ASYMPTOTIC HOMOLOGICAL CONJECTURES IN MIXED CHARACTERISTIC

HANS SCHOUTENS

1. INTRODUCTION

In the last three decades, all the so-called Homological Conjectures have been settled completely for Noetherian local rings containing a field by work of Szpiro-Peskine, Hochster-Roberts, Hochster, Evans-Griffith, et. al. (some of the main papers are [9, 13, 14, 20, 26]). More recently, Hochster-Huneke have given more simplified proofs of most of these results by means of their tight closure theory, including their canonical construction of big Cohen-Macaulay algebras in positive characteristic (see [17, 18, 19, 22]; for further discussion and proofs, see [6, 89] or [45]).

In sharp contrast is the development in mixed characteristic, where only sporadic results (often in low dimensions) are known, apart from a single break-through [28] by Roberts, settling the New Intersection Theorem for all Noetherian local rings. Some attempts have been made by Hochster, either by finding a suitable substitute for tight closure in mixed characteristic [15], or by finding big Cohen-Macaulay modules in mixed characteristic [12]. These approaches have yet to bear fruit and the best result to date in this direction is the existence of big Cohen-Macaulay algebras in dimension three [16], which in turn relies on the positive solution of the Direct Summand Conjecture in dimension three by Heitmann [11].

In this paper, we will follow the big Cohen-Macaulay algebra approach, but instead of trying to work with rings of Witt vectors, we will use the Ax-Kochen-Ershov Principle [4, 7, 8], linking complete discrete valuation rings in mixed characteristic with complete discrete valuation rings in positive characteristic via an equicharacteristic zero (non-discrete) valuation ring (see Theorem 2.3 below). This intermediate valuation ring is obtained by a construction which originates from logic, but is quite algebraic in nature, to wit, the ultraproduct construction. Roughly speaking, this construction associates to an infinite collection of rings $C_w$ their ultraproduct $C_\infty$, which should be thought of as a kind of “limit” or “average” (realized as a certain homomorphic image of the product). An ultraproduct inherits many of the algebraic properties of its components. The correct formulation of this transfer principle is Łos’ Theorem, which makes precise when a property carries over (namely, when it is first order definable in some suitable language). Properties that carry over are those of being a domain, a field, a valuation ring, local, Henselian; among the properties that do not carry over is Noetherianity, so that almost no ultraproduct...
is Noetherian (except an ultraproduct of fields or of Artinian rings of bounded length). This powerful tool is used in \[30, 31, 33, 38,\] to obtain uniform bounds in polynomial rings over fields; in \[33, 34, 50, 42,\] to transfer properties from positive to zero characteristic; and in \[39, 40, 41,\] to give an alternative treatment of tight closure theory in equicharacteristic zero. The key fact in the first set of papers is a certain flatness result about ultraproducts (see Theorem 2.2 below for a precise formulation), and in the two last sets, the so-called Lefschetz Principle for algebraically closed fields (the ultraproduct of the algebraic closures of the \(p\)-element fields \(\mathbb{F}_p\) is isomorphic to \(\mathbb{C}\)).

The Ax-Kochen-Ershov Principle is a kind of Lefschetz Principle for Henselian valued fields, and its most concrete form states that the ultraproduct of all \(\mathbb{F}_p[[t]]\), with \(t\) a single variable, is isomorphic to the ultraproduct of all rings of \(p\)-adic integers \(\mathbb{Z}_p\). We will identify both ultraproducts and denote the resulting ring by \(\mathfrak{O}\). It follows that \(\mathfrak{O}\) is an equicharacteristic zero Henselian valuation ring with principal maximal ideal, whose separated quotient (=the reduction modulo the intersection of all powers of the maximal ideal) is an equicharacteristic zero excellent complete discrete valuation ring.

To explain the underlying idea in this paper, for a (not necessarily Noetherian) local ring \((\mathbb{Z}, \mathfrak{p})\), let us denote by \(\text{Aff}(\mathbb{Z})\) the category of local \(\mathbb{Z}\)-affine algebras, that is to say, \(\mathbb{Z}\)-algebras of the form \((\mathbb{Z}[X]/I)_m\), with \(X\) a finite tuple of variables, \(I\) a finitely generated ideal and \(m\) a prime ideal containing \(\mathfrak{p}\) and \(I\). The objective is to transfer algebraic properties (such as the homological Conjectures) from the positive characteristic categories \(\text{Aff}(\mathbb{F}_p[[t]])\) to the mixed characteristic categories \(\text{Aff}(\mathbb{Z}_p)\). This will be achieved through the intermediate equicharacteristic zero category \(\text{Aff}(\mathfrak{O})\). As this latter category consists mainly of non-Noetherian rings, we will have to find analogues in this setting of many familiar notions from commutative algebra, such as dimension, depth, Cohen-Macaulayness or regularity (see §§5 and 6).

The following example is paradigmatic: let \(X\) be a finite tuple of variables and let \(A^\infty\) be the ultraproduct of all \(\mathbb{F}_p[[t]][X]\), and \(A^\infty_{\text{max}}\), the ultraproduct of all \(\mathbb{Z}_p[X]\). Note that both rings contain \(\mathfrak{O}\), and in fact, contain \(\mathfrak{O}[X]\). The key algebraic fact, which follows from a result on effective bounds by Aschenbrenner, is that both inclusions are flat. Suppose we have in each \(\mathbb{F}_p[[t]][X]\) a polynomial \(f_p\), and let \(f_\infty\) be their ultraproduct. A priori, we can only say that \(f_\infty \in A^\infty_{\text{max}}\). However, if all \(f_p\) have degree \(d\), for some \(d\) independent from \(p\), then \(f_\infty\) itself is a polynomial over \(\mathfrak{O}\) of degree \(d\) (since an ultraproduct commutes with finite sums by Łos’ Theorem). Hence, as \(f_\infty\) lies in \(\mathfrak{O}[X]\), we can also view it as an element in \(A^\infty_{\text{max}}\). Therefore, there are polynomials \(f_p \in \mathbb{Z}_p[X]\) whose ultraproduct is equal to \(f_\infty\). The choice of the \(f_p\) is not unique, but any two choices will be equal for almost \(p\), by Łos’ Theorem. In conclusion, to a collection of polynomials defined over the various \(\mathbb{F}_p[[t]]\), of uniformly bounded degree, we can associate, albeit not uniquely, a collection of polynomials defined over the various \(\mathbb{Z}_p\) (of uniformly bounded degree), and of course, this also works the other way. Instead of doing this for just one polynomial in each component, we can now do this for a finite tuple of polynomials of fixed length. If at the same time, we can maintain certain algebraic relations among them (characterizing one of the properties we seek to transfer), we will have achieved our goal.

Unfortunately, it is the nature of an ultraproduct that it only captures the “average” property of its components. In the present context, this means that the desired property does not necessarily hold in all \(\mathbb{Z}_p[X]\), but only in almost all. In conclusion, we cannot hope for a full solution of the Homological Conjectures by this method, but only an asymptotic solution. In view of the above, the following definition is natural.
Complexity. Let $Z \to C$ be a local homomorphism between Noetherian local rings. We say that $C$ has $Z$-complexity at most $c$, if we can write $C$ as $(Z[X]/I)_m$ (so that $C$ is a member of $\text{Aff}(Z)$), with $X$ a tuple of at most $c$ variables and $m$ a prime ideal containing $I$ and the maximal ideal of $Z$, such that $I$ and $m$ are generated by polynomials of degree at most $c$.

An element $r \in C$ is said to have $Z$-complexity at most $c$, if $C$ has $Z$-complexity at most $c$ and if there exist polynomials $P, Q \in Z[X]$ of degree at most $c$ with $Q \notin m$, such that $r = P/Q$ in $C$. Sometimes we might say that a tuple or a matrix has $Z$-complexity at most $c$, to indicate that each entry has $Z$-complexity at most $c$ and the dimensions are also bounded by $c$.

An ideal $J$ of $C$ has $Z$-complexity at most $c$, if $C$ has $Z$-complexity at most $c$ and $J$ is generated by elements of $Z$-complexity at most $c$. A homomorphism $C \to D$ is said to have $Z$-complexity at most $c$, if both $C$ and $D$ have $Z$-complexity at most $c$ and the homomorphism is given by sending $X_i$ to an element in $S$ of $Z$-complexity at most $c$ (where $C = (Z[X]/I)_m$).

Asymptotic Properties. Let $\mathcal{P}$ be a property of Noetherian local rings (possibly involving some additional data). We will use the phrase $\mathcal{P}$ holds asymptotically in mixed characteristic, to express that for each $c$, we can find a bound $c'$, such that if $V$ is a complete discrete valuation ring of mixed characteristic and $C$ a $V$-algebra of $V$-complexity at most $c$ (and a similar bound on the data), then property $\mathcal{P}$ holds for $C$, provided the characteristic of the residue field of $V$ is at least $c'$. Sometimes, we have to control some additional invariants in terms of the bound $c$. In this paper, we will prove that in this sense, many Homological Conjectures hold asymptotically in mixed characteristic.

A Final Note. Its asymptotic nature is the main impediment of the present method to carry out Hochster’s program of obtaining tight closure and big Cohen-Macaulay algebras in mixed characteristic. For instance, despite the fact that we are able to define an analogue of a balanced big Cohen-Macaulay for $\mathcal{D}$-affine domains, this object cannot be realized as an ultraprodut of $\mathbb{Z}_p$-algebras, so that there is no candidate so far for a big Cohen-Macaulay in mixed characteristic. Although I will not pursue this line of thought in this paper, one could also define some non-standard closure operation on ideals in $\mathcal{D}$-affine algebras, but again, such an operation will only partially descend to any component.

Notation. A tuple $x$ over a ring $Z$ is always understood to be finite. Its length is denoted by $|x|$ and the ideal it generates is denoted $xZ$. When we say that $(Z, p)$ is local, we mean that $p$ is its maximal ideal, but we do not imply that $Z$ has to be Noetherian.

For a survey of the results and methods in this paper, see also [37].

2. Ultraproducts

In this preliminary section, I state some generalities about ultraproducts and then briefly review the situation in equicharacteristic zero and the Ax-Kochen-Ershov Principle. The next section lays out the essential tools for conducting the transfer discussed in the introduction, to wit, approximations, restricted ultraproducts and non-standard hulls, whose properties are then studied in §5 and §6. The subsequent sections contain proofs of various asymptotic results, using these tools.

Whenever we have an infinite index set $W$, we will equip it with some (unnamed) non-principal ultrafilter; ultraproducts will always be taken with respect to this ultrafilter and we will write $\ulim_{w \to \infty} O_w$ or simply $O_\infty$ for the ultraproduct of objects $O_w$ (this will apply to rings, ideals and elements alike). A first introduction to ultraproducts, including
Łos’ Theorem, sufficient to understand the present paper, can be found in [41 §2]; for a more detailed treatment, see [21]. Łos’ Theorem states essentially that if a fixed algebraic relation holds among finitely many elements $f_{1w}, \ldots, f_{sw}$ in each ring $C_{w}$, then the same relation holds among their ultraproducts $f_{1\infty}, \ldots, f_{s\infty}$ in the ultraproduct $C_{\infty}$, and conversely, if such a relation holds in $C_{\infty}$, then it holds in almost all $C_{w}$. Here almost all means “for all $w$ in a subset of the index set which belongs to the ultrafilter” (the idea is that sets belonging to the ultrafilter are large, whereas the remaining sets are small).

An immediate, but important application of Łos’ Theorem is that the ultraproduct of algebraically closed fields of different prime characteristics is an (uncountable) algebraically closed field of characteristic zero, and each uncountable algebraically closed field of characteristic zero, including $\mathbb{C}$, can be realized thus. This simple observation, in combination with work of van den Dries on non-standard polynomials (see below), was exploited in [41] to define an alternative version of tight closure for a field by Łos’ Theorem. Let $A_{\infty}$ be a fixed finite tuple of variables and set $A := C_{\infty}[X]$ and $A_{w} := K_{w}[X]$. Let $A_{\infty}$ be the ultraproduct of the $A_{w}$. As in the example discussed in the introduction, we have a canonical embedding of $A$ inside $A_{\infty}$. In fact, the following easy observation, valid over arbitrary rings, describes completely the elements in $A_{\infty}$ that lie in $A$ (the proof is straightforward and left to the reader).

**2.1. Lemma.** Let $X$ be a finite tuple of variables. Let $C_{w}$ be rings and let $C_{\infty}$ be their ultraproduct. If $f_{w}$ is a polynomial in $C_{w}[X]$ of degree at most $c$, for each $w$ and for some $c$ independent from $w$, then their ultraproduct in $\lim_{w \to \infty} C_{w}[X]$ belongs already to the subring $C_{\infty}[X]$, and conversely, every element in $C_{\infty}[X]$ is obtained in this way.

This result also motivates the notion of complexity from the introduction. Returning to the Schmidt-van den Dries results, the following two properties of the embedding $A \subset A_{\infty}$ do not only imply the uniform bounds from [31] [33], but play also an important theoretical role in the development of non-standard tight closure [41].

**2.2. Theorem** (Schmidt-van den Dries). The embedding $A \subset A_{\infty}$ is faithfully flat and every prime ideal in $A$ extends to a prime ideal in $A_{\infty}$.

To carry out the present program, we have to replace the base fields $K_{w}$ by complete discrete valuation rings $\mathcal{O}_{w}$. Unfortunately, we now have to face the following complications. Firstly, the ultraproduct $\mathcal{O}_{\infty}$ of the $\mathcal{O}_{w}$ is no longer Noetherian, and so in particular the corresponding $A := \mathcal{O}_{\infty}[X]$ is non-Noetherian. Moreover, the embedding $A \subset A_{\infty}$, where $A_{\infty}$ is now the ultraproduct of the $A_{w} := \mathcal{O}_{w}[X]$, although flat (see Theorem 4.2 below), is no longer faithfully flat (this is related to Dedekind’s problem; see [2] or [44] for details). Moreover, not every prime ideal extends to a prime ideal, and in order to preserve this, we will have to work locally (see Remark 4.3).

To address the first of these problems, we will realize $\mathcal{O}_{\infty}$ in two different ways, as an ultraproduct of complete discrete valuation rings in positive characteristic and as an ultraproduct of complete discrete valuation rings in mixed characteristic, and then pass from one set to the other via $\mathcal{O}_{\infty}$, as explained in the introduction. This is the celebrated Ax-Kochen-Ershov Principle [3] [7] [8], and I will discuss this now. For each $w$, let $\mathcal{O}_{w}^{\text{mix}}$ be a complete discrete valuation ring of mixed characteristic with residue field $\kappa_{p}$ of characteristic $p$. To each $\mathcal{O}_{w}^{\text{mix}}$, we associate a corresponding equicharacteristic complete discrete
valuation ring with the same residue field, by letting
\[ D_p^{eq} := \kappa_p[[t]] \]
where \( t \) is a single variable.

2.3. Theorem (Ax-Kochen-Ershov). The ultraproduct of the \( D_p^{eq} \) is isomorphic (as a local ring) with the ultraproduct of the \( D_p^{\text{mix}} \).

2.4. Remark. As stated, we need to assume the continuum hypothesis. Otherwise, by the Keisler-Shelah Theorem [21 Theorem 9.5.7], one might need to take further ultrapowers, that is to say, take a larger index set endowed with a (non-\( \omega \)-complete) non-principal ultrafilter. In order to not complicate the exposition, I will nonetheless make the set-theoretic assumption, so that our index set can always be taken to be the set of prime numbers.

To conclude this section, I state a variant of Prime Avoidance which will be used in the form discussed in the remark following it.

