An effective gluon potential and hybrid approach to Yang-Mills thermodynamics

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We derive the partition function for the SU(3) Yang-Mills theory in the presence of a uniform gluon field within the background field method. We show, that the n-body gluon contributions in the partition function are characterized solely by the Polyakov loop. We express the effective action through characters of different representations of the color gauge group resulting in a form deduced in the strong-coupling expansion. A striking feature of this potential is that at low temperature gluons are physically disfavored and therefore they do not yield the correct thermodynamics. We suggest a hybrid approach to Yang-Mills thermodynamics, combining the effective gluon potential with glueballs implemented as dilaton fields.

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1. INTRODUCTION

The SU(N_c) pure gauge theory has a global Z(N_c) symmetry which is dynamically broken in the high temperature phase. The Polyakov loop, defined from the temporal gauge field integrated over the Euclidean time, plays a role of an order parameter of the Z(N_c) global symmetry \cite{1}. Effective Polyakov-loop models \cite{2,3} have been suggested as a macroscopic approach to the pure SU(3) gauge theory. The thermodynamics that comes out from such models is qualitatively in agreement with that obtained in lattice gauge theories \cite{4}. Alternative approaches are based on the quasi-particle picture of thermal gluons \cite{5}. A natural extension was carried out from such models is qualitatively in agreement with the Polyakov-loop effective potentials used in the literature anchored to the field theoretical basis. We show, that in this approach the calculated gluon potential, exhibits the correct asymptotic behavior at high temperatures, whereas at low temperatures, it disfavors gluons as appropriate dynamical degrees of freedom. We derive its correspondence to the strong-coupling expansion, of which the relevant coefficients of the gluon energy distribution are specified solely by characters of the SU(3) group.

The paper is organized as follows: In Section \textsuperscript{4} we introduce a hybrid approach that matches gluons with glueballs at deconfinement transition and study its thermodynamics. Our concluding remarks are given in Section \textsuperscript{5}.

2. MODELING GLUONS IN HOT MATTER

To formulate thermodynamics of the SU(3) Yang-Mills theory, we start with the partition function for gluon A_\mu and ghost C fields,

\[
Z = \int \mathcal{D}A_\mu \mathcal{D}C \mathcal{D}\bar{C} \exp \left[ i \int d^4x \mathcal{L} \right],
\]

\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}},
\]

with the gauge fixing (GF) and the Faddeev-Popov ghost (FP) terms. The kinetic term is given by

\[
\mathcal{L}_{\text{kin}} = -\frac{1}{2g^2} \text{tr} \left[ A_{\mu\nu} A^{\mu\nu} \right],
\]

\[
A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu],
\]

where A_\mu = A^a_\mu T^a and tr [$T^a T^b$] = \delta^{ab}/2.

Following \cite{2,11} we employ the background field gauge to evaluate the functional integrals. The gluon field is decomposed into background (classical) \bar{A}_\mu and the quantum \tilde{A}_\mu contribution

\[
A_\mu = \bar{A}_\mu + g \tilde{A}_\mu.
\]

We fix the background field gauge with the following form

\[
\mathcal{L}_{\text{GF}} = -\frac{1}{\alpha} \text{tr} \left[ (\tilde{D}_\mu \bar{A}_\mu)^2 \right],
\]

where \alpha is the gauge fixing parameter and \tilde{D}_\mu A_\nu = \partial_\mu A_\nu - i [\bar{A}_\mu, A_\nu]. Expanding the Lagrangian and collecting the terms quadratic in the quantum fields one gets

\[
\mathcal{L}^{(2)} = \frac{1}{2} \bar{A}_\alpha \left( [D_\mu D_\nu]_{ab}^{\alpha\beta} + \Sigma_{\alpha\beta} \right) \bar{A}_\beta
\]

\[
+ i C_\alpha \left( D_\mu D_\nu \right)_{ab}^{(\alpha \beta)} C^b,
\]

