AN OKA PRINCIPLE FOR STEIN $G$-MANIFOLDS

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ABSTRACT. Let $G$ be a reductive complex Lie group acting holomorphically on Stein manifolds $X$ and $Y$. Let $p_X: X \rightarrow Q$ and $p_Y: Y \rightarrow Q$ be the quotient mappings. Assume that we have a biholomorphism $Q := Q_X \rightarrow Q_Y$ and an open cover $\{U_i\}$ of $Q$ and $G$-biholomorphisms $\Phi_i: p_X^{-1}(U_i) \rightarrow p_Y^{-1}(U_i)$ inducing the identity on $U_i$. There is a sheaf of groups $\mathcal{A}$ on $Q$ such that the isomorphism classes of all possible $Y$ is the cohomology set $H^1(Q, \mathcal{A})$. The main question we address is to what extent $H^1(Q, \mathcal{A})$ contains only topological information. For example, if $G$ acts freely on $X$ and $Y$, then $X$ and $Y$ are principal $G$-bundles over $Q$, and Grauert’s Oka principle says that the set of isomorphism classes of holomorphic principal $G$-bundles over $Q$ is canonically the same as the set of isomorphism classes of topological principal $G$-bundles over $Q$. We investigate to what extent we have an Oka principle for $H^1(Q, \mathcal{A})$.

1. Introduction

Contents

1. Introduction 1
2. Background 3
3. Strongly continuous homeomorphisms and vector fields 3
4. Logarithms in $\mathcal{A}$ 5
5. Homotopies in $H^1(Q, \mathcal{A})$ 7
6. $H^1(Q, \mathcal{A}) \rightarrow H^1(Q, \mathcal{A}_c)$ is a bijection 10
References 11

Let $X$ be a Stein $G$-manifold where $G$ is a complex reductive group. There is a quotient space $Q_X = X/G$ (or just $Q$ if $X$ is understood) and surjective morphism $p_X$ (or just $p$) from $X$ to $Q$. Then $Q$ is a reduced normal Stein space and the fibers of $p$ are canonically affine $G$-varieties (generally, neither reduced nor irreducible) containing precisely one closed $G$-orbit. For $S$ a subset of $Q$ we denote $p^{-1}(S)$ by $X_S$ and we abbreviate $X_{\{q\}}$ as $X_q$, $q \in Q$. We have a sheaf of groups $\mathcal{A}^X$ (or just $\mathcal{A}$) on $Q$ where $\mathcal{A}(U) = \text{Aut}_U(X_U)^G$ is the group of holomorphic $G$-automorphisms of $X_U$ which induce the identity map $\text{Id}_U$ on $X_U/G = U$.

Let $Y$ be another Stein $G$-manifold. In [KLS15, KLS] we determined sufficient conditions for $X$ and $Y$ to be equivariantly $G$-biholomorphic. Clearly we need that $Q_Y$ is biholomorphic to $Q_X$, so let us assume that we have fixed an isomorphism of $Q_Y$ with $Q = Q_X$. Let us also suppose that there are no local obstructions to a $G$-biholomorphism of $X$ and $Y$ covering $\text{Id}_Q$. (See [KLS, Theorem 1.3] for sufficient conditions for vanishing of the local obstructions.) Then there is an open cover $U_i$ of $Q$ and $G$-biholomorphisms $\Phi_i: X_{U_i} \rightarrow Y_{U_i}$ inducing $\text{Id}_{U_i}$. We say that $X$ and $Y$ are locally $G$-biholomorphic over $Q$. Set $\Phi_{ij} = \Phi_i^{-1}\Phi_j$. Then the $\Phi_{ij} \in \mathcal{A}(U_i \cap U_j)$ are a 1-cocycle, i.e., an element of $Z^1(Q, \mathcal{A})$ (we repress explicit mention of the open cover). Conversely, given $\Psi_{ij} \in Z^1(Q, \mathcal{A})$ (for the same open cover) we can construct a corresponding complex $G$-manifold $Y$ from the disjoint union of the $X_{U_i}$ by identifying $X_{U_j}$ and $X_{U_i}$ over

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$$U_i \cap U_j$$ via $$\psi_{ij}$$. By [KLS, Theorem 5.11] the manifold $$Y$$ is Stein, and it is obviously locally $$G$$-biholomorphic to $$X$$ over $$Q$$. Let $$\Psi'_{ij}$$ be another cocycle for $$\{U_i\}$$ corresponding to the Stein $$G$$-manifold $$Y'$$. If $$Y'$$ is $$G$$-biholomorphic to $$Y$$ (inducing $$\text{Id}_Q$$), then $$\Psi_{ij}$$ and $$\Psi'_{ij}$$ give the same class in $$H^1(Q, \mathcal{A})$$. Thus $$H^1(Q, \mathcal{A})$$ is the set of $$G$$-isomorphism classes of Stein $$G$$-manifolds $$Y$$ which are locally $$G$$-biholomorphic to $$X$$ over $$Q$$ where the $$G$$-isomorphisms are required to induce the identity on $$Q$$.

A fundamental question is whether or not $$H^1(Q, \mathcal{A})$$ contains more than topological information. For example, suppose that $$G$$ acts freely on $$X$$ so that $$X \to Q$$ is a principal $$G$$-bundle. Then $$X$$ corresponds to an element of $$H^1(Q, \mathcal{E})$$ where $$\mathcal{E}$$ is the sheaf of germs of holomorphic mappings of $$Q$$ to $$G$$. By Grauert’s famous Oka principle [Gra58], $$H^1(Q, \mathcal{E}) \simeq H^1(Q, \mathcal{E}^c)$$ where $$\mathcal{E}^c$$ is the sheaf of germs of continuous mappings of $$Q$$ to $$G$$. In other words, the set of isomorphism classes of holomorphic principal $$G$$-bundles over $$Q$$ is the same as the set of isomorphism classes of topological principal $$G$$-bundles over $$Q$$. The main point of this note is to establish a similar Oka principle in our setting.

We define another sheaf of groups $$\mathcal{A}_c$$ on $$Q$$. For $$U$$ open in $$Q$$, $$\mathcal{A}_c(U)$$ consists of “strongly continuous” families $$\sigma = \{\sigma_q\}$$ of $$G$$-automorphisms of the affine $$G$$-varieties $$X_q$$, $$q \in U$$. We define the notion of strongly continuous family in §3. The sheaf $$\mathcal{A}$$ is a subsheaf of $$\mathcal{A}_c$$.

Fix an open cover $$\{U_i\}$$ of $$Q$$. Our main theorems are the following (the first of which is a consequence of [KLS, Theorem 1.4]).

**Theorem 1.1.** Let $$\Phi_{ij}$$, $$\psi_{ij} \in Z^1(Q, \mathcal{A})$$ and suppose that there are $$c_i \in \mathcal{A}_c(U_i)$$ satisfying $$\Phi_{ij} = c_i \psi_{ij} c_j^{-1}$$. Then there are $$c_i' \in \mathcal{A}(U_i)$$ satisfying the same equation.

**Theorem 1.2.** Let $$\Phi_{ij} \in Z^1(Q, \mathcal{A}_c)$$. Then there are $$c_i \in \mathcal{A}_c(U_i)$$ such that $$c_i \Phi_{ij} c_j^{-1} \in Z^1(Q, \mathcal{A})$$.