2.5. Proposition (Controlled Ideal Avoidance). Let \( Z \) be a local ring with infinite residue field \( \kappa \) and let \( C \) be an arbitrary \( Z \)-algebra. Let \( I := (f_1, \ldots, f_n)C \) be a finitely generated ideal in \( C \) and let \( \mathfrak{a}_1, \ldots, \mathfrak{a}_t \) be arbitrary ideals in \( C \) not containing \( I \). If \( W \) denotes the \( Z \)-submodule of \( C \) generated by the \( f_i \), then we can find \( f \in W \) not contained in any of the \( \mathfrak{a}_i \).

Proof. We induct on the number \( t \) of ideals to be avoided. If \( t = 1 \), then some \( f_i \not\in \mathfrak{a}_1 \), since \( I \subsetneq \mathfrak{a}_1 \). Hence assume \( t > 1 \). By induction, we can find elements \( g_1 \in W \), for \( i = 1, 2 \), which lie outside any \( \mathfrak{a}_j \) for \( j \neq i \). If either \( g_1 \not\in \mathfrak{a}_1 \) or \( g_2 \not\in \mathfrak{a}_2 \), we are done, so assume \( g_1 \in \mathfrak{a}_1 \). Therefore, every element of the form \( g_1 + zg_2 \) with \( z \) a unit in \( Z \) does not lie in \( \mathfrak{a}_1 \) nor in \( \mathfrak{a}_2 \). Since \( \kappa \) is infinite, we can find \( t-1 \) units \( z_1, z_2, \ldots, z_{t-1} \) in \( Z \) whose residues in \( \kappa \) are all distinct. I claim that at least one of the \( g_1 + z_i g_2 \) lies outside all \( \mathfrak{a}_j \), so that we found our desired element in \( W \). Indeed, if not, then each \( g_1 + z_i g_2 \) lies in one of the \( t-1 \) ideals \( \mathfrak{a}_3, \ldots, \mathfrak{a}_t \), by our previous remark. By the Pigeon Hole Principle, for some \( j \) and some \( l \neq k \), we have that \( g_1 + z_k g_2 \) and \( g_1 + z_l g_2 \) both lie in \( \mathfrak{a}_j \). Hence so does their difference \((z_k - z_l)g_2\). However, \( z_k - z_l \) is a unit in \( Z \), by choice of the \( z_i \), so that \( g_2 \in \mathfrak{a}_j \), contradiction. \( \square \)

2.6. Remark. In particular, if \( Z \) is Noetherian and both \( C \) and \( I \) have \( Z \)-complexity at most \( c \), then we can find an element \( f \in I \) of \( Z \)-complexity at most \( c \), outside any finite set of ideals not containing \( I \). Indeed, every element in the module \( W \) has \( Z \)-complexity at most \( c \).

3. Approximations, Restricted Ultraproducts and Non-standard Hulls

In this section, some general results on ultraproductions of finitely generated algebras over discrete valuation rings will be derived. We start with introducing some general terminology, over arbitrary Noetherian local rings, but once we start proving some non-trivial properties in the next sections, we will specialize to the case that the base rings are discrete valuation rings. For some results in the general case, we refer to [18, 44].

For each \( w \), we fix a Noetherian local ring \( \mathcal{O}_w \) and let \( \mathcal{O} \) be its ultraproduct. If the \( \mathfrak{p}_w \) are the maximal ideals of the \( \mathcal{O}_w \), then their ultraproduct \( \mathfrak{p} \) is the maximal ideal of \( \mathcal{O} \). We will write \( \varpi \) for the ideal of \emph{infinitesimals} of \( \mathcal{O} \), that is to say, the intersection of all the powers \( \mathfrak{p}^k \) (note that in general \( \varpi \neq (0) \) and therefore, \( \mathcal{O} \) is in particular non-Noetherian).

By saturatedness of ultraproducts, \( \mathcal{O} \) is quasi-complete in its \( \mathfrak{p} \)-adic topology in the sense that any Cauchy sequence has a (non-unique) limit. Hence the completion of \( \mathcal{O} \) is
\[\Omega/\varpi\] (see also Lemma \[5.4\] below). Moreover, we will assume that all \(\Omega_w\) have embedding dimension at most \(\epsilon\). Hence so do \(\Omega\) and \(\Omega/\varpi\). Since a complete local ring with finitely generated maximal ideal is Noetherian (\[25\] Theorem 29.4), we showed that \(\Omega/\varpi\) is a Noetherian complete local ring. For more details in the case of interest to us, where each \(\Omega_w\) is a discrete valuation ring or a field, see \[41\].

We furthermore fix throughout a tuple of variables \(X = (X_1, \ldots, X_n)\) and, we let \(A := \Omega[X]\) and \(A_w := \Omega_w[X]\).

### 3.1. Definition

The **non-standard hull** of \(A\) is by definition the ultraproduct of the \(A_w\) and is denoted \(A_\infty\).

By Łos’ Theorem, we have an inclusion \(\Omega \subset A_\infty\). Let us continue to write \(X_i\) for the ultraproduct in \(A_\infty\) of the constant sequence \(X_i \in A_w\). By Łos’ Theorem, the \(X_i\) are algebraically independent over \(\Omega\). In other words, \(A\) is a subring of \(A_\infty\). In the next section, we will prove the key algebraic property of the extension \(A \subset A_\infty\) when the base rings \(\Omega_w\) are discrete valuation rings, to wit, its flatness. We start with extending the notions of non-standard hull and approximation from \[41\], to arbitrary local \(\Omega\)-affine algebras (recall that a local \(\Omega\)-affine algebra is a localization of a finitely presented \(\Omega\)-algebra at a prime ideal containing \(p\)).

**Approximations and non-standard hulls.** An approximation of a polynomial \(f \in A\) is a sequence of polynomials \(f_w \in A_w\), such that their ultraproduct is equal to \(f\), viewed as an element in \(A_\infty\). Note that according to Lemma \[2.1\] we can always find such an approximation. Moreover, any two approximations are equal for almost all \(w\), by Łos’ Theorem. Similarly, an approximation of a finitely generated ideal \(I := fA\) with \(f\) a finite tuple, is a sequence of ideals \(I_w := f_w A_w\), where \(f_w\) is an approximation of \(f\) (meaning that each entry in \(f_w\) is an approximation of the corresponding entry in \(f\)). Łos’ Theorem gives once more that any two approximations are almost all equal. Moreover, if \(I_w\) is some approximation of \(I\) then

\[
\ulim_{w \to \infty} I_w = IA_\infty.
\]

Assume now that \(C\) is a finitely presented \(\Omega\)-algebra, say \(C = A/I\) with \(I\) a finitely generated ideal. We define an approximation of \(C\) to be the sequence of finitely generated \(\Omega_w\)-algebras \(C_w := A_w/I_w\), where \(I_w\) is some approximation of \(I\). We define the non-standard hull of \(C\) to be the ultraproduct of the \(C_w\) and denote it \(C_\infty\). It is not hard to show that \(C_\infty\) is uniquely defined up to \(C\)-algebra isomorphism (for more details see \[41\] or \[38\]). From \[2.1\], it follows that \(C_\infty = A_\infty/A_\infty\). In particular, there is a canonical homomorphism \(C \to C_\infty\) obtained from the base change \(A \to A_\infty\).

**Some Caveats.** When \(I\) is not finitely generated, \(IA_\infty\) might not be realizable as an ultraproduct of ideals, and consequently, has no approximation. Although one can find special cases of infinitely generated ideals admitting approximations, we will never have to do this in the present paper. Similarly, we only define approximations for finitely presented algebras.

Although \(A \to A_\infty\) is injective, this is not necessarily the case for \(C \to C_\infty\), if the \(\Omega_w\) are not fields. For instance, if \(W\) is the set of prime numbers, \(\Omega_p := \mathbb{Z}_p\) for each \(p \in W\) and \(I = (1 - \pi X, \gamma)A\) where \(\pi = \ulim_{p \to \infty} p\) and \(\gamma = \ulim_{p \to \infty} p^p\), then \(I \neq (1)\) but \(IA_\infty = (1)\).

However, when the \(\Omega_w\) are discrete valuation rings, we will see shortly, that this phenomenon disappears if we localize at prime ideals containing \(p\). Next we define a process which is converse to taking approximations.
**Restricted Ultraproducts.** Fix some $c$. For each $w$, let $I_w$ be an ideal in $A_w$ of $\mathcal{O}_w$-complexity at most $c$. In other words, we can write $I_w = f_w A_w$, for some tuple $f_w$ of $\mathcal{O}_w$-complexity at most $c$. Let $f$ be the ultraproduct of these tuples. By Lemma 2.1, the tuple $f$ is already defined over $\mathcal{O}$. We call $I := fA$ the restricted ultraproduct of the $I_w$. It follows that the $I_w$ are an approximation of $I$ and that $IA_{\infty}$ is the ultraproduct of the $I_w$.

With $C_w := A_w/I_w$ and $C := A/I$, we call $C$ the restricted ultraproduct of the $C_w$. The $C_w$ are an approximation of $C$ and their ultraproduct $C_{\infty}$ is the non-standard hull of $C$. We can now extend the previous definition to the image in $C_w$ of an element $c_w \in A_w$ (respectively, to the extension $J_wC_w$ of a finitely generated ideal $J_w \subset A_w$) of $\mathcal{O}_w$-complexity at most $c$ and define similarly their restricted ultraproduct $c \in C$ and $JC$ as the image in $C$ of the respective restricted ultraproduct of the $c_w$ and the $J_w$.

**Functoriality.** We have a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & D \\
\downarrow \varphi_{\infty} & & \downarrow \varphi_{\infty} \\
C_{\infty} & \xrightarrow{\varphi_{\infty}} & D_{\infty}
\end{array}
\]

where $C \to D$ is an $\mathcal{O}$-algebra homomorphism of finite type between finitely presented $\mathcal{O}$-algebras and $C_{\infty} \to D_{\infty}$ is its base change over the non-standard hulls $C_{\infty} := A_{\infty} \otimes C$ and $D_{\infty} := A_{\infty} \otimes D$, or alternatively, where $C \to D$ (respectively, $C_{\infty} \to D_{\infty}$) are the restricted ultraproduct (respectively, ultraproduct) of $\mathcal{O}_w$-algebras homomorphisms $C_w \to D_w$ of $\mathcal{O}_w$-complexity at most $c$, for some $c$ independent from $w$.

**3.2. Lemma.** Any prime ideal $m$ of $A$ containing $p$ is finitely generated and its extension $mA_{\infty}$ is again prime.

**Proof.** Since $A/pA = \kappa[X]$ is Noetherian, where $\kappa$ is the residue field of $\mathcal{O}$, the ideal $m(A/pA)$ is finitely generated. Therefore so is $m$, since by assumption $p$ is finitely generated. Moreover, $A_{\infty}/pA_{\infty}$ is the ultraproduct of the $\kappa_w[X]$, so that by Theorem 2.2, the extension $m(A_{\infty}/pA_{\infty})$ is prime, whence so is $mA_{\infty}$. \(\Box\)

In particular, if $m_w$ is an approximation of $m$, then almost all $m_w$ are prime ideals. Therefore, the following notions are well-defined (with the convention that we put $B_n$ equal to zero whenever $n$ is not a prime ideal of the ring $B$). Let $R$ be a local $\mathcal{O}$-affine algebra, say, of the form $C_m$, with $C$ a finitely presented $\mathcal{O}$-algebra and $m$ a prime ideal containing $p$.

**3.3. Definition.** We call $(C_{\infty})_{mC_{\infty}}$ the non-standard hull of $R$ and denote it $R_{\infty}$. Moreover, if $C_w$ and $m_w$ are approximations of $C$ and $m$ respectively, then the collection $R_w := (C_w)_{m_w}$ is an approximation of $R$.

One easily checks that the ultraproduct of the approximations $R_w$ is precisely the non-standard hull $R_{\infty}$.

**4. Flatness of Non-standard Hulls**

In this section, we specialize the notions from the previous result to the situation where each $\mathcal{O}_w$ is a discrete valuation ring. We fix throughout the following notation. For each
$w$, let $\Omega_w$ be a discrete valuation ring with uniformizing parameter $\pi_w$ and with residue field $k_w$. Let $\Omega$, $\pi$ and $k$ be their respective ultraproducts, so that $\pi\Omega$ is the maximal ideal of $\Omega$ and $k$ its residue field. The intersection of all $\pi^m\Omega$ is called the ideal of infinitesimals of $\Omega$ and is denoted $\varpi$. Using [32], one sees that $\Omega/\pi^m\Omega$ is an Artinian local Gorenstein $k$-algebra of length $m$.

Fix a finite tuple of variables $X$ and let $A_\infty$ be the ultraproduct of the $A_w := \Omega_w[X]$. Set $A := \Omega[X]$ and view it as a subring of $A_\infty$.

4.1. Proposition. For $I$ an ideal in $A$, the residue ring $A/I$ is Noetherian if, and only if, $\varpi \subseteq I$. In particular, every maximal ideal of $A$ contains $\varpi$ and is of the form $\varpi A + J$ with $J$ a finitely generated ideal.

Proof. Let $C := A/I$ for some ideal $I$ of $A$. If $C$ is Noetherian, then the intersection of all $\pi^n C$ is zero by Krull’s Intersection Theorem. Hence $\varpi \subseteq I$. Conversely, if $\varpi \subseteq I$, then since $A/\varpi A = (\Omega/\varpi)[X]$ is Noetherian, so is $C$. The last assertion is now clear. $\square$

In spite of Lemma 4.2 there are even maximal ideals of $A$ (necessarily not containing $\pi$) which do not extend to a proper ideal in $A_\infty$. For instance with $X$ a single variable, the ideal $\varpi A + (1 - \pi X)A$ is maximal (with residue field the field of fractions of $\Omega/\varpi$), but $\varpi A_\infty + (1 - \pi X)A_\infty$ is the unit ideal. Indeed, let $f_\infty$ be the ultraproduct of the

$$f_w := \frac{(1 - (\pi w)X)^w}{(1 - \pi w)X}.\)$$

Since $(1 - \pi w X)f_w \equiv 1$ modulo $(\pi w)^wA_w$, we get by Łos’ Theorem that $(1 - \pi X)f_\infty \equiv 1$ modulo $\varpi A_\infty$. Therefore, we cannot hope for $A \rightarrow A_\infty$ to be faithfully flat. Nonetheless, using for instance a result of Aschenbrenner on bounds of syzygies, we do have this property for local affine algebras. This result will prove to be crucial in what follows.

4.2. Theorem. The canonical homomorphism $A \rightarrow A_\infty$ is flat. In particular, the canonical homomorphism of a local $\Omega$-affine algebra to its non-standard hull is faithfully flat, whence in particular injective.

Proof. The last assertion is clear from the first, since the homomorphism $R \rightarrow R_\infty$ is obtained as a base change of $A \rightarrow A_\infty$ followed by a suitable localization, for any local $\Omega$-affine algebra $R$. I will provide two different proofs for the first assertion.

