The width of a chaotic layer

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Abstract

A model of nonlinear resonance as a periodically perturbed pendulum is considered, and a new method of analytical estimating the width of a chaotic layer near the separatrices of the resonance is derived for the case of slow perturbation (the case of adiabatic chaos). The method turns out to be successful not only in the case of adiabatic chaos, but in the case of intermediate perturbation frequencies as well.

1 Introduction

The extent of chaotic domains, and, in particular, the width of chaotic layers, is one of the most important characteristics of the chaotic motion of Hamiltonian systems. Until now, several aspects of the problem of analytical estimation of the width of a chaotic layer were considered in Refs. [1, 2, 3, 4, 5, 6, 7]. Potentially, the ability of estimating the extent of chaos in phase space of Hamiltonian systems has a wide field of applications in physics and dynamical astronomy. Wisdom et al. [8] and Wisdom [9] estimated the width of the chaotic layer near the separatrices of spin-orbit resonances in the rotational dynamics of planetary satellites and Mercury. Yamagishi [10] made estimates of the width of the chaotic layer near the magnetic separatrix in poloidal diverter tokamaks. In these both applications, Chirikov’s approach [2] based on the separatrix map theory was used. Chirikov derived approximate formulas for the width in the assumption of high-frequency perturbation of non-linear
resonance; however, as follows from these same formulas, the chaotic layer is exponentially thin with the ratio of perturbation frequency to the frequency of small-amplitude phase oscillations on the resonance. This means that the cases of intermediate and low frequencies of perturbation are most actual in applications. So, analysis of the problem of estimation of the width of a chaotic layer in these cases is definitely necessary.

In this paper, a method of analytical estimation of the width of a chaotic layer, especially aimed at the case of slow, or adiabatic, chaos, is proposed. It is based on the theory of separatrix maps. The nonlinear resonance is modelled by the Hamiltonian of a perturbed nonlinear pendulum. There are two fundamental parameters: the ratio of the frequency of perturbation to the frequency of small-amplitude phase oscillations on the resonance, and the parameter characterizing strength of the perturbation.

The applicability of the theory of separatrix maps for description of the motion near the separatrices of the perturbed nonlinear resonance in the full range of the relative frequency of perturbation, including its low values, was discussed and shown to be legitimate in Ref. [11].

The field of applications of the derived method is rather wide due to generic character [2] of the perturbed pendulum model of nonlinear resonance. The method can be used in any application where a separatrix map is derived for description of chaotic motion. Many of such applications are described, e.g., in Ref. [12].

Analytical and numerical approaches to measuring the width of a chaotic layer have different merits and different demerits. The inherent shortcoming of any analytical approach consists in that it implies an idealization of the phenomenon, and the estimates are inherently approximate. The precision of the estimates is hard to evaluate, due to a number of approximations involved. On the other hand, the numerical methods are applicable in a rather narrow range of values of parameters: they cannot be used in the case of very low relative frequencies of perturbation (due to limitations on computation time), also in the case of high relative frequencies of perturbation (because the width of the chaotic layer is exponentially thin with the perturbation frequency), and in the case of very small amplitudes of perturbation, due to limitations on the arithmetic precision. Therefore only analytical methods can give the global picture. Their another advantage is that the analytical estimation is easy and fast, as soon as the theoretical model is shown to be valid. Finally, the most important advantage, perhaps, is in the physical insight that the analytical methods provide, making the role of each parameter
2 The model of nonlinear resonance and the separatrix map

Under general conditions \cite{13,2,14}, a model of nonlinear resonance is provided by the Hamiltonian of the nonlinear pendulum with periodic perturbations. A number of problems on nonlinear resonances in mechanics and physics is described by the Hamiltonian

\[ H = \frac{G p^2}{2} - F \cos \varphi + a \cos(k \varphi - \tau) + b \cos(k \varphi + \tau) \]  

(1)

(see, e.g., Ref. \cite{11}). The first two terms in Eq. (1) represent the Hamiltonian \( H_0 \) of the unperturbed pendulum; \( \varphi \) is the pendulum angle (the resonance phase angle), \( p \) is the momentum. The periodic perturbations are given by the last two terms; \( \tau \) is the phase angle of perturbation: \( \tau = \Omega t + \tau_0 \), where \( \Omega \) is the perturbation frequency, and \( \tau_0 \) is the initial phase of the perturbation. The quantities \( F, \ G, \ a, \ b, \ k \) are constants. We assume that \( F > 0, \ G > 0, \ k \) is integer, and \( a = b \). We use the notation \( \varepsilon \equiv a/F = b/F \) for the relative amplitude of perturbation.

