Monotone Paths in Geometric Triangulations

Adrian Dumitrescu1 · Ritankar Mandal1 · Csaba D. Tóth2,3

Published online: 28 February 2018
© Springer Science+Business Media, LLC, part of Springer Nature 2018

Abstract (I) We prove that the (maximum) number of monotone paths in a geometric triangulation of $n$ points in the plane is $O(1.7864^n)$. This improves an earlier upper bound of $O(1.8393^n)$; the current best lower bound is $\Omega(1.7003^n)$. (II) Given a planar geometric graph $G$ with $n$ vertices, we show that the number of monotone paths in $G$ can be computed in $O(n^2)$ time.

Keywords Monotone path · Triangulation · Counting algorithm

This article is part of the Topical Collection on Special Issue on Combinatorial Algorithms

The work of Csaba D. Tóth was supported in part by the NSF awards CCF-1422311 and CCF-1423615.

An extended abstract of this paper appeared in the Proceedings of the 27th International Workshop on Combinatorial Algorithms (IWOCA 2016), LNCS 9843, pp. 411–422, Springer International Publishing, 2016.

✉ Ritankar Mandal
rmandal@uwm.edu
Adrian Dumitrescu
dumitres@uwm.edu
Csaba D. Tóth
cdtoth@acm.org

1 Department of Computer Science, University of Wisconsin–Milwaukee,
Milwaukee, WI, USA
2 Department of Mathematics, California State University, Northridge, Los Angeles, CA, USA
3 Department of Computer Science, Tufts University, Medford, MA, USA
1 Introduction

A directed polygonal path $\xi$ in $\mathbb{R}^d$ is **monotone** if there exists a nonzero vector $u \in \mathbb{R}^d$ that has a positive inner product with every directed edge of $\xi$. The study of combinatorial properties of monotone paths is motivated by the classical simplex algorithm in linear programming, which finds an optimal solution by tracing a monotone path in the 1-skeleton of a $d$-dimensional polytope of feasible solutions. It remains an elusive open problem whether there is a pivoting rule for the simplex method that produces a monotone path whose length is polynomial in $d$ and $n$ [1].

Let $S$ be a set of $n$ points in the plane. A **geometric graph** $G$ is a graph drawn in the plane so that the vertex set consists of the points in $S$ and the edges are drawn as straight line segments between the corresponding points in $S$. A **plane geometric graph** is one in which edges intersect only at common endpoints. In this paper, we are interested in the maximum number of monotone paths over all plane geometric graphs with $n$ vertices; it is easy to see that triangulations maximize the number of such paths (since adding edges can only increase the number of monotone paths).

**Our results** We first show that the number of monotone paths (over all directions) in a triangulation of $n$ points in the plane is $O(1.8193^n)$, using a fingerprinting technique in which incidence patterns of 8 vertices are analyzed. We then give a sharper bound of $O(1.7864^n)$ using the same strategy, by enumerating fingerprints of 11 vertices using a computer program.

**Theorem 1** The number of monotone paths in a geometric triangulation on $n$ vertices in the plane is $O(1.7864^n)$.

It is often challenging to determine the number of configurations (i.e., count) faster than listing all such configurations (i.e., enumerate). In Section 6 we show that monotone paths can be counted in polynomial time in plane graphs.

**Theorem 2** Given a plane geometric graph $G$ with $n$ vertices, the number of monotone paths in $G$ can be computed in $O(n^2)$ time. The monotone paths can be enumerated in an additional $O(1)$-time per edge, i.e., in $O(n^2 + K)$ time, where $K$ is the sum of the lengths of all monotone paths.

**Related previous work.** We derive a new upper bound on the maximum number of monotone paths in geometric triangulations of $n$ points in the plane. Analogous problems have been studied for cycles, spanning cycles, spanning trees, and matchings [4] in $n$-vertex edge-maximal planar graphs, which are defined in purely graph theoretic terms. In contrast, the monotonicity of a path depends on the embedding of the point set in the plane, i.e., it is a geometric property. The number of geometric configurations contained (as a subgraph) in a triangulation of $n$ points have been considered only recently. The maximum number of convex polygons is known to be between $\Omega(1.5028^n)$ and $O(1.5029^n)$ [9, 16]. For the number of monotone paths, Dumitrescu et al. [5] gave an upper bound of $O(1.8393^n)$; we briefly review their proof in Section 2. A lower bound of $\Omega(1.7003^n)$ is established in the same paper. It can be deduced
Fig. 1 Left: a graph on $n = 2^\ell + 2$ vertices (here $\ell = 3$) that contains a Hamiltonian path $\xi_0 = (v_1, \ldots, v_n)$. Right: an isomorphic plane monotone graph where corresponding vertices are in the same order by $x$-coordinate; and edges above (resp., below) $\xi_0$ remain above (resp., below) $\xi_0$. For $n$ sufficiently large, a graph in this family contains $\Omega(1.7003^n)$ $x$-monotone paths

from the following construction illustrated in Fig. 1. Let $n = 2^\ell + 2$ for an integer $\ell \in \mathbb{N}$; the plane graph $G$ has $n$ vertices $V = \{v_1, \ldots, v_n\}$, it contains the Hamiltonian path $\xi_0 = (v_1, \ldots, v_n)$, and it has edge $(v_i, v_{i+2^\ell})$, for $1 \leq i \leq n-2^\ell$, iff $i-1$ or $i-2$ is a multiple of $2^\ell$.

Every $n$-vertex triangulation contains $\Omega(n^2)$ monotone paths, since there is a monotone path between any two vertices (by a straightforward adaptation of [6, Lemma 1] from convex subdivisions to triangulations). The minimum number of monotone paths in an $n$-vertex triangulation lies between $\Omega(n^2)$ and $O(n^{3.17})$ [5].

The number of several common crossing-free structures (such as matchings, spanning trees, spanning cycles, triangulations) on a set of $n$ points in the plane is known to be exponential [2, 7, 10, 19, 22–25]; see also [8, 26]. Early upper bounds in this area were obtained by multiplying an upper bound on the maximum number of triangulations on $n$ points with an upper bound on the maximum number of desired configurations in an $n$-vertex triangulation; valid upper bounds result since every plane geometric graph can be augmented into a triangulation.

For a polytope $P \subset \mathbb{R}^d$, let $G(P)$ denote its 1-skeleton, which is the graph consisting of the vertices and edges of $P$. The efficiency of the simplex algorithm and its variants hinges on extremal bounds on the length of a monotone paths in the 1-skeleton of a polytope. For example, the monotone Hirsch conjecture [29] states that for every $u \in \mathbb{R}^d \setminus \{0\}$, the 1-skeleton of every $d$-dimensional polytope with $n$ facets contains a $u$-monotone path with at most $n - d$ edges from any vertex to a $u$-maximal vertex (i.e., a vertex whose orthogonal projection onto $u$ is maximal). Klee [15] verified the conjecture for 3-dimensional polytopes, but counterexamples have been found in dimensions $d \geq 4$ [27] (see also [20]). Kalai [13, 14] gave a subexponential upper bound for the length of a shortest monotone path between any two vertices (better bounds are known for the diameter of the 1-skeleta of polyhedra [28], but the shortest path between two vertices need not be monotone). However, even in $\mathbb{R}^3$, no deterministic pivot rule is known to find a monotone path of length $n - 3$ [12], and the expected length of a path found by randomized pivot rules requires averaging over all $u$-monotone paths [11, 17]. See also [21] for a summary of results of the polymath 3 project on the polynomial Hirsch conjecture.
2 Preliminaries

A polygonal path \( \xi = (v_1, v_2, \ldots, v_t) \) in \( \mathbb{R}^d \) is monotone in direction \( u \in \mathbb{R}^d \setminus \{0\} \) if every directed edge of \( \xi \) has a positive inner product with \( u \), that is, \( \langle v_i, v_{i+1}, u \rangle > 0 \) for \( i = 1, \ldots, t - 1 \); here \( 0 \) is the origin. A path \( \xi = (v_1, v_2, \ldots, v_t) \) is monotone if it is monotone in some direction \( u \in \mathbb{R}^d \setminus \{0\} \). A path \( \xi \) in the plane is \( x \)-monotone, if it is monotone with respect to the positive direction of the \( x \)-axis, i.e., monotone in direction \( u = (1, 0) \).

Let \( S \) be a set of \( n \) points in the plane. A (geometric) triangulation of \( S \) is a plane geometric graph with vertex set \( S \) such that the bounded faces are triangles that jointly tile of the convex hull of \( S \). Since a triangulation has at most \( 3n - 6 \) edges for \( n \geq 3 \), and the \( u \)-monotonicity of an edge \((a, b)\) depends on the sign of \( \langle \overrightarrow{ab}, u \rangle \), it is enough to consider monotone paths in at most \( 2(3n - 6) = 6n - 12 \) directions (one direction between any two consecutive unit normal vectors of the edges). In the remainder of the paper, we obtain an upper bound on the number of monotone paths in a fixed direction, which we may assume to be the positive direction of the \( x \)-axis.

Let \( G = (S, E) \) be a plane geometric graph with \( n \) vertices. An \( x \)-monotone path \( \xi \) in \( G \) is maximal if \( \xi \) is not a proper subpath (consisting of consecutive vertices) of some \( x \)-monotone path in \( G \). Every \( x \)-monotone path in \( G \) contains at most \( n \) vertices, hence it contains at most \( \binom{n}{2} \) \( x \)-monotone subpaths. Conversely, every \( x \)-monotone path can be extended to a maximal \( x \)-monotone path. Let \( \lambda_n \) denote the maximum number of \( x \)-monotone paths in an \( n \)-vertex triangulation (summed over all directions \( u \)). Let \( \mu_n \) denote the maximum number (over all directions \( u \)) of maximal \( u \)-monotone paths in an \( n \)-vertex triangulation. As such, we have

\[
\lambda_n \leq (6n - 12) \binom{n}{2} \cdot \mu_n = O(n^3 \mu_n).
\]

We prove an upper bound for a broader class of graphs, plane monotone graphs, in which every edge is an \( x \)-monotone Jordan arc. Consider a plane monotone graph \( G \) on \( n \) vertices with a maximum number of \( x \)-monotone paths. We may assume that the vertices have distinct \( x \)-coordinates; otherwise we can perturb the vertices without decreasing the number of \( x \)-monotone paths. Since inserting new edges can only increase the number of \( x \)-monotone paths, we may also assume that \( G \) is fully triangulated [18, Lemma 3.1], i.e., it is an edge-maximal planar graph. Conversely, every plane monotone graph is isomorphic to a plane geometric graph in which the \( x \)-coordinates of the corresponding vertices are the same [18, Theorem 2]. Consequently, the number of maximal \( x \)-monotone paths in \( G \) equals \( \mu_n \).

Denote the vertex set of \( G \) by \( W = \{w_1, w_2, \ldots, w_n\} \), ordered by increasing \( x \)-coordinates; and direct each edge \( w_iw_j \in E(G) \) from \( w_i \) to \( w_j \) if \( i < j \); we thereby obtain a directed graph \( G \). By [5, Lemma 3], all edges \( w_iw_{i+1} \) must be present, i.e., \( G \) contains a Hamiltonian path \( \xi_0 = (w_1, w_2, \ldots, w_n) \). If \( T(i) \) denotes the number of maximal (w.r.t. inclusion) \( x \)-monotone paths in \( G \) starting at vertex \( w_{n-i+1} \), it was shown in the same paper that \( T(i) \) satisfies the recurrence \( T(i) \leq T(i - 1) + T(i - 2) + T(i - 3) \) for \( i \geq 4 \), with initial values \( T(1) = T(2) = 1 \) and \( T(3) = 2 \).
(one-vertex paths are also counted). This recurrence solves to $T(n) = O(α^n)$, where $α = 1.8392\ldots$ is the unique real root of the cubic equation $x^3 - x^2 - x - 1 = 0$. Consequently, any $n$-vertex geometric triangulation admits at most $O(n^3 T(n)) = O(1.8393^n)$ monotone paths. Theorem 1 improves this bound to $O(1.7864^n)$.

**Fingerprinting technique** An $x$-monotone path can be represented uniquely by the subset of visited vertices. This unique representation gives the trivial upper bound of $2^n$ for the number of $x$-monotone paths. For a set of $k$ vertices $V \subseteq W$, an incidence pattern of $V$ (pattern, for short) is a subset of $V$ that appears in a monotone path $ξ$ (i.e., the intersection between $V$ and a monotone path $ξ$). Denote by $I(V)$ the set of all incidence patterns of $V$; see Fig. 2. For instance, $v_1 v_3 \in I(V)$ implies that there exists a monotone path $ξ$ in $G$ that is incident to $v_1$ and $v_3$ in $V$, but no other vertices in $V$. The incidence pattern $\emptyset \in I(V)$ denotes an empty intersection between $ξ$ and $V$, i.e., a monotone path that has no vertices in $V$.

We now describe a divide & conquer application of the fingerprinting technique we use in our proof. For $k \in \mathbb{N}$, let $p_k = \max_{|V|=k} |I(V)|$ denote the maximum number of incidence patterns for a set $V$ of $k$ consecutive vertices in a plane monotone triangulation. We trivially have $p_k \leq 2^k$, and it immediately follows from the definition that $p_k \leq p_i p_j$ for all $i, j \geq 1$ with $i + j = k$; in particular, we have $p_{2k} \leq p_k^2$. Assuming that $n$ is a multiple of $k$, the product rule yields $μ_n \leq p_k^{n/k}$. For arbitrary $n$ and constant $k$, we obtain $μ_n \leq p_k^{[n/k]} 2^{n-k[n/k]} \leq p_k^{[n/k]} 2^k = O\left(p_k^{[n/k]}\right)$. Table 1 summarizes the upper bounds obtained by this approach.

