Fuzzy Hermite-Hadamard type inequality for \( r \)-preinvex and \( (\alpha, m) \)-preinvex function

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Abstract

The purpose of this paper is to study the Hermite-Hadamard type inequality for \( r \)-preinvex and \( (\alpha, m) \)-preinvex function which is based on Sugeno integral.

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1 Introduction

In 1974 Sugeno stared study of theory of fuzzy integral [13]. It is very useful tool for various application in theoretical and applied statistics which is based on non-additive measures. Hanson [15] introduced a generalization of convex functions in terms of invex function. In [14], [18], [26] authors studied the basic properties and role of preinvex functions in optimization, equilibrium problems and variational inequalities.

The study of inequalities for Sugeno integral was initiated by Roman-Flores et.al., [1] [9]. H. Agahi et. al., [7], [8] proved the general Barnes-Godunova-Levin and new general extensions of Chebyshev type inequalities for Sugeno integrals. J. Caballero and K. Sadarangani [11], [12] studied the Cauchy-Schwarz type inequality and Chebyshev type inequality for Sugeno integrals. N. Okur [23] proved the Hermite-Hadamard type inequality for log-preinvex function using Sugeno integral. In [3] [4], [5], [25] authors proved the fuzzy Hermite-Hadamard type inequality for convex functions.

Motivated by above results in this paper we obtain the Hermite-Hadamard type inequality for \( r \)-preinvex function and \( (\alpha, m) \)-preinvex function with respect to Sugeno integral.

2 Preliminaries

In this section we give some basic definitions and properties of Sugeno integral, [13], [30], [33].
Suppose that \( \wp \) is a \( \sigma \)-algebra of subsets of \( X \) and \( \mu : \wp \rightarrow [0, \infty) \) be a non-negative, extended real valued set function. We say that \( \mu \) is a fuzzy measure if it satisfies:

1. \( \mu(\emptyset) = 0 \).
2. \( E, F \in \wp \) and \( E \subset F \) imply \( \mu(E) \leq \mu(F) \).
3. \( \{E_n\} \subseteq \wp, E_1 \subset E_2 \subset \ldots \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \).
4. \( \{E_n\} \subseteq \wp, E_1 \supset E_2 \supset \ldots, \mu(E_1) < \infty \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) \).

If \( f \) is non-negative real-valued function defined on \( X \), we denote the set \( \{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\} \) by \( F_\alpha \) for \( \alpha \geq 0 \). Note that if \( \alpha \leq \beta \) then \( F_\beta \subseteq F_\alpha \).

Let \((X, \wp, \mu)\) be a fuzzy measure space, we denote \( M^+ \) the set of all non-negative measurable functions with respect to \( \wp \).

**Definition 2.1** (Sugeno [13]). Let \((X, \wp, \mu)\) be a fuzzy measure space, \( f \in M^+ \) and \( A \in \wp \), the Sugeno integral (or fuzzy integral) of \( f \) on \( A \), with respect to the fuzzy measure \( \mu \), is defined as

\[
(s) \int_A f \, d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)],
\]

when \( A = X \),

\[
(s) \int_X f \, d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)],
\]

where \( \bigvee \) and \( \wedge \) denote the operations sup and inf on \([0, \infty)\), respectively.

The properties of Sugeno integral are well known and can be found in [34] as follows.

**Proposition 2.1** Let \((X, \wp, \mu)\) be fuzzy measure space, \( A, B \in \wp \) and \( f, g \in M^+ \) then:

1. \( (s) \int_A f \, d\mu \leq \mu(A) \).
2. \( (s) \int_A k \, d\mu = k \wedge \mu(A) \), \( k \) for non-negative constant.
3. \( (s) \int_A f \, d\mu \leq (s) \int_A g \, d\mu \), for \( f \leq g \).
4. \( \mu(A \cap \{f \geq \alpha\}) \geq \alpha \implies (s) \int_A f \, d\mu \geq \alpha \).
5. \( \mu(A \cap \{f \geq \alpha\}) \leq \alpha \implies (s) \int_A f \, d\mu \leq \alpha \).
6. \( (s) \int_A f \, d\mu > \alpha \iff \) there exists \( \gamma > \alpha \) such that \( \mu(A \cap \{f \geq \gamma\}) > \alpha \).
7. \( (s) \int_A f \, d\mu < \alpha \iff \) there exists \( \gamma < \alpha \) such that \( \mu(A \cap \{f \geq \gamma\}) < \alpha \).

