The overlap phases of SIC-POVMs

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Abstract. A symmetric informationally complete positive-operator-valued measure (SIC-POVM) is a special type of generalized quantum measurements that possesses a high degree of symmetry. It plays an important role in quantum tomography, quantum cryptography and foundations of quantum mechanics. The most important and difficult problem about SIC-POVM is to show its existence in every dimension \( n \), which is suggested by many numerical evidences. In this paper, this existence problem of SIC-POVMs is investigated by considering their overlap phases. Specifically, the symmetric requirements of SIC-POVMs are encoded into the overlap phases between the fiducial project and the \( n^2 \) displacement operators. In this way, an equivalent condition of the existence of Weyl-Heisenberg group covariant SIC-POVMs is obtained in terms of these phases, so that a set of complex equations is established, which enables one to study the existence problem in a more explicit and direct fashion. In particular, we show that all the numbers appearing in the overlap phases have to be algebraic numbers. The significance of Zauner’s conjecture in our equivalent condition is also discussed, and therefore the SIC-POVMs in dimension four is constructed explicitly. Actually, our system of equations always admits a solution, the real difficulty lies in how to ensure these complex number solutions are of norm one, and therefore indeed phases. Our result shows that constructing SIC-POVMs from the overlap phases is possible, and symmetries can be applied directly to reduce the number of equations, so that it provide new insights into the difficult problem of SIC-POVMs, especially about the algebraic properties of these overlap phases.

1. Introduction
A symmetric informationally complete positive-operator-valued measure (SIC-POVM) [1, 2] in an \( n \)-dimensional Hilbert space is defined to a set of \( n^2 \) subnormalized rank one projectors \( \{ \Pi_{j,k}; j, k = 0, 1, \ldots, n - 1 \} \) such that

\[
\text{tr} \, \{ \Pi_{j,k} \Pi_{l,m} \} = n \delta_{j,l} \delta_{k,m} + \frac{1}{n^2(n + 1)}.
\]

Due to the high symmetry it possesses, it plays an important role in quantum tomography [3-6] and in foundations of quantum mechanics [7-11]. Beside these, it is also useful in quantum information technologies, such as in quantum cryptography [12-16] and in classical signal processing [17]. The investigation over it in the physics society was started by Zauner in 1999 [1], and the central problem, which turns out to be an extremely difficult one, is to prove the existence of SIC-POVMs in every finite dimension. After many decades’ efforts, this existence is only established in a few cases: we have analytical solutions in the special dimensions 1-21, 23, 24, 28, 30, 31, 35, 37, 39, 43, 48, 124 and 323, and numerical solutions in every dimension...
up to 151 and in several other higher dimensions [18-22]. Among these solutions, with the only exception of the Hoggar lines in dimension eight [23], all SIC-POVMs are covariant with the Weyl-Heisenberg group \( \{ D_{i,j}; i, j = 0, \ldots, n-1 \} \). In other words, there exists a fiducial projector \( |\phi\rangle\langle\phi| \) such that its orbit under the action of the Weyl-Heisenberg group provides us a SIC-POVM. Therefore the problem of constructing a SIC-POVM turns into the problem of constructing such a fiducial vector \( |\phi\rangle \). This group covariance property makes the Clifford group, which is the normalizer of the Weyl-Heisenberg group [3], important in consideration of the existence problem of SIC-POVMs [3, 24].

