THE COHOMOLOGICAL INDEX OF FREE $\mathbb{Z}/p$-ACTIONS IS NOT ADDITIVE WITH RESPECT TO JOIN

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Abstract. Let $p$ denote an odd prime. We show by example that the inequalities obtained in [GKPS] for the behaviour of the cohomological index of a join of free $\mathbb{Z}/p$-actions are sharp. Namely, for all odd integers $k, l$ at least one of which is greater than one, we give examples of finite free $\mathbb{Z}/p$-CW complexes of cohomological indices $k$ and $l$ whose join has index $k + l + 1$ and also examples where the join has index $k + l - 1$.

1. Introduction

Let $p$ be an odd prime, $\tilde{X}$ be a free $\mathbb{Z}/p$-CW complex and $X = \tilde{X}/\mathbb{Z}/p$ be its orbit space. Let $f: X \to B\mathbb{Z}/p$ be a map classifying the covering $\tilde{X} \to X$. The cohomological index of the free $\mathbb{Z}/p$-action is the dimension of the $\mathbb{F}_p$-vector space $\text{Im} f_* \subset H^*(B\mathbb{Z}/p; \mathbb{F}_p)$. We will use the notation $\text{ind}(X)$ for the cohomological index so as not to clash with the notation in [GKPS]. It will always be clear from the context which free action having $X$ as an orbit space we are considering.

Using the product structure on $H^*(B\mathbb{Z}/p)$ and the action of the Bockstein operation, it is easy to check that $\text{Im} f_*$ is of the form $\oplus_{i=0}^{k-1} H_i(B\mathbb{Z}/p; \mathbb{F}_p)$ for some $0 \leq k \leq \infty$ (i.e. there are no "gaps" in the image of $f_*$). Hence one may also define $\text{ind}(X)$ as the least possible degree of a non-zero class in $\text{ker} f^*: H^*(B\mathbb{Z}/p) \to H^*(X)$.

Cohomological indices of this type for free $G$-spaces were introduced and studied by Fadell and Rabinowitz [FR]. Hara and Kishimoto [HK] call $\text{ind}(X) - 1$ the height of the covering $\tilde{X} \to X$. The height is a lower bound for the category (in the sense of Lusternik-Schnirrelman) of the classifying map $f: X \to B\mathbb{Z}/p$ (see [HK, Section 3]).

Let $\tilde{X}$ and $\tilde{Y}$ be free $\mathbb{Z}/p$-CW complexes. The join $\tilde{X} \ast \tilde{Y}$ has a natural diagonal $\mathbb{Z}/p$-action and we write $X \ast_p Y = (\tilde{X} \ast \tilde{Y})/\mathbb{Z}/p$ for the orbit space. This paper is concerned with the relation between $\text{ind}(X \ast_p Y)$ and the cohomological indices of $X$ and $Y$.

The following is known regarding this question.

Theorem 1.1. [GKPS, Proposition 3.9] Let $\tilde{X}$ and $\tilde{Y}$ be free $\mathbb{Z}/p$-CW complexes. Then

\[ \text{ind}(X \ast_p Y) = \text{ind}(X) + \text{ind}(Y) \quad \text{when either } \text{ind}(X) \text{ or } \text{ind}(Y) \text{ are even} \]

When $\text{ind}(X)$ and $\text{ind}(Y)$ are both odd, the following inequalities hold

\[ \text{ind}(X) + \text{ind}(Y) - 1 \leq \text{ind}(X \ast_p Y) \leq \text{ind}(X) + \text{ind}(Y) + 1. \]

The paper [GKPS] dealt only with free $\mathbb{Z}/p$-actions which can be embedded in an odd sphere with a free linear $\mathbb{Z}/p$-action but it is easy to see that no loss of generality results from making that assumption.
The purpose of this note is to show that the inequalities in the previous statement are sharp by giving examples of \( \mathbb{Z}/p \)-free CW complexes realizing the upper and lower bounds for any given pair of odd \( \text{ind}(X) \) and \( \text{ind}(Y) \) not both equal to one. The case where both indices are one is trivial (and additivity holds).

