Coproducts of distributive lattice-based algebras

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Abstract. This paper presents a systematic study of coproducts. This is carried out principally, but not exclusively, for finitely generated quasivarieties \( A \) that admit a (term) reduct in the variety \( \mathcal{D} \) of bounded distributive lattices. In this setting we present necessary and sufficient conditions on \( A \) for the forgetful functor \( U_A \) from \( A \) to \( \mathcal{D} \) to preserve coproducts. We also investigate the possible behaviours of \( U_A \) as regards coproducts in \( A \) under weaker assumptions. Depending on the properties exhibited by the functor, different procedures are then available for describing these coproducts. We classify a selection of well-known varieties within our scheme, thereby unifying earlier results and obtaining some new ones.

The paper’s methodology draws heavily on duality theory. We use Priestley duality as a tool and our descriptions of coproducts are given in terms of this duality. We also exploit natural duality theory, specifically multisorted piggyback dualities, in our analysis of the behaviour of the forgetful functor into \( \mathcal{D} \). In the opposite direction, we reveal that the type of natural duality that the class \( A \) can possess is governed by properties of coproducts in \( A \) and the way in which the classes \( A \) and \( U_A(A) \) interact.

1. Introduction

Coproducts in varieties of algebras have been extensively studied and a multitude of papers related to this topic can be found in the literature: [35, 13, 14, 8, 32, 21], and more. Often the characterisation of coproducts has been motivated by interest in amalgamation properties, determination of free objects, axiomatisations, or colimits more widely. Whatever the motivation, the analysis of coproducts has generally been variety-specific, relying on tools tailored to particular classes of algebras. A recurring theme, however, is the use of a categorical duality. The main objective of this paper is to give a uniform treatment, based on duality theory, of coproducts in classes of algebras that admit a bounded distributive lattice reduct. Within this setting, we contribute new theoretical results and thereby provide a unified perspective on characterisations (most already known, a few new) of coproducts in particular classes.

Let \( \mathcal{D} \) denote the category whose objects are bounded distributive lattices and whose morphisms are the lattice homomorphisms which preserve the bounds. We will say that a class of algebras \( A \) with language \( \mathcal{L} \) is a \( \mathcal{D} \)-based...
3.6 gives necessary and sufficient conditions for the first giving conditions for each map \( \chi \) for any set \( K \). It is obtained by combining two results, Theorems 3.3 and 3.5, the first giving conditions for each map \( \chi \) to be surjective (property (S)) and the second conditions for each map \( \chi \) to be an embedding (property (E)). Theorem 3.6 then gives two sets of conditions, each of which is equivalent to a (forgetful) functor \( U_A \) from \( A \) into \( \mathcal{D} \). We will concentrate on the problem of determining when \( U_A \) preserves coproducts. We give the precise (categorical) definition of this property in Section 3 (see also [26, Section V.4]). We note here that it implies, but is stronger than, the requirement that \( U_A(\bigcup \{A(\mathfrak{r})\}) \cong \bigcup \mathcal{D} U_A(\mathfrak{r}) \), for each set \( \mathfrak{r} \subseteq A \). Henceforth, we shall omit the subscript from the forgetful functor \( U \) when working with some fixed class \( A \).

We shall, generally but not exclusively, restrict attention to the situation in which \( A \) is a finitely generated quasivariety, that is, there exists a finite set of finite algebras \( \mathfrak{M} \subseteq A \) such that \( A = \text{ISP}(\mathfrak{M}) \). Such a set \( \mathfrak{M} \) exists whenever \( A \) is a \( \mathcal{D} \)-based variety expressible as \( \text{HSP}(\mathfrak{M}) \), where \( \mathfrak{M} \) is a finite set of finite algebras. This is a consequence of Jónsson’s Lemma: we have \( A = \text{ISP}(\mathfrak{M}) \), where \( \mathfrak{M} \subseteq \text{HSP}(\mathfrak{M}) \) (see [5, Corollaries 6.9 and 6.10]). Working with quasivarieties certainly ensures the existence of coproducts for each set of algebras in the class.

The principal result of this paper is Theorem 3.6 (Coproduct Preservation Theorem). Given \( A \) and \( U \) as above we show that, for any set \( \mathfrak{r} \) in \( A \), there is a natural \( \mathcal{D} \)-homomorphism \( \chi_\mathfrak{r} : \bigcup U(\mathfrak{r}) \to U(\bigcup \mathfrak{r}) \). Theorem 3.6 gives necessary and sufficient conditions for \( \chi_\mathfrak{r} \) to be an isomorphism for any set \( \mathfrak{r} \). It is obtained by combining two results, Theorems 3.3 and 3.5, the first giving conditions for each map \( \chi_\mathfrak{r} \) to be surjective (property (S)) and the second conditions for each map \( \chi_\mathfrak{r} \) to be an embedding (property (E)). Theorem 3.6 then gives two sets of conditions, each of which is equivalent to the satisfaction of both (E) and (S). The first set can be viewed as specifying a particular, and very special, form of interaction between \( A \) and the subclass \( U(A) \) of \( \mathcal{D} \). There is required to be an algebra \( M \) such that \( A = \text{ISP}(M) \) and a \( \mathcal{D} \)-homomorphism \( \omega \) from \( U(M) \) into the two-element algebra \( 2 \) in \( \mathcal{D} \) with two properties, which we now introduce. The first is a separation condition, \( (\text{Sep})_{M,\omega} \): for \( a \neq b \) in \( M \), there is \( u \in A(M,M) \) such that \( \omega(u(a)) \neq \omega(u(b)) \). The second property demands that for each algebra \( A \in A \) and each subalgebra \( L \) of \( U(A) \) there exists a subalgebra of \( A \) maximal for the inclusion order among the subalgebras of \( A \) contained in \( L \). Interestingly, this property has a syntactic counterpart, to the effect that an arbitrary term in the language \( L \) of \( A \) is equivalent in \( A \) to a term built from unary terms in the language \( \mathcal{L} \).
and terms in the language of $\mathcal{D}$ (see Theorems 3.3 and 4.1). The second, alternative, set of conditions in Theorem 3.6 is included principally to allow us to demonstrate that whether or not coproducts are preserved is a decidable problem; see the discussion following the theorem.

Theorems 3.3 and 3.5 are of independent interest. They allow us to analyse the behaviour of $\mathcal{U}$ under weaker conditions than the ones presented in the preceding paragraph (see the flowchart in Fig. 8). Therefore even when the functor $\mathcal{U}$ fails to commute with coproducts we may be able to describe coproducts. Table 2 summarises the strategies we have available for doing this.

In Section 2 we set out the duality theory on which we shall rely. In particular we set up, briefly and in a self-contained way, the framework for the Multisorted Piggyback Duality Theorem (stated as Theorem 2.1). We conclude Section 2 with Theorem 2.3. This allows us to relate the dual structures supplied by Theorem 2.1 for the algebras in a finitely generated $\mathcal{D}$-based quasivariety to the Priestley dual spaces of their $\mathcal{D}$-reducts. We then have in place the machinery we need to investigate coproducts via duality theory. Theorem 2.3 is new, though what it tells us about dualities for particular quasivarieties was already known in certain cases. Section 3—the core of the paper—presents the statements and proofs of our main results, as outlined above. Anyone conversant with the piggybacking method [15, 16, 12] will have recognised our first set of conditions for preservation of coproducts as statements relating to piggyback dualities. Indeed, the classification of classes $\mathcal{A}$ according to whether property (S) and/or property (E) holds or fails governs the form a piggyback duality for $\mathcal{A}$ will take. Moreover, via property (E), we are able easily to prove that $\mathcal{A}$ admits free products if and only if $\mathcal{A}$ is generated as a quasivariety by a single finite algebra (this specialises a result known to hold more generally). Details are given in Theorem 3.7 and Theorem 3.8.

In Section 4 we venture beyond the confines of finitely generated $\mathcal{D}$-based quasivarieties, with the aim of revealing how far certain results in Section 3 hold in greater generality. Finally, in Section 5, we apply our results and techniques to particular well-known classes of finitely generated varieties. Primarily our catalogue unifies pre-existing descriptions of coproducts, both as regards the descriptions and as regards the methodology for obtaining them.

A few comments should be made on our assumption that the quasivarieties we consider be finitely generated. This comes into play to guarantee that we can set up the dualities we shall require (see Section 2). In self-justification, we note that many interesting quasivarieties are finitely generated. Moreover, in any class $\mathcal{A}$ of algebras which is locally finite (that is, finitely generated algebras are finite), any quasivariety generated by a finitely generated free object in $\mathcal{A}$ is finitely generated. This simple observation demonstrates that our results have interesting consequences beyond the finitely generated setting (see Section 4).
2. Preliminaries: duality theories

Take, as above, $\mathcal{A}$ to be a $\mathcal{D}$-based and finitely generated quasivariety. In this section we outline the results on which our analysis of coproducts in $\mathcal{A}$ will rest. Underlying our strategy throughout will be duality theory, in two forms. Our main results are obtained by operating with these two forms in tandem, and toggling between them. First we briefly discuss the role played by Priestley duality as a platform on top of which dualities for classes of $\mathcal{D}$-based algebras can be built. For many such classes, this technique gives a valuable, set-based, representation theory. However, although isolated results exist in the literature for particular classes, such representations do not lend themselves well in general to the description of coproducts. Secondly, we venture a little way into the theory of (multisorted) natural dualities as it applies to finitely generated $\mathcal{D}$-based quasivarieties, since such dualities, in common with Priestley duality itself, have good categorical properties. The key theorem on which we shall rely is the Multisorted Piggyback Duality Theorem. We present the bare minimum of the theory necessary to state it (Theorem 2.1). It allows us immediate access to coproducts, via dual structures which are (multisorted) cartesian products. This would be of little assistance were it not possible directly to retrieve the coproduct, or at least the Priestley dual space of its $\mathcal{D}$-reduct, from the dual structure. Theorem 2.3 provides exactly the translation tool we need. (We remark that the usefulness of Theorem 2.3 potentially extends well beyond the applications to coproducts given in this paper.)

Priestley duality for (bounded) distributive lattices establishes a dual equivalence between the category $\mathcal{D}$ and the category $\mathcal{P}$ of Priestley spaces, that is, compact totally order-disconnected spaces with continuous order-preserving maps between such spaces as the morphisms. We shall below assume familiarity with Priestley duality (summaries can be found in, for example, [19, Chapter 11] and [20]) but do indicate here how the duality is set within its rightful categorical framework. We recall that $\mathcal{D} = \text{ISP}(2)$, the class of isomorphic copies of subalgebras of products of $2$; and $\mathcal{P} = \text{ISP}^{+}(2_{\tau})$, the class of isomorphic copies of closed substructures of powers of $2_{\tau}$, where $2_{\tau}$ denotes the 2-element Priestley space $\left\{\begin{array}{c} 0 \\ 1 \end{array}\right\}$, $\leq$, $\tau$, in which $0 < 1$ and $\tau$ is the discrete topology. The duality between $\mathcal{D}$ and $\mathcal{P}$ is set up by the natural hom-functors, $H = \mathcal{D}(\cdot, 2)$ and $K = \mathcal{P}(\cdot, 2_{\tau})$. The accounts in [20] and, in [12, Chapter 5] highlight the highly amenable properties this duality possesses, and why. We note two key facts. Firstly, products in the category $\mathcal{P}$ are concrete (that is, cartesian) products. This implies that $H$ maps coproducts to cartesian products. Secondly, in the terminology of [12], the duality is strong. Consequently, for a $\mathcal{D}$-morphism $f$, the dual map $H(f)$ is surjective if and only if $f$ is injective and that $H(f)$ is an embedding in $\mathcal{P}$ if and only if $f$ is surjective.

Now assume that $\mathcal{A}$ is a variety or quasivariety of $\mathcal{D}$-based algebras and assume that we wish to find a category $\mathcal{Z}$ of structured spaces we can use.
to represent the algebras and morphisms of $\mathcal{A}$ in terms of $\mathcal{Z}$. One way to try to proceed is first to take the class $U(\mathcal{A})$ of $\mathcal{D}$-reducts of members of $\mathcal{A}$ and to equip the associated class $HU(\mathcal{A})$ of Priestley spaces with additional (relational or functional) structure to enable each $KHU(\mathcal{A})$ to be made into an algebra in $\mathcal{A}$ isomorphic to $\mathcal{A}$; this gives the objects of $\mathcal{Z}$. Then an associated subclass of morphisms needs to be identified to serve as the $\mathcal{Z}$-morphisms. When an equivalence between $\mathcal{A}$ and $\mathcal{Z}$ is constructed in this manner we shall say we have a $\mathcal{D}$-$\mathcal{P}$-based duality between $\mathcal{A}$ and $\mathcal{Z}$. Dualities of this type occur widely in the literature, for many different varieties of $\mathcal{D}$-based algebras, not necessarily finitely generated, and have close affinities with the discrete dualities associated with Kripke-style relational semantics for various propositional logics. We recall some preliminary examples to highlight some salient points. The examples generalise Boolean algebras in different ways: we consider a unary operation modelling a non-classical negation and a binary operation modelling intuitionistic implication.

Let $\mathcal{DM}$ be the variety of De Morgan algebras (see for example [1, 13]). The dual category $\mathcal{Z}_{\mathcal{DM}}$, of De Morgan spaces, consists of the Priestley spaces $\mathcal{Z}$ carrying a continuous order-reversing involution, $g: \mathcal{Z} \to \mathcal{Z}$. The lattice $K(\mathcal{Z})$ obtained from a De Morgan space $\mathcal{Z}$ carries a De Morgan negation given by $(\neg a)(z) = 1$ if and only if $a(g(z)) = 0$. The De Morgan space morphisms are the continuous order-preserving maps commuting with $g$. The variety $\mathcal{K}$ of Kleene algebras is a subvariety of $\mathcal{DM}$ and the associated dual category $\mathcal{Z}_{\mathcal{K}}$, of Kleene spaces, consists of those De Morgan spaces $(\mathcal{Z}, \leq, g, T)$ such that for each $z \in \mathcal{Z}$ we have $z \leq g(z)$ and/or $g(z) \leq z$ [14]. The varieties $\mathcal{DM}$ and $\mathcal{K}$, besides being of interest to logicians, have been influential in the development of the theory of coproducts originating in [13, 14], and of the theory we develop below.

For a contrasting example we consider Heyting algebras. This time the dual category consists of the so-called Esakia spaces, namely Priestley spaces $\mathcal{Z}$ with the property that, for each open set $U$, the down-set $\downarrow U$ generated by $U$ is open. With $K(\mathcal{Z})$ identified with the clopen up-sets of $\mathcal{Z}$, the Heyting implication is given by $a \to b = \mathcal{Z} \setminus \downarrow(a \setminus b)$. A morphism of Esakia spaces is a continuous order-preserving map commuting with the $\downarrow$ operator. (See for example [18] for details.)

The very different forms taken by the dualities for De Morgan and Kleene algebras and for Heyting algebras signal that dualities built on top of Priestley duality may be very diverse, and suggest that discovering them is more of an art than a science. (Indeed, we are dealing here with a topic closely akin to correspondence theory, as the term is used in modal logic, and so should not expect a simple, all-embracing, method to be available for setting up dualities for $\mathcal{D}$-based algebras in general.) Already in the examples above we can see a dichotomy emerging. In the case of $\mathcal{DM}$ and $\mathcal{K}$, the additional structure needed to capture negation is given by extra structure on the underlying Priestley spaces, in the guise of the operation $g$, whereas for Heyting algebras,
the implication is uniquely determined by the order relation which is already present. Thus in some cases coproducts in a class $\mathcal{A}$ of $D$-based algebras will be determined by their underlying lattices; in other cases additional work will be needed to describe coproducts completely.

Our second observation is even more important. Given a dual equivalence between a $D$-based category of algebras $\mathcal{A}$ and a $P$-based category $\mathcal{Z}$, we have no guarantee that the cartesian product of $\mathcal{Z}$-objects will be a $\mathcal{Z}$-object (a classic example is provided by $\mathcal{A} = \mathcal{K}$); and even when this problem does not arise the projection maps might not be $\mathcal{Z}$-morphisms. Therefore we contend that $D$-$P$-based dualities may be useful in analysing coproducts, but it cannot always be expected that the product in the dual category will be as simple as a cartesian product.

