A Dual Ramsey Theorem for Finite Ordered Oriented Graphs

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October 24, 2017

Abstract

In contrast to the abundance of “direct” Ramsey results for classes of finite structures (such as finite ordered graphs, finite ordered metric spaces and finite posets with a linear extension), in only a handful of cases we have a meaningful dual Ramsey result. In this paper we prove a dual Ramsey theorem for finite ordered oriented graphs. Instead of embeddings, which are crucial for “direct” Ramsey results, we consider a special class of surjective homomorphisms between finite ordered oriented graphs. Since the setting we are interested in involves both structures and morphisms, all our results are spelled out using the reinterpretation of the (dual) Ramsey property in the language of category theory.

Key Words: dual Ramsey property, finite oriented graphs, category theory

AMS Subj. Classification (2010): 05C55, 18A99

1 Introduction

Generalizing the classical results of F. P. Ramsey from the late 1920’s, the structural Ramsey theory originated at the beginning of 1970s in a series of papers (see [9] for references). We say that a class $K$ of finite structures has the Ramsey property if the following holds: for any number $k \geq 2$ of colors and all $A, B \in K$ such that $A$ embeds into $B$ there is a $C \in K$
such that no matter how we color the copies of $A$ in $C$ with $k$ colors, there is a *monochromatic* copy $B'$ of $B$ in $C$ (that is, all the copies of $A$ that fall within $B'$ are colored by the same color). In this parlance the Finite Ramsey Theorem takes the following form:

**Theorem 1.1 (Finite Ramsey Theorem [12])** The class of all finite chains has the Ramsey property.

In [3] Graham and Rothschild proved their famous Graham-Rothschild Theorem, a powerful combinatorial statement about words intended for dealing with the Ramsey property of certain geometric configurations. The fact that it also implies the following dual Ramsey statement was recognized almost a decade later.

**Theorem 1.2 (Finite Dual Ramsey Theorem [3, 10])** For all positive integers $k$, $a$, $m$ there is a positive integer $n$ such that for every $n$-element set $C$ and every $k$-coloring of the set $\left[ C \atop a \right]$ of all partitions of $C$ with exactly $a$ blocks there is a partition $\beta$ of $C$ with exactly $m$ blocks such that the set of all partitions from $\left[ C \atop a \right]$ which are coarser than $\beta$ is monochromatic.

In its original form, the Ramsey theorem is a statement about coloring $k$-element subsets of $\omega = \{0, 1, 2, \ldots\}$. A dual statement about coloring $k$-element partitions of $\omega$ was proved in [2]. These results actually marked the beginning of a search for “dual” Ramsey statements, where instead of coloring substructures we are interested in coloring “quotients” of structures.

Going back to the Finite Dual Ramsey Theorem, it was observed in [11] that each partition of a finite linearly ordered set can be uniquely represented by the rigid surjection which takes each element of the underlying set to the minimum of the block it belongs to (see Subsection 2.1 for the definition of a rigid surjection). Hence, Finite Dual Ramsey Theorem is a structural Ramsey result about finite chains and special surjections between them. This result was then generalized to trees in [14], and, using a different set of techniques, to finite permutations in [5]. In the setting of finite algebras, Dual Ramsey theorems for finite boolean algebras and for finite distributive lattices endowed with a particular linear order were proved in [7].

A possible direction for a general treatment of structural dual Ramsey results was outlined in [6] where the first author proved dual Ramsey theorems for several classes of finite relational structures which include finite posets with a linear extension, finite ordered metric spaces, finite ordered graphs and hypergraphs, and finite acyclic digraphs with a linear extension. One of the great frustrations the first author had had while working
on [6] was his inability to prove a dual Ramsey statement for finite ordered oriented graphs, although all the other related classes of structures could be covered by the approach of [6]. In this paper we finally present a dual Ramsey statement for finite ordered oriented graphs.

In contrast to the on-going Ramsey classification projects (see for example [1]) where the research is focused on fine-tuning the objects, in [6] we advocate the idea that fine-tuning the morphisms is the key to proving dual Ramsey results. Since the setting we are interested in involves both structures and morphisms, all the results in [6] are spelled out using the categorical reinterpretation of the Ramsey property as proposed in [8]. Actually, it was Leeb who pointed out already in 1970 that the use of category theory can be quite helpful both in the formulation and in the proofs of results pertaining to structural Ramsey theory [4]. In [6], but also in the present paper, we argue that this is even more the case when dealing with the dual Ramsey property.

In Section 2 we give a brief overview of certain technical notions referring to linear orders and oriented graphs.

In Section 3 we provide basics of category theory and give a categorical reinterpretation of the Ramsey property as proposed in [8]. We define the Ramsey property and the dual Ramsey property for a category and illustrate these notions using some well-known examples.

Finally, in Section 4 we prove a dual Ramsey theorem for finite ordered oriented graphs, which is the main result of the paper.

2 Preliminaries

In order to fix notation and terminology in this section we give a brief overview of certain notions referring to linear orders and oriented graphs.

