MAXIMALITY AND FINITENESS OF TYPE 1
SUBDIAGONAL ALGEBRAS

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ABSTRACT. Let $\mathfrak{A}$ be a type 1 subdiagonal algebra in a $\sigma$-finite von Neumann algebra $\mathcal{M}$ with respect to a faithful normal conditional expectation $\Phi$. We give necessary and sufficient conditions for which $\mathfrak{A}$ is maximal among the $\sigma$-weakly closed subalgebras of $\mathcal{M}$. In addition, we show that a type 1 subdiagonal algebra in a finite von Neumann algebra is automatically finite which gives a positive answer of Arveson’s finiteness problem in 1967 in type 1 case.

1. INTRODUCTION

There are fruitful theorems in classical Hardy space theory. For example, A well-known classical result on bounded analytic function algebra $H^\infty(\mathbb{T})$ is that it is maximal as a $w^*$-closed subalgebras in $L^\infty(\mathbb{T})$. A noncommutative analogue is obtained by replacing $L^\infty(\mathbb{T})$ by a von Neumann algebra $\mathcal{M}$ and $H^\infty(\mathbb{T})$ by a unital $\sigma$-weakly closed subalgebra $\mathfrak{A}$ of $\mathcal{M}$. There are many successful noncommutative extensions of classical $H^p$ space theory from now on. One very important notion is subdiagonal algebras introduced by Arveson in [1]. Based on subdiagonal algebras, noncommutative Hardy spaces are developed(cf. [2, 3, 4, 5, 9, 10, 11, 12, 13]). For example, Marsalli and West [16] gave a Riesz factorization theorem for finite noncommutative $H^p$ spaces. Blecher and Labuschagne established Beurling type invariant subspace theorems for a finite subdiagonal algebra in [6] and Labuschagne in [15] extended their results to noncommutative $H^2$ for maximal subdiagonal algebras in a $\sigma$-finite von Neumann algebra. On the other
hand, several authors are of interest in the maximality of an analytic subalgebra as a $\sigma$-weakly closed subalgebra in a von Neumann algebra. McAsey, Muhly and Saito(\cite{17, 18}) considered the maximality of analytic crossed products. A necessary and sufficient condition for the maximality of analytic operator algebra $H^\infty(\alpha)$ determined by a flow $\alpha$ on a von Neumann algebra $\mathcal{M}$ is given by Solel in \cite{20}. Very recently, Peligrad gave a complete solution of the maximality problem for one-parameter dynamical systems in \cite{19}. Subdiagonal algebras are very important classes of analytic operator algebras. It becomes interesting to determine the maximality of these algebras. We consider related problems in this paper. We firstly recall some notions.

Let $\mathcal{M}$ be a $\sigma$-finite von Neumann algebra acting on a complex Hilbert $\mathcal{H}$. $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ is the center of $\mathcal{M}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. If $Z(\mathcal{M}) = \mathbb{C}I$, the multiples of identity $I$, then $\mathcal{M}$ is said to be a factor. We denote by $\mathcal{M}_*$ the space of all $\sigma$-weakly continuous linear functionals of $\mathcal{M}$. Let $\Phi$ be a faithful normal conditional expectation from $\mathcal{M}$ onto a von Neumann subalgebra $\mathcal{D}$. Arveson \cite{1} gave the following definition. A subalgebra $\mathcal{A}$ of $\mathcal{M}$, containing $\mathcal{D}$, is called a subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ if

(i) $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$,
(ii) $\Phi$ is multiplicative on $\mathcal{A}$, and
(iii) $\mathcal{A} + \mathcal{A}^*$ is $\sigma$-weakly dense in $\mathcal{M}$.

The algebra $\mathcal{D}$ is called the diagonal of $\mathcal{A}$. we may assume that subdiagonal algebras are $\sigma$-weakly closed without loss generality(cf.\cite{11}).

We say that $\mathcal{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\Phi$ in case that $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\Phi$. Put $\mathcal{A}_0 = \{X \in \mathcal{A} : \Phi(X) = 0\}$ and $\mathcal{A}_m = \{X \in \mathcal{M} : \Phi(AXB) = \Phi(BXA) = 0, \forall A \in \mathcal{A}, B \in \mathcal{A}_0\}$. By \cite{1, Theorem 2.2.1}, we recall that $\mathcal{A}_m$ is a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ containing $\mathcal{A}$. $\mathcal{A}$ is said to be finite if there is a faithful normal finite trace $\tau$ on $\mathcal{M}$ such that $\tau \circ \Phi = \tau$. Finite subdiagonal algebras are maximal subdiagonal(cf.\cite{7}). A well known
problem given by Arveson in [11] is whether a subdiagonal algebra in a finite von Neumann algebra is automatically finite. It is still open now.

We next recall Haagerup’s noncommutative $L^p$ spaces associated with a general von Neumann algebra $\mathcal{M}$ (cf. [8, 22]). Let $\varphi$ be a faithful normal state on $\mathcal{M}$ and let $\{\sigma_t^\varphi : t \in \mathbb{R}\}$ be the modular automorphism group of $\mathcal{M}$ associated with $\varphi$ by Tomita-Takesaki theory. We consider the crossed product $\mathcal{N} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ of $\mathcal{M}$ by $\mathbb{R}$ with respect to $\sigma^\varphi$. We denote by $\theta$ the dual action of $\mathbb{R}$ on $\mathcal{N}$. Then $\{\theta_s : s \in \mathbb{R}\}$ is an automorphisms group of $\mathcal{N}$. Note that $\mathcal{M} = \{X \in \mathcal{N} : \theta_s(X) = X, \forall s \in \mathbb{R}\}$. $\mathcal{N}$ is a semifinite von Neumann algebra and there is the normal faithful semifinite trace $\tau$ on $\mathcal{N}$ satisfying

$$\tau \circ \theta_s = e^{-s} \tau, \quad \forall s \in \mathbb{R}.$$  

According to Haagerup [8, 22], the noncommutative $L^p$ spaces $L^p(\mathcal{M})$ for each $0 < p \leq \infty$ is defined as the set of all $\tau$-measurable operators $x$ affiliated with $\mathcal{N}$ satisfying

