Kink propagation in inhomogeneous systems driven by spatiotemporal perturbations

M. A. García-Ñustes¹, I. Rondón¹, Jorge A. González¹, and R. Chacón²

¹ Centro de Física, Instituto Venezolano de Investigaciones Científicas, Apartado 21827, Caracas 1020-A, Venezuela
² Departamento de Física Aplicada, Escuela de Ingenierías Industriales, Universidad de Extremadura, Apartado Postal 382, E-06071 Badajoz, Spain

E-mail: mogarcia@ivic.gob.ve

Abstract.
We investigate the propagation of kinks in inhomogeneous media. We show that the extended character of the kink, the internal mode instabilities and the phenomenon of disappearance of the translational mode can affect the kink motion in the presence of space-dependent external perturbations. We apply the results to the analysis of kink ratchets and the propagation of kinks driven by wave fields.

1. Introduction
Unlike point-like particles kink-solitons solutions are extended objects. The differences in their stability were theoretically demonstrated in several studies [1–3]. Moreover, effects such as length scale competition, i.e., the non-propagation of solitons under the action of external forces when the perturbation wavelength is of the order of the soliton width [4], can be explained from the point of view of instabilities of the internal modes of the soliton induced by external perturbations [5, 6]. In other words, the extended character of the soliton plays a crucial role in its propagation properties in inhomogeneous media. Besides, propagation of particles in inhomogeneous potentials have been studied on the context of many physical phenomena as spatiotemporal pattern dynamics [7–9], wave fields [10, 11], stochastic resonances [12, 13] and ratchet motion [14–17]. These studies have lead to multiple applications in models of molecular motors [14, 18–20], vortex dynamics in superconductors [21] and fluxons motion in Josephson junctions [22, 23].

In the present work, we show that under certain conditions of the parameters of the external force different phenomena, linked to the extended character of the soliton, could exist: the length scale competition, the non-existence of the translational mode, and the kink-soliton break-up. We discuss how all those phenomena can influence the propagation of kink-solitons in inhomogeneous media as periodic and ratchet-like potentials. We also investigate an optimization process of soliton transport when they are forced by the presence of wave fields.

In the following section, we discuss the main differences in stability conditions if we consider a point-like particle and a extended object, the length scale competition and the phenomenon of non-existence of translational mode and kink break-up by the presence of internal instabilities.
In section 3.1, we show how the kink motion in a periodic medium depends on the distance between the zeros of the external force. Next, in section 3.2 and 3.3, we analyze kink ratchets and propagation driven by a wave field, based on the results mentioned in previous sections. Finally, in section 4 we discuss our results and present some concluding remarks.

2. The kink as an extended object
2.1. Differences in stability conditions
The perturbed Klein-Gordon equation with dissipation is

\[ \phi_{tt} - \phi_{xx} + \gamma \phi_t + \frac{\partial U}{\partial \phi} = F(x), \] (1)

where where \( U(\phi) \) is a potential that possesses at least two minima and \( \gamma \) is the damped coefficient. The sine-Gordon and \( \phi^4 \) equations are particular examples of (1).

It is known that the external force \( F(x) \) creates an effective potential \( V(x) \) for the kink-soliton solution. When the soliton is treated as a point-like particle, the zeroes of \( F(x) \) are generally considered equilibrium positions for the motion of the kink [1, 24]. However, the stability conditions for these equilibrium positions are quite different if we analyze a point-like particle and a extended object. In fact, the soliton behaves as a particle only when the width of the potential makes the soliton appear pointlike. Otherwise, the soliton can have wavelike extended character [1, 3, 25].

Suppose we have a static kink whose center of mass is placed at an equilibrium point created by a zero of \( F(x) \). When \( F(x) \) are suitably chosen functions, the stability problem for the kink [1, 2] can be solved exactly leading to the following eigenvalue problem:

\[ \mathcal{L} f = \Gamma f, \] (2)

where \( \phi(x, t) = \phi_k(x) + f(x)e^{i\lambda t} \), \( \mathcal{L} = -\partial^2_x + \left\{ \frac{\partial^2 U}{\partial \phi^2} |_{\phi=\phi_k(x)} \right\} \), \( \Gamma = -\lambda^2 - \gamma \).

