SIX GENERATED ACM BUNDLE ON A HYPERSURFACE IS SPLIT

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ABSTRACT. Let $X$ be a smooth projective hypersurface. In this note we show that any six generated arithmetically Cohen-Macaulay vector bundle over $X$ splits if $\dim X \geq 6$.

1. Introduction

We work over an algebraically closed field of characteristic 0.

Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface. We say that a vector bundle $E$ on $X$ is split if it is a direct sum of line bundles on $X$. We say that a vector bundle is $k$-generated if $k$ is the smallest integer such that there exists a surjection $\bigoplus_{i=1}^k \mathcal{O}_X(a_i) \to E$. A convenient notation for a coherent sheaf $\mathcal{F}$ on $X$ is:

$$H^i_*(X, \mathcal{F}) := \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{F}(m))$$

We say a bundle is split if it is a direct sum of line bundles. An arithmetically Cohen-Macaulay (ACM) bundle on $X$ is a vector bundle $E$ satisfying

$$H^i(X, E(m)) = 0, \forall m \in \mathbb{Z} \text{ and } 0 < i < \dim X$$

This definition is equivalent to saying that $\Gamma_*(X, E)$ is a maximal Cohen-Macaulay as $S_X$-module where $S_X$ is the graded ring corresponding to $X$.

A split bundle on a hypersurface or a projective space is obviously an ACM bundle. A theorem of Horrocks [12] tells that over projective spaces, the converse is also true - A vector bundle on a projective space splits if and only if it is an arithmetically Cohen-Macaulay bundle. One can ask if Horrocks’s criterion is true for arithmetically Cohen-Macaulay bundles over hypersurfaces?

The answer is no as there are examples of indecomposable arithmetically Cohen-Macaulay bundles over hypersurfaces (see [15] or [17]). Though there is a conjecture in this direction:

Conjecture (Buchweitz, Greuel and Schreyer [2]): Let $X \subset \mathbb{P}^n$ be a hypersurface. Let $E$ be an ACM bundle on $X$. If $\text{rank } E < 2^e$, where $e = \left[ \frac{n-2}{2} \right]$, then $E$ splits. (Here $[q]$ denotes the largest integer $\leq q$.)

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This conjecture can not be strengthened further, as there exists indecomposable arithmetically Cohen-Macaulay bundles of rank $2^e$ on such hypersurfaces for all degrees (see the construction in [21]).

For low degree cases, we refer the reader to [14] for $d = 2$ and to [4] for the case of $d = 3$ surfaces in $\mathbb{P}^3$. For rank 2 on a general hypersurface of low degree in $\mathbb{P}^4$ and $\mathbb{P}^5$ we suggest [5], [6], [7].

For general hypersurfaces, Sawada [19] found a sufficient condition for a arithmetically Cohen-Macaulay bundle to split depending upon the dimension as well as the degree of the hypersurface and rank of the vector bundle. His method uses matrix factorization (see Eisenbud [9] for a background).

Rank two case is understood fairly well. A rank 2 arithmetically Cohen-Macaulay bundle on a hypersurface $X$ splits if:

1. $\dim(X) \geq 5$ (see [13] and [15]).
2. $\dim(X) = 4$ and $X$ is general hypersurface and $d \geq 3$ (see [15] and [18]).
3. $\dim(X) = 3$ and $X$ is general hypersurface and $d \geq 6$ (see [16] and [18]).

In a previous work [20], we showed that any rank 3 (resp. rank 4) ACM bundle over a hypersurface of $\dim(X) \geq 7$ (resp. $\dim(X) \geq 9$) is split. Our method employed a cohomological chase over a Koszul complex. In this note, we find an improvement for six generated ACM bundles,

**Theorem 1.1.** Let $E$ be any ACM bundle on a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ where $n \geq 6$. If $E$ is six generated then $E$ is a split bundle.

## 2. Preliminaries

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d \geq 2$. Let $E$ be a rank $r$ ACM bundle on $X$ which is $k$-generated. The following facts are well known and will be used several times. The proofs for the same can be found (for instance) in [20] (section 2).

