POINTEWISE DECAY OF SOLUTIONS TO THE ENERGY CRITICAL NONLINEAR SCHRÖDINGER EQUATIONS

ZHIZHU GA, CHUNYAN HUANG, AND LIANG SONG

Abstract. In this note, we prove pointwise decay in time of solutions to the 3D energy-critical nonlinear Schrödinger equations assuming data in $L^1 \cap H^3$. The main ingredients are the boundness of the Schrödinger propagators in Hardy space due to Miyachi [9] and a fractional Leibniz rule in the Hardy space. We also extend the fractional chain rule to the Hardy space.

1. Introduction

Consider the nonlinear Schrödinger equation
\[
\begin{cases}
i \partial_t u + \Delta u = \mu |u|^4 u, & (x, t) \in \mathbb{R}^3 \times \mathbb{R}, \\
u(x, 0) = u_0(x)
\end{cases}
\] (1.1)
with $\mu \in \{1, -1\}$. It is well-known that the solutions of the linear Schrödinger equation satisfy the following dispersive estimates
\[
\|e^{it\Delta} u_0\|_{L^\infty_x} \leq C |t|^{-3/2} \|u_0\|_{L^1_x}. 
\] (1.2)

It is natural to ask whether one can obtain global solutions $u$ to the nonlinear Schrödinger equation (1.1) with the same time decay, namely
\[
\|u(t)\|_{L^\infty_x} \leq C |t|^{-3/2}. 
\] (1.3)

The equation (1.1) is energy-critical. More precisely, (1.1) is invariant under the following scaling transform
\[
u(t, x) \rightarrow \lambda^{1/2} u(\lambda x, \lambda^2 t), \quad u_0(x) \rightarrow \lambda^{1/2} u_0(\lambda x)
\] (1.4)
and $\|\lambda^{1/2} u_0(\lambda x)\|_{H^1} = \|u_0\|_{H^1}$. Global well-posedness and scattering theory for (1.1) were extensively studied in the last two decades (e.g. see [1, 7] for the introduction). A global solution $u$ of (1.1) scatters in the energy space means there exists $\phi_{\pm} \in H^1$ such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} \phi_{\pm}\|_{H^1} = 0.
\] (1.5)

Even if we have scattering in $H^1$, to get pointwise-in-time decay for solutions of (1.1) is not trivial. On one hand, one does not expect pointwise-in-time decay if assuming initial data only in $H^1(\mathbb{R}^3)$ in view of the linear Schrödinger equation. On the other hand, assuming $u_0 \in L^1$ one cannot ensure the scattering state $\phi_{\pm} \in L^1$. Thus to obtain pointwise decay requires some extra effort. Recently, by contradiction argument Fan and Zhao [3] proved that scattering solution of (1.1) satisfies (1.3) assuming $u_0 \in L^1 \cap H^k$ for some $k$.

In this note, we consider the finer asymptotic behaviour based on scattering results in the energy space, and hence give a direct and simpler proof of Fan-Zhao’s result for (1.1). Moreover, our result is more quantitative. We can show that the scattering state $\phi_{\pm} \in L^1$ and reversely the wave operator is also defined in $L^1$. The main result is

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Theorem 1.1. (1) Let $u$ be a global solution of (1.1) with finite scattering norm
\[ \|u\|_{L_{t,x}^{10}(\mathbb{R} \times \mathbb{R}^3)} \leq K \] (1.6)
for some $K > 0$. Assume in addition $u(0) \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Then $u \in C(\mathbb{R} : H^3)$, $e^{-it\Delta}u(t) \in C(\mathbb{R} ; L^1)$ and
\[ \|u(t,x)\|_{L^\infty} \leq C(K, u_0)|t|^{-3/2}. \] (1.7)
Moreover, there exists $\phi_\pm \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ such that
\[ \lim_{t \to \pm \infty} (\|u(t) - e^{it\Delta} \phi_\pm\|_{H^3} + \|e^{-it\Delta} u(t) - \phi_\pm\|_{L^1}) = 0 \] (1.8)
\[ \|u(t) - e^{it\Delta} \phi_\pm\|_{L^\infty} \leq C|t|^{-5}, \quad \pm t > 0. \]

