Persistence and extinction of a modified Leslie–Gower Holling-type II two-predator one-prey model with Lévy jumps

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ABSTRACT

This paper is concerned with a modified Leslie–Gower and Holling-type II two-predator one-prey model with Lévy jumps. First, we use an Ornstein–Uhlenbeck process to describe the environmental stochasticity and prove that there is a unique positive solution to the system. Moreover, sufficient conditions for persistence in the mean and extinction of each species are established. Finally, we give some numerical simulations to support the main results.

1. Introduction

The relationship between prey and predator is one of the most important and interesting topics in biomathematics. Functional response is a significant component of the predator–prey relationship. The famous predator–prey framework with modified Leslie–Gower and Holling-type II schemes proposed by Aziz-Alaoui and Okiye [4] can be denoted as follows:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t)(r_1 - ax(t) - \frac{cy(t)}{h + x(t)}), \\
\frac{dy(t)}{dt} &= y(t)(r_2 - \frac{fy(t)}{h + x(t)}),
\end{align*}
\]

(1)

where \(x(t)\) and \(y(t)\) represent the population sizes of the prey and the predator, respectively. \(r_1, r_2, a, c, f\) and \(h\) are positive constants. \(r_1\) and \(r_2\) are the growth rates of the prey and the predator, respectively, \(a\) represents the competitive strength among individuals of the prey, \(c\) stands for the per capita reduction rate of prey \(x\), the meaning of \(f\) is similar to \(c\), and \(h\) describes the protection of the environment. Aziz-Alaoui and Okiye [4] studied the boundedness and global stability of model (1). From then on, many authors have paid attention to model (1) and its generalized forms (see, e.g. [1, 2, 5, 10–14, 26, 27, 30, 31, 33, 35]).

The above studies have focused on two-species models. However, it is a common phenomenon that several predators compete for a prey in the natural world. At the same time, the growth of the population is inevitably affected by environmental fluctuations in real situations. Suppose that the growth rate \(r_i\) is affected by white noise (see, e.g. [8, 15–17,
with \( r_i \to r_i + \sigma_i \hat{W}_i(t) \), Xu et al. [32] proposed a stochastic two-predator one-prey system with modified Leslie–Gower and Holling-type II schemes:

\[
\begin{align*}
\dot{x}(t) &= x(t)(r_1 - ax(t) - \frac{c_1 y_1(t)}{h_1 + x(t)} - \frac{c_2 y_2(t)}{h_2 + x(t)})dt + \sigma_1 x(t)dW_1(t), \\
\dot{y}_1(t) &= y_1(t) \left( r_2 - \frac{f_1 y_1(t)}{h_1 + x(t)} \right) dt + \sigma_2 y_1(t)dW_2(t), \\
\dot{y}_2(t) &= y_2(t) \left( r_3 - \frac{f_2 y_2(t)}{h_2 + x(t)} \right) dt + \sigma_3 y_2(t)dW_3(t),
\end{align*}
\]

where \( W_i(t) \) is a standard Brownian motion defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions and \( \sigma_i^2 \) stands for the intensity of the white noise.

However, the growth of species in the real world is often affected by sudden random perturbations, such as epidemics, harvesting, earthquakes, and so on; and these phenomena cannot be described by white noise. Bao and Yuan [6] and Bao et al. [7] suggested that these phenomena can be described by a Lévy jump process. Therefore, we can obtain the following two-predator one-prey model with white noise and Lévy jumps, which introduced Lévy noise into the population model in the same way as [22]:

\[
\begin{align*}
\dot{x}(t) &= x(t^-)(r_1 - ax(t) - \frac{c_1 y_1(t)}{h_1 + x(t)} - \frac{c_2 y_2(t)}{h_2 + x(t)})dt + \sigma_1 x(t^-)dW_1(t) \\
&\quad + \int_{\mathbb{Y}} \lambda_1(u)x(t^-)\tilde{N}(dt, du), \\
\dot{y}_1(t) &= y_1(t^-) \left( r_2 - \frac{f_1 y_1(t^-)}{h_1 + x(t^-)} \right) dt + \sigma_2 y_1(t^-)dW_2(t) \\
&\quad + \int_{\mathbb{Y}} \lambda_2(u)y_1(t^-)\tilde{N}(dt, du), \\
\dot{y}_2(t) &= y_2(t^-) \left( r_3 - \frac{f_2 y_2(t^-)}{h_2 + x(t^-)} \right) dt + \sigma_3 y_2(t^-)dW_3(t) \\
&\quad + \int_{\mathbb{Y}} \lambda_3(u)y_2(t^-)\tilde{N}(dt, du),
\end{align*}
\]

with initial data \( x(0) > 0, y_1(0) > 0 \) and \( y_2(0) > 0 \), where \( x(t^-), y_1(t^-) \) and \( y_2(t^-) \) are the left limit of \( x(t), y_1(t) \) and \( y_2(t) \), respectively. \( N \) is a Poisson counting measure with characteristic measure \( \eta \) on a measurable subset \( \mathbb{Y} \) of \((0, +\infty)\) with \( \eta(\mathbb{Y}) < +\infty \), \( \tilde{N}(dt, du) = N(dt, du) - \eta(du)dt \), \( \lambda_i : \mathbb{Y} \times \Omega \to \mathbb{R} \) is bounded and continuous with respect to \( \eta \), and is \( \mathcal{B}(\mathbb{Y}) \times \mathcal{F}_t \)-measurable, \( i = 1, 2, 3 \).

Model (2) assumes that the growth rate is linearly dependent on the Gaussian white noise in the random environments

\[
\tilde{r}_i(t) = r_i + \sigma_i \frac{dW_i(t)}{dt}, \quad i = 1, 2, 3.
\]

Integrating on the interval \([0, T]\), we can see that

\[
\tilde{r}_i = \frac{1}{T} \int_0^T \tilde{r}_i(t)dt \to r_i + \sigma_i \frac{W_i(T)}{T} \sim N(r_i, \sigma_i^2 / T).
\]
Hence, the variance of the average per capita growth rate \( \bar{r}_i \) over an interval of length \( T \) tends to \( \infty \) as \( T \rightarrow 0 \). According to this point, we can see that model (2) cannot accurately describe the real situation. Therefore, many authors (see [9, 34]) have proposed that using the mean-reverting Ornstein–Uhlenbeck process is a more appropriate way to incorporate the environmental perturbations. On account of this approach, one has

\[
d\bar{r}_i(t) = \alpha_i(r_i - \bar{r}_i(t))dt + \xi_i dW_i(t), \quad i = 1, 2, 3,
\]
i.e.

\[
\bar{r}_i(t) = r_i + (r_{i0} - r_i)e^{-\alpha_i t} + \xi_i \int_0^t e^{-\alpha_i (t-s)} dW_i(s)
\]

\[
= r_i + (r_{i0} - r_i)e^{-\alpha_i t} + \sigma_i(t) \frac{dW_i(t)}{dt}, \quad i = 1, 2, 3,
\]

where \( r_{i0} = \bar{r}_i(0), \sigma_i(t) = \frac{\xi_i}{\sqrt{2\alpha_i}} \sqrt{1 - e^{-2\alpha_i t}}, \xi_i^2 \) means the intensity of stochastic perturbations and \( \alpha_i > 0 \) characterizes the speed of reversion. As a result, Zhou et al. [36] considered the following stochastic model:

\[
\begin{aligned}
\text{dx}(t) &= x(t)(r_1 + (r_{10} - r_1)e^{-\alpha_1 t} - ax(t) - \frac{cy(t)}{h + x(t)})dt + \sigma_1(t)x(t)dW_1(t), \\
\text{dy}_1(t) &= y_1(t)(r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{fy_1(t)}{h + x(t)})dt + \sigma_2(t)y_1(t)dW_2(t).
\end{aligned}
\]

Motivated by these, according to model (2), we can derive the following stochastic two-predator one-prey model with modified Leslie–Gower and Holling-type II schemes with Lévy jumps:

\[
\begin{aligned}
\text{dx}(t) &= x(t^-)(r_1 + (r_{10} - r_1)e^{-\alpha_1 t} - ax(t) - \frac{c_1y_1(t)}{h_1 + x(t)} - \frac{c_2y_2(t)}{h_2 + x(t)})dt \\
&\quad + \sigma_1(t)x(t^-)dW_1(t) + \int_{\mathbb{V}} \lambda_1(u)x(t^-)\tilde{N}(dt, du), \\
\text{dy}_1(t) &= y_1(t^-)(r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f_1y_1(t)}{h_1 + x(t)})dt \\
&\quad + \sigma_2(t)y_1(t^-)dW_2(t) + \int_{\mathbb{V}} \lambda_2(u)y_1(t^-)\tilde{N}(dt, du), \\
\text{dy}_2(t) &= y_2(t^-)(r_3 + (r_{30} - r_3)e^{-\alpha_3 t} - \frac{f_2y_2(t)}{h_2 + x(t)})dt \\
&\quad + \sigma_3(t)y_2(t^-)dW_3(t) + \int_{\mathbb{V}} \lambda_3(u)y_2(t^-)\tilde{N}(dt, du).
\end{aligned}
\]

To the best of our knowledge, there are few studies related to model (4), so we mainly study the properties of model (4) in this paper.

