Zero-index Weyl metamaterials

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Depending on the geometry of their Fermi surfaces, Weyl semimetals and their analogues in classical systems have been classified into two types. In type I Weyl semimetals (WSMs), the cone-like spectrum at the Weyl point (WP) is not tilted, leading to a point-like closed Fermi surface. In type II WSMs, on the contrary, the energy spectrum around the WP is strongly tilted such that the Fermi surface transforms from a point into an open surface. Here, we demonstrate, both theoretically and experimentally, a new type of (classical) Weyl semimetal whose Fermi surface is neither a point nor a surface, but a flat line. The distinctive Fermi surfaces of such semimetals, dubbed as type III or zero-index WSMs, gives rise to unique physical properties: one of the edge modes of the semimetal exhibits a zero index of refraction along a specific direction, in stark contrast to types I and II WSMs for which the index of refraction is always non-zero. We show that the zero-index response of such topological phases enables exciting applications such as extraordinary wave transmission (EOT).
Weyl semimetals (WSMs) have recently attracted enormous research interest because of their unconventional band structures and topological features [1-4]. In the Brillouin zone of a WSM, the valence and conduction bands touch each other at a set of points, called Weyl points (WPs) [5-11]. These points are sources and drains of Berry curvature in the momentum space and, thus, can be associated with a topological charge [12]. The topological nature of WPs protects them against annihilation in case of small perturbations, and leads to a large variety of intriguing physical phenomena such as superconductivity [13,14], chiral anomaly [14,16], and topological negative refraction [17]. Although it is not possible to remove Weyl nodes because of their topological character, the energy dispersion at a WP might, in principle, be tilted along a certain direction. While such a tilting of the band structure does not affect the essential topology of the WP, it can change the bulk properties of the semi-metallic phase by modifying the geometry of the Fermi surface [18]. As such, depending on whether their cone-like band structure is tilted or not, WSMs and their classical analogues have been grouped into two types. In a standard (type I) WSM [19,20], the energy dispersion at the WP is not tilted, leading to a point-like Fermi surface. In type II WSMs [21-27], on the other hand, the energy spectrum around the Weyl node is strongly tilted (by an angle larger than a specific angle $\theta_c$, at which parts of the conduction and valence bands start to coexist in energy). As a result, the cone-like dispersion around the WP tips over and the Fermi surface transforms from a single point into an open surface. This gives rise to distinctive physical properties for type II WSMs such as orientation dependent chiral anomaly and anomalous Hall effects [28,29].

So far, Weyl semimetals have been studied both when the tilting angle at the WP is smaller than $\theta_c$ (type I WSMs), or larger (type II WSMs). Here, on the contrary, we investigate in theory, simulation and experiment, the exotic properties of classical Weyl semimetals when their energy spectrum is tilted exactly by $\theta_c$. We show that the Fermi surface of such kinds of
semi-metallic phases is neither a point, as in type I WSMs, nor a surface, as in type II WSMs. Instead, it is a line connecting a pair of WPs. Since the geometry of the Fermi surface plays a very important role in determining the bulk properties of WSMs, we categorize these new kinds of semimetals into a distinct group, referred to as type III or “zero-index” WSMs. Such an appellation is inspired by the fact that, as opposed to type I and II WSMs possessing a non-zero index of refraction in all directions, the topological surface states of the proposed WSM exhibit a zero group velocity, or more precisely, a zero-index response along a specific direction. This behavior generalizes to three-dimensions the zero-index response of the edge modes of type III Dirac semimetals [23]. We demonstrate that the static-like behavior of type III Weyl semimetals can be used to achieve extra-ordinary wave transmission, akin to anomalous tunneling in the so-called zero-index metamaterials [30-33].

To start, we consider the tight-binding crystal shown in Fig. 1a, described by the tight-binding Hamiltonian

\[ H = 2\lambda_z \cos k_z \sigma_0 + (2\lambda_x \cos k_x + 2\lambda_y \cos k_y)\sigma_x + 2\delta_x \sin k_x \sigma_y + 2\delta_z \cos k_z \sigma_z \]  

(1)

in which \( \sigma_0, \sigma_{x,y,z} \) are identity and Pauli matrices, respectively, and \( k_x, k_y, \) and \( k_z \) are the Bloch wave numbers. We assume \( \lambda_x = 1, \lambda_y = 2, \lambda_z = 2, \) and \( \delta_x = 0.5. \) Furthermore, for now, we suppose that the parameter \( \delta_z \) is set to be zero, i.e. \( \delta_z = 0. \) With these choices of parameters, the tight-binding Hamiltonian \( H \) vanishes at \( P: (k_x, k_y, k_z) = (0, \pm 2\pi/3, \pm \pi/2) \), implying that the dispersion bands of the two-level system cross each other at these four points. As an example, we report in Fig. 1b the energy band structure of the corresponding Hamiltonian at the plane \( k_y = 2\pi/3, \) which includes two of these crossing points at \( (k_x, k_z) = (0, \pm \pi/2). \) These crossing points are Weyl nodes, possessing opposite chiralities (see [34]). The energy dispersion around these Weyl nodes is not tilted, leading to a point-like closed Fermi surface (characteristic of type I WSMs).
Fig. 1: Zero-index (type III) Weyl semimetals. a, We consider a tight-binding model consisting of a set of resonators coupled to each other with specified hopping strengths. b, Band structure of the crystal for $\lambda_x=1$, $\lambda_y=2$, $\lambda_z=2$, $\delta_x = 0.5$ and $\delta_z = 0$, calculated at the plane $k_y = 2\pi/3$. The band structure supports a pair of gap-closing Weyl points (WPs). The conical dispersion around the WP is not tilted, leading to a point-like Fermi surface (type I WSM). c, Same as panel b except that the parameter $\delta_z$ is chosen to be $\delta_z = 2\lambda_z$. The energy spectrum at the WP is strongly tilted in this case, so that the Fermi surface transforms from a point into an open surface (type II WSM). d, Same as b and c but for $\delta_z = \lambda_z$. In this case, the canonical spectrum around the WP is tilted by the critical angle $\theta_c$, at which the valence and conduction bands start to coexist in energy. The Fermi surface of the corresponding semi-metallic phase is neither a point nor a surface, but is a completely flat line. This gives rise to a zero-index response for the semimetal.

