SERIES EXPANSION FOR THE FOURIER TRANSFORM OF A RATIONAL FUNCTION IN THREE DIMENSIONS

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Abstract. In Rashba–Dresselhaus spin-orbit coupled systems, the calculation of Green’s function requires the knowledge of the inverse Fourier transform of rational function $P(p)/Q(p)$, where $P(p)$ takes the values 1 and $p^2$, and where

$$Q(p) = (p^2 - \zeta)^2 - \alpha^2 \left(p_1^2 + p_2^2\right) - \beta^2$$

with suitable parameters $\alpha, \beta \geq 0, \zeta \in \mathbb{C}$. While a two-dimensional problem, with $p = (p_1, p_2)$, has been recently solved [J. Brüning et al, J. Phys. A: Math. Theor. 40 (2007)], its three-dimensional analogue, with $p = (p_1, p_2, p_3)$, remains open. In this paper, a hypergeometric series expansion for the triple integral is provided. Convergence of the series dependent on the parameters is studied in detail.

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1. Introduction

The spectral analysis of ultracold atomic gases is closely related to the calculation of the integral kernel (Green’s function) for Rashba–Dresselhaus spin-orbit coupled Hamiltonian. In momentum representation the Hamiltonian is the operator of multiplication by

\[
H_R = \begin{pmatrix}
    p^2 + \beta & -\alpha (p_2 + ip_1) \\
    -\alpha (p_2 - ip_1) & p^2 - \beta
\end{pmatrix},
\]

(1.1a)

\[
H_D = \begin{pmatrix}
    p^2 + \beta & -\alpha (p_1 - ip_2) \\
    -\alpha (p_1 + ip_2) & p^2 - \beta
\end{pmatrix}.
\]

(1.1b)

The subscript \( R \) (resp. \( D \)) indicates Rashba (resp. Dresselhaus) type spin-orbit interaction. Vector \( p = (p_1, p_2) \in \mathbb{R}^2 \) or \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \), depending on the model; see eg [1] and the citation therein. The parameter \( \alpha \geq 0 \) has a meaning of spin-orbit-coupling strength and \( \beta \geq 0 \) characterizes the magnetic Zeeman field.

The representatives \( H_R \) and \( H_D = U H_R U^* \),

\[
U \in \left\{ (-1)^n e^{i \delta} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} : \delta \geq 0, n \in \mathbb{N}_0 \right\} \subset U(2),
\]

of the Hamiltonian (1.1) are unitarily equivalent. The Green’s function

\[
G_R(x) = G_R(x; \alpha, \beta, \zeta), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 - \{0\}
\]

for \( H_R \) is

\[
G_R(x) = \begin{pmatrix}
    G_2(x) - \beta G_1(x) & -\alpha D_+ G_1(x) \\
    \alpha D_+ G_1(x) & G_2(x) + \beta G_1(x)
\end{pmatrix}
\]

(1.2)

where

\[
G_1(x) \equiv G_1(x; \alpha, \beta, \zeta) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{-ip \cdot x} \frac{1}{Q(p)}
\]

(1.3a)

\[
G_2(x) \equiv G_2(x; \alpha, \beta, \zeta) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{-ip \cdot x} \frac{p^2 - \zeta}{Q(p)}
\]

(1.3b)

where

\[
D_\pm = \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2}, \quad Q(p) = \left(p^2 - \zeta\right)^2 - \alpha^2 \left(p_1^2 + p_2^2\right) - \beta^2.
\]

where we have used

\[
i \frac{\partial}{\partial x_j} G_1(x) = \int_{\mathbb{R}^3} \frac{d^3 p}{(2\pi)^3} e^{-ip \cdot x} \frac{p_j}{Q(p)}, \quad j = 1, 2.
\]

The complex number \( \zeta \in \mathbb{C} \) is in the resolvent set of \( H (= H_R, H_D) \). When considered separately, \( G_1 \) is defined for \( x \in \mathbb{R}^3 \), and \( G_2 \) for \( x \in \mathbb{R}^3 - \{0\} \). The associated Green’s function for \( H_D \) is \( U G_R U^* \).

The resolvent set of \( H \) consists of complex numbers \( \zeta \) for which the denominator \( Q(p) \), \( \forall p \in \mathbb{R}^3 \), is nonzero:

\[
\zeta \in \mathbb{C} - [-\Sigma, \infty), \quad \Sigma = \begin{cases} \beta, & \text{if } \beta > \frac{1}{2} \alpha^2, \\ \left(\frac{\beta}{\alpha}\right)^2 + \left(\frac{\alpha}{2}\right)^2, & \text{if } \beta \leq \frac{1}{2} \alpha^2. \end{cases}
\]

(1.4)

Stated otherwise, \([-\Sigma, \infty)\) is the essential spectrum of \( H \).

In this paper, our principle goal is to examine (1.2)–(1.3) with \( \alpha, \beta \geq 0 \) (due to physical reasons) and \( \zeta \) as in (1.4) (to ensure the existence of \((H - \zeta)^{-1}\)).
While for the spinless systems, that is, for $\alpha = 0$, the integrals (1.3) are easy to compute for suitable $x$, $\alpha$, $\beta$, $\zeta$ [rewrite $p$ in spherical coordinates, integrate over the angles and then apply \[ Eq. (3.728.2)\]], the case when $\alpha > 0$ is much more involved.

No evidence of successful algebraic treatment of the integrals (1.3) has been found so far. The special case when $\alpha = 0$ has been discussed in [3]. A two dimensional equivalent of (1.3), with $p = (p_1, p_2) \in \mathbb{R}^2$ and $x \in \mathbb{R}^2$, has been computed in [4][5][6]. In fact, the derivation of the Green’s function in dimension two does not require an explicit calculation of (1.3), for one explores the fact that the square of spin-orbit term in the Hamiltonian is just the two-dimensional Laplace operator. In other words, the equality $p^2 = p_1^2 + p_2^2$ is critical in dimension two. This is no longer the case in dimension three, and thus the problem stands in need of different computational methods.

In this paper, a hypergeometric series expansion for (1.2)–(1.4) is provided (§§1.6, Eq. (44), §1.7, Eq. (16), Eq. (18)), the case when $\alpha = 0$ is much more involved.

Convergence conditions on $\alpha$, $\beta$, $\zeta$ are studied in detail. Some properties and special cases of the series are also discussed (§§1.6, 1.7).

2. Notation and terminology

The hypergeometric series to be used are the Kampé de Fériet function $F_{p,q;k}^{p,q;k}$, the generalized Lauricella function of two variables $F_{A,B;C,D}^{A,B} \subset C,D$, the complete and confluent Horn, Appell, Humbert functions.

The Kampé de Fériet function $F_{p,q;k}^{p,q;k}$ is defined according to [7] §1.7, Eq. (16)]

$$F_{p,q;k}^{p,q;k}(a_1, \ldots, a_p; \beta_1, \ldots, \beta_m; \gamma_1, \ldots, \gamma_n; \zeta_1, \zeta_2) = \sum_{\epsilon_1, \epsilon_2=0}^\infty \prod_{j=1}^p \epsilon_1^{a_j} \prod_{j=1}^q \epsilon_2^{b_j} \prod_{j=1}^k \epsilon_3^{c_j} \zeta_1^\epsilon_1 \zeta_2^\epsilon_2 \prod_{j=1}^m \epsilon_1^{d_j} \prod_{j=1}^m \epsilon_2^{e_j} \prod_{j=1}^n \epsilon_3^{f_j} \zeta_1^\epsilon_3 \zeta_2^\epsilon_4 \frac{\epsilon_1! \epsilon_2! \epsilon_3! \epsilon_4!}{r!s!}$$

where the parentheses indicate the Pochhammer symbol. In the present paper, the Kampé de Fériet functions to be used fulfill $p + q < l + m + 1$, $p + k < l + n + 1$. This ensures the convergence for all $|\zeta_1|, |\zeta_2| < \infty$.

The generalized Lauricella function of two variables $F_{A,B;C,D}^{A,B}$ is due to Srivastava–Daoust [7] §1.7, Eq. (18),

$$F_{A,B;C,D}^{A,B}(\epsilon_1, \epsilon_2; \zeta_1, \zeta_2) = \sum_{\epsilon_1, \epsilon_2=0}^\infty \prod_{j=1}^A \epsilon_1^{a_j} \prod_{j=1}^B \epsilon_2^{b_j} \prod_{j=1}^C \epsilon_1^{c_j} \zeta_1^\epsilon_1 \zeta_2^\epsilon_2 \prod_{j=1}^D \epsilon_1^{d_j} \prod_{j=1}^D \epsilon_2^{e_j} \prod_{j=1}^D \epsilon_1^{f_j} \zeta_1^\epsilon_3 \zeta_2^\epsilon_4 \frac{\epsilon_1! \epsilon_2! \epsilon_3! \epsilon_4!}{r!s!}$$

where the coefficients

$$(\theta_j; j = 1, \ldots, A), \quad (\theta_j'; j = 1, \ldots, A), \ldots, (\theta_j'; j = 1, \ldots, D')$$

are real and positive, and $(a)$ abbreviates the array of $A$ parameters $a_1, \ldots, a_A$, with similar interpretation for $(b), \ldots, (d')$. In the present paper, the coefficients are such that the convergence is ensured for all $|\zeta_1|, |\zeta_2| < \infty$. For other cases, one is referred to [8][9].

