On the equivalence among some chiral-boson theories

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Abstract

We make a comparative study of chiral-boson theories in the Florenani-Jackiw (FJ) and linear constraint formulations. A special attention is given to the case with an improved way of implementing the linear constraint. We show that it has the same spectrum of the FJ formulation.

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1 Introduction

Chiral-boson theories are two-dimensional scalar theories where the scalar field $\phi$ exhibits the so-called chiral condition $\dot{\phi} = \phi'$ as a solution. It is usually known four formulations of chiral-boson theories in literature. The one that has called much attention is due to Floreanini and Jackiw (FJ), described by the non-covariant Lagrangian

$$L_1 = \dot{\phi}\phi' - \phi'\phi'.$$  \hspace{1cm} (1)

This theory has a constraint that is second-class at all points of momentum space except one, where it is first-class. This fact deserves some care when calculating the spectrum, because raising and lowering operators have to be defined at all points.

Another formulation is due to Srivastava, where the chiral condition is introduced linearly by means of a Lagrange multiplier. The corresponding Lagrangian reads

$$L_2 = -\frac{1}{2}\eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + \lambda (\dot{\phi} - \phi').$$  \hspace{1cm} (2)

This theory has been criticized by some authors. The main arguments are that it does not lead to a positive definite Hamiltonian and that its physical spectrum, after introducing ghost degrees of freedom, is just the vacuum state. In fact, the theory described by (2) is not equivalent to the FJ one, even though both of them contain the same chiral condition $\dot{\phi} - \phi' = 0$ as a classical equation of motion. We deserve Sec. 2 to discuss these points with details and to review the obtainment of the spectrum of these two theories. We also take the opportunity to introduce some new ingredients and to fix the notation and convention we shall use throughout the paper.

More recently, it has been introduced another way of implementing the linear constraint in the chiral-boson theory. This is described by the Lagrangian

$$L_3 = -\frac{1}{2}\eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + \lambda (\dot{\phi} - \phi') + \frac{1}{2} \lambda^2.$$  \hspace{1cm} (3)

Here, the equation of motion for $\lambda$ does not lead to the chiral condition, but using this equation back to (3) the FJ Lagrangian is obtained. They are consequently on-shell classically equivalent (as it was in some sense FJ and Srivastava formulations).

The main purpose of our work is to study the theory described by the Lagrangian above in the quantum point of view. This will be done in Sec. 3. We shall see that it does not have the inconsistencies of the previous formulation and, more than that, has the same spectrum of the FJ theory. In other words, they are also quantically equivalent.

\footnote{Dot means time derivative and prime means derivative with respect to the spatial coordinate $x$. We shall also consider the metric convention $\eta_{00} = -\eta_{11} = -1, \eta_{01} = 0$ throughout this paper.}
To conclude this introduction we just mention the formulation due to Siegel [10], where the chiral condition is obtained by projecting out one of the components of the energy-momentum tensor, which results in a Lagrangian where the chiral relation appears quadratically

\[ L_4 = -\frac{1}{2} \eta_{\mu\nu} \partial^\mu \phi \partial^\nu \phi + \lambda (\dot{\phi} - \phi')^2. \] (4)

This system is anomalous and its quantization is only consistently achieved by introducing a Wess-Zumino term, resulting a theory where chiral-bosons are coupled to gravity [11].

2 Spectra of FJ and Srivastava chiral-boson theories

The Lagrangian of the FJ theory leads to the constraint

\[ \Omega_1(x, t) = \pi(x, t) - \phi'(x, t), \] (5)

where \( \pi \) is the canonical momentum conjugate to \( \phi \). The quantity \( \Omega_1 \) represents, in fact, an infinite number of constraints, one for each value of the space coordinate \( x \). Another important particularity concerns with the constraint matrix

\[ C(x, y) = \{ \Omega_1(x, t), \Omega_1(y, t) \} = -2 \delta'(x - y). \] (6)

