Dynamical delocalization of Majorana edge states by sweeping across a quantum critical point

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\textbf{Abstract.} We study the adiabatic dynamics of Majorana fermions across a quantum phase transition. We show that the Kibble–Zurek scaling, which describes the density of bulk defects produced during the critical point crossing, is not valid for edge Majorana fermions. Therefore, the dynamics governing an edge state quench is nonuniversal and depends on the topological features of the system. Besides, we show that the localization of Majorana fermions is a necessary ingredient to guarantee robustness against defect production.

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1. Introduction

When a certain control parameter is varied, quantum mechanical fluctuations may drive a critical change of the system ground state [1]. Even though such phase transitions occur at zero temperature, they have a profound influence on phenomena as diverse as high-$T_c$ superconductivity [2], magnetism or quantum Hall effects (QHEs). The potential in both fundamental and applied research has stimulated an increasing interest among the physical community in studying the effects of the dynamical crossing of quantum critical points [3]–[30]. At such points, the correlation length of the system diverges, and the characteristic energy gap between the ground state and the lowest lying excitations vanishes. Accordingly, adiabatic evolution is precluded, and any dynamical quench across the critical point is accompanied by a production of excitations in the system. As a result, the final state will only be partially ordered, displaying a nonvanishing density of defects imported from the quantum disordered phases into the ordered phases. Such a density of defects is uniquely determined by the universality class of the system and can be accurately described by the so-called Kibble–Zurek (KZ) scaling [3, 4].

KZ scaling was formerly proposed as the mechanism underlying topological defect production in a cosmological scenario [3]–[5] or in classical phase transitions occurring at finite temperature [6, 7]. The extension of the KZ mechanism to the quantum domain was proposed in [8], and numerically confirmed in the transverse Ising model, a cornerstone in the theory of quantum phase transitions. This result paved the way to a considerable amount of work dealing with defect production in the zero-temperature regime [9]–[17], which showed that the KZ scenario also holds for quantum phase transitions (see [18] for an experiment). Modifications to the characteristic scaling laws have been studied in order to account for nonlinear quenches [19, 20], crossing of multi-critical points [21] (see [22] for a recent proposal on the implementation of tri-critical points in trapped ions), quenches across a critical region [23, 24], or quenches along a gapless line [25]. The effect of classical noise on the driving parameter has also been studied in [26]. In [27]–[29], it was shown that the density of defects scales with the rate of the quench even at finite temperatures in a regime of incipient criticality.

Recently, we evidenced that the ground state topology of the system modifies the quench dynamics in a subtle manner [30]. In particular, we showed how single fermions bound to the edges of the Creutz ladder [31] induce an anomalous defect production as the magnetic flux is swept across a quantum critical point. In this paper, we reinforce the scenario by providing further evidence of anomalous defect production in the adiabatic dynamics of edge states. We consider a set of Majorana fermions (i.e. real fermions) arranged in a one-dimensional chain [32, 33]. We describe the deviations of standard KZ scaling, and identify the necessary ingredients to obtain robustness of edge states even in a critical region by resorting to the experience gained with the Creutz ladder [30]. The sensitivity to boundary conditions generically indicates that a nontrivial topological order is encoded into the system. This is specified by the presence of states localized at the boundaries of the system, the so-called edge states [34]. Note also that topological edge states arise naturally in a wide variety of systems, such as one-dimensional spin models [35] and the integer and fractional QHEs [36, 37], and have been experimentally realized on topological insulators (TIs) [38]–[40].

