REGULARITY AND QUANTITATIVE GRADIENT ESTIMATE OF P-HARMONIC MAPPINGS BETWEEN RIEMANNIAN MANIFOLDS

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ABSTRACT. Let $M$ be a $C^2$-smooth Riemannian manifold with boundary and $N$ a complete $C^2$-smooth Riemannian manifold. We show that each stationary $p$-harmonic mapping $u: M \to N$, whose image lies in a compact subset of $N$, is locally $C^{1,\alpha}$ for some $\alpha \in (0, 1)$, provided that $N$ is simply connected and has non-positive sectional curvature. We also prove similar results for each minimizing $p$-harmonic mapping $u: M \to N$ with $u(M)$ being contained in a regular geodesic ball. Moreover, when $M$ has non-negative Ricci curvature and $N$ is simply connected and has non-positive sectional curvature, we deduce a quantitative gradient estimate for each $C^1$-smooth weakly $p$-harmonic mapping $u: M \to N$. Consequently, we obtain a Liouville-type theorem for $C^1$-smooth weakly $p$-harmonic mappings in the same setting.

Keywords: Non-positive curvature; regular geodesic ball; $p$-harmonic mappings; interior regularity; gradient estimate; Liouville theorem

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1. INTRODUCTION AND MAIN RESULTS

1.1. Background. For any $C^1$ mapping $u: (M, g) \to (N, h)$ between two Riemannian manifolds with $\dim M = n$ and $1 < p < \infty$, there is a natural concept of $p$-energy associated to $u$, namely, the functional

$$\int_M |\nabla u|^p d\mu,$$

where $\nabla u$ is the gradient of $u$ and $\mu$ is the Riemannian volume on $M$, that generalizes the classical Dirichlet energy when $p = 2$. Throughout this paper, critical point/stationary point/minimizers of such energy functionals are referred to as weakly/stationary/minimizing $p$-harmonic mappings and weakly/stationary/minimizing harmonic mappings in case $p = 2$.

The research on (weakly/stationary/minimizing) harmonic mappings has a long and distinguished history, making it one of the most central topics in geometric analysis on manifolds [31, 45]. In his pioneering work [34], Morrey proved the Hölder continuity of harmonic mappings when $n = 2$ (and smooth if $M$ and $N$ are smooth). The breakthrough of higher dimensional theory for harmonic mappings was made by Eells and Sampson [8], where they proved that every homotopy class of mappings from a closed manifold $M$ into...
$N$ has a smooth harmonic representative, if $N$ has non-positive (sectional) curvature. Important progress were made later by Hartman [24] and Hamilton [21]. When the image of a (weakly) harmonic mapping $u$ is contained in a regular geodesic ball of $N$, the existence, uniqueness and regularity theory were substantially developed by Hildebrandt and Widman [28] and Hildebrandt, Kaul and Widman [27]. In particular, it was proved in [27] that each (weakly) harmonic mapping $u: M \to N$ is smooth whenever $u(M)$ is contained in a regular geodesic ball $B_R(P)$ of $N$ (see Definition 1.2 below for the precise definition of regular geodesic ball). This result is optimal in the sense that the result fails if we enlarge the radius $R$ of the geodesic ball $B_R(P)$ (so that $B_R(P)$ fails to be regular). In the Euclidean setting, important results were obtained by Giaquinta and Giusti [16] for the case that the image of harmonic mappings lies in a coordinate chart. The regularity theory for (minimizing) harmonic mappings into general target Riemannian manifolds was later developed by Schoen and Uhlenbeck in their seminal paper [41] (see also [42] for boundary regularity theory and [43] for the case $N = S^n$). In particular, Schoen and Uhlenbeck proved that minimizing harmonic mappings are smooth away from a small singular set with Hausdorff dimension no more than $n - 3$, where the structure of singular sets (of minimizing and stationary harmonic mappings) has gained deeper understanding in the recent works [2, 32, 35]; see also [39] for an elegant new approach for the regularity result of weakly harmonic mappings.

Parallel results for the general $p$-harmonic mappings, $1 < p < \infty$, also gained growing interest in the past decades; see for instance [5, 6, 9, 10, 11, 12, 13, 14, 15, 19, 22, 23, 33, 36]. In particular, Duzaar and Fuchs [6], and Wei [47] generalized the existence result of Eells and Sampson [8] to $p$-harmonic mappings for $p \geq 2$; see also the very recent work [38]. Relying on the fundamental work of Struwe [46], Fardoun and Regbaoui [9, 10] developed the theory of $p$-harmonic mapping flow and partially extended the results of Eells and Sampson [8] to $p$-harmonic mappings. Concerning the (partial) regularity result for general Riemannian targets, Hardt and Lin [22], Luckhaus [33], and Fuchs [14] have extended the regularity result of Schoen and Uhlenbeck [41] to minimizing $p$-harmonic mappings ($1 < p < \infty$). More precisely, they proved that minimizing $p$-harmonic mappings (between compact smooth Riemannian manifolds) are locally $C^{1,\alpha}$ away from a singular set with Hausdorff dimension at most $n - \lceil p \rceil - 1$, where the singular set is defined as

$$S_u := \left\{ a \in M : \limsup_{r \to 0} r^{\frac{p-n}{p-1}} \int_{B_r(a)} |\nabla u|^p d\mu > 0 \right\}. \quad (1.1)$$

The structure of singular set has gained deeper understanding more recently in [23, 2, 36].

In the spirit of Schoen and Uhlenbeck [41], Hardt and Lin [22], Luckhaus [33] and Fuchs [14], it is natural to find geometric restrictions that exclude the singular set $S_u$ of a minimizing $p$-harmonic mapping $u: M \to N$. That is, we look for geometric conditions to ensure that each minimizing $p$-harmonic mapping is regular everywhere on $M$. In [41, Theorem IV] and [22, Theorem 4.5], the authors have developed some criteria to exclude the singular set for (minimizing) harmonic and $p$-harmonic mappings. As a corollary of their main results, Schoen and Uhlenbeck [41, Corollary] proved that if either the target manifold $N$ has non-positive curvature or the image of a harmonic mapping lies in a strict convex ball in $N$, then the harmonic mapping is smooth. This is closely related to the
earlier work of Eells and Sampson [8] and Hildebrandt, Kaul and Widman [27]. In [22, Theorem 4.5], it was proved that if each $p$-minimizing tangent mapping from the unit ball in $\mathbb{R}^l$ into $N$ is constant for $l = 1, 2, \ldots, n$, then $S_u = \emptyset$ for each minimizing $p$-harmonic mapping $u: M \to N$.

On the other hand, if we impose certain geometric restrictions on the manifold $N$ or on the image of $M$ under $u$, then some partial results for $S_u = \emptyset$ are well-known. In particular, when the image of $M$ of a $p$-harmonic mapping $u$ is contained in a smaller regular geodesic ball in $N$, Fuchs proved in [13] that $S_u = \emptyset$ for each $p$-harmonic mapping $u: M \to N$ with $p \geq 2$. If $N$ is simply connected and has non-positive sectional curvature, Wei and Yau [48] proved that each $p$-minimizing tangent mapping of $u$ from the unit ball in $\mathbb{R}^l$ into $N$ is constant for each $l = 1, 2, \ldots, n$ whenever it enjoys certain a priori regularity for $p \geq 2$ and so $S_u = \emptyset$ in this case by the criteria of Hardt and Lin.

In view of the above-mentioned works, two interesting and basic questions regarding the regularity theory of (minimizing) $p$-harmonic mapping between Riemannian manifolds can be formulated as follows:

**Regularity Question (NPC):** Are $p$-harmonic mappings $u: M \to N$, $1 < p < \infty$, necessarily locally $C^{1,\alpha}$ if $N$ is simply connected and has non-positive sectional curvature?

**Regularity Question (Regular ball):** Are $p$-harmonic mappings $u: M \to N$, $1 < p < \infty$, necessarily locally $C^{1,\alpha}$ if $u(M)$ is contained in a regular geodesic ball $B_R(P) \subset N$?

In the present paper, we shall provide affirmative answers to these two questions. Before turning to the statement of our main results, let us point out that in the case when $N$ has non-positive curvature, the regularity method of Hardt and Lin [22] (and also [33, 14]) necessarily generates singular sets for minimizing $p$-harmonic mappings $u: M \to N$, and the criteria mentioned above (to deduce that the singular set $S_u$ is empty) seems not working directly without any further a priori regularity assumption for $u$. The argument of Schoen and Uhlenbeck [41, Corollary] also fails in our setting as composition of (square of) the distance function with a $p$-harmonic mapping fails in general to be a sub-$p$-harmonic function. On the other hand, the method from [27, 13] relies crucially on the fact that the range of $u$ lies in a (smaller) regular geodesic ball of $N$. Thus, to obtain optimal regularity results, new idea is needed to handle the regularity question in this case.

### 1.2. Main results.

We first fix the basic setting before the statement of our main result. Let $M$ be an $n$-dimensional $C^2$-smooth Riemannian manifold with boundary $\partial M$ and $N$ a complete $C^2$-smooth Riemannian manifold. For simplicity, we assume that $N = (N, h)$ is isometrically embedded into some Euclidean space $\mathbb{R}^k$. Throughout this paper, we assume that $p \in (1, \infty)$.

