Understanding Distributional Ambiguity via Non-robust Chance Constraint

Qi WU*
School of Data Science
City University of Hong Kong
qiwu55@cityu.edu.hk

Shumin MA
School of Data Science
City University of Hong Kong
shuminma@cityu.edu.hk

LEUNG Cheuk Hang
School of Data Science
City University of Hong Kong
chleung87@cityu.edu.hk

Wei LIU
Tencent AI Lab
wl2223@columbia.edu

Abstract

The choice of the ambiguity radius is critical when an investor uses the distributionally robust approach to address the issue that the portfolio optimization problem is sensitive to the uncertainties of the asset return distribution. It cannot be set too large because the larger the size of the ambiguity set the worse the portfolio return. It cannot be too small either; otherwise, one loses the robust protection. This tradeoff demands a financial understanding of the ambiguity set. In this paper, we propose a non-robust interpretation of the distributionally robust optimization (DRO) problem. By relating the impact of an ambiguity set to the impact of a non-robust chance constraint, our interpretation allows investors to understand the size of the ambiguity set through parameters that are directly linked to investment performance. We first show that for general $\phi$-divergences, a DRO problem is asymptotically equivalent to a class of mean-deviation problem, where the ambiguity radius controls investor’s risk preference. Based on this non-robust reformulation, we then show that when a boundedness constraint is added to the investment strategy, the DRO problem can be cast as a chance-constrained optimization (CCO) problem without distributional uncertainties. If the boundedness constraint is removed, the CCO problem is shown to perform uniformly better than the DRO problem, irrespective of the radius of the ambiguity set, the choice of the divergence measure, or the tail heaviness of the center distribution. Our results apply to both the widely-used Kullback-Leibler (KL) divergence which requires the distribution of the objective function to be exponentially bounded, as well as those more general divergence measures which allow heavy tail ones such as student $t$ and lognormal distributions.

1 Introduction

Decision making in many business and financial applications can be formulated as problems of maximizing expectations. These problems require an estimation of the state variables’ distribution. In a data-driven context, the estimated distribution is often time either not accurate due to the limited amount of observations or changing in law as new data comes in. The optimal decisions and the corresponding outcomes are thus not stable. A case in point is the asset allocation problem in finance,

*The corresponding author.

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also called the portfolio optimization problem. It starts from Harry Markowitz’s mean-variance setting \([25]\) and is now known to suffer from issues related to estimating the return distribution.

For example, when the ‘true distribution’ of asset return is not known, estimating it from historical data is prone to error. For instance, Merton \([26]\) analyzes three models of equilibrium expected market returns, while estimations of covariance can be found in \([1], [11],\) and \([27]\). The point is that estimation error is inevitable irrespective to the method used. Further, a small difference in the estimation could have a great impact on both the allocation decision and the optimal portfolio value. Meanwhile, to minimize the impact of the estimation errors, a rather large estimation window is required, which suggests that the estimated mean and covariance of assets returns converge to their true values only when the number of observations is sufficiently large. The more assets there are, the longer period of data is needed. For example, as stated in \([6]\), for a portfolio of 25 assets, around 3,000 months of historical data are needed to give accurate approximations. In reality, it is not possible.

One consequence of the resulting instabilities, meaning the optimality is sensitive to the state variables’ distribution, is that it causes unnecessary buy or sell decisions to re-balance the portfolio, which increases transaction costs. Another consequence is that the decision maker would have an erroneous investment outlook if optimality is prone to model uncertainty. One way to address this issue is to formulate the original optimization problem as a distributionally robust optimization (DRO) problem, by adding an extra layer of inner optimization over a set of distributions, called ambiguity set. The shape and size of such an ambiguity set are typically defined by a divergence or distance measure between two distributions and a radius parameter.

The main idea to solve a DRO problem is to find an equivalent reformulation where the ambiguity of the state variable’s distribution can be removed. For example, Hu and Hong \([19]\) suggests removing ambiguity by the Lagrangian dual when the divergence measure associated with the ambiguity set is the Kullback-Leibler (KL) divergence. On the other hand, one can remove ambiguity by considering series expansion similar to Henry Lam’s approach \([23]\) for the KL divergence and the approach used by Gotoh et al. \([17]\) for more general divergence measure on objective function with a penalty term.

In this paper, we would like to understand the ambiguity set in the financial context and further find a non-robust interpretation of the radius parameter. In particular, we find that the DRO problem in the asset allocation context can be reformulated as a class of optimization problems with mean-deviation as the objective function. With this result, we further derive conditions under which such a DRO problem is equivalent to a chance-constrained optimization (CCO) problem. This equivalence opens a door for financial practitioners to interpret the radius parameter through parameters that are directly linked to investment performance.

The rest of this paper is organized as follows. In Section 3 we provide background information and motivation on the optimization problems being considered, including notations and \(\phi\)-divergences in defining the ambiguity set. Section 2 summarizes literature relevant to us. In Section 4 we reformulate the DRO problem and CCO problem, respectively, and solve the corresponding optimal decisions and optimal values. Section 5 gives numerical experiments, while Section 6 concludes our findings. All detailed proofs are deferred to the Appendix.

## 2 Literature Review

Our work is closely related to the literature on DRO problems. The usual ways of constructing ambiguity sets for underlying distribution in DRO problems include but not limited to by shape, by moments, by \(\phi\)-divergence, and by Wasserstein distance, etc. For example, Scarf \([29]\) shows that a closed-form solution can be obtained in the newsvendor problem context when a distribution set with a fixed mean and covariance is chosen. Delage and Ye \([9]\) takes into account the knowledge of the distribution’s support and a confidence region for its mean and second-order matrix to show the associated distributionally robust stochastic program can be solved efficiently. Zymler et al. \([34]\) considers the distributionally robust individual and joint chance constraints where the first- and second-order moments, as well as the support of the uncertain parameters, are given. The same form of distribution set is also used in Calafiore and El Ghaoui \([5]\) and Goh and Sim \([16]\). Hu et al. \([20]\) studies ambiguous probabilistic programs where the ambiguity is modeled by the \(\phi\)-divergence. Jiang and Guan \([21]\) studies stochastic programs with distributionally robust chance constraints where the ambiguity set is based on a general \(\phi\)-divergence measure. Wang et al. \([31]\)
defines the distribution set to contain only those distributions that make the observed data achieve a certain level of likelihood and proves that the proposed model is easily solvable. Esfahani and Kuhn [12] demonstrates that DRO problems over Wasserstein balls can be reformulated as finite convex programs.

DRO problems are studied in the financial context previously. Qiu et al. [27] builds on a quantile-based scatter matrix estimator to propose a portfolio optimization approach. Zhu et al. [33] considers robust portfolio selection problem under a risk measure based on the lower-partial moment associated with various types of distributional uncertainties, such as multimode, polytopic, box, and ellipsoidal uncertainties. Zhu and Fukushima [32] considers minimizing the worst-case CVaR under mixture distribution uncertainty, box uncertainty, and ellipsoidal uncertainty. Kim et al. [22] focuses on an ellipsoidal uncertainty set for expected returns to investigate the behavior of robust portfolios. Chen et al. [7] derives tight bounds for the robust portfolio selection problem where the ambiguous distributions share the same first two moments. Calafiore [4] proposes a methodology to optimize the worst-case risk of a portfolio (variance or absolute deviation) under distributional uncertainty characterized by KL divergence. Glasserman and Xu [14] develops portfolio control rules that are robust to distributional uncertainty. In addition to the above works, some researches pay special attention to heavy tail distributions. Dey and Juneja [10] points out that the optimal solution to minimize KL divergence with respect to a known measure may not exist when constraints involve fat tail distributions. They emphasize that such drawback can be corrected if polynomial-divergence entropy distance is used. Glasserman and Xu [15] develops a general approach for quantifying model risk and bounding the impact of model error in mean-variance portfolio optimization, CVaR, etc. Moreover, they extend the ideas to heavy tail distributions with α-divergence replacing relative entropy.

There are also other applications for DRO formulations. Shapiro [30] discusses the law invariance of the associated worst-case functional to distributionally robust stochastic programming, taking into consideration two basic constructions of uncertainty sets (distance approach and φ-divergence approach). Hashimoto et al. [18] applies DRO to control the risk of the minority group and demonstrate that DRO presents disparity amplification on examples where empirical risk minimization fails. Chen and Paschalidis [8] presents a novel l1-loss based robust learning procedure using DRO in a linear regression framework.

3 Notations and Motivation

In this section, we introduce the notations used throughout this paper, and give a brief introduction of the φ-divergence for defining the ambiguity set. Furthermore, we put forward the motivation to explain why we focus on the ambiguity radius in the DRO problem.

3.1 Notations

Let $\mathbf{r} \in \mathbb{R}^n$, an $n$-dimensional real-valued random vector, be the vector of asset returns and suppose the joint probability distribution of $\mathbf{r}$ is $P$. Let $P_0$ be the nominal probability distribution of $\mathbf{r}$. Let $\mathbf{x} \in \mathbb{R}^n$ be the asset allocation strategy and $\mathbf{e} \in \mathbb{R}^n$ be a vector with all entries equal to 1. We assume that $\mathbf{x}$ lies in a convex set $\mathcal{X}$ and $P$ lies in an ambiguity set $\mathcal{U}$. The expectation and variance of a random variable under $P$ are represented by $E_{P}[\cdot]$ and $V_{P}[\cdot]$, respectively.

3.2 φ-divergence

φ-divergence is a commonly used statistical distance to describe the ambiguity set $\mathcal{U}$. It quantifies how one probability distribution diverges from another. Such a divergence is treated as “distance” between two distributions. We begin with the definition of the φ-divergence.

Definition 3.1. Assume that $\phi(t)$ is convex for $t \geq 0$, and set $\phi(t) = \infty$ for $t < 0$. Then the φ-divergence $D(Q||P)$ for two distributions is defined as:

$$D(Q||P) := \int_{\Omega} \phi \left( \frac{dQ}{dP} \right) P(dt) = E_{P} \left[ \phi \left( \frac{dQ}{dP} \right) \right] := E_{P} \left[ \phi \left( L \right) \right], \quad (1)$$

where

$$\phi(1) = 0; \quad 0 \cdot \phi \left( \frac{a}{0} \right) := a \cdot \lim_{t \to 0} \frac{\phi(t)}{t}, \quad \forall \ a > 0; \quad 0 \phi \left( \frac{0}{0} \right) := 0. \quad (2)$$
Table 1: The two $\phi$-divergences used in this paper. The KL divergence applies to light-tail distributions, while the Cressie-Read divergence is compatible with heavy tail distributions.

| Divergence       | $\phi(t), t \geq 0$ | $\phi^*(s)$ |
|------------------|----------------------|-------------|
| Kullback-Leibler | $t \log(t) - t + 1$ | $e^s - 1$   |
| Cressie-Read     | $\frac{1-\theta+\theta t}{\theta(1-\theta)}$, $\theta \neq 0$, $1 < s < \frac{1}{1-\theta}$ | $\frac{(1-s(1-\theta))^{\frac{1}{\theta}}}{\theta} - \frac{1}{\theta}$ |

The quantity $L$ in Eq. (1) is called the Radon Nikodym derivative (or likelihood ratio) such that $L \geq 0$ almost surely and $\mathbb{E}_P[L] = 1$. Given the function $\phi$ defined in Definition 3.1, its conjugate $\phi^*$ is defined as

$$
\phi^*(s) := \sup_{t \geq 0} \{st - \phi(t)\}.
$$

(3)

Table 1 lists the two divergences used in this paper. The KL divergence is commonly used because of the nice properties its conjugate function has. In order to use the KL divergence, the distribution of the objective function needs to be exponentially bounded, which excludes important heavy tail distributions used ubiquitously for financial asset returns, especially the lognormal distribution and the student $t$-distribution. The Cressie-Read divergence, also called the $\alpha$-divergence in [15], overcomes this limitation and can well accommodate heavy tail distributions. However, our interpretation of the ambiguity radius $\rho$ as a chance constraint applies to all the $\phi$-divergences, including Burg entropy, $J$-divergence, $\chi^2$-distance, modified $\chi^2$-distance, and Hellinger distance.