Our motivation comes from contact topology. In [GKPS], following Givental [Gi], the cohomological index was used to construct an integer valued quasi-morphism on the universal cover of the contactomorphism group of a lens space (with its standard contact structure). If the cohomological index were exactly additive, the arguments in that paper would prove Sandon’s contact version of the Arnold conjecture for lens spaces. Unfortunately additivity does not hold, so we are not able to improve on the lower bounds in [GKPS] for the number of translated points of a contactomorphism of a lens space which is contact isotopic to the identity.

1.2. Notation and conventions. Homology and cohomology is with \( \mathbb{F}_p \) coefficients for \( p \) an odd prime, unless otherwise noted. We will generally denote free \( \mathbb{Z}/p \)-CW complexes by letters decorated with a tilde and remove the tilde to denote their orbit spaces. We will write \( \{\ast\} \) for a one point space, the orbit space of \( \mathbb{Z}/p \).

We write \( R = \mathbb{F}_p[\mathbb{Z}/p] \) for the group algebra of \( \mathbb{Z}/p \), \( g \) for the canonical generator of \( \mathbb{Z}/p \) and \( \tau = g^0 - g \in R \) so that \( R = \mathbb{F}_p[\tau]/\tau^p \). The indecomposable finitely generated \( R \)-modules are the submodules \( \tau^i R \subset R \) for \( i = 0, \ldots, p - 1 \). Any finitely generated \( R \)-module is isomorphic to a direct sum of indecomposable submodules.

\( C_*(\tilde{X}) \) denotes the cellular chain complex of the free \( \mathbb{Z}/p \)-CW complex \( \tilde{X} \) with \( \mathbb{F}_p \) coefficients. This is a chain complex of free \( R \)-modules. \( C_*(X) \) denotes the cellular chain complex of the orbit space also with \( \mathbb{F}_p \)-coefficients. Note that \( C_*(X) = C_*(\tilde{X}) \otimes_R \mathbb{F}_p \) where \( \mathbb{F}_p \) is regarded as an \( R \)-algebra via the augmentation (which sends \( \tau \) to 0).

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2. Generalities on the homology of a join

If \( X \) and \( Y \) are CW complexes, we give \( X \ast Y = (X \times [0, 1] \times Y)/\sim \) its natural CW structure. This has \( X \) and \( Y \) as subcomplexes, while the remaining cells are of the form \( e \ast f \) for \( e \) a cell of \( X \) and \( f \) a cell of \( Y \). If \( e \) is an \( m \)-cell and \( f \) an \( n \)-cell, the boundary map of the generator \( e \ast f \) in \( C_*(X \ast Y) \) is given by the expression

\[
\partial(e \ast f) = \begin{cases} 
  f - e & \text{if } m = n = 0 \\
  f - e \ast (\partial f) & \text{if } m = 0, n > 0 \\
  \partial(e) \ast f + (-1)^{m+1}e & \text{if } m > 0, n = 0 \\
  \partial(e) \ast f + (-1)^{m+1}e \ast \partial(f) & \text{otherwise.}
\end{cases}
\]

When \( M \) and \( N \) are \( R \)-modules, the vector space \( M \otimes_{\mathbb{F}_p} N \) has a natural diagonal \( \mathbb{Z}/p \)-action and therefore becomes an \( R \)-module. In terms of the presentation of \( R \) as \( \mathbb{F}_p[\tau]/\tau^p \), the module structure on a decomposable element of \( M \otimes_{\mathbb{F}_p} N \) is described by

\[
\tau(a \otimes b) = (\tau a) \otimes b + a \otimes (\tau b) - (\tau a) \otimes (\tau b)
\]
Consequently, for $\tilde{X}$ and $\tilde{Y}$ free $\mathbb{Z}/p$-CW complexes, the action of $\tau$ on an element of the form $e * f \in C_{s}(\tilde{X} \ast \tilde{Y})$ is given by
\[ \tau(e * f) = (\tau e) * f + e * (\tau f) - (\tau e) * (\tau f). \]