Recently, the link between SIC-POVMs and algebraic number theory, e.g., with Galois field theory, ray class fields, unit groups, the Hilberts 12th problem, and etc., is discovered [22-28]. It is interesting to see that although the explicit forms of the overlap phases (will be defined in Section 2) are extremely complicated, the numbers appearing in them are always algebraic numbers. It is thus our aim here to investigate the existence problem of SIC-POVMs in terms of these overlap phases. We will discuss the basic properties of these phases, and examine the effect of Zauner’s symmetry, which is suggested by the numerics and is already explicit in Zauner’s work [1], on them. By doing so, we find an equivalent condition for the existence of a group covariant SIC-POVMs in terms of these phases, which provides us an direct and easy way to solve the existence problem in dimension four analytically without any use of computer. Typically as any problems about SIC-POVMs, both the number of independent equations and the order of the equations grow quickly as dimension goes up, and one has to use computer softwares to solve them even for dimension five. But any way, our result provides not only concrete ways to understand the known abstract results about SIC-POVMs, but also direct links of the existence problem of SIC-POVMs to number theoretical problems. Since the complex number field is algebraic close, our system of equations always admits a solution. The problem is that among all these solutions how can one ensure that there always exists one solution consisting of only norm one numbers. The numerical evidence that all the overlap phases are algebraic numbers suggests that such norm one solutions are always possible, which implies the existence of Weyl-Heisenberg group covariant SIC-POVMs in every finite dimension.

The rest of the paper is organized as follows. In Section 2, we give a brief introduction to Weyl-Heisenberg groups, and discuss some related properties of the fiducial vector. We then establish the equivalent condition in Section 3, and discuss how to use Zauner symmetry to reduce independent variables and equations. In Section 4, we use last section’s result to construct SIC-POVMs in dimension four analytically. The paper is closed with a summary.

2. The Weyl-Heisenberg group
The Weyl-Heisenberg group is discussed in many literatures, such as [3, 18, 24]. Here we briefly collect the facts which will be important in our following discussions. In an \( n \)-dimensional Hilbert space, the Weyl-Heisenberg group is generated by the two elements \( X \) and \( Z \), which can be represented as

\[
X = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \quad \quad Z = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}
\]

with \( \omega = \exp (2\pi i / n) \). Define the displacement operator \( D_{i,j} \) as

\[
D_{i,j} = \tau^{ij} X^i Z^j
\]

with \( \tau = -\exp (\pi i / n) \). Then it is straightforward to verify the multiplication rule

\[
D_{i,j} D_{k,l} = \tau^{il-jk} D_{i+k,j+l}.
\]
Identifying the pair of number \((i,j)\) as a vector \(p \in \mathbb{Z}_n^2\), we may rewrite (4) compactly as \(D_pD_q = \tau^{(p,q)}D_{p+q}\), where \(\cdot\) denotes the symplectic product. Note that (4) implies immediately that \(D_0^2 = D_{-p}\) since both \(X\) and \(Z\) are unitary. Note that due to choice of \(\tau\), we do not have the identity \(D_p = D_{p+q}\) for arbitrary indexes \(p\) and \(q\) in even dimensions, instead

\[
D_{p+q} = (-1)^{(p,q)}D_p.
\]  

(5)

Define the projector \(\Pi = |\phi\rangle\langle\phi|\) with \(|\phi\rangle\) a fiducial vector, then the defining property (1) becomes

\[
\text{tr}(\Pi D_p) = \sqrt{n\delta_{p,0} + \frac{1}{n+1}} e^{i\theta_p}, \ p \in \mathbb{Z}_n^2
\]

(6)

where \(\theta_p\), with the convention that \(\theta_0 = 0\), is the overlap phases we are interested in. Taking complex conjugation of (6), we obtain the relation \(\exp(i\theta_p) = \exp(-i\theta_{-p})\), since \(\Pi\) is hermitian by definition. Therefore if \(n\) is even, then we have \(\exp(i\theta_{n/2,n/2}) = 1\) or \(-1\). On the other hand, the multiplication rule (4) implies that

\[
\text{tr}(D_p^\dagger D_q) = n\delta_{p,q}.
\]

(7)

Hence the Weyl-Heisenberg group form an orthogonal basis of the operator space with respect to the Hilbert-Schmidt norm, so that we may express \(\Pi\) in terms of the overlap phases uniquely as

\[
\Pi = \frac{1}{n} \sum_{p \in \mathbb{Z}_n^2} \text{tr}(\Pi D_p)D_p = \frac{1}{n} \mathbb{1} + \frac{1}{n\sqrt{n} + 1} \sum_{p \neq 0} e^{i\theta_p} D_p.
\]