It is clear that $p_1 = 2$ and $p_2 = 4$, and it is not difficult to see that $p_3 = 7$ (note that $p_3 < p_1 p_2$). We prove $p_4 = 13$ (and so $p_4 < p_2^2 = 16$) by analytic methods (Section 3); this yields the upper bounds $μ_n = O(13^{n/4})$ and consequently $λ_n = O(n^3 13^{n/4}) = O(1.8989^n)$. A careful analysis of the edges between two consecutive groups of 4 vertices shows that $p_8 = 120$, and so $p_8$ is significantly smaller than $p_4^2 = 13^2 = 169$ (Lemma 10), hence $μ_n = O(120^{n/8})$ and $λ_n = O(n^3 120^{n/8}) = O(1.8193^n)$. Computer search shows that $p_{11} = 591$, and so $μ_n = O(591^{n/11})$ and $λ_n = O(n^3 591^{n/11}) = O(1.7864^n)$ (Section 5).

The analysis of $p_k$, for $k \geq 12$, using the same technique is expected to yield further improvements. Handling incidence patterns on 12 or 13 vertices is still realistic (although time consuming), but working with larger groups is currently prohibitive, both by analytic methods and with computer search. Significant improvement over our results may require new ideas.
Table 1 Upper bounds obtained via the fingerprinting technique for \( k \leq 11 \)

| \( k \) | \( p_k \) | \( \mu_n = O\left(p_k^{n/k}\right) \) | \( \lambda_n = O(n^3\mu_n) \) |
|---|---|---|---|
| 2 | 4 | \( 2^n \) | \( O(n^3 2^n) \) |
| 3 | 7 | \( O(7^{n/3}) \) | \( O(n^3 7^{n/3}) = O(1.913^n) \) |
| 4 | 13 | \( O(13^{n/4}) \) | \( O(n^3 13^{n/4}) = O(1.8989^n) \) |
| 5 | 23 | \( O(23^{n/5}) \) | \( O(n^3 23^{n/5}) = O(1.8722^n) \) |
| 6 | 41 | \( O(41^{n/6}) \) | \( O(n^3 41^{n/6}) = O(1.8570^n) \) |
| 7 | 70 | \( O(70^{n/7}) \) | \( O(n^3 70^{n/7}) = O(1.8348^n) \) |
| 8 | 120 | \( O(120^{n/8}) \) | \( O(n^3 120^{n/8}) = O(1.8193^n) \) |
| 9 | 201 | \( O(201^{n/9}) \) | \( O(n^3 201^{n/9}) = O(1.8027^n) \) |
| 10 | 346 | \( O(346^{n/10}) \) | \( O(n^3 346^{n/10}) = O(1.7944^n) \) |
| 11 | 591 | \( O(591^{n/11}) \) | \( O(n^3 591^{n/11}) = O(1.7864^n) \) |

**Definitions and notations for a single group** Let \( G \) be a directed plane monotone triangulation that contains a Hamiltonian path \( \xi_0 = (w_1, w_2, \ldots, w_n) \). Denote by \( G^- \) (resp., \( G^+ \)) the path \( \xi_0 \) together with all edges below (resp., above) \( \xi_0 \). Let \( V = \{v_1, \ldots, v_k\} \) be a set of \( k \) consecutive vertices of \( \xi_0 \). For the purpose of identifying the edges relevant for the incidence patterns of \( V \), the edges between a vertex \( v_i \in V \) and any vertex preceding \( V \) (resp., succeeding \( V \)) are equivalent since they correspond to the same incidence pattern. We therefore apply a graph homomorphism \( \varphi \) on \( G^- \) and \( G^+ \), respectively, that maps all vertices preceding \( V \) to a new node \( v_0 \), and all vertices succeeding \( V \) to a new node \( v_{k+1} \). The path \( \xi_0 \) is mapped to a new path \( (v_0, v_1, \ldots, v_k, v_{k+1}) \). Denote the edges in \( \varphi(G^- \setminus \xi_0) \) and \( \varphi(G^+ \setminus \xi_0) \), respectively, by \( E^-(V) \) and \( E^+(V) \); they are referred to as the upper side and the lower side; and let \( E(V) = E^-(V) \cup E^+(V) \). The incidence pattern of the vertex set \( V \) is determined by the triple \((V, E^-(V), E^+(V))\). We call this triple the group induced by \( V \), or simply the group \( V \).

The edges \( v_iv_j \in E(V) \), \( 1 \leq i < j \leq k \), are called inner edges. The edges \( v_0v_i \), \( 1 \leq i \leq k \), are called incoming edges of \( v_i \in V \); and the edges \( v_iv_{k+1} \), \( 1 \leq i \leq k \), are outgoing edges of \( v_i \in V \) (note that \( v_0 \) and \( v_{k+1} \) are not in \( V \)). An incoming edge \( v_0v_i \) for \( 1 < i \leq k \) (resp., and outgoing edge \( v_iv_{k+1} \) for \( 1 \leq i < k \)) may be present in both \( E^-(V) \) and \( E^+(V) \). Denote by \( \text{In}(v) \) and \( \text{Out}(v) \), respectively, the number of incoming and outgoing edges of a vertex \( v \in V \); and note that \( \text{In}(v) \) and \( \text{Out}(v) \) can be 0, 1 or 2.

For \( 1 \leq i \leq k \), let \( V_{si} \) denote the set of incidence patterns in the group \( V \) ending at \( i \). For example in Fig. 2 (right), \( V_{s3} = \{v_1v_2v_3, v_1v_3, v_2v_3, v_3\} \). By definition we have \( |V_{si}| \leq 2^{k-i} \). Similarly \( V_{si} \) denotes the set of incidence patterns in the group \( V \) starting at \( i \). In Fig. 2 (left), \( U_{s2} = \{u_2, u_2u_3, u_2u_3u_4, u_2u_4\} \). Observe that \( \sum_{i=1}^{k} |V_{si}| \leq 2^{k-i} \). Note that

\[
|I(V)| = 1 + \sum_{i=1}^{k} |V_{si}| \quad \text{and} \quad |I(V)| = 1 + \sum_{i=1}^{k} |V_{si}|. \tag{1}
\]
Reflecting all components of a triple \((V, E^-(V), E^+(V))\) with respect to the \(x\)-axis generates a new group denoted by \((V, E^-(V), E^+(V))^R\), or \(V^R\) for a shorthand notation. By definition, both \(V\) and \(V^R\) have the same set of incidence patterns.

Remark Our counting arguments pertain to maximal \(x\)-monotone paths. Suppose that a maximal \(x\)-monotone path \(\xi\) has an incidence pattern in \(V_{*i}\) for some \(1 \leq i < k\). By the maximality of \(\xi\), \(\xi\) must leave the group after \(v_i\), and so \(v_i\) must be incident to an outgoing edge. Similarly, the existence of a pattern in \(V_{i*}\) for \(1 < i \leq k\), implies that \(v_i\) is incident to an incoming edge.

3 Groups of 4 Vertices

In this section we analyze the incidence patterns of groups with 4 vertices. We prove that \(p_4 = 13\) and find the only two groups with 4 vertices that have 13 patterns (Lemma 5). We also prove important properties of groups that have exactly 11 or 12 patterns, respectively (Lemmata 2, 3 and 4).

Lemma 1 Let \(V\) be a group of 4 vertices with at least 10 incidence patterns. Then there is

(i) an outgoing edge from \(v_2\) or \(v_3\); and

(ii) an incoming edge into \(v_2\) or \(v_3\).

Proof (i) There is at least one outgoing edge from \(\{v_1, v_2, v_3\}\), since otherwise \(V_{*1} = V_{*2} = V_{*3} = \emptyset\) implying \(|I(V)| = |V_{*4}| + 1 \leq 9\). Assume there is no outgoing edge from \(v_2\) and \(v_3\); then \(V_{*1} = \{v_1\}\) and \(V_{*2} = V_{*3} = \emptyset\). From (1), we have \(|V_{*4}| = 8\) and this implies \(\{v_1v_3v_4, v_2v_4, v_3v_4\} \subset V_{*4}\). The patterns \(v_1v_3v_4\) and \(v_2v_4\), respectively, imply that \(v_1v_3, v_2v_4 \in E(V)\). The patterns \(v_2v_4\) and \(v_3v_4\), respectively, imply there are incoming edges into \(v_2\) and \(v_3\). Refer to Fig. 3. Without loss of generality, an outgoing edge from \(v_1\) is in \(E^+(V)\). By planarity, all incoming edges into \(v_2\) or \(v_3\) have to be in \(E^-(V)\). Then \(v_1v_3\) and \(v_2v_4\) both have to be in \(E^+(V)\), which by planarity is impossible.

(ii) By symmetry in a vertical axis, there is an incoming edge into \(v_2\) or \(v_3\).

Fig. 3 \(v_1\) cannot be the last vertex with an outgoing edge from a group \(V = \{v_1, v_2, v_3, v_4\}\) with at least 10 incidence patterns

© Springer
Lemma 2 Let $V$ be a group of 4 vertices with at least 11 incidence patterns. Then there is

(i) an incoming edge into $v_2$; and
(ii) an outgoing edge from $v_3$.

Proof (i) Assume $\text{In}(v_2) = 0$. Then $|V_{2s}| = 0$. By Lemma 1(ii), we have $\text{In}(v_3) > 0$. By definition $|V_{3s}| \leq 2$. We distinguish two cases.

Case 1: $\text{Iz}(v_4) = 0$. In this case, $|V_{4s}| = 0$. Refer to Fig. 4 (left). By planarity, the edge $v_1v_4$ and an outgoing edge from $v_2$ cannot coexist with an incoming edge into $v_3$. So either $v_1v_4$ or $v_1v_2$ is not in $V_{1s}$, which implies $|V_{1s}| < 8$. Therefore, (1) yields $|I(V)| = |V_{1s}| + |V_{3s}| + 1 < 8 + 2 + 1 = 11$, which is a contradiction.

Case 2: $\text{Iz}(v_4) > 0$. In this case, $|V_{4s}| = 1$. If the incoming edges into $v_3$ and $v_4$ are on opposite sides (see Fig. 4 (center)), then by planarity there are outgoing edges from neither $v_1$ nor $v_2$, which implies that the patterns $v_1$ and $v_1v_2$ are not in $V_{1s}$, and so $|V_{1s}| \leq 8 - 2 = 6$. If the incoming edges into $v_3$ and $v_4$ are on the same side (see Fig. 4 (right)), then by planarity either the edges $v_1v_4$ and $v_2v_4$ or an outgoing edge from $v_3$ cannot exist, which implies that either $v_1v_4$ and $v_1v_2v_4$ are not in $V_{1s}$ or $v_1v_3$ and $v_1v_2v_3$ are not in $V_{1s}$. In both cases, $|V_{1s}| \leq 8 - 2 = 6$.

Therefore, irrespective of the relative position of the incoming edges into $v_3$ and $v_4$ (on the same side or on opposite sides), (1) yields $|I(V)| = |V_{1s}| + |V_{3s}| + 1 \leq 6 + 2 + 1 + 1 = 10$, which is a contradiction.

(ii) By symmetry in a vertical axis, $\text{Out}(v_3) > 0$.

Lemma 3 Let $V$ be a group of 4 vertices with exactly 11 incidence patterns. Then the following hold.

(i) If $\text{In}(v_3) = 0$, then all the incoming edges into $v_2$ are on the same side of $\xi_0$, $|V_{1s}| \geq 5$, and $|V_{2s}| \geq 3$.
(ii) If $\text{In}(v_3) > 0$, then all the incoming edges into $v_3$ are on the same side of $\xi_0$, $|V_{1s}| \geq 4$, $|V_{2s}| \geq 2$, and $|V_{3s}| = 2$.

Proof By Lemma 2, $\text{In}(v_2) \neq 0$ and $\text{Out}(v_3) \neq 0$. Therefore $\{v_2v_3, v_2v_3v_4\} \subseteq V_{2s}$, implying $|V_{2s}| \geq 2$. By definition $|V_{4s}| \leq 1$. 
(i) Assume that \( \mathcal{I}(v_3) = 0 \). Then we have \( |V_{3*}| = 0 \). By (1), we obtain \( |V_{1*}| + |V_{2*}| \geq 9 \). By definition \( |V_{2*}| \leq 4 \), implying \( |V_{1*}| \geq 5 \). All incoming edges into \( v_2 \) are on the same side, otherwise the patterns \( \{v_1, v_1v_3, v_1v_3v_4, v_1v_4\} \) cannot exist, which would imply \( |V_{1*}| < 5 \). If \( |V_{2*}| < 3 \), then \( v_2 \) and \( v_2v_4 \) are not in \( V_{2*} \) implying that \( v_1v_2v_4 \) and \( v_1v_2v_4v_2v_4 \) are not in \( V_{1*} \); hence \( |V_{1*}| \leq 6 \) and thus \( |V_{1*}| + |V_{2*}| < 9 \), which is a contradiction. We conclude that \( |V_{2*}| \geq 3 \).

(ii) Assume that \( \mathcal{I}(v_3) > 0 \). Then we have \( \{v_3, v_3v_4\} \subseteq V_{3*} \), hence \( |V_{3*}| = 2 \). By (1), we obtain \( |V_{1*}| + |V_{2*}| \geq 7 \). If \( |V_{1*}| < 4 \), then \( |V_{2*}| \geq 4 \) and so \( \{v_2, v_2v_3, v_2v_4, v_2v_3v_4\} \subseteq V_{2*} \). This implies \( \{v_1v_2, v_1v_2v_3, v_1v_2v_4, v_1v_2v_3v_4\} \subseteq V_{1*} \), hence \( |V_{1*}| \geq 4 \), which is a contradiction. We conclude that \( |V_{1*}| \geq 4 \). All incoming edges into \( v_3 \) are on the same side, otherwise the patterns \( \{v_1, v_1v_2, v_1v_2v_4, v_1v_4, v_2, v_2v_4\} \) cannot exist, and thus \( |I(V)| \leq 10 \), which is a contradiction. 

