DEFECT IN CYCLOTONIC HECKE ALGEBRAS

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Abstract. The complexity of a block of a symmetric algebra can be measured by the notion of defect, a numerical datum associated with each of the simple modules contained in the block. Geck showed that the defect is a block invariant for Iwahori–Hecke algebras of finite Coxeter groups in the equal parameter case, and speculated that a similar result should hold in the unequal parameter case. We prove that the defect is a block invariant for all cyclotomic Hecke algebras associated with the complex reflection groups of the infinite series \( G(l, p, n) \), which include the Weyl groups of type \( B_n \) in the unequal parameter case. In particular, for the groups \( G(l, 1, n) \), we show that the defect corresponds to the notion of weight in the sense of Fayers. We thus also obtain a new way of computing the weight, which uses a generalisation of the notion of hook lengths. We further show computationally that the defect is a block invariant for all cyclotomic Hecke algebras of exceptional type for which the blocks are known, and we conjecture that the result should hold for all complex reflection groups. Finally, we obtain that the defect is also a block invariant for cyclotomic Yokonuma–Hecke algebras.

1. Introduction

Iwahori–Hecke algebras associated with Weyl groups appear as endomorphism algebras of induced representations in the study of the representation theory of finite reductive groups. They can also be defined independently as one-parameter deformations of the Weyl group algebras. Cyclotomic Hecke algebras generalise the notion of Iwahori–Hecke algebras from Weyl groups (and more generally, real reflection groups) to complex reflection groups, and they can be also seen as one-parameter deformations of the corresponding group algebras. Cyclotomic Hecke algebras associated with complex reflection groups appear naturally in the generalised Harish-Chandra series of finite reductive groups, as well as in the ordinary Harish-Chandra series of Spetese, the mysterious yet objects that are expected to be generalisations of finite reductive groups. Since their first appearance in the works of Broué, Malle, Michel and Rouquier \[\text{BroMa, BMR, BMM}\], the study of cyclotomic Hecke algebras associated with complex reflection groups has grown as a subject on its own right, with numerous connections to other research areas in algebra, geometry, topology and even theoretical physics.

Let \( W \) be a complex reflection group. The irreducible representations of a cyclotomic Hecke algebra associated with \( W \) are in bijection with the irreducible representations of \( W \). When the parameter of the algebra is specialised to a complex number, we obtain a decomposition matrix that records how the irreducible representations of the cyclotomic Hecke algebra (labelling the rows of the matrix) decompose into irreducible representations of the specialised algebra (labelling the columns of matrix). The decomposition matrix has a block diagonal form and we say that two irreducible representations of \( W \) are in the same block if they label rows with non-zero entries in the same block of the matrix. If the specialised algebra is semisimple, then the decomposition matrix is the identity matrix.

The decomposition matrix is a tool used first by Brauer for the study of the representation theory of finite groups in characteristic \( p > 0 \). If \( G \) is a finite group and \( V \) is an irreducible representation of \( G \) over an algebraically closed field of characteristic 0, with character \( \chi_V \), then one can define the \( p \)-defect of \( V \) to be the \( p \)-part of the integer \( |G|/\chi_V(1) \). The \( p \)-defect of a block is the maximal \( p \)-defect of the representations it contains, and it is a good way to measure its complexity. Blocks of \( p \)-defect 0 are simply singletons, while the structure of blocks of \( p \)-defect 1 is well-understood. However, the \( p \)-defect is not a block invariant, that is, not all representations in the same block have the same \( p \)-defect.

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If $G$ is a finite group as above and $\text{Irr}(G)$ denotes the set of its irreducible representations over an algebraically closed field of characteristic 0, then the linear map $\tau := \sum_{V \in \text{Irr}(G)} (\chi_V(1)/|G|)\chi_V$ is a symmetrising trace on $\mathbb{Z}[G]$, i.e., the bilinear map defined by $\tau$ is symmetric and non-degenerate. We say that $\mathbb{Z}[G]$ is a symmetric algebra and we call $\tau$ the canonical symmetrising trace on $\mathbb{Z}[G]$. In the context of symmetric algebras, the integers $|G|/\chi_V(1)$ are the Schur elements with respect to $\tau$.

Let $W$ be a complex reflection group as before. A generalisation of the canonical symmetrising trace of the group algebra of $W$ to the cyclotomic Hecke algebras associated with $W$ was conjectured to exist in [BMM]. Its existence was a known fact for real reflection groups, but it remains an open problem for non-real ones. Any complex reflection group is isomorphic to a direct product of irreducible ones, which either belong to the infinite series $G(l, p, n)$ or to the exceptional groups $G_4, G_5, \ldots, G_{37}$, following the classification by Shephard and Todd [ST]. A list of groups for which the so-called “BMM symmetrising trace conjecture” has been proved is given in Subsection 2.3 right after stating the conjecture. However, the Schur elements with respect to the conjectural canonical symmetrising trace have been completely determined for all complex reflection groups. These Schur elements are products of $K$-cyclotomic polynomials, where $K$ is the splitting field of $W$. Using the bijection between the irreducible representations of $W$ and those of any cyclotomic Hecke algebra associated with $W$, we denote by $s_V$ the Schur element of $V \in \text{Irr}(W)$. If $\Phi$ is a $K$-cyclotomic polynomial, then we define the $\Phi$-defect of $V$ to be the multiplicity of $\Phi$ as a factor of $s_V$.

Now, Schur elements yield a semisimplicity criterion for symmetric algebras: a symmetric algebra is semisimple if and only if all its Schur elements are non-zero. Therefore, if we specialise the parameter of a cyclotomic Hecke algebra to a complex number, and this complex number is not a root of unity (and more specifically, not a root of one of the $K$-cyclotomic polynomials appearing in the factorisation of its Schur elements), then the decomposition matrix is the identity matrix. If on the other hand we specialise the parameter to a root of a $K$-cyclotomic polynomial $\Phi$, we can define the $\Phi$-defect of a block to be the maximal $\Phi$-defect of the representations contained in the block.

In [Ge2] Geck showed that, contrary to $p$-defect for finite groups, the $\Phi$-defect is a block invariant for Iwahori–Hecke algebras of Weyl groups in the equal parameter case, that is, all representations of a block share the same $\Phi$-defect. This result was generalised to all real reflection groups with the explicit calculation of the decomposition matrices for the exceptional groups in [GeP]. He later conjectured that this result should also hold in the unequal parameter case for all real reflection groups. In the current paper, we prove that the $\Phi$-defect is a block invariant for all cyclotomic Hecke algebras associated with the complex reflection groups of the infinite series $G(l, p, n)$, including thus the Weyl groups of type $B_n$ in the unequal parameter case. We also computationally verify that the same result holds for all exceptional groups for which the decomposition matrices have been explicitly calculated. Based on all this evidence, we conjecture that the defect is a block invariant for all cyclotomic Hecke algebras associated with complex reflection groups.

Hecke algebras associated with the groups of type $G(l, 1, n)$ are also known as Ariki–Koike algebras. Their simple modules are parametrised by the $l$-partitions of $n$. Using the Schur element formula for the Ariki–Koike algebras that we developed in [ChJo2], we are able to give a combinatorial expression for the defect of the simple module $V^\lambda$, where $\lambda$ is an $l$-partition of $n$. This expression introduces the notion of charged hook lengths of a multipartition, which generalises the notion of generalised hook lengths introduced in [ChJo2]. Thanks to the Morita equivalence by Dipper and Mathas [DiMa], we can restrict to a specific setting, where the $\Phi$-defect of a module is given by the number of charged hook lengths that are congruent to 0 modulo $e$, where $e$ is the order of the root of unity whose minimal polynomial is $\Phi$. We compute this number using the abacus decomposition of $\lambda$ and we prove that it is equal to the weight of $\lambda$, another combinatorial datum attached to a multipartition by Fayers [Fa]. Since the weight is a block invariant, we are able to deduce that the defect is a block invariant. This concludes the proof of our main result for the groups $G(l, 1, n)$ (Theorem 3.4.4) to which the whole Section 3 is devoted. At the same time, we obtain a new way of calculating the weight, using the easy-to-compute charged hook lengths. In Section 4, we extend the result of the defect being a block invariant to the groups $G(l, p, n)$ with the use of Clifford theory.

Furthermore, in Section 5, we prove that the defect is a block invariant for all exceptional cyclotomic Hecke algebras studied in [ChM]. These are the only cyclotomic Hecke algebras associated with exceptional complex reflection groups for which the decomposition matrices have been explicitly calculated (besides [GeP] where only real groups were considered). Our proof is computational and is based on a GAP3 program that we created to calculate all possible blocks and the defects of the representations they contain.
Finally, in Section 6, we generalise our results to the case of cyclotomic Yokonuma–Hecke algebras, introduced in [CPA2] as generalisations of the Yokonuma–Hecke algebra of type $A$ and of the Ariki–Koike algebras. Through their connection to Ariki–Koike algebras, we are able to prove that the defect is a block invariant for these algebras as well. We conclude by pondering the question whether the defect being a block invariant is a property of essential algebras, that is, algebras whose Schur elements have a specific form (see Definition 2.2.1) and include Hecke algebras and Yokonuma–Hecke algebras as particular cases. This is why, in Section 2, which contains all preliminary notions needed for understanding this paper, we discuss Hecke algebras in the general context of essential algebras (which were in fact introduced in [Chl1] in order again to study the blocks of Hecke algebras in a more general context).

2. Schur elements for Hecke algebras

In this section, we will give a quick overview of the theory of symmetric algebras and the related results on Hecke algebras.

2.1. Symmetric algebras. Let $R$ be a commutative integral domain and let $A$ be an $R$-algebra, free and finitely generated as an $R$-module. If $R'$ is a commutative integral domain containing $R$, we will write $R'\otimes_RA$ and we will denote by $\text{Irr}(R')$ the set of irreducible representations of $R'$.

Let $B = (b_i)_{i \in I}$ be an $R$-basis of $A$. A symmetrising trace on the algebra $A$ is a linear map $\tau : A \to R$ such that the bilinear form $A \times A \to R$, $(a,b) \mapsto \tau(ab)$ is symmetric and non-degenerate, that is, the matrix $(\tau(b_ib_j))_{b_i,b_j \in B}$ is symmetric and has a determinant in $R$. If there exists a symmetrising trace on $A$, we say that $A$ is a symmetric algebra.

Example 2.1.1. Let $G$ be a finite group. The linear map $\tau : \mathbb{Z}[G] \to \mathbb{Z}$ defined by $\tau(1) = 1$ and $\tau(g) = 0$ for all $g \in G \setminus \{1\}$ is a symmetrising trace on $\mathbb{Z}[G]$; it is called the canonical symmetrising trace on $\mathbb{Z}[G]$.

Suppose that there exists a symmetrising trace $\tau$ on $A$. Let $B' = (b_i')_{i \in I}$ denote the dual basis to $B$ with respect to $\tau$, which is uniquely determined by the property $\tau(b_i'b_j') = \delta_{ij}$. Let $K$ be a field containing $R$ such that the algebra $KA$ is split. The map $\tau$ can be extended to $KA$ by extension of scalars. Let $V \in \text{Irr}(KA)$ with character $\chi_V$. The element $\sum_{i \in I} \chi_V(b_i)b_i'$ belongs to the centre of $KA$ [GePf] Lemma 7.1.7 and, by Schur’s lemma, acts as a scalar on $V$. This scalar, denoted by $s_V$, is called the Schur element associated with $V$. We have $s_V \in R_K$, where $R_K$ denotes the integral closure of $R$ in $K$ [GePf] Proposition 7.3.9]. Moreover, the Schur element $s_V$ satisfies the following equation (which also yields a formula for calculating $s_V$ when $\text{char}K \nmid \chi_V(1)$):

$$s_V \chi_V(1) = \sum_{i \in I} \chi_V(b_i) \chi_V(b_i').$$

The algebra $KA$ is semisimple if and only if $s_V \neq 0$ for all $V \in \text{Irr}(KA)$ [GePf] Theorem 7.2.6]. If this is the case, we have (see [CuRe] or [GePf] Theorem 7.2.6]):

$$\tau = \sum_{V \in \text{Irr}(KA)} \frac{1}{s_V} \chi_V.$$

Example 2.1.2. Let $G$ be a finite group and let $\tau$ be the canonical symmetrising trace on $A := \mathbb{Z}[G]$. The set $\{g\}_{g \in G}$ forms a basis of $A$ over $\mathbb{Z}$, with $\{g^{-1}\}_{g \in G}$ the dual basis of $A$ with respect to $\tau$. If $K$ is an algebraically closed field of characteristic 0, then $KA$ is a split semisimple algebra and $s_V = |G|/\chi_V(1) \in \mathbb{Q}$ for all $V \in \text{Irr}(KA)$. Because of the integrality of the Schur elements, we must have $|G|/\chi_V(1) \in \mathbb{Z}_K \cap \mathbb{Q} = \mathbb{Z}$ for all $V \in \text{Irr}(KA)$. Thus, we have also recovered the fact that $\chi_V(1)$ divides $|G|$ in $\mathbb{Z}$.

From now on, we assume that $R$ is integrally closed in $K$ and that $KA$ is split semisimple. Let $\theta : R \to L$ be a ring homomorphism into a field $L$ such that $L$ is the field of fractions of $\theta(R)$. We call such a ring homomorphism a specialisation of $R$. Let us also assume that the algebra $LA := L \otimes_R A$ is split. We then obtain a decomposition matrix $D_\theta = ([V : M])_{V \in \text{Irr}(KA), M \in \text{Irr}(LA)}$ that records how the simple modules of $KA$ decompose after the specialisation $\theta$ (for the definition of the decomposition matrix, the reader may refer to [GePf] Theorem 7.4.3]). The Brauer graph associated with $\theta$ has vertices labelled by $\text{Irr}(KA)$ and an edge joining $V, V' \in \text{Irr}(KA)$ if there exists $M \in \text{Irr}(LA)$ such that $[V : M] \neq 0$ and $[V' : M] \neq 0$. A connected component of the Brauer graph is called a $\theta$-block.
Since $L$ is a field, the kernel of $\theta$ is a prime ideal of $R$. Let $O = \{a/b \mid a, b \in R, \theta(b) \neq 0\}$ be the localisation of $R$ at $\text{Ker} \theta$. We say that $\theta$ is a principal specialisation if $O$ is a discrete valuation ring. Note that $\theta$ extends naturally to a map from $O$ to the residue field of $O$ and that the decomposition matrix does not change if we consider our algebra $A$ over $O$ instead of $R$. If $\Phi$ is a generator of the maximal ideal of $O$, we will write $\nu_\Phi : K^\times \to \mathbb{Z}$ for the exponential valuation associated with $O$. Thus, any $x \in K^\times$ can be written uniquely in the form $x = \Phi^{\nu_\Phi(x)}u$, where $u \in O^\times$. Furthermore, we have $O = \{x \in K^\times \mid \nu_\Phi(x) \geq 0\} \cup \{0\}$.