As a consequence we have the following Oka principle:

**Corollary 1.3.** The canonical map $$H^1(Q, \mathcal{A}) \to H^1(Q, \mathcal{A}_c)$$ is a bijection.

**Remark 1.4.** Suppose that $$X$$ is a smooth affine $$G$$-variety and that $$Z \to Q$$ is a morphism of affine varieties. Then $$G$$ acts on the fiber product $$Z \times_Q X$$ and we have the group Aut_{Z,alg}(Z \times_Q X)^G of algebraic $$G$$-automorphisms of $$Z \times_Q X$$ which induce the identity on the quotient $$Z$$. A scheme $$\mathcal{G}$$ with projection $$\pi: \mathcal{G} \to Q$$ such that the fibers of $$\mathcal{G}$$ are groups whose structure depends algebraically on $$q \in Q$$ is called a group scheme over $$Q$$. (See [KS92, Ch. III] for a more precise definition.) We say that the automorphism group scheme of $$X$$ exists if there is a group scheme $$\mathcal{G}$$ over $$Q$$ together with a canonical isomorphism of $$\Gamma(Z, Z \times_Q \mathcal{G})$$ and Aut_{Z,alg}(Z \times_Q X)^G for all $$Z \to Q$$. The automorphism group scheme of $$X$$ exists (and is an affine variety) if, for example, $$p: X \to Q$$ is flat [KS92, Ch. III Proposition 2.2]. Assuming $$\mathcal{G}$$ exists, now consider $$X$$ as a Stein $$G$$-manifold and $$\mathcal{G}$$ as an analytic variety. Then for $$U$$ open in $$Q$$, $$\mathcal{A}(U) \simeq \Gamma(U, \mathcal{G})$$ and one can show that $$\mathcal{A}_c(U)$$ is the set of continuous sections of $$\mathcal{G}$$ over $$U$$. Thus, in this case, our theorems reduce to the precise analogues of Grauert’s for the cohomology of $$\mathcal{G}$$ using holomorphic or continuous sections.

For $$U$$ an open subset of $$Q$$ we have a topology on $$\mathcal{A}_c(U)$$ and $$\mathcal{A}(U)$$ and we define the notion of a continuous path (or homotopy) in $$\mathcal{A}_c(U)$$ or $$\mathcal{A}(U)$$. We establish a result which is well-known in the case of principal bundles but rather non-trivial in our situation.

**Theorem 1.5.** Let $$\Phi_{ij}(t)$$ be a homotopy of elements in $$Z^1(Q, \mathcal{A}_c)$$, $$t \in [0, 1]$$. Then there are homotopies $$c_i(t) \in \mathcal{A}_c(U_i)$$, $$t \in [0, 1]$$, such that $$\Phi_{ij}(t) = c_i(t) \Phi_{ij}(0) c_j(t)^{-1}$$. Hence $$\Phi_{ij}(t) \in H^1(Q, \mathcal{A}_c)$$ is independent of $$t$$.

**Theorem 1.6.** Let $$\Phi_{ij}(t) \in Z^1(Q, \mathcal{A}_c)$$ be a homotopy, $$t \in [0, 1]$$, where the $$\Phi_{ij}(0)$$ and $$\Phi_{ij}(1)$$ are holomorphic. Then there is a homotopy $$\psi_{ij}(t) \in Z^1(Q, \mathcal{A})$$ with $$\psi_{ij}(0) = \Phi_{ij}(0)$$ and $$\psi_{ij}(1) = \Phi_{ij}(1)$$.
Here is an outline of this paper. In §2 we recall Luna’s slice theorem and related results. In §3 we define the sheaf of groups \( \mathcal{A}_c \) as well as a corresponding sheaf of Lie algebras \( \mathcal{L}\mathcal{A}_c \). In §4 we show that sections of \( \mathcal{A}_c \), sufficiently close to the identity are the exponentials of sections of \( \mathcal{L}\mathcal{A}_c \). In §5 we establish our main technical result (Theorem 5.1) about homotopies in \( \mathcal{A}_c \). We prove Theorem 1.1 and Theorem 1.6 as well as a preliminary version of Theorem 1.5. In §6 we establish Theorem 1.2 and use it to prove Theorem 1.5. Finally, let \( X \) and \( Y \) be locally \( G \)-biholomorphic over \( Q \). We establish a theorem giving necessary and sufficient conditions for a \( G \)-biholomorphism from \( X_U \to Y_U \) over \( \text{Id}_U \), where \( U \subset Q \) is Runge, to be the limit of the restrictions to \( X_U \) of \( G \)-biholomorphisms from \( X \) to \( Y \) over \( \text{Id}_Q \).

**Remark 1.7.** In [KLS15, KLS] we also consider \( G \)-diffeomorphisms \( \Phi \) of \( X \) which induce the identity over \( Q \) and are strict. This means that the restriction of \( \Phi \) to \( X_q, q \in Q \), induces an algebraic \( G \)-automorphism of \( (X_q)_{\text{red}} \) where “red” denotes reduced structure. One can adapt the techniques developed here to prove the analogues of our main theorems for strong \( G \)-homeomorphisms replaced by strict \( G \)-diffeomorphisms.

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## 2. Background

For details of what follows see [Lun73] and [Sno82, Section 6]. Let \( X \) be a Stein manifold with a holomorphic action of a reductive complex Lie group \( G \). The categorical quotient \( Q_X = X//G \) of \( X \) by the action of \( G \) is the set of closed orbits in \( X \) with a reduced Stein structure that makes the quotient map \( p_X : X \to Q_X \) the universal \( G \)-invariant holomorphic map from \( X \) to a Stein space. The quotient \( Q_X \) is normal. When \( X \) is understood, we drop the subscript \( X \) in \( p_X \) and \( Q_X \). If \( U \) is an open subset of \( Q \), then \( p^* \) induces isomorphisms of \( \mathbb{C} \)-algebras \( \mathcal{O}_X(X_U)^G \simeq \mathcal{O}_Q(U) \) and \( C^0(X_U)^G \simeq C^0(U) \). We say that a subset of \( X \) is \( G \)-saturated if it is a union of fibers of \( p \). If \( X \) is an affine \( G \)-variety, then \( Q \) is just the complex space corresponding to the affine algebraic variety with coordinate ring \( \mathcal{O}_{\text{alg}}(X)^G \).

Let \( H \) be a reductive subgroup of \( G \) and let \( B \) be an \( H \)-saturated neighborhood of the origin of an \( H \)-module \( W \). We always assume that \( B \) is Stein, in which case \( B//H \) is also Stein. Let \( G \times^H B \) (or \( T_B \)) denote the quotient of \( G \times B \) by the (free) \( H \)-action sending \( (g, w) \) to \( (gh^{-1}, hw) \) for \( h \in H, g \in G \) and \( w \in B \). We denote the image of \( (g, w) \) in \( G \times^H B \) by \([g, w]\).

Let \( Gx \) be a closed orbit in \( X \). Then the isotropy group \( G_x \) is reductive and the slice representation at \( x \) is the action of \( H = G_x \) on \( W = T_xX/T_x(Gx) \). By the slice theorem, there is a \( G \)-saturated neighborhood of \( Gx \) which is \( G \)-biholomorphic to \( T_B \) where \( B \) is an \( H \)-saturated neighborhood of \( 0 \in W \).

## 3. Strongly continuous homeomorphisms and vector fields

The group \( G \) acts on \( \mathcal{O}(X) \), \( f \mapsto g \cdot f \), where \( (g \cdot f)(x) = f(g^{-1}x), x \in X, g \in G, f \in \mathcal{O}(X) \).

Let \( \mathcal{O}_{\text{fin}}(X) \) denote the set of holomorphic functions \( f \) such that the span of \( \{g \cdot f \mid g \in G \} \) is finite dimensional. They are called the \( G \)-finite holomorphic functions on \( X \) and obviously form an \( \mathcal{O}(Q) = \mathcal{O}(X)^G \)-algebra. If \( X \) is a smooth affine \( G \)-variety, then the techniques of [Sch80, Proposition 6.8, Corollary 6.9] show that for \( U \subset Q \) open and Stein we have

\[
\mathcal{O}_{\text{fin}}(X_U) \simeq \mathcal{O}(U) \otimes_{\mathcal{O}_{\text{alg}}(Q)} \mathcal{O}_{\text{alg}}(X).
\]

Let \( V \) be the direct sum of pairwise non-isomorphic non-trivial \( G \)-modules \( V_1, \ldots, V_r \). Let \( \mathcal{O}(X)_V \) denote the elements of \( \mathcal{O}_{\text{fin}}(X) \) contained in a copy of \( V \). If \( H \) is a reductive subgroup of \( G \) and \( W \) an \( H \)-module, we similarly define \( \mathcal{O}_{\text{alg}}(T_W)_V \). Then for \( B \) an \( H \)-saturated neighborhood of \( 0 \in W \), \( \mathcal{O}_{\text{alg}}(T_W)_V \) generates \( \mathcal{O}(T_B)_V \) over \( \mathcal{O}(B)^H \). By Nakayama’s Lemma,
\(f_1, \ldots, f_m \in \mathcal{O}(X)_V\) restrict to minimal generators of the \(\mathcal{O}(U)\)-module \(\mathcal{O}(X_U)_V\) for some neighborhood \(U\) of \(q \in Q\) if and only if the restrictions of the \(f_i\) to \(X_q\) form a basis of \(\mathcal{O}(X_q)_V = \mathcal{O}_{\text{alg}}(X_q)_V\). Thus by the slice theorem, the sheaf of algebras of \(G\)-finite holomorphic functions is locally finitely generated as an algebra over \(\mathcal{O}_{Q}\).

