Sub-eikonal corrections to scattering amplitudes at high energy

Giovanni Antonio Chirilli

Institut für Theoretische Physik, Universität Regensburg,
D-93040 Regensburg, Germany

E-mail: giovanni.chirilli@ur.de

ABSTRACT: Most of the progress in high-energy Quantum Chromodynamics has been obtained within the eikonal approximation and infinite Wilson-line operators. Evolution equations of Wilson lines with respect to the rapidity parameter encode the dynamics of the hadronic processes at high energy. However, even at high energy many interesting aspects of hadron dynamics are not accessible within the eikonal approximation, the spin physics being an obvious example. The higher precision reached by the experiments and the possibility to probe spin dynamics at future Electron Ion Colliders make the study of deviations from eikonal approximation especially timely. In this paper I derive the sub-eikonal quark and gluon propagators which can serve as a starting point of studies of these effects.
1 Introduction

It is well known that high-energy behavior of QCD amplitudes can be described by the evolution of relevant Wilson-line operators. The typical example is the deep inelastic scattering (DIS) at low Bjorken $x_B$, where the T-product of two electromagnetic currents can be approximated by a perturbative expansion in terms of coefficient functions (photon impact factors) and matrix elements of Wilson-line operators evaluated in the proton or
nucleus state. The evolution equation of the Wilson-line operators with respect to the rapidity parameter provides the energy dependence of the cross section. This procedure takes the name of high-energy Operator Product Expansion (OPE) (see ref. [1] for a review on Wilson-line formalism in high-energy QCD).

The operator describing the DIS scattering amplitude is a trace of two Wilson lines. Its evolution equation with respect to the rapidity of the fields generates a hierarchy, the Balitsky-hierarchy [2], of evolution equations. The Balitsky-hierarchy is equivalent to the Jalilian-Marian–Iancu–McLerran–Weigert–Leonidov–Kovner (JIMWLK) [3–6] evolution equation thus, they are referred to as B–JIMWLK equation. The Balitsky-Kovchegov (BK) equation [7, 8] is the first of the Balitsky hierarchy and it is obtained in the mean field approximation. Its linearization coincides with the BFKL [9, 10] equation (for a review, see ref. [11]).

The OPE in terms of composite Wilson-line operators [12] has been applied up to NLO accuracy in processes like DIS [13, 14] and proton-nucleus collisions [15, 16]. In ref.[17] the analytic expression of the $\gamma^*\gamma^*$ cross section has been computed using the NLO OPE in linearized (BFKL) form.

A relevant part of the program of the proposed Electron Ion Collider [18–20] is dedicated to the study of DIS with spin. Since the high-energy OPE formalism developed so far is suitable only for unpolarized scattering processes, the extension of such formalism to high-energy spin-dynamics for Transverse Momentum Distribution (TMD) functions and $g_1$ structure function is in strong demand.

The high-energy OPE in terms of infinite Wilson lines is based on a semi-classical description of the process. In eikonal-approximation, the quantum free wave function of a particle propagating in a classical external field is modified by a simple phase factor in QED or gravity [21, 22], and by a path-ordered exponential in QCD [23–25]. In high-energy QCD the rapidity parameter serves as a discriminator between classical and quantum fields and the propagation of fast moving particles in an external filed is described by eikonal interactions. Unfortunately, eikonal interactions are insensitive to the spin content of the process. So, in order to bring in spin information, it is necessary to include sub-eikonal contributions. The subject of this paper is the derivation of these corrections to the quark and gluon propagator in the background of a shock-wave. These results represent the first necessary step to include spin dynamics in the high-energy OPE formalism.

In refs. [26, 27] sub-eikonal corrections to scalar and gluon propagators have been calculated in order to construct a formalism that provides evolution equations of gluon Transverse Momentum Distribution (TMD) from low to moderate $x_B$. In this paper, I calculate the same result for the scalar and gluon propagators but including also the contributions coming from the transverse gauge fields neglected in refs. [26, 27]. In addition, I compute the sub-eikonal corrections to the quark propagator. The results obtained in this paper can be used to study, for example, quark-TMDs with spin and to check results obtained in refs. [28–34].

Corrections to the eikonal formalism have been considered also in the context of spin asymmetries in proton-nucleus collision. In ref. [35], for example, sub-eikonal corrections for the retarded gluon propagator have been calculated.
Sub-eikonal corrections have also been considered in the context of high-energy QCD at fixed angle and resummation of threshold logarithms [36, 38].

The paper is structured as follows. In section 2.1 and in Appendix A we introduce the formalism and derive the leading-eikonal scalar and quark propagator. The sub-eikonal corrections to the scalar propagator are derived in section 3. This result is a necessary step to derive the sub-eikonal corrections to the quark propagator, in section 4, and to the gluon propagator in light-cone gauge, in section 5. In the Appendix we consider a modification of the shock-wave picture, that is, we consider the case in which the particle starts or ends its propagation inside the external field. In the Appendix we also include an alternative derivation of the eikonal quark propagator and we calculate the sub-eikonal corrections to the gluon propagator in background-Feynman gauge.

2 Eikonal approximation for scalar and quark propagators

We start our analysis with the derivation of the scalar and quark propagator in the background of a shock-wave in eikonal approximation. We will use this preliminary step in order to introduce the formalism that will be used throughout the paper.

2.1 Scalar propagator in the eikonal approximation

The idea of the shock-wave formalism is based on the observation that high-energy regime can be reached not only rescaling the longitudinal momenta of the projectile-particle by a large parameter, but also performing a longitudinal boost of the fields generated by the target-particle. We will study the propagation of a particle in the background of a highly boosted gluon field.

Let \( p_1^\mu \) and \( p_2^\mu \) be two light-cone vectors such that \( p_1 \cdot p_2 = \frac{s}{2} \). We assume that the projectile-particle is propagating along \( p_1 \) direction, while the target particle is moving along \( p_2 \) direction. Using the two light-cone vectors we can perform a Sudakov decomposition of the momentum \( p = x p_1^\mu + \beta p_2^\mu + p_\perp^\mu \). We also define the light-cone components \( x^\pm = x^0 \pm x^3 \sqrt{2} \).

The gauge field \( A^\mu \) generated by the target, under a boost, gets rescaled by a large parameter \( \lambda \) as follows

\[
\begin{align*}
A_\bullet(x_\bullet, x_s, x_\perp) & \to \lambda A_\bullet(\lambda^{-1} x_\bullet, x_s, x_\perp), \\
A_\ast(x_\bullet, x_s, x_\perp) & \to \lambda^{-1} A_\ast(\lambda^{-1} x_\bullet, x_s, x_\perp), \\
A_\perp(x_\bullet, x_s, x_\perp) & \to A_\perp(\lambda^{-1} x_\bullet, x_s, x_\perp) \quad (2.1)
\end{align*}
\]

and the field strength tensor as

\[
\begin{align*}
F_\bullet(x_\bullet, x_s, x_\perp) & \to \lambda F_\bullet(\lambda^{-1} x_\bullet, x_s, x_\perp), \\
F_\ast(x_\bullet, x_s, x_\perp) & \to \lambda^{-1} F_\ast(\lambda^{-1} x_\bullet, x_s, x_\perp), \\
F_\circ(x_\bullet, x_s, x_\perp) & \to F_\circ(\lambda^{-1} x_\bullet, x_s, x_\perp), \\
F_{ij}(x_\bullet, x_s, x_\perp) & \to F_{ij}(\lambda^{-1} x_\bullet, x_s, x_\perp). \quad (2.2)
\end{align*}
\]
In Schwinger representation the free scalar propagator is
\[ \langle x | \frac{i}{p^2 + i\epsilon} | y \rangle = i \int d^4k \frac{e^{-ik(x-y)}}{p^2 + i\epsilon}, \tag{2.3} \]
with \( \langle k|x \rangle = e^{i x\cdot k} \). In (2.3) we used the \( \hbar \)-inspired notation \( d^4k \equiv \frac{d^3k}{(2\pi)^3} \) and \( \delta^{(4)}(x-y) = (2\pi)^4\delta^{(4)}(x-y) = 1 \).

Because of the infinite boost, in first approximation, we can assume that the field operator \( \hat{A}_\mu \) commutes with the \( \hat{a} = \frac{i}{\hbar c} \) operator where, as we already mentioned above, \( \alpha \) is the longitudinal component along the light-cone vector \( p_1 \). The eikonal approximation is based on the observation that, the only component surviving the boost is \( A_\bullet \), so \( \hat{P}^2 \simeq \hat{p}^2 + 2ag\hat{A}_\bullet \). If \( A_\mu(x) \) is small in comparison with the typical distance \( (x-y) \), we can represent the propagator as a series
\[ \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle \simeq \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle + g \int d^4z \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | z \rangle 2i\alpha A_\bullet(z_\perp) \langle z | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle + \ldots. \tag{2.4} \]

Alternatively, expansion (2.4) can also be expressed through Schwinger’s proper time integral
\[ \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle = -i \int_0^{t^\prime} dt \langle x | e^{i(\hat{p}^2 + 2\alpha\hat{A}_\bullet + i\epsilon)t} | y \rangle = -i \int_0^{t^\prime} dt \left[ \langle x | e^{i\hat{P}^2 + \hat{A}_\bullet e^{i\hat{P}^2 + \hat{A}_\bullet + i\epsilon}t} | y \rangle \right] + \ldots. \tag{2.5} \]

Using either (2.4) or (2.5), the expansion of the scalar propagator reduces to
\[ \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle = \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_\infty^0 \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \right] e^{-i\alpha(x_\perp - y_\perp)} \times \left[ \langle x_\perp | e^{-i\frac{\hat{p}^2}{2\alpha}(x_\perp - y_\perp)} | y_\perp \rangle \right] \times \left[ \int_{x_\perp}^x dz_\perp \langle x_\perp | e^{-i\frac{\hat{p}^2}{2\alpha}(x_\perp - z_\perp)} \right] e^{-i\frac{\hat{P}^2}{2\alpha}(z_\perp - y_\perp)} | y_\perp \rangle + \ldots \right]. \tag{2.6} \]

We are interested in the shock-wave picture relevant for high-energy scattering so, we assume that the particle starts and ends its propagation outside the interval in which the field strength tensor is different then zero (see Fig. 1). With this assumption we can rewrite expansion (2.6) as
\[ \langle x | \frac{i}{\hat{p}^2 + i\epsilon} | y \rangle = \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_\infty^0 \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \right] e^{-i\alpha(x_\perp - y_\perp)} \times \int d^2z d^2z' \langle x_\perp | e^{-i\frac{\hat{p}^2}{2\alpha}x_\perp} | z_\perp \rangle \times \langle z_\perp | \text{Pexp} \left\{ ig \int_{y_\perp}^x \frac{d\omega}{s} e^{i\frac{\hat{p}^2}{2\alpha}\omega} A_\bullet(\omega) e^{-i\frac{\hat{P}^2}{2\alpha}\omega} \right\} | z_\perp \rangle \times \langle z_\perp' | e^{i\frac{\hat{p}^2}{2\alpha}y_\perp} | y_\perp \rangle. \tag{2.7} \]
So far, to arrive at the propagator given in Eq. (2.7), we have implemented only two consequences of the longitudinal Lorentz boost of the external field: the commutation relation $[\hat{\alpha}, \hat{A}] = 0$ and the fact that the most dominant component of the gauge external field is $A_\bullet$. Indeed, since we are considering an infinite Lorentz boost, we can perform further approximations.

Propagator (2.7) describes the propagation, following any path from point $z_\perp + \frac{2}{s} x_s p_1$ to point $z'_\perp + \frac{2}{s} y_s p_1$ in the external field $A^\mu(x) = (A_\bullet(x_s, x_\perp), 0, 0)$, of a spinless particle. Notice that, the particle propagates between points $(x_\bullet, x_s, x_\perp)$ and $(y_\bullet, y_s, y_\perp)$, while the field strength tensor, because of the longitudinal boost, is defined within an infinitesimal interval in the longitudinal direction. In other words, $F^{\mu\nu}(\omega_s, \omega_\perp) \neq 0$ for $\omega_s \in [-\epsilon_s, \epsilon_s]$ with $0 < \epsilon_s \ll 1$ and, since we are in the shock-wave case, $x_s, y_s \notin [-\epsilon_s, \epsilon_s]$.

Since we are boosting the coordinates, the longitudinal distance traveled by the particle in the external field is

$\frac{2}{s} x_s p_1$.

Deviation from the straight-line propagation are taken into account by the further approximations.

So far, to arrive at the propagator given in Eq. (2.7), we have implemented only two consequences of the longitudinal Lorentz boost of the external field: the commutation relation $[\hat{\alpha}, \hat{A}] = 0$ and the fact that the most dominant component of the gauge external field is $A_\bullet$. Indeed, since we are considering an infinite Lorentz boost, we can perform further approximations.

Propagator (2.7) describes the propagation, following any path from point $z_\perp + \frac{2}{s} x_s p_1$ to point $z'_\perp + \frac{2}{s} y_s p_1$ in the external field $A^\mu(x) = (A_\bullet(x_s, x_\perp), 0, 0)$, of a spinless particle. Notice that, the particle propagates between points $(x_\bullet, x_s, x_\perp)$ and $(y_\bullet, y_s, y_\perp)$, while the field strength tensor, because of the longitudinal boost, is defined within an infinitesimal interval in the longitudinal direction. In other words, $F^{\mu\nu}(\omega_s, \omega_\perp) \neq 0$ for $\omega_s \in [-\epsilon_s, \epsilon_s]$ with $0 < \epsilon_s \ll 1$ and, since we are in the shock-wave case, $x_s, y_s \notin [-\epsilon_s, \epsilon_s]$.

Since we are boosting the coordinates, the longitudinal distance traveled by the particle in the external field is rescaled under a boost as $\omega_s \to \frac{2}{s} \omega_s$ while the gauge field is rescaled as $A_\bullet \to \lambda A_\bullet$ with $\lambda \gg 1$ the boost parameter. So, we can make a further approximation in Eq. (2.7) and write

$$e^{i \frac{2}{s} \omega_s A_\bullet(\omega_s)} e^{-i \frac{2}{s} \omega_s A_\bullet} = A_\bullet(\omega_s) + O(\lambda^0).$$

Making use of (2.8) in propagator (2.7), we obtain

$$\langle x | \frac{i}{P^2 + i\epsilon} | y \rangle = \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)}$$

$$\times \int d^2 z \langle x_\perp | e^{-i \frac{2}{s} \omega_s A_\bullet(\omega_s, z_\perp)} | y \rangle \langle x_\parallel | \frac{i}{s} \hat{\alpha} \cdot \hat{A}_\bullet(\omega_s, z_\perp) | z_\parallel \rangle.$$  (2.9)

where we have defined the gauge link at fixed transverse position $z_\perp$ as

$$[x_s, y_s]_z = \text{Pexp} \left\{ i g \frac{2}{s} \int_{y_s}^{x_s} d\omega_s A_\bullet(\omega_s, z_\perp) \right\}.$$  (2.10)

Propagator (2.9) describes the propagation of the particle in the external field along a straight-line. Deviation from the straight-line propagation are taken into account by the higher order terms neglected in Eq. (2.8). We will consider them in the next section.

In the shock wave approximation we can trade the finite gauge link with the infinite Wilson line because, under the infinite boost, the dominant component of the field strength tensor, $F_\bullet$, has an infinitesimal thin support in $x_s$ coordinate. We assumed that $F_\bullet$ is peaked at the origin and outside the infinitesimal interval $[-\epsilon_s, \epsilon_s]$, $F_\bullet = 0$ (see Fig. 1).

In the gauge rotated field $A^\Omega$ the gauge field outside the external field is zero so, we can trade the gauge link $[\epsilon_s, -\epsilon_s]$ with the infinite Wilson line $[\infty p_1, -\infty p_1]$. In this gauge we can then write

$$\langle x | \frac{i}{P^2 + i\epsilon} | y \rangle = \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)}$$

$$\times \int d^2 z \langle x_\perp | e^{-i \frac{2}{s} \omega_s A_\bullet(\omega_s, z_\perp)} | z_\perp \rangle U_z \langle z_\perp | e^{i \frac{2}{s} \omega_s A_\bullet(\omega_s, z_\perp)} | y \rangle,$$  (2.11)
where we have defined the infinite Wilson line $U_z$ at fixed transverse position $z_\perp$ as

$$U_z \equiv [p_1, -p_1] = \text{Pexp} \left\{ i \frac{g_2}{s} \int_{-\infty}^{+\infty} dz_s A_s \left( \frac{2}{s} p_1 z_s + z_\perp \right) \right\} . \quad (2.12)$$

Notice that, because of the infinite boost, we have set $A^{\mu}_{\perp}$ component to zero, so, since it is a pure gauge, it can be restored, for example, as a transverse gauge link (see Fig. 1).

### 2.2 Quark propagator in eikonal approximation

We consider now the quark propagator in the eikonal approximation. From now on we work in the gauge rotated field $A^\Omega$. This allows us to set to zero the transverse fields at the edges of the gauge fields, i.e., $A_i(x^\ast) = A_i(y^\ast) = 0$.

We proceed in the same way as in the scalar propagator case. After boosting the fields, we consider only the dominant component of the gauge fields and write

$$\langle x | \frac{i}{P} + i \epsilon | y \rangle \simeq \langle x | \hat{P} \frac{i}{P^2 + ig_2 \gamma^\rho p_2 F^\rho \bullet + i \epsilon} | y \rangle$$

$$= \left( i \hat{\phi}^x + g_2 \frac{2}{s} \phi_2 A_s(x^\ast, x_\perp) \right)$$

$$\times \langle x | \left[ \frac{i}{P^2 + i \epsilon} - \frac{i}{P^2 + ig_2 \gamma^\rho p_2 F^\rho \bullet + i \epsilon} \right] y \rangle , \quad (2.13)$$

where $P^2 \simeq p^2 + 2g\alpha A^\bullet$. Notice that, because $\phi_2 \phi_2 = 0$, there is only one term surviving the expansion in (2.13). In appendix B we will consider an alternative way of expanding the quark propagator in the external field. However, to study the sub-eikonal corrections the expansion used in this section is more convenient.