2.1 Linear orders

A chain is a pair \((A, <)\) where \(<\) is a linear order on \(A\). In case \(A\) is finite we shall simply write \(\{a_1 < a_2 < \ldots < a_n\}\) instead of \((A, <)\).

Let \((A, <)\) and \((B, \sqsubseteq)\) be chains such that \(A \cap B = \emptyset\). Then \((A \cup B, < \oplus \sqsubseteq)\) denotes the concatenation of \((A, <)\) and \((B, \sqsubseteq)\), which is a chain on \(A \cup B\) such that every element of \(A\) is smaller than every element of \(B\), the elements in \(A\) are ordered linearly by \(<\), and the elements of \(B\) are ordered linearly by \(\sqsubseteq\).

Following [11] we say that a surjection \(f : \{a_1 < a_2 < \ldots < a_n\} \to \{b_1 < b_2 < \ldots < b_k\}\) between two finite chains is rigid if \(\min f^{-1}(x) < \min f^{-1}(y)\)
whenever $x < y$. Equivalently, a rigid surjection maps each initial segment of \{a_1 < a_2 < \ldots < a_n\} onto an initial segment of \{b_1 < b_2 < \ldots < b_k\}; other than that, a rigid surjection is not required to respect the linear orders in question.

Every finite chain $(A, <)$ induces the \textit{anti-lexicographic order} on $A^2$ as follows: $(a_1, a_2) <_{\text{alex}} (b_1, b_2)$ if and only if $a_2 < b_2$, or $a_2 = b_2$ and $a_1 < b_1$. It also induces the \textit{anti-lexicographic order} on $\mathcal{P}(A)$ as follows. For $X \in \mathcal{P}(A)$ let $\vec{X} \in \{0, 1\}^{|A|}$ denote the characteristic vector of $X$. (As $A$ is linearly ordered, we can assign a string of 0’s and 1’s to each subset of $A$.) Then for $X, Y \in \mathcal{P}(A)$ we let $X <_{\text{alex}} Y$ if and only if $\vec{X} <_{\text{alex}} \vec{Y}$, where the vectors are compared with respect to the usual ordering $0 < 1$. It is easy to see that for $X, Y \in \mathcal{P}(A)$ we have that $X <_{\text{alex}} Y$ if and only if $X < Y$, or $\max(X \setminus Y) < \max(Y \setminus X)$ in case $X$ and $Y$ are incomparable.

Finally, for a finite chain $(A, <)$ let us define the linear order $<_{\text{sal}}$ on $A^2$ as follows (“sal” in the subscript stands for “special anti-lexicographic”; cf. the definition of $<_{\text{sal}}$ in [6]). Take any $(a_1, a_2), (b_1, b_2) \in A^2$.

- If $a_1 = a_2$ and $b_1 = b_2$ then $(a_1, a_2) <_{\text{sal}} (b_1, b_2)$ if and only if $a_1 < b_1$;
- if $a_1 = a_2$ and $b_1 \neq b_2$ then $(a_1, a_2) <_{\text{sal}} (b_1, b_2)$;
- if $a_1 \neq a_2$, $b_1 \neq b_2$ and $\{a_1, a_2\} = \{b_1, b_2\}$ then $(a_1, a_2) <_{\text{sal}} (b_1, b_2)$ if and only if $(a_1, a_2) <_{\text{alex}} (b_1, b_2)$;
- finally, if $a_1 \neq a_2$, $b_1 \neq b_2$ and $\{a_1, a_2\} \neq \{b_1, b_2\}$ then $(a_1, a_2) <_{\text{sal}} (b_1, b_2)$ if and only if $(a_1, a_2) <_{\text{alex}} (b_1, b_2)$.

### 2.2 Oriented graphs

An \textit{oriented graph} $\mathcal{V} = (V, \rho)$ is a set $V$ together with a reflexive binary relation $\rho$ on $V$ such that $(v_1, v_2) \in \rho \Rightarrow (v_2, v_1) \notin \rho$ whenever $v_1 \neq v_2$. (Note that all the graph-like structures in this paper will be reflexive because our principal structure maps will be special surjective homomorphisms.) Let $\Delta_V = \{(v, v) : v \in V\}$.

Let $\mathcal{V} = (V, \rho)$ and $\mathcal{W} = (W, \sigma)$ be oriented graphs. A mapping $f : V \rightarrow W$ is a \textit{homomorphism} from $\mathcal{V}$ to $\mathcal{W}$, and we write $f : \mathcal{V} \rightarrow \mathcal{W}$, if $(v_1, v_2) \in \rho \Rightarrow (f(v_1), f(v_2)) \in \sigma$ for all $v_1, v_2 \in V$. A homomorphism $f : \mathcal{V} \rightarrow \mathcal{W}$ is an \textit{embedding} if $f$ is injective and $(f(v_1), f(v_2)) \in \sigma \Rightarrow (v_1, v_2) \in \rho$ for all $v_1, v_2 \in V$. A homomorphism $f : \mathcal{V} \rightarrow \mathcal{W}$ is a \textit{quotient map} if $f$ is surjective and for every $(w_1, w_2) \in \sigma$ there exists a pair $(v_1, v_2) \in \rho$ such that $f(v_1) = w_1$ and $f(v_2) = w_2$. 

