ON NEW INEQUALITIES FOR $h$–CONVEX FUNCTIONS VIA RIEZMANN-LIOUVILLE FRACTIONAL INTEGRATION

MEVLÜT TUNÇ

Abstract. In this paper, some new inequalities of the Hermite-Hadamard type for $h$–convex functions via Riemann-Liouville fractional integral are given.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and let $a, b \in I$, with $a < b$. The following inequality;

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

In [1], Varošanec introduced the following class of functions.

Definition 1. Let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be a positive function. We say that $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is $h$–convex function or that $f$ belongs to the class $SX(h, I)$, if $f$ is nonnegative and for all $x, y \in I$ and $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y).$$

If the inequality in (1.2) is reversed, then $f$ is said to be $h$–concave, i.e., $f \in SV(h, I)$.

Obviously, if $h(\lambda) = \lambda$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\lambda) = \frac{1}{\lambda}$, then $SX(h, I) = Q(I)$; if $h(\lambda) = 1$, then $SX(h, I) \supseteq P(I)$ and if $h(\lambda) = \lambda^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_2^s$. For some recent results for $h$–convex functions we refer to the interested reader to the papers [3]–[4].

Definition 2. [See [2]] A function $h : J \rightarrow \mathbb{R}$ is said to be a superadditive function if

$$h(x + y) \geq h(x) + h(y)$$

for all $x, y \in J$.

In [3], Sarıkaya et al. proved the following Hadamard type inequalities for $h$–convex functions.

2000 Mathematics Subject Classification. 26D15, 41A51, 26D10.

Key words and phrases. Riemann-Liouville fractional integral, $h$–convex function, Hadamard’s inequality.
Theorem 1. Let \( f \in SX(h, I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then

\[
1 \leq 2h(\frac{1}{2}) \int_a^b f(x)dx \leq \frac{1}{b - a} \left( f(a) + f(b) \right)
\]

In [10], Sarıkaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.

Theorem 2. Let \( f : [a, b] \rightarrow \mathbb{R} \) be positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
f \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^{\alpha}} [J_{a^+}^\alpha (b) + J_{b^-}^\alpha (a)] \leq \frac{f(a) + f(b)}{2}
\]

with \( \alpha > 0 \).

Now we give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t)dt, \quad x > a
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t)dt, \quad x < b
\]

respectively where \( \Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha - 1}du \). Here is \( J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

For some recent results connected with fractional integral inequalities see [8]-[15] and [10].

In [10], Sarıkaya et al. proved a variant of the identity is established by Dragomir and Agarwal in [7] Lemma 2.1 for fractional integrals as the following.

Lemma 1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f' \in L[a, b] \), then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)]
\]

\[
= \frac{b - a}{2} \int_0^1 [(1 - t)^\alpha - t^\alpha] f'(t(a + (1 - t)b)dt.
\]

The aim of this paper is to establish Hadamard type inequalities for \( h \)-convex functions via Riemann-Liouville fractional integral.
2. MAIN RESULTS

**Theorem 3.** Let \( f \in SX(h, I) \), \( a, b \in I \) with \( a < b \) and \( f \in L_1[a, b] \). Then one has inequality for \( h \)-convex functions via fractional integrals

\[
\frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J_{a^+}^\alpha(b) + J_{b^-}^\alpha(a) \right] 
\leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} \left[ h(t) + h(1-t) \right] dt
\leq 2 \frac{[f(a) + f(b)]}{(\alpha p - p + 1)^\frac{1}{q}} \left( \int_0^1 (h(t))^q dt \right)^\frac{1}{q}
\]

where \( p^{-1} + q^{-1} = 1 \).