2.1. Homology of $X \ast_p \{\ast\}$. Let $\tilde{X}$ be a free $\mathbb{Z}/p$-CW complex. The join $\tilde{X} \ast \mathbb{Z}/p$ is a union of $p$ cones on a common base $\tilde{X}$, hence
\[ \tilde{X} \ast \mathbb{Z}/p \simeq \vee_{i=1}^{p-1} \Sigma \tilde{X}. \]

**Lemma 2.2.** Let $\tilde{X}$ be a free $\mathbb{Z}/p$-CW complex. For each $k \geq 1$ we have the following isomorphism of $R$-modules
\[ H_{k+1}(\tilde{X} \ast \mathbb{Z}/p) \cong H_{k}(\tilde{X}) \otimes_{\mathbb{F}_p} \tau R \]
(where the tensor product is given the $R$-module structure described by (1)).

**Proof.** Given a cellular cycle $z \in C_{k}(\tilde{X})$, the isomorphism
\[ H_{k}(\tilde{X}) \otimes_{\mathbb{F}_p} \tau R \to H_{k+1}(\tilde{X} \ast \mathbb{Z}/p) \]
sends $[z] \otimes \tau$ to the homology class $[z * g^0 - z * g] \in H_{k+1}(\tilde{X} \ast \mathbb{Z}/p)$. \qed

For later use we record the following computation which follows easily from (1) (see for instance [Be, Section 2.3]).

**Lemma 2.3.** For the $R$-module structure on the tensor product described by (1) we have
\[ \tau^{p-1} R \otimes_{\mathbb{F}_p} \tau R \cong \tau R, \quad \tau^{p-2} R \otimes_{\mathbb{F}_p} \tau R \cong \tau \tau R \oplus R \]

The following observation follows immediately from the definitions.

**Lemma 2.4.** Let $\tilde{X}$ be a free $\mathbb{Z}/p$-CW complex. There is a cofiber sequence
\[ \tilde{X} \to X \to X \ast_p \{\ast\} \]
and hence a short exact sequence
\[ 0 \to \text{coker} q_{s} \to \tilde{H}_{s}(X \ast_p \{\ast\}) \to \Sigma \ker q_{s} \to 0 \]

3. Counterexamples to additivity of the index

We consider the standard model for the universal principal $\mathbb{Z}/p$ bundle
\[ S^{\infty} \to L_{p}^{\infty} \]
where $\mathbb{Z}/p$ acts diagonally on $S^{\infty} = \text{colim} \ S^{2n+1}$. The infinite lens space $L_{p}^{\infty}$ has a standard cell decomposition (described for instance in [Ha, Example 2.43]) with one cell in each dimension. This in turn gives rise to a $\mathbb{Z}/p$-free cell decomposition of $S^{\infty} = L_{p}^{\infty} = E\mathbb{Z}/p$ whose cellular chain complex $C_{s}(L_{p}^{\infty})$ is the standard (periodic) resolution of $\mathbb{F}_p$ over $R$:
\[ \cdots \to R \overset{\tau^{p-1}}{\to} R \overset{\tau}{\to} R \overset{\tau^{p-1}}{\to} R \overset{\tau}{\to} R \]
We write $\tilde{L}_p^k$ for the $k$-skeleton of this decomposition and note that
\[ \tilde{L}_p^k = \begin{cases} S^k & \text{if } k \text{ is odd} \\ S^{k-1} \ast \mathbb{Z}/p & \text{if } k \text{ is even} \end{cases} \]
Proposition 3.4. Let sequence differential $d$ ind \((\text{is trivial if and only if}) \) Lemma 3.3. Let Definition 3.2. \(\pi\) momorphism sequence where ˜ \(\text{kernel of the augmentation homomorphism}\) \(Z\) □ the statement follows. \(Z\) Since, again by Theorem 1.1, we have ind \((\text{for the index of})\) \(X\) It follows from Theorem 1.1 that \(X\) have \(\text{Since the join is associative and commutative up to natural homeomorphism we}\) \(\text{Proof.}\) \(Z\) \(\text{Lemma 3.1.}\) Let \(k \geq 1\) and \(l \geq 0\) be integers. Let \(\tilde{X}\) be a free \(\mathbb{Z}/p\)-CW complex and let \(Z = X \ast_p L_p^{2k-1}\). Then additivity holds for the index of \(Z \ast_p L_p^{2l}\) if and only if it holds for the index of \(X \ast_p \{\ast\}\). \[L_p^k \ast_p L_p^l \cong \begin{cases} L_p^{k+l+1} & \text{if } k \text{ or } l \text{ are odd} \\ L_p^{k+l-1} \ast_p \{\ast\} & \text{if both } k \text{ and } l \text{ are even} \end{cases} \]
and one checks easily that \(\text{ind}(\{\ast\} \ast_p \{\ast\}) = 2\).

