Abstract. Recently Ozsváth and Szabó defined an invariant of contact structures with values in the Heegaard-Floer homology groups. They also proved that a version of the invariant with twisted coefficients is non trivial for weakly symplectically fillable contact structures. In this article we show that their non vanishing result does not hold in general for the contact invariant with untwisted coefficients. As a consequence of this fact Heegaard-Floer theory can distinguish between weakly and strongly symplectically fillable contact structures.

1. Introduction

Recently Ozsváth and Szabó showed how to associate to any contact manifold \((Y, \xi)\) an isotopy invariant \(c(\xi) \in \widehat{HF}(-Y)/\pm 1\) in the Heegaard-Floer homology of \(-Y\) reduced modulo \(\pm 1\). They also proved that \(c(\xi) = 0\) if \(\xi\) is an overtwisted contact structure, and \(c(\xi)\) is a primitive element of \(\widehat{HF}(-Y)/\pm 1\) if \(\xi\) is Stein fillable. One can get rid of the sign indeterminacy in the definition of \(c(\xi)\) by working with the Heegaard–Floer homology with coefficients in \(\mathbb{Z}/2\mathbb{Z}\). This is the choice we will do throughout this article. The Ozsváth-Szabó contact invariant has already been useful in proving tightness of contact structures which resisted to all previously known techniques: see for example [17, 16, 15].

In this article we study the relation between the Ozsváth-Szabó contact invariant and the symplectic fillability of contact structures. There are two different notions of symplectic fillability. A contact manifold \((Y, \xi)\) is said to be weakly symplectically fillable if \(Y\) oriented by \(\xi\) is the oriented boundary of a symplectic 4-manifold \((X, \omega)\) such that \(\omega|_{\xi} > 0\). A contact manifold \((Y, \xi)\) is said to be strongly symplectically fillable if \(\xi\) is the kernel of a 1-form \(\alpha\) such that \(d\alpha = \omega|_{\xi}\). Strong fillability implies weak fillability, but the converse is not true. The first example of a weakly but not strongly fillable contact manifold was discovered on \(T^3\) by Eliashberg [2], and more examples were constructed by Ding and Geiges [1] on torus bundles over \(S^1\) building on Eliashberg’s.

We will construct infinitely many weakly fillable contact structures whose contact invariant is trivial. These are the first examples of tight contact structures with vanishing Ozsváth–Szabó invariant over \(\mathbb{Z}/2\mathbb{Z}\). More precisely, let

\[
M_0 = T^2 \times [0, 1]/(v, 1) = (Av, 0)
\]
be the mapping torus of the map $A : T^2 \to T^2$ induced by the matrix $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$. Giroux constructed a family of weakly symplectically fillable contact structures $\xi_n$ on $M_0$ for $n \in \mathbb{N}^+$ as follows. Put coordinates $(x, y, t)$ on $T^2 \times \mathbb{R}$ and fix a function $\phi : \mathbb{R} \to \mathbb{R}$. For any $n > 0$ the 1-form

$$\alpha_n = \sin(\phi(t))dx + \cos(\phi(t))dy$$

on $T^2 \times \mathbb{R}$ defines a contact structure $\xi_n$ on $M_0$ provided that

1. $\phi'(t) > 0$ for any $t \in \mathbb{R}$
2. $\alpha_n$ is invariant under the action $(v, t) \mapsto (Av, t - 1)$
3. $(2n - 1)\pi \leq \sup_{t \in \mathbb{R}}(\phi(t + 1) - \phi(t)) < 2n\pi$.

The main result of this article is the following theorem.

**Theorem 1.1.** If $n$ is even, then the Ozsváth–Szabó contact invariant $c(\xi_n)$ is trivial.

Theorem 1.1 should be contrasted with a recent non-vanishing result for the contact invariant with twisted coefficients proved by Ozsváth and Szabó. Associated to any module $A$ over the group ring $\mathbb{Z}[H^1(M, \mathbb{Z})]$ of $H^1(M, \mathbb{Z})$ there is a Heegaard–Floer homology group “with twisted coefficients” $\hat{HF}(M; A)$. The ordinary “untwisted” Heegaard–Floer group is a particular case of this construction with $A = \mathbb{Z}/2\mathbb{Z}$. See [19], Section 8. In this setting the contact invariant $c(\xi)$ can be generalised to an invariant $c(\xi; A)$ with values in $\hat{HF}(-M; A)/\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$, where $\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$ denotes the multiplicative group of the invertible elements in $\mathbb{Z}[H^1(M, \mathbb{Z})]$.

Let $(W, \omega)$ be a weak symplectic filling of the contact manifold $(M, \xi)$. Following [23], we define a $\mathbb{Z}[H^1(M, \mathbb{Z})]$-module structure on $\mathbb{Z}[\mathbb{R}]$ via the ring homomorphism $H^1(M, \mathbb{Z}) \to \mathbb{Z}[\mathbb{R}]$ defined as

$$\gamma \mapsto T^r \gamma \wedge \omega$$

where $T^r$ denotes the group-ring element associated to the real number $r$. The Heegaard-Floer homology group with twisted coefficients in the module $\mathbb{Z}[\mathbb{R}]$ will be denoted by $\hat{HF}(M; [\omega])$. The contact invariant with twisted coefficients of weakly symplectically fillable contact structures satisfies the following non-vanishing theorem.

**Theorem 1.2.** ([23], Theorem 4.2). Let $(W, \omega)$ be a weak symplectic filling of $(M, \xi)$. Then the associated contact invariant $c(\xi, [\omega]) \in \hat{HF}(M; [\omega])/\mathbb{Z}[H^1(M, \mathbb{Z})]^\times$ is non-torsion and primitive.

Theorem 1.2 implies that the “untwisted” Ozsváth–Szabó invariant of a strongly symplectically fillable contact structure is non-trivial, therefore the contact manifolds $(M_0, \xi_n)$ are not strongly symplectically fillable if $n$ is even. Theorem 1.1 shows that, in general, the use of twisted coefficients in the non-triviality theorem for weakly symplectically fillable contact structures cannot be avoided, and that the Heegaard-Floer theory is subtle enough to distinguish between weakly and strongly symplectically fillable contact structures.