Fix a domain $\Omega \subset M$. The Sobolev space $W^{1,p}(\Omega, N)$, $1 < p < \infty$, is defined as

$$W^{1,p}(\Omega, N) := \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^k) : u(x) \in N \quad \text{for almost every } x \in \Omega \right\},$$

where $W^{1,p}(\Omega, \mathbb{R}^k)$ is the Banach space of $\mathbb{R}^k$-valued $L^p$ functions on $\Omega$ with first distributional derivatives in $L^p$ as well. For $u, v \in W^{1,1}(\Omega, \mathbb{R}^k)$, the inner product $\langle \nabla u, \nabla v \rangle$ is
well-defined for almost every point on $\Omega$ by
\[
\langle \nabla u, \nabla v \rangle = \sum_{\alpha, \beta} g^{\alpha\beta} \frac{\partial u}{\partial x^\alpha} \cdot \frac{\partial v}{\partial x^\beta}
\]
where $g^{\alpha\beta} = [g_{\alpha\beta}]^{-1}$ is the inverse of the matrix representing the metric $g$ of $M$ in local coordinates $x_1, \cdots, x_n$. The energy density for $u \in W^{1,p}(\Omega, N)$ is defined as
\[
e_p(u) = |\nabla u|^p := \langle \nabla u, \nabla u \rangle^{p/2}
\]
and the $p$-energy of $u$ is
\[
E_p(u) = \int_{\Omega} |\nabla u|^p \, d\mu.
\]
A mapping $u \in W^{1,p}(\Omega, N)$ is said to be \textit{weakly} $p$-harmonic if it is a weak solution of the $p$-Laplace equation
\[
-\text{div}(|\nabla u|^{p-2} \nabla u) = |\nabla u|^{p-2} A(u) \langle \nabla u, \nabla u \rangle,
\]
where $A$ is the second fundamental form of $N$ in $\mathbb{R}^k$. In particular, for any compactly supported $W^{1,p}_0(M, \mathbb{R}^k) \cap L^\infty(M, \mathbb{R}^k)$-vector field $\psi$ along $u$ and $u_t = \exp_{u(x)} (t\psi(x))$, it holds
\[
0 = \left. \frac{d}{dt} \right|_{t=0} E_p(u_t) = \int_{\Omega} |\nabla u|^{p-2} \nabla u, \nabla \psi| \, d\mu. \tag{1.2}
\]
If in addition, $u$ is a critical point with respect to variations in its domain of definition, then it is called a \textit{stationary} $p$-harmonic mapping. In particular, a stationary $p$-harmonic mapping is a weakly $p$-harmonic mapping satisfying
\[
\left. \frac{d}{dt} \right|_{t=0} E_p(u(\exp_x(t\xi(x)))) = \left. \frac{d}{dt} \right|_{t=0} \int_{\Omega} |\nabla u(\exp_x(t\xi(x)))|^p \, d\mu = 0.
\]
A mapping $u \in W^{1,p}(\Omega, N), 1 < p < \infty$, is called \textit{minimizing} $p$-harmonic, if
\[
E_p(u|_{\Omega'}) \leq E_p(v|_{\Omega'})
\]
for every relatively compact domain $\Omega' \subset \Omega$ and every $v \in W^{1,p}(\Omega, N)$ with the same trace as $u$ on $\partial \Omega'$. The definition of trace can be found for instance in [30, Section 1.12]. The requirement that $v$ has the same trace as $u$ is equivalent to $u - v \in W^{1,p}_0(\Omega', \mathbb{R}^k)$.

Our first main result of this paper reads as follows.

\textbf{Theorem 1.1.} Each stationary $p$-harmonic mapping $u: \Omega \to N$, whose image lies in a compact subset of $N$, is locally $C^{1,\alpha}$ for some $\alpha \in (0,1)$ if $N$ is simply connected and has non-positive sectional curvature.

As commented earlier, Theorem 1.1 can be viewed as a natural extension of the regularity result of Eells and Sampson [8] or [41, Corollary] for harmonic mappings into Riemannian manifolds with non-positive curvature.

To prove Theorem 1.1, the main idea is to derive a Morrey type estimate (see Lemma 2.4) of a $p$-harmonic mapping $u: \Omega \to N$. Then it follows immediately that $u$ is locally $C^{0,\alpha}$ and the standard regularity theory (see e.g. Hardt-Lin [22, Section 3]) gives the desired local $C^{1,\alpha}$ regularity. The approach is inspired by an idea for proving Lipschitz regularity of harmonic mappings into singular metric spaces, due to Gromov and Schoen [18]. More precisely, we follow the idea of Gromov-Schoen [18] to consider
the composed function \( f_Q := d^2(u, Q) \), and derive certain weak differential inequality (see Lemma 2.1 below) that relates the \( p \)-energy of \( u \) and the gradient of \( f_Q \), which allows us to control the \( p \)-energy from above by (a constant multiple of) the integration of \( |\nabla u|^{p-2} |\nabla f_Q| \) over \( \partial B(a, r) \). Then we use Hölder’s inequality and Poincaré inequality to estimate the \( p \)-energy of \( u|_{B(a, r)} \) from above. A crucial technical point here is to use (a Riemannian version of) the monotonicity formula for stationary \( p \)-harmonic mappings due to Hardt-Lin [22] (see Lemma 2.3 below).

We next recall the definition of regular geodesic ball from [26].

**Definition 1.2** (Regular geodesic ball). Let \( B_R(P) \subset N \) be a geodesic ball centered at \( P \) with radius \( R \). Let \( C(P) \) be the cut locus of its center \( P \). We say that \( B_R(P) \) is a regular geodesic ball if \( B_R(P) \cap C(P) = \emptyset \) and \( R < \frac{\pi}{2\sqrt{\kappa}} \), where \( \kappa \geq 0 \) is an upper bound for the sectional curvature of \( N \) on the ball \( B_R(P) \).

Our second main result answers affirmatively the second regularity question.

**Theorem 1.3.** Each minimizing \( p \)-harmonic mapping \( u : \Omega \to N \) is locally \( C^{1,\alpha} \) for some \( \alpha \in (0,1) \) if \( u(M) \) is contained in a regular geodesic ball \( B_R(P) \subset N \).

The main arguments leading to Theorem 1.3 are due to Hardt-Lin [22] and Fuchs [12]. More precisely, in [12], Fuchs has shown that each \( p \)-harmonic mapping \( u \) from \( \mathbb{R}^l \) to a regular geodesic ball \( B_R(P) \subset N \) is constant for \( l = 1,2,\ldots \) when \( p \geq 2 \). We will actually extend his idea in parallel to the case \( p \in (1,2) \) and then conclude that each \( p \)-minimizing tangent map of \( u \) from the unit ball in \( \mathbb{R}^l \) into \( N \) is constant for each \( l = 1,2,\ldots,n \) and hence \( S_u = \emptyset \) by the criteria of Hardt and Lin [22, Theorem 4.5]. The main difference, compared with the proof of [12], is that we will apply some delicate estimates of [27] to derive an important Caccioppoli inequality.

Next we further estimate the gradient of stationary \( p \)-harmonic mappings. In [40, Theorem 2.2], Schoen proved that there exists an \( \varepsilon > 0 \) depending only on \( n, g \) and \( N \) such that if \( u : B_r \to N \) is (minimizing) harmonic with \( r^{2-n} \int_{B_r} |\nabla u|^2 d\mu < \varepsilon \), then

\[
\sup_{B_{r/2}} |\nabla u|^2 \leq C \int_{B_r} |\nabla u|^2 d\mu.
\]

When \( N \) is assumed to be non-positively curved, the gradient estimate as above still holds if we drop the smallness assumption on the normalized energy; see [18, Theorem 2.4].

This result was improved later by Korevaar and Schoen [30, Theorem 2.4.6] in the following form: Let \( \Omega \) be a smooth bounded domain of a Riemannian manifold \( M \) and \( N \) non-positively curved in the sense of Alexandrov. Suppose \( u : \Omega \to N \) is minimizing harmonic. Then for any ball \( B_{2R}(o) \) with \( B_{2R}(o) \subset \subset \Omega \), there exists a constant \( C \) depending only on \( n = \dim(M), R, \) the injectivity radius of \( o \) and the \( C^1 \)-norm of \( g \) on \( B_{2R}(o) \) such that

\[
\sup_{B_{R}(o)} |\nabla u| \leq C \int_{B_{2R}} |\nabla u| d\mu \leq C \left( \int_{B_{2R}} |\nabla u|^2 d\mu \right)^{1/2}.
\]
The dependence of the constant $C$ was further improved by Zhang, Zhong and Zhu in their very recent work \cite{49}.