$$\theta_s(x) = e^{-\frac{s}{p}} x, \quad \forall s \in \mathbb{R}.$$  

There is a linear bijection between the predual $\mathcal{M}_*$ of $\mathcal{M}$ and $L^1(\mathcal{M})$: $f \rightarrow h_f$. If we define $\text{tr}(h_f) = f(I), f \in \mathcal{M}_*$, then

$$\text{tr}(|h_f|) = \text{tr}(|h_{|f|}|) = |f|(I) = \|f\|$$

for all $f \in \mathcal{M}_*$ and

$$\|\text{tr}(x)\| \leq \text{tr}(|x|)$$

for all $x \in L^1(\mathcal{M})$. Note that for any $x \in L^p(\mathcal{M}), \|x\|_p = (\text{tr}(|x|^p))^\frac{1}{p}$ is the norm of $x$. As in [8], we define the operator $L_A$ and $R_A$ on $L^p(\mathcal{M})(1 \leq p < \infty)$ by $L_Ax = Ax$ and $R_Ax = xA$ for all $A \in \mathcal{M}$ and $x \in L^p(\mathcal{M})$. Note that $L^2(\mathcal{M})$ is a Hilbert space with the inner product $\langle a, b \rangle = \text{tr}(b^* a), \forall a, b \in L^2(\mathcal{M})$ and $A \rightarrow L_A$ (resp. $A \rightarrow R_A$) is a faithful normal $*$-representation (resp. $*$-anti-representation) of $\mathcal{M}$ on $L^2(\mathcal{M})$. We may identify $\mathcal{M}$ with $L(\mathcal{M}) = \{L_A : A \in \mathcal{M}\}$.

Let $\mathfrak{A}$ be a maximal subdiagonal algebra with respect to $\Phi$ such that $\varphi \circ \Phi = \varphi$. It is known that the noncommutative $H^p$ space $H^p(\mathcal{M})$ and
\( H^p_0(\mathcal{M}) \) in \( L^p(\mathcal{M}) \) for any \( 1 \leq p < \infty \) is \( H^p = H^p(\mathcal{M}) = [h_0^p \mathfrak{A} h_0^p]_p \) and \( H^p_0 = H^p_0(\mathcal{M}) = [h_0^p \mathfrak{A}_0 h_0^p]_p \) for any \( \theta \in [0, 1] \) from [9, Definition 2.6] and [10, Proposition 2.1]. If \( p = \infty \), then we identify \( H^\infty \) as \( \mathfrak{A} \) and \( H^\infty_0 \) as \( \mathfrak{A}_0 \).

In this paper, we consider type 1 subdiagonal algebras introduced in [11]. We give necessary and sufficient conditions for a type 1 subdiagonal algebra to be maximal in \( \sigma \)-weakly closed subalgebras of a von Neumann algebra. In addition, we devote to Arveson’s finiteness problem of subdiagonal algebras in a finite von Neumann algebra.

2. Maximality of type 1 subdiagonal algebras

A \( \sigma \)-weakly closed proper subalgebra \( \mathcal{A} \subseteq \mathcal{M} \) is said to be maximal if it can not be contained in any other \( \sigma \)-weakly closed proper subalgebras of \( \mathcal{M} \). We consider conditions for which a type 1 subdiagonal algebra to be maximal. We need certain invariant space representations of a maximal subdiagonal algebras in noncommutative \( L^p \) spaces. So we firstly recall that the notion of column \( L^p \)-sum of noncommutative \( L^p \)-space \( L^p(\mathcal{M})(1 \leq p \leq \infty) \) for a \( \sigma \)-finite von Neumann algebra \( \mathcal{M} \) studied by Junge and Sherman [14]. Assume that \( X \) is a closed subspace of \( L^p(\mathcal{M}) \). If \( \{X_i : i \in \Lambda\} \) is a family of closed subspaces of \( X \) such that \( X = \vee\{X_i : i \in \Lambda\} \) with the property that \( X^*_i X_i = \{0\} \) if \( i \neq j \), then we say that \( X \) is the internal column \( L^p \)-sum \( \oplus^{\text{col}}_{i \in \Lambda} X_i \).

If \( p = \infty \), we assume that \( X \) and \( \{X_i : i \in \Lambda\} \) are \( \sigma \)-weakly closed, and the closed linear span is taken with the \( \sigma \)-weak topology. For symmetry, if \( X_j X^*_i = \{0\} \) if \( i \neq j \) and \( X = \vee\{X_i : i \in \Lambda\} \), then we say that \( X \) is the internal row \( L^p \)-sum \( \oplus^{\text{row}}_{i \in \Lambda} X_i \). In addition, we also give the following definition which in fact is used in [6, 15]. Let \( \mathfrak{A} \) be a maximal subdiagonal algebra with respect to \( \Phi \) and \( \mathfrak{D} \) is the diagonal.

**Definition 2.1.** Let \( \mathcal{U} = \{U_n : n \geq 1\} \) in \( \mathcal{M} \) be a family of partial isometries. If \( U^*_n U_m = 0 \) (resp. \( U^*_n U^*_m = 0 \)) for \( n \neq m \) and \( U^*_n U_n \in \mathfrak{D} \) (resp. \( U^*_n U^*_n \in \mathfrak{D} \)) for all \( n \), then we say that \( \{U_n : n \geq 1\} \) is column (resp. row) orthogonal.
Note that $U = \{ U_n : n \geq 1 \}$ is column orthogonal if and only if $U^* = \{ U_n^* : n \geq 1 \}$ is row orthogonal.

We recall that a closed subspace $\mathcal{M}$ of $L^p(\mathcal{M})$ is right (resp. left) invariant if $\mathcal{M}A \subseteq \mathcal{M}$ (resp. $AM \subseteq \mathcal{M}$). If it is both left and right invariant, then we say that it is two-side invariant. Following [6, 15], we define the right wandering subspace of $\mathcal{M}$ to be the space $K = \mathcal{M} \ominus [\mathcal{M}A_0]_2$ when $p = 2$, where $[S]_p$ is the closed linear span of a subset $S$ in $L^p(\mathcal{M})$. We say that $\mathcal{M}$ is of type 1 if $K$ generates $\mathcal{M}$ as an $A$-module (that is, $\mathcal{M} = [KA]_2$). We say that $\mathcal{M}$ is of type 2 if $K = \{ 0 \}$. Note that every right invariant subspace $\mathcal{M}$ is an $L^2$-column sum $\mathcal{M} = N_1 \oplus \text{col} N_2$, where $N_i$ is of type $i$ for $i = 1, 2$ from [6, Theorem 2.1] and [15, Theorem 2.3]. In particular, if $\mathcal{M}$ is of type 1, then $\mathcal{M}$ is of the Beurling type, that is, there exists a family of column orthogonal partial isometries $\{ U_n : n \geq 1 \}$ such that $\mathcal{M} = \oplus_{n \geq 1} U_n H^2$. We refer to [6, 15] for more details. Symmetrically, a type 1 left invariant subspace $\mathcal{M}$ may be represented as $\mathcal{M} = \oplus_{n \geq 1}^\text{row} H^2 V_n$ by a family of row orthogonal partial isometries $\{ V_n : n \geq 1 \}$ and this fact will be used frequently.