Let $V \in \text{Irr}(KA)$. Recall that $s_V \neq 0$. The integer $\nu_\Phi(s_V)$ is called the $\Phi$-defect of $V$. We define the $\Phi$-defect of a $\theta$-block $B$ to be the highest among the values of $\Phi$-defects of the elements of $B$. The following result about blocks of $\Phi$-defect 0 is a consequence of [GePi] Theorem 7.5.11 and [GeRo] Proposition 4.4:

**Theorem 2.1.3. (Blocks of defect 0)** Let $V \in \text{Irr}(KA)$. We have that $\theta(s_V) \neq 0$ if and only if $\{V\}$ is a $\theta$-block and the corresponding decomposition matrix block is the $(1 \times 1)$-matrix with entry 1.

In particular, we recover the following known semisimplicity criterion for the algebra $LA$ [GePi] Theorem 7.4.7]: $LA$ is semisimple if and only if $\theta(s_V) \neq 0$ for all $V \in \text{Irr}(KA)$. If this is the case, Tits’s deformation theorem (see, for example, [GePi] Theorem 7.4.6) yields a bijection between $\text{Irr}(KA)$ and $\text{Irr}(LA)$.

### 2.2. Essential algebras.

The notion of essential algebras was introduced by the first author in [Chl1] in order to study the block theory of Hecke algebras in positive characteristic. Let $A$ be an $R[x, x^{-1}]$-algebra, where $R$ is a Noetherian integrally closed domain and $x = (x_j)_{0 \leq j \leq m-1}$ is a set of $m$ indeterminates over $R$. We assume that $A$ is free and finitely generated as an $R[x, x^{-1}]$-module and that $A$ is symmetric. We also assume that the algebra $K(x)A$ is split semisimple, where $K$ is the field of fractions of $R$.

**Definition 2.2.1.** We say that the algebra $A$ is essential if, for each $V \in \text{Irr}(K(x)A)$, the Schur element $s_V(x)$ is of the form

$$s_V(x) = \xi_V N_V \prod_{i \in I_V} \Psi_{V,i}(M_{V,i})^{n_{V,i}}$$

where

(a) $\xi_V$ is an element of $R \setminus \{0\}$,
(b) $N_V$ is a monomial in $R[x, x^{-1}]$,
(c) $I_V$ is a finite index set,
(d) $(\Psi_{V,i})_{i \in I_V}$ is a family of monic polynomials in one variable with coefficients in $R$, irreducible over $K$, such that $\Psi_{V,i}(0) \in R^\times$,
(e) $(M_{V,i})_{i \in I_V}$ is a family of primitive monomials in $R[x, x^{-1}]$, that is, if $M_{V,i} = \prod_{j=0}^{m-1} x_j^{a_j}$, then $\gcd(a_j) = 1$,
(f) $(n_{V,i})_{i \in I_V}$ is a family of positive integers.

By [Chl1] Theorem 1.5.6], the Laurent polynomials $\Psi_{V,i}(M_{V,i})$ are irreducible in $K[x, x^{-1}]$, so Equation (2.1) yields the (unique) factorisation of $s_V(x)$ inside that ring. What is more, the monomials $(M_{V,i})_{i \in I_V}$ are unique up to inversion [Chl1 Proposition 3.1.2].

Let $y$ be another indeterminate over $R$, and let $\varphi : R[x, x^{-1}] \to R[y, y^{-1}]$ be an $R$-algebra morphism such that $\varphi(x_j) = y^{r_j}$, where $r_j \in \mathbb{Z}$ for all $j = 0, 1, \ldots, m - 1$. We denote by $A_\varphi$ the algebra obtained as a specialisation of $A$ via the morphism $\varphi$. As long as $\varphi(s_V(x)) \neq 0$ for all $V \in \text{Irr}(K(x)A)$, the algebra $K(y)A_\varphi$ is split semisimple (this is quite straightforward, but one can also refer to the proof of [Chl1 Proposition 4.3.3]). If this is the case, then, by Tits’s deformation theorem, we have a bijection between $\text{Irr}(K(x)A)$ and $\text{Irr}(K(y)A_\varphi)$. Moreover, the specialisation of the symmetrising trace of $A$ via $\varphi$ is a symmetrising trace on $A_\varphi$. In particular, the algebra $A_\varphi$ is essential. If we now consider a specialisation $\theta$ of $A_\varphi$ as in (2.1), then the representation theory of the specialised algebra becomes “interesting” when $\theta(y)$ is a root of one of the irreducible factors of the Schur elements of $A_\varphi$ (but not only then).

**Remark 2.2.2.** In fact, the algebra $K(y)A_\varphi$ is not semisimple if and only if $\varphi(\Psi_{V,i}(M_{V,i})) = 0$ for some $V \in \text{Irr}(K(x)A)$ and some $i \in I_V$. This in turn can happen if and only if $\varphi(M_{V,i}) = 1$ and $\Psi_{V,i}(1) = 0$. This is why in our original definition of essential algebras [Chl1 Definition 3.1.1], we also asked that the polynomials $\Psi_{V,i}$ satisfy $\Psi_{V,i}(1) \neq 0$. 


2.3. Hecke algebras. Let \( V \) be a finite dimensional complex vector space. A pseudo-reflection is a non-trivial element \( s \in \text{GL}(V) \) that fixes a hyperplane pointwise, that is, \( \dim(\text{Ker}(s - \text{id}_V)) = \dim(V) - 1 \). The hyperplane \( \text{Ker}(s - \text{id}_V) \) is the reflecting hyperplane of \( s \). A complex reflection group is a finite subgroup of \( \text{GL}(V) \) generated by pseudo-reflections. If \( W \) acts irreducibly on \( V \), then \( W \) is called an irreducible complex reflection group and the dimension of \( V \) is called the rank of \( W \). Every complex reflection group is isomorphic to a direct product of irreducible complex reflection groups; the latter have been classified by Shephard and Todd [ShTo].

Theorem 2.3.1. Let \( W \subset \text{GL}(V) \) be an irreducible complex reflection group. Then

- either \( (W, V) \cong (G(l, p, n), \mathbb{C}^{n-1}) \), where \( (l, p, n) \in (\mathbb{N})^3 \setminus \{(2, 2, 1)\} \) and \( G(l, p, n) \) is the group of all \( n \times n \) monomial matrices whose non-zero entries are \( l \)-th roots of unity, while the product of all non-zero entries is an \((l/p)\)-th root of unity;
- \((W, V)\) is isomorphic to one of the 34 exceptional groups \( G_n \) \((n = 4, \ldots, 37)\).

From now on, let \( W \) be an irreducible complex reflection group. Let \( A \) be the set of reflecting hyperplanes of \( W \) and let \( V_{\text{reg}} := V \setminus \bigcup_{H \in A} H \). We define \( P(W) := \pi_1(V_{\text{reg}}, x_0) \) and \( B(W) := \pi_1(V_{\text{reg}}/W, x_0) \), where \( x_0 \in V_{\text{reg}} \) is some fixed basepoint, to be respectively the pure braid group and the braid group of \( W \). It is known by [Bes2] Theorem 12.8 that the centre of \( B(W) \) is cyclic, generated by some element \( z \). We set \( \pi := (z^{[\omega(W)]}) \in P(W) \), where \( Z(W) \) denotes the centre of \( W \). For every orbit \( C \) of the action of \( W \) on \( A \), let \( C \) be the common order of the subgroups \( W_H \) where \( H \) is any element of \( C \) and \( W_H \) is the pointwise stabiliser of \( H \). Note that \( W_H \) is cyclic, for all \( H \in C \). Set \( R := \mathbb{Z}[u, u^{-1}] \) to be the Laurent polynomial ring in a set of indeterminates \( u = (u_{c,j})_{(c \in A/W)(0 \leq j \leq \epsilon_c-1)} \). The generic Hecke algebra \( H(W) \) of \( W \) is the quotient of the group algebra \( R[B(W)] \) by the ideal generated by the elements of the form

\[
(s - u_{c,0})(s - u_{c,1}) \cdots (s - u_{c,\epsilon_c-1}),
\]

where \( C \) runs over the set \( A/W \) and \( s \) runs over the set of monodromy generators around the images in \( V_{\text{reg}}/W \) of the elements of \( C \) (see [BMR] §2 for their definition).

This definition of generic Hecke algebras of complex reflection groups is due to Broué, Malle and Rouquier, who also conjectured that the algebra \( H(W) \) is a free \( R \)-module of rank \( |W| \) [BMR] §4. This conjecture is known as the “BMR freeness conjecture”, and was then known to hold for the real reflection groups [Bou IV, §2], the groups \( G(l, p, n) \) [ArKo, BroMa, Ar2] and \( G_4 \) [BroMa]. The proof of this conjecture for the exceptional irreducible complex reflection groups was completed only a couple of years ago, thanks to the works of Chavli [Cha1, Cha2], Marin [Mar1, Mar2, Mar3], Marin–Pfeiffer [MarPl] and Tsuchioka [Tsu].

Let now \( K \) be the field generated by the traces on \( V \) of all the elements of \( W \). Benard [Ben] and Bessis [Bes1] have proved, using a case-by-case analysis, that \( K \) is a splitting field for \( W \). The field \( K \) is called the field of definition of \( W \). If \( K \subset \mathbb{R} \), then \( W \) is a finite Coxeter group, and if \( K = \mathbb{Q} \), then \( W \) is a Weyl group. Malle [Ma3 5.2] has shown that, given that the BMR freeness conjecture holds, we can always find \( N_W \in \mathbb{Z}_{>0} \) such that if we take \( v := (v_{c,j})_{(c \in A/W)(0 \leq j \leq \epsilon_c-1)} \) defined by

\[
v_{c,j}^{N_W} := \eta_{c,j}^i u_{c,j},
\]

where \( \eta_{c,j} := \exp(2\pi i/e_c) \), then the \( K(v) \)-algebra \( K(v)H(W) \) is split semisimple. Taking \( N_W \) to be the number of roots of unity in \( K \) works every time, but sometimes it is enough to take \( N_W \) to be even as small as 1 (for example, if \( W = G(l, 1, n) \) or \( W = G_4 \)). Following Tits's deformation theorem, the specialisation \( v_{c,j} \mapsto 1 \) induces a bijection between \( \text{Irr}(K(v)H(W)) \) and the set \( \text{Irr}(W) \) of irreducible representations of \( W \).

The following conjecture is [BMM 2.1]:

Conjecture 2.3.2. “The BMM symmetrising trace conjecture” There exists a symmetrising trace \( \tau : H(W) \to R \) that satisfies the following two conditions:

1. \( \tau \) specialises to the canonical symmetrising trace on the group algebra of \( W \) when \( u_{c,j} \mapsto \eta_{c,j}^i \).
2. \( \tau \) satisfies

\[
\tau(T_{\beta^{-1}}^*) = \frac{\tau(T_{\beta \pi})}{\tau(T_{\pi})} \quad \text{for all } \beta \in B(W),
\]
where \( \beta \mapsto T_\beta \) denotes the natural surjection \( R[B(W)] \to \mathcal{H}(W) \) and \( x \mapsto x^* \) the automorphism of \( R \) given by \( u \mapsto u^* \).

If \( \tau \) exists, then \( \tau \) is unique \([BMM] \ 2.1\), and it is called the canonical symmetrising trace on \( \mathcal{H}(W) \). When Broué, Malle and Michel stated the above conjecture, it was known to hold for real reflection groups, while a symmetrising trace satisfying Condition (1) existed for the complex reflection groups of the infinite series \( G(l, p, n) \) (defined by Bremke and Malle in \([BreMa]\) and shown to be non-degenerate over \( R \) by Malle and Mathas in \([MaMi]\)). The non-real exceptional complex reflection groups for which the BMM symmetrising trace conjecture is known to hold are:

- \( G_4 \) \([MaMi, MaWa, BCCK]\) (3 independent proofs),
- \( G_5, G_6, G_7, G_8 \) \([BCCK]\),
- \( G_{12}, G_{22}, G_{24} \) \([MaMi]\),
- \( G_{13} \) \([BCC]\).

Nevertheless, the Schur elements with respect to the (conjectural) canonical symmetrising trace have been determined for all complex reflection groups. For real reflection groups, a complete list of bibliographical references can be found in \([Chl2] \ §2.2\). For the groups of type \( G(l, 1, n) \), two independent descriptions of the Schur elements with respect to the symmetrising trace of Bremke and Malle have been given by Geck–Iancu–Malle \([GIM]\) and Mathas \([Mat]\), whereas a simpler description has been subsequently given in \([ChJa2]\) using Mathas’s formula. Moreover, Geck, Iancu and Malle have shown that these Schur elements satisfy a certain palindromicity property \([GIM] \ Theorem 5.2\), which, by \([BMM] \ Lemma 2.7\), amounts to proving Condition (2) of the BMM symmetrising trace conjecture. From the Schur elements of \( G(l, 1, n) \), one recovers the Schur elements of \( G(l, p, n) \) when \( n > 2 \) or \( n = 2 \) and \( p \) is odd, with the use of Clifford theory. All remaining irreducible complex reflection groups have been dealt with by Malle: the groups of rank 2 in \([Ma2]\), the groups of superior rank in \([Mal]\).

Using a case-by-case analysis, we have shown that the algebra \( \mathcal{H}(W) \), defined over the ring \( \mathbb{Z}_K[v, v^{-1}] \), is essential \([Chl1] \ Theorem 4.2.6\). In fact, the irreducible polynomials \( \Psi_{V,i} \) appearing in the factorisation of the Schur elements of \( \mathcal{H}(W) \) are always \( K \)-cyclotomic polynomials, that is, minimal polynomials of roots of unity over the field \( K \). \([Chl1] \ Theorem 4.2.5\). They also satisfy \( \Psi_{V,i}(1) \neq 0 \).

Let \( y \) be an indeterminate, and let \( \varphi : \mathbb{Z}_K[v, v^{-1}] \to \mathbb{Z}_K[y, y^{-1}] \) be a \( \mathbb{Z}_K \)-algebra morphism such that \( \varphi(v_{C,j}) = y^{r_{C,j}} \), where \( r_{C,j} \in \mathbb{Z} \) for all \( C, j \). We call \( \varphi \) a cyclotomic specialisation of \( \mathcal{H}(W) \) and the algebra \( \mathcal{H}_\varphi(W) \) a cyclotomic Hecke algebra. By \([Chl1] \ Proposition 4.3.3\), the algebra \( K(y)\mathcal{H}_\varphi(W) \) is split semisimple (since none of the Schur elements is mapped to 0 by \( \varphi \)), and by Tits’s deformation theorem, we have:

\[
\text{Irr}(K(y)\mathcal{H}_\varphi(W)) \leftrightarrow \text{Irr}(K(y)\mathcal{H}_\varphi(W)).
\]

Furthermore, the specialisation \( y \mapsto 1 \) yields a bijection:

\[
\text{Irr}(K(y)\mathcal{H}_\varphi(W)) \leftrightarrow \text{Irr}(K(y)\mathcal{H}_\varphi(W)).
\]

**Remark 2.3.3.** Let \( \zeta \in K \) be a root of unity and set \( x := \zeta y \). The cyclotomic specialisation \( \varphi \) can be also given by \( \varphi(v_{C,j}) = (\zeta^{-1} x)^{r_{C,j}} \) for all \( C, j \). Then the algebra \( \mathcal{H}_\varphi(W) \) can be regarded as a \( \mathbb{Z}_K[x, x^{-1}] \)-algebra, which specialises to the group algebra of \( W \) when \( x \mapsto \zeta \). If that is the case, we refer to \( \varphi \) as a \( \zeta \)-cyclotomic specialisation.

