THE REAL LOCUS OF AN INVOLUTION MAP ON THE MODULI SPACE OF FLAT CONNECTIONS ON A RIEMANN SURFACE

NAN-KUO HO

ABSTRACT. It is known that every nonorientable surface $\Sigma$ has an orientable double cover $\tilde{\Sigma}$. The covering map induces an involution on the moduli space $\mathcal{M}$ of gauge equivalence classes of flat $G$-connections on $\tilde{\Sigma}$. We identify the relation between the moduli space $\mathcal{M}$ and the fixed point set of the moduli space $\tilde{\mathcal{M}}$. In particular, $\mathcal{M}$ is isomorphic to the fixed point set of $\tilde{\mathcal{M}}$ if and only if the order of the center of $G$ is odd. One important application is that we give a way to construct a minimal Lagrangian submanifold of the moduli space $\tilde{\mathcal{M}}$.

1. Introduction

We know that every nonorientable surface $\Sigma$ has an orientable double cover $\tilde{\Sigma}$. The covering map induces an involution on the space of all flat $G$-connections on $\tilde{\Sigma}$ such that the space of all flat $G$-connections on $\Sigma$ sits in it as the fixed point set of the induced involution. Again, there is an induced involution on the moduli space $\tilde{\mathcal{M}}$ of gauge equivalence classes of flat $G$-connections on $\tilde{\Sigma}$. However, we do not know if we will still have that the moduli space $\mathcal{M}$ of flat $G$-connections on $\Sigma$ sits in $\tilde{\mathcal{M}}$ as the fixed point set of the involution since now we take the quotient by gauge transformations on both spaces.

We show in this paper that when the order of the center of $G$ is odd, there exists an isomorphism between (the smooth part which is an open dense set of) the moduli space $\mathcal{M}$ and (the smooth part which is an open dense set of) the fixed point set of the involution on the moduli space $\tilde{\mathcal{M}}$.

The benefit of this result is that since the moduli space of gauge equivalence classes of flat $G$-connections on an orientable surface is Kähler, the fixed point set of this involution, which is an anti-symplectic, anti-holomorphic isometry, is a totally geodesic Lagrangian submanifold of the Kähler manifold.

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We constructed a model similar to the standard model used to identify the moduli space of flat $G$-connections on a surface with the space of representations from the fundamental group of the surface into $G$, divided by the conjugate action (for example: [Go1]) by introducing two base points on the surface instead of one base point. This model is constructed in such a way that it is clear how exactly the involution works on both the surface $\tilde{\Sigma}$ and the moduli space $\tilde{\mathcal{M}}$.

Throughout this paper, denote $\Sigma$ a compact, closed surface; $G$ a compact, connected, semisimple Lie group; and $e$ the identity element of the Lie group $G$. Our main result is, (note that the smooth part of the moduli space is an open dense subset)

**Theorem 1.** Let $\Sigma$ be a compact, closed nonorientable surface and $\tilde{\Sigma}$ its orientable double cover. Then the natural map from the moduli space of flat connections on $\Sigma$, $\mathcal{M}$, into the fixed point set of the involution on the moduli space of flat connections on $\tilde{\Sigma}$, $\tilde{\mathcal{M}}$, is a $|Z(G)/2Z(G)|$ to one map. In particular, (the smooth part of) the moduli space $\mathcal{M}$ is isomorphic to (the smooth part of) the fixed point set of the involution on the moduli space $\tilde{\mathcal{M}}$ if and only if $|Z(G)|$ is odd.

The paper is organized as follows: section two explains what is the fixed point set of the involution on $\tilde{\mathcal{M}}$; section three gives the relation between the fixed point set of $\tilde{\mathcal{M}}$ and $\mathcal{M}$; Examples are given in section four, and the last section gives an application of the theorem to the moduli space of semistable vector bundles on a Riemann surface.

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2. The fixed point set of the involution of $\tilde{\mathcal{M}}$

Consider $\Sigma$, a nonorientable surface, and $\tilde{\Sigma}$ its orientable double cover. Let $\mathcal{M}$ denote the moduli space of flat $G$-connections on $\Sigma$ and $\tilde{\mathcal{M}}$ denote the moduli space of flat $G$-connections on $\tilde{\Sigma}$. The $\mathbb{Z}_2$ action on $\tilde{\Sigma}$ gives the quotient space $\Sigma$, and this $\mathbb{Z}_2$-action induces an involution $\tau$ on the space of all $\mathfrak{g}$-valued one forms on $\tilde{\Sigma}$ and the fixed point set of the space of of all $\mathfrak{g}$-valued one forms on $\tilde{\Sigma}$ is equivalent to the space of all $\mathfrak{g}$-valued one forms on $\Sigma$.

Let $\tau : \Omega^1(\tilde{\Sigma}, \mathfrak{g}) \to \Omega^1(\tilde{\Sigma}, \mathfrak{g})$ denote the involution ($\mathbb{Z}_2$-action) and $A$ is a one form on $\tilde{\Sigma}$. Then $A$ is a fixed point of $\tau$ if $\tau(A)(x) = A(x)$ for all $x \in \tilde{\Sigma}$, i.e. $A(\tau(x)) = A(x)$,
i.e. $A$ has the same value at $x$ and $\tau(x)$ which means that $A$ corresponds to a one form in $\Omega^1(\Sigma, g)$.

Thus, the the fixed point set of the involution on the flat $G$-connections on the orientable double cover is the same as the space of all flat $G$-connections on the nonorientable surface, i.e. $(\mathcal{A}_{flat}(\tilde{\Sigma}))^\tau = \mathcal{A}_{flat}(\Sigma)$.

Fix a metric $h$ on $\Sigma$ and let $\pi: \tilde{\Sigma} \to \Sigma$ be the covering map. Let $\pi^*h$ be the (corresponding) pullback metric on $\tilde{\Sigma}$ then $\pi^*h$ determines a complex structure $J$ on $\tilde{M}$ such that $(\tilde{M}, \omega)$ is a Kähler manifold. In our situation here the $J$ is the Hodge star $*$ on $\Omega^1(\tilde{\Sigma}, g)$, and $\omega$ is the usual symplectic structure on $\tilde{M}$ i.e. $\omega(a, b) = \int_{\tilde{\Sigma}} Tr(a \wedge b)$ where $Tr$ stands for the Killing form on the Lie algebra $g$.

We have the following two observations:

**Lemma 2.** The isometric involution $\tau: \mathcal{A}_{flat}(\tilde{\Sigma}) \to \mathcal{A}_{flat}(\tilde{\Sigma})$ is anti-holomorphic.