We now turn our attention to natural duality theory and to multisorted piggyback dualities in particular. We outline, in black box fashion, the framework we need in order to state the Multisorted Piggyback Duality Theorem in the situation in which we employ it. The theory is presented in full generality in [16, 12]. We remark in passing that Kleene algebras supplied the motivation and the trail-blazer example for the development of dualities employing multisorted dual structures [16].

We fix until further notice a quasivariety $\mathcal{A} = \text{ISP}(\mathcal{M})$ of $D$-based algebras, where $\mathcal{M}$ is a finite set of finite algebras. The dual category $\mathcal{X}$ is built using an alter ego $\mathcal{M}^\sim$ for $\mathcal{M}$; this will be a multisorted topological structure. We let the underlying set of $\mathcal{M}^\sim$ be the (disjointified) union $\bigcup \mathcal{M}$ of the sets $M$, for $M \in \mathcal{M}$, and its topology $T$ be the disjoint union topology obtained by equipping each set $M$ with the discrete topology. In addition, $\bigcup \mathcal{M}$ is equipped with a set $R$ of binary relations and a set $G$ of unary maps between the members of $\mathcal{M}$. Here we impose the restriction that the relations in $R$ are taken to be algebraic, meaning that each $r \in R$ is a subalgebra of $M_1 \times M_2$ for some $M_1, M_2 \in \mathcal{M}$ and that each member of $G$ is a homomorphism from $M_1$ to $M_2$, for some $M_1, M_2$.

We must now describe how $\mathcal{M}^\sim$, as above, is used to generate a category $\mathcal{X}$ of $\mathcal{M}$-sorted topological structures. Every element $X$ of $\mathcal{X}$ will be a union $\bigcup_{M \in \mathcal{M}} X_M$, where each sort $X_M$ carries a topology and the union is equipped with the disjoint union topology. Moreover, $X$ will be equipped with a set of relations $R^X$ and operations $G^X$ indexed by the members of $R$ and $G$. Obviously $\mathcal{M}^\sim$ is of this type. A morphism from $X$ to $Y$, where $X, Y \in \mathcal{X}$, is a map which preserves the sorts (that is, it maps $X_M$ into $Y_M$ for each $M \in \mathcal{M}$) and is continuous and structure-preserving. For a non-empty set $S$, the $S$-fold power of $X \in \mathcal{X}$ has underlying set $\bigcup_{M \in \mathcal{M}} (X_M)^S$, with each $(X_M)^S$ equipped with the product topology; the relational structure is lifted pointwise in the obvious manner. The candidate dual category $\mathcal{X} := \text{ISP}^+(\mathcal{M})$ for $\mathcal{A}$ then has as objects the $\mathcal{M}$-sorted topological structures in $\mathcal{X}$ which are isomorphic copies of closed substructures of powers of $\mathcal{M}^\sim$ and the morphisms are as described above; the empty structure is included.
Finally in this preamble we specify the natural hom-functors, \( D : \mathcal{A} \to \mathcal{X} \) and \( E : \mathcal{X} \to \mathcal{A} \) based on \( \mathfrak{M} \) and \( \mathfrak{M} \). We define
\[
D(A) = \bigcup \{ A(A, M) \mid M \in \mathfrak{M} \};
\]
the sorts of \( \mathcal{X} := D(A) \) are the hom-sets \( X_M := A(A, M) \), for \( M \in \mathfrak{M} \), and \( A(A, M) \) is a closed subspace of the power \( M^A \), where \( M \) is equipped with the discrete topology. Each \( r \in R \) is lifted pointwise to a relation \( r^X \), and \( R^X := \{ r^X \mid r \in R \} \), and \( G^X \) is defined likewise. Then \( D(A) \) belongs to \( \mathcal{X} \).

Now consider any \( X \in \mathcal{X} \). A map from \( X \) to \( \mathfrak{M} \) which preserves the sorts can be seen as defining an element of the product
\[
\prod \{ M^X \mid M \in \mathfrak{M} \}
\]
and the \( M \)th factor of this product can be structured pointwise from \( M \) to give an element of \( \mathcal{A} \). Hence we can define
\[
E(X) = X(X, \mathfrak{M}),
\]
with \( E \) regarded as mapping \( \mathcal{X} \) into \( \mathcal{A} \). We define \( D \) and \( E \) to act on morphisms by composition in the obvious way. It is then straightforward to verify that \( D \) and \( E \) are functors setting up a dual adjunction between \( \mathcal{A} \) and \( \mathcal{X} \).

For each \( A \in \mathcal{A} \) we have a **multisorted evaluation map** \( e_A : A \to ED(A) \) given by
\[
e_A(a)(M)(x) = x(a) \quad \text{for } a \in A, \ x \in A(A, M).
\]
The map \( e_A \) can then be shown to be an embedding. We say that the alter ego \( \mathfrak{M} \) of \( \mathfrak{M} \) yields a **multisorted duality** on \( \mathcal{A} \) if, for each \( A \in \mathcal{A} \), the map \( e_A \) is an isomorphism. In these circumstances, any algebra \( A \) in \( \mathcal{A} \), and in particular the coproduct of a family of algebras in \( \mathcal{A} \), is determined up to isomorphism by its dual \( D(A) \), and can be recaptured from its dual as a family of continuous multisorted structure-preserving maps. An important special case concerns free algebras: the dual \( D(F_A(S)) \) of the free algebra in \( \mathcal{A} \) on a set \( S \) of free generators is the structure \( \mathfrak{M}^S \) \([12, \text{Lemma } 2.2.1]\).

(For the purposes of this paper we need only that \( D \) and \( E \) set up a duality between \( \mathcal{A} \) and the subcategory \( D(\mathcal{A}) \) of \( \mathcal{X} = \mathfrak{M}^+ \) \([-\mathfrak{M}] \), and not the stronger requirement that they set up a dual equivalence between \( \mathcal{A} \) and \( \mathcal{X} \). For this reason we do not need to include partial operations in our alter ego \( \mathfrak{M} \).)

The preceding introduction does not tell us how to choose the structure of the alter ego so as to make each evaluation map \( e_A \) an isomorphism. The piggybacking technique supplies an answer. (The term ‘piggyback’ is used because the proof relies on the known dual equivalence between \( \mathcal{D} \) and \( \mathcal{P} \) to set up a suitable alter ego.) The piggybacking theorem, essentially as we give it below, originated in [16, Section 2] and is reproduced as [12, Theorem 7.2.1]. Before stating the theorem we introduce notation to describe the relations we shall use. For \( M_i \in \mathfrak{M} \) and each \( \omega_i \in D(U(M_i), 2) \) \((i = 1, 2)\), we have a bounded sublattice
\[
(\omega_1, \omega_2)^{-1}(\leq) := \{ (a, b) \in M_1 \times M_2 \mid \omega_1(a) \leq \omega_2(b) \}.
\]
of \( M_1 \times M_2 \). It will contain a (possibly empty) set \( R_{\omega_1,\omega_2} \) of algebraic relations which are maximal with respect to the property of being contained in \( (\omega_1,\omega_2)^{-1}(\subseteq) \).

**Theorem 2.1.** (Multisorted Piggyback Duality Theorem, for \( \mathcal{D} \)-based algebras) Let \( A = \mathbb{ISP}(\mathfrak{M}) \), where \( \mathfrak{M} \) is a finite set of finite \( \mathcal{D} \)-based algebras of common type. For each \( M \in \mathfrak{M} \) let \( \Omega_M \) be a (possibly empty) subset of \( \mathcal{D}(U(M),2) \). Let

\[
\mathfrak{M} = \left( \bigcup_{M \in \mathfrak{M}} M; R,G,T \right)
\]

be the topological structure in which

(i) \( T \) is the discrete topology;
(ii) \( R \) is the union, for \( \omega_i \in \Omega_M \) (\( i \in \{1,2\} \)) of the sets \( R_{\omega_1,\omega_2} \), as defined above, and

\[
G = \bigcup \{ A(M_1, M_2) \mid M_1, M_2 \in \mathfrak{M} \}.
\]

Let \( \Omega = \bigcup_{M \in \mathfrak{M}} \Omega_M \) and assume that the sets \( \Omega_M \) have been chosen so that the following separation condition is satisfied:

(Sep)\( \mathfrak{M}, \Omega \): for all \( M \in \mathfrak{M} \), given \( a, b \in M \) with \( a \neq b \), there exist \( M' \in \mathfrak{M} \), \( u \in A(M, M') \) and \( \omega \in \Omega_M \) such that \( \omega(u(a)) \neq \omega(u(b)) \).

Then \( \mathfrak{M} \) yields a multisorted duality on \( A \).

Given \( \mathfrak{M} \), the separation condition (Sep)\( \mathfrak{M}, \Omega \) can always be satisfied by taking \( \Omega_M = \mathcal{D}(U(M),2) \) for each \( M \in \mathfrak{M} \).

The elements of \( \Omega \) are called *carrier maps*. In the separation condition we shall omit set brackets when \( \mathfrak{M} \) contains a single element, and likewise for \( \Omega \).

The term *simple piggybacking* is used when, given \( A \), there exists a finite algebra \( M \) with \( A = \mathbb{ISP}(M) \) and a single carrier map \( \omega \in \mathcal{D}(U(M),2) \) such that (Sep)\( M, \omega \) holds. Examples of simple piggyback dualities are given in [15] and of multisorted piggyback dualities in [16, 12], as well as in Section 5.

Simple piggybacking is not always possible—the variety of Kleene algebras provides an archetypal example; see [16]. When \( |\mathfrak{M}| = 1 \), we shall say that the piggyback duality supplied by Theorem 2.1 is *single-sorted*.

Assume that \( \mathfrak{M} \) and \( \Omega = \bigcup \Omega_M \) are such that \( A = \mathbb{ISP}(\mathfrak{M}) \) and (Sep)\( \mathfrak{M}, \Omega \) holds.

There is an intimate relationship between the elements \( M \in \mathfrak{M} \) and the associated carrier maps as regards the representation of \( A \) as a quasivariety. Whenever \( M \in \mathfrak{M} \) is such that \( \Omega_M = \emptyset \), condition (Sep)\( \mathfrak{M}, \Omega \) implies, on the one hand, that (Sep)\( \mathfrak{M} \setminus \{M\}, \Omega \) holds. On the other hand, for each \( a \neq b \) in \( M \), we can find an algebra \( M' \in \mathfrak{M} \setminus \{M'\} \), and maps \( u \in A(M, M') \) and \( \omega \in \Omega_M \) such that \( \omega(u(a)) \neq \omega(u(b)) \) and hence necessarily \( u(a) \neq u(b) \).

Then \( M \in \mathbb{ISP}(\mathfrak{M} \setminus \{M'\}) \), and this together with \( A = \mathbb{ISP}(\mathfrak{M}) \) imply that \( A = \mathbb{ISP}(\mathfrak{M} \setminus \{M'\}) \). In particular we see that if (Sep)\( \mathfrak{M}, \Omega \) holds, then only those members \( M \) of \( \mathfrak{M} \) for which \( \Omega_M \) is non-empty are needed when \( \mathfrak{M} \) is used to represent \( A \) as a finitely generated quasivariety, and to develop a piggyback duality as in Theorem 2.1. We shall make crucial use of this elementary observation in the proof of Theorem 3.5 below. (In Theorem 2.1
we took, as the default, $G$ to contain all homomorphisms between members of $\mathcal{M}$. Depending on the choice of $\Omega$, a smaller set $G$ may suffice.)

We are now almost ready to achieve our stated goal of relating the Priestley space $\text{HU}(A)$ to the dual space $\text{D}(A)$ of $A \in \mathcal{A}$ under the duality set up as in Theorem 2.1. In the proof of Theorem 2.3 and several later proofs we make use of the Joint Surjectivity Lemma (see [12, Lemma 7.2.2]) which is a key ingredient in the proof of the Multisorted Piggybacking Theorem. For convenience we recall this lemma below. Observe that condition (3) says that every element of the natural dual $\text{HU}(A)$ can be realised as the image of an element in $\text{D}(A)$ under a map of the form $\omega \circ -$. Thus the lemma can be seen as a first step towards relating $\text{HU}(A)$ to the multisorted dual $\text{D}(A)$.

**Lemma 2.2.** (Joint Surjectivity) Let $A = \text{ISP}(\mathcal{M})$ and let $\Omega = \bigcup \Omega_M$, where $\Omega_M$ is a (possibly empty) subset of $\text{D}(U(M), \mathcal{2})$, for each $M \in \mathcal{M}$. Then the following conditions are equivalent:

1. condition (Sep)$_{\mathcal{M}, \Omega}$, as given in Theorem 2.1, is satisfied;
2. for every $A \in \mathcal{A}$ and every $a, b \in A$ with $a \neq b$ there exist $M \in \mathcal{M}$, $\omega \in \Omega_M$ and $x \in A(A, M)$ such that $\omega(x(a)) \neq \omega(x(b))$;
3. for every $A \in \mathcal{A}$ and each $y \in \text{HU}(A)$ there exist $M \in \mathcal{M}$, $\omega \in \Omega_M$ and $x \in A(A, M)$ such that $y = \omega \circ x$.

For the remainder of this section we assume that $\Omega$ and $R$ have been chosen so that the conditions of Theorem 2.1 are satisfied. We establish some additional notation. For a fixed algebra $A \in \text{ISP}(\mathcal{M})$ and $X = \text{D}(A)$, we let

$$Y = \bigcup \{ X_M \times \Omega_M \mid M \in \mathcal{M} \}$$

and equip it with the topology $T_Y$ having as a base of open sets

$$\{ U \times \{ \omega \} \mid U \text{ open in } X_M \text{ and } \omega \in \Omega_M \}$$

and with the binary relation $\preceq \subseteq Y^2$ defined by

$$(x, \omega_1) \preceq (y, \omega_2) \text{ if } (x, y) \in r^X \text{ for some } r \in R_{\omega_1, \omega_2}.$$  

**Theorem 2.3.** For each $A \in \mathcal{A}$, the structure $(Y, \preceq, T_Y)$ defined above has the following properties.

(i) The space $(Y, T_Y)$ is compact and Hausdorff.

(ii) The binary relation $\preceq \subseteq Y^2$ is a pre-order on $Y$.

(iii) Let $\approx = \preceq \cap \succeq$ be the equivalence relation on $Y$ determined by $\preceq$. Then

$$(Y/\approx, \preceq/\approx, T_Y/\approx)$$

is a Priestley space isomorphic to $\text{HU}(A)$, where $T_Y/\approx$ denotes the quotient topology.

**Proof.** We recall that for each for $M \in \mathcal{M}$ the $M$th sort of the multisorted structure $\text{D}(A)$ is $X_M := A(A, M)$. The topology of $Y$ coincides with the
topology of the finite disjoint union of the topological spaces \( X_M \times \Omega_M \), considering \( X_M \) as carrying the induced topology from \((X, T^X)\) and \( \Omega_M \) the discrete topology. Then (i) follows from the fact that \( \Omega \) is finite.

Now consider (ii). For each \( M_1, M_2 \in \mathfrak{M} \), each \( x \in X_{M_1} \) and \( y \in X_{M_2} \), and each \( \omega_i \in \Omega_{M_i} \) (with \( i \in \{1, 2\} \)),

\[
(x, \omega_1) \preceq (y, \omega_2) \iff (x, y) \in r^X \text{ for some } r \in R_{\omega_1, \omega_2}^X
\]

\[
\iff \{(x(a), y(a)) \mid a \in A\} \subseteq r \text{ for some } r \in R_{\omega_1, \omega_2}
\]

\[
\iff (\omega_1 \circ x)(a) \leq (\omega_2 \circ y)(a) \text{ for each } a \in A
\]

\[
\iff \omega_1 \circ x \preceq \omega_2 \circ y \text{ in } \text{HU}(A).
\]

It is straightforward to check that \( \preceq_{Y_A} \) is reflexive and transitive, and therefore a pre-order, which proves (ii).