Its inverse reads

\[ C^{-1}(x, y) = -\frac{1}{2} \epsilon(x - y) + f(t), \] (7)

where \( \epsilon(x - y) \) is the step function and \( f(t) \) is some arbitrary function of time. We see that the inverse is not unique. This lack of uniqueness is related to the fact that there is one first-class constraint among all the infinite constraints represented by \( \Omega_1 \) [2, 3]. This is easily seen by considering Fourier transformations of fields and constraints. They are defined through the relations

\[ \phi(x, t) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\phi}(k, t), \]
\[ \pi(x, t) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\pi}(k, t), \]
\[ \Omega_1(x, t) = \frac{1}{\sqrt{2\pi}} \int dk e^{ikx} \hat{\Omega}_1(k, t). \] (8)
Using the relations above and (5) we see that

\[ \tilde{\Omega}_1(k, t) = \tilde{\pi}(k, t) - i k \tilde{\phi}(k, t). \]  

(9)

Further, the fundamental Poisson bracket relation, written in momentum space,

\[ \{ \tilde{\phi}(k, t), \tilde{\pi}(p, t) \} = \delta(k + p), \]  

(10)

gives for the constraint matrix

\[ \tilde{C}(k, p) = -2 i k \delta(k + p). \]  

(11)

We notice that

\[ \tilde{\Omega}_1(0, t) \equiv \tilde{\Omega}_2 = \tilde{\pi}(0, t) \]  

(12)

is first-class. This explains why the inverse \( C^{-1}(x,y) \) given by (6) is not unique.

To discuss the spectrum of the FJ theory, we have to be aware of the quantization for all \( k \). Let us then choose a gauge-fixing condition for the first-class constraint \( \tilde{\pi}(0, t) \approx 0 \). A natural choice is

\[ \tilde{\Omega}_3 = \tilde{\phi}(0, t) \approx 0. \]  

(13)

Let us calculate the Dirac brackets in order to perform the quantization. The matrix of the Poisson brackets of constraints reads

\[
\tilde{C}(\bar{k}, \bar{p}) = \begin{pmatrix}
-2 i k \delta(\bar{k} + \bar{p}) & 0 & 0 \\
0 & 0 & -\delta(0) \\
0 & \delta(0) & 0
\end{pmatrix}.
\]  

(14)

We are using the notation \( \bar{k} \) to mean any \( k \neq 0 \). Considering the inverse of this matrix, the Dirac brackets can be directly calculated. The result is

\[
\{ \tilde{\phi}(\bar{k}, t), \tilde{\phi}(\bar{p}, t) \}_D = \frac{i}{2k} \delta(\bar{k} + \bar{p}),
\]

\[
\{ \tilde{\phi}(0, t), \tilde{\phi}(\bar{p}, t) \}_D = 0,
\]

\[
\{ \tilde{\phi}(0, t), \tilde{\phi}(0, t) \}_D = 0.
\]  

(15)

Before replacing the above Dirac brackets by commutators, it is convenient to have a better view of the meaning of \( \tilde{\phi}(k, t) \). First we notice that the Lagrangian (1) leads to the equation of motion

\[ \dot{\phi}' - \phi'' = 0. \]  

(16)
Using the Fourier transformations (8), we get

\[ k (i \dot{\phi} + k \phi) = 0. \]  

(17)

For \( k \neq 0 \) the solution is

\[ \tilde{\phi}(\bar{k}, t) = A(\bar{k}) e^{i\bar{k}t}, \]

(18)

where \( A \) is some generic function of \( \bar{k} \). For \( k = 0 \), nothing can be concluded from eq. (17). However, from the gauge condition we have chosen, the field \( \phi \) is also defined at this point (see expression (13)). So, one can consider the expression above for all \( k \), once one takes

\[ A(0) = 0. \]  

(19)

Replacing back these results into (15), and defining \( A^\dagger(\bar{k}) \equiv A(-\bar{k}) \equiv a(\bar{k})/\sqrt{2\bar{k}} \) \((\bar{k} > 0)\), \( a(0) = 0 \), we obtain

\[ [a(\bar{k}), a^\dagger(\bar{p})] = \delta(\bar{k} - \bar{p}), \]
\[ [a(0), a^\dagger(0)] = [a(0), a^\dagger(\bar{p})] = [a(0), a(\bar{p})] = 0 \]  

(20)

where Dirac brackets have been replaced by \( i \) times the corresponding commutators. It is easily seen that \( a^\dagger(\bar{k}) \) and \( a(\bar{k}) \) are raising and lowering operators respectively. One can also write the Fourier expansion for \( \phi(x, t) \) as

\[ \phi(x, t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} \left[ a(k) e^{-ik(x+t)} + a^\dagger(k) e^{ik(x+t)} \right]. \]  

(21)

It is interesting to notice that \( \phi(x, t) = \phi(x^+) \) with \( x^+ = x + t \). This reflects the presence of the chiral condition \( \dot{\phi} - \phi' = 0 \) in the solution above.