\( \square \)

**Lemma 4** Let \( V \) be a group of 4 vertices with exactly 12 incidence patterns. Then the following hold.

(i) For \( i = 1, 2, 3 \), all outgoing edges from \( v_i \), if any, are on the same side of \( \xi_0 \).

(ii) If \( V \) has outgoing edges from exactly one vertex, then this vertex is \( v_3 \) and we have \( |V_{3*}| = 4 \) and \( |V_{4*}| = 7 \). Otherwise there are outgoing edges from \( v_2 \) and \( v_3 \), and we have \( |V_{3*}| = 2, |V_{4*}| = 3 \) and \( |V_{4*}| = 5 \).

(iii) For \( i = 2, 3, 4 \), all incoming edges into \( v_i \), if any, are on one side of \( \xi_0 \).

(iv) If \( V \) has incoming edges into exactly one vertex, then this vertex is \( v_2 \) and we have \( |V_{2*}| = 4 \) and \( |V_{1*}| = 7 \). Otherwise there are incoming edges into \( v_3 \) and \( v_2 \), and we have \( |V_{3*}| = 2, |V_{2*}| = 3 \) and \( |V_{1*}| = 5 \).

**Proof** (i) By Lemma 2(i), there is an incoming edge into \( v_2 \). So by planarity, all outgoing edges from \( v_1 \), if any, are on one side of \( \xi_0 \).

If there are outgoing edges from \( v_2 \) on both sides, then by planarity the edges \( v_1v_3, v_1v_4 \) and any incoming edge into \( v_3 \) cannot exist, hence the five patterns \( \{v_1v_3, v_1v_3v_4, v_1v_4, v_3, v_3v_4\} \) are not in \( I(V) \) and thus \( |I(V)| \leq 16 - 5 = 11 \), which is a contradiction.

If there are outgoing edges from \( v_3 \) on both sides (see Fig. 5(a)), then by planarity the edges \( v_1v_4, v_2v_4 \) and an incoming edge into \( v_4 \) cannot exist, hence the four patterns \( \{v_1v_2v_4, v_1v_4, v_2v_4, v_4\} \) are not in \( I(V) \). Without loss of generality, an incoming edge into \( v_2 \) is in \( E^+(V) \). Then by planarity, any outgoing edge of \( v_1 \) and the edge \( v_1v_3 \) (which must be present) are in \( E^-(V) \). Also by planarity, either an incoming edge into \( v_3 \) or an outgoing edge from

\[ \begin{align*}
\text{(a) Having outgoing edges from } v_1 \text{ on both sides is impossible. (b) Existence of outgoing edges only from } \{v_1v_3\} \text{ is impossible. (c) } |V_{3*}| \geq 3. \text{ (d) } |V_{4*}| \geq 5
\end{align*} \]
If \( V \) has outgoing edges from exactly one vertex, then by Lemma 2 (ii), this vertex is \( v_3 \). Consequently, \( V_{e_1} = V_{e_2} = \emptyset \). Using (1), \(|V_{e_3}| + |V_{e_4}| = 11\). Therefore \(|V_{e_4}| \geq 7\), since by definition \(|V_{e_3}| \leq 4\). If \(|V_{e_4}| = 8\), then we have \( \{v_1v_2v_3v_4, v_1v_3v_4, v_2v_3v_4, v_3v_4\} \subseteq V_{e_4}\). Existence of these four patterns along with an outgoing edge from \( v_3 \) implies \( \{v_1v_2v_3, v_1v_3, v_2v_3, v_3\} \subseteq V_{e_3} \) and thus \(|V_{e_3}| + |V_{e_4}| = 4 + 8 = 12\), which is a contradiction. Therefore \(|V_{e_4}| = 7\) and \(|V_{e_3}| = 4\).

If \( V \) has outgoing edges from more than one vertex, the possible vertex sets with outgoing edges are \( \{v_1, v_3\}, \{v_2, v_3\} \), and \( \{v_1, v_2, v_3\} \). We show that it is impossible that all outgoing edges are from \( \{v_1, v_3\} \), which will imply that there are outgoing edges from both \( v_2 \) and \( v_3 \).

If there are outgoing edges from \( \{v_1, v_3\} \) only, we may assume the ones from \( v_1 \) are in \( E^-(V) \) and then by planarity all incoming edges into \( v_2 \) are in \( E^+(V) \), see Fig. 5 (b). Then by planarity, either \( v_1v_3 \) or \( v_2v_4 \) or an incoming edge into \( v_3 \) cannot exist implying that \( \{v_1v_3, v_1v_3v_4\} \) or \( \{v_1v_2v_4, v_2v_4\} \) or \( \{3, v_3v_4\} \) is not in \( I(V) \). By the same token, depending on the side the outgoing edges from \( v_3 \) are on, either the edge \( v_1v_4 \) or an incoming edge into \( v_4 \) cannot exist, implying that either \( v_1v_4 \) or \( v_4 \) is not in \( I(V) \). Since \( V_{e_2} = \emptyset \), \( \{v_1v_2, v_2\} \) are not in \( I(V) \). So \(|I(V)| \leq 16 - (2 + 1 + 2) = 11\), which is a contradiction. Therefore the existence of outgoing edges only from \( v_1 \) and \( v_3 \) is impossible.

If there are outgoing edges from (precisely) \( \{v_2, v_3\} \) or \( \{v_1, v_2, v_3\} \), then we have \( \{v_1v_2, v_2\} \subseteq V_{e_2} \) and \( \{v_1v_2v_3, v_2v_3\} \subseteq V_{e_3} \), since \( \mathbb{I}(v_2) \neq 0 \) and \( \mathbb{O}(v_3) \neq 0 \) by Lemma 2. Therefore \(|V_{e_2}| = 2 \) and \(|V_{e_3}| \geq 2\). If \(|V_{e_3}| < 3\), then \( v_1v_3, v_3 \notin V_{e_3} \), which implies that \( v_1v_3 \) and an incoming edge into \( v_3 \) are not in \( E(V) \). Consequently, \( v_1v_3v_4, v_3v_4 \notin I(V) \). Observe Fig. 5 (c).

By planarity the edge \( v_1v_4 \), an incoming edge into \( v_4 \) and an outgoing edge from \( v_1 \) cannot exist with an incoming edge into \( v_2 \) and an outgoing edge from \( v_3 \). So at least one of the patterns \( \{v_1, v_1v_4, v_4\} \) is missing implying \(|I(V)| \leq 16 - (2 + 2 + 1) = 11\), which is a contradiction. So \(|V_{e_3}| \geq 3\). If \(|V_{e_4}| < 5\), then (1) yields \(|V_{e_3}| = 4\), \(|V_{e_2}| = 2\) and \(|V_{e_1}| = 1\). We may assume that all outgoing edges from \( v_1 \) are in \( E^+(V) \); see Fig. 5 (d). By planarity, the incoming edges into \( v_2 \) are in \( E^-(V) \). Depending on the side the outgoing edges from \( v_2 \) are on, either \( v_1v_3 \) or an incoming edge into \( v_3 \) cannot exist, implying that either \( v_1v_3 \) or \( v_3 \) is not in \( V_{e_3} \), therefore \(|V_{e_3}| < 4\), creating a contradiction. We conclude that \(|V_{e_4}| \geq 5\).

(iii) By symmetry, (iii) immediately follows from (i).

(iv) By symmetry, (iv) immediately follows from (ii).

\[ \square \]
Proof Observe that group $A$ in Fig. 6 has 13 patterns. Let $V$ be a group of 4 vertices with at least 13 patterns.

We first claim that $V$ has an incoming edge into $v_3$ and an outgoing edge from $v_2$. Their existence combined with Lemma 2 implies that $\{v_2v_4, v_3\} \subset I(V)$ and $\{v_1v_2, v_2\} \subset I(V)$, respectively. At least one of these two edges has to be in $E(V)$, otherwise $V$ has at most $16 - (2 + 2) = 12$ patterns. Assume that one of the two, without loss of generality, the outgoing edge from $v_2$ is not in $E(V)$. Then $\{v_1v_3, v_2v_4\} \subset E(V)$, otherwise either patterns $\{v_1v_3, v_1v_3v_4\}$ or $\{v_1v_2v_4, v_2v_4\}$ are not in $I(V)$ and there are at most $16 - (2 + 2) = 12$ patterns. By Lemma 2, there is an incoming edge into $v_2$ and an outgoing edge from $v_3$. Without loss of generality, the outgoing edge from $v_3$ is in $E^-(V)$. So by planarity $v_2v_4$ is in $E^+(V)$, which implies that $v_1v_3$ and the incoming edge into $v_3$ are in $E^-(V)$. By the same token, the incoming edge into $v_2$ is in $E^+(V)$. So by planarity the edge $v_1v_4$ and an outgoing edge from $v_1$ cannot be in $E(V)$. Then the patterns $\{v_1v_4, v_1\}$ are not in $I(V)$, thus $V$ has at most $16 - (2 + 2) = 12$ patterns, which is a contradiction. This completes the proof of the claim.

We may assume without loss of generality (by applying a reflection in the $x$-axis if necessary) that the incoming edge into $v_3$ is in $E^-(V)$. By planarity, the outgoing edge from $v_2$ is in $E^+(V)$. By the same token the edge $v_1v_4$ cannot be in $E(V)$, which implies that $v_1v_4 \notin I(V)$. So $I(V) \leq 16 - 1 = 15$. By Lemma 2, there is an incoming edge into $v_2$ and an outgoing edge from $v_3$. By planarity, if the incoming edge into $v_2$ is in $E^+(V)$ then the outgoing edge from $v_1$ cannot be in $E(V)$, therefore the pattern $v_1$ is not in $I(V)$. But if the incoming edge into $v_2$ is in $E^-(V)$ then the edge $v_1v_3$ cannot be in $E(V)$ therefore neither $v_1v_3$ nor $v_1v_3v_4$ is in $I(V)$. By a similar argument, if the outgoing edge from $v_3$ is in $E^-(V)$ then the incoming edge into $v_4$ cannot be in $E(V)$, therefore the pattern $v_4$ is not in $I(V)$. But if the outgoing edge from $v_3$ is in $E^+(V)$ then the edge $v_2v_4$ cannot be in $E(V)$, therefore neither $v_2v_4$ nor $v_1v_2v_4$ is in $I(V)$. Since $I(V) \geq 13$, the only solution is $v_1 \notin I(V)$ and $v_4 \notin I(V)$. Therefore $V$ induces the group $A$ and has exactly 13 patterns.

If the incoming edge into $v_3$ is in $E^+(V)$, then $V$ induces $A^R$ (with exactly 13 patterns).

4 Groups of 8 Vertices

In this section, we analyze two consecutive groups, $U$ and $V$, each with 4 vertices, and show that $p_8 = 120$ (Lemma 10). Let $U \equiv \{u_1, u_2, u_3, u_4\}$ and
$V = \{v_1, v_2, v_3, v_4\}$, and put $UV = U \cup V$ for short. We may assume that $|I(V)| \leq |I(U)|$ (by applying a reflection about the vertical axis if necessary), and we have $|I(U)| \leq 13$ by Lemma 5. This yields a trivial upper bound $|I(UV)| \leq |I(U)| \cdot |I(V)| \leq 13^2 = 169$. It is enough to consider cases in which $10 \leq |I(V)| \leq |I(U)| \leq 13$, otherwise the trivial bound is already less than 120.

In all cases where $|I(U)| \cdot |I(V)| > 120$, we improve on the trivial bound by finding edges between $U$ and $V$ that cannot be present in the group $UV$. If edge $u_iv_j$ is not in $E(UV)$, then any of the $|U_{\ast i}| \cdot |V_{\ast j}|$ patterns that contain $u_iv_j$ is excluded. Since every maximal $x$-monotone path has at most one edge between $U$ and $V$, distinct edges $u_iv_j$ exclude disjoint sets of patterns, and we can use the sum rule to count the excluded patterns. We continue with a case analysis.

**Lemma 6** Consider a group $UV$ consisting of two consecutive groups of 4 vertices, where $|I(U)| \geq 10$ and $|I(V)| = 10$. Then $UV$ allows at most 120 incidence patterns.

**Proof** If $U$ has at most 12 patterns, then $UV$ has at most $12 \times 10 = 120$ patterns, and the proof is complete. We may thus assume that $U$ has 13 patterns. By Lemma 5, $U$ is either $A$ or $A^R$. We may assume, by reflecting $UV$ about the horizontal axis if necessary, that $U$ is $A$. Refer to Fig. 7 (left). Therefore $|U_{\ast 2}| = 2$, $|U_{\ast 3}| = 4$ and $|U_{\ast 4}| = 6$, according to Fig. 6. The cross product of the patterns of $U$ and $V$ produce $13 \times 10 = 130$ possible patterns. We show that at least 10 of them are incompatible in each case. It follows that $|I(UV)| \leq 130 - 10 = 120$. Let $v_i$ denote the first vertex with an incoming edge in $E(V)$, where $i \neq 1$. By Lemma 1 (ii), $i = 2$ or 3.