**Definition 3.1.** Let \(U \subset Q\) be relatively compact. Then there is a \(V\) as above such that the \(\mathcal{O}(X_U)_V\) are finitely generated over \(\mathcal{O}(U)\) and generate \(\mathcal{O}_{\text{fin}}(X_U)\) as \(\mathcal{O}(U)\)-algebra. Let \(f_1, \ldots, f_n\) be a generating set of \(\bigoplus \mathcal{O}(X_U)_V\) with each \(f_i\) in some \(\mathcal{O}(X_U)_V\). Then we call \(\{f_i\}\) a standard generating set of \(\mathcal{O}_{\text{fin}}(X_U)\). When \(U = TB/G\) as before, we always assume that our standard generators are the restrictions of homogeneous elements of \(\mathcal{O}_{\text{alg}}(T_W)\).

Let \(U \subset Q\), \(V\) and \(\{f_1, \ldots, f_n\}\) be as above. We say that a \(G\)-equivariant homeomorphism \(\Psi : X_U \to X_U\) is strong if it lies over the identity of \(U\) and \(\Psi^* f_i = \sum_j a_{ij} f_j\) where the \(a_{ij}\) are in \(C^0(X_U)^G \simeq C^0(U)\). We also require that the \(a_{ij}(q)\) induce a \(G\)-isomorphism of \(\mathcal{O}(X_q)_V\) for all \(q \in U\). Then \(\Psi\) induces an algebraic isomorphism \(\Psi_q : X_q \to X_q\) for all \(q \in U\); it is easy to see that the definition does not depend on our choice of \(V\) and the generators \(f_i\). We call \(\{a_{ij}\}\) a matrix associated to \(\Psi\). Using a partition of unity on \(U\) it is clear that \(\Psi\) is strong if and only if it is strong in a neighborhood of every \(q \in Q\). In a neighborhood of any particular \(q\), we may assume that the \(f_i\) restrict to a basis of \(\mathcal{O}(X_q)_V\), in which case \(\{a_{ij}\}\) is invertible in a neighborhood of \(q\). Then \(\Phi^{-1}\) has matrix \((a_{ij})^{-1}\) near \(q\). Thus if \(\Phi\) is strong, so is \(\Phi^{-1}\). Let \(\mathcal{A}_c(U)\) denote the group of strong \(G\)-homeomorphisms of \(X_U\) for \(U\) open in \(Q\). Then \(\mathcal{A}_c\) is a sheaf of groups on \(Q\).

We say that a vector field \(D\) on \(X_U\) is formally holomorphic if it annihilates the antiholomorphic functions on \(X_U\). Let \(D\) be a continuous formally holomorphic vector field on \(X_U\), \(G\)-invariant, annihilating \(\mathcal{O}(X_U)^G\). We say that \(D\) is strongly continuous (and write \(D \in \mathcal{L}\mathcal{A}_c(U)\)) if for any \(q \in U\) there is a neighborhood \(U'\) of \(q\) in \(U\) and a standard generating set \(f_1, \ldots, f_n\) for \(\mathcal{O}_{\text{fin}}(X_{U'})\) such that \(D(f_i) = \sum d_{ij} f_j\) where the \(d_{ij}\) are in \(C^0(U')\). We say that \(D\) has matrix \((d_{ij})\) over \(U'\). The matrix is usually not unique. Clearly our definition of \(\mathcal{L}\mathcal{A}_c(U)\) is independent of the choices made. We denote the corresponding sheaf by \(\mathcal{L}\mathcal{A}_c\).

**Remark 3.2.** Let \(D \in \mathcal{L}\mathcal{A}_c(U)\) and \(q \in Q\). Then \(D\) is tangent to \(F = X_q\) and acts algebraically on \(\mathcal{O}_{\text{alg}}(F)\), hence lies in the space of \(G\)-invariant derivations \(\text{Der}_{\text{alg}}(F)^G\) of \(\mathcal{O}_{\text{alg}}(F)\). Since \(\text{Der}_{\text{alg}}(F)^G\) is the Lie algebra of the algebraic group \(\text{Aut}(F)^G\), the restriction of \(D\) to \(F\) can be integrated for all time. It follows that \(D\) is a complete vector field.

Let \(U\) be open in \(Q\), let \(\epsilon > 0\) and let \(K\) be a compact subset of \(U\). Let \(f = \{f_1, \ldots, f_n\}\) be a standard generating set of \(\mathcal{O}_{\text{fin}}(X_{U'})\) where \(U'\) is a neighborhood of \(K\). Define

\[
\Omega_{K, \epsilon, t} = \{\Phi \in \mathcal{A}_c(U) : \| (a_{ij}) - I \|_K < \epsilon \}
\]

where \(\{a_{ij}\}\) is some matrix associated to \(\Phi\). Here \(\| (a_{ij}) - I \|_K\) denotes the supremum of the matrix norm of \((a_{ij}) - I\) over \(K\). Let \(f' = \{f'_1, \ldots, f'_m\}\) be another standard generating set defined on a neighborhood of \(K\) in \(U\).

**Lemma 3.3.** Let \(\epsilon' > 0\). Then there is an \(\epsilon > 0\) such that \(\Omega_{K, \epsilon, t} \subset \Omega_{K, \epsilon', t'}\).

**Proof.** We may assume that the \(f_i\) and \(f'_j\) are standard generating sets of \(\mathcal{O}_{\text{fin}}(U)\). There are polynomials \(h_i\) with coefficients in \(\mathcal{O}(U)\) such that \(f'_i = h_i(f_1, \ldots, f_n), 1 \leq i \leq m\). We may assume that \(\{f'_1, \ldots, f'_m\}\) are the \(f'_j\) corresponding to an irreducible \(G\)-module \(V_i\). Let \(\Phi \in \Omega_{K, \epsilon, t}\) with corresponding matrix \((a_{uv})\) such that \(\| (b_{uv}) \|_K < \epsilon\) where \((b_{uv}) = (a_{uv}) - I\). Let \(r_i\) be the degree of \(h_i\). Then for \(1 \leq i \leq s\) we have

\[
(\Phi^* f'_i - f'_i = h_i(\Phi^* f_1, \ldots, \Phi^* f_n) - h_i(f_1, \ldots, f_n) = \sum_{k, l=1}^n b_{kl} p_{kl} M_{kl}(f_1, \ldots, f_n)
\]
where the $p_{kl}$ are polynomials in the $a_{uv}$ of degree at most $r_i - 1$ and the $M_{kl}$ are polynomials in the $f_j$ with coefficients in $\mathcal{O}(U)$ which are independent of the $a_{uv}$ and $b_{uv}$. Since $\Phi^* f'_i$ is a covariant corresponding to $V_i$, we can project the $M_{kl}$ to $\mathcal{O}(X_U)_{V_i}$ in which case we get
\[
\sum_{j=1}^n N_{jkl} f'_j \text{ where the } N_{jkl} \text{ are in } \mathcal{O}(U) \text{ and independent of the } a_{uv} \text{ and } b_{uv}. \]

Hence
\[
(\Phi^* f'_i) - f'_i = \sum_{j=1}^s \sum_{k,l=1}^n b_{kl} N_{jkl} p_{kl} f'_j.
\]

Since the $N_{jkl} p_{kl}$ are bounded on $K$, choosing $\epsilon$ sufficiently small, we can force the terms $\sum_{k,l=1}^n b_{kl} N_{jkl} p_{kl}$ to be close to 0. Hence there is an $\epsilon > 0$ such that $\Omega_{K,\epsilon,t} \subset \Omega_{K,\epsilon',t'}$. □

By the lemma, we get the same neighborhoods of the identity in $\mathcal{A}_c(U)$ from any standard generating set of $\mathcal{O}_{\text{fin}}(X_{U'})$ where $U'$ is a neighborhood of $K$. Thus we can talk about neighborhoods of the identity without specifying the $\mathfrak{f}$ in question. We then have a well-defined topology on $\mathcal{A}_c(U)$ where $\Phi$ is close to $\Phi'$ if $\Phi' \Phi^{-1}$ is close to the identity.