The complete Horn $H_1$, function [10] §5.7.1, Eq. (15)], the confluent Horn $H_2$ and $H_0$ functions [10] §5.7.1, Eqs. (31) and (38)], the confluent Appell (or Humbert) $\Xi_2$ function [7] §1.6, Eq. (44)] obey the following series representation

$$H_3(a, b; c; \zeta_1, \zeta_2) = \sum_{m,n=0}^\infty \frac{(a)_{2m+n}(b)_{2n}}{(c)_{m+n}} \zeta_1^m \zeta_2^n \frac{m! n!}{(m+n)!}, \quad |\zeta_1| < R, \quad |\zeta_2| < S,$$
\[ R + \left( S - \frac{1}{2} \right)^2 = \frac{1}{4}, \]

\[ H_3(a, b; c; \zeta_1, \zeta_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m \xi_1^m \xi_2^n}{(c)_m m! n!}, \quad |\zeta_1| < 1, \quad |\zeta_2| < \infty, \]

\[ H_{10}(a; c; \zeta_1, \zeta_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m-n} \xi_1^m \xi_2^n}{(c)_{m} m! n!}, \quad |\zeta_1| < \frac{1}{4}, \quad |\zeta_2| < \infty, \]

\[ \Xi_2(a, b; c; \zeta_1, \zeta_2) = \sum_{m,n=0}^{\infty} \frac{(a)_{m}(b)_{m} \xi_1^m \xi_2^n}{(c)_{m+n} m! n!}, \quad |\zeta_1| < 1, \quad |\zeta_2| < \infty, \]

for suitable parameters \( a, b, c \in \mathbb{C} \).

Various properties of the Appell and Horn confluent functions are derived in [7][10][11].

Throughout the paper the absence of parameters in either series is left blank and in that case the value of an empty product is unity.

### 3. Main Results

**Lemma 3.1.** Let \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \) and define the triple series

\[ X(a, b; c) = \sum_{m,n,p=0}^{\infty} \frac{\xi_1^m \xi_2^n \xi_3^p}{m! p! (a)_{2m+n+p} (b)_{m+n}}, \]

\( \forall a, b \in \mathbb{C} - \{ -n : n \in \mathbb{N}_0 \} \). Then, the series (3.1) takes the following equivalent representations

\[ X(a, b; c) = \sum_{n=0}^{\infty} \frac{\xi_1^n}{n!(a)_{2n}(b)_n} F_{0,1:0}^{2,1:0} \left( \begin{array}{c} \zeta_1 \\
\end{array} \right| \begin{array}{c} \zeta_2, \zeta_3 \\
\end{array} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{\xi_2^n}{n!(a)_{n}(b)_n} F_{0,0:0}^{1,1:0} \left( \begin{array}{c} \zeta_1 + n \\
\end{array} \right| \begin{array}{c} \zeta_2, \zeta_3 \\
\end{array} \right) \]

\[ = \sum_{n=0}^{\infty} \frac{\xi_3^n}{n!(a)_{n}(b)_n} F_{0,0:0}^{1,1:0} \left( \begin{array}{c} \zeta_1 + n \\
\end{array} \right| \begin{array}{c} \zeta_2, \zeta_3 \\
\end{array} \right) \]

\( \forall |\zeta| < \infty \). Moreover, \( X \) fulfills the recurrence relation

\[ X(a, b; c) - \frac{\xi_2}{ab} X(a + 1, b + 1; c) = F_{0,0:0}^{2,0:0} \left( \begin{array}{c} \zeta_1 + n \\
\end{array} \right| \begin{array}{c} \zeta_2, \zeta_3 \\
\end{array} \right) \]

**Lemma 3.2.** Let \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \) and define the triple series

\[ X'(a, b; c) = \sum_{m,n,p=0}^{\infty} \frac{\xi_1^m \xi_2^n \xi_3^p}{(m-n)! p! (b)_{m+n}}, \]

\( \forall a \in \mathbb{C} - \mathbb{N}, \forall b \in \mathbb{C} - \{ -n : n \in \mathbb{N}_0 \} \). Then, the series (3.4) is absolutely convergent if:

\[ X'(a, b; c) = \sum_{n=0}^{\infty} \frac{\xi_1^n (a)_{2n}}{n!(b)_n} \Xi_2(1, -n; 1 - a - 2n; \zeta_2, -\zeta_3), \]

\( |\zeta_1| < \frac{1}{4}, \quad |\zeta_2| < 2 \) or

\[ |\zeta_1| = \frac{1}{4}, \quad |\zeta_2| < 2, \quad \text{Re} \left( a - b - \frac{1}{2} \right) < 0 \]
\[
\sum_{n=0}^{\infty} \frac{(\zeta_1 \zeta_2)^n(a)}{(b)_n} h_{10}(a + n; b + n; \zeta_1, \zeta_3),
\]

\[|\zeta_1| < \frac{1}{4}, \quad |\zeta_2| < \frac{1 + \sqrt{1 - 4|\zeta_1|}}{2|\zeta_1|} \quad \text{or} \]

\[|\zeta_1| < \frac{1}{4}, \quad |\zeta_2| = \frac{1 + \sqrt{1 - 4|\zeta_1|}}{2|\zeta_1|}, \quad \text{Re}(a - b) < 0 \]

(3.5b)

\[= \sum_{n=0}^{\infty} \frac{(-\zeta_3)^n}{n!(1 - a)_n} h_3(a - n, 1; b; \zeta_1, \zeta_1 \zeta_2), \]

(3.5c)

\[|\zeta_1| < R, \quad |\zeta_1 \zeta_2| < S, \quad R + \left( S - \frac{1}{2} \right)^2 = \frac{1}{4}. \]

\[\forall |\zeta_3| < \infty. \quad \text{Moreover, } X' \text{ fulfills the recurrence relation} \]

\[X'(a, b; \zeta) - \frac{a}{b} \zeta_1 \zeta_2 X'(a + 1, b + 1; \zeta) = h_{10}(a; b; \zeta_1, \zeta_3), \]

(3.6)

\[|\zeta_1| < \frac{1}{4} \quad \text{The confluence on a pair } (\zeta_1, \zeta_2) \text{ implies that} \]

\[\lim_{\epsilon \to 0} X' \left( a, b; \epsilon \zeta_1, \frac{\zeta_2}{\epsilon}, \zeta_3 \right) = h_3(a, 1; b; \zeta_1 \zeta_2, \zeta_3), \]

|\zeta_1 \zeta_2| < 1.

**Theorem 3.3.** Let \( \alpha, \beta \geq 0, \ p = (p_1, p_2, p_3) \in \mathbb{R}^3, \ x \in \mathbb{R}^3, \ r = |x|. \) Suppose that \( \mathbb{C} \setminus [-\Sigma, \infty) \ni \zeta \) meets at least one set of the following three:

(a) \( 2\beta > \alpha^2, \quad \beta \leq |\zeta| < 2 \left( \frac{\beta}{\alpha^2} \right)^{\frac{3}{2}} \)

(b) \( |\zeta| \geq \Sigma; \text{ the equality is available only if } 0 \leq 2\beta < \alpha^2 \)

(c) \( |\zeta| > \max \left( \frac{\beta}{2 \sqrt{R}}, \frac{\alpha^2}{4S} \right), \ R + \left( S - \frac{1}{2} \right)^2 = \frac{1}{4}. \)

Then

\begin{align*}
G_1(x; \alpha, \beta, \zeta) &= \frac{1}{8\pi \sqrt{-\zeta}} X' \left( \frac{1}{2}, \frac{3}{2}; \frac{\beta^2}{4\zeta^2}, -\frac{\zeta \alpha^2}{\beta^2}, \frac{\zeta r^2}{4} \right) \\
&\quad - \frac{r}{8\pi X} \left( \frac{1}{2}, \frac{3}{2}; \frac{\beta^2}{64}, -\frac{\alpha^2 r^2}{16}, -\frac{\zeta r^2}{4} \right), \quad r \geq 0,
\end{align*}

(3.8)

\begin{align*}
G_2(x; \alpha, \beta, \zeta) &= \frac{1}{4\pi r} X' \left( \frac{1}{2}, \frac{1}{2}; \frac{\beta^2}{64}, -\frac{\alpha^2 r^2}{16}, -\frac{\zeta r^2}{4} \right) \\
&\quad - \frac{\sqrt{-\zeta}}{4\pi} X' \left( \frac{1}{2}, \frac{1}{2}; \frac{\beta^2}{4\zeta^2}, -\frac{\zeta \alpha^2}{\beta^2}, -\frac{\zeta r^2}{4} \right), \quad r > 0.
\end{align*}

(3.9)

Conditions (a), (b) and (c) indicate that \( X' \) is given by (3.5a), (3.5b) and (3.5c), respectively. The series representation for \( X \) admits any form given in (3.2).

**Remark 3.1.** The series expansion \((3.3) - (3.9)\) for the triple integrals (1.3) remains valid for complex \( \alpha, \beta \): In this case, parameters \( \alpha, \beta \) in (a)–(c) must be replaced by their corresponding absolute values and the domain (1.4) would change due to the solutions to \( Q(p) = 0, \forall p \in \mathbb{R}^3; \) these have to be excluded from the whole plane \( \mathbb{C} \).
For particular values of parameters, the series representation (3.3)–(3.9) of integrals (1.3) can be considerably simplified. For illustrative purposes, we examine the limits $\alpha \to 0$, $\beta \to 0$, and $r \to 0$.