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An **ordered oriented graph** is a structure $V = (V, \varrho, <)$ where $(V, \varrho)$ is an oriented graph and $<$ is a linear order on $V$.

A **digraph with a linear extension** $V = (V, \varrho, <)$ is a set $V$ together with a reflexive binary relation $\varrho$ and a linear order $<$ on $V$ such that $(v_1, v_2) \in \varrho \Rightarrow v_1 < v_2$ whenever $v_1 \neq v_2$. Note that such a digraph is necessarily acyclic.

It has been amply demonstrated in [6] that the key to providing a structural dual Ramsey result is the right choice of morphisms. In case of digraphs with linear extensions the following notion has been suggested. Let $V = (V, \varrho, <)$ and $W = (W, \sigma, \sqsubseteq)$ be two digraphs with linear extensions. Then each homomorphism $f : (V, \varrho) \to (W, \sigma)$ induces a mapping $\hat{f} : \varrho \to \sigma$ by: $\hat{f}(v_1, v_2) = (f(v_1), f(v_2))$. A homomorphism $f : (V, \varrho) \to (W, \sigma)$ is a **strong rigid quotient map** from $V$ to $W$ [6] if $\hat{f} : (\varrho, <_{sal}) \to (\sigma, \sqsubseteq_{sal})$ is a rigid surjection. It is rather easy to see that a strong rigid quotient map is a rigid surjection and a quotient map (see [6, Lemma 5.2]).

Let us now present the corresponding notion for ordered oriented graphs. Let $V = (V, \varrho, <)$ be an ordered oriented graph. Let

$$\varrho_< = \Delta_V \cup \{(v_1, v_2) \in \varrho : v_1 < v_2\},$$

and

$$\varrho_> = \Delta_V \cup \{(v_1, v_2) \in \varrho : v_1 > v_2\}.$$ 

Note that both $(V, \varrho_<, <)$ and $(V, (\varrho_>)^{-1}, <)$ are digraphs with linear extensions.

**Definition 2.1** Let $V = (V, \varrho, <)$ and $W = (W, \sigma, \sqsubseteq)$ be finite ordered oriented graphs and $f : (V, \varrho) \to (W, \sigma)$ a homomorphism. Then $f$ is a **strong rigid quotient map** between $V$ and $W$ if $\hat{f} : (\varrho, <_{sal}) \to (\sigma, \sqsubseteq_{sal})$ is a rigid surjection, and $\hat{f} : ((\varrho_>)^{-1}, <_{sal}) \to ((\sigma_<)^{-1}, \sqsubseteq_{sal})$ is a rigid surjection.

**Lemma 2.2** A strong rigid quotient map between two finite ordered oriented graphs is a rigid surjection and a quotient map.

**Proof.** Let $V = (V, \varrho, <)$ and $W = (W, \sigma, \sqsubseteq)$ be two finite ordered oriented graphs, and let $f : V \to W$ be a strong rigid quotient map between them. What we aim to prove is that $f : (V, <) \to (W, \sqsubseteq)$ is a rigid surjection whereas at the same time $f : (V, \varrho) \to (W, \sigma)$ is a quotient map.

We begin the proof by showing that $f$ is indeed surjective. Take any $v \in W$. Then $(v, v) \in \sigma$ due to the fact that $\sigma$ is reflexive. Therefore, there exists an $(u_1, u_2) \in \varrho$ (regardless of whether $(u_1, u_2)$ belongs to $\varrho_<$ or $\varrho_>$) such that $\hat{f}(u_1, u_2) = (v, v)$ because $\hat{f}$ is surjective (in both cases). But then $f(u_1) = v$ which is exactly what we needed, as $u_1 \in V$. 

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Since $f$ is a homomorphism and both $\hat{f} : \sigma_\prec \rightarrow \sigma_\subset$ and $\hat{f} : (\sigma_\succ)^{-1} \rightarrow (\sigma_\subset)^{-1}$ are surjective it follows immediately that $f$ must be a quotient map.

