Solution-giving formula to Cauchy problem for multidimensional parabolic equation with variable coefficients

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We present a general method of solving the Cauchy problem for multidimensional parabolic (diffusion type) equation with variable coefficients which depend on spatial variable but do not change over time. We assume the existence of the $C_0$-semigroup (this is a standard assumption in the evolution equations theory, which guarantees the existence of the solution) and then find the representation of the solution in terms of coefficients of the equation and initial condition.

Our main technical tool is the Chernoff theorem on $C_0$-semigroups. We recall theorem’s statement, provide it’s elementary version to show the core idea, explain how the theorem helps in obtaining solutions of evolution equations. Then we apply the Chernoff theorem to a specially constructed family of translation operators and obtain a sequence of functions that converge uniformly to the exact solution of the equation studied. We also represent the solution as a Feynman formula (i.e. as a limit of a multiple integral with multiplicity tending to infinity) with generalized functions appearing in the integral kernel. The paper is written to be self-contained and understandable to a wide mathematical audience.

Keywords: Cauchy problem; parabolic PDE; diffusion equation; $C_0$-semigroup; approximation of solution; Feynman formula; translation operator; Chernoff theorem

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Problem Setting and Approach Proposed

At the present time we know of a relatively small number of situations where a short formula gives the solution of a partial differential equation with variable coefficients in terms of these coefficients and initial/boundary conditions. In the present paper, we provide such formula for a diffusion-type equation (1), see below. Let us first describe the Cauchy problem and then provide the necessary background and references.

Consider integer dimension $d \geq 1$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $t \geq 0$, $u : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ and set the Cauchy problem for a second-order parabolic partial differential equation

$$
\begin{align*}
\left\{ \begin{array}{ll}
u_t'(t, x) = \sum_{j=1}^d (a_j(x))^2 u_{x_jx_j}(t, x) + \langle b(x), \nabla u(t, x) \rangle + c(x)u(t, x) = Hu(t, x), \\
u(0, x) = u_0(x).
\end{array} \right.
\end{align*}
$$

The coefficients of (1) are: an $\mathbb{R}^1$-valued function $c$, an $\mathbb{R}^d$-valued function $b$, and $\mathbb{R}^1$-valued functions $a_j$ for each $j = 1, \ldots, d$. We also represent function $b$ as a vector of $d$ $\mathbb{R}^1$-valued functions $b_j$ as follows: $b(x) = (b_1(x), \ldots, b_d(x))$. We assume that all coefficients are bounded and uniformly continuous. Parenthesis $\langle \cdot, \cdot \rangle$ are used for the scalar product in $\mathbb{R}^d$. We use symbol $\nabla u(t, x)$ for the gradient vector with respect to $x$ as follows: $\nabla u = \left( \partial u/\partial x_1, \ldots, \partial u/\partial x_d \right)$. For each $j = 1, \ldots, d$ we write the multiplier of higher derivative $\partial^2/\partial x_j^2$ in the form $(a_j(x))^2$ for two reasons: to show that the multiplier is non-negative, and to shorten some of the formulas.

This paper is dedicated to deriving an explicit formula that gives the solution of (1) in terms of $a_j$, $b_j$, $c$, $u_0$ assuming that the operator $H$ is an infinitesimal generator of the $C_0$-semigroup $\left(e^{tH} \right)_{t \geq 0}$. This assumption is standard in studies of evolution equations, which is the class of equations that the considered equation belongs to. According to general theory of $C_0$-semigroups \[ this assumption implies that the solution of the Cauchy problem (1) exists,
is bounded and uniformly continuous with respect to \( x \) for each \( t \), depends on \( u_0 \) continuously, and can be represented in a form \( u(t, x) = (e^{tH}u_0)(x) \). We apply the Chernoff theorem [1, 2] to a specially constructed family of operators \( (S(t))_{t \geq 0} \), and express \( e^{tH} \) in terms of \( a_j, b_j, c \) reaching the proposed goal. We do not discuss the problem of finding the class of functions in which the solution is unique under certain assumptions on functions \( a, b, c, u_0 \), but keep in mind that for a heat equation there are known unbounded solutions.

The formula that provides the solution of (1) is given in theorem 3.