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2. Contact Ozsváth–Szabó invariants

2.1. Heegaard–Floer homology. Heegaard–Floer homology is a family of topological quantum field theories for Spin$^c$ three–manifolds introduced by Ozsváth and Szabó in [20, 19, 21]. They associate $\mathbb{Z}/2\mathbb{Z}$–graded Abelian groups $HF(Y, t)$, $HF^\infty(Y, t)$, $HF^-(Y, t)$, and $HF^+(Y, t)$ to any closed oriented Spin$^c$ 3–manifold $(Y, t)$, and homomorphisms

$$F^\circ_{W, s}: HF^\circ(M, t_1) \to HF^\circ(M, t_2)$$

to any oriented Spin$^c$ cobordism $(W, s)$ between two Spin$^c$ manifolds $(M, t_1)$ and $(M, t_2)$. Here $HF^\circ$ denotes any of the four functors $HF$, $HF^+$, $HF^-$, and $HF^\infty$. We write $HF^\circ(Y)$ for the direct sum $\bigoplus_{t \in Spin^c(Y)} HF^\circ(Y)$ and $F^\circ_W$ for the sum $\sum_{s \in Spin^c(W)} F^\circ_{W, s}$. $F^\circ_W$ is a well defined map because $F^\circ_{W, s} \neq 0$ only for finitely many Spin$^c$–structures on $W$. The homomorphisms between Heegaard–Floer homology groups satisfy the following composition rule.

**Theorem 2.1.** ([21, Theorem 3.4). Let $(W_1, s_1)$ be a Spin$^c$–cobordism between $(Y_1, t_1)$ and $(Y_2, t_2)$, and let $(W_2, s_2)$ be a Spin$^c$–cobordism between $(Y_2, t_2)$ and $(Y_3, t_3)$. Denote by $W$ the cobordism between $Y_1$ and $Y_2$ obtained by gluing $W_1$ and $W_2$ along $Y_2$. Then

$$F^\circ_{W_2, s_2} \circ F^\circ_{W_1, s_1} = \sum_{s \in Spin^c(W)} F^\circ_{W, s},$$

where $s|_{W_1} = s_1$ and $s|_{W_2} = s_2$.

The groups $HF^\circ(Y, t)$ are linked to each other by the exact triangles

(1) $$\xymatrix{ & HF^-(Y, t) \ar[r] & HF^\infty(Y, t) \ar[r] & HF^+(Y, t) & }$$

(2) $$\xymatrix{ & \widetilde{HF}(Y, t) \ar[r] & HF^+(Y, t) \ar[r] & HF^+(Y, t) & }$$

These exact triangles are natural, in the sense that they commute with the maps induced by cobordisms.

The Heegaard-Floer homology groups $HF^\circ(Y, t)$ have a natural $\mathbb{Z}/\text{div}(t)$ relative grading, where $\text{div}(t)$ is the divisibility of $c_1(t)$ in $H^2(Y, \mathbb{Z})$. It was shown in [22] that, when $c_1(t)$ is a torsion element, the relative $\mathbb{Z}$–grading admits a natural lift to an absolute $\mathbb{Q}$–grading. In conclusion, for a torsion Spin$^c$–structure $t$ on $Y$ the Ozsváth–Szabó homology groups $HF^\circ(Y, t)$ split as

$$HF^\circ(Y, t) = \bigoplus_{d \in \mathbb{Q}} HF^\circ_d(Y, t).$$

When $t \in Spin^c(Y)$ has torsion first Chern class, there is an isomorphism between the homology groups $\widetilde{HF}_d(Y, t)$ and $\widetilde{HF}_{-d}(-Y, t)$. 
Proposition 2.2. (See [18], Theorem 7.1). Let \((W, \mathfrak{s})\) be a Spin\(^c\) cobordism between two Spin\(^c\) manifolds \((Y_1, \mathfrak{t}_1)\) and \((Y_2, \mathfrak{t}_2)\). If the Spin\(^c\) structures \(\mathfrak{t}_i\) have both torsion first Chern class and \(x \in HF^0(Y_1, \mathfrak{t}_1)\) is a homogeneous element of degree \(d(x)\), then \(F_{W, \mathfrak{s}}(x) \in HF^0(Y_2, \mathfrak{t}_2)\) is also homogeneous of degree
\[
d(x) + \frac{1}{4}(c_1^2(\mathfrak{s}) - 3\sigma(W) - 2\chi(W)).
\]

Notice that \(F_{W}^0\) might map a homogeneous element \(x \in HF^0(Y_1, \mathfrak{t}_1)\) into a non homogeneous element \(F_{W}^0(x) \in HF^0(Y_2)\).

2.2. Definition of the contact invariants. The Ozsváth–Szabó contact invariant is defined using the correspondence between contact structures and open book decompositions of three–manifolds recently discovered by Giroux. An open book decomposition of a 3–manifold \(Y\) is a fibred link \(B \subset Y\) together with a fibration \(\pi : Y \setminus B \to S^1\). The link \(B\) is called the binding of the open book decomposition and the union of a fibre of \(\pi : Y \setminus B \to S^1\) with \(B\) is called a page.

Definition 2.3. ([9], Definition 1). Let \((Y, \xi)\) be a contact 3–manifold. An open book decomposition \((B, \pi)\) of \(Y\) is said to be adapted to \(\xi\) if:

1. \(B\) is transverse to \(\xi\),
2. \(\xi\) is defined by a contact form \(\alpha\) such that \(d\alpha\) is a symplectic form on any fibre of \(\pi\),
3. the orientation of \(B\) induced by the contact structure coincides with the orientation as boundary of the fibres of \(\pi\) oriented by \(d\alpha\).

By [9] Theorem 3 any contact structure on a three manifold admits an adapted open book decomposition. This open book decomposition is not unique, in fact two open book decompositions which differ by the positive plumbing of an annulus are adapted to isotopic contact structures. See [9] Section B. After positive plumbing, we can assume that the binding is connected and pages have genus \(g \geq 2\). Adding a 2–handle along \(B\) with the framing induced by a page we form a cobordism \(V\) between \(Y\) and \(Y_0\), where \(Y_0\) is a 3-manifold fibred over \(S^1\) with fibres of genus \(g \geq 2\). On \(Y_0\) there is a canonical Spin\(^c\)–structure \(\mathfrak{t}_0\) induced by the fibration. \(\widehat{HF}(-Y_0, \mathfrak{t}_0) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) with the summands lying in different degrees for the absolute \(\mathbb{Z}/2\mathbb{Z}\) grading, while \(HF^+(Y_0, \mathfrak{t}_0) = \mathbb{Z}/2\mathbb{Z}\).

See [18] Section 3. We fix a distinguished element \(c_0 \in \widehat{HF}(-Y_0, \mathfrak{t}_0)\) as the homogeneous element of \(\widehat{HF}(-Y_0, \mathfrak{t}_0)\) which is mapped to the non zero element of \(HF^+(Y_0, \mathfrak{t}_0)\) by the natural map \(\widehat{HF}(-Y_0, \mathfrak{t}_0) \to HF^+(Y_0, \mathfrak{t}_0)\). We denote by \(\overline{V}\) the cobordism \(V\) turned upside–down, so that \(\overline{V}\) is a cobordism between \(-Y_0\) and \(-Y\).

