Effects of Lifshitz transitions in ferromagnetic superconductors: the case of URhGe

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(Dated: August 18, 2018)

In ferromagnetic superconductors, like URhGe, superconductivity co-exists with magnetism near zero field, but then re-appears again in a finite field range, where the system also displays mass enhancement in the normal state. We present the theoretical understanding of this non-monotonic behavior. We explore the multi-band nature of URhGe and associate re-entrant superconductivity and mass enhancement with the topological transition (Lifshitz) in one of the bands in a finite magnetic field. We found excellent agreement between our theory and a number of experimental results for URhGe, such as weakly first order reentrant transition, the dependence of superconducting $T_c$ on a magnetic field, and the field dependence of the effective mass, the specific heat and the resistivity in the normal state. Our theory can be applied to other ferromagnetic multi-band superconductors.

Ferromagnetic superconductors are exciting systems to study the interplay of magnetism and superconductivity, contrary to the common wisdom that the presence of a ferromagnetic order destroys superconductivity. The coexistence of superconductivity and magnetism has been realised experimentally for uranium-based heavy-fermion compounds, like UGe$_2$ [1], UCoGe [2] and URhGe [3]. The materials exhibit a wealth of exotic properties, including, e.g., the appearance of non-Landau damping in magnetic excitations [4].

Among these systems, URhGe has attracted much attention both experimentally [5–17] and theoretically [18–20]. In zero applied magnetic field, it displays ferromagnetism with magnetic moment oriented along the c-axis, and spin-triplet superconductivity at a lower temperature [6]. In an external magnetic field along the b-axis (b ⊥ c), superconductivity disappears at about $B=2T$. This is believed to be caused by the orbital effect of the field [6]. However, at higher magnetic fields, in the range from 8T to 13.5T, it reappears again [5] (see Fig. 1).

Ferromagnetic spin fluctuations are believed to provide the pairing glue for superconductivity in a ferromagnetic metal [19, 20]. Indeed, NMR spin-spin relaxation measurements indicate that uniform longitudinal spin fluctuations (the ones in the direction of a magnetic field) are strongly enhanced in the field range where superconductivity has been observed [14]. Measurements of the specific heat [9, 10], electric conductivity [7, 9, 14, 15], and magnetisation [9] indicate that the increase of spin fluctuations is accompanied by the increase of the effective mass of fermions. This is indicative of a critical behavior near a ferromagnetic instability.

In this communication we address the origin of the ferromagnetic instability in a finite field. We argue that it is due to a Lifshitz transition observed [8] in one of the bands which form the electronic structure of URhGe. This Lifshitz transition pushes the system closer to the magnetic instability and enhances the magnetic fluctuations. This in turn leads to re-entrance of superconductivity (RSC) at a finite field.

The minimal model of the electronic structure of URhGe has two bands with non-equal dispersions (different masses $m_1$ and $m_2$) and band minima shifted by $K_0$, (see Fig. 1). In a ferromagnetic state at zero field spin-up and spin-down states in both bands are split by an effective exchange field. Both branches of band 1 cross the chemical potential $\mu$, while both branches of band 2 are above the chemical potential $\mu$ (see the left inset in Fig. 1). At a finite $H$ the bands experience additional Zeeman splitting. The exchange field was reported [5] to be rather weak ($\sim 0.1T$), hence at fields near 10T Zeeman splitting dominates. The dispersions of the two bands with Zeeman splitting are $\epsilon_{1,\sigma}(k) = \frac{k^2}{2m_1} - \mu - \sigma \mu_B H$ and $\epsilon_{2,\sigma}(k) = \frac{(k - K_0)^2}{2m_2} - \mu + \delta \mu - \sigma \mu_B H$, where $\delta \mu$ is the energy shift between bands 1 and 2 and $\sigma$ is the projected value of the spin. As the field increases, the splitting grows, and at some critical field the system experiences a Lifshitz transition, in which spin-up branch of band 2 crosses the chemical potential (the middle inset in Fig. 1). We add the Hubbard four-fermion repulsive interaction $U$, the same for inter-band and intra-band scattering, and analyze the tendency towards magnetic order and magnetically-mediate superconductivity in this field range. As we show [21], the contribution of the inter-band scattering to the susceptibility is weak given the large momentum separation of the two bands. The fitting parameters relevant to URhGe are presented in [22].