**Case 1:** $(u_4, v_i) \in E(UV)$. We first show that $|V_{\ast i}| \geq 3$. By definition $|V_{\ast 3}| \leq 2$ and $|V_{\ast 4}| \leq 1$. By (1), $|V_{\ast 1}| + |V_{\ast 2}| \geq 9 - (2 + 1) = 6$. If $|V_{\ast 2}| \leq 3$, then $|V_{\ast 1}| \geq 3$. Otherwise $|V_{\ast 2}| = 4$, implying $V_{\ast 1} = \{v_2v_3v_4, v_2v_3, v_2v_4, v_2\}$. This implies there are outgoing edges from $v_2$ and $v_3$ in $E(V)$. Therefore $\{v_1v_2v_3v_4, v_1v_2v_3v_1v_2\} \subset V_{\ast 1}$ and $|V_{\ast i}| \geq 3$.

**Case 1.1:** $(u_4, v_i) \in E^-(UV)$; see Fig. 7 (right). As $i = 2$ or 3, by planarity $(u_3, v_1) \notin E(UV)$. Hence at least $|U_{\ast 3}| \cdot |V_{\ast 1}| \geq 4 \times 3 = 12$ combinations are incompatible.

**Case 1.2:** $(u_4, v_i) \in E^+(UV)$; see Fig. 8 (right). Then by planarity $(u_2, v_1) \notin E(UV)$ and $|U_{\ast 2}| \cdot |V_{\ast 1}| \geq 2 \times 3 = 6$ combinations are incompatible.

An incoming edge into $v_i$ in $E(V)$ implies $|V_{\ast i}| \geq 1$. If $u_3v_i \notin E(UV)$, then $|U_{\ast 3}| \cdot |V_{\ast i}| \geq 4 \times 1 = 4$ combinations are incompatible. Hence there are at least $6 + 4 = 10$ incompatible patterns. If $u_3v_i \in E(UV)$, then by planarity an incoming

![Fig. 7](image_url)

Fig. 7 Left: $|I(U)| = 13$ and $|I(V)| = 10$. Right: $(u_4, v_i) \in E^-(UV); i = 2$ here
Fig. 8 Left: $|I(U)| = 13$ and $|I(V)| = 10$. Right: $(u_4, v_i) \in E^+(UV)$; $i = 2$ here

edge into $v_1$ in $E(UV)$ cannot exist and $|[\emptyset]| |V_1*| \geq 1 \times 3 = 3$ combinations are incompatible. If $u_2v_i \in E(UV)$, then by planarity an outgoing edge from $u_4$ cannot exist. So $|U_4*||\emptyset| \geq 6 \times 1 = 6$ combinations are incompatible. So there are at least $6 + 3 + 6 = 15$ incompatible patterns. If $u_2v_i \notin E(UV)$, then by planarity an outgoing edge from $u_4$ cannot exist. So $|U_4*||\emptyset| \geq 6 \times 1 = 6$ combinations are incompatible. Hence there are at least $6 + 3 + 2 = 11$ incompatible patterns.

Case 2: $(u_4, v_i) \notin E(UV)$. By showing $|V_i*| \geq 2$ for all possible values of $i$ (i.e., 2 and 3), we can conclude that at least $|U_4*||V_i*| \geq 6 \times 2 = 12$ combinations are incompatible.

If $i = 2$, then $v_2v_3v_4 \in V_2*$. By Lemma 1 (i), there is an outgoing edge from $v_2$ or $v_3$ in $E(V)$, which implies $v_2 \in V_2*$ or $v_2v_3 \in V_2*$. Hence $|V_2*| \geq 2$.

If $i = 3$ and there is no outgoing edge from $v_3$ in $E(V)$, then by Lemma 1 (i), there is an outgoing edge from $v_2$. In that case by planarity, there are only 7 possible incidence patterns $\{\emptyset, v_1v_2v_3v_4, v_1v_2v_4, v_1v_3v_4, v_1v_2, v_3v_4\}$ in $V$, which is a contradiction. So if $i = 3$, then there is an outgoing edge from $v_3$ in $E(V)$, which implies $\{v_3v_4, v_3\} \subset V_3*$ therefore $|V_3*| \geq 2$.

Lemma 7 Consider a group $UV$ consisting of two consecutive groups of 4 vertices, where $|I(U)| \geq 11$ and $|I(V)| = 11$. Then $UV$ allows at most 120 incidence patterns.

Proof We distinguish three cases depending on $|I(U)|$.

Case 1: $|I(U)| = 11$. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 11 \times 11 = 121$, it suffices to show that at least one of these patterns is incompatible. By Lemma 2, there is an outgoing edge from $u_3$ in $E(U)$ and an incoming edge into $v_2$ in $E(V)$. Therefore $u_1u_2u_3 \in U_3$ and $v_2v_3v_4 \in V_2*$. Refer to Fig. 9 (left). If $(u_3v_2) \notin E(UV)$, then $u_1u_2u_3v_2v_3v_4$ is not in $I(UV)$. If $(u_3v_2) \in E(UV)$, then by planarity either an outgoing edge from $u_4$ w.r.t. $UV$, or an incoming edge into $v_1$ w.r.t. $UV$, cannot be in $E(UV)$, implying that either $u_1u_2u_3u_4$ or $v_1v_2v_3v_4$ is not in $I(UV)$.

Fig. 9 Left: $|I(U)| = |I(V)| = 11$. Right: outgoing edge from $u_3$ is in $E^-(UV)$ and outgoing edge from $u_2$ is in $E^+(UV)$.
Case 2: $|I(U)| = 12$. By Lemma 4 (ii), if $U$ has outgoing edges from exactly one vertex, then they are from $u_3$ and we have $|U_{s3}| = 4$, $|U_{s4}| = 7$, otherwise $|U_{s3}| \geq 3$ and $|U_{s4}| \geq 5$. By Lemma 4 (i), all the outgoing edges from $u_3$ in $E(U)$ are on one side of $U$. For simplicity assume those are in $E^-(U)$. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 12 \times 11 = 132$, it suffices to show that at least $132 - 120 = 12$ of these patterns are incompatible.

Case 2.1: There is no incoming edge into $v_3$ in $E(V)$. Then by Lemma 3 (i), all the incoming edges into $v_2$ in $E(V)$ are on one side of $V$ and we have $|V_{1*}| \geq 5$ and $|V_{2*}| \geq 3$.

Case 2.1.1: The incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$. So by planarity $u_3v_2 \notin E(UV)$ and at least $|U_{s3}||V_{2*}|$ patterns are incompatible. If $U$ has outgoing edges from exactly one vertex, then $|U_{s3}||V_{2*}| \geq 4 \times 3 = 12$ and we are done. Otherwise $U$ has outgoing edges from $u_2$, where $|U_{s2}| = 2$ and at least $|U_{s3}||V_{2*}| \geq 3 \times 3 = 9$ patterns are incompatible. Also by Lemma 4 (i), all the outgoing edges from $u_2$ in $E(U)$ are on one side of $U$. If the outgoing edges from $u_2$ w.r.t. $U$ are in $E^+(U)$, see Fig. 9 (right), then $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$; by planarity both edges cannot be in $E(UV)$ and thus at least $\min(|U_{s2}||V_{1*}|, |U_{s4}||V_{2*}|) \geq \min(2 \times 5, 5 \times 3) = 10$ patterns are incompatible. If the outgoing edges from $u_2$ w.r.t. $U$ are in $E^-(U)$, then by planarity $u_2v_2 \notin E(UV)$ and thus at least $|U_{s2}||V_{2*}| \geq 2 \times 3 = 6$ patterns are incompatible. Therefore irrespective of the relative position of the outgoing edge from $u_2$ in $E(U)$, at least $9 + \min(10, 6) = 15$ patterns are incompatible and we are done.

Case 2.1.2: The incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$. Therefore $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{s3}||V_{1*}|, |U_{s4}||V_{2*}|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible.

Case 2.2: There is an incoming edge into $v_3$ in $E(V)$. By Lemma 3 (ii), all the incoming edges into $v_3$ in $E(V)$ are on one side of $V$, $|V_{1*}| \geq 4$, $|V_{2*}| \geq 2$ and $|V_{3*}| = 2$.

Case 2.2.1: The incoming edges into $v_3$ in $E(V)$ are on both sides of $V$.

If the incoming edges into $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 10 (left), then by planarity $u_3v_3 \notin E(UV)$. So at least $|U_{s3}||V_{3*}| \geq 3 \times 2 = 6$ patterns are incompatible. By planarity an outgoing edge from $u_4$ w.r.t. $UV$, an incoming edge into $v_3$ w.r.t. $UV$ and $u_3v_3$ cannot coexist in $E(UV)$. Therefore at least $\min(|\emptyset||V_{3*}|, |U_{s4}||\emptyset|, |U_{s3}||V_{2*}|) \geq \min(1 \times 2, 5 \times 1, 3 \times 2) = 2$ patterns are incompatible. By the same argument, the edges $u_3v_2$, $u_4v_2$ and an

---

**Fig. 10** Left: incoming edges into $v_2$ are in both $E^+(V)$ and $E^-(V)$ and incoming edge into $v_3$ is in $E^+(V)$. Right: incoming edges into $v_2$ are in both $E^+(V)$ and $E^-(V)$ and incoming edge into $v_3$ is in $E^-(V)$.
incoming edge into $v_1$ w.r.t. $UV$ cannot be in $E(UV)$ together. Hence at least
\[ \min(|U_3||V_2|, |U_4||V_2|, |\emptyset||V_1|) = \min(3 \times 2, 5 \times 2, 1 \times 4) = 4 \]
patterns are incompatible. Therefore at least $6 + 2 + 4 = 12$ patterns are incompatible.

If incoming edges into $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 10 (right), then either an outgoing edge from $u_3$ w.r.t. $UV$ or an incoming edge into $v_3$ w.r.t. $UV$ cannot be in $E(UV)$. So at least $\min(|U_3||V_1|, |U_4||V_3|) \geq \min(3 \times 4, 5 \times 2) = 10$ patterns are incompatible. Therefore at least $2 + 10 = 12$ patterns are incompatible.

**Case 2.2.2:** All the incoming edges into $v_2$ in $E(V)$ are on one side of $V$ and the incoming edges into $v_2$ and $v_3$ in $E(V)$ are on same side of $V$.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 11 (left), then by planarity $u_3v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U_3||V_2| + |U_3||V_3| \geq 3 \times 2 + 3 \times 2 = 12$ patterns are incompatible.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 11 (right), then $u_3v_1$ and both $u_4v_2$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_3||V_1|, |U_4||V_3|) \geq \min(3 \times 4, 5 \times 2) = 12$ patterns are incompatible.

**Case 2.2.3:** All the incoming edges into $v_2$ in $E(V)$ are on one side of $V$ and all the incoming edges into $v_3$ in $E(V)$ are on the opposite side of $V$.

Let the incoming edges w.r.t. $V$ in $E^+(V)$ are into $v_i$ and the incoming edges w.r.t. $V$ in $E^-(V)$ are into $v_j$. So either $i = 2$, $j = 3$ or $i = 3$, $j = 2$, see Fig. 12. Therefore $|V_1|, |V_j|$ are at least $\min(|V_2|, |V_3|) = 2$. By planarity $u_3v_i \notin E(UV)$.

So at least $|U_3||V_i| \geq 3 \times 2 = 6$ patterns are incompatible. Also $u_3v_1$ and $u_4v_j$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_3||V_1|, |U_4||V_j|) = \min(3 \times 4, 5 \times 2) = 10$ patterns are incompatible. Therefore at least $6 + 10 = 16$ patterns are incompatible.
**Case 3:** \(|I(U)| = 13\). By Lemma 5, \(U\) is either \(A\) or \(A^R\). We may assume, by reflecting \(UV\) about the horizontal axis if necessary, that \(U\) is \(A\). Therefore \(|U_{s2}| = 2, |U_{s3}| = 4\) and \(|U_{s4}| = 6\), see Fig. 6. Since \(|I(UV)| \leq |I(U)| \cdot |I(V)| = 13 \times 11 = 143\), it suffices to show that at least \(143 - 120 = 23\) of these patterns are incompatible.

**Case 3.1:** There is no incoming edge into \(v_3\) in \(E(V)\). Then by Lemma 3 (i), all the incoming edges into \(v_2\) in \(E(V)\) are on one side of \(V\), \(|V_{1*}| \geq 5\) and \(|V_{2*}| \geq 3\).

If the incoming edges into \(v_2\) w.r.t. \(V\) are in \(E^-(V)\), see Fig. 13 (left), by planarity \(u_2v_2 \notin E(UV)\). So at least \(|U_{s3}||V_{2*}| \geq 2 \times 3 = 6\) patterns are incompatible. Also \(u_4v_2\) and \(u_3v_1\) can only be in \(E^-(UV)\). By planarity both edges cannot be in \(E(UV)\). Hence at least \(\min(|U_{s4}||V_{2*}|, |U_{s3}||V_{1*}|) = \min(6 \times 3, 4 \times 5) = 18\) patterns are incompatible. Therefore at least \(6 + 18 = 24\) patterns are incompatible.

Similarly if the incoming edges into \(v_2\) w.r.t. \(V\) are in \(E^+(V)\), cf. Fig. 13 (right), by planarity \(u_3v_2 \notin E(UV)\). So at least \(|U_{s3}||V_{2*}| \geq 4 \times 3 = 12\) patterns are incompatible. Also \(u_4v_2\) and \(u_2v_1\) can only be in \(E^+(UV)\). By planarity both edges cannot be in \(E(UV)\). If \(u_4v_2 \notin E(UV)\), then at least \(|U_{s4}||V_{2*}| \geq 6 \times 3 = 18\) patterns are incompatible. Otherwise \(u_2v_1 \notin E(UV)\) and either an incoming edge into \(v_1\) w.r.t. \(UV\) or an outgoing edge from \(u_3\) w.r.t. \(UV\) cannot be in \(E(UV)\). Hence at least

\[|U_{s2}||V_{1*}| + \min(|\emptyset||V_{1*}|, |U_{s3}||\emptyset|) \geq 2 \times 5 + \min(1 \times 5, 4 \times 1) = 14\]

patterns are incompatible. Therefore at least \(12 + \min(18, 14) = 26\) patterns are incompatible.

