\tilde{\varrho}^{-1})].
\]

What is left is to express the \( \lambda_i \)'s in term of the \( \nu_i \)'s and the hard part is to obtain the limits of integration in the double integral (22). We note when inverting the transformation (20) that the \( \lambda_i \)'s can be evaluated by solving the cubic polynomial equation
\[
\lambda^3 - D^{-1}(1 - \nu_1 + \nu_2)\lambda^2 + D^{-2}(\nu_2 + \nu_3)\lambda - D^{-3}\nu_3 = 0 .
\]
The problem of defining the range of integration in (22) is thus reduced to finding where all three roots of Equation (23) are real. In practice we simply set \( Q_\nu = 0 \) when the roots are not all real.

We finally obtain after some algebra (see Appendix A for details)
\[
Q(\tilde{\varrho}, t) = \frac{N}{\tilde{\varrho}^2 \sigma^4} \int_{\frac{3\tilde{\varrho}^{-1/3}}{1}}^{\infty} ds \ e^{-(s-3)^2/2\sigma^2} \left( 1 + e^{-6s/\sigma^2} \right) \left( e^{-\beta_1^2/2\sigma^2} + e^{-\beta_2^2/2\sigma^2} - e^{-\beta_3^2/2\sigma^2} \right),
\]
\[
\beta_n(s) \equiv s^{5^{1/2}} \left( \frac{1}{2} + \cos \left[ \frac{2}{3}(n - 1)\pi + \frac{1}{3} \arccos \left( \frac{54}{\tilde{\varrho}s^3} - 1 \right) \right] \right),
\]

\[\text{(25a)}\]

\[\text{(25b)}\]

14
where $\sigma(t) \equiv D(t)\sigma_{in}$ is the standard deviation of $\tilde{\phi} = \phi/\bar{\rho}$ at time $t$ according to linear theory [given $\sigma_{in}$ at $t_{in}$ and $D(t)$ being the growing solution between $t_{in}$ and $t$]. The shape of $Q(\tilde{\phi})$ depends on time only via the parameter $\sigma$. The numerical factor is $N = 9 \cdot 5^{3/2}/4\pi \approx 8.007328$. $\int Q(\tilde{\phi})d\tilde{\phi} = 1$. In the pure laminar regime $A = 1$. The complicated expression (25) for $Q(\tilde{\phi})$ indeed reduces to a simple Gaussian distribution when $\sigma \ll 1$ (see Appendix B for a proof). Otherwise, the one-dimensional integral of equation (25) has to be performed numerically for a given $\sigma$ and $\tilde{\phi}$.

The desired single-stream Eulerian distribution can now be evaluated by Eqs. (4) and (5):

$$P(\tilde{\phi}) = \frac{1}{\tilde{\phi}N_s}Q(\tilde{\phi}) .$$

In the laminar regime equations (25) and (26) give the Zel’dovich approximations for $Q(\rho/\bar{\rho})$ and $P(\rho/\bar{\rho})$.

We can use the derived $Q(\tilde{\phi})$ of the Zel’dovich approximation to estimate the mean number of streams at each position, $N_s(\sigma)$, using Eq. (6),

$$N_s(\sigma) = \int \frac{1}{\tilde{\phi}}Q(\tilde{\phi})d\tilde{\phi} .$$

The resultant $N_s$ as a function of $\sigma$ is shown in Figure 1. $N_s$, much like $A$, remains flat at unity until $\sigma \sim 1$, and then, when orbit crossing becomes severe and the Zel’dovich approximation breaks down in certain places, it shoots off to large values. This growth is supposed to be proportional to $D^2$ for large $\sigma$ for the following reason. Replace the integration over $Q(\phi)d\phi$ in Eq. (27) by

$$N_s = \int \tilde{\phi}^{-1}Q_\lambda(\lambda_1, \lambda_2, \lambda_3)d\lambda_1d\lambda_2d\lambda_3 .$$

Recall from Eq. (18) that $\tilde{\phi}^{-1} = |1 - \mu_1D + \mu_2D^2 - \mu_3D^3|$. A simple symmetry argument guarantees that $Q(\lambda_1, \lambda_2, \lambda_3)$ is invariant under $(\lambda_1, \lambda_2, \lambda_3) \rightarrow (-\lambda_1, -\lambda_2, -\lambda_3)$, so $\langle \mu_1 \rangle = \langle \mu_3 \rangle = 0$. The deviation from $N_s = 1$ is therefore only due to the term which grows in time $\propto D^2$. 

15
As a bonus from this analysis we can gain, for example, some insight into the formation of pancakes in the Zel’dovich approximation. Simulations indicate that the process of pancaking is rather typical (Melott and Shandarin 1989; Nusser and Dekel 1990; Kofman et al. 1992). Assume that there is a universal density fall off from the nearest caustic plane, \( \rho(x) \). Such a one-to-one correspondence between \( x \) and \( \rho \) implies \( P(\rho) \, d\rho \propto dx \). Since we find in Eq. (25) that for large densities \( P(\rho) \propto \rho^{-3} \), we get by integration \( x \propto \rho^{-2} \), i.e. \( \rho \propto x^{-1/2} \). This is indeed consistent with the density profile of caustics in developed pancakes (see Shandarin and Zel’dovich 1989 for a review). Kofman (1991b), Coles and Jones (1991) and Coles and Frenk (1991) have also made a similar point.

The mean number of streams at a point can tell us in turn about the rate of pancake formation. One can write \( N_s = p(1) + 3p(3) + \ldots \), where \( p(N) \) is the probability for \( N \) streams at a point (as in Eq. 8), since only an odd number of streams is possible. The normalization, on the other hand, guaranties that \( p(1) + p(3) + \ldots = 1 \). If \( N_s \) is not too big, we can ignore the occurrence of 5 streams or more and estimate the probability for (three-stream) pancakes to occur at any Eulerian point: \( p(3) \approx (N_s - 1)/2 \).