(2) Assume $\phi_+ \in H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. Then there exists a unique solution $u \in L_{t,x}^{10}$ such that $u \in C(\mathbb{R} : H^3)$, $e^{-it\Delta}u(t) \in C(\mathbb{R} ; L^1)$ and
\[ \lim_{t \to \infty} (\|u(t) - e^{it\Delta} \phi_+\|_{H^3} + \|e^{-it\Delta} u(t) - \phi_+\|_{L^1}) = 0, \]
\[ \|u(t) - e^{it\Delta} \phi_+\|_{L^\infty} \leq C|t|^{-5}, \quad t > 0. \] (1.9)
Similar results hold for $t \to -\infty$.

It is known that global well-posedness and scattering of (1.1) holds in $\dot{H}^1(\mathbb{R}^3)$ and
\[ \|u\|_{L_{t,x}^{10}(\mathbb{R} \times \mathbb{R}^3)} \leq C\|u_0\|_{\dot{H}^1} \] (1.10)
provided that either of the following conditions hold:

- in the defocusing case $\mu = 1$ (see [1]);
- in the focusing case $\mu = -1$ (see [2]): $E(u_0) < E(W)$, $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $u_0$ radial, where
\[ E(u) = \int \frac{1}{2} |\nabla u|^2 + \frac{1}{6} |u|^6 \, dx \] (1.11)
and $W = (1 + |x|^2/3)^{-1/2}$. Moreover, assuming additional conditions $u_0 \in H^3(\mathbb{R}^3)$, one can obtain the persistence (scattering in $H^3$) and
\[ \|\langle \nabla \rangle^q u\|_{L_{t,x}^{r}(\mathbb{R} \times \mathbb{R}^3)} \leq C\|u_0\|_{H^3} \] (1.12)
where $(q, r)$ satisfies the admissible conditions: $2 \leq q, r \leq \infty$ and $\frac{1}{q} = \frac{3}{2}(\frac{1}{2} - \frac{1}{r})$.

It is known (see [3]) that for nice initial data $f$ we have for $\gamma > \frac{1}{4} + 2\beta$
\[ e^{it\Delta} f(x) = \frac{1}{(i2\eta)^{n/2}} e^{\frac{|x|^2}{4\eta}} \hat{f}(x/t) + \frac{1}{t^{n/2 + \beta}} O(\|\langle x \rangle^\gamma f\|_{L^2}), \quad t \to \pm \infty. \] (1.13)
Scattering and the wave operator in $L^1$ thus provide a precise asymptotic profile of the nonlinear solutions.

Our main ingredient of the proof is the boundedness for the Schrödinger propagator in Hardy space which was proved by Miyachi [4] and a fractional Leibniz rule in Hardy space.

2. Proof of the main Theorem

Throughout this note, we use $C$ to denote some universal constant which may change from line to line. For $X, Y \in \mathbb{R}$, $X \lesssim Y$ means $X \leq CY$ for some $C > 0$, similarly for $X \gtrsim Y$. $X \sim Y$ means $X \lesssim Y$ and $X \gtrsim Y$.

Fix a bump function $\eta \in C_0^\infty(\mathbb{R}^d)$ such that $\eta$ is non-negative, radial and radially decreasing, supp $\eta \subset \{|x| \leq 1.1\}$ and $\eta(x) \equiv 1$ for $|x| \leq 1$. Let $\chi(x) = \eta(x) - \eta(2x)$. For $k \in \mathbb{Z}$, define the operators
\[ \hat{P}_k = F^{-1} \chi \left( \frac{x}{2^k} \right) F, \quad P_{\leq k} = F^{-1} \eta \left( \frac{x}{2^k} \right) F, \quad P_{> k} = I - P_{\leq k} \] (2.1)
and

\[
P_k = \begin{cases} 
\mathcal{F}^{-1} \chi(\frac{x}{k})\mathcal{F}, & k > 0; \\
\mathcal{F}^{-1} \eta(x)\mathcal{F}, & k = 0; \\
0, & k \leq -1.
\end{cases}
\] (2.2)

Here \( \mathcal{F} \) is the Fourier transform: \( \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx \). We define \( D^s = \mathcal{F}^{-1}|\xi|^s\mathcal{F} \) and \( J^s = \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2}\mathcal{F} \).