Let $U$, $K$ and the $f_i$ be as above. Define
\[
\Omega_{K,\epsilon,t} = \{ D \in \mathcal{L} \mathcal{A}_c(U) \mid D(f_i) = \sum d_{ij} f_j \text{ and } ||(d_{ij})||_K < \epsilon \}
\]
where $D$ has (continuous) matrix $(d_{ij})$ defined on a neighborhood of $K$. As before, the $\Omega'_{K,\epsilon,t}$ give a basis of neighborhoods of $0$ and define a topology on $\mathcal{L} \mathcal{A}_c(U)$, independent of the choice of $\mathfrak{f}$.

**Proposition 3.4.** Let $U$ be open in $Q$ and let $\{f_1, \ldots, f_n\}$ be a standard generating set for $\mathcal{O}_{\text{fin}}(U)$.

1. Let $D$ be a $G$-invariant formally holomorphic vector field on $X_U$ which annihilates $\mathcal{O}(U)$ such that $D(f_i) = \sum d_{ij} f_j$ for $d_{ij} \in C^0(U)$. Then $D$ is continuous, i.e., $D \in \mathcal{L} \mathcal{A}_c(U)$.
2. $\mathcal{L} \mathcal{A}_c$ is a sheaf of Lie algebras and a module over the sheaf of germs of continuous functions on $Q$.
3. exp: $\mathcal{L} \mathcal{A}_c(U) \to \mathcal{A}_c(U)$ is continuous.
4. $\mathcal{L} \mathcal{A}_c(U)$ is a Fréchet space.

**Proof.** Let $D$ be as in (1) and let $x_0 \in X_U$. There is a subset, say $f_1, \ldots, f_r$, of the $f_i$ and holomorphic invariant functions $h_{r+1}, \ldots, h_s$ such that the $z_i = f_i - f_i(x_0)$ and $z_j = h_j - h_j(x_0)$ are local holomorphic coordinates at $x_0$. Then, near $x_0$, $D$ has the form $\sum a_i \partial/\partial z_i$ where each $a_i = D(f_i)$ is continuous. Hence $D$ is continuous giving (1). Let $D, D' \in \mathcal{L} \mathcal{A}_c(U)$ with matrices $(d_{ij})$ and $(d'_{ij})$. Let $(e_{ij})$ be their matrix bracket. Then $[D, D']$ is $G$-invariant, annihilates $\mathcal{O}(U)$ and sends $f_i$ to $\sum e_{ij} f_j$. Hence we have (2). Part (3) is clear.

The topology on $\mathcal{L} \mathcal{A}_c(U)$, $U$ open in $Q$, is defined by countably many seminorms, hence $\mathcal{L} \mathcal{A}_c(U)$ is a metric space and it is Fréchet if it is complete. Let $D_k$ be a Cauchy sequence in $\mathcal{L} \mathcal{A}_c(U)$. Let $K \subset U$ be a compact neighborhood of $q \in U$. There are matrices $(d^K_{ij})$ of elements of $C^0(K)$ such that $D_k(f_i) = \sum d^K_{ij} f_j$ over $K$. Since $\{D_k\}$ is Cauchy, we may assume that $||(d^{l}_{ij}) - (d^K_{ij})||_K < 1/m$ for $k, l > N_m, m \in \mathbb{N}$. Then $\lim_{k \to \infty} d^K_{ij} = d_{ij} \in C^0(K)$ for all $i, j$. It follows that the pointwise limit of the $D_k$ exists and is a formally holomorphic vector field $D$ annihilating the invariants such that $D(f_i) = \sum d_{ij} f_j$. By (1), $D$ is of type $\mathcal{L} \mathcal{A}_c$ over the interior of $K$. It follows that $\mathcal{L} \mathcal{F}(U)$ is complete. □

4. **Logarithms in $\mathcal{A}_c$**

Let $U$ be an open subset of $Q$ isomorphic to $T_B = G \times^H B$ where $H$ is a reductive subgroup of $G$ and $B$ is an $H$-saturated neighborhood of the origin in an $H$-module $W$. Let $f_1, \ldots, f_n$ be a standard generating set for $\mathcal{O}_{\text{fin}}(X_U)^G$ consisting of the restrictions to $X_U$ of homogeneous polynomials in $\mathcal{O}_{\text{alg}}(T_W)$. Consider polynomial relations of the $f_i$ with coefficients in $\mathcal{O}(U)$. 

These are generated by the relations with coefficients in \( \mathcal{O}_{\text{alg}}(T_W)^G \). Let \( h_1, \ldots, h_m \) be generating relations of this type. Let \( N \) be a bound for the degree of the \( h_j \). Now take the covariants which correspond to all the irreducible \( G \)-representations occurring in the span of the monomials of degree at most \( N \) in the \( f_i \). Let \( \{ f_\alpha \} \) be a set of generators for these covariants and let \( K \subset U \) be compact. Let \( \Phi \in \mathcal{A}_c(U) \). Then \( \Phi^* f_\alpha = \sum c_{\alpha,\beta} f_\beta \) where the \( c_{\alpha,\beta} \in C^0(U) \). We also have that \( \Phi^* f_i = \sum a_{ij} f_j \) where the \( a_{ij} \in C^0(U) \). We fix a neighborhood \( \Omega \) of the identity in \( \mathcal{A}_c(U) \) such that \( \Phi \in \Omega \) implies that \( ||(c_{\alpha,\beta}) - I||_K < 1/3 \) and that \( ||(a_{ij}) - I||_K < 1/2 \). For \( \Phi \in \Omega \) let \( \Lambda' \) denote the restriction of \( \Lambda \) to \( K \). Then the formal power series \( S(\Lambda') \) for log \( \Phi^* \) is \( -\Lambda' - (1/2)(\Lambda')^2 - 1/3(\Lambda')^3 - \ldots \).

Now we restrict to a fiber \( F = X_q, q \in K \). Let \( M \) denote the span of the covariants \( f_\alpha \) restricted to \( F \). Then \( M \) is finite dimensional and we give it the usual euclidean topology. Let \( \Lambda \) denote the restriction of \( \Lambda' \) to \( M \).

**Lemma 4.1.** Let \( m \in M \). Then the series \( S(\Lambda)(m) \) converges in \( M \).

**Proof.** We have \( m = \sum a_{\alpha} f_\alpha |_F \) where the \( a_{\alpha} \in \mathbb{C} \). Then \( \Lambda(m) = \sum \alpha_{\beta}(\delta_{\alpha,\beta} - c_{\alpha,\beta}(q)) a_\beta f_\beta |_F \).

Let \( C \) denote \( (c_{\alpha,\beta}(q)) \). Then \( ||I - C|| < 1/3 \). By induction, \( \Lambda^k \) acts on \( \sum a_{\alpha} f_\alpha |_F \) via the matrix \( (I - C)^k \), where \( ||(I - C)^k|| < (1/3)^k \). Let

\[
C' = - \sum_{k=1}^{\infty} (I - C)^k.
\]

Then \( S(\Lambda)(m) \) converges to \( \sum \alpha_{\beta} C'_{\alpha,\beta} a_\beta f_\beta |_F \in M \). \( \square \)

For \( f \in M \), define \( D(f) \) to be the limit of \( S(\Lambda)(f) \). Then \( D \) is a \( G \)-equivariant linear endomorphism of \( M \).