For (iii) we need to set up the required isomorphism between the quotient structure obtained in (ii) and \( \text{HU}(A) \). Let \( \Phi: Y \to \text{HU}(A) \) be defined by \( \Phi(x, \omega) = \omega \circ x \).

From the characterisation of \( \preceq \) above it follows that

\[
(x, \omega_1) \preceq_{Y_A} (y, \omega_2) \iff \Phi(x, \omega_1) \preceq \Phi(y, \omega_2),
\]

for each \( (x, \omega_1), (y, \omega_2) \in Y \). Thus \( \ker(\Phi) \) equals \( \approx \). The unique map

\[
\Phi': Y/\approx \to \text{HU}(A)
\]

such that \( \Phi = \Phi' \circ \rho \) is order-preserving and order-reversing; here \( \rho: Y \to Y/\approx \) denotes the quotient map \( \rho(x, \omega) = [(x, \omega)]_\approx \); see Fig. 1. It follows that \( \Phi' \) is injective.

\[
\begin{array}{ccc}
(Y, \preceq, T^Y) & \xrightarrow{\Phi} & \text{HU}(A) \\
\rho \downarrow & & \Phi' \downarrow \\
(Y/\approx, \preceq/\approx, T^{Y/\approx}) & & \\
\end{array}
\]

**Figure 1.** The proof of Theorem 2.3

To prove that \( \Phi' \) is also surjective, we appeal to Lemma 2.2. Condition (2) in the lemma holds because \( A = \mathbb{ISP}(\mathfrak{M}) \) and condition (Sep)_{\mathfrak{M}_1, \Omega_1} is assumed to hold. Therefore \( \Phi' \) and \( \Phi \) are surjective maps.

Given \( a \in A \), let \( \pi_a: y \mapsto y(a) \) be the evaluation map from \( \text{HU}(A) \) into \( 2 \). Then

\[
(\pi_a \circ \Phi)^{-1}(\{1\}) = \{(x, \omega) \in Y \mid (\omega \circ x)(a) = 1\}
\]

\[
= \{(x, \omega) \in Y \mid x \in (e_A(M)(a))^{-1}(\omega^{-1}(\{1\})) \text{ and } \omega \in \Omega_M\}
\]

\[
= \bigcup \{e_A(M)(a)^{-1}(\omega^{-1}(\{1\})) \times \{\omega\} \mid M \in \mathfrak{M} \text{ and } \omega \in \Omega_M\}.
\]
Since \( e_A(a) \) is continuous for each \( a \in A \) it follows that \( e_A(a)^{-1}(\omega^{-1}(\{1\})) \) is open in \( X_M \), and therefore \( (\pi_n \circ \Phi)^{-1}(\{1\}) \) is open in \( T^Y \). Similarly, we have that \( (\pi_n \circ \Phi)^{-1}(\{0\}) \) is open, proving that \( \pi_n \circ \Phi \) is continuous for each \( a \in A \).

Since \( \mathbf{HU}(A) \) inherits its topology from \( \mathcal{D}^\omega \), it follows that \( \Phi \) is continuous.

By the definition of the quotient topology, \( \Phi' \) is continuous.

Combining (i) with the fact that the quotient map \( \rho \) is continuous, we obtain that \( (Y/\sim, T^Y/\sim) \) is compact. Since \( \Phi' \) is an injective continuous map from \( (Y/\sim, T^Y/\sim) \) onto the Hausdorff space \( \mathbf{HU}(A) \) and hence that \( \Phi' \) is a homeomorphism. Therefore \( \Phi' \) is an isomorphism of Priestley spaces. \( \square \)

A special case of Theorem 2.3 deserves to be recorded as a corollary. It describes the situation which pertains, in particular, whenever coproducts are preserved, as Theorems 3.6 and 3.8 will show.

**Corollary 2.4.** Assume \( \mathcal{A} = \mathbb{ISP}(M) \) where \( M \) is a finite \( \mathcal{D} \)-based algebra and that \( (\text{Sep})_M,\omega \) holds for some \( \omega \in \mathbf{HU}(M) \), so that \( \mathcal{A} \) has a simple piggyback duality. If \( |R_{\omega,\omega}| = 1 \), then for each \( A \in \mathcal{A} \), the Priestley space \( \mathbf{HU}(A) \) is isomorphic to the ordered space obtained from the natural dual space \( X = \mathcal{D}(A) \) by equipping its underlying set with the partial order relation \( R_{X,\omega} \) and its existing topology.

The corollary can be regarded as asserting that, under the stipulated conditions, the quasivariety \( \mathcal{A} \) has a natural duality which is, in a certain sense, also a \( \mathcal{D} \)-\( \mathcal{P} \)-based duality. In addition, if we have already derived a \( \mathcal{D} \)-\( \mathcal{P} \)-based duality for \( \mathcal{A} \), then we can regard the structure on the Priestley dual spaces which is used to capture any additional operations as living also on the natural dual spaces. Thus there is a very close relationship between a \( \mathcal{D} \)-\( \mathcal{P} \)-based duality for \( \mathcal{A} \) and the piggyback duality mentioned in the corollary.

One additional remark deserves to be made about the special situation covered by Corollary 2.4. In general, some of the sets \( R_{\omega_1,\omega_2} \), for \( \omega_1,\omega_2 \in \Omega \), arising in Theorem 2.3 may be empty (and we shall see instances of this in Section 5). But when Corollary 2.4 applies, then \( R_{\omega,\omega} \) is necessarily non-empty.

We note that in case we have \( \mathcal{A} = \mathbb{ISP}(M) \) but of necessity more than one carrier map, there is a choice as to how to formulate a natural duality for \( \mathcal{A} \). If we base the natural duality on the single algebra \( M \) then the dual spaces will be structured Boolean spaces, with the structure lifted from an alter ego with underlying set \( M \). The construction in Theorem 2.3 then tells us that, for \( A \in \mathcal{A} \) we get the Priestley dual space of the underlying lattice \( U(A) \) by first taking multiple copies of \( \mathcal{D}(A) \). If, instead, we opt to set up the natural duality with multiple copies of \( M \) and a single carrier map associated with each, then we essentially build the first stage of the translation process into the natural duality. Which of these two approaches is to be preferred is largely a matter of taste. In Section 5 we adopt the former approach, noting that the second has been used in the literature in certain cases.
3. Coproducts

In this section we embark on discussion of the central topic of this paper and present our main results. Specifically, we shall describe the structure of coproducts in finitely generated \( \mathcal{D} \)-based classes of algebras in terms of the corresponding coproducts in \( \mathcal{D} \). Background material on coproducts in a categorical context can be found in [26].

Let \( \mathcal{A} \) be a quasivariety of algebras, not yet assumed to be finitely generated. Assume that \( \mathcal{A} \) is \( \mathcal{D} \)-based, and that \( U: \mathcal{A} \to \mathcal{D} \) is the associated forgetful functor. Let \( \mathcal{R} \) be a set of algebras in \( \mathcal{A} \). Since \( \mathcal{A} \) is a quasivariety, the coproduct \( \bigsqcup_{\mathcal{A}} \mathcal{R} \) in \( \mathcal{A} \) always exists. In what follows, we shall omit the subscript in \( \bigsqcup_{\mathcal{A}} \) when the class where we are considering the coproduct is clearly determined by context. Let \( \{ \varepsilon_B: B \to \bigsqcup \mathcal{R} \mid B \in \mathcal{R} \} \) be the universal co-cone that determines the coproduct \( \bigsqcup \mathcal{R} \) up to isomorphism (Fig. 2(a)). Then \( U \) preserves coproducts if and only if, for each set \( \mathcal{R} \subseteq \mathcal{A} \),

\[
\{ U(\varepsilon_A): U(A) \to U(\bigsqcup(\mathcal{R})) \mid A \in \mathcal{R} \}
\]

is a universal co-cone in \( \mathcal{D} \). There always exists a unique map

\[
\chi_\mathcal{R}: \bigsqcup U(\mathcal{R}) \to U(\bigsqcup \mathcal{R})
\]

such that \( \chi_\mathcal{R} \circ \varepsilon_B = U(\varepsilon_B) \) for each \( B \in \mathcal{R} \), where the family of maps \( \{ \varepsilon_B: U(B) \to \bigsqcup U(\mathcal{R}) \mid B \in \mathcal{R} \} \) is the universal co-cone in \( \mathcal{D} \) for \( \bigsqcup U(\mathcal{R}) \) (Fig. 2(b)). Proving that \( U \) preserves coproducts is equivalent to proving that, for each set of algebras \( \mathcal{R} \), the homomorphism \( \chi_\mathcal{R} \) is an isomorphism. This in turn is equivalent to proving that \( H(\chi_\mathcal{R}) \) is an isomorphism (in the category of Priestley spaces) between \( \text{HU}(\bigsqcup \mathcal{R}) \) and \( \text{HU}(\bigsqcup U(\mathcal{R})) \).

Since \( H: \mathcal{D} \to \mathcal{P} \) sends coproducts into cartesian products, we can identify \( H(\bigsqcup U(\mathcal{R})) \) with \( \prod \text{HU}(\mathcal{R}) \) for each set \( \mathcal{R} \subseteq \mathcal{A} \). Under this identification,

\[
\iota_\mathcal{R} = H(\chi_\mathcal{R}): \text{HU}(\bigsqcup \mathcal{R}) \to \prod \text{HU}(\mathcal{R})
\]

is the unique map such that \( \pi_{\text{HU}(B)} \circ \iota_\mathcal{R} = H(\varepsilon_B) \) for each \( B \in \mathcal{R} \), where

\[
\pi_{\text{HU}(B)} = H(\varepsilon_B): \text{HU}(\mathcal{R}) \to \text{HU}(B)
\]

denotes the projection map (Fig. 2(c)). In order to determine when \( \iota_\mathcal{R} \) is an isomorphism of Priestley spaces we shall first describe this map using Theorem 2.3 (see Lemma 3.1).

We fix until further notice a \( \mathcal{D} \)-based quasivariety \( \mathcal{A} \) which is expressible as \( \mathcal{A} = \text{ISP}(\mathcal{M}) \), where \( \mathcal{M} \) is a finite set of finite algebras. Adopting the notation introduced in Theorem 2.3, we have

\[
Y_{\bigsqcup \mathcal{R}} = \bigcup \{ D(\bigsqcup \mathcal{R})_M \times \Omega_M \mid M \in \mathcal{M} \}.
\]

We now use the fact that \( D(\bigsqcup \mathcal{R}) \cong \prod D(\mathcal{R}) = \prod \{ D(B) \mid B \in \mathcal{R} \} \) and that, under this isomorphism, \( D(\varepsilon_B): \prod \{ D(B) \mid B \in \mathcal{R} \} \to D(B) \) is the projection map. Hence for each \( B \in \mathcal{R} \) we have a map \( \xi_B: Y_{\bigsqcup \mathcal{R}} \to Y_B \) defined by

\[
\xi_B(\bar{x}, \omega) = (D(\varepsilon_B)(\bar{x}), \omega) = (x_B, \omega).
\]
Since $D(\epsilon_B)$ preserves the relations in $R$, it follows that, if $(\vec{x}, \omega) \preceq_{\coprod \mathcal{R}} (\vec{y}, \omega')$ then $\xi_B(\vec{x}, \omega) \preceq_B \xi_B(\vec{y}, \omega')$. Moreover $\xi_B$ is continuous for each $B \in \mathcal{R}$. We can then define a map $\Psi_{\coprod \mathcal{R}}: Y_{\coprod \mathcal{R}} \to \prod \text{HU}(\mathcal{R})$ by

$$
(\vec{x}, \omega) \in \prod D(\mathcal{R}) \times \Omega_M \mapsto \vec{y} \in \prod \text{HU}(\mathcal{R}),
$$

where $y_B = \omega \circ x_B = \Phi_B(\xi_B(\vec{x}, \omega))$ for each $B \in \mathcal{R}$ and for each $M \in \mathcal{M}$. It follows that

$$
(\vec{x}_1, \omega_1) \preceq_{\coprod \mathcal{R}} (\vec{x}_2, \omega_2) \implies \Psi_{\coprod \mathcal{R}}(\vec{x}_1, \omega_1) \preceq \Psi_{\coprod \mathcal{R}}(\vec{x}_2, \omega_2).
$$

In fact, if $(\vec{x}_1, \omega_1) \preceq_{\coprod \mathcal{R}} (\vec{x}_2, \omega_2)$, then there exists $r \in R_{\omega_1, \omega_2}$ for which $(\vec{x}_1, \vec{x}_2) \in r$ on $D(\coprod \mathcal{R})$. Therefore $((x_1)_B, (x_2)_B) \in r$ on $D(B)$ for each $B \in \mathcal{R}$. Then, for each $B \in \mathcal{R}$, the maps $\xi_B$ and $\Psi_{\coprod \mathcal{R}}$ are such that the diagram in Fig. 3 commutes.

$$
\begin{array}{ccc}
Y_{\coprod \mathcal{R}} & \xrightarrow{\Psi_{\coprod \mathcal{R}}} & \prod \text{HU}(\mathcal{R}) \\
\xi_B \downarrow & & \downarrow \pi_{\text{HU}(B)} \\
Y_B & \xrightarrow{\Phi_B} & \text{HU}(B)
\end{array}
$$

**Figure 3.** The definition of $\Psi_{\coprod \mathcal{R}}$

By applying Theorem 2.3 to $\coprod \mathcal{R}$ we can obtain the following lemma.

**Lemma 3.1.** For each set of algebras $\mathcal{R} \subseteq \mathcal{A}$, the map

$$
\iota_{\mathcal{R}}: \text{HU}(\coprod \mathcal{R}) \to \prod \text{HU}(\mathcal{R})
$$

satisfies $\iota_{\mathcal{R}} \circ \Phi_{\coprod \mathcal{R}} = \Psi_{\coprod \mathcal{R}}$. That is, the diagram in Fig. 4 commutes.

**Proof.** Let $(\vec{x}, \omega) \in Y_{\coprod \mathcal{R}}$, then $(\vec{x}, \omega) \in (\prod D(\mathcal{R})_M) \times \Omega_M$ for some $M \in \mathcal{M}$. Then, for each $B \in \mathcal{R}$,

$$
(\pi_{\text{HU}(B)}) \circ \iota_{\mathcal{R}} \circ \Phi_{\coprod \mathcal{R}}(\vec{x}, \omega) = (\pi_{\text{HU}(B)} \circ \iota_{\mathcal{R}})(\omega \circ \vec{x}) = \text{HU}(\epsilon_B)(\omega \circ \vec{x})
$$

$$
= (\pi_{\text{HU}(B)} \circ \Psi_{\coprod \mathcal{R}})(\vec{x}, \omega).
$$

□
By Lemma 2.2, \( \{ \omega \circ x \mid x \in D(B) \text{ and } \omega \in \Omega \} = HU(B) \). Hence the assignment
\[
f \mapsto \Lambda_B(f) := \{ \omega \in \Omega \mid f = \omega \circ x \text{ for some } x \in D(B) \}\]
gives a well-defined map \( \Lambda_B \) from \( HU(B) \) into the family of non-empty subsets of \( \Omega \). With this notation the following result follows directly from Theorem 2.3.