The specialisation of the canonical symmetrising trace on \( \mathcal{H}(W) \) is a symmetrising trace on \( \mathcal{H}_\varphi(W) \), whose Schur elements are the images of the Schur elements of \( \mathcal{H}(W) \) via \( \varphi \). Therefore, the form of the Schur elements of \( \mathcal{H}_\varphi(W) \) is given by the following proposition \([Chl1] \ Proposition 4.3.5\):

**Proposition 2.3.4.** Let \( V \in \text{Irr}(W) \). The associated Schur element \( s_V(y) \) of \( \mathcal{H}_\varphi(W) \) is of the form

\[
s_V(y) = \xi_V y^{a_V} \prod_{\Phi \in C_V} \Phi(y)^{n_{V,\Phi}}
\]

where \( \xi_V \in \mathbb{Z}_K \), \( a_V \in \mathbb{Z} \), \( C_V \) is a finite set of \( K \)-cyclotomic polynomials, and \( n_{V,\Phi} \in \mathbb{N} \).

Let now \( \eta \) be a non-zero complex number, and let \( \theta : \mathbb{Z}_K[y, y^{-1}] \to K(\eta) \), \( y \mapsto \eta \) be a specialisation of \( \mathbb{Z}_K[y, y^{-1}] \). The specialised algebra \( K(\eta)\mathcal{H}_\varphi(W) \) is split. It is also semisimple unless \( \eta \) is the root of one of the \( K \)-cyclotomic polynomials appearing in the factorisation of the Schur elements of \( \mathcal{H}_\varphi(W) \). If \( s_V(y) \)
is a Schur element of $\mathcal{H}_c(W)$, with a form given by Equation (2.3), and $\eta$ is a root of some $\Phi \in C_V$, then $n_{V,\eta} = \nu_\Phi(s_V(y))$, the $\Phi$-defect of $V$. Furthermore, if $\Phi = \Phi_c$, the $c$-th cyclotomic polynomial over $\mathbb{Q}$ for some $c > 0$, then we may also refer to the $\Phi_c$-defect of $V$ as the $c$-defect of $V$.

**Remark 2.3.5.** We have that $s_V(1)$ is the Schur element of $V \in \text{Irr}(W)$ with respect to the canonical symmetrising trace on $\mathbb{Z}[W]$ (see Example 2.1.2). Thus, $s_V(1) \neq 0$ and $\Phi_1(y)$ is not a factor of $s_V(y)$. We deduce that the 1-defect of $V$ is 0 for all $V \in \text{Irr}(W)$.

Geck has shown that if $W$ is a finite Coxeter group and we are in the equal parameter case (that is, $r_{C,j} = r_{C',j}$ for all $C, C' \in \mathcal{A}/W$ and $j = 0, 1$), then any two representations that are in the same $\Phi$-block have the same $\Phi$-defect. This is proved in [Ge2, Proposition 7.4] for Weyl groups, using an analogous result for blocks of finite groups of Lie type [Ge1, 1.4], and it can be verified by inspection for $H_3$ and $H_4$, using the explicit knowledge of the $\theta$-blocks given in [GePf, Appendix F]. In [GecJa, Remark 3.3.16], it is conjectured that such a property may hold for all cyclotomic Hecke algebras associated to finite Coxeter groups. We will go one step further and conjecture that this property holds for all cyclotomic Hecke algebras associated to complex reflection groups.

**Conjecture 2.3.6.** Let $V, V' \in \text{Irr}(W)$. If $V$ and $V'$ belong to the same $\theta$-block of $K(\eta)\mathcal{H}_c(W)$, then they have the same $\Phi$-defect.

In the rest of this paper, we are going to prove the above conjecture in the following cases:

- $W = G(l, p, n)$, where $n \neq 2$, or $n = 2$ and $p$ is odd;
- $W$ is an exceptional complex reflection group of rank 2 and $\mathcal{H}_c(W)$ has “distinguished” parameters motivated by generalised Harish-Chandra theory.

These cases cover the unequal parameter case for finite Coxeter groups of type $B_n \cong G(2, 1, n)$, as well as the dihedral groups $I_2(p) \cong G(p, p, 2)$ where $p$ is odd. We will also show that the validity of the conjecture transfers to some other types of essential algebras, the cyclotomic Yokonuma–Hecke algebras, which are isomorphic to direct sums of matrix algebras over tensor products of cyclotomic Hecke algebras.

**Remark 2.3.7.** Let $W$ be a Weyl group, let $V \in \text{Irr}(W)$ and let $p$ be a prime number. As we mentioned in the introduction, when studying the representation theory of $W$ over a field $\mathbb{F}$ of characteristic $p$, we are interested in the traditional $p$-defect of $V$, which is defined as the $p$-part of the integer $|W|/\chi_V(1)$ and is not a block invariant. Since $|W|/\chi_V(1) = s_V(1)$ and $K = \mathbb{Q}$, that number is equal to the sum of $\nu_{\Phi_{p^k}}(s_V(y))$ for all $k \in \mathbb{Z}_{\geq 0}$. If now $\nu_{\Phi_{p^k}}(s_V(y)) = 0$ for all $k \in \mathbb{Z}_{>1}$, then that $p$-defect coincides with the $\Phi_p$-defect of $V$, which is a block invariant of the Hecke algebra when $y$ specialises to a primitive $p$th root of unity (or conjectured to be for $I_2(6)$ in the unequal parameter case). For example, if $W$ is the symmetric group $S_n$, then this condition holds if and only if $p^2 > n$ (for the form of the Schur elements in this case, see Formula (3.2) for $l = 1$). Note that James’s conjecture [Jam] also makes a connection between the decomposition matrix of the group algebra of $S_n$ over $\mathbb{F}$ and the one of its corresponding cyclotomic Hecke algebra when $y$ specialises to a primitive $p$th root of unity with the assumption $p^2 > n$ (the conjecture has been recently disproved, cf. [Wi]). Furthermore, in Theorem 3.7.1 we establish that the $\Phi_p$-defect is equal to the $p$-weight, which is also a known block invariant of $F[S_n]$.

3. **Defect in cyclotomic Ariki–Koike algebras**

Hecke algebras associated with complex reflection groups of type $G(l, 1, n)$ are also known as Ariki–Koike algebras. In this section, we give the description of the Schur elements for Ariki-Koike algebras obtained in [ChJa2]. This description uses a generalised version of hook lengths for partitions called “generalised hook lengths”. We give an easy way to compute variations of these elements, called “charged hook lengths”, using the notion of abaci. We produce counting formulas for the number of charged hook lengths equal to 0 in $\mathbb{Z}$ and in $\mathbb{Z}/e\mathbb{Z}$ for $e \in \mathbb{Z}_{>1}$, which allow us to establish a connection between the notions of defect and weight for a multipartition (see (3.7)). This connection leads to the proof of Conjecture 2.3.6 for type $G(l, 1, n)$.

Throughout this section, let $n \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}_{>0}$.
3.1. Generic Ariki-Koike algebras. Let \( q := (Q_0, \ldots, Q_{l-1}; q) \) be a set of \( l + 1 \) indeterminates and set \( R := \mathbb{Z}[q, q^{-1}] \). The Ariki-Koike algebra \( H_q^G \) is the associative \( R \)-algebra (with unit) with generators \( T_0, T_1, \ldots, T_{n-1} \) and relations:

\[
\begin{align*}
(T_0 - Q_0)(T_0 - Q_1) \cdots (T_0 - Q_{l-1}) &= 0 \\
(T_i - q)(T_i + 1) &= 0 \quad \text{for } 1 \leq i \leq n - 1 \\
T_0T_1T_0 &= T_1T_0T_1 \\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for } 1 \leq i \leq n - 2 \\
T_iT_j &= T_jT_i \quad \text{for } 0 \leq i < j \leq n - 1 \text{ with } j - i > 1.
\end{align*}
\]

The generic Hecke algebra of \( G(l, 1, n) \) has a presentation very similar to the presentation of the Ariki–Koike algebra, with the only difference being that the generators \( T_i \), for \( i \neq 0 \), satisfy quadratic relations of the form \((T_i - q_0)(T_i - q_1) = 0\), where \( q_0, q_1 \) are two indeterminates. If we set \( q = -q_0q_1^{-1} \), then \( H(G(l, 1, n)) \) can be defined over \( R \) and is isomorphic to \( H_q^G \).

The representation theory of \( H_q^G \) has first been studied by Ariki and Koike \([\text{ArKo}]\) and is governed by the combinatorics of partitions. A partition \( \lambda \) is a non-increasing sequence \( \lambda = (\lambda_1, \ldots, \lambda_m) \) of non-negative integers. One can assume this sequence is infinite by adding parts equal to zero. The rank of the partition is defined to be the number \( |\lambda| := \sum_{1 \leq i \leq m} \lambda_i \). If \( |\lambda| = n \in \mathbb{N} \), we say that \( \lambda \) is a partition of \( n \). By convention, the unique partition of 0 is the empty partition \( \emptyset \). More generally, for \( l \in \mathbb{Z}_{>0} \), an \( l \)-partition \( \lambda \) of \( n \) is a sequence \((\lambda^0, \lambda^1, \ldots, \lambda^{l-1}) \) of \( l \) partitions such that \( \sum_{0 \leq i \leq l-1} |\lambda^i| = n \). The number \( n \) is called the rank of \( \lambda \) and it is denoted by \( |\lambda| \). The set of all partitions is denoted by \( \Pi \) and the set of \( l \)-partitions of rank \( n \) is denoted by \( \Pi^l(n) \) (if \( l = 1 \), the letter \( l \) is omitted). It follows from \([\text{ArKo}]\) and Ariki’s semisimplicity criterion \([\text{ArI}]\) that the algebra \( \mathbb{Q}(q)H_q^G \) is split semisimple. We have a bijection \( \Pi^l(n) \leftrightarrow \text{Irr}(\mathbb{Q}(q)H_q^G) \), \( \lambda \leftrightarrow V^\lambda \).

3.2. Generalised hook lengths and Schur elements. As stated in the previous section, two independent descriptions of the Schur elements of \( H_q^G \) have been given by Geck–Iancu–Malle \([\text{GIM}]\) and Mathas \([\text{Mat}]\). In both articles, the Schur elements are given as fractions in \( \mathbb{Q}(q) \). However, since the Schur elements belong to the Laurent polynomial ring \( R \), we know that the denominator always divides the numerator. In \([\text{ChJa2}]\), we have given a cancellation-free formula for these Schur elements, that is, we have explicitly described their irreducible factors in \( R \). This is the formula that we are going to use in this paper. In order to present it, we need some further combinatorial notions.

Let \( \lambda \in \Pi \). We define the set of nodes \( [\lambda] \) of \( \lambda \) to be the set

\[
[\lambda] := \{(i, j) \mid i \geq 1, \ 1 \leq j \leq \lambda_i \}.
\]

Each node \((i, j)\) represents a box in the \( i \)-th row and the \( j \)-the column of the Young diagram of \( \lambda \), which we define as a left-justified array. We identify partitions with their Young diagrams.

The conjugate partition of \( \lambda \) is the partition \( \lambda' \) defined by

\[
\lambda'_k := \# \{ i \mid i \geq 1 \text{ such that } \lambda_i \geq k \}.
\]

The set of nodes of \( \lambda' \) satisfies

\[
(i, j) \in [\lambda'] \iff (j, i) \in [\lambda]
\]

and the Young diagram of \( \lambda' \) is obtained from the one of \( \lambda \) by transposition with respect to the main diagonal. If \( x = (i, j) \in [\lambda] \) and \( \mu \in \Pi \), we define the generalised hook length of \( x \) with respect to \( (\lambda, \mu) \) to be the integer:

\[
(3.1) \quad h_{i,j}^{\lambda,\mu} := \lambda_i - i + \mu'_j - j + 1.
\]

For \( \mu = \lambda \), the above formula becomes the classical hook length formula, which yields the length of the hook of \( \lambda \) that \( x \) belongs to (the hook consists of \( x \), the boxes below \( x \), and the boxes to its right).

Finally, we set

\[
N(\lambda) := \sum_{i \geq 1} (i-1)\lambda_i = \frac{1}{2} \sum_{i \geq 1} (\lambda'_i - 1)\lambda'_i = \sum_{i \geq 1} \left( \frac{\lambda'_i}{2} \right).
\]

The result below is the main result of \([\text{ChJa2}]\).
Theorem 3.3.1. Let $\lambda = (\lambda^0, \lambda^1, \ldots, \lambda^{l-1}) \in \Pi^l(n)$. The Schur element $s_\lambda(q)$ associated to the simple module $V^{\lambda}$ is given by

$$s_\lambda(q) = (-1)^{n(l-1)} q^{-N(\lambda)} (q - 1)^{-n} \prod_{0 \leq a \leq l-1} \prod_{(i,j) \in [\lambda^a]} \prod_{0 \leq b \leq l-1} (q^{k_{i,j}^a \lambda} Q_a Q_b^{-1} - 1)$$

where $\lambda$ is the partition of $n$ obtained by reordering all the numbers in $\lambda$. If we want to get rid of the term $(q - 1)^{-n}$, we can rewrite the formula as follows:

$$s_\lambda(q) = (-1)^{n(l-1)} q^{-N(\lambda)} \prod_{0 \leq a \leq l-1} \prod_{(i,j) \in [\lambda^a]} \prod_{0 \leq b \leq l-1} \prod_{b \neq a} (q^{k_{i,j}^a \lambda} Q_a Q_b^{-1} - 1)$$

where $[k]_q := (q^k - 1)/(q - 1) = q^{k-1} + 2 + \ldots + q + 1$ for any $k \in \mathbb{N}^*$. 

Remark 3.3.2. Let $\lambda \in \Pi$. The content of a node $(i,j) \in [\lambda]$ is defined to be the difference $c(i,j) := j - i$. For $s \in \mathbb{Z}$, we define the $s$-charged content of the node $(i,j)$ to be the integer $c(i,j) + s + 1$. If now we look at Formula (3.1) for the generalised hook length of $(i,j)$ with respect to $(\lambda, \mu) \in \Pi \times \Pi$, we observe that $\lambda_i - i$ is the content of the rightmost box in the $i$-th row of the Young diagram of $\lambda$, while $j - \mu_j$ is the content of the box at the bottom of the $j$-th column of the Young diagram of $\mu$; in order for the latter to hold even when the $j$-th column is empty, we may assume that the Young diagram of $\mu$ has an imaginary 0-th row with an infinite amount of boxes (see the definition of extended Young diagram in [Ja]). This approach to the definition of the generalised hook length will serve us later on when we work with charged hook lengths.