- There exist split bundles $\widetilde{F}_1, \widetilde{F}_0$ of rank $k$ on $\mathbb{P}^{n+1}$ and a minimal resolution of $E$ on $\mathbb{P}^{n+1}$,

\[ 0 \to \widetilde{F}_1 \to \widetilde{F}_0 \to E \to 0 \]

- Restricting the above resolution to $X$ gives,

\( (1) \quad 0 \to G \to F_0 \to E \to 0 \)

\( (2) \quad 0 \to E(-d) \to F_1 \to G \to 0 \)

where $F_1, F_0$ are split bundles over $X$ of rank $k$ and $G$ is an arithmetically Cohen-Macaulay bundle.
Let $k$ be a positive integer. For any sequence $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$ of vector bundles on a projective variety $Z$, there exists a resolution of $k$-th exterior power of $\mathcal{F}_2$,

$$0 \to \text{Sym}^k(\mathcal{F}_0) \to \cdots \to \text{Sym}^{k-i}(\mathcal{F}_0) \otimes \wedge^i \mathcal{F}_1 \to \cdots \to \wedge^k \mathcal{F}_1 \to \wedge^k \mathcal{F}_2 \to 0$$

In [20] we have called this the Sym $- \wedge$ sequence of index $k$ associated to the given short exact sequence. Similarly there exists a resolution of $k$-th symmetric power of $\mathcal{F}_2$ which we denoted as $\wedge - \text{Sym}$ sequence of index $k$ associated to the given sequence.

$$0 \to \wedge^k(\mathcal{F}_0) \to \cdots \to \wedge^{k-i}(\mathcal{F}_0) \otimes \text{Sym}^i \mathcal{F}_1 \to \cdots \text{Sym}^k \mathcal{F}_1 \to \text{Sym}^k \mathcal{F}_2 \to 0$$

For details we refer to [3] and references provided therein.

Following lemma is from [20]:

**Lemma 2.1.** Let $E$ be any bundle (not necessarily ACM) on a hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geq 3$. Assume further that $H^1_*(X, E^\vee) = 0$. Let the exact sequence $0 \to G \to F_0 \to E \to 0$ be a minimal (1-step) resolution of $E$ on $X$. If $G$ admits a line bundle as a direct summand, then $E$ is split.

### 3. Proof of the theorem

**Proof of theorem [1,7] :** Let $E$ be a six generated arithmetically Cohen-Macaulay bundle on $X$ where $\text{dim}(X) \geq 6$. We take a minimal (1-step) resolution of $E$ on $X$:

$$0 \to G \to F_0 \to E \to 0$$

By assumption $\text{rk}(F_0) = 6$ therefore $\text{rk}(E) \leq 6$. The following cases are easily resolved:

1. If $\text{rk}(E) = 6$ then $E \cong F_0$ is split.
2. If $\text{rk}(E) = 5$ then $G$ is a line bundle whence by lemma 2.1 $E$ splits.
3. If $\text{rk}(E) = 4$ then $G$ splits as it is then an ACM bundle of rank 2 on $X$ (see [15]) whence $E$ splits (again by lemma 2.1).
4. If $\text{rk}(E) = 2$ then splitting is by results from [15].

This leaves the case $\text{rk}(E) = 3$ open. $G$ (an ACM bundle) is also of rank 3 for this case. We have a minimal resolution of $E$ on $\mathbb{P}^{n+1}$:

$$0 \to \widetilde{F}_1 \to \widetilde{F}_0 \to E \to 0$$

Taking exterior product, we get

$$0 \to \wedge^3 \widetilde{F}_1 \to \wedge^3 \widetilde{F}_0 \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a coherent sheaf with support on $X$. It can be verified that $\mathcal{F}$ is arithmetically Cohen-Macaulay sheaf which means that it is (infact) an ACM vector bundle on $X$ as $X$ is smooth. Restricting the above sequence to $X$ gives:

$$0 \to \text{Tor}^1(\mathcal{O}_X, \mathcal{F}) \to \wedge^3 F_1 \to \wedge^3 F_0 \to \mathcal{F} \to 0$$

\footnote{We were unable to find any standard terminology in the literature for the given resolution.}
To compute the \( Tor \) term, we tensor the following sequence by \( F \):

\[
0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0
\]
to get \( Tor^1(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(-d) \). Thus we get the sequence:

\[
0 \to \mathcal{F}(-d) \to \wedge^3 F_1 \to \wedge^3 F_0 \to \mathcal{F} \to 0
\]

The map \( F_1 \to F_0 \) factors via \( G \) (see section 2). By functoriality of exterior product, the map \( \wedge^3 F_1 \to \wedge^3 F_0 \) will factor via \( \wedge^3 G \). Thus the sequence above breaks up into 2 short exact sequences:

\[
0 \to F(-d) \to \wedge^3 F_1 \to \wedge^3 G \to 0 \quad (4)
\]

\[
0 \to \wedge^3 G \to \wedge^3 F_0 \to F \to 0 \quad (5)
\]

Above sequences along with the fact that \( G \) is rank 3 and \( F_1, F_0 \) are split bundles imply that \( F \) is a split bundle - for example by verifying that \( H^1_*(\mathcal{F}) = 0 \) which implies that equation (4) splits.

