ANALYSIS ON SEMIHYPERGROUPS: FUNCTION SPACES, HOMOMORPHISMS AND IDEALS

CHOITI BANDYOPADHYAY

ABSTRACT. The main purpose of this article is to initiate a systematic study of Semihypergroups, first introduced by C. Dunkl [4], I. Jewett [13] and R. Spector [20] independently around 1972. We introduce and study several natural algebraic and analytic structures on semihypergroups, which are well-known in the case of topological groups and semigroups. In particular, we first study almost periodic and weakly almost periodic function spaces (basic properties, their relation to the compactness of the underlying space, introversion and Arens product on their duals among others). We then introduce homomorphisms and ideals, and thereby examine their behaviour (basic properties, structure of the kernel and relation of amenability to minimal ideals) in order to gain insight into the structure of a Semihypergroup itself. In the process, we further investigate where and why this theory deviates from the classical theory of semigroups.

1. Introduction

The theory of topological groups and semigroups have been an important area of research in mathematics, specially from early 1960’s. But in practice, one often comes across certain objects arising from groups (coset and double-coset spaces, orbit spaces etc. for example), which although have a structure somewhat similar to groups, are not exactly groups.

Consider the double-coset space $GL_n(\mathbb{R})//O(n)$ for example. Although it retains some group-like structures from $GL_n(\mathbb{R})$, it fails to be a group anymore. In fact, given any locally compact group $G$ and a compact subgroup $H$ of it, the double-coset space $G//H$ fails to be a group, hence forcing us to treat it individually whenever we come across these kinds of objects in research. We experience similar difficulties while investigating the orbit space of a continuous affine action of a compact group on a locally compact group (more discussion on this will follow later) as well.

Around 1972, three mathematicians, C. Dunkl [4], I. Jewett [13] and R. Spector [20] independently came up with a unified theory involving convolution of measures, in pursuit of moulding these kinds of objects into an unified class of objects called Hypergroups.

A hypergroup may essentially be seen as a generalized form of a locally compact group. Only here, the product of two points is a probability measure, rather than a single point;

2010 Mathematics Subject Classification. Primary 43A62, 43A60, 43A10, 43A99, 20E06; Secondary 46G12, 46J20, 46E27.

Key words and phrases. semihypergroups, hypergroups, ideals, homomorphisms, kernel, almost periodic functions, weakly almost periodic functions, free products.

This work is part of the author’s PhD thesis at the University of Alberta.
and as expected, it contains an identity and a certain ‘involution’, which is an interesting
generalization for inverses in the group theory [13].

It turns out that all the double-coset spaces and the orbit spaces where the individual
mappings \( g \mapsto h.g : G \rightarrow G \) are automorphisms, can be given a natural hypergroup-structure;
and hence their properties can directly be studied and derived from the study of the unified
theory of hypergroups. Since the initiation of systematic studies on it, there has been
rapid progress in the research on several areas involving hypergroups, including the area
of commutative and weighted hypergroups, amenability and several function spaces and
algebras on it.

The next question is, what happens when the orbit spaces of continuous affine actions
do not permit all the individual maps \( g \mapsto h.g : G \rightarrow G \) to be automorphisms. For
example, what happens when instead of double-coset spaces, we encounter homogeneous
spaces or simply left-coset spaces like \( GL_n(\mathbb{R})/O_n(\mathbb{R}) \), \( GL_n(\mathbb{R})/SO_n(\mathbb{R}) \), \( S_4/D_8 \), \( S_4/S_3 \), or
for that matter, any left-coset space \( G/H \), \( G \) being any locally compact group and \( H \) being
a compact non-normal subgroup of it. It is apparent that these spaces, although retaining
some group-like structures, are not topological groups. More importantly, none of these
spaces can even be given a hypergroup-structure (reasons explained in the third section).
Hence these kinds of objects that frequently appear while studying the classical theory of
topological groups, fall out of the parent category and can not quite be studied or analysed
with the existing general theory for topological groups and hypergroups.

The solution to this problem can be attained through a detailed study of a more general
category of objects involving convolution algebra of measures, called Semihypergroups. They
were introduced at the same time as hypergroups [4, 13, 20] and in a sense, served as building
blocks for hypergroup-axioms. A semihypergroup, as one would expect, can also be seen as a
generalized form of a locally compact semigroup, where the product of two points is a certain
probability measure, rather than a single point. But unlike hypergroups, it does not need to
have an identity or involution. Hence in a nutshell, a semihypergroup is essentially a locally
compact topological space where the measure space is equipped with a certain convolution
product, turning it into an associative algebra; whereas in the case of a hypergroup, the
measure algebra is also equipped with an identity and an involution.

It can be seen (details in the third section) that all the orbit spaces and coset spaces
discussed above can naturally be given a semihypergroup structure, though not a hypergroup
structure. The fact that semihypergroups contain more generalized objects arising from
different fields of research than classical group and hypergroup theory and yet sustains
enough structure to allow an independent theory to develop, makes it an intriguing and
useful area of study with essentially a broader area of applications.

Unlike hypergroups, there is a severe lack of any extensive prior research on Semihyper-
groups since its inception. The significant examples it contains, opens up a number of new
intriguing paths of research on semihypergroups. The lack of an algebraic structure on the
underlying space poses a serious challenge in extending results from semigroups to semi-
hypergroups. Also unlike hypergroups, the fact that a semihypergroup structure lacks the
existence of a Haar measure or an involution in its measure algebra, creates a serious obstacle to the extension of most of the important group and hypergroup theories and ideas naturally to semihypergroups.

Hence the main motivation of this paper is to initiate and develop a systematic study of semihypergroups. We will base our work on semihypergroups on Jewett’s definition of ‘semiconvos’ [13]. In this particular paper, we will restrict ourselves to the discussion of some particular functions spaces, ideals and homomorphisms of a semihypergroup. The rest of the paper is designed as below.

In the next, i.e., second section of this article, we recall some preliminary definitions and notations given by Jewett in [13], and introduce some new definitions required for further work. In the third section, we list some important useful examples of semihypergroups and hypergroups.

In our fourth section we discuss two of the most important function spaces on semihypergroups, namely the spaces of almost periodic and weakly almost periodic functions. Here we explore the relation between the left and right counterparts of these function spaces (Theorem 4.2, Corollary 4.4) and show how the compactness of the underlying space influences amenability and the structure of these function spaces, depending on the kind of continuity we have on the measure algebra of the underlying space (Proposition 4.10, Theorems 4.11, 4.13). We then introduce introversion on the function spaces and conclude the section with examining Arens regularity on the duals of these function spaces (Corollary 4.18, Theorem 4.19).

In the fifth section, we deal with the inner structure of a semihypergroup itself. We introduce the concept of an ideal in (semitopological) semihypergroups and explore some of its basic properties as well as its relation to a more general form of homomorphism between semihypergroups. Furthermore, we investigate the structure of the kernel of a compact (semitopological) semihypergroup (Theorem 5.14), giving us insight into the algebraic structure of a semihypergroup. The discussion also allows us insight into where exactly this theory starts deviating from the classical theory of topological semigroups (for example, Remark 5.12). Finally, we conclude the section with exploring the connection between minimal left ideals and amenability on a compact semihypergroup (Theorem 5.16).

Finally, this article is the initiation of a systematic study of this class of objects with intriguing scope for application in the theory of coset and orbit spaces of locally compact groups, homogeneous spaces and Lie groups. We conclude with some potential problems and areas which we intend to work on and explore further in near future, in addition to introducing and investigating similar structures as in [5], [6], [7], [9], [10], [11], [14], [15], [18], [21], [22] for the case of semihypergroups and explore further where and why the theory deviates from the classical theory of topological semigroups and groups.

2. Notations and Definitions

Here we first list some of the preliminary set of basic notations that we will use throughout, followed by a brief introduction to the tools and concepts needed for the formal definition of a
semi-hypergroup [13]. All the topologies throughout this article are assumed to be Hausdorff, unless otherwise specified.

For any locally compact Hausdorff topological space $X$, we denote by $M(X)$ the space of all regular complex Borel measures on $X$, where $M^+(X)$ and $P_+(X)$ denote the subsets of $M(X)$ consisting of all finite non-negative regular Borel measures on $X$ and all probability measures with compact support on $X$ respectively. Moreover, $B(X), B(X), C(X)$ and $C^*_c(X)$ denote the function spaces of all bounded functions, Borel measurable functions, bounded continuous functions and non-negative compactly supported continuous functions on $X$ respectively.

Next, we introduce two very important topologies on the positive measure space and the space of compact subsets for any locally compact topological space $X$. Unless mentioned otherwise, we will always assume these two topologies on the respective spaces.

The cone topology on $M^+(X)$ is defined as the weakest topology on $M^+(X)$ for which the maps $\mu \mapsto \int_X f \, d\mu$ is continuous for any $f \in C^*_c(X) \cup \{1_X\}$ where $1_X$ denotes the characteristic function of $X$. Note that if $X$ is compact then it follows immediately from the Riesz representation theorem that the cone topology coincides with the weak*-topology on $M^+(X)$ in this case.

We denote by $\mathfrak{C}(X)$ the set of all compact subsets of $X$. The Michael topology on $\mathfrak{C}(X)$ is defined to be the topology generated by the sub-basis $\{C_{UV} : U, V \text{ are open sets in } X\}$, where

$$C_{UV} = \{C \in \mathfrak{C}(X) : C \cap U \neq \emptyset, C \subset V\}.$$ 

Note that $\mathfrak{C}(X)$ actually becomes a locally compact Hausdorff space with respect to this natural topology. Moreover if $X$ is compact then $\mathfrak{C}(X)$ is also compact.

For any locally compact Hausdorff space $X$ and any element $x \in X$, we denote by $p_x$ the point-mass measure or the Dirac measure at the point $\{x\}$.

For any three locally compact Hausdorff spaces $X, Y, Z$, a bilinear map $\Psi : M(X) \times M(Y) \to M(Z)$ is called positive continuous if the following holds:

1. $\Psi(\mu, \nu) \in M^+(Z)$ whenever $\mu \in M^+(X), \nu \in M^+(Y)$.
2. The map $\Psi|_{M^+(X) \times M^+(Y)}$ is continuous.

