GENERALIZED ARTIN PATTERN OF HETEROGENEOUS MULTIPLETS
OF DIHEDRAL FIELDS AND PROOF OF SCHOLZ’S CONJECTURE

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Abstract. The concept of Artin pattern ((ker(TK,Ni)),(Clp(Ni))) for homogeneous multiplets (N1, ..., Nm) of unramified cyclic prime degree p extensions N/K of a base field K with p-class transfer homomorphisms TK,Ni : Clp(K) → Clp(Ni) for an arbitrary prime p ≥ 2. The strategy of pattern recognition via Artin transfers [23] is a method for using number theoretic information on the unramified cyclic extensions N/K of relative degree p in a group theoretic search for G = G2(K) with the aid of the p-group generation algorithm [25, 26]. According to the Artin reciprocity law of class field theory [3], the kernels and targets of the p-class extension homomorphisms TK,N : Clp(K) → Clp(N), a · P_K → aCN · TN, correspond to the kernels and targets of the transfer homomorphisms T_G,H : G/G' → H/H' from G to its maximal subgroups H < G of index p [4, 22]. In [25] we defined the Artin pattern of K as the collection AP(K) = (ζ(K), τ(K)) of kernels ζ(K) = (ker(TK,N))N, and targets τ(K) = (Clp(N))N of p-class extensions, where N varies over all unramified cyclic extensions N/K of relative degree p.

In the present article, our principal aim is to use extended Artin patterns for a systematic classification of ramified cyclic extensions N/K of odd prime degree p over quadratic or cyclotomic fields K with the aid of transfer data and principal factorization types, rather than for finding a suitable automorphism group.

The precise definition of homogeneous and heterogeneous multiplets of dihedral fields is given in §2, using concepts of the multiplicity theory of dihedral discriminants.

After the illustration of two important special situations involving heterogeneous multiplets, which are crucial for verifying the truth of several hypotheses of Scholz in §3, §4 introduces the generalized Artin pattern of a heterogeneous multiplet. In §5 we distinguish between multiplets of objects and associated multiplets of invariants.

For obtaining deeper structural insight, we sort the components of the Artin pattern by differential principal factorization (DPF) types in §6 where we prove that the Fp-vector space P_N/K/P_K of primitive ambiguous principal ideals of a number field extension N/K can be endowed with a natural trichotomic direct product structure.

Finally, as an application of the notions of multiplets, Artin patterns and DPF types, the Conjecture of Scholz [31] is stated, refined, and proved completely in §6.
2. Homogeneous and heterogeneous multiplets of dihedral fields

In the sequel we fix an odd prime number \( p \geq 3 \).

**Definition 2.1.** Let \((N_1, \ldots, N_m)\) be a multiplet of \( m \) pairwise non-isomorphic dihedral fields of degree \( 2p \) which share a common quadratic subfield \( K < N_i \) (\( 1 \leq i \leq m \)) with discriminant \( d_K \).

1. The multiplet \((N_1, \ldots, N_m)\) is called **homogeneous** if:
   a. all members share a common class field theoretic conductor \( f = f(N_i/K) \), and
   b. the number \( m = m_p(d_K, f) \) is the \( p \)-multiplicity of \( f \) with respect to \( K \).

2. The multiplet \((N_1, \ldots, N_m)\) is called **heterogeneous** if:
   a. all members possess conductors \( f_i = f(N_i/K) \) which are divisors of an assigned \( p \)-admissible conductor \( f_0 \) for \( K \), and
   b. the number \( m = \sum f_i f_0 m_p(d_K, f) \) is the sum of all \( p \)-multiplicities of divisors \( f \) of \( f_0 \) with respect to \( K \).

3. Generalized Artin pattern

Since the extended Artin pattern of a composite conductor has a structure with high complexity, we abstain from maximal possible generality and restrict ourselves to the simplest non-trivial situation where \( f_0 = q \in \mathbb{P} \) is a \( p \)-admissible prime conductor for a quadratic field \( K \) with \( p \)-class rank \( \rho_p = \text{rank}_p(\text{Cl}(K)) \geq 1 \), that is, \( K \) has class number divisible by \( p \). Then \( f_0 \) has only two divisors, \( f = 1 \) corresponding to unramified abelian extensions \( N_{1,i}/K \), \( 1 \leq i \leq m_p(d_K, 1) \), and \( f = q \) corresponding to ramified abelian extensions \( N_{q,j}/K \), \( 1 \leq j \leq m_p(d_K, q) \).

**Example 3.1.** Before we give an abstract definition of the extended Artin pattern, we put \( p = 3 \) the smallest odd prime, and we start with two illuminating examples for heterogeneous multiplets.

- Suppose \( K \) has minimal positive 3-class rank \( \rho_3 = 1 \), then a heterogeneous multiplet with free prime conductor \( q \), having \( p \)-defect \( \delta_p(q) = 0 \), possesses

\[
m = m_3(d_K, 1) + m_3(d_K, q) = \frac{1}{2}(3^{3\rho_3+1} - 1) = \frac{1}{2}(3^{1+1} - 1) = \frac{1}{2}(9 - 1) = 4
\]

members, being a quartet, and consists of an unramified homogeneous singulet \( N_{1,1} \), since \( m_3(d_K, 1) = \frac{1}{2}(3^{3\rho_3} - 1) = \frac{1}{2}(3^1 - 1) = 1 \), and a ramified homogeneous triplet \((N_{q,1}, N_{q,2}, N_{q,3})\), since \( m_3(d_K, q) = 4 - 1 = 3 \). The situation is illustrated in Figure 4.

- Assume that \( K \) has 3-class rank \( \rho_3 = 2 \) and \( q \) is free with \( p \)-defect \( \delta_p(q) = 0 \), then a heterogeneous multiplet with conductor \( q \) possesses

\[
m = m_3(d_K, 1) + m_3(d_K, q) = \frac{1}{2}(3^{3\rho_3+1} - 1) = \frac{1}{2}(3^{2+1} - 1) = \frac{1}{2}(27 - 1) = 13
\]

members, being a tridecuplet, and consists of an unramified homogeneous quartet \((N_{1,1}, \ldots, N_{1,4})\), since \( m_3(d_K, 1) = \frac{1}{2}(3^{3\rho_3} - 1) = \frac{1}{2}(3^2 - 1) = 4 \), and a ramified homogeneous nonet \((N_{q,1}, \ldots, N_{q,9})\), since \( m_3(d_K, q) = 13 - 4 = 9 \). The situation is illustrated in Figure 2.

**Definition 3.1.** The generalized or extended Artin pattern \( \text{AP}(M) \) of the heterogeneous multiplet \( M = ((N_{1,i}), (N_{q,j})) \) consists of the kernels and targets of the \( p \)-class extension homomorphisms \( T_{K,N_{1,i}} : \text{Cl}_p(K) \to \text{Cl}_p(N_{1,i}) \), \( 1 \leq i \leq m(1) \), and \( T_{K,N_{q,j}} : \text{Cl}_p(K) \to \text{Cl}_p(N_{q,j}) \), \( 1 \leq j \leq m(q) \), where we abbreviate \( m(f) = m_p(d_K, f) \), since \( p \) and \( K \) are fixed. Explicitly:

\[
\text{AP}(M) = \left( \left( \kappa_u(M), \tau_u(M), \kappa_r(M), \tau_r(M) \right) \right),
\]

where

\[
(3.3) \quad \kappa_u(M) = (\ker(T_{K,N_{1,i}})), \quad \tau_u(M) = (\text{Cl}_p(N_{1,i})) (u \ldots \text{unramified}),
\]

\[
\kappa_r(M) = (\ker(T_{K,N_{q,j}})), \quad \tau_r(M) = (\text{Cl}_p(N_{q,j})) (r \ldots \text{ramified}).
\]

**Example 3.2.** Let \( p = 3 \) be the smallest odd prime, \( K \) be a real quadratic field with 3-class rank \( \rho_3 = 1 \), and \( f = 2 \) be a free admissible prime conductor for \( K \). According to Formula (3.1) in Example 3.1, \( K \) gives rise to a singulet \( N_{1,1} \) of unramified absolute 3-class fields, and a triplet \((N_{2,1}, N_{2,2}, N_{2,3})\) of ramified 3-ring class fields modulo 2. The generalized Artin pattern of the heterogeneous multiplet \((N_{1,1}; (N_{2,j}))_{j=1}^{3} \) is \( (\ker(T_{K,N_{1,1}})), \text{Cl}_3(N_{1,1}); (\ker(T_{K,N_{2,j}})), \text{Cl}_3(N_{2,j}))_{j=1}^{3} \).
Since we want to supplement our description of dihedral fields with differential principal factorization (DPF) types, additionally to the generalized Artin patterns, we briefly explain that we shall have to deal with two distinct kinds of closely related multiplets. Suppose, we have a mapping,

\[ I : \text{Objects} \rightarrow \text{Invariants}, \ X \mapsto I(X), \]

from certain objects to invariants of these objects, then a \textit{multiplet of objects}, \( (X_1, \ldots, X_m) \), with some non-negative integer \( m \), will be mapped to an \textit{associated multiplet of invariants}, \( I(X_1, \ldots, X_m) = (I(X_1), \ldots, I(X_m)) \).