where \alpha is the gauge fixing parameter and \tilde{D}_\mu A_\nu = \partial_\mu A_\nu - i [\bar{A}_\mu, A_\nu]. Expanding the Lagrangian and collecting the terms quadratic in the quantum fields one gets

\[
\mathcal{L}^{(2)} = \frac{1}{2} A_\alpha \left( [D_\mu D_\nu]^{\alpha\beta}_{ab} + \Sigma_{\alpha\beta} \right) A^\beta
\]

\[
+ i C_\alpha \left( D_\mu D_\nu \right)^{(\alpha \beta)}_{ab} C^b,
\]

\[
\Sigma_{\alpha\beta} = \text{tr} \left( [T^a T^b] \delta_{\alpha\beta} \right).
\]
where we define
\[ (D^\mu_{\alpha \beta})_{ab} = -g^{\alpha \beta} \delta_{ab} \partial_\mu + \Gamma^{\alpha \beta}_{\mu,ab}, \]
\[ \Gamma^{\alpha \beta}_{\mu,ab} = -2 \text{tr} [\hat{A}_\mu [T_a, T_b]] g^{\alpha \beta}, \]
\[ \Sigma^{\alpha \beta}_{ab} = -4 \text{tr} [\hat{A}^{\alpha \beta} [T_a, T_b]], \]
\[ (D^\mu_{\alpha \beta})^{(CC)}_{ab} = \delta_{ab} \partial_\mu + 2 \text{tr} [\hat{A}_\mu [T_a, T_b]]. \] (2.6)

Here 't Hooft-Feynman gauge (\( \alpha = 1 \)) was taken \(^1\). Note that the ghost term does not contain \( \hat{A}_\mu \) and therefore the Gaussian integral over the ghost fields can easily be carried out. In the following we keep only the terms quadratic in \( \hat{A}_\mu \) and rewrite Eq. (2.5) as
\[ \mathcal{L}^{(2)} = \frac{1}{2} A_\mu^a (\delta^{\alpha \beta} \partial_\mu)^2 - f_{abc} \left( \partial^\beta A^\alpha _{,c} + 2 g^{\alpha \beta} A^\mu _{,\mu} \right) \]
\[ + f_{a \epsilon x} f_{\epsilon b d} \delta^{\alpha \beta} A^\epsilon _{,d} + 2 f_{abc} A^{\alpha \beta} + (2.7) \]

In the above Lagrangian we consider a constant uniform background \( \bar{A}_0 \):
\[ \bar{A}_\mu^a = \bar{A}_0^a \delta_{\mu 0}. \] (2.8)

It is convenient to take a diagonal and traceless generators, i.e. \( a = 3,8 \). We first consider the simplest case where only \( \bar{A}_0^a \) contributes. Then, in the momentum space, the Lagrangian \((2.8)\) is of the following form
\[ \mathcal{L}^{(2)} = \frac{1}{2} \bar{A}_\mu^a (D^{-1})_{ab} \bar{A}_\mu^b, \]
\[ (D^{-1})_{ab} = \delta_{ab} p^2 + 2i f_{a \epsilon x} A_{,\epsilon x}^3 p_0 - f_{a \epsilon x} f_{\epsilon x b} (\bar{A}_0^3)^2 \] (2.9)

where non-vanishing elements of the inverse propagator are given by
\[ (D^{-1})_{11} = (D^{-1})_{22} = p^2 + (\bar{A}_0^3)^2, \]
\[ (D^{-1})_{33} = (D^{-1})_{88} = p^2, \]
\[ (D^{-1})_{44} = (D^{-1})_{55} = (D^{-1})_{66} = (D^{-1})_{77} = p^2 + \frac{1}{4} (\bar{A}_0^3)^2, \]
\[ (D^{-1})_{12} = - (D^{-1})_{21} = 2i \bar{A}_0^3 p_0, \]
\[ (D^{-1})_{45} = - (D^{-1})_{54} = -(D^{-1})_{67} = (D^{-1})_{76} = i \bar{A}_0^3 p_0. \] (2.10)

