A fast method of reconstruction for X-ray phase contrast imaging with arbitrary Fresnel number.

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Abstract. New lensless diffractive X-ray technic for micro-scale imaging of biological tissue is based on quantitative phase retrieval schemes. By incorporating refraction, this method yields improved contrast compared to purely absorption-based radiography but involves a phase retrieval problem since of physical limitation of detectors. A general method is proposed in this paper for one step reconstruction of the ray integral of complex refractive index of an optically weak object from intensity distribution of the hologram.

Key words. refractive index, phase contrast imaging, Fresnel propagator interpolation

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1 Introduction

A sample is illuminated by a parallel beam of coherent X-rays; the intensity of the diffracted pattern is registered at a distant plane detector for determination of distribution of attenuation and refractivity in the sample. By incorporating refraction, this method yields improved contrast compared to purely absorption-based radiography but involves a phase retrieval problem. The linearized version of this problem is known as the contrast transfer function model (CTF). This model is applied to optically weak objects and the Helmholtz equation is replaced by the paraxial approximation \[1, 6\]. In \[2\] the near-field phase retrieval problem was considered for general compactly
supported objects, given two independent intensity patterns recorded at different distances or frequencies. A version of the Newton method was applied for 2D reconstruction from one hologram [4]. The method of phase contrast imaging was applied for resolving the refractive index in three dimensions [8]. Methods of complex analysis were applied in [5] for proving stability of the inversion. An exact estimate of the norm of the inversion is given showing exponential growth for large Fresnel numbers [5]. In this paper a theoretically exact method is proposed for reconstruction of ray integrals of the complex refraction index of object from the Fourier transform of a single hologram in the frame of CTF model with arbitrary Fresnel number.

2 CTF-model

Intensity of the radiation is the measurable quantity

\[ I(\psi) = |D(\exp(\psi))|^2. \]

Here \( D \) is the Fresnel propagator (paraxial approximation) in 3D space generating the near field hologram,. \( \psi = k \int (1 - n) dz = k \int (\delta - i\beta) dz, \)

\( n = 1 - \delta + i\beta \) is the refraction index of the object, \( k \) is the space frequency, and \( z \) is the coordinate along the central ray. For small \( \psi \),

\[ D(\exp(\psi)) = 1 + D(\psi) + O(\|\psi\|^2) \]

which yields

\[ I(\psi) = 1 + 2T(\psi) + O(\|\psi\|)\psi \]

for an appropriate norm. Operator \( T(\psi) = \text{Re\,} D(\psi) \) is the linearisation of \( I \) at \( \psi = 0 \) (weak-object approximation). According to [7] the propagator can be written in the form

\[ D(\psi) \approx F^{-1}(m_\delta F(\psi)), \quad m_\delta(\xi) = \exp \left( \frac{-id|\xi|^2}{2k} \right), \quad (1) \]

\( ^{2}T \) is the operator denoted by \( T \) in [5].
where \( z = d \) is the distance to detector, and \(|\xi|^2 = \xi_1^2 + \xi_2^2\) where \( \xi = (\xi_1, \xi_2) \) are coordinates on the frequency plane of the object. We use notation

\[
\hat{a}(\xi) = F(a) = \int \exp(-2\pi i (x_1\xi_1 + x_2\xi_2)) a(x) \, dx_1 dx_2
\]

for the Fourier transform of a function \( a \) on \( \mathbb{R}^2 \) where \( x = (x_1, x_2) \) are coordinates vanishing on the central ray. The dimensionless Fresnel number is

\[
f = \frac{k b^2}{2\pi d},
\]

where \( b \) is the diameter of a central disc \( \Omega_b \) in the plane \( \{z = 0\} \) that contains the support \( S \) of the projection of the sample (or of its projection). Integrals

\[
\mu = k \int \delta dz, \quad \varphi = k \int \beta dz
\]