The so-called separatrix (or “whisker”) map

\[ w_{i+1} = w_i - W \sin \tau_i, \]
\[ \tau_{i+1} = \tau_i + \lambda \ln \frac{32}{|w_{i+1}|} \pmod{2\pi}, \]  

(2)

written in the present form and explored in Refs. \cite{13,11,2} and first introduced in Ref. \cite{15}, describes the motion in the vicinity of the separatrices of Hamiltonian (1). The quantity \( w \) denotes the relative (with respect to the unperturbed separatrix value) pendulum energy \( w \equiv \frac{H_0}{E} - 1 \), and \( \tau \) retains its meaning of the phase angle of perturbation. The constants \( \lambda \) and \( W \) are the two basic parameters, already mentioned in the Introduction. The parameter \( \lambda \) is the ratio of \( \Omega \), the perturbation frequency, to \( \omega_0 = (F \mathcal{G})^{1/2} \), the frequency of the small-amplitude pendulum oscillations. The parameter \( W \) in the case of \( k = 1 \) and \( a = b \) has the form \cite{16}:

\[ W = \varepsilon \lambda \left( A_2(\lambda) + A_2(-\lambda) \right) = \frac{4\pi \varepsilon \lambda^2}{\sinh \frac{\pi \lambda}{2}}. \]  

(3)
Here $A_2(\lambda) = 4\pi\lambda \frac{\exp(\pi\lambda/2)}{\sinh(\pi\lambda)}$ is the value of the Melnikov–Arnold integral as defined in Ref. [2]. Formula (3) differs from that given in Refs. [2, 14] by the term $A_2(-\lambda)$, which is small for $\lambda \gg 1$. However, its contribution is significant for $\lambda$ small [16], i.e., in the case of adiabatic chaos. Expression (3) for the parameter $W$ needs to be modified at very high relative frequencies of perturbation (see Refs. [17, 5]). Analytical expressions for $W$ at different values of $k$ are given in Refs. [2, 11] and at arbitrary $a$, $b$ in Ref. [11].

The accuracy of separatrix map (2) in describing the behaviour of original system (1) can be estimated by the order of magnitude as $\sim \varepsilon$ (see Refs. [5, 12]). Measurement of the chaotic layer width allows one to estimate the accuracy directly, as demonstrated below in Section 5.

Note that the expression for the increment of the phase $\tau$ in map (2) is a rough approximation. It is valid for a low strength of perturbation, i.e., at $\omega \ll 1$. According to Refs. [16, 18], one can improve the accuracy of the map by means of replacing the logarithmic approximation of the phase increment by the original expression through the elliptic integrals. For the sake of brevity we do not explore the advantages of this improvement further in estimating the width. This can be straightforwardly accomplished if one needs to improve precision of estimating the width at increasing the magnitude of perturbation.

One iteration of map (2) corresponds to one period of the pendulum rotation or a half-period of its libration. The motion of system (1) is mapped by Eqs. (2) asynchronously [16]: the relative energy variable $w$ is taken at $\varphi = \pm \pi$, while the perturbation phase $\tau$ is taken at $\varphi = 0$. The desynchronization can be removed by a special procedure [16, 11]. The synchronized separatrix map gives correct representation of the sections of the phase space of the near-separatrix motion both at high and low perturbation frequencies; this was found in Ref. [11] by direct comparison of phase portraits of the separatrix map to the corresponding sections obtained by numerical integration of the original systems. This testifies good performance of both the separatrix map theory and the Melnikov theory (that describes the splitting of the separatrices).

The asymptotic expression for $W$ that ensues from Eq. (3) at $\lambda \sim 0$ is $W \approx 8\varepsilon\lambda$. A good correspondence of this expression to the actual amplitude of the separatrix map derived numerically by integration of the original system was found in Ref. [6].

An equivalent form of Eqs. (2), used, e.g., in Refs. [19, 16], is
\[ y_{i+1} = y_i + \sin x_i, \]
\[ x_{i+1} = x_i - \lambda \ln |y_{i+1}| + c \pmod{2\pi}, \]  
where \( y = w/W, \ x = \tau + \pi; \) and
\[ c = \lambda \ln \frac{32}{|W|}. \]  

Note that while the second line of Eqs. (4) is taken modulo \( 2\pi \), the quantity \( c \) given by Eq. (5) is taken modulo \( 2\pi \) in the following analytical treatment only when explicitly stated.

### 3 The case of the least perturbed border

We obtain the dependence of the half-width \( y_b \) of the main chaotic layer of the separatrix map in a numerical experiment with Eqs. (4). The border value \( y_b \), corresponding to the maximum energy deviation (from the unperturbed separatrix) of a chaotic trajectory inside the layer, is determined as the maximum of \(|y_i|\) obtained during computation of a single chaotic trajectory. At each step (equal to 0.05) in the segment \( \lambda \in [0, 10] \), the values of \( y_b \) are computed for 100 values of \( c \) equally spaced in the interval \([0, 2\pi]\). The number of iterations for each trajectory is \( n_{it} = 10^8 \). This has been checked to be sufficient to saturate the computed values of \( y_b \). Following the approach of Ref. [7], at each step in \( \lambda \) we find the value of \( c \) corresponding to the minimum \( y_b \) (the case of the least perturbed border), and plot the value of \( y_b \) corresponding to this case; such \( y_b \) value is denoted in what follows as \( y_{lb} \).

The initial fragment of thus obtained “\( \lambda - y_b \)” relation is plotted in Fig. 1. The observed dependence apparently follows the piecewise linear law
\[ y_{lb} \approx \begin{cases} 
1, & \text{if } 0 \leq \lambda \leq 1 - a, \\
\lambda + a, & \text{if } \lambda > 1 - a, 
\end{cases} \]  
where \( a \approx 1/2 \) apparently.

