SUBGEOMETRIC RATES OF CONVERGENCE FOR DISCRETE
TIME MARKOV CHAINS UNDER DISCRETE TIME
SUBORDINATION

CHANG-SONG DENG

ABSTRACT. In this note, we are concerned with the subgeometric rate of convergence
of a Markov chain with discrete time parameter to its invariant measure in the $f$-norm.
We clarify how three typical subgeometric rates of convergence are inherited under a
discrete time version of Bochner’s subordination. The crucial point is to establish the
corresponding moment estimates for discrete time subordinators under some reasonable
conditions on the underlying Bernstein function.

1. INTRODUCTION

This note is a continuation of the very recent work [9], where subgeometric rates of
convergence are established for continuous time Markov processes under subordination in
the sense of Bochner, and it aims to derive the analogous result when the time parameter
is discrete. Readers are urged to refer to [6, 17] for some background on the topic
of convergence rates of Markov processes. For recent developments on subgeometric
ergodicity, see e.g. [5, 10, 11, 12, 13, 21].

First, we recall the notion of discrete time subordinator, which is a discrete time
counterpart of the classical continuous time subordinator (i.e. nondecreasing Lévy process
on $[0, \infty)$) and was initiated in [11]; see also [12, 13, 18, 19] for further developments on
random walks under discrete time subordination. A function $\phi : (0, \infty) \to [0, \infty)$ is a
Bernstein function if $\phi$ is a $C^\infty$-function satisfying $(-1)^{n-1}\phi^{(n)} \geq 0$ for all $n \in \mathbb{N}$ (here
$\phi^{(n)}$ denotes the $n$-th derivative of $\phi$). It is well known, see e.g. [22, Theorem 3.2], that
every Bernstein function enjoys a unique Lévy–Khintchine representation

$$\phi(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xy}) \, \nu(dy), \quad x > 0,$$

where $a \geq 0$ is the killing term, $b \geq 0$ is the drift term and $\nu$ is a Radon measure
on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge y) \, \nu(dy) < \infty$. As usual, we make the convention that
$\phi(0) := \phi(0+) = a$. For our purpose, we will assume that $\phi$ has no killing term (i.e.
$a = 0$); that is, $\phi$ is of the form

$$\phi(x) = bx + \int_{(0,\infty)} (1 - e^{-xy}) \, \nu(dy), \quad x \geq 0,$$

where $b$ and $\nu$ are as above. Without loss of generality, we also assume that $\phi(1) = 1$;
otherwise, we replace $\phi$ by $\phi/\phi(1)$.

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acknowledged.
For \( m \in \mathbb{N} \), we set
\[
c(\phi, m) = \frac{1}{m!} \int_{(0, \infty)} y^m e^{-y} \nu(dy) + \begin{cases} 
0, & \text{if } m = 1, \\
b, & \text{if } m \geq 2. 
\end{cases}
\]

Since
\[
\sum_{m=1}^{\infty} c(\phi, m) = b + \sum_{m=1}^{\infty} \frac{1}{m} \int_{(0, \infty)} y^m e^{-y} \nu(dy) = b + \int_{(0, \infty)} (1 - e^{-y}) \nu(dy) = \phi(1) = 1,
\]
we know that \( \{c(\phi, m) : m \in \mathbb{N}\} \) gives rise to a probability measure on \( \mathbb{N} \), and hence we can define a random walk \( T = \{T_n : n \in \mathbb{N}\} \) on \( \mathbb{N} \) by \( T_n := \sum_{k=1}^{n} R_k \), where \( \{R_k : k \in \mathbb{N}\} \) is a sequence of independent and identically distributed random variables with \( \mathbb{P}(R_k = m) = c(\phi, m) \) for \( k, m \in \mathbb{N} \). As a strictly increasing process, the random walk \( T \) is called a discrete time subordinate associated with the Bernstein function \( \phi \). Obviously, \( T_n \geq n \), and for \( m, n \in \mathbb{N} \) with \( m \geq n \),
\[
\mathbb{P}(T_n = m) = \sum_{m_1 + \ldots + m_n = m} \prod_{i=1}^{n} c(m_i, \phi).
\]