### 3. EVOLUTION OF PDF’S IN AN N-BODY SIMULATION

To test the nonlinear effects including the effects of multistreaming, the PDFs have been computed in a cosmological N-body simulation. We use a particle-mesh code (Bertschinger and Gelb 1991) with \( 256^3 \) grid cells and \( 128^3 \) particles in a periodic cubic box of comoving size \( 200 \, h^{-1}\text{Mpc} \). The initial conditions for the simulation are a random Gaussian realization of the “standard” CDM spectrum (Davis et al. 1985), assuming \( \Omega = 1 \) and \( h = 0.5 \). The spectrum is normalized such that today, based on linear growth, \( \sigma_8^2 \equiv \langle \delta^2 \rangle \) in spheres of radius \( 8 \, h^{-1}\text{Mpc} \) is unity. The expansion factor at that time is set to \( a = 1 \). Figure 2 shows the particle distribution in the simulation at \( a = 1 \) in an arbitrary slice of thickness \( 25 \, h^{-1}\text{Mpc} \).

Continuous density and velocity fields do not exist in an N-body simulation, where instead one has a point process. However, continuous fields may be defined by replacing
each particle with a smoothing kernel. The statistical properties of the point process are then reflected in those of the continuous density and velocity fields, as discussed in detail by Scherrer & Bertschinger (1991) and Scherrer (1992).

The desired density and velocity fields are evaluated on a $80^3$ grid of comoving spacing $2.5 \, h^{-1}\text{Mpc}$. The first operations are on the grid-cell scale: a trilinear interpolation (cloud-in-cell assignment) of mass and momentum to a grid point from the particles in its neighboring cells, followed by small-scale Gaussian smoothing of the mass density and momentum density with a Gaussian window of radius $2.5 \, h^{-1}\text{Mpc}$. The velocity at a grid point is then defined as the ratio of momentum to mass density there. The purpose of this small-scale smoothing is to obtain meaningful velocity values in grid points that reside in a neighborhood of empty cells so that they are not spuriously assigned zero velocity.

The density and velocity fields are then smoothed further on a larger scale using a spherical Gaussian window of radius $R_s$. The purpose of this smoothing is to reduce the effects of nonlinearities by diminishing the density contrasts. A range of $R_s$ is considered in order to span a range of nonlinear effects. Note that smoothing after nonlinear evolution does not fully remove nonlinear effects, although the longest wavelengths are expected to evolve according to linear theory. We will see, however, that on sufficiently large scales, smoothing restores approximately the linear evolution of the large-scale initial conditions.

The smoothed velocity and density fields on the grid, at different times and with different smoothing lengths, are used to construct the PDFs.

Figure 3 shows the Eulerian velocity PDF in the most nonlinear case studied with the simulation: $a = 1$ and $R_s = 6 \, h^{-1}\text{Mpc}$, with $\sigma_v = 277 \, \text{km s}^{-1}$ (and $\sigma_\delta = 0.55$). The error bars are the standard deviation of the mean in eight octants of side $100 \, h^{-1}\text{Mpc}$ each. These are only rough estimates of the true statistical uncertainties because only eight subvolumes were used. The PDFs at earlier times and larger smoothings are very similar and are therefore not shown. They are all very much Gaussian, as predicted by the laminar Zel’dovich approximation. The apparent excess in the positive tail beyond $3\sigma_v$ is
at least partly due to a random deviation in the initial conditions due to the limited volume of the box; a similar excess shows up at all times. Thus, the N-body simulation confirms the Zel’dovich prediction that the velocity PDF of initially Gaussian fluctuations remains Gaussian in the quasilinear regime, and it extends this result into the multistreaming regime.

Figure 4 shows the Eulerian density PDFs at $a = 1$ for three different smoothing lengths: $R_s = 18, 12$ and $6 \, h^{-1}\text{Mpc}$, corresponding to $\sigma = 0.11, 0.26$ and $0.55$ respectively. The symbols are the means from eight octants in the simulation and the error bars are the corresponding standard deviations of the mean in the eight octants. A range of $\sigma$ values could similarly be spanned by a sequence of time steps with a fixed smoothing length. The dependence of the PDF on $\sigma$ is similar but not identical (see the discussion of moments and Figure 5 below). We show the smoothing sequence here because it is what one can obtain observationally.

The dashed lines are based on the laminar Zel’dovich approximation $[\mathcal{P}(\tilde{\rho})$ from $\mathcal{Q}(\tilde{\rho})$ of Eq. 25 and 26] for the corresponding $\sigma$ values. The laminar Zel’dovich approximation indeed provides an excellent approximation to the simulated PDF in the intermediate, slightly nonlinear case with $\sigma = 0.26$. At $\sigma = 0.55$ the Zel’dovich approximation is still a very good approximation out to $\rho/\bar{\rho} = 3$ ($\approx 5.5\sigma$), but it starts to overestimate the positive tail beyond that. The Zel’dovich power-law tail, $P \propto \rho^{-3}$, reflects the collapse of the highest peaks (in $\lambda_1$) into caustics — caustics which are smeared out by the smoothing applied to the N-body simulation.

This example shows how smoothing and nonlinear evolution do not always commute. If the initial density field is smoothed first, nonlinear evolution produces caustics of high density. However, when an unsmoothed field is evolved and then smoothed, the smoothing reduces the high densities. Equation (25) is appropriate in the former case but not the latter. Nonlinear evolution erases some memory of the initial conditions on small scales but it also generates small-scale structure from the collapse of long waves (Kofman et al.)
The solid curves are lognormal distributions with the same \( \sigma \),

\[
P(\tilde{\rho}) = \frac{1}{(2\pi \sigma_l^2)^{1/2}} \exp \left[\frac{(\ln \tilde{\rho} - \mu_l)^2}{\sigma_l^2}\right] \cdot \frac{1}{\tilde{\rho}},
\]

where \( \mu_l \) and \( \sigma_l \) are the mean and standard deviation of \( \ln \tilde{\rho} \). They are related to the corresponding moments of \( \tilde{\rho} \), \( \mu \) and \( \sigma \), via

\[
\mu_l = \ln \mu - (1/2)\sigma_l^2, \quad \sigma_l^2 = \ln(1 + \sigma^2/\mu^2)
\]

(and recall that in fact \( \mu = 1 \) for \( \tilde{\rho} \)). As argued by Coles and Jones (1991) and noted earlier by Hamilton (1985), the lognormal distribution turns out to be an excellent approximation to the actual density PDF. The way it fits the simulation at \( \sigma = 0.55 \) over the whole range tested, out to \( \sim 10\sigma \), is striking!