Proof. Since $\tau$ is an orientation reversing isometry, we have $< a, b > = < \tau_*a, \tau_*b >$ and $a \wedge b = -\tau_*a \wedge \tau_*b$. Thus

$$< a, b > = \int Tr(a \wedge Jb) = \int -Tr(\tau_*a \wedge \tau_*Jb)$$

$$< \tau_*a, \tau_*b > = \int Tr(\tau_*a \wedge J\tau_*b)$$

for all $a, b \in T_{\mathcal{A}}\mathcal{A}_{flat}(\tilde{\Sigma})$, thus we have $J\tau_* = -\tau_*J$ which means $\tau$ is anti-holomorphic. □

**Proposition 3.** Let $M$ be a Kähler manifold. If $f: M \to M$ is an anti-holomorphic isometric involution on $M$, and if the fixed point set of $f$, $N$, is nonempty, then $N$ is a totally geodesic, totally real, Lagrangian submanifold.

Proof. We refer to [Oh].

(1) We would like to show that a fixed point set of an isometry is a totally geodesic submanifold.

If $N$ is the fixed point set of an isometry $f : M \to M$, and suppose $r(t)$ is a geodesic in $M$ where $r(0)$ is in $N$ and $r'(0) \in T_{r(0)}N$, then $f \circ r(0) = r(0)$ since $r(0) \in N$ and $(f \circ r)'(0) = f_*r'(0) = r'(0)$ since $r'(0) \in T_{r(0)}N$. Thus $f \circ r$ is a geodesic (since $f$ is an isometry and $r$ is a geodesic) with $f \circ r(0) = r(0)$ and $f \circ r'(0) = r'(0)$. It follows that $r(t) = f \circ r(t)$ by the uniqueness of geodesics, which means $r(t)$ is fixed by $f$ so $r(t) \in N$. This shows that $N$ is a totally geodesic submanifold.

(2) The fixed point set of an anti-holomorphic involution is totally real.
Suppose $f$ is the anti-holomorphic involution, and $N$ is the fixed point set. Let $M$ and $J$ be given. Now since $f$ is anti-holomorphic, $Jf = -fJ$. Now, if $v \in T_xN$ then $f_*v = v$ and $Jv = Jf_*v = -f_*(Jv)$ so $Jv$ is not fixed by $f_*$. Thus $Jv \notin T_xN$, so $T_xN \cap J(T_xN) = 0$ and $N$ is totally real. □

Since the moduli space of flat $G$-connections on a Riemann surface is a Kähler manifold (ref: [AB]), and $\tilde{M}^r$ is nonempty (since the set of $(A_{flat}(\tilde{\Sigma}))^r$ is nonempty) we have the following corollary:

**Corollary 4.** The real locus of the involution on the moduli space $\tilde{M}$ is a totally real, totally geodesic Lagrangian submanifold of $\tilde{M}$.

### 3. The map between the fixed point set of the involution of $\tilde{M}$ and $M$

In this section we discuss the relation between $M$ and the fixed point set of $\tilde{M}$ according to its topological type of the base space $\Sigma$.

**Case 1.** $\Sigma = \Sigma_{\ell} \# \mathbb{RP}^2$

Let $\Sigma$ be a compact nonorientable surface whose topological type is the connected sum of a Riemann surface of genus $\ell$ and the real projective plane. Let $\tilde{\Sigma}$ be the orientable double cover of $\Sigma$. Then $\tilde{\Sigma}$ is a compact orientable surface of genus $2\ell$.

Let us look at the picture below,

![Figure 1. Holonomy on the double cover](image)

where $a_i, b_i$ (resp. $\bar{a}_i, \bar{b}_i$) are the holonomies of flat connections around loops $A_i$ and $B_i$ (resp. $\bar{A}_i$ and $\bar{B}_i$) based at $P_+$ (resp. $P_-$), while $c$ is the parallel transport along an arc from $P_+$ to $P_-$ and $\bar{c}$ is the parallel transport along an arc from $P_-$ to $P_+$. 
We can then define the moduli space of flat connections on \( \tilde{\Sigma} \) by introducing two base points \( P_+, P_- \),
\[
\mathcal{M}(\tilde{\Sigma}) = \left[ \mathcal{A}_F(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}, P_+, P_-) \right]/G^2
\]
\[
= \left\{ (a_1, b_1, \cdots, a_\ell, b_\ell, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, c) \in G^{2(2\ell + 1)} \mid \prod_{i=1}^\ell [a_i, b_i] = c\bar{c}, \prod_{i=1}^\ell [\bar{a}_i, \bar{b}_i] = \bar{c}\bar{c} \right\}/G \times G
\]
where \( G \times G \) acts as
\[
(g_1, g_2) \cdot (a_1, b_1, \cdots, a_\ell, b_\ell, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, c) \equiv (g_1 a_1 g_1^{-1}, g_1 b_1 g_1^{-1}, \cdots, g_1 a_\ell g_1^{-1}, g_1 b_\ell g_1^{-1}, g_1 c\bar{g}_1^{-1},
\]
\[g_2 \bar{a}_1 g_2^{-1}, g_2 \bar{b}_1 g_2^{-1}, \cdots, g_2 \bar{a}_\ell g_2^{-1}, g_2 \bar{b}_\ell g_2^{-1}, g_2 \bar{c}\bar{g}_2^{-1}\].

and the moduli space of flat connections on \( \Sigma \),
\[
\mathcal{M}(\Sigma) = \left[ \mathcal{A}_F(\Sigma)/\mathcal{G}(\Sigma, P) \right]/G
\]
\[
= \left\{ (a_1, b_1, \cdots, a_\ell, b_\ell, c) \in G^{2\ell + 1} \mid \prod_{i=1}^\ell [a_i, b_i] = c^2 \right\}/G
\]
where \( G \) action is the conjugate action.

**Case 2.** \( \Sigma = \Sigma_{\ell +} \textbf{Klein bottle} \).

Let \( \Sigma \) be a compact nonorientable surface whose topological type is the connected sum of a Riemann surface of genus \( \ell \) and a Klein bottle. Let \( \tilde{\Sigma} \) be the orientable double cover of \( \Sigma \). Then \( \tilde{\Sigma} \) is a compact orientable surface of genus \( 2\ell + 1 \).

Figure 2 is the picture of holonomy on the double cover \( \Sigma_3 \), where \( a_i, b_i, d \) (resp. \( \bar{a}_i, \bar{b}_i, \bar{d} \)) are the holonomies of flat connections around loops \( A_i, B_i, \) and \( D \) (resp. \( \bar{A}_i, \bar{B}_i, \) and \( \bar{D} \)) based at \( P_+ \) (resp. \( P_- \)), while \( c \) is the parallel transport along an arc from \( P_+ \) to \( P_- \) and \( \bar{c} \) is the parallel transport along an arc from \( P_- \) to \( P_+ \).