Let \( X = \{X_n : n \in \mathbb{N}\} \) be a Markov chain on a general measurable state space \( E \), and denote by \( P^n(x, dy) \) the \( n \)-step transition kernel. Throughout this note, we always assume that \( X \) has an invariant measure \( \pi \) and that \( X \) and \( T \) are independent. The subordinate process is given by the random time-change \( X^\phi_n := X_{T_n} \). The process \( X^\phi = \{X^\phi_n : n \in \mathbb{N}\} \) is again a Markov chain, and it follows easily from the independence of \( X \) and \( T \) that the \( n \)-step transition kernel of \( X^\phi \) is
\[
P^n_\phi(x, dy) = \sum_{m=n}^{\infty} P^m(x, dy) \mathbb{P}(T_n = m).
\]

This implies that \( \pi \) is also invariant for the time-changed chain \( X^\phi \).

For a (measurable) control function \( f : E \to [1, \infty) \), the \( f \)-norm of a signed measure \( \mu \) on \( E \) is defined as \( \|\mu\|_f := \sup_{|g| \leq f} \|\mu(g)\| \), where the supremum ranges over all measurable functions \( g : E \to \mathbb{R} \) with \( |g| \leq f \), and \( \mu(g) = \int_E g \, d\mu \). If \( f \equiv 1 \), then the \( f \)-norm reduces to the total variation norm \( \|\cdot\|_{TV} \); since \( f \geq 1 \), we always have \( \|\cdot\|_f \geq \|\cdot\|_{TV} \); if furthermore \( f \) is bounded then these two norms are equivalent.

It is said that the process \( X \) has subgeometric convergence in the \( f \)-norm if
\[
\|P^n(x, \cdot) - \pi\|_f \leq C(x)r(n), \quad x \in E, \ n \in \mathbb{N},
\]
where \( C(x) \) is a positive constant depending on \( x \in E \) and \( r : \mathbb{N} \to (0, 1] \) is a non-increasing function with \( r(n) \downarrow 0 \) and \( \log r(n)/n \uparrow 0 \) as \( n \to \infty \). Here, \( r \) is called the subgeometric rate. In many specific models, the convergence rate \( r \) can be explicitly given and typical examples contain
\[
r(n) = e^{-\theta n^\delta}, \quad r(n) = n^{-\beta}, \quad r(n) = \log^{-\gamma}(2 + n),
\]
where \( \theta > 0, \ \delta \in (0, 1) \) and \( \beta, \gamma > 0 \) are some constants. For instance, the backward recurrence time chain admits such convergence rates (subexponential, polynomial and logarithmic) under some assumptions, see [11] Subsection 3.1 for details.

In recent years, there has been an increasing interest in the stability of properties of continuous time Markov processes and their semigroups under Bochner’s subordination. See [15] for the dimension-free Harnack inequality for subordinate semigroups, [7] for shift Harnack inequality for subordinate semigroups, [8] for the quasi-invariance property of Brownian motion under random time-change, and [9] for subgeometric rates of
convergence for continuous time Markov processes under continuous time subordination. Subordinate functional inequalities can be found in [14] [23].

It is a natural question whether subgeometric rates of convergence can be preserved under discrete time subordination. If $P^n$ is subgeometrically convergent to $\pi$ in the $f$-norm, is it possible to derive quantitative bounds on the convergence rates of the subordinate Markov chain $X^\phi$? What we are going to do is to find some function $r_\phi : \mathbb{N} \rightarrow (0, 1]$ such that $\lim_{n \rightarrow \infty} r_\phi(n) = 0$ and

$$\|P^n(\cdot, \cdot) - \pi\|_f \leq C(n)r_\phi(n), \quad x \in E, n \in \mathbb{N}$$

(1.5)

for some constant $C(x) > 0$ depending only on $x \in E$. As in [9], it turns out that if the convergence rates of the original chain $X$ are of the three typical forms in (1.4), then we are able to obtain convergence rates for the subordinate Markov chain $X^\phi$ under some reasonable assumptions on the underlying Bernstein function.