Next, suppose that the parameter $\delta_z$ is set to be $\delta_z = 2\lambda_z$. The corresponding band structure at the plane $k_y = 2\pi/3$ is shown in Fig. 1c. Like the previous case, the band structure exhibits a pair of Weyl points located at $(k_x, k_z) = (0, \pm\pi/2)$. However, as opposed to the previous case, the cone-spectrum is significantly tilted around these nodes, so that some modes from the valence band coexist in energy with others in the conduction band. Such a
strong tilting modifies the geometry of the Fermi surface: it becomes an open surface (characteristic of type II WSMs [21]).

Now we assume $\delta_z = \lambda_z$. Fig. 1d represents the corresponding band structure at $k_y = 2\pi/3$. Similar to the previous two cases, the band structure exhibits two Weyl nodes at $(k_x, k_z) = (0, \pm\pi/2)$. Around these points, the energy band structure is tilted, similar to what we had in the latter case. However, the tilting angle is exactly equal to the critical angle $\theta_c$, leading to a Fermi surface which is neither a single point, like type I WSMs, nor an open surface, like type II WSMs, but a straight line connecting a pair of Weyl points [34]. Such a unique Fermi surface gives rise to unique physical properties for the semi-metallic phase (referred to as type III WSM), as demonstrated below.

We first investigate the topological boundary states carried by zero-index Weyl semimetals. To this end, we consider a $100\times1\times1$ supercell of the crystal and calculate the corresponding dispersion surfaces [34]. The resulting band structure is represented in Fig. 2a, clearly showing the existence of two helical gap-closing topological edge states. In order to further analyze this result, we report the dispersion curves corresponding to three different plane sections, namely $k_y = \pi$, $k_y = 2\pi/3$ and $k_y = 0$ (Figs. 2b-d, respectively). As explained earlier, the plane $k_y = 2\pi/3$ is the one at which the Weyl transition occurs, so the band structure is gapless at this plane (Fig. 2c). As confirmed by direct calculations, the constant-$k_y$ planes before ($k_y = 0$) and after ($k_y = \pi$) this critical point possess different $\mathbb{Z}_2$ topological invariants, making the system a $\mathbb{Z}_2$ Weyl semimetal [34]. In particular, $k_y = \pi$ has a zero invariant [34], consistent with the fully gapped dispersion diagram of Fig. 2b. Conversely, $k_y = 0$ is associated with a non-vanishing invariant [34], explaining the two helical topological surface states observed in Fig. 2d. Remarkably, in contrast to type I and II WSMs for which the gap-less states have always a non-zero group velocity, one of the
associated helical topological surface states of the obtained WSM has a zero group velocity along \( k_z \) (note that in directions other than \( k_z \), the edge mode remains dispersive). The origin of this behavior is linked to the balance between the \( \sigma_0 \) and \( \sigma_z \) terms in Eq. 1, canceling one of the diagonal elements of the Hamiltonian at the critical angle.

Fig. 2: Topological boundary states of the proposed zero-index WSM. a, Dispersion surfaces of the 100x1x1 super-cell of the zero-index WSM. The band structure supports two helical topological boundary states (yellow surfaces). b, Band structure of the super cell at \( k_y = \pi \), which is completely gapped because the plane section has a zero topological invariant. c, Same as panel b but for the plane \( k_y = 2\pi/3 \). The band structure is gap-less at this plane. d, Same as panels b and c except that the band structure is calculated at the plane \( k_y = 0 \), corresponding to a non-vanishing topological invariant. The band structure exhibits gap-closing topological boundary states. Importantly, one of these boundary states possesses a completely flat dispersion, i.e. a zero group velocity along \( z \).

Next, we investigate an implementation of the proposed zero-index WSM using the sonic crystal shown in Fig. 3a, consisting of Helmholtz resonators connected to each other via
acoustic channels with varying widths (see [34] for a geometrical description). Full-wave simulations [34] confirm that the band structure of the crystal displays a type III Weyl semi-metallic phase (Fig. 3b). Fig. 3c represents the band structure of a $11 \times 1 \times 1$ super-cell of the crystal. Consistent with our prior findings, the band structure of the super-cell consists of plane sections with different topologies. At the topological plane ($k_y = k_{y3}$), the crystal supports two helical surface states, one of which has a near-zero index of refraction along $z$ (note that, due to longer range hopping, the associated effective index is not exactly zero, but near it). To obtain more insights into the physics of the near-zero-index topological edge mode, we plot in Fig. 3d (left panel) the associated acoustic pressure field, obtained via eigen-frequency simulations [34]. It is seen that, along the $z$ direction, this mode has a quasi-static pressure distribution. This is equivalent to an infinite phase velocity along the edge of the semimetal. Interestingly, such a zero-index behavior also occurs at another cut-plane of the band structure, namely the transition plane $k_y = k_{y2}$. This is demonstrated in Fig. 3d (right), reporting the associated acoustic field profile at this plane for $k_z = 0$.

The static-like behavior of type III (zero-index) topological phases enables exciting physical phenomena. Fig. 3e, for example, demonstrates the possibility of achieving extra-ordinary transmission of sound [35] by direct anomalous matching of an external waveguide to the near-zero topological edge mode of the $k_y = 0$ 2D topological cut-plane of the Weyl semimetal. Due to the small value of both the effective density and bulk modulus of the edge channel, the edge mode is perfectly impedance-matched to the external waveguides (see [34]), leading to the transmission amplitude of near unity (0.988) with almost no phase lag (0.052 rad). Note that, markedly different from Fabry-Pérot resonances, the obtained anomalous resonance tunneling does not depend on the length of the semi-metallic connection [33,34].
Fig. 3: Topological acoustic super-coupling induced by zero-index WSMs. a, We consider a three-dimensional sonic crystal, with Helmholtz resonators connected to each other via acoustic channels with specific widths, realizing a zero-index acoustic WSM. b, Band structure of the infinite sonic crystal, resembling the one of the type III WSM described before. c, Dispersion surfaces of a supercell of the crystal. Similar to our previous observations, the obtained band structure consists of plane sections with different topological properties. d, Left: mode profile of the crystal at the frequency $f_2$. The edge mode has a static-like distribution along the $z$ axis. Right: Pressure distribution at the frequency $f_1$. The bulk mode is also uniform along $z$. e, Demonstration of extra-ordinary transmission of sound through the zero-index topological boundary state.