The linear fit \( y_{lb}(\lambda) = a + b\lambda \) of the observed dependence at \( \lambda \in [0.5, 10] \) gives the following values for the coefficients and their standard errors: \( a = 0.5351 \pm 0.0041, \ b = 1.0059 \pm 0.0007 \). The \( y_{lb}(\lambda) \) dependence represents the sum of two addends. The first addend, \( a \), is the half-amplitude of the last rotational invariant curve of the standard map at the critical value of the
stochasticity parameter, because the separatrix map at high values of $\lambda$ is locally (in $y$) approximated by the standard map. The theoretical value of this addend, which can be found by direct computation with the standard map, is $\approx 0.508$. The second addend, $b\lambda$, is the border value of $y$ averaged over $x \in [0, 2\pi]$. The theoretical value of this addend approximately equals $\lambda$, see Ref. [2]. So, the deviations of the observed values of $a$ and $b$ from their theoretical values are rather small, 0.027 and 0.006 respectively.

One can eliminate the constant component, arising due to the fact that the border curve is not a horizontal straight line, by subtracting the half-amplitude of the border curve. The resulting value coincides with the value of $y$ time-averaged on the trajectory following the boundary curve. We designate this value by $\bar{y}_{lb}$. At $\lambda \geq \frac{1}{2}$, one has $\bar{y}_{lb} \approx \lambda$ apparently. At $\lambda \leq \frac{1}{2}$, $\bar{y}_{lb}$ equals one half of the maximum $y$ value; this follows from the form of the border curve derived in the next Section. So, one obtains for the time-averaged half-width

$$\bar{y}_{lb} \approx \begin{cases} \frac{1}{2}, & \text{if } 0 \leq \lambda \leq \frac{1}{2}, \\ \lambda, & \text{if } \lambda > \frac{1}{2}. \end{cases} \quad (7)$$

The clear break at the point $\lambda \approx 1/2$ manifests a physical distinction between two types of dynamics, namely, slow and fast chaos. While in the case of slow chaos the chaotization of motion can be physically explained by the sporadic encounters with the singular line $y = 0$ (such mechanism was originally evoked in an analysis of the so-called “relativistic” map in Ref. [20]), in the case of fast chaos the natural traditional explanation consists in the phenomenon of resonance overlapping [2].

Approximating the separatrix map by the standard one locally in $y$, Chirikov [2] derived the linear $\lambda$ dependence for the width of the chaotic layer at $\lambda \gg 1$. By means of a rigourous mathematical argument, Ahn et al. [4] were able to set a lower bound on the width of the chaotic layer. In our notation, this lower bound is given by the formula $y_b > \frac{3}{4}\lambda$ (see Eq. (5.8) in Ref. [4]). Eq. (7) is in accord with these theoretical findings.

The formula for the time-averaged half-width in the original energy variable $w$ is

$$\bar{w}_{lb} = |W|\bar{y}_{lb} = \frac{4\pi|\epsilon|\lambda^2\bar{y}_{lb}}{\sinh \frac{\lambda^2}{2}}, \quad (8)$$

where $\bar{y}_{lb}$ is given by Eq. (7). Formula (8) is valid for any frequency of perturbation in system (1) with $k = 1, a = b$, provided that the separatrix map
correctly describes the behaviour of the original system, and the analytical representation of \( W \) is correct. At very high frequencies of perturbation, expression (3) for the parameter \( W \) needs correction, as noted above. At low frequencies of perturbation, formula (8) demonstrates that the chaotic layer width, expressed in \( w \), decreases linearly with \( \lambda \). This is due to the fact that we consider the case of the least perturbed border, and therefore \( \varepsilon \) is not fixed.

If one fixes \( \varepsilon \), the low frequency limit of the width is a non-zero constant. This is a trivial consequence of the usual “slowly pulsating separatrix” approach considered e.g. in Refs. [21, 22]. Indeed, Hamiltonian (1) with \( k = 1, a = b \) is naturally set in the form of that of a pendulum with modulated frequency of small-amplitude oscillations:

\[
H = \frac{G p^2}{2} - (F - 2a \cos \tau) \cos \varphi
\]

(see, e.g., Ref. [2]). Considering the relative full energy \( w_H \equiv \frac{H}{F} - 1 \) instead of \( w = \frac{H}{F} - 1 \), one can get a simple heuristic estimate for the width of the chaotic layer in the limit \( \lambda \to 0 \). Indeed, as follows from representation (9), the energy \( w_H \) on the slowly pulsating separatrix varies from \(-2\varepsilon\) to \(2\varepsilon\); so, the half-width of the layer is equal to \(2\varepsilon\). This quantity, however, includes the amplitude of the chaotic layer bending (the effect, described in [11]). Let us note again that this “zero \( \lambda \)” half-width estimate is made in the units of the relative full energy \( w_H \). Below we shall see that, in the \( w \) units, the “zero \( \lambda \)” half-width incorporating bending is equal to \(4\varepsilon\), and the “zero \( \lambda \)” sheer half-width is equal to \(\varepsilon\) (for the same case of \( k = 1 \) and \( a = b \), of course; in other cases the widths are generally different).