Finally, let us prove that $f$ is also a rigid surjection. In other words, taking any $u, v \in W$ such that $u \subset v$ let us show that $\min f^{-1}(u) < \min f^{-1}(v)$. From $u \subset v$ it is clear that $(u, u) \subset_{sal} (v, v)$ whence $\min \hat{f}^{-1}(u, u) <_{sal} \min \hat{f}^{-1}(v, v)$ owing to $\hat{f} : \sigma_\prec \rightarrow \sigma_\subset$ being a rigid surjection. Now, let $\min \hat{f}^{-1}(u, u) = (x_1, x_2)$, where $x_1 \neq x_2$. Then $\hat{f}(x_1, x_2) = (u, u)$ which implies $f(x_1) = u = f(x_2)$. Hence, $\hat{f}(x_1, x_1) = (u, u)$. However, it would then appear that $(x_1, x_1) \subset_{sal} (x_1, x_2) = \min \hat{f}^{-1}(u, u)$ which is a clear contradiction. So, $\min f^{-1}(u, u)$ must be of the form $(x, x)$ for some (adequate) $x \in V$. We shall show that $x = \min f^{-1}(u)$. Assuming the opposite that there exists a $t \in V, t \neq x$ such that $t = \min f^{-1}(u)$, but $f(x) = u$, we would once again encounter a problem as it would mean that $t < x \Rightarrow (t, t) <_{sal} (x, x) = \min \hat{f}^{-1}(u, u)$, even though $\hat{f}(t, t) = (f(t), f(t)) = (u, u)$, which is an obvious contradiction. Analogously, we have every right to denote $\min \hat{f}^{-1}(v, v)$ with $(y, y)$, where $y = \min f^{-1}(v)$. At last we see that from $(x, x) = \min \hat{f}^{-1}(u, u) <_{sal} \min \hat{f}^{-1}(v, v) = (y, y)$ our conclusion $\min f^{-1}(u) = x < y = \min f^{-1}(v)$ follows. □

Because the definition of strong rigid quotient maps for ordered oriented graphs is far from intuitive let us give a simple example.

**Example 2.1** Let $\mathcal{A} = \{1, 2, 3\}, \alpha, \prec$ and $\mathcal{B} = \{1, 2, 3, 4, 5, 6\}, \beta, \prec$ be ordered oriented graphs where the non-loops in $\mathcal{A}$ are 12, 23 and 31, the non-loops in $\mathcal{B}$ are 12, 23, 34, 45, 56 and 61, and $\prec$ is the usual ordering of the integers. (In this example only we shall write $ij$ instead of $(i, j)$.) Consider the following surjective homomorphisms $\mathcal{B} \rightarrow \mathcal{A}$:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 1 & 2 & 3 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

Then $f$ is not a strong rigid quotient map $\mathcal{B} \rightarrow \mathcal{A}$ because $\hat{f} : \beta_\prec \rightarrow \alpha_\prec$ is not well defined. Namely, $34 \in \beta_\prec$ but $\hat{f}(34) = 31 \notin \alpha_\prec$. On the other hand, $g$ is a strong rigid quotient map $\mathcal{B} \rightarrow \mathcal{A}$ as both

$$\hat{g} : \beta_\prec \rightarrow \alpha_\prec : \begin{pmatrix} 11 & 22 & 33 & 44 & 55 & 66 \\ 11 & 22 & 22 & 33 & 33 & 33 \end{pmatrix}$$

and

$$\hat{g} : (\beta_\succ)^{-1} \rightarrow (\alpha_\succ)^{-1} : \begin{pmatrix} 11 & 22 & 33 & 44 & 55 & 66 & 16 \\ 11 & 22 & 22 & 33 & 33 & 33 & 13 \end{pmatrix}$$

are well-defined rigid surjections.
3 Category theory and the Ramsey property

In order to specify a category $C$ one has to specify a class of objects $\text{Ob}(C)$, a set of morphisms $\text{hom}_C(A,B)$ for all $A, B \in \text{Ob}(C)$, the identity morphism $\text{id}_A$ for all $A \in \text{Ob}(C)$, and the composition of morphisms $\cdot$ so that $\text{id}_B \cdot f = f \cdot \text{id}_A = f$ for all $f \in \text{hom}_C(A,B)$, and $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ whenever the compositions are defined. A morphism $f \in \text{hom}_C(B,C)$ is monic or left cancellable if $f \cdot g = f \cdot h$ implies $g = h$ for all $g, h \in \text{hom}_C(A,B)$ where $A \in \text{Ob}(C)$ is arbitrary. A morphism $f \in \text{hom}_C(B,C)$ is epi or right cancellable if $g \cdot f = h \cdot f$ implies $g = h$ for all $g, h \in \text{hom}_C(C,D)$ where $D \in \text{Ob}(C)$ is arbitrary.

Example 3.1 Finite chains and embeddings constitute a category that we denote by $\text{Ch}_{\text{emb}}$.

Example 3.2 The composition of two rigid surjections is again a rigid surjection, so finite chains and rigid surjections constitute a category which we denote by $\text{Ch}_{\text{rs}}$.

Example 3.3 Finite digraphs with linear extensions together with strong rigid quotient maps constitute a category which we denote by $\text{EDig}_{\text{srq}}$.

Lemma 3.1 Finite ordered oriented graphs together with strong rigid quotient maps as introduced in Definition 2.1 constitute a category which we denote by $\text{OOgra}_{\text{srq}}$.

Proof. We only have to show that the composition of strong rigid quotient maps of ordered oriented graphs (Definition 2.1) is again a strong rigid quotient map of ordered oriented graphs.

So, let us consider the composition $g \cdot f$ of two strong rigid quotient maps $f : (V, \varrho, <) \to (W, \sigma, \sqsubset)$ and $g : (W, \sigma, \sqsubset) \to (T, \psi, \subset)$. Knowing that both $f$ and $g$ are homomorphisms it follows easily that their composition $g \cdot f : (V, g) \to (T, \psi)$ is a homomorphism, too. What remains to be shown is that $\hat{g} \cdot \hat{f} : (\rho <, <_{\text{sal}}) \to (\psi <, \subset_{\text{sal}})$ and $\hat{g} \cdot \hat{f} : ((\rho >)^{-1}, <_{\text{sal}}) \to ((\psi >)^{-1}, \subset_{\text{sal}})$ are rigid surjections.