3.3. Specialisations and decomposition matrices. Let $k$ be a field and let $\theta: R \to k$ be a specialisation of $R$. Set $\xi_i := \theta(Q_j)$, for $i = 0, 1, \ldots, l - 1$, $u := \theta(q)$ and $u := (\xi_0, \ldots, \xi_{l-1}; u)$. Assume that the algebra $k\mathcal{H}_n^u$ is split. In [ChJa2, Theorem 4.2], we have shown that the semisimplicity criterion for symmetric algebras given at the end of [24] in combination with the form of the Schur elements given by Theorem 3.2.1 allows us to recover Ariki’s semisimplicity criterion for Ariki–Koike algebras, which is the following [Ar] Main theorem:

Theorem 3.3.1. The algebra $k\mathcal{H}_n^u$ is semisimple if and only if

$$\prod_{1 \leq i \leq n} (1 + u + \ldots + u^{i-1}) \prod_{0 \leq a < b \leq l-1} \prod_{-n < c < n} (u^c \xi_a - \xi_b) \neq 0.$$

In any case, we have a well-defined decomposition matrix $D_0 = ([V^\lambda : M])_{\lambda \in \Pi^l(n), M \in \text{Irr}(k\mathcal{H}_n^u)}$. There is a useful result by Dipper and Mathas [DiMa] which allows us to restrict ourselves to a very specific situation in order to study $D_0$. In order to do this, we set $U := \{0, 1, \ldots, l - 1\}$ and we assume that we have a partition

$$U = U_1 \cup U_2 \cup \ldots \cup U_t$$

which is the finest with respect to the property

$$\prod_{1 \leq a < b \leq l} \prod_{(a,b) \in U_a \times U_b} \prod_{-n < c < n} (u^c \xi_a - \xi_b) \neq 0.$$

For $i = 1, \ldots, t$, write $U_i := \{a_{i,1}, \ldots, a_{i,m_i}\}$ with $a_{i,1} < \ldots < a_{i,m_i}$. Whenever $f = (f_0, \ldots, f_{t-1})$ is a sequence indexed by $U$, we will write $f[i]$ for the sequence $(f_{a_{i,1}}, \ldots, f_{a_{i,m_i}})$.

Theorem 3.3.2. (The Morita equivalence of Dipper and Mathas) For $i = 1, \ldots, t$, we set $u_i := ((\xi_0, \ldots, \xi_{l-1}); u)$. The algebra $k\mathcal{H}_n^u$ is Morita equivalent to the algebra

$$\bigoplus_{n_1 + \ldots + n_t = n} k\mathcal{H}_{n_1}^u \otimes_k k\mathcal{H}_{n_2}^u \otimes_k \ldots \otimes_k k\mathcal{H}_{n_t}^u.$$ 

Remark 3.3.3. Recently, Rostam [Ros] has produced an explicit isomorphism between $k\mathcal{H}_n^u$ and

$$\bigoplus_{n_1 + \ldots + n_t = n} \text{Mat}_{n_1 \times n_2} (k\mathcal{H}_{n_1}^u \otimes_k k\mathcal{H}_{n_2}^u \otimes_k \ldots \otimes_k k\mathcal{H}_{n_t}^u),$$

which implies the Morita equivalence of Dipper and Mathas.
If, for each $i = 1, \ldots, t$, we take $M_i$ to be a simple $k\mathcal{H}_n^{\mu_i}$-module, then, under the Morita equivalence of Dipper and Mathas, we get a simple module $M \in \text{Irr}(k\mathcal{H}_n^{\mu})$ associated to $M_1 \otimes \cdots \otimes M_t$. For all $\lambda \in \Pi^*(n)$, we have:

$$[V^\lambda : M] = [V^{\lambda[1]} : M_1] \cdots [V^{\lambda[t]} : M_t].$$

Moreover, all the remaining entries of $D_0$ must be equal to 0.

Let us now consider the Brauer graph associated with $\theta$. Let $\lambda, \mu \in \Pi(n)$. By (3.3), we have $[V^\lambda : M] \neq 0 \neq [V^\mu : M]$ if and only if $[V^{\lambda[1]} : M_1] \neq 0 \neq [V^{\mu[1]} : M_1]$ for all $i = 1, \ldots, t$. This can obviously only happen if $\lambda[i]$ and $\mu[i]$ have the same rank, say $n_i$, for all $i = 1, \ldots, t$. We deduce that $V^\lambda$ and $V^\mu$ are in the same $\theta$-block if and only if $\lambda[i]$ and $\mu[i]$ are in the same block (of $k\mathcal{H}_n^{\mu_i}$) for all $i = 1, \ldots, t$. Sometimes, for simplicity, we say that $\lambda$ and $\mu$ are in the same block to mean that $V^\lambda$ and $V^\mu$ are in the same block.

Let $O$ be the localisation of $R$ at $\text{Ker} \theta$ and assume that $O$ is a discrete valuation ring. Let $\Phi$ be a generator of the maximal ideal of $O$. Let $\lambda \in \Pi(n)$. For $1 \leq \alpha < \beta \leq t$ and $(a, b) \in \mathcal{U}_a \times \mathcal{U}_b$, we have:

$$u(a \xi^a, \xi^{c+d}) - 1 \neq 0 \quad \text{for all } (i, j) \in [\lambda^a],$$

whence

$$\nu_\Phi(s_{\lambda} (q_{i})) = \sum_{i=1}^{l} \nu_{\Phi}(s_{\lambda[i]} (q_{i})).$$

Thanks to the form of the Schur elements of the Ariki–Koike algebras given by (3.2), we deduce the following formula for the calculation of the $\Phi$-defect of $V^\lambda$:

$$\nu_\Phi(s_{\lambda} (q_{i})) = \sum_{i=1}^{l} \nu_{\Phi}(s_{\lambda[i]} (q_{i})).$$

where $q_{i} = ((Q_{0}, \ldots, Q_{t-1})[i] : q)$. That is, we have that the $\Phi$-defect of $V^\lambda$ is the sum of the $\Phi$-defects of the $V^{\lambda[i]}$'s.

3.4. Cyclotomic Ariki–Koike algebras. Let $\eta := \exp(2 \pi i / l)$ and $K := \mathbb{Q}(\eta)$. Let $y$ be an indeterminate, and let $\varphi : \mathbb{Z}_K[q, q^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ be a cyclotomic specialisation of $\mathcal{H}_n^{\mu}$ such that

$$\varphi(Q_i) = \eta_i^r y^i \quad \text{for } i = 0, 1, \ldots, l - 1,$$

$$\varphi(q) = y^r$$

where $(r_0, \ldots, r_{t-1}, r) \in \mathbb{Z}^{t+1}$. The cyclotomic Ariki–Koike algebra $K(y)(\mathcal{H}_n^{\mu})_{\varphi}$ is split semisimple.

Let now $\eta$ be a non-zero complex number, and let $\theta : \mathbb{Z}_K[y, y^{-1}] \to K(\eta), y \mapsto \eta$ be a specialisation of $\mathbb{Z}_K[y, y^{-1}]$ such that $\theta(\eta)$ is a primitive $l$-th root of unity (for simplicity, we may assume that $\theta$ is a $\mathbb{Z}_K$-algebra morphism, whence $\theta(\eta^p) = \eta^p$). Let $\Phi$ denote the minimal polynomial of $\eta$ over the field $K$. The aim of Section 3 will be to prove the following result:

**Theorem 3.4.1.** Let $\lambda, \mu \in \Pi(n)$. If $V^\lambda$ and $V^\mu$ are in the same $\theta$-block, then they have the same $\Phi$-defect, that is, $\nu_\Phi(\varphi(s_{\lambda}(q_i))) = \nu_\Phi(\varphi(s_{\mu}(q_i)))$.

Set $\xi_i := \eta_i^r \eta^r = \theta(\varphi(Q_i))$, for $i = 0, 1, \ldots, l - 1$, $u := \eta^r = \theta(\varphi(q))$ and $u := (\xi_0, \ldots, \xi_{l-1}; u)$. It follows from Proposition 2.5.3 that the algebra $K(\eta)(\mathcal{H}_n^{\mu})_{\varphi}$ is semisimple unless $\eta$ is a root of unity, which is true if and only if $u$ is a root of unity.

From now on, let us assume that $u$ is a primitive root of unity of order $e > 0$, and set $k := K(\eta)$. Using the notation of the previous subsection, we have that $k\mathcal{H}_n^{\mu}$ is Morita equivalent to the algebra

$$\bigoplus_{n_i + \cdots + n_t = n} k\mathcal{H}_n^{u_1} \otimes k\mathcal{H}_n^{u_2} \otimes \cdots \otimes k\mathcal{H}_n^{u_t}.$$

For $i = 1, \ldots, t$, by definition of the sets $\mathcal{U}_i$, there exists $(s_{a_1, \ldots, a_{m_i}}) \in \mathbb{Z}^{m_i}$ such that $\xi_{a_{i,j}}/x_{a_{i,j}} = u^{s_{a_{i,j}}}$ for all $j = 1, \ldots, m_i$. In fact, for any $n_i \geq 0$, the algebra $k\mathcal{H}_n^{u_i}$ is isomorphic to the algebra $k\mathcal{H}_n^{\tilde{u}_i}$ where $\tilde{u}_i := (u^{s_{a_{1,1}}, \ldots, u_{s_{a_{m_{1},m_{1}}}}; u})$.

Let $\lambda \in \Pi(n)$ and $i \in \{1, \ldots, t\}$. For all $a, b \in \mathcal{U}_i$ and $(c, d) \in [\lambda^a]$, we have:

$$\theta(\varphi(q^c_{a,d} Q_a Q_b^{-1}) - 1) = \theta(\eta_i^r y^r u^{s_{a,b}} \eta_i^{r} y^{r} u^{s_{a,b}} - 1) = \eta_i^{r} y^r u^{s_{a,b}} \eta_i^{r} y^{r} u^{s_{a,b}} - 1 = u^{s_{a,b} + s_{a,b} - 1},$$

- 10
whence we obtain that

\[
(3.5) \quad \eta_1^{-a-b} \eta_1^{b} h_{c,d}^{\lambda} + r_a - r_b - 1 = 0 \Leftrightarrow h_{c,d}^{\lambda} + s_a - s_b \equiv 0 \mod e.
\]

Set \( M := r h_{c,d}^{\lambda} + r_a - r_b \). If \( M \neq 0 \), consider the following polynomial inside \( K[y] \):

\[
P(y) := \begin{cases}
\eta_1^{-a-b} y^M - 1 & \text{if } M > 0, \\
\eta_1^{b-a} y^M - 1 & \text{if } M < 0.
\end{cases}
\]

Since \( \gcd(P(y), P'(y)) = 1 \), all roots of \( P(y) \) are simple. Thus, we have that \( P(\eta) = 0 \) if and only if \( \Phi \) divides \( P(y) \) exactly once. That is,

\[
(3.6) \quad \eta_1^{-a-b} \eta_1^{b} - 1 = 0 \Leftrightarrow \nu_\Phi(\eta_1^{-a-b} y^M - 1) = 1.
\]

If \( M = 0 \) and \( a \neq b \), then \( \eta_1^{-a-b} - 1 \neq 0 \) and \( \nu_\Phi(\eta_1^{-a-b} - 1) = 0 \). If \( M = 0 \) and \( a = b \), then we must have \( r = 0 \), because the classical hook length of a node is always non-zero. However, in this case, we are covered by the case \( u = e = 1 \) below. Combining (3.5) and (3.6) with (3.2), we deduce that \( \nu_\Phi(\varphi(s_\lambda(q_i))) \) is equal to the number of elements of the multiset

\[
H(\lambda)_i := \begin{cases}
\{h_{c,d}^{\lambda} + s_a - s_b | a, b \in U_i, (c, d) \in [\lambda^a]\} & \text{if } e > 1, \\
\{h_{c,d}^{\lambda} + s_a - s_b | a, b \in U_i, a \neq b, (c, d) \in [\lambda^a]\} & \text{if } e = 1.
\end{cases}
\]

that are congruent to 0 modulo \( e \). If \( e = 1 \), this number is simply the cardinality of the multiset \( H(\lambda)_i \), which is equal to \( |\lambda|/(m_1 - 1) \).

**Remark 3.4.2.** The value of \( \nu_\Phi(\varphi(s_\lambda(q_i))) \) is not affected by our choice of \((s_{a_1,1}, \ldots, s_{a_1,m_1}) \). If \((s_{a_1,1}', \ldots, s_{a_1,m_1}') \in \mathbb{Z}^m_1 \) also satisfies \( \xi_{a_1,i}/\xi_{a_1} = u^{s_{a_1,i}'}, \) then \( s_{a_1,i} = s_{a_1,i}' \mod e \), for all \( j = 1, \ldots, m_i \).

We can now use Formula (3.21) to calculate \( \nu_\Phi(\varphi(s_\lambda(q))) \). We obtain:

\[
(3.7) \quad \nu_\Phi(\varphi(s_\lambda(q))) = \sum_{i=1}^{t} \sharp\{h \in H(\lambda)_i \mid h \equiv 0 \mod e\}.
\]

Therefore, in order to study the defect in Ariki–Koike algebras, it is enough to focus on specialised Ariki–Koike algebras of the form \( \mathcal{H}_n^u \) where \( u = (u_0, \ldots, u_{l-1}) \) for some \((s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l_1 \). In particular, in order to prove Theorem 3.3.1 it is enough to prove the following: for \( \lambda, \mu \in \Pi(n) \), we have

\[
(3.8) \quad \lambda, \mu \text{ are in the same block of } \mathcal{H}_n^u \Rightarrow \sharp\{h \in H(\lambda) \mid h \equiv 0 \mod e\} = \sharp\{h \in H(\mu) \mid h \equiv 0 \mod e\},
\]

where

\[
H(\lambda) := \begin{cases}
\{h_{c,d}^{\lambda} + s_a - s_b | a, b \in \{0, 1, \ldots, l-1\}, (c, d) \in [\lambda^a]\} & \text{if } e > 1, \\
\{h_{c,d}^{\lambda} + s_a - s_b | a, b \in \{0, 1, \ldots, l-1\}, a \neq b, (c, d) \in [\lambda^a]\} & \text{if } e = 1.
\end{cases}
\]

For \( e = 1 \), (3.5) is automatically true, since \( \sharp\{h \in H(\lambda) \mid h \equiv 0 \mod e\} = \sharp H(\lambda) = n(l-1) \) for all \( \lambda \in \Pi(n) \). Therefore, in the rest of the paper, we are only going to consider the case \( e > 1 \).

**Remark 3.4.3.** Let us choose \((s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l_1 \) so that all elements of the multiset \( H(\lambda) \) are non-zero, for all \( \lambda \in \Pi(n) \). If we now consider the \( \mathbb{Z} \)-algebra morphism \( \varphi : \mathbb{Z}[q, q^{-1}] \to \mathbb{Z}[y, y^{-1}] \) such that

\[
\varphi(Q_i) = y^{s_i} \quad \text{for } i = 0, 1, \ldots, l-1,
\]

then the algebra \( \mathbb{Q}(y)(\mathcal{H}_n^u)_\varphi \) is split semisimple and the \( c \)-defect of \( V^{\lambda} \) is equal to the number of elements of \( H(\lambda) \) that are congruent to 0 modulo \( e \), for all \( \lambda \in \Pi(n) \).
Remark 3.4.4. One could ask, especially after seeing Formula (3.7), whether we can replace $\Phi$-defect with $e$-defect when $\eta = u$, for any cyclotomic Ariki–Koike algebra. We saw that it is the case when the Ariki–Koike algebra is as in the previous remark, and it is also the case when we deal with real reflection groups. However, in the complex case, the $e$-defect is not always well-defined, because $K$-cyclotomic polynomials may not be the same as the cyclotomic polynomials over $\mathbb{Q}$.