A more quantitative measure of the deviations of the PDFs from Gaussian is provided by their third and forth irreducible moments (Figure 5). Given a set of random measurements of the random variable \( x \), define as usual the mean, \( \mu \equiv \langle x \rangle \), and the variance, \( \sigma^2 \equiv \langle (x - \mu)^2 \rangle \). Then the dimensionless skewness and kurtosis relative to the mean are defined by

\[
s \equiv \frac{\langle (x - \mu)^3 \rangle}{\sigma^3} \quad \text{and} \quad k \equiv \frac{\langle (x - \mu)^4 \rangle}{\sigma^4} - 3.
\]

As a reference, a Gaussian distribution has \( s = k = 0 \).

Based on the Zel’’dovich approximation, we expect that any deviation from a normal shape are predominantly a function of \( \sigma \). We therefore plot in Figure 5 the skewness and kurtosis of the PDFs as a function of \( \sigma \). Note though that a range of \( \sigma \) values could be obtained either by analyzing the system at different times or by using a variety of smoothing lengths. Each panel shows three curves corresponding to three different times \( (a) \), with the Gaussian smoothing length \( (R_s \text{ in comoving } h^{-1}\text{Mpc}) \) varying along each curve. The error bars mark the standard deviation of the mean for the moments as evaluated in eight octants.
Because of the expected symmetry between positive and negative velocities, there is no surprise in the fact that the velocity skewness is consistent with zero; any deviation must be a result of the limited volume sampled. But the fact that the velocity kurtosis remains constant and consistent with Gaussian is a meaningful confirmation of the laminar Zel’ dovich approximation. One can see that the apparent small deviation from zero of both $s$ and $k$, which is probably associated with the apparent tail in Figure 3, is indeed similar at all times, indicating that it must be due to the finite volume sampled rather than nonlinear evolution.

While the moments of velocity remain constant over the whole range of $\sigma_v$ tested, the moments of density gradually deviate from Gaussian in the corresponding range of $\sigma_{\delta}$. This deviation is significant already at relatively low $\sigma_{\delta}$ values.

We can also see from Figure 5 that the growth of $\sigma_{\delta}$ in the simulations is very similar to the prediction of linear theory: $\sigma_{\delta} \propto a$ to within 3% in the tested range. This is saying that the faster nonlinear growth in high density regions is roughly compensated for by the slower deepening of low-density regions (limited by $\delta \geq -1$). This allows one to use in the Zel’dovich approximation (25) the true, observable $\sigma$ instead of the linear $\sigma$ which we can’t measure.

The actual values of $a$ and $R_s$ that boil down to a given value of $\sigma$ make some difference, which is significant (in view of the errors) only for the density kurtosis. This difference depends on the power spectrum used. Since observationally $\sigma$ can vary only due to the smoothing used, we limited ourselves in Figures 3 and 4 to “observing” the simulation only at one time while the smoothing length is varied. Assuming a universal galaxy biasing factor of unity ($b = \sigma^{-1}_{s} = 1$), we used the time step in the simulation $a = 1$. If the true biasing factor is different from unity, then a different time step in the simulation should have been used to resemble the present universe (e.g. $a = 0.5$ for $b = 2$). Based on Figure 5, this would not affect much the dependence of skewness on $\sigma$ but the density kurtosis would behave somewhat differently.
Second-order perturbation theory predicts that the ratio of density skewness to its standard deviation is constant, \( s/\sigma = 34/7 \) (Peebles 1980), which is marked by the dotted line in the top-right panel of Figure 5. We can see that this is a good approximation for \( R_s \geq 12 \, h^{-1}\text{Mpc} \), independently of how \( \sigma \) was changed (by \( a \) or by \( R_s \)). At smaller smoothing lengths the N-body results tend towards \( s/\sigma \) values in the range 4 to 3, in general agreement with the various models discussed by Coles and Frenk (1991). In particular, it has been predicted based on second order theory that this ratio would depend on the effective logarithmic slopes of the power spectrum at the smoothing scale, \( n \), roughly as \( s/\sigma = 34/7 - (n + 3) \) (Juszkiewicz, Bouchet and Colombi 1992, but note they assumed top-hat smoothing). In our case of a CDM spectrum and Gaussian smoothing lengths in the range 6 – 21 \( h^{-1}\text{Mpc} \) the effective slope is in the vicinity of \( n \approx -1 \), which indeed predicts \( s/\sigma \approx 3 \).

4. TENTATIVE COMPARISON WITH OBSERVATIONS

Equipped with the above results concerning the PDFs of initially Gaussian fluctuations for a given \( \sigma \), we now make first attempts to reconstruct the one-point spatial distribution functions of the smoothed velocity and density fields as estimated from galaxies in a finite volume around us. The following comparison is tentative as it is based on limited data in a relatively small volume. The pilot data are provided by either the early POTENT analysis of observed velocities or the analysis of the 1.9Jy redshift survey of IRAS galaxies.

The POTENT analysis used the observed radial peculiar velocities of about 1000 galaxies in a sphere of radius \( \sim 60 \, h^{-1}\text{Mpc} \) about the Local Group. The observed radial velocities were first smoothed into a radial velocity field on a grid, in a way that minimizes the effects of sparse sampling and measurement errors. POTENT then imposes the requirement of potential flow, \( \mathbf{v} = -\nabla \phi \), which is a natural outcome of gravitational instability, to reconstruct the missing two components of the velocity field (Bertschinger and Dekel 1989; Dekel et al. 1990; Bertschinger et al. 1990). Assuming that the galaxies are fair tracers of the smoothed velocity field independent of their specific type, the resultant
velocity field is independent of galaxy “biasing”. Finally, assuming a value for \( \Omega \), and using a quasilinear approximation for the equations governing the evolution of fluctuations (Nusser et al. 1991), POTENT yielded the mass-density fluctuation field that has given rise to the peculiar velocities. The output is provided on a cubic grid of spacing \( 5 \, h^{-1} \text{Mpc} \) inside a spherical volume of radius \( 60 \, h^{-1} \text{Mpc} \). A Monte Carlo analysis of distance-measurement errors provided estimates of the uncertainties of each field, which enables us to use for the reconstruction of the PDFs only points where the uncertainty is smaller than a certain conservative limit, and to estimate the resultant uncertainties in the PDFs.