**State of the Art**

The diffusion equation and heat equation have a long history of research since the beginning of 19-th century [7], and there are many publications devoted to it. The reader can see textbooks [5, 8, 9], recent papers [21, 30, 10, 18, 25, 33, 6, 32] and references therein, but still this list will be very incomplete. In the present paper, we are focused on the one particular question: how we can express the solution of a diffusion equation with variable coefficients in terms of these coefficients? There are not so many publications answering it, but in the last decade the number of them increases rapidly. This growth is achieved mainly via an approach based on the Chernoff theorem (theorem 1 below), and we also follow this way. Let us explain the idea of our method and mention some of the papers where the Chernoff theorem was used to obtain formulas (representations) of solutions of evolution equations.

It is known (and will be discussed with more details in the “Technique Employed” section below), that if operator \( H \) generates a \( C_0 \)-semigroup \( \{e^{tH}\}_{t \geq 0} \), then the solution of Cauchy problem \( u'(t, x) = Hu(t, x); u(0, x) = u_0(x) \) can be represented in the form \( u(t, x) = (e^{tH}u_0)(x) \). The Chernoff theorem [1, 2] allows us to reduce the problem of finding \( e^{tH} \) to the problem of finding an appropriate operator-valued function \( S(t) = I + tH + o(t) \), which is called the Chernoff function, and then use the Chernoff formula

\[
 e^{tH} = \lim_{n \to \infty} S(t/n)^n
\]

One advantage of that step is that we can define \( S(t) \) by an explicit formula that depends on the coefficients of the operator \( H \). Another advantage is that for each \( t \) the operator \( S(t) \) is a linear bounded operator, which allows us to define analytic functions of argument \( S(t) \) via power series (see examples [31, 20, 29, 22]) to obtain a semigroup \( \{e^{-itH}\}_{t \geq 0} \) that solves Schrödinger equation with Hamiltonian \( H \). This idea was introduced in [31] where we defined \( R(t) = \exp \left( -i(S(t) - I) \right) \) and proved that \( e^{-itH} = \lim_{n \to \infty} R(t/n)^n \). Members of O.G.Smolyanov’s group employed Chernoff’s theorem using integral operators as Chernoff functions to find solutions to parabolic equations in many cases during the last 15 years: see the pioneering papers [3, 12], overview [13], several examples [4, 27, 14, 15, 16, 17] to see the diversity of applications, and recent papers [20, 24, 22, 23, 19, 26]. The solutions obtained were represented in the form of a Feynman formula, i.e. as a limit of a multiple integral as the multiplicity goes to infinity. Indeed, if \( S(t) \) is an integral operator for each \( t > 0 \), then \( S(t/n)^n \) is an \( n \)-tuple integral operator and equality \( e^{tH}u_0 = \lim_{n \to \infty} S(t/n)^n u_0 \) is a Feynman formula. See also [31] for applications of so-called quasi-Feynman formulas which allow to solve Schrödinger equation in situations where it is difficult to obtain a Feynman formula for solution.

The specific feature of the presented research is that we use translation operators instead of integral operators when constructing the Chernoff function \( S(t) \), see [28] for integral-based approach to similar equation; however, we allow all \( a_j(x) \) to be zero for some \( x \), this case is not covered by methods of [28]. For this reason the solution of (1) is now represented via a new type of formulas that do not include integrals. However, one may interpret these formulas as Feynman formulas with Dirac delta-functions in the integral kernel, which is discussed later in remark 7 in the end of the paper. The present paper may also be considered as a generalization of [11] to a multi-dimensional case.
**Technique Employed**

Let $\mathcal{F}$ be a Banach space. Let $\mathcal{L}(\mathcal{F})$ be a set of all bounded linear operators in $\mathcal{F}$. Suppose we have a mapping $V: [0, +\infty) \to \mathcal{L}(\mathcal{F})$, i.e. $V(t)$ is a bounded linear operator $V(t): \mathcal{F} \to \mathcal{F}$ for each $t \geq 0$. The mapping $V$ is called [1] a $C_0$-semigroup, or a strongly continuous one-parameter semigroup if it satisfies the following conditions:

1) $V(0)$ is the identity operator $I$, i.e. $\forall \varphi \in \mathcal{F}: V(0)\varphi = \varphi$;