**Proposition 4.2.** Suppose that \( m_1, m_2 \) and \( m_1 m_2 \) are in \( M \). Then

\[
D(m_1 m_2) = D(m_1) m_2 + m_1 D(m_2).
\]

**Proof.** By [Pra86, Proof of Theorem 4]

\[
\Lambda^k(m_1 m_2) = \sum_{l=0}^{2k} \sum_{n=0}^{l} c_{kln} \Lambda^n(m_1) \Lambda^{l-n}(m_2)
\]

where \( c_{kln} \) is the coefficient of \( x^n y^{l-n} \) in \( (x + y - xy)^k \). We know that \( \Lambda^k \) is given by the action of the matrix \( (I - C)^k \) where \( ||I - C|| < 1/3 \). The series \( \sum 1/k(x + y - xy)^k \) converges absolutely when \( x \) and \( y \) have absolute value at most 1/3. Thus we may make a change in the order of summation:

\[
D(m_1 m_2) = \sum_{k=1}^{\infty} \sum_{l=0}^{2k} \sum_{n=0}^{l} 1/k c_{kln} \Lambda^n(m_1) \Lambda^{l-n}(m_2) = \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} 1/k c_{kln} \Lambda^n(m_1) \Lambda^{l-n}(m_2).
\]

By [Pra86, Proof of Theorem 4] the (actually finite) sum \( \sum_{k=1}^{\infty} (1/k) c_{kln} \) equals 0 unless we have \( l > 0 \) and \( (n = 0 \text{ or } n = l) \), in which case the value is \( 1/l \). Hence

\[
D(m_1 m_2) = - \sum_{l=1}^{\infty} \frac{1}{l}(\Lambda^l(m_1) m_2 + m_1 \Lambda^l(m_2)) = D(m_1) m_2 + m_1 D(m_2).
\]

\( \square \)

**Proposition 4.3.** Let \( D : M \to M \) be as above. Then \( D \) extends to a \( G \)-equivariant derivation of \( \mathcal{O}_{\text{alg}}(F) \).
Proof. Let \( R = \mathbb{C}[z_1, \ldots, z_n] \). We have a surjective morphism \( \rho: R \to \mathcal{O}_{\text{alg}}(F) \) sending \( z_i \) to \( f_i|_F, i = 1, \ldots, n \). The kernel \( J \) of \( \rho \) is generated by polynomials of degree at most \( N \) (the \( h_j \) considered as elements of \( R \)). Let \( E \) denote the derivation of \( R \) which sends \( z_i \) to \( \sum d'_{ij}z_j \) where \( (d'_{ij}) \) is the logarithm of \( (a_{ij}(q)) \). Recall that \( \| (a_{ij}) - I \|_K < 1/2 \). By construction, \( E \) on the span of the \( z_i \) is the pull-back of \( D \) on the span of the \( f_i|_F \). By Proposition 4.2, \( E \) restricted to polynomials of degree at most \( N \) is the pull-back of \( D \) restricted to polynomials of degree at most \( N \) in the \( f_i|_F \). Hence \( E \) preserves the span of the elements of degree at most \( N \) in \( J \). Since these elements generate \( J \), we see that \( E \) preserves \( J \). Hence \( E \) induces a derivation of \( R/J \), i.e., \( D \) extends to a \( G \)-invariant derivation of \( \mathcal{O}_{\text{alg}}(F) \). \( \square \)

Corollary 4.4. Let \( \Phi \in \Omega \) and let \( U' \) denote the interior of our compact subset \( K \subset U \). There is a \( D \in \mathcal{L}A_c(U') \) such that \( \exp(D) = \Phi|_{X_{U'}} \). The mapping \( \Omega \ni \Phi \mapsto D \in \mathcal{L}A_c(U') \) is continuous.

Proof. For \( q \in U' \), let \( D_q \) be the \( G \)-equivariant derivation of \( \mathcal{O}_{\text{alg}}(X_q) \) constructed above. Let \( D \) be the vector field on \( X_{U'} \) whose value on \( X_q \) is \( D_q \), \( q \in U' \). Then \( D(f_i) = \sum d_{ij}f_j \), where \( (d_{ij}) = \log(a_{ij}) \). By Proposition 3.4, \( D \in \mathcal{L}A_c(U') \). By construction, \( \exp(D_q) = \Phi_q \) for all \( q \in U' \). Hence \( \exp D = \Phi|_{X_{U'}} \). The continuity of \( \Phi \mapsto D \) is clear since \( (d_{ij}) = \log(a_{ij}) \). \( \square \)

Definition 4.5. Let \( U \subset Q \) be open and let \( f_1, \ldots, f_n \) be a standard generating set of \( \mathcal{O}_{\text{fin}}(X_U) \). Let \( U' \subset U \) be open with \( U' \subset U \). We say that \( \Phi \in A_c(U) \) admits a logarithm in \( \mathcal{L}A_c(U') \) if the following hold.

1. \( \Phi^* f_i = \sum a_{ij}f_j \) where \( \| (a_{ij}) - I \|_{U'} < 1/2 \).
2. There is a \( D \in \mathcal{L}A_c(U') \) such that \( D(f_i) = \sum d_{ij}f_j \) on \( X_{U'} \), where \( (d_{ij}) = \log(a_{ij}) \).

Note that \( (a_{ij}) \) is not unique. The condition is that some \( (a_{ij}) \) corresponding to \( \Phi \) satisfies (1) and (2).

Remarks 4.6. The formal series \( \log \Phi^* \), when applied to any \( f_i \), converges to \( D(f_i) \). Hence \( D \) is independent of the choice of \( (a_{ij}) \). Properties (1) and (2) imply that \( \exp D = \Phi \) over \( U' \). Note that \( \| (d_{ij}) \|_{U'} < \log 2 \) and \( (d_{ij}) \) is the unique matrix satisfying this property whose exponential is \( (a_{ij}) \).

Corollary 4.4 and its proof imply the following result.

Theorem 4.7. Let \( K \subset U \subset Q \) where \( K \) is compact and \( U \) is open. Then there is a neighborhood \( \Omega \) of the identity in \( A_c(U) \) and a neighborhood \( U' \) of \( K \) in \( U \) such that every \( \Phi \in \Omega \) admits a logarithm \( D = \log \Phi \) in \( \mathcal{L}A_c(U') \). The mapping \( \Phi \mapsto \log \Phi \) is continuous.

Corollary 4.8. Let \( \Phi_n \) be a Cauchy sequence in \( A_c(U) \). Then \( \Phi_n \to \Phi \in A_c(U) \).

Proof. Since this is a local question, we can assume that we have a standard generating set \( \{f_i\} \) for \( \mathcal{O}_{\text{fin}}(U) \). Let \( q \in U \) and let \( U' \) be a relatively compact neighborhood of \( q \) in \( U \). Then there is a neighborhood \( \Omega \) of the identity in \( A_c(U) \) such that any \( \Psi \in \Omega \) admits a logarithm in \( \mathcal{L}A_c(U') \). Let \( \Omega_0 \) be a smaller neighborhood of the identity with \( \overline{\Omega_0} \subset \Omega \). There is an \( N \in \mathbb{N} \) such that \( n \geq N \) implies that \( \Phi^{-1}_N \Phi_n \in \Omega_0 \), hence \( \log(\Phi^{-1}_N \Phi_n) = D_n \in \mathcal{L}A_c(U') \), and \( D_n \) converges to an element \( D \in \mathcal{L}A_c(U') \) by Proposition 3.4. Set \( \Phi = \Phi_N \exp D \in A(U') \). Since \( \exp D_n = \Phi^{-1}_N \Phi_n \) over \( U' \) we have \( \Phi_n \to \Phi \) in \( A_c(U') \). \( \square \)

5. Homotopies in \( H^1(Q, A_c) \)

We establish our main technical result concerning homotopies in \( H^1(Q, A_c) \). We give proofs of Theorems 1.1 and 1.6 and a special case of Theorem 1.5.