This means
\[
aba^{-1}b^{-1}d^{-1}cd^{-1}c^{-1} = \text{id}.
\]
thus
\[
aba^{-1}b^{-1} = cd^{-1}d
\]
We can then define the moduli space of flat connections on $\tilde{\Sigma}$ by introducing two base points $P_+, P_-$

$$\mathcal{M}(\tilde{\Sigma}) = [\mathcal{A}_F(\tilde{\Sigma})/\mathcal{G}(\tilde{\Sigma}, P_+, P_-)]/G^2$$

$$= \{ (a_1, b_1, \ldots, a_\ell, b_\ell, d, c, \overline{a}_1, \overline{b}_1, \ldots, \overline{a}_\ell, \overline{b}_\ell, \overline{d}, \overline{c}) \in G^{2(2\ell+2)} \mid \prod_{i=1}^\ell [a_i, b_i] = c \overline{d} c^{-1} \overline{d}, \prod_{i=1}^\ell [\overline{a}_i, \overline{b}_i] = \overline{c} \overline{d} \overline{c}^{-1} \overline{d} \} / G \times G$$

where $G \times G$ acts as

$$(g_1, g_2) \cdot (a_1, b_1, \ldots, a_\ell, b_\ell, d, c, \overline{a}_1, \overline{b}_1, \ldots, \overline{a}_\ell, \overline{b}_\ell, \overline{d}, \overline{c})$$

$$= (g_1 a_1 g_1^{-1}, g_1 b_1 g_1^{-1}, \ldots, g_1 a_\ell g_1^{-1}, g_1 b_\ell g_1^{-1}, g_1 d g_1^{-1}, g_1 c g_1^{-1},$$
$$g_2 a_1 g_2^{-1}, g_2 b_1 g_2^{-1}, \ldots, g_2 a_\ell g_2^{-1}, g_2 b_\ell g_2^{-1}, g_2 d g_2^{-1}, g_2 c g_2^{-1})$$. 

**Figure 2.** Holonomy on the double cover $\Sigma_3$
and the moduli space of flat connections on $\Sigma$,

$$
\mathcal{M}(\Sigma) = \left[ A_F(\Sigma)/G(\Sigma, P) \right]/G
$$

$$
= \left\{ (a_1, b_1, \cdots, a_\ell, b_\ell, d, c) \in G^{2\ell+2} \mid \prod_{i=1}^{\ell} a_i, b_i ] = cdc^{-1} \right\} /G
$$

where $G$ action is the conjugate action.

Thus the natural involution from $\mathcal{M}(\Sigma)$ to $\mathcal{M}(\tilde{\Sigma})$ is either

$$
\mathcal{M}(\Sigma) \to \mathcal{M}(\tilde{\Sigma})
$$

$$
[(a_1, b_1, \cdots, c)] \mapsto [(a_1, b_1, \cdots, c, a_1, b_1, \cdots, c)]
$$

or

$$
\mathcal{M}(\Sigma) \to \mathcal{M}(\tilde{\Sigma})
$$

$$
[(a_1, b_1, \cdots, d, c)] \mapsto [(a_1, b_1, \cdots, d, c, a_1, b_1, \cdots, d, c)]
$$

We would like to introduce some notations for later convenience:

**Case 1: $\Sigma = \Sigma_{\ell \sharp} RP^2$**

Set

$$
M = \left\{ (a_1, b_1, \cdots, a_\ell, b_\ell, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}) \in G^{2(2\ell+1)} \mid \prod_{i=1}^{\ell} [a_i, b_i] = \bar{c}c, \prod_{i=1}^{\ell} [\bar{a}_i, \bar{b}_i] = \bar{c}\bar{c} \right\}
$$

$$
K = G \times G \text{ acts on } M \text{ by }
$$

$$
(g_1, g_2) \cdot (a_1, b_1, \cdots, a_\ell, b_\ell, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}) = (g_1a_1g_1^{-1}, g_1b_1g_1^{-1}, \cdots,
$$

$$
g_1a_\ell g_1^{-1}, g_1b_\ell g_1^{-1}, g_1c g_2^{-1}, g_2\bar{a}_1 g_2^{-1}, g_2\bar{b}_1 g_2^{-1}, \cdots, g_2\bar{a}_\ell g_2^{-1}, g_2\bar{b}_\ell g_2^{-1}, g_2\bar{c} g_1^{-1})
$$

There is an involution $\tau : K \to K$ given by

$$
\tau(g_1, g_2) = (g_2, g_1),
$$

and an involution $\tau : M \to M$ given by

$$
\tau(a_1, b_1, \cdots, a_\ell, b_\ell, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}) = (\bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}, a_1, b_1, \cdots, a_\ell, b_\ell, c).
$$

**Case 2: $\Sigma = \Sigma_{\ell \sharp} $ Klein bottle**

Set

$$
M = \left\{ (a_1, b_1, \cdots, a_\ell, b_\ell, d, c, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{d}, \bar{c}) \in G^{2(2\ell+2)} \mid \prod_{i=1}^{\ell} [a_i, b_i] = \bar{d}d, \prod_{i=1}^{\ell} [\bar{a}_i, \bar{b}_i] = \bar{d}\bar{c} \right\}
$$
There is an involution \( \tau : K \to K \) given by

\[ \tau(g_1, g_2) = (g_2, g_1), \]

and an involution \( \tau : M \to M \) given by

\[ \tau(a_1, b_1, \ldots, a_\ell, b_\ell, d, c, a_1, b_1, \ldots, a_\ell, b_\ell, d, c) \]

\[ = (\bar{a}_1, \bar{b}_1, \ldots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}, a_1, b_1, \ldots, a_\ell, b_\ell, d, c). \]

In both cases, we have \( \tau(k \cdot x) = \tau(k) \cdot \tau(x) \) for \( k \in K \), \( x \in M \) and it induces an involution \( \tau : M/K \to M/K \),

\[ \tau([x]) = [\tau(x)], \]

which can be easily checked to be well-defined.

Notice that the action defined by equation (2) and (6) is just a combination of conjugate action and an action as follows: \( (g, h) \cdot (x, y) = (gxh^{-1}, hyg^{-1}) \). Thus equation (2) and (6) can be rewritten as

\[ (g_1, g_2) \cdot (V, c, \bar{V}, \bar{c}) = (g_1Vg_1^{-1}, g_1cg_2^{-1}, g_2\bar{V}g_2^{-1}, g_2\bar{c}g_1^{-1}) \]

if we denote \( (a_1, b_1, \ldots, a_\ell, b_\ell) \) by \( V \), and \( (ha_1h^{-1}, hb_1h^{-1}, \ldots, ha_\ell h^{-1}, hb_\ell h^{-1}) \) by \( hvh^{-1} \) for equation (2); \( (a_1, b_1, \ldots, a_\ell, b_\ell, d) \) by \( V \), and \( (ha_1h^{-1}, hb_1h^{-1}, \ldots, ha_\ell h^{-1}, hb_\ell h^{-1}, hdh^{-1}) \) by \( hvh^{-1} \) for equation (6).