\textbf{Example 4.1.} In Theorem 5.5 we shall see that the DPF type of pure cubic fields \( L = \mathbb{Q}(\sqrt[3]{d}) \) is a mapping \( L \mapsto T(L) \in \{\alpha, \beta, \gamma\} \). Therefore, a multiplet \( (L_1, \ldots, L_m) \) of pure cubic fields has an associated multiplet of DPF types \( (T(L_1), \ldots, T(L_m)) \equiv (\alpha^x, \beta^y, \gamma^z) \) with \( x + y + z = m \).

\section{5. Trichotomy of primitive ambiguous principal ideals}

Our intention in this section is to supplement the generalized Artin pattern by \textit{differential principal factorization (DPF) types} and to establish a common theoretical framework for the classification
• of dihedral fields $N/Q$ of degree $2p$ with an odd prime $p$, viewed as $p$-ring class fields over a quadratic field $K$, and
• of pure metacyclic fields $N = K(\sqrt{D})$ of degree $(p - 1) \cdot p$ with an odd prime $p$, viewed as Kummer extensions of a cyclotomic field $K = \mathbb{Q}(\zeta_p)$,

by the following arithmetical invariants:

1. the $\mathbb{F}_p$-dimensions of subspaces of the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of primitive ambiguous principal ideals, which are also called differential principal factors, of $N/K$,
2. the capitulation kernel $\ker(T_{N/K})$ of $T_{N/K} : \text{Cl}_p(K) \to \text{Cl}_p(N)$, the transfer homomorphism of $p$-classes from $K$ to $N$, and
3. the Galois cohomology $H^0(G, U_N)$, $H^1(G, U_N)$ of the unit group $U_N$ as a module over the automorphism group $G = \text{Gal}(N/K) \simeq C_p$.

**Figure 3.** Dihedral and Metacyclic Situation

5.1. **Primitive ambiguous ideals.** Let $p \geq 2$ be a prime number, and $N/K$ be a relative extension of number fields with degree $p$ (not necessarily Galois).

**Definition 5.1.** The group $\mathcal{I}_N$ of fractional ideals of $N$ contains the *subgroup of ambiguous ideals* of $N/K$, denoted by the symbol $\mathcal{I}_{N/K} := \{ \mathfrak{A} \in \mathcal{I}_N \mid \mathfrak{A}^p \in \mathcal{I}_K \}$. The quotient $\mathcal{I}_{N/K}/\mathcal{I}_K$ is called the $\mathbb{F}_p$-vector space of *primitive ambiguous ideals* of $N/K$.

**Proposition 5.1.** Let $\Sigma_1, \ldots, \Sigma_t$ be the totally ramified prime ideals of $N/K$, then a basis and the dimension of the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ over $\mathbb{F}_p$ are finite and given by

$$\mathcal{I}_{N/K}/\mathcal{I}_K \cong \prod_{i=1}^t ((\Sigma_i)/\langle\Sigma_i^p\rangle) \cong \mathbb{F}_p^t,$$

whereas $\mathcal{I}_{N/K}$ is an infinite abelian group containing $\mathcal{I}_K$.

**Proof.** According to the definition of $\mathcal{I}_{N/K}$, the quotient $\mathcal{I}_{N/K}/\mathcal{I}_K$ is an elementary abelian $p$-group. By the decomposition law for prime ideals of $K$ in $N$, the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ is generated by the *totally ramified* prime ideals (with ramification index $e = p$) of $N/K$, that is to say $\mathcal{I}_{N/K} = \langle \Sigma \in \mathcal{P}_N \mid \Sigma^p \in \mathcal{P}_K \rangle \mathcal{I}_K$. According to the theorem on prime ideals dividing the discriminant, the number $t$ of totally ramified prime ideals $\Sigma_1, \ldots, \Sigma_t$ of $N/K$ is finite. \[ \square \]

If $L$ is another subfield of $N$ such that $N = L \cdot K$ is the compositum of $L$ and $K$, and $N/L$ is of degree $q$ coprime to $p$, then the relative norm homomorphism $N_{N/L}$ induces an epimorphism

$$N_{N/L} : \mathcal{I}_{N/K}/\mathcal{I}_K \to \mathcal{I}_{L/F}/\mathcal{I}_F,$$

where $F := L \cap K$ denotes the intersection of $L$ and $K$ in $N$ in Figure 4. Thus, by the isomorphism theorem, we have proved:
**Theorem 5.1.** There are the following two isomorphisms between $\mathbb{F}_p$-vector spaces:

\[
\begin{align*}
(I_{N/K}/I_K) / \ker(N_{N/L}) & \simeq I_{L/F}/I_F \quad \text{(quotient),} \\
I_{N/K}/I_K & \simeq (I_{L/F}/I_F) \times \ker(N_{N/L}) \quad \text{(direct product).}
\end{align*}
\]

**Definition 5.2.** Since the relative different of $N/K$ is essentially given by $D_{N/K} = \prod_{i=1}^{s} \mathbb{Q}_i^{p-1}$ the space $I_{N/K}/I_K \simeq \prod_{i=1}^{s} (\langle \mathfrak{D}_i \rangle / (\mathfrak{D}_i^p))$ of primitive ambiguous ideals of $N/K$ is also called the space of differential factors of $N/K$. The two subspaces in the direct product decomposition of $I_{N/K}/I_K$ in formula (5.3) are called subspace $I_{L/F}/I_F$ of absolute differential factors of $L/F$ and subspace $\ker(N_{N/L})$ of relative differential factors of $N/K$.

5.2. **Splitting off the norm kernel.** The second isomorphism in formula (5.3) causes a dichotomic decomposition of the space $I_{N/K}/I_K$ of primitive ambiguous ideals of $N/K$ into two components, whose dimensions can be given under the following conditions:

**Theorem 5.2.** Let $p \geq 3$ be an odd prime and put $q = 2$. Among the prime ideals of $L$ which are totally ramified over $F$, denote by $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ those which split in $N$, $\mathfrak{p}_i \mathfrak{O}_N = \mathfrak{P}_i \mathfrak{P}'_i$ for $1 \leq i \leq s$, and by $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ those which remain inert in $N$, $\mathfrak{q}_j \mathfrak{O}_N = \mathfrak{Q}_j$ for $1 \leq j \leq n$. Then the space $I_{N/K}/I_K$ of primitive ambiguous ideals of $N/K$ is the direct product of the subspace $I_{L/F}/I_F$ of absolute differential factors of $L/F$ and the subspace $\ker(N_{N/L})$ of relative differential factors of $N/K$, whose dimensions over $\mathbb{F}_p$ are given by

\[
\begin{align*}
I_{L/F}/I_F & \simeq \prod_{i=1}^{s} (\langle \mathfrak{p}_i \rangle / (\mathfrak{p}_i^p)) \times \prod_{j=1}^{n} (\langle \mathfrak{q}_j \rangle / (\mathfrak{q}_j^p)) \simeq \mathbb{F}_p^{s+n}, \\
\ker(N_{N/L}) & \simeq \prod_{i=1}^{s} (\langle \mathfrak{P}_i (\mathfrak{P}'_i)^{p-1} \rangle / (\langle \mathfrak{P}_i (\mathfrak{P}'_i)^{p-1} \rangle^p)) \simeq \mathbb{F}_p^s.
\end{align*}
\]

Consequently, the complete space of differential factors has dimension $\dim_{\mathbb{F}_p} (I_{N/K}/I_K) = n + 2s$.