The subtle difference between trivial insulators and TIs is rooted in the nontrivial topological features of the bulk bands of the latter, such as nonvanishing Chern numbers in the integer QHE [41]. TIs represent an intriguing state of matter, where edge transport exists even in the presence of an insulating bulk gap [42]. These edge states are usually localized in...
the interface that separates two topologically distinct insulators, the simplest example being the interface between an integer quantum Hall effect sample and vacuum [36]. Besides, they connect the conduction and valence bands and allow for conduction along the interface. Other interesting TIs are the anomalous half-integer QHE in the honeycomb [43] or square lattices [44], and those characterized by \( \mathbb{Z}_2 \) topological invariants [45]–[50], which lead to the quantum spin Hall effect (see [51] for a recent experimental proposal with ultracold atoms). Depending on the symmetries of the Hamiltonian and the dimension of the system, TIs can be classified in a periodic table [52, 53]. Accordingly, one of the simplest TIs is the one-dimensional chain of Majorana fermions [32], where edge states correspond to bound Majorana fermions localized at the chain ends. The possibility to isolate Majorana fermions [54] represents an important step towards topological quantum computation [55, 56] and can, in principle, be achieved in the vortex core of two-dimensional \( p + ip \) superconductors [57]. A recent proposal is to create, fuse and transport Majoranas by proximity effects in the tri-junctions of superconductor–TI–superconductor [58, 59]. Therefore, the results on the quench dynamics of the idealized Majorana chain discussed in this work might have applications in the more realistic scenario described in [58]–[64].

This paper is organized as follows. In section 2, we describe the nature of Majorana edge states in a one-dimensional lattice system with broken U(1) gauge symmetry. In section 3, we present the results for the quench dynamics of the Majorana chain across a quantum phase transition. In particular, we show that bulk states fulfill the KZ prediction, whereas topological edge states provide a clear anomaly. Finally, we present the conclusions of this work in the light of topological band insulators in section 4. In appendix A, we describe the main properties of bulk states in the periodic chain.

2. The Majorana chain

In this section, we review the properties of the Majorana chain [32], a fermionic one-dimensional system that breaks time-reversal and U(1) gauge symmetries and presents Majorana fermions bound to its edges. We present a detailed description of the different phases where edge states occur, and discuss the nature of the quantum phase transition connecting the aforementioned topological phases. A discussion on the bulk spectrum is presented in appendix A.

Let us consider a system of spinless fermions hopping in a one-dimensional chain according to the following Hamiltonian:

\[
H = \sum_j \left( -w a_j^\dagger a_{j+1} + \Delta a_j a_{j+1} - \frac{\mu}{2} a_j^\dagger a_j + \text{h.c.} \right),
\]

(1)

where \( a_j \) (\( a_j^\dagger \)) represent spinless fermionic annihilation (creation) operators satisfying canonical anticommutation relations \( \{ a_j, a_k^\dagger \} = a_j a_k^\dagger + a_k^\dagger a_j = \delta_{jk} \). As a second quantized Hamiltonian, equation (1) describes a mean-field Bardeen–Cooper–Schrieffer (BCS) superconductor [65], where \( w \) is the hopping amplitude, \( \Delta = |\Delta| e^{i\theta} \) is the superconducting gap and \( \mu \) is the chemical potential. Due to the U(1) symmetry breaking inherent to superconductors (i.e. absence of fermion number conservation), it is more appropriate to describe the system in terms of Majorana fermions

\[
c_{2j-1} = \frac{1}{\sqrt{2}} \left( e^{-i\theta/2} a_j^\dagger + e^{i\theta/2} a_j \right),
\]

\[
c_{2j} = \frac{1}{\sqrt{2}} \left( e^{-i\theta/2} a_j^\dagger - e^{i\theta/2} a_j \right),
\]

(2)
Figure 1. Scheme of the Majorana chain represented as a virtual ladder. The initial fermionic chain is mapped onto the Majorana fermions by means of equation (2), which can then be represented in a virtual ladder, where the \( j \)th rung contains two Majoranas \( c_{2j-1}, c_{2j} \) and corresponds to the \( j \)th site of the original chain. In the ladder scheme, the Majorana couplings in equation (3) become simple vertical and diagonal links.

which are Hermitian operators \( c_{2j-1}^\dagger = c_{2j-1}, c_{2j}^\dagger = c_{2j} \) satisfying the Majorana fermionic algebra \( \{c_j, c_k\} = \delta_{jk} \). Although Majorana excitations are commonly paired to constitute a standard fermion, very recent research activity proposes to isolate and manipulate them, topological quantum computation being the major motivation for doing so [55, 56]. Here, \( v = \frac{\nu}{2} \) fractional QHE [66]–[68] and time-reversal TIs [58] are the most promising candidates.