1. In the limit $\alpha \to 0$,

$$G_1(x;0,\beta,\zeta) = \frac{1}{8\pi} \sqrt{-\zeta} H_{10} \left( \frac{1}{2} : \frac{1}{2} ; \frac{\beta^2}{4\zeta^2} \right)$$

(4.1)

$$- \frac{r}{8\pi} F_{1;0}^{0;1,0} \left( \frac{1}{2} : 2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\zeta^2}{4} \right)$$

$\forall \beta \geq 0$, $\forall \zeta \in \mathbb{C} - [-\beta, \infty)$, $|\zeta| > \beta$, $\forall r \geq 0$. In comparison, a direct calculation of (1.3a) with $\alpha = 0$ gives (see also (3))

(4.2)

$$G_1(x;0,\beta,\zeta) = \frac{1}{8\pi\beta r} \left( e^{-r\sqrt{-\beta\zeta}} - e^{-r\sqrt{-\beta\zeta}} \right)$$

$\forall \beta \geq 0$, $\forall \zeta \in \mathbb{C} - [-\beta, \infty)$, $\forall r \geq 0$. Equation (4.2) requires a milder condition on $\zeta$ than (4.1). Define, for convenience,

$$\zeta_1 = \frac{\beta^2}{4\zeta^2}, \quad \zeta_2 = \frac{\zeta^2}{4}.$$ 

It is not hard to see from elementary algebra that, $|\zeta_1| < \frac{1}{4}$,

(4.3)

$$H_{10} \left( \frac{1}{2} : \frac{3}{2} ; \zeta_1, \zeta_2 \right) = \sum_{\sigma=1}^3 \sqrt{1 + 2\sigma \sqrt{\zeta_1}} \frac{\gamma \left( \frac{3}{2} ; \zeta_2 \right)}{2\sqrt{\zeta_1}} \frac{\varphi \left( \frac{3}{2} ; -\zeta_2 \right)}{\sqrt{\zeta_1}}.$$ 

The function $\varphi (\cdot)$ is entire. If one substituted the right-hand side of (4.3) in (4.1), the condition $|\zeta| > \beta$ would be removed by analytic continuation.

Similar considerations apply to $G_2$ at $\alpha = 0$. By Theorem 3.3,

(4.4)

$$G_2(x;0,\beta,\zeta) = \frac{1}{8\pi r} \left( e^{-r\sqrt{-\beta\zeta}} + e^{-r\sqrt{-\beta\zeta}} \right)$$

$\forall \beta \geq 0$, $\forall \zeta \in \mathbb{C} - [-\beta, \infty)$, $|\zeta| > \beta$, $\forall r > 0$. A direct calculation of (1.3b) with $\alpha = 0$ gives

(4.5)

$$G_2(x;0,\beta,\zeta) = \frac{1}{8\pi r} \left( e^{-r\sqrt{-\beta\zeta}} + e^{-r\sqrt{-\beta\zeta}} \right)$$

$\forall \beta \geq 0$, $\forall \zeta \in \mathbb{C} - [-\beta, \infty)$, $\forall r > 0$. Again, in view of

$$H_{10} \left( -\frac{1}{2} : \frac{1}{2} ; \zeta_1, \zeta_2 \right) = \sum_{\sigma=1}^3 \frac{\sqrt{1 + 2\sigma \sqrt{\zeta_1}}}{2} \frac{\gamma \left( \frac{3}{2} ; \zeta_2 \right)}{2\sqrt{\zeta_1}} \frac{\varphi \left( \frac{3}{2} ; -\zeta_2 \right)}{\sqrt{\zeta_1}}$$

$|\zeta_1| < \frac{1}{4}$, the additional condition $|\zeta| > \beta$ in (4.4) can be relaxed.

2. In the limit $\beta \to 0$, $\forall \alpha \geq 0$, $\forall \zeta \in \mathbb{C} - [-\frac{1}{4}\alpha^2, \infty)$, $|\zeta| > \frac{1}{4}\alpha^2$,

(4.6)

$$G_1(x;\alpha,0,\zeta) = \frac{1}{8\pi \sqrt{-\zeta}} H_3 \left( \frac{1}{2} : \frac{3}{2} : \frac{3}{2} ; \frac{\alpha^2}{4\zeta} \frac{\zeta^2}{4} \right)$$

$$- \frac{r}{8\pi} F_{1;0}^{0;1,0} \left( \frac{3}{2} : \frac{3}{2} ; \frac{3}{2} ; \frac{\alpha^2}{16} \frac{\zeta^2}{4} \right), \quad r \geq 0,$$
\[ G_2(x; \alpha, 0, \zeta) = \frac{1}{4\pi r} F_{0;1;0}^{1;1;0} \left( \begin{array}{c} 1; 1; \frac{\alpha^2 r^2}{4\zeta^2} \\ \zeta^2 \end{array} ; -\frac{\zeta^2}{4} \right) - \frac{\sqrt{\zeta}}{4\pi} H_3 \left( \begin{array}{c} 1/2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\alpha^2}{4\zeta^2} \\ \zeta \end{array} ; -\frac{\zeta^2}{4} \right), \quad r > 0. \]

(4.7)

Taking both \( \alpha = \beta = 0 \), we find from (4.1)–(4.7) that

\[ G_1(x; 0, 0, \zeta) = \frac{e^{-r\sqrt{-\zeta}}}{8\pi \sqrt{-\zeta}} \quad (r \geq 0), \quad G_2(x; 0, 0, \zeta) = \frac{e^{-r\sqrt{-\zeta}}}{4\pi r} \quad (r > 0) \]

\( \forall \zeta \in \mathbb{C} - [0, \infty) \). As is seen, function \( G_2(x; 0, 0, \zeta) \) is the Green’s function for the three-dimensional kinetic energy operator (recall (1.1)).

3. In physical applications, the limit \( r \to 0 \) in Green’s function is usually necessary for the calculation of point spectrum of the operator perturbed by the point-interaction \([14, 15, 16, 17]\).

By the theorem,

\[ G_1(0; \alpha, \beta, \zeta) = \frac{1}{8\pi \sqrt{-\zeta}} H_3 \left( \begin{array}{c} 1/2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\alpha^2}{4\zeta^2} \\ \zeta \end{array} ; -\frac{\zeta^2}{4} \right) \]

where \( \zeta \) meets (1.4) and (c) in Theorem 3.3. An additional condition (c) ensuring the convergence of the complete Horn \( H_3 \) can be relaxed on account of

\[ H_3 \left( \begin{array}{c} 1/2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\alpha^2}{4\zeta^2} \\ \zeta \end{array} ; -\frac{\zeta^2}{4} \right) = \frac{2}{\alpha} \sqrt{-\zeta} \arctanh \left( \frac{\alpha}{\beta} \sqrt{\frac{-\zeta}{2} \left( 1 - \sqrt{1 - \left( \frac{\beta}{\zeta} \right)^2} \right)} \right). \]

The inverse hyperbolic tangent \( \arctanh \) is defined on \( \mathbb{C} - [-1, 1] \); the latter is always satisfied for \( \zeta \) as in (1.4).

Function \( G_2 \) exists \( \forall x \in \mathbb{R}^3 - \{0\} \), but its renormalized counterpart

\[ G_2^{\text{ren}}(x; \alpha, \beta, \zeta) = G_2(x; \alpha, \beta, \zeta) - \frac{e^{-r\sqrt{-\zeta}}}{4\pi r} \]

exists \( \forall x \in \mathbb{R}^3 \). Indeed, by the theorem,

\[ G_2^{\text{ren}}(0; \alpha, \beta, \zeta) = \frac{\sqrt{-\zeta}}{4\pi} - \sqrt{-\zeta} H_3 \left( \begin{array}{c} 1/2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\alpha^2}{4\zeta^2} \\ \zeta \end{array} ; -\frac{\zeta^2}{4} \right) \]

provided that \( \zeta \) is as in (1.4) and Theorem 3.3(c). Likewise, condition (c) can be omitted if noting that

\[ H_3 \left( \begin{array}{c} 1/2; 1; \frac{\beta^2}{4\zeta^2}; \frac{\alpha^2}{4\zeta^2} \\ \zeta \end{array} ; -\frac{\zeta^2}{4} \right) = \frac{1}{2} \left( 1 + \sqrt{1 - \left( \frac{\beta}{\zeta} \right)^2} \right) \]

\[ - \frac{\alpha}{\sqrt{-\zeta}} \arctanh \left( \frac{\alpha}{\beta} \sqrt{\frac{-\zeta}{2} \left( 1 - \sqrt{1 - \left( \frac{\beta}{\zeta} \right)^2} \right)} \right). \]

The function \( G_2^{\text{ren}} \) (4.9)–(4.10) appears explicitly in the theory of singular perturbations [16] when dealing with self-adjoint extensions of operators with point-interaction; see also §8.
5. Demonstration of main results

The proof of Lemma 3.1 is straightforward and thus omitted: It requires nothing more than the definition of the Kampé de Fériet and Lauricella functions and elementary rearrangement of summands due to Pochhammer symbol.

The proof of Lemma 3.2 is more involved.

Proof.

Step (Series representation). To prove (3.5a), substitute

\[
(a)_{2m-n-p} = \frac{(-1)^p(a)_{2m}}{(1 - a - 2m)_{n+p}}, \quad (m-n)! = (1)_{m-n} = \frac{(-1)^p(1)_{m}}{(-m)_n}
\]

in (3.4) and apply the series representation of \( \Xi_2 \).