**Corollary 3.2.** For every set of algebras \( \mathcal{R} \subseteq \mathcal{A} \),
\[
\iota_{\mathcal{R}}(HU(\prod \mathcal{R})) = \{ \bar{y} \in \prod HU(\mathcal{R}) \mid \bigcap \{ \Lambda_B(y_B) \mid B \in \mathcal{R} \} \neq \emptyset \}.
\]

We now have in place the relationships which we can establish in general about the maps \( \chi_{\mathcal{R}} : \prod U(\mathcal{R}) \to U(\prod \mathcal{R}) \), for a subset \( \mathcal{R} \) of our given quasivariety \( \mathcal{A} \). We progress to an investigation of conditions under which these maps are necessarily isomorphisms. We split the problem into two natural parts. We say that the functor \( U \) satisfies
\begin{itemize}
  \item[(S)] if for each set \( \mathcal{R} \subseteq \mathcal{A} \), the map \( \chi_{\mathcal{R}} \) is surjective;
  \item[(E)] if for each set \( \mathcal{R} \subseteq \mathcal{A} \), the map \( \chi_{\mathcal{R}} \) is injective.
\end{itemize}

In preparation for the following results we establish some conventions. Given a class of algebras \( \mathcal{B} \), for each \( B \in \mathcal{B} \) and \( C \subseteq B \), we write \( \langle C \rangle_B \) to denote the subalgebra of \( B \) generated by \( C \). We shall repeatedly encounter sets of subalgebras of some specified algebra \( \mathcal{A} \). We shall always order such sets by inclusion. Under this assumption, the assertion that \( Y \subseteq S(\mathcal{A}) \) has a top element is equivalent to saying that \( \bigcup Y \) is the universe of a subalgebra of \( \mathcal{A} \).

Our next step is to determine when \( \chi_{\mathcal{R}} \) maps \( \prod U(\mathcal{R}) \) onto \( U(\prod \mathcal{R}) \) or, equivalently, when \( \iota_{\mathcal{R}} \) is an order embedding.

**Theorem 3.3.** Assume that \( \mathcal{A} \) is a finitely generated \( \mathcal{D} \)-based quasivariety. Assume that \( \mathcal{M} \) is such that \( \mathcal{A} = ISP(\mathcal{M}) \) and that \( \Omega \subseteq \bigcup_{M \in \mathcal{M}} HU(M) \) satisfies (Sep)\( \mathcal{M}, \Omega \). Then the following statements are equivalent:
\begin{enumerate}
  \item[(1)] \( U \) satisfies (S), that is, for every set \( \mathcal{R} \) of algebras in \( \mathcal{A} \), the image of \( \chi_{\mathcal{R}} \) is \( U(\prod \mathcal{R}) \);
  \item[(2)] for every finite set \( \mathcal{R} \) of disjoint copies of \( F_A(1) \), the image of \( \chi_{\mathcal{R}} \) is \( U(\prod \mathcal{R}) \);
\end{enumerate}
for every \( n \geq 1 \) and every \( n \)-ary term \( t \) in the language \( L \) of \( \mathcal{A} \) there exist unary terms \( t_1, \ldots, t_n \) in \( L \) and an \( n \)-ary term \( s \) in the language of \( \mathcal{D} \) such that

\[
t^A(a_1, \ldots, a_n) = s^A(t_1^A(a_1), \ldots, t_n^A(a_n)),
\]

for every \( A \in \mathcal{A} \) and every \( a_1, \ldots, a_n \in A \);

(4) for every \( A \in \mathcal{A} \) and every bounded sublattice \( L \) of \( U(A) \), then either \( L \) contains no \( \mathcal{A} \)-subalgebra of \( A \) or \( \{ B \in S(A) \mid B \subseteq L \} \) has a top element;

(5) for every \( \omega_1, \omega_2 \in \Omega \) it is the case that \( |R_{\omega_1, \omega_2}| \leq 1 \);

(6) for every set of algebras \( \mathfrak{R} \subseteq \mathcal{A} \), the map \( t_{\mathfrak{R}} \) is an order embedding.

Proof. The equivalence of (1) and (6) is a standard fact about Priestley duality. We obviously have (1) \( \Rightarrow \) (2).

We first prove that (2) \( \Rightarrow \) (3). Let \( t \) be a \( n \)-ary term in the language \( L \) and let \( \mathfrak{R} \) be a set of \( n \) disjoint copies of \( F_A(1) \). Then \( F_A(n) = \bigsqcup \mathfrak{R} \). Let \( a_1, \ldots, a_n \in F_A(n) \) denote the \( n \) free generators of \( F_A(n) \). Here \( a_i \) will also denote the free generator of the \( i \)-th copy of \( F_A(n) \) in \( \mathfrak{R} \). We let \( a = t_{F_A(n)}(a_1, \ldots, a_n) \). By assumption \( \chi_{\mathfrak{R}} \) maps onto \( F_A(n) \), so that there exists \( b \in \bigsqcup U(\mathfrak{R}) \) such that \( \chi_{\mathfrak{R}}(b) = a \). Since \( \bigsqcup U(\mathfrak{R}) = (\bigcup U(\mathfrak{R}))_{\mathcal{D}} \), there exist \( b_i \) in the \( i \)-th copy of \( F_A(1) \), for each \( i \in \{ 1, \ldots, n \} \) and an \( n \)-ary term \( s \) in the language of \( \mathcal{D} \) such that \( s_{\bigcup U(\mathfrak{R})}(b_1, \ldots, b_n) = b \). Then there exist unary terms \( t_1, \ldots, t_n \) in the language \( L \) such that \( t_i = t_{\mathfrak{R}}(a_i) \). This implies that

\[
t^A(a_1, \ldots, a_n) = s^A(t_1^A(a_1), \ldots, t_n^A(a_n)).
\]

Since \( a_1, \ldots, a_n \) are the free generators of \( F_A(n) \) it follows that the equation \( t(x_1, \ldots, x_n) \approx s(t_1(x_1), \ldots, t_n(x_n)) \) is valid in \( \mathcal{A} \). This proves (3).

We now prove (3) \( \Rightarrow \) (4). Fix \( A \in \mathcal{A} \). Let \( L^o = \{ a \in A \mid \langle a \rangle_A \subseteq L \} \). Certainly \( \bigcup \{ B \in S(A) \mid B \subseteq L \} = L^o \). Now let \( n \geq 1 \), let \( a_1, \ldots, a_n \in L^o \) and let \( t \) be an \( n \)-ary term in the language \( L \). Take \( t_1, \ldots, t_n \) and \( s \) as in the statement of (4). Since \( t^A(a_i) \in \langle a_i \rangle_A \subseteq L \) for each \( i \in \{ 1, \ldots, n \} \) and \( L \) is closed under the lattice operations, we see that \( a \in L \). This proves that \( \langle a_1, \ldots, a_n \rangle_A \subseteq L \) and hence that \( \langle a_1, \ldots, a_n \rangle_A \subseteq L^o \). We conclude that \( L^o \) is the universe of a subalgebra of \( A \) and hence that (4) holds.

To prove (4) \( \Rightarrow \) (5) we only need to observe that for every \( A_1, A_2 \in \mathcal{A} \) and \( \omega_j \in \text{HU}(A_j) \) for \( j = 1, 2 \), the set \( (\omega_1, \omega_2)^{-1}(\leq) \) is a bounded sublattice of \( \text{HU}(A_1) \times \text{HU}(A_2) \).

Now assume that (5) holds and let \( \mathfrak{R} \) be a subset of \( \mathcal{A} \). To prove that (6) holds we appeal to Theorem 2.3, applied with \( A = \bigsqcup \mathfrak{R} \), and also Lemma 3.1. Let \( \langle x_1, \omega_1 \rangle, \langle x_2, \omega_2 \rangle \in Y_{\mathcal{R}} \) be such that \( \Psi_{\mathcal{R}}(x_1, \omega_1) \leq \Psi_{\mathcal{R}}(x_2, \omega_2) \). Then, for each \( B \in \mathfrak{R} \), we have \( \omega_1 \circ (x_1)_B \leq \omega_2 \circ (x_2)_B \). Applying Theorem 2.3 to each \( B \in \mathfrak{R} \), there exists a relation \( r^B \in R_{\omega_1, \omega_2} \) such that \( ((x_1)_B, (x_2)_B) \in r^B \) on \( \text{D}(B) \). By assumption \( |R_{\omega_1, \omega_2}| \leq 1 \), so there is a single relation \( r \) such that \( r^B r \) for all \( B \in \mathfrak{R} \). Therefore \( r^B \) \( \text{D}(\mathfrak{R}) \) and \( \langle x_1, \omega_1 \rangle \leq (x_2, \omega_2) \in \text{D}(\mathfrak{R}) \). Hence (6) holds. \( \square \)
In Theorem 4.1 we will reveal that the equivalence of (1), (2), (3), and (4) in Theorem 3.3 is independent of the fact that \( A \) is finitely generated.

In connection with condition (3) in Theorem 3.3 we should comment on when it happens that an algebra \( A \) is always such that for any \( L \in S(U(A)) \) there is at least one subalgebra of \( A \) contained in \( L \). It is easy to see that this is the case if and only if \( \{0^A, 1^A\} \) is the universe of a subalgebra of \( S(A) \) for each \( A \in \mathcal{A} \), or equivalently if and only if \( F_A(\emptyset) \) has only two elements.

Theorem 3.3 tells us under what conditions the Priestley space of a coproduct is embeddable in the product of the Priestley spaces of the original algebras. When the theorem can be applied, we can combine it with Corollary 3.2 to determine the subspace of the product of Priestley spaces that is isomorphic to the Priestley space of the coproduct.

In applications it is often simple to verify or to refute condition (5) in Theorem 3.3. But we note also that, under restricted conditions on the operations in \( \mathcal{A} \) and the equations these satisfy, the syntactic condition (3) may also be simple to check. An instance is provided by [16, Lemma 3.5]; a slightly more general version, though still restricted to operations which are at most unary, is given by Talukder [36, Lemma 6.1.2]. In both cases direct proofs were given of the existence of a unique element in a set of the form \( R_{\omega_1, \omega_2} \) as part of the work needed to set up certain piggyback dualities. For use later we record here a minor variant of these earlier results. We stress that the scope of our syntactic condition (3) is much wider; in particular we have been able to remove the restriction to operations of arity at most 1.

**Lemma 3.4.** Let \( \mathcal{A} \) be a \( \mathcal{D} \)-based quasivariety such that the members of \( \mathcal{A} \) take the form \( A = (A, \land, \lor, 0, 1, \{f_i \mid i \in I\}) \), where \( I \) is a finite set and each \( f_i \) is a lattice endomorphism or a dual lattice endomorphism of \( U(A) \). Then, for any \( A \in \mathcal{A} \) and any bounded sublattice \( L \) of \( U(A) \), either \( L \) contains no subalgebra of \( A \) or \( \{B \in S(A) \mid B \subseteq L\} \) has a top element. Moreover, if each \( f_i \) preserves the bounds 0 and 1, then \( \{B \in S(A) \mid B \subseteq L\} \) is non-empty and so has a top element.

Our next result, Theorem 3.5, gives necessary and sufficient conditions for \( U \) to satisfy (E), that is, for \( \chi_{\mathcal{A}} \) to be an embedding, or equivalently, for \( \iota_{\mathcal{A}} \) to be surjective, for each \( \mathcal{A} \subseteq \mathcal{A} \). In order to present item (4) in the theorem, we need to recall some definitions and facts about quasivarieties (for details see [24, Chapter 3]). Let \( \mathcal{Q} \) be a quasivariety. For each \( A \in \mathcal{Q} \), let \( \text{Con}_\mathcal{Q}(A) \) denote the set of congruences \( \theta \) on \( A \) for which \( A/\theta \in \mathcal{Q} \); we order \( \text{Con}_\mathcal{Q}(A) \) by inclusion. An algebra \( B \in \mathcal{Q} \) is said to be subdirectly irreducible relative to \( \mathcal{Q} \) if whenever \( B \) is isomorphic to a subdirect product of algebras in \( \mathcal{Q} \), then it is isomorphic to at least one of the components in the product. Let \( \text{Si}(\mathcal{Q}) \) denote the class of subdirectly irreducible algebras relative to \( \mathcal{Q} \). An algebra \( B \) belongs to \( \text{Si}(\mathcal{Q}) \) if and only if \( B \in \mathcal{Q} \) and \( \Delta_B \neq \bigcap \{\text{Con}_\mathcal{Q}(B) \setminus \{\Delta_B\}\} \), where \( \Delta_B = \{ (b, b) \mid b \in B \} \). Moreover, since \( \mathcal{Q} \) is determined by a set of quasi-equations, it follows that, if \( D \) is an up-directed subset of \( \text{Con}_\mathcal{Q}(A) \), then
Let \( D \in \text{Con}_q(A) \). As a consequence, for each \( A \in \Omega \) and each \( a, b \in A \) such that \( a \neq b \), the set \( \{ \theta \in \text{Con}_q(A) \mid (a, b) \notin \theta \} \) has a maximal element. Then every algebra in \( \Omega \) is a subdirect product of algebras in \( \text{Si}(\Omega) \). If \( \Omega = \text{ISP}(\mathfrak{M}) \), where \( \mathfrak{M} \) is a finite set of finite algebras \( \mathfrak{M} \), then \( \text{Si}(\Omega) \subseteq \text{ISP}(\mathfrak{M}) \).

Below we refer to an algebra as being non-trivial if its has at least two elements. We denote by \( \mathcal{A}_{\text{fin}} \) the class of finite algebras in \( \mathcal{A} \).

**Theorem 3.5.** Let \( \mathcal{A} \) be a finitely generated \( \mathcal{D} \)-based quasivariety. Then the following statements are equivalent:

1. \( U \) satisfies (E), that is, for every set \( \mathfrak{R} \) of algebras in \( \mathcal{A} \), the map \( x_\mathfrak{R} \) is injective;
2. there exists a finite set of finite algebras \( \mathfrak{M} \) such that
   - (i) \( \mathcal{A} = \text{ISP}(\mathfrak{M}) \);
   - (ii) for every set \( \mathfrak{R} \) of disjoint copies of algebras in \( \mathfrak{M} \), the map \( x_\mathfrak{R} \) is injective;
3. there exists \( M \in \mathcal{A}_{\text{fin}} \) and \( \omega \in \text{HU}(M) \) such that
   - (i) \( \mathcal{A} = \text{ISP}(M) \);
   - (ii) \( (\text{Sep})_{M,\omega} \) holds;
4. there exists \( M \in \text{Si}(\mathcal{A}) \) and \( \omega \in \text{HU}(M) \) such that
   - (i) \( \mathcal{A} = \text{ISP}(M) \) and
   - (ii) \( (\text{Sep})_{M,\omega} \) holds;
5. for every set of algebras \( \mathfrak{R} \subseteq \mathcal{A} \), the image of the map \( i_\mathfrak{R} \) is \( \prod \text{HU}(\mathfrak{R}) \).

**Proof.** The equivalence of (1) and (5) is a standard fact about Priestley duality and the implication (1) \( \Rightarrow \) (2) holds trivially. Since \( \mathcal{A} \) is finitely generated, each \( M \in \text{Si}(\mathcal{A}) \) is finite. Therefore (4) \( \Rightarrow \) (3). To complete the proof we establish that (2) \( \Rightarrow \) (3) \( \Rightarrow \) (5) and (1) \( \Rightarrow \) (4).

Assume (2). Let \( \mathfrak{R} \) be a set of disjoint copies of non-trivial algebras from \( \mathfrak{M} \) which, for each \( M \in \mathfrak{M} \), contains one copy \( M_\omega \) of \( M \) for each \( \omega \in \text{HU}(M) \). Observe that the assumption that \( M \) is non-trivial ensures (it is actually equivalent to) \( \text{HU}(M) \neq \emptyset \). By hypothesis, \( i_\mathfrak{R} \) maps \( \text{HU}(\prod \mathfrak{R}) \) onto \( \prod \text{HU}(\mathfrak{R}) \). Let \( \tilde{y} \in \prod \text{HU}(\mathfrak{R}) \) be such that the \( M_\omega \)-coordinate of \( \tilde{y} \) is \( \omega \). Then there exists \( (\tilde{x}, \omega_0) \in \text{Y}_\mathfrak{R} \), where \( \text{Y}_\mathfrak{R} \) is as in Theorem 2.3, such that

\[ i_\mathfrak{R}(\Phi_{\text{Y}_\mathfrak{R}}(\tilde{x}, \omega_0)) = \tilde{y}. \]

That is, \( \omega_0 \circ x_{M_\omega} = \omega \) for each non-trivial \( M \in \mathfrak{M} \) and each \( \omega \in \Omega_M \).