The rest of this paper is organized as follows. In Section 2, we give some lemmas for our main results and obtain sufficient conditions for persistence in the mean and extinction for each species. In Section 3, we introduce some simulation figures to illustrate our main theoretical results. Some concluding remarks are given in Section 4.
2. Main results

For convenience and simplicity, we define some notations as follows:

\[ R^3_+ = \{ z \in R^3 \mid z_i > 0, i = 1, 2, 3 \}, \quad \langle f(t) \rangle = t^{-1} \int_0^t f(s) ds, \]

\[ \langle f \rangle^* = \limsup_{t \to +\infty} t^{-1} \int_0^t f(s) ds, \quad \langle f \rangle_* = \liminf_{t \to +\infty} t^{-1} \int_0^t f(s) ds, \]

\[ b_i(t) = r_i - \frac{\xi_i^2}{4\alpha_i} + \frac{\xi_i^2}{4\alpha_i} e^{-2\alpha_i t}, \quad \bar{b}_i = \lim_{t \to +\infty} t^{-1} \int_0^t b_i(s) ds = r_i - \frac{\xi_i^2}{4\alpha_i}, \]

\[ \beta_i = \int_{\mathbb{Y}} [\lambda_i(u) - \ln(1 + \lambda_i(u))] \eta(du), \]

\[ k_i(t) = \int_0^t \int_{\mathbb{Y}} [\ln(1 + \lambda_i(u))] \tilde{N}(ds, du), \quad i = 1, 2, 3. \]

First, we give the following assumption and definition.

**Assumption 2.1:** There exists a constant \( m > 0 \) such that

\[ 1 + \lambda_i(u) > 0, \quad \int_{\mathbb{Y}} [\ln(1 + \lambda_i(u))]^2 \eta(du) < m, \quad i = 1, 2, 3, \]

which means that the jump noise is not too strong.

**Definition 2.1 ([22]):**
- \( x(t) \) is said to be extinctive if \( \lim_{t \to +\infty} x(t) = 0 \) a.s.
- \( x(t) \) is said to be persistent in the mean if \( \liminf_{t \to +\infty} t^{-1} \int_0^t x(s) ds > 0 \) a.s.

Before we state and prove our main results, we recall some lemmas which will be used later.

**Lemma 2.1 ([23]):** Suppose that \( f(t) \in C(\Omega \times [0, +\infty), [0, +\infty)) \), where \( C(\Omega \times [0, +\infty), [0, +\infty)) \) denotes the family of all positive-valued functions defined on \( \Omega \times [0, +\infty) \).

(I) If there are three positive constants \( T, \lambda_0, \lambda \) such that for all \( t \geq T \),

\[ \ln f(t) \leq \lambda t - \lambda_0 \int_0^t f(s) ds + F(t), \]

where \( F(t)/t \to 0 \) as \( t \to +\infty \), then

\[ \langle f \rangle^* = \limsup_{t \to +\infty} t^{-1} \int_0^t f(s) ds \leq \lambda/\lambda_0 \quad \text{a.s.} \]

(II) If there are three positive constants \( T, \lambda_0, \lambda \) such that for all \( t \geq T \),

\[ \ln f(t) \geq \lambda t - \lambda_0 \int_0^t f(s) ds + F(t), \]

where \( F(t)/t \to 0 \) as \( t \to +\infty \), then

\[ \langle f \rangle_* = \liminf_{t \to +\infty} t^{-1} \int_0^t f(s) ds \geq \lambda/\lambda_0 \quad \text{a.s.} \]
Lemma 2.2 ([20]): Suppose that $M(t), t \geq 0$, is a local martingale vanishing at time zero. Then
\[
\lim_{t \to +\infty} \rho_M(t) < +\infty \Rightarrow \lim_{t \to +\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.}
\]
where
\[
\rho_M(t) = \int_0^t \frac{d\langle M, M \rangle(s)}{(1 + s)^2}, \quad t \geq 0
\]
and $\langle M, M \rangle(t)$ is Meyer’s angle bracket process (see, e.g. [3, 18]).

Lemma 2.3: For any given initial value $(x(0), y_1(0), y_2(0)) \in \mathbb{R}^3$, model (4) has a unique solution $(x(t), y_1(t), y_2(t)) \in \mathbb{R}^3$ for all $t \geq 0$ almost surely.

**Proof:** To begin with, let us consider the following system:
\[
\begin{align*}
du(t) &= [b_1(t) - \beta_1 + (r_{10} - r_1)e^{-\alpha_1 t} - ae^{\mu(t)} - \frac{c_1 e^{\nu_1(t)}}{h_1 + e^{\mu(t)}} - \frac{c_2 e^{\nu_2(t)}}{h_2 + e^{\mu(t)}}]dt \\
&\quad + \sigma_1(t)dW_1(t) + \int_Y \ln(1 + \lambda_1(u))\tilde{N}(dt, du), \\
&\quad + \int_Y \ln(1 + \lambda_2(u))\tilde{N}(dt, du), \\
dv_1(t) &= \left[b_2(t) - \beta_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f_1 e^{\nu_1(t)}}{h_1 + e^{\mu(t)}} \right]dt + \sigma_2(t)dW_2(t) \\
&\quad + \int_Y \ln(1 + \lambda_2(u))\tilde{N}(dt, du), \\
dv_2(t) &= [b_3(t) - \beta_3 + (r_{30} - r_3)e^{-\alpha_3 t} - \frac{f_2 e^{\nu_2(t)}}{h_2 + e^{\mu(t)}}]dt + \sigma_3(t)dW_3(t) \\
&\quad + \int_Y \ln(1 + \lambda_3(u))\tilde{N}(dt, du),
\end{align*}
\]
on $t \geq 0$ with initial data $u(0) = \ln x(0)$, $v_1(0) = \ln y_1(0)$, $v_2(0) = \ln y_2(0)$. The coefficients of system (5) satisfy the local Lipschitz condition, then there is a unique local solution on $[0, \tau_c)$ (see Theorems 3.15–3.17 in [25]), where $\tau_c$ means the explosion time. Therefore, it follows from Itô’s formula that on $[0, \tau_c)$ model (4) has a unique solution $(x(t), y_1(t), y_2(t)) = (e^{\mu(t)}, e^{\nu_1(t)}, e^{\nu_2(t)})$ which is positive. Now we validate $\tau_c = +\infty$. Consider the following systems:
\[
\begin{align*}
d\Phi(t) &= \Phi(t)[r_1 + (r_{10} - r_1)e^{-\alpha_1 t} - a\Phi(t)]dt + \sigma_1(t)\Phi(t)dW_1(t) \\
&\quad + \int_Y \lambda_1(u)\Phi(t)\tilde{N}(dt, du), \quad \Phi(0) = x(0); \\
d\Psi_1(t) &= \Psi_1(t)[r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f_1}{h_1}\Psi_1(t)]dt + \sigma_2(t)\Psi_1(t)dW_2(t) \\
&\quad + \int_Y \lambda_2(u)\Psi_1(t)\tilde{N}(dt, du), \quad \Psi_1(0) = y_1(0); \\
d\Psi_2(t) &= \Psi_2(t)[r_2 + (r_{20} - r_2)e^{-\alpha_2 t} - \frac{f_1}{h_1 + \Phi(t)}\Psi_2(t)]dt + \sigma_2(t)\Psi_2(t)dW_2(t)
\end{align*}
\]
According to Lemma 4.2 in [7], Equation (6) has the explicit formula

$$
\Phi(t) = \frac{\int_0^t b_1(s) ds - \beta_1 t - \frac{\tau_0 - \tau_1}{a_1} (e^{-a_1 t} - 1) + \int_0^t \sigma_1(s) dW_1(s) + k_1(t)}{x^{-1}(0) + a \int_0^t e^{\int_0^\tau b_1(\tau) d\tau - \beta_1 s - \frac{\tau_0 - \tau_1}{a_1} (e^{-a_1 \tau} - 1) + \int_0^\tau \sigma_1(\tau) dW_1(\tau) + k_1(\tau)} ds}.
$$

