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A note on the simultaneous edge coloring *

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Abstract

Let \( G = (V,E) \) be a graph. A (proper) \( k \)-edge-coloring is a coloring of the edges of \( G \) such that any pair of edges sharing an endpoint receive distinct colors. A classical result of Vizing [3] ensures that any simple graph \( G \) admits a \((\Delta(G) + 1)\)-edge coloring where \( \Delta(G) \) denotes the maximum degree of \( G \). Recently, Cabello raised the following question: given two graphs \( G_1, G_2 \) of maximum degree \( \Delta \) on the same set of vertices \( V \), is it possible to edge-color their (edge) union with \( \Delta + 2 \) colors in such a way the restriction of \( G \) to respectively the edges of \( G_1 \) and the edges of \( G_2 \) are edge-colorings? More generally, given \( \ell \) graphs, how many colors do we need to color their union in such a way the restriction of the coloring to each graph is proper?

In this short note, we prove that we can always color the union of the graphs \( G_1, \ldots, G_\ell \) of maximum degree \( \Delta \) with \( \Omega(\sqrt{\ell} \cdot \Delta) \) colors and that there exist graphs for which this bound is tight up to a constant multiplicative factor. Moreover, for two graphs, we prove that at most \( \frac{\ell}{2} \Delta + 4 \) colors are enough which is, as far as we know, the best known upper bound.

1 Introduction

All along the paper, we only consider simple loopless graphs. In his seminal paper, Vizing proved in g [3] that any simple graph \( G \) can be properly edge-colored using \( \Delta(G) + 1 \) colors (where \( \Delta(G) \) denotes the maximum degree of \( G \)). The union of two graphs \( G_1 \) and \( G_2 \) on vertex set \( V \) is the (simple) graph \( G \) with vertex set \( V \) and where \( uv \) is an edge if and only if \( uv \) is an edge of \( G_1 \) or an edge of \( G_2 \). An edge coloring of \( G \) is simultaneous with respect to \( G_1 \) and \( G_2 \) if its restrictions to the edge set of \( G_1 \) and to the edge set of \( G_2 \) are proper edge-colorings. Recently, Cabello raised the following question:\(^1\): given two graphs \( G_1, G_2 \) of maximum degree \( \Delta \) on the same set of vertices \( V \), does it always exist a simultaneous \((\Delta + 2)\)-edge coloring with respect to \( G_1 \) and \( G_2 \)? Cabello proved that this property is satisfied if the intersection of \( G_1 \) and \( G_2 \) is regular [1]. Using Vizing’s theorem, one can easily notice that there exists a simultaneous \((2\Delta + 1)\)-edge coloring. From a lower bound perspective, no graph where \( \Delta + 2 \) colors are needed is known.

Cabello introduced a generalization of this notion. Let \( \ell \) graphs \( G_1, G_2, \ldots, G_\ell \) and \( G \) be their (edge) union. In other words, \( uv \) is an edge of \( G \) if and only if \( uv \) is an edge of at least one graph \( G_i \) with \( i \leq \ell \). An edge-coloring of \( G \) is simultaneous with respect to \( G_1, \ldots, G_\ell \) if its restriction to each graph \( G_i \) is a proper edge-coloring. Cabello asked how many colors are needed to ensure the existence of a simultaneous coloring of \( G \) with respect to each \( G_i \). Let us denote by \( \chi(\ell, \Delta) \) the minimum number of colors needed to obtain a simultaneous coloring. And let \( \chi'(\ell, \Delta) \) be the largest integer \( k \) such that \( k = \chi'(G_1, \ldots, G_\ell) \) for some graphs \( G_1, \ldots, G_\ell \) of maximum degree (at most) \( \Delta \). Vizing’s theorem ensures that \( \chi'(\ell, \Delta) \leq \ell \Delta + 1 \) and Cabello exhibit a graph for which \( \chi'(3, \Delta) \geq \Delta + 5 \) (with \( \Delta = 10 \)) [1]. In this note, we prove that the order of magnitude of \( \chi'(\ell, \Delta) \) is \( \Theta(\sqrt{\ell\Delta}) \). More precisely, we prove that the following statement holds:

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Theorem 1
\[ \chi'(\ell, \Delta) \leq 2\sqrt{2\ell \Delta} - \sqrt{2\ell} + 2. \]

We claim that this bound is tight up to a constant multiplicative factor. Let \( \ell \in \mathbb{N} \) and \( \Delta \) be an even value. Let \( G := S_{1,k,\Delta} \) be the star with \( k\Delta \) leaves, where \( k = \lceil \sqrt{2} \rceil \). Partition the edges of \( G \) into \( 2k \) sets \( A_1, \ldots, A_{2k} \) of size \( \frac{\Delta}{2} \). For every pair \( i, j \), create the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Note that each graph \( G_{i,j} \) has maximum degree \( \Delta \) since by construction the set of edges \( A_i \) induces a graph of maximum degree \( \Delta/2 \). Moreover the total number of graphs \( G_{i,j} \) is \( 2k(2k-1)/2 \leq \ell \). Finally, by construction, every pair of edges of \( G \) appears in at least one graph \( G_{i,j} \). So in order to obtain a simultaneous coloring, we need to color all the edges of \( G \) with different colors (since all the edges are incident to the center of the star). So:

Proposition 2
\[ \chi'(\ell, \Delta) \geq \left\lceil \frac{\sqrt{2}}{2} \Delta \right\rceil. \]

Note that, for \( \ell = 3 \) and the graph \( S_{1,3,\Delta/2} \), a similar construction ensures that \( \chi'(3, \Delta) \geq 3\left\lceil \frac{\Delta}{2} \right\rceil \), improving the lower bound of \( \Delta + 5 \). Indeed, let us partition the edges of the star into three sets \( A_1, A_2, A_3 \) of size \( \lceil \Delta/2 \rceil \). We similarly define for every \( i \neq j \) the graph \( G_{i,j} \) with edge set \( A_i \cup A_j \). Each graph \( G_{i,j} \) has maximum degree \( \Delta \) and every pair of edges appears in at least one graph \( G_{i,j} \). So an edge coloring of \( S_{1,3,\Delta/2} \) simultaneous with respect to \( G_{1,2}, G_{1,3} \) and \( G_{2,3} \) is a proper edge coloring of \( S_{1,3,\Delta/2} \). When \( \ell = 2 \), a careful reading of the proof of Theorem 1 permits to remark that we can improve the trivial \((2\Delta + 1)\) upper bound into \(2\Delta\). We prove the following much better upper bound with a different technique:

Theorem 3
\[ \chi'(2, \Delta) \leq \left\lfloor \frac{3}{2} \Delta + 4 \right\rfloor. \]

As far as we know, it is the best known upper bound.

2 Proof of Theorem 1

Let \( G_1, \ldots, G_\ell \) be \( \ell \) graphs of maximum degree \( \Delta \). Let us partition the set of edges of \( G = \bigcup_{i=1}^{\ell} G_i \) into two sets (all along the paper, the notation \( \cup \) stands for edge union). The multiplicity of an edge \( e \) is the number of graphs \( G_i \) with \( i \leq \ell \) on which \( e \) appears. For some fixed \( k \), the set \( E_1 \) is the set of edges with multiplicity at least \( k \) and \( E_2 \) is the set of edges with multiplicity less than \( k \). We will optimize the value of \( k \) later. (Note that we do not necessarily assume that \( k \) is an integer.) For every \( i \in \{1,2\} \), let us denote by \( H_i \) the graph \( G \) restricted to the edges of \( E_i \). Note that \( G = H_1 \cup H_2 \).

We claim that the graph \( H_1 \) has degree at most \( \ell \Delta/k \). Indeed, let \( u \) be a vertex and \( E_1(u) \) be the set of edges of \( H_1 \) incident to it. Since every edge of \( H_1 \) has multiplicity at least \( k \) and at most \( \ell \Delta \) edges (with multiplicity) are incident to \( u \) in \( G \), at most \( \frac{\ell}{k} \Delta \) different edges are in \( E_1(u) \). So \( H_1 \) has maximum degree \( \frac{\ell}{k} \Delta \). By Vizing’s theorem, \( H_1 \) can be properly edge-colored with \( (\frac{\ell}{k} \Delta + 1) \) colors.

Let us now prove that \( H_2 \) can be simultaneously edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let us prove it by induction on the number of edges of \( H_2 \). The empty graph can indeed be edge-colored with \( 2k(\Delta - 1) + 1 \) colors. Let \( e = uv \) be an edge of \( H_2 \). By induction, there exists a simultaneous coloring \( c' \) of \( H_2 \setminus e \) with \( 2k(\Delta - 1) + 1 \) colors. Let us prove that \( c' \) can be extended to a simultaneous coloring of \( H_2 \). Without loss of generality, we can assume that \( e \) is an edge of the graphs \( G_1, \ldots, G_r \) with \( r < k \) and is not an edge of \( G_{r+1}, \ldots, G_\ell \). Let \( F \) be the set of edges of \( G_1, \ldots, G_r \) incident to \( u \) or to \( v \) distinct from \( e \). By assumption, there are at most \( 2r(\Delta - 1) \) such edges \( (2(\Delta - 1) \) in each graph). Since \( r < k \), at most \( 2k(\Delta - 1) \) edges are in \( F \). So there exists a color \( a \) that does not appear in \( F \). The edge \( e \) can be colored with \( a \) without violating any constraints. It holds by choice of \( a \) for \( G_i \) with \( i \leq r \) and it holds since \( e \notin G_i \) for \( i > r \).
So $\chi'(\ell, \Delta) \leq \frac{\ell}{2} \Delta + 2k(\Delta - 1) + 2$ colors. We finally optimize the integer $k$ which minimize the number of colors. We want to minimize $\frac{\ell}{2} + 2k$ which is minimal when $k = \sqrt{\frac{\ell}{2}}$. It finally ensures that $\chi'(\ell, \Delta) \leq 2\sqrt{2\ell\Delta} - \sqrt{2\ell} + 2$, which completes the proof of Theorem 1.