In (2.13), for each factor $\frac{1}{P^2 + i \epsilon}$ we use the scalar propagator (2.9), and using

$$\langle x^\ast | e^{-i \alpha(x^\ast - y^\ast)} \langle x_\perp | e^{-i \frac{P^2}{2s^2} x_\perp} | z_\perp \rangle$$

$$= e^{-i \alpha(x^\ast - y^\ast)} \langle x_\perp | e^{-i \frac{P^2}{2s^2} x_\perp} \left( \frac{1}{\alpha s^2} \hat{\phi}^x \hat{\phi}^x + \frac{2}{s} \phi_2 D_s^\bullet \right) | z_\perp \rangle , \quad (2.14)$$
together with the identity
\[ iD_\ast^x [x_s, y_s] = \left( i\partial_x^x + gA_\ast(x_s) \right) [x_s, y_s] = 0 \]  
(2.15)

we arrive at
\[
\begin{align*}
\langle x | \frac{i}{p} + i \epsilon | y \rangle &= \left[ \int_{0}^{\infty} \frac{d \alpha}{2 \alpha^2} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d \alpha}{2 \alpha^2} \theta(y_s - x_s) \right] e^{-i \theta(x_s - y_s)} \\
&\quad \times \frac{1}{\alpha^2} | x_\perp | e^{-\frac{\alpha^2}{2} x_\perp \cdot \hat{p}_1 \cdot \hat{p}_2} \\
&\quad \times \hat{p} \{ x_s, y_s \} - \frac{2}{s} \gamma_\perp \int_{y_s}^{x_s} d \omega_s [ x_s, \omega_s ] F_{s \rho} [ \omega_s, y_s ] e^{i \frac{\alpha^2}{2} y_\perp | y_\perp \rangle}.
\end{align*}
\]
(2.16)

Note that, in using Eq. (2.14) we have neglected the delta function \( \delta(x_s - y_s) \) coming from the differentiation of the Theta functions \( \theta(x_s - y_s) \) and \( \theta(y_s - x_s) \) of the scalar propagator Eq. (2.9). This is because we are assuming that \( x_s \neq y_s \).

Quark propagator (2.16) is in covariant gauge form. An equivalent expression of the quark propagator (2.16) is
\[
\begin{align*}
\langle x | \frac{i}{p} + i \epsilon | y \rangle &= \frac{1}{s} \left[ \int_{0}^{\infty} \frac{d \alpha}{2 \alpha^2} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d \alpha}{2 \alpha^2} \theta(y_s - x_s) \right] e^{-i \theta(x_s - y_s)} \\
&\quad \times \langle x_\perp | e^{-\frac{\alpha^2}{2} x_\perp \cdot \hat{p}_1 \cdot \hat{p}_2} \hat{p} \{ x_s, y_s \} - \frac{2}{s} \gamma_\perp \int_{y_s}^{x_s} d \omega_s [ x_s, \omega_s ] F_{s \rho} [ \omega_s, y_s ] \rangle e^{i \frac{\alpha^2}{2} y_\perp | y_\perp \rangle}.
\end{align*}
\]
(2.17)

To arrive at Eq. (2.17) from Eq. (2.16) we have carried one of the \( \hat{p} \) operator to the right of the gauge-link. The quark propagator in Eq. (2.17) is not in gauge covariant form but it is suitable for the shock-wave picture. Indeed, in the shock-wave approximation, \( A_\perp(x_s) \) and \( A_\perp(y_s) \) are zero in the gauge rotated field \( A_\perp \) we are working in. Moreover, the gauge-link \( [x_s, y_s] \), reduces to \( [\epsilon_s, -\epsilon_s] \). At this point we can extend the limit of integration \( \epsilon_s \to \infty \) and \( -\epsilon_s \to -\infty \) thus, (2.17) becomes
\[
\langle x | \frac{i}{p} + i \epsilon | y \rangle = \frac{1}{s} \left[ \int_{0}^{\infty} \frac{d \alpha}{2 \alpha^2} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d \alpha}{2 \alpha^2} \theta(y_s - x_s) \right] e^{-i \theta(x_s - y_s)} \\
&\quad \times \langle x_\perp | e^{-\frac{\alpha^2}{2} x_\perp \cdot \hat{p} \{ z_\perp \} \hat{p}_2 U_{z_\perp} \hat{p} \cdot \hat{p}_2 U_{z_\perp} e^{i \frac{\alpha^2}{2} y_\perp | y_\perp \rangle}.
\end{align*}
\]
(2.18)

In (2.18) the shock-wave picture is now evident: we have free propagation from point \( x \) to the point of interaction \( z \) with the shock wave, eikonal interaction with the shock wave, which is represented by the infinite Wilson line multiplied by \( \hat{p}_2 \), and then again free propagation from point \( z \) to point \( y \) (see Fig. 1). Propagator (2.18) is the one that has been used so far in all the results that have been obtained in the high-energy Wilson-lines formalism. In this paper we are interested in the sub-eikonal corrections to (2.18).
3 Sub-eikonal corrections to the scalar propagator

In the previous section, to obtain the quark propagator in the eikonal approximation, we have used the scalar propagator as intermediate step. For the sub-eikonal corrections we proceed in a similar way: we first derive the sub-eikonal correction to the scalar propagator, and then use this result as a useful intermediate step in order to get the sub-eikonal corrections to the quark propagator.

We consider a background gauge field with all components different then zero: $A^a_\mu(x_\perp, x_\parallel) = (A^a_\mu(x_\perp, x_\parallel), A^a_\perp(x_\perp, x_\parallel), A^a_\parallel(x_\perp, x_\parallel))$. From now on we shall omit the superscript “cl” from the classical field. The shock-wave now has a finite width, so we have to identify the sub-dominant components of the momentum operator $\hat{P}^2 = \{\hat{p}^2_\perp, \hat{A}_\perp\} + \{2 \hat{p}_\perp, \hat{A}_\parallel\} - g\hat{A}_\parallel^2$. First, let us notice that we can get rid of the term $\{\hat{P}_\perp, A_\perp\}$ observing that

$$\langle x | \frac{1}{p^2 + 2agA_\perp + i\epsilon} A_\perp \hat{P}_\perp \frac{1}{p^2 + 2agA_\perp + i\epsilon} | y \rangle$$

$$= \left[ - \int_0^{+\infty} \frac{d\alpha}{4\alpha^2} \theta(x_\perp - y_\perp) + \int_{-\infty}^{0} \frac{d\alpha}{4\alpha^2} \theta(y_\perp - x_\perp) \right] e^{-i\alpha(x_\perp - y_\perp)}$$

$$\times \int_{y_\perp}^{x_\perp} \frac{d\lambda}{\lambda^2} \langle x_\perp | e^{-i\frac{\lambda}{2\perp} x_\perp} \rangle \left( A_\perp + \frac{i\lambda}{\alpha_s} [\hat{p}^2_\perp, A_\perp] \right)$$

$$\times \left( \frac{i\lambda}{\alpha_s} [\hat{p}^2_\perp, A_\perp] + \frac{\hat{p}^2_\perp}{2\alpha} + iD^\perp_\perp \right) [z_\perp, y_\perp] e^{i\frac{\lambda^2}{2\perp} y_\perp} | y_\perp \rangle = O(\lambda^{-2})\), \quad (3.1)$$

where in the first step we inserted $\int d^4z |z\rangle \langle z| = 1$ and used

$$e^{i\frac{\lambda^2}{2\perp} y_\perp} A_\perp e^{-i\frac{\lambda^2}{2\perp} y_\perp} \approx A_\perp + \frac{i\lambda}{\alpha_s} [\hat{p}^2_\perp, A_\perp] \quad (3.2)$$

and

$$(i\partial^\perp_\perp + gA_\perp(x_\perp, x_\perp)) \langle z_\perp | e^{-i\frac{\lambda^2}{2\perp} x_\perp} \rangle | y_\perp \rangle = \langle z_\perp | e^{-i\frac{\lambda^2}{2\perp} x_\perp} \left( \frac{i\lambda}{\alpha_s} \hat{p}^2_\perp, A_\perp \right) + \frac{\hat{p}^2_\perp}{2\alpha} + iD^\perp_\perp \right) [z_\perp, y_\perp] e^{i\frac{\lambda^2}{2\perp} y_\perp} | y_\perp \rangle, \quad (3.3)$$

while in the last step we used $(i\partial^\perp_\perp + gA_\perp(x_\perp, x_\perp)) | x_\perp, y_\perp \rangle = 0$. Similarly, we can drop the term $\hat{P}_\perp A_\perp$. In Appendix C, following an alternative procedure, we will show that $\{\hat{P}_\perp, A_\perp\}$ contributes only as sub-sub-eikonal correction.

Let us define $\hat{O} = \{\hat{p}^2_\perp, \hat{A}_\perp\} - g\hat{A}_\perp^2$ so, $\hat{P}^2 = \hat{p}^2 + 2g\alpha A_\perp + g\hat{O}$. The scalar propagator in the boosted external filed can now be written as

$$\langle x | \frac{i}{\hat{P}^2 + i\epsilon} | y \rangle = \langle x | \frac{i}{\hat{p}^2 + 2gagA_\perp + g\hat{O} + i\epsilon} | y \rangle$$

$$= \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \right] e^{-i\alpha(x_\perp - y_\perp)}$$

$$\times \langle x_\perp | e^{-i\frac{\lambda^2}{2\perp} x_\perp} \rangle \text{Pexp} \left\{ ig \int_{y_\perp}^{x_\perp} \frac{d\omega}{\omega} e^{i\frac{\lambda^2}{2\perp} \omega} \left( \hat{A}_\perp(\omega) + \hat{O}(\omega) \right) e^{-i\frac{\lambda^2}{2\perp} \omega} \right\}$$

$$\times e^{i\frac{\lambda^2}{2\perp} y_\perp} | y_\perp \rangle. \quad (3.4)$$
We are interested in corrections to the eikonal propagator that go like \( \frac{1}{\lambda} \), so in the following expansion it is sufficient to consider the first sub-dominant contribution which goes like \( \lambda^0 \)

\[
e^{-\frac{i\Phi^2}{2\alpha}} \left( \hat{A}_* + \frac{\hat{O}}{2\alpha} \right) e^{-\frac{i\Phi^2}{\alpha S}} = \hat{A}_* + \frac{\hat{O}}{2\alpha} + \frac{i}{\alpha S} \{ \hat{p}_\perp^2, \hat{A}_* \} + O(\lambda^{-1}). \tag{3.5}
\]

Making use of (3.5) and of the following identity

\[
\frac{\hat{O}}{2\alpha} + \frac{i}{\alpha S} \{ \hat{p}_\perp^2, \hat{A}_* \} = \frac{1}{2\alpha} \left( \{ \hat{p}_\perp^2, \frac{2}{s} \omega_p F_{\bullet} + (D_{\bullet} \frac{2}{s} \omega_p \hat{A}_\perp) \} - g A_{\perp}^2 \right), \tag{3.6}
\]

after some algebra, we arrive at the desired expression of the scalar propagator with sub-eikonal corrections

\[
\langle x | \frac{i}{\hbar} \frac{\hat{p}_\perp^2}{2 + i\epsilon | y \rangle = \int_0^{\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \left[ e^{-i\alpha(x_\perp - y_\perp)} \right. \times \langle x_\perp | e^{-\frac{i\Phi^2}{\alpha S}} \left[ \{ x_\perp, y_\perp \} + \frac{ig}{2\alpha} \left\{ \left( \hat{P}_\perp \perp, \hat{P}_\perp \perp \right) \{ x_\perp, y_\perp \} - g A_{\perp}^2 \right\} \right] \right] \left. e^{-\frac{i\Phi^2}{\alpha S}} \right]
\]

\[
+ \frac{\hat{O}}{2\alpha} + \frac{i}{\alpha S} \{ \hat{p}_\perp^2, \hat{A}_\perp \} = \frac{1}{2\alpha} \left( \{ \hat{p}_\perp^2, \frac{2}{s} \omega_p F_{\bullet} + (D_{\bullet} \frac{2}{s} \omega_p \hat{A}_\perp) \} - g A_{\perp}^2 \right), \tag{3.6}
\]

(3.7)

In the shock-wave limit, the fields outside the shock-wave are pure gauge and the scalar propagator takes the following form

\[
\langle x | \frac{i}{\hbar} \frac{\hat{p}_\perp^2}{2 + i\epsilon | y \rangle = \int_0^{\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \left[ e^{-i\alpha(x_\perp - y_\perp)} \langle x_\perp | e^{-\frac{i\Phi^2}{\alpha S}} \right. \times \langle x_\perp | e^{-\frac{i\Phi^2}{\alpha S}} \left[ \{ x_\perp, y_\perp \} + \frac{ig}{2\alpha} \left\{ \left( \hat{P}_\perp \perp, \hat{P}_\perp \perp \right) \{ x_\perp, y_\perp \} - g A_{\perp}^2 \right\} \right] \right] \left. e^{-\frac{i\Phi^2}{\alpha S}} \right]
\]

\[
+ \frac{\hat{O}}{2\alpha} + \frac{i}{\alpha S} \{ \hat{p}_\perp^2, \hat{A}_\perp \} = \frac{1}{2\alpha} \left( \{ \hat{p}_\perp^2, \frac{2}{s} \omega_p F_{\bullet} + (D_{\bullet} \frac{2}{s} \omega_p \hat{A}_\perp) \} - g A_{\perp}^2 \right), \tag{3.6}
\]

(3.8)

Except for the terms with the transverse fields at the edges (i.e. at the points \( x_\perp \) and \( y_\perp \)), propagator (3.8) coincides with the one obtained in Ref. [26].

In next section we will derive, following a similar procedure, the sub-eikonal corrections to the quark propagator. To this end the result obtained in this section, Eq. (3.7), will be an essential intermediate step. It is useful to notice that, the terms with transverse fields at points \( x_\perp \) and \( y_\perp \) in Eq. (3.7) cannot be set to zero when we use the scalar propagator as intermediate step because, as we will see in next section, the points \( x_\perp \) and \( y_\perp \) might just be the points in the middle of the shock-wave.
4 Sub-eikonal corrections to the quark propagator

In this section we derive the sub-eikonal correction to the quark propagator in the background of gluon filed and quark anti-quark field. We will start with the gluon field case first.

4.1 Quark propagator in the background of gluon field

As already mentioned before, the sub-eikonal corrections to the scalar propagator, Eq. (3.7) will be an essential intermediate step.

Our starting point is

\[
\langle x | \frac{i}{p + ie} | y \rangle = \langle x | \hat{P} \frac{i}{p^2 + 2\alpha g A_\perp + g B + i2g F_\perp \bar{p}_2 \gamma^i + ie} | y \rangle 
\]

\[
= \left( i\hat{\partial}_x + g^2 A_\perp(x_\perp, x) \bar{p}_2 + gA \perp(x_\perp, x) \right) \times \left[ \int_0^{\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp) \right] e^{-i\alpha(x_\perp - y_\perp)} (x_\perp | e^{-i\frac{p^2}{m^2} x_\perp} 
\]

\[
\times \text{Exp} \left\{ ig \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( A_\perp(\omega_\perp) + \frac{B(\omega_\perp)}{2\alpha} + \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 \gamma^i \right) \right\} 
\]

\[
\times e^{-i\frac{p^2}{m^2} \omega_\perp} \rangle e^{i\frac{p^2}{m^2} y_\perp} | y_\perp \rangle, 
\]

where we have defined \( B \equiv O + \frac{4}{s} g F_{\perp} \sigma_\perp + \frac{i}{2} F_{ij} \sigma_{ij} \).

Expanding again up to \( \lambda^0 \) contributions, we get

\[
e^{i\frac{p^2}{m^2} \omega_\perp} \left( A_\perp(\omega_\perp) + \frac{B(\omega_\perp)}{2\alpha} + \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 \gamma^i \right) e^{-i\frac{p^2}{m^2} \omega_\perp} \]

\[
\simeq A_\perp + \frac{B}{2\alpha} + \frac{i}{2\alpha} F_{\perp} \bar{p}_2 \gamma^i + \frac{i\omega_\perp}{\alpha s} [p_\perp^2, A_\perp] + \frac{i\omega_\perp}{\alpha s} [p_\perp^2, \frac{i}{2\alpha} F_{\perp} \bar{p}_2 \gamma^i] .
\]