Concerning the quantitative gradient estimate for stationary $p$-harmonic mappings, Duzaar and Fuchs proved in \cite[Theorem 2.1]{5} that, there exist $\varepsilon$ and $C$ depending only on $n$, $p$ and the curvature bound of $N$, such that if $u : B_r \to N$ is $C^1$-smooth weakly $p$-harmonic ($p \geq 2$) with the smallness condition $r^{p-n} \int_{B_r} |\nabla u|^p d\mu < \varepsilon$, then

$$\sup_{B_{r/2}} |\nabla u|^p \leq C \int_{B_r} |\nabla u|^p d\mu.$$

In this paper, we establish the quantitative gradient estimate for $C^1$-smooth weakly $p$-harmonic mappings when $M$ has non-negative Ricci curvature and $N$ is simply connected and has non-positive sectional curvature. As in the harmonic case \cite{30}, the smallness condition for the normalized $p$-energy is unnecessary. Our third main result of this paper reads as follows.

**Theorem 1.4.** Assume that $M$ has non-negative Ricci curvature and $N$ is simply connected and has non-positive sectional curvature. Let $u : M \to N$ be a $C^1$-smooth weakly $p$-harmonic mapping. Then there exists a constant $C$, depending only on $n = \text{dim} \, M$, such that for each ball $B_r : = B_r(o)$ with $B_{2r}(o) \subset M$, we have

$$\sup_{B_r} |\nabla u|^{p-1} \leq C \int_{B_{2r}} |\nabla u|^{p-1} d\mu \leq C \left( \int_{B_{2r}} |\nabla u|^p d\mu \right)^{(p-1)/p}. \quad (1.3)$$

The proof of Theorem 1.4 follows closely the idea of Schoen and Yau \cite{44}, which relies crucially on the Bochner-Weitzenböck formula (due to Eells and Sampson \cite{8}). However, the degeneracy of $p$-harmonicity for $p \neq 2$ causes some extra technical difficulty. We tackle this difficulty by adapting some ideas from Duzaar and Fuchs \cite[Proof of Theorem 2.1]{5}, where the authors deal mainly with the case $p \geq 2$.

As an immediate consequence of Theorem 1.4, we obtain the following Liouville-type theorem, which extends the classical result of Schoen and Yau \cite[Theorem 1.4]{44} for harmonic mappings to the setting of $p$-harmonic mappings.

**Corollary 1.5.** Let $M = (M, g)$ be an $n$-dimensional complete non-compact Riemannian manifold with nonnegative Ricci curvature and $N$ a simply connected Riemannian manifold with non-positive sectional curvature. Then any $C^1$-smooth weakly $p$-harmonic mapping $u : M \to N$ with finite $(p - 1)$ or $p$-energy must be constant.

**Proof.** As $\mu(M) = \infty$, the result follows from (1.3) by sending $r$ to infinite. \hfill \Box

Note that, under the assumption of Corollary 1.5, Nakauchi \cite{37} proved that any $C^1$-smooth weakly $p$-harmonic mapping $u : M \to N$ with finite $p$-energy must be constant for $p \geq 2$. Corollary 1.5 extends this result to all $p \in (1, \infty)$.

\footnote{Indeed, the authors obtained quantitative gradient estimates for minimizing harmonic mappings from Riemannian manifolds with non-negative Ricci curvature into metric spaces with non-positive curvature in the sense of Alexandrov, which is much more general than the setting of Korevaar and Schoen.}
1.3. **Structure of the paper.** This paper is structured as follows. The proofs of Theorem 1.1 and Theorem 1.3 are given in Section 2 and Section 3, respectively. In Section 4, we prove Theorem 1.4. The final section, Section 5, contains some comments about our general method and possible extensions to mappings into more general metric spaces. We also include an appendix, establishing $W^{2,2}$ regularity estimates for weakly $p$-harmonic mappings in the case $1 < p < 2$, and as a byproduct, we extend the main results of Duzaar and Fuchs [5] on gradient estimates and removable singularity of weakly $p$-harmonic mappings to the case $1 < p < 2$.

Our notation of various concepts is rather standard. Whenever we write $A(r) \lesssim B(r)$, it means that there exists a positive constant $C$, independent of $r$, such that $A(r) \leq CB(r)$.

2. **Proof of Theorem 1.1**

In this section we assume that $N$ is simply connected and has non-positive sectional curvature and $\Omega \subset M$ is a domain.

Given a weakly $p$-harmonic mapping $u: M \to N$ whose image is contained in a compact subset of $N$, we will show in the following lemma that the composed function $d^2(u, Q)$ satisfies a weak differential inequality that relates the $p$-energy of $u$ and the gradient of $d^2(u, Q)$. In the (minimizing) harmonic case, this is due to Gromov and Schoen [18, Proposition 2.2].

**Lemma 2.1.** If $u: \Omega \to N$ is weakly $p$-harmonic with $u(\Omega)$ being contained in a compact subset of $N$, then for each $Q \in N$, the function $d^2(u, Q)$ satisfies the differential inequality

$$
\int_{\Omega} |\nabla u|^{p-2}(2\eta |\nabla u|^2 + \nabla \eta \cdot \nabla d^2(u, Q))d\mu \leq 0
$$

for any $\eta \in C^\infty_0(\Omega)$.

**Proof.** Since $u: \Omega \to N$ is weakly $p$-harmonic, for any compactly supported $W^{1,p}_0(M, \mathbb{R}^k) \cap L^\infty(M, N)$-smooth vector field $\psi: \Omega \to \mathbb{R}^k$ and $u_t(x) = \exp_{u(x)}(t\psi(x))$, we have

$$
0 = d \frac{d}{dt} \big|_{t=0} E_p(u_t) = \int_{\Omega} \langle |\nabla u|^{p-2}\nabla u, \nabla \psi \rangle d\mu. \tag{2.1}
$$

Given $\eta \in C^\infty(\Omega)$ and $Q \in N$, we denote by $f := d^2(x, Q)$. As $f \in C^2(N, \mathbb{R}) \cap W^{2,p}_{loc}(N, \mathbb{R})$, we may select a sequence of compactly supported smooth function $f_i \in C^\infty_0(N, \mathbb{R})$ on $N$ such that $f_i \to f$ in $W^{2,p}_{loc}(N, \mathbb{R})$. Since $u \in W^{1,p}(\Omega, N)$, $u$ is colo-cally weakly differentiable, which means that $F \circ u$ is weakly differentiable whenever $F \in C^1_0(N, \mathbb{R})$ (see [3, Proposition 2.1 and Proposition 2.6]). Moreover, we have $\nabla f_i \circ u \in W^{1,p}_{loc}(M, \mathbb{R}^k)$ by [3, Proposition 2.5] as $u(M)$ is contained in a compact subset of $N$ and $\nabla f_i \in C^1_0(N, \mathbb{R})$. Set

$$
\psi_i = \eta(x)(\nabla f_i) \circ u(x).
$$

Then $\psi_i \in W^{1,p}_0(M, \mathbb{R}^k) \cap L^\infty(M, \mathbb{R}^k)$ is an admissible test vector field.
Substitute $\psi_i$ in (2.1) and we obtain

$$0 = \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla (\eta(x) \nabla f_i) \rangle$$

$$= \int_\Omega |\nabla u(x)|^{p-2} \langle \nabla u(x), \eta(x) \nabla \frac{\partial}{\partial x} f_i + \nabla \eta \otimes \nabla f_i \rangle d\mu$$

$$= \int_\Omega |\nabla u|^{p-2} \left( \eta(x) \nabla (\nabla f_i) (\nabla u, \nabla u) + \langle \nabla u(x), \nabla \eta \otimes \nabla f_i \rangle \right) d\mu$$

$$\text{as } i \to \infty \int_\Omega |\nabla u|^{p-2} \left( \eta(x) \nabla (\nabla f)(\nabla u, \nabla u) + \langle \nabla u(x), \nabla \eta \otimes \nabla f \rangle \right) d\mu$$

$$= \int_\Omega |\nabla u|^{p-2} \left( \eta(x) \nabla (\nabla f)(\nabla u, \nabla u) + \langle \nabla \eta(x), \nabla (f \circ u) \rangle \right) d\mu,$$

where the last second limit follows because $f_i \to f$ in $W^{2,p}_{loc}(N, \mathbb{R})$ and $u(M)$ is contained in a compact subset of $N$. On the other hand, Note that since $N$ is simply connected and non-positively curved,

$$\nabla (\nabla f)(\nabla u, \nabla u) \geq 2||\nabla u||^2;$$

see e.g. [29, Lemma 5.8.2]. Inserting this into (2.2) yields

$$\int_\Omega |\nabla u|^{p-2} (2\eta |\nabla u|^2 + \nabla \eta \cdot \nabla d^2(u, Q)) d\mu \leq 0.$$

\[\square\]

Remark 2.2. Note that the assumption $N$ being non-positively curved is crucial in the above arguments as it implies that

$$\nabla (\nabla f)(v, v) \geq 2||v||^2$$

for the squared distance function $f = d^2(x, Q)$ (with any given $Q \in N$).