**Proof.** Whereas the qualitative formula (5.3) is valid for any prime $p \geq 2$ and any integer $q > 1$ with $\gcd(p, q) = 1$, the quantitative description of the norm kernel $\ker(N_{N/L})$ is only feasible if we put $q = 2$ and therefore have to select an odd prime $p \geq 3$. Replacing $N$ by $L$ and $K$ by $F$ in formula (5.3), we get $t = n + s$ and thus the first isomorphism of formula (5.3). For $N$ and $K$, however, we obtain $t = n + 2s$. We point out that, if $s = 0$, that is, if none of the totally ramified primes of $L/F$ splits in $N$, then the induced norm mapping $N_{N/L}$ in formula (5.2) is an isomorphism. For the constitution of the norm kernel, see [26] Thm. 3.4 and Cor. 3.3(3)].
5.3. Primitive ambiguous principal ideals. The preceding result concerned \textit{primitive ambiguous ideals} of $N/K$, which can be interpreted as ideal factors of the \textit{relative different} $\mathfrak{D}_{N/K}$. Formula (5.1) and Theorem 5.2 show that the $F_p$-dimension of the space $\mathcal{I}_{N/K}/\mathcal{I}_K$ increases indefinitely with the number $t$ of totally ramified primes of $N/K$.

Now we restrict our attention to the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of \textit{primitive ambiguous principal ideals} or \textit{differential principal factors} (DPF) of $N/K$. We shall see that fundamental constraints from Galois cohomology prohibit an infinite growth of its dimension over $F_p$, for quadratic fields $K$.

5.4. Splitting off the capitulation kernel. We have to cope with a difficulty which arises in the case of a non-trivial class group $\text{Cl}(K) = \mathcal{I}_K/\mathcal{P}_K > 1$, because then $\mathcal{P}_{N/K}/\mathcal{P}_K$ cannot be viewed as a subgroup of $\mathcal{I}_{N/K}/\mathcal{I}_K$. Therefore we must separate the \textit{capitulation kernel} of $N/K$, that is the kernel of the \textit{transfer} homomorphism $T_{N/K} : \text{Cl}(K) \to \text{Cl}(N)$, $\mathfrak{a} : \mathcal{P}_K \mapsto (\mathfrak{a}\mathcal{O}_N) : \mathcal{P}_N$, which extends classes of $K$ to classes of $N$:

\begin{equation}
\ker(T_{N/K}) = \{ \mathfrak{a} : \mathcal{P}_K \mid (\exists A \in N) \mathfrak{a}\mathcal{O}_N = A\mathcal{O}_N\} = (\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K.
\end{equation}

On the one hand, $\ker(T_{N/K}) = (\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K$ is a subgroup of $\mathcal{I}_K/\mathcal{P}_K = \text{Cl}(K)$, consisting of capitulating ideal classes of $K$. On the other hand, since $\mathcal{I}_K \leq \mathcal{I}_{N/K}$ consists of ambiguous ideals of $N/K$, $\ker(T_{N/K}) = (\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K$ is a subgroup of $\mathcal{P}_{N/K}/\mathcal{P}_K$, consisting of special primitive ambiguous principal ideals of $N/K$, and we can form the quotient

\begin{equation}
(\mathcal{P}_{N/K}/\mathcal{P}_K)/(\mathcal{I}_K \cap \mathcal{P}_N)/\mathcal{P}_K \simeq \mathcal{P}_{N/K}/(\mathcal{I}_K \cap \mathcal{P}_N) = \mathcal{P}_{N/K}/\mathcal{P}_K \simeq (\mathcal{P}_{N/K}\mathcal{I}_K)/\mathcal{I}_K.
\end{equation}

This quotient relation of $F_p$-vector spaces is equivalent with a direct product relation

\begin{equation}
\mathcal{P}_{N/K}/\mathcal{P}_K \simeq (\mathcal{P}_{N/K}\mathcal{I}_K)/\mathcal{I}_K \times \ker(T_{N/K}).
\end{equation}

Since $(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K \leq \mathcal{I}_{N/K}/\mathcal{I}_K$ is an actual inclusion, the factorization of $\mathcal{I}_{N/K}/\mathcal{I}_K$ in formula (5.6) restricts to a factorization

\begin{equation}
(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K \simeq (\mathcal{P}_{L/F}/\mathcal{P}_F) \times \left(\ker(N_{N/L}) \cap \{(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K\}\right),
\end{equation}

provided that $F$ is a field with trivial class group $\text{Cl}(F)$, that is $\mathcal{I}_F = \mathcal{P}_F$ and thus $\mathcal{P}_{L/F}/\mathcal{P}_F \leq \mathcal{I}_{L/F}/\mathcal{I}_F$. Combining the formulas (5.7) and (5.8) for the rational base field $F = \mathbb{Q}$, we obtain:

\begin{theorem}
There is a \textit{trichotomic decomposition} of the space $\mathcal{P}_{N/K}/\mathcal{P}_K$ of differential principal factors of $N/K$ into three components,

\begin{equation}
\mathcal{P}_{N/K}/\mathcal{P}_K \simeq \mathcal{P}_{L/Q}/\mathcal{P}_Q \times \left(\ker(N_{N/L}) \cap \{(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K\}\right) \times \ker(T_{N/K}),
\end{equation}

the \textit{absolute principal factors}, $\mathcal{P}_{L/Q}/\mathcal{P}_Q$, of $L/Q$, the \textit{relative principal factors}, $\ker(N_{N/L}) \cap \{(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K\}$, of $N/K$, and the \textit{capitulation kernel}, $\ker(T_{N/K})$, of $N/K$.
\end{theorem}

5.5. Galois cohomology. For establishing a quantitative version of the qualitative formula (5.9), we suppose that $N/K$ is a cyclic relative extension of odd prime degree $p$ and we use the Galois cohomology of the unit group $U_N$ as a module over the automorphism group $G = \text{Gal}(N/K) = \langle \sigma \rangle \simeq C_p$. In fact, we combine a theorem of Iwasawa [11] on the first cohomology $H^1(G, U_N)$ with a theorem of Hasse [11] on the Herbrand quotient of $U_N$ [11], and we use Dirichlet’s theorem on the torsion-free unit rank of $K$:

\begin{equation}
H^1(G, U_N) \simeq (U_N \cap \ker(N_{N/K}))/U_N^{-1} \simeq \mathcal{P}_{N/K}/\mathcal{P}_K \quad \text{(Iwasawa)},
\end{equation}

\begin{equation}
H^0(G, U_N) \simeq U_K/N_{N/K}(U_N), \quad \text{with} \quad (U_K : N_{N/K}(U_N)) = p^U, \quad 0 \leq U \leq r_1 + r_2 - \theta,
\end{equation}

\begin{equation}
\#H^1(G, U_N)/H^0(G, U_N) = \left[ N : K \right] = p \quad \text{(Hasse)},
\end{equation}

where $(r_1, r_2)$ is the signature of $K$, and $\theta = 0$ if $K$ contains the $p$th roots of unity, but $\theta = 1$ else.

\begin{corollary}
If $N/K$ is cyclic of odd prime degree $p \geq 3$, then the $F_p$-dimensions of the spaces of differential principal factors in Theorem 5.3 are connected by the \textit{fundamental equation}

\begin{equation}
U + 1 = A + R + C, \quad \text{where}
\end{equation}

\begin{align*}
U & = \mathcal{P}_{L/Q}/\mathcal{P}_Q, \\
A & = \ker(N_{N/L}) \cap \{(\mathcal{P}_{N/K}\cdot\mathcal{I}_K)/\mathcal{I}_K\}, \\
R & = \ker(T_{N/K}), \\
C & = \mathcal{P}_{L/Q}/\mathcal{P}_Q, \quad \text{of } L/Q,
\end{align*}

of $N/K$.\end{corollary}
be an odd prime. We recall the classification theorem for Differential principal factorization (DPF) types of complex dihedral fields.

\[ A := \dim_{\mathbb{F}_p}(\mathcal{P}_{L/Q}/\mathcal{P}_Q) \] is the dimension of absolute principal factors,

\[ R := \dim_{\mathbb{F}_p}\left( \ker(N_{N/L}) \cap \left( (\mathcal{P}_{N/K}/\mathcal{I}_K)/(\mathcal{I}_K) \right) \right) \] is the dimension of relative principal factors, and

\[ C := \dim_{\mathbb{F}_p}(\ker(T_{N/K})) \] is the dimension of the capitulation kernel.

**Corollary 5.2.** Under the assumptions \( p \geq 3, q = 2 \) of Theorem 5.1, in particular for \( N \) dihedral of degree \( 2p \), the dimensions in Corollary 5.1 are bounded by the following estimates

\[ 0 \leq A \leq \min(n + s, m), \quad 0 \leq R \leq \min(s, m), \quad 0 \leq C \leq \min(g_p, m), \]

where \( g_p := \text{rank}_p(\text{Cl}(K)) \), and \( m := 1 + r_1 + r_2 - \theta \) denotes the cohomological maximum of \( U + 1 \).