Let \( \mathcal{F}_1 = \ker(\wedge^3 F_0 \rightarrow \wedge^3 E) \). We have the following pullback diagram:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^3 G & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^3 G & \longrightarrow & \wedge^3 F_0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \wedge^3 E & \longrightarrow & \wedge^3 E & & & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & & & \\
\end{array}
\]

Here the map \( \wedge^3 G \rightarrow \mathcal{F}_1 \) is coming from the filtration diagram for \( \wedge^3 F_0 \) as induced by

the sequence \( 0 \to G \to F_0 \to E \to 0 \) and \( \mathcal{E} = \text{coker}(\wedge^3 G \rightarrow \mathcal{F}_1) \).

The above diagram (and the fact that \( \mathcal{F} \) is split) will imply that \( \mathcal{E} \) is split which in turn means that \( \mathcal{F}_1 \) is a split bundle. Lemma \( \text{[3.1]} \) (below) will now imply that \( E \) is split. \( \square \)

We complete the proof of the above theorem with following lemma which uses similar form of cohomological chase as done in [20]:

**Lemma 3.1.** Let \( E \) be an ACM vector bundle of rank 3 on a hypersurface \( X \) of dimension \( \geq 6 \). Let \( G, F_0 \) denote the vector bundles on \( X \) coming from the 1-step resolution of \( E \) as in sequence (1). Let \( \mathcal{F}_1 = \ker(\wedge^3 F_0 \rightarrow \wedge^3 E) \) (as assumed in the proof of theorem [1.1]). If \( \mathcal{F}_1 \) is split then \( E \) is split.

**Proof.** Consider the following \( \wedge - \text{Sym} \) sequence for index 3 associated with sequence (2):

\[
0 \rightarrow \wedge^3 E(-d) \rightarrow \wedge^2 E(-d) \otimes F_1 \rightarrow E(-d) \otimes \text{Sym}^2 F_1 \rightarrow \text{Sym}^3 F_1 \rightarrow \text{Sym}^3 G \rightarrow 0
\]
Breaking it into short exact sequences and using the fact that as $E$ is rank 3 therefore $\wedge^3 E(-d), \wedge^2 E(-d), E(-d)$ are all arithmetically Cohen-Macaulay bundle and $\text{Sym}^i F_1$ is a split bundle for all $i$, we get

$$H^i_\ast (\text{Sym}^3 G) = 0 \text{ for } i = 1, 2, \ldots, \dim(X) - 4$$

Similarly we write the $\wedge - \text{Sym}$ sequence for index 2:

$$0 \to \wedge^2 E(-d) \to E(-d) \otimes F_1 \to \text{Sym}^2 F_1 \to \text{Sym}^2 G \to 0$$

which gives $H^i_\ast (\text{Sym}^2 G) = 0$ for $i = 1, 2, \ldots, \dim(X) - 3$.

Now we write the $\text{Sym} - \wedge$ sequence for index 3 for the short exact sequence (1):

$$0 \to \text{Sym}^3 G \to \text{Sym}^2 G \otimes F_0 \to G \otimes \wedge^2 F_0 \to \wedge^3 F_0 \to \wedge^3 E \to 0$$

Breaking it up we get $0 \to \text{Sym}^3 G \to \text{Sym}^2 G \otimes F_0 \to J_1 \to 0$. The vanishing results for cohomologies of $\text{Sym}^3 G$, $\text{Sym}^2 G$ implies that if $\dim(X) \geq 6$ then $H^i_\ast (X, J_1) = 0$. Now $J_1$ further fits into the following short exact sequence:

$$0 \to J_1 \to G \otimes \wedge^2 F_1 \to F_1 \to 0 \quad (7)$$

where $F_1$ is split by assumption. Therefore (7) is a split sequence and hence $G$ admits a line bundle as a direct summand. Lemma 2.1 tells us that $E$ is split. \qed

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