Simulation of pulse wave propagation using one-dimensional models of hemodynamics

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Abstract. The models of hemodynamics, corresponding to the inviscid, Newtonian, and non-Newtonian models, are compared. The models are constructed by the averaging of the hydrodynamic system on the vessel cross-section. For the inviscid case, the analytical solution of the problem for pulse propagation is obtained. As the result of the comparison, the deviations of the solutions for non-Newtonian models from the Newtonian and inviscid cases are demonstrated.

1. Introduction
The one-dimensional (1D) models of blood flow, obtained by the averaging of the incompressible Navier–Stokes system, are widely used as the predictive tool in clinical applications [1, 2]. As it is demonstrated in [3], the results obtained using 1D models, are consistent with the averaged results, obtained after the application of 3D models.

The blood is considered as a heterogeneous suspension of different cells in plasma. From the physical viewpoint, it is a viscous incompressible fluid with a non-Newtonian behavior [4]. But in most works on 1D models, the non-Newtonian property is ignored, and blood is modeled as a Newtonian fluid [2, 5, 6]. Moreover, in some papers (e.g. see [7, 8, 9, 10]), the viscosity is ignored and it is considered as an inviscid fluid. The non-Newtonian models are used in a relatively small number of works [11, 12]. In the presented paper, new non-Newtonian 1D models of blood flow are considered and compared with the inviscid and Newtonian models on the problem of pulse wave propagation. For the inviscid model, the analytical solution is obtained by the perturbation method. For the viscous models, the numerical solution is obtained by the Lax–Wendroff scheme.

2. Mathematical models
The hyperbolic system, obtained after the averaging of incompressibility condition and motion equation, is written as [5, 6]:

\[ \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \alpha \frac{Q^2}{A} \right) + \frac{A}{\rho} \frac{\partial P}{\partial z} = f(A, Q), \]

where \( t \) is a time, \( z \) is a cylindrical coordinate, \( A(t, z) \) is the cross-sectional area, \( Q(t, z) \) is the flow rate, \( P(t, z) \) is the pressure, \( \alpha \) is a momentum correction coefficient, \( \rho \) is a constant...
density and $f$ is a frictional term. The system is closed by the so-called equation-of-state, which represents the dependence of $P$ on $A$. For the arteries, the following equation is used [7]:

$$P - P_{ext} = P_d + \frac{\beta}{A_d}(\sqrt{A} - \sqrt{A_d}),$$

where $P_{ext}$ is the external pressure, $A_d$ and $P_d$ are the diastolic cross-sectional area and pressure, $\beta = \frac{4}{3} \sqrt{\pi} Eh$, where $E$ is the Young’s modulus and $h$ is the vessel wall thickness. So the following system is obtained:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \alpha \frac{Q^2}{A} + \frac{\beta}{2} \sqrt{A} \frac{\partial A}{\partial z} \right) = f(A, Q), \quad (1)$$

where $\gamma = \beta/(2A_d \rho)$.

The particular model of blood is determined by the expression of $f$. The case of $f \equiv 0$ corresponds to the inviscid fluid. For the viscid models it is defined as [12]:

$$f(A, Q) = \frac{2}{3} \gamma \sqrt{A} T_{rz}|r=R,$$

where $R$ is a vessel radius, $T_{rz}$ is the component of the tangential stress tensor, defined by the following dependence on the component $D_{rz}$ of the strain rate tensor $\mathbf{D}$: $T_{rz} = 2\mu(I_2)D_{rz}$, where $I_2$ is a second invariant of $\mathbf{D}$.

The value of $\alpha$ is defined by the expression of the dimensionless velocity profile $s(y)$, where $y \in [0, 1]$ is a dimensionless radius [5, 13]. Inviscid model corresponds to $\alpha = 1$, Newtonian model — to $\alpha = 4/3$. For the non-Newtonian models, the following representation of $s(y)$ is used:

$$s(y) = \frac{d + 2}{d} \left( 1 - y^d \right).$$

The flattening of the profile, typical for the blood, can be obtained by the variation of $d$.

In the dimensionless variables, system (1) in the inviscid case is rewritten as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \alpha \frac{Q^2}{A} + \frac{\beta}{2} \sqrt{A} \frac{\partial A}{\partial z} \right) = 0, \quad (2)$$

where $\chi = \frac{2}{3} \gamma \sqrt{A} C / U^2 C$, where $A_C$ and $U_C$ are the characteristic cross-sectional area and velocity, respectively. In the viscid case, the system is rewritten as:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \alpha \frac{Q^2}{A} + \chi A^{3/2} \right) = f(A, Q). \quad (3)$$

The expressions of $f$ for the models, used in the paper, are presented in Table 1.