In particular,

- \( m = 2 \) for real quadratic \( K \) with \( (r_1, r_2) = (2, 0) \) or \( K = \mathbb{Q}(\sqrt{-3}) \) if \( p = 3 \),
- \( m = 1 \) for imaginary quadratic \( K \) with \( (r_1, r_2) = (0, 1) \), except \( K = \mathbb{Q}(\sqrt{-3}) \) when \( p = 3 \).

**Remark 5.1.** For \( N \) pure metacyclic of degree \( (p-1)p \), the space \( \mathcal{P}_{L/Q}/\mathcal{P}_Q \) of absolute principal factors contains the one-dimensional subspace \( \Delta = (\sqrt{D}) \) generated by the radicals, and thus

\[ 1 \leq A \leq \min(t, m), \quad 0 \leq R \leq m - 1, \quad 0 \leq C \leq \min(g_p, m - 1), \]

where \( m = \frac{p+1}{2} \) for cyclotomic \( K \) with \( (r_1, r_2) = (0, \frac{p-1}{2}) \).

In particular \( C = 0 \) for a regular prime \( p \), for instance \( p < 37 \).

**Remark 5.2.** We mentioned that in general \( \mathcal{P}_{N/K}/\mathcal{P}_K \) cannot be viewed as a subgroup of \( \mathcal{I}_{N/K}/\mathcal{I}_K \). In fact, for a dihedral field \( N \) which is unramified with conductor \( f = 1 \) over \( K \), we have \( n = s = 0 \), consequently \( A = R = 0 \), and \( \mathcal{I}_{N/K}/\mathcal{I}_K \simeq 0 \) is the subspace, whereas \( \mathcal{P}_{N/K}/\mathcal{P}_K \simeq \ker(T_{N/K}) \) is at least one-dimensional, according to Hilbert’s Theorem 94 \([15]\), and at most two-dimensional, and the estimate \( C \leq \min(g_p, m) \leq \min(g_p, 2) \leq 2 \).

In the next two sections, we apply the results of \( \S \S \) 5.1–5.5 to various extensions \( N/K \).

### 5.6. Differential principal factorization (DPF) types of complex dihedral fields.

Let \( p \) be an odd prime. We recall the classification theorem for pure cubic fields \( L = \mathbb{Q}(\sqrt{D}) \) and their Galois closure \( N = \mathbb{Q}(\zeta_3, \sqrt{D}) \), that is the metacyclic case \( p = 3 \). The coarse classification of \( N \) according to the cohomological invariants \( U \) and \( A \) alone is closely related to the classification of simply real dihedral fields of degree \( 2p \) with any odd prime \( p \) by Nicole Moser \([27]\) Dfn. III.1 and Prop. III.3, p. 61], as illustrated in Figure 5. The coarse types \( \alpha \) and \( \beta \) are completely analogous in both cases. The additional type \( \gamma \) is required for pure cubic fields, because there arises the possibility that the primitive cube root of unity \( \zeta_3 \) occurs as relative norm \( N_{N/K}(Z) \) of a unit \( Z \in U_N \). Due to the existence of radicals in the pure cubic case, the \( \mathbb{F}_p \)-dimension \( A \) of the vector space of absolute DPF exceeds the corresponding dimension for simply real dihedral fields by one.

![Figure 5. Classification of Simply Real Dihedral and Pure Cubic Fields](image)

The fine classification of \( N \) according to the invariants \( U, A, R \) and \( C \) in the simply real dihedral situation with \( U + 1 = A + R + C \) splits type \( \alpha \) with \( A = 0 \) further in type \( \alpha_1 \) with \( C = 1 \) (capitulation) and type \( \alpha_2 \) with \( R = 1 \) (relative DPF). In the pure cubic situation, however, no further splitting occurs, since \( C = 0 \), and \( R = U + 1 - A \) is determined uniquely by \( U \) and \( A \) already. We oppose the two classifications in the following theorems.
Theorem 5.4. Each simply real dihedral field \(N/\mathbb{Q}\) of absolute degree \([N : \mathbb{Q}] = 2p\) with an odd prime \(p\) belongs to precisely one of the following 3 differential principal factorization types, in dependence on the triplet \((A, R, C)\):

| Type | \(U\) | \(U + 1 = A + R + C\) | \(A\) | \(R\) | \(C\) |
|------|-------|------------------------|------|------|------|
| \(\alpha_1\) | 0     | 1                      | 0    | 0    | 1    |
| \(\alpha_2\) | 0     | 1                      | 0    | 1    | 0    |
| \(\beta\)     | 0     | 1                      | 1    | 0    | 0    |

Proof. Consequence of Cor. 5.1 and 5.2. See [27, Dfn. III.1 and Prop. III.3, p. 61] and [18]. \(\square\)

Theorem 5.5. Each pure metacyclic field \(N = \mathbb{Q}(\zeta_3, \sqrt[3]{D})\) of absolute degree \([N : \mathbb{Q}] = 6\) with cube free radicand \(D \in \mathbb{Z}, D \geq 2\), belongs to precisely one of the following 3 differential principal factorization types, in dependence on the invariant \(U\) and the pair \((A, R)\):

| Type | \(U\) | \(U + 1 = A + R\) | \(A\) | \(R\) |
|------|-------|-------------------|------|------|
| \(\alpha\) | 1     | 2                 | 1    | 1    |
| \(\beta\) | 1     | 2                 | 2    | 0    |
| \(\gamma\) | 0     | 1                 | 1    | 0    |

Proof. A part of the proof is due to Barrucand and Cohn [5] who distinguished 4 different types, \(I = \beta, II, III = \alpha,\) and \(IV = \gamma\). However, Halter-Koch [10] showed the impossibility of one of these types, namely type II. Our new proof with different methods is given in [1, Thm. 6.2]. \(\square\)

5.7. Differential principal factorization (DPF) types of real dihedral fields. Now we state the classification theorem for pure quintic fields \(L = \mathbb{Q}(\sqrt[p]{D})\) and their Galois closure \(N = \mathbb{Q}(\zeta_5, \sqrt[p]{D})\), that is the metacyclic case \(p = 5\). The coarse classification of \(N\) according to the invariants \(U\) and \(A\) alone is closely related to the classification of totally real dihedral fields of degree \(2p\) with any odd prime \(p\) by Nicole Moser [27, Thm. III.5, p. 62], as illustrated in Figure 6. The coarse types \(\alpha, \beta, \gamma, \delta, \varepsilon\) are completely analogous in both cases. Additional types \(\zeta, \eta, \vartheta\) are required for pure quintic fields, because there arises the possibility that the primitive fifth root of unity \(\zeta_5\) occurs as relative norm \(N_{N/K}(\mathbb{Z})\) of a unit \(Z \in U_N\). Due to the existence of radicals in the pure quintic case, the \(\mathbb{F}_p\)-dimension \(A\) of the vector space of absolute DPF exceeds the corresponding dimension for totally real dihedral fields by one (see Remark 5.1).

**Figure 6. Classification of Totally Real Dihedral and Pure Quintic Fields**

![Diagram](image)

The fine classification of \(N\) according to the invariants \(U, A, R\) and \(C\) in the totally real dihedral situation with \(U + 1 = A + R + C\) splits type \(\alpha\) with \(U = 1, A = 0\) further in type \(\alpha_1\) with
$C = 2$ (double capitulation), type $\alpha_2$ with $C = R = 1$ (mixed capitulation and relative DPF), type $\alpha_3$ with $R = 2$ (double relative DPF), type $\beta$ with $U = A = 1$ in type $\beta_1$ with $C = 1$ (capitulation), type $\beta_2$ with $R = 1$ (relative DPF), and type $\delta$ with $U = A = 0$ in type $\delta_1$ with $C = 1$ (capitulation), type $\delta_2$ with $R = 1$ (relative DPF). In the pure quintic situation with $U + 1 = A + I + R$ [26], however, we arrive at the second of the following theorems where we oppose the two classifications.