Now we are ready to state the formal definition for a semi-hypergroup. Note that we follow Jewett’s notion [13] in terms of the definitions and notations, in most cases.

**Definition 2.1. (Semihypergroup)** A pair $(K, *)$ is called a (topological) semihypergroup if it satisfies the following properties:

(A1): $K$ is a locally compact Hausdorff space and $*$ defines a binary operation on $M(K)$ such that $(M(K), *)$ becomes an associative algebra.

(A2): The bilinear mapping $*: M(K) \times M(K) \to M(K)$ is positive continuous.

(A3): For any $x, y \in K$ the measure $p_x * p_y$ is a probability measure with compact support.

(A4): The map $(x, y) \mapsto \text{supp}(p_x * p_y)$ from $K \times K$ into $\mathfrak{C}(K)$ is continuous.

A semi-hypergroup $(K, *)$ is called a hypergroup if $K$ admits an identity element $e \in K$ and a certain involution map (analogous to inverses in groups). Note that for any $A, B \subseteq K$ the
ANALYSIS ON SEMIHYPERSUBGROUPS

5
convolution of subsets is defined as the following:

\[ A \ast B := \bigcup_{x \in A, y \in B} \text{supp}(p_x \ast p_y) \]

Now we conclude this section by introducing the concept of left (or right) topological and semitopological semihypergroups, similar to the concepts of left (or right) topological and semitopological semigroups.

**Definition 2.2.** A pair \((K, \ast)\) is called a left topological semihypergroup if it satisfies all the conditions of Definition 2.1 with property \((A2)\) replaced by the following:

\[(A2'): \text{The map } (\mu, \nu) \mapsto \mu \ast \nu \text{ is positive and for each } \omega \in M^+(K) \text{ the map } L_\omega : M^+(K) \to M^+(K) \text{ given by } L_\omega(\mu) = \omega \ast \mu \text{ is continuous.} \]

**Definition 2.3.** A pair \((K, \ast)\) is called a right topological semihypergroup if it satisfies all the conditions of Definition 2.1 with property \((A2)\) replaced by the following:

\[(A2''): \text{The map } (\mu, \nu) \mapsto \mu \ast \nu \text{ is positive and for each } \omega \in M^+(K) \text{ the map } R_\omega : M^+(K) \to M^+(K) \text{ given by } R_\omega(\mu) = \mu \ast \omega \text{ is continuous.} \]

**Definition 2.4.** A pair \((K, \ast)\) is called a semitopological semihypergroup if it is both left and right topological semihypergroup.

3. Examples

In this section, we list some well known examples \([13, 24]\) of semihypergroups and hypergroups. Thus we explore how the shortcomings explained in the introduction are overcome by the category of semihypergroups and hypergroups, as well as why most of the structures discussed there, although they attain semihypergroup structures, fail to be hypergroups.

**Example 3.1.** If \((S, \cdot)\) is a locally compact topological semigroup, then \((S, \ast)\) is a semihypergroup where \(p_x \ast p_y = p_{xy}\) for any \(x, y \in S\).

Similarly, if \((G, \cdot)\) is a locally compact topological group, then \((G, \ast)\) is a hypergroup with the same bilinear operation \(\ast\), identity element \(e\) where \(e\) is the identity of \(G\) and the involution on \(G\) defined as \(x \mapsto x^{-1}\).

**Example 3.2.** Take \(T = \{e, a, b\}\) and equip it with the discrete topology. Define

\[
\begin{align*}
p_e \ast p_a &= p_a \ast p_e = p_a \\
p_e \ast p_b &= p_b \ast p_e = p_b \\
p_a \ast p_b &= p_b \ast p_a = z_1 p_a + z_2 p_b \\
p_a \ast p_a &= x_1 p_e + x_2 p_a + x_3 p_b \\
p_b \ast p_b &= y_1 p_e + y_2 p_a + y_3 p_b 
\end{align*}
\]

where \(x_i, y_i, z_i \in \mathbb{R}\) such that \(x_1 + x_2 + x_3 = y_1 + y_2 + y_3 = z_1 + z_2 = 1\) and \(y_1 x_3 = z_1 x_1\). Then \((T, \ast)\) is a commutative hypergroup with identity \(e\) and the identity function on \(T\) taken as involution.
Example 3.3. Let $G$ be a locally compact topological group and $H$ be a compact subgroup of $G$. Also, let $\mu$ be the normalized Haar measure of $H$. Consider the left quotient space $S := G/H = \{xH : x \in G\}$ and equip it with the quotient topology. For any $x, y \in G$, define

$$p_{xH} * p_{yH} = \int_H p_{(xy)tH} \, d\mu(t).$$

Then $(S, \ast)$ is a semihypergroup.

For instance, take $G$ to be the symmetric group $S_4$ and take $H$ to be the dihedral group $D_8$. We know that $D_8$ is not a normal subgroup of $S_4$. Consider the left coset space $G/H = \{H, s_1H, s_2H\}$ where $s_1 = (124)$ and $s_2 = (142)$. Then the above formulation gives us that

$$p_{xH} * p_{yH} = \frac{1}{8} \sum_{h \in H} p_{(xy)tH}.$$

Hence a direct computation gives us that the left coset space $S_4/D_8$ is a discrete semihypergroup where the convolution is given by the following table:

| $\ast$ | $p_H$ | $p_{s_1H}$ | $p_{s_2H}$ |
| --- | --- | --- | --- |
| $p_H$ | $p_H$ | $\frac{1}{2}(p_{s_1H} + p_{s_2H})$ | $\frac{1}{2}(p_{s_1H} + p_{s_2H})$ |
| $p_{s_1H}$ | $p_{s_1H}$ | $\frac{1}{2}(p_H + p_{s_2H})$ | $\frac{1}{2}(p_H + p_{s_1H})$ |
| $p_{s_2H}$ | $p_{s_2H}$ | $\frac{1}{2}(p_H + p_{s_1H})$ | $\frac{1}{2}(p_H + p_{s_2H})$ |

Note that here the element $x = H$ fails to be a right identity, and hence $S_4/D_8$ does not attain a hypergroup structure. Thus in general, for a left coset space $G/H$ where $H$ is not normal, $H$ fails to serve as a right identity.

The next example [13] of double-coset spaces overcomes this barrier and becomes a hypergroup using a similar convolution product.

Example 3.4. Consider $G$ and $H$ as in the previous example and equip the space of double cosets $K := G//H = \{HxH : x \in G\}$ with the usual quotient topology. For any $x, y \in G$, define

$$p_{HxH} * p_{HyH} = \int_H p_{H(xy)tH} \, d\mu(t).$$

Then $(K, \ast)$ is a hypergroup with the identity element $e = H$ and involution function $HxH \mapsto Hx^{-1}H$.

Finally, the next example [13] of orbit spaces includes [13] both Examples 3.3, 3.4. Recall that a continuous action of a topological group $T$ on a Hausdorff space $X$ is a continuous map $(t, x) \mapsto x^t : T \times X \to X$ such that $x^e = x$ and $x^{st} = (x^s)^t$ for each $x \in X$, $s, t \in T$.

Example 3.5. Let $G$ be a locally compact topological group and $H$ be any compact group. Recall that a map $\phi : G \to G$ is called affine if there exists a scaler $\alpha$ and an automorphism $\Psi$ of $G$ such that $\phi(x) = \alpha \Psi(x)$ for each $x \in G$. 
A continuous action $\pi$ of $H$ on $G$ given by $\pi(h, g) = g^h$ for each $g \in G, h \in H$ is called a continuous affine action if the map $g \mapsto g^h : G \to G$ is affine for each $h \in H$. For any continuous affine action $\pi$ of $H$ on $G$, consider the orbit space

$$O := \{x^H : x \in G\},$$

where $x^H = O(x) = \{\pi(h, x) : h \in H\}$.

Let $\sigma$ be the normalized Haar measure of $H$. Consider $O$ with the quotient topology and the following convolution

$$p_{x^H} \ast p_{y^H} := \int_H \int_H p_{\pi(s,x)\pi(\pi(t,y))} \, d\sigma(s) \, d\sigma(t).$$

Then $(O, \ast)$ becomes a semihypergroup.

### 4. Almost Periodic and Weakly Almost Periodic Functions

In this section, we start with testing some basic properties of the spaces of almost periodic and weakly almost periodic functions for a (semitopological) semihypergroup, and investigate the relation between these spaces and compactness of the underlying space. Finally, we conclude the section with examining Arens regularity [1] on the dual of the space of almost periodic functions.

Recall [13] that for any measurable function $f$ on a (semitopological) semihypergroup $K$ and each $x, y \in K$ we define the left and right translates of $f$ (denoted as $L_x f$ and $R_x f$ respectively) as follows.

$$L_x f(y) = R_y f(x) = f(x \ast y) = \int_K f \, d(p_x \ast p_y)$$

For any function $f \in C(K)$ we define the right orbit $O_r(f)$ of $f$ as

$$O_r(f) := \{R_x f : x \in K\}.$$

A function $f \in C(K)$ is called right almost periodic if $O_r(f)$ is relatively compact in $C(K)$ with respect to the norm topology. Similarly, a function $f \in C(K)$ is called right weakly almost periodic if $O_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology on $C(K)$.

We denote these two classes of functions as:

$$AP_r(K) := \text{Space of all right almost periodic functions on } K.$$  
$$WAP_r(K) := \text{Space of all right weakly almost periodic functions on } K.$$
both left and right uniformly continuous. The space consisting of such functions is denoted by \( UC(K) \).

Let \( K \) be a (semitopological) semihypergroup with identity and \( \mathcal{F} \) a linear subspace of \( C(K) \) containing constant functions. A function \( m \in \mathcal{F}^* \) is called a mean of \( \mathcal{F} \) if we have that \( ||m|| = 1 = m(1) \). If \( \mathcal{F} \) is a translation-invariant linear subspace of \( C(K) \) containing constant functions, a mean \( m \) of \( \mathcal{F} \) is called a left invariant mean (LIM) if \( m(L_x f) = m(f) \) for any \( x \in K \), \( f \in \mathcal{F} \). We can similarly define a right invariant mean (RIM) for \( \mathcal{F} \). We say that a (semitopological) semihypergroup \( K \) is left (resp. right) amenable if \( C(K) \) admits a LIM (resp. RIM).