**Remark 5.**

With the definition of \( M, K, \tau \) given in [1], [2], [3] and [4], or [5], [6], [7], and [8], the moduli space \( \mathcal{M}(\Sigma) \) of gauge equivalence classes of flat \( G \)-connections on \( \tilde{\Sigma} \) is identified with \( M/K \), while the moduli space \( \mathcal{M}(\Sigma) \) of gauge equivalence classes of flat \( G \)-connections on \( \Sigma \) is identified with \( M'/K' \). Also, we have the same action formulas for Case 1 and Case 2 if we use the simplified notation \( (V, c, \bar{V}, \bar{c}) \) to write points in
where $V$ represents either $(a_1, b_1, \cdots, a_\ell, b_\ell)$ or $(a_1, b_1, \cdots, a_\ell, b_\ell, d)$. Thus in this section, we will be using $M, K, \tau$ and $(V, c, \bar{V}, \bar{c})$ to show the relation between $\mathcal{M}(\Sigma)$ and $\mathcal{M}((\Sigma)).$

There is a natural map $I$ from the moduli space $M^\tau/K^\tau$ of flat $G$-connections on $\Sigma$ to $(M/K)^\tau$ the fixed point set of the involution on the moduli space of flat $G$-connections on $\bar{\Sigma}$,

$$I : M^\tau/K^\tau \to (M/K)^\tau$$

$$[[[V, c, V, c]]] \mapsto [(V, c, V, c)]$$

Here $[[:]]$ represents the equivalence class in $M^\tau/K^\tau$ and $[.]$ represents the equivalence class in $(M/K)^\tau$.

This map is well defined: If $[[[V, c, V, c]]] = [[[V', c', V', c']]]$ in $M^\tau/K^\tau$ then there exists $(g, g)$ in $K^\tau$ such that $(g, g) \cdot (V, c, V, c) = (V', c', V', c')$. Of course $[[V, c, V, c]] \in (M/K)^\tau$ and $[[V', c', V', c']] \in (M/K)^\tau$. Let $k = (g, g)$ thus $k \cdot (V, c, V, c) = (V', c', V', c')$ and they are the same element in $(M/K)^\tau$.

We want to know if the map $I$ is surjective or injective.

3.1. Surjectivity. We know that points which have center as their stabilizer correspond to the smooth part of the moduli space of flat connections which is open dense (ref: [Go1] and [AB]). For this reason, we give the following definition:

**Definition 6.** A point $x$ is called generic if its stabilizer is equal to the center.

Suppose that $x = (V, c, \bar{V}, \bar{c}) \in M$ satisfies $[x] \in (M/K)^\tau$. Then there exists $k = (g_1, g_2) \in K$ such that $k \cdot x = \tau(x)$. Recall that the action $k \cdot x$ is the conjugate action on the $V, \bar{V}$ part and $(g_1, g_2) \cdot (c, \bar{c}) = (g_1c g_2^{-1}, g_2 c g_1^{-1})$, and $\tau(x)$ just switches $(V, c)$ and $(\bar{V}, \bar{c})$. Thus we have

$$(g_1V g_1^{-1}, g_1c g_2^{-1}, g_2 \bar{V} g_2^{-1}, g_2 c g_1^{-1}) = (\bar{V}, \bar{c}, V, c)$$

$$\iff$$

$$(g_2 g_1 V g_1^{-1} g_2^{-1}, g_2 g_1 c g_2^{-1} g_1^{-1}) = (V, c)$$

For generic $[x] \in (M/K)^\tau$ we may assume that $r^{-1} = g_2 g_1 \in Z(G)$. Then $k = (g, r^{-1} g^{-1})$, where $g = g_1 \in G$, $r \in Z(G)$ and we have

$$x = (V, c, gV g^{-1}, gcgr).$$

Let us define

$$\bar{x} := (e, g^{-1}) \cdot x = (V, cg, V, cgr).$$

Note that $[\bar{x}] = [x]$, and $\bar{x} \in N_r$, for $N_r$ is the subset of $M$ defined by
Case 1: $\Sigma = \Sigma_2^2 \mathbb{RP}^2$

$$N_r = \{ (V,c,V,cr) = (a_1, b_1, \ldots, a_\ell, b_\ell, c, a_1, b_1, \ldots, a_\ell, b_\ell, cr) \mid (a_1, b_1, \ldots, a_\ell, b_\ell, c) \in G^{2\ell+1}, \prod_{i=1}^{\ell} [a_i, b_i] = c^2 r \}$$

or

Case 2: $\Sigma = \Sigma_2^2$ Klein bottle

$$N_r = \{ (V,c,V,cr) = (a_1, b_1, \ldots, a_\ell, b_\ell, c, a_1, b_1, \ldots, a_\ell, b_\ell, d, cr) \mid (a_1, b_1, \ldots, a_\ell, b_\ell, c) \in G^{2\ell+2}, \prod_{i=1}^{\ell} [a_i, b_i] = cdc^{-1} dr \}$$

In this way, we choose a nice representative $\tilde{x} \in M$ for any generic point $[x] \in (M/K)^r$ where "generic" was defined in Definition 6.

Let $P : M \to M/K$ be the natural projection. We have

$$S := \bigcup_{r \in Z(G)} P(N_r) \subset (M/K)^r,$$

and $S$ is a dense subset of $(M/K)^r$. Clearly $\bigcup_{r \in Z(G)} P(N_r) \subset (M/K)^r$. The reason for $S$ being dense in $(M/K)^r$ is that, for generic $[x] \in (M/K)^r$, we can always find some $h$ such that $h.x \in N_r$ for some $r \in Z(G)$ (from the previous argument) such that $P(h.x) = [h.x] = [\tilde{x}] = [x]$, which means $\bigcup_{r \in Z(G)} P(N_r)$ covers generic points in $(M/K)^r$.

Thus, there are at most $Z(G)$ copies of $P(N_r)$ which cover generic points in $(M/K)^r$. We would like to know if there is any relation between $P(N_{r_1})$ and $P(N_{r_2})$ for $r_1 \neq r_2$ in $Z(G)$.

**Lemma 7.** Let $2Z(G) \subset Z(G)$ denote the subgroup $\{g^2 \mid g \in Z(G)\}$ of the finite abelian group $Z(G)$. If $[r_2] = [r_1] \in Z(G)/2Z(G)$, then $P(N_{r_1}) = P(N_{r_2})$. Moreover, if $[r] \neq [e]$, then for generic $[x] \in P(N_r)$, $[x] \notin P(N_e)$.

**Proof.** If $[r_2] = [r_1] \in Z(G)/2Z(G)$, then $r_2 = r_1 s^2$ for some $s \in Z(G)$, and

$$(s,c) \cdot (V,c,V,cr_2) = (V,sc,V,scr_1),$$

which means

$$[(V,c,V,cr_2)] = [(V,sc,V,scr_1)],$$

where the left hand side is a point in $P(N_{r_2})$ and the right hand side is a point in $P(N_{r_1})$, i.e. $P(N_{r_2}) \subset P(N_{r_1})$ which implies $P(N_{r_1}) = P(N_{r_2})$. Thus, if $[r_1] = [r_2] \in Z(G)/2Z(G)$, then $P(N_{r_1}) = P(N_{r_2})$ (in $(M/K)^r$).