Let \( \Phi(t) \in A_c(U), t \in C \), where \( C \) is a topological space. We say that \( \Phi(t) \) is continuous if relative to a standard generating set, \( \Phi(t) \) has corresponding matrices \( (a_{ij}(t, q)) \) where each \( a_{ij} \) is continuous in \( t \) and \( q \in U \). (It is probably false that every continuous map \( C \to A_c(U) \)
is continuous in our sense.) Let \( A_c(U) \) denote the set of all continuous paths \( \Phi(t) \in A_c(U) \), \( t \in [0, 1] \), starting at the identity. We have a topology on \( A_c(U) \) as in §3 and \( A_c(U) \) is a topological group. When we talk of homotopies in \( A_c(U) \) we mean that the corresponding families with parameter space \([0, 1]^2\) are continuous as above. We define continuous families of elements of \( \mathcal{L}A_c(U) \) similarly. One defines \( A(U) \) similarly to \( A_c(U) \) where, of course, the relevant \( a_{ij}(t, q) \) are required to be holomorphic in \( q \) and continuous in \( t \).

Here is our main technical result about \( A_c \).

**Theorem 5.1.**  
(1) The topological group \( A_c(Q) \) is pathwise connected.  
(2) If \( U \subset Q \) is open, then \( A_c(Q) \) is dense in \( A_c(U) \).  
(3) \( H^1(Q, A_c) = 0 \).

**Proof.** Let \( \Phi(t) \) be an element of \( A_c(Q) \). Since \( \{0\} \) is a deformation retract of \([0, 1]\), there is a homotopy \( \Phi(s, t) = \Phi(t) \) of \( \Phi(0, t) = \Phi(t) \) the identity automorphism. Hence we have (1). For (2), let \( \Phi \in A_c(U) \). Let \( K \) be a compact subset of \( U \) and \( U' \) a relatively compact neighborhood of \( K \) in \( U \). It follows from Theorem 4.7 that there are \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) and continuous families \( D_j(s) \) in \( \mathcal{L}A_c(U') \) for \( s \in [0, t_{j+1} - t_j] \) such that, over \( U' \), \( \Phi(s + t_j) = \Phi(t_j) \exp(D_j(s)) \), \( s \in [0, t_{j+1} - t_j], j = 0, \ldots, m - 1 \). Multiplying by a cutoff function, we can assume that the \( D_j \) are in \( \mathcal{L}A_c(Q) \). Then our formula gives an element of \( A_c(Q) \) which restricts to \( \Phi \) on a neighborhood of \( K \) and we have (2).

Let \( K \subset Q \) be compact which is of the form \( K' \cup K'' \) where \( K' \) and \( K'' \) are compact. Let \( U = U' \cup U'' \) be a neighborhood of \( K \) where \( K' \subset U' \), \( K'' \subset U'' \). Let \( \Phi(t) \) be in \( Z^1(U, A_c) \) for the open covering \( \{U', U''\} \) of \( U \). Then \( \Phi(t) \) is just an element in \( A_c(U' \cap U'') \). By (2) we can write \( \Phi = \Psi_1 \Psi_2^{-1} \) where \( \Psi_1 \) is defined over \( Q \) (hence over \( U' \)) and \( \Psi_2 \) is close to the identity over \( K' \cap K'' \). Then \( \Psi_2(t) = \exp(D(t)) \) where \( D(t) \in \mathcal{L}A_c(U' \cap U'') \) is a continuous family and \( D(0) = 0 \). Using a cutoff function again, we can find \( D_0(t) \in \mathcal{L}A_c(Q) \) which equals \( D(t) \) in a neighborhood of \( K' \cap K'' \) and vanishes when \( t = 0 \). We have \( \Phi = \Psi_1 \Psi_2^{-1} \) where \( \Psi_2^{-1} \) is the exponential of \( D_0(t) \). Thus the cohomology class of \( \Phi \) becomes trivial if we replace \( U' \) and \( U'' \) by slightly smaller neighborhoods of \( K' \) and \( K'' \). Let \( H^1(K, A_c) \) denote the direct limit of \( H^1(U, A_c) \) for \( U \) a neighborhood of \( K \). As in [Car58, §5], our result above shows that there is a sequence of compact sets \( K_1 \subset V_2 \subset K_2 \cdots \) with \( V_n \) the interior of \( K_n \), \( Q = \bigcup K_n \) and \( H^1(K_n, A_c) = 0 \) for all \( n \).

Let \( \{U_i\} \) be an open cover of \( Q \) and \( \Phi_{ij} \in A_c(U_i \cap U_j) \) a cocycle. There are \( c^n_i \in A_c(U_i \cap V_n) \) such that \( \Phi_{ij} = (c^n_i)^{-1}c^n_j \) on \( U_i \cap U_j \cap V_n \). Thus \( c_i^{n+1}(c_i^n)^{-1} = c_j^{n+1}(c_j^n)^{-1} \) on \( U_i \cap U_j \cap V_n \). The \( c_i^{n+1}(c_i^n)^{-1} \) define a section \( d \in A_c(V_n) \). By (2) there is a section \( d' \) of \( A_c(Q) \) which is arbitrarily close to \( d \) on \( K_{n-1} \). Replace each \( c_i^{n+1} \) by \( (d')^{-1}c_i^{n+1} \). Then \( c_i^{n+1} \) is very close to \( c_i^n \) on \( K_{n-1} \) and we can arrange that the limit as \( n \to \infty \) of the \( c_i^n \) converges on every compact subset to \( c_i \in A_c(U_i) \) such that \( \Phi_{ij} = c_i^{-1}c_j \). We have used Corollary 4.8. This completes the proof of (3). \( \square \)

Note that (3) says that for any homotopy of a cocycle \( \Phi_{ij}(t) \) starting at the identity there are \( c_i(t) \in A_c(U_i) \) such that \( \Phi_{ij}(t) = c_i(t)^{-1}c_j(t) \) for all \( t \in [0, 1] \). Hence \( \Phi_{ij}(t) \) is the trivial element in \( H^1(Q, A_c) \) for all \( t \). We now use a trick to show a similar result if we only assume that \( \Phi_{ij}(0) \in Z^1(Q, A) \).

Let \( \Psi_{ij} \in Z^1(Q, A) \) for some open cover \( \{U_i\} \) of \( Q \). By [KLS, Theorem 5.11], there is a Stein \( G \)-manifold \( Y \) with quotient \( Q \) corresponding to the \( \Psi_{ij} \). Let \( X_i = X_{U_i} \) and \( Y_i = Y_{U_i} \). Then there are \( G \)-biholomorphisms \( \Psi_i : X_i \to Y_i \) over the identity of \( U_i \) such that \( \Psi_i^{-1} \Psi_j = \Psi_{ij} \).

Here is an analogue of the twist construction in Galois cohomology. We leave the proof to the reader.
Lemma 5.2. Let $\Psi_{ij} \in Z^1(Q, \mathcal{A})$ and $\Phi_{ij} \in Z^1(Q, \mathcal{A}_c)$ be cocycles for the open cover $\{U_i\}$ of $Q$. Let $Y$ and $\Psi_i: X_i \rightarrow Y_i$ be as above. The mapping $\Phi_{ij} \mapsto \Psi_i \Phi_{ij} \Psi_j^{-1}$ induces an isomorphism of $H^1(Q, \mathcal{A}_c)$ and $H^1(Q_\mathcal{A}_c^\mathcal{Y})$ which sends the class $\Psi_{ij}$ to the trivial class of $H^1(Q, \mathcal{A}_c^\mathcal{Y})$.

Corollary 5.3. Let $\Phi_{ij}(t)$ be a homotopy of cocycles with values in $\mathcal{A}_c(U_i \cap U_j)$ where $\{U_i\}$ is an open cover of $Q$. Suppose that $\Phi_{ij}(0)$ is holomorphic. Then there are $c_i \in \mathcal{A}_c(U_i)$ such that $\Phi_{ij}(t) = c_i(t)^{-1} \Phi_{ij}(0) c_j(t)$ for all $t$.

Proof. By Lemma 5.2 we may reduce to the case that $\Phi_{ij}(0)$ is the identity, so we can apply Theorem 5.1. □

Let $X$, $Y$ and the $\Psi_i$ be as above. We say that a $G$-homeomorphism $\Phi: X \rightarrow Y$ is strong if $\Psi_i^{-1} \circ \Phi: X_i \rightarrow X_i$ is strong for all $i$, i.e., in $\mathcal{A}_c(U_i)$. It is easy to see that this does not depend upon the particular choice of the $\Psi_i$. Similarly one can define what it means for a family $\Phi(t)$ of strong $G$-homeomorphisms to be continuous, $t \in [0, 1]$. Then we have the following nice result [KLS, Theorem 1.4].