are the phase shift and the attenuation of a ray \( \{x = \text{const}\} \), respectively. The coordinates \( y_i = x_i / b, \ i = 1, 2 \) are normalized in such way that support of \( \psi \) is contained in the disc \( \{y : |y| \leq 1/2\} \). By (1) and (2) we have

\[
\hat{T}(\psi) = \cos\left(\frac{\pi |\eta|^2}{f}\right) \hat{\mu} + \sin\left(\frac{\pi |\eta|^2}{f}\right) \hat{\varphi}
\]

where \( \eta_i = b \xi_i, \ i = 1, 2 \) are coordinates dual to \( y \). Equation (3) has the unique solution [3]. The norm of \( T \) is bounded by 1 on \( L_2(\mathbb{R}^2) \) since operator \( D \) is unitary. The norm of the operator \( T^{-1} \) is estimated by \( \exp(\text{const } f) \) according to [5].

3 Reconstruction of the phase shift

**Definition.** A function \( Z \) defined on the complex plane is said to be of *sine-type* if it is entire and

(i) the zero set \( \Lambda \) of \( Z \) is separated that is there exists \( c > 0 \) such that

\[
|\lambda - \mu| \geq c
\]

for any \( \lambda, \mu \in \Lambda \),
(ii) there are constants $A, B, H$ such that

$$A \exp (\pi |\text{Im} \lambda|) \geq |Z(\lambda)| \geq B \exp (\pi |\text{Im} \lambda|)$$

(5)

for any $\lambda$ such that $|\text{Im} \lambda| \leq H$.

We call $Z$ generating function for phase shift at a Fresnel number $f$ if it is of sine-type with real zeros and each zero $\lambda$ satisfies

$$\lambda^2 = f \left( l(\lambda) + \frac{1}{2} \right)$$

(6)

where $l(\lambda)$ is an integer.

**Theorem 3.1** If there exists a generating function $Z$ at a Fresnel number $f$ then for arbitrary functions $\varphi, \mu \in L^2(\mathbb{R}^2)$ supported by $\Omega_b$, the phase shift $\varphi$ can be found explicitly from (3).

**Remark.** Note that $b$ in (2) is arbitrary number that fulfils $b \geq b(S)$ where $b(S)$ is the minimal diameter of a central disc that contains $S$. Therefore the reconstruction can be done for a sample with an arbitrary Fresnel number $f$ just by choosing closest odd $f_{\text{odd}} \geq f$ and applying Theorem 3.1 at the number $f_{\text{odd}}$.

**Proof.** Let $P$ denote (Paley-Wiener) space of functions $g \in L^2(\mathbb{R})$ such that $F(g)$ is supported in $[-1/2, 1/2]$.

**Lemma 3.2** Let $Z$ be a generating function at Fresnel number $f$. Then for an arbitrary function $g \in P$ can be interpolated from the set $\Lambda$ of roots of $Z$ by

$$g(t) = \sum_{\lambda \in \Lambda} \frac{Z(t)g(\lambda)}{(t - \lambda)Z'(\lambda)},$$

(7)

where the series converges uniformly on each bounded interval.

The proof will be given in the next section.

**Proof of Theorem.** Let $\theta$ be a unit vector in the frequency plane. For any zero $\lambda$, vector $\eta = \pm \lambda \theta$ satisfies $|\eta|^2 = f(l(\lambda) + 1/2)$. It follows that $\cos \left( \pi |\eta|^2 / f \right) = 0$ which yields $\hat{\varphi}(\lambda \theta) = (-1)^l(\lambda) \hat{\Psi}(\lambda \theta)$. By the slice theorem for any unit vector $\theta$,

$$\varphi(t\theta) = \int \exp(-2\pi ipt) R \varphi(p, \theta) \, dp, \ t \in \mathbb{R},$$