To prove (3.5b), let \( m = n + p + q \). Then \( q = -n - p, -n - p + 1, \ldots \) But then

\[
\frac{1}{(1)_{m-n}} = \frac{1}{(1)_{p+q}} = 0 \quad \text{for} \quad p + q = -1, -2, \ldots
\]

Turns out that \( q = -p, -p + 1, \ldots \) and, by (3.4),

\[
X'(a; b; \zeta) = \sum_{n,p=0}^{\infty} \sum_{q=-p}^{p} \frac{(\zeta_1 \zeta_2)^p (\zeta_1 \zeta_3)^q}{n!} \frac{(1)_{n}(a)_{2q+n+p}}{p! (b)_{n+p+q}(1)_{p+q}}.
\]

Let \( q = l - p \). Then \( l = 0, 1, \ldots \) and

\[
X'(a; b; \zeta) = \sum_{n=0}^{\infty} \sum_{l,p=0}^{\infty} \frac{(\zeta_1 \zeta_2)^p (\zeta_1 \zeta_3)^q}{l! p!} \frac{(a)_{2l+n-p}}{(b)_{n+p} (a)_{2l+n}}.
\]

The double sum over \( l, p \) represents the confluent Horn \( H_{10} \), thus showing (3.5b).

To prove (3.5c), note that

\[
\frac{(a)_{2l+n-p}}{(b)_{n+p}} = \frac{(-1)^p(a-p)_{2l+n}}{(1-a)_{p}(b)_{n+p}}.
\]

Substitute the right-hand side in (5.1) and get that the double sum over \( l, n \) is the complete Horn \( H_3 \) as in (3.5c).

Step (Convergence). The formula to be used is [18 §2.11, Eq. (4)]

\[
\ln \Gamma(z + a) = \left(z + a - \frac{1}{2}\right) \ln(z) - z + \frac{1}{2} \ln(2\pi) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} B_{k+1}(a)}{k(k+1)} z^{-k},
\]

\[|\arg(z)| \leq \pi - \epsilon, \quad \epsilon > 0, \quad a, z \in \mathbb{C}, \]

where \( B_{k+1}(a) \) is the Bernoulli polynomial in \( a \) of degree \( k + 1 \); \( B_k(0) \equiv B_k \) is the Bernoulli number.

Another asymptotic formula to be used in the proof follows from (5.2): see also [8 Eq. (1.4)], [18 §2.11, Eq. (11)]:

\[
\frac{\Gamma(a+z)}{\Gamma(b+z)} = z^{a-b} \left(1 + \frac{(a-b)(a+b-1)}{2z} + O\left(z^{-2}\right)\right).
\]
The confluent Appell \( E_2 \) function in (3.5a) is a polynomial in \( \zeta_2 \) of degree \( n \). As a result, the condition \(|\zeta_2| < 1\) is slightly relaxed. Indeed, by (5.3), the Pochhammer symbol \((-n)_m \sim (-n)^m\) for \( m = 0, 1, \ldots, n \) as \( n \to \infty \). Also,

\[
\frac{1}{(1 - a - 2n)_{m+p}} = (-1)^{m+p}(2n)^{-m-p} \left( 1 + O\left( n^{-1}\right) \right), \quad \forall m, p \in \mathbb{N}_0.
\]

Subsequently,\[
\lim_{n \to \infty} E_2(1, -n; 1 - a - 2n; \zeta_2, -\zeta_3) = \frac{1}{1 - \zeta_2/2}, \quad |\zeta_2| < 2.
\]

Define

\[
a_n = \frac{\zeta_1^n(a)_{2n}}{n!(b)_n} E_2(1, -n; 1 - a - 2n; \zeta_2, -\zeta_3).
\]

Then, \( \forall|\zeta_2| < 2 \),

\[
a_n \sim \frac{\Gamma(b)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} (4\zeta_1)^n n^{a-b-\frac{1}{2}} \quad \text{as} \quad n \to \infty.
\]

But then, \( \lim a_n = 0 \) if either \(|\zeta_1| < \frac{1}{4}\) or \(|\zeta_1| = \frac{1}{4}\), \( \Re\left(a - b - \frac{1}{2}\right) < 0 \). The ratio test gives \( a_n+1/a_n \sim 4\zeta_1 \) as \( n \to \infty \). One deduces that the condition given in (3.5a) is necessary and sufficient for the absolute convergence of the series \( \sum a_n \), hence (3.5a).

The confluent Horn \( H_{10} \) in (3.5b) is of the form

\[
H_{10}(a + n; b + n; \zeta_1, \zeta_3) = \frac{\Gamma(b + n)}{\Gamma(a + n)} \sum_{m,p=0}^{\infty} \frac{\zeta_1^m \zeta_3^p}{m! p!} \frac{\Gamma(a + 2m - p + n)}{\Gamma(b + m + n)}.
\]

\(|\zeta_1| < \frac{1}{4}\). Apply (5.2) to \( \Gamma(a + 2m - p + n) \) and \( \Gamma(b + m + n) \) to obtain

\[
H_{10}(a + n; b + n; \zeta_1, \zeta_3) \sim \frac{\Gamma(b + n)}{\Gamma(a + n)} e^{-b} \sum_{m,p=0}^{\infty} \frac{(n\zeta_1)^m (\zeta_3/n)^p}{m! p!} \left( B_{k+1}(a + 2m - p) - B_{k+1}(b + m) \right).
\]

In the limit \( n \to \infty \), \( (\zeta_3/n)^p/p! \sim 0 \) for \( p \in \mathbb{N} \), and \( = 1 \) for \( p = 0 \). Hence, \( \text{put } p = 0 \) in the above equation and deduce that

\[
H_{10}(a + n; b + n; \zeta_1, \zeta_3) \sim F_1\left(\frac{a + n + b + n}{b+n}; \zeta_1; 4\zeta_1\right) \quad \text{as} \quad n \to \infty.
\]

Define, for convenience,

\[
\alpha = \frac{a}{2}, \quad \beta = 1 + \frac{a}{2} - b, \quad \gamma = \frac{1}{2}, \quad \zeta = 4\zeta_1, \quad \lambda = \frac{n}{2}.
\]

In [13] §7.2, Eq. (11) it was shown that, for complex \( \lambda \),

\[
F_1\left(\frac{\alpha + \lambda, 1 + \alpha - \gamma + \lambda}{1 + \alpha - \beta + 2\lambda}; \zeta\right) \sim \frac{\zeta^{\lambda-1}}{\sqrt{1 - \zeta}} \frac{\sqrt{\Gamma(1 + \alpha - \beta + 2\lambda)}}{\sqrt{\Gamma(1 + \alpha - \gamma + \lambda)\Gamma(\gamma - \beta + \lambda)}} \times \frac{1 - \zeta}{(1 - \zeta)^{\gamma-\frac{1}{2}}} \quad \text{as} \quad |\lambda| \to \infty.
\]

\(|\arg(\lambda)| \leq \pi - \delta, \delta > 0 \). Substitute (5.5), (5.6) in (5.4) and get that, \( \forall|\zeta_1| < \frac{1}{4}, \)

\[
H_{10}(a + n; b + n; \zeta_1, \zeta_3) \sim 2^{1-b-\frac{1}{2}} \zeta_1^{-1-b} \left( 4\zeta_1 - 1 + \sqrt{1 - 4\zeta_1} \right)^{b-a-1}
\]
as \( n \to \infty \). Define
\[
  b_n = \frac{(\zeta_1 \zeta_2)^n (a)_n}{(b)_n} H_{10}(a + n; b + n; \zeta_1, \zeta_2),
\]
apply (5.3) to \((a)_n/(b)_n\), and deduce from the above asymptotic formula that the series \( \sum b_n \) is absolutely convergent if
\[
  |\zeta_2 \sqrt{1 - 2\zeta_1} - \sqrt{1 - 4 \zeta_1}| \leq \sqrt{2},
\]
the equality holds only if \( \text{Re}(a - b) < 0 \), hence (3.5b). The condition is necessary and sufficient.

The complete Horn function \( H_3 \) can be represented as follows
\[
  H_3(a - n, 1; b; x, y) = \sum_{m,p=0}^{\infty} C_{mp} x^m y^p,
\]
\[
  x = \zeta_1, \quad y = \zeta_1 \zeta_2, \quad |x| < R, \quad |y| < S,
\]
\[
  R + \left( S - \frac{1}{2} \right)^2 = \frac{1}{4}.
\]
It will be shown that the condition for the convergence of \( H_3 \) is necessary and sufficient for the absolute convergence of the third series (5.5c).