For instance, if the distance between the zeros of \( F(x) \) is much larger than \( l_k \), where \( l_k \) is the kink’s width, the kink feels the zeros of \( F(x) \) as equilibrium positions. Let \( x_1 \) be a zero of \( F(x) \) \( (F(x_1) = 0) \). If \( \frac{\partial F(x)}{\partial x} \big|_{x=x_1} > 0 \), the kink’s equilibrium position is stable. If \( \frac{\partial F(x)}{\partial x} \big|_{x=x_1} < 0 \), the equilibrium position is unstable for the kink. Nevertheless, if the distance between the zeros of \( F(x) \) is comparable with or less than \( l_k \), then the situation can change drastically. Let us say that \( F(x) \) has three zeros: \( x_1, x_2 \) and \( x_3 \) such that \( \frac{\partial F(x)}{\partial x} \big|_{x=x_{1,3}} > 0 \), \( \frac{\partial F(x)}{\partial x} \big|_{x=x_2} < 0 \), from the point of view of the effective potential, this would imply the existence of a barrier between two stable potential wells for the kink. But when \( |x_2 - x_1| < l_k \), \( |x_3 - x_2| < l_k \), the kink does not feel the point \( x_2 \) as an unstable equilibrium position, i.e., the kink does not feel the barrier at point \( x_2 \). Conversely, if \( \frac{\partial F(x)}{\partial x} \big|_{x=x_1,3} < 0 \), \( \frac{\partial F(x)}{\partial x} \big|_{x=x_2} > 0 \), the effective potential has two unstable points and one stable between them. For the “point-like” kink placed in \( x_2 \) exists a potential well between two barriers. But, when \( l_k \) is larger than the distance between zeros, the kink will not be trapped by the potential well.

2.2. Length scale competition
Let us consider the following equation,

\[ \phi_{tt} - \phi_{xx} + [1 + g(x)] \frac{\partial U}{\partial \phi} = 0. \] (3)

In this case, the extrema of \( g(x) \) are generally considered equilibrium positions for the movement of the kink. In Ref. [4], the authors considered a periodic function with zero average of the form
\( g(x) = \varepsilon \cos(kx) \) and they observed that when the distance between the extrema of \( g(x) \) is very close to the kink’s width, the kink propagation in an inhomogeneous medium is impossible. This phenomenon was called length-scale competition [4–6].

The length scale competition phenomenon has been analyzed from the point of view of collective coordinate approach and stability conditions [4–6]. Next, we show the main ideas of both approaches in order to explain the phenomenon and how it can affect the kink dynamics in the presence of perturbations. The starting point of the collective coordinate method is to derive expressions for the two functions \( x_{cm}(t) \) and \( l(t) \) that represent, respectively, the center and the width of the kink. For this, it is proposed the following ansatz:

\[
\phi(x, t) = \phi_k \left\{ \frac{x - x_{cm}(t)}{l(t)} \right\},
\]

where \( \phi_k \) is the kink solution.

It is possible to obtain dynamical equations for the variables \( x_{cm}(t) \) and \( l(t) \) by using the method based on the Lagrangian [4–6]. The soliton dynamics obtained is equivalent to the particle motion in an effective potential [4]:

\[
V(x_{cm}, l) = \frac{M_0}{2} \left( \frac{l_0}{l_0} + \frac{l}{l_0} \right) + \frac{\varepsilon}{k} \cos(kx) W(kl),
\]

where \( M_0 \) and \( l_0 \) are the kink “mass” and natural width, respectively, without perturbation [5,6]. Their specific values depend on the function \( U(\phi) \) considered. For example for the sine-Gordon equation \( M_0 = 8, l_0 = 1 \) and \( W(y) = \frac{2\pi y^2}{\sinh(\frac{\pi y}{2})} \).

It can be proved that \( F(x_{cm}) = \frac{\partial V_{cm}}{\partial x_{cm}} \) and the effective force that acts on the kink is

\[
F(x_{cm}) = -2 \int_{-\infty}^{\infty} F(x) f_0(x - x_{cm}) dx,
\]

where \( f_0(x) \) is the translational or Goldstone mode (see [6] and references therein for more details). We should remark that other approaches as the Fredholm-alternative based technique (solvability conditions) reports similar results where the kink dynamics is obtained through the Goldstone modes around the center of mass of the kink [9,26,27].

