Approximate controllability for fractional diffusion equations by Dirichlet boundary control

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Abstract. In this paper, we consider the approximate controllability of partial differential equations with time derivatives of non-integer order via boundary control. First we show the unique existence and regularity of the solution by using the eigenfunction expansion. Next we also study the dual system and show the unique continuation property. Finally we apply it to prove our main result.

1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^d$ with $C^2$ boundary $\Gamma = \partial \Omega$. We consider the following initial value/boundary value problem of partial differential equation:

\[
\begin{aligned}
& \partial_t^\alpha u + \mathcal{L}u = 0 \quad \text{in} \quad \Omega \times (0, T), \\
& u = g \quad \text{on} \quad \Gamma \times (0, T), \\
& u(\cdot, 0) = u_0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

In (1.1), $u = u(x, t)$ is the state to be controlled and $g = g(x, t)$ is the control which is localized on a subboundary $\Gamma_0$ of $\Gamma$. We will act by $g$ to drive the initial state $u_0 = u_0(x)$ to some target function $u_1 = u_1(x)$. Here $\mathcal{L}$ denotes a symmetric and uniformly elliptic operator, which is specified later and $T > 0$ is a fixed value. The Caputo fractional derivative $\partial_t^\alpha$ is defined by

\[
\partial_t^\alpha h(t) := \begin{cases} 
\frac{d^n h}{dt^n}(t), & \alpha = n \in \mathbb{N}, \\
\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n h}{d\tau^n}(\tau)d\tau, & n-1 < \alpha < n, \ n \in \mathbb{N},
\end{cases}
\]

for $\alpha > 0$ (see [11], [18]). If $\alpha = 1$, then equation (1.1) is a classical diffusion equation. Equation (1.1) with $0 < \alpha < 1$ is called a fractional diffusion equation and regarded as a model of anomalous diffusion in heterogeneous media. In the present paper, we consider the case of $0 < \alpha < 1$.

According to Adams and Gelhar [1], the field data in a highly heterogeneous aquifer cannot be described well by classical advection diffusion equations. Hatano and Hatano [9] applied the continuous-time random walk (CTRW) as a microscopic model of the diffusion of ions in heterogeneous media. From the CTRW model, Metzler and Klafter [16] derived equation

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(1.1) with $0 < \alpha < 1$ as a macroscopic model. Concerning the mathematical analysis of fractional differential equations, there are many works. For the study of ordinary differential equations with fractional orders, see [11], [18] and [21] for example. As for partial differential equations with time fractional derivatives, we can refer to Gejji and Jafari [8], Agarwal [3] and Luchko [14] for example.

The aim of this article is to study the boundary control problem for fractional diffusion equations. We say that equation (1.1) is approximately controllable if for any $u_1 \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control $g$ such that the solution $u$ of (1.1) satisfies

$$
\|u(\cdot, T) - u_1\|_{L^2(\Omega)} \leq \varepsilon. 
$$

We can refer to [5], [17] and [19] for the general theory of control problems for partial differential equations. These surveys deal with controllability of equations with integer order and the relations with other concepts—observability, stabilizability, pole assignability, etc. There are various works about control problems for equations with integer orders. In particular, for the boundary control of heat equations, see MacCamy, Mizel and Seidman [15], Schmidt and Weck [22], and the references therein. However, to the author’s best knowledge, there are few works on the control problems for fractional diffusion equations, especially on the boundary control problems.

The remainder of this paper is composed of five sections and an appendix. In Section 2, we state the main result. In Section 3, we give a representation of the solution by Fourier’s method and discuss its fundamental properties. In Section 4, we define the weak solution by using the representation obtained in Section 3. In Section 5, we study the dual system of (1.1) and prove the unique continuation property, which plays an essential role in the proof of our main result. In Section 6, we complete the proof of the main result. In the appendix, we discuss the related boundary value problem for an elliptic equation.

### 2. Main result

In this section, we prepare the notations and state our main results. We denote by $L^p(\Omega)$, $1 \leq p \leq \infty$, a usual $L^p$-space. In particular, $L^2(\Omega)$ denotes the $L^2$-space equipped with the scalar product $(\cdot, \cdot)$. As for the inner product in $L^2(\Gamma)$, we denote it by $(\cdot, \cdot)$. Moreover $H^l(\Omega)$ and $H^m_0(\Omega)$, $l, m \in \mathbb{N}$, are the Sobolev spaces (see Adams [2] for example). In equation (1.1), let the differential operator $\mathcal{L}$ be given by

$$
\mathcal{L} u(x) = -\sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j}(x) \right) + c(x) u(x), \quad x \in \Omega,
$$

where the coefficients satisfy the following:

$$
a_{ij} = a_{ji}, \quad a_{ij} \in C^1(\overline{\Omega}), \quad 1 \leq i, j \leq d, \quad \sum_{i,j=1}^d a_{ij}(x)\xi_i \xi_j \geq \mu |\xi|^2, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^d,
$$

$$
c \in C(\overline{\Omega}), \quad c(x) \geq 0, \quad x \in \overline{\Omega},
$$

(1.3)
where $\mu > 0$ is constant. We define the operator $L : L^2(\Omega) \to L^2(\Omega)$ as $\mathcal{L}$ equipped with the homogeneous Dirichlet boundary condition:

$$
\mathcal{D}(L) := H^1_0(\Omega) \cap H^2(\Omega),
\quad Lu := \mathcal{L}u, \quad u \in \mathcal{D}(L).
$$

Since $L$ is a symmetric and uniformly elliptic operator, the spectrum of $L$ is entirely composed of countable number of eigenvalues and we can set with finite multiplicities:

$$
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots.
$$

By $\varphi_n \in H^2(\Omega) \cap H^1_0(\Omega)$, we denote the orthonormal eigenfunction corresponding to $\lambda_n$:

$$
L\varphi_n = \lambda_n \varphi_n, \quad n = 1, 2, \cdots.
$$

Then the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ is an orthonormal basis in $L^2(\Omega)$. We can represent the fractional power of $L$ as follows;

$$
\mathcal{D}(L^\theta) = \left\{ u \in L^2(\Omega); \sum_{n=1}^{\infty} \lambda_n^{2\theta} |(u, \varphi_n)|^2 < \infty \right\},
\quad L^\theta u = \sum_{n=1}^{\infty} \lambda_n^\theta (u, \varphi_n) \varphi_n, \quad u \in \mathcal{D}(L^\theta),
$$

where $\theta > 0$. Then $\mathcal{D}(L^\theta)$ is a Hilbert space equipped with the norm $\| \cdot \|_{\mathcal{D}(L^\theta)}$ defined by

$$
\|u\|_{\mathcal{D}(L^\theta)} := \|L^\theta u\|_{L^2(\Omega)} = \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta} |(u, \varphi_n)|^2 \right)^{1/2}, \quad u \in \mathcal{D}(L^\theta).
$$

The domain $\mathcal{D}(L^\theta)$ with $0 \leq \theta \leq 1$, $\theta \neq 1/4$, is expressed by using the Sobolev spaces with norm equivalence;

$$
\mathcal{D}(L^\theta) = \begin{cases} 
H^{2\theta}(\Omega), & 0 \leq \theta < 1/4, \\
H^2_D(\Omega), & 1/4 < \theta \leq 1,
\end{cases}
\quad C^{-1}\|u\|_{H^{2\theta}} \leq \|u\|_{\mathcal{D}(L^\theta)} \leq C\|u\|_{H^{2\theta}}, \quad u \in \mathcal{D}(L^\theta),
$$

where $H^s_D(\Omega) := \{ u \in H^s(\Omega) \mid \gamma_0 u = 0 \}$ and the operator $\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ maps a function $u$ to its restriction $u|_\Gamma$ to the boundary $\Gamma$ for $s > 1/2$. For the details of $\mathcal{D}(L^\theta)$ and the Sobolev spaces with fractional powers, see Fujiwara [7] and Yagi [25] for example. The operator $\partial_{\nu_L} : H^s(\Omega) \to H^{s-3/2}(\Gamma)$, $s > 3/2$, is defined as

$$
\partial_{\nu_L} u(x) = \frac{\partial u}{\partial \nu_L}(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \nu_j(x),
$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\Gamma$ at $x$. In particular, $\partial_{\nu_L} \varphi_n$ belongs to $L^2(\Gamma)$ since $\varphi_n \in H^2(\Omega)$. We define the Mittag-Leffler function by

$$
E_{\alpha, \beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},
$$
where \( \alpha > 0 \) and \( \beta \in \mathbb{R} \) are arbitrary constants. We can directly verify that \( E_{\alpha, \beta}(z) \) is an entire function of \( z \in \mathbb{C} \). Henceforth \( C \) denotes the positive generic constant which is independent of \( g \), but may depend on \( \alpha \) and the coefficients of the operator \( L \).

According to Theorem 2.1 in [20] and Proposition 3.1 in the next section, for any \( u_0 \in L^2(\Omega) \) and \( g \in C_0^\infty(\Gamma_0 \times (0, T)) \), equation (1.1) admits a unique solution \( u \in C([0, T]; L^2(\Omega)) \) with the representation as;

\[
\begin{align*}
  u(x,t) &= \sum_{n=1}^{\infty} (u_0, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n(x) \\
          &- \sum_{n=1}^{\infty} \left( \int_0^t \langle g(\cdot, \tau) \partial_t \varphi_n \rangle \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x).
\end{align*}
\]

(2.1)

In particular, the value \( u(\cdot, T) \) at time \( t = T \) makes sense in \( L^2(\Omega) \) and consequently we can discuss problems such as whether (1.3) is possible or not.