Definition 2.4. The Ozsváth–Szabó contact invariant of a contact 3–manifold \((Y, \xi)\) is the element \(c(\xi) \in \widehat{HF}(-Y)\) defined by
\[
c(\xi) = \widehat{F}_{\overline{\pi}}(c_0).
\]

By [18] Theorem 1.3 \(c(\xi)\) is independent of the choice of the open book decomposition adapted to \(\xi\) and is an isotopy invariant. The Ozsváth–Szabó contact invariant is non trivial and detects important topological properties of the contact structures, in fact
Theorem 2.5. ([18], Theorem 1.4 and Theorem 1.5). If \((Y, \xi)\) is overtwisted, then \(c(\xi) = 0\). If \((Y, \xi)\) is Stein fillable, then \(c(\xi) \neq 0\).

The Ozsváth–Szabó contact invariant \(c(\xi)\) encodes the homotopy invariants of \(\xi\), see [18], Proposition 4.6. Any contact structure \(\xi\) on a 3–manifold \(Y\) determines a \(\text{Spin}^c\)–structure \(t_\xi\) on \(Y\), then \(c(\xi) \in \text{HF}^{\text{cc}}(-Y, t_\xi)\). If the first Chern class of \(\xi\) is torsion, by [10] Theorem 4.16 the homotopy type of \(\xi\) is determined by the \(\text{Spin}^c\)–structure \(t_\xi\) and by the \(\mathbb{Q}\)–valued Gompf invariant \(d_3(\xi)\) defined as follows.

**Definition 2.6.** (See [10], Definition 4.2). Let \(\xi\) be an oriented tangent plane field on the 3–manifold \(Y\) with torsion first Chern class, and let \((X, J)\) be a almost complex 4–manifold such that \(Y\) is the boundary of \(X\) and \(\xi = TY \cap J(TY)\) is the field of complex lines in \(TY\). Then we define

\[
d_3 = \frac{1}{4}(c_1(J)^2 - 2\chi(X) - 3\sigma(X))
\]

where \(\chi\) denote the Euler characteristic, \(\sigma\) the signature, and \(c_1(J)^2\) is defined because \(c_1(\xi) = c_1(J)|_Y\) is torsion.

By [18], Proposition 4.6, if \(c_1(\xi)\) is a torsion element of \(H^2(Y, \mathbb{Z})\), then \(c(\xi)\) is an homogeneous element of degree \(-d_3(\xi) - \frac{1}{2}\).

**Theorem 2.7.** ([18], Theorem 4.2 and [17], Theorem 2.3). If the contact manifold \((Y', \xi')\) is obtained from the contact manifold \((Y, \xi)\) by Legendrian surgery along a Legendrian knot \(L\), and \(W\) is the cobordism between \(Y\) and \(Y'\) obtained by adding a 2–handle to \(Y \times [0, 1]\) along \(L \times \{1\}\) with framing \(-1\) with respect to the contact framing, then

\[
\overline{F}_W(c(\xi')) = c(\xi)
\]

where \(\overline{W}\) denotes the cobordism \(W\) turned upside–down.

The space of oriented contact structures on \(Y\) has a natural involution.

**Definition 2.8.** For any contact structure \(\xi\) on a 3–manifold \(Y\) we denote by \(\overline{\xi}\) the contact structure on \(Y\) obtained from \(\xi\) by inverting the orientation of the planes.

This operation is compatible with the conjugation of the \(\text{Spin}^c\)–structure defined by the contact structure, in fact \(t_\xi = \overline{t_\xi}\). There is an isomorphism \(\overline{\mathfrak{J}} : HF^0(-Y, s) \to HF^0(-Y, \overline{s})\) defined in [19], Theorem 2.4. We recall that the isomorphism \(\overline{\mathfrak{J}}\) preserves the \(\mathbb{Z}/2\mathbb{Z}\)–grading of the Heegaard–Floer homology groups and is a natural transformation in the following sense.

**Proposition 2.9.** ([21], Theorem 3.6) Let \((W, s)\) be a \(\text{Spin}^c\)–cobordism between \((Y_1, t_1)\) and \((Y_2, t_2)\). Then the following diagram

\[
\begin{array}{ccc}
HF^0(Y_1, t_1) & \xrightarrow{F^0_{W, s}} & HF^0(Y_2, t_2) \\
\downarrow \mathfrak{J} & & \downarrow \mathfrak{J} \\
HF^0(Y_1, \overline{t_1}) & \xrightarrow{F^0_{W, \overline{s}}} & HF^0(Y_2, \overline{t_2})
\end{array}
\]

commutes.
The isomorphism $\mathfrak{J}$ commutes also with the maps in the exact triangles (1) and (2) relating the different Heegaard–Floer homology groups.

**Theorem 2.10.** Let $(Y, \xi)$ be a contact manifold, then

$$c(\xi) = \mathfrak{J}(c(\xi)).$$

**Proof.** If $(B, \pi)$ is an open book decomposition adapted to $\xi$, then the open book decomposition $(-B, \bar{\pi})$, where $-B$ denotes the binding $B$ with opposite orientation and $\bar{\pi}$ is the composition of $\pi$ with the complex conjugation on $S^1$, is adapted to $\bar{\xi}$. The pages of $(-B, \bar{\pi})$ are the pages of $(B, \pi)$ with opposite orientation, so the fibration on $Y_0$ induced by $(-B, \bar{\pi})$ differs from the fibration induced by $(B, \pi)$ for the orientation of the fibres, therefore its canonical $\text{Spin}^c$–structure is the conjugate of $t_0$. The commutative diagram

$$\begin{array}{ccc}
\widehat{HF}(\bar{Y}_0', \bar{t}_0) & \longrightarrow & HF^+(\bar{Y}_0, \bar{t}_0) \\
\mathfrak{J} & \downarrow & \mathfrak{J} \\
\widehat{HF}(Y_0', t_0) & \longrightarrow & HF^+(Y_0, t_0)
\end{array}$$

together with the fact that $\mathfrak{J}$ is an isomorphism and preserves the $\mathbb{Z}/2\mathbb{Z}$–grading of the Heegaard–Floer homology groups shows that the distinguished element of $\widehat{HF}(\bar{Y}_0', \bar{t}_0)$ is $\bar{t}_0 = \mathfrak{J}(c_0)$, therefore

$$c(\xi) = \widehat{F}_{\bar{\nu}}(\bar{t}_0) = \widehat{F}_{\bar{\nu}}(\mathfrak{J}(c_0)) = \mathfrak{J}(\widehat{F}_{\bar{\nu}}(c_0)) = \mathfrak{J}(c(\xi)).$$

$\square$

### 2.3. Ozsváth–Szabó contact invariants of strongly symplectically fillable contact structures.

In this section we prove a non vanishing theorem for the Ozsváth–Szabó contact invariant of strongly symplectically fillable contact structures. This theorem can be easily derived as a corollary of the more general non vanishing Theorem 1.2 proved by Ozsváth and Szabó using the twisted coefficients, however it is also possible to adapt the proof of Theorem 1.2 so that we do not need to use Heegaard-Floer homologies with twisted coefficients. We choose this second option, but the proof requires some more Heegaard–Floer machinery.