In order to understand how the Majorana chain supports unpaired Majorana fermions at the boundaries, we need to rewrite the Hamiltonian of equation (1) in the Majorana picture:

\[
H = \sum_j \left( tc_{2j}c_{2j+1} + uc_{2j-1}c_{2j+2} + vc_{2j-1}c_{2j} \right),
\]

where \( t = w + |\Delta|, u = -w + |\Delta| \) and \( v = -\mu \) represent the different couplings between Majoranas, which might be interpreted as simple links in the ladder configuration of figure 1.

We shall be interested in the following regimes [51]:

1. **Left-handed regime**: This regime occurs at \((t, u, v) = (0, 2|\Delta|, 0)\) and presents the following zero-energy edge states, completely localized within the boundary rungs

\[
|l_1\rangle = \left| \begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
\circ & \circ & \cdots & \circ \\
\end{array} \rightangle = c_2|\Omega\rangle, \quad |r_1\rangle = \left| \begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
\circ & \circ & \cdots & \circ \\
\end{array} \rightangle = c_{2L-1}|\Omega\rangle,
\]

where \( |\Omega\rangle \) is the Bogoliubov vacuum (see equation (A.7) in appendix A). As represented in figure 2(a), this is the only possibility to obtain localized Majoranas, since the remaining bulk eigenstates consist of paired Majoranas with energies \( \epsilon_{j\pm} = \pm 2|\Delta| \), and eigenstates

\[
|\epsilon_{j\pm}\rangle_1 = \left| \begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
\circ & \circ & \cdots & \circ \\
\end{array} \rightangle = \frac{1}{\sqrt{2}}(c_{2j-1}c_{2j+2}|\Omega\rangle).
\]

The bulk solution consists of a flat band of plaquettes (i.e. two rungs) that contain a couple of Majoranas dimerized into a standard complex fermion.

2. **Right-handed regime**: In this regime \((t, u, v) = (2|\Delta|, 0, 0)\), we obtain the following zero-energy edge states completely localized within the boundary rungs:

\[
|l_1\rangle = \left| \begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
\circ & \circ & \cdots & \circ \\
\end{array} \rightangle = c_2|\Omega\rangle, \quad |r_1\rangle = \left| \begin{array}{cccc}
\circ & \circ & \cdots & \circ \\
\circ & \circ & \cdots & \circ \\
\vdots & \vdots & \ddots & \vdots \\
\circ & \circ & \cdots & \circ \\
\end{array} \rightangle = c_{2L}|\Omega\rangle.
\]
Figure 2. Scheme of unpaired Majorana fermions bound at the edges, and dimerized Majoranas at the bulk. (a) In the left-handed regime, only the left-tilted diagonal links survive, and thus a couple of unpaired Majoranas appear at sites $2, 2L - 1$ of the ladder. (b) In the right-handed regime, only the right-tilted diagonal links survive, and thus a couple of unpaired Majoranas appear at sites $1, 2L$. (c) In the topologically trivial regime, every Majorana fermion is paired within a single rung, and thus no edge can exist.

In figure 2(b), we represent the localized Majoranas, and the bulk plaquette-like eigenstates with energies $\varepsilon_{j\pm} = \pm 2|\Delta| \pm i c \Delta$ become

$$|\varepsilon_{j\pm}\rangle = \left| \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{array} \right| \frac{1}{\sqrt{2}}(\pm ic_{2j} + c_{2j+1})|\Omega\rangle. \quad (7)$$

Note how the bulk dimers containing a couple of paired Majoranas have been tilted with respect to the left-handed regime in figure 2(a).

3. Topologically trivial regime: In the regime $|t|, |u| \ll |v|$, every Majorana is paired up with its rung partner, and forms plaquette-like eigenstates in flat bands $\varepsilon_{j\pm} = \pm \mu$:

$$|\varepsilon_{j\pm}\rangle = \left| \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \end{array} \right| \frac{1}{\sqrt{2}}(\mp ic_{2j-1} + c_{2j})|\Omega\rangle. \quad (8)$$

Therefore, in this trivial regime, edge states are forbidden (see figure 2(c)).