Extract \( H_3 \) as a sum \( X_n + Y_n \), where
\[
  X_n = \sum_{m=0}^{\infty} \sum_{p=0}^{n} C_{mp} x^m y^p, \quad Y_n = \sum_{m=0}^{\infty} \sum_{p=n+1}^{\infty} C_{mp} x^m y^p.
\]
The first term explicitly reads
\[
  X_n = \frac{\Gamma(b)}{\Gamma(a - n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \left( \frac{\Gamma(a + 2m - n)}{\Gamma(b + m + 1)} y + \frac{\Gamma(a + 2m + 1 - n)}{\Gamma(b + m + 2)} y^2 + \ldots + \frac{\Gamma(a + 2m - 1)}{\Gamma(b + m + n - 1)} y^{n-1} + \frac{\Gamma(a + 2m)}{\Gamma(b + m + n)} y^n \right)
\]
\[
  \sim \frac{\Gamma(b)}{\Gamma(a - n)} \sum_{m=0}^{\infty} \frac{x^m}{m!} \frac{\Gamma(a + 2m - n)}{\Gamma(b + m)} \text{ as } n \to \infty
\]
for it holds \( |y| < 1 \). Hence,
\[
  X_n \sim \binom{a-n}{b} \binom{a+1+n}{b+1} : 4x \quad \text{as } n \to \infty, \quad |x| < \frac{1}{4},
\]
By the Kummer transformation formula [18] §3.8, Eqs. (1)–(3),
\[
  \binom{a-n}{b} \binom{a+1+n}{b+1} : 4x = (1 - 4x)^{\frac{a+n}{2}} \binom{b - \frac{a}{2}}{b} \binom{a+1+n}{b+1} : 4x - 1.
\]
Define, for convenience,
\[
  \alpha = b - \frac{a}{2}, \quad \beta = \frac{a+1}{2}, \quad \gamma = b, \quad \zeta = \frac{1 + 4x}{1 - 4x}, \quad \lambda = \frac{n}{2}.
\]
Then, for large complex \( \lambda \), it was shown that [18 §7.2, Eq. (8)],

\[
zF_1 \left( \alpha + \lambda, \beta - \lambda ; \frac{1 - \zeta}{2} \right) \sim \frac{2^{\alpha + \beta - 1} \Gamma(1 - \beta + \lambda) \Gamma(\gamma)}{\sqrt{\pi} \lambda \Gamma(\beta - \lambda) \\
\times \left( 1 + \zeta - \sqrt{\zeta^2 - 1} \right)^{\gamma - \alpha - \beta - \frac{1}{2}} \left( 1 - \zeta + \sqrt{\zeta^2 - 1} \right)^{\gamma - \beta} \right)
\]

\[(5.10) \quad + e^{\pm i(\gamma - \frac{1}{2})(\zeta + \sqrt{\zeta^2 - 1})^{\frac{1}{2} - \alpha}} \quad \text{as} \quad |\lambda| \to \infty
\]

where the upper (lower) sign in the exponent is taken if \( \text{Im} \zeta > 0 \) (\( \text{Im} \zeta < 0 \)), and

\[-\frac{\pi}{2} - \omega_2 + \delta < \arg(\lambda) < \frac{\pi}{2} + \omega_1 - \delta, \quad \delta > 0,
\]

\[
\omega_1 = -\arctan \left( \frac{\nu - \pi}{\mu} \right), \quad \omega_2 = \arctan \left( \frac{\nu}{\mu} \right), \quad \nu \geq 0,
\]

\[
\omega_1 = -\arctan \left( \frac{\nu}{\mu} \right), \quad \omega_2 = \arctan \left( \frac{\nu + \pi}{\mu} \right), \quad \nu \leq 0,
\]

and \( \mu = \text{Re} \xi, \nu = \text{Im} \xi, \zeta = \cosh \xi \).

In the second term, \( Y_n \), make a substitution \( p = n + 1 + q, q = 0, 1, \ldots \) and apply

\[
\Gamma(a - n) = \frac{(-1)^n \Gamma(a) \Gamma(1 - a)}{\Gamma(1 - a + n)}.
\]

Then

\[
Y_n = \frac{(-1)^n y^a \Gamma(b)}{\Gamma(a) \Gamma(1 - a)} \sum_{m, q = 0}^{\infty} \frac{\zeta^m}{m!} \frac{y^q n^{-a - b - m - q}}{\Gamma(a + 1 + 2m + q)}.
\]

Apply (5.3) and get that

\[
Y_n \sim \frac{(-1)^n y^{a+1}}{\Gamma(1 - a)} \sum_{m, q = 0}^{\infty} \frac{x^m y^q n^{-a-b-m-q}}{m!} \Gamma(a + 1 + 2m + q)
\]

as \( n \to \infty \). Hence,

\[(5.11) \quad Y_n \sim \frac{a \Gamma(b)}{\Gamma(1 - a)} (-1)^n y^{a+1} n^{-a-b} \quad \text{as} \quad n \to \infty.
\]

Substitute (5.8)-(5.10) in (5.7) and exploit (5.3) to get the asymptotic for \( X_n \), then substitute this formula in \( H_3 = X_n + Y_n \), where \( Y_n \) is as in (5.11), and get that

\[
H_3(a - n, 1; b; x, y) \sim \frac{2^{2b-1} \Gamma(b) (1 - 4x)^{\frac{a-b}{2}} \left( \frac{1}{2} + \sqrt{x} \right)^{b+\frac{1}{2}}}{\sqrt{\pi} \left( 2 \sqrt{x} \right)^{b-\frac{1}{2}}}
\]

\[
\times n^{\frac{1}{2} - b} \left( \eta(x)^{\frac{a-b}{2}} + \varphi(x) \eta(x)^{\frac{a-b}{2}} \right)
\]

\[
+ \frac{a \Gamma(b)}{\Gamma(1 - a)} (-1)^n y^{a+1} n^{-a-b} \quad \text{as} \quad n \to \infty,
\]

\[
\eta(x) = \frac{1}{2} + \sqrt{x}, \quad \varphi(x) = \exp \left( i \pi \left( b - \frac{1}{2} \right) \text{sgn} \text{Im} \left( 1 + 4x \right) \right).
\]

Define

\[
c_n = \frac{(-\zeta)^n}{n!(1 - a)n} H_3(a - n, 1; b; x, y)
\]
and thus deduce that $c_n \to 0$ as $n \to \infty$ for all $x, y$ in the cone $R + \left(S - \frac{1}{4}\right)^2 = \frac{1}{4}$. This completes the proof of (3.5).

The proof of (3.6)–(3.7) is straightforward. □

The proof of Theorem 3.3 relies on the following result.

**Lemma 5.1.**

$$\sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{m!x^m y^n}{(2m+1)!(m-n)!} K_{2m-n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} \left( \frac{1}{2} \right) X^\nu \left( \frac{1}{2} \right) \left( \frac{x^2}{2} \frac{y z}{2} \frac{y z}{16} - \frac{x y z}{8} - \frac{z^2}{4} \right) - \sqrt{\frac{\pi}{2z}} X^\nu \left( \frac{1}{2} \right) \left( \frac{x^2}{2} \frac{y z}{2} \frac{y z}{16} - \frac{x y z}{8} - \frac{z^2}{4} \right).$$

The variables $(x, y, z) \in \mathbb{C}^3$, with $|z| < \infty$, fulfill at least one of the following three conditions:

(i) $4|x| \leq |x|^2$, $|y| < 4$

(ii) $4|x| < |x|^2$, $|y| \leq |z| + \sqrt{|z|^2 - 4|x|}$

(iii) $|x| < R|z|^2$, $|y| < 2S|z|$, $R + \left(S - \frac{1}{4}\right)^2 = \frac{1}{4}$.

Items (i), (ii) and (iii) indicate that $X'$ obeys the series representation given in (3.5a), (3.5b) and (3.5c), respectively. The series representation for $X$ admits any form given in (3.2).

Here $K_{2m-n+\frac{1}{2}}$ is the Macdonald function.

**Proof.** First, note that (i), (ii) and (iii) are due to (3.5a), (3.5b) and (3.5c), respectively, by setting $\zeta_1 = \frac{x}{2}$, $\zeta_2 = \frac{x}{2}$, $\zeta_3 = -\frac{y}{4}$. The conditions ensure that both series in the lemma are absolutely convergent.

Substitute the series representation [19] §V.5.3, Eq. (3)] of Macdonald function

$$K_{\nu}(z) = \frac{1}{2} \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\nu-l)}{l!} \left( \frac{z}{2} \right)^{-\nu+2l} + \frac{1}{2} \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\nu-l)}{l!} \left( \frac{z}{2} \right)^{\nu+2l}$$

($\nu = 2m - n + \frac{1}{4}$) in the left-hand side of the first series in the lemma and get the expression

$$\sum_{m,n,l=0}^{\infty} \frac{(4x/2)^m (yz/2)^n (-z^2/4)^l}{m! n! l!} \left( \frac{(m!)^2 n! \Gamma \left( 2m - n + \frac{1}{2} - l \right)}{(2m+1)! \Gamma \left( m - n + 1 \right)} \right)$$

$$+ \frac{1}{2} \sum_{m,n,l=0}^{\infty} \frac{(x z^2/4)^m (y z/2)^n (-z^2/4)^l}{m! n! l!} \left( \frac{(m!)^2 n! \Gamma \left( 2m + n - \frac{1}{2} - l \right)}{(2m+1)! \Gamma \left( m - n + 1 \right)} \right).$$

Write

$$\frac{(m!)^2 n! \Gamma \left( 2m - n + \frac{1}{2} - l \right)}{(2m+1)! \Gamma \left( m - n + 1 \right)} = \frac{(1)n(n)\left(\frac{3}{2}\right)_{2m-n-l}}{4^m \left(\frac{3}{2}\right) n(n)\left(\frac{3}{2}\right)_{2m-n-l}} \sqrt{n},$$

$$\frac{(m!)^2 n! \Gamma \left( 2m + n - \frac{1}{2} - l \right)}{(2m+1)! \Gamma \left( m - n + 1 \right)} = \frac{2 \sqrt{n} (1)n(n)\left(\frac{3}{2}\right) n(n)\left(\frac{3}{2}\right)_{2m-n-l}}{4^m \left(\frac{3}{2}\right) n(n)\left(\frac{3}{2}\right)_{2m-n-l}}.$$
and get the expression
\[
\sqrt{\frac{\pi}{2z}} \sum_{m,n,l=0}^{\infty} \frac{(x/z^2)^m (yz/2)^n (-z^2/4)^l}{m! n! l!} \frac{(1/2)^m (1/2)^{2m-n-l}}{(1/2)_m (1/2)_{m-n}}.
\]