**Remark 2.1** Consider the distribution function \( F \) associated to \( f \) on \( A \), that is, \( F(\alpha) = \mu(A \cap \{f \geq \alpha\}) \). Then due to (4) and (5) of Proposition 2.1, we have \( F(\alpha) = \alpha \implies (s) \int_A f \, d\mu = \alpha \). Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation \( F(\alpha) = \alpha \).
Let \( f : K \rightarrow \mathbb{R} \) and \( \eta(.,.) : K \times K \rightarrow \mathbb{R}^n \) be continuous functions, where \( K \subset \mathbb{R}^n \) is a nonempty closed set.

We use the notations, \( \langle .,. \rangle \) and \( ||.|| \), for inner product and norm, respectively. Now we give some definitions and condition which will be useful subsequent discussion.

**Definition 2.2** ([14, 26]) Let \( u \in K \). Then, the set \( K \) is said to be invex at \( u \in K \) with respect to \( \eta(.,.) \) if
\[
  u + t\eta(v, u) \in K, \quad \forall u, v \in K, t \in [0,1].
\] (2.1)

The invex set \( K \) is also called a \( \eta \) connected set.

Invex set has a clear geometric interpretation which says that there is a path starting from a point \( u \) which is contained in \( K \). It does not require that the point \( v \) should be one of the end points of the path. This observation plays an important role [27].

If \( v \) is an end point of the path for every pair of points, \( u, v \in K \), then \( \eta(v, u) = v - u \) and consequently invexity reduces to convexity. Thus every convex set is also an invex set with respect to \( \eta(v, u) = v - u \), but its converse is not necessarily true [26], [32].

**Definition 2.3** (T. Weir [26]). The function \( f \) on the invex set \( K \) is said to be preinvex with respect to \( \eta \) if
\[
  f(u + t\eta(v, u)) \leq (1-t)f(u) + tf(v), \quad \forall u, v \in K, t \in [0,1].
\] (2.2)

In [19], Mohan and Neogy has given the condition for function \( \eta \) known as,

**Condition C**: Let \( K \subseteq \mathbb{R} \) be an open invex subset to \( \eta : K \times K \rightarrow \mathbb{R} \). For any \( x, y \in K \) and any \( t \in [0,1] \),
\[
  \eta(y, y + t\eta(x, y)) = -t\eta(x, y),
  \eta(x, y + t\eta(x, y)) = (1-t)\eta(x, y).
\]

If for any \( x, y \in K \) and \( t_1, t_2 \in [0,1] \), we have
\[
  \eta(y + t_2\eta(x, y), y + t_1\eta(x, y)) = (t_2 - t_1)\eta(x, y).
\]

The concepts of the invex and preinvex functions have played very important roles in the development of generalized convex programming, see [2, 16, 17, 19, 20, 24].

In [28, 29], Antczak introduced the concept of \( r \)-invex and \( r \)-preinvex functions which as follows.

**Definition 2.4** A positive function \( f \) on the invex set \( K \) is said to be \( r \)-preinvex with respect to \( \eta \) if, for each \( u, v \in K \), \( t \in [0,1] \).
\[
  f(u + t\eta(v, u)) \leq \begin{cases} 
  ((1-t)f^r(u) + tf^r(v))^{1/r}, & r \neq 0 \\
  (f(u))^{1-t}(f(v))^t, & r = 0.
  \end{cases}
\]

Note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are classical preinvex functions. If \( f \) is \( r \)-preinvex function then \( f^r \) is preinvex function \( (r > 0) \).
The $m$-preinvex function is defined as

**Definition 2.5** [21] The function $f$ on the invex set $K \subseteq [0, b^*]$, $b^* > 0$ is said to be $m$-preinvex with respect to $\eta$ if

$$f(u + t\eta(v, u)) \leq (1 - t)f(u) + mtf\left(\frac{v}{m}\right),$$

holds for all $u, v \in K$, $t \in [0, 1]$ and $m \in (0, 1]$. The function $f$ is said to be $m$-preconcave if and only if $-f$ is $m$-preinvex.