Now we have to expand the path ordered exponential of the right-hand-side of (4.2) up to the desired order. We have

\[
\text{Pexp} \left\{ ig \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( A_\perp(\omega_\perp) + \frac{B(\omega_\perp)}{2\alpha} + \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 \gamma^i \right) \right\} e^{-i\frac{p^2}{m^2} \omega_\perp} \}
\]

\[
= \int [x, y] + ig \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( \frac{O(\omega_\perp)}{2\alpha} + \frac{i\omega_\perp}{\alpha s} [p_\perp^2, A_\perp] \right) \left[ \omega_\perp, y_\perp \right] 
\]

\[
+ \left( \frac{ig}{2\alpha} \right) \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 \gamma^i + \frac{i\omega_\perp}{\alpha s} [p_\perp^2, \frac{i}{2\alpha} F_{\perp} \bar{p}_2 \gamma^i] \right) \left[ \omega_\perp, y_\perp \right] 
\]

\[
+ \left( ig \right)^2 \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( \frac{B(\omega_\perp)}{2\alpha} + \frac{i\omega_\perp}{\alpha s} [p_\perp^2, A_\perp] \right) \left[ \omega_\perp, y_\perp \right] 
\]

\[
\int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 \gamma^i \right) \left[ \omega_\perp, y_\perp \right] 
\]

\[
+ \left( ig \right)^2 \int_{y_\perp}^{x_\perp} \frac{d}{s_\perp} e^{i\frac{p^2}{m^2} \omega_\perp} \left( \frac{i}{2\alpha} F_{\perp}(\omega_\perp) \bar{p}_2 ^i \right) \left[ \omega_\perp, y_\perp \right] 
\]

\[
+ O(\lambda^{-2}) .
\]
With the help of some algebra (see Appendix D for some details of the derivation), we can rewrite Eq. (4.3) in a gauge invariant form as

$$P_{\text{exp}} \left\{ ig \int_{y_s}^x \frac{2}{s} \omega_s e^{i \frac{2}{s} \omega_s} \left( A_\bullet(\omega_s) + \frac{B(\omega_s)}{2\alpha} + \frac{i}{2\alpha} \frac{2}{s} F_\bullet(\omega_s) \hat{p}_2 \gamma^3 \right) e^{-i \frac{2}{s} \omega_s} \right\}$$

$$= \left( 1 - \frac{1}{2\alpha} \frac{2}{s} \hat{p}_2 i \mathcal{D}_\perp [x_s, y_s] + \frac{ig}{2\alpha} \int_{y_s}^x \frac{2}{s} \omega_s [x_s, \omega_s] B_1[\omega_s, y_s] \right)$$

$$+ \frac{1}{4\alpha^2} \int_{y_s}^x \frac{d}{s} \omega_s \left[ i(\mathcal{D}_\perp \frac{2}{s} \hat{p}_2 [x_s, z_s]) i g B_1[z_s, y_s] + [x_s, z_s] i g B_1(i \mathcal{D}_\perp \frac{2}{s} \hat{p}_2 [z_s, y_s]) \right]$$

$$+ \frac{ig}{2\alpha} \left[ \int_{y_s}^x \frac{2}{s} \omega_s \left\{ p^i, [x_s, \omega_s] \frac{2}{s} \omega_s F_\bullet(\omega_s) [\omega_s, y_s] \right\} \right] - \frac{ig}{4\alpha^2} \gamma^2 \frac{2}{s} \hat{p}_2 \int_{y_s}^x \frac{d}{s} \omega_s \frac{2}{s} \omega_s$$

$$\times \left[ (i \mathcal{D}_\parallel^i [x_s, \omega_s])(i D_1 F_\bullet)[\omega_s, y_s] - [x_s, \omega_s][(i D_1^i F_\bullet)(i \mathcal{D}_1)[\omega_s, y_s]) - \{ p^i, [x_s, \omega_s]((i D_1^i F_\bullet)[\omega_s, y_s]) \} \right]$$

$$+ \frac{g}{4\alpha^2} \frac{2}{s} \hat{p}_2 \int_{y_s}^x \frac{d}{s} \omega_s \left[ \frac{2}{s} \omega_s (\mathcal{D}_\perp [x_s, \omega_s]) F_\bullet(i \mathcal{D}_\parallel^i [\omega_s, y_s]) - \frac{2}{s} \omega_s (i \mathcal{D}_\parallel^i [x_s, \omega_s]) F_\bullet(\mathcal{D}_\perp [\omega_s, y_s]) \right]$$

$$+ \{ p^i, [x_s, \omega_s] \frac{2}{s} \omega_s F_\bullet(\mathcal{D}_\perp [\omega_s, y_s]) \} + \{ p^i, (\mathcal{D}_\perp [x_s, \omega_s]) \frac{2}{s} \omega_s F_\bullet[\omega_s, y_s] \}$$

$$+ [x_s, \omega_s] \frac{2}{s} \omega_s F_\bullet i \mathcal{D}_\parallel^i (\mathcal{D}_\perp [x_s, \omega_s]) - (i \mathcal{D}_\parallel^i (\mathcal{D}_\perp [x_s, \omega_s])) \frac{2}{s} \omega_s F_\bullet [\omega_s, y_s]$$

$$+ O(\lambda^{-2}),$$

(4.4)

where we have defined $B_1 \equiv \frac{4}{\alpha^2} F_\bullet \sigma_{\bullet\bullet} + \frac{1}{\alpha^2} F_{ij} \sigma^{ij}$.

To continue our analysis we observe that

$$\left( i \hat{\phi}^x + \frac{2}{s} \hat{p}_2 A_\bullet(x_s, x_\perp) + g A_\perp(x_s, x_\perp) \right) e^{-i \alpha(x_s - y_s)} \langle x_\perp | e^{-i \frac{2}{s} \omega_s^2} | z_\perp \rangle$$

$$= e^{-i \alpha(x_s - y_s)} \langle x_\perp | e^{-i \frac{2}{s} \omega_s^2} x_\perp | z_\perp \rangle$$

$$\times \left( \frac{1}{\alpha^2} \hat{p}_2 \hat{p}_2 + \frac{2}{s} \hat{p}_2 D_\perp^i \right) + \frac{i x_s}{\alpha^2} \left[ \frac{2}{s} \hat{p}_2 A_\bullet(x_s), g \right] + g A_\perp(x_s) \right) | z_\perp \rangle,$$

(4.5)

and note that $i [p^2_\perp, g A_\bullet(x_s)] = g \{ p^i, F_\bullet(x_s) + D_\bullet A_\bullet(x_s) \}$. The field strength tensor $F_\bullet(x_s) = 0$ since $x_s$ is outside the shock-wave (see Fig. 1). Similarly, we can set all the transverse fields at the edges of the gauge-link (outside the shock-wave) to zero since they are pure gauge. So, (4.5) reduces to

$$\left( i \hat{\phi}^x + \frac{2}{s} \hat{p}_2 A_\bullet(x_s, x_\perp) + g A_\perp(x_s, x_\perp) \right) e^{-i \alpha(x_s - y_s)} \langle x_\perp | e^{-i \frac{2}{s} \omega_s^2} | z_\perp \rangle$$

$$= e^{-i \alpha(x_s - y_s)} \langle x_\perp | e^{-i \frac{2}{s} \omega_s^2} \left( \frac{1}{\alpha^2} \hat{p}_2 \hat{p}_2 + \frac{2}{s} \hat{p}_2 D_\perp^i \right) | z_\perp \rangle,$$

(4.6)

Using (4.6) and the identity $i D_\perp^i [x_s, y_s] = 0$, we arrive at the following expression for
the quark propagator

\[
\begin{align*}
\langle x | \frac{i}{\not{P} + i\epsilon} | y \rangle &= \left[ \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) - \int_{0}^{\infty} \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_* - y_*)} \frac{1}{\alpha s} (x_\perp | e^{-i\frac{\beta_1}{\alpha s}} ) \\
&\times \{ \not{p} \not{p} \not{p} \not{p} (1 - \frac{1}{2\alpha s} \frac{\not{p}}{2} \not{D}_\perp) [x_*, y_*] + \frac{ig}{2\alpha} \int_{x_*}^{y_*} d\frac{2}{s} \not{p} \not{p} \not{p} \not{p} [x_*, z_\perp] \not{D} \not{B} [z_\perp, y_*] + [x_*, z_\perp] i \not{D} \not{B} [i \not{D} \not{D}_\perp p [z_\perp, y_*]] \} \\
&+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
&+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
&+ \{ P^i_\perp, [i \not{D} \not{D}_\perp [z_\perp, y_*]] \} \not{D} \not{B} [i \not{D} \not{D}_\perp [z_\perp, y_*]] + \{ P^i_\perp, [i \not{D} \not{D}_\perp [z_\perp, y_*]] \} \not{D} \not{B} [i \not{D} \not{D}_\perp [z_\perp, y_*]] \\
&- \{ P^i_\perp, [i \not{D} \not{D}_\perp [z_\perp, y_*]] \} \not{D} \not{B} [i \not{D} \not{D}_\perp [z_\perp, y_*]]
\end{align*}
\]

In order to have \( \not{p} \) to the left and to the right of the gauge link, which is suitable for the shock-wave picture, we will perform similar steps we performed in going from Eq. (2.16) to Eq. (2.17), and set again to zero the fields \( A_\parallel \) at the point \( x_* \) and \( y_* \). To this end, we notice that

\[
\begin{align*}
\frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right)
\end{align*}
\]

\[
\begin{align*}
\frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right) \\
+ \frac{ig}{2\alpha} \int_{y_*}^{x_*} d2 \frac{2}{s} \omega_\perp \not{p} \not{p} \not{p} \not{p} \left( \{ P^i_\perp, [x_\perp, \omega_\perp] \frac{2}{s} \omega_\perp F^i_\perp (\omega_\perp) [w_\perp, y_*] \} \right)
\end{align*}
\]
where we have used $F_{ij}(x_s) = F_{ij}(y_s) = 0$, as points $x_s$ and $y_s$ are outside the shock-wave (see Fig. 1). To arrive at Eq. (4.8) we have used Eqs. (A.5), (A.6), (A.7), and the fact that $\hat{\alpha}$ commutes with all fields.

Using Eq. (4.8) in Eq. (4.7) we arrive at

$$\langle x | \frac{i}{\not{P} + i\epsilon} | y \rangle = \left[ \int_{0}^{\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)} \frac{1}{\alpha s} (x_{\perp} - y_{\perp}) e^{-i \frac{\alpha^2}{4} (x_{\perp} - y_{\perp})^2} $$

$$\times \left\{ \hat{\not{p}}_2[x_s, y_s]\hat{p} + \frac{ig}{2\alpha} \int_{y_s}^{x_s} \frac{d^2 z_s^2}{s} \hat{\not{p}}_2 [x_s, z_s] B_1 [z_s, y_s] \right\}$$

$$+ \frac{1}{4\alpha^2} \int_{y_s}^{x_s} \frac{d^2 \omega_s^2}{s} \hat{\not{p}}_2 \left( \{ p^i, [x_s, \omega_s] \} \frac{2}{s} \omega_s F_{\bullet}(\omega_s) [\omega_s, y_s] \right)$$

$$+ g \int_{\omega_s}^{x_s} \frac{d^2 \omega_s'}{s} \left( \omega_s - \omega_s' \right) [x_s, \omega_s'] F_{\bullet} [\omega_s', \omega_s] F_{\bullet} [\omega_s, y_s] \right\} e^{i \frac{\alpha^2}{4} (y_{\perp})^2}$$

(4.9)

We now use the following results

$$\int \frac{d^2 z_s}{s} \hat{\not{p}}_2 [x_s, z_s] \frac{1}{2} F_{ij} \sigma^{ij} \left[ z_s, y_s \right] \frac{ig}{2\alpha} \int_{y_s}^{x_s} \frac{d^2 \omega_s^2}{s} \left( \{ p^i, [x_s, \omega_s] \} \frac{2}{s} \omega_s F_{\bullet}(\omega_s) [\omega_s, y_s] \right)$$

(4.10)

and

$$\int \frac{d^2 z_s}{s} \hat{\not{p}}_2 [x_s, z_s] \frac{4}{s^2} F_{ij} \sigma^{ij} \left[ z_s, y_s \right] \frac{ig}{2\alpha} \int_{y_s}^{x_s} \frac{d^2 \omega_s^2}{s} \left( \{ p^i, [x_s, \omega_s] \} \frac{2}{s} \omega_s F_{\bullet}(\omega_s) [\omega_s, y_s] \right)$$

(4.11)
and Eq. (4.9) becomes

\[
\langle x | \frac{i}{p + ic} | y \rangle = \left[ \int_0^{\infty} \frac{d \alpha}{2 \alpha} \theta(x - y_s) - \int_0^{\infty} \frac{d \alpha}{2 \alpha} \theta(y_s - x_s) \right] e^{-i \alpha (x_s - y_s)} \frac{1}{\alpha s} (x_\perp | e^{-i \frac{p^2}{2m} x_\perp} \\
\times \left\{ \hat{p} \hat{p}_2(x_s, y_s) \hat{p} + \frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \hat{p} \hat{p}_2 \left( \left\{ p^i, [x_s, \omega_x] \frac{2}{s} \omega_x F_s(\omega_s) [\omega_s, y_s] \right\} \right) \right. \\
+ [x_s, \omega_s] 2 F_{ij} \sigma^{ij} [\omega_s, y_s] + g \int_{x_s}^{x_s} \frac{2}{s} \omega_x \frac{2}{s} (\omega_s - \omega_s') [x_s, \omega_s'] F_s(\omega_s') [\omega_s', y_s] F_s(\omega_s, y_s) \hat{p} \\
\left. + \frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \hat{p} \hat{p}_2 \left[ [x_s, \omega_s] \frac{i}{4} \left( (i \mathcal{D}^k [x_s, \omega_s]) F^k(\omega_s) \{- F_s(\omega_s, y_s) \} \right) \hat{p} \\
+ [x_s, \omega_s] i F_{k}\gamma (i \mathcal{D}^k [x_s, \omega_s]) F_s(\omega_s, y_s) \right) \hat{p} \\
+ (\hat{\alpha} \hat{p}_1 - \hat{\beta}_\perp, [x_s, \omega_s] i \hat{F}_s(\mathcal{D}^k [x_s, \omega_s]) \right) \right\} e^{-i \frac{p^2}{2m} y_\perp} \right).
\]

(4.12)

To underline the structure of result (4.12), we define the following two operators

\[
\hat{O}_1(x_s, y_s; p_\perp) = \frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \left[ [x_s, \omega_s] \frac{i}{4} \left( (i \mathcal{D}^k [x_s, \omega_s]) F^k(\omega_s) \{- F_s(\omega_s, y_s) \}ight) \hat{p} + [x_s, \omega_s] i F_{k}\gamma (i \mathcal{D}^k [x_s, \omega_s]) F_s(\omega_s, y_s) \right) \hat{p} + (\hat{\alpha} \hat{p}_1 - \hat{\beta}_\perp, [x_s, \omega_s] i \hat{F}_s(\mathcal{D}^k [x_s, \omega_s]))
\]

(4.13)

and

\[
\hat{O}_2(x_s, y_s; p_\perp) = \frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \left[ [x_s, \omega_s] \frac{i}{4} \left( (i \mathcal{D}^k [x_s, \omega_s]) F^k(\omega_s) \{- F_s(\omega_s, y_s) \}ight) \hat{p} + [x_s, \omega_s] i F_{k}\gamma (i \mathcal{D}^k [x_s, \omega_s]) F_s(\omega_s, y_s) \right) \hat{p} + (\hat{\alpha} \hat{p}_1 - \hat{\beta}_\perp, [x_s, \omega_s] i \hat{F}_s(\mathcal{D}^k [x_s, \omega_s]))
\]

(4.14)

where

\[
\frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \left[ [x_s, \omega_s] i F_{k}\gamma (i \mathcal{D}^k [x_s, \omega_s]) F_s(\omega_s, y_s) \right) \hat{p} + (\hat{\alpha} \hat{p}_1 - \hat{\beta}_\perp, [x_s, \omega_s] i \hat{F}_s(\mathcal{D}^k [x_s, \omega_s]))
\]

(4.15)

and

\[
\frac{ig}{2 \alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_x \left[ [x_s, \omega_s] i F_{k}\gamma (i \mathcal{D}^k [x_s, \omega_s]) F_s(\omega_s, y_s) \right) \hat{p} + (\hat{\alpha} \hat{p}_1 - \hat{\beta}_\perp, [x_s, \omega_s] i \hat{F}_s(\mathcal{D}^k [x_s, \omega_s]))
\]

(4.16)
and result (4.9) becomes

\[
\langle x | \frac{i}{\not{P} + i\epsilon} | y \rangle = \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)} \frac{1}{\alpha s} \langle x_\perp | e^{-\frac{\not{P}^2}{2\alpha} y_s} \langle y \rangle \langle y_\perp | e^{\frac{\not{P}^2}{2\alpha} x_s} \langle x \rangle \\
\times \left\{ \hat{\not{P}}_2[x_s, y_s] \hat{p} + \hat{\not{P}}_2 \hat{O}_1(x_s, y_s; p_\perp) \hat{p} + \hat{\not{P}}_2 \hat{O}_2(x_s, y_s; p_\perp) \hat{p} \right\} e^{\frac{\not{P}^2}{2\alpha} y_\perp} | y \rangle + O(\lambda^{-2}).
\]

Equation (4.17) is the final result for the quark propagator with sub-eikonal corrections in a non symmetric form. In next section we will rewrite it in a symmetric form with respect to left and right of the shock-wave.

4.1.1 Symmetrizing the quark-sub-eikonal corrections

We observe that the sub-eikonal contribution \( \hat{O}_1 \) has, like the leading-eikonal term, the operator \( \hat{p} \) to its left and to its right, while the sub-eikonal contribution \( \hat{O}_2 \) has the operator \( \hat{p} \) only to its left. We can eliminate the asymmetry between the propagation to the left and to the right of the shock-wave considering the following symmetrization

\[
\langle x | \frac{i}{\not{P} + i\epsilon} | y \rangle = \frac{1}{2} \langle x | \left[ \frac{\hat{P}}{P^2 + i\epsilon} + \frac{i}{P^2 + i\epsilon} \hat{P} \right] | y \rangle.
\]

We have to consider Eq. (4.10) and Eq. (4.11) with \( \hat{\not{P}}_2 \hat{p} \) to the right and the final result for the quark propagator with sub-eikonal corrections can be written in terms of operators \( \hat{O}_1 \) and \( \hat{O}_2 \) as

\[
\langle x | \frac{i}{\not{P} + i\epsilon} | y \rangle = \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)} \frac{1}{\alpha s} \langle x_\perp | e^{-\frac{\not{P}^2}{2\alpha} x_s} \langle x \rangle \langle y_\perp | e^{\frac{\not{P}^2}{2\alpha} y_s} \langle y \rangle \\
\times \langle x_\perp | e^{-\frac{\not{P}^2}{2\alpha} x_s} \langle x \rangle \langle y_\perp | e^{\frac{\not{P}^2}{2\alpha} y_s} \langle y \rangle \rangle \langle x \rangle \langle y \rangle \\
\times \left\{ \hat{\not{P}}_2[x_s, y_s] \hat{p} + \hat{\not{P}}_2 \hat{O}_1(x_s, y_s; p_\perp) \hat{p} + \hat{\not{P}}_2 \hat{O}_2(x_s, y_s; p_\perp) \hat{p} \right\} e^{\frac{\not{P}^2}{2\alpha} y_\perp} | y \rangle + O(\lambda^{-2}).
\]

The operators \( \hat{O}_1, \hat{O}_2 \) are the sub-eikonal corrections to the quark propagator that measure the deviation from the straight-line due to the finite width of the shock-wave.