The first triple series represents \( X' \) (3.4) with
\[
a = \frac{1}{2}, \quad b = \frac{3}{2}, \quad \zeta_1 = \frac{x}{z^2}, \quad \zeta_2 = \frac{yz}{2}, \quad \zeta_3 = -\frac{z^2}{4}.
\]

The second triple series can be rewritten thus: Make a substitution \( l \to l - m \) for \( l = m, m+1, \ldots \) and get the series
\[
\sum_{m,n,l=0}^{\infty} \frac{(x/4)^m (-2y/z)^n (z^2/4)^l}{m! n! l!} \frac{(1/2)_m (1/2)_{m-n}}{(1/2)^{m-n}_m (1/2)_{m-n+l}}.
\]

But \( 1/(1/2)_1 = 0 \) for \( l = 0, 1, \ldots, m-1 \). Thus, the sum over \( l = m, m+1, \ldots \) can be replaced with the sum over \( l = 0, 1, \ldots \). Next, let \( m = p + n, l = m + q \). Then \( l = n + p + q \), \( p = -n, -n + 1, \ldots, q = 0, 1, \ldots \) and the general term of the above series obey the form
\[
\frac{a^n b^p c^q}{m! n! l!} \frac{(1/2)_m (1/2)_{m-n}}{(1/2)^{m-n}_m (1/2)_{m-n+l}} = \frac{(ac)^p (abc)^q}{p! n! q!} \frac{(1/2)_l (1/2)_{m-n+l}}{(1/2)_m (1/2)_{m+n}}
\]
\((a = x/4, b = -2y/z, c = z^2/4)\). But \( 1/p! = 0 \) for \( p = -n, -n + 1, \ldots, -1 \). Hence, the sum over \( p = -n, -n + 1, \ldots \) can be replaced with the sum over \( p = 0, 1, \ldots \) Relabeling \( p \) with \( m \) and \( q \) with \( l \), one derives the series
\[
\sum_{m,n,l=0}^{\infty} \frac{(xz^2/16)^m (-xyz/8)^n (z^2/4)^l}{m! n! l!} \frac{(1/2)^m (1/2)_{m+n}}{(1/2)_m (1/2)_{m+n}}
\]
which is \( X \) (3.1) with
\[
a = \frac{3}{2}, \quad b = \frac{3}{2}, \quad \zeta_1 = \frac{x^2}{16}, \quad \zeta_2 = -\frac{xyz}{8}, \quad \zeta_3 = -\frac{z^2}{4}.
\]

This demonstrates the first series in the lemma. The proof of the second one is omitted. For, the derivation of the second series requires no additional ideas but those presented above (in the series representation of Macdonald function put \( \nu = 2m - n - \frac{1}{2} \) and proceed identically as before).

To accomplish the proof of the theorem, one needs to show that the integrals (3.8)–(3.9) obey the series representation given in Lemma 5.1.

For this, rewrite \( Q(p^{-1}) \) with the help of binomial series
\[
\frac{1}{(p^2 - \zeta)^2 - a^2 (p_1^2 + p_2^2) - b^2} = \sum_{m=0}^{\infty} \frac{(\beta^2 + a^2 (p_1^2 + p_2^2))^m}{(p^2 - \zeta)^{2m+2}},
\]
\[
(\beta^2 + a^2 (p_1^2 + p_2^2))^m = \sum_{n=0}^{m} \binom{m}{n} a^{2n} b^{2m-2n} (p_1^2 + p_2^2)^n, \quad \forall m \in \mathbb{N}_0.
\]

The convergence of the series is ensured by the convergence of \( X' \) as it will be shown below.
Rewrite \( p = (p_1, p_2, p_3) \in \mathbb{R}^3 \) in spherical coordinates \((k, \vartheta, \varphi), k = |p|\), and substitute the series representation of \( Q(p)^{-1} \) in the left-hand side of (3.8). Then

\[
(3.8) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\pi d\vartheta \sin \vartheta \int_0^\infty dk k^2 e^{-ikr \cos \vartheta} \times \sum_{m=0}^{\infty} \frac{1}{(k^2 - \zeta)^{2m+2}} \sum_{n=0}^{m} \binom{m}{n} a^{2n} b^{2m-2n} k^{2n} \sin 2n \vartheta.
\]

But

\[
\int_0^{2\pi} d\varphi \ e^{-ikr \cos \vartheta} \sin 2n+1 \vartheta = \frac{\sqrt{\pi n!}}{\Gamma(n + \frac{1}{2})} \, \mathbf{F}_1 \left( n + \frac{3}{2}, \frac{1}{4} k^2 r^2 \right),
\]

\[
\int_0^\infty dk k^{2n+2} \mathbf{F}_1 \left( n + \frac{1}{2}, \frac{k^2 r^2}{4} \right) = \frac{\Gamma(n + \frac{1}{2})}{(2m + 1)!} (-\zeta)^{\frac{1}{2} - m + n/2} \left( \frac{r}{2} \right)^{\frac{1}{2} + 2m - n} \times K_{2m-n+\frac{1}{2}} \left( r \sqrt{-\zeta} \right).
\]

Hence,

\[
(3.8) = \frac{\sqrt{r}}{4\pi \sqrt{2\pi} \sqrt{\zeta}} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{m! x^m y^n}{(2m + 1)! (m - n)!} K_{2m-n+\frac{1}{2}} (\zeta)
\]

with

\[
(5.12) \quad x = \frac{\beta^2 r^2}{4\zeta}, \quad y = \frac{2a^2 \sqrt{-\zeta}}{\beta^2 r}, \quad \zeta = r \sqrt{-\zeta}.
\]

Apply the first series in Lemma 5.1 and get the expression as in the right-hand side of (3.8).

The calculation of the second integral, (3.9), is similar, but in this case one infers the integral

\[
\int_0^\infty dk k^{2n+2} \mathbf{F}_1 \left( n + \frac{1}{2}, \frac{k^2 r^2}{4} \right) = \frac{\Gamma(n + \frac{1}{2})}{(2m + 1)!} (-\zeta)^{\frac{1}{2} - m + n/2} \left( \frac{r}{2} \right)^{\frac{1}{2} + 2m - n} \times K_{2m-n+\frac{1}{2}} \left( r \sqrt{-\zeta} \right)
\]
due to the numerator \( p^2 - \zeta \). Hence,

\[
(3.9) = \frac{\sqrt{-\zeta}}{2\pi \sqrt{2\pi r}} \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{m! x^m y^n}{(2m + 1)! (m - n)!} K_{2m-n+\frac{1}{2}} (\zeta)
\]

with \( x, y, \zeta \) as in (5.12). Apply the second series in Lemma 5.1 and get the expression as in right-hand side of (3.9).

Next, substitute

\[
(5.13) \quad \zeta_1 = \frac{\beta^2}{4\zeta^2}, \quad \zeta_2 = \frac{\zeta a^2}{\beta^2}, \quad \zeta_3 = \frac{\zeta r^2}{4}
\]
in the conditions in (3.5) and get that (3.5a) \( \Rightarrow \) (a), (3.5b) \( \Rightarrow \) (b), (3.5c) \( \Rightarrow \) (c). This completes the proof of the theorem and the main results as a whole.

6. **Further properties and corollaries**

In this paragraph, some further properties of the series defined in Lemmas 5.1–5.2 are discussed. In particular, the results will be useful for the derivation of the series representation for the off-diagonal terms of Green’s function (1.2). These terms will be discussed in the subsequent paragraph 8.7.
Proposition 6.1. Consider \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \) and define the differential operators \( \partial_j = \partial/\partial \zeta_j, \partial_{jk} = \partial_j \partial_k, \forall j, k = 1, 2, 3 \). Then, \( \forall a, b \in \mathbb{C} - \{ -n : n \in \mathbb{N}_0 \} \),

\[
\begin{align*}
(6.1) \quad & \partial_1 X(a, b; \zeta) = \frac{1}{ab(a + 1)} X(a + 2, b + 1; \zeta), \\
(6.2) \quad & \partial_2 X(a, b; \zeta) = \frac{1}{ab} X(a + 1, b + 1; \zeta) + \frac{\zeta_2}{ab} \partial_2 X(a + 1, b + 1; \zeta), \\
(6.3) \quad & \partial_3 X(a, b; \zeta) = \frac{1}{a} X(a + 1, b; \zeta), \\
(6.4) \quad & (\partial_1 + \zeta_2 \partial_{12} - \partial_{23}) X(a, b; \zeta) = 0.
\end{align*}
\]

Proof. To prove equations (6.1)–(6.3), explore the definition (3.1) and elementary properties of Pochhammer symbol; (6.1) \(\Rightarrow\) (6.4). \(\Box\)

Corollary 6.2. Let \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 \). Then, \( \forall a, b \in \mathbb{C} - \{ -n : n \in \mathbb{N}_0 \} \),

\[
0 = \sum_{n=1}^{\infty} \frac{\zeta_2^n}{(a)_n(b)_n} X(a, b; \zeta)
\]

\[
\begin{equation}
(6.5) \quad - n F^{a, 1, 0; 0, 1, 0}_{a, 2, 1} \left( (a + n; b + 1; \zeta_1, \zeta_2, \zeta_3) \right),
\end{equation}
\]

provided each series involved is absolutely convergent.