Now we examine the relation between the spaces of left and right (weakly) almost periodic functions. But before we prove the first result, let us first recall an important result on the existence of a vector integral on the function space of a general locally compact Hausdorff space, proved in [19].

**Theorem 4.1.** Suppose that \((X, \tau)\) is a Hausdorff topological vector space where \( X^* \) separates points, and \( \lambda \in P_c(Y) \) where \( Y \) is a locally compact Hausdorff space. If a function \( F : Y \to X \) is continuous and \( \overline{co}(F(Y)) \) is compact in \( X \) then the vector integral \( \omega := \int_Y F d\lambda \) exists and \( \omega \in \overline{co}(F(Y)) \).

We divide the proof of our next theorem in a series of key steps for convenience, due to the length of the proof.

**Theorem 4.2.** Let \( K \) be a semihypergroup. Then

\[
WAP_r(K) \cap UC(K) = WAP_l(K) \cap UC(K).
\]

**Proof.** Pick any \( f \in WAP_l(K) \cap UC(K) \). We need to show that \( f \in WAP_r(K) \), i.e, \( \mathcal{O}_r(f) \) is relatively weakly compact in \( C(K) \). We will show this in five steps.

**Step I:** Embed \( K \) into \( P_c(K) \) through the homomorphism \( x \mapsto p_x \) [13]. Now define a map \( \Phi : C(K) \to C(P_c(K)) \) by \( \Phi(f) = \hat{f} \) for any \( f \in C(K) \) where

\[
\hat{f}(\mu) := \int f \, d\mu \quad \forall \, \mu \in P_c(K).
\]

We know that \( \Phi \) is an isometry [13]. Thus \( \Phi \) is continuous in the norm topologies of \( C(K) \) and \( C(P_c(K)) \) and hence \( \Phi \) is continuous in the weak topologies of \( C(K) \) and \( C(P_c(K)) \).

**Step II:** For any \( \mu \in P_c(K) \) define a function \( \hat{L}_\mu f \) on \( K \) by

\[
\hat{L}_\mu f(x) := \int_K f(y \ast x) \, d\mu(y).
\]

Since \( f \in UC(K) \) the map \( x \mapsto R_x f \) is continuous, and \( \hat{L}_\mu f(x) = \int_K R_x f \, d\mu \) by the above construction. Hence \( \hat{L}_\mu f \in C(K) \) for any \( \mu \in P_c(K) \). Now define a subset in \( C(K) \) as the following.

\[
\mathcal{O}_l(f) := \{ \hat{L}_\mu f : \mu \in P_c(K) \}.
\]
Now consider the left-translation map \( \psi : K \rightarrow C(K) \) given by \( \psi(x) := L_x f \). Since \( f \in UC(K) \) \( \psi \) is continuous on \( K \). Also since \( f \in WAP(K) \) the left orbit of \( f \) namely \( \psi(K) \) is relatively weakly compact in \( C(K) \).

Hence by the Krein-Smulian Theorem we have that \( \overline{\sigma}(\psi(K)) \) is weakly compact in \( C(K) \), i.e, \( \overline{\sigma}(\mathcal{O}_l(K)) \) is weakly compact in \( C(K) \).

**Step III:** Now can use Theorem 4.1 by letting \((X, \tau) = (C(K), \text{weak topology}), Y = K, F = \psi \) and \( \lambda = \mu \) for any \( \mu \in P_c(K) \). Thus we see that \( \omega_0 := \int_K \psi \ d\mu \in \overline{\sigma}(\psi(K)) \). Now for any \( x \in K \) we have that

\[
\omega_0(x) = \int_K (\psi(y))(x) \ d\mu(y)
= \int_K L_y f(x) \ d\mu(y)
= \int_K f(y \ast x) \ d\mu(y) = \hat{L}_\mu f(x).
\]

Since \( \psi(K) = \mathcal{O}_l(f) \) by construction, we see that \( \hat{L}_\mu f \in \overline{\sigma}(\mathcal{O}_l(K)) \) for any \( \mu \in P_c(K) \). Hence \( \mathcal{O}_l(f) \subset \overline{\sigma}(\mathcal{O}_l(K)) \). But from Step II we know that \( \overline{\sigma}(\mathcal{O}_l(K)) \) is weakly compact in \( C(K) \). Hence \( \mathcal{O}_l(f) \) is relatively weakly compact in \( C(K) \).

**Step IV:** Note that \( (\hat{L}_\mu f) = L_\mu \tilde{f} \) for any \( \mu \in P_c(K) \). This is true since for any \( \nu \in P_c(K) \) we have that

\[
(\hat{L}_\mu f)\nu = \int_K \hat{L}_\mu f(x) \ d\nu(x)
= \int_K \int_K f(y \ast x) \ d\mu(y) \ d\nu(x)
= \int_K f \ d(\mu \ast \nu)
= \tilde{f}(\mu \ast \nu) = L_\mu \tilde{f}(\nu).
\]

where the third equality follows from [13 Theorem 3.1E]. Thus we see that

\[
(\mathcal{O}_l(f)) = \{ \tilde{f} : f \in \mathcal{O}_l(f) \subset C(K) \} = \mathcal{O}_l(\tilde{f}).
\]

From Step III we know that \( \mathcal{O}_l(f) \) is relatively weakly compact in \( C(K) \), and from Step I we see that the map \( f \mapsto \tilde{f} \) from \( C(K) \) to \( C(P_c(K)) \) is continuous when both spaces are equipped with weak topology. Hence the set \( (\mathcal{O}_l(f)) \) is also relatively weakly compact in \( C(P_c(K)) \). Thus \( \mathcal{O}_l(\tilde{f}) \) is relatively weakly compact in \( C(P_c(K)) \), i.e, \( \tilde{f} \in WAP(P_c(K)) \).

**Step V:** We know that \((P_c(K), \ast)\) is a topological semigroup. Hence \( WAP_r(P_c(K)) = WAP_r(P_c(K)) \). Thus we have that \( \tilde{f} \in WAP_r(P_c(K)) \), i.e, \( \mathcal{O}_r(\tilde{f}) \) is relatively weakly compact in \( C(P_c(K)) \).

Hence the set \( N := \{ R_{\mu} \tilde{f} : x \in K \} \subset \mathcal{O}_r(\tilde{f}) \) is also relatively weakly compact in \( C(P_c(K)) \).
Now consider the map \( \phi : C(P_c(K)) \to C(K) \) given by \( h \mapsto \tilde{h} \) where \( \tilde{h}(x) := h(p_x) \) for any \( x \in K \). The fact that \( \tilde{h} \) is continuous on \( K \) follows directly from the fact that the map \( x \mapsto p_x \) is continuous. Also, \( \phi \) is continuous since \( ||\tilde{h}|| = \sup_{x \in K} |h(p_x)| \leq ||h|| \) for any \( h \in C(P_c(K)) \).

Note that for each \( h \in C(K) \), \( x \in K \), \( \phi(R_{p_x}h) = R_xh \) since for any \( y \in K \) we have

\[
\phi(R_{p_x}h)(y) = (R_{p_x}h)(y) = \tilde{h}(p_y * p_x) = h(y * x) = R_xh(y).
\]

Thus in particular, we have that \( \phi(N) = O_r(f) \). Since \( \phi \) is continuous and \( N \) is relatively weakly compact in \( C(P_c(K)) \), we finally have that \( O_r(f) \) is relatively weakly compact in \( C(K) \), i.e., \( f \in WAP_r(K) \) as required.

Similarly for any \( f \in WAP_r(K) \cap UC(K) \) we can show that \( f \in WAP_l(K) \). Hence the proof is complete. \( \square \)

**Remark 4.3.** In the proof of Theorem 4.2 the ideas involved in steps I-III are similar to [23, Theorem 2.4] as no specific involution properties are required. However, the above proof of those steps contains several important details, not found in the aforementioned text.

**Corollary 4.4.** Let \( K \) be a semihypergroup. Then \( AP_l(K) = AP_r(K) \).

**Proof.** This can be proved in exactly similar manner as in Theorem 4.2. The only modification needed is that, we need to consider the norm topologies on \( C(K) \) and \( C(P_c(K)) \) as opposed to the fact that we had to consider weak topologies on these two spaces in the proof of Theorem 4.2. \( \square \)

The following facts, regarding the relation between the function spaces in question, are proved for hypergroups in [23]. The same proof works for semihypergroups as well.

**Proposition 4.5.** Let \( K \) be a semihypergroup. Then the following statements hold true.

1. \( WAP_l(K), WAP_r(K), AP_l(K), AP_r(K) \) are norm-closed, conjugate-closed subsets of \( C(K) \) containing the constant functions.
2. \( AP_l \subset UC(K) \) and \( AP_r(K) \subset UC(K) \).
3. \( AP_l(K) \subset WAP_l(K) \) and \( AP_r(K) \subset WAP_r(K) \).

Next we examine the translation-invariance properties of the spaces of (weakly) almost periodic functions on a (semitopological) semihypergroup. Recall that a function space \( \mathcal{F} \) on a (semitopological) semihypergroup \( K \) is called left (or right) translation-invariant if we have that \( L_xf \in \mathcal{F} \) (or \( R_xf \in \mathcal{F} \)) for any \( f \in \mathcal{F} \) and \( x \in K \). Also, \( \mathcal{F} \) is simply said to be translation-invariant if it is both left and right translation-invariant.