Suppose \( \{ \psi_j \} \) is a sequence of functions in \( \mathcal{Z} \). We define

\[
\hat{P}_k = \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.
\] (2.6)

The maximal operator can control many \( L^1 \)-average types operators. We recall the well-known pointwise maximal function estimate, see [10].

**Lemma 2.1.** Let \( g(x) \) be a nonnegative radial decreasing integrable function, suppose \( |\psi(x)| \leq g(x) \) almost everywhere and \( f \in L^1_{loc}(\mathbb{R}^d) \), then

\[
|\psi_{\epsilon} \ast f(x)| \leq C\|g\|_{L^1} \cdot M(f)(x), \quad \forall \epsilon > 0,
\]

where \( \psi_{\epsilon}(x) = \epsilon^{-d}\psi(\epsilon^{-1}x) \).

We will also need the following result concerning the boundedness of \( M \) acting on vector-valued functions (see [10]).

**Lemma 2.2** (Maximal inequality). Let \( (p,q) \in (1, \infty) \times (1, \infty) \) or \( p = q = \infty \) be given. Suppose \( \{f_j\}_{j \in \mathbb{Z}} \) is a sequence of functions in \( L^p(\mathbb{R}^d) \) satisfying \( \|f_j\|_{L^p(\mathbb{Z})} \leq \|f_j\|_{L^p(\mathbb{R}^d)} \), then

\[
\|M(f_j)(x)\|_{L^q_{\xi j}} \leq C\|f_j\|_{L^p_{\xi j}}
\] (2.7)

for some constant \( C = C(p,q) \).

It is worth noting that the above Lemma fails for \( p = 1 \), which causes some difficulty when dealing with \( L^1 \)-based space (e.g. Hardy space). This was usually overcome by the following variant maximal estimate.

**Lemma 2.3** ([12]). Assume \( 0 < r < \infty \). There exists \( C > 0 \) such that for all \( R > 0 \)

\[
\sup_{y \in \mathbb{R}^d} \frac{|f(x - y)|}{(1 + |Ry|)^r} \leq C\left[M(|f|)(x)\right]^\frac{r}{2}, \quad \forall x \in \mathbb{R}^d
\] (2.8)

holds for all \( f \) with \( \text{supp} \hat{f} \subset B(R) \).

The above lemma is useful for linear estimate. Motivated by [3], we derive the following improvement which is useful for the multilinear estimates.
Lemma 2.4. Let $L > 0$, $j, k \in \mathbb{Z}$, $j > k - L$ and $r \in (0, 1]$. Assume $\psi_k \in L^1(\mathbb{R}^d)$ satisfies for some $A > 0$ and any $f \in L^1_{\text{loc}}(\mathbb{R}^d)$
\[
|\psi_k(y)|(1 + |2^k y|^{\frac{1}{r} - 1}) \ast f(x) | \leq A \cdot M(f)(x), \quad \forall x \in \mathbb{R}^d. \tag{2.9}
\]
Then there exists a constant $C = C(A, L)$ such that
\[
|\psi_k \ast f(x)| \leq C\left\{ C_2(j-k)(\frac{1}{r} - 1) d \left[ M(|f|)(x) \right]^{\frac{1}{r}} \right. \tag{2.10}
\]
holds for all $f$ with $\text{supp} \hat{f} \subset B(c2^j)$.

