A SCALAR CURVATURE FORMULA FOR THE NONCOMMUTATIVE 3-TORUS

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Abstract. We compute the scalar curvature of a curved noncommutative 3-torus. To perturb the flat metric, the standard volume form on the noncommutative 3-torus is conformally perturbed and the corresponding perturbed Laplacian is analyzed. Using Connes’ pseudodifferential calculus for the noncommutative 3-torus, we explicitly compute the first three terms of the small time heat kernel expansion for the perturbed Laplacian. The third term of the expansion gives a local formula for the scalar curvature. Finally, we show that in the classical limit when the deformation parameters vanish, our formula coincides with the formula for the commutative case.

1. Introduction

From the very beginning of noncommutative geometry in [4], noncommutative tori have proved to be an invaluable model to understand and test many aspects of noncommutative geometry. Curvature, one of the most important geometric invariants, is among those aspects. Defining a suitable curvature concept in the noncommutative setting is an important problem at the heart of noncommutative geometry. More precisely, we are interested in curvature invariants of noncommutative Riemannian manifolds. In contrast, it should be noted that curvature of connections and the corresponding Chern-Weil theory in the noncommutative setting has already been defined in [4].

In their pioneering work [7], Connes and Tretkoff (cf. also [3] for a preliminary version) took a first step in this direction and proved a Gauss-Bonnet theorem for a curved noncommutative 2-torus equipped with a conformally deformed metric. In fact, they gave a spectral definition of curvature and computed its trace. This result was extended in [14] to noncommutative tori equipped with an arbitrary translation invariant complex structure and conformal perturbation of its metric. The full computation of curvature in these examples was done
independently and simultaneously in [6] and [15]. This line of work has been followed up and extended in different directions in many papers [2, 16] [8, 9, 10, 11, 12, 13, 21, 17, 19].

The approach used in the aforementioned papers is based on the heat kernel techniques and Connes’ pseudodifferential calculus on noncommutative tori. In this paper using a similar technique we will give a formula for the scalar curvature of a curved noncommutative 3-torus. This would be the first odd dimensional case that has been studied among the noncommutative tori. In [22] a general pattern for the scalar curvature of even dimensional noncommutative tori is found which in some sense repeats the two dimensional case [6, 15]. A similar question in the odd dimensional case needs a close study of the three dimensional case first. The concept of curvature in the noncommutative setting has also been studied through an algebraic approach and a noncommutative analogue of the Levi-Civita connection in [11, 23].

This paper is organized as follows. In Section 2, we recall some facts about the heat kernel expansion in the commutative case. In Section 3, we recall basic facts about higher dimensional noncommutative tori and their flat geometry. Then we perturb the standard volume form on this space conformally and analyse the corresponding perturbed Laplacian. In Section 4, we recall the pseudodifferential calculus of [5] for $\mathbb{T}_3^g$. In Section 5, we review the derivation of the small time heat kernel expansion for the perturbed Laplacian, using the pseudodifferential calculus. Then we perform the computation of the scalar curvature for $\mathbb{T}_3^g$, and find explicit formulas for the local functions that describe the curvature in terms of the modular automorphism of the conformally perturbed volume form and derivatives of the Weyl factor.

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2. Heat Kernel Expansion and Scalar Curvature

To motivate the definition of scalar curvature in our noncommutative setting, let us first recall Gilkey’s theorem on asymptotic expansion of heat kernels for the special case of Laplacians. Let $(M, g)$ be a closed, oriented Riemannian manifold of dimension $n$, endowed with the metric $g$ and let $\triangle$ denote the scalar Laplacian acting on $\mathcal{C}^\infty(M)$, the algebra of smooth functions on $M$. If $C$ is a contour going counterclockwise around the nonnegative part of the real axis without touching it, then using the Cauchy integral formula

$$e^{-t\triangle} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(\triangle - \lambda)^{-1}d\lambda$$
and approximating the operator \((\triangle - \lambda)^{-1}\) by a pseudodifferential operator \(R(\lambda)\) one can find an asymptotic expansion for the smooth kernel \(K(t, x, y)\) of \(e^{-t\triangle}\) on the diagonal \([18]\).

More precisely, using the formula for the symbol of the product of two pseudo differential operators one can inductively find an asymptotic expansion \(\sum_{j=0}^{\infty} r_j(x, \xi, \lambda)\) for the symbol of \(R(\lambda)\) such that \(r_j(x, \xi, \lambda)\) is a symbol of order \(-2 - j\) depending on the complex parameter \(\lambda\), where \(j \in \mathbb{N} \cup \{0\}\), \(x \in M\) and \(\xi \in \mathbb{R}^n\). Then one can see that for \(t > 0\), the operator \(e^{-t\triangle}\) has a smooth kernel \(K(t, x, y)\) and as \(t \to 0^+\), there exist an asymptotic expansion

\[
K(t, x, x) \sim t^{-n/2} \sum_{m=0}^{\infty} a_{2m}(x)t^m,
\]

where

\[(1) \quad a_{2m}(x) = \frac{1}{2\pi i} \int \int e^{-\lambda r_{2m}(x, \xi, \lambda)}d\lambda d\xi.
\]

It follows that we have an asymptotic expansion for the heat trace

\[
\text{Tr}_{L^2} e^{-t\triangle} \sim t^{-n/2} \sum_{m=0}^{\infty} a_{2m} t^m \quad t \to 0,
\]

where

\[
a_{2m} = \int_M a_{2m}(x) \, d\text{vol}(x).
\]

Moreover, it is known that \(a_2(x)\) is a constant multiple of the scalar curvature of \(M\) at the point \(x\), so that \(a_2\) is (a multiple of) the total scalar curvature \([18]\). In what follows we will explain how we exploit these facts to define the scalar curvature of the curved noncommutative 3-torus by analogy.

3. Curved Noncommutative 3-tori

Let \(\theta = (\theta_{k\ell}) \in M_3(\mathbb{R})\) be a skew symmetric matrix. The universal unital \(C^*\)-algebra generated by three unitary elements \(u_1, u_2, u_3\) subject to the relations

\[
u_k u_\ell = e^{2\pi i \theta_{k\ell}} u_\ell u_k, \quad k, \ell = 1, 2, 3
\]

is called the noncommutative 3-torus and is denoted by \(A^3_\theta\). It has a positive faithful normalized trace denoted by \(\tau\). This \(C^*\)-algebra is indeed a noncommutative deformation of \(C(\mathbb{T}^3)\), the algebra of continuous functions on the 3-torus.
For \( r = (r_1, r_2, r_3) \in \mathbb{Z}^3 \) we set
\[
u^r = \exp(\pi i (r_1 \theta_{12} r_2 + r_1 \theta_{13} r_3 + r_2 \theta_{23} r_3)) u_1^{r_1} u_2^{r_2} u_3^{r_3}
\]
There is an action \( \alpha \) of the 3-torus \( T^3 \) on \( A^3_\theta \) which is defined by
\[
\alpha_z(u^r) = z^r u^r \] where \( z = (z_1, z_2, z_3) \in T^3 \) and \( z^r = z_1^{r_1} z_2^{r_2} z_3^{r_3} \). Let \( T^3_\theta \) be the set of all elements \( a \in A^3_\theta \) for which the map
\[
\alpha(a) : T^3 \longrightarrow A^3_\theta, \quad z \mapsto \alpha_z(a),
\]
is a smooth map. This set is a unital dense subalgebra of \( A^3_\theta \) and it is called the algebra of smooth elements of \( A^3_\theta \). In fact, it is the analogue of \( C^\infty(T^3) \), the algebra of smooth functions on the 3-torus. It is known that
\[
T^3_\theta = \left\{ \sum_{r \in \mathbb{Z}^3} a_r u^r : (a_r) \text{ is a rapidly decreasing function on } \mathbb{Z}^3 \right\}.
\]
By rapidly decreasing we mean for all \( k \in \mathbb{N} \),
\[
\sup (1 + |r|^2)^k |a_r|^2 < \infty.
\]
The trace \( \tau \) on \( A^3_\theta \) plays the role of integration in the noncommutative setting and extracts the constant term of the elements of \( T^3_\theta \), i.e.
\[
\tau \left( \sum_{r \in \mathbb{Z}^3} a_r u^r \right) = a_0.
\]
The algebra \( T^3_\theta \) also possesses three derivations, uniquely defined by the relations
\[
\delta_j \left( \sum_{r \in \mathbb{Z}^3} a_r u^r \right) = \sum_{r \in \mathbb{Z}^3} r_j a_r u^r, \quad j = 1, 2, 3.
\]
One can see that for \( a \in T^3_\theta \),
\[
(\delta_j(a))^* = -\delta_j(a^*).
\]
These derivations are noncommutative counterparts of the partial derivatives on \( C^\infty(T^3) \) and they satisfy the integration by parts relation i.e.
\[
\tau(a \delta_j(b)) = -\tau(\delta_j(a) b), \quad a, b \in T^3_\theta.
\]
Our next goal is to introduce \( \Delta \), the Laplace operator on \( T^3_\theta \). Then perturbing the metric in a conformal class we will define the perturbed Laplace operator \( \Delta_\varphi \). Let \( k \in T^3_\theta \) be a positive element representing
the conformal class of the metric on $A^3_θ$. We shall show that $\triangle ϕ$ is anti-unitarily equivalent to the operator $P$ where