Now let \( r \) be an element in \( Z(G) \) such that \( r \notin 2Z(G) \). We want to show that \( P(N_r) \cap P(N_e) \) is generically empty, i.e. for generic \( [x] \in P(N_r) \), \([x] \notin P(N_e) \). Let
\[
x = (V, c, V, cr) \in N_r.
\]

Suppose that \([x] \in P(N_e) \). Then there exists \( k = (g_1, g_2) \in K \) such that \( k \cdot x \in N_e \). We have
\[
(g_1 V g_1^{-1}, g_1 c g_2^{-1}) = (g_2 V g_2^{-1}, g_2 c g_1^{-1})
\]
\[\iff (V, c) = (g_1^{-1} g_2 V g_2^{-1} g_1, g_1^{-1} g_2 g_1^{-1} g_2 r)
\]
For generic \([x] \in P(N_r) \) we have \( g_1^{-1} g_2 = s \in Z(G) \), and \( c = g_1^{-1} g_2 c g_1^{-1} g_2 r \) gives us \( s^2 r = e \), which contradicts the fact that \( r \notin 2Z(G) \).
\( \square \)

This simplifies the relation between \( \bigcup_{r \in Z(G)} P(N_r) \) and \((M/K)^r \) as follows:

**Lemma 8.** Given \([r] \in Z(G)/2Z(G) \), set \( P_{[r]} = P(N_r) \). We have
\[
I(M^r/K^r) = P_{[e]} \subset S := \bigcup_{[r] \in Z(G)/2Z(G)} P_{[r]} = (M/K)^r,
\]
where \( e \) is the identity element of \( Z(G) \) and the union is (generically) disjoint union.

Proof. We know that \( S \) is a dense subset of \((M/K)^r \). The set \( S \) is compact since it is the image of the compact subset \( \bigcup_{r \in Z(G)} N_r \) of \( M \) under the continuous map \( p : M \to M/K \).
In particular, \( S \) is closed because \( \bigcup_{r \in Z(G)} N_r \) is. Thus we have
\[
(M/K)^r = S = \bigcup_{[r] \in Z(G)/2Z(G)} P_{[r]},
\]
and \( P_{[e]} = I(M^r/K^r) \)
\( \square \)

Thus we have the following theorem:

**Theorem 9.** The map \( I : M^r/K^r \to (M/K)^r \) is surjective if and only if \(| Z(G) | \) is odd.

Proof. The above Lemma says \( (M/K)^r = \bigcup_{[r] \in Z(G)/2Z(G)} P_{[r]} \) where \( P_{[e]} \) is the image of \( M^r/K^r \) under \( I \). (This applies to all points, not only generic points.) This tells us that when \( 2Z(G) = Z(G) \) the map \( I \) is surjective (not only that the image is a dense set). On the other hand, each set \( P_{[r]} \) is compact and closed, and they are generically disjoint and thus the set \( P_{[e]} \) need not be dense in \( M^r/K^r \) if \( Z(G)/2Z(G) \neq \{e\} \). This implies that
the map $I$ is surjective if and only if $2Z(G) = Z(G)$. This is the same as $|Z(G)|$ being odd because $Z(G)$ is a finite abelian group, and since $2Z(G)$ is a subgroup of $Z(G)$, the order of $2Z(G)$ divides the order of the group $Z(G)$. This means that $2Z(G) = Z(G)$ if and only if the order of $Z_{n_i}$ is never even.

\[
\square
\]

3.2. Injectivity. Suppose that we have $x, \bar{x} \in M^\tau$ such that $[x] = [\bar{x}] \in (M/K)^\tau$. Then we have $\bar{x} = k \cdot x$ for some $k = (g_1, g_2) \in K$.

Suppose that

\[
\begin{align*}
    x &= (V, c, V, c) \\
    \bar{x} &= (\bar{V}, \bar{c}, \bar{V}, \bar{c})
\end{align*}
\]

We have

\[
\begin{align*}
    (\bar{V}, \bar{c}) &= (g_1 V g_1^{-1}, g_1 c g_2^{-1}) \\
    &= (g_2 V g_2^{-1}, g_2 c g_1^{-1})
\end{align*}
\]

\[\iff\]

\[
(V, c) = (g_1^{-1} g_2 V g_2^{-1} g_1, g_1^{-1} g_2 c g_1^{-1} g_2)
\]

For the generic case, we have $g_1^{-1} g_2 = r \in Z(G)$ and $r^2 = e$ (from $c = g_1^{-1} g_2 c g_1^{-1} g_2$), where $e$ is the identity element of $Z(G)$. If we consider the homomorphism $\phi : Z(G) \to Z(G)$ given by $g \mapsto g^2$, then $r \in \text{Ker}\phi$. The element $k = (g_1, g_2)$ is in $K^\tau$ if and only if $r = e$. Let $d = |\text{Ker}\phi| = |Z(G)/2Z(G)|$. Then $I$ is generically $d$ to 1. $I$ is generically injective if and only if $2Z(G) = Z(G)$. In other words, if $x$ and $\bar{x}$ in $M^\tau$ maps to the same class in $(M/K)^\tau$ then $r \in Z(G)$ and $r^2 = e$, and the condition for $[x] = [\bar{x}]$ in $M^\tau/K^\tau$ is that $r = e$. Thus for the same class in $(M/K)^\tau$, we could have (number of preimages of $e$ in $Z(G)$) classes in $M^\tau/K^\tau$, i.e. $|Z(G)/2Z(G)|$ classes. We have the following theorem:

**Theorem 10.** The map $I : M^\tau/K^\tau \to (M/K)^\tau$ is generically $|Z(G)/2Z(G)|$ to 1. In particular, $I$ is injective at generic points of $M^\tau/K^\tau$ if and only if $|Z(G)|$ is odd.

**Remark 11.**

As we see now, the topological type of the nonorientable surface does not affect the injectivity and surjectivity of the map $I$. 
4. Examples

Let $I : M^\tau/K^\tau \to (M/K)^\tau$ be defined as in Section 3, where $M^\tau/K^\tau$ is the moduli space of gauge equivalence classes of flat $G$-connections on a nonorientable compact surface $\Sigma$, and $(M/K)^\tau$ is the fixed point set of the involution on the moduli space of gauge equivalence classes of flat $G$-connections on its orientable double cover $\tilde{\Sigma}$. Also, recall that the generic points are the smooth part of the manifold $M$.

1. $G = SU(n)$: If $n$ is odd, we have $Z(G) = \mathbb{Z}_n = 2\mathbb{Z}(G)$, so $I$ is surjective and generically injective. If $n$ is even, we have $Z(G)/2Z(G) = \{[e], [-e]\} \cong \mathbb{Z}/2\mathbb{Z}$, so $I$ is not surjective and is generically 2 to 1.

2. $G = Sp(n)$: We have $Z(G) = \{e, -e\}$ and $2Z(G) = \{e\}$, so $I$ is not surjective and is generically 2 to 1.

3. $G = Spin(n)$: If $n$ is odd, we have $Z(G) = \{1, -1\}$ and $2Z(G) = \{1\}$, so $I$ is not surjective and is generically 2 to 1. If $n$ is even, we have $Z(G) = \{1, -1, e_1e_2 \cdots e_n, -e_1e_2 \cdots e_n\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $2Z(G) = \{1\}$, so $I$ is not surjective and is generically 4 to 1.