Now, we experimentally investigate such anomalous tunneling through the edge mode at the topological cut-plane $k_y = 0$, based on the fabricated prototype of the sonic crystal, shown in Fig. 4a-b. This particular cut-plane choice allows us to simplify the experiment since the field of the edge mode is invariant by crystalline translations along $y$. Therefore, one can restrict the fabrication to a single inversion-symmetric layer, because the acoustic hard wall boundary conditions at the top and bottom are equivalent to an infinite system along $y$. Two
external waveguides are connected to the edge of the fabricated prototype, with a loudspeaker at the entrance of the first and an anechoic termination in the second. Four microphones probe the acoustic pressure distribution along the edge of the semimetal. Full wave simulations (neglecting losses) predict zero-index tunneling of sound through the edge at the frequency \( f_0 = 2.92 \, kHz \). In practice, due to the presence of absorption, the transmission coefficient of the fabricated structure does not reach unity at this frequency. However, the existence of the edge mode at \( f_0 \) and its static-like phase profile are not affected by the losses, and can be probed in experiment [34]. Fig. 4c (top) reports the sound pressure level spectrum measured by microphone number 3. Near \( f_0 \), a resonance peak is observed, corresponding to the near-zero topological edge mode. The quasi-static nature of the edge mode is confirmed in Fig. 4c (bottom), reporting the phases of the pressure field at all four microphones versus frequency. As observed, near \( f_0 \), the corresponding phase spectra converge to almost the same value. To further illustrate the near-zero-index response of the topological edge mode, we report in Fig. 4d the variation of the pressure phase at \( f_0 \) along the edge of the crystal (red region), together with the standard phase variation in waveguides, i.e. \( \phi = 2\pi f_0 L/c \) (green region). This figure confirms the quasi-static sound propagation along the edge of the semimetal with almost no phase variations along a distance of 1.4 acoustic wavelengths.
Fig. 4, Experimental demonstration of anomalous tunneling through the topological zero-index edge mode, a. Fabricated prototype of a zero-index type III semimetal, b, Experimental setup used to verify the anomalous tunneling through the zero-index topological edge mode. c, (Top) Spectrum of the sound pressure level inside the zero-index crystal, measured by the microphone 3. The spectrum exhibits a resonance peak at the frequency of $f_0 = 2920$ Hz, corresponding to the near-zero index anomalous tunneling. (Bottom) Variation of the pressure phases, measured by each of the four microphones, as a function of frequency. Around $f_0$, the four measured spectra are almost identical, indicating the zero-index character of the crystal within this frequency range. d, Variation of the pressure phase along the edge of the crystal (red region), compared to the phase variation in external waveguides ($\varphi = 2\pi f L/c$).

Finally, leveraging the concept of synthetic dimensions [36-39], we measure the complete band structure of the type III Weyl semimetal. To this end, we consider a one-dimensional sonic crystal made from evanescently coupled acoustic bound states in continuum (BIC) [40-44] with on-site frequencies $\omega_n$ and hopping rates $k_n$ following the equations: $\omega_n = 2\lambda_z \cos \varphi_z (1 + (-1)^n)$ and $k_n = \lambda_x + (-1)^n \delta_x + \lambda_y \cos \varphi_y$ [34]. It is straightforward to verify that, upon replacing $\varphi_y$ with $k_y$ and $\varphi_z$ with $k_z$, the tight-binding Hamiltonian of this
one-dimensional chain is identical to the one of Eq. 1. Therefore, the complete semi-metallic band structure can be probed considering the family of one-dimensional sonic systems generated when sweeping the parameters \( \varphi_y \) and \( \varphi_z \) in a synthetic 3D Brillouin zone. We report in Figs. 5a-c the band structures of the chains versus \( k_z(\varphi_x) \) at three different \( k_y(\varphi_y) \), namely \( k_y = \pi, k_y = 2\pi/3, \) and \( k_y = 0, \) respectively, obtained by full-wave eigen-frequency simulations. Consistent with our previous tight-binding studies, at \( k_y = \pi, \) the energy spectrum is gapped. Contrarily, at \( k_y = 2\pi/3, \) it is gap-less and exhibits a pair of Weyl points, connected to each other with a flat line. At \( k_y = 0, \) which is a plane with a non-zero \( \mathbb{Z}_2 \) charge, the two gap-closing helical surface states are indeed found, one of them with a zero group velocity.

To extract the band structure of the crystal from far-field scattering tests, we excite the system under study with a plane-wave and measure the corresponding transmission spectra by performing standard standing-wave pattern analysis [34]. The evolution of the transmission coefficient of the structure as a function of both frequency and \( k_z \) is plotted in Figs. 5d-f, corresponding respectively to \( k_y = 0 \) (panel d), \( k_y = 2\pi/3 \) (panel e), and \( k_y = \pi \) (panel f). It is observed that the distinctive characteristics of the band structure of the crystal, including the peculiar dispersion of the topological edge modes (Fig. 5a-c), translate into clear features in the scattering spectrum (Fig. 5d-f), namely transmission maxima or minima.

These findings are verified in experiment, based on one-dimensional sonic crystals of plastic rods arranged inside an acoustic waveguide [34]. Each rod of the crystal supports a BIC mode. By changing the radii of the rods and the distance between them, the on-site frequencies and hopping rates of the 1D BIC chain were tuned according to the prescribed relations, so as to resolve the dispersion bands of the proposed type III Weyl semimetal. The results of our experiment, shown in Fig. 5g-I, agree well with simulation.
To conclude, we explored the properties of a new kind of WSMs, dubbed as type III or zero-index Weyl semimetals, and demonstrated their intriguing dispersion properties in theory, simulation, and experiment. The unconventional character of the topological boundary states of zero-index WSM may open exciting venues for a large variety of engineering-oriented metamaterial applications, for example energy concentration, supercoupling, wavefront shaping and subwavelength imaging.