4.2 Quark propagator in the background of quark and anti-quark fields

Up to this point the background fields considered was made only by gluons. In this section we consider the propagation of a quark in an external field made of quarks and anti-quarks as well (see Fig 2). Our starting point is

\[
\langle T \psi(x) \bar{\psi}(y) \rangle = -g^2 \int d^4 z d^4 z' \langle x | \frac{i}{\not{P} + i\epsilon} | z \rangle \gamma^\mu t^a \psi(z) G_{ab}^\mu(z, z') \not{\gamma}^\nu(z') | \frac{i}{\not{P} + i\epsilon} | y \rangle.
\]

Propagator (4.20) is made of three terms: two quark propagators in the background of gluon field and a gluon propagator in the background of gluon field. Moreover, at point \( z \) and \( z' \)
we have the insertion of the background quark fields. The leading eikonal contribution of propagator \((4.20)\) is

\[
\langle \mathcal{T} \psi(x) \bar{\psi}(y) \rangle_{\psi, \bar{\psi}} = \frac{ig^2}{\alpha^4 s^4} \int_0^{\infty} \frac{d\alpha}{2\alpha} \int_{y_\perp}^{x_\perp} dz_\perp \int_{z_\perp}^{x_\perp} dz'_\perp - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \int_{y_\perp}^{x_\perp} dz_\perp \int_{z_\perp}^{x_\perp} dz'_\perp \right] e^{-i\alpha(x_\perp - y_\perp)}
\]

\[
\times \langle x_\perp | e^{-i\frac{\hat{p}^2}{2\alpha s^2} x_\perp} \hat{p} \phi_2 \left[ \bar{\psi}[x_\perp, z_\perp] \gamma^\mu t^a \psi(z_\perp) G^{ab}_{\mu\nu}(z_\perp, z'_\perp) \bar{\psi}(z'_\perp) t^b \gamma^\nu [z'_\perp, y_\perp] \right] \hat{p} 
\]

\[
+ g \frac{2}{s} \gamma^i \int_{z_\perp}^{x_\perp} d\omega_\perp [x_\perp, \omega_\perp] F_{\omega_\perp} [\omega_\perp, z_\perp] \gamma^\mu t^a \psi(z_\perp) G^{ab}_{\mu\nu}(z_\perp, z'_\perp) \times \bar{\psi}(z'_\perp) t^b \gamma^\nu [z'_\perp, y_\perp] 
\]

\[
- g \frac{2}{s} \gamma^i \int_{z_\perp}^{x_\perp} d\omega_\perp [x_\perp, \omega_\perp] F_{\omega_\perp} [\omega_\perp, z_\perp] \gamma^\mu t^a \psi(z_\perp) G^{ab}_{\mu\nu}(z_\perp, z'_\perp) \bar{\psi}(z'_\perp) t^b \gamma^\nu [z'_\perp, y_\perp] \hat{p} 
\]

\[
- \bar{\psi}[x_\perp, z_\perp] \gamma^\mu t^a \psi(z_\perp) G^{ab}_{\mu\nu}(z_\perp, z'_\perp) \bar{\psi}(z'_\perp) t^b \gamma^\nu [z'_\perp, y_\perp] \hat{p} \left[ \frac{2}{g s} \gamma^i \int_{y_\perp}^{z'_\perp} d\omega_\perp [z_\perp, \omega_\perp] F_{\omega_\perp} [\omega_\perp, y_\perp] \right] \times \hat{p} \phi_2 e^{-i\frac{\hat{p}^2}{2\alpha s^2} y_\perp} + O(\lambda^{-3}),
\]

where \(G^{ab}_{\mu\nu}\) is the gluon propagator in the background of gluon field, and it can be either in the light-cone gauge or background-Feynman gauge. Notice that the quark propagator in the background of quark fields starts at \(\lambda^{-2}\).

5 Sub-eikonal corrections to the gluon propagator

In this section we calculate the sub-eikonal corrections to the gluon propagator in the in the Light-Cone gauge \(p_{2\mu} A^\mu = A_\perp = 0\) in the shock-wave limit. In the Appendix F we will calculate the gluon propagator in the background-Feynman gauge.

These corrections have been calculated in refs. \([26, 27]\) for an external field \(A^\mu = (A_\perp, 0, 0)\). We will consider, instead, a more general external filed with all components different than zero, namely, \(A^\mu = (A_\perp, A_\perp, 0)\).
5.1 Gluon propagator in the light-cone gauge

We start considering the gluon propagator in the background of gluon field in functional form

\[
(T\{A^a_\mu(x)A^b_\nu(y)\}) = \lim_{w \to 0} N^{-1} \int DAA^a_\mu(x)A^b_\nu(y) \\
\times e^{i \int dz \text{Tr}(A^a_\mu(z)(D^2 g^{\alpha\beta} - D^\alpha D^\beta - 2ig^\alpha_\beta \frac{1}{2} p^2_\mu p^2_\beta)A^b_\nu(z))},
\]

(5.1)

where \(D_\mu = \partial_\mu - igA^c_\mu\). We omit the superscript “cl” from the external field from now on.

The issue of fixing properly the sub-gauge conditions will not be discussed here. For this, we refer the reader to ref. [39].

In Schwinger notation the gluon propagator (5.1) can be written as

\[
i\langle T\{A^a_\mu(x)A^b_\nu(y)\}\rangle = iC^{ab}(x,y) = \langle x|\frac{1}{\Box^{\mu\nu} - P_\mu P_\nu + \frac{1}{w}p^2_\mu p^2_\nu}|y\rangle^{ab},
\]

(5.2)

where we defined \(\Box^{\mu\nu} = P^2 g^{\mu\nu} + 2ig F^{\mu\nu}\). It easy to show that (5.2) satisfies a recursion formula (we omit the symbol \(\Box\) on top of operators, from now on)

\[
\frac{1}{\Box^{\mu\nu} - P_\mu P_\nu + \frac{p^2_\mu p^2_\nu}{\omega}} = \left(\delta^\xi_\mu - P_\mu \frac{p^2_\xi}{P_\xi}\right) \frac{1}{\Box^{\eta\nu}} \left(\delta^\eta_\nu - \frac{p^2_\eta}{P_\eta} P_\eta \right) + P_\mu \frac{\omega}{P_\omega} P_\nu
\]

\[
- \frac{g}{\Box^{\mu\alpha} - P_\mu P_\alpha + \frac{p^2_\mu p^2_\alpha}{\omega}} \left(D_\lambda F^{\alpha\lambda} P_\alpha - P_\alpha \frac{1}{P^2} D_\lambda F^{\lambda\beta}\right) \frac{1}{\Box^{3\gamma}} \left(\delta^\gamma_\nu - \frac{p^2_\gamma}{P_\gamma} P_\gamma \right)
\]

\[
+ \frac{g}{\Box^{\mu\alpha} - P_\mu P_\alpha + \frac{p^2_\mu p^2_\alpha}{\omega}} D_\lambda F^{\alpha\lambda} \omega^{\lambda} P_\lambda P_\nu.
\]

(5.3)

We need only the terms which are linear and square in the source \(D^\mu F^{\alpha\nu}(x) = -\bar{\psi}(x)\gamma_\nu t^a\psi(x)\) so, from (5.3) we get (now we can also set \(w \to 0\))

\[
\frac{1}{\Box^{\mu\nu} - P_\mu P_\nu + \frac{p^2_\mu p^2_\nu}{\omega}} = \left(\delta^\xi_\mu - P_\mu \frac{p^2_\xi}{P_\xi}\right) \frac{1}{\Box^{\eta\nu}} \left(\delta^\eta_\nu - \frac{p^2_\eta}{P_\eta} P_\eta \right)
\]

\[
- \left(\delta^\xi_\mu - P_\mu \frac{p^2_\xi}{P_\xi}\right) \frac{1}{\Box^{\eta\nu}} \left(D_\lambda F^{\alpha\lambda} P_\alpha - P_\alpha \frac{1}{P^2} D_\lambda F^{\lambda\beta}\right) \frac{1}{\Box^{3\gamma}} \left(\delta^\gamma_\nu - \frac{p^2_\gamma}{P_\gamma} P_\gamma \right)
\]

\[
- \left(\delta^\eta_\nu - \frac{p^2_\eta}{P_\eta} P_\eta \right) \frac{1}{\Box^{\xi\mu}} \left(D_\lambda F^{\alpha\lambda} P_\alpha - P_\alpha \frac{1}{P^2} D_\lambda F^{\lambda\beta}\right) \frac{1}{\Box^{3\gamma}} \left(\delta^\gamma_\nu - \frac{p^2_\gamma}{P_\gamma} P_\gamma \right)
\]

\[
\times \left(D_\lambda F^{\alpha\lambda} P_\alpha - P_\alpha \frac{1}{P^2} D_\lambda F^{\lambda\beta}\right) \frac{1}{\Box^{3\gamma}} \left(\delta^\gamma_\nu - \frac{p^2_\gamma}{P_\gamma} P_\gamma \right).
\]

(5.4)
We are interested in calculating corrections up to \( \frac{1}{\lambda} \). It is easy to check that we need to expand \( \Box_{\mu\nu} \) in terms of \( F_{\mu\nu} \) up to \( F^3 \) terms (in subsequent algebra we will omit the \( +i\epsilon \) prescription for each of the \( \frac{1}{p^2} \) factor)

\[
iG^{ab}_{\mu\nu}(x, y) = \langle x | (\delta^x_{\mu} - P_{\mu} \frac{p_{\mu}}{P^2}) \left[ g_{\xi\eta} \frac{1}{P^2} - 2i \frac{1}{p^2} F_{\xi\eta} \frac{1}{P^2} + O_{\xi\eta} \right] (\delta^y_{\nu} - P_{\nu} \frac{p_{\nu}}{P^2}) | y \rangle + O(\lambda^{-2}),
\]

(5.5)

where we defined the operator \( O_{\mu\nu} \) as

\[
O_{\mu\nu} = -4g^2 \frac{1}{P^2} J^\mu F^\xi \frac{1}{P^2} F_{\xi\nu} \frac{1}{P^2} + 8ig^3 \frac{1}{p^2} F_{\mu} \xi \frac{1}{P^2} F_{\xi\nu} \frac{1}{P^2} \frac{1}{p^2} F_{\eta} \frac{1}{P^2} F_{\eta\nu} \frac{1}{P^2} \frac{1}{p^2}
\]

\[
- g \frac{1}{P^2} \left( D^\alpha F_{\alpha\mu} \frac{p_{\mu}}{p_s} + \frac{p_{\mu}}{p_s} D^\alpha F_{\alpha\nu} - \frac{p_{\mu}}{p_s} D^\beta F_{\alpha\nu} \frac{p_{\nu}}{p_s} \right) \frac{1}{P^2}
\]

\[
+ 2i \frac{1}{P^2} \left( \frac{p_{\mu}}{P_s} \frac{p_{\nu}}{P_s} D^\lambda F_{\alpha\beta} \frac{1}{P^2} F_{\beta\nu} + F_{\mu} \frac{1}{P^2} D_{\alpha} F_{\lambda\nu} \frac{p_{\nu}}{P_s} \frac{1}{P^2} \frac{1}{P^2} \right)
\]

\[
+ \frac{1}{P^2} \left( \frac{p_{\mu}}{P_s} D_{\mu} F_{\lambda} \frac{1}{P^2} \frac{1}{P^2} \right)
\]

(5.6)

Keeping only terms up to \( \lambda^{-1} \), Eq.(5.5) becomes

\[
iG^{ab}_{\mu\nu}(x, y) = \left[ g_{\mu\nu} - 2i \frac{1}{P^2} \left( \frac{2}{s^2} p_{\mu} F_{\nu} - \frac{2}{s^2} p_{\nu} F_{\mu} \right) \frac{1}{P^2} - \frac{1}{P^2} \frac{p_{\mu}}{p_s} F_{\nu} - \frac{1}{P^2} \frac{p_{\nu}}{p_s} F_{\mu} \right]
\]

\[
+ \frac{1}{s^2} g_{\alpha\mu} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} F^{\cdot j} \frac{1}{P^2} \frac{1}{P^2} - 2i \frac{1}{P^2} \frac{p_{\mu}}{p_s} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} \frac{1}{P^2}
\]

\[
+ \frac{1}{s^2} \frac{g_{\alpha\mu} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} F^{\cdot j} \frac{1}{P^2} \frac{1}{P^2} - 2i \frac{1}{P^2} \frac{p_{\mu}}{p_s} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} \frac{1}{P^2}
\]

\[
+ \frac{1}{s^2} \frac{g_{\alpha\mu} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} F^{\cdot j} \frac{1}{P^2} \frac{1}{P^2} - 2i \frac{1}{P^2} \frac{p_{\mu}}{p_s} p_{\nu} \frac{1}{P^2} F^{\cdot i} \frac{1}{P^2} \frac{1}{P^2}
\]

(5.7)

Note that the terms in square bracket contain both eikonal and sub-eikonal terms, while the terms outside the square bracket are only sub-eikonal terms. After a bit of algebra we
can rewrite (5.7) as

\[
\begin{align*}
iG_{\mu\nu}^{ab}(x, y) &= \langle x \bigg| \left( \left( \frac{\delta_\mu - P_{\mu}^{a}}{p_s} \right)
\frac{g_{20}}{p^2} \left( \frac{\delta^\nu - P_{\nu}^{b}}{p_s} \right) - \frac{p_\mu P_\nu}{p_s^2} \\
&+ \frac{4g^2 p_\mu p_\nu}{s p_s} \frac{1}{p^2} F^i \frac{1}{p^2} (P^i, F_2) \frac{1}{p^2} F^i \frac{1}{p^2} + \frac{4g^2 p_\mu p_\nu}{s p_s} \frac{1}{p^2} (P^i, F_2) \frac{1}{p^2} F^i \frac{1}{p^2} \\
&+ \frac{2g_\mu p_\nu}{p_s^2} 1 \frac{1}{p^2} P_i D_j F^i \frac{1}{p^2} + \frac{g^2 p_\mu p_\nu}{s p_s^2} \frac{1}{p^2} F^i \frac{1}{p^2} F^i \frac{1}{p^2} - 2g \frac{1}{p^2} g_{\alpha\mu} g_{\beta\nu} F_{\alpha\beta} \frac{1}{p^2} \\
&- \frac{32i g^2 p_\mu p_\nu}{s^2} \frac{1}{p^2} F^i \frac{1}{p^2} F^i \frac{1}{p^2} - \frac{8g^2 g_{\alpha\mu} p_\nu}{s p_s} \frac{1}{p^2} F^i \frac{1}{p^2} F^i \frac{1}{p^2} - 8g^2 g_{\alpha\mu} p_\nu \frac{1}{p_s} \frac{1}{p^2} F^i \frac{1}{p^2} F^i \frac{1}{p^2} \\
&- \frac{1}{p^2} gD_i F_\alpha \frac{1}{p^2} g_{\alpha\mu} \frac{1}{p_s} \frac{1}{p^2} - \frac{1}{p^2} gD_i F_\alpha \frac{1}{p^2} g_{\alpha\mu} \frac{1}{p_s} \frac{1}{p^2} \\
&+ \frac{4i p_\mu p_\nu}{s p_s^2} \frac{1}{p^2} \left( D_i F^i \frac{1}{p^2} F^i \frac{1}{p^2} + D_i F^i \frac{1}{p^2} F^i \frac{1}{p^2} \right) \frac{1}{p^2} \\
&+ \frac{8g_\mu p_\nu}{s^2} \frac{1}{p^2} D_i F^i \frac{1}{p^2} D_j F^j \frac{1}{p^2} \frac{1}{p^2} \right) \bigg| y \rangle. \tag{5.8}
\end{align*}
\]

In Eq. (5.8) we need to make an extension of the operator \( P^2 \). We have to use leading-eikonal scalar propagator (in the adjoint representation) for the terms that are already sub-eikonal i.e. all terms in the square bracket, while we need to use the scalar propagator with sub-eikonal corrections, Eq. (3.7), for the first term right after equal sign in Eq. (5.8). Since we are considering the shock-wave case we can disregard the terms with fields at the edges of the gauge links (i.e. at points \( x_\ast \) and \( y_\ast \)) in the scalar propagator (3.7). Taking all this into account, Eq. (5.8) becomes

\[
\begin{align*}
G_{\mu\nu}^{ab}(x, y) &= \left[ - \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\ast - y_\ast) + \int_0^{-\infty} \frac{d\alpha}{2\alpha} \theta(y_\ast - x_\ast) \right] e^{-i\alpha(x_\ast - y_\ast)} \langle x_\perp | e^{-\frac{i\alpha}{2\alpha} x_\perp} \\
&\times \left( \frac{\delta_\mu - P_{\mu}^{a}}{p_s} \right) O_\alpha(x_\ast, y_\ast) \left( \frac{g_{2\nu} - p_\nu^{a}}{p_s} \right) e^{-\frac{i\alpha}{2\alpha} y_\ast} \langle y_\perp | e^{\frac{i\alpha}{2\alpha} y_\perp} \rangle^{ab} + i \langle x | \frac{p_\mu P_\nu}{p_s^2} | y \rangle^{ab} \\
&+ \left[ - \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\ast - y_\ast) + \int_0^{-\infty} \frac{d\alpha}{2\alpha} \theta(y_\ast - x_\ast) \right] e^{-i\alpha(x_\ast - y_\ast)} \\
&\times \left\{ | x_\perp | e^{-\frac{i\alpha}{2\alpha} x_\perp} \left[ \mathcal{O}_1(x_\ast, y_\ast) + \mathcal{O}_2(x_\ast, y_\ast) + \mathcal{O}_3(x_\ast, y_\ast) + \mathcal{O}_4(x_\ast, y_\ast) \right] \right. \\
&\left. \times e^{\frac{i\alpha}{2\alpha} y_\ast} \langle y_\perp | \right\}^{ab} + O(\lambda^{-2}), \tag{5.9}
\end{align*}
\]

where we defined

\[
\begin{align*}
O_\alpha(x_\ast, y_\ast) &\equiv | x_\ast, y_\ast | + \frac{i g}{2\alpha} \int_{y_\ast}^{x_\ast} d\omega \omega \left\{ p^i, [x_\ast, \omega] - \omega F_\ast(\omega) [x_\ast, \omega ] \right\} \\
&+ g \int_{\omega}^{\omega_\ast} d\omega \omega \left\{ F_\ast(\omega) [x_\ast, \omega_\ast] F_\ast(\omega) [x_\ast, \omega ] \right\}. \tag{5.10}
\end{align*}
\]
and
\[
G_{1 \mu \nu}(x\star, y\star) = -\frac{g P_{2 \mu} P_{2 \nu}}{s^2 \alpha^2} \int_{y\star}^{x\star} \frac{2}{s} d\omega s \left[ 4p_i [x\star, \omega \star] F_{ij}[\omega \star, y\star] p^j \right] \\
+ \bigg( \int_{\omega \star}^{x\star} \frac{2}{s} d\omega s \bigg) \left[ (\omega \star - \omega s) [x\star, \omega \star] iD^j F_{ij}[\omega \star, \omega s] iD^j F_{ij}[\omega \star, y\star] \right], 
\]
(5.11)

\[
G_{2 \mu \nu}(x\star, y\star) = -\frac{g \delta^i_j}{\alpha s} \int_{y\star}^{x\star} \frac{2}{s} d\omega s [x\star, \omega \star] F_{ij}[\omega \star, y\star], 
\]
(5.12)

\[
G_{3 \mu \nu}(x\star, y\star) = \frac{g}{\alpha s} \left( \delta^i_j [2 + \delta^i_j] + \delta^i_j [2 - \delta^i_j] \right) \int_{y\star}^{x\star} \frac{2}{s} d\omega s [x\star, \omega \star] iD^j F_{ij}[\omega \star, y\star], 
\]
(5.13)

\[
G_{4 \mu \nu}(x\star, y\star) = -\frac{2g^2}{\alpha^2 s} \int_{y\star}^{x\star} \frac{2}{s} d\omega s \int_{\omega \star}^{x\star} \frac{2}{s} d\omega s \left( \delta^i_j [2 + \delta^i_j] [x\star, \omega \star'] F_{ij}[\omega \star', \omega s] F_{ij}[\omega \star', y\star] \right) \\
+ \delta^i_j [2 + \delta^i_j] [x\star, \omega \star'] F_{ij}[\omega \star', \omega s] F_{ij}[\omega \star', y\star]. 
\]
(5.14)

Equation (5.9) is our final result for the gluon propagator with sub-eikonal corrections in the light-cone gauge. At this point we can send \(x\star \to \infty\) and \(y\star \to -\infty\) and obtain the gluon propagator in the light-cone gauge with sub-eikonal corrections in the shock-wave case.