Similarly

$$
\Psi_1(t) = \frac{\int_0^t b_2(s) ds - \beta_2 t - \frac{\tau_0 - \tau_2}{a_2} (e^{-a_2 t} - 1) + \int_0^t \sigma_2(s) dW_2(s) + k_2(t)}{y_1^{-1}(0) + \int_0^t \frac{f_1}{h_1} e^{\int_0^\tau b_2(\tau) d\tau - \beta_2 s - \frac{\tau_0 - \tau_2}{a_2} (e^{-a_2 \tau} - 1) + \int_0^\tau \sigma_2(\tau) dW_2(\tau) + k_2(\tau)} ds},
$$

$$
\Psi_2(t) = \frac{\int_0^t b_2(s) ds - \beta_2 t - \frac{\tau_0 - \tau_2}{a_2} (e^{-a_2 t} - 1) + \int_0^t \sigma_2(s) dW_2(s) + k_2(t)}{y_1^{-1}(0) + \int_0^t \frac{f_1}{h_1} e^{\int_0^\tau b_2(\tau) d\tau - \beta_2 s - \frac{\tau_0 - \tau_2}{a_2} (e^{-a_2 \tau} - 1) + \int_0^\tau \sigma_2(\tau) dW_2(\tau) + k_2(\tau)} ds},
$$

$$
\varphi_1(t) = \frac{\int_0^t b_3(s) ds - \beta_3 t - \frac{\tau_0 - \tau_3}{a_3} (e^{-a_3 t} - 1) + \int_0^t \sigma_3(s) dW_3(s) + k_3(t)}{y_2^{-1}(0) + \int_0^t \frac{f_2}{h_2} e^{\int_0^\tau b_3(\tau) d\tau - \beta_3 s - \frac{\tau_0 - \tau_3}{a_3} (e^{-a_3 \tau} - 1) + \int_0^\tau \sigma_3(\tau) dW_3(\tau) + k_3(\tau)} ds},
$$

$$
\varphi_2(t) = \frac{\int_0^t b_3(s) ds - \beta_3 t - \frac{\tau_0 - \tau_3}{a_3} (e^{-a_3 t} - 1) + \int_0^t \sigma_3(s) dW_3(s) + k_3(t)}{y_2^{-1}(0) + \int_0^t \frac{f_2}{h_2} e^{\int_0^\tau b_3(\tau) d\tau - \beta_3 s - \frac{\tau_0 - \tau_3}{a_3} (e^{-a_3 \tau} - 1) + \int_0^\tau \sigma_3(\tau) dW_3(\tau) + k_3(\tau)} ds}.
$$

Due to the fact that $\Phi(t), \Psi_1(t), \Psi_2(t), \varphi_1(t),$ and $\varphi_2(t)$ are existent on $t \geq 0$, then we can obtain $\tau_\epsilon = +\infty$.

**Lemma 2.4:** Let $\bar{b}_1 > \beta_1$. If $\bar{b}_2 > \beta_2$ (respectively, $\bar{b}_3 > \beta_3$), then

$$
\lim_{t \to +\infty} t^{-1} \ln y_1(t) = 0 \quad \text{(respectively,} \quad \lim_{t \to +\infty} t^{-1} \ln y_2(t) = 0) \quad \text{a.s.}
$$

**Proof:** Here we only prove the case $\bar{b}_2 > \beta_2$, the proof of $\bar{b}_3 > \beta_3$ is similar.
For sufficiently small $\varepsilon > 0$, there is sufficiently large $T$ such that, for $t \geq T$,

$$(\bar{b}_i - \varepsilon)t \leq \int_0^t b_i(s)ds \leq (\bar{b}_i + \varepsilon)t, \quad i = 1, 2; \quad e^{(\bar{b}_1 - \beta_1 - \varepsilon)t} \geq 2;$$

and for $t \geq T_1 = T + \ln 2/(\bar{b}_2 - \beta_2 - \varepsilon)$,

$$e^{(\bar{b}_2 - \beta_2 - \varepsilon)t} \geq 2e^{(\bar{b}_2 - \beta_2 - \varepsilon)T}.$$

Then when $t \geq T$, by (12),

$$\Phi(t) = \frac{e^{\int_0^t b_1(s)ds - \beta_1 t - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + \int_0^t \sigma_1(s)dW_1(s) + k_1(t)}}{x^{-1}(0) + a \int_0^t e^{\int_0^t b_1(s)ds - \beta_1 s - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 s} - 1) + \int_0^s \sigma_1(\tau)dW_1(\tau) + k_1(s)} ds \leq \frac{e^{\int_0^t b_1(s)ds - \beta_1 t - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + \int_0^t \sigma_1(s)dW_1(s) + k_1(t)}}{a \int_0^t e^{\int_0^t b_1(\tau)d\tau - \beta_1 s - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 s} - 1) + \int_0^s \sigma_1(\tau)dW_1(\tau) + k_1(s)} ds \leq \frac{e^{(\bar{b}_1 - \beta_1 - \varepsilon)t - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + \int_0^t \sigma_1(s)dW_1(s) + k_1(t)}}{ae \min_{\theta \leq \varepsilon \leq \ell}(\int_0^\theta \sigma_1(\tau)dW_1(\tau) - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 \tau} - 1) + k_1(\tau)) \int_0^t e^{(\bar{b}_1 - \beta_1 - \varepsilon)\tau} ds \leq \frac{2(\bar{b}_1 - \beta_1 - \varepsilon) e^{(\bar{b}_1 - \beta_1 - \varepsilon)t - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + \int_0^t \sigma_1(s)dW_1(s) + k_1(t)}}{ae(\bar{b}_1 - \beta_1 - \varepsilon)t - 1)} e^{\min_{\theta \leq \varepsilon \leq \ell}(\int_0^\theta \sigma_1(\tau)dW_1(\tau) - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 \tau} - 1) + k_1(\tau)) \int_0^t e^{(\bar{b}_1 - \beta_1 - \varepsilon)\tau} ds \leq \frac{2(\bar{b}_1 - \beta_1 - \varepsilon) e^{2\varepsilon t} G_1(t)}{a} \quad e^{2\varepsilon t} G_1(t),$$

where

$$G_1(t) = \frac{e^{\int_0^t \sigma_1(s)dW_1(s) - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + k_1(t)}}{e^{\min_{\theta \leq \varepsilon \leq \ell}(\int_0^\theta \sigma_1(\tau)dW_1(\tau) - \frac{\tau_0 - \tau_1}{\alpha_1} (e^{-\alpha_1 \tau} - 1) + k_1(\tau)) \int_0^t e^{(\bar{b}_1 - \beta_1 - \varepsilon)\tau} ds \leq \frac{2(\bar{b}_1 - \beta_1 - \varepsilon) e^{2\varepsilon t} G_1(t)}{a} \quad e^{2\varepsilon t} G_1(t).$$

Note that $G_1(t) \geq 1$. Consequently,

$$\int_T^t \frac{f_1}{h_1 + \Phi(s)} e^{\int_0^t b_2(s)ds - \frac{\tau_0 - \tau_2}{\alpha_2} (e^{-\alpha_2 s} - 1) + \int_0^t \sigma_2(s)dW_2(s) + k_2(s)} ds \geq \int_T^t \frac{f_1 e^{(\bar{b}_1 - \beta_1 - \varepsilon)s - \frac{\tau_0 - \tau_2}{\alpha_2} (e^{-\alpha_2 s} - 1) + \int_0^t \sigma_2(s)dW_2(s) + k_2(s)}}{h_1 + \frac{2(\bar{b}_1 - \beta_1 - \varepsilon)}{a} e^{2\varepsilon s} G_1(s)} ds \geq \int_T^t \frac{f_1 e^{(\bar{b}_1 - \beta_1 - \varepsilon)s - \frac{\tau_0 - \tau_2}{\alpha_2} (e^{-\alpha_2 s} - 1) + \int_0^t \sigma_2(s)dW_2(s) + k_2(s)}}{(h_1 + \frac{2(\bar{b}_1 - \beta_1 - \varepsilon)}{a}) e^{2\varepsilon s} G_1(s)} ds \geq \int_T^t e^{(\bar{b}_1 - \beta_1 - 3\varepsilon)s - \frac{\tau_0 - \tau_2}{\alpha_2} (e^{-\alpha_2 s} - 1) + \int_0^t \sigma_2(s)dW_2(s) + k_2(s)} G_1^{-1}(s) ds.$$
where
\[ G_2(t) = G_1^{-1}(t) e^{\int_0^t \sigma_2(s) \, dW_2(s) + k_2(t) - \frac{20}{\alpha_2} (e^{-\alpha_2 t} - 1)}, \]
\[ G_3(t) = \frac{af_1}{ah_1 + 2(\bar{b}_1 - \beta_1 - \varepsilon)} \frac{1}{\bar{b}_2 - \beta_2 - 3\varepsilon} \min_{0 \leq v \leq t} \{ G_2(v) \}. \]

Moreover, for \( t \geq T_1 \), substituting the above inequalities into (14) leads to

\[ \frac{1}{\Psi_2(t)} \geq e^{-\int_0^t b_2(s) \, ds + \beta_2 (t-T) + \frac{20}{\alpha_2} (e^{-\alpha_2 (t-T)} - 1) - \int_T^t \sigma_2(s) \, dW_2(s) - (k_2(t) - k_2(T))} \]
\[ \times G_3(t) (e^{(\bar{b}_2 - \beta_2 - 3\varepsilon) t} - e^{(\bar{b}_2 - \beta_2 - 3\varepsilon) T}) \]
\[ \geq e^{-\int_0^t b_2(s) \, ds + \beta_2 (t-T) + \frac{20}{\alpha_2} (e^{-\alpha_2 (t-T)} - 1) - \int_T^t \sigma_2(s) \, dW_2(s) - (k_2(t) - k_2(T))} \]
\[ \times \frac{1}{2} G_3(t) e^{(\bar{b}_2 - \beta_2 - 3\varepsilon) t} \]
\[ \geq G_4(t) \times e^{-4\varepsilon t}, \]

where
\[ G_4(t) = \frac{1}{2} G_3(t) e^{\int_0^T b_2(s) \, ds - \beta_2 T + \frac{20}{\alpha_2} (e^{-\alpha_2 (t-T)} - 1) - \int_T^t \sigma_2(s) \, dW_2(s) - (k_2(t) - k_2(T))}. \]