Let \( M_0 \in \mathfrak{M} \) be the algebra such that \( \omega_0 \in \Omega_{M_0} \). Now assume \( M \in \mathfrak{M} \) and let \( a, b \in M \) be such that \( a \neq b \). Then there exists \( \omega \in \text{HU}(M) \) such that \( \omega(a) \neq \omega(b) \). Therefore \( x_{M_\omega} \in \mathcal{A}(M, M_0) \) satisfies

\[ \omega_0 \circ x_{M_\omega}(a) = \omega(a) \neq \omega(b) = \omega_0 \circ x_{M_\omega}(b). \]

This proves that \( M \in \text{ISP}(M_0) \) for each \( M \in \mathfrak{M} \) and hence \( \mathcal{A} = \text{ISP}(M_0) \). In addition, putting \( M = M_0 \) we see that \( (\text{Sep})_{M_\omega} \) holds.

We now prove that (3) \( \Rightarrow \) (5). Let \( \mathfrak{M} = \{ M \} \) and \( \Omega = \{ \omega \} \). By assumption, \( \mathcal{A} = \text{ISP}(M) \) and \( (\text{Sep})_{M,\omega} \) holds. Then we can apply Theorem 2.1. Now,
let $\mathfrak{R} \subseteq \mathcal{A}$. By Lemma 2.2, $\{\omega\} = \Lambda_B(f)$ for each $B \in \mathfrak{R}$ and each $f \in \text{HU}(B)$. By Corollary 3.2,

$$\iota_{\mathfrak{R}}(\{ y \in \prod \text{HU}(\mathfrak{R}) \mid \bigcap \{ \Lambda_B(y_B) \mid B \in \mathfrak{R} \} \neq \emptyset \}) = \prod \text{HU}(\mathfrak{R}).$$

Finally we prove (1) $\Rightarrow$ (4). Let $\mathfrak{M}$ be a finite set of algebras such that $\mathcal{A} = \text{ISP}(\mathfrak{M})$. Let $\mathfrak{N} = \text{Si}(\mathcal{A}) \cap \text{Si}(\mathfrak{M})$. Clearly $\mathfrak{N}$ is a finite set of finite algebras. Since $\mathcal{A} = \text{ISP}(\mathfrak{M})$ we have, on the one hand, $\mathcal{A} = \text{ISP}(\mathfrak{N})$. By assumption $U$ satisfies condition (E), and in particular for every set $\mathfrak{K}$ of disjoint copies of algebras in $\mathfrak{N}$, the map $\chi_{\mathfrak{K}}$ is injective. Now the proof of the implication (2) $\Rightarrow$ (3) above, applied to $\mathfrak{N}$, implies that there exists $M \in \mathfrak{N} \subseteq \text{Si}(\mathcal{A})$ and $\omega \in \text{HU}(M)$ satisfying (i) and (ii) in (4). $\square$

We now present our main theorem, as advertised in Section 1. This amalgamates the results from Theorems 3.3 and 3.5.

**Theorem 3.6.** (Coproduct Preservation Theorem) Let $\mathcal{A}$ be a finitely generated $\mathcal{D}$-based quasivariety and $U: \mathcal{A} \to \mathcal{D}$ the associated forgetful functor. Then the following statements are equivalent:

(A) $U: \mathcal{A} \to \mathcal{D}$ preserves coproducts;

(B) the following conditions hold:

(i) there exist a finite algebra $M \in \mathcal{A}$ and $\omega \in \mathcal{D}(U(M), 2)$ such that $\mathcal{A} = \text{ISP}(M)$ and (Sep)$_{M, \omega}$ holds (that is, for all $a, b \in M$, if $\omega(u(a)) = \omega(u(b))$ for each $u \in \mathcal{A}(M, M)$, then $a = b$),

(ii) for every $A \in \mathcal{A}$ and every bounded sublattice $L$ of $U(A)$ the subposet

$$\{ C \in S(A) \mid C \subseteq L \}$$

of $S(A)$ is empty or has a top element;

(C) there exists a finite algebra $M \in \mathcal{A}$ which is subdirectly irreducible relative to $\mathcal{A}$ and there exists $\omega \in \mathcal{D}(U(M), 2)$ such that

(i) every algebra which is subdirectly irreducible relative to $\mathcal{A}$ belongs to $\text{ISP}(M)$,

(ii) for every $a, b \in M$, if $\omega(u(a)) = \omega(u(b))$ for each $u \in \mathcal{A}(M, M)$, then $a = b$,

(iii) the subposet

$$\{ C \in S(M^2) \mid \forall (c_1, c_2) \in C, \omega(c_1) \leq \omega(c_2) \}$$

of $S(M^2)$ has a top element.

Proof. The functor $U$ preserves coproducts if and only if it satisfies conditions (E) and (S). Condition (B)(i) is exactly item (3) in Theorem 3.5. Then $U$ satisfies (E) if and only if (B)(i) holds. Likewise, item (4) in Theorem 3.3 tells us that $U$ satisfies (S) if and only if (B)(ii) holds.

Assume that (A) holds. Observe that it is a straightforward consequence of Theorem 3.5 that (C)(i) and (C)(ii) hold. Also Theorem 3.3 implies that (C)(iii) holds. We have proved that (A) $\Rightarrow$ (C).
To prove (C) ⇒ (A), we proceed as follows. Let us fix $\mathfrak{M} = \{ M \}$ and $\Omega = \{ \omega \}$. Condition (C)(i) gives $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$, and condition (C)(ii) implies that $(\text{Sep})_{M,\omega}$ holds. Then, by Theorem 3.5, conditions (C)(i)–(ii) imply that $U$ satisfies $(E)$. Now observe that (C)(iii) is equivalent to the assertion that $|R_{\omega,\omega}| = 1$. Since $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$, and $(\text{Sep})_{M,\omega}$ holds, Theorem 3.3 implies that $U$ satisfies $(S)$. □

The equivalence of (A) and (B) spells out the characteristic properties of the class $\mathcal{A}$ that hold if and only if $U$ preserves coproducts. More precisely, it tells us how $\mathcal{A}$ interacts with the subclass $U(\mathcal{A})$ of $\mathcal{D}$. The equivalence of (A) and (C) brings benefits of a different kind. Assume that we are presented with a finite family of finite algebras $M$ such that $\mathcal{A} = \mathbb{ISP}(M)$ is $\mathcal{D}$-based. Then condition (C) enables us to assert that the decision problem “$U$ preserves coproducts” is decidable. To see this, first note that $\text{Si}(\mathcal{A}) \subseteq \mathbb{ISP}(\mathfrak{M})$. Then (C) implies that we only need to check (i)–(iii) on the finite set of pairs $(M, \omega)$ where $M \in S(\mathfrak{M})$ and $\omega \in HU(M)$. Having $\mathfrak{M}$ to hand, to check (C)(i) on a particular pair $(M, \omega)$ amounts to proving that $N \in \mathbb{ISP}(M)$ for each $N \in M$. This is decidable since each $N \in M$ is finite. This, together with the fact that Items (C)(ii)–(iii) are clearly decidable for each pair $(M, \omega)$, confirms our decidability claim concerning preservation of coproducts by $U$. We remark that the restriction to subdirectly irreducible algebras is not itself pertinent to the decidability question. What is essential is that $\mathcal{A}$ should be generated by a single finite algebra $M$. Moreover, since $\text{Si}(\mathbb{ISP}(\mathfrak{M})) \subseteq \mathbb{ISP}(\mathfrak{M})$, the finite algebra $M$ can be chosen to be a member of $\mathfrak{M}$. There are two main reasons for (C) to be formulated as it is. Firstly, it reflects the way the required property would customarily be verified in practice. Secondly, maximal subdirectly irreducible algebras of $\mathcal{A}$ are absolute retracts within $\mathcal{A}$, and hence, if $M \in \text{Si}(\mathcal{A})$ satisfies (C)(i), then $M$ is a retract of any $A \in \mathcal{A}$ satisfying (C)(i). Therefore, performing a full check of (C)(ii)–(iii) is faster in a subdirectly irreducible algebra satisfying (C)(i) than in a non-subdirectly irreducible one.

Up to this point, we have used natural dualities to study the behaviour of coproducts in finitely generated $\mathcal{D}$-based quasivarieties. Theorems 3.3 and 3.5 can also be used in the reverse direction. By this we mean that the type of natural duality that we can obtain for a given class of algebras is governed by the properties of coproducts in that class. Most of the information we require in order to make this assertion precise can be extracted from what we have proved already, but before we can reveal the full picture we need to recall some definitions.

A coproduct in a class $\mathcal{A}$ of a set $\mathfrak{R}$ of non-trivial algebras in $\mathcal{A}$ is a free product if the universal co-cone $\{ \epsilon_A : A \to \bigsqcup \mathfrak{R} \mid A \in \mathfrak{R} \}$ is such that each $\epsilon_A$ is an embedding. When this is the case for every such $\mathfrak{R}$, then $\mathcal{A}$ is said to admit free products. A class of algebras $\mathcal{A}$ is said to have the embedding property if for each set of non-trivial algebras $\mathfrak{R}$ in $\mathcal{A}$ there exists an algebra
into which all the algebras in \( \mathcal{R} \) can be embedded. In the case that a class of algebras admits coproducts, the two properties—admitting free products and the embedding property—are known to be equivalent.

Observe that \( \mathcal{D} \) admits free products. Therefore, if \( \mathcal{A} \) is a class of \( \mathcal{D} \)-based algebras such that \( U \) satisfies (E), then \( \mathcal{A} \) also admits free products. In Theorem 3.5 we proved that, given a finitely generated \( \mathcal{D} \)-based quasivariety \( \mathcal{A} \), the functor \( U \) satisfies (E) if and only if there exists a finite algebra generating \( \mathcal{A} \), together with a particular lattice homomorphism from (the reduct of) this algebra into \( 2 \). This shows that there is a deep connection between free products and condition (E), since a quasivariety admits free products (or, equivalently, has the embedding property) if and only if it is generated by a single algebra [24, Proposition 2.1.19]. By modifying part of the argument used to prove Theorem 3.5 we can present a different proof of the equivalence between free products and single generation for the case of \( \mathcal{D} \)-based quasivarieties which reveals more overtly the connection between piggyback dualities and free products.

**Theorem 3.7.** Let \( \mathcal{A} \) be a finitely generated \( \mathcal{D} \)-based quasivariety. Then \( \mathcal{A} \) admits free products if and only if there exists a (finite) algebra \( M \in \mathcal{A} \) such that \( \mathcal{A} = \text{ISP}(M) \).

*Proof.* For the forward implication, let \( \mathcal{M} \) be a finite set of finite algebras such that \( \mathcal{A} = \text{ISP}(\mathcal{M}) \). Since \( \mathcal{A} \) admits free products, \( \mathcal{M} \subseteq \text{ES}(\biguplus \mathcal{M}) \) and hence \( \mathcal{A} = \text{ISP}(\biguplus \mathcal{M}) \). Moreover, since \( \mathcal{A} \) is locally finite, \( \biguplus \mathcal{M} \) is finite.

Assume now that \( \mathcal{A} = \text{ISP}(M) \) for some algebra \( M \). There is no loss of generality in assuming that \( M \) is finite. Now \( \text{Sep}_{M, \Omega} \) holds, where \( \Omega = HU(M) \). Let \( \mathcal{R} \) be a set of non-trivial algebras in \( \mathcal{A} \) and let \( \{ e_B \mid B \in \mathcal{R} \} \) be the universal co-cone that determines the coproduct \( \biguplus \mathcal{R} \). Proving that each \( e_B \) is an embedding is equivalent to proving that the image of \( HU(e) \) is \( HU(B) \).

Let \( h \in HU(B) \). By Theorem 2.3 (referring back to Lemma 2.2) there exists \( (x, \omega) \in Y_B \) such that \( \Phi_B(x, \omega) = h \). Since \( \mathcal{M} = \{ M \} \) and \( \Omega = HU(M) \), we have \( x \in A(B, M) \) and \( \omega \in HU(M) \). Let \( x_B = x \) and, for each \( A \in \mathcal{R} \setminus \{ B \} \), let \( x_A \) be an arbitrary element of \( A(A, M) \) (since \( A \) is non-trivial, this set is non-empty). Then \( \langle \bar{x}, \omega \rangle \in Y_{\biguplus \mathcal{R}} \). From Fig. 4,

\[
HU(e_B)(\Psi_{\biguplus \mathcal{R}}(\bar{x}, \omega)) = \Phi_B \circ \xi_B(\bar{x}, \omega) = \omega \circ x_B = \omega \circ x = h.
\]

It follows that the image of \( HU(e_B) \) is \( HU(B) \). \( \square \)

The following theorem collects together the connections between the properties of coproducts in a finitely generated \( \mathcal{D} \)-based quasivariety and the type of natural duality that it admits. Its proof is a direct application of Theorems 2.1, 3.3, 3.5 and 3.7 and Corollary 2.4.

**Theorem 3.8.** Let \( \mathcal{A} \) be a finitely generated \( \mathcal{D} \)-based quasivariety and let \( U : \mathcal{A} \to \mathcal{D} \) be the corresponding forgetful functor.
(i) \( \mathcal{A} \) admits free products if and only if it admits a single-sorted natural duality.

(ii) \( U \) satisfies (E) if and only if \( \mathcal{A} \) admits a simple piggyback duality.

(iii) \( U \) satisfies (S) if and only if \( \mathcal{A} \) admits a piggyback duality, which may be single-sorted or multisorted, but which is such that \( |\omega_1, \omega_2| \leq 1 \) for each pair of carrier maps \( \omega_1, \omega_2 \). Moreover, if any piggyback duality for \( \mathcal{A} \) has this feature then all such dualities do.

(iv) \( U \) preserves coproducts if and only if \( \mathcal{A} \) admits a simple piggyback duality that is also a \( \mathcal{D} \)-\( \mathcal{P} \)-based duality.

4. Stability properties of term reducts

Until now we have focused on finitely generated \( \mathcal{D} \)-based quasivarieties, and made full use of the duality theory that is thereby available. However it is striking that in our surjectivity theorem (Theorem 3.3) certain of the equivalent conditions would be meaningful in a wider setting. We accordingly now seek to ascertain how far the results of the previous section depend critically on the assumptions we made about \( \mathcal{A} \). There are two respects in which we may extend the setting. First of all we demanded that our quasivarieties be finitely generated in order to exploit techniques from natural duality theory; this is not a necessary requirement for coproducts to be available. Secondly, we may seek to remove the restriction that our base variety be \( \mathcal{D} \).

In what follows it will be appropriate to consider the following generalisation of the notion of a quasivariety. A class of algebras \( \mathcal{A} \) is called a prevariety if \( ISP(\mathcal{A}) = \mathcal{A} \). For each algebra \( A \) in a given prevariety \( \mathcal{A} \) the set of congruences \( Con_{A}(\mathcal{A}) = \{ \theta \in Con(\mathcal{A}) \mid A/\theta \in \mathcal{A} \} \) is a complete lattice for the usual inclusion order. We note that, as a consequence, any prevariety \( \mathcal{A} \) admits free objects and coproducts.

We shall extend our earlier definition so as to encompass prevarieties \( \mathcal{A} \) from which there exists an appropriate natural forgetful functor \( U_{A, \mathcal{C}} \) to a specified class \( \mathcal{C} \) of algebras. The properties of the functor \( U_{A, \mathcal{C}} \) with respect to coproducts will depend on both \( \mathcal{A} \) and \( \mathcal{C} \). We remain interested principally in the case that \( \mathcal{C} = \mathcal{D} \) and make no claim to give an exhaustive analysis of the properties of \( U_{A, \mathcal{C}} \) with respect to coproducts. However we do take our study far enough to reveal which properties of the forgetful functor are preserved when we restrict to subprevarieties of \( \mathcal{A} \). This enables us to enlarge the range of classes of algebras in which coproducts can be identified.

