NOBODIES ARE PERFECT, THEIR SEMIGROUPS ARE NOT

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ABSTRACT. We provide a combinatorial criterion for the finite generation of a valuation semigroup associated with an ample divisor on a smooth toric surface and a non-toric valuation of maximal rank.

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1. INTRODUCTION

Finite generation of semigroups or rings arising from geometric situations has been a question of interest for a long time. As a salient example we can recall the finite generation of canonical or adjoint rings from birational geometry, which motivated the field through the minimal model program for several decades [BCHM10]. The question of finite generation of valuation semigroups arising from Newton–Okounkov theory appears to be equally difficult in general, with little progress beyond the completely toric situation, but potentially great benefits such as the existence of toric degenerations [And13] and completely integrable systems [HK15] to name but a few.

In this article, we make a few steps away from the situation where every participant is toric: we consider valuation semigroups associated with torus-invariant divisors on toric surfaces with respect to a non-toric valuation.

The main idea behind Newton–Okounkov theory is to attach combinatorial/convex-geometric objects to geometric situations to facilitate their analysis, in other words, to partially replicate the setup of toric geometry in settings without any useful group action. The basis for the theory was developed by Kaveh–Khovanskii [KK12] and Lazarsfeld–Mustaţă [LM09] building on earlier work of Okounkov [Oko96], but the subject has seen substantial growth in the last decade. By now applications...
of Newton–Okounkov theory range from combinatorics and representation theory through birational geometry [KL17a, KL17b, KL19, KL18a] to mirror symmetry [RW19] and geometric quantization in mathematical physics.

Given a projective variety $X$ and a divisor or line bundle $D$ on $X$, Newton–Okounkov theory associates to $(X, D)$ a semigroup $S_{Y^*}(D)$, the valuation semigroup, and a convex body $\Delta_{Y^*}(D)$, the Newton–Okounkov body of $D$. Both the valuation semigroup and the Newton–Okounkov body depend however on a maximal rank valuation of the function field of $X$ coming from an admissible flag $Y^*$ of subvarieties.

The Newton–Okounkov body $\Delta_{Y^*}(D)$ is an asymptotic version of $S_{Y^*}(D)$ and is, accordingly, a lot easier to determine. Newton–Okounkov bodies on surfaces end up being almost rational polygons [KLM12]. If the section ring of $L$ is finitely generated then a suitably general flag valuation will yield a rational simplex as its Newton–Okounkov body [AKL14]. In the case of a toric variety $X$ with torus-invariant $D$ and $Y^*$, the associated Newton–Okounkov body recovers the moment polytope of the polarized toric variety.

In this paper, we will focus on the valuation semigroup $S_{Y^*}(D)$, more concretely, on the question whether or not it is finitely generated. It is a classical fact that $S_{Y^*}(D)$ is often not finitely generated even if $X$ is a smooth projective curve.

It is known that $S_{Y^*}(D)$ is a finitely generated semigroup if $X$ is a simplicial toric variety, $D$ a torus-invariant divisor, and $Y^*$ an admissible flag of torus-invariant subvarieties [IM19]. We consider the next open question, namely, the case of toric surfaces and non-torus-invariant flag valuations. Although the divisorial geometry of toric surfaces themselves is not particularly complicated, this relative simplicity ceases to exist once we start blowing up non-toric points. Blowing up many general points quickly leads to notoriously difficult situations like Nagata’s conjecture. Nevertheless, as recent research of Castravet–Laface–Tevelev–Ugaglia [CLTU20] illustrated, blowing up just one point on a toric surface can lead to surfaces with infinitely many negative curves on them.

1.1. Newton–Okounkov bodies. Assume that $X = \mathbb{T}V(\Sigma)$ is a smooth projective toric variety of dimension $n$. Then ample torus-invariant divisors $D$ or their associated line bundles $\mathcal{O}_X(D)$ can be understood as lattice polytopes $\Delta(D)$ in the character lattice $M \cong \mathbb{Z}^n$ of the torus $\mathbb{T}$ acting on $X$. Using this description, the starting fan $\Sigma$ can be recovered as the normal fan of these polytopes.

Moreover, it is a well-known feature of the toric theory that the vector space $\Gamma(X, \mathcal{O}_X(D))$ has the set of lattice points $\Delta(D) \cap M$ as its distinguished basis. That is, the polytope $\Delta(D)$ gives $\Gamma(X, \mathcal{O}_X(D))$ not just a dimension but also a shape.

In [LM09] and [KK12], this concept was generalized to arbitrary varieties $X$ (still of dimension $n$). If we are given a so-called admissible flag

$$Y^* : X = Y_0 \supseteq Y_1 \supseteq \ldots \supseteq Y_n$$
of nested (irreducible) varieties with $\dim Y_i = n - i$ and being smooth in the special point $Y_n$, then for every ample divisor $D$ there is an associated polytope $\Delta_{Y^*}(D)$ ("NObody") in $\mathbb{R}^n$ reflecting many properties of $D$, see Subsection 2.1 for details.

Note that $\mathbb{Z}^n$ has ceased to be a character lattice because there is no longer a torus around. While the NObody $\Delta_{Y^*}(D)$ does not depend on $D$ but only on its numerical class, the dependence on the chosen flag is striking.

If, for instance, $X$ is toric as in the very beginning, then the construction of $\Delta_{Y^*}(D)$ recovers the correspondence between divisors and polytopes we have mentioned above. But to make this true, it requires a toric flag, i.e., all subvarieties $Y_i$ are supposed to be orbit closures.

The fact that toric varieties with toric flags lead to well-known polytopes is not a one-way street. In fact, if, for a general variety $X$ the semigroups $S_{Y^*}(D)$ are finitely generated, then they provide a toric degeneration of $X$. This was shown by [And13]. Observe that in general the finite generation of the semigroup $S_{Y^*}(D)$ is much stronger than $\Delta_{Y^*}(D)$ being polyhedral. This finite generation of the valuation semigroup $S_{Y^*}(D)$ is the main point of this paper.

1.2. Results. In [IM19] the finite generation was shown for complexity one $\mathbb{T}$-varieties with toric flags. Note that this implies the toric case. Whenever one is only interested in the NObody (instead of the semigroup), then there are more results: The most general one solves the question for surfaces using Zariski decomposition [LM09, Theorem 6.4]. In particular, the NObodies are polyhedral in this case. Specializing this situation, [HKW20] have provided an explicit combinatorial description of $\Delta_{Y^*}(D)$ for toric surfaces with certain non-toric flags.

In the present paper, partially inspired by [AP20], we consider the very same setup of toric surfaces, namely with the following admissible flag:

$$Y^* : X = Y_0 \supseteq Y_1 \supseteq Y_2,$$

where $Y_1$ is the closure of a one-parameter subgroup of the torus, which is non-torus-invariant and $Y_2$ is a general smooth point. Then we prove that, in dependence on $Y_1$, the semigroups $S_{Y^*}(D)$ can be both finitely generated and not finitely generated.

The main result of the paper (Theorem 6.8) comes from understanding the relationship between the Newton–Okounkov body of $D$ and the Newton polygon associated with the non-toric flag curve $Y_1$. The significance of our contribution lies in the fact that we infer the finite generation of valuation semigroups from asymptotic/convex geometric data and provide a combinatorial criterion.

In order to be able to state our Theorem, we introduce a bit of terminology. For our flag $Y^*$, we consider a non-torus-invariant curve $Y_1$ given as the closure of the one-parameter subgroup determined by a primitive vector $v_N \in N$, our flag point is going to be $0 \otimes \mathbb{Z} 1$ on the torus $\mathbb{T} \cong \mathbb{N} \otimes \mathbb{Z} \mathbb{C}^\times$. For a strongly convex cone $\sigma \subseteq N_\mathbb{R}$ and a lattice point $u \in \text{int}(\sigma) \cap N$ we say that $u$ is strongly decomposable in $\sigma$ if $u = u' + u''$ for suitable $u', u'' \in \text{int}(\sigma) \cap N$. Given a divisor $D$ on $X$, we construct
strongly convex cones $\sigma^+$ and $\sigma^-$ associated with $v_N$ that are spanned by certain rays of the fan of $X$ (see Definition 6.7). With this said, our main result goes as follows:

**Theorem** (Theorem 6.8). Let $X$ be a smooth toric surface associated with a fan $\Sigma$ and $D$ an ample divisor on $X$. The valuation semigroup $S_{Y^*}(D)$ is finitely generated if and only if $v_N$ is not strongly decomposable in $\sigma^+$ and $-v_N$ is not strongly decomposable in $\sigma^-$. 

To illustrate the combinatorial content, Figure 1 pictures the situation in the case of the 7-gon of [CLTU20], where the blow-up surface $X = \mathbb{T}V(\Sigma)$ accomodates infinitely many negative curves (cf. Figure 10 for a complete picture).

![Figure 1. The good 7-gon $\Delta(D)$.](image)

(A) The polytope $\Delta(D)$ and the rational line segment $\Delta(D)^{v_N}$. (B) The normal fan $\Sigma$ of $\Delta(D)$ together with the two cones $\sigma^+ \ni v_N$ and $\sigma^- \ni -v_N$.

2. Notation and preliminaries

Let $X$ be a 2-dimensional smooth projective variety and assume that we are given an admissible flag

$$Y^* : X = Y_0 \supseteq Y_1 \supseteq Y_2,$$

as in Subsection 1.1, and an ample divisor $D$ on $X$.

2.1. Newton–Okounkov bodies and valuation semigroups. Following [LM09], we obtain a rank 2 valuation-like function (or, equivalently, a rank 2 valuation of the function field of $X$, see [KMR21])

$$\text{val}_{Y^*} : \Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} \to \mathbb{Z}^2$$

as follows: For any non-trivial section $s \in \Gamma(X, \mathcal{L})$ of an invertible sheaf $\mathcal{L}$, e.g., $\mathcal{L} = \mathcal{O}_X(D)$, we define

$$\text{val}_{Y^*}(s) := (\text{val}_1(s), \text{val}_2(s))$$
using the following inductive procedure: \( \text{val}_i(s) := \text{ord}_{Y_1}(s_i) \), where \( \mathcal{L}_1 := \mathcal{L}, s_1 := s \) and \( (\mathcal{L}, s_i) := (\mathcal{L}_{i-1} \otimes \mathcal{O}_X(-\text{val}_{i-1}(s) \cdot Y_{i-1})) \mid_{Y_1, s_{i-1}(Y_1)} \). More concretely, since we work on a surface, we obtain

\[
\text{val}_1(s) = \text{ord}_{Y_1}(s) \quad \text{and} \quad \text{val}_2(s) = \text{ord}_{Y_2}(\tilde{s} \mid_{Y_1}),
\]

where \( \tilde{s} := s \) are understood as global sections of \( \mathcal{L} \) and of \( \mathcal{L} \otimes \mathcal{O}_X(-\text{val}_1(s) \cdot Y_1) \subseteq \mathcal{L} \), respectively.