The main result of this note is the following. As usual, denote by $\phi^{-1}$ the inverse function of the (strictly increasing) Bernstein function $\phi$.

**Theorem 1.1.** Let $X$ be a discrete time Markov chain and $T$ an independent discrete time subordinator associated with Bernstein function $\phi$ given by (1.1) such that $\phi(1) = 1$.

a) Assume that (1.3) holds with rate $r(n) = e^{-\theta n^\delta}$ for some constants $\theta > 0$ and $\delta \in (0, 1]$. If $\nu(dy) \geq cy^{-1-\alpha}dy$ for some constants $c > 0$ and $\alpha \in (0, 1)$, then (1.5) holds with rate

$$r_\phi(n) = \exp \left[ -C n^{\frac{\delta}{1-\alpha}} \right],$$

where $C = C(\theta, \delta, c, \alpha) > 0$.

b) Assume that (1.3) holds with rate $r(n) = n^{-\beta}$ for some constant $\beta > 0$. If

$$\liminf_{x \rightarrow \infty} \frac{\phi(x)}{\log x} > 0 \quad \text{and} \quad \limsup_{x \rightarrow 0} \frac{\phi(\lambda x)}{\phi(x)} > 1 \quad \text{for some} \lambda > 1,$$

then (1.5) holds with rate

$$r_\phi(n) = \left( \frac{n}{\lambda} \right)^\beta.$$

c) Assume that (1.3) holds with rate $r(n) = \log^{-\gamma}(2 + n)$ for some constant $\gamma > 0$. If $\nu(dy) \geq cy^{-1-\alpha}dy$ for some constants $c > 0$ and $\alpha \in (0, 1)$, then (1.5) holds with rate

$$r_\phi(n) = \log^{-\gamma}(2 + n).$$

**Remark 1.2.** a) According to [9] Lemma 2.2 (ii)], the second condition in (1.6) is equivalent to

$$\limsup_{x \rightarrow 0} \frac{\phi(\lambda x)}{\phi(x)} > 1 \quad \text{for all} \lambda > 1.$$

b) As pointed out in [9] Remark 1.1], typical examples for Bernstein function $\phi$ satisfying (1.6) are

- $\phi(x) = \log(1 + x)/\log 2$;
- $\phi(x) = x^\alpha \log^\beta(1 + x)/\log^\beta 2$ with $\alpha \in (0, 1)$ and $\beta \in [0, 1 - \alpha)$;
- $\phi(x) = x^\alpha \log^{-\beta}(1 + x)/\log^{-\beta} 2$ with $0 < \beta < \alpha < 1$;
- $\phi(x) = 2^\alpha x(1 + x)^{-\alpha}$ with $\alpha \in (0, 1)$.

See [22] for more examples of such Bernstein functions.
The rest of this note is organized as follows. Section 2 is devoted to three types of moment estimates for discrete time subordinators, which will be crucial for the proof of Theorem 1.1; we stress that this part is of some interest on its own. In Section 3, we present the proof of Theorem 1.1. Finally, we give in the Appendix an elementary inequality, which has been used in Section 2.

2. Moment estimates for discrete time subordinators

Recall that a continuous time subordinator \( S = \{ S_t : t \geq 0 \} \) associated with Bernstein function \( \phi \) is a nondecreasing Lévy process taking values in \([0, \infty)\) and with Laplace transform
\[
E e^{-u S_t} = e^{-t \phi(u)}, \quad u, t \geq 0.
\]
The following result concerning moment estimates for continuous time subordinators is taken from \([9, \text{Theorem 2.1}]\).

**Lemma 2.1.** Let \( S \) be a continuous time subordinator associated with Bernstein function \( \phi \) given by \((1.1)\).