(8)

Therefore as already observed by some authors [29, 30], it is easy to construct a projector \(\Pi\) such that (6) is satisfied. In factor, any set of phases \(\{\theta_p; p \in \mathbb{Z}_n^2 \setminus \{0\}\}\) will provide us such a projector. The difficulty is how to ensure this projector is of rank one. In other words, a generic choice of phase \(\{\theta_p\}\) will not give us a fiducial vector, instead, the correct choice of the phases corresponds to a solution of a system of nonlinear equations, which we will discuss in the coming section.

3. An equivalent condition

\(\Pi\) is of rank one is equivalent to \(\Pi \cdot \Pi = \Pi\), or in terms of the phases, we should have

\[
\frac{1}{n} \mathbb{1} + \frac{1}{n\sqrt{n} + 1} \sum_{p \neq 0} e^{i\theta_p} D_p = \frac{1}{n^2} \mathbb{1} + \frac{2}{n^2\sqrt{n} + 1} \sum_{p \neq 0} e^{i\theta_p} D_p
\]

\[
+ \frac{1}{n^2(n+1)} \sum_{p,q \neq 0} e^{i(\theta_p + \theta_q)}\tau^{(p,q)}D_{p+q}.
\]

(9)

Since \(\{D_p\}\) are orthogonal, we may obtain equations by comparing coefficients. The equation

\[
\frac{1}{n^2} + \frac{n^2 - 1}{n^2(n+1)} = \frac{1}{n}
\]

implies that the coefficient for \(\mathbb{1}\) is always the same. By equaling the coefficient before \(D_p\), we obtain

\[
\sum_{q \in \mathbb{Z}_n^2 \setminus \{0,p\}} e^{i(\theta_q + \theta_{p-q} - \theta_p)}\tau^{(q,p)} = (n-2)\sqrt{n+1},
\]

(11)

or, since \(\theta_0 = 0\) by convention, equivalently we have

\[
\sum_{q \in \mathbb{Z}_n^2} e^{i(\theta_q + \theta_{p-q} - \theta_p)}\tau^{(q,p)} = (n-2)\sqrt{n+1} + 2.
\]

(12)
In conclusion, we have the following equivalent condition for the existence of a Weyl-Heisenberg group covariant SIC-POVM is dimension $n$: For any $p \in \mathbb{Z}_n^2 \setminus \{0\}$, (11) or (12) is satisfied. Since $\theta_p = -\theta_{-p}$ and $\theta_{n/2,n/2} = 0$ or $\pi$ for even $n$ by construction, we have in total $[(n^2 - 1)/2]$ number of equations, with also $[(n^2 - 1)/2]$ number of independent variables, where the bracket $[x]$ denotes the largest integer smaller than $x$. Note that since the coefficients in (11) are all algebraic numbers and the corresponding ideal is zero-dimensional, the solutions to this system of equations are necessarily algebraic numbers. Moreover, since the field of complex number is algebraically close, such set of equations always admit solutions, but we require that there exists at least one solution such that all numbers are of norm one, and therefore are indeed phase factors. The existence of such phase factor solutions implies the existence of SIC-POVMs. In other words, all the information about Weyl-Heisenberg group covariant SIC-POVMs is encoded into the system of equations (11), so that one may study the symmetry properties and the algebraic properties of these numbers directly by considering these equations. As an example, we will interpret the Clifford symmetry on our equations in the following.