Let \( \mathcal{A} \) be a class of algebras with language \( \mathcal{L} \). For each set \( \mathcal{T} \) of \( \mathcal{L} \)-terms, the \( \mathcal{T} \)-reduct of an algebra \( A \in \mathcal{A} \) is defined to be the algebra \( (A, \{ t^A \mid t \in \mathcal{T} \}) \), where \( A \) is the universe of \( A \). The algebra \( U_{A, \mathcal{T}}(\mathcal{A}) = (A, \{ t^A \mid t \in \mathcal{T} \}) \) is called a term-reduct of \( \mathcal{A} \). This notion of term-reduct encompasses the usual notion of reduct in the following way. Let \( \mathcal{L}' \subseteq \mathcal{L} \) and, for an \( n \)-ary operation \( f \in \mathcal{L}' \), let \( \mathcal{T} = \{ t_f \mid f \in \mathcal{L}' \} \), where for an \( n \)-ary operation \( f \) the term \( t_f \) is
$f(x_1, \ldots, x_n)$ for variables $x_1, \ldots, x_n$. Then

$$U_{\mathcal{A}, \mathcal{T}}(A) = (A, \{ t^A_f \mid f \in \mathcal{L} \}) = (A, \{ f^A \mid f \in \mathcal{L} \}).$$

Since each $\mathcal{L}$-term has a well-defined arity, we can consider $\mathcal{T}$ as a language. If $\mathcal{C}$ is a class of $\mathcal{T}$-algebras containing $U_{\mathcal{A}, \mathcal{T}}(A)$ for each $A \in \mathcal{A}$, we shall write $U_{\mathcal{A}, \mathcal{C}}(A)$ when we wish to highlight that we are considering the algebra $U_{\mathcal{A}, \mathcal{T}}(A)$ as an algebra in $\mathcal{C}$. In this case we say that $\mathcal{A}$ is $\mathcal{C}$-based. Observe that if $\mathcal{C}$ is a variety and a class of algebras $\mathcal{A}$ is $\mathcal{C}$-based, then $\forall(\mathcal{A})$, the variety generated by $\mathcal{A}$, is also $\mathcal{C}$-based.

Assume that a given class $\mathcal{A}$ of algebras is $\mathcal{C}$-based. We wish to define $U_{\mathcal{A}, \mathcal{C}}$ on morphisms so as to make it a functor from $\mathcal{A}$ into the class $\mathcal{C}$. For every $n$-ary $\mathcal{L}$-term $t$ and every homomorphism $h: A \to B$, we have $h(t^A(a_1, \ldots, a_n)) = t^B(h(a_1), \ldots, h(a_n))$ for each $a_1, \ldots, a_n \in A$. Then the assignment $U_{\mathcal{A}, \mathcal{C}}(h) = h$ does indeed make $U_{\mathcal{A}, \mathcal{C}}$ into a functor. In the case that $\mathcal{A}$ is $\mathcal{D}$-based, we omit the subscripts and write $U$ instead of $U_{\mathcal{A}, \mathcal{D}}$.

Let $\mathcal{R}$ be a set of algebras in $\mathcal{A}$ and let $\coprod_{\mathcal{A}} \mathcal{R}$ be its coproduct in $\mathcal{A}$. Let $\{ \varepsilon_B : B \to \coprod_{\mathcal{A}} \mathcal{R} \mid B \in \mathcal{R} \}$ be the universal co-cone that determines the coproduct $\coprod_{\mathcal{A}} \mathcal{R}$ up to isomorphism. There exists a unique map

$$\chi_{\mathcal{R}} : \coprod_{\mathcal{C}} U_{\mathcal{A}, \mathcal{C}}(\mathcal{R}) \to U_{\mathcal{A}, \mathcal{C}}(\coprod_{\mathcal{A}} \mathcal{R})$$

such that $\chi_{\mathcal{R}} \circ \varepsilon_B = U_{\mathcal{A}, \mathcal{C}}(\varepsilon_B)$ for each $B \in \mathcal{R}$, where the family of homomorphisms $\{ \varepsilon_B : U_{\mathcal{A}, \mathcal{C}}(B) \to \coprod_{\mathcal{C}} U_{\mathcal{A}, \mathcal{C}}(\mathcal{R}) \mid B \in \mathcal{R} \}$ is the universal co-cone in $\mathcal{C}$.

In the previous section we investigated when the functor $U$, for a fixed $\mathcal{D}$-based quasivariety $\mathcal{A}$, preserves coproducts. We extend our earlier usage and say that the functor $U_{\mathcal{A}, \mathcal{C}}$ satisfies

\begin{align*}
(\text{S}) & \text{ if for each set } \mathcal{R} \subseteq \mathcal{A}, \text{ the map } \chi_{\mathcal{R}} \text{ is surjective;} \\
(\text{E}) & \text{ if for each set } \mathcal{R} \subseteq \mathcal{A}, \text{ the map } \chi_{\mathcal{R}} \text{ is injective.}
\end{align*}

A remark is in order here. The only result of this section in which we study the properties of condition (E) is Theorem 4.6. The more amenable properties of condition (S) are mainly due to the fact that we can relate it to a property of the free objects in $\mathcal{A}$ as in Theorem 3.3(2), while condition (E) depends on the set of generating algebras, or the set $\text{Si}(\mathcal{A})$, as can be seen from Theorem 3.5.

The first result of this section generalises Theorem 3.3(1)–(4).

**Theorem 4.1.** Let $\mathcal{A}$ and $\mathcal{C}$ be prevarieties such that $\mathcal{A}$ is $\mathcal{C}$-based. Then the following statements are equivalent:

1. $U_{\mathcal{A}, \mathcal{C}}$ satisfies (S);
2. for every finite set $\mathcal{R}$ of disjoint copies of $F_\mathcal{A}(1)$, the image of $\chi_{\mathcal{R}}$ is $U_{\mathcal{A}, \mathcal{C}}(\coprod_{\mathcal{A}} \mathcal{R})$;
3. for every $n \geq 1$ and every $n$-ary term $t$ in the language of $\mathcal{A}$ there exist unary terms $t_1, \ldots, t_n$ in the language of $\mathcal{A}$ and an $n$-ary term $s$ in the language of $\mathcal{C}$ such that $t^A(a_1, \ldots, a_n) = s^A(t_1^A(a_1), \ldots, t_n^A(a_n))$. 


for every \( A \in \mathcal{A} \) and every \( a_1, \ldots, a_n \in A \);  
(4) for every \( A \in \mathcal{A} \) and \( B \in \mathcal{S}(\bigcup_{A \in \mathcal{A}} A) \), the set \( \{ C \in \mathcal{S}(A) \mid C \subseteq B \} \) is empty or has a top element.

**Proof.** The implication (1) \( \Rightarrow \) (2) is straightforward. To prove the implications (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) it is enough to replace \( \mathcal{D} \) by \( \mathcal{E} \) in the proofs of the corresponding implications in Theorem 3.3, since the latter do not depend on the fact that \( \mathcal{A} \) is finitely generated.

We now prove (4) \( \Rightarrow \) (1). Let us consider a set \( \mathfrak{R} \) of algebras in \( \mathcal{A} \) and let \( \{ \epsilon_A : A \to \prod_A \mathfrak{R} \mid A \in \mathfrak{R} \} \) be the universal co-cone that determines \( \prod_A \mathfrak{R} \).

Now let \( \mathcal{B} = \langle \bigcup \{ \epsilon_A(A) \mid A \in \mathfrak{R} \} \rangle \subseteq \prod A \mathfrak{R} \). Then \( \mathcal{B} \in \mathcal{S}(\bigcup_{A \in \mathfrak{R}} A) \) and \( \epsilon_A(A) \subseteq \mathcal{B} \) for each \( A \in \mathfrak{R} \). If \( \{ \epsilon_A : A \to \prod_A \mathcal{U}_{A,e}(\mathfrak{R}) \mid A \in \mathfrak{R} \} \) denotes the universal co-cone that determines \( \prod_A \mathcal{U}_{A,e}(\mathfrak{R}) \), it follows that \( \prod_A \mathcal{U}_{A,e}(\mathfrak{R}) = \langle \bigcup \{ \epsilon_A(A) \mid A \in \mathfrak{R} \} \rangle \). By (Chi) we have \( \mathcal{R} \langle \prod A \mathcal{U}_{A,e}(\mathfrak{R}) \rangle = B \). By assumption, \( \{ C \in \mathcal{S}(\bigcup \mathfrak{R}) \mid C \subseteq B \} \) has a top element, \( D \) say. Then \( \epsilon_A(A) \subseteq D \) for each \( A \in \mathfrak{R} \). Finally, since \( \langle \bigcup \{ \epsilon_A(A) \mid A \in \mathfrak{R} \} \rangle \mathcal{A} = \prod A \mathfrak{R} \), we have that \( D = \prod A \mathfrak{R} \) and that the universes \( D \) and \( B \) coincide. We conclude that \( \mathcal{R} \langle \prod A \mathcal{U}_{A,e}(\mathfrak{R}) \rangle = \prod A \mathfrak{R} \). \( \square \)

**Corollary 4.2.** Let \( \mathcal{A} \) and \( \mathcal{E} \) be prevarieties such that the variety \( \forall \mathcal{A} \) generated by \( \mathcal{A} \) is \( \mathcal{E} \)-based. Then the following statements are equivalent:

1. \( \bigcup \mathcal{A}, e \) satisfies (S);
2. \( \bigcup \mathcal{B}, e \) satisfies (S) for each prevariety \( \mathcal{B} \) such that \( \mathcal{ISP}(\mathcal{F}(\mathcal{N}_0)) \subseteq \mathcal{B} \subseteq \forall \mathcal{A} \).

**Proof.** The result follows directly from Theorem 4.1 and the observation that \( \mathcal{F}(\mathcal{A})(\lambda) = \mathcal{F}(\mathcal{A}) = \mathcal{F}(\mathcal{B})(\lambda) \) for any cardinal \( \lambda \) and each \( \mathcal{B} \) such that \( \mathcal{ISP}(\mathcal{F}(\mathcal{N}_0)) \subseteq \mathcal{B} \subseteq \forall \mathcal{A} \). \( \square \)

We observe that we can relax the assumption of finite generation in Theorem 3.3 to local finiteness and still have a condition similar to condition (5).

**Theorem 4.3.** Let \( \mathcal{A} \) be a locally finite \( \mathcal{D} \)-based quasivariety. Then the following statements are equivalent:

1. \( \bigcup \) satisfies (S);
2. for each \( n \geq 1 \) and each \( \omega_1, \omega_2 \in \mathcal{HU}(\mathcal{F}(\mathcal{A}(n))) \), the set \( \{ C \in \mathcal{S}(\mathcal{F}(\mathcal{A}(n))^2) \mid C \subseteq (\omega_1, \omega_2)^{-1}(\leq) \} \) is either empty or has a top element.

**Proof.** Here we shall need to consider other quasivarieties besides \( \mathcal{A} \) itself and their forgetful functors to \( \mathcal{D} \). We add subscripts to indicate the domains of the functors.

The implication (1) \( \Rightarrow \) (2) follows from Theorem 4.1 and the observation that \( (\omega_1, \omega_2)^{-1}(\leq) \) is a sublattice of \( \bigcup \mathcal{F}(\mathcal{A}(n)) \).

Let us now prove (2) \( \Rightarrow \) (1). Fix \( n \geq 1 \) and let \( \mathcal{B} = \mathcal{ISP}(\mathcal{F}(\mathcal{A}(n))) \). Let \( \mathfrak{M} = \{ \mathcal{F}(\mathcal{A}(n)) \} \) and \( \Omega = \mathcal{HU}(\mathcal{F}(\mathcal{A}(n))) \). By assumption \( |R_{\omega_1, \omega_2}| \leq 1 \) for each
ω₁, ω₂ ∈ Ω. By Theorem 3.3, the functor U₂ satisfies (S). Now let ℝ be a set with n disjoint copies of FA(1). Then \[ \prod A ℝ = FA(n) = \prod B ℝ. \] It follows that the map \( χ_ℝ: \prod B U oblivious(ℝ) \to U oblivious(\prod A ℝ) \) coincides with the map \( χ_ℝ': \prod B U oblivious(ℝ) \to U oblivious(\prod B ℝ) \), as defined in (Chi). The latter map is surjective by hypothesis. Since this argument is valid for every \( n \geq 1 \), Theorem 4.1 proves that U oblivious satisfies (S). □

We now wish to pursue the idea that a given prevariety Q may sometimes be contained in another prevariety Q’ which has better properties as regards coproducts and that this may assist us in describing coproducts in Q. Let Q and Q’ be prevarieties such that Q ⊆ Q’. The category Q can be viewed as a reflective subcategory of Q’. Indeed, for each A ∈ Q, the set of congruences θ of A such that A/θ ∈ Q’ has a bottom element; we denote this by \( \theta_Q(A) \). Moreover, given a homomorphism h: A → B there is a unique homomorphism \( h': A/\theta_Q(A) → B/\theta_Q(A) \) that makes the diagram in Fig. 5 commute. The assignment \( A ↦ A/\theta_Q(A) \) and \( h ↦ h' \) is then a well-defined functor \( R_Q: Q → Q' \). It follows that \( R_Q \) is left adjoint to the inclusion functor from Q’ into Q. (See [24, Theorem 2.1.8] or [27, p. 235] for the existence of the minimal congruence and [29, Corollary 4.22] for its categorical properties.) Therefore Q’ is a reflective subcategory of Q. The following proposition now tells us that we can obtain coproducts in Q provided we have descriptions of coproducts in Q’ and of the congruence \( \theta_Q \). This tactic was employed for example in [8, 14] and we shall use it several times in Section 5.

**Figure 5.** Functoriality of the reflector \( R_Q' \)

**Proposition 4.4.** Let Q and Q’ be prevarieties such that Q′ ⊆ Q. Let ℝ be a set of algebras in Q’. Then \( \prod ℝ_Q \cong R_Q'(\prod ℝ) = (\prod ℝ)/\theta_Q'(\prod ℝ) \). Now we assemble some consequences of Proposition 4.4.

**Theorem 4.5.** Let A and ℂ be prevarieties such that A is ℂ-based. Then U A, ℂ satisfies (S) if and only if U A’, ℂ satisfies (S) for each prevariety A’ ⊆ A.

**Proof.** For the non-trivial implication, let A’ be a prevariety contained in A. Since A is ℂ-based, A’ is also ℂ-based, and we can consider the functor U A’, ℂ: A → ℂ. Let ℝ ⊆ A’ be a set of algebras. We will use t to distinguish \( χ_ℝ: \prod ℂ U oblivious(A, ℂ)(ℝ) → U oblivious(A, ℂ)(\prod A ℝ) \) from \( χ_ℝ': \prod ℂ U oblivious(A', ℂ)(ℝ) → U oblivious(A', ℂ)(\prod A ℝ) \).
By Proposition 4.4 we have $\prod_e U_A.e(\mathcal{R}) \cong (\prod_e U_A.(\mathcal{R}))/\theta_{A'}(\prod_e U_A.(\mathcal{R}))$ Let $\rho: \prod_e U_A.(\mathcal{R}) \to \prod_{A'} \mathcal{R}$ denote the quotient map. Since $U_{A'.e'}(\prod_{A'} \mathcal{R}) = U_{A'.e'}(\prod_{A'} \mathcal{R})$, the diagram in Fig. 6 commutes. Thus, if $\chi_{\mathcal{R}}$ is surjective, so is $\chi_{\mathcal{R}'}$. □

**Theorem 4.6.** Let $\mathcal{A}$ and $\mathcal{C}$ be prevarieties such that $\mathcal{A}$ is $\mathcal{C}$-based. If $\mathcal{C}'$ is a prevariety such that $\mathcal{C} \subseteq \mathcal{C}'$, then the following statements hold:

(i) if $U_{A'.e}$ satisfies (S) then $U_{A.e'}$ satisfies (S);  
(ii) if $U_{A.e'}$ satisfies (E) then $U_{A.e}$ satisfies (E).

**Proof.** Let $\mathcal{R} \subseteq \mathcal{A}$ be a set of algebras. Now we will use $'$ to distinguish $\chi_{\mathcal{R}}: \prod_e U_A.e(\mathcal{R}) \to U_A.e(\prod_e \mathcal{R})$ from $\chi_{\mathcal{R}'}: \prod_{e'} U_{A'.e'}(\mathcal{R}) \to U_{A.e'}(\prod_{e'} \mathcal{R})$.