For the first proof, we use a result of Aschenbrenner [2] in order to verify the equational criterion for flatness, that is to say, given a linear equation $L = 0$, with $L$ a linear form over $A$, and given a solution $f_\infty$ over $A_\infty$, we need to show that there exist solutions $b_i$ in $A$ such that $f_\infty$ is an $A_\infty$-linear combination of the $b_i$. Choose $L_w$ and $f_w$ with respective ultraproducts $L$ and $f_\infty$. In particular, almost all $L_w$ have $\Omega_w$-complexity at most $c$, for some $c$ independent from $w$. By Łos’ Theorem, $f_w$ is a solution of the linear equation $L_w = 0$, for almost all $w$. Therefore, by [2 Corollary 4.27], there is a bound $c'$, only depending on $c$, such that $f_w$ is an $A_w$-linear combination of solutions $b_{1,w}, \ldots, b_{s,w}$ of $\Omega_w$-complexity at most $c$. Note that $s$ can be chosen independent from $w$ as well by 38 Lemma 1]. In particular, the ultraproduct $b_i$ of the $b_{i,w}$ lies in $A$ by Lemma 2.1. By Łos’ Theorem, each $b_i$ is a solution of $L = 0$ in $A_\infty$, whence in $A$, and $f_\infty$ is an $A_\infty$-linear combination of the $b_i$, proving flatness.

If we want to avoid the use of Aschenbrenner’s result, we can reason as follows. By Theorem 2.2 both extensions $A/\pi A \rightarrow A_\infty/\pi A_\infty$ and $A \otimes Q \rightarrow A_\infty \otimes Q$ are faithfully flat, where $Q$ is the field of fractions of $\Omega$. Let $M$ be an $A$-module. Since $\pi$ is $A$-regular, the standard spectral sequence

$$\text{Tor}^A_p(A_\infty/\pi A_\infty, \text{Tor}^A_q(M, A/\pi A)) \Longrightarrow \text{Tor}^A_{p+q}(A_\infty/\pi A_\infty, M)$$


degenerates into short exact sequences

$$\text{Tor}^A_i (A_\infty / \pi A_\infty, (0 : M \pi)) \rightarrow \text{Tor}^A_i (A_\infty / \pi A_\infty, M) \rightarrow$$

$$\text{Tor}^A_i (A_\infty / \pi A_\infty, M/\pi M),$$

for all \(i \geq 2\). In particular, the flatness of \(A/\pi A \rightarrow A_\infty / \pi A_\infty\) implies the vanishing of \(\text{Tor}^A_i (A_\infty / \pi A_\infty, M)\). Applying this to the short exact sequence

$$0 \rightarrow A_\infty \xrightarrow{\pi} A_\infty \rightarrow A_\infty / \pi A \rightarrow 0$$

we get a short exact sequence

$$(4) \quad 0 = \text{Tor}^A_2 (A_\infty / \pi A_\infty, M) \rightarrow \text{Tor}^A_1 (A_\infty, M) \xrightarrow{\pi} \text{Tor}^A_1 (A_\infty, M).$$

On the other hand, flatness of \(A \otimes Q \rightarrow A_\infty \otimes Q\) yields

$$(5) \quad \text{Tor}^A_1 (A_\infty, M) \otimes Q = \text{Tor}^A_1 (A_\infty \otimes Q, M \otimes Q) = 0.$$

In order to prove flatness, it suffices by [25 Theorem 7.8] to show that \(\text{Tor}^A_1 (A_\infty, A/I)\) vanishes, for every finitely generated ideal \(I\) of \(A\). Towards a contradiction, suppose that \(\text{Tor}^A_1 (A_\infty, A/I)\) contains a non-zero element \(\tau\). By (5), we have \(\tau \pi = 0\), for some non-zero \(a \in \mathfrak{A}\). As observed in [29 Proposition 3], every polynomial ring over a valuation ring is coherent, so that in particular \(I\) is finitely presented (namely, since \(I\) is torsion-free over \(\mathfrak{A}\), it is \(\mathfrak{A}\)-flat, and therefore finitely presented by [27 Theorem 3.4.6]). Hence we have some exact sequence

$$A^{a_2} \xrightarrow{\varphi_2} A^{a_1} \xrightarrow{\varphi_1} A \rightarrow A/I \rightarrow 0.$$

Therefore \(\text{Tor}^A_1 (A_\infty, A/I)\) is calculated as the homology of the complex

$$(A_\infty)^{a_2} \xrightarrow{\varphi_2} (A_\infty)^{a_1} \xrightarrow{\varphi_1} A_\infty.$$

Suppose \(\tau\) is the image of a tuple \(x \in (A_\infty)^{a_1}\) with \(\varphi_1 (x) = 0\). Hence \(x\) does not belong to \(\varphi_2 ((A_\infty)^{a_2})\) but \(ax\) does. Choose \(x_w, a_w\) and \(\varphi_i w\) with respective ultraproduct \(x, a\) and \(\varphi_i\). By Łos’ Theorem, almost all \(x_w\) lie in the kernel of \(\varphi_1\) but not in the image of \(\varphi_2\), yet \(a_w x_w\) lies in the image of \(\varphi_2\). Choose \(n_w \in \mathbb{N}\) maximal such that \(y_w := (\pi_w)^{n_w} x_w\) does not lie in the image of \(\varphi_2\). Since almost all \(a_w\) are non-zero, this maximum exists for almost all \(w\). Therefore, if \(y\) is the ultraproduct of the \(y_w\), then \(\varphi_1 (y) = 0\) and \(y\) does not lie in \(\varphi_2 ((A_\infty)^{a_2})\), but \(\pi y\) lies in \(\varphi_2 ((A_\infty)^{a_2})\). Therefore, the image of \(y\) in \(\text{Tor}^A_1 (A_\infty, A/I)\) is a non-zero element annihilated by \(\pi\), contradicting (4).

4.3. Remark. In [44], I exhibit a general connection between the flatness of an ultraproduct over certain canonical subrings and the existence of bounds on syzygies. In particular, using these ideas, the second argument in the above proof of flatness reproves the result in [2]. In fact, the role played here by coherence is not accidental either; see [1] or [44] for more details.

4.4. Theorem. Let \(R\) be a local \(\mathfrak{A}\)-affine algebra with non-standard hull \(R_\infty\) and approximation \(R_w\).

- Almost all \(R_w\) are flat over \(\mathfrak{A}_w\) if, and only if, \(R\) is torsion-free over \(\mathfrak{A}\) if, and only if, \(\pi\) is \(R\)-regular.
- Almost all \(R_w\) are domains if, and only if, \(R\) is.
Proof. Suppose first that almost all $R_w$ are flat over $\Omega_w$, which amounts in this case, to almost all $R_w$ being torsion-free over $\Omega_w$. By Łos’ Theorem, $R_\infty$ is torsion-free over $\Omega$, and since $R \subset R_\infty$, so is $R$. Conversely, assume $\pi$ is $R$-regular. By faithful flatness, $\pi$ is $R_\infty$-regular, whence almost all $\pi_w$ are $R_w$-regular by Łos’ Theorem. Since the $\Omega_w$ are discrete valuation rings, this means that almost all $\Omega_w \to R_w$ are flat.

If almost all $R_w$ are domains, then so is $R_\infty$ by Łos’ Theorem, and hence so is $R$, since it embeds in $R_\infty$. Conversely, assume $R$ is a domain. If $\pi = 0$ in $R$, then $R_\infty$ is a domain by Lemma 5.2, whence so are almost all $R_w$ by Łos’ Theorem. So assume $\pi$ is non-zero in $R$, whence $R$-regular. By what we just proved, $R$ is then torsion-free over $\Omega$. Let $Q$ be the field of fractions of $\Omega$. Write $R$ in the form $S/p$, where $S$ is some localization of $A$ at a prime ideal containing $\pi$ and $p$ is a finitely generated prime ideal in $S$. Since $S/p$ is torsion-free over $\Omega$, the extension $p(S_\infty \otimes_\Omega Q)$ is again prime and its contraction in $S$ is $p$. By Theorem 5.2, since we are now over a field, $p(S_\infty \otimes_\Omega Q)$ is a prime ideal, where $S_\infty$ is the non-standard hull of $S$ (note that $S_\infty \otimes_\Omega Q$ is then the non-standard hull of $S \otimes_\Omega Q$ in the sense of [11]). Moreover, since $S/p$ is torsion-free over $\Omega$, so is $S_\infty/pS_\infty$ by the first assertion. This in turn means that $$pS_\infty = p(S_\infty \otimes_\Omega Q) \cap S_\infty,$$ showing that $pS_\infty$ is prime. It follows then from Łos’ Theorem that almost all $p_w$ are prime, where $p_w$ is an approximation of $p$, and hence almost all $R_w$ are domains. \qed

4.5. Remark. The last assertion is equivalent with saying that any prime ideal in $R$ extends to a prime ideal in $R_\infty$. Indeed, let $q$ be a prime ideal in $R$ with approximation $q_w$. By the above result (applied to $R/q$ and its approximation $R_w/q_w$), we get that almost all $q_w$ are prime, whence so is their ultrapower $qR_\infty$, by Łos’ Theorem.

5. Pseudo-dimension

In this and the next section, we will study the local algebra of the category $\text{Aff}(\Omega)$. Although part of the theory can be developed for arbitrary base rings $\Omega$, we will only deal with the case that $\Omega$ is a local domain of embedding dimension one. Recall that the embedding dimension of a local ring $(Z, \mathfrak{p})$ is by definition the minimal number of generators of $\mathfrak{p}$, and its ideal of infinitesimals $\mathfrak{w}$ is the intersection of all powers $\mathfrak{p}^n$. Of course, if $Z$ is moreover Noetherian, then its ideal of infinitesimals is zero. In general, we call $Z/\mathfrak{w}$ the separated quotient of $Z$.

For the duration of the next two sections, let $\Omega$ denote a local domain of embedding dimension one, with generator of the maximal ideal $\pi$, with ideal of infinitesimals $\mathfrak{w}$ and with residue field $\kappa$.

5.1. Lemma. The separated quotient $\Omega/\mathfrak{w}$ of $\Omega$ is a discrete valuation ring with uniformizing parameter $\pi$.

Proof. For each element $a \in \Omega$ outside $\mathfrak{w}$, there is a smallest $e \in \mathbb{N}$ for which $a \notin \pi^{e+1}\Omega$. Hence $a = u\pi^e$ with $u$ a unit in $R$. It is now straightforward to check that the assignment $a \mapsto e$ induces a discrete valuation on $\Omega/\mathfrak{w}$. \qed

Note that we do not even need $\Omega$ to be domain, having positive depth (that is to say, assuming that $\pi \Omega$ is not an associated prime of $\Omega$; see [6] Proposition 9.1.4) would suffice, for then $\pi$ is necessarily $\Omega$-regular. However, we do not need this amount of generality as in all our applications $\Omega$ will be of the following special form.
5.2. Definition. We say that \( \mathcal{D} \) is an ultra-DVR, if it is realized as the ultraproduct of some discrete valuation rings \( \mathcal{D}_w \).

Note that \( \pi \) and \( \kappa \) are then the respective ultraproducts of the uniformizing parameter \( \pi_w \) and the residue field \( \kappa_w \) of \( \mathcal{D}_w \).

We will work in the category \( \text{Aff}(\mathcal{D}) \) of local \( \mathcal{D} \)-affine algebras (see the Introduction), that is to say, the category of algebras of the form \( (A/I)_m \), with \( I \) a finitely generated ideal and \( m \) a prime ideal containing \( \pi \) (and \( I \)). Nonetheless, some results can be stated even for local algebras which are locally finitely generated over \( \mathcal{D} \), that is without the assumption that \( I \) is finitely generated. We call \( R \) a torsion-free \( \mathcal{D} \)-algebra if it is torsion-free over \( \mathcal{D} \) (that is to say, if \( ar = 0 \) for some \( r \in R \) and some non-zero \( a \in \mathcal{D} \), then \( r = 0 \)).

Recall from Theorem 5.4 that a local \( \mathcal{D} \)-affine algebra \( R \) is torsion-free if, and only if, \( \pi \) is \( R \)-regular.

If \( \mathcal{D} \) is an ultra-DVR, then we set \( A_w := \mathcal{D}_w[X] \) and let \( A_\infty \) be their ultraproduct. If \( R \) belongs to \( \text{Aff}(\mathcal{D}) \), that is to say, is a local \( \mathcal{D} \)-affine algebra of the form \( (A/I)_m \), then \( R_\infty := (A_\infty/I_\infty)_m \), denotes its non-standard hull and we let \( R_w := (A_w/I_w)_m \), be an approximation of \( R \), where \( I_w \) and \( m_w \) are approximations of \( I \) and \( m \) respectively. Note that \( m \) is finitely generated, as it contains by definition \( \pi \).

5.3. Lemma. Let \( R \) be a local ring which is locally finitely generated over \( \mathcal{D} \). If \( I \) is a proper ideal in \( R \) containing some power \( \pi^m \), then the intersection of all \( I^N \) is equal to \( \varpi R \).

Proof. Suppose \( \pi^m \in I \subset m \). Let \( J \) be the intersection of all \( I^N \). Since \( \pi^m \in I \), we get that \( \varpi R \subset J \). Since \( S := R/\varpi R \) is locally finitely generated over the discrete valuation ring \( \mathcal{D}/\varpi \) (see Lemma 5.1), it is itself Noetherian. Applying Krull’s Intersection Theorem (see for instance [23, Theorem 8.10]), we get that \( JS = (0) \), and hence that \( J = \varpi R \). \( \square \)

5.4. Lemma. Let \( \mathcal{D} \) be an ultra-DVR (see Definition 5.2) and let \( (R, m) \) be a local \( \mathcal{D} \)-affine algebra with approximation \( R_w \). The \( m \)-adic completion \( \hat{R} \) of \( R \) is isomorphic to \( R_\infty/i \), where \( i \) is the ideal of infinitesimals of \( R_\infty \).

In particular, \( \hat{R} \) is Noetherian.

Proof. Clearly, \( R \) is \( m \)-adically dense in \( R_\infty \), so that they have a common completion. On the other hand, by saturatedness of ultraproducts, \( R_\infty \) is quasi-complete in the sense that every Cauchy sequence has a (non-unique) limit. Therefore, its separated quotient \( R_\infty/i \) is complete. \( \square \)

Our first goal is to introduce a good notion of dimension. Below, the dimension of a ring will always mean its Krull dimension. Recall that it is always finite for Noetherian local rings.

5.5. Theorem. For a local ring \( (R, m) \) which is locally finitely generated over \( \mathcal{D} \), the following numbers are all equal:

- the minimal length \( d \) of a tuple in \( R \) generating an \( m \)-primary ideal;
- the dimension \( d \) of the completion \( \hat{R} \);
- the dimension \( d \) of \( R/\varpi R \);
- the degree \( d \) of the Hilbert-Samuel polynomial \( \chi_R \), where \( \chi_R \) is the unique polynomial with rational coefficients for which \( \chi_R(n) \) equals the length of \( R/m^{n+1} \) for all large \( n \).

Moreover, if \( \pi \) is \( R \)-regular, then \( R/\pi R \) has dimension \( d - 1 \).
If $\mathcal{O}$ is an ultra-DVR and $R$ is a torsion-free local $\mathcal{O}$-affine algebra with approximation $R_w$, then almost all $R_w$ have dimension $d$.

**Proof.** Let $\tilde{R} := R/\varpi R$, so that $\tilde{R}$ is a Noetherian local ring (see the proof of Lemma 5.3) and $\tilde{d}$ is finite. Since $R/m^n \cong \tilde{R}/m^n\tilde{R}$, the completion of $\tilde{R}$ is $\hat{R}$ and hence, $\tilde{d} = \hat{d}$.

Moreover, $\chi_R = \chi_{\tilde{R}}$, so that by the Hilbert-Samuel theory, $\hat{d} = \tilde{d}$.