3.3 Motivation

The goal is to maximize the expected return over a set of admissible allocation strategies $\mathcal{X}$, namely,

$$
\max_{x \in \mathcal{X}} \mathbb{E}_P[x^T r].
$$

(4)

Both the optimal value and the optimal solution depend strongly on assumptions and properties of the asset return $r$, the true distribution of which is usually unknown. A DRO formulation addresses this issue by constructing an ambiguity set of distributions centered at the nominal distribution $P_0$ and then maximizing the worst-case return among all distribution within the ambiguity set. We focus on ambiguity set $\mathcal{U}$ which is defined by the $\phi$-divergence and controlled by a radius parameter $\rho > 0$, that is, $\mathcal{U} := \{P : D(P\|P_0) \leq \rho\}$.

Thus, the distributionally robust counterpart of problem (4) is:

$$
\max_{x \in \mathcal{X}} \min_{P \in \mathcal{U}} \mathbb{E}_P[x^T r].
$$

(OPT1)

For an investor, the choice of ambiguity radius $\rho$ is critical. One cannot set it too large since the optimal portfolio return decreases in $\rho$. However, if it is too small, one loses the robust protection. There is a trade-off in choosing its magnitude in the financial context. This situation demands a financial interpretation of the ambiguity radius $\rho$. Pardo in [15] proves that, assuming the true distribution $P$ and the nominal distribution $P_0$ belong to the same parameterized distribution family with parameter dimension $d$, then the normalized estimated $\phi$-divergence $\frac{1}{d}\phi^2(x^T P\|P_0)$ asymptotically follows a $\chi^2_d$-distribution. It thus relates the ambiguity radius $\rho$ to a confidence level at which the true distribution $P$ falls within this ambiguity set. However, in financial practice with real data, the assumption that the true distribution is in the same parameterized family with the center distribution of an ambiguity set is too strong. A wrong guess of the nominal distribution might lead to a meaningless confidence level interpretation of the ambiguity radius $\rho$. With the financial motivation and focusing on the impact of the ambiguity radius $\rho$ on the optimal return, we realize that, through a CCO problem, the ambiguity radius $\rho$ can be explained by the parameters of a chance-constrained problem, using only the nominal one: $P_0$. 

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Let us define the CCO problem considered:

\[
\begin{align*}
\max_{x \in X} & \quad \mathbb{E}_{P_0}[x^T r] \\
\text{s.t.} & \quad \Pr_{x \sim P_0}(x^T r \leq -\delta) \leq \epsilon.
\end{align*}
\] (OPT1')

The CCO problem shares the same objective function as that of problem (4). The expectation is taken under the nominal distribution \(P_0\), not subject to distributional robustness. The new component is the probability constraint with the parameters \(\delta\) and \(\epsilon\) characterizing an investor’s tolerance to losses.

### 4 Interpret \(\rho\) as a chance constraint

In this section, we show that for general \(\phi\)-divergences, [OPT1] can be reformulated as a class of mean-deviation optimization problems with the investor’s risk preference parameter controlled by the ambiguity radius \(\rho\). By theoretically solving the mean-deviation optimization problem with an unbounded feasibility set, we can determine a threshold for the ambiguity radius \(\rho\). It is only beyond this threshold that the distributional uncertainty is indeed effective. Furthermore, by solving the equivalent reformulation of [OPT1], we find that when both the DRO problem and the CCO problem achieve finite optimal values, the CCO problem performs uniformly better than its DRO counterpart.

#### 4.1 Reformulation of [OPT1]

Consider the inner optimization problem in [OPT1]

\[
\operatorname{min}_{P \in \mathcal{U}} \mathbb{E}_P[x^T r].
\] (5)

The Lagrangian dual to problem (5) is:

\[
\begin{align*}
\max_{\eta_1, \eta_2 \in \mathbb{R}, \eta_2 \geq 0} & \quad \left\{ -\frac{1}{\eta_2} \max_L \left\{ \mathbb{E}_{\phi_0}[-\eta_2 (x^T r + \eta_1) L - \phi(L)] \right\} - \eta_1 - \frac{\rho}{\eta_2} \right\}.
\end{align*}
\] (6)

Through the conjugate function of \(\phi\)-divergence defined in Eq. (3), we see that the Lagrangian dual in problem (6) is equivalent to

\[
\max_{\eta_1, \eta_2 \in \mathbb{R}, \eta_2 \geq 0} \left\{ -\frac{1}{\eta_2} \mathbb{E}_{\phi_0}[\phi^*(\eta_2 (x^T r + \eta_1))] - \eta_1 - \frac{\rho}{\eta_2} \right\}.\] (7)

Difficulty in solving problem (7) lies in the term \(\mathbb{E}_{\phi_0}[\phi^*(\eta_2 (x^T r + \eta_1))]\). We hereby follow the idea in [17] to express optimization (5) in terms of Regular Measure of Deviation. The results are summarized in Theorem 4.1.

**Theorem 4.1.** Let \(\phi\) be a closed proper convex function satisfying the conditions in Eq. (2) and the conjugate function \(\phi^*\) defined in Eq. (3). Suppose that under mild conditions, the strong duality holds. Then, the following two objective functions are equal:

1. **Obj1.** \(\min_{P \in \mathcal{U}} \mathbb{E}_P[x^T r];\)
2. **Obj2.** \(\mathbb{E}_{\phi_0}[x^T r] - \min_{\eta_2 \geq 0} \left\{ \frac{\rho}{\eta_2} + D_{\eta_2, \phi, \phi_0}(x^T r | \mathbb{E}_{\phi_0}[x^T r]) \right\} \)

where \(D_{\eta_2, \phi, \phi_0}(x^T r | \mathbb{E}_{\phi_0}[x^T r]) := \min_{\eta_1} \left\{ \eta_1 + \frac{1}{\eta_2} \mathbb{E}_{\phi_0} \left[ \phi^* \left( \eta_2 (\mathbb{E}_{\phi_0}[x^T r] - x^T r - \eta_1) \right) \right] \right\}\) is a regular measure of deviation.

Theorem 4.1 says that the inner optimization in [OPT1] is equivalent to Obj2. While, the quantity \(D_{\eta_2, \phi, \phi_0}(x^T r | \mathbb{E}_{\phi_0}[x^T r])\) can be further expanded as a series of terms, the coefficients of which can be computed under the nominal distribution \(\mathbb{E}_{\phi_0}\). In such way, we can reformulate [OPT1] as a single-layer maximization problem. Before we present the expansion of \(D_{\eta_2, \phi, \phi_0}(x^T r | \mathbb{E}_{\phi_0}[x^T r])\), let us
introduce two notations. To shorten the notation, we denote \( \mathbb{E}_{\mathbb{P}_0}[r] \) and \( \mathbb{V}_{\mathbb{P}_0}[r] \) by \( \mu \) and \( \Sigma \) throughout the rest of the paper. Naturally we would have \( \mathbb{E}_{\mathbb{P}_0}[x^T r] = x^T \mu \) and \( \mathbb{V}_{\mathbb{P}_0}[x^T r] = x^T \Sigma x \).

**Lemma 4.2.** Suppose that \( n \) is an even number, \( \phi \in \mathcal{C}^n \) is a convex function which satisfies \( \phi(1) = 0 \) and \( \phi(2) > 0 \). Define \( X := x^T r - x^T \mu \), then

\[
D_{\eta_2, \phi, \mathbb{P}_0}(x^T r | x^T \mu) = \sum_{k=1}^{n-1} b_k \mathbb{E}_{\mathbb{P}_0} \left[ (X + \eta_1^*)^{k+1} \right] \eta_2^k + o(\eta_2^{-1}),
\]

where \( b_k = \frac{(-1)^{k+1} \phi^{(k)}(0)}{(k+1)!} \). \( \eta_1^* \) is the optimal solution to \( \min_{\eta_1} \sum_{k=1}^{n-1} b_k \mathbb{E}_{\mathbb{P}_0} \left[ (X + \eta_1)^{k+1} \right] \eta_2^k \).

Specifically, \( z(\cdot) \) is a function satisfying \( z(0) = 1 \), \( z^{(1)}(\cdot) = \frac{1}{\phi^{(2)}(z^{(1)} \cdot)} \) and \( z^{(k)}(\cdot) \) can be obtained recursively for \( k \geq 2 \).

Note that most of the \( \phi \)-divergences (KL divergence, Cressie-Read divergence, Burg entropy, J-divergence, \( \chi^2 \)-distance, modified \( \chi^2 \)-distance, and Hellinger distance) satisfy the smoothness conditions. Take KL divergence and Cressie Read divergence as example, for \( n = 4 \), we can explicitly solve the terms in Eq. (8), as shown in the following corollary.

**Corollary 4.3.** Consider the case \( n = 4 \). We have

\[
D_{\eta_2, \phi, \mathbb{P}_0}(x^T r | x^T \mu) = \sum_{k=1}^{3} b_k \mathbb{E}_{\mathbb{P}_0} \left[ (X + \eta_1^*)^{k+1} \right] \eta_2^k + o(\eta_2^3),
\]

with \( \eta_1^* \) being the real root to the 3\(^{rd} \) order equation

\[
\sum_{k=1}^{3} (k+1)b_k \eta_2^{k+1} \cdot \eta_1^* + 12b_3 \eta_2^3 \mathbb{E}_{\mathbb{P}_0}[X^2] \cdot \eta_1 + (4b_3 \eta_2^3 \mathbb{E}_{\mathbb{P}_0}[X^3] + 3b_2 \eta_2^2 \mathbb{E}_{\mathbb{P}_0}[X^2]) = 0.
\]

- For KL divergence, the coefficients are \( b_1 = \frac{1}{2} \), \( b_2 = -\frac{1}{6} \), \( b_3 = \frac{1}{24} \).
- For Cressie Read divergence with \( \theta > 2 \), the coefficients are \( b_1 = \frac{1}{2} \), \( b_2 = \frac{\theta - 2}{6} \), \( b_3 = \frac{(\theta - 2)(2\theta - 3)}{24} \).

Gotth et al. in [17] gives a similar expansion of \( D_{\eta_2, \phi, \mathbb{P}_0}(x^T r | \mathbb{E}_{\mathbb{P}_0}[x^T r]) \) in Proposition 3.5. The main difference between our expansion in Eq. (9) and the expansion given by Gotth et al. lies in the calculation of \( \eta_1^* \). In Eq. (9), \( \eta_1^* \) is directly solved through the polynomial equation, while in [17], \( \eta_1^* \) is approximated as a function of \( \eta_2 \). Our expansion is shown to be more accurate and we defer the comparison results through numerical tests to the last section.

In the sequel, we take the expansion terms up to \( o(\eta_2) \) in Eq. (8) and ignore the higher order terms, which gives

\[
\min_{\mathbb{P} \in \mathcal{U}} \mathbb{E}_{\mathbb{P}}[x^T r] \approx \mathbb{E}_{\mathbb{P}_0}[x^T r] - \min_{\eta_2 \geq 0} \left\{ \frac{\rho}{\eta_2} + \frac{\eta_2}{2\phi^{(2)}(1)} x^T \Sigma x \right\} = x^T \mu - \sqrt{\frac{2\rho x^T \Sigma x}{\phi^{(2)}(1)}}.
\]

The last equality comes as a result of \( \min_{\eta_2 \geq 0} \left\{ \frac{\rho}{\eta_2} + \frac{\eta_2}{2\phi^{(2)}(1)} x^T \Sigma x \right\} = \sqrt{\frac{2\rho x^T \Sigma x}{\phi^{(2)}(1)}} \) and the minimum is achieved at \( \eta_2 = \sqrt{\frac{2\rho x^T \Sigma x}{\phi^{(2)}(1)}} \). This suggests, when \( \rho \) is small, the optimal Lagrangian multiplier \( \eta_2 \) is also small and the expansion in (8) is accurate. By taking \( \max_{x \in \mathcal{X}} \) on both sides, we finally achieve the reformulation of OPT1 in Theorem 4.4.

**Theorem 4.4.** Suppose that \( \phi \) is convex, twice continuously differentiable, and that \( \phi(1) = \phi^{(1)}(1) = 0 \) and \( \phi^{(2)}(1) > 0 \). Given \( \mathbb{E}_{\mathbb{P}_0}[r] = \mu \) and \( \mathbb{V}_{\mathbb{P}_0}[r] = \Sigma \), the DRO problem in OPT1
We begin with the unbounded feasible set $X$ whether the two optimizations are convex problems. It can be seen that, function in the form of $\rho(11)$ and (12) to interpret the impact of the ambiguity radius $\gamma$. This suggests that OPT1' can be reformulated as

$$\max_{x \in X} \left\{ x^T \mu - \sqrt{2\rho x^T \Sigma x} \right\}.$$

(11)

Theorem 4.4 tells that the ambiguity radius $\rho$ controls the investor’s risk preference.