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where \( R \) means the Radon transform and function \( R\varphi (\cdot, \theta) \in L^2(\mathbb{R}) \) is supported in \([-1/2, 1/2]\). Therefore \( g \in P \) where \( g_{\theta}(t) = \hat{\varphi}(t\theta) \). Lemma 3.2 can be applied to \( g_{\theta} \) for any vector \( \theta \). Therefore function \( \hat{\varphi} \) is reconstructed by (7) on the whole frequency plane by the formula

\[
\hat{\varphi}(\eta) = Z(|\eta|) \sum_{\lambda \in \Lambda, \lambda > 0} (-1)^l(\lambda) \frac{\hat{\Psi}(\lambda\theta)}{(|\eta| - \lambda) Z'(\lambda\theta)}
\]

where \( \theta = |\eta|^{-1} \eta \) and \( \Psi = T(\psi) \). Application of the inverse FT recovers the phase shift \( \varphi \). \( \square \)

Remark. If a generating function \( Z \) is even function (8) can be written in form

\[
\hat{\varphi}(\eta) = Z(|\eta|) \sum_{\lambda \in \Lambda, \lambda > 0} (-1)^l(\lambda) \left( \frac{\hat{\Psi}(\lambda\theta)}{(|\eta| - \lambda) Z'(\lambda\theta)} + \frac{\hat{\Psi}(-\lambda\theta)}{(|\eta| + \lambda) Z'(\lambda\theta)} \right)
\]

since \( Z(0) \neq 0 \).

4 Interpolation in Paley-Wiener space

Lemma 4.1 \[9\] If \( Z \) is a sine-type function, then for arbitrary \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( l \) that is not in \( \varepsilon \) neighborhood of a zero of \( Z \) inequality holds

\[
|Z(l + ih)| \geq \delta \exp (\pi |h|).
\]

Proof of Lemma 3.2. Let \( \Gamma_{\theta} = \{\text{Im} \lambda = h\} \). We have

\[
\int_{\Gamma_{h}} |g(\lambda)|^2 \, d\lambda \leq \|g\|^2 \exp (2\pi |h|).
\]

since \( g(\xi + ih) = F(\exp(2\pi ih) F^{-1}(f)) \). By (5), the Cauchy inequality and (11) we get for any real \( t \),

\[
B^2 \left| \int_{\Gamma_{h}} \frac{g(\lambda)}{(\lambda - t) Z(\lambda)} \, d\lambda \right|^2 \leq \exp (-2\pi |h|) \left( \int_{\Gamma_{h}} \left| \frac{g(\lambda)}{\lambda - t} \right| \, d\lambda \right)^2 \leq \exp (-2\pi |h|) \int_{\Gamma_{h}} \frac{\, d\lambda}{|\lambda - t|^2} \int_{\Gamma_{h}} |g(\lambda)|^2 \, d\lambda \leq \frac{\pi}{|h|} \|g\|^2.
\]
that is
\[ \left| \int_{\Gamma_h} \frac{g(\lambda)}{(\lambda - t) Z(\lambda)} \, d\lambda \right| \leq \frac{1}{B} \left( \frac{\pi}{h} \right)^{1/2} \|g\|. \]

For any \( l > 0 \), one can replace here \( \Gamma_h \) by \( \Gamma_{h,l} = \{ |\text{Re} \lambda| \leq l, \ |\text{Im} \lambda| = h, \} \).
This implies
\[ \left| \int_{\Gamma_{h,l}} \frac{g(\lambda)}{(\lambda - t) Z(\lambda)} \, d\lambda \right| \to 0 \]
as \( h \to \infty \) uniformly with respect to \( l \). Set \( \Gamma_{l,h} = \{ |\text{Re} \lambda| = \pm l, \ |\text{Im} \lambda| \leq h \} \) and choose a sequence \( l = l_m \to \infty \) such that \( l_m \) and \(-l_m\) are kept at a distance \( \geq c/4 \) from \( \Lambda \) for any \( m \) where \( c \) is the constant in (4). We have then
\[ \left| \int_{\Gamma_{l,h}} \frac{g(\lambda)}{(\lambda - t) Z(\lambda)} \, d\lambda \right| \to 0 \]
as \( k \to \infty \) uniformly for \( h \). This property follows from (10) if we take \( \varepsilon = c/4 \).