Let $\overline{C}(n)$ be the projective Clifford group in dimension $n$, i.e., identifying elements in the Clifford group (the normalizer of the Weyl-Heisenberg group) with only phase differences, and define

$$\pi = \begin{cases} n, & \text{when } n \text{ is odd;} \\ 2n, & \text{when } n \text{ is even.} \end{cases} \quad (13)$$

then as established in [3], there is a subjective morphism from the semi-direct product of $\text{SL}(2;\mathbb{Z}_\pi) \ltimes \mathbb{Z}_n^2$ to $\overline{C}(n)$ which maps $(F,\chi)$ to $U$ such that

$$UD_pU^\dagger = \omega^{\langle x,Fp \rangle}D_{Fp} \quad (14)$$

It is clear that the action of any element $U$ in the Clifford group on $\Pi$ will also produce a fiducial projector $U\Pi U^\dagger$. This fact is easily interpreted in our (11): By (14), such action amounts to a linear transform of the indexes and a multiplication of phase factors, since $F$ is linear, we have

$$F(p - q) = Fp - Fq, \text{ and } \langle Fp,Fq \rangle = \langle p,q \rangle, \quad (15)$$

which implies that if the set of phases $\{\theta_p\}$ satisfies (11), then also does $\{\theta_{Fp}\omega^{\langle x,Fp \rangle}\}$. Moreover, since the phases are paired by conjugation, complex conjugation will also not change (11). Therefore one should also consider the anti-unitary operators that stabilizes the Weyl-Heisenberg group, and thus the extended Clifford group, where similarly the subjective map can also be constructed [3]. By the above discussion, the fiducial projectors on the same orbit of the extended Clifford group correspond to essentially the same set of phases $\{\theta_p\}$ that satisfies (11). For example, in dimension four, we have found only one such orbit, which suggests the fact that there is essentially only one set of phases that satisfies (11) for any $p \in \mathbb{Z}_4^2 \setminus \{0\}$.

One may also use this symmetry to reduce the number of equations. If there exists one element $U \in \overline{C}(n)$ such that $U\Pi U^\dagger = \Pi$, then the corresponding matrix $S \in \text{SL}(2;\mathbb{Z}_\pi)$ satisfies $\theta_p = \theta_{Sp}\omega^{\langle x,Fp \rangle}$ for any $p$, since $\{D_p\}$ is a orthogonal basis. From the extensive numerical searches, in all the cases that one finds a numerical solution, this is the so-called Zauner symmetry [1, 3, 18], which suggests that such one element $U$ is mapped by $(S_z,0)$, where $S_z$, which is called Zauner’s matrix, is of the form

$$S_z = \begin{pmatrix} 0 & n - 1 \\ n + 1 & n - 1 \end{pmatrix}. \quad (16)$$

So that we have for any $p$, $\theta_p = \theta_{Sp}$. Since $S_z$ is of order three, i.e., $S_z^3 = 1$, the index $\{p\}$ is divided into disjoint orbits with three elements or one elements. Moreover it is easy to see...
that $S_z$ stabilized indexes only when $3 \mid n$, which implies that in any dimension $n$ which is not a multiple of three, all orbits have three different elements; whereas when $n = 3k$, there are two orbits with single element. So that the number of independent equations after considering Zauner symmetry is as

$$\begin{align*}
\# \text{ of equations} &= \begin{cases} 
(n^2 - 1)/6, & 2 \nmid n \& 3 \nmid n \\
(n^2 - 4)/6, & 2 \mid n \& 3 \nmid n \\
(n^2 - 3)/6, & 2 \nmid n \& 3 \mid n \\
(n^2 - 6)/6, & 2 \mid n \& 3 \mid n.
\end{cases} 
\end{align*}$$

(17)

For example, in dimension four, there are only two variables and two equations to solve, which is possible to solve by hand. More symmetry is also discovered from the numerics in some special dimensions, e.g., Scott and Grassl [18] suggest that there are more symmetry matrices in the form

$$S_b = \begin{pmatrix} -k & n \\ n & n-k \end{pmatrix}, \text{ for } n = k^2 - 1;$$

(18)

$$S_c = \begin{pmatrix} \kappa & n - 2\kappa \\ n + 2\kappa & n - \kappa \end{pmatrix}, \text{ for } n = (3k \pm 1)^2 + 3$$

(19)

with $\kappa = 3k^2 \pm k + 1$. These symmetries further reduce the number of equations and simplify the problem in the corresponding dimensions.