For this reason,
\[ t^{-1} \ln \Psi_2(t) \leq -t^{-1} \ln G_4(t) + 4\varepsilon. \]  

(17)

According to Assumption 2.1, we get
\[ \langle k_i(t), k_i(t) \rangle = t \int_Y (\ln(1 + \lambda_i(u)))^2 \eta(du) < mt, \quad i = 1, 2, 3. \]

In view of Lemma 2.2, then
\[ \lim_{t \to +\infty} \frac{k_i(t)}{t} = 0 \quad \text{a.s.}, \quad i = 1, 2, 3. \]

We then deduce from \( \lim_{t \to +\infty} t^{-1} \int_0^t \sigma_i(s) \, dW_i(s) = 0 \) (\( i = 1, 2, 3 \)) that if \( \bar{b}_2 > \beta_2 \), \( \lim_{t \to +\infty} t^{-1} \ln G_4(t) = 0 \) \( \text{a.s.} \)
Substituting the above identities into (17) leads to
\[
\limsup_{t \to +\infty} t^{-1} \ln y_1(t) \leq \limsup_{t \to +\infty} t^{-1} \ln \Psi_2(t) \leq 0 \quad \text{a.s.}
\]
Now let us prove \( \liminf_{t \to +\infty} t^{-1} \ln y_1(t) \geq 0 \) a.s. Making use of Itô’s formula to (7) deduces
\[
d \ln \Psi_1(t) = [r_2 + (r_2 - r_2)e^{-\alpha_2 t} - \frac{f_1}{h_1} \Psi_1(t) - \frac{1}{2} \sigma_2^2(t)]dt + \sigma_2(t)dW_2(t)
- \int_Y \left[ \lambda_2(u) - \ln(1 + \lambda_2(u)) \right] \eta(du)dt + \int_Y \ln(1 + \lambda_2(u)) N(dt, du)
- [b_2(t) + (r_2 - r_2)e^{-\alpha_2 t} - \frac{f_1}{h_1} \Psi_1(t) - \beta_2]dt + \sigma_2(t)dW_2(t)
+ \int_Y \ln(1 + \lambda_2(u)) N(dt, du).
\]
That is to say
\[
t^{-1} \ln \Psi_1(t) = t^{-1} \ln y_1(0) + t^{-1} \int_0^t b_2(s)ds - \frac{(r_2 - r_2)}{\alpha_2 t} (e^{-\alpha_2 t} - 1)
- \frac{f_1}{h_1} t^{-1} \int_0^t \Psi_1(s)ds - \beta_2 + t^{-1} \int_0^t \sigma_2(s)dW_2(s) + \frac{k_2(t)}{t}.
\]
For arbitrary given \( \varepsilon > 0 \), there exists \( T > 0 \) such that, for \( t \geq T \),
\[
\bar{b}_2 - 2\varepsilon \leq t^{-1} \ln y(0) + t^{-1} \int_0^t b_2(s)ds - \frac{(r_2 - r_2)}{\alpha_2 t} (e^{-\alpha_2 t} - 1) + \frac{k_2(t)}{t} \leq \bar{b}_2 + 2\varepsilon.
\]
We then deduce from (18) that, for \( t \geq T \),
\[
t^{-1} \ln \Psi_1(t) \leq (\bar{b}_2 - \beta_2 + 2\varepsilon) - \frac{f_1}{h_1} t^{-1} \int_0^t \Psi_1(s)ds + t^{-1} \int_0^t \sigma_2(s)dW_2(s),
\]
(19)
\[
t^{-1} \ln \Psi_1(t) \geq (\bar{b}_2 - \beta_2 - 2\varepsilon) - \frac{f_1}{h_1} t^{-1} \int_0^t \Psi_1(s)ds + t^{-1} \int_0^t \sigma_2(s)dW_2(s),
\]
(20)
where \( \varepsilon \) is sufficiently small such that \( 0 < \varepsilon < \frac{1}{2}(\bar{b}_2 - \beta_2) \). According to Lemma 2.1, we can obtain
\[
\frac{h_1(\bar{b}_2 - \beta_2 - 2\varepsilon)}{f_1} \leq (\Psi_1)_* \leq (\Psi_1)_* \leq \frac{h_1(\bar{b}_2 - \beta_2 + 2\varepsilon)}{f_1} \quad \text{a.s.}
\]
We then deduce from the arbitrariness of \( \varepsilon \) that
\[
\lim_{t \to +\infty} t^{-1} \int_0^t \Psi_1(s)ds = \frac{h_1(\bar{b}_2 - \beta_2)}{f_1} \quad \text{a.s.}
\]
which indicates that \( \lim_{t \to +\infty} t^{-1} \ln \Psi_1(t) = 0 \) a.s. In accordance with (11),
\[
\liminf_{t \to +\infty} t^{-1} \ln y_1(t) \geq \lim_{t \to +\infty} t^{-1} \ln \Psi_1(t) = 0 \quad \text{a.s.} \quad (21)
\]
The proof of Lemma 2.4 is completed.
Now we are in the position to give our main result.

**Theorem 2.1:** For model (4), the following conclusions hold:

(i) If \( \tilde{b}_1 < \beta_1, \tilde{b}_2 < \beta_2 \) and \( \tilde{b}_3 < \beta_3 \), then all the populations go to extinction, i.e. \( \lim_{t \to +\infty} x(t) = 0, \lim_{t \to +\infty} y_1(t) = 0, \lim_{t \to +\infty} y_2(t) = 0 \) a.s.

(ii) If \( b_1 < \beta_1, b_2 < \beta_2 \) and \( b_3 > \beta_3 \), then both \( x \) and \( y_1 \) become extinct, and \( y_2 \) is persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_2(b_3-\beta_3)}{f_2} \) a.s.

(iii) If \( \tilde{b}_1 < \beta_1, \tilde{b}_2 > \beta_2 \) and \( \tilde{b}_3 < \beta_3 \), then both \( x \) and \( y_2 \) become extinct, and \( y_1 \) is persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(b_3-\beta_3)}{f_1} \) a.s.

(iv) If \( \tilde{b}_1 < \beta_1, \tilde{b}_2 > \beta_2 \) and \( \tilde{b}_3 > \beta_3 \), then \( x \) becomes extinct, and \( \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(b_3-\beta_3)}{f_1}, \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_2(b_3-\beta_3)}{f_2} \) a.s.

(v) If \( \tilde{b}_1 < \beta_1, \tilde{b}_2 < \beta_2 \) and \( \tilde{b}_3 < \beta_3 \), then both \( y_1 \) and \( y_2 \) become extinct, and \( x \) is persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t x(s) ds = \frac{b_1-\beta_1}{a} \) a.s.

(vi) If \( \tilde{b}_1 < \beta_1, \tilde{b}_2 < \beta_2 \) and \( \tilde{b}_3 > \beta_3 \), then \( y_1 \) becomes extinct, (a) if \( \tilde{b}_1 < \beta_1 + \frac{c_1(b_3-\beta_3)}{f_1} \), then \( x \) becomes extinct and \( y_2 \) is persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_2(b_3-\beta_3)}{f_2} \) a.s. (b) if \( \tilde{b}_1 > \beta_1 + \frac{c_2(b_3-\beta_3)}{f_2} \), then both \( x \) and \( y_2 \) are persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_1-\beta_1}{a} - \frac{c_2(b_3-\beta_3)}{a f_2}, \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_1(b_3-\beta_3)}{f_1} \) a.s.

(vii) If \( \tilde{b}_1 > \beta_1, \tilde{b}_2 > \beta_2 \) and \( \tilde{b}_3 < \beta_3 \), then \( y_2 \) becomes extinct, (c) if \( \tilde{b}_1 < \beta_1 + \frac{c_1(\tilde{b}_2-\beta_2)}{f_1} \), then \( x \) becomes extinct and \( y_1 \) is persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(\tilde{b}_2-\beta_2)}{f_1} \) a.s. (d) if \( \tilde{b}_1 > \beta_1 + \frac{c_1(b_2-\beta_2)}{f_1} \), then both \( x \) and \( y_1 \) are persistent in the mean, i.e. \( \lim_{t \to +\infty} t^{-1} \int_0^t x(s) ds = \frac{b_1-\beta_1}{a} - \frac{c_1(b_2-\beta_2)}{a f_1}, \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(\tilde{b}_2-\beta_2)}{f_1} \) a.s.

(viii) If \( \tilde{b}_1 > \beta_1, \tilde{b}_2 > \beta_2 \) and \( \tilde{b}_3 > \beta_3 \), (e) if \( \tilde{b}_1 < \beta_1 + \frac{c_1(\tilde{b}_2-\beta_2)}{f_1} + \frac{c_2(\tilde{b}_1-\beta_3)}{f_2} \), then \( x \) becomes extinct and \( y_1 \) is persistent in the mean: \( \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(b_i+1-\beta_{i+1})}{f_i} \) a.s., \( i = 1, 2 \); (f) if \( \tilde{b}_1 > \beta_1 + \frac{c_1(b_2-\beta_2)}{f_1} + \frac{c_2(b_3-\beta_3)}{f_2} \), then all the populations are persistent in the mean: \( \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{b_1-\beta_1}{a} - \frac{c_1(b_2-\beta_2)}{a f_1} - \frac{c_2(b_3-\beta_3)}{a f_2}, \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_1-\beta_1}{f_1}, \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{b_1-\beta_1}{f_2} \) a.s.