The sum
\[ \int_{\Gamma_{l,h}} + \int_{\Gamma_{h,l}} \frac{g(\lambda)}{(\lambda - t) Z(\lambda)} \, d\lambda \quad (12) \]
is the integral over the perimeter of the rectangle of size \( 2l \times 2h \). If \( l_k \to \infty \) and \( h \to \infty \), the sum (12) tends to zero. If the orientation of the perimeter is counterclockwise the sum can be calculated by the Residue theorem for the poles \( \lambda = t, \ |t| < l, \ \lambda \in \Lambda, \ |\lambda| < l_m \). This yields
\[ \text{res}_t + \sum \text{res}_{\lambda_k} = \frac{g(t)}{Z(t)} + \sum_{\lambda \in \Lambda, |\lambda| < l} \frac{g(\lambda)}{(\lambda - t) Z'(\lambda)} \to 0 \]
as \( l \to \infty \). The limit of the partial sum vanishes and (7) follows. □

5 Small Fresnel numbers

Here generating functions at \( f = 1, 2, 3, 4, 5 \) are constructed.

Case \( f = 1 \)

Check that \( Z_1(\lambda) = \cos \left( \pi \sqrt{\lambda^2 - 1/4} \right) \) is a generating function. The numbers \( \lambda_k = \pm \sqrt{k^2 + k + 1/2}, \ k = 0, 1, \ldots \) are all zeros and \( \lambda_k^2 = l(k) + 1/2 \).
where \( l(k) = k^2 + k \). By Lemma 3.2 we obtain formula

\[
\hat{\varphi}(\eta) = \cos \left( \frac{\pi}{4} \sqrt{|\eta|^2 - \frac{1}{4}} \right) \sum_0^\infty (-1)^{l(k)+k} \frac{k + 1/2}{\pi \lambda_k} \left( \frac{\hat{\Psi}(\lambda_k \theta) + \hat{\Psi}(-\lambda_k \theta)}{|\eta| - \lambda_k + |\eta| + \lambda_k} \right)
\]

since \( Z_1'(\lambda_k) = (-1)^k \pi \lambda_k (k + 1/2)^{-1} \).

**Case \( \mathfrak{f} = 2 \)**

The function \( Z_2(\lambda) = \cos \pi \sqrt{\lambda^2 - 3/4} \) is of the sine-type. The roots of \( Z \) fulfil \( \lambda_k = (k + 1/2)^2 + 3/4 = k^2 + k + 1 = 2 \left( l(k) + 1/2 \right) \) for \( k = 0, 1, 2, \ldots \) where \( l(k) = (k^2 + k)/2 \) is integer for any \( k \). This implies (6) at \( \mathfrak{f} = 2 \) hence \( Z_2 \) is a generating function hence Lemma 3.2 can be applied.

**Case \( \mathfrak{f} = 3 \)**

The function

\[
Z_0(\lambda) = \cos \left( \frac{\pi}{3} \sqrt{\lambda^2 + \frac{3}{4}} \right) \cos \frac{\pi}{3} \left( \sqrt{\lambda^2 - \frac{5}{4}} + 1 \right) \cos \frac{\pi}{3} \left( -\sqrt{\lambda^2 - \frac{5}{4}} + 2 \right)
\]