**Proof.** Since \( f \in SX(h, I) \), we have

\[
f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)
\]

and

\[
f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y).
\]

By adding these inequalities we get

\[
f(tx + (1-t)y) + f((1-t)x + ty) \leq [h(t) + h(1-t)] [f(x) + f(y)].
\]

By using (2.2) with \( x = a \) and \( y = b \) we have

\[
f(ta + (1-t)b) + f((1-t)a + tb) \leq [h(t) + h(1-t)] [f(a) + f(b)].
\]

Then multiplying both sides of (2.3) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0, 1] \), we get

\[
\int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt 
\leq \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] [f(a) + f(b)] dt,
\]

\[
\frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J_{a^+}^\alpha(b) + J_{b^-}^\alpha(a) \right] 
\leq [f(a) + f(b)] \int_0^1 t^{\alpha-1} [h(t) + h(1-t)] dt
\]

and thus the first inequality is proved.
To obtain the second inequality in (2.1), by using Hölder inequality for the right hand side of (2.4), we obtain

\[
\int_0^1 t^{\alpha-1} [h(t) + h(1-t)] \, dt \\
\leq \left( \int_0^1 (t^{\alpha-1})^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^q \, dt \right)^{\frac{1}{q}} \\
= \left( \frac{t^{\alpha p-p+1} |_{0}^{1}}{\alpha p-p+1} \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^q \, dt \right)^{\frac{1}{q}} \\
= \left( \frac{1}{\alpha p-p+1} \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^q \, dt \right)^{\frac{1}{q}}
\]

Then using Minkowski inequality

\[
\left( \frac{1}{\alpha p-p+1} \right)^{\frac{1}{p}} \left( \int_0^1 (h(t) + h(1-t))^q \, dt \right)^{\frac{1}{q}} \\
\leq \left( \frac{1}{\alpha p-p+1} \right)^{\frac{1}{p}} \left[ \left( \int_0^1 (h(t))^q \, dt \right)^{\frac{1}{q}} + \left( \int_0^1 (h(1-t))^q \, dt \right)^{\frac{1}{q}} \right] \\
= \frac{2}{(\alpha p-p+1)^{\frac{1}{p}}} \left( \int_0^1 (h(t))^q \, dt \right)^{\frac{1}{q}}
\]

where the proof is completed. \(\square\)

**Remark 1.** If we choose \(\alpha = 1\) in Theorem 1, we obtain

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq [f(a) + f(b)] \int_0^1 h(t) \, dt \\
\leq [f(a) + f(b)] \left( \int_0^1 (h(t))^q \, dt \right)^{\frac{1}{q}}.
\]

**Corollary 1.** (1) If we choose \(h(\lambda) = \lambda\) in Remark 1, we get

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2} = \frac{f(a) + f(b)}{(q + 1)^{\frac{1}{q}}}
\]

for ordinary convex functions.

(2) If we choose \(h(\lambda) = 1\) in Remark 1, we get

\[
\frac{2}{b-a} \int_a^b f(x) \, dx \leq 2 (f(a) + f(b))
\]

for \(P\)-functions. This inequality is refinement of right hand side of (1.1) for \(P\)-functions.

(3) If we choose \(h(\lambda) = \lambda^s\) in Remark 1, we get

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1} \leq \frac{f(a) + f(b)}{(sq + 1)^{\frac{1}{q}}}
\]

for \(s\)-convex functions in the second sense with \(s \in (0, 1]\).
Theorem 4. Let \( f \in SX(h, I), \) \( a, b \in I \) with \( a < b \), \( h \) is superadditive on \( I \) and \( f \in L_1[a, b], \) \( h \in L_1[0,1]. \) Then one has inequality for \( h \)-convex functions via fractional integrals

\[
(2.5) \quad \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} [J^\alpha_a (b) + J^\alpha_b (a)] \leq \frac{h(1)}{\alpha} [f(a) + f(b)].
\]

Proof. Since \( f \in SX(h, I), \) we have

\[
f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)
\]

and

\[
f((1-t)x + ty) \leq h(1-t)f(x) + h(t)f(y).
\]

By adding these inequalities we get

\[
(2.6) \quad f(tx + (1-t)y) + f((1-t)x + ty) \leq [h(t) + h(1-t)] [f(x) + f(y)].
\]

By using (2.6) with \( x = a \) and \( y = b \) and \( h \) is superadditive, we get

\[
(2.7) \quad f(ta + (1-t)b) + f((1-t)a + tb) \leq h(1) [f(a) + f(b)].
\]

Then multiplying both sides of (2.7) by \( t^{\alpha-1} \) and integrating the resulting inequality with respect to \( t \) over \([0,1], \) we get

\[
\int_0^1 t^{\alpha-1} [f(ta + (1-t)b) + f((1-t)a + tb)] dt
\]

\[
\leq \int_0^1 t^{\alpha-1} h(1) [f(a) + f(b)] dt,
\]

\[
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} [J^\alpha_a (b) + J^\alpha_b (a)]
\]

\[
\leq h(1) [f(a) + f(b)] \int_0^1 t^{\alpha-1} dt.
\]

This completes the proof. \( \square \)

Remark 2. If we choose \( \alpha = 1 \) in Theorem 4, then (2.7) reduce to special version of right hand side of (1.7).