Fig. 5: Experimental observation of zero-index Weyl semimetals. We map the tight-binding model represented in Fig. 1 into a one-dimensional phononic crystal [34] with two additional phason degrees of freedom, $\varphi_y$ and $\varphi_z$, taking the role of synthetic Bloch wavenumbers ($k_y$ and $k_z$). a, Eigen-frequency spectrum of the sonic crystal at the plane $k_y = 0$, possessing a zero topological index. b, Same as panel a but for $k_y = 2\pi/3$ (the plane on transition). c, Same as panels a and b except that we set $k_y = \pi$. The band structure exhibits a zero-index topological boundary state. d,e,f, Resolution of the obtained eigen-frequency spectra based on the corresponding transmission spectra plotted as a function of frequency and $k_z$, for $k_y = 0$ (panel d), $k_y = 2\pi/3$ (panel e) and $k_y = \pi$ (panel f). g,h,i, Corresponding experimental measurements, showing good agreement with simulations.
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Supplementary information for the paper

Zero-index Weyl metamaterials

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This supplementary information contains the following sections

I. Topology of the proposed type III Weyl semimetal

II. Geometrical description

III. Numerical methods

IV. Experimental methods

V. Anomalous tunneling based on type III semimetals

VI. Effect of losses on the super-coupling effect

VII. Fermi surface of the proposed type III semimetal

VIII. Observation of zero-index Weyl semimetals based on electromagnetic waves

IX. Observation of type I Weyl semimetals in the proposed sonic crystal

X. Observation of type II Weyl semimetals in the proposed sonic crystal
Supplementary Note I. Topology of the proposed type III Weyl semimetal

The proposed system belongs to a class of three-dimensional systems known as $\mathbb{Z}_2$ Weyl semimetals. In such systems, possessing preserved time-reversal symmetry, the Hamiltonian is invariant under the operation of two operators: (i) an anti-unitary one related to time-reversal symmetry, such that $T H(k_x, k_y, k_z) T^{-1} = H(-k_x, -k_y, -k_z)$; and (ii) a unitary reflection symmetry operator $R_y$ which is related to a mirror plane, here in the $y$ direction, such that $R_y H(k_x, k_y, k_z) R_y^{-1} = H(k_x, -k_y, k_z)$. Let us point out that our system indeed supports these two symmetries with the simplest possible $R_y$, namely $R_y = \sigma_0$, due to the fact that the Hamiltonian of equation (1) (in the main text) is an even function of $k_y$, and $T = i \sigma_0 \mathcal{K}$, which is indeed anti-unitary and satisfies $T^2 = -1$ ($\mathcal{K}$ is complex conjugation).

These two symmetries have several implications about the number of allowed Weyl points and the definition of the topology of 2D cut-planes. In particular, if a Weyl point exists with a given chirality at $(k_x, k_y, k_z)$, then there exists three other Weyl points: one at $(-k_x, -k_y, -k_z)$, with the same chirality (due to $T$), and two others at $(k_x, -k_y, k_z)$ and $(-k_x, k_y, -k_z)$, with the opposite chirality (due to $R_y$). Supplementary Figure 1a shows these four Weyl points together with the Fermi arcs, which connect them along the $k_y$ direction. We performed a direct calculation of the distribution of the Berry curvature around these four points, demonstrating that their respective chiralities indeed obey the above symmetry constraints (see Supplementary Figure 1b-e).
Supplementary Figure 1: The four Weyl points of our system, their chirality, and the topological invariants of 2D cut planes with fixed \( k_y \). 

a, Location of the Weyl points and Fermi arc in the \( k_x = 0 \) plane. The red and blue points indicate whether they are sources or drains of Berry curvature. The Fermi arcs are represented as blue lines connecting a pair of Weyl points. The dashed lines represent two possible cut planes with the calculated values of their \( \nu_1 \) invariants. 

b-e, Vector plot of the Berry curvature obtained via a direct calculation, and plotted on a small spherical surface around the Weyl points, confirming their chiralities. The coordinates of the Weyl points in momentum space are shown above the plots.

The topology of 2D cut planes with fixed \( k_y \) is defined in the following way. Noting that the anti-unitary operator \( \tilde{T} = R_z T \) satisfies \( \tilde{T} H(k_x,k_y,k_z) \tilde{T}^{-1} = H(-k_x,k_y,-k_z) \), with \( \tilde{T}^2 = -1 \), we see that \( \tilde{T} \) links a Weyl point at \((k_x,k_y,k_z)\) to another one at\((-k_x,k_y,-k_z)\). It therefore operates like a time-reversal symmetry operator at fixed \( k_y \). Since \( \tilde{T}^2 = -1 \), the fixed-\( k_y \) Hamiltonians \( H(k_x,k_z;k_y) \) belong to the AII class of the Atland-Zirnbauer
classification of free-fermion Hamiltonians. Therefore, on each 2D cut planes with fixed $k_y$, one can define the $\mathbb{Z}_2$ index $v_2(k_y)$ in the usual way as:

$$(-1)^{v_2(k_y)} = \prod_{(k_x,k_z)\in TRIM_2} \frac{P_f(\omega(k_x,k_z;k_y))}{\sqrt{\det(\omega(k_x,k_z;k_y))}}$$ (S1)

where the matrix $\omega$ is defined as $\omega_{ij}(k_x,k_z;k_y) = \langle \phi_i(-k_x,-k_z;k_y) | \tilde{T} | \phi_j(k_x,k_z;k_y) \rangle$, with $| \phi_j(k_x,k_z;k_y) \rangle$ is the wave function of the $j$th band, smoothly defined over $(k_x,k_z)$, and $TRIM_2$ denotes the ensemble of momenta that are invariant under the action of $\tilde{T}$ on the 2D cut plane.