4. $G = SO(n)$: If $n$ is odd, we have $Z(G) = \{e\} = 2Z(G)$, so $I$ is surjective and generically injective. If $n$ is even, we have $Z(G) = \{e, -e\}$ and $2Z(G) = \{e\}$, so $I$ is not surjective and is generically 2 to 1.

5. moduli space of semistable vector bundles on a Riemann surface

There is a very important relation between the moduli space of flat $SU(k)$-connections over a Riemann surface and the moduli space of semistable holomorphic vector bundles of rank $k$, degree 0 on a Riemann surface as shown in [NS]. Using our result, we find an interesting Lagrangian submanifold of the moduli space of semistable holomorphic vector bundles of rank $k$ and degree $d$ on a Riemann surface of genus $\ell$,

$$\mathcal{M}(k, d) = \{(a_1, b_1, \cdots, a_\ell, b_\ell) \in G^{2\ell} \mid \prod_{i=1}^{\ell} [a_i, b_i] = \exp^{2\pi id/k} \mathbb{I}_k \}/G.$$  

For $k = 2n + 1$, the moduli space of flat $SU(2n + 1)$-connections on a non-orientable surface is a minimal Lagrangian submanifold of the moduli space of flat $SU(2n + 1)$-connections on its orientable double cover which is a Riemann surface. Thus the moduli space of flat $SU(2n + 1)$-connections on a nonorientable surface is a minimal Lagrangian submanifold of the moduli space of semistable holomorphic vector bundles of rank $2n+1$ and degree 0 on its orientable double cover.
For \( k = 2n \), we know that the moduli space of flat \( PSU(2n) \)-connections on a non-orientable surface is a minimal Lagrangian submanifold of the moduli space of flat \( PSU(2n) \)-connections on its orientable double cover because the center of \( PSU(2n) \) is the trivial group. We look at the commutative diagram below,

\[
\begin{array}{ccc}
SU(2n)^{2\ell} & \xrightarrow{\tilde{\mu}} & SU(2n) \\
\pi^{2\ell} & \downarrow & \downarrow \pi \\
PSU(2n)^{2\ell} & \xrightarrow{\mu} & PSU(2n)
\end{array}
\]

where \( \pi \) is the projection from \( SU(2n) \) to \( PSU(2n) \) and \( \mu \) is the product of commutators. If \( h \in PSU(2n)^{2\ell} \) is an element such that \( \mu(h) = e \) the identity element in \( PSU(2n) \), then there is a lifting \( \tilde{h} \in SU(2n)^{2\ell} \) of \( h \) such that the whole diagram commutes. Thus, \( \mu(\tilde{h}) \in Kern = Z(SU(2n)) \). Thus \( \tilde{\mu}(\tilde{h}) = exp^{2\pi id/2n} I_2 \) for some integer \( 0 \leq d \leq 2n \) which means \( \tilde{h} \) represents a semistable vector bundle of rank \( 2n \) and degree \( d \). The question is which \((d,2n)\) this \( \tilde{h} \) represents.

First, we need to go back to two earlier papers [Li] [HL] about the connected components of the moduli spaces. In [Li] we see that the moduli space of flat \( PSU(2n) \)-connections on a Riemann surface has \( 2n \) connected components since the order of \( \pi_1(PSU(2n)) \) is \( 2n \). Thus the moduli space of flat \( PSU(2n) \)-connections on a Riemann surface (equivalently representations of \( \pi_1(\Sigma) \) into \( PSU(2n) \) mod conjugacy) is equivalent under \( \pi^{2\ell} \) to the disjoint union of the moduli spaces \( M(2n,d) \) of semistable holomorphic vector bundles of rank \( 2n \) and of degree \( d \), \( d \) from 0 to \( 2n - 1 \), by the Narasimhan-Seshadri theorem [NS]. In [HL], we can see that the moduli space of flat \( PSU(2n) \)-connections on a nonorientable surface has only two connected components since the order of \( \pi_1(PSU(2n))/2\pi_1(PSU(2n)) \) is 2.

Let us recall the map from our model of two base points for the moduli space to the standard model of one base point for the moduli space:

Map \( \Phi \) for genus \( 2\ell \),

\[
\Phi : \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \in G^{4\ell + 2} \mid \prod_{i=1}^{\ell} [a_i, b_i] = cc', \prod_{i=1}^{\ell} [a'_i, b'_i] = c'c\}/G \times G
\]

\[
\to \{(a_1, b_1, \cdots, a_\ell, b_\ell, a'_1, b'_1, \cdots, a'_\ell, b'_\ell) \in G^{4\ell} \mid \prod_{i=1}^{\ell} [a_i, b_i] = e\}/G
\]

is defined as

\[
\Phi : \[(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c')\]
\]

\[
\to \[(a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b'_\ell c', c^{-1}a'_\ell c', \cdots, c^{-1}b'_1 c', c^{-1}a'_1 c')\].
\]
Notice that this map is a well defined bijection:

If

\[(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \mapsto (a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b'_\ell c', c^{-1}a'_\ell c', \cdots, c^{-1}b'_1 c', c^{-1}a'_1 c')\]

then

\[(e, g)(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \mapsto (a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b'_\ell c', c^{-1}a'_\ell c', \cdots, c^{-1}b'_1 c', c^{-1}a'_1 c')\]

which are the same elements, and

\[(g, e)(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c') \mapsto (ga_1 g^{-1}, gb_1 g^{-1}, \cdots, ga_\ell g^{-1}, gb_\ell g^{-1}, gc, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c' g^{-1})\]

\[\mapsto (ga_1 g^{-1}, gb_1 g^{-1}, \cdots, ga_\ell g^{-1}, gb_\ell g^{-1}, (c' g^{-1})^{-1}b'_\ell (c' g^{-1}), (c' g^{-1})^{-1}a'_1 (c' g^{-1}))\]

\[= g(a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b'_\ell c', c^{-1}a'_\ell c', \cdots, c^{-1}b'_1 c', c^{-1}a'_1 c') g^{-1}\]

which means they are members of the same equivalence classes in the moduli space.

Thus

\[[(a_1, b_1, \cdots, a_\ell, b_\ell, c, a'_1, b'_1, \cdots, a'_\ell, b'_\ell, c')] \mapsto [(a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b'_\ell c', c^{-1}a'_\ell c', \cdots, c^{-1}b'_1 c', c^{-1}a'_1 c')].\]

is well defined.