Because of the limited volume analyzed, and the zero-point uncertainty in the distance indicators, the mean density and velocity within this volume have finite values different from the universal zero means: the mean density contrast is \( \mu_\delta = 0.11 \) and the mean one-dimensional velocity is \( \mu_v = 223 \, \text{km s}^{-1} \). We use the fact that if \( x \) is a Gaussian field then the conditional probability of \( x \) in a neighborhood where the local mean is given is also Gaussian with a displaced mean and somewhat reduced dispersion (depending on the two-point correlation function; cf. Dekel 1981, the appendix). We therefore compute and plot the PDFs of \( (\delta - \mu_\delta)/\sigma_\delta \) and \( (v - \mu_v)/\sigma_v \).

These PDFs of POTENT velocities and densities, assuming \( \Omega = 1 \), are shown in Figure 6. We use only points with Monte Carlo measurement errors in the three-dimensional velocities and in the densities smaller than \( 300 \, \text{km s}^{-1} \) and \( 0.3 \) respectively. The resultant effective volume corresponds to a sphere of radius \( 37 \, h^{-1} \text{Mpc} \). The error bars are the standard deviation of the PDF in 30 Monte Carlo noise simulations of POTENT. Also marked are the derived first four moments.

Errors due to the limited volume sampled can be estimated using the CDM N-body simulation, because the power-spectrum deduced from POTENT is not significantly different from the standard CDM spectrum with \( b = 1 \) (Kolatt, Seljak, Bertschinger and Dekel 1993). In Figure 7 we show the mean PDF of eight independent spheres of radius \( 50 \, h^{-1} \text{Mpc} \) from the simulation and the associated standard deviations are marked by
the error bars. Also marked are the first four moments and their standard deviations. The relative volume errors for POTENT should be larger by a factor which is roughly the square root of the volume ratio, i.e. 1.57. For a conservative error estimate, the volume errors and the measurement errors shown in figure 7 should be added in quadrature.

We see from figure 6 that the PDFs based on POTENT are consistent with Gaussian initial fluctuations. However, the errors are very big, making this preliminary comparison only marginally interesting. Things are expected to get better though because the rapid progress in peculiar velocity surveys allow a POTENT reconstruction with smaller uncertainties in a larger volume.

The IRAS analysis (Strauss et al. 1990; 1992; Yahil et al. 1990) translated the redshift catalog of 1.9Jy IRAS galaxies into a uniform galaxy-density map, whose first approximation was given in redshift space. The predicted peculiar velocities, and the corresponding corrections to the galaxy positions from redshift space to configuration space, were reconstructed via a self-consistent iterative scheme using linear dynamical theory of gravity with small-scale smoothing and quasilinear corrections (Nusser et al. 1991), after assuming linear biasing between the density fluctuations of galaxies and mass. The resultant peculiar velocity depends both on Ω and on the “biasing” parameter b between the density fluctuations of galaxies and mass, but the obtained density field depends only weakly on the assumed value of Ω through the correction from redshifts to real positions. This analysis provided estimates for the density and velocity fields within a sphere of radius 80 h⁻¹Mpc about the local group. In the present, preliminary application we take these fields as provided to us by the authors of the IRAS analysis; we do not make an attempt to carefully estimate the errors involved in the sampling and in the analysis beyond the volume errors.

A strong correlation is found between the IRAS and POTENT fields, both featuring as extended structures the Great Attractor, an adjacent large void, and the Pisces part of the Perseus-Pisces supercluster. The comparison yields Ω⁰.⁶/b = 1.3 ± 0.7 (Dekel et al. 1993,
the error bar is 95% confidence level). This allows one to adopt the simple linear biasing assumption as a working hypothesis and associate the PDFs derived from *IRAS* galaxies with the desired PDFs of the underlying dynamical mass distribution.

The distributions of the *IRAS* densities and velocities, for two different smoothing lengths, are shown in Figure 8. The means are again removed and the deduced first four moments are marked. The errors due to the finite volume should be about twice the error bars shown in Figures 3 and 4 from the CDM N-body simulation, or about one half of the errors in Figure 7. Within these uncertainties, the *IRAS* data are consistent with Gaussian initial fluctuations. The ratio $s/\sigma$ is consistent with being a constant as a function of smoothing scale, in the range $s/\sigma \approx 1.5 - 1.8$

Note that although *IRAS* galaxies are underrepresented in rich cluster cores, so that they are biased against high density, this has little effect on $P(\rho)$ because clusters occupy a very small fraction of the volume. This is an advantage of the one-point Eulerian density distribution over other statistics that strongly weight high-density regions.

The more recently completed 1.2Jy *IRAS* survey (Fisher 1992) allows a more reliable determination of the *IRAS* PDFs in a larger volume and with more quantitative error analysis. The moments of the density PDF from this survey indeed seem to behave as expected from an initially Gaussian field subject to gravity and $n \approx 0$ near the smoothing scale, with $s/\sigma = 34/7 - (n + 3) \sim$ roughly constant as a function of smoothing scale, in the range $1 - 2$ (Bouchet, Davis and Strauss 1992). The skewness measured from the QDOT study of counts in their 1-in-6 redshift survey of *IRAS* galaxies (Saunders et al. 1991), given their errors, is also consistent with the above measurements (G. Efstathiou, private communication).

5. DISCUSSION AND CONCLUSION

We investigated the quasilinear effects on the one-point probability density functions (PDFs) of initially Gaussian fluctuation fields. The laminar Zel’dovich approximation provides useful analytic expressions that are confirmed and extended into
the multistreaming regime using a standard CDM N-body simulation. We found that the velocity PDF smoothed on scales $\geq 6 \, h^{-1}\text{Mpc}$ is hardly affected by nonlinear evolutionary effects while the density PDF develops a lognormal shape.

The observed velocity and density fields, based on POTENT reconstruction from radial velocities with $\Omega = 1$ or on an analysis of a redshift survey of IRAS galaxies, have PDFs that are apparently consistent with Gaussian initial conditions. The data used here, however, are still limited in volume and they still carry large errors. Noting that random errors can produce a spurious Gaussian PDF, we were careful to estimate the uncertainties due to measurement errors and to conservatively restrict ourselves to low-error regions at the expense of sampling a larger volume. New data are expected to allow a significantly more accurate determination of the PDFs.