One important point is that the $v_2$ invariant is a topological index and therefore cannot change unless the band gap of the cut-plane closes, which happens only when $k_y$ crosses a pair of Weyl points (which explains the topological transitions when crossing the two Weyl points at $k_y = \pm 2\pi/3$). By direct calculation, we find $v_2(k_y) = 1$ for $|k_y| < 2\pi/3$, and 0 otherwise.

**Supplementary Note II: Geometrical description**

- **Geometry of the 3D sonic crystal**

The unit cell of the three-dimensional sonic crystal (Fig. 3 of the main text) is shown in Supplementary Figure 2. As observed, it is composed of Helmholtz resonators connected to each other via acoustic channels with specific widths. The width of the channels determines the coupling between the resonators. In this configuration, the diameters of the spheres are $d = 2.1 \text{ cm}$. Along $y$ and $z$ directions, the width of the tubes are $w_y = 1.1 \text{ cm}$ and $w_z = 0.9 \text{ cm}$, respectively. Along $x$ direction, the corresponding thick and thin channels have the widths of $w_{x1} = 0.7 \text{ cm}$, and $w_{x2} = 0.4 \text{ cm}$, respectively. The lattice constant is $a = 3.8 \text{ cm}$. 
The super-cell of the crystal (consisting of $7 \times 11$ unitcells) was three-dimensional (3D) printed in an ABS polymer.

**Supplementary Figure 2:** Geometry of the unit-cell of the sonic crystal shown in Fig. 3 of the article. The unit cell consists of Helmholtz resonators (air volume represented in grey) connected to each other via acoustic channels with specific widths.

**-Geometry of the 1D synthetic Weyl semimetal**

Here, we provide the geometrical description of the 1D sonic crystal used to explore the dispersion properties of the proposed zero-index WSM in synthetic dimension (used to obtain Fig. 5 of the main text). The geometry of the crystal is represented in Supplementary Figure 3a. It is composed of an acoustic waveguide with the width and height of $w = 7 \ cm$, $h = 7 \ cm$, respectively, and the length of $l = 1.8 \ m$. Inside the waveguide, a phononic crystal with detuned on-site potentials and couplings is implemented. Note that, as schematically shown in the figure, the on-site potentials of the resonators and the associated hopping rates can be tuned by changing the diameters of the cylinders ($2R_n$), and the distance ($d_n$) between them.
Supplementary Figure 3b shows the experimental realization of this 1D sonic crystal. As mentioned in the main text, the structure consists of an acrylic square pipe, with the width, height and length of \( w = 7 \text{ cm} \), \( h = 7 \text{ cm} \), and \( l = 1.8 \text{ m} \), respectively. Inside the pipe, the 1D sonic crystal was formed based on commercially available cast plastic rods.

Supplementary Figure 3: Example of sonic crystal geometry used to explore the dispersion properties of three-dimensional crystals in synthetic dimensions: a, Geometry used in simulation. The structure consists of an acoustic waveguide and a set of cylindrical scatters arranged inside the waveguide. b, Realization of the sonic crystal described in panel a in practice. The configuration includes an acrylic square pipe, serving as the acoustic waveguide, and cast plastic rods arranged inside the pipe.

Supplementary Note III: Numerical Methods

-Semi-analytical methods

The results of Figures 1 of the article were obtained based on the tight-binding Hamiltonian given in Eq.1, which corresponds to the crystal shown in Fig. 1a of the article. In order to obtain the band structure of a single unit-cell (results of Fig. 1 of the article), we calculated the eigen-values of the Hamiltonian, while sweeping the wavenumbers \( k_x \), \( k_y \) and \( k_z \) in the Brillouin zone. On the other hand, the dispersion diagrams shown in Fig. 2 correspond to a finite chain of the crystal, composed of 100 unit-cells connected along the x direction, but
periodic (infinite) in y and z directions. The first and last couplings in the x direction are zero. The dispersion surfaces of this super-cell were obtained by varying $k_y$ and $k_z$ in the associated 2D Brillouin zone, and calculating the eigen-values of the tight-binding Hamiltonian. Note that we considered the following parameters in our calculations: $\lambda_x=1$, $\lambda_y=2$, $\lambda_z=2$, $\delta_x=0.5$ and $\delta_z=\lambda_z$.

**Finite-element simulations**

All full-wave finite-element-based simulations throughout the manuscript were performed using Comsol Multiphysics 5.3a. We considered sound hard-wall boundary conditions to be applied on the external boundaries of all scatterers and the waveguide. The volume between the cylindrical rods and the waveguide was filled with a material with the defined density of $\rho=1.2 \text{ kg/m}^3$ and a bulk modulus of $\beta=142 \text{ kPa}$, which corresponds to the acoustic properties of air in audible range.

In order to obtain the band structure shown in Fig. 3b, we considered a single unit cell of the sonic crystal and applied Floquet boundary condition to its boundaries. Fig. 3c assumes a supercell with 11 cells along x, terminated with hard wall boundary conditions in x, and periodic boundary conditions in the other directions. The dispersion bands were obtained from the FEM eigenvalue solver for all of the associated Bloch wave-numbers.

The results of Fig. 3d of the manuscript are obtained based on a super-cell of the proposed sonic crystal, consisting of $7 \times 11$ unit-cells, terminated along x by hard wall boundary conditions, and periodic boundary conditions in the other directions. The profile of the corresponding bulk and edge modes were obtained directly from the FEM eigenvalue solver.

In order to obtain the results of Fig. 3e, we excited one of the external waveguides of the system using a plane-wave with unitary amplitude. The other external waveguide was terminated by a plane-wave radiation boundary condition, but with no incident pressure field. The system was assumed to be lossless. Therefore, the corresponding transmission spectrum
was readily obtained by plotting the amplitude of the transmitted field as a function of frequency.