Proof. We have
\[
|\psi_k \ast f(x)| \leq \int_{\mathbb{R}^d} |\psi_k(y)| \cdot |f(x - y)| dy \leq \int_{\mathbb{R}^d} 2^{-kd} |\psi_k(2^{-k} y)| \cdot |f(x - 2^{-k} y)| dy \leq \int_{\mathbb{R}^d} 2^{-kd} |\psi_k(2^{-k} y)| \cdot |f(x - 2^{-k} y)|^{\gamma} (1 + 2^{j-k} |y|)^{\frac{1}{r} - 1} d \cdot \sup_{y \in \mathbb{R}^d} \frac{|f(x - 2^{-k} y)|^{1-r}}{(1 + 2^{j-k} |y|)^{\frac{1}{r} - 1}} dy. \tag{2.11}
\]
In view of Lemma 2.3 one can see
\[
\sup_{y \in \mathbb{R}^d} \frac{|f(x - 2^{-k} y)|^{1-r}}{(1 + 2^{j-k} |y|)^{\frac{1}{r} - 1}} \leq C[M(|f|)(x)]^{\frac{1}{1-r}}.
\]
Then we get
\[
|\psi_k \ast f(x)| \leq \int_{\mathbb{R}^d} 2^{-kd} |\psi_k(2^{-k} y)| \cdot |f(x - 2^{-k} y)|^{\gamma} (1 + 2^{j-k} |y|)^{\frac{1}{r} - 1} dy \cdot [M(|f|)(x)]^{\frac{1}{1-r}} \leq 2^{(j-k)(\frac{1}{r} - 1) - 1} \int_{\mathbb{R}^d} |\psi_k(2^{-k} y)| \cdot |f(x - 2^{-k} y)|^{\gamma} (1 + |y|)^{\frac{1}{r} - 1} dy \cdot [M(|f|)(x)]^{\frac{1}{1-r}} \tag{2.12}
\]
\[
\leq 2^{(j-k)(\frac{1}{r} - 1) - 1} \int_{\mathbb{R}^d} |\psi_k(y)| \cdot |f(x - y)|^{\gamma} (1 + |2^{k} y|)^{\frac{1}{r} - 1} dy \cdot [M(|f|)(x)]^{\frac{1}{1-r}} \leq 2^{(j-k)(\frac{1}{r} - 1) - 1} [M(|f|)(x)]^{\frac{1}{r}}.
\]
Thus we complete the proof. \hfill \Box

Remark 2.1. \eqref{2.10} is crucial for us deriving the nonlinear estimates in the next subsection. By \eqref{2.12} we obtain for $r < 1$
\[
|\psi_k \ast (\hat{P}_j f \cdot \hat{P}_j g)| \lesssim 2^{(j-k)(\frac{1}{r} - 1)} [M(|\hat{P}_j f \cdot \hat{P}_j g|)(x)]^{\frac{1}{r}}. \tag{2.13}
\]
In particular, the above estimate is useful to handle the high-high to low frequency interactions.

A key ingredient in our proof is the boundedness of the Schrödinger propagator on Hardy space $H^1$. This was proved by Miyachi \cite{9}.

Lemma 2.5 (Hardy space boundness). We have
\[
\| e^{it \Delta} f \|_{L^1_{\text{loc}}(\mathbb{R}^d)} \lesssim (1 + |t|)^{d/2} \| f \|_{L^2(\mathbb{R}^d)},
\]
\[
\| e^{it \Delta} P_j f \|_{L^1_{\text{loc}}(\mathbb{R}^d)} \lesssim (1 + |t|)^{d/2} \| f \|_{L^2(\mathbb{R}^d)}.
\]
(2.14)

The second ingredient is the nonlinear estimates in Hardy space. First we derive a Leibniz rule in the Hardy space.
Lemma 2.6. Let $s > 0$, $p_i, q_i \in (1, \infty)$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1, 2$. Then
\begin{equation}
\|D^s(fg)\|_{H^s} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}}.
\end{equation}

Proof. By Bony’s paraproduct decomposition, we have
\begin{equation}
f g = \sum_{|j-m| \leq 3} \hat{P}_m(\hat{P}_j f \cdot \hat{P}_{\leq m-5g}) + \sum_{|j-m| \leq 3} \hat{P}_m(\hat{P}_{\leq m-5f} \cdot \hat{P}_j g) + \sum_{j \geq m-3} \hat{P}_m(\hat{P}_j f \cdot \hat{P}_j g) := I + II + III.
\end{equation}