$$P = k \triangle k^3 + \sum_{j=1}^{3} k^3 \delta_j(k^{-2}) \delta_j k^3$$

and we will use the latter to define the scalar curvature of $A^3_θ$ with a conformally perturbed metric. In fact, by analogy with (1), we define the scalar curvature of $A^3_θ$ with the perturbed metric to be

$$(4)\quad \frac{k^6}{2\pi i} \int_{\mathbb{R}^3} \int_C e^{-\lambda} b_2(\xi, \lambda) d\lambda d\xi,$$

where $C$ is a contour going counterclockwise around the nonnegative part of the real axis and $b_2(\xi, \lambda)$ is the third term in the asymptotic expansion of the symbol of $(P - \lambda)^{-1}$. We will find the first three terms of this asymptotic expansion by Connes’ pseudodifferential calculus [4] and finally we will compute (4).

Let $\langle \cdot, \cdot \rangle_τ$ be the inner product on $A^3_θ$ defined by

$$\langle a, b \rangle_τ = \tau(b^* a), \quad a, b \in A^3_θ.$$

We denote the completion of $A^3_θ$ with respect to this inner product by $H_τ$. It is indeed the representation space in the GNS construction associated to $τ$. We define an unbounded operator

$$d : \mathbb{T}^3_θ \subset H_τ \longrightarrow H_τ \times H_τ \times H_τ$$

by $d(a) = (\delta_1(a), \delta_2(a), \delta_3(a))$ for $a \in \mathbb{T}^3_θ$. This operator is defined using an analogy with the classic case. Indeed, for $\mathbb{T}^3$, the classic 3-torus, the de Rham operator is an operator from $C^∞(\mathbb{T}^3)$ to $Ω^1(\mathbb{T}^3) = C^∞(\mathbb{T}^3) \otimes \mathbb{C}^3$, where $C^∞(\mathbb{T}^3)$ is the space of smooth functions and $Ω^1(\mathbb{T}^3)$ is the space of 1-forms on $\mathbb{T}^3$.

Let $x, y, z, b \in \mathbb{T}^3_θ$ and $\langle \cdot, \cdot \rangle_0$ be the inner product of $H_τ \times H_τ \times H_τ$. We have, for the adjoint $d^*$,

$$\langle d^*(x, y, z), b \rangle_τ = \langle (x, y, z), d(b) \rangle_0 = \langle (x, y, z), (\delta_1(b), \delta_2(b), \delta_3(b)) \rangle_0 =$$

$$-\tau(\delta_1(b^*) x) - \tau(\delta_2(b^*) y) - \tau(\delta_3(b^*) z).$$

Using integration by parts (3), we obtain

$$\langle d^*(x, y, z), b \rangle_τ = \tau(b^* \delta_1(x)) + \tau(b^* \delta_2(y)) + \tau(b^* \delta_3(z)) =$$

$$\tau(b^* (\delta_1(x) + \delta_2(y) + \delta_3(z))) = \langle \delta_1(x) + \delta_2(y) + \delta_3(z), b \rangle_τ.$$

Therefore, $d^*(x, y, z) = \delta_1(x) + \delta_2(y) + \delta_3(z)$. Now we define the Laplace operator

$$\triangle : \mathbb{T}^3_θ \subset H_τ \longrightarrow H_τ,$$
by $\Delta = d^*d$. Note that
\[
\triangle(a) = d^*d(a) = d^* (\delta_1(a), \delta_2(a), \delta_3(a)) = \delta_1^2(a) + \delta_2^2(a) + \delta_3^2(a),
\]
so $\triangle = \delta_1^2 + \delta_2^2 + \delta_3^2$.

Next we will conformally perturb the metric on $T^3_\theta$. Let $h \in T^3_\theta$ be a self adjoint element, we define a positive linear functional $\varphi$ on $A^3_\theta$ by
\[
\varphi(a) = \tau(ae^{-3h}), \quad a \in A^3_\theta.
\]

Let $\Delta$ be the modular operator for $\varphi$, i.e.
\[
\Delta(a) = e^{-3h}ae^{3h}, \quad a \in A^3_\theta,
\]
and $\{\sigma_t\}, t \in \mathbb{R}$ be a 1-parameter group of automorphisms of $A^3_\theta$ defined by
\[
\sigma_t(a) = \Delta^{-it}(a), \quad a \in A^3_\theta.
\]

Unlike $\tau$, $\varphi$ is not a trace. But it satisfies the KMS condition at $\beta = 1$ for $\{\sigma_t\}$. In other words,
\[
\varphi(ab) = \varphi(b\sigma_t(a)), \quad a, b \in A^3_\theta.
\]

Now we define an inner product on $A^3_\theta$ by
\[
\langle a, b \rangle _\varphi = \varphi(b^*a), \quad a, b \in A^3_\theta
\]
and we denote the Hilbert space completion of $A^3_\theta$ with this inner product by $H_\varphi$.

To define the Laplace operator on $H_\varphi$, again we mimic the classic case. Let $g$ be the flat metric on the 3-torus $T^3$ and
\[
\tilde{g} = e^{-2h}g.
\]

Clearly $\tilde{g}^{-1} = e^{2h}g^{-1}$ and $d\text{vol}_{\tilde{g}} = e^{-3h}d\text{vol}_g$. Therefore, for $\alpha, \beta \in \Omega^1(T^3)$ we have
\[
\langle \alpha, \beta \rangle _{\tilde{g}} = \int \tilde{g}^{-1}(\alpha_p, \beta_p) \ d\text{vol}_{\tilde{g}} = \int e^{2h}g^{-1}(\alpha_p, \beta_p)e^{-3h} d\text{vol}_g = \int g^{-1}(\alpha_p, \beta_p)e^{-h} d\text{vol}_g.
\]

We also define a positive linear functional $\psi$ on $A^3_\theta$ by
\[
\psi(a) = \tau(ae^{-h}), \quad a \in A^3_\theta,
\]
and an inner product on $A^3_\theta$ by
\[
\langle a, b \rangle _\psi = \psi(b^*a), \quad a, b \in A^3_\theta.
\]

We also denote the Hilbert space completion of $A^3_\theta$ with this inner product by $H_\psi$. Let
\[
d_\psi : T^3_\theta \subset H_\varphi \longrightarrow H_\psi \times H_\psi \times H_\psi
\]
be defined by $d_{\varphi}(a) = (\delta_1(a), \delta_2(a), \delta_3(a))$ for $a \in \mathbb{T}_0^3$.