2) $V$ maps the addition of numbers in $[0, +\infty)$ into the composition of operators in $\mathcal{L}(\mathcal{F})$, i.e. $\forall t \geq 0, \forall s \geq 0: V(t+s) = V(t) \circ V(s)$, where for each $\varphi \in \mathcal{F}$ the notation $(A \circ B)(\varphi) = A(B(\varphi)) = AB\varphi$ is used;

3) $V$ is continuous with respect to the strong operator topology in $\mathcal{L}(\mathcal{F})$, i.e. $\forall \varphi \in \mathcal{F}$ function $t \mapsto V(t)\varphi$ is continuous as a mapping $[0, +\infty) \to \mathcal{F}$.

The definition of a $C_0$-group is obtained by the substitution of $[0, +\infty)$ by $\mathbb{R}$ in the paragraph above.

It is known [1] that if $(V(t))_{t \geq 0}$ is a $C_0$-semigroup in Banach space $\mathcal{F}$, then the set

$$\left\{ \varphi \in \mathcal{F} : \exists \lim_{t \to +0} \frac{V(t)\varphi - \varphi}{t} \right\}$$

is dense in $\mathcal{F}$. The operator $L$ defined on the domain $\text{Dom}(L)$ by the equality

$$L\varphi = \lim_{t \to +0} \frac{V(t)\varphi - \varphi}{t}$$

is called an infinitesimal generator (or just generator to make it shorter) of the $C_0$-semigroup $(V(t))_{t \geq 0}$.

One of the reasons for the study of $C_0$-semigroups is their connection with differential equations. If $Q$ is a set, then the function $u: [0, +\infty) \times Q \to \mathbb{C}$, $u: (t, x) \mapsto u(t, x)$ of two variables $(t, x)$ can be considered as a function $u: t \mapsto [x \mapsto u(t, x)]$ of one variable $t$ with values in the space of functions of the variable $x$. If $u(t, \cdot) \in \mathcal{F}$ then one can define $Lu(t, x) = (Lu(t, \cdot))(x)$. If there exists a $C_0$-semigroup $(e^{tL})_{t \geq 0}$ then the Cauchy problem for a linear evolution equation

$$\begin{cases} u'_t(t, x) = Lu(t, x) \quad \text{for } t > 0, x \in Q \\ u(0, x) = u_0(x) \quad \text{for } x \in Q \end{cases}$$

has a unique (in sense of $\mathcal{F}$, where $u(t, \cdot) \in \mathcal{F}$ for every $t \geq 0$) solution

$$u(t, x) = (e^{tL}u_0)(x)$$

depending on $u_0$ continuously. Compare also different meanings of the solution [1], including mild solution which solves the corresponding integral equation. Note that if there exists a strongly continuous group $(e^{tL})_{t \in \mathbb{R}}$ then in the Cauchy problem the equation $u'_t(t, x) = Lu(t, x)$ can be considered not only for $t > 0$, but for $t \in \mathbb{R}$, and the solution is provided by the same formula $u(t, x) = (e^{tL}u_0)(x)$.

**Definition 1 (Introduced in [31])**. Let $\mathcal{L}(\mathcal{F})$ be the set of all linear bounded operators in a Banach space $\mathcal{F}$. Let the operator $L: \mathcal{F} \supset \text{Dom}(L) \to \mathcal{F}$ be linear and closed. The function $G$ is called Chernoff-tangent to the operator $L$ iff:

(CT1) $G$ is defined on $[0, +\infty)$, takes values in $\mathcal{L}(\mathcal{F})$, and the function $t \mapsto G(t)f$ is continuous for each $f \in \mathcal{F}$.

(CT2) $G(0) = I$, i.e. $G(0)f = f$ for each $f \in \mathcal{F}$.

(CT3) There exists such a dense subspace $\mathcal{D} \subset \mathcal{F}$ that for each $f \in \mathcal{D}$ there exists a limit

$$G'(0)f = \lim_{t \to 0} \frac{G(t)f - f}{t}.$$
The closure of the operator \((G^t(0), D)\) is equal to \((L, \text{Dom}(L))\).