From the exact triangle (1) we define a fifth group $HF^{red}(Y, t)$ as the kernel of the map

$$HF^{-}(Y, t) \rightarrow HF^\infty(Y, t)$$

or, equivalently, as the cokernel of the map

$$HF^\infty(Y, t) \rightarrow HF^+(Y, t).$$

The group $HF^{red}(Y, t)$ is always finitely generated. Let $W$ be an oriented cobordism between the 3–manifolds $Y_1$ and $Y_2$. An admissible cut of $W$ ([21], Definition 8.3) is a 3–manifold $N \subset W$ which divides $W$ into two pieces $W_1$ and $W_2$ such that $b_2^+(W_i) > 0$ for $i = 1, 2$, and the connecting homomorphism $\delta : H^1(N, \mathbb{Z}) \rightarrow H^2(W, \partial W)$ of the Meyer–Vietoris sequence of the pair $(W_1, W_2)$ is trivial. It is shown in [21], Example 8.4 that an admissible cut of $W$ always exists if $b_2^+(W) > 1$. By [21] Lemma 8.2 the maps

$$F_{W_1, s}^\infty : HF^\infty(Y_1, s|_{Y_1}) \rightarrow HF^\infty(N, s|_N)$$

$$F_{W_1, s}^\infty : HF^\infty(N, s|_N) \rightarrow HF^\infty(Y_2, s|_{Y_2})$$
vanish for any $Spin^c$--structure $s$ on $W$, therefore an easy diagram chase on the exact triangle (1) allows us to define a “mixed” homomorphism $F_{W,s}^{mix} : HF^-(Y_1, t_1) \to HF^+(Y_2, t_2)$ which factors through $HF^{red}(N, s)$. By [21], Theorem 8.5 the mixed map $F_{W,s}^{mix}$ does not depend on the particular admissible cut used to define it.

The mixed map can be used to define a numerical invariant of smooth four–manifolds with $b^+_2 > 1$ which is conjecturally equal to the Seiberg–Witten invariant. If $X$ is a closed oriented 4–manifold, after removing two balls we can view it as a cobordism from $X$ where $\Phi$ bundles over $Y$. Let $W$ be the cobordism

\[ \Theta^+ = \Theta^+ \left( Y_0, \xi \right) = \Theta^+ = \Theta^+ \left( Y_1, t_1 \right) \]

\[ \Theta^- = \Theta^- \left( Y_0, \xi \right) = \Theta^- = \Theta^- \left( Y_1, t_1 \right) \]

is the map

\[ \Phi_X : Spin^c(X) \to \mathbb{Z}/2\mathbb{Z} \]

where $\Phi_X(s)$ is defined as the coefficient of $\Theta^+$ in $F_{X,s}^{mix}(\Theta^-)$.

We denote by $c^+(\xi)$ the image of $c(\xi)$ in $HF^+(-Y)$. Theorem [27] can be refined in the following way.

**Lemma 2.11.** Suppose that $(Y', \xi')$ is obtained from $(Y, \xi)$ by Legendrian surgery on a Legendrian link $L$ and that $(W, \omega)$ is the symplectic cobordism from $(Y, \xi)$ to $(Y', \xi')$ induced by this surgery. Then we have

\[ F^{+}_{W, \xi}(c^+(\xi')) = c^+(\xi) \]

for the canonical spin$^c$-structure $\xi$ associated to the symplectic structure on $W$, and

\[ F^{+}_{W,s}(c^+(\xi')) = 0 \]

for any spin$^c$-structure $s$ on $W$ with $s \neq \xi$.

**Proof.** As in the proof of [17] Theorem 2.3 there exists an open book decomposition of $Y$ adapted to the contact structure $\xi$ so that the surgery link lies on a page. We can also assume that the binding is connected and the pages have genus $g > 1$. An open book decomposition adapted to $\xi'$ is obtained from the open book decomposition adapted to $\xi$ by composing the monodromy with right–handed Dehn twists along the surgery link. Let $Y_0$ and $Y'_0$ be the 3–manifolds obtained from $Y$ and $Y'$ respectively by 0-surgery on the binding, and let $V, V'$ be the induced cobordisms. The surgery on $L$ induces cobordisms $W$ between $Y$ and $Y'$ and $W_0$ from $Y_0$ to $Y'_0$. Both $Y_0$ and $Y'_0$ are surface bundles over $S^1$, and $W_0$ admits a Lefschetz fibration over the annulus. Let $t_0$ and $t'_0$ be the Spin$^c$-structures on $Y_0$ and $Y'_0$ respectively determined by the fibration, and let $t_0$ be the canonical Spin$^c$-structure on $W_0$ determined by the Lefschetz fibration. By [21], Theorem 5.3,

\[ F^{+}_{W_0,t_0} : HF^+(-Y'_0, t'_0) \to HF^+(-Y_0, t_0) \]

is an isomorphism, while the maps

\[ F^{+}_{W_0,s} : HF^+(-Y'_0, t') \to HF^+(-Y_0, t) \]

are trivial when $s \neq t_0$.

Let $W'$ be the cobordism $W' = W_0 \cup_{Y_0} V' = V \cup_Y W$ from $Y_0$ to $Y'$. Since the cobordism $V'$ is obtained by adding a unique 2-handle along a homologically non trivial curve, the
restriction map $H^2(W', \mathbb{Z}) \to H^2(W_0, \mathbb{Z})$ is an isomorphism, therefore there is a unique Spin$^c$-structure $\xi'_0$ on $W$ which extends $\xi_0$. By the composition formula [21] Theorem 3.4 $F_{W'}^+\xi'_0 = F_{V'}^+ \circ F_{W_0, b_0}^+$ and for any other Spin$^c$-structure $s \neq \xi'_0$ the map $F_{X, s}$ is trivial. Let $s'$ be the restriction of $\xi'_0$ to $W$, then the diagram

$$
\begin{array}{ccc}
HF^+(-Y', \xi'_0) & \xrightarrow{F_{W', \xi'_0}^+} & HF^+(-Y_0, t_0) \\
F_{\xi'_0}^- & & F_{\xi}^-
\end{array}
$$

commutes and $F_{W_0, s}^+ = 0$ for any $s \neq s'$. To finish the proof, we have to identify $s'$ with $\xi$.