Let $l = 3$ and consider the cyclotomic Ariki–Koike algebra $(\mathcal{H}_2^3)_{\varphi}$, where

$$\varphi(Q_0) = 1, \varphi(Q_1) = \eta_3, \varphi(Q_2) = \eta_3^2 y \quad \text{and} \quad \varphi(q) = y.$$ 

We have $K = \mathbb{Q}(\eta_3)$ and, over $K$, the minimal polynomials of the third roots of unity are of degree 1. For $\lambda = (2, 0, 0)$, we have

$$\varphi(s_{\lambda}(q)) = -3\eta_3^2 y^{-1}(y + 1)(y - \eta_3)^2,$$

and so $\nu_{y - \eta_3}(\varphi(s_{\lambda}(q))) = 2$, while $\nu_{y - \eta_3^2}(\varphi(s_{\lambda}(q))) = 0$. The 2-defect of $V^\lambda$ is equal to 1.

Nevertheless, thanks to the work of Brundan and Kleshchev on cyclotomic quiver Hecke algebras [BrKl], we know that specialising $y$ to roots of unity of the same order yields isomorphic Ariki–Koike algebras. Therefore, the representation theory remains the same, even though the labelling of the representations might change. In this particular example, for $\mu = (0, 2, 0)$, we have

$$\varphi(s_{\mu}(q)) = -3\eta_3 y^{-1}(y + 1)(y - \eta_3^2)^2,$$

and so the roles of $\lambda$ and $\mu$ interchange, depending on whether $y \mapsto \eta_3$ or $y \mapsto \eta_3^2$.

3.5. Charged hook lengths and abaci. We begin with several classical definitions. An abacus is a subset $L$ of $\mathbb{Z}$ such that $-i \in L$ and $i \not\in L$ for all $i$ large enough. In a less formal way, each $i \in L$ corresponds to the position of a black bead on the horizontal abacus which is full of black beads on the left and empty on the right. One can associate to $\lambda \in \Pi$ and $s \in \mathbb{Z}$ an abacus $L_s(\lambda)$ such that $k \in L_s(\lambda)$ if and only if there exists $j \in \mathbb{Z}_{\geq 0}$ such that $k = \lambda_j - j + s + 1$ (note that $\lambda$ is assumed to have an infinite number of zero parts). The elements of $L_s(\lambda)$ are called the $s$-charged $\beta$-numbers of $\lambda$. Given an abacus $L$, one can easily find the unique partition $\lambda$ and the integer $s \in \mathbb{Z}$ such that $L_s(\lambda) = L$. Indeed, each part of $\lambda$ corresponds to a black bead of the abacus and is equal to the number of empty positions at its left, while $s$ is the integer labelling the position obtained by the rightmost black bead after sliding all the black beads in the abacus to the left. If now we fix $m \in \mathbb{Z}_{\geq 0}$ such that $\lambda_{m+s} = 0$, we denote by $X[\lambda, m]$ the tuple of $s$-charged $\beta$-numbers $(\beta_1, \beta_2, \ldots, \beta_{m+s})$, where $\beta_j := \lambda_j - j + s + 1$ for all $j \in \mathbb{Z}_{\geq 0}$. Using the terminology of Remark 3.4.2, $\beta_j$ is the $s$-charged content of the rightmost box in the $j$-th row of the Young diagram of $\lambda$ (again, in order for this to hold even when the $j$-th row is empty, we may assume that the Young diagram of $\lambda$ has an imaginary 0-th column with an infinite amount of boxes). Moreover, note that $\beta_1 > \beta_2 > \cdots > \beta_{m+s}$ and that $\beta_{m+s} = -m + 1$.

Example 3.5.1. Let us take the partition $\lambda = (5, 4, 2, 1, 1)$ and $s = 0$. The associated abacus $L_0(\lambda)$ may be represented as follows, where the positions to the right of the dashed vertical line are labelled by the non-negative integers:

```
● ● ● ● ● ● O ● ● ● O O O O O O O O O O
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
```

We have $X[\lambda, 6] = (5, 3, 0, -2, -3, -5)$.

Note that the abacus may be recovered from the datum of $X[\lambda, m]$ by setting black beads in the positions given in $X[\lambda, m]$ and in all the positions to the left of $\beta_{m+s}$.

Now, let $\lambda \in \Pi^l$ and $s = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$; we refer to $s$ as a multicharge. One can associate to $\lambda$ and $s$ an $l$-abacus defined as the $l$-tuple $(L_{s_0}(\lambda^0), \ldots, L_{s_{l-1}}(\lambda^{l-1}))$. This $l$-abacus is pictured by stacking the abaci so that $L_{s_i}(\lambda^i)$ is right on top of $L_{s_{i-1}}(\lambda^{i-1})$, for $i = 1, \ldots, l - 1$, and the positions are aligned vertically. If we fix $m \in \mathbb{Z}_{\geq 0}$ such that $\lambda_{i, m+s_i} = 0$ for all $i = 0, 1, \ldots, l - 1$, then we denote by $X[\lambda, m]$ the $l$-tuple $(X[\lambda^0, m], \ldots, X[\lambda^{l-1}, m])$. We will simply write $X^i$ for $X[\lambda^i, m]$ and we will denote by $(\beta_1^i, \ldots, \beta_{m+s_i}^i)$ the elements of $X^i$, for all $i = 0, 1, \ldots, l - 1$.

Example 3.5.2. Let $l = 3$, $\lambda = (2, 1, 1, 1)$ and $s = (0, 1, 2)$. The associated $l$-abacus can be represented as follows:
We have $X[\lambda, 3] = ((2, -1), (2, 0, -1), (3, 2, 0, -1), (-2)).$

Obviously, the $l$-abacus can be easily recovered from the datum of $X[\lambda, m]$. From now on, we will often switch from one representation using abaci to the one using $\beta$-numbers.

Let $a, b \in \{0, 1, \ldots, l - 1\}$ and $(c, d) \in [\lambda^a]$. We define the $\mathbf{s}$-charged hook length of $(c, d)$ with respect to $(\lambda^a, \lambda^b)$ to be the integer

$$ch_{c,d}^{\lambda^a, \lambda^b} := h_{c,d}^{\lambda^a, \lambda^b} + s_a - s_b.$$

We have

$$ch_{c,d}^{\lambda^a, \lambda^b} = \lambda^a_c - c + \lambda^b_d - d + 1 + s_a - s_b = (\lambda^a_c - c + s_a + 1) - (d - \lambda^b_d + s_b + 1) + 1$$

We observe that $\lambda^a_c - c + s_a + 1 = \beta^a_c$, while $d - \lambda^b_d + s_b + 1$ is the $s_a$-charged content of the box at the bottom of the $d$-th column of the Young diagram of $\lambda^b$ (cf. Remark 3.2.2). The latter coincides with the leftmost position in the component $b$ of the abacus that has exactly $d$ empty positions at its left. In fact, $d - \lambda^b_d + s_b$ is the $d$-th empty position in the component $b$ of the abacus, reading from left to right. Therefore, we can use the set of $\beta$-numbers and the associated abacus decomposition of our $l$-partition in order to easily compute the elements of the multiset

$$H(\lambda) := \{ch_{c,d}^{\lambda^a, \lambda^b} \mid a, b \in \{0, 1, \ldots, l - 1\}, (c, d) \in [\lambda^a]\}$$

as follows:

- Let $\beta$ be the position of a black bead.
- Let $\delta(\beta)$ be the number of empty positions to the left of $\beta$.
- For $b = 0, 1, \ldots, l - 1$ and $d = 1, \ldots, \delta(\beta)$, let $y^b_d$ be the $d$-th empty position in the component $b$ of the abacus, reading from left to right.
- For $b = 0, 1, \ldots, l - 1$, define $H^b(\beta)$ to be the set $\{\beta - y^b_d \mid d = 1, \ldots, \delta(\beta)\}$. Note that:
  - if $\delta(\beta) = 0$, then $H^b(\beta) = \emptyset$;
  - if $\delta(\beta) \neq 0$ and $\beta = \beta^a_c$, then $\beta - y^b_d = ch_{c,d}^{\lambda^a, \lambda^b}$.
- Denote by $H(\beta)$ the multiset defined by the union $\bigcup_{b=0}^{l-1} H^b(\beta)$.
- The multiset $H(\lambda)$ is the union of the multisets $H(\beta)$, where $\beta$ runs over the positions of black beads (that is, the $\beta$-numbers $\beta^a_c$). It is enough to consider the black beads that have at least one empty position at their left (that is, $\lambda^a_c \neq 0$), since otherwise $H(\beta) = \emptyset$.

**Example 3.5.3.** Let $l = 2$, $\lambda = (3, 1, 2, 1, 1)$ and $\mathbf{s} = (0, 2)$. We write the associated 2-abacus as follows:

```
 0 1 2 3 4 5 6 7 8 9 10
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1
```

We begin with $\beta^0_1 = 3$. We observe that $\beta^0_1$ has 3 empty positions to its left, and so $\delta(\beta^0_1) = 3$. We have

$$y^0_1 = -1, \quad y^0_2 = 1, \quad y^0_3 = 2, \quad y^0_4 = 0, \quad y^0_5 = 3, \quad y^0_6 = 5$$

and so

$$H(\beta^0_1) = \{\beta^0_1 - y^0_1, \beta^0_1 - y^0_2, \beta^0_1 - y^0_3 \mid b = 0, 1\} = \{3 + 1, 3 - 1, 3 - 2, 3 - 3, 3 - 3, 3 - 5\} = \{4, 2, 1, 3, 0, -2\}$$

One may find it more convenient to compute this set the other way round. We take the leftmost empty position in the component 0, labelled by $-1$, and the leftmost empty position in the component 1, labelled by 0.
So at this stage, we have \(\{\beta_2^0 - y_1^0, \beta_1^0 - y_1^1\} = \{3 + 1 = 4, 3 - 0 = 3\}\). We then consider the second leftmost empty position in each of the two components.

We obtain \(\{\beta_0^0 - y_2^0, \beta_1^0 - y_2^1\} = \{3 - 1 = 2, 3 - 3 = 0\}\). We finally do the same for the third leftmost empty positions.

We obtain \(\{\beta_0^0 - y_3^0, \beta_0^0 - y_3^1\} = \{3 - 2 = 1, 3 - 5 = -2\}\) and so \(H(\beta_0^0) = \{4, 3, 2, 0, 1, -2\}\).

Continuing this way, we get:

- \(H(\beta_2^0) = \{\beta_2^0 - y_1^0 \mid b = 0, 1\} = \{0 + 1, 0 - 0\} = \{1, 0\}\),
- \(H(\beta_1^0) = \{\beta_1^0 - y_1^1, \beta_1^0 - y_2^1 \mid b = 0, 1\} = \{4 + 1, 4 - 1, 4 - 0, 4 - 3\} = \{5, 3, 4, 1\}\),
- \(H(\beta_2^1) = \{\beta_2^1 - y_1^0 \mid b = 0, 1\} = \{2 + 1, 2 - 0\} = \{3, 2\}\),
- \(H(\beta_3^1) = \{\beta_3^1 - y_1^0 \mid b = 0, 1\} = \{1 + 1, 1 - 0\} = \{2, 1\}\).

We obtain

\[H(\lambda) = \{-2, 0, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 5\}.

**Example 3.5.4.** Let us also work out Example 3.5.2. We have

\[y_0^0 = 0, \ y_2^0 = 1, \ y_1^1 = 1, \ y_2^1 = 3, \ y_1^2 = 1, \ y_2^2 = 4\]

and so

- \(H(\beta_0^0) = \{\beta_0^0 - y_1^0, \beta_0^0 - y_2^0 \mid b = 0, 1, 2\} = \{2 - 0, 2 - 1 - 2, 2, 2 - 3, 2 - 1, 2 - 4\} = \{2, 1, 1, -1, 1, -2\}\),
- \(H(\beta_1^0) = \{\beta_1^0 - y_1^1, \beta_1^0 - y_2^1 \mid b = 0, 1, 2\} = \{2 - 0, 2 - 1, 2 - 1\} = \{2, 1, 1\}\),
- \(H(\beta_2^1) = \{\beta_2^1 - y_1^0 \mid b = 0, 1, 2\} = \{3 - 0, 3 - 1, 3 - 1\} = \{3, 2, 2\}\),
- \(H(\beta_3^1) = \{\beta_3^1 - y_1^0 \mid b = 0, 1, 2\} = \{2 - 0, 2 - 1, 2 - 1\} = \{2, 1, 1\}\).

We obtain

\[H(\lambda) = \{-2, -1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 3\}.

### 3.6 Residue classes of charged hook lengths.

Let \(\lambda \in \Pi^l\), \(s = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l\) and \(e \in \mathbb{Z}_{>1}\). We consider the \(l\)-abacus associated to the pair \((\lambda, s)\) and the set of \(\beta\)-numbers \(X := X[\lambda, m] = (X^0, \ldots, X^{l-1})\). Recall that \(\min(X^e) = 1 - m\) for all \(c = 0, \ldots, l - 1\). The aim of this subsection is to show how one can use \(X\) to compute the number of charged hook lengths congruent to 0 modulo \(e\). In order to do this, we will assume that \(s\) belongs to

\[A_e^l = \{(s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l \mid s_0 \leq s_1 \leq \ldots \leq s_{l-1} \leq s_0 + e\},\]

since permuting the entries of the multicharge or taking other representatives of their congruence classes modulo \(e\) does not affect the Ariki–Koike algebra.

We use the notation of the previous subsection for the description of the elements of \(H(\lambda)\). We have

\[
(3.9) \quad H(\lambda) = \bigcup_{c_1=0}^{l-1} \bigcup_{c_2=0}^{l-1} \bigcup_{x \in X'^{c_1}} H^{c_2}(x),
\]

where \(H^{c_2}(x) = \{x - y_0^{c_2} \mid d = 1, \ldots, \delta(x)\}\). Note that \(H^{c_2}(x)\) contains \(\delta(x)\) distinct elements. The following lemma, which gives a necessary and sufficient condition for 0 to belong to \(H^{c_2}(x)\), will be useful later on.
Lemma 3.6.1. Let \((c_1, c_2) \in \{0, \ldots, l-1\}^2\) and let \(x \in X^{c_1}\). We have \(0 \notin H^{c_2}(x)\) if and only if the following two conditions are satisfied:

(i) \(x \notin X^{c_2}\);

(ii) \(\sharp\{y \in X^{c_1} \mid y < x\} < \sharp\{y \in X^{c_2} \mid y < x\}\).

Proof. If \(x \in X^{c_2}\), then \(0 \notin H^{c_2}(x)\). If \(x \notin X^{c_2}\) and \(\sharp\{y \in X^{c_1} \mid y < x\} \geq \sharp\{y \in X^{c_2} \mid y < x\}\), then

\[
\sharp\{y \in \mathbb{Z} \setminus X^{c_1} \mid 1 - m < y < x\} \leq \sharp\{y \in \mathbb{Z} \setminus X^{c_2} \mid 1 - m < y < x\}.
\]

We have \(\delta(x) = \sharp\{y \in \mathbb{Z} \setminus X^{c_1} \mid 1 - m < y < x\}\), whence \(y_d^{c_2} < x\) for all \(d = 1, \ldots, \delta(x)\). We conclude that \(0 \notin H^{c_2}(x)\).

