Recovery of damaged information and the out-of-time-ordered correlators

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A time-reversed dynamics unwinds information scrambling, which is induced during the time-forward evolution with a complex Hamiltonian. We show that if the scrambled information is, in addition, partially damaged by a local measurement, then such a damage can still be treated by application of the time-reversed protocol. This information recovery is described by the long-time saturation value of a certain out-of-time-ordered correlator of local variables. We also propose a simple test that distinguishes between quantum and reversible classical chaotic information scrambling.

In complex strongly correlated systems, local information spreads quickly over the whole system, hindering the scrambled information from local measurements. This process is characterized by exponentially fast changes of the out-of-time-ordered correlators (OTOCs) [1–4], such as the following correlator of local operators $W$ and $V$ with specific time-ordering:

$$F(t) = \langle W^\dagger(t)V^\dagger(0)W(t)V(0) \rangle.$$  \hspace{1cm} (1)

Recovering the scrambled information is a problem of considerable interest [5–7]. Scrambling can encode information into global entanglement, and protect it against local perturbations. For instance, a single qubit thrown into a black hole is quickly dispersed and lost behind the horizon. With the resource of early Hawking radiation, only a few qubits of information emitted from the black hole are needed to reconstruct the lost qubit [8]. However, there is usually no simple recipe for how to recover the desired information.

Here, we consider a more accessible scenario for information scrambling and unscrambling. Let us describe it as a hypothetical application of a quantum processor, such as the one in the quantum supremacy test [9], for hiding quantum information. Our processor can be simpler than that in [9] because we will assume that only the state of one of the qubits can be prepared and measured, which is suitable for experiments with liquid-NMR quantum computers [10, 11].

Let Alice have such a processor that implements a reversible but pseudochaotic evolution of many interacting qubits. She uses it to hide an original state of one of her qubits, which we will call the central qubit. In order to recover the initial state of this qubit, Alice can apply a time-reversed protocol.

Let Bob be an intruder who can measure the state of the central qubit in any basis unknown to Alice, as shown in Fig. 1. If her processor has already scrambled the information, Alice is sure that Bob cannot get anything useful. However, Bob’s measurement changes the state of the central qubit and also destroys all quantum correlations between this qubit and the rest of the system. According to the no-hiding theorem [12], the useful information is still in the bath, but Alice does not have knowledge of the bath state at any time. How can she recover the useful information in this case?

In this Letter, we show that even after Bob’s measurement, Alice can still recover her information by application of the time-reversed protocol. This effect is enabled by the no-hiding theorem and hence distinguishes quantum information scrambling from the scrambling that could be achieved using classical chaotic evolution. In addition, we reveal the importance of the long-time OTOC saturation values for quantifying such effects, suggesting a new domain for the OTOC applications.

To explore the game played by Alice and Bob, we consider the control sequence in Fig. 1 applied to a system of qubits (spins-1/2s). The system starts from an initial product state, $\rho_0 = |i\rangle\langle i| \otimes \rho_B$. Here, the central qubit state $|i\rangle$ encodes the information to be scrambled and recovered, whereas the state $\rho_B$ of the bath qubits can be arbitrary.

After a unitary evolution during time $t_1$, a projective measurement along a random axis is applied to the central spin, without collecting any data from the measurement outcome. Then, the system evolves backward in time during $t_2$, followed by a state tomography for the central spin. We claim that when $t_2 = t_1$, the final mea-
measurements contain information that can fully reconstruct the initial state of the central qubit.

Two remarks are in order. First, we assume that the unitary evolution is complex enough to scramble the information. In other words, at long times, the reduced density matrix of the central qubit becomes maximally mixed. For instance, the unitary can be drawn from an ensemble of random few-qubit unitaries. Without a large spin bath the described effect is suppressed. Second, we recover the initial information at the time $t_2 = t_1$, i.e., right after the backward unitary becomes conjugated to the forward unitary but it will be useful to explore what happens for $t_2 \neq t_1$.