4.2 Reformulation of OPT1'

Notice that, the chance constraint in [OPT1] is in the same form as the definition of VaR, which focuses on the probability of losses. This motivates us to reorganize the tail chance constraint in OPT1'.

The VaR is defined as the minimal level $\gamma$ such that the probability that the portfolio loss $-x^T r$ exceeds $\gamma$ is below $\epsilon$:

$$V_\epsilon(x) := \inf \{ \gamma \in \mathbb{R} : Pr_{\sim \mathbb{P}_0} \{-x^T r \geq \gamma \} \leq \epsilon \}.$$  

The equivalent form of the chance constraint in OPT1' $Pr_{\sim \mathbb{P}_0} \{-x^T r \geq \delta \} \leq \epsilon$ implies that, $\delta$ is included in the set $\{ \gamma \in \mathbb{R} : Pr_{\sim \mathbb{P}_0} \{-x^T r \geq \gamma \} \leq \epsilon \}$. That is to say, the chance constraint can be reorganized with $V_\epsilon(x)$, namely,

$$Pr_{\sim \mathbb{P}_0} \{-x^T r \geq \delta \} \leq \epsilon \iff V_\epsilon(x) \leq \delta.$$  

Hence, given $E_{\mathbb{P}_0}[x^T r] = x^T \mu$, OPT1' can be reformulated as

$$\max_{x \in X} x^T \mu \quad \text{s.t.} \quad V_\epsilon(x) \leq \delta.$$  

If $\mathbb{P}_0$ is a normal distribution, then

$$V_\epsilon(x) = \kappa(\epsilon) \sqrt{x^T \Sigma x - x^T \mu},$$

where $\kappa(\epsilon) = -\Phi^{-1}(\epsilon)$. For general elliptical distributions, Lesniewski et al. in [24] gave an asymptotic expansion of $V_\epsilon(x)$, which takes the form $\kappa(\epsilon) \sqrt{x^T \Sigma x - x^T \mu}$ asymptotically when $\epsilon \to 0$. For example, if $\mathbb{P}_0$ is student $t$-distribution with degree of freedom parameter $\nu$, then

$$\kappa(\epsilon) = De^{-\frac{1}{2}},$$

where $D = \left( \frac{c_d \nu^{\frac{d+1}{2}} \Gamma\left(\frac{\nu+1}{2}\right)}{\pi^{\frac{d}{2}} \Gamma\left(\frac{\nu}{2}\right)} \right)^{\frac{1}{2}}$ and $c_d = \frac{I\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} \nu^\frac{\nu}{2} \Gamma\left(\frac{\nu}{2}\right)$. For distributions other than elliptical distributions, [13] suggests an approximation of $\kappa(\epsilon)$: $\kappa(\epsilon) = \sqrt{1-\epsilon}$. This suggests that OPT1 can be reformulated as

$$\max_{x \in X} x^T \mu \quad \text{s.t.} \quad \kappa(\epsilon) \sqrt{x^T \Sigma x - x^T \mu} \leq \delta.$$  

(12)

4.3 Solutions

It has been shown that OPT1 and OPT1' can be reformulated to a deterministic mean-deviation problem in Eq. (11) and a linear optimization with a second-order cone constraint in Eq. (12), respectively. In this section, we look into the optimal solution and the optimal value to optimizations (11) and (12) to interpret the impact of the ambiguity radius $\rho$. In the later context, we would use $x^*$ and $v^*$ to denote respectively the optimal solution and optimal value to optimization (11). And the optimal solution and optimal value to optimization (12) are denoted by $x^*$ and $v^*$, respectively.

We begin with the unbounded feasible set $X := \{ x \in \mathbb{R}^n \mid x^T e = 1 \}$. We need to first verify whether the two optimizations are convex problems. It can be seen that, function in the form of $\alpha \sqrt{x^T \Sigma x} - x^T \mu$ appears in both the objective function of optimization (11) and constraint of optimization (12). We prove in Lemma 4.3 that this function is convex in the decision variable $x$.  

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Lemma 4.5. Given $a > 0$, the function $a\sqrt{x^T \Sigma x - x^T \mu}$ is a convex function.

As a result, problem (11) is a convex optimization and problem (12) is a convex optimization only when $\kappa(\epsilon) > 0$. Recall that in a convex optimization, any local optimum is also a global optimum. This motivates us to study the optimal solution to problem (11), $x^*$, and the optimal solution to problem (12), $\tilde{x}^*$, through KKT conditions. The results for $(x^*, v^*)$ and $(\tilde{x}^*, \tilde{v}^*)$ are summarized in Theorem 4.6 and Theorem 4.7, respectively.

Theorem 4.6. Suppose $\phi^{(2)}(1) > 0$. Define $A := e^T \Sigma^{-1} e$, $B := \mu^T \Sigma^{-1} e$, and $C := \mu^T \Sigma^{-1} \mu$. Then for optimization (11) with the feasibility set $\mathcal{X} := \{x \in \mathbb{R}^n | x^T e = 1\}$, we have:

- If $\rho > \frac{\phi^{(2)}(1)}{2}(C - \frac{B^2}{A})$, then there is a local optimal solution for problem (11) at

$$x^* = \frac{\Sigma^{-1}(\mu - \lambda^* e)}{\mu^T \Sigma^{-1} e - \lambda^* e^T \Sigma^{-1} e},$$

with the optimal value

$$v^* = \lambda^*,$$

where $\lambda^* = \frac{B - 2\sqrt{A^2 - BC}}{A}.\]

- If $\rho \leq \frac{\phi^{(2)}(1)}{2}(C - \frac{B^2}{A})$, then no local optimal solution for problem (11) and the optimal value $v^* = +\infty$.

Notice that, for the original formulation of [OPT1] with an unbounded feasibility set, the optimal value is $+\infty$ when there is no distributional uncertainty (equivalently, $\rho = 0$). Thus, Theorem 4.6 establishes a threshold for the ambiguity radius $\rho$, beyond which the distributional uncertainty is indeed effective.

Theorem 4.7. Given $A$, $B$, and $C$ defined in Theorem 4.6. Suppose $\kappa(\epsilon) > 0$, and then for optimization (12) with the feasibility set $\mathcal{X} := \{x | x^T e = 1\}$, the following conclusions hold:

- If $(\epsilon, \delta)$ satisfies $\frac{AC - B^2}{A} < (\kappa(\epsilon))^2 < \delta^2 A + 2\delta B + C$ and $B + \delta A > 0$, then there is a local optimal solution at

$$\tilde{x}^* = \frac{\Sigma^{-1}(1 + \tilde{\lambda})\mu - \tilde{\theta} e}{\mu^T \Sigma^{-1} e - \lambda^* e^T \Sigma^{-1} e},$$

with the finite optimal value

$$\tilde{v}^* = \tilde{\lambda} \delta + \tilde{\theta},$$

where $\tilde{\lambda} = \frac{AC - B^2}{A \kappa(\epsilon)^2 - AC + B^2} + \frac{\kappa(\epsilon)(B + A\delta)\sqrt{AC - B^2}}{(A \kappa(\epsilon)^2 - AC + B^2)\sqrt{A^2 + 2B\delta + C - \kappa(\epsilon)^2}}$ and $\tilde{\theta} = \frac{(AC - B^2)(\tilde{\lambda} + 1) - \kappa(\epsilon)\epsilon^2}{B + A \delta + \kappa(\epsilon)\epsilon^2}$. Furthermore, we can verify that when the optimal value $v^*$ for problem (11) is finite, $\tilde{v}^* \geq v^*$.

- If $B > 0$ and if $(\epsilon, \delta)$ satisfies $\frac{AC - B^2}{A} < (\kappa(\epsilon))^2 < \delta^2 A + 2\delta B + C$ and $B + \delta A < 0$, then the feasibility set for problem (12) is $\emptyset$.

- If $(\epsilon, \delta)$ satisfies $(\kappa(\epsilon))^2 \leq C - \frac{B^2}{A}$, then there exists no local optimal solution for problem (12) and the optimal value $\tilde{v}^* = +\infty$.

In Theorem 4.7, we first provide the sufficient and necessary conditions of $(\delta, \epsilon)$ for the optimization problem in Eq. (12) to have a finite optimal value. We further compare the finite optimal values to problem (12) and to problem (11), $\tilde{v}^*$ and $v^*$, and find that the CCO reformulation performs uniformly better than the DRO reformulation. In addition, we identify one sufficient condition under which optimization (12) is infeasible. We also identify one sufficient condition for the chance constraint to be redundant in optimization (12), and the optimal value tends to be positive infinity.
5 Experiments

In Section 4 we have analytically proven that for an unbounded strategy set, the CCO reformulation uniformly performs better than the DRO reformulation when the optimal values are both finite. For a bounded strategy set, theoretical analysis is difficult, so we resort to numerical analysis. In Section 5.1 we numerically test the reformulation accuracy of optimization (11) to the original robust problem OPT1. In Section 5.2 we conduct experiments to understand the ambiguity radius $\rho$ via the chance constraints and find that the tail heaviness indeed affects the interpretation of $\rho$. In Section 5.3 we establish the relationship between the ambiguity radius $\rho$ and the sample size $N$. In Section 5.4 we test the financial interpretation of the ambiguity radius based on empirical data.

5.1 Reformulation accuracy of optimization (11) to OPT1

In this section, we test the reformulation accuracy of the optimization (11) numerically with respect to the original robust problem in OPT1. The experiment is conducted with 5 assets under a bounded nonnegative set of allocation strategies $\mathcal{X} = \{x \in \mathbb{R}^n \mid x^T e = 1, \ x \geq 0\}$. We consider two types of nominal distributions $\mathcal{P}_0$: a multivariate normal distribution (abbreviated by Normal) and a joint distribution of 5 independent exponential distributions (abbreviated by Exp.). The two joint distributions share the same expectation parameters, where the maximal expected return of a single asset is no larger than 20%. The $\phi$-divergence we take in this experiment is KL divergence and the ambiguity radius $\rho$ ranges from 0.001 to 0.1. Under KL divergence, the robust optimization in OPT1 can be exactly solved. For problem (11), it is convex and can thus be easily solved globally using interior-point techniques for convex second-order cone programing.

Table 2 records the norm difference of optimal solutions and the absolute difference of optimal values to the robust problem in OPT1 and its reformulation in problem (11). Here, similar to the notations of $(x^*, v^*)$ for problem (11), we use $(x_{opt1}, v_{opt1})$ to denote the optimal solution and optimal value for OPT1. In Table 2 we summarize the results of 10 cases of $\rho$.

| $\rho$ | 0.006 | 0.016 | 0.026 | 0.036 | 0.046 | 0.056 | 0.066 | 0.076 | 0.086 | 0.096 |
|-------|------|------|------|------|------|------|------|------|------|------|
| $\|x_{opt1} - x^*\|_2$ Nornal (e-3) | 0.03 | 0.005 | 0.04 | 0.03 | 0.03 | 0.02 | 0.01 | 0.03 | 0.04 | 0.05 |
| Ex (e-3) | 54.39 | 55.45 | 50.80 | 48.70 | 47.52 | 46.71 | 46.14 | 45.71 | 45.37 | 45.08 |
| $\|v_{opt1} - v^*\|_2$ Nornal (e-6) | 0.16 | 0.08 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.08 | 0.80 |
| Ex (e-6) | 824.59 | 1178.64 | 1602.59 | 2037.89 | 2478.83 | 2919.90 | 3362.38 | 3805.02 | 4247.57 | 4689.87 |

From Table 2 we have two observations. The first observation is that the reformulation is rather accurate for multivariate normal distributions compared to exponential distributions. This is within expectation since the skewness of normal distribution is 0, which eliminates the effect of higher order terms in the expansion of regular measure of deviation. The second observation is for distributions with non-zero higher order moments: the reformulation accuracy decays as the ambiguity radius $\rho$ increases. So, under small values of $\rho$, problem (11) is in general a good reformulation to OPT1.