4 The general formulas at slow perturbation

Let us assume that \( \lambda \ll 1 \). In this case the diffusion across the layer is slow, and on a short time interval the phase point follows close to some current curve. We call this curve guiding. Let us derive an analytical expression for the guiding curve with an irrational winding number far enough from the main rationals. We approximate the winding number \( Q \) by the rationals \( m/n \). Thus \( c \approx 2\pi m/n \). Noticing that at an iteration \( n \) of the map the phase point hits in a small neighbourhood of the starting point, one obtains for the derivative
\[
\frac{dy}{dx} = \frac{1}{nc - 2\pi m} \sum_{k=0}^{n-1} \sin(x + kc) = \\
= \frac{1}{nc - 2\pi m} \sin \left( \frac{nc}{2} \cosec \left( \frac{c}{2} \sin \left( x + \frac{n - 1}{2} c \right) \right) \right) 
\]
(10)

(the second equality follows from formula (1.341.1) in Ref. [23]). Integrating and passing to the limit \( n \to \infty \), one obtains finally

\[
y = -\frac{1}{2} \cosec \left( \frac{c}{2} \right) \cos \left( x - \frac{c}{2} \right) + C, 
\]
(11)

where \( C \) is an arbitrary constant of integration. Description (11) inside the chaotic layer is approximately valid on short intervals of time, for which the slow diffusion across the layer can be neglected.

The motion is chaotic only when curve (11) crosses the line \( y = 0 \) which is the inverse image of the singular curve \( y = -\sin x \). Hence the half-width of the chaotic layer is

\[
y_b = \left| \cosec \left( \frac{c}{2} \right) \right|, 
\]
(12)

or, equivalently, the half-width in the original energy variable \( w \) is

\[
w_b = \left| W \cosec \left( \frac{\lambda \ln |32/|W|)}{2} \right) \right|. 
\]
(13)

The quantities \( y_b \) and \( w_b \) are the maximum relative energy deviations inside the layer; the time-averaged half-widths \( \bar{y}_b \) and \( \bar{w}_b \), as follows from the geometry of the boundary curve, Eq. (11), are two times less.

A different approach for passing from map (2) to a differential equation in the case of slow chaos was used in Ref. [6]. That approach assumes that the increments in the map (2) variables are small at each iteration of the map. From our representation (1) of the separatrix map it is clear that such condition can be satisfied only when \( y \gg 1 \), and therefore \( y_b \gg 1 \); from Eq. (12) it follows then that the value of \( c \) should be close to the “main resonance” value \( \approx \lambda \text{mod} 2\pi \). On this condition the increment in \( x \) at each iteration of Eqs. (1) is small as well.

What are the restrictions for the validity of our approximation (11)? Since, in deriving the increment in \( x \), we have neglected the term \( \lambda \ln |y_{i+1}| \) in the second equation of Eqs. (1), the inequality \( c \text{mod} 2\pi \gg \lambda \ln |y_b| \) should hold, i.e.,
\[ c \mod 2\pi \gg \lambda \ln \left| \csc \frac{c}{2} \right|. \] (14)

So, since \( \lambda \) is small, the value of \( c \) should be far enough from the main resonance \( c \approx 0 \mod 2\pi \). Besides, the value of \( c \) should not correspond to other resonances, the role of which, however, is much less.

Expression (12) is compared to the numerical experimental data for \( \lambda = 0.1 \) in Fig. 2. The maximum relative energy deviations have been found for the chaotic trajectories of Eqs. (11) in the interval \( c \in [0, 2\pi) \) with the step in \( c \) equal to 0.001; each trajectory has been computed on the time interval \( n_{it} = 10^8 \). This has been checked to be sufficient to saturate the computed values of \( y_b \). It is evident that the theoretical curve follows closely the numerical data at all values of \( c \) except the resonant ones, where discontinuities are observed. The latter ones are conditioned by appearance of regular islands inside the chaotic layer at the resonant values of \( c \). These perturbations are similar to those in the behaviour of the standard map, for which the small local peaks in the dependence of the maximum Lyapunov exponent on the stochasticity parameter are conditioned by the local depressions in the measure of the chaotic component of phase space, due to appearance of regular islands [24].

The deviations of the theory from numerics are most visible near the main resonance, i.e., at \( c \) near 0 mod 2\( \pi \). If \( \lambda \) and \( c \) are close to zero, the relative increments of \( w \) and \( \tau \) in Eqs. (2) are small. Then, as mentioned above, approach [6] for passing from map (2) to a differential equation is applicable, and Eqs. (2) can be approximated by the differential equation

\[ \frac{dw}{d\tau} = -\frac{W \sin \tau}{\lambda \ln \frac{32}{|w|}} \] (15)

analogous to that derived in Ref. [6], except that we use homogeneous variables here and take into account the condition on \( c \). Similarly to Eq. (10), Eq. (15) describes a guiding curve with an arbitrary constant of integration \( C \):

\[ w \ln \frac{32e}{|w|} = \frac{W}{\lambda} (\cos \tau + C). \] (16)

As in derivation of Eq. (12), we note that the motion is chaotic only when curve (16) crosses the line \( w = 0 \). Then the constant of integration for the boundary curves of the chaotic layer is \( C = \pm 1 \), and the equation for the half-width \( w_b \) of the chaotic layer is
\[ w_b \ln \frac{32e}{w_b} = \frac{2|W|}{\lambda}. \quad (17) \]

The approximation of \( W \), which should be substituted here in the case of \( k = 1 \), follows from Eq. (3): if \( \lambda \ll 1 \), one has \( W \approx 8\varepsilon\lambda \), then

\[ w_b \ln \frac{32e}{w_b} = 16|\varepsilon|. \quad (18) \]

So, at small values of \( \lambda \) the half-width \( w_b \) depends solely on \( \varepsilon \); the \( \lambda \) dependence expires. For a different value of \( k \), the formula for \( W \) and its approximation are different (see Refs. [11, 6]), but the \( \lambda \) dependence in Eq. (17) expires all the same. Eq. (18) (or (17)) is easily solved numerically by iterations.

