Arbitrage of the first kind and filtration enlargements in semimartingale financial models

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(based on a joint work with C. Fontana and C. Kardaras)
Outline of the talk

- Problem formulation and motivation
- Progressive enlargement of filtration
- Initial enlargement of filtration
- Examples
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The problem:

- Consider a market without arbitrage profits.
- Suppose some agents have additional information.
- Can they use this information to realize arbitrage profits?
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Mathematically:

- market: \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, S)\), with \(\mathbb{F}\) satisfying the usual conditions, \(S = (S^i)_{i=1,...,d}\) non-negative semimartingale, \(S^0 \equiv 1\).
- additional information:
  - progressive enlargement of filtration (with any random time)
  - initial enlargement of filtration
- arbitrage profits: ...(some motivation first)
The basic example

- Let $W$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})$.
- Let $S$ represent the discounted price of an asset and be given by
  \[ S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right), \quad \sigma > 0 \text{ given.} \]
- Let $S_t^* := \sup \{ S_u, \ u \leq t \}$ and define the random time \[ \tau := \sup \{ t : S_t = S^*_\infty \} = \sup \{ t : S_t = S^*_t \} \]
- An agent with information $\tau$ can follow the arbitrage strategy
  
  “buy at $t = 0$ and sell at $t = \tau$”
The basic example

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$$\tau := \sup \{t : S_t = S^*_\infty\} = \sup \{t : S_t = S^*_t\}$$

An agent with information $\tau$ can follow the arbitrage strategy

“buy at $t = 0$ and sell at $t = \tau$”

Remark. Here $\tau$ is an honest time: $\forall \ t \geq 0 \ \exists \ \xi_t \ \mathcal{F}^W_t$-measurable s.t. $\tau = \xi_t$ on $\{\tau \leq t\}$ (e.g., $\xi_t := \sup \{u \leq t : S_u = \sup_{r \leq t} S_r\}$).

See: Fontana, Jeanblanc, Song (2013), Imkeller (2002), Zwierz (2007)
Different notions of arbitrage

Admissible wealth processes $\mathcal{X}(\mathbb{F}, S)$: class of all non-negative processes of the type $X^{x,H} := x + \int_0^t H_t dS_t$.

We recall the notions of:

- **arbitrage**: $\exists X^{1,H} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}[X^{1,H}_\infty \geq 1] = 1$, $\mathbb{P}[X^{1,H}_\infty > 1] > 0$. If such strategies do not exist we say that NA($\mathbb{F}, S$) holds.

- **free lunch with vanishing risk**: $\exists \epsilon > 0$, $0 \leq \delta_n \uparrow 1$, $X^{1,H^n} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $\mathbb{P}[X^{1,H^n}_\infty > \delta_n] = 1$, $\mathbb{P}[X^{1,H^n}_\infty > 1 + \epsilon] \geq \epsilon$. If such strategies do not exist we say that NFLVR($\mathbb{F}, S$) holds.

- **arbitrage of the first kind**: $\exists \xi \geq 0$ with $\mathbb{P}[\xi > 0] > 0$ s.t. for all $x > 0$, $\exists X \in \mathcal{X}(\mathbb{F}, S)$ with $X_0 = x$ s.t. $\mathbb{P}[X_\infty \geq \xi] = 1$. If such strategies do not exist we say that NA1($\mathbb{F}, S$) holds.

**Remark.** NA1 (Kardaras, 2010) $\iff$ BK (Kabanov, 1997) $\iff$ NUPBR (Karatzas, Kardaras 2007)
NFLVR $\iff$ NA + NA1
Martingale measures and deflators

- NFLVR $\iff$ NA + NA1

- NFLVR $\iff$ $\exists$ equivalent local martingale measure for $S$
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- NFLVR $\iff$ $\exists$ equivalent local martingale measure for $S$

- NA1 $\iff$ $\exists$ supermartingale deflator (Karatzas, Kardaras 2007):
  \[ Y > 0, \ Y_0 = 1 \ 	ext{s.t.} \ YX \text{ is a supermartingale } \forall X \in \mathcal{X} \]

  $\iff$ $\exists$ local martingale deflator (Takaoka 2013, Song 2013):
  \[ Y > 0, \ Y_0 = 1 \ 	ext{s.t.} \ YX \text{ is a local martingale } \forall X \in \mathcal{X} \]

  $\iff$ $\exists$ readable local martingale deflator (A.F.K. 2014):
  \[ Y \text{ local martingale deflator s.t. } 1/Y \in \mathcal{X} \ (\text{up to } Q \sim P) \]
As seen in the basic example, NA and NFLVR easily fail under additional information.

NA1 is completely characterized in terms of the semimartingale characteristics of $S$.

NA1 is the minimal condition in order to proceed with utility maximization.

NA1 is stable under change of numéraire.

NA1 is also equivalent to the existence of a numéraire portfolio $X^*$ (= growth optimal portfolio = log optimal portfolio), in which case $1/X^*$ is a supermartingale deflator.
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Whereas when an arbitrage exists we are in general not able to spot it, when an arbitrage of the first kind exists we are able to construct (and hence exploit) it

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**NA1** is also equivalent to the existence of a numéraire portfolio \( X^* \) (growth optimal portfolio = log optimal portfolio), in which case \( 1/X^* \) is a supermartingale deflator.
Why NA1? - Let me try to convince you

- As seen in the basic example, **NA and NFLVR easily fail under additional information**
- Whereas when an arbitrage exists we are in general not able to spot it, when an arbitrage of the first kind exists we are able to construct (and hence exploit) it
- NA1 is completely characterized in terms of the semimartingale characteristics of $S$
- NA1 is the minimal condition in order to proceed with utility maximization
- NA1 is stable under change of numéraire

NA1 is also equivalent to the existence of a numéraire portfolio $X^*$ (= growth optimal portfolio = log optimal portfolio), in which case $1/X^*$ is a supermartingale deflator.
Some related work (NA1 preservation)

On progressive enlargement:
- Aksamit, Choulli, Deng, Jeanblanc 2013:
  \( S \) quasi-left-continuous local martingale, using optional stochastic integral

- Fontana, Jeanblanc, Song 2013:
  \( S \) continuous, PRP, \( \tau \) honest and avoids all \( \mathbb{F} \)-stopping times, NFLVR in the original market. Then in the enlarged market:
  - on \([0, \infty)\): NA1, NA and NFLVR all fail;
  - on \([0, \tau]\): NA and NFLVR fail, but NA1 holds.

