Further remarks on the higher dimensional Suita conjecture

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Abstract. For a domain $D \subset \mathbb{C}^n$, $n \geq 2$, let $F_k^D(z) = K_D(z)\lambda(I_k^D(z))$, where $K_D(z)$ is the Bergman kernel of $D$ along the diagonal and $\lambda(I_k^D(z))$ is the Lebesgue measure of the Kobayashi indicatrix at the point $z$. This biholomorphic invariant was introduced by Błocki. We study its limiting boundary behaviour on two classes of domains: $h$-extendible and strongly pseudoconvex polyhedral domains.

1. Introduction. We continue the study of $F_k^D$, a biholomorphic invariant defined by Błocki [3] in his work on Suita’s conjecture. Recall that for a domain $D \subset \mathbb{C}^n$,

$$F_k^D(z) = K_D(z)\lambda(I_k^D(z))$$

where $K_D(z)$ is the Bergman kernel of $D$ along the diagonal and $\lambda(I_k^D(z))$ is the Lebesgue measure of the Kobayashi indicatrix at $z \in D$. More generally, for any invariant pseudo-metric $\tau_D = \tau_D(z,v)$, the invariant function $F^\tau_D$ is analogously defined. As usual, we will denote by $a_D, c_D, k_D$ respectively the Azukawa, Carathéodory, and Kobayashi metric on $D$. Błocki–Zwonek [4] have shown that

$$1 \leq F^a_D(z) \leq C^n$$

where $C = 4, 16$ according as $D$ is convex or $\mathbb{C}$-convex respectively. Furthermore, their work also contains a detailed discussion of this invariant on convex egg domains in $\mathbb{C}^2$. These results were supplemented in [1] where this invariant was considered on strongly pseudoconvex domains in $\mathbb{C}^n$ and a few other observations were made about its boundary behaviour on egg domains in $\mathbb{C}^2$. In particular, even on smoothly bounded convex eggs of the form

$$E_{2\mu} = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{2\mu} < 1\}$$

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for integers $\mu > 1$, $F_{E_{2\mu}}^k$ does not admit a limit at any of the weakly pseudo-convex points of $\partial E_{2\mu}$. In fact, the full range of all possible values of $F_{E_{2\mu}}^k$ at points of $E_{2\mu}$ show up as possible limits near any of the weakly pseudo-convex points on $\partial E_{2\mu}$. By the well-known work of Lempert, all invariant metrics on bounded convex domains $D$ coincide; so in particular for any invariant metric $\tau$, $F_D^k \equiv F_D^\tau$. While such an identity need not hold on strongly pseudoconvex domains in general, it was shown in [1] that on any smoothly bounded strongly pseudoconvex domain $D$, the boundary limits of $F_D^\tau$ exist and give rise to the same value: $F_D^\tau(z) \to 1$ as $z$ approaches $\partial D$. A different approach has been suggested recently in [5], where the focus was the invariant metric $\tau = a$ of Azukawa. In that article, Błocki–Zwonek have also raised questions about the boundary behaviour of $F_D^a$ both for bounded convex domains and for smoothly bounded pseudoconvex domains. While the aforementioned (convex) egg domains settle the non-existence of boundary limits of $F_D^a$ at non-strongly pseudoconvex boundary points in general, it is possible to make certain definite statements about the possible limiting boundary values.

The purpose of this note is to record some general properties of $F_D^k$ and to compute its possible limiting boundary values on $h$-extendible and strongly pseudoconvex polyhedral domains.

2. Some observations

(i) Bounds on $\mathbb{C}$-convex domains. Let $D$ be a $\mathbb{C}$-convex domain in $\mathbb{C}^n$. By [16] Corollary 2, $a_D \leq k_D \leq 4c_D \leq 4a_D$. It follows from (1.1) that

$$\frac{1}{16^n} \leq F_D^k(z) \leq 16^n.$$ 

(ii) Removable singularities. For a bounded domain $G \subset \mathbb{C}^n$ and a subvariety $V \subset G$ of codimension at most 2, it is known that $k_G(z,v) = k_{G\setminus V}(z,v)$ for $(z,v) \in (G \setminus V) \times \mathbb{C}^n$. Further, the Bergman kernels along the diagonal of $G$ and $G\setminus V$ are equal since $V$ is a removable singularity for $L^2$-holomorphic functions. Hence $F_G^k = F_{G\setminus V}^k$.

If $V$ has codimension 1, this is no longer the case despite the fact that $V$ is still removable for $L^2$-holomorphic functions. In general, $k_G \leq k_{G\setminus V}$, as can be seen by taking $G$ to be the unit disc $\mathbb{D} \subset \mathbb{C}$ and $V = \{0\}$. Thus, the most that can be said in general is that $F_{G\setminus V}^k \leq F_G^k$. Examples in higher dimensions can be constructed by observing that $F_D^k$ is multiplicative as a function of the domain $D$, that is,

$$F_{D\times G}^k((p,q)) = F_D^k(p)F_G^k(q)$$

for $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$, and hence $F_{D\times(\mathbb{D}\setminus\{0\})}^k \leq F_{D\times\mathbb{D}}^k$. Finally, using the multiplicative property, this invariant for the Hartogs triangle $\Omega \subset \mathbb{C}^2$ (which
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is biholomorphic to $\mathbb{D} \times (\mathbb{D} \setminus \{0\})$ can be computed as

$$F^k_{\Omega}(z, w) = F^k_{\mathbb{D}}\left(\frac{w}{z}\right) F^k_{\mathbb{D} \setminus \{0\}}(z) = 4 \left(\frac{|z| \log |z|}{1 - |z|^2}\right)^2.$$ 

In particular, $F^k_{\Omega}(z, w) \to 0$ as $(z, w)$ approaches the origin from within $\Omega$.