Now we are ready to state one of our main results;

**Theorem 2.1.** Let \( 0 < \alpha < 1 \) and \( \Gamma_0 \) be an open set in \( \Gamma \). Then equation (1.1) is approximately controllable for arbitrarily given \( T > 0 \). That is,

\[
\{ u(\cdot, T); g \in C_0^\infty(\Gamma_0 \times (0, T)) \} = L^2(\Omega),
\]

(2.2)

where \( u \) is the solution to (1.1) and the closure on the left-hand side is taken in \( L^2(\Omega) \).

In order to prove this theorem, we also need to consider the dual system for (1.1), which is a usual strategy for partial differential equations of integer order (see Section 8 in [19] or Chapters 2 and 3 in [17] for example). The dual system for (1.1) corresponds to the following initial value/boundary value problem with a different type of fractional derivative;

\[
\begin{aligned}
  D_t^\alpha v + \mathcal{L} v &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
  v &= 0 \quad \text{on} \quad \Gamma \times (0, T), \\
  I_{T-}^{1-\alpha} v(\cdot, T) &= v_0 \quad \text{in} \quad \Omega.
\end{aligned}
\]

(2.3)

Here \( D_t^\alpha \) denotes the backward Riemann-Liouville derivative and is defined by

\[
D_t^\alpha h(t) = \begin{cases} 
  \left( \frac{-d}{dt} \right)^n h(t), & \alpha = n \in \mathbb{N}, \\
  \frac{1}{\Gamma(n-\alpha)} \left( \frac{-d}{dt} \right)^n \int_t^T (\tau - t)^{n-\alpha-1} h(\tau) d\tau, & n-1 < \alpha < n, \ n \in \mathbb{N},
\end{cases}
\]

(2.4)

for \( \alpha > 0 \) (see [18]). Moreover \( I_{T-}^\alpha \) is the backward integral operator, which is defined by

\[
I_{T-}^\alpha h(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (\tau - t)^{\alpha-1} h(\tau) d\tau
\]

for \( \alpha > 0 \) (see Section 3 for details). In particular, for \( 0 < \alpha < 1 \), we have

\[
D_t^\alpha h(t) = -\frac{d}{dt} I_{T-}^{1-\alpha} h(t).
\]

(2.5)
We also note that the third equation in (2.3) means that

\[
I_1^{-\alpha}v(x, T) := \lim_{t \to T} \frac{1}{\Gamma(1-\alpha)} \int_t^T (\tau - t)^{-\alpha} v(x, \tau) d\tau = v_0(x), \quad 0 < \alpha < 1.
\]

In Section 5, we will study problem (2.3). In Section 6, we will see that the unique continuation property for (2.3) is equivalent to the approximate controllability for (1.1) stated in Theorem 2.1. Moreover, by the variational approach, we can construct the control and show that it is also finite-approximately controllable.

Let \( E \) be a finite dimensional subspace of \( L^2(\Omega) \) and fix \( \varepsilon > 0 \) and \( u_1 \in L^2(\Omega) \) arbitrarily. We introduce the functional \( J_\varepsilon \) on \( L^2(\Omega) \) defined by

\[
J_\varepsilon(v_0) := \frac{1}{2} \int_0^T \int_{\Gamma_0} (T - t)^2|\partial_{\nu_L} v|^2 d\sigma_t + \varepsilon \| (I - \pi_E) v_0 \|_{L^2(\Omega)} + (v_0, u_1) - (I_1^{-\alpha}v(\cdot, 0), u_0), \quad v_0 \in L^2(\Omega), \tag{2.6}
\]

where \( v \) is the solution of (2.3) and \( \pi_E \) denotes the orthogonal projection to \( E \). By Proposition 4.1 in [6], for any \( v_0 \in L^2(\Omega) \) equation (2.3) possesses a unique solution \( v \) with \( I_1^{-\alpha}v \in C([0, T]; L^2(\Omega)) \). Moreover \( v \) is represented by

\[
v(x, t) = \sum_{n=1}^{\infty} (v_0, \varphi_n)(T - t)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n(T - t)^{\alpha}) \varphi_n(x).
\]

Therefore \( I_1^{-\alpha}v(\cdot, 0) \) belongs to \( L^2(\Omega) \) and \( (T - t)^2|\partial_{\nu_L} v|^2 \) is integrable in \( \Gamma \times (0, T) \). Thus the functional \( J_\varepsilon \) is well-defined.

Then we obtain the following result:

**Theorem 2.2.** The functional \( J_\varepsilon \) defined in (2.6) has a unique minimizer \( \overline{v}_0 \in L^2(\Omega) \). Moreover, let \( \overline{v} \) be the solution of (2.3) with \( v_0 = \overline{v}_0 \), then the solution \( u \) of (1.1) with \( g = (T - t)^2 \partial_{\nu_L} \overline{v} \) satisfies

\[
\| u(\cdot, T) - u_1 \|_{L^2(\Omega)} \leq \varepsilon \quad \text{and} \quad \pi_E(u(\cdot, T)) = \pi_E(u_1).
\]

In the above theorem, we take \( g = (T - t)^2 \partial_{\nu_L} \overline{v} \) as the control. However, in order to do this, we have to verify that (1.1) has a solution in \( C([0, T]; L^2(\Omega)) \) for non-smooth \( g \). In Section 4, therefore, we will define the **weak solution** of (1.1) for \( g \in L^p(0, T; L^2(\Omega)) \) with large \( p \) and study its regularity.

As for the variational method introduced here, we can refer to Lions [12] and Zuazua [26].

We finally note that in Theorems 2.1 and 2.2, we may assume \( u_0 = 0 \) without loss of generality. Indeed, consider the following two problems

\[
\begin{cases}
\partial_t^\alpha u + \mathcal{L} u = 0 & \text{in } \Omega \times (0, T), \\
u = g & \text{on } \Gamma \times (0, T), \\
u(\cdot, 0) = 0 & \text{in } \Omega
\end{cases} \tag{2.7}
\]
and
\[
\begin{aligned}
\partial_t^\alpha \tilde{u} + L\tilde{u} &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\tilde{u} &= 0 \quad \text{on} \quad \Gamma \times (0, T), \\
\tilde{u}(\cdot, 0) &= u_0 \quad \text{in} \quad \Omega
\end{aligned}
\] (2.8)

and let \( u_1 \in L^2(\Omega) \) be the given target function. By Theorem 2.1 in [20], (2.8) has a unique solution \( \tilde{u} \in C([0, T]; L^2(\Omega)) \) and hence \( \tilde{u}(\cdot, T) \in L^2(\Omega) \). If (2.7) is approximately controllable, then for any \( \varepsilon > 0 \) there exists \( g \in C^\infty_0(\Gamma_0 \times (0, T)) \) such that the solution \( u \) of (2.7) satisfies
\[
\|u(\cdot, T) - (u_1 - \tilde{u}(\cdot, T))\|_{L^2(\Omega)} < \varepsilon.
\]

We see that \( u + \tilde{u} \) solves equation (1.1) and satisfies
\[
\|(u + \tilde{u})(\cdot, T) - u_1\|_{L^2(\Omega)} < \varepsilon.
\]

Thus approximate controllability for (2.7) immediately implies Theorem 2.1. In the following, therefore, we will mainly consider (2.7) instead of (1.1).

3. Representation of the solution

In order to obtain the representation of the solution to (2.7), we first prepare the notations.

**Proposition 3.1.** Let \( 0 < \alpha < 1 \) and \( g \in C^\infty_0(\Gamma_0 \times (0, T)) \), then there exists a unique solution \( u \in C^\infty([0, T]; H^2(\Omega)) \) to (2.7) represented as

\[
u(x, t) = -\sum_{n=1}^\infty \left( \int_0^t \gamma(t - \tau, \partial_{x\nu}\varphi_n) \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x).
\]

The series in (3.1) is convergent in \( C^m([0, T]; H^2(\Omega)) \) and

\[
\|\partial_t^m u(\cdot, t)\|_{H^2(\Omega)} \leq C \left( t^{\alpha(\theta-1)+1} \|\partial_t^{m+1} g\|_{L^\infty(0, T; H^{3/2}(\Gamma))} + \|\partial_t^m g(\cdot, t)\|_{H^{3/2}(\Gamma)} \right)
\]

for \( m = 0, 1, 2, \ldots \), where \( 0 < \theta < 1/4 \).