Theorem 5. Let \( h : J \subset \mathbb{R} \to \mathbb{R} \) and \( f : [a, b] \to \mathbb{R} \) be positive functions with \( 0 \leq a < b \) and \( h^q \in L_1[0,1], \) \( f \in L_1[a, b], \) \( f' \) is an \( h \)-convex mapping on \([a, b], \) then the following inequality for fractional integrals holds,

\[
(2.8) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J^\alpha_{a+} f(b) + J^\alpha_{b-} f(a)] \right|
\]

\[
\leq \frac{(b-a) [f''(a)] + |f''(b)|}{2} \left[ \left( \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right)^{\frac{1}{p}} - \left( \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right)^{\frac{1}{p}} \right]
\]

\[
\times \left[ \left( \int_0^{\frac{1}{2}} (h(t))^q dt \right)^{\frac{1}{q}} + \left( \int_{\frac{1}{2}}^1 (h(t))^q dt \right)^{\frac{1}{q}} \right]
\]

where \( \alpha > 0, p > 1 \) and \( p^{-1} + q^{-1} = 1. \)
Proof. From Lemma \[\text{I}\] and using the properties of modulus, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{\alpha}^a f(b) + J_{\alpha}^b f(a) \right] \right|
\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| \, dt.
\]

Since \(|f'| is \(h\)-convex on \([a,b]\), we have

\[
(2.9) \quad \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{\alpha}^a f(b) + J_{\alpha}^b f(a) \right] \right|
\leq \frac{b-a}{2} \left\{ \int_0^\frac{1}{2} |(1-t)^\alpha - t^\alpha| |h(t)| f'(a)\, dt + \int_0^\frac{1}{2} (\alpha - 1)(t^\alpha) |h(t)| f'(b)\, dt \right\}
\]

\=
\frac{b-a}{2} \left\{ |f'(a)| \int_0^\frac{1}{2} (1-t)^\alpha h(t) \, dt - |f'(a)| \int_0^\frac{1}{2} t^\alpha h(t) \, dt + |f'(b)| \int_0^\frac{1}{2} (1-t)^\alpha h(t) \, dt - |f'(b)| \int_0^\frac{1}{2} t^\alpha h(t) \, dt \right\}.

In the right hand side of above inequality by using Hölder inequality for \(p^{-1} + q^{-1} = 1\) and \(p > 1\), we get

\[
\int_0^\frac{1}{2} (1-t)^\alpha h(t) \, dt = \int_\frac{1}{2}^1 t^\alpha h(1-t) \, dt \leq \left[ \frac{2^{\alpha p + 1} - 1}{2^{\alpha p + 1} (\alpha p + 1)} \right]^\frac{1}{p} \left( \int_0^\frac{1}{2} [h(t)]^q \, dt \right)^\frac{1}{q},
\]

\[
\int_0^\frac{1}{2} (1-t)^\alpha h(1-t) \, dt = \int_\frac{1}{2}^1 t^\alpha h(t) \, dt \leq \left[ \frac{2^{\alpha p + 1} - 1}{2^{\alpha p + 1} (\alpha p + 1)} \right]^\frac{1}{p} \left( \int_\frac{1}{2}^1 [h(t)]^q \, dt \right)^\frac{1}{q},
\]

\[
\int_0^\frac{1}{2} t^\alpha h(t) \, dt = \int_\frac{1}{2}^1 (1-t)^\alpha h(t) \, dt \leq \left[ \frac{1}{2^{\alpha p + 1} (\alpha p + 1)} \right]^\frac{1}{p} \left( \int_0^\frac{1}{2} [h(t)]^q \, dt \right)^\frac{1}{q},
\]