By [11, Definition 2.1], $\mathfrak{A}$ is said to be type 1 if every right invariant subspace of $\mathfrak{A}$ in $H^2$ is of type 1. Then there exists a column orthogonal family of partial isometries $\{ U_n : n \geq 1 \}$ in $\mathcal{M}$ such that

\[(2.1) \quad H_0^2 = \oplus_{n \geq 1}^\text{col} U_n H^2.\]

To consider the maximality of $\mathfrak{A}$, we firstly consider invariant subspaces in non commutative $L^1(\mathcal{M})$ and $\mathcal{M}$.

For a family of column orthogonal partial isometries $\mathcal{W} = \{ W_n : n \geq 1 \}$, we define right invariant subspaces $\mathcal{M}_{\mathcal{W}}^1 = \oplus_{n \geq 1}^\text{col} W_n H^1$ in $L^1(\mathcal{M})$ and $\mathcal{M}_{\mathcal{W}}^\infty = \oplus_{n \geq 1}^\text{col} W_n H^\infty$ in $\mathcal{M}$ respectively.

**Theorem 2.2.** Let $\mathfrak{A}$ be a type 1 subdiagonal algebra with respect to $\Phi$.

1. If $\mathcal{M} \subseteq L^1(\mathcal{M})$ is a closed right invariant subspace, then there exist a family of column orthogonal partial isometries $\mathcal{W}$ and a projection $E$ in $\mathcal{M}$ and such that $\mathcal{M} = \mathcal{M}_{\mathcal{W}}^1 \oplus^\text{col} EL^1(\mathcal{M})$. 
(2) If $\mathcal{M} \subseteq \mathcal{M}$ is a $\sigma$-weakly closed right invariant subspace, then there exist a family of column orthogonal partial isometries $\mathcal{W}$ and a projection $E$ in $\mathcal{M}$ and such that $\mathcal{M} = \mathcal{M}_W^\infty \oplus^\text{col} E\mathcal{M}$.

Proof. (1) Put $\mathcal{M}_2 = \cap_{n \geq 1}[\mathcal{M}_W^0]_1$, where $\mathcal{A}_0$ is the $\sigma$-weakly closed ideal of $\mathcal{A}$ generated by $\{a_1a_2 \cdots a_n : a_j \in \mathcal{A}_0\}$. Then $\mathcal{M}_2 \subseteq \mathcal{M}$ is a right invariant subspace of $\mathcal{A}$. We show that $\mathcal{M}_2 = EL^1(\mathcal{M})$ for a projection $E \in \mathcal{M}$. By [11] Theorem 2.7, $\mathcal{A}_0 = \vee\{D_{U_1U_2 \cdots U_n} : i_j \geq 1\}$, where $\{U_i : i \geq 1\}$ is a family of column orthogonal partial isometries as in (2.1). Note that $\mathcal{M}$ is $\mathcal{D}$ right invariant and $[\mathcal{M}_2\mathcal{A}_0]_1 = \mathcal{M}_2$. Since $\vee\{\mathcal{M}_2U_n : n \geq 1\} \supseteq \vee\{\mathcal{M}_2U_iU_2 \cdots U_n : i_k \geq 1, n \geq 1\}$, $\mathcal{M}_2 = \vee\{\mathcal{M}_2U_n : n \geq 1\}$. For any $m, n \geq 1$ and $x \in \mathcal{M}_2$, we have $R_{U_m}R_{U_n}x = xU_nU_m^* \in \mathcal{M}_2$ since $U_nU_m^* \in \mathcal{D}$ from [11] Proposition 2.3. Then $\mathcal{M}$ is right $\mathcal{M}$ invariant. By [11] Chapter III, Theorem 2.7, there is a projection $E \in \mathcal{M}$ such that $\mathcal{M}_2 = EL^1(\mathcal{M})$. Put $\mathcal{M}_1 = (I - E)\mathcal{M} \subseteq \mathcal{M}$. This mean that $\mathcal{M}_1$ is closed and right invariant such that $\mathcal{M} = \mathcal{M}_1 \oplus^\text{col} \mathcal{M}_2$. Note that $\cap_{n \geq 1}[\mathcal{M}_1\mathcal{A}_0]_1 \subseteq \cap_{n \geq 1}[\mathcal{M}_W^0]_1 \subseteq \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$.

Put

$$\mathcal{P} = \{\mathcal{W} = \{W_n\}_{n \geq 1} \subseteq \mathcal{M} : \text{column orthogonal partial isometries such that } \mathcal{M}_W^2H^2 \subseteq \mathcal{M}_1\},$$