In order to prove this proposition, we briefly describe some properties concerning convolutions, fractional integrals and the Mittag-Leffler functions. First we state the following well known lemma;

**Lemma 3.2** (Young’s inequality). Let \( 1 \leq p, q, r \leq \infty \) satisfy \( 1/p + 1/q = 1 + 1/r \). If \( f \in L^p(0, T) \) and \( g \in L^q(0, T) \), then the function \( f \ast g \) defined by

\[
(f \ast g)(t) = \int_0^t f(t - \tau) g(\tau) d\tau
\]

belongs to \( L^r(0, T) \) and satisfies the estimate

\[
\|f \ast g\|_{L^r(0, T)} \leq \|f\|_{L^p(0, T)} \|g\|_{L^q(0, T)}.
\]
In particular, if \( r = \infty \), then \( f * g \) belongs to \( C[0, T] \) (not only \( L^\infty(0, T) \)) and
\[
\|(f * g)(t)\| \leq \|f\|_{L^p(0,t)} \|g\|_{L^q(0,t)}, \quad t \in [0, T].
\]

For the above lemma, see Appendix A in Stein [23] for example.

For the convenience of calculation, we introduce the notation of fractional integrals. For \( \alpha > 0 \) and \( f \in L^1(0, T) \), we define \( \alpha \)-th order forward and backward integrals of \( f \) by
\[
I_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau,
\]
\[
I_{T-}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (\tau - t)^{\alpha-1} f(\tau) d\tau.
\]

In other words, the forward integral operators of \( \alpha \)-th order is the convolution with \( t^{\alpha-1}/\Gamma(\alpha) \) and consequently \( I_{0+}^\alpha f \) also belongs to \( L^1(0, T) \). The same argument is also valid for the backward integrals. In particular, we have
\[
\partial_t^\alpha f(t) = I_{0+}^{1-\alpha} f'(t)
\] (3.3)
if \( 0 < \alpha < 1 \) and \( f \in H^1(0, T) \).

A straightforward calculation yields
\[
I_{0+}^\alpha [t^\nu] = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + \alpha + 1)} t^{\nu + \alpha}
\]
for \( \nu > -1 \) and \( \alpha > 0 \). Therefore, by the termwise integration, we have
\[
I_{0+}^{1-\alpha} \left( t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) \right) = E_{\alpha,1}(-\lambda t^\alpha), \quad t > 0.
\] (3.4)
for \( 0 < \alpha < 1 \) and \( \lambda > 0 \), which is a particular case of (1.100) in [18].

We also have the following formula for fractional integration by parts.

**Lemma 3.3.** Let \( \alpha > 0 \) and \( 1 < p, q < \infty \) satisfy \( 1/p + 1/q \leq 1 + \alpha \). If \( f \in L^p(0, T) \) and \( g \in L^q(0, T) \), then
\[
\int_0^T I_{0+}^\alpha f(t)g(t) dt = \int_0^T f(t) I_{T-}^\alpha g(t) dt.
\]
In particular, we have
\[
(I_{0+}^\alpha f) * g(t) = f * (I_{0+}^\alpha g)(t).
\] (3.5)

This lemma is derived from Theorem 3.5 in [21] as its corollary (see pp.66-67 in [21]).

As for the Mittag-Leffler functions, we have the following two lemmata.

**Lemma 3.4.** Let \( 0 < \alpha < 2 \) and \( \beta \in \mathbb{R} \) be arbitrary and \( \mu \) satisfy \( \pi \alpha/2 < \mu < \min\{\pi, \pi \alpha\} \).
Then there exists a constant \( C = C(\alpha, \beta, \mu) > 0 \) such that
\[
|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|^\mu}, \quad \mu \leq |\arg(z)| \leq \pi.
\] (3.6)
The proof of Lemma 3.4 can be found on p. 35 in [18].

**Lemma 3.5.** Let $\lambda, \alpha > 0$. For positive integer $m \in \mathbb{N}$,
\[
\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0.
\]

**Proof.** Since $E_{\alpha,\beta}(z)$ is an entire function of $z$, the function $E_{\alpha,\beta}(x)$ is real analytic and the series $\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = E_{\alpha,\beta}(z)$ is termwise differentiable in $\mathbb{R}$. Since $t^\alpha$ is also real analytic in $t > 0$, so is $E_{\alpha,\beta}(-\lambda t^\alpha)$ in $t > 0$. Therefore equation (3.7) can be obtained by termwise differentiation. \qed

Now we are ready to show Proposition 3.1.

**Proof of Proposition 3.1.** Step 1. First we prove the unique existence of the solution to (2.7). Since the uniqueness can be shown similarly to Theorem 2.1 in [20], it is sufficient to show that the solution $u$ of (2.7) is given by (3.1).

We split $u$ into $w + \Lambda g$ where $w$ solves
\[
\begin{cases}
\partial_t^\alpha w + \mathcal{L} w = -\partial_t^\alpha \Lambda g & \text{in } \Omega \times (0,T), \\
w = 0 & \text{on } \Gamma \times (0,T), \\
w(\cdot,0) = 0 & \text{in } \Omega
\end{cases}
\]
and
\[\Lambda g \in C^\infty_0((0,T);H^2(\Omega)).\] (3.8)

Then $u = w + \Lambda g$ satisfies (2.7). By Theorem 2.2 in [20] (or Proposition 3.1 in [6]), $w$ is given by
\[
w(x,t) = -\sum_{n=1}^{\infty} \left( \int_0^t (\partial_\tau^\alpha \Lambda g)(\cdot,t-\tau), \varphi_n \right) E_{\alpha,a}(-\lambda_n \tau^\alpha) d\tau \varphi_n(x).
\]

Then by equations (3.3), (3.4) and (3.5), we have
\[
w(x,t) = -\sum_{n=1}^{\infty} \left( \int_0^t (\partial_\tau^\alpha \Lambda g)(\cdot,t-\tau), \varphi_n \right) E_{\alpha,a}(-\lambda_n \tau^\alpha) d\tau \varphi_n(x)
\]
\[
= -\sum_{n=1}^{\infty} \left( \int_0^t (\partial_\tau \Lambda g)(\cdot,t-\tau), \varphi_n \right) \cdot I_{1+}^{\alpha}(\tau^\alpha E_{\alpha,a}(-\lambda_n \tau^\alpha)) d\tau \varphi_n(x)
\]
\[
= -\sum_{n=1}^{\infty} \left( \int_0^t (\partial_\tau \Lambda g)(\cdot,t-\tau), \varphi_n \right) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \varphi_n(x).
\]

Since $\Lambda g(\cdot,0) = 0$ by (3.8), the integration by parts yields
\[
w(x,t) = \sum_{n=1}^{\infty} \left( \int_0^t \frac{\partial}{\partial \tau} (\Lambda g(\cdot,t-\tau), \varphi_n) \cdot E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x)
\]
\[ = -\Lambda g(x, t) - \sum_{n=1}^{\infty} \left( \int_{0}^{t} (\Lambda g(\cdot, t - \tau), \varphi_n) \cdot \frac{\partial}{\partial \tau} \left( E_{\alpha,1}(-\lambda_n \tau^\alpha) \right) d\tau \right) \varphi_n(x). \]

By (A.9) and Lemma 3.5, we have
\[ u(x, t) = w(x, t) + \Lambda g(x, t) = \sum_{n=1}^{\infty} \left( \int_{0}^{t} (\Lambda g(\cdot, t - \tau), \varphi_n) \lambda_n \tau^{-\alpha} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x) \]
\[ = -\sum_{n=1}^{\infty} \left( \int_{0}^{t} \langle g(\cdot, t - \tau), \partial_{\alpha L} \varphi_n \rangle \tau^{-\alpha} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n(x). \]

Thus the solution \( u \) of (2.7) is given by (3.1).

**Step 2.** Next we prove that the function \( u \) given by (3.1) satisfies estimate (3.2). Using representation (3.9), we have
\[
\|Lw(\cdot, t)\|_{L^2(\Omega)} = \left\| \sum_{n=1}^{\infty} \lambda_n \left( \int_{0}^{t} \left( (\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n \right) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right\|_{L^2(\Omega)}
\]
\[
= \left\| \int_{0}^{t} \left( \sum_{n=1}^{\infty} \lambda_n \left( (\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n \right) E_{\alpha,1}(-\lambda_n \tau^\alpha) \varphi_n \right) d\tau \right\|_{L^2(\Omega)}
\]
\[
\leq \int_{0}^{t} \left\| \sum_{n=1}^{\infty} \lambda_n \left( (\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n \right) E_{\alpha,1}(-\lambda_n \tau^\alpha) \varphi_n \right\|_{L^2(\Omega)} d\tau
\]
\[
\leq \int_{0}^{t} \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta} \left| \left( (\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n \right) \right|^2 \cdot |\lambda_n^{1-\theta} E_{\alpha,1}(-\lambda_n \tau^\alpha)|^2 \right)^{1/2} d\tau.
\]

By using Lemma 3.4 again, we have
\[
|\lambda_n^{1-\theta} E_{\alpha,1}(-\lambda_n \tau^\alpha)| \leq \lambda_n^{1-\theta} \cdot \frac{C}{1 + \lambda_n \tau^\alpha} = C \cdot \frac{(\lambda_n \tau^\alpha)^{1-\theta}}{1 + \lambda_n \tau^\alpha} \cdot \tau^{\alpha(\theta-1)} \leq C \tau^{\alpha(\theta-1)}.
\]

