ON A SOLUTION TO THE BASEL PROBLEM BASED ON THE FUNDAMENTAL THEOREM OF CALCULUS

ALESSIO DEL VIGNA

1. Introduction

Let $s$ be a real number. Everyone knows that the infinite series
\[ \sum_{n=1}^{\infty} \frac{1}{n^s} \]
is convergent if $s > 1$ and that it diverges if $s \leq 1$. The sum of this series is denoted by $\zeta(s)$ and it is known as the Riemann zeta function. In 1743 Euler computed the value of $\zeta(2)$ by proving the identity
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \] (1)
The problem of evaluating the sum of the reciprocals of the squares, also known under the name of Basel problem, was first posed by Mengoli in the mid-seventeenth century and attacked by many prominent mathematicians of the time without success, until Euler.

Over the years, several solutions to the Basel problem have been found using a vast variety of techniques. The proof in [5] relies on a recurrence equation obtained by cleverly evaluating certain trigonometric integrals and then by telescoping. Some other proofs are based on evaluations of double integrals: the proof by Apostol [2] and that by Beukers, Kolk, and Calabi [3] use simple double integral and ingenious substitutions, for which they deserved a place in the wonderful text “Proofs from the book” [1]. The more recent proof in [7] uses the double integral of a rational function with the lowest degree among the functions used in other similar proofs. Moreover, many textbooks in Fourier analysis contain proofs or guided exercises about the Basel problem, based on the evaluation of the Fourier series of certain functions or on the Parseval identity. In some complex analysis textbooks the proof is instead given via contour integrals and their evaluation through the residue theorem. Lastly, [6] contains an original proof based on elementary probability tools. This list is not intended to be exhaustive, but it is just a way to show that original proofs can come from different area of mathematics.

Here we give another way of solving the Basel problem from the area of the mathematical analysis. The main ingredients for our proof are the differentiation under the integral sign, a trick which is recurrent in series evaluation, and the fundamental theorem of calculus, which to our knowledge has never been used in this context.

\[ \text{The Riemann zeta function is actually defined for } s \text{ being complex. In this setting it can be proven that the series } \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ converges if and only if } \Re(s) > 1 \text{ and that } \zeta(s) \text{ can be extended to a meromorphic function on the whole complex plane, which is holomorphic everywhere but a simple pole at } s = 1. \]
2. The proof

We consider the function \( f : (0, \frac{\pi}{2}) \times [0, 1] \rightarrow \mathbb{R} \) defined to be

\[
f(x, t) = \arccos \left( \frac{t - \tan^2 x}{t + \tan^2 x} \right).
\]

Since the integral of \( f(x, t) \) with respect to \( x \) exists for every \( t \in [0, 1] \), we are allowed to define \( g : [0, 1] \rightarrow \mathbb{R} \), function of just the variable \( t \), to be

\[
g(t) = \int_0^\frac{\pi}{2} \arccos \left( \frac{t - \tan^2 x}{t + \tan^2 x} \right) \, dx.
\]

Here it comes the differentiation under the integral sign because we would like to compute the derivative of the function \( g \).

**Lemma 1.** The function \( g \) is differentiable on \((0, 1)\) and it holds

\[
g'(t) = \frac{\log t}{2\sqrt{t(1-t)}}.
\]

**Proof.** The function \( g \) is defined through an integral and the conclusion is just a computation provided that we are allowed to differentiate under the integral sign. To do this it suffices to show that the conditions stated in Theorem 3 hold. For all \( x \in (0, \frac{\pi}{2}) \) and for all \( t \in (0, 1) \) we have the existence of the partial derivative

\[
\frac{\partial f}{\partial t}(x, t) = -\frac{\tan x}{\sqrt{t(t + \tan^2 x)}}.
\]

For the domination condition we can use the inequality between the arithmetic and the geometric mean of non-negative numbers:

\[
\left| \frac{\partial f}{\partial t}(x, t) \right| = \frac{1}{t} \cdot \frac{\sqrt{t} \tan x}{t + \tan^2 x} \leq \frac{1}{2t}.
\]