We next derive the monotonicity formula for stationary $p$-harmonic mappings $u: M \to N$. Fix an arbitrary point $a \in M$ and set $E(r) = \int_{B_r(a)} |\nabla u|^p d\mu$. Note that $E'(r) = \int_{\partial B_r(a)} |\nabla u|^p d\Sigma$ for almost every $r$. When $M = \mathbb{R}^n$, the monotonicity formula (see [22, Lemma 4.1]) for (minimizing) $p$-harmonic mappings implies that for almost every $r \in (0, r_0)$,

$$\frac{d}{dr} \left( r^{p-n} \int_{B_r(a)} |\nabla u|^p dx \right) = pr^{p-n} \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma,$$

or we may equivalently formulate as

$$E'(r) = \frac{n-p}{r} E(r) + p \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma.$$

Lemma 2.3 (Monotonicity formula). If $u: \Omega \to N$ is stationary $p$-harmonic, then for each $a \in M$, there exists a radius $r_0 > 0$ such that for almost every $r \in (0, r_0)$, we have

$$E'(r) = (1 + O(r)) \left( \frac{n-p+O(r)}{r} E(r) + p \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma \right).$$

Proof. The proof is similar to the case $p = 2$ from [18, Section 2, Page 192-193]; see also [4, Lemma 3.1 and Lemma 3.2].
Let \( \eta \) be a smooth function with support in a small neighborhood of \( a \). For \( t \) small consider the diffeomorphism of \( \Omega \) given in a normal coordinates by \( F_t(x) = (1 + t\eta(x))x \) in a neighborhood of \( 0 \) with \( F_t = \text{id} \) outside this neighborhood. Consider the comparison mappings \( u_t = u \circ F_t \). Then \( u_t \) has the same trace and regularity as \( u \). Since \( u \) is stationary \( p \)-harmonic, \( \frac{d}{dt}|_{t=0}E(u_t) = 0 \). Direct computation (see [18, Section 2, Page 192] and [11, Page 270]) implies that

\[
0 = \int_{\Omega} |\nabla u|^p - 2(\eta - |\nabla u|^2 \sum_i \frac{\partial \eta}{\partial x_i} + p \sum_{i,j,k} g^{ik} \frac{\partial \eta}{\partial x_i} x_j \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} d\mu) + A,
\]

where \( A \) is the reminder term given by

\[
\int_{\Omega} |\nabla u|^p - 2(\sum_{i,j,k} \frac{\partial g^{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \sqrt{g} + |\nabla u|^2 \sum_i \frac{\partial \sqrt{g}}{\partial x_i}) dx.
\]

Choosing \( \eta \) to approximate the characteristic function of \( B_r(a) \), we obtain

\[
0 = rE'(r) - (n - p + O(r))E(r) - pr \int_{\partial B_r(a)} |\nabla u|^p - 2 \sum_{i,j,k} g^{ik} \frac{\partial \eta}{\partial x_i} x_j \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} d\Sigma,
\]

where we have used the fact that the reminder term \( |A| \leq crE(r) \) (because \( |\frac{\partial g^{ij}}{\partial x_k}|, |\frac{\partial \sqrt{g}}{\partial x_i}| \) are bounded from above by some constant \( c \)). Since \( g^{ik} \leq \delta^{ik} + cr \) when \( r \) is sufficiently small, we get

\[
\sum_{i,j,k} g^{ik} \frac{\partial \eta}{\partial x_i} x_j \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} \leq |\frac{\partial u}{\partial r}|^2 + cr |\nabla u|^2,
\]

from which the claim follows. \( \square \)

We would like to point out that the non-positive curvature assumption for \( N \) was only used in Lemma 2.1, while the conclusion of Lemma 2.3 remains valid for general Riemannian manifold \( N \) (without any curvature restriction). With the aid of Lemma 2.1 and Lemma 2.3, we are able to derive the following important monotonicity inequality.

**Lemma 2.4.** There exist \( r_1 > 0 \) and \( \gamma > 0 \) depending on \( B_{r_0}(a) \), the Lipschitz bound and the ellipticity constant of \( g \) such that

\[
r \mapsto \frac{E(r)}{r^{n-p+\gamma}}, \quad r \in (0, r_1)
\]

is non-decreasing.

**Proof.** Set \( I_Q(r) = \int_{\partial B_r(a)} |\nabla u|^{p} \, d\Sigma \), where \( r > 0 \) is small. Recall that the Poincaré inequality for \( B_{r_0}(a) \) implies that

\[
\inf_{Q \in N} I_Q(r) \leq Cr^p \int_{\partial B_r(a)} |\nabla u|^p \, d\Sigma,
\]

where the constant \( C \) depends only on \( B(a, r_0) \) and the ellipticity constant of \( g \). We will fix \( Q \in N \) such that the above Poincaré inequality holds for \( u \).
We first consider the case $p \geq 2$. Choosing $\eta$ to approximate $\chi_{B_r(a)}$ in Lemma 2.1 and then applying the Hölder’s inequality and Poincaré inequality, we infer that
\[ E(r)^p \lesssim \left( \int_{\partial B_r(a)} |\nabla u|^{p-2}d(u, Q)\frac{\partial}{\partial r}d(u, Q)d\Sigma \right)^p \]
\[ \leq I_Q(r)\left( \int_{\partial B_r(a)} |\nabla u|^{p}d\Sigma \right)^{p-2}\left( \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma \right)^{p/2} \]
\[ \lesssim r^{p/2}(rE'(r))^{p/2}\left( \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma \right)^{p/2}. \]

Set $A = \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial u}{\partial r} \right|^2 d\Sigma$. Lemma 2.3 and the above inequality imply that
\[ E(r)^p \lesssim (1 + O(r))\left( (n - p + O(r))E(r) + prA \right)^{p/2} (rA)^{p/2} \]
\[ \lesssim \left( (E(r))^{p/2} + (rA)^{p/2} \right) (rA)^{p/2}. \]

Note that the constant in the above estimate depends only on the constant from the Poincaré inequality and the ellipticity constant of $g$. Applying the Young’s inequality $ab \leq \varepsilon a^2 + C_\varepsilon b^2$ (with $\varepsilon$ sufficiently small), we obtain from the previous inequality that
\[ E(r) \leq KrA \]
for some constant $K > 0$ independent of $r$.

Now using Lemma 2.3 again, we have
\[ rE'(r) = (n - p + O(r))E(r) + (pr + O(r^2))A \]
\[ \geq (n - p + O(r))E(r) + \frac{p + O(r)}{K}E(r) \]
\[ = (n - p + \frac{p}{K} + O(r))E(r) \]
\[ \geq (n - p + r\gamma)E(r) \]
for some $\gamma > 0$ when $r$ is sufficiently small. This implies
\[ \frac{d}{dr}(\log \frac{E(r)}{r^{n-p+p\gamma}}) \geq 0 \]
and so the claim follows in this case.

Next we consider the case $1 < p < 2$. Similarly as in the previous case, we have
\[ E(r)^p \lesssim \left( \int_{\partial B_r(a)} |\nabla u|^{p-2}d(u, Q)\left| \frac{\partial}{\partial r}d(u, Q) \right| d\Sigma \right)^p \]
\[ \lesssim \left( \int_{\partial B_r(a)} d^p(u, Q)d\Sigma \right)\left( \int_{\partial B_r(a)} \left| \nabla u \right|^{p-1-\varepsilon} \left| \frac{\partial}{\partial r}d(u, Q) \right| \right)^{\frac{p}{p-\varepsilon}} d\Sigma \]
\[ \lesssim I_Q(r)\left( \int_{\partial B_r(a)} |\nabla u|^{p-2} |\nabla u|^{2-p\varepsilon} \left| \frac{\partial}{\partial r}d(u, Q) \right| d\Sigma \right)^{p-1}. \]
Here, \( p' = p/(p-1) \), \( \epsilon > 0 \) is chosen such that \( p - 1 - \epsilon > 0 \) and \( 2 - p'\epsilon > 0 \). Applying Hölder’s inequality and Poincaré inequality, we deduce

\[
E(r)^p \leq I_Q(r) \left( \int_{\partial B_r(a)} |\nabla u|^p d\Sigma \right)^{p-1-\frac{2}{p'}} \left( \int_{\partial B_r(a)} |\nabla u|^{p-2} \left| \frac{\partial}{\partial r} d(u,Q) \right|^2 d\Sigma \right)^{\frac{2}{p'}}
\]

\[
\leq r^p (E'(r))^{p-\frac{2}{p'}} A_r^{\frac{2}{p'}},
\]

which, according to Lemma 2.3 and Young’s inequality, implies that

\[
E(r) \leq (rA)^{\epsilon/2} (rE'(r))^{1-\epsilon/2}
\]

\[
\leq (rA)^{\epsilon/2} (E(r) + rA)^{1-\epsilon/2} \leq \frac{1}{2} E(r) + KrA
\]

for some \( K > 0 \) independent of \( r \). The rest arguments are the same as in the previous case. This completes the proof. \( \Box \)

**Proof of Theorem 1.1.** Since by Lemma 2.4, \( u \) is locally Hölder continuous. The local \( C^{1,\alpha} \)-regularity follows by the standard regularity theory of elliptic PDEs; see e.g. [22, Section 3]. \( \Box \)

### 3. Proof of Theorem 1.3

#### 3.1. The \( p \)-minimizing tangent maps

Following [41] (for the case \( p = 2 \)) and [22], we introduce the definition of minimizing tangent maps.