Let $x$ be a tuple of length $\tilde{d}$ such that its image in $\tilde{R}$ is a system of parameters of $\tilde{R}$. Hence, for some $n$, we have that $m^n \subset xR + \varpi R$. In particular, since $\varpi R \subset \pi^{n+1}R$ by Lemma 5.3, we can find $x \in xR$ and $r \in R$, such that $\pi^n = x + r\pi^{n+1}$. Therefore, $\pi^n \in xR$, since $(1 - r\pi)$ is a unit. Since $\varpi \subset \pi^n\mathcal{O}$, we get that $m^n \subset xR$, showing that $xR$ is an $m$-primary ideal and hence that $\tilde{d} \leq \hat{d}$. On the other hand, if $y$ is a tuple of length $\hat{d}$ such that $yR$ is $m$-primary, then $y\tilde{R}$ is an $m\tilde{R}$-primary ideal, and hence $\hat{d} \leq \tilde{d}$.

This concludes the proof of the first assertion.

Assume that $\pi$ is moreover $\hat{R}$-regular. I claim that $\pi$ is $\tilde{R}$-regular. Indeed, suppose $\pi\tilde{r} = 0$, for some $\tilde{r} \in \tilde{R}$. Take a pre-image $r \in R$, so that $\pi r \in \varpi R \subset \pi^n R$, for every $n$.

Since $\pi$ is $\hat{R}$-regular, we get that $r \in \pi^{n+1}R$, for all $n$. Therefore $r \in \varpi R$, whence $\tilde{r} = 0$ in $\tilde{R}$, as we needed to show. Since $\pi$ is $\tilde{R}$-regular and $\tilde{R}/\pi R = R/\pi R$, the dimension of $R/\pi R$ is $\tilde{d} - 1$.

Suppose finally that $\mathcal{O}$ is moreover an ultra-DVR. We already observed that $R_w/\pi_w R_w$ is an approximation of $R/\pi R$ in the sense of [41]. In particular, by [41] Theorem 4.5], almost all $R_w/\pi_w R_w$ have dimension $\tilde{d} - 1$. Since $\pi$ is $R_\infty$-regular by flatness, whence $\pi_w$ is $R_w$-regular by Los’ Theorem, we get that $R_w$ has dimension $\tilde{d}$, for almost all $w$. □

5.6. **Definition** (Pseudo-dimension). The minimal number of generators of an $m$-primary ideal is called the pseudo-dimension of $R$. We call a tuple $x$ in $R$ generic, if it generates an $m$-primary ideal and has length equal to the pseudo-dimension of $R$.

5.7. **Corollary.** In a local ring $(R, m)$ which is locally finitely generated over $\mathcal{O}$, every tuple generating an $m$-primary ideal can be trimmed to a generic sequence by omitting some of its entries.

**Proof.** Let $\tilde{R} := R/\varpi R$ and let $\tilde{d}$ be the pseudo-dimension of $R$. Let $x$ be a tuple generating an $m$-primary ideal of $R$. By a well-known prime avoidance argument in the Noetherian ring $\tilde{R}$, we can rearrange this tuple so that it has the form $(y, z)$ with $y$ a system of parameters in $\tilde{R}$. In particular, $|y| = \tilde{d}$ by Theorem 5.5. Let $S := R/yR$ and $\tilde{S} := S/\varpi S$. By Theorem 5.5, the pseudo-dimension of $S$ is equal to the dimension of $\tilde{S}$, whence is zero since $\tilde{S} = R/y\tilde{R}$. Therefore, the empty tuple is a generic sequence in $S$, and hence $yR$ is $m$-primary. Since $y$ has length equal to the pseudo-dimension of $R$, it is therefore a generic sequence.

In fact the above proof shows that there is a one-one correspondence between generic sequences in $R$ and systems of parameters in $R/\varpi R$. In general, the last assertion in Theorem 5.5 is false when $\tilde{R}$ is not torsion-free. For instance, let $R := D/\gamma D$ with $\gamma$ a non-zero infinitesimal, so that each $R_w = D_w/\gamma_w D_w$ has dimension zero, but $R/\varpi R$ is the (one-dimensional) discrete valuation ring $D/\varpi$.

In the following definition, let $\mathcal{O}$ be an ultra-DVR and let $R$ be a local $\mathcal{O}$-affine algebra of pseudo-dimension $d$, with approximation $R_w$. Note that the $R_w$ have almost all dimension at most $d$. Indeed, if $y$ has length $\tilde{d}$ and generates an $m$-primary ideal, then almost all $y_w$ are $m_w$-primary by Los’ Theorem, for $y_w$ an approximation of $y$. 


5.8. Definition. We say that $R$ is isodimensional if almost all $R_w$ have dimension equal to the pseudo-dimension of $R$.

Theorem 5.5 shows that every torsion-free local $\mathcal{O}$-affine algebra is isodimensional. In particular, over an ultra-DVR, the restricted ultraproduct $R$ of domains $R_w$ of uniformly bounded $\mathcal{O}_w$-complexity is isodimensional, since $R_{\infty}$ is then a domain by Łos’ Theorem, whence so is $R$ as it embeds in $R_{\infty}$. The next result shows that generic sequences in an isodimensional ring are the analog of systems of parameters.

5.9. Corollary. Let $\mathcal{O}$ be an ultra-DVR and $R$ an isodimensional local $\mathcal{O}$-affine algebra with approximation $R_w$. Let $x$ be a tuple in $R$ with approximation $x_w$.

If $x$ is generic, then $x_w$ is a system of parameters of $R_w$, for almost all $w$. Conversely, if $(\pi_w)^c \in x_w R_w$, for some $c$ and almost all $w$, then $x$ is generic.

Proof. Let $m$ be the maximal ideal of $R$, with approximation $m_w$. Let $d$ be the pseudo-dimension of $R$, so that almost all $R_w$ have dimension $d$. Suppose first that $x$ is generic, so that $|x| = d$ and $xR$ is $m$-primary. Since $xR_{\infty}$ is then $mR_{\infty}$-primary, $x_w R_w$ is $m_w$-primary by Łos’ Theorem, showing that $x_w$ is a system of parameters for almost all $w$.

Conversely, suppose $x_w$ is a system of parameters of $R_w$, generating an ideal containing $(\pi_w)^c$. If $x$ is the ultraproduct of the $x_w$, then it is defined over $R$ by Lemma 2.1. By Łos’ Theorem and faithful flatness, $\pi^c \in xR$. Applying [35 Corollary 4] to the Artinian base ring $\mathcal{O}_w/(\pi_w)^c$, we can find a bound $c'$, only depending on $c$, such that $(m_w)^{c'} \subset x_w R_w$, for almost all $w$. Hence $m^{c'} R_{\infty} \subset xR_{\infty}$, so that by faithful flatness, $xR$ is $m$-primary. This shows that $x$ is generic.

The additional requirement in the converse is necessary: indeed, for arbitrary $n_w > 0$, the element $(\pi_w)^{n_w}$ is a parameter in $\mathcal{O}_w$ and has $\mathcal{O}_w$-complexity zero, but if $n_w$ is unbounded, its ultraproduct is an infinitesimal whence not generic. To characterize isodimensional rings, we use the following notion introduced in [35].

5.10. Definition (Parameter Degree). The parameter degree of a Noetherian local ring $C$ is by definition the smallest possible length of a residue ring $C/\pi C$, where $\pi$ runs over all systems of parameters of $C$.

In general, the parameter degree is larger than the multiplicity, with equality precisely when $C$ is Cohen-Macaulay (see [25 Theorem 17.11]). The homological degree of $C$ is an upper bound for its parameter degree (see [35 Corollary 4.6]). A priori, being isodimensional is a property of the approximations of $R$, of for that matter, of its non-standard hull. However, the last equivalent condition in the next result shows that it is in fact an intrinsic property.

5.11. Proposition. Let $\mathcal{O}$ be an ultra-DVR and let $R$ be a local $\mathcal{O}$-affine algebra with approximation $R_w$. The following are equivalent:

1. $R$ is isodimensional;
2. there exists a $c \in \mathbb{N}$, such that for almost all $w$, we can find a system of parameters $x_w$ of $R_w$ of $\mathcal{O}_w$-complexity at most $c$, generating an ideal containing $(\pi_w)^c$;
3. there exists an $e \in \mathbb{N}$, such that almost all $R_w$ have parameter degree at most $e$;
4. for every generic sequence in $R$ of the form $(\pi, y)$, we have that $y R \cap \mathcal{O}$ is the zero ideal.

Proof. Let $m$ be the maximal ideal of $R$, with approximation $m_w$. Let $d$ be the pseudo-dimension of $R$ and let $d'$ be the dimension of almost all $R_w$. Suppose first that $d = d'$. 

Let $x$ be any generic sequence in $R$ with approximation $x_w$. By Łos’ Theorem, almost all $x_w$ generate an $m_w$-primary ideal. Since their length is equal to the dimension of $R_w$, they are almost all systems of parameters of $R_w$. Choose $c$ large enough so that $\pi^c \in xR$. Enlarging $c$ if necessary, we may moreover assume by Lemma 5.3 that almost all $x_w$ have $\mathcal{O}_w$-complexity at most $c$. By Łos’ Theorem, $(\pi_w)^c \in x_w R_w$, so that (3) holds.

Assume next that $c$ and the $x_w$ are as in (2). Let $T_w := R_w / (\pi_w)^c R_w$. We can apply Corollary 2 over $\mathcal{O}_w / (\pi_w)^c \mathcal{O}_w$ to the $m_w T_w$-primary ideal, to conclude that there is some $c'$ depending only on $c$, such that $T_w / x_w T_w$ has length at most $c'$. Since the latter residue ring is just $R_w / x_w R_w$ by assumption, the parameter degree of $R_w$ is at most $c'$, and hence (3) holds.

To show that (3) implies (1), assume that almost all $R_w$ have parameter degree at most $c$. Let $y_w$ be a system of parameters of $R_w$ such that $R_w / y_w R_w$ has length at most $c$, for almost all $w$. It follows that $(m_w)^c$ is contained in $y_w R_w$. Let $y_\infty$ be the ultraproduct of the $y_w$. By Łos’ Theorem, $m^\infty R_\infty \subset y_\infty R_\infty$. Since the completion $\widehat{R}$ is a homomorphic image of $R_\infty$ by Lemma 5.3, we get that $y_\infty \widehat{R}$ is a $m\widehat{R}$-primary. Since $y_\infty$ has length at most $d'$ (some entries might be zero in $\widehat{R}$), the dimension of $\widehat{R}$ is at most $d'$. Since we already remarked that $d' \leq d$, we get from Theorem 4.4 that $d' = d$.

So remains to show that (3) is equivalent to the other conditions. Assume first that it holds but that $R$ is not isodimensional. Since we have inequalities $d - 1 \leq d' \leq d$, this means that $d' = d - 1$. Moreover, $R / \pi R$ must have pseudo-dimension also equal to $d - 1$, for if not, its pseudo-dimension would be $d$, whence almost all $R_w / \pi_w R_w$ would have dimension $d$ by [41] Theorem 4.5, which is impossible. Since there is a uniform bound $c$ on the $\mathcal{O}_w$-complexity of each $R_w$, we can choose using Remark 2.6, a system of parameters $y_w$ of $R_w$ of $\mathcal{O}_w$-complexity at most $c$ (and hence of length $d - 1$). In particular, some power of $\pi_w$ lies in $y_w R_w$. Let $a \in \mathcal{O}$ be the ultraproduct of these powers. If $y$ is the ultraproduct of the $y_w$, then $y$ is already defined over $R$ by Lemma [2.1]. By Łos’ Theorem, $a \in y R_\infty$, whence by faithful flatness, $a$ is a non-zero element in $y R \cap \mathcal{O}$. Therefore, to reach the desired contradiction with (4), we only need to show that $(\pi, y)$ is generic. As we already established, $R_w / \pi_w R_w$ has dimension $d - 1$, so that $y_w$ is also a system of parameters in that ring. Therefore, $y$ is a system of parameters in $R / \pi R$ by [41] Theorem 4.5. This in turn implies that $(\pi, y)$ generates an $m$-primary ideal in $R$. Since this tuple has length $d$, it is therefore generic, as we wanted to show.

Finally, assume $R$ is isodimensional, and suppose $(\pi, y)$ is generic. Let $a \in y R \cap \mathcal{O}$ and choose approximations $a_w$ of $a$ and $y_w$ of $y$ respectively. By Łos’ Theorem, $a_w \in y_w R_w$. However, if $a$ is non-zero, then $a_w$ is a power of $\pi_w$, which contradicts the fact that $(\pi_w, y_w)$ is a system of parameters by Corollary 5.9. So $a = 0$, as we needed to show. □

5.12. Corollary. For each $c$, there exists a bound $PD(c)$ with the following property. Let $V$ be a discrete valuation ring and let $R$ be a local $V$-algebra of $V$-complexity at most $c$. If $R$ is torsion-free over $V$, then the parameter degree of $R$ is at most $PD(c)$.

Proof. If the statement is false for some $c$, then we can find for each $w$ a discrete valuation ring $\mathcal{O}_w$ and a torsion-free local $\mathcal{O}_w$-algebra $R_w$ of $\mathcal{O}_w$-complexity at most $c$, whose parameter degree is at least $w$. Let $\overline{R}$ be the restricted ultraproduct of the $R_w$ and let $R_\infty$ be their ultraproduct. Since $\pi_w$ is $R_w$-regular, $\pi$ is $R_\infty$-regular, whence $R$-regular. Hence $R$ is isodimensional by Theorem 5.5. Therefore, there is a bound on the parameter degree of almost all $R_w$ by Proposition 5.11, contradicting our assumption. □

Our next goal is to introduce a notion similar to height. Let $I$ be an arbitrary ideal of $R$. 
5.13. **Definition** (Pseudo-height). We call the *pseudo-height of* $I$ the maximum of all $h$ such that there exists a generic sequence with its first $h$ entries in $I$.

For Noetherian rings, we cannot expect a good relationship between the height of an ideal and the dimension of its residue ring, unless the ring is a catenary domain; the following is a pseudo-analogue.

5.14. **Theorem.** Let $\mathcal{O}$ be an ultra-DVR and let $R$ be a local $\mathcal{O}$-affine domain with approximation $R_w$. Let $I$ be a finitely generated ideal in $R$ with approximation $I_w$.

If $R/I$ is isodimensional, then the pseudo-height of $I$ is equal to the pseudo-dimension of $R$ minus the pseudo-dimension of $R/I$, and this is also equal to the height of almost all $I_w$.

**Proof.** Let $d$ be the pseudo-dimension of $R$ and $\overline{d}$ the pseudo-dimension of $R/I$. Since a domain is isodimensional, almost all $R_w$ have dimension $d$ by Theorem 5.8, and by assumption, almost all $R_w/I_w$ have pseudo-dimension $\overline{d}$. Let $h$ be the pseudo-height of $I$. Let $z$ be a generic sequence in $R$ with its first $h$ entries in $I$, and let $z_w$ be an approximation of $z$. By Corollary 5.9, almost all $z_w$ are a system of parameters in $R_w$. By Łos’ Theorem the first $h$ entries of $z_w$ lie in $I_w$, we get that $R_w/I_w$ has dimension at most $d-h$. In other words, $h \leq d-\overline{d}$. Since almost all $R_w$ are catenary domains, almost all $I_w$ have height $d-\overline{d}$.