We first compute the longitudinal and transverse susceptibilities at $T = 0$. Within RPA we have [23]:

$$\chi_\parallel(q) = \sum_\sigma \frac{\chi^0_\parallel(q)}{1 - U^2 \chi^0_\parallel(q) \chi^{-\sigma}_\parallel(q)},$$

$$\chi_\perp(q) = \sum_\sigma \frac{\chi^{-\sigma}_\parallel(q)}{1 - U \chi^{-\sigma}_\parallel(q)},$$

where $\chi^0_\sigma = \sum_{i=1,2} \chi_{0,i,\sigma,\sigma}$ and $\chi^{-\sigma}_\sigma = \sum_{i=1,2} \chi_{i,i,\sigma,\sigma}$ are particle-hole susceptibilities of free fermions from the
where \( H \) is the magnetic field, \( \mu \) is the chemical potential, \( \epsilon \) is the energy, \( \Delta \) is the superconducting gap, \( \sigma \) is the spin projection, \( N_{1\sigma}(0) \) is the density of states at the Fermi surface of a sub-band \( i = 1, 2 \) with spin projection \( \sigma \). At \( H > H_L \), \( N_{2\uparrow}(0) \) becomes non-zero, \( D \) decreases and \( \chi_{\parallel}(q = 0) \) increases. At higher fields \( N_{1\uparrow}(0) \) decreases (see right insert in Fig. 1) and \( \chi_{\parallel}(q = 0) \) decreases. At even higher magnetic field the spin-down sub-band of band 1 undergoes the second Lifshitz transition and becomes unoccupied, in agreement with [8]. The transverse \( \chi_{\perp}(0) \) has much weaker dependence on \( N_{2\uparrow}(0) \) and does not show a peak around \( H_R \). This agrees with the NMR results [14].

The effective mass of a conduction electron is given by

\[
m^*_\sigma = \left[ 1 - \partial_\omega \Sigma_{\sigma\sigma}(p_{i\sigma}, \omega) \right]_{\omega = 0},
\]

The electron self-energy \( \Sigma \) can be written as a sum of longitudinal and transverse components. The corresponding ring and ladder diagrams [23, 24] are shown in Fig. 3 a and b, respectively:

\[
\Sigma_{\sigma\sigma}^L(p, i\omega) = \int_q G_{i\sigma}(p - q, i(\omega - \xi))V^-_{L\sigma}(q, i\xi),
\]

\[
\Sigma_{\sigma\sigma}^T(p, i\omega) = \int_q G_{i\sigma}(p - q, i(\omega - \xi))V^\tau_{\sigma\sigma}(q, i\xi),
\]

where the effective interactions are

\[
V^-_{L\sigma}(q, i\omega) = \frac{U^2\chi_0^{-\sigma}(q, i\omega)\chi_0^{-\sigma}(q, i\omega)}{1 - U^2\chi_0^{-\sigma}(q, i\omega)\chi_0^{-\sigma}(q, i\omega)},
\]

\[
V^\tau_{\sigma\sigma}(q, i\omega) = \frac{U^3\chi_0^{-\sigma}(q, i\omega)^2}{1 - U\chi_0^{-\sigma}(q, i\omega)},
\]

Again neglecting the weak inter-band scattering (i.e., the scattering between band 1 and band 2 [21]) in the calculation of susceptibilities, but retaining the dependence of

\[
D = 1 - U^2(N_{2\uparrow}(0) + N_{1\uparrow}(0))N_{1\downarrow}(0),
\]
self-energies on all diagonal susceptibilities, we perform the frequency integration and, following [23, 24], we obtain

\[ \frac{m^*_i}{m_i} = 1 + \lambda_i^{\sigma} + \lambda_i^{\overline{\sigma}}, \]

where

\[ \lambda_i^{\sigma} = \frac{m_i}{(2\pi)^2 p_{Fi\sigma}} \int_{0}^{2p_{Fi\sigma}} dq dq V_{Li\sigma}(q, 0), \]

\[ \lambda_i^{\overline{\sigma}} = \frac{m_i}{(2\pi)^2 p_{Fi\sigma}} \int_{0}^{p_u} dq dq V_{Li\sigma}^{\overline{\sigma}}(q, 0). \]

Here \( p_{Fi\sigma} \) is the Fermi momentum of the sub-band \( \{i, \sigma\} \), and the integration limits for the transverse component are \( p_l = \max\{p_{Fi\sigma} - p_{Fi,\overline{\sigma}}, 0\} \) and \( p_u = p_{Fi\sigma} + p_{Fi,\overline{\sigma}} \).