Therefore,
\[
\|Lw(\cdot, t)\|_{L^2(\Omega)} \leq C \int_{0}^{t} \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta} \left| \left( (\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n \right) \right|^2 \right)^{1/2} \tau^{\alpha(\theta-1)} d\tau
\]
\[
= C \int_{0}^{t} \| (\partial_t \Lambda g)(\cdot, t - \tau) \|_{D(L^\theta)^{\tau^{\alpha(\theta-1)}}} d\tau
\]
\[
\leq C \int_{0}^{t} \| (\partial_t g)(\cdot, t - \tau) \|_{H^{3/2}(\Gamma)^{\tau^{\alpha(\theta-1)}}} d\tau \leq C \| \partial_t g \|_{L^\infty(0,T;H^{3/2}(\Gamma))} \int_{0}^{t} \tau^{\alpha(\theta-1)} d\tau
\]
\[
\leq C t^{\alpha(\theta-1)+1} \| \partial_t g \|_{L^\infty(0,T;H^{3/2}(\Gamma))}. \]

Since \( u = w + \Lambda g \), we have
\[
\|u(\cdot, t)\|_{H^2(\Omega)} \leq \|w(\cdot, t)\|_{H^2(\Omega)} + \|\Lambda g(\cdot, t)\|_{H^2(\Omega)} \leq C\|Lw(\cdot, t)\|_{L^2(\Omega)} + \|g(\cdot, t)\|_{H^{3/2}(\Omega)}
\]
\[
\leq C \left( t^{\alpha(\theta-1)+1} \| \partial_t g \|_{L^\infty(0,T;H^{3/2}(\Gamma))} + \|g(\cdot, t)\|_{H^{3/2}(\Gamma)} \right). \]
Similarly we can also show
\[ \| \partial_t^m u(\cdot, t) \|_{H^2(\Omega)} \leq C \left( t^{\alpha(\theta - 1) + 1} \| \partial_t^{m+1} g \|_{L^\infty(0, T; H^{3/2}(\Gamma))} + \| \partial_t^m g(\cdot, t) \|_{H^{3/2}(\Gamma)} \right) \]
for any \( m \in \mathbb{N} \).

**Step 3.** We prove that the series in (3.1) converges in \( C^m([0, T]; H^2(\Omega)) \) for \( m = 0, 1, 2, \ldots \). Since \( \Lambda g \) clearly belongs to \( C^\infty([0, T]; H^2(\Omega)) \), it is sufficient to show the convergence of (3.9). By the similar calculation to Step 2, we have
\[
\left\| \sum_{n=M}^{N} \left( \int_0^t ((\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right\|_{H^2(\Omega)}
\]
\[
\leq C \left\| \sum_{n=M}^{N} \left( \int_0^t ((\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right\|_{D(L)}
\]
\[
= C \left( \sum_{n=M}^{N} \lambda_n \left( \int_0^t ((\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right)_{L^2(\Omega)}
\]
\[
\leq C \int_0^t \left( \sum_{n=M}^{N} \lambda_n^{2\theta} \left| ((\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n) \right|^2 \right)^{1/2} \tau^{\alpha(\theta - 1)} d\tau
\]
\[
\leq C t^{\alpha(\theta - 1) + 1} \sup_{0 \leq t \leq T} \left( \sum_{n=M}^{N} \lambda_n^{2\theta} \left| ((\partial_t \Lambda g)(\cdot, t), \varphi_n) \right|^2 \right)^{1/2}
\]
Since \( \partial_t \Lambda g \in C([0, T]; D(L^\theta)) \), we have
\[
\sup_{0 \leq t \leq T} \left\| \sum_{n=M}^{N} \left( \int_0^t ((\partial_t \Lambda g)(\cdot, t - \tau), \varphi_n) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right\|_{H^2(\Omega)}
\]
\[
\leq C T^{\alpha(\theta - 1) + 1} \sup_{0 \leq t \leq T} \left( \sum_{n=M}^{N} \lambda_n^{2\theta} \left| ((\partial_t \Lambda g)(\cdot, t), \varphi_n) \right|^2 \right)^{1/2} \rightarrow 0 \quad \text{as} \quad M, N \rightarrow \infty.
\]
Thus the series in (3.9) is convergent in \( H^2(\Omega) \) uniformly in \( t \in [0, T] \). In the same way, we can also show the uniform convergence of
\[
\partial_t^m w(\cdot, t) = -\sum_{n=1}^{\infty} \left( \int_0^t ((\partial_t^{m+1} \Lambda g)(\cdot, t - \tau), \varphi_n) E_{\alpha,1}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n
\]
for any \( m \in \mathbb{N} \). \( \square \)
4. Weak solution

In this section,

As we have seen in Proposition 3.1, the function $u$ defined by (3.1) is the solution of (1.1) with $u_0 = 0$ when $g$ is restricted in $C_0^\infty(\Gamma_0 \times (0, T))$. However, the domain of the map $g \mapsto u$ can be extended keeping $u$ belonging to $C([0, T]; L^2(\Omega))$.

**Proposition 4.1.** Let $0 < \alpha < 1$ and $g \in L^p(0, T; L^2(\Gamma))$ with $p > 4/\alpha$. Then the function $u$ given by (3.1) belongs to $C([0, T]; L^2(\Omega))$ and satisfies

\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C_2 t^{\alpha - 1/p}\|g\|_{L^p(0,T;L^2(\Gamma))},
\]

where $1/(p\alpha) < \theta < 1/4$. Moreover for any $0 < \delta < \theta - 1/(p\alpha)$, we have

\[
\|u(\cdot, t)\|_{D(L^\delta)} \leq C t^{\alpha(\theta - \delta) - 1/p}\|g\|_{L^\infty(0,T;H^{3/2}(\Gamma))}.
\]

**Remark 4.1.** If $\alpha = 1$, then the similar result holds for $p > 4$ (see [24]).

**Proof of Proposition 4.1.** Step 1. By a simple calculation, we have

\[
\|u(\cdot, t)\|_{L^2(\Omega)} = \left\{ \sum_{n=1}^{\infty} \left( \int_0^t \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) d\tau \right) \varphi_n \right\}_{L^2(\Omega)}
\]

\[
= \left\{ \int_0^t \left( \sum_{n=1}^{\infty} \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \varphi_n \right) d\tau \right\}_{L^2(\Omega)}
\]

\[
\leq \int_0^t \left\{ \sum_{n=1}^{\infty} \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \varphi_n \right\}_{L^2(\Omega)} d\tau
\]

\[
= \int_0^t \left( \sum_{n=1}^{\infty} \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \varphi_n \right)^2_{L^2(\Omega)} \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta - 2} \| \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \| \right)^2 \left( \sum_{n=1}^{\infty} \lambda_n^{1-\theta} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \right)^2_{L^2(\Omega)} d\tau.
\]

Similarly to Proposition 3.1, we use Lemma 3.4 to obtain

\[
|\lambda_n^{1-\theta} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)| \leq \lambda_n^{1-\theta} \tau^{\alpha-1} \frac{C}{1 + \lambda_n \tau^\alpha} = C \cdot \frac{(\lambda_n \tau^\alpha)^{1-\theta}}{1 + \lambda_n \tau^\alpha} \cdot \tau^\alpha \leq C \tau^\theta.
\]

Therefore

\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C \int_0^t \left( \sum_{n=1}^{\infty} \lambda_n^{2\theta - 2} \| \langle g(\cdot, t - \tau), \partial_{\nu_L} \varphi_n \rangle \|^2 \right)^{1/2} \tau^{\alpha - 1} d\tau
\]

\[
\leq C \int_0^t \|g(\cdot, t - \tau)\|_{L^2(\Gamma)} \tau^{\alpha - 1} d\tau.
\]
Let \( q \in \mathbb{R} \) satisfy \( 1/p + 1/q = 1 \), then (4.3) and Lemma (3.2) yields
\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C\|g\|_{L^p(0, t, L^2(\Gamma))} \left( \int_0^t \tau^{q(\alpha^\theta - 1)} \, dt \right)^{1/q} \leq C t^{\alpha^\theta - 1/p} \|g\|_{L^p(0, T; L^2(\Gamma))}.
\]
Thus we have proved estimate (4.1). Moreover, by the similar calculation, we have
\[
\left\| - \sum_{n=M}^N \left( \int_0^t \langle g(\cdot, t - \tau), \partial_{v_L} \varphi_n \rangle \tau^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n^\theta \tau^\theta) \, d\tau \right) \varphi_n \right\|_{L^2(\Omega)} \\
\leq C \int_0^t \left( \sum_{n=M}^N \lambda_n^{2\theta} \| (Ag(\cdot, \tau), \varphi_n) \|^2 \right)^{1/2} \tau^{\alpha\theta - 1} \, d\tau \\
\leq C \left[ \int_0^t \left( \sum_{n=M}^N \lambda_n^{2\theta} \| (Ag(\cdot, \tau), \varphi_n) \|^2 \right)^{p/2} \, d\tau \right]^{1/p} \left( \int_0^t \tau^{(\alpha - 1)\theta} \, d\tau \right)^{1/q} \\
\leq C t^{\alpha\theta - 1/p} \left[ \int_0^t \left( \sum_{n=M}^N \lambda_n^{2\theta} \| (Ag(\cdot, \tau), \varphi_n) \|^2 \right)^{p/2} \, d\tau \right]^{1/p}.
\]
Therefore
\[
\sup_{0 \leq t \leq T} \left\| - \sum_{n=M}^N \left( \int_0^t \langle g(\cdot, t - \tau), \varphi_n \rangle E_{\alpha, 1}(-\lambda_n \tau^\theta) \, d\tau \right) \varphi_n \right\|_{H^2(\Omega)} \\
\leq C T^{\alpha\theta - 1/p} \left[ \int_0^T \left( \sum_{n=M}^N \lambda_n^{2\theta} \| (Ag(\cdot, \tau), \varphi_n) \|^2 \right)^{p/2} \, d\tau \right]^{1/p} \rightarrow 0 \quad \text{as} \quad M, N \rightarrow \infty.
\]
Thus the series in (3.1) is convergent in \( L^2(\Omega) \) uniformly in \( t \in [0, T] \). Therefore \( u \) belongs to \( C([0, T]; L^2(\Omega)) \).