Regarding the numerical results presented in Fig. 5, the procedure is the following. Each cylinder of the one-dimensional sonic crystal supports a bound state in the continuum (BIC), which is odd-symmetric with respect to the waveguide center line in the direction perpendicular to the cylinder axis, and is highly confined to the cylinder. The resonance frequency of this BIC can be regulated by changing the diameter of the cylinder: the larger the diameter of the cylinder, the higher the resonance frequency of the BIC mode. The evanescent coupling strength between the BIC modes, on the other hand, can be desirably tuned by changing the distance between the centers of the scatterers: in general, the larger the distance between two scatterers, the weaker the coupling between them. To map the proposed tight binding toy model into such a 1D system, we considered (in a synthetic dimension) an ensemble of such one-dimensional sonic crystals with resonance frequencies and coupling strengths modulated according to the relations prescribed in the main text. We performed eigen-frequency simulations for each of the one-dimensional sonic crystals and recorded the results. The band structure of the proposed zero-index WSM was demonstrated by assembling the recorded results.

As explained in the main text, in order to resolve the band structure of the crystal in scattering tests, one needs to excite the system (the waveguide) with a plane-wave and analyze the corresponding transmission spectrum. In order to be able to excite the BIC modes from the far field, the mirror symmetry of the structure was broken by slightly shifting all cylinders to the top. Plane-wave radiation boundary conditions were applied to the ends of the waveguide, and an incident pressure field with unitary amplitude was sent from the left. A visco-thermal acoustic loss coefficient of 1.2 dB/m was included in the simulations to take into account the losses present in the experiment and obtain the plots of Fig. 5d-f. The scattering coefficients,
including the transmission coefficient through the structure, were numerically obtained by extracting the acoustic pressure field (amplitude and phase) at three different points, one located at the transmission side and the other two at the reflection side.

**Supplementary Note IV: Experimental methods**

**-Super-coupling experiment**

In our 3D experiment (the results of Fig. 4), we connected the two external sides of the sonic crystal to two big circular waveguides with the cross-section area of 32.3 cm². One of the external waveguides (the one on the left) was excited with a loudspeaker, whereas the other one on the right side was terminated with an absorbent material (foam). The amplitude (Fig. 4c, top) and phase (Fig. 4c, bottom) of the acoustic pressure field were measured by four different ICP® PCB 130F20 microphones, placed at four specific locations inside the crystal, and the signal processed by a Dataphysics Quattro acquisition card, performing an average over 10 measurements.

**-Experiment on 1D synthetic Weyl semimetal**

Our proposed one-dimensional sonic crystal is composed of two ingredients: an acoustic waveguide and a set of cylindrical scatterers with different radii. In our experiment, we used a very simple square pipe, made of transparent acrylic glass, as the acoustic waveguide. On the other hand, commercially available cast nylon rods were inserted into the waveguide to construct the sonic crystal. The rods were cut at specific diameters and placed at their specified locations. A home-made anechoic acoustic termination was also designed and put at the end of the waveguide in order to minimize the amount of unwanted reflection.

To extract the band structure of the proposed zero-index WSM, we used the same technique as in the numerical simulations: we excited the mode of the waveguide with a loudspeaker, driven with a burst noise voltage, and calculate the corresponding transmission spectrum by
doing standing wave-pattern analysis. The standing wave pattern analysis was performed based on the corresponding sound pressure level at three different points, measured by three different ICP® PCB 130F20 microphones. The whole setup was controlled by our Data physics Quattro signal analyzer, connected to a standard computer.

**Supplementary Note V: Anomalous tunneling based on type III semimetals**

It is possible to support the claim of perfect matching between the external waveguide and the edge mode by a direct quantitative demonstration. For an external waveguide of cross-sectional area $S_{\text{ext}}$, the impedance is defined as $Z_{\text{ext}} = \sqrt{\frac{\rho_0 B_0}{S_{\text{ext}}}}$, where $\rho_0$ and $B_0$ are the density and bulk modulus of air. Taking $\rho_0 = 1.225 \text{ kg/m}^3$, $B_0 = 144 \text{ kPa}$, and $S_{\text{ext}} = \pi r_{\text{ext}}^2 = 30.6 \text{ cm}^2$, one finds $Z_{\text{ext}} = 137.255 \text{ kg s}^{-1} \text{ m}^{-4}$. On the other hand, the acoustic impedance of the edge mode $Z_{\text{crys}} = \sqrt{\frac{\rho_{\text{eff}} B_{\text{eff}}}{S_{\text{crys}}}}$ can be evaluated after extracting the effective parameters $\rho_{\text{eff}} = 0.0169 \text{ kg/m}^3$ and $B_{\text{eff}} = 4.43 \text{kPa}$ (we used the standard extraction techniques based on the measurement of the reflection and transmission coefficients). Given that the geometrical parameter $S_{\text{crys}} = \pi r_{\text{crys}}^2 = 0.63 \text{ cm}^2$ is known, one finds $Z_{\text{crys}} = 137.342 \text{ kg s}^{-1} \text{ m}^{-4}$, which differs from $Z_{\text{ext}}$ by less than 1‰, proving our point. This matching condition is expressed as:

$$\frac{\sqrt{\rho_{\text{eff}} B_{\text{eff}}}}{S_{\text{crys}}} = \frac{\sqrt{\rho_0 B_0}}{S_{\text{ext}}} \quad \text{(S2)}$$

Eq. S2 reveals the underlying physics of anomalous tunneling. From this equation, it can be inferred that the anomalously small refractive index of the zero-index crystal compensates the geometrical mismatch between the crystal and the external channels (i.e. the large difference between $S_{\text{ext}}$ and $S_{\text{crys}}$). Indeed, the effective mass of the crystal is $\rho = 0.0169 \text{ kg/m}^3$, i.e. 1.3% of the one of air, and the bulk modulus is $B_{\text{eff}} = 4.43 \text{kPa}$, corresponding to 2.8% of the
one of air. This matching effect is drastically different from the conventional Fabry-Perot ones. In fact, due to the quasi-static nature of the wave propagation inside the zero-index crystal, the matching and, as such, the tunneling, do not depend on the length or shape of the zero-index crystal. This effect is akin to electrical cables at kHz frequencies, for which the electrical wavelength is so large such that they can be bent. To demonstrate that the zero-index tunneling does not depend on the length of the crystal, we report the corresponding mode profile of the structure for two different lengths, namely $L_1 = 2.3\lambda$, and $L_2 = 1.7\lambda$. The results are shown in Supplementary Figures 4a and b.