**Remark 1.** Let us consider one-dimensional example \(F = \mathcal{L}(F) = \mathbb{R}\). Then \(g: [0, +\infty) \to \mathbb{R}\) is Chernoff-tangent to \(l \in \mathbb{R}\) iff \(g(t) = 1 + tl + o(t)\) as \(t \to +0\).

**Theorem 1** (P. R. Chernoff (1968), see [1] and [2]). Let \(F\) and \(\mathcal{L}(F)\) be as above. Suppose that the operator \(L: F \ni \text{Dom}(L) \to F\) is linear and closed, and function \(G\) takes values in \(\mathcal{L}(F)\). Suppose that these assumptions are fulfilled:

- There exists a \(G_0\)-semigroup \((e^{Lt})_{t \geq 0}\) with the infinitesimal generator \((L, \text{Dom}(L))\).
- \(G\) is Chernoff-tangent to \((L, \text{Dom}(L))\).
- There exists such a number \(\omega \in \mathbb{R}\), that \(\|G(t)\| \leq e^{\omega t}\) for all \(t \geq 0\).

Then for each \(f \in F\) we have \((G(t/n))nf \to e^{Lt}f\) as \(n \to \infty\) with respect to norm in \(F\) uniformly with respect to \(t \in [0, T]\) for each \(T > 0\), i.e.

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| e^{Lt}f - (G(t/n))nf \right\| = 0.
\]

**Remark 2.** In our one-dimensional example \((F = \mathcal{L}(F) = \mathbb{R})\) the Chernoff theorem says that \(e^{Lt} = \lim_{n \to \infty} g(t/n)^n = \lim_{n \to \infty} (1 + tl/n + o(t/n))^n\), which is a simple fact of calculus.

**Definition 2.** Let \(F, \mathcal{L}(F), L\) be as above. If \(G\) is Chernoff-tangent to \(L\) and the equation \(\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| e^{Lt}f - (G(t/n))nf \right\| = 0\) holds, then \(G\) is called a Chernoff function for the operator \(L\), and the expression \((G(t/n))nf\) is called a Chernoff approximation to \(e^{Lt}f\).

**Remark 3.** If \(L\) is a linear bounded operator in \(F\), then \(e^{Lt} = \sum_{k=0}^{+\infty} (L)^k / k!\) where the series converges in the usual operator norm topology in \(\mathcal{L}(F)\). When \(L\) is not bounded (such as Laplacian and many other differential operators), expressing \((e^{Lt})_{t \geq 0}\) in terms of \(L\) is not an easy problem that is equivalent to the problem of finding (for each \(u_0 \in F\)) the \(F\)-valued function \(U\) that solves the Cauchy problem \(U'(t) = L U(t); U(0) = u_0\). If one finds this solution, then \(e^{Lt}\) is obtained for each \(u_0 \in F\) and each \(t \geq 0\) in the form \(e^{Lt}u_0 = U(t)\).

**Remark 4.** In the definition of the Chernoff tangency the family \((G(t))_{t \geq 0}\) usually does not have a semigroup composition property, i.e. \(G(t_1 + t_2) \neq G(t_1)G(t_2)\), while \((e^{Lt})_{t \geq 0}\) has it:

\[e^{Lt_1}e^{Lt_2} = e^{(t_1 + t_2)L}\].

However, each \(C_0\)-semigroup \((e^{Lt})_{t \geq 0}\) is Chernoff-tangent to its generator \(L\) and appears to be it’s Chernoff function. When coefficients of the operator \(L\) are variable, usually there is no simple formula for \(e^{Lt}\) due to the remark 3. On the other hand, even in this case one can find rather simple formulas to construct Chernoff function \(G\) for the operator \(L\), because there is no need to worry about the composition property, and then obtain \(e^{Lt}\) in the form \(e^{Lt} = \lim_{n \to \infty} G(t/n)^n\) via the Chernoff theorem. This is what we do in the present paper for \(L = H\) and what people have done for different operators described in papers cited above.

**Chernoff Function for Operator \(H\)**

**Remark 5.** Let us denote the set of all (real-valued and defined on \(\mathbb{R}^d\)) bounded continuous functions as \(C_b(\mathbb{R}^d)\), the set of all bounded functions with bounded derivatives of all orders as \(C_b^\infty(\mathbb{R}^d)\), and the set of all bounded, uniformly continuous functions as \(UC_b(\mathbb{R}^d)\).