where $\mathcal{M}_W^2H^2$ is the closed right invariant subspace generated by $\{xy : x \in \mathcal{M}_W^2, y \in H^2\}$. We define a partial order in $\mathcal{P}$ by $\mathcal{W} \leq \mathcal{V}$ if $\mathcal{M}_W^2 \subseteq \mathcal{M}_V^2$ for any $\mathcal{W}, \mathcal{V} \in \mathcal{P}$. Let $\{\mathcal{W}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{P}$ be a totally ordered family in $\mathcal{P}$. Put $\mathcal{M} = \vee\{\mathcal{M}_{W_\lambda}^2 : \lambda \in \Lambda\}$. Note that $\mathcal{M} \subseteq L^2(\mathcal{M})$ is a right invariant subspace in $L^2(\mathcal{M})$. Then $\mathcal{M} = \mathcal{M}_W^2 \oplus^\text{col} \mathcal{M}_2$ for a family of column orthogonal partial isometries $\mathcal{W}$ and a right invariant subspace $\mathcal{M}_2$ of type 2 in $L^2(\mathcal{M})$. Then $[\mathcal{M}_2H^2]_1$ is also a right invariant subspace in $L^1(\mathcal{M})$ such that $[\mathcal{M}_2H^2\mathcal{A}_0]_1 = [\mathcal{M}_2H^2]_1$. This implies that $[\mathcal{M}_2H^2]_1 = \cap_{n \geq 1}[\mathcal{M}_2H^2\mathcal{A}_0]_1 \subseteq \cap_{n \geq 1}[\mathcal{M}_W^2\mathcal{A}_0]_1 = \{0\}$. Hence $\mathcal{M}_2 = \{0\}$. Since $\mathcal{M}_{W_\lambda}^2H^2 \subseteq \mathcal{M}_1$, $\mathcal{M}H^2 \subseteq \mathcal{M}_1$. Thus $\mathcal{W} \in \mathcal{P}$ with $\mathcal{W}_\lambda \leq \mathcal{W}$ for any $\lambda \in \Lambda$. That is, $\mathcal{W}$ is an upper bound of $\{\mathcal{W}_\lambda : \lambda \in \Lambda\}$. By
Zorn’s lemma, there exists a maximal element \( W \in \mathcal{P} \). We show that \( \mathcal{M}_1 = \mathcal{M}_W^{1} \).

Otherwise, assume that there is an \( h \in \mathcal{M}_1 \) such that \( h \notin \mathcal{M}_W^{1} \). Then by considering the polar decomposition of \( h \), there are \( h_1 \in L^2(\mathcal{M}) \) and an outer element \( h_2 \in H^2 \) such that \( h = h_1 h_2 \) by Theorem 3.1].

This implies that \( \mathcal{M}_1 \supseteq [h\mathcal{A}]_1 = [h_1\mathcal{A}]_2 H^2 \). Note that \( [h_1\mathcal{A}]_2 \subseteq L^2(\mathcal{M}) \) is a right invariant subspace. By Lemma 3.3, \([h_1\mathcal{A}]_2 = VH^2 \oplus \text{col } N_2 \) for a partial isometry \( V \in \mathcal{M} \) such that \( V^*V \in \mathcal{D} \) and a right invariant subspace \( N_2 \) of type 2. Note that \( [h_1\mathcal{A}]_2 H^2 \subseteq \mathcal{M}_1 \). Then \( N_2 = 0 \).

Since \( h = h_1 h_2 \notin \mathcal{M}_W^{1} \), \( V H^2 \nsubseteq \mathcal{M}_W^{2} \). Now \( \tilde{\mathcal{N}} = \vee \{ \mathcal{M}_W^{2}, VH^2 \} \) is also a right invariant subspace in \( L^2(\mathcal{M}) \) with \( \tilde{\mathcal{N}} H^2 \subseteq \mathcal{M}_1 \). In this case, we have \( \tilde{\mathcal{N}} = \mathcal{M}_W^{2} \) for a family of column orthogonal partial isometries \( V \in \mathcal{P} \). It is trivial that \( \mathcal{M}_W^{2} \subseteq \mathcal{M}_W^{1} \) by a similar treatment. This is a contradiction. Thus \( \mathcal{M}_1 = \mathcal{M}_W^{2} H^2 = \mathcal{M}_W^{1} \).

(2) Let \( p = \infty \). Note that \( [\mathcal{M}_W^{1}]_2 = \mathcal{M} H^2 \subseteq L^2(\mathcal{M}) \) is a right invariant invariant subspace of \( \mathcal{A} \) in \( L^2(\mathcal{M}) \) since \( [\mathcal{M} \mathcal{A} \mathcal{A}]_n = \mathcal{M} \). Thus there are a family of column orthogonal partial isometries \( \mathcal{W} = \{ W_n : n \geq 1 \} \) and a projection \( E \) in \( \mathcal{M} \) such that \( \mathcal{M} H^2 = \mathcal{M}_W^{2} \oplus \text{col } EL^2(\mathcal{M}) = (\mathcal{M}_W^{\infty} \oplus \text{col } E \mathcal{M}) H^2 \).

We next show that \( \mathcal{M} = \mathcal{M}_W^{\infty} \oplus \text{col } E \mathcal{M} \). For any \( x \in \mathcal{M} \), it is elementary that \( xh_0^{\frac{q}{2}} = \oplus_{n \geq 1} W_n W_n^* xh_0^{\frac{q}{2}} + Exh_0^{\frac{q}{2}} \) with \( W_n x h_0^{\frac{q}{2}} \in H^2 \).

Then \( W_n x \in \mathcal{A} \) and \( x = \oplus_{n \geq 1} W_n W_n^* x \oplus Ex \in \mathcal{M}_W^{\infty} \oplus E \mathcal{M} \). Therefore \( \mathcal{M} \subseteq \mathcal{M}_W^{\infty} \oplus E \mathcal{M} \).

For any closed subspace \( K \subseteq L^p(\mathcal{M}) \), we put \( K^\perp = \{ y \in L^q(\mathcal{M}) : tr(xy) = 0, \forall x \in K \} \), where \( p \) and \( q \) are conjugate exponents, that is, \( \frac{1}{p} + \frac{1}{q} = 1 \). It is known that \( K^\perp \) is left invariant when \( K \) is right invariant.