Let $x$ be a tuple of length $\overline{d}$ whose image in $R/I$ is generic. Put $S := R/xR$ and let $e$ be its pseudo-dimension. If $x_w$ is an approximation of $x$, then almost all $x_w$ are a system of parameters in $(R_w/I_w)$. By Corollary 5.9, since $x_w$ is therefore part of a system of parameters of $R_w$, we get that $S_w := R_w/x_wR_w$ has dimension $d-\overline{d}$ by [25, Theorem 14.1]. Using Corollary 5.7, we can find a tuple $y$ of length $e$ with entries in $I$, so that its image in $S$ is a generic sequence. It follows that $xR + yR$ is $m$-primary. We already observed that there are only two possibilities for $e$, to wit, $d-\overline{d}$ or $d-\overline{d}+1$. If $e = d-\overline{d}$, then $(x, y)$ has length $d$ and hence is generic. Since $y$ has all its entries in $I$, we get that $h$ is at least $d-\overline{d}$ and we are done in this case.

Hence assume that $e = d-\overline{d}+1$, so that $(x, y)$ has length $d+1$. By Corollary 5.7, we can trim this tuple to a generic sequence, by omitting an appropriately chosen entry. Therefore, $I$ contains at least $e-1$ entries of this trimmed sequence, so that $h \geq e-1 = d-\overline{d}$, and again we are done.

6. Pseudo Singularities

In this section, we maintain the notation introduced in the previous section. Our goal is to extend several singularity notions of Noetherian local rings to the category of local $\mathcal{O}$-affine algebras.

**Grade and Depth.** Let $B$ be an arbitrary ring and $I := (x_1, \ldots, x_n)B$ a finitely generated ideal. The grade of $I$, denoted $\text{grade}(I)$, is by definition equal to $n-h$, where $h$ is the largest value $i$ for which the $i$-th Koszul homology $H_i(x_1, \ldots, x_n)$ is non-zero. For a local ring $R$ of finite embedding dimension, we define its depth as the grade of its maximal ideal.

If $B$ is moreover Noetherian, then we can define the grade of $I$ alternatively as the minimal $i$ for which $\text{Ext}^i_B(B/I, B)$ is non-zero (for all this see for instance [26, §9.1]). An arbitrary local ring has positive depth if, and only if, its maximal ideal is not an associated prime. Grade, and hence depth, deforms well, in the sense that the

$$\text{grade}(I(B/xB)) = \text{grade}(I) - |x|$$
for every $B$-regular sequence $x$. For a locally finitely generated $\mathcal{O}$-algebra $(R, m)$, its depth never exceeds its pseudo-dimension. Indeed, by definition, the grade of a finitely generated ideal never exceeds its minimal number of generators, and by [6, Proposition 9.1.3], the depth of $R$ is equal to the grade of any of its $m$-primary ideals. It follows that the depth of $R$ is at most its pseudo-dimension.

In general, the grade of a finitely generated ideal might be positive without it containing a $B$-regular element. However, the next lemma shows that this is not the case for ultraproducts of Noetherian local rings.

6.1. Lemma. Let $C_\infty$ be the ultraproduct of Noetherian local rings $C_w$ and let $I_\infty$ be a finitely generated ideal of $C_\infty$ obtained as the ultraproduct of ideals $I_w \subset C_w$.

If $I_\infty$ has grade $n$, then there exists a $C_\infty$-regular sequence of length $n$ with all of its entries in $I_\infty$. Moreover, any permutation of a $C_\infty$-regular sequence is again $C_\infty$-regular.

Proof. By [6, Proposition 9.1.3], there exists a finite tuple of variables $Y$ and a $C_\infty[Y]$-regular sequence $f_\infty$ of length $n$, with all of its entries in $I_\infty C_\infty[Y]$. Choose tuples $f_w$ in $C_w[Y]$ so that their ultraproduct is $f_\infty$. By Łos’ Theorem, $f_w$ is $C_w[Y]$-regular and has all of its entries in $I_w C_w[Y]$, for almost all $w$. This shows that $I_w C_w[Y]$ has grade at least $n$. Since $C_w \to C_w[Y]$ is faithfully flat, $I_w$ has grade at least $n$ by [6 Proposition 9.1.2]. Hence, since $C_w$ is Noetherian, we can find a $C_w$-regular sequence $x_w$ of length $n$ with all of its entries in $I_w$. By Łos’ Theorem, the ultraproduct $x_\infty$ of the $x_w$ is $C_\infty$-regular and has all of its entries in $I_\infty$.

The last assertion follows from Łos’ Theorem and the fact that in a Noetherian local ring, any permutation of a regular sequence is again regular ([25 Theorem 16.3]).

Recall that a Noetherian local ring for which its dimension and its depth (respectively, its dimension and its embedding dimension) coincide is Cohen-Macaulay (respectively, regular). We will shortly see that upon replacing dimension by pseudo-dimension, we get equally well behaved notions. Let us therefore make the following definitions, for $R$ a locally finitely generated $\mathcal{O}$-algebra.

6.2. Definition. We say that $R$ is pseudo-Cohen-Macaulay, if its pseudo-dimension is equal to its depth, and pseudo-regular, if its pseudo-dimension is equal to its embedding dimension.

Clearly, a pseudo-regular local ring is pseudo-Cohen-Macaulay.

6.3. Theorem. Let $\mathcal{O}$ be an ultra-DVR and let $R$ be an isodimensional local $\mathcal{O}$-affine algebra with approximation $R_w$. In order for $R$ to be pseudo-Cohen-Macaulay it is necessary and sufficient that almost all $R_w$ are Cohen-Macaulay.

Proof. Let $d$ be the pseudo-dimension of $R$ and $\delta$ its depth. Suppose first that $d = \delta$. Since $R \to R_\infty$ is faithfully flat, $R_\infty$ has depth $\delta$ as well by [6 Proposition 9.1.2]. By Lemma [6.1] there is an $R_\infty$-regular sequence $x_\infty$ of length $d$. If $x_w$ is an approximation of $x_\infty$, then almost all $x_w$ are $R_w$-regular by Łos’ Theorem. Since almost all $R_w$ have dimension $d$ by assumption, we showed that they are Cohen-Macaulay.

Conversely, assume almost all $R_w$ are Cohen-Macaulay. It follows by reversing the above argument that $R_\infty$ has depth $d$ and hence, so has $R$, by faithful flatness.

Since every system of parameters is a regular sequence in a local Cohen-Macaulay ring, we expect a similar behavior for generic sequences, and this indeed holds.

6.4. Theorem. Let $\mathcal{O}$ be an ultra-DVR and let $R$ be an isodimensional local $\mathcal{O}$-affine algebra. If $R$ is pseudo-Cohen-Macaulay, then any generic sequence is $R$-regular.
Proof. Let \( \mathfrak{x} \) be a generic sequence with approximation \( x_w \). Almost all \( x_w \) are a system of parameters in \( R_w \), by Corollary 5.9. Since almost all \( R_w \) are Cohen-Macaulay by Theorem 6.3, almost all \( x_w \) are \( R_w \)-regular. Hence \( \mathfrak{x} \) is \( R_\infty \)-regular, by Łos’ Theorem, whence \( R \)-regular, by faithful flatness. 

6.5. Theorem. Let \( \mathcal{O} \) be an ultra-DVR. An isodimensional local \( \mathcal{O} \)-affine algebra \( R \) with approximation \( R_w \), is pseudo-regular if, and only if, almost all \( R_w \) are regular local rings.

Proof. Let \( \mathfrak{m} \) be the maximal ideal of \( R \), with approximation \( m_w \). Let \( R_\infty \) be the non-standard hull of \( R \). Let \( \epsilon \) be the embedding dimension of \( R \) and \( d \) its pseudo-dimension. Suppose that \( R \) is pseudo-regular, that is to say, that \( \epsilon = d \). Hence \( \mathfrak{m} = xR \) for some \( d \)-tuple \( x \) (necessarily generic). Since \( \mathfrak{m}R_\infty = xR_\infty \), Łos’ Theorem yields that \( m_w = x_w \mathfrak{m} \), where \( x_w \) is an approximation of \( x \). Since almost all \( R_w \) have dimension \( d \), it follows that they are regular local rings.

Conversely, suppose almost all \( R_w \) are regular. Since the \( \mathcal{O}_w \)-complexity of almost all \( R_w \) is at most \( c \), for some \( c \), we can find a regular system of parameters \( x_w \) of \( \mathcal{O}_w \)-complexity at most \( c \). By Corollary 5.9, their ultraproduct \( x \) is a generic sequence, which generates \( \mathfrak{m} \), by Łos’ Theorem and faithful flatness. Therefore, \( \epsilon \leq d \). Since always \( d \leq \epsilon \) (use Theorem 5.5), we get that \( R \) is pseudo-regular. 

The following is now immediate from the previous result and Theorem 4.4.

6.6. Corollary. Let \( \mathcal{O} \) be an ultra-DVR. If \( R \) is an isodimensional pseudo-regular local \( \mathcal{O} \)-affine algebra, then \( R \) is a domain and every localization of \( R \) with respect to a prime ideal containing \( \pi \) is again pseudo-regular.

In fact, the restricted ultraproduct \( R \) of regular local \( \mathcal{O}_w \)-algebras \( R_w \) of uniformly bounded \( \mathcal{O}_w \)-complexity is pseudo-regular and isodimensional. Indeed, we already observed that then \( R \) is isodimensional, and therefore by Theorem 6.5, pseudo-regular. For a homological characterization of pseudo-regularity, see Corollary 11.3 below.

6.7. Example. If \( R \) denotes the localization of \( \mathcal{O}[X, Y]/(X^2 + Y^3 + \pi) \) at the maximal ideal generated by \( X, Y \) and \( \pi \), then \( R \) is pseudo-regular (namely \( X \) and \( Y \) generate the maximal ideal, so \( \epsilon = 2 \), and since \( R/\pi R \) has dimension one, \( d = 2 \) as well). Note though that \( R/\pi R \) is not regular.

Transfer. Let me elaborate on why the results in this section are instances of transfer between positive and mixed characteristic. Suppose \( \mathcal{O} \) is alternatively realized as the ultraproduct of discrete valuation rings \( \mathcal{O}_w \). Note that this not imply that \( \mathcal{O}_w \) and \( \mathcal{O}_w \) are almost all pair-wise isomorphic. In fact, in the next sections, one set of discrete valuation rings will be of mixed characteristic and the other set of prime characteristic. Let us put \( \bar{A}_w := \mathcal{O}_w[X] \) and let \( \bar{A}_\infty \) denote their ultraproduct, so that we have also a canonical embedding \( A \to \bar{A}_\infty \). With notation as above, \( R \) has also a non-standard hull and approximations with respect to this second set of discrete valuation rings: let us denote them by \( \bar{R}_\infty \) and \( \bar{R}_w \), respectively. Suppose \( \mathcal{O}_w \) and \( \mathcal{O}_w \) have pair-wise isomorphic residue fields (as will be the case). Since the \( R_w/\pi_w R_w \) are an approximation of the \( \kappa \)-algebra \( R/\pi R \) (in the sense of [11]) and, mutatis mutandis, so are the \( \bar{R}_w/\bar{\pi}_w \bar{R}_w \), where \( \bar{\pi}_w \) is a uniformizing parameter of \( \mathcal{O}_w \), we get from [11] 3.2.3] that almost all \( R_w/\pi_w R_w \) are isomorphic to \( R_w/\pi_w R_w \). Therefore, if we assume that there is no torsion, then \( R_w \) and \( \bar{R}_w \) have the same dimension, and one set consists of almost all Cohen-Macaulay local rings if, and only if, the other set does (note that this argument does not yet use the above pseudo notions). However, this argument breaks down in the presence of torsion, or, when
we want to transfer the regularity property. This can be overcome by using the notions defined in this section, provided we have a uniform upper bound on the parameter degree.

Suppose, for some $d, e \in \mathbb{N}$, that almost all $R_w$ have dimension $d$ and parameter degree at most $e$. Note that in view of Corollary 5.12 this last condition is automatically satisfied if almost all $R_w$ are torsion-free over $\mathcal{O}_w$; and that it is implied by the assumption that almost all $R_w$ have uniformly bounded homological multiplicity (see [35 Corollary 4.6]). Applying Proposition 5.11 twice gives first that $R$ is isodimensional, with pseudo-dimension $d$, and then that almost all $R_w$ have dimension $d$ and uniformly bounded parameter degree. Now, Theorems 6.3 and 6.5 tell us that almost all $R_w$ are respectively Cohen-Macaulay or regular, if, and only if, almost all $\tilde{R}_w$ are.

7. Big Cohen-Macaulay Algebras

In [39], ultraproducts of absolute integral closures in characteristic $p$ were used to define big Cohen-Macaulay algebras over $\mathbb{C}$. This same process can be used in the current mixed characteristic setting. Recall that for an arbitrary domain $B$, we define its absolute integral closure as the integral closure of $B$ in some algebraic closure of its field of fractions and denote it $B^\circ$. This is uniquely defined up to $B$-algebra isomorphism.

For each prime number $p$, let $\mathcal{O}_p^{\text{mix}}$ be a mixed characteristic complete discrete valuation ring with uniformizing parameter $\pi_p$ and residue field $\kappa_p$ of characteristic $p$, and let $\mathcal{O}$, $\pi$ and $\kappa$ be their respective ultraproducts. Put $\mathcal{O}_p^{\text{eq}} := \kappa_p[[t]]$, for $t$ a single variable. By the Ax-Kochen-Ershov Theorem, $\mathcal{O}$ is isomorphic to the ultraproduct of the $\mathcal{O}_p^{\text{eq}}$. Let $\mathcal{O}$ denote the ideal of infinitesimals of $\mathcal{O}$, that is to say, the intersection of all $\pi^n\mathcal{O}$. As before, we put $A := \mathcal{O}[X]$, for a fixed tuple of variables $X$, and let $A^{\text{eq}}_\infty$ and $A^{\text{mix}}_\infty$ be its respective equicharacteristic and mixed characteristic non-standard hull, that is to say, the ultraproduct of respectively the $A^{\text{eq}}_p := \mathcal{O}_p^{\text{eq}}[X]$ and the $A^{\text{mix}}_p := \mathcal{O}_p^{\text{mix}}[X]$.

Throughout, $R$ will be a local $\mathcal{O}$-affine domain with $R^{\text{eq}}_p$ and $R^{\text{mix}}_\infty$ respectively an equicharacteristic approximation and the equicharacteristic non-standard hull of $R$ (so that $R^{\text{eq}}_\infty$ is the ultraproduct of the $R^{\text{eq}}_p$). By Theorem 5.12 almost all $R^{\text{eq}}_p$ are local domains.

7.1. Definition. Define $B(R)$ as the ultraproduct of the $(R^{\text{eq}}_p)^\circ$.

Since $(R^{\text{eq}}_p)^\circ$ is well-defined up to $R^{\text{eq}}_p$-algebra isomorphism, we have that $B(R)$ is well-defined up to $R$-algebra isomorphism. Moreover, this construction is weakly functorial in the following sense. Let $R \rightarrow S$ be an $\mathcal{O}$-algebra homomorphism between local $\mathcal{O}$-affine domains. This induces $\mathcal{O}$-algebra homomorphisms $R^{\text{eq}}_p \rightarrow S^{\text{eq}}_p$ of the corresponding equicharacteristic approximations. These in turn yield homomorphisms $(R^{\text{eq}}_p)^\circ \rightarrow (S^{\text{eq}}_p)^\circ$ between the absolute integral closures. Taking ultraproducts, we get an $\mathcal{O}$-algebra homomorphism $B(R) \rightarrow B(S)$ and a commutative diagram

$$\begin{array}{ccc}
R & \rightarrow & S \\
\downarrow & & \downarrow \\
B(R) & \rightarrow & B(S).
\end{array}$$

(6)

7.2. Theorem. If $R$ is a local $\mathcal{O}$-affine domain, then any generic sequence in $R$ is $B(R)$-regular.
Proof. Let $R_p^{eq}$ and $R_p^{eq}_p$ be respectively, the equicharacteristic non-standard hull and an equicharacteristic approximation of $R$. Let $x$ be a generic sequence, and let $x_p$ be an approximation of $x$. By Corollary 5.9 almost all $x_p$ are systems of parameters in $R_p^{eq}$, whence are $(R_p^{eq})^+$-regular by [17]. By Łos’ Theorem, $x$ is $\mathcal{B}(R)$-regular. □

8. IMPROVED NEW INTERSECTION THEOREM

The remaining sections will establish various asymptotic versions in mixed characteristic of the Homological Conjectures listed in the abstract. We start with discussing Intersection Theorems. By [28], we now know that the New Intersection Theorem holds for all Noetherian local rings. However, this is not yet known for the Improved New Intersection Theorem. We need some terminology and notation (all taken from [6]).