**Proposition 4.6.** Let \( K \) be any semitopological semihypergroup. Then \( AP(K) \) is translation-invariant.

**Proof.** Pick any \( f \in AP(K), \ x_0 \in K \). Now consider the map \( \Phi : C(K) \to C(K) \) given by \( g \mapsto L_{x_0}g \). Note that for any \( x, y, t \in K \) we have that

\[
R_xL_yl(f(t)) = L_yf(t * x) = f(y * t * x) = R_xf(y * t) = L_yR_xf(t).
\]
Proof. Pick any \( g \in C(K) \) we have that

\[
\| \Phi(g) - \Phi(h) \| = \sup_{y \in K} \left| L_{x_0}(g - h)(y) \right|
\]

\[
\leq \| g - h \| \sup_{y \in K} |(p_{x_0} \ast p_y)(K)| = \| g - h \|.
\]

Thus we see that \( \Phi \) is continuous in the norm topology on \( C(K) \). Hence \( \Phi(\overline{O_r(f)}) \) is compact since \( f \in AP_r(K) \). Also as noted before, since \( O_r(L_{x_0}f) = \Phi(O_r(f)) \subset \Phi(\overline{O_r(f)}) \) we have that \( O_r(L_{x_0}f) \) is relatively compact in \( C(K) \) with respect to the norm-topology and hence \( L_{x_0}f \in AP_r(K) = AP(K) \). A similar argument, using the map \( \Psi : C(K) \to C(K) \) given by \( g \mapsto R_{x_0}g \), shows that \( O_t(R_{x_0}f) \) is relatively compact in \( C(K) \) so that \( R_{x_0}f \in AP_t(K) = AP(K) \) as required. \( \square \)

Proposition 4.7. Let \( K \) be any semitopological semihypergroup. Then

(1) \( WAP_r(K) \) is left translation-invariant.
(2) \( WAP_t(K) \) is right translation-invariant.

Proof. Pick any \( f \in WAP_r(K), x_0 \in K \). As in the proof of the above theorem, consider the map \( \Phi : C(K) \to C(K) \) given by \( g \mapsto L_{x_0}g \). Note that \( \overline{O_r(f)} \) is compact in \( C(K) \). But the weak topology on \( C(K) \) is stronger than the topology of pointwise-convergence where the latter topology is Hausdorff. Hence these two topologies coincide on any compact subset on \( C(K) \), in particular on \( \overline{O_r(f)} \).

Now let \( g \in \overline{O_r(f)} \) and \( \{ g_\alpha \} \) be a net in \( \overline{O_r(f)} \) such that \( g_\alpha \xrightarrow{w} g \) on \( K \). Then we have

\[
\int_K g_\alpha \ d(p_{x_0} \ast p_y) \rightarrow \int_K g \ d(p_{x_0} \ast p_y) \text{ for each } y \in K.
\]

\[
\Rightarrow \quad L_{x_0}g_\alpha \rightarrow L_{x_0}g \text{ pointwise on } K.
\]

\[
\Rightarrow \quad L_{x_0}g_\alpha \xrightarrow{w} L_{x_0}g \text{ on } C(K).
\]

Thus \( \Phi \) is weak-weak continuous on \( \overline{O_r(f)} \) and hence \( \Phi(\overline{O_r(f)}) \) is weakly compact in \( C(K) \). Now the fact that \( L_{x_0}f \in WAP_r(K) \) follows from the fact that

\[
O_r(L_{x_0}f) = \{ R_xL_{x_0}f : x \in K \}
\]

\[
= \{ L_{x_0}R_xf : x \in K \} = \Phi(O_r(f)) \subset \Phi(\overline{O_r(f)}) = \overline{O_r(f)}.
\]

In a similar manner we can also see that \( WAP_t(K) \) is right translation-invariant. \( \square \)

Next we examine the behavior of the (weakly) almost periodic function spaces when the underlying (semitopological) semihypergroup is compact. Before we proceed further, let us first quickly recall a widely-used result proved by A. Grothendieck [12].
Theorem 4.8 (Grothendieck). Let $X$ be a compact Hausdorff space. Then a bounded set in $C(X)$ is weakly compact if and only if it is compact in the topology of pointwise convergence.

The next Proposition also follows easily from [13, Section 3].

**Proposition 4.9.** Let $K$ be a semitopological semihypergroup and $f$ a continuous function on $C(K)$. Then the map $(x, y) \mapsto f(x * y)$ is separately continuous.

**Proposition 4.10.** If $K$ is a compact semitopological semihypergroup, then

$$\text{WAP}_r(K) = C(K).$$

**Proof.** Pick any $f \in C(K)$. We need to show that $\mathcal{O}_r(f)$ is weakly compact in $C(K)$.

Let $\{x_\alpha\}$ be a net in $K$ converging to $x$. By Proposition 4.9 we see that for each $y \in K$ the map $x \mapsto f(x * y) = R_x f(y)$ is continuous and hence $R_{x_\alpha} f(y) \to R_x f(y)$. Thus the map $x \mapsto R_x f$ from $K$ into $C(K)$ is continuous where $C(K)$ is equipped with the topology of pointwise convergence.

By Theorem 4.8 we have that $\mathcal{O}_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology.

It turns out that the result becomes much stronger if we have joint-continuity on the convolution product $*$ on $M(K)$, as opposed to separate continuity as in the above case. The semigroup versions for both the above proposition and the following theorem can be found in [3].

**Theorem 4.11.** If $K$ is a compact semihypergroup, then

$$\text{AP}(K) = \text{WAP}_r(K) = \text{WAP}_t(K) = C(K).$$

**Proof.** Pick any $f \in C(K)$. We need to show that $\mathcal{O}_r(f)$ is relative compact in $C(K)$ with respect to the strong (norm) topology.

Consider the map $\Phi : K \to C(K)$ given by $x \mapsto R_x f$. Since the map $(\mu, \nu) \mapsto \mu * \nu$ is continuous on $M^+(K) \times M^+(K)$ we have that the map $(x, y) \mapsto f(x * y)$ is continuous on $K \times K$.

Fix $y_0 \in K$ and pick any $\varepsilon > 0$. Then for each $x \in K$ we will get open neighborhoods $V_x$ of $x$ and $W_x$ of $y_0$ such that

$$|f(x * y_0) - f(s * t)| < \varepsilon$$

where $(s, t) \in V_x \times W_x$.

Since $\{V_x\}_{x \in K}$ is an open cover of the compact space $K$, we will have a finite subcover $\{V_{x_i}\}_{i=1}^n$ that covers $K$. Set $W := \cap_{i=1}^n W_{x_i}$. Then for any $x \in K, t \in W$

$$|R_{y_0} f(x) - R_t f(x)| = |f(x * y_0) - f(x * t)| < \varepsilon.$$

Thus we get an open neighborhood $W$ of $y$ such that $||R_{y_0} f - R_t f||_\infty < \varepsilon$ for any $t \in W$.

Hence the map $\Phi$ is continuous and so $\mathcal{O}_r(f)$ is compact.

Since $\text{AP}(K) \subset \text{WAP}_t(K)$ and $\text{AP}(K) \subset \text{WAP}_t(K)$ the result follows. \qed
The immediate question that naturally rises now is whether converses to Proposition 4.10 and Theorem 4.11 also hold. The answer is yes for hypergroups. For semihypergroups, the converse is partially true, which needs the introduction of some more classes and techniques regarding semihypergroups. This will be discussed in details in another upcoming article by the author. Before we proceed further, let us first note the following property of (weakly) almost periodic functions on a semitopological semihypergroup.

**Proposition 4.12.** Let $K$ be a semitopological semihypergroup and $f \in C(K)$. If $O_r(f)$ is (weakly) relatively compact in $C(K)$ then the map $x \mapsto R_x f$ is (weakly) continuous on $K$.

**Proof.** First assume that $O_r(f)$ is relatively compact in $C(K)$ in norm topology. Let $\{x_\alpha\}$ be a net in $K$ converging to $x \in K$. Then by Proposition 4.10 we have that $f(y * x_\alpha) \to f(y * x)$ for each $y \in K$ and hence $R_{x_\alpha} f \to R_x f$ pointwise in $C(K)$.

If the net $\{R_{x_\alpha} f\}$ has a limit point in $C(K)$, it has to be $R_x f$. But $\overline{O_r(f)}$ is compact in $C(K)$ and hence $R_{x_\alpha} f \to R_x f$, as required.

The case where $O_r(f)$ is relatively compact in $C(K)$ with respect to the weak topology, can be proved proceeding along the same lines. \[\square\]

Note that all the above results will also hold true for the spaces of left almost periodic functions. The next result gives us a sufficient condition for the existence of a left-invariant mean on a semihypergroup. Recall that a (semitopological) semihypergroup $K$ is called *commutative* whenever we have that $\mu * \nu = \nu * \mu$ for all $\mu, \nu \in M(K)$, or equivalently, whenever $p_x * p_y = p_y * p_x$ for all $x, y \in K$.

**Theorem 4.13.** Let $K$ be a commutative semihypergroup. Then there exists a LIM on $C(K)$.

**Proof.** Consider the function $\Phi : C(K) \to C(P_C(K))$ introduced in the proof of Theorem 4.2 which was defined as $f \mapsto \tilde{f}$ where $\tilde{f}(\mu) := \int_K f d\mu$ for each $\mu \in P_C(K)$.

We know that $(P_C(K),\ast)$ is a commutative semigroup. There exists a LIM $m$ on $P_C(K)$ [3, 17]. define $\tilde{m} := m \circ \Phi : C(K) \to \mathbb{C}$, i.e., $\tilde{m}(f) = m(\tilde{f})$ for each $f \in C(K)$.

Note that for any $f \in C(K)$ such that $f \geq 0$ we have that $\tilde{f}(\mu) = \int_K f d\mu \geq 0$ for any $\mu \in P_C(K)$. Therefore $\tilde{m}(f) = m(\tilde{f}) \geq 0$. Moreover, $\tilde{m}(1) = m(\tilde{1}) = m(1) = 1$. Hence $\tilde{m}$ is a mean on $C(K)$.

Now pick any $x \in K$, $f \in C(K)$. For any $\mu \in P_C(K)$ we have that

$$\langle L_x f \rangle(\mu) = \int_K L_x f \ d\mu$$

$$= \int_K f \ d(p_x * \mu)$$

$$= \tilde{f}(p_x * \mu) = L_{p_x} \tilde{f}(\mu).$$

Now the left invariance of $\tilde{m}$ follows since

$$\tilde{m}(L_x f) = m(\langle L_x f \rangle) = m(L_{p_x} \tilde{f}) = m(\tilde{f}) = \tilde{m}(f).$$

\[\square\]
In the final part of this section, we introduce the concept of introversion on a function space and explore how the introversion operators help us in acquiring an algebraic structure on \( AP(K)^* \).