Suppose now that Conditions (i) and (ii) are satisfied. Then

\[
\sharp\{y \in \mathbb{Z} \setminus X^{c_1} \mid 1 - m < y < x\} > \sharp\{y \in \mathbb{Z} \setminus X^{c_2} \mid 1 - m < y < x\}.
\]

Since \(\delta(x) = \sharp\{y \in \mathbb{Z} \setminus X^{c_1} \mid 1 - m < y < x\}\) and \(x \notin X^{c_2}\), there exists \(d \in \{1, \ldots, \delta(x)\}\) such that \(y_d^{c_2} = x\). Hence, \(0 \in H^{c_2}(x)\).

Let \(c \in \{0, 1, \ldots, l-1\}\) and \(x \in X^c\). Then, for \(k \in \mathbb{Z}_{\geq 0}\), we set:

\[
\mathcal{N}_k(x) := \begin{cases} \sharp\{t \in \{c+1, \ldots, l-1\} \mid x \notin X^t\} & \text{if } k = 0; \\ \sharp\{t \in \{0, \ldots, l-1\} \mid x - kz \notin X^t \cup \mathbb{Z}_{\leq -m}\} & \text{if } k > 0. \end{cases}
\]

The main result of this section is Proposition 3.6.3, which produces the number of charged hook lengths in \(H(\lambda)\) that are congruent to 0 modulo \(e\) in terms of the numbers \(\mathcal{N}_k(x)\). In order to prove it, we will first compute the number of charged hook lengths equal to 0 — we can refer to this as the case \(e = \infty\).

Proposition 3.6.2. The number of charged hook lengths in \(H(\lambda)\) equal to 0 is

\[
\sum_{0 \leq s \leq l-1} \sum_{x \in X^s} \mathcal{N}_0(x).
\]

Proof. Let \(c \in \{0, \ldots, l-1\}\). First, we observe that if \(x \in X^c\), then \(0 \notin H^c(x)\).

Now let \((c_1, c_2) \in \{0, \ldots, l-1\}^2\) such that \(c_1 < c_2\) and let \(\mathcal{N}_0(X^{c_1}, X^{c_2})\) be the number of elements equal to 0 inside

\[
H(X^{c_1}, X^{c_2}) := \left( \bigcup_{x \in X^{c_1}} H^{c_2}(x) \right) \cup \left( \bigcup_{x \in X^{c_2}} H^{c_1}(x) \right).
\]

If \(x \in X^{c_1} \cap X^{c_2}\), then \(0 \notin H^{c_2}(x)\) for \(i = 1, 2\). Moreover, since \(s_{c_1} \leq s_{c_2}\), the set \(X^{c_2}\) has \(s_{c_2} - s_{c_1}\) more elements than \(X^{c_1}\).

Set \(M := s_{c_2} - s_{c_1}\). Let \(x_1 > x_2 > \cdots > x_N\) be the elements of \(X^{c_1} \setminus X^{c_2}\) and \(z_1 > z_2 > \cdots > z_{M+N}\) be the elements of \(X^{c_2} \setminus X^{c_1}\). Lemma 3.6.1 implies the following:

- For all \(i = 1, \ldots, M\), we have \(0 \notin H^{c_1}(z_i)\).
- For all \(i = M+1, \ldots, M+N\), we have
  - \(0 \notin H^{c_2}(x_{i-M})\) and \(0 \in H^{c_1}(z_i)\), if \(x_{i-M} < z_i\);
  - \(0 \in H^{c_2}(x_{i-M})\) and \(0 \notin H^{c_1}(z_i)\), if \(x_{i-M} > z_i\).

Therefore, \(\mathcal{N}_0(X^{c_1}, X^{c_2}) = N = \sharp\{x \in X^{c_1} \mid x \notin X^{c_2}\}\).

The result of the proposition follows from (3.9) and the fact that

\[
\sum_{0 \leq c_1 < c_2 \leq l-1} \mathcal{N}_0(X^{c_1}, X^{c_2}) = \sum_{0 \leq c \leq l-1} \sum_{x \in X^c} \mathcal{N}_0(x).
\]

Proposition 3.6.3. The number of charged hook lengths in \(H(\lambda)\) congruent to 0 modulo \(e\) is

\[
\sum_{0 \leq c \leq l-1} \sum_{x \in X^c} \sum_{k \geq 0} \mathcal{N}_k(x).
\]
Proof. Let $k \in \mathbb{Z}_{\geq 0}$. For all $c \in \{0, \ldots, l-1\}$, we have $X^c = (\beta_1^c, \beta_2^c, \ldots, \beta_m^c)$, and we will write $X^c[ke]$ for the tuple $(\beta_1^c + ke, \beta_2^c + ke, \ldots, \beta_m^c + ke, -m + ke - 1, -m + ke - 2, \ldots, -m + 1)$. Let $(c_1, c_2) \in \{0, \ldots, l-1\}^2$ such that $c_1 < c_2$. Note that, since $s \in A_c$, we have 
\[ s_{c_1} \leq s_{c_2} + ke \quad \text{and} \quad s_{c_2} \leq s_{c_1} + ke. \]
Using again here the notation of the proof of Proposition 3.6.2 we obtain that the number of elements equal to $\pm ke$ inside $H(X^{c_1}, X^{c_2})$ is 
\[ N_0(X^{c_1}, X^{c_2}[ke]) + N_0(X^{c_2}, X^{c_1}[ke]). \]
In a similar way, we obtain that the number of elements equal to $\pm ke$ inside $\bigcup_{x \in X^c} H^c(x)$ is $N_0(X^c, X^c[ke])$ for all $c \in \{0, \ldots, l-1\}$. Following (3.9), we deduce that the number of charged hook lengths in $H(\lambda)$ equal to $\pm ke$ is 
\[ \sum_{c_1=0}^{l-1} \sum_{c_2=0}^{l-1} N_0(X^{c_1}, X^{c_2}[ke]). \]
We now observe that, for all $(c_1, c_2) \in \{0, \ldots, l-1\}^2$, we have 
\[ N_0(X^{c_1}, X^{c_2}[ke]) = \sharp\{x \in X^{c_1} \mid x \notin X^{c_2}[ke]\} = \sharp\{x \in X^{c_1} \mid x - ke \notin X^{c_2} \cup \mathbb{Z}_{\leq -m}\}. \]
Therefore, the number of charged hook lengths in $H(\lambda)$ equal to $\pm ke$ is 
\[ \sum_{0 \leq c \leq l-1} \sum_{x \in X^c} N_k(x). \]
We conclude that the number of charged hook lengths in $H(\lambda)$ congruent to 0 modulo $e$ is 
\[ \sum_{0 \leq c \leq l-1} \sum_{x \in X^c} \sum_{k \geq 0} N_k(x). \]
\[ \square \]

Example 3.6.4. Let us look at the Example 3.5.2 whose $H(\lambda)$ has been calculated in Example 3.5.4. We have $X[\lambda, 3] = ((2, -1, -2), (2, 0, -1, -2), (3, 2, 0, -1, -2))$. Since $X^0 \subset X^1 \subset X^2$, we have $N_0(x) = 0$ for all $x \in X^c$ with $c \in \{0, 1, 2\}$, and so $0 \notin H(\lambda)$.

Take $e = 2$. For $k > 1$, we have $N_k(x) = 0$ for all $x \in X^c$ with $c \in \{0, 1, 2\}$. For $k = 1$, we have $N_k(x) = 3$ for all $x \in X^c$ with $c \in \{0, 1, 2\}$ except for $x \in \beta_1^0, \beta_1^1, \beta_2^1$. We have $N_1(\beta_1^0) = 1, N_1(\beta_1^1) = 1, N_1(\beta_2^1) = 3$ and $N_1(\beta_2^0) = 1$. We deduce that the total number of even charged hook lengths in $H(\lambda)$ is equal to 6.

Take $e = 3$. For $k > 1$, we have $N_k(x) = 0$ for all $x \in X^c$ with $c \in \{0, 1, 2\}$. For $k = 1$, we have $N_k(x) = 0$ for all $x \in X^c$ with $c \in \{0, 1, 2\}$ except for $x = \beta_1^0$. We have $N_1(\beta_1^1) = 1$, and so the number of charged hook lengths in $H(\lambda)$ congruent to 0 modulo 3 is equal to 1.

For $e > 3$, we have no charged hook lengths congruent to 0 modulo $e$.

Example 3.6.5. Let us look at the Example 3.5.3. We have $X[\lambda, 3] = ((3, 0, -2), (4, 2, 1, -1, -2))$. By Proposition 3.6.2 the number of charged hook lengths equal to 0 is 
\[ \sum_{c=0}^{1} \sum_{x \in X^c} N_0(x) = \sharp\{x \in X^0 \mid x \notin X^1\} = 2. \]
Take $e = 2$. Let $k \in \mathbb{Z}_{>0}$ and let $x \in X^0 \cup X^1$. We have $N_k(x) = 0$ except in the following cases where we have $N_k(x) = 1$
- $k = 1$ and $x \in \{\beta_1^0, \beta_1^1, \beta_2^1, \beta_3^1\}$;
- $k = 2$ and $x \in \{\beta_1^0, \beta_1^1\}$.
By Proposition 3.6.3 the number of even charged hook lengths in $H(\lambda)$ is equal to 8.
3.7. Defect and weight for Ariki-Koike algebras. Let $\lambda \in \Pi^l$, $s = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{>1}$. From now on, we will refer to the number of elements of $H(\lambda)$ congruent to 0 modulo $e$ as the $e$-defect of $\lambda$ attached to $s$. For $l = 1$, this number is known as the $e$-weight of a partition.

In [Fa], Fayers introduced a notion of weight for a charged multipartition. We have now at least three different ways to compute the weight $p_{(e,s)}(\lambda)$ of $\lambda$ attached to $(e,s)$. In this subsection, we will show that $p_{(e,s)}(\lambda)$ is equal to the $e$-defect of $\lambda$. One consequence of this equality will be the proof of our main result in the case of cyclotomic Ariki–Koike algebras.

Here are the three equivalent definitions of weight:

- **The original definition [Fa]:**
  
  $$p_{(e,s)}(\lambda) = \sum_{0 \leq i \leq l-1} c_{e,i}(\lambda) - \frac{1}{2} \sum_{i \in \mathbb{Z}/e\mathbb{Z}} (c_{e,i}(\lambda) - c_{e,i-1}(\lambda))^2$$

  where $c_{e,i}(\lambda)$ denotes the number of nodes in $\lambda$ with content congruent to $i$ modulo $e$.

- **The quantum group definition [Fa]:**
  
  $$p_{(e,s)}(\lambda) = \|\emptyset\|(e,s) - ||\lambda||^{(e,s)}$$

  where $||.||$ is given in [JaLe] §3.1 (we will not need the precise definition here).

- **The abaci definition [JaLe]:**
  
  $$p_{(e,s)}(\lambda) = \omega_e(\tau_{e,s}(\lambda))$$

  where $\omega_e$ is the usual $e$-weight of a partition and $\tau_{e,s} : \Pi^l \to \Pi$ is Uglow’s map, which is easy to define using abaci (see [JaLe]).

  The definition we will use in this paper is the last one. According to it, the weight $p_{(e,s)}(\lambda)$ can be defined recursively as follows: Consider the set of $\beta$-numbers $X := X[\lambda, m] = (X^0, \ldots, X^{l-1})$ of $\lambda$ and set $w(X) := 0$

  **Step 1:** If $X^{c-1} \subseteq X^{c}$ for all $c = 1, \ldots, l-1$, then we move to Step 3. Otherwise, if there exists $x \in X^{c-1}$ such that $x \notin X^c$ for some $c \in \{1, \ldots, l-1\}$, then we define a set of $\beta$-numbers $Y = (Y^0, \ldots, Y^{l-1})$ such that $Y^{c-1} := X^{c-1} \setminus \{x\}$, $Y^c := X^c \cup \{x\}$ and $Y^j := X^j$ for all $j \neq c-1, c$. We set $w(Y) := w(X) + 1$ and we move to Step 2.

  **Step 2:** We set $X := Y$ and $w(X) := w(Y)$, and we go back to Step 1.

  **Step 3:** If $\{x - e \mid x \in X^{l-1} \cap \mathbb{Z}_{e=m}\} \subseteq X^0$, then we move to Step 5. Otherwise, if there exists $x \in X^{l-1}$ such that $x - e \notin X^0 \cup \mathbb{Z}_{\leq -m}$, then we define a set of $\beta$-numbers $Y = (Y^0, \ldots, Y^{l-1})$ such that $Y^{l-1} := X^{l-1} \setminus \{x\}$, $Y^0 := X^0 \cup \{x-e\}$ and $Y^j := X^j$ for all $j \neq 0, l-1$. We set $w(Y) := w(X) + 1$ and we move to Step 4.

  **Step 4:** We set $X := Y$ and $w(X) := w(Y)$, and we go back to Step 1.

  **Step 5:** We set $p_{(e,s)}(\lambda) := w(X)$.

  It is now straightforward to see that the weight is given by the same formula as in Proposition 3.6.3 and so we have the following:

  **Theorem 3.7.1.** Let $\lambda \in \Pi^l$, $s \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{>1}$. We have that the $e$-weight $p_{(e,s)}(\lambda)$ of $\lambda$ is equal to the $e$-defect of $\lambda$ attached to $s$, that is, the number of charged hook lengths in $H(\lambda)$ congruent to 0 modulo $e$.

  Now, by [Fa] (see also [JaLe] for another proof), any two $l$-partitions that are in the same block have the same $e$-weight, and thus the same $e$-defect. We have hence proved (3.8), which in turn implies the validity of Theorem 3.4.1.

3.8. $e$-cores and Schur elements. Let $e \in \mathbb{Z}_{>1}$. Another fundamental notion associated to the blocks of Ariki-Koike algebras is the notion of $e$-core. Since Theorem 3.7.1 allows us to define the notion of $e$-weight using the Schur elements, it is natural to ask whether the same thing can be done with the $e$-core.

The $e$-core of a partition $\lambda \in \Pi$ is the partition obtained after removing all the $e$-hooks, that is, all hooks of length $e$. Two simple modules $V^\lambda$ and $V^\mu$ (with $\lambda$ and $\mu$ of the same rank) are in the same block if and only if they have the same $e$-core.

**Proposition 3.8.1.** If $\lambda \in \Pi$ and $\mu$ is the $e$-core of $\lambda$, then the Schur element $s_\mu(q)$ of $V^\mu$ divides the Schur element $s_\lambda(q)$ of $V^\lambda$ in $\mathbb{Z}[q, q^{-1}]$. 

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Proof. The proof is based on a result by Bessenrodt, Gramain and Olsson. By [BGO] Theorem 4.4, the multiset $H(\mu)$ of hook lengths of $\mu$ is contained in the multiset $H(\lambda)$ of hook lengths of $\lambda$. The result is a straightforward consequence of the description of Schur elements given by Formula (3.2).

Remark 3.8.2. If $\lambda \in \Pi(n)$ and $q = 1$, then we are in the group algebra case and $s_\lambda(1) = n!/\chi_{\nu\lambda}(1)$. Using the famous hook length formula for $\chi_{\nu\lambda}(1)$ yields that $s_\lambda(1)$ is equal to the product of the hook lengths of $\lambda$. Therefore, if $\mu$ is the $e$-core of $\lambda$, then $s_\lambda(1)/s_\mu(1)$ is equal to the product of the elements of $H(\lambda) \setminus H(\mu)$. This is stated in [BGO] Corollary 4.12, where it is also observed that the relative hook formula obtained in [GaLaMa] Theorem 9.1 is a particular case of this result.