Let $C$ be an arbitrary Noetherian local ring and $\varphi : C^a \to C^b$ a linear map between finite free $C$-modules. We will always think of $\varphi$ as an $(a \times b)$-matrix over $C$. For $r > 0$, recall that the $r$-th Fitting ideal of $\varphi$, denoted $I_r(\varphi)$, is the ideal in $C$ generated by all $(r \times r)$ minors of $\varphi$; if $r$ exceeds the size of the matrix, we put $I_r(\varphi) := (0)$.

With a finite free complex over $C$ we mean a complex

$$(F_\bullet) : 0 \to C^{a_s} \xrightarrow{\varphi_s} C^{a_{s-1}} \xrightarrow{\varphi_{s-1}} \cdots \xrightarrow{\varphi_2} C^{a_1} \xrightarrow{\varphi_1} C^{a_0} \to 0.$$ 

We call $s$ the length of the complex, and for each $i$, we define $r_i := \sum_{j=1}^{s} (-1)^{j-i}a_j$.

We will refer to $r_i$ as the expected rank of $\varphi_i$. We will call the residue ring $C/I_{r_i}(\varphi_i)$ the $i$-th Fitting ring of $F_\bullet$ and we will denote it $\mathfrak{R}_i(F_\bullet)$.

The $i$-th homology of $F_\bullet$ is by definition the quotient module

$$H_i(F_\bullet) := \ker(\varphi_i)/\im(\varphi_{i+1}).$$

We call $F_\bullet$ acyclic, if all $H_i(F_\bullet) = 0$ for $i > 0$. In that case, $F_\bullet$ yields a finite free resolution of $H_0(F_\bullet)$.

In case $C$ is $Z$-algebra with $Z$ a Noetherian local ring, we say that $F_\bullet$ has $Z$-complexity at most $c$, if its length $s$ is at most $c$, if all $a_i < c$, and if every entry of each $\varphi_i$ has $Z$-complexity at most $c$. Below we will say that an element $\tau$ in a homology module $H_i(F_\bullet)$ has $Z$-complexity at most $c$, if it is the image of a tuple in $\ker(\varphi_i)$ of $Z$-complexity at most $c$ (for more details, see [11] below).

8.1. Theorem (Asymptotic Improved New Intersection Theorem). For each $c$, there exists a bound $\text{INIT}(c)$ with the following property. Let $V$ be a mixed characteristic discrete valuation ring and let $(C, \mathfrak{m})$ be a $d$-dimensional local $V$-affine domain. Let $F_\bullet$ be a finite free complex over $C$, such that the $i$-th Fitting ring $\mathfrak{R}_i(F_\bullet)$ has dimension at most $d - i$. Assume $H_0(F_\bullet)$ has a minimal generator $\tau$ (that is to say, $\tau \notin \mathfrak{m}H_0(F_\bullet)$), such that $C\tau$ has finite length.

Assume that $c$ simultaneously bounds the $V$-complexity of $C$, $\tau$ and $F_\bullet$, the parameter degree of each Fitting ring $\mathfrak{R}_i(F_\bullet)$, and the length of $C\tau$. If the characteristic of the residue field of $\Omega$ is bigger than $\text{INIT}(c)$, then $d$ is at most the length of the complex $F_\bullet$.

Proof. If $\pi C = 0$, then $C$ contains the residue field of $V$ and in that case the Theorem is known (see for instance [6] Theorem 9.4.1) or [9][13]). So we may moreover assume that $C$ is flat over $V$. By faithful flat descent, we may replace $V$ and $C$ by $\hat{V}$ and $\hat{C}$, where $\hat{V}$ is the completion of $V$. In other words, we only need to prove the result for a
torsion-free local domain over a complete discrete valuation ring of mixed characteristic. Suppose this last assertion is false for some $c$, so that for each prime number $p$, we can find a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring $\Omega_p^\text{mix}$ with uniformizing parameter $\pi_p$, whose residue field has characteristic $p$;
- a local $\Omega_p^\text{mix}$-affine domain $R_p^\text{mix}$ of $\Omega_p^\text{mix}$-complexity at most $c$;
- a finite free complex $(F_p^\text{mix})_0 \rightarrow (F_p^\text{mix})_1 \rightarrow \cdots$

of length $s$ and of $\Omega_p^\text{mix}$-complexity at most $c$, such that the $i$-th Fitting ring $\mathcal{R}_i(F_p^\text{mix})$ has dimension at most $d - i$ and parameter degree at most $c$;

- a minimal generator $\tau_p$ of $H_0(F_p^\text{mix})$ of $\Omega_p^\text{mix}$-complexity at most $c$, generating a module of length at most $c$,

but such that $s$ is strictly less than the dimension of $R_p^\text{mix}$. Note that, without loss of generality, we may assume that the dimension of $R_p^\text{mix}$ and that all the ranks of $F_p^\text{mix}$ are independent from $p$, since there are only finitely many possibilities, so that precisely one such possibility almost always holds. In particular, the expected ranks do not depend on $p$.

Let $\mathcal{O}$ and $\pi$ be the respective ultraproduct of the $\Omega_p^\text{mix}$ and the $\pi_p$. Let $R$ and $R_\infty$ be the respective restricted ultraproduct and ultraproduct of the $R_p^\text{mix}$. It follows from Theorem 4.2 that $R$ is a local $\mathcal{O}$-affine domain, and from Theorem 4.2 that $R \rightarrow R_\infty$ is faithfully flat. Let $d$ be the pseudo-dimension of $R$, so that almost all $R_p^\text{mix}$ have dimension $d$ by Theorem 5.2. Let $\tau$ be the ultraproduct of the $\tau_p$. It follows from Lemma 5.1 that each $\tau_i$ is already defined over $R$. Hence by Łos’ Theorem, we get that

$$(F_\bullet) \quad 0 \rightarrow R^a_{i+1} \xrightarrow{\varphi_{i+1}} R^a_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} R^a_1 \xrightarrow{\varphi_1} R^a_0 \rightarrow 0$$

is a finite free complex. Let $M$ denote its zero-th homology. Fix some $i$. By Łos’ Theorem, $I_{\tau_i}(\varphi_i, p)$ is an $\Omega_p^\text{mix}$-approximation of $I_{\tau_i}(\varphi_i)$. By the uniform boundedness of the parameter degrees, $\mathcal{R}_i(F_{\bullet})$ is isodimensional by Proposition 5.11. If $d_i$ is the pseudo-dimension of $\mathcal{R}_i(F_{\bullet})$, then $d - d_i$ is equal to the height of almost all $I_{\tau_i}(\varphi_i, p)$ and to the pseudo-height of $I_{\tau_i}(\varphi_i)$, by Theorem 5.14. In particular, by assumption, $i \leq d - d_i$, and therefore, by definition of pseudo-height, we can find a generic sequence $x_i$ in $R$ whose first $i$ entries belong to $I_{\tau_i}(\varphi_i)$.

Let $B := B(R)$. Since $x_i$ is $B$-regular by Theorem 7.2, the grade of $I_{\tau_i}(\varphi_i, p)$ is at least $i$.

Since this holds for all $i$, the Buchsbaum-Eisenbud-Northcott Acyclicity Theorem (6 Theorem 9.1.6)) proves that $F_{\bullet} \otimes_R B$ is acyclic. Since $B$ has depth at least $d$, it follows from [5 Theorem 9.1.2] that the zero-th homology of $F_{\bullet} \otimes_R B$, that is to say, $M \otimes_R B$, has depth at least $d - s$.

Let $\tau$ be the ultraproduct of the $\tau_p$. Note that each $\tau_p$ is by assumption the image of a tuple in $(R_p^\text{mix})_{ao}$ of $\Omega_p^\text{mix}$-complexity at most $c$, so that $\tau$ is already defined over $R$ by Lemma 2.1. By Łos’ Theorem, $\tau$ is a minimal generator of $H_0(F_{\bullet} \otimes R_\infty) = M \otimes R_\infty^\text{mix}$, and by [52 Proposition 1.1] or [53 Proposition 9.1], the length of $R_\infty^\text{mix}$ is at most $c$. By faithful flatness, $\tau \in M - mM$, where $m$ is the maximal ideal of $R$, and $R\tau$ has length at most $c$. In particular, the image of $\tau \otimes 1$ in $M/mM \otimes B/mB$ is non-zero, and therefore...
\[ \tau \otimes 1 \text{ itself is a non-zero element of } M \otimes B. \] Since \( m^c \) annihilates \( \tau \otimes 1 \), we get that \( M \otimes B \) has depth zero. Together with the conclusion from the previous paragraph, we get that \( d \leq s \), contradiction.

9. Monomial and Direct Summand Conjectures

We keep notation as in the previous section, so that in particular \( \mathcal{D} \) will denote the ultraproduct of mixed characteristic complete discrete valuation rings \( \mathcal{D}^\text{mix}_p \). In order to formulate a non-standard version of the Monomial Conjecture, we need some terminology.

Let each \( B \) be a local ring, \( X := (X_1, \ldots, X_d) \) variables and \( A_\infty \) the ultraproduct of the \( C_\nu[X] \). Although each \( C_\nu[X] \) is \( \mathbb{N} \)-graded, it is not true that \( A_\infty \) is \( \mathbb{N}_\infty \)-graded, since we might have infinite sums of monomials in \( A_\infty \). Nonetheless, for each \( \nu_\infty \in \mathbb{N}_\infty \), the element \( X^{\nu_\infty} \) is well-defined, namely, if \( \nu_\infty \) is the ultraprodut of elements \( \nu_w \in \mathbb{N} \), then

\[ X^{\nu_\infty} := \lim_{w \to \infty} X^{\nu_w}. \]

In particular, if \( B_\infty \) is an arbitrary ultraproduct of rings \( B_w \) and if \( x \) is a \( d \)-tuple in \( B_\infty \), then \( x^{\nu_\infty} \) is a well-defined element of \( B_\infty \).

With a cone \( H \) in a semi-group \( \Gamma \) (e.g., \( \Gamma = \mathbb{N}^d \) or \( \Gamma = \mathbb{N}_\infty^d \)), we mean a subset \( H \) of \( \Gamma \) such that \( \nu + \gamma \in H \), for every \( \nu \in H \) and every \( \gamma \in \Gamma \). A cone \( H \) is finitely generated, if there exist \( \nu_1, \ldots, \nu_s \in H \), called generators of the cone, such that

\[ H = \bigcup_i \nu_i + \Gamma. \]

If \( H \) is a cone in \( \mathbb{N}^d \), we denote by \( J_H \) the monomial ideal in \( \mathbb{Z}[Y] \), generated by all \( Y^\nu \) with \( \nu \in H \), where \( Y \) is a \( d \)-tuple of variables. If \( H \) is generated by \( \nu_1, \ldots, \nu_s \), then \( J_H \) is generated by \( X^{\nu_1}, \ldots, X^{\nu_s} \). Conversely, if \( J \) is a monomial ideal in \( \mathbb{Z}[Y] \), then the collection of all \( \nu \) for which \( Y^\nu \in J \), is a cone in \( \mathbb{N}^d \). Since \( \mathbb{Z}[Y] \) is Noetherian, every cone in \( \mathbb{N}^d \) is finitely generated. This is no longer true for a cone in \( \mathbb{N}_\infty^d \).

Let \( B \) be an arbitrary ring. We will use the following well-known fact about regular sequences. If \( x \) is a \( B \)-regular sequence (in fact, it suffices that \( x \) is quasi-regular), \( H \) a cone in \( \mathbb{N}^d \) and \( \nu \notin H \), then \( x^\nu \) does not lie in the ideal \( J_H(x) \) generated by all \( x^\theta \) with \( \theta \in H \).

9.1. Corollary. Let \( R \) be a local \( \mathcal{D} \)-affine domain with equicharacteristic non-standard hull \( R^\text{eq}_\infty \). Let \( x \) be a generic sequence in \( R \), let \( H \) be a cone in \( \mathbb{N}_\infty^d \) and let \( \nu_\infty \in \mathbb{N}_\infty^d \). If \( \nu_\infty \notin H \), then

\[ x^{\nu_\infty} \notin J_H(x) := (x^\mu \mid \mu \in H) R^\text{eq}_\infty. \]

Proof. Suppose \( \text{(7)} \) is false for some choice of cone \( H \) of \( \mathbb{N}_\infty^d \) and some \( \nu_0 \infty \notin H \). In other words, we can find \( f_i_\infty \) in \( R^\text{eq}_\infty \) and tuples \( \nu_i \infty \) in \( H \), such that

\[ x^{\nu_0 \infty} = f_1 x^{\nu_1 \infty} + \cdots + f_s x^{\nu_s \infty}. \]

In order to derive a contradiction, we will argue that such a relation \( \text{(8)} \) cannot hold in \( B(R) \). Indeed, suppose it does. Let \( R^\text{eq}_p \) be an equicharacteristic approximation of \( R \), so that \( B(R) \) is the ultraproduct of the \( (R^\text{eq}_p)^+ \). Choose tuples \( \nu_i \in \mathbb{N} \), elements \( f_i \in (R^\text{eq}_p)^+ \) and tuples \( x_p \) in \( R^p \) whose respective ultraproducts are \( \nu_i \infty \), \( f_i \infty \) and \( x \). By Los’ Theorem, we get that

\[ x^{\nu_0 \infty}_p = f_1 x^{\nu_1 \infty}_p + \cdots + f_s x^{\nu_s \infty}_p. \]
in \((R^{eq}_p)^+\), for almost all \(p\). Łos’ Theorem also yields that \(\nu_{0p}\) does not lie in the cone of \(\mathbb{N}^d\) generated by \(\nu_{1p}, \ldots, \nu_{sp}\), for almost all \(p\). However, \(x\) is \(B(\hat{R})\)-regular by Theorem 7.2, whence, almost all \(x_p\) are \((R^{eq}_p)^+\)-regular by Łos’ Theorem. It follows that (9) cannot hold for those \(p\).

9.2. Theorem (Asymptotic Monomial Conjecture). For each pair \((c, s)\), there exists a bound \(MC(c, s)\) with the following property. Let \(Y\) be a \(d\)-tuple of variables. \(J\) a monomial ideal in \(\mathbb{Z}[Y]\) generated by \(s\) monomials and \(Y^\nu\) a monomial not belonging to \(J\). Let \(V\) be a mixed characteristic discrete valuation ring and let \(C\) be a \(d\)-dimensional local \(V\)-affine domain. Let \(y\) be a system of parameters in \(V\).

If \(C\) and \(y\) have \(V\)-complexity at most \(c\) and \(\pi^e\in yc\), and if the characteristic of the residue field of \(V\) is bigger than \(MC(c, s)\), then

\[y^\nu \notin J(y)C\]

where \(J(y)C\) is the ideal in \(C\) obtained from \(J\) by the substitution \(Y \mapsto y\).