For the term $I$, by Lemma 2.4 we have
\begin{equation}
\|D^s(I)\|_{H^s} \lesssim \|2^{ms} \hat{P}_m(\hat{P}_j f \cdot \hat{P}_{\leq m-5g})\|_{L^1} \lesssim \|M(2^{ms} \hat{P}_m f \cdot \hat{P}_{\leq m-5g}|^{1/2})\|_{L^1} \lesssim \|2^{ms} \hat{P}_m f\|_{L^2} M(g)\|_{L^1} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}}.
\end{equation}

The term $II$ is similar to the term $I$ and we can get
\begin{equation}
\|D^s(II)\|_{H^s} \lesssim \|f\|_{L^{p_2}} \|D^s g\|_{L^{q_2}}.
\end{equation}

For the term $III$, applying (2.13) by taking $r < 1$ such that $d(1/r - 1) < s$, we get
\begin{equation}
\|D^s(III)\|_{H^s} \lesssim \|\sum_{j \geq m-3} 2^{(m-j)s} \hat{P}_m(2^{js} \hat{P}_j f \cdot \hat{P}_j g)\|_{L^1} \lesssim \|\sum_{j \geq m-3} 2^{(m-j)s} 2^{(j-m)d(1/r - 1)} [M(2^{js} \hat{P}_j f \cdot \hat{P}_j g)]^{1/r}\|_{L^1} \lesssim \|M(2^{js} \hat{P}_j f \cdot \hat{P}_j g)^{1/r}\|_{L^{1/r}, 1_2/r}.
\end{equation}

Then by Lemma 2.2 and the Hölder inequality we have
\begin{equation}
\|D^s(III)\|_{H^s} \lesssim \|2^{js} \hat{P}_j f \cdot M g\|_{L^{1_2}} \lesssim \|D^s f\|_{L^{p_1}} \|g\|_{L^{q_1}}.
\end{equation}

Therefore, we complete the proof. \hfill \square

Remark 2.2. It seems to us Lemma 2.6 is new\footnote{By private communication, the authors learned that some generalized Leibniz rules in Hardy space were obtained independently in \cite{a}.}. The fractional Leibniz rule has been extensively studied, see \cite{b} for a comprehensive survey of the current results. The classical inequality reads
\begin{equation}
\|D^s(fg)\|_{r} \lesssim \|D^s f\|_{p_1} \|g\|_{q_1} + \|f\|_{p_2} \|D^s g\|_{q_2}
\end{equation}
where $s > 0$, $1 < p_i, q_i \leq \infty$ with $1/r = 1/p_i + 1/q_i$: $i = 1, 2$, and $1/(1 + s) < r \leq \infty$.

The same argument of the proof of Lemma 2.4 can be applied to the Hardy space boundedness of a class of bilinear operators. This is known as compensated compactness. See \cite{c}. Consider the bilinear operator of the form
\begin{equation}
B_m(f, g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} m(\xi_1, \xi_2) \hat{f}(\xi_1) \hat{g}(\xi_2) e^{ix(\xi_1 + \xi_2)} d\xi_1 d\xi_2.
\end{equation}
We decompose
\begin{equation}
m = m_{HL} + m_{LH} + m_{HH},
\end{equation}
where

\[
\begin{align*}
    m_{HL}(\xi, \eta) &= \sum_{j,k:j \geq k+5} \chi_j(\xi)\chi_k(\eta) \\
    m_{LH}(\xi, \eta) &= \sum_{j,k:j \geq k+5} \chi_j(\xi)\chi_k(\eta) \\
    m_{HH}(\xi, \eta) &= \sum_{j,k:j \geq k+5} \chi_j(\xi)\chi_k(\eta).
\end{align*}
\]  

(2.24)

By the similar argument as proving Lemma 2.6, we can obtain the following proposition, although we will not need it in this note.

**Proposition 2.7.** Assume \( B_\sigma(f, g) : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d) \) is a bounded bilinear operator, for \( \sigma \in \{m_{HL}, m_{LH}, m_{HH}\} \), and some cancellation property: there exists \( \gamma > 0 \) such that

\[
\| \dot{P}_k B_m(\dot{P}_j f, \dot{P}_l g) \|_{L^1(\mathbb{R}^d)} \lesssim 2^{(k-j)\gamma} \| \dot{P}_j f \|_{L^2} \| \dot{P}_l g \|_{L^2}
\]

(2.25)

for all \( k, j, l \in \mathbb{Z} \) with \( 2^k \ll 2^j \sim 2^l \). Then \( \| B_m(f, g) \|_{H^1(\mathbb{R}^d)} \lesssim \| f \|_{L^2} \| g \|_{L^2} \).