Diagonalizing \( D^{-1} \) into \( \tilde{D}^{-1} = U D^{-1} U^T \) using unitary

\(^1\) The partition function must be independent of gauge, i.e. \( d \ln \mathcal{Z} / d \alpha = 0 \). Since the running coupling depends also on \( \alpha \), the condition reads, \( d \ln \mathcal{Z} / d \alpha = (\partial / \partial \alpha + \partial \alpha / \partial \alpha) (\partial / \partial \alpha) \ln \mathcal{Z} = 0 \) [12]. In this paper we work in Feynman gauge since the standard partition function of a free boson gas is readily obtained in the high temperature limit.

\(^2\) Equation \((2.12)\) can be generalized to any \( N_c \). See e.g. [1].
Due to this change of variables, the partition function is rewritten as

\[
\ln Z = V \int \frac{d^3 p}{(2\pi)^3} \ln \det \left( 1 - \hat{L}_A e^{-|\vec{p}|/T} \right) + \ln M(\phi_1, \phi_2),
\]

(2.17)

with \( M \) being the Haar measure for a fixed volume \( V \) given by

\[
M(\phi_1, \phi_2) = \frac{8}{9\pi^2} \sin^2 \left( \frac{\phi_1 - \phi_2}{2} \right) \sin^2 \left( \frac{2\phi_1 + \phi_2}{2} \right) \times \sin^2 \left( \frac{\phi_1 + 2\phi_2}{2} \right),
\]

(2.18)

which is normalized such that

\[
\int_0^{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 M(\phi_1, \phi_2) = 1.
\]

(2.19)

The first term in Eq. (2.17) yields the following thermodynamics potential:

\[
\Omega_g = 2T \int \frac{d^3 p}{(2\pi)^3} \ln \left( 1 - \hat{L}_A e^{-E_g/T} \right),
\]

(2.20)

where in the quasi-gluon energy \( E_g = \sqrt{|p|^2 + M_g^2} \) the effective gluon mass \( M_g \) is introduced from phenomenological reasons \(^3\). To calculate the thermodynamic potential (2.20) one still needs to perform the trace in a color space. We define gauge invariant quantities, normalized by dimensions of representations, as

\[
\Phi = \frac{1}{N_c} \text{tr} \hat{L}_F, \quad \bar{\Phi} = \frac{1}{N_c} \text{tr} \hat{L}_F^\dagger, \quad \Phi_A = \frac{1}{N_c^2 - 1} \text{tr} \hat{L}_A,
\]

(2.21)

where \( \hat{L}_F \) is the Polyakov loop matrix in the fundamental representation,

\[
\hat{L}_F = \text{diag} \left( e^{i\phi_1}, e^{i\phi_2}, e^{-i(\phi_1 + \phi_2)} \right),
\]

(2.22)

and \( \Phi_A \) is related with \( \Phi \) and \( \bar{\Phi} \) via

\[
(N_c^2 - 1) \Phi_A = N_c^2 \Phi \bar{\Phi} - 1.
\]

(2.23)

Carrying out the trace over colors and expressing it in terms of \( \Phi \) and its conjugate \( \bar{\Phi} \), one finally finds

\[
\Omega_g = 2T \int \frac{d^3 p}{(2\pi)^3} \ln \left( 1 + \sum_{n=1}^{8} C_n e^{-nE_g/T} \right),
\]

(2.24)

with the coefficients \( C_n \) given by

\[
C_8 = 1, \\
C_1 = C_7 = 1 - 9\Phi \bar{\Phi}, \\
C_2 = C_6 = 1 - 27\Phi \bar{\Phi} + 27 (\bar{\Phi}^3 + \Phi^3), \\
C_3 = C_5 = -2 + 27\Phi \bar{\Phi} - 81 (\bar{\Phi} \Phi)^2, \\
C_4 = 2 \left( -1 + 9\Phi \bar{\Phi} - 27 (\bar{\Phi}^3 + \Phi^3) + 81 (\bar{\Phi} \Phi)^2 \right).
\]

(2.25)

Thus, the \( n \)-body gluon contributions to the thermodynamic potential (2.24) are characterized solely by the Polyakov loop, i.e. the characters of the fundamental and the conjugate representations of the color SU(3) gauge group.