(iii) Regularity of $F^k_F$. Let $G \subset \mathbb{C}^n$ be an arbitrary domain. Then $F^k_F(z)$ is always lower semicontinuous; it is continuous when the Kobayashi metric $k_G(z, v)$ is continuous on $G \times \mathbb{C}^n$ and non-degenerate in the sense that $k_G(z, v) > 0$ for all $v \neq 0$.

It suffices to show the lower semicontinuity of the function $z \mapsto \lambda(I^k_G(z))$. For this, fix a $z^0 \in G$ and let $z^\nu$ be a sequence in $G$ converging to $z^0$. We claim that

$$I^k_G(z^0) \subset \liminf_{\nu \to \infty} I^k_G(z^\nu) = \bigcup_{\mu=1}^\infty \bigcap_{\nu=\mu}^\infty I^k_G(z^\nu).$$

Indeed, let $v^0 \in I^k_G(z_0)$. Then $\epsilon = 1 - k_G(z^0, v^0) > 0$ and by the upper semicontinuity of $k_G(\cdot, v^0)$, there exists $N \in \mathbb{N}$ (depending possibly on both $z^0$ and $v^0$) such that for all $\nu \geq N$ we have

$$k_G(z^\nu, v^0) \leq k_G(z^0, v^0) + \epsilon/2 = 1 - \epsilon/2 < 1.$$ 

This implies that $v^0 \in I^k_G(z^\nu)$ for all $\nu \geq N$, proving our claim (2.1). Now by Fatou’s lemma for measurable sets,

$$\lambda(I^k_G(z^0)) \leq \lambda(\liminf_{\nu \to \infty} I^k_G(z^\nu)) \leq \liminf_{\nu \to \infty} \lambda(I^k_G(z^\nu))$$

which establishes the lower semicontinuity of $z \mapsto \lambda(I^k_G(z))$.

Now, we restrict attention to those domains $G$ for which $k_G$ is continuous and non-degenerate. To show the continuity of $F^k_F$, it suffices to show that $z \mapsto \lambda(I^k_G(z))$ is upper semicontinuous. For this, pick a $z^0 \in G$ and let $z^\nu$ be a sequence in $D$ converging to $z^0$. Let $\epsilon > 0$. By the continuity of $k_G$, there exists $N \in \mathbb{N}$ such that

$$k_G(z^\nu, v) > k_G(z^0, v) - \epsilon$$

for all $\nu \geq N$ and $v \in S = \partial \mathbb{B}^n$, the standard Euclidean unit sphere. Now, for any non-zero vector $v \in I^k_G(z^\nu)$ with $\nu \geq N$,

$$1 > k_G(z^\nu, v) = |v| k_G(z^\nu, v/|v|)$$

and thus

$$1 + \epsilon |v| > k_G(z^0, v).$$

The continuity of $k_G$ together with its non-degeneracy also implies that there is a positive constant $c$ depending only on $z^0$ such that $k_G(z^\nu, v) \geq c |v|$ for
all \( \nu \geq N \) and \( v \in \mathbb{C}^n \), with \( N \) modified if necessary. This implies that
\[ I^k_G(z^\nu) \subset c^{-1}\mathbb{B}^n \] for all \( \nu \geq N \).
It follows that any vector \( v \) picked from any of the indicatrices \( I^k_G(z^\nu) \) for \( \nu \geq N \) satisfies
\[ k_G(z^0, v) < 1 + c^{-1}\epsilon \]
and this means that \( I^k_G(z^\nu) \subset (1 + c^{-1}\epsilon)I^k_G(z^0) \). Therefore,
\[ \lambda(I^k_G(z^\nu)) \leq (1 + c^{-1}\epsilon)^2\lambda(I^k_G(z^0)) \]
for all \( \nu \geq N \). As \( c \) depends only on \( z^0 \) but not on \( \epsilon \), this implies
\[ \limsup_{\nu \to \infty} \lambda(I^k_G(z^\nu)) \leq \lambda(I^k_G(z^0)) \]
proving the upper semicontinuity of \( z \mapsto \lambda(I^k_G(z)) \).
To conclude, note that the non-degeneracy of \( k_G \) entails the boundedness of the indicatrices and its continuity implies the following geometric property: every ray emanating from the origin in \( T_z(G) \) intersects the (usual topological) boundary \( \partial I^k_G(z) \) of the indicatrix in exactly one point. This then leads to \( \partial I^k_G(z) \) being homeomorphic to \( \partial \mathbb{B}^n \), as the ‘graph’ of a (uniformly) continuous function on \( \partial \mathbb{B}^n \), thereby ensuring that the 2n-dimensional Lebesgue measure of \( \partial I^k_G(z) \) is zero.
All of the aforementioned remarks about regularity hold good with the Kobayashi metric replaced by any upper semicontinuous infinitesimal invariant metric \( \tau \), and in particular for the Azukawa metric. More precisely, for any domain \( G \subset \mathbb{C}^n \) and any upper semicontinuous invariant metric \( \tau_G(z, v) \), the corresponding invariant function \( F^{\tau}_{G}(z) \) is lower semicontinuous; it is continuous provided \( \tau(z, v) \) is continuous on \( G \times \mathbb{C}^n \) and non-degenerate in the sense that \( \tau(z, v) > 0 \) for all \( v \neq 0 \). The proof of this follows by the above arguments verbatim with \( k \) replaced by \( \tau \).