We have thus shown that the Majorana chain Hamiltonian introduced in equation (2) entails different phases where edge states are pinned to the system boundary, but also trivial phases where unpaired Majoranas cannot exist. We comment that the Hamiltonian in equation (1) can be realized with junctions involving s-wave superconductors and TIs [58]. In particular, these different phases can be engineered in superconductor–TI tri-junctions, where the key parameter that determines the existence of unpaired Majoranas is the relative phase of the superconductors. In the following section, we describe the quantum phase transitions between the phases introduced above, and study how the edge and bulk states behave when the system is driven across a quantum critical point.

3. Quench between edge states

In this section, we study the quantum phase transition between the left- and right-handed phases (see (a) and (b) above). Such a quantum phase transition can be characterized by $t = 0$ and...
a relative parameter $\zeta = w/|\Delta|$, so that the left-handed regime is realized at $\zeta = -1$ and the right-handed at $\zeta = 1$. In figure 3, we depict the energy spectrum of the Majorana Hamiltonian in equation (3) for $\zeta \in [-1, 1]$, which clearly shows a quantum critical point at $\zeta_c = 0$ where the energy gap vanishes. Note also how two zero-energy modes, corresponding to the unpaired Majoranas introduced above, are present for arbitrary $\zeta$. Let us remark here that such edge modes are absent in the periodic chain (see equation (A.6) and figure A.1 in appendix A).

The critical properties of this quantum phase transition are characterized by the critical exponents $\nu = z = 1$, which describe how the coherence length $\xi_l$ diverges and the energy gap $\Delta \varepsilon$ vanishes at criticality $\xi_l \sim \Delta \varepsilon^{-1} \sim |\zeta - \zeta_c|^{-1}$. We shall consider the adiabatic dynamics of a state initially localized within the bulk or edge of the system for $w(t_0) = -|\Delta|$. At the end of the quench $w(t_f) = -|\Delta|$, the system shall be found in a state $|\Psi(t_f)\rangle$ given by the superposition of the adiabatically connected state, which is still localized, and delocalized defects that have been excited close to the critical point. Since we are concerned with single-particle edge or bulk states, we shall refer to the probability to create a defect $P_{\text{def}} = \sum_{E > 0} |\langle E |\Psi(t_f)\rangle|^2$, with $\langle E |$ being the eigenvectors of the Hamiltonian. Interestingly enough, for a linear adiabatic quench $\zeta(t) = -1 + v_Q t$ with rate $v_Q \ll 1$ and lasting for $t \in [0, 2/v_Q]$, the KZ mechanism predicts that such probability scales as

$$P_{\text{def}}^{KZ} \sim \frac{v_Q}{v_Q^{\frac{1}{\nu z}}},$$

where $d$ is the dimension of the system [3, 4]. In the present phase transition, the KZ mechanism predicts a scaling $P_{\text{def}}^{KZ} \sim \sqrt{v_Q}$. In the following sections, we shall confront the KZ prediction with the exact dynamics of the Majorana chain.

### 3.1. Quench dynamics in the periodic chain

In appendix A, we describe the energy spectrum of the translationally invariant Majorana chain, and show how the single-particle energy levels can be expressed as excitations over the Bogoliubov vacuum (i.e. $\gamma^\dagger_{\omega} |\Omega\rangle$). Here, we study the quench dynamics of the periodic chain when the parameter $\zeta$ is adiabatically modified across the corresponding critical point. In the translationally invariant case, no edge states exist, and the only localized state in the negative
energy band $E = -2|\Delta|$ at $\zeta = -1$ is a plaquette-like fermion:

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (-\imath c_{2j-1} + c_{2j+2}) |\Omega\rangle = \sum_q \Psi_q^c(t_0) c_q |\Omega\rangle,$$

where $c_q(t_0) = \frac{\imath e^{\imath \omega_q t}}{2\sqrt{L}} (e^{-\imath \theta/2} (-1 + e^{\imath \theta}) - e^{\imath \theta/2} (1 + e^{\imath \theta}))$ determines such an initial state in momentum Nambu representation. Remarkably, the whole quench dynamics can be expressed as a collection of uncoupled two-level systems, each associated to a Nambu spinor $\Psi_q$. Therefore, the dynamics under $\zeta(t) = -1 + v_Q t$ entails an ensemble of two-level Landau–Zener processes \cite{11, 69, 70}, and the final excitation probability (compare to equation (12)) can be simply calculated as