Let $x, y, z, b \in \mathbb{T}_0^3$, $k = e^{h/2}$ and $\langle \cdot, \cdot \rangle_1$ be the inner product of $H_\psi \times H_\psi$. We have

$$\langle d_{\varphi}^*(x, y, z), b \rangle_\varphi = \langle (x, y, z), d_{\varphi}(b) \rangle_1 = \langle (x, y, z), (\delta_1(b), \delta_2(b), \delta_3(b)) \rangle_1 =$$

$$-\tau(\delta_1(b^*)xk^{-2}) - \tau(\delta_2(b^*)yk^{-2}) - \tau(\delta_3(b^*)zk^{-2}).$$

Using integration by parts (3), we obtain

$$\langle d_{\varphi}^*(x, y, z), b \rangle_\varphi = \tau(b^*\delta_1(xk^{-2})) + \tau(b^*\delta_2(yk^{-2})) + \tau(b^*\delta_3(zk^{-2})) =$$

$$\tau(b^*(\delta_1(xk^{-2}) + \delta_2(yk^{-2}) + \delta_3(zk^{-2}))) =$$

$$\langle (\delta_1(xk^{-2}) + \delta_2(yk^{-2}) + \delta_3(zk^{-2}))k^6, b \rangle_\varphi .$$

Therefore,

$$d_{\varphi}^*(x, y, z) = \delta_1(xk^{-2})k^6 + \delta_2(yk^{-2})k^6 + \delta_3(zk^{-2})k^6 .$$

Now we define the perturbed Laplace operator

$$\triangle_\varphi : \mathbb{T}_0^3 \subset H_\varphi \rightarrow H_\varphi,$$

by $\triangle_\varphi = d_{\varphi}^*d_{\varphi}$. Note that

$$\triangle_\varphi(a) = d_{\varphi}^*d_{\varphi}(a) = d_{\varphi}^*(\delta_1(a), \delta_2(a), \delta_3(a)) =$$

$$\delta_1(\delta_1(a)k^{-2})k^6 + \delta_2(\delta_2(a)k^{-2})k^6 + \delta_3(\delta_3(a)k^{-2})k^6 .$$

Since

$$\delta_j(\delta_j(a)k^{-2})k^6 = (\delta_j^2(a)k^{-2} + \delta_j(a)\delta_j(k^{-2}))k^6 = \delta_j^2(a)k^4 + \delta_j(a)\delta_j(k^{-2})k^6 ,$$

we have

$$\triangle_\varphi = \sum_{j=1}^{3} R_k^*\delta_j^2 + R(\delta_j(k^{-2})k^6)\delta_j ,$$

where for any element $x \in A_3^3$, by $R_x$ we mean the right multiplication operator by $x$.

Moreover, since

$$\langle R_k^*a, R_k^*b \rangle_\varphi = \langle ak^3, bk^3 \rangle_\varphi =$$

$$\varphi(k^3b^*ak^3) = \tau(k^3b^*ak^3k^{-6}) = \tau(b^*a) = \langle a, b \rangle_\tau ,$$

$R_k^*$ extends to a unitary operator $W : H_\tau \rightarrow H_\varphi$. Let

$$J : H_\tau \rightarrow H_\tau$$

be the anti unitary operator defined by $J(a) = a^*$. Then

$$WJ : H_\tau \rightarrow H_\varphi.$$
is an anti unitary operator and obviously \( \triangle_\varphi \) is anti unitarily equivalent to
\[
(5) \quad JW^* \triangle_\varphi WJ = JR_{k^{-3}}J J \triangle_\varphi J JR_{k^3}J.
\]
One can see that
\[
JR_{k^{-3}}J = k^{-3}, \quad JR_{k^3}J = k^3, \quad JR_{(\delta_j(k^{-2})k^0)}J = -k^6 \delta_j(k^{-2}).
\]
By \( k^{-3}, k^3 \) and \( -k^6 \delta_j(k^{-2}) \) we mean the left multiplication operator by these elements. Moreover, using (2), we see that \( J \) anticommutes with \( \delta_j \). So
\[
JR_{k^{-3}}J = k^{-3}, \quad JR_{k^3}J = k^3, \quad JR_{(\delta_j(k^{-2})k^0)}J = -k^6 \delta_j(k^{-2}).
\]
Using (5), we see that \( \triangle_\varphi \) is anti unitarily equivalent to
\[
k^{-3} \sum_{j=1}^3 k^4 \delta_j^2 + k^6 \delta_j(k^{-2}) \delta_j = \sum_{j=1}^3 k^4 \delta_j^2 + k^6 \delta_j(k^{-2}) \delta_j.
\]

4. Connes’ Pseudodifferential Calculus

In this section we will recall Connes’ pseudodifferential calculus that was introduced in [4]. We shall be primarily working in dimension three.

For \( n \in \mathbb{N} \cup \{0\} \), a differential operator on \( \mathbb{T}^3 \) of order \( n \) is a polynomial in \( \delta_1, \delta_2, \delta_3 \) of the form
\[
P(\delta_1, \delta_2, \delta_3) = \sum_{|j|\leq n} a_j \delta_1^{i_1} \delta_2^{i_2} \delta_3^{i_3}
\]
where \( j = (j_1, j_2, j_3) \in \mathbb{Z}^3_{\geq 0}, |j| = j_1 + j_2 + j_3 \) and \( a_j \in \mathbb{T}^3_0 \). Now we extend this definition to pseudodifferential operators.

**Definition 4.1.** A smooth function \( \rho : \mathbb{R}^3 \to \mathbb{T}^3_0 \) is called a symbol of order \( n \in \mathbb{Z} \) if for all nonnegative integers \( i_1, i_2, i_3, j_1, j_2, j_3 \) there exists a constant \( C \), such that
\[
\|\delta_1^{i_1} \delta_2^{i_2} \delta_3^{i_3} (\partial_1^{j_1} \partial_2^{j_2} \partial_3^{j_3} \rho(\xi))\| \leq C(1 + |\xi|)^{n - |j|},
\]
and if there exists a smooth function \( k : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \to \mathbb{T}^3_0 \) such that
\[
\lim_{\lambda \to \infty} \lambda^{-n} \rho(\lambda \xi_1, \lambda \xi_2, \lambda \xi_3) = k(\xi_1, \xi_2, \xi_3).
\]
In the last definition by \( \partial_1, \partial_2, \partial_3 \) we mean partial derivatives, i.e.
\[
\partial_1 = \partial/\partial \xi_1, \quad \partial_2 = \partial/\partial \xi_2, \quad \partial_3 = \partial/\partial \xi_3.
\]
The space of symbols of order \( n \) is denoted by \( S_n \). To any symbol \( \rho \in S_n \), an operator \( P_\rho \) on \( T^3_\theta \) is associated which is given by
\[
P_\rho(a) = (2\pi)^{-3} \iint e^{-iz.\xi} \rho(\xi) \alpha_z(a) dz d\xi, \quad a \in T^3_\theta,
\]
and is called a pseudodifferential operator.

**Definition 4.2.** Let \( \rho \) and \( \rho' \) be symbols of order \( k \). They are called equivalent if and only if \( \rho - \rho' \in S_n \) for all \( n \in \mathbb{Z} \). This equivalence relation is denoted by \( \rho \sim \rho' \).

The next proposition which plays a key role in our computations in this paper, shows that the space of pseudodifferential operators is an algebra. Given the pseudodifferential operators \( P \) and \( Q \), by the next proposition we can find the symbols of \( PQ \) and \( P^* \) up to the equivalence relation \( \sim \), where \( P^* \) is the adjoint of \( P \) with respect to the inner product \( \langle \cdot, \cdot \rangle_\tau \) on \( H_\tau \) (See [7]).

**Proposition 1.** Let \( \rho \) and \( \rho' \) be the symbols of the pseudodifferential operators \( P \) and \( Q \). Then \( PQ \) and \( P^* \) are pseudodifferential operators, and \( \sigma(PQ) \) and \( \sigma(P^*) \), symbols of \( PQ \) and \( P^* \) respectively, can be obtained by the following formulas
\[
\sigma(PQ) \sim \sum_{(\ell_1, \ell_2, \ell_3) \in (\mathbb{Z} \geq 0)^3} \frac{1}{\ell_1! \ell_2! \ell_3!} \partial^{\ell_1}_1 \partial^{\ell_2}_2 \partial^{\ell_3}_3 (\rho(\xi)) \delta^{\ell_1}_1 \delta^{\ell_2}_2 \delta^{\ell_3}_3 (\rho'(\xi)),
\]
\[
\sigma(P^*) \sim \sum_{(\ell_1, \ell_2, \ell_3) \in (\mathbb{Z} \geq 0)^3} \frac{1}{\ell_1! \ell_2! \ell_3!} \partial^{\ell_1}_1 \partial^{\ell_2}_2 \partial^{\ell_3}_3 (\rho(\xi))^*,
\]

**5. The Main Result**

In this section using Connes’ pseudodifferential calculus we will define the scalar curvature of \( A^3_\theta \) with a perturbed metric and we will compute it.