We then fix \( \delta \) with \( 0 < \delta < 1 \) and we restrict \( t \) to the interval \((\delta, 1)\), so that the previous bound yields

\[
\left| \frac{\partial f}{\partial t}(x, t) \right| \leq \frac{1}{2\delta},
\]

with the bounding function being integrable over \((0, \frac{\pi}{2})\). Thus \( g \) is differentiable on the interval \((\delta, 1)\) for every \( \delta \), and since \( 0 < \delta < 1 \) we have the differentiability of \( g \) over the whole \((0, 1)\). We are thus allowed to differentiate \( g \) by passing the derivative under the integral sign. This yields

\[
g'(t) = -\frac{1}{\sqrt{t}} \int_0^{\frac{\pi}{2}} \frac{\tan x}{t + \tan^2 x} \, dx = -\frac{1}{\sqrt{t}} \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{(t - 1) \cos^2 x + 1} \, dx = \frac{1}{2\sqrt{t(t-1)}} \log((t-1)\cos 2x + 1) \bigg|^{\frac{\pi}{2}}_{x=0} = \frac{\log t}{2\sqrt{t(1-t)}}.
\]

and Lemma 1 is proved. \( \square \)
We are now in a position to prove (1) by applying the fundamental theorem of calculus to the function \( g' \) over the interval \((0, 1)\). As in many solutions to the Basel problem, we do not directly show (1) but the equivalent
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.
\] (2)
Indeed we immediately have
\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.
\]
**Theorem 2.** It holds that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]
**Proof.** In Lemma 1 we proved that \( g \) is differentiable on \((0, 1)\), which implies that \( g \) is continuous on \((0, 1)\). We claim that \( g \) is also continuous in \( t = 0 \). Let \((t_n)_{n=0}^{\infty}\) be a sequence in \((0, 1)\) such that \( t_n \to 0 \). Since for all \( x \in (0, \pi/2) \) and for all \( t \in [0, 1] \) holds \( |f(x, t)| \leq \pi \) and since \((0, \pi/2) \times [0, 1] \) has finite measure we can apply the dominated convergence theorem to obtain
\[
\lim_{n \to +\infty} g(t_n) = \lim_{n \to +\infty} \int_{\pi/2}^{0} f(x, t_n) \, dx = \int_{\pi/2}^{0} \lim_{n \to +\infty} f(x, t_n) \, dx = g(0).
\]
An analogous argument holds for \( t = 1 \), so that \( g \) turns out to be continuous on the whole \([0, 1]\). We now claim that \( g' \) is integrable on \((0, 1)\): \( g' \) is continuous on \((0, 1)\), it can be extended by continuity in \( t = 1 \), and moreover
\[
\frac{\log t}{\sqrt{t(1-t)}} \sim \frac{\log t}{\sqrt{t}} \quad \text{for } t \to 0^+.
\]
Thus we can apply the fundamental theorem of calculus to \( g' \) to write
\[
g(1) - g(0) = \int_{0}^{1} g'(t) \, dt. \tag{3}
\]
We now evaluate the quantities of the above identity. We have \( g(0) = \int_{0}^{\pi/2} \pi \, dx = \pi^2/2 \) and
\[
g(1) = \int_{0}^{\pi/2} \arccos \left( \frac{1 - \tan^2 x}{1 + \tan^2 x} \right) \, dx = \int_{0}^{\pi/2} 2x \, dx = \frac{\pi^2}{4},
\]
where we used the trigonometric identity \((1 - \tan^2 x)/(1 + \tan^2 x) = \cos 2x\). For the right-hand side of (3) we use the expression of \( g' \) and series expansion to evaluate the resulting integral:
\[
\int_{0}^{1} g'(t) \, dt = \frac{1}{2} \int_{0}^{1} \frac{\log t}{\sqrt{t(1-t)}} \, dt = 2 \int_{0}^{1} \frac{\log u}{1 - u^2} \, du =
\]
\[
= 2 \int_{0}^{1} \log u \left( \sum_{n=0}^{\infty} u^{2n} \right) \, du = 2 \sum_{n=0}^{\infty} \left( \int_{0}^{1} u^{2n} \log u \, du \right) =
\]
\[
= -2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.
\]
By equating the quantities involved in (3), we immediately obtain (2). \(\square\)
Appendix A. Differentiation under the integral sign

Differentiation under the integral sign concerns integrals depending on a parameter and it is often a powerful tool to evaluate definite integrals and series. Over analysis and measure theory textbooks one can find plenty of versions, with slight differences in the hypotheses or in the conclusions. To avoid misunderstandings, we decided to explicitly state the version we used: see, for instance, [4, Theorem 6.2.6].