**Case 3.2:** There is an incoming edge into \(v_3\) in \(E(V)\). By Lemma 3 (ii), all the incoming edges into \(v_3\) are on one side of \(V\), \(|V_{1*}| \geq 4\), \(|V_{2*}| \geq 2\) and \(|V_{3*}| = 2\).

**Case 3.2.1:** The incoming edges into \(v_2\) in \(E(V)\) are on both sides of \(V\).

If the incoming edges into \(v_3\) w.r.t. \(V\) are in \(E^+(V)\), see Fig. 14 (left), then by planarity \(u_3v_3 \notin E(UV)\). So at least \(|U_{s3}||V_{3*}| \geq 4 \times 2 = 8\) patterns are incompatible. Also \(u_2v_1\) and \(u_4v_3\) can only be in \(E^+(UV)\). By planarity both edges cannot be in \(E(UV)\). Hence at least \(\min(|U_{s2}||V_{1*}|, |U_{s4}||V_{3*}|) = \min(2 \times 4, 6 \times 2) = 8\) patterns are incompatible. By the same token, an outgoing edge from \(u_4\) w.r.t. \(UV\) and the edges \(u_2v_3\) and \(u_3v_2\) cannot exist together in \(E(UV)\). Therefore at least

\[\min(|\emptyset||V_{1*}|, |U_{s2}||V_{3*}|, |U_{s3}||V_{2*}|) = \min(6 \times 1, 2 \times 2, 4 \times 2) = 4\]

patterns are incompatible. Similarly, by planarity, an incoming edge into \(v_1\) w.r.t. \(UV\) and the edges \(u_3v_2\) and \(u_4v_3\) cannot coexist in \(E(UV)\). Therefore at least

\[\min(|\emptyset||V_{1*}|, |U_{s3}||V_{2*}|, |U_{s4}||V_{3*}|) = \min(1 \times 4, 4 \times 2, 6 \times 2) = 4\]
patterns are incompatible. Hence at least \(8 + 8 + 4 + 4 = 24\) patterns are incompatible.

If the incoming edges into \(v_3\) w.r.t. \(V\) are in \(E^-(V)\), see Fig. 14 (right), then by planarity \(u_2v_3 \notin E(UV)\). So at least \(|U_{a2}| |V_{3s}| \geq 2 \times 2 = 4\) patterns are incompatible. Also \(u_3v_1\) and \(u_4v_3\) can only be in \(E^-(UV)\). By planarity both the edges cannot be in \(E(UV)\). Hence at least \(\min(|U_{a3}| |\emptyset|, |U_{a4}| |V_{3s}|) = \min(4 \times 4, 6 \times 2) = 12\) edges are incompatible. By planarity an outgoing going edge from \(u_3\) w.r.t. \(UV\) and an incoming edge into \(v_3\) w.r.t. \(UV\) cannot exist together in \(E(UV)\). Therefore at least \(\min(|U_{a2}| |V_{1s}|, |U_{a3}| |V_{1s}|, |U_{a4}| |\emptyset|) = \min(2 \times 4, 4 \times 4, 6 \times 1) = 6\) patterns are incompatible. So at least \(4 + 12 + 2 + 6 = 24\) patterns are incompatible.

**Case 3.2.2:** All the incoming edges into \(v_2\) in \(E(V)\) are on one side of \(V\) and all the incoming edges into \(v_2\) and \(v_3\) in \(E(V)\) are on the same side of \(V\).

If the incoming edges into \(v_2\) and \(v_3\) w.r.t. \(V\) are in \(E^+(V)\), see Fig. 15 (left), by planarity \(u_3v_2\) and \(u_3v_3\) are not in \(E(UV)\). So at least \(|U_{a3}| |V_{2s}| + |U_{a3}| |V_{3s}| = 4 \times 2 + 4 \times 2 = 16\) patterns are incompatible. Also \(u_2v_1\) and \(u_4v_2\) can only be in \(E^+(UV)\). By planarity both the edges cannot be in \(E(UV)\). Hence at least \(\min(|U_{a2}| |V_{1s}|, |U_{a4}| |V_{2s}|) = \min(2 \times 4, 6 \times 2) = 8\) patterns are incompatible. Therefore at least \(16 + 8 = 24\) patterns are incompatible.

If the incoming edges into \(v_2\) and \(v_3\) w.r.t. \(V\) are in \(E^-(V)\), see Fig. 15 (right), by planarity \(u_2v_3 \notin E(UV)\). So at least \(|U_{a2}| |V_{3s}| \geq 2 \times 2 = 4\) patterns are incompatible. Also \(u_3v_1, u_3v_2, u_4v_2, u_4v_3\) can only be in \(E^-(UV)\). By planarity either \(u_3v_1\) or \(u_4v_2\) and either \(u_3v_2\) or \(u_4v_3\) can be in \(E(UV)\). Hence at least

\[
\min(|U_{a3}| |V_{1s}|, |U_{a4}| |V_{2s}|) + \min(|U_{a3}| |V_{2s}|, |U_{a4}| |V_{3s}|) = \min(4 \times 4, 6 \times 2) + \min(4 \times 2, 6 \times 2) = 20
\]

**Fig. 14** Left: incoming edges into \(v_2\) are on both sides and incoming edges into \(v_3\) are in \(E^+(V)\). Right: incoming edges into \(v_2\) are on both sides and incoming edges into \(v_3\) are in \(E^-(V)\)

**Fig. 15** Left: incoming edges into \(v_2, v_3\) are in \(E^+(V)\). Right: incoming edges into \(v_2, v_3\) are in \(E^-(V)\)
combinations are incompatible. Therefore at least $4 + 20 = 24$ patterns are incompatible.

**Case 3.2.3:** All the incoming edges into $v_2$ are on one side of $V$ and all the incoming edges into $v_3$ are on the opposite side of $V$.

Let the incoming edges w.r.t. $V$ in $E^+(V)$ are into $v_i$ and the incoming edges w.r.t. $V$ in $E^-(V)$ are into $v_j$. So either $i = 2$, $j = 3$ or $i = 3$, $j = 2$, see Fig. 16. Therefore $|V_{i*}|, |V_{j*}|$ are at least $\min(|V_{2*}|, |V_{3*}|) = 2$. By planarity $u_3v_i \notin E(UV)$. So at least $|U_{3*}||V_{i*}| \geq 4 \times 2 = 8$ patterns are incompatible. Also $u_2v_1$ and $u_4v_i$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{3*}||V_{i*}|, |U_{4*}||V_{j*}|) = \min(2 \times 4, 6 \times 2) = 8$ patterns are incompatible. Similarly both $u_3v_1$ and $u_4v_j$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{3*}||V_{i*}|, |U_{4*}||V_{j*}|) = \min(4 \times 4, 6 \times 2) = 12$ patterns are incompatible. Therefore at least $8 + 8 + 12 = 28$ patterns are incompatible.

**Lemma 8** Consider a group $UV$ consisting of two consecutive groups of 4 vertices, where $|I(U)| \geq 12$ and $|I(V)| = 12$. Then $UV$ allows at most 120 incidence patterns.

**Proof** We distinguish two cases depending on $|I(U)|$.

**Case 1:** $|I(U)| = 12$. Then by Lemma 4 (i) & (ii), for each vertex $u_i$, all the outgoing edges from $u_i$, if any, are on one side of $U$. Since $|I(UV)| \leq |I(U)| \cdot |I(V)| = 12 \times 12 = 144$, it suffices to show that at least $144 - 120 = 24$ of these patterns are incompatible. We distinguish three cases depending on which vertex in $U$ have outgoing edges and which sides are containing those outgoing edges.

**Case 1.1:** $U$ has outgoing edges from exactly one vertex. By Lemma 4 (ii), they are from $u_3$ and we have $|U_{3*}| = 4$ and $|U_{4*}| = 7$. For simplicity assume they are in $E^-(U)$.

**Case 1.1.1:** $V$ has incoming edges into exactly one vertex. By Lemma 4 (iv), they are into $v_2$ and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$, see Fig. 17 (left), then $u_3v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{3*}||V_{1*}|, |U_{4*}||V_{2*}|) = \min(4 \times 7, 7 \times 4) = 28$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$, see Fig. 17 (right), then by planarity $u_3v_2$ is not in $E(UV)$. So $|U_{3*}||V_{2*}| = 4 \times 4 = 16$ patterns are incompatible. By planarity the edge $u_3v_1$, an outgoing edge from $u_4$ w.r.t. $UV$ and an incoming edge into $v_2$ w.r.t. $UV$ cannot be in $E(UV)$ together. Therefore at least $\min(|U_{3*}||V_{1*}|, |U_{4*}||[\emptyset]|, |[\emptyset]|V_{2*}|) = \min(4 \times 7, 7 \times 1, 1 \times 4) = 4$.
patterns are incompatible. By the same token, the edge \( u_4v_2 \), an incoming edge into \( v_1 \) w.r.t. \( UV \) and an outgoing edge from \( u_3 \) w.r.t. \( UV \) cannot coexist in \( E(UV) \). Therefore at least

\[
\min(|U_{u4}|, |\emptyset|, |V_{1*}|, |U_{u4}|, |V_{1*}|) = \min(7 \times 4, 1 \times 7, 4 \times 1) = 4
\]

patterns are incompatible. So \( 16 + 4 + 4 = 24 \) patterns are incompatible. Observe that this group, \( UV \), has 120 patterns, which is the maximum number of patterns.

Case 1.1.2: \( V \) has incoming edges into more than one vertex. Then by Lemma 4 (iv), there are incoming edges into \( v_3 \) and \( v_2 \) and we have \(|V_{3*}| = 2, |V_{2*}| \geq 3 \) and \(|V_{1*}| \geq 5\). We distinguish four scenarios based on which sides of \( V \) are containing the incoming edges into \( v_2 \) and \( v_3 \).

If the incoming edges into \( v_2 \) and \( v_3 \) w.r.t. \( V \) are in \( E^- (V) \), see Fig. 18 (left), then both \( u_3v_1 \) and \( u_4v_2 \) can only be in \( E^- (UV) \). By planarity both the edges cannot be in \( E(UV) \). So at least \( \min(|U_{u3}|, |V_{1*}|, |U_{u4}|, |V_{2*}|) = \min(4 \times 5, 7 \times 3) = 20 \) patterns are incompatible. By the same token, both \( u_3v_2 \) and \( u_4v_3 \) can only be in \( E^- (UV) \). By planarity both edges cannot be in \( E(UV) \). So at least \( \min(|U_{u3}|, |V_{2*}|, |U_{u4}|, |V_{3*}|) = \min(4 \times 3, 7 \times 2) = 12 \) other patterns are incompatible. Overall, at least \( 20 + 12 = 32 \) patterns are incompatible.

If the incoming edges into \( v_2 \) and \( v_3 \) w.r.t. \( V \) are in \( E^+ (V) \), see Fig. 18 (right), then by planarity both \( u_3v_2 \) and \( u_3v_3 \) are not in \( E(UV) \). So at least \( |U_{u3}|, |V_{2*}| + |U_{u4}|, |V_{3*}| = 4 \times 3 + 4 \times 2 = 20 \) patterns are incompatible. By planarity incoming edges into \( v_2 \) and \( v_3 \) w.r.t. \( UV \) cannot coexist with outgoing edges from \( u_4 \) w.r.t. \( U \) and the edge \( u_3v_1 \). So at least

\[
\min(|\emptyset|, |V_{2*}| + |\emptyset|, |V_{3*}|, |U_{u4}|, |\emptyset|, |U_{u3}|, |V_{1*}|) = \min(1 \times 3 + 1 \times 2, 7 \times 1, 4 \times 5) = 5
\]

patterns are incompatible. So at least \( 20 + 5 = 25 \) patterns are incompatible.

If the incoming edges into \( v_2 \) w.r.t. \( V \) are in \( E^- (V) \) and the incoming edges into \( v_3 \) w.r.t. \( V \) are in \( E^+ (V) \), see Fig. 19 (left), then by planarity \( u_3v_3 \) is not in \( E(UV) \). So at least \( |U_{u3}|, |V_{3*}| = 4 \times 2 = 8 \) patterns are incompatible. Also both \( u_3v_1 \) and \( u_4v_2 \)

\[
\min(|\emptyset|, |V_{2*}| + |\emptyset|, |U_{u4}|, |\emptyset|, |U_{u3}|, |V_{1*}|) = \min(1 \times 3 + 1 \times 2, 7 \times 1, 4 \times 5) = 5
\]

patterns are incompatible. So at least \( 20 + 5 = 25 \) patterns are incompatible.
20 patterns are incompatible. By planarity an outgoing edge from $v_1$ and $v_3$ are into at least 12 and at least 8 outgoing edges are in incompatible. So at least $20 + 20 = 28$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$ and the incoming edges into $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 19 (right), then by planarity $u_3v_2$ is not in $E(UV)$. So at least $|U_3||V_{2^*}| = 4 \times 3 = 12$ patterns are incompatible. Also, both $u_3v_1$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $|U_3||V_{1^*}| = 3 \times 7 = 20$ patterns are incompatible. So at least $12 + 14 = 26$ patterns are incompatible.

**Case 1.2:** $U$ has outgoing edges from $u_2$ and $u_3$ and both are on the same side. By Lemma 4 (ii), $|U_{a2}| = 2$, $|U_{a3}| \geq 3$ and $|U_{a4}| \geq 5$. For simplicity assume that the outgoing edges are in $E^-(U)$.