\(\text{a)}\) Let \( \theta > 0 \) and \( \delta \in (0, 1] \). If \( \nu(dy) \geq c y^{-1-\alpha} dy \) for some constants \( c > 0 \) and \( \alpha \in (0, 1) \), then there exists a constant \( C = C(\theta, \delta, c, \alpha) > 0 \) such that
\[
E e^{-\theta S_t} \leq \exp \left[ -C t^{\frac{1-\alpha}{\alpha}} \right] \quad \text{for all sufficiently large } t > 1.
\]

\(\text{b)}\) Let \( \beta > 0 \). If the Bernstein function \( \phi \) satisfies \((1.6)\), then there exists a constant \( C = C(\beta) > 0 \) such that
\[
E S_t^{-\beta} \leq C \left( \frac{1}{n} \right)^{\beta} \quad \text{for all sufficiently large } t > 1.
\]

\(\text{c)}\) Let \( \gamma > 0 \). If \( \nu(dy) \geq c y^{-1-\alpha} dy \) for some constants \( c > 0 \) and \( \alpha \in (0, 1) \), then there exists a constant \( C = C(\gamma, c, \alpha) > 0 \) such that
\[
E \log^{-\gamma}(1 + S_t) \leq C \log^{-\gamma} \left( 1 + t^{1/\alpha} \right) \quad \text{for all } t > 0.
\]

Analogous to Lemma 2.1, we shall establish the corresponding results for discrete time subordinators. For related moment estimates for general Lévy(-type) processes, we refer to \([7] [16]\).

Our main contribution in this section is the following result.

**Theorem 2.2.** Let \( T \) be a discrete time subordinator associated with Bernstein function \( \phi \) given by \((1.1)\) such that \( \phi(1) = 1 \).

\(\text{a)}\) Let \( \theta > 0 \) and \( \delta \in (0, 1] \). If \( \nu(dy) \geq c y^{-1-\alpha} dy \) for some constants \( c > 0 \) and \( \alpha \in (0, 1) \), then there exists a constant \( C = C(\theta, \delta, c, \alpha) > 0 \) such that
\[
E e^{-\theta T_n^\delta} \leq \exp \left[ -C n^{\frac{1-\alpha}{\alpha}} \right] \quad \text{for all sufficiently large } n \in \mathbb{N}.
\]

\(\text{b)}\) Let \( \beta > 0 \). If the Bernstein function \( \phi \) satisfies \((1.6)\), then there exists a constant \( C = C(\beta) > 0 \) such that
\[
E T_n^{-\beta} \leq C \left( \frac{1}{n} \right)^{\beta} \quad \text{for all } n \in \mathbb{N}.
\]

\(\text{c)}\) Let \( \gamma > 0 \). If \( \nu(dy) \geq c y^{-1-\alpha} dy \) for some constants \( c > 0 \) and \( \alpha \in (0, 1) \), then there exists a constant \( C = C(\gamma, c, \alpha) > 0 \) such that
\[
E \log^{-\gamma}(1 + T_n) \leq C \log^{-\gamma} \left( 1 + n \right) \quad \text{for all } n \in \mathbb{N}.
\]
In order to prove Theorem 2.2 we first present a general result to bound the completely monotone moment of a discrete time subordinator by that of a continuous time subordinator.

A function \( g : (0, \infty) \to \mathbb{R} \) is called a completely monotone function if \( g \) is of class \( C^\infty \) and \((-1)^ng^{(n)} \geq 0 \) for all \( n = 0, 1, 2, \ldots \), see [22, Chapter 1]. By the celebrated theorem of Bernstein (cf. [22, Theorem 1.4]), every completely monotone function is the Laplace transform of a unique measure on \([0, \infty)\). More precisely, if \( g \) is a completely monotone function, then there exists a unique measure \( \mu \) on \([0, \infty)\) such that

\[
(2.1) \quad g(x) = \int_{[0, \infty)} e^{-xt} \mu(dt) \quad \text{for all } x > 0.
\]