Let, for the central qubit, $\hat{P}_r$ be the projection operator for Bob’s measurement. There are two complementary histories in which, after Bob’s measurement, the central spin state is projected to the subspace described by either $\hat{P}_r$ or $\hat{1} - \hat{P}_r$. The probability of a nonzero result for Alice’s projective measurement $\hat{P}_f$ at the final time moment is

$$\text{Prob}(\hat{P}_f) = \int [d\tau] \text{Prob}(\hat{P}_f, \hat{P}_r) + \text{Prob}(\hat{P}_f, \hat{1} - \hat{P}_r),$$

where $\text{Prob}(\hat{P}_f, \hat{P}_r)$ is the joint probability that both Alice and Bob find unit measurement outcomes for their measurement operators. Note that, since we do not collect results of the Bob’s intermediate measurements, his complementary measurement outcome contributes to $\text{Prob}(\hat{P}_f)$ as well. The integral over $r$ in (2) accounts for averaging over an arbitrary distribution of possible directions for Bob’s measurement axes.

Denote $\hat{U}(t_1)$ the evolution operator for the time-forward protocol during time $t_1$ and $\hat{U}^\dagger(t_2)$ the evolution operator for the time-reversed protocol during time $t_2$. The probability of the nonzero outcome for the intermediate measurement $\hat{P}_r$ and the corresponding post measurement state $\rho_r$, is then given by

$$\text{Prob}(\hat{P}_r) = \text{Tr} \rho_r \rho(r(t_1) \hat{P}_r / \text{Prob}(\hat{P}_r)),$$

where $\rho(r(t_1) = \hat{U}(t_1) \rho_0 \hat{U}^\dagger(t_1)$ is the system state at time $t_1$. The probability for the final measurement $\hat{P}_f$, conditioning on the system being projected to the post measurement state $\rho_r$, is

$$\text{Prob}(\hat{P}_f | \hat{P}_r) = \text{Tr} \big( \hat{P}_f \hat{U}(t_2) \rho_r \hat{U}^\dagger(t_2) \hat{P}_f \big).$$

This gives the desired joint probability $\text{Prob}(\hat{P}_f, \hat{P}_r) = \text{Prob}(\hat{P}_f / \hat{P}_r) \text{Prob}(\hat{P}_r)$. Since the system is initially in the state $\rho_0 = |\cdot\rangle \langle \cdot| \otimes \rho_B$, the joint probability can be expressed in a compact form:

$$\text{Prob}(\hat{P}_f, \hat{P}_r) = \langle \hat{P}_r(t_1) \hat{P}_f(t_1 - t_2) \hat{P}_r(t_1) \hat{P}_f \rangle.$$  \hspace{1cm} (5)

Here, the ensemble average is defined as $\langle \cdot \rangle \equiv \text{Tr}(\cdot \hat{1} \otimes \rho_B)$. Equation (5) shows that the effect of Bob’s interference on the information that Alice obtains after applying the time-reversed protocol is described by a two-time OTOC of projection operators.

The fact that it is the projection rather than the Pauli operator that describes Bob’s interference in (5) is important. However, it is useful now to express this OTOC in terms of Pauli operators, using the identity $\hat{\sigma}_\phi = 2\hat{P}_\phi - \hat{1}$.

We are interested in long scrambling times. The second order spin correlators, such as $\langle \hat{\sigma}_f(t_1) \hat{\sigma}_i \rangle$, decay quickly with time. Hence, we can safely neglect all such correlators except the ones that depend on $t_1 - t_2 \ll t_1$. The joint probability is then

$$\text{Prob}(\hat{P}_f, \hat{P}_r) = \frac{1}{4} + \frac{1}{16} \langle \hat{\sigma}_f(t_1 - t_2) \hat{\sigma}_i \rangle + \frac{1}{16} \langle \hat{\sigma}_r(t_1) \hat{\sigma}_f(t_1 - t_2) \hat{\sigma}_r(t_1) \hat{\sigma}_i \rangle.$$  \hspace{1cm} (6)

For $t_1 = t_2$, the second term on the right hand side in (6) is independent of the evolution unitary. All such details are hidden in the third term. At $t_1 = t_2$, this four-point correlator becomes a standard spin OTOC, i.e.,

$$F(t) = \langle \hat{\sigma}_r(t) \hat{\sigma}_r(t) \hat{\sigma}_f \rangle.$$  \hspace{1cm} (7)