**Theorem 5.6.** Each totally real dihedral field $N/\mathbb{Q}$ of absolute degree $[N : \mathbb{Q}] = 2p$ with an odd prime $p$ belongs to precisely one of the following 9 differential principal factorization types, in dependence on the invariant $U$ and the triplet $(A, R, C)$.

| Type | $U$ | $U + 1 = A + R + C$ | $A$ | $R$ | $C$ |
|------|-----|---------------------|-----|-----|-----|
| $\alpha_1$ | 1 | 2 | 0 | 0 | 2 |
| $\alpha_2$ | 1 | 2 | 0 | 1 | 1 |
| $\alpha_3$ | 1 | 2 | 0 | 2 | 0 |
| $\beta_1$ | 1 | 2 | 1 | 0 | 1 |
| $\beta_2$ | 1 | 2 | 1 | 1 | 0 |
| $\gamma$ | 1 | 2 | 2 | 0 | 0 |
| $\delta_1$ | 0 | 1 | 0 | 0 | 1 |
| $\delta_2$ | 0 | 1 | 0 | 1 | 0 |
| $\varepsilon$ | 0 | 1 | 1 | 0 | 0 |

Proof. Consequence of the Corollaries 5.1 and 5.2. See also [27, Thm. III.5, p. 62] and [18]. □

**Theorem 5.7.** Each pure metacyclic field $N = \mathbb{Q}(\zeta_5, \sqrt[5]{D})$ of absolute degree $[N : \mathbb{Q}] = 20$ with 5-th power free radicand $D \in \mathbb{Z}, D \geq 2$, belongs to precisely one of the following 13 differential principal factorization types, in dependence on the invariant $U$ and the triplet $(A, I, R)$.

| Type | $U$ | $U + 1 = A + I + R$ | $A$ | $I$ | $R$ |
|------|-----|---------------------|-----|-----|-----|
| $\alpha_1$ | 2 | 3 | 1 | 0 | 2 |
| $\alpha_2$ | 2 | 3 | 1 | 1 | 1 |
| $\alpha_3$ | 2 | 3 | 1 | 2 | 0 |
| $\beta_1$ | 2 | 3 | 2 | 0 | 1 |
| $\beta_2$ | 2 | 3 | 2 | 1 | 0 |
| $\gamma$ | 2 | 3 | 3 | 0 | 0 |
| $\delta_1$ | 1 | 2 | 1 | 0 | 1 |
| $\delta_2$ | 1 | 2 | 1 | 1 | 0 |
| $\varepsilon$ | 1 | 2 | 2 | 0 | 0 |
| $\zeta_1$ | 1 | 2 | 1 | 0 | 1 |
| $\zeta_2$ | 1 | 2 | 1 | 1 | 0 |
| $\eta$ | 1 | 2 | 2 | 0 | 0 |
| $\theta$ | 0 | 1 | 1 | 0 | 0 |

The types $\delta_1, \delta_2, \varepsilon$ are characterized additionally by $\zeta_5 \notin N_{N/K}(U_N)$, and the types $\zeta_1, \zeta_2, \eta$ by $\zeta_5 \in N_{N/K}(U_N)$.

Proof. The proof is given in [26, Thm. 6.1]. □
6. Complete verification of the conjecture of Scholz

Let \( L \) be a non-cyclic totally real cubic field. Then \( L \) is non-Galois over the rational number field \( \mathbb{Q} \) with two conjugate fields \( L' \) and \( L'' \). The Galois closure \( N \) of \( L \) is a totally real dihedral field of degree 6, which contains a unique real quadratic field \( K \), as illustrated in Figure 7.

\[
\text{Figure 7. Hasse Subfield Diagram of } \mathbb{Q}/\mathbb{Q}
\]

![Hasse Subfield Diagram](image)

In 1930, Helmut Hasse \[12\] determined the discriminants of \( L \) \[12\] pp. 567 and 575] and \( N \) \[12\] p. 566], in dependence on the discriminant of \( K \) and on the class field theoretic conductor \( f = f(N/K) \) of the cyclic cubic, and thus abelian, relative extension \( N/K \):

\[
d_L = f^2 \cdot d_K, \quad \text{and} \quad d_N = f^4 \cdot d_K^2.
\]

Three years later, in 1933, Arnold Scholz \[31\] p. 216] determined the relation

\[
h_N = \frac{I}{9} \cdot h_L^2 \cdot h_K
\]

between the class numbers of the fields \( N, L \) and \( K \), in dependence on the index of subfield units, \( I = (U_N : U_0) = 3^E \), where \( U_0 = \langle U_K, U_L, U_{K'}, U_{L''} \rangle \) and \( E \in \{0, 1, 2\} \).

Note that \( E = 0 \), resp. \( I = 1 \), is the distinguished case where the unit group \( U_N \) of the normal field \( N \) is entirely generated by all proper subfield units, that is \( U_N = U_0 \).

Scholz was able to give explicit numerical examples \[31\] p. 216], for \( E = 1 \) (\( d_L = 229 \)), resp. \( E = 2 \) (\( d_L = 148 \)), but not for \( E = 0 \), and he formulated the following hypothesis.

**Conjecture 6.1.** (The Conjecture of Scholz, 1933, illustrated in Figure 8)

There should exist non-Galois totally real cubic fields \( L \) whose Galois closure \( N \) is either

1. unramified, with conductor \( f = 1 \), over some real quadratic field \( K \) with 3-class rank \( \varrho_3(K) = 2 \) whose complete 3-elementary class group capitulates in \( N \), and such that \( U_N = U_0 \) \[31\] p. 219], or
2. ramified, with conductor \( f > 1 \), over some real quadratic field \( K \) such that also \( U_N = U_0 \) \[31\] p. 221] (here, Scholz calls \( N \) a ring class field over \( K \), by abuse of language).

We point out that, in the unramified situation \( f = 1 \), \( d_L = d_K \) is a quadratic fundamental discriminant, and \( d_N = d_K^2 \) is a perfect cube, according to formula \[6.1\]. In this unramified case, the verification of Conjecture 6.1 can be obtained from a more general theorem, since any real quadratic field \( K \) with 3-class rank \( \varrho_3(K) = 2 \) possesses a multiplet of four unramified cyclic cubic extensions \( N_1, \ldots, N_4 \), that is a quartet of absolutely dihedral fields of degree 6 \[21\] with non-Galois totally real subfields \( L_1, \ldots, L_4 \), each of them selected among three conjugate fields.

For such a quartet, Chang and Foote \[8\] introduced the concept of the capitulation number \( 0 \leq \nu(K) \leq 4 \), defined as the number of those members of the quartet in which the complete 3-elementary class group of \( K \) capitulates. For this number \( \nu(K) \), the following theorem holds.

**Theorem 6.1.** For each value \( 0 \leq \nu \leq 4 \), there exists a real quadratic field \( K \) with 3-class rank \( \varrho_3(K) = 2 \) and capitulation number \( \nu(K) = \nu \). It is even possible to restrict the claim to fields with elementary 3-class group of type \( \text{Cl}_3(K) \cong C_3 \times C_3 \).
Proof. From the viewpoint of finite $p$-group theory, this theorem is a proven statement about the possible transfer kernel types of finite metabelian 3-groups $G$ with abelianization $G/G' \simeq (3,3)$ applied to the second 3-class group $G := \text{Gal}(F_3(K)/K)$ of $K$. However, it is easier to give explicit numerical paradigms for each value of $\nu(K)$. We have the following minimal occurrences:

- $\nu(K) = 4$ for $d_K = 62501$,
- $\nu(K) = 3$ for $d_K = 32009$,
- $\nu(K) = 2$ for $d_K = 710652$,
- $\nu(K) = 1$ for $d_K = 534824$,
- $\nu(K) = 0$ for $d_K = 214712$,

which have been computed by ourselves in [21]. This completes the proof of Theorem 6.1.

Remark 6.1. We have the priority of discovering the first examples of real quadratic fields $K$ with $\nu(K) \in \{0, 1, 2\}$ in [21]. However, the first examples of real quadratic fields $K$ with $\nu(K) \in \{3, 4\}$ are due to Heider and Schmithals [13], who performed a mainframe computation on the CDC Cyber of the University of Cologne, and thus the following corollary is proven since 1982 already.

**Corollary 6.1.** (Verification of Conjecture 6.1, (1) for unramified extensions; see Figure 9)
There exist non-Galois totally real cubic fields $L$ whose Galois closure $N$ is unramified, with conductor $f = 1$, over a real quadratic field $K$ with 3-class rank $\nu_3(K) = 2$ whose complete 3-elementary class group capitulates in $N$, and which therefore has $U_N = U_0$. The minimal discriminant of such a field $L$ is $d_L = 32009$ (three of four fields, [13], 49 years after [31]).
Proof. It suffices to take a real quadratic field $K$ with $1 \leq \nu(K) \leq 4$ in Theorem 6.1. In view of the minimal discriminant, we select $\nu(K) = 3$ and obtain $U_N = U_0$ for $d_L = d_K = 32009$.