We easily have \( \mathcal{M}^\perp \subseteq L^1(\mathcal{M}) \) is a left invariant subspace. As just proved, by symmetry, we have that \( \mathcal{M}^\perp = \ominus_{n \geq 1} H^1 V_n \oplus^{\text{row}} L^1(\mathcal{M}) F \) for a family of row orthogonal partial family \( \{ V_n : n \geq 1 \} \) and a projection \( F \in \mathcal{M} \). It is elementary that \( \mathcal{M}^\perp = H^2 (\ominus_{n \geq 1} H^2 W_n \oplus^{\text{row}} L^2(\mathcal{M}) F) = H^2 \mathcal{N} \), where \( \mathcal{N} = \ominus_{n \geq 1} H^2 W_n \oplus^{\text{row}} L^2(\mathcal{M}) F \subseteq L^2(\mathcal{M}) \) is left invariant.
We claim that \((\mathcal{M}H^2)^\perp = \mathfrak{N}\). It is clear that \((\mathcal{M}H^2)^\perp \supseteq \mathfrak{N}\). Take any \(y \in (\mathcal{M}H^2)^\perp\). Then \(H^2 y \subseteq \mathfrak{M}^\perp = \bigoplus_{n \geq 1} H^1 V_n \bigoplus \text{row } L^1(\mathcal{M})F\). By a similar treatment as above, we have \(h_{n_0}^2 y = \bigoplus_{n \geq 1} h_{n_0}^2 y V_n \bigoplus \text{row } h_{n_0}^2 y F\) and thus \(y = \bigoplus_{n \geq 1} y V_n \bigoplus \text{row } y F \in \mathfrak{N}\). Thus, \(\mathfrak{M}^\perp = H^2 \mathfrak{N} = H^2(\mathcal{M}H^2)^\perp\). On the other hand, replacing \(\mathcal{M}\) by \(\mathcal{M}^\infty + E\mathcal{M}\), we have \((\mathcal{M}^\infty + E\mathcal{M})^\perp = H^2((\mathcal{M}^\infty + E\mathcal{M})H^2)^\perp = H^2(\mathcal{M}H^2)^\perp = \mathfrak{M}^\perp\). It follows that \(\mathfrak{M} = \mathcal{M}^\infty + E\mathcal{M}\). □

We now consider the maximality of a type 1 subdiagonal algebra \(\mathfrak{A}\) as a \(\sigma\)-weakly closed subalgebra in a von Neumann algebra \(\mathcal{M}\). We next assume that \(\mathcal{B}\) is a \(\sigma\)-weakly closed proper subalgebra of \(\mathcal{M}\) such that \(\mathfrak{A} \subseteq \mathcal{B} \subsetneq \mathcal{M}\).

**Proposition 2.3.** \(\mathcal{B} = \{T \in \mathcal{M} : [h_{00}^2 \mathcal{B}]_2 T \subseteq [h_{00}^2 \mathcal{B}]_2\} = \{T \in \mathcal{M} : T[\mathcal{B}h_{00}^2]_2 \subseteq [\mathcal{B}h_{00}^2]_2\}\).

**Proof.** As in the proof of Theorem 2.2, we put \(\mathcal{B}^\perp = \{h \in L^1(\mathcal{M}) : \text{tr}(hB) = 0, \forall B \in \mathcal{B}\}\) is the pre-annihilator of \(\mathcal{B}\). It is known that \((\mathcal{B}^\perp)^\perp = \mathcal{B}\) since \(\mathcal{B}\) is \(\sigma\)-weakly closed. Note that \(\mathcal{B}^\perp \subseteq \mathcal{B}_{10}^{\perp}\) is a two-side \(\mathcal{B} \supseteq \mathfrak{A}\) invariant subspace. By Theorem 2.2, \(\mathcal{B}^\perp = \mathcal{M}_{\mathcal{W}}^\perp = \mathcal{M}_{\mathcal{W}}^\perp \mathcal{B}^2\) for a family of column orthogonal partial isometries \(\mathcal{W}\) since \([\mathcal{B}^\perp \mathfrak{A}]_1 \subseteq [\mathcal{B}_{10}^{\perp} \mathfrak{A}]_1 = \{0\}\). Take a \(T \in \mathcal{M}\) such that \([h_{00}^2 \mathcal{B}]_2 T \subseteq [h_{00}^2 \mathcal{B}]_2\). For any \(h_1 \in \mathcal{M}_{\mathcal{W}}^\perp\) and \(h_2 \in \mathcal{H}^2 \subseteq [h_{00}^2 \mathcal{B}]_2\), \(h_2 T \in [h_{00}^2 \mathcal{B}]_2\) and then \(h_2 T = \lim_{m \to \infty} h_{00}^2 B_m\) for a sequence \(\{B_m : m \geq 1\} \subseteq \mathcal{B}\). Now \(\text{tr}(Th_1 h_2) = \text{tr}(h_2 Th_1) = \lim_{m \to \infty} \text{tr}(h_{00}^2 B_m h_1) = \lim_{m \to \infty} \text{tr}(B_m h_1) = 0\). It follows that \(T \in (\mathcal{B}^\perp)^\perp = \mathcal{B}\). By symmetry, we also have the second equality. □

**Corollary 2.4.** Put \(Q = \vee\{E \in \mathcal{M} : E\mathcal{M} \subseteq \mathcal{B}\}\). Then \((I - Q)BQ = 0\).

**Proof.** Note that \(Q\mathcal{M} \subseteq \mathcal{B}\). Put \(\mathcal{N} = [(I - Q)BQ\mathcal{M}h_{00}^2]_2 \subseteq [\mathcal{B}h_{00}^2]_2\). Then \(\mathcal{N}\) is right reducing and \(\mathcal{N} = F\mathcal{L}(\mathcal{M})\) for a projection \(F \in \mathcal{M}\). It follows that \(F\mathcal{M} \subseteq \mathcal{B}\) by Proposition 2.3 since \(F\mathcal{M}[\mathcal{B}h_{00}^2]_2 \subseteq \mathcal{N} \subseteq \mathcal{M}\).
[Bl^2_{\theta}]_2 and thus $F \leq Q$. However, $(I - Q)F = F$. Hence $F = 0$ and $(I - Q)BQ = 0$.

We now consider main result in this section. If $\mathcal{M} \subseteq L^2(\mathcal{M})$ is a type 1 right invariant subspace of $\mathfrak{A}$, then the projection on the wandering subspace $K = \mathcal{M} \ominus [\mathfrak{M}A_0]_2$ is in the commutant $(R(\mathfrak{D}))'$ of the von Neumann algebra $R(\mathfrak{D}) = \{R_D : D \in \mathfrak{D}\}$. For any two projections $E$ and $F$ in a von Neumann algebra $\mathcal{M}$, $E \preceq F$ means that there is a partial isometry $V \in \mathcal{M}$ such that $V^*V = E$ and $VV^* \leq F$. If $E \preceq F$ and $F \preceq E$, then $E \sim F$. The following proposition holds for a maximal subdiagonal algebra.

**Proposition 2.5.** Let $\mathcal{M}_i(i = 1, 2)$ be two type 1 right invariant subspaces of $\mathfrak{A}$ with wandering subspaces $K_i(i = 1, 2)$. If $p_i$ are the projections on $K_i(i = 1, 2)$ and $p_1 \preceq p_2$ in $(R(\mathfrak{D}))'$, then there is a partial isometry $W \in \mathcal{M}$ such that $\mathcal{M}_2 = W\mathcal{M}_1 \ominus \text{col}(I - WW^*)\mathcal{M}_2$ and $\mathcal{M}_1 = W^*\mathcal{M}_2$.