- Kreher 2014:
  all \( \mathbb{F} \)-martingales are continuous, \( \tau \) avoids all \( \mathbb{F} \)-stopping times, NFLVR in the original market

On initial enlargement: nothing in the literature that we are aware of. Some work in progress by Jeanblanc et al.
• Problem formulation and motivation
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• Examples
Let $\tau$ be a **random time** ($= \text{positive, finite, } \mathcal{F}\text{-measurable r.v.}$).

Consider the **progressively enlarged filtration** $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$,

$\mathcal{G}_t := \{ B \in \mathcal{F} \mid B \cap \{ \tau > t \} = B_t \cap \{ \tau > t \} \text{ for some } B_t \in \mathcal{F}_t \}$. 

[Jeulin-Yor theorem ensures that $\mathcal{G}'$-hypothesis holds up to $\tau$: every $\mathcal{F}$-semimartingale remains a $\mathcal{G}$-semimartingale up to time $\tau$ (in particular $S_{\tau}$ is a $\mathcal{G}$-semimartingale).]
Progressive enlargement of filtrations

- Let $\tau$ be a **random time** ($= \text{positive, finite, } \mathcal{F}\text{-measurable r.v.}$).

- Consider the **progressively enlarged filtration** $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$,

$$\mathcal{G}_t := \{ B \in \mathcal{F} \mid B \cap \{\tau > t\} = B_t \cap \{\tau > t\} \text{ for some } B_t \in \mathcal{F}_t \}.$$ 

- Jeulin-Yor theorem ensures that $\mathcal{H}'$-hypothesis holds up to $\tau$: every $\mathcal{F}$-semimartingale remains a $\mathcal{G}$-semimartingale up to time $\tau$ (in particular $S^\tau$ is a $\mathcal{G}$-semimartingale).
Our main tools

Consider the two $\mathbb{F}$-supermartingales associated to $\tau$:

$$Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P}[\tau \geq t \mid \mathcal{F}_t]$$

($Z = \text{Azéma supermartingale}$ associated to $\tau$)
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  \[ Z_t := \mathbb{P} [\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P} [\tau \geq t \mid \mathcal{F}_t] \]
  ($Z =$ Azéma supermartingale associated to $\tau$)

- $A := \mathbb{F}$-dual optional projection of $\mathbb{1}_{[\tau, \infty[}$, $M_t := \mathbb{E} [A_\infty \mid \mathcal{F}_t]$, 
  \[ Z_t = M_t - A_t, \quad \tilde{Z}_t = M_t - A_{t-}, \quad \Delta A_t = \tilde{Z}_t - Z_t, \]
  \[ \Delta A_\sigma = \mathbb{P} [\tau = \sigma \mid \mathcal{F}_\sigma] \text{ for all } \mathbb{F}\text{-stopping times } \sigma \]
Our main tools

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  \[ Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t], \quad \tilde{Z}_t := \mathbb{P}[\tau \geq t \mid \mathcal{F}_t] \]
  
  ($Z = \text{Azéma supermartingale}$ associated to $\tau$)

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  \[ \Delta A_\sigma = \mathbb{P}[\tau = \sigma \mid \mathcal{F}_\sigma] \quad \text{for all } \mathbb{F}\text{-stopping times } \sigma \]

- Define the stopping time
  \[ \zeta := \inf \{ t \in \mathbb{R}_+ \mid Z_t = 0 \}. \]

  Note that $\tau \leq \zeta$. 
Recall that $Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t]$ and $\zeta := \inf\{t \in \mathbb{R}_+ \mid Z_t = 0\}$.

Define $\Lambda := \{\zeta < \infty, Z_{\zeta-} > 0, \Delta A_\zeta = 0\} \in \mathcal{F}_\zeta$

$= \text{set where } Z \text{ jumps to zero after } \tau$
Our main tools

Recall that $Z_t := \mathbb{P}[\tau > t \mid \mathcal{F}_t]$ and $\zeta := \inf \{ t \in \mathbb{R}_+ \mid Z_t = 0 \}$.

- Define $\Lambda := \{ \zeta < \infty, Z_{\zeta-} > 0, \Delta A_\zeta = 0 \} \in \mathcal{F}_\zeta$
  
  = set where $Z$ jumps to zero after $\tau$

- and define

  $\eta := \zeta \mathbb{1}_\Lambda + \infty \mathbb{1}_{\Omega \setminus \Lambda}$

Note that $\tau < \eta$; $\eta =$ time when $Z$ jumps to zero after $\tau$. 
Representation pair associated with $\tau$

Theorem (Itô, Watanabe 1965, Kardaras 2014).
The following multiplicative optional decomposition holds for the Azéma supermartingale $Z$:

$$Z_t = \mathbb{P} [\tau > t \mid \mathcal{F}_t] = L_t (1 - K_t), \quad t \geq 0,$$

where:

- $L$ is a nonnegative $\mathbb{F}$-local martingale with $L_0 = 1$,
- $K$ is a nondecreasing $\mathbb{F}$-adapted process with $0 \leq K \leq 1$,

$\triangleright$ The local martingale $L$ coming from this decomposition will play a main role in our results.
Back to the basic example

Asset price process: \( S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right) \)

Random time: \( \tau := \sup \{ t : S_t = S^*_\infty \} \)

In this case
\[
Z_t = \mathbb{P} [ \tau > t \mid \mathcal{F}_t] = \frac{S_t}{S_t^*}
\]

(that is, \( L = S \) in the previous multiplicative decomposition)

and \( Y := 1/L^\tau = 1/S^\tau \) is a local martingale deflator for \( S^\tau \) in \( \mathbb{G} \).