Concerning the ramified situation $f > 1$ in Conjecture 6.1 (2), Scholz does not explicitly impose any conditions on the underlying real quadratic field $K$. We suppose that he also tacitly assumed a real quadratic field $K$ with 3-class rank $\varrho_3(K) = 2$. However, more recent extensions of the theory of dihedral fields by means of differential principal factorizations and Galois cohomology, two concepts which we have expanded thoroughly in the preparatory sections §5.2, 5.3 and 5.5, revealed that for $U_N = U_0$ no constraints on the $p$-class rank $\varrho_p(K)$ are required. In 1975, Nicole Moser [24] used the Galois cohomology $H^0(G, U_N) \cong U_K/N_{N/K}(U_N)$ of the unit group $U_N$ of the normal closure $N$ as a module over $G = \text{Gal}(N/K)$ to establish a fine structure with five possible types $\alpha, \beta, \gamma, \delta, \varepsilon$ on the coarse classification by three possible values of the index of subfield units:

\[ (U_N : U_0) = 1 \iff \text{type } \alpha \iff (U_K : N_{N/K}(U_N)) = 3, \]

\[ (U_N : U_0) = 3 \iff \text{type } \beta \iff (U_K : N_{N/K}(U_N)) = 3 \text{ or type } \delta \iff (U_K : N_{N/K}(U_N)) = 1, \]

\[ (U_N : U_0) = 9 \iff \text{type } \gamma \iff (U_K : N_{N/K}(U_N)) = 3 \text{ or type } \varepsilon \iff (U_K : N_{N/K}(U_N)) = 1. \]

Thus, Moser’s refinement does not illuminate the situation $U_N = U_0$ (⇒ type $\alpha$) of Scholz’s conjecture more closely. Meanwhile, Barrucand and Cohn [5] had coined the concept of differential principal factorization (DPF) for pure cubic fields. In 1991, we generalized the theory of DPFs for dihedral fields of both signatures [13], and we obtained a hyperfine structure by splitting Moser’s types further according to the $F_p$-dimensions $C$ of the capitulation kernel $\ker(T_{K,N})$ and $R$ of the space of relative DPFs of $N/K$, which we recalled in the preparatory section §5.7. In particular, type $\alpha$ with $U_N = U_0$ splits into three subtypes:

- type $\alpha_1$ ⇐⇒ $C = 2$, $R = 0$, which implies $\varrho_p(K) \geq 2$,
- type $\alpha_2$ ⇐⇒ $C = 1$, $R = 1$, which implies $\varrho_p(K) \geq 1$ and a split prime divisor of $f$ ($s \geq 1$),
- type $\alpha_3$ ⇐⇒ $C = 0$, $R = 2$, which is compatible with any $\varrho_p(K) \geq 0$, but requires $s \geq 2$.

Consequently, we were led to the following refinement of Conjecture 6.1 (2).

**Conjecture 6.2.** (Conjecture of D. C. Mayer, 1991)
Non-Galois totally real cubic fields $L$ whose Galois closure $N$ is ramified, with conductor $f > 1$, over some real quadratic field $K$, and is of type $\alpha$, with $U_N = U_0$, should exist for each of the following three situations:

1. $\alpha_1$ with $(\text{dim}_{F_3}(\ker(T_{K,N}))) = 2$ and $\varrho_3(K) = 2$, $s = 0$,
2. $\alpha_2$ with $(\text{dim}_{F_3}(\ker(T_{K,N}))) = 1$ and $\varrho_3(K) = 1$, $s = 1$,
3. $\alpha_3$ with $(\text{dim}_{F_3}(\ker(T_{K,N}))) = 0$ and $\varrho_3(K) = 0$, $s = 2$,

where $T_{K,N} : \text{Cl}_3(K) \to \text{Cl}_3(N)$, $a \cdot \mathcal{P}_K \to (a\mathfrak{O}_N) \cdot \mathcal{P}_N$, denotes the transfer homomorphism of 3-classes from $K$ to $N$, and $s$ counts the prime divisors of the conductor $f$ which split in $K$.

![Figure 10. Ring Class Field modulo $f = 63 = 3^2 \cdot 7$ over $K$](image)

**Theorem 6.2.** (Verification of Conjecture 6.2, (3), and Conjecture 6.1, (2); see Figure 10)
There exist non-Galois totally real cubic fields $L$ whose Galois closure $N$ is ramified, with conductor $f > 1$ divisible by two prime divisors which split in $K$, over a real quadratic field $K$ with 3-class rank $\varrho_3(K) = 0$, without capitulation in $N$, but which nevertheless has $U_N = U_0$. The minimal discriminant of such a field $L$ is $d_L = 146853 = (7 \cdot 9)^2 \cdot 37$ (singulet, 19), 58 years after 31.
Proof. This was proved in the numerical supplement [19] of our paper [18] by computing a gapless list of all 10 015 totally real cubic fields $L$ with discriminants $d_L < 200000$ on the AMDAHL mainframe of the University of Manitoba. There occurred the minimal discriminant $d_L = 146853 = f^2 \cdot d_K$ with $d_K = 37$ and conductor $f = 63 = 3^2 \cdot 7$ divisible by two primes which both split in $K$, i.e. $s = 2$. This is a necessary requirement for a two-dimensional relative principal factorization with $R = 2$ and is unique up to $d_L < 200000$. (The next is $d_L = 240149$ with $f = 7 \cdot 13$.) There is only a single field $L$ with this discriminant $d_L = 146853$ (forming a singulet). 

Our discovery of the truth of Theorem 6.2 with the aid of the list [19] was a random hit without explicit intention to find a verification of Scholz’s conjecture. Unfortunately, [19] does not contain examples of the unique missing DPF type $\alpha_2$. It required more than 25 years until we focused on an attack against this lack of information. In contrast to the techniques of [19], we did not use the Voronoi algorithm [33] after cumbersome preparation of generating polynomials for totally real cubic fields, but rather the class field theory routines of Magma [6, 7, 17] for a direct generation of the fields as subfields of 3-ray class fields modulo conductors $f > 1$.

Theorem 6.3. (Verification of Conjecture 6.2 (2.2), and Conjecture 6.1 (2); see Figure 11) There exist non-Galois totally real cubic fields $L$ whose Galois closure $N$ is ramified, with conductor $f > 1$ divisible by a single prime divisor that splits in $K$, i.e. $s = 1$, over a real quadratic field $K$ with 3-class rank $q_3(K) = 1$, with complete capitulation of the elementary 3-class group in $N$, but nevertheless with $U_N = U_0$. The minimal discriminant of such a field $L$ is $d_L = 966397 = 19^2 \cdot 2677$ (two of three fields, 19 November 2017, 84 years after [31], 1933).

Proof. The proof is conducted in the following section §6.1.

Theorem 6.4. (Verification of Conjecture 6.2 (2.1), and Conjecture 6.1 (2); see Figure 12) There exist non-Galois totally real cubic fields $L$ whose Galois closure $N$ is ramified, with conductor $f > 1$ divisible only by prime divisors which do not split in $K$, over a real quadratic field $K$ with 3-class rank $q_3(K) = 2$, with complete capitulation of the elementary 3-class group in $N$, and thus with $U_N = U_0$. The minimal discriminant of such a field $L$ is $d_L = 18251060 = 2^2 \cdot 4562765$ (five of nine fields, 23 November 2017, 84 years after [31], 1933).

Proof. The proof is conducted in the following section §6.2.

The proof of Theorem 6.4 and Theorem 6.3 is conducted in the following sections on real quadratic base fields with 3-class rank 1 and 2.
6.1. Real quadratic base fields with 3-class rank 1. In Table 1 we present the results of our search for the minimal discriminant $d_L$, resp. $d_N$, of a non-Galois totally real cubic field $L$, resp. its normal closure $N$, with differential principal factorization type $\alpha_2$. The unramified component is a singulet which must be of DPF type $\delta_1$. For each member of the ramified triplet the DPF types $\alpha_2, \beta_1, \beta_2, \delta_1, \delta_2, \varepsilon$ would be possible, but only the types $\alpha_2, \delta_1, \delta_2$ occur usually.