We also compare the expansion under different orders with OPT1. The results are summarized in Table 3. It shows that the higher order improvement is particularly notable when data exhibits a heavier tail than normal. Here, we assume the ambiguity set under the KL divergence so that the nominal distribution $\mathcal{P}_0$ centers at a six-dimensional multivariate exponential distribution with mean= 20%, std= 20%, skewness= 2, and kurtosis= 6. We set the dimensions to be i.i.d to see a clean impact from the heavy tail. The 3rd & 4th row reports relative errors w.r.t the exact optimal value (the 2nd row). We see that the larger the size of the ambiguity set, i.e. large $\rho$, the better the improvement. In fact, the error reduction is about 10 folds in this example. However, using 2nd order equivalent formulation is good enough to study relations between OPT1 and OPT1 when $\rho$ is small.
We read from Figure 1(a) that, first, the ambiguity radius \( N \) represents the maximum radius of the ambiguity set that can achieve the same optimal value as optimization (11). For all the three distributions, \( N = 20\% \) is equivalent to that of problem (12) under the same distribution reformulation. We focus on three distributions for 5 assets: multivariate normal distribution, multivariate lognormal distribution, and student \( t_3 \)-distribution. The set of allocation strategies is bounded \( \mathcal{X} = \{ x \in \mathbb{R}^n \mid x^T e = 1, x \geq -5 \} \). We say that the ambiguity radius \( \rho \) can be explained by a chance constraint with parameters \((\delta, \epsilon)\) if the optimal value of problem (11) is equal to that of problem (12) under the same distribution \( \mathbb{P}_0 \). So the experiment is conducted in the following way: first solve optimization (11), and then solve optimization (12) for the pair of \((\delta, \epsilon)\) that can achieve the same optimal value as optimization (11). For all the three distributions, \( \rho \) is the same and fixed at 0.27. We plot the results of equivalent \((\delta, \epsilon)\) in Figure 1(a).

We read from Figure 1(a) that, first, the ambiguity radius \( \rho \) is equivalent to a set of pairs \((\delta, \epsilon)\) in terms of the impact on the optimal value. Second, tail heaviness affects the interpretation of \( \rho \). Distributions with heavier tail result in a larger loss threshold for a given loss probability \( \epsilon \). Furthermore, for given \( \rho \), the loss probability should be lower for a larger loss threshold.

**5.2 Interpretation of \( \rho \) under distributions with different tail heavinesses**

This experiment shows that tail heaviness of the nominal distribution \( \mathbb{P}_0 \) can indeed affect the interpretation of the ambiguity radius \( \rho \). We focus on three distributions for 5 assets: multivariate normal, multivariate lognormal, and student \( t_3 \)-distribution. The set of allocation strategies is bounded \( \mathcal{X} = \{ x \in \mathbb{R}^n \mid x^T e = 1, x \geq -5 \} \). We say that the ambiguity radius \( \rho \) can be explained by a chance constraint with parameters \((\delta, \epsilon)\) if the optimal value of problem (11) is equal to that of problem (12) under the same distribution \( \mathbb{P}_0 \). So the experiment is conducted in the following way: first solve optimization (11), and then solve optimization (12) for the pair of \((\delta, \epsilon)\) that can achieve the same optimal value as optimization (11). For all the three distributions, \( \rho \) is the same and fixed at 0.27. We plot the results of equivalent \((\delta, \epsilon)\) in Figure 1(a).

We read from Figure 1(a) that, first, the ambiguity radius \( \rho \) is equivalent to a set of pairs \((\delta, \epsilon)\) in terms of the impact on the optimal value. Second, tail heaviness affects the interpretation of \( \rho \). Distributions with heavier tail result in a larger loss threshold for a given loss probability \( \epsilon \). Furthermore, for given \( \rho \), the loss probability should be lower for a larger loss threshold.

![Figure 1](a) (b)

**Figure 1**: (a): Given \( \rho = 0.27 \), tail heaviness affects the equivalent loss threshold \( \delta \). (b) The corresponding optimal \( \rho \) varies with the sample size \( N \), with the distribution form of \( \mathbb{P}_0 \), and with the parameters \((\delta, \epsilon)\).

**5.3 Relative error of \( \rho \) with respect to the sample size \( N \)**

In this section, we numerically test the relationship of the ambiguity radius \( \rho \) with sample size \( N \), and with \( \mathbb{P}_0 \) in different tail heavinesses. In financial markets, available data is always limited. The largest window of daily financial data spans at most 40 years, which incorporates roughly \( 10^4 \) data points. Not least to say the emerging markets, where reliable data source only consists of \( 10^2 \) or at most \( 10^3 \) data points. Limited data aggravates the error in estimating the nominal distribution \( \mathbb{P}_0 \) and imposes a difficulty on the suitable choice of ambiguity radius \( \rho \). In this section, we design an experiment to test the error of \( \rho \) with respect to sample size \( N \). Also, we look into the effect of tail heaviness on the magnitude of \( \rho \).

The experiments are as follows. First, with given \( N \) samples, we numerically fit the mean and covariance with a given distribution form of \( \mathbb{P}_0 \). In this experiment, 5 assets and two distribution forms

![Table 3](Relative error (abbreviated as R.E. in the table) of expansion under different orders with \( \nu_{opt1} \). The nominal distribution is a six-dimensional multivariate exponential distribution with mean \( \mu = 20\% \), std \( = 20\% \), skewness \( = 6 \), and kurtosis \( = 6 \) so that each dimension is i.i.d. and \( \mathcal{X} = \{ x \mid x^T e = 1, x \geq -1 \} \).]

| \( \nu_{opt1} \) | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| R.E. of \( N = 4 \) reformulation | 0.0002% | 0.00038% | 0.00128% | 0.00274% | 0.00479% | 0.00748% | 0.00948% | 0.01082% | 0.01483% | 0.01951% |
| R.E. of \( N = 6 \) reformulation | 0.1112% | 0.2397% | 0.3859% | 0.4951% | 0.5220% | 0.5613% | 0.5979% | 0.637% | 0.778% |

- Normal, \( (\delta = 5\%, \epsilon = 8\%) \)
- Lognormal, \( (\delta = 20\%, \epsilon = 3\%) \)
- Student, \( (\delta = 5\%, \epsilon = 8\%) \)
- Lognormal, \( (\delta = 20\%, \epsilon = 3\%) \)
are considered: multivariate normal distribution (light tail) and multivariate lognormal distribution (heavy tail). Then, with specified \((\delta, \epsilon)\), we solve optimization (12) for different \(P_0\) and take the corresponding optimal value as the benchmark for the optimal value of optimization (11) with the same \(P_0\). Finally, we solve the optimal \(\rho\) that makes problem (11) achieve the optimal value equal to the benchmark. So the optimal \(\rho\) is highly related to the distribution \(P_0\), and to the sample size \(N\). We plot the results of \(\rho\) under two pairs of \((\delta, \epsilon)\) and under two distributions in Figure 1(b). In addition, we record the optimal \(\rho\) and its relative error under fixed \(\epsilon = 8\%\) and ranging \(\delta\) from 5\% to 35\% in Table 4.

Table 4: Given \(\epsilon = 8\%\), the corresponding optimal \(\rho\) under different \(\delta\), different sample size \(N\), and different distribution forms of \(P_0\) (normal or lognormal). The bold numbers display the relative error to the \(\rho\) with \(N = 10^6\).

|                | Normal | Lognormal |
|----------------|--------|-----------|
|                | \(N = 250\) | \(2.5 \times 10^3\) | \(1 \times 10^4\) | \(10^6\) | \(N = 250\) | \(2.5 \times 10^3\) | \(1 \times 10^4\) | \(10^6\) |
| \(\delta = 5\%\) | 0.267 | 0.284 | 0.277 | 0.276 | 0.357 | 0.293 | 0.297 | 0.301 |
| \(3.6\%\) | 2.90\% | 0.36\% | 0 | 0 | 18.60\% | 2.66\% | 1.33\% | 0 |
| \(\delta = 15\%\) | 0.215 | 0.232 | 0.226 | 0.225 | 0.31 | 0.275 | 0.279 | 0.283 |
| \(4.44\%\) | 3.11\% | 0.44\% | 0 | 0 | 9.54\% | 2.83\% | 1.41\% | 0 |
| \(\delta = 25\%\) | 0.168 | 0.185 | 0.18 | 0.18 | 0.32 | 0.262 | 0.266 | 0.271 |
| \(6.67\%\) | 2.78\% | 0 | 0 | 0 | 18.08\% | 3.32\% | 1.85\% | 0 |
| \(\delta = 35\%\) | 0.127 | 0.144 | 0.141 | 0.141 | 0.304 | 0.251 | 0.254 | 0.258 |
| \(9.93\%\) | 2.13\% | 0 | 0 | 0 | 17.83\% | 2.71\% | 1.55\% | 0 |

Figure 1(b) and Table 4 tell that, the relative error of \(\rho\) gets smaller as the sample size \(N\) increases. Although the general trend of decreasing \(\rho\) as the loss threshold \(\delta\) increases, the relative error of \(\rho\) increases. Furthermore, we see that, under a heavy tail distribution, the optimal ambiguity radius is larger than its light tail counterpart to achieve the corresponding optimal return with the same chance constraint.

5.4 Empirical studies

To see more clearly the financial interpretation of the ambiguity radius \(\rho\), we undergo experiments based on empirical data. We extract past 40 years’ daily simple returns of four major asset classes: Equity indexes (DAX, FTSE, HSI, NASDAQ, NIKKEI250, SP500), US Treasuries (2year, 10year, 30year), Currencies (AUD/USD, CHF/USD, EUR/USD, GBP/USD, JPY/USD) and Commodities (Crude oil, Silver, Gold). For the DRO problem, we use the Cressie-Read divergence instead of KL divergence since all data exhibits quite heavy tail. For the CCO problem, we choose the negative daily return threshold \(-\delta\) to be the 1\%, 3\% and 5\% empirical quantiles of the daily simply return series for a given asset class rather than fixed values so that they can differ across assets. We choose the chance level \(\epsilon\) to be 2\%, 5\% and 20\%, mincing (rounded) event frequencies at quarterly (4 out of 252), monthly (12 out of 252) and weekly (52 out of 252) so that investors can relate \(\epsilon\) to the degree of event rareness. The portfolio weights are constrained to be bounded below by \(-100\%\) so that shorting is allowed. Both multivariate \(t\)- and normal distribution are tested when fitting data to obtain the center \(P_0\) of the ambiguity set \(\mathcal{U}\).

Table 5 reports the equivalent ambiguity radius \(\rho\) of the DRO problem, together with the corresponding optimal portfolio return (annualized), at a given pair of the CCO parameters \(-\delta\) (the threshold of negative portfolio return investors hate to go below) and \(\epsilon\) (the probability investors can tolerate such undesirable outcomes), i.e. those in OPT1'. By equivalent, we mean the optimal portfolio returns of the two problems are equal. To read the table, use equity as an example, when the CCO parameters are \((\epsilon = 2\%, -\delta = -2.68\%)\), the equivalent ambiguity radius \(\rho\) in the DRO problem is at top left \(\rho_t=1.4e-3\) when we fit \(P_0\) to data assuming it’s multivariate \(t\)-distributed. By contrast, the top right entry \(\rho_n=2.4e-4\) corresponds to estimating \(P_0\) by assuming data is multivariate normal distributed. The annualized optimal portfolio returns are \(R_t = 24.4\%\) (bottom left) for the \(t\)-assumption and \(R_n = 11.7\%\) (bottom right) for the normal assumption. We use bold fonts to highlight the winners.

We read from Table 5 that, the DRO problem is highly sensitive to the size of the ambiguity set \(\mathcal{U}\) and yet it’s not financially interpretable. By relating it to the CCO chance parameters, it then becomes
tangible, without which even the appropriate order is hard to guess. In our tests, its magnitude can range from $10^{-3}$ to $10^{-15}$ depending on asset classes and on the investor’s tolerance level. What’s more, the heavy tail nature of financial data demands the usage of divergence measures such as the Cressie-Read divergence that allow heavy tail distribution if one takes the robust approach for portfolio optimization. Ambiguity sets constructed by the KL divergence, however, require the objective function to be exponentially bounded, which exclude important heavy tail distributions used ubiquitously for financial asset returns, e.g. the student $t$-distribution. Among the 36 tests in the above table, 29 favors fitting data by assuming $\mathbb{P}_0$ is multivariate $t$-distributed.