**Proof:** (i). Taking advantage of Itô’s formula to model (4) gives

\[
\begin{align*}
\text{d} \ln x(t) &= [r_1 + (r_{10} - r_1)e^{-\sigma_1 t} - ax(t) - \frac{c_1 y_1(t)}{h_1 + x(t)} - \frac{c_2 y_2(t)}{h_2 + x(t)} - \frac{1}{2} \sigma_1^2(t)] \text{d}t \\
&+ \sigma_1(t) \text{d}W_1(t) - \int_\mathbb{V} \left[ \lambda_1(u) - \ln(1 + \lambda_1(u)) \right] \eta(du) \text{d}t \\
&+ \int_\mathbb{V} \ln(1 + \lambda_1(u)) \tilde{N}(dt, du)
\end{align*}
\]
Integrating both sides from 0 to $t$, one can see that

\[
\ln x(t) - \ln x(0) = \int_0^t b_1(s)ds - \beta_1 t - \frac{r_{10} - r_1}{\alpha_1} (e^{-\alpha_1 t} - 1) - a \int_0^t x(s)ds
\]

\[
- c_1 \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds - c_2 \int_0^t \frac{y_2(s)}{h_2 + x(s)} ds
\]

\[
+ \int_0^t \sigma_1(s)dW_1(s) + k_1(t),
\]

\[
\ln y_1(t) - \ln y_1(0) = \int_0^t b_2(s)ds - \beta_2 t - \frac{r_{20} - r_2}{\alpha_2} (e^{-\alpha_2 t} - 1) + k_2(t)
\]

\[
- f_1 \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds + \int_0^t \sigma_2(s)dW_2(s),
\]

\[
\ln y_2(t) - \ln y_2(0) = \int_0^t b_3(s)ds - \beta_3 t - \frac{r_{30} - r_3}{\alpha_3} (e^{-\alpha_3 t} - 1) + k_3(t)
\]

\[
- f_2 \int_0^t \frac{y_2(s)}{h_2 + x(s)} ds + \int_0^t \sigma_3(s)dW_3(s).
\]

It follows from (22) that, for sufficiently large $t$,

\[
t^{-1} \ln \frac{x(t)}{x(0)} \leq \tilde{b}_1 - \beta_1 + \varepsilon + t^{-1} \int_0^t \sigma_1(s)dW_1(s) - \frac{r_{10} - r_1}{\alpha_1} (e^{-\alpha_1 t} - 1) + \frac{k_1(t)}{t}. \tag{25}
\]

Note that $\lim_{t \to +\infty} t^{-1} \int_0^t \sigma_1(s)dW_1(s) = 0$, $\lim_{t \to +\infty} t^{-1} k_1(t) = 0$ and $\tilde{b}_1 - \beta_1 + \varepsilon < 0$, then we have $\lim_{t \to +\infty} x(t) = 0$ a.s. In the same way, if $\tilde{b}_2 < \beta_2$, it follows from (23) that $\lim_{t \to +\infty} y_1(t) = 0$ a.s.; if $\tilde{b}_3 < \beta_3$, it follows from (24) that $\lim_{t \to +\infty} y_2(t) = 0$ a.s.

(ii). Since $\tilde{b}_1 < \beta_1$, $\tilde{b}_2 < \beta_2$, (i) implies $\lim_{t \to +\infty} x(t) = 0$, $\lim_{t \to +\infty} y_1(t) = 0$ a.s. Then for sufficiently large $t$,

\[
\ln y_2(t) \leq (\tilde{b}_3 - \beta_3 + 2\varepsilon)t - \frac{f_2}{h_2 + \varepsilon} \int_0^t y_2(s)ds + \int_0^t \sigma_3(s)dW_3(s) + k_3(t), \tag{26}
\]
\[
\ln y_2(t) \geq (\bar{b}_3 - \beta_3 - 2\varepsilon)t - \frac{f_2}{h_2 - \varepsilon} \int_0^t y_2(s)ds + \int_0^t \sigma_3(s)dW_3(s) + k_3(t). 
\] (27)

Making use of Lemma 2.1 to (26) and (27), we can obtain that

\[
\frac{(h_2 - \varepsilon)(\bar{b}_3 - \beta_3 - 2\varepsilon)}{f_2} \leq \langle y_2 \rangle_* \leq \frac{(h_2 + \varepsilon)(\bar{b}_3 - \beta_3 + 2\varepsilon)}{f_2} \quad \text{a.s.}
\]

According to the arbitrariness of \(\varepsilon\), we have \(\lim_{t \to +\infty} t^{-1} \int_0^t y_2(s)ds = \frac{h_2(\bar{b}_3 - \beta_3)}{f_2}\) a.s.

The proof of (iii) and (iv) is similar to (ii), hence is omitted.

(v). Since \(\bar{b}_2 < \beta_2, \bar{b}_3 < \beta_3\), (i) implies \(\lim_{t \to +\infty} y_1(t) = 0\) a.s., \(i = 1, 2\). Besides, \(\bar{b}_1 > \beta_1\), for sufficiently large \(t\), by (22), we obtain

\[
\ln x(t) \leq (\bar{b}_1 - \beta_1 + 2\varepsilon)t - a \int_0^t x(s)ds + \int_0^t \sigma_1(s)dW_1(s) + k_1(t),
\] (28)

\[
\ln x(t) \geq (\bar{b}_1 - \beta_1 - 2\varepsilon)t - a \int_0^t x(s)ds + \int_0^t \sigma_1(s)dW_1(s) + k_1(t). 
\] (29)

Making use of Lemma 2.1 to (28) and (29) results in

\[
\frac{\bar{b}_1 - \beta_1 + 2\varepsilon}{a} \leq \langle x \rangle_* \leq \frac{\bar{b}_1 - \beta_1 - 2\varepsilon}{a} \quad \text{a.s.}
\]

Making use of the arbitrariness of \(\varepsilon\) gives \(\lim_{t \to +\infty} t^{-1} \int_0^t x(s)ds = \frac{\bar{b}_1 - \beta_1}{a}\) a.s.

(vi) Since \(\bar{b}_2 < \beta_2\), by (i) we know \(\lim_{t \to +\infty} y_1(t) = 0\).

(a) Computing \(22 \times f_2 - (24) \times c_2\) yields

\[
f_2 t^{-1} \ln \frac{x(t)}{x(0)} = c_2^{-1} \ln \frac{y_2(t)}{y_2(0)} + f_2 t^{-1} \int_0^t b_1(s)ds - c_2 t^{-1} \int_0^t b_3(s)ds
\]

\[\quad - \frac{r_{10} - r_1}{\alpha_1 t}(e^{-\alpha_1 t} - 1)f_2 + \frac{r_{30} - r_3}{\alpha_3 t}(e^{-\alpha_3 t} - 1)c_2 \]

\[\quad - af_2 t^{-1} \int_0^t x(s)ds - c_1 f_2 t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)}ds
\]

\[\quad + f_2 t^{-1} \int_0^t \sigma_1(s)dW_1(s) - c_2 t^{-1} \int_0^t \sigma_3(s)dW_3(s)
\]

\[\quad + f_2 t^{-1} k_1(t) - c_2 t^{-1} k_3(t) - \beta_1 f_2 + \beta_3 c_2. \] (30)

On the basis of Lemma 2.4, for arbitrary \(\varepsilon > 0\), there is a constant \(T > 0\) such that \(c_2 t^{-1} \ln \frac{y_2(t)}{y_2(0)} \leq \varepsilon/5\), for \(t \geq T\). For this reason,

\[
f_2 t^{-1} \ln x(0) \leq \varepsilon/5,
\]

\[
f_2 t^{-1} k_1(t) - c_2 t^{-1} k_3(t) \leq \varepsilon/5,
\]

\[
\frac{r_{30} - r_3}{\alpha_3 t}(e^{-\alpha_3 t} - 1)c_2 - \frac{r_{10} - r_1}{\alpha_1 t}(e^{-\alpha_1 t} - 1)f_2 \leq \varepsilon/5,
\]
for \( t \geq T \). Substituting the above inequalities into (30) yields

\[
f_{2} t^{-1} \ln x(t) \leq (1 + c_{2} + f_{2}) \epsilon + f_{2} (\bar{b}_{1} - \beta_{1}) - c_{2} (\bar{b}_{3} - \beta_{3}),
\]

for all \( t \geq T \) almost surely. Let \( \epsilon \) be sufficiently small such that \( 0 < \epsilon < \frac{c_{2}(\bar{b}_{3} - \beta_{3}) - f_{2}(\bar{b}_{1} - \beta_{1})}{1 + c_{2} + f_{2}} \), thus \( \lim_{t \to +\infty} x(t) = 0 \) a.s. Then similar to the proof of (ii), we can prove

\[
\lim_{t \to +\infty} t^{-1} \int_{0}^{t} y_{2}(s)ds = \frac{h_{2}(\bar{b}_{3} - \beta_{3})}{f_{2}} \quad \text{a.s.}
\]