The Haar measure (2.18) is also expressed in terms of \( \Phi \) and \( \bar{\Phi} \) as

\[
M(\phi_1, \phi_2) = \frac{8}{9\pi^2} \left[ 1 - 6\Phi \bar{\Phi} + 4 (\Phi^3 + \bar{\Phi}^3) - 3 (\bar{\Phi} \Phi)^2 \right].
\]

(2.26)

A complete effective thermodynamic potential of gluons in the large volume limit is obtained from Eq. (2.17) as follows:

\[
\Omega = \Omega_g + \Omega_\Phi + c_0,
\]

(2.27)

where \( \Omega_g \) is given by Eq. (2.24) and the Haar measure contribution

\[
\Omega_\Phi = -a_0 T \ln \left( 1 - 6\Phi \bar{\Phi} + 4 (\Phi^3 + \bar{\Phi}^3) - 3 (\bar{\Phi} \Phi)^2 \right).
\]

(2.28)

In (2.28) we have neglected the normalization factor of the Haar measure which gives sub-leading contribution to thermodynamics. The \( a_0 \) and \( c_0 \) and/or gluon mass are free parameters which have to be fixed through certain external conditions. They can be e.g. chosen to reproduce the equation of state of the SU(3) pure gauge theory obtained on the lattice through Monte Carlo calculations.

### 3. ASYMPTOTIC EXPANSIONS OF THE POTENTIAL

The potential for the SU(3) Yang-Mills theory (2.27) obtained in the previous section provides an effective description of gluon thermodynamics. In particular, in asymptotically high temperatures, the \( \Omega_g \) should reproduce the ideal gas limit. Indeed, taking the limit \( \Phi, \bar{\Phi} \to 1 \), corresponding to \( A_0 \to 0 \), one finds from (2.24), that

\[
\Omega_g(\Phi = \bar{\Phi} = 1) = 16T \int \frac{d^3 p}{(2\pi)^3} \ln \left( 1 - e^{-E_g/T} \right).
\]

(3.1)
Thus, the standard expression for a non-interacting gas of massive/massless gluons is recovered.

On the other hand, having in mind a quasi-particle approach, where gluons are considered as massive particles with a temperature-dependent mass $M_g(T)$, one can expand the logarithm in Eq. (2.24). For a sufficiently large $M_g(T)/T$ one approximates the logarithm by the first term of the expansion, resulting in the following form of the potential:

$$
\Omega_g \simeq \frac{T^2 M_g^2}{\pi^2} \sum_{n=1}^{8} \frac{C_n}{n} K_2(n \beta M_g),
$$

(3.2)

where $C_n$ are as in Eq. (2.25) and $K_2(x)$ is the Bessel function. The above can also be considered as a strong coupling expansion regarding the relation $M_g(T) = g(T)T$, where $g(T)$ is an effective gauge coupling.