Proof. Note that since \(C\) has \(V\)-complexity at most \(c\), its dimension \(d\) is at most \(c\). By faithful flat descent, we may replace \(V\) and \(C\) by \(\hat{V}\) and \(\hat{V} \otimes_V C\), where \(\hat{V}\) is the completion of \(V\). In other words, we only need to prove the result for complete discrete valuation rings of mixed characteristic. Suppose this is false for some \((c, s)\), so that we can find for each prime number \(p\)

- a mixed characteristic complete discrete valuation ring \(\mathcal{O}_p^{mix}\) with uniformizing parameter \(\pi_p\), whose residue field has characteristic \(p\),
- a local \(\mathcal{O}_p^{mix}\)-affine domain \(R_p^{mix}\) of \(\mathcal{O}_p^{mix}\)-complexity at most \(c\),
- tuples \(\nu_{0p}, \ldots, \nu_{sp}\) with \(\nu_{0p}\) not in the cone generated by the remaining tuples,
- a system of parameters \(y_p\) of \(\mathcal{O}_p^{mix}\)-complexity at most \(c\) generating an ideal containing \((\pi_p)^c\),

such that

\[y_p^{\nu_{0p}} \in (y_p^{\nu_{1p}}, \ldots, y_p^{\nu_{sp}})R_p^{mix}\]

Let \(\mathcal{D}\) be the ultraproduct of the \(\mathcal{O}_p^{mix}\) and let \(\hat{R}\) and \(R^{mix}_\infty\) be the respective restricted ultraproduct and ultraproduct of the \(R_p^{mix}\). Since \(\hat{R}\) is then a domain, it is isodimensional. Let \(y\) and \(\nu_{\infty}\) be the respective ultraproducts of \(y_p\) and \(\nu_{ip}\). The sequence \(y\) is defined over \(R\), by Lemma 4.1 and is generic in \(R\), by Corollary 5.9. By Łos’ Theorem and Theorem 4.2 we get that

\[y^{\nu_{0\infty}} \in (y^{\nu_{1\infty}}, \ldots, y^{\nu_{s\infty}})R\]

However, this contradicts Corollary 5.1 for \(H\) the cone of \(\mathbb{N}^d\) generated by \(\nu_{1\infty}, \ldots, \nu_{s\infty}\).

9.3. Remark. Using some results from [44], we can remove the restriction on \(C\) to be a domain. Namely, by the usual argument, we reduce to the domain case by killing a minimal prime \(p\) of \(C\) of maximal dimension (that is to say, so that \(\dim C = \dim C/p\)). However, in order to apply the theorem to the domain \(C/p\), we must be guaranteed that its \(V\)-complexity is at most \(c’\), for some \(c’\) only depending on \(c\). Such a bound does indeed exists by [44] Theorems 9.2 and 9.12].

9.4. Theorem (Asymptotic Direct Summand Conjecture). For each \(c\), we can find a bound \(DSI(c)\) with the following property. Let \(V\) be a mixed characteristic discrete valuation ring and let \(C \to D\) be a finite injective local homomorphism between local \(V\)-affine algebras, with \(C\) regular.
If $C \to D$ has $V$-complexity at most $c$ and if the characteristic of the residue field of $V$ is bigger than the bound $DS(c)$, then $C$ is a direct summand of $D$ (as a $C$-module).

Proof. If $\pi C = 0$, we are in the equicharacteristic case and the result is well-known. So we may assume that $V \subset C$. We leave it to the reader to make the reduction to the case that $V$ is complete and $D$ is torsion-free over $V$. Choose a regular system of parameters $x := (x_1, \ldots, x_d)$ of $C$ of $V$-complexity at most $c$. Since $C \to D$ is finite, $x$ is a system of parameters in $D$ with $\pi \in xD$. Since $C \to D$ has $V$-complexity at most $c$, the image of $x$ in $D$ has $V$-complexity at most $c^2$. Hence if the characteristic of the residue field is bigger than $MC(c^2, d)$, then by Theorem 10.2

$$(x_1x_2 \cdots x_d)^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})D,$$

for any $t$. By [6 Lemma 9.2.2], this implies that $C$ is a direct summand of $D$. $\square$

10. PURE SUBRINGS OF REGULAR RINGS

We keep notation as in the previous section, so that in particular $\mathcal{O}$ will denote the ultraproduct of mixed characteristic complete discrete valuation rings $\mathcal{O}^{\text{mix}}$. Our goal is to show an asymptotic version of the Hochster-Roberts Theorem in [20]. Recall that a ring homomorphism $C \to D$ is called cyclically pure if every ideal $I$ in $C$ is extended from $D$, that is to say, if $I = ID \cap C$.

10.1. Theorem. If $R$ is a pseudo-regular isodimensional local $\mathcal{O}$-affine algebra, then $R \to B(R)$ is faithfully flat.

Proof. Let $L$ be a linear form in a finite number of variables $Y$ with coefficients in $R$ and let $b$ be a solution in $B := B(R)$ of $L = 0$. Let $R^\text{eq}_{p,s}, L^\text{eq}_p$ and $b^\text{eq}_p$ be equicharacteristic approximations of $R, L$ and $b$ respectively. By Łos’ Theorem, $b^\text{eq}_p$ is a solution in $(R^\text{eq}_p)^+$ of the linear equation $L^\text{eq}_p = 0$. By [2 Corollary 4.27], we can find tuples $a^\text{eq}_1, \ldots, a^\text{eq}_s$ over $R^\text{eq}_p$ generating the module of solutions of $L^\text{eq}_p = 0$, all of $\mathcal{O}^\text{eq}$-complexity at most $c$, for some $c$ independent from $p$ and $s$. Let $a_1, \ldots, a_s$ be the respective ultraproducts, which are then defined over $R$ by Lemma 2.1. By Łos’ Theorem, $L(a_i) = 0$, for each $i$. On the other hand, almost all $R^\text{eq}_p$ are regular, by Theorem 6.3. Therefore, $R^\text{eq}_p \to (R^\text{eq}_p)^+$ is flat by [24 Theorem 9.1]. Hence we can write $b^\text{eq}_p$ as a linear combination over $(R^\text{eq}_p)^+$ of the $a^\text{eq}_i$. By Łos’ Theorem, $b$ is a $B$-linear combination of the solutions $a_i$, showing that $R \to B$ is flat whence faithfully flat. $\square$

10.2. Proposition. Let $R \to S$ be an injective homomorphism of local isodimensional $\mathcal{O}$-affine algebras. If $R/\pi R \to S/\pi S$ is cyclically pure and $S$ is a pseudo-regular local ring, then $R$ is pseudo-Cohen-Macaulay.

Proof. Since $S$ is a domain by Corollary 6.6, so is $R$ by cyclic purity. If $\pi R = 0$, we are in an equicharacteristic Noetherian situation and the statement becomes the Hochster-Roberts Theorem 20. Therefore, we may assume $\pi$ is $R$-regular, so that we can choose a generic sequence $x := (x_1, \ldots, x_d)$ in $R$ with $x_1 = \pi$. For each $n$, let $I_n := (x_1, \ldots, x_n)R$. Suppose $rx_{n+1} \in I_n$, for some $r \in R$. By Theorem 7.2 we have that $x$ is a $B(R)$-regular sequence. Therefore, $r \in I_nB(R)$. Since the homomorphism $R \to S$ induces a homomorphism $B(R) \to B(S)$, we get that $r \in I_nB(S)$. By Theorem 10.1 we have $I_nB(S) \cap S = I_nS$, so that $r \in I_nS$. Using finally that $R/\pi R \to S/\pi S$ is cyclically pure, we get $r \in I_n$. This shows that $x$ is $R$-regular, so that $R$ has depth at least $d$ and hence is pseudo-Cohen-Macaulay. $\square$
10.3. **Theorem (Asymptotic Hochster-Roberts Theorem).** For each \( c \), we can find a bound \( HR(c) \) with the following property. Let \( V \) be a mixed characteristic discrete valuation ring and let \( C \to D \) be a cyclically pure local homomorphism of local \( V \)-algebras with \( D \) regular.

If \( C \to D \) has complexity at most \( c \), then \( C \) is Cohen-Macaulay, provided the characteristic of the residue field of \( V \) is at least \( HR(c) \).

**Proof.** As before, we may reduce to the case that \( V \) is complete and that \( V \subset C \). Suppose this assertion is then false for some \( c \), so that we can find for each prime number \( p \), a mixed characteristic complete discrete valuation ring \( \mathcal{O}_{p}^{\text{mix}} \) with residue field of characteristic \( p \) and a cyclically pure homomorphism \( \mathcal{O}_{p}^{\text{mix}} \to S_{p}^{\text{mix}} \) of \( \mathcal{O}_{p}^{\text{mix}} \)-complexity at most \( c \) between local \( \mathcal{O}_{p}^{\text{mix}} \)-algebras, such that \( S_{p}^{\text{mix}} \) is regular but \( R_{p}^{\text{mix}} \) is not Cohen-Macaulay.

Let \( R \to S \) and \( R_{\infty}^{\text{mix}} \to S_{\infty}^{\text{mix}} \) be respectively the restricted ultraproduct and the ultraproduct of the \( R_{p}^{\text{mix}} \to S_{p}^{\text{mix}} \). **Theorem 6.5** implies that \( R \) is not pseudo-Cohen-Macaulay and **Theorem 6.3** that \( S \) is pseudo-regular. I claim that \( R/\pi R \to S/\pi S \) is cyclically pure.

Assuming this claim, we get from **Proposition 10.2** that \( R \) is pseudo-Cohen-Macaulay, contradiction.

To prove the claim, let \( I \) be an arbitrary ideal in \( R \) containing \( \pi \). Let \( r \in IS \cap R \), so that we need to show that \( r \in I \). Note that \( I \) is finitely generated, as \( R/\pi R \) is Noetherian.

Let \( I_{p}^{\text{mix}} \) and \( r_{p}^{\text{mix}} \) be mixed characteristic approximations in \( R_{p}^{\text{mix}} \) of \( I \) and \( r \) respectively. By Łos’ Theorem, almost all \( I_{p}^{\text{mix}} \) lie in \( I_{p}^{\text{mix}} \cap R_{p}^{\text{mix}} \), whence in \( I_{p}^{\text{mix}} \) by cyclical purity. Hence by Łos’ Theorem \( r \in IR_{\infty}^{\text{mix}} \), so that \( r \in I \) by faithful flatness, as we needed to prove. \( \square \)

11. **Asymptotic vanishing for maps of Tor**

11.1. **Proposition.** If \( R \to S \) is an integral extension of local \( \mathcal{O} \)-affine domains, then \( B(R) = B(S) \).

**Proof.** Since any integral extension is a direct limit of finite extensions, we may assume that \( R \to S \) is finite. Choose an equicharacteristic approximation \( R_{p}^{\text{eq}} \to S_{p}^{\text{eq}} \) of \( R \to S \).

By **Theorem 4.4** and Łos’ Theorem, almost all \( R_{p}^{\text{eq}} \) and \( S_{p}^{\text{eq}} \) are domains and the extension \( B_{p}^{\text{eq}} \to S_{p}^{\text{eq}} \) is finite. Therefore, \((R_{p}^{\text{eq}})^{+} = (S_{p}^{\text{eq}})^{+} \), so that in the ultraproduct, we get \( B(R) = B(S) \). \( \square \)

11.2. **Theorem.** Let \( R \to S \to T \) be local \( \mathcal{O} \)-algebra homomorphisms between local \( \mathcal{O} \)-affine domains. Assume that \( R \) and \( T \) are pseudo-regular and that \( R \to S \) is integral and injective. For every \( R \)-module \( M \), the induced map \( \text{Tor}_{i}^{R}(S, M) \to \text{Tor}_{i}^{R}(T, M) \) is zero, for all \( i \geq 1 \).

**Proof.** Since \( R \to S \) is integral, we have that \( B(R) = B(S) \) by **Proposition 11.1**. Therefore, \( \text{Tor}_{i}^{R}(B(S), M) = 0 \), for all \( i \geq 1 \), by **Theorem 10.1**. By weak functoriality, we have, for each \( i \geq 1 \), a commutative diagram

\[
\begin{array}{ccc}
\text{Tor}_{i}^{R}(S, M) & \xrightarrow{f} & \text{Tor}_{i}^{R}(T, M) \\
\downarrow & & \downarrow \\
0 & = & \text{Tor}_{i}^{R}(B(S), M) \\
& & \xrightarrow{g} \\
& & \text{Tor}_{i}^{R}(B(T), M).
\end{array}
\]

(10)
In particular, the composite map in this diagram is zero, so that the statement follows once we have shown that the last vertical map is injective. However, this is clear, since $T \to B(T)$ is faithfully flat by Theorem 10.1. □

To make use of this theorem, we need to incorporate modules in our present setup. I will not provide full details, since many results are completely analogous to the case where we work over a field, and this has been treated in detail in [33]. Of course, we do not have the full equivalent of Theorem 2.2 to our disposal, but for most purposes, the flatness result in Theorem 4.2 suffices.

Let $C$ be an arbitrary Noetherian local ring and $M$ a finitely generated module over $C$. We say that a finite free complex $F_{\bullet}$ is a finite free resolution of $M$ up to level $n$, if $H_0(F_{\bullet}) = M$ and all $H_j(F_{\bullet}) = 0$, for $j = 1, \ldots, n$. Hence, if $n$ is strictly larger than the length of $F_{\bullet}$, then this just means that $F_{\bullet}$ is a finite free resolution of $M$ (compare with the terminology introduced in the beginning of §).

Suppose moreover that $Z$ is a Noetherian local ring and $C$ is a local Z-affine algebra. We say that $M$ has Z-complexity at most $c$, if $C$ has Z-complexity at most $c$ and if $M$ can be realized as the cokernel of a matrix of Z-complexity at most $c$ (meaning that its size is at most $c$ and all its entries have Z-complexity at most $c$).

11.3. Proposition. For each pair $(c, n)$, there exist bounds $\text{RES}(c, n)$ and $\text{HOM}(c)$ with the following property. Let $V$ be a mixed characteristic discrete valuation ring and let $C$ be a local Z-affine algebra of V-complexity at most $c$.

- Any finitely generated $C$-module of V-complexity at most $c$, admits a (minimal) finite free resolution up to level $n$ of $V$-complexity at most $\text{RES}(c, n)$.
- Any finite free complex over $C$ of $V$-complexity at most $c$, has homology modules of $Z$-complexity at most $\text{HOM}(c)$.

Proof. The first assertion follows by induction from the already quoted [2 Corollary 4.27] on bounds of syzygies (compare with the proof of [33 Theorem 4.3]). It is also clear that we may take this resolution to be minimal (=every tuple in one of the kernels has its entries in the maximal ideal), if we choose to do so. The second assertion is derived from the flatness of the non-standard hull in exactly the same manner as the corresponding result for fields was obtained in [33 Lemma 4.2 and Theorem 4.3]. □

Recall that the weak global dimension of a ring $C$ is by definition the supremum (possibly infinite) of the weak homological dimensions (=flat dimensions) of all $C$-modules, that is to say, the supremum of all $n$ for which $\text{Tor}_n^C(\cdot, \cdot)$ is not identically zero.