(iv) Localization. It is possible to localize this invariant near peak points as follows:

**Proposition 2.1.** Let \( G \subset \mathbb{C}^n \) be a pseudoconvex domain and let \( p \in \partial G \) be a local holomorphic peak point. Then for a sufficiently small neighbourhood \( U \) of \( p \),
\[ \lim_{U \cap G \ni z \to p} \frac{F^k_{U \cap G}(z)}{F^k_G(z)} = 1. \]

It should be mentioned that this holds for \( F^\tau_D \) where \( \tau = a \) is the Azukawa metric as well; the proof is immediate when the already known localization properties of the Azukawa and the Kobayashi metrics (cf. [14], [15], [10]) are combined with that of the Bergman kernel (cf. [11], [13]).
3. \(h\)-extendible domains. Recall that a pseudoconvex domain \(D \subset \mathbb{C}^{n+1}\) is said to be \(h\)-extendible near a smooth, finite-type point \(p \in \partial D\) if the Catlin multitype \((1, m_1, \ldots, m_n)\) of \(D\) at \(p\) satisfies \(m_{n-q+1} = \Delta_q < \infty\) for \(1 \leq q \leq n\), where \(\Delta_q\) is the \(q\)-type of \(p\). In this case, there are local coordinates \(z = (z_0, z') = (z_0, z_1, \ldots, z_n)\) around \(p = 0\) and a real-valued, plurisubharmonic, \((1/m_1, \ldots, 1/m_n)\) weighted homogeneous polynomial \(P\) of total weight 1 with no pluriharmonic terms such that \(D\) is defined locally near \(p\) by

\[
\rho(z) = \text{Re} z_0 + P(z', z') + R(z)
\]

where \(R(z) \lesssim (|z_0| + |z_1|^{m_1} + \cdots + |z_n|^{m_n})^\gamma\) for some \(\gamma > 1\). Call the local model for \(D\) at \(p\). Then \(D_\infty\) is a taut domain. It is known that \(h\)-extendability of \(D\) at \(p\) is equivalent to the existence of a positive, \(C^\infty\)-smooth function \(a(z')\) on \(\mathbb{C}^n \setminus \{0\}\) such that \(a\) is weighted homogeneous with the same weights as for \(P\) and \(P(z') - \epsilon a(z')\) is strictly plurisubharmonic on \(\mathbb{C}^n \setminus \{0\}\) when \(0 < \epsilon \leq 1\). Note that Levi corank 1, convex finite type and decoupled finite type domains are all examples of \(h\)-extendible domains. More details can be found in [6] and [19].

**Theorem 3.1.** Let \(D \subset \mathbb{C}^{n+1}\) be a bounded pseudoconvex domain that is \(h\)-extendible at \(p \in \partial D\) with multitype \((1, m_1, \ldots, m_n)\) and whose associated local model is \(D_\infty\). If \(\Gamma\) is a non-tangential cone in \(D\) with vertex at \(p\), then

\[
\lim_{\Gamma \ni z \to p} F^k_D(z) = F^k_{D_\infty}(b)
\]

where \(b = (-1, 0, \ldots, 0) \in D_\infty\).

**Proof.** The boundary behaviour of \(K_D(z)\) as \(z \to 0\) within \(\Gamma\) is known. Indeed, Theorem 1 in [6] shows that

\[
\lim_{\Gamma \ni z \to 0} K_D(z)|\rho(z)|^\beta = K_{D_\infty}(b)
\]

where \(\beta = \sum_{j=0}^{n} 2/m_j\).

To handle the Kobayashi indicatrices, first fix an \(\epsilon \in (0, 1)\) and let \(U_\epsilon\) be a neighbourhood of \(p = 0\) such that the bumped model

\[
D_\epsilon = \{\text{Re} z_0 + P(z') - \epsilon a(z') < 0\}
\]

contains \(D \cap U_\epsilon\). By [19], \(D_\epsilon\) is taut. For \(t > 0\), let

\[
\pi_t(z) = (tz_0, t^{1/m_1}z_1, \ldots, t^{1/m_n}z_n)
\]

and note that the scaled domains \(D_z = \pi_1/|\rho(z)|\{D \cap U_\epsilon\}\) converge to \(D_\infty\) in the Hausdorff sense. Also, if \(z \to 0\) within \(\Gamma\), the base points \(\zeta(z) = \pi_1/|\rho(z)|(z)\) converge to a compact subset of the line \(\{\text{Re} z_0 = -1, z' = 0\} \subset D_\infty\). This is so since non-tangential convergence implies that \(|\text{Re} z_0| \approx |\rho(z)|\). Finally,
note that \( \pi_t \in \text{Aut}(D_\infty) \) for all \( t > 0 \) and hence \( D_z \subset D_{\epsilon} \) for all \( z \in \Gamma \) close to the origin. Theorem 2.1 of \([19]\) shows that for all fixed \( w \in D_\infty \), the Kobayashi metrics \( k_{D_z}(w,v) \) tend to \( k_{D_\infty}(w,v) = 0 \) as \( z \to 0 \) within \( \Gamma \). Moreover, the convergence is uniform on compact subsets of \( D_\infty \times \mathbb{C}^n \). Hence, the indicatrices \( I_{D_z}^k(w) \) converge to \( I_{D_\infty}^k(w) \) in the Hausdorff sense and \( \lambda(I_{D_z}^k(w_j)) \to \lambda(I_{D_\infty}^k(w_0)) \) if \( w_j \to w_0 \in D_\infty \). In particular, as \( z \to 0 \) within \( \Gamma \),

\[
\lambda(I_{D_z}^k(\zeta(z))) \to \lambda(I_{D_\infty}^k(\tilde{\zeta}))
\]

where \( \tilde{\zeta} \) is a possible limit point of \( \zeta(z) \). But as noted above, \( \tilde{\zeta} \) lies on the line \( \{ \Re z_0 = -1, z' = 0 \} \) and since \( D_\infty \) is invariant under volume preserving translations of the form \( z \mapsto z + i\alpha, \alpha \in \mathbb{R} \), it follows that \( \lambda(I_{D_\infty}^k(\tilde{\zeta})) = \lambda(I_{D_\infty}^k(b)) \).

To conclude, it remains to note that

\[
\lambda(I_{D_z}^k(\zeta(z))) = |\rho(z)|^{-\beta} \lambda(I_{D}^k(z))
\]

by the change of variables formula, and

\[
F^k_D(z) = K_D(z)|\rho(z)|^{\beta} \lambda(I_{D_z}^k(z))|\rho(z)|^{-\beta} \to K_{D_\infty}(b)\lambda(I_{D_\infty}^k(b)) = F^k_{D_\infty}(b)
\]

as \( z \to 0 \) within \( \Gamma \). \( \blacksquare \)

It should be noted that the non-tangential condition cannot be dropped as the example of \( F^k_{E_{2\mu}} \) shows. More precisely, for \( D = E_{2\mu} \) and \( q \) one of its weakly pseudoconvex points (say \( q = (0,1) \)), note that

\[
D_\infty = \{(z, w) \in \mathbb{C}^2 : 2\Re z + |w|^{2\mu} < 0 \}.
\]

Indeed, an analogue of the Cayley transform maps \( D_\infty \) biholomorphically onto \( E_{2\mu} \) with \( b = (-1,0) \) mapped to the origin where the value of \( F^k_{E_{2\mu}} \) is 1. So,

\[
\lim_{\Gamma \ni z \to q} F^k_{E_{2\mu}} = 1
\]

whereas we know from \([11]\) that every value attained by \( F^k_{E_{2\mu}}(z) \) as \( z \) varies in \( E_{2\mu} \) is also attained as a boundary limiting value at \( q \); in particular (as \( F^k_{E_{2\mu}} \) is a constant function only for \( \mu = 1 \)), there are sequences \( \{p_n\} \subset E_{2\mu} \) approaching \( q \) non-tangentially along which \( F^k_{E_{2\mu}} \) has a limit and the limiting value differs from 1—for instance, follow any orbit of a point of the form \((0, p)\) with \( 0 < p < 1 \) under the action of the automorphism group \( \text{Aut}(E_{2\mu}) \). But then it turns out that ‘highly tangential sequences’ again yield boundary limit 1—to record this peculiar feature of the boundary behaviour of \( F^k_{E_{2\mu}} \) at the weakly pseudoconvex points of \( F^k_{E_{2\mu}} \) a bit more precisely but briefly, let \( q_n \) be a sequence of points in \( E_{2\mu} \) converging to \( q \) such that (i) the inner products of \((q_n - q)/|q_n - q|\) with the unit inner normal to \( \partial E_{2\mu} \) converge to 0 and (ii) the \( q_n \)'s belong to mutually distinct orbits of \( \text{Aut}(E_{2\mu}) \). Then

\[
\lim_{n \to \infty} F^k_{E_{2\mu}}(q_n) = 1.
\]
It is possible to obtain global bounds for this invariant on Levi co-rank 1 domains. This follows from the following where $\tau$ denotes any distance decreasing metric or the Bergman metric.

**Lemma 3.2.** Let $\Omega$ be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ whose boundary $\partial \Omega$ is of (finite type and of) Levi corank at most one at $p \in \partial \Omega$. Then there exist positive constants $c, C$ and a neighborhood $U$ of $p$ such that $c \leq F^\tau_\Omega(z) \leq C$ for all $z \in \Omega \cap U$.

**Proof.** This will follow from the well-known boundary estimates of Catlin and Cho for $\tau$ equal to the Carathéodory, Kobayashi or the Bergman metric. We recall the relevant ideas briefly. Let $r$ be a local defining function for $\partial \Omega$ in a neighbourhood $U$ of $p = 0$. By shrinking this neighbourhood if needed, we may assume that the orthogonal projection onto the boundary $\partial D$ is well-defined on $U$ and that the normal vector field, given at any $\zeta \in U$ by

$$\nu(\zeta) = \left( \partial r / \partial z_1(\zeta), \partial r / \partial z_2(\zeta), \ldots, \partial r / \partial z_n(\zeta) \right),$$

has no zeros in $U$; this is normal to the hypersurface $\Gamma_\zeta = \{ r(z) = r(\zeta) \}$.

For each $\zeta \in U$, there is a uniform radius $R > 0$ and an injective holomorphic mapping $\Phi^\zeta : B(\zeta, R) \to \mathbb{C}^n$ such that the transformed defining function $\rho^\zeta = r^\zeta \circ (\Phi^\zeta)^{-1}$ reads

$$\rho^\zeta(w) = r(\zeta) + 2 \operatorname{Re} w_n + \sum_{l=2}^{2m} P_l(\zeta; w_1) + |w_2|^2 + \cdots + |w_{n-1}|^2$$

$$+ \sum_{\alpha=2}^{n-1} \sum_{j+k \leq m \atop j,k > 0} \operatorname{Re}(b^\zeta_{jk}(\zeta) w_1^j \bar{w}_1^k w_\alpha) + R(\zeta; w)$$

where

$$P_l(\zeta; w_1) = \sum_{j+k=l} a^l_{jk}(\zeta) w_1^j \bar{w}_1^k$$

are real-valued homogeneous polynomials of degree $l$ without harmonic terms and the error function $R(\zeta, w)$ tend to 0 as $w \to 0$ faster than one of the monomials of weight 1. Further, the map $\Phi^\zeta$ is actually a holomorphic polynomial automorphism of weight 1 of the form

$$\Phi^\zeta(z) = (z_1 - \zeta_1, G_\zeta(\bar{z} - \bar{\zeta}) - Q_2(z_1 - \zeta_1), \langle \nu(\zeta), z - \zeta \rangle - Q_1(\langle z' - \zeta \rangle))$$

where $G_\zeta \in \text{GL}_{n-2}(\mathbb{C})$, $\bar{z} = (z_2, \ldots, z_{n-1})$, $z' = (z_1, \ldots, z_{n-1})$ and $Q_2$ is a vector-valued polynomial whose $\alpha$th component is a polynomial of weight at most $1/2$ of the form

$$Q_2^\alpha(t) = \sum_{k=1}^m b^\alpha_k(\zeta)t^k$$
for $t \in \mathbb{C}$ and $2 \leq \alpha \leq n - 1$. Finally, $Q_1('z - \zeta')$ is a polynomial of weight at most 1 and is of the form $\hat{Q}_1(z_1 - \zeta_1, G_\zeta(\bar{z} - \zeta))$ with $Q_1$ of the form

$$
\hat{Q}_1(t_1, \ldots, t_{n-1}) = \sum_{k=2}^{2m} a_k(\zeta)t_1^k - \sum_{\alpha=2}^{n-1} \sum_{k=1}^{m} a^\alpha_k(\zeta)t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha(\zeta)t_\alpha^2.
$$

Since $G_\zeta$ is just a linear map, $Q_1('z - \zeta')$ also has the same form when considered as an element of the algebra of holomorphic polynomials $\mathbb{C}'[z - \zeta']$, when $\zeta$ is held fixed. The coefficients of all the polynomials mentioned above are smooth functions of $\zeta$. Note that $\Phi_\zeta(\zeta) = 0$ and

$$
\Phi_\zeta(\zeta_1, \ldots, \zeta_{n-1}, \zeta_n - \epsilon) = (0, \ldots, 0, -\epsilon \partial_r / \partial \zeta_n(\zeta)).
$$

Define, for each $\delta > 0$, the special radius

$$
(3.4) \quad \tau(\zeta, \delta)
$$

$$
= \min\{((\delta / |P_l(\zeta, \cdot)|)^{1/l}, (\delta^{1/2} / B_{l'}(\zeta))^{1/l'} : 2 \leq l \leq 2m, 2 \leq l' \leq m\},
$$

where

$$
B_{l'}(\zeta) = \max\{|b^\alpha_{jk}(\zeta)| : j + k = l', 2 \leq \alpha \leq n - 1\}, \quad 2 \leq l' \leq m.
$$

Here, the norm of the homogeneous polynomial $P_l(\zeta, \cdot)$ of degree $l$ is taken according to the following convention: for a homogeneous polynomial

$$
p(v) = \sum_{j+k=l} a_{j,k} v^j \bar{v}^k,
$$

define $|p(\cdot)| = \max_{\theta \in \mathbb{R}} |p(e^{i\theta})|$. It was shown in [9] that the coefficients $b^\alpha_{jk}$ in the above definition of $\tau(\zeta, \delta)$ are insignificant and may be ignored, so that

$$
\tau(\zeta, \delta) = \min\{((\delta / |P_l(\zeta, \cdot)|)^{1/l} : 2 \leq l \leq 2m\}.
$$

Set

$$
\tau_1(\zeta, \delta) = \tau(\zeta, \delta) = \tau, \quad \tau_2(\zeta, \delta) = \cdots = \tau_{n-1}(\zeta, \delta) = \delta^{1/2}, \quad \tau_n(\zeta, \delta) = \delta
$$

define the dilations

$$
\Delta_\zeta^\delta(z) = (z_1 / \tau_1(\zeta, \delta), \ldots, z_n / \tau_n(\zeta, \delta)).
$$

The scaling maps are defined by the composition

$$
S_\zeta^\delta(z) = \Delta_\zeta^\delta \circ \Phi_\zeta
$$

and they induce the so-called $M$-metric defined on the one-sided neighbourhood $U \cap D$:

$$
M_D(\zeta, v) = \sum_{k=1}^{n} |(D\Phi_\zeta(\zeta)v)_k| / |\tau_k(\zeta, \epsilon(\zeta))| = |D(S_\zeta^\delta(\zeta))(v)|_{11}
$$

where $\epsilon(\zeta) > 0$ is such that $\tilde{\zeta} = \zeta + (0, \ldots, \epsilon(\zeta))$ lies on $\partial D$. The significance of this metric is that it is uniformly comparable to the Kobayashi metric [18],
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in the sense that

\[(3.5) \quad k_D(\zeta, v) \approx M_D(\zeta, v) \approx \|D(S^\delta_\zeta(\zeta))(v)\|,\]

where \(\| \cdot \|\) denotes any norm on \(\mathbb{C}^n\) and the suppressed constants are independent of \(v\) and \(\zeta\) (depending only on the domain \(U \cap D\)). In particular, taking \(\| \cdot \|\) to be the \(l^\infty\)-norm, we may translate this estimate on the Kobayashi metric into one about its indicatrix and its dilates:

\[(3.6) \quad cR(\zeta) \subset I^k_D(\zeta) \subset CR(\zeta),\]

where \(c, C\) are positive constants independent of \(\zeta\), and \(R(\zeta)\) is the polydisc centered at the origin of polyradius

\[\left(\tau_1(\zeta, \epsilon(\zeta)), \sqrt{\epsilon(\zeta)}, \ldots, \sqrt{\epsilon(\zeta)}, \epsilon(\zeta)\right).\]

When \(D\) is additionally bounded and globally pseudoconvex, it follows from [8, Theorem 1] that for all \(\zeta\) in some tubular neighbourhood of \(U \cap \partial D\),

\[K_D(\zeta, \bar{\zeta}) \approx (\text{Vol}(R(\zeta)))^{-1}\]

where the implied constants depend only on \(D\) and are independent of \(\zeta\). Combining this with (3.6) finishes the verification of Lemma 3.2.

We now note that as a consequence of Proposition 2.1, the boundedness assumption on \(\Omega\) in the above lemma can be dropped, provided we restrict to \(\tau = a\) or \(k\) the Azukawa or Kobayashi metric. Indeed, to see that the global smoothness assumption of the above lemma can be circumvented when we combine it with Proposition 2.1 (for \(\tau = a,k\)) to drop the boundedness assumption, we recall a technique explained by Bell in the final section of [2], for completing a small piece of \(\partial \Omega\) (in case \(\Omega\) is unbounded) about \(p\) into a smooth pseudoconvex finite type hypersurface so that the resulting smoothly bounded domain \(G\) is a (small) subdomain of \(\Omega\); the lemma above applies to \(G\) and then Proposition 2.1 will compare \(F^G_\tau\) with \(F^\tau_{\Omega}\) to yield the version of the above lemma for unbounded \(\Omega\) as desired in case \(\tau = a,k\). Further, the just mentioned technique of Bell also enables us to drop the global smoothness and boundedness assumption to deduce

**Theorem 3.3.** Let \(D \subset \mathbb{C}^{n+1}\) be a pseudoconvex domain whose boundary is smooth and of finite type near \(p \in \partial D\). Suppose \(p\) is an \(h\)-extendible point with \(D_\infty\) being the associated local model. If \(\Gamma\) is a non-tangential cone in \(D\) with vertex at \(p\), then

\[\lim_{F_{\Omega} \to p} F^k_D(z) = F^k_{D_\infty}(b)\]

where \(b = (-1,0,\ldots,0) \in D_\infty\).
4. Piecewise smooth strongly pseudoconvex domains

**Definition 4.1.** A bounded domain $D$ in $\mathbb{C}^n$ is said to be a **strongly pseudoconvex polyhedral domain** with piecewise smooth boundary if there are $C^2$-smooth real-valued functions $\rho_1, \ldots, \rho_k : \mathbb{C}^n \to \mathbb{R}$, $k \geq 2$, such that

(i) $D = \{ z \in \mathbb{C}^n : \rho_1(z) < 0, \ldots, \rho_k(z) < 0 \}$,

(ii) for $\{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$, the gradient vectors $\nabla \rho_{i_1}(p), \ldots, \nabla \rho_{i_l}(p)$ are linearly independent over $\mathbb{C}$ for every point $p$ such that $\rho_{i_1}(p) = \cdots = \rho_{i_l}(p) = 0$, and

(iii) $\partial D$ is strongly pseudoconvex at every smooth boundary point,

where for each $i = 1, \ldots, k$ and $z \in \mathbb{C}^n$,

$$\nabla \rho_i(z) = 2 \left( \frac{\partial \rho_i}{\partial z_1}(z), \ldots, \frac{\partial \rho_i}{\partial z_n}(z) \right).$$

Since the intersection of finitely many domains of holomorphy is a domain of holomorphy, it follows that the polyhedral domain $D$ as in Definition 4.1 is pseudoconvex.

Let $D \subset \mathbb{C}^2$ be a strongly pseudoconvex polyhedral domain with piecewise smooth boundary as above defined by

$$D = \{ z \in \mathbb{C}^2 : \rho_1(z) < 0, \ldots, \rho_k(z) < 0 \}.$$ 

Let $p^0 \in \partial D$ be a singular boundary point, i.e., $\partial D$ is not smooth at $p^0$. We study $F^k_D(z)$ as $z \to p^0$. It is evident from Definition 4.1 that exactly two of the hypersurfaces $\{ z \in \mathbb{C}^2 : \rho_j(z) = 0 \}$ (where $j = 1, \ldots, k$) intersect at the point $p^0$. Without loss of generality, we may assume that

$$\rho_1(p^0) = \rho_2(p^0) = 0.$$ 

Let $p^j$ be a sequence of points in $D$ converging to $p^0$. Denote

$$\lambda_j = \text{dist}(p^j, \{ \rho_1 = 0 \}), \quad \mu_j = \text{dist}(p^j, \{ \rho_2 = 0 \})$$

for each $j$. Note that both $\lambda_j$ and $\mu_j$ tend to zero as $j \to \infty$.

Following [12], there are three cases to consider:

(I) The sequence $p^j$ is of **radial type**, i.e., there is a positive constant $C$ (independent of $j$) such that $1/C \leq \mu_j^{-1} \lambda_j \leq C$ for all $j$.

(II) The sequence $p^j$ is of **$q$-tangential type**, i.e., either $\lim_{j \to \infty} \mu_j^{-1} \sqrt{\lambda_j} = 0$ or $\lim_{j \to \infty} \lambda_j^{-1} \sqrt{\mu_j} = 0$.

(III) The sequence $p^j$ is of **mixed type**, i.e., it is neither radial type nor $q$-tangential type. Here, there are two further cases:

(a) $\lim_{j \to \infty} \lambda_j/\mu_j = 0$ and $\lim_{j \to \infty} \sqrt{\lambda_j}/\mu_j = m > 0$,

(b) $\lim_{j \to \infty} \lambda_j/\mu_j = 0$ and $\lim_{j \to \infty} \sqrt{\lambda_j}/\mu_j = \infty$. 


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THEOREM 4.2. Let $D$ be a strongly pseudoconvex polyhedral bounded domain in $\mathbb{C}^2$ with piecewise smooth boundary. Let $p^0 \in \partial D$ be a singular boundary point and $p^j$ be a sequence of points in $D$ converging to $p^0$.

(i) If \( \{p^j\} \) is of radial type, then \( F^k_D(p^j) \to F^k_{\Delta \times \Delta}((0, 0)) = 1 \).

(ii) If \( \{p^j\} \) is of $q$-tangential type, then \( F^k_D(p^j) \to F^k_{\mathbb{S}^2}((0, 0)) = 1 \).

(iii) If \( \{p^j\} \) is of mixed type, then

\[
F^k_D(p^j) \to \begin{cases} 
F^k_{D_1,\infty}((0, 0)) & \text{in case (III)(a),} \\
F^k_{\Delta \times \Delta}((0, 0)) = 1 & \text{in case (III)(b),}
\end{cases}
\]

where $D_{1,\infty}$ is the model domain defined by

\[
D_{1,\infty} = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 + 1 > Q_1(z_2)/m^2, \Im z_2 > -1\},
\]

and $Q_1$ is a strictly subharmonic polynomial of degree 2.

It should be noted that if $p^0 \in \partial D$ is a smooth boundary point, then the proof of Theorem 1.1 of [1] implies that \( F^k_D(z) \to 1 \) as $z \to p^0$.

We adapt the scaling method from [12] to understand $F^k_D(p^j)$ in each of the above cases. To begin, apply a complex linear change of coordinates $A$ so that $A(p^0) = (0, 0)$ and the gradient vector to the hypersurface $A(\{\rho_1 = 0\})$ and $A(\{\rho_2 = 0\})$ at the origin is parallel to the $\Im z_1$ and $\Im z_2$ axis respectively. Write $A(p^j) = \tilde{p}^j$ for each $j$.

CASE (I): The smoothness of $\rho_1$ and $\rho_2$ implies that for each $j$, there is a unique point $s^j$ on $A(\{\rho_1 = 0\})$ and $t^j$ on $A(\{\rho_2 = 0\})$ such that

\[
\text{dist}(\tilde{p}^j, A(\{\rho_1 = 0\})) = |\tilde{p}^j - s^j|, \\
\text{dist}(\tilde{p}^j, A(\{\rho_2 = 0\})) = |\tilde{p}^j - t^j|.
\]

There exists a sequence \( \{B^j\} \) of affine automorphisms of $\mathbb{C}^2$ such that $B^j(\tilde{p}^j) = (0, 0)$ for each $j$ and the domains $B^j \circ A(U \cap D)$ (for a sufficiently small neighbourhood $U$ of $p^0$) are defined by

\[
\{(z_1, z_2) : \Im(z_1 - s^j_1) > Q_1(z_2, \bar{z}_2) + o(|z_1 - s^j_1| + |z_2|^2), \\
\Im(z_2 - t^j_2) > Q_2(z_1, \bar{z}_1) + o(|z_2 - t^j_2| + |z_1|^2)\},
\]

where $Q_1$ and $Q_2$ are real-valued quadratic polynomials.

Define the dilations $L^j(z_1, z_2) = (z_1/\lambda_j, z_2/\mu_j)$, and the dilated domains $D^j = L^j \circ B^j \circ A(U \cap D)$. Note that $L^j \circ B^j \circ A(p^j) = (0, 0)$ for all $j$. The following two claims were proved in [12]. First, $D^j$ converges to

\[
D_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 > -c_1, \Im z_2 > -c_2\},
\]
where $c_1$ and $c_2$ are positive constants. Secondly, for all $j$ large, the scaled domains $D^j$ are contained in $D_0$, where

$$D_0 = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 > -c_1 - r, \Im z_2 > -c_2 - r\},$$

and $r > 0$ is fixed. It should be noted that there is a biholomorphism from the limit domain $D_\infty$ onto the unit bidisc $\Delta \times \Delta$ that preserves the origin.

**Case (II):** Assume that the sequence $p^j$ is of $q$-tangential type to $\{\rho_1 = 0\}$, i.e., $\lim_{j \to \infty} \mu_j^{-1} \sqrt{|\lambda_j|} = 0$.

For a sufficiently small neighbourhood $U$ of $p^0$, we may assume that $p^j$ is in $U$ for all $j$. The domain $A(U \cap D)$ is given by

\[
\{ (z_1, z_2) : \Im(z_1 - \tilde{p}_1^j) + \lambda_j > Q_1(z_2 - \tilde{p}_2^j) + o(|z_1 - \tilde{p}_1^j| + |z_2 - \tilde{p}_2^j|) , \\
\Im(z_2 - \tilde{p}_2^j) + \mu_j > Q_2(z_1 - \tilde{p}_1^j) + o(|z_2 - \tilde{p}_2^j| + |z_1 - \tilde{p}_1^j|) \},
\]

where $Q_1$ are $Q_2$ are strictly subharmonic quadratic polynomials. Let $L^j : \mathbb{C}^2 \to \mathbb{C}^2$ be the dilations given by

$$L^j(z_1, z_2) = \left( \frac{z_1 - \tilde{p}_1^j}{\lambda_j}, \frac{z_1 - \tilde{p}_2^j}{\sqrt{|\lambda_j|}} \right).$$

It follows that $L^j \circ A(p^j) = (0, 0)$ and the scaled domains $D^j = L^j \circ A(U \cap D)$ converge to

$$D_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 + 1 > Q_1(z_2) \},$$

which is biholomorphically equivalent to $\mathbb{B}^2$.

**Case (III):** Here, the sequence $p^j$ is of mixed type. Consider the dilations

$$L^j(z_1, z_2) = \left( \frac{z_1 - \tilde{p}_1^j}{\lambda_j}, \frac{z_1 - \tilde{p}_2^j}{\mu_j} \right)$$

and note that $L^j \circ A(p^j) = (0, 0)$. It follows that the dilated domains $D^j = L^j \circ A(U \cap D)$ converge to

$$D_\infty = \left\{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 + 1 > \lim_{j \to \infty} \frac{\mu_j^2}{\lambda_j} Q_1(z_2) , \Im z_2 > -1 \right\}.$$

More specifically, the limit domain turns out to be

$$D_{1, \infty} = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 + 1 > Q_1(z_2)/m^2, \Im z_2 > -1 \}$$

in case III(a), and

$$D_{2, \infty} = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_1 > -1, \Im z_2 > -1 \}$$

in case III(b).

Note that the limiting domain $D_{1, \infty}$ is a Siegel domain of second kind (refer [17] for more details) and hence complete Kobayashi hyperbolic. Evidently, $D_{1, \infty}$ can be written as the intersection of an open ball with a half-space in $\mathbb{C}^2$. Moreover, $D_{1, \infty}$ is an unbounded convex domain. Furthermore,
according to [17], $D_{1,\infty}$ is biholomorphic to a bounded domain in $\mathbb{C}^2$. In particular, the Bergman kernel $K_{D_{1,\infty}}$ is non-vanishing along the diagonal. Also, note that the limit domain $D_{2,\infty}$ is biholomorphic to the unit bidisc $\Delta \times \Delta$ via a map that preserves the origin.

The stability of the infinitesimal Kobayashi metric under scaling can be proved using similar ideas [13, Lemma 5.2]. The following two ingredients will be required in the proof. First, the limit domain $D_{\infty}$ is complete Kobayashi hyperbolic and hence taut in each of the cases (I)–(III). The next step is to consider the mappings $f^j : \Delta \to D^j$ that almost realize $k_{D^j}(\cdot, \cdot)$ and establish that $\{f^j\}$ is normal. Recall that, in each of the three cases listed above, the scaled domains $D^j$ are all contained in the taut domain $2D_{\infty}$ for large $j$. Hence, it is possible to pass to a subsequence of $\{f^j\}$ that converges to a holomorphic mapping $f : \Delta \to D_{\infty}$ uniformly on compact subsets of $\Delta$. It follows that the limit map $f$ provides a candidate in the definition of $k_{D_{\infty}}(\cdot, \cdot)$.

**Lemma 4.3.** For $(z,v) \in D_{\infty} \times \mathbb{C}^2$,

$$k_{D^j}(z,v) \to k_{D_{\infty}}(z,v).$$

Moreover, the convergence is uniform on compact subsets of $D_{\infty} \times \mathbb{C}^2$.

The next step is a stability statement for the Kobayashi indicatrices of the scaled domains $D^j$.

**Lemma 4.4.** For $z$ in any compact subset $S$ of $D_{\infty}$,

(i) $I_{D^j}(z)$ is uniformly compactly contained in $\mathbb{C}^n$ for all $j$ large, and

(ii) the indicatrices $I_{D^j}(z)$ converge uniformly in the Hausdorff sense to $I_{D_{\infty}}(z)$.

Finally, for each $z \in D_{\infty}$, the functions $\lambda(I_{D^j}(z))$ converge to $\lambda(I_{D_{\infty}}(z))$.

For the proof, repeat the arguments provided earlier along with the following observation: the limit domain $D_{\infty}$ is biholomorphically equivalent to a bounded domain in $\mathbb{C}^2$ in each of the cases (I)–(III), which implies that there is a uniform positive constant $C$ (depending only on $S$) such that for $z \in S$,

$$k_{D_{\infty}}(z,v) \geq C|v| \quad \text{for all } v \in \mathbb{C}^2.$$

**Proof of Theorem 4.2.** Observe that

$$F_{U \cap D}^k(p^j) = F_{D^j}^k((0,0)) = K_{D^j}((0,0))\lambda(I_{D^j}^k((0,0)))$$

for each $j$. To control the Bergman kernels $K_{D^j}$ on the scaled domains, note first that the limit domain $D_{\infty}$ is biholomorphic to a convex domain in each of the cases (I)–(III), which implies that

$$K_{D^j}((0,0)) \to K_{D_{\infty}}((0,0))$$
by [1, Lemma 2.1]. Moreover, from Lemma 4.4 it follows that 
\[ F_k^D(p_j) \to F_k^{D_\infty}((0,0)). \] 
Finally, to conclude, note that the domain \( D \) as in Definition 4.1 supports a local holomorphic peak function at each boundary point. It follows that \( F_k^D \) can be localized near \( p^0 \in \partial D \). This observation together with (4.3) yields 
\[ F_k^D(p_j) \to F_k^{D_\infty}((0,0)), \]
where \( D_\infty \) is the model domain at \( p^0 \). The result follows by recalling that the limit domain \( D_\infty \) is biholomorphic to \( \Delta \times \Delta \) in cases (I) and (IIIb) and to \( \mathbb{B}^2 \) in case (II). ■

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