ALGEBRAS AND HOPF ALGEBRAS

IN BRAIDED CATEGORIES

SHAHN MAJID

Department of Applied Mathematics & Theoretical Physics
University of Cambridge, Cambridge CB3 9EW, U.K.

ABSTRACT

This is an introduction for algebraists to the theory of algebras and Hopf algebras in braided categories. Such objects generalise super-algebras and super-Hopf algebras, as well as colour-Lie algebras. Basic facts about braided categories $\mathcal{C}$ are recalled, the modules and comodules of Hopf algebras in such categories are studied, the notion of `braided-commutative' or `braided-cocommutative' Hopf algebras (braided groups) is reviewed and a fully diagrammatic proof of the reconstruction theorem for a braided group $\text{Aut}(\mathcal{C})$ is given. The theory has important implications for the theory of quasitriangular Hopf algebras (quantum groups). It also includes important examples such as the degenerate Sklyanin algebra and the quantum plane.

One of the main motivations of the theory of Hopf algebras is that they provide a generalization of groups. Hopf algebras of functions on groups provide examples of commutative Hopf algebras, but it turns out that many group-theoretical constructions work just as well when the Hopf algebra is allowed to be non-commutative. This is the philosophy associated to some kind of non-commutative (or so-called quantum) algebraic geometry. In a Hopf algebra context one can say the same thing in a dual way: group algebras and enveloping algebras are cocommutative but many constructions are not tied to this. This point of view has been highly successful in recent years, especially in regard to the quasitriangular Hopf algebras of Drinfeld. These are non-cocommutative but the non-cocommutativity is controlled by a quasitriangular structure $\mathcal{R}$. Such objects are commonly called quantum groups. Coming out of physics, notably associated to solutions of the Quantum Yang-Baxter Equations (QYBE) is a rich supply of quantum groups.

Here we want to describe some kind of rival or variant of these quantum groups, which we call braided groups. These are motivated by an earlier revolution that was very popular some decades ago in mathematics and physics, namely the theory of super or...
\(Z_2\)-graded algebras and Hopf algebras. Rather than make the algebras non-commutative etc one makes the notion of tensor product \(\otimes\) non-commutative. The algebras remain commutative with respect to this new tensor product (they are super-commutative). Under this point of view one has super-groups, super-manifolds and super-differential geometry. In many ways this line of development was somewhat easier than the notion of quantum geometry because it is conceptually easier to make an entire shift of category from vector spaces to super-vector spaces. One can study Hopf algebras in such categories also (super-quantum groups).

In this second line of development an obvious (and easy) step was to generalise such constructions to the case of symmetric tensor categories\([27]\). These have a tensor product \(\otimes\) and a collection of isomorphisms \(\Psi\) generalizing the transposition or super-transposition map but retaining its general properties. In particular, one keeps \(\Psi^2 = \text{id}\) so that these generalized transpositions still generate a representation of the symmetric group. Since only such general properties are used in most algebraic constructions, such as Hopf algebras and Lie algebras, these notions immediately (and obviously) generalise to this setting. See for example Gurevich \([19]\), Pareigis \([61]\), Scheunert \([67]\) and numerous other authors. On the other hand, the theory is not fundamentally different from the super-case.

Rather more interesting is the further generalization to relax the condition that \(\Psi^2 = \text{id}\).

Now \(\Psi\) and \(\Psi^{-1}\) must be distinguished and are more conveniently represented by braid-crossings rather than by permutations. They generate an action of the Artin braid group on tensor products. Such quasitensor or braided-tensor categories have been formally introduced into category theory in \([24]\) and also arise in the representation theory of quantum groups. The study of algebras and Hopf algebras in such categories is rather more non-trivial than in the symmetric case. It is this theory that we wish to describe here. It has been introduced by the author under the heading ‘braided groups’ as mentioned above. Introduced were the relevant notions (not all of them obvious), the basic lemmas (such as a braided-tensor product analogous to the super-tensor product of super-algebras) and a construction leading to a rich supply of examples.

On the mathematical level this project of ‘braiding’ all of mathematics is, I believe, a deep one (provided one goes from the symmetric to the truly braided case). Much of mathematics consists of manipulating symbols, making transpositions etc. The situation appears to be that in many constructions the role of permutation group can (with care) be played equally well by the braid group. Not only the algebras and braided groups to be described here, but also braided differential calculus, braided-binomial theorems and braided-exponentials are known\([57]\) as well as braided-Lie algebras\([58]\). Much more can be expected. Ultimately we would like some kind of braided geometry comparable to the high-level of development in the super case.

Apart from this long-term philosophical motivation, one can ask what are the more immediate applications of this kind of braided geometry? I would like to mention five of them.

1. Many algebras of interest in physics such as the degenerate Sklyanin algebra, quantum planes and exchange algebras are not naturally quantum groups but turn out to be braided ones\([52]\)\([56]\). There are braided-matrices \(B(R)\) and braided-vectors \(V(R')\) associated to \(R\)-matrices.

2. The category of Hopf algebras is not closed under quotients in a good sense. For
example, if $H \subset H_1$ is covered by a Hopf algebra projection then $H_1 \cong B \rtimes H$ where $B$ is a braided-Hopf algebra. This is the right setting for Radford’s theorem as we have discovered and explained in detail in \[52\].

3. Braided groups are best handled by means of braid diagrams in which algebraic operations ‘flow’ along strings. This means deep connections with knot theory and is also useful even for ordinary Hopf algebras. For example, you can dualise theorems geometrically by turning the diagram-proof up-side-down and flip conventions by viewing in a mirror.

4. A useful tool in the theory of quasitriangular Hopf algebras (quantum groups) via a process of transmutation. By encoding their non-cocommutativity as braiding in a braided category they appear ‘cocommutative’. Likewise, dual quasitriangular Hopf algebras are rendered ‘commutative’ by this process \[45\][49].

5. In particular, properties of the quantum groups $O_q(G)$ and $U_q(g)$ are most easily understood in terms of their braided versions $B_q(G)$ and $BU_q(g)$. This includes an Ad-invariant ‘Lie algebra-like’ subspace $\mathcal{L} \subset U_q(g)$ and an isomorphism $B_q(G) \cong U_q(g)$ \[50\][52].

An outline of the paper is the following. In Section 1 we recall the basic notions of braided tensor categories and how to work in them, and some examples. We recall basic facts about quasitriangular and dual quasitriangular Hopf algebras and the braided categories they generate. In Section 2 we do diagrammatic Hopf-algebra theory in this setting. In Section 3 we give a new diagrammatic proof of our generalised Tannaka-Krein-type reconstruction theorem. In Section 4 we explain the results about ordinary quantum groups obtained from this braided theory. In Section 5 we end with basic examples of braided matrices etc associated to an $R$-matrix. Although subsequently of interest in physics, the braided matrices arose quite literally from the Tannaka-Krein theorem mentioned above. This is an example of pure mathematics feeding back into physics rather than the other way around (for a change).

Our work on braided groups (or Hopf algebras in braided categories) was presented to the Hopf algebra community at the Euler Institute in Leningrad, October 1990 and at the Biannual Meeting of the American Maths Society in San Francisco, January 1991 and published in \[18\][19]. The result presented at these meetings was the introduction of Hopf algebras living in the braided category of comodules of a dual quasitriangular Hopf algebra. The connection between crossed modules (also called Drinfeld-Yetter categories) and the quantum double as well as the connection with Radford’s theorem were introduced in \[38\] in early 1990. The braided interpretation of Radford’s theorem was introduced in detail in \[52\] and circulated at the start of 1992. Dual quasitriangular (or coquasitriangular) Hopf algebras themselves were developed in connection with Tannaka-Krein ideas in \[36\][43][49, Appendix] (and earlier in other equivalent forms). A related Tannaka-Krein theorem in the quasi-associative dual quasitriangular setting was obtained in \[43\] at the Amherst conference and circulated in final form in the Fall of 1990.

It is a pleasure to see that some of these ideas have subsequently proven of interest in Hopf algebra circles (directly or indirectly). I would also like to mention some constructions of Lyubashenko\[30\][31] relating to our joint work\[32\]. Also in joint work with Gurevich\[21\] the transmutation construction is related to Drinfeld’s process of twisting\[13\]. Several other
papers can be mentioned here. On the whole I have resisted the temptation to give a full survey of all results obtained so far. Instead, the aim here is a more pedagogical exposition of the more elementary results, with proofs.

Throughout this paper we assume familiarity with usual techniques of Hopf algebras such as in the book of Sweedler [34]. In this sense the style (and also the motivation) is somewhat different from our braided-groups review article for physicists [42]. We work over a field $k$. With more care one can work here with a ring just as well. When working with matrix or tensor components we will use the convention of summing over repeated indices. Some of the elementary quantum groups material should appear in more detail in my forthcoming book.

1 Braided Categories

Here we develop the braided categories within which we intend to work, namely those coming from (co)modules of quantum groups. In fact, the theory in Sections 2, 3 is not tied to quantum groups and works in any braided category. The material in the present section is perfectly standard by now.

1.1 Definition and General Constructions

Symmetric monoidal (=tensor) categories have been known for some time and we refer to [27] for details. The model is the category of $k$-modules. The notion of braided monoidal (=braided tensor=quasitensor) category is a small generalization if this.

Briefly, a monoidal category means $(\mathcal{C}, \otimes, \Phi_{V,W,Z}, 1, l, r)$ where $\mathcal{C}$ is a category with objects $V, W, Z$ etc, $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor and $\Phi$ is a natural transformation between the two functors $(\otimes(\otimes))$ and $(\otimes)\otimes$ from $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. This means a functorial collection of isomorphisms $\Phi_{V,W,Z}: V \otimes (W \otimes Z) \to (V \otimes W) \otimes Z$. These are in addition required to obey the 'pentagon' coherence condition of Mac Lane. This expresses equality of two ways to go via $\Phi$ from $U \otimes (V \otimes (W \otimes Z)) \to (((U \otimes V) \otimes W) \otimes Z$. Once this is assumed Mac Lane's theorem ensures that all other re-bracketing operations are consistent. In practice this means we can forget $\Phi$ and brackets entirely. We also assume a unit object $1$ for the tensor product and associated functorial isomorphisms $l_V: V \to 1 \otimes V, r_V: V \to V \otimes 1$ for all objects $V$, which we likewise suppress.

A monoidal category $\mathcal{C}$ is rigid (=has left duals) if for each object $V$, there is an object $V^*$ and morphisms $ev_V: V^* \otimes V \to 1$, $coev_V: 1 \to V \otimes V^*$ such that

$$
V^*_{\text{coev}}(V \otimes V^*) \otimes V \cong V \otimes (V^* \otimes V)_{\text{coev}} V
$$

$$
V^*_{\text{coev}}V^* \otimes (V \otimes V^*) \cong (V^* \otimes V) \otimes V^*_{\text{coev}} V
$$

compose to $\text{id}_V$ and $\text{id}_{V^*}$ respectively. A single object has a left dual if $V^*$, $ev_V$, $coev_V$ exist. The model is that of a finite-dimensional vector space (or finitely generated projective module when $k$ is a ring).

Finally, the monoidal category is braided if it has a quasisymmetry or ‘braiding’ $\Psi$ given as a natural transformation between the two functors $\otimes$ and $\otimes^\text{op}$ (with opposite product) from $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. This is a collection of functorial isomorphisms $\Psi_{V,W}: V \otimes W \to W \otimes V$ obeying two ‘hexagon’ coherence identities. In our suppressed notation these are

$$
\Psi_{V \otimes W, Z} = \Psi_{V, Z} \circ \Psi_{W, Z}, \quad \Psi_{V, W \otimes Z} = \Psi_{V, Z} \circ \Psi_{V, W}
$$
while identities such as
\[ \Psi_{V,V} = \text{id}_V = \Psi_{V,V} \]
(4)
can be deduced. If \( \Psi^2 = \text{id} \) then one of the hexagons is superfluous and we have an ordinary symmetric monoidal category.

Let us recall that the functoriality of maps such as those above means that they commute in a certain sense with morphisms in the category. For example, functoriality of \( \Psi \) means
\[
\Psi_{Z,W}(\phi \otimes \text{id}) = (\text{id} \otimes \phi)\Psi_{V,W} \forall \phi \downarrow, \quad \Psi_{V,Z}(\text{id} \otimes \phi) = (\phi \otimes \text{id})\Psi_{V,W} \forall \phi \downarrow.
\]
(5)

These conditions (3)-(5) are just the obvious properties that we take for granted when transposing ordinary vector spaces or super-vector spaces. In these cases \( \Psi \) is the twist map \( \Psi_{V,W}(v \otimes w) = w \otimes v \) or the supertwist
\[
\Psi_{V,W}(v \otimes w) = (-1)^{|v||w|}w \otimes v
\]
on homogeneous elements of degree \(|v|, |w|\). The form of \( \Psi \) in these familiar cases does not depend directly on the spaces \( V, W \) so we often forget this. But in principle there is a different map \( \Psi_{V,W} \) for each \( V, W \) and they all connect together as explained.

In particular, note that for any two \( V, W \) we have two morphisms \( \Psi_{V,W}, \Psi_{W,V}^{-1} : V \otimes W \to W \otimes V \) and in the truly braided case these can be distinct. A convenient notation in this case is to write them not as permutations but as braid crossings. Thus we write morphisms pointing downwards (say) and instead of a usual arrow, we use the shorthand
\[
\Psi_{V,W} = \frac{V}{W} W V \quad \Psi_{W,V}^{-1} = \frac{W}{V} V W.
\]
(7)

In this notation the hexagons (3) appear as
\[
\begin{align*}
\frac{V}{W} W Z & = \frac{V}{W} W Z \quad \frac{V}{W} W Z \quad \frac{V}{W} W Z \quad \frac{V}{W} W Z
\end{align*}
\]
(8)
The doubled lines refer to the composite objects \( V \otimes W \) and \( W \otimes Z \) in a convenient extension of the notation. The coherence theorem for braided categories can be stated very simply in this notation: if two series of morphisms built from \( \Psi, \Phi \) correspond to the same braid then they compose to the same morphism. The proof is just the same as Mac Lane’s proof in the symmetric case with the action of the symmetric group replaced by that of the Artin braid group.

This notation is a powerful one. We can augment it further by writing any other morphisms as nodes on a string connecting the inputs down to the outputs. Functoriality
then says that a morphism \( \phi : V \to Z \) say can be pulled through braid crossings,

\[
\begin{array}{c}
V & W \\
\theta & \theta \\
W & Z \\
\end{array} = \begin{array}{c}
V & W \\
\theta & \theta \\
W & Z \\
\end{array}
\]

Similarly for \( \Psi^{-1} \) with inverse braid crossings. An easy lemma using this notation is that for any braided category \( C \) there is another mirror-reversed braided monoidal category \( \bar{C} \) with the same monoidal structure but with braiding

\[
\bar{\Psi}_{V,W} = \Psi_{W,V}^{-1}
\]

in place of \( \Psi_{V,W} \), i.e with the interpretation of braid crossings and inverse braid crossings interchanged.

Finally, because of (11) we can suppress the unit object entirely so the evaluation and co-evaluation appear simply as \( ev = \cup \) and \( coev = \cap \). Then (1)-(2) appear as

\[
\begin{array}{c}
ev_{V} = V & V^* \\
\cap & \cup \\
V & V^* \\
\end{array} = \begin{array}{c}
ev_{V} = V & V^* \\
\cap & \cup \\
V & V^* \\
\end{array}
\]

There is a similar notion of right duals \( V^\sim \) and \( \bar{ev}_{V}, \bar{coev}_{V} \) for which the mirror-reflected double-bend here can be likewise straightened.

**Example 1.1** Let \( R \in M_n(k) \otimes M_n(k) \) be invertible and obey the QYBE

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\]

then the monoidal category \( C(V, R) \) generated by tensor products of \( V = \mathbb{C}^n \) is braided.

**Proof** This is an elementary exercise (and extremely well-known). The notation is \( R_{12} = R \otimes \text{id} \) and \( R_{23} = \text{id} \otimes R \) in \( M_n^\otimes 3 \). The braiding on basis vectors \( \{ e_i \} \) is

\[
\Psi(e_i \otimes e_j) = e_b \otimes e_a R_{i,b}^{a,j}
\]

extended to tensor products according to (12). The morphisms in the category are linear maps such that \( \Psi \) is functorial with respect to them in the sense of (13). The associativity \( \Phi \) is the usual one on vector spaces. □

If \( R \) obeys further conditions then \( C(V, V^*, R) \) generated by \( V, V^* \) is rigid. One says that such an \( R \) is dualizable. For this there should exist among other things a ‘second-inverse’

\[
\tilde{R} = (R^{t_2})^{-1} t_2
\]

where \( t_2 \) is transposition in the second \( M_n \) factor. This defines one of the mixed terms in the braiding

\[
\Psi_{V^*, V^*}(f^i \otimes f^j) = R_{i,b}^{j,a} f^b \otimes f^a
\]
Ψ_{V,V^*}(e_i \otimes f^j) = \widetilde{R}_{i}^{a} f_{b}^{j} \otimes e_{a} \tag{15}
Ψ_{V^*,V}(f^i \otimes e_j) = e_{a} \otimes f^{b} R^{-1}_{b}^{a} f_{i}^{j} \tag{16}

where \( V^* = \{ f^i \} \) is a dual basis. The evaluation and coevaluation are given by the usual morphisms

\[ \text{ev}_V(f^i \otimes e_j) = \delta^i_j, \quad \text{coev}_V(1) = \sum_i e_i \otimes f^i. \tag{17} \]

One needs also the second-inverse \( \widetilde{R}^{-1} \) for \( \Psi \) to be invertible. In this way one translates the various axioms into a linear space setting. We see in particular that the QYBE are nothing other than the braid relations in matrix form.

We turn now to some general categorical constructions. One construction in [37][40] is based on the idea that a pair of monoidal categories \( C \rightarrow V \) connected by a functor behaves in many ways like a bialgebra with \( \otimes \) in \( C \) something like the product. In some cases this is actually true as we shall see in Section 3 in the form of a Tannaka-Krein-type reconstruction theorem, but we can keep it in general as motivation. Motivated by this we showed that for every pair \( C \rightarrow V \) of monoidal categories there is a dual one \( C^\circ \rightarrow V \) where \( C^\circ \) is the Pontryagin dual monoidal category [37]. This generalised the usual duality for Abelian groups and bialgebras to the setting of monoidal categories. We also proved such things as a canonical functor

\[ C \rightarrow \circ(C^\circ). \tag{18} \]

Of special interest to us now is the case \( C \rightarrow C \) where the functor is the identity one. So associated to every monoidal category \( C \) is another monoidal category \( C^\circ \) of ‘representations’ of \( \otimes \). This special case can also be denoted by \( C^\circ = \mathcal{Z}(C) \) the ‘center’ or ‘inner double’ of \( C \) for reasons that we shall explain shortly. This case was found independently by V.G. Drinfeld who pointed out that it is braided.