In summary, formulas (12) and (13) for the half-width of the chaotic layer are applicable in the case of generic values of the \( c \) parameter (excluding the main resonance case), and Eqs. (17) and (18) are applicable in the main resonance case \( c \approx 0 \mod 2\pi \). In the first case the chaotic layer width depends on both \( \lambda \) and \( \varepsilon \), while in the second case the dependence on \( \lambda \) expires and the width depends solely on \( \varepsilon \).

5 Analytical estimates versus numerical experiment

To check the theory, the width of the chaotic layer near the separatrices of Hamiltonian (1) has been directly computed. The integration of the equations of motion has been performed by the integrator by Hairer et al. [25]. It is an explicit 8th order Runge–Kutta method due to Dormand and Prince, with the step size control.

At each value of \( \lambda \) the half-width has been measured by two methods. The first one was proposed in Refs. [1, 2] and developed and extensively used in Ref. [6]. It is based on calculation of the minimum period \( T_{min} \) of the motion in the chaotic layer. The half-width is determined by the formula [1, 2, 6]:

\[ w_b = 32 \exp(-\omega_0 T_{min}). \quad (19) \]

The minimum period corresponds to the maximum energy deviation from the unperturbed separatrix value. This formula directly follows from the second line of Eqs. (2).
The second method consists in the direct continuous measuring of the relative energy deviation from the unperturbed separatrix \( w = \frac{H_0}{F} - 1 \) in the course of integration, and fixing the extremum one.

The integration time interval has been chosen to be \( 10^3 \), in the units of periods of perturbation. Each unit is divided in \( 10^5 \) equal segments; the trajectories have been output at the end of each segment, to provide, with such time resolution, the calculation of the time period and the relative energy deviation of the motion. Further increasing the integration time interval or decreasing the length of the segments have been checked to leave the estimates of the width unchanged within 3–4 significant digits; i.e., the estimates are saturated enough.

We set \( k = 1, a = b, \varepsilon = 10^{-5} \). The results of calculation of \( w_b \) by the first method are shown in Fig. 3, by the second method in Fig. 4. Logarithmic horizontal scale is used; decimal logarithms are implied by “log” throughout this paper. The experimental values are shown by dots. The theoretical dependence, given by Eq. (13), is shown in both figures by solid curve. General qualitative agreement is observed between the theory and the experimental data up to \( \lambda \approx 1 \), i.e., up to the intermediate values of the frequency of perturbation; even the sharp variations are in qualitative accord. Theoretical dependence (13) visually follows the experimental one even at high values of \( \lambda \). This is solely a visual effect: deviations in locations of resonance peaks increase with increasing \( \lambda \). Eq. (13) is applicable at low and intermediate relative frequencies of perturbation; at high frequencies a different formula, namely Eq. (8), should be used. This latter formula describes the behaviour averaged over \( c \).

Before analyzing the quantitative agreement of the theory and the experimental data, let us consider a notable feature of the observed dependence: the sharp peaks. They are conditioned by the process of encountering the main resonances \( c \approx 2\pi m, m = 1, 2, \ldots \), as \( \lambda \) increases. According to Eqs. (12, 13), at such values of \( c \) the width goes to infinity; the real width is finite, of course. The approximate location of the peaks in \( \lambda \) is determined by the equation

\[
  c = \lambda \ln \frac{4}{|\varepsilon| \lambda} = 2\pi m. \tag{20}
\]

The location is practically insensitive to the value of \( k \). If \( |\varepsilon| \ll \lambda \), the location can be estimated by the formula \( \lambda_m \approx -2\pi m / \ln \varepsilon \) very approximately; i.e., on decreasing \( \varepsilon \) the peaks move slowly to the left.

The abscissas \( \lambda_m \) of the peaks in Figs. 3 and 4 are all greater than 0.5, i.e.,
the peaks are situated in the domain of fast chaos. In such circumstances, the “tangency condition” \[16\] for the marginal integer resonances should provide better precision for estimating the location of the peaks. An opportunity of sporadic strong variations of the relative energy \(w\) in the motion in the chaotic layer at \(\lambda > 1\) depends on the structure of the border of the chaotic layer [2, 16]. Excursions to high values of the relative energy \(w\) become possible, when, with variation of a parameter, the border of the chaotic layer starts to overlap with the narrow chaotic layer near the separatrices of an integer resonance, i.e., a heteroclinic connection emerges between them. Since the first layer is very narrow, this phenomenon can be approximately described as “tangency” of the unperturbed separatrix of the marginal resonance and the border of the main chaotic layer. The equation for the tangency condition [16] is

\[ W = W_t^{(m)}(\lambda), \]  

(21)

where \(W\) at \(k = 1, a = b\), is given by Eq. (3) and

\[ W_t^{(m)}(\lambda) = \frac{32}{\lambda^2} \left( (1 + \lambda^2)^{1/2} - 1 \right)^2 \exp \left( -\frac{2\pi m}{\lambda} \right). \]  

(22)

Eq. (21), solved numerically, does provide good precision: the calculated abscissas of the first five peaks \(\log \lambda_m\) \((m = 1, 2, 3, 4, 5)\) deviate from the observed in the numerical experiment not greater than by 0.02. The deviations are all positive. The accuracy of the observed values is determined by the horizontal resolution of the experimental plots, which is \(\log 1.01 \approx 0.004\).