**Definition 2.6** [21] The function $f$ on the invex set $K \subseteq [0, b^*]$, $b^* > 0$ is said to be $(\alpha, m)$-preinvex function with respect to $\eta$ if

$$f(u + t\eta(v, u)) \leq (1 - t^\alpha)f(u) + mt^{\alpha}f\left(\frac{v}{m}\right),$$

holds for all $u, v \in K$, $t \in [0, 1]$ and $(\alpha, m) \in (0, 1] \times (0, 1]$. The function $f$ is said to be $(\alpha, m)$-preincave if and only if $-f$ is $(\alpha, m)$-preinvex.

**Remark 2.2** [21] If we put $m = 1$ in Definition 2.5, then we get the definition of preinvexity. If we put $\alpha = m = 1$, then Definition 2.6 becomes the definition of preinvex function. Every $m$-preinvex function and $(\alpha, m)$-preinvex functions are $m$-convex and $(\alpha, m)$-convex with respect to $\eta(v, u) = v - u$ respectively.

### 3 Fuzzy Hadamard type inequality for $r$-preinvex function

Now in this section we give results obtained on Hadamard type inequality for $r$-preinvex function with respect to Sugeno integral.

In [31] W. Ul-Haq and J. Iqbal proved the following Hermite-Hadamard type inequality for $r$-preinvex function.

**Theorem 3.1** Let $f : K = [a, a + \eta(b, a)] \rightarrow (0, \infty)$ be an $r$-preinvex function on the interval of real numbers $K^\circ$ (interior of $K$) and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequalities holds

$$\frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \left[\frac{f'(a) + f'(b)}{2}\right]^{1/r}, \quad r \geq 1. \quad (3.1)$$

Now we consider an example.

**Example 3.1** Consider $X = [0, \eta(1, 0)]$ and let $\mu$ be the Lebesgue measure on $X$. If we take $f(x) = \frac{x^4}{2}$ be a non-negative and $\frac{1}{2}$-preinvex function on $[0, \eta(1, 0)]$. From Remark [2.7], we have

$$F(\beta) = \mu([0, \eta(1, 0)] \cap \{x|\frac{x^4}{2} \geq \beta\})$$

$$= \mu((2\beta)^{1/4}, \eta(1, 0))$$

$$= \eta(1, 0) - (2\beta)^{1/4}, \quad (3.2)$$
and the solution of (3.2) is

\[ \eta(1,0) - (2\beta)^{1/4} = \beta, \]

where \(0 \leq \eta(1,0) \leq 1\) to \(0 \leq \beta \leq 0.2023\), we get

\[ 0 \leq (s) \int_{0}^{\eta(1,0)} f d\mu = (s) \int_{0}^{\eta(1,0)} \frac{x^4}{2} d\mu \leq 0.2023. \]

\[ 0.2023 \leq \frac{1}{\eta(1,0)} (s) \int_{0}^{\eta(1,0)} \frac{x^4}{2} d\mu < \infty. \quad (3.3) \]

On the other hand

\[ \left\lceil \frac{f^{1/2}(0) + f^{1/2}(\eta(1,0))}{2} \right\rceil^2 = \frac{\eta(1,0)^4}{8} = \frac{1}{8} = 0.125, \quad (3.4) \]

where \(0 \leq \eta(1,0) \leq 1\).

From inequalities (3.3), (3.4) and Boolean operator on real numbers, it is seen that Hermite-Hadamard type inequality (3.1) is not valid in fuzzy context.

Now we give the Hermite-Hadamard type inequality for Sugeno integral with respect to \(r\)-preinvex function.

**Theorem 3.2** Let \(r > 0\) and \(\mu\) be the Lebesgue measure on \(\mathbb{R}\), \(f : [0, \eta(1,0)] \rightarrow [0, \infty)\) be \(r\)-preinvex function with \(f(0) \neq f(\eta(1,0))\).

**Case 1.** If \(f(\eta(1,0)) > f(0)\), then

\[ (s) \int_{0}^{\eta(1,0)} f d\mu \leq \min\{\beta, \eta(1,0)\}, \]

where \(\beta\) satisfies the following equation

\[ (f^r(\eta(1,0)) - f^r(0))\beta + \eta(1,0)\beta^r - \eta(1,0)f^r(\eta(1,0)) = 0. \quad (3.5) \]

**Case 2.** If \(f(0) > f(\eta(1,0))\), then

\[ (s) \int_{0}^{\eta(1,0)} f d\mu \leq \min\{\beta, \eta(1,0)\}, \]

where \(\beta\) satisfies the following equation

\[ (f^r(\eta(1,0)) - f^r(0)) - \beta^r \eta(1,0) + \eta(1,0)f^r(0) = 0. \quad (3.6) \]