Theorem 5.4. Let $\Phi: X \rightarrow Y$ be strongly continuous. Then there is a continuous family $\Phi(t)$ of strong $G$-homeomorphisms from $X$ to $Y$ with $\Phi(0) = \Phi$ and $\Phi(1)$ holomorphic.

Proof of Theorem 1.1. We have $\Phi_{ij}$, $\Psi_{ij} \in Z^1(Q, \mathcal{A})$ and $c_i \in \mathcal{A}_c(U_i)$ satisfying $\Phi_{ij} = c_i \Psi_{ij} c_j^{-1}$. Using Lemma 5.2 we may assume that $\Psi_{ij}$ is the trivial class. Then the $c_i$ are the same as a strong $G$-homeomorphism $\Theta: X \rightarrow Y$ where $Y$ is the Stein $G$-manifold corresponding to the $\Psi_{ij}$ (after our twisting). By Theorem 5.4 not only are there $d_i \in \mathcal{A}(U_i)$ such that $\Psi_{ij} = d_i \Psi_{ij}$, but the $d_i$ are $e_i(1)$ where $e_i(t)$ is a path in $\mathcal{A}_c(U_i)$ starting at $c_i$ and ending at $d_i$. The $d_i$ correspond to a $G$-biholomorphism of $X$ and $Y$ over $Q$.

We now prepare to prove Theorem 1.6.

Lemma 5.5. Let $\Phi \in \mathcal{A}_c(Q)$ such that $\Phi(1)$ is holomorphic. Then $\Phi$ is homotopic to $\Phi' \in \mathcal{A}(Q)$ where $\Phi'(1) = \Phi(1)$.

Proof. We have to make use of a sheaf of groups $\mathcal{F}$ on $Q$ which is a subsheaf of the sheaf of $G$-diffeomorphisms of $X$ which induce the identity on $Q$ and are algebraic isomorphisms on the fibers of $p$. See [KLS, Ch. 6]. We give $\mathcal{F}(U)$ the usual $C^\infty$-topology. Let $\mathfrak{F}(U)$ denote the sheaf of homotopies $\Psi(t)$ of elements of $\mathcal{F}(U)$, $t \in [0, 1]$, where $\Psi(0)$ is the identity and $\Psi(1)$ is holomorphic. Then [KLS, Theorem 10.1] tells us that $\mathfrak{F}(Q)$ is pathwise connected. Hence for $\Phi \in \mathfrak{F}(Q)$ there is a homotopy $\Psi(s) \in \mathfrak{F}(Q)$ such that $\Psi(0) = \Phi$ and $\Psi(1)$ is the identity. Then $\Psi(s)$ evaluated at $s = 1$ is a homotopy from $\Phi(1)$ to the identity in $\mathcal{A}(Q)$, establishing the lemma when $\Phi \in \mathfrak{F}(Q)$.

We now use a standard trick. Let $\Delta$ denote a disk in $\mathbb{C}$ containing $[0, 1]$ with trivial $G$-action. Then $\Delta \times X$ has quotient $\Delta \times Q$ with the obvious quotient mapping. Let $\rho: \Delta \rightarrow [0, 1]$ be continuous such that $\rho$ sends a neighborhood of 0 to 0 and a neighborhood of 1 to 1. For $(z, x) \in \Delta \times X$, define $\Psi(z, x) = (z, \Phi(\rho(z), x))$. Then $\Psi \in \mathcal{A}_c(\Delta \times Q)$ where $\mathcal{A}_c = \mathcal{A}(\Delta \times X)$. Moreover, $\Psi$ is the identity on the inverse image of a neighborhood of $\{0\} \times Q$ and is holomorphic on the inverse image of a neighborhood of $\{1\} \times Q$. By [KLS, Theorem 8.7] we can find a homotopy $\Psi(s)$ which starts at $\Psi$ and ends up in $\mathcal{F}(\Delta \times Q)$. Moreover, the proof shows that we can assume that the elements of the homotopy are unchanged over a neighborhood of $\{0, 1\} \times Q$. Restricting $\Psi(1)$ to $[0, 1] \subset \Delta$ we have an element in $\mathfrak{F}(Q)$ which at time 1 is still $\Phi(1)$. Then we can apply the argument above. □

Proof of Theorem 1.6. By Lemma 5.2 we may assume that $\Phi_{ij}(0)$ is the identity cocycle. Since $H^1(Q, \mathcal{A}_c)$ is trivial, there are $c_i \in \mathcal{A}_c(U_i)$ such that $\Phi_{ij}(t) = c_i(t)^{-1} \Phi_{ij}(0) c_j(t)$ for $t \in [0, 1]$. Now the $c_i(1)$ define a strongly continuous $G$-homeomorphism from $X$ to the Stein $G$-manifold $Y$.
corresponding to \( \Phi_{ij}(1) \). By Theorem 5.4 there is a homotopy \( c_i(t), 1 \leq t \leq 2 \), such that the \( c_i(2) \) are holomorphic and split \( \Phi_{ij}(1) \). Reparameterizing, we may reduce to the case that the original \( c_i(t) \) are holomorphic for \( t = 1 \). Now apply Lemma 5.5 to \( \Psi_{ij}(t) = c_i(t)c_j(t)^{-1} \).

6. \( H^1(Q, A) \rightarrow H^1(Q, A_c) \) is a bijection

We give a proof of Theorem 1.2. We are given an open cover \( \{U_i\} \) of \( Q \) and \( \Phi_{ij} \in Z^1(Q, A_c) \). We want to find \( c_i \in A_c(U_i) \) such that \( c_i^{-1}\Phi_{ij}c_j \) is holomorphic. We may assume that the \( U_i \) are relatively compact, locally finite and Runge. We say that an open set \( U \subset Q \) is good if there are sections \( c_i \in A_c(U_i \cap U) \) such that \( c_i^{-1}\Phi_{ij}c_j \) is holomorphic on \( U \) for all \( i \) and \( j \) where \( U_{ij} \) denotes \( U_i \cap U_j \). This says that \( \{\Phi_{ij}\} \) is cohomologous to a holomorphic cocycle on \( U \). The goal is to show that \( Q \) is good. It is obvious that small open subsets of \( Q \) are good.

**Lemma 6.1.** Suppose that \( Q = Q' \cup Q'' \) where \( Q' \) and \( Q'' \) are good and \( Q' \cap Q'' \) is Runge in \( Q \). Then \( Q \) is good.

**Proof.** By hypothesis, we have \( c'_i \in A_c(Q' \cap U_i) \) and \( c''_i \in A_c(Q'' \cap U_i) \) such that

\[ \Psi'_{ij} = (c'_i)^{-1}\Phi_{ij}c'_j, \quad \text{and} \quad \Psi''_{ij} = (c''_i)^{-1}\Phi_{ij}c''_j \]

are holomorphic. Then on \( U_{ij} \cap Q' \cap Q'' \) we have

\[ \Psi''_{ij} = h_i^{-1}\Psi'_{ij} h_j \text{ where } h_i = (c'_i)^{-1}c''_i. \]

The \( \Psi'_{ij} \) are a holomorphic cocycle for the covering \( U_i \cap Q' \) of \( Q' \), hence they correspond to a Stein G-manifold \( X' \) with quotient \( Q' \). Similarly the \( \Psi''_{ij} \) give us \( X'' \), and \( X' \) and \( X'' \) are locally G-biholomorphic to \( X \) over \( Q' \) and \( Q'' \), respectively. The \( h_i \) give us a strong G-homeomorphism \( h: X' \rightarrow X'' \), everything being taken over \( Q' \cap Q'' \). By Theorem 5.4 there is a homotopy \( h(t, x) \) with \( h(0, x) = h(x) \) and \( h(1, x) \) holomorphic. Let \( k(x) \) denote \( h(1, x) \). Then \( k \) corresponds to a family \( k_i \) homotopic to the family \( h_i \).