Therefore: \( \text{NA1 holds while NA and NFLVLR fail.} \)
Back to the basic example

Asset price process: \( S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right) \)

Random time: \( \tau := \sup \{ t : S_t = S_\infty^* \} \)

In this case
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Therefore: NA1 holds while NA and NFLVR fail.

Remarks.
1) An analogous situation occurs if we consider the random time \( \tau' := \sup \{ t : S_t = a \} \), for some \( 0 < a < 1 \).

2) The decomposition \( Z_t = L_t/L_t^* \) holds for a wide class of honest times (see Nikeghbali,Yor 2006, Kardaras 2013, A.,Penner 2014)
Remember: \( \eta \) is the time when \( Z \) jumps to zero after \( \tau \).

**Proposition.** Let \( X \) be a nonnegative \( \mathcal{F} \)-local martingale such that \( X = 0 \) on \([\eta, \infty[J\). Then \( X^{\tau}/L^{\tau} \) is a \( \mathcal{G} \)-local martingale.

Main tool in the proof:
- For any nonnegative optional processes \( V \) on \((\Omega, \mathcal{F})\),

\[
\mathbb{E}[V_{\tau}] = \mathbb{E} \left[ \int_{\mathbb{R}^+} V_t L_t \, dK_t \right],
\]

where \( L, K \) come from the multiplicative decomposition of \( Z \).
Remember: $\eta$ is the time when $Z$ jumps to zero after $\tau$.

**Proposition.** Let $X$ be a nonnegative $F$-local martingale such that $X = 0$ on $[\eta, \infty]$. Then $X^\tau/L^\tau$ is a $G$-local martingale.

Main tool in the proof:
- For any nonnegative optional processes $V$ on $(\Omega, F)$,
  
  $$\mathbb{E}[V_\tau] = \mathbb{E}\left[\int_{\mathbb{R}_+} V_t L_t dK_t\right],$$

  where $L, K$ come from the multiplicative decomposition of $Z$.

As an immediate consequence we have the following

**Key-Proposition.** Suppose there exists a local martingale deflator $M$ for $S$ in $F$ such that $M = 0$ on $[\eta, \infty]$. Then $M^\tau/L^\tau$ is a local martingale deflator for $S^\tau$ in $G$. 
To have preservation of the NA1 property, given a deflator for $S$ in $\mathbb{F}$, we want to “kill it” from $\eta$ on.

We will do it with the help of the following lemma.

**Lemma.** Let $D$ be the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta, \infty[}$. Then:

- $\Delta D < 1$ $\mathbb{P}$-a.s. ($\Rightarrow \mathcal{E}(-D) > 0$ and nonincreasing);
- $\mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta[}$ is a local martingale on $(\Omega, \mathbb{F}, \mathbb{P})$.

Main idea: for any predictable time $\sigma$ on $(\Omega, \mathbb{F})$,

$$
\Delta D_\sigma = \mathbb{P}[\eta = \sigma \mid \mathcal{F}_{\sigma^-}] < 1 \text{ on } \{\sigma < \infty\}.
$$
Theorem (one fixed $S$). Suppose that $\mathbb{P} [\eta < \infty, \Delta S_\eta \neq 0] = 0$. If NA1($\mathcal{F}, S$) holds, then NA1($\mathcal{G}, S^\tau$) holds.

That is: $S$ does not jump when $Z$ jumps to zero.
**Theorem (one fixed $S$).** Suppose that $\mathbb{P}[\eta < \infty, \Delta S_\eta \neq 0] = 0$. If $\text{NA1}(\mathcal{F}, S)$ holds, then $\text{NA1}(\mathcal{G}, S^\tau)$ holds.

That is: $S$ does not jump when $Z$ jumps to zero.

**Corollary.** If $\text{NA1}(\mathcal{F}, S^{\eta-})$ holds, then $\text{NA1}(\mathcal{G}, S^\tau)$ holds.

**Remark.** Condition $\mathbb{P}[\eta < \infty, \Delta S_\eta \neq 0] = 0$ is equivalent to evanescence of the set $\{Z_- > 0, \bar{Z} = 0, \Delta S \neq 0\}$. (See Aksamit et al. (2013), where $S$ is a quasi-left-continuous local martingale.)
Proof of the theorem

Recall: $D$ is the $\mathcal{F}$-predictable compensator of $\mathbb{I}_{[\eta,\infty]}$.

- $\text{NA1}(\mathcal{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathcal{F}, S)$ s.t. $Y := (1/\hat{X})$ is a local martingale deflator for $S$ in $\mathcal{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathcal{F}$ that vanishes on the set $[\eta, \infty]$.
Recall: $D$ is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta, \infty]}$.

- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathbb{F}, S)$ s.t. $Y := (1/\hat{X})$ is a local martingale deflator for $S$ in $\mathbb{F}$ ($\Rightarrow \Delta Y = 0$ when $\Delta S = 0$).

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathbb{F}$ that vanishes on the set $[\eta, \infty[$.

- Let $M := Y\mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta[} \Rightarrow \{M > 0\} = [0, \eta[)$.

- By the Lemma, $MS - [\mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta[, YS] \mathbb{F}$-local martingale.
Proof of the theorem

Recall: $D$ is the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta, \infty[}.$

- $\text{NA1}(\mathbb{F}, S) \Rightarrow \exists \hat{X} \in \mathcal{X}(\mathbb{F}, S) \text{ s.t. } Y := (1/\hat{X}) \text{ is a local martingale deflator for } S \text{ in } \mathbb{F} (\Rightarrow \Delta Y = 0 \text{ when } \Delta S = 0).$

- In order to apply the Key-Proposition, we need a deflator for $S$ in $\mathbb{F}$ that vanishes on the set $[\eta, \infty[.$

- Let $M := Y\mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta]} \Rightarrow \{M > 0\} = [0, \eta[.$

- By the Lemma, $MS - [\mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta]}, YS] \text{ is } \mathbb{F}-\text{local martingale}.$

- We want $M$ to be a deflator for $S$ in $\mathbb{F},$ so we need to show that the quadratic covariation part is an $\mathbb{F}$-local martingale.