The character expansion of the potential (3.2) corresponds to that obtained in the Polyakov loop models on the lattice, derived using strong coupling techniques for the non-Abelian gauge group SU(3). Indeed, in the strong-coupling expansion the effective action to the next-to-leading order is obtained in terms of group characters as $^{10} #4$:

$$
S_{\text{eff}}^{(SC)} = \lambda_{10} S_{10} + \lambda_{20} S_{20} + \lambda_{11} S_{11} + \lambda_{21} S_{21},
$$

(3.3)

with products of characters $S_{pq}$, specified by two integers $p$ and $q$ counting the numbers of fundamental and conjugate representations, and couplings $\lambda_{pq}$ being real functions of temperature. One readily finds those couplings from Eq. (3.2), as well as the correspondence between $S_{pq}$ and $C_n$ from Eqs. (3.2) and (2.25), as

$$
C_{1,7} = S_{10}, \quad C_{2,6} = S_{21},
C_{3,5} = S_{11}, \quad C_{4} = S_{20}.
$$

(3.4)

Taking only the contribution of a single-gluon distribution $\exp[-M_g(T)]$ in the expansion Eq. (3.2) yields the “minimal model” described by

$$
\Omega_g \simeq - F(T, M_g) \Phi \bar{\Phi},
$$

(3.5)

with the negative sign needed to get a first-order transition as studied in $^{10}$. Here, an explicit form of the function $F$ relies on approximations used in evaluation of the momentum integration and parameterization of $M_g(T)$. Assuming appropriate temperature dependence of $F$ so that the thermodynamic potential (2.24) yields the phase transition with thermal expectation value of the Polyakov loop $\langle \Phi \rangle$ as the order parameter of $Z(3)$ symmetry, the form widely used in the PNJL model $^{13-16}$ is recovered.

In addition, the logarithm of the Haar measure part $\Omega_4$ can also be expanded in powers of $Z(3)$-invariant operators. In this case, the effective gluon potential is found in the polynomial form $^{3,10}$. Such form of $\Omega$ can, however, be applied only to a weak first-order phase transition. The polynomial form applied in the PNJL model was also shown to cause some problems in behaviors of charge fluctuations $^{15}$ as well as with the phase structure and symmetry properties of the potential at complex chemical potential $^{17}$.

4. A HYBRID APPROACH

The model described by Eqs. (2.24), (2.27) and (2.28) works fairly well when the thermal expectation $\langle \Phi \rangle$ of the Polyakov loop is non-vanishing. However, at any temperatures below $T_c$, where $\langle \Phi \rangle = 0$ is dynamically favored, it causes some unphysical results on the equation of state.

The model parameters $a_0$ and $c_0$ are fixed such that the model reproduces the value of $T_c$ and the pressure at $T_c$ calculated from SU(3) Lattice Gauge Theory. In such formulation, the model applied to the phase below $T_c$ yields a positive pressure, however the entropy and energy densities turn out to be negative. In fact, keeping at low temperatures only the $C_1$ term as a main contribution, one gets the potential

$$
\Omega_g(\Phi = \bar{\Phi} = 0) \simeq 2T \int \frac{d^3p}{(2\pi)^3} \ln \left(1 + e^{-E_g/T} \right) ,
$$

(4.1)

which does not posses the correct sign in front of $\exp[-E_g/T]$ expected from the Bose-Einstein statistics. One immediately finds that the entropy and energy densities calculated from Eq. (4.1) are negative. Therefore, the model cannot be naively applied to the phase below $T_c$. This problem appears not only with the complete potential (2.24), but also with its minimal form (3.5) which is frequently used in the literature. There, the equation of state is entirely zero, at any temperature below $T_c$. This is clearly unphysical as there are color-singlet hadrons, glueballs, contributing to thermodynamics in a pure Yang-Mills theory and they must generate a non-vanishing pressure.

The above aspects are in a striking contrast to the quark sector of the thermodynamics obtained in the presence of a background gluon field $A_0$. There, the thermodynamic potential for quarks and anti-quarks with $N_f$ flavors is obtained as,

$$
\Omega_{q+\bar{q}} = - 2N_f T \int \frac{d^3p}{(2\pi)^3} \text{tr} \ln \left[ 1 + \tilde{L}_F e^{-(E_q-\mu)/T} \right] + (\mu \rightarrow -\mu).
$$

(4.2)

The trace over color indices in this case is easily performed and the potential is expressed by characters of fundamental $\Phi$ and the conjugate $\bar{\Phi}$ representation.
as \[13, 18\]
\[
\Omega_{q+\bar{q}} = -2N_f T \int \frac{d^3 p}{(2\pi)^3} \ln \left[ 1 + N_c \left( \Phi + \Phi e^{-E^+/T} \right) e^{-E^+/T} + e^{-3E^+/T} \right] + (\mu \rightarrow -\mu),
\]
(4.3)

with \(E^\pm = E_q \mp \mu\) being the energy of a quark (antiquark).