### Table 5

| $(\rho_c, \rho_s) = (R_t, R_s)$ | Equity: 1988Jan04 - 2018Jun29 | Bond: 1988Feb29 - 2018Jul13 | FX: 1975Jan03 - 2018Jul09 | Commodity: 1983Apr04 - 2018Jul13 |
|-----------------------------|-------------------------------|-----------------------------|-----------------------------|----------------------------------|
| $\delta = 2\%$             | 1.4e-3, 1.4e-4, 1.1e-3, 1.9e-4 | 6.4e-4, 1.1e-4, 3.5e-4, 1.3e-4, 1.4e-14 | 7.7e-6, 1e-14, 8.6e-4, 1.2e-4, 3.3e-4, 3.6e-5 | 3e-4, 4.3e-5, 2.8e-4, 4.9e-9, 3e-4, 2.3e-9 |
| $\delta = 5\%$             | 1e-4, 1.8e-4, 7.3e-4, 1.8e-4, 3.1e-4, 5.6e-5 | 9.4e-6, 1e-14, 5.2e-6, 1e-14, 1.9e-7, 4e-10 | 4.9e-4, 8e-5, 3.3e-4, 5.3e-5, 1.5e-4, 1.9e-5, 1e-4, 3.1e-9, 5.5e-5, 1.4e-9 |
| $\delta = 20\%$            | 5.2e-4, 3.3e-4, 2.6e-4, 1.6e-4, 5.9e-5, 2.6e-5, 1e-10, 1.3e-10, 3.1e-10, 1e-14, 4.1e-10 | 2e-4, 5.1e-6, 8e-5, 1.1e-7, 7.5e-5, 3.8e-10, 1.2e-5, 2.1e-10 |
|                            | 4.62%, 32.7%, 57.0%, 40.1%, 75.3%, 49.1% | 3.2%, -2.6%, 3.2%, -2.6%, 3.2%, -2.6% | 6.2%, 8.4%, 8.2%, 9.6%, 12.2%, 10.5%, 27.7%, 4.6%, 31.8%, 4.7%, 41.2%, 4.7% |

### 6 Conclusions

We delved into the ambiguity radius for DRO problems with a distribution ambiguity set controlled by $\phi$-divergence. We showed that for general $\phi$-divergences, a DRO portfolio optimization problem is asymptotically equivalent to a mean-deviation problem, where the ambiguity radius controls an investor’s risk preference parameter. Theoretical analysis over the mean-deviation problem establishes a threshold for the ambiguity radius, across which the optimal value suffers from a drastic phase transition. It is only beyond that radius threshold can the distributional uncertainty take effect. We also showed both numerically and theoretically that, when the investment strategy is bounded, the ambiguity radius can be cast as a chance constraint in a deterministic optimization with the same objective. Otherwise, within the set of unbounded investment strategies, a chance-constrained deterministic optimization consistently performs better than the DRO problem.

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Appendix

This section includes all the proofs of lemmas and theorems appearing in the main paper.

We first introduce the definition of Regular Measure of Deviation, which is useful in the proof of Theorem 4.1.

**Definition 1.** Given any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $L^2(\Omega)$ denote the space of square-integrable random variables, i.e., $\mathbb{E}_\mathbb{P}[X^2] < \infty$. A functional $D : L^2(\Omega) \to [0, \infty]$ is said to be a regular measure of deviation if it is closed convex and satisfies:

1. $D(c) = 0$ for any constant $c \in \mathbb{R}$;
2. $D(Z) > 0$ for any (non-constant) random variable $Z \in L^2(\Omega)$.

**Proof of Theorem 4.1** We need to check that the quantity $D_{\eta_2, \phi, \rho_0}(x^T r | \mathbb{E}_{\rho_0}[x^T r])$ defined in Theorem 4.1 satisfies the conditions stated in definition 1. It is easy and is omitted.

Write $\min_{x \in \mathcal{U}} \mathbb{E}_\mathbb{P}[x^T r]$ as

$$\min_{x} \mathbb{E}_\mathbb{P}[x^T r]$$

subject to $D(\mathbb{P}) \leq \rho$. (A.1)

Applying changing of measure, it is equivalent to solving

$$\min_{L} \mathbb{E}_{\rho_0} \left[ (x^T r)L \right]$$

subject to $\mathbb{E}_{\rho_0}[\phi(L)] \leq \rho$, $\mathbb{E}_{\rho_0}[L] = 1$. (A.2)

Under mild conditions to make strong duality hold, (A.2) is equivalent to solving its dual:

$$\max_{\eta_1, \eta_2 \geq 0} \min_{L} \{ \mathbb{E}_{\rho_0}[x^T r] L + \eta_2 (\mathbb{E}_{\rho_0}[\phi(L)] - \rho) + \eta_1 (\mathbb{E}_{\rho_0}[L] - 1) \}.$$ (A.3)

Now,

$$\max_{\eta_1, \eta_2 \geq 0} \min_{L} \{ \mathbb{E}_{\rho_0}[x^T r] L + \eta_2 (\mathbb{E}_{\rho_0}[\phi(L)] - \rho) + \eta_1 (\mathbb{E}_{\rho_0}[L] - 1) \}$$

$$= \max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 + \eta_2 \min_{L} \{ \mathbb{E}_{\rho_0}[x^T r] L + \eta_2 \mathbb{E}_{\rho_0}[L \phi(L)] + \eta_1 \mathbb{E}_{\rho_0}[L] \}$$

$$= \max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 + \eta_2 \min_{L} \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L \right] + \mathbb{E}_{\rho_0}[\phi(L)] \right]$$

$$= \max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 - \eta_2 \min_{L} \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L \right] - \mathbb{E}_{\rho_0}[\phi(L)] \right]$$.

We apply Result 2 so that the maximization sign ‘max’ and the expectation sign $\mathbb{E}$ can be interchanged, giving

$$\max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 - \eta_2 \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right]$$

Now,

$$\max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 - \eta_2 \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right]$$

$$= \max_{\eta_1, \eta_2 \geq 0} -\eta_2 \rho - \eta_1 - \eta_2 \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right]$$

$$= -\min_{\eta_1, \eta_2 \geq 0} \eta_2 \rho + \eta_1 + \eta_2 \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right]$$

$$= -\min_{\eta_1, \eta_2 \geq 0} \eta_2 \rho + \eta_1 + \eta_2 \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right]$$

$$= -\min_{\eta_2 \geq 0} \eta_2 \rho + \min_{\eta_1} \left( \eta_1 + \frac{1}{\eta_2} \mathbb{E}_{\rho_0} \left[ \frac{(x^T r + \eta_1)}{\eta_2} L - \phi(L) \right] \right)$$

$$= \min_{\eta_2 \geq 0} \left( \frac{\rho}{\eta_2} + \mathbb{D}_{\eta_2, \phi, \rho}(x^T r | \mathbb{E}_{\rho_0}[x^T r]) - \mathbb{E}_{\rho_0}[x^T r] \right)$$

$$= \mathbb{E}_{\rho_0}[x^T r] - \min_{\eta_2 \geq 0} \left( \frac{\rho}{\eta_2} + \mathbb{D}_{\eta_2, \phi, \rho}(x^T r | \mathbb{E}_{\rho_0}[x^T r]) - \mathbb{E}_{\rho_0}[x^T r] \right).$$
The second last equality holds, since if we start from the definition of \( D_{\eta_2, \phi, \mu_0}(x^T r|E_{\mu_0}[x^T r]) \) and let \( \bar{\eta}_1 = \eta_1 - E_\mu[x^T r] \), we have

\[
D_{\eta_2, \phi, \mu_0}(x^T r|E_{\mu_0}[x^T r]) := \min_{\eta_1} \left\{ \eta_1 + \frac{1}{\eta_2} E_{\mu_0} \left[ \phi^* \left( \eta_2(E_{\mu_0}[x^T r] - x^T r - \eta_1) \right) \right] \right\}
\]

\[
= \min_{\eta_1} \left\{ \bar{\eta}_1 + \frac{1}{\eta_2} E_{\mu_0} \left[ \phi^* \left( \eta_2(-x^T r - \bar{\eta}_1) \right) \right] + E_{\mu_0}[x^T r] \right\}.
\]

\( \square \)

**Result 2.** [2], [27] Let \( \Omega \) be a \( \sigma \)-finite measure space, and let \( \mathcal{Y} := L^p(\Omega, \mathcal{F}, \mathbb{P}), p \in [1, +\infty] \). Let \( g : \mathbb{R} \times \Omega \rightarrow (-\infty, +\infty) \) be a normal integrand, and define on \( \mathcal{Y} \) the integral functional

\[
I_g(y) := \int_{\Omega} g(y(\omega), \omega) \mathbb{P}(d\omega).
\]

Then,

\[
\inf_{x \in \mathcal{Y}} \int_{\Omega} g(y(\omega), \omega) \mathbb{P}(d\omega) = \int_{\Omega} \inf_{s \in \mathbb{R}} g(s, \omega) \mathbb{P}(d\omega),
\]

provided that the left-hand side is finite. Moreover,

\[
\bar{y} \in \arg \min_{y \in \mathcal{Y}} I_g(y) = \bar{y}(\omega) \in \arg \min_{s \in \mathbb{R}} g(s, \omega), \text{ a.e. on } \omega \in \Omega.
\]

**Proof of Result** [2] The proof can be found in Theorem 14.60 of [28]. \( \square \)

**Proof of Lemma** [4,2] Denoting \( X = x^T r - E_\mu[x^T r] \) and using Taylor expansion on \( \phi^*(\cdot) \) around 0, we recall that

\[
D_{\eta_2, \phi, \mu_0}(x^T r|X_\mu) = \min_{\eta_1} \left\{ \eta_1 + \frac{1}{\eta_2} E_{\mu_0} \left[ \phi^* \left( \eta_2(E_{\mu_0}[x^T r] - x^T r - \eta_1) \right) \right] \right\}
\]

\[
= \min_{\eta_1} \left\{ \eta_1 + \frac{1}{\eta_2} E_{\mu_0} \left[ \phi^* \left( \eta_2(-X - \eta_1) \right) \right] \right\}.
\]

Now, considering the expansion of \( \phi^*(\eta_2(-X - \eta_1)) \) up to order \( N \), which gives

\[
\sum_{k=1}^{n} \frac{n!}{k!} \left( \phi^* \right)^{(k)}(0) (X + \eta_1)^k
\]

\[
= \sum_{k=1}^{n} \frac{1}{\eta_2^k} \left( \phi^* \right)^{(k+1)}(0) \eta_1^{k+1}
\]

It remains to compute \( \left( \phi^* \right)^{(k+1)}(\xi) \). We start from the definition of \( \phi^*(\xi) = \sup_{z \geq 0} \{ z \xi - \phi(z) \} \). Let \( z(\xi) \) be the optimizer, then we obtain \( \phi^*(\xi) = z(\xi) \xi - \phi(z(\xi)) \). Differentiating it gives

\( (\phi^*)^{(1)}(\xi) = z^{(1)}(\xi) \xi + z(\xi) - \phi^{(1)}(z(\xi))z^{(1)}(\xi) = z^{(1)}(\xi) \). The last equality follows from the first order optimality condition, which gives that \( \xi - \phi^{(1)}(z(\xi)) = 0 \). As a result, \( (\phi^*)^{(k+1)}(\xi) = z^{(k)}(\xi) \) and it suffices to find \( z^{(k)}(\xi) \).