Carrying out the calculation for the sine-Gordon case, it is possible to show that stable equilibrium positions satisfy the following conditions: For the center of mass coordinate we have \( x_{cm} = \frac{(2n+1)\pi}{k} \) and for \( l \), the expression is given by the parameter \( z \):

\[
l(z) = \frac{l_0}{\sqrt{1 - \frac{4\pi\varepsilon l_0 z}{M_0} \left[ \frac{2\tanh(\frac{\pi z}{2}) - \frac{\pi z}{2}}{\sinh(\frac{\pi z}{2}) \tanh(\frac{\pi z}{2})} \right]}},
\]

\[
k(z) = \frac{z}{l_0} \sqrt{1 - \frac{4\pi\varepsilon l_0 z}{M_0} \left[ \frac{2\tanh(\frac{\pi z}{2}) - \frac{\pi z}{2}}{\sinh(\frac{\pi z}{2}) \tanh(\frac{\pi z}{2})} \right]}},
\]

where \( z \in [0, \infty) \) [5].

Here, the most interesting aspect of the procedure is that it is possible also calculate the “curvature” of the effective potential \( V(x_{cm}, l) \) along the direction \( x_{cm} \) evaluated in the stable equilibrium points:

\[
|\kappa| = \frac{2\pi\varepsilon l^2 k^3}{\sinh(\frac{\pi l}{2})}\frac{1}{3}.
\]
It is not difficult to show theoretically that $|c|$ depends on $k$ and has a maximum for a value of $k$ such that the distance between the perturbation extrema $\varepsilon \cos(kx)$ is comparable with $l_0$.

In other words, for values of $k$ over a certain finite interval ($1.5 \lesssim k \lesssim 4$), the kink moves in a curved potential with extraordinary large barriers. In fact, the kink motion is very difficult for values of $k$ where $|c|$ is maximum ($k_m = 1.6$). For large and small values of the the wavelength of the perturbation outside of this interval, the kink moves in a effective potential almost flat.

Now, let us discuss the length scale competition phenomenon using the stability analysis approach. As we mentioned before, the stability analysis can be solved exactly for a certain function $F(x)$ with several zeros \cite{1}. However, we are going to consider the case when $F(x)$ is also a periodic function. Let us analyze the stability conditions when the potential is given by $U(\phi) = (\phi^2 - 1)^2/8$ and $\gamma = 0$ in (1).

We define

$$F_{\text{II}}(x) = \frac{1}{2} A \tanh(Bx) \left[ \varepsilon_1 + \varepsilon_2 \cosh^{-2}(Bx) \right]. \quad (9)$$

where $\varepsilon_1 = A^2 - 1$ and $\varepsilon_2 = 4B^2 - A^2$. Depending on the values of the parameters, $F_{\text{II}}(x)$ can have either one or three zeros. When $A^2 > 1$ and $4B^2 < 1$, $F_{\text{II}}(x)$ has three zeros that correspond to one unstable equilibrium point at $x = 0$ and two stable equilibria at $x = \pm d$ ($A^2 > 1, 4B^2 < 1$) with

$$d = \frac{1}{B} \arccosh \left( \sqrt{\frac{A^2 - 4B^2}{A^2 - 1}} \right).$$

In this case, we are able of construct a periodic function $F(x)$ using $F_{\text{II}}(x)$ of the form

$$F(x) = F_{\text{II}}(x + 2n d), \quad \text{for} \quad (2n-1)d < x \leq (2n+1)d,$$

with $n = 0, \pm 1, \pm 2, \ldots$.

Based on the discussion in the previous section (see Sec.2.1), we can see that if the distance between the zeros (given by $d$) is much smaller than the kink’s width ($l_0 = 2$), the kink practically does not feel the potential barriers and it moves in almost flat potential. On the other hand, if $d$ is much larger than $l_0$, then the kink moves in a periodic potential \cite{3,25}.

However, for certain middle values of $d$, comparable with the kink’s width, the kink remains trapped inside some potential wells created by $F(x)$. In this case, the kink behaves as a finite size body into a “rectangular hole” with high walls and the same size of the body. Thus, the mobility of the kink is practically zero. It is clear that these results are in agreement with those obtained by means of the collective coordinate approach.

2.3. Non-existence of the translational mode

Let us consider again the $\phi^4$ potential $U(\phi) = (\phi^2 - 1)^2/8$ in (1) but now we will discuss the phenomena of non-existence of the translational mode for some values of the parameters of the external force $F(x)$.

Let us take a function $F(x)$ that possess, at least, one zero $x_*$ between a minimum $x_*^m$ and a maximum $x_*^M$ with the properties that $\frac{\partial F(x)}{\partial x}|_{x_*} > 0$ and $F(x_*^m)$ and $F(x_*^M)$,

$$F(x) = \frac{1}{2} (4B^2 - 1) \sinh(Bx)/ \cosh^3(Bx). \quad (10)$$

The exact solution for a kink which center of mass is placed in an equilibrium position $x = 0$ when $F(x)$ is given by (10) is $\phi_k = \tanh(Bx)$.