Besides the fractional Leibniz rule, the fractional chain rule is also important in applications, especially in dealing with the non-algebraic nonlinearity \( F(u) = |u|^{p-1}u \) when \( p \) is not odd. The classical fractional chain rule says (see Tao [11]): if \( s \in (0, 1) \), \( 1 < m, t, q < \infty \) with \( \frac{1}{m} = \frac{p-1}{r} + \frac{1}{q} \),

\[
\| D^s F(u) \|_{L^m(\mathbb{R}^d)} \lesssim \| u \|_{L^r}^{p-1} \| D^s u \|_{L^q}.
\]

(2.26)

In the lemma below, we extend the fractional chain rule to the Hardy space when \( m = 1 \), although we do not need it in this note.

**Lemma 2.8** (Fractional chain rule in Hardy space). Let \( F(u) \) be a power-type nonlinearity with exponent \( p \geq 1 \), namely of the form \( |u|^p \). Assume \( s \in (0, 1) \), \( 1 < t, q < \infty \) with \( 1 = \frac{t}{p} + \frac{1}{q} \). Then

\[
\| D^s F(u) \|_{F^s_{1,t}(\mathbb{R}^d)} \lesssim \| u \|_{L^r}^{p-1} \| D^s u \|_{L^q}.
\]

(2.27)

**Proof.** We assume first the pointwise estimate: for \( r \in (0, 1) \)

\[
|\dot{P}_j [F(u)](x)| \lesssim \sum_k \min(2^k, 1) \left[ M(|Mu|^{p-1})(x) M(\dot{P}_{j+k} u)(x) \right. \\
+ \left. 2^{kd(\frac{1}{q}-1)} \left( M \left( |Mu|^{p-1} M(\dot{P}_{j+k} u) \right)^r (x) \right)^{1/r} \right].
\]

(2.28)

Then

\[
\| D^s F(u) \|_{F^s_{1,t}(\mathbb{R}^d)} \sim \| 2^j \dot{P}_j F(u) \|_{L^{1,t}_{j}^r} \\
\lesssim \sum_k \min(2^k, 1) 2^{-ks} 2^{j+k+s} M(|Mu|^{p-1}) \cdot M(\dot{P}_{j+k} u) \|_{L^{1,t}_{j}^r} \\
+ \left\| \sum_k \min(2^k, 1) 2^{-ks} 2^{j+k+s} 2^{kd(\frac{1}{q}-1)} \left( M \left( |Mu|^{p-1} M(\dot{P}_{j+k} u) \right)^r (x) \right)^{1/r} \right\|_{L^{1,t}_{j}^r} \\
:= A + B.
\]

(2.29)

For the term \( A \), since \( s > 0 \) we get by Lemma 2.2 and the Hölder inequality that

\[
A \lesssim \| M(|Mu|^{p-1}) \cdot \| D^{2^s} \dot{P}_j u \|_{L^r_j} \lesssim \| u \|_{L^r}^{p-1} \| D^s u \|_{L^q}.
\]

(2.30)
For the term $B$, since $s \in (0, 1)$, we choose $r < 1$ but sufficiently close to 1 such that $s - d(\frac{1}{r} - 1) > 0$. Then we get by Lemma [2.22] and the Hölder inequality that

$$B \leq \left\| M \left( \left| M(\langle |u|^p \rangle - 1) M(2^{2s} \hat{P}_j u) \right|^r \right)^{1/r} \right\|_{L^1 L^2} \leq \| u \|_{L^p}^{p-1} \| D^s u \|_{L^q}.$$  \hfill (2.31)