**Case 1.2.1:** $V$ has incoming edges into exactly one vertex. By Lemma 4 (iv), these are into $v_2$ and we have $|V_{2^*}| = 4$ and $|V_{1^*}| = 7$.

If the incoming edges into $v_2$ are in $E^- (V)$, see Fig. 20 (left), then both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $|U_3||V_{1^*}| = 3 \times 7 = 20$ patterns are incompatible. By planarity an outgoing edge from $u_4$ w.r.t. $UV$, an incoming edges into $v_1$ w.r.t. $UV$ and the edge $u_2v_2$ cannot coexist in $E(UV)$. So at least

$$\min(|U_{a4}||\emptyset|, |\emptyset||V_{1^*}|, |U_{a2}||V_{2^*}|) = \min(5 \times 1, 1 \times 7, 2 \times 4) = 5$$

patterns are incompatible. So in total at least $20 + 5 = 25$ incidence patterns are incompatible.

If the incoming edges into $v_2$ are in $E^+(V)$, see Fig. 20 (right), then by planarity $u_2v_2$ and $u_3v_2$ are not in $E(UV)$. So at least $|U_{a2}||V_{2^*}| + |U_{a3}||V_{2^*}| = 2 \times 4 + 3 \times 4 = 20$ patterns are incompatible. By planarity an outgoing edge from $u_4$ w.r.t. $UV$, an incoming edges into $v_2$ w.r.t. $UV$ and the edge $u_3v_1$ cannot exist together in $E(UV)$. So at least

$$\min(|U_{a4}||\emptyset|, |\emptyset||V_{2^*}|, |U_{a2}||V_{1^*}|) = \min(5 \times 1, 1 \times 4, 3 \times 7) = 4$$

![Fig. 19](image1.png) Left: the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$. Right: the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$

![Fig. 20](image2.png) Left: the incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$. Right: the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$
patterns are incompatible. So in total at least $20 + 4 = 24$ incidence patterns are incompatible.

Case 1.2.2: $V$ has incoming edges into more than one vertex. Then by Lemma 4 (iv), there are incoming edges into $v_3$ and $v_2$ and we have $|V_3\ast| = 2$, $|V_2\ast| \geq 3$ and $|V_1\ast| \geq 5$. We distinguish four scenarios based on which sides of $V$ are containing the incoming edges into $v_2$ and $v_3$.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 21 (left), then both $u_2v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_2\ast||V_1\ast|, |U_4\ast||V_2\ast|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible. By the same token, both $u_2v_1$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_2\ast||V_1\ast|, |U_4\ast||V_3\ast|) = \min(2 \times 5, 5 \times 2) = 10$ patterns are incompatible. So at least $15 + 10 = 25$ patterns are incompatible.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 21 (right), then by planarity $u_2v_2$, $u_2v_3$, $u_3v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U_2\ast||V_2\ast| + |U_2\ast||V_3\ast| + |U_3\ast||V_2\ast| + |U_3\ast||V_3\ast| = 2 \times 3 + 2 \times 2 + 3 \times 3 + 3 \times 2 = 25$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$ and the incoming edges into $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 22 (left), then by planarity $u_2v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U_2\ast||V_3\ast| + |U_3\ast||V_3\ast| = 2 \times 2 + 3 \times 2 = 10$ patterns are incompatible. Both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_3\ast||V_1\ast|, |U_4\ast||V_2\ast|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible. So at least $10 + 15 = 25$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$ and the incoming edges into $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 22 (right), then by planarity $u_2v_2$ and $u_3v_2$ are not in $E(UV)$. So at least $|U_2\ast||V_2\ast| + |U_3\ast||V_2\ast| = 2 \times 3 + 3 \times 3 = 15$ patterns are incompatible. Both $u_3v_1$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both the edges cannot be in $E(UV)$. So at least $\min(|U_3\ast||V_1\ast|, |U_4\ast||V_3\ast|) = \min(3 \times 5, 5 \times 2) = 10$ patterns are incompatible. So at least $15 + 10 = 25$ patterns are incompatible.
Case 1.3: $U$ has outgoing edges from $u_2$ and $u_3$ and both are on opposite sides. By Lemma 4(ii), $|U_{a2}| = 2$, $|U_{a3}| \geq 3$ and $|U_{a4}| \geq 5$. For simplicity assume that the outgoing edges from $u_3$ w.r.t. $U$ are in $E^+(U)$.

Case 1.3.1: $V$ has incoming edges into exactly one vertex. By Lemma 4(iv), they are into $v_2$ and we have $|V_{2*}| = 4$ and $|V_{1*}| = 7$. If the incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$, see Fig. 23 (left), then by planarity $u_2v_2$ is not in $E(UV)$. So at least $|U_{a2}||V_{2*}| = 2 \times 4 = 8$ patterns are incompatible. Also both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a3}||V_{1*}|, |U_{a4}||V_{2*}|) = \min(3 \times 7, 5 \times 4) = 20$ patterns are incompatible. So at least $8 + 20 = 28$ patterns are incompatible. If the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$, see Fig. 23 (right), then by planarity $u_3v_2$ is not in $E(UV)$. So at least $|U_{a3}||V_{2*}| = 3 \times 4 = 12$ patterns are incompatible. Also both $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a2}||V_{1*}|, |U_{a4}||V_{2*}|) = \min(2 \times 7, 5 \times 4) = 14$ patterns are incompatible. So at least $12 + 14 = 26$ patterns are incompatible.

Case 1.3.2: $V$ has incoming edges into more than one vertex. Then by Lemma 4(iii), there are incoming edges into $v_3$ and $v_2$ and we have $|V_{3*}| = 2$, $|V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of $V$ are containing the incoming edges into $v_2$ and $v_3$.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 24 (left), then by planarity $u_2v_2$ and $u_2v_3$ are not in $E(UV)$. So at least $|U_{a2}||V_{2*}| + |U_{a2}||V_{3*}| = 2 \times 3 + 2 \times 2 = 10$ patterns are incompatible. Also both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a3}||V_{1*}|, |U_{a4}||V_{2*}|) = \min(3 \times 5, 5 \times 3) = 15$ patterns are incompatible. So at least $10 + 15 = 25$ patterns are incompatible.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 24 (right), then by planarity $u_3v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U_{a3}||V_{2*}| + |U_{a3}||V_{3*}| = 3 \times 3 + 3 \times 2 = 15$ patterns are incompatible. Also both $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a2}||V_{1*}|, |U_{a4}||V_{2*}|) = \min(2 \times 5, 5 \times 3) = 10$ patterns are incompatible. So at least $15 + 10 = 25$ patterns are incompatible.

Fig. 23 Left: the incoming edges into $v_2$ w.r.t. $V$ are in $E^-(V)$. Right: the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$

Fig. 24 Left: the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^-(V)$. Right: the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^+(V)$
If the incoming edges into \( v_2 \) w.r.t. \( V \) is in \( E^-(V) \) and the incoming edges into \( v_3 \) w.r.t. \( V \) are in \( E^+(V) \), see Fig. 25 (left), then both \( u_2v_1 \) and \( u_4v_3 \) can only be in \( E^+(UV) \). By planarity both edges cannot be in \( E(UV) \). So at least
\[
\min(|U_{u2}| |V_{v1}|, |U_{u4}| |V_{v3}|) = \min(2 \times 5, 5 \times 2) = 10 \text{ patterns are incompatible.}
\]
By similar token both \( u_3v_1 \) and \( u_4v_2 \) can only be in \( E^-(UV) \). By planarity both edges cannot be in \( E(UV) \). So at least \( \min(|U_{u3}| |V_{v1}|, |U_{u4}| |V_{v2}|) = \min(3 \times 5, 5 \times 3) = 15 \) patterns are incompatible. So at least 10 + 15 = 25 patterns are incompatible.

If the incoming edges into \( v_2 \) w.r.t. \( V \) is in \( E^+(V) \) and the incoming edges into \( v_3 \) w.r.t. \( V \) are in \( E^-(V) \), see Fig. 25 (right), then both \( u_2v_1 \) and \( u_4v_2 \) can only be in \( E^+(UV) \). By planarity both edges cannot be in \( E(UV) \). So at least
\[
\min(|U_{u2}| |V_{v1}|, |U_{u4}| |V_{v2}|) = \min(2 \times 5, 5 \times 3) = 10 \text{ patterns are incompatible.}
\]
Also both \( u_3v_1 \) and \( u_4v_3 \) can only be in \( E^-(UV) \). By planarity both edges cannot be in \( E(UV) \). So at least \( \min(|U_{u3}| |V_{v1}|, |U_{u4}| |V_{v3}|) = \min(3 \times 5, 5 \times 2) = 10 \) patterns are incompatible. By planarity an outgoing edge from \( u_4 \) w.r.t. \( UV \), the edge \( u_2v_2 \) and the edge \( u_3v_3 \) cannot exist together in \( E(UV) \). So at least
\[
\min(|U_{u4}| |\emptyset|, |U_{u2}| |V_{v2}|, |U_{u3}| |V_{v3}|) = \min(5 \times 1, 2 \times 3, 3 \times 2) = 5
\]
patterns are incompatible. So in total at least 10 + 10 + 5 = 25 incidence patterns are incompatible.

**Case 2:** \( |I(U)| = 13 \). By Lemma 5, \( U \) is either \( A \) or \( AR \). If \( U \) is \( AR \), then after reflecting \( UV \) about the horizontal axis, \( U \) is \( A \). We analyze the cases based on this assumption. Since \( |I(UV)| \leq |I(U)| \cdot |I(V)| = 13 \times 12 = 156 \), it suffices to show that at least 156 - 120 = 36 of these patterns are incompatible.

**Case 2.1:** \( V \) has incoming edges into exactly one vertex. Then by Lemma 4 (iv), they are into \( v_2 \) and we have \( |V_{v2}| = 4 \) and \( |V_{v1}| = 7 \).

If the incoming edges into \( v_2 \) w.r.t. \( V \) are in \( E^-(V) \), see Fig. 26 (left), then by planarity \( u_2v_2 \) is not in \( E(UV) \). So at least \( |U_{u2}| |V_{v2}| = 2 \times 4 = 8 \) patterns are incompatible.

\[
\min(|U_{u2}| |V_{v2}|, |U_{u3}| |V_{v3}|) = \min(2 \times 4, 5 \times 3) = 8
\]
patterns are incompatible. So in total at least 8 + 8 = 16 of these patterns are incompatible.
patterns are incompatible. Both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{s3}||V_{1*}|, |U_{s4}||V_{2*}|) = \min(4 \times 7, 6 \times 4) = 24$ patterns are incompatible. By planarity an outgoing edge from $u_4$ w.r.t. $UV$, the edge $u_2v_1$ and the edge $u_3v_2$ cannot exist together in $E(UV)$. So at least

$$\min(|U_{s4}||\emptyset|, |U_{s2}||V_{1*}|, |U_{s3}||V_{2*}|) = \min(6 \times 1, 2 \times 7, 4 \times 4) = 6$$

patterns are incompatible. So in total at least $8 + 24 + 6 = 38$ incidence patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ are in $E^+(V)$, see Fig. 26 (right), then by planarity $u_3v_2$ is not allowed. So at least $|U_{s3}||V_{2*}| = 4 \times 4 = 16$ patterns are incompatible. Both $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{s2}||V_{1*}|, |U_{s4}||V_{2*}|) = \min(2 \times 7, 6 \times 4) = 14$ patterns are incompatible. By planarity an outgoing edge from $u_4$ w.r.t. $UV$, the edge $u_2v_2$ and the edge $u_3v_1$ cannot exist together in $E(UV)$. So at least

$$\min(|U_{s4}||\emptyset|, |U_{s2}||V_{2*}|, |U_{s3}||V_{1*}|) = \min(6 \times 1, 2 \times 4, 4 \times 7) = 6$$

patterns are incompatible. So at least $16 + 14 + 6 = 36$ patterns are incompatible.

Case 2.2: $V$ has incoming edges into more than one vertex. By Lemma 4 (iv), there are incoming edges into $v_3$ and $v_2$ and we have $|V_{3*}| = 2, |V_{2*}| \geq 3$ and $|V_{1*}| \geq 5$. We distinguish four scenarios based on which sides of $V$ are containing the incoming edges into $v_2$ and $v_3$.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 27 (left), then by planarity $u_2v_2$ is not in $E(UV)$. So at least $|U_{s2}||V_{2*}| = 2 \times 3 = 6$ patterns are incompatible. Both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{s3}||V_{1*}|, |U_{s4}||V_{2*}|) = \min(4 \times 5, 6 \times 3) = 18$ patterns are incompatible. Similarly both $u_3v_2$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{s3}||V_{2*}|, |U_{s4}||V_{3*}|) = \min(4 \times 3, 6 \times 2) = 12$ patterns are incompatible. So at least $6 + 18 + 12 = 36$ patterns are incompatible.