6 Conclusions and Outlook

In the high-energy OPE formalism, in the eikonal approximation, the evolution equations of the relevant operators, the infinite Wilson lines, are the non-linear evolution BK or B-JIMWLK evolution equations. When sub-eikonal terms are included, new operators appear and, consequently, new non-linear evolution equations need to be derived. The new evolution equations will describe, for example, the high-energy dynamics of scattering processes with spin. These sub-eikonal corrections to high-energy OPE are similar to higher-twist corrections to the usual light-ray OPE [40] which are now important part of QCD phenomenology [41–44].

The advantage of the operatorial formalism adopted in this paper is provided by both the gauge invariant representation of the sub-eikonal operators, and by the systematics provided by the OPE formalism. Indeed, given an operatorial definition of the observable, like, for example, the T-product of two electromagnetic currents in DIS, the application of the high-energy OPE will provide a systematic description of the observable in terms of coefficient functions and matrix elements of the relevant operators evaluated in the target state. The relevant operators are provided by the propagation of the projectile-particle in the background of the target-external field like the propagator in Eq. (4.19).

The main results obtained in this paper are the sub-eikonal corrections to the quark propagator given in Eq. (4.19), and to the gluon propagator in the light-cone gauge given in Eq. (5.9).

The first step we made was the calculation of the scalar propagator and its sub-eikonal corrections given in Eq. (3.7). To obtained this first result, we showed that the term
\{\hat{P}_\bullet, A_\bullet\}, which is related to the possibility for the fields to have dependence also on \(x_\bullet\), is actually a sub-sub-eikonal correction. We proved it in two different ways. We have also shown that the non gauge-invariant terms at the edges of the gauge fields in Eq. (3.7), which can be put to zero being pure gauges, actually play an important role: when the scalar propagator is used in intermediate steps, these non gauge-invariant terms may be in the middle of the shock-wave and therefore cannot be put to zero.

In the quark propagator Eq. (4.19), the new gauge invariant operators are given in the definition of \(\hat{O}_1, \hat{O}_2\) in eqs. (4.13) and (4.14) respectively.

To get the sub-eikonal terms we had to assume that the external field has a finite width. As a consequence one has to consider also the possibility that the scattering particle starts (or ends) its propagation inside the target-filed. To this end, we have derived scalar propagator (E.4) and the quark propagator (E.9) which take into account this possibility.

In section 5.1 we have calculated the sub-eikonal corrections to the gluon propagator in the light-cone gauge. In ref. [27] such correction have been calculated for a background field \(A^\mu(x) = (A_\bullet(x_\bullet, x_\perp), 0, 0)\). Here we considered an external field where all the components of the external field are different then zero. The result is given in Eq. (5.9). The operators \(\mathcal{G}_{1\mu\nu}, \mathcal{G}_{2\mu\nu}, \mathcal{G}_{3\mu\nu}\), and \(\mathcal{G}_{4\mu\nu}\), given in equations (5.11), (5.12), (5.13) and (5.14), respectively, are the result of an external gluon field with all components different then zero. The sub-eikonal corrections to the gluon propagator in background-Feynman gauge, given in Eq. (F.3), are all included in the operators \(\mathcal{B}_{1\mu\nu}, \mathcal{B}_{2\mu\nu}, \mathcal{B}_{3\mu\nu}, \text{ and } \mathcal{B}_{4\mu\nu}\) in Eqs. (F.4), (F.5), (F.6), and (F.7) respectively.

In ref. [32, 33] it was shown that both the gluon and the quark distribution contribute equally to the spin structure function at low-\(x_B\). This result was obtained within the Leading Log Approximation (LLA). In order to consider also the quark distribution within the Wilson-line formalism we have considered in section 4.2 also the sub-eikonal corrections due to quark and anti-quark in the target-external field. The quark propagator with such sub-eikonal corrections is given in Eq. (4.21). This result will not only be relevant for high-energy spin-dynamics but also relevant to obtain sub-eikonal corrections to the BK equation. Although such corrections to the BK equations are energy suppressed, one may obtain for the first time the Regge limit of scattering amplitudes with two-fermions in the t-channel [34] within the Wilson-line formalism.

The applications of the results derived in this paper does not end with high-energy spin-dynamics. The TMD formalism developed so far (for a review see, [45]) does not describe data at sufficiently low-\(x_B\). If one wants to have at hand a formalism that can be applicable at a wider range of \(x_B\), one has to consider sub-eikonal corrections as a way to connect to lower energies and thus moderate \(x_B\). In the case of gluon TMDs this connection has been already made in refs. [26, 27]. In the case of quark-TMDs, instead, one my use the results derived in this paper, although here we have not included terms coming from twist expansion since we considered classical fields and quantum fields with comparable transverse momenta. Corrections due to the hierarchy between the transverse momenta of the quantum and classical field is left for future publication.

The author is grateful to I. Balitsky and V.M. Braun for valuable discussions.
A Notation

In this section we explain some of the notations used throughout the paper.

Given two light-cone vectors \( p_1^\mu \) and \( p_2^\mu \), with \( p_1^\mu p_{2\mu} = \frac{2}{s} \), we can decompose any coordinate as \( x^\mu = \frac{2}{s} x_s p_1^\mu + \frac{2}{s} x_p p_2^\mu + x_\perp^\mu \) with \( x_s = x_\mu p_1^\mu = \sqrt{\frac{s}{2}} x^+ \), \( x_\perp = x_\mu p_1^\mu = \sqrt{\frac{s}{2}} x^- \) and \( x^\pm = \frac{x^+ x^-}{s^2} \). We use the notation \( x_\perp^\mu = (0, x^1, x^2, 0) \) and \( x^\mu = (x^1, x^2) \) such that \( x^1 x_1 = x_\perp^\mu x_\perp^\mu = -x_2^2 \). So, Latin indexes assume values 1, 2, while Greek indexes run from 0 to 3.

We define the gauge link at fixed transverse position as

\[
[u_{p1}, v_{p2}]_z \equiv [u_{p1} + z_\perp, v_{p1} + z_\perp] \equiv \text{Pexp}\left\{ ig \int_v^u dt \ A_\star (tp_1 + z_\perp) \right\}. \tag{A.1}
\]

The derivative of the gauge link with respect to the transverse position is

\[
\frac{\partial}{\partial z_\perp} [u_{p1}, v_{p1}]_z = igA_i(u_{p1} + z_\perp)[u_{p1}, v_{p1}]_{z_\perp} - ig[u_{p1}, v_{p1}]_{z_\perp} A_i(v_{p1} + z_\perp) + ig \int_v^u ds [u_{p1}, sp_1]_{z_\perp} F_\star (p_1 s + z_\perp)[p_1 s, p_1 v]_z, \tag{A.2}
\]

with index \( i = 1, 2 \). From (A.2) we may formally define the transverse covariant derivative \( D_i \) that acts on a non-local operator as

\[
iD_i [u_{p1}, v_{p1}]_z \equiv i \frac{\partial}{\partial z_\perp} [u_{p1}, v_{p1}]_z + g [A_i(z_\perp), [u_{p1}, v_{p1}]_z]
\quad = ig \int_v^u ds [u_{p1}, sp_1]_{z_\perp} F_\star (p_1 s + z_\perp)[p_1 s, p_1 v]_z, \tag{A.3}
\]

where we have used the implicit notation \([A_i(z_\perp), [u_{p1}, v_{p1}]_z] = A_i(z_\perp + u_{p1})[u_{p1}, v_{p1}]_z - [u_{p1}, v_{p1}]_z A_i(z_\perp + v_{p1})\).

Given a gauge link \([x_s, y_s]_z \equiv \left[ \frac{2}{s} x_s p_1 + z_\perp, \frac{2}{s} y_s p_1 + z_\perp \right] \), in Schwinger notation we have

\[
\langle x_\perp | [x_s, y_s]_z \rangle = [x_s, y_s]_z \delta^{(2)}(x - y). \tag{A.4}
\]

The transverse momentum operator \( P_i = \tilde{p}_i + gA_i \) acts on the gauge link as

\[
\langle x_\perp | [\tilde{P}_i, [x_s, y_s]]_z \rangle = \langle x_\perp | i\tilde{D}_i [x_s, y_s]_z \rangle = \langle x_\perp | g \frac{2}{s} \int_{y_s}^{x_s} ds \omega_s [x_s, \omega_s] F_\star (\omega_s, y_s) [\omega_s, y_s] \rangle, \tag{A.5}
\]

where we used again the short-hand notation \([x_s, \omega_s] F_\star (\omega_s, y_s) = [x_s, \omega_s] F_\star (\omega_s) [\omega_s, y_s] \).

The point to make regarding (A.5) is that the covariant derivative \( i\tilde{D}_i \) acts on the gauge link even though the transverse coordinate has not been specified yet and, as matter of fact, it does not have to in order to know how it acts on the gauge link. Therefore, through out the paper we will make quite a bit of algebra involving the gauge link, the momentum operator \( P_i \) and the covariant derivative \( i\tilde{D}_i \) without specifying the bra \( [x_s, y_s]_z \) and the ket \( | y_s \rangle \).

At this point, one is tempted to identify the covariant derivative \( i\tilde{D}_i \) as the usual covariant derivative which acts on a local operator. This identification would not be correct. Indeed, one can easily check that

\[
[i\tilde{D}_i, i\tilde{D}_j] [x_s, y_s] = igF_{ij} [x_s, y_s] - ig[x_s, y_s] F_{ij}. \tag{A.6}
\]
To arrive at (A.6) we have implemented the definition of \( iD \) given in (A.3).

Another identity involving \( D \) that we will often use is

\[
iD_j([x_\omega, y_\omega]) = (iD_j[x_\omega, y_\omega])F_{\omega}([\omega_\omega, y_\omega]) + [x_\omega, y_\omega](iD_jF_{\omega})(\omega_\omega, y_\omega)
\]

\[
= g^2 \int \frac{d^2\omega}{s} [x_\omega, z_\omega]F_{\omega}(z_\omega, \omega_\omega)(\omega_\omega, y_\omega)
\]

\[
+ [x_\omega, y_\omega](iD_jF_{\omega})(\omega_\omega, y_\omega).
\]

Notice that in Eq. (A.7) derivative \( D \) acts on the gauge link, while the usual covariant derivative \( D \) acts on the local field operator \( F \) with respect to the transverse coordinate that will be specified in a second step according to Schwinger notation (A.4). So, as we can see from the first line after the equal sign in Eq. (A.7), \( D \) follows the usual Leibnitz rule for derivative of product of functions with the exception that when \( D \) acts on a gauge link it acts as in Eq. (A.3), while when it acts on a local filed operator it becomes the usual covariant derivative. It is now easy to show that

\[
D^2_{\perp}[x_\omega, y_\omega] = 2g^2 \int \frac{d^2\omega}{s} [x_\omega, \omega_\omega] \int \frac{d^2\omega'}{s} [x_\omega, \omega_\omega']F_{\omega}(\omega_\omega, \omega_\omega')(\omega_\omega', y_\omega)
\]

\[
+ [x_\omega, \omega_\omega](iD_jF_{\omega})(\omega_\omega, y_\omega),
\]

where \( D^2_{\perp} = -D^iD_i \).

\[\text{B} \quad \text{An alternative type of expansion for the quark propagator}\]

The procedure we adopted in section 2.2 to obtain the quark propagator in the eikonal approximation is certainly not the only one. In this section we consider the following type of expansion

\[
\langle x | \frac{1}{p + i\epsilon} | y \rangle = \langle x | \frac{1}{p + i\epsilon} | y \rangle + \langle x | \frac{1}{p + i\epsilon} gA \frac{1}{p + i\epsilon} | y \rangle
\]

\[
+ \langle x | \frac{1}{p + i\epsilon} gA \frac{1}{p + i\epsilon} gA \frac{1}{p + i\epsilon} | y \rangle + \ldots ,
\]

which is diagrammatically shown in Fig. 3. Now we are interested only in the eikonal contribution, so we can use, in Schwinger notation, \( \langle x | A | y \rangle = \frac{1}{2}p_2A_{\omega}(x_\omega, x_\omega)\delta^{(4)}(x - y) \) and
Eq. (B.1) becomes

\[
\langle x | \frac{1}{p + i \epsilon} | y \rangle = \frac{\hat{p} - \hat{y}}{2\pi^2[(x - y)^2 - i\epsilon]^2} - \frac{i}{s} \int_{y_s}^{x_s} d\alpha \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha^2} \theta(x_s - z_s)\theta(z_s - y_s) \right. \\
- \int_{-\infty}^{0} \frac{d\alpha}{2\alpha^2} \theta(y_s - z_s)\theta(z_s - x_s) \left. \right] e^{-i\alpha(x - y^\star)} \\
\times \langle x^\perp | \hat{p} e^{-i\frac{\alpha^2}{\alpha^2}(x^\star - z^\star)} \int g^2_s \hat{A}_\star(z_s) \hat{p}^\star | y^\perp \rangle \\
- \frac{i}{s} \int_{y_s}^{x_s} d\alpha \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha^2} \theta(x_s - z_s)\theta(z_s - y^\star)\theta(z^\star - y_s) \right. \\
- \int_{-\infty}^{0} \frac{d\alpha}{2\alpha^2} \theta(y_s - z^\star)\theta(y_s - x_s) \left. \right] e^{-i\alpha(x - y^\star)} \\
\times \langle x^\perp | \hat{p} e^{-i\frac{\alpha^2}{\alpha^2}(x^\star - z^\star)} \int g^2_s \hat{A}_\star(z^\star) \hat{p}^\star | y^\perp \rangle .
\]  

(B.2)

We can rewrite expansion (B.2) as a path ordered exponential

\[
\langle x | i \frac{1}{p + i \epsilon} | y \rangle = \frac{1}{s} \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha^2} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha^2} \theta(y_s - x_s) \right] e^{-i\alpha(x - y^\star)} \langle x^\perp | \hat{p} e^{-i\frac{\alpha^2}{\alpha^2}x^\star} \hat{A}_\star(z_s) e^{i\frac{\alpha^2}{\alpha^2}z^\star} | y^\perp \rangle .
\]  

(B.3)

Figure 3. Diagrammatic expansion of the quark propagator in the external field. Quantum fields are in blue, while classical fields are in red as usual.

\[
\langle x | i \frac{1}{p + i \epsilon} | y \rangle = \hat{p} \exp \left\{ i g \int_{y_s}^{x_s} d\alpha \int_{-\infty}^{0} \frac{d\alpha}{2\alpha^2} \hat{A}_\star(z_s) e^{i\frac{\alpha^2}{\alpha^2}z^\star} \right\} e^{-i\alpha(x - y^\star)} \langle x^\perp | \hat{p} e^{-i\frac{\alpha^2}{\alpha^2}x^\star} \hat{A}_\star(z_s) e^{i\frac{\alpha^2}{\alpha^2}z^\star} | y^\perp \rangle .
\]  

(B.4)

We may perform a further approximation due to the infinite boost (we are interest only in the eikonal contribution)

\[
e^{-i\frac{\alpha^2}{\alpha^2}x^\star} \hat{A}_\star(z_s) e^{i\frac{\alpha^2}{\alpha^2}z^\star} = \hat{A}_\star(z_s) + O(\lambda^0),
\]  

(B.4)

where \( \lambda \) is the large parameter of the Lorentz boost. With this approximation we arrive at

\[
\langle x | i \frac{1}{p + i \epsilon} | y \rangle = \hat{p} \exp \left\{ i g \int_{y_s}^{x_s} d\alpha \int_{-\infty}^{0} \frac{d\alpha}{2\alpha^2} \hat{A}_\star(z_s) e^{i\frac{\alpha^2}{\alpha^2}z^\star} \right\} e^{-i\alpha(x - y^\star)} \\
\times \langle x^\perp | \hat{p} e^{-i\frac{\alpha^2}{\alpha^2}x^\star} \hat{p}^2 [x_s, y_s] e^{i\frac{\alpha^2}{\alpha^2}y^\star} \hat{p} | y^\perp \rangle .
\]  

(B.5)

Notice that to arrive at propagator (B.5), which is already in the shock-wave form, we did not need to perform the gauge rotation we performed in section 2.2, because expansion (B.1), assumes from the start a free propagation before and after the interaction with the external field, as can also be seen from its diagrammatic representation in Fig.3. In other words, when we start drawing Feynman diagrams, like in Fig.3, we are implicitly fixing a gauge for the external field.
C The $\{\hat{P}, A_\ast\}$ term

In section 3 we showed that the term $\{\hat{P}, A_\ast\}$ is actually a sub-sub-eikonal correction. We now demonstrate the same thing following an alternative procedure. To this end, we notice that (for simplicity we omit the $+i\epsilon$ prescription from each $1\hat{p}^2 + 2\alpha g A_\ast$ and the $\hat{}$ symbol on top of operators)

$$\frac{1}{p^2 + 2\alpha g A_\ast} \{P_\ast, A_\ast\} \frac{1}{p^2 + 2\alpha g A_\ast}$$

$$= \frac{1}{2\alpha} \{A_\ast, \frac{1}{p^2 + 2\alpha g A_\ast}\} + \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} \{p^2_\perp, A_\ast\} \frac{1}{p^2 + 2\alpha g A_\ast}$$

$$= \frac{1}{2\alpha} \{A_\ast, \frac{1}{p^2 + 2\alpha g A_\ast}\} + O(\lambda^{-2}). \quad (C.1)$$