These results sound encouraging for the “standard” model but can we actually use them to reject any non-Gaussian model of interest? Besides the fact that the current errors are large, we are facing several fundamental limitations. For example, there is a general reason for the velocity field to become Gaussian under a wide range of conditions, even when it came from non-Gaussian initial fluctuations (Scherrer 1992). Recall that the peculiar velocity at a given point is related to the net peculiar force there (directly proportional to it in the linear regime and in the Zel’довich approximation), which is the integral of the forces from all the mass fluctuations around it. Assume, for example, that the non-Gaussianity is expressed as excessively large density peaks or wells which dominate the large-scale force. If the characteristic separation between these structures is not larger than the typical range over which the force converges, then the velocity is practically a sum over a few independent random fluctuations and as such, based on the central limit theorem, it becomes a Gaussian variable. Thus, only certain non-Gaussian models would show a non-Gaussian velocity PDF, so it has to be individually evaluated for each model before the model can be rejected based on an observed Gaussian velocity PDF.

For instance, the Texture model cannot be rejected based on the velocity PDF.
(Gooding et al. 1992). But it still remains to be seen whether the model could be rejected based on the density PDF. As another example, it is clear that the non-Gaussianity scale in certain versions of the cosmic string model is small enough for it not to have any noticeable trace in the velocity PDF. A string model that could probably be rejected is the version where the structure is dominated by $\sim 40 \, h^{-1}\text{Mpc}$ wakes formed behind long, well-separated strings which accrete “hot” dark matter (e.g. Brandenberger 1991). In this model the velocity at a point is typically determined by the nearest wake and is therefore expected to be very non-Gaussian.

The discriminatory power of the density PDF is also limited for several reasons. First, the deviation from Gaussianity depends on the $rms$ fluctuation of the dynamical mass density, $\sigma$, which is deduced from the observed $\sigma$ of galaxies (e.g. in the case of IRAS) only via a certain assumption concerning biasing — the relation between the galaxy density and the underlying mass density fluctuations which is not well determined (see Dekel and Rees 1987 for a review). Second, the actual deviation from Gaussianity, and in particular its dependence on the smoothing scale, is somewhat dependent on the shape of the power spectrum of fluctuations. We assumed above a “standard” CDM spectrum which seems to fit the data used here reasonably well, but recall that it is not the Gaussian CDM model that one is trying to reject here. The moral is, again, that the density PDF has to be evaluated individually for each specific non-Gaussian model.

The main purpose of this paper was to provide useful tools for addressing the question of Gaussian versus non-Gaussian initial fluctuations. The current preliminary comparison of observation with theory is encouraging for the “standard” model but it is certainly far from being conclusive. We hope to be able to more quantitatively rule out certain non-Gaussian models of interest with data that are becoming available.

**ACKNOWLEDGEMENT**

We thank M. Davis, J. Huchra, M. Strauss and A. Yahil and our POTENT colleagues for allowing the use of the 1.9Jy IRAS and POTENT data. Supercomputer time was provided
by the Cornell National Supercomputer Facility. This research has been supported by US-Israel Binational Science Foundation grant 89-00194, by NSF grant AST90-01762, and by an Alfred P. Sloan Foundation Fellowship to E.B. LK especially thanks the hospitality of the Hebrew University and MIT.
APPENDIX A: Derivation of equation (25)

In this Appendix we derive equation (25) for the density PDF from the double integral (22). The main work is to find the appropriate integration limits in (22) in terms of \((\nu_1, \nu_2, \nu_3)\), for which all three roots \(\lambda(\nu_1, \nu_2, \nu_3)\) of the cubic polynomial equation (24) are real.

We rewrite equation (24) in the canonical form:

\[
\lambda^3 + A\lambda^2 + B\lambda + C = 0. \tag{A1}
\]

First, let us find the region of the \((A, B, C)\)-space, for which the three roots of equation (A1) are real. To analyze the properties of the roots of this cubic equation, we need its discriminant

\[
\Delta = \frac{1}{4} \left[ 2 \left( \frac{A}{3} \right)^3 - \frac{AB}{3} + C \right]^2 - \frac{1}{27} \left( \frac{A^2}{3} - B \right)^3. \tag{A2}
\]

All three roots \(\lambda\) are real if this determinant is negative:

\[
\Delta \leq 0. \tag{A3}
\]

To satisfy this condition, at least the second term on the right-hand side of equation (A2) has to be negative, i.e. \(A^2 - 3B \geq 0\). Then we can write the determinant in the form

\[
\Delta = \frac{1}{4} \left[ C - C^{(-)}(A, B) \right] \left[ C - C^{(+)}(A, B) \right], \quad 27C^{(\pm)}(A, B) = -2A^3 + 9AB \pm 2(A^2 - 3B)^{3/2}. \tag{A4}
\]

The condition (A3) is satisfied if

\[
A^2 \geq 3B \quad \text{and} \quad C^{(-)}(A, B) \leq C \leq C^{(+)}(A, B). \tag{A5}
\]

These are the conditions required for equation (A1) to have real roots.

Now we find the corresponding region in the \((\nu_1, \nu_2, \nu_3)\) space where all three \(\lambda\) are real. Comparing equations (24) and (A1), we have

\[
A = (\nu_1 - \nu_2 - 1)D^{-1}, \quad B = (\nu_2 + \nu_3)D^{-2}, \quad C = -\nu_3D^{-3}. \tag{A6}
\]
Substituting equations (A6) into (A5), we get two constraints in terms of \((\nu_1, \nu_2, \nu_3)\):

\[
9(\nu_1 - \nu_2 + 2)\nu_3 + 9(\nu_1 - \nu_2 - 1)\nu_2 - 2(\nu_1 - \nu_2 - 1)^3 + 2 \left[(\nu_1 - \nu_2 - 1)^2 - 3(\nu_2 + \nu_3)\right]^{3/2} \geq 0,
\]

(A7)

and

\[
9(\nu_1 - \nu_2 + 2)\nu_3 + 9(\nu_1 - \nu_2 - 1)\nu_2 - 2(\nu_1 - \nu_2 - 1)^3 - 2 \left[(\nu_1 - \nu_2 - 1)^2 - 3(\nu_2 + \nu_3)\right]^{3/2} \leq 0.
\]