We will now give examples in which the index is not additive. We first note that in order to produce examples in all possible cases (i.e. for arbitrary odd indices of the factors, not both equal to one), it suffices to give examples of spaces \(X\) with index 3 such that additivity does not hold for \(X \ast_p \{\ast\}\):

**Lemma 3.1.** Let \(k \geq 1\) and \(l \geq 0\) be integers. Let \(\tilde{X}\) be a free \(\mathbb{Z}/p\)-CW complex and let \(Z = X \ast_p L_p^{2k-1}\). Then additivity holds for the index of \(Z \ast_p L_p^{2l}\) if and only if it holds for the index of \(X \ast_p \{\ast\}\).

**Proof.** Since the join is associative and commutative up to natural homeomorphism we have
\[Z \ast_p L_p^{2l} \cong X \ast_p L_p^{2k-1} \ast_p L_p^{2l-1} \ast_p \{\ast\} \cong (X \ast_p \{\ast\}) \ast_p L_p^{2k+2l-1} \]
It follows from Theorem 1.1 that
\[\text{ind}(Z \ast_p L_p^{2l}) = \text{ind}(X \ast_p \{\ast\}) + 2k + 2l\]
Since, again by Theorem 1.1 we have \(\text{ind}(Z) = \text{ind}(X) + 2k\) and \(\text{ind}(L_p^{2l}) = \text{ind}(\{\ast\}) + 2l\) the statement follows.

Recall that \(\tilde{L}^2 = S^1 \ast \mathbb{Z}/p \cong \bigvee_{i=1}^{p-1} S^2\). As a \(\mathbb{Z}[\mathbb{Z}/p]\)-module, \(H_2(\tilde{L}^2; \mathbb{Z})\) is naturally the kernel of the augmentation homomorphism \(\mathbb{Z}[\mathbb{Z}/p] \to \mathbb{Z}\).

**Definition 3.2.** Let \(\tilde{U}\) be the free \(\mathbb{Z}/p\) cell complex obtained from \(\tilde{L}^2\) by attaching a free \(\mathbb{Z}/p\) 2-cell by a map \(S^2 \xrightarrow{a} \tilde{L}^2\) with \(a_*([S^2]) = g^0 - 2g + g^2 \in H_2(\tilde{L}^2; \mathbb{Z})\).

Note that the attaching map \(a\) in the previous definition exists as the Hurewicz homomorphism \(\pi_2(\tilde{L}^2) \to H_2(\tilde{L}^2; \mathbb{Z})\) is an isomorphism. By definition, \(\tilde{U}\) sits in a cofiber sequence
\[\mathbb{Z}/p \times S^2 \xrightarrow{\tilde{a}} \tilde{L}^2 \to \tilde{U}\]
where \(\tilde{a}\) is the equivariant extension of the map \(a\).

We will make use of the following observation whose proof is left to the reader.

**Lemma 3.3.** Let \(\tilde{X}\) be a \((m - 2)\)-connected free \(\mathbb{Z}/p\)-cell complex with \(m \geq 2\). The differential \(d_m : E_{m,0}^m \to E_{0,m-1}^m\) in the Serre spectral sequence of the homotopy fiber sequence
\[\tilde{X} \to X \to B\mathbb{Z}/p\]
is trivial if and only if \(\text{ind}(X) \geq m + 1\).