As expected, Eq. (20) is less successful: it predicts \(\log \lambda_m = -0.34, -0.01, 0.18, 0.32, 0.42 (m = 1, 2, \ldots, 5)\) versus the observed values \(-0.28, 0.02, 0.19, 0.30, 0.39\), i.e., the deviation in absolute value is not greater than 0.06.

The tangency condition can be employed for analytical estimating the height of the peaks; see Eq. (10) in Ref. [16].

While the main resonances with \(m = 1, 2, \ldots\) manifest themselves in the peaks, that with \(m = 0\) results in the asymptotic horizontal plateau at \(\lambda \to 0\). This plateau was found and discussed in Ref. [6], and an approximate heuristic formula was proposed for its asymptotic height: \(w_b/\varepsilon \approx 0.22 \cdot 8 = 1.76\) for \(k = 1\) (see Eq. (14) of that paper; a misprint (missing \(\varepsilon\)) is corrected here). This estimate differs from our data less than by a factor of 2.

Our experimental \(w_b/\varepsilon\) value at \(\lambda = 0.01\) is equal to 1.00585. Numerical solving Eq. (18) gives \(w_b/\varepsilon = 1.00142\), i.e., a value perfectly close to the experimental one; the difference is only 0.4%. Formula (13), giving \(w_b/\varepsilon = \)
0.915, is also quite successful, notwithstanding the fact that this part of dependence is beyond the scope of its applicability ($c$ is close to zero).

The reason for the plateau emerging at small values of $\lambda$ is clear: if one fixes $\varepsilon$, no matter how small this fixed value is, on decreasing $\lambda$ the value of $c = \lambda \ln \frac{32}{|W|}$ also decreases and inevitably finds itself near the main resonance $c \approx 0$. The transition point to the main resonance domain is determined by the abscissa of the point of intersection between the curve given by formula (13) (this curve goes asymptotically to zero, though slowly) and the horizontal line defined by Eq. (17).

In Fig. 3, this transition point is situated at log $\lambda \approx -1.44$. To the left of this point, the deviations of the observed $w_b$ data from the theoretical plateau level 1.00142 do not exceed 2%. To the right of the point, the deviations of the solid curve (given by formula (13)) from the observed data is less than 10% in absolute value typically, except that at narrow resonant intervals in $\lambda$ (fractional resonances manifest themselves as small peaks) the deviations rise up to about 30%. This agreement continues up to log $\lambda \approx -0.4$. Then the first integer resonance comes into play, and, due to incomplete correspondence in real and theoretical locations of the peak, the deviation rises sharply. At greater values of $\lambda$ the agreement is solely qualitative.

The “$\lambda-w_b$” dependence constructed by the second method (in Fig. 4) at small values of $\lambda$ lies notably higher than the experimental data. This reveals an interesting phenomenon of the chaotic layer bending, described in Ref. [11]. This strictly geometrical phenomenon is absent in the previous graph, because it averages out in that case. According to Ref. [11], the relative energetic amplitude of bending at $k = 1$ in the limit $\lambda \to 0$ at the section of phase space $\varphi = 0 \mod 2\pi$ is equal to $4\varepsilon$. The experimentally observed value of $w_b/\varepsilon$ at $\lambda = 0.01$ is equal to 4.00158, in perfect agreement with this theoretical prediction: the deviation is only 0.04%.

One should emphasize that the theoretical value of bending refers to the section $\varphi = 0 \mod 2\pi$; at any other value of $\varphi \mod 2\pi$ the bending is, generally speaking, different. Our experimental procedure gives the value of the maximum bending. In the considered case ($k = 1$) both values coincide, i.e., the maximum bending is achieved at $\varphi = 0 \mod 2\pi$.

The numerical data considered above refer all to the case $k = 1$. The cases of different values of $k$ can be considered in a similar way. To obtain the theoretical dependence “$\lambda-w_b$”, one should simply substitute the relevant expression for $W$ in Eq. (13). E.g., in the case $k = 2$, $a = b$, the expression is
\[ W = \frac{8\pi\varepsilon\lambda^2(\lambda^2 - 2)}{3 \sinh \frac{\pi\lambda}{2}} \]  

(23)

In Fig. 5, the dependences “\(\lambda-w_b\)" for this case are shown: the theoretical one by solid curve, while those obtained by direct integration of system \((1)\) are represented by dots. The visual agreement between the theory and numerical experiment at low and intermediate frequencies of perturbation is as good as in the case \(k = 1\). The theoretical plateau level of \(w_b/\varepsilon\) in the case of \(k = 2\), given by Eq. (17), is 1.36140. The characteristics of quantitative agreement in the full range of \(\lambda\) are the same as in the case of \(k = 1\) (see above), only the location of the transition point, \(\log \lambda \approx -1.42\), is slightly different.