By Proposition 4.4, $\prod_e U_{A'.e'}(\mathcal{R}) \cong (\prod_{e'} U_{A.e'}(\mathcal{R}))/\theta_{A}.(\prod_{e'} U_{A'.e'}(\mathcal{R}))$.

Let $\rho: \prod_{e'} U_{A'.e'}(\mathcal{R}) \to \prod_{e'} U_{A.e'}(\mathcal{R})$ denote the quotient map. Then the diagram in Fig. 7 commutes. Consequently, on the one hand, if $\chi_{\mathcal{R}}$ is surjective, then $\rho$ is surjective, which proves (i). On the other hand, if $\chi_{\mathcal{R}}' is injective, then $\rho is injective, and therefore an isomorphism. This implies that $\chi_{\mathcal{R}} is injective, proving (ii). □

**Figure 6.** The proof of Theorem 4.5  

**Figure 7.** The proof of Theorem 4.6

### 5. Applications

Finitely generated quasivarieties $\mathcal{A}$ of $\mathcal{D}$-based algebras can be classified into four types according to whether the forgetful functor $U: \mathcal{A} \to \mathcal{D}$ satisfies or fails to satisfy the conditions (S) and (E). Theorems 3.3 and 3.5 suggest...
Figure 8. Flowchart for testing conditions (E) and (S)

a strategy for analysing the properties of $\mathcal{U}$. This is depicted in the flowchart in Fig. 8. The flowchart comes in three parts: an input section in which we assemble the relevant information about the class $\mathcal{A}$ to be considered; a series of questions; and an output section in which the class $\mathcal{A}$ is classified according to the answers to the questions. At the outset, we assume that, or arrange that, $\mathcal{A}$ is expressed in ‘simplified’ form, that is, that it is presented as $\mathcal{A} = \mathcal{ISP}(\mathcal{M})$, where $\mathcal{M} \subseteq \text{Si}(\mathcal{A})$ and $\mathcal{M}$ has minimal cardinality.
Table 1. A medley of examples

(see the remarks following Theorem 3.6). The first question posed—whether $|\mathcal{M}| = 1$—could be bypassed. Still it is useful to know the answer: $|\mathcal{M}| = 1$ is exactly the condition for $\mathcal{A}$ to admit a single-sorted natural duality (Theorem 3.8(i)), and this property is advantageous; moreover, $|\mathcal{M}| = 1$ is also the condition for $\mathcal{A}$ to admit free products (Theorem 3.7). In the output section of the flowchart, and in Table 2 below, we adopt an abbreviated notation for results: $E^\vee S^\times$ indicates that condition (E) holds and condition (S) fails, and so on.

In this section we will present examples of varieties for the four possible combinations of properties. We summarise our examples in Table 1; definitions of the listed classes, all of which are $\mathcal{D}$-based varieties, are recalled below. In the table, we leave it tacit that the quasivarieties are of the form $\text{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras. When we refer to a quasivariety being *singly generated*, we mean that $\mathcal{M}$ contains a single algebra. Our catalogue of examples is by no means exhaustive and there are many other classes of algebras we could equally well have used to illustrate our methods.

For each of our selected examples, we shall answer the questions in each section of the flowchart, giving references for known results and including proofs only when we could not find these in the literature. The output section of Fig. 8 will then tell us the properties of $\mathcal{U}$ for each variety $\mathcal{A}$. Once we have
these properties to hand in a given case, we want to proceed to describe coproducts in the variety concerned. When the algebraic structure is completely determined by the lattice reduct, as in the case of Heyting algebras or pseudocomplemented lattices, the description of the behaviour of $U$ on coproducts, completely determines coproducts in the corresponding class. Even when this is not the case, we may still be able to describe coproducts. Table 2 indicates some general strategies (motivated by Theorem 3.8) for achieving this.

In case coproducts are preserved, Theorem 3.8(iv) tells us that we are dealing with a quasivariety for which a simple piggyback duality is available and, moreover, this duality can equivalently be viewed as a $\mathcal{D}-\mathcal{P}$-based duality. When $U_\mathcal{A}$ does not preserve coproducts, the primary idea is to use the results in Section 4 to find a quasivariety $\mathcal{B}$ containing $\mathcal{A}$ such that $U_\mathcal{B}$ has better properties than $U_\mathcal{A}$ (here we annotate the functor to indicate the class to which it refers). Such an enveloping quasivariety for $\mathcal{A}$ is not in general unique and which choice we make will depend on how much information we have about coproducts in the possible quasivarieties $\mathcal{B}$. Depending on the properties of coproducts of this enveloping quasivariety $\mathcal{B}$ that the class $\mathcal{A}$ does not have, we have simpler natural dualities for $\mathcal{B}$ than for $\mathcal{A}$; see Theorem 3.8. Again we will cite earlier literature where appropriate.
We now discuss in turn the varieties listed in Table 1. For complete clarity we shall write $U_{\mathcal{A}}$, for each given choice of $\mathcal{A}$, rather than $U$.

**De Morgan Algebras, DM.**

An algebra $\mathbf{A} = (A, \land, \lor, \neg, 0, 1)$ is a *De Morgan algebra* if its lattice reduct $(A, \land, \lor, 0, 1)$ is in $\mathcal{D}$ and $\neg$ is a unary operation satisfying the equations $x \approx \neg \neg x$ and $\neg (x \land y) \approx \neg x \lor \neg y$ and $\neg 0 \approx 1$. Then $\mathcal{DM} = ISP(4)$, where $4 = \{\{0, a, b, 1\}, \land, \lor, \neg, 0, 1\}$ denotes the four-element De Morgan algebra with two $\neg$-fixpoints. Let $\eta$ be the automorphism of $4$ which interchanges $a$ and $b$ and let $\omega: U_{\mathcal{DM}}(4) \to 2$ be defined by $\omega(a) = \omega(1) = 1$ and $\omega(b) = \omega(0) = 0$. Then $(\text{Sep})_{4,\omega}$ is satisfied. The answers to (1) and (2) in Fig. 8 are yes. Note that Lemma 3.4 is applicable here, so the answer to (3) is also yes. Therefore, the functor $U_{\mathcal{DM}}$ preserves coproducts.

By Theorem 2.1, the structure $M = (\{0, a, b, 1\}, r, e, \mathcal{P})$, where \[ r = \{0, 0\}, (0, a), (a, a), (b, 0), (b, a), (b, b), (0, 1), (1, a), (1, 1) \}, \] yields a natural duality for $\mathcal{DM}$ and coproducts in $\mathcal{DM}$ correspond to cartesian products in the dual category. The duality and the description of coproducts stemming from it were already developed in [13]. There the duality was introduced as a $\mathcal{D} \times \mathcal{P}$-based one. In [12, Section 3.15] this same duality is presented from the natural duality perspective, in accord with Corollary 2.4.

**Kleene algebras, $\mathcal{K}$.**

An algebra $\mathbf{A}$ is a *Kleene algebra* if it is a De Morgan algebra satisfying the Kleene condition $x \land \neg x \triangleleft y \lor \neg y$. We have $\mathcal{K} = ISP(3)$ where $3$ denotes the subalgebra $\{\{0, a, 1\}, \land, \lor, \neg, 0, 1\}$ of $4$ determined by $\{0, a, 1\}$. Then the answer to (1) is yes. The only endomorphism of $3$ is the identity. Thus the only $\Omega$ that satisfies $(\text{Sep})_{3,\Omega}$ is $\Omega = HU_{\mathcal{K}}(3) = \{\omega_1, \omega_2\}$, where $\omega_1, \omega_2: U_{\mathcal{K}}(3) \to 2$ are defined by $\omega_1(a) = \omega_1(1) = 1$, $\omega_1(0) = 0$ and $\omega_2(1) = 1$, $\omega_2(a) = \omega_2(0) = 0$. Therefore the answer to (2) is no. By Lemma 3.4, for each pair $\omega_i, \omega_j$ ($i, j = 1, 2$), we have $|R_{\omega_i, \omega_j}| = 1$, so the answer to (3) is yes. Hence Kleene algebras admit free products and $U_{\mathcal{K}}$ has the property (S) but not property (E).

Guided by Table 2, to describe coproducts in $\mathcal{K}$ fully we seek an enveloping variety $\mathcal{B}$ such that $U_{\mathcal{B}}$ preserves coproducts and then determine the retraction $R_{\mathcal{K}}$. This is exactly the procedure followed in [8] and in [14] to describe coproducts in $\mathcal{K}$ using $\mathcal{DM}$ as enveloping variety. An alternative proof can be obtained using Lemma 2.2, Theorem 3.3 and Corollary 3.2. Theorem 2.1 can then be used to prove that $(\{0, a, 1\} \cup \{0, a, 1\}; \{R_{\omega_i, \omega_j} \mid i, j \in \{1, 2\}\}; \mathcal{P})$ yields a multisorted natural duality on $\mathcal{K}$. (See [16], where this piggyback duality for Kleene algebras was first developed.)

**Pseudocomplemented distributive lattices.**

We consider the countable chain of non-trivial finitely generated subvarieties $\mathcal{B}_n$ of the variety $\mathcal{B}_\omega$ of pseudocomplemented distributive lattices. An
algebra $A = (A, \land, \lor, *, 0, 1)$ is in $\mathcal{B}_\omega$ if $(A, \land, \lor, 0, 1) \in \mathcal{D}$ and $*$ is a unary operation satisfying $x \land y \approx 0$ if and only if $x \leq y^*$. For $0 \leq n < \omega$ the variety $\mathcal{B}_n$ is expressible as $\mathbb{ISP}(\mathcal{B}_n)$ where $\mathcal{B}_n$ has as underlying lattice the Boolean lattice with $n$ atoms with a new top element adjoined; here $\mathcal{B}_0$ and $\mathcal{B}_1$ correspond to the varieties of Boolean algebras and of Stone algebras, respectively. (See for example [1] for details.)

Piggyback dualities dualities for the classes $\mathcal{B}_n$ were studied in [17], building on earlier work in [15]. We already noted that the answer to our question (1) is yes for each $\mathcal{B}_n$. In [17, p. 48] it is observed that for each $n$ there exists $\omega_n$ such that $(\text{Sep})_{\mathcal{B}_n, \omega_n}$ holds, and so the answer to (2) is yes for each $n \geq 0$. In [17, Theorem 3.6], it is proved that $|R_{\omega_n, \omega_n}|$ is equal to the number of partitions of $n$. Thus the answer to (3) is yes if and only if $n \leq 1$. In summary, $U_{\mathcal{B}_n}$, for $n \leq 1$, preserves coproducts and $U_{\mathcal{B}_n}$, for $n \geq 2$, satisfies condition (E) but not condition (S).

**Quasivarieties of Heyting algebras generated by finite chains.**

We recall that an algebra $A = (A, \land, \lor, \to, 0, 1)$ is a Heyting algebra if $(A, \land, \lor, 0, 1) \in \mathcal{D}$ and $\to$ is a binary operation satisfying $x \land y \leq z$ if and only if $x \leq y \to z$. The implication $\to$ is uniquely determined by the lattice structure. Therefore determining the properties of $U$ with respect to coproducts suffices for a full understanding of coproducts in quasivarieties of Heyting algebras.

For each $n \in \{2, 3, \ldots\}$, let $C_n$ denote the $n$-element Heyting chain. and let $\mathcal{G}_n = \mathbb{ISP}(C_n)$. The algebras in $\mathcal{G}_n$ are known as $n$-valued Gödel algebras, since they form the algebraic counterpart of the $n$-valued Gödel logic (see [23, 25]). It is easy to see that $\mathcal{G}_2$ is term-equivalent to the variety of Boolean algebras. So we restrict to the case $n \geq 3$. For each $n$ the condition $(\text{Sep})_{C_n, \omega_n}$ is satisfied if we take $\omega_n : U_{\mathcal{G}_n}(C_n) \to \mathcal{2}$ to be the map determined by $\omega_n^{-1}(1) = \{1\}$. Therefore in each $\mathcal{G}_n$ the answer to (1) and (2) is yes. The algebra $C_3$ belongs to $\mathcal{G}_n$ and it can easily be checked that $|R_{\omega_3, \omega_3}| = \{r_1, r_2\}$, where $r_1 = \{(0,0), (d, d), (1,1)\}$ and $r_2 = \{(0,0), (d, 1), (1,1)\}$. Thus Theorem 3.3(4) proves that the answer to (3) is no for each $\mathcal{G}_n$. Thus, for each $n \geq 3$, the functor $U_{\mathcal{G}_n}$ satisfies (E) but not (S).

Coproducts can be described completely by using the simple piggyback dualities dualities for the classes $\mathcal{G}_n$ derived in [15] (or see [12, Section 7.3]). In [6] we undertake an in-depth study of coproducts in the classes $\mathcal{G}_n$. A different approach to coproducts of finite Gödel algebras $\mathcal{V}(\{C_n \mid n \geq 1\})$ has been developed in [21]. The results in [21] combined with Proposition 4.4 provide another description of coproducts for finite algebras in $\mathcal{G}_n$ for each $n \geq 2$.

**MV-algebras.**

An algebra $(A, \oplus, \neg, 0)$ is an **MV-algebra** if $(A, \oplus, 0)$ is a commutative monoid satisfying $\neg(-x) \approx x, \neg 0 \oplus x \approx 0$ and $\neg(-x \oplus y) \oplus y \approx \neg(y \oplus x) \oplus x$. The variety of MV-algebras is the algebraic counterpart of Łukasiewicz infinite-valued logic (see [11]). The terms $\neg(-x \oplus y) \oplus y, \neg(x \oplus y) \oplus y, 0$ and $0$
determine a bounded distributive lattice structure on any MV-algebra \( A \). For each finitely generated variety \( \mathcal{A} \) of MV-algebras there is a finite set \( \mathfrak{M} \) of finite MV-chains such that \( \mathcal{A} = \text{ISP}(\mathfrak{M}) \) [11, Chapter 8]. The \( n \)-element MV-chain is denoted by \( L_{n-1} \). (We deviate here from the notation in [11, Section 3.5] where the \( n \)-element MV-chain is denoted by \( L_n \). We do this so that \( L_m \in \text{ISP}(L_n) \) if and only if \( m \) divides \( n \).) The variety \( \text{ISP}(L_1) \) is term-equivalent to the variety of Boolean algebras [11, Corollary 8.2.4]. Therefore \( U_{\text{ISP}(L_n)} \) preserves coproducts. Now assume that \( n \geq 2 \). Since the only endomorphism of \( L_n \) is the identity, (Sep)\( L_n, \Omega \) can be satisfied only by taking \( \Omega = HU_{\text{ISP}(L_n)}(L_n) \). Then the answer to (1) is yes but to (2) is no. If \( n \) is a power of a prime, it can be proved that \( |R_{\omega_1, \omega_2}| = 1 \) for each \( \omega_1, \omega_2 \in HU_{\text{ISP}(L_n)} \) and in these cases the answer to (3) is yes and \( U_{\text{ISP}(L_n)} \) satisfies (S) but not (E). In case \( n \) has at least two distinct prime divisors, it can be checked that there exist \( \omega_1, \omega_2 \in HU_{\text{ISP}(L_n)} \) such that \( |R_{\omega_1, \omega_2}| > 1 \) and then the answer to (3) is no. Thus \( U_{\text{ISP}(L_n)} \) satisfies neither (E) nor (S).

If a quasivariety of MV-algebras is generated by a finite family of MV-chains \( \{L_{n_1}, \ldots, L_{n_m}\} \) with \( m > 1 \), then the answer to (1) and (2) is no, and the answer to (3) is yes if and only if \( n_i \) is a prime power for \( 1 \leq i \leq m \), that is, if no \( L_{p^r} \) with \( p \) and \( q \) distinct primes belongs to \( \text{ISP}(L_{n_1}, \ldots, L_{n_m}) \).

