Robust Polarimetry via Convex Optimization

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Abstract: We present mathematical methods, based on convex optimization, for correcting non-physical coherency matrices measured in polarimetry. We also develop the method for recovering the coherency matrices corresponding to the smallest and largest values of the degree of polarization given the experimental data and a specified tolerance. We use experimental non-physical results obtained with the standard polarimetry scheme and a commercial polarimeter to illustrate these methods. Our techniques are applied in post-processing, which compliments other experimental methods for robust polarimetry.

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1. Introduction

Polarization describes the trajectory of the electric field vector of light as it oscillates. Polarimetry and polarization imaging enable technologies in many fields, such as machine vision [1, 2], remote sensing [3, 4], biomedical optics [5], astronomy [6, 7], and free-space optical communication [8–10]. Many quantum information protocols also depend on the determination of polarization states [11–14]. A new generation of polarization imaging cameras is currently under development, which will further accelerate the application of polarimetry in many fields [15].

The state of polarization can be described by the Stokes parameters [16, 17] or the coherency matrix [18], which is a generalization of the Jones calculus [19]. The Stokes parameters, s0, s1, s2, s3, and the coherency matrix, J, are related by

\[ J = \frac{1}{2} \begin{pmatrix} s_0 + s_1 & s_2 + is_3 \\ s_2 - is_3 & s_0 - s_1 \end{pmatrix}. \]  

(1)

The coherency matrix provides all second-order statistical information about the polarization state.

As shown in later sections, non-physical coherency matrices (for example, with negative eigenvalues) arise in common polarimetry schemes due to experimental errors, such as fluctuations of the light source, imperfect alignment of optical components, and spectral bandwidth of the source. Several techniques based on pre-processing, calibration, and in-situ optimization [20–22] or novel polarimetric schemes [23–26] have been developed to reduce the effect of such experimental errors.

In this article, we present a method of correcting these non-physical results by finding the closest physical coherency matrix via convex optimization. This method is applied in post-processing, and does not depend on a priori information or the experimental setup. Having such a method is especially useful when dealing with other degrees of freedom in addition to polarization. For example, when measuring the polarization profile of a vector beam, we might have the result where only a few points are non-physical due to experimental errors [27]. Using our method, we can find the closest approximate physical coherency matrix for these points rather than invalidating all points and repeating the entire measurement. It is also potentially
Fig. 1. Experimental polarimetry setup: (a) modified standard method; (b) polarimeter method. H: half-wave plate; Q: quarter-wave plate; Pol: linear polarizer.

useful when dealing with measurements that cannot be easily repeated, such as the polarimetry of single photons. This method can be easily generalized to be used for multi-photon Stokes parameters [28].

2. Experimental Setup

We used the two independent polarimetry schemes shown in Fig. 1 to measure the coherency matrices of both linearly and elliptically polarized light to verify the validity of the developed methods. In both schemes, the light from the laser is vertically polarized, and passes through either a half-wave plate (HWP) or a quarter-wave plate (QWP). The HWP preserves the linearity of the laser light, but changes the angle of polarization, while the QWP changes linear polarization to elliptical polarization. The exact polarization state after the waveplates depends on $\theta$, which is the angle between the fast axis of the waveplates and the horizontal axis. In Fig. 1 (a), we use a modified version of the standard method for measuring the Stokes parameters [17, 29]. The detection scheme consists of another QWP, a linear polarizer, and an intensity detector. The following four intensity measurements are required to measure all four Stokes parameters: $I(0^\circ, 0^\circ)$, $I(0^\circ, 90^\circ)$, $I(0^\circ, 45^\circ)$, and $I(45^\circ, 45^\circ)$, where $I(\psi, \phi)$ is the intensity measured by the detector when the fast axis of the QWP (in the detection scheme) is at angle $\psi$ w.r.t. the horizontal axis and the axis of transmission of the polarization is at angle $\phi$ w.r.t. the horizontal axis. The Stokes parameters can be calculated from the intensity measurements using the following equations,

$$s_0 = I(0^\circ, 0^\circ) + I(0^\circ, 90^\circ),$$

$$s_1 = I(0^\circ, 0^\circ) - I(0^\circ, 90^\circ),$$

$$s_2 = 2I(45^\circ, 45^\circ) - s_0,$$

$$s_3 = 2I(0^\circ, 45^\circ) - s_0.$$

Non-physical results produced via this method mostly come from fluctuations of the laser light between the four intensity measurements. In the polarimeter method (Fig. 1 (b)), the measurement is done solely with a polarimeter, and the Stokes parameters are given automatically. The polarimeter employs a spinning waveplate and curve-fitting technique to obtain the Stokes parameters [30]. While this method is fast (sampling rate up to 400 Hz), the polarimeter
We want to find the corrected coherency matrix, \( J \), which satisfies Eq. (8) with equality. Thus, our problem is convex. Since the solution to a problem where the objective and constraint functions are convex, i.e. they satisfy the condition that for all \( x, y \in C \)

\[
f_i(tx + (1-t)y) \leq t f_i(x) + (1-t) f_i(y),
\]

