A Study of Equicontinuous Maps On Uniform G –Spaces

By

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Abstract. In this paper we shall study some new properties of equicontinuous maps on uniform G –Spaces. Here the phase space consider as a uniform space. Also we show the relationship among the equicontinuous maps with the distal dynamical system and expansive dynamical system.

1.Introduction

One of the most significant in the investigation of the hypothesis of dynamical framework is equicontinuous dynamical framework. Numerous creators have been examined the dynamical ideas in a measurement space or in topological space.

R. Das (2012) [1] characterize and study the mayhem of a grouping of maps in "a metric G–space". Additionally, he [2] characterize while research a idea of G–transitive subset for a ceaseless guide upon the smaller metrical G–space.

R. Das [3] (2013) get enough status beneath that consequence from pair maps, at that singular is "Devaney's" G1-befuddled while else is "Devaney's" G2-scattered, is "Devaney's" G1 × G2-chaotic.

R. Das , T. Das [4] (2012) describe and research the thoughts from determinedly and antagonistically "G-asymptotic" spotlights at a homeomorphism by a "metric G-space". Furthermore, in [5] (2012) they describe and research a possibility of "topological transitivity" of an industrious self- chart during the "metric G–space" named like "topologically G-transitive" guide and secure hers depiction.

P. Das and T. Das [6] (2019) show that the course of action of concentrates dual asymptotic into a dot hold measure zero concerning every expansive outside common mensuration to a bi-quantifiable guide on a discernable "uniform space".
I. J. Kadhim and S. K. Jebur [7] (2017) they study the some acclaimed dynamical thoughts, for instance, tricky transitive mixture while equicontinuous at a general topological.

E. Shah and T. Das [8] (2013) portray while research the idea from inconsequentality while detail for self a homeomorphism from a "metric G-space X". utilize "G-minimality", they get a category concerning maps that don't contain a "G-shadowing property". Further, get the enough event into "G-expansive homeomorphisms" and "G-shadowing property" to have "G-specification property".

Here, we will concentrate some new properties of equicontinuous maps on uniform G-Spaces. In Sec.2, a few ideas related with the uniform space a few properties of a uniform space that required in our work are state. Sec. 3 comprises of the primary aftereffects of our work.

2. Uniform space

Disregard X a set. mean by Δ,X a corner to corner in (X×X), in order to is a set Δ,X={x,x}:x∈X}. a regressive U^(-1) of the subcategory U⊂X×X is a subcategory of X×X described via U^(-1)={(x,y):(y,x)∈U}. On decision in order to U is symmetric if U^(-1)=U, we get U∩U^(-1) symmetric into each U⊂X×X. We describe the combined U∩V of pair subsets (U ,V ) of X×X by U∩V={(x,y):found z∈X to that a degree, such (x,z)∈U and (z,y)∈V}⊂X×X.

Definition 2.1[10] suppose X is a set. A "uniform structure" on X be a invalid combination U containing subsets of the Cartesian square (X×X) satisfactory a going with situations:

[UN-1] if U∈U, then Δ[X]⊂U;

[UN-2] if U∈U and U⊂V⊂X×X, then V∈U;

[UN-3] if U∈U and V∈U, then U∩V∈U;

[UN-4] if U∈U, then U^(-1)∈U;

[UN-5] if U∈U, then there exists V∈U such that V∩V⊂U.

a segments of U is known as the escort of the "uniform structure" while the set X is known as a "uniform space". The consistency U is called segregating (and X is said to be disconnected) if ∩{U:U∈U}=Δ.

annotation in order to the events [UN-3], [UN-4] and [UN-5] propose that, into each organization U found a symmetric escort V with the ultimate objective in order to V∩V⊂U. Disregard X a set while put U⊂X×X. specified a point x∈X, describe a subset U_1([x])⊂X by U_1([x])={y∈X:(x,y)∈U}.

In case X is a "uniform space", thither a started topology on X charactrized via a way in order to the regions from an emotional dot x∈X include the sets U_1([x]) , wherever U works onto every organizations of X. This topology is "Hausdorff" if while just if the interchange purpose of the impressive number of escorts of X is reduced to one side Δ,X.