If the incoming edges into $v_2$ and $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 27 (right), then by planarity both $u_3v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U_{s3}||V_{2*}| + |U_{s3}||V_{3*}| = 4 \times 3 + 4 \times 2 = 20$ patterns are incompatible. Also both $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{s2}||V_{1*}|, |U_{s4}||V_{2*}|) = \min(2 \times 5, 6 \times 3) = 10$ patterns are incompatible. Similarly both $u_2v_2$ and $u_4v_3$ can only be in $E^+(UV)$. By planarity

\[\text{Fig. 27 Left: the incoming edges into } v_2 \text{ and } v_3 \text{ w.r.t. } V \text{ are in } E^-(V). \text{ Right: the incoming edges into } v_2 \text{ and } v_3 \text{ w.r.t. } V \text{ are in } E^+(V).\]
both edges cannot be in $E(UV)$. So at least $\min(|U_{a2}|V_{2a}|, |U_{a4}|V_{3a}|) = \min(2 \times 3, 6 \times 2) = 6$ patterns are incompatible. So at least $20 + 10 + 6 = 36$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ is in $E^-(V)$ and the incoming edges into $v_3$ w.r.t. $V$ are in $E^+(V)$, see Fig. 28(left), then by planarity $u_2v_3$ is not in $E(UV)$. So at least $|U_{a3}|V_{3a}| = 4 \times 2 = 8$ patterns are incompatible. Both $u_2v_1$ and $u_4v_3$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a2}|V_{1a}|, |U_{a4}|V_{3a}|) = \min(2 \times 5, 6 \times 2) = 10$ patterns are incompatible. Similarly both $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a3}|V_{1a}|, |U_{a4}|V_{2a}|) = \min(4 \times 5, 6 \times 3) = 18$ patterns are incompatible. So at least $8 + 10 + 18 = 36$ patterns are incompatible.

If the incoming edges into $v_2$ w.r.t. $V$ is in $E^+(V)$ and the incoming edges into $v_3$ w.r.t. $V$ are in $E^-(V)$, see Fig. 28(right), then by planarity both $u_2v_3$ and $u_4v_2$ are not in $E(UV)$. So at least $|U_{a2}|V_{3a}| + |U_{a3}|V_{2a}| = 2 \times 2 + 4 \times 3 = 16$ patterns are incompatible. Both $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a2}|V_{1a}|, |U_{a4}|V_{2a}|) = \min(2 \times 5, 6 \times 3) = 10$ patterns are incompatible. Similarly both $u_3v_1$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. So at least $\min(|U_{a3}|V_{1a}|, |U_{a4}|V_{3a}|) = \min(4 \times 5, 6 \times 2) = 12$ patterns are incompatible. So at least $16 + 10 + 12 = 38$ patterns are incompatible.

Lemma 9 Consider a group $UV$ consisting of two consecutive groups of 4 vertices, where $|I(U)| = |I(V)| = 13$. Then $UV$ allows at most 120 incidence patterns.

Proof By Lemma 5, $U$ is either $A$ or $A^R$. We may assume, after reflecting $UV$ about a horizontal axis, that $U$ is $A$. Therefore $|U_{a2}| = 2$, $|U_{a3}| = 4$ and $|U_{a4}| = 6$, see Figure 6. Similarly, Lemma 5 implies that $V$ is either $A$ or $A^R$. We distinguish two cases depending on whether $V$ is $A$ or $A^R$. The cross product of $I(U)$ and $I(V)$ yields $13 \times 13 = 169$ possible patterns. It suffices to show that at least $169 - 120 = 49$ of these patterns are incompatible.

Case 1: $V$ is $A$, see Fig. 29(left). By planarity $u_2v_3$ and $u_3v_2$ are not in $E(UV)$. So at least $|U_{a2}|V_{3a}| + |U_{a3}|V_{2a}| = 2 \times 2 + 4 \times 4 = 20$ patterns are incompatible. Further, $u_2v_1$ and $u_4v_2$ can only be in $E^+(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U_{a2}|V_{1a}|, |U_{a4}|V_{2a}|) = \min(2 \times 6, 6 \times 4) = 12$ patterns are incompatible. Similarly $u_3v_1$ and $u_4v_3$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Therefore at least $\min(|U_{a3}|V_{1a}|, |U_{a4}|V_{3a}|) = \min(4 \times 6, 6 \times 2) = 12$ patterns are incompatible. By planarity an outgoing edge
from $u_4$ w.r.t. $UV$ and the edges $u_2v_2$ and $u_3v_1$ cannot exist together in $E(UV)$. So at least
\[
\min(|U^*_{s+4}| \emptyset|, |U^*_{s+2}|V^*_2|, |U^*_{s+3}|V^*_3|) = \min(6 \times 1, 2 \times 4, 4 \times 2) = 6
\]
patterns are incompatible. Overall, at least $20 + 12 + 12 + 6 = 50$ patterns are incompatible.

Case 2: $V$ is $A^R$, see Fig. 29 (right). By planarity $u_2v_2$ and $u_3v_3$ are not in $E(UV)$. So at least $|U^*_{s+2}|V^*_2| + |U^*_{s+3}|V^*_3| = 2 \times 4 + 4 \times 2 = 16$ patterns are incompatible. Also $u_2v_1$ and $u_4v_3$ can only be in $E^+ (UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U^*_{s+2}|V^*_1|, |U^*_{s+4}|V^*_3|) = \min(2 \times 6, 6 \times 2) = 12$ patterns are incompatible. Similarly $u_3v_1$ and $u_4v_2$ can only be in $E^-(UV)$. By planarity both edges cannot be in $E(UV)$. Hence at least $\min(|U^*_{s+3}|V^*_1|, |U^*_{s+4}|V^*_2|) = \min(4 \times 6, 6 \times 4) = 24$ patterns are incompatible. Overall, at least $16 + 12 + 24 = 52$ patterns are incompatible. □

Lemma 10 Every group on 8 vertices has at most 120 incidence patterns, and this bound is the best possible. Consequently, $p_8 = 120$.

Proof A group of 8, denoted by $UV$, where $U$ and $V$ are the groups induced by the first and last four vertices of $UV$, respectively. If $|I(U)| \leq 9$ or $|I(V)| \leq 9$, then $|I(UV)| \leq |I(U)| \cdot |I(V)| \leq 9 \times 13 = 117$ by Lemma 5. Otherwise, Lemmas 6–9 show that $|I(UV)| \leq 120$.

Consider the group $(U, E^-(U), E^+(U))$ of 8 vertices depicted in Fig. 30 (right). The first and second half of $U$ are the groups $B_2$ and $B_3$ in Fig. 30 (left), each with 12 patterns. Observe that exactly 24 patterns are incompatible, thus $U$ has exactly $|I(B_2)| \cdot |I(B_3)| - 24 = 12 \times 12 - 24 = 120$ patterns. Aside from reflections, the extremal group of 8 vertices in Fig. 30 (right) is unique. □

Fig. 30 $U$ has 120 patterns. The 24 missing patterns are 123678, 12367, 12368, 1236, 123, 13678, 1367, 1368, 136, 13, 23678, 2367, 2368, 236, 23, 3678, 367, 368, 36, 3, 678, 67, 68, 6.
Groups \(U\) and \(V\) (hence also \(U^R\) and \(V^R\)) are the only groups of 9 vertices with 201 incidence patterns. Observe that \(V\) is the reflection of \(U\) in the \(y\)-axis.

5 Groups of 9, 10, and 11 Vertices Via Computer Search

The application of the same fingerprinting technique to groups of 9, 10, and 11 vertices via a computer program\(^1\) shows that

- A group of 9 vertices allows at most 201 incidence patterns; the extremal configuration appears in Fig. 31. This yields the upper bound of \(O(n^{3\cdot 201}/9) = O(1.8027^n)\) for the number of monotone paths in an \(n\)-vertex triangulation. Aside from reflections, the extremal group of 9 vertices in Fig. 31 (left) is unique.

- A group of 10 vertices allows at most 346 incidence patterns; the extremal configuration appears in Fig. 32. This yields the upper bound of \(O(n^{3\cdot 346}/10) = O(1.7944^n)\) for the number of monotone paths in an \(n\)-vertex triangulation, as given in Theorem 1. Aside from reflections, the extremal group of 10 vertices in Fig. 32 is unique.

- A group of 11 vertices allows at most 591 incidence patterns; the extremal configuration appears in Fig. 33. This yields the upper bound of \(O(n^{3\cdot 591}/11) = O(1.7864^n)\) for the number of monotone paths in an \(n\)-vertex triangulation. Aside from reflections, the extremal group of 11 vertices in Fig. 33 (left) is unique.

To generate all groups of \(k\) vertices, the program first generates all possible sides of \(k\) vertices, essentially by brute force. A side of \(k\) vertices \(V = \{v_1, \ldots, v_k\}\) is represented by a directed planar graph with \(k + 2\) vertices, where the edges \(v_0v_1\) and

\(^1\)Refer to the .c file and the Appendix within the source at arXiv:1608.04812.
Fig. 32 Group $U$ (hence also $U^R$) is the only group of 10 vertices with 346 incidence patterns. Observe that the reflection of $U$ in the y-axis is $U^R$.

$v_i v_{i+1}$, for $1 \leq i \leq k$, denote an incoming edge into $v_i$ and an outgoing edge from $v_i$, respectively. The edge $v_0 v_{k+1}$ represents the $\emptyset$ pattern. Note that $\xi_0 \cup v_0 v_{k+1}$ forms a plane cycle on $k + 2$ vertices in the underlying undirected graph. Therefore, $E^+(V)$ and $E^-(V)$ can each have at most $(k + 2) - 3 = k - 1$ edges. After all possible sides are generated, the program combines all pairs of sides with no common inner edge to generate a group $(V, E^-(V), E^+(V))$. For each generated group, the program calculates the corresponding number of patterns and in the end returns the group with the maximum number of patterns.

Remark It is interesting to observe how the structure of the unique extremal groups of 9, 10 and 11 vertices (depicted in Figs. 31, 32 and 33) match that of the current best lower bound construction illustrated in Fig. 1 (right).

Fig. 33 Groups $U$ and $V$ (hence also $U^R$ and $V^R$) are the only groups of 11 vertices with 591 incidence patterns. Observe that $V$ is the reflection of $U$ in the y-axis.
6 Counting and Enumerating Monotone Paths

Counting and enumerating $x$-monotone paths Let $G = (V, E)$ be a plane geometric graph with $n$ vertices. We first observe that the number of $x$-monotone paths in $G$ can be computed by a sweep-line algorithm. For every vertex $v \in V$, denote by $m(v)$ the number of (directed) nonempty $x$-monotone paths that end at $v$. Sweep a vertical line $\ell$ from left to right, and whenever $\ell$ reaches a vertex $v$, we compute $m(v)$ according to the relation

$$m(v) = \sum_{q \in L(v)} [m(q) + 1],$$

where $L(v)$ denotes the set of neighbors of vertex $v$ in $G$ that lie to the left of $v$. The total number of $x$-monotone paths in $G$ is $\sum_{v \in V} m(v)$. For every $v \in V$, the computation of $m(v)$ takes $O(\deg(v))$ time, thus computing $m(v)$ for all $v \in V$ takes $\sum_{v \in V} O(\deg(v)) = O(n)$ time. The algorithm for computing the number of $x$-monotone paths in $G$ takes $O(n)$ time if the vertices are in sorted order, or $O(n \log n)$ time otherwise.

The sweep-line algorithm can be adapted to enumerate the $x$-monotone paths in $G$ in $O(n \log n + K)$ time, where $K$ is the sum of the lengths of all $x$-monotone paths. For every vertex $v \in V$, denote by $M(v)$ the set of (directed) nonempty $x$-monotone paths that end at $v$. When the vertical sweep-line $\ell$ reaches a vertex $v$, we compute $M(v)$ according to the relation

$$M(v) = \bigcup_{q \in L(v)} \{(q, v)\} \cup \{p \oplus (q, v) : p \in M(q)\},$$

where the concatenation of two paths, $p_1$ and $p_2$, is denoted by $p_1 \oplus p_2$. The set of all $x$-monotone paths is $\bigcup_{v \in V} M(v)$.

The number (resp., set) of $u$-monotone paths in $G$ for any direction $u \in \mathbb{R}^2 \setminus \{0\}$ can be computed in a similar way, using the counts $m_u(v)$ (resp., sets $M_u(v)$) and the neighbor sets $L_u(v)$, instead; the overall time per direction is $O(n)$ (resp., $O(n + K_u)$) if the vertices are in sorted order, or $O(n \log n)$ (resp., $O(n \log n + K_u)$) otherwise.

Computing all monotone paths For computing the total number of monotone paths over all directions $u \in \mathbb{R}^2 \setminus \{0\}$, some care is required. As shown subsequently, it suffices to consider monotone paths in at most $2|E|$ directions, one direction between any two consecutive directions orthogonal to the edges in $E$ (each edge yields two opposite directions). It is worth noting however, that it does not suffice to consider only directions parallel or orthogonal to the edges of $G$, i.e., the union of monotone paths over all such directions may not contain all monotone paths. Second, observe that once a sufficient set of directions is established, we cannot simply sum up the number of monotone paths over all such directions, since a monotone path in $G$ may be monotone in several of these directions.

---

2The algorithm has been revised, as some of the ideas were implemented incorrectly in the earlier conference version.
We first compute a set of directions that is sufficient for our purpose, i.e., every monotone path is monotone with respect to at least one of these directions. We then consider these directions sorted by angle, and for each new direction, we compute the number of new paths. In fact, we count directed monotone paths, that is, each path will be counted twice, as traversed in two opposite directions. We now proceed with the details.

Partition the edge set $E$ of $G$ into subsets of parallel edges; note that each subset yields two opposite directions. Since $|E| \leq 3n - 6$, the edges are partitioned into at most $|E| \leq 3n - 6$ subsets. Let $D$ be a set of direction vectors $\overrightarrow{ab}$ of the edges $(a, b) \in E$, one from each subset of parallel edges. Let $D^\perp$ be a set of vectors obtained by rotating each vector in $D$ counterclockwise by $\pi/2$ and by $-\pi/2$; note that $|D^\perp| = 2|D| \leq 2|E| \leq 6n - 12$. Sort the vectors in $D^\perp$ by their arguments in cyclic order, and let $U$ be a set of vector sums of all pairs of consecutive vectors in $U^\perp$; clearly $|U| \leq |D^\perp| \leq 6n - 12$. We show that $U$ is a sufficient set of directions. Indeed, consider a path $\xi$ monotone with respect to some direction $u \in \mathbb{R}^2 \setminus \{0\}$. Observe that the edges in $\xi$ cannot be orthogonal to $u$; it follows that $\xi$ is still monotone with respect to at least one of the two adjacent directions in $U$ (closest to $u$).