In the limit of \(\Phi, \Phi \rightarrow 0\), which is expected at low temperatures, the contribution of one- and two-quark states is suppressed and only three-quark (baryonic) states, \(\exp(-3E^{(3)}/T)\), survives. This, on a qualitative level, is similar to confinement in QCD thermodynamics \[13\]. One should however keep in mind that this model yields only colored quarks being statistically suppressed at low temperatures. On the other hand, unphysical thermodynamics that comes out below \(T_c\) from the gluon sector \[2.27\], apparently indicates that gluons are physically forbidden below \(T_c\). We note that in the mean field approximation higher representations of the Polyakov loop are not condensed when one evaluates their group averages with the Haar measure. Such “hidden” physics of the higher representations can be embedded in the mean field approach when all the gluon energy distributions are expressed in terms of the fundamental Polyakov loop \(\Phi\). In this way all the colored gluons are suppressed and therefore the correct physics interpretation is recovered.

This property is not affected by quarks. Let us consider massive gluons and quarks at zero chemical potential. From Eqs. \(4.1\) and \(4.3\) applied to \(T < T_c\), the thermodynamic potential is approximated as
\[
\Omega_g + \Omega_{q+\bar{q}} \simeq \frac{T^2}{\pi^2} \left[ M_g^2 K_2 \left( \frac{M_g}{T} \right) - \frac{2N_f}{3} K_2 \left( \frac{3M_g}{T} \right) \right].
\]
(4.4)

Assuming that the glueball and nucleon are made from two gluons and three quarks respectively and putting empirical numbers \(M_{\text{glueball}} = 1.7\, \text{GeV} \) \[19\] and \(M_{\text{nucleon}} = 0.94\, \text{GeV}\), one finds \(M_g = 0.85\, \text{GeV}\) and \(M_q = 0.31\, \text{GeV}\). Given those numbers, Eq. \(4.4\) is positive for either \(N_f = 2\) or 3. Consequently, the entropy density is negative at any temperature as found in a pure Yang-Mills case. Therefore, thermodynamics remains unphysical, unless additional terms responsible for the non-perturbative effects in confined phase are considered.

### 4.1. Modeling glueballs as dilaton fields

Below \(T_c\), thermodynamics needs to be described in terms of physical degrees of freedom, i.e. glueballs. We introduce a glueball as a dilaton field \(\chi\) representing the gluon composite \(\langle A_{\mu\nu} A^{\mu\nu} \rangle\), which is responsible for the QCD trace anomaly \[20\]. The Lagrangian that we use is of a standard form given by
\[
\mathcal{L}_{\chi} = \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V_{\chi},
\]
\[
V_{\chi} = \frac{B}{4} \left( \frac{\chi}{\chi_0} \right)^4 \left[ \ln \left( \frac{\chi}{\chi_0} \right) \right] - 1,
\]
(4.5)

where \(B\) is the bag constant and \(\chi_0\) is a dimensionful quantity. The two parameters \(B\) and \(\chi_0\) can be fixed using the vacuum energy density \(\mathcal{E} = 4B = 0.6\, \text{GeV\,fm}^{-3} \) \[21\] and the vacuum glueball mass \(M_\chi = 1.7\, \text{GeV} \) \[19\] with the following definition:
\[
M_\chi^2 = \left. \frac{\partial^2 V}{\partial \chi^2} \right|_{\chi = \chi_0} = \frac{4B}{\chi_0^2}.
\]
(4.6)

One finds that \(B = (0.368\, \text{GeV})^4\) and \(\chi_0 = 0.16\, \text{GeV}\).