**Proof.** As \(f\) is a \(r\)-preinvex function for \(x \in [0, \eta(1,0)]\) we have

\[ f(0 + x\eta(\eta(1,0))) \leq (1 - x)f^r(0) + xf^r(\eta(1,0)))^{1/r}, \]

and from Condition C, we have

\[ \eta(\eta(1,0), 0) = \eta(0 + 1.\eta(1,0), 0 + 0.\eta(1,0)) = \eta(1,0). \]
Therefore,

\[
  f(x) = f\left(\frac{x}{\eta(1,0) \cdot \eta(1,0)}\right) \\
  \leq \left[\left(1 - \frac{x}{\eta(1,0)}\right)f^r(0) + \left(\frac{x}{\eta(1,0)}\right)f^r(\eta(1,0))\right]^{1/r} \\
  = g(x).
\]

By Proposition (2.1), we have

\[
  (s) \int_0^{\eta(1,0)} f(x) d\mu = (s) \int_0^{\eta(1,0)} f\left(\frac{x}{\eta(1,0) \cdot \eta(1,0)}\right) d\mu \\
  \leq (s) \int_0^{\eta(1,0)} \left[\left(1 - \frac{x}{\eta(1,0)}\right)f^r(0) + \left(\frac{x}{\eta(1,0)}\right)f^r(\eta(1,0))\right]^{1/r} d\mu \\
  = (s) \int_0^{\eta(1,0)} g(x) d\mu. \tag{3.7}
\]

To calculate the right hand side of (3.7), we consider the distribution function \( F \) given by

\[
  F(\beta) = \mu\left([0, \eta(1,0)] \cap \left\{ x \mid g(x) \geq \beta \right\}\right) \\
  = \mu\left([0, \eta(1,0)] \cap \left\{ x \left[\left(1 - \frac{x}{\eta(1,0)}\right)f^r(0) + \left(\frac{x}{\eta(1,0)}\right)f^r(\eta(1,0))\right]^{1/r} \geq \beta \right\}\right). \tag{3.8}
\]

**Case 1.** If \( f(\eta(1,0)) > f(0) \), then from (3.8), we have

\[
  F(\beta) = \mu\left([0, \eta(1,0)] \cap \left\{ x \mid x \geq \eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)} \right\}\right) \\
  = \mu\left(\eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)}, \eta(1,0)\right) \\
  = \eta(1,0) - \eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)}. \tag{3.9}
\]

and the solution of (3.9) is \( F(\beta) = \beta \), where \( \beta \) satisfies the following equation

\[
  \beta(f^r(\eta(1,0)) - f^r(0)) + \beta^r \eta(1,0) - \eta(1,0)f^r(\eta(1,0)) = 0.
\]

By Proposition (2.1) and Remark (2.1), we have

\[
  (s) \int_0^{\eta(1,0)} f(x) d\mu \leq \min\{\beta, \eta(1,0)\}.
\]
Case 2. If \( f(0) > f(\eta(1,0)) \), then from (3.8), we have

\[
F(\beta) = \mu \left( [0, \eta(1,0)] \cap \left\{ x \mid x \leq \eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)} \right\} \right)
\]

\[
= \mu \left( 0, \eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)} \right)
\]

\[
= \eta(1,0) \frac{\beta^r - f^r(0)}{f^r(\eta(1,0)) - f^r(0)},
\]

and the solution (3.10) is \( F(\beta) = \beta \), where \( \beta \) satisfies the following equation

\[
\beta (f^r(\eta(1,0)) - f^r(0)) - \eta(1,0) \beta^r + \eta(1,0) f^r(0) = 0.
\]

By Proposition 2.1 and Remark 2.1, we have

\[
(s) \int_0^{\eta(1,0)} f(x) dx \leq \min \{ \beta, \eta(1,0) \}.
\]

Example 3.2 Consider \( X = [0, \eta(1,0)] \) and let \( \mu \) be the Lebesgue measure on \( X \). If we take \( f(x) = \frac{x^3}{3} \) be the \( \frac{1}{2} \)-preinvex function, where \( 0 \leq \eta(1,0) \leq 1 \) from Remark 2.1, we have

\[
(s) \int_0^{\eta(1,0)} \frac{x^3}{3} d\mu = 0.1847.
\]

From Theorem 3.2 we have

\[
0.1847 = (s) \int_0^{\eta(1,0)} \frac{x^3}{3} d\mu \leq \min \{ 0.2087, \eta(1,0) \} = 0.2087.
\]

Remark 3.1 In case if we take \( f(0) = f(\eta(1,0)) \) in Theorem 3.2, then we get

\[
(s) \int_0^{\eta(1,0)} f(x) dx \leq (s) \int_0^{\eta(1,0)} g(x) dx = (s) \int_0^{\eta(1,0)} f(0) d\mu = f(0) \land \eta(1,0).
\]

Now we obtain the results for inequality on \( r \)-preinvex function.