When $l \in \mathbb{Z}_{> 1}$, there is a similar (but a little bit more complicated) notion of $e$-core for $\lambda \in \Pi^l$, which has been given in [JaLe] and yields the same properties as far as blocks are concerned. In fact, if $s \in \mathbb{Z}^l$, we say that $\lambda \in \Pi^l$ is an $(e,s)$-core if $p_{(e,s)}(\lambda) = 0$. Following Theorem 3.7.1, the $(e,s)$-cores are exactly the $l$-partitions with no charged hook lengths divisible by $e$ (this is a well-known result for $l = 1$). Therefore, Theorem 3.7.1 allows us to determine whether an $l$-partition is an $(e,s)$-core by looking at its Schur element:

**Proposition 3.8.3.** Let $\lambda \in \Pi^l(n)$ and $s = (s_0, \ldots, s_{l-1}) \in \mathbb{Z}^l$. We have that $\lambda$ is an $(e,s)$-core if and only if $\Phi_e(y)$ does not divide $s_\lambda(y)$, the Schur element associated with the simple module $V^\lambda$ of the Ariki–Koike algebra $H_n^\lambda$ where $y = (y^{s_0}, \ldots, y^{s_{l-1}}; y)$.

Proof. By Theorem 3.7.1, $p_{(e,s)}(\lambda)$ is equal to the $e$-defect of $\lambda$ attached to $s$. The latter is equal to the $e$-defect of $V^\lambda$ (that is, the maximal $N \in \mathbb{N}$ such that $\Phi_e(y)^N$ divides $s_\lambda(y)$) as long as $s_\lambda(y) \neq 0$.

If $\lambda$ is an $(e,s)$-core, then $0 \notin H(\lambda)$ and so $s_\lambda(y) \neq 0$. Thus, the $e$-defect of $V^\lambda$ is equal to 0, whence $\Phi_e(y)$ does not divide $s_\lambda(y)$. Conversely, if $\Phi_e(y)$ does not divide $s_\lambda(y)$, then $s_\lambda(y) \neq 0$. Thus, the $e$-defect of $\lambda$ attached to $s$ is equal to 0, whence $p_{(e,s)}(\lambda) = 0$.

4. Defect in cyclotomic Hecke algebras of $G(l,p,n)$

We can now deduce our main result for type $G(l,p,n)$ from $G(l,1,n)$ with the use of Clifford theory. For more details about Clifford theory in general the reader may refer to [Ch1, §2.3], and in this particular setting to [GenLa], [ChJa2] or [Mall].

4.1. **Generic Hecke algebras of $G(l,p,n)$**. Let $l, p, n$ be three positive integers such that $d := l/p \in \mathbb{Z}$. By definition, $G(l,p,n)$ is the group of all $n \times n$ monomial matrices whose non-zero entries are $l$-th roots of unity, while the product of all non-zero entries is a $d$-th root of unity. Therefore, for $n = 1$, we have $G(l,p,1) \cong G(d,1,1) \cong \mathbb{Z}/d\mathbb{Z}$, and this case has been covered in the previous section. Moreover, for $n = 2$, the case where $p$ is even cannot be treated with the use of Clifford theory, and it is an unfortunate but usual exception to this kind of results. This is why, from now on, we assume that

- either $n > 2$,
- or $n = 2$ and $p$ is odd.

Let $x := (X_0, \ldots, X_{d-1}; x)$ be a set of $d + 1$ indeterminates and set $R := \mathbb{Z}[x, x^{-1}]$. The **generic Hecke algebra** $H_{l,p,n}$ of $G(l,p,n)$ is the $R$-algebra with generators $t_0, t_1, \ldots, t_n$ and relations:

- $(t_0 - X_0)(t_0 - X_1) \cdots (t_0 - X_{d-1}) = 0$,
- $(t_j - x)(t_j + 1) = 0$ for $j = 1, \ldots, n$,
- $t_1t_1t_1 = t_3t_1t_3$, $t_3t_1t_3 = t_1t_3t_1$, $t_2j+1t_2j = t_2j+1t_2j+1$ for $j = 2, \ldots, n - 1$,
- $t_1t_1t_1t_3t_1t_3 = t_3t_1t_1t_3t_1t_3$,
- $t_1t_1 = t_2t_1$ for $j = 4, \ldots, n$,
- $t_3t_1 = t_2t_1$ for $2 \leq i < j \leq n$ with $j - i > 1$,
- $t_0t_1 = t_1t_0$ for $j = 3, \ldots, n$,
- $t_0t_1 = t_1t_0$,
- $t_2t_1t_2t_1t_2t_3 \cdots = t_0t_1t_2t_1t_2t_2 \cdots$ (with $p+1$ factors on both sides).
4.2. Cyclotomic Hecke algebras of $G(l, p, n)$ and Clifford theory. Let $\eta_l := \exp(2\pi i/l)$, $\eta_d := \eta_d^p$ and $K := \mathbb{Q}(\eta_l)$. Let $y$ be an indeterminate, let $m \in \mathbb{Z}$ and let $\varphi_m : \mathbb{Z}_K[x, x^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ be a cyclotomic specialisation of $\mathcal{H}_{l,p,n}$ such that
\[
\varphi_m(X_j) = \eta_l^j y^m r_j \text{ for } j = 0, 1, \ldots, d - 1,
\]
\[
\varphi_m(x) = y^m r
\]
where $(r_0, \ldots, r_{d-1}, r) \in \mathbb{Z}^{d+1}$. The cyclotomic Hecke algebra $K(y)(\mathcal{H}_{l,p,n})_{\varphi_m}$ is split semisimple. Note that the representation theory of $(\mathcal{H}_{l,p,n})_{\varphi_m}$, and especially its block theory, does not depend much on $m$ (see also [Chll] §5.5.1).

The algebra $(\mathcal{H}_{l,p,n})_{\varphi_p}$ can be viewed as a subalgebra of the cyclotomic Ariki–Koike algebra $(\mathcal{H}_n^\eta)$ of type $G(l, 1, n)$ where $\varphi : \mathbb{Z}_K[q, q^{-1}] \to \mathbb{Z}_K[y, y^{-1}]$ is the cyclotomic specialisation given by
\[
\varphi(Q_i) = \eta_l^i y^i \text{ for } i = 0, 1, \ldots, l - 1,
\]
\[
\varphi(q) = y^{pr}
\]
with $r_{kd+j} := r_j$ for all $j = 0, \ldots, d - 1$ and $k = 1, \ldots, p - 1$. In particular, if we denote by $G$ the cyclic group of order $p$, $(\mathcal{H}_n^\eta)_{\varphi}$ is a “twisted symmetric algebra” of $G$ over $(\mathcal{H}_{l,p,n})_{\varphi_p}$ (see again [Chll] §5.5.1). This implies an action of $G$ on $\text{Irr}(K(y)(\mathcal{H}_n^\eta))$ which corresponds to the action generated by the cyclic permutation by $d$-packages on the $l$-partitions of $n$:
\[
\sigma : \lambda = (\lambda^0, \ldots, \lambda^{d-1}, \lambda^d, \ldots, \lambda^{2d-1}, \ldots, \lambda^{pd-d}, \ldots, \lambda^{pd-1})
\]
\[
\mapsto \sigma(\lambda) = (\lambda^{pd-d}, \ldots, \lambda^{pd-1}, \lambda^0, \ldots, \lambda^{d-1}, \ldots, \lambda^{pd-2d}, \ldots, \lambda^{pd-d-1}).
\]

From now on, to make the notation lighter, we will write $\mathcal{H}$ for $K(y)(\mathcal{H}_n^\eta)$ and $\tilde{\mathcal{H}}$ for $K(y)(\mathcal{H}_{l,p,n})_{\varphi_p}$. Let $M \in \text{Irr}(\mathcal{H})$. By Clifford theory, there exists $V^\lambda \in \text{Irr}(\mathcal{H})$ such that $M$ is a composition factor of $\text{Res}^H_{\mathcal{H}}(V^\lambda)$. Moreover, there is an action of $G$ on $\text{Irr}(\tilde{\mathcal{H}})$ such that, if we denote by $\Omega_\lambda$ the orbit of $M$ under the action of $G$, we have
\[
[\text{Res}^H_{\mathcal{H}}(V^\lambda)] = \sum_{E \in \Omega_\lambda} [E]
\]
in the Grothendieck group of the category of finite dimensional $\tilde{\mathcal{H}}$-modules.

Let $V \in \text{Irr}(\tilde{\mathcal{H}})$. The elements of $\Omega_\lambda$ appear as composition factors in $\text{Res}^H_{\mathcal{H}}(V)$ if and only if $V = V^\varphi(\lambda)$ for some $g \in G$. In particular, we have
\[
[\text{Res}^H_{\mathcal{H}}(V^\varphi(\lambda))] = [\text{Res}^H_{\mathcal{H}}(V^\lambda)].
\]
We deduce that
\[
\text{Irr}(\tilde{\mathcal{H}}) = \{ E \mid E \in \Omega_\lambda, \lambda \in \Pi^l(n) \}.
\]

4.3. Blocks and defect. Let $\theta : \mathbb{Z}_K[y, y^{-1}] \to K(\eta), y \mapsto \eta$ be a specialisation of $\mathbb{Z}_K[y, y^{-1}]$ such that $\eta$ is a non-zero complex number and $\theta(\eta)$ is a primitive $l$-th root of unity. The $\theta$-blocks of $\tilde{\mathcal{H}}$ have been determined by Wada [Wa], and they can be described as follows: Let $E, F \in \text{Irr}(\tilde{\mathcal{H}})$ with $E \in \Omega_\lambda$ and $F \in \Omega_\mu$ for some $\lambda, \mu \in \Pi^l(n)$.

- If $\{V^\lambda\}$ is a $\theta$-block, then $\{E\}$ is a $\theta$-block.
- Otherwise, $E$ and $F$ are in the same $\theta$-block if and only if $V^\lambda$ and $V^\varphi(\mu)$ are in the same $\theta$-block for some $g \in G$.

On the other hand, thanks to Clifford theory, the Schur elements of $\tilde{\mathcal{H}}$ can be easily obtained from the Schur elements of $\mathcal{H}$ as follows (see [ChII] Proposition 2.3.15):
\[
s_E(y) = \frac{|\Omega_\lambda|}{p} \varphi(s_\lambda(q)).
\]
The above equation, in combination with (4.1), implies that
\[
\varphi(s_\lambda(q)) = \varphi(s_{\lambda(\sigma)}(q)).
\]
In fact, the last equality can be obtained independently of Clifford theory by [ChII] Proposition 2.5.

We can now prove Conjecture [ChJa] for the groups $G(l, p, n)$.
Parameters of $\mathcal{H}_{\psi}(W)$.

| $W$      | $K$            | Parameters of $\mathcal{H}_{\psi}(W)$ |
|----------|----------------|----------------------------------------|
| $G_4$    | $\mathbb{Q}(\eta_3)$ | $1, y, y^2$                         |
| $G_5$    | $\mathbb{Q}(\eta_3)$ | $1, y, y^2; 1, y, y^2$              |
|          |                 | $1, y, y^2; 1, y^2, y^4$             |
|          |                 | $1, y, y^2; 1, y^4, y^8$             |
| $G_8$    | $\mathbb{Q}(t)$  | $1, \eta_8^s y, \eta_8^s y, y^2$     |
|          |                 | $1, y, -y, y^2$                      |
|          |                 | $1, y, y^2, y^3$                     |
|          |                 | $1, y, -y, -y^4$                     |
|          |                 | $1, y^3, -y^3, -y^4$                 |
|          |                 | $1, -1, -y, y^5$                     |
|          |                 | $1, -y^4, y^5, -y^5$                 |
|          |                 | $1, -y, -y^4, y^5$                   |
| $G_9$    | $\mathbb{Q}(\eta_8)$ | $1, y^2; 1, y^2, y^4, y^6$          |
| $G_{10}$ | $\mathbb{Q}(\eta_{12})$ | $1, -y^2, y^4; 1, y^2, -y^3, y^6$   |
| $G_{12}$ | $\mathbb{Q}(i/\sqrt{2})$ | $1, y^2$                           |
| $G_{16}$ | $\mathbb{Q}(\eta_5)$ | $1, y, y^2, y^3, y^4$               |
| $G_{20}$ | $\mathbb{Q}(\eta_3, \sqrt{5})$ | $1, y, y^2$                        |
| $G_{22}$ | $\mathbb{Q}(i, \sqrt{5})$ | $1, y^2$                           |

TABLE 1. The “Zyklotomische Heckealgebren” of the rank 2 exceptional groups

5. DEFECT IN CYCLOTOMIC HECKE ALGEBRAS OF EXCEPTIONAL GROUPS

As far as the exceptional complex reflection groups are concerned, there are no general results about decomposition matrices and blocks in the case of $G(l, p, n)$. For the exceptional real reflection groups, one can also verify the conjecture for these groups by inspecting the corresponding decomposition matrices and blocks as in the case of $G(l, p, n)$. In that paper, the cyclotomic Hecke algebras considered are the ones arising from generalised Harish-Chandra theory, for which the term “cyclotomic Hecke algebras” was originally conceived. The choices of parameters for these cyclotomic Hecke algebras are given by Broué and Malle in [BroMa] and for the exceptional complex reflection groups of rank 2 are as follows:

$$
D_\theta = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
* & * & \ldots & * \\
* & * & \ldots & * \\
\end{pmatrix}
$$

In [ChMi], the authors calculated the decomposition matrices for all the algebras above and for all specialisations $\theta : y \mapsto \eta$ that yield non-semisimple algebras in characteristic 0. They also showed that all modular irreducible representations can be lifted to ordinary ones, that is, the decomposition matrix is of the form:
The subset $B_\theta$ of $\text{Irr}(W)$ labelling the rows of the identity matrix at the top of $D_\theta$ is called an optimal basic set. More formally, $B_\theta$ is a subset of $\text{Irr}(W)$ such that

1. there exists a bijection $B_\theta \leftrightarrow \text{Irr}(K(\eta)\mathcal{H}_\varphi(W))$, $V \mapsto M_V$;
2. we have $[V : M_{V'}] = \delta_{V,V'}$ for all $V, V' \in B_\theta$.

To save space, the information given in [ChMi] is the shape of each block of $D_\theta$, as well as the representations of the block that belong to $B_\theta$. This is why, in order to verify Conjecture 2.3.6 in all these cases, we created a computer function that first recovered all other representations inside the block, and then calculated their defect.

5.1. The algorithm. The algorithm is very elementary and we chose to use GAP3 to implement it, because the package CHEVIE [Mi] [GHLMP] contains the irreducible representations and the Schur elements in factorised form for any Hecke algebra associated to a complex reflection group. However, any language with this information could do. The function, named “DefBlock”, and its outputs can be found on the project’s webpage [Web].