where \( t \in [0, 1] \) and \( C \) is some convex set [31]. A convex set \( C \) is defined as a set where, for all \( x, y \in C \) and \( t \in [0, 1] \), \( tx + (1-t)y \in C \). The constraint presented in Eq. (7b) restricts the possible choices for \( J_{\text{corrected}} \) to the set of positive semi-definite matrices, which are known to form a convex set [32]. The objective function Eq. (7a) in our problem is convex due to the definition of norms, namely that they are subadditive and absolutely scalable. Finally, we know that Eq. (7c) is convex because of the linearity of trace, \( \text{Tr}(aA + \beta B) = a \text{Tr}(A) + \beta \text{Tr}(B) \), which satisfies Eq. (8) with equality. Thus, our problem is convex. Since the solution to a convex optimization problem is unique and provides a lower bound on more general optimization problems, the ability to construct and solve a convex optimization problem has proven useful in a wide variety of topics such as the reconstruction of quantum channels [33], the selection of sensors to minimize error in a measurement [34], and multi-period trading [35]. As such, a number of tools and techniques for solving convex optimization problems efficiently have been developed. In particular we opted to use Matlab’s CVX library [36, 37] due to its ability to handle complex matrices and its ease of use compared to other options.

We also want to determine the upper and lower bounds on the DOP for \( J_{\text{corrected}} \) given some tolerance \( \epsilon \) for the acceptable difference between \( J_{\text{corrected}} \) and \( J_{\text{measured}} \). Let \( J_{\text{min}} \) denote the value for \( J_{\text{corrected}} \) with the lowest DOP for the specified tolerance; likewise, \( J_{\text{max}} \) be the value for \( J_{\text{corrected}} \) with the largest DOP. If we consider the representation of coherence matrices on the Poincaré sphere as shown in Fig. 2, the Stokes vectors for \( J_{\text{min}} \) and \( J_{\text{max}} \) must lie within a ball of radius \( \epsilon \) centered on \( J_{\text{measured}} \). The DOP of a coherency matrix corresponds to the length of its Stokes vector. Thus, if a ray (labelled \( \tau \) in Fig. 2) is drawn from the origin of the Poincaré sphere into the direction of the Stokes vector for \( J_{\text{measured}} \), then \( J_{\text{max}} \) and \( J_{\text{min}} \) are found where this ray
Fig. 2. Schematic for the construction of a convex optimization problem to determine the coherency matrices with the highest and lowest DOP (respectively $J_{\text{max}}$ and $J_{\text{min}}$) for a given $J_{\text{measured}}$ and error tolerance $\epsilon$. The blue arrows are the Stokes vectors of the corresponding coherency matrices. $\mathbf{I}/2$ corresponds to the zero vector, where $\mathbf{I}$ is the identity matrix.

Intersects the $\epsilon$-ball. To make this problem easier to formalize, we parameterize the ray in Fig. 2 by $\tau$ and require that the sought $J_{\text{min}}$ and $J_{\text{max}}$ lie on the ray. This is achieved by the following expression:

$$\left\| J_{\text{min, max}} - \tau J_{\text{measured}} - \frac{1 - \tau}{2} \mathbf{I} \right\| = 0. \quad (9)$$

The smaller the $\tau$, the closer the sought coherency matrices to the origin. Hence, $J_{\text{max}}$ and $J_{\text{min}}$ can be found by maximizing and minimizing $\tau$, respectively. It is important to note that the value of $\tau$ not only depends upon $J_{\text{measured}}$ and $\epsilon$, but also on the constraint both $J_{\text{min, max}}$ be non-negative as in the problem (7a–(7c). Because the l.h.s. of Eq. (9) is a norm of a linear expression over a convex set, the constrain function (9) is convex. Now we can combine the constraints (7b), (7c), and (9) with the requirement that $J_{\text{min, max}}$ be $\epsilon$-close to $J_{\text{measured}}$ to formulate the following convex optimization problems for obtaining physically corrected coherency matrices $J_{\text{min, max}}$ with the minimal and maximal DOP:

To recover the physically-constrained coherency matrix $J_{\text{min}}$ with the minimal DOP that is $\epsilon$-close to the measured $J_{\text{measured}}$, we solve the convex optimization problem

$$\min_{\tau, J_{\text{min}}} \quad \tau$$

subject to

$$\left\| J_{\text{min}} - \tau J_{\text{measured}} - \frac{1 - \tau}{2} \mathbf{I} \right\| = 0,$$

$$\left\| J_{\text{min}} - J_{\text{measured}} \right\| \leq \epsilon,$$

$$\text{Tr}(J_{\text{min}}) = 1,$$

$$J_{\text{min}} \geq 0. \quad (10)$$

Likewise, to obtain the physically-constrained coherency matrix $J_{\text{max}}$ with the maximal DOP
that is $\epsilon$-close to the measured $J_{\text{measured}}$, we solve

$$\begin{align*}
\text{maximize} & \quad \tau, J_{\text{max}} \\
\text{subject to} & \quad \|J_{\text{max}} - \tau J_{\text{measured}} - \frac{1 - \tau}{2} I\| = 0, \\
& \quad \|J_{\text{max}} - J_{\text{measured}}\| \leq \epsilon, \\
& \quad \text{Tr}(J_{\text{max}}) = 1, \\
& \quad J_{\text{max}} \geq 0.
\end{align*}$$