To describe coproducts for singly-generated quasivarieties of MV-algebras we have available a special kind of natural duality since each \( L_n \) is a discriminator algebra (see [31] and [12, Section 3.12]). If a quasivariety of MV-algebras is generated by \( \{L_{n_1}, \ldots, L_{n_m}\} \) then each \( L_{n_i} \in S(L_k) \), where \( k = n_1 \ldots n_m \). We can then combine the duality for \( \text{ISP}(L_k) \) and Proposition 4.4 (see [28], where this approach was used to develop a duality for finitely generated quasivarieties of MV-algebras). Coproducts of MV-algebras have also been studied, using different tools, in [32] and [30, Chapter 7].

**Q-lattices.**

An algebra \( A = (A, \land, \lor, \nabla, 0, 1) \) is a **distributive lattice with a quantifier** (\( Q \)-lattice for short) if \( (A, \land, \lor, 0, 1) \in \mathcal{D} \) and \( \nabla \) is a unary operation satisfying the conditions:

\[
\nabla 0 \approx 0, \quad x \leq \nabla x, \quad \nabla(x \lor y) \approx \nabla x \lor \nabla y \quad \text{and} \quad \nabla(x \land y) \approx \nabla x \land \nabla y.
\]

The variety of \( Q \)-lattices was introduced in [9]. There it is proved that the lattice of subvarieties of \( Q \)-lattices is a chain. Each proper non-trivial subvariety is generated by a finite algebra \( D_{pq} \) where \( p, q \in \{0, 1, \ldots\} \), with \( p = q = 0 \) excluded (we refer to [9, Section 4] for precise definitions).

In [33], the second author presented piggyback natural dualities for these varieties \( \mathcal{D}_{pq} = \forall(D_{pq}) \). By [33, Theorem 3.6], the answer to (1) and to (2) is yes if and only if \( q \leq 1 \) and the answer to both questions is no otherwise. From [33, Theorem 3.10], it follows that the answer to (3) is yes only for \( \mathcal{D}_{10} \) and \( \mathcal{D}_{01} \) (see also [33, Table 1]). We deduce that \( U_{\mathcal{D}_{10}} \) and \( U_{\mathcal{D}_{01}} \) preserve coproducts, that \( U_{\mathcal{D}_{pq}} \) and \( U_{\mathcal{D}_{pq}} \), for \( p \geq 2 \) and \( q \geq 1 \), satisfy condition (E).
but not condition (S), and that $U_{D_{pq}}$ with $q \geq 2$ satisfies neither (E) nor (S). (To see that $U_{D_{00}}$ and $U_{D_{03}}$ preserve coproducts it suffices to observe that $D_{10}$ and $D_{01}$ are term-equivalent to $D$ and to Stone algebras, respectively.)

Coproducts of $Q$-lattices have not been studied in the literature, apart from what is included in this subsection, and the observation made in [10] that coproducts of $Q$-lattices do not correspond to cartesian products in the $D$-$P$-based so coproducts are not preserved. The natural duality developed in [33] can be used to describe coproducts on $D_{pq}$, but the the proliferation of relations in the alter ego as $p$ and $q$ increase makes a naive application of this technique unfeasible when $p + q$ is large.

**n-valued pre-Moisil and pre-Lukasiewicz–Moisil algebras.**

Our terminology here largely follows that in [4], which provides our primary source for facts about the classes of algebras we now consider. An algebra $A = (A, \wedge, \lor, 0, 1, \{f_i, \overline{f}_i | i \in \{1, \ldots, n - 1\})$ is called an $n$-valued pre-Lukasiewicz–Moisil algebra (or pre-LM$n$-algebra for short) if it satisfies the following:

1. $(A, \wedge, \lor, 0, 1) \in D$;
2. each $f_i$ is a $D$-endomorphism;
3. $f_i(a) \land \overline{f}_i(a) = 0$ and $f_i(a) \lor \overline{f}_i(a) = 1$, for each $i \in \{1, \ldots, n - 1\}$ and each $a \in A$;
4. $f_i \circ f_j = f_j$, for each $i, j \in \{1, \ldots, n - 1\}$;
5. if $i \leq j$, then $f_i(a) \leq f_j(a)$ for each $a \in A$.

An algebra $(A, \wedge, \lor, \neg, 0, 1, \{f_i | i \in \{1, \ldots, n\})$ is an $n$-valued pre-Moisil algebra (pre-M$n$-algebra for short) if its reduct $(A, \wedge, \lor, \neg, 0, 1)$ is a De Morgan algebra, and the algebra $(A, \wedge, \lor, 0, 1, \{f_i, \neg \circ f_i | i \in \{1, \ldots, n - 1\})$, is a pre-LM$n$-algebra. Following [4], we let $\mathcal{LM}_n^0$ and $\mathcal{M}_n^0$ denote the classes of pre-LM$n$-algebras and pre-M$n$-algebras, respectively.

In [4, Chapter 6], $D$-$P$-based dualities for $\mathcal{LM}_n^0$ and $\mathcal{M}_n^0$ were set up. In [4, Lemmas 5.20 and 5.24], it is proved that this duality sends coproducts into cartesian products. From this we infer that $U_{\mathcal{LM}_n^0}$ and $U_{\mathcal{M}_n^0}$ preserve coproducts. If it happens that $\mathcal{LM}_n^0$ is finitely generated then Theorem 3.6 implies that there exist an algebra $L_n^0 \in \mathcal{LM}_n^0$ and a lattice homomorphism $\omega: U_{\mathcal{LM}_n^0}(L_n^0) \rightarrow 2$, such that $\mathcal{ISP}(L_n^0) = \mathcal{LM}_n^0$ and (Sep)$L_n^0, \omega$. Similarly, if $\mathcal{M}_n^0$ is finitely generated, then there exist $M_n^0 \in \mathcal{M}_n^0$ and a lattice homomorphism $\omega: U_{\mathcal{M}_n^0}(M_n^0) \rightarrow 2$, such that $\mathcal{ISP}(M_n^0) = \mathcal{M}_n^0$ and (Sep)$M_n^0, \omega$ holds. Such algebras are not, however, exhibited in the literature on $n$-valued Moisil and Lukasiewicz–Moisil algebras. We shall remedy this omission.

The algebra $L_n^0 = \{0, 1\} \times \{0, 1, \ldots, n - 1\}, \wedge, \lor, 0, n - 1, \{e_i, \overline{e}_i | i \in \{1, \ldots, n - 1\})$
whose lattice order is the product order of \( \{0, 1\} \times \{0, 1, \ldots, n - 1\} \) and its operations are defined as follows:

\[
e_i(j, k) = \begin{cases} 
(0, 0) & \text{if } k < n - i, \\
(1, 1) & \text{otherwise;}
\end{cases}
\]

\[
\tau_i(j, k) = \begin{cases} 
(1, 1) & \text{if } k < n - i, \\
(0, 0) & \text{otherwise,}
\end{cases}
\]

is a pre-LM\(_n\)-algebra. Similarly, we obtain a pre-M\(_n\)-algebra

\[
M^0_n = (\{0, a, b, 1\} \times \{0, 1, \ldots, n - 1\}, \land, \lor, \neg, 0, n - 1, \{f_i \mid i \in \{1, \ldots, n - 1\}\})
\]

by equipping \( \{0, a, b, 1\} \times \{0, 1, \ldots, n - 1\} \) with the product lattice order and operations

\[
f_i(j, k) = \begin{cases} 
(0, 0) & \text{if } k < n - i, \\
(1, 1) & \text{otherwise,}
\end{cases}
\]

and \( \neg(j, k) = (\neg j, n - 1 - k) \), where \( \neg \) on the right-hand side has its interpretation in the De Morgan algebra 4.

We will now prove that the algebras L\(_n^0\) and M\(_n^0\) satisfy conditions (B)(i)–(ii) of Theorem 3.6. Observe that condition (B)(iii) follows from Lemma 3.4.

**Theorem 5.1.** For each \( n \), the following statements hold:

(i) \( \mathbb{ISP}(L^0_n) = \mathbb{LM}^0_n \);

(ii) if \( \omega_{L^0_n} : L^0_n \to 2 \) is given by \( \omega_{L^0_n}(x, y) = x \), then \( (\text{Sep})_{L^0_n}^{\omega_{L^0_n}} \) holds;

(iii) \( \mathbb{ISP}(M^0_n) = M^0_n \);

(iv) if \( \omega_{M^0_n} : L^0_n \to 2 \) is given by

\[
\omega_{L^0_n}(x, y) = \begin{cases} 
1 & \text{if } x \in \{a, 1\}, \\
0 & \text{otherwise,}
\end{cases}
\]

then \( (\text{Sep})_{M^0_n}^{\omega_{M^0_n}} \) holds.

**Proof.** To prove (i) let \( A = (A, \land, \lor, 0, 1, \{f_i, \tau_i \mid i \in \{1, \ldots, n - 1\}\}) \in \mathbb{LM}^0_n \) and let \( a, b \in A \) be such that \( a \neq b \). There is a lattice homomorphism \( h : U_{\mathbb{LM}^0_n}(A) \to 2 \) such that \( h(a) \neq h(b) \). Let \( h' : A \to \{1, \ldots, n - 1\} \) be defined by

\[
h'(a) = \begin{cases} 
0 & \text{if } h(f_i(a)) = 0 \text{ for } 1 \leq i \leq n - 1, \\
\max\{n - i \mid h(f_i(a)) = 1\} & \text{otherwise.}
\end{cases}
\]

It follows that the map \( \overline{h} : A \to K^0_n \) defined by \( \overline{h}(a) = (h(a), h'(a)) \) is a homomorphism of pre-LM\(_n\)-algebras. Since \( \overline{h}(a) \neq \overline{h}(b) \) we have that \( A \in \mathbb{ISP}(L^0_n) \). This concludes the proof of (i).

The proof of (iii) follows by a similar argument but using the fact that if \( A \in \mathbb{M}^0_n \) and \( a, b \in A \) are such that \( a \neq b \), there exists a homomorphism \( h \) of De Morgan algebras from \( A \) to 4 such that \( h(a) \neq h(b) \).
The proof of (ii) follows from the observation that for each \( i \in \{1, \ldots, n-1\} \) the maps \( \eta_i : L_n^0 \to L_n^0 \) defined by

\[
\eta_i(j, k) = \begin{cases} 
(0, k) & \text{if } k < i, \\
(1, k) & \text{otherwise}
\end{cases}
\]

are endomorphisms of \( L_n^0 \). If \( (j, k) \neq (j', k') \), then \( j \neq j' \) or \( k \neq k' \). If \( j \neq j' \), then \( \omega_{L_n^0}(j, k) \neq \omega_{L_n^0}(j', k') \). If \( k \neq k' \), we may assume without loss of generality that \( k < k' \). Then \( \omega_{L_n^0}(\eta_i(j, k)) \neq \omega_{L_n^0}(\eta_i(j', k')) \).

The proof of (iv) follows similar lines. \( \square \)

Here we have an example in which we use the flowchart in Fig. 8 in the reverse direction from hitherto. That is, knowing that the forgetful functor into \( n \) only if \( \Omega = HU \) generalizes that \( L \) satisfies (Sep) \( M \). The answer to question (1) is yes. The only endomorphisms \( L \) and \( M \) be the algebras, in \( LM_n \) and \( M_n \) respectively, in which the operations are defined as follows:

\[
d_i(j) = \begin{cases} 
0 & \text{if } j < n - i, \\
n - 1 & \text{otherwise;}
\end{cases}
\]

\[
\tilde{d}_i(j) = \begin{cases} 
n - 1 & \text{if } j < n - i, \\
0 & \text{otherwise;}
\end{cases}
\]

and \( \neg(j) = n - 1 - j \). Then \( \mathcal{M}_n = ISP(L_n) \) and \( M_n = ISP(M_n) \) [4, Corollary 6.1.9]. The answer to question (1) is yes. The only endomorphisms of \( L_n \) and \( M_n \) are the corresponding identity maps [4, Theorem 6.1.6]. Thus to satisfy \( (\text{Sep})_{LM_n}^{\Omega} \) we must take \( \Omega = HU_{LM_n}(L_n) \). Similarly \( (\text{Sep})_{M_n}^{\Omega} \) holds only if \( \Omega = HU_{M_n}(M_n) \). Then the answer to (2) is yes if and only if \( n \leq 2 \). Lemma 3.4 is applicable, so the answer to (3) is yes for each \( n \geq 1 \).

To describe coproducts in \( LM_n \) and \( M_n \) fully we can use the piggyback duality developed by the second author in [34]. Alternatively, we can apply the strategy of Table 2 with \( LM_n^0 \) and \( M_n^0 \) as the enveloping varieties. It can be

Moisil and Lukasiewicz–Moisil algebras.

An algebra \( A = (A, \land, \lor, 0, 1, \{f_i, \overline{f}_i \mid i \in \{1, \ldots, n\}) \) is called an \( n \)-valued Lukasiewicz–Moisil algebra if it satisfies the following:

(1) \( A \) is an \( n \)-valued pre-Lukasiewicz–Moisil algebra;

(2) if \( f_i(a) = f_i(b) \) for each \( i \in \{1, \ldots, n\} \), then \( a = b \) for each \( a, b \in A \).

Similarly, an algebra \( A = (A, \land, \lor, \neg, 0, 1, \{f_i \mid i \in \{1, \ldots, n\}) \) is called an \( n \)-valued Moisil algebra if it is an pre-Mn-algebra and satisfies (2).

Let \( LM_n \) and \( M_n \) denote the classes of \( n \)-valued Lukasiewicz–Moisil and Moisil algebras, respectively. Let

\[
L_n = (\{0, 1, \ldots, n - 1\}, \min, \max, 0, n, \{d_i, \overline{d}_i \mid i \in \{1, \ldots, n-1\})
\]

and

\[
M_n = (\{0, 1, \ldots, n - 1\}, \min, \max, \neg, 0, n, \{d_i \mid i \in \{1, \ldots, n-1\})
\]

be the algebras, in \( LM_n \) and \( M_n \) respectively, in which the operations are defined as follows:

\[
d_i(j) = \begin{cases} 
0 & \text{if } j < n - i, \\
n - 1 & \text{otherwise;}
\end{cases}
\]

\[
\overline{d}_i(j) = \begin{cases} 
n - 1 & \text{if } j < n - i, \\
0 & \text{otherwise;}
\end{cases}
\]

and \( \neg(j) = n - 1 - j \). Then \( LM_n = ISP(L_n) \) and \( M_n = ISP(M_n) \) [4, Corollary 6.1.9]. The answer to question (1) is yes. The only endomorphisms of \( L_n \) and \( M_n \) are the corresponding identity maps [4, Theorem 6.1.6]. Thus to satisfy \( (\text{Sep})_{LM_n}^{\Omega} \) we must take \( \Omega = HU_{LM_n}(L_n) \). Similarly \( (\text{Sep})_{M_n}^{\Omega} \) holds only if \( \Omega = HU_{M_n}(M_n) \). Then the answer to (2) is yes if and only if \( n \leq 2 \). Lemma 3.4 is applicable, so the answer to (3) is yes for each \( n \geq 1 \).

To describe coproducts in \( LM_n \) and \( M_n \) fully we can use the piggyback duality developed by the second author in [34]. Alternatively, we can apply the strategy of Table 2 with \( LM_n^0 \) and \( M_n^0 \) as the enveloping varieties. It can be
proved that the dual counterparts of the retraction functors $R_{L\mathcal{M}_n}$ and $R_{\mathcal{M}_n}$ under the Priestley dualities for $L\mathcal{M}_n$ and $\mathcal{M}_n$ correspond to the assignment to $(X, \psi, \mathcal{J})$ of the closed subspace

$$Y = \{ x \in X \mid \psi_i(x) = x \text{ for some } i \in \{1, \ldots, n\} \}.$$ 

Our conclusions here strengthen the results detailed in [4, Theorem 7.5.9] where duals of coproducts of only two algebras were calculated.

Finally, we remark that a translation between the natural dualities for $L\mathcal{M}_n$ and $\mathcal{M}_n$ and the Priestley-style dualities for these classes was provided in [34, Theorem 3.9]. The process that is described there for obtaining $HU_{L\mathcal{M}_n}$ and $HU_{\mathcal{M}_n}$ from the corresponding natural duals is exactly the one described in Theorem 2.3 applied to (Lukasiewicz-)Moisil algebras.

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