The desired minimum is clearly given by $d_L = 19^2 \cdot 2677 = 966397$ with two occurrences of ramified extensions with DPF type $\alpha_2$. For $f = 3^2$, the condition $d_K \equiv 1 \pmod{3}$ is required.

Table 1. Heterogeneous Quartets of Dihedral Fields with a Splitting Prime $f$

| $f$ | $d_K$ | $d_L = f^2 \cdot d_K$ | unramified component | ramified components |
|-----|------|----------------------|----------------------|---------------------|
| $3^2$ | 14197 | 1149957 | $\delta_1$ | $\alpha_2, \delta_1, \delta_2$ |
| 7 | 21781 | 1067269 | 1 | 3 0 0 |
| 13 | 9749 | 1647581 | 1 | 2 1 0 |
| 19 | 2677 | 966397 | 1 | 2 1 0 |
| 31 | 3877 | 3725797 | 1 | 2 0 1 |
| 37 | 5477 | 7498013 | 1 | 1 0 2 |
| 43 | 4933 | 9121117 | 1 | 3 0 0 |
| 61 | 3981 | 14813301 | 1 | 3 0 0 |
| 67 | 4493 | 20169077 | 1 | 2 0 1 |
| 73 | 10733 | 57196157 | 1 | 3 0 0 |

Since we know a small candidate $d_L = 966397$ for the minimal discriminant, and since the smallest quadratic discriminant with $\Delta_3(K) = 1$ is $d_K = 229$, we only have to investigate prime and composite conductors $f = \sqrt{\frac{d_L}{d_K}}$ with $s \geq 1$ and

$$f \leq \sqrt{\frac{966397}{229}} \approx \sqrt{4220} \approx 64.9,$$

which are divisible by a split prime, that is,

$$f \in \{7, 9 = 3^2, 13, 14 = 2 \cdot 7, 18 = 2 \cdot 3^2, 19, 21 = 3 \cdot 7, 26 = 2 \cdot 13, 31, 35 = 5 \cdot 7, 37, 38 = 2 \cdot 19, 39 = 3 \cdot 13, 42 = 2 \cdot 3 \cdot 7, 43, 45 = 5 \cdot 3^2, 57 = 3 \cdot 19, 61, 62 = 2 \cdot 31, 63 = 7 \cdot 3^2\}.$$
Table 2. Heterogeneous Quartets of Dihedral Fields with Conductor $f$

| $f$ | condition | $d_K$ | $d_L = f^2 \cdot d_K$ | unramified component | ramified components $\alpha_2$, $\beta_1$, $\beta_2$, $\delta_1$, $\delta_2$ |
|-----|-----------|-------|------------------------|----------------------|--------------------------------|
| 7   | $d_K = 1 \ (3)$ | 21781 | 1067 269               | 1                    | 2 0 0 1 0                       |
| $3^2$ | $d_K = 1 \ (3)$ | 14197 | 1149 957               | 1                    | 3 0 0 0 0                       |
| 13  | $d_K = 3 \ (9)$ | 9749  | 1647 581               | 1                    | 2 0 0 0 0                       |
| 19  | $d_K = 6 \ (9)$ | 2677  | 966 397                | 1                    | 2 0 0 0 1                       |
| 31  | $d_K = 6 \ (9)$ | 3877  | 3725 797               | 1                    | 2 0 0 0 0                       |
| 37  | $d_K = 6 \ (9)$ | 5477  | 7498 013               | 1                    | 1 0 0 0 2                       |
| 43  |             | 4933  | 9121 117               | 1                    | 3 0 0 0 0                       |
| 61  |             | 3981  | 14813 301              | 1                    | 3 0 0 0 0                       |
| 2·7 | $d_K = 1 \ (3)$ | 6997  | 1371 412               | 1                    | 3 0 0 0 0                       |
| 2·$3^2$ | $d_K = 1 \ (3)$ | 16141 | 5229 684               | 1                    | 3 0 0 0 0                       |
| 3·7 | $d_K = 3 \ (9)$ | 28137 | 12408 417              | 1                    | 3 0 0 0 0                       |
| $3^2$·$7$ | $d_K = 6 \ (9)$ | 57516 | 2536 556               | 1                    | 3 0 0 0 0                       |
| 2·13|             | 21557 | 14572 532              | 1                    | 3 0 0 0 0                       |
| 5·7 |             | 14457 | 17709 825              | 1                    | 3 0 0 0 0                       |
| 2·19|             | 13765 | 19876 660              | 1                    | 3 0 0 0 0                       |
| 3·13| $d_K = 3 \ (9)$ | 51528 | 78374 088              | 1                    | 3 0 0 0 0                       |
| 3·13| $d_K = 6 \ (9)$ | 37176 | 56544 696              | 1                    | 3 0 0 0 0                       |
| $2^2$·$3^2$ | $d_K = 3 \ (9)$ | 891237 | 1572 142 068          | 1                    | 4 1 1 0 0                       |
| $2^2$·$3^2$ | $d_K = 6 \ (9)$ | 474261 | 836 596 404            | 1                    | 2 0 1 0 0                       |
| 5·$3^2$ | $d_K = 3 \ (9)$ | 24952 | 50257 800              | 1                    | 1 0 0 1 1                       |
| 3·19| $d_K = 3 \ (9)$ | 24393 | 79252 857              | 1                    | 3 0 0 0 0                       |
| 3·19| $d_K = 6 \ (9)$ | 39417 | 128065 833             | 1                    | 3 0 0 0 0                       |
| 2·31|             | 7573  | 29110 612              | 1                    | 3 0 0 0 0                       |
| 7·$3^2$ | $d_K = 1 \ (3)$ | 2941  | 11672 829             | 1                    | 3 0 0 0 0                       |
| 7·$3^2$ | $d_K = 2 \ (3)$ | 23993 | 95228 217             | 1                    | 3 0 0 0 0                       |

The result of the investigations is summarized in Table 2 which clearly shows that $d_L = 966 397$, for $d_K = 2677$ and splitting prime conductor $f = 19$ bigger than the conductor $f = 1$ of unramified extensions $N/K$, is the desired minimal discriminant of a totally real cubic field with ramified extension $N/K$, DPF type $\alpha_2$ and $U_N = U_0$. The information has been computed with class field theoretic routines of Magma [17].

6.2. Real quadratic base fields with 3-class rank 2. In this situation, the unramified quartet is non-trivial, since the two DPF types $\alpha_1$ and $\delta_1$ are possible. These quartets have been studied thoroughly in [21], and in the Tables 3 and 4, we use the corresponding notation for capitulation types.

In Table 3, we present the results of the crucial search for the minimal discriminant $d_L$, resp. $d_N$, of a non-Galois totally real cubic field $L$, resp. its normal closure $N$, with differential principal factorization type $\alpha_1$ such that $N/K$ is a ramified extension of a real quadratic field $K$ with 3-class rank $\beta_1 = 2$. We tried to fix the minimal possible conductor $f > 1$, namely $f = 2$. This experiment was motivated by the fact that the conductor $f$ enters the expression $d_L = f^2 \cdot d_K$ in its second power, whereas the quadratic discriminant $d_K$ enters linearly. Consequently, the probability to find the minimum of $d_L$ is higher for small $f$ than for small $d_K$.

The table is ordered by increasing quadratic fundamental discriminants $d_K$ and gives $d_L = 2^2 \cdot d_K$ and the extended Artin pattern of the heterogeneous tridecuplet of cyclic cubic relative extensions $N/K$ consisting of an unramified quartet $(N_{1,1}, \ldots, N_{1,4})$ with conductor $f = 1$ and a ramified nonet $(N_{2,1}, \ldots, N_{2,9})$ with conductor $f = 2$, grouped by the possible two, resp. four,
DPF types. Capitulation kernels \( \kappa \) are abbreviated by digits, 0 for two-dimensional and 1, \ldots, 4 for one-dimensional principalization, and an asterisk * for a trivial kernel. Transfer targets \( \tau \) are abbreviated by logarithmic abelian type invariants of 3-class groups. Symbolic exponents always denote repetition.

The desired minimum is given by \( d_L = 4 \cdot 4562765 = 18251060 \) with five occurrences of ramified extensions with DPF type \( \alpha_1 \). Generally, there is an abundance of ramified extensions with two-dimensional capitulation kernel: at least three and at most all nine of a nonet.