Thus, we are left with sub-eikonal corrections with the $A_\ast$ (a gauge dependent term) only at the edges of the gauge link. When we use the scalar propagator in intermediate steps with the terms (C.1), we will have terms like

$$\frac{1}{p^2 + 2\alpha g A_\ast} \{A_\ast, igF_\ast\} \frac{1}{p^2 + 2\alpha g A_\ast}$$

$$= \frac{1}{p^2 + 2\alpha g A_\ast} \left( P_i P_\ast A_\ast - A_\ast P_i P_\ast \right) \frac{1}{p^2 + 2\alpha g A_\ast} + \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} A_\ast P_i - \frac{1}{2\alpha} P_i A_\ast \frac{1}{p^2 + 2\alpha g A_\ast}$$

$$+ \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} A_\ast P_i P_\ast \frac{1}{p^2 + 2\alpha g A_\ast} - \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} p^2_\perp P_i A_\ast \frac{1}{p^2 + 2\alpha g A_\ast}$$

$$= \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} A_\ast P_i - \frac{1}{2\alpha} P_i A_\ast \frac{1}{p^2 + 2\alpha g A_\ast} + O(\lambda^{-2}). \quad (C.2)$$

So, once again, we arrived at an expression which has the non gauge-invariant terms only at the edges of the gauge links and they could be eliminated by a proper gauge choice since are outside the shock-wave. To arrive at Eq. (C.2), we noticed that the term

$$\frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} A_\ast P_i p^2_\perp \frac{1}{p^2 + 2\alpha g A_\ast} - \frac{1}{2\alpha} \frac{1}{p^2 + 2\alpha g A_\ast} p^2_\perp P_i A_\ast \frac{1}{p^2 + 2\alpha g A_\ast} \quad (C.3)$$
is sub-sub-eikonal and therefore can be disregarded. Moreover, we used

\[
\langle z | \frac{1}{p^2 + 2\alpha g A_s} A_s P_x P_y \frac{1}{p^2 + 2\alpha g A_s} | y \rangle
\]

\[
= \int dz \langle x | \frac{1}{p^2 + 2\alpha g A_s} | z \rangle A_s(z, \alpha) iD_\alpha^2 D_\alpha^1 \langle z | \frac{1}{p^2 + 2\alpha g A_s} | y \rangle
\]

\[
= \left[ -\int_0^{+\infty} \frac{d\alpha}{4\alpha^2} \theta(x_s - y_s) + \int_0^{0} \frac{d\alpha}{4\alpha^2} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)}
\]

\[
\times \int_{y_s}^{x_s} \frac{2}{s} z_s \langle x, z | e^{-i\frac{p^2}{\alpha s} x} [x_s, z_s] e^{i\frac{p^2}{\alpha s} z_s} A_s(z) iD_\alpha^2 D_\alpha^1 e^{-i\frac{p^2}{\alpha s} z_s} \rangle y_s \rangle
\]

\[
= \left[ -\int_0^{+\infty} \frac{d\alpha}{4\alpha^2} \theta(x_s - y_s) + \int_0^{0} \frac{d\alpha}{4\alpha^2} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)}
\]

\[
\times (-g) \int_{y_s}^{x_s} \frac{2}{s} z_s \langle x, z | e^{-i\frac{p^2}{\alpha s} x} [x_s, z_s] \left( A_s(z) + \frac{iz_s}{\alpha s} [p^2, A_s] \right) \left( \frac{iz_s}{\alpha s} [p^2, A_s] + \frac{p_s^2}{2\alpha} + iD_\alpha^2 \right) \]

\[
\times \int_{y_s}^{x_s} \frac{2}{s} z_s \langle z_s, y_s | F_s[z_s, y_s] e^{i\frac{p^2}{\alpha s} y_s} | y \rangle = O(\lambda^{-2}),
\]

(C.4)

where we used the identity \((iD_\alpha^2 + gA_s(x_s))[x_s, z_s] = 0\). Clearly, the advantage of the procedure adopted in section 3 is that one does not have to deal with non-gauge-invariant terms even though they are at the edges of the gauge links.

D  Some calculation details of the sub-eikonal corrections to the quark propagator

In this section we present some details of the calculation to obtain Eq. (4.4). Our starting point is Eq. (4.3) which we report here

\[
\begin{align*}
P_{\text{exp}} & \left\{ ig \int_{y_s}^{x_s} \frac{2}{s} \omega_s \left( \frac{p^2}{\alpha s} \omega_s \left( A_s(\omega_s) + \frac{B(\omega_s)}{2\alpha} + \frac{i}{2\alpha} F_{\gamma'}(\omega_s) \right) \right) e^{-i\frac{p^2}{\alpha s} \omega_s} \right\} \\
& = [x_s, y_s] + ig \int_{y_s}^{x_s} \frac{2}{s} \omega_s [x_s, \omega_s] \left( \frac{O(\omega_s)}{2\alpha} + i \frac{\omega_s}{\alpha s} [p^2, A_s] \right) [\omega_s, y_s] \\
& + \frac{ig}{2\alpha} \int_{y_s}^{x_s} \frac{2}{s} \omega_s [x_s, \omega_s] \left( \frac{2}{\alpha s} F_{\gamma'}(\omega_s) \right) [\omega_s, y_s] \\
& + ig \int_{y_s}^{x_s} \frac{2}{s} \omega_s [x_s, \omega_s] \left( \frac{2}{\alpha s} F_{\gamma'}(\omega_s) \right) [\omega_s, y_s] \\
& + (ig)^2 \int_{y_s}^{x_s} \frac{2}{s} \omega_s \int_{y_s}^{x_s} \frac{2}{s} \omega_s [x_s, \omega_s] \left( \frac{B(\omega_s)}{2\alpha} + i \frac{\omega_s}{\alpha s} [p^2, A_s] \right) [\omega_s, \omega_s'] \left( \frac{2}{\alpha s} F_{\gamma'}(\omega_s') \right) [\omega_s', y_s] \\
& + (ig)^2 \int_{y_s}^{x_s} \frac{2}{s} \omega_s \int_{y_s}^{x_s} \frac{2}{s} \omega_s [x_s, \omega_s] \left( \frac{2}{\alpha s} F_{\gamma'}(\omega_s) \right) [\omega_s, \omega_s'] \left( \frac{B(\omega_s')}{2\alpha} + i \frac{\omega_s'}{\alpha s} [p^2, A_s] \right) [\omega_s', y_s] \\
& + O(\lambda^{-2}).
\end{align*}
\]
Our goal is to rewrite (D.1) in a gauge invariant form. To this end, we divide the right-hand-side of Eq. (D.1) in four pieces, simplify them separately, and then sum them up.

Using result (3.7), the first piece in (4.3) reduces to

\[
[x_s, y_s] + ig \int_{y_S}^{x_s} d^2 \omega_s \left[ x_s, \omega_s \right] \left( \frac{Q(\omega_s)}{2s} + i \frac{\omega_s}{\alpha s} [p_{\perp}^2, A_s] \right) \left[ \omega_s, y_s \right] \\
+ \frac{ig}{2s} \int_{y_s}^{x_s} d^2 \omega_s \left[ x_s, \omega_s \right] \left( \frac{4}{s^2} F_{s\gamma} \sigma_{s\gamma} + \frac{1}{2} F_{i\gamma} \sigma_{i\gamma} \right) \left[ \omega_s, y_s \right] \\
= [x_s, y_s] + \frac{ig}{2s} \left\{ \frac{2}{s} x_s \left( \{P_i, A^i(x_s)\} - gA_i(x_s)A^i(x_s) \right) \right\} [x_s, y_s] \\
- [x_s, y_s] \frac{2}{s} y_s \left( \{P_i, A^i(y_s)\} - gA_i(y_s)A^i(y_s) \right) \\
+ \int_{y_s}^{x_s} d^2 \omega_s \left( \{P_i, [x_s, \omega_s] \frac{2}{s} \omega_s F_{\gamma} [\omega_s, \omega_s] F_{\gamma} \left[ \omega_s, y_s \right] \} \right) \\
+ \frac{ig}{2s} \int_{y_s}^{x_s} d^2 \omega_s \left[ x_s, \omega_s \right] \left( \frac{4}{s^2} F_{\gamma} \sigma_{\gamma} + \frac{1}{2} F_{i\gamma} \sigma_{i\gamma} \right) \left[ \omega_s, y_s \right].
\]  

(D.2)

Next, making use of the following identity

\[
[p_{\perp}^2, F_{\gamma}] = g \left( \{P_i, A_i\} - gA_i A_i \right) F_{\gamma} \\
r - gF_{\gamma} \left( \{P_i, A_i\} - gA_i A_i \right) - \{P_i, (iD_i F_{\gamma}) \},
\]

(D.3)

the second piece reduces to

\[
ig \int_{y_s}^{x_s} d^2 \omega_s \left[ x_s, \omega_s \right] \left( \frac{i}{2s} \frac{2}{s} F_{\gamma} \phi_2 \gamma \right) + i \frac{\omega_s}{\alpha s} [p_{\perp}^2, \frac{i}{2s} F_{\gamma} \phi_2 \gamma] \right) \left[ \omega_s, y_s \right] \\
= - \frac{i}{2s} \frac{2}{s} \phi_2 \left( \{P_i, A_i\} - gA_i A_i \right) F_{\gamma} \\
g \left( \{P_i, A_i\} - gA_i A_i \right) F_{\gamma} \\
- \frac{2}{s} \omega_s \left\{ \frac{2}{s} \omega_s \left[ x_s, \omega_s \right] (iD_i F_{\gamma}) [\omega_s, y_s] \right\} \\
- \frac{2}{s} \omega_s \left\{ \{P_i, A_i\} - gA_i A_i \right\} [x_s, \omega_s] (iD_i F_{\gamma}) [\omega_s, y_s] \\
+ \frac{2}{s} \omega_s \left( iD_i [x_s, \omega_s] \right) (iD_i F_{\gamma}) [\omega_s, y_s].
\]

(D.4)

Note that identity (D.3) highlights the importance of the transverse fields at the edges of the gauge-link in the scalar propagator (3.7). These terms can be set to zero only when they are at the point \( x_s \) and \( y_s \), but when we use the scalar propagator as an intermediate step, these terms will not always be at the edges. So, setting them to zero once and for all, will lead to a wrong result.
We now turn our attention to the third piece. After some algebra it reduces to

\[
(i g)^2 \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \frac{B(\omega_s)}{2\alpha} + i \frac{\omega_s}{\alpha s} \right] \left[ \alpha_s \left( \frac{p_{1\perp}^2}{2\alpha} + i \frac{\omega_\perp s}{\alpha s} \right) \right] \left[ \omega_s, \omega_y \right]
\]

\[
= \left( \frac{ig}{2\alpha} \right)^2 \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \frac{2}{s} x_s \left\{ \{P_1, A^i(x_s)\} - gA_i A^i \right\} \right] \left[ \omega_s, \omega_y \right]
\]

\[
- \left[ x_s, \omega_y \right] 2 \omega_y \left\{ \{P_1, A^i(\omega_y')\} - gA_i A^i \right\} i \frac{2}{s} F_\bullet(\omega_y') \partial_2 \gamma^i \left[ \omega_y', y_s \right]
\]

\[
+ \frac{g^2}{4\alpha^2 s} \overline{\partial_2} \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \alpha_\perp \left( \omega_s \right) \right]
\]

\[
B_1 \left( \overline{\partial_\perp} \left[ \omega_s, y_s \right] \right)
\]

\[
+ \frac{g^2}{4\alpha^2 s} \overline{\partial_2} \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \left\{ P^i, \left[ x_s, \omega_s \right] \right\} \right] \left[ \omega_s, \omega_y \right] F_\bullet \left( \overline{\partial_\perp} \left[ \omega_s, y_s \right] \right)
\]

\[
+ \frac{g^2}{4\alpha^2 s} \overline{\partial_2} \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \left\{ P^i, \left[ x_s, \omega_s \right] \right\} \right] \left[ \omega_s, \omega_y \right] \left[ \omega_s, y_s \right] F_\bullet \left( \overline{\partial_\perp} \left[ \omega_s, y_s \right] \right)
\]

\[
+ \left[ x_s, \omega_s \right] \frac{2}{s} \omega_s F_\bullet \left( \overline{\partial_\perp} \left[ \omega_s, y_s \right] \right) \left[ \omega_s, y_s \right]
\]

\[
\tag{D.5}
\]

Finally, let us consider the fourth piece

\[
(i g)^2 \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \frac{B(\omega_s)}{2\alpha} + i \frac{\omega_s}{\alpha s} \right] \left[ \alpha_s \left( \frac{p_{1\perp}^2}{2\alpha} + i \frac{\omega_\perp s}{\alpha s} \right) \right] \left[ \omega_s, y_s \right]
\]

\[
= \left( \frac{ig}{2\alpha} \right)^2 \int_{y_s}^{x_s} d\omega_x d\omega_y \left[ \frac{2}{s} x_s \left\{ \{P_1, A^i\} - gA_i A^i \right\} \right] \left[ \omega_s, y_s \right]
\]

\[
- \frac{2}{s} y_s \left[ \omega_s, y_s \right] \left\{ \{P_1, A^i\} - gA_i A^i \right\} + \frac{2}{s} \overline{\partial_2} \left( i \overline{\partial_\perp} \left[ x_s, \omega_s \right] \right) \frac{2}{s} \omega_s F_\bullet \left[ \omega_s, y_s \right]
\]

\[
- \left\{ P^i, \left[ x_s, \omega_s \right] \right\} \frac{2}{s} \omega_s F_\bullet \left[ \omega_s, y_s \right] - \left( \overline{\partial_\perp} \left[ x_s, \omega_s \right] \right) B_1 \left[ \omega_s, y_s \right]
\]

\[
+ g \int_{\omega_s}^{x_s} d\omega_y \left[ \frac{2}{s} \omega_y \left\{ \{P_1, A^i\} \right\} \right] \left[ \omega_s, y_s \right] F_\bullet \left[ \omega_s, y_s \right]
\]

\[
\tag{D.6}
\]

We can now sum the four terms (D.2), (D.4), (D.5) and (D.6) and, since we are interested in the shock-wave limit, we can disregard the fields that are at the edges of the gauge-links.
that is at points \( x_\ast \) and \( y_\ast \). After some algebra, we obtain

\[
\text{Pexp}\left\{ ig \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} e^{i \frac{g}{4\alpha} \omega^i} \left( A_\ast(\omega_s) + \frac{B(\omega_s)}{2\alpha} + \frac{i}{2\alpha} F_{\ast}(\omega_s) \not{\!\!}_2 \gamma^i \right) e^{-i \frac{g}{4\alpha} \omega^i} \right\}
= \left( 1 - \frac{1}{2\alpha} \frac{g}{2\alpha} \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \right) [x_\ast, y_\ast] + \frac{ig}{2\alpha} \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} [x_\ast, \omega_s] B_1[\omega_s, y_\ast] + \frac{ig}{2\alpha} \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \not{\!\!}_2 \gamma^i (i \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \not{\!\!}_2 \gamma^i) \left\{ p^i, [x_\ast, \omega_s] \frac{2}{s} \omega_s F_{\ast}(\omega_s) [\omega_s, y_\ast] \right\} - \frac{ig}{4\alpha} \frac{2}{s} \gamma^i \not{\!\!}_2 \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \omega_s \times \left( i \int \not{\!\!}_1 [x_\ast, \omega_s] (i D_i F_{\ast})[\omega_s, y_\ast] - [x_\ast, \omega_s] (i D^i F_{\ast}) (i \int \not{\!\!}_1 [\omega_s, y_\ast]) - \{ p^i, [x_\ast, \omega_s] (i D_i F_{\ast})[\omega_s, y_\ast] \} \right.
+ \frac{g}{4\alpha} \frac{2}{s} \not{\!\!}_2 \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \frac{2}{s} \omega_s F_{\ast}(\not{\!\!}_1 [x_\ast, \omega_s]) \left\{ p^i, (i \int \not{\!\!}_1 [x_\ast, \omega_s]) \frac{2}{s} \omega_s F_{\ast}[\omega_s, y_\ast] \right\} + \{ p^i, \omega_s \frac{2}{s} \omega_s F_{\ast}(i \int \not{\!\!}_1 [\omega_s, y_\ast]) \} \left. + \{ p^i, (i \int \not{\!\!}_1 [\omega_s, y_\ast]) \frac{2}{s} \omega_s F_{\ast}[\omega_s, y_\ast] \} \right. + O(\lambda^{-2}), \tag{D.7}
\]

where we have defined \( B_1 \equiv \frac{ig}{s} F_{\ast} \sigma_{\ast} \). \( \frac{2}{s} F_{\ast} \gamma^i \). .

Note that, to get to Eq. (D.7) we could have started, alternatively, from the following type of expansion

\[
\langle x | P | \frac{p^2}{s} + \frac{i}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu | y \rangle
= \langle x | P \left[ \frac{i}{p^2} - \frac{i}{p^2} (ig \frac{2}{s} F_{\ast} \gamma^i \not{\!\!}_2 + g B_1) \left( p^2 \right) \int_{y_\ast}^{x_\ast} \frac{d^2 \omega_s}{s} \right] \left\{ p^i, \omega_s \not{\!\!}_2 \frac{2}{s} \omega_s F_{\ast}(i \int \not{\!\!}_1 [\omega_s, y_\ast]) \right\} + \{ p^i, \omega_s \frac{2}{s} \omega_s F_{\ast}(i \int \not{\!\!}_1 [\omega_s, y_\ast]) \} \left\{ p^i, (i \int \not{\!\!}_1 [\omega_s, y_\ast]) \right\} + O(\lambda^{-2}), \tag{D.8}
\]

and, substituting the scalar propagator with sub-eikonal corrections, Eq. (3.7), for each \( \frac{i}{p^2 + 2gaA_\ast \gamma O} \) factor, and the eikonal scalar propagator, Eq. (2.9), for each \( \frac{i}{p^2 + 2gaA_\ast} \) factor, we would get to the same gauge invariant expression, Eq. (D.7), with the help of steps similar to Eqs. (D.2), (D.4), (D.5) and (D.6). -- 29 --
Figure 4. Particle starts its propagation within the shock-wave. In this case the pure gauge is only to the left of the shock-wave.