The valuation semigroup \( S_{Y_1}(D) \) of \( D \) (with respect to the flag \( Y_1 \)) is defined as

\[
S_{Y_1}(D) := \{ (\ell, \text{val}_{Y_1}(s)) \mid s \in \Gamma(X, \mathcal{O}_X(\ell D)) \setminus \{0\}, \ell \in \mathbb{N} \} \subseteq \mathbb{N}^3.
\]

The Newton–Okounkov body of \( D \) (with respect to the flag \( Y_1 \)) is defined as the set

\[
\Delta_{Y_1}(D) := \bigcup_{\ell \geq 1} \frac{1}{\ell} \{ \text{val}_{Y_1}(s) \mid s \in \Gamma(X, \mathcal{O}_X(\ell D)) \setminus \{0\}, \ell \in \mathbb{N} \} \subseteq \mathbb{R}^2.
\]

For a detailed account of Newton–Okounkov theory on surfaces, see [KL18b].

2.2. Toric setup. Let \( N \cong \mathbb{Z}^2 \) be a two-dimensional lattice with dual lattice \( M \) and \( \Sigma \) a smooth complete fan associated with the toric surface \( X = \mathbb{TV}(\Sigma) \). We may assume that our ample divisor \( D \) is toric, hence being represented by a polytope \( \Delta(D) \). Recall that \( \mathbb{T} = \text{Spec}(\mathbb{C}[M]) \) is our torus acting on \( X \), hence \( M \) becomes its character lattice and \( N \) the associated lattice of one-parameter subgroups.

Next, fix an admissible flag \( Y_1 : X \supseteq Y_1 \supseteq Y_2 \) as follows: we take \( Y_1 := C_v \) to be the closure of the one-parameter subgroup \( \lambda^v : \mathbb{C}^* \to \mathbb{T} \) given by a primitive element \( v_N \in N \), that is,

\[
C_v := \lambda^v(\mathbb{C}^*) = (v_N \otimes_{\mathbb{Z}} \mathbb{C}^*) \cdot (0 \otimes_{\mathbb{Z}} 1),
\]

where \( 0 \otimes_{\mathbb{Z}} 1 \in \mathbb{T} \cong N \otimes_{\mathbb{Z}} \mathbb{C}^* \) denotes the unit element (cf. Subsection 1.2). The flag curve \( C_v \) is an irreducible curve containing \( 0 \otimes_{\mathbb{Z}} 1 \) and we pick \( Y_2 := \{0 \otimes_{\mathbb{Z}} 1\} \).

Since we are in the surface case, \( C_v \) is a hypersurface. Within the torus \( \mathbb{T} \), it is given by the binomial equation \( f := x^M - 1 \) with \( v_M \in M \) being one of the two primitive elements of \( v_N^\perp \subseteq M_{\mathbb{R}} \). The associated Newton polytope

\[
\Delta_{\text{newt}} := \text{newt}(f)
\]

is the line segment \([0, v_M]\) connecting \( 0 \) and \( v_M \) in \( M_{\mathbb{R}} \).

2.3. Torifying the curve. Note that \( C_v \) is also a prime (Cartier) divisor on \( X \), which intersects all torus-invariant curves, i.e., all boundary curves of \( X \), properly. Therefore, \( C_v \) is nef and

\[
C'_v := C_v - \text{div}(f)
\]

is its \( \mathbb{T} \)-invariant representative.

Lemma 2.1. The polygon \( \Delta_{\text{newt}} := \Delta(C'_v) \) is given by

\[
\Delta_{\text{newt}} = \{ m \in M_{\mathbb{R}} \mid \langle m, \rho \rangle \geq \min\{0, \langle v_M, \rho \rangle\} \text{ for all } \rho \in \Sigma(1) \},
\]

where the rays \( \rho \in \Sigma(1) \) are identified with their first (hence primitive) lattice points.
Proof. The prime divisor $D_\rho = \overline{\text{orb}(\rho)}$ associated with a ray $\rho \in \Sigma(1)$ appears in the torus-invariant Weil divisor $C'_v$ as often as it does in the (non-equivariant) principal divisor $- \text{div}(f)$. Thus, $\Delta_{\text{nef}} = \{ m \in M_\mathbb{R} \mid \langle m, \rho \rangle \geq \text{ord}_\rho(f) \text{ for all } \rho \in \Sigma(1) \}$ and it remains to discuss $\text{ord}_\rho(f)$. If $\rho \neq \pm v_N$, then the $\rho$-orders of the two summands of $f = x^{v_M} - 1$ are different, hence
$$\text{ord}_\rho(f) = \min\{\text{ord}_\rho(1), \text{ord}_\rho(x^{v_M})\} = \min\{0, \langle v_M, \rho \rangle\}.$$ If $\rho = \pm v_N$, then the situation looks locally like $\text{ord}_y(x - 1) = 0$ in $A^2$. □

Compare [HKW20, Proposition 3.1] for a different proof.

Example 2.2. Let $X = \text{TV}(\Sigma)$ be the toric surface associated with the fan $\Sigma$, where $\Sigma(1) = \{ \rho_i \mid 0 \leq i \leq 3 \}$ with $\rho_0 = (-1,0)$, $\rho_1 = (0,-1)$, $\rho_2 = (1,2)$, and $\rho_3 = (0,1)$.

We again, as we already did in Lemma 2.1, identify the rays $\rho_i$ with their generating lattice points (cf. Figure 2A). We denote by $D_i := \overline{\text{orb}(\rho_i)}$ ($0 \leq i \leq 3$) the toric prime divisors on $X$. Then, $D := 8D_2 + 3D_3$ is an ample divisor on $X$, which corresponds to the polytope $\Delta(D) = \text{conv}([0,0], [-8,0], [-2,-3], [0,-3])$.

We take $C_v = \{(t^{\frac{1}{2}}), t \in \mathbb{C}\}$ as our curve for the non-toric flag. This means that $v_N = (-2,3) \in N$ and $v_M = [-3,-2] \in M$, hence $\Delta_{\text{newt}} = \text{conv}([0,0], [-3,-2])$.

Since the boundary part of $\text{div}(f)$ equals $-7D_2 - 2D_3$, we obtain $C'_v = 7D_2 + 2D_3$ with nef polytope (cf. Figure 2B)
$$\Delta_{\text{nef}} = \text{conv}([0,0], [-7,0], [-3,-2], [0,-2]) .$$

![Figure 2](image-url)
2.4. An alternative view on $\Delta_{\text{nef}}$. Beside the explicit description of Lemma 2.1, it is possible to describe the shape of $\Delta_{\text{nef}}$ in the following more combinatorial way:

2.4.1. The relation $v_M \in v_N^\perp$ among our curve parameters means

$$\langle 0, v_N \rangle = \langle v_M, v_N \rangle = 0,$$

i.e., $\Delta_{\text{newt}} = \text{conv}(0, v_M)$ is contained in the level set $[v_N = 0]$.

2.4.2. We denote by $r_{\text{max}}, r_{\text{min}} \in M$ the vertices of $\Delta(D)$, where $\langle \Delta(D), v_N \rangle$ admits its extremal values (cf. Figure 3A). Moreover, we define $\sigma_{\text{max}}, \sigma_{\text{min}}$ to be the two-dimensional cones generated by the two edges of $\Delta(D)$ that contain the vertices $r_{\text{max}}$ and $r_{\text{min}}$, respectively.

2.4.3. We take the line segment $\Delta_{\text{newt}}$ and fit it inside the cone $\sigma_{\text{max}}$ until it hits both rays of this cone. In this way, we construct a lattice triangle $\Delta_{\text{max}}$ with base $\Delta_{\text{newt}}$ and top vertex $r_{\text{max}}$. In a similar way, we construct $\Delta_{\text{min}}$ (cf. Figure 3A). In other words, the cones $\sigma_{\text{max}}$ and $\sigma_{\text{min}}$ are cut off along $v_N$-constant lines producing edges of $\Delta_{\text{max}}$ and $\Delta_{\text{min}}$ of lattice length one, respectively. Note that both cut lines are parallel translates. Gluing $\Delta_{\text{max}}$ and $\Delta_{\text{min}}$ along $\Delta_{\text{newt}}$ (cf. Figure 3B) yields

$$\Delta_{\text{nef}} = \Delta_{\text{max}} \cup \Delta_{\text{newt}} \cup \Delta_{\text{min}}.$$

2.4.4. Note that $\Delta_{\text{nef}} \supseteq \Delta_{\text{newt}}$ is the smallest polytope containing $\Delta_{\text{newt}}$ and having $\Sigma$ as a refinement of its normal fan. Actually, $\Delta_{\text{nef}}$ is either a quadrangle with $\Delta_{\text{newt}}$ serving as one of its diagonals, or it is a triangle with $\Delta_{\text{newt}}$ as a side.

![Figure 3](image_url)

Figure 3. Alternative view on $\Delta_{\text{nef}}$. (A) The two cones $\sigma_{\text{max}}, \sigma_{\text{min}}$ and polytopes $\Delta_{\text{max}}, \Delta_{\text{min}}$ constructed as in Subsubsection 2.4.3 with $v_M = [-3, -2] \in M$. (B) Gluing $\Delta_{\text{max}}$ and $\Delta_{\text{min}}$ along $\Delta_{\text{newt}}$ yields $\Delta_{\text{nef}}$.

3. Valuation semigroups associated with non-toric flags

In this section we determine the valuation semigroup $S_{Y^\bullet}(D)$ associated with an ample (Cartier) divisor $D$ and a non-toric flag $Y^\bullet$ as a subset of $\mathbb{N}^3$. The main result is Theorem 3.11, where the abstract semigroup $S_{Y^\bullet}(D)$ is described in terms of lattice points coming from a polyhedral construction in $M$.
Let us fix \( \ell \geq 1 \). Although we are actually interested in the vanishing order \( k \) of global sections \( s \in \Gamma(X, \mathcal{O}_X(\ell D)) \) along \( C_v \), it turns out to be fruitful to look at the question the way around: we fix \( k \) and are going to find all sections \( s \in \Gamma(X, \mathcal{O}_X(\ell D)) \) that have at least the given vanishing order \( k \) along \( C_v \).