**Step 2.** Next we prove (4.2). By a simple calculation, we have
\[
\|u(\cdot, t)\|_{\mathcal{D}(L^2)} = \left\| \sum_{n=1}^\infty \lambda_n^\delta \left( \int_0^t \langle g(\cdot, t - \tau), \partial_{v_L} \varphi_n \rangle \tau^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n \tau^\theta) \, d\tau \right) \varphi_n \right\|_{L^2(\Omega)} \\
= \left\| \int_0^t \left( \sum_{n=1}^\infty \lambda_n^\delta \langle g(\cdot, t - \tau), \partial_{v_L} \varphi_n \rangle \tau^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n \tau^\theta) \varphi_n \right) \, d\tau \right\|_{L^2(\Omega)} \\
\leq \int_0^t \left\| \sum_{n=1}^\infty \lambda_n^\delta \langle g(\cdot, t - \tau), \partial_{v_L} \varphi_n \rangle \tau^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n \tau^\theta) \varphi_n \right\|_{L^2(\Omega)} \, d\tau \\
= \int_0^t \left( \sum_{n=1}^\infty \lambda_n^\delta \langle g(\cdot, t - \tau), \partial_{v_L} \varphi_n \rangle \tau^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n \tau^\theta) \right)^{1/2} \, d\tau \\
= \int_0^t \left( \sum_{n=1}^\infty \lambda_n^{2\theta} \| (Ag(\cdot, t - \tau), \varphi_n) \|^2 \right)^{1/2} \, d\tau.
\]
By Lemma 3.4, we have
\[ |\lambda_n^{1+\delta - \tau^\alpha - 1} E_{\alpha,\alpha} (-\lambda_n \tau^\alpha) | \leq \lambda_n^{1+\delta - \tau^\alpha - 1} . \frac{C}{1 + \lambda_n \tau^\alpha} = C \frac{(\lambda_n \tau^\alpha)^{1+\delta - \theta}}{1 + \lambda_n \tau^\alpha} \cdot \tau^{-(\theta - \delta)} \leq C \tau^{-(\theta - \delta)}. \]

Therefore combining this with (A.8), we have
\[ \|u(\cdot, t)\|_{D(L^q)} \leq C \int_0^t \left( \sum_{n=1}^\infty \lambda_n^{2q} |\Lambda g(\cdot, t - \tau), \varphi_n| | \right)^{1/2} \tau^{(\theta - \delta) - 1} d\tau \]
\[ = C \int_0^t \|\Lambda g(\cdot, t - \tau)\|_{D(L^q)} \tau^{(\theta - \delta) - 1} d\tau \leq C \int_0^t \|g(\cdot, t - \tau)\|_{L^2(\Gamma)} \tau^{(\theta - \delta) - 1} d\tau. \]

By taking \( q \in \mathbb{R} \) as before, we apply Lemma 3.2 again and have
\[ \|u(\cdot, t)\|_{D(L^q)} \leq C \|g\|_{L^p(0, t; L^2(\Gamma))} \left( \int_0^t \tau^{(\alpha - \delta) - 1} d\tau \right)^{1/q} \leq C \tau^{(\theta - \delta) - 1/p} \|g\|_{L^p(0, t; L^2(\Gamma))}. \]

Thus the proof of Proposition 4.1 is completed. \( \square \)

5. Dual system

We prove the following propositions;

**Proposition 5.1.** Let \( 0 < \alpha < 1 \) and \( v_0 \in L^2(\Omega) \). Then there exists a unique solution \( v \in C([0, T); H^2(\Omega) \cap H_0^1(\Omega)) \) to (2.3) which is represented as
\[ v(x, t) = \sum_{n=1}^\infty (v_0, \varphi_n)(T - t)^{\alpha - 1} E_{\alpha,\alpha} (-\lambda_n (T - t)^\alpha) \varphi_n(x). \]

and has the following estimate for any \( 0 < \delta \leq 1; \)
\[ \|v(\cdot, t)\|_{D(L^{1-\delta})} \leq C(T - t)^{\alpha \delta - 1} \|v_0\|_{L^2(\Omega)}. \]

Moreover the mapping \([0, T) \ni t \mapsto \partial_{\nu_L} v(\cdot, t) \in L^2(\Gamma)\) is analytically extended to \( S_T := \{ z \in \mathbb{C}; \text{Re} \ z < T \} \).

**Proposition 5.2.** Let \( \Gamma_0 \) be open in \( \Gamma \) and \( v \in C([0, T); H^2(\Omega) \cap H_0^1(\Omega)) \) be the solution of (2.3) corresponding to \( v_0 \in L^2(\Omega) \). If \( \partial_{\nu_L} v = 0 \) on \( \Gamma_0 \times (0, T), \) then \( v = 0 \) in \( \Omega \times (0, T). \)

**Proof of Proposition 5.1.** By Proposition 4.1 in [6], it is already known that (2.3) has a unique solution and that it is given by (5.1).

We first show estimate (5.2). By (5.1), we have
\[ \|v(\cdot, t)\|_{D(L^{1-\delta})}^2 = \left\| \sum_{n=1}^\infty \lambda_n^{1-\delta} (v_0, \varphi_n)(T - t)^{\alpha - 1} E_{\alpha,\alpha} (-\lambda_n (T - t)^\alpha) \varphi_n \right\|_{L^2(\Omega)}^2 \]
\[ = \sum_{n=1}^\infty |\lambda_n^{1-\delta} (v_0, \varphi_n)(T - t)^{\alpha - 1} E_{\alpha,\alpha} (-\lambda_n (T - t)^\alpha)|^2. \]
We use Lemma 3.4 to obtain
\[
|\lambda_n^{1-\delta}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})| \leq \lambda_n^{1-\delta}(T-t)^{\alpha-1} \cdot \frac{C}{1+\lambda_n(T-t)^{\alpha}} = C \cdot \frac{(\lambda_n(T-t)^{\alpha})^{1-\delta}}{1+\lambda_n(T-t)^{\alpha}} \cdot (T-t)^{\alpha\delta-1} \leq C(T-t)^{\alpha\delta-1}.
\]
Therefore,
\[
\|v(\cdot, t)\|_{D(L^{1-\delta})} \leq C(T-t)^{\alpha\delta-1} \left( \sum_{n=1}^{\infty} |(v_0, \varphi_n)|^2 \right)^{1/2} = C(T-t)^{\alpha\delta-1} \|v_0\|_{L^2(\Omega)}.
\]

Next we show the analyticity of \(\partial_{v_L} v(\cdot, t)\) in \(t \in S_T\). Since \(\partial_{v_L} : H^2(\Omega) \rightarrow L^2(\Gamma)\) is bounded, we have
\[
\frac{\partial v}{\partial v_L}(\cdot, t) = \sum_{n=1}^{\infty} (v_0, \varphi_n)(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \frac{\partial \varphi_n}{\partial v_L}
\]
and the right-hand side of the above is convergent in \(L^2(\Gamma)\) for any \(t \in (0, T)\).

We note that \(E_{\alpha,\alpha}(-\lambda_n z)\) is an entire function (see Section 1.8 in [11] for example) and consequently each \((T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\) is analytic in \(z \in S_T\). Therefore \(S_T \ni z \mapsto \sum_{n=1}^{N}(v_0, \varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\partial_{v_L} \varphi_n \in L^2(\Gamma)\) is also analytic. If we fix \(\delta' > 0\) arbitrarily, then for \(z \in \mathbb{C}\) with \(\text{Re} \, z \leq T - \delta'\), we have
\[
\left\| \sum_{n=M}^{N} (v_0, \varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\partial_{v_L} \varphi_n \right\|_{L^2(\Gamma)}^2 
\leq C \left\| \sum_{n=M}^{N} (v_0, \varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\varphi_n \right\|_{H^2(\Omega)}^2 
\leq C \left\| \sum_{n=M}^{N} (v_0, \varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})\varphi_n \right\|_{D(L)}^2 
= C \sum_{n=M}^{N} |\lambda_n(v_0, \varphi_n)(T-z)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-z)^{\alpha})|^2 
\leq C \sum_{n=M}^{N} |(v_0, \varphi_n)|^2 |T-z|^{-2} \left( \frac{\lambda_n|T-z|^\alpha}{1+\lambda_n|T-z|^\alpha} \right)^2 
\leq C\delta'^{-2} \sum_{n=M}^{N} |(v_0, \varphi_n)|^2 \rightarrow 0 \quad \text{as} \quad M, N \rightarrow \infty.
\]
That is, (5.3) is uniformly convergent in \(\{z \in \mathbb{C}; \text{Re} \, z \leq T - \delta'\}\). Hence \(\partial_{v_L} v(\cdot, t)\) is also analytic in \(t \in S_T\).
Proof of Proposition 5.2. Since $\partial_{\nu_L} v = 0$ in $\Gamma_0 \times (0,T)$ and $\partial_{\nu_L} v : [0,T) \to L^2(\Gamma)$ can be analytically extended to $S_T$, we have

$$\frac{\partial v}{\partial \nu_L}(x,t) = \sum_{n=1}^{\infty} (v_0, \varphi_n)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \frac{\partial \varphi_n}{\partial \nu_L}(x) = 0, \quad x \in \Gamma_0, \ t \in (-\infty,T).$$