Proof. For convenience, define

\[
F(a, b; \zeta_1, \zeta_2, \zeta_3) = F^{a, 1, 0; 0, 1, 0}_{a, 2, 1} \left( (a + n; b + 1; \zeta_1, \zeta_2, \zeta_3) \right).
\]

By (6.2),

\[
\begin{align*}
\partial_2 X(a, b; \zeta) &= \frac{1}{ab} X(a + 1, b + 1; \zeta) + \frac{\zeta_2}{a(a + 1)b(b + 1)} X(a + 2, b + 2; \zeta) + \ldots \\
&= \frac{1}{ab} \sum_{n=0}^{\infty} \frac{\zeta_2^n}{(a + 1)_n(b + 1)_n} X(a + 1 + n, b + 1 + n; \zeta).
\end{align*}
\]

Then, by (6.3),

\[
\begin{align*}
\partial_2 X(a, b; \zeta) &= \frac{1}{ab} \sum_{n=0}^{\infty} \frac{\zeta_2^n}{(a + 1)_n(b + 1)_n} \frac{(a + n)(b + n)}{\zeta_2} \\
& \times \left( X(a + n, b + n; \zeta) - F(a + n, b + n; \zeta_1, \zeta_3) \right) \\
&= \frac{1}{\zeta_2} \sum_{n=0}^{\infty} \frac{\zeta_2^n}{(a)_n(b)_n} X(a + n, b + n; \zeta) - \frac{1}{\zeta_2} X(a, b; \zeta) \\
& \Rightarrow \partial_2 X(a, b; \zeta) = \sum_{n=1}^{\infty} \frac{\zeta_2^{n-1}}{(a)_n(b)_n} X(a + n, b + n; \zeta).
\end{align*}
\]

But also

\[
(6.6) \quad \partial_2 X(a, b; \zeta) = \sum_{n=1}^{\infty} \frac{n \zeta_2^{n-1}}{(a)_n(b)_n} F(a + n, b + n; \zeta_1, \zeta_3)
\]

by (3.2b). Hence, one derives (6.5). \(\Box\)
Example. Substitute $X(\xi,\eta)$ in (6.5) and get that
\[
\sum_{m,n=0}^{\infty} \frac{\xi^m \eta^n}{m!(a)_{m+n}(b)_{m+n}} \frac{t^{m+n}}{1:t} \binom{a+n+2m}{b+n+m; \xi_2, \xi_3} \\
= \sum_{n=0}^{\infty} \frac{(n+1)^2 \xi_n}{(a)_n(b)_n} \frac{t^{n}}{1:t} \binom{a+n+2}{b+n+1; \xi_1, \xi_3}.
\]
In particular, set $\xi_1 = \xi_3 = 0, \xi_2 = \xi$ in (6.7) and get a well-known series identity
\[
\sum_{n=1}^{\infty} \frac{\xi^n}{(a)_n(b)_n} \frac{1}{I_2} \binom{1}{a+n, b+n; \xi} = \frac{\xi}{ab} \frac{2}{a+b+1; \xi}.
\]
[The above equation is easy to prove by representing $I_2$ on the left-hand side as a series and then by applying the sum rule.]

Set $\xi_2 = 0$ in (6.7) and get, after relabeling the parameters (that is, $\xi_1 \to \frac{1}{2} t z^2, \sqrt{\xi_3} \to \frac{1}{2} z, a \to v + 1, b \to a$), that
\[
\sum_{n=0}^{\infty} \frac{t_n^{2n}}{n!(a)_n} \frac{1}{I_2} \binom{1}{a+n, b+n; \xi} = \frac{\xi}{ab} \frac{2}{a+b+1; \xi},
\]
where $I_2$ is the modified Bessel function. Many more series identities can be deduced from (6.7) if one notes that, for $\xi_2 = -\xi_3$, the Kampé de Fériet function $\frac{2}{1;1;1}$ reduces to the hypergeometric function $\frac{2}{1;1;1}$.

Proposition 6.3. Let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3$ and define the differential operators $\partial_j = \partial/\partial\xi_j, D_j = \xi_j \partial_j, \forall j = 1, 2, 3$. Then, $\forall a \in \mathbb{C} - \mathbb{N}, \forall b \in \mathbb{C} - \{-n: n \in \mathbb{N}_0\}$,
\[
\partial_1 X(a, b; \xi) = \frac{a}{b} \xi_2 (1 + D_1) X(a + 1, b + 1; \xi)
\]
\[
+ \frac{a(a + 1)}{b} H_1(a + 2, b + 1; \xi_1, \xi_3),
\]
\[
\partial_2 X(a, b; \xi) = \frac{a}{b} \xi_1 (1 + D_2) X(a + 1, b + 1; \xi)
\]
\[
\partial_3 X(a, b; \xi) = \frac{1}{a - 1} X(a - 1, b; \xi),
\]
\[
0 = ((D_3 + D_2 - 2 D_1 + 1 - a) \partial_3 + 1) X(a, b; \xi).
\]

Proof. To prove equations (6.8)–(6.10), explore the definition (3.4) and elementary properties of Pochhammer symbol. The implication (6.8)–(6.10)⇒(6.11) is not obvious, and thus the proof of (6.11) will be outlined below.

The easiest way to show (6.11) is to begin with the equation
\[
((D_3 + 1 - a) \partial_3 + 1) H_1(a + n; b + n; \xi_1, \xi_3)
\]
\[
= (n + 2 D_1) \partial_3 H_1(a + n; b + n; \xi_1, \xi_3)
\]
which proceeds from \cite{10} §5.9, Eq. (41)], \cite{13} Appendix A.2, Eq. (A.19'). Then, apply the operator $(D_3 + 1 - a) \partial_3 + 1$ to (3.5b) to obtain the expression
\[
((D_3 + 1 - a) \partial_3 + 1) X(a, b; \xi) = \partial_3 \sum_{n=1}^{\infty} \frac{n^2 \xi_2 \xi_3^n(a)_{n}}{(b)_n} H_1(a + n; b + n; \xi_1, \xi_3)
\]
\[
+ 2 \xi_1 \partial_3 \sum_{n=0}^{\infty} \frac{\xi_1 \xi_2 \xi_3^n(a)_{n}}{(b)_n} \partial_1 H_1(a + n; b + n; \xi_1, \xi_3).
\]
Corollary 6.4. This proves (6.11).

The first sum on the right-hand side is just \(D_2 X'\), as it is seen from (3.5b). In the second sum, substitute

\[
\xi_1^a \partial_1 H_{10} = \partial_1 \left( \xi_1^a H_{10} \right) - n \xi_1^{a-1} H_{10}
\]

and get that the sum ends up as \(\partial_1 X' - \frac{1}{\xi_1} D_2 X'\). Hence,

\[
((D_3 + 1 - a) \partial_3 + 1) X'(a, b; \xi) = \partial_1 D_2 X'(a, b; \xi)
\]

\[
+ 2 \xi_1 \partial_1 \left( \partial_1 X'(a, b; \xi) - \frac{1}{\xi_1} D_2 X'(a, b; \xi) \right).
\]

This proves (6.11).

Corollary 6.4. Let \(\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3\). Then, \(\forall a \in \mathbb{C} - \mathbb{N}, \forall b \in \mathbb{C} - \{-n: n \in \mathbb{N}_0\}\),

\[
0 = \sum_{n=1}^{\infty} \left( \frac{\xi_1 \xi_2}{b} \right)^a n (a + n, b + n; \xi) - n H_{10}(a + n; b + n; \xi_1, \xi_3),
\]

provided each series involved is absolutely convergent.

Proof. Similar to the proof of Corollary 6.2 exploit (6.9) to obtain

\[
\partial_2 X'(a, b; \xi) = \frac{1}{\xi_2} \sum_{n=1}^{\infty} \left( \frac{\xi_1 \xi_2}{b} \right)^a (a + n, b + n; \xi).
\]

But

\[
\partial_2 X'(a, b; \xi) = \frac{1}{\xi_2} \sum_{n=1}^{\infty} \left( \frac{\xi_1 \xi_2}{b} \right)^a H_{10}(a + n; b + n; \xi_1, \xi_3)
\]

by (3.5b); hence the result.

7. Series representation for Green’s function

The diagonal terms \(G_2 \pm \beta G_1\) of Green’s function \(G_R(1.2)\) obey the series representation due to Theorem 3.3. The series representation for the off-diagonal entries \(\pm \alpha D_n G_1\) can be obtained from Theorem 3.3 and Propositions 6.1 and 6.3.

In this paragraph, we shall concentrate on the functions \(D_2 G_1\) and their particular values in the limit \(r \to 0, a \to 0, \) and \(\beta \to 0\), where as before, \(r = |x|, x = (x_1, x_2, x_3) \in \mathbb{R}^3\).