4. Example of dimension four

In this section, we explicitly solve the equations in dimension four. By consider the Zauner symmetry, we obtain the five disjoint orbits as

$$\{(0,1),(-1,-1),(1,0)\}, \{(0,1),(1,1),(-1,0)\},$$

$$\{(1,-2),(2,3),(-3,-1)\}, \{(-1,2),(-2,-3),(3,1)\},$$

$$\{(0,2),(2,2),(2,0)\}.$$

Note that since $(0,1), (0,-1)$ are conjugate, we have $\theta_{0,1} = -\theta_{0,-1}$; and similarly $\theta_{1,-2} = -\theta_{-1,2}$. Also we know that $\theta_{2,2} = 0$ or $\pi$, hence there leaves only two independent cases: $p = (0,1)$ and $p = (1,-2)$ and therefore two equations. Note that due to (5), we need to be careful about the indexes, e.g., we do not have $D_{1,-2} = D_{1,2}$, instead $D_{1,-2} = -D_{1,2}$. Let $a = e^{-i\theta_{0,1}}$ and $b = e^{-i\theta_{1,-2}}$, by setting $\theta_{0,2} = \pi$ (one may also try $\theta_{0,2} = 0$, which produces no valid solutions), then after some work, we obtain

$$\begin{cases} 
\sqrt{2} (-b^{-1} + a^3 - b - a) - 2a^2 = 2\sqrt{5}, \\
\sqrt{2} (-a^{-1} + b^3 - a - b) - 2b^2 = 2\sqrt{5}.
\end{cases}$$

(20)

Since in the above (20), $a$ and $b$ are symmetric, setting $a = b$ further reduces one equation, which is also suggested by the symmetry $S_c$ in (19), since clearly $S_c(0,1)^T = (2,3)^T$. Then after some simplifications, finally we obtain

$$(a^2 + \frac{\sqrt{5} - 1}{\sqrt{2}}a + 1)(a^2 - \frac{\sqrt{5} + 1}{\sqrt{2}}a - 1) = 0.$$

(21)

Note that the first multiplier in (21) is reciprocal, and with constant norm one, we immediately conclude that the two solutions of the second order equation

$$a^2 + \frac{\sqrt{5} - 1}{\sqrt{2}}a + 1 = 0,$$

(22)
which is
\[ a = \frac{\sqrt{5} - 1}{\sqrt{2}} \pm \frac{i\sqrt{\sqrt{5} + 1}}{2} \]  
(23)
provide us the valid overlap phases in the construction of SIC-POVMs.

The case of dimension four is exceptional in the sense that in the end there is only one independent variable. According to the known numerics, it is the only such case. Therefore our method necessarily involves computer softwares in higher dimensions. Hence instead of construct the SIC-POVMs explicitly, it is more interesting to consider its algebraic properties and try to relate the problem with number theoretical techniques. Since the essential part is that the solutions to the system of equations of the form (11) are all phases factors, it links to the problem of unit groups naturally, as observed in [26, 28].

5. Conclusion
We have analyzed the overlap phases in Weyl-Heisenberg group covariant SIC-POVMs and established an equivalent condition for their existence. We have shown how to interpret some abstract results about SIC-POVMs concretely by considering equation (11). In particular, we show how one can use the Zauner symmetry to reduce independent variables and therefore solved the problem in dimension four explicitly. We note that the solutions to our system of equations always exist, the difficulty is how to prove that these solutions are indeed phase factors. And hence this problem naturally relates to many problems in number theory. Our explicit condition (11) therefore not only provides a direct way of constructing Weyl-Heisenberg group covariant SIC-POVMs, but also suggests how to apply various algebraic techniques over the existence problem of SIC-POVMs.

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