**Proposition 1.2** [37][10] Let \( C \) be a monoidal category. There is a braided monoidal category \( C^\circ = \mathcal{Z}(C) \) defined as follows. Objects are pairs \( (V, \lambda_V) \) where \( V \) is an object of \( C \) and \( \lambda_V \) is a natural isomorphism in \( \text{Nat}(V \otimes \text{id}, \text{id} \otimes V) \) such that

\[ \lambda_{V \otimes Z} = \text{id}, \quad (\text{id} \otimes \lambda_{V,Z})(\lambda_{V,W} \otimes \text{id}) = \lambda_{V,W \otimes Z}. \]

and morphisms are \( \phi : V \rightarrow W \) such that the modules are intertwined in the form

\[ (\text{id} \otimes \phi) \lambda_{V,Z} = \lambda_{W,Z}(\phi \otimes \text{id}), \quad \forall Z \text{ in } C. \]

The monoidal product and braiding are

\[ (V, \lambda_V) \otimes (W, \lambda_W) = (V \otimes W, \lambda_{V \otimes W}), \quad \lambda_{V \otimes W,Z} = (\lambda_{V,Z} \otimes \text{id})(\text{id} \otimes \lambda_{W,Z}) \]

\[ \Psi_{(V,\lambda_V),(W,\lambda_W)} = \lambda_{V,W}. \]
Proof  The monoidal structure was found in the author’s paper [37] where full proofs were also given. We refer to this for details. Its preprint was circulated in the Fall of 1989. The braiding was pointed out by Drinfeld[10] who had considered the construction from a very different and independent point of view to our duality one, namely in connection with the double of a Hopf algebra as we shall explain below. Another claim to the construction is from the direction of tortile categories[25]. See also [40] for further work from the duality point of view. □

The ‘double’ point of view for this construction is based on the following example cf[10].

Example 1.3 Let $H$ be a bialgebra over $k$ and $C = _H \mathcal{M}$ the monoidal category of $H$-modules. Then an object of $\mathcal{Z}(C)$ is a vector space $V$ which is both a left $H$-module and an invertible left $H$-comodule such that

$$\sum h_{(1)} v^{(1)} \otimes h_{(2)} \triangleright v^{(2)} = \sum (h_{(1)} \triangleright v)^{(1)} h_{(2)} \otimes (h_{(1)} \triangleright v)^{(2)}, \quad \forall h \in H, \; v \in V.$$ 

In this form $\mathcal{Z}(H, \mathcal{M})$ coincides with the category $^H \mathcal{H}, \mathcal{M}$ of $H$-crossed modules[72] with an additional invertibility condition. The braiding is

$$\Psi_{V,W}(v \otimes w) = \sum v^{(i)} \triangleright w \otimes v^{(2)}.$$ 

The invertibility condition on the comodules ensures that $\Psi^{-1}$ exists, and is automatic if the bialgebra $H$ has a skew-antipode.

Proof  The proof is standard from the point of view of Tannaka-Krein reconstruction methods (which we shall come to later). From $C$ we can reconstruct $H$ as the representing object for a certain functor. This establishes a bijection $\text{Lin}(V, H \otimes V) \cong \text{Nat}(V \otimes \text{id}, \text{id} \otimes V)$ under which $\lambda_V$ corresponds to a map $V \to H \otimes V$. That $\lambda_V$ represents $\otimes$ corresponds then to the comodule property of this map. That $\lambda_V$ is a collection of morphisms corresponds to the stated compatibility condition between this coaction and the action on $V$ as an object in $C$. To see this in detail let $H_L$ denote $H$ as an object in $C$ under the left action. Given $\lambda_V$ a natural transformation we define

$$\sum v^{(i)} \otimes v^{(2)} = \lambda_{V,H_L}(v \otimes 1) \quad (19)$$

and check

$$(\text{id} \otimes \lambda_{V,H_L})(\lambda_{V,H_L} \otimes \text{id})(v \otimes 1 \otimes 1) = \lambda_{V,H_L} \otimes H_L(v \otimes (1 \otimes 1))$$

$$= \lambda_{V,H_L} \otimes H_L(v \otimes \Delta(1)) = (\Delta \otimes \text{id}) \circ \lambda_{V,H_L}(v \otimes 1)$$

where the first equality is the fact that $\lambda_V$ ‘represents’ $\otimes$ and the last is that $\lambda_V$ is functorial under the morphism $\Delta : H_L \to H_L \otimes H_L$. The left hand side is the map $V \to H \otimes V$ in (19) applied twice so we see that this map is a left coaction. Moreover,

$$\sum h_{(1)} v^{(i)} \otimes h_{(2)} \triangleright v^{(2)} = h \triangleright \lambda_{V,H_L}(v \otimes 1) = \lambda_{V,H_L}(h \triangleright (v \otimes 1))$$

$$= \sum \lambda_{V,H_L}(h_{(1)} \triangleright v \otimes R_{h_{(2)}}(1)) = \sum (\lambda_{V,H_L}(h_{(1)} \triangleright v \otimes 1))(h_{(2)} \otimes 1)$$

where the first equality is the definition (19) and the action of $H$ on $H_L \otimes V$. The second equality is that $\lambda_{V,H_L}$ is a morphism in $C$. The final equality uses functoriality under
the morphism $R_{h[2]} : H_L \rightarrow H_L$ given by right-multiplication to obtain the right hand side of the compatibility condition. The converse directions are easier. Given a coaction $V \rightarrow H \otimes V$ define $\lambda_{V,W}(v \otimes w) = \sum v^{(1)} \triangleright w \otimes v^{(2)}$. This also implies at once the braiding $\Psi = \lambda$ as stated.

Finally we note that in Proposition 1.2 the definition assumes that the $\lambda_V$ are invertible. If we were to relax this then we would have a monoidal category which is just that of crossed modules as in [2], but then $\Psi$ would not necessarily be invertible and hence would not be a true braiding. The invertible $\lambda_V$ correspond to left comodules which are invertible in the following sense: there exists a linear map $\lambda_V^{-1}(w) \otimes v = \sum v^{[2]} \otimes v^{[1]}$ say, such that

$$\sum v^{[2]} \otimes v^{[1]} = 1 \otimes v = \sum v^{(2)} \otimes v^{(1)}, \quad \forall v \in V. \quad (20)$$

One can see that if such an ‘inverse’ exists, it is unique and a right comodule. Moreover, it is easy to see that the invertible comodules are closed under tensor products. They correspond to $\lambda_V^{-1}$ in a similar way to (19) and with $\lambda_{V,W}^{-1}(w \otimes v) = \sum v^{[2]} \otimes v^{[1]} \triangleright w$ for the converse direction. In the finite-dimensional case they provide left duals $V^*$ with left coaction $\beta_{V^*}(f)(v) = \sum v^{[1]} f(v^{[2]})$. If the bialgebra $H$ has a skew-antipode then every left comodule is invertible by composing with the skew-antipode. So in this case the condition becomes empty.

From the categorical point of view in Proposition 1.2, if $C$ has right duals then every $\lambda_{V,W}$ is invertible, cf [3]. The inverse is the right-adjoint of $\lambda_{V,W^*}$, namely $\lambda_{V,W}^{-1} = (ev_W \otimes id) \circ \lambda_{V,W} \circ (id \otimes coev_W)$. When $C = H \mathcal{M}$ then the finite-dimensional left modules have right duals if the bialgebra $H$ has a skew-antipode, so in this case the invertibility of $\lambda_V$ is automatic. On the other hand, we do not need to make these suppositions here.

This completes our computation of $Z(H \mathcal{M})$. Apart from the invertibility restriction we see that it consists of compatible module-comodule structures as stated. □

Note that the notion of a crossed module is an immediate generalisation of the notion of a crossed $G$-module[7] with $H = kG$, the group algebra of a finite group $G$. In this case the category of crossed $G$-modules is well-known to be braided[8]. Moreover, because the objects can be identified with underlying vector spaces, we know by the Tannaka-Krein reconstruction theorem[66] that there must exist a bialgebra $coD(H)$ such that our braided-category is equivalent to that of right $coD(H)$-comodules

$$\mathcal{M}_{f.d.}^{coD(H)} = H \mathcal{M}_{f.d.}. \quad (21)$$

Here we take the modules to be finite-dimensional as a sufficient (but not necessary) condition for a Tannaka-Krein reconstruction theorem to apply and the co-double $coD(H)$ to exist. In the nicest case the category is also $D(H)\mathcal{M}_{f.d.}$ for some $D(H)$. This is an abstract definition of Drinfeld’s quantum double and works for a bialgebra.

If it happens that $H$ is a Hopf algebra with invertible antipode then one can see from the above that $H \mathcal{M}_{f.d.}$ is rigid and so $coD(H)$ and $D(H)$ will be Hopf algebras. The categorical reason is that $H \mathcal{M}_{f.d.}$ is rigid and this duality extends to $Z(C)$ with the dual of $\lambda_V$ defined by the left-adjoint of $\lambda_V^{-1}$, namely $\lambda_V^{-1} \circ (ev_V \otimes id) \circ \lambda_V^{-1} \circ (id \otimes coev_V)$. We will study details about categories of modules and comodules and the reconstruction theorems later in this section and in Section 3. The point is that these categorical methods are very powerful.
Proposition 1.4 If \( H \) is a finite-dimensional Hopf algebra then \( \mathcal{D}(H) \) (the quantum double Hopf algebra of \( H \)) is built on \( H^* \otimes H \) as a coalgebra with the product

\[
(a \otimes h)(b \otimes g) = \sum b_{(2)}a \otimes h_{(2)}g < Sh_{(1)}, b_{(1)}> h_{(3)}, b_{(3)}>, \quad h, g \in H, \ a, b \in H^*
\]

where \(<, >\) denotes evaluation.

Proof The quantum double \( \mathcal{D}(H) \) was introduced by Drinfeld as a system of generators and relations built from the structure constants of \( H \). The formula stated on \( H^* \otimes H \) is easily obtained from this as done in [33]. We have used here the conventions introduced in [38] that avoid the use of the inverse of the antipode. Also in [38] we showed that the modules of the double were precisely the crossed modules category as required. To see this simply note that \( H \) and \( H^\text{op} \) are sub-Hopf algebras and hence a left \( \mathcal{D}(H) \)-module is a left \( H \)-module and a suitably-compatible right \( H^* \)-module. The latter is equally well a left \( H \)-comodule compatible as in Example 1.3. See [38] for details. \(\square\)

In [33] we introduced a further characterization of the quantum double as a member of a class of double cross product Hopf algebras \( H_1 \bowtie H_2 \) (in which \( H_i \) are mutually acting on each other). Thus, \( \mathcal{D}(H) = H^\text{op} \bowtie H \) where the actions are mutual coadjoint actions. In this form it is clear that the role of \( H^* \) can be played by \( H^\circ \) in the infinite dimensional Hopf algebra case. We will not need this further here.

1.2 Quasitriangular Hopf Algebras

We have already described one source of braided categories, namely as modules of the double \( \mathcal{D}(H) \) (or comodules of the codouble) of a bialgebra. Abstracting from this one has the notion, due to Drinfeld, of a quasitriangular Hopf algebra. These are such that their category of modules is braided.

Definition 1.5 A quasitriangular bialgebra or Hopf algebra is a pair \((H, R)\) where \( H \) is a bialgebra or Hopf algebra and \( R \in H \otimes H \) is invertible and obeys

\[
(\Delta \otimes \text{id})R = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}. \tag{22}
\]

\[
\tau \circ \Delta h = R(\Delta h)R^{-1}, \quad \forall h \in H. \tag{23}
\]

Here \( R_{12} = R \otimes 1 \) and \( R_{23} = 1 \otimes R \) etc, and \( \tau \) is the usual twist map.

Thus these Hopf algebras are like cocommutative enveloping algebras or group algebras but are cocommutative now only up to an isomorphism implemented by conjugation by an element \( R \). Some elementary (but important) properties are

Lemma 1.6 If \((H, R)\) is a quasitriangular bialgebra then \( R \) as an element of \( H \otimes H \) obeys

\[
(\varepsilon \otimes \text{id})R = (\text{id} \otimes \varepsilon)R = 1. \tag{24}
\]

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{25}
\]

If \( H \) is a Hopf algebra then one also has

\[
(S \otimes \text{id})R = R^{-1}, \quad (\text{id} \otimes S)R^{-1} = R, \quad (S \otimes S)R = R \tag{26}
\]

\[
\exists S^{-1}, u, v, \quad S^2(h) = uhv^{-1}, \quad S^{-2}(h) = vhv^{-1}, \quad \forall h \in H \tag{27}
\]
Proof For \([24]\) apply \(\epsilon\) to \([22]\), thus \((\epsilon \otimes \text{id} \otimes \text{id})(\Delta \otimes \text{id})R = R_{23} = (\epsilon \otimes \text{id} \otimes \text{id})R_{13}R_{23}\) so that (since \(R_{23}\) is invertible) we have \((\epsilon \otimes \text{id})R = 1\). Similarly for the other side. For \([23]\) compute \((\text{id} \otimes \tau \circ \Delta)R\) in two ways: using the second of axioms \([22]\) directly or using axiom \([24]\), and then the second of \([22]\). For \([20]\) consider \(\sum \mathcal{R}^{(1)}_1 S \mathcal{R}^{(2)}_2 \otimes \mathcal{R}^{(2)}_2 = 1\) by the property of the antipode and equation \([24]\) already proven, but equals \(R(S \otimes \text{id})R\) by axiom \([22]\). Similarly for the other side, hence \((\text{id} \otimes \text{id})R = R^{-1}\). Finally, we return to our basic construction, \([27]\) the relevant expressions are

\[
u = \sum (S \mathcal{R}^{(2)}_2) \mathcal{R}^{(1)}_1, \quad \nu^{-1} = \sum \mathcal{R}^{(2)}_2 S^2 \mathcal{R}^{(1)}_1, \quad v = Su
\]

which one can verify to have the right properties. In addition one can see that \(\Delta u = (\mathcal{R}_{21} \mathcal{R}_{12})^{-1} (u \otimes u)\) and similarly for \(v\) so that \(uv^{-1}\) is group-like (and implements \(S^4\)). For details of the computations see \([12]\) or reviews by the author. □

Here \([23]\) is the reason that Physicists call \(R\) the 'universal R-matrix' (compare Example 1.1). Indeed, in any finite-dimensional representation the image of \(R\) is such an \(R\)-matrix. There are well-known examples such as \(U_q(\mathfrak{sl}_2)\) and \(U_q(\mathfrak{g})\). Here we give perhaps the simplest known quasitriangular Hopf algebras

Example 1.7 \([13]\) Let \(\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}\) be the finite cyclic group of order \(n\) and \(k\mathbb{Z}_n\) its group algebra with generator \(g\). Let \(q\) be a primitive \(n\)-th root of unity. Then there is a quasitriangular Hopf algebra \(\mathbb{Z}_n^\prime\) consisting of this group algebra and

\[
\Delta g = g \otimes g, \quad \epsilon g = 1, \quad Sg = g^{-1}, \quad \mathcal{R} = n^{-1} \sum_{a,b=0}^{n-1} q^{-ab} g^a \otimes g^b. \tag{28}
\]

Proof We assume that \(k\) is of suitable characteristic. To verify the non-trivial quasitriangular structure we use that \(n^{-1} \sum_{a,b=0}^{n-1} q^{ab} = \delta_{a,0}\). Then \(\mathcal{R}_{13} \mathcal{R}_{23} = n^{-2} \sum q^{-(ab+cd)} g^a \otimes g^c \otimes g^{b+d} = n^{-2} \sum q^{-(a-c)d} g^a \otimes g^c \otimes g^{b} = n^{-1} \sum q^{-ab} g^a \otimes g^a \otimes g^{b} = (\Delta \otimes \text{id})\mathcal{R}\) where \(b' = b + d\) was a change of variables. Similarly for the second of \([22]\). The remaining axiom \([24]\) is automatic because the Hopf algebra is both commutative and cocommutative. □

Example 1.8 \([3],[27]\) Let \(G\) be a finite Abelian group and \(k(G)\) its function Hopf algebra. Then a quasitriangular structure on \(k(G)\) means a function \(\mathcal{R} \in H \otimes H\) obeying

\[
\mathcal{R}(gh, f) = \mathcal{R}(g, f) \mathcal{R}(h, f), \quad \mathcal{R}(g, hf) = \mathcal{R}(g, h) \mathcal{R}(g, f), \quad \mathcal{R}(g, e) = 1 = \mathcal{R}(e, g)
\]

for all \(g, h, f\) in \(G\) and \(e\) the identity element. I.e., a quasitriangular structure on \(k(G)\) means precisely a bicharacter of \(G\).

Proof We identify \(k(G) \otimes k(G)\) with functions on \(G \times G\), with pointwise multiplication. Using the comultiplication given by multiplication in \(G\) we have at once that \([22]\) corresponds to the first two displayed equations. Axiom \([22]\) becomes \(hg \mathcal{R}(g, h) = \mathcal{R}(g, h)gh\) and so is automatic because the group is Abelian. Given these first two of the stated conditions, the latter two hold iff \(\mathcal{R}\) is invertible. □

The \(\mathbb{Z}_n\) example here also has an immediate generalization to the group algebra \(kG\) of a finite Abelian group equipped with a bicharacter on \(\hat{G}\). This just coincides with the last example applied to \(k(\hat{G}) = kG\). Finally, we return to our basic construction,
Example 1.9 [11] Let $H$ be a finite-dimensional bialgebra. Then $D(H)$ is quasitriangular. In the Hopf algebra case the quasitriangular structure is $R = \sum_a (f^a \otimes 1) \otimes (1 \otimes e_a)$ where $H = \{e_a\}$ is a basis and $\{f^a\}$ a dual basis.

Proof The result is due to Drinfeld. A direct proof in the abstract Hopf algebra setting appeared in [33]. The easiest way to show that $R$ is invertible is to verify in view of (24) that $(S \otimes \text{id})(R)$ is the inverse. □

Theorem 1.10 e.g. [33] Let $(H, R)$ be a quasitriangular bialgebra. Then the category $H M$ of modules is braided. In the Hopf algebra case the finite-dimensional modules are rigid. The braiding and the action on duals are

$$\Psi_{V,W}(v \otimes w) = \sum R^{(2)} \triangleright w \otimes R^{(1)} \triangleright v, \quad h \triangleright f = f((Sh)\triangleright( ))$$

with $e_\text{v}, e_\text{coev}$ as in (12).

Proof $h \triangleright \Psi(v \otimes w) = (\Delta h) \triangleright (\Psi(v \otimes w)) = \tau((\Delta^\text{op})h)R \triangleright (v \otimes w) = \tau(R(\Delta h)\triangleright (v \otimes w)) = \Psi(h \triangleright (v \otimes w))$ in virtue (23). It is easy to see that (22) likewise just correspond to the hexagons (3) or (8). Functoriality is also easily shown. For an early treatment of this topic see [33, Sec. 7]. Note that if $R_{21} = R^{-1}$ (the triangular rather than quasitriangular case) we have $\Psi$ symmetric rather than braided. This was the case treated in [11] though surely the general quasitriangular case was also known to some experts at the time or shortly thereafter. □

Proposition 1.11 [50, Sec. 6] Let $H = \mathbb{Z}_2'$ denote the quantum group in Example 1.7 with $n = 2$. Then $C = \mathbb{Z}_2' M = \text{SuperVec}$ the category of super-vector spaces.