(A8)

In addition, the expression in square brackets must be nonnegative. We denote it henceforth as

\[t^2 = (\nu_1 - \nu_2 - 1)^2 - 3\nu_2 - 3\nu_3,\]  

(A9)

It is also convenient to introduce

\[s = \nu_1 - \nu_2 + 2.\]  

(A10)

The constraints become simpler in terms of \(\nu_1\) and the new variables \(s\) and \(t\):

\[
|t|^3 - \frac{3}{2}st^2 + \frac{1}{2}(s^3 - 27\nu_1) \geq 0, \quad -|t|^3 - \frac{3}{2}st^2 + \frac{1}{2}(s^3 - 27\nu_1) \leq 0.
\]

(A11)

The intervals of \(t\) which satisfy to both of these constraints simultaneously are

\[t^2 > t^2_1(s, \nu_1) \quad \text{or} \quad t^2_3(s, \nu_1) < t^2 < t^2_2(s, \nu_1),\]  

(A12)

Here the functions \(t_n(s, \nu_1)\) are defined by

\[t_n(s, \nu_1) = s \left(\frac{1}{2} + \cos \theta_n\right), \quad \theta_n = \frac{1}{3} \arccos(54s^{-3}\nu_1 - 1).\]  

(A13)

There are three roots because one may add an integer multiple of \(2\pi/3\) to \(\theta\). For definiteness, we label the roots according to the phase \(\theta\): \(0 \leq |\theta_1| \leq \pi/3, \pi/3 \leq |\theta_2| \leq 2\pi/3, 2\pi/3 \leq |\theta_3| \leq \pi\), so that \(t^2_3 \leq t^2_2 \leq t^2_1\). In addition to the constraints imposed on \(t\) by equation (A12), we require that \(t_n(s, \nu_1)\) be real, implying

\[s \geq 3\nu_1^{1/3}, \quad \nu_1 \geq 0,\]  

(A14)
and
\[ s \leq -3|\nu_1|^{1/3}, \nu_1 \leq 0. \]  \hspace{1cm} (A15)

The constraints (A12)–(A15) give us the limits in terms of \( t \) and \( s \). They are easily expressed in terms of \( \nu_2 \) and \( \nu_3 \) using equations (A9) and (A10) to get \( \nu_2 = \nu_1 - s + 2 \) and
\[ \nu_3(t, s, \nu_1) = \frac{1}{3} (s^2 - t^2) - s - \nu_1 + 1. \]  \hspace{1cm} (A16)

Now we are ready to treat the integral (22). We have from equations (19)–(21) (the Jacobian in eq. 21 simply removes the three \( \lambda \)-dependent factors from eq. 19):
\[
Q_{\nu_1} = \int d\nu_2 \int d\nu_3 Q_{\nu}(\nu_1, \nu_2, \nu_3)
\]
\[= \frac{5^{5/2}27}{8\pi(\sigma_{in}D)^6} \int d\nu_2 \int d\nu_3 \exp \left[ -\frac{3(\nu_1 - \nu_2 - 1)^2}{(\sigma_{in}D)^2} + \frac{15}{2(\sigma_{in}D)^2}(\nu_2 + \nu + 3) \right]. \]  \hspace{1cm} (A17)

The first integral over \( \nu_3 \) has limits given by equations (A12) and (A16) and is trivial. With two ranges of integration (one an unbounded interval) this integral yields three exponentials. The integral over \( \nu_2 \) is performed using the variable \( s \) through equation (A10). Its limits are given by equations (A14) and (A15). Their are two terms, one for \( \nu_1 > 0 \) and the other for \( \nu_1 < 0 \) in equation (23). The integration of \( s \) finally reduces to equation (25) of the main text.

**APPENDIX B: Asymptotics of the density PDF for \( \sigma \ll 1 \)**

In this appendix we show that the density PDF given as the integral (25) reduces to the Gaussian distribution (1) in the limit of small density dispersion \( \sigma \ll 1 \).

To see this, we investigate the properties of the integral (25) as \( \sigma \ll 1 \). The expression (25) can be represented as an algebraic combination of six integrals
\[
Q(\tilde{\sigma}) = \frac{N}{\tilde{\sigma}^2 \sigma^4} \sum_{k=1}^{6} \left( \pm I_k(p) \right), \quad I_k(p) = \int_{s_0}^{\infty} ds \, e^{-pF_k(s)}. \]  \hspace{1cm} (B1)
Here \( p = 1/\sigma^2 \gg 1 \) is a large parameter and \( s_0 = 3\tilde{\sigma}^{-1/3} \) denotes the lower limit of integration. The six integrals come from the six terms in equation (25) that one gets by multiplying out the exponentials. Four of these terms have a positive sign and two have a negative sign. The formulas for \( F_k(s) \) are rather tedious; some that we will use are given by equations (B3) and (B4) below.

The asymptotic expansions of the integrals in equation (B1) for \( p \to \infty \) are

\[
I_k(p) \sim e^{-pF_k(s_0)} \cdot \text{(asymptotic series in powers of } p^{-1}) . \tag{B2}
\]

The asymptotic series have to be defined based on the analytic properties of the functions \( F_k(s) \). Using equation (25b) for the functions \( \beta_n(s) \), after straightforward but tedious analysis of the functions \( F_k(s) \) in the vicinity of \( s_0 \), one can show that two following terms in the sum (B1) are of leading order as \( p \to \infty \): the positive term we denote \( I_2 \) with

\[
F_2(s) = (s - 3)^2/2 + \beta_2^2(s)/2 , \tag{B3}
\]

and the negative term we denote \( I_3 \) with

\[
F_3(s) = (s - 3)^2/2 + \beta_3^2(s)/2 . \tag{B4}
\]