This is encoded by
\[
\Gamma(X, \mathcal{O}_X(\ell D - k C_v)) \hookrightarrow \Gamma(X, \mathcal{O}_X(\ell D)) \subseteq \mathbb{C}(X) .
\]
We denote \( \mathcal{L}(\ell, k) := \mathcal{O}_X(\ell D - k C_v) \) and \( L(\ell, k) := \Gamma(X, \mathcal{L}(\ell, k)) \subseteq \mathbb{C}(X) \). Note that understanding \( s \) as an element of \( L(\ell, k) \) instead of just being a section of \( \Gamma(X, \mathcal{O}_X(\ell D)) \) means that we have implicitly done the weird innocent step \( \tilde{s} := s \) being explained at the end of Subsection 2.3. Continuing this programme, it remains to restrict \( \tilde{s} \in L(\ell, k) \) to \( Y_1 = C_v \), i.e., to consider
\[
\tilde{s}|_{C_v} \in \Gamma(C_v, L(\ell, k)|_{C_v}) .
\]

3.1. Return to toric geometry. Our goal is to understand the restriction of global sections via toric geometry. Therefore, we implement two changes. First, we will shift the linear series of the flag curve, which enables us to replace some of the line bundles we study by torus-invariant ones. Second, we will normalize the restriction.

We are going to use \( C'_v = C_v - \text{div}(f) \) from Subsection 2.3. In terms of the associated sheaves, this means
\[
\mathcal{O}_X(C'_v) = f \cdot \mathcal{O}_X(C_v) ,
\]
where \( f \) is the equation \( x^{\text{wt}} - 1 \) of \( C_v \) mentioned earlier. This leads to the possibility of replacing \( \mathcal{L}(\ell, k) \) by the isomorphic, but torus-invariant line bundle
\[
\mathcal{L}'(\ell, k) := \mathcal{O}_X(\ell D - k C'_v) = f^{-k} \cdot \mathcal{L}(\ell, k) \subseteq \mathbb{C}(X) .
\]
Accordingly, we denote
\[
L'(\ell, k) := \Gamma(X, \mathcal{L}'(\ell, k))
\]
and replace \( \tilde{s} \in L(\ell, k) \) by \( s' := f^{-k} \cdot \tilde{s} \in L'(\ell, k) \).

Recall that the nef invertible sheaves \( \mathcal{O}_X(\ell D) \) and \( \mathcal{O}_X(k C'_v) \) correspond to the polytopes \( \ell \Delta(D) \) and \( k \Delta^\text{nef} \), respectively. This implies that \( L'(\ell, k) \) has a monomial base provided by
\[
\Theta(\ell, k) := (\ell \Delta(D) : k \Delta^\text{nef}) := \{ m \in M_\mathbb{R} \mid m + k \Delta^\text{nef} \subseteq \ell \Delta(D) \}
\]
\[
= (\ell \Delta(D) : k \Delta^{\text{newt}}) := \{ m \in M_\mathbb{R} \mid m + k \Delta^{\text{newt}} \subseteq \ell \Delta(D) \}
\]
see, e.g., the \((i = 0)\)-case of [AP20, Theorem 2].

Example 3.1. Continuing Example 2.2, we obtain
\[
\Theta(\ell, k) = \{ m \in M_\mathbb{R} \mid \langle m, (-1, 0) \rangle \geq 0, \langle m, (0, -1) \rangle \geq 0, \langle m, (1, 2) \rangle \geq 7k - 8\ell, \langle m, (0, 1) \rangle \geq 2k - 3\ell \} .
\]
In particular, \( \Theta(1, 1) = \text{conv}([0, 0], [-1, 0], [0, -1/2]) \) (cf. yellow polytope in Figure 4).
Next, we describe how we normalize the restriction. The projection map

\[ \pi : M \rightarrow \overline{M} := M/\mathbb{Z}v_M \cong \mathbb{Z} \]

is dual to the embedding

\[ \iota : \overline{N} := \mathbb{Z}v_N \hookrightarrow N. \]

The latter induces a map of fans, hence a toric map \( \iota : \mathbb{P}^1 \rightarrow X = \mathbb{T}^V(\Sigma) \). Note that \( \iota(\mathbb{P}^1) = C_v \) and that \( \iota : \mathbb{P}^1 \rightarrow C_v \) is the normalization map. Hence, we are going to replace the restrictions of \( \mathcal{L}'(\ell,k) \) and \( s' \) to \( C_v \) by taking the pullback \( \iota^* \).

**Example 3.2.** Continuing Example 3.1, the projection looks like \( \pi : M_\mathbb{R} \rightarrow \overline{M}_\mathbb{R} = (M/\mathbb{Z}v_M)_\mathbb{R} \). This map (cf. Figure 4) can be identified with \( v_N : M \rightarrow \mathbb{Z} \), that is

\[ \pi : [x,y] \mapsto -2x + 3y. \]

**Figure 4.** Projection map. \( \pi : M \rightarrow \overline{M} = M/\mathbb{Z}[-3,-2] : [x,y] \mapsto -2x + 3y. \)

### 3.2. An alternative view on \( \Theta(\ell,k) \)

In general, \( \Sigma \) is not the normal fan of \( \Theta(\ell,k) \) as it was of \( \Delta(D) \). Geometrically, this means that \( \Theta(\ell,k) \) is, in general, *not* encoding a nef Cartier divisor on \( X \). While \( \Theta(\ell,k) \) is defined as some kind of a difference of polytopes, it is in general not true that the inclusions

\[ \Theta(\ell,k) + k\Delta^{\text{ned}} \subseteq \ell\Delta(D) \quad \text{and} \quad \Theta(\ell,k) + k\Delta^{\text{newt}} \subseteq \ell\Delta(D) \]

become equalities (cf. Example 3.4). We present a suggestion how to overcome this.

Recall from Subsection 2.4 that we had denoted by \( r^\text{max}, r^\text{min} \in M \) the vertices of \( \Delta(D) \), where \( \langle \Delta(D), v_N \rangle \) takes on its extremal values. Similarly, we denote by \( r^\text{\prime max}(\ell,k), r^\text{\prime min}(\ell,k) \in M_\mathbb{R} \) the \( v_N \)-extremal vertices of \( \Theta(\ell,k) \). The latter leads to
the line segments $r'_{\text{max}}(\ell, k) + k\Delta_{\text{newt}}$ and $r'_{\text{min}}(\ell, k) + k\Delta_{\text{newt}}$, which cut the polytope $\ell\Delta(D)$ into three subpolytopes, called $\square_{\text{max}}(\ell, k)$, $\Delta(D)C^{(\ell, k)}$, and $\square_{\text{min}}(\ell, k)$.

More concretely, here is how we obtain $\square_{\text{max}}(\ell, k)$ and $\square_{\text{min}}(\ell, k)$: we take the line segment $k\Delta_{\text{newt}}$, and fit it inside the polytope $\ell\Delta(D)$ until it hits the boundary twice. This way, we construct the lattice polygon (not necessarily a triangle) $\square_{\text{max}}(\ell, k)$ such that $k\Delta_{\text{newt}}$ is one of its edges and $r'_{\text{max}}(\ell, k)$ is one of its vertices. In a similar way, we construct $\square_{\text{min}}(\ell, k)$ using $r'_{\text{min}}(\ell, k)$.

As $\ell\Delta(D)$ before, the polytope $\Delta(D)C^{(\ell, k)}$ just defined still fulfills the equality $\Theta(\ell, k) = (\ell\Delta(D)C^{(\ell, k)} : k\Delta_{\text{newt}})$, however, now we also have the equality $\Theta(\ell, k) + k\Delta_{\text{newt}} = \ell\Delta(D)C^{(\ell, k)}$.

**Remark 3.3.** After this point we will use the shorter notation $r'_{\text{max}} = r'_{\text{max}}(\ell, k)$, $r'_{\text{min}} = r'_{\text{min}}(\ell, k)$, $\square_{\text{max}} = \square_{\text{max}}(\ell, k)$, $\Delta(D)C = \Delta(D)C^{(\ell, k)}$, and $\square_{\text{min}} = \square_{\text{min}}(\ell, k)$.

Nevertheless, one should keep in mind that all of these quantities depend on $l$ and $k$.

![Figure 5. Alternative view on $\Theta(\ell, k)$](image)

(A) The Minkowski sum $\Theta(\ell, k) + k\Delta_{\text{nef}}$ and $\Theta(\ell, k) + k\Delta_{\text{newt}}$. (B) The cut of $\Delta(D)$ along $r'_{\text{max}} + k\Delta_{\text{newt}}$ and $r'_{\text{min}} + k\Delta_{\text{newt}}$ into $\square_{\text{max}}$, $\square_{\text{min}}$, and $\Delta(D)C$ with $\Delta(D)C = \Theta(\ell, k) + k\Delta_{\text{newt}}$.

**Example 3.4.** Continuing Example 3.2, Figure 5A shows that the inclusions

$$\Theta(\ell, k) + k\Delta_{\text{nef}} = \text{conv}([0, 0], [-8, 0], [-3, -5/2], [0, -5/2]) \subset \Delta(D)$$

and

$$\Theta(\ell, k) + k\Delta_{\text{newt}} = \text{conv}([0, 0], [-1, 0], [-4, -2], [-3, -5/2], [0, -1/2]) \subset \Delta(D)$$

are strict in general for $(\ell, k) = (1, 1)$.

We cut the polytope $\Delta(D)$ along the line segments $r'_{\text{max}} + k\Delta_{\text{newt}}$ and $r'_{\text{min}} + k\Delta_{\text{newt}}$ into the subpolytopes

$$\square_{\text{max}} = \text{conv}([-1, 0], [-8, 0], [-4, -2]), \quad \Delta(D)C = \Theta(\ell, k) + k\Delta_{\text{newt}},$$
and
\[ \square_{\min} = \text{conv}([0, -1/2], [0, -3], [-2, -3], [-3, -5/2]) , \]
where \( r'_{\max} = [-1, 0] \) and \( r'_{\min} = [0, -1/2] \), respectively (cf. Figure 5B).