- $\Delta S_{\eta} = 0 \Rightarrow \Delta(YS)_{\eta} = 0 \Rightarrow [.., ..] = [\mathcal{E}(-D)^{-1}, YS], \text{ which is indeed an } \mathbb{F}\text{-local martingale}.$
Theorem (general stability). TFAE:

1) for any $S$ s.t. $\text{NA1}({\mathbb{F}}, S)$ holds, $\text{NA1}({\mathbb{G}}, S^\tau)$ holds;

2) $\eta = \infty$ $\mathbb{P}$-a.s.;

3) For every nonnegative local martingale $X$ on $(\Omega, {\mathbb{F}}, \mathbb{P})$, the process $X^\tau / L^\tau$ is a local martingale on $(\Omega, {\mathbb{G}}, \mathbb{P})$;

4) The process $1/L^\tau$ is a local martingale on $(\Omega, {\mathbb{G}}, \mathbb{P})$.

Remark. Condition 2) is equivalent to evanescence of the set
\[
\{Z_- > 0, \tilde{Z} = 0\} = \{Z_- > 0, Z = 0, \Delta A = 0\}.
\]
(See Aksamit et al. (2013).)
Proof of the theorem

2) $\Rightarrow$ 1): from previous Theorem.

1) $\Rightarrow$ 2): suppose $\mathbb{P}[\eta < \infty] > 0$. Define

$$S := \mathcal{E}(-D)^{-1}\mathbb{1}_{[0,\eta]}.$$ 

Then $S$ is a $\mathbb{F}$-local martingale, and $S^\tau$ is nondecreasing with $\mathbb{P}[S_\tau > 1] > 0$. Hence NA1($\mathbb{F}, S$) holds, but NA1($\mathbb{G}, S^\tau$) fails.

2) $\Rightarrow$ 3): from the Proposition.

3) $\Rightarrow$ 4): trivial.

4) $\Rightarrow$ 2): uses properties of the processes $L$ and $K$ appearing in the multiplicative decomposition of $Z$. 
On the $\mathcal{H}'$-hypothesis

**Proposition.** Let $X$ be a nonnegative $\mathbb{F}$-supermartingale. Then, the process $X^\tau/L^\tau$ is a $\mathbb{G}$-supermartingale.

**Remark.** This can be used to establish that for any semimartingale $X$ on $(\Omega, \mathbb{F}, \mathbb{P})$, the process $X^\tau$ is a semimartingale on $(\Omega, \mathbb{G}, \mathbb{P})$. Indeed:

- By the Proposition, $\forall X$ nonnegative bounded $\mathbb{F}$-local martingale $\Rightarrow X^\tau/L^\tau$ and $1/L^\tau$ are $\mathbb{G}$-semimartingales $\Rightarrow X^\tau$ is a $\mathbb{G}$-semimartingale.
- From the semimartingale decomposition + localisation, same result for any $\mathbb{F}$-semimartingale $X$. 
A common assumption is that $\tau$ avoids all $\mathbb{F}$-stopping times: 
$\mathbb{P}[\tau = \sigma < \infty] = 0$ for all stopping times $\sigma$ on $(\Omega, \mathbb{F})$.

**Theorem.** Suppose that $\tau$ avoids all stopping times on $(\Omega, \mathbb{F}, \mathbb{P})$. If there exists a local martingale deflator for $S^\tau$ on $\mathcal{G}$, then there is a local martingale deflator for $S$ on $\mathbb{F}$ that vanishes on $[\eta, \infty[$.

**Proof.**
A partial converse

A common assumption is that \( \tau \) avoids all \( \mathbb{F} \)-stopping times: \( \mathbb{P}[\tau = \sigma < \infty] = 0 \) for all stopping times \( \sigma \) on \( (\Omega, \mathbb{F}) \).

**Theorem.** Suppose that \( \tau \) avoids all stopping times on \( (\Omega, \mathbb{F}, \mathbb{P}) \). If there exists a local martingale deflator for \( S^\tau \) on \( \mathbb{G} \), then there is a local martingale deflator for \( S \) on \( \mathbb{F} \) that vanishes on \([\eta, \infty[\).

**Proof.**

- Let \( M \) be a local martingale deflator for \( S^\tau \) on \( \mathbb{G} \).
- Let \( C \) be the \( \mathbb{G} \)-predictable compensator of \( I_{[\tau, \infty[} \).
- Then also \( U := ME(-C)^{-1}I_{[0, \tau[} \) is a local martingale deflator for \( S^\tau \) on \( \mathbb{G} \).
- Let \( Y \) be the optional projection of \( U \) on \( (\Omega, \mathbb{F}, \mathbb{P}) \). Then \( Y \) is a local martingale deflator for \( S \) on \( \mathbb{F} \), with \( Y = 0 \) on \([\eta, \infty[\).
Problem formulation and motivation

Progressive enlargement of filtration

Initial enlargement of filtration

Examples
Let $J$ be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel $\sigma$-field of $E$.

Let $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ be the right-continuous augmentation of the filtration $\mathcal{G}^0 = (\mathcal{G}_t^0)_{t \in \mathbb{R}_+}$ defined by

$$\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}_+.$$
Let $J$ be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel $\sigma$-field of $E$.

Let $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ be the right-continuous augmentation of the filtration $\mathcal{G}^0 = (\mathcal{G}_t^0)_{t \in \mathbb{R}^+}$ defined by

$$\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}^+.$$ 

Let $\gamma : \mathcal{B}_E \mapsto [0, 1]$ be the law of $J$ ($\gamma[B] = \mathbb{P}[J \in B], B \in \mathcal{B}_E$).