(5.4)

Let $\{\mu_k\}_{k \in \mathbb{N}}$ be all spectra of $L$ without multiplicities and we denote by $\{\varphi_k\}_{1 \leq j \leq m_k}$ an orthonormal basis of Ker($\mu_k - L$). By using these notations, we can rewrite (5.4) by

$$\sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} (v_0, \varphi_k) \frac{\partial \varphi_k}{\partial \nu_L}(x) \right) (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) = 0, \quad x \in \Gamma_0, \ t \in (-\infty,T). \quad (5.5)$$

We regard $\partial_{\nu_L}$ as a bounded operator from $H^{2\varepsilon+3/2}(\Omega)$ to $H^{2\varepsilon}(\Gamma)$ with $0 < \varepsilon < 1/4$. Then for any $z \in \mathbb{C}$ with $\text{Re} \ z = \xi > 0$ and $N \in \mathbb{N}$, we have

$$\left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} (v_0, \varphi_k) \frac{\partial \varphi_k}{\partial \nu_L}(x) \right) e^{\xi(t-T)} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right\|^2_{L^2(\Gamma)}$$

$$\leq \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} (v_0, \varphi_k) \frac{\partial \varphi_k}{\partial \nu_L}(x) \right) e^{\xi(t-T)} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right\|^2_{H^{2\varepsilon}(\Gamma)}$$

$$\leq C \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} (v_0, \varphi_k) \varphi_k \right) e^{\xi(t-T)} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right\|^2_{H^{2\varepsilon+3/2}(\Omega)}$$

$$\leq C \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} (v_0, \varphi_k) \varphi_k \right) e^{\xi(t-T)} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right\|^2_{D(L^{\varepsilon/4})}$$

$$= C \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} |(v_0, \varphi_k)|^2 \right) e^{2\xi(t-T)} \left| \mu_k^{\varepsilon+3/4} (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right|^2.$$
where $\beta := (1/4 - \varepsilon)\alpha > 0$. The right-hand side of the above is integrable on $(-\infty, T)$;

$$\int_{-\infty}^{T} e^{\xi(T-t)}(T-t)^{\beta-1} dt = \frac{\Gamma(\beta)}{\xi^\beta}.$$ 

Hence the Lebesgue theorem yields that

$$\int_{-\infty}^{T} e^{z(T-t)} \left( \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \frac{\partial \varphi_{kj}}{\partial \nu_L}(x)(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) \right) dt$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \frac{\partial \varphi_{kj}}{\partial \nu_L}(x) \left( \int_{-\infty}^{T} e^{z(T-t)}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) dt \right)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \frac{\partial \varphi_{kj}}{\partial \nu_L}(x), \text{ a.e. } x \in \Gamma, \text{ Re } z > 0,$$

(5.6)

where we have used the Laplace transform formula;

$$\int_{0}^{\infty} e^{-zt}t^{\alpha-1} E_{\alpha,\alpha}(-\mu_k t^{\alpha}) dt = \frac{1}{z^\alpha + \mu_k}, \text{ Re } z > 0$$

(see (1.80) in p.21 of [18]). By (5.5) and (5.6), we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \frac{\partial \varphi_{kj}}{\partial \nu_L}(x) \left( \int_{-\infty}^{T} e^{z(T-t)}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\mu_k(T-t)^{\alpha}) dt \right)$$

$$= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} (v_0, \varphi_{kj}) \frac{\partial \varphi_{kj}}{\partial \nu_L}(x) = 0, \text{ a.e. } x \in \Gamma_0, \text{ Re } z > 0,$$

that is,

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{(v_0, \varphi_{kj})}{\eta + \mu_k} \frac{\partial \varphi_{kj}}{\partial \nu_L}(x) = 0, \text{ a.e. } x \in \Gamma_0, \text{ Re } \eta > 0.$$

By using analytic continuation in $\eta$, we have

$$\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{(v_0, \varphi_{kj})}{\eta + \mu_k} \frac{\partial \varphi_{kj}}{\partial \nu_L}(x) = 0, \text{ a.e. } x \in \Gamma_0, \eta \in \mathbb{C} \setminus \{-\mu_k\}_{k \in \mathbb{N}}.$$

(5.7)

Then we can take a suitable disk which includes $-\mu_\ell$ and does not include $\{-\mu_k\}_{k \neq \ell}$. By integrating (5.7) in the disk, we have

$$\sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \frac{\partial \varphi_{\ell j}}{\partial \nu_L}(x) = 0, \text{ a.e. } x \in \Gamma_0.$$

By setting $\tilde{v}_\ell := \sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \varphi_{\ell j}$, we have

$$(L - \mu_\ell)\tilde{v}_\ell = 0 \text{ in } \Omega \text{ and } \frac{\partial \tilde{v}_\ell}{\partial \nu_L} = 0 \text{ on } \Gamma_0.$$

Therefore the unique continuation result for eigenvalue problem of elliptic operator (see Corollary 2.2 in [22] or Chapter 3 in [10] for example) implies

$$\tilde{v}_\ell(x) = \sum_{j=1}^{m_\ell} (v_0, \varphi_{\ell j}) \varphi_{\ell j}(x) = 0, \text{ } x \in \Omega$$
for each \( \ell \in \mathbb{N} \). Since \( \{ \varphi_{\ell j} \}_{1 \leq j \leq m_\ell} \) is linearly independent in \( \Omega \), we see that
\[
(v_0, \varphi_{\ell j}) = 0, \quad 1 \leq j \leq m_\ell, \ell \in \mathbb{N}.
\]
This implies \( v = 0 \) in \( \Omega \times (0, T) \). \( \square \)

6. Proof of main results

In this section, we complete the proof of our main theorems.

**Proof of Theorem 2.1. Step 1.** We first show that for any \( g \in C_0^\infty(\Gamma_0 \times (0, T)) \) and \( v_0 \in L^2(\Omega) \), the following identity holds;
\[
\int_\Omega u(x,T)v_0(x)dx + \int_0^T \int_{\Gamma_0} g(x,t) \frac{\partial v}{\partial \nu_L}(x,t)d\sigma_x dt = 0, \tag{6.1}
\]
where \( u \) and \( v \) are the corresponding solutions of (2.7) and (2.3) respectively. Since the first equation in (2.7) holds in \( C_0^\infty([0, T]; L^2(\Omega)) \) by Proposition 3.1 and \( v \in L^1(0, T; L^2(\Omega)) \) by (5.2) with \( \delta = 1 \), we see that
\[
0 = \int_0^T \int_\Omega (\partial_t^\alpha u + \mathcal{L}u) v dx dt = \int_0^T \int_\Omega (\partial_t^\alpha u) v dx dt + \int_0^T \int_\Omega (\mathcal{L}u) v dx dt.
\]
In the above equation, the first term is calculated as follows;
\[
\int_0^T \int_\Omega (\partial_t^\alpha u) v dx dt = \int_0^T \int_\Omega I_{t=0}^{1-\alpha} \frac{\partial u}{\partial t} v dx dt = \int_0^T \int_\Omega \frac{\partial u}{\partial t} I_{t=0}^{1-\alpha} v dx dt
\]
\[
= \int_\Omega u(t) I_{t=0}^{1-\alpha} v |_{t=0}^{t=T} - \int_0^T \int_\Omega u(\cdot) \frac{\partial}{\partial t} I_{t=0}^{1-\alpha} v dx dt
\]
\[
= \int_\Omega u(\cdot, T)v_0 dx + \int_0^T \int_\Omega u(\cdot, t) v_0 dx dt.
\]
Here we have used Lemma 3.3, the integration in \( t \) by parts and the initial conditions in (2.7) and (2.3). In terms of \( u \in C_0^\infty([0, T]; H^2(\Omega)) \) and \( v \in C([0, T); H^2(\Omega) \cap H_0^1(\Omega)) \) by Propositions 3.1 and 5.1, we apply the Green formula to have
\[
\int_0^T \int_\Omega (\mathcal{L}u)v dx dt = \int_0^T \int_\Omega u(\mathcal{L}v) dx dt + \int_0^T \int_\Gamma \left( u \frac{\partial v}{\partial \nu_L} - \frac{\partial u}{\partial \nu_L} v \right) d\sigma_x dt
\]
\[
= \int_0^T \int_\Omega u(\mathcal{L}v) dx dt + \int_0^T \int_{\Gamma_0} g \frac{\partial v}{\partial \nu_L} d\sigma_x dt.
\]
In the above calculation, we have used boundary conditions in (2.7) and (2.3). We also note that by (4.2) and (5.2), the function
\[
t \mapsto \int_\Omega u(x,t)\mathcal{L}v(x,t) dx = (u(\cdot, t), Lv(\cdot, t))
\]
is integrable in \( t \in (0, T) \). Therefore we have
\[
0 = \int_0^T \int_\Omega (\partial_t^\alpha u)dxdt + \int_0^T \int_\Omega (\mathcal{L}u)v dx dt
\]
\[
\frac{\int u(\cdot, T)v_0dx + \int_0^T \int_\Omega u(D^\alpha_t v)dxdt}{\|v_0\|_{L^2(\Omega)}} = \frac{\int u(\cdot, T)v_0dx + \int_0^T \int_\Omega u(\mathcal{L}v)dxdt + \int_0^T \int_{\Gamma_0} g \frac{\partial v}{\partial \nu_L}d\sigma_xdt}{2}
\]

Thus we have proved (6.1).