**Definition 3.1.** A mapping \( v \in W^{1,p}_{loc}(\mathbb{R}^l, N) \) is said to be a \( p \)-minimizing tangent map if \( v: \mathbb{R}^l \to N \) is locally \( p \)-harmonic and is homogeneous of degree 0, that is, the radial derivative \( \frac{\partial v}{\partial r} = 0 \) almost everywhere.

Fix a \( p \)-harmonic mapping \( u: \Omega \to N \) and an integer \( l \in \{1, 2, \cdots, n\} \). We consider the blow-up mappings \( u_{x,r}(y) := u(x + ry): \mathbb{B} \to N \), where \( \mathbb{B} \subset \mathbb{R}^l \) is the unit open ball. By [22, Corollary 4.4], there exists a sequence \( r_i \to 0 \) such that \( u_{x,r_i} \) converges strongly in \( W^{1,p}(\mathbb{B}, N) \) to a mapping \( u_0 \in W^{1,p}(\mathbb{B}, N) \) which is homogeneous of degree 0. By homogeneity, we may then extend \( u_0 \) to all of \( \mathbb{R}^l \) (and we still denote by \( u_0 \) the extended mapping) so that \( u_0: \mathbb{R}^l \to N \) is a \( p \)-minimizing tangent map. We call such \( u_0 \) a \( p \)-minimizing tangent map of \( u \).

Note that if \( u(M) \subset B_R(P) \), then \( u_{x,r}(\mathbb{B}) \subset B_R(P) \) for each \( i \in \mathbb{N} \). The strong convergence of \( u_{x,r_i} \) to \( u_0 \) then implies that \( u_0(\mathbb{B}) \subset B_R(P) \). As \( u_0 \) is homogeneous of degree 0, \( u_0(\mathbb{R}^l) \subset B_R(P) \) as well. Consequently, Theorem 1.3 follows immediately from [22, Theorem 4.5] and the following Liouville’s theorem for \( p \)-harmonic mappings from Euclidean space \( \mathbb{R}^l \) into regular geodesic balls.

**Theorem 3.2.** There is no non-constant \( p \)-harmonic mapping \( u: \mathbb{R}^l \to B_R(P) \subset N \) for each \( l = 1, 2, \cdots \).

As commented earlier in the introduction, the case \( p \geq 2 \) has been proved by Fuchs [12] and later the proof was extended to stationary \( p \)-harmonic mappings \( (p \geq 2) \) in [13], where the image is required to be contained in a smaller geodesic ball. We will give the proof
of Theorem 3.2 in the next section, where we essentially extend the original arguments of Fuchs [12] in combination with some arguments from [27] to the case \( p \in (1, 2) \).

3.2. Proof of Theorem 3.2. We fix a \( p \)-harmonic mapping \( u : \mathbb{R}^l \to B_R(P) \subset N \), where \( h \) is the Riemannian metric on \( N \). Before turning to the proof of Theorem 3.2, we recall some elementary facts about \( p \)-harmonic mappings. In the following calculation, we will use the standard Einstein summation convention.

Let \( v \) denote the representative of \( u \) with respect to the normal coordinates centered in \( B_R(P) \) and recall that

\[
|\nabla v|^{p-2} = \left( \delta_{\alpha\beta} h_{ij}(v) \frac{\partial v^i}{\partial x_\alpha} \frac{\partial v^j}{\partial x_\beta} \right)^{\frac{p}{2}-1}.
\]

Fix a ball \( B \subset \mathbb{R}^l \). The Euler system for \( v \) reads as

\[
\int_B |\nabla v|^{p-2} \left( h_{ij}(v) \frac{\partial v^i}{\partial x_\alpha} \frac{\partial v^j}{\partial x_\beta} + \frac{1}{2} \left( \frac{\partial h_{ij}}{\partial x_k} \circ v \right) \frac{\partial v^i}{\partial x_\alpha} \frac{\partial v^j}{\partial x_\beta} \Phi^k \right) \delta_{\alpha\beta} dx = 0, \tag{3.1}
\]

for all bounded \( \Phi \in W_0^{1,p}(B, \mathbb{R}^k) \). If we take \( \Phi^k = h^{ki}(v) \Psi^i \), then

\[
\frac{\partial \Phi^k}{\partial x_\beta} = h^{ki}(v) \frac{\partial \Psi^i}{\partial x_\beta} + \frac{\partial h^{ki}}{\partial v^m} \frac{\partial v^m}{\partial x_\beta} \Psi^i.
\]

Plug this in (3.1), we finally arrive at

\[
\int_B |\nabla v|^{p-2} \left( \frac{\partial v^i}{\partial x_\alpha} \frac{\partial \Psi^j}{\partial x_\beta} \delta_{ij} - \Gamma^l_{ij}(v) \frac{\partial v^i}{\partial x_\alpha} \frac{\partial v^j}{\partial x_\beta} \Psi^l \right) \delta_{\alpha\beta} dx = 0, \tag{3.2}
\]

where \( \Gamma^l_{ij} \) denotes the Christoffel symbols on the manifold \( N \).

For each \( x \in B \) with \( r < d(x, \partial B)/2 \), we define

\[
\bar{V} := \int_{B_2r(x)} Vdz \quad \text{and} \quad \bar{P} := (\exp_p)^{-1}(\bar{V}),
\]

where \( V \) is the representation of \( u \) with respect to the normal coordinates centered at \( P \). Since \( \bar{P} \in B_R(P) \), we may introduce another normal coordinates with center \( \bar{P} \) and denote by \( v \) the representation of \( u \) with respect to this normal coordinates.

Let \( \eta \in C_0^\infty(B_{2r}(x)) \) be a cut-off function which satisfies \( \eta = 1 \) on \( B_r(x) \), \( 0 \leq \eta \leq 1 \) in \( B_{2r}(x) \) and \( |\nabla \eta| \leq cr^{-1} \) for some constant \( c = c(n) \). Set

\[
\theta(v, \nabla v) := |\nabla v|^2 - \Gamma^l_{ij}(v) \nabla v^i \cdot \nabla v^j v^l.
\]

Note that

\[
|v| = d(u, \bar{P}) \leq d(u, P) + d(P, \bar{P}) \leq 2R < \frac{\pi}{\sqrt{c}}
\]

and so by [27, Estimate (4.7)]

\[
\theta(v, \nabla v) \geq a_\kappa(2R) h_{ij}(v) |\nabla v|^2 - \nabla v^i \cdot \nabla v^j = a_\kappa(2R) |\nabla v|^2, \tag{3.3}
\]

where \( a_\kappa > 0 \) is defined as in [27, Section 2]. Inserting (3.3) into (3.2) and taking \( \Psi = \eta^p v \), we arrive at

\[
\int_B \eta^p a_\kappa(2R) |\nabla v|^2 dx \leq \sum_{a, i} \int_B |\nabla v|^{p-2} \frac{\partial v^i}{\partial x_\alpha} \eta^p \frac{\partial \eta^p}{\partial x_\alpha} dx. \tag{3.4}
\]
Note that by [27, Lemma 1], we have
\[ b_\kappa^2(|y|)|\xi|^2 \leq h_{ij}(y)\xi^i\xi^j \leq b_\omega(|y|)|\xi|^2 \tag{3.5} \]
for all $\xi \in \mathbb{R}^k$, where $b_\kappa$ and $b_\omega$ are defined as in [27, Lemma 1]. Applying (3.5) with $y = v(x)$, we deduce
\[ c_2|\nabla v(x)|^2 \leq h_{ij}(v(x))\nabla v^i(x) \cdot \nabla v^j(x) \leq c_3|\nabla v(x)|^2 \tag{3.6} \]
for almost every $x \in B$. Applying (3.4), (3.5) and $\varepsilon$-Young’s inequality, we obtain
\[
a_\kappa(2R) \int_B \eta^p|\nabla v|^p dx \leq \sum_{\alpha,i} \int_B |\nabla v|^{p-2} D_\alpha v^i v^i D_\alpha (\eta^p) dx
\leq \varepsilon \int_B |\nabla v|^p \eta^p dx + c'(\varepsilon) \sum_i \int_B |\nabla \eta|^p |v^i|^p dx.
\]
Absorbing the first term into the left-hand side of the previous inequality, we obtain
\[
\int_B \eta^p|\nabla v|^p dx \leq c_4 \int_B |\nabla \eta|^p |v|^p dx. \tag{3.7}
\]
Note that
\[
\int_{B_r(x)} |\nabla v|^p dz = E_p(u|_{B_r(x)}) \geq c_5 \int_{B_r(x)} |\nabla V|^p dz,
\]
where we have used the fact that an inequality of the form (3.5) remains valid in normal coordinates centered at $P$. Observe that (see [27, Page 11, footnote (1)])
\[
|v(x)| = d(u(x), P) \leq b_\omega(2R)|u(x) - \bar{u}|.
\]
Combining all these estimates, we arrive at the following Caccioppoli inequality for the coordinate representative of $u$
\[
\int_{B_r(x)} |\nabla V|^p dz \leq c_6 r^{-p} \int_{B_{2r}(x)} |V - \tilde{V}|^p dz. \tag{3.8}
\]

**Remark 3.3.** 1) The Caccioppoli inequality (3.8) was first obtained by Fuchs [12, Page 412], where he assumed $p \geq 2$ and refers to the book of Giaquinta [17]. The proofs we are using here make use of some delicate estimates from [27, Proof of Theorem 3]. In particular, the Caccioppoli inequality (3.8) holds for weakly $p$-harmonic mappings.