Firstly, notice that $\hat{g} \cdot \hat{f} = \hat{g} \cdot \hat{f}$ and that $\hat{g} \cdot \hat{f}^{-1} = \hat{f}^{-1} \cdot \hat{g}^{-1}$. Since both $\hat{f}$ and $\hat{g}$ are surjections it follows that their composition $\hat{g} \cdot \hat{f} = g \cdot f$ is surjective. Now, take any $(x, y), (z, t) \in \psi <$ such that $(x, y) \subset_{\text{sal}} (z, t)$. Since $\hat{g} : (\sigma <, \subset_{\text{sal}}) \to (\psi <, \subset_{\text{sal}})$ is a rigid surjection, we have $\min \hat{g}^{-1}(x, y) \subset_{\text{sal}}$.
min \( g^{-1}(z, t) \). Similarly, due to the fact that \( \hat{f} : (g_{<}, <_{sal}) \rightarrow (\sigma_{<}, \subset_{sal}) \) is a rigid surjection it follows that

\[
\min \hat{f}^{-1}(\min g^{-1}(x, y)) <_{sal} \min \hat{f}^{-1}(\min g^{-1}(z, t)).
\]

It is easy to show that for any \( S \subseteq g_{<} \) we have \( \min \hat{f}^{-1}(\min S) = \min \hat{f}^{-1}(S) \). Therefore,

\[
\min \hat{f}^{-1}(\min g^{-1}(x, y)) = \min \hat{f}^{-1}(\min g^{-1}(z, t))
\]

which confirms the claim that \( \hat{g} \cdot \hat{f} : (g_{<}, <_{sal}) \rightarrow (\psi_{<}, \subset_{sal}) \) is a rigid surjection. By the same argument, \( g \cdot f : ((g_{<})^{-1}, <_{sal}) \rightarrow ((\psi_{<})^{-1}, \subset_{sal}) \) is a rigid surjection.

\[\square\]

For a category \( C \), the opposite category, denoted by \( C^{op} \), is the category whose objects are the objects of \( C \), morphisms are formally reversed so that \( \text{hom}_{C^{op}}(A, B) = \text{hom}_C(B, A) \), and so is the composition: \( f \cdot_{C^{op}} g = g \cdot_C f \).

A category \( D \) is a subcategory of a category \( C \) if \( \text{Ob}(D) \subseteq \text{Ob}(C) \) and \( \text{hom}_D(A, B) \subseteq \text{hom}_C(A, B) \) for all \( A, B \in \text{Ob}(D) \). A category \( D \) is a full subcategory of a category \( C \) if \( \text{Ob}(D) \subseteq \text{Ob}(C) \) and \( \text{hom}_D(A, B) = \text{hom}_C(A, B) \) for all \( A, B \in \text{Ob}(D) \).

A functor \( F : C \rightarrow D \) from a category \( C \) to a category \( D \) maps \( \text{Ob}(C) \) to \( \text{Ob}(D) \) and maps morphisms of \( C \) to morphisms of \( D \) so that \( F(f) \in \text{hom}_D(F(A), F(B)) \) whenever \( f \in \text{hom}_C(A, B) \), \( F(f \cdot g) = F(f) \cdot F(g) \) whenever \( f \cdot g \) is defined, and \( F(\text{id}_A) = \text{id}_{F(A)} \).

Categories \( C \) and \( D \) are isomorphic if there exist functors \( F : C \rightarrow D \) and \( G : D \rightarrow C \) which are inverses of one another both on objects and on morphisms.

The product of categories \( C_1 \) and \( C_2 \) is the category \( C_1 \times C_2 \) whose objects are pairs \((A_1, A_2)\) where \( A_1 \in \text{Ob}(C_1) \) and \( A_2 \in \text{Ob}(C_2) \), morphisms are pairs \((f_1, f_2) : (A_1, A_2) \rightarrow (B_1, B_2)\) where \( f_1 : A_1 \rightarrow B_1 \) is a morphism in \( C_1 \) and \( f_2 : A_2 \rightarrow B_2 \) is a morphism in \( C_2 \). The composition of morphisms is carried out componentwise: \((f_1, f_2) \cdot (g_1, g_2) = (f_1 \cdot g_1, f_2 \cdot g_2)\).