It remains to prove (2.28), we may assume $j = 0$ by scaling and $x = 0$ by translation. Then by fundamental theorem of calculus we have

$$F(u) = F(P_{\leq 0} u) + \int_0^1 F'(P_{\leq 0} u + tP_{>0} u) dt \cdot P_{>0} u$$  \hfill (2.32)

and thus

$$\hat{P}_0 F(u)(0) = \hat{P}_0 [F(P_{\leq 0} u)](0) + \hat{P}_0 \left[ \int_0^1 F'(P_{\leq 0} u + tP_{>0} u) dt \cdot P_{>0} u \right](0)$$  \hfill (2.33)

$$:= I + II.$$

For the term $II$, we have

$$II = \sum_{k > 0} \sum_{|j - k| \leq 5} \hat{P}_0 \left( \hat{P}_j \left[ \int_0^1 F'(P_{\leq 0} u + tP_{>0} u) dt \cdot \hat{P}_k u \right] \right).$$  \hfill (2.34)

By (2.13) we have

$$|II| \lesssim \sum_{k > 0} \sum_{|j - k| \leq 5} 2^{kd(\frac{1}{2} - 1)} \left( M \left( \left| \hat{P}_j \left[ \int_0^1 F'(P_{\leq 0} u + tP_{>0} u) dt \cdot \hat{P}_k u \right]^r \right)^{1/r} \right) \right)(0)$$  \hfill (2.35)

For the term $I$, we have

$$I = \hat{P}_0 [F(P_{\leq 0} u) - F(P_{\leq 0} u)(0)](0)$$

$$= \hat{P}_0 \left( \int_0^1 F'(P_{\leq 0} u(0) + t[P_{\leq 0} u - P_{\leq 0} u(0)]) dt \cdot [P_{\leq 0} u - P_{\leq 0} u(0)] \right)$$

$$= \sum_{k \leq 0} \hat{P}_0 \left( \left[ \int_0^1 F'(P_{\leq 0} u(0) + t[P_{\leq 0} u - P_{\leq 0} u(0)]) dt \cdot [\hat{P}_k u - \hat{P}_k u(0)] \right) \right)$$

$$= \sum_{k \leq 0} \int \left( \left[ \int_0^1 F'(P_{\leq 0} u(0) + t[P_{\leq 0} u - P_{\leq 0} u(0)]) dt \right] \cdot \int_0^1 \nabla \hat{P}_k u(sy) \cdot y ds \right) \chi(y) dy.$$

By Lemma [2.31] we have

$$|\nabla \hat{P}_k u(sy)| \lesssim 2^{k} M \left( |\hat{P}_k u|^r \right)(0)^{1/r} (1 + |y|)^{d/r}.$$  \hfill (2.37)

Therefore we get

$$I \lesssim \sum_{k \leq 0} 2^{k} M \left( |M u|^{p-1} \right)(0) M \left( |\hat{P}_k u|^r \right)(0)^{1/r}.$$  \hfill (2.38)

The proof is completed. \hfill \Box

**Lemma 2.9** (Nonlinear estimate in Hardy space).

$$\| u \|^4 u \|_{L^1 L^2(\mathbb{R}^3)} \lesssim \| u \|_{L^\infty} \| \hat{F}^3 u \|_2 \| u \|_2.$$  \hfill (2.39)
Proof. For the low frequency component, it’s easy to see

\[ \|P_{\leq 0}(|u|^4u)\|_{L^1_t} \lesssim \|u\|^3_{\infty} \|u\|_2. \]  

(2.40)

Thus it remains to prove

\[ \|P_{\geq 1}(|u|^4u)\|_{F^1_{t,2}} \lesssim \|u\|^2_{\infty} \|J^3u\|_2. \]  

(2.41)

The above inequality follows from Lemma 2.6 and (2.21). \( \square \)

2.1. Proof of Theorem 1.1. From the Duhamel formula, we have

\[ u = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta}|u(s)|^4u(s)ds. \]  

(2.42)