Then \(C_b^\infty(\mathbb{R}^d) \subset UC_b(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)\), and with respect to the uniform (Chebyshev) norm \(\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|\) the first inclusion is dense, and the last two spaces are Banach spaces.

**Theorem 2.** Let \(e_j \in \mathbb{R}^d\) be a constant \(d\)-dimensional vector with 1 at position \(j\) and 0 at other \(d - 1\) positions. For each \(x \in \mathbb{R}^d\), \(t \geq 0\), \(f \in C_b(\mathbb{R}^d)\) and \(\varphi \in C_b^\infty(\mathbb{R}^d)\) set

\[
(S(t)f)(x) = \frac{1}{4d} \sum_{j=1}^{d} \left( f \left( x + 2\sqrt{d}a_j(x)\sqrt{t}e_j \right) + f \left( x - 2\sqrt{d}a_j(x)\sqrt{t}e_j \right) + \frac{1}{2} f(x + 2tb(x)) + tc(x) f(x) \right)
\]

(2)
\[(H\varphi)(x) = \sum_{j=1}^{d} (a_j(x))^2 \varphi'_{x,j}(x) + \langle b(x), \nabla \varphi(x) \rangle + c(x)\varphi(x) \quad (3)\]

Then, with respect to the norm \(\|g\| = \sup_{x \in \mathbb{R}^d} |g(x)|\), the following holds:

I) for each \(t \geq 0\) and \(f \in C_b(\mathbb{R})\) we have \(\|S(t)f\| \leq (1 + \|c\|t)\|f\|\).

II) for each \(\varphi \in C^2_b(\mathbb{R}^d)\) we have \(\lim_{t \to +0} \|S(t)\varphi - \varphi - tH\varphi\|/t = 0\).

III) if \(t_n \to t_0, t_n \geq 0\) and \(f \in UC_b(\mathbb{R}^d)\), then \(\lim_{t \to t_0} \|S(t)f - S(t_0)f\| = 0\) for each \(t_0 \geq 0\).

IV) if \(a_j, b_j, c, f \in UC_b(\mathbb{R}^d)\), then \(S(t)f \in UC_b(\mathbb{R})^d\) for each \(t \geq 0\).

**Proof.** I) Let us write sup instead of \(\sup_{x \in \mathbb{R}^d}\) in the proof of this item to make it shorter. Indeed, for each \(t, x\) in (2) and consider \((S(t)\varphi)(x)\) as a smooth function of \(\sqrt{t}\). Then we use Taylor’s expansion in powers of \(\sqrt{t}\) in first two sums and in powers of \(t\) in the third summand to show that \((S(t)\varphi)(x) = \varphi(x) + t\langle H\varphi(x), t\sqrt{R(t,x)}\rangle\). As the derivatives of \(\varphi\) are bounded, and functions \(a_j, b_j, c\) are also bounded, one can represent the remainder in Lagrange’s form and see that \(\sup_{t \in [0, t_0]} \sup_{x \in \mathbb{R}^d} |R(t, x)| < \infty\) for each fixed \(t_0 > 0\). We skip the exact formula for \(R(t, x)\) to make the proof shorter. Indeed, for each \(j\) we have

\[
\varphi \left( x + 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) = \varphi(x) + 2\sqrt{a_j(x)}\sqrt{\mathcal{E}} \langle \nabla \varphi(x), e_j \rangle + \frac{1}{2}(2\sqrt{a_j(x)}\sqrt{\mathcal{E}})^2 \langle \varphi''(x)e_j, e_j \rangle + o(t)
\]

so

\[
\varphi \left( x + 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) + \varphi \left( x - 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) = 2\varphi(x) + 4d(a_j(x))^2 \langle \varphi''(x)e_j, e_j \rangle t + o(t),
\]

and after summation we obtain

\[
\frac{1}{4d} \sum_{j=1}^{d} \left( \varphi \left( x + 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) + \varphi \left( x - 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) = \frac{1}{2} \varphi(x) + t \sum_{j=1}^{d} (a_j(x))^2 \varphi'_{x,j}(x) + o(t).\right.
\]