The term \( z^{(k)}(\xi) \) can be found from the first order optimality condition. If we differentiate it on both sides w.r.t. \( \xi \), we obtain \( \phi^{(2)}(z(\xi))z^{(1)}(\xi) = 1 \). This means that \( z^{(1)}(\xi) = \frac{1}{\phi^{(2)}(z(\xi))} \) and hence \( z^{(k)}(\xi) = \left[ \frac{1}{\phi^{(2)}(z(\xi))} \right]^{(k-1)} \) for \( k \geq 2 \). Denoting \( \eta_1^* \) be the optimal solution to

\[
\min_{\eta_1} \sum_{k=1}^{n} b_k E_{\mu_0}[X^k + \eta_1^{k+1}] \eta_1^k.
\]

We then show that \( z(0) = 1 \). From the definition of \( \phi^*(\xi) \), we know that \( \phi^*(0) = 0 \) so that \( \phi(z(0)) = 0 \). If \( z(0) \neq 1 \), then \( \phi(x) = 0 \) for any \( x \) lying between 1 and \( a \) according to the definition of \( \phi \). This means that \( \phi^{(2)}(x) = 0 \) for any \( x \in (1, a) \) and \( x \) is a point lying between 1 and \( a \). Note that \( \phi^{(2)}(1) \neq 0 \) and \( \phi \) is \( \infty \)-differentiable function, we can find a point \( \bar{x} \) lying in between 1 and \( a \) (indeed, close to 1) such that \( \phi^{(2)}(\bar{x}) \neq 0 \), contradiction arises. The proof of Lemma is completed. \( \square \)
Proof of Corollary 4.3} Finally, we prove for the case when $\phi$-divergence is chosen as Cressie Read divergence with $\theta > 2$. The case when $\phi$-divergence is chosen as KL divergence is similar and is omitted. First, we compute the first four derivatives of $\phi$-divergence when it is a Cressie Read divergence, which gives

$$\phi^{(1)}(t) = \frac{1 - t^\theta}{1 - \theta}, \quad \phi^{(2)}(t) = t^{\theta - 2}, \quad \phi^{(3)}(t) = (\theta - 2)t^{\theta - 3}, \quad \phi^{(4)}(t) = (\theta - 2)(\theta - 3)t^{\theta - 4}.$$  

We need to compute $z^{(1)}(\xi)$, $z^{(2)}(\xi)$ and $z^{(3)}(\xi)$. We obtain

$$z^{(1)}(\xi) = \frac{1}{\phi^{(2)}(z(\xi))} \left[ g(z(\xi)) \right]^{(1)}[z^{(1)}(\xi)]$$

$$z^{(2)}(\xi) = -\frac{\phi^{(3)}(z(\xi))}{[\phi^{(2)}(z(\xi))]^2} \left[ g(z(\xi)) \right]^{(2)}[z^{(2)}(\xi)]$$

$$z^{(3)}(\xi) = -\frac{\phi^{(3)}(z(\xi))}{[\phi^{(2)}(z(\xi))]^3} \left[ g(z(\xi)) \right]^{(3)}[z^{(3)}(\xi)]$$

Hence, we obtain

$$z^{(1)}(0) = 1, \quad z^{(2)}(0) = -\theta, \quad z^{(3)}(0) = (\theta - 2)(2\theta - 3)$$

This means that

$$D_{\eta_2, \phi_{P_0}}(x^T r | x^T \mu) = \min_{\eta_1} \left\{ \frac{\eta_2 E_{P_0} \left[ (X + \eta_1)^2 \right]}{2!} + \frac{\eta_2^2 (\theta - 2) E_{P_0} \left[ (X + \eta_1)^3 \right]}{3!} + \frac{\eta_2^3 (\theta - 2)(2\theta - 3) E_{P_0} \left[ (X + \eta_1)^4 \right]}{4!} \right\}$$

We need to show that the local optimizer $\eta_1$ of (A.6) is a root of a third order polynomial equation. Indeed, the objective function is a polynomial of $\eta_1$ with degree 4, which can be rewritten as $a_0 + a_1 \eta_1 + a_2 \eta_1^2 + a_3 \eta_1^3 + a_4 \eta_1^4$ where

$$a_0 = \frac{\eta_2 E_{P_0} \left[ X^2 \right]}{2} + \frac{(\theta - 2) \eta_2^2 E_{P_0} \left[ X^3 \right]}{6} + \frac{(\theta - 2)(2\theta - 3) \eta_2^3 E_{P_0} \left[ X^4 \right]}{24}$$

$$a_1 = \frac{(\theta - 2) \eta_2 E_{P_0} \left[ 3X^2 \right]}{6} + \frac{(\theta - 2)(2\theta - 3) \eta_2^3 E_{P_0} \left[ 4X^3 \right]}{24}$$

$$a_2 = \frac{\eta_2^2}{2} + \frac{(\theta - 2)(2\theta - 3) \eta_2^3 E_{P_0} \left[ 6X^2 \right]}{24}$$

$$a_3 = \frac{(\theta - 2) \eta_2^2}{6}$$

$$a_4 = \frac{\eta_2^3}{24}$$

Since $\theta > 2$, we have $(\theta - 2)(2\theta - 3) > 0$ and so $(\theta - 2)(2\theta - 3) \eta_2^3 > 0$. We can therefore find a local minimum (global minimum). The local optimal points should satisfies $a_1 + 2a_2 \eta_1 + 3a_3 \eta_1^2 + 4a_4 \eta_1^3 = 0$. It is not difficult to obtain (10) using the definition (4.3) together with (A.8), (A.9), (A.10) and (A.11).
It remains to show that the local optimal obtained indeed a local minimum. This can be done by examining if \( 2a_2 + 6a_3\eta_1 + 12a_4\eta_1^2 \geq 0 \) for local optimal \( \eta_1 \). Indeed,

\[
2a_2 + 6a_3\eta_1 + 12a_4\eta_1^2 = 2\eta_2 \left( \frac{(\theta - 2)(2\theta - 3)}{4} \eta_1^2 + \frac{\theta - 2}{2} \eta_1 \eta_2 + \frac{1}{2} + \frac{(\theta - 2)(2\theta - 3)\eta_2}{4} \xi \phi_0 \right) \sin^2 [X^2]
\]

\[
= \frac{2\eta_2}{4} \left( \frac{(\theta - 2)(2\theta - 3)\eta_1^2}{4} + 2(\theta - 2)\eta_1 \eta_2 + 2 + (\theta - 2)(2\theta - 3)\eta_2 \xi \phi_0 \right) \sin^2 [X^2]
\]

\[
= \frac{2\eta_2}{4} \left( (\theta - 2)(2\theta - 3) \left[ \eta_1 \eta_2 + \frac{1}{2\theta - 3} \right]^2 + \frac{3\theta - 4}{2\theta - 3} + (\theta - 2)(2\theta - 3)\eta_2 \xi \phi_0 \right) \sin^2 [X^2] \geq 0.
\]

As a result, we can further simplify \( D_{\eta_2, \phi_0} (x^T r | x^T \mu) \) if \( \eta_1 \) exists for a real solution of \( a_1 + 2a_2\eta_1 + 3a_3\eta_1^2 + 4a_4\eta_1^3 = 0 \) and can be expressed explicitly.

The remaining in the proof shows how to find the explicit formula of local optimizer \( \eta_1 \). We examine the \( \Delta \) for a triple polynomial equation. When \( \Delta > 0 \), we know that the local optimal \( \eta_1 \) can be expressed as \( u = \frac{-q + \sqrt{\Delta}}{2} \) such that

\[
p = \frac{a_2}{2a_4} - \frac{3a_3^2}{16a_4^2}, \quad q = \frac{a_3^3}{32a_4^3} - \frac{a_2a_4}{8a_4^2} + \frac{a_1}{4a_4}, \quad \Delta = q^2 + \frac{4p^3}{27}, \quad u = \sqrt{-q + \sqrt{\Delta}},
\]

which follows from Cardano formula for a cubic equation. This follows from the fact that \( p \geq 0 \), since

\[
p = \frac{a_2}{2a_4} - \frac{3a_3^2}{16a_4^2} = \frac{8a_2a_4}{16a_4^2} - \frac{3a_3^2}{16a_4^2} = \frac{1}{16a_4^2} \left( 8a_2a_4 - 3a_3^2 \right)
\]

\[
= \frac{\eta_2^4}{192a_4^2} \left( 2(\theta - 2)(2\theta - 3) + (\theta - 2)^2(2\theta - 3)^2 \xi \phi_0 + X^2 \right) - (\theta - 2)^2
\]

\[
= \frac{\eta_2^4}{192a_4^2} \left[ (\theta - 2)(3\theta - 4) + (\theta - 2)^2(2\theta - 3)^2 \xi \phi_0 + X^2 \right] \geq 0.
\]

This completes the proof.

**Proof of Theorem 4.4** We state the relation

\[
\min_{\xi \in U} \xi \phi_0 \left[ x^T r \right] = \xi \phi_0 \left[ x^T r \right] - 2 \sqrt{\frac{\xi \phi_0 \left[ x^T r \right]}{2\phi_0'(1)}} = x^T \mu - \sqrt{\frac{2px \Sigma x}{\phi_0'(1)}} (A.12)
\]

in the main paper. We provide a detailed derivation here. From the proof given in [17], we can obtain

\[
\min_{\xi \in U} \xi \phi_0 \left[ x^T r \right] = \xi \phi_0 \left[ x^T r \right] - \min_{\eta_2 \geq 0} \left( \frac{\rho}{\eta_2} + \frac{\eta_2}{2\phi_0'(1)} \xi \phi_0 \left[ x^T r \right] + o(\eta_2) \right),
\]

where \( \xi \phi_0 \left[ x^T r \right] \) is the variance of \( x^T r \). To complete the proof, we use Result [3] to find the term

\[
\min_{\eta_2 \geq 0} \left( \frac{\rho}{\eta_2} + \frac{\eta_2}{2\phi_0'(1)} \xi \phi_0 \left[ x^T r \right] \right) = 2 \sqrt{\frac{\xi \phi_0 \left[ x^T r \right]}{2\phi_0'(1)}}, \quad \text{which gives (A.12).}
\]

Recalling the definition that \( \xi \phi_0 \left[ r \right] = \mu, \xi \phi_0 \left[ r \right] = \Sigma \) and \( \xi \phi_0 \left[ x^T r \right] = x^T \Sigma x \), we obtain

\[
\min_{\xi \in U} \xi \phi_0 \left[ x^T r \right] \approx x^T \mu - \sqrt{\frac{2px \Sigma x}{\phi_0'(1)}}.
\]

Taking \( \max_{x \in X} \) on both sides, we obtain the answer stated in Theorem 4.4. \( \square \)
Result 3. Let \( f(z) = ax + bz \), where \( a \) and \( b \) are positive. Then \( f(z) \) attains its minimum when \( z > 0 \) at \( \sqrt{\frac{a}{b}} \), which equals \( 2\sqrt{ab} \).

Proof of Result 3 Differentiating \( f(z) \) gives \( f'(z) = -\frac{a}{z} + b \) and \( f''(z) = \frac{2a}{z^2} \). Applying the second derivative test, we know that \( f(z) \) attains its minimum at \( \sqrt{\frac{a}{b}} \) with value \( 2\sqrt{ab} \).

Proof of Lemma 4.5 Let \( \lambda \in (0, 1) \). For any \( x \) and \( y \), we have

\[
a\sqrt{(\lambda x + (1-\lambda)y)^T \Sigma (\lambda x + (1-\lambda)y) - (\lambda x + (1-\lambda)y)^T b} = a\sqrt{[\lambda^2 x^T \Sigma x + 2\lambda(1-\lambda)x^T \Sigma y + (1-\lambda)^2 y^T \Sigma y] - (\lambda x + (1-\lambda)y)^T b}.
\]

Note that

\[
(\lambda \sqrt{x^T \Sigma x} + (1-\lambda)\sqrt{y^T \Sigma y})^2 - (\sqrt{[\lambda^2 x^T \Sigma x + 2\lambda(1-\lambda)x^T \Sigma y + (1-\lambda)^2 y^T \Sigma y]})^2 =\lambda^2 x^T \Sigma x + 2\lambda(1-\lambda)\sqrt{x^T \Sigma y \sqrt{y^T \Sigma y}} + (1-\lambda)^2 y^T \Sigma y - [\lambda^2 x^T \Sigma x + 2\lambda(1-\lambda)x^T \Sigma y + (1-\lambda)^2 y^T \Sigma y] = 2\lambda(1-\lambda)\sqrt{x^T \Sigma x \sqrt{y^T \Sigma y} - [\lambda^2 x^T \Sigma x + 2\lambda(1-\lambda)x^T \Sigma y + (1-\lambda)^2 y^T \Sigma y]\sqrt{y^T \Sigma y} - x^T \Sigma y].
\]

Since \( \Sigma > 0 \), let \( \tilde{x} = \Sigma^{\frac{1}{2}} x \) and \( \tilde{y} = \Sigma^{\frac{1}{2}} y \). The last equality can be written as

\[2\lambda(1-\lambda)[\sqrt{x^T \Sigma x} \sqrt{y^T \Sigma y} - x^T \Sigma y] = 2\lambda(1-\lambda)[\sqrt{x^T \tilde{x} \sqrt{\tilde{y}^T \tilde{y}} - \tilde{x}^T \tilde{y}] \geq 0.\]

The last inequality holds by Cauchy Schwartz inequality. Hence,

\[
a\sqrt{(\lambda x + (1-\lambda)y)^T \Sigma (\lambda x + (1-\lambda)y) - (\lambda x + (1-\lambda)y)^T b} \leq \lambda \left( a\sqrt{x^T \Sigma x - x^T b} \right) + (1-\lambda) \left( a\sqrt{y^T \Sigma y - y^T b} \right).
\]

As a result, the statement is true and our proof is thus completed.