3.3. Projections of polytopes. We start pulling back the sheaf \( L'(\ell, k) \). To this end, we define
\[ \overline{d}(\ell, k) := \ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta_{\text{nef}}) , \]
where \( \text{width}_{v_N}(\cdot) \) denotes the lattice width of a polytope with respect to the linear functional \( v_N \in \mathbb{N} \), i.e., if \( \Delta \subseteq M_{\mathbb{R}} \) is a polytope, then \( \text{width}_{v_N}(\Delta) := \max_{m, m' \in \Delta} |\langle m, v_N \rangle - \langle m', v_N \rangle| \). Note that this equals the length of the line segment \( \overline{\Delta} := \pi(\Delta) \), i.e.,
\[ \overline{d}(\ell, k) = \ell \cdot \text{length}(\Delta(D)) - k \cdot \text{length}(\Delta_{\text{nef}}) \].

Proposition 3.5. The pullback \( i^*L'(\ell, k) \) is a line bundle on \( \mathbb{P}^1 \) of degree \( \overline{d}(\ell, k) \).

Proof. We obtain
\[
i^*L'(\ell, k) = i^*O_X(\ell \Delta(D) - k \Delta_{\text{nef}})
= O_{\mathbb{P}^1}(\ell \cdot (D, C'_v) - k \cdot (C'_v, C'_v))
= O_{\mathbb{P}^1}(\ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta_{\text{nef}})) .
\]

Remark 3.6. Altogether this yields the sequence of inclusions
\[ \pi(\Theta(\ell, k) \cap M) \subseteq \Theta(\ell, k) = (\ell \Delta(D) : k \Delta_{\text{nef}}) \subseteq (\ell \Delta(D) : k \Delta_{\text{nef}}) =: \Xi(\ell, k) , \]
which might be strict, where \( \overline{\Delta} = \pi(\Delta) \) is the projection of any polytope \( \Delta \subseteq M \) along \( v_M \).

Example 3.7. Continuing Example 3.4, let us fix \( (\ell, k) = (1, 1) \). Then the projected polytopes along \( \pi \) are
\[ \overline{\Delta(D)} = \text{conv}([-9], [16]) , \quad \overline{\Delta_{\text{nef}}} = \text{conv}([-6], [14]) , \]
\[ \overline{\Theta(\ell, k)} = \text{conv}([-3/2], [2]) , \quad \overline{\Xi(\ell, k)} = \text{conv}([-3], [2]) , \]
where \( \overline{d}(\ell, k) = 25 - 20 = 5 \) (cf. Figure 6).

\[ \overline{\Xi(\ell, k)} \quad \overline{\Psi(\ell, k)} \]
\[ \overline{\Delta(D)} \quad \pi(v_M) \quad \overline{\Delta_{\text{nef}}} \]

Figure 6. Projected polytopes.
Recall that the (torus-equivariant) global sections of $L'(\ell, k)$ are encoded by the elements of $\Theta(\ell, k) \cap M$. Under this identification their pullback via $\iota^*$ is given by their images under $\pi$. Denote their number by
\[ e(\ell, k) := \# \pi(\Theta(\ell, k) \cap M). \]

Summarizing what we have done so far, we obtain

**Proposition 3.8.** The pullback $\iota^* L'(\ell, k) = \mathcal{O}_{\mathbb{P}^1}(\Xi(\ell, k))$ is a line bundle on $\mathbb{P}^1$ of degree $d(\ell, k)$. Its global sections correspond to the elements of $\Xi(\ell, k) \cap M$. Under this identification, the subspace $\iota^* L'(\ell, k)$ coincides with $\pi(\Theta(\ell, k) \cap M)$. In particular,
\[ \dim(\iota^* L'(\ell, k)) = e(\ell, k). \]

**Proof.** Proposition 3.5 yields the degree $d(\ell, k)$ of the pullback of $L'(\ell, k)$. What is missing is to show that this sheaf is precisely given via $\Xi(\ell, k)$ as $\mathcal{O}_{\mathbb{P}^1}(\Xi(\ell, k))$ which is a slightly finer information. The statement holds since if $\Delta \subseteq M_{\mathbb{R}}$ is a nef polytope (e.g., $\ell \Delta(D)$ or $k\Delta_{\text{nef}}$), then
\[ \iota^* \mathcal{O}_X(\Delta) = \mathcal{O}_{\mathbb{P}^1}(\pi(\Delta)) = \mathcal{O}_{\mathbb{P}^1}(\Delta). \]

This claim is valid for any toric map and does not depend on having $\mathbb{P}^1$ as a target. \[ \square \]

**Example 3.9.** Continuing Example 3.7, we obtain $\Theta(\ell, k) \cap M = \{[0, 0], [-1, 0]\}$ and therefore $\pi(\Theta(\ell, k) \cap M) = \{0, 2\}$, i.e., $e(\ell, k) = 2$ for $(\ell, k) = (1, 1)$.

### 3.4. Shape of the semigroup.

As we did before, let us fix a pair $(\ell, k)$. We know from Subsection 2.1 that we are supposed to collect the values $\text{ord}_{Y_2}(\tilde{s}|_{C_v})$ for all possible sections $\tilde{s}$, where $Y_2 = \{1\}$ is a smooth point on $C_v$. In Subsection 3.1, we have transferred this setup to $\text{ord}_{1 \in \mathbb{P}^1}(\iota^* s')$, where $s'$ runs through all global sections represented by the polytope $\Theta(\ell, k) \subseteq M_{\mathbb{R}}$.

Proposition 3.8 implies that the pullbacks $\iota^* s'$ run through all $e(\ell, k)$ elements of $\pi(\Theta(\ell, k) \cap M) \subseteq \overline{M} \cong \mathbb{Z}$. Each element of $\mathbb{Z}$ represents a rational monomial function on $\mathbb{P}^1$. We are supposed to find the orders of all of their linear combinations.

**Lemma 3.10.** Let $Z \subset \mathbb{Z}$ be a finite subset with $e$ elements leading to the $e$-dimensional vector space
\[ \mathbb{C}[Z] := \{ f \in \mathbb{C}[t, t^{-1}] \mid \text{supp}(f) \subseteq Z \}. \]

Then $\text{ord}_1 \mathbb{C}[Z] = \{0, 1, \ldots, e - 1\} = \text{ord}_c \mathbb{C}[Z]$ for all $c \in \mathbb{C}^* \subseteq \mathbb{P}^1$. 


Proof. Set \( Z = \{p_1, \ldots, p_e\} \). For an element \( f \in \mathbb{C}[Z] \) with \( f = \lambda_1 \cdot t^{p_1} + \ldots + \lambda_e \cdot t^{p_e} \) and \( d \in \mathbb{N} \), the rows of the matrix

\[
P := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
p_1 & p_2 & \cdots & p_e \\
p_1(p_1 - 1) & p_2(p_2 - 1) & \cdots & p_e(p_e - 1) \\
p_1(p_1 - 1)(p_1 - 2) & p_2(p_2 - 1)(p_2 - 2) & \cdots & p_e(p_e - 1)(p_e - 2) \\
\vdots & \vdots & \ddots & \vdots \\
p_1 \cdots (p_1 - (d - 1)) & p_2 \cdots (p_2 - (d - 1)) & \cdots & p_e \cdots (p_e - (d - 1))
\end{pmatrix}
\]

encode \( f(1) = \lambda_1 \cdot 1 + \ldots + \lambda_e \cdot 1 \), \( f'(1) = \lambda_1 \cdot p_1 \cdot 1 + \ldots + \lambda_e \cdot p_e \cdot 1 \), \( f''(1), \ldots, f^{(d)}(1) \). Let \( p \) be an arbitrary variable. Then the linear spaces

\[
L_1(p) := \text{span}_\mathbb{Q}\{1, p, p^2, \ldots, p^d\} \subseteq \mathbb{Q}[p]
\]

and

\[
L_2(p) := \text{span}_\mathbb{Q}\{0!\binom{p}{0}, 1!\binom{p}{1}, 2!\binom{p}{2}, \ldots, d!\binom{p}{d}\} \subseteq \mathbb{Q}[p]
\]

coincide because \( L_1(p) \supseteq L_2(p) \) and \( \dim(L_1(p)) = \dim(L_2(p)) \). In particular, \( P \) is equivalent to \( V \), i.e., there exists a lower triangular matrix \( C \) such that \( P = C \cdot V \), where \( V := (p_j^k)_{0 \leq i \leq d, 1 \leq j \leq e} \) is the transposed Vandermonde matrix. If we choose \( d = e - 1 \), the matrices \( V \) and thus \( P \) are regular. Hence the system of linear equations

\[
P \cdot (\lambda_1, \ldots, \lambda_e)^T = 0
\]

together with the linear equations \( f(1) = f'(1) = f''(1) = \ldots = f^{(e - 1)}(1) = 0 \) has no non-trivial solution, i.e., \( f = 0 \).

On the other hand, deleting the \( k \)-th \((1 \leq k \leq e)\) row of \( P \), i.e., dealing with \( P' \) being a \((e - 1) \times e\)-matrix, yields the linear system

\[
P' \cdot (\lambda_1, \ldots, \lambda_k, \ldots, \lambda_e)^T = 0
\]

having a non-trivial solution with \( f(1) = f'(1) = \ldots = f^{(k - 2)}(1) = 0, f^{(k - 1)}(1) \neq 0, f^{(k)}(1) = \ldots = f^{(e - 1)}(1) = 0 \). The corresponding (essentially uniquely determined) \( f \) satisfies \( \text{ord}_1(f) = k - 1 \).

As a direct consequence, we obtain the following statement:

**Theorem 3.11.** The valuation semigroup is given as

\[
S_{\mathbf{V}}(D) = \{(\ell, k, \delta) \in \mathbb{N}^2 \mid 0 \leq \delta \leq \text{e}(\ell, k) - 1\}
\]

and \( \text{e}(\ell, k) = 0 \) for large \( k \gg \ell \).

Proof. The definition of the valuation semigroup can be reformulated as

\[
S_{\mathbf{V}}(D) = \{(\ell, k, \delta) \in \mathbb{N}^2 \mid s' \in \Gamma(X, \mathcal{L}'(\ell, k)) \setminus \{0\}, \text{ord}_{1}\mathbf{E}_1(u^*s') = \delta\}
\]

Then everything follows with Proposition 3.8. \( \square \)
4. Shape of the Newton–Okounkov body

Building on Section 3, we determine the Newton–Okounkov body $\Delta Y \cdot (D)$ in Theorem 4.3. Consider the assignment

$$d(\ell, k) := \text{width}_{v_N}(\Theta(\ell, k)) = \text{length}(\pi(\Theta(\ell, k))) = \text{length}(\Theta(\ell, k)) \in \mathbb{Q} \sqcup \{-\infty\}.$$ 

We extend this definition to all $\ell, k \in \mathbb{R}_{\geq 0}$ using the convention $\text{width}_{v_N}(\emptyset) = -\infty$. This becomes necessary when $k \cdot \Delta_{\text{nef}}$ does not fit inside $\ell \cdot \Delta(D)$, which will happen for $k \gg \ell$.