If (X,d) be an estimation space, thither a trademark uniform build upon X whom organization are the sets U⊂X×X satisfactory the going with situation: found an authentic numeral ε>0such that U involves every paire (x,y)∈X×X with the ultimate objective in order to d(x,y)<ε. The topology related together and such orderly build is subsequently comparable to the topology started by the estimation.
Theorem 2.2 [10] (a) For every $x \in X$, the assortment $\mathcal{K}_x = \{ U_x([x]) : U \in U \}$ structure a local base at $x \in X$, making $X$ a topological space. A similar topology is delivered if any base $B$ is utilized instead of $U$. (b) the topology is Hausdorff if and just if $U$ is isolated.

Theorem 2.3[10] The consistency $U$ is isolated if and just if for each $x,y \in X$ with $x \neq y$, there exists $U \in U$ to such an extent that $(x,y) \notin U$.

Corollary 2.4[10] The topology is Hausdorff if and just if for every $x,y \in X$ with $x \neq y$, found $U \in U$ to this an extent that $(x,y) \notin U$.

Definition 2.5. [10] Let $(X,U)$ while $(Y,V)$ be "uniform spaces". A capacity $f:X \to Y$ is told into be uniform persistent if for every $V \in V$, there is some $U \in U$ with the end goal that $(x,y) \in U$ implis that $(f(x),f(y)) \in V$. On the off chance that $f$ is one-one, onto and both $f$ and $f^{-1}$ are uniform persistent, we consider $f$ a uniform isomorphism (uniform comparability) and state $X$ and $Y$ are consistently isomorphic ( consistently proportionate). Each consistently constant capacity is nonstop and thus every uniform isomorphism is "homeomorphism".

Definition 2.6. [10] suppose $(X,U)$ and $(Y,V)$ are duo uniform spaces. A mapping $f:X \to X$ is told into be uniform equicontinuous on $X$ if for each company $V \in V$ and for each positive number $n$, fund an escort $U \in U$ to such an extent that $(x,y) \in U$ infers $(f^n(x),f^n(y)) \in V$.

Obviously that any self-nonstop guide is uniform equicontinuous however the opposite need not be valid.

Definition 2.7 [10] let $(X,U)$ and $(Y,V)$ are "uniform spaces". By subsequently the consequence from $(X,U)$ and $(Y,V)$ is a "uniform space" $(Z,W)$ together the concealed set $Z=X \times Y$ while the consistency $W$ on $Z$ whom basis involves the sets

$$W_\_ (U,V) = \{ ((x,y),(x^\lambda,y^\lambda )) \in Z \times Z : (x,x^\lambda ) \in U, (y,y^\lambda ) \in V \},$$

wherever $U \in U$ and $V \in V$. a consistency $W$ is known as the consequence of $U , V$ and is made as $W=U \times V$.

3- Main Results

Right now idea of equicontinuous, sweeping and distal maps in a uniform G-space are presented and some new properties of such ideas are demonstrated.

Definition 3.1[9] through a "G-space" we purpose a triplex $(X,G,\theta)$, wherever $X$ is a "Hausdorff space", $G$ is a topological social occasion and $\theta : G \times X \to X$ is a perpetual movement of $G$ on $X$.

Definition 3.2 The 4-tuple $(X,G,U,\theta)$ is said to be Uniform G-space if $(X,G,\theta)$ is G-space and $(X,U)$ is uniform space.

For simplest, we shall indicate for $(X, G, U, \theta)$ by $\theta$.

Definition 3.3 The pair of maps

$$(\mu,\psi):(G_1,X,U,\theta_1) \to (G_2,Y,V,\theta_2)$$

is said to be uniform homomorphism between the two uniform spaces $(G_1,X,U,\theta_1)$ and $(G_2,Y,V,\theta_2)$ if
(I) $\mu: G_1 \rightarrow G_2$ is topological gathering homomorphism,

(ii) $\psi: X \rightarrow Y$ is uniform consistent guide and

(iii) $\psi(\theta_1(g,x)) = \theta_2(\mu(g),\psi(x))$.

**Definition 3.4** Suppose $X$ is a uniform $G$-space. A uniform ceaseless mapping $f: X \rightarrow X$ is said to be uniform $G$-equicontinuous on $X$ if for each escort $V \in U$ and for each positive whole number $n$, there exists a company $U \in U$ to such an extent that

$(x,y) \in U$ infers $(f^n (\theta(g,x)), f^n (\theta(p,y))) \in V$, $g,p \in G$.