**Proof.** Let $w \in (R(\mathfrak{D}))'$ be a partial isometry such that $w^*w = p_1$ and $ww^* \leq p_2$. Note that $[K_i\mathcal{M}]_2 = [K_i\mathfrak{A}_0]_2 \oplus [K_i\mathfrak{D}]_2 \oplus [K_i\mathfrak{A}_0]_2$ are right reducing subspaces. For any $x \in K_1$, we have $wx \in K_2$ and $x^*x, (wx)^*(wx) \in L^1(\mathfrak{D})$ by [15, Theorem 2.3]. Note that $R_Dx = xD \in \mathcal{M}_1$ and $wR_Dx = R_D(wx) = (wx)D$ and therefore $\|wR_Dx\|_2 = \|R_Dx\|_2 = \|xD\|_2$. Take any $A \in \mathcal{M}$. Recall that $\Phi_1$ is the contraction from $L^1(\mathcal{M})$ onto $L^1(\mathfrak{D})$. Since $\text{tr} \circ \Phi_1 = \text{tr}$ from [9, Proposition 2.1], we have that

\[
\|R_A(wx)\|_2^2 = \|(wx)A\|_2^2 = \text{tr}(A^*(wx)^*(wx)A) = \text{tr}(\|wx\|^2AA^*)
\]

\[= \text{tr}(\|wx\|^2\Phi(AA^*)) = \text{tr}((\Phi(AA^*))^{\frac{1}{2}}(wx)^*(wx)(\Phi(AA^*))^{\frac{1}{2}})
\]

\[= \|(wx)(\Phi(AA^*))^{\frac{1}{2}}\|^2_2 = \|(R_{\Phi(AA^*)})^{\frac{1}{2}}(wx)\|^2_2
\]

\[= \|wR_{\Phi(AA^*)}^{\frac{1}{2}}x\|^2_2 = \|R_{\Phi(AA^*)}^{\frac{1}{2}}x\|^2_2
\]

\[= \text{tr}((\Phi(AA^*)^{\frac{1}{2}}x^*x(\Phi(AA^*))^{\frac{1}{2}}) = \text{tr}(x^*x\Phi(AA^*))
\]

\[= \text{tr}(x^*xAA^*) = \|xA\|_2^2.
\]
We define $W(xA) = (wx)A$ for any $x \in K_1$ and $A \in \mathcal{M}$. Then $W$ is an well defined isometry on $[K_1\mathcal{M}]_2$ with range $[w(K_1)\mathcal{M}]_2$. Put $Wx = 0$ for any $x \in [K_1\mathcal{M}]_2$. Then $W$ is a partial isometry. Note that $WR_B(xA) = W(xAB) = (wx)AB = R_B(R_A(wx)) = R_BW(xA)$ for all $A, B \in \mathcal{M}$ and $x \in K_1$. On the other hand, if $x \in [K_1\mathcal{M}]_2$, then $WR_Bx = W(xB) = 0 = R_B(Wx)$ since $[K_1\mathcal{M}]_2$ is right reducing. This implies that $W \in R(\mathcal{M})' = L(\mathcal{M})$. Note that $W(K_1) = w(K_1) \subseteq K_2$ is a right $\mathcal{D}$ module, so is $W_2 \ominus W(K_1)$. It is elementary that $W\mathcal{A}_1 = [W(K_1)\mathcal{A}]_2 \subseteq \mathcal{M}_2$ and $[(K_2 \ominus W(K_1))\mathcal{A}]_2$ are right invariant subspaces of $\mathcal{A}$ of type 1 such that $\mathcal{M}_2 = W\mathcal{A}_1 \oplus col [(K_2 \ominus W(K_1))\mathcal{A}]_2$. It is trivial that $\mathcal{M}_1 = [K_1\mathcal{A}]_2 = [W(W(K_1))W(K_1)\mathcal{A}]_2 = W^*[W(K_1)\mathcal{A}]_2$. It is known that $W^*(yA) = 0$ for all $y \in K_2 \ominus wK_1$ and $A \in \mathcal{M}$. Therefore $\mathcal{M}_1 = W^*\mathcal{M}_2$ and $[(K_2 \ominus W(K_1))\mathcal{A}]_2 = (I - WW^*)\mathcal{M}_2$. □

**Theorem 2.6.** Let $\mathfrak{a} \subset \mathcal{M}$ be a type 1 subdiagonal algebra with respect to $\Phi$. Then $\mathfrak{a}$ is a maximal $\sigma$-weakly closed subalgebra of $\mathcal{M}$ if and only if one of following assertions holds.

1. There is a projection $E \in \mathcal{M}$ which is not in the center $Z(\mathcal{M})$ such that $\mathfrak{a} = EM + (I - E)\mathcal{M}(I - E) = \{ A \in \mathcal{M} : (I - E)AE = 0 \}$.

2. There is a projection $E \in Z(\mathcal{M}) \cap \mathfrak{d}$ such that $EM = E\mathfrak{a}$ and $(I - E)\mathfrak{d}$ is a factor.

In particular, if $\mathfrak{d}$ is a factor, then $\mathfrak{a}$ is maximal.

**Proof.** If assertion (1) holds, then it is trivial that $\mathfrak{a}$ is maximal.

We assume that assertion (2) holds. Then $\mathfrak{a} = EM + E\mathfrak{a}$. In this case, $E\mathfrak{a}$ is a type 1 subdiagonal algebra of $EM$ with respect to $\Phi_E$, where $\Phi_E(EM) = E\Phi(A)$ for all $A \in \mathcal{M}$. It is sufficient to prove that $E\mathfrak{a}$ is a maximal subalgebra in $EM$. Without loss of generality, we may assume that $\mathfrak{d}$ itself is a factor.