By [3], Theorem 1.1, the symplectic structure induced by the Lefschetz fibration on $W_0$ extends over the 2-handle $V'$, thus we obtain a symplectic structure $\omega'$ on $W'$ with canonical Spin$^c$-structure $\xi'_0$. The restriction of $\omega'$ to $W$ coincides with the symplectic structure on $W$ induced by the Legendrian surgery, therefore $s' = \xi$.

We have stated Lemma 2.11 in the form in which we are going to use it, however it can be proved in the same way for the stronger contact invariant in $\overline{HF}(-Y)$ with integer coefficients.

**Lemma 2.12.** Let $(Y, \xi)$ be a contact manifold, then there exists a concave symplectic filling $(W', \omega_{W'})$ of $(Y, \xi)$ with canonical Spin$^c$-structure $\xi_{W'}$ such that $b_2^+(W') > 1$ and

$$c^+(\xi) = F_{\overline{W'},\xi_{W'}}^{\text{mix}}(\Theta^-).$$

**Proof.** Combining [3], Theorem 1.1 and [5] Lemma 3.1 there is a Stein fillable contact manifold $(Y', \xi')$ and a symplectic cobordism $(V_1, \omega_{V_1})$ from $(Y, \xi)$ to $(Y', \xi')$ so that $Y'$ is a rational homology sphere and $V_1$ is composed by 2-handles attached in a Legendrian way. By [25] Lemma 1 there is a concave filling $(V_2, \omega_{V_2})$ of $(Y', \xi')$ with canonical Spin$^c$-structure $\xi_{V_2}$ such that $b_2^+(V_2) > 1$ and $c^+(\xi') = F_{\overline{V_2},\xi_{V_2}}^{\text{mix}}(\Theta^-)$.

Let $(W', \omega_{W'})$ be the concave filling of $(Y, \xi)$ obtained by gluing $(V_1, \omega_{V_1})$ and $(V_2, \omega_{V_2})$ along $(Y', \xi')$, and let $\xi_{V_1}$ be the canonical Spin$^c$-structures of $(V_1, \omega_{V_1})$. Since $Y'$ is a rational homology sphere, $H^2(W', \mathbb{Z}) = H^2(V_1, \mathbb{Z}) \oplus H^2(V_2, \mathbb{Z})$ therefore there exists a unique Spin$^c$-structure $\xi_{W'}$ on $W'$ which restricts to $\xi_{V_1}$ on $V_1$ and to $\xi_{V_2}$ on $V_2$. The composition formula [21], Theorem 3.4, together with Lemma 2.11 yields

$$c^+(\xi) = F_{\overline{V_1},\xi_{V_1}}^+, F_{\overline{V_2},\xi_{V_2}}^{\text{mix}}(\Theta^-) = F_{\overline{W'},\xi_{W'}}^{\text{mix}}(\Theta^-).$$

**Theorem 2.13.** Let $(Y, \xi)$ be a strongly symplectically fillable contact manifold, then $c(\xi) \neq 0$.

**Proof.** Let $(W_1, \omega_1)$ be a strong symplectic filling of $(Y, \xi)$, and let $(W_2, \omega_2)$ be the concave symplectic filling considered in Lemma 2.12 Gluing $(W_1, \omega_1)$ and $(W_2, \omega_2)$ we obtain a closed symplectic manifold $(X, \omega)$ with $b_2^+(X) > 1$. The composition formula [21] Theorem
Remark 2.14. Actually the proof of Theorem 2.13 proves the stronger fact that, if we see \( \partial \mathcal{W}_0 \).

Depend on the function \( \varphi \) structure on \( M \) giving \( \text{Spin}^c \) for the canonical \( X \) for any other \( \text{Spin}^c \)-structure in the sum we have \( c_1(s) - c_1(\xi_X) \in \delta(\alpha(s)) \) for \( \alpha(s) \in H^1(Y, \mathbb{Z}) \), where \( \delta \) is the homomorphism \( H^1(Y) \to H^2(X) \) in the Meyer–Vietoris exact sequence for the pair \((W_1, W_2)\), therefore

\[
\langle c_1(s) - c_1(\xi_X), [\omega]\rangle_X = \langle \alpha(s), [\omega]_Y \rangle_Y = 0
\]

in fact \( \omega|_Y \) is exact because \( W_1 \) is a strong filling.

By [24] Theorem 1.1 the only non zero term in the sum is \( \Phi_X(\xi_X) = 1 \), therefore

\[
F^+_{W_1, W_2}(c(\xi)) = c(\xi_0)
\]

for the canonical \( \text{Spin}^c \)-structure of \((W_1, \omega_1)\) and

\[
F^+_S(c(\xi)) = 0
\]

for any other \( \text{Spin}^c \)-structure on \((W_1, \omega_1)\) with

\[
\langle c_1(s), [\omega_1]\rangle_{W_1} = \langle c_1(\xi_{W_1}), [\omega_1]\rangle_{W_1}.
\]

3. Weakly fillable contact structures with trivial untwisted \( \mathbb{Z}/2\mathbb{Z} \)

Ozsváth–Szabó contact invariant

3.1. Tight contact structures on \( M_0 \). Let \( M_0 \) be the \( T^2 \)-bundle over \( S^1 \) with monodromy map \( A : T^2 \times \{1\} \to T^2 \times \{0\} \) given by \( A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \).

Put coordinates \((x, y, t)\) on \( T^2 \times \mathbb{R} \) and fix a function \( \phi : \mathbb{R} \to \mathbb{R} \). For any \( n > 0 \) the 1-form

\[
\alpha_n = \sin(\phi(t))dx + \cos(\phi(t))dy
\]

on \( T^2 \times \mathbb{R} \) defines a contact structure \( \xi_n \) on \( M_0 \) provided that

(1) \( \phi'(t) > 0 \) for any \( t \in \mathbb{R} \)
(2) \( \alpha_n \) is invariant under the action \((v, t) \mapsto (Av, t - 1)\)
(3) \( (2n - 1)\pi \leq \sup_{t \in \mathbb{R}} (\phi(t + 1) - \phi(t)) < 2n\pi \)

The main results about this family of contact structures are the following.

Theorem 3.1. ([8], Proposition 2 and Theorem 6). The contact structures \( \xi_n \) do not depend on the function \( \phi \) up to isotopy, and are all universally tight and distinct.

Theorem 3.2. ([13], Theorem 0.1). The tight contact structures \( \xi_n \) are the only tight contact structures on \( M_0 \) up to isotopy.
Theorem 3.3. (\cite{H}, Theorem 1). For any $n \in \mathbb{N}$, $\xi_n$ is weakly symplectically fillable. There is a number $n_0$ such that, for any $n > n_0$, $\xi_n$ is not strongly symplectically fillable.