**Proposition 3.4.** Let \(\tilde{U}\) be the space of Definition 3.2. Then \(\text{ind}(\tilde{U}) = 3\) and \(\text{ind}(U \ast_p \{\ast\}) = 5\).
Proof. Since \( \tau^2 = g^0 - 2g + g^2 \in R \), the chain complex \( C_\ast(\tilde{U}) \) is given by

\[
\cdots \to 0 \to R \xrightarrow{\tau^2} R \xrightarrow{\tau^{p-1}} R \xrightarrow{\tau} R \to \cdots
\]

and hence, as \( R \)-modules we have

\[
H_k(\tilde{U}) \cong \begin{cases} 
\tau^{p-1}R & \text{if } k = 0, 2 \\
\tau^{p-2}R & \text{if } k = 3 \\
0 & \text{otherwise.}
\end{cases}
\]

From Lemma 2.3 we conclude that

\[
(2) \quad H_k(U \ast \mathbb{Z}/p) \cong \begin{cases} 
\tau R & \text{if } k = 0, 3 \\
\tau^2 R \oplus R & \text{if } k = 4 \\
0 & \text{otherwise.}
\end{cases}
\]

On the other hand, \( C_\ast(U) = C_\ast(\tilde{U}) \otimes_R \mathbb{F}_p \) is the complex

\[
\cdots \to 0 \to \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p 
\]

hence the projection map \( q: \tilde{U} \to U \) induces the 0 homomorphism in positive degrees. Lemma 2.4 implies that

\[
\tilde{H}_k(U \ast_p \{\ast\}) = \begin{cases} 
\mathbb{F}_p & \text{if } k = 1, 2 \\
\mathbb{F}_p \oplus \mathbb{F}_p & \text{if } k = 3, 4 \\
0 & \text{otherwise.}
\end{cases}
\]

Now consider the Leray-Serre spectral sequence of the Borel construction on \( \tilde{U} \ast \mathbb{Z}/p \)

\[
(3) \quad \tilde{U} \ast \mathbb{Z}/p \xrightarrow{i} (\tilde{U} \ast \mathbb{Z}/p)_{h\mathbb{Z}/p} \to B\mathbb{Z}/p
\]

and recall that the natural projection \((\tilde{U} \ast \mathbb{Z}/p)_{h\mathbb{Z}/p} \xrightarrow{\pi} U \ast_p \{\ast\}\) is a homotopy equivalence making the following diagram commute:

\[
\begin{array}{ccc}
\tilde{U} \ast \mathbb{Z}/p & \xrightarrow{i} & (\tilde{U} \ast \mathbb{Z}/p)_{h\mathbb{Z}/p} \\
\downarrow \quad q & & \downarrow \pi \\
U \ast_p \{\ast\} & & B\mathbb{Z}/p
\end{array}
\]

We have

\[
E^2_{p,q} = H_p(B\mathbb{Z}/p; \{H_q(\tilde{U} \ast \mathbb{Z}/p)\}) = H_p(\mathbb{Z}/p; H_q(\tilde{U} \ast \mathbb{Z}/p)) \Rightarrow H_{p+q}(U \ast_p \{\ast\})
\]

In view of (2), we see that \( E^2_{p,q} = 0 \) for \( q = 1, 2 \) and \( E^2_{0,3} \cong \mathbb{F}_p \). As \( H_3(U \ast_p \{\ast\}) \cong \mathbb{F}_p \) it follows that the differential \( d^4: E^4_{1,0} \to E^4_{0,3} \) must be 0 and hence, by Lemma 3.3 we have \( \text{ind}(U \ast_p \{\ast\}) \geq 5 \). As \( \dim U \ast_p \{\ast\} = 4 \), it follows that \( \text{ind}(U \ast_p \{\ast\}) = 5 \).