**Definition 4.14.** Let \( \mathcal{F} \) be a translation-invariant linear subspace of \( C(K) \). For each \( \mu \in \mathcal{F}^* \) the left introversion operator \( T_\mu \) determined by \( \mu \) is the map \( T_\mu : \mathcal{F} \to B(K) \) defined as

\[
T_\mu f(x) := \mu(L_x f)
\]

for each \( x \in K \).

Similarly, the right introversion operator \( U_\mu \) determined by \( \mu \) is the map \( U_\mu : \mathcal{F} \to B(K) \) given by

\[
U_\mu f(x) := \mu(R_x f).
\]

Here \( \mathcal{F} \) is called left-introverted if \( T_\mu f \in \mathcal{F} \) for each \( \mu \in \mathcal{F}^* \), \( f \in \mathcal{F} \). Similarly, \( \mathcal{F} \) is called right-introverted if \( U_\mu f \in \mathcal{F} \) for each \( \mu \in \mathcal{F}^* \), \( f \in \mathcal{F} \). We denote by \( \mathcal{B}_1 \) the closed unit ball of \( AP(K)^* \). Before we proceed further to define an algebraic structure on \( AP(K)^* \), let us first explore some basic properties of introversion operators on \( AP(K) \).

The following property gives us a necessary and sufficient condition for a function to be almost periodic, in terms of left and right introversion operators. Let us first quickly recall the following version of Mazur’s Theorem and another important result from [17].

**Theorem 4.15** (Mazur). Let \( A \) be a compact subset of a Banach space \( X \). Then the closed circled convex hull of \( A \), denoted as \( \text{cco}(A) \), is also compact.

**Theorem 4.16.** Let \( K \) be a semihypergroup and \( \mathcal{F} \) be a translation-invariant conjugation-closed linear subspace of \( B(K) \) containing constant functions. For any \( f \in \mathcal{F} \), the set \( \{ T_\mu f : ||\mu|| \leq 1 \} \) is the closure of \( \text{cco}(\mathcal{O}_r(f)) \) in \( B(K) \) with respect to the topology of pointwise convergence.

A proof for the above version of Mazur’s Theorem can be found in [17, Appendix A]. The above theorem is proved for topological semigroups in [17, Section 2.2]. The proof for semihypergroups follows along similar lines.

**Theorem 4.17.** Let \( K \) be a semitopological semihypergroup. Then \( f \in AP(K) \) if and only if the map \( \mu \mapsto T_\mu f : \mathcal{B}_1 \to B(K) \) is \( \sigma(\mathcal{B}_1, AP(K)) \)-norm continuous.

**Proof.** Let \( f \in AP(K) \). Let \( \Psi : \mathcal{B}_1 \to B(K) \) be the map given by

\[
\Psi(\mu) := T_\mu f \text{ for each } \mu \in \mathcal{B}_1.
\]

Let \( \mu_\alpha \to \mu \) in \( \mathcal{B}_1 \) with respect to the topology \( \sigma(\mathcal{B}_1, AP(K)) \), i.e., we have that \( \mu_\alpha(g) \to \mu(g) \) for each \( g \in AP(K) \). Also by Proposition 4.6 we know that \( L_x f \in AP(K) \) for each \( x \in K \). Hence in particular for each \( x \in K \) we have that

\[
\mu_\alpha(L_x f) \to \mu(L_x f) \text{ for each } x \in K
\]

\[
\Rightarrow T_{\mu_\alpha} f(x) \to T_{\mu} f(x) \text{ for each } x \in K
\]
Hence the map $\Psi$ is continuous when $B(K)$ is equipped with the topology of pointwise convergence.

Also, $\Psi(B_1)$ is the closure of $cco(O_r(f))$ with respect to topology of pointwise convergence. Since $O_r(f)$ is compact in $B(K)$, we have that $\Psi(B_1)$ is compact. Hence the topology of pointwise convergence coincides with the norm topology on $\Psi(B_1)$.

Conversely, suppose the map $\Psi : B_1 \to B(K)$ defined above is $\sigma(B_1, AP(K))$-norm continuous. Then it follows immediately that $f \in AP_r(K) = AP(K)$ since $\Psi(B_1)$ is norm-compact and the following inclusion holds:

$$O_r(f) \subset cco(O_r(f)) = \Psi(B_1).$$

□

Of course, the right-counterpart of the above theorem holds true in a similar manner, i.e., we can also show that a function $f$ is almost periodic if and only if the map $\mu \mapsto U_\mu f : B_1 \to B(K)$ is $\sigma(B_1, AP(K))$-norm continuous.

Now we see that $AP(K)$ is left and right introverted, enabling us to introduce left and right Arens product [1] on $AP(K)^*$. 

**Corollary 4.18.** Let $K$ be any semitopological semihypergroup. Then $AP(K)$ is left and right introverted.

**Proof.** We know that $AP(K)$ is a translation-invariant linear subspace of $C(K)$. Now pick any $f \in AP(K)$ and again consider the function $\Psi : B_1 \to B(K)$ given by

$$\Psi(\mu) := T_\mu f$$

for each $\mu \in B_1$.

As pointed out in the preceding proof, we have that the norm topology coincides with the topology of pointwise convergence on $\Psi(B_1)$ and hence finally we have that

$$\Psi(B_1) = cco(O_r(f)) = AP(K).$$

Hence after scaling by proper scalars we have that $T_\mu f \in AP(K)$ for each $\mu \in AP(K)^*$, as required.

The proof for the right-counterpart follows similarly. □

**Theorem 4.19.** Let $K$ be a semitopological semihypergroup and $f \in AP(K)$. Then the map $\Phi : B_1 \times B_1 \to C$ given by

$$\Phi(\mu, \nu) := \mu(T_\nu f)$$

is continuous with respect to the topology $\sigma(B_1, AP(K)) \times \sigma(B_1, AP(K))$.

Moreover, for any $\mu, \nu \in AP(K)^*$ we have that

$$\mu(T_\nu f) = \nu(U_\mu f).$$

**Proof.** Pick any $(\mu_0, \nu_0) \in B_1 \times B_1$ and consider the function $\Psi$ as in Theorem 4.17. Then for any $\mu, \nu \in B_1$ we have that

$$|\Phi(\mu, \nu) - \Phi(\mu_0, \nu_0)| = |\mu(T_\nu f) - \mu_0(T_{\nu_0} f)|$$

$$\leq ||\Psi(\nu) - \Psi(\nu_0)|| + ||(\mu - \mu_0)(\Psi(\nu_0))||.$$
Since $\Psi$ is $\sigma(B_1, AP(K))$-norm continuous, we have that both the terms in the last inequality tends to zero whenever $(\mu, \nu)$ tends to $(\mu_0, \nu_0)$ in $B_1 \times B_1$ with respect to the topology $\sigma(B_1, AP(K)) \times \sigma(B_1, AP(K))$.

In a similar way, we can show that the map $\Phi': B_1 \times B_1 \to \mathbb{C}$ given by

$$\Phi'(\mu, \nu) := \nu(U_\mu f)$$

is continuous with respect to the topology $\sigma(B_1, AP(K)) \times \sigma(B_1, AP(K))$.

Now for any $x \in K$ consider the evaluation map $E_x \in AP(K)^*$ given by

$$E_x(f) := f(x) \quad \text{for each } f \in AP(K).$$

We denote the set of all evaluation maps in $AP(K)^*$ as $E(K)$, i.e., we have

$$E(K) := \{E_x : x \in K\}.$$ 

Pick any $\mu = E_x, \nu = E_y$ for some $x, y \in K$. Then we have

$$\mu(T_\nu f) = E_y(L_x f) = f(x \ast y) = E_x(R_y f) = E_y(U_\mu f) = \nu(U_\mu f).$$

Since $E(K)$ serves as the set of extreme points, we have that the weak* closure of $cco(E(K))$ equals $B_1$ \cite{17}. Hence it follows from the continuity of the maps $\Phi$ and $\Phi'$ that

$$\mu(T_\nu f) = \nu(U_\mu f)$$

for any $\mu, \nu \in B_1$ and hence by proper scaling, for any $\mu, \nu \in AP(K)^*$.

Note that the above theorem holds true even if we replace $AP(K)$ by any translation invariant conjugation-closed linear subspace $F$ of $C(K)$ containing constant functions.

Now let us define a product $\star$ on $AP(K)^*$ given by

$$\mu \star \nu(f) := \mu(T_\nu f).$$

for any $(\mu, \nu) \in AP(K)^* \times AP(K)^*$.

Now recall the construction of the left Arens product $\hat{\diamond}$ for any Banach algebra $X$ \cite{11} and apply it on $AP(K)^*$. For any $\mu, \nu \in AP(K)^*, f \in AP(K)$ and $x, y \in K$ we have that

$$f \hat{\diamond} x(y) := f(x \ast y).$$

$$\nu \hat{\diamond} f(x) := \nu(f \hat{\diamond} x).$$

$$\mu \hat{\diamond} \nu(f) := \mu(\nu \hat{\diamond} f).$$

Hence in particular, we have that

$$f \hat{\diamond} x(y) = L_x f(y).$$

$$\nu \hat{\diamond} f(x) = \nu(L_x f) = T_\nu f(x).$$

$$\mu \hat{\diamond} \nu(f) = \mu(T_\nu f) = \mu \star \nu(f).$$
Note that here the second step is possible since $AP(K)$ is left translation-invariant and the last step is possible since $AP(K)$ is left introverted. Thus the left Arens product coincides with $\star$ on $AP(K)^*$, and hence $(AP(K)^*, \star)$ becomes a Banach algebra.

Now in a similar manner, consider the right Arens product $\Box$ on $AP(K)^*$. Then we have that

\[
\begin{align*}
 x \Box f(y) &= f(y \ast x) = R_x f(y), \\
 f \Box \mu(x) &= \mu(x \Box f) = \mu(R_x f) = U_{\mu} f(x), \\
 \mu \Box \nu(f) &= \nu(f \Box \mu) = \nu(U_{\mu} f).
\end{align*}
\]

Thus in view of Theorem 4.19 we see that the left and right Arens products coincide on $AP(K)^*$, i.e, $AP(K)^*$ is Arens regular.