(b) By (24), we have

\[
t^{-1} \ln \frac{y_{2}(t)}{y_{2}(0)} = t^{-1} \int_{0}^{t} b_{3} ds - \beta_{3} - \frac{r_{30} - r_{3}}{\alpha_{3} t} (e^{-\alpha_{3} t} - 1) + t^{-1} k_{3}(t)
\]

\[
- f_{2} t^{-1} \int_{0}^{t} \frac{y_{2}(s)}{h_{2} + x(s)} ds + t^{-1} \int_{0}^{t} \sigma_{3}(s)dW_{3}(s).
\]

We then deduce from Lemmas 2.2, 2.4 and \( \lim_{t \to +\infty} t^{-1} \int_{0}^{t} \sigma_{3}(s)dW_{3}(s) = 0 \) that

\[
\lim_{t \to +\infty} t^{-1} \int_{0}^{t} \frac{y_{2}(s)}{h_{2} + x(s)} ds = \frac{\bar{b}_{3} - \beta_{3}}{f_{2}} \quad \text{a.s.}
\]

As a result, for any \( \epsilon > 0 \), we can find out \( T > 0 \) such that, for \( t \geq T \),

\[
- \frac{c_{2}(\bar{b}_{3} - \beta_{3})}{f_{2}} - \epsilon \leq t^{-1} \ln x(0) - \frac{r_{10} - r_{1}}{\alpha_{1}} (e^{-\alpha_{1} t} - 1) - c_{2} \int_{0}^{t} \frac{y_{2}(s)}{h_{2} + x(s)} ds
\]

\[
\leq - \frac{c_{2}(\bar{b}_{3} - \beta_{3})}{f_{2}} + \epsilon.
\]

Substituting (34) into (22), one can derive that

\[
t^{-1} \ln x(t) \geq \bar{b}_{1} - \beta_{1} - \frac{c_{2}(\bar{b}_{3} - \beta_{3})}{f_{2}} - 2 \epsilon - at^{-1} \int_{0}^{t} x(s)ds
\]

\[
+ t^{-1} \int_{0}^{t} \sigma_{1}(s)dW_{1}(s) + t^{-1} k_{1}(t),
\]
for sufficiently large $t$, where $\epsilon > 0$ obeys $\frac{1}{2}(\bar{b}_1 - \beta_1 - \frac{c_2(\bar{b}_3 - \beta_3)}{f_2}) > \epsilon > 0$. Then by Lemma 2.1,

$$\frac{\bar{b}_1 - \beta_1 - 2\epsilon}{a} - \frac{c_2(\bar{b}_3 - \beta_3)}{af_2} \leq (x)_* \leq (x)^* \leq \frac{\bar{b}_1 - \beta_1 + 2\epsilon}{a} - \frac{c_2(\bar{b}_3 - \beta_3)}{af_2}.$$ 

Then the arbitrariness of $\epsilon$ means

$$\lim_{t \to +\infty} t^{-1} \int_0^t x(s)ds = \frac{\bar{b}_1 - \beta_1}{a} - \frac{c_2(\bar{b}_3 - \beta_3)}{af_2} \text{ a.s.}$$

The proof of (vii) is analogous to that of (vi) and hence is left out.

(viii) (e) Multiplying (22), (23) and (24) by $f_1 f_2$, $-c_1 f_2$ and $-c_2 f_1$, respectively, and then adding them, one gets that for sufficiently large $t$,

$$f_1 f_2 t^{-1} \ln \frac{x(t)}{x(0)} = c_1 f_2 t^{-1} \ln \frac{y_1(t)}{y_1(0)} + c_2 f_1 t^{-1} \ln \frac{y_2(t)}{y_2(0)} + f_1 f_2 t^{-1} \int_0^t b_1(s)ds$$

$$- c_1 f_2 t^{-1} \int_0^t b_2(s)ds - c_2 f_1 t^{-1} \int_0^t b_3(s)ds - \beta_1 f_1 f_2$$

$$+ \beta_2 c_1 f_2 + \beta_3 c_2 f_1 - \frac{r_{10} - r_1}{\alpha_1 t} (e^{-\alpha_1 t} - 1)f_1 f_2$$

$$+ \frac{r_{20} - r_2}{\alpha_2 t} (e^{-\alpha_2 t} - 1)c_1 f_2 + \frac{r_{30} - r_3}{\alpha_3 t} (e^{-\alpha_3 t} - 1)c_2 f_1$$

$$- af_1 f_2 t^{-1} \int_0^t x(s)ds + f_1 f_2 t^{-1} \int_0^t \sigma_1(s)dW_1(s)$$

$$- c_1 f_2 t^{-1} \int_0^t \sigma_2(s)dW_2(s) - c_2 f_1 t^{-1} \int_0^t \sigma_3(s)dW_3(s)$$

$$+ f_1 f_2 t^{-1} k_1(t) - c_1 f_2 t^{-1} k_2(t) - c_2 f_1 t^{-1} k_3(t).$$

By virtue of Lemma 2.4, we can observe that for arbitrary $\epsilon > 0$, $c_1 f_2 t^{-1} \ln \frac{y_1(t)}{y_1(0)} + c_2 f_1 t^{-1} \ln \frac{y_2(t)}{y_2(0)} \leq \epsilon/5$, and

$$f_1 f_2 t^{-1} \ln x(0) \leq \epsilon/5,$$

$$f_1 f_2 t^{-1} k_1(t) - c_1 f_2 t^{-1} k_2(t) - c_2 f_1 t^{-1} k_3(t) \leq \epsilon/5,$$

$$\frac{r_{20} - r_2}{\alpha_2 t} (e^{-\alpha_2 t} - 1)c_1 f_2 + \frac{r_{30} - r_3}{\alpha_3 t} (e^{-\alpha_3 t} - 1)c_2 f_1 - \frac{r_{10} - r_1}{\alpha_1 t} (e^{-\alpha_1 t} - 1)f_1 f_2 \leq \epsilon/5,$$

$$f_1 f_2 t^{-1} \int_0^t \sigma_1(s)dW_1(s) - c_1 f_2 \int_0^t \sigma_2(s)dW_2(s) - c_2 f_1 t^{-1} \int_0^t \sigma_3(s)dW_3(s) \leq \epsilon/5,$$
\[ f_1 f_2 t^{-1} \int_0^t b_1(s)ds - c_1 f_2 t^{-1} \int_0^t b_2(s)ds - c_2 f_1 t^{-1} \int_0^t b_3(s)ds \]
\[ \leq f_1 f_2 \tilde{b}_1 - c_1 f_2 \tilde{b}_2 - c_2 f_1 \tilde{b}_3 + (f_1 f_2 + c_1 f_2 + c_2 f_1)\varepsilon. \]

As a result, for \( t \geq T \),
\[ f_1 f_2 t^{-1} \ln x(t) \leq f_1 f_2 (\tilde{b}_1 - \beta_1) - c_1 f_2 (\tilde{b}_2 - \beta_2) - c_2 f_1 (\tilde{b}_3 - \beta_3) + (1 + f_1 f_2 + c_1 f_2 + c_2 f_1)\varepsilon, \]

where \( \varepsilon \) satisfies \( \frac{c_1 f_2 (\tilde{b}_2 - \beta_2) + c_2 f_1 (\tilde{b}_3 - \beta_3) - f_1 f_2 (\tilde{b}_1 - \beta_1)}{1 + f_1 f_2 + c_1 f_2 + c_2 f_1} > \varepsilon > 0 \), thus \( \lim_{t \to +\infty} x(t) = 0 \) a.s. The proof of
\[ \lim_{t \to +\infty} t^{-1} \int_0^t y_1(s)ds = \frac{h_1 (\tilde{b}_2 - \beta_2)}{f_1}, \quad \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s)ds = \frac{h_2 (\tilde{b}_3 - \beta_3)}{f_2} \]
a.s.
is similar to (iv) and hence is omitted.

(f) Similar to the proof of (b) in (vi), we can get
\[ \lim_{t \to +\infty} t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)}ds = \frac{\tilde{b}_2 - \beta_2}{f_1}, \quad \lim_{t \to +\infty} t^{-1} \int_0^t \frac{y_2(s)}{h_2 + x(s)}ds = \frac{\tilde{b}_3 - \beta_3}{f_2} \]
a.s.

Dividing both sides of (22) by \( t \), we can get
\[ t^{-1} \ln x(t) \geq (\tilde{b}_1 - \beta_1) - \frac{c_1 (\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{f_2} - 2\varepsilon \]
\[ - at^{-1} \int_0^t x(s)ds + t^{-1} \int_0^t \sigma_1(s)dW_1(s) + t^{-1}k_1(t), \]
\[ t^{-1} \ln x(t) \leq (\tilde{b}_1 - \beta_1) - \frac{c_1 (\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{f_2} + 2\varepsilon \]
\[ - at^{-1} \int_0^t x(s)ds + t^{-1} \int_0^t \sigma_1(s)dW_1(s) + t^{-1}k_1(t). \]

Choose \( 0 < \varepsilon < \frac{1}{2}(\tilde{b}_1 - \beta_1) - \frac{c_1 (\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{f_2} \); on the basis of Lemma 2.1,
\[ \frac{\tilde{b}_1 - \beta_1 - 2\varepsilon}{a} - \frac{c_1 (\tilde{b}_2 - \beta_2)}{af_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{af_2} \leq \langle x \rangle_* \leq \langle x \rangle^* \]
\[ \leq \frac{\tilde{b}_1 - \beta_1 + 2\varepsilon}{a} - \frac{c_1 (\tilde{b}_2 - \beta_2)}{af_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{af_2}. \]

Using the arbitrariness of \( \varepsilon \), one can observe that
\[ \lim_{t \to +\infty} t^{-1} \int_0^t x(s)ds = \frac{\tilde{b}_1 - \beta_1}{a} - \frac{c_1 (\tilde{b}_2 - \beta_2)}{af_1} - \frac{c_2 (\tilde{b}_3 - \beta_3)}{af_2} \]
a.s.