Let
\[
P = k \triangle + \sum_{j=1}^3 k^3 \delta_j (k^{-2}) \delta_j k^3.
\]

As we mentioned in Section 3, \( \triangle_\phi \) is antiunitarily equivalent to the operator \( P \) on \( H_\tau \). So to study the spectral geometry of \( A^3_\theta \) with a perturbed metric we work with the operator \( P \). Exploiting the formula in Proposition 1 and considering \( k, k^3, k^3 \delta_j (k^{-2}) \) as pseudodifferential
operators of order 0 with symbols $\sigma(k) = k$, $\sigma(k^3) = k^3$, $\sigma(k^3\delta_j(k^{-2})) = k^3\delta_j(k^{-2})$, plus the fact that the symbols of $\triangle$ and $\sum_{j=1}^{3}\delta_j$ are

$$\sigma(\triangle) = \sum_{i=1}^{3}\xi_i^2, \quad \sigma(\sum_{j=1}^{3}\delta_j) = \sum_{i=1}^{3}\xi_i$$

we can find the symbol of $P$. Indeed, we can show that

$$\sigma(P) = a_0(\xi) + a_1(\xi) + a_2(\xi),$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and

$$a_0(\xi) = \sum_{i=1}^{3}(k\delta_i^2(k^3) + k^3\delta_i(k^{-2})\delta_i(k^3)),$$

$$a_1(\xi) = \sum_{i=1}^{3}\xi_i(2k\delta_i(k^3) + k^3\delta_i(k^{-2})k^3)$$

$$= \sum_{i=1}^{3}\xi_i(2k\delta_i(k^3) + k^3\delta_i(k) - k\delta_i(k^3))$$

$$= \sum_{i=1}^{3}\xi_i(k\delta_i(k^3) + k^3\delta_i(k)),$$

$$a_2(\xi) = \sum_{i=1}^{3}k^4\xi_i^2.$$

Let $\lambda \in \mathbb{C}$. As mentioned in the introduction, to define the scalar curvature of $A^3_{\theta}$ with a conformally perturbed metric, we need to find an asymptotic expansion of the symbol of $(P - \lambda)^{-1}$. Indeed, we have to find an operator $R_{\lambda}$ such that

$$\sigma(R_{\lambda} \cdot (P - \lambda)) \sim \sigma(I)$$

where $I$ is the identity operator. Using the formula in Proposition 11 and following the steps in page 52 of [18], we can find a recursive formula for the terms of an asymptotic expansion of $(P - \lambda)^{-1}$. In fact, one can show that

$$\sigma(P - \lambda)^{-1} \sim \sum_{n=0}^{\infty} b_n(\xi, \lambda),$$
where \( b_n(\xi, \lambda) \) is a symbol of order \(-2 - n\) given by the following recursive formula:

\[
b_0(\xi, \lambda) = (k^4 \sum_{i=1}^{3} \xi_i^2 - \lambda)^{-1},
\]

\[
b_n(\xi, \lambda) = -\sum_{2+j+\ell_1+\ell_2+\ell_3-m=n} \frac{1}{\ell_1!\ell_2!\ell_3!} \partial_{\ell_1} \partial_{\ell_2} \partial_{\ell_3} (b_j) \delta_{\ell_1} \delta_{\ell_2} \delta_{\ell_3} (a_m) b_0,
\]

for \( n \geq 1 \).

Now we are able to define the scalar curvature of \( A^3_\theta \) with a conformally perturbed metric. Indeed, using the notations that we have introduced, (1) motivates us to define the scalar curvature of \( A^3_\theta \) with a conformally perturbed metric as follows:

**Definition 5.1.** Let \( C \) be a contour going counterclockwise around the nonnegative part of the real axis, and \( b_2(\xi, \lambda) \), for \( \lambda \in \mathbb{C} \), be the third term in the asymptotic expansion of the symbol of \((P - \lambda)^{-1}\). Then the scalar curvature of \((A^3_\theta, \varphi)\) is defined to be the element \( S \in A^3_\theta \) given by

\[
S = \frac{k^6}{2\pi i} \int_{\mathbb{R}^3} \int_C e^{-\lambda} b_2(\xi, \lambda) d\lambda d\xi.
\]

Let

\[
\alpha(\lambda) = \int_{\mathbb{R}^3} b_2(\xi, \lambda) d\xi.
\]

The function \( \alpha \) is homogeneous of degree \(-1/2\) with respect to \( \lambda \). We also define

\[
\beta(\lambda) = \lambda^{-1/2} \alpha(\lambda).
\]

The function \( \beta \) is homogeneous of degree \(-1\) with respect to \( \lambda \). For the square root function we consider the nonnegative part of the real axis as the branch cut. Then we have

\[
S = \frac{k^6}{2\pi i} \beta(-1) \int_C \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda.
\]

To compute the latter, we consider the contour \( C = C_1 + C_2 + C_3 \), where \( C_1 = re^{i\pi/4} \) for \( r \in (\infty, 1) \), \( C_2 = e^{i\theta} \) for \( \theta \in (\pi/4, 7\pi/4) \) and \( C_3 = re^{7i\pi/4} \) for \( r \in (1, \infty) \). One can see that

\[
\int_{C_1} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = (-1)^{7/8} c_{\pi} e^{i\pi/4} \sqrt{\pi} \text{Erfc}\left[(-1)^{1/8}\right],
\]

\[
\int_{C_2} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = \sqrt{\pi} \left(-\text{Erf}\left[(-1)^{1/8}\right] + \text{Erf}\left[(-1)^{7/8}\right]\right),
\]

\[
\int_{C_3} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = \sqrt{\pi} \left(-\text{Erf}\left[(-1)^{1/8}\right] + \text{Erf}\left[(-1)^{7/8}\right]\right).
\]
and
\[ \int_{C_3} e^{-\lambda} \frac{1}{\lambda^{1/2}} d\lambda = (\pi^{-1/2} e^{7i\pi/8} (1 + \text{Erf} (1^{7/8}))). \]

Therefore,
\[ \int_{C} e^{-\lambda} \frac{1}{\lambda^{1/2}} d\lambda = -2\sqrt{\pi} \]
and this implies that
\[ S = \frac{k^6}{2\pi i} \int_{\mathbb{R}^3} \int_{C} e^{-\lambda} b_2(\xi, \lambda) d\xi d\lambda = \frac{-k^6}{\sqrt{\pi}} \alpha(-1). \]

By this argument to find \( S \), it suffices to work with \( \lambda = -1 \) and compute
\[ \alpha(-1) = \int_{\mathbb{R}^3} b_2(\xi, -1) d\xi. \]

We devote the rest of the paper to the calculation of \( \alpha(-1) \).

6. The Computation of \( b_2(\xi, -1) \)

In this section we will use the recursive formula (6) to find \( b_2(\xi, -1) \).

In what follows, we set \( b_n = b_n(\xi, -1) \) for \( n \in \mathbb{N} \).

We know that
\[ b_0 = (k^4 \sum_{i=1}^{3} \xi_i^2 + 1)^{-1}. \]

Now using (3), we have
\[ b_1 = -b_0 a_0 b_0 - (\sum_{i=1}^{3} \partial_i(b_0) \delta_i(a_1)) b_0. \]

Computing the above formula and using the result in (3) we obtain \( b_2 \).