5. Homomorphisms and Ideals

In this section, we first introduce a more general form of homomorphism in (semitopological) semihypergroups. Later we introduce the concept of an ideal in a semihypergroup and obtain some basic properties and relations between (closed) ideals and homomorphisms. Finally, we investigate the structure of the kernel of a compact (semitopological) semihypergroup and the connection between minimal left ideals and amenability in compact semihypergroups. Unless otherwise mentioned, $K$ and $H$ will denote semitopological semihypergroups.

**Definition 5.1.** A continuous map $\phi : K \to H$ is called a homomorphism if for any Borel measurable function $f$ on $H$ and for any $x, y \in K$ we have that

\[ f \circ \phi(x \ast y) = f(\phi(x) \ast \phi(y)). \]

A homomorphism $\phi$ between $K$ and $H$ is called an isomorphism if the map $\phi$ is bijective.

**Remark 5.2.** In 1975 Jewett introduced the concept of orbital morphisms for hypergroups in [13]. If $K$ and $J$ are two hypergroups, then a map $\phi : K \to J$ is called proper if $\phi^{-1}(C)$ is compact in $K$ for every compact set $C$ in $J$. A recomposition of $\phi$ was defined to be a continuous map from $J \to M^+(K)$ defined as $x \mapsto q_x$ such that $\text{supp}(q_x) = \phi^{-1}(\{x\})$.

Roughly speaking, a proper open map $\phi : K \to J$ is called an orbital homomorphism if for any $x, y \in J$ we have that $p_x \ast p_y = \phi(q_x \ast q_y)$ as measures, i.e, for any Borel measurable function $f$ on $J$ we have that $\int_J f \, dp_x \ast p_y = \int_K f \circ \phi \, dq_x \ast q_y$.

Jewett defined homomorphism between hypergroups as a special case of orbital morphisms. According to the definition in [13] a continuous open map $\phi : K \to J$ is called a homomorphism if $\phi(p_x \ast p_y) = p_{\phi(x)} \ast p_{\phi(y)}$ for any $x, y \in K$.

Note that our definition of homomorphism is equivalent to that of Jewett’s apart from the fact that in Definition 5.1 we do not need the map $\phi$ to be open and proper, and there the map was defined between two (semitopological) semihypergroups, instead of hypergroups. Otherwise these two definitions coincide since similar to the definition of orbital morphisms, Jewett’s definition of homomorphism implies that for any measurable function $f$ on $J$, $x, y \in J$. In this context, closed left ideals play a dual role to the open right ideals in hypergroups.
We have that \( \int_{\mathcal{J}} f d(p_{\phi(x)} * p_{\phi(y)}) = \int_K f \circ \phi d(p_x * p_y) \). Hence by definition \( f(\phi(x) * \phi(y)) = f \circ \phi(x * y) \).

Now we check whether some of the basic properties of the classical case hold for this definition as well. Since in the classical case the multiplication of two points simply gives us another point, these properties hold trivially in that case.

**Lemma 5.3.** Let \( K \) and \( H \) be (semitopological) semihypergroups, and \( \phi : K \to H \) a homomorphism. Then for any \( x, y \in K \), for almost all \( z \in \text{supp}(p_x * p_y) \) with respect to the measure \( (p_x * p_y) \) we have

\[
\phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}).
\]

Conversely, for any \( x, y, z \in K \), for almost all \( \phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \) with respect to the measure \( (p_{\phi(x)} * p_{\phi(y)}) \) we have

\[
z \in \phi^{-1} \phi(\text{supp}(p_x * p_y)).
\]

**Proof.** Pick any \( x, y \in K \) and let \( z \in \text{supp}(p_x * p_y) \). Now set \( A := \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \subset H \) and define a measurable function \( g \) on \( H \) such that \( g \equiv 0 \) on \( A \), \( g \equiv 1 \) on \( A^c \). Then we have that

\[
0 = \int_A g \, d(p_{\phi(x)} * p_{\phi(y)}) = \int_H g \, d(p_{\phi(x)} * p_{\phi(y)}) = g(\phi(x) * \phi(y)) = g \circ \phi(x * y) = \int_K g \circ \phi \, d(p_x * p_y).
\]

For any \( z \in \text{supp}(p_x * p_y) \), for the relation \( \int_K (g \circ \phi) \, d(p_x * p_y) = 0 \) to hold true we must have that \( g \circ \phi(z) = 0 \) almost everywhere on \( \text{supp}(p_x * p_y) \) with respect to \( (p_x * p_y) \). Thus by construction of \( g \) we have that \( \phi(z) \in A \) for almost all \( z \in \text{supp}(p_x * p_y) \) as required.

Now to prove the converse, pick any \( x, y \in K \) and let \( z \in K \) be such that \( \phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \).

Set \( A := \text{supp}(p_x * p_y) \) and define a measurable function \( h \) on \( H \) such that \( h \equiv 0 \) on \( \phi(A) \) and \( h \equiv 1 \) on \( \phi(A)^c \). Note that for any \( u \in A \) we have that \( \phi(u) \in \phi(A) \) and hence \( h \circ \phi(u) = 0 \).

Thus we have

\[
0 = \int_A h \circ \phi \, d(p_x * p_y) = \int_K h \circ \phi \, d(p_x * p_y) = h(\phi(x) * \phi(y)) = \int_H h \, d(p_{\phi(x)} * p_{\phi(y)}).
\]

where the first equality follows from the construction of \( h \) and the second equality follows from the construction of \( A \).

Now if \( z \in A \), then the result follows trivially. Suppose \( z \) does not lie in \( A \). Since \( \phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \subset H \), for the relation \( \int_H h \, d(p_{\phi(x)} * p_{\phi(y)}) = 0 \) to be true we must have that \( h(\phi(z)) = 0 \) for almost all \( \phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \) with respect to \( (p_{\phi(x)} * p_{\phi(y)}) \). Thus almost all \( \phi(z) \in \phi(A) \) and so \( z \in \phi^{-1} \phi(A) \) as required. \( \square \)
Remark 5.4. Roughly speaking, the above lemma implies that for an isomorphism \( \phi : K \to H \) we have the following:

For any \( x, y \in K \) we have that \( z \in \text{supp}(p_x * p_y) \) if and only if \( \phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(y)}) \), up to a set of measure zero.

**Proposition 5.5.** Let \( \phi : K \to H \) be a homomorphism. Then \( \phi(K) \) is a semihypergroup.

**Proof.** In order to verify this, we only need to check that \( \phi(K) \) is closed under convolution, i.e., \( \phi(K) \ast \phi(K) \subset \phi(K) \).

Pick \( a, b \in K \) and set \( A := \text{supp}(p_{\phi(a)} * p_{\phi(b)}) \). If possible, suppose \( A \not\subseteq \phi(K) \) so that \( A \setminus \phi(K) \neq \emptyset \). Now define a measurable function \( f \) on \( H \) such that \( f \equiv 1 \) on \( A \setminus \phi(K) \) and \( f \equiv 0 \) on \( A \cap \phi(K) \). Then we have

\[
\int_{A \setminus \phi(K)} f \ d(p_{\phi(a)} * p_{\phi(b)}) = \int_A f \ d(p_{\phi(a)} * p_{\phi(b)}) = \int_H f \ d(p_{\phi(a)} * p_{\phi(b)}) = f(\phi(a) \ast \phi(b)) = f \circ \phi(a \ast b) = \int_K f \circ \phi \ d(p_a * p_b) = 0.
\]

where the first equality holds since \( f \equiv 0 \) on \( A \cap \phi(K) \), the second equality follows from the construction of \( A \) and the last equality holds since for almost all \( z \in \text{supp}(p_a * p_b) \) we have from Lemma 5.3 that \( \phi(z) \in A \). Thus \( \phi(z) \in A \cap \phi(K) \) and hence \( f \circ \phi(z) = 0 \).

But \( f \equiv 1 \) on \( A \setminus \phi(K) \) and \( A \setminus \phi(K) \subset \text{supp}(p_{\phi(a)} * p_{\phi(b)}) \). So \( A \setminus \phi(K) \) can not be a set of \( (p_{\phi(a)} * p_{\phi(b)})\)-measure zero and hence \( \int_{A \setminus \phi(K)} f \ d(p_{\phi(a)} * p_{\phi(b)}) \neq 0 \), and we arrive at a contradiction. \( \square \)

We now introduce and study ideals in (semitopological) semihypergroups.

**Definition 5.6.** Let \( K \) be a (semitopological) semihypergroup. A subset \( I \subset K \) is called a left ideal of \( K \) if for any \( a \in I \), \( x \in K \) we have that \( p_x * p_a(I) = 0 \).

Equivalently, a subset \( I \subset K \) is called a left ideal of \( K \) if for any \( a \in I \), \( x \in K \) and any Borel measurable set \( E \subset K \) such that \( E \cap I = \emptyset \), we have that \( p_x * p_a(E) = 0 \).

Note that a closed subset \( I \subset K \) is a left ideal in \( K \) if and only if for any \( x \in K \), \( a \in I \) we have that \( \text{supp}(p_x * p_a) \subset I \). Similar definitions and comments apply to right ideals in \( K \).

Throughout this section, we discuss the properties of left ideals and their relations to several algebraic and analytic objects associated to \( K \). All these results have similar counterparts for right ideals too. Henceforth all ideals will be assumed to be closed, unless otherwise specified.

**Proposition 5.7.** Let \( K \) be a semitopological semihypergroup. Pick any \( a \in K \). Then \( I := K \ast \{a\} \) is a left ideal in \( K \).

**Proof.** Pick any \( x \in K \) and \( b \in I \). Since \( I = \cup_{y \in K} \text{supp}(p_y * p_a) \) there exists some \( y_0 \in K \) such that \( b \in \text{supp}(p_{y_0} * p_a) = \{y_0\} \ast \{a\} \). Now the result follows as

\[
\text{supp}(p_x * p_b) = \{x\} \ast \{b\} \subset \{x\} \ast (\{y_0\} \ast \{a\}) = (\{x\} \ast \{y_0\}) \ast \{a\} \subset K \ast \{a\} = I.
\]
Proposition 5.8. Let $\phi : K \to H$ be a homomorphism and $I \subset K$ a left ideal. Then $\phi(I)$ is also a left ideal in $\phi(K)$.