(13)
is single-valued and of sine-type. Its zeros are real and equal

\[
\alpha_k^2 = 9 \left( k + \frac{1}{2} \right)^2 - \frac{3}{4} = 3 \left( l(k) + \frac{1}{2} \right), \quad l(k) = 3k^2 + 3k,
\]

\[
\beta_k^2 = \left( 3 \left( k + \frac{1}{2} \right) - 1 \right)^2 + \frac{5}{4} = 3 \left( l(k) + \frac{1}{2} \right), \quad l(k) = 3k^2 + k,
\]

\[
\gamma_k^2 = \left( 3 \left( k + \frac{1}{2} \right) - 2 \right)^2 + \frac{5}{4} = 3 \left( l(k) + \frac{1}{2} \right), \quad l(k) = 3k^2 - k,
\]

where \( k = 0, 1, 2, \ldots \) for the cosine factors in (13), respectively. They satisfy (6) at \( \mathfrak{f} = 3 \) however the zeros \( \lambda = \pm \sqrt{3/2} \) have multiplicity 2. We set

\[
Z_3(\lambda) = Z_0(\lambda) R(\lambda), \quad R(\lambda) = \frac{\lambda^2 - 9/2}{\lambda^2 - 3/2}.
\]

Function \( Z_3 \) is still of sine-type since \( R(\lambda) \to 1 \) as \( |\text{Im} \lambda| \to \infty \), the denominator reduces multiplicity to 1. The numerator has zero \( \lambda^2 = 9/2 \) which also satisfies (6). It follows that \( Z \) is a generating function at \( \mathfrak{f} = 3 \).

**Case \( \mathfrak{f} = 4 \)**

Function

\[
Z_0(\lambda) = \sin \left( \frac{\pi}{2} \sqrt{\lambda^2 - 2} \right) \cos \frac{\pi}{2} \sqrt{\lambda^2 - 1}
\]
is of sine-type whose zeros are

\[ \alpha_k = \pm \sqrt{4 k^2 + 2}; \quad \beta_k = \pm \sqrt{4 (k^2 + k) + 2}, \quad k = 0, 1, 2, \ldots \]

The zeros fulfil (6) and are simple except the double zeros \( \alpha_0 = \beta_0 = \pm \sqrt{2} \). Therefore the product

\[ Z_4 = Z_0 R, \quad R(\lambda) = \frac{\lambda^2 - 14}{\lambda^2 - 2} \]

is still of sine-type since \( R(\lambda) \to 1 \) as \(|\text{Im} \, \lambda| \to \infty\). The denominator reduces multiplicity of the double zeros to 1. The product is a generating function since all zeros are real and fulfil (6). We call \( R \) correction factor.

**Case \( f = 5 \)**

Take

\[ Z_5(\lambda) = \cos \frac{\pi}{5} \sqrt{\lambda^2 + \frac{15}{4}} \cos \frac{\pi}{5} \left( \sqrt{\lambda^2 - \frac{1}{4}} + 1 \right) \cos \frac{\pi}{5} \left( -\sqrt{\lambda^2 - \frac{1}{4}} + 4 \right) \]

\[ \times \cos \frac{\pi}{5} \left( \sqrt{\lambda^2 - \frac{9}{4}} + 2 \right) \cos \frac{\pi}{5} \left( -\sqrt{\lambda^2 - \frac{9}{4}} + 3 \right) R(\lambda). \]

The zeros of the first five factors are

\[ \alpha_k^2 = \left( 5 \left( k + \frac{1}{2} \right) \right)^2 - \frac{15}{4} = 5 \left( l + \frac{1}{2} \right), \quad l = 5k^2 + 5k, \]

\[ \beta_k^2 = \left( 5 \left( k + \frac{1}{2} \right) - 1 \right)^2 + \frac{1}{4} = 5 \left( l + \frac{1}{2} \right), \quad l = 5k^2 + 3k, \]

\[ \gamma_k^2 = \left( 5 \left( k + \frac{1}{2} \right) - 2 \right)^2 + \frac{9}{4} = 5 \left( l + \frac{1}{2} \right), \quad l = 5k^2 + k, \]

\[ \delta_k^2 = \left( 5 \left( k + \frac{1}{2} \right) - 3 \right)^2 + \frac{9}{4} = 5 \left( l + \frac{1}{2} \right), \quad l = 5k^2 - k, \]