Solving the stability problem we get that the discrete eigenvalues of the spectral problem (2) are given by the expansion:

$$\Gamma_n = -\frac{1}{2} + B^2(\Lambda + 2\Lambda n - n^2), \quad (11)$$
where \( \Lambda(\Lambda+1) = \frac{3}{2B^2} \). The integer part of \( \Lambda \) ([\Lambda]) yields the number of the internal kink modes.

When \( B^2 = 1/4 \), the perturbation \( F(x) \) disappears and we have \( \Lambda = 2 \) which corresponds to a two famous internal modes of \( \phi^4 \) equation: the translational mode and the internal shape mode [1]. But when \( B^2 > 3/4 \) we can show that \( \Lambda < 1 \) and therefore, the translational mode does not exist anymore [5]. This is due to the parameter \( B \) controls the slope of the function \( F(x) \) in the zero \( x = 0 \) and the values of \( F(x^m) \) and \( F(x^M) \). When \( B \) grows, the slope of \( F(x) \) in the zero and the values of \( |F(x^m)| \) and \( |F(x^M)| \) are increased. In other words, for larger values of \( B \), the kink will be “tighter” inside the potential well created by \( F(x) \). In fact, there exists a value when the kink has not possibility of motion. The critical value for \( |F(x^m)| \) and \( F(x^M) \) is

\[ |F_c(x^m)| = |F_c(x^M)| = \frac{2}{\sqrt{3}}. \]

This phenomena can be appeared even when the function \( F(x) \) possess more than one zero. Consider the function (9) as the external function \( F(x) = F_{II}(x) \) but when \( A^2 < 1 \) and \( 4B^2 > 1 \). In this later case, \( F(x) \) possess three zeros. The zero \( x = 0 \) has the property that \( \frac{\partial F}{\partial x}|_{x=0} > 0 \) which represents a stable equilibrium position for the kink.

The stability problem produces the following condition for the discrete eigenvalues, \( \Lambda(\Lambda+1) = 3A^2/2B^2 \), while the distance between the zeros is given by

\[ d = \frac{1}{B} \arccosh \left[ \sqrt{\frac{4B^2 - A^2}{1 - A^2}} \right]. \]  

(12)

It is clear that the condition for the non-existence of the translational mode is,

\[ 4B^2 > 3A^2. \]  

(13)

Thus, we can conclude that even if \( F(x) \) possess several zeros, we can find some values for the parameters such that the translational mode does not exist and therefore the kink can be trapped firmly inside one of the wells created by \( F(x) \).

2.4. Kink break-up

Another phenomena related with the extended character of the kink-soliton and its internal dynamics is the kink break-up [28]. This phenomenon is produced by an instability in the internal shape mode of the kink and it has been reported in connection with the possibility of kink escape of potential wells by an additional kink–antikink pair creation and soliton explosions [28,29]. In order to illustrate the breaking-up process, we will return to the same equation of the former section,

\[ \phi_t + \gamma \phi_x - \phi_{xx} - \frac{1}{2} \phi + \frac{1}{2} \phi^3 = F(x), \]

where

\[ F(x) = \frac{1}{2}(4B^2 - 1) \sinh(Bx)/ \cosh^3(Bx). \]  

(14)

When \( 4B^2 < 1 \), the zero of the function \( F(x) \) \( (x = 0) \) has the following property \( \frac{\partial F}{\partial x}|_{x=0} < 0 \) (see Fig.1). Thus, the point \( x = 0 \) is an unstable equilibrium position. Now, if the translational mode is unstable but the shape modes are stables the kink center of mass is pulled out of its equilibrium position and it will move away of this position without any change on its shape structure. However, if

\[ B^2 < \frac{11 - \sqrt{117}}{8}, \]  

(15)

the first internal shape mode can be unstable \( (\Gamma_1 < 0) \) leading to a “kink break-up”. In fact, the kink breaks into one kink and two antikinks [2,24,28]. In general, when \( F(x) \) has a zero \( x^s \) such as
that $\frac{\partial F}{\partial x}_{x=x^*} < 0$ and this point is between the values $x_M$ and $x_m$ that are a maximum and a minimum of $F(x)$ respectively, then exists a critical value $F_c$ such that if $|F(x_M) - F(x_m)| > F_c$, the kink will break up.