Let \( T \) be sufficiently large. For simplicity, we define \( \|u\|_{X_T} := \|t^{3/2}u\|_{L^\infty_{x,t}([T,\infty])} \). Then applying the decay estimate for the Schrödinger equation (1.2), we obtain that

\[ \|u\|_{X_T} \leq \|u_0\|_{L^1_x} + \int_0^\infty \|e^{-is\Delta}|u(s)|^4u(s)\|_{L^1_x} ds \]
\[ \leq \|u_0\|_{L^1_x} + \int_0^T \|e^{-is\Delta}|u(s)|^4u(s)\|_{L^1_x} ds + \int_T^\infty \|e^{-is\Delta}|u(s)|^4u(s)\|_{L^1_x} ds \]
\[ := \|u_0\|_{L^1_x} + I + II. \]  

(2.43)

For the term \( I \), using Strichartz estimates we can easily get

\[ I \leq C_T. \]  

(2.44)

It remains to estimate the term \( II \). By Lemma 2.5, we have

\[ II \lesssim \int_T^\infty \|e^{-is\Delta}|u(s)|^4u(s)\|_{F^1_{t,2}} ds \]
\[ \lesssim \int_T^\infty s^{3/2} \|u(s)|^4u(s)\|_{F^1_{t,2}} ds \]
\[ \lesssim \int_T^\infty s^{3/2} \|D^3u(s)\|_{L^2_x} \|u(s)\|_{L^2_x} \|u(s)\|_{L^\infty_x} ds \]
\[ \lesssim \|u\|^2_{L^2_{t,x}([T,\infty])} \|u\|_{X_T}. \]  

(2.45)

where we used \( \|u\|_{L^\infty_{t,x}} \leq C \). Thus we obtain

\[ \|u\|_{X_T} \leq \|u_0\|_{L^1_x} + C_T + \|u\|^2_{L^2_{t,x}([T,\infty])} \|u\|_{X_T}. \]  

(2.46)

Since \( \|u\|_{L^2_{t,x}([T,\infty])} < \infty \) by (1.12) and Sobolev embedding, then \( \|u\|_{L^2_{t,x}([T,\infty])} \to 0 \) as \( T \to \infty \).

Taking \( T > 0 \) sufficiently large, we get \( \|u\|_{X_T} \lesssim 1 \).

Now we show that \( \lim_{t \to +\infty} e^{-it\Delta}u(t) = \phi_+ \) in \( L^1 \). It suffices to show \( \int_0^\infty e^{-is\Delta}|u|^4uds \) is convergent in \( L^1 \), which can be obtained by the previous argument. Moreover, we have

\[ \|u(t) - e^{it\Delta}\phi_+\|_{L^\infty} \lesssim \left\| \int_t^\infty e^{i(t-s)\Delta}|u(s)|^4u(s)ds \right\|_{L^\infty} \]
\[ \lesssim \left\| \int_t^{2t} e^{i(t-s)\Delta}|u(s)|^4u(s)ds \right\|_{L^\infty} + \left\| \int_{2t}^\infty e^{i(t-s)\Delta}|u(s)|^4u(s)ds \right\|_{L^\infty} \]
\[ \lesssim \int_t^{2t} \left| e^{i(t-s)\Delta}|u(s)|^4u(s) \right|_{H^2} ds + \int_{2t}^\infty |t-s|^{-3/2} \|u(s)|^4u(s)\|_{L^1} ds \]
\[ \lesssim \int_t^{2t} \|u(s)|^4 \|_{L^\infty} \|u(s)\|_{H^2} ds + \int_{2t}^\infty |t-s|^{-3/2} \|u(s)|^3 \|u(s)\|_{L^2}^2 ds \lesssim t^{-5}. \]  

(2.47)
Conversely, given final data $\phi_+ \in H^3 \cap L^1$, by the classical results one can get a global solution $u \in C(\mathbb{R} : H^3) \cap L_{t,x}^{10}$ such that $e^{-it\Delta} u(t) \to \phi_+$ in $H^3$ as $t \to \infty$. Moreover,

$$u = e^{it\Delta} \phi_+ - i\mu \int_0^\infty e^{i(t-s)\Delta} |u(s)|^4 u(s) ds.$$  \hfill (2.48)

Repeating the previous arguments, we see $u \in X_T$ and $e^{-it\Delta} u(t) \to \phi_+$ in $L^1$ as $t \to \infty$. Thus the proof is completed.

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