For all $t \in \mathbb{R}^+$, let $\gamma_t : \Omega \times \mathcal{B}_E \mapsto [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of $J$. 

Jacod's hypothesis. We assume $\gamma_t \ll \gamma_{\mathbb{P}}$-a.s., $t \in \mathbb{R}^+$. This ensures the $H'$-hypothesis and that we can apply Stricker&Yor calculus with one parameter $(L^1(\Omega, \mathcal{F}, \mathbb{P}))$ separable.)
Initial enlargement of filtrations

Let $J$ be an $\mathcal{F}$-measurable random variable taking values in a Lusin space $(E, \mathcal{B}_E)$, where $\mathcal{B}_E$ denotes the Borel $\sigma$-field of $E$.

Let $\mathcal{G} = (\mathcal{G}_t)_{t \in \mathbb{R}^+}$ be the right-continuous augmentation of the filtration $\mathcal{G}^0 = (\mathcal{G}^0_t)_{t \in \mathbb{R}^+}$ defined by

$$\mathcal{G}_t^0 := \mathcal{F}_t \vee \sigma(J), \quad t \in \mathbb{R}^+. $$

Let $\gamma : \mathcal{B}_E \mapsto [0, 1]$ be the law of $J$ ($\gamma[B] = \mathbb{P}[J \in B], B \in \mathcal{B}_E$).

For all $t \in \mathbb{R}^+$, let $\gamma_t : \Omega \times \mathcal{B}_E \mapsto [0, 1]$ be a regular version of the $\mathcal{F}_t$-conditional law of $J$.

**Jacod's hypothesis.** We assume

$$\gamma_t \ll \gamma \quad \mathbb{P}\text{-a.s.}, \quad t \in \mathbb{R}^+. $$

This ensures the $\mathcal{H}'$-hypothesis and that we can apply Stricker& Yor calculus with one parameter ($\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ separable).
Our main tools

\( \mathcal{O}(\mathbb{F}) \) (resp. \( \mathcal{P}(\mathbb{F}) \)) is the \( \mathbb{F} \)-optional (resp. pred.) \( \sigma \)-field on \( \Omega \times \mathbb{R}_+ \).

**Lemma.** There exists a \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable function \( E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p_t^x(\omega) \in [0, \infty) \), càdlàg in \( t \in \mathbb{R}_+ \) s.t.:

- \( \forall t \in \mathbb{R}_+, \gamma_t(dx) = p_t^x \gamma(dx) \) holds \( \mathbb{P} \)-a.s;
- \( \forall x \in E, p^x = (p_t^x)_{t \in \mathbb{R}_+} \) is a martingale on \( (\Omega, \mathbb{F}, \mathbb{P}) \).
Our main tools

\(O(\mathbb{F})\) (resp. \(P(\mathbb{F})\)) is the \(\mathbb{F}\)-optional (resp. pred.) \(\sigma\)-field on \(\Omega \times \mathbb{R}_+\).

**Lemma.** There exists a \(B_E \otimes O(\mathbb{F})\)-measurable function

\[E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p_t^x(\omega) \in [0, \infty),\] càdlàg in \(t \in \mathbb{R}_+\) s.t.:

- \(\forall t \in \mathbb{R}_+, \gamma_t(dx) = p_t^x \gamma(dx)\) holds \(\mathbb{P}\)-a.s;
- \(\forall x \in E, p^x = (p_t^x)_{t \in \mathbb{R}_+}\) is a martingale on \((\Omega, \mathbb{F}, \mathbb{P})\).

\(\triangleright\) For every \(x \in E\) define

\[\zeta^x := \inf\{t \in \mathbb{R}_+ \mid p_t^x = 0\}.\]
Our main tools

\( \mathcal{O}(\mathbb{F}) \) (resp. \( \mathcal{P}(\mathbb{F}) \)) is the \( \mathbb{F} \)-optional (resp. pred.) \( \sigma \)-field on \( \Omega \times \mathbb{R}_+ \)

**Lemma.** There exists a \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable function

\[ E \times \Omega \times \mathbb{R}_+ \ni (x, \omega, t) \mapsto p^x_t(\omega) \in [0, \infty), \text{ càdlàg in } t \in \mathbb{R}_+ \text{ s.t.:} \]

- \( \forall t \in \mathbb{R}_+, \gamma_t(dx) = p^x_t(\gamma(dx)) \) holds \( \mathbb{P} \)-a.s;
- \( \forall x \in E, p^x = (p^x_t)_{t \in \mathbb{R}_+} \) is a martingale on \( (\Omega, \mathbb{F}, \mathbb{P}) \).

▷ For every \( x \in E \) define

\[ \zeta^x := \inf \{ t \in \mathbb{R}_+ \mid p^x_t = 0 \}. \]

▷ Let \( \Lambda^x := \{ \zeta^x < \infty, p^x_{\zeta^x} > 0 \} \in \mathcal{F}_{\zeta^x} \) and define

\[ \eta^x := \zeta^x 1_{\Lambda^x} + \infty 1_{\Omega \setminus \Lambda^x}, \quad x \in E \]

Note that \( \eta^x \) (\( = \text{ time at which } p^x \text{ jumps to zero} \)) is a stopping time on \( (\Omega, \mathbb{F}) \).
NA1 under initial enlargement

Similar results (see) for the martingale deflators lead to:

**Theorem (one fixed $S$).** Let $\mathbb{P} [\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0 \gamma$-a.e. If NA1($\mathbb{F}, S$) holds, then NA1($\mathbb{G}, S$) holds.

**Theorem (general stability).** TFAE:

1) $\eta^x = \infty$ $\mathbb{P}$-a.s. for $\gamma$-a.e $x \in E$.

2) for all $X \geq 0$ $\mathcal{B}_E \otimes \mathcal{O}(\mathbb{F})$-meas. s.t. $X^x$ $\mathbb{F}$-loc.martingale vanishing on $[\eta^x, \infty]$ $\gamma$-a.e., $X^J/p^J$ is a $\mathbb{G}$-loc.martingale

3) The process $1/p^J$ is a $\mathbb{G}$-loc.martingale

And 1) $\Rightarrow$ For any $S$ s.t. NA1($\mathbb{F}, S$) holds, NA1($\mathbb{G}, S$) also holds.