The latter consideration is important for a kink moving in a medium where $F(x)$ has several zeros, maxima, and minima. If the properties described above are satisfy, the kink could develop an instability near a zero of $F(x)$ and create an additional kink-antikink pair.

3. Kink propagation in inhomogeneous media

The former section gives us a vision of how the internal modes instabilities, the extended character of the kink-soliton, and the not appearance of some internal modes can affect the transport properties of solitons in inhomogeneous media. All the above considerations allow us to analyze the propagation of kink-solitons when are perturbed by different external forces.

3.1. Kink propagation in periodic media

In this section we investigate the equation

$$\phi_{tt} + \gamma \phi_t - \phi_{xx} - \frac{1}{2} \phi + \frac{1}{2} \phi^3 = A \sin(kx),$$

as an example of the phenomena formerly discussed. We will show that the kink motion depends on the parameter $k$ that defines the distance between the zeros of $F(x)$.

The Cauchy problem is the following

$$\phi(x, 0) = \tanh \left[ \frac{x}{2\sqrt{1 - v_0^2}} \right],$$

$$\phi_t(x, 0) = -\frac{v_0}{2\sqrt{1 - v_0^2}} \cosh^{-2} \left[ \frac{x}{2\sqrt{1 - v_0^2}} \right].$$

These initial conditions describe a kink with an initial velocity of center of mass $v_0$. This problem can not be treated directly with the stability analysis approach but the qualitative behavior can be described based on the exactly solvable cases instead. For values of $k >> 1$, the kink moves as if $F(x)$ does not exist, i.e. the kink moves in an effective potential completely flat. An example of this, is shown in Fig. 2. This is due to the fact that the kink width is larger than the potential wells. Therefore, the potential wells does not affect the kink.
Figure 2. A kink moves in an effective potential created by $F(x) = A \sin(kx)$ for a value of $k = 10$ in Eq. (16).

Figure 3. Comparative scheme of the effective potential generated by an external force $F(x) = A \sin(kx)$ (pale line) and the kink soliton width (bold line). Left: If $k >> 1$ the kink soliton does not feel the potential wells due to its width is larger that the separation between potential wells. Middle: If $k << 1$ the kink soliton behaves as a point particle in an effective potential. Right: The kink width is comparable to the distance between equilibrium points for middle values of $k$.

If $k << 1$, the kink moves in a periodic effective potential. However, we must remark that the kink has mobility. It can move freely inside potential wells and with enough velocity to jump between them. This phenomenon can be seen in Fig.3 (Middle) from the point of the effective potential. Figure 4 shows the numerical simulation of (16) with $k = 0.01$.

Nevertheless, for certain middle values of $k$, the kink remains trapped inside a potential well with a similar extension to the kink width (Fig.3(right)). Figure 5 shows a numerical simulation of this effect. In this case, the kink remains “stuck” inside the potential well. The kink has no possibility of motion.
3.2. Ratchet-like motion
In this section we will investigate the following equation:

$$\phi_{tt} + \gamma \phi_t - \phi_{xx} - \frac{1}{2} \phi + \frac{1}{2} \phi^3 = F_1(x) + F_2(t),$$

(17)

where $F_1(x) = A \sin(kx) - 2\mu A \sin(2kx + \theta)$, and $F_2(t) = f_0 \sin(\omega t + \delta)$.

Note that, in the case of a point particle, the force $F_1(x)$ generates an asymmetrical effective potential that would allow ratchet-like motion (for example with $\mu = -1/4, \theta = \pi/2$) [3, 14].

Let us see first what happen when $f_0 = 0$. $F_1(x)$ is slightly different of $F(x) = A \sin(kx)$, but still is a periodic function with zeros and extrema. As on the simple case $F(x) = A \sin(kx)$, when $k >> 1$, the kink moves as if the effective potential was flat (See Fig. 6 and Fig.3)(Left)). In Fig.7 we observe that the kink remains trapped in a potential well for $k = 0.5$ with $F(x) = A \sin(kx)$, as we have seen in section 3.1.

If $A$ is larger, the zeros of $F_1(x)$ that correspond to unstable equilibrium points for the kink, can be also points where the internal modes are unstable. This leads to a kink destruction
Figure 6. For \( k = 10 \) in \( F_1(x) \) (Eq. (17)), the kink moves freely as if the effective potential was flat.

Figure 7. The kink remains trapped in the potential for \( k = 0.5 \) (Eq.(17)).

when the kink moves in this medium. But still when \( F_1(x) \) is not sufficient to make the kink unstable, the internal mode instabilities can be activated by introducing a time dependent force \( F_2(t) = f_0 \sin(\omega t + \delta) \) (See Fig.8).