Proof. Pick any $a \in I$, $x \in K$. Set $B := \text{supp}(p_{\phi(x)} * p_{\phi(a)})$ and define a function $f$ on $H$ such that $f \equiv 1$ on $\phi(I) \cap B$ and $f \equiv 0$ elsewhere. Then we have

$$p_{\phi(x)} * p_{\phi(a)}(\phi(I)) = \int_{\phi(I) \cap B} 1 \, d(p_{\phi(x)} * p_{\phi(a)}) = \int_H f \, d(p_{\phi(x)} * p_{\phi(a)})$$

$$= f(\phi(x) * \phi(a)) = f \circ \phi(x * a) = \int_K (f \circ \phi) \, d(p_x * p_a) = \int_{\phi^{-1}(I)} 1 \, d(p_x * p_a) = (p_x * p_a)(\phi^{-1}(I)) = (p_x * p_a)(I) = 1.$$

where the sixth equality follows from the fact that for almost all $z \in \text{supp}(p_x * p_a)$ Lemma 5.3 gives us that $\phi(z) \in B$ and hence $f(\phi(z)) = 0$ whenever $\phi(z)$ does not lie in $\phi(I)$, i.e., whenever $z$ lies outside $\phi^{-1}(I)$. Also, the second last equality follows since $I \subset \phi^{-1}(I)$ and $\text{supp}(p_x * p_a) \subset I$ as $I$ is a left ideal in $K$.

Proposition 5.9. Let $\phi : K \to H$ be a homomorphism and $J \subset H$ a left ideal in $H$. Then $\phi^{-1}(J)$ is also a left ideal in $K$.

Proof. Pick any $a \in \phi^{-1}(J)$, $x \in K$. Then $\phi(a) \in J$ and hence $\text{supp}(p_{\phi(x)} * p_{\phi(a)}) \subset J$ and $(p_{\phi(x)} * p_{\phi(a)})(J) = 1$. Define a function $f$ on $H$ by $f \equiv 1$ on $J$ and $f \equiv 0$ elsewhere. Now for almost all $z \in \text{supp}(p_x * p_a)$ using Lemma 5.3 we have that

$$\phi(z) \in \text{supp}(p_{\phi(x)} * p_{\phi(a)}) \subset J.$$

Hence $f \circ \phi(z) = 1$ for almost all $z \in \text{supp}(p_x * p_a)$. Hence the result follows as

$$(p_x * p_a)(\phi^{-1}(J)) = \int_{\phi^{-1}(J)} 1 \, d(p_x * p_a) = \int_J f \circ \phi \, d(p_x * p_a)$$

$$= f \circ \phi(x * a) = f(\phi(x) * \phi(a)) = \int_H f \, d(p_{\phi(x)} * p_{\phi(a)}) = \int_J 1 \, d(p_{\phi(x)} * p_{\phi(a)}) = (p_{\phi(x)} * p_{\phi(a)})(J) = 1.$$

□
A left (resp. right) ideal $I \subset K$ is called a minimal left (resp. right) ideal of $K$ if $I$ does not contain any proper left (resp. right) ideal of $K$. An ideal $I$ of $K$ which is both a minimal left and right ideal, is called a minimal ideal of $K$. We now start off with examining some equivalence criteria for the minimality of a left ideal. Next we examine some basic properties of minimal left (resp. right) ideals that hold trivially for semigroups, for reasons explained in the previous section.

**Proposition 5.10.** For any left ideal $I \subset K$ the following are equivalent:

1. $I$ is a minimal left ideal.
2. $K \ast \{a\} = I$ for any $a \in I$.
3. $I \ast \{a\} = I$ for any $a \in I$.

**Proof.** (1) $\Rightarrow$ (3): Let $I$ be a minimal left ideal of $K$ and $a \in I$. Then $I \ast \{a\} \subset I \ast I \subset I$ since $I$ is a left ideal. Also, $I \ast \{a\}$ is a left ideal since $K \ast (I \ast \{a\}) = (K \ast I) \ast \{a\} \subset I \ast \{a\}$. Hence the minimality of $I$ gives us that $I \ast \{a\} = I$.

(3) $\Rightarrow$ (2): Since $I$ is a left ideal, for each $a \in I$ we have $I = I \ast \{a\} \subset K \ast \{a\} \subset I$, which forces that $K \ast \{a\} = I$.

(2) $\Rightarrow$ (1): Let $J$ be a left ideal contained in $I$. Pick any $b \in J$. Since $b \in I$ as well, (2) gives us $I = K \ast \{b\} \subset K \ast J \subset J$. Therefore $I = J$ implying the minimality of $I$ in $K$. \qed

**Corollary 5.11.** For any (semitopological) semihypergroup $K$ the following assertions hold.

1. If $I_1, I_2$ are minimal left ideals in $K$, then either $I_1 = I_2$ or $I_1 \cap I_2 = \emptyset$.
2. Let $I$ be a minimal left ideal in $K$. Then any minimal left ideal in $K$ will be of the form $I \ast \{x\}$ for some $x \in K$. Moreover, we have that $K \ast J = J$ for any minimal left ideal $J$ in $K$.
3. $K$ can have at most one minimal ideal, namely the intersection of all ideals in $K$.

**Proof.** The proofs of (1) and (3) are straight-forward. To prove (2): let $J$ be any minimal left ideal of $K$. Pick $y \in J$. Then $I \ast \{y\} \subset I \ast J \subset J$ and the minimality of $J$ forces that $J = I \ast \{y\}$. Now pick any $x_0 \in K$ and consider the left ideal $I_0 := I \ast \{x_0\}$. It follows readily from Proposition 5.10 that $K \ast I = \cup_{a \in I} K \ast \{a\} = \cup_{a \in I} I = I$. Therefore $K \ast I_0 = K \ast (I \ast \{x_0\}) = (K \ast I) \ast \{x_0\} = I \ast \{x_0\} = I_0$. \qed

**Remark 5.12.** Unlike in the semigroup setting [2, 3], the family of sets $\{I \ast \{x\} : x \in K\}$ does not serve as an exhaustive family of minimal left ideals for a semihypergroup, where $I$ is a minimal left ideal in $K$. For any $x \in K$ the set $I \ast \{x\}$ is of course a left ideal of $K$, but it need not be minimal. For example, simply consider the finite semihypergroup $(K, \ast)$ where $K = \{a, b, c\}$ and the operation $\ast$ given in the following table.

| *   | $p_a$ | $p_b$ | $p_c$ |
|-----|-------|-------|-------|
| $p_a$ | $p_a$ | $\frac{1}{2}(p_a + p_b)$ | $\frac{1}{2}(p_a + p_c)$ |
| $p_b$ | $\frac{1}{2}(p_a + p_b)$ | $p_b$ | $\frac{1}{2}(p_a + p_c)$ |
| $p_c$ | $\frac{1}{2}(p_a + p_c)$ | $\frac{1}{2}(p_a + p_c)$ | $p_c$ |

Set $I = \{a\}$. Then $I$ is a minimal left ideal as the left multiplication by $\ast$ leaves $p_a$ fixed. But $I \ast \{b\} = \{a, b\}$ and $I \ast \{c\} = \{a, c\}$ are both left ideals that fail to be minimal.
Proposition 5.13. Let $K$ be a compact semitopological semihypergroup. Then each left ideal of $K$ contains at least one minimal left ideal and each right ideal contains at least one minimal right ideal of $K$.

Moreover, each minimal left and right ideal of $K$ is closed.

Proof. We will prove the statement only for minimal left ideals.

Let $I$ be a left ideal of $K$. Consider the following collection of left ideals in $K$

$$Q := \{ J \subset K : J \text{ is a left ideal in } K \text{ and } J \subset I \}.$$ 

If $a \in I$, then $K \cdot \{a\} \subset K \cdot I \subset I$ is a left ideal in $K$, by Proposition 5.7. Hence $Q$ is non-empty and non-trivial. Equip $Q$ with the partial order of reverse inclusion. Let $\mathcal{C}$ be a linearly ordered sub-collection of $Q$. Since $K$ is compact, the ideal $\cap_{J \in \mathcal{C}} J$ is non-empty. Hence we can use Zorn’s Lemma to deduce that there exists a minimal element $J_0$ in $Q$, which serves as a minimal left ideal of $K$ contained in $I$.

Now let $a \in J_0$. Then $K \cdot \{a\}$ is a left ideal in $K$ contained in $J_0$. Hence by minimality of $J_0$ we get that $J_0 = K \cdot \{a\}$, which is closed as $K$ is compact. \[\square\]

Finally in the last part of this section, we define and explore the characteristics of the kernel of a (semitopological) semihypergroup. We restrict ourselves to the case where the underlying space is compact and investigate the interplay between the structures of a kernel and the exhaustive set of minimal left (resp. right) ideals. We conclude with a result outlining the relationship between amenability and the existence of a unique minimal left ideal for a compact semihypergroup.

The kernel of a (semitopological) semihypergroup $K$ denoted as $\text{Ker}(K)$ is defined to be the intersection of all (two-sided) ideals in $K$, i.e, we define

$$\text{Ker}(K) := \bigcap_{I \subset K \text{ is an ideal}} I.$$ 

The set of all minimal left and right ideals of $K$ are denoted as $\mathcal{L}(K)$ and $\mathcal{R}(K)$ respectively. Finally, the following theorems give us an explicit idea of the structure of the kernel of a compact semihypergroup.