We have
\[ b_2 = -b_0 a_0 - b_1 a_1 \]
\[ -\partial_1(b_0) \delta_1(a_1) - \partial_2(b_0) \delta_2(a_1) - \partial_3(b_0) \delta_3(a_1) \]
\[ -\partial_1(b_1) \delta_1(a_2) - \partial_2(b_1) \delta_2(a_2) - \partial_3(b_1) \delta_3(a_2) \]
\[ -\partial_{12}(b_0) \delta_1(\delta_2(a_2)) - \partial_{13}(b_0) \delta_1(\delta_3(a_2)) - \partial_{23}(b_0) \delta_2(\delta_3(a_2)) \]
\[ -(1/2)\partial_{11}(b_0) \delta_1^2(a_2) - (1/2)\partial_{22}(b_0) \delta_2^2(a_2) - (1/2)\partial_{33}(b_0) \delta_3^2(a_2). \]

After doing computations we get a simplified formula for \( b_2 \) which has more than 800 terms. In the next section we will use that simplified formula for \( b_2 \).
7. Integrating $b_2(\xi, -1)$ over $\mathbb{R}^3$

In this section, first we will change the variables and then we will use a rearrangement lemma to integrate $b_2(\xi, -1)$ over $\mathbb{R}^3$.

To integrate $b_2(\xi, -1)$ with respect to $\xi = (\xi_1, \xi_2, \xi_3)$, we apply the spherical change of coordinates

$$\xi_1 = r \sin \Phi \cos \theta, \quad \xi_2 = r \sin \Phi \sin \theta, \quad \xi_3 = r \cos \Phi,$$

where $0 \leq \theta \leq 2\pi, 0 \leq \Phi \leq \pi$ and $0 \leq r \leq \infty$. Considering this change of coordinates and integrating with respect to $\theta$ and $\Phi$, one finds that

$$\int_{\mathbb{R}^3} b_2(\xi, -1) d\xi,$$

up to an overall factor of $4\pi/3$ is

$$\int_0^\infty B(r) dr$$

where

$$B(r) = -6r^2 b_0 k^2 \delta_j(k)^2 b_0 - 3r^2 b_0 k^3 \delta_j(k) (\delta_j(k)) b_0 + 12r^4 b_0^2 k^6 \delta_j(k)^2 b_0$$

$$+ 7r^4 b_0^3 k^7 \delta_j(k) (\delta_j(k)) b_0 - 8r^6 b_0^3 k^{10} \delta_j(k) b_0 - 4r^6 b_0^3 k^{11} \delta_j(k) b_0$$

$$- 6r^2 b_0 k \delta_j(k)^2 b_0 - 3r^2 b_0 k \delta_j(k) \delta_j(k) b_0 k^2 - 3r^2 b_0 k^2 \delta_j(k) \delta_j(k) b_0$$

$$+ 8r^4 b_0^2 k^4 \delta_j(k) b_0 k^2 + 3r^4 b_0^2 k^4 \delta_j(k) b_0 k^3 + 10r^4 b_0^2 k^5 \delta_j(k) b_0$$

$$+ 5r^4 b_0^2 k^5 \delta_j(k) b_0 k^2 + 5r^4 b_0^2 k^6 \delta_j(k) b_0$$

$$- 8r^6 b_0^3 k^8 \delta_j(k)^2 b_0 k^2 - 4r^6 b_0^3 k^8 \delta_j(k) b_0 k^3$$

$$- 8r^6 b_0^3 k^8 \delta_j(k) b_0 k^2 - 4r^6 b_0^3 k^8 \delta_j(k) b_0$$

$$- 4r^6 b_0^3 k^8 \delta_j(k) b_0 k^2 - 6r^6 b_0 k \delta_j(k) k \delta_j(k) b_0$$

$$+ 10r^4 b_0^2 k^4 \delta_j(k) k^2 \delta_j(k) b_0 + 12r^4 b_0^2 k^5 \delta_j(k) k \delta_j(k) b_0$$

$$- 8r^6 b_0^3 k^8 \delta_j(k) k^2 \delta_j(k) b_0 - 8r^6 b_0^3 k^8 \delta_j(k) k \delta_j(k) b_0$$

$$+ 5r^4 b_0 k \delta_j(k) b_0 k^5 \delta_j(k) b_0 - 2r^6 b_0 k \delta_j(k) b_0^2 k^9 \delta_j(k) b_0$$

$$+ 5r^4 b_0 k \delta_j(k) b_0 k^4 \delta_j(k) b_0 - 2r^6 b_0 k^2 \delta_j(k) b_0^2 k^8 \delta_j(k) b_0$$

$$+ 10r^4 b_0 k^3 \delta_j(k) b_0 k^3 \delta_j(k) b_0 + 6r^6 b_0 k^3 \delta_j(k) b_0 \delta_j(k) b_0 k^3$$

$$- 4r^6 b_0 k^3 \delta_j(k) b_0 k^7 \delta_j(k) b_0 + 8r^4 b_0^2 k^4 \delta_j(k) k \delta_j(k) b_0 k$$

$$- 14r^6 b_0^2 k^4 \delta_j(k) b_0 k \delta_j(k) b_0 + 4r^8 b_0^2 k^4 \delta_j(k) b_0^2 k^{10} \delta_j(k) b_0$$

$$- 16r^6 b_0^2 k^5 \delta_j(k) b_0 k^5 \delta_j(k) b_0 + 4r^8 b_0^2 k^5 \delta_j(k) b_0^2 k^5 \delta_j(k) b_0$$

$$- 16r^6 b_0^2 k^6 \delta_j(k) b_0 k^4 \delta_j(k) b_0 + 4r^8 b_0^2 k^6 \delta_j(k) b_0^2 k^8 \delta_j(k) b_0$$

$$- 18r^6 b_0^2 k^7 \delta_j(k) b_0 k^3 \delta_j(k) b_0 + 14r^8 b_0^2 k^7 \delta_j(k) b_0 \delta_j(k) b_0 k^3$$

$$- 8r^6 b_0^3 k^8 \delta_j(k) k \delta_j(k) b_0 + 4r^8 b_0^3 k^8 \delta_j(k) b_0 k^8 \delta_j(k) b_0$$

$$+ 8r^6 b_0^3 k^8 \delta_j(k) k \delta_j(k) b_0 + 8r^8 b_0^3 k^{10} \delta_j(k) b_0 k^4 \delta_j(k) b_0$$
\[ \begin{align*}
+8r^8b^3k^{11}\delta_j(k)b_kk^3\delta_j(k)b_0 + 8r^8b^3k^{11}\delta_j(k)b_0\delta_j(k)b_0k^3 \\
+3r^4b_0k\delta_j(k)b_0k^2\delta_j(k)b_0k^3 + 4r^4b_0k\delta_j(k)b_0k^3\delta_j(k)b_0k^2 \\
+4r^4b_0k\delta_j(k)b_0k^4\delta_j(k)b_0k^3 - 2r^6b_0k\delta_j(k)b^6\delta_j(k)b_0k^3 \\
-2r^6b_0k\delta_j(k)b^6k^7\delta_j(k)b_0k^2 - 2r^6b_0k^2\delta_j(k)b^6k^7\delta_j(k)b_0k \\
+3r^4b_0k^2\delta_j(k)b_0k^3\delta_j(k)b_0k^2 + 4r^4b_0^2k^2\delta_j(k)b_0k^3\delta_j(k)b_0k^2 \\
-2r^6b_0k^2\delta_j(k)b^6k^7\delta_j(k)b_0k^3 - 2r^6b_0k^2\delta_j(k)b^6k^7\delta_j(k)b_0k \\
+8r^4b_0^3k^3\delta_j(k)b_0k^3\delta_j(k)b_0k^2 + 8r^4b_0^3k^3\delta_j(k)b_0k^3\delta_j(k)b_0k^2 \\
-2r^6b_0k^3\delta_j(k)b^6k^7\delta_j(k)b_0k^3 - 2r^6b_0k^3\delta_j(k)b^6k^7\delta_j(k)b_0k \\
+4r^4b_0^3k^3\delta_j(k)b^6k^7\delta_j(k)b_0k^3 - 4r^4b_0^3k^3\delta_j(k)b^6k^7\delta_j(k)b_0k^2 \\
+4r^4b_0^3k^3\delta_j(k)b^6k^7\delta_j(k)b_0k^3 - 4r^4b_0^3k^3\delta_j(k)b^6k^7\delta_j(k)b_0k^2
\end{align*} \]