With the Lagrangian \(4.5\), the thermodynamic poten-
tial of effective glueball fields is found to be
\[ \Omega = \Omega_\chi + V_\chi + \frac{B}{4}, \]
\[ \Omega_\chi = T \int \frac{d^3p}{(2\pi)^3} \ln \left(1 - e^{-E_\chi/T}\right), \]
\[ E_\chi = \sqrt{p^2 + M_\chi^2}, \quad M_\chi^2 = \frac{\partial^2 V_\chi}{\partial \chi^2}, \quad (4.7) \]
where a constant \(B/4\) is added so that \(\Omega = 0\) at zero temperature.

### 4.2. The hybrid thermodynamic potential

To avoid problems of unphysical equations of state in confined phase we adopt a hybrid approach which accounts for gluons and glueballs degrees of freedom by combining Eqs. (2.27) and (4.7) as follows:
\[ \Omega = \Theta(T_c - T) \Omega(\chi) + \Theta(T - T_c) \Omega(\Phi). \quad (4.8) \]

The model parameters are constrained by requiring that
- \(\Omega(\Phi)\) yields a first-order phase transition at \(T_c = 270\) MeV as found in SU(3) lattice calculations [4, 27].
- \(\Omega(\chi)\) and \(\Omega(\Phi)\) match at \(T_c\).

When the gluon effective mass is assumed to be zero, one finds the following model parameters:
\[ M_g = 0: \quad \langle \Phi \rangle_{T_c} = 0.395, \quad a_0 = (0.197 \text{ GeV})^3, \]
\[ c_0 = -(0.180 \text{ GeV})^4. \quad (4.9) \]

One can also assume that the gluon becomes massive via non-vanishing gluon condensation \(\langle \chi \rangle \neq 0\) [23]. Requiring that a glueball is composed of two constituent gluons yields \(M_g = \chi / 2\). Since \(\langle \chi \rangle\) little varies with temperature around \(T_c\) [24], we treat \(M_g\) as a constant and choose \(M_g = 1.7 \text{ GeV}/2 = 0.85\) GeV. The parameters in this case are found as
\[ M_g = 0.85 \text{ GeV}: \quad \langle \Phi \rangle_{T_c} = 0.439, \quad a_0 = (0.125 \text{ GeV})^3, \]
\[ c_0 = -(0.130 \text{ GeV})^4. \quad (4.10) \]

A more general case, not considered in this paper, would include the temperature-dependent effective gluon mass which is fixed such that the present model quantifies thermodynamics calculated on the lattice in the SU(3) gauge theory.

### 4.3. Thermodynamics

Thermodynamic properties of the hybrid model and its phase structure can be quantified directly from the potential (4.8). Fig. 1 shows the thermal expectation value of the Polyakov loop \(\langle \Phi \rangle\) obtained from Eq. (1.8) as the solution of the stationary condition, \(\partial \Omega / \partial \Phi = 0\). There is a trivial solution \(\langle \Phi \rangle = 0\) at any temperatures and it becomes degenerate with a non-trivial solution \(\langle \Phi \rangle\) at some \(T_c\), indicating a first-order deconfinement transition.

The Polyakov loop expectation is weakly changing with \(M_g\) and approaches unity rather quickly as seen in Fig. 1. The temperature dependence of \(\langle \Phi \rangle\) just above \(T_c\) and its value at \(T_c\) are consistent with lattice results and can be still improved by introducing a thermal gluon mass \(M_g(T)\) as done e.g. in [3]. There, the effective mass was parameterized as in the standard quasi-particle approaches at high temperature, \(M_g(T) = g(T)T\), with the effective running coupling \(g(T)\). The lattice data on the renormalized \(\langle \Phi \rangle\) are known to exceed unity at \(T/T_c \sim 3/2\). This property of lattice data, which is associated with uncertainties of the renormalization procedure [2, 26], can never appear in effective Polyakov loop models where \(\Phi\) is the character of the fundamental representation and restricts the target space, so that \(\langle \Phi \rangle\) is not allowed to go beyond unity.