Proof of Theorem 4.6 According to the symbols defined and let \( b = \sqrt{\frac{2p}{\phi(\ell)}} \) for notation simplification, problem (11) in the main paper is equivalent to

\[-\min_{x \in X} \left\{ -x^T \mu + b \sqrt{x^T \Sigma x} \right\} \Leftrightarrow \left\{ \begin{array}{l} -\min_{s.t.} \left\{ -x^T \mu + b \sqrt{x^T \Sigma x} \right\} \\ x^T e = 1 \end{array} \right. \quad \text{(A.13)}
\]

Applying Lagrangian multiplier method, let us examine

\[
-\mu + b \frac{\Sigma x}{\sqrt{x^T \Sigma x}} + \lambda e = 0, \quad \text{(A.13)}
\]

\[
x^T e = 1. \quad \text{(A.14)}
\]

Let \( x(\lambda) = \frac{\Sigma^{-1}(\mu - \lambda e)}{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e} \). From the assumption of \( \lambda, \lambda \) satisfies

\[
e^T \Sigma^{-1} e \cdot \lambda^2 - 2\lambda \mu^T \Sigma^{-1} e + (\mu^T \Sigma^{-1} e \cdot \mu - b^2) = 0.
\]

for two real roots, where we denote them as

\[
\lambda_+ = \frac{\mu^T \Sigma^{-1} e + \sqrt{(\mu^T \Sigma^{-1} e)^2 - (e^T \Sigma^{-1} e)(\mu^T \Sigma^{-1} e \cdot \mu - b^2)}}{e^T \Sigma^{-1} e}, \quad \text{(A.15)}
\]

\[
\lambda_- = \frac{\mu^T \Sigma^{-1} e - \sqrt{(\mu^T \Sigma^{-1} e)^2 - (e^T \Sigma^{-1} e)(\mu^T \Sigma^{-1} e \cdot \mu - b^2)}}{e^T \Sigma^{-1} e}.
\]
If \((\mu^T \Sigma^{-1} e)^2 - (e^T \Sigma^{-1} e)(\mu^T \Sigma^{-1} \mu - b^2) > 0\), then
\[
- \mu + \frac{b \Sigma x(\lambda)}{\sqrt{x(\lambda)^T \Sigma x(\lambda)}} + \lambda e = 0
\]
(A.16)
iff \(\frac{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e}{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e} = 1\). Note that \(\mu^T \Sigma^{-1} e - \lambda_+ e^T \Sigma^{-1} e < 0 < \mu^T \Sigma^{-1} e - \lambda_- e^T \Sigma^{-1} e\), so the point \(\mu^T \Sigma^{-1} e - \lambda_+ e^T \Sigma^{-1} e\) is rejected. Therefore, there is one local optimal solution and it is at \(x(\lambda_-)\) with optimal values \(\frac{\mu^T \Sigma^{-1} (\mu - e^T \Sigma^{-1} e)}{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e} - b e = \sqrt{\frac{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e}{\mu^T \Sigma^{-1} e - \lambda e^T \Sigma^{-1} e} = \lambda_-\), where the last equality follows from (A.13).

If \((\mu^T \Sigma^{-1} e)^2 - (e^T \Sigma^{-1} e)(\mu^T \Sigma^{-1} \mu - b^2) < 0\), no real roots for \(\lambda_+\) and \(\lambda_-\) exist. This means that no local optimal solutions exist.

We can draw the same conclusion as in Theorem 4.6 using the same notations in Theorem 4.6.

**Proof of Theorem 4.7** According to the symbols defined, problem (12) in the main paper is equivalent to

\[
\begin{cases}
-\min_{x \in K} \{-x^T \mu\} \\
\text{s.t. } \kappa(\epsilon) \sqrt{x^T \Sigma x} - x^T \mu \leq \delta
\end{cases} \iff \begin{cases}
-\min_{x} \{-x^T \mu\} \\
\text{s.t. } x^T e = 1 \\
\kappa(\epsilon) \sqrt{x^T \Sigma x} - x^T \mu \leq \delta.
\end{cases}
\]

Applying KKT, it is equivalent to examine
\[
-(1 + \tilde{\lambda}) \mu + \frac{\tilde{\lambda} \alpha x}{\sqrt{x^T \Sigma x}} + \tilde{\theta} e = 0,
\]
(A.17)
\[
\tilde{\theta} (x^T e - 1) = 0,
\]
(A.18)
\[
\tilde{\lambda} \{\kappa(\epsilon) \sqrt{x^T \Sigma x} - x^T \mu - \delta\} = 0,
\]
(A.19)
\[
\tilde{\lambda} \geq 0.
\]
(A.20)

From (A.17), we know that \(x\) is linearly dependent of \((1 + \tilde{\lambda}) \mu - \tilde{\theta} e\). Let \(x = x(\tilde{\lambda}, \tilde{\theta}) = \frac{\Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}{e^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}\). What is left is to find \(\tilde{\lambda}\) and \(\tilde{\theta}\). From (A.19), \(\tilde{\lambda} \neq 0\), otherwise we obtain \(\tilde{\theta} e = \mu\) from (A.17), which is a contradiction in general. Hence, we only need to consider
\[
-(1 + \tilde{\lambda}) \mu + \frac{\tilde{\lambda} \alpha x}{\sqrt{x^T \Sigma x}} + \tilde{\theta} e = 0,
\]
(A.21)
\[
\kappa(\epsilon) \sqrt{x^T \Sigma x} - x^T \mu - \delta = 0.
\]
(A.22)

Now, we compute the terms \(x^T \mu\), \(\Sigma x\), and \(\sqrt{x^T \Sigma x}\), respectively.
\[
x^T \mu = \frac{\mu^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}{e^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]},
\]
\[
\Sigma x = \frac{[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}{e^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]},
\]
\[
x^T \Sigma x = \frac{[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}{e^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]^2}.
\]

This suggests that (A.21)
\[
-(1 + \tilde{\lambda}) \mu + \frac{\tilde{\lambda} \kappa(\epsilon) [(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}{e^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]} + \frac{[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]^2}{\sqrt{[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]^T \Sigma^{-1}[(1 + \tilde{\lambda}) \mu - \tilde{\theta} e]}} + \tilde{\theta} e = 0,
\]
(A.23)
We obtain the first relation between $\tilde{\lambda}$ and $\tilde{\theta}$ by substituting (A.24) into (A.23):

$$\tilde{\lambda}(\kappa(\epsilon))^2 = \mu^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] + \delta e^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon].$$

Now, from (A.24), together with the constraint that $x^T\epsilon = 1$, we have the second relation

$$0 = -(1 + \tilde{\lambda}) \left( \frac{\kappa(\epsilon)}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right) \left( \frac{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right) + \tilde{\lambda}(\kappa(\epsilon))^2 = -(1 + \tilde{\lambda}) \tilde{\lambda}(\kappa(\epsilon))^2.$$

We would like to simplify the equation. This gives

$$0 = -(1 + \tilde{\lambda}) \left( \frac{\kappa(\epsilon)}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right) \left( \frac{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right) + \tilde{\lambda}(\kappa(\epsilon))^2.$$

Squaring on both sides, we have

L.H.S

$$\left\{ \frac{(1 + \tilde{\lambda}) \left( \mu^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] \right) \left( \epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] \right)}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right\}^2 = \left\{ \frac{(1 + \tilde{\lambda}) \left( \mu^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] \right) \left( \epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] \right) - \tilde{\theta}(\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon])^2}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right\}^2.$$

R.H.S

$$\left( \frac{\lambda(\kappa(\epsilon))^2}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} \right)^2 = \left( \lambda(\kappa(\epsilon))^2 \right)^2.$$

Both sides have the term $(\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon])^2$ which can be eliminated. So it reduces to have

$$\left\{ (1 + \tilde{\lambda}) \left( \mu^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] \right) - \tilde{\theta}(\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]) \right\}^2.$$

Lastly, from the L.H.S of (A.23) and the fact that $[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]^T \Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] = (\lambda(\kappa(\epsilon))^2$ proved, it can be seen that

$$\frac{|\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]|}{\epsilon^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon]} = 1$$

and $e^T\Sigma^{-1}[(1 + \tilde{\lambda})\mu - \tilde{\theta}\epsilon] > 0$. 21
Moreover, if we can solve \( \tilde{\lambda} \) and \( \tilde{\theta} \), from (A.21) and (A.22), it can be seen that the optimal value, which equals to \( \lambda x^T \mu \), can be expressed as \( \lambda (\lambda x^T \mu + \kappa(e) \sqrt{\Sigma x}) + \tilde{\theta} = \lambda \delta + \tilde{\theta} \).

Now, applying the given notations that \( A = e^T \Sigma^{-1} e \), \( B = \mu^T \Sigma^{-1} e \), and \( C = \mu^T \Sigma^{-1} \mu \), we can summarize the following conditions:

\[
\begin{align*}
\tilde{\lambda} \kappa(e)^2 &= [(1 + \tilde{\lambda})C - \tilde{\theta}B] + \delta[(1 + \tilde{\lambda})B - \tilde{\theta}A], \\
(\tilde{\lambda} \kappa(e))^2 &= [(1 + \tilde{\lambda})^2 C - 2\tilde{\theta}(1 + \tilde{\lambda})B + \tilde{\theta}^2 A], \\
\tilde{\lambda} &\geq 0, \\
(1 + \tilde{\lambda})B - \tilde{\theta}A &> 0.
\end{align*}
\]

(A.27) \hspace{1cm} (A.28) \hspace{1cm} (A.29) \hspace{1cm} (A.30)

We then solve \( \tilde{\lambda} \) explicitly.