Theorem 3.3 Let \( r > 0 \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \), \( f : [a, a + \eta(b,a)] \rightarrow [0, \infty) \) be \( r \)-preinvex function with \( f(a) \neq f(a + \eta(b,a)) \).

Case 1. If \( f(a + \eta(b,a)) > f(a) \), then

\[
(s) \int_a^{a+\eta(b,a)} f(x) dx \leq \min \{ \beta, \eta(b,a) \},
\]

where \( \beta \) satisfies the following equation

\[
\beta (f^r(a + \eta(b,a)) - f^r(a)) + \beta^r \eta(b,a) - \eta(b,a) f^r(a + \eta(b,a)) = 0.
\]
**Case 2.** If \( f(a + \eta(b, a)) < f(a) \), then

\[
(s) \int_{a}^{a+\eta(b,a)} f(x) d\mu \leq \min\{\beta, \eta(b, a)\},
\]

where \( \beta \) satisfies the following equation

\[
\beta(f^r(a + \eta(b, a)) - f^r(a)) - \beta^r \eta(b, a) + \eta(b, a) f^r(a) = 0.
\]

(3.12)

**Remark 3.2** If we take \( f(a) = f(a + \eta(b, a)) \) in Theorem 3.3, we have \( g(x) = f(a) \) and by Proposition 2.1, we have

\[
(s) \int_{a}^{a+\eta(b,a)} f(x) d\mu \leq (s) \int_{a}^{a+\eta(b,a)} g(x) d\mu
\]

\[
= (s) \int_{a}^{a+\eta(b,a)} f(a) d\mu
\]

\[
= f(a) \wedge \mu([a, a + \eta(b, a)]).
\]

**Corollary 3.1** Let \( r < 0 \), and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Let \( f : [a, a + \eta(b, a)] \rightarrow [0, \infty) \) be the \( r \)-preinvex with \( f(a) \neq f(a + \eta(b, a)) \), then

**Case 1.** If \( f(a + \eta(b, a)) > f(a) \), then

\[
(s) \int_{a}^{a+\eta(b,a)} f(x) d\mu \leq \min\{\beta, \eta(b, a)\},
\]

where \( \beta \) satisfies the following equation

\[
\beta(f^r(a + \eta(b, a)) - f^r(a)) - \beta^r \eta(b, a) + \eta(b, a) f^r(a) = 0.
\]

(3.13)

**Case 2.** If \( f(a + \eta(b, a)) < f(a) \), then

\[
(s) \int_{a}^{a+\eta(b,a)} f(x) d\mu \leq \min\{\beta, \eta(b, a)\},
\]

where \( \beta \) satisfies the following equation

\[
\beta(f^r(a + \eta(b, a)) - f^r(a)) + \beta^r \eta(b, a) - \eta(b, a) f^r(a + \eta(b, a)) = 0.
\]

(3.14)

4 **Fuzzy Hermite-Hadamard type inequality for \((\alpha, m)\)-preinvex function**

Now in this section we give the Hermite-Hadamard type inequality for \((\alpha, m)\)-preinvex function. In [22] Noor proved following the Hermite-Hadamard type inequality for preinvex functions.

**Theorem 4.1** Let \( f : [a, a + \eta(b, a)] \rightarrow (0, \infty) \) be a preinvex function on the interval of real numbers \( K^\circ \) (the interior of \( K \)) and \( a, b \in K^\circ \) with \( a < a + \eta(b, a) \). Then the following inequality holds,

\[
f\left(\frac{2a + \eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b,a)} f(x) dx \leq \frac{f(a) + f(b)}{2}.
\]

(4.1)
Now consider an examples.