Let us now pass to consider the Hamiltonian in terms of the operators \( a(\bar{k}) \) and \( a^\dagger(\bar{k}) \). Since commutators are obtained from Dirac brackets, the quantum evolution ought to take constraints in a strong way. Consequently, the Hamiltonian we have to consider is the canonical one and we use the constraints, when necessary, as strong relations. Hence

\[ H_c = \int dx (\pi \dot{\phi} - \mathcal{L}), \]
\[ = \int dx \left[ (\pi - \phi') \dot{\phi} + \phi' \phi'' \right], \]
\[ \rightarrow \int dx \phi' \phi''. \]  

(22)
where the constraint (3) was used in the last step. The combination of (21) and (22) gives

\[ H_c = \int_0^\infty dk \ k \ a^\dagger(k) a(k), \]  

rewritten in normal order. Since the Hamiltonian above is positively defined, we have that the action of the lowering operator in a state cannot be done indefinitely. So, we can introduce a vacuum state from where all states can be generated. Consequently, the FJ chiral boson theory has a spectrum of bosonic massless particles just with right movers.

Let us now discuss on the spectrum of the theory described by Lagrangian (2). It does not describe the same theory of the FJ one as it will become patent during the development we are going to present. However, there is a simple argument we can mention that makes the affirmative above quite evident. To see this, we refer to the constraints of (2):

\[ \Omega_1 = q \approx 0, \]  
\[ \Omega_2 = \pi - \phi' - \lambda \approx 0, \]  

where \( q \) is the canonical momentum conjugated to \( \lambda \). These are second-class. Consequently, the theory has two physical (continuous) degrees of freedom whereas the FJ has just one.

The nonvanishing brackets involving \( \Omega_1 \) and \( \Omega_2 \) are

\[ \{\Omega_1(x, t), \Omega_2(y, t)\} = \delta(x - y), \]  
\[ \{\Omega_2(x, t), \Omega_2(y, t)\} = -2\delta'(x - y). \]

Also here, let us make Fourier transformations of fields and constraints. We obtain

\[ \tilde{\Omega}_1(k, t) = \tilde{q}(k, t), \]  
\[ \tilde{\Omega}_2(k, t) = \tilde{\pi}(k, t) - i k \tilde{\phi}(k, t) - \tilde{\lambda}. \]

The corresponding Poisson brackets matrix reads

\[ C(k, p) = \begin{pmatrix} 0 & 1 \\ -1 & -2ik \end{pmatrix} \delta(k + p), \]

and we notice that \( \det C \neq 0 \), independently of \( k \). There is no point where constraints can be first-class. This is another difference with respect the FJ theory. The inverse \( C^{-1} \) is immediately calculated.
The Dirac brackets involving $\tilde{\phi}$ and $\tilde{\lambda}$ are

\[
\begin{align*}
\{\tilde{\phi}(k, t), \tilde{\phi}(p, t)\}_D &= 0, \\
\{\tilde{\lambda}(k, t), \tilde{\lambda}(p, t)\}_D &= 2 i k \delta(k + p), \\
\{\tilde{\phi}(k, t), \tilde{\lambda}(p, t)\}_D &= \delta(k + p).
\end{align*}
\]

As it was done in the FJ case, let us look at equations of motion before quantizing this theory. From Lagrangian (2) we get

\[
\begin{align*}
\Box \phi - \dot{\lambda} + \lambda' &= 0, \\
\dot{\phi} - \phi' &= 0.
\end{align*}
\]