With the parameters as in Theorem 3.3 \(\forall j = 1, 2, 3,\)

\[
\frac{\partial}{\partial x_j} G_1(x) = \frac{1}{8\pi} \frac{\partial}{\partial x_j} X' \left( \frac{1}{2} \frac{3}{2}; v \right) - \frac{x_j}{8\pi} X \left( \frac{3}{2} \frac{3}{2}; u \right) - \frac{r}{8\pi} \frac{\partial}{\partial x_j} X \left( \frac{3}{2} \frac{3}{2}; u \right)
\]

where the triplets \(u = (u_1, u_2, u_3) \in \mathbb{C}^3, v = (v_1, v_2, v_3) \in \mathbb{C}^3\) are given by

\[
u = \left( \frac{\beta^2 r^4}{16} - \frac{\alpha^2 r^2}{4} - \frac{\xi r^2}{4} \right), \quad v = \left( \frac{\beta^2 r^4}{16} - \frac{\alpha^2 r^2}{4} - \frac{\xi r^2}{4} \right).
\]

By (6.10) in Proposition 6.3

\[
\frac{\partial}{\partial x_j} X' \left( \frac{1}{2} \frac{3}{2}; v \right) = \frac{\partial v_3}{\partial x_j} \frac{\partial}{\partial v_3} X' \left( \frac{1}{2} \frac{3}{2}; v \right) = -\xi x_j X' \left( \frac{1}{2} \frac{3}{2}; v \right).
\]

By (6.1) and (6.3) in Proposition 6.1

\[
\frac{\partial}{\partial x_j} X \left( \frac{3}{2} \frac{3}{2}; u \right) = \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_j} \frac{\partial}{\partial u_k} X \left( \frac{3}{2} \frac{3}{2}; u \right)
\]

\[
= x_j \left( \frac{\beta^2 r^4}{45} X \left( \frac{7}{2} \frac{5}{2}; u \right) - \frac{2\xi}{3} X \left( \frac{5}{2} \frac{3}{2}; u \right) - \frac{\alpha^2}{4} \frac{\partial}{\partial u_2} X \left( \frac{3}{2} \frac{3}{2}; u \right) \right)
\]
where $\partial_2 \equiv \partial/\partial u_2$ and, by (6.6),
\begin{equation}
\partial_2 X \left( \frac{3}{2}, \frac{3}{2}; u \right) = \sum_{n=1}^{\infty} n u_2^{-1} F_{0,0}^{0,0} \left( \left( \frac{3}{2} + n : 0, 1 \right); \left( \frac{3}{2} + n : 1 \right); u_1, u_3 \right)
\end{equation}
and the series is absolutely convergent.
Combining all together,
\[
D_x G_t(x, \alpha, \beta, \zeta) = \left( \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2} \right) G_t(x, \alpha, \beta, \zeta)
\]
\[
= \frac{x_1 \pm ix_2}{8\pi} \left( \sqrt{\zeta X} \left( -1, 3/2 \cdot \frac{\beta^2 r^2}{2} - \frac{\alpha^2 r^2}{4} - \frac{\xi^2}{4} \right) \right.
\]
\[
- \frac{1}{r} \left( \frac{3}{2} \cdot \frac{\beta^2 r^2}{45} - \frac{\alpha^2 r^2}{16} - \frac{\xi^2}{4} \right)
\]
\[
- \frac{\alpha^2 r^2}{4} \left( \frac{3}{2} \cdot \frac{\beta^2 r^2}{64} - \frac{\alpha^2 r^2}{16} - \frac{\xi^2}{4} \right)
\]
\begin{equation}
\left. \right) \left( \frac{3}{2} : 2, 1 \right); \left( \frac{3}{2} : 1 \right); u_1, u_3 \right)
\end{equation}
where $\partial_2 X$ is given by (7.1); the parameters are as in Lemmas 3.1, 3.2.
1. Let $U_\varepsilon$ be an $\varepsilon$-neighborhood of the origin $0 \in \mathbb{R}^3$. By (7.2),
\begin{equation}
D_x G_t(x; \alpha, \beta, \zeta) = -\hat{x}_j \pm i \hat{x}_j \frac{8\pi}{8\pi}, \quad \forall x \in U_\varepsilon, \quad \hat{x}_j = \frac{x_j}{r}, \quad j = 1, 2
\end{equation}
for $\varepsilon > 0$ sufficiently small. Turns out that the off-diagonal entries $\pm \partial_x G_t(x)$ of Green’s function are well-defined $\forall x \in \mathbb{R}^3 \setminus \{0\}$.
2. By (7.2), in the limit $\alpha \to 0$,
\begin{equation}
D_x G_t(x; 0, \beta, \zeta) = \frac{x_1 \pm ix_2}{8\pi} \left( \sqrt{\zeta} H_{10} \left( -1, 3/2 \cdot \frac{\beta^2 r^2}{2} - \frac{\alpha^2 r^2}{4} - \frac{\xi^2}{4} \right) \right.
\]
\[
- \frac{1}{r} F_{1,1,0}^{0,0,0} \left( \left( \frac{3}{2} : 2, 1 \right); \left( \frac{3}{2} : 1 \right); \left( \frac{3}{2} : 1 \right); \frac{\beta^2 r^2}{64} - \frac{\alpha^2 r^2}{16} - \frac{\xi^2}{4} \right)
\]
\[
- \frac{\alpha^2 r^2}{4} \left( \frac{3}{2} \cdot \frac{\beta^2 r^2}{64} - \frac{\alpha^2 r^2}{16} - \frac{\xi^2}{4} \right)
\]
\begin{equation}
\left. \right) \left( \frac{3}{2} : 2, 1 \right); \left( \frac{3}{2} : 1 \right); u_1, u_3 \right)
\end{equation}
$\forall \beta \geq 0, \forall \zeta \in \mathbb{C} \setminus [-\beta, \infty), |\zeta| > \beta, \forall x \in \mathbb{R}^3 \setminus \{0\}$. In view of
\[
H_{10} \left( -1, 3/2 \cdot \zeta, \zeta_1, \zeta_2 \right) = \sum_{a=1}^{\infty} a^c \left( \frac{1 + 2\sigma \sqrt{c}}{2} \right)^{\zeta} \left( \frac{3}{2} : \zeta \right) F_{0,0}^{0,0} \left( \left( \frac{3}{2} : \zeta \right); \left( \frac{3}{2} : \zeta \right); u_1, u_3 \right).
\]
$\forall |\zeta| < \frac{1}{2}$, the additional condition $|\zeta| > \beta$ in (7.4) can be omitted. Note that $D_x G_t(x; 0, \beta, \zeta)$ is also easy to obtain from (4.2):
\begin{equation}
D_x G_t(x; 0, \beta, \zeta) = \frac{x_1 \pm ix_2}{8\pi \beta r^2} \left( \sqrt{\beta - \zeta} + \frac{1}{r} \right) e^{-r \sqrt{\beta - \zeta}} - \left( \sqrt{\beta - \zeta} + \frac{1}{r} \right) e^{-r \sqrt{\beta - \zeta}}
\end{equation}
∀β ≥ 0, ∀ζ ∈ C − [−β, ∞), ∀x ∈ R³ − {0}.

By (7.4)-(7.5), the Green’s function $G_{H}(x; 0, β, ζ)$ is diagonal.

3. In the limit β → 0, ∀α ≥ 0, ∀ζ ∈ C − [α², ∞), |ζ| > α², ∀x ∈ R³ − {0},

$$D_{x}G_{1}(x; α, 0, ζ) = \frac{\hat{x}_1 ± i\hat{x}_2}{8\pi} \left( \sqrt{-ζ}H_3 \left( \frac{1}{2}; \frac{3}{2}; \frac{α^2}{4}\frac{ζr^2}{4} \right) \right.$$

$$- \frac{1}{r}F^{0:1:0}_{1:1:0} \left( \frac{3}{2}; \frac{3}{2}; \frac{α^2r^2}{16}; \frac{ζr^2}{4} \right)$$

$$+ \frac{r}{2} \left[ F^{0:1:0}_{1:1:0} \left( \frac{3}{2}; \frac{3}{2}; \frac{α^2r^2}{16}; \frac{ζr^2}{4} \right) \right.$$

$$\left. + \frac{α^2}{9}F^{0:1:0}_{1:1:0} \left( \frac{3}{2}; \frac{3}{2}; \frac{α^2r^2}{16}; \frac{ζr^2}{4} \right) \right].$$

(7.6)

Taking both α = β = 0, one finds from (7.4)-(7.6) that

$$D_{x}G_{1}(x; 0, 0, ζ) = \frac{\hat{x}_1 ± i\hat{x}_2}{8\pi} e^{-r\sqrt{-ζ}}$$

∀ζ ∈ C − [0, ∞), ∀x ∈ R³ − {0}.

8. Discussion

The series representation of Green’s function is well-suited for further spectral analysis of $H$ (1.1). Suppose that $H_0$ is the operator $H$ restricted to the set of compactly supported smooth functions that vanish at the origin. The operator $H_0$ is symmetric but not self-adjoint. Self-adjoint extensions of $H_0$, say $\tilde{H}$, can be found by applying the singular perturbation theory [16]. From this point of view, $H$ is a trivial extension of $H_0$. The extensions incorporate the operators that are usually referred to as the Hamiltonians with point-interaction. In physical applications, for example in ultracold atomic gases, operator $\tilde{H}$ would describe the spin-orbit coupled Hamiltonian considered in the presence of magnetic field with the impurity scattering treated via the zero-range interaction. Without spin-orbit coupling, that is, for α = 0, self-adjoint extensions and their spectral properties have been examined in [3]; in this case, the ability to obtain exact eigenvalues is clear from a simple structure of Green’s function, see (4.2) and (4.5). For a general α ≥ 0, however, it is convenient to represent the resolvent of $\tilde{H}$ in terms of Krein’s $Q$-matrix function. One can show that, since $D_{x}G_{1}(x; α, β, ζ)$ is independent of ζ (1.4) in the neighborhood of the origin 0 ∈ R³ (7.3), the $Q$-matrix function is made up of only $G_{1}(0; α, β, ζ)$ (4.3) and $G_{2}(0; α, β, ζ)$ (4.10). The analysis of these functions leads to the transcendental equation with respect to the eigenvalue of $\tilde{H}$. The results subsequent to the algebraic treatment of the present discussion will be announced elsewhere.

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