**Step 2.** We note that the assertion of Theorem 2.1 is equivalent to
\[
\{u(\cdot, T); \ g \in C^\infty_0(\Gamma_0 \times (0, T))\}^\perp = \{0\},
\]
where the orthogonal complement is taken in \(L^2(\Omega)\). Suppose that \(v_0 \in L^2(\Omega)\) satisfies
\[
(u(\cdot, T), v_0) = 0
\]
for any \(g \in C^\infty_0(\Gamma_0 \times (0, T))\). Then, by (6.1), we have
\[
\int_0^T \int_{\Gamma_0} g(x, t) \frac{\partial v}{\partial \nu_L}(x, t)d\sigma_xdt = 0
\]
for any \(g \in C^\infty_0(\Gamma_0 \times (0, T))\). By the fundamental lemma of the calculus of variations, we have
\[
\frac{\partial v}{\partial \nu_L}(x, t) = 0, \ (x, t) \in \Gamma_0 \times (0, T),
\]
from which Proposition 5.2 implies
\[
v_0 \equiv 0.
\]
Thus we have shown (6.2) and completed the proof of Theorem 2.1.

---

**Proof of Theorem 2.2.** **Step 1.** First we show that \(J_\varepsilon\) admits a unique minimizer. Since \(J_\varepsilon\) is clearly convex and lower semi-continuous, it suffices to show its coercivity.

Let \(\{v_0^j\}\) be a sequence in \(L^2(\Omega)\) such that
\[
\lim_{j \to \infty} \|v_0^j\|_{L^2(\Omega)} = \infty.
\]
We set \(w_0^j := \frac{v_0^j}{\|v_0^j\|_{L^2(\Omega)}}\) and denote by \(w^j\) the solution of (2.3) with \(v_0 = w_0^j\). That is,
\[
w^j(x, t) = \sum_{n=1}^{\infty} (w_0^j, \varphi_n)(T - t)^{\alpha - 1} \tilde{E}_{\alpha, \alpha}(-\lambda_n(T - t)^{\alpha})\varphi_n(x)
\]
(see (5.1)). Then we have
\[
\frac{J_\varepsilon(v_0^j)}{\|v_0^j\|_{L^2(\Omega)}} = \frac{\|v_0^j\|_{L^2(\Omega)}}{2} \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L}w^j|^2d\sigma_xdt + \varepsilon\|I - \pi_E\|_{L^2(\Omega)}w_0^j, u_1 = \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L}w^j|^2d\sigma_xdt > 0.
\]
If \(\lim_{j \to \infty} \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L}w^j|^2d\sigma_xdt > 0\), then we immediately obtain
\[
\lim_{j \to \infty} \frac{J_\varepsilon(v_0^j)}{\|v_0^j\|_{L^2(\Omega)}} = \infty,
\]
Moreover, by the representation of (6.3), we have
\[ \lim_{j \to \infty} \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu L} w^j|^2 d\sigma_x dt = 0. \] (6.4)
Since \( \|w_0^j\|_{L^2(\Omega)} = 1 \), there exists a subsequence (denoted by \( \{w_0^j\} \) again without any confusion) weakly converging to some \( \overline{w}_0 \in L^2(\Omega) \). Let \( \overline{w} \) be the solution of (2.3) wth \( v_0 = \overline{w}_0 \), that is,
\[ \overline{w}(x, t) = \sum_{n=1}^{\infty} (\overline{w}_0, \varphi_n)(T - t)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n(T - t)^\alpha)\varphi_n(x). \] (6.5)
Then we see that
\[ \frac{\partial w^j}{\partial \nu_L}(\cdot, t) \to \frac{\partial \overline{w}}{\partial \nu_L}(\cdot, t) \] in \( L^2(\Gamma) \) for any \( t \in (0, T) \). Indeed, by (6.3) and (6.5), we use the similar calculation to the proof of Proposition 5.1 and have
\[
\left\| \frac{\partial w^j}{\partial \nu_L}(\cdot, t) - \frac{\partial \overline{w}}{\partial \nu_L}(\cdot, t) \right\|_{L^2(\Gamma)}^2 \leq C \left\| w^j(\cdot, t) - \overline{w}(\cdot, t) \right\|_{H^2(\Omega)}^2 \leq C \left\| w^j(\cdot, t) - \overline{w}(\cdot, t) \right\|_{D(L)}^2 \\
= C \sum_{n=1}^{\infty} \lambda_n^2 |(w_0^j - \overline{w}_0, \varphi_n)|^2 (T - t)^{2\alpha - 2} |E_{\alpha, \alpha}(-\lambda_n(T - t)^\alpha)|^2 \\
\leq C \sum_{n=1}^{\infty} |(w_0^j - \overline{w}_0, \varphi_n)|^2 (T - t)^{-2} \left| \frac{\lambda_n(T - t)^\alpha}{1 + \lambda_n(T - t)^\alpha} \right|^2 \\
\leq \frac{C}{(T - t)^2} \sum_{n=1}^{\infty} |(w_0^j - \overline{w}_0, \varphi_n)|^2 \to 0
\]
for any \( t \in (0, T) \), from which (6.6) follows. Here we have used Lebesgue’s convergence theorem regarding the summation as an integral. Now we set
\[ \psi^j(t) := (T - t)^2 \left\| \frac{\partial w^j}{\partial \nu_L}(\cdot, t) \right\|_{L^2(\Gamma)}^2 \quad \text{and} \quad \overline{\psi}(t) := (T - t)^2 \left\| \frac{\partial \overline{w}}{\partial \nu_L}(\cdot, t) \right\|_{L^2(\Gamma)}^2. \]
Then by (6.6),
\[ \lim_{j \to \infty} \psi^j(t) = \overline{\psi}(t), \quad t \in (0, T). \]
Moreover, by the representation of (6.3), we have
\[
\psi^j(t) = (T - t)^2 \left\| \frac{\partial w^j}{\partial \nu_L}(\cdot, t) \right\|_{L^2(\Gamma)}^2 \leq C(T - t)^2 \left\| \psi^j(\cdot, t) \right\|_{H^2(\Omega)}^2 \leq C(T - t)^2 \left\| \psi^j(\cdot, t) \right\|_{D(L)}^2 \\
= C(T - t)^2 \sum_{n=1}^{\infty} \lambda_n^2 |(w_0^j, \varphi_n)|^2 (T - t)^{2\alpha - 2} |E_{\alpha, \alpha}(-\lambda_n(T - t)^\alpha)|^2 \\
\leq C \sum_{n=1}^{\infty} |(w_0^j, \varphi_n)|^2 \left| \frac{\lambda_n(T - t)^\alpha}{1 + \lambda_n(T - t)^\alpha} \right|^2 \leq C \|w_0^j\|^2_{L^2(\Gamma)} = C.
\]
Therefore, by Lebesgue’s convergence theorem, we have
\[
\int_0^T \psi^j(t) dt \to \int_0^T \bar{\psi}(t) dt,
\]
that is,
\[
\int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L} w^j|^2 d\sigma_x dt \to \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L} \bar{w}|^2 d\sigma_x dt.
\]
Combining this with (6.4), we have
\[
\int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L} \bar{w}|^2 d\sigma_x dt = 0.
\]
Hence we have
\[
\partial_{\nu_L} \bar{w} = 0 \quad \text{on } \Gamma_0 \times (0, T),
\]
from which we deduce \(\bar{w}_0 \equiv 0\) by Proposition 5.2. That is, \(\{w^j_0\}\) weakly converges to 0 in \(L^2(\Omega)\) and consequently we have
\[
\lim_{j \to \infty} (w^j_0, u_1) = 0 \quad \text{and} \quad \lim_{j \to \infty} \|I - \pi_E\| w^j_0 \|_{L^2(\Omega)} = 1
\]
since \(\pi_E\) is a compact operator. Therefore we obtain
\[
\lim_{j \to \infty} \frac{J_\varepsilon(w^j_0)}{\|w^j_0\|_{L^2(\Omega)}} \geq \varepsilon.
\]
Thus we have shown the coercivity of \(J_\varepsilon\).