According to the argument appeared in [H], the moduli space $\mathcal{M}_{PSU(2n)}$ of flat $SU(2n)$-connections on the connected sum of $\Sigma_\ell$ and $RP^2$ has two connected components:

\[\mathcal{M}_{PSU(2n)} = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in PSU(2n)^{2\ell+1} | \prod_{i=1}^{\ell} [a_i, b_i]^2 = e\} / PSU(2n)\]

\[= \pi^{2\ell+1}\{(\bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}) \in SU(2n)^{2\ell+1} | \prod_{i=1}^{\ell} [\bar{a}_i, \bar{b}_i]^2 = e\} / SU(2n)\]

\[\cup \pi^{2\ell+1}\{(\bar{a}_1, \bar{b}_1, \cdots, \bar{a}_\ell, \bar{b}_\ell, \bar{c}) \in SU(2n)^{2\ell+1} | \prod_{i=1}^{\ell} [\bar{a}_i, \bar{b}_i]^2 = \exp\frac{2\pi i}{2n}\} / SU(2n)\]

where the map

\[\pi : SU(2n) \mapsto PSU(2n) = SU(2n) / Z(SU(2n))\]
is the projection map.

If we look at this moduli space $\mathcal{M}_{PSU(2n)}$ on the connected sum of $\Sigma_\ell$ and $RP$ as the fixed point set of the moduli space $\tilde{\mathcal{M}}_{PSU(2n)}$ of flat $PSU(2n)$ connections on $\Sigma_{2\ell}$, it becomes

$$
\mathcal{M}_{PSU(2n)} = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in PSU(2n)^{4\ell+2} | \prod_{i=1}^{\ell} [a_i, b_i]c^2 = e \}/(PSU(2n) \times PSU(2n))
$$

$$
\subset \tilde{\mathcal{M}}_{PSU(2n)}
$$

where

$$
\tilde{\mathcal{M}}_{PSU(2n)} = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in PSU(2n)^{4\ell+2} | \prod_{i=1}^{\ell} [a_i, b_i] = cc', \prod_{i=1}^{\ell} [a'_i, b'_i] = c'c \}/(PSU(2n) \times PSU(2n))
$$

So if a point $(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in \tilde{\mathcal{M}}_{PSU(2n)}$ is in $\mathcal{M}_{PSU(2n)}$ then it must either satisfy $\prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2 = e$ or $\prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2 = k$ for some $k \in Z(SU(2n))$ such that $[k] \neq [e] \in Z(SU(2n))/2Z(SU(2n))$ where $\tilde{a}_i, \tilde{b}_i, \tilde{c}$ are the lifting of $a, b, c$ from $PSU(2n)$ to $SU(2n)$.

On the other hand, $\tilde{\mathcal{M}}_{PSU(2n)}$ has another description

$$
\{(a_1, b_1, \cdots, a_\ell, b_\ell, a'_1, b'_1, \cdots, a'_\ell, b'_\ell) \in PSU(2n)^{4\ell} | \prod_{i=1}^{\ell} [a_i, b_i] \prod_{i=1}^{\ell} [a'_i, b'_i] = e \}/PSU(2n).
$$

which is the standard model for the moduli space with one base point and these two models are identified by the map $\Phi$ we defined by equation (9).

Thus we only need to know what is the product of the commutators of the lifting in $\tilde{\mathcal{M}}_{SU(2n)}$ of a point in $\mathcal{M}_{PSU(2n)} \subset \tilde{\mathcal{M}}_{PSU(2n)}$.

Suppose $(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in \mathcal{M}_{PSU(2n)} \subset PSU(2n)^{4\ell+2}$. If $a_i$ lifts to $\tilde{a}_i$, $b_i$ lifts to $\tilde{b}_i$, and $c$ lifts to $\tilde{c}$, then $\prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2 = e$ or $k$ for some $k \in Z(SU(2n))$ such that $[k] \neq [e] \in Z(SU(2n))/2Z(SU(2n))$ and the map $\Phi$ sends

$$
(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in \mathcal{M}_{PSU(2n)} \subset \tilde{\mathcal{M}}_{PSU(2n)}
\mapsto (a_1, b_1, \cdots, a_\ell, b_\ell, c^{-1}b_\ell c, c^{-1}a_\ell c, \cdots, c^{-1}b_1 c, c^{-1}a_1 c) \in \tilde{\mathcal{M}}_{PSU(2n)}.
$$
Thus the product of the commutators of the lifting is
\[
\prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i] \prod_{i=1}^{\ell} [\tilde{c}^{-1}\tilde{b}_i\tilde{c}, \tilde{c}^{-1}\tilde{a}_i\tilde{c}] = \prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]\tilde{c}^{-1} \prod_{i=1}^{\ell} [\tilde{b}_i, \tilde{a}_i] = e^{\sum_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2} = e^{\sum_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2 = e^{kk^{-1}}} = e^{\sum_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i]c^2 = k}
\]

This means that the lifting of the moduli space of flat $PSU(2n)$-connections over the connected sum of $\Sigma_\ell$ and $RP^2$ is (in) the moduli space of semi-stable vector bundle of rank $2n$ and degree $0$ over $\Sigma_{2\ell}$. Thus a natural Lagrangian submanifold of $\mathcal{M}(2n,0)$ over $\Sigma_{2\ell}$ is (equivalent to) the moduli space of flat $PSU(2n)$-connections over the connected sum of $\Sigma_{2\ell}$ and $RP^2$ and a natural Lagrangian submanifold of $\mathcal{M}(2n,0)$ over $\Sigma_{2\ell+1}$ is the moduli space of flat $PSU(2n)$-connections over the connected sum of $\Sigma_{\ell}$ and the Klein bottle.

Another way to look at the problem is that:

For $SU(2n)$, the fixed point set of the moduli space $\tilde{\mathcal{M}}_{SU(2n)}$ is isomorphic to the generically disjoint union of $P_0$ and $P_1$ where

\[
P_0 = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in SU(2n)^{4\ell+2} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2\}/SU(2n) \times SU(2n)
\]

and choosing any $r \in Z(SU(2n))$, $[r] \neq [c],

\[
P_1 = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, cr) \in SU(2n)^{4\ell+2} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2r\}/SU(2n) \times SU(2n).
\]

The piece $P_0$ is in fact the moduli space of $SU(2n)$-bundles over the connected sum of $\Sigma_\ell$ and $RP^2$ which is

\[
\{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in SU(2n)^{2\ell+1} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2\}/SU(2n),
\]

thus we have left only $P_1$ unidentified.

If we look at the moduli space of $PSU(2n)$-bundles over the connected sum of $\Sigma_\ell$ and $RP^2$, according to $[\mathfrak{Y}]$, we also have two pieces

\[
\pi^{2\ell+1}(\{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in SU(2n)^{2\ell+1} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2\}/SU(2n))
\]

and

\[
\pi^{2\ell+1}(\{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in SU(2n)^{2\ell+1} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2r\}/SU(2n))
\]

for $r \in Z(SU(2n))$ and $[r] \neq [c] \in Z(SU(2n))/2Z(SU(2n))$, where $\pi$ is the projection map from $SU(2n)$ to $PSU(2n)$. 


Now the first piece is identified with $\pi^{2\ell+1}(P_0)$. What about the second piece? Note that the space

$$\{(a_1, b_1, \cdots, a_\ell, b_\ell, c) \in G^{2\ell+1} | \prod_{i=1}^\ell [a_i, b_i] = c^2 \}/G$$

is identified with the space

$$\{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, cr) | \prod_{i=1}^\ell [a_i, b_i] = c^2 r \}/G \times G$$

by the isomorphism

$$(a_1, b_1, \cdots, a_\ell, b_\ell, c) \mapsto (a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, cr)$$

so the second piece can be identified with $\pi^{2\ell+1}(P_1)$.