and

\[
\int_0^\frac{1}{2} t^\alpha h(1-t) \, dt = \int_\frac{1}{2}^1 (1-t)^\alpha h(t) \, dt \leq \left[ \frac{1}{2^{\alpha p + 1} (\alpha p + 1)} \right]^\frac{1}{p} \left( \int_\frac{1}{2}^1 [h(t)]^q \, dt \right)^\frac{1}{q}.
\]
Then using the above inequalities in the right hand side of (2.9), we get
\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha (b) + J_{b-}^\alpha (a) \right] \right|
\leq \frac{b-a}{2} \left\{ |f'(a)| \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_0^{\frac{1}{2}} [h(t)]^q \, dt \right)^{\frac{1}{q}} \right\}
\]
\[
+ \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_{\frac{1}{2}}^1 [h(t)]^q \, dt \right)^{\frac{1}{q}} \left| f'(b) \right| \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_0^{\frac{1}{2}} [h(t)]^q \, dt \right)^{\frac{1}{q}} \]
\[
+ \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_{\frac{1}{2}}^1 [h(t)]^q \, dt \right)^{\frac{1}{q}} \left( \frac{b-a}{2} \right) \left\{ |f'(a)| \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \right\} \left( \int_0^{\frac{1}{2}} [h(t)]^q \, dt \right)^{\frac{1}{q}}
\]
\[
+ \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_{\frac{1}{2}}^1 [h(t)]^q \, dt \right)^{\frac{1}{q}} \left| f'(b) \right| \left( \left[ \frac{2^{\alpha p+1} - 1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} - \left[ \frac{1}{2^{\alpha p+1} (\alpha p + 1)} \right]^{\frac{1}{p}} \right) \left( \int_0^{\frac{1}{2}} [h(t)]^q \, dt \right)^{\frac{1}{q}}
\]
\[
\times \left( \int_0^{\frac{1}{2}} [h(t)]^q \, dt \right)^{\frac{1}{q}} + \left( \int_{\frac{1}{2}}^1 [h(t)]^q \, dt \right)^{\frac{1}{q}} \right\}
\]
which is the desired result. The proof is completed. \[\square\]

References

[1] S. Varošanec, On h-convexity, J. Math. Anal. Appl., 326 (2007), 303-311.
[2] H. Alzer, A superadditive property of Hadamard’s gamma function, Abh. Math. Semin. Univ. Hambg., 79 (2009), 11-23.
[3] M. Z. Sarıkaya, A. Sağlam and H. Yıldırım, On some Hadamard-type inequalities for h-convex functions. Journal of Mathematical Inequalities, 2 (3) (2008), 335-341.
[4] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities, Computers and Mathematics with Applications, 58 (2009), 1869-1877.
[5] M.Z. Sarıkaya, E. Set and M.E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions, Acta Math. Univ. Comenianae, Vol. LXXVII, 2 (2010), pp. 265-272.
[6] P. Burai and A. Házy, On approximately h-convex functions, Journal of Convex Analysis, 18 (2) (2011).
[7] S.S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11 (5) (1998), 91-95.
[8] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
[9] Z. Dahmani, *New inequalities in fractional integrals*, International Journal of Nonlinear Science, 9(4) (2010), 493-497.

[10] M.E. Ozdemir, H. Kavurmaci and M. Avci, *New inequalities of Ostrowski type for mappings whose derivatives are (α, m)-convex via fractional integrals*, RGMIA Research Report Collection, 15(2012), Article 10, 8 pp.

[11] M.E. Özdemir, H. Kavurmacı and Ç. Yıldız, *Fractional integral inequalities via s-convex functions*, arXiv:1201.4915v1 [math.CA] 24 Jan 2012.

[12] E. Set, *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl., In Press, Corrected Proof, 29 December 2011.

[13] Z. Dahmani, *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(1) (2010), 51-58.

[14] Z. Dahmani and L. Tabharit, S. Taf, *Some fractional integral inequalities*, Nonl. Sci. Lett. A., 1(2) (2010), 155-160.

[15] Z. Dahmani, L. Tabharit and S. Taf, *New generalizations of Grüss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.

[16] M. Z. Sarıkaya, E. Set, H. Yaldiz and N. Başak, *Hermite-Hadamard’s inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, In Press.

University of Kilis 7 Aralık, Faculty of Science and Arts, Department of Mathematics, 79000, Kilis, Turkey

E-mail address: mevluttunc@kilis.edu.tr