Proof One can easily check that this $\mathbb{Z}_2'$ is indeed a quasitriangular (in fact, triangular) Hopf algebra. Hence we have a (symmetric) tensor category of representations. Writing $p = \frac{1}{\sqrt{2}}$ we have $p^2 = p$ hence any representation $V$ splits as $V_0 \oplus V_1$ according to the eigenvalue of $p$. We can also write $R = 1 - 2p \otimes p$ and hence from Theorem 1.10 we compute $\Psi(v \otimes w) = \tau(R \triangleright (v \otimes w)) = (1 - 2p \otimes p)(w \otimes v) = (-1)^{|v||w|}w \otimes v$ as in (3). □

So this non-standard quasitriangular Hopf algebra $\mathbb{Z}_2'$ (non-standard because of its non-trivial $R$) recovers the category of super-spaces with its correct symmetry $\Psi$. In just the same way the category $C_n = \mathbb{Z}_n' M$ consists of vector spaces that split as $V = \oplus_{a=0}^{n-1} V_a$ with the degree of an element defined by the action $g \triangleright v = q^{|v|} v$ where $q$ is a primitive $n$-th root of unity. From Theorem 1.10 and (28) we find

$$\Psi_{V,W}(v \otimes w) = q^{|v||w|}w \otimes v.$$  \hspace{1cm} (29)

Thus we call $C_n$ the category of anyonic vector spaces of fractional statistics $\frac{1}{n}$, because just such a braiding is encountered in anyonic physics. For $n > 2$ the category is strictly braided in the sense that $\Psi \neq \Psi^{-1}$. There are natural anyonic traces and anyonic dimensions generalizing the super-case[17]

$$\dim(V) = \sum_{a=0}^{n-1} q^{-a^2} \dim V_a, \quad \text{Tr}(f) = \sum_{a=0}^{n-1} q^{-a^2} \text{Tr} f|_{V_a}. \hspace{1cm} (30)$$
We see that this anyonic category is generated by the quantum group $\mathbb{Z}_n'$.

Obviously we can take this idea for generalising super-symmetry to the further case of Example 1.8. In this case a $k(G)$-module just means a $G$-graded space where $f \triangleright v = f(|v|)v$ on homogeneous elements of degree $|v| \in G$. This is well-known to Hopf algebraists for some time: the new ingredient is that a bicharacter gives our $G$-graded spaces a natural braided-transposition $\Psi$. We have given plenty of other less obvious examples of braided categories generated in this way from quasitriangular Hopf algebras [50, Sec. 6] [13]. Our idea in this work is not to use Hopf algebras in connection with deformations (the usual setting) but rather as the ‘generator’ of a category within which we shall later make algebraic constructions. This is how quantum groups are naturally used to generalise supersymmetry. In this context they are typically discrete.

1.3 Dual Quasitriangular Structures

In this section we describe the dual results to those above. If a quasitriangular Hopf algebra is almost cocommutative up to conjugation then its dual Hopf algebra should be almost commutative up to ‘conjugation’ in the convolution algebra. The relevant axioms are obtained by dualizing in the standard way by writing out the axioms of a quasitriangular Hopf algebra as diagrams and then reversing all the arrows (and a left-right reversal). Obviously it is the axioms that are being dualised and not any specific Hopf algebra. This is important because in the infinite-dimensional case the dual axioms are weaker. This is a rigorous way to work with the standard quantum groups over a field as appreciated in [36] among other places.

We will always denote our dual quasitriangular bialgebras and Hopf algebras by $A$ (to avoid confusion). These are equipped now with a map $\mathcal{R} : A \otimes A \to k$ which should be invertible in Hom($A \otimes A, k$) in the sense that there exists a map $\mathcal{R}^{-1} : A \otimes A \to k$ such that

$$
\sum \mathcal{R}^{-1}(a_{(1)} \otimes b_{(1)})\mathcal{R}(a_{(2)} \otimes b_{(2)}) = \epsilon(a)\epsilon(b) = \sum \mathcal{R}(a_{(1)} \otimes b_{(1)})\mathcal{R}^{-1}(a_{(2)} \otimes b_{(2)}).
$$

Keeping such considerations in mind, it is easy to dualize the remainder of Drinfeld’s axioms to obtain the following definition.

**Definition 1.12** A dual quasitriangular bialgebra or Hopf algebra $(A, \mathcal{R})$ is a bialgebra or Hopf algebra $A$ and a convolution-invertible map $\mathcal{R} : A \otimes A \to k$ such that

$$
\mathcal{R}(ab \otimes c) = \sum \mathcal{R}(a \otimes c_{(1)})\mathcal{R}(b \otimes c_{(2)}), \quad \mathcal{R}(a \otimes bc) = \sum \mathcal{R}(a_{(1)} \otimes c)\mathcal{R}(a_{(2)} \otimes b)
$$

(31)

$$
\sum b_{(1)}a_{(1)}\mathcal{R}(a_{(2)} \otimes b_{(2)}) = \sum \mathcal{R}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}
$$

(32)

for all $a, b, c \in A$.

This looks a little unfamiliar but is in fact obtained by replacing the multiplication in Definition 1.5 by the convolution product and the comultiplication by the multiplication in $A$. Axiom (32) is the dual of (23) and says, as promised, that $A$ is almost commutative – up to $\mathcal{R}$. Axioms (31) are the dual of (22) and say that $\mathcal{R}$ is a ‘bialgebra bicharacter’. They should be compared with Example 1.14 below. We also have analogues of the various results in Section 1.2. Again, the new language is perhaps unfamiliar so we give some of the proofs in this dual form in detail.
Lemma 1.13  If \((A, \mathcal{R})\) is a dual quasitriangular bialgebra then

\[
\mathcal{R}(a \otimes 1) = \epsilon(a) = \mathcal{R}(1 \otimes a).
\] (33)

\[
\sum \mathcal{R}(a_{(1)} \otimes b_{(1)})\mathcal{R}(a_{(2)} \otimes c_{(1)})\mathcal{R}(b_{(2)} \otimes c_{(2)}) = \sum \mathcal{R}(b_{(1)} \otimes c_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(2)})\mathcal{R}(a_{(2)} \otimes b_{(2)})
\] (34)

for all \(a, b, c\) in \(A\). If \(A\) is a Hopf algebra then in addition,

\[
\mathcal{R}(Sa \otimes b) = \mathcal{R}^{-1}(a \otimes b), \quad \mathcal{R}^{-1}(a \otimes Sb) = \mathcal{R}(a \otimes b), \quad \mathcal{R}(Sa \otimes Sb) = \mathcal{R}(a \otimes b) \quad (35)
\]

\[\exists S^{-1}, \quad v(a) = \sum \mathcal{R}(a_{(1)} \otimes Sa_{(2)}), \quad \sum a_{(1)}v(a_{(2)}) = \sum v(a_{(1)})S^2a_{(2)} \quad (36)\]

Proof  Using (33) we have

\[
\mathcal{R}(a \otimes 1) = \sum (\mathcal{R}^{-1}(a_{(1)} \otimes 1)\mathcal{R}(a_{(2)} \otimes 1))\mathcal{R}(a_{(3)} \otimes 1)
\]

\[= \sum \mathcal{R}^{-1}(a_{(1)} \otimes 1)(\mathcal{R}(a_{(2)} \otimes 1)\mathcal{R}(a_{(3)} \otimes 1)) = \sum \mathcal{R}^{-1}(a_{(1)} \otimes 1)\mathcal{R}(a_{(2)} \otimes 1.1) = \epsilon(a)
\]

as in (33). Likewise on the other side. Also, if \(\mathcal{R}^{-1}\) exists it is unique. Hence for \(A\) a Hopf algebra it is given by \(\mathcal{R}^{-1}(a \otimes b) = \mathcal{R}(Sa \otimes b)\) (use axioms (31)). In this case \(a \otimes b \mapsto \mathcal{R}(Sa \otimes Sb)\) is convolution inverse to \(\mathcal{R}^{-1}\) because \(\sum \mathcal{R}(Sa_{(1)} \otimes Sb_{(1)})\mathcal{R}(Sa_{(2)} \otimes b_{(2)}) = \sum \mathcal{R}(Sa \otimes (Sb_{(1)}b_{(2)})) = \mathcal{R}(Sa \otimes 1)\epsilon(b) = \epsilon(a)\epsilon(b)\) etc. Hence \(\mathcal{R}(Sa \otimes Sb) = \mathcal{R}(a \otimes b)\), proving the other side and the third part of (35). For (36) we apply the second of (31), (32) and the second of (31) again,

\[
\sum (\mathcal{R}(a_{(1)} \otimes b_{(1)})\mathcal{R}(a_{(2)} \otimes c_{(1)}))\mathcal{R}(b_{(2)} \otimes c_{(2)}) = \sum \mathcal{R}(a \otimes c_{(1)}b_{(1)})\mathcal{R}(b_{(2)} \otimes c_{(2)})
\]

\[= \sum \mathcal{R}(b_{(1)} \otimes c_{(1)})\mathcal{R}(a \otimes b_{(2)}c_{(2)}) = \sum \mathcal{R}(b_{(1)} \otimes c_{(1)})\mathcal{R}(a_{(1)} \otimes c_{(2)})\mathcal{R}(a_{(2)} \otimes b_{(2)})
\].

For (36) one defines \(v : A \rightarrow k\) as shown (and similarly a map \(u : A \rightarrow k\)) and checks the relevant facts analogous to Lemma 1.6. This is done in complete detail in [43, Appendix] to which we refer the reader. \(\square\)

Note that if \(\mathcal{R}\) is a linear map obeying (33) and (31) and if \(A\) is a Hopf algebra, we can use (33) as a definition of \(\mathcal{R}^{-1}\). Some authors in defining similar notions have made (33) an axiom in the bialgebra case (this is the case in [22] and in first versions of some other works). For this and other reasons we stick to our original terminology from [36, 43, 48] with axioms and properties as above.

Example 1.14  Let \(G\) be an Abelian group and \(kG\) its group algebra. This is dual quasitriangular iff there is a function \(\mathcal{R} : G \times G \rightarrow k\) obeying the bicharacter conditions in Example 1.8.

Proof  In the group algebra we can work with group-like elements (these form a basis). On such elements the axioms in (31)-(32) simplify: simply drop the \((1), (2)\) suffixes! This immediately reduces to the bimultiplicativity while invertibility corresponds once again (given this) to (33). \(\square\)
This example is clearly identical in content to Example 1.8. Because \( kG \) is dual to \( k(G) \) by evaluation, it is obvious that a dual quasitriangular structure on \( kG \) is just the same thing as a quasitriangular structure on \( k(G) \), namely as we see, a bicharacter. The result is however, more transparent from the dual quasitriangular point of view and slightly more general. This example is the reason that we called \( R \) obeying \((31)\) a bialgebra bicharacter\([36]\). For a concrete example, one can take \( G = \mathbb{Z}_n \) and \( R(a,b) = q^{ab} \) with \( q \) a primitive \( n \)-th root of unity to give a non-standard dual quasitriangular structure on \( k\mathbb{Z}_n \). It is just Example 1.7 after a \( \mathbb{Z}_n \)-Fourier transform.

Now let \( R \) be an invertible matrix solution of the QYBE as in Example 1.1. There is a by-now standard bialgebra \( A(R) \)\([11]\)\([14]\) defined by generators 1 and \( t = \{t^i_j \} \) (regarded as an \( n \times n \) matrix) and the relations, comultiplication and counit

\[
R^{i,j}_{a,b} t^a_k t^b_l = t^i_b t^a_l R^{i,j}_{k,b}, \quad \Delta t^i_j = t^a_i \otimes t^a_j, \quad \epsilon t^i_j = \delta^i_j, \quad \text{i.e.} \quad R t_1 t_2 = t_2 t_1 R, \quad \Delta t = t \otimes t, \quad \epsilon t = \text{id}
\]

where \( t_1 \) and \( t_2 \) denote \( t \otimes \text{id} \) and \( \text{id} \otimes t \) in \( M_n \otimes M_n \) with values in \( A(R) \).

The power of this matrix notation lies in the fact that \( \otimes \) is used only to refer to the abstract tensor product of copies of the algebra (as in defining the axioms of a Hopf algebra etc). The matrix tensor product as in \( M_n \otimes M_n \) is suppressed and its role is replaced by the suffices \( 1,2 \) etc when needed. Thus \( R = R_{12} \) (with the indices suppressed when there are only two \( M_n \) in the picture) while \( t \) means that the \( t \) is viewed as a matrix in the first \( M_n \) (with values in \( A(R) \)). The rules of the notation are that matrices are understood as multiplied in the usual order, independently in the \( 1,2 \) etc copies of \( M_n \). Using this notation (or directly with indices) one can see at once that \( A(R) \) has two fundamental representations \( \rho^\pm \) in \( M_n \) defined by

\[
\rho^+(t^i_j)^k_l = R^i_j k_l, \quad \rho^-(t^i_j)^k_l = R^{-1} k_l i_j, \quad \text{i.e.} \quad \rho^+_2(t_1) = R_{12}, \quad \rho^-_2(t_1) = R^{-1}_{21}.
\]

In the compact notation the proof reads

\[
\rho^+_3(R_{12} t_1 t_2) = R_{12} \rho^+_3(t_1) \rho^+_3(t_2) = R_{12} R_{13} R_{23},
\]

\[
= R_{23} R_{13} R_{12} = \rho^+_3(t_2) \rho^+_3(t_1) R_{12} = \rho^+_3(t_2 t_1 R_{12})
\]

\[
\rho^-_3(R_{12} t_1 t_2) = R_{12} \rho^-_3(t_1) \rho^-_3(t_2) = R_{12} R_{31}^{-1} R_{32}^{-1},
\]

\[
= R_{32}^{-1} R_{31}^{-1} R_{12} = \rho^-_3(t_2) \rho^-_3(t_1) R_{12} = \rho^-_3(t_2 t_1 R_{12}).
\]

Note that for \( \rho^- \) we need \( R_{12} R_{31}^{-1} R_{32}^{-1} = R_{32}^{-1} R_{31}^{-1} R_{12} \), i.e. \( R_{31} R_{32} R_{12} = R_{12} R_{32} R_{31} \) which is again the QYBE after a relabeling of the positions in \( M_n \otimes 3 \).

This bialgebra \( A(R) \) is important because for the standard \( R \)-matrices one has a convenient construction of the quantum function algebras \( O_q(G) \) deforming the ring of representative functions on compact simple group \( G \)\([14]\). One has to quotient the bialgebra by suitable further relations (or localise a determinant) to obtain a Hopf algebra. In the standard case of course one knew that the result was dual quasitriangular because of Drinfeld’s result that \( U_q(g) \) was quasitriangular (over formal power-series). The question for a general \( R \)-matrix was not so clear at the time and was resolved in \([33]\)\([34]\) where we showed that there is always some kind of quasitriangular structure in the form of a map \( R : A(R) \rightarrow A(R)^* \). In the more modern setting our result reads as follows. One can also see subsequent works such as \([28]\) but I retain here the strategy (which comes from physics) of my original proofs, this time with full pedagogical details.
Proposition 1.15  Let $R$ be an invertible solution of the QYBE in $M_n \otimes M_n$. Then the associated matrix bialgebra $A(R)$ is dual quasitriangular with $\mathcal{R} : A(R) \otimes A(R) \to k$ given by $\mathcal{R}(t \otimes 1) = id = \mathcal{R}(1 \otimes t)$ and $\mathcal{R}(t_1 \otimes t_2) = R$ extended as a bialgebra bicharacter according to (31). Explicitly,

$$\mathcal{R}(t_{i_1}^{i_1} t_{i_2}^{i_2} \cdots t_{i_M}^{i_M} \otimes t_{k_N}^{k_1} t_{l_N}^{k_2} t_{t_{l_N}}^{k_{N-1}} \cdots t_{t_{k_1}}^{k_1}) = R_{i_1}^{m_{11}} n_{11} R_{m_{11}}^{m_{12}} n_{12} \cdots R_{m_{1N-1}}^{m_{1N}} n_{N1},$$

where the last notation is as a partition function [35, Sec. 5.2]. Here $I = (i_1, \ldots, i_M)$ and $K = (k_1, \ldots, k_N)$ etc. There is a similar expression for $\mathcal{R}^{-1}$. If we adopt the notation $\bar{K} = (k_N, \ldots, k_1)$ and $t_{i_1}^{i_1} \cdots t_{i_M}^{i_M} = t^I J$ then

$$\mathcal{R}(t^I J \otimes t^K L) = Z_R(I \square J), \quad \mathcal{R}^{-1}(t^I J \otimes t^K L) = Z_R^{-1}(J \square I).$$

Proof  Note that $\mathcal{R}(t_1 \otimes t_2) = \rho^+(t_1)$ and the proof that this extends in its first input as a bialgebra bicharacter is exactly the proof above that $\rho^+$ extends to products as a representation. Thus we have $\mathcal{R} (a \otimes t) = \rho^+(a)$ for all $a$ and $\mathcal{R}(ab \otimes t) = \mathcal{R}(a \otimes t) \mathcal{R}(b \otimes t)$ as we require for the first of (31). In particular,

$$\mathcal{R}(t_1 t_2 \cdots t_M \otimes t_{M+1}) = \rho^+_{M+1}(t_1 t_2 \cdots t_M) = R_{1M+1} \cdots R_{MM+1}$$

is well-defined. Next the tensor product of representations is also a representation (because $A(R)$ is a bialgebra), hence there is a well-defined algebra map $\rho^{+ \otimes N} : A(R) \to M_n^N$ given by $\rho^{+ \otimes N}(a) = (\rho_1^+ \otimes \rho_2^+ \otimes \cdots \otimes \rho_N^+) \circ \Delta^{N-1}(a)$. In particular,

$$R_{1M+1} R_{1M+2} \cdots R_{1M+N},$$

$$R_{2M+1} R_{2M+2} \cdots R_{2M+N} = \rho^+_{M+1}(t_1 \cdots t_M) \cdots \rho^+_{M+N}(t_1 \cdots t_M) = (\rho^+) \otimes N(t_1 \cdots t_M)$$

also depends only on $t_1 t_2 \cdots t_M$ as an element of $A(R)$. The array on the left can be read (and multiplied up) column after column (so that the first equality is clear) or row after row (like reading a book). The two are the same when we bear in mind that $R$ living in distinct copies of $M_n \otimes M_n$ commute. The expression is just the array $Z_R$ in our compact notation. If we define $\mathcal{R}(t_1 t_2 \cdots t_M \otimes t_{M+N} \cdots t_{M+1})$ as this array, we know that the second of (31) will hold and that $\mathcal{R}$ is well defined in its first input.