The combination of these two equations gives

\[
\dot{\lambda} - \lambda' = 0,
\]

that is, $\lambda$ also satisfies the chiral condition. Considering the Fourier transformations for $\phi$ and $\lambda$ into (32) and (33) we may rewrite $\tilde{\phi}$ and $\tilde{\lambda}$ by

\[
\begin{align*}
\tilde{\phi}(k, t) &= A(k) e^{ikt}, \\
\tilde{\lambda}(k, t) &= \Lambda(k) e^{ikt}.
\end{align*}
\]

With these results, the initial expressions of Fourier transformations turns to be

\[
\begin{align*}
\phi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \left[ A(k) e^{-ik(x+t)} + A^\dagger(k) e^{ik(x+t)} \right], \\
\lambda(x, t) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \left[ \Lambda(k) e^{-ik(x+t)} + \Lambda^\dagger(k) e^{ik(x+t)} \right].
\end{align*}
\]

The presence of the on-shell chiral conditions for both $\phi$ and $\lambda$ is expressed in their dependence of $x^+ = x + t$.

Let us now consider the canonical Hamiltonian. The final result is (using constraints strongly)

\[
H_c = \int dx \phi' \left( \phi' + \lambda \right),
\]
In terms of operators $A$, $A^\dagger$, $\Lambda$ and $\Lambda^\dagger$ we get (in the normal order)

\[
H_c = \int_0^\infty dk \left[ 2k^2 A^\dagger(k)A(k) + ik A^\dagger(k)\Lambda(k) - ik \Lambda^\dagger(k)A(k) \right].
\]  

(38)

The quantization leads to

\[
\begin{align*}
[A(k), A^\dagger(p)] &= 0 = [A(k), A(p)] = [A^\dagger(k), A^\dagger(p)], \\
[\Lambda(k), \Lambda^\dagger(p)] &= -2k \delta(k - p), \\
[\Lambda(k), \Lambda(p)] &= 0 = [\Lambda^\dagger(k), \Lambda^\dagger(p)], \\
[A(k), \Lambda^\dagger(p)] &= i \delta(k - p), \\
[A(k), \Lambda(p)] &= 0 = [A^\dagger(k), \Lambda^\dagger(p)].
\end{align*}
\]  

(39)

We can verify that $A$ and $A^\dagger$ are raising and lowering operators, respectively (whereas $\Lambda^\dagger$ and $\Lambda$ are mixing ones). However, we cannot talk on the spectrum of the field $\phi$ because eigenvalues of $H_c$ are not necessarily positive and, consequently, this does not permit us to introduce a vacuum state. At this stage, it may be opportune to mention the paper of ref. [7], where ghost fields have been conveniently introduced to solve this problem and it was then possible to define a vacuum state, but it was shown that this vacuum state is the only physical state of the theory.

### 3 Improved version of chiral-boson with linear constraints

We now refer to the theory described by Lagrangian [3]. Its canonical momenta are the same as the Srivastava’s model but we have now three (continuous) constraints

\[
\begin{align*}
\Omega_1 &= q, \\
\Omega_2 &= \pi - \phi', \\
\Omega_3 &= \lambda'.
\end{align*}
\]  

(40-42)

They satisfy the algebra (only nonvanishing brackets)

\[
\begin{align*}
\{\Omega_1(x, t), \Omega_3(y, t)\} &= \delta'(x - y), \\
\{\Omega_2(x, t), \Omega_2(y, t)\} &= -2 \delta'(x - y).
\end{align*}
\]  

(43)

It is opportune to observe that this theory, contrarily to what occurs with the previous one, has just one physical degree of freedom, the same number of the FJ theory.
Considering Fourier transformations of fields and constraints, we have

\[
\begin{align*}
\tilde{\Omega}_1(k, t) & = \tilde{q}(k, t), \\
\tilde{\Omega}_2(k, t) & = \tilde{\pi}(k, t) - i k \tilde{\phi}(k, t), \\
\tilde{\Omega}_3(k, t) & = i k \tilde{\lambda}(k, t).
\end{align*}
\] (44)

The corresponding matrix \( C(k, p) \) is

\[
C(k, p) = \begin{pmatrix} 
0 & 0 & ik \\
0 & -2ik & 0 \\
iki & 0 & 0 \\
\end{pmatrix} \delta(k + p). \] (47)