3 Proof of Theorem 3

Let $G_1, G_2$ be two graphs of maximum degree $\Delta$ and let $G$ be their union. Let $E_2$ be the edges that appear in both graphs and, for every $i \in \{1, 2\}$, let $E_i$ be the set of edges that appears only in $G_i$. Let us denote by $H_2$ (resp. $H_1^i$) the graph restricted to the edges of $E_2$ (resp. $E_i$).

For every vertex $v$ and every graph $H$, we denote by $deg_H(v)$ the degree of $v$ in $H$. Let $H$ be a graph and $f, g$ be two functions from $V(H)$ to $\mathbb{R}^+$. A $(g, f)$-factor of $H$ is an edge-subgraph $H'$ of $H$ such that every vertex $v$ satisfies $g(v) \leq deg_{H'}(v) \leq f(v)$. Kano and Saito proved in [2] that the graph $H$ admits a $(g, f)$-factor if

(i) $f$ and $g$ are two integer valued functions, and

(ii) for every vertex $v$, $g(v) < f(v)$, and

(iii) there exists a real number $\theta$ such that $0 \leq \theta \leq 1$ and for every vertex $v$, $g(v) \leq \theta \cdot deg_H(v) \leq f(v)$.

Let $1 \leq i \leq 2$. We will extract from $H_1^i$ a $(g, f)$-factor where $g(v) = \lfloor \frac{deg_{H_1^i}(v)}{2} \rfloor - 1$ and $f(v) = \lfloor \frac{deg_{H_1^i}(v)}{2} \rfloor$. The points (i) and (ii) are satisfied. Moreover, by choosing $\theta = \frac{1}{2}$, (iii) is also satisfied. Thus by [2], the graphs $H_1^i$ and $H_2^i$ admit $(g, f)$-factors. For $i \leq 2$, let $K_1^i$ be a $(g, f)$-factor of $H_1^i$. For every $i$, let $L_1^i = H_1^i \setminus K_1^i$. Let $L = L_1^1 \cup L_1^2$ and $R = H_2 \cup K_1^1 \cup K_1^2$. Note that $G = L \cup R$. Let us now color these two graphs.

Let us first prove that $L$ can be colored with $\left\lfloor \frac{\Delta}{2} \right\rfloor + 2$ colors. For every $i$, the graph $L_1^i$ has maximum degree at most $\left\lfloor (\Delta/2) + 1 \right\rfloor$ since every vertex $v$ of $K_1^i$ has degree at least $\lfloor (deg_{H_1^i}(v)/2) - 1 \rfloor$. By Vizing’s theorem, the graph $L_1^i$ can be colored with at most $\left\lfloor \frac{\Delta}{2} \right\rfloor + 1 = \left\lfloor \frac{\Delta}{2} \right\rfloor + 2$ colors. Since the edges of $L_1^i$ and $L_2^i$ are disjoint, $L = L_1^1 \cup L_1^2$ can be simultaneously colored with $\left\lfloor \frac{\Delta}{2} \right\rfloor + 2$ colors (the same set of colors can be re-used for each graph).

Let us now color the graph $R$. Let $v$ be a vertex of $R$. Let us denote by $d$ the degree of $v$ in $H_2$. Since edges of $H_2$ are in both $G_1$ and $G_2$, the vertex $v$ has degree at most $\Delta - d$ in both graphs $H_1^1$ and $H_2^2$. Since the graphs $K_1^1$ and $K_2^1$ are $(g, f)$-factors of respectively $H_1^1$ and $H_2^2$, the degree of the vertex $v$ is at most $\left\lfloor \frac{\Delta - d}{2} \right\rfloor$ in each graph. So the degree of $v$ in the graph $R$ is at most $d + 2\left\lfloor \frac{\Delta - d}{2} \right\rfloor \leq \Delta + 1$. By Vizing’s theorem, the graph $R$ can be colored using at most $\Delta + 2$ colors.

Since $G = L \cup R$, we can find a simultaneous edge-coloring with respect to $G_1$ and $G_2$ using at most $\left\lfloor \frac{\Delta}{2} \right\rfloor + 2 + \Delta + 2 = \left\lfloor \frac{3\Delta}{2} \right\rfloor + 4$ colors.

4 Conclusion

Theorem 1 and Proposition 2 ensures that the following holds:

$$\left\lfloor \sqrt{\frac{\ell}{2}} \right\rfloor \Delta \leq \chi'(\ell, \Delta) \leq 2\sqrt{2\ell\Delta} - \sqrt{2\ell} + 2.$$

Closing the multiplicative gap of 4 between lower and upper bound is an interesting open problem. For $\ell = 2$, we still do not know any graph for which $\chi'(2, \Delta) > \Delta + 1$. Cabello asked the following question that is still widely open despite the progress obtained in Theorem 3:

**Question 4 (Cabello)** Is it true that $\chi'(2, \Delta) \leq \Delta + 2$ ?

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3
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