Fig. 1 right shows the pressure calculated from the effective gluon (2.27) and from the effective glueball (1.7) potential for massless and massive gluons. Although the presence of a constant \(c_0\) in Eq. (2.27) makes the pressure positive below \(T_c\), as mentioned in the previous section, Eq. (2.27) unavoidably leads to a negative entropy and energy densities. Consequently, in the hybrid model (4.8) and for \(T < T_c\), the pressure must be quantified by the glueball potential (1.7). The cusp at \(T_c\) in pressure implies a discontinuity in its temperature derivative. The energy density and the interaction measure \(\Delta = (E - 3P)/T^4\) are presented in Fig. 2. The energy density has a jump from glueballs to gluons thermodynamics at \(T_c\), whereas the interaction measure exhibits a maximum just above \(T_c\).

Even though, the qualitative behaviors of \(E/T^4\) and \(\Delta\) follow general trends seen in lattice data, the EoS is apparently more sensitive to \(M_g\) than \(\langle \Phi \rangle\). The model calculations with massless gluons converge too quickly to asymptotic Stefan-Boltzmann limit. The EoS for massive gluons, on the other hand, exhibits a better agreement with lattice data. This clearly indicates the presence of some residual interactions above \(T_c\) which can be incorporated into \(M_g(T)\). Thus, to quantitate lattice results one would need to include the temperature-dependent gluon mass. In simplified quasi-particle models, where \(\Phi = 1\) for any \(T\), the \(M_g(T)\) was shown to be strongly increasing when approaching \(T_c\) from above. This behavior, however, can be modified if the contribution of the background gauge field is included in the quasiparticle model formulated in Eq. (2.24).

For \(T \leq T_c\) the hybrid model includes a glueball as the relevant degree of freedom, similarly as seen in lattice calculations. However, in the present treatment, the model contains only the lowest-lying glueball, which might not be sufficient to quantify lattice thermodynamics in a con-
fined phase. One way out would be to deal with a gluon degeneracy factor as an additional parameter.

5. CONCLUSIONS

We have derived the thermodynamic potential in the SU(3) pure Yang-Mills theory in the presence of a uniform gluon field within the background field method. We have shown that such effective gluon potential, which accounts for quantum statistics and reproduces an ideal gas limit at high temperatures, is formulated in terms of the Polyakov loop in the fundamental representation.

The gluon distributions are found to be specified solely by the Polyakov loop and therefore there is one-to-one correspondence to the effective action in the strong-coupling expansion. We have shown that effective models of the Polyakov loop used so far to describe pure gauge theory thermodynamics appear as limiting cases of our result.

Our main observation is that the effective Polyakov-loop potential can not be applied to the phase where the thermal expectation value of the Polyakov-loop vanishes. There, in confined phase, the equation of state is unphysical resulting in negative entropy and energy densities. This property of gluon potential is in remarkable contrast to the description of “confinement” within a class of chiral models with Polyakov loops. There, colored quarks are suppressed only statistically at low temperatures.

The gluonic model considered here indicates that colored gluons are forbidden below $T_c$ as dynamical degrees of freedom. This feature is unchanged by the presence of quarks. To avoid problems of unphysical thermodynamics in confined phase we proposed a hybrid approach which matches at deconfinement critical temperature the effective model of gluons to the one of glueballs constrained by the QCD trace anomaly.

The approach developed in this work is open to further investigations of the SU(3) gluodynamics guided by available lattice results, as well as to a more realistic description of an effective QCD thermodynamics with quarks.

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