Now we repeat the steps above for the second input of $\mathcal{R}$. Thus $\mathcal{R}(t_1 \otimes t_2) = R = \bar{\rho}^+_1(t_2)$ extends in its second input as a bialgebra bicharacter since this $\bar{\rho}^+$ extends as an antirepresentation $A(R) \to M_n$ (proof similar to that for $\rho^+$). Thus we define $\mathcal{R}(t \otimes a) = \bar{\rho}^+_1(a)$ and in particular,

$$\mathcal{R}(t_M \otimes t_{M+N} \cdots t_{M+2} t_{M+1}) = \bar{\rho}^+_M(t_{M+N} \cdots t_{M+2} t_{M+1}) = R_{MM+1} R_{MM+2} \cdots R_{MM+N}$$

16
is well-defined. Likewise, we can take tensor products of $\hat{\rho}^+$ and will again have well-defined anti-representations. Hence

$$R_{1M+1} R_{1M+2} \cdots R_{1M+N}$$

$$R_{2M+1} R_{2M+2} \cdots R_{2M+N} = \hat{\rho}^+_1 (t_{M+N} \cdots t_{M+1}) \cdots \hat{\rho}^+_M (t_{M+N} \cdots t_{M+1}) = (\hat{\rho}^+)^\otimes M (t_{M+N} \cdots t_{M+1})$$

... 

$$R_{MM+1} \cdots R_{MM+N}$$

depends only on $t_{M+N} \cdots t_{M+2} t_{M+1}$ as an element of $A(R)$. The first equality comes from writing out the $\hat{\rho}^+$ and rearranging the resulting array (bearing in mind that copies of $R$ in distinct $M_n$ tensor factors commute). The resulting array of matrices then coincides with that above, which we have already defined as $\mathcal{R}(t_1 \cdots t_M \otimes t_{M+N} \cdots t_{M+1})$. We see that this array then is well defined as a map $A(R) \otimes A(R) \to k$, in its first input (for fixed $t_{M+N}, \cdots, t_{M+1}$) by its realization as a tensor power of $\hat{\rho}^+$ and in its second input (for fixed $t_1, \cdots, t_M$) by its realization as a tensor power of $\hat{\rho}^+$. By its construction, it obeys (31).

Next, we note that when $R$ is invertible there is a similar construction for $\mathcal{R}^{-1}$ to that for $\mathcal{R}$ above. Here $\mathcal{R}^{-1}$ obeys equations similar to (31) but with its second input multiplicative and its first input antimultiplicative. We use $R^{-1}$ in the role of $R$, for example, $\mathcal{R}^{-1}(t \otimes a) = \hat{\rho}^- (a)$ extends as a representation, while $\mathcal{R}^{-1}(a \otimes t)$ extends as an antirepresentation. The steps are entirely analogous to those above, and we arrive at the partition function $Z_{R^{-1}}$. We have to show that $\mathcal{R}, \mathcal{R}^{-1}$ are inverse in the convolution algebra of maps $A(R) \otimes A(R) \to k$. Explicitly, we need,

$$\mathcal{R}(t^I_A \otimes t^K_B) \mathcal{R}^{-1}(t^A_J \otimes t^B_L) = \delta^I_J \delta^K_L, \text{ i.e. } Z_{R}(t^I_A \otimes t^K_B) Z_{R^{-1}}(t^A_J \otimes t^B_L) = \delta^I_J \delta^K_L$$

and similarly on the other side. Writing the arrays in our compact notation we have

$$R_{1M+N} \cdots R_{1M+1}$$

$$\cdots$$

$$R_{MM+N} \cdots R_{MM+1} R_{MM+1}^{-1} \cdots R_{MM+2}^{-1} \cdots R_{MM+N}^{-1}$$

$$R_{M-1M+N} \cdots R_{M-1M+1}$$

$$\cdots$$

Here the copies of $M_n$ numbered 1 · · · $M$ on the left correspond to the index $I$, on the right to $J$ (they occur reversed). The copies of $M_n$ numbered $M+1$ · · · $M+N$ correspond on the top to $K$ (they occur reversed) and on the bottom to $L$. In between they are matrix-multiplied as indicated, corresponding to the sum over $A, B$. The overlapping line here collapses after cancellation of inverses ending in id in the copy of $M_n$ numbered $M$, and results in a similar picture with one row less. Repeating this, the whole thing collapses to the identity in all the copies of $M_n$.

Finally, we check (32), which now takes the form

$$t^K_B t^I_A Z_{R}(t^B_A \otimes t^A_J) = Z_{R}(t^K_B \otimes t^I_A) t^A_J t^K_B.$$  

(40)
In the compact notation we compute
\[ t_{M+N} \cdots t_{M+1} t_M R_{1M+1} \cdots R_{1M+N} \]
\[ \cdots \cdots \]
\[ R_{MM+1} \cdots R_{MM+N} \]
\[ = R_{1M+1} \cdots R_{1M+N} t_1 t_{M+N} \cdots t_{M+1} t_{M+2} \cdots R_{2M+1} \cdots R_{2M+N} \]
\[ \cdots \cdots \]
\[ R_{MM+1} \cdots \cdots R_{MM+N} t_1 t_{M+N} \cdots t_{M+1}. \]

Here the copies of \(M_n\) numbered 1, \(\ldots\), \(M\) on the left correspond to the index \(I\), and the copies of \(M_n\) numbered \(M+1, \ldots, M+N\) correspond on the top to \(\bar{K}\) (they occur reversed), etc. The first equality makes repeated use of the relations (37) of \(A(R)\) to give
\[ t_{M+N} \cdots t_{M+1} t_1 R_{1M+1} \cdots R_{1M+N} = R_{1M+1} \cdots R_{1M+N} t_1 t_{M+N} \cdots t_{M+1}. \]

The \(t_2 \cdots t_M\) move past the \(R_{1M+1} \cdots R_{1M+N}\) freely since they live in different matrix spaces. This argument for the first equality is then applied to move \(t_{M+N} \cdots t_{M+1}\), and so on. The arguments in this proof may appear complicated, but in fact this kind of repeated matrix multiplication (multiplication of entire rows or columns of matrices) is quite routine in the context of exactly solvable statistical mechanics (where the QYBE originated).

Let us note that while the algebra relations (37) of \(A(R)\) do not depend on the normalization of \(R\), the dual quasitriangular structure does. The elements \(t^{i_1}_{j_1} \cdots t^{i_M}_{j_M}\) of \(A(R)\) have a well-defined degree \(|t^{i_1}_{j_1} \cdots t^{i_M}_{j_M}| = M\) (the algebra is graded), and if \(R' = \lambda R\) is a non-zero rescaling of our solution \(R\) then the corresponding dual quasitriangular structure is changed to
\[ \mathcal{R}'(a \otimes b) = \lambda^{|a||b|} \mathcal{R}(a \otimes b) \] (41)
on homogeneous elements. This is evident from the expression in terms of \(Z_R\) that we have obtained in the last proposition. Finally, in view of the reasons that we passed to the dual setting it is obvious that

**Theorem 1.16** Let \((A, \mathcal{R})\) be a dual quasitriangular bialgebra. Then \(\mathcal{M}^A\) the category of right \(A\)-comodules is braided. In the Hopf algebra case the finite-dimensional comodules are rigid,
\[ \Psi_{V,W}(v \otimes w) = \sum w^{(i)} \otimes v^{(i)} \mathcal{R}(v^{(2)} \otimes w^{(2)}), \quad \beta_{V^*}(f) = (f \otimes S) \circ \beta_V \]
where \(\beta_{V^*}(f) \in V^* \otimes A\) is given as a map \(V \to A\).
Proof  This is an entirely trivial dualization of the proof of Theorem 1.10 above. For example, $\Psi$ is an intertwiner because  

$$
\Psi_{V,W}(v \otimes w) \mapsto \sum w^{(1)}(1) \otimes v^{(1)}(1) \otimes w^{(1)}(2) v^{(2)}(2) R(v^{(2)}(2) \otimes w^{(2)}(2)) \\
= \sum w^{(1)}(1) \otimes v^{(1)}(1) \otimes w^{(2)}(1) v^{(2)}(1) R(v^{(2)}(2) \otimes w^{(2)}(2)) \\
= \sum w^{(1)}(1) \otimes v^{(1)}(1) \otimes R(v^{(2)}(1) \otimes w^{(2)}(1)) v^{(2)}(2) w^{(2)}(2) \\
= \sum w^{(1)}(1) \otimes R(v^{(2)}(1) \otimes w^{(2)}(1)) v^{(2)}(2) w^{(2)}(2)
$$

where the arrow is the tensor product $W \otimes V$ coaction. We used (32). The result is $\Psi$ applied to the result of the tensor product coaction $V \otimes W$. The hexagons (3) correspond in a similarly trivial way to (31). $\square$

Note that $C(V, R)$ in Example 1.1 forms a subcategory of $M^A(R)$. Moreover, in the dualizable case there is a Hopf algebra $GL(R) \supset A(R)$ such that $C(V, V^*, R)$ is a subcategory of $M^{GL(R)}$. The relevant coactions are

$$
e_i \mapsto e_j \otimes t^i_j, \quad f^i \mapsto f^j \otimes S t^i_j
$$

and we recover from Theorem 1.16 the braidings quoted. Also, in [36][43][48][49] we regarded this proposition as a starting point and set out to prove something further, namely its converse. If $M^A$ is braided then $A$ has induced on it by Tannaka-Krein reconstruction a dual quasitriangular structure. We will see this in Section 3. [43] generalised the Tannaka-Krein theorem to the setting of dual quasi-Hopf algebras (associative up to an isomorphism cf[13]) while [48][49] generalised it to the braided setting. It is more or less the sine qua non for the work here.

2  Braided Tensor Product Algebra and Braided Hopf Algebras

So far we have described braided monoidal or quasitensor categories and ways to obtain them. Now we begin our main task and study algebraic structures living in such categories. For this we use the diagrammatic notation of Section 1.1. Detailed knowledge of quantum groups etc is not required in this section.

The idea of an algebra $B$ in a braided category is just the usual one. Thus there should be product and unit morphisms

$$
\cdot : B \otimes B \to B, \quad \eta : \mathbb{1} \to B
$$

obeying the usual associativity and unity axioms but now as morphisms in the category. Note that the term ‘algebra’ is being used loosely since we have not discussed direct sums and linearity under a field or ring. These notions are perfectly compatible with what follows but do not play any particular role in our general constructions.

The fundamental lemma for the theory we need is the generalization to this setting of the usual $\mathbb{Z}_2$-graded or super-tensor product of superalgebras:

**Lemma 2.1** [44][40] Let $B, C$ be two algebras in a braided category. There is a braided tensor product algebra $B \otimes C$, also living in the braided category. It has product $(\cdot_B \otimes \cdot_C) \circ (\text{id} \otimes \Psi_{C,B} \otimes \text{id})$ and tensor product unit morphism.
Proof. We repeat here the diagrammatic proof. The box is the braided tensor product multiplication,

\[
\begin{align*}
B & \otimes C \quad B & \otimes C \quad B & \otimes C \\
\Delta & \quad \Delta & \quad \Delta \\
B & \otimes C \\
\end{align*}
\]

The first step uses functoriality as in (9) to pull the product morphism through the braid crossing. The second equality uses associativity of the products in \(B, C\) and the third equality uses functoriality again in reverse. The product is manifestly a morphism in the category because it is built out of morphisms. Finally, the unit is the tensor product one because the braiding is trivial on \(1\).

In the concrete case the braided tensor product is generated by \(B = B \otimes 1\) and \(C = 1 \otimes C\) and an exchange law between the two factors given by \(\Psi\). This is because \((b \otimes 1)(1 \otimes c) = (b \otimes c)\) while \((1 \otimes c)(b \otimes 1) = \Psi(c \otimes b)\). Another notation is to label the elements of the second copy in the braided tensor product by \('\). Thus \(b \equiv (b \otimes 1)\) and \(c' \equiv (1 \otimes c)\). Then if \(\Psi(c \otimes b) = \sum b_k \otimes c_k\) say, we have the braided-tensor product relations

\[
c' b \equiv (1 \otimes c)(b \otimes 1) = \Psi(c \otimes b) = \sum b_k \otimes c_k \equiv \sum b_k c'_k.
\]

This makes clear why the lemma generalizes the notion of \(\mathbb{Z}_2\)-graded or super-tensor product. Note also that there is an equally good opposite braided tensor product with the inverse braid crossing in Lemma 2.1. This is simply the braided tensor product algebra constructed in the mirror-reversed category \(\bar{C}\) but with the result viewed in our original category.

2.1 Braided Hopf Algebras

Armed with the braided tensor product of algebras in a braided category we can formulate the notion of Hopf algebra.

Definition 2.2. A Hopf algebra in a braided category or braided-Hopf algebra is \((B, \Delta, \epsilon, S)\) where \(B\) is an algebra in the category and \(\Delta : B \to B \otimes B\), \(\epsilon : B \to 1\) are algebra homomorphisms where \(B \otimes B\) has the braided tensor product algebra structure. In addition, \(\Delta, \epsilon\) obey the usual coassociativity and counity axioms to form a coalgebra in the category, and \(S : B \to B\) obeys the usual axioms of an antipode. If there is no antipode then we speak of a braided-bialgebra or bialgebra in a braided category.

In diagrammatic form the algebra homomorphism and braided-antipode axioms read

\[
\begin{align*}
\Delta & = \Delta \\
\epsilon & = \epsilon \\
S & = S
\end{align*}
\]

(45)
One then proceeds to develop the usual elementary theory for these Hopf algebras. For example, recall that the usual antipode is an antialgebra map.

**Lemma 2.3** For a braided-Hopf algebra $B$, the braided-antipode obeys $S(b \cdot c) = \cdot \Psi(Bb \otimes Sc)$ and $S(1) = 1$, or more abstractly, $S \circ \cdot = \cdot \circ \Psi(B, B \circ (S \otimes S))$ and $S \circ \eta = \eta$.

**Proof** In diagrammatic form the proof is

In the first two equalities we have grafted on some circles containing the antipode, knowing they are trivial from (45). We then use the coherence theorem to lift the second $S$ over to the left, and associativity and coassociativity to reorganise the branches. The fifth equality uses the axioms (45) for $\Delta$.

Here we want to mention a powerful *input-output symmetry* of these axioms. Namely, turn the pages of this book upside down and look again at these diagrams. The axioms of a braided-Hopf algebra (45) are unchanged except that the roles of product/coproduct and unit/counit morphisms are interchanged. The proof of Lemma 2.1 becomes the proof of a new lemma expressing coassociativity of the braided-tensor product of two coalgebras. Meanwhile the proof of Lemma 2.3 reads as the proof of a new lemma that the braided-antipode is a braided-anti-coalgebra map.

This applies therefore to all results that we prove about bialgebras or Hopf algebras in braided categories provided all notions are suitably turned upside-down. This is completely rigorous and nothing to do with finite-dimensionality or individual dual objects. In addition, there is a *left-right symmetry* of the axioms consisting of reflecting in mirror about a vertical axis combined with reversal of all braid crossings. These symmetries of the axioms can be taken together so that we obtain precisely four theorems for the price of one when we use the diagrammatic method.

An endemic problem for those working in Hopf algebras is that every time something is proven one has to laboriously figure out its input-output-reversed version or its version with opposite left-right conventions. This problem is entirely solved by reflecting in a mirror or turning up-side-down.

### 2.2 Dual Braided Hopf Algebras

Suppose now that the category has dual objects (is rigid) in the sense explained in Section 1.1. In this case the input-output symmetry of the axioms of a Hopf algebra becomes realised concretely as the construction of a dual Hopf algebra.

**Proposition 2.4** If $B$ is a braided-Hopf algebra, then its left-dual $B^*$ is also a braided-Hopf
algebra with product, coproduct, antipode, counit and unit given by

\[
\eta \quad \Delta \quad \varepsilon
\]

**Proof**  Associativity and coassociativity follow at once from coassociativity and associativity of \( B \). Their crucial compatibility property comes out as

\[
\Delta \quad \varepsilon
\]

where we use the double-bend axiom (11) for dual objects. The antipode property comes out just as easily. □

We see that in diagrammatic form the dual bialgebra or Hopf algebra is obtained by rotating the desired structure map in an anticlockwise motion and without cutting any of the attaching strings. For right duals the motion should be clockwise. Let us stress that this dual-Hopf algebra construction of an individual object should not be confused with the rather more powerful input-output symmetry for the axioms introduced above.

### 2.3 Braided Actions and Coactions

Another routine construction is the notion of module and its input-output-reversed notion of comodule. These are just the obvious ones but now as morphisms in the category. Let us check that the tensor product of modules of a bialgebra is a module. If \( V, \alpha_V : B \otimes V \to V \) and \( W, \alpha_W : B \otimes W \to W \) are two left modules then

\[
\alpha_V \otimes \alpha_W : B \otimes (V \otimes W) \to V \otimes W
\]

is the definition (in the box) of tensor product module and proof that it is indeed a module. The first equality is the homomorphism property of \( \Delta \). Likewise for right modules by left-right reflecting the proof in a mirror and also reversing all braid crossings.
As for usual Hopf algebras, the left-right symmetry can be concretely realised via the antipode. Thus if \( V, \alpha^R : V \otimes B \to V \) is a right module then

\[
\begin{align*}
V & \otimes B \to V \\
\alpha & \quad \tilde{\alpha} \\
V & \to V
\end{align*}
\]

shows the construction of the corresponding left module. This is shown in the box.

Also if the left-module \( V \) has a left-dual \( V^* \) then this becomes a right-module with

\[
\begin{align*}
V^* & \otimes B \to V^* \\
\alpha & \quad \alpha^* \\
V^* & \to V^*
\end{align*}
\]

Combining these two constructions we conclude (with the obvious definition of intertwiners or morphisms between braided modules):

**Proposition 2.5** \([49]\) Let \( B \) be a bialgebra in the braided category \( C \). Then the category \( _B C \) of braided left-modules is a monoidal category. If \( B \) is a Hopf algebra in \( C \) and \( C \) is rigid then \( _B C \) is rigid.