Since the functions \( x \mapsto x^\delta \ (\delta \in (0, 1]) \) and \( x \mapsto \log(1 + x) \) are (complete) Bernstein functions, it follows easily from [22, Theorem 3.7] that the following functions

\[
(2.2) \quad x \mapsto e^{-\theta x^\delta}, \ x \mapsto x^{-\beta}, \ x \mapsto \log^{-\gamma}(1 + x)
\]

are completely monotone functions, where \( \theta > 0, \delta \in (0, 1], \) and \( \beta, \gamma > 0. \) Indeed, one has for \( \theta > 0 \) and \( \delta \in (0, 1] \) (see [20]),

\[
e^{-\theta x^\delta} = \int_0^\infty e^{-xt} \psi(\theta, \delta, t) \, dt, \quad x > 0,
\]

where

\[
\psi(\theta, \delta, t) = \pi^{-1} \theta^{-1/\delta} \int_0^\infty e^{-\theta^{1/\delta} u} e^{-u^\delta \cos \pi \delta} \sin \left( u^\delta \sin \pi \delta \right) \, du;
\]

moreover, for \( \beta > 0, \)

\[
x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-xt} t^{\beta-1} \, dt, \quad x > 0,
\]

and (by changing the order of integration) for \( \gamma > 0, \)

\[
\log^{-\gamma}(1 + x) = \frac{1}{\Gamma(\gamma)} \int_0^\infty e^{-xt} \left( \int_0^\infty \frac{1}{\Gamma(s)} t^{s-1} s^{-\gamma-1} \, ds \right) \, dt, \quad x > 0.
\]

**Lemma 2.3.** Let \( T \) be a discrete time subordinator with Bernstein function \( \phi \) given by (1.1) such that \( \phi(1) = 1. \) Let \( S \) be a continuous time subordinator with the same Bernstein function \( \phi. \) If \( g : (0, \infty) \to \mathbb{R} \) is a completely monotone function, then

\[
\mathbb{E} g(T_n) \leq \mathbb{E} g(S_n) \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** By the representation formula (2.1) and Tonelli’s theorem,

\[
\mathbb{E} g(T_n) = \mathbb{E} \left[ \int_{(0, \infty)} e^{-T_n t} \mu(dt) \right]
\]

\[
= \int_{(0, \infty)} \mathbb{E} e^{-T_n t} \mu(dt)
\]

\[
= \int_{(0, \infty)} \prod_{k=1}^n \mathbb{E} e^{-tR_k} \mu(dt)
\]

\[
= \int_{(0, \infty)} (e^{-tR_1})^n \mu(dt).
\]

Note that for \( t \geq 0, \)

\[
\mathbb{E} e^{-tR_1} = \sum_{m=1}^\infty e^{-tm} c(\phi, m)
\]
\[
= b e^{-t} + \sum_{m=1}^{\infty} e^{-tm} \frac{1}{m!} \int_{(0,\infty)} y^m e^{-y} \nu(dy)
\]
\[
= b e^{-t} + \int_{(0,\infty)} (e^{be^{-t}} - 1) e^{-y} \nu(dy)
\]
\[
= \left[ b + \int_{(0,\infty)} (1 - e^{-y}) \nu(dy) \right] - \left[ b (1 - e^{-t}) + \int_{(0,\infty)} (1 - e^{-(1-e^{-t})y}) \nu(dy) \right]
\]
\[
= \phi(1) - \phi \left( 1 - e^{-t} \right)
\]
which does not exceed \( e^{-\phi(t)} \) according to Lemma 4.1 in the Appendix. Then we have for all \( n \in \mathbb{N} \),
\[
E g(T_n) \leq \int_{(0,\infty)} e^{-n\phi(t)} \mu(dt)
= \int_{(0,\infty)} E e^{-tS_n} \mu(dt)
= E \left[ \int_{(0,\infty)} e^{-tS_n} \mu(dt) \right]
= E g(S_n),
\]
which was to be proved.

\textbf{Proof of Theorem 2.2.} Since the functions given in (2.2) are completely monotone functions, we only need to combine Lemma 2.3 with Lemma 2.1 to get the desired estimates.