The other integrals \( I_1, I_4, I_5, \) and \( I_6 \) are exponentially suppressed compared with \( I_2 \) and \( I_3 \). Thus we have to consider the combination \( (I_2 - I_3) \) for \( p \to \infty \). The decompositions of the functions \( F_{2,3} \) in the vicinity of \( s_0 \) are:

\[
\begin{align*}
F_2(s) &= \frac{(s_0 - 3)^2}{2} + \left( \frac{7s_0 - 3}{2} \right) \cdot (s - s_0) - \frac{5\sqrt{s_0}}{3} 
\cdot (s - s_0)^{3/2} - \frac{1}{3} \cdot (s - s_0)^2 + ..., \tag{B5} \\
F_3(s) &= \frac{(s_0 - 3)^2}{2} + \left( \frac{7s_0 - 3}{2} \right) \cdot (s - s_0) + \frac{5\sqrt{s_0}}{3} \cdot (s - s_0)^{3/2} - \frac{1}{3} \cdot (s - s_0)^2 + .... \tag{B6}
\end{align*}
\]

To obtain an asymptotic series (B2), the functions \( F_k \) involved in the integrals \( I_k \), have to be represented as analytic functions of the variable of integration. However, the series given by (B5) and (B6) show that \( F_2(s) \) and \( F_3(s) \) are not analytic functions of \( s \). Therefore we must change variables of integration. For this purpose we introduce

\[
v = \sqrt{s - s_0} . \tag{B7}
\]
Now the integrals read as

$$I_k(p) = 2 \int_0^\infty dv v e^{-p \tilde{F}_k(v)}, \quad (B8)$$

where the functions of the new variable have the following analytic series in the vicinity of $v = 0$:

$$\tilde{F}_{2,3}(v) = \left(\frac{s_0 - 3}{2}\right)^2 + \left(\frac{7s_0 - 3}{2}\right) \cdot v^2 \pm \frac{5\sqrt{s_0}}{3} \cdot v^3 - \frac{1}{3} \cdot v^4 + ..., \quad (B9)$$

The upper sign corresponds to $k = 3$ and the lower to $k = 2$. For these analytic functions we can use the general formula of the asymptotic expansion (e.g. Olver 1974)

$$I_k(p) \simeq 2e^{-p \tilde{F}_k(0)} \sum_{n=0}^{\infty} \frac{\Gamma(n+2)}{2n} \frac{a_n^{(k)}}{p^{n+2/2}}, \quad (B10)$$

where $\Gamma(x)$ is the Gamma function. The coefficients $a_n^{(2,3)}$ are expressed through the coefficients of the series (B9). The first two of them, which we will use, are

$$a_0^{(2,3)} = (7s_0 - 6)^{-1}, \quad (B11a)$$

and

$$a_1^{(2,3)} = (\pm 1) \frac{-\sqrt{10s_0}}{(7s_0 - 6)^{5/2}}. \quad (B11b)$$

In the sum (B1) we have the combination $(I_2 - I_3)$. Substituing in this combination the asymptotic series (B10) and using (B11), we find that the first terms with coefficients $a_0$ cancel. The leading remaining terms with coefficients $a_1^{(2,3)}$ give

$$I_2 - I_3 = \frac{2\sqrt{10\pi s_0}}{(7s_0 - 6)^{5/2}} \sigma^3 e^{-(s_0 - 3)^2 / 2\sigma^2}, \quad (B12)$$

where we have replaced $p$ by the original the small parameter $\sigma$.

Now we are ready to reproduce the result of the linear theory. Let us recall that $s_0 = 3\delta^{-1/3} \approx 3 - D(t)\delta$ for small density fluctuations $\delta \ll 1$ and $\sigma = D(t)\sigma_{in}$. Substituing these formulas into (B12), we obtain

$$I_2 - I_3 = \left(9 \cdot 5^{3/2} / 4\pi\right)^{-1} \frac{1}{\sqrt{2\pi\sigma_{in}}} e^{-\delta^2 / 2\sigma_{in}}, \quad (B13)$$

Substituing this expression into equation (B1) we get the normal distribution for the density fluctuations $\delta$. 

32
REFERENCES

Bahcall, N. 1988, Ann. Rev. Astr. Ap. 26, 631.

Bertschinger, E. 1989, Ann. Ny. Acad. Sci. 57, 151.

Bertschinger, E. and Dekel, A. 1989, ApJ 336, L5.

Bertschinger, E., Dekel, A., Faber, S.M., Dressler, A. and Burstein, D. 1990, ApJ 364, 370.

Bertschinger, E. and Gelb, G. 1991, Computers in Physics Mar/Apr, 164.

Bertschinger, E., Gorski, K. and Dekel, A. 1990, Nature 345, 507.

Bouchet, F. R., Davis, M. and Strauss, M. 1992, in: Proc. of the 2nd DAEC Meeting, ed. Mamon, G. and Gerbal, D., (Paris Observatory), p.287.

Weinberg, D. and Cole, S. 1992, M.N.R.A.S. 259, 652.

Brandenberger, R. 1991, Physica Scripta, T36, 114.

Coles, P. and Frenk, C. S. 1991, M.N.R.A.S. 253, 727.

Coles, P. and Jones, B. 1991, M.N.R.A.S. 248, 1.

Davis, M., Efstathiou, G., Frenk, C. and White, S.D.M. 1985, ApJ 292, 371.

Dekel, A. 1981, A&A 101, 79.

Dekel, A., Bertschinger, E. and Faber, S.M. 1990, ApJ 364, 349.

Dekel, A., Bertschinger, E., Yahil, A., Strauss, M. Davis, M. and Huchra, J. 1993, ApJ , in press.

Dekel, A. and Rees, M.J. 1987, Nature 326, 455.

Doroshkevich, A.G. 1970, Astrofizica, 6, 581

Efstathiou, G. 1990, in: Physics of the Early Universe, eds. J. Peacock, A. Heavens, and A. Davies, (Inst. of Physics Publishing, Bristol), p.361.
Efstathiou, G., Kaiser, N., Saunders, W., Lawrence, A., Rowan-Robinson M., Ellis, R.S., and Frenk, C.S. 1990, M.N.R.A.S. 247, 10p.

Fisher, K.B. 1992, PhD. thesis, University of California, Berkeley

Gooding, A., Park, C., Spergel, D., Turok, N. and Gott, J.R., III. 1992, ApJ 393, 42.

Hamilton, A. 1985, ApJ, 292, L35.

Juszkiewicz, R., Bouchet, F. R. and Colombi, S. 1993, Ap.J., submitted.

Kofman, L. 1991a, Physica Scripta, T36, 108.

Kofman, L. 1991b, in: Primordial Nucleosynthesis and Evolution of Early Universe, eds. Sato, K. and Audouze, J. (Dordrecht: Kluwer), p.495.