Supplementary Figure 4: Anomalous tunneling based on the proposed type III semimetal, a, Extra-ordinary transmission through the zero-index edge mode of the proposed type III semimetal. The length is the crystal is approximately $L = 2.3\lambda$ ($\lambda$ is the wavelength at which the tunneling occurs), b, Same as panel a except that the length of the crystal is reduced to $L = 1.7\lambda$ (the tunneling frequency is the same). The results of the figure demonstrate that the tunneling is independent of the length of the crystal.

Supplementary Note VI: V. Effect of losses on the super-coupling effect
In this section, we discuss the effect of losses on the super-coupling effect. The numerical results of Fig. 3e of the manuscript were obtained by ignoring the dissipation loss of the system. Dissipation, however, breaks the impedance matching condition, leading to a transmission amplitude less than unity (at the frequency at which the tunneling occurs). However, losses destroy neither the presence of the zero-index topological edge mode nor its constant phase profile. In order to demonstrate these points, we repeated our simulations for two different values of acoustic losses introduced to the system, namely 0.2 \( \text{dB/m} \) and 0.4 \( \text{dB/m} \). The corresponding results are shown in Supplementary Figures 5a and b, respectively. It is seen that the amplitude of the transmitted pressure field decreases when the amount of loss is increased. However, the quasi-static nature of the wave propagation inside the crystal is not affected, and can be safely observed in experiments by probing the phase of the field along the edge (Fig. 4 of the main text).

**Supplementary Figure 5: Effect of dissipation losses on the anomalous tunneling.** The figure repeats the analysis of Fig. 3e of the main text except that some dissipation is introduced to the system. The associated amounts of losses are 0.2 \( \text{dB/m} \) in panel a and 0.4 \( \text{dB/m} \) in panel b.

**Supplementary Note VII. Fermi surface of the proposed type III Weyl semimetal**
In this section, we study the Fermi surfaces of the proposed Weyl semimetal. The Fermi surfaces of our proposed zero-index semimetal are shown in Supplementary Figure 6. It is seen that the associated Fermi surfaces are two straight lines, connecting a pair of Weyl nodes in the momentum space. This is in contrast to previously proposed type I and II Weyl semimetals, whose Fermi surfaces are closed single points and open surfaces, respectively (see Ref. 21 of the article).

Supplementary Figure 6: Fermi Surface of the proposed type III Weyl semimetal at the critical angle. The Fermi surfaces consist of straight lines connecting a pair of Weyl nodes to each other.

Supplementary Note VIII: Observation of photonic zero-index Weyl semimetals based on synthetic dimensions

While in the main text we based our analysis on acoustic signals for the sake of a simpler practical realization, the Hamiltonian of the proposed type III Weyl semimetals (Eq. 1 of the main text), can be realized in other areas of classical wave physics. Here, for instance, we study how it can be realized in electromagnetism.
Supplementary Figure 7: Observation of zero-index Weyl semimetals in electromagnetics. We vary the on-site potentials and coupling coefficients of a 1D photonic crystal according to the formulas specified in the main text. \( a \), Dispersion surfaces of a single unit cell of the corresponding photonic crystal, calculated at the plane \( k_y = 2\pi/3 \). \( b \), Band structure of the photonic crystal for \( k_y = \pi \). The plane section has a zero topological charge, leading to a gapped energy spectrum, as it is observed. \( c \), Same as \( b \) but for \( k_y = 2\pi/3 \). Notice that the Fermi surface of the obtained Weyl semimetal is a flat line. \( d \), Same as panels \( b \) and \( c \) except that \( k_y \) is chosen to be \( k_y = 0 \). It is seen that the band structure exhibits topological surface states, one of which has a completely flat dispersion along \( k_z \).

Consider a one-dimensional photonic crystal made of dielectric rods. As we explained in the main text, in order to observe a zero-index WSM in such a one-dimensional system, one should vary the values of the on-site potentials and hopping strengths of the chain according to formulas \( \omega_n = 2\lambda_z \cos \varphi_z (1 + (-1)^n) \) and \( k_n = \lambda_x + (-1)^n \delta_x + \lambda_y \cos \varphi_y \). We first investigate the dispersion behavior of a single unit cell of the photonic crystal.
Supplementary Figure 7a shows the band structure of the photonic crystal at the plane section $k_y = 2\pi/3$. It is seen that there exists a gap-closing point (Weyl point) at $(k_x, k_z) = (0, \pi/2)$. Supplementary Figures 7b-d represent the eigen-frequency spectrum of a 1×1×8 supercell at different plane sections, namely $k_y = \pi$, $k_y = 2\pi/3$ and $k_y = 0$. As usual, the band structure is gapped at $k_y = \pi$, since it is a topologically trivial plane. At $k_y = 2\pi/3$, the phase transition occurs. Notice that the corresponding Fermi surface is a flat line, signature of a zero-index (type III) WSM. This gives rise to distinctive physical properties for the obtained semi-metallic phase as it was discussed in the main text. Finally, at the plane $k_y = \pi$ (which is a plane with a non-zero topological index), one realizes the presence of two helical surface states, possessing a zero group velocity (or a static-like field distribution) along $z$ direction.

**Supplementary Note IX: Observation of type I Weyl sonic semimetals in synthetic dimensions**

In this section, we discuss the possibility of observing a standard (type I) WSM in our one-dimensional sonic crystal. We first remark that the tight-binding toy model in Fig. 1 of the main text is nothing but a type I Weyl semimetal when $\delta_z = 0$. Suppose that we have modulated the two available parameters of the 1D phononic crystal according to the relations $\omega_n = 2\lambda_z \cos \varphi_z (-1)^n$, and $k_n = \lambda_x + (-1)^n\delta_x + \lambda_y \cos \varphi_y$. It can be easily verified that the corresponding (2×2) Hamiltonian of the chain will be identical to that of the type I WSM described by our toy model. As such, the dispersion characteristics of this semimetal can be probed by sweeping the modulation parameters $k_y (\varphi_y)$ and $k_z (\varphi_z)$ in the (synthetic) Brillouin zone. To demonstrate this point, we first report the dispersion surfaces of a single
Supplementary Figure 8: Realization of a type I Weyl semimetal in a one-dimensional sonic crystal. We modulate the hopping strengths and on-site potentials of the chain based on the relations described in the text. a, Dispersion surfaces of a single unit cell of the corresponding sonic crystal, calculated at the plane $k_y = 2\pi/3$. b, Band structure of a $1\times1\times8$ super cell of the crystal at $k_y = \pi$. The spectrum is gapped in this plane section since its topological charge is zero. c, Same as panel b but for $k_y = 2\pi/3$. It is observed that the band gap is closed and the Weyl transition occurs. d, Same as b and c but for $k_y = 0$, which is a topological plane. The existence of two helical topological surface states is obvious in the calculated band structure.

unit-cell of the phononic crystal (see Supplementary Figure 8a). The band structure is calculated at the plane $k_y = 2\pi/3$, which is the one at which the phase transition occurs.