E Quark and scalar propagators with one end-point in the external field

E.1 Scalar propagator

In section 3 we derived the scalar propagator for the shock-wave case, that is, for the case in which the particle starts and ends its propagation outside the shock-wave as shown in Fig. 1. We expanded around a point which is in the middle of the external field, Eq. (3.5). We are now interested in the scalar propagator for a particle that starts or ends its propagation inside the external field. We consider the case in which the point $x_*$ is in the shock-wave, i.e. inside the interval in which the field strength tensor is different then zero. In this case, expansion (2.6) can be written as

$$\langle x | \frac{i}{P^2 + i\epsilon} | y \rangle = \left[ \int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_* - y_*) - \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_* - x_*) \right] e^{-i\alpha(x_* - y_*)}$$

$$\times \langle x_* \perp | \text{Pexp} \left\{ ig \int_{y_*}^{x_*} d\omega_\perp \frac{P^2}{8} e^{\frac{i}{\alpha\omega_\perp}(\omega_\perp - x_*)} \left( A_*(\omega_\perp) + \frac{O(\omega_\perp)}{2\alpha} \right) e^{-i\frac{P^2}{\alpha\omega_\perp}(\omega_\perp - x_*)} \right\} \right.$$}

$$\times e^{-i\frac{P^2}{\alpha\omega_\perp}(x_* - y_*)} | y_\perp \rangle.$$

(E.1)

In Eq. (E.1), we observe that the coordinate $x_*$, which is the end point of the particle’s propagation, is now in the path ordered exponential. This means that both, particle and external field end at $x_*$. We can now repeat the same steps we performed in the previous
section and arrive at

\[ \langle x | \frac{i}{P^2 + i\epsilon} | y \rangle = \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)} \]

\[ \times \langle x_y \rangle \left\{ \left[ x_y, y_y \right] + \frac{ig}{2\alpha} \left[ x_y, y_y \frac{2}{s}(x_s - y_s) \right] \left\{ \{ P_i, A^i(y_s) \} - gA_i(y_s)A^i(y_s) \right\} \right. \]

\[ + \int_{y_s}^{x_s} \frac{d^2 \omega}{s} \left[ \left\{ P_i, [x_y, \omega_s] \right\} - gA_i(y_s)A^i(y_s) \right] F^i_{\omega_s} \omega_s [\omega_s, y_s] \]

\[ \left. + g \int_{\omega_s}^{x_s} \frac{d^2 \omega}{s} \left[ \left\{ [x_y, \omega_s], F^i_{\omega_s} \right\} [\omega_s, y_s] F^i_{\omega_s} \right] \right\} \}

\times e^{-i\frac{\alpha^2}{\alpha^2} (x_s - y_s) | y_s \rangle + O(\lambda^{-2}). \] (E.2)

We will now show that, since the particle ends its propagation inside the external field at point \( x_s \), the non gauge-invariant terms, the transverse field \( A_i \), will be only at point \( y_s \). To this end we use

\[ \int_{x_s}^{\infty} d^2 \omega_s \left\{ P_i, [x_y, \omega_s] \right\} \]

\[ = \int_{y_s}^{x_s} \frac{d^2 \omega}{s} \left[ [x_y, \omega_s] \frac{2}{s}(x_s - x_s)(iD^i F^i_{\omega_s})[\omega_s, y_s] \right] + 2[x_y, \omega_s] \frac{2}{s}(x_s - x_s) F^i_{\omega_s} \omega_s [\omega_s, y_s] \]

\[ - g \int_{\omega_s}^{x_s} \frac{d^2 \omega}{s} \int_{y_s}^{x_s} \frac{d^2 \omega}{s} \left[ [x_y, \omega_s], F^i_{\omega_s} \right] [\omega_s, y_s] \frac{2}{s}(x_s - x_s) F^i_{\omega_s} \omega_s [\omega_s, y_s] \]

\[ + \int_{x_s}^{\infty} d^2 \omega_s \left\{ P_i, [x_y, \omega_s] \right\} \]

\[ \cdot \left\{ [x_s, \omega_s], F^i_{\omega_s} \right\} [\omega_s, y_s] F^i_{\omega_s} \omega_s [\omega_s, y_s] \] (E.3)

To arrive at Eq. (E.3) we have pushed the operator \( P_i \) to the right of the gauge link up to point \( y_s \) which is the point outside the range in which \( F^i_{\mu
u} \neq 0 \). Hence, result (E.2) can be written as

\[ \langle x | \frac{i}{P^2 + i\epsilon} | y \rangle = \left[ \int_{0}^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_s - y_s) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_s - x_s) \right] e^{-i\alpha(x_s - y_s)} \]

\[ \times \langle x_y \rangle \left\{ \left[ x_y, y_y \right] + \frac{ig}{2\alpha} \left[ x_y, y_y \frac{2}{s}(x_s - y_s) \right] \left\{ \{ P_i, A^i(y_s) \} - gA_i(y_s)A^i(y_s) \right\} \right. \]

\[ + \int_{y_s}^{x_s} \frac{d^2 \omega}{s} \left[ [x_y, \omega_s] \frac{2}{s}(x_s - x_s)(iD^i F^i_{\omega_s})[\omega_s, y_s] \right] \]

\[ - 2g \int_{\omega_s}^{x_s} \frac{d^2 \omega}{s} \int_{y_s}^{x_s} \frac{d^2 \omega}{s} \left[ [x_y, \omega_s], F^i_{\omega_s} \right] [\omega_s, y_s] F^i_{\omega_s} \omega_s [\omega_s, y_s] \]

\[ + 2[x_y, \omega_s] \frac{2}{s}(x_s - x_s) F^i_{\omega_s} [\omega_s, y_s] F^i_{\omega_s} \omega_s [\omega_s, y_s] \}

\[ \left. + \left\{ P_i, [x_s, \omega_s] \right\} [\omega_s, y_s] F^i_{\omega_s} \omega_s [\omega_s, y_s] \right\} \}

\times e^{-i\frac{\alpha^2}{\alpha^2} (x_s - y_s) | y_s \rangle + O(\lambda^{-2}). \] (E.4)

All non gauge-invariant terms in (E.4) are now only to the right of the gauge links, that is to the point \( y_s \). This is opposite to what we have in Eq. (3.7) which has the transverse field \( A^i \) on both edges of the gauge link (i.e. at points \( x_s \) and \( y_s \)). In the gauge-rotated field \( A^\Omega \), we can set such non gauge-invariant terms to zero.
In a similar way we can expand with respect to \(y_\perp + \frac{2}{s}p_1y_\perp\) and get
\[
\langle x|\frac{i}{\not P + i\epsilon}|y\rangle = \left[\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp)\right] e^{-i\alpha(x_\perp - y_\perp)}
\times \langle x_\perp|e^{-i\frac{\epsilon^2}{\alpha}(x_\perp - y_\perp)}
\times \left\{[x_\perp, y_\perp] + \frac{ig}{2\alpha} \left(\{P, A_\perp(x_\perp)\} - gA_i(x_\perp)A_\perp(x_\perp)\right)[x_\perp, y_\perp] \frac{2}{s}(x_\perp - y_\perp)
+ \int_{y_\perp}^{x_\perp} \frac{d\omega_\perp}{s} \left(- [x_\perp, \omega_\perp] \frac{2}{s}(\omega_\perp - y_\perp)(iD_\perp F_\perp(\omega_\perp))[\omega_\perp, y_\perp]
+ 2g \int_{\omega_\perp}^{x_\perp} \frac{d\omega_\perp}{s} \omega_\perp [x_\perp, \omega_\perp] F_\perp(\omega_\perp)[\omega_\perp, \omega_\perp] \frac{2}{s}(\omega_\perp - y_\perp) F_\perp(\omega_\perp)[\omega_\perp, y_\perp]
+ 2P_\perp [x_\perp, \omega_\perp] \frac{2}{s}(\omega_\perp - y_\perp) F_\perp(\omega_\perp)[\omega_\perp, y_\perp]\right]\right\}|y_\perp\rangle + O(\lambda^{-2}). \tag{E.5}
\]

In Eq. (E.5) we have the non-gauge-invariant terms only at point \(x_\perp\).

### E.2 Quark propagator

In this section we derive the sub-eikonal corrections to the quark propagator with one end-point inside the external field. To this end we need to use result (E.4). Let us suppose that \(x_\perp + \frac{2}{s}p_1x_\perp\) is the end-point of the quark propagator in the external field as shown in Fig. 4. It is convenient to start from the following expression of the quark propagator
\[
\langle x|\frac{i}{\not P + i\epsilon}|y\rangle = \langle x|\frac{i}{\not P + 2\alpha g A_\perp + g B + i \frac{2}{s}g F_\perp \not P y + i\epsilon}|y\rangle. \tag{E.6}
\]

and we need to include the point \(x_\perp\) in the path-ordered exponential. So, we have
\[
\langle x|\frac{i}{\not P + i\epsilon}|y\rangle = \left[\int_0^{+\infty} \frac{d\alpha}{2\alpha} \theta(x_\perp - y_\perp) - \int_{-\infty}^0 \frac{d\alpha}{2\alpha} \theta(y_\perp - x_\perp)\right] e^{-i\alpha(x_\perp - y_\perp)} \langle x_\perp|e^{-i\frac{\epsilon^2}{\alpha}x_\perp}
\times \text{Pexp} \left\{ig \int_{y_\perp}^{x_\perp} \frac{d\omega_\perp}{s} \frac{2}{s} \omega_\perp \left[A_\perp(\omega_\perp) + B(\omega_\perp) \frac{2}{2\alpha} + \frac{i}{2\alpha} s F_\perp(\omega_\perp) \not P y \right]\right\}
\times e^{-i\frac{\epsilon^2}{\alpha}x_\perp} \langle y_\perp|\left(-i \not \phi + \frac{2}{s} A_\perp(y_\perp, y_\perp) \not p_2 + g A_\perp(y_\perp, y_\perp)\right)\right]. \tag{E.7}
\]

We also need
\[
\langle y_\perp|e^{\frac{i\not p_2}{\alpha s} (y_\perp - x_\perp)}|y_\perp\rangle e^{-i\alpha(x_\perp - y_\perp)} \left(i \not \phi + \frac{2}{s} \not p_2 A_\perp y_\perp + A(y_\perp)\right)\]
\[
= \langle y_\perp|\left(\frac{1}{\alpha s} \not p_2 \not p_2 y_\perp - \frac{2}{s} \not p_2 D_\perp y_\perp - \frac{i}{\alpha s} [x_\perp - y_\perp] \frac{2}{s} \not p_2 A_\perp y_\perp \right) + A(y_\perp)\right)\]
\times e^{\frac{i\not p_2}{\alpha s} (y_\perp - x_\perp)}|y_\perp\rangle e^{-i\alpha(x_\perp - y_\perp)}. \tag{E.8}
\]
At this point the procedure we have to adopt is the same as the one performed in the previous section. Therefore, we arrive at

$$\langle x | \frac{i}{\hat{P}} + i|\epsilon \rangle y = \left[ \int_{-\infty}^{\infty} \frac{d\alpha}{2\alpha} \theta(x_\mu - y_\mu) - \int_{-\infty}^{0} \frac{d\alpha}{2\alpha} \theta(y_\mu - x_\mu) \right] e^{-i\alpha(x_\mu - y_\mu)}$$

$$\times \left\{ \left[ x_\mu, y_\mu \right]  \hat{p} - g \int_{y_\mu}^{x_\mu} \frac{2}{s} \omega_s \gamma^i [x_\mu, \omega_s] F_{i\mu}(\omega_s, y_\mu) \right\} \frac{1}{2\alpha} \hat{p} \hat{p}$$

$$+ \frac{ig}{2\alpha} \int_{y_\mu}^{x_\mu} \frac{2}{s} \omega_s \left( \left[ x_\mu, \omega_s \right] \right) \frac{1}{2} \sigma^{ij} F_{ij} + \frac{4}{s^2} F_{i\mu} \sigma_{\mu} \left[ \omega_s, y_\mu \right]$$

$$- 2g \int_{\omega_s}^{x_\mu} \frac{2}{s} \omega_s \left( \left[ x_\mu, \omega_s \right] \right) 2 \left( \omega_s - x_\mu \right) F_{i\mu}(\omega_s, y_\mu) P^i \right\} \frac{1}{2\alpha} \hat{p} \hat{p}$$

$$+ \left( 2 \left( \omega_s - x_\mu \right) F_{i\mu}(\omega_s, y_\mu) P^i \right) \frac{1}{2\alpha} \hat{p} \hat{p}$$

$$+ \frac{ig}{4\alpha^2} \int_{y_\mu}^{x_\mu} \frac{2}{s} \omega_s \left( i \bar{D}_\mu [x_\mu, \omega_s] \right) B_1 (i \bar{D}_\mu [\omega_s, y_\mu])$$

$$+ \frac{ig}{4\alpha^2} \int_{y_\mu}^{x_\mu} \frac{2}{s} \omega_s \left( \bar{D}_\mu [x_\mu, \omega_s] \right) B_1 [\omega_s, y_\mu] + \frac{ig}{4\alpha^2} \int_{y_\mu}^{x_\mu} \frac{2}{s} \omega_s \left( \bar{D}_\mu [\omega_s, y_\mu] \right)$$

$$\times \left( - 2 \left( \omega_s - x_\mu \right) \left( i \bar{D}_\mu \gamma^i F_{j\mu} \right) \left[ \omega_s, y_\mu \right] P^i - 2 \left( \omega_s - x_\mu \right) \left( i \bar{D}_\mu \gamma^i F_{j\mu} \right) \left( i \bar{D}_\mu [\omega_s, y_\mu] \right) \right)$$

$$\times e^{i \alpha/y_\mu}(y_\mu - x_\mu)|y_\mu \rangle. \tag{E.9}$$

Equation (E.9) is the final result for the quark propagator with sub-eikonal corrections with point $x_\mu^0 + 2z_\mu p_\mu^0$ in the external field. Note that, in (E.9), the non gauge-invariant terms are those with gauge field $A_\mu$ only at point $y_\mu$ which, being outside the shock-wave, can be set to zero (see Fig. (4)). In Eq. (E.9) the action of the covariant derivative $\bar{D}_\mu$ can be performed using its definition (A.3).

F Gluon propagator in the background-Feynman gauge

Let us consider gluon propagator in the background-Feynman gauge $D_\mu A_\mu = 0$ where again $D_\mu = \partial_\mu - ig A_\mu$. The superscript “cl” will be omitted again from now on. The propagator in Schringer formalism can be written as

$$i \left\langle T \left\{ A_\mu(x) A_{\nu}^\dagger(y) \right\} \right\rangle = \langle x | \frac{1}{P^2 - 2igF + i\epsilon} | y \rangle_{\mu\nu} \tag{F.1}$$

where $\hat{P}^2 = \hat{p}^2 + \frac{2}{s} \alpha \hat{A}_\mu + g \left( \hat{p}_\mu, \hat{A}_\mu \right) + \frac{2}{s} g \left( \hat{P}_\mu, \hat{A}_\mu \right) - g^2 \hat{A}_\mu^2 \hat{A}_\mu$. We will omit the symbol $\hat{\cdot}$ from the operators, for simplicity. We define again the operator $O \equiv \left\{ \hat{p}_\mu, \hat{A}_\mu \right\} + \frac{2}{s} \left( \hat{P}_\mu, \hat{A}_\mu \right) - g \hat{A}_\mu^2 \hat{A}_\mu$ and get rid of the term $\left\{ \hat{P}_\mu, \hat{A}_\mu \right\}$ as we did in section 3.
Let us expand Eq. (F.1) in $F_{\mu
u}$ up to $F^3$ terms, which are the ones relevant to get the $\lambda^{-1}$ corrections (we will omit the $+i\epsilon$ prescription for each $\frac{1}{p^2}$ factor)

\[
\left(\frac{1}{p^2 + 2igF + i\epsilon}\right)_{\mu\nu} = \frac{g_{\mu\nu}}{p^2} - 2ig\frac{1}{p^2}F_{\mu\nu}\frac{1}{p^2} - 4g^2\frac{1}{p^2}F_{\mu\nu}\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2} + 8ig^3\frac{1}{p^2}F_{\mu\nu}\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\tau\eta}\frac{1}{p^2} + \ldots
\]

\[
= \frac{g_{\mu\nu}}{p^2} - \frac{4ig}{s}(p_{\nu\mu}g_{\rho\sigma} - p_{\mu\rho}g_{\nu\sigma})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2} + \frac{16g^2}{s^2}(p_{\nu\mu}p_{\rho\sigma} - p_{\mu\rho}p_{\nu\sigma})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\nu\mu}\frac{1}{p^2} + \ldots
\]

\[
+ \frac{8g^2}{s^2}(p_{\mu\nu}g_{\rho\sigma} - p_{\rho\sigma}g_{\mu\nu})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2} + \frac{16g^2}{s^2}(p_{\nu\mu}g_{\rho\sigma} - p_{\mu\rho}g_{\nu\sigma})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\nu\mu}\frac{1}{p^2} + \ldots
\]

\[
- \frac{64g^3}{s^3}(p_{\mu\nu}p_{\rho\sigma} - p_{\rho\sigma}p_{\mu\nu})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\nu\mu}\frac{1}{p^2} + \frac{64g^3}{s^3}(p_{\nu\mu}p_{\rho\sigma} - p_{\rho\sigma}p_{\nu\mu})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\nu\mu}\frac{1}{p^2} + \ldots
\]

\[
- \frac{32g^3}{s^3}(p_{\mu\nu}p_{\rho\sigma} - p_{\rho\sigma}p_{\mu\nu})\frac{1}{p^2}F_{\rho\sigma}\frac{1}{p^2}F_{\nu\mu}\frac{1}{p^2} \right) + O(\lambda^{-2}).
\]