### 3. Initial-boundary value problem

Let the semi-infinite interval $z \in [0, +\infty)$ is considered. The initial values of $A$ and $Q$ are written as:

$$A(0, z) = \psi(z), \quad Q(0, z) = \varphi(z), \quad (4)$$

where $|\psi(z)|, |\varphi(z)| < +\infty$ at $z \to +\infty$. At the left boundary $z = 0$ the pulse is initiated by the following way:

$$Q(t, 0) = m(t). \quad (5)$$
Table 1. Expressions for the dimensionless frictional term $f(A, Q)$.

| Model           | $f(A, Q)$                                                                 |
|-----------------|---------------------------------------------------------------------------|
| Carreau         | $f(A, Q) = -\varepsilon \frac{Q}{A} - \delta \left(1 + \frac{Q^2}{A^2}\right)^{n-1} \frac{Q}{A}$ |
| Karreau–Yasuda  | $f(A, Q) = -\varepsilon \frac{Q}{A} - \delta \left(1 + \frac{Q^2}{A^2}\right)^{n-1} \frac{Q}{A}$ |
| Cross           | $f(A, Q) = -\varepsilon \frac{Q}{A} - \frac{1}{1+\varepsilon} \frac{Q}{A}$ |
| Powell–Eyring   | $f(A, Q) = -\varepsilon \frac{Q}{A} - \delta \tanh\left(\frac{|Q|}{A^2}\right) \text{sign}(Q)\sqrt{A}$ |

3.1. Perturbation method for the inviscid system

Let the functions, which represent the conditions (4) and (5), are written as:

$$
\psi(z) = A^0 + \varepsilon \psi_1(z) + \varepsilon^2 \psi_2(z) + \ldots, \quad \varphi(z) = \varepsilon \varphi_1(z) + \varepsilon^2 \varphi_2(z) + \ldots,
$$

$$
m(t) = \varepsilon m_1(t) + \varepsilon^2 m_2(t) + \ldots,
$$

where $\varepsilon$ is a small dimensionless parameter, $A^0 > 0$ is a given constant. Presented formulas can be considered as the perturbation of the rest state $(A^0, 0)$, corresponding to the zero initial mean velocity (so-called 'dead man' state [14]). According to these representations, the solution of problem (2), (4), (5) is presented in the following form:

$$
A(t, z) = A_0 + \varepsilon A_1(t, z) + \varepsilon^2 A_2(t, z) + \ldots, \quad Q(t, z) = \varepsilon Q_1(t, z) + \varepsilon^2 Q_2(t, z) + \ldots
$$

The equations for $A_i$ and $Q_i$ are obtained after the substitution of (6) into (2). It is easy to demonstrate, that $A_0 = A^0$. For $i \geq 1$ the following equations take place:

$$
\frac{\partial A_i}{\partial t} + \frac{\partial Q_i}{\partial z} = 0, \quad \frac{\partial Q_i}{\partial t} + \frac{3\chi}{2} \sqrt{A_0} \frac{\partial A_i}{\partial z} = F_i(t, z),
$$

where $F_i \equiv 0$ and for $i \geq 2, F_i$ are dependent on $A_{i-1}, Q_{i-1}, \ldots, A_1, Q_1$. For example, at $i = 2$ it is presented as:

$$
F_2 = -\left(\frac{3\chi}{8\sqrt{A_0}} \frac{\partial A_2}{\partial z} + \frac{1}{A_0} \frac{\partial Q_2}{\partial z}\right).
$$
Figure 1. Plots of $Q(t, z)$ at selected time moments for the problem at semi-infinite interval at $\varepsilon \approx 0.0552$: black line — inviscid model; blue line — Newtonian model

From system (7) the following equation for $Q_i$ is obtained:

$$\frac{\partial^2 Q_i}{\partial t^2} = c^2 \frac{\partial^2 Q_i}{\partial z^2} + \nu_i(t, z),$$

where $c^2 = 3\sqrt{A_0}/2$ and $\nu_i(t, z) = \frac{\partial F_i}{\partial t}$. For the wave equation (8) the following conditions are stated:

$$Q_i(0, z) = \varphi_i(z), \quad \frac{\partial Q_i}{\partial t}(0, z) = \kappa_i(z), \quad Q_i(t, 0) = m_i(t),$$

where $\kappa_i(z) = -c^2 \frac{\partial \varphi_i(z)}{\partial z} + F_i(0, z)$. The expression for $A_i(t, z)$ is obtained from the first equation of system (7). The solution of problem (8)–(9) is written as:

$$Q_i(t, z) = \overline{Q}_i(t, z) + \frac{1}{2c} \overline{\varphi}_i(t, z),$$

where

$$\overline{Q}_i(t, z) = \begin{cases} \frac{1}{2}(\varphi_i(z + ct) + \varphi_i(z - ct)) + \frac{1}{2c} \int_{z-ct}^{z+ct} \kappa_i(\xi)d\xi, & t < \frac{z}{c}, \\ \frac{1}{2}(\varphi_i(z + ct) - \varphi_i(ct - z)) + \frac{1}{2c} \int_{z+ct}^{z-ct} \kappa_i(\xi)d\xi + m_i(t - \frac{z}{c}), & t > \frac{z}{c}, \end{cases}$$