The fibration on $M_0$ admits a transverse 1–dimensional foliation induced by the foliation by segments on $T^2 \times [0, 1]$. Let $F$ be the image of $\{0\} \times [0, 1]$ in $M_0$, then $F$ is Legendrian with respect to the contact structure $\xi_n$ for all $n$.

The manifold $M_0$ has a presentation as 0-surgery on the right-handed trefoil knot $K$, in fact the complement of $K$ in $S^3$ fibres over $S^1$ with fibre the holed torus and the monodromy acts on the homology of the fibre as $A = \left( \begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right)$ for some choice of coordinates in the fibre. Moreover the identification between $M_0$ and the 0–surgery on $K$ can be chosen so that the complement of a tubular neighbourhood of $K$ in $S^3$ is mapped diffeomorphically into the complement of a tubular neighbourhood of $F$ in $M_0$ and the meridian of $K$ is mapped to a longitude of $F$.

We perform a change of coordinates in a neighbourhood of $F$ to determine what longitude of $F$ corresponds to the meridian of $K$ and to compute the twisting number of $\xi_n$ along $F$ induced by this longitude.

Lemma 3.4. Let $R = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ be the rotation by angle $-\frac{\pi}{3}$. Then $A$ is conjugate to $R$ in $GL^+(2, \mathbb{R})$.

Proof. $A$ and $R$ are conjugated in $GL(2, \mathbb{C})$ because they have the same characteristic polynomial with distinct roots, therefore they are conjugate also in $GL(2, \mathbb{R})$ because they are both real. Let $B \in GL(2, \mathbb{R})$ be a matrix such that $BAB^{-1} = R$. For any $x \in \mathbb{R}^2 \setminus \{0\}$ we have $x \wedge Ax \neq 0$ because $A$ has no real eigenvalues, therefore, after identifying $\mathbb{A}^2 \mathbb{R}^2$ to $\mathbb{R}$ using the canonical basis, $x \wedge Ax$ has constant sign as a function $\mathbb{R}^2 \to \mathbb{R}$. A direct computation at $x = (0, 1)$ shows that $x \wedge Ax$ is negative. For the same reason, $x \wedge Rx$ is also negative, therefore $\det B > 0$ because $x \wedge Rx = B^{-1}Bx \wedge B^{-1}ABx = (\det B)^{-1}Bx \wedge ABx$. \hfill $\Box$

Lemma 3.5. The twisting number of $\xi_n$ along the Legendrian curve $F$ is $tn(F, \xi_n) = -n$.

Proof. Let $U$ be a small $A$-invariant neighbourhood of $(0, 0)$ in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ so that

$$V = U \times [0, 1]/(v, 1) = (Av, 0)$$

is a standard neighbourhood of $F$. Then $B^{-1}$ is defined on $U$ and $U_0 = B^{-1}(U)$ is a $R$-invariant neighbourhood of $(0, 0)$, i.e. a disc centred in $(0, 0)$. In the coordinates $(x', y', t)$ of $U_0 \times \mathbb{R}$ the 1-form $\alpha_n$ can be written as

$$\alpha_n = \sin(2\pi(n + \frac{5}{6})t)dx' + \cos(2\pi(n + \frac{5}{6})t)dy'.$$

By Lemma 3.4 the leaves of the transverse foliation in the boundary of the neighbourhood of $K$ have slope $-\frac{1}{6}$, therefore they intersect the meridian of $K$ once. If we put coordinates $(\theta, t)$ on $\partial U_0 \times I$, then the longitude of $F$ corresponding to the meridian of $K$ is the image in $\partial V$ of the arc $t \mapsto (e^{\frac{2\pi}{6}t}, t)$ (the dotted curve in Figure 1) because it intersect the leaves of the transverse foliation only once. A dividing curve of $\xi_n$ is isotopic to the image of the arc $t \mapsto (e^{-2\pi(n + \frac{5}{6})t}, t)$ therefore the twisting number of $\xi_n$ along $F$, which is the algebraic

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The boundary of $V$. The inner circle is glued to the outer one after a rotation of $-\frac{\pi}{3}$. The dotted line closes to a longitude of $V$, the radial lines close to a leaf of the transverse foliation and the bold line closes to a dividing curve for $\xi_0$.

intersection of a dividing curve with the longitude, is $-n$. Figure 1 shows what happens for $n = 1$. □

Lemma 3.6. If $L \subset M_0$ is a Legendrian curve which is smoothly isotopic to $F$, then $tn(L, \xi_n) \leq tn(F, \xi_n)$

Proof. Since $A^6 = I$, $M_0$ has a six-fold cover with total space $T^3$ induced by a cover of $S^1$. Let $\hat{F}_3$ and $\hat{L} \subset T^3$ be the pre-images of $F$ and $L$ respectively. By [12], Theorem 7.6, $\hat{F}_3$ maximises the twisting number in its smooth isotopy class. The lemma follows from the obvious monotonicity of the twisting number under finite coverings. □

Since the right-handed trefoil can be put in Legendrian form with Thurston-Bennequin invariant 1, this surgery presentation yields a Stein fillable contact structure on $M_0$.

Proposition 3.7. The Stein fillable contact structure on $M_0$ described by the presentation of $M$ as 0-surgery on the right-handed trefoil knot $K$ is $\xi_1$.

Proof. By Theorem 3.2, the Stein fillable contact structure on $M_0$ is isotopic to $\xi_k$ for some $k \in \mathbb{N}$.

It is easy to make the meridian of $K$ Legendrian with Thurston–Bennequin invariant $-1$ in the standard tight contact structure of $S^3$, therefore $tn(F, \xi_k) \geq -1$ because the image of the meridian of $K$ is isotopic to $F$ as a framed knot in $M_0$. By Lemma 3.5 and Lemma 3.6, this is possible only if $k = 1$. □

3.2. Tight contact structures on $-\Sigma(2, 3, 6n + 5)$. The manifold $-\Sigma(2, 3, 6n + 5)$ is obtained from $M_0$ by $-(n + 1)$–surgery on $F$. For any $n \in \mathbb{N}$ and $n \geq 2$ we define $\mathcal{P}_n^* = \{-n + 1, -n + 3, \ldots, n - 3, n - 1\}$. If $n$ is even, then $0 \not\in \mathcal{P}_n^*$ and we define $\mathcal{P}_n = \mathcal{P}_n^* \cup \{0\}$. In the following we will always consider $n$ even, although some of the facts that we are going to prove are true for any $n$. 