For finite $t$ and for a small bath this correlator has a nontrivial system-specific behavior that obscures the contribution of $\langle \hat{\sigma}_f(t_1 - t_2) \hat{\sigma}_i \rangle$ in (6). However, we are interested in the typical complex unitary evolution that scrambles information. Hence, we claim that this correlator saturates to a universal value that is described by its average over an ensemble of random unitaries. This average can be evaluated as an integral over all unitaries with respect to the Haar measure [13, 14], i.e.,

$$\bar{F} = \int dU \text{ Tr} \left[ U^\dagger \sigma_i U \sigma_f U^\dagger \sigma_f \rho_B \right],$$

where $\rho_B$ is the initial state of bath qubits. The integral can be further calculated using the identity for Haar unitaries [6]:

$$\begin{align*}
U_{m_1 n_1} U^*_{m_1' n_1'} U_{m_2 n_2} U^*_{m_2' n_2'} \\
\delta_{m_1, m_1'} \delta_{m_2, m_2'} \delta_{n_1, n_1'} \delta_{n_2, n_2'} + \delta_{m_1, m_2} \delta_{m_1', m_2'} \delta_{n_1, n_1'} \delta_{n_2, n_2'} \\
+ \delta_{m_2, m_1} \delta_{m_2', m_1'} \delta_{n_2, n_1'} \delta_{n_2, n_2'} \\
= \frac{N^2 - 1}{N (N^2 - 1)},
\end{align*}$$

where $N$ is the dimension of the Hilbert space. After summing over all indices and using a trivial identity $\text{tr}(\hat{\sigma}_{f,i,r}) = 0$, the average reduces to

$$\bar{F} = \langle \sigma_i \sigma_f \rangle / (N^2 - 1) \equiv \text{tr}(\sigma_i \sigma_f \otimes \rho_B) / (N^2 - 1).$$

We need this formula only for $N \gg 1$ because then a single typical unitary produces the effect that coincides with the average of the OTOC. The large denominator, for $N \to \infty$, makes this OTOC decay to zero. Since this happens for random unitary evolution, the same is true for sufficiently long scrambling times and sufficiently large baths, so the fourth order correlator in Eq. (6) is also vanishing.
After averaging over random unitaries, the joint probability \( \text{Prob}(\hat{P}_f, \hat{P}_r) \) is the same as \( \text{Prob}(\hat{P}_f, -\hat{P}_r) \), so the final probability is twice of \( \text{Prob}(\hat{P}_f, \hat{P}_r) \) for any distribution of Bob’s measurement axes. Equation (6) reduces then to

\[
\text{Prob}(\hat{P}_f) = \frac{1}{2} + \frac{1}{8} \langle \hat{\sigma}_f(t_1 - t_2) \hat{\sigma}_f \rangle. \tag{11}
\]

This is the main result of our work. It shows that Alice’s measurement probability at \( t_2 = t_1 \) depends only on the initial state of the central qubit. In addition, Eq. (11) shows that this information returns to the central qubit during a short central spin lifetime, as it is described by a second order spin correlator.

Thus an echo signal shows up in Alice’s measurement probability near \( t_2 = t_1 \). The full information about the initial state, i.e., \( \hat{\sigma}_f \), can be reconstructed by detecting this echo.

Finally, we note that the central qubit has not been reversed to a completely separable state – this qubit ends up in a partially mixed state entangled with the bath. What has been unscrambled is the initial state information, which can be fully extracted either by repeating the experiment or by working with a relatively small ensemble of our system with the same initial state of the central qubit but not necessarily of the bath and scrambling unitaries.

**The model of a nuclear spin bath.** In order to verify that the Haar random unitary provides correct description of the universal behavior of OTOCs in realistic systems, we studied a spin-bath model numerically. This model has been frequently used to describe interactions of a solid state qubit with nuclear spins [15, 16]. Its Hamiltonian is

\[
H = \sum_{i=1}^{N_s} \sum_{\alpha = \{x,y,z\}} J_i^{\alpha} S_i^{\alpha} s_i^{\alpha}, \quad \alpha = x,y,z, \tag{12}
\]

where the couplings \( J_i^{\alpha} \) are independent Gaussian distributed random numbers with zero mean and variance \( J \). Here, the central spin-1/2, \( S \), interacts with the bath of \( N_s \) spins-1/2’s, \( s_i \). We simulated the Schrödinger equation with this Hamiltonian numerically to obtain the effect of the unitary evolution.

In our simulations, the central spin starts from the \(| \uparrow \rangle \) state, i.e., \( \sigma_i = \sigma_+ \) in Eq. (11). The bath spins are prepared in a maximally mixed state. This corresponds to a common situation of practically infinite temperature for nuclear spins. Figure 2 (top left) shows the obtained final Alice’s probability of a nonzero measurement result for \( \sigma_f = \sigma_+ \) and a fixed randomly chosen \( \sigma_r \). It does show the echo effect along \( t_1 = t_2 \) line, in perfect quantitative agreement with Eq. (11). No additional averaging over Bob’s measurement axes and parameters \( J_i^{\alpha} \) is needed for quantitative agreement with Eq. (11). We also verified numerically for \( N_s = 10 \) (not shown) that the same universal result is obtained for several other choices of the initial density matrix of the spin bath.