We can generalize this idea from $G = SU(n)$ to any compact connected semisimple Lie group $G$, and draw the following relations:

$$\mathcal{M}_G(\Sigma^{2\ell}_2 R^2) = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in G^{4\ell+2} | \prod_{i=1}^\ell [a_i, b_i] = c^2 \}/(G \times G)$$

$$\downarrow$$

the fixed point set of $\tilde{\mathcal{M}}_G(\Sigma_{2\ell}) = P_0 \cup P_1$ (or more pieces)

where

$$P_0 = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, c) \in G^{4\ell+2} | \prod_{i=1}^\ell [a_i, b_i] = c^2 \}/(G \times G),$$

$$P_1 = \{(a_1, b_1, \cdots, a_\ell, b_\ell, c, a_1, b_1, \cdots, a_\ell, b_\ell, cr) \in G^{4\ell+2} | \prod_{i=1}^\ell [a_i, b_i] = c^2 r \}/(G \times G),$$

for $r \in Z(G)$ such that $[r] \neq [e] \in Z(G)/2Z(G)$, and if we have more pieces then we have $r_i \in Z(G)$ such that $[r_i] \neq [r_j] \neq [e] \in Z(G)/2Z(G)$. 
Since the center of $G/Z(G)$ is trivial, the fixed point set of $\mathcal{M}_{G/Z(G)}(\Sigma_{2\ell})$ is equivalent to $\mathcal{M}_{G/Z(G)}(\Sigma_{2\ell} \# RP^2)$, we have

The fixed point set of $\mathcal{M}_{G/Z(G)}(\Sigma_{2\ell})$

$$= \{(a_1, b_1, \ldots, a_\ell, b_\ell, c, a_1, b_1, \ldots, a_\ell, b_\ell, c) \in (G/Z(G))^{4\ell+2} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2 \}/(G/Z(G) \times G/Z(G))$$

$$= \{(a_1, b_1, \ldots, a_\ell, b_\ell, c) \in (G/Z(G))^{2\ell+1} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2 \}/G$$

$$\cup \pi^{2\ell+1}(\{(\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_\ell, \tilde{b}_\ell, \tilde{c}) \in G^{2\ell+1} \mid \prod_{i=1}^{\ell} [\tilde{a}_i, \tilde{b}_i] = \tilde{c}^2 \}/G)$$

$$= \pi^{2\ell+1}(P_0) \cup \pi^{2\ell+1}(P_1)$$

**Conclusion:**

(1) For $G$ for which the order of its center is not odd, the fixed point set of the involution on the moduli space of flat $G$-bundles over $\Sigma_{2\ell}$ has more than one piece say $P_1$ (or $P_2$, $P_3$) where $P_0$ is identified with the moduli space of flat $G$-bundles over the connected sum of $\Sigma$ and $RP^2$, then this extra piece $P_1$ or these extra pieces $P_1, P_2, P_3$ can be identified with the other connected components of the moduli space of flat $G/Z(G)$-bundles over the connected sum of $\Sigma$ and $RP^2$, while the connected component $\pi^{2\ell+1}(\{(a_1, b_1, \ldots, a_\ell, b_\ell, c) \in G^{2\ell+1} \mid \prod_{i=1}^{\ell} [a_i, b_i] = c^2 \}/G)$ is identified with $\pi^{2\ell+1}(P_0)$.

or

(2) The lifting from $G/Z(G)$ to $G$ of the moduli space of $G/Z(G)$-bundles over $\Sigma_{2\ell} \# RP^2$, $\mathcal{M}_{G/Z(G)}(\Sigma_{2\ell} \# RP^2)$, is the fixed point set of the moduli space of $G$-bundles over $\tilde{\Sigma}$, $\mathcal{M}_G(\tilde{\Sigma})$. In other words, the fixed point set of the moduli space of $G$-bundles over $\tilde{\Sigma}$ is identified (through the projection $\pi$) with the fixed point set of the moduli space of $G/Z(G)$-bundles over $\tilde{\Sigma}$.

**Remark 12.**

We know that the moduli space of flat $SU(2)$-connections on the genus 2 Riemann surface is the smooth $CP^3$ as showed in [NR]. We want to show that, $N$, the fixed point set of the involution which we referred before is the natural $RP^3$ with the metric on $RP^3$ inherited from the Fubini-Study metric on $CP^3$. 
Notice that $RP^3$ is a totally real, totally geodesic submanifold in $CP^3$. Fix one point $p \in CP^3$, the tangent space at $p$ decomposes as $T_p CP^3 = T_p RP^3 \oplus JT_p RP^3$. Since $N$ and $RP^3$ are both real in $CP^3$, it is possible to send $T_q N$ to $T_p RP^3$. In fact, we can find an element in $PSU(4)$ the isometry group of $CP^3$ such that it sends $T_q N$ to $T_p RP^3$.

It suffices to show that any element of $GL(3, \mathbb{C})$ maps one totally real vector space in $\mathbb{C}^3$ to another. Suppose $W$ is a totally real vector space in $\mathbb{C}$. Let $\{w_1, w_2, w_3\}$ denote a basis of $W$. Then $\{Jw_1, Jw_2, Jw_3\}$ is a basis of $JW$ and $W \cap JW = \{0\}$. Suppose $k$ is any element of $GL(3, \mathbb{C})$. We want to show that $kW \cap kJW = \{0\}$. Suppose $p \in kW$ then $p = akw_1 + bkw_2 + ckw_3$ for some constant $a, b, c$. Thus $p = k(aw_1 + bw_2 + cw_3)$. Now if $p \in JW$ then $p = AJw_1 + BJw_2 + CJw_3$ for some constant $A, B, C$. Recall that $GL(3, \mathbb{C}) = \{k \in GL(6, \mathbb{R}), kJ = Jk\}$. Thus $p = k(AJw_1 + BJw_2 + CJw_3)$. Since $\det(k) \neq 0$, we have $aw_1 + bw_2 + cw_3 = AJw_1 + BJw_2 + CJw_3$ which gives all the constants 0 because of $W \cap JW = \{0\}$.

Now apply the exponential map to both $T_p N$ and $T_q RP^3$, by the uniqueness of geodesics, $N \simeq RP^3$ and the metric on $N$ is the induced metric on $RP^3$ since we know $\mathcal{M} \simeq CP^3$ with the standard Fubini-Study metric. The reason that $N$ is smooth is that $N$ is the same as $RP^3$ by the exponential map, so it is smooth because $RP^3$ is. $N$ is not a proper subset of $RP^3$ because $N$ is also compact and has the same dimension as $RP^3$ so it is not possible for it to be a proper subset.

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**Department of Mathematics, National Cheng-Kung University**

*E-mail address: nankuo@math.toronto.edu, nankuo@mail.ncku.edu.tw*