Let \( C \) be a category and \( S \) a set. We say that \( S = \bigcup \{X_i \mid i \in I\} \) is a \( k \)-coloring of \( S \) if \( X_i \cap X_j = \emptyset \) whenever \( i \neq j \). Equivalently, a \( k \)-coloring of \( S \) is any map \( \chi : S \rightarrow \{1, 2, \ldots, k\} \). For an integer \( k \geq 2 \) and \( A, B, C \in \text{Ob}(C) \) we write \( C \rightarrow (B)_k^A \) to denote that for every \( k \)-coloring \( \text{hom}_C(A, C) = \bigcup X_i \) there is an \( i \in \{1, \ldots, k\} \) and a morphism \( w \in \text{hom}_C(B, C) \) such that \( w \cdot \text{hom}_C(A, B) \subseteq X_i \).
**Definition 3.2** A category $C$ has the *Ramsey property* if for every integer $k \geq 2$ and all $A, B \in \text{Ob}(C)$ such that $\text{hom}_C(A, B) \neq \emptyset$ there is a $C \rightarrow (B)^A_k$.

A category $C$ has the *dual Ramsey property* if $C^{\text{op}}$ has the Ramsey property.

Clearly, if $C$ and $D$ are isomorphic categories and one of them has the (dual) Ramsey property, then so does the other. Actually, even more is true: if $C$ and $D$ are equivalent categories and one of them has the (dual) Ramsey property, then so does the other. We refrain from providing the definition of (the fairly standard notion of) categorical equivalence as we shall have no use for it in this paper, and for the proof we refer the reader to [8].

**Example 3.4** The category $\text{Ch}_{emb}$ (see Example 3.1) has the Ramsey property. This is just a reformulation of the Finite Ramsey Theorem (Theorem 1.1).

**Example 3.5** The category $\text{Ch}_{rs}$ (see Example 3.2) has the dual Ramsey property. This is just a reformulation of the Finite Dual Ramsey Theorem (Theorem 1.2; see also the discussion in the Introduction.)

**Example 3.6** The category $\text{EDig}_{sq}$ (see Example 3.3) has the dual Ramsey property [6].

## 4 The main result

Our goal in this paper is to prove that the category $\text{OOGra}_{sq}$ has the dual Ramsey property. In order to do so, we shall employ a strategy devised in [5]. Let us recall two technical statements from [5].

A *diagram* in a category $C$ is a functor $F : \Delta \rightarrow C$ where the category $\Delta$ is referred to as the *shape of the diagram*. Given a diagram $F : \Delta \rightarrow C$, an object $C \in \text{Ob}(C)$ and a family of morphisms $(h_\delta : F(\delta) \rightarrow C)_{\delta \in \text{Ob}(\Delta)}$ form a *commuting cocone in $C$ over $F$* if $h_\gamma \cdot F(g) = h_\delta$ for every morphism $g : \delta \rightarrow \gamma$ in $\Delta$:

$$
\begin{array}{c}
C \\
\downarrow h_\delta \\
F(\delta) \rightarrow F(\gamma) \\
\end{array}
\quad F(g)
$$

We then say that the diagram $F$ has a *commuting cocone in $C$*. 
A binary category is a finite, acyclic, bipartite digraph with loops where all the arrows go from one class of vertices into the other and the out-degree of all the vertices in the first class is 2 (modulo loops):

A binary diagram in a category $C$ is a functor $F: \Delta \to C$ where $\Delta$ is a binary category, $F$ takes the bottom row of $\Delta$ onto the same object, and takes the top row of $\Delta$ onto the same object, Fig. 1. A subcategory $D$ of a category $C$ is closed for binary diagrams if every binary diagram $F: \Delta \to D$ which has a commuting cocone in $C$ has a commuting cocone in $D$.

**Theorem 4.1** [5] Let $C$ be a category such that every morphism in $C$ is monic and such that $\operatorname{hom}_C(A, B)$ is finite for all $A, B \in \operatorname{Ob}(C)$, and let $D$ be a (not necessarily full) subcategory of $C$. If $C$ has the Ramsey property and $D$ is closed for binary diagrams, then $D$ has the Ramsey property.

We shall also need a categorical version of the Product Ramsey Theorem for Finite Structures of M. Sokić [13]. We proved this statement in the categorical context in [5] where we used this abstract version to prove that the class of finite permutations has the dual Ramsey property.

**Theorem 4.2** [5] Let $C_1$ and $C_2$ be categories such that $\operatorname{hom}_{C_i}(A, B)$ is finite for all $A, B \in \operatorname{Ob}(C_i), i \in \{1, 2\}$. If $C_1$ and $C_2$ both have the Ramsey property then $C_1 \times C_2$ has the Ramsey property.
The following dual Ramsey theorem for finite ordered oriented graphs is the main result of the paper.