Some care for the converse (see); we can derive $\mathcal{H}'$-hyp. (see).
• Problem formulation and motivation
• Progressive enlargement of filtration
• Initial enlargement of filtration
• Examples
Example 1: progressively enlarged filtration

- Consider $\zeta: \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall \ t \in \mathbb{R}_+$.
- Let $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.
- Define $\tau := \zeta/2$. 

\[ Z_t := \mathbb{P} \{ \tau > t | \mathcal{F}_t \} = \exp(-t) \mathbb{1}_{\{ t < \zeta \} } \] for all $t \in \mathbb{R}_+$.

\[ \zeta = \inf \{ t \geq 0 | Z_t = 0 \} =: \eta < \infty \operatorname{P-a.s.} \]

The $\mathcal{F}_t$-pred. comp. of $I_{[\eta, \infty[}$ is $\mathcal{D} := (\eta \wedge t)_{t \in \mathbb{R}_+}$.

\[ S := \mathbb{E}(\exp(-\mathcal{D}))^{-1} I_{[0, \eta[} \] that is, $S_t = \exp(t) I_{\{ t < \zeta \} }$.

$S$ nonnegative $\mathcal{F}_t$-martingale $\implies$ $\text{NA}_1(\mathcal{F}, S)$.

But $S$ is strictly increasing up to $\tau$ $\implies$ $\text{NA}_1(\mathcal{G}, S_\tau)$ fails.
Example 1: progressively enlarged filtration

- Consider $\zeta : \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall \ t \in \mathbb{R}_+$.
- Let $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.
- Define $\tau := \zeta/2$.

- Note that $Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t] = \exp(-t)I_{\{t < \zeta\}}$ for all $t \in \mathbb{R}_+$.
- Note that $\zeta = \inf\{t \geq 0 \mid Z_t = 0\} =: \eta < \infty$ $\mathbb{P}$-a.s.
- The $\mathbb{F}$-pred. comp. of $I_{[\eta, \infty]}$ is $D := (\eta \land t)_{t \in \mathbb{R}_+}$. 
Example 1: progressively enlarged filtration

Consider $\zeta : \Omega \mapsto \mathbb{R}_+$ such that $\mathbb{P}[\zeta > t] = \exp(-t), \forall \ t \in \mathbb{R}_+$.

Let $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.

Define $\tau := \zeta/2$.

Note that $Z_t := \mathbb{P}[\tau > t|\mathcal{F}_t] = \exp(-t)\mathbb{I}_{\{t < \zeta\}}$ for all $t \in \mathbb{R}_+$.

Note that $\zeta = \inf\{t \geq 0 \mid Z_t = 0\} =: \eta < \infty \ \mathbb{P}$-a.s.

The $\mathbb{F}$-pred. comp. of $\mathbb{I}_{[\eta, \infty]}$ is $D := (\eta \wedge t)_{t \in \mathbb{R}_+}$.

$S := \mathcal{E}(-D)^{-1}\mathbb{I}_{[0, \eta]} = \exp(D)\mathbb{I}_{[0, \eta]}$, that is, $S_t = \exp(t)\mathbb{I}_{\{t < \zeta\}}$.

$S$ nonnegative $\mathbb{F}$-martingale $\Rightarrow$ NA1($\mathbb{F}, S$).

But $S$ is strictly increasing up to $\tau$ $\Rightarrow$ NA1($\mathcal{G}, S^\tau$) fails.
Consider a Poisson(\(\lambda\)) process \(N\) stopped at time \(T \in (0, \infty)\).

Let \(\mathbb{F}\) be the right-cont. filtration generated by \(N\) and \(J := N_T\).

Then (Grorud, Pontier 2001) \(p_T^x = e^{-\lambda T} x!/(\lambda T)^x \mathbb{I}_{\{N_T = x\}}\) and

\[
p_t^x = e^{-\lambda t} \frac{(\lambda(T-t))^{x-N_t}}{(\lambda T)^x} \frac{x!}{(x-N_t)!} \mathbb{I}_{\{N_t \leq x\}}, \quad \forall \ t \in [0, T).
\]
Example 2: initially enlarged filtration

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathbb{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud,Pontier 2001) $p^x_T = e^{-\lambda T}x!/(\lambda T)^x\mathbb{I}_{\{N_T=x\}}$ and $p^x_t = e^{-\lambda t}\frac{(\lambda(T-t))^{x-N_t}}{(\lambda T)^x}x!\mathbb{I}_{\{N_t\leq x\}}, \ \forall \ t \in [0, T]$.

- $S_t := \exp(N_t - \lambda t(e - 1))$, for all $t \in [0, T]$.
- $S$ is a strictly positive $\mathbb{F}$-martingale $\Rightarrow$ NA1($\mathbb{F}, S$) holds.
Example 2: initially enlarged filtration

- Consider a Poisson($\lambda$) process $N$ stopped at time $T \in (0, \infty)$.
- Let $\mathbb{F}$ be the right-cont. filtration generated by $N$ and $J := N_T$.
- Then (Grorud, Pontier 2001) $p^x_T = e^{-\lambda T}x!/(\lambda T)^x \mathbb{I}_{\{N_T=x\}}$ and
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p^x_t = e^{-\lambda t} \frac{(\lambda(T-t))^{x-N_t}}{(\lambda T)^x} \frac{x!}{(x-N_t)!} \mathbb{I}_{\{N_t\leq x\}}, \quad \forall \ t \in [0, T).
\]
- $S_t := \exp(N_t - \lambda t(e-1))$, for all $t \in [0, T]$.
- $S$ is a strictly positive $\mathbb{F}$-martingale $\Rightarrow$ NA1($\mathbb{F}$, $S$) holds.
- Define the $\mathbb{G}$-stopping time $\sigma := \inf \{ t \in [0, T] \mid N_t = N_T \}$.
- For all $t \in [0, T]$, we get
  \[
  (-\mathbb{I}_{[\sigma, T]} \cdot S)_t = \mathbb{I}_{\{t>\sigma\}} \exp(N_\sigma - \lambda \sigma(e-1)) \left( 1 - \exp(-\lambda(t-\sigma)(e-1)) \right).
  \]
- $-\mathbb{I}_{[\sigma, T]} \cdot S$ is nondecreasing, $\mathbb{P} [\sigma < T] = 1 \Rightarrow$ NA1($\mathbb{G}$, $S$) fails.
Example 2: initially enlarged filtration