3.3. Kink propagation driven by a wave field
In this section we will be interested in what happen when the inhomogeneous periodic perturbation as \( F(x) = A \sin(kx) \) “moves” in the space with a certain velocity. This is equivalent to say that the kink is being perturbed by an external wave. We consider the perturbed \( \phi^4 \) equation with dissipation and the external force is given by \( A \sin[k(x - wt)] \),

\[
\phi_{tt} + \gamma \phi_t - \phi_{xx} + \frac{\partial U}{\partial u} = A \sin[k(x - wt)].
\]  

(18)

The potential wells and barriers created by the perturbation \( F(x) = A \sin(kx) \) now are
moving. We are interesting in the following Cauchy problem:

\[
\begin{align*}
\phi(x, t = 0) &= \tanh \left[ \frac{x}{2} \right], \\
\phi_t(x, t = 0) &= 0.
\end{align*}
\]

In other words, the kink is initially static. We want to study if \( F(x, t) = A \sin[k(x - wt)] \) produces kink motion or not. In fact, the kink can be carried by a external wave [10, 11]. To obtain the “optimal conditions” for this transport we must take into account all the phenomena discussed above.

\[\text{Figure 8.} \text{ Kink destruction by internal modes instabilities when } k = 0.5, \omega = 1, \text{ and } f_0 = 0.4 \text{ in Eq.}(17).\]

\[\text{Figure 9.} \text{ The kink is transported by the wave field when } k = 0.09 \text{ in Eq. (18).}\]

The plane wave is a periodic function and it can create equilibrium positions where the kink can be trapped during the transportation. The distance between the zeros of the function must be larger than the characteristic kink width. Thus, the potential wells are sufficiently wide as to assure that the kink feels them as a equilibrium states. However the periodic wave can create unstable equilibrium points too. If the wave amplitude is too large, these points are not only
unstable for the center of mass of the kink, but the kink shape modes can be unstable too. Therefore, the kink cannot be transported in a stable way.

We have seen also that the kink can be trapped by a well due to the length-scale competition phenomenon. We have also seen that when a kink is trapped in a zero of $F(x)$ between a minimum and a maximum (if the difference between these extrema is large enough), the translational mode does not exist. Hence, we must increase the wave amplitude $A$, if we want to assure that the kink remains inside the potential well during the transportation by an external wave. In short, in order to obtain an instability on the internal modes, we must have an amplitude $A$ large but not so much. Numerically we have encountered that the following parameters allow an efficient transport of the kink: $A = 0.2$, $k = 0.09$ and $w = 1$. The real velocity of the kink is going to depend of the damping coefficient $\gamma$.

We must mention that a perturbation like $F(x, t) = A\sin[k(x - wt)]$ produces a spatiotemporal symmetry breaking [10, 11] in a similar way as a ratchet system. In fact, there are some values for external wave parameters that cause the kink remain static without any kind of motion. For example, for large values of $k$, the kink does not feel the potential wells and barriers created by the perturbation. Hence, the kink remains at the initial position where it has been placed. In Fig.9 we observe that for a $k$ small enough to make the kink remains trapped inside a well, a moving external wave transports a kink with it. However in Fig. 10 we observe that for a $k >> 1$, the external wave cannot transport the kink.

**Figure 10.** For $k = 3$ in Eq. (18) the kink remains at rest.

4. Discussion and conclusions

The propagation of a kink in a inhomogeneous medium can be affected by different phenomena that are connected with the fact that a kink is a deformable extended object with a rich internal dynamics. In general, the soliton could be considered as a deformable object that could be trapped by holes of finite size or could move freely even when an effective potential with wells and barriers exists.

Even when the effective potential is such that a point particle could move in a ratchet-like process, the kink would be broken-up in a system with several kink and antikink, if the internal modes are unstable.

But the most extraordinary phenomenon is the kink propagation forced by the presence of a wave field. The results presented here allow to optimize this process. The wave parameters can be selected in such way that the kink stay stabilized in a potential well created by the wave.
In this case, the kink can move at an extraordinary speed. This velocity can be in the order of the maximum velocity of the kink in the Klein-Gordon equation without external perturbations and no dissipation.

Acknowledgements
We thank Jhoan Toro-Mendoza and J.J. Suárez for their help on the manuscript preparation.

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Received xxxx 20xx; revised xxxx 20xx.