Let $\theta : \mathbb{Z}_K[y,y^{-1}] \to K(\eta)$, $y \mapsto \eta$ be a $\mathbb{Z}_K$-algebra morphism such that the specialised algebra $K(\eta)\mathcal{H}_\varphi(W)$ is not semisimple. Then $\eta$ is a root of unity of order $e > 0$ and there exists $r \in \{1, 2, \ldots, e\}$ with $\gcd(e, r) = 1$ such that $\eta = \eta_r^e$. Let $\Phi$ denote the minimal polynomial of $\eta$ over $K$. Our function takes as inputs the cyclotomic Hecke algebra $\mathcal{H}_\varphi(W)$ with parameter $y$, the integers $e$ and $r$, and the list $l$ of irreducible representations in a block $B$ that belong to the optimal basic set $B_\theta$. We only need to consider blocks of non-zero defect.

The first step is to determine all irreducible representations in the block $B$ (this is information that the first author should have stored somewhere, but did not). We calculate all possible linear combinations of the characters of the representations in the list $l$, with $y$ specialised to $\eta_r^e$. Since the dimensions of the irreducible representations of the exceptional complex reflection groups we consider are no higher than 6, taking linear combinations of up to 6 characters is enough. It turns out that this suffices to uniquely determine which irreducible representations belong to the block $B$.

The second step is straightforward, given that one has the Schur elements in factorised form. For each irreducible representation in the block $B$, we calculate its $\Phi$-defect as the multiplicity of $\eta_r^e$ as a root of its Schur element. We then determine the set of all $\Phi$-defect values inside $B$.

The outputs of the function are a list with the set of all $\Phi$-defect values and all representations inside the block $B$. Let us see an example:

Example 5.1.1. We would like to calculate the defect of the block of $\mathcal{H}_\varphi(G_4)$ generated by $\chi_{1,0}$ and $\chi_{1,8}$ for $\eta = \eta_{12}$. We type

```
gap> W:=ComplexReflectionGroup(4);;
gap> H:=Hecke(W,[[1,y,y^2]]);;
gap> DefBlock(H,12,1,[[1,0],[1,8]]);
[ [ 1 ], [ [ [ 1, 0 ] ], [ [ 1, 8 ] ], [ [ 2, 3 ] ] ] ]
```

We know from [ChMi] that the corresponding block of the decomposition matrix is of the form

$$
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1
\end{pmatrix}
$$

where the first two rows are labelled by $\chi_{1,0}$ and $\chi_{1,8}$. We now obtained that the last row is labelled by $\chi_{2,3}$ and that all representations in this block are of defect 1.

As we said in the beginning of this subsection, we ran this function for all blocks of non-zero defect of the cyclotomic Hecke algebras of Table 1, and stored the results on [Web]. If the cardinality of the set of all $\Phi$-defect values of a block is equal to 1, then all irreducible representations in the block have the same $\Phi$-defect. This, in combination with our results, allowed us to deduce the following theorem:

Theorem 5.1.2. Let $W \in \{G_4, G_5, G_6, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22}\}$ and let $\mathcal{H}_\varphi(W)$ be one of the cyclotomic Hecke algebras of Table 1. If $V, V' \in \text{Irr}(W)$ belong to the same $\theta$-block, then they have the same $\Phi$-defect.
Corollary 5.1.3. Let $W \in \{G_{12}, G_{22}\}$ and let $\mathcal{H}_\varphi(W)$ be any cyclotomic Hecke algebra associated with $W$. If $V, V' \in \text{Irr}(W)$ belong to the same $\Theta$-block, then they have the same $\Phi$-defect.

Proof. If $W \in \{G_{12}, G_{22}\}$, then $W$ is generated by elements of order 2 (that is, $W$ is a 2-reflection group) that belong to a single conjugacy class. Therefore, the generic Hecke algebra $\mathcal{H}(W)$, defined over a suitable base ring, is isomorphic to the algebra $\mathcal{H}_\varphi(W)$ of Table 1, and every possible specialisation has already been considered by [ChMa] and our program.

Before we finish this section, we should mention here that there is another work on decomposition matrices of exceptional complex reflection groups, Chavli’s paper [Cha3] on the generic Hecke algebras of $G_4, G_8$ and $G_{16}$, where she proves the existence of optimal basic sets for all possible specialisations of these algebras. The decomposition matrices of all cyclotomic Hecke algebras associated with these three groups could theoretically be studied through Chavli’s paper, but there are too many options to be considered if we want to verify Conjecture 2.3.6 case-by-case. We will return to this point when we discuss future perspectives in §6.2.

6. Defect in other types of essential algebras

In §2.2 we discussed about essential algebras, which are symmetric algebras whose Schur elements are Laurent polynomials of a specific form. Hecke algebras are particular cases of essential algebras. One may ask whether Conjecture 2.3.10 holds for other types of essential algebras. The answer is positive for Yokonuma–Hecke algebras (of type $A$), which were introduced by Yokonuma [Yo] as generalisations of Iwahori–Hecke algebras.

6.1. The Yokonuma–Hecke algebra of type $A$ and the cyclotomic Yokonuma–Hecke algebra. The Yokonuma–Hecke algebra of type $A$ is a generalisation of the Iwahori–Hecke algebra of type $A$ and can be seen as a particular case of a cyclotomic Yokonuma–Hecke algebra, introduced in [CPA2]. The term “cyclotomic” here does not refer to a cyclotomic specialisation, but to the fact that the cyclotomic Yokonuma–Hecke algebra generalises the Yokonuma–Hecke algebra of type $A$ in the same way that the Ariki–Koike algebra, which is also called “cyclotomic Hecke algebra” by several people, generalises the Hecke algebra of type $A$. In order not to repeat everything twice, we will work directly with cyclotomic Yokonuma–Hecke algebras, but, for all results used, we will also give the references for the Yokonuma–Hecke algebra of type $A$, because it was studied first.

Let $n \in \mathbb{Z}_{\geq 0}$ and $d, l \in \mathbb{Z}_{>0}$. Let $q := (Q_0, \ldots, Q_{l-1}; q)$ be a set of $l$ indeterminates and set $R := \mathbb{Q}[q, q^{-1}]$. We define the cyclotomic Yokonuma–Hecke algebra $Y(d, l, n)$ as the associative $R$-algebra (with unit) with a presentation by generators:

$$g_1, g_2, \ldots, g_{n-1}, t_1, \ldots, t_n, X_1$$

and relations:

$$
g_i g_j = g_j g_i \quad \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1,
$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for all } i = 1, \ldots, n-2,$$

$$t_i t_j = t_j t_i \quad \text{for all } i, j = 1, \ldots, n,
$$

$$g_i t_j = t_{s(i)} g_i \quad \text{for all } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, n,$$

$$t_i^2 = 1 \quad \text{for all } j = 1, \ldots, n,$$

$$g_i^2 = q + (q-1) e_i g_i \quad \text{for all } i = 1, \ldots, n-1,$$

$$X_1 g_1 X_1 g_1 = g_1 X_1 g_1 X_1 \quad \text{for all } i = 2, \ldots, n-1,$$

$$X_1 t_j = t_j X_1 \quad \text{for all } j = 1, \ldots, n,$$

$$(X_1 - Q_0) \cdots (X_1 - Q_{l-1}) = 0$$

where, for all $i = 1, \ldots, n-1$, $s_i$ is the transposition $(i, i + 1)$ and

$$e_i = \frac{1}{d} \sum_{0 \leq s \leq d-1} t_i^{s}t_i^{-s}.$$

For $d = 1$, the generators $t_j$ disappear and $Y(1, l, n)$ is isomorphic to the Ariki–Koike algebra $R\mathcal{H}_\varphi$. For $l = 1$, the generator $X_1$ disappears and $Y(d, 1, n)$ is the Yokonuma–Hecke algebra of type $A$, which in turn
becomes the group algebra of $G(d, 1, n)$ for $q = 1$. The presentation of $Y(d, 1, n)$ by generators and relations is due to [Ju1, JuKa, Ju2], with the simplified quadratic relation for the $g_i$ being due to [CPA1] (see also [ChPo] Remark 3.1) for the quadratic relation given above.

The representation theory of the Yokonuma–Hecke algebra $Y(d, 1, n)$ of type $A$ was first studied in [Th1, Th2, Th3], but its irreducible representations were explicitly constructed in [CPA1] and adapted in [ChPo] for the quadratic relation given above. The irreducible representations of $Y(d, l, n)$ were constructed in [CPA2] and can be adapted similarly. It turns out that the algebra $Y(d, l, n)$ is split semisimple over the field $K(q)$, where $K := \mathbb{Q}(q_1)$ (the splitting field is $\mathbb{Q}(q_d, q)$). Moreover, there exists a bijection $\Pi^d(n) \leftrightarrow \text{Irr}(K(q)Y(d, l, n))$, $\lambda \mapsto V^\lambda$.

A symmetrising trace $\tau$ was defined in [CPA2] on $K(q)Y(d, l, n)$; this map satisfies $\tau(b) = \delta_{ib}$ for all $b$ in a certain basis of $Y(d, l, n)$. The map $\tau$ coincides with the canonical symmetrising trace on $H^q_n$ for $d = 1$ and with the symmetrising trace defined on the Yokonuma–Hecke algebra of type $A$ in [CPA1] for $l = 1$. By [CPA2] Proposition 7.4, the Schur element of $V^\lambda \in \text{Irr}(K(q)Y(d, l, n))$ is equal to

$$(6.1) \quad d^n s_{\lambda[1]}(q)s_{\lambda[2]}(q) \ldots s_{\lambda[d]}(q)$$

where, for all $i = 1, \ldots, d$, $\lambda[i] = (\lambda^{(i-1)}l, \lambda^{(i-1)}l+1, \ldots, \lambda^{(i-1)}l+1) \in \Pi^l(n)$ and $s_{\lambda[i]}(q)$ is given by Formula (3.2).

Our notation for the $\lambda[i]$ is in agreement with the notation used in [Ko] if we take $U = \{0, 1, \ldots, d-1\}$, $t = d$ and $\mathcal{U}_r = \{(i-1)l+k | k = 0, 1, \ldots, l-1\}$ for all $i = 1, \ldots, d$.

This connection between the Schur elements of $Y(d, l, n)$ and those of Ariki–Koike algebras was subsequently explained by the following isomorphism of $R$-algebras proved independently in [PdA1] and in [Res]:

$$(6.2) \quad Y(d, l, n) \cong \bigoplus_{n_1 + \ldots + n_d = n, n_1, \ldots, n_d \geq 0} \text{Mat}_{n_1 \times n_1} \cdots \text{Mat}_{n_d \times n_d} \left( H_{n_1}^q \otimes_R H_{n_2}^q \otimes_R \ldots \otimes_R H_{n_d}^q \right).$$

This isomorphism is the case of the Yokonuma–Hecke algebra of type $A$, that is, the case $l = 1$, had first been obtained in [Ju1], subsequently in [JP A], and adapted in [ChPo] for the quadratic relation given above and the ring $R$. Another proof for $l = 1$ was later given in [PsKv]. The existence of the isomorphism (6.2) implies that the map $\tau$ above is indeed a symmetrising trace on $Y(d, l, n)$ over $R$.

Let $y$ be an indeterminate and and let $\varphi : K(q, q^{-1}) \rightarrow K[y, y^{-1}]$ be a $K$-algebra morphism such that

$$\varphi(Q_i) = \eta_i y^r \quad \text{for } i = 0, 1, \ldots, l-1,$$

$$\varphi(q) = y^r$$

where $(r_0, \ldots, r_{l-1}, r) \in \mathbb{Z}^{l+1}$. Let also $\theta : K[y, y^{-1}] \rightarrow K(\eta)$, $y \mapsto \eta$ be a specialisation of $K[y, y^{-1}]$ such that $\eta \in \mathbb{C}^*$ and $\theta(\eta)$ is a primitive $l$-th root of unity (for simplicity, we may assume that $\theta$ is a $K$-algebra morphism, whence $\theta(\eta_i) = \eta_i$). Let $\Phi$ denote the minimal polynomial of $\eta$ over the field $K$. It follows from Equations (6.1) and (6.2) that, for $\lambda, \mu \in \Pi^d(n)$,

- the $\Phi$-defect of $V^\lambda$ is equal to $\sum_{i=1}^d \nu_{\Phi}(\varphi(s_{\lambda[i]}(q)))$, and
- $V^\lambda$ and $V^\mu$ are in the same $\theta$-block if and only $V^\lambda[i]$ and $V^\mu[i]$ are in the same $\theta$-block of $K(\eta)(H^q_{n_i})_{\varphi}$, where $n_i = |\lambda[i]| = |\mu[i]|$, for all $i = 1, \ldots, d$.

Given the above, the following result is a direct consequence of Theorem 3.4.1

**Theorem 6.1.1.** Let $\lambda, \mu \in \Pi^d(n)$. If $V^\lambda$ and $V^\mu$ are in the same $\theta$-block, then they have the same $\Phi$-defect.

Therefore, the analogue of Conjecture 2.3.6 is also true for the cyclotomic Yokonuma–Hecke algebra $Y(d, l, n)$, and in particular for the Yokonuma–Hecke algebra $Y(d, 1, n)$ of type $A$.

**6.2. Other essential algebras.** We have stated Conjecture 2.3.6 for all cyclotomic Hecke algebras associated with complex reflection groups and we have seen that it holds in all cases for which information on blocks and decomposition matrices is known, that is, for the groups of the infinite series $G(l, p, n)$ and for certain exceptional groups. In the previous subsection we saw that its analogue holds for the Yokonuma–Hecke algebra of type $A$ and for the cyclotomic Yokonuma–Hecke algebra, but one could argue that this is because of the connection, established by the isomorphism (6.2), with the Iwahori–Hecke algebra of type $A$ and the Ariki–Koike algebra respectively. However, one could also wonder whether a similar result holds for all one-parameter essential algebras (including the ones obtained as specialisations of multi-parameter
ones as in the end of §2.2. One could go a step further and wonder whether a similar result holds for any essential algebra, with Φ-defect being replaced by \( Ψ_{\psi,1}(M_{\psi,1}) \)-defect, using the notation of Definition 2.2.1. Unfortunately, we do not have any data or examples outside the ones treated in this paper to back up this claim. Proving the claim on a generic level would help a lot though in avoiding to do a huge case-by-case analysis in order to prove Conjecture 2.3.10 for the exceptional complex reflection groups.

Another perspective to consider is the case of positive characteristic. Throughout this paper we only consider specialisations to subfields of \( C \). Moreover, one can use classical modular representation theory of finite groups to see that not all simple modules inside a \( p \)-block (where \( p \) is a prime number) have the same defect. However, if we consider a cyclotomic Ariki–Koike algebra as in Remark 3.4.3 and a specialisation \( θ : Z_K[y, y^{-1}] → k, y → η \), where \( k \) is any field and \( η \in k \) is a root of unity of order \( e > 1 \), then Theorem 3.4.1 is still valid with a proof similar to the one we provided if the characteristic of \( k \) is large enough (to be precise, larger than \( e + n - 1 \)). It is also valid if \( η = 1 \) and \( k \) is a field of characteristic \( e > 0 \). These statements can, through the Morita equivalence of Dipper and Mathas, generalise to any cyclotomic Ariki–Koike algebra. We decided not to include these results for the sake of uniformity, and in order to restrict ourselves to the proof of Conjecture 2.3.10 in its current form. However, we do not exclude the possibility that other Hecke algebras and, more generally, other essential algebras might share similar properties.

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