This should not be confused with $d(\ell, k)$ which was defined on page 11 as

$$d(\ell, k) = \ell \cdot \text{width}_{v_N}(\Delta(D)) - k \cdot \text{width}_{v_N}(\Delta_{\text{nef}}).$$

The chain of inclusions at the end of Subsection 3.3 gives rise to the inequalities

$$e(\ell, k) - 1 \leq d(\ell, k) \leq \overline{d}(\ell, k).$$

**Example 4.1.** Continuing Example 3.9, i.e., $(\ell, k) = (1, 1)$, $d(\ell, k) = 5$, and $e(\ell, k) = 2$, we obtain $d(\ell, k) = 7/2$ satisfying the inequalities $1 \leq 7/2 \leq 5$.

Moreover, we observe

**Lemma 4.2.** For $q \in \mathbb{R}_{\geq 0}$, the assignment $d(q) := d(\ell, q)/\ell$ does not depend on $\ell$. In particular, $d(q) = d(1, q)$.

**Proof.** The width function is linear in its polyhedral argument. □

Note that the same statement holds true for $\overline{d}(\ell, k)$, but not for $\overline{e}(\ell, k)$.

From now on, we return to $(\ell, k) \in \mathbb{N}^2$.

**Theorem 4.3.** The Newton–Okounkov body $\Delta Y \cdot (D)$ coincides with the convex hull of the set

$$\{[q, t] \in (\mathbb{R}_{\geq 0})^2 \mid 0 \leq t \leq d(q)\} = \bigcup_{q \in \mathbb{R}_{\geq 0}, d(q) \geq 0} \text{conv}([q, 0], [q, d(q)]).$$

Moreover, $d(q)$ is a decreasing piecewise linear function with $d(q) = -\infty$ for $q \gg 0$.

**Proof.** Let $\epsilon > 0$ and denote by $A, B$ the vertices of $\pi(\Theta(\ell, k)) = \overline{\Theta(\ell, k)}$. Assume first that the dimension of $\Theta(\ell, k)$ equals two. Then the two fibers

$$\pi^{-1}(T \cap \Theta(\ell, k))$$

with $T = A + \epsilon$ or $B - \epsilon$ have positive lengths greater (or equal) than some $\mu > 0$. In particular, all fibers in between do so as well. Hence, setting $\lambda = 1/\mu$, the fibers

$$\pi^{-1}(T' \cap \lambda \Theta(\ell, k))$$

have at least length 1 for $\lambda(A + \epsilon) \leq T' \leq \lambda(B - \epsilon)$. If in addition $T' \in \overline{M}$, then all of these fibers have to contain lattice points in $M$. Thus, we obtain

$$\text{conv}(\lambda(A + \epsilon), \lambda(B - \epsilon)) \subseteq \pi(\lambda \Theta(\ell, k) \cap M).$$
Keeping \( q = k/\ell \) constant and using Lemma 4.2, \( e(\ell, k) \) behaves like \( d(\ell, k) \) asymptotically with respect to dilations. The result then follows by recalling the fact that Newton–Okounkov bodies are closed.

It remains to consider the pathological case of \( \dim(\Theta(1, q)) = 1 \). Here, we can approximate \([q, t]\) by \([q-\epsilon, t]\) so that the resulting \( \Theta(1, q-\epsilon) \) is full-dimensional and \( t \leq d(q) \leq d(q-\epsilon) \). Then the previous argument shows that \([q-\epsilon, t] \in \Delta(Y(D)) \). Newton–Okounkov bodies are closed by definition, \([q, t] \in \Delta(Y(D)) \). □

We remark that the case of \( \dim(\Theta(1, q)) = 1 \) from the previous proof requires special \( v_N \) and unique \( a_q = k/\ell \). This configuration is characterized by the fact that (a shift of) \( q_0 \) connects two parallel edges of \( \Delta(D) \). Note that \( \Theta(\ell, k) \) is also parallel to these edges. In contrast to the general case, for \( k/\ell = q_0 \) the number \( e(\ell, k) \) behaves asymptotically like \( 1/g \cdot d(\ell, k) \), where \( g := d(\ell, k)/\text{length}_{M}(\Theta(\ell, k)) \).

Despite that \( e(\ell, k) \) for \( k/\ell = q_0 \) does not approach \( d(\ell, k) \) at all, this does not cause a problem: as we have seen in the proof, for \( q \leq q_0 \) the general case applies and for \( q > q_0 \), we have \( d(q) = -\infty \) anyway.

**Example 4.4.** We continue Example 4.1 and apply Theorem 4.3. The Newton–Okounkov body \( \Delta(Y(D)) \) (cf. Figure 7) is given as

\[
\Delta(Y(D)) = \text{conv}(\{0, 0\}, \{0, 25\}, \{1/10, 23\}, \{2/10, 21\}, \{3/10, 19\}, \{4/10, 17\}, \\
\{5/10, 15\}, \{6/10, 13\}, \{7/10, 10.85\}, \{8/10, 8.4\}, \{9/10, 5.95\}, \\
\{1, 3.5\}, \ldots, \{8/7, 0\}).
\]

In particular, it has vertices \([0, 0]\), \([0, 25]\), \([8/7, 0]\), and \([2/3, 35/3]\) (cf. [HKW20]).

**Figure 7.** Newton–Okounkov body \( \Delta(Y(D)) \) with flipped coordinates.

This example already gives an instance of a vertex that does not lift to the semigroup (cf. Definition 5.3) when building the Newton–Okounkov body in question. Let us consider the vertex \([2/3, 35/3]\), and fix \( \ell = 3, k = 2 \). The respective polyhedra \( 3\Delta(D) \), \( 2\Delta_{\text{newt}} \), \( 2\Delta_{\text{nef}} \), and \( \Theta(\ell, k) = (3\Delta(D) : 2\Delta_{\text{nef}}) \) are pictured in Figure 8. To hit the vertex \([2/3, 35/3]\), the value of \( e(3, 2) \) would have to coincide with \( d(3, 2) = 35 \). However, we only obtain \( e(3, 2) = \#(\pi(\Theta(\ell, k) \cap M)) = 30 \). The red lines in Figure 8 indicate the gaps. No matter how big of a multiple of \( (\ell, k) \) we consider, the gaps will not be closed in any scaled version of the situation. Hence, the vertex \([2/3, 35/3]\) is never hit and the associated valuation semigroup \( S_Y(D) \) is therefore not finitely generated.
5. Criterion for the finite generation of certain valuation semigroups

We provide a criterion for the finite generation of strictly positive (with respect to their height functions) semigroups in terms of their limit polyhedra.

5.1. Semigroups with polyhedral limit. We start with a free abelian group $M$ of rank $n$, i.e., $M \cong \mathbb{Z}^n$ and a linear form $h : M \rightarrow \mathbb{Z}$ which we call a height function. This induces $h_\mathbb{R} : M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \rightarrow \mathbb{R}$ which we will often denote by $h$ as well.

Let $S \subseteq h^{-1}(\mathbb{N})$ be a semigroup that is strictly positive, i.e., $S \cap \ker(h) = \{0\}$. In order to refer to the individual layers of a given height, we will write

$$S_k := S \cap h^{-1}(k), \text{ i.e., we have } S = \bigcup_{k \in \mathbb{N}} S_k \text{ with } S_0 = \{0\}.$$ 

This setup allows us to define the enveloping cone

$$C_S := \text{cone } S \subseteq M_\mathbb{R}$$

as well as the convex limit figure

$$\Delta_S := C_S \cap \widetilde{M_\mathbb{R}} \supseteq S_1,$$

where $\widetilde{M} := h^{-1}(1)$ and $\widetilde{M_\mathbb{R}} := h_\mathbb{R}^{-1}(1)$.

In the case of a valuation semigroup $S_{\bullet}(D)$ the height function $h : \mathbb{Z}^3 \rightarrow \mathbb{Z}$ is the projection on $\ell$, which then leads to the Newton–Okounkov body $\Delta_{\bullet}(D) = C_{S_{\bullet}(D)} \cap \widetilde{M_\mathbb{R}} \subseteq \mathbb{R}^2$.

Definition 5.1. We say that $S$ has a polyhedral limit if $\Delta_S$ is a polytope, i.e., if $\Delta_S$ equals the convex hull of its (finitely many) vertices.
Obviously, this property is fulfilled whenever the semigroup $S$ is finitely generated because it is strictly positive by assumption. In this case $\Delta_S$ has even rational vertices, it is a rational polytope. However, the following standard example shows that the converse implication does not hold in general:

**Example 5.2.** Let $h : \mathbb{Z}^2 \to \mathbb{Z}, [x,y] \mapsto x + y$ be the summation map. Then $S := \{0\} \cup (\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1})$ is not finitely generated, but $C_S = \mathbb{R}_{\geq 0}$ and $\Delta_S$ equals the line segment connecting the points $[1,0]$ and $[0,1]$.

5.2. **Equivalent conditions for finite generation.** We assume that $S \subseteq M$ is a strictly positive (with respect to $h$) semigroup that has polyhedral limit $\Delta_S$.

**Definition 5.3.** We say that a point $p \in \Delta_S$ lifts to the semigroup $S$ (i.e., is a valuation point) if there exists some scalar $c \in \mathbb{R}_{>0}$ with $c \cdot p \in S$.

Note that, if this is the case, then $p$ as well as $c$ have to be rational, i.e., $p \in \Delta_S \cap M_{\mathbb{Q}}$ and $c \in \mathbb{Q}_{>0}$. Hence, it is not a surprise that the assumption of the next lemma is automatically fulfilled if the semigroup $S$ is finitely generated.

**Lemma 5.4.** If all vertices of $\Delta_S$ lift to $S$, then they are rational (i.e., $\Delta_S$ is a rational polytope) and every rational point $p \in \Delta_S \cap M_{\mathbb{Q}}$ lifts to $S$.

**Proof.** Let $v^1, \ldots, v^d \in \Delta_S$ be linearly independent (rational) vertices such that $p$ is contained in their convex hull. Then the unique coefficients $\lambda_i$ in the representation $p = \sum_{i=1}^d \lambda_i v^i$ have to be rational, too. Thus, we may choose an integer $\mu$ such that $\mu \cdot \lambda_i \in \mathbb{N}$. On the other hand, there is a joint factor $c \in \mathbb{Z}_{\geq 1}$ such that all multiples $c \cdot v^i$ belong to $S$. This implies

$$
\mu c \cdot p = \sum_{i=1}^d \mu \lambda_i \cdot (c \cdot v^i) \in S.
$$

□

Next, we formulate the main point of this section.