Let $S_+$ and $S_-$ denote the operations of positive and negative stabilisation defined, for example, in [11], Section 2.7. Given $i \in \mathcal{P}_n^*$, denote the contact structure on $-\Sigma(2, 3, 6n+5)$ obtained by Legendrian surgery on $(M_0, \xi_1)$ along the Legendrian knot $S_+^{(n-1+i)/2}S_-^{(n-1-i)/2}(F)$ by $\eta_i$. We denote the tight contact structure on $-\Sigma(2, 3, 6n+5)$ obtained by Legendrian surgery on $(M_0, \xi_n)$ along $F$ by $\eta_0$.

The contact manifolds $(-\Sigma(2, 3, 6n+5), \eta_i)$ for $i \in \mathcal{P}_n^*$ are the Stein fillable contact manifolds considered in [13], in fact $(M_0, \xi_1)$ is the Stein fillable contact manifold obtained by Legendrian surgery on a positive trefoil knot in $S^3$ with Thurston-Bennequin invariant 0 by Proposition 3.7 and performing Legendrian surgery on a stabilisation of $F$ is equivalent to performing Legendrian surgery on a stabilisation of a meridian of the trefoil knot.

**Proposition 3.8.** Let $\overline{\eta}_i$ be the contact structure obtained from $\eta_i$ by reversing the orientation of the contact planes. Then $\overline{\eta}_i$ is isotopic to $\eta_{-i}$.

**Proof.** For any $n \in \mathbb{N}^+$ $(M_0, \xi_n)$ is isotopic to $(M_0, \xi_n)$. The isotopy is induced by a translation in the $t$ direction in the cover $T^2 \times \mathbb{R}$, therefore it fixes $F$. We denote $S_+^{(n-1+i)/2}S_-^{(n-1-i)}(F)$ thought of as a Legendrian knot in $(M_0, \xi_n)$ by $S_+^{(n-1+i)/2}S_-^{(n-1-i)/2}(F)$. Since changing the orientation of the planes changes positive stabilisations into negative ones and vice versa, $S_+^{(n-1+i)/2}S_-^{(n-1-i)/2}(F)$ is Legendrian isotopic to $S_+^{(n-1-i)/2}S_-^{(n-1-i)/2}(F)$, therefore inverting the orientation of the planes transforms Legendrian surgery on $S_+^{(n-1+i)/2}S_-^{(n-1-i)}(F)$ into Legendrian surgery on $S_+^{(n-1-i)/2}S_-^{(n-1-i)/2}(F)$. \qed

**Theorem 3.9.** The contact structures $\eta_i$ on $-\Sigma(2, 3, 6n+5)$, with $i \in \mathcal{P}_n$, are all pairwise non isotopic.

**Proof.** By [13], Theorem 4.2, and [14], Corollary 4.2, the contact structures $\eta_i$ with $i \in \mathcal{P}_n^*$ are pairwise non isotopic. In particular, since we are considering $n$ even, $\eta_i$ is never isotopic to $\eta_{-i}$ if $i \in \mathcal{P}_n^*$ because $0 \notin \mathcal{P}_n^*$. Suppose by contradiction that $\eta_0$ is isotopic to $\eta_i$ for some $i \in \mathcal{P}_n^*$. Inverting the orientation of the contact planes and applying Proposition 3.8 we obtain that $\eta_0$ is also isotopic to $\eta_{-i}$. From this it would follow that $\eta_i$ is isotopic to $\eta_{-i}$. \qed

**Remark 3.10.** Using methods from [7] one can prove that $-\Sigma(2, 3, 17)$ admits at most three tight contact structures up to isotopy, therefore Proposition 3.8 gives the classification of the tight contact structures on $-\Sigma(2, 3, 17)$.

### 3.3. Computation of the homotopy invariants.

In this subsection we will compute the Gompf’s three-dimensional homotopy invariant $d_3(\eta_i)$. This computation will show that all $\eta_i$ are homotopic and therefore all their Ozsváth-Szabó contact invariants belong to the same factor of $HF(-M)$.

By [10], Theorem 4.5 (for an easy proof of this theorem for integer homology spheres see also [13], Proposition 2.2), $\eta_i$ is homotopic to $\eta_{i_2}$ as a plane field if and only if $d_3(\eta_i) = d_3(\eta_{i_2})$, where

$$d_3(\eta_i) = \frac{1}{4}(c_1^2(J_i) - 2\chi(X_i) - 3\sigma(X_i))$$

and $(X_i, J_i)$ is an almost complex manifold such that $\partial X_i = M$ and $\eta_i = TM \cap J(TM)$. 

As almost complex manifold for the computation of \(d_3(\eta_i)\) we will take symplectic fillings of \((M, \eta_i)\) endowed with an adapted almost complex structure. More precisely, let \((X_0, \omega)\) be the weak symplectic filling of \((M_0, \xi_0)\) for any \(n \in \mathbb{N}\) constructed in [11] Proposition 15. If \(T \subset M_0\) is a fibre of the torus bundle \(M_0 \to S^1\), then we can assume that \(\int_T \omega = 1\). In the setting of symplectic fillings Legendrian surgery corresponds to adding symplectic 2–handles, so adding symplectic 2–handles to \((X_0, \omega)\) as explained in the definition of \((M, \eta_i)\), we obtain symplectic manifolds \((X, \omega_i)\) which fill \((M, \eta_i)\) for \(i \in \mathcal{P}_n\). We choose almost complex structures \(J_i\) adapted to \(\omega_i\) so that the contact structure \(\eta_i\) is \(J_i\)–invariant for any \(i \in \mathcal{P}_n\), all \(J_i\) coincide on \(X_0\) and the fibre \(T\) in \(M_0 = \partial X_0\) are quasi–complex submanifolds.

In \(M_0\), the homology class represented by \(F\) is Poincaré dual of \([\omega_0|_{M_0}]\), because \(F \cdot T = 1 = \int_T \omega_0\) and \([T]\) generates \(H^2(M_0)\), therefore \(F\) bounds a surface \(\Sigma \subset X_0\) which represents the Poincaré dual of \([\omega]\). Applying the homology long exact sequence to the pair \((X, X_0)\) we obtain \(H_2(X) = H_2(X_0) \oplus \mathbb{Z}[\Sigma]\), where \(\Sigma \subset X\) is the surface obtained by capping \(\Sigma\) with the core of the 2–handle attached along \(F^3\). Analogously, the cohomology exact sequence yields \(H^2(X) \cong H^2(X_0) \oplus \mathbb{Z}\), where the isomorphism is given by \(\alpha \mapsto (\iota^* \alpha, [\alpha, [\Sigma]])\).

**Lemma 3.11.** Let \(\alpha \in H^2(X)\) be the 2–dimensional cohomology class determined by \(\iota^* (\alpha) = 0\) and \(\langle \alpha, [\Sigma]\rangle = 1\). Then, up to torsion, \(\alpha\) is the Poincaré dual of \([T] \in H_2(X) \cong H_2(X, \partial X)\).