This completes the proof.
3. Discussions and numerical simulations

Now we test the functions of the mean-reverting Ornstein–Uhlenbeck process on the persistence and extinction of Model (4). We note that the speed of reversion $\alpha_i$ and the intensity of the perturbation $\xi^2_i$ are two key parameters in the Ornstein–Uhlenbeck process. Theorem 2.1 shows that the persistence and extinction of system (4) are entirely dominated by the signs of $\tilde{b}_i - \beta_i$, $\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1}$, $\tilde{b}_1 - \beta_1 - \frac{c_2(\tilde{b}_3 - \beta_3)}{f_2}$ and $\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2) - c_2(\tilde{b}_3 - \beta_3)}{f_1}$. Obviously,

$$\frac{\partial(\tilde{b}_1 - \beta_1)}{\partial \alpha_i} > 0, \quad \frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1})}{\partial \alpha_1} > 0, \quad \frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1})}{\partial \alpha_2} < 0, \quad \frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1})}{\partial \alpha_3} > 0,$$

$$\frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2(\tilde{b}_3 - \beta_3)}{f_2})}{\partial \alpha_1} > 0, \quad \frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2(\tilde{b}_3 - \beta_3)}{f_2})}{\partial \alpha_2} < 0, \quad \frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1} - \frac{c_2(\tilde{b}_3 - \beta_3)}{f_2})}{\partial \alpha_3} > 0.$$

Hence, as $\alpha_i$ (respectively, $\xi^2_i$) increases, species $i$ tends to be persistent (respectively, extinct), $i = 1, 2, 3$. Moreover, due to the fact that $\frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1})}{\partial \alpha_2} < 0$ (respectively, $\frac{\partial(\tilde{b}_1 - \beta_1 - \frac{c_1(\tilde{b}_2 - \beta_2)}{f_1})}{\partial \xi^2_i} > 0$), so sufficiently large $\alpha_2$ (respectively, $\xi^2_i$) could make $x$ extinct (respectively, persistent) if $\tilde{b}_1 > \beta_1$ and $\tilde{b}_2 > \beta_2$. Similarly, sufficiently large $\alpha_3$ (respectively, $\xi^2_i$) could make $x$ extinct (respectively, persistent) if $\tilde{b}_1 > \beta_1$ and $\tilde{b}_3 > \beta_3$.

Now we use the Euler scheme offered in [29] to prove our theoretical results numerically (here we only provide the functions of $\alpha_i$ since the functions of $\xi^2_i$ can be proffered...
analogously). Consider the following model:

\[
\begin{align*}
\frac{dx(t)}{dt} &= x(t^-)(0.6 - 0.2e^{-\alpha_1 t} - 0.4x(t)) - \frac{0.4y_1(t)}{1 + x(t)} - \frac{0.3y_2(t)}{1 + x(t)} dt \\
&+ \frac{\xi_1}{\sqrt{2\alpha_1}} \sqrt{1 - e^{-2\alpha_1 t}x(t^-)} dW_1(t) + \int_Y \lambda_1(u)x(t^-)\tilde{N}(dt, du), \\
\frac{dy_1(t)}{dt} &= y_1(t^-)(0.35 - 0.15e^{-\alpha_2 t} - \frac{0.4y_1(t)}{1 + x(t)}) dt \\
&+ \frac{\xi_2}{\sqrt{2\alpha_2}} \sqrt{1 - e^{-2\alpha_2 y_1(t^-)}y_2(t^-)} dW_2(t) + \int_Y \lambda_2(u)y_1(t^-)\tilde{N}(dt, du), \\
\frac{dy_2(t)}{dt} &= y_2(t^-)(0.15 - 0.05e^{-\alpha_3 t} - \frac{0.3y_2(t)}{1 + x(t)}) dt \\
&+ \frac{\xi_3}{\sqrt{2\alpha_3}} \sqrt{1 - e^{-2\alpha_3 y_2(t^-)}y_2(t^-)} dW_3(t) + \int_Y \lambda_3(u)y_2(t^-)\tilde{N}(dt, du),
\end{align*}
\]

where \(\xi_1 = 0.2, \xi_2 = 0.13, \xi_3 = 0.1, Y = (0, +\infty), \eta(Y) = 1\), and we suppose the initial data are \(x(0) = 0.5, y_1(0) = 0.3\) and \(y_2(0) = 0.1\).

- Choose \(\alpha_1 = 0.0172, \alpha_2 = 0.0124, \alpha_3 = 0.0171, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034\). Then \(\tilde{b}_1 = 0.0186, \tilde{b}_2 = 0.0093, \tilde{b}_3 = 0.0038, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005\). Thus by (i) in Theorem 2.1, all the species become extinct. Figure 1 confirms these.

- Choose \(\alpha_1 = 0.0172, \alpha_2 = 0.0124, \alpha_3 = 0.455, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034\). Then \(\tilde{b}_1 = 0.0186, \tilde{b}_2 = 0.0093, \tilde{b}_3 = 0.1445, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005\). It, therefore, follows from (ii) in Theorem 2.1 that both \(x\) and \(y_1\) become extinct and \(\lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_2(\tilde{b}_3 - \beta_3)}{\tilde{f}_2} = 0.465 > 0\). Figure 2 confirms these.

- Choose \(\alpha_1 = 0.0172, \alpha_2 = 0.4225, \alpha_3 = 0.0171, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034\). Then \(\tilde{b}_1 = 0.0186, \tilde{b}_2 = 0.34, \tilde{b}_3 = 0.0038, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005\). It, therefore, follows from (iii) in Theorem 2.1 that both \(x\) and \(y_2\) become extinct and \(\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(\tilde{b}_3 - \beta_3)}{\tilde{f}_1} = 0.82 > 0\). Figure 3 confirms these.

- Choose \(\alpha_1 = 0.0172, \alpha_2 = 0.4225, \alpha_3 = 0.455, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034\). Then \(\tilde{b}_1 = 0.0186, \tilde{b}_2 = 0.34, \tilde{b}_3 = 0.1445, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005\). It, therefore, follows from (iv) in Theorem 2.1 that \(x\) becomes extinct and \(\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1(\tilde{b}_3 - \beta_3)}{\tilde{f}_1} = 0.82 > 0, \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{h_2(\tilde{b}_3 - \beta_3)}{\tilde{f}_2} = 0.465 > 0\). Figure 4 confirms these.

- Choose \(\alpha_1 = 0.625, \alpha_2 = 0.0124, \alpha_3 = 0.0171, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034\). Then \(\tilde{b}_1 = 0.584, \tilde{b}_2 = 0.0093, \tilde{b}_3 = 0.0038, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005\). Thus by (v) in Theorem 2.1, both \(y_1\) and \(y_2\) become extinct, and \(\lim_{t \to +\infty} t^{-1} \int_0^t x(s) ds = \frac{\tilde{b}_1 - \tilde{b}_1}{\tilde{a}} = 1.41 > 0\). Figure 5 confirms these.

Comparing Figure 1 with Figure 2, we can see that with the rise of \(\alpha_3\), \(y_2\) tends to be persistent. Similarly, comparing Figure 1 with Figure 3 (respectively, Figure 1 with Figure 5), we can see that with the rise of \(\alpha_2\) (respectively, \(\alpha_1\), \(y_1\) (respectively, \(x\)) tends to be persistent.
Figure 1. All the species become extinct almost surely.

Figure 2. \( x \) and \( y_1 \) become extinct, and the species \( y_2 \) is persistent in the mean almost surely.

- Choose \( \alpha_1 = 0.0228, \alpha_2 = 0.0124, \alpha_3 = 0.75, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034 \). Then \( \overline{b}_1 = 0.1614, \overline{b}_2 = 0.0093, \overline{b}_3 = 0.1467, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005, \beta_1 + \frac{c_2(\overline{b}_3 - \beta_3)}{f_2} = 0.1617 \). Thus by (a) of (vi) in Theorem 2.1, both \( x \) and \( y_1 \) become extinct, and \( \lim_{t \to +\infty} t^{-1} \int_0^t y_2(s)ds = \frac{h_2(\overline{b}_3 - \beta_3)}{f_5} = 0.4722 > 0 \). Figure 6 confirms these.