**Proposition 5.5.** A semigroup $S$ with a polyhedral limit $\Delta_S$ is finitely generated if and only if all vertices of $\Delta_S$ lift to $S$.

We have already seen that this condition is necessary for the finite generation. Now we will show that it is sufficient, too. Note that, in Example 5.2, the two vertices of the line segment $\Delta_S$ indeed do not lift to the semigroup.

Let $S \subseteq M$ be a subsemigroup with rational polyhedral limit $\Delta(S)$ (with respect to some height function $h : M \to \mathbb{Z}$). Assume that the vertices and thus, by Lemma 5.4, all rational points of $\Delta_S$ lift to $S$. Moreover, we may assume that $S$ generates $M$ as a group, hence $C_S$ is a full-dimensional cone.

**Proof of Proposition 5.5.** Assume $S$ is not finitely generated. Then $S$ has infinitely many irreducible elements, i.e.,

$$
\mathcal{H} := S \setminus ((S \setminus \{0\}) + (S \setminus \{0\}))
$$
is infinite [CLS11, Proposition 1.2.23].

By taking a simplicial subdivision we may, w.l.o.g., assume that the cone $C_S$ is simplicial and given as $C_S = \text{cone}(s_1, \ldots, s_n)$. Consider the lattice $\Lambda$ generated by $s_1, \ldots, s_n$. As $M/\Lambda$ is finite, there must be a coset $m + \Lambda$ which contains infinitely many elements of $H$. Here we may choose $m$ to be a minimal representative in $C_S$:

$m \in C_S \cap M$ so that $m - s_i \notin C_S$ for $i = 1, \ldots, n$.

As the elements in $C_S \cap H$ were indecomposable in $S$, they certainly are irreducible in $C_S \cap S$. In particular, if we identify $(m + \Lambda) \cap C_S$ with $\mathbb{N}^n$, we obtain an infinite set of pairwise incomparable elements, in contradiction to Dickson’s Lemma [CLO15, Chapter 4, Theorem 5].

6. Finite generation criterion

6.1. Characterising the lifting property. The following theorem gives a purely combinatorial criterion to check if our valuation semigroup $S_Y(\mathcal{D})$ (in the language of Subsection 2.2) is finitely generated.

**Theorem 6.1.** The point $(1, k/\ell, d(k/\ell)) = (1, q, d(q))$ is a valuation point (i.e., a multiple of it lies in $S_Y(\mathcal{D})$) if and only if there exists a $\lambda \in \mathbb{N}$ such that

$$
\pi : \lambda \Theta(1, q) \cap M \rightarrow \pi(\lambda \Theta(1, q)) \cap \overline{M}
$$

is surjective (i.e., $\lambda \cdot d(q) = \lambda \cdot e(1, q) - 1$) and $\pi(\lambda \Theta(1, q))$ is a lattice polytope.

**Proof.** By definition, $(1, q, d(q))$ is a valuation point if and only if there exists an $\lambda \in \mathbb{N}$ such that $\lambda \cdot (1, q, d(q)) \in S_Y(\mathcal{D}) \subseteq \mathbb{N}^3$. Using Theorem 3.11, the latter happens exactly if there exists $\lambda$ such that

$$
0 \leq \lambda \cdot d(q) \leq \lambda \cdot e(1, q) - 1,
$$

where $e(1, q) = \#\pi(\Theta(1, q) \cap M)$ and $\lambda \cdot e(1, q) = \#\pi(\lambda \Theta(1, q) \cap M)$. In addition, we see that

$$
\lambda \cdot d(q) = \lambda \cdot d(1, q) = \lambda \cdot \text{length}(\pi(\Theta(1, q)))
= \text{length}(\lambda \pi(\Theta(1, q))) = \text{length}(\lambda \Theta(1, q))
\geq \#\pi(\lambda \Theta(1, q)) \cap \overline{M} - 1
\geq \#\pi(\lambda \Theta(1, q) \cap M) - 1.
$$

Combining all these inequalities, we obtain the equalities

(1) \quad $\lambda \cdot d(q) = \#\pi(\lambda \Theta(1, q)) \cap \overline{M} - 1$

and

(2) \quad $\#\pi(\lambda \Theta(1, q) \cap M) = \#\pi(\lambda \Theta(1, q) \cap M)$,

where Equation (1) is equivalent to $\pi(\lambda \Theta(1, q))$ being a lattice polytope, and Equation (2) to

$$
\pi : \lambda \Theta(1, q) \cap M \rightarrow \pi(\lambda \Theta(1, q)) \cap \overline{M}
$$
being surjective (i.e., \( \pi \) meets all possible lattice points in \( \pi(\lambda \Theta(1,q)) \cap \overline{M} \)). □

**Lemma 6.2.** Let \((1, k/\ell, d(k/\ell)) = (1, q, d(q))\) be a point. If there exists a \( \lambda \in \mathbb{N} \) such that \( \pi(\lambda \Theta(1,q)) \) is a lattice polytope and the two lattice points in \( \pi(\lambda \Theta(1,q)) \) next to its vertices are being hit by \( \pi \) (i.e., the fibers over these points contain at least one lattice point), then

\[
\pi : \lambda \Theta(1,q) \cap M \to \pi(\lambda \Theta(1,q)) \cap \overline{M}
\]

is surjective.

**Proof.** If \( \dim(\Theta(1,q)) = 2 \), then the two vertices of \( \pi(\Theta(1,q)) \) are the projections of the two \( v_2 \)-extremal vertices \( r'_{\min} \) and \( r'_{\max} \) of \( \Theta(1,q) \) having \( \sigma'_{\min} \) and \( \sigma'_{\max} \), respectively, as tangent cones. We consider the tangent cone \( \sigma'_{\min} \) and detect when it leaves \( \Theta(1,q) \) (besides \( r'_{\min} \)). Now, we look at the length of the fiber of \( \pi \) at exactly this point and choose \( \lambda'_{\min} \) such that the length is at least 1 and \( \pi(\lambda'_{\min} \Theta(1,q)) \) is a lattice polytope.

Analogously, we choose \( \lambda'_{\max} \) and set \( \lambda = \max\{\lambda'_{\min}, \lambda'_{\max}\} \). We call the lattice points in \( \pi(\lambda \Theta(1,q)) \subseteq \overline{M}_\mathbb{R} \) revealing the above-mentioned fibers \( m'_{\min}, m'_{\max} \). Our condition on the length guarantees that all fibers between these two points contain at least one lattice point. Therefore, all lattice points in \( \pi(\lambda \Theta(1,q)) \) between these are being hit.

If there exists \( r_1 \in \lambda \Theta(1,q) \cap M \) such that \( \pi(r_1) \in \overline{M} \) has lattice distance 1 to \( \pi(r'_{\min}) \in \overline{M} \), we obtain surjectivity in the following way: By shifting the cone \( \sigma'_{\min} \) to \( r_1 + \sigma'_{\min} \) and applying the same argument, there exists \( r_2 \in \lambda \Theta(1,q) \cap M \) such that \( \pi(r'_{\min}) \neq \pi(r_2) \in \overline{M} \) has lattice distance 1 to \( \pi(r_1) \in \overline{M} \). By induction all lattice points of \( \pi(\lambda \Theta(1,q)) \cap \overline{M} \) between its vertices and \( m'_{\min}, m'_{\max} \) are being hit.

If \( \dim(\Theta(1,q)) = 1 \), the statement remains true, but the asymptotic behavior of getting closer to surjectivity from the 2-dimensional case is not existing anymore. □

Together with Theorem 6.1 Lemma 6.2 leads to the following

**Corollary 6.3.** The point \((1, k/\ell, d(k/\ell)) = (1, q, d(q)) \in S_{V_\mathbb{R}}(D)\) is a valuation point if and only if there exists a \( \lambda \in \mathbb{N} \) such that \( \pi(\lambda \Theta(1,q)) \) is a lattice polytope and the two lattice points in \( \pi(\lambda \Theta(1,q)) \) next to its vertices are hit by \( \pi \), i.e., the fibers over these points contain at least one lattice point.

### 6.2. Strong decomposability

Our aim is to decide whether the valuation semigroup \( S_{V_\mathbb{R}}(D) \) is finitely generated. We give a combinatorial criterion for this which can be read off the input data. To this end, we introduce the following notion:

**Definition 6.4.** Let \( \sigma \subseteq N_\mathbb{R} \) be a cone. A lattice point \( u \in \text{int}(\sigma) \cap N \) is called strongly decomposable in \( \sigma \) if \( u = u' + u'' \) for suitable \( u', u'' \in \text{int}(\sigma) \cap N \).

**Lemma 6.5.** Let \( \sigma \subseteq N_\mathbb{R} \) be a cone and \( u \in \text{int}(\sigma) \cap N \) a direction. Then the following statements are equivalent:

1. \( 1 \notin \langle \mathcal{H}_\sigma, u \rangle \), where \( \mathcal{H}_\sigma \) denotes the Hilbert basis of \( \sigma \).
(ii) \( u \) is strongly decomposable in \( \sigma \)
(iii) the closure of the 1-parameter subgroup \( \lambda^u(\mathbb{C}^\ast) \subseteq \mathbb{T} \) in \( \mathbb{T} \mathbb{V}(\sigma) \) is singular.

Proof. One checks quickly that (i) and (iii) are equivalent: The 1-parameter subgroup represented by \( u \) can always be extended to \( \lambda^u : \mathbb{C} \rightarrow \mathbb{T} \mathbb{V}(\sigma) \). On the dual level of regular functions, however, this corresponds to
\[
\langle \cdot, u \rangle : \mathbb{C}[\sigma^\vee \cap M] \rightarrow \mathbb{C}[N].
\]
The latter map is surjective if and only if \( 1 \in \langle \sigma^\vee \cap M, u \rangle \).

\( (i) \Rightarrow (ii) \): By assumption, there exist primitive lattice points \( s^0, s^1 \in M \setminus \sigma^\vee \) such that the line segment between them lying on \([u = 1]\) contains no interior lattice point. Note that \( \{s^0, s^1\} \) is a \( \mathbb{Z} \)-basis because the lattice triangle \( \text{conv}(0, s^0, s^1) \) is unimodular. Moreover, we obtain
\[
\text{cone}(s^0, s^1)^\circ \supset \sigma^\vee \quad \text{and} \quad \text{cone}(t^0, t^1) \subset \sigma^\circ,
\]
where \( \{t^0, t^1\} \) denotes the dual basis to \( \{s^0, s^1\} \). By definition of the dual basis, this yields \( \langle s^i, u \rangle = 1 = \langle s^i, t^0 + t^1 \rangle \) for all \( i \), i.e., the two linear functionals coincide on the basis. Therefore, \( u = t^0 + t^1 \).