Now just consider the space \( U_i \) covered by the two open sets \( U_i \cap Q' \) and \( U_i \cap Q'' \). Then \( h_i \) and \( k_i \) are defined on the intersection of the two open sets and are homotopic where \( h_i \) is cohomologous to the trivial cocycle since \( h_i = (c'_i)^{-1}c''_i \). By Corollary 5.3 and Theorem 1.1 the cohomology class represented by \( k_i(x) \) is holomorphically trivial. Hence there are holomorphic sections \( h'_i \) and \( h''_i \) such that \( k_i = (h'_i)^{-1}h''_i \) on \( U_i \cap Q' \cap Q'' \). Then \( h'_i\Psi'_i h''_i^{-1} \) and \( h''_i\Psi''_i(h''_i)^{-1} \) on \( U_{ij} \cap Q' \cap Q'' \). We construct a holomorphic cocycle \( \Psi_{ij} \) on \( U_{ij} \) by \( \Psi_{ij} = h'_i\Psi'_i(h''_i)^{-1} \) on \( U_{ij} \cap Q' \) and \( h''_i\Psi''_i(h''_i)^{-1} \) on \( U_{ij} \cap Q'' \).

Using Lemma 5.2 we may reduce to the case that \( \Psi_{ij} \) is the trivial cocycle. As in the beginning of the proof there are \( c'_i \in A_c(Q' \cap U_i) \) and \( c''_i \in A_c(Q'' \cap U_i) \) such that

\[ \Phi_{ij}|_{X'} = c'_i(c'_j)^{-1} \text{ and } \Phi_{ij}|_{X''} = c''_i(c''_j)^{-1} \]

where \( X' = X_{Q'} \) and \( X'' = X_{Q''} \). Let \( h_i = (c'_i)^{-1}c''_i \). Then \( h_i = h_j \) on \( U_{ij} \cap Q' \cap Q'' \), hence we have a section \( h \in A_c(Q' \cap Q'') \), and this section gives the same cohomology class as \( \Phi_{ij} \) (use the open cover \( \{Q' \cup U_i, Q'' \cup U_i\} \)). By Theorem 5.4, \( h \) is homotopic to an element \( \tilde{h} \in A(Q' \cap Q'') \), and this holomorphic section gives the same cohomology class by Corollary 5.3. Since going to a refinement of an open cover is injective on \( H^1 \), we see that our original \( \Phi_{ij} \) differs from a holomorphic cocycle by a coboundary. Thus \( Q \) is good.

**Proof of Theorem 1.2.** Using Lemma 6.1 as in [Car58, §5] we can show that there is a cover of \( Q \) by compact subsets \( K_n \) such that \( K_1 \subset V_2 \subset K_2 \ldots \) where \( V_j \) is the interior of \( K_j \) and such that a neighborhood of every \( K_n \) is good. We can assume that \( U_i \cap V_n \neq \emptyset \) implies that \( U_i \subset V_{n+1} \). This is possible by replacing \( \{K_n\} \) by a subsequence. For each \( n \) we choose \( c^n_i \in A_c(U_i \cap V_n) \) such that

\[ (c^n_i)^{-1}\Phi_{ij}c^n_j = \Psi^n_{ij} \]

is holomorphic on \( U_{ij} \cap V_n \).
Then
\[ \Psi^n_{ij} = (d^n_i)^{-1} \Psi^{n+1}_{ij} d^n_j \] on \( U_i \cap V_n \)
where \( d^n_i = (c^{n+1}_i)^{-1} c^n_i \) gives a strongly continuous map from the Stein \( G \)-manifold \( Y_n \) over \( V_n \) obtained using the \( \Psi^n_{ij} \) to the Stein \( G \)-manifold \( Y_{n+1} \) obtained using the \( \Psi^{n+1}_{ij} \). We know that the map is homotopic to a holomorphic one. Hence there are homotopies \( d^n_i(t) \) on \( U_i \cap V_n \) such that

1. \( \Psi^n_{ij} = (d^n_i(t))^{-1} \Psi^{n+1}_{ij} d^n_j(t) \) on \( U_i \cap U_j \cap V_n \), for all \( t \).
2. \( d^n_i(0) = (c^n_i)^{-1} c^n_i \).
3. The \( d^n_i(1) \) give a \( G \)-equivariant biholomorphic map from \( Y_n \) to \( Y_{n+1} \) over \( \text{Id}_{V_n} \).

Without changing the \( c^n_i \) we may replace the \( c^{n+1}_i \) by sections \( c_i^{n+1} \) such that

4. \( \tilde{\Psi}^{n+1}_{ij} = (\tilde{c}_i^{n+1})^{-1} \Phi^i_{ij} c^{n+1}_j \) is holomorphic on \( U_i \cap U_j \cap V_{n+1} \).
5. \( \tilde{c}_i^{n+1} = c^n_i \) on \( U_i \cap V_{n-2} \).

It suffices to set \( \tilde{c}_i^{n+1} = c_i^{n+1} \) if \( U_i \cap V_{n-1} = \emptyset \) and if not, then \( U_i \subset V_n \), and we can set

\[ \tilde{c}_i^{n+1} = c_i^{n+1} \cdot d^n_i(\lambda(x)), \]

where \( \lambda: V_n \to [0,1] \) is continuous, 0 for \( x \in V_{n-2} \) and 1 for \( x \not\in V_{n-1} \). Then one has (4) and (5). Thus we can arrange that \( c_i^{n+1} = c_i^n \) in \( U_i \cap V_{n-2} \), hence we obviously have convergence of the \( c_i^n \) to a continuous section \( c_i \) such that \( (c_i)^{-1} \Phi^i_{ij} c_j \) is holomorphic.

We now have Theorems 1.1 and 1.2 which imply Corollary 1.3, i.e., that \( H^1(Q, \mathcal{A}) \to H^1(Q, \mathcal{A}) \) is a bijection.

Proof of Theorem 1.5. This is immediate from Corollary 5.3 and Theorem 1.2.

We end with the analogue of an approximation theorem of Grauert.

Theorem 6.2. Let \( U \subset Q \) be Runge. Suppose that \( \Phi: X_U \to Y_U \) is biholomorphic and \( G \)-equivariant inducing \( \text{Id}_U \). Here \( X \) and \( Y \) are locally \( G \)-biholomorphic over \( Q \). Then \( \Phi \) can be arbitrarily closely approximated by \( G \)-biholomorphisms of \( X \) and \( Y \) over \( \text{Id}_Q \) if and only if this is true for strong \( G \)-homeomorphisms of \( X \) and \( Y \).

Proof. Let \( K \subset U \) be compact and let \( \Phi \in \text{Mor}(X_U, Y_U)^G \) be our holomorphic \( G \)-equivariant map inducing \( \text{Id}_U \). We can find a relatively compact open subset \( U' \) of \( U \) which contains \( K \) and is Runge in \( Q \). By hypothesis, there is a strong \( G \)-homeomorphism \( \Psi: X \to Y \) which is arbitrarily close to \( \Phi \) over \( \overline{U'} \). Then \( \Psi^{-1} \Phi = \exp D \) where \( D \in \mathcal{L}(\mathcal{A}(U')) \), hence \( \Psi' \) and \( \Phi' \) are homotopic, where \( \Phi' \) is the restriction of \( \Phi \) to \( U' \) and similarly for \( \Psi' \). Now \( \Psi \) is homotopic to a biholomorphic \( G \)-equivariant map \( \Theta: X \to Y \) inducing \( \text{Id}_Q \), and \( \Psi' \) is homotopic to the restriction \( \Theta' \) of \( \Theta \) to \( U' \). Then \( (\Phi')^{-1} \Theta' \) is holomorphic and homotopic to the identity section over \( U' \). Since the end points of the homotopy are holomorphic, by Theorem 1.6 we can find a homotopy all of whose elements are holomorphic. By [KLS, Theorem 10.1] there is a section \( \Delta \in \mathcal{A}(Q) \) which is arbitrarily close to \( (\Phi')^{-1} \Theta' \) on \( U' \). Then \( \Theta \Delta^{-1}, \text{ restricted to } U' \), is arbitrarily close to \( \Phi' \), hence this is true over \( K \). This establishes the theorem.

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