2) As a consequence of the Caccioppoli inequality (3.8) and [7, Lemma 5], we infer that if a weakly $p$-harmonic mapping $u: B_{2r} \to N$ satisfies $u(B_{2r}) \subset B_R(P)$ for a regular geodesic ball in $N$ and $E_p(u) \leq \varepsilon$ for some $\varepsilon$ depending only on $n$, $p$, and $N$, then $u \in C^{1,\alpha}(B_r, N)$ for some $\alpha$ depending only on $n$, $p$, and $N$.

3) Since (3.8) holds for all balls $B_{2r}(x) \subset B$, we may apply the standard reverse Hölder inequality (see Giaquinta [17]) to deduce that there is $q > p$ such that $\nabla V \in L^1_{\text{loc}}$. Moreover,
\[
\left( \int_{B_r(x)} |\nabla V|^q dz \right)^{1/q} \leq c_7 \left( \int_{B_{2r}(x)} |\nabla V|^p dz \right)^{1/p}. \tag{3.9}
\]

Now we prove Theorem 3.2.
Proof of Theorem 3.2. For each $k \in \mathbb{N}$, we set $u_k(x) := u(kx)$ and let $v$ and $v_k$ be the coordinate representation of $u$ and $u_k$ with respect to the normal coordinates centered at $P$.

We first consider the case $p \leq l$. By the Caccioppoli inequality (3.8) we know
\[
\sup_k \|\nabla v_k\|_{L^p(B_t)} \leq c(t)
\]
for all $t \in (0, \infty)$ with some constant $c(t)$ independent of $k$, where $B_t = B_t(0) \subset \mathbb{R}^l$. In particular, by the weak compactness of Sobolev spaces, we infer that there exists a $v_0 \in W^{1,p}_{\text{loc}}(\mathbb{R}^l, \mathbb{N})$ such that $v_k$ converges to $v_0$ weakly in $W^{1,p}_{\text{loc}}(\mathbb{R}^l, \mathbb{N})$ and $v_k \to v$ pointwise almost everywhere. Thus $v_k$ converges to $v_0$ strongly in $W^{1,p}(\mathbb{R}^l, \mathbb{N})$ as well (by [12, Lemma 2] or [33, Proposition 2]). Moreover, $v_0$ is homogenous of degree 0, i.e., $\frac{\partial v_0}{\partial r} = 0$ almost everywhere by the arguments of Fuchs [12, Page 413], where only the monotonicity formula [22, Lemma 4.1] is needed; see also [33, Proof of Proposition 2]. Note that the strong convergence of $v_k$ to $v_0$ implies that $v_0$ also satisfies (3.2) and so we may select $\Phi(x) := \eta(|x|)v_0(x)$ with $\eta \in C^0_0((0,1))$ and $\eta \geq 0$ to deduce that
\[
0 = \int_{B_t} |\nabla v_0|^{p-2}(\nabla v_0)^2 - \Gamma_{ij}^l \nabla v_0^i \cdot \nabla v_0^j \eta dx
\]
\[
\geq a_\alpha(2R) \int_{B_t} |\nabla v_0|^{p-2} h_{ij} \nabla v_0^i \cdot \nabla v_0^j dx
\]
\[
\geq ca_\alpha(2R) \int_{B_t} |\nabla v_0|^p dx,
\]
where in the first equality we have used the fact that
\[
\sum_{\beta} \frac{x}{|x|} D_\beta v_0^i = 0 \quad \text{almost everywhere for all } i
\]
as $v_0$ is homogenous of degree 0. Therefore, $\nabla v_0 = 0$ on $B_t$. Sending $t$ to infinite, we conclude that $\nabla v_0 = 0$ on $\mathbb{R}^l$. Now, using the monotonicity inequality again, we have for any $t \in (0, \infty)$
\[
t^{p-l} \int_{B_t} |\nabla u|^p dx \leq (kt)^{p-l} \int_{B_{kt}} |\nabla u|^p dx = t^{p-l} \int_{B_t} |\nabla u_k|^p dx \to 0
\]
as $k \to \infty$. Thus $\nabla u = 0$ on $B_t$ and hence also on $\mathbb{R}^l$.

When $p > l$, the Liouville theorem follows directly from the Caccioppoli inequality (3.8):
\[
\int_{B_t} |\nabla u|^p dx \leq c_0 t^{-p} \int_{B_{2t}} |u - \bar{u}|^p dz \leq R^p c_1 t^{-p+l} \to 0
\]
as $t \to \infty$. Thus $\nabla u = 0$ on $\mathbb{R}^l$. This completes our proof.

□

Remark 3.4. 1). The compactness of (minimizing) $p$-harmonic mappings is not really necessary in the proof above. Indeed, since by Remark 3.3 3), $v_k$ has locally uniformly

\[\text{In fact, it was proved there that if a sequence of } p\text{-harmonic mappings } u_i \text{ converges weakly in } W^{1,p} \text{ to some mapping } u, \text{ then the convergence is strong and } u \text{ is a } p\text{-harmonic mapping as well.}\]
bounded $W^{1,q}$-norm for some $q > p$, we may (use the Vitali’s convergence theorem) directly conclude that a subsequence $v_{k_j}$ of $v_k$ converges in strongly in $W^{1,p'}_{\text{loc}}$ to $v_0$ for any $p' < q$. Then the remaining arguments are identical.

2). It would be interesting to know whether in the setting of Theorem 1.3, each weakly $p$-harmonic mapping $u: \Omega \to N$ is continuous. For $p = 2$, this is the well-known result of Hildebrandt, Kaul and Widman [27], and for $p = n$, this follows immediately from the reverse Hölder inequality (3.9).

4. Quantitative gradient estimates for stationary $p$-harmonic mappings

In this section we assume $M$ has nonnegative Ricci curvature and $N$ has nonpositive sectional curvature. Recall that the Bochner-Weitzenböck formula for $C^3$-smooth maps $u: M \to N$ reads as follows (see for instance [37, Lemma 1]):

\[
\frac{1}{2} \Delta(|du|^{2(p-1)}) = \langle \Delta(|du|^{p-2}du), |du|^{p-2}du \rangle + \|\nabla(|du|^{p-2}du)\|^2 + |du|^{2(p-2)}R(du),
\]

where the reminder term

\[
R(du) = \sum_i \langle \text{Ric}(du(e_i)), du(e_i) \rangle - \sum_{i,j} \langle \text{R}^{N}(du(e_i), du(e_j)) du(e_i), du(e_j) \rangle
\]

and $\Delta = -(dd^* + d^*d)$ is the Hodge-Laplace operator. Note that

\[
\frac{1}{2} \Delta(|du|^{2(p-1)}) = |du|^{p-1} \Delta(|du|^{p-1}) + \|\nabla|du|^{p-1}\|^2 \\
\leq |du|^{p-1} \Delta(|du|^{p-1}) + \|\nabla(|du|^{p-2}du)\|^2.
\]

Thus it follows from (4.1) that

\[
|du|^{p-1} \Delta(|du|^{p-1}) \geq \langle \Delta(|du|^{p-2}du), |du|^{p-2}du \rangle + |du|^{2(p-2)}R(du).
\]

or equivalently,

\[
|du| \Delta(|du|^{p-1}) \geq \langle \Delta(|du|^{p-2}du), du \rangle + |du|^{(p-2)}R(du)
\]

Note that if $\text{Ric}_M \geq 0$ and $\text{R}^N \leq 0$, then $|du|^{2(p-2)}R(du) \geq 0$ and so (4.3) reduces to

\[
|du| \Delta(|du|^{p-1}) \geq \langle \Delta(|du|^{p-2}du), du \rangle
\]

We now turn to the proof of Theorem 1.4. Set $\Omega_+ = \{x \in \Omega : |\nabla u| > 0\}$. We claim that for any non-negative $\eta \in C^1_0(\Omega_+)$, we have

\[
\int_{\Omega} \eta \langle \Delta(|du|^{p-2}du), du \rangle d\mu = 0.
\]

Indeed, since $u$ is smooth $p$-harmonic in $\Omega_+$ and since $d^* \eta = 0$, we have

\[
\int_{\Omega} \eta \langle \Delta(|du|^{p-2}du), du \rangle d\mu = -\int_{\Omega} \eta((dd^* + d^*d)(|du|^{p-2}du), du) d\mu \\
= -\int_{\Omega} (d^*(\eta d(|du|^{p-2}du)), du) d\mu \\
= -\int_{\Omega} \eta d(|du|^{p-2}du), d(du)) d\mu = 0.
\]
Using a simple approximation argument, we may extend (4.5) to all non-negative \( \eta \in W^{1,2}_0(\Omega_+) \). Now we may divide \( |du| \) on both side of (4.4) to obtain that
\[
\Delta(|du|^{p-1}) \geq |du|^{-1}\langle \Delta(|du|^{p-2}du), du \rangle.
\]

We next observe that \( |du|^{-1} \in W^{1,2}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega_+) \). Indeed, for \( p \geq 2 \), this follows directly from Duzaar and Fuchs [5, Page 391, -4 line], and for \( p \in (1,2) \), it follows from Proposition A.1 below. Now for any non-negative \( \eta \in C^0_0(\Omega_+) \), we have \( |du|^{-1}\eta \in W^{1,2}_0(\Omega_+) \) and so it follows from (4.5) that
\[
\int_\Omega \Delta_g(|\nabla u|^{p-1})\eta dx \geq 0,
\]
where \( \Delta_g \) is the standard Laplace-Beltrami operator on \( M \). By [5, Lemma 2.4], (4.6) holds for all non-negative functions \( \eta \in C^0_0(\Omega) \).