Theorem 5.14. Let $K$ be a compact semitopological semihypergroup and $I$ be a minimal left ideal in $K$. Then we have that

$$\bigcup_{M \in \mathcal{L}(K)} M \subset \text{Ker}(K) \subset \bigcup_{x \in K} (I \cdot \{x\})$$

Proof. Let $I \in \mathcal{L}(K)$ and consider the following union

$$I_0 := \bigcup_{x \in K} I \cdot \{x\}.$$ 

We know from Corollary 5.11 that $I_0$ is a union of left ideals and hence is a left ideal of $K$ itself. Now pick any $a \in I_0$. There exists some $x_0 \in K$ such that $a \in I \cdot \{x_0\}$. Thus for any
$y \in K$ we have
\[
\text{supp}(p_a * p_y) = \{a\} \ast \{y\} \subseteq (I \ast \{x_0\}) \ast \{y\} \\
= I \ast (\{x_0\} \ast \{y\}) \\
= \cup_{x \in \{x_0\} \ast \{y\}} (I \ast \{x\}) \subset \cup_{x \in K} (I \ast \{x\}) = I_0.
\]
Hence $I_0$ is a right ideal of $K$ as well and therefore $\text{Ker}(K) \subseteq I_0$.

Now let $J$ be any ideal in $K$ and $M \in \mathcal{L}(K)$. Then
\[
K \ast (J \ast M) = (K \ast J) \ast M \subset J \ast M \subset M.
\]
Thus $J \ast M$ is a left ideal of $K$ contained in $M$ and hence by minimality of $M$, we have that $M = J \ast M$. But $J$ is a two-sided ideal and hence $M = J \ast M \subset J$.

This is true for any $M \in \mathcal{L}(K)$ and any ideal $J$ in $K$, and hence finally we see that $\cup_{I \in \mathcal{L}(K)} I \subset \text{Ker}(K)$ as required. □

The result holds true similarly for minimal right ideals as well, i.e., for any minimal right ideal $J \in \mathcal{R}(K)$ we have that
\[
\bigcup_{N \in \mathcal{R}(K)} N \subset \text{Ker}(K) \subset \bigcup_{x \in K} (\{x\} \ast J)
\]

**Corollary 5.15.** Let $K$ be a compact semitopological semihypergroup. Then $\text{Ker}(K)$ is non-empty.

**Proof.** We know by Proposition 5.7 that $K \ast \{a\}$ is a left ideal for any $a \in K$. Since $K$ is compact, it follows from Proposition 5.13 that it contains at least one minimal left ideal. Hence the result follows immediately from the above theorem. □

Note that the above Corollary only implies that $K$ does not contain any two disjoint ideals. But it may very well be the case that $\text{Ker}(K) = K$ as demonstrated in Remark 5.12 and Example 3.2.

**Theorem 5.16.** Let $K$ be a compact semihypergroup. If $K$ is right amenable, i.e., if $C(K)$ admits a right invariant mean, then $K$ has a unique minimal left ideal.

**Proof.** Let $m$ be a right invariant mean on $C(K)$. Since $K$ is compact, it follows from Proposition 5.13 that there exists at least one minimal left ideal of $K$.

If possible, let $I_1$ and $I_2$ be two distinct minimal left ideals of $K$. Then by Corollary 5.11 and Proposition 5.13 we have that $I_1 \cap I_2 = \emptyset$ and both $I_1$ and $I_2$ are closed in $K$. Since $K$ is compact, it is normal. Hence we can use Urysohn’s Lemma to get a continuous function $f \in C(K)$ such that $f \equiv 0$ on $I_1$ and $f \equiv 1$ on $I_2$.

Now pick $a \in I_1$. For any $x \in K$ we have that
\[
R_a f(x) = f(x \ast a) = \int_{\text{supp}(p_x \ast p_a)} f \ d(p_x \ast p_a) = 0
\]
where the last equality follows since $\text{supp}(p_x \ast p_a) \subset I_1$ and hence $f \equiv 0$ on $\text{supp}(p_x \ast p_a)$.
Similarly for any $b \in I_2$, $x \in K$ we have that
\[ R_b f(x) = f(x \ast b) = \int_{\text{supp}(p_x \ast p_b)} f \ d(p_x \ast p_b) = (p_x \ast p_b)(K) = 1 \]
where as before, the fourth equality follows since $\text{supp}(p_x \ast p_b) \subset I_2$ and hence $f \equiv 1$ on $\text{supp}(p_x \ast p_b)$.
Thus for any $a \in I_1$, $b \in I_2$ we have that $R_a f \equiv 0$ and $R_b f \equiv 1$. This leads to a contradiction since
\[ 1 = m(1) = m(R_b f) = m(f) = m(R_a f) = m(0). \]
Hence the minimal left ideal of $K$ must be unique. □

6. OPEN QUESTIONS

As discussed briefly in the introduction, the lack of prior extensive research since its inception and the significant examples available in coset and orbit spaces of locally compact groups, Lie groups and homogeneous spaces, makes way to a number of exciting new avenues of research on semihypergroups.

The author has already shown (in an upcoming article) that a concept of free products can naturally be introduced to semihypergroups, giving rise to a new class of examples. What makes the structure more intriguing is the fact that the natural algebraic free-product structure, which does not work for topological groups, indeed works in this case for a vast class of semihypergroups, including nontrivial coset and orbit spaces.

Here we list some of the immediate potential problems and areas of semihypergroups that we are currently working on and/or intend to work on in the near future.

**Problem 1:** Use the algebraic structure imposed on $AP(K)^*$ to acquire a general compactification of semihypergroups.

**Problem 2:** Investigate the set of minimal ideals on a (semitopological) semihypergroup more closely, finally to explore its relation to the space of almost periodic and weakly almost periodic functions.

**Problem 3:** Investigate the idempotents of a compact semihypergroup and explore their relation to the space of minimal left ideals, kernel and amenability of a semihypergroup.

**Problem 4:** Explore if results equivalent to isomorphism theorems hold true for a semihypergroup, in addition to exploring the structure of the kernel of a homomorphism for a (semitopological) semihypergroup with identity, as in Example 3.3 and Example 3.5.

**Problem 5:** Study different notions of amenability on (semitopological) semihypergroups, specially for the non-commutative case.

**Problem 6:** Investigate the $F$-algebraic structure (defined by A. T. Lau in [14]) on the measure algebra of a semihypergroup.

7. ACKNOWLEDGEMENT

The author would like to thank her doctoral thesis advisor Dr. Anthony To-Ming Lau for suggesting the topic of this paper and for the helpful discussions during the course of this
work. She would also like to thank the referee for the valuable suggestions that led to a better presentation of the article.

References

[1] Richard Arens The Adjoint of a Bilinear Operation. Proc. Amer. Math. Soc., Vol. 2, 1951, 839-848.
[2] K. Deleeuw, I. Glicksberg Almost Periodic Functions on Semigroups. Acta Math. 105, 1961, 99-140.
[3] K. Deleeuw, I. Glicksberg Applications of Almost Periodic Compactifications. Acta Math. 105, 1961, 63-97.
[4] Charles F. Dunkl The Measure Algebra of a Locally Compact Hypergroup. Trans. Amer. Math. Soc., vol. 179, 1973, 331-348.
[5] B. Forrest Amenability and Bounded Approximate Identities in Ideals of $A(G)$. Illinois J. Math. 34, no. 1, 1990, 1-25.
[6] B. Forrest Amenability and Ideals in $A(G)$. J. Austral. Math. Soc. Ser. A 53, no. 2, 1992, 143-155.
[7] B. Forrest, N. Spronk and E. Samei Convolutions on Compact Groups and Fourier Algebras of Coset Spaces. Studia Math. 196 no. 3, 2010, 223-249.
[8] F. Ghahramani and A. R. Medgalchi Compact Multipliers on Weighted Hypergroup Algebras. Math. Proc. Chembridge Philos. Soc. 98, 1985, 493-500.
[9] F. Ghahramani, A. T. Lau, V. Losert Isometric Isomorphisms between Banach Algebras Related to Locally Compact Groups. Trans. Amer. Math. Soc. 321 no. 1, 1990, 273-283.
[10] F. Ghahramani, J. P. McClure The Second Dual Algebra of the Measure Algebra of a Compact Group. Bull. London Math. Soc. 29 no. 2, 1997, 223-226.
[11] F. Ghahramani, H. Farhadi Involutions on the Second Duals of Group Algebras and a Multiplier Problem. Proc. Edinb. Math. Soc. (2) 50, no. 1, 2007, 153-161.
[12] A. Grothendieck, Critères de compaîté dans les espaces fonctionnels généraux. Amer.J. Math., 74, 1952, 168-186.
[13] Robert I. Jewett Spaces with an Abstract Convolution of Measures. Advances in Mathematics. 18, no. 1, 1975, 1-101.
[14] Anthony T. Lau Analysis On a Class of Banach Algebras with Applications to Harmonic Analysis on Locally Compact Groups and Semigroups. Fund. Math. 118, no.3, 1983, 131-175.
[15] Anthony T. Lau, H. G. Dales and Dona P. Strauss Banach Algebras on Semigroups and on their Compactifications. Mem. Amer. Math. Soc. 205, no. 966, 2010, vi+165 pp.
[16] E. Michael Topologies on Spaces of Subsets. Trans. Amer. Math. Soc. 71, 1951, 152-182.
[17] John F. Berglund, Hugo D. Junghenn and Paul Milnes Analysis on Semigroups. Canadian Mathematical Society Series of Monographs and Advanced Texts, Wiley Interscience Publication. 24, 1989.
[18] Leopoldo Nachbin On the Finite Dimensionality of Every Irreducible Representation of a Compact Group. Proc. Amer. Math. Soc. 12, 1961, 11-12.
[19] W. Rudin Functional Analysis. Springer-Verlag 2nd Ed.
[20] R. Spector Apercu de la theorie des hypergroups, (French) Analyse harmonique sur les groupes de Lie (Sém. Nancy-Strasbourg, 1973-75). Lecture Notes in Math., vol. 497, Springer-Verlag, New York, 1975, 643-673.
[21] G. Willis, Approximate Units in Finite Codimensional Ideals of Group Algebras. J. London Math. Soc. (2) 26, no. 1, 1982, 143-154.
[22] G. Willis, Factorization in Codimension Two Ideals of Group Algebras. Proc. Amer. Math. Soc. 89, no. 1, 1983, 95-100.
[23] S. Wolfenstetter Weakly Almost Periodic functions on Hypergroups. Monatsh. Math. 96, no. 1, 1983, 67-79.
[24] H. Zeuner One Dimensional Hypergroups. Advances in Mathematics 76, no. 1, 1989, 1-18.