$$P_{\text{def}} = \sum_q |c_q(t_f)\rangle^2,$$

where $c_q(t_f)$ is the final state after the quench evolution and $c_q^{t=1}$ corresponds to the positive-energy eigenstates at $\zeta = +1$. The evaluation of the excitation probabilities is simpler in this translationally invariant case, and we can thus treat larger chains and slower quenches. In figure 4, we observe that the scaling fully agrees with the KZ prediction $P_{\text{def}} \sim \sqrt{v_Q}$.

In this section, we have considered a single-particle initial state localized within a rung of the Majorana ladder. Let us note that KZ scaling also holds for an initial ground state, which consists of all the negative-energy modes being occupied. For such states, one usually refers to the density of defects $n_{\text{def}}^{\text{KZ}} \sim \sqrt{v_q}$, which is finite in the thermodynamical limit. However, since we are interested in the comparison of initial single-particle bulk or edge states, we shall refer to the probability to create defects $P_{\text{def}}$ rather than to a thermodynamical density.

### 3.2. Quench dynamics in the open chain

The objective of this section is to test the prediction for an initial plaquette-like state in the bulk and for an initial edge Majorana fermion (see figures 5(a) and (b) for the adiabatic evolution of such initial states).
3.2.1. Adiabatic quench for an initial plaquette. In this case, the initial state corresponds to a negative energy $\varepsilon = -2|\Delta|$ localized in the bulk $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}(-ic_{L-1} + c_{L+2})|\Omega\rangle$ for $\zeta = -1$. We evaluate the defects produced close to the critical point as follows:

$$P_{\text{def}} = \sum_{\varepsilon > 0} |\langle \varepsilon |\Psi(t_f)\rangle|^2. \quad (12)$$

Let us emphasize that the initial bulk fermion can be expressed as a linear combination of every periodic eigenstate in the negative-energy band (see equation (A.3)). In particular, the modes responsible for universal KZ scaling (i.e. those around the gapless mode $q \sim 0$) shall be populated, and one thus expects the KZ prediction to hold. Indeed, the numerical results in figure 6(a) are in clear agreement with the predicted scaling $P_{\text{def}}^{\text{KZ}} \sim \sqrt{vQ}$ at the thermodynamical limit $L \rightarrow \infty$ (see also figure 4 for KZ scaling in a periodic chain).

3.2.2. Adiabatic quench for an initial edge Majorana. In this case, the initial state corresponds to a zero-energy Majorana fermion bound to the left edge $|\Psi(0)\rangle = |l\rangle = c_2|\Omega\rangle$ for $\zeta = -1$. Let us note that this state also crosses the critical point at the gapless mode energy, and thus one would expect a similar KZ scaling to hold, $P_{\text{def}}^{\text{KZ}} \sim \sqrt{vQ}$. Note, however, that the obtained scaling (figure 6(b)) completely disagrees, $P_{\text{def}} \sim (vQ)^0$, and thus offers further evidence of the anomalous defect production for localized edge states [30]. The departure from universal KZ scaling is drastic for such Majorana fermions. Besides, it predicts that the edge state robustness is lost as the critical point is crossed (i.e. $P_{\text{def}} = \frac{1}{2}$ for both positive- and negative-energy bands). The underlying reason for such a departure is that the edge state at the critical point can be expressed as a linear combination of the gapless modes $q = \{0, \pi\}$, and thus fuses into the bulk bands. Indeed, we show in the next section that in the thermodynamical limit

$$|l(\zeta_c)\rangle_l \sim (c_2 + c_6 + \cdots + c_{2L-1})|\Omega\rangle \sim \frac{1}{\sqrt{2}} \left(\gamma_{0,+}^\dagger + \gamma_{0,-}^\dagger\right)|\Omega\rangle, \quad (13)$$

where $\gamma_{q,\pm}^\dagger$ are the fermionic creation operators of positive- and negative-energy solutions associated with the mode $q$ (see equation (16)). Therefore, the initially unpaired Majorana at