**Remark 3.5** Beneath the paltry activity of $G$ on $X$ the thoughts of "uniform equicontinuous" and "uniform $G$-equicontinuous" are agreed.

**Theorem 3.6** Suppose $X$ and $Y$ is a "uniform $G$-spaces" and $h_1:X\rightarrow X$, h_1:Y→Y be "equivariant topologically" conjugate by means of $\phi:X\rightarrow Y$. In the event that $h_1$ is "uniform $G$-equicontinuous", at that point so is $h_2$.

**Proof.** Let $h_1$ is "uniform $G$-equicontinuous". Let $V \in V$. Since $\phi$ is uniform isomorphism, so we found $U \in U$ Like that

$$(x_1,x_2) \in U \implies (\phi(x_1), \phi(x_2)) \in V.$$  \hspace{1cm} (1)

while $h_2: X \rightarrow X$ is uniform $G$-equicontinuous, so we found an entourage $\tilde{U} \in U$ and $g,p \in G$ such that

$$(\tilde{x}_1, \tilde{x}_2) \in \tilde{U} \implies (h_1^n (\theta(g,\tilde{x}_1)), h_1^n (\theta(p,\tilde{x}_2))) \in U.$$  \hspace{1cm} (2)

Since $\phi^{-1}: Y \rightarrow X$ is uniform continuous, subsist $\tilde{V} \in V$ Like that

$$(\tilde{y}_1, \tilde{y}_2) \in \tilde{V} \implies \text{denote}(\phi^{-1}(\tilde{y}_1), \phi^{-1}(\tilde{y}_2)) \in U.$$  \hspace{1cm} (3)

By (2) we have

$$(h_1^n (\theta(g, \phi^{-1}(\tilde{y}_1))), h_1^n (\theta(p, \phi^{-1}(\tilde{y}_2)))) \in U.$$  \hspace{1cm}

By (1) we have

$$(\phi h_1^n (\theta(g, \phi^{-1}(\tilde{y}_1))), \phi h_1^n (\theta(p, \phi^{-1}(\tilde{y}_2))) \in U.$$  \hspace{1cm}

Since $h_1$, $h_2$ be equivariant topologically conjugate via $\phi$, then

$h_1^n (\theta(g, \phi^{-1}(y))) = \phi^{-1}(\sigma(\tilde{g}, h_1^n (y)))$, for every $y \in Y$ and $\tilde{g} \in G_1$

$$= \phi^{-1}(h_2^n (\sigma(\tilde{g}, y)))$$

Thus
\[(\varphi \varphi^{-1}(h_2^n(\sigma(\bar{g}, \bar{y}_2))), \varphi \varphi^{-1}(h_2^n(\sigma(\bar{g}, \bar{y}_2)))) \in U.\]

This means that \((h_2^n(\sigma(\bar{g}, \bar{y}_2)), h_2^n(\sigma(\bar{g}, \bar{y}_2))) \in U.\) Consequently \(h_1\) is uniform \(G\) -equicontinuous.

**Theorem 3.7.** Suppose \(X\) and \(Y\) be uniform spaces and \(h_1:X \to X\), \(h_1:Y \to Y\) be equivariant topologically conjugate by means of \(\varphi:X \to Y\). In the event that \(h_1\) is uniform equicontinuous, at that point so is \(h_2\).

**Proof.** Let \(h_1\) is uniform equicontinuous. Let \(V \in \mathcal{V}\). Since \(\varphi\) is uniform isomorphism, at that point there exists an escort \(U \in \mathcal{U}\) with the end goal that

\[(x_1, x_2) \in U \quad \text{implies} \quad (\varphi(x_1), \varphi(x_2)) \in V. \quad (1)\]

Since \(h_1:X \to X\) is uniform equicontinuous, at that point there exists an escort \(\bar{U} \in \mathcal{U}\) with the end goal that

\[(\bar{x}_1, \bar{x}_2) \in \bar{U} \quad \text{implies} \quad (h_1^n(\bar{x}_1), h_1^n(\bar{x}_2)) \in U. \quad (2)\]