**Example 4.1** Consider \( X = [0, \eta(1, 0)] \) and let \( \mu \) be the Lebesgue measure on \( X \). If we take the function \( f(x) = \frac{x^2}{2} \) then \( f(x) \) is \((1/2, 1/3)\)-preinvex function. From Remark 2.7 we have

\[
F(\beta) = \mu([0, \eta(1, 0)] \cap \{x | \frac{x^2}{2} \geq \beta\})
= \eta(1, 0) - \sqrt{2\beta}, \tag{4.2}
\]

and the solution of (4.2) is
\[
\eta(1, 0) - \sqrt{2\beta} = \beta,
\]
where \( 0 \leq \eta(1, 0) \leq 1 \) to \( 0 \leq \beta \leq 0.2679 \), we get
\[
0 \leq (s) \int_0^{\eta(1, 0)} f d\mu = (s) \int_0^{\eta(1, 0)} \frac{x^2}{2} d\mu \leq 0.2679.
\]

\[
0.2679 \leq \frac{1}{\eta(1, 0)} (s) \int_0^{\eta(1, 0)} \frac{x^2}{2} d\mu < \infty.
\]

From right hand side of (4.1) and for \( 0 \leq \eta(1, 0) \leq 1 \), we have
\[
\frac{f(0) + f(\eta(1, 0))}{2} = 0.25.
\]

This proves that the right hand side of (4.1) is not satisfied for Sugeno integral.

**Example 4.2** Consider \( X = [0, \eta(1, 0)] \) and let \( \mu \) be the Lebesgue measure on \( X \). If we take the function \( f(x) = 3x^2 \) then \( f(x) \) is \((1/2, 1/3)\)-preinvex function. From Remark 2.7 we have

\[
F(\beta) = \mu([0, \eta(1, 0)] \cap \{x | 3x^2 \geq \beta\})
= \eta(1, 0) - \sqrt{\frac{\beta}{3}}, \tag{4.3}
\]

and the solution of (4.3) is
\[
\eta(1, 0) - \sqrt{\frac{\beta}{3}} = \beta,
\]
where \( 0 \leq \eta(1, 0) \leq 1 \) and \( 0 \leq \beta \leq 0.5657 \), we get
\[
0 \leq (s) \int_0^{\eta(1, 0)} f d\mu = (s) \int_0^{\eta(1, 0)} 3x^2 d\mu \leq 0.5657.
\]

\[
0.5657 \leq \frac{1}{\eta(1, 0)} (s) \int_0^{\eta(1, 0)} 3x^2 d\mu < \infty.
\]

From left hand side of (4.1) and for \( 0 \leq \eta(1, 0) \leq 1 \), we have
\[
f \left( \frac{2a + \eta(b, a)}{2} \right) = 0.75.
\]

This proves that the left hand side of (4.1) is not satisfied for Sugeno integral.
Now in next theorem we prove Hermite-Hadamard type inequality for Sugeno integral with respect to \((\alpha, m)\)-preinvex function.

**Theorem 4.2** Let \( f : [0, \eta(1,0)] \to [0, \infty) \) be \((\alpha, m)\)-preinvex function, \((\alpha, m) \in (0,1)^2\), \( f(0) \leq f(\eta(1,0)) \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

\[
\left( s \right) \int_0^{\eta(1,0)} f(x) \, d\mu \leq \min \{ \beta, \eta(1,0) \},
\]

where \( \beta \) satisfies the following equation

\[
(\eta(1,0) - \beta)^\alpha mf\left(\frac{\eta(1,0)}{m}\right) - (\eta(1,0) - \beta)^\alpha f(0) - \eta(1,0)^\alpha (\beta - f(0)) = 0. \tag{4.4}
\]

Now we give the Hermite-Hadamard type inequality for decreasing \((\alpha, m)\)-preinvex function.

**Theorem 4.3** Let \( f : [0, \eta(1,0)] \to [0, \infty) \) be \((\alpha, m)\)-preinvex function, \((\alpha, m) \in (0,1)^2\), \( f(0) > f(\eta(1,0)) \) and \( \mu \) be the Lebesgue measure on \( \mathbb{R} \). Then

**Case 1.** If \( m \in (0, \frac{f(\eta(1,0))}{f(0)}) \), then

\[
\left( s \right) \int_0^{\eta(1,0)} f(x) \, d\mu \leq \min \{ \beta, \eta(1,0) \},
\]

where \( \beta \) satisfies the following equation

\[
(\eta(1,0) - \beta)^\alpha mf\left(\frac{\eta(1,0)}{m}\right) - (\eta(1,0) - \beta)^\alpha f(0) - \eta(1,0)^\alpha (\beta - f(0)) = 0. \tag{4.5}
\]