Let $\mathcal{B}$ be a $\sigma$-weakly closed subalgebra of $\mathcal{M}$ such that $\mathfrak{a} \subset \mathcal{B} \subset \mathcal{M}$. We show that $\mathcal{B} = \mathfrak{a}$. Since $\mathcal{B}$ is two-sided $\mathfrak{a}$ invariant, by Theorem 2.2, there is a family of column orthogonal partial isometries $W = \{ W_m : m \geq 1 \}$ and a projection $P$ in $\mathcal{M}$ such that $\mathcal{B} = \oplus_{m \geq 1} W_m\mathfrak{a} \oplus col P\mathcal{M}$. Note that $P < I$ by Proposition 2.3 since $\mathcal{B}$ is a proper subalgebra.
Thus $W$ is an isometry since $W$ is invariant for any $n$, $PL^2(M)$ is also left $B$ invariant for any $n$. $PL^2(M)$ is also left $B$ invariant. This implies that $PBP = BP$ for any $B \in B$. In particular, $PD = DP$ for all $D \in B \cap B^\ast$. Note that $W_m \in B \cap B^\ast$ and $PW_m = 0$ for any $m$. Thus $PW_m = W_mP = 0$ and $W_m^*W_m \leq I - P$ for any $m$. On the other hand, $P \in (B \cap B^\ast)^\prime \subseteq \mathcal{D}$. We have that $Φ(P) \in Z(\mathcal{D}) = CI$. This implies that $0 \leq Φ(P) < 1$ is a scalar. However, $W_m^*W_m = Φ(W_m^*W_m) \leq I - Φ(P)$. It follows that $Φ(P) = 0$ since $W_m^*W_m$ is a nonzero projection for any $m$. Thus $P = 0$ and $[Bh_0^{\frac{1}{2}}]_2$ is a type 1 right invariant subspace.

Let $p$ and $q$ be the projections on $L^2(\mathcal{D})$ and $[Bh_0^{\frac{1}{2}}]_2 \oplus [Bh_0^{\frac{1}{2}}_0A_0]_2$, the wandering subspaces of $H^2$ and $[Bh_0^{\frac{1}{2}}]_2$ respectively. Then $p$ and $q$ are in the factor $(R(\mathcal{D}))^\prime$. Thus $p \preceq q$ or $q \preceq p$ in $(R(\mathcal{D}))^\prime$.

Case 1. $p \preceq q$. Then $H^2 = W^*[Bh_0^{\frac{1}{2}}]_2$ for a partial isometry $W \in M$ such that $WW^*H^2 = \left[w(L^2(\mathcal{D})\mathfrak{A})\right]_2 = WH^2 \subseteq [Bh_0^{\frac{1}{2}}]_2$ and $[Bh_0^{\frac{1}{2}}]_2 = WH^2 \oplus_{c\text{od}} (I - WW^*)[Bh_0^{\frac{1}{2}}]_2$ by Proposition 2.5. It is trivial that $W$ is an isometry since $h_0^{\frac{1}{2}}$ is in the initial space of $W$. On the other hand, $W^*B \subseteq \mathfrak{A}$ and $W\mathfrak{A} \subseteq B$ by Proposition 2.3. Thus $\mathfrak{A} = W^*B = W^*BB = \mathfrak{A}B = B$.

Case 2. $q \preceq p$. Then $[Bh_0^{\frac{1}{2}}]_2 = W^*H^2$ for a partial isometry $W \in M$. Thus $W$ is a co-isometry. However $W^*W[Bh_0^{\frac{1}{2}}]_2 = W^*H^2 = [Bh_0^{\frac{1}{2}}]$. Then $W$ is unitary and again $B = \mathfrak{A}$.

Conversely, we assume that $\mathfrak{A}$ is maximal in $M$. We claim that for any $E \in Z(\mathcal{D})$, either $EM \subseteq \mathfrak{A}$ or $(I - E)M \subseteq M$.

Let $0 < E < I$. If $EU^*_n \in \mathfrak{A}$ for all $n$, then $E\mathfrak{A}^* \subseteq \mathfrak{A}$ and thus $EM \subseteq \mathfrak{A}$. Assume that $EU^*_n \notin \mathfrak{A}$ for some $n$. Then the $\sigma$-weakly closed subalgebra generated by $\{EU^*_n : n \geq 1\}$ and $\mathfrak{A}$ is $M$. Thus $E\mathfrak{A}^* + \mathfrak{A}$ is $\sigma$-weakly dense in $M$. In particular, $(I - E)M = (I - E)\mathfrak{A} \subseteq \mathfrak{A}$.

Put $Q = \vee \{E \in M : EM \subseteq \mathfrak{A}\}$. By Lemma 3.1, $QM \subseteq \mathfrak{A}$ and $(I - Q)\mathfrak{A}Q = 0$. In particular, $Q \in Z(M)$.

Case 1. $Q \notin Z(M)$. Then $QM(I - Q) \neq 0$. In this case we have that $\mathfrak{A} = QM + (I - Q)M(I - Q)$ since $\mathfrak{A}$ is maximal. Thus (1) holds.
Case 2. \( Q \in Z(\mathcal{M}) \). Then \((I - Q)\mathfrak{A}\) is a maximal subalgebra of \((I - Q)\mathcal{M}\). In this case, \((I - Q)\mathcal{D}\) is a factor. Otherwise, as the above claim, there is a central projection \( E \in Z((I - Q)\mathcal{D}) \) such that \( E(I - Q) \neq 0 \) and \( E(I - Q)\mathcal{M} \subseteq (I - Q)\mathfrak{A} \). This is a contradiction. Therefore (2) holds. \( \square \)

3. FINITUDE OF TYPE 1 SUBDIAGONAL ALGEBRAS

We recall that a subdiagonal algebra with respect to \( \Phi \) in \( \mathcal{M} \) is finite if there is a faithful normal finite trace \( \tau \) on \( \mathcal{M} \) such that \( \tau \circ \Phi = \tau \). A longstanding open problem given in [1] by Arveson is whether a subdiagonal algebra in a finite von Neumann algebra is automatically finite. It is still open from now on. We answer this problem for type 1 case.

We recall that all results for type 1 subdiagonal algebras in general von Neumann algebras hold for a finite von Neumann algebra if we replace Haagerup’s noncommutative \( L^p(\mathcal{M}) \) by the noncommutative \( L^p \) space \( L^p(\mathcal{M}, \tau) \) for a von Neumann algebra \( \mathcal{M} \) with a a faithful normal finite trace \( \tau \).

**Theorem 3.1.** Let \( \mathcal{M} \) be a finite von Neumann algebra with a faithful normal finite trace \( \tau \). If \( \mathfrak{A} \) is a type 1 subdiagonal algebra with respect to \( \Phi \) in \( \mathcal{M} \), then \( \mathfrak{A} \) is finite, that is, there exists a faithful normal finite trace \( \rho \) on \( \mathcal{M} \) such that \( \rho \circ \Phi = \rho \).