- Consider a Poisson(\(\lambda\)) process \(N\) stopped at time \(T \in (0, \infty)\).
- Let \(\mathbb{F}\) be the right-cont. filtration generated by \(N\) and \(J := N_T\).
- Then (Grorud, Pontier 2001) \(p_T^x = e^{-\lambda T} x!/(\lambda T)^x I\{N_T=x\}\) and
  \[
p_t^x = e^{-\lambda t} \frac{(\lambda(T-t))^{x-N_t} x!}{(\lambda T)^x (x-N_t)!} I\{N_t \leq x\}, \quad \forall \ t \in [0, T).
\]
- \(S_t := \exp(N_t - \lambda t(e-1))\), for all \(t \in [0, T]\).
- \(S\) is a strictly positive \(\mathbb{F}\)-martingale \(\Rightarrow\) NA1(\(\mathbb{F}, S\)) holds.
- Define the \(\mathbb{G}\)-stopping time \(\sigma := \inf \{t \in [0, T] \mid N_t = N_T\}\).
- For all \(t \in [0, T]\), we get
  \[
  (-\mathbb{I}_{\sigma,T} \cdot S)_t = \mathbb{I}_{\{t > \sigma\}} \exp(N_\sigma - \lambda \sigma(e-1)) \left(1 - \exp(-\lambda(t-\sigma)(e-1))\right).
  \]
  \(-\mathbb{I}_{\sigma,T} \cdot S\) is nondecreasing, \(\mathbb{P}[\sigma < T] = 1 \Rightarrow\) NA1(\(\mathbb{G}, S\)) fails.

Note: \(p^x\) have positive probability to jump to zero exactly in correspondence of the jump times of the Poisson process \(N\)
(condition \(\mathbb{P}[\eta^x < \infty, \Delta S_{\eta^x} \neq 0] = 0\) \(\gamma\)-a.e. fails).
Thank you for your attention!
An example with $\eta$ accessible

- Let $\zeta : \Omega \mapsto \mathbb{N}$ s.t. $p_k := \mathbb{P} [\zeta = k] \in (0, 1) \ \forall k \in \mathbb{N}$, $\sum_k p_k = 1$.
- Set $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ to be the smallest filtration that satisfies the usual hypotheses and makes $\zeta$ a stopping time.
- Since $\zeta$ is $\mathbb{N}$-valued, it is an accessible time on $(\Omega, \mathcal{F}, \mathbb{P})$.
- Define $\tau := \zeta - 1$.
- $Z_t = 0$ holds on $\{\zeta \leq t\}$. Moreover, with $q_k := \sum_{n=k+1}^{\infty} p_n \ \forall k \in \{0, 1, \ldots\}$, and denoting $\lceil \cdot \rceil$ the integer part, on $\{t < \zeta\}$

$$Z_t = \mathbb{P} [\tau > t \mid \mathcal{F}_t] = \mathbb{P} [\zeta > t + 1 \mid \mathcal{F}_t] = \mathbb{P} [\zeta > \lceil t + 1 \rceil \mid \mathcal{F}_t] = \frac{q_{\lceil t + 1 \rceil}}{q_{\lceil t \rceil}}.$$

- $\zeta = \inf \{t \in \mathbb{R}_+ \mid Z_{t-} = 0 \text{ or } Z_t = 0\}$.
- $Z_{\zeta-} = \frac{q_{\zeta}}{q_{\zeta-1}} > 0$.
- $\eta = \zeta$; in particular, $\eta$ is accessible on $(\Omega, \mathcal{F}, \mathbb{P})$.  

Another example where NA fails and NA1 holds

\[ S_t = \exp \left( \sigma W_t - \frac{1}{2} \sigma^2 t \right) . \]

For a given constant \( a \in (0, 1) \), define \( \tau := \sup \{ t : S_t = a \} \). Then

\[ Z_t := \mathbb{P} [ \tau > t | \mathcal{F}_t ] = \left( \frac{S_t}{a} \right) \wedge 1 = \frac{N_t}{N^*_t}, \quad t \geq 0 \]

\[ N = \mathcal{E} \left( \frac{1}{a} \int \frac{1}{Z} 1_{\{S < a\}} dS \right) . \]

Note: \( \tau := \sup \{ t : N_t = N^*_t \} \).

▷ Since \( S \) is continuous, NA1(\( \mathbb{G} \), \( S^\tau \)) holds.

▷ On the other hand, the following strategy realizes a classical arbitrage in the enlarged filtration at time \( \tau \) (see Aksamit et al.):

\[ \psi = \frac{1}{a} 1_{\{S < a\}} . \]
Let \( S = \mathcal{E}(\sigma W) \) and \( \tau := \sup\{ t \leq 1 : S_1 - 2S_t = 0 \} \), that is, the last time before 1 when \( S \) equals half of its value at time 1.

Here both \( \text{NA}(\mathbb{G}, S^\tau) \) and \( \text{NA1}(\mathbb{G}, S^\tau) \) hold \( \Rightarrow \) \( \text{NFLVR}(\mathbb{G}, S^\tau) \).