In the above sum \( b_0 = (k^4r^2 + 1)^{-1} \). One can see that for \( x \in \mathbb{T}_0^2 \), \( xk^n = k^2\Delta_{n/6}(x) \). Using this relation plus the fact that \( kb_0 = b_0k \) we can see that

\[ B(r) = -6r^2k^2b_0\delta_j(k)^2b_0 - 3r^2k^3b_0\delta_j(k)\delta_j(k)b_0 + 12r^4k^6b^2\delta_j(k)^2b_0 \]

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\[+7 r^4 k^7 b_0^3 \delta_j(\delta_j(k)) b_0 - 8 r^6 k^{10} b_0^3 \delta_j(\delta_j(k))^2 b_0 - 4 r^6 k^{11} b_0^3 \delta_j(\delta_j(k)) b_0
\]
\[-6 k^2 r^2 b_0 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 - 3 r^2 k^3 b_0 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0
\]
\[-3 r^2 k^3 b_0 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 + 8 r^4 k^6 b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0
\]
\[+3 r^4 k^7 b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 + 10 r^4 k^6 b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0
\]
\[+5 r^4 k^7 b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0 + 5 r^4 k^7 b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0
\]
\[-8 r^6 k^{10} b_0^3 \Delta^{1/3}(\delta_j(\delta_j(k)))^2 b_0 - 4 r^6 k^{11} b_0^3 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0
\]
\[-8 r^6 k^{10} b_0^3 \Delta^{1/6}(\delta_j(\delta_j(k)))^2 b_0 - 4 r^6 k^{11} b_0^3 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0
\]
\[-4 r^6 k^{11} b_0^3 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 - 6 r^2 k^2 b_0 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0
\]
\[+10 r^4 k^6 b_0^3 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0 + 12 r^4 k^6 b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0
\]
\[+8 r^4 k^6 b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0 - 14 r^4 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0
\]
\[-4 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 + 4 r^8 k^{14} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0\]
\[-16 r^6 k^{10} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0 + 8 r^8 k^{14} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0\]
\[-18 r^6 k^{10} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 - 14 r^4 k^{10} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0\]
\[-8 r^8 k^{10} b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k)))^2 b_0 + 8 r^8 k^{14} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0\]
\[+8 r^8 k^{14} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0 - 8 r^8 k^{14} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0\]
\[+8 r^8 k^{14} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0 + 8 r^8 k^{14} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0\]
\[+8 r^8 k^{14} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 + 8 r^8 k^{14} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[+3 r^4 k^6 b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 + 4 r^4 k^6 b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0\]
\[+4 r^4 k^6 b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0 - 2 r^6 k^{10} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0\]
\[-2 r^6 k^{10} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0 + 2 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[+3 r^4 k^6 b_0^2 \Delta^{2/3}(\delta_j(\delta_j(k))) b_0 + 4 r^4 k^6 b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0\]
\[+4 r^4 k^6 b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 + 2 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[-2 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 + 2 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[+8 r^4 k^{10} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0 + 8 r^8 k^{14} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0\]
\[-16 r^6 k^{10} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0 - 2 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[-4 r^6 k^{10} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 + 4 r^8 k^{14} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0\]
\[-2 r^6 k^{10} b_0^2 \Delta^{5/6}(\delta_j(\delta_j(k))) b_0 - 2 r^6 k^{10} b_0^2 \Delta^{3/2}(\delta_j(\delta_j(k))) b_0\]
\[+8 r^8 k^{14} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 + 8 r^8 k^{14} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0\]
\[+4 r^8 k^{14} b_0^2 \Delta^{1/6}(\delta_j(\delta_j(k))) b_0 - 2 r^6 k^{10} b_0^2 \Delta^{7/6}(\delta_j(\delta_j(k))) b_0\]
\[+4 r^8 k^{14} b_0^2 \Delta^{7/6}(\delta_j(\delta_j(k))) b_0 - 4 r^6 k^{10} b_0^2 \Delta^{7/6}(\delta_j(\delta_j(k))) b_0\]
\[-10 r^4 k^{10} b_0^2 \Delta^{1/2}(\delta_j(\delta_j(k))) b_0 - 12 r^6 k^{10} b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0\]
\[-12 r^6 k^{10} b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0 + 4 r^8 k^{14} b_0^2 \Delta^{5/3}(\delta_j(\delta_j(k))) b_0\]
\[+4 r^8 k^{14} b_0^2 \Delta^{5/3}(\delta_j(\delta_j(k))) b_0 + 4 r^8 k^{14} b_0^2 \Delta^{1/3}(\delta_j(\delta_j(k))) b_0\]
\[-12r^6k^{10}b_0^2 \Delta^{5/6}(\delta_j(k))b_0 \Delta^{1/2}(\delta_j(k))b_0 - 14r^6k^{10}b_0^2 \Delta^{5/6}(\delta_j(k))b_0 \Delta^{1/3}(\delta_j(k))b_0 - 14r^6k^{10}b_0^2 \Delta^{5/6}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 + 4r^8k^{12}b_0^2 \Delta^{3/2}(\delta_j(k))b_0 \Delta^{1/3}(\delta_j(k))b_0 + 4r^8k^{12}b_0^2 \Delta^{3/2}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 12r^6k^{10}b_0^2 \Delta^{2/3}(\delta_j(k))b_0 \Delta^{1/2}(\delta_j(k))b_0 - 14r^6k^{10}b_0^2 \Delta^{2/3}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 + 4r^8k^{14}b_0^2 \Delta^{4/3}(\delta_j(k))b_0 \Delta^{1/3}(\delta_j(k))b_0 + 4r^8k^{14}b_0^2 \Delta^{4/3}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 + 4r^8k^{14}b_0^2 \Delta^{7/6}(\delta_j(k))b_0 \Delta^{1/2}(\delta_j(k))b_0 + 4r^8k^{14}b_0^2 \Delta^{7/6}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 + 8r^8k^{14}b_0^3 \Delta^{2/3}(\delta_j(k))b_0 \Delta^{1/2}(\delta_j(k))b_0 + 8r^8k^{14}b_0^3 \Delta^{2/3}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 + 8r^8k^{14}b_0^3 \Delta^{5/6}(\delta_j(k))b_0 \Delta^{1/3}(\delta_j(k))b_0 + 8r^8k^{14}b_0^3 \Delta^{5/6}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 16r^6k^{10}b_0^2 \Delta^{1/2}(\delta_j(k))b_0 \Delta^{1/3}(\delta_j(k))b_0 - 16r^6k^{10}b_0^2 \Delta^{1/2}(\delta_j(k))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/3}(\delta_j(k-1))b_0 \Delta^{1/3}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/6}(\delta_j(k-1))b_0 \Delta^{1/3}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/3}(\delta_j(k-1))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/6}(\delta_j(k-1))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/6}(\delta_j(k-1))b_0 \Delta^{1/6}(\delta_j(k))b_0 - 3r^2k^4b_0 \Delta^{1/6}(\delta_j(k-1))b_0 \Delta^{1/6}(\delta_j(k))b_0\]
Proof. Let $G_n$ and $G_{n,\alpha}$ be the inverse Fourier transforms of the functions defined respectively by
\[
g_n(t) = (e^{t/2} + e^{-t/2})^{-n}
\]
and
\[
H_{n,\alpha}(t) = e^{(n-\alpha)t} (e^t + 1)^{-n},
\]
where $n \in \mathbb{N}$ and $\alpha \in (0, n)$. Then $G_{n,\alpha}(s) = G_n(s - i(n/2 - \alpha))$. So we have
\[
(7) \quad H_{n,\alpha}(t) = \int_{-\infty}^{\infty} G_n(s - i(n/2 - \alpha))e^{-ist}ds.
\]
Let $J$ be the integral in the left hand side of the equation in the lemma. Now we use the substitutions $u = e^s$ and $k = e^{j/4}$ to compute $J$. Therefore, we have
\[
J = 
\int_{-\infty}^{\infty} (e^{(s+f) + 1})^{-m_0} \rho_1(e^{(s+f) + 1})^{-m_1} \cdots \rho_l(e^{(s+f) + 1})^{-m_l} e^{i\sum_{j=0}^{l-1} m_j s} e^{s/2} ds
\]
Then for $j = 0, 1, 2, \ldots, l$, we pick a positive real number $\alpha_j$ such that $\sum_{j=0}^{l} \alpha_j = 1$. We also set $\beta_j = -\sum_{i=j}^{l} (m_i - \alpha_i)$. Replacing $(e^{(s+f) + 1})^{-m_j}$ by $e^{(m_j - \alpha_j)(f+s)}(e^{s+f} + 1)^{-m_j}$ in $J$, we get
\[
J = e^{-i\sum_{j=0}^{l} m_j s} \rho_j (s + f) \Delta^\beta (\rho_1) H_{m_1,\alpha_1}(s + f) \cdots \Delta^\beta (\rho_l) H_{m_l,\alpha_l}(s + f) e^{s/2} ds.
\]
Let $\rho_j' = \Delta^\beta_j (\rho_j)$. Using (7), $J$ can be written as an integral of the form
\[
e^{-i\sum_{j=0}^{l} m_j s} H_{m_0,\alpha_0}(s + f) \rho_1' e^{-i(s+f)t_1} \rho_2' \cdots e^{-i(s+f)t_l} \rho_l' e^{-i(s+f)t_l} e^{s/2}
\]
with respect to the measure $\prod_{j=1}^{l} G_{m_j,\alpha_j}(t_j) dt_j ds$.