This implies that \( |\nabla u|^{p-1} \) is a subharmonic function on \( M \) and so the standard theory for elliptic PDEs implies that there exists a positive constant \( C \), depending only on \( n \), such that
\[
\sup_{B_r}|\nabla u|^{p-1} \leq C\int_{B_{2r}}|\nabla u|^{p-1}d\mu.
\]
The desired inequality (1.3) follows by applying the Hölder’s inequality. This completes the proof of Theorem 1.4.

5. Concluding remarks

In Theorem 1.1, we have assumed that \( u(M) \) is contained in a compact subset of \( N \) and this assumption was used only in Lemma 2.1. This extra assumption can be dropped by a standard approximation argument if \( W^{1,p}(M,N) \cap C^\infty(M,N) \) is dense in \( W^{1,p}(M,N) \) (or actually even the under weaker density condition \( W^{1,p}(M,N) \cap L^\infty(M,N) \) is dense in \( W^{1,p}(M,N) \)). This technical issue appears here because of the definition of Sobolev spaces and the choice of density for Sobolev mappings.

Let us recall the following definition of Sobolev spaces from [3]. A mapping \( u: M \to N \) is said to be colocally weakly differentiable if \( u \) is measurable and \( f \circ u \) is weakly differentiable for every smooth compactly supported function \( f \in C^0_0(N,\mathbb{R}) \). For a colocally weakly differentiable mapping \( u: M \to N \), a mapping \( Du: TM \to TN \) is a colocal weak derivative of \( u \) if \( Du \) is a measurable bundle morphism that covers \( u \) and
\[
D(f \circ u) = Df \circ Du
\]
holds almost everywhere in \( M \) for every \( f \in C^0_0(N,\mathbb{R}) \). A mapping \( u: M \to N \) belongs to the Sobolev space \( W^{1,p}_{cs}(M,N) \) if \( u \in L^p(M,N) \) is colocally weakly differentiable and the norm of the colocal weak differential \( |Du|_{g^*_M \otimes g_N} \in L^p(M) \).

In many aspects, colocal weak derivatives behave as nicely as weak derivatives of mappings between Euclidean spaces. In particular, for a \( C^1 \)-smooth mapping \( u: M \to N \), the colocal weak derivative coincides with the classical weak derivative almost everywhere. Moreover, one can show that the Sobolev space \( W^{1,p}_{cs}(M,N) \) is equivalent to the Sobolev space \( W^{1,p}(M,N) \) defined as in Section 1.2; see [3, Proposition 2.6]. Thus we can develop a theory for \( p \)-harmonic mappings based on the colocal weak derivative as \( Du \) (and thus \( |Du| \)) is well-defined. In this case, one would expect Lemma 2.1 holds with \( Du \) in
place of \( \nabla u \) as \( f = d^2(x, Q) \) is Lipschitz on \( N \) and \( f \circ u \) would be weakly differentiable; see [3, Proposition 2.1]. Moreover, Lemma 2.3 and Lemma 2.4 remain valid as only nice computation law for “derivatives” are needed.

For simplicity of our exposition, we did not consider this issue in the current paper, but we will present all the details in a forthcoming work, together with extensions of \( p \)-harmonic mappings into general metric spaces with non-positively curved spaces in the sense of Alexandrov (NPC). Let us comment that in case of metric-valued mappings, there are many equivalent definitions of Sobolev spaces (see the monograph [25] for a comprehensive presentation). However, the energy densities corresponding to different Sobolev spaces are only mutually comparable and thus the technique for the regularity theory of minimizing \( p \)-harmonic mapping could be in principle very different, except for the case \( p = n \) (see [20]). The advantage of our approach for Theorem 1.1 is that they are quite general and can be applied to establish a regularity theory of minimizing \( p \)-harmonic mappings into NPC spaces. Note however that one can not directly generalize the approach of Korevaar and Schoen [30] or Zhang and Zhu [50] to the \( p \)-harmonic case as Lemma 2.3 fails for the energy density of Korevaar and Schoen.

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Appendix A. \( W^{2,2} \) regularity and removable singularities of \( p \)-harmonic mappings: \( 1 < p < 2 \)

To derive the boundedness of gradients of \( C^1 \)-smooth weakly \( p \)-harmonic mappings (which is needed in Section 4), we need a \( W^{2,2} \) regularity estimate. In the case \( p \geq 2 \), this type of result has been established by Duzaar and Fuchs [5]. We believe the corresponding results, for the case \( 1 < p < 2 \), are also well-known among specialists in the field. But, since we do not find a precise reference for such a result, we decide to include a sketch of proof below. We will apply the method of Acerbi and Fusco [1], where, among other results, \( W^{2,2} \) regularity estimates for \( p \)-harmonic mappings \( (1 < p < 2) \) between Euclidean spaces were established.

From now on, we stick to the assumption \( 1 < p < 2 \). Let \( \Omega \subset \mathbb{R}^n \) be an open set and \( N \) a smooth Riemannian manifold that is isometrically embedded in some Euclidean space \( \mathbb{R}^k \) with \( k \in \mathbb{N} \). Let \( u : \Omega \to N \) be a \( C^1 \)-smooth weakly \( p \)-harmonic mapping, that is, \( u \) satisfies the \( p \)-Laplace equation

\[
\int_{\Omega} |\nabla u|^p - 2 \nabla u \cdot \nabla \varphi + \int_{\Omega} |\nabla u|^{p-2} A(u)(\nabla \alpha u, \nabla \alpha u) \cdot \varphi = 0, \quad \forall \varphi \in C^1_0(\Omega, \mathbb{R}^k), \tag{A.1}
\]

where \( A(q)(\cdot, \cdot) : T_q N \times T_q N \to (T_q N)^\perp \) is the second fundamental form of \( N \) at \( q \in N \). Note that Einstein summation convention over \( \alpha \) from 1 to \( n \) is applied above. We further
assume that $N$ satisfies the curvature assumptions (1.4) and (1.5) of $N$ prescribed in Duzaar and Fuchs [5].

$W^{2,2}$ regularity of $p$-harmonic mappings for $1 < p < 2$. In this section, we will establish an interior $W^{2,2}$ regularity estimate of $u$, and then in the next section, extend the main result of Duzaar and Fuchs [5] with a sketch of proof.

We will use the following elementary inequality, which is a consequence of Lemma 2.2 of Acerbi and Fusco [1]: for any $l \geq 1$, there exists a constant $c = c(l, p) > 0$ such that for any $a, b \in \mathbb{R}^l$,

$$c \frac{|a - b|^2}{(|a|^2 + |b|^2)^{\frac{2-p}{2}}} \geq |a|^{p-2}a - |b|^{p-2}b, a - b) \geq (p - 1) \frac{|a - b|^2}{(|a|^2 + |b|^2)^{\frac{2-p}{2}}}, \quad (A.2)$$

The main result of this section reads as follows.

**Proposition A.1.** Each mapping $u \in W^{1,p}(\Omega, N) \cap C^1(\Omega, N)$ that satisfies (A.1) belongs to $W^{2,2}_{\text{loc}}(\Omega_+, N)$, where $\Omega_+ = \{x \in \Omega : |\nabla u(x)| > 0\}$. Moreover, there exists a constant $C > 0$ depending only on $n, k, p$ and the curvature assumptions on $N$, such that for any $B_r \subset \subset \Omega$, it holds

$$\int_{B_{r/2}} |\nabla^2 u|^2 \leq C (r^2 + M_r^2) M_r^{2-p} \int_{B_r} |\nabla u|^p,$$

where $M_r = \sup_{B_r} |\nabla u|$.