The result of the calculation of \( m_{i\sigma}/m_i \) for the sub-band 1 ↓ is shown in Fig. 2 b. As expected, the mass enhancement is peaked at \( H \approx H_R \), where the uniform susceptibility is the largest. The effective masses for other sub-bands show similar enhancement [21]. The theoretical result \( m^*/m \) agrees well with the mass ratio extracted from magnetisation measurements [9] (see Fig. 2 b). Using the result for \( m^*(H) \), we computed the specific heat and resistivity. The main contribution to the specific heat comes from the 1 ↓ sub-band, and the Sommerfeld coefficient \( \gamma = C(T)/T \) can be estimated as [25] \( \gamma \propto N_{1\downarrow}(0)m^*_1/m_1 \). In Fig. 4a we show the calculated \( \gamma \) and the experimental one from Ref. [9]. Clearly, both are peaked around \( H_R \) and show similar behavior at smaller and higher fields. In Fig. 4 b we show theo-

![FIG. 4. Enhancement of (a) Sommerfeld coefficient \( \gamma \) measured in [9] (triangles) and (b) conductivity coefficient \( \Lambda \) measured in Ref [15] (squares) and calculated with our model (solid line) for the same parameters as in Fig. 1.]

We next turn to superconductivity. The reduction and subsequent disappearance of superconductivity at small fields has been argued to be due to the orbital effect in a field [6, 27]. The reduction of \( T_c \) due to a vector potential close to \( T_c \) follows [27]

\[ T_c(H) = T_c(H = 0)(1 - H/H_c), \quad H_c = \frac{20\pi k_B T_c^2(0)}{\zeta(3) e^2 \hbar^2 c \mu_0} \]

where \( k_B \) is the Boltzmann constant, \( \mu_0 \) is the vacuum permeability. This form of \( T_c(H) \) agrees with the data at small fields [6]. When \( H > H_c, T_c(H) \) vanishes.

To study the re-entrant superconductivity, we do the analysis in two steps. First, we compute \( T_{c}^{Eli} \) within Eliashberg spin-fluctuation formalism, without including the orbital effect of a field. We then use the result for \( T_{c}^{Eli} \) as an effective \( T_c(0) \) to estimate \( H_c \) from (13). We then use the full non-linear dependence of \( H_c \) on \( T_c(0) \) to obtain the actual \( T_c \) for the polar p-wave state.

An exchange by ferromagnetic spin fluctuations enhances the pairing vertex in p-wave channel, and below we search for superconducting order with p-wave symmetry. We analyze all fields and keep both Zeeman and exchange splitting. To keep calculations under control, we neglect the feedback from ferromagnetic order on the pairing interaction. This feedback is relevant near a ferromagnetic quantum-critical point [28], but less relevant away from criticality, where our analysis holds.

In Eliashberg theory one needs to solve the equations for quasiparticle \( Z_{\sigma}(\mathbf{q}, i\omega_n) = 1 - \Sigma_{\sigma}(\mathbf{q}, i\omega_n) \) and the pairing vertex, \( W_{\sigma}(\mathbf{p}) \):

\[ (1 - Z_{\sigma}(\mathbf{p}))(i\omega_n) = T \sum_{\mathbf{p}'} \frac{V_{\sigma}(\mathbf{p} - \mathbf{p}')Z_{\sigma}(\mathbf{p}')(i\omega_n'),}{(i\omega_n', \Sigma_{\sigma}(\mathbf{p}'))^2 - \epsilon_{i\sigma}(\mathbf{p}')^2}, \]

\[ W_{\sigma}(\mathbf{p}) = T \sum_{\mathbf{p}'} \frac{V_{\sigma}(\mathbf{p} - \mathbf{p}')W_{\sigma}(\mathbf{p}')}{(i\omega_n, \Sigma_{\sigma}(\mathbf{p}'))^2 - \epsilon_{i\sigma}(\mathbf{p}')^2}, \]

where \( \sum_{\mathbf{p}'} = \sum_{\omega_{n'}} \int d^3 p' ..., \omega_n = \pi(2n + 1)T, p = \{\mathbf{p}, i\omega_n\} \), and the interactions are \( V_{\sigma}(\mathbf{p}) = V_{\sigma}^+ + V_{\sigma}^- \) and \( V_{\sigma}^+ = V_{\sigma}^- \). We use a standard trick and reduce Eliashberg set to a single equation by introducing \( \Phi(p) = W(p)/|\omega_n, \Sigma(p)|. \) Then \( \Phi(p) \) is expanded in spherical harmonics and only the p-wave piece, \( \Phi_{\sigma}(\mathbf{p}) \) is retained. Integrating over momenta as \( \int d^3 p ... = \int d\Omega \int d\epsilon N_{\sigma}(\mathbf{p}) \), \( \omega_n \) is the solid angle, we obtain an integral equation of \( \Phi_{\sigma}(\omega_n) \) in the form

\[ \sum_{n \geq 0} K_{mn}(\omega_m, \omega_n) \Phi_m(\omega_n) = 0, \]

where

\[ K_{mn} = \lambda^{(1)}_{\sigma}(\omega_m - \omega_n) + \lambda^{(1)}_{\sigma}(\omega_m + \omega_n) - \frac{\delta_{mn}}{\pi^2} \]