**Step 2.** Let \(\bar{v}_0\) be the minimizer of \(J_\varepsilon\), then for any \(h > 0\) and \(v_0 \in L^2(\Omega)\), we have
\[
0 \leq J_\varepsilon(\bar{v}_0 + hv_0) - J_\varepsilon(\bar{v}_0)
\]
\[
\leq h \int_0^T \int_{\Gamma_0} (T - t)^2 (\partial_{\nu_L} \bar{v})(\partial_{\nu_L} v) d\sigma_x dt + \frac{h^2}{2} \int_0^T \int_{\Gamma_0} (T - t)^2 |\partial_{\nu_L} v|^2 d\sigma_x dt
\]
\[
+ \varepsilon h \|I - \pi_E\| v_0 \|_{L^2(\Omega)} + h(v_0, u_1).
\]
Dividing the above inequality by \(h\) and letting \(h \to 0\), we have
\[
0 \leq \int_0^T \int_{\Gamma_0} (T - t)^2 (\partial_{\nu_L} \bar{v})(\partial_{\nu_L} v) d\sigma_x dt + \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)} - (v_0, u_1)
\]
\[
= \int_0^T \int_{\Gamma_0} g(\partial_{\nu_L} v) d\sigma_x dt + \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)} + (v_0, u_1).
\]
By the density argument, we can verify (6.1) for \(g \in L^p(0, T; L^2(\Omega))\) with \(p > 4/\alpha\). Then we obtain
\[
0 \leq -\langle u(\cdot, T), v_0 \rangle + \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)} - (u_1, v_0),
\]
that is,
\[
(u(\cdot, T) - u_1, v_0) \leq \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)}.
\]
By taking \(h < 0\) and repeating the same argument, we also have \((u_1 - u(\cdot, T), v_0) \leq \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)}\). Therefore
\[
\|(u(\cdot, T) - u_1, v_0) \leq \varepsilon \|I - \pi_E\| v_0 \|_{L^2(\Omega)}.
\]
(6.7)
Since $v_0 \in L^2(\Omega)$ was arbitrary, we take $v_0 \in E^\perp$ and obtain

$$|(u(\cdot, T) - u_1, v_0)| \leq \varepsilon \|v_0\|_{L^2(\Omega)},$$

that is,

$$\|u(\cdot, T) - u_1\|_{L^2(\Omega)} \leq \varepsilon.$$ 

Moreover, by taking $v_0 \in E$ in (6.7), we have

$$|(u(\cdot, T) - u_1, v_0)| = 0.$$

Since $v_0 \in E$ can be taken arbitrarily, we have

$$u(\cdot, T) - u_1 \in E^\perp,$$

that is,

$$\pi_E(u(\cdot, T)) = \pi_E(u_1).$$

\[\square\]

**Appendix A. Regularity of the elliptic problem**

In this section, we consider the following elliptic boundary value problem:

$$\begin{cases}
\mathcal{L}u = 0 & \text{in } \Omega, \\
u = g & \text{on } \Gamma,
\end{cases} \quad (A.1)$$

where $g$ is given on $\Gamma$. For $g \in H^{3/2}(\Gamma)$, by using the trace theorem and lifting and applying the well known results for the elliptic boundary value problems with homogeneous data (see Theorems 8.1 and 9.8 in Agmon [4] for example), we see that (A.1) has a unique solution $u \in H^2(\Omega)$ satisfying

$$\|u\|_{H^2(\Omega)} \leq C\|g\|_{H^{3/2}(\Gamma)}. \quad (A.2)$$

In the following, we will discuss (A.1) for non-smooth $g$ by the transposition method.

We first consider the dual system for (A.1):

$$\begin{cases}
\mathcal{L}v = f & \text{in } \Omega, \\
v = 0 & \text{on } \Gamma,
\end{cases} \quad (A.3)$$

where $f$ is given in $\Omega$. It is well known that for any $f \in L^2(\Omega)$, (A.3) posseses a unique solution $v \in H^2(\Omega)$ satisfying

$$\|v\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \quad (A.4)$$

Henceforth we will denote this solution by $v_f$. Now we can define the solution of (A.1) in a weaker sense.

**Definition A.1.** A function $u$ is a weak solution of (A.1) if

$$\int_{\Omega} u(x)f(x)dx + \int_{\Gamma} g(x)\frac{\partial v_f}{\partial \nu_L}(x)d\sigma_x = 0 \quad (A.5)$$

holds for any $f \in L^2(\Omega)$.
According to this definition, the solution \( u \in H^2(\Omega) \) obtained before is also a weak solution. Indeed, by the Green’s formula, we have
\[
0 = \int_{\Omega} \mathcal{L} u(x) v_f(x) dx = \int_{\Omega} u(x) \mathcal{L} v_f(x) dx + \int_{\Gamma} \left( u(x) \frac{\partial v_f}{\partial \nu_L}(x) - \frac{\partial u}{\partial \nu_L}(x)v(x) \right) d\sigma_x
\]
\[
= \int_{\Omega} u(x) f(x) dx + \int_{\Gamma} g(x) \frac{\partial v_f}{\partial \nu_L}(x)d\sigma_x.
\]
Thus condition (A.5) is satisfied. We also see that (A.1) has a weak solution if \( g \) is a distribution;

**Proposition A.1.** For any \( g \in H^{-1/2}(\Gamma) \), there exists a unique weak solution \( u \in L^2(\Omega) \) satisfying
\[
\|u\|_{L^2(\Omega)} \leq C\|g\|_{H^{-1/2}(\Gamma)}.
\] (A.6)

**Proof.** As we have seen, for \( f \in L^2(\Omega) \), the solution \( v_f \) of (A.3) belongs to \( H^2(\Omega) \). Therefore, by the trace theorem, \( \partial_{\nu_L} v_f \in H^{1/2}(\Gamma) \) and
\[
\left\| \frac{\partial v_f}{\partial \nu_L} \right\|_{H^{1/2}(\Gamma)} \leq C\|v_f\|_{H^2(\Omega)}.
\]
Combining this with (A.4), we obtain
\[
\left\| \frac{\partial v_f}{\partial \nu_L} \right\|_{H^{1/2}(\Gamma)} \leq C\|f\|_{L^2(\Omega)}.
\] (A.7)
Thus the mapping
\[
L^2(\Omega) \ni f \mapsto \frac{\partial v_f}{\partial \nu_L} \in H^{1/2}(\Gamma)
\]
is bounded, and so is
\[
L^2(\Omega) \ni f \mapsto - \int_{\Gamma} g(x) \frac{\partial v_f}{\partial \nu_L}(x)d\sigma_x \in C.
\]
Hence the Riesz’s representation theorem yields that there exists a unique \( u \in L^2(\Omega) \) such that
\[
\int_{\Omega} u(x) f(x) dx = - \int_{\Gamma} g(x) \frac{\partial v_f}{\partial \nu_L}(x)d\sigma_x.
\]
Moreover, by the above equation and (A.7), we have
\[
|\langle u, f \rangle| \leq \|g\|_{H^{-1/2}(\Gamma)}\|\partial_{\nu_L} v_f\|_{H^{1/2}(\Gamma)} \leq C\|g\|_{H^{-1/2}(\Gamma)}\|f\|_{L^2(\Omega)},
\]
for any \( f \in L^2(\Omega) \), from which estimate (A.6) follows. \( \square \)

Let \( \Lambda \) be a linear map which maps \( g \) to the unique weak solution of (A.1). Then we have seen that
\[
\Lambda \in \mathcal{L}(H^{3/2}(\Gamma); H^2(\Omega)) \cap \mathcal{L}(H^{-1/2}(\Gamma); L^2(\Omega)),
\]
where \(\mathcal{L}(X;Y)\) denotes a set of bounded linear operators from a Banach space \(X\) to another one \(Y\). Then by the interpolation (see Theorem 5.1 in Chapter 1 of [13]), we have

\[
\Lambda \in \mathcal{L}([H^{3/2}(\Gamma), H^{-1/2}(\Gamma)]_\theta; [H^2(\Omega); L^2(\Omega)]_\theta).
\]

for any \(\theta \in [0, 1]\). In particular, we choose \(\theta = 3/4\) and obtain

\[
\Lambda \in \mathcal{L}(L^2(\Gamma); H^{1/2}(\Omega)).
\]

That is, for any \(g \in L^2(\Gamma)\), \(\Lambda G\) belongs to \(H^{1/2}(\Omega)\) and satisfies

\[
\|\Lambda g\|_{H^{1/2}(\Omega)} \leq C\|g\|_{L^2(\Gamma)}.
\]

In particular, for any \(0 \leq \theta < 1/4\), \(\Lambda G\) belongs to \(\mathcal{D}(L^\theta)\) and satisfies

\[
\|\Lambda g\|_{\mathcal{D}(L^\theta)} \leq C\|g\|_{H^{1/2}(\Gamma)}.
\]

(A.8)

By substituting \(f = \varphi_n\) in (A.5), we obtain

\[
\lambda_n(\Lambda g, \varphi_n) = -\langle g, \partial_\nu \varphi_n \rangle, \quad n = 1, 2, \ldots.
\]

(A.9)

For the arguments used here, we can refer to Chapter 2 in [13], in which more general elliptic operator of order \(2m\) is dealt with by assuming \(C^\infty\)-regularity for the coefficients \(a_{ij}\) and the boundary \(\Gamma\).

In Section 3, we apply the above results to the calculation of the eigenfunction expansion for the solution of (2.7).

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