(11)

4. Results

To solve Eqs. (7a)–(7c), (10), and (11), we had to specify a norm. We chose to use the Frobenius norm, which is defined by

$$\|A\|_F = \sqrt{\text{Tr}(A^\dagger A)},$$

(12)

because, according to Eq. (6), the Frobenius norm of a coherency matrix is related to its DOP. While the use of the Frobenius norm may have been the natural choice given our interest in the DOP, the convex optimization problems outlined in Eqs. (7a)–(7c), (10), and (11) can be solved using any norm.

The results of our program, shown in Code 1 [38], applied to four sets of measured coherency matrices of linearly polarized light and four sets of measured coherency matrices of elliptically polarized light are respectively displayed in Fig. 3 and Fig. 4. In the non-physical cases where the DOP of $J_{\text{measured}}$ is greater than 1, the DOP of $J_{\text{corrected}}$ obtained from solving the minimization problem (7a)–(7c) is exactly 1. In the cases where $J_{\text{measured}}$ is physical, the obtained $J_{\text{corrected}}$ is equal to $J_{\text{measured}}$. In both measurement schemes shown in Fig. 1, we found that measuring the vertically polarized light through a QWP gave more non-physical results than that of a HWP.

We also found $J_{\text{min}, \text{max}}$ with the minimum and maximum DOP given a tolerance parameter of $\epsilon = 0.1$, which was done by solving the optimization problems defined in Eqs. (10) and (11), respectively. In most cases, the maximum DOP is found to be 1 and the minimum DOP is a constant value, which depends on $\epsilon$, lower than the measured DOP. Given the constraints in Eqs. (10) and (11), this is to be expected. There are a few exceptional points. In Fig. 4(c) at $\theta = 30^\circ$, 40°, the maximum DOP is lower than 1. This is caused by the constraint that $J_{\text{max}}$ be $\epsilon$-close to $J_{\text{measured}}$, which makes the Stoke vector for $J_{\text{max}}$ lie inside the Poincaré sphere. Another exceptional case is demonstrated by the missing points on both the max DOP and min DOP plots in Fig. 4(a) at $\theta = 70^\circ$. Here, due to the same constraint above, the vectors corresponding to both $J_{\text{max}}$ and $J_{\text{min}}$ lie outside the Poincaré sphere, and thus, no solutions for both Eqs. (10) and (11) can be found.

The normalized Stokes vectors corresponding to $J_{\text{corrected}}$ and $J_{\text{measured}}$ for each of the data points are displayed in the right column of Fig. 3 for linearly polarized light and in the right column of Fig. 4 for elliptically polarized light. In every case, the vectors for $J_{\text{corrected}}$ and $J_{\text{measured}}$ are parallel. In the cases where the vector of $J_{\text{measured}}$ is outside the Poincaré sphere, the vector of $J_{\text{corrected}}$ ends on the surface of the Poincaré sphere. This indicates that our method is successful at preserving the direction of the measured Stoke vectors while correcting for experimental errors.

5. Conclusion

We presented the convex optimization methods for the purpose of robust polarimetry as described in Sec. 3. We have demonstrated the validity of these methods using the experimentally measured results obtained for different polarization states and via different polarimetry schemes described in Sec. 2. The performance of the developed techniques are discussed in Sec. 4. The presented
Fig. 3. DOP of $J_{\text{measured}}$ and $J_{\text{corrected}}$ (left column) and the location of their corresponding vectors on the Poincaré sphere (right column) for linearly polarized light: (a), (b) high intensity light measured with the standard method; (c), (d) high intensity light measured with the polarimeter method; (e), (f) low intensity light measured with the standard method; (g), (h) low intensity light measured with the polarimeter method. In each case, $J_{\text{corrected}}$ was obtained by solving Eqs. (7a)–(7c). The minimum and maximum DOP were calculated from the solutions to Eqs. (10) and (11) using a tolerance of $\epsilon = 0.1$. 
Fig. 4. DOP of $\mathbf{J}_\text{measured}$ and $\mathbf{J}_\text{corrected}$ (left column) and the location of their corresponding vectors on the Poincaré sphere (right column) for elliptically polarized light: (a), (b) low intensity light measured with the standard method; (c), (d) high intensity light measured with the polarimeter method; (e), (f) low intensity light measured with the standard method; (g), (h) low intensity light measured with the polarimeter method. In each case, $\mathbf{J}_\text{corrected}$ was obtained by solving Eqs. (7a)–(7c). The minimum and maximum DOP were calculated from the solutions to Eqs. (10) and (11) using a tolerance of $\epsilon = 0.1$. 
methods do not depend on any \textit{a priori} information or calibration of the components nor on the type of experimental noise or error, and can be easily integrated into the post-processing of many polarimetry protocols.

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\textbf{Disclosures}

The authors declare no conflicts of interest.
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