- Choose \( \alpha_1 = 0.0228, \alpha_2 = 0.0124, \alpha_3 = 0.0262, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034 \). Then \( \overline{b}_1 = 0.1614, \overline{b}_2 = 0.0093, \overline{b}_3 = 0.0546, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005, \beta_1 + \frac{c_2(\overline{b}_3 - \beta_3)}{f_2} = 0.0696 \). Thus by (b) of (vi) in Theorem 2.1, \( y_1 \) becomes
Figure 3. $x$ and $y_2$ become extinct, and the species $y_1$ is persistent in the mean almost surely.

Figure 4. $y_1$ and $y_2$ are persistent in the mean, and $x$ becomes extinct almost surely.

extinct, and $\lim_{t \to +\infty} t^{-1} \int_0^t x(s)ds = \frac{\tilde{b}_1 - \tilde{\beta}_1}{a} - \frac{c_2(\tilde{b}_3 - \tilde{\beta}_3)}{a \tilde{f}_2} = 0.2296 > 0$, $\lim_{t \to +\infty} t^{-1} \int_0^t y_2(s)ds = \frac{\tilde{b}_3 - \tilde{\beta}_3}{\tilde{f}_2} = 0.1653 > 0$. Figure 7 confirms these.

Comparing Figure 6 with Figure 7, we can see that with the rise of $\alpha_3$, the prey population tends to become extinct.

- Choose $\alpha_1 = 0.04$, $\alpha_2 = 0.55$, $\alpha_3 = 0.0171$, $\lambda_1(u) = 0.2136$, $\lambda_2(u) = 0.1631$, $\lambda_3(u) = 0.1034$. Then $\tilde{b}_1 = 0.35$, $\tilde{b}_2 = 0.3423$, $\tilde{b}_3 = 0.0038$, $\beta_1 = 0.02$, $\beta_2 = 0.012$, $\beta_3 = 0.005$, $\beta_1 + \frac{c_1(\tilde{b}_2 - \tilde{b}_3)}{\tilde{f}_1} = 0.3503$. Thus by (c) of (vii) in Theorem 2.1, both $x$ and $y_2$ become
Figure 5. $y_1$ and $y_2$ become extinct, and $x$ is persistent in the mean almost surely.

Figure 6. $x$ and $y_1$ become extinct, and $y_2$ is persistent in the mean almost surely.

extinct, and $\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) \, ds = \frac{h_1(\bar{b}_2 - \beta_2)}{f_1} = 0.8258 > 0$. Figure 8 confirms these.

- Choose $\alpha_1 = 0.04, \alpha_2 = 0.0306, \alpha_3 = 0.0171, \lambda_1(u) = 0.2136, \lambda_2(u) = 0.1631, \lambda_3(u) = 0.1034$. Then $\bar{b}_1 = 0.35, \bar{b}_2 = 0.2119, \bar{b}_3 = 0.0038, \beta_1 = 0.02, \beta_2 = 0.012, \beta_3 = 0.005, \beta_1 + \frac{c_1(\bar{b}_2 - \beta_2)}{f_1} = 0.2199$. Thus by (d) of (vii) in Theorem 2.1, $y_2$ becomes extinct, and $\lim_{t \to +\infty} t^{-1} \int_0^t x(s) \, ds = \frac{\bar{b}_2 - \beta_2}{f_1} - c_1(\bar{b}_2 - \beta_2) = 0.3252 > 0, \lim_{t \to +\infty} t^{-1} \int_0^t \frac{y_1(s)}{n_1 + x(s)} \, ds = \frac{\bar{b}_2 - \beta_2}{f_1} = 0.4998 > 0$. Figure 9 confirms these.
Figure 7. $x$ and $y_2$ are persistent in the mean, and $y_1$ becomes extinct almost surely.

Figure 8. $x$ and $y_2$ become extinct, and $y_1$ is persistent in the mean almost surely.

Comparing Figure 8 with Figure 9, we can see that with the rise of $\alpha_2$, the prey population tends to become extinct.

- Choose $\alpha_1 = 0.09$, $\alpha_2 = 0.8$, $\alpha_3 = 0.55$, $\lambda_1(u) = 0.2136$, $\lambda_2(u) = 0.1631$, $\lambda_3(u) = 0.1034$. Then $\bar{b}_1 = 0.4889$, $\bar{b}_2 = 0.3447$, $\bar{b}_3 = 0.1455$, $\beta_1 = 0.02$, $\beta_2 = 0.012$, $\beta_3 = 0.005$, $\beta_1 + \frac{c_1(\bar{b}_2-\beta_2)}{f_1} + \frac{c_2(\bar{b}_3-\beta_3)}{f_2} = 0.4932$. Thus by (e) of (viii) in Theorem 2.1, $x$ becomes extinct, and $\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) \, ds = \frac{h_1(\bar{b}_2-\beta_2)}{f_1} = 0.8318 > 0$, $\lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) \, ds = \frac{h_2(\bar{b}_3-\beta_3)}{f_2} = 0.4682 > 0$. Figure 10 confirms these.

- Choose $\alpha_1 = 0.09$, $\alpha_2 = 0.0729$, $\alpha_3 = 0.089$, $\lambda_1(u) = 0.2136$, $\lambda_2(u) = 0.1631$, $\lambda_3(u) = 0.1034$. Then $\bar{b}_1 = 0.4889$, $\bar{b}_2 = 0.292$, $\bar{b}_3 = 0.1219$, $\beta_1 = 0.02$, $\beta_2 = 0.012$, $\beta_3 =$
Figure 9. $x$ and $y_1$ are persistent in the mean, and $y_2$ becomes extinct almost surely.

Figure 10. $y_1$ and $y_2$ are persistent in the mean, and $x$ becomes extinct almost surely.

$0.005, \beta_1 + \frac{c_1(\bar{b}_3 - \beta_3)}{f_1} + \frac{c_2(\bar{b}_3 - \beta_3)}{f_2} = 0.417$. It therefore follows from (f) of (viii) in Theorem 2.1 that $\lim_{t \to +\infty} t^{-1} \int_0^t x(s) ds = \frac{\bar{b}_1 - \beta_1}{a} - \frac{c_1(\bar{b}_2 - \beta_2)}{a_1} - \frac{c_2(\bar{b}_3 - \beta_3)}{a_2} = 0.18 > 0$, $\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{\bar{b}_2 - \beta_2}{f_1} = 0.7 > 0$, $\lim_{t \to +\infty} t^{-1} \int_0^t y_2(s) ds = \frac{\bar{b}_3 - \beta_3}{f_2} = 0.3897 > 0$. Figure 11 confirms these.

In the following, we discuss the effect of Lévy jumps on model (4).
Figure 11. All the species are persistent in the mean almost surely.

Figure 12. $x$ and $y_1$ become extinct, and $y_2$ is persistent in the mean almost surely.

- In Figure 12, we choose $\lambda_2(u) = 1.358$ (i.e. $\beta_2 = 0.5$) and assume that all other parameters are the same as those in Figure 4. It follows from (ii) in Theorem 2.1 that both $x$ and $y_1$ become extinct and

$$\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_2}{f_2} \left( \bar{b}_3 - \beta_3 \right) = 0.465.$$ 

Comparing Figure 12 with Figure 4, we can see that with the rise of $\lambda_2(u), y_1$ becomes extinct.

- In Figure 13, we choose $\lambda_3(u) = 0.7722$ (i.e. $\beta_3 = 0.2$) and assume that all other parameters are the same as those in Figure 4. It follows from (iii) in Theorem 2.1 that both $x$ and $y_2$ become extinct and

$$\lim_{t \to +\infty} t^{-1} \int_0^t y_1(s) ds = \frac{h_1}{f_1} \left( \bar{b}_2 - \beta_2 \right) = 0.82.$$
Figure 13. $x$ and $y_2$ become extinct, and $y_1$ is persistent in the mean almost surely.

Figure 14. All the species are become extinct almost surely.

Comparing Figure 13 with Figure 4, we can see that with the rise of $\lambda_3(u)$, $y_2$ becomes extinct.

- In Figure 14, we choose $\lambda_1(u) = 1.527$ (i.e. $\beta_1 = 0.6$) and assume that all other parameters are the same as those in Figure 5. It follows from (i) in Theorem 2.1 that all the species become extinct.

Comparing Figure 14 with Figure 5, we can see that with the rise of $\lambda_1(u)$, $x$ becomes extinct.
By analysing Figures 12–14, we can see that Lévy noise can change the properties of the population systems, and it can force the population to become extinct when $\lambda_i(u)$ is sufficiently large.

4. Concluding remarks

In this paper, we take advantage of a mean-reverting Ornstein–Uhlenbeck process to describe the random perturbations in the environment and formulate a stochastic three-species predator–prey system with Lévy jumps, which might be more appropriate to depict reality than model (2). We obtain sharp sufficient conditions for persistence in the mean and extinction for each species of model (4) and uncovered some significant functions of Ornstein–Uhlenbeck process: sufficiently large $\alpha_i$ (the speed of reversion) could make species $i$ persistent, $i = 1, 2, 3$; moreover, in some situations, sufficiently large $\alpha_2$ and $\alpha_3$ could make $x$ become extinct.

Some interesting questions deserve further investigation. The present article probed into the white noises and Lévy noise, one could examine other random noises such as the telephone noise (see [21]), etc. Besides, one could consider and investigate model (4) in higher dimensions. All these considerations are left for future study.

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