We introduced \( \lambda^{(1)}_{\sigma} = -N_{\sigma}(0) \int d\Omega Y_1(\cos(\theta))V_{\sigma}^-(\theta) \) and \( \lambda^{(0)}_{\sigma} = N_{\sigma}(0) \int d\Omega V_{\sigma}^-(\theta) \), where \( Y_1(\theta) \) is the first spherical harmonic and \( \theta \) is the angle between \( \mathbf{p} \) and \( \mathbf{p}' \). Keeping only the interaction with small momentum transfer, we factorize the pairing between three bands: up and down sub-bands of band 1 and spin-up sub-band.
of band 2. We recall, however, that effective p-wave pairing interaction between fermions on a given band is the sum of contributions from particle-hole bubbles from all three bands. We solve Eq. (17) for all three bands and find the largest $T_c$ (see [21] for details). The result is shown as a solid line in Fig. 5a. At smaller fields $H < H^*$, where $H^*$ is slightly above $H_L$, superconductivity develops on the 1 $\uparrow$ sub-band. At $H > H^*$ it switches to sub-band 1 $\downarrow$, and $T_c$ for superconductivity on this band has a maximum at $H \sim H_{R1}$, where the effective mass on this band is also maximal. We next include the orbital effect. In Fig. 5b we show $H_c$ from Eq. 13 as a function of external $H$. The orbital effect destroys superconductivity when $H_c$ (blue line) is smaller than $H$ (thin black line). The value of $H_c$ changes discontinuously at $H = H^*$, when superconductivity switches from sub-band 1 $\uparrow$ to sub-band 1 $\downarrow$, where the effective mass is larger. This gives rise to a jump in the actual $T_c$ in panel (a). The effective mass over the whole range is presented in the inset.

![FIG. 5. (a) Solid line – $T_c^{\text{Eli}}(H)$, obtained within Eliashberg formalism without including the orbital effect of the field. Dashed line – the actual $T_c$, with both Zeeman (exchange) and orbital effects. The actual $T_c$ is always smaller than $T_c^{\text{Eli}}(H)$ due to the orbital effect of the external and the exchange fields. (b) The effective field $H_c$ from Eq. 13 as a function of $H$. The orbital effect destroys superconductivity when $H_c$ changes discontinuously at $H = H^*$, when superconductivity switches from sub-band 1 $\uparrow$ to sub-band 1 $\downarrow$, where the effective mass is larger. This gives rise to a jump in the actual $T_c$ in panel (a). The effective mass over the whole range is presented in the inset.]

behavior of the system near a field when the bottom of the spin-up branch of previously unoccupied band 2 sinks below the Fermi level. We first computed the transverse and longitudinal spin susceptibilities and argued that the longitudinal susceptibility dramatically enhances in some field range above the Lifshitz transition, while the transverse susceptibility remains flat. This fully agrees with the behavior of longitudinal and transverse susceptibilities, extracted from NMR measurements of the relaxation times, $1/T_1$ and $1/T_2$ [14]. We next computed the one-loop self-energy due to magnetically-mediated interaction and obtained the enhancement of the effective mass. The theoretical result for $m^*/m$ agrees with the experimental data extracted from magnetisation measurements [9, 29]. There were also very good agreements for the Sommerfeld coefficient and the prefactor for the $T^2$ term in the resistivity [9, 15].

Then, we turned to the analysis of the superconductivity. We first solved the Eliashberg equation for magnetically-mediated superconductivity without the orbital effect of the applied field and obtained $T_c^{\text{Eli}}$ with a maximum at a field where the effective mass is the largest. Superconductivity resides on 1 $\uparrow$ sub-band at smaller fields and on 1 $\downarrow$ sub-band at higher fields. We then took into account the pair-breaking orbital effect and found that superconductivity exists at small fields, gets destroyed by orbital effect at intermediate fields, and re-appears discontinuously roughly at a field of Lifshitz transition. This behavior fully agrees with the data [5] (Fig. 1). The reduction of theoretical $T_c$ at higher fields is somewhat slower than in the data. One reason could be a re-orientational transition, detected at 12T [5], in which the magnetic moment rotates towards the field direction, leaving its magnitude unchanged. This spin re-orientation does not increase longitudinal fluctuations but complicates the field dependence of $T_c$ above 12T. Overall, it looks increasingly likely that topological Fermi-surface transitions can account for much of the puzzling physics in nearly magnetic itinerant systems [30, 31].

We thank R. Fernandes and M. Greven for useful conversations. This work was supported by the EPSRC (YS and JJB) through the grant EP/P002811/1. and by the U.S. Department of Energy through the University of Minnesota Center for Quantum Materials, under award DE-SC-0016371 (A.V.C.).

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