As expected, the two bands of the two-level system touch each other at $(k_x, k_z) = (0, \pi/2)$ (Weyl point). More importantly, the cone spectrum is not tilted around these points, so the corresponding semi-metallic phase belongs to the group of type I WSMs.
Next, we calculate the band structure of a 1×1×8 super cell of such a crystal at three different plane sections, namely \(k_y = \pi\), \(k_y = 2\pi/3\), and \(k_y = 0\). The corresponding results are displayed in Supplementary Figures 8b-d, respectively.

**Supplementary Figure 9**: Observing type I Weyl semimetals in far-field scattering tests.

We suppose that the phononic crystal is excited with a plane wave with unit amplitude and calculate the corresponding transmission spectra as a function of \(k_z\) for a, \(k_y = \pi\), b, \(k_y = 2\pi/3\), and c, \(k_y = 0\). The resolved band structures are in perfect agreement with Supplementary Figure 8.

It is seen that at \(k_y = \pi\), the band structure is completely gapped, since this plane section includes no topological charge. At \(k_y = 2\pi/3\), the band structure is closed and the Weyl transition occurs. Lastly, at \(k_y = 0\), corresponding to a non-zero topological index, the dispersion diagram exhibits two helical gap-closing surface states. Note that, in contrast to
the type III (zero-index) WSM demonstrated in the main text, both of the topological boundary states have a non-zero group velocity in this case.

The eigen-frequency spectrum of the obtained semi-metallic phase can be resolved in scattering tests. To demonstrate this point, we excite the sonic crystal with a plane wave, calculate the corresponding transmission spectra as a function of $k_z$, and plot the results in Supplementary Figure 9a-c. The panels correspond to the plane sections $k_y = \pi, k_y = 2\pi/3, k_y = 0$, respectively. It is observed that, in all three cases, the band structure of the semi-metallic phase can be resolved perfectly by straightforward image processing (extracted band structure is shown in the right panels).

**Supplementary Note X: Observation of type II Weyl sonic semimetals in synthetic dimensions**

In this section, we discuss the possibility of probing the dispersion characteristics of a type II Weyl semimetal in our one-dimensional sonic crystal. We start with reminding that, for $\delta_z = 2\lambda_z$, the tight binding Hamiltonian given in Eq. 1 of the main text is identical to the Hamiltonian of a type II WSM. In order to realize this tight-binding Hamiltonian in our one-dimensional sonic system, we modulate the on-site energies and the coupling coefficients of the array according to the equations $\omega_n = 2\lambda_z \cos \varphi_z (1 + (-2)^n)$ and $k_n = \lambda_x + (-1)^n \delta_x + \lambda_y \cos \varphi_y$. It is easy to verify that the Hamiltonian of the corresponding 1D chain will then be identical to that of the type II WSM described by our tight binding toy model. In order to illustrate this point, we represent in Supplementary Figure 10a the dispersion surfaces of a single unit cell of the sonic crystal at $k_y = 2\pi/3$. It is seen that the band structure resembles that of a type II WSM. Supplementary Figures 10b-d represent the band structure of a 1×1×8 super cell of this single unit cell, calculated at three different plane sections, namely, $k_y = \pi, k_y = 2\pi/3, \text{and } k_y = 0$. As usual, the band structure is gapped at
Supplementary Figure 10: Realization of a type II Weyl semimetal in a one-dimensional sonic crystal. We vary the on-site potentials and coupling coefficients of the sonic crystal according to the formula specified in the main text. 

a, Dispersion surfaces of a single unit cell of the corresponding sonic crystal, calculated at the plane $k_y = 2\pi/3$.

b, Band structure of a $1\times1\times8$ super cell at the plane $k_y = \pi$, which is a plane with vanishing topological charge. 

c, Same as b but at $k_y = 2\pi/3$.

d, Same as panels b and c except that the band structure is calculated at the plane $k_y = 0$.

$k_y = \pi$. At $k_y = 2\pi/3$, the Weyl transition occurs. Notably, the Fermi surface of this semimetallic phase is an open surface, since the cone has been completely tipped over and the valence and conduction bands are coexisting in energy. At $k_y = 0$, the band structure exhibits two topological surface states with non-zero group velocities.

We also demonstrate these findings using two-port scattering analysis. To that end, we excite the system with a plane wave and report, in Supplementary Figures 11a-c, the corresponding
transmission spectra vs frequency and $k_z$ for different $k_y$, namely $k_y = 0, k_y = 2\pi/3, k_y = \pi$. Being in perfect agreement with what we observed in Supplementary Figure 10, the results of these figures show how our one-dimensional acoustic system is capable of mimicking the dispersion properties of a type II WSM.

**Supplementary Figure 11: Observation of a type II Weyl semimetal in the proposed sonic crystal.** We excite the sonic crystal with a plane wave and resolve the band structure of the chain based on the corresponding transmission spectrum. **a,** Transmission spectrum of the waveguide at the plane $k_y = \pi$. **b,** Transmission spectrum of the waveguide at the plane $k_y = 2\pi/3$. **c,** Transmission spectrum of the waveguide at the plane $k_y = 0$. Being in perfect agreement with our previous findings in Supplementary Figure 10, the results of this figure demonstrate the possibility of probing the band structure of a type II Weyl semimetal in the proposed 1D acoustic system.