It is instructive to compare the echo in OTOC with another type of spin echo, which is induced by a similar protocol, in which the backward evolution, \( \hat{U}^{-1}(t_2) \), is replaced by the forward one, \( \hat{U}(t_2) \). The joint probability is then described by the correlator (5) with \( t_2 \) replaced by \( -t_2 \), i.e.,

\[
\text{Prob}_2(\hat{P}_f, \hat{P}_r) = \langle \hat{P}_f(t_1 + t_2) \hat{P}_r(t_1) \hat{P}_f \rangle. \tag{13}
\]

This correlator can be measured in solid state systems by standard means, for example, it was studied experimentally in semiconductor quantum dots [17] (see also Ref. [18]). As we show in Figure 2 (top right), the final probability (13) also shows an echo effect near \( t_2 = t_1 \). However, this echo originates from a finite scrambling rate and therefore decays at large times, in sharp contrast to the echo in the OTOC.

**Quantum vs classical scrambling.** Our observation can be used to additionally validate quantum supremacy tests such as in [9]. There is only a finite depth for precise simulations of classical chaotic motion with classical digital computers because of the exponential increase of round-off errors [19]. Therefore, a classical analog computer with nonlinear interactions between its components can evolve reversibly to a state that cannot be predicted with classical digital computers. Hence, to validate quantum supremacy, an additional test may be needed to prove that we deal with a quantum scrambled state at the end rather than with a state generated by classical chaos [20].

In the classical case, a small change of the state vector at the end of the forward protocol would be quickly magnified during the backward time evolution. Thus, the state at the end of our control protocol would be strongly different from the initial one.

**FIG. 2.** Top: The left and right panels correspond to the joint measurement probabilities (5) and (13), respectively, in the model of a quantum spin bath (12). Bottom: Dynamics of Z-component of the central spin vector in a model (12) with all classical spins. The green bar marks time \( t_1 \) with invasive measurement of the central spin. Black dashed curve and blue solid curve correspond to the cases with and without intermediate invasive measurement, respectively. \( N_s = 10 \) and \( N_s = 30 \) for quantum and classical simulations, respectively.
To illustrate this, we simulated the evolution of interacting classical spins with the same Hamiltonian (12) subject to the classical Landau-Lifshitz equations [21]. The state of a classical spin is specified by a three-dimensional unit vector. Our initial state for the central spin is at \( |i \rangle \) of the target qubit to the maximally mixed state, provided that the input states of the bath qubits are \(+| = (|0\rangle + |1\rangle)/\sqrt{2}\). This unitary is not a typical Haar random unitary, since its scrambling effect depends on the bath initial state. Consequently, the four point correlator at \( t_2 = t_1 \) in Eq. (6), i.e., \( \langle \hat{U} \hat{\sigma}_x \hat{U}^\dagger \hat{\sigma}_y \hat{U} \hat{\sigma}_x \hat{U}^\dagger \hat{\sigma}_z \rangle \), is not zero, even after averaging \( \hat{\sigma}_r \) over random Pauli matrices (corresponding to random intermediate projective measurements). Instead, it makes the above four point correlator vanish over \( \hat{\sigma}_z \) is averaged over the Pauli group. Thus, if we include the identity operator as part of the intermediate measurement, the rest of the protocol and the final result shall not change.

The central qubit starts at \(|0\rangle\). We perform the final measurements along three orthogonal directions, corresponding to \(|0\rangle\), \((|0\rangle + |1\rangle)/\sqrt{2}\), and \((|0\rangle + i|1\rangle)/\sqrt{2}\). The theoretical probabilities for these final measurements are 0.75, 0.5, and 0.5, respectively. Figure (3) shows the statistics of the final measurements, obtained with the IBM 5-qubit processor, from which we inferred the initial state \( 0.992|0\rangle - (0.082 - 0.101i)|1\rangle \). This corresponds to 0.983 fidelity, which means that natural decoherence was not detrimental.

In conclusion, OTOCs are unusual correlators that have no counterpart in classical physics. We showed that such correlators quantify not only the information scrambling rate but also the purely quantum effect of damaged information recovery. This effect must be testable using experimental capabilities for the generation of random unitary matrices.

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