Theorem 3. Let $X \subseteq \mathbb{R}^n$ be a measurable set and $A \subseteq \mathbb{R}$ be an open set. Let $f : X \times A \to \mathbb{R}$ be a measurable function such that

(i) the function $x \mapsto f(x, t)$ is integrable on $X$ for all $t \in A$;
(ii) the partial derivative $\partial f/\partial t(x, t)$ exists for a.e. $x \in X$ and for all $t \in A$;
(iii) there exists an integrable function $h : X \to \mathbb{R}$ such that $|\partial f/\partial t(x, t)| \leq h(x)$ for a.e. $x \in X$ and for all $t \in A$.

Then for all $t \in A$ holds

$$\frac{d}{dt} \left( \int_X f(x, t) \, dx \right) = \int_X \frac{\partial f}{\partial t}(x, t) \, dx.$$

Proof. Let $t \in A$ and let $r > 0$ such that $B_r(t) = (t - r, t + r) \subseteq A$. Let $(t_n)_{n=0}^{\infty}$ be a sequence in $B_r(t)$ converging to $t$ and consider the ratio

$$\int_X f(x, t_n) \, dx - \int_X f(x, t) \, dx \bigg/ \frac{t_n - t}{t_n - t} = \int_X \frac{f(x, t_n) - f(x, t)}{t_n - t} \, dx.$$

Condition (ii) implies that the last integrand function converges to $\partial f/\partial t(x, t)$ for a.e. $x \in X$. From the mean value theorem there exists $\xi \in B_r(t)$ such that for a.e. $x \in X$ holds

$$\frac{f(x, t_n) - f(x, t)}{t_n - t} = \frac{\partial f}{\partial t}(x, \xi),$$

and hence condition (iii) implies that for a.e. $x \in X$

$$\left| \frac{f(x, t_n) - f(x, t)}{t_n - t} \right| \leq h(x).$$

We can thus apply the Lebesgue dominated convergence theorem to obtain

$$\frac{d}{dt} \left( \int_X f(x, t) \, dx \right) = \lim_{n \to +\infty} \int_X \frac{f(x, t_n) - f(x, t)}{t_n - t} \, dx = \int_X \left( \lim_{n \to +\infty} \frac{f(x, t_n) - f(x, t)}{t_n - t} \right) \, dx = \int_X \frac{\partial f}{\partial t}(x, t) \, dx,$$

so that Theorem 3 is proved. \qed

References

[1] M. Aigner and G. M. Ziegler, Proofs from THE BOOK, 4th, Springer Publishing Company, Incorporated, 2009
[2] T. Apostol, A proof that Euler missed: Evaluating $\zeta(2)$ the easy way, Math. Intelligencer 5: 59–60, 1983
[3] E. C. F. Beukers J. A. C. Kolk, *Sums of Generalized Harmonic Series and Volumes*, Nieuw Arch. Wisk. 11: 217–224, 1993

[4] O. Hijab, *Introduction to Calculus and Classical Analysis*, Undergraduate Texts in Mathematics, Springer, New York/Heidelberg/Berlin, 2016

[5] Y. Matsuoka, *An Elementary Proof of the Formula* $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, The American Mathematical Monthly 68: 485–487, 1961

[6] L. Pace, *Probabilistically Proving that* $\zeta(2) = \frac{\pi^2}{6}$, The American Mathematical Monthly 118(7): 641–643, 2011

[7] D. Ritelli, *Another Proof of* $\zeta(2) = \frac{\pi^2}{6}$ *Using Double Integrals*, The American Mathematical Monthly 120(7): 642–645, 2013