Let us clarify what is the role of resonances by means of constructing phase portraits of the motion. The main (integer) resonances \(m = 0, 1, 2, \ldots\) result in stretching the layer in the \(y\) direction; the motion in these resonances is quite simple: it follows the guiding curves \((16)\). The role of fractional resonances is more intricate. The phase portrait of separatrix map \((4)\) for the case of the fractional resonance with winding number \(Q = 4/5\) is shown in Fig. 6a; \(\lambda = 0.01\) and \(c = 5.0189 \approx 2\pi Q\). Only the chaotic component is shown. The choice of the value of \(c\) corresponds to the minimum measure of the chaotic component inside the borders of the layer. The phase portrait has been obtained by iterating Eqs. \((4)\) \(10^7\) times. Further increasing \(n_t\) does not give any detectable increase of chaotic component. Instead of visualizing each iteration (this would produce a file of incredible volume), Fig. 6a represents a “rasterized” phase portrait: it is comprised by the set of pixels explored by a chaotic trajectory on the grid of \(400 \times 400\) pixels.

The “porous” structure of the layer at resonance is clearly seen in the Figure. The main pattern is formed by 5 curves of sinusoidal form, embedded in broad strips of generic chaos. These curves are nothing but the singular curve \(y = -\sin x\) and its four consecutive images. The figure demonstrates how the resonant structure with large amount of inner regular component is formed: in the case of resonance with winding number equal to the rational \(p/q\), the \(q + 1\)th image of the singular curve \(y = -\sin x\) coincides with the initial singular curve exactly; so, the strips of generic chaos in the neighbourhood of the singular curves are not blurred.

A small positive or negative shift (by \(\approx 0.002\)) in \(c\) establishes complete visual ergodicity of the motion inside the layer, i.e., upon the shift, the layer
in Fig. 6a would be just a black band. However, the proximity of resonance influences the form of the borders of the layer. On further shifting $c$ away from the resonance, this influence diminishes, and the borders become closer and closer in their form to the guiding curve (11). In Fig. 6b, the value of the $c$ parameter is shifted from the resonant case by 0.01 (i.e., $c = 5.0289$). No regular islands are seen inside the layer. This visual impression is deceptive: as follows from our numerical experiments, very small islands can always be found by implementing special numerical techniques, such as computation of Lyapunov exponents for a set of trajectories on a fine grid of initial data. So, no complete ergodicity of motion inside the layer is achieved in reality. This interesting phenomenon will be discussed elsewhere.

The fractional resonances are barely seen (as small peaks) in Figs. 3, 4 and 5. On decreasing $\lambda$, no matter what the value of $\varepsilon$ is, one reaches the plateau corresponding to the main resonance $m = 0$. Instead, if $\lambda$ is fixed and $\varepsilon$ is decreased, one finds much more intricate behaviour. According to formula (5), no matter how small the value of $\lambda$ is, one can achieve any value of $c$ by diminishing $\varepsilon$. In other words, all the set of resonances is traversed once and once again, if one decreases $\varepsilon$ steadily. Due to logarithmic dependence on $\varepsilon$, at small values of $\lambda$ (already at $\lambda = 0.01$) the encounters with prominent resonances take place at microscopic values of $\varepsilon$. Setting $c = 2\pi(Q + m)$ (where $Q$ is taken modulo 1, and $m = 0, 1, 2, \ldots$) and rearranging Eq. (5), one has for the resonant value of $\varepsilon$:

$$
\varepsilon_{\text{res}} = \frac{4}{\lambda} \exp \left( -\frac{2\pi(Q + m)}{\lambda} \right).
$$

(24)

If $\lambda = 0.01$, the $4/5$ (mod 1) resonance considered above is located at $\varepsilon_{\text{res}} \approx 2.004 \cdot 10^{-216}$ ($m = 0$), $\varepsilon_{\text{res}} \approx 2.670 \cdot 10^{-489}$ ($m = 1$), $\varepsilon_{\text{res}} \approx 3.559 \cdot 10^{-762}$ ($m = 2$), \ldots. This is why it is impossible to observe the resonant structure presented in Fig. 6a by means of constructing sections of phase space of original system (1): the relative magnitude of perturbation is too small. Such situation is typical; this tentatively explains why the chaotic layers in phase space sections of slowly perturbed Hamiltonian systems usually do not show any sign of large inner regular component (for examples see Ref. [26]): very fine tuning of the values of the parameters is necessary to make the resonant structure visible, and the resonant values of the relative magnitude of perturbation for the most prominent resonances are usually microscopic.

Finally, let us consider the influence of the magnitude of perturbation on precision of our theoretical estimates as compared to the numerical data.
These data have been obtained by computing the half-width $w_b$ of the chaotic layer of original system (1) with $k = 1$, $a = b$, in dependence on the magnitude of perturbation $\varepsilon$. Three values of $\lambda$, namely, $\lambda = 0.01$, 0.1, and 0.3, have been chosen, while the range of variation in $\varepsilon$ is six orders of magnitude. The method utilizing Eq. (19) (the “first method”, see above) has been employed for calculating $w_b$. The computation results are shown in Fig. 7 as dots. Small peaks “perturbing” the smooth curves are notable features of the constructed dependences, especially in the case of the relatively large value $\lambda = 0.3$. They are conditioned by the process of encountering fractional resonances (such as the resonance the phase portrait of which is presented in Fig. 6a), as $\varepsilon$ is varied.