11.4. Corollary. A pseudo-regular local $\mathcal{O}$-affine domain has finite weak global dimension.

Proof. Let $R$ be a pseudo-regular local $\mathcal{O}$-affine domain. Given an arbitrary $R$-module $M$, we have to show that $M$ has finite flat dimension, that is to say, admits a finite flat resolution. Assume first that $M$ is finitely presented. Hence we can realize $M$ as the cokernel of some matrix $\Gamma$. Let $R_{\infty}$ be the non-standard hull of $R$ and let $R_w$ and $\Gamma_w$ be approximations of $R$ and $\Gamma$ respectively. Let $M_w$ be the cokernel of $\Gamma_w$. Let $d$ be the pseudo-dimension of $R$. By Proposition 11.3 we can find a finite free resolution $F_{\bullet, w}$ up to level $d$ of each $M_w$, of $\mathcal{O}_w$-complexity at most $c$, for some $c$ depending only on $\Gamma$, whence independent from $w$. Since almost all $R_w$ are regular by Theorem 5.5 and have dimension $d$ by Theorem 5.5, each $M_w$ has projective dimension at most $d$, so that we can even assume that $F_{\bullet, w}$ is a finite free resolution of $M_w$. Let $F_{\bullet}$ be the restricted ultraproduction of
the $F_{w \bullet}$ (that is to say, the finite free complex over $R$ given by the restricted ultraproduct of the matrices in $F_{w \bullet}$). By Łos’ Theorem, $F_{w \bullet} \otimes_R R_\infty$ is a free resolution of $M \otimes_R R_\infty$, and therefore by faithful flat descent, $F_{w \bullet}$ is a free resolution of $M$, proving that $M$ has projective dimension at most $d$.

Assume now that $M$ is arbitrary. By what we just proved, we have for every finitely generated ideal $I$ of $R$ that $\text{Tor}^R_{d+1}(M, R/I)$ vanishes. Hence, if $H$ is a $d$-th syzygy of $M$, then $\text{Tor}^R_d(H, R/I) = 0$. Since this holds for every finitely generated ideal of $R$, we get from [25, Theorem 7.7] that $H$ is flat over $R$. Hence $M$ has finite flat dimension (at most $d$).

By [24], any flat $R$-module has projective dimension less than the finitistic global dimension of $R$ (the supremum of all projective dimensions of modules of finite projective dimension). Therefore, if, moreover, the finitistic global dimension of $R$ is finite, then so is its global dimension. For a Noetherian local ring, its global dimension is finite if, and only if, its residue field has finite projective dimension (if, and only if, it is regular). The following is the pseudo analogue of this.

11.5. Corollary. A local $\mathcal{O}$-affine domain is pseudo-regular if, and only if, it is a coherent regular ring in the sense of [5] if, and only if, its residue field has finite projective dimension.

Proof. In [5] or [10, §5], a local ring $R$ is called a coherent regular ring, if every finitely generated ideal of $R$ has finite projective dimension. If $R$ is pseudo-regular local $\mathcal{O}$-affine domain, then this property was established in the course of the proof of Corollary 11.4. Conversely, suppose $R$ is a local $\mathcal{O}$-affine domain in which every finitely generated ideal has finite projective dimension. In particular, its residue field $K$ admits a finite projective resolution, say of length $n$. Let $R_w$ and $k_w$ be approximations of $R$ and $K$ respectively. Since the $k_w$ have uniformly bounded $\mathcal{O}_w$-complexity, Proposition 11.3 allows us to take a minimal finite free resolution $F_{w \bullet}$ of $k_w$ up to level $n$, with the property that each $F_{w \bullet}$ has $\mathcal{O}_w$-complexity at most $c$, for some $c$ independent from $w$. Let $F_{w \bullet}$ be the restricted ultraproduct of these resolutions. By Łos’ Theorem, $F_{w \bullet}$ is a minimal finite free resolution of $k$ up to level $n$. Since $F_{w \bullet}$ is minimal and since $k$ has by assumption projective dimension $n$, it follows that the final morphism (that is to say, the left most arrow) in $F_{w \bullet}$ is injective. By Łos’ Theorem, so are almost all final morphisms in $F_{w \bullet}$, showing that almost all $k_w$ have finite projective dimension. By Serre’s characterization of regular local rings, we conclude that almost all $R_w$ are regular. Theorem 6.5 then yields that $R$ is pseudo-regular, as we wanted to show. \qed

Closer inspection of the above argument shows that the residue field of a pseudo-regular local $\mathcal{O}$-affine domain $R$ has projective dimension equal to the pseudo-dimension of $R$. In particular, the weak global dimension of $R$ is equal to its pseudo-dimension.

11.6. Theorem (Asymptotic Vanishing for Maps of Tors). For each $c$, we can find a bound $VT(c)$ with the following property. Let $V$ be a mixed characteristic discrete valuation ring, let $C \to D \to E$ be local $V$-algebra homomorphisms of local domains and let $M$ be a finitely generated $R$-module. Assume $C$ and $E$ are regular and $C \to D$ is integral and injective.

If $M$ and all homomorphisms have $V$-complexity at most $c$, then the natural map $\text{Tor}^C_n(D, M) \to \text{Tor}^C_n(E, M)$ is zero, for all $n \geq 1$, provided the characteristic of the residue field of $V$ is at least $VT(c)$. 
Proof. Note that $C$ has dimension at most $c$ and therefore $\text{Tor}_n^C(\cdot, \cdot)$ vanishes identically for all $n > c$ and the assertion trivially holds for these values of $n$. If $\pi C = 0$, we are in the equicharacteristic case and the result is known in that case (22, Theorem 9.7). Hence we may assume that all rings are torsion-free over $\mathcal{O}$. Moreover, without loss of generality, we may assume that $V$ is complete. Suppose even in this restricted setting, there is no such bound for $c$ and some $1 \leq n < c$. Hence, for each prime number $p$, we can find a counterexample consisting of the following data:

- a mixed characteristic complete discrete valuation ring $\mathcal{O}^{\text{mix}}_p$ of residual characteristic $p$;
- local homomorphisms $R^{\text{mix}}_p \to S^{\text{mix}}_p \to T^{\text{mix}}_p$ of torsion-free local domains of $\mathcal{O}^{\text{mix}}_p$-complexity at most $c$, with $R^{\text{mix}}_p$ and $T^{\text{mix}}_p$ regular and $R^{\text{mix}}_p \to S^{\text{mix}}_p$ integral;
- a finitely generated $R^{\text{mix}}_p$-module $M^{\text{mix}}_p$ of $\mathcal{O}^{\text{mix}}_p$-complexity at most $c$;

such that

$$\text{Tor}_n^{R^{\text{mix}}_p}(S^{\text{mix}}_p, M^{\text{mix}}_p) \to \text{Tor}_n^{T^{\text{mix}}_p}(T^{\text{mix}}_p, M^{\text{mix}}_p)$$

is non-zero.

Let $\mathcal{O}$ be the ultraproduct of the $\mathcal{O}^{\text{mix}}_p$ and let $M$ be the restricted ultraproduct of the $M^{\text{mix}}_p$ (that is to say, $M$ is the cokernel of the restricted ultraproduct of matrices whose cokernel is $M^{\text{mix}}_p$). Let $R \to S \to T$ and $R^{\text{mix}}_\infty \to S^{\text{mix}}_\infty \to T^{\text{mix}}_\infty$ be the respective restricted ultraproduct and mixed characteristic ultraproduct of the homomorphisms $R^{\text{mix}}_p \to S^{\text{mix}}_p \to T^{\text{mix}}_p$. It follows from Corollary 6.6 and Theorems 4.4 and 6.5 that $R$, $S$ and $T$ are local $\mathcal{O}$-affine domains with $R$ and $T$ pseudo-regular. By Łos’ Theorem, $R^{\text{mix}}_\infty \to S^{\text{mix}}_\infty$ is integral, whence so is $R \to S$ by faithful flat descent. By Theorem 11.2, the natural homomorphism $\text{Tor}_n^R(S, M) \to \text{Tor}_n^R(T, M)$ is therefore zero.

By Proposition 11.3 we can find a finite free resolution $F^{\text{mix}}_p$ of $M^{\text{mix}}_p$ up to level $n$, of $\mathcal{O}^{\text{mix}}_p$-complexity at most $c'$, for some $c'$ only depending on $c$ and $n$. By definition of Tor, we have isomorphisms

$$\text{Tor}_n^{R^{\text{mix}}_p}(S^{\text{mix}}_p, M^{\text{mix}}_p) \cong H_n(F^{\text{mix}}_p \otimes^{\text{mix}}_p S^{\text{mix}}_p)$$

$$\text{Tor}_n^{T^{\text{mix}}_p}(T^{\text{mix}}_p, M^{\text{mix}}_p) \cong H_n(F^{\text{mix}}_p \otimes^{\text{mix}}_p T^{\text{mix}}_p)$$

In particular, by Proposition 11.4 both modules have $\mathcal{O}^{\text{mix}}_p$-complexity at most $c''$, for some $c''$ only depending on $c'$, whence only on $(c, n)$. Let $H_M$ and $H_T$ be their respective restricted ultraproduct, so that by Łos’ Theorem and our assumptions, $H_M \to H_T$ is non-zero. Let $F_*$ be the restricted ultraproduct of the $F^{\text{mix}}_p$. By Łos’ Theorem and faithful flatness, $H_M$ and $H_T$ are isomorphic respectively with $H_M(F_* \otimes_R S)$ and $H_M(F_* \otimes_R T)$. However, since $F_*$ is a finite free resolution of $M$ up to level $n$ by another application of Łos’ Theorem and faithful flatness, these two modules are isomorphic respectively with $\text{Tor}_n^R(S, M)$ and $\text{Tor}_n^R(T, M)$. Hence the natural map between these two modules is non-zero, contradiction. \(\square\)

References

1. M. Aschenbrenner, Remarks on a paper by Sabbagh, manuscript.
2. ______, Ideal membership in polynomial rings over the integers, Ph.D. thesis, University of Illinois, Urbana-Champaign, 2001.
3. J. Ax and S. Kochen, Diophantine problems over local fields I, II, Amer. J. Math. 87 (1965), 605–630, 631–648.
4. J. Becker, J. Denef, L. van den Dries, and L. Lipshitz, Ultraproducts and approximation in local rings I, Invent. Math. 51 (1979), 189–203.
HANS SCHOUTENS

[5] J. Bertin, *Anneaux coherents reguliers*, C. R. Acad. Sci. Paris 273 (1971), 1–2.

[6] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

[7] Y. Eršov, *On the elementary theory of maximal normed fields I*, Algebra i Logika 4 (1965), 31–69.

[8] Y. Eršov, *On the elementary theory of maximal normed fields II*, Algebra i Logika 5 (1966), 8–40.

[9] E. Evans and P. Griffith, *The syzygy problem*, Ann. of Math. 114 (1981), 323–333.

[10] S. Glaz, *Commutative coherent rings: Historical perspective and current developments*, Nieuw Arch. Wisk. 10 (1992), 37–56.

[11] R. Heitmann, *The direct summand conjecture in dimension three*, Ann. of Math. (to appear).

[12] M. Hochster, *Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors*, Proceedings of the conference on commutative algebra, Kingston 1975, Queen’s Papers in Pure and Applied Math., vol. 42, 1975, pp. 106–195.

[13] M. Hochster, *Topics in the homological theory of modules over commutative rings*, CBMS Regional Conf. Ser. in Math, vol. 24, Amer. Math. Soc., Providence, RI, 1975.

[14] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. Algebra 84 (1983), 503–553.

[15] M. Hochster, *Solid closure*, Commutative Algebra: Syzygies, Multiplicities and Birational Algebra (Providence), Contemp. Math., vol. 159, Amer. Math. Soc., 1994, pp. 103–172.

[16] M. Hochster, *Big Cohen-Macaulay Algebras in Dimension Three via Heitmann’s theorem*, J. Algebra 254 (2002), 395–408.

[17] M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Ann. of Math. 135 (1992), 53–89.

[18] M. Hochster, *Tight closure*, Commutative Algebra, vol. 15, 1997, pp. 305–338.

[19] M. Hochster, *Tight closure in equal characteristic zero*, preprint on http://www.math.lsa.umich.edu/~hochster/tcz.ps.Z, 2000.

[20] M. Hochster and J. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. in Math. 13 (1974), 115–175.

[21] W. Hodges, *Model theory*, Cambridge University Press, Cambridge, 1993.

[22] C. Huneke, *Tight closure and its applications*, CBMS Regional Conf. Ser. in Math, vol. 88, Amer. Math. Soc., 1996.

[23] C. Jensen and H. Lenzing, *Model theoretic algebra*, Gordon and Breach Science Publishers, 1989.

[24] C.U. Jensen, *Vanishing of Ext*, J. Algebra 15 (1970), 151–166.

[25] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[26] C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie etale*, Inst. Hautes Études Sci. Publ. Math. 42 (1972), 47–119.

[27] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, Invent. Math. 13 (1971), 1–89.

[28] P. Roberts, *Le théorème d’intersections*, C. R. Acad. Sci. Paris 304 (1987), 177–180.

[29] G. Sabbagh, *Coherence of polynomial rings and bounds in polynomial ideals*, J. Algebra 31 (1974), 499–507.

[30] K. Schmidt, *Bounds and definability over fields*, J. Reine Angew. Math. 377 (1987), 18–39.

[31] K. Schmidt and L. van den Dries, *Bounds in the theory of polynomial rings over fields: A non-standard approach*, Invent. Math. 76 (1984), 77–91.

[32] H. Schoutens, *Existentially closed models of the theory of Artinian local rings*, J. Symbolic Logic 64 (1999), 825–845.

[33] H. Schoutens, *Bounds in cohomology*, Israel J. Math. 116 (2000), 125–169.

[34] H. Schoutens, *Uniform bounds in algebraic geometry and commutative algebra*, Connections between Model Theory and Algebraic and Analytic Geometry (A. Macintyre, ed.), Quaderni di Mathematica, vol. 6, 2000, pp. 43–93.

[35] H. Schoutens, *Absolute bounds on the number of generators of Cohen-Macaulay ideals of height at most 2*, preprint on http://www.math.ohio-state.edu/~schoutens, 2001.

[36] H. Schoutens, *Lefschetz principle applied to symbolic powers*, will appear in J. of Algebra and its Appl., preprint on http://www.math.ohio-state.edu/~schoutens, 2001.

[37] H. Schoutens, *Mixed characteristic homological theorems in low degrees*, will appear in Comp. Rend. Acad. Sci., preprint on http://www.math.ohio-state.edu/~schoutens, 2002.

[38] H. Schoutens, *Bounds in polynomial rings over Artinian local rings*, manuscript, 2003.

[39] H. Schoutens, *Canonical big Cohen-Macaulay algebras and rational singularities*, preprint on http://www.math.ohio-state.edu/~schoutens, 2003.

[40] H. Schoutens, *Closure operations and pure subrings of regular rings*, preprint on http://www.math.ohio-state.edu/~schoutens, 2003.
ASYMPTOTIC HOMOLOGICAL CONJECTURES IN MIXED CHARACTERISTIC

In this paper, various Homological Conjectures are studied for local rings which are locally finitely generated over a discrete valuation ring $V$ of mixed characteristic. Typically, we can only conclude that a particular Conjecture holds for such a ring provided the residual characteristic of $V$ is sufficiently large in terms of the complexity of the data, where the complexity is primarily given in terms of the degrees of the polynomials over $V$ that define the data, but possibly also by some additional invariants such as (homological) multiplicity. Thus asymptotic versions of the Improved New Intersection Theorem, the Monomial Conjecture, the Direct Summand Conjecture, the Hochster-Roberts Theorem and the Vanishing of Maps of Tors Conjecture are given.

That the results only hold asymptotically, is due to the fact that non-standard arguments are used, relying on the Ax-Kochen-Ershov Principle, to infer their validity from their positive characteristic counterparts. A key role in this transfer is played by the Hochster-Huneke canonical construction of big Cohen-Macaulay algebras in positive characteristic via absolute integral closures.