As one observes, \( \det C = 0 \) for \( k = 0 \). Constraints \( \tilde{\Omega}_1 \) and \( \tilde{\Omega}_2 \) are first-class at this point (constraint \( \tilde{\Omega}_3 \) is just the identity \( 0 = 0 \)). In order to quantize the theory for all \( k \), we have to fix the gauge. Natural choices are

\[
\begin{align*}
\tilde{\lambda}(0, t) & = 0, \\
\tilde{\phi}(0, t) & = 0.
\end{align*}
\] (48)

(49)

Thus, the full set of constraints we have is

\[
\begin{align*}
\tilde{\Omega}_1(\bar{k}, t) & = \tilde{q}(\bar{k}, t), \\
\tilde{\Omega}_2(\bar{k}, t) & = \tilde{\pi}(\bar{k}, t) - i \bar{k} \tilde{\phi}(\bar{k}, t), \\
\tilde{\Omega}_3(\bar{k}, t) & = i \bar{k} \tilde{\lambda}(\bar{k}, t), \\
\tilde{\Omega}_4 & = \tilde{q}(0, t), \\
\tilde{\Omega}_5 & = \tilde{\pi}(0, t), \\
\tilde{\Omega}_6 & = \tilde{\lambda}(0, t), \\
\tilde{\Omega}_7 & = \tilde{\phi}(0, t).
\end{align*}
\] (50)

The constraint matrix is then enlarged to

\[
C(k, p) = \begin{pmatrix} 
0 & 0 & ik\delta(\bar{k} + \bar{p}) & 0 & 0 & 0 & 0 \\
0 & -2ik\delta(\bar{k} + \bar{p}) & 0 & 0 & 0 & 0 & 0 \\
iki & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta(0) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta(0) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta(0) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\] (51)
that is not singular. The Dirac brackets are directly calculated

\[
\{\tilde{\phi}(\vec{k}, t), \tilde{\phi}(\vec{p}, t)\}_D = \frac{i}{2k} \delta(\vec{k} + \vec{p}),
\]

\[
\{\tilde{\lambda}(\vec{k}, t), \tilde{\lambda}(\vec{p}, t)\}_D = 0 = \{\tilde{\phi}(\vec{k}, t), \tilde{\lambda}(\vec{p}, t)\}_D.
\] (52)

Other brackets involving \(\tilde{\phi}(0)\) and \(\tilde{\lambda}(0)\) are also zero. We notice here that mixing brackets involving \(\lambda\) and \(\phi\) are zero and that the first bracket above is the same as the FJ theory.

To obtain the spectrum we have to calculate the canonical Hamiltonian. Taken into account the same comments made at Sec. 3, we get

\[
H_c = \int dx \left( \phi' \phi' + \lambda' \phi \right),
\]

\[
\rightarrow \int dx \phi' \phi'.
\] (53)

In the last step it was used the constraint \(\Omega_3\). As this is also the same canonical Hamiltonian of the FJ theory, it is not difficult to see that the field \(\phi\) has the same spectrum of the FJ theory and that \(\lambda\) does not generate any state. We can thus conclude that \(L_1\) and \(L_3\) describe the same theory, classically and quantically, contrarily to what occurs with \(L_1\) and \(L_2\) that are equivalently just in terms of equation of motion.

4 Conclusion

We have made a comparative study among chiral-boson theories, mainly in their quantum aspects. In the case of the FJ formulation there is a zero mode where the constraint becomes first-class \(\Omega_3\). We have chosen a convenient gauge-fixing in order to define creation and destruction operators at all points of the momentum space. Concerning chiral-boson with a linear constraint, we have also reviewed the obtainment of the spectrum and discuss the equivalence with the FJ theory. Here, constraints are second-class at all points but it is not possible to define a spectrum \(\Omega_3\).

The main part of our work was a discussion of a improved version of the chiral boson with linear constraint. We have shown that it has the same physical degrees of freedom of the FJ theory and also exhibits a point where constraints are first-class. We have concluded by carefully studying its spectrum and showing that it is also the same of the FJ theory.

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