\[ \varepsilon_k^2 = \left( 5 \left( k + \frac{1}{2} \right) - 4 \right)^2 + \frac{1}{4} = 5 \left( l + \frac{1}{2} \right), \quad l = 5k^2 - 3k, \]

where \( k = 0, 1, 2, \ldots \) and the points \( \lambda = \pm \sqrt{5/2} (k = 0) \) are zeros of multiplicity 3. We define the correction factor by

\[ R(\lambda) = \frac{(\lambda^2 - 15/2) (\lambda^2 - 35/2)}{(\lambda^2 - 5/2)^2}. \]
This makes the product (14) a generating function at \( f = 5 \).

6 Generating function at arbitrary odd \( f \)

For an arbitrary odd \( f = 2p + 1, \ p > 0 \), we take

\[
Z_0(\lambda) = \cos \left( \frac{\pi}{f} \sqrt{\rho_0} \right) \prod_{q=1}^{p} \cos \left( \frac{\pi}{f} \left( \sqrt{\rho_q} + q \right) \right) \cos \left( \frac{\pi}{f} \left( \sqrt{\rho_{f-q}} + f - q \right) \right),
\]

where

\[
\rho_q(\lambda) = \rho_{f-q}(\lambda) = \lambda^2 - \frac{f}{2} + \left( \frac{f}{2} - q \right)^2.
\]

The first factor is a single-valued entire function. The same is true for the products

\[
\cos \left( \frac{\pi}{f} \left( \sqrt{\rho_q} + q \right) \right) \cos \left( \frac{\pi}{f} \left( \sqrt{\rho_{f-q}} + f - q \right) \right), \ q = 1, \ldots, p
\]

since

\[
\cos \left( \frac{\pi}{f} \left( -\sqrt{\rho_q} + q \right) \right) = -\cos \left( \frac{\pi}{f} \left( \sqrt{\rho_{f-q}} + f - q \right) \right).
\]

Therefore the product of all \( f \) factors is also an single-valued entire function. Its zeros are

\[
\lambda_{k,q}^2 = f \left( l + \frac{1}{2} \right), \ l = f k^2 + (f - 2q) k, \ q = 0, \ldots, p, \ k = 0, 1, 2, \ldots
\]

where the zero \( \lambda = \pm \sqrt{\frac{f}{2}} \) appears \( p + 1 \) times. We define

\[
Z_f = Z_0 R.
\]

The correction factor

\[
R(\lambda) = \frac{\prod_{q=1}^{p} \lambda^2 - f(2q + 1/2)}{\left( \lambda^2 - f/2 \right)^p}
\]

has zeros \( \mu_q = \pm \sqrt{f(2q + 1/2)}, \ q = 1, 2, \ldots, p \) that also fulfil (10). Check that no of these zeros is a zero of \( Z_0 \). Suppose the opposite, let \( \lambda_{k,r}^2 = f(2q + 1/2) \) for some \( 1 \leq r, q \leq p \). Then

\[
f k^2 + (f - 2r) k = 2q
\]
for some $k$. It is not the case if $k = 0$ since $q > 0$. In the case $k \geq 1$ we have

$$fk^2 = 2q - (f - 2r)k \leq 2qk - (f - 2r)k = (2(q + r) - f)k < fk$$

since $2(q + r) \leq 4p < 2f$. This yields $k^2 < k$ which is the contradiction. It follows that $Z_f$ is an even generating function at $f$ for phase shift. The zeros of the main series with $k > 0$ fulfil

$$..., \lambda_{k,p} < \lambda_{k,p-1} < ... < \lambda_{k,0} < \lambda_{k+1,p} < \lambda_{k+1,p+1} < ...$$

and

$$\lambda_{k,q} = f \left( k + \frac{1}{2} \right) - q + O(k^{-1}).$$

It follows that the gap between adjacent zeros tends to 1 as $k \to \infty$.

Formulas for the cases $f = 1, 3, 5$ as above are particular cases of (15).