For the third summand we have

\[
\frac{1}{2} \varphi(x + 2tb(x)) = \frac{1}{2} \varphi(x) + t \langle \nabla \varphi(x), b(x) \rangle + o(t).
\]

The term \(tc(x)\varphi(x)\) is already in the form we need. Summing up we have

\[
(S(t)\varphi)(x) = \frac{1}{4d} \sum_{j=1}^{d} \left( \varphi \left( x + 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) + \varphi \left( x - 2\sqrt{a_j(x)}\sqrt{\mathcal{E}}_j \right) + \frac{1}{2} \varphi(x + 2tb(x)) + tc(x)\varphi(x)\right.
\]

\[
= \left( \frac{1}{2} \varphi(x) + t \sum_{j=1}^{d} (a_j(x))^2 \varphi''_{x,j}(x) + o(t) \right) + \left( \frac{1}{2} \varphi(x) + t \langle \nabla \varphi(x), b(x) \rangle + o(t) \right) + tc(x)\varphi(x)
\]

\[
= \varphi(x) + t \sum_{j=1}^{d} (a_j(x))^2 \varphi''_{x,j}(x) + t \langle \nabla \varphi(x), b(x) \rangle + tc(x)\varphi(x) + o(t) = \varphi(x) + t(H\varphi)(x) + o(t).
\]
III) In this item we assume that \( t_n \rightarrow t_0 \) and prove that \((S(t_n)f)(x) \rightarrow (S(t_0)f)(x)\) uniformly with respect to \( x \in \mathbb{R}^d \) as \( n \rightarrow \infty \). Indeed, functions \( a_j \) are bounded, so for each \( j \) we have \( x + 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}} \rightarrow x + 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}} \) uniformly with respect to \( x \). Function \( f \) is uniformly continuous, so \( f(x + 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}}) \rightarrow f(x + 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}}) \) uniformly with respect to \( x \). In the same manner we use the fact that functions \( b_j, c \) are bounded, and then sum all the limit conditions obtained.

IV) If \( t \geq 0 \) is fixed, then \( \left[ z \mapsto f(z+2\sqrt{\text{da}_j(z)\sqrt{\text{te}_j}}) \right] \in UC_b(\mathbb{R}^d) \) because \( a_j, f \in UC_b(\mathbb{R}^d) \). All the summands of \( S(t)f \) are processed in this manner. \( \square \)

Main Result

Theorem 3. Suppose that functions \( a_j, b_j, c \) belong to the space \( UC_b(\mathbb{R}^d) \) endowed with the norm \( \|f\| = \sup_{x \in \mathbb{R}^d} |f(x)| \). Suppose that operator \( H \) is defined by equation (3) on the domain \( C_b^\infty(\mathbb{R}^d) \subset UC_b(\mathbb{R}^d) \), and the closure of this operator: a) exists; b) is an infinitesimal generator of a \( C_0-\)semigroup \( (e^{tH})_{t \geq 0} \) in \( UC_b(\mathbb{R}^d) \).

Then for each \( u_0 \in UC_b(\mathbb{R}^d) \) there exists a bounded (and uniformly continuous with respect to \( x \in \mathbb{R}^d \) for each \( t \geq 0 \)) solution \( u \) of the Cauchy problem (1), it depends on \( u_0 \) continuously and uniformly with respect to \( x \in \mathbb{R}^d \) for each \( t \geq 0 \). For each \( x \in \mathbb{R}^d \) and \( t \geq 0 \) this solution is given by the formula

\[
u(t, x) = (e^{tH}u_0)(x) = \lim_{n \to \infty} \left( \left( S(t/n) \right)^n u_0 \right)(x), \tag{4}\]

where \( S(t/n) \) is obtained by substitution of \( t \) by \( t/n \) in the equation (2), and the \( n \)-th power in the expression \( (S(t/n))^n \) means the composition of \( n \) copies of linear bounded translation operator \( S(t/n) \). The limit (4) for each fixed \( t > 0 \) is taken in the space \( UC_b(\mathbb{R}^d) \) and appears to be uniform with respect to \( t \in [0, t_0] \) for each \( t_0 > 0 \).