We leave to the reader the similar, but easier, argument showing that \( \text{ind}(U) = 3 \). \( \square \)

Remark 3.5. Let \( k \geq 1 \), \( \alpha_k \) denote a generator of \( H^k(B\mathbb{Z}/p) \) and \( Y_k \) denote the homotopy fiber of \( B\mathbb{Z}/p \xrightarrow{\alpha_k} K(\mathbb{Z}/p, k) \). Then \( Y_k \) is the orbit space of a free \( \mathbb{Z}/p \)-action on a complex with the homotopy type of \( K(\mathbb{Z}/p, k-1) \). This action is a "versal" action of index \( k \) in the sense that, for any free \( \mathbb{Z}/p \)-CW complex \( \tilde{X} \) with \( \text{ind}(X) = k \), the classifying map \( X \to B\mathbb{Z}/p \) factors through \( Y_k \).
It was shown in [Go] that for \(k, l\) odd, not both equal to one, \(\text{ind}(Y_k \ast_p Y_l) = k + l + 2\). The space \(\tilde{U}\) of Definition 3.2 is the essential part of the 3-skeleton of a \(\mathbb{Z}/p\)-free cellular approximation of \(K(\mathbb{Z}/p, 2) = \tilde{Y}_3\).

**Lemma 3.6.** There exists a two dimensional free \(\mathbb{Z}/p\)-cell complex \(\tilde{V}\) such that \(C_*(\tilde{V})\) is isomorphic to the complex

\[
\cdots \to 0 \to R \xrightarrow{\tau p^{-1}} R \oplus R \xrightarrow{-\tau} R
\]

*Proof.* The 1-skeleton of \(\tilde{V}\) is obtained by attaching two 1-cells \(e_1, e_2\) to \(\tilde{V}_0 = \mathbb{Z}/p\) using the attaching map of the 1-cell in \(\tilde{L}_p^1 = S^1\). We define the isomorphism \(\psi: R \oplus R \to C_1(\tilde{V})\) by \(\psi(1, 0) = e_1\) and \(\psi(0, 1) = e_1 - e_2\).

Since \(\tilde{V}_1\) is homotopy equivalent to a wedge of circles, it is clear we can pick an attaching map \(S^1 \to \tilde{V}_1\) so as to realise the desired differential \(\partial_2\). □

**Proposition 3.7.** Let \(\tilde{V}\) be one of the spaces provided by Lemma 3.6. Then \(\text{ind}(\tilde{V}) = 3\) and \(\text{ind}(\tilde{V} \ast_p \{\ast\}) = 3\).

*Proof.* The proof is analogous to that of Proposition 3.4 so we just indicate the main steps. We have

\[
H_k(\tilde{V}) \cong \begin{cases}
\tau^{p-1}R & \text{if } k = 0, 2 \\
\tau^{p-2}R & \text{if } k = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

From Lemma 2.3 we conclude that

\[
H_k(\tilde{V} \ast \mathbb{Z}/p) \cong \begin{cases}
\tau^{p-1}R & \text{if } k = 0, 3 \\
\tau^2R \oplus R & \text{if } k = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

The complex \(C_*(\tilde{V})\) is isomorphic to

\[
\cdots \to 0 \to \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \oplus \mathbb{F}_p \xrightarrow{0} \mathbb{F}_p
\]

and Lemma 2.3 implies

\[
\tilde{H}_k(\tilde{V} \ast_p \{\ast\}) = \begin{cases}
\mathbb{F}_p & \text{if } k = 1, 3 \\
\mathbb{F}_p \oplus \mathbb{F}_p & \text{if } k = 2 \\
0 & \text{otherwise.}
\end{cases}
\]

In the Leray-Serre spectral sequence of

\[
\tilde{V} \ast \mathbb{Z}/p \to V \ast_p \{\ast\} \to B\mathbb{Z}/p
\]

there must be a differential \(d^3: E^3_{3, 0} \to E^3_{0, 2} \cong \mathbb{F}_p^2\). For otherwise we would have \(H_2(V \ast \mathbb{Z}/p) \cong \mathbb{F}_p^3\). It follows from Lemma 3.3 that \(\text{ind}(V \ast_p \mathbb{Z}/p) = 3\). We leave it to the reader to check that \(\text{ind}(V)\) is also 3. □

**Remark 3.8.** The complexes \(\tilde{V}\) were found in the following way: they are the simplest examples of a space of index \(k\) such that the generator of \(H_{k+1}(B\mathbb{Z}/p)\) does not transgress in the Serre spectral sequence of \(\tilde{V} \to V \to B\mathbb{Z}/p\). This can be shown using the cellular model for the Borel construction described in [AP].
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