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Figure 6. Scaling of the defects produced by adiabatically crossing the \( \zeta_c = 0 \) critical point for a lattice with number sites \( L \in \{10, 20, 30, 40, 50, 60, 70, 80, 160\} \). (a) Defects \( P_{\text{def}} \) for an initial negative-energy plaquette localized at the bulk. (b) Defects \( P_{\text{def}} \) for an initial zero-energy Majorana localized at the left edge of the sample.

the left edge becomes completely delocalized along the upper chain of the rung (see figure 5(b)) at criticality \( \zeta_c = 0 \). Being uniformly delocalized, it overlaps with bulk excitations with well-defined momenta \( q = \{0, \pi\} \), and therefore completely fuses in the continuum bands. In this regard, the Majorana character of the initial state is completely lost across the quantum critical point, and one cannot further transport the Majorana to the right-hand side (i.e. \( |l\rangle_1 \not\rightarrow |r\rangle_2 \)).

Due to the finite number of modes coupled to the edge states, this anomalous dynamics might be viewed as a saturation effect, such as that studied in the XY model [26]. Nevertheless, we remark that the effect presented in this work traces back indeed to the topological order of the system, since it only occurs for the unpaired Majorana edge states. In a trivial insulator, where in-gap edge states are simply absent, we should have restricted ourselves to bulk plaquette-like excitations, which do not lead to any anomaly in KZ scaling.

Let us now compare these results with the edge-state defect production in the Creutz model, a ladder of usual fermions pierced by a perpendicular magnetic field [30]. In this system,
quench of the magnetic flux drives the fermions across a quantum phase transition of the same universality class $v = z = d = 1$. Similarly to the Majorana chain, the defect production for initial bulk states fulfills $P_{\text{def}} \sim \sqrt{vQ}$ in accordance with KZ scaling. This result emphasizes the fact that both models, being microscopically different, belong to the same universality class.

Conversely, edge states in the Creutz ladder fulfill $P_{\text{def}} \sim v^{1.35}$, which completely deviates from the KZ prediction, and from the defect production of a Majorana chain $P_{\text{def}} \sim v_0^0$. This result gives further evidence of the conjecture on the nonuniversality of the edge-state dynamics across a critical point, as raised in [30]. Moreover, the result on the Creutz ladder highlights the resilience of edge states due to a quantum interference phenomenon that allows them to remain well-localized at the boundaries of the lattice. Conversely, Majorana edge states become completely delocalized at the critical point, and this leads to a higher sensitivity at criticality $P_{\text{def}} \sim v_0^0$. Therefore, we can conclude that edge state robustness is nonuniversal and is intimately related to edge state localization.

### 3.3. Adiabatic path for edge states

Here, we show how the unpaired Majoranas, initially localized at the chain edges, become completely delocalized over the whole chain when the critical point $\zeta_c = 0$ is achieved. Besides, we show that such delocalized states overlap with the continuum modes in the thermodynamic limit $L \to \infty$, and thus justify the scaling $P_{\text{def}} \sim \frac{1}{2}(vQ)^0$. Remarkably, the zero-energy modes associated with unpaired Majoranas can be analytically obtained in the whole regime $-1 \leq \zeta \leq 0$, $\eta = 0$, and read as follows:

$$|l(\zeta)\rangle = \frac{1}{N_l} \left( c_2 + rc_6 + \cdots + r^{(L+1)/2}c_{2L} \right) |\Omega\rangle,$$

$$|r(\zeta)\rangle = \frac{1}{N_r} \left( c_{2L-1} + rc_{2L-5} + \cdots + r^{(L+1)/2}c_1 \right) |\Omega\rangle,$$

where $r = (1 + \zeta)/(1 - \zeta)$ and $N_l, N_r$ are normalization constants. Note that the exponential tail of the edge states fulfills $0 \leq r \leq 1$ in the regime of interest. In particular, at the initial point $\zeta = -1$, one can easily check that these edge states become completely localized at the edge rungs and coincide with equations (4). Conversely, at criticality $\zeta_c = 0$, the Majoranas become uniformly delocalized along each chain forming the virtual rung:

$$|l(\zeta_c)\rangle = \frac{1}{\sqrt{L}} \left( c_2 + c_6 + \cdots + c_{2L} \right) |\Omega\rangle,$$

$$|r(\zeta_c)\rangle = \frac{1}{\sqrt{L}} \left( c_1 + c_5 + \cdots + c_{2L-1} \right) |\Omega\rangle.$$