Now we can write (8) as
\[
e^{-i\sum_{j=0}^{l} m_j s} H_{m_0,\alpha_0}(s + f) e^{-i\sum_{j=0}^{l-1} t_j}(s + f) \prod_{h=1}^{l} \Delta^{-i\sum_{j=h}^{l-1} t_j} (\rho_h) e^{s/2}.
\]
We also have
\[
\int_{\mathbb{R}} H_{m_0,\alpha_0}(s + f) e^{-i\sum_{j=0}^{l-1} t_j}(s + f) e^{s/2} ds =
\]
\[
e^{-f/2} e^{(s+f)/2} H_{m_0,\alpha_0}(s + f) e^{-i\sum_{j=0}^{l-1} t_j}(s + f) ds = 2\pi e^{-f/2} \rho_{m_0,\alpha_0}(\sum_{j=1}^{l} t_j),
\]
\[
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\]
where $P_{m_0,a_0}$ is the inverse Fourier transform of the function $e^{s/2}H_{m_0,a_0}(s)$. So we have

$$J = 2\pi e^{-f/2}e^{-\left(\sum_{j=0}^l m_j - 1\right)f} \int \prod_{h=1}^l \Delta^{-i} \sum_{j=h}^l t_j (\rho'_h) P_{m_0,a_0}(-\sum_{j=1}^l t_j) \prod_j G_{m_j,a_j}(t_j) dt_j.$$

Replacing $\rho'_j$ by $\Delta^\beta_j(\rho_j)$ we have

$$\Delta^{-i} \sum_{j=h}^l (\rho'_h) = \Delta^{-i} \sum_{j=h}^l t_j + \beta_h(\rho_h).$$

Now we replace the last term by $u_h^{-i} \sum_{j=h}^l t_j + \beta_h$. We define

$$F_{m_0,m_1,m_2,\ldots,m_l}(u_1,u_2,\ldots,u_l) = \int \prod_{h=1}^l \Delta^{-i} \sum_{j=h}^l t_j + \beta_h P_{m_0,a_0}(-\sum_{j=1}^l t_j) \prod_j G_{m_j,a_j}(t_j) dt_j.$$

Moreover, we can write

$$2\pi P_{m_0,a_0}(-\sum_{j=1}^l t_j) = \int_{-\infty}^\infty e^{s/2}H_{m_0,a_0}(s)e^{-i(\sum_{j=1}^l t_j)(s)} ds.$$

Using this and assuming that $u_h = e^{s_h}$, we can do the integration. Then the coefficient of $t_j$ in the exponent is

$$-is - i \sum_{h=1}^j s_h.$$

So integrating in $t_j$ gives the Fourier transform of $G_{m_j,a_j}$ at $s + \sum_{h=1}^j s_h$. On the other hand we have

$$e^{(m_j-a_j)(s+\sum_{h=1}^j s_h)}(e^{(s+\sum_{h=1}^j s_h) + 1} - m_j) =$$

$$e^{(m_j-a_j)s} \left( \prod_{h=1}^j u_h \right)^{(m_j-a_j)} \left( e^{s} \prod_{h=1}^j u_h + 1 \right)^{-m_j}.$$

When we multiply these terms from $j = 1$ to $j = l$, the exponent of $u_h$ is $\sum_{j=h}^l (m_j - a_j)$. So $u_h^{\beta_h}$ disappears and we get

$$F_{m_0,m_1,m_2,\ldots,m_l}(u_1,u_2,\ldots,u_l) =$$

$$\int_{-\infty}^\infty (e^{s} + 1)^{-m_0} \prod_{j=1}^l \left( e^{s} \prod_{h=1}^j u_h + 1 \right)^{-m_j} e^{(\sum_{j=0}^l m_j - 1)s} e^{s/2} ds.$$
In Lemma 7.1 it is clear that
\[ F_{n_0,n_1,n_2,...,n_l}(u_1, u_2, \ldots, u_l) = H_{n_0,n_1,n_2,...,n_l}(u_1, u_1 u_2, \ldots, u_1 \cdots u_l), \]
where
\[ H_{n_0,n_1,n_2,...,n_l}(u_1, u_2, \ldots, u_l) = \int_0^\infty (u + 1)^{-n_0} \prod_{j=1}^l (u u_j + 1)^{-m_j} u^{(\sum_{j=0}^l m_j - 3)/2} du. \]
We only need some of these functions:
\[ H_{1,1}(x) = \int_0^\infty (u + 1)^{-1} (ux + 1)^{-1} u^{1/2} du = \frac{\pi}{x + \sqrt{x}}, \]
\[ H_{1,1,1}(x, y) = \int_0^\infty (u + 1)^{-1} (ux + 1)^{-1} (uy + 1)^{-1} u^{3/2} du = \frac{\pi (\sqrt{x} + \sqrt{y} + 1)}{(\sqrt{x} + 1) \sqrt{x} (\sqrt{y} + \sqrt{y}) (\sqrt{x} + \sqrt{y})}, \]
\[ H_{2,1}(x) = \int_0^\infty (u + 1)^{-2} (ux + 1)^{-1} u^{3/2} du = \frac{2\pi}{2 (\sqrt{x} + 1)^2}, \]
\[ H_{2,1,1}(x, y) = \int_0^\infty (u + 1)^{-2} (ux + 1)^{-1} (uy + 1)^{-1} u^{5/2} du = \frac{\pi (\sqrt{x} (\sqrt{y} + 2)^2 + x (\sqrt{y} + 2) + 2 (\sqrt{y} + 1)^2)}{2 (\sqrt{x} + 1)^2 \sqrt{x} (\sqrt{y} + 1)^2 \sqrt{y} (\sqrt{x} + \sqrt{y})}, \]
\[ H_{1,2,1}(x, y) = \int_0^\infty (u + 1)^{-1} (ux + 1)^{-2} (uy + 1)^{-1} u^{5/2} du = \frac{\pi (2x^{3/2} + 4x (\sqrt{y} + 1) + 2\sqrt{x} (\sqrt{y} + 1)^2 + y + \sqrt{y})}{2 (\sqrt{x} + 1)^2 x^{3/2} (\sqrt{y} + 1) \sqrt{y} (\sqrt{x} + \sqrt{y})^2}, \]
\[ H_{2,2,1}(x, y) = \int_0^\infty (u + 1)^{-2} (ux + 1)^{-2} (uy + 1)^{-1} u^{7/2} du = \frac{\pi (2 (x^{3/2} + 4x + 4\sqrt{x} + 1) y + (7x^{3/2} + x^2 + 13x + 7\sqrt{x} + 1) \sqrt{y})}{2 (\sqrt{x} + 1)^3 x^{3/2} (\sqrt{y} + 1)^2 \sqrt{y} (\sqrt{x} + \sqrt{y})^2} + \frac{\pi ((x + 3\sqrt{x} + 1) y^{3/2} + 2 (\sqrt{x} + 1)^3 \sqrt{x})}{2 (\sqrt{x} + 1)^3 x^{3/2} (\sqrt{y} + 1)^2 \sqrt{y} (\sqrt{x} + \sqrt{y})^2}, \]
\[ H_{3,1}(x) = \int_0^\infty (u + 1)^{-3} (ux + 1)^{-1} u^{5/2} du = \frac{\pi (3x + 9\sqrt{x} + 8)}{8 (\sqrt{x} + 1)^3 \sqrt{x}}, \]
\[ H_{3,1,1}(x, y) = \int_0^\infty (u + 1)^{-3}(ux + 1)^{-1}(uy + 1)^{-1}u^{7/2}du = \]
\[ \frac{\pi}{8(x - y)} \left( -\frac{3x + 9\sqrt{\pi} + 8}{(\sqrt{\pi} + 1)^2\sqrt{\pi}} - \frac{8}{\sqrt{\pi} + 1} - \frac{5}{(\sqrt{\pi} + 1)^2} - \frac{2}{(\sqrt{\pi} + 1)^3} + \frac{8}{\sqrt{\pi}} \right). \]