Since \(\varphi^{-1}:Y \to X\) is uniform continuous, subsist \(\bar{V} \in \mathcal{V}\) same that

\[(\bar{y}_1, \bar{y}_2) \in \bar{V} \quad \text{suggest} \quad (\varphi^{-1}(\bar{y}_1), \varphi^{-1}(\bar{y}_2)) \in \bar{U}. \quad (3)\]

By (2) we have

\[(h_1^n(\varphi^{-1}(\bar{y}_1)), h_1^n(\varphi^{-1}(\bar{y}_2))) \in U.\]

By (1) we have

\[(\varphi h_1^n(\varphi^{-1}(\bar{y}_1)), \varphi h_1^n(\varphi^{-1}(\bar{y}_2))) \in U.\]

Since \(h_1, h_2\) be equivariant topologically conjugate via \(\varphi\), then

\[h_1^n(\varphi^{-1}(y)) = \varphi^{-1}(h_2^n(y)), \text{ for every } y \in Y.\]

Thus

\[(\varphi \varphi^{-1}(h_2^n(\bar{y}_1)), \varphi \varphi^{-1}(h_2^n(\bar{y}_2))) \in U.\]

This means that \((h_2^n(\bar{y}_1), h_2^n(\bar{y}_2)) \in U.\) Consequently \(h_1\) is uniform equicontinuous.

Here the relation between the \((G-\) equicontinuous and \((G-\) expansive is studied in uniform space.

First we shall introduce the concepts of expansive and \(G\)- expansive in uniform space.
Definition 3.8 In the event that \((X,U)\) is a "uniform space" and \(h \in H(X)\) at that point \(h\) is called far reaching, on the off chance that there exists an escort \(U \in U\) with the end goal that at whatever point \(x,y \in X, x \neq y\), at that point found a whole number \(n\) fulfilling
\[
(h^n(x), h^n(y)) \notin U;
\]
\(U\) is then named a far reaching escort for \(h\).

Definition 3.9 Suppose \((X,U)\) be a uniform space and \(h \in H(X)\) at that point \(h\) is called uniform \(G\)-far reaching, in the event that there exists an escort \(U \in U\) with the end goal that at whatever point \(x,y \in X, G(x) \neq G(y)\) at that point found a number \(n\) fulfilling
\[
(h^n(u), h^n(v)) \notin U, \text{ for all } u \in G(x) \text{ and } v \in G(y).
\]

Theorem 3.10 Put \((X,U)\) be a uniform space and \(f \in H(X)\). On the off chance that \(f\) is equicontinuous map, at that point its sweeping.

Proof. Assume that \(f\) is uniform equicontinuous. Let \(x,y \in X\) with \(x \neq y\). Let \(V \in U\) be a non-symmetric escort. By speculation there exists an escort \(U \in U\) with the end goal that
\[
(x,y) \in U \quad \text{implies} \quad (f^n(x), f^n(y)) \in V, \text{ for every integer } n.
\]
while \(V\) is non- symmetric and \(V^{-1} \in \mathcal{U}\), then \((f^n(x), f^n(y)) \notin V^{-1}.
\]
This means that \(f\) is expansive.

Theorem 3.11 Leave \(X\) alone a uniform \(G\)-space and \(f \in H(X)\). In the event that \(f\) is \(G\)-equicontinuous map, at that point its \(G\)-extensive.

Proof. Assume that \(f\) is uniform \(G\)-equicontinuous. Let \(x,y \in X\) with \(G(x) \neq G(y)\). Let \(V \in U\) be a non-symmetric escort. By theory there exists an escort \(U \in U\) and \(g,p \in G\) to such an extent that
\[
(x,y) \in U \quad \text{implies} \quad (f^n(\theta(g,x)), f^n(\theta(q,y))) \in V, \text{ for every integer } n.
\]
While \(V\) is not symmetric and \(V^{-1} \in \mathcal{U}\), then for every integer \(n\)
\[
(f^n(\theta(g,x)), f^n(\theta(q,y))) \notin V^{-1} \quad (1)
\]
Let \(u \in G(x)\) and \(v \in G(y)\). Then there exist \(g,q \in G\) such that \(u = \theta(g,x), v = \theta(q,y)\). Thus we have
\[
(f^n(u), f^n(v)) \notin V^{-1}, \text{ for every integer } n.
\]
This means that $f$ is $G$-expansive. This completes the proof.