\( (ii) \Rightarrow (i) \): Let \( u \) be strongly decomposable in \( \sigma \). Then \( u = u' + u'' \) for some \( u', u'' \in \text{int}(\sigma) \cap N \). Therefore, we have \( \langle m, u \rangle = \langle m, u' \rangle + \langle m, u'' \rangle \in \mathbb{Z} \) for \( m \in \mathbb{H}_{\sigma^\vee} \). Since \( u', u'' \) lie in the interior of \( \sigma \), both summands are positive. Thus, \( 1 \notin \langle \mathbb{H}_{\sigma^\vee}, u \rangle \).

\[ \square \]

**Figure 9.** Strongly decomposable primitive element. (A) The polytope \( \Theta(3, 2) \) with tangent cone \( \sigma'_{\text{min}} \) at the \( v_N \)-extremal vertex \( r'_{\text{min}} \). (B) The dual cone \( (\sigma'_{\text{min}})^\vee \) together with the strong decomposition \( (-2, 3) = (-2, 1) + (0, 2) \) of \( v_N \) inside it.

**Example 6.6.** We continue Example 4.4 and fix \( (\ell, k) = (3, 2) \). Then
\[
\Theta(\ell, k) = \text{conv}([0, 0], [-10, 0], [0, -5])
\]
(cf. Figure 8). Its two \( v_N \)-extremal vertices are \( r'_{\text{min}} = [-10, 0] \) and \( r'_{\text{max}} = [0, -5] \). Thus, the corresponding tangent cones \( \sigma'_{\text{min}} \) and \( \sigma'_{\text{max}} \) are given as \( \text{cone}([0, 1], [-2, 1]) \) and \( \text{cone}([1, 0], [2, -1]) \), respectively (cf. Subsection 3.2). Hence the direction \( v_N = (-2, 3) \) is contained in the interior of \( (\sigma'_{\text{min}})^\vee = \text{cone}((-1, 0), (1, 2)) \). It is strongly
decomposable in \((\sigma'_\text{min})^\vee\) because \(v_N = (-2, 1) + (0, 2)\) (cf. Figure 9). An application of Lemma 6.5 yields that we do not obtain 1 via \(\langle H_{\sigma'_\text{min}}, v_N \rangle\).

**Definition 6.7.** Given \(\Delta(D)\) and \(v_N\), we define the rational line segment \(\Delta(D)^{v_N} \subseteq \Delta(D)\) as the line segment \(\Delta(D) \cap [v_N = c] \) of maximal length orthogonal to \(v_N\), and call its vertices \(v_1\) and \(v_2 \in R\). Moreover, we denote by \(e_1^+\) the part of the edge of \(\Delta(D)\) with vertex \(v_1\) lying in the half plane \([v_N \geq c]\), and by \(e_2^+\) the part of the edge of \(\Delta(D)\) with vertex \(v_2\) lying also in the half plane \([v_N \geq c]\). The cone \(\sigma^- \subseteq N\) is the cone generated by the inner normal vectors of \(e_1^+\) and \(e_2^+\).

In the same manner, we define the line segments \(e_1^-\) and \(e_2^-\) contained in \([v_N \leq c]\), which yield the cone \(\sigma^+\).

**Theorem 6.8.** The valuation semigroup \(S_Y(D)\) is finitely generated if and only if \(v_N\) is not strongly decomposable in \(\sigma^+\) and \(-v_N\) is not strongly decomposable in \(\sigma^-\).

**Proof.** Let \(S_Y(D)\) be finitely generated, i.e., for every fixed \(q\) the point \((1, q, d(q))\) lifts to the semigroup. According to Corollary 6.3, this holds if and only if there exists a multiple \(\lambda \in \mathbb{N}\) such that \(\pi(\lambda \Theta(1, q))\) is a lattice polytope and the fibers over the two lattice points in \(\pi(\lambda \Theta(1, q))\) next to its vertices contain at least one lattice point.

Note that the fibers over the two vertices of \(\pi(\lambda \Theta(1, q))\) contain exactly one lattice point, \(r'_\text{max}\) and \(r'_\text{min}\), corresponding to the tangent cones \(\sigma'_\text{max}\) and \(\sigma'_\text{min}\), respectively. This in mind, the latter is true if and only if both tangent cones contain at least one lattice point at height 1, i.e.,

\[
1 \in \langle H_{\sigma'_\text{max}}, v_N \rangle \quad \text{and} \quad 1 \in \langle H_{\sigma'_\text{min}}, -v_N \rangle.
\]

Using Lemma 6.5, this is equivalent to the condition that \(v_N\) is not strongly decomposable in \((\sigma'_\text{max})^\vee\) and \(-v_N\) is not strongly decomposable in \((\sigma'_\text{min})^\vee\) \(\subseteq N\).

For convenience only, let us assume \(\lambda = 1\). In the next step, we will detect how \(\Delta(D)\) yields the tangent cone \(\sigma'_\text{min}\): In the polytope cut off \(\Delta(D)^C \subseteq \Delta(D)\) (cf. Subsection 3.2) the vertex \(r'_\text{min} \in \Theta(1, q)\) corresponds to the rational line segment

\[
\Delta(D)^{v_N}_{\text{min}} := r'_\text{min} + q\Delta^\text{newt}
\]

being orthogonal to \(v_N\), i.e., lying in \([v_N = c]\). Moreover, it is an edge of \(\Delta(D)^C\) having two vertices \(v_1^\text{min}, v_2^\text{min} = r'_\text{min}\) lying on the boundary of \(\Delta(D)\). We call \(e_1^\text{min}, e_2^\text{min}\) the two part edges of \(\Delta(D)\) contained in \([v_N \leq c]\) with vertices \(v_1^\text{min}, v_2^\text{min}\), respectively. Now, \(\sigma'_\text{min}\) equals the cone generated by the two edge directions \(e_1^\text{min}, e_2^\text{min}\) with \(1 \in \langle H_{\sigma'_\text{min}}, -v_N \rangle\). Analogously, we obtain \(\sigma'_\text{max}\).

To sum it up, \(S_Y(D)\) is finitely generated if and only if \(v_N\) is not strongly decomposable in \((\sigma'_\text{max})^\vee\) and \(-v_N\) is not strongly decomposable in \((\sigma'_\text{min})^\vee\) for all \(q\), where these two cones are obtained by looking on \(\Delta(D)\) instead of \(\Theta(1, q)\).

Let \(q\) be maximal such that \(q\Delta^\text{newt} \subseteq \Delta(D)\) \((q\Delta^\text{newt} \) possibly shifted), i.e., \(\Delta(D)^{v_N} = q\Delta^\text{newt}\). In order to obtain finitely generation, it is sufficient to check
failure of the two strongly decomposable conditions in this maximal case, i.e., for $\sigma^+$ and $\sigma^-$. The reason for this lies in the fact that $\Delta(D)$ is a convex polytope, i.e.,

$$(\sigma^+)^\vee \subseteq \sigma'_{\min} \quad \text{and} \quad (\sigma^-)^\vee \subseteq \sigma'_{\max}$$

for all $q$ smaller than the maximal one. Therefore, $(\sigma'_{\min})^\vee \subseteq \sigma^+$ and $(\sigma'_{\max})^\vee \subseteq \sigma^-$. If $v_N$ and $-v_N$ are not strongly decomposable in $\sigma^+$ and $\sigma^-$, respectively, they are especially not for any smaller cone contained in them.

If the maximal length of the rational line segment being orthogonal to $v_N$ is obtained where $\Delta(D)$ has two parallel edges, then there is no unique $\Delta(D)^{v_N}$. In this case, we will consider two copies of the rational line segment sitting at the borders of the parallel edges to construct $\sigma^+$ and $\sigma^-$.  

**Corollary 6.9.** The valuation semigroup $S_Y(D)$ is finitely generated if and only if the morphism $\mathbb{P}^1 \rightarrow X'$ given by $v_N$ is a smooth embedding, where $X'$ is the toric variety associated with the fan generated by $\sigma^+$ and $\sigma^-$.  

**Example 6.10.** We apply Theorem 6.8 to the 7-gon $\Delta(D)$ (cf. Figure 10A) with vertices $[4, 1], [7, 2], [9, 3], [6, 5], [1, 8], [1, 7], [2, 4]$, and $[4, 1]$ from Example 4.8 in [CLTU20], which is a good polytope in the language of loc. cit.

![Figure 10](image.png)

**Figure 10. The good 7-gon $\Delta(D)$.** (A) The polytope $\Delta(D)$ having seven vertices together with the rational line segment $\Delta(D)^{v_N} = \Delta(D) \cap [v_N = 3]$ and its two vertices $v_1 = [8/3, 3], v_2 = [9, 3]$. (B) The normal fan $\Sigma$ of $\Delta(D)$ having seven rays $\rho_i$ ($0 \leq i \leq 6$) together with the two cones $\sigma^+ = \text{cone}(\rho_4, \rho_6) \ni v_N$ and $\sigma^- = \text{cone}(\rho_0, \rho_4) \ni -v_N$.

In loc. cit. the authors construct examples of projective toric surfaces whose blow-ups at the general point have a non-polyhedral pseudo-effective cone. In particular, this is the case for projective toric surfaces associated with good polytopes [CLTU20, Definition 4.3, Theorem 4.4].
Consider the projective toric surface $X = \mathbb{T}V(\Sigma)$ associated with the normal fan $\Sigma$ of $\Delta(D)$ with rays

$\rho_0 = (-2, -3), \rho_1 = (-3, -5), \rho_2 = (1, 0), \rho_3 = (3, 1), \rho_4 = (3, 2), \rho_5 = (-1, 3),$ and $\rho_6 = (-1, 2)$

(cf. Figure 10B) and an admissible flag $Y_\bullet: X \supseteq C_v \supseteq \{1\}$ on $X$ with $v_N = (0, 1)$.