**Theorem 4.3** The category \( \text{OOgra}_{\text{srq}} \) has the dual Ramsey property.

**Proof.** The category \( \mathbf{C}^{\text{op}} = \text{EDig}_{\text{srq}}^{\text{op}} \times \text{EDig}_{\text{srq}}^{\text{op}} \) has the Ramsey property (Example 3.6 and Theorem 4.2). Let \( \mathbf{D} \) be the following subcategory of \( \mathbf{C} \). For each \( (V, \varrho, <) \in \text{Ob}(\text{OOgra}_{\text{srq}}) \) the pair \( ((V, \varrho, <), (V, (\varrho_{\uparrow})^{-1}, <)) \) is an object in \( \mathbf{D} \) and these are the only objects in \( \mathbf{D} \). Morphisms of \( \mathbf{D} \) are pairs of morphisms from \( \text{EDig}_{\text{srq}} \) of the form

\[
(f, f) : ((V, \varrho, <), (V, (\varrho_{\uparrow})^{-1}, <)) \to ((W, \varrho_{\uparrow}, \sqsubset), (W, (\varrho_{\uparrow})^{-1}, \sqsubset)).
\]

Clearly, a pair \( ((V, \varrho, <), (W, \varrho_{\uparrow}, \sqsubset)) \in \text{Ob}(\mathbf{C}) \) belongs to \( \text{Ob}(\mathbf{D}) \) if and only if \( V = W, \varrho = \sqsubset, \varrho_{\uparrow} = \varrho_{\uparrow} = \Delta_V \).

Following Theorem 4.1 it suffices to show that \( \mathbf{D}^{\text{op}} \) is a subcategory of \( \mathbf{C}^{\text{op}} \) closed for binary diagrams. Take any \( \mathcal{A} = ((A, \alpha_{\uparrow}, <), (A, (\alpha_{\uparrow})^{-1}, <)) \) and \( \mathcal{B} = ((B, \beta_{\uparrow}, <), (B, (\beta_{\uparrow})^{-1}, <)) \in \text{Ob}(\mathbf{D}) \) and let \( F : \Delta \to \mathbf{D}^{\text{op}} \) be a binary diagram which takes the top row of \( \Delta \) to \( \mathcal{B} \), takes the bottom row of \( \Delta \) to \( \mathcal{A} \) and which has a commuting cocone in \( \mathbf{C}^{\text{op}} \). Let \( ((V, \varrho, \uparrow), (W, \sigma, <)) \) together with the morphisms \( e_i = (f_i, g_i), 1 \leq i \leq k \), be a commuting cocone in \( \mathbf{C}^{\text{op}} \) over \( F \):

The arrows in the diagram are reversed because the diagram depicts a situation in \( \mathbf{C} \). Without loss of generality we may assume that \( V \cap W = \emptyset \). Recall that \( \hat{f}_i : (\varrho, \varrho_{\text{sal}}) \to (\beta_{\varrho_{\uparrow}}, \varrho_{\text{sal}}) \) and \( \hat{g}_i : (\sigma, \varrho_{\text{sal}}) \to ((\beta_{\varrho_{\uparrow}})^{-1}, \varrho_{\text{sal}}) \) are rigid surjections.

Now, let \( D = V \cup W, \delta = \varrho \cup \varrho_{\uparrow}^{-1} \) and \( \sqsubset = \uparrow \oplus \downarrow \). For each \( i \in \{1, \ldots, k\} \) define \( \varphi_i : D \to B \) as follows:

\[
\varphi_i(x) = \begin{cases} f_i(x), & x \in V, \\ g_i(x), & x \in W. \end{cases}
\]

The next step would be to prove that \( D = ((D, \delta_{\sqsubset}, \sqsubset), (D, (\delta_{\sqsubset})^{-1}, \sqsubset)) \) belongs to \( \text{Ob}(\mathbf{D}) \) and that \( (\varphi_i, \varphi_i) \in \text{hom}_{\mathbf{D}}(D, B) \), for all \( i \). The former is
easily verifiable bearing in mind that $\delta_\subset = \varrho \cup \Delta_D$ and $(\delta_\subset)^{-1} = \sigma \cup \Delta_D$. To prove the latter note that $\varphi_i : (D, \delta) \to (B, \beta)$, $\varphi_i : (D, \delta_\subset) \to (B, \beta_\subset)$ and $\varphi_i : (D, (\delta_\subset)^{-1}) \to (B, (\beta_\subset)^{-1})$ are homomorphisms, so we need to show that both $\tilde{\varphi}_i : (\delta_\subset, \subseteq_{\varrho}) \to (\beta_\subset, \subseteq_{\varrho})$ and $\tilde{\varphi}_i : ((\delta_\subset)^{-1}, \subseteq_{\varrho}) \to ((\beta_\subset)^{-1}, \subseteq_{\varrho})$ are rigid surjections. As the proof of rigidity in the second case does not differ substantially from the one in the first case, aside from some minor technical details, we shall present only the first one.

Basically, what we need to show is that whenever $a \triangleleft \varrho b$ for some $a = (a_1, a_2), b = (b_1, b_2) \in \beta_\subset$, it immediately follows that $x \subseteq_{\varrho} y$, where $x = \min \tilde{\varphi}_i^{-1}(a)$ and $y = \min \tilde{\varphi}_i^{-1}(b)$. Clearly, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ belong to $\delta_\subset = \varrho \cup \Delta_D = \varrho \cup \Delta_W$.

There are four cases to consider, but before we begin, let us first notice a trivial, yet useful, fact that since $\kappa \subseteq \sqsubseteq$ and $\varrho \subseteq \subseteq_{\varrho}$ we have that $\kappa \subseteq_{\varrho} \subseteq_{\varrho}$.