**Proof.** Let \( L^2(\mathcal{M}, \tau) \) be the noncommutative \( L^2 \) space associated with \( \tau \). Then \( \mathcal{M} \subseteq L^2(\mathcal{M}, \tau) \). We choose a faithful normal state \( \varphi \) on \( \mathcal{M} \) such that \( \varphi \circ \Phi = \varphi \). If \( h_0 \) is the image of \( \varphi \) in \( L^1(\mathcal{M}, \tau) \), that is \( \varphi(A) = \tau(Ah_0) \) for all \( A \in \mathcal{M} \), then noncommutative \( H^2 \) and \( H^2_0 \) are defined similarly. Note that \( \mathfrak{A} \) is of type 1. \( H^2_0 = \bigoplus_n^{\text{col}} U_n H^2 \) for a family of column orthogonal partial isometries \( \{U_n : n \geq 1\} \) in \( \mathcal{M} \) such that \( U_n^* U_m = 0 \) and \( U_n^* U_n \in \mathcal{D} \) as in formula (2.1). Therefore

\[
I = x^* + d + x = (\oplus_{n \geq 1} U_n x_n)^* + d + \oplus_{n \geq 1} U_n x_n,
\]

(3.1)
where \( d \in L^2(\mathfrak{D})_+ \), \( \{x_n : n \geq 1\} \subset H^2 \) and \( x = \oplus_{n \geq 1} U_n x_n \). It follows that

\[
D = D(\oplus_{n \geq 1} U_n x_n)^* + Dd + D(\oplus_{n \geq 1} U_n x_n) = (\oplus_{n \geq 1} U_n x_n)^* D + dD + (\oplus_{n \geq 1} U_n x_n)D
\]

for any \( D \in \mathfrak{D} \). Since \( D(\oplus_{n \geq 1} U_n x_n), (\oplus_{n \geq 1} U_n x_n)D \in H^2_0 \), we have \( Dd = dD \) for all \( D \in \mathfrak{D} \) by (3.1).

Again by (3.1), we have for any \( m \geq 1 \),

\[
U_m = U_m(\oplus_{n \geq 1} U_n x_n)^* + U_mD + U_m(\oplus_{n \geq 1} U_n x_n)
\]

\[
= (\oplus_{n \geq 1} U_n x_n)^* U_m + dU_m + (\oplus_{n \geq 1} U_n x_n)U_m.
\]

It follows that

\[
U_mD - dU_m = U_m(\oplus_{n \geq 1} U_n x_n)^* + U_m(\oplus_{n \geq 1} U_n x_n) - (\oplus_{n \geq 1} U_n x_n)^* U_m - (\oplus_{n \geq 1} U_n x_n)U_m.
\]

We recall that \( \tau(y^* x) = \langle x, y \rangle \) denote the inner product for any \( x, y \in L^2(\mathcal{M}, \tau) \). It is elementary that \( \langle U_m d, U_m U_n x_n \rangle = \langle d, U_m^* U_m U_n x_n \rangle = 0 \) since \( U_m^* U_m \in \mathfrak{D} \). Similarly, we have \( \langle U_m d, U_n x_n U_m \rangle = \langle U_n^* U_m d, x_n U_m \rangle = 0 \). We also note that

\[
\langle U_m d, U_m (x_n U_m)^* \rangle = \langle U_m^* U_m d, (x_n U_n)^* \rangle = 0
\]

and

\[
\langle U_m d, (U_n x_n)^* U_m \rangle = \langle U_m d, x_n^* U_n^* U_m \rangle = \langle d, U_m^* x_n^* U_n^* U_m \rangle = 0.
\]

This implies that

\[
\langle U_m d, U_m (\oplus_{n \geq 1} U_n x_n)^* + U_m (\oplus_{n \geq 1} U_n x_n) - (\oplus_{n \geq 1} U_n x_n)^* U_m - (\oplus_{n \geq 1} U_n x_n)U_m \rangle = 0.
\]

By the same way, we have

\[
\langle dU_m, U_m (\oplus_{n \geq 1} U_n x_n)^* + U_m (\oplus_{n \geq 1} U_n x_n) - (\oplus_{n \geq 1} U_n x_n)^* U_m - (\oplus_{n \geq 1} U_n x_n)U_m \rangle = 0.
\]

This means that \( \langle U_m d - dU_m, U_m d - dU_m \rangle = 0 \) and thus \( U_m d - dU_m = 0 \).

By [III] Theorem 2.7, we know that \( \mathcal{M} \) is the von Neumann algebra generated by \( \mathfrak{D} \) and \( \{U_m : m \geq 1\} \). It now follows that \( Ad = dA \) and therefore

\[
(3.2) \quad Ad^2 = d^2 A
\]
for all $A \in \mathcal{M}$. Now let $E$ be the support projection of $d^2 \in L^1(\mathfrak{D})$ (cf. [21, Chapter III, Definition 3.7]). Then $E \in \mathfrak{D}$ and $(I-E)d = d(I-E) = 0$.

However, 

$$I - E = (I - E)x^*(I - E) + (I - E)d(I - E) + (I - E)x(I - E) = (I - E)x^*(I - E) + (I - E)x(I - E)$$

by (3.1). Note that $I - E \geq 0$ and $(I - E)x^*(I - E) + (I - E)x(I - E) \in (H_0^2)^* + H_0^2$. It follows that $I - E = 0$. Thus $d \in L^2(\mathfrak{D})$ is both left and right separating. Put $\rho(A) = \langle Ad, d \rangle = \tau(Ad^2), \forall A \in \mathcal{M}$. Then $\rho$ is a faithful normal state on $\mathcal{M}$. It is trivial that $\rho \circ \Phi = \rho$ since $d^2 \in L^1(\mathfrak{D})$. Now for any $A, B \in \mathcal{M}$, we have $\rho(AB) = \tau(ABd^2) = \tau(Bd^2A) = \tau(d^2BA) = \rho(BA)$ by (3.2). Thus $\rho$ is a faithful normal trace on $\mathcal{M}$ such that $\rho \circ \Phi = \rho$ and $\mathfrak{A}$ is finite. \hfill \Box

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