To apply Theorem 6.8, we compute the following data:

$\Delta(D)^{v_N} = \Delta(D) \cap [v_N = 3] = \text{conv}(\{8/3, 3\}, [9, 3])$,

i.e., $v_1 = [8/3, 3]$ and $v_2 = [9, 3]$. The inner normal vectors of $e_1^+, e_2^+$ and $e_1^-, e_2^-$ are $\rho_4$, $\rho_0$ and $\rho_4$, $\rho_6$, respectively. Therefore,

$\sigma^+ = \text{cone}(\rho_4, \rho_6) \ni (0, 1) = v_N$ and $\sigma^- = \text{cone}(\rho_4, \rho_0) \ni (0, -1) = -v_N$.

Thus the associated semigroup $S_{Y_\bullet}(D)$ is finitely generated because $v_N = (0, 1)$ is not strongly decomposable in $\sigma^+$ and $-v_N = (0, -1)$ is not strongly decomposable in $\sigma^-$.  

6.3. Varying of $v_N$ and $D$. The strategy to obtain a finitely generated semigroup by choosing an appropriate direction $v_N$ does not always work. For the following example, there is no $v_N$ that works.

Example 6.11. Consider the ample divisor $D$ associated with the polytope $\Delta(D)$ depicted in Figure 11A on the toric variety $X = \mathbb{T}V(\Sigma)$ corresponding to the fan $\Sigma$ depicted in Figure 11B. We claim that no matter what $v_N \in \mathbb{N}$ we pick, the resulting semigroup $S_{Y_\bullet}(D)$ will not be finitely generated.

![Figure 11](image-url)

**Figure 11.** $S_{Y_\bullet}(D)$ non-finitely generated for all $v_N$. (A) The polytope $\Delta(D)$ associated with an ample divisor $D$ on $X = \mathbb{T}V(\Sigma)$. (B) The normal fan $\Sigma$ of $\Delta(D)$ with 16 rays.

We will use our characterization in Theorem 6.8. As $\Delta(D)$ is centrally symmetric, the longest line segment $\Delta(D)^{v_N}$ in Definition 6.7 will pass through the origin, whatever $v_N$.

We distinguish two cases: either the endpoints of the segment $\Delta(D)^{v_N}$ are vertices of $\Delta(D)$ or they belong to the interior of an edge. Up to symmetry, there are four
vertices and four edges to consider. We will carry out the argument for one vertex
and for one edge. The others are left to the reader.

If $\Delta(D)^{v_N}$ hits the interior of the edges $e_1^\pm, e_2^\pm$ indicated in Figure 12A, $v_N$ must
belong to the interior of the red region in Figure 12B. In this case, the cones $\sigma^\pm$
from Definition 6.7 will be the two half planes bounded by the dotted line which is
perpendicular to the direction of $e_1^\pm, e_2^\pm$. But all lattice vectors in the red region are
strongly decomposable in their half plane as they all have lattice distance $> 1$ from
the dotted line. (The vectors which are not strongly decomposable are the vectors
at distance one, \textit{i.e.}, they lie on the dashed lines.)

If, on the other hand, $\Delta(D)^{v_N}$ contains the vertices $v_1, v_2$ indicated in Figure 12C,
$v_N$ is determined up to sign, as are $\sigma^\pm$ (cf. Figure 12D). Again, we see that $v_N$ is
strongly decomposable. (And again, the vectors which are not strongly decomposable
are the vectors which lie on the dashed lines.)

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure12a.png}
\caption{(A) The line segment $\Delta(D)^{v_N}$ containing the interior of the edges $e_1^\pm, e_2^\pm$.}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure12b.png}
\caption{(B) The possible region for $\pm v_N$ in red together with the two cones $\sigma^\pm$.}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure12c.png}
\caption{(C) The line segment $\Delta(D)^{v_N}$ hitting two vertices $v_1, v_2$ of $\Delta(D)$.}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\textwidth]{figure12d.png}
\caption{(D) The cones $\sigma^\pm$ containing $\pm v_N$.}
\end{subfigure}
\caption{Illustration of Example 6.11. (A) The line segment $\Delta(D)^{v_N}$ containing the interior of the edges $e_1^\pm, e_2^\pm$. (B) The possible region for $\pm v_N$ in red together with the two cones $\sigma^\pm$. (C) The line segment $\Delta(D)^{v_N}$ hitting two vertices $v_1, v_2$ of $\Delta(D)$. (D) The cones $\sigma^\pm$ containing $\pm v_N$.}
\end{figure}
Proposition 6.12. Given a fan $\Sigma$ and a direction $v_N$, the valuation semigroup $S_{Y_\ell}(D)$ is finitely generated for all ample divisors $D$ on $X$ if and only if $v_N$ and $-v_N$ are not strongly decomposable in $\sigma$ for all cones $\sigma \subseteq \mathbb{N}_{\mathbb{R}}$ generated by rays of $\Sigma$.

Proof. ”$\Leftarrow$”: Theorem 6.8. ”$\Rightarrow$”: Assume there exists a cone $\sigma$ built from rays of $\Sigma$ such that $v_N$ is strongly decomposable in $\sigma$. We will construct an ample divisor $D$ such that $S_{Y_\ell}(D)$ is not finitely generated:

In a first step, we build a torus-invariant divisor $D_{\Theta(\ell,k)} = \sum_{\rho \in \Sigma(1)} b_\rho D_\rho$ whose associated polytope $\Theta(\ell,k)$ has a vertex $r'_{\min}$ with tangent cone $\sigma'_{\min} = \sigma^\vee$. For that we choose coefficients $b_\rho \in \mathbb{Z}$ such that $D_{\Theta(\ell,k)}$ has positive intersection numbers with all curves corresponding to rays $\rho \not\in \text{int}(\sigma)$. Then we relax the inequalities at $r'_{\min}$ by choosing the coefficients $b_\rho \gg 0$ for all $\rho \not\in \text{int}(\sigma)$. This guarantees that $\sigma^\vee$ is the tangent cone of $\Theta(\ell,k)$ at its vertex $r'_{\min}$.

Now, we want to define an ample torus-invariant divisor $D$ such that $\Theta(\ell,k) = (\Delta(D) : \Delta^\text{nef})$. As an intermediate step set $D' = \sum_{\rho \in \Sigma(1)} a'_\rho D_\rho := C'_v + D_{\Theta(\ell,k)}$. Then by construction, the associated polytope $\Delta(D')$ contains the Minkowski sum $\Theta(\ell,k) + \Delta^\text{nef}$. Moreover, the defining inequalities of $\Theta(\ell,k) + \Delta^\text{nef}$ coincide with the ones for the polytope $\Delta(D')$ for all rays $\rho \not\in \text{int}(\sigma)$.

In general, $D'$ is not nef and in particular not ample because it might have negative intersection with curves associated with remaining rays $\rho \not\in \text{int}(\sigma)$. Hence, as a last step, we define a torus-invariant divisor $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ with $a_\rho = a'_\rho$ for all $\rho \in \Sigma(1) \setminus \text{int}(\sigma)$. For the remaining rays $\rho \not\in \text{int}(\sigma)$, choose coefficients $a_\rho$ small enough such that $D$ is ample and big enough such that $\Theta(\ell,k) + \Delta^\text{nef} \subseteq \Delta(D)$. Then we have $\Theta(\ell,k) = (\Delta(D) : \Delta^\text{nef})$ and $\Theta(\ell,k)$ has a vertex $r'_{\min}$ with tangent cone $\sigma'_{\min} = \sigma^\vee$. Since $v_N$ is strongly decomposable in $\sigma$ it follows with Lemma 6.5 and Corollary 6.3 that the semigroup $S_{Y_\ell}(D)$ is not finitely generated.

Example 6.13. We illustrate the construction from the proof of Proposition 6.12 for a modification of our running Example 6.6 with $v_N = (-2,3) \in \mathbb{N}$. Let $X' = \mathbb{T}
abla_\Sigma'$ be the toric surface associated with the fan $\Sigma'$, where $\Sigma'(1) = \Sigma(1) \cup \{\rho_4, \rho_5\}$ and $\rho_4 = (1,0)$, $\rho_5 = (-1,1)$. Then (as in Example 6.6) $v_N = (-2,1) + (0,2)$ is strongly decomposable in $\sigma = \text{cone}(\rho_0, \rho_2)$ (cf. Figure 13A). Moreover,

$$C'_v = 7D_2 + 2D_3 + 3D_4$$

in $X' = \mathbb{T}
abla_\Sigma'$ with $\Delta^\text{nef} = \text{conv}([0,0], [-3,0], [-3,-2], [-2,-2])$ (cf. Figure 13B). Choose the divisor $D_{\Theta(\ell,k)}$ as

$$D_{\Theta(\ell,k)} = 6D_2 + 4D_3 + 2D_4 + 6D_5,$$

i.e., $\Theta(\ell,k) = \text{conv}([0,0], [-2,0], [-2,-2], [0,-3])$ (cf. Figure 13C). We obtain the divisor

$$D' = D_{\Theta(\ell,k)} + C'_v = 13D_2 + 6D_3 + 5D_4 + 6D_5$$
with $\Delta(D') = \text{conv}([0, 0], [-5, 0], [-5, -4], [-1, -6], [0, -6])$ and $\Theta(\ell, k) + \Delta_{\text{nef}} \subseteq \Delta(D')$ (cf. Figure 13D). Note that $D'$ is not ample, since we have $D'.D_5 = 0$.

To make it ample, adjust the coefficient of $D_5$ to $5.5$. Then the resulting divisor $D$ is ample with $\Delta(D) = \text{conv}([0, 0], [-5, 0], [-5, -4], [-1, -6], [-0.5, -6], [0, -5.5])$ (cf. Figure 13E). Moreover, we have $\Theta(\ell, k) = (\Delta(D) : \Delta_{\text{nef}})$.

Figure 13. Illustration of Proposition 6.12. (A) The set of rays $\Sigma'(1) = \Sigma(1) \cup \{\rho_4, \rho_5\}$, a primitive element $v_N = (-2, 3) \in \mathbb{N}$, and the cone $\sigma$ in which $v_N$ is strongly decomposable. (B) The Newton polytope $\Delta_{\text{newt}}$ given by $v_M = [-3, -2]$ and the polytope $\Delta_{\text{nef}}$ corresponding to $C_\rho$ in $X' = \mathbb{T}\mathbb{V}(\Sigma')$. (C) The polytope $\Theta(\ell, k)$ having a vertex $r_{\min}'$ with tangent cone $\sigma'^\vee$. (D) The polytope $\Delta(D')$ corresponding to the non-ample divisor $D'$ containing $\Theta(\ell, k) + \Delta_{\text{nef}}$. (E) The polytope $\Delta(D)$ corresponding to the ample divisor $D$ with $\Theta(\ell, k) = (\Delta(D) : \Delta_{\text{nef}})$.

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