**Proof.** Let $B_r \subset \subset \Omega_+$ and $h > 0$ be sufficiently small. For fixed $1 \leq \beta \leq n$, we denote

$$\Delta_h f(x) = \frac{1}{h}(f(x + he_\beta) - f(x))$$

and set

$$V = |\nabla u|^{\frac{p-2}{2}} \nabla u.$$

By (A.1), we have

$$\int_\Omega \Delta_h (|\nabla u|^{p-2} \nabla_\alpha u) \cdot \nabla_\alpha \varphi = - \int_\Omega \Delta_h (A(u)(V, V)) \cdot \varphi \quad (A.3)$$

for any $\varphi \in C_0^1(\Omega, \mathbb{R}^k)$. It is easy to see that the above equation holds for all $\varphi \in W^{1,p}_0 \cap L^\infty(\Omega_+, \mathbb{R}^k)$ as well. Substitute $\varphi = \eta^2 \Delta_h u$ into (A.3) for $\eta \in C_0^2(\Omega_+)$ and we obtain

$$\int_\Omega \Delta_h (|\nabla u|^{p-2} \nabla_\alpha u) \cdot \nabla_\alpha \varphi = \int \eta^2 \Delta_h (|\nabla u|^{p-2} \nabla_\alpha u) \cdot \Delta_h \nabla_\alpha u + \int 2\eta \Delta_h (|\nabla u|^{p-2} \nabla_\alpha u) \cdot \Delta_h \nabla_\alpha \eta.$$

Applying (A.2), we deduce

$$\int \eta^2 \Delta_h (|\nabla u|^{p-2} \nabla_\alpha u) \cdot \Delta_h \nabla_\alpha u \geq c \int \eta^2 (|\nabla u(x)| + |\nabla u(x + he_\beta)|)^{p-2} |\Delta_h \nabla u|^2$$
and
\[
\left| \int 2\eta \Delta_h \left( |\nabla u|^{p-2} \nabla \alpha u \right) \cdot \Delta_h u \nabla \alpha \eta \right|
\leq c' \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h \nabla u|^2 \, |\Delta_h u| |\nabla \eta| \, |\nabla \eta|
\]
for some constants \( c, c' > 0 \) depending only on \( p \). Combining it with Young’s inequality gives us
\[
\int \Delta_h \left( |\nabla u|^{p-2} \nabla \alpha u \right) \cdot \nabla \alpha \varphi \geq c_1 \int \eta^2 \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h \nabla u|^2
\]
\[
- c_2 \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h u|^2 |\nabla \eta|^2
\]
for some constants \( c_1, c_2 > 0 \) depending only on \( p \) and \( k \).

On the other hand, by estimate (2.6) of Duzaar and Fuchs [5], we have
\[
\left| \int \Delta_h \left( A(u)(V, V) \right) \cdot \eta^2 \Delta_h u \right| \leq \frac{c_1}{2} \int \eta^2 \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h \nabla u|^2
\]
\[
+ c \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^p |\Delta_h u|^2 \eta^2
\]
for some \( c > 0 \) depending only on \( n, p, k \) and the curvature assumptions on \( N \). Hence, combining (A.3), (A.4) and (A.5) yields
\[
\int \eta^2 \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h \nabla u|^2
\]
\[
\leq c \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h u|^2 |\nabla \eta|^2
\]
\[
+ c \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^p |\Delta_h u|^2 \eta^2.
\]

Now choose \( \eta \in C_0^\infty(B_{3r/4}) \) such that \( \eta \equiv 1 \) on \( B_{r/2} \) and \( |\nabla \eta| \leq 8/r \) and \( |\nabla^2 \eta| \leq 8/r^2 \). Recall that \( u \in C^1(\Omega, N) \) and we obtain from the above that
\[
\int_{B_{r/2}} |\Delta_h \nabla u|^2 \leq cM_r^{2-p} \int \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^{p-2} |\Delta_h u|^2
\]
\[
+ cM_r^{2-p} M_r^2 \int_{B_{3r/4}} \left( |\nabla u(x)| + |\nabla (u + he_\beta)| \right)^p.
\]

Letting \( h \rightarrow 0 \) yields \( \nabla^2 u \in L^2(B_{r/2}) \) and the desired estimate. The proof is complete. \( \Box \)

**Removable singularities of \( p \)-harmonic mappings for \( 1 < p < 2 \).** We next point out that the main result of Duzaar and Fuchs [5, Theorem, page 386] holds for the case \( 1 < p < 2 \) as well.

**Theorem A.2.** Let \( n \geq 2 \) and \( 1 < p < 2 \). Suppose \( u \in C^1(B_1 \setminus \{0\}, N) \cap W^{1,p}(B_1, N) \) is a weakly \( p \)-harmonic mapping (that is, \( u \) solves (A.1)). Then, there exists a constant \( \epsilon_0 > 0 \) depending only on \( n, k, p \) and the geometry of \( N \), such that if the \( p \)-energy of \( u \) satisfies
\[
E_p(u) \equiv \int_{B_1} |\nabla u|^p \leq \epsilon_0,
\]
then $u \in C^{1,\gamma}(B_1, N)$ for some $\gamma \in (0, 1)$. Moreover, the Hölder exponent $\gamma$ depends only on $n, k, p$ and the geometry of $N$.

The proof of Theorem A.2 follows closely the arguments of Duzaar and Fuchs [5] with any minor modifications. In below, we list the main ingredients and point out the corresponding modifications.

The first ingredient is the following quantitative gradient estimates for $p$-harmonic mappings, which extend Theorem 2.1 of Duzaar and Fuchs [5] to the case $1 < p < 2$.

**Proposition A.3.** Let $1 < p < 2$. Assume that $u \in C^1(B_r, N)$ is a weakly $p$-harmonic mapping. Then, there exist constants $\epsilon_1, C_1 > 0$ depending only on $n, k, p$ and the geometry of $N$, such that if $r^{p-n} \int_{B_r} |\nabla u|^p \leq \epsilon_1$, then

$$\sup_{B_{r/2}} |\nabla u|^p \leq C_1 \int_{B_r} |\nabla u|^p.$$  

In the case $p \geq 2$, the above result is Theorem 2.1 of Duzaar and Fuchs [5]. The key ingredient in the proof of Theorem 2.1 is to derive $W^{2,2}$ type regularity estimates for $p$-harmonic mappings; see Lemma 2.2 of Duzaar and Fuchs [5]. In our case, one can easily check that, with the $W^{2,2}$ regularity estimates (Proposition A.1) at hand, the rest arguments of Duzaar and Fuchs [5] can be applied without changes.

The second ingredient is the following proposition, which extend Proposition 3.1 of Duzaar and Fuchs [5] to the case $1 < p < 2$.

**Proposition A.4.** There exist constants $\epsilon_0 > 0$, $\sigma \in (0, 1)$, depending only on $n, k, p$ and the curvature assumptions of $N$, such that for any weakly $p$-harmonic mapping $u \in C^1(B_1 \setminus \{0\}, N) \cap W^{1,p}(B_1, \mathbb{R}^k)$ with $\int_{B_1} |\nabla u|^p \leq \epsilon_0$, it holds

$$\sigma^{p-n} E(\sigma) \leq \frac{1}{2} E(1),$$

where we used the notation $E(r) = \int_{B_r} |\nabla u|^p$.

To establish this result for $1 < p < 2$, we only need to show that similar estimates as equations (3.5) and (3.10) of Duzaar and Fuchs [5] holds for the case $1 < p < 2$ as well.

Let $\{v_i\}$ be defined as that of [5, Page 397]. Then, by the same arguments as that of [5], we have

$$\int_{B_{1/2} \setminus B_r} \left( |\nabla v_i|^{p-2} \nabla \alpha v_i - |\nabla v_j|^{p-2} \nabla \alpha v_j \right) \cdot (\nabla \alpha v_i - \nabla \alpha v_j) \eta^p \to 0$$

as $i, j \to \infty$. Then, (A.2) implies $3$

$$\int_{B_{1/2} \setminus B_r} (|\nabla v_j|^2 + |\nabla v_i|^2)^{p-2} |\nabla v_i - \nabla v_j|^2 \eta^p \to 0,$$

as $i, j \to \infty$.

$3$Note that there is a print error in (3.5) of Duzaar and Fuchs [5]: the first symbol $\infty$ in (3.5) should be 0.
from which we deduce that \( v_i \to v_\infty \) strongly in \( W^{1,p}(B_{1/2} \setminus B_r) \) for some \( v_\infty \) in \( W^{1,p}(B_{1/2} \setminus B_r) \), in view of the following Hölder inequality

\[
\int |\nabla v_i - \nabla v_j|^p \eta^p \leq \left( \int \frac{|\nabla v_i - \nabla v_j|^2}{(|\nabla v_j|^2 + |\nabla v_i|^2)^{\frac{p-2}{2}}} \eta^p \right)^{\frac{p}{2}} \left( \int \left( |\nabla v_j|^2 + |\nabla v_i|^2 \right)^{\frac{p}{2}} \eta^p \right)^{\frac{2}{p}}
\]

and the fact that \( \{v_j\} \) is uniformly bounded in \( W^{1,p}(B_1, \mathbb{R}^k) \). Hence (3.5) of Duzaar and Fuchs [5] holds for \( 1 < p < 2 \) as well.

As to the estimate (3.10) of Duzaar and Fuchs [5], it has been established in the case \( 1 < p < 2 \) by Acerbi and Fusco [1], Proposition 2.7.

The rest arguments of Duzaar and Fuchs [5] remains valid for \( 1 < p < 2 \), and so Proposition A.4 holds.

With Propositions A.3 and A.4 at hand, Theorem A.2 follows by the same arguments as that of Duzaar and Fuchs [5] and so we omit the details.

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