**Case 2.** If \( m = \frac{f(\eta(1,0))}{f(0)} \), then

\[
\left( s \right) \int_0^{\eta(1,0)} f(x) \, d\mu \leq \min \{ \beta, \eta(1,0) \},
\]

where \( \beta \) satisfies the following equation

\[
(\eta(1,0) - \beta)^\alpha \frac{f(\eta(1,0))}{f(0)} f\left(\frac{f(0)\eta(1,0)}{f(\eta(1,0))}\right) - (\eta(1,0) - \beta)^\alpha f(0) - \eta(1,0)^\alpha (\beta - f(0)) = 0. \tag{4.6}
\]

**Case 3.** If \( m \in (\frac{f(\eta(1,0))}{f(0)}, \eta(1,0)) \), then

\[
\left( s \right) \int_0^{\eta(1,0)} f(x) \, d\mu \leq \min \{ \beta, \eta(1,0) \},
\]

where \( \beta \) satisfies the following equation

\[
\beta^\alpha \left( mf\left(\frac{\eta(1,0)}{m}\right) - f(0) \right) - \eta(1,0)^\alpha (\beta - f(0)) = 0. \tag{4.7}
\]

Now we give the general case of Theorem 4.2 and 4.3.
Theorem 4.4 Let $f : [a, a + \eta(b, a)] \to [0, \infty)$ be $(\alpha, m)$-preinvex function, $(\alpha, m) \in (0, 1)^2$, $f(a) \leq f(a + \eta(b, a))$ and $\mu$ be the Lebesgue measure on $\mathbb{R}$. Then
\[
(s) \int_{a}^{a+\eta(b,a)} f(x) \, d\mu \leq \min\{\beta, \eta(b,a)\},
\]
where $\beta$ satisfies the following equation
\[
(\eta(b,a) - \beta)\alpha f\left(\frac{a + \eta(b,a)}{m}\right) - (\eta(b,a) - \beta)^{\alpha} f(a) - \eta(b,a)^{\alpha}(\beta - f(a)) = 0. \quad (4.8)
\]

Theorem 4.5 Let $f : [a, a + \eta(b, a)] \to [0, \infty)$ be $(\alpha, m)$-preinvex function, $(\alpha, m) \in (0, 1)^2$, $f(a) > f(a + \eta(b, a))$ and $\mu$ be the Lebesgue measure on $\mathbb{R}$. Then

Case 1. If $m \in (0, \frac{f(a + \eta(b,a))}{f(a)})$, then
\[
(s) \int_{a}^{a+\eta(b,a)} f(x) \, d\mu \leq \min\{\beta, \eta(b,a)\},
\]
where $\beta$ satisfies the following equation
\[
(\eta(b,a) - \beta)\alpha f\left(\frac{a + \eta(b,a)}{m}\right) - (\eta(b,a) - \beta)^{\alpha} f(a) - \eta(b,a)^{\alpha}(\beta - f(a)) = 0. \quad (4.9)
\]

Case 2. If $m = \frac{f(a + \eta(b,a))}{f(a)}$, then
\[
(s) \int_{a}^{a+\eta(b,a)} f(x) \, d\mu \leq \min\{\beta, \eta(b,a)\},
\]
where $\beta$ satisfies the following equation
\[
(\eta(b,a) - \beta)\alpha f\left(\frac{a + \eta(b,a)}{f(a)}\right) f\left(\frac{a + \eta(b,a)f(a)}{f(a + \eta(b,a))}\right) - (\eta(b,a) - \beta)^{\alpha} f(a) - \eta(b,a)^{\alpha}(\beta - f(a)) = 0. \quad (4.10)
\]

Case 3. If $m \in (\frac{f(a + \eta(b,a))}{f(a)}, \eta(1,0))$, then
\[
(s) \int_{a}^{a+\eta(b,a)} f(x) \, d\mu \leq \min\{\beta, \eta(b,a)\},
\]
where $\beta$ satisfies the following equation
\[
\beta^\alpha (mf\left(\frac{a + \eta(b,a)}{m}\right)) - \beta^\alpha f(a) - \eta(b,a)^\alpha(\beta - f(a)) = 0. \quad (4.11)
\]

Similarly we can prove the above results as Theorem 3.2, 3.3.

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