Proof. Let us check the conditions of the Chernoff theorem and thus show that \( S \) is a Chernoff function for \( L \). In theorem 1 and definition 1 we set \( \mathcal{F} = UC_b(\mathbb{R}), G(t) = S(t), L = H, \mathcal{D} = C_b^\infty(\mathbb{R}), \omega = \|c\| \). Condition \( (E) \) is an assumption from theorem 3, condition \( (N) \) is provided by item I) of theorem 2: \( \|S(t)\| \leq 1 + \|c\|t \leq e^{\|c\|t} \). Let us check the Chernoff tangency: (CT1) follows from items IV) and III) of theorem 2, (CT2) is follows directly from formula (2), (CT3) follows from item II) of theorem 2, (CT4) is an assumption of theorem 3. Therefore the statement of theorem 3 is true thanks to the statement of the Chernoff theorem and standard facts of the \( C_0 \)-semigroup theory \( [1] \). \( \square \)

Remark 6. Formula (4) proven in theorem 3 contains \( \lim_{n \to \infty} \). After the limit is taken we obtain the exact solution to Cauchy problem (1). For each fixed \( n \) the expression under the limit sign is an approximation of the solution. With growth of \( n \) such approximations converge to the exact solution uniformly with respect to \( x \in \mathbb{R}^d \) and \( t \in [0, t_0] \) for each fixed \( t_0 > 0 \).

Feynman Formulas with Generalized Functions

Remark 7. Formula (2) can be rewritten in terms of generalized functions (=distributions) in \( \mathbb{R}^d \), based on the fact that for Dirac’s \( \delta \)-function the following equation holds by definition of the integral in the right-hand side: \( f(w) = \int_{\mathbb{R}^d} \delta(y - w)f(y)dy \). We fix \( x \in \mathbb{R}^d \) and start from

\[
\int_{\mathbb{R}^d} \delta \left( y - x - 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}} \right) f(y)dy,
\]

then, after the substitution \( y = x + z, z = y - x, dy = dz \) in the integral above, we get

\[
\int_{\mathbb{R}^d} \delta \left( z - 2\sqrt{\text{da}_j(x)\sqrt{\text{te}_j}} \right) f(x + z)dz.
\]
In the same way we have

\[ f \left( x - 2\sqrt{d}a_j(x)\sqrt{te_j} \right) = \int_{\mathbb{R}^d} \delta \left( z + 2\sqrt{d}a_j(x)\sqrt{te_j} \right) f(x + z)dy, \]

\[ f(x + 2tb(x)) = \int_{\mathbb{R}^d} \delta(z - 2tb(x)) f(x + z)dy, \quad c(x) f(x) = \int_{\mathbb{R}^d} c(x) \delta(z) f(x + z)dy. \]

Finally

\[ (S(t)f)(x) = \int_{\mathbb{R}^d} \Phi(z, x, t)f(x + z)dz, \quad (5) \]

where

\[ \Phi(z, x, t) = \frac{1}{4d} \sum_{j=1}^{d} \left( \delta \left( z - 2\sqrt{d}a_j(x)\sqrt{te_j} \right) + \delta \left( z + 2\sqrt{d}a_j(x)\sqrt{te_j} \right) \right) + \frac{1}{2} \delta(z - 2tb(x)) + tc(x) \delta(z). \]

Then (4) can be rewritten as a Feynman formula, i.e. as a representation of the function \( u \) in a form of a limit of multiple integral where multiplicity tends to infinity:

\[ u(t, x) = \lim_{n \to \infty} (S(t/n)^n u_0)(x) = \lim_{n \to \infty} \int_{\mathbb{R}^d} \Phi(z_1, x, t/n) \int_{\mathbb{R}^d} \Phi(z_2, x + z_1, t/n) \int_{\mathbb{R}^d} \Phi(z_3, x + z_1 + z_2, t/n) \ldots \]

\[ \ldots \int_{\mathbb{R}^d} \Phi(z_n, x + z_1 + \cdots + z_{n-1}, t/n) u_0(x + z_1 + \cdots + z_n)dz_n \ldots dz_1. \quad (6) \]

**Remark 8.** Right-hand sides of equations (5) and (6) are formal expressions, but left-hand sides are well-defined thanks to theorem 2 and theorem 3. However, it’s interesting to study such expressions using methods of generalized functions theory and find the physical meaning of generalized function \( \Phi \) which in regular case is a transitional density of the diffusion process.

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