It is precisely at this critical point where they overlap with the bulk gapless modes (see equation (A.3), and thus

$$|l(\zeta_c)\rangle = \frac{1}{\sqrt{2}} \left( \gamma_{0,+}^\dagger + \gamma_{-\pi,-}^\dagger \right) |\Omega\rangle,$$

$$|r(\zeta_c)\rangle = \frac{1}{\sqrt{2}} \left( -\gamma_{0,+}^\dagger + \gamma_{-\pi,-}^\dagger \right) |\Omega\rangle.$$

Consequently, the edge states will not follow the adiabatic path that would connect them to the edge Majoranas at $\zeta = +1$, but rather fuse into the positive- and negative-energy bulk bands.  

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This explains the anomalous scaling $P_{\text{def}} = \frac{1}{2}$ for edge states (note also that the probability to decay into the negative-energy band would be $P_{\text{neg}} = \frac{1}{2}$).

4. Conclusions

In this paper, we have described the quench dynamics of a one-dimensional-lattice model with unpaired Majorana fermions bound to the boundaries [32]. The transport of Majorana fermions between the edges of the chain is hampered by the loss of adiabaticity at the quantum critical point. The main conclusion drawn from this work is two-fold: edge-state dynamics across a critical point is not universal, and the robustness/fragility of the latter relies on their localization at criticality.

We have found that the scaling of defects deviates from the usual KZ paradigm, and is nonuniversal. Indeed, the Creutz ladder [30] and the Majorana chain present different scalings within the same bulk universality class. With respect to our results on the Creutz ladder [30], we conclude that such deviations are due to the localization dynamics of edge states. We note, in particular, that Majorana edge states produce a peculiar scaling of excitations due to their interesting delocalization dynamics. At the quantum critical point, Majorana edge states ‘fuse’ with the bulk excitations, and thus become delocalized over the system. This should be contrasted with the phenomenology evidenced for the Creutz ladder, where edge states stay localized for any value of the control parameter. Such a difference, in turn, traces back to the physical content of such states: for the Creutz ladder, standard fermions are pinned to the ladder edges due to interference effects caused by additional magnetic fields; for the Majorana chain, they are induced by the U(1) symmetry-breaking term provided by a superconducting gap. Despite this quantitative difference in the microscopic origin, we note that both systems belong to the class of topological band insulators [52, 53]. Therefore, we may conclude that the quench dynamics of boundary states in TIs should provide additional anomalies in the defect production rate.

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Appendix A. Bulk spectrum of the Majorana chain

In this appendix, we describe the bulk spectrum of the Majorana chain for periodic boundary conditions ($a_{L+1} = a_1$). Let us first point out that the Hamiltonian in equation (1), after a standard Jordan–Wigner transformation [71], describes a set of localized spins governed by

$$H = -\frac{1}{2} |\Delta| \cos \theta \sum_{j=1}^{N} (\zeta \sec \theta + 1) \sigma_j^x \sigma_{j+1}^x + (\zeta \sec \theta - 1) \sigma_j^y \sigma_{j+1}^y$$

$$+ \tan \theta (\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+),$$

where $h = \mu/|\Delta| \cos \theta$ plays the role of an external transverse magnetic field, and the superconducting phase $\theta$ leads to local lattice anisotropy in the exchange couplings,
namely, \( J_z = (\zeta \sec \theta + 1) \), and \( J_y = (\zeta \sec \theta - 1) \) with \( \zeta = w/|\Delta| \). Notice also the appearance of additional spin coupling terms \( \sigma_j^+ \sigma_{j+1}^- \), which are absent in standard anisotropic XY models \([72, 73]\). Accordingly, a spin picture offers an alternative perspective on the U(1) symmetry-broken fermion system in equation (1). In momentum space, the fermionic operators become \( a_j = \frac{1}{\sqrt{L}} \sum_q a_q e^{-i q j} \), where \( q \in [-\pi, \pi] \) lies in the Brillouin zone and we have assumed unit lattice spacing. The initial Hamiltonian in equation (1) can be expressed as follows:

\[
H = \sum_q \Psi_q^\dagger H_q \Psi_q, \quad H_q = \begin{pmatrix} \zeta_q & \Delta_q \\ \Delta_q^* & -\zeta_q \end{pmatrix}, \tag{A.2}
\]

where we have introduced the Nambu spinor \( \Psi_q = (a_q, a_{-q}^\dagger)^\dagger \), and \( \zeta_q = -(w \cos q + \mu/2) \), \( \Delta_q = i|\Delta|e^{-i\nu \sin q} \). The energy spectrum is obtained after a Bogoliubov transformation is performed, in a similar procedure as the BCS Hamiltonian is diagonalized [65]. Transforming the fermion operators according to

\[
\gamma_q^+ = u_q a_q + v_q a_{-q}^\dagger, \quad \gamma_q^- = -v_q^* a_q + u_q a_{-q}^\dagger, \tag{A.3}
\]

one obtains the energies \( \epsilon_q^\pm = \pm \epsilon_q = \pm \sqrt{\zeta_q^2 + |\Delta_q|^2} \) corresponding to the positive- and negative-energy solutions \( \gamma_q^\pm \), defined through the following parameters:

\[
u_q = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\zeta_q}{\epsilon_q}}, \quad \epsilon_q = \pm \sqrt{\frac{1 - \zeta_q}{\epsilon_q}}. \tag{A.4}
\]

Besides, the solutions fulfill \( \gamma_{-q}^\pm = \gamma_{q \mp}^\dagger \), which allows us to restrict the Hamiltonian to half-Brillouin zone:

\[
H = \sum_{q > 0} \left( 2 \epsilon_q \gamma_q^\dagger \gamma_q + 2 \epsilon_q \gamma_q^\dagger \gamma_q - 2 \epsilon_q (\gamma_q^\dagger - \gamma_q) \right). \tag{A.5}
\]

Therefore, the final energy spectrum for a periodic Majorana chain is

\[
E_{q \pm} = \pm 2 \epsilon_q = \pm 2 |\Delta| \sqrt{(\eta + \zeta \cos q)^2 + \sin^2 q}, \tag{A.6}
\]

where we have introduced the relative parameters \( \zeta = w/|\Delta| \), \( \eta = \mu/2|\Delta| \) used to study the quantum phase transitions in the open chain. In figure A.1, we represent the energy spectrum as a function of \( \zeta \) for \( \eta = 0 \). We clearly identify a nontopological two-band insulator since no mid-gap edge states occur at zero energies. This fact should be compared to the open ladder figures 3, where zero-energy edge states localized at the interface between the TI and vacuum arise. In these figures, it is also clear that the energy gap vanishes at the critical points \( \zeta_c = 0 \), indicating a quantum phase transition. Furthermore, from the exact energy dispersion in equation (A.6), one can show that the critical exponents of both transitions are \( z = \nu = 1 \).

Let us finally comment on the system vacuum and single-particle states. One may define the Bogoliubov vacuum as a state satisfying \( \gamma_q^\dagger |\Omega\rangle = 0 \), \( \forall q \in \text{BZ} \). This vacuum can be constructed from the Fock state vacuum \( |0\rangle \) as follows:

\[
|\Omega\rangle = \prod_q \gamma_q^\dagger \gamma_q |0\rangle = \prod_q (u_q + v_q a_{-q}^\dagger a_q^\dagger) |0\rangle, \tag{A.7}
\]

which shows that the Bogoliubov vacuum contains the usual Fock vacuum, into which a number of exotic spinless Cooper pairs have been condensed. One can check that in the limit of
vanishing gap ($\Delta \to 0$), the standard Fock vacuum is recovered. The positive- and negative-energy single-particle levels can be built from the Bogoliubov vacuum as $|\varepsilon_{q\pm}\rangle = \gamma_{q\pm}^\dagger |\Omega\rangle$.

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