Indeed,

\[
\{ \tau \leq t \} = \left\{ \inf_{t \leq s \leq 1} 2 \frac{S_s}{S_t} \geq \frac{S_1}{S_t} \right\}.
\]

Therefore,

\[
P[\tau \leq t | \mathcal{F}_t] = P\left[ \inf_{t \leq s \leq 1} 2S_{s-t} \geq S_{1-t} \right] = F(1 - t),
\]

where \( F(u) = P[\inf_{s \leq u} 2S_s \geq S_u] \). Then \( Z_t \) deterministic, decreasing \( \Rightarrow \) \( \tau \) pseudo-stopping time and \( S^\tau \) is a \( \mathbb{G} \)-martingale.

\( \uparrow \) On the other hand:

after \( \tau \) there are arbitrages and arbitrages of the first kind: at \( \tau \) we know the value of \( S_1 \), and \( S_t > S_\tau \ \forall t \in (\tau, 1] \).
Local martingales in the initially enlarged filtration

For \( x \in E \), \( D^x \) denotes the \( \mathbb{F} \)-predictable compensator of \( \mathbb{I}_{[\eta^x, \infty]} \).

**Lemma.** \( D \) can be chosen \( \mathcal{B}_E \otimes \mathcal{P}(\mathbb{F}) \)-measurable and:

- \( \Delta D^x < 1 \) \( \mathbb{P} \)-a.s. (\( \Rightarrow \mathcal{E}(-D^x) > 0 \) and nonincreasing);
- \( \mathcal{E}(-D^x)^{-1}\mathbb{I}_{[0, \eta^x]} \) is a \( \mathbb{F} \)-local martingale.

**Proposition.** Let \( X \geq 0 \) be \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable, such that \( X^x \) \( \mathbb{F} \)-local martingale vanishing on \( [\eta^x, \infty] \) \( \gamma \)-a.e. Then \( X^J/p^J \) is a \( \mathcal{G} \)-local martingale.

As an immediate consequence we have the following

**Key-Proposition.** Suppose there is \( M \geq 0 \), \( \mathcal{B}_E \otimes \mathcal{O}(\mathbb{F}) \)-measurable s.t. \( M_0^x = 1 \), \( M^x \) and \( M^x S \) are \( \mathbb{F} \)-local martingales vanishing on \( [\eta^x, \infty] \) \( \gamma \)-a.e. Then, \( M^J/p^J \) is a \( \mathcal{G} \)-local martingale deflator.
Some converse implication

Recall that $D^x$ denotes the $\mathbb{F}$-predictable compensator of $\mathbb{I}_{[\eta^x, \infty[}$ and define $S^x := \mathcal{E}(-D^x)^{-1} \mathbb{I}_{[0, \eta^x[}$, $x \in E$.

**Theorem.** Let $\int_E \mathbb{P}[\eta^x < \infty] \gamma(dx) > 0$. Then NA1($\mathbb{F}, S^x$) holds for every $x \in E$, but NA1($\mathbb{G}, S^J$) fails.

Indeed, $S^x$ are $\mathbb{F}$-local martingales, $S^J = \mathcal{E}(-D^J)^{-1}$ is nondecreasing and $\mathbb{P}[S^J_t = S^J_0, \forall t \in \mathbb{R}_+] < 1$.

▶ An insider with knowledge of $J$ takes at time zero a position on a single unit of the stock with index $J$, and keeps it indefinitely. (The insider identifies from the beginning a single asset in the family $(S^x)_{x \in E}$ which will *not* default and can therefore arbitrage.)

Some particular cases depending on the law of $J$, see [here](#).

Back to [NA1.in](#)
If \( \sum_{k \in \mathbb{N}} \mathbb{P}[J = x_k] = 1 \) holds for some family \( (x_k)_{k \in \mathbb{N}} \subseteq E \),

- \( \int_E \mathbb{P}[\eta^x < \infty] \gamma(dx) > 0 \Rightarrow \exists \kappa : \mathbb{P}[\eta^{x_\kappa} < \infty] > 0; \)
- since \( \mathbb{P}[\zeta^J < \infty] = 0 \), then \( \mathbb{P}[J = x_\kappa, \eta^{x_\kappa} < \infty] = 0; \)
- the buy-and-hold strategy \( \mathbb{I}_{\{J=x_\kappa\}} \) in the single asset \( S^{x_\kappa} \) results in the arbitrage \( \mathbb{I}_{\{J=x_\kappa\}} \cdot S^{x_\kappa} \).

\( (\text{NA1}(\mathbb{F}, S^{x_\kappa}) \text{ holds while NA1}(\mathbb{G}, S^{x_\kappa}) \text{ fails}) \)

If the law \( \gamma \) has a diffuse component, one can still obtain an arbitrage of the first kind, under the stronger hypothesis:

- \( \exists B \in \mathcal{B}_E \) with \( \gamma[B] > 0 \) s.t. \( \mathbb{P}[\eta^B < \infty] > 0 \), where \( \eta^B \) is the time when the martingale \( (\gamma_t[B])_{t \in \mathbb{R}^+} \) jumps to zero.

Indeed, denoting \( D^B \) the \( \mathbb{F} \)-predictable compensator of \( I_{[\eta^B, \infty[} \),

- \( S := \mathcal{E}(-D^B)^{-1}I_{[0, \eta^B]} \) is a \( \mathbb{F} \)-local martingale, \( I_{\{J \in B\}} \cdot S \) is nondecreasing, and \( \mathbb{P}[S_t = S_0, \forall t \in \mathbb{R}^+] < 1. \)

\( (\text{NA1}(\mathbb{F}, S) \text{ holds while NA1}(\mathbb{G}, S) \text{ fails}) \).
Proposition. Let $X \geq 0$ be $\mathcal{B}_E \otimes \mathcal{O}(\mathcal{F})$-measurable, such that $X^\times$ $\mathcal{F}$-supermartingale $\gamma$-a.e. Then $X^J/p^J$ is a $\mathcal{G}$-supermartingale.

(cf. concept of 'universal supermartingale density' in Imkeller, Perkowski 2013)

Remark. This can be used to establish that any semimartingale $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ remains a semimartingale on $(\Omega, \mathcal{G}, \mathbb{P})$. 