In Eq. (F.2) the terms in the square bracket are sub-eikonal terms so, for those terms, for each of the $\frac{1}{p^2}$ factors, we need the leading-eikonal scalar propagator. The terms outside the square bracket, instead, contain both eikonal and sub-eikonal corrections so, for those terms, for each of the $\frac{1}{p^2}$, we need the scalar propagator with up to sub-eikonal corrections, Eq. (3.7), in the adjoint representation. Therefore, neglecting the terms with fields at the edges of the gauge link, that is, at point $x_\ast$ and $y_\ast$, we arrive, after some algebra,

\[
\langle A^a_\mu(x)A^b_\nu(y)\rangle = \left[ -\int_{-\infty}^{+\infty} d\alpha \frac{e^{-i\alpha|x_\ast - y_\ast|}}{2\kappa} \right] e^{-i\alpha(x_\ast - y_\ast)}
\]

\[
\times \langle x_\perp \rangle e^{-i\frac{2\pi}{N}x_\perp} \left\{ g_{\mu\nu}[x_\ast, y_\ast] - \frac{2g}{\alpha s} \int_{y_\ast}^{x_\ast} d\omega \left[ (p_{\mu\nu} - p_{\nu\mu})[x_\ast, \omega_\ast]F_{\mu\nu}[\omega_\ast, y_\ast] \right. \]

\[
+ \frac{2g}{\alpha s}(p_{\mu\nu}p_{\rho\sigma} - p_{\rho\sigma}p_{\mu\nu}) \int_{\omega_\ast}^{x_\ast} d\omega_\prime \left[ x_\ast, \omega_\prime \right] F_{\mu\nu}[\omega_\prime, \omega_\ast]F_{\mu\nu}[\omega_\ast, y_\ast] \]

\[
+ \mathcal{B}_{1\mu\nu}(x_\ast, y_\ast; p_\perp) + \mathcal{B}_{2\mu\nu}(x_\ast, y_\ast; p_\perp) + \mathcal{B}_{3\mu\nu}(x_\ast, y_\ast; p_\perp) + \mathcal{B}_{4\mu\nu}(x_\ast, y_\ast; p_\perp) + O(\lambda^{-2}) \right) e^{i\frac{2\pi}{N}y_\perp} \langle y_\perp \rangle^{ab}
\]

(F.3)

where we defined

\[
\mathcal{B}_{1\mu\nu}(x_\ast, y_\ast; p_\perp) = g_{\mu\nu} \frac{i\kappa}{2\alpha} \left[ \int_{y_\ast}^{x_\ast} d\omega \left( \frac{1}{2} [p^\prime, \omega_\ast] \frac{2}{s} \omega_\ast F_{\mu\nu}[\omega_\ast, \omega_\ast] [x_\ast, \omega_\ast] \right) \right]
\]

\[
+ g \int_{\omega_\ast}^{x_\ast} d\omega \omega_\ast \left( \frac{2}{s} (\omega_\ast - \omega_\prime) [x_\ast, \omega_\ast] F_{\mu\nu}[\omega_\ast, \omega_\ast] F_{\mu\nu}[\omega_\ast, y_\ast] \right)
\]

(F.4)
\[ \mathfrak{M}_{2\mu \nu}(x_s, y_s; p_{\perp}) = \frac{1}{\alpha} \left( \frac{2}{s} p_{2\mu} \delta^\mu_\nu - \frac{2}{s} p_{2\mu} \delta^\mu_\nu \right) \frac{i g}{2 \alpha} \int \frac{d^2 z_{1s}}{s} \left\{ \frac{2}{s} z_{1s} \{ p^i, [x_s, z_{1s}] \} \right\} \\
+ \frac{2}{s} (\omega_s - z_{1s}) [x_s, \omega_s] F^i \star \{ [\omega_s, z_{1s}] \} F_j \star \{ z_{1s}, y_s \} \\
+ \frac{2}{s} (\omega_s - z_{1s}) [x_s, \omega_s] F^i \star \{ [\omega_s, z_{1s}] \} F_j \star \{ z_{1s}, y_s \} \\
+ \frac{2}{s} (\omega_s - z_{1s}) [x_s, \omega_s] F^i \star \{ [\omega_s, z_{1s}] \} F_j \star \{ z_{1s}, y_s \} \\
+ \frac{2}{s} (\omega_s - z_{1s}) [x_s, \omega_s] F^i \star \{ [\omega_s, z_{1s}] \} F_j \star \{ z_{1s}, y_s \} \right\} \\
(F.5) \]
\[ \mathfrak{B}_{3\mu
u}(x_s, y_s; p_\perp) = \frac{4g p_\mu p_\nu}{\alpha^2 s^2} \int\frac{d^2 z_1}{s} \int\frac{d^2 z_2}{s} \left\{ \frac{i g}{2\alpha} \int\frac{d^2 z_3}{s} \right. \\
\times \left[ \frac{2}{s} \left\{ p^i, [x_s, z_1] iD_i F^j \right\} [z_1, y_s] F_j [z_2, y_s] \right] \\
+ \frac{2}{s} \left\{ p^i, [x_s, z_1] F^j \right\} [z_1, z_2] F_j [z_2, y_s] \\
+ \frac{2}{s} \left( z_1 - z_2 \right) [x_s, z_1] iD_i F^j \right\} [z_1, z_2] iD_i F_j [z_2, y_s] \\
- g \int\frac{d^2 z_3}{s} \left\{ \left[ p^i, [x_s, z_1] F^j \right\} [z_1, z_2] F_j [z_2, y_s] \right\} \\
+ \left\{ p^i, [x_s, z_1] F^j \right\} [z_1, z_2] F_j [z_2, y_s] \right\} \\
+ \frac{2}{s} \left( \omega_s - \omega_2 \right) [x_s, \omega_s] F^j \right\} [\omega_s, z_1] F^j \right\} [z_1, z_2] iD_i F_j [z_2, y_s] \\
+ \frac{2}{s} \left( z_1 - z_2 \right) [x_s, \omega_s] F^j \right\} [\omega_s, z_1] F^j \right\} [z_1, z_2] iD_i F_j [z_2, y_s] \\
+ \frac{2}{s} \left( z_1 - z_2 \right) [x_s, \omega_s] F^j \right\} [\omega_s, z_1] iD_i F_j [z_1, z_2] F^i \right\} [z_2, y_s] \\
+ \frac{2}{s} \left( \omega_s - \omega_2 \right) [x_s, \omega_s] iD_i F^j \right\} [\omega_s, z_1] F_j \right\} [z_1, z_2] F^i \right\} [z_2, y_s] \\
+ \frac{2}{s} \left( \omega_s - \omega_1 \right) [x_s, \omega_s] iD_i F^j \right\} [\omega_s, z_1] F_i \right\} [z_1, z_2] F_j \right\} [z_2, y_s] \\
+ \frac{2}{s} \left( \omega_s - \omega_1 \right) [x_s, \omega_s] iD_i F^j \right\} [\omega_s, z_1] \right\} [z_1, z_2] F_j \right\} [z_2, y_s] \\
+ \frac{2}{s} \left( z_2 - z_2 + \omega_s - \omega'_s \right) [x_s, \omega'_s] F^i \right\} [\omega'_s, \omega_s] F_i \right\} [z_1, z_2] F_j [z_2, y_s] \\
+ \frac{2}{s} \left( z_2 + z_1 - \omega_s - \omega'_s \right) [x_s, \omega'_s] F^i \right\} [\omega'_s, \omega_s] F_i \right\} [z_1, z_2] F_j [z_2, y_s] \\
\left\{ \left[ \omega_s, z_1 \right] F^j \right\} [z_1, z_2] F_j [z_2, y_s] \right\} \right\} \right\} \right\} (F.6) \]
\[ m_{4\mu}(x, y, p_\perp) = \int_{\frac{x s}{S}}^{-\frac{2}{S}} d\omega_s \left[ -\frac{g^2}{\alpha s^2} \delta^\mu_\nu [x, \omega_s] F_{ij}[\omega_s, y_s] - \frac{4g}{\alpha s^2} (p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu}) [x, \omega_s] F_\bullet[\omega_s, y_s] \right. \\
\left. - \frac{2g^2}{\alpha^2 s} p_{2\mu} \delta^\mu_\nu \int_{\omega_s}^{x_s} \frac{2}{S} d\omega_s' [x, \omega_s'] F_{ij}[\omega_s', \omega_s] F_{ij}[\omega_s', y_s] \right. \\
\left. - \frac{2g^2}{\alpha^2 s} p_{2\mu} \delta^\mu_\nu \int_{\omega_s}^{x_s} \frac{2}{S} d\omega_s' [x, \omega_s'] F_{j\nu}[\omega_s', \omega_s] F_{i\bullet}[\omega_s', y_s] \right. \\
\left. - \frac{4g^2}{\alpha^2 s^2} p_{2\mu} \delta^\mu_\nu \int_{\omega_s}^{x_s} \frac{2}{S} d\omega_s' [x, \omega_s'] F_{ji}[\omega_s', \omega_s] F_{n\bullet}[\omega_s', y_s] \right. \\
\left. + \frac{4g^3 p_{2\mu} p_{2\nu}}{\alpha^3 s^4} \int_{\omega_s}^{x_s} \frac{2}{S} d\omega_s' \int_{\omega_{s'}}^{x_s} \frac{2}{S} d\omega_s'' \left( \frac{2}{s} [x, \omega_s'' F_{ij}[\omega_s'' F_{i\bullet}[\omega_s'', \omega_s] F_{\bullet}[\omega_s'', \omega_s] F_{ij}[\omega_s'', y_s] \right. \\
\left. + [x, \omega_s'' F_{ij}[\omega_s'', \omega_s] F_{i\bullet}[\omega_s'', \omega_s] F_{ij}[\omega_s'', y_s] \right) \right] \quad (F.7) \]

Equation (F.3) is the final result of the gluon propagator with sub-eikonal corrections in the background-Feynman gauge. The operators \( m_{1\mu}, m_{2\mu}, m_{3\mu}, \) and \( m_{4\mu} \) are the sub eikonal corrections to the gluon propagator.

References

[1] I. Balitsky, “High-Energy QCD and Wilson Lines”, In Shifman, M. (ed.): At the frontier of particle physics, vol. 2a, p. 1237-1342 (World Scientific, Singapore, 2001) [hep-ph/0101042].

[2] I. Balitsky, Nucl. Phys. B 463 (1996) 99 doi:10.1016/0550-3213(95)00638-9 [hep-ph/9509384].

[3] J. Jalilian-Marian, A. Kovner and H. Weigert, The Wilson renormalization group for low x physics: Gluon evolution at finite parton density, Phys. Rev. D59 (1998) 014015, [hep-ph/9709432].

[4] J. Jalilian-Marian, A. Kovner, A. Leonidov and H. Weigert, The Wilson renormalization group for low x physics: Towards the high density regime, Phys. Rev. D59 (1998) 014014, [hep-ph/9706377].

[5] E. Iancu, A. Leonidov and L. D. McLerran, The renormalization group equation for the color glass condensate, Phys. Lett. B510 (2001) 133–144.

[6] E. Iancu, A. Leonidov and L. D. McLerran, Nonlinear gluon evolution in the color glass condensate. I, Nucl. Phys. A692 (2001) 583–645, [hep-ph/0011241].

[7] Y. V. Kovchegov, Small-x F2 structure function of a nucleus including multiple pomeron exchanges, Phys. Rev. D60 (1999) 034008, [hep-ph/9901281].

[8] Y. V. Kovchegov, Unitarization of the BFKL pomeron on a nucleus, Phys. Rev. D61 (2000) 074018, [hep-ph/9905214].

[9] E. A. Kuraev, L. N. Lipatov and V. S. Fadin, The Pomeronchuk singularity in non-Abelian gauge theories, Sov. Phys. JETP 45 (1977) 199–204.
[10] I. Balitsky and L. Lipatov, *The Pomeranchuk Singularity in Quantum Chromodynamics*, Sov. J. Nucl. Phys. **28** (1978) 822–829.

[11] Y. V. Kovchegov and E. Levin, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. **33** (2012).

[12] I. Balitsky and G. A. Chirilli, Phys. Lett. B **687**, 204 (2010) doi:10.1016/j.physletb.2010.02.084 [arXiv:0911.5192 [hep-ph]].

[13] I. Balitsky and G. A. Chirilli, Phys. Rev. D **83** (2011) 031502 doi:10.1103/PhysRevD.83.031502 [arXiv:1009.4729 [hep-ph]].

[14] I. Balitsky and G. A. Chirilli, Phys. Rev. D **87** (2013) no.1, 014013 doi:10.1103/PhysRevD.87.014013 [arXiv:1207.3844 [hep-ph]].

[15] G. A. Chirilli, B. W. Xiao and F. Yuan, Phys. Rev. Lett. **108** (2012) 122301 doi:10.1103/PhysRevLett.108.122301 [arXiv:1112.1061 [hep-ph]].

[16] G. A. Chirilli, B. W. Xiao and F. Yuan, Phys. Rev. D **86** (2012) 054005 doi:10.1103/PhysRevD.86.054005 [arXiv:1203.6139 [hep-ph]].

[17] G. A. Chirilli and Y. V. Kovchegov, JHEP **1405** (2014) 099 Erratum: [JHEP **1508** (2015) 075] doi:10.1007/JHEP05(2014)099, 10.1007/JHEP08(2015)075 [arXiv:1403.3384 [hep-ph]].

[18] D. Boer *et al.*, arXiv:1108.1713 [nucl-th].

[19] A. Accardi *et al.*, Eur. Phys. J. A **52** (2016) no.9, 268 doi:10.1140/epja/i2016-16268-9 [arXiv:1212.1701 [nucl-ex]].

[20] E. C. Aschenauer, R. Sassot and M. Stratmann, Phys. Rev. D **92** (2015) no.9, 094030 doi:10.1103/PhysRevD.92.094030 [arXiv:1509.06489 [hep-ph]].

[21] J. D. Bjorken, J. B. Kogut and D. E. Soper, Phys. Rev. D **3** (1971) 1382. doi:10.1103/PhysRevD.3.1382

[22] G. ’t Hooft, Phys. Lett. B **198** (1987) 61. doi:10.1016/0370-2693(87)90159-6

[23] J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **263** (1986) 37. doi:10.1016/0550-3213(86)90026-X

[24] H. Cheng and T. T. Wu, CAMBRIDGE, USA: MIT-PR. (1987) 285p

[25] O. Nachtmann, Annals Phys. **209** (1991) 436. doi:10.1016/0003-4916(91)90036-8

[26] I. Balitsky and A. Tarasov, JHEP **1510** (2015) 017 doi:10.1007/JHEP10(2015)017 [arXiv:1505.02151 [hep-ph]].

[27] I. Balitsky and A. Tarasov, JHEP **1606** (2016) 164 doi:10.1007/JHEP06(2016)164 [arXiv:1603.06548 [hep-ph]].

[28] Y. V. Kovchegov, D. Pitonyak and M. D. Sievert, JHEP **1601** (2016) 072 Erratum: [JHEP **1610** (2016) 148] doi:10.1007/JHEP01(2016)072, 10.1007/JHEP10(2016)148 [arXiv:1511.06737 [hep-ph]].

[29] Y. V. Kovchegov, D. Pitonyak and M. D. Sievert, Phys. Rev. D **95** (2017) no.1, 014033 doi:10.1103/PhysRevD.95.014033 [arXiv:1610.06197 [hep-ph]].

[30] Y. V. Kovchegov, D. Pitonyak and M. D. Sievert, Phys. Lett. B **772** (2017) 136 doi:10.1016/j.physletb.2017.06.032 [arXiv:1703.05809 [hep-ph]].

[31] Y. V. Kovchegov, D. Pitonyak and M. D. Sievert, JHEP **1710**, 198 (2017) doi:10.1007/JHEP10(2017)198 [arXiv:1706.04236 [nucl-th]].
[32] J. Bartels, B. I. Ermolaev and M. G. Ryskin, Z. Phys. C 70 (1996) 273 [hep-ph/9507271].
[33] J. Bartels, B. I. Ermolaev and M. G. Ryskin, Z. Phys. C 72 (1996) 627
doi:10.1007/s002880050285, 10.1007/BF02909194 [hep-ph/9603204].
[34] V. G. Gorshkov, V. N. Gribov, L. N. Lipatov and G. V. Frolov, Sov. J. Nucl. Phys. 6 (1968)
95 [Yad. Fiz. 6 (1967) 129].
[35] T. Altinoluk, N. Armesto, G. Beuf, M. Martinez and C. A. Salgado, JHEP 1407 (2014) 068
doi:10.1007/JHEP07(2014)068 [arXiv:1404.2219 [hep-ph]].
[36] E. Laenen, G. Stavenga and C. D. White, JHEP 0903 (2009) 054
doi:10.1088/1126-6708/2009/03/054 [arXiv:0811.2067 [hep-ph]].
[37] E. Laenen, L. Magnea and G. Stavenga, Phys. Lett. B 669 (2008) 173
doi:10.1016/j.physletb.2008.09.037 [arXiv:0807.4412 [hep-ph]].
[38] E. Laenen, L. Magnea, G. Stavenga and C. D. White, JHEP 1101 (2011) 141
doi:10.1007/JHEP01(2011)141 [arXiv:1010.1860 [hep-ph]].
[39] G. A. Chirilli, Y. V. Kovchegov and D. E. Wertepny, JHEP 1512 (2015) 138
doi:10.1007/JHEP12(2015)138 [arXiv:1508.07962 [hep-ph]].
[40] I. I. Balitsky and V. M. Braun, Nucl. Phys. B 311, 541 (1989).
doi:10.1016/0550-3213(89)90168-5
[41] V. M. Braun, G. P. Korchemsky and A. N. Manashov, Phys. Lett. B 476 (2000) 455
doi:10.1016/S0370-2693(00)00131-3 [hep-ph/0001130].
[42] V. M. Braun, A. N. Manashov and J. Rohrwild, Nucl. Phys. B 807, 89 (2009)
doi:10.1016/j.nuclphysb.2008.08.012 [arXiv:0806.2531 [hep-ph]].
[43] V. M. Braun, A. N. Manashov and B. Pirnay, Phys. Rev. D 80 (2009) 114002 Erratum:
[Phys. Rev. D 86 (2012) 119902] doi:10.1103/PhysRevD.80.114002,
10.1103/PhysRevD.86.119902 [arXiv:0909.3410 [hep-ph]].
[44] V. M. Braun, Y. Ji and A. N. Manashov, JHEP 1705 (2017) 022
doi:10.1007/JHEP05(2017)022 [arXiv:1703.02446 [hep-ph]].
[45] J.C. Collins, Foundations of Perturbative QCD, Cambridge University Press, Cambridge
U.K. (2011).