**Proof** For the second part we feed the result of (48) into (47). A more traditional-style proof with commuting diagrams is in \([49]\) for comparison. It should convince the reader of the power of the diagrammatic method. \( \square \)

Naturally, a braided left \( B \)-module algebra is by definition an algebra living in the category \( _B C \). This means an algebra \( C \) such that

\[
\begin{align*}
B & \otimes C \to C \\
\Delta & \quad \alpha \\
C & \to C
\end{align*}
\]

Likewise for other constructions familiar for actions of bialgebras or Hopf algebras. For example, a coalgebra \( C \) in the category \( _B C \) is a coalgebra such that

\[
\begin{align*}
B & \otimes C \to C \\
\Delta & \quad \alpha \\
C & \to C
\end{align*}
\]

For the right-handed theory reflect the above in a mirror and reverse all braid crossings. This gives the notion of right \( B \)-module algebras etc. Next, by turning the pages of this book upside down we have all the corresponding results for comodules in place of modules. Thus the category \( C^B \) of right-comodules had a tensor product and in the Hopf algebra and rigid case is also rigid. Likewise for left comodules.
Proposition 2.6 [46] If \( C \) is a left \( B \)-module algebra then there is a braided cross product or semidirect product algebra \( C \triangleright B \) built on the object \( C \otimes B \). Likewise for right \( B \)-module algebras, left \( B \)-module coalgebras and right \( B \)-comodule coalgebras. The semidirect (co)product maps are

\[
\begin{array}{c}
\text{Cross Product} \\
\text{by left action}
\end{array}
\quad
\begin{array}{c}
\text{Cross Product} \\
\text{by right action}
\end{array}
\quad
\begin{array}{c}
\text{Cross Coproduct} \\
\text{by left coaction}
\end{array}
\quad
\begin{array}{c}
\text{Cross Coproduct} \\
\text{by right coaction}
\end{array}
\]

Proof We only need to prove one of these by our diagrammatic means to conclude all four. Full details are in [46]. \( \square \)

Example 2.7 [51, Appendix] Let \( B \) be a braided-Hopf algebra. Then \( B \) is a left \( B \)-module algebra by the braided adjoint action \( \cdot \circ (\id \otimes \Psi_{B,B}) \circ (\id \otimes S \otimes \id) \circ (\Delta \otimes \id) \).

Example 2.8 [58] Let \( B \) be a braided-Hopf algebra with left dual \( B^* \). Then \( B \) is a right \( B^* \)-module algebra by a braided right regular action \( \text{ev}_B \circ (\id \otimes S \otimes \id) \circ (\id \otimes \Delta) \circ \Psi_{B,B^*} \).

The verification of these examples is a nice demonstration of the techniques above. In diagrammatic form they read

\[
\text{Ad} = \quad \text{Reg} = \qquad (51)
\]

The adjoint action leads to a notion of braided Lie algebra [58] among other applications, while the right regular action corresponds to the action of fundamental vector fields. It can also be used to construct a braided Weyl algebra cf [57].

2.4 Braided-(Co)-Commutativity

Next we come to the question of commutativity or cocommutativity in a braided category. Again, we only have to work with one of these and turn our diagrams up-side-down for the other. The main problem is that the naive opposite-coproduct

\[
\tilde{\Delta} = \Psi_{B,B}^{-1} \circ \Delta
\]

(52)
does not make \( B \) into a bialgebra in our original braided category \( \mathcal{C} \), but rather gives a bialgebra in the mirror-reversed category \( \mathcal{C}^{\text{op}} \). Thus there is a notion of opposite bialgebra.
(and if \( B \) is a Hopf algebra with invertible antipode then \( S^{-1} \) provides an antipode) but it forces us to leave the category.

Hence there is no way to consider bialgebras or Hopf algebras that are cocommutative in the sense that they coincide with their opposite. There is so far no intrinsic notion of braided-cocommutative Hopf algebras for this reason. On the other hand we have introduced in [44] the notion of a braided-cocommutativity with respect to a module. This is a property of a module on which \( B \) acts.

**Definition 2.9** [44] A braided left module \((V, \alpha_V)\) is braided-cocommutative (or \( B \) is braided-cocommutative with respect to \( V \)) if

\[
\Delta_B V = \alpha_V \Delta_B V
\]

**Braided – Cocommutativity**

To understand this notion suppose that the category is symmetric not strictly braided. In this case \( \Psi^2 = \text{id} \) and we see that the condition is implied by \( \bar{\Delta} = \Delta \). But in a general braided category we cannot disentangle \( V \) and must work with this weaker notion. Moreover, as far as such modules are concerned the bialgebra \( B \) has all the usual representation-theoretic features of usual cocommutative Hopf algebras. One of these is that their tensor product is symmetric under the usual transposition of the underlying vector spaces of modules. The parallel of this is

**Proposition 2.10** [45] Let \( B \) be a bialgebra in a braided category and define \( \mathcal{O}(B) \subset B C \) the subcategory of braided-cocommutative modules. Then \( \mathcal{O}(B) \) is closed under \( \otimes \). Moreover, the tensor product in \( \mathcal{O} \) is braided with braiding induced by the braiding in \( C \),

\[
\begin{align*}
\Delta_B V W &= \alpha_V \Delta_B V W \\
\alpha_V &\text{ Braided – Commutativity of Product of Modules}
\end{align*}
\]

**Proof** The first part is given in detail in [45] in a slightly more general context. The second part follows from Definition 2.9 by adding an action on \( W \) to both sides. \( \square \)

The trivial representation is always braided-cocommutative. In many examples the adjoint representation in Example 2.7 is also braided-cocommutative. In this case one can formulate properties like those of an enveloping algebra of a braided-Lie algebra [58]. One can formally define a *braided group* as a pair consisting of a Hopf algebra in a braided category and a class of braided-cocommutative modules. This turns out to be a useful notion because in many situations it is only this weak notion of cocommutativity that is needed. For example

**Theorem 2.11** [47] Let \( B, C \) be Hopf algebras in a braided category and \( C \) a braided-cocommutative-\( B \)-module algebra and coalgebra. Then \( C \triangleright \triangleleft B \) forms a Hopf algebra in the braided category with the braided tensor product coalgebra structure.
2.5 Quantum-Braided Groups

We can go further and consider bialgebras that are quasi-cocommutative in some sense, analogous to the idea of a quasitriangular bialgebra in Section 1.2. To do this we require a second coproduct which we denote $\Delta^{\text{op}} : B \to B \otimes B$ also making $B$ into a bialgebra. In this case we have the notion of a braided $B$-module with respect to which $\Delta^{\text{op}}$ behaves like an opposite coproduct. This is just as in Definition 2.9 but with the left hand $\Delta$ replaced by $\Delta^{\text{op}}$. The class of such $B$-modules is denoted $\mathcal{O}(B, \Delta^{\text{op}})$.

The second ingredient that we need is a quasitriangular structure which is understood now as a convolution-invertible morphism $\mathcal{R} : 1 \to B \otimes B$. With these ingredients the analogue of Definition 1.5 is

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (A) at (0,0) {$B$};
\node (B) at (2,0) {$B$};
\node (C) at (4,0) {$B$};
\node (D) at (6,0) {$B$};
\node (E) at (8,0) {$B$};
\node (F) at (10,0) {$B$};
\node (G) at (12,0) {$B$};
\node (H) at (14,0) {$B$};
\node (I) at (16,0) {$B$};
\node (J) at (18,0) {$B$};
\node (K) at (20,0) {$B$};
\node (L) at (22,0) {$B$};
\node (M) at (24,0) {$B$};
\node (N) at (26,0) {$B$};
\node (O) at (28,0) {$B$};
\node (P) at (30,0) {$B$};
\node (Q) at (32,0) {$B$};
\node (R) at (34,0) {$B$};
\node (S) at (36,0) {$B$};
\node (T) at (38,0) {$B$};
\node (U) at (40,0) {$B$};
\node (V) at (42,0) {$B$};
\node (W) at (44,0) {$B$};
\node (X) at (46,0) {$B$};
\node (Y) at (48,0) {$B$};
\node (Z) at (50,0) {$B$};
\node (AA) at (52,0) {$B$};
\node (BB) at (54,0) {$B$};
\node (CC) at (56,0) {$B$};
\node (DD) at (58,0) {$B$};
\node (EE) at (60,0) {$B$};
\node (FF) at (62,0) {$B$};
\node (GG) at (64,0) {$B$};
\node (HH) at (66,0) {$B$};
\node (II) at (68,0) {$B$};
\node (JJ) at (70,0) {$B$};
\node (KK) at (72,0) {$B$};
\node (LL) at (74,0) {$B$};
\node (MM) at (76,0) {$B$};
\node (NN) at (78,0) {$B$};
\node (OO) at (80,0) {$B$};
\node (PP) at (82,0) {$B$};
\node (QQ) at (84,0) {$B$};
\node (RR) at (86,0) {$B$};
\node (SS) at (88,0) {$B$};
\node (TT) at (90,0) {$B$};
\node (UU) at (92,0) {$B$};
\node (VV) at (94,0) {$B$};
\node (WW) at (96,0) {$B$};
\node (XX) at (98,0) {$B$};
\node (YY) at (100,0) {$B$};
\node (ZZ) at (102,0) {$B$};
\node (AAA) at (104,0) {$B$};
\node (BBB) at (106,0) {$B$};
\node (CCC) at (108,0) {$B$};
\node (DDD) at (110,0) {$B$};
\node (EEE) at (112,0) {$B$};
\node (FFF) at (114,0) {$B$};
\node (GGG) at (116,0) {$B$};
\node (HHH) at (118,0) {$B$};
\node (III) at (120,0) {$B$};
\node (JJJ) at (122,0) {$B$};
\node (KKK) at (124,0) {$B$};
\node (LLL) at (126,0) {$B$};
\node (MMM) at (128,0) {$B$};
\node (NNN) at (130,0) {$B$};
\node (OOO) at (132,0) {$B$};
\node (PPP) at (134,0) {$B$};
\node (QQQ) at (136,0) {$B$};
\node (RRR) at (138,0) {$B$};
\node (SSS) at (140,0) {$B$};
\node (TTT) at (142,0) {$B$};
\node (UUU) at (144,0) {$B$};
\node (VVV) at (146,0) {$B$};
\node (WWW) at (148,0) {$B$};
\node (XXX) at (150,0) {$B$};
\node (YYY) at (152,0) {$B$};
\node (ZZZ) at (154,0) {$B$};
\node (AAAA) at (156,0) {$B$};
\node (BBBB) at (158,0) {$B$};
\node (CCCC) at (160,0) {$B$};
\node (DDDD) at (162,0) {$B$};
\node (EEEE) at (164,0) {$B$};
\node (FFFF) at (166,0) {$B$};
\node (GGGG) at (168,0) {$B$};
\node (HHHH) at (170,0) {$B$};\end{tikzpicture}
\end{array}
\end{align*}
\]

The braided analogue of Theorem 1.10 is then

**Theorem 2.12** [45] Let $(B, \Delta^{\text{op}}, \mathcal{R})$ be a quasitriangular bialgebra in a braided category. Then $\mathcal{O}(B, \Delta^{\text{op}}) \subset \mathcal{B} \mathcal{C}$ is a braided monoidal category with braiding $\Psi_{V,W}^{O} = \Psi_{V,W} \circ (\alpha_{V} \otimes \alpha_{W}) \circ \Psi_{B,V} \circ (\mathcal{R} \otimes \text{id})$.

**Proof** Diagrammatic proofs are in [45, Sec. 3]. □

The dual theory with comodules and an opposite product is developed in [48, 49]. We mention here only that turning (53) up-side-down and then setting the category to be the usual one of vector spaces returns not the axioms of a dual quasitriangular structure as in Section 1.3 but its inverse. This reversal is due to the fact that the categorical dualization in Section 2.2 yields in the vector space category the opposite coproduct and product to the usual dualization.

3 Reconstruction Theorem

In this section we give a construction (not the only one) for bialgebras and Hopf algebras in braided categories. There is such a bialgebra associated to a pair of braided categories $\mathcal{C} \to \mathcal{V}$ or even to a single braided category $\mathcal{C}$. The idea behind this is the theory of Tannaka-Krein reconstruction generalised to the braided setting.

The Tannaka-Krein reconstruction theorems should be viewed as a generalization of the simple notion of Fourier Transform. The idea is that the right notion of representation of an algebraic structure should itself have enough structure to reconstruct the original algebraic object. On the other hand many constructions may appear very simple in terms of the representation theory and highly non-trivial in terms of the original algebraic object, and vice versa.

In the present setting we know that quantum groups give rise to braided categories as their representations, while conversely we will see that the representations or endomorphisms of a category $\mathcal{C}$ in a category $\mathcal{V}$ gives rise to a quantum group in $\mathcal{V}$.
### 3.1 Usual Tannaka-Krein Theorem

The usual Tannaka-Krein theorem for Hopf algebras says that a monoidal category $\mathcal{C}$ equipped with a functor to the category of vector spaces (i.e., whose objects can be identified in a strict way with vector spaces) is equivalent to that of the comodules of a certain bialgebra $A$ reconstructed from $\mathcal{C}$. All our categories $\mathcal{C}$ are assumed equivalent to small ones.

**Theorem 3.1** Let $F : \mathcal{C} \rightarrow \text{Vec}$ be a monoidal functor to the category of vector spaces with finite-dimensional image. Then there exists a bialgebra $A$ uniquely determined as universal with the property that $F$ factors through $\mathcal{M}^A$. If $\mathcal{C}$ is braided then $A$ is dual quasitriangular. If $\mathcal{C}$ is rigid then $A$ has an antipode.

**Proof** We defer this to Theorem 3.11 below. Just set $\mathcal{V} = \text{Vec}$ there. □

An early treatment of the bialgebra case is in [16]. See also [3]. The part concerning the antipode was shown in [70]. That a symmetric category gives a dual-triangular structure was pointed out in the modules setting in [11]. See also [29]. It is a trivial step to go from there to the braided case in which case the result is dual quasitriangular. This has been done by the author, while at the same time (in order to say something new) generalising in two directions. One is to the quasi-Hopf algebra setting [33] [43] and the other to the braided-Hopf algebra setting [39] [49].

This theorem tells us that (dual)quasitriangular Hopf algebras are rather more prevalent in mathematics (and physics) than we might have otherwise suspected. It also gives us a useful perspective on any Hopf algebra construction, by allowing us to go backwards and forwards between representations and the algebra itself. For example, if we are already given a bialgebra $A$ then coming out of the reconstruction theorem one has associated to any subcategory

$$\mathcal{O} \subset \mathcal{M}^A$$

closed under tensor product, a sub-bialgebra

$$A_\mathcal{O} = \bigcup_{(V, \beta_V) \in \mathcal{O}} \text{image} (\beta_V) \subset A,$$

$$\text{image} (\beta_V) = \{(f \otimes \text{id}) \circ \beta_V (v); v \in V, f \in V^*\}.$$  \hspace{1cm} (55)

If the sub-category is braided then $A_\mathcal{O}$ is dual quasitriangular etc. So this is a concrete form of the reconstruction theorem in the case where $\mathcal{O}$ is already in the context of a bialgebra.

For example if $A = A(R)$ and $\mathcal{O} = \mathcal{C}(V, R)$ in Section 1 then $A_\mathcal{O} = A(R)$ again. This is because the image of tensor powers of $V$ for the coaction in (12) is clearly any monomial in the generators $t$ of $A(R)$. Hence in this case the subcategory reconstructs all of $A(R)$. The result is due to Lyubashenko though the proof in [24] is different (and stated in the triangular case).

For another example let $A$ be a bialgebra and $\mathcal{O}$ the category of comodules which are commutative in the sense $\sum v^{(i)} \otimes a v^{(2)} = \sum v^{(i)} \otimes v^{(2)} a$ for all $a \in A$ and $v$ in the comodule. Then $A_\mathcal{O}$ is a bialgebra contained in the center of $A$. This is therefore a canonically associated ‘bialgebra centre’ construction.

There are analogous results to these for modules. At the level of Theorem 3.1 the module theory is less powerful only if one functors (as usual) into familiar finite-dimensional vector spaces.
3.2 Braided Reconstruction Theorem

In this section we come to the fully-fledged braided Tannaka-Krein-type reconstruction theorem. We follow for pedagogical reasons the original module version [39] [45], mainly because the comodule version was already given in complete detail in [49] and we do not want to repeat it. Also, we give here for the first time a fully diagrammatic proof.

Throughout this section we fix $F : \mathcal{C} \to \mathcal{V}$ a monoidal functor between monoidal categories. At least $\mathcal{V}$ should be braided. In this case there is an induced functor $\mathcal{V} \mapsto \text{Nat}(\mathcal{V} \otimes F, F)$. We suppose that this functor is representable. So there is an object $B \in \mathcal{V}$ such that $\text{Nat}(\mathcal{V} \otimes F, F) \cong \text{Hom}_{\mathcal{V}}(\mathcal{V}, B)$ by functorial bijections. Let $\{\alpha_X : B \otimes F(X) \to F(X); X \in \mathcal{C}\}$ be the natural transformation corresponding to the identity morphism $B \to B$. Then using $\alpha$ and the braiding we get an induced map

$$\text{Hom}_{\mathcal{V}}(\mathcal{V}, B \otimes^n) \to \text{Nat}(\mathcal{V} \otimes F^n, F^n) \quad (56)$$

and we assume that these are likewise bijections. This is the representability assumption for modules and we assume it in what follows.

**Theorem 3.2** [45] Let $F : \mathcal{C} \to \mathcal{V}$ obey the representability assumption for modules. Then $B$ is a bialgebra in $\mathcal{V}$, uniquely determined as universal with the property that $F$ factors through $B \mathcal{V}$. If $\mathcal{C}$ is braided then $B$ is quasitriangular in the braided category with $R$ given by the ratio of the braidings in $\mathcal{C}$ and $\mathcal{V}$. If $\mathcal{C}$ is rigid then $B$ has a braided-antipode.

We will give the proof in diagrammatic form. The bijections (57) and the structure maps in the theorem are characterized by

Here the assumption that $F$ is monoidal means that there are functorial isomorphisms $F(X \otimes Y) \cong F(X) \otimes F(Y)$ and in the rigid case $F(X^*) \cong F(X)^*$. The latter follow from the uniqueness of duals up to isomorphism (for example one can define $F(X)^* = F(X^*)$ etc. and any other dual is isomorphic). These isomorphisms are used freely and suppressed in the notation. The solid node $\alpha_{X \otimes Y}$ is $\alpha$ on the composite object $X \otimes Y$ but viewed via the first of these isomorphisms as a morphism $B \otimes F(X) \otimes F(Y) \to F(X) \otimes F(Y)$. Similarly for $\alpha_X$. In this way all diagrams refer to morphisms in $\mathcal{V}$. The unit $1 \to B$ corresponds to the identity natural transformation and the counit to $\alpha_1$. Their proofs are suppressed.
**Lemma 3.3** The product on \( B \) defined in (57) is associative.