Case 1: $x, y \in \varrho \subseteq V^2$. Then $a = \tilde{\varphi}_i(x) = \hat{f}_i(x)$ and $b = \tilde{\varphi}_i(y) = \hat{f}_i(y)$, bearing in mind (at all times) the very definition of $\varphi_i$. Consequently, a $\triangleleft_{\varrho}$ $b$ implies $x = \min \tilde{\varphi}_i^{-1}(a) = \min \hat{f}_i^{-1}(a) \kappa_{\varrho} \min \hat{g}_i^{-1}(b) = \min \tilde{\varphi}_i^{-1}(b) = y$, since $\hat{f}_i$ is a rigid surjection. Finally, $x \subseteq_{\varrho} y$ because $\kappa \subseteq_{\varrho} \subseteq_{\varrho}$.

Case 2: $x, y \in \Delta_W \subseteq W^2$. Similarly as in Case 1 we have that $a \triangleleft_{\varrho} b$ implies $x = \min \tilde{\varphi}_i^{-1}(a) = \min \hat{g}_i^{-1}(a) \triangleleft_{\varrho} \min \hat{g}_i^{-1}(b) = \min \tilde{\varphi}_i^{-1}(b) = y$, only this time it is the fact that $\hat{g}_i$ is a rigid surjection which yields $x \subseteq_{\varrho} y$.

Case 3: $x \in \varrho \subseteq V^2$ and $y \in \Delta_W \subseteq W^2$. Note, first, that $b_1 = b_2$ since $b = \tilde{\varphi}_i(y) = (\varphi_i(y_1), \varphi_i(y_2))$, and $y_1 = y_2$ due to $y \in \Delta_W$. The assumption $a \triangleleft_{\varrho} b$ now implies that $a_1 = a_2$ as well, whence $x \in \Delta_V$. The fact that $\sqsubseteq = \kappa \oplus \varrho$ immediately leads to $x \subseteq_{\varrho} y$.

Case 4: $x \in \Delta_W \subseteq W^2$ and $y \in \varrho \subseteq V^2$. Similarly as in Case 3 we have that $a_1 = a_2$. If $b_1 \neq b_2$ then as a consequence we have that $y_1 \neq y_2$, so $x \subseteq_{\varrho} y$ by definition of $\subseteq_{\varrho}$.

Assume, therefore, that $b_1 = b_2$. Let us show that then $y_1 = y_2$. Assume this is not the case. Then $\tilde{\varphi}_i(y_1, y_1) = (b_1, b_1) = (b_1, b_2) = \tilde{\varphi}_i(y_1, y_2)$. Therefore, $(y_1, y_1) \in \tilde{\varphi}_i^{-1}(b)$. On the other hand, we know that $y = \min \tilde{\varphi}_i^{-1}(b)$, whence $(y_1, y_2) = y \subseteq_{\varrho} (y_1, y_1)$, which is impossible by definition of $\subseteq_{\varrho}$.

So, we have shown that $y_1 = y_2$, whence $y \in \Delta_V$ and thus $y \subseteq_{\varrho} x$. Since $y = \min \tilde{\varphi}_i^{-1}(b) \in \Delta_V \subseteq V^2$ it follows that $\min \tilde{\varphi}_i^{-1}(b) = \min \hat{f}_i^{-1}(b)$. Knowing that $\hat{f}_i$ is a rigid surjection (or in other words that it maps an initial segment of a chain onto an initial segment of the other chain) we may now claim the existence of $z \in \Delta_V$ such that $z \subseteq_{\varrho} y$ for which $\hat{f}_i(z) = a$. Clearly $\tilde{\varphi}_i(z) = a$, but bearing in mind that $z \subseteq_{\varrho} y \subseteq_{\varrho} x$ we come to a contradiction with the fact that $x = \min \tilde{\varphi}_i^{-1}(a)$. 


Finally, what remains to be checked is that \((u,u) \cdot (\varphi_i, \varphi_i) = (v,v) \cdot (\varphi_j, \varphi_j)\) whenever \((u,u) \cdot e_i = (v,v) \cdot e_j\).

Assume that \((u,u) \cdot e_i = (v,v) \cdot e_j\). Then \(u \cdot f_i = v \cdot f_j\) and \(u \cdot g_i = v \cdot g_j\). Now, take any \(x \in D\). If \(x \in V\) then \(u \cdot \varphi_i(x) = u(\varphi_i(x)) = u(f_i(x)) = (u \cdot f_i)(x) = (v \cdot f_j)(x) = v(f_j(x)) = v(\varphi_j(x)) = v \cdot \varphi_j(x)\). If, on the other hand, \(x \in W\) then \(u \cdot \varphi_i(x) = u(\varphi_i(x)) = u(g_i(x)) = (u \cdot g_i)(x) = (v \cdot g_j)(x) = v(g_j(x)) = v(\varphi_j(x)) = v \cdot \varphi_j(x)\). This concludes the proof.

\[\square\]

5 Acknowledgements

The authors gratefully acknowledges the support of the Grant No. 174019 of the Ministry of Education, Science and Technological Development of the Republic of Serbia.

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