The theoretical dependences given by Eq. (13) (for $\lambda = 0.3$) and Eq. (18) are shown in Fig. 7 as solid curves. Eq. (13) is valid for description of the case of $\lambda = 0.3$ because $0.79 < c < 4.92$ in the given range of $\varepsilon$ (i.e., the $c$ values are far from main resonance); and Eq. (18) is suitable for $\lambda = 0.01$ because $0.06 < c < 0.20$, i.e., $c$ is close to zero (main resonance). The theoretical curve for $\lambda = 0.1$ is not shown, because the numerical data for this case are very close to that for the case of $\lambda = 0.01$. Theoretical dependence (18) visually coincides with the numerical data for the case of $\lambda = 0.01$ at $\log \varepsilon < -2$.

The resonant peaks are not present in the theoretical curves, due to the nature of approximation; so, the deviation from the observed data increases locally at the resonances. Fig. 8 demonstrates the $\varepsilon$ dependence of the deviation $\delta = (w_{b}^{num} - w_{b}^{theor})/w_{b}^{theor}$ of the observed data at $\lambda = 0.01$ from theoretical dependence (18). At $\log \varepsilon < -3$ there is a plateau at the level $\log \delta \approx -1.5$ with some resonant drops. Most probably its emergence is due to the limit in precision of the numerical determination of $w_b$ by the method used. The level $\log \delta \approx -1.5$ corresponds to the accuracy of $\approx 3\%$. Beginning at $\log \varepsilon \approx -3$, there is a linear rise of $\log \delta$ with $\log \varepsilon$: the variation of $\log \varepsilon$ by three orders of magnitude results in the variation of $\log \delta$ by approximately one and a half orders of magnitude, i.e., $\delta$ behaves like $\sim \varepsilon^{1/2}$. This is in contrast with the expected precision of the separatrix map (2), which is $\sim \varepsilon$ (see Section 2). The different power law index is no wonder, since the numerical data are not exact. However, detailed understanding of the $\varepsilon$ dependence of the precision in determining the chaotic layer width apparently needs further numerical experiments and theoretical work.
6 Conclusions

We have considered the problem of estimation of the width of a chaotic layer near the separatrices of nonlinear resonance in a Hamiltonian system. The model of a perturbed nonlinear pendulum, representing, according to [2], a universal model of nonlinear resonance, has been adopted for the analysis.

In a numerical experiment on determination of the width of the chaotic layer, it has been shown that a sharp physical borderline exists between slow and fast chaos at \( \lambda \approx 1/2 \) (Fig. 1).

A new method of description of the chaotic motion in the neighbourhood of the separatrices of nonlinear resonance with periodic perturbations of low frequency (the case of slow chaos) has been presented. The method is based on the theory of separatrix maps. Within the framework of the new method, formulas (12), (13) and Eqs. (17), (18) for the width of the chaotic layer have been obtained.

The developed theory has been compared to the results of numerical experiments with original system (1); close agreement has been observed at low and intermediate frequencies of perturbation (Figs. 3, 5) and at small relative magnitudes of perturbation (Figs. 7, 8). The phenomenon of bending of the chaotic layer \[ \text{[11]} \] at low values of \( \lambda \) has been directly observed (Figs. 4, 5).

An important inference can be brought from the graphs in Figs. 3, 4, and 5: very sharp changes of the width of the chaotic layer are possible upon only a small change in a parameter of the system. However, fine tuning is necessary: large width of the layer is reached inside narrow intervals of a parameter variation. Such fine tuning may provide a potentially effective tool for controlling the properties of the chaotic motion in Hamiltonian systems.

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Figure 1: The $\lambda$ dependence of the chaotic layer half-width $y_{lb}$ in the case of the least perturbed border of the layer.
Figure 2: The dependence of $y_b$ on $c$, computed (circles) and theoretical (solid curve), $\lambda = 0.1$. 
Figure 3: The half-width $w_b$ of the chaotic layer of system (11) with $k = 1$, $a = b$, in dependence on $\lambda$: the results of direct computation by the first method (dots) and the theoretical curve given by Eq. (13).
Figure 4: The same as in Fig. 3, but the half-width is computed by the second method (i.e., as the maximum energy deviation).
Figure 5: The half-width $w_b$ of the chaotic layer of system (11) with $k = 2$, $a = b$, in dependence on $\lambda$: the results of direct computation by the first method (big dots) and by the second method (small dots), and the theoretical curve given by Eq. (13).
Figure 6: The phase portrait of separatrix map [4], an example. Only the chaotic component is shown. (a) The case of the 4/5 resonance ($\lambda = 0.01$, $c = 5.0189$). (b) The value of the $c$ parameter is shifted away from the resonant case by 0.01 (i.e., $c = 5.0289$).
Figure 7: The half-width $w_b$ of the chaotic layer of system (11) with $k = 1$, $a = b$, in dependence on the magnitude of perturbation $\varepsilon$: the results of direct computation (dots) and the theoretical curves given by Eq. (13) (for $\lambda = 0.3$) and Eq. (18).
Figure 8: The $\varepsilon$ dependence of the deviation $\delta$ of the theoretical curve given by Eq. (18) from the observed data, $\lambda = 0.01$. 