**Definition 3.12** A uniform $G$—we can state **distal** whether, for every pair space $x, y \in X$ with $x \neq y$, the closure of the set $\{(\theta(g, x), \theta(g, y)) : g \in G\}$ is disjoint from the diagonal $\Delta = \{(x, x) : x \in X\}$ in $X \times X$.

**Theorem 3.13** If $(G, X, \theta)$ is equicontinuous, then it is distal.

**Proof** Let $x, y \in X$ with $x \neq y$. Then found an index $\beta$ on $X$ with $(x, y) \notin \beta$. By equicontinuity found an index $\alpha$ like that $(u, v) \in \alpha$ implies

$$(\theta(h, x), \theta(h, y)) \in \beta \text{ for all } h \in G.$$  

It pursue that

$$(\theta(g, x), \theta(g, y)) \notin \alpha \text{ for all } g \in G.$$  

Otherwise, we could let $u = \theta(g, x), v = \theta(g, y), h = g^{-1}$, and reach a contradiction. Thus $\{(\theta(g, x), \theta(g, y)) : g \in G\}$ is disjoint from the diagonal $\Delta = \{(x, x) : x \in X\}$ in $X \times X$. Since $\Delta \subseteq \alpha$, and $\alpha$ is open in the product topology, it follows that $(G, X, \theta)$ is distal. $\blacksquare$

**Theorem 2.9.** Let $, Y$ be $G$—spaces and $f_1: X \to X$, $f_2: Y \to Y$ be maps. Then $f_1 \times f_2: X \times Y \to X \times Y$ is uniform $G_1 \times G_2$—equicontinuous iff $f_1$ is uniform $G_1$—equicontinuous and $f_2$ is uniform $G_2$—equicontinuous.

**Proof.** Assume that $f = f_1 \times f_2$ is a uniform $G_1 \times G_2$—equicontinuous on $X \times Y$. We will show that $f_1$ is uniform $G_1$—equicontinuous on $X$ and correspondingly we can show that $f_2$ is $G_2$—equaicontinuous on $Y$. Let $V \subseteq U_X$ and $n$ be a positive whole number. Since $Y \times Y \subseteq U_Y$ at that point

$$V \times (Y \times Y) = W \subseteq U_{X \times Y}.$$  

By hypothesis, found $U \subseteq U_{X \times Y}$ like that if $(x, y) \in U$, after that

$$(f^n(\theta(g, x), f^n(\theta(p, y))) \in W = V \times (Y \times Y), g, p \in G = G_1 \times G_2$$  

Since $(x, y) \in U$, then found $U_1 \subseteq U_X$ and $U_2 \subseteq U_X$ like that

$$x = (x_1, x_2) \in U_1 \text{ and } y = (y_1, y_2) \in U_2.$$  

But $f^n(\theta(g, x)) = (f_1^n(\theta_1(g_1, x_1)), f_2^n(\theta_1(p_1, x_2))) \in V$  

This means that $f_1$ is uniform $G_1$—equicontinuous. Conversely, suppose that $f_1$ is uniform $G_1$—equicontinuous and $f_2$ is uniform $G_2$—equaicontinuous. Let $W \subseteq U_{X \times Y}$. Then there exist $W_1 \subseteq U_X$ and $W_2 \subseteq U_Y$ such that $= W_1 \times W_2$. By hypothesis, there exist $U_1 \subseteq U_X$ and $U_2 \subseteq U_Y$ like that if $(x, x') \in U_1$ and $(y, y') \in U_2$, after that

$$(f^n_1(\theta_1(g, x), f^n_1(\theta_1(g', x'))) \in W_1$$  

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and

\[(f_2^n(\theta_2(p, y), f_2^n(\theta_2(p', y')) \in W_2\]

for all \((g, g') \in G_1\) and \((p, p') \in G_2\) Set \(U_1 \times U_2 = U\). Then \(U \in \mathcal{U}_{XY}\). Thus we have

\[(f^n(\theta(g, x), f^n(\theta(p, y)) \in W \quad \forall g \in G_1 \times G_2.\]

This means that \(f_1 \times f_2 : X \times Y \to X \times Y\) is uniform \(G_1 \times G_2\)-equicontinuous. This complete the proof.

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