**Conclusion 6.1** Interpolation formula for the phase shift like (9) holds for $Z_f$ for any odd $f$.

**Remark.** A similar construction can be applied for even Fresnel numbers.

### 7 Reconstruction of the attenuation

The similar method can be applied to reconstruction of the attenuation $\mu$. We call a function $W$ generating at a Fresnel number $f$ for attenuation if it is of sine-type, all zeros $\lambda$ are real and fulfil the equation

$$|\lambda|^2 = fl(\lambda)$$

where $l$ is an integer for any $\lambda$. Function

$$W_1(\lambda) = \cos \left( \pi \sqrt{\lambda^2 + \frac{1}{4}} \right)$$

is generating at Fresnel number $f = 1$ for the attenuation with zeros $\lambda_k = \pm \sqrt{k^2 + k}$, $k = 0, 1, 2, ...$. A generating function for attenuation can be constructed at any odd Fresnel number $f$ as follows. We can take

$$W_f(\lambda) = \cos \frac{\pi}{f} \sqrt{\sigma_0} \prod_{q=1}^{p} \cos \frac{\pi}{f} (\sqrt{\sigma_q} + q) \cos \frac{\pi}{f} (\sqrt{\sigma_q + f} + q) R(\lambda)$$
where
\[ \sigma_q(\lambda) = \lambda^2 + \left( \frac{f}{2} - q \right)^2, \quad q = 0, \ldots, f - 1. \]

The zeros of the first \( f \) factors are
\[ \lambda_{k,q}^2 = f l, \quad l = f k^2 + (f - 2q) k, \quad q = 0, \ldots, p, \quad k = 0, 1, 2, \ldots \quad (19) \]
They fulfil (18) but number \( \lambda = 0 \) appears \( p + 1 \) times. The factor
\[ R(\lambda) = \prod_{q=0}^{p-1} \frac{(\lambda^2 - (2q + 1)f)}{\lambda^{2p}} \]
corrects to 1 the multiplicity of this zero. The zeros of the numerator satisfies (18) and do not coincide with points (19). Therefore function \( W_f \) generates attenuation and the arguments of Theorem 3.1 work for reconstruction of attenuation \( \mu \). Note that points (19) satisfies inequalities (16) and (17).

**Conclusion 7.1** Interpolation formula for attenuation like (9) holds for \( W_f \) for any odd \( f \).

### 8 Convergence and truncation error

**Theorem 8.1** If \( \Lambda \) is the set of zeros of a function \( Z \) of sine-type, then the functions
\[ \frac{Z(t)}{(t - \lambda) Z'(\lambda)}, \quad \lambda \in \Lambda \]
form a Riesz basis in the space \( P \). This means that \{\( g(\lambda) \)\} \( \in l_2 \) for any function \( g \in P \) and the sequence (7) converges to \( g \) in \( L_2(\mathbb{R}) \). Vice versa for any sequence \{\( c_\lambda \)\} \( \in l_2 \) the series
\[ \sum_{\lambda \in \Lambda} g(\lambda) \frac{Z(t)}{(t - \lambda) Z'(\lambda)} \quad (20) \]
converges to an element of \( P \).

Pointwise convergence of series (20) can be evaluated for the classical Whittleberger-Kotelnikov-Shannon series which is the particular case for \( Z(t) = \sin \pi t \). Take the indicator function \( f \) of the set \([-1, -2/3] \cup [2/3, 1]\) as a model.
for the space of piecewise smooth functions. Compare function \( g = \hat{f} \) of Paley-Wiener type with the truncated WKS sum

\[
g_N(t) = \sum_{k=-N}^{N} g(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.
\]

A calculation shows that the partial sum \( g_8 \) gives rather good approximation:

\[
\max_{|t| \leq 6} |g(t) - g_8(t)| \leq 0.006.
\]

The series \([20]\) is of the same as the WKS series. Therefore the convergence of these series is expected to have the same rate.

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