**Table 3. Heterogeneous multiplets of Artin patterns for \( f = 2 \)**

| \( d_K \) | Type | \( \alpha_1 \) | \( \delta_1 \) | \( \alpha_1 \) | \( \beta_1 \) | \( \delta_1 \) | \( \varepsilon \) |
|---|---|---|---|---|---|---|---|
| 4562765 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^2 \) | 1 \( 1^4 \) | 1 \( 1^4 \) | 12 \( 1^5 \) | 21^3 |
| 7339397 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 2 \( (1^4)^7 \) | 2 \( 21^3 \) | 1 \( 21^3 \) |
| 7601461 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 2 \( (1^4)^7 \) | 2 \( 21^3 \) | 234 \( (21^3)^3 \) |
| 7657037 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 1 \( (1^4)^6 \) | 1 \( 21^3 \) | 12 \( 1^5 \) | 21^3 |
| 7736749 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 3 \( (1^4)^7 \) | 23 \( 21^3 \) |
| 8102053 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 4 \( (1^4)^7 \) | 7 \( (21^3)^2 \) |
| 9182229 | a.2 | \( 0^3 (1^2) \) | 4 \( 21 \) | 8 \( 21^2 (1^4)^7 \) | 2 \( 21^3 \) |
| 9500453 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 8 \( 21^2 (1^4)^7 \) | 2 \( 21^3 \) |
| 9533357 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 6 \( (1^4)^6 \) | 23 \( (21^3)^2 \) |
| 1003845 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 4 \( (1^4)^7 \) | 21^3 \( 7 \( 21^3 \) |
| 12071253 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 7 \( (1^4)^7 \) | 3 \( 21^3 \) | 2 \( 21^3 \) |
| 14266853 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 8 \( 21^2 (1^4)^7 \) | 23 \( (21^3)^2 \) |
| 14308421 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 4 \( (1^4)^4 \) | 4 \( 21^3 \) |
| 1435775 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 7 \( (1^4)^7 \) | 23 \( (21^3)^2 \) |
| 14395013 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 6 \( (1^4)^6 \) | 1 \( 21^3 \) | 23 \( (21^3)^2 \) |
| 1513149 | D.10 | 1 \( 1^3 \) | 21 \( 21^3, 21^3 \) | 23 \( (21^3)^2 \) |
| 16385471 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 4 \( (1^4)^4 \) | 3 \( 21^3 \) | 2 \( 21^3 \) |

Table 3 shows analogous results for the conductor \( f = 5 \), that is, \( d_L = 5^2 \cdot d_K \). The minimum \( d_L = 25 \cdot 1049512 = 26237800 \) is clearly beaten by the minimum \( 4 \cdot 4562765 = 18251060 \) in Table 3.

**Table 4. Heterogeneous multiplets of Artin patterns for \( f = 5 \)**

| \( d_K \) | Type | \( \alpha_1 \) | \( \delta_1 \) | \( \alpha_1 \) | \( \beta_1 \) | \( \delta_1 \) | \( \varepsilon \) |
|---|---|---|---|---|---|---|---|
| 1049512 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 0 \( (1^4)^4 \) | 234 \( (21^3)^5 \) |
| 2461937 | a.2 | \( 0^3 (1^2) \) | 4 \( 21 \) | 0 \( (1^4)^4 \) | 12 \( (21^3)^2 \) |
| 2811613 | a.3* | \( 0^3 (1^2) \) | 1 \( 1^3 \) | 5 \( 21^2 (1^4)^4 \) | 2 \( 21^3 \) | 123 \( (21^3)^3 \) |
| 3091133 | a.3 | \( 0^3 (1^2) \) | 1 \( 21 \) | 0 \( (1^4)^4 \) | 4 \( 21^3 \) | 123 \( (21^3)^4 \) |
| 5858753 | G.19 | 2143 \( (21^4) \) | 0 \( (1^4)^4 \) | 3 \( 21^3 \) | * \( 21^4 \) |
| 6036188 | D.10 | 3431 \( (21^3) \) | 0 \( (1^4)^8 \) | * \( 21^2 \) |

Since we know a small candidate \( d_L = 18251060 \) for the minimal discriminant, and since the smallest quadratic discriminant with \( \varphi_3(K) = 2 \) is \( d_K = 32009 \), we only have to investigate prime
and composite conductors \( f = \sqrt{\frac{d_K}{d_L}} \) with

\[
f \leq \sqrt{\frac{18251060}{32009}} \approx \sqrt{570.2} \approx 23.9,
\]

that is,

\[
f \in \{2, 3, 5, 6 = 2 \cdot 3, 7, 9 = 3^2, 10 = 2 \cdot 5, 11, 13, 14 = 2 \cdot 7,
15 = 3 \cdot 5, 17, 18 = 2 \cdot 3^2, 19, 21 = 3 \cdot 7, 22 = 2 \cdot 11, 23\}.
\]

Table 5. Heterogeneous tridecuplets of dihedral fields with conductor \( f \)

| \( f \) | condition | \( d_K \) | \( d_L = f^2 \cdot d_K \) | unramified components \( \alpha_1 \) \( \delta_1 \) | \( f \) | \( \alpha_1 \) \( \beta_1 \) \( \delta_1 \) \( \varepsilon \) |
|-------|-----------|-----------|---------------------|-----------------|---|-------------|-------------|-------------|-------------|
| 2     |            | 4562765   | 18251060            | 3              | 1  | 5           | 1           | 3           | 0           |
| 3     | \( d_K \equiv 3 \pmod{9} \) | 9964821   | 8968389             | 3              | 1  | 4           | 0           | 4           | 1           |
| 5     |            | 1049512   | 26237800            | 3              | 1  | 4           | 0           | 5           | 0           |
| 7     |            | 966053    | 47336597            | 3              | 1  | 4           | 0           | 4           | 1           |
| 3^2   | \( d_K \equiv 1 \pmod{3} \) | 1482568   | 120088008           | 3              | 1  | 5           | 1           | 2           | 1           |
| 3^2   | \( d_K \equiv 2 \pmod{3} \) | 2515388   | 203746428           | 3              | 1  | 6           | 1           | 2           | 0           |
| 3^2   | \( d_K \equiv 6 \pmod{9} \) | 621429    | 50335749            | 3              | 1  | 6           | 0           | 3           | 0           |
| 11    |            | 476152    | 57614392            | 3              | 1  | 7           | 0           | 2           | 0           |
| 13    |            | 1122573   | 189714837           | 3              | 1  | 7           | 0           | 2           | 0           |
| 17    |            | 665832    | 192425848           | 3              | 1  | 7           | 0           | 2           | 0           |
| 19    |            | 635909    | 229563149           | 3              | 1  | 5           | 3           | 1           | 0           |
| 23    |            | 390876    | 206773404           | 3              | 1  | 7           | 1           | 1           | 0           |

\( 2 \cdot 3 \) \( d_K \equiv 3 \pmod{9} \) | 5963493 | 214685748 | 3 | 1 | 7 | 2 | 0 | 0 |
| \( 2 \cdot 3 \) \( d_K \equiv 6 \pmod{9} \) | 4305957 | 155014452 | 0 | 4 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 5 \) | 363397 | 36339700 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 7 \) | 358285 | 70223860 | 4 | 0 | 7 | 2 | 0 | 0 |
| \( 3^5  d_K \equiv 3 \pmod{9} \) | 4845432 | 109022200 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 3^5  d_K \equiv 6 \pmod{9} \) | 1646817 | 370533825 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 3^2 \) \( d_K \equiv 1 \pmod{3} \) | 2142445 | 694152180 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 3^2 \) \( d_K \equiv 2 \pmod{3} \) | 635909 | 206034516 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 3^2 \) \( d_K \equiv 6 \pmod{9} \) | 2538285 | 822404340 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 3 \cdot 7 \) \( d_K \equiv 3 \pmod{9} \) | 3597960 | 1586700360 | 3 | 1 | 6 | 3 | 0 | 0 |
| \( 3 \cdot 7 \) \( d_K \equiv 6 \pmod{9} \) | 3122232 | 1376904312 | 0 | 4 | 6 | 3 | 0 | 0 |
| \( 2 \cdot 11 \) | 2706373 | 1309884532 | 3 | 1 | 6 | 3 | 0 | 0 |

The result of the investigations is summarized in Table 5, which clearly shows that \( d_L = 18251060 \), for \( d_K = 4562765 \) and the smallest possible conductor \( f = 2 \) bigger than the conductor \( f = 1 \) of unramified extensions \( N/K \), is the desired minimal discriminant of a totally real cubic field with ramified extension \( N/K \), DPF type \( \alpha_1 \) and \( U_N = U_0 \). The information has been computed with class field theoretic routines of Magma [17].

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