**Proof** We use the definition of \( \cdot \) twice in terms of its corresponding natural transformations and then in reverse

\[
\begin{array}{c}
\xymatrix{
B \otimes B \otimes B 
& F(X) \ar[rd]_{\alpha_X} \ar[ld]^{\alpha_X} \\
& F(X) \\
& F(X)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \otimes B \otimes B 
& F(X) \ar[rd]_{\alpha_X} \ar[ld]^{\alpha_X} \\
& F(X) \\
& F(X)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \otimes B \otimes B 
& F(X) \ar[rd]_{\alpha_X} \ar[ld]^{\alpha_X} \\
& F(X) \\
& F(X)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \otimes B \otimes B 
& F(X) \ar[rd]_{\alpha_X} \ar[ld]^{\alpha_X} \\
& F(X) \\
& F(X)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \otimes B \otimes B 
& F(X) \ar[rd]_{\alpha_X} \ar[ld]^{\alpha_X} \\
& F(X) \\
& F(X)
}
\end{array}
\]

Hence the natural transformations corresponding to the two morphisms \( B \otimes B \otimes B \to B \) coincide and we have an algebra in the category. \( \square \)

**Lemma 3.4** The coproduct \( \Delta \) on \( B \) defined in (57) is coassociative.

**Proof** We use the definition of \( \Delta \) twice and then in reverse, using in the middle that \( F \) is monoidal and hence compatible with the (suppressed) associativity in the two categories

\[
\begin{array}{c}
\xymatrix{
B \ar[r]_{\Delta} & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y) \otimes F(Z)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \ar[r]_{\Delta} & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y) \otimes F(Z)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \ar[r]_{\Delta} & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y) \otimes F(Z)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \ar[r]_{\Delta} & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y) \otimes F(Z)
}
\end{array}
= \begin{array}{c}
\xymatrix{
B \ar[r]_{\Delta} & F(X) \otimes F(Y) \otimes F(Z) \\
& F(X) \otimes F(Y) \otimes F(Z)
}
\end{array}
\]

The key step is the third equality which follows from functoriality of \( \alpha \) under the associativity morphism \( X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \) and that \( F \) is monoidal. If \( F \) is not monoidal but merely multiplicative one has here a quasi-associative coproduct as explained in \([33][34]\). \( \square \)

**Proposition 3.5** The product and coproduct in the last two lemmas fit together to form a bialgebra in \( \mathcal{V} \).
Proof. We use the definitions of $\cdot$ and $\Delta$

\[
\begin{align*}
B & \B F(X) \ F(Y) \ \Delta \\
\alpha_X & \alpha_Y \\
F(X) & \ F(Y)
\end{align*}
\]

\[
\begin{align*}
\Delta & \Delta \\
\alpha_X & \alpha_Y \\
F(X) & \ F(Y)
\end{align*}
\]

\[
\begin{align*}
B & \B F(X \otimes Y) \\
\alpha_X & \alpha_Y \\
F(X \otimes Y)
\end{align*}
\]

\[
\begin{align*}
B & \B F(X \otimes Y) \\
\alpha_X & \alpha_Y \\
F(X \otimes Y)
\end{align*}
\]

Lemma 3.6 The second coproduct $\Delta^\text{op}$ on $B$ defined in (57) is coassociative.

Proof. This is similar to the proof of Lemma 3.4

\[
\begin{align*}
\Delta^\text{op} & B \\
\alpha_X & \alpha_Y \\
F(X) & \ F(Y) \ F(Z)
\end{align*}
\]

\[
\begin{align*}
\Delta^\text{op} & B \\
\alpha_X & \alpha_Y \\
F(X) & \ F(Y) \ F(Z)
\end{align*}
\]

\[
\begin{align*}
\Delta^\text{op} & B \\
\alpha_X & \alpha_Y \\
F(X) & \ F(Y) \ F(Z)
\end{align*}
\]

Proposition 3.7 The product and the second coproduct fit together to form a second bialgebra in $\mathcal{V}$.
Proof  This is similar to the proof of Proposition 3.5

\[
\begin{align*}
\Delta & \quad \Delta^\text{op} \\
\alpha & \quad \alpha^\text{op} \\
F(X) & \quad F(Y) \\
\end{align*}
\]

\[
\begin{align*}
B & \quad B \\
F(X) & \quad F(Y) \\
\end{align*}
\]

\[
\begin{align*}
\alpha & \quad \alpha \\
X & \quad Y \\
\end{align*}
\]

\[
\begin{align*}
F(X) & \quad F(Y) \\
\end{align*}
\]

\[
\begin{align*}
\eta \circ \epsilon & \quad \\ \\
\end{align*}
\]

Proposition 3.8  If \( \mathcal{C} \) is rigid then \( S \) defined in (57) is an antipode for the coproduct \( \Delta \).

Proof  The first, second and fourth equalities are the definitions of \( \cdot, S, \Delta \). The fifth uses functoriality of \( \alpha \) under the evaluation \( X^* \otimes X \to 1 \)

\[
\begin{align*}
\begin{array}{cccc}
\Delta & \quad \Delta^\text{op} & \quad \Delta & \quad \Delta^\text{op} \\
\alpha & \quad \alpha^\text{op} & \quad \alpha & \quad \alpha^\text{op} \\
F(X) & \quad F(X) & \quad F(X) & \quad F(X) \\
\alpha_X & \quad \alpha_Y & \quad \alpha_X & \quad \alpha_Y \\
F(X) & \quad F(Y) & \quad F(X) & \quad F(Y) \\
\end{array}
\end{align*}
\]

The result is the natural transformation corresponding to \( \eta \circ \epsilon \). Similarly for the second line using functoriality under the coevaluation morphism \( 1 \to X \otimes X^* \).

Proposition 3.9  If \( \mathcal{C} \) is braided then \( \mathcal{R} \) defined in (57) makes \( B \) into a quasitriangular bialgebra.
**Proof** To prove the first of (53) we evaluate the definitions and use $F$ applied to the hexagon identity in $C$ for the fifth equality, and then in reverse.

The same strategy works for the second of (53)

Finally, to prove the last of (53) we use in the third equality the functoriality of $\alpha$ under the morphism $\Psi_{X,Y}$.
The construction of $R^{-1}$ is based in the inverse natural transformation to that for $R$ and the proof that this then is inverse in the convolution algebra $\mathbb{1} \rightarrow B \otimes B$ is straightforward using the same techniques. □

Clearly the definition of the product in (57) is such that $\alpha_X$ become modules. So we have a functor $C \rightarrow B\mathcal{V}$. The universal property of $B$ follows easily from its role as representing object for natural transformations $[5X]$.

**Corollary 3.10** If $C$ is braided and $F$ is a tensor functor in the sense that the braiding of $C$ is mapped on to the braiding of $\mathcal{V}$ then $\Delta^{op} = \Delta$, $R$ is trivial and $B$ is a braided group (braided-cocommutative) with respect to the image of the functor $C \rightarrow B\mathcal{V}$.

**Proof** This follows at once from the form of $R, \Delta^{op}$ in (57). □

For example if $F = id$ (or the canonical functor into a suitable completion of $C$) then to every rigid braided $C$ we have an associated braided Hopf algebra $B$ and a large class of braided-cocommutative modules $\{\alpha_X\}$. The ratio of the braidings is trivial and this is why we have from this point of view some kind of braided group rather than braided quantum group.

Finally, given a monoidal functor $F : C \rightarrow \mathcal{V}$ we can equally well require representability of the functor $\mathcal{V} \mapsto \text{Nat}(F, F \otimes V)$ and its higher order products i.e., bijections

$$\text{Hom}_\mathcal{V}(B^{\otimes n}, V) \rightarrow \text{Nat}(F^n, F^n \otimes V)$$

This is the representability assumption for comodules and is always satisfied if $\mathcal{V}$ is cocomplete and if the image of $F$ is rigid. In this case one can write $B$ as a coend $B = \int^X F(X)^* \otimes F(X)$.

**Theorem 3.11** Let $F : C \rightarrow \mathcal{V}$ obey the representability assumption for comodules. Then $B$ is a bialgebra in $\mathcal{V}$, uniquely determined as universal with the property that $F$ factors through $\mathcal{V}^B$. If $C$ is braided then $B$ is dual quasitriangular in the braided category with $R$ given by the ratio of the braidings in $C$ and $\mathcal{V}$. If $C$ is rigid then $B$ has a braided-antipode.
Proof. Literally turn the above proofs up-side-down. [44] has more traditional proofs. Note also the slightly different conventions there which are chosen so as to ensure that the dual quasitriangular structure reduces for $\mathcal{V} = \text{Vec}$ to the usual notion as in Section 1.3 rather than its convolution-inverse. ☐

By taking the identity functor to a cocompletion we obtain a canonical braided-Hopf algebra $B = \text{Aut} (\mathcal{C})$ associated to a rigid braided monoidal category $\mathcal{C}$ [44]. By turning Corollary 3.10 up-side-down the braided-Hopf algebra this time is braided-commutative with respect to a class of comodules. In this sense $\text{Aut} (\mathcal{C})$ is a braided group of function algebra type.

4 Applications to Ordinary Hopf Algebras

In this section we give some applications of the above braided theory to ordinary Hopf algebras. In this case there is either a background quasitriangular bialgebra or Hopf algebra $H$ and we work in the braided category $H \mathcal{M}$ or a background dual quasitriangular Hopf algebra $A$ and we work in the braided category $\mathcal{M}^A$. See Sections 1.2 and 1.3 respectively. The latter theory is the dual theory to the former and given by turning the diagram-proofs in this book upside down. There need be no relationship between $H$ or $A$ since we are dualizing the theory and not any specific Hopf algebra.

In this context the content of Lemma 2.1 is immediate: it says that two $H$-module algebras or $A$-comodule algebras have a braided tensor product which is also an $H$-module algebra or $A$-comodule algebra respectively. From [44] and Theorem 1.10 or Theorem 1.16 we have obviously

$$B, C, B \otimes C \in H \mathcal{M} : \quad (a \otimes c)(b \otimes d) = \sum a(\mathcal{R}^{(2)} b \otimes \mathcal{R}^{(1)} c)d \quad (59)$$

$$B, C, B \otimes C \in \mathcal{M}^A : \quad (a \otimes c)(b \otimes d) = \sum ab^{(1)} \otimes c^{(1)} d \mathcal{R}(c^{(2)} \otimes b^{(2)}) \quad (60)$$

for all $a, b \in B, \ c, d \in C$. We have introduced this construction in [44]–[50] and explained there that it is exactly a generalization of the $\mathbb{Z}_2$-graded or super-tensor product of super-algebras. This in turn has specific applications and spin-offs. An amusing one is

**Proposition 4.1** Let $H$ be a Hopf algebra. Then the usual $n$-fold tensor product $H^{\otimes n}$ is an $H$-module algebra under the action

$$h \triangleright (b_1 \otimes \cdots \otimes b_n) = \sum h_{(1)} b_1 Sh_{(2n)} \otimes h_{(2)} b_2 Sh_{(2n-1)} \otimes \cdots \otimes h_{(n-1)} b_{n-1} Sh_{(n+2)} \otimes h_{(n)} b_n Sh_{(n+1)}.$$

**Proof.** This arises from the general theorems below as the $n$-fold braided tensor product of $H$ as an $H$-module algebra and in the case where $H$ is quasitriangular. This $n$-fold braided tensor product turns out to be isomorphic to the usual tensor product in a non-trivial way using the quasitriangular structure. Computing the resulting action through this isomorphism and using the axioms [23] etc one finds the stated action. On the other hand the resulting formula does not require any quasitriangular structure at all and can then be checked directly to work for any Hopf algebra as stated. ☐

This is a typical example in that the braided theory leads one to unexpected formulas (as far as I know the last proposition is unexpected) which can then be verified directly. We will come to another such spin-off in Section 4.3.
4.1 Transmutation

In this section we shall show how to obtain non-trivial examples of bialgebras and Hopf algebras in braided categories. The key construction is one that we have called transmutation because it asserts that an ordinary Hopf algebra can be turned by this process into a braided one. This is achieved as an application of the generalised reconstruction theorem in Section 3 and is actually part of a rather general principle: by viewing an algebraic structure in terms of its representations, and targeting these by means of a functor into some new category, we can reconstruct our algebraic object in this new category. In this way the category in which an algebraic structure lives can be changed. Moreover, this ‘mathematical alchemy’ can be useful in that the structure may look more natural and have better properties after transmutation to the new category.

In the present setting the data for our transmutation is a pair $H_1 \xrightarrow{f} H$ where $H_1$ is a quasitriangular Hopf algebra, $H$ is at least a bialgebra, and $f$ a bialgebra map.

**Theorem 4.2** $H$ can be viewed equivalently as a bialgebra $B(H_1, H)$ living in the braided category $H_1\mathcal{M}$ by the adjoint action induced by $f$. Here

$$B(H_1, H) = \begin{cases} H & \text{as an algebra} \\ \Delta, \, S, \, R & \text{modified coproduct, antipode, quasitriangular structure} \end{cases}$$

where $B$ has a braided-antipode if $H$ has an antipode and a braided-quasitriangular structure if $H$ is quasitriangular.

**Proof** We let $\mathcal{C} = H\mathcal{M}$ and $\mathcal{V} = H_1\mathcal{M}$ and $F$ the functor by pull-back along $f$. Then Theorem 3.2 tells us there is a braided-Hopf algebra $B$. 

Explicit formulæ for the transmuted structure are

$$\Delta b = \sum b_{(1)} f(SR_1^{(2)} \otimes R_1^{(1)} \triangleright b_{(2)}), \quad Sb = \sum f(R_1^{(2)}) S(R_1^{(1)} \triangleright b)$$

$$R = \sum \rho^{(1)} f(SR_1^{(2)} \otimes R_1^{(1)} \triangleright \rho^{(2)}), \quad \rho = f(R_1^{-1}) R.$$  \hfill (61)  \hfill (62)

There is also an opposite coproduct characterised by

$$\sum \Psi(b_{(1)\text{op}} \otimes Q_1^{(1)} \triangleright b_{(2)\text{op}}^{(2)}) f(Q_1^{(2)}) = \sum b_{(1)} \otimes b_{(2)}$$

where $Q_1 = (R_1)_{21}(R_1)_{12}$ and $f(Q_1^{(2)})$ right-multiplies the second tensor factor of the output of $\Psi$. The underlines in the superscripts are to remind us that we intend here the braided-coproducts $\Delta$ and $\Delta^{\text{op}}$. The equation can also be inverted to give an explicit formula for $\Delta^{\text{op}}$. That these formulæ obey the axioms (15) and (53) of Section 2 is verified explicitly in [45].

**Corollary 4.3** $H$ be a Hopf algebra containing a group-like element $g$ of order $n$. Then $H$ has a corresponding anyonic version $B$. It has the same algebra and

$$\Delta b = \sum b_{(1)} g^{-|b_{(2)}|} \otimes b_{(2)} , \quad eb = eb, \quad Sb = g^{b} Sb$$

$$\Delta^{\text{op}} b = \sum b_{(2)} g^{-2|b_{(1)}|} \otimes g^{-|b_{(2)}|} b_{(1)}, \quad R = R_n^{-1} \sum R^{(1)} g^{-|R^{(2)}|} \otimes R^{(2)}.$$
We apply the transmutation theorem, Theorem 4.2 and compute the form of $B = B(z'_n, H)$. Here $z'_n$ is the non-standard quasitriangular Hopf algebra in Example 1.7 with quasitriangular structure $\mathcal{R}_{z'_n}$. The action of $g$ on $H$ is in the adjoint representation $g \triangleright b = gbg^{-1}$ for $b \in B$ and defines the degree of homogeneous elements by $g \triangleright b = q^{|b|}b$. □

**Corollary 4.4** Let $H$ be a quasitriangular Hopf algebra containing a group-like element $g$ of order 2. Then $H$ has a corresponding super-version $B$.

**Proof** The formulae are as in Corollary 4.3 with $n = 2$. The commutation relations with $g$ define the grading of $B$. □

The first corollary was applied, for example to $H = u_q(g)$ at a root of unity to simplify its structure. It led to a new simpler form for its quasitriangular structure by finding its anyonic quasitriangular structure and working back\[47\]. The second corollary was usefully applied in [59] to superise the non-standard quantum group associated to the Alexander-Conway polynomial. In these examples, a sub-quasitriangular Hopf algebra is used to generate the braided category in which the entire quasitriangular Hopf algebra is then viewed by transmutation. In the process its quasitriangular structure becomes reduced because the part from the sub-Hopf algebra is divided out. This means that the part corresponding to the sub-Hopf algebra is made in some sense cocommutative.

**Corollary 4.5** [44] Every quasitriangular Hopf algebra $H$ has a braided-group analogue $B(H, H)$ which is braided-cocommutative in the sense that $\mathcal{R} = 1 \otimes 1$ and $\Delta^{\text{op}} = \Delta$. The latter means

$$\sum \Psi(b_{(1)} \otimes Q^{(1)} \triangleright b_{(2)})Q^{(2)} = \sum b_{(1)} \otimes b_{(2)}.$$ 

We call $B(H, H)$ the braided group of enveloping algebra type associated to $H$. It is also denoted by $\mathcal{H}$.

**Proof** Here we take the transmutation principle to its logical extreme and view any quasitriangular Hopf algebra $H$ in its own braided category $H\mathcal{M}$, by $H \subseteq H$. This is a bit like using a metric to determine geodesic co-ordinates. In that co-ordinate system the metric looks locally linear. Likewise, in its own category (as a braided group) our original quasitriangular Hopf algebra looks braided-cocommutative. From Corollary 3.10 we know that $\Delta^{\text{op}} = \Delta$ and this gives the formula stated. □

This completely shifts then from one point of view (quantum=non-cocommutative object in the usual category of vector spaces) to another (classical=‘cocommutative’ but braided object), and means that the theory of ordinary quasitriangular Hopf algebras is contained in the theory of braided-groups.

The braided-Hopf algebra $B$ in Theorem 4.2 is equivalent to the original one in that spaces and algebras etc on which $H$ act also become transmuted to corresponding ones for $B$. Partly, this is obvious since $B = H$ as an algebra, so any $H$-module $V$ of $H$ is also a braided $B$-module. The key point is that $V$ is also acted upon by $H_1$ through the mapping $H_1 \to H$. So the action of $H$ is used in two ways, both to define the corresponding action of $B$ and to define the ‘grading’ of $V$ as an object in a braided category $H_1\mathcal{M}$. This extends the process of transmutation to modules.
Proposition 4.6 \cite{Fre, Prop 3.2} If $C$ is an $H$-module algebra then its transmutation is a braided $B(H, H)$-module algebra in the sense of \cite{Fre}. Similarly an $H$-module coalgebra becomes a braided $B(H, H)$-module-coalgebra. Here the transmutation does not change the action, but simply views it in the braided category.

**Proof** An elementary computation from the form of \cite{Fre} and the braiding in Theorem 1.10. $\square$

For example, the adjoint action of $H$ on itself transmutes to the braided-adjoint action of $B = B(H, H)$ on itself in Example 2.7. Moreover, it means that $B(H, H)$ is braided-cocommutative with respect to Ad. Indeed

**Proposition 4.7** \cite{Fre, Fre} For $B(H_1, H)$ the $\Delta^\op$ behaves like an opposite coproduct on all $B(H_1, H)$-modules that arise from transmutation. In particular, $B(H, H)$ is cocommutative in the sense of Definition 2.9 with respect to all braided-modules that arise from transmutation.

**Proof** Writing the braids in Definition 2.9 in terms of the quasitriangular structure as explained in Theorem 1.10, we see that the condition for all $V$ is implied by (and essentially equivalent to) the intrinsic braided-cocommutativity formula in Corollary 4.5. $\square$

Finally, we mention a different aspect of this transmutation theory, namely a result underlying the direct proof that $B(H, H)$ is a braided-Hopf algebra (if one does not like the categorical one).