From (A.25), we have

\[
\tilde{\theta} = -\frac{-\tilde{\lambda} \kappa(e)^2 + (C + \delta B) + \tilde{\lambda}(C + \delta B)}{B + \delta A},
\]

(A.31)

Substituting the expression of \( \tilde{\theta} \) into (A.28), we acquire a quadratic equation as follows

\[
(\tilde{\lambda} \kappa(e))^2 = (1 + \tilde{\lambda})^2 C - 2 \left( \frac{-\tilde{\lambda} \kappa(e)^2 + (C + \delta B) + \tilde{\lambda}(C + \delta B)}{B + \delta A} \right) (1 + \tilde{\lambda})B \\
+ \left( \frac{-\tilde{\lambda} \kappa(e)^2 + (C + \delta B) + \tilde{\lambda}(C + \delta B)}{B + \delta A} \right)^2 A
\]

\[
\Rightarrow (\tilde{\lambda} \kappa(e))^2 (B + \delta A)^2 = (1 + \tilde{\lambda})^2 (B + \delta A)^2 C \\
- 2(B + \delta A)[-\tilde{\lambda} \kappa(e)^2 + (C + \delta B)(1 + \tilde{\lambda})](1 + \tilde{\lambda})B \\
+ \left( -\tilde{\lambda} \kappa(e)^2 + (C + \delta B)(1 + \tilde{\lambda}) \right)^2 A
\]

\[
\Rightarrow \tilde{\lambda}^2 \kappa(e)^2 (B + \delta A)^2 = (1 + 2\tilde{\lambda} + \tilde{\lambda}^2)(B + \delta A)^2 C \\
+ 2B(B + \delta A)\tilde{\lambda}(1 + \tilde{\lambda}) \kappa(e)^2 \\
- 2(B + \delta A)(1 + \tilde{\lambda})^2(C + \delta B)B \\
+ \left( -\tilde{\lambda} + \kappa(e)^2 + (C + \delta B)(1 + \tilde{\lambda}) \right)^2 A
\]

\[
\Rightarrow M \tilde{\lambda}^2 - 2G \tilde{\lambda} + H = 0,
\]

(A.32)

where

\[
M = (B + \delta A)^2(\kappa(e)^2 - C) - 2B(B + \delta A)\kappa(e)^2 \\
+ 2B(B + \delta A)(C + \delta B) - A\kappa(e)^4 - A(C + \delta B)^2 + 2A(C + \delta B)\kappa(e)^2, \\
G = (B + \delta A)^2C + B(B + \delta A)\kappa(e)^2 \\
- 2(B + \delta A)(C + \delta B)B + A(C + \delta B)^2 - \kappa(e)^2 A(C + \delta B), \\
H = -(B + \delta A)^2C - 2B(B + \delta A)(C + \delta B) - A(C + \delta B)^2.
\]
We simplify the terms $M$, $G$, and $H$:

\[M = (B + \delta A)^2(\kappa(\epsilon)^2 - C) - 2B(B + \delta A)\kappa(\epsilon)^2\]
\[\quad + 2B(B + \delta A)(C + \delta B) - A\kappa(\epsilon)^4 - A(C + \delta B)^2 + 2A(C + \delta B)\kappa(\epsilon)^2\]
\[= (B^2 + 2\delta AB + \delta^2 A^2)\kappa(\epsilon)^2 - C(B^2 + 2\delta AB + \delta^2 A^2) - 2B^2\kappa(\epsilon)^2 - 2\delta AB\kappa(\epsilon)^2\]
\[\quad + 2B(BC + \delta B^2 + \delta AC + \delta^2 AB) - A\kappa(\epsilon)^4 - AC^2 + 2\delta ABC + \delta^2 AB^2 + 2A(C + \delta B)\kappa(\epsilon)^2\]
\[= [A(\delta B^2 + 2\delta B + C) + AC - B^2]\kappa(\epsilon)^2 + (B^2 - AC)(\delta A^2 + 2B\delta + C) - A\kappa(\epsilon)^4\]
\[= \kappa(\epsilon)^2 - (\delta A^2 + 2B\delta + C)[AC - B^2 - A\kappa(\epsilon)^2],\]

\[G = (B + \delta A)^2C + B(B + \delta A)\kappa(\epsilon)^2\]
\[\quad - 2(B + \delta A)(C + \delta B)B + A(C + \delta B)^2 - \kappa(\epsilon)^2 A(C + \delta B)\]
\[= (B^2C + 2\delta ABC + \delta^2 A^2C) + B(B + \delta A)\kappa(\epsilon)^2\]
\[\quad - 2B(BC + \delta B^2 + \delta AC + \delta^2 AB) + A(C^2 + 2\delta BC + \delta^2 B^2) - \kappa(\epsilon)^2 AC - \kappa(\epsilon)^2 \delta AB\]
\[= (B^2C + 2\delta ABC + \delta^2 A^2C) + B^2A^2 + \delta AB\kappa(\epsilon)^2\]
\[\quad - 2B(BC + \delta B^2 + \delta AC + \delta^2 AB) + A(C^2 + 2\delta BC + \delta^2 B^2) - \kappa(\epsilon)^2 AC - \kappa(\epsilon)^2 \delta AB\]
\[= (B^2 - AC)\kappa(\epsilon)^2 - B^2(A\delta^2 + 2B\delta + C) + AC(\delta A^2 + 2B\delta + C)\]
\[\quad - (B^2 - AC)[\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C)],\]

\[H = -(B + \delta A)^2C + 2B(B + \delta A)(C + \delta B) - A(C + \delta B)^2\]
\[= -B^2C - 2\delta ABC - \delta^2 A^2C + 2B(BC + \delta B^2 + \delta AC + \delta^2 AB) - A(C^2 + 2\delta BC + \delta^2 B^2)\]
\[= B^2C - \delta^2 A^2C + 2\delta B^3 + \delta^2 AB^2 - AC^2 - 2\delta ABC\]
\[= (B^2 - AC)(A\delta^2 + 2B\delta + C).\]

Now, the roots of $\lambda$ can be computed as
\[\hat{\lambda}_+ = \frac{G + \sqrt{G^2 - 4MH}}{M},\]
\[\hat{\lambda}_- = \frac{G - \sqrt{G^2 - 4MH}}{M}.\]

Simplifying the term $G^2 - MH$ using the definition that $K = (B^2 - AC)$ and $L = A\delta^2 + 2B\delta + C$, we obtain
\[G^2 - MH = K^2[\kappa(\epsilon)^2 - L^2] - KL\{[AL + AC - B^2]\kappa(\epsilon)^2 + KL - A\kappa(\epsilon)^4\}
\[= K^2[\kappa(\epsilon)^2 - 2K^2 + K^2 L^2 - AKL\kappa(\epsilon)^2 - 4KLK^2 - K^2 L^2 + AKL^2\kappa(\epsilon)^2]
\[= K\kappa(\epsilon)^2[(K + AL)(\kappa(\epsilon)^2 - L) - (K + AC - B^2)L]
\[= K\kappa(\epsilon)^2[(B + \delta A)^2(\kappa(\epsilon)^2 - L)] = \kappa(\epsilon)^2(B^2 + \delta A)^2G.\]

The second last equality follows since $K + AL = B^2 - AC + \delta^2 A^2 + 2\delta AB + AC = (B + \delta A)^2$ and $(K + AC - B^2) = 0$. We can get $\hat{\theta}_+$ and $\hat{\theta}_-$ by substituting $\hat{\lambda}_+$ and $\hat{\lambda}_-$ in (A.33), respectively.

$\hat{\lambda}_+$ and $\hat{\lambda}_-$ are given in (A.35) and (A.34). Note that $G^2 - MH = \kappa(\epsilon)^2((B^2 - AC)(\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C))$. Hence $G^2 - MH > 0 \Rightarrow (B^2 - AC)(\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C)) > 0 \Rightarrow 0 \leq \kappa(\epsilon)^2 < A\delta^2 + 2B\delta + C$ since $B^2 - AC \leq 0$ by Cauchy inequality.

Before considering case by case, let us study the signs of $G$ and $H$ first:

\[G = (B^2 - AC)[\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C)] \geq 0,\]
\[H = (B^2 - AC)(A\delta^2 + 2B\delta + C) \leq 0.\]

Now, we are able to prove the statement case by case.

Case 1: $B + \delta A > 0$ and $\frac{AC - B^2}{A} < \kappa(\epsilon)^2 < A\delta^2 + 2B\delta + C$. 

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In this case, $M = [\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C)] [AC - B^2 - A\kappa(\epsilon)^2] > 0$. Now, we can consider the signs of $\lambda_+$ and $\lambda_-$. Clearly, $G^2 - MH \geq G^2 \Rightarrow \sqrt{G^2 - MH} \geq G$. This gives that $\lambda_+ \geq 0$ and $\lambda_- \leq 0$ so that the only choice for $\tilde{\lambda}$ is $\lambda_+$. Substituting $\lambda_+$ obtained into (A.31), we obtain

$$\tilde{\theta}_+ = -\frac{\tilde{\lambda}_+\kappa(\epsilon)^2 + (1 + \tilde{\lambda}_+)(C + \delta B)}{B + \delta A}.$$ 

Also, from the given conditions that $B + \delta A > 0$, (A.31) and (A.33), it can be seen that

$$(1 + \tilde{\lambda}_+)B - \tilde{\theta}_+A = (1 + \tilde{\lambda}_+)B - \frac{\tilde{\lambda}_+\kappa(\epsilon)^2 + (1 + \tilde{\lambda}_+)(C + \delta B)}{B + \delta A} = \frac{(1 + \tilde{\lambda}_+)B(B + \delta A) + \tilde{\lambda}_+\kappa(\epsilon)^2 - (1 + \tilde{\lambda}_+)A(C + \delta B)}{B + \delta A} = \frac{(B^2 - AC) + \tilde{\lambda}_+(B^2 - AC + A\kappa(\epsilon)^2)}{B + \delta A} = \frac{B^2 - AC + \frac{G + \sqrt{G^2 - MH}}{B + \delta A}[(B^2 - AC + A\kappa(\epsilon)^2)]}{B + \delta A} = \frac{(B^2 - AC)\{[A\delta^2 + 2B\delta + C] - \kappa(\epsilon)^2\} + G + \sqrt{G^2 - MH} [B + \delta A]\{[A\delta^2 + 2B\delta + C] - \kappa(\epsilon)^2\} = 0.$$

This means that the optimal solution and optimal value of (12) are obtained into (A.31), we obtain

$$\lambda_+ = \frac{\tilde{\lambda}_+\kappa(\epsilon)^2 + (1 + \tilde{\lambda}_+)(C + \delta B)}{B + \delta A}.$$ 

In this case, we can compare the optimal value of problem (12) and the optimal value of problem (11). This is done by considering

$$\tilde{\lambda}_+\delta + \tilde{\theta}_+ - \lambda^* = \lambda_+\delta + \frac{(1 + \lambda_+)(C + \delta B) - \lambda_+\kappa(\epsilon)^2}{B + \delta A} - \frac{B - \sqrt{\Delta}}{A} = \frac{\lambda_+\delta(B + \delta A) + \lambda_+(C + \delta B) - \lambda_+\kappa(\epsilon)^2}{B + \delta A} + \frac{\lambda_+\kappa(\epsilon)^2}{B + \delta A} = \frac{\lambda_+(\delta^2A + 2\delta B + C - \kappa(\epsilon)^2) + CA - B^2}{B + \delta A} + \frac{\sqrt{\Delta}}{A} > 0.$$ 

**Case 2:** $B > 0$, $B + \delta A < 0$, and $\frac{\delta^2A - B^2}{A} < \kappa(\epsilon)^2 < \delta^2A + 2\delta B + C$.

In this case, we check the equation (A.30)

$$(1 + \tilde{\lambda}_+)B - \tilde{\theta}_+A = \frac{\sqrt{G^2 - MH}}{B + \delta A}\{[A\delta^2 + 2B\delta + C] - \kappa(\epsilon)^2\} < 0.$$ 

This means that the conditions (A.27), (A.28), (A.29), and (A.30) are not satisfied. We cannot have a local optimal solution.

Next, we prove that under the given conditions, we cannot find $x$ such that $x^T e = 1$ and $\kappa(\epsilon)\sqrt{x^T \Sigma x} - x^T \mu \leq \delta$ are satisfied. We consider

$$\min \kappa(\epsilon)\sqrt{x^T \Sigma x} - x^T \mu$$

s.t. $x^T e = 1$.

From the Proof of Theorem 4.6 and the given condition that $\kappa(\epsilon)^2 > \frac{B^2 - AC}{A}$, we find that the optimal solution is $\frac{B - \sqrt{B^2 - AC + A\kappa(\epsilon)^2}}{A}$. Since $\kappa(\epsilon)^2 < \delta^2A + 2\delta B + C$ and $B + \delta A < 0$, we know
that
\[
\frac{B - \sqrt{B^2 - AC + A\kappa(\epsilon)^2}}{A} > \frac{B - \sqrt{B^2 + 2\delta AB + \delta^2 A^2}}{A} = \frac{B - |B + \delta A|}{A} = \frac{2B + \delta A}{A} = \frac{2B}{A} + \delta > \delta
\]
when \(B > 0\). This means that no feasible solution in this case exists.

**Case 3:** \(\kappa(\epsilon)^2 < \frac{AC - B^2}{A}\). We check the signs of \(\tilde{\lambda}_+\) and \(\tilde{\lambda}_-\). \(M = [\kappa(\epsilon)^2 - (A\delta^2 + 2B\delta + C)][AC - B^2 - A\kappa(\epsilon)^2] < 0\). Then \(0 < G^2 - MH \leq G^2\). As a result,
\[
\tilde{\lambda}_+ = \frac{G + \sqrt{G^2 - MH}}{M} < 0,
\tilde{\lambda}_- = \frac{G - \sqrt{G^2 - MH}}{M} < \frac{G - \sqrt{G^2}}{M} = 0.
\]
This means that (A.27), (A.28), (A.29), and (A.30) are not satisfied. Hence, no local optimal solution for problem (12) exists.