**Proof.** Any 2–dimensional homology class can be represented as a closed, oriented embedded surface. Let \(K\) be a surface representing a homology class in \(H_2(X_0)\), then \(K \cdot T = 0\) because \(K\) can be made disjoint from \(\partial X_0 = M_0\) and \(\langle \alpha, [K]\rangle = \langle \iota^* \alpha, [K]\rangle = 0\). On the other hand, \(\Sigma \cdot T = F \cdot T = 1 = \langle \alpha, [\Sigma]\rangle\). \(\square\)

**Theorem 3.12.** The contact structures \(\eta_i\) with \(i \in \mathcal{P}_n\) are pairwise homotopic and \(d_3(\eta_i) = -\frac{2}{7}\).

**Proof.** To prove that the contact structures are homotopic we will show that they have the same three dimensional invariant \(d_3\). Since in the computation of \(d_3(\eta_i)\) we use the almost complex manifolds \((X, J_i)\) which are smoothly diffeomorphic, it is enough to prove that \(c_1^3(J_i)\) does not depend on \(i\). Given \(i_1, i_2 \in \mathcal{P}_n\) we can decompose

\[
c_1^3(J_{i_1}) - c_1^3(J_{i_2}) = \langle (c_1(J_{i_1}) + c_1(J_{i_2})), PD(c_1(J_{i_1}) - c_1(J_{i_2})) \rangle.
\]

By the functoriality of the Chern classes for any \(i \in \mathcal{P}_n\) we have \(\iota^*(c_1(J_i)) = c_1(J_i|_{X_0})\), then \(\iota^*(c_1(J_{i_1}) - c_1(J_{i_2})) = 0\), because all \(J_i\) agree on \(X_0\). Lemma 3.11 implies that \(PD(c_1(J_{i_1}) - c_1(J_{i_2}))\) is a multiple of \([T]\). Since \(T\) is a complex submanifold of \((X, J_i)\), the adjunction equality gives \(\langle c_1(J_i), [T]\rangle = \chi(T) + T \cdot T = 0\), then \(c_1^3(J_{i_1}) - c_1^3(J_{i_2}) = 0\).

\(d_3(\eta_i)\) can be computed for any of the Stein fillable contact structures \(\eta_i\) with \(i \in \mathcal{P}_n^*\) using the Stein filling \((W, J_i)\) described in [13], Figure 2. One can immediately check that \(c_1^3(J_i) = 0\), \(\chi(W) = 3\) and \(\sigma(W) = 0\). \(\square\)

We stress the point that the Stein manifolds \((W, J_i)\) used to compute \(d_3(\eta_i)\) are different from the almost complex manifolds \((X, J_i)\) used in the first part of Theorem 3.12 to show that all \(\eta_i\) are homotopic.
3.4. Computation of the Ozsváth-Szabó invariants. In [22, Section 8, $HF^+(\Sigma(2, 3, 6n + 5))$ is computed. Applying the long exact sequence relating $HF^+$ and $\widehat{HF}$ and the isomorphism between $\widehat{HF}_d(Y)$ and $\widehat{HF}_{-d}(-Y)$ it is easy to show that $\widehat{HF}(-\Sigma(2, 3, 6n + 5)) = (\mathbb{Z}/2\mathbb{Z})_{n+1}^{n+1} \oplus (\mathbb{Z}/2\mathbb{Z})^{2n+1}$. The degree of $c(\xi)$ is +1 because $d_3(\eta_i) = -\frac{3}{2}$. By [25, Section 4 $\widehat{HF}(+1)(-\Sigma(2, 3, 6n + 5))$ is freely generated by the elements $c(\eta_i)$ for $i \in P^*_n$.

**Proof of Theorem 1.1** The fix space $\text{Fix}(\xi) \subset \widehat{HF}(+1)(-\Sigma(2, 3, 6n + 5))$ is generated by elements of the form $c(\eta_i) + c(\eta_{-i})$ for $i \in P^*_n$. Let $W$ be the smooth cobordism between $M_0$ and $-\Sigma(2, 3, 6n + 5)$ constructed by attaching a 2-handle to $M_0$ along $F$, then by [18, Theorem 4.2

$$
\hat{F}_{\text{HF}}(c(\eta_i) + c(\eta_{-i})) = \hat{F}_{\text{HF}}(c(\eta_i)) + \hat{F}_{\text{HF}}(c(\eta_{-i})) = 2c(\xi) = 0.
$$

Consequently $\text{Fix}(\xi) \subset \ker \hat{F}_{\text{HF}}$, in particular

$$
c(\xi) = \hat{F}_{\text{HF}}(\eta_0) = 0
$$

because $c(\eta_0) \in \text{Fix}(\xi)$ by Proposition 3.8 and Theorem 2.10.

In view of Theorem 2.13 we have the following corollary.

**Corollary 3.13.** The contact manifolds $(M_0, \xi_n)$ are not strongly symplectically fillable if $n$ is even.

This is a new non fillability result, because the integer $n_0$ in Theorem 3.3 is not given explicitly.

4. A remark on integer coefficients

Unfortunately Theorem 1.1 does not imply that the Ozsváth–Szabó contact invariants $c(\xi_n)$ for $n$ even with untwisted integer coefficients are zero, but only that they are the double of some elements of $\widehat{HF}(-M_0)/ \pm 1$. Fix an open book decomposition of $M_0$ adapted to $\xi_n$ for an even $n$. We denote by $M_0'$ the 3–manifold obtained by 0–surgery on the binding and by $M_0''$ the 3–manifold obtained by 1–surgery on the binding. Of course the manifolds $M_0'$ and $M_0''$ depend on $n$. By [19, Theorem 9.1 there is a surgery exact triangle

$$
\begin{array}{ccc}
\widehat{HF}(-M_0') & \xrightarrow{F} & \widehat{HF}(-M_0) \\
& & \downarrow \widehat{HF}(-M_0'') \\
\end{array}
$$

The group $\widehat{HF}(-M_0')$ is generated by $c_0$, therefore if $F(c_0) = c(\xi_2) \neq 0$, the exact triangle becomes a short exact sequence

$$
0 \rightarrow \widehat{HF}(-M_0') \rightarrow \widehat{HF}(-M_0) \rightarrow \widehat{HF}(-M_0'') \rightarrow 0
$$

If $c(\xi_n)$ is non primitive there are torsion elements in $\widehat{HF}(-M_0'')$. Since all Heegaard–Floer homology groups known so far are free, it is reasonable to expect that $c(\xi_n) = 0$ also in the Heegaard–Floer homology group with integer coefficients.
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