Kofman, L., Melott, A. Pogosyan, D. Yu. and Shandarin , S. 1992, ApJ 393, 437.

Kolatt, T., Seljak, U., Dekel, A. and Bertschinger, E. 1993, in preparation.

Linde, A. 1990, Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland).

Lynden-Bell, D., Faber, S.M., Burstein, D., Davies, R.L., Dressler, A., Terlevich, R.J. and Wegner, G. 1988, ApJ 326, 19.

Maddox, S.J, Efstathiou, G., Sutherland, W.J. and Loveday, J. 1990, M.N.R.A.S. 242, 43p.

Melott, A. L. and Shandarin, S. F. 1989, ApJ 343, 26.

Monin, A. and Yaglom, A., 1971, Statistical fluid mechanics (Cambridge: MIT Press).

Nityananda, R. and Ostriker, J. 1984, J. Astr. Ap. 5, 235.

Nusser, A., Dekel, A., Bertschinger, E. and Blumenthal, G. R. 1991, ApJ 379, 6.

Olivier et al. 1993, preprint.

Olver, F.W.J. 1974, Introduction to Asymptotics and Special Functions (NY: Academic
Ostriker, J. P. 1988, in: IAU Symp. No. 130, ed. J. Audouze and A. Szalay (Dordrecht: Reidel).

Peebles, P.J.E. 1980, The Large-Scale Structure of the Universe (Princeton: Princeton University Press).

Rowan-Robinson, M., Lawrence, A., Saunders, W., Crawford, J., Ellis, R.S., Frenk, C.S., Parry, I., Xiaoyang, X., Allington-Smith, J., Efstathiou, G. and Kaiser, N. 1990, M.N.R.A.S. 247, 1.

Saunders, W., Frenk, C.S., Rowan-Robinson, M., Ellis, R., Lawrence, A., Kaiser, N., Efstathiou, G., Crawford, J., Xia, X.Y., and Parry, I. 1991, Nature 349, 32.

Scherrer, R. & Bertschinger, E. 1991, ApJ 381, 349.

Scherrer, R. 1992, ApJ 390, 330.

Shandarin, S.F. and Zel’dovich, Ya.B. 1989, Rev.Mod.Phys. 61, 185.

Strauss, M. A., Davis, M., Yahil, A., and Huchra, J. P. 1990, ApJ 361, 49.

Strauss, M. A., Davis, M., Yahil, A., and Huchra, J. P. 1992, ApJ 385, 421.

Trimble, V. 1987, ARAA, 25, 425.

Turok, N. 1991, Physica Scripta, T36, 135.

Yahil, A., Strauss, M. A., Davis, M., and Huchra, J. P. 1991, ApJ 372, 380.

Zel’dovich, Ya.B. 1970, A&A 5, 20.
FIGURE CAPTIONS

Figure 1: The mean number of streams at a point according to the Zel’dovich approximation, \( N_s \), as functions of the linear standard deviation of density fluctuations, \( \sigma_\delta = a(t)\sigma_{in} \).

Figure 2: The projected distribution of matter in a slice of the standard CDM N-body simulation used in this paper at time corresponding to linear \( \sigma_8 = 1 \). The box side is 200 \( h^{-1}\text{Mpc} \) and the slice thickness is 25 \( h^{-1}\text{Mpc} \).

Figure 3: Velocity PDF \( P(v/\sigma_v) \) in the N-body simulation (at \( a = \sigma_8 = 1 \)). The distribution has been assembled from 80\(^3\) cubic grid points inside the box of side 200 \( h^{-1}\text{Mpc} \). The error bars are the standard deviations of \( P \) in the eight octants of the volume. Shown is the most nonlinear case, \( a = 1 \) and \( R_s = 6 h^{-1}\text{Mpc} \), in comparison with a Gaussian distribution.

Figure 4: Density PDF \( P(\rho/\bar{\rho}) \) in the N-body simulation (as in the previous figure, \( a = 1 \)) for three different smoothing scales: \( R_s = 18, 12, 6 h^{-1}\text{Mpc} \), corresponding to \( \sigma = 0.11, 0.26, 0.55 \). The solid curves are lognormal distributions with the corresponding \( \sigma \) and the dashed curves are laminar Zel’dovich distributions with the same \( \sigma \).

Figure 5: Moments of the distribution of density and velocity in the N-body simulation. Shown are the skewness and kurtosis as functions of the standard deviation. Each curve corresponds to a given time in the simulation, \( a = 0.5, 0.7, 1 \), and the variable along each curve is the Gaussian smoothing radius in the comoving range \( 6 \leq R_s \leq 21 h^{-1}\text{Mpc} \). The error bars are ± standard deviation in the eight octants of the volume. The dotted line marks the second-order prediction (Peebles 1980) \( s_\delta = (34/7)\sigma_\delta \).

Figure 6: PDFs for potent velocity and density fields (\( \Omega = 1 \)) with \( R_s = 12 h^{-1}\text{Mpc} \) smoothing (Dekel et al. 1990; Bertschinger et al. 1990) in a volume (selected to have small errors) comparable to a sphere of radius 37 \( h^{-1}\text{Mpc} \). Also shown are the Gaussian and lognormal curves with the same \( \sigma \). The error bars correspond to
distance measurement errors as derived by Monte Carlo simulations. The relative errors associated with the limited volume sampled are roughly 57% larger than the errors shown in the next figure based on the N-body simulation.

**Figure 7:** Mean and standard deviation of the PDFs in eight disjoint spheres of radius $50 \, h^{-1}\text{Mpc}$ in the N-body simulation with smoothing $R_s = 12 \, h^{-1}\text{Mpc}$.

**Figure 8:** PDFs for *IRAS* 1.9Jy density and velocity fields (Strauss et al. 1991, Yahil et al. 1991) in a sphere of radius $80 \, h^{-1}\text{Mpc}$. (a) $R_s = 12 \, h^{-1}\text{Mpc}$ and (b) $R_s = 6 \, h^{-1}\text{Mpc}$. Also shown are the Gaussian and lognormal curves with the same $\sigma$. The errors associated with the limited volume sampled should be roughly twice the errors shown in figures 3 and 4 based on the whole N-body simulation, or half the errors in figure 7.