Now with the notations that we have set up, we can state and prove the main result of this paper:

**Theorem 7.2.** The scalar curvature of \( \mathbb{R}^3 \), with the perturbed metric, up to a factor of \(-4\sqrt{\pi}/3\) is the element \( S \in \mathbb{R}^3 \), given by

\[
S = k^2(-3H_{1,1} + 6H_{2,1} - 4H_{3,1})(\Delta(1)(\delta_i(k))^2)
+ k^3(-3/2H_{1,1} + 7/2H_{2,1} - 2H_{3,1})(\Delta(1)(\delta_i(k))
+ k^3(-3H_{1,1} + 5H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/3}(k))^2)
+ k^3(-3/2H_{1,1} + 5/2H_{2,1} - 2H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^3(-3/2H_{1,1} + 5/2H_{2,1} - 2H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^2(4H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^3(3/2H_{2,1} - 2H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^2(-3H_{1,1} + 6H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/3}(k)))
+ k^2(5H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/3}(k)))
\]

With the perturbed metric, the scalar curvature is given by

\[
S = k^2(-3H_{1,1} + 6H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^2(-3H_{1,1} + 6H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^2(4H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
+ k^2(5H_{2,1} - 4H_{3,1})(\Delta(1)(\Delta^{1/2}(k)))
\]
\[+k^2(-H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{3/2}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+k^2(3/2H_{1,1,1} + 6H_{2,1,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{2/3}(\delta_i(k))\Delta^{1/2}(\delta_i(k)))\]
\[+k^2(2H_{1,1,1} - 7H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{2/3}(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+k^2(2H_{1,1,1} - 7H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{2/3}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+k^2(-H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{1/3}(\delta_i(k))\Delta^{1/2}(\delta_i(k)))\]
\[+k^2(-H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{1/3}(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+k^2(-H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{1/3}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+k^2(3H_{1,1,1} - 8H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{1/2}(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+k^2(3H_{1,1,1} - 8H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{1/2}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+k^2(-2H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{7/6}(\delta_i(k))\Delta^{1/2}(\delta_i(k)))\]
\[+k^2(-2H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{7/6}(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+k^2(-2H_{1,2,1} + 2H_{2,2,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta^{7/6}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+k^2(-5H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta(\delta_i(k))\Delta^{1/2}(\delta_i(k)))\]
\[+k^2(-6H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+k^2(-6H_{2,2,1} + 4H_{3,1,1})(\Delta(1), \Delta(1)\Delta(2))(\Delta(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[+2k^2H_{2,2,1}(\Delta(1), \Delta(1)\Delta(2))(\Delta^{5/3}(\delta_i(k))\Delta^{1/2}(\delta_i(k)))\]
\[+2k^2H_{2,2,1}(\Delta(1), \Delta(1)\Delta(2))(\Delta^{5/3}(\delta_i(k))\Delta^{1/3}(\delta_i(k)))\]
\[+2k^2H_{2,2,1}(\Delta(1), \Delta(1)\Delta(2))(\Delta^{5/3}(\delta_i(k))\Delta^{1/6}(\delta_i(k)))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/3}(\delta_i(k^{-1}))\delta_i(k))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/3}(\delta_i(k^{-1}))\Delta^{1/3}(\delta_i(k)))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/6}(\delta_i(k^{-1}))\delta_i(k))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/6}(\delta_i(k^{-1}))\Delta^{1/3}(\delta_i(k)))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/6}(\delta_i(k^{-1}))\Delta^{1/6}(\delta_i(k)))\]
\[-3/2k^4H_{1,1,1}(\Delta(1)) (\Delta^{1/6}(\delta_i(k^{-1}))\Delta^{1/6}(\delta_i(k)))\]

**Proof.** It suffices to find
\[\int_0^\infty B(r)dr.\]

For that we only need to use the substitution \(r^2 = u\), and then apply Lemma □
\[11\]

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Now we shall show that the formula in Theorem 7.2 is compatible with the formula in the commutative case. In fact, in the commutative case the modular operator is the identity operator. So it suffices to find the limit of $S$ in Theorem 7.2. After simplification we see that

$$S = \lim_{x \to 1} k^3(-9/2H_{1,1} + 10H_{2,1} - 8H_{3,1})(x) (\delta_i(\delta_i(k)))$$

$$+ \lim_{(x,y) \to (1,1)} k^2(-9H_{1,1} + 30H_{2,1} - 24H_{3,1} + 32H_{1,1,1} - 112H_{2,1,1}$$

$$+ 64H_{3,1,1} - 16H_{1,2,1} + 32H_{2,2,1})(x, y)(\delta_i(k)\delta_i(k))$$

$$- \lim_{x \to 1} 9k^4H_{1,1}(x) (\delta_i(k^{-1})\delta_i(k)).$$

Using the derivation property in the last term, we get

$$S = \lim_{x \to 1} k^3(-9/2H_{1,1} + 10H_{2,1} - 8H_{3,1})(x) (\delta_i(\delta_i(k)))$$

$$+ \lim_{(x,y) \to (1,1)} k^2(-9H_{1,1} + 30H_{2,1} - 24H_{3,1} + 32H_{1,1,1} - 112H_{2,1,1}$$

$$+ 64H_{3,1,1} - 16H_{1,2,1} + 32H_{2,2,1})(x, y)(\delta_i(k)\delta_i(k))$$

$$+ \lim_{x \to 1} 9k^2H_{1,1}(x)(\delta_i(k)\delta_i(k)).$$

Therefore,

$$S = k^3(-9\pi/4 + 15\pi/4 - 5\pi/2) (\delta_i(\delta_i(k)))$$

$$+ k^2(-9\pi/2 + 45\pi/4 - 30\pi/4 + 12\pi - 35\pi)$$

$$+ 35\pi/2 - 5\pi + 35\pi/4 + 9\pi/2)(\delta_i(k)\delta_i(k)).$$

So

$$S = -\pi k^3 (\delta_i(\delta_i(k))) + 2\pi k^2(\delta_i(k)\delta_i(k)).$$

On the other hand, by applying Lemma 5.1 in [15], we obtain

$$k^{-2}\delta_i(k)\delta_i(k) = \delta_i(\log k)\delta_i(\log k)$$

and

$$k^{-1}\delta_i(\delta_i(k)) = \delta_i(\delta_i(\log k)) + \delta_i(\log k)\delta_i(\log k).$$

Using (10) and (11) in (9), we see that

$$S = -\pi k^4\delta_i(\delta_i(\log k)) + \pi k^4\delta_i(\log k)\delta_i(\log k),$$

which is the same formula as in the classic case up to a normalization factor.
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