\section{3. Proof of Theorem 1.1}

\textbf{Lemma 3.1.} If (1.3) holds with some rate \( r(n) \), then so does (1.5) with rate \( r_{\phi}(n) = E r(T_n) \).

\textbf{Proof.} It holds from (1.2) and (1.3) that
\[
\| P^n_{\phi}(x, \bullet) - \pi \|_f = \left\| \sum_{m=n}^{\infty} [P^m_{\phi}(x, \bullet) - \pi] \mathbb{P}(T_n = m) \right\|_f
\leq \sum_{m=n}^{\infty} \| P^m_{\phi}(x, \bullet) - \pi \|_f \mathbb{P}(T_n = m)
\leq C(x) \sum_{m=n}^{\infty} r(m) \mathbb{P}(T_n = m)
= C(x) E r(T_n),
\]
and hence the claim follows.

\textbf{Proof of Theorem 1.1.} The assertion follows immediately by combining Lemma 3.1 with the moment estimates derived in Theorem 2.2.
4. Appendix

If $\phi : [0, \infty) \to [0, \infty)$ is a concave function, then it is easy to see that

$$\phi(tx) \geq t\phi(x) \quad \text{for all } t \in [0, 1] \text{ and } x \geq 0. \quad (4.1)$$

**Lemma 4.1.** Let $\phi : [0, \infty) \to [0, \infty)$ be a concave function such that $\phi(1) = 1$ and $\phi$ is differentiable on $(0, 1)$ with $\phi'(0, 1) \geq 0$. Then

$$e^{-\phi(x)} + \phi \left( (1 - e^{-x}) \cdot 1 \right) \geq 1 \quad \text{for all } x \geq 0.$$

In particular, the above inequality holds if $\phi$ is a Bernstein function (not necessarily with $\phi(0) = 0$).

**Proof.** Let

$$\Phi(x) := e^{-\phi(x)} + \phi \left( (1 - e^{-x}) \cdot 1 \right), \quad x \geq 0.$$

It follows from (4.1) that for $x \geq 1$,

$$1 = \phi(1) = \phi \left( \frac{1}{x} \cdot x \right) \geq \frac{1}{x} \phi(x),$$

whence

$$e^{-\phi(x)} \geq e^{-x}, \quad x \geq 1. \quad (4.2)$$

Now we obtain from (4.1) and (4.2) that for $x \geq 1$,

$$\Phi(x) = e^{-\phi(x)} + \phi \left( (1 - e^{-x}) \cdot 1 \right) \geq e^{-\phi(x)} + (1 - e^{-x}) \phi(1) = 1 + e^{-\phi(x)} - e^{-x} \geq 1.$$

It remains to consider the case that $x \in [0, 1)$. Since $\phi$ is concave and differentiable on $(0, 1)$, we know that $\phi'$ is nonincreasing on $(0, 1)$. For $x \in (0, 1)$, by the elementary inequality that $1 - e^{-x} < x$, we obtain

$$\phi' \left( (1 - e^{-x}) \cdot 1 \right) \geq \phi'(x) \geq 0. \quad (4.3)$$

Moreover, by (4.1) one has for $x \in (0, 1)$,

$$\phi(x) = \phi(x \cdot 1) \geq x \phi(1) = x,$$

which yields that

$$e^{-x} \geq e^{-\phi(x)}, \quad x \in (0, 1).$$

Combining this with (4.3), we find for $x \in (0, 1)$,

$$\Phi'(x) = e^{-x} \phi' \left( 1 - e^{-x} \right) - e^{-\phi(x)} \phi'(x) \geq 0.$$

This implies that $\Phi$ is nondecreasing on $(0, 1)$ and thus for all $x \in [0, 1)$,

$$\Phi(x) \geq \Phi(0) = e^{-\phi(0)} + \phi(0) \geq 1,$$

which completes the proof. \[\square\]
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(C.-S. Deng) School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: dengcs@whu.edu.cn