$$\overline{\varphi}_i(t, z) = \begin{cases} \int_{0}^{t} \int_{0}^{z+ct} \nu_i(\tau, \xi)d\xi d\tau, & t < \frac{z}{c}, \\ \int_{t - \frac{z}{c}}^{t} \int_{z+ct}^{z} \nu_i(\tau, \xi)d\xi d\tau + \int_{t - \frac{z}{c}}^{t} \int_{z+ct}^{z} \nu_i(\tau, \xi)d\xi d\tau, & t > \frac{z}{c}. \end{cases}$$
Figure 2. Plots of $Q(t,z)$ at selected time moments for the problem at semi-infinite interval at $\varepsilon \approx 0.1087$ (corresponding to $d = 6$): 1 — inviscid model; 2 — Carreau model; 3 — Carreau–Yasuda model; 4 — Cross model; 5 — Powell–Eyring model

Figure 3. Plots of $I$ (a) and $Q_{rel}$ (b) at the case of the problem for the semi-infinite interval: 1 — Newtonian model; 2 — Carreau model; 3 — Carreau–Yasuda model; 4 — Cross model; 5 — Powell–Eyring model

3.2. Numerical schemes
The problem for the viscid models (3),(4),(5) is solved numerically. The semi-infinite interval is reduced to the bounded interval $[0,L]$. System (3) is discretized by the Lax–Wendroff finite-difference scheme. For the computation of both variables at $z = 0$ condition (5) is combined with the compatibility condition which is discretized with second-order finite differences. At $z = L$ the compatibility condition and non-reflecting condition are stated.
4. Numerical results
In the presented paper, only terms up and including the second order in (6) are considered. Let the following functions for the initial and boundary conditions are used:

\[ \varphi_i = \psi_i = 0, \forall i, m_i = 0 \]

at \( i \geq 2 \), and \( m_1(t) \), which simulates the pulse, is presented as:

\[ m_1(t) = te^{-(t-1)^2} \]

For the numerical calculations, the spatial interval \([0, 10]\) and the same time interval are used. The values of dimensionless parameters are estimated by the values of \( A_C, L_C, U_C, \) and \( T_C \) for the carotid artery, taken from [3]. For the viscous models, parameters of rheological relations are taken from [4, 15, 16]. The value of \( \varepsilon \) for the Newtonian model is estimated as 0.0552, for the non-Newtonian model it is equal to 0.0679 for \( d = 2 \), to 0.0815 for \( d = 4 \) and to 0.1087 for \( d = 6 \). For the comparison with the inviscid model, such values are used in the analytical solution.

For the comparison of solutions, the following criterions are used:

\[ I = \| U(t,z) - U_v(t,z) \|, \quad Q_{rel} = \frac{\max_z |Q(t^*, z)|}{\max_{(t,z)} |Q(t,z)|} \]

where \( U = (A,Q)^T \) is the solution for the inviscid model, \( U_v \) is the solution for the viscous model, and \( t^* \) is a selected time moment. Criterion \( I \) estimates the deviation of the solution from the inviscid case, and \( Q_{rel} \) estimates the magnitude of damping of the solution, corresponding to the presence of the viscosity.

For the considered conditions, the maximum of \( |Q| \) is realized at \( t \approx 1.3642 \). The value of \( t^* \) is taken as 8.7484 — at this value, the propagating pulse has its maximum near the right boundary \( z = 10 \). In Fig. 1, the plots of solutions for the inviscid and Newtonian models are presented. In Fig. 2, the same plots are presented for the inviscid and non-Newtonian models.

In Fig. 3, the plots of \( I \) and \( Q_{rel} \) at different values of \( d \) are presented. As it can be seen, the values of \( I \) for the non-Newtonian models are larger, than for the Newtonian case. The values of this criterion are increased with the increase of \( d \). The values of \( Q_{rel} \) for the non-Newtonian models are lower than for the Newtonian case, so the inclusion of non-Newtonian effects in the model leads to the more intense damping of the solution, than in the Newtonian case. With the increase of \( d \) the values of \( Q_{rel} \) are decreased, so the effect of damping becomes significant with the increasing of the flattening of the velocity profile.

5. Conclusion
In the paper, the 1D models of hemodynamics for the inviscid and viscous (Newtonian and non-Newtonian) models are compared on the solution of problem for the single pulse propagation. For the inviscid model, the analytical solution is obtained. As the result of the comparison of solutions, obtained by different models, it is demonstrated, that for the non-Newtonian models the strongest damping in comparison with the Newtonian case takes place. Also, it is demonstrated, that the flattening of the velocity profile leads to the larger deviations of solutions in non-Newtonian case from the inviscid and Newtonian case.

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