We next present the algorithm. Sort the vectors in $U$ by their arguments, in $O(n \log n)$ time. Let $u_0 \in U$ be an vector of minimum argument in $U$. We first compute the number of $u_0$-monotone directed paths in $G$, $\sum_{v \in V} m_{u_0}(v)$, by the sweep-line algorithm described above in $O(n \log n)$ time.

Consider the directions $u \in U \setminus \{u_0\}$, sorted by increasing arguments. For each $u$, we maintain the number of directed paths in $G$ that are monotone in some direction between $u_0$ and $u$ (implicitly, by the sum of parameters $\gamma$ defined below). For each new direction $u$, exactly one subset of parallel edges, denoted $E_u$, becomes $u$-monotone (i.e., it consists of $u$-monotone edges). Therefore, it is enough to count the number of $u$-monotone paths that traverse some edge in $E_u$.

**Counting the monotone paths in $G$** These paths can be counted by sweeping $G$ with a line $\ell$ orthogonal to $u$: Sort the vertices in direction $u$ (ties are broken arbitrarily). Compute two parameters for every vertex $v \in V$:

- the number $m_u(v)$ of $u$-monotone paths that end at $v$,
- the number $\gamma_u(v)$ of $u$-monotone paths that end at $v$ and contain some edge from $E_u$.

When reaching vertex $v$, the sweep-line algorithm computes the first parameter $m_u(v)$ according to the relation:

$$m_u(v) = \sum_{q \in L_u(v)} [m_u(q) + 1].$$

The second parameter $\gamma_u(v)$ is computed as follows:

$$\gamma_u(v) = \begin{cases} 
1 + m_u(a) + \sum_{q \in L_u(v) \setminus \{a\}} \gamma_u(q) & \text{if } \exists (a, v) \in E_u \text{ with } \langle \overrightarrow{av}, u \rangle > 0, \\
\sum_{q \in L_u(v)} \gamma_u(q) & \text{otherwise}.
\end{cases}$$

3The argument of a vector $u \in \mathbb{R}^2 \setminus \{0\}$ is the angle measure in $[0, 2\pi)$ of the minimum counterclockwise rotation that carries the positive $x$-axis to the ray spanned by $u$. Springer
The total number of monotone paths, returned by the algorithm in the end is

\[ \sum_{v \in V} \left( m_{u_0}(v) + \sum_{u \in U \setminus \{u_0\}} \gamma_u(v) \right). \]

We next show that the sorted order of vertices in the \( O(n) \) directions in \( U \) can be computed in \( O(n^2) \) time. Consider the duality transform, where every point \( v = (a, b) \) is mapped to a dual line \( v^* : y = ax - b \), and every line \( \ell : y = ax - b \) is mapped to a dual point \( \ell^* = (a, b) \). It is known that the duality preserves the above-below relationship between points and lines [3, Ch. 8]. Note that the lines of slope \( a \) are mapped to dual points on the vertical line \( x = a \). Consequently, when we sweep \( V \) by a line of slope \( a \) in direction \( u = (-a, 1) \), we encounter the points in \( V \) in the order determined by \( y \)-coordinates of the intersections of the vertical line \( x = a \) with the dual lines in \( V^* = \{v^* : v \in V\} \).

Let \( A \) be the arrangement of the \( n \) dual lines in \( V^* \) plus the \( O(n) \) vertical lines corresponding to the slopes of the vectors in \( U \). The arrangement \( A \) of these \( O(n) \) lines has \( O(n^2) \) vertices, and can be computed in \( O(n^2) \) time [3, Ch. 8]. By tracing the vertical line corresponding to each vector \( u \in U \) in \( A \), we find its intersection points with the dual lines in \( V^* \), sorted by \( y \)-coordinates, in \( O(n) \) time. Since \( |U| = O(n) \), the total running time of the algorithm is \( O(n^2) \), as claimed.

**Enumerating the monotone paths in \( G \)** To this end we adapt the formulae (2) and (3) to sets of monotone paths in a straightforward manner. For every vertex \( v \in V \), we compute two sets:

- the set \( M_u(v) \) of \( u \)-monotone paths that end at \( v \),
- the set \( \Gamma_u(v) \) of \( u \)-monotone paths that end at \( v \) and contain some edge from \( E_u \).

When reaching vertex \( v \), the sweep-line algorithm computes \( M_u(v) \) according to the relation:

\[ M_u(v) = \bigcup_{q \in L_u(v)} ((q, v)) \cup \{p \oplus (q, v) : p \in M(q)\}. \tag{4} \]

The set \( \Gamma_u(v) \) is computed as follows:

\[ \Gamma_u(v) = \begin{cases} 
\{(a, v)\} \cup \{p \oplus (a, v) : p \in M_u(a)\} \cup \\
\bigcup_{q \in L_u(v) \setminus \{a\}} \{p \oplus (q, v) : p \in \Gamma_u(q)\} & \text{if } \exists(a, v) \in E_u \text{ with } (\overrightarrow{av}, u) > 0, \\
\bigcup_{q \in L_u(v)} \{p \oplus (q, v) : p \in \Gamma_u(q)\} & \text{otherwise}. 
\end{cases} \tag{5} \]

Now the set of directed monotone paths in \( G \) is

\[ \bigcup_{v \in V} \left( M_{u_0}(v) \cup \bigcup_{u \in U \setminus \{u_0\}} \Gamma_u(v) \right), \]
where every undirected monotone path appears twice: once in each direction. This completes the proof of Theorem 2.

7 Concluding Remarks

A path is simple if it has no repeated vertices; obviously every monotone path is simple. A directed polygonal path $\xi = (v_1, v_2, \ldots, v_t)$ in $\mathbb{R}^d$ is weakly monotone if there exists a nonzero vector $u \in \mathbb{R}^d$ that has a nonnegative inner product with every directed edge of $\xi$, that is, $\langle v_i v_{i+1}, u \rangle \geq 0$ for $i = 1, \ldots, t - 1$. In many applications such as local search, a weakly monotone path may be as good as a monotone one, since both guarantee that the objective function is nondecreasing.

It therefore appears as a natural problem to find a tight asymptotic bound on the maximum number of weakly monotone simple paths over all plane geometric graphs with $n$ vertices. As for monotone paths, it is easy to see that triangulations maximize the number of such paths. Recall that $\mu_n$ denotes the maximum number (over all directions $u$) of maximal $u$-monotone paths in an $n$-vertex triangulation. Let $\beta_n$ denote the maximum number (over all directions $u$) of maximal weakly $u$-monotone simple paths in an $n$-vertex triangulation.

We clearly have $\beta_n \geq \mu_n$, and so $\beta_n = \Omega(1.7003^n)$. However, $\beta_n$ could in principle grow faster than $\mu_n$. Let $n = 4$ and consider the three vertices and the center of an equilateral triangle, and the unique triangulation of these four points; shown in Fig. 34. Observe that: (i) the 5 paths 132, 1342, 142, 1432, and 12 are weakly $u$-monotone and maximal, where $u = (1, 0)$ and yield $\beta_4 = 5$; (ii) the 4 paths 143, 142, 12, and 13 are $u$-monotone and maximal, where $u = (\cos \pi/6, \sin \pi/6)$ and yield $\mu_4 = 4$; and so $\beta_4 > \mu_4$.

We conclude with the following open problems.

1. What upper and lower bounds can be derived for $\beta_n$? Is $\beta_n = \omega(\mu_n)$?
2. What can be said about counting and enumeration of weakly monotone paths in a given plane geometric graph?

Appendix A: Extremal configurations

The groups of 4 vertices with 12 and 11 patterns There are exactly 4 groups with exactly 12 incidence patterns (modulo reflections about the x-axis); see Fig. 35.
B1–B4 are the only four groups with 12 incidence patterns:

- $I(B1): \emptyset, 1234, 123, 12, 134, 13, 234, 23, 2, 3, 4.$
- $I(B2): \emptyset, 1234, 123, 124, 134, 13, 234, 23, 24, 34, 3, 4.$
- $I(B3): \emptyset, 1234, 123, 124, 12, 134, 13, 1, 23, 234, 24, 2.$
- $I(B4): \emptyset, 1234, 123, 124, 12, 1, 23, 234, 24, 2, 34, 3.$

There are exactly 20 groups with exactly 11 incidence patterns (modulo reflections about the $x$-axis); see Fig. 36:

C1–C20 are the only 20 groups with 11 incidence patterns.
References

1. Adler, I., Papadimitriou, C., Rubinstein, A.: On simplex pivoting rules and complexity theory. In: Proceedings of the 17th IPCO, LNCS 8494, Springer (2014)
2. Ajtai, M., Chvátal, V., Newborn, M., Szemerédi, E.: Crossing-free subgraphs. Ann. Discret. Math. 12, 9–12 (1982)
3. de Berg, M., Cheong, O., van Kreveld, M., Overmars, M. Computational Geometry, 3rd edn. Springer, Berlin (2008)
4. Buchin, K., Knauer, C., Kriegel, K., Schulz, A., Seidel, R.: On the number of cycles in planar graphs. In: Proceedings of the 13th COCOON, LNCS 4598, Springer (2007)
5. Dumitrescu, A., Löffler, M., Schulz, A., Tóth, C.S.D.: Counting carambolas. Graphs Combin. 32(3), 923–942 (2016)
6. Dumitrescu, A., Rote, G., Tóth, C.S.D.: Monotone paths in planar convex subdivisions and polytopes. In: Discrete Geometry and Optimization, vol. 69 of Fields Institute of Communications, Springer, pp. 79–104 (2013)
7. Dumitrescu, A., Schulz, A., Sheffer, A., Tóth, C.S.D.: Bounds on the maximum multiplicity of some common geometric graphs. SIAM J. Discret. Math. 27(2), 802–826 (2013)
8. Dumitrescu, A., Tóth, C.S.D.: Computational Geometry Column 54. SIGACT News Bullet. 43(4), 90–97 (2012)
9. Dumitrescu, A., Tóth, C.S.D.: Convex polygons in geometric triangulations. Combin. Probab. Comput. 26(5), 641–659 (2017)
10. García, A., Noy, M., Tejel, A.: Lower bounds on the number of crossing-free subgraphs of $K_N$. Comput. Geom. 16(4), 211–221 (2000)
11. Gärtner, B., Kaibel, V.: Two new bounds for the random-edge simplex-algorithm. SIAM J. Discret. Math. 21(1), 178–190 (2007)
12. Kaibel, V., Mechtel, R., Sharir, M., Ziegler, G.M.: The simplex algorithm in dimension three. SIAM J. Comput. 34(2), 475–497 (2005)
13. Kalai, G.: Upper bounds for the diameter and height of graphs of convex polyhedra. Discret. Comput. Geom. 8(4), 363–372 (1992)
14. Kalai, G.: Polytope skeletons and paths. In: Handbook of Discrete and Computational Geometry, Goodman, J., O’Rourke, J., Tóth, C. D. (eds), Chapter 19, pp. 505–532, 3rd edn, CRC Press, Boca Raton (2017)
15. Klee, V.: Paths on polyhedra I. J. SIAM 13(4), 946–956 (1965)
16. van Kreveld, M., Löffler, M., Pach, J.: How many potatoes are in a mesh?, in Proc. 23rd ISAAC, LNCS 7676, Springer, pp. 166–176 (2012)
17. Matoušek, J., Szabó, T.: RANDOM EDGE can be exponential on abstract cubes. Adv. Math. 204(1), 262–277 (2006)
18. Pach, J., Tóth, G.: Monotone drawings of planar graphs. J. Graph Theory 46, 39–47 (2004). Corrected version: arXiv:1101.0967, 2011
19. Razen, A., Snoeyink, J., Welzl, E.: Number of crossing-free geometric graphs vs. triangulations. Electron. Notes Discret. Math. 31, 195–200 (2008)
20. Santos, F.: A counterexample to the Hirsch conjecture. Ann. Math. 176(1), 383–412 (2012)
21. Santos, F.: Recent progress on the combinatorial diameter of polytopes and simplicial complexes. TOP 21(3), 426–460 (2013)
22. Sharir, M., Sheffer, A.: Counting triangulations of planar point sets. Electron. J. Combin. 18, P70 (2011)
23. Sharir, M., Sheffer, A.: Counting plane graphs: cross-graph charging schemes. Combin. Probab. Comput. 22(6), 935–954 (2013)
24. Sharir, M., Sheffer, A., Welzl, E.: Counting plane graphs: perfect matchings, spanning cycles, and Kasteleyn’s technique. J. Combin. Theory, Ser. A 120(4), 777–794 (2013)
25. Sharir, M., Welzl, E.: On the number of crossing-free matchings, cycles, and partitions. SIAM J. Comput. 36(3), 695–720 (2006)
26. Sheffer, A.: Numbers of plane graphs, https://adamsheffer.wordpress.com/numbers-of-plane-graphs/ (version of April, 2016)
27. Todd, M.J.: The monotonic bounded Hirsch conjecture is false for dimension at least 4. Math. Oper. Res. 5(4), 599–601 (1980)
28. Todd, M.J.: An improved Kalai-Kleitman bound for the diameter of a polyhedron. SIAM J. Discret. Math. 28, 1944–1947 (2014)
29. Ziegler, G.M.: Lectures on Polytopes, vol. 152 of GTM, Springer, pp. 83–93 (1994)