**Lemma 4.8** Let $H$ be quasitriangular and $C$ be an algebra in $H\mathcal{M}$. Then there is an algebra isomorphism

$$\theta_{H,C} : H \otimes C \to B(H, H) \otimes C$$

where $\otimes$ is the braided-tensor-product in Lemma 2.1.

**Proof** This is provided by $\theta_{H,C}(h \otimes c) = \sum h S R^{(2)} \otimes R^{(1)} \triangleright c$ and shows at once that $\Delta = \theta_{H,H} \circ \Delta$ in \cite{Fre} is an algebra homomorphism. The proof that $\theta$ is an algebra homomorphism is

$$\begin{align*}
\theta_{H,C}(h \otimes c) \theta_{H,C}(g \otimes d) &= \sum h (S R^{(2)})(R''^{(2)} \triangleright (g S R'^{(2)})) \otimes (R''^{(1)} R^{(1)} \triangleright c)(R'^{(1)} \triangleright d) \\
&= \sum h (S R^{(2)}) R''^{(2)} g (S R'^{(2)}) S R''^{(2)} \otimes (R''^{(1)} R'^{(1)} \triangleright c)(R'^{(1)} \triangleright d) \\
&= \sum h g S (R''^{(2)} R'^{(2)}) \otimes (R''^{(1)} R'^{(1)} \triangleright c)(R'^{(1)} \triangleright d) = \theta_{H,C}(h g \otimes c d)
\end{align*}$$

using the definition \cite{Fre} and the axioms \cite{Fre}.

As an immediate example one has that $H^{\otimes n} \cong B(H, H) \hat{\otimes}^n$, for n-fold tensor products (iterate the lemma). Since $B(H, H) \hat{\otimes}^n$ lives in $H\mathcal{M}$ as an algebra (an $H$-module algebra) via the adjoint action, it follows that $H^{\otimes n}$ does also. Computing this gives Proposition 4.1 as an amusing spin-off.

The transmutation theory obviously has a dual version for $A \rightarrow A_1$ a bialgebra map where $A_1$ is a dual quasitriangular Hopf algebra and $A$ is at least a bialgebra.
Theorem 4.9 A can be viewed equivalently as a bialgebra \( B(A, A_1) \) living in the braided category \( \mathcal{M}^{A_1} \) by the right adjoint coaction induced by \( f \). Here

\[
B(A, A_1) = \begin{cases} A & \text{as a coalgebra} \\ \omega \cdot \mathcal{S} \cdot \mathcal{R} & \text{modified product, antipode, dual quasitriangular structure} \end{cases}
\]

where \( B \) has a braided-antipode if \( A \) has an antipode and a braided-dual quasitriangular structure if \( A \) is dual quasitriangular.

Proof We let \( \mathcal{C} = \mathcal{M}^A \) and \( \mathcal{V} = \mathcal{M}^{A_1} \) and \( F \) the functor by push-out along \( f \). Then Theorem 3.11 tells us there is a braided-Hopf algebra \( B \). The exact conventions for \( \mathcal{R} \) most useful here are in \[49\]. \( \square \)

Explicit formulae for the transmuted structure are\[49\] \[48\]

\[
a \cdot b = \sum a_{(2)} b_{(3)} \mathcal{R}((S a_{(1)}) a_{(4)} \otimes S b_{(1)}), \quad \mathcal{S} a = \sum S a_{(2)} \mathcal{R}((S^2 a_{(3)}) S a_{(1)} \otimes a_{(4)}) \quad (64)
\]

where for simplicity we concentrate on the case where \( f \) is the identity (the formulae in the general case are similar). The analogue of Corollary 4.5 is that \( B(A, A) \) is braided-commutative in the sense of Definition 2.9 turned up-side-down, for all comodules \( V \) that come from transmutation of comodules of \( A \). This reduces to an intrinsic form of commutativity dual to Corollary 4.5. This comes out explicitly as

\[
b \cdot a = \sum a_{(3)} b_{(3)} \mathcal{R}(S a_{(2)} \otimes b_{(1)}) \mathcal{R}(a_{(4)} \otimes b_{(2)}) \mathcal{R}(b_{(5)} \otimes S a_{(1)}) \mathcal{R}(b_{(4)} \otimes a_{(5)}). \quad (65)
\]

We call \( \mathcal{A} = B(A, A) \) the braided group of function algebra type associated to \( A \). A direct proof that these formulae define a braided-Hopf algebra as in \[45\] appears in \[13\], Appendix.

Finally, we suppose that \( A \) is actually dual to \( H \) in a suitable sense (for example, one can suppose they are finite dimensional). Until now we have not assumed anything like this. Then from the two theorems above we have two Hopf algebras in braided categories, and moreover the two categories can be identified in the usual way. Thus a right \( A \)-comodule defines a left \( H \)-comodule and viewing everything in this way in \( _H\mathcal{M} \) we have two Hopf algebras \( B(H, H) \) and \( B(A, A) \) in the same braided category.

Proposition 4.10 If \( A \) is dual to \( H \) then the corresponding braided groups \( B(A, A) \) and \( B(H, H) \) are dual in the braided category, \( B(A, A)^* = B(H, H) \).

Proof Explicitly, the duality is given by \( b \in B(H, H) \) mapping to a linear functional \( \langle S b, ( \cdot ) \rangle \) on \( B(A, A) \), where \( S \) is the usual antipode of \( H \). See \[23\] for full details. \( \square \)

Thus the usual duality if it exists becomes the categorical duality as in Section 2.2. This is to be expected. More remarkable is the fact, also verified explicitly\[22\] that there is a canonical homomorphism of Hopf algebras in the braided category

\[
Q: B(A, A) \rightarrow B(H, H), \quad Q(a) = (a \otimes \text{id})(\mathcal{R}_{21}\mathcal{R}_{12}) \quad (66)
\]

given by evaluation against \( \mathcal{R}_{21}\mathcal{R}_{12} \). In the standard examples \( H = U_q(g) \) one has a formal expansion \( \mathcal{R}_{21}\mathcal{R}_{12} = 1 + 2\hbar K^{-1} + O(\hbar^2) \) where \( K^{-1} \) is the inverse Killing form \( q^* \rightarrow g \). So \( (66) \) is a version for braided groups of this linear map provided by \( Q \). This point of view has already been developed at the level of linear maps \( A \rightarrow H \) in \[64\] where the Hopf algebra is called factorizable if the map \( Q \) is a linear isomorphism. What we have in \( (66) \) is a much stronger statement: in the factorizable case \( B(A, A) \cong B(H, H) \) as Hopf algebras in a braided category. Since the first of these is the braided version of the quantum function algebra \( O_q(G) \) and the second of the enveloping algebra \( U_q(g) \), the isomorphism of their braided versions is remarkable.
4.2 Bosonization

In this section we prove results going the other way, turning any Hopf algebra in the braided category $H \mathcal{M}$ or $\mathcal{M}A$ into an ordinary Hopf algebra. This process has been introduced by the author under the heading *bosonization*. The origin of this term is from physics where $\mathbb{Z}_2$-graded algebras etc are called ‘fermionic’ while ordinary ungraded ones are called ‘bosonic’. Not all braided groups are of the type coming from the transmutation in the last section, so bosonization is not simply transmutation in reverse.

**Theorem 4.11** [46, Thm 4.1] Suppose that $B$ is a Hopf algebra living in a braided category of the form $H \mathcal{M}$. Then there is an ordinary Hopf algebra $\text{bos}(B) = B > \bowtie H$.

**Proof** The abstract way that the result arose in [46] is as follows (but once the result is known a direct proof is also easy). Since $B$ lives in $H \mathcal{M}$ it is in particular an $H$-module algebra. From Proposition 4.6 we see that the same linear map makes $B$ a braided $B(H,H)$-module algebra. Likewise it is a braided module coalgebra and from Proposition 4.7 this braided-module structure is braided-cocommutative. Hence from Theorem 2.11 we have a semidirect product Hopf algebra $B > \bowtie B(H,H)$. This contains $B(H,H)$ and it is easy to see that this is indeed the transmutation of $H \rightarrow H_2$ where $H_2$ is some ordinary Hopf algebra. It computes explicitly as follows. As an algebra it is the semidirect or smash product by the action of $H$ on $B$. So $(1 \otimes h)(b \otimes 1) = \sum h_{(1)} \bowtie b \otimes h_{(2)}$ where $\bowtie$ is the action of $H$. As a coalgebra it is

$$\Delta(b \otimes h) = \sum b_{(1)} \otimes R^{(2)} h_{(1)} \otimes R^{(1)} \bowtie b_{(2)} \otimes h_{(2)}, \quad (67)$$

Once the result and formula (67) is known it is not hard to verify it directly. The key lemma for this direct verification is

**Lemma 4.12** [38] Let $H$ be a quasitriangular bialgebra or Hopf algebra and $B$ a left $H$-module with action $\bowtie$. Then

$$\beta(b) = \sum R^{(2)} \otimes R^{(1)} \bowtie b$$

makes $B$ into a left $H$-comodule. Moreover, it is compatible with $\bowtie$ in the sense of Example 1.3 and invertible so $B \in H_H \mathcal{M}$.

**Proof** Using the axioms of a quasitriangular bialgebra one sees at once that this defines a coaction and this is compatible in the sense of Example 1.3. In the case when $H$ is only a bialgebra we have to check invertibility in the sense of (20). The required inverse is provided by $R^{-1}$ in place of $R$ in the definition of $\beta$. □

As we explained in [38], this defines a functor $H \mathcal{M} \rightarrow H_H \mathcal{M} = Z(H \mathcal{M})$. Since the functor takes morphisms to morphisms (or by direct computation) it is easy to see that if $B$ is an $H$-module (co)algebra then it becomes in this way an $H$-comodule (co)algebra. It is completely clear then that $\text{bos}(B) = B > \bowtie H$ has the structure of a semidirect (co)product both as an algebra by $\bowtie$, and as a coalgebra by the coaction $\beta$ from Lemma 4.12. This is the direct interpretation of (67). Simultaneous semidirect products and coproducts have been studied in [52] but the present construction of examples of them is of course new and due to the author.
One thing that we learn from the categorical point of view is that this ordinary Hopf algebra \( \text{bos}(B) \) is equivalent to the original \( B \) in the sense that its ordinary representations correspond to the braided-representations of \( B \). This applies as much to super-Hopf algebras as to Hopf algebras on other categories, so we recover a result known to experts working with super-Lie algebras and super-Hopf algebras that they can be reduced to ordinary ones. See [15] for an example of this strategy.

**Corollary 4.13** [14, Cor. 4.3] Any super-quasitriangular super-Hopf algebra can be bosonised to an equivalent ordinary quasitriangular Hopf algebra. It consists of adjoining an element \( g \) with relations \( g^2 = 1 \), \( gb = (-1)^{|b|}bg \) and

\[
\Delta g = g \otimes g, \quad \Delta b = \sum b_{(1)} g^{(|b|)} \otimes b_{(2)}, \quad Sb = g^{-|b|}Sb, \quad \mathcal{R} = \mathcal{R}_{22} \sum \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \otimes \mathcal{R}^{(3)}.
\]

**Proof** We have seen in Proposition 1.11 that the category of super-vector spaces is of the required form, with \( H = \mathbb{Z}_2 \). Here the \( \mathbb{Z}_2 \)-graded modules of the original super-Hopf algebra are in one-to-one correspondence with the usual representations of the bosonised ordinary Hopf algebra. We suppose that we work over characteristic not 2. Also, we have written the formulae in a way that works in the anyonic case with 2 replaced by \( n \) and \( -1 \) by a primitive \( n \)-th root of unity for the Hopf algebra structure. \( \Box \)

This means that the theory of super-Lie algebras (and likewise for colour-Lie algebras, anyonic quantum groups etc) is in a certain sense redundant – we could have worked with their bosonized ordinary Hopf algebras. This is especially true in the super or colour case where there is no braiding to complicate the picture.

As usual the above theory has exactly a version for a Hopf algebra living in the braided category \( \mathcal{M}^A \) where \( A \) is dual quasitriangular.

**Theorem 4.14** Suppose that \( B \) is a Hopf algebra living in a braided category of the form \( \mathcal{M}^A \). Then there is an ordinary Hopf algebra \( \text{cobos}(B) = A \ltimes B \).

**Proof** We turn our diagram-proofs in the theory leading to Theorem 4.11 up-side-down. This time the coproduct is the semidirect one by the coaction whereby \( B \) is an object in \( \mathcal{M}^A \), and this also defines a right action \( \rhd \) (the dual version of Lemma 4.12) with respect to which we have a semidirect product algebra on \( A \otimes B \),

\[
b \rhd a = \sum b^{(1)} \mathcal{R}(b^{(2)} \otimes a), \quad (1 \otimes b)(a \otimes 1) = \sum a_{(1)} \otimes b \rhd a_{(2)}.
\]

(68)

Just as one has a direct proof of Theorem 4.11, one can verify directly that this \( \text{cobos}(B) \) is a Hopf algebra. \( \Box \)

It is rather hard to consider this result and its attendant lemma as new results since they are exactly the dual construction (by turning proofs as diagrams up-side-down) of our bosonization Theorem 4.11. Equally well one could reflect in a mirror about a horizontal axis (or simply reverse the arrows in the more conventional commutative diagrams). In this case left modules/comodules become left comodules/modules and the Hopf algebra is \( \text{cobos}(B) = B \ltimes A \) by a left handed semidirect product and coproduct. Equally well we could reflect in a vertical axis turning left modules to right modules etc in Theorem 4.11 and giving a right-handed version \( \text{bos}(B) = H \ltimes B \).
As before there is no suggestion here that $A$ is dual to $H$ since it is the construction that is being reversed and not any specific Hopf algebra. But if $A$ is dual to $H$ (say finite-dimensional) then we can make both constructions. Indeed, if $B^*$ is the categorical dual as in Section 2.2 then

$$\text{cobos}(B^*) = A \triangleleft B^* \cong (B \triangleright \! H)^* = \text{bos}(B)^*.$$  \hspace{1cm} (69)

Details and (more importantly) an application may be found in [53]. Another application of the bosonization theorem in its original form and in the dual form can be found in [7] and [16], to prove nice double-centraliser theorems for various kinds of Lie algebras in symmetric categories.

### 4.3 Radford’s Theorem

In [52, Appendix] we have explained in detail how the above ideas provide a new braided interpretation of Radford’s theorem [62] about Hopf algebras with projections. This theorem asserts that if $H, H_1$ are ordinary Hopf algebras and if

$$H_1 \xrightarrow{\text{p}} \leftarrow \xrightarrow{i} H$$

are bialgebra maps with $p \circ i = \text{id}$ (a Hopf algebra projection), then there is an algebra and coalgebra $B$ such that $H_1 \cong B \triangleright \! H$ as a simultaneous semidirect product and semidirect coproduct. Radford called such simultaneous semidirect (co)products where the result is a Hopf algebra ‘biproduts’ and showed that they correspond to projections. We have already introduced some examples in the last section (arising from the bosonization process) but now we consider the general situation.

At the time of [62] the notion of braided categories was yet to be invented. Because of this the algebra and coalgebra $B$ in Radford’s theorem was simply some exotic object where the algebra and coalgebra did not form an ordinary Hopf algebra. We have pointed out in [52] and cf. [38] that $B$ is in fact nothing other than a Hopf algebra in the braided category $H^{H,M} = D(H)^{H,M}$ of Example 1.3 (we stressed the latter in [52] for pedagogical reasons but explained that the former was more useful in the infinite-dimensional case). Thus we arrived at the following interpretation of Radford’s theorem.

**Proposition 4.15** [52, Prop. A.2] Let $H_1 \xrightarrow{\text{p}} \leftarrow \xrightarrow{i} H$ be a Hopf algebra projection and let $H$ have invertible antipode. Then there is a Hopf algebra $B$ living in the braided category $H^{H,M}$ such that $B \triangleright \! H \cong H_1$.

**Proof** Explicitly, $B$ is a subalgebra of $H_1$ and in $H^{H,M}$ by action $\triangleright$ and coaction $\beta$,

$$B = \{ b \in H_1 \mid \sum b_{(1)} \otimes p(b_{(2)}) = b \otimes 1 \}, \quad h \triangleright b = \sum i(h_{(1)}) b S \circ i(h_{(2)}), \quad \beta(b) = p(b_{(1)}) \otimes b_{(2)}$$

where $h \in H$. The braided-coproduct, braided-antipode and braiding of $B$ are

$$\Delta b = \sum b_{(1)} S \circ i \circ p(b_{(2)}) \otimes b_{(3)}, \quad S b = \sum i \circ p(b_{(1)}) S b_{(2)}, \quad \Psi_{B,B}(b \otimes c) = \sum p(b_{(1)}) \triangleright c \otimes b_{(2)}.$$  \hspace{1cm} (72)

The isomorphism $\theta : B \triangleright \! H \to H_1$ is $\theta(b \otimes h) = bi(h)$, with inverse $\theta^{-1}(a) = \sum a_{(1)} S \circ i \circ p(a_{(2)}) \otimes p(a_{(3)})$ for $a \in H_1$. The only new part beyond [52] is the identification of the ‘twisted Hopf algebra’ $B$ now as a Hopf algebra living in a braided category, and some
slightly more explicit formulae for its structure. The set $B$ coincides with the image of the projection $\Pi : H_1 \rightarrow H_1$ defined by $\Pi(a) = \sum a(1) S \circ i \circ p(a(2))$ in [62], while the pushed-out left adjoint coaction of $H$ on $B$ then reduces to the left coaction as stated. The braiding is from Example 1.3. The axioms of a Hopf algebra in a braided category require that $\Delta : B \rightarrow B \otimes B$ is an algebra homomorphism with respect to the braided tensor product algebra structure on $B \otimes B$. Writing $\Delta b = \sum b(1) \otimes b(2)$, this reads

$$\Delta(bc) = \sum b(1) \Psi(b(2) \otimes c(1)) c(2) = \sum b(1) (b(2)^{\langle 1 \rangle} \triangleright c(1)) \otimes b(2)^{\langle 2 \rangle} c(2)$$

(73)

which indeed derives the condition in [62]. The structure of $B \triangleright H$ is the standard left-handed semidirect one by the action and coaction stated. Applying $\theta$ to these structures and evaluating further at once gives $\theta$ as a Hopf algebra isomorphism. Of course if $H_1$ is only a bialgebra then $B$ is only a bialgebra in a braided category. In this case one can use the convolution inverse $i \circ S$ in the above. Also, the restriction to invertible antipode on $H$ is needed only to ensure that $\Psi$ is invertible as explained in Example 1.3. It is part of our interpretation of $B$ as a braided-Hopf algebra rather than part of Radford’s theorem itself.

Such Hopf algebra projections have a geometrical interpretation as examples of trivial quantum principal bundles [3] and at the same time as quantum mechanics [53]. These papers also make some limited contact with more established ideas of non-commutative geometry as in [8]. Also note that in the above the Hopf algebra $H$ need not be quasitriangular or dual quasitriangular. If it is then the above construction becomes related to the bosonization theorems of the last subsection, as mentioned there.

5 Braided Linear Algebra

In this section we describe some general constructions for examples of bialgebras and Hopf algebras in braided categories associated to a general matrix solution of the QYBE as in Example 1.1. This includes some interesting algebras for ring theorists. The first of these, $B(R)$, has a matrix of generators with matrix coproduct and includes a degenerate form of the Sklyanin algebra for the usual $GL_q(2)$ $R$-matrix. The second has a vector or covector of generators with linear coproduct, and includes the famous quantum plane $yx = qxy$ for this $R$-matrix. Thus the quantum plane does have a linear addition law provided we work in a braided category. Finally, we mention some recent developments such as a notion of braided-Lie algebra.

5.1 Braided Matrices

Let $R$ be an invertible matrix solution of the QYBE and $A(R)$ the associated dual quasitriangular bialgebra as in Section 1.3. In the nicest case we can quotient $A(R)$ to obtain a dual quasitriangular Hopf algebra $A$. We say in this case that $R$ is regular. This is true for the standard $R$ matrices and one obtains $A = O_q(G)$ as shown in [14]. Another way to obtain a dual quasitriangular Hopf algebra is if $R$ is dualizable. In this case we have an associated dual quasitriangular Hopf algebra $A = GL(R)$. Either way we have a canonical bialgebra map $A(R) \rightarrow A$ and can apply the transmutation theorem, Theorem 4.9 to obtain a bialgebra $B(R) = B(A(R), A)$ in the category $M^A$. This gives the following construction, which we verify directly.
**Proposition 5.1** [48][50] Let $R$ be a bi-invertible solution of the QYBE with $v^i_j = \tilde{R}^i_a_j$ also invertible. Then there is a bialgebra $B(R)$ in the braided category of $A$-comodules, with matrix generators $u = \{u^i_j\}$ and relations, braiding and coalgebra

$$R^k_{a b} u^b_c R^{-1}_{c d} u^d_l = u^k_{a b} R^a_{c d} R^b_{c d}, \quad \text{i.e.} \quad R_{21} u_1 R_{12} u_2 = u_2 R_{21} u_1 R_{12}.$$ 

$\Psi(u^i_j \otimes u^k_l) = u^0_p \otimes u^m_n R^i_p_{a p} R^{-1}_{a p} R^d_{a p} R^{-1}_{a p} R^c_{a p}, \quad \text{i.e.} \quad \Psi(R^{-1} u_1 \otimes R u_2) = u_2 R^{-1} \otimes u_1 R$

$$\Delta u^i_j = u^i_a \otimes u^a_j, \quad \epsilon u^i_j = \delta^i_j \quad \text{i.e.} \quad \Delta u = u \otimes u, \quad \epsilon u = id$$

**Proof** The formulae are obtained from Theorem 4.9 and then verified directly. Bi-invertible means $R^{-1}$ and the second-inverse $\tilde{R}$ exist and is all that is needed to verify the braided-bialgebra axioms. The existence of $\nu^{-1}$ is needed in defining $\Psi^{-1}$ and is equivalent to demanding that $R$ is dualizable. The commutation relations come from (65) using Proposition 1.15 to evaluate the formulae in matrix form. This gives a matrix equation of the form $uu = uuR^{-1}RR\tilde{R}$ with appropriate indices, see [48][50]. Putting two of the $R$’s to the left (or rearranging (65)) gives the formula stated. The same applies to the braiding which comes out as a product of 4 $R$-matrices as shown but is also conveniently written in the compact form stated. Its extension to products is by definition in such a way that the product is a morphism in the category generated by this braiding $\Psi$ and one can verify that this extension is consistent with the relations. To verify that the result is indeed a braided bialgebra an even more compact notation is useful. Namely as in (44) we can label the second copy of $B(R)$ we can label the second copy of $B(R)$ in the braided tensor product $B(R) \otimes B(R)$ are $R^{-1} u'_1 R u'_2 = u_2 R^{-1} u'_1 R$ and we compute

$$R_{21} u_1 u'_1 R u_2 u'_2 = R_{21} u_1 R(u^{-1}_1 R u_2) u'_2 = (R_{21} u_1 R u_2) R^{-1} R_{21}^{-1} u'_1 R u'_2$$

$$= u_2 R_{21} (u_1 R^{-1} u'_1 R_{21}) u'_1 R = u_2 R_{21} R_{21}^{-1} u'_2 R_{21} u_1 u'_1 R = u_2 u'_2 R_{21} u_1 u'_1 R$$

as required for $\Delta$ to extend to $B(R)$ as a bialgebra in a braided category. In each expression, the brackets indicate how to apply the relevant relation to obtain the next expression. □

Note that the braided category here is generated by the matrix in $M_{n^2} \otimes M_{n^2}$ corresponding to $\Psi$ as stated. On the other hand in the regular or dualizable case we can identify this category as contained in that of $A$-comodules, under the induced adjoint coaction in Theorem 4.9. In the present setting this is

$$\beta(u^i_j) = u^m_n \otimes (St^i_m) r^m_j, \quad \text{i.e.} \quad u \to t^{-1} u t. \quad (74)$$

Note also that if $R_{21} R_{12} = 1$ (the triangular case) then the braiding and the commutation relations co-incide so that $\cdot = \Psi^{-1} \cdot$ which is the naive notion of braided-commutativity. In general however this will not do and instead the braided-commutativity relations are different from the braiding itself.

Finally, it is good to know that the algebra $B(R)$ has at least one canonical representation. This is provided by

$$\rho(u^i_j)^k_l = Q^i_j k_l, \quad \text{i.e.} \quad \rho_2(u_1) = Q_{12}; \quad Q = R_{21} R_{12} \quad (75)$$
In the compact notation the proof reads

\[
\rho_3(R_{21}u_1R_{12}u_2) = R_{21}\rho_3(u_1)R_{12}\rho_3(u_2) = R_{21}Q_{13}R_{12}Q_{23} = Q_{23}R_{21}Q_{13}R_{12} = \rho_3(u_2)R_{21}\rho_3(u_1)R_{12} = \rho_3(u_2R_{21}u_1R_{12}).
\]

The middle equality follows from repeated use of the QYBE. Note that this representation is trivial in the triangular case. Other useful representations can be built from this.

From the theory above we obtain also that these braided matrices are related to usual quantum matrices by transmutation. The formula for the modified product comes out from (64) as (76)

\[
\begin{align*}
    u^i_j &= t^i_j, \\
    u^i_j u^k_l &= t^{a} b^{d} c^{c} a^{a} d^{d} R^{i}_{a} c^{c} a^{a} d^{d} R^{b}_{j} k^{k}, \\
    u^i_j u^k_l u^m_n &= t^{d} t^{s} t^{z} n^{n} R^{i}_{a} p^{p} R^{a}_{d} w^{w} \tilde{R}^{b}_{c} v^{v} \tilde{R}^{c}_{j} k^{k} p^{p} R^{q}_{s} y^{y} \tilde{R}^{l}_{v} u^{m}_{v}.
\end{align*}
\]

etc. Here the products on the left are in \(B(R)\) and are related by transmutation to the products on the right which are in \(A(R)\). If we write some or all of the \(R\)-matrices over to the left hand side we have equally well the compact matrix form (76)

\[
\begin{align*}
    u &= t, \\
    R_{12}^{-1} u_1 R_{12} u_2 &= t_1 t_2, \\
    R_{23}^{-1} R_{13}^{-1} R_{12}^{-1} u_1 R_{12} u_2 R_{13} R_{23} u_3 &= t_1 t_2 t_3
\end{align*}
\]

etc. This is just a rearrangement of (76) or our universal formula (74). For the transmuted product of multiple strings the universal formula from (14) involves a kind of partition function made from products of \(R\) to transmute the bosonic \(A(R)\) to the braided \(B(R)\).

We believe these braided matrices deserve more study as certain well-behaved quadratic algebras. For the standard \(R\)-matrices they have quotients giving the braided versions \(B_q(G)\) say of the quantum function algebras \(O_q(G)\), and are at the same time isomorphic via (66) and for generic \(q\) to the braided versions of the corresponding quantum enveloping algebras \(U_q(g)\). This is related to constructions in physics[65]. On the other hand these \(B(R)\) are interesting even at the quadratic level. The case of \(BM_q(2)\) for the \(GL_q(2)\) \(R\)-matrix was studied in [54] and shown in [52] to be a degenerate form of the Sklyanin algebra. Some remarkable homological properties of these braided matrices have recently been obtained in [3].

### 5.2 Braided Planes

If the algebras \(B(R)\) are like ‘braided matrices’ because they have a matrix of generators with matrix coproduct, one can complete the picture with some notion of ‘braided vectors’ and ‘braided covectors’. The usual algebra suggested here from physics is the Zamalodchikov or ‘exchange algebra’ of the form \(x_1 x_2 = \lambda x_2 x_1 R\) where \(x\) say is a row vector of
Proposition 5.2 [56] Let \( R \in M_n \otimes M_n \) be an invertible solution of the QYBE and \( R' \in M_n \otimes M_n \) an invertible matrix obeying

\[
R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}, \quad R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}, \quad (PR + 1)(PR' - 1) = 0
\]

where \( P \) is the permutation matrix. Then there are braided-bialgebras \( V^{-}(R') \) and \( V(R') \) with row and column vectors of generators \( x = \{x_i\} \) and \( v = \{v^i\} \) respectively and with relations and braiding

\[
x_ix_j = x_bx_aR^{a}_{\phantom{a}i \phantom{a}}^{b}_{\phantom{b}j}, \quad \Psi(x_i \otimes x_j) = x_b \otimes x_aR^{a}_{\phantom{a}i \phantom{a}}^{b}_{\phantom{b}j}, \quad \text{i.e.} \quad x_1x_2 = x_2x_1R', \quad \Psi(x_1 \otimes x_2) = x_2 \otimes x_1R
\]

\[
v^iv^j = R'^{a}_{\phantom{a}i \phantom{a}}^{b}_{\phantom{b}j}v^b \otimes v^a, \quad \Psi(v^i \otimes v^j) = R'^{a}_{\phantom{a}i \phantom{a}}^{b}_{\phantom{b}j}v^b \otimes v^a, \quad \text{i.e.} \quad v_1v_2 = R'v_2v_1, \quad \Psi(v_1 \otimes v_2) = Rv_2 \otimes v_1
\]

and linear coalgebra

\[
\Delta x = x \otimes 1 + 1 \otimes x, \quad \xi x = 0, \quad \Delta v = v \otimes 1 + 1 \otimes v, \quad \xi v = 0.
\]

If \( R'_{21} = R'_{21}R_{12} \) then these are braided-Hopf algebras with \( \tilde{S}x = -x \) and \( \tilde{S}v = -v \)

Proof One can see that \( \Psi \) extends to products of generators in such a way that the product is a morphism in the braided category generated by \( R \). For details see [56]. To see that the result forms a bialgebra we use the compact notation as in the proof of Proposition 5.1, \((x_1 + x'_1)(x_2 + x'_2) = x_1x_2 + x'_1x_2 + x_1x'_2 + x'_1x'_2 = x_2x_1R' + x_1x'_2(PR_{12} + 1) + x'_2x'_1R'_{12}\) while \((x_2 + x'_2)(x_1 + x'_1)R'_{12}\) has the same outer terms and the cross terms \(x'_2x_1R_{12} + x_2x'_1R_{12} = x_1x'_2(R_{21} + P)R'\). These are equal since \( PR + 1 = PRPR' + PR' \) from the assumption on \( R' \). Similarly for the other details. \( \Box \)

There are lots of ways to satisfy the auxiliary equation for the matrix \( R' \). If \( R \) is a Hecke symmetry we can simply take \( R' = \lambda R \) where \( \lambda \) is a suitable normalization. This is the familiar case. For example the standard \( GL_q(n) \) \( R \)-matrix gives the algebra

\[
x_ix_j = qx_jx_i \quad \text{if} \quad i > j
\]

which is the \( n \)-dimensional quantum plane. Thus we see that it forms a Hopf algebra in the braided category generated by \( R \). There is another normalization \( \lambda \) giving another quantum plane algebra and again a braided-Hopf algebra.

Another solution is \( R' = P \) the permutation matrix. In this case the relations of the algebra are free (no relations). The \( R \)-matrix still enters into the braiding. This free-braided plane is therefore canonically associated to any invertible matrix solution of the QYBE. The simplest member of this family is the braided line. This is \( B = k < x > \) (one generator) and

\[
\Psi(x \otimes x) = qx \otimes x, \quad \Delta x = x \otimes 1 + 1 \otimes x, \quad \xi x = 0, \quad \tilde{S}x = -x.
\]
Finally, any $R$ will obey an equation of the form $\prod_i (PR - \lambda_i) = 0$ and for each $\lambda_i$ we can rescale $R$ so that one of these factors become $PR + 1$. Then $PR' - 1$ defined as a multiple of the remaining factors gives us an $R'$ obeying the required equations. Thus there are canonical braided Hopf algebras $V^\vee(R')$ and $V(R')$ for each eigenvalue $\lambda_i$ in the decomposition. In the Hecke case there are by definition two such eigenvalues corresponding to the two normalizations mentioned, but in general $R'$ will not be a multiple of $R$ and need not obey the QYBE.

The algebra $V(R')$ can be called the **braided vector** algebra. Likewise the algebra $V^\vee(R')$ can be called the **braided covector** algebra. Another notation is $V^\vee(R')$ but we have avoided this here in order to not suggest that it is the left dual of the braided vector algebra in the sense of Section 2.2. In fact, the braiding is such that in the dualizable case the vector space generating the vector algebra is the left dual of that generating the covector algebra (compare with Section 1.1). Together with the braided matrices $B(R)$, these algebras form a kind of braided linear algebra. We refer to [51] for more details.

### 5.3 New Directions

In the above we have described the basic theory of Hopf algebras in braided categories and some canonical canonical constructions for them which have been established for the most part in the period 1989 – 1991. Some more recent directions are as follows.

One direction is that of **braided differential calculus**. The idea is that just as the differential structure on a group is obtained by making an infinitesimal group translation, now we can use our braided-coproducts on the braided-planes and braided-matrices to obtain corresponding differential operators.

**Proposition 5.3** [57] The operators $\partial^i : V^\vee(R') \to V^\vee(R')$ defined by

$$\partial^i x_{i_1} \cdot \cdot \cdot \cdot x_{i_m} = \delta^i_{j_1} x_{j_1} \cdot \cdot \cdot \cdot x_{j_m} [m; R]_{j_1 \cdot \cdot \cdot \cdot j_m}$$

$$[m; R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12} \cdots (PR)_{m-1,m}$$

obey the relations of $V(R')$ and the braided-Leibniz rule

$$\partial^i(ab) = (\partial^i a)b + \Psi^{-1}(\partial^i \otimes a)b, \quad \forall a, b \in V^\vee(R').$$

The result says that the braided-covector algebra (which is like the algebra of co-ordinate functions on some kind of non-commutative algebraic variety) is a left $V(R')$-module algebra in the category with reversed braiding. For the braided-line we obtain the usual Jackson $q$-derivative $(\partial f)(x) = \frac{f(qx) - f(x)}{(q-1)x}$. For the famous quantum plane we recover its well-known two dimensional differential calculus obtained usually by other means. We have also introduced in this context a braided-binomial theorem for the ‘counting’ of braided partitions. This is achieved by means of the **braided integers** $[m, R]$ in the proposition. One can also define a braided exponential map $\exp_R(x|y)$ and prove a braided Taylor’s theorem at least in the free case where $R' = P$. We refer to [57] for details.

Likewise, one has some natural right-handed differential operators on the braided matrices obtained from Example 2.8 applied to $B(R)$ and its braided-Hopf algebra quotients (the result lifts to the bialgebra setting). These are computed in detail in [58].

Related to this, one can formalise at least one notion of **braided Lie algebra** [58]. In a symmetric (not braided) category the notion of a Lie algebra and its enveloping algebra are...
just the obvious ones, see [19]. But as soon as one tries to make this work in the braided setting there are problems with the naive approach. One solution is based on the notion of braided-cocommutativity in Definition 2.9 with respect to the braided-adjoint action. Based on its properties in this case one can extract the axioms of a braided-Lie algebra as the following: a coalgebra \((\mathcal{L}, \Delta, \epsilon)\) in the braided category and a bracket \([\cdot, \cdot] : \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}\) obeying

\[
\begin{align*}
\Delta \chi & = \chi \otimes 1 + 1 \otimes \chi + \chi \otimes \chi \\
R_{21} \chi_1 R_{12} \chi_2 - \chi_2 R_{21} \chi_1 R_{12} & = Q_{12} \chi_2 - \chi_2 Q_{12}.
\end{align*}
\]

We show that such a braided-Lie algebra has an enveloping bialgebra living in the category. The braided matrices \(B(R)\) (e.g. the degenerate Sklyanin algebra) double up in this role as braided-enveloping bialgebras. Here \(\mathcal{L} = \text{span}\{u^i_j\}\) with the matrix coproduct as in Section 5.1. These generators are a mixture of ‘group-like’ and ‘primitive-like’ generators. If one wants something more classical one can work equally well with the space \(\mathcal{X} = \text{span}\{\chi^i_j\}\), where \(\chi^i_j = u^i_j - \delta^i_j\). In these terms the braiding has the same form as between the \(u\) generators, while the relations and coproduct become

\[
R_{21} \chi_1 R_{12} \chi_2 - \chi_2 R_{21} \chi_1 R_{12} = Q_{12} \chi_2 - \chi_2 Q_{12}, \quad \Delta \chi = \chi \otimes 1 + 1 \otimes \chi + \chi \otimes \chi
\]

in the compact notation. This is more like a ‘Lie algebra’ and for the standard \(R\) matrices, as the deformation parameter \(q \to 1\) the matrix \(Q = R_{21} R_{12} \to \text{id}\) and the commutator on the right hand side vanishes. This means that it is the rescaled generators \(\bar{\chi} = h^{-1} \chi\) that tend to a usual Lie algebra and indeed the extra non-primitive term in the coproduct of the \(\bar{\chi}\) now tends to zero. For details see [58]. We recall also that for the standard \(R\) matrices the algebras \(B(R)\) have a quotient which is \(U_q(g)\) at least formally, so these are understood as (the quotient of) the braided-enveloping algebra of a braided-Lie algebra.

We have not tried here to survey a number of more specific applications of this braided work. These include [4][21][32][53] as well as more physically-based applications. In addition among many relevant and interesting recent works by other mathematicians and that I have not had a chance to touch upon, I would like to at least mention [17][26][60][68] in the categorical direction and [6][63] as well as [2][7][16][3] already mentioned for works in an algebraic direction. What we have shown is that a number of general mathematical constructions can be braided. There is clearly plenty of scope for further work in this programme. Some potential areas are applications